

Probability for Statistics

Problem Sheet 2

1. Consider a probability space $(\Omega, \mathcal{F}, \Pr)$ in which

$$\Omega = \{1, 2, 3, 4, 5, 6\}, \quad \mathcal{F} = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}.$$

Determine whether each of the two functions $X_1, X_2 : \Omega \rightarrow \mathbf{R}$ defined below is a random variable with respect to \mathcal{F} .

$$X_1(s) = s, \quad X_2(s) = \begin{cases} 0 & s \text{ even} \\ 1 & s \text{ odd} \end{cases} \quad \forall s \in \Omega$$

2. (a) Let $X : \Omega \rightarrow \mathbf{R}$ be a random variable, and let \mathcal{B} be the Borel sigma algebra on \mathbf{R} . Show that $\mathcal{F}_X = \{X^{-1}(B) : B \in \mathcal{B}\}$ is a sigma algebra on Ω .
- (b) Consider an experiment in which a fair coin is flipped twice, so that the sample space is $\Omega = \{HH, HT, TH, TT\}$. Let $X : \Omega \rightarrow \mathbf{R}$ take the value 1 if precisely one flip comes up heads, and 0 otherwise. Determine the sigma algebra \mathcal{F}_X .
- (c) For Ω as in the previous part, give an example of a function $Y : \Omega \rightarrow \mathbf{R}$ and a function g (with suitable domain) such that $X = g(Y)$ and $\mathcal{F}_X \subset \mathcal{F}_Y$.
3. Suppose P and Q are two probability functions defined on the same sample space Ω and sigma algebra \mathcal{F} .
- (a) Show that if $P(A) = Q(A)$ for all $A \in \mathcal{F}$ such that $P(A) \leq \frac{1}{2}$, then in fact $P(A) = Q(A)$ for all $A \in \mathcal{F}$.
- (b) Show by means of an explicit example that if instead we only have $P(A) = Q(A)$ for all $A \in \mathcal{F}$ such that $P(A) < \frac{1}{2}$, then P and Q need not agree on all of \mathcal{F} .
4. Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space and let X and Y be random variables with respect to \mathcal{F} . If $A \in \mathcal{F}$, define $Z : \Omega \rightarrow \mathbf{R}$ by
- $$Z(\omega) = \begin{cases} X(\omega) & \omega \in A \\ Y(\omega) & \omega \notin A. \end{cases}$$
- (a) Show that Z is a random variable with respect to \mathcal{F} .
- (b) Show that if instead $A \subseteq \Omega$ is not an event, i.e. $A \notin \mathcal{F}$, Z need not be a random variable.
5. On the probability space $(\Omega, \mathcal{F}, \Pr)$, let Z be a random variable such that $\Pr(Z > 0) > 0$. Explain carefully why there exists $\delta > 0$ such that $\Pr(Z \geq \delta) > 0$.
6. In this question, you will derive the mean and variance of the hypergeometric distribution.
- (a) If $X \sim \text{BINOMIAL}(n, p)$, we can write $X = \sum_{i=1}^n Z_i$, where $Z_i \sim \text{BERNOULLI}(p)$ are independent. Use this representation to show that $E(X) = np$ and $\text{Var}(X) = np(1-p)$.

Suppose now that X is hypergeometric, representing the distribution of the number of red balls in a sample of size n drawn without replacement from an urn containing r red and w white balls, $N = r + w$. In this case,

$$\Pr(X = x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}.$$

As in the binomial case, we can represent X as a sum of Bernoulli variables: $X = \sum_{i=1}^n Z_i$, where Z_i takes the value 1 if the i th ball is red and 0 otherwise.

- (b) What is the distribution of the Z_i ? Are they independent?
- (c) Show that $E(X) = n \frac{r}{N}$.
- (d) (Harder) Show that $\text{Var}(X) = n \frac{r}{N} \frac{w}{N} \frac{N-n}{N-1}$.

For discussion

7. For real numbers $s > 1$, define the Riemann zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Let $s > 1$ be fixed, and let the random variable X have probability mass function

$$f_X(x) = \Pr(X = x) = \frac{1}{x^s} \frac{1}{\zeta(s)}, \quad x \geq 1.$$

Let D_k by the event that X is divisible by k , for $k \geq 2$.

- (a) What is $\Pr(D_k)$?
- (b) Show that the events $\{D_p : p \text{ is prime}\}$ are independent.
- (c) Prove Euler's formula for the zeta function in terms of the prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Hint: You may assume that whenever a collection $\{A_i : i \in I\}$ of events is independent, so is the collection $\{A_i^c : i \in I\}$. Recall also that for a countable collection of independent events,

$$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \prod_{i=1}^{\infty} \Pr(A_i).$$

8. In this question, we look what happens to the geometric distribution when we pass from discrete to continuous time. Let T have the waiting time geometric distribution with parameter p , so that

$$\Pr(T \geq j) = (1-p)^j, \quad j = 0, 1, 2, \dots$$

We think of T , which takes non-negative integer values, as the number of units of time we need to wait for an event to occur. When p is very small, T typically takes very large values, so we seek to rescale time, so that the waiting times are given in more reasonable units. Let M be a large number, such that $a = pM$ and $t = \frac{j}{M}$ are both small relative to M . What is the distribution of $U = \frac{T}{M}$, in terms of the parameter a ? What important property has been preserved in this limit?