1(a). With the labelling of nodes given in the figure the Laplacian is

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

The system to solve is

$$\mathbf{Kx} = \mathbf{f}, \tag{1}$$

where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ X \\ Y \\ Z \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where X, Y, Z are the unknown potentials at nodes $4, 5$ and 6 and f_1 is the required effective conductance.

1(b). We can merge nodes 2 and 3 because they are both grounded, and by the symmetry of the graph and forcing we see that nodes 5 and 6 must be at the same potential/voltage so we can also merge those. Keeping all edges intact, the circuit becomes the "equivalent" circuit shown in the figure.

With the labelling of nodes given in the figure the Laplacian is

$$\hat{\mathbf{K}} = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 4 & -2 \\ -2 & -2 & -2 & 6 \end{bmatrix}.$$

The system to solve is

$$\hat{\mathbf{K}}\mathbf{x} = \mathbf{f}, \tag{2}$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ \hat{X} \\ \hat{Y} \end{bmatrix}, \qquad \hat{\mathbf{f}} = \begin{bmatrix} \hat{f}_1 \\ -\hat{f}_1 \\ 0 \\ 0 \end{bmatrix},$$

and $\hat{X}, \hat{Y}$ are the unknown potentials at nodes 4 and 5/6 and $\hat{f}_1$ is the effective conductance.

1(c). Introduce the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is easily checked that

$$\mathbf{KS} = \mathbf{SK} \tag{3}$$

that is, matrices $\mathbf{S}$ and $\mathbf{K}$ commute. This means that if $\mathbf{x}$ satisfies

$$\mathbf{Kx} = \mathbf{f} \tag{4}$$

then

$$\mathbf{SKx} = \mathbf{Sf},$$

or since $\mathbf{K}$ and $\mathbf{S}$ commute,

$$\mathbf{KSx} = \mathbf{Sf}. \tag{5}$$

Note that, unlike the example considered in lectures, $\mathbf{Sf} \neq \mathbf{f}$. However, we know that

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{Sf} = \begin{bmatrix} f_1 \\ f_3 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Let us introduce the following subblock decomposition of $\mathbf{K}$:

$$\mathbf{K} = \begin{bmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{bmatrix}, \tag{6}$$

where all subblock matrices are 3-by-3, and $\mathbf{R}$ is positive definite (and hence invertible). We will also write

$$\mathbf{x} = \begin{bmatrix} \mathbf{e}_1 \\ \hat{\mathbf{x}} \end{bmatrix}, \qquad \mathbf{Sx} = \begin{bmatrix} \mathbf{e}_1 \\ \hat{\mathbf{y}} \end{bmatrix},$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

It is then easy to check that (4) implies

$$\mathbf{Q}\mathbf{e}_1 + \mathbf{R}\hat{\mathbf{x}} = 0 \tag{8}$$

while (5) implies

$$\mathbf{Q}\mathbf{e}_1 + \mathbf{R}\hat{\mathbf{y}} = 0. \tag{9}$$

These equations imply that

$$\hat{\mathbf{x}} = -\mathbf{R}^{-1}\mathbf{Q}\mathbf{e}_1 = \hat{\mathbf{y}}.$$

Since

$$\hat{\mathbf{y}} = \begin{bmatrix} X \\ Z \\ Y \end{bmatrix}, \tag{10}$$

then we have proved that the potentials at nodes 5 and 6 are the same.

However (4) also implies

$$\mathbf{P}\mathbf{e_1} + \mathbf{Q}^T\hat{\mathbf{x}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

while (5) implies

$$\mathbf{P}\mathbf{e_1} + \mathbf{Q}^T\hat{\mathbf{y}} = \begin{bmatrix} f_1 \\ f_3 \\ f_2 \end{bmatrix}$$

However since $\hat{\mathbf{x}} = \hat{\mathbf{y}}$ we can also conclude that

$$f_2 = f_3.$$

This means that

$$\mathbf{S}\mathbf{f} = \mathbf{f}$$

and hence, on use of this in (4) and (5), we see that both $\mathbf{x}$ and $\mathbf{S}\mathbf{x}$ are solutions of the same circuit problem.

It follows that we can now add columns 5 and 6 in the linear system meaning that

$$\mathbf{K}\mathbf{x} = \mathbf{f}$$

becomes

$$\begin{bmatrix} 2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 4 & -2 \\ -1 & -1 & 0 & -1 & 3 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can also remove columns 2 and 3 since they do not contribute:

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & -2 \\ -1 & -1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can now eliminate the last row since it is a repeated equation; also, with $f_2 = f_3$, we can drop the third row since it is also a repeated equation:

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & 4 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix}$$

On multiplying rows 2 and 4 by 2 this system becomes

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} f_1 \\ 2f_2 \\ 0 \\ 0 \end{bmatrix}$$

We also know that since

$$f_1 + f_2 + f_3 = 0, \qquad f_2 = f_3$$

then

$$2f_2 = -f_1.$$

However this is the **same** as system (2) once the second column of that system is removed (because it does not contribute), i.e., system (2) becomes

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ -\hat{f}_1 \\ 0 \\ 0 \end{bmatrix}.$$

We have therefore systematically shown that the two linear systems are equivalent.

2(a). We notice that

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{K}_3 & 0 \\ 0 & 0 & \mathbf{K_2} \end{bmatrix}, \tag{11}$$

where a zero denotes a zero block matrix of appropriate size and $\mathbf{K}_n$ denotes an n-by-n circulant matrix. Since we know that $\mathbf{K_2}^2 = \mathbf{I}_2, \mathbf{K_3}^3 = \mathbf{I}_3$ it is clear that $\mathbf{S}^6 = \mathbf{I}_6$ (here $\mathbf{I}_n$ denotes the n-by-n identity matrix).

2(b). Suppose

$$\mathbf{S}\mathbf{x} = \lambda \mathbf{x} \tag{12}$$

then

$$\mathbf{S}^6\mathbf{x} = \lambda^6 \mathbf{x} = \mathbf{x} \tag{13}$$

which implies that

$$\lambda^6 = 1 = e^{2\pi \mathrm{i} m}, \tag{14}$$

where m is any integer. It follows that any eigenvalues of $\mathbf{S}$ are among the 6th roots of unity (note however that it does *not* mean that every 6th root of unity is an eigenvalue). Indeed it is clear from the block decomposition above that the eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ \omega_3 \\ \omega_3^2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ \omega_3^2 \\ \omega_3^4 \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

$$\mathbf{x}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \tag{16}$$

where $\omega_3 = e^{2\pi \mathrm{i}/3}$. The corresponding eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = \omega_3, \qquad \lambda_4 = \omega_3^2, \qquad \lambda_5 = 1, \qquad \lambda_6 = -1. \tag{17}$$

3. (a) If we write

$$\mathbf{X}_k = \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/n}, \qquad k = 0, 1, \cdots, n-1,$$

then, for $m \neq k$,

$$\overline{\mathbf{X}}_m^T \mathbf{X}_k = \begin{pmatrix} 1 \\ \omega^{-m} \\ \omega^{-2m} \\ \cdot \\ \cdot \\ \omega^{-(n-1)m} \end{pmatrix}^T \begin{pmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{pmatrix} = \sum_{j=1}^{n} \omega^{(j-1)(k-m)}$$

Using the sum of a finite geometric progression we find that

$$\overline{\mathbf{X}}_m^T \mathbf{X}_k = \frac{1-\omega^{(k-m)n}}{1-\omega} = 0 \tag{18}$$

since $\omega^{(k-m)n} = 1$ by the definition of ω.

(b). By the same arguments as in part (a) we see that if $m = k$ then

$$\overline{\mathbf{X}}_k^T \mathbf{X}_k = n. \tag{19}$$

It follows that we should pick

$$A_k = \frac{1}{\sqrt{n}} \tag{20}$$

to ensure that the vectors

$$\mathbf{x}_k = A_k \mathbf{X}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/n}, \qquad k = 0, 1, \cdots, n-1,$$

satisfy the orthonormality condition

$$\overline{\mathbf{x}}_k^T \mathbf{x}_m = \delta_{km}. \tag{21}$$

4. We want to use the results of the previous question to find the inverse of the matrix $\mathbf{K}_0$. For an $(n-1)$-by-$(n-1)$ matrix the relevant set of orthonormal vectors is

$$\mathbf{x}_k = \frac{1}{\sqrt{n-1}} \begin{pmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-2)k} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/(n-1)}, \qquad k = 0, 2, \cdots, n-2.$$

where we have simply set $n \mapsto n-1$ in the results of question 3. The eigenvalues of $\mathbf{K}_0$ are easy to find by simply computing $\mathbf{K}_0\mathbf{x}_k$:

$$\lambda_k = (n-1) - \sum_{j=1}^{n-2} \omega^{jk} = \begin{cases} 1, & k=0, \\ n, & k \neq 0. \end{cases} \tag{22}$$

Here we have used the fact that

$$\sum_{j=1}^{n-2} \omega^{jk} = -1$$

which is readily established using the formula for the sum of a finite geometric progression.

Suppose we write the required inverse in terms of its columns, i.e.,

$$\mathbf{K}_0^{-1} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \ldots \ \mathbf{c}_{n-2} \ \mathbf{c}_{n-1}]. \tag{23}$$

We will attempt to find these column vectors. To do so, since we know that

$$\mathbf{K}_0\mathbf{K}_0^{-1} = \mathbf{I} \tag{24}$$

then, for any $p = 1, \ldots, n-1$,

$$\mathbf{K}_0\mathbf{c}_p = \mathbf{e}_p, \qquad p = 1, \ldots, n-1, \tag{25}$$

where $\mathbf{e}_p$ is the column vector with 1 in element p and zeroes everywhere else. Now write the column vector $\mathbf{c}_p$ in terms of the eigenvectors:

$$\mathbf{c}_p = \sum_{j=0}^{n-2} a_j \mathbf{x}_j \tag{26}$$

Then

$$\mathbf{K}_0\mathbf{c}_p = \sum_{j=0}^{n-2} a_j \mathbf{K}_0 \mathbf{x}_j = \sum_{j=0}^{n-2} a_j \lambda_j \mathbf{x}_j = \mathbf{e}_p, \qquad p = 1, \ldots, n-1. \tag{27}$$

This equation can be multiplied by $\overline{\mathbf{x}}_k^T$:

$$\sum_{j=0}^{n-2} a_j \lambda_j \overline{\mathbf{x}}_k^T \mathbf{x}_j = \overline{\mathbf{x}}_k^T \mathbf{e}_p, \qquad p = 1, \ldots, n-1. \tag{28}$$

It follows, on use of the orthonormality property, that

$$a_k \lambda_k = \frac{1}{\sqrt{n-1}} \overline{\omega}^{(p-1)k} = \frac{1}{\sqrt{n-1}} \omega^{(1-p)k}.$$

This gives

$$a_k = \frac{1}{\sqrt{n-1}} \frac{\omega^{(1-p)k}}{\lambda_k}$$

and, consequently,

$$\mathbf{c}_p = \sum_{j=0}^{n-2} \frac{1}{\sqrt{n-1}} \frac{\omega^{(1-p)j}}{\lambda_j} \mathbf{x}_j = \sum_{j=0}^{n-2} \frac{1}{n-1} \frac{\omega^{(1-p)j}}{\lambda_j} \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ . \\ . \\ \omega^{(n-2)j} \end{pmatrix} \tag{29}$$

or

$$\mathbf{c}_p = \frac{1}{n-1} \begin{pmatrix} \sum_{j=0}^{n-2} \frac{\omega^{(1-p)j}}{\lambda_j} \\ \sum_{j=0}^{n-2} \frac{\omega^{(2-p)j}}{\lambda_j} \\ \sum_{j=0}^{n-2} \frac{\omega^{(3-p)j}}{\lambda_j} \\ . \\ . \\ \sum_{j=0}^{n-2} \frac{\omega^{(n-1-p)j}}{\lambda_j} \end{pmatrix}. \tag{30}$$

The q-th term in this vector has the form

$$\sum_{j=0}^{n-2} \frac{\omega^{(q-p)j}}{\lambda_j} \tag{31}$$

But, using (22), this is

$$\begin{aligned} \sum_{j=0}^{n-2} \frac{\omega^{(q-p)j}}{\lambda_j} &= 1 + \frac{1}{n} \left[\omega^{q-p} + \omega^{2(q-p)} + \ldots \omega^{(n-2)(q-p)} \right] \\ &= \begin{cases} 1 + \frac{1}{n}(n-2), & q = p, \\ \\ 1 - \frac{1}{n}, & q \neq p \end{cases} \end{aligned} \tag{32}$$

where we have used the facts that, for $q = p$,

$$\omega^{q-p} + \omega^{2(q-p)} + \ldots \omega^{(n-2)(q-p)} = (n-2)$$

and that when $q \neq p$,

$$\omega^{q-p} + \omega^{2(q-p)} + \ldots \omega^{(n-2)(q-p)} = -1,$$

where the latter follows using the formula for a finite geometrical progression.

On simplifying the result (32), and substituting into (30), for any choice of $p = 1, \ldots, n-1$, column $\mathbf{c}_p$ of $\mathbf{K}_0^{-1}$ has $2/n$ as its p-th element with all other elements equal to $1/n$. In other words, we have therefore established that the inverse is

$$\mathbf{K}_0^{-1} = \frac{1}{n} \begin{bmatrix} 2 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 2 & \cdots & \cdots & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & \cdots & \cdots & 1 & 2 \end{bmatrix}. \tag{33}$$

This is the same answer as obtained, using different arguments, in question 9 of Problem Sheet 1.

5. (a) Write the unnormalized $\mathbf{\Phi}_k$ as

$$\mathbf{\Phi}_k = \mathrm{Im} \begin{pmatrix} \omega_k \\ \omega_k^2 \\ . \\ . \\ \omega_k^n \end{pmatrix} = \frac{1}{2\mathrm{i}} \begin{pmatrix} \omega_k - \omega_k^{-1} \\ \omega_k^2 - \omega_k^{-2} \\ . \\ . \\ \omega_k^n - \omega_k^{-n} \end{pmatrix}, \qquad k = 1, 2, \cdots, n,$$

where

$$\omega_k = e^{k\pi\mathrm{i}/(n+1)}. \tag{34}$$

Here Im[.] means take the imaginary part. It follows that

$$\mathbf{\Phi}_k{}^T \mathbf{\Phi}_k = \frac{1}{4} \sum_{j=1}^{n} \left(\omega_k^j - \omega_k^{-j}\right) \left(\omega_k^{-j} - \omega_k^j\right)$$

since we must sum the squared modulus of each of the elements. However this is

$$\mathbf{\Phi}_k{}^T \mathbf{\Phi}_k = \frac{1}{4} \sum_{j=1}^{n} \left(2 - \omega_k^{2j} - \omega_k^{-2j}\right) = \frac{1}{4} \left(2n - \sum_{j=1}^{n} \omega_k^{2j} - \sum_{j=1}^{n} \omega_k^{-2j}\right). \tag{35}$$

Now on summing a finite geometric progression,

$$\sum_{j=1}^{n} \omega_k^{2j} = \frac{\omega_k^{2n+2} - \omega_k^2}{\omega_k^2 - 1} = \omega_k^2 \left[\frac{\omega_k^{2n} - 1}{\omega_k^2 - 1}\right].$$

But it follows from (34)

$$\omega_k^{2n} = \omega_k^{2(n+1)-2} = \frac{1}{\omega_k^2}.$$

Hence

$$\sum_{j=1}^{n} \omega_k^{2j} = \omega_k^2 \left[\frac{\omega_k^{-2} - 1}{\omega_k^2 - 1} \right] = -1$$

and, on taking a complex conjugate,

$$\sum_{j=1}^{n} \omega_k^{-2j} = -1.$$

Substitution of these results into (35) leads to

$$\mathbf{\Phi}_k{}^T \mathbf{\Phi}_k = \frac{1}{4}(2n+2) = \frac{n+1}{2}. \tag{36}$$

Note that this result is independent of k (i.e. it is the same for every eigenvector). We therefore need to take

$$A_k = \sqrt{\frac{2}{n+1}} \tag{37}$$

in order to normalize the eigenvectors.

(b) Given

$$\mathbf{\Phi}_n = \begin{pmatrix} \sin(n\pi/(N+1)) \\ \sin(2n\pi/(N+1)) \\ . \\ . \\ \sin(nN\pi/(N+1)) \end{pmatrix}, \qquad n = 1, 2, \cdots, N$$

then

$$\mathbf{\Phi}_n^T \mathbf{\Phi}_m = \sum_{k=1}^{N} \sin\left(\frac{n\pi k}{N+1}\right) \sin\left(\frac{m\pi k}{N+1}\right) \tag{38}$$

Using a trigonometric identity we can write this as

$$\begin{aligned} \mathbf{\Phi}_n^T \mathbf{\Phi}_m &= \frac{1}{2} \sum_{k=1}^{N} \left[\cos\left(\frac{(n-m)\pi k}{N+1}\right) - \cos\left(\frac{(n+m)\pi k}{N+1}\right) \right] \\ &= \frac{1}{2} \mathrm{Re} \left\{ \sum_{k=1}^{N} \left[e^{\pi \mathrm{i}(n-m)k/(N+1)} - e^{\pi \mathrm{i}(n+m)k/(N+1)} \right] \right\}. \end{aligned} \tag{39}$$

Now using the sum of a finite geometric progression we know that

$$\sum_{k=1}^{N} x^k = \frac{x(1-x^N)}{1-x} \tag{40}$$

Hence

$$\begin{aligned}\sum_{k=1}^{N} e^{\pi \mathrm{i}(n-m)k/(N+1)} &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1-e^{\pi \mathrm{i}(n-m)N/(N+1)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1-e^{\pi \mathrm{i}(n-m)(N+1-1)/(N+1)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= e^{\pi \mathrm{i}(n-m)/(N+1)} \frac{1-(-1)^{(n-m)}e^{-\pi \mathrm{i}(n-m)/(N+1)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)}-(-1)^{(n-m)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}}\end{aligned} \tag{41}$$

A similar formula holds with $m-n \mapsto m+n$. Now $m-n$ and $m+n$ will either both be even or odd. If they are both even then

$$\begin{aligned}\frac{e^{\pi \mathrm{i}(n-m)/(N+1)}-(-1)^{(n-m)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} &= \frac{e^{\pi \mathrm{i}(n+m)/(N+1)}-(-1)^{(n+m)}}{1-e^{\pi \mathrm{i}(n+m)/(N+1)}} \\ &= -1\end{aligned} \tag{42}$$

hence

$$\boldsymbol{\Phi}_n^T \boldsymbol{\Phi}_m = 0. \tag{43}$$

If $m-n$ and $m+n$ are both odd then

$$\begin{aligned}\frac{e^{\pi \mathrm{i}(n-m)/(N+1)}-(-1)^{(n-m)}}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)}+1}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} \\ &= \frac{e^{\pi \mathrm{i}(n-m)/(N+1)}+1}{1-e^{\pi \mathrm{i}(n-m)/(N+1)}} \left[\frac{1-e^{-\pi \mathrm{i}(n-m)/(N+1)}}{1-e^{-\pi \mathrm{i}(n-m)/(N+1)}}\right]\end{aligned} \tag{44}$$

which is purely imaginary, as will the same quantity with $m-n \mapsto m+n$. Hence in this case too

$$\boldsymbol{\Phi}_n^T \boldsymbol{\Phi}_m = \frac{1}{2}\mathrm{Re}\left\{\sum_{k=1}^{N}\left[e^{\pi \mathrm{i}(n-m)k/(N+1)} - e^{\pi \mathrm{i}(n+m)k/(N+1)}\right]\right\} = 0. \tag{45}$$

6. A circulant matrix of dimension $n+1$ has the form

$$C_{n+1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

This can be written in the sub-block decomposition

$$C_{n+1} = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix}. \tag{46}$$

where

$$K_n = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

is the matrix considered in lectures. We know eigenvectors $\mathbf{x}$ of C_{n+1} are

$$\mathbf{x}_m = \begin{bmatrix} 1 \\ \omega^m \\ \omega^{2m} \\ \cdot \\ \cdot \\ \omega^{nm} \end{bmatrix}, \qquad m = 0, 1, \cdots, n, \tag{47}$$

where $\omega = e^{2\pi i/(n+1)}$ with eigenvalue

$$\lambda_m = 2\left[1 - \cos(2\pi m/(n+1))\right].$$

The imaginary part of this is also an eigenvector

$$\hat{\mathbf{x}}_m = \mathrm{Im}[\mathbf{x}_m] = \begin{bmatrix} 0 \\ \mathrm{Im}[\omega^m] \\ \mathrm{Im}[\omega^{2m}] \\ \cdot \\ \cdot \\ \mathrm{Im}[\omega^{nm}] \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{X}_m \end{bmatrix}, \qquad m = 0, 1, \cdots, n. \tag{48}$$

Of these $n+1$ possible vectors $\hat{x}_0$ is identically zero (the case $m = 0$), while the vectors for each $m = j$ and $m = n+1-j$ for $j = 1, \ldots, n$ can be shown to be linearly dependent, meaning that there are only $n/2$ linearly independent vectors of this kind. But any such vector has the property

$$C_{n+1}\hat{\mathbf{x}}_m = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix} \hat{\mathbf{x}}_m = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & K_n \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{X}_m \end{bmatrix} = \lambda_m \hat{\mathbf{x}}_m, \tag{49}$$

or equivalently,

$$\mathbf{q}^T\mathbf{X}_m = 0, \qquad K_n\mathbf{X}_m = \lambda_m\mathbf{X}_m. \tag{50}$$

This means that the n-dimensional subvector $\mathbf{X}_m$ is an eigenvector of K_n, and there are $n/2$ of these (recall that we have restricted n to be even). Therefore, this construction produces *half* of the eigenvectors of K_n (and gives further rationale for considering the "doubled circulant" matrix in lectures).

7. Writing the Fourier sine series as

$$1 - \frac{x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx \tag{51}$$

then we multiply by $\sin mx$ and integrate between 0 and π:

$$\int_0^{\pi} \sin mx \left[1 - \frac{x}{\pi}\right] dx = \int_0^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin mx dx. \tag{52}$$

Now on use of integration by parts we find

$$\int_0^{\pi} \sin mx \left[1 - \frac{x}{\pi}\right] dx = \frac{1}{m}.$$

Also, from a trigonometric identity,

$$\int_0^{\pi} \sin mx \sin nx dx = \frac{1}{2} \int_0^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$$

and the right hand side equals 0 unless $m = n$ when it equals $\pi/2$. On substitution of these results into (52) above we find

$$\frac{\pi}{2} b_m = \frac{1}{m}$$

so that, from (51),

$$1 - \frac{x}{\pi} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin nx$$

as found in lectures by a limiting process of a $n+1$ spring-mass system.

8. Notice that

$$\tilde{\mathbf{K}}_n = \begin{bmatrix} a & -b & 0 & . & . & 0 \\ -b & a & -b & . & . & 0 \\ 0 & -b & a & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -b & a & -b \\ 0 & 0 & . & . & -b & a \end{bmatrix}$$

$$= b \begin{bmatrix} a/b & -1 & 0 & . & . & 0 \\ -1 & a/b & -1 & . & . & 0 \\ 0 & -1 & a/b & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -1 & a/b & -1 \\ 0 & 0 & . & . & -1 & a/b \end{bmatrix}$$

Now this can be written

$$\tilde{\mathbf{K}}_n = b \begin{bmatrix} (\frac{a}{b}-2)+2 & -1 & 0 & . & . & 0 \\ -1 & (\frac{a}{b}-2)+2 & -1 & . & . & 0 \\ 0 & -1 & (\frac{a}{b}-2)+2 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -1 & (\frac{a}{b}-2)+2 & -1 \\ 0 & 0 & . & . & -1 & (\frac{a}{b}-2)+2 \end{bmatrix}$$

$$= b\left[\left(\frac{a}{b}-2\right)\mathbf{I}_n + \mathbf{K}_n\right]$$

where $\mathbf{K}_n$ is the matrix considered in lectures, and in question 5. It is now clear that the vectors $\mathbf{\Phi}_j$ for $j = 1, \ldots, n$ are also eigenvectors of $\tilde{\mathbf{K}}_n$ since

$$\tilde{\mathbf{K}}_n \mathbf{\Phi}_j = b\left[\left(\frac{a}{b}-2\right)\mathbf{I}_n + \mathbf{K}_n\right]\mathbf{\Phi}_j = b\left[\left(\frac{a}{b}-2\right) + \lambda_j\right]\mathbf{\Phi}_j$$

where

$$\lambda_j = 2\left(1 - \cos\left(\frac{\pi j}{n+1}\right)\right), \qquad j = 1, \ldots, n$$

are the eigenvalues of $\mathbf{K}_n$. In summary, the eigenvectors of $\tilde{\mathbf{K}}_n$ are the vectors $\mathbf{\Phi}_j$ with eigenvalues

$$\tilde{\lambda}_j = a - 2b\cos\left(\frac{\pi j}{n+1}\right), \qquad j = 1, \ldots, n.$$