**Exercise 6.1.** Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a,b])$:

(i)
$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

(ii)
$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$$

(iii)
$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{1/2}$$

*Hint: to show that* $\|\cdot\|_2$ *is a norm, you need to use the Cauchy-Schwarz inequality and the definition of the integral as the limit of certain sums.*

**Exercise 6.2.** Show that if $V$ is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2\,\mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric $\mathrm{d}_{\text{disc}}$ on $\mathbb{R}^2$.

**Exercise 6.3.** Let $(X, \mathrm{d})$ be a metric space.

(i) Show that for every $x$, $y$, and $z$ in $X$, we have
$$|\,\mathrm{d}(x,z) - \mathrm{d}(y,z)| \le \mathrm{d}(x,y).$$

(ii) Show that for all $x$, $y$, $z$ and $t$ in $X$, we have
$$|\,\mathrm{d}(x,y) - \mathrm{d}(z,t)| \le \mathrm{d}(x,z) + \mathrm{d}(y,t).$$

(iii) Show that for all $x_1$, $x_2$, $\dots$, $x_n$ in $X$, we have
$$\mathrm{d}(x_1,x_n) \le \mathrm{d}(x_1,x_2) + \mathrm{d}(x_2,x_3) + \cdots + \mathrm{d}(x_{n-1},x_n).$$

**Exercise 6.4.** Let $(X, \mathrm{d})$ be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By an example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have
$$\bigcap_{n\in\mathbb{N}} B_{1/n}(x) = \{x\}.$$

**Exercise 6.5.** (i) Show that for all $x$ and $y$ in $\mathbb{R}^n$, we have
$$\mathrm{d}_\infty(x,y) \le \mathrm{d}_2(x,y) \le \sqrt{n}\cdot \mathrm{d}_\infty(x,y).$$

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

(ii) Show that for all $x$ and $y$ in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x,y) \leq \mathrm{d}_1(x,y) \leq n \cdot \mathrm{d}_\infty(x,y).$$

(iii) Show/conclude that for all $x$ and $y$ in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}}\,\mathrm{d}_2(x,y) \leq \mathrm{d}_1(x,y) \leq n\,\mathrm{d}_2(x,y).$$

(iv) Conclude that the metrics $\mathrm{d}_1$, $\mathrm{d}_2$ and $\mathrm{d}_\infty$ on $\mathbb{R}^n$ are topologically equivalent.

**Exercise 6.6.** Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in $X$. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\text{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

**Exercise 6.7.** Let $(X, \mathrm{d})$ be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in $X$. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set $U$ in $(X, \mathrm{d})$ with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

*Hint: $U$ can be the ball $B_r(x)$.*

**Exercise 6.8.** Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space. Show that every set in $X$ is closed.

*Hint: First show that every set in $X$ is open with respect to* $\mathrm{d}_{\text{disc}}$.

**Unseen Exercise.** Let $E = \{1, 2, 3, 4, 5, 6\}$, and let $\mathcal{P}(E)$ be the set of all subsets of $E$. Consider the metric $\mathrm{d}_{\text{card}}$ on $\mathcal{P}(E)$ (see typed lecture notes). Let $e = \{1, 2, 3\} \in \mathcal{E}$. What is $B_{1/2}(e)$? What is $B_1(e)$? What is $B_{3/2}(e)$?