MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Guidance for the exam

This document gives you guidance on how to prepare for the exam. In general, the best preparation for the exam is to repeat and understand the exercises, understand the lecture notes well and practise using past exams. More specific guidance is given as follows.

Structure of the exam. The differential equations part of the exam consists of three questions (Questions 4, 5 and 6). Question 4 concerns basic understanding and focusses on the chapter on existence and uniqueness; Question 5 is about linear systems, and Question 6 focusses on nonlinear systems. All three questions check knowledge of very basic and elementary material, as well as more advanced understanding. Note this exam will contain a part where you have to decide on whether a statement is true or false. Each statement gives three marks, where one mark is given on the correct choice and two marks are given for the justification.

Relevance of the past exams. I am teaching this course now for the sixth time, and the exams from the last years are a good guidance for you. Please note that while due to the curriculum review, some parts of the course have been removed in 2020/21 (such as metric spaces, space of continuous functions, Banach's fixed point theorem, criteria for global existence and some parts of linear/nonlinear systems), these parts have not been covered in the exams in the both 2018/19 and 2019/20. You may notice that the exams from the last five years are quite consistent with the exams from the previous years, which gives you a lot of material to practise, but note that when teaching the course the first time in 2018/19, I have done some changes to what was taught in the years before. I will give more details below what questions from these past exams are relevant for this years exam. Note that this is a rough guidance, where I say which parts are relevant in principle, but some questions may contain material not covered this year.

Details that need to be provided in your answers. You may have noticed that on problem sheets, my solutions were sometimes more detailed, and sometimes less. For example, one-dimensional questions were treated quite thoroughly, while arguments in higher-dimensions, in particular in phase plane analysis, were a bit more sketchy in places (making them rigorous is possible, but not the point of the question). I guess you will get a good feeling what level of detail is required, but in addition to this, I have specifically tried to make things clear in the questions in the exam. Sometimes I may ask you to justify or explain something. In these cases, the best answer is normally less formal than a rigorous proof, or it is a short proof, or a short argument involving different facts, or citation of a result from the course. When I write that you should prove or show a certain statement, more details are normally needed, and as always, please try to keep the line of your arguments clear. You also may be guided by the marks you can get for such questions.

Additional detailed comments on the material covered in the lecture notes.

<u>Chapter 1. Introduction</u>

§1 This introductory section focusses on basic definitions and properties of differential equations.

§2 This section addresses issues arising when looking at initial value problems (remind yourself that there are examples where such problems have no solutions, many solutions, and solutions do not need to exist for all times). Via separation of variables, you can solve certain one-dimensional differential equations, and consequences with regard to uniqueness and non-uniqueness have been addressed also on problem sheets. I expect you to be in good command of this elementary material.

§3 This section describes how to visualise ordinary differential equations. This is a very useful part of the course, and the ideas established here help to understand the material of this course better.

Chapter 2. Existence and uniqueness

§1 Picard iterates are very fundamental to obtain local solutions to initial value problems. It motivates the general approach we use later in §4 of this chapter. In particular, you should understand why Picard iterates are important.

§2 Lipschitz continuity is very important to establish existence and uniqueness of solutions. You should be very confident to understand the building blocks involving the mean value theorem and the mean value inequality as well as being able to apply this in practical settings.

§3 The main result in the whole chapter on existence and uniqueness is given the Picard–Lindelöf theorem, which was stated and proved for didactic reasons first for the global Lipschitz continuous case and then for locally Lipschitz right hand sides. I would like to stress that the proof of this theorem and the connections to the tools from analysis and the material covered earlier (such as Lipschitz continuity and Picard iterates) are an extremely important part of the course.

§4 It is shown in this section that local solutions that have been obtained via the Picard–Lindelöf theorem can be extended to maximal solutions. Maximal solutions to initial value problems are maximal in the sense that their domain (i.e. the maximal existence interval) cannot be extended. I expect you to understand the arguments well and that you are able to check maximality of solutions of differential equations in practical examples.

§5 The concept of a general solution or a flow is a bit abstract, but clear if the results from the previous section have been understood well, since these two objects parameterise maximal solutions. Our thinking about differential equations now becomes more dynamical, in the sense that we think of how the dynamics evolves when we start at time t_0 in x_0 (for a nonautonomous differential equation) or in x_0 (for an autonomous differential equation, where the initial time does not matter). Given that flows will appear regularly until the end of the course, the understanding of this part of the course is crucial.

Chapter 3. Linear systems

§1 The matrix exponential function is fundamental for solving autonomous linear systems. I expect you to be confident with this part of the course. All the properties shown in this part have been used later and are very fundamental.

§2 This section describes the different types of phase portraits for two-dimensional linear systems. The understanding of this part is very important, since some of these phase portraits appear typically in nonlinear systems as well, locally in neighbourhoods of hyperbolic equilibria. Spend some time to think about details in the pictures you see and understand why they are looking like this. The last part of this section is on exponential growth (Lyapunov exponents) and rotation behaviour (imaginary part of eigenvalues). It is very important to understand this well; locally we observe this also in nonlinear systems.

§3 This part concerns Jordan normal forms, with focus on the real Jordan normal form. Understanding how to compute Jordan normal forms and what the columns of the transformation matrix T mean is important. I expect you to understand theoretical aspects and calculations involving Jordan normal forms only in dimension two, see examples on the problem sheets and what was explained in the lectures.

§4 This part explains how to compute matrix exponential functions explicitly. You need to understand this thoroughly. Think about the different parts of the matrix exponential function and what they mean for the behaviour of the system.

§5 This section gives a more detailed analysis on exponential growth in linear systems, extending what you have seen in §2 for two-dimensional systems. It is important that you understand the philosophy behind this and that you are able to apply this in a practical situation (as explained above, theoretical and practical situations are restricted to two dimensions in the exam).

§6 The variation of constants formula is very important, both for linear systems and nonlinear systems (it is used in the proof of linearised stability).

Chapter 4. Nonlinear systems

§1 This is a very long section on stability. I expect you to be very confident with the basic definitions and all the results in this section; its importance cannot be overstated. I expect you to also understand how to analyse nonlinear systems locally around equilibria and that you know in which cases the linear approximation has a meaning for the nonlinear system locally.

§2 Invariant sets and limit sets are important for the asymptotic (long-term) behaviour of differential equation. The material provided here is fundamental and I expect you to be fully confident with all the details.

§3 Lyapunov functions are important tools to detect (asymptotic) stability of equilibria, leading to more global results than we can expect to achieve via linearisation. Underlying is the concept of an orbital derivative, which indicates decay (or growth) of a real-valued function along the flow of the differential equation. I expect you to understand and be able to apply the results of this section in a practical setting.

§4 The last part of this course deals with the Poincaré–Bendixson theorem. I expect you to understand the result (and the corollary) well, and to confidently apply the theorem in mathematical arguments.

Some comments on the exams from 2004 to 2018:

- 2018. Part 1, 2 and 4 are relevant.
- 2017. Part 1, 2 and 4 are relevant.
- 2016. Part 2 and 3 are relevant.
- 2015. All parts are relevant, except 2(d), 3(a), 3(b)(iii)–(iv).
- 2014. Part 1, 2 and 4 are relevant, except 2(c)–(d).
- 2013. Part 1, 2 and 4 are relevant, except 2(c)–(d).
- 2011–2012. These exams only deal with linear differential equations. Most of the material is not relevant.
- 2008–2010. The vast majority of questions is relevant, and looking at these exams is helpful as preparation.
- 2004–2007. There is only little intersection of the material in these years with the course this year.

I note that I use different notation sometimes, but this should not cause much confusion.

Good luck with the exam!