

We further note that if the short-run fixed factors  $\underline{x}_F$  are fixed at their long-run conditional factor demand for a given output,  $y^*$  (i.e.  $\underline{x}_F = \underline{x}_F^*(\underline{w}, y^*)$ ), then

$$LMC(y^*) = SMC(y^*)$$

Proof

The argument is as follows:

$$C^*(\underline{w}, y) = C_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y) \quad \forall y > 0$$

That means, we can take the total derivative on both sides. That is

$$\frac{dC^*(\underline{w}, y)}{dy} = \frac{dC_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}{dy}$$

$$= \sum_{j=1}^n \frac{\partial C_s^*(\underline{w}, \underline{x}_F, y)}{\partial (\underline{x}_F)_j} \left| \frac{d(\underline{x}_F^*(\underline{w}, y))}{dy} \right|_{\underline{x}_F = \underline{x}_F^*(\underline{w}, y)}$$

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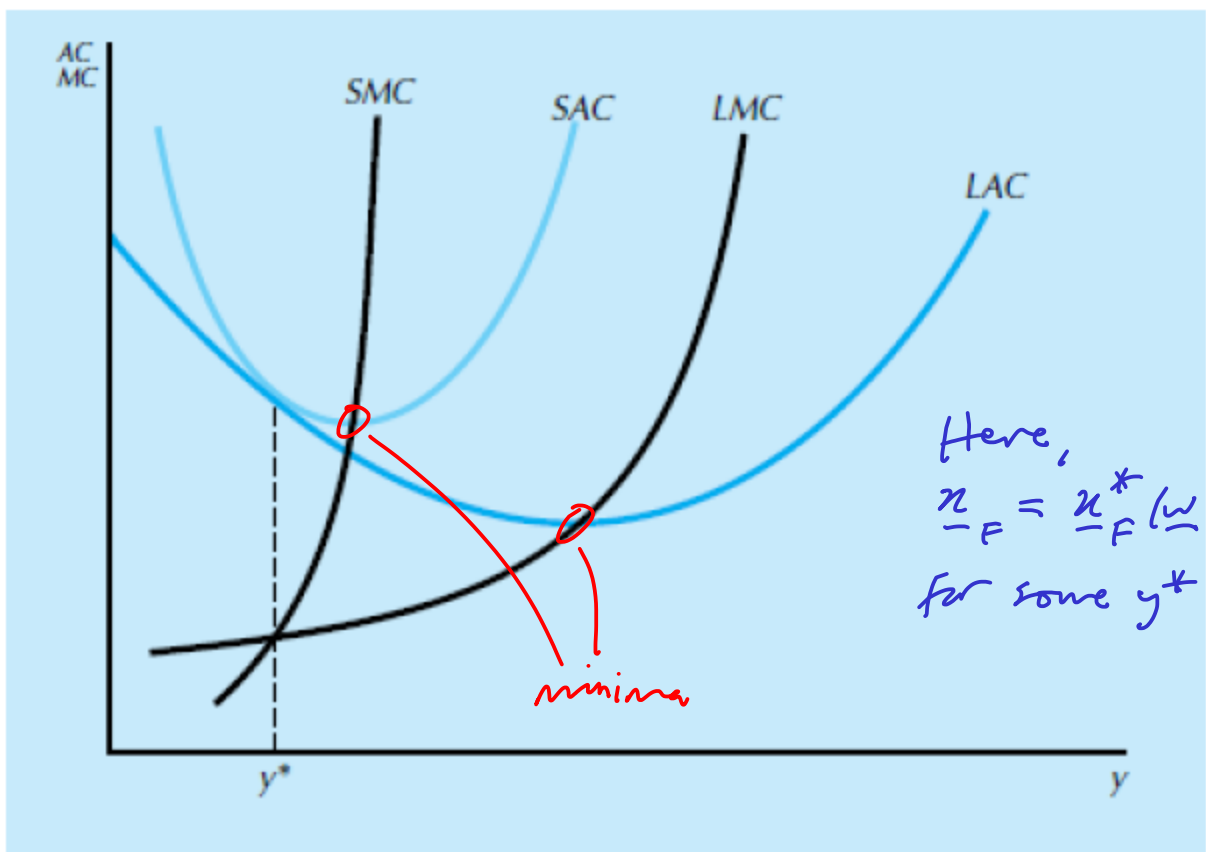
0 since  $\underline{x}_F = \underline{x}_F^*(\underline{w}, y)$  minimizes  $C_s^*(\underline{w}, \underline{x}_F, y)$

$$+ \frac{\partial C_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= \frac{\partial C_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= SMC(y)$$

but  $\frac{dC^*(\underline{w}, y)}{dy} = \frac{\partial C^*(\underline{w}, y)}{\partial y}$  since  $\underline{w}$  is constant. //



So, if  $\pi_F = \pi_F^*(\underline{w}, y^*)$ , then when  $y = y^*$  the additional cost per additional unit of output (i.e. the marginal cost) is the same in the short-run as in the long-run (i.e., whether or not any of the inputs are constrained).

## Profit maximisation given minimised costs

We have now considered the choices that a firm must make to minimise its costs, given knowledge of its factor prices  $\underline{w}$  and a given level of output  $y$ . As mentioned previously, we now consider how a firm should subsequently choose an optimal level of output  $y$  in order to maximise profits conditional on minimised costs.

To set up the profit-maximisation question in this conditional framework, we initially maintain the assumption of perfect competition; we also assume to begin with that we are operating in the short-run.

Recall that previously, profit maximisation was framed as a question of how much input to use, and that the output of the firm was specified by the production function  $f$ . Now, all of the firm's technical constraints are implicitly specified by the cost function.

We therefore reformulate our profit maximisation problem:

We wish to solve

$$\operatorname{argmax}_{y \geq 0} \left\{ py - c_s^*(\underline{w}, \underline{r}_F, y) \right\}$$

for  $y$ . Note that in the short-run,  $\underline{r}_F$  is fixed. And  $p$  and  $\underline{w}$  are also fixed (by the assumption of perfect competition). Hence  $py - c_s^*(\underline{w}, \underline{r}_F, y)$  is simply a function of  $y$ .

First- and second-order conditions for the optimal level of output given minimised costs are given by:

$$\frac{\partial}{\partial y} (py - c_s^*(\underline{w}, \underline{r}_F, y)) = 0 \Rightarrow p = SMC(y)$$

$$\frac{\partial^2}{\partial y^2} (py - c_s^*(\underline{w}, \underline{r}_F, y)) \leq 0 \Rightarrow \frac{\partial^2 c_s^*(\underline{w}, \underline{r}_F, y)}{\partial y^2} \geq 0$$

$$\text{i.e., } \frac{\partial SMC(y)}{\partial y} \geq 0.$$

These conditions suggest that, in order to maximise profits, the output should be such that the corresponding short-run marginal cost is increasing and equal to the output price  $p$ .

For a cost-minimising competitive firm, this specifies a relationship between the market-defined output price  $p$  and the quantity of output that the firm should provide.

### Example 1:

Suppose that a firm's short-run cost function for a good is specified as

$$C_S^*(w_1, w_2, r_F, y) = 2 \sqrt{w_1 w_2} y^2 + \underbrace{FC(w_F, r_F)}_{\text{fixed costs}}$$

If the market price for the good is  $\text{£}16$  and each input costs the firm  $\text{£}4$ , how many units of the good should the firm produce in the short run, and what is their maximised profit if fixed costs are  $\text{£}12$ ?

Solution:

$$SMC(y) = \frac{\partial C_S^*(w_1, w_2, r_F, y)}{\partial y} = 4 \sqrt{w_1 w_2} y \quad (\text{a linear function of } y)$$

Then

$$\text{F.O.C.: } SMC(y) = p \Rightarrow y = \frac{p}{4 \sqrt{w_1 w_2}} = 1, = \hat{y}, \text{ say}$$

$$\text{S.O.C.: } \frac{\partial SMC(y)}{\partial y} = 4 \sqrt{w_1 w_2} = 16 \geq 0 \quad \forall y$$

So  $\hat{y} = 1$  gives the maximum profit. And this is:

$$\begin{aligned}
 \text{maximum profit} &= p\hat{y} - c_s^*(w_1, w_2, \underline{x}_F, \hat{y}) \\
 &= 16.1 - (2\sqrt{w_1 w_2} \cdot (1)^2 + 12) \\
 &= -4
 \end{aligned}$$

One may deduce that...

In the short-run, i.e. when there are fixed costs, the most profitable position for a firm may be one that returns negative profit.

### Example 2:

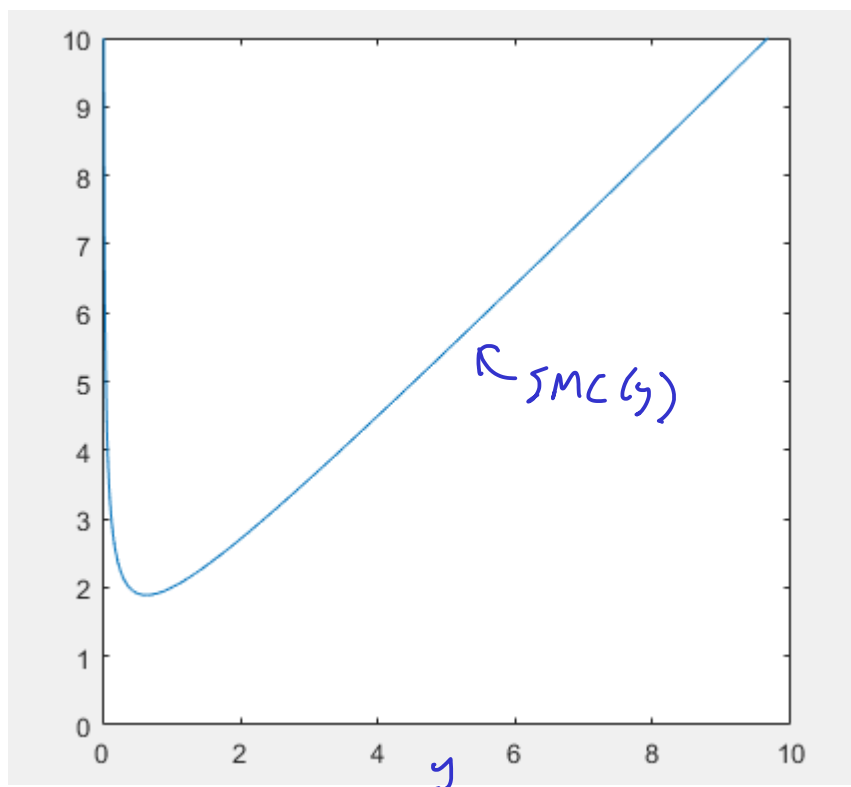
Consider the cost function

$$c_s^*(\underline{w}, \underline{x}_F, y) = w_1 y^{1/2} + w_2 y^2 + FC(\underline{w}_F, \underline{x}_F).$$

What is the maximised profit here, when  $w_1 = 2$ ,  $w_2 = \frac{1}{2}$ , and  $p = 2$ ?

Solution

$$SMC(y) = \frac{\partial c_s^*(\underline{w}, \underline{x}_F, y)}{\partial y} = \frac{w_1}{2} y^{-1/2} + 2w_2 y = y^{-1/2} + y$$



$$\Sigma . \text{F.O.C.} : \quad \text{SMC}(y) = p \Rightarrow y^{-1/2} + y = 2$$

$$\Rightarrow \frac{1}{y} = (2-y)^2$$

$$\Rightarrow y^3 - 4y^2 + 4y - 1 = 0$$

$$\Rightarrow (y-1)(y^2 - 3y + 1) = 0$$

$$\Rightarrow y = 1, \frac{3 \pm \sqrt{5}}{2}$$

Let note (check!),  $\frac{3+\sqrt{5}}{2}$  doesn't give  $\text{SMC}(y) = 2$ , so discard it. Next,

$$\text{S.O.C.} : \quad \frac{\partial \text{SMC}(y)}{\partial y} = -\frac{y^{-3/2}}{2} + 1$$

$$= \begin{cases} \frac{1}{2} & \text{for } y=1 \\ < 0 & \text{for } y = \frac{3-\sqrt{5}}{2} \end{cases}$$

$\Rightarrow y = 1 = \hat{y}$ , say, maximises the profit (for  $y > 0$ ).

$$\Rightarrow \text{max. profit} = p\hat{y} - C_s^*(\underline{w}, \underline{x}_F, \hat{y})$$

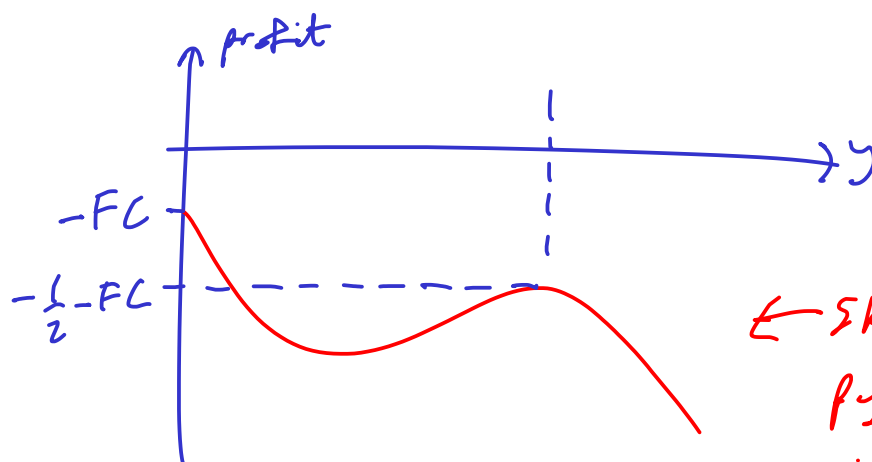
$$= 2 - \left(2 + \frac{1}{2} + FC\right)$$

$$= -\frac{1}{2} - FC$$

$$< \textcircled{-FC}$$

" profit if  $y=0$ .

Our earlier analysis didn't pick up the global maximum of the profit at  $y=0$ , since the F.O.C. is not satisfied there:



← Sketch of profit curve  $p(y) - C_s^*(\underline{w}, \underline{x}_F, y)$  in this case.

(Note that in Example 1, the profit function  $p(y) - C_s^*(\underline{w}, \underline{x}_F, y)$  is a quadratic in  $y$ , and its value at  $\hat{y} = 1$  is greater than at  $y = 0$ ).

So, as illustrated in Example 2, in some circumstances it may be preferable for a firm to go out of business rather than provide  $y > 0$ .

(that is, produce no output, i.e., set  $y = 0$ )

(i.e., produce no output)

Indeed, we can generalise: it will be preferable to go out of business when

the profit for  $y = 0$  exceeds  $p(y) - C_s^*(\underline{w}, \underline{x}_F, y) \forall y > 0$ ,  
i.e., when

$$\underbrace{-w_F x_F^T}_{\text{profit for } y=0} > py - (w_F x_F^T + w_V x_S^*(w, x_F, y)) \quad \forall y > 0$$

$$\Leftrightarrow \frac{w_V x_S^*(w, x_F, y)}{y} > p \quad \forall y > 0$$

$$\Leftrightarrow \text{SAVC}(y) > p \quad \forall y > 0$$

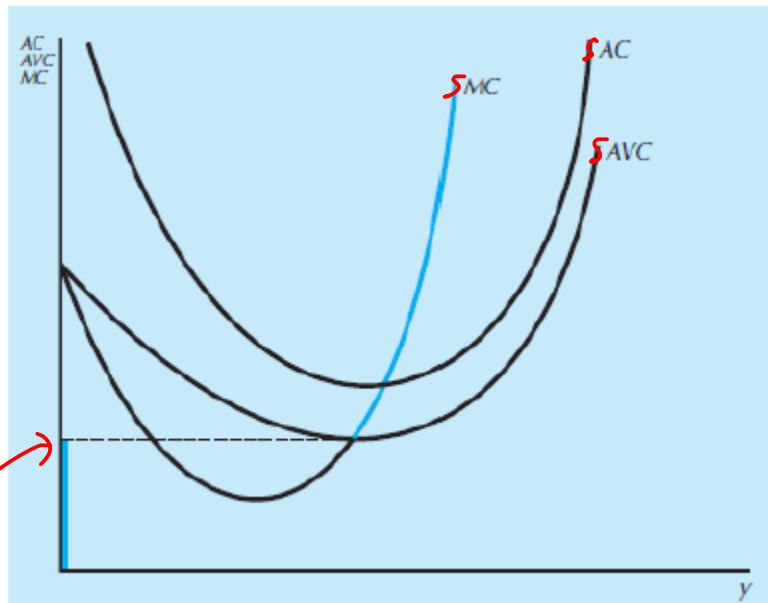
This is known as the **shutdown condition**; when satisfied, it is preferable for the firm to go out of business (i.e., produce nothing).

So we must refine our definition of the firm's chosen short-run supply. The competitive cost-minimising firm should choose a positive level of output  $y$  such that:

- $SMC(y) = p$ ;
- $SMC(y)$  is increasing in  $y$ ;
- and  $SAVC(y) \leq p$ .

If no such  $y > 0$  exists for the given  $p$ , then the firm should set  $y = 0$ .

These conditions are satisfied by the portion of the  $SMC$  curve that is increasing in  $y$  and that lies on or above the  $SAVC$  curve:



If  $p$  is less than this value then the firm

should shut down production (i.e., produce nothing) (since in this case there exists no  $y > 0$  such that  $SMC(y) = p$ ,  $SAVC(y)$ ).

In the long run, we have a very similar story. Neither the first- nor second-order conditions above explicitly require the costs to be dependent on fixed factors of production; these translate to the long-run scenario as would be expected. The long-run profit-maximising supply for a cost-minimising firm is given by  $y$  such that

- $LMC(y) = p$
- $LMC(y)$  must be increasing in  $y$ .
- $LAC(y) \leq p$

One arrives at the above by noting that in the long-run, the profit is given by

$$py - c^*(\underline{w}, y)$$

which is maximised for  $y$  s.t.

$$\underbrace{\frac{\partial c^*(\underline{w}, y)}{\partial y}}_{LMC(y)} = p \quad \text{and} \quad \frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} > 0$$

and  $py - c^*(\underline{w}, y) \geq \underbrace{-c^*(\underline{w}, 0)}_{\text{profit for } y=0, = 0 \text{ (} \pi^*(\underline{w}, 0) = 0 \text{)}}$

i.e.,  $\underbrace{\frac{c^*(\underline{w}, y)}{y}}_{LAC(y)} \leq p$

Once more, if no such  $y > 0$  exists for the given  $p$ , then the firm should choose to go out of business. (i.e., produce nothing).

### Profit maximisation for a noncompetitive firm

To contrast, we consider the profit maximisation problem for a cost-minimising monopolist. Whilst monopolists have more control over output prices than in a competitive market, they cannot choose price and output independently of one another; they must respect the market demand for their product. We therefore assume that the monopolist chooses the amount of output to provide,  $y$ , and the output price is determined according to the market demand for this output, i.e. as a function of  $y$ ,  $p(y)$ .

The function  $p(y)$  is the inverse of the market's demand function and is referred to as the inverse demand function "facing the firm"; we note that it may be dependent on other determinants, but assume these to be held constant in our analysis.

$$\text{i.e., } p(y) = D^{-1}(y), \text{ or, } y = D(p(y)).$$

To maximise profits, we therefore seek :

$$\arg \max_{y \geq 0} \{ p(y)y - c_s^*(\underline{w}, \underline{x}_F, y) \}.$$

First- and second-order conditions for finding a profit-maximising position for a monopolist facing an inverse demand function are therefore given by

$$\frac{\partial}{\partial y} (p(y)y - c_s^*(\underline{w}, y)) = 0 \Rightarrow \frac{\partial p(y)}{\partial y} y + p(y) = SMC(y) \quad (FOC)$$

$$\frac{\partial^2}{\partial y^2} (p(y)y - c_s^*(\underline{w}, y)) \leq 0 \Rightarrow \frac{\partial^2 c_s^*(\underline{w}, \underline{x}_F, y)}{(\partial y)^2} \geq \frac{\partial^2 p(y)}{(\partial y)^2} y + 2 \frac{\partial p(y)}{\partial y} \quad (SOC)$$

We can rearrange the FOC as follows:

$$p(y) \left[ 1 + \frac{1}{\epsilon_D(y)} \right] = SMC(y) \quad (*)$$

where  $\epsilon_D(y) = \frac{\partial y}{\partial p(y)} \cdot \frac{p(y)}{y}$  is the price elasticity of demand.

But with  $y = D(p(y))$  we have (by differentiating wrt  $y$ ):

$$1 = D'(p(y)) \cdot p'(y)$$

$$\Rightarrow p'(y) = \frac{1}{D'(p(y))}$$

Now note that  $\epsilon_D(y) < 0$  (demand  $y$  decreases with increasing price  $p$ ), and  $SMC(y) > 0$  ( $SMC(y) = \frac{\partial c_S^*(\underline{w}, \underline{x}_F, y)}{\partial y}$  and  $c_S^*(\underline{w}, \underline{x}_F, y)$  increases with increasing output  $y$ ).

Then, it follows from  $(*)$  that a necessary condition for the firm to maximize profit is that  $|\epsilon_D| \geq 1$  (so that the LHS of  $(*)$  is also  $> 0$ ), i.e., it should face elastic demand.

### Example:

Consider the monopolist faced with a linear inverse demand

$$p(y) = a_1 - a_2 y \quad a_1, a_2 > 0$$

and ~~Cobb-Douglas~~ variable costs in the short term:

$$c_S^*(\underline{w}, \underline{x}_F, y) = 2\sqrt{w_1 w_2} y^2 + FC(\underline{w}_F, \underline{x}_F).$$

What is the maximum profit that this monopolist can achieve?

### Solution

First,

$$\frac{1}{\epsilon_D(y)} = \frac{\frac{\partial p(y)}{\partial y} \cdot y}{p(y)} = \frac{-a_2 y}{a_1 - a_2 y} \quad \left( \begin{array}{l} \text{note this is } \leq 0 \\ \text{since } a_1, a_2 > 0, \\ 0 \leq y \leq a_1/a_2 \end{array} \right)$$

Then  $|\epsilon_D| \geq 1$  provided

$$|a_2 y| \leq |a_1 - a_2 y| \quad \left\{ \begin{array}{l} \text{since } a_1, a_2 > 0, \\ 0 \leq y \leq a_1/a_2 \end{array} \right.$$

$$\text{i.e. provided } y \leq \frac{a_1}{2a_2}.$$

So any profit-maximising level of output must be below  $\frac{a_1}{2a_2}$ .

Now solve the FOC:

$$p(\hat{y}) \left[ 1 + \frac{1}{\varepsilon_D(\hat{y})} \right] = SMC(\hat{y}) \quad \left( = \frac{\partial C_S^*(\underline{w}, \underline{x}_F, \hat{y})}{\partial \hat{y}} \right)$$

$$\Rightarrow a_1 - 2a_2 \hat{y} = 4\sqrt{w_1 w_2} \hat{y}$$

$$\Rightarrow \hat{y} = \frac{a_1}{2a_2 + 4\sqrt{w_1 w_2}} \leq \frac{a_1}{2a_2} \text{ as required.}$$

Evaluating the SOC verifies that this is a maximum (exercise: check).

$$\text{Then } p(\hat{y}) = a_1 - a_2 \hat{y}$$

and the maximum profit is

$$p(\hat{y}) \cdot \hat{y} - C_S^*(\underline{w}, \underline{x}_F, \hat{y}) =$$

$$= (a_1 - a_2 \hat{y}) \hat{y} - (2\sqrt{w_1 w_2} \hat{y}^2 + FC(\underline{w}_F, \underline{x}_F))$$

$$\Rightarrow \dots$$

$$= \frac{a_1^2}{4(a_2 + 2\sqrt{w_1 w_2})} - FC(\underline{w}_F, \underline{x}_F).$$

We can see from this example that it is also possible for profit-maximising monopolists to experience losses in the short-run; this is not a phenomenon unique to competitive markets. (i.e., the above maximum profit could be  $< 0$  if  $FC$  is large enough).

The above optimisation assumes that  $y > 0$ . Just as for competitive firms, however, we note that the profit-maximising (loss-minimising) position for a monopolist may be to go out of business, i.e. to set  $y = 0$ . This happens when the losses incurred by setting output according to the above first- and second-order conditions are greater than the fixed costs, i.e. when

$$\Delta AVC(y) > p(y) \quad \forall y > 0.$$

One can see this as follows.

$$\text{profit} = p(y) \cdot y - C_s^*(\underline{w}, \underline{x}_F, y)$$

For  $y = 0$ , this reduces to minus the fixed costs. So it is best to set  $y = 0$  if these fixed costs are greater than  $p(y) \cdot y - C_s^*(\underline{w}, \underline{x}_F, y) \quad \forall y > 0$ , i.e., if

$$0 > p(y) \cdot y - \underline{x}_v \underline{x}_s^*(\underline{w}, \underline{x}_F, y) \quad \forall y > 0$$

i.e., if  $\frac{\underline{x}_v \underline{x}_s^*(\underline{w}, \underline{x}_F, y)}{y} > p(y) \quad \forall y > 0.$

$\Delta AVC(y)$

So, in summary, for a cost-minimising monopolist, the short run profit-maximising output  $y$  will satisfy the following conditions:

$$\cdot) p(y) \left[ 1 + \frac{1}{\varepsilon_D(y)} \right] = SMC(y)$$

$$\cdot) \frac{\partial^2 C^*(\underline{w}, \underline{x}_F, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{\partial y^2} y + 2 \frac{\partial p(y)}{\partial y}$$

$$\cdot) SAVC(y) \leq p(y)$$

If no such  $y > 0$  exists, then the firm should set  $y = 0$ .

We also note that, as for competitive firms, the extension to the long-run is trivial. For a cost-minimising monopolist, the long-run profit-maximising output  $y$  will satisfy the following conditions:

$$\cdot) p(y) \left[ 1 + \frac{1}{\varepsilon_D(y)} \right] = LMC(y)$$

$$\cdot) \frac{\partial^2 C^*(\underline{w}, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{\partial y^2} y + 2 \frac{\partial p(y)}{\partial y}$$

$$\cdot) LAC(y) \leq p(y)$$