Imperial College
London

Department of Mathematics

M3S11 Games, Risks & Decisions

Question	Examiner's Comments
Q 1	Q1 was intended to be straightforward and on basic definitions and procedures. In (a) marks were lost if the range of x values for a pssp or a pair of equaliser strategies was not made clear, or if the value of the game was omitted. Parts (b) and (c) was done well by most students. Part (d) was staightforward but some students made slips in solving the resultng linear equation.
Q 2	It is good practice to check that the obtained solution represents a randomised strategy. Some students did not extend the subgame solution to the whole game.
Q 3	A significant number of students missed to specify the Pareto optimal set, the negotiation set or prove that the Shapley solution is in the negotiation set in part b.
Q 4	Two different strategies were possible. Students who solved the question by either strategy correctly, were awarded full marks.

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M4S11 Games, Risks & Decisions

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Q 5	Disappointingly, only a couple of you did well on this question. The rest had not learned the material sufficiently well or had made slips in the calculation.

Imperial College
London

M4/5S11

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Games, Risks and Decisions

Date: Wednesday, 16 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. (a) In the pay-off matrix below the (i, j) entry is the pay-off to player A in a zero-sum game against an opponent B in which A has pure strategies a_i $(i = 1,2)$ and B has pure strategies b_j $(j = 1, 2)$.

		B	
		b_1	b_2
	a_1	4	x
A	a_2	x	2

Solve the game for all real values of x.

(b) Define (i) an *equilibrium pair* of strategies and (ii) the *Pareto Optimal set* in a two- person cooperative game.

Give a numerical example in each case to show that an element of the Pareto Optimal set is not necessarily in equilibrium and that an equilibrium pair of strategies is not necessarily in the Pareto Optimal set.

(c) In a two-person non-cooperative game the first entry in each ordered pair in the table below is the pay-off to player A and the second entry is the pay-off to player B. A has pure strategies a_i $(i = 1,2)$ and B has pure strategies b_j $(j = 1, 2)$.

		B	
		b_1	b_2
A	a_1	(0, 6)	(3, 1)
	a_2	(2,4)	(4,5)

Sketch the pay-off set.

(d) An individual X wishes to share a sum of money £M between two adult children, A and B. The children currently have assets £a and £b respectively, and utility $u(z)$ for money £z of the form

$$u(z) = 1 - e^{-\theta z}$$

where $z > 0$ and $\theta = \theta_A > 0$ for A and $\theta = \theta_B > 0$ for B. Individual X wants to do this in such a way that the utility of the money child A will have after the gift has been given to them will be the same as that for child B. How should X's money be split?

2. (a) Show *from first principles* that if both players have equaliser strategies, α^* and β^* respectively, in a two-person zero-sum game, then one of these strategies is maximin and the other is minimax.

(b) Army A has 2 regiments and army B has 3 regiments. They are fighting over two battlefields X and Y where battlefield X is worth a (>1) times as much as battlefield Y to both armies. Each army must decide how to allocate their regiments to the two battlefields.

Army A can send both regiments to battlefield X (pure strategy a_1), or they can send one regiment to each of X and Y (pure strategy a_2), or they can send both regiments to battlefield Y (pure strategy a_3).

Similarly, army B can decide to send any number of its regiments to X and any remaining regiments to Y. At each battlefield the winner is the army with the most regiments. They gain the value of the battlefield and the loser loses the value of the battlefield. If both send the same number of regiments to the battlefield then neither gets anything there. The overall pay-off to each army is the sum of their pay-offs at the two battlefields.

(i) Model this situation as a zero-sum game. Show that B should never place all their regiments at Y and that in the subgame which excludes this pure strategy for B, A has an equaliser strategy in which their pure strategies (a_1, a_2, a_3) are played with probabilities

$$\frac{1}{a^2+1}, \frac{a-1}{a^2+1}, \frac{a^2-a+1}{a^2+1}.$$

respectively.

(ii) Solve the game.

3. The pay-off table for a two-person cooperative game is

		B	
		b_1	b_2
	a_1	$(0,0)$	$(\lambda a, \mu b)$
A	a_2	$(a,0)$	$(0,b)$

where a and b are positive constants and $0 < \lambda < 1$, $0 < \mu < 1$ and $\lambda + \mu < 1$. The first entry in each ordered pair is the pay-off to A and the second is the pay-off to B.

(a) Show that the pay-off set is a triangle and its interior.
(b) Find the pay-offs to A and B which give the Shapley solution to this game.

4. A piece of equipment has a fault which is either minor (with prior probability 0.8) or serious (with prior probability 0.2). However it is not easy to discover which kind of fault it is. If a correct diagnosis is made then no extra costs are incurred (above those needed to repair the piece of equipment). If a fault is diagnosed as serious when in fact it is minor, there is an extra cost of £1000. If the fault is serious but is diagnosed as minor then an extra cost of £3000 is incurred.

It is possible to pay for a test which will indicate the correct nature of the fault. However if the fault is serious the test will only identify it as such with probability 0.7. If the fault is minor it will also identify it as such with probability 0.7.

The owner of the equipment must decide whether or not to have the test conducted, and must then make a diagnosis following any result (or otherwise). The owner's aim is to minimise any extra cost involved.

(a) Draw the relevant decision tree together with all its associated probabilities.
(b) Determine the maximum amount the owner should be prepared to pay for the test.

5. (a) In the context of an n-person cooperative game define *an imputation.*

(b) Prove that the core of an n-person cooperative game is a convex set.

(c) In a 3-person game each player has 2 pure strategies 0 and 1. The pay-offs are as follows:

Pure strategy for A	Pure strategy for B	Pure strategy for C		Pay-off to A	Pay-off to B	Pay-off to C
0	0	0		0	1	1
0	0	1		1	0	1
0	1	0		1	1	0
0	1	1		0	0	0
1	0	0		0	1	0
1	0	1		1	1	1
1	1	0		1	0	0
1	1	1		0	0	a

Show that the game has a non-empty core for all $a \in (0,1)$.

Q1
p1

M34S11
2017-18

(a) When $2 \le x \le 4$ the game has a pure strategy saddle-point at (a_1, b_2) and the value of the game is x.

2 B

If $x < 2$ or $x > 4$ the game has a pair of equaliser strategies:

unseen (similar seen)

$$\alpha^* = \left(\frac{|2-x|}{|x-4|+|2-x|}, \frac{|x-4|}{|x-4|+|2-x|} \right) = \beta^*$$

3 B

and the value of the game is

$$v = \frac{4|2-x| + x|x-4|}{|x-4| + |2-x|}.$$

(b) (i) A pair of strategies (α^*, β^*) is in equilibrium if

1 seen B

$$\begin{cases} g_A(\alpha^*, \beta^*) \geq g_A(\alpha, \beta^*) \\ g_B(\alpha^*, \beta^*) \geq g_B(\alpha^*, \beta) \end{cases}$$

for all strategies $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$

(ii) A pair of strategies (α^*, β^*) is in the Pareto optimal set if $\nexists\ (\alpha, \beta)$ s.t.

2. seen B

$$\begin{cases} g_A(\alpha, \beta) \geq g_A(\alpha^*, \beta^*) \\ g_B(\alpha, \beta) \geq g_B(\alpha^*, \beta^*) \end{cases}$$

(with at least one strict inequality)

	b_1	b_2
a_1	(1,4)	(3,3)
a_2	(1,1)	(4,1)

(3,3) is in the Pareto opt. set but is not in equilibrium.

3 seen S

	b_1	b_2
a_1	(2,2)	(1,2)
a_2	(2,1)	(3,3)

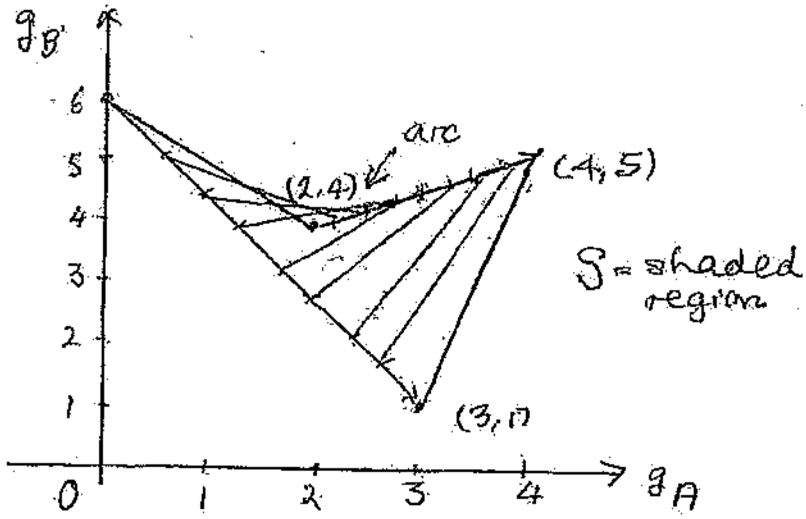
(2,2) is in equilibrium but is dominated by (3,3)

3 seen S

Q1
p2

M34511
2017-18

(c)



unseen

(similar seen)

3 M

(d) Let X give £m to A and £n to B

Then $m + n = M$ and

$$1 - e^{-\theta_A (m+a)} = 1 - e^{-\theta_B (n+b)}$$

Hence $\theta_A (m+a) = \theta_B (n+b)$, giving

unseen

$$\begin{cases} m = \dfrac{M\theta_B - a\theta_A + b\theta_B}{\theta_A + \theta_B} \\ \\ n = \dfrac{M\theta_A - b\theta_B + a\theta_A}{\theta_A + \theta_B} \end{cases}$$

3

Q2 p1

(a) Let $g(a,b)$ = gain to A when A plays a and B plays b.

If α^* and β^* are ESs then

$$\inf_{\beta} g(\alpha^*, \beta) = g(\alpha^*, \beta^*) \quad \text{since } \alpha^* \text{ is an ES}$$

5 Seen

$$= g(\alpha, \beta^*) \ \forall \alpha \quad \text{since } \beta^* \text{ is ES}$$

$$\geqslant \inf_{\beta} g(\alpha, \beta) \ \forall \alpha.$$

Hence α^* is maximin. Similarly β^* is minimax

rest unseen (but similar seen)

(b) (i)

		B			
		$b_1 = XXX$	$b_2 = XXY$	$b_3 = XYY$	$b_4 = YYY$
	$a_1 = XX$	$-a$	-1	$a-1$	$a-1$
A	$a_2 = XY$	$-a+1$	$-a$	-1	$a-1$
	$a_3 = YY$	$-a+1$	$-a+1$	$-a$	-1

3 B

Add a to each entry:

		B		
	0	$a-1$	$2a-1$	$2a-1$
A	1	0	$a-1$	$2a-1$
	1	1	0	$a-1$

We see that YYY for B is inadmissible (compare it with XYY) if $a > 1$.

2 B

Let $\alpha^* = (p, q, r)$.

Q2 b2

M34S11 2017-18

For the subgame (ignoring b_4):

all unseen but similar methods seen

$$g(\alpha^*, b_1) = \frac{a^2}{a^2+1}, \quad g(\alpha^*, b_2) = \frac{a^2}{a^2+1},$$

$$g(\alpha^*, b_3) = \frac{2a-1+(a-1)^2}{a^2+1} = \frac{a^2}{a^2+1}$$ 3 B

(ii) We note that (p, q, r) is a rand. strat. So α^* is an ES in the subgame. We look for an ES strat $\beta^* = (s, t, u)$ for B in the subgame: 1 M

$$\begin{cases} t(a-1) + u(2a-1) = \dfrac{a^2}{a^2+1} & \text{(i)} \\ s + u(a-1) = \dfrac{a^2}{a^2+1} & \text{(ii)} \\ s + t = \dfrac{a^2}{a^2+1} & \text{(iii)} \end{cases}$$ 3 S

(iii) − (ii) $\Rightarrow$ $t = u(a-1)$

(i) $\Rightarrow$ $u(a-1)^2 + u(2a-1) = \dfrac{a^2}{a^2+1}$

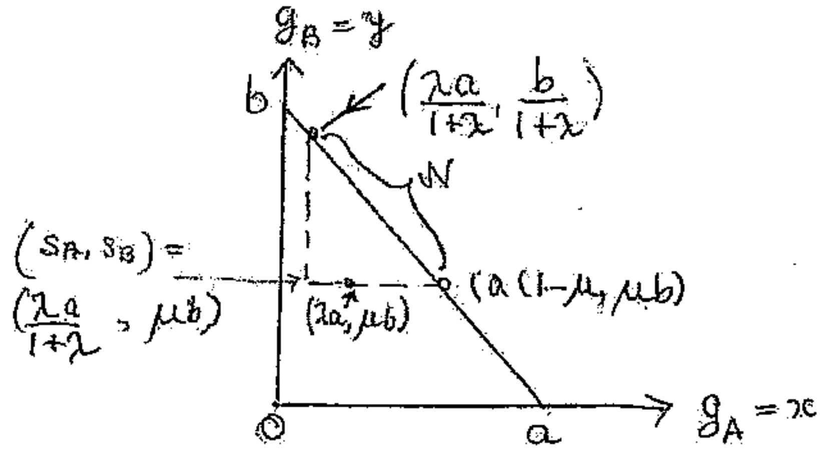
So $u = \dfrac{1}{a^2+1}$, $t = \dfrac{a-1}{a^2+1}$ and $s = \dfrac{a^2-a+1}{a^2+1}$ by (iii). Clearly (s, t, u) represents a randomised strategy. 1 B

From part (a), we see that α^* and $\beta_s^* = \left(\dfrac{a^2-a+1}{a^2+1}, \dfrac{a-1}{a^2+1}, \dfrac{1}{a^2+1}\right)$ is a simple solution to the subgame. Extending this to the whole game we see that 1 M

α^* and $\beta^* = \left(\dfrac{a^2-a+1}{a^2+1}, \dfrac{a-1}{a^2+1}, \dfrac{1}{a^2+1}, 0\right)$ are maximin and minimax and the value is $v = -a + \dfrac{a^2}{1+a^2} = \dfrac{(a-1-a^2)a}{1+a^2}$ 1 B

Q3 p1

M34S11
2017-18



all unseen but method seen

The line joining $(a,0)$ and $(0,b)$ is $ay + bx = ab$. Since $a\mu b + \lambda ab = ab(\lambda+\mu) < ab$, the pt. $(\lambda a, \mu b)$ lies below the line joining $(a,0)$ and $(0,b)$ 4 S

(a) The pay-off set is the triangle with vertices $(0,0), (0,a)$ and $(b,0)$ i.e. the convex hull of the 4 pts. representing pure strategies.

(b) The Pareto optimal set is the line joining $(a,0)$ and $(0,b)$. We look for the security levels for A and B; 1 B

A's pay-offs are
$$\begin{matrix} 0 & \lambda a \\ a & 0 \end{matrix}$$

A's maximin strategy $= \left(\frac{a}{a+\lambda a}, \frac{\lambda a}{a+\lambda a}\right)$
$= \left(\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right)$ 2 B

Hence $s_A = \frac{\lambda a}{1+\lambda}$ $\left(\in (0, \lambda a)\right)$

Q3 p2

M34S11
2017-18

all unseen

B's pay-offs are ⓪ μb
⓪ b — 2 M

Hence $s_B = \mu b$ since $0 < \mu < 1$. — method seen

The negotiation set N consists of those pts (x, y) with $s_1 \geqslant \frac{\lambda a}{1+\lambda}$, $s_2 \geqslant \mu b$. — 2

The Shapley solution $(s_1^* s_2^*)$ maximises

$$\Delta = \left(s_1 - \frac{\lambda a}{1+\lambda}\right)(s_2 - \mu b) \text{ subject to } as_2 + bs_1 = ab$$

$$\Delta = \left(s_1 - \frac{\lambda a}{1+\lambda}\right)\left(\frac{ab - bs_1}{a} - \mu b\right),$$ — 3 S

a quadratic in s_1 ($\cap$), is maximised when $\frac{d\Delta}{ds_1} = 0$; giving

$$\begin{cases} s_1^* = \frac{a}{2}\left(1 - \mu + \frac{\lambda}{\lambda+1}\right) \\ s_2^* = \frac{b}{2}\left(1 + \mu - \frac{\lambda}{\lambda+1}\right). \end{cases}$$ — 2

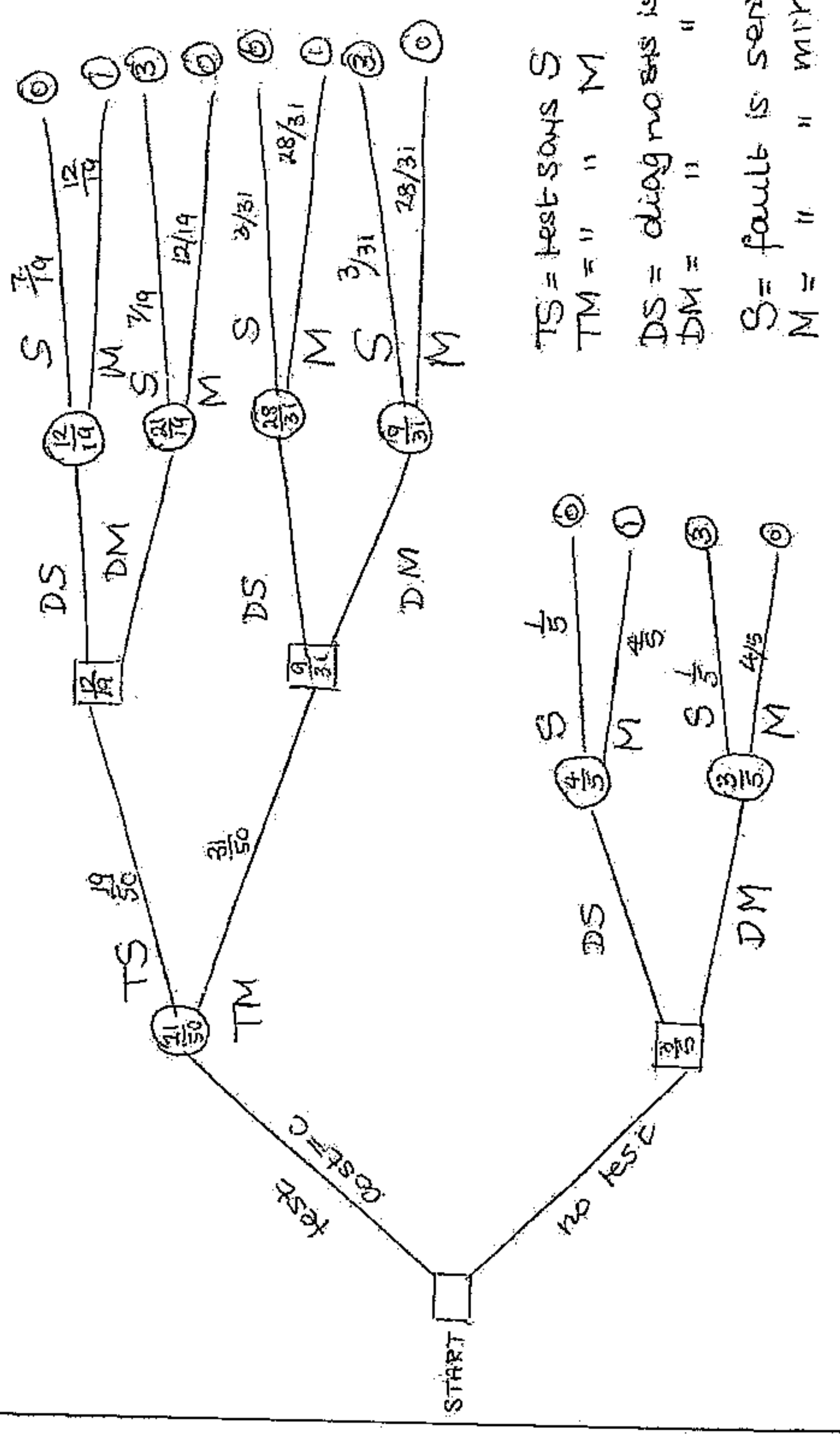
$(s_1^*, s_2^*) \in N$ because $1 - \mu > \frac{\lambda}{\lambda+1}$ (otherwise $\mu\lambda + \mu > 1$, contradicting $\lambda + \mu < 1$, $0 < \lambda < 1$, $0 < \mu < 1$) which tells us that $\lambda a < s_1^* < a(1-\mu)$. — 4 M

Hence (s_1^*, s_2^*) are the pay-offs for A and B which give the Shapley solution.

Q4 p1

(a)

all unseen



4 for basic tree S

method seen

2 for terminal costs

B

Q4 p2

M34S11 2017-18

$P(TS) = P(TS|S)P(S) + P(TS|M)P(M)$

$= \frac{7}{10}\cdot\frac{1}{5} + \frac{3}{10}\cdot\frac{4}{5} = \frac{19}{50}$

$P(TM) = 1 - P(TS) = \frac{31}{50}$

all unseen

$P(S|TS) = P(TS|S)P(S) / P(TS)$

$= \dfrac{\frac{7}{10}\cdot\frac{1}{5}}{\frac{19}{50}} = \frac{7}{19}$

6 B

$P(M|TS) = 1 - P(S|TS) = \frac{12}{19}$

method seen

$P(S|TM) = P(TM|S)P(S) / P(TM)$

$= \dfrac{\frac{3}{10}\cdot\frac{1}{5}}{\frac{31}{50}} = \frac{3}{31}$

$P(M|TM) = 1 - P(S|TM) = \frac{28}{31}$

(b) By folding back the tree, the expected extra cost if the test is used is $c + \frac{21}{50} \times 1000$. If the test is not used it is $\frac{3}{5} \times 1000$. So the maximum amount the owner should pay for the test is given by $c + 420 < 600$ i.e. if the cost of the test is less than £180, then the owner should use the test.

4 M

unseen

4

(a) If $P = \{P_1, \dots, P_n\}$ is the set of players then the vector $(x_1, \dots, x_n)$ is an imputation if seen

$$x_i \geq v(P_i) \quad \text{for } i = 1, \dots, n$$

and $\sum_i^n x_i = v(P)$ 2

(b) $(x_1, \dots, x_n)$ is in the core $\iff$ unseen

$$\begin{cases} \sum_i x_i = v(P) \text{ and} \\ \sum_{P_i \in S} x_i \geq v(S) \text{ for every coalition } S \end{cases}$$ 2

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be in the core. Let $\lambda \in [0,1]$. Then $\lambda x + (1-\lambda) y$ is also in the core because 2

$$\sum_i [\lambda x_i + (1-\lambda) y_i] = \lambda v(P) + (1-\lambda) v(P) = v(P)$$

and $\sum_{P_i \in S} [\lambda x_i + (1-\lambda) y_i)$ 2

$$= \lambda \sum_{P_i \in S} x_i + (1-\lambda) \sum_{P_i \in S} y_i \geq$$

$$\geq \lambda v(S) + (1-\lambda) v(S) = v(S) \ \forall S$$

Hence the core is convex.

Q5 p2

M34511
2017-18

(c) we calculate the characteristic function of the game:

all unseen but method seen

		BC 00	01	10	11
A	0	0	1	1	0
	1	0	1	1	0

01 and 10 are inadmissible for BC so $V(A) = 0$ 1

		AC 00	01	10	11
B	0	1	0	1	1
	1	1	0	0	0

01 for AC dominates 00, 10 and 11 so $V(B) = 0$ 1

		AB 00	01	10	11
C	0	1	0	0	0
	1	1	0	1	a

Since $a > 0$, 01 for AB dominates 00, 10 and 11, so $V(C) = 0$ 1

		A 0	1
BC	00	2	1
	01	1	2
	10	1	0
	11	0	a

since $a < 1$, 00 dominates 10 and 11 for BC. So $V(BC) = 3/2$ 1

		B 0	1
AC	00	1	1
	01	2	0
	10	0	1
	11	2	a

01 is dominated by 11 for AC as $a > 0$. 00 dominates 10 for AC. So $v(AC) = 1$ 1

		C 0	1
AB	00	1	1
	01	2	0
	10	1	2
	11	1	0

11 for AB is dominated 00 (and) 10. 00 is dominated by 10, leaving $\begin{matrix} 2 & 0 \\ 1 & 2 \end{matrix}$

so $v(AB) = \frac{4}{3}$ 1

Q5 p3

M34S11 2017-18

$\underline{v(ABC) = 3}$ $\underline{v(\phi) = 0}$ 1

We know that (x_1, x_2, x_3) is in the core of the game $\Leftrightarrow$

$$\sum_i x_i = v(ABC) = 3$$

and

$$\sum_{i \in S} x_i \geqslant v(S) \text{ for every coalition}$$

S.

all unseen but similar seen

Hence (x_1, x_2, x_3) is in the core $\Leftrightarrow$

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 \geqslant 0 \\ x_1 + x_2 \geqslant \frac{4}{3}, \ x_1 + x_3 \geqslant 1, \ x_2 + x_3 \geqslant \frac{3}{2} \end{cases}$$

2

If $x_1 = \frac{3}{2}$, $x_2 = x_3 = \frac{3}{4}$, then

$$x_1 + x_2 = \frac{3}{2} + \frac{3}{4} = \frac{9}{4} \geqslant \frac{4}{3}$$

$$x_1 + x_3 = \frac{3}{2} + \frac{3}{4} > 1, \quad x_2 + x_3 = \frac{3}{2}$$

3

Hence the core is non-empty.

(other answers possible)