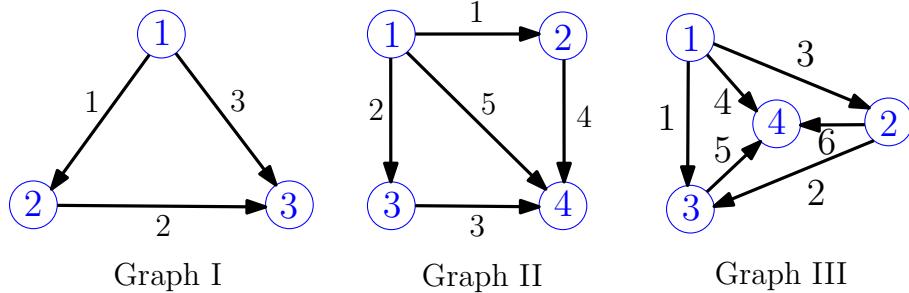


Linear algebra review and basic graph theory

1. Consider the following three graphs I, II and III:



- (a) Find the incidence matrix \mathbf{A} for each graph (use the numbering of nodes and edges shown in the figures to label columns/rows and use the edge directions indicated by arrows).
- (b) For each graph, find all vectors in the right null spaces of \mathbf{A} and \mathbf{A}^T .
- (c) Find the degree matrix \mathbf{D} for each graph.
- (d) Find the adjacency matrix \mathbf{W} for each graph.
- (e) Find the Laplacian matrix \mathbf{K} for each graph.
- (f) Are any of the graphs complete?

2. Suppose that graphs I, II and III are the disconnected pieces of a **single** graph with 11 nodes and 14 edges.

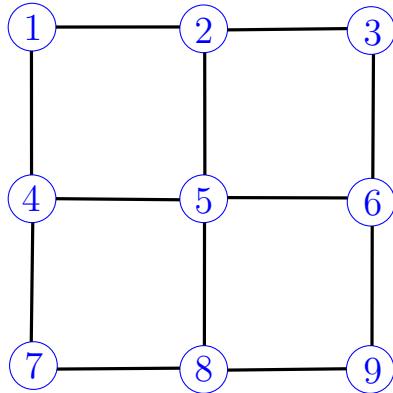
- (a) What is the rank of the incidence matrix \mathcal{A} of this new single graph?
- (b) Find all linearly independent solutions of

$$\mathcal{A}\mathbf{x} = 0.$$

- (c) Find all linearly independent solutions of

$$\mathcal{A}^T\mathbf{w} = 0.$$

3. Consider a 3-by-3 square grid with $n = 9$ nodes and $m = 12$ edges as shown in the figure. Let its Laplacian matrix be \mathbf{K} .



- (a) How many of the 81 entries of \mathbf{K} are zero?
- (b) Write down the degree matrix \mathbf{D} .
4. In a graph with n nodes and n edges argue that there must be a loop.
5. In the June 2020 issue of the magazine **SIAM News** (<https://sinews.siam.org>) an article on modelling the spreading of COVID-19 using “networked epidemiology” suggests three different mathematical interpretations of “reducing contacts by half” (such considerations have been critical in guiding governmental decisions on lock-down actions). In the figure below a graph representing contacts (“edges”) between actors (“nodes”) is shown at the top, and graphs (a), (b) and (c) show three different ways one might *mathematically* “reduce contacts by half”.

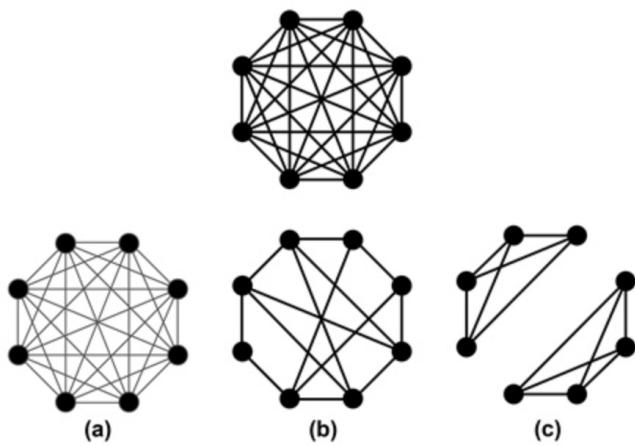


Figure 1. Three interpretations of “reducing contacts by half” in the top network.

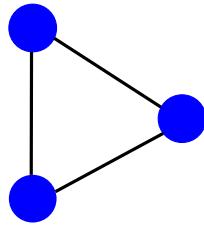
1a. Reducing the strength of each contact by half. **1b.** Reducing the number of contacts by half. **1c.** Reducing the size of connected components by half. Figure courtesy of Stephen Eubank.

[Click to see SIAM News article](#)

- (i) Find the rank of the incidence matrices of each of the graphs **(a)**, **(b)** and **(c)**.
- (ii) Find the dimension of the left null space of each incidence matrix.
- (iii) **Without** writing down any vectors, can you give a *geometrical* representation (i.e., draw some sketches) of a basis of the left null space of the incidence matrix of each of the two graphs **(b)** and **(c)**? [Note: only the intersections of edges given by bold dots represent *nodes* of the graph].

Note: in this question, it is not necessary to write down any incidence matrices.

6. Consider the 3-node graph



- (a) Write down the incidence matrix \mathbf{A} and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T \mathbf{A}$.
- (b) Let $\omega = e^{2\pi i/3}$ be a third root of unity so that

$$\omega^3 = 1.$$

Now introduce the vectors

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Show that, for $n = 0, 1$ and 2 ,

$$\mathbf{K}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$

and find the values of the constants λ_0, λ_1 and λ_2 (the “eigenvalues”).

- (c) Why did we only consider the three possible values $n = 0, 1, 2$ in part (b)?
- (d) The inner product (think “dot product”) of two complex-valued vectors \mathbf{u} and \mathbf{v} is defined to be

$$\bar{\mathbf{u}}^T \mathbf{v},$$

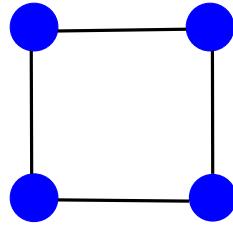
where $\bar{\mathbf{u}}$ is the complex conjugate of the complex vector \mathbf{u} . Show that the vectors \mathbf{x}_n for $n = 0, 1, 2$ are orthogonal with respect to this inner product (meaning that the inner product of any two vectors is zero).

7. Consider the matrix

$$\mathbf{C} = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix}.$$

- (a) Find its eigenvalues and the corresponding eigenvectors.
- (b) Do you notice anything about the eigenvectors? [Hint: refer back to Q6]
- (c) Can you find *another* (non-identity) matrix, different from \mathbf{C} , having the same eigenvectors as \mathbf{C} ?

8. Consider the 4-node graph



- (a) Using a natural numbering of consecutive nodes as you cycle around the graph, write down the incidence matrix \mathbf{A} and find the Laplacian matrix $\mathbf{K} = \mathbf{A}^T \mathbf{A}$.
- (b) Let $\omega = e^{2\pi i/4}$ be a fourth root of unity so that

$$\omega^4 = 1.$$

Now introduce the vectors

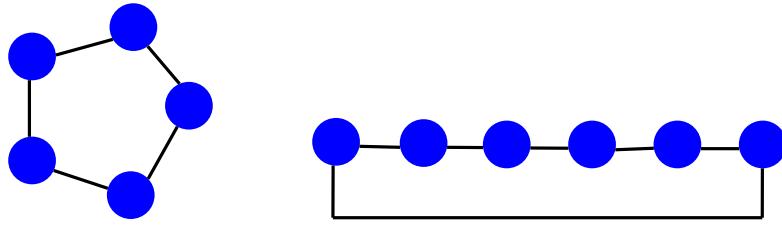
$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Show that, for $n = 0, 1, 2$ and 3 ,

$$\mathbf{K}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$

and find the values of the constants $\lambda_0, \lambda_1, \lambda_2$ and λ_3 .

- (c) Show that the vectors \mathbf{x}_n for $n = 0, 1, 2$ and 3 are mutually orthogonal in the same sense as in question 6(d).
- (d) Now consider the 5-node and 6-node graphs



Again, using a natural numbering of consecutive nodes as you cycle around each graph, use information from parts (a)–(c), and the results of question 6, to investigate the eigenvectors and eigenvalues of the Laplacian matrices associated with these two graphs.

9. Consider a **complete** graph with $n \geq 2$ nodes and edges between all pairs of nodes.

- (a) Write down the general form of the Laplacian matrix $\mathbf{K} = \mathbf{A}^T \mathbf{A}$.
- (b) Can you find n distinct non-zero vectors \mathbf{x} satisfying the relation

$$\mathbf{K}\mathbf{x} = \lambda\mathbf{x}$$

for some value of λ ? Find the corresponding values of λ .

- (c) Suppose now that one of the nodes is grounded. Find the general form of the corresponding reduced Laplacian matrix \mathbf{K}_0 .
- (d) Can you find $n - 1$ distinct non-zero vectors \mathbf{x} satisfying the relation

$$\mathbf{K}_0\mathbf{x} = \hat{\lambda}\mathbf{x}$$

for some value of $\hat{\lambda}$? Find the corresponding values of $\hat{\lambda}$.

- (e) Unlike \mathbf{K} , the reduced Laplacian matrix \mathbf{K}_0^{-1} is invertible. By directly computing \mathbf{K}_0^{-1} for small values of $n = 2, 3, 4, \dots$ and trying to spot a pattern (or indeed by any other method), can you propose a general formula for \mathbf{K}_0^{-1} for general n ?