Imperial College London

**MATH97040 MATH97149**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

# Measure and Integration

Date: Tuesday, 18 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. (a) We endow the set $\mathbb{N} = \{1, 2, \dots\}$ of strictly positive integers with the discrete $\sigma$-algebra $2^{\mathbb{N}}$. We define $\mu : 2^{\mathbb{N}} \to [0, +\infty]$ by setting, for all $A \subset \mathbb{N}$,

$$\mu(A) = \sum_{n \in A} \frac{1}{n^2}$$

if $A$ is finite, and $\mu(A) = +\infty$ if $A$ is infinite. Does $\mu$ define a measure on $(\mathbb{N}, 2^{\mathbb{N}})$? Carefully justify your answer. (6 marks)

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be the non-decreasing and right-continuous function defined by $F(x) = 2x^3 + 3\mathbf{1}_{[0,+\infty)}(x)$ for all $x \in \mathbb{R}$, and let $\mu_F$ be the corresponding Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(i) Compute $\mu_F(\{0\})$, $\mu_F(\{1\})$, $\mu_F((-1, 0))$ and $\mu_F((0, 1])$. (8 marks)

(ii) Is $\mu_F$ a finite measure? (2 marks)

(iii) Is $\mu_F$ absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$? (4 marks)

(Total: 20 marks)

2. (a) Are the following statements true or false? Prove your claim or give a counter-example:

(i) Denoting by $\lambda^2$ the Lebesgue measure on $([0,1]^2, \mathcal{B}([0,1]^2))$, if $f : [0,1]^2 \to \mathbb{R}$ is continuous, then $f$ is integrable with respect to $\lambda^2$. (4 marks)

(ii) If $(f_n)_{n\geq 1}$ is a sequence of functions in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ converging pointwise to a function $f : \mathbb{R} \to \mathbb{R}$, and if $\int |f_n| \, d\lambda \leq 1$ for all $n \geq 1$, then $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. (3 marks)

(iii) If $(f_n)_{n\geq 1}$ is a sequence of functions in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ converging pointwise to a function $f : \mathbb{R} \to \mathbb{R}$, and if $\int |f_n| \, d\lambda \leq 1$ for all $n \geq 1$, then $\int f_n \, d\lambda \underset{n\to\infty}{\longrightarrow} \int f \, d\lambda$. (3 marks)

(iv) If $(f_n)_{n\geq 1}$ is a non-increasing sequence of measurable real-valued functions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ converging pointwise to a function $f : \mathbb{R} \to \mathbb{R}$, then

$$\int_{\mathbb{R}} e^{-f_n(x)} \, d\lambda(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} e^{-f(x)} \, d\lambda(x).$$

(4 marks)

(b) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \mathbf{1}_{[0,1]}(x) + \cos(4\pi x), \qquad x \in \mathbb{R}.$$

Is $f$ measurable from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$? Justify your answer. (3 marks)

(ii) Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be a metric space endowed with its Borel $\sigma$-algebra. Let $f : \mathcal{X} \to \mathbb{R}$ be a function such that, for all $y \in \mathbb{R}$, $\{x \in \mathcal{X} : f(x) > y\}$ is an open subset of $\mathcal{X}$. Show that $f$ is Borel measurable.

(3 marks)

(Total: 20 marks)

3. (a) Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that $X_n \underset{n\to\infty}{\longrightarrow} 1$ a.s. Show that $\mathbb{P}(X_n \geq 0)$ converges as $n \to \infty$ to a limit that you shall determine.

(6 marks)

(b) Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of strictly positive integers. We endow $\mathbb{N}^2$ with the discrete $\sigma$-algebra $2^{\mathbb{N}^2}$, and recall that any map $f : \mathbb{N}^2 \to \mathbb{R}$ is measurable. We define the measure $\mu$ on $(\mathbb{N}^2, 2^{\mathbb{N}^2})$ by setting

$$\mu(A) = \sum_{(n,m)\in A} 2^{-n-m}, \qquad A \subset \mathbb{N}^2.$$

**We shall admit that $\mu$ is indeed a measure**.

(i) Explain whether the following statements are true or false (carefully justify your answer):
- $\mu$ is a finite measure,
- $\mu$ is a probability measure,
- $\mu$ is a $\sigma$-finite measure.

(6 marks)

(ii) For $f : \mathbb{N}^2 \to [0, +\infty)$, give an expression for $\int_{\mathbb{N}^2} f \, d\mu$ in terms of the sum of an infinite series. Carefully justify your answer. (4 marks)

(iii) We define $f : \mathbb{N}^2 \to [0, +\infty)$ by setting

$$f(n, m) = (-1)^{n+m} \, m, \qquad (n, m) \in \mathbb{N}^2.$$

For which values of $p \in [1, +\infty)$ do we have $f \in \mathcal{L}^p(\mathbb{N}^2, 2^{\mathbb{N}^2}, \mu)$? (4 marks)

(Total: 20 marks)

4. (a) Let $X$ be a strictly positive, real-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We recall that $\int_0^{+\infty} e^{-t}\, t^{-1/2}\, dt = \sqrt{\pi}$.

(i) Show that, for all $t \geq 0$, $f(t) := \mathbb{E}(e^{-tX})$ is a well-defined, finite number. Carefully justify your answer. (5 marks)

(ii) We admit that the map $F : \Omega \times (0, +\infty) \to \mathbb{R}$ defined by

$$F(\omega, t) = e^{-t\, X(\omega)}\, t^{-1/2}, \qquad (\omega, t) \in \Omega \times (0, +\infty),$$

is measurable from the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}((0, +\infty))$ to $\mathcal{B}(\mathbb{R})$. By computing $\int_{\Omega \times (0,+\infty)} F\, d(\mathbb{P} \times \lambda)$ in two different ways, establish the relation

$$\mathbb{E}(X^{-1/2}) = \frac{1}{\sqrt{\pi}} \int_{(0,+\infty)} f(t)\, t^{-1/2}\, dt.$$

(6 marks)

(b) Let $X$ be a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the probability distribution $P_X$ of $X$ has a density $f$ with respect to the Lebesgue measure, i.e. $dP_X = f\, d\lambda$, where

$$f(x) = \frac{1}{2} e^{-|x|}, \qquad x \in \mathbb{R}.$$

(i) Show that the random variable $X$ is integrable with respect to $\mathbb{P}$. (3 marks)

(ii) Let $Y : \Omega \to \mathbb{Z}$ be the random variable defined by $Y = \lfloor X \rfloor$, that is for all $\omega \in \Omega$, $Y(\omega) = n$, where $n$ is the unique integer such that $n \leq X(\omega) < n + 1$. Recall briefly why there exists a map $\varphi : \mathbb{Z} \to \mathbb{R}$ such that $\mathbb{E}(X|Y) = \varphi(Y)$, and compute $\varphi(n)$ for all $n \in \mathbb{Z}$ (you may leave the answer as a quotient of integrals). (6 marks)

(Total: 20 marks)

5. (a) Let $N \geq 1$ be an integer. We consider the set $\mathcal{X}$ of integers $k$ such that $1 \leq k \leq 2N$. We endow $\mathcal{X}$ with the discrete $\sigma$-algebra $2^{\mathcal{X}}$, the counting measure $\mu$ on $(\mathcal{X}, 2^{\mathcal{X}})$, and the transformation $T : \mathcal{X} \to \mathcal{X}$ given by $T(x) = x + 2 \bmod 2N$, for $x \in \mathcal{X}$. We admit that $\mu$ is invariant for $T$. Is it also ergodic? Carefully justify your answer. (4 marks)

(b) Let $\mathcal{X}$ be a non-empty metric space and $\mathcal{B}(\mathcal{X})$ its Borel $\sigma$-algebra. We endow $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with a finite measure $\mu$ and a measure-preserving transformation $T : \mathcal{X} \to \mathcal{X}$. For all $n \geq 1$, we shall denote by $T^n : \mathcal{X} \to \mathcal{X}$ the map $T$ composed $n$ times with itself (thus $T^1 = T$, $T^2 = T \circ T$, etc.). We assume that $\mu$ has full support, i.e. that, for all non-empty open subset $U$, $\mu(U) > 0$.

Let $U$ be a non-empty open subset. Let $Y$ be the set of points $x \in \mathcal{X}$ entering $U$ infinitely often under the action of $T$. That is,

$$Y = \{x \in \mathcal{X} : \forall m \geq 1, \exists n \geq m, T^n(x) \in U\}.$$

(i) Show that $Y \in \mathcal{B}(\mathcal{X})$. (2 marks)

(ii) Show that $\mu(Y) > 0$. (6 marks)

(iii) We further assume that $\mu$ is ergodic for $T$. Show that $Y$ is dense in $\mathcal{X}$. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

# MATH96031/MATH97040/MATH97149

# Measure and Integration (Solutions)

| Setter's signature | Checker's signature | Editor's signature |
|---|---|---|
| . . . . . . . . . . . . . . . . | . . . . . . . . . . . . . . . . | . . . . . . . . . . . . . . . . |

1. (a) $\mu$ is not a measure. Indeed, the measurable subsets $\{n\}$, for $n \in \mathbb{N}$, are disjoint and their union is the whole set $\mathbb{N}$, however $\boxed{\text{sim. seen} \Downarrow}$

$$\sum_{n\in\mathbb{N}} \mu(\{n\}) = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < +\infty = \mu(\mathbb{N}),$$

so $\mu$ is not $\sigma$-additive. $\boxed{6, \mathrm{B}}$

(b) (i) We have

$$\mu_F(\{0\}) = F(0) - F(0-) = 3 - 0 = 3.$$

$F$ is continuous at $1$, so $\mu_F(\{1\}) = 0$. Moreover,

$$\mu_F((-1,0)) = F(0-) - F(-1) = 0 - (-2) = 2$$

while

$$\mu_F((0,1]) = F(1) - F(0) = 5 - 3 = 2.$$

$\boxed{8, \mathrm{A}}$

(ii) $\mu_F$ is not a finite measure, as

$$\mu_F(\mathbb{R}) = F(+\infty) - F(-\infty) = +\infty - (-\infty) = \infty.$$

$\boxed{2, \mathrm{A}}$

(iii) $\mu_F$ is not absolutely continuous with respect to the Lebesgue measure $\lambda$, as $\lambda(\{0\}) = 0$ but $\mu_F(\{0\}) > 0$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{4, \mathrm{A}}$

2. (a) (i) True. Since $f$ is continuous $[0,1]^2$, it is Borel measurable. Since $[0,1]^2$ is compact, $f$ is also bounded by some constant $M < \infty$, so

sim. seen $\Downarrow$

$$\int_{[0,1]^2} |f| \, d\lambda^2 \leq \int_{[0,1]^2} M \, d\lambda^2 = M < \infty.$$

Hence $f$ is integrable with respect to $\lambda^2$.

4, A

(ii) True. Indeed, since the sequence of non-negative, measurable functions $(|f_n|)_{n\geq 1}$ converges pointwise to $f$, by Fatou's Lemma,

$$\int |f| \, d\lambda = \int (\liminf_n |f_n|) \, d\lambda \leq \liminf_n \int |f_n| \, d\lambda \leq 1.$$

In particular, $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

3, B

(iii) False. Consider $f_n = \mathbf{1}_{[n,n+1]}$ for $n \geq 1$. Then $f_n$ converges pointwise to $0$, $\int |f_n| \, d\lambda = 1$ for all $n \geq 1$, however $\int f_n \, d\lambda = 1$ does not converge to $\int 0 \, d\lambda = 0$.

3, B

(iv) True. Indeed, by the assumptions, the non-decreasing sequence of non-negative measurable functions $e^{-f_n}$ converges pointwise to $e^{-f}$, so the conclusion follows by the monotone convergence theorem.

unseen $\Downarrow$

4, D

(b) (i) $f$ is measurable from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Indeed, the function $x \mapsto \cos(4\pi x)$ is continuous hence Borel measurable on $\mathbb{R}$. Moreover $[0,1]$ is a Borel subset of $\mathbb{R}$, so $\mathbf{1}_{[0,1]}$ is a Borel measurable function on $\mathbb{R}$. Hence $f$ is Borel measurable as a sum of Borel measurable functions.

sim. seen $\Downarrow$

3, A

(ii) Let $\mathcal{C} = \{(-\infty, y], y \in \mathbb{R}\}$. Then by assumption, for all $A \in \mathcal{C}$, $f^{-1}(A^c)$ is an open, hence Borel subset of $\mathcal{X}$, so $f^{-1}(A) = f^{-1}(A^c)^c \in \mathcal{B}(\mathcal{X})$. Since we know from lectures that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, the claim follows.

unseen $\Downarrow$

3, C

3. (a) Almost-surely, $X_n \underset{n\to\infty}{\longrightarrow} 1$ so that $\mathbf{1}_{\{X_n \geq 0\}} = 1$ for some $n$ large enough. In particular, $\mathbf{1}_{\{X_n\geq 0\}} \underset{n\to\infty}{\longrightarrow} 1$ a.s. Since, for all $n \geq 1$, we have $0 \leq \mathbf{1}_{\{X_n\geq 0\}} \leq 1$, and since $1$ is integrable with respect to $\mathbb{P}$, by the dominated convergence theorem

unseen ⇓

$$\mathbb{P}(X_n \geq 0) = \mathbb{E}(\mathbf{1}_{\{X_n\geq 0\}}) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(1) = 1.$$

6, D

(b) (i) We have

sim. seen ⇓

$$\mu(\mathbb{N}^2) = \sum_{(n,m)\in\mathbb{N}^2} 2^{-n-m} = \sum_{n\in\mathbb{N}} 2^{-n} \sum_{m\in\mathbb{N}} 2^{-m} = 1,$$

so that $\mu$ is a probability measure. In particular it is also a finite and a $\sigma$-finite measure.

6, A

(ii) For $p \geq 1$, let $f_p := \sum_{n,m\leq p} f(n,m)\,\mathbf{1}_{\{(n,m)\}}$. Note that $(f_p)_{p\geq 1}$ forms a non-decreasing sequence of non-negative simple functions converging pointwise to $f$ from below. Hence, by the monotone convergence theorem,

$$\int_{\mathbb{N}^2} f\,d\mu = \lim_{p\to\infty} \int_{\mathbb{N}^2} f_p\,d\mu = \lim_{p\to\infty} \sum_{n,m\leq p} f(n,m)\,\mu(\{(n,m)\}).$$

Since $\mu(\{(n,m)\}) = 2^{-n-m}$ for all $(n,m) \in \mathbb{N}^2$, we obtain

$$\int_{\mathbb{N}^2} f\,d\mu = \sum_{n,m\in\mathbb{N}} 2^{-n-m}\,f(n,m).$$

4, B

(iii) By (ii), for all $p \in [1,+\infty)$, we have

$$\int |f|^p\,d\mu = \sum_{(n,m)\in\mathbb{N}^2} 2^{-n-m}|f(n,m)|^p = \sum_{(n,m)\in\mathbb{N}^2} 2^{-n-m}m^p = \sum_{n\in\mathbb{N}} 2^{-n} \sum_{m\in\mathbb{N}} 2^{-m}\,m^p.$$

Now the sums $\sum_{n\in\mathbb{N}} 2^{-n}$ and $\sum_{m\in\mathbb{N}} 2^{-m}\,m^p$ both converge, so $\int |f|^p\,d\mu < \infty$. Hence $f \in \mathcal{L}^p(\mathbb{N}^2, 2^{\mathbb{N}^2}, \mu)$ for all $p \in [1,+\infty)$.

4, B

4. (a) (i) $X$ is a random variable and $x \mapsto e^{-tx}$ is continuous hence Borel measurable from $\mathbb{R}$ to $\mathbb{R}$, hence, by the composition rule, $e^{-tX}$ is a random variable. Moreover, we have $0 \le e^{-tX} \le 1$, so $e^{-tX}$ is integrable, so that $\mathbb{E}(e^{-tX})$ is a well-defined, finite number.

unseen $\Downarrow$

$\boxed{5, \mathrm{A}}$

(ii) Since $F$ is measurable with respect to the product $\sigma$-algebra, and takes non-negative values, by Fubini-Tonnelli

$$\int_\Omega \int_{(0,+\infty)} F(\omega,t)\,dt\,d\mathbb{P}(\omega) = \int_{\Omega\times(0,+\infty)} F\,d(\mathbb{P}\times\lambda) = \int_{(0,+\infty)} \int_\Omega F(\omega,t)\,d\mathbb{P}(\omega)\,dt.$$

Ther right-most term above is $\int_{(0,+\infty)} f(t)\,t^{-1/2}\,dt$. We now compute the left-most term. For all $\omega \in \Omega$,

$$\int_{(0,+\infty)} F(\omega,t)\,dt = \int_0^{+\infty} e^{-tX(\omega)}\,t^{-1/2}\,dt = X(\omega)^{-1/2} \int_0^{+\infty} e^{-s}\,s^{-1/2}\,ds = \sqrt{\pi}X(\omega)^{-1/2},$$

where we performed the change of variable $s = X(\omega)\,t$ in the middle equality. Therefore

$$\int_\Omega \int_{(0,+\infty)} F(\omega,t)\,dt\,d\mathbb{P}(\omega) = \sqrt{\pi} \int_\Omega X(\omega)^{-1/2}\,d\mathbb{P}(\omega) = \sqrt{\pi}\,\mathbb{E}(X^{-1/2}).$$

Hence $\sqrt{\pi}\,\mathbb{E}(X^{-1/2}) = \int_{(0,+\infty)} f(t)\,t^{-1/2}\,dt$, yielding the result.

$\boxed{6, \mathrm{C}}$

(b) (i) By the transfer formula, we have

$$\mathbb{E}(|X|) = \int_\mathbb{R} |x|\,dP_X(x) = \frac{1}{2}\int_\mathbb{R} |x|\,e^{-|x|}\,\lambda(dx).$$

Since $x \mapsto |x|\,e^{-|x|}$ is even, the latter can be rewritten as $\int_{\mathbb{R}_+} x\,e^{-x}\,\lambda(dx)$, and the last integral is finite as $x\,e^{-x}$ is bounded by $Ce^{-x/2}$ for some $C > 0$, hence integrable. Therefore $X$ is integrable with respect to $\mathbb{P}$.

$\boxed{3, \mathrm{C}}$

(ii) Since the random variable $\mathbb{E}(X|Y)$ is $\sigma(Y)$-measurable, by the factorisation lemma there exists $\varphi : \mathbb{Z} \to \mathbb{R}$ such that $\mathbb{E}(X|Y) = \varphi(Y)$. Note that, for all $n \in \mathbb{Z}$,

$$\mathbb{P}(Y = n) = \mathbb{P}(n \le X < n+1) = \frac{1}{2}\int_n^{n+1} e^{-|x|}\,\lambda(dx) > 0.$$

Now, for all $n \in \mathbb{Z}$, since the set $\{Y = n\}$ is $\sigma(Y)$-measurable, by the definition of the conditional expectation, we have

$$\mathbb{E}(X\,\mathbf{1}_{\{Y=n\}}) = \mathbb{E}(\varphi(Y)\,\mathbf{1}_{\{Y=n\}}) = \varphi(n)\,\mathbb{P}(Y = n),$$

whence

$$\varphi(n) = \frac{\mathbb{E}(X\,\mathbf{1}_{\{Y=n\}})}{\mathbb{P}(Y=n)} = \frac{\int_n^{n+1} x\,e^{-|x|}\,\lambda(dx)}{\int_n^{n+1} e^{-|x|}\,\lambda(dx)}.$$

$\boxed{6, \mathrm{D}}$

5. (a) $\mu$ is not ergodic for $T$. Indeed, the subset $A = \{2k,\, k = 1, \ldots, N\}$ is invariant under $T$. However, we have $\mu(\mathcal{X}) = 2N$, while $\mu(A) = N \notin \{0, 2N\}$, precluding ergodicity. $\boxed{4, \text{M}}$

(b) (i) We have $Y = \cap_{m\geq 1} \cup_{n\geq m} T^{-n}(U)$. Since $U$ is an open set, it is Borel measurable, hence $T^{-n}(U) \in \mathcal{B}(\mathcal{X})$ for all $n \geq 1$. Since $\mathcal{B}(\mathcal{X})$ is closed under countable unions and intersections, $Y \in \mathcal{B}(\mathcal{X})$. $\boxed{2, \text{M}}$

(ii) We have $\mu(Y) \geq \mu(U \cap Y)$. Now,

$$U \cap Y = \{x \in U : \forall m \geq 1, \exists n \geq m, T^n(x) \in U\},$$

and since $\mu$ is a finite measure, by Poincaré's recurrence theorem the latter set has measure $\mu(U)$. Therefore $\mu(Y) \geq \mu(U) > 0$ as $\mu$ has full support, yielding the result. $\boxed{6, \text{M}}$

(iii) First note that $Y$ is invariant. Indeed,

$$T^{-1}(Y) = \cap_{m\geq 1} \cup_{n\geq m} T^{-(n+1)}(U) = \cap_{m\geq 2} \cup_{n\geq m} T^{-n}(U).$$

But the sequence of subsets $(\cup_{n\geq m} T^{-n}(U))_{m\geq 1}$ is non-increasing, so we may start from $m = 1$ instead of $m = 2$ in the intersection above. Hence

$$T^{-1}(Y) = \cap_{m\geq 1} \cup_{n\geq m} T^{-n}(U) = Y.$$

By ergodicity of $\mu$, and since by 5)b)ii) $\mu(Y) > 0$, therefore $\mu(Y) = \mu(\mathcal{X})$. Since $\mu(\mathcal{X}) < \infty$, this implies $\mu(\mathcal{X} \setminus Y) = 0$. As a consequence, $\mathcal{X} \setminus Y$ has empty interior, otherwise it would contain a non-empty open set $V$, so we would have $\mu(\mathcal{X} \setminus Y) \geq \mu(V) > 0$ as $\mu$ has full support, which is a contradiction. Therefore $\mathcal{X} \setminus Y$ has empty interior, so $Y$ is dense in $\mathcal{X}$. $\boxed{8, \text{M}}$

**If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.**
**Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the  Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.**

| ExamModuleCode | QuestionNumber | Comments for Students |
|---|---|---|
| MATH97040/97149 & MATH96031 | 1 | Very good/excellent answers. |
| MATH97040/97149 & MATH96031 | 2 | Overall good/very good answers. A common mistake in a)i) was to bound the integral of f (instead of \|f\|) to prove integrability.  Often use of Fatou's lemma or MCT were not sufficiently detailed or justified. |
| MATH97040/97149 & MATH96031 | 3 | The computational questions were done well, but often justifications were missing, e.g. in b)ii).  In b)iii) there was typo in the definition of the range of the function f, which takes real values (rather than non-negative ones). As a consequence, students who fogot to insert an absolute value in the definition of the p-norm of f were not penalised for that. |
| MATH97040/97149 & MATH96031 | 4 | The computational questions were done well, but often justifications were missing or insufficient. In a)i), measurability was often omitted. In b)i) many students bounded the expectation of X (instead of \|X\|) to prove its integrability. There were also many mistake in the use of the transfer lemma. |
| MATH97040/97149 & MATH96031 | 5 | Often the statement of the Poincaré recurrence theorem did not seem to be known. |