Imperial College London
MATH 50004/50015 Multivariable Calculus
Mid-Term Examination Date: 16th November 2023
SOLUTIONS

Part (a) solution

(i)

$$
\begin{aligned}
(\text{curl}\mathbf{A})\cdot(\text{curl}\mathbf{A}) &= (\text{curl}\mathbf{A})_i(\text{curl}\mathbf{A})_i \\
&= \varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}\varepsilon_{ilm}\frac{\partial A_m}{\partial x_l} \\
&= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\frac{\partial A_k}{\partial x_j}\frac{\partial A_m}{\partial x_l} \\
&= \left(\frac{\partial A_k}{\partial x_j}\right)^2-\frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k},
\end{aligned}
$$

as required. **[3 marks]**
(ii)

$$
\begin{aligned}
(\mathbf{A}\times\text{curl}\mathbf{A})_j &= \varepsilon_{jkl}A_k(\text{curl}\mathbf{A})_l \\
&= \varepsilon_{jkl}A_k\varepsilon_{lmn}\frac{\partial A_n}{\partial x_m} \\
&= A_k(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})\frac{\partial A_n}{\partial x_m} \\
&= A_k\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right),
\end{aligned}
$$

as required. **[3 marks]**
(iii)

$$
\begin{aligned}
\text{div}(\mathbf{A}\times\text{curl}\mathbf{A}) &= \frac{\partial}{\partial x_j}(\mathbf{A}\times\text{curl}\mathbf{A})_j \\
&= \frac{\partial}{\partial x_j}\left(A_k\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right)\right) \\
&= \frac{\partial A_k}{\partial x_j}\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right)+A_k\frac{\partial}{\partial x_j}\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right) \\
&= \left(\frac{\partial A_k}{\partial x_j}\right)^2-\frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k}+A_k\frac{\partial^2 A_k}{\partial x_j^2}-A_k\frac{\partial}{\partial x_k}\left(\frac{\partial A_j}{\partial x_j}\right).
\end{aligned}
$$

The final term is zero because the field is solenoidal (i.e. $\text{div}\,\mathbf{A}=0$) **[3 marks]**
Therefore putting this together with part (i) we have

$$
\begin{aligned}
\text{div}(\mathbf{A}\times\text{curl}\mathbf{A})-(\text{curl}\mathbf{A})\cdot(\text{curl}\mathbf{A}) &= A_k\frac{\partial^2 A_k}{\partial x_j^2} \\
&= \mathbf{A}\cdot\nabla^2\mathbf{A}.\ \textbf{[1 mark]}
\end{aligned}
$$

Part (b) solution

(i) The perimeter of the triangle should be traversed anti-clockwise, keeping the finite interior region to the left. **[1 mark]**
(ii) We can split C up into C_1 (the part along the $x-$axis, i.e. $y = 0, x = t$ with t from 0 to 1), C_2 ($y = 1 - t$, $x = t$ starting at $t = 1$ ending at $t = 1/2$), and C_3 ($y = x = t$ with t starting at 1/2 and ending at 0.) **[3 marks]**
(iii) Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ where $F_1(x, y) = e^{x+y}, F_2(x, y) = xy$. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{3} \int_{C_i} (F_1 dx + F_2 dy) .$$

Now

$$\int_{C_1} F_1 dx + F_2 dy = \int_0^1 F_1(t, 0) dt = \int_0^1 e^t dt = e - 1, \;\; \textbf{[2 marks]}$$

since $dy = 0$ on C_1.

$$\begin{aligned} \int_{C_2} F_1 dx + F_2 dy &= \int_1^{1/2} F_1(t, 1-t) dt - F_2(t, 1-t) dt = \int_1^{1/2} e - t + t^2 dt \\ &= \left[et - \frac{t^2}{2} + \frac{t^3}{3} \right]_1^{1/2} = -\frac{e}{2} + \frac{1}{12}, \;\; \textbf{[2 marks]} \end{aligned}$$

where we have used the fact that $dy = -dx = -dt$ on C_2.

$$\begin{aligned} \int_{C_3} F_1 dx + F_2 dy &= \int_{1/2}^{0} F_1(t, t) dt + F_2(t, t) dt = \int_{1/2}^{0} e^{2t} + t^2 dt \\ &= \left[\frac{e^{2t}}{2} + \frac{t^3}{3} \right]_{1/2}^{0} = \frac{1}{2} - \frac{e}{2} - \frac{1}{24}, \;\; \textbf{[2 marks]} \end{aligned}$$

with $dy = dx = dt$ on C_3. Adding the three contributions together: $\oint_C \mathbf{F} \cdot d\mathbf{r} = -11/24$.
(iv)

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{x+y} & xy & 0 \end{vmatrix} = (y - e^{x+y})\mathbf{k}. \;\; \textbf{[1 mark]}$$

We can cover R by horizontal strips with x starting at y and ending at $1-y$, and then considering all values of y between 0 and 1/2. Thus:

$$\begin{aligned} \int_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \; dx \, dy &= \int_{y=0}^{y=1/2} \int_{x=y}^{x=1-y} (y - e^x e^y) \, dx \, dy \\ &= \int_{y=0}^{y=1/2} [xy - e^x e^y]_{x=y}^{x=1-y} \, dy \\ &= \int_0^{1/2} y(1-y) - e - y^2 + e^{2y} \, dy \\ &= \left[\frac{y^2}{2} - \frac{2y^3}{3} - ey + \frac{e^{2y}}{2} \right]_0^{1/2} \\ &= -\frac{11}{24}. \;\; \textbf{[3 marks]} \end{aligned}$$

in agreement with part (iii). If we do the y integration first then the top boundary is described by $y = x$ for $0 < x < 1/2$ and $y = 1 - x$ for $1/2 < x < 1$. Therefore the double integral needs to be evaluated in two parts. **[1 mark]**