Imperial College London

MATH70135

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analytic Methods in Partial Differential Equations

Date: 5 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Give an example of a second order PDE which is

   (i) Linear (1 mark)

   (ii) Semilinear, but not linear (1 mark)

   (iii) Quasilinear, but not semilinear (1 mark)

   (iii) Fully nonlinear (not semilinear or quasilinear) (1 mark)

   (b) Consider the first order PDE

$$\frac{\partial u}{\partial y} - u^2 \frac{\partial u}{\partial x} = 5u \ . \tag{1}$$

   (i) Write down the system of characteristic ODE for $(x(t), y(t), z(t))$ for this system (where u takes values in the z coordinate) (3 marks)

   (ii) Let Γ be a curve parameterized by some $\gamma(s)$, state what it means for Γ to be non-characteristic at $\gamma(s_0)$ for a general first order PDE of the form

$$a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u) \ .$$

   Additionally, if we take the particular case of $\gamma(s) = (s, 0, \sin(s))$, verify that Γ is not characteristic for (1) at any point. (3 marks)

   (iii) From theorems in lecture we know that we can build local solutions $u(x, y)$ to (1) around any point of Γ. Show that any such solution satisfies, locally near Γ, the implicit equation

$$u(x, y) = \sin\left(x - \frac{u(x, y)^2}{10} + \frac{u(x, y)^2}{10e^{10y}}\right)e^{5y} \ .$$

   (10 marks)

(Total: 20 marks)

2. (a) Consider the Burgers type equation $\partial_t u + u^3 \partial_x u = 0$ on $\{(t, x) \in \mathbb{R}^2 : t \geq 0\}$ with initial data

$$u(0, x) = h(x) = \text{sign}(x)(1 - e^{|x|}) \ ,$$

   where $\text{sign}(x) = 1$ for $x > 0$ and $\text{sign}(x) = -1$ for $x < 0$.

   (i) Give a definition for what it would mean for $u \in L^\infty_{\text{loc}}$ to be a weak/integral solution for this initial value problem. (4 marks)

   (ii) Compute (with some justification) the time $t > 0$ at which the first shock forms. (6 marks)

   (b) (i) Suppose u solves the 1-d heat equation $\partial_t u - \partial_x^2 u = 0$ for all times with initial data $u(0, x) = g(x^4)$ for some g analytic in a neighborhood of 0 but not entire. Show that $u(t, x)$ cannot be analytic near $(t, x) = 0$. (8 marks)

   (ii) Explain why part (i) above does not violate the Cauchy-Kovalevskaya theorem (you can answer this even if you haven't done part (i) above). (2 marks)

(Total: 20 marks)

3. (a) Suppose that $u \in C^2\Big((0,2) \times (-2,2)^2\Big) \cap C\Big([0,2) \times (-2,2)^2\Big)$ satisfies

$\partial_t u - \Delta u = u(\partial_x u + \partial_y u)$ on $(0,2)\times(-2,2)^2$ and $u(0,x,y) = f(x,y)$ for all $(x,y) \in [-1,1]^2$.

Here we write $(t,x,y) \in (0,2) \times (-2,2)^2$ and take $f \in C([-1,1]^2)$. Prove the following weak maximum principle:

$$\max_{(t,x,y)\in[0,1]\times[-1,1]^2} u(t,x,y) = \max\Big(\max_{(t,x,y)\in[0,1)\times\{-1,1\}^2} u(t,x,y), \max_{(x,y)\in[-1,1]^2} f(x,y)\Big) .$$

(10 marks)

(b) Let $u \in C^\infty(\mathbb{R}^3)$.

(i) For any $r > 0$, define

$$S(r) = (4\pi r^2)^{-1} \int_{S_r} u \text{ and } V(r) = (4\pi r^3/3)^{-1} \int_{B_r} \Delta u .$$

where $B_r = \{w \in \mathbb{R}^3 : |w| \leq r\}$ is the closed ball of radius r centered at 0 and $S_r = \partial B_r$ is the sphere of radius r centered at the origin. Prove, using Green's formula, that

$$\frac{d}{dr} S(r) = rV(r)/3 .$$

(4 marks)

(ii) Suppose additionally that u is biharmonic, that is

$$\Delta^2 u = \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)^2 u = 0 .$$

Show that

$$S(r) = u(0) + \frac{r^2(\Delta u)(0)}{6} .$$

You may use part (i) above even if you haven't proved it. A hint - remember the mean value property of harmonic functions. (6 marks)

(Total: 20 marks)

4. (a) (i) Define the Fourier-analytic Sobolev space $H_s(\mathbb{R}^d)$ for $s \in \mathbb{R}$. (4 marks)

(ii) Show that for $s > \frac{d}{2}$, there exists $c(s,d)$ such that for $u \in H_s(\mathbb{R}^d)$ we have

$$\|\hat{u}\|_{L^1} \leq c(s,d)\|u\|_{H_s}$$

where $\hat{u}$ denotes the Fourier transform of u. (4 marks)

(b) Show that for fixed $d \geq 1$, $s > d/2$, and for any $R > 0$, there exists $T > 0$ such that, for any $f \in H_s(\mathbb{R}^d)$ with $\|f\|_s \leq R$, there exists a solution $u \in C([-T,T]; H_s)$ to

$$i\partial_t u = \Delta u + u|u|^2 , \quad u(0,\cdot) = f(\cdot) \in H_s .$$

You can use the fact mentioned in lecture that for $s > d/2$ and any $g, \tilde{g} \in H_s(\mathbb{R}^d)$, there exists $\tilde{c}(s,d)$ such that

$$\|g\tilde{g}\|_{H_s} \leq \tilde{c}(s,d)\|g\|_{H_s}\|\tilde{g}\|_{H_s} .$$

You may also use facts about the Schroedinger evolution group without proof, just clearly state them. (12 marks)

(Total: 20 marks)

5. (a) Let $U \subset \mathbb{R}^d$ be a bounded domain. Let

$$Lu = -\sum_{i,j} \partial_i\big(a_{i,j}(x)\partial_j u(x)\big) ,$$

be a uniformly elliptic operator (with the $a^{i,j}$ symmetric in i,j and smooth) and let $u \in H^1_0(U)$ be a weak solution to

$$Lu = f$$

for some $f \in L^2(U)$, that is for every $v \in H^1_0(U)$ one has

$$B(u,v) := \int_U \sum_{i,j} a_{i,j}\partial_j u \partial_i v = \int_U f .$$

Show that, for any $V \subset\subset U$ (that is $\mathrm{dist}(V, \partial U) > \delta$ for some $\delta > 0$) there exists some C (depending on U, V, but not f) such that

$$\|Du\|_{L^2(V)} \leq C\Big(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}\Big) .$$

You may assume existence of a smooth η with $0 \leq \eta \leq 1$, $\mathrm{supp}(\eta) \subset U$, and $\eta = 1$ on V. (12 marks)

(b) (i) State the (local) Holmgren's uniqueness theorem. (5 marks)

(ii) Give an example that indicates why non-characteristic condition in Holmgren's uniqueness theorem is essential. (3 marks)

(Total: 20 marks)

1. (a) (i) The heat equation $\partial_t u - \partial_x^2 u = 0$

(ii) $\partial_t u - \partial_x^2 u = u^2$.

(iii) $\partial_t u - u\partial_x^2 u = 0$

(iii) $(\partial_t u) + (\partial_x u)^2 = 0$

(b) (i) The characteristic ODEs are given by (1 mark for each):

$$\dot{x} = -z^2 \quad \dot{y} = 1 \,, \quad \dot{z} = 5z \,.$$

(ii) Writing $\gamma(s) = (f(s), g(s), h(s))$ in order for Γ to be non-characteristic at $\gamma(s_0) = (x_0, y_0, z_0)$ we should have (2 marks for general criterion)

$$f'(s_0)b(x_0, y_0, z_0) - g'(s_0)a(x_0, y_0, z_0) \neq 0 \,.$$

In our particular case the right hand side is given by (1 mark for specific case)

$$1 \cdot 1 - 0 \cdot \sin(s_0)^2 \neq 0 \,.$$

(iii) We solve the characteristic ODE in t for $x(s,t), y(s,t), z(s,t)$ where for fixed s we use initial data $\big(x(s,0), y(s,0), z(s,0)\big) = \gamma(s)$. (3 marks for formulating method of characteristics correctly)

For the y component we clearly have $y(s,t) = t$. Next, for the z component we have

$$\frac{\partial}{\partial t} z(s,t) = 5z \text{ and } z(s,0) = \sin(s) \quad \Rightarrow \quad z(s,t) = \sin(s)e^{5t} \,.$$

Finally we have

$$\frac{\partial}{\partial t} x(s,t) = -\sin^2(s)e^{10t} \text{ and } x(s,0) = s \quad \Rightarrow \quad x(s,t) = -\frac{\sin^2(s)e^{10t}}{10} + \frac{\sin^2(s)}{10} + s \,.$$

(3 marks for solving the characteristic ODE)

To get the desired implicit formula we write (s,t) in terms of (x,y,z) (4 marks for inversion). We have $t(x,y,z) = y$. Turning to s we have $\sin(s) = ze^{-5y}$, and therefore from our solution for $x(s,t)$ we have

$$s = x - \frac{z^2}{10} + \frac{z^2}{10e^{10y}}$$

It follows that

$$z = \sin\left(x - \frac{z^2}{10} + \frac{z^2}{10e^{10y}}\right)e^{5y}$$

which gives us the desired implicit formula.

(Total: 0 marks)

2. (a) (i) We say u is an integral solution to the given initial value problem if, for every smooth $v : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ (1 mark for formulating via duality) with compact support we have

$$\int_0^\infty \int_{-\infty}^\infty \Big(u(t,x)\partial_t v(t,x) + \frac{u(t,x)^4}{4}\partial_x v(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^\infty h(x)v(0,x)\,\mathrm{d}x = 0 \,.$$

(3 marks for rest of solution)

(ii) The characteristic starting from $(0, s)$ is given by

$$x = h(s)^3 t + s \ .$$

Therefore two characteristics starting from $(0, s_1)$ and $(0, s_2)$ would cross at time

$$t = -\frac{s_2 - s_1}{h(s_2)^3 - h(s_1)^3} \ ,$$

if the t above satisfies $t \in (0, \infty)$. (3 marks for recalling condition for shock, either characteristics intersecting or ∇u failing to exist).
We note that for such a time t, the expression of $-\frac{1}{t}$ is a difference quotient for $h^3(s)$, therefore we look for negative minimum of

$$g(s) = 3h^2(s)h'(s) = 3e^{-|x|}(1 - e^{-|x|})^2 \ ,$$

(2 marks for finding criterion to identify minimal time) We have

$$g'(s) = -3e^{-|x|}(1 - e^{-|x|})(1 - 3e^{-|x|}) \ ,$$

which gives $g'(\tilde{s}) = 0 \Rightarrow \tilde{s} \in \{0, \pm \log(3)\}$, from which we can infer the minimum value of g is -3, so the time of the first shock is $t = 1/3$. (1 mark for finding time)

(b) (i) Since $g(x^4)$ is analytic near $x = 0$ but not entire, there exists $0 < R < \infty$ and a power series

$$\sum_{n=0}^{\infty} \frac{g_{4n}}{(4n)!} x^{4n} \ .$$

which is absolutely convergent and equal to $g(x^4)$ for $|x| < R$, and on the other hand we must have

$$\limsup_n \left| \frac{g_{4n}}{(4n)!} \right| (R+1)^{4n} = \infty \ .$$

(3 marks for using analytic + not entire in the right way)

Suppose by contradiction that u analytic near 0, so we can write

$$u = \sum_{m,n} \frac{u_{m,n}}{m!n!} t^m x^n \ , \tag{2}$$

we have that the PDE gives us $u_{m+1,n} = u_{m,n+2}$. Our initial data tells us that $u_{0,4n} = g_{4n}$. It follows that

$$u_{m,2n} = u_{m-1,2(n+1)} = u_{m-2,2(n+2)} = u_{0,2(m+n)} = g_{2(m+n)}$$

(3 marks for using PDE to work out coefficients)

Next observe that for any $\tilde{R} > 0$

$$\limsup_n \left| \frac{u_{n,2n}}{(n!)(2n!)} \tilde{R}^n \right| = \limsup_n \left| \frac{g_{4n}}{(n!)(2n!)} \tilde{R}^n \right| > \limsup_n \left| \frac{g_{4n}}{(4n)!} (R+1)^{4n} \right| = \infty$$

where we used the fact that

$$\frac{(4n)!}{(n!)(2n!)} K^n \to \infty \ .$$

for any $K > 0$, in particular for $K = \tilde{R}/(R+1)^4$. Therefore, the series $\sum_{n=0}^{\infty} \frac{u_{n,2n}}{(n!)(2n!)} r^n$ has zero radius of convergence and the same would have to be true of (2), which is a contradiction. (2 marks for finishing argument)

(ii) This does not violate the conditions of the Cauchy-Kovaleskaya theorem because the $\{t = 0\}$ hyperplane is characteristic for the differential operator $\partial_t - \partial_x^2$. (2 marks)

(Total: 0 marks)

3. (a) Let $w = u + \epsilon t$ for $\epsilon > 0$ (3 marks for shifting function to use weak maximum principle argument), so that on the space-time domain $\Omega = (0,2) \times (-2,2)^2$ we have

$$\partial_t w + \epsilon = w(\partial_x w + \partial_y w) + \epsilon t(\partial_x w + \partial_y w) + \Delta w \ .$$

It follows that if w has relative extrema located at z in the interior of Ω then

$$\partial_t w(z) = \partial_x w(z) = \partial_y w(z) = 0 \ ,$$

which forces $\Delta w(z) = \epsilon > 0$ so that z is the location of a minimum. This means the maximum of w on $\tilde{\Omega} = [0,1] \times [-1,1]^2$ cannot only occur on $(0,1) \times (-1,1)^2$ (3 marks for ruling out interior).

Next, we suppose that w obtains its maximum on $\tilde{\Omega}$ at some $z = (1,p)$ with $p \in (1,1)^2$. It follows that $\partial_x w(z) = \partial_y w(z) = 0$ and $\partial_t w(z) \geq 0$, from which we infer $\Delta w(z) \geq \epsilon > 0$ - but this is in contradiction with z being the location of such a maximum. This rules out the maximum of w on $\tilde{\Omega}$ being obtained only at some point of the form $(1,p)$ with $p \in (1,1)^2$ (4 marks for ruling out terminal time boundary), so we have

$$\max_{(t,x,y)\in[0,1]\times[-1,1]^2} w(t,x,y) = \max\Big(\max_{(t,x,y)\in[0,1]\times\{-1,1\}^2} w(t,x,y), \max_{(x,y)\in[-1,1]^2} f(x,y) \Big) \ .$$

However, we are done since $\sup_{z\in\tilde{\Omega}} |u(z) - w(z)| = \epsilon$ and so, as $\epsilon \downarrow 0$,

$$\max_{(t,x,y)\in[0,1]\times\{-1,1\}^2} w(t,x,y) \vee \max_{(x,y)\in[-1,1]^2} f(x,y) \to \max_{(t,x,y)\in[0,1]\times\{-1,1\}^2} u(t,x,y) \vee \max_{(x,y)\in[-1,1]^2} f(x,y) \ .$$

(b) (i) We have $S(r) = \frac{1}{4\pi} \int_{S_1} u(rx)\mathrm{d}x$ from which it follows that

$$\frac{dS}{dr} = \frac{1}{4\pi} \int_{S_1} \frac{\partial u}{\partial n}(rx)\mathrm{d}x = \frac{1}{4\pi r^2} \int_{S_r} \frac{\partial u}{\partial n}(x)\mathrm{d}x = \frac{1}{4\pi r^2} \int_{B_r} u(x)\mathrm{d}x = rV(r)/3 \ .$$

where we used Green's formula in second to last equality above. (4 marks)

(ii) Since u is biharmonic, that means Δu is harmonic (2 marks), in particular one has

$$V(r) = \Delta u(0)$$

(3 marks for correctly using/stating mean value property) which gives us

$$\frac{d}{dr} S(r) = r\Delta u(0)/3 \Rightarrow S(r) = \frac{r^2}{6}\Delta u(0) + S(0) = \frac{r^2}{6}\Delta u(0) + u(0) \ ,$$

since $\lim_{r\downarrow 0} S(r) = u(0)$ since u is continuous at 0 (1 mark for finishing argument).

(Total: 0 marks)

4. (a) (i) We say $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $H_s(\mathbb{R}^d)$ if $\hat{u}$ is given by a measurable function and $\hat{u}(\xi)(1+|\xi|^2)^{s/2} \in L^2(\mathbb{R}^d)$, we define a norm on H_s by setting

$$\|u\|_{H_s} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \, \mathrm{d}\xi$$

(4 marks)

(ii) We have, by Cauchy-Schwartz,

$$\begin{aligned} \|\hat{u}\|_{L^1} &= \int_{\mathbb{R}^d} |\hat{u}(\xi)| \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} |\hat{u}(\xi)|(1+|\xi|)^{s/2}(1+|\xi|)^{-s/2} \, \mathrm{d}\xi \\ &\leq \Big(\int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 (1+|\xi|)^s \, \mathrm{d}\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^d} (1+|\xi|)^{-s} \, \mathrm{d}\xi \Big)^{1/2} = c(s,d)\|u\|_{H_s} \, , \end{aligned}$$

since $c(s,d)^2 = \int_{\mathbb{R}^d} (1+|\xi|)^{-s} \, \mathrm{d}\xi$ is finite for $s > d$. (4 marks)

(b) We set $\tilde{R} = (2R+1)$ and choose $T = 1/\big(12\tilde{c}(s,d)^2\tilde{R}^3\big)$.

We obtain the desired solution by using Banach's fixed point theorem on the Banach space $\mathcal{X}_T = C\Big([-T,T]; H_s\Big)$. For the norm on $\mathcal{X}_T$ we set, for $F : [-T,T] \to H_s$,

$$\|F\|_{\mathcal{X}_T} = \sup_{-T \leq r \leq T} \|F(r)\|_{H_s} \, .$$

We write $B_{\tilde{R}}$ for the ball of radius $\tilde{R}$ in $\mathcal{X}_T$.

Writing $S(t)$ for the Schroedinger group $S(t) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ with $(d/dt)S(t)u = (\partial_t - i\Delta)u$, we recall that $S(t)$ extends to a unitary map on H_s.

We will now define a map $\mathcal{K} : B_{\tilde{R}} \to B_{\tilde{R}}$ by setting

$$\mathcal{K}(F) = S(t)f - i\int_{-T}^{t} S(t-r)F(r)F(r)\bar{F}(r) \, \mathrm{d}r$$

By Duhamel's formula, we see that a fixed point of this map would be the desired solution. (4 marks for identifying right fixed point problem formulation)

To obtain such a fixed point, we need to show that $\mathcal{K}$ is indeed well-defined as a map form $B_{\tilde{R}}$ to itself and is contractive.

To show that it is well defined we note that, for $F \in B_{\tilde{R}}$,

$$\begin{aligned} \|\mathcal{K}(F)\|_{\mathcal{X}_T} \leq& \|S(\bullet)f\|_{\mathcal{X}_T} + \| \int_{-T}^{\bullet} S(\bullet - r)F(r)F(r)\bar{F}(r) \, \mathrm{d}r \|_{\mathcal{X}_T} \\ \leq& \|S(\bullet)f\|_{\mathcal{X}_T} + 2T \sup_{-T \leq r \leq t \leq T} \|S(t-r)F(r)F(r)\bar{F}(r)\|_{H_s} \\ =& \|f\|_{H_s} + 2T \sup_{-T \leq r \leq T} \|F(r)F(r)\bar{F}(r)\|_{H_s} \\ \leq& R + 2T\tilde{c}(s,d)^2 \sup_{-T \leq r \leq T} \|F(r)\|_{H_s} \leq R + 2T\tilde{c}(s,d)^2\tilde{R}^3 < 2R \, . \end{aligned}$$

In the third equality we used that $S(\cdot)$ preserves the H_s norm and in going to the final line we used the multiplicative inequality (and the fact that complex conjugation preserves the H_s norm). (4 marks for showing map is well defined)

Now we show $\mathcal{K}$ is contractive. Fixing $F, G \in B_{\tilde{R}}$, we have

$$\begin{aligned}\|\mathcal{K}(F-G)\|_{\mathcal{X}_T} \leq & \| \int_{-T}^{\bullet} S(\bullet - r)\Big(F(r)F(r)\bar{F}(r) - G(r)G(r)\bar{G}(r)\Big)\, \mathrm{d}r \|_{\mathcal{X}_T} \\ \leq & 2T \|F(\cdot)F(\cdot)\bar{F}(\cdot) - G(\cdot)G(\cdot)\bar{G}(\cdot)\|_{\mathcal{X}_T} \ .\end{aligned}$$

Now note that, thanks to the identity

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} \Big(\prod_{1 \leq j < i} b_j \Big) (a_i - b_i) \Big(\prod_{i < j \leq n} b_j \Big) \ ,$$

we have

$$\|F(r)F(r)\bar{F}(r) - G(r)G(r)\bar{G}(r)\|_{H_s} \leq 3\tilde{c}(s,d)^2 \tilde{R}^2 \|F - G\|_{H_s} \ .$$

It follows that

$$\|\mathcal{K}(F-G)\|_{\mathcal{X}_T} \leq 6T\tilde{c}(s,d)^2\tilde{R}^2 \|F-G\|_{\mathcal{X}_T} < \frac{1}{2}\|F-G\|_{\mathcal{X}_T} \ .$$

(4 marks for contractive bound)

(Total: 0 marks)

5. (a) We fix a test function η as in the statement of the theorem, and we take $v = \eta^2 u$ in the definition of weak solution (3 marks).

We then have

$$B(u,v) = \sum_{i,j=1}^{n} \int_U a^{i,j} \partial_j u \partial_i (\eta^2 u) = \sum_{i,j=1}^{n} \int_U a^{i,j} \partial_j u (2\eta \partial_i \eta u + \eta^2 \partial_i u)$$

We set $A_1 = \sum_{i,j=1}^{n} \int_U \eta^2 a^{i,j} \partial_j u \partial_i u$ and $A_2 = \sum_{i,j=1}^{n} \int_U a^{i,j} 2\eta u \partial_j u \partial_i \eta$.

Uniform ellipticity tells us that that there exists $\theta > 0$ such that

$$A_1 \geq \theta \int_U \eta^2 |Du|^2 \ .$$

(3 marks)

On the other hand, by estimating $2\sup_x |D\eta(x)| \leq C$ (which depends only on U, V) combined with Young's inequality,

$$|A_2| \leq C \int_U \eta |Du||u| \leq C_1 \epsilon \int_U \eta^2 |Du|^2 + \frac{C_2}{\epsilon} \int_U |u|^2 \ .$$

(3 marks)

Choosing $\epsilon = \frac{\theta}{2C_1}$ we have

$$B(u,v) = A_1 + A_2 \geq A_1 - |A_2| \geq \frac{\theta}{2} \int_U \eta^2 |Du|^2 - \frac{2C_1 C_2}{\theta} \int_U |u|^2$$

We also have

$$|B(u,v)| = \Big| \int_U vf \Big| = \Big| \int_U \eta^2 u f \Big| \leq \frac{1}{2} \int_U u^2 + \frac{1}{2} \int_U f^2 \ .$$

It follows that

$$\frac{\theta}{2} \int_V |Du|^2 \leq \frac{\theta}{2} \int_U \eta^2 |Du|^2 \leq \frac{2C_1 C_2}{\theta} \int_U u^2 + \frac{1}{2} \int_U u^2 + \int_U f^2 \ ,$$

which gives us the desired bound. (3 marks for putting things together)

(b) (i) Let $P(x, \partial)$ be a linear partial differential operator of order m whose coefficients are analytic in a neighborhood of a point $\bar{x} \in \mathbb{R}^d$. Let Σ be an analytic hypersurface which is non-characteristic for P at $\bar{x}$. Then, if $u \in C^m$ satisfies

$$Pu = 0 \text{ and } u\Big|_{\Sigma} = 0 \ , \tag{3}$$

it follows that u vanishes on a neighborhood of $\bar{x}$ in $\mathbb{R}^d$. (5 marks)

(ii) Let $P = \partial_t$, and let $\Sigma = \{x = 0\} \subset \mathbb{R}^2$. Note that every point of Σ is characteristic for P. The statement fails since, for any $c \in \mathbb{R}$, the function $u(t,x) = cx$ satisfies (3), but need not vanish in a neighborhood of any point on Σ. (3 marks)

(Total: 0 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70135	1	Students did well on this problem, it was relatively straightforward.
MATH70135	2	2)a)i) Students did well on this 2)a)ii) This was one of the more challenging questions on the exam, all students knew how to identify when a shock happens and nearly all were able to get far in computing the first shock 2)b)i) This is the closest to a really unseen problem, most students didn't do well on it - in particular many students didn't really present a plausible strategy for finding the proof. A key fact is that a function that if f is analytic at a point z but not entire then the power series of f at z must have a positive, finite radius of convergence. 2)b)ii) Most students got this, again fairly straightforward if you remember the CK Theorem.
MATH70135	3	3)a) Most students didn't do well on this problem, but it was quite close to the maximum principle arguments we discussed in lectures. 3)b) Most students did well on this problem
MATH70135	4	l4)a) i) This was a straightforward question, most people did well4)a)ii) Students overthought this is a bit, trying to use Sobolev embedding (which is possible but kind of indirect as this estimate is similar to one argument that can be used in the proof of Sobolev embedding). You can get away with using Cauchy-Schwarz and it is a quick proof4) b) Long but relatively straightforward, I think some students were running short on time
MATH70135	5	5)a) Essentially a proof on interior regularity we used in lectures, students knew they should test the localised solution against the eqution but only half were able to complete the proof5)b) Students did well on this problem