IMPERIAL

MATH70029

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Functional Analysis

Date: Wednesday, May 28, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Consider ℓ^∞, the space of bounded sequences, and the spaces

$$c_0 := \left\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \mid \lim_{k\to\infty} x_k = 0\right\}, \quad c := \left\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \mid \lim_{k\to\infty} x_k \text{ exists}\right\},$$

with norm $\|\cdot\|_{\ell^\infty}$, and dual spaces c_0^*, c^*. All sequences are assumed to be real-valued.

(a) Show that $c \subset \ell^\infty$ is closed. (8 marks)

(b) (i) Construct a continuous linear isomorphism $\Phi : c^* \to c_0^*$ with

$$\|\Phi\|_{\mathcal{L}(c^*,c_0^*)} \leq 2.$$

[*Hint: use (the inverse of) the linear map* $T : c \to c_0$, $Tx = (\bar{x}, x_1 - \bar{x}, x_2 - \bar{x}, \dots)$ *where* $\bar{x} = \lim_n x_n$.] (8 marks)

(ii) Is c^* isomorphic to ℓ^1? [*Hint:* use (i).] (4 marks)

(Total: 20 marks)

2. Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$, and let $A : H \to H$ be a symmetric linear operator.

(a) Show that A has closed graph. (8 marks)

(b) Is A continuous? Justify your answer. (2 marks)

(c) Assume now in addition that A has the following property: there exists $\lambda > 0$ such that

$$(Ax, x) \geq \lambda\|x\|^2 \quad \text{for every } x \in H.$$

(i) Show that A is injective. (2 marks)

(ii) Show that $\mathrm{im}(A)$ is closed in H. (8 marks)

(Total: 20 marks)

3. Consider ℓ^∞, the space of bounded real-valued sequences, endowed with its norm $\|\cdot\|_{\ell^\infty}$, and the subspace $c \subset \ell^\infty$ of convergent sequences.

(a) Let $f : c \to \mathbb{R}$ be the functional defined by

$$f(x) = \lim_n x_n \quad \text{where } x = (x_n) \in c.$$

Show that f extends to a bounded linear functional $l : \ell^\infty \to \mathbb{R}$.
[*Hint: consider the functional* $p(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n x_k$.] (8 marks)

(b) Show that $\|l\| = 1$. (4 marks)

(c) Compute $l(x)$ where $x = (1, 0, 1, 0, \dots)$. (4 marks)

(d) Is l multiplicative, i.e. does $l(xy) = l(x)l(y)$ hold for $x, y, \in \ell^\infty$? Here xy denotes the sequence obtained by coordinate-wise multiplication, that is, $(xy)_n = x_n y_n$. (4 marks)

(Total: 20 marks)

4. Let $H = L^2([0,1])$ with functions having values in $\mathbb{C}$. For $f \in C = C([0,1],\mathbb{C})$ consider the operator $M_f : H \to H$ given by

$$M_f(g) = f \cdot g, \quad g \in H.$$

(a) Show that M_f is continuous and that $M_f \circ M_g = M_{f \cdot g}$ for all $f, g \in C$. (4 marks)

(b) Determine the adjoint operator $(M_f)^*$. Under which condition on f is M_f self-adjoint? (4 marks)

(c) Show that M_f is invertible in $\mathcal{L}(H)$ if and only if $f(x) \neq 0$ for all $x \in [0,1]$. (8 marks)

(d) Determine the spectrum $\sigma(M_f)$. (4 marks)

(Total: 20 marks)

5. Let $a < b$ be real numbers and $I = (a, b)$. Let $1 \leq p < \infty$.

(a) Define the Sobolev space $W^{1,p}(I)$. Specify the meaning of the derivative in this context. (4 marks)

(b) Let $u \in W^{1,p}(I)$.

(i) Let $G : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Prove that the composition $G \circ u$ is in $W^{1,p}(I)$ and that

$$(G \circ u)' = (G' \circ u)u'$$

holds. [*Hint: consider a suitable sequence* (u_n) *of smooth functions.*] (10 marks)

(ii) Prove that $|u| \in W^{1,p}(I)$ and compute its weak derivative.
[*Hint: consider for* $\varepsilon > 0$ *the function* $G_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$, $x \in \mathbb{R}$.] (6 marks)

(Total: 20 marks)

Module:	MATH60029/MATH70029
Setter:	Rodriguez
Checker:	Krasovsky
Editor:	editor
External:	external
Date:	April 10, 2025
Version:	Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2025

MATH60029/MATH70029 Functional Analysis **SOLUTIONS**

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 8 marks; 1(b)(i) 8 marks; 1(b)(ii) 4 marks; 2(b) 2 marks; 2(c)(i) 2 marks; 4(a) 4 marks; 4(b) 4 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2(a) 8 marks; 3(b) 4 marks; 3(c) 4 marks; 3(d) 4 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(b)(ii) 8 marks; 4(d) 4 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
3(a) 8 marks; 4(c) 8 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
....	....	....

Temporary front page

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Functional Analysis **SOLUTIONS**

Date: Wednesday, 28th May 2025
Time: 10:00 – 12:00/12:30
Time Allowed: 2 Hours for MATH60029 paper; 2.5 Hours for MATH70029 paper
This paper has *5 Questions*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Consider ℓ^∞, the space of bounded sequences, and the spaces

$$c_0 := \left\{ (x_k)_{k\in\mathbb{N}} \in \ell^\infty \mid \lim_{k\to\infty} x_k = 0 \right\}, \quad c := \left\{ (x_k)_{k\in\mathbb{N}} \in \ell^\infty \mid \lim_{k\to\infty} x_k \text{ exists} \right\},$$

with norm $\|\cdot\|_{\ell^\infty}$. All sequences are assumed to be real-valued.

(a) (SEEN SIMILAR) Show that $c \subset \ell^\infty$ is closed. (8 marks)

Solution: Let $x = (x_n)_{n\in\mathbb{N}} \in c$. Then there exists a sequence $(x^k)_{k\in\mathbb{N}}, x^k = (x^k_n)_{n\in\mathbb{N}} \in c$ such that:

$$\sup_{n\in\mathbb{N}} |x^k_n - x_n| = \|x^k - x\|_{\ell^\infty} \to 0 \text{ as } k \to \infty.$$

We show that x is Cauchy, which is enough. Let $\varepsilon > 0$. First pick k_ε such that

$$\|x^{k_\varepsilon} - x\|_{\ell^\infty} < \varepsilon/3.$$

By definition $x^{k_\varepsilon} \in c$, hence it is Cauchy, so there exists N_ε such that for all $n, m \geq N_\varepsilon$, $|x^{k_\varepsilon}_n - x^{k_\varepsilon}_m| < \varepsilon/3$. Hence, for all such n, m,

$$|x_n - x_m| \leq |x_n - x^{k_\varepsilon}_n| + |x^{k_\varepsilon}_n - x^{k_\varepsilon}_m| + |x^{k_\varepsilon}_m - x_m| < \varepsilon,$$

as desired.

(b) (i) (UNSEEN) Construct a continuous linear isomorphism $\Phi : c^* \to c_0^*$ with

$$\|\Phi\|_{\mathcal{L}(c^*,c_0^*)} \leq 2.$$

[*Hint: use (the inverse of) the linear map* $T : c \to c_0$, $Tx = (\bar{x}, x_1 - \bar{x}, x_2 - \bar{x}, \dots)$ *where* $\bar{x} = \lim_n x_n$.] (8 marks)

Solution: the map T is well-defined, i.e. $\lim_n (Tx)_n = 0$. The inverse of T is the map $S : c_0 \to c$ given by $Sy = (y_2 + y_1, y_3 + y_1, \dots)$. Indeed, $STx = x$ is immediate and $TSy = y$ follows using that $\lim_n (y_n + y_1) = y_1$. So we define

$$\Phi(f) = f \circ S, \quad f \in c^*.$$

This is clearly linear as a composition of linear maps. It is also continuous with

$$\|\Phi\|_{\mathcal{L}(c^*,c_0^*)} = \sup_{\|f\|_{c^*}=1} \|\Phi(f)\|_{c_0^*} = \sup_{\|f\|_{c^*}=1} \sup_{y\in c_0 : \|y\|_{\ell^\infty}=1} |\langle \Phi(f), y\rangle| \leq 2$$

using that

$$|\langle \Phi(f), y\rangle| = |\langle f, Sy\rangle| \leq \|f\|_{c^*} \|Sy\|_{\ell^\infty} \leq 2\|f\|_{c^*} \|y\|_{\ell^\infty}.$$

(ii) (UNSEEN) Is c^* is isomorphic to ℓ^1? [*Hint*: use (i).] (4 marks)

Solution: Yes, combining (i) and the seen fact that ℓ^1 and c_0^* are isometrically isomorphic.

(Total: 20 marks)

2. Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$, and let $A : H \to H$ be a symmetric linear operator.

(a) (SEEN SIMILAR) Show that A has closed graph. (8 marks)

Solution: Consider $(x_k, y_k = Ax_k)$ a sequence in $H \times H$ with $x_k \to x$ and $y_k \to y$ as $k \to \infty$. We need to argue that $Ax = y$. Using symmetry of A and continuity of $(\cdot,\cdot)$, we have on the one hand $(Ax_k, z) \to (y, z)$ for all $z \in H$ and on the other $(x_k, Az) \to (x, Az) = (Ax, z)$, whence

$$(Ax - y, z) = 0, \quad \text{for all } z \in H.$$

Choosing $z = Ax - y$ gives $Ax = y$.

(b) (SEEN) Is A continuous? Justify your answer. (2 marks)

Solution: Yes, by the closed graph theorem.

(c) Assume now in addition that A has the following property: there exists $\lambda > 0$ such that

$$(Ax, x) \geq \lambda \|x\|^2 \quad \text{for every } x \in H.$$

(i) (SEEN SIMILAR) Show that A is injective. (2 marks)

Solution: If $Ax = 0$ then the assumption implies

$$\lambda \|x\|^2 \leq (Ax, x) = 0,$$

whence $x = 0$.

(ii) (UNSEEN) Show that $\mathrm{im}(A)$ is closed in H. (8 marks)

Solution: Let $(y_n) \subset \mathrm{im}(A)$ be a sequence converging to some $y \in H$. Let x_n be such that $Ax_n = y_n$. Then (x_n) is Cauchy because

$$\|x_n - x_m\|^2 \leq \lambda^{-1}(Ax_n - Ax_m, x_n - x_m)$$
$$= (y_n - y_m, x_n - x_m) \leq \|y_n - y_m\| \cdot \|x_n - x_m\|$$

using the assumption and Cauchy-Schwarz. Hence (x_n) converges to some $x \in H$. By item (b), A is continuous, which implies that

$$Ax = \lim_n Ax_n = \lim y_n = y.$$

(Total: 20 marks)

3. Consider ℓ^∞, the space of bounded real-valued sequences, endowed with its norm $\|\cdot\|_\infty$, and the subspace $c \subset \ell^\infty$ of convergent sequences.

(a) (UNSEEN) Let $f : c \to \mathbb{R}$ be the functional defined by

$$f(x) = \lim_n x_n \quad \text{where } x = (x_n) \in c.$$

Show that f extends to a bounded linear functional $l : \ell^\infty \to \mathbb{R}$.
[*Hint: consider the functional* $p(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k$.] (8 marks)

Solution: The idea is to apply (a corollary of) the Hahn-Banach theorem. It is straightforward to see that f defines a bounded linear functional on c since $|f(x)| = |\lim_n x_n| \leq \|x\|_\infty$. Moreover, one checks that the functional $p(\cdot)$ is sub-linear. Therefore, H-B immediately yields the existence of l, provided we show that

$$f \leq p \text{ on } c.$$

In fact, we show that the two are equal on c. Let $x \in c$ and $\varepsilon > 0$. There exists $N = N(\varepsilon)$ such that $|x_n - f(x)| < \varepsilon$ for all $n \geq N$. Hence, for $n \geq N$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k - f(x)\right| \leq \frac{1}{n} N \left(f(x) + \max_{1\leq k\leq N} |x_k|\right) + \frac{1}{n}\sum_{k=N}^{n} |x_k - f(x)| \leq \frac{C(\varepsilon)}{n} + \varepsilon.$$

Letting first $n \to \infty$, then $\varepsilon \to 0$, the claim follows.

(b) (SEEN SIMILAR) Show that $\|l\| = 1$. (4 marks)

Solution: If $x \in \ell^\infty$ satisfies $\|x\|_\infty \leq 1$, then $x_n \leq 1$ for all n and so $l(x) \leq p(x) \leq 1$. Hence $\|l\| \leq 1$. The lower bound $\|l\| \geq 1$ follows by considering the sequence $x = (1, 1, \dots)$. Overall, $\|l\| = 1$.

(c) (SEEN SIMILAR) Compute $l(x)$ where $x = (1, 0, 1, 0, \dots)$. (4 marks)

Solution: $l(x) = \frac{1}{2}$. One way to obtain this is to observe that l is invariant under shifts, i.e. $l \circ T = l$, where $T : \ell^\infty \to \ell^\infty$ is defined by $(Tx)_n = x_{n+1}$. Then since $Tx + x = (1, 1, \dots)$ for the given sequence x it follows by linearity that

$$1 = l(Tx + x) = l(Tx) + l(x) = 2l(x).$$

(d) (UNSEEN) Is l multiplicative, i.e. does $l(xy) = l(x)l(y)$ hold for $x, y, \in \ell^\infty$? Here xy denotes the sequence obtained by coordinate-wise multiplication, that is, $(xy)_n = x_n y_n$. (4 marks)

Solution: No. Take x as in part c), $y = Tx$. Then $xy = (0, 0, \dots)$ hence $l(xy) = 0$ but $l(x)l(y) = l(x)^2 = \frac{1}{4}$.

(Total: 20 marks)

4. Let $H = L^2([0,1])$ with functions having values in $\mathbb{C}$. For $f \in C = C([0,1],\mathbb{C})$ consider the operator $M_f : H \to H$ given by

$$M_f(g) = f \cdot g, \quad g \in H.$$

(a) (SEEN SIMILAR) Show that M_f is continuous and that $M_f \circ M_g = M_{f \cdot g}$ for all $f, g \in C$. (4 marks)

Solution: Since f is continuous on a compact set it is bounded so continuity of M_f follows from the estimate

$$\|M_f(h)\|_2 = \|fh\|_2 \leq \|f\|_\infty \|h\|_2.$$

The second property is checked readily from the definition.

(b) (SEEN SIMILAR) Determine the adjoint operator $(M_f)^*$. Under which condition on f is M_f self-adjoint? (4 marks)

Solution: For all $g, h \in H$, one has

$$\langle M_f(g), h\rangle_2 = \int_0^1 (fg)\bar{h} d\lambda = \int_0^1 g\overline{(\bar{f}h)} d\lambda = \langle g, M_{\bar{f}}(h)\rangle_2$$

so $(M_f)^* = M_{\bar{f}}$. Moreover since $M_f = M_g$ if and only if $f = g$ for $f, g \in C$ one has that M_f is self-adjoint, i.e. $M_f = (M_f)^* = M_{\bar{f}}$ if and only if $f = \bar{f}$.

(c) (UNSEEN) Show that M_f is invertible in $\mathcal{L}(H)$ if and only if $f(x) \neq 0$ for all $x \in [0,1]$. (8 marks)

Solution: If $f(x) \neq 0$ for all $x \in [0,1]$ then $1/f$ exists and $(M_f)^{-1} = M_{1/f}$ on account of part (a). If on the contrary $f(x) = 0$ for some x then by continuity of f the set $A_n = \{|f| < \frac{1}{n}\}$ is open and non-empty for all n and thus $1_{A_n} \neq 0$ a.e. Moreover,

$$\|M_f(1_{A_n})\|_2 \leq \|f 1_{A_n}\|_\infty \|1_{A_n}\|_2 \leq \frac{1}{n}\|1_{A_n}\|_2$$

which implies that M_f is not bounded from below and therefore not invertible.

(d) (UNSEEN) Determine the spectrum $\sigma(M_f)$. (4 marks)

Solution: First note that $\lambda \in \sigma(M_f)$ is equivalent to $M_f - \lambda 1 = M_{f-\lambda 1}$ being not invertible. By (c) this is in turn equivalent to $(f - \lambda 1)(x) = 0$, or, $f(x) = \lambda$ for some $x \in [0,1]$. Hence,

$$\sigma(M_f) = \mathrm{im}(f) = f([0,1]).$$

(Total: 20 marks)

5. Let $a < b$ be real numbers and $I = (a, b)$. Let $1 \leq p < \infty$.

(a) Define the Sobolev space $W^{1,p}(I)$. Specify the meaning of the derivative in this context. (4 marks)

Solution: This is the space of all u such that both u and its weak derivative u' belong to $L^p(I)$. The weak derivative is defined up to a.e. equivalence by the requirement that

$$\int_I u\varphi' d\lambda = -\int_I u'\varphi d\lambda,$$

for any test function $\varphi \in C_c^\infty(I)$.

(b) Let $u \in W^{1,p}(I)$.

(i) Let $G : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Prove that the composition $G \circ u$ is in $W^{1,p}(I)$ and that

$$(G \circ u)' = (G' \circ u)u'$$

holds. [*Hint: consider a suitable sequence* (u_n) *of smooth functions.*] (10 marks)

Solution: From the bound

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}$$

we deduce that u is bounded, hence so is $G \circ u$ by assumptions on G and so in particular $G \circ u \in L^p(I)$. To compute the weak derivative, we take u_n smooth compactly supported such that $u_n \to u$ in $W^{1,p}(I)$. In particular

$$\|u - u_n\|_{L^\infty(I)} \to 0$$

by above inequality so the convergence is uniform and in particular u_n is uniformly bounded on I, say by M. Now for a test function $\varphi \in C_c^\infty(I)$, we have for every n that

$$\int (G \circ u_n)\varphi' dx = -\int (G' \circ u_n)u_n'\varphi dx \tag{1}$$

by the usual chain rule and integration by parts. We now let $n \to \infty$ on both sides of (1). On the one hand $|(G \circ u_n)\varphi'| \leq \sup_{t \in [-M,M]} |G(t)||\varphi'|$, so the left-hand side converges pointwise and is bounded uniformly in n by a function in L^p. Applying dominated convergence, it follows that the LHS of (1) tends to $\int (G \circ u)\varphi' dx$ in the limit $n \to \infty$. For the right-hand side, we use the pointwise bound

$$|(G' \circ u_n)u_n'\varphi| \leq \sup_{t \in [-M,M]} |G(t)||u_n'||\varphi| \leq C|\varphi|(|u_n' - u'| + |u'|)$$

where $u' \in L^p(I)$ refers to the weak derivative of u. Since $u_n' \to u$ in $L^p(I)$, the integrand on the RHS of (1) is bounded up to a negligible term by $C|u'| \in L^p(I)$. A similar argument using dominated convergence then implies that the RHS converges to $\int (G' \circ u)u'\varphi dx$, as desired.

(ii) Prove that $|u| \in W^{1,p}(I)$ and compute its weak derivative.
[*Hint: consider for* $\varepsilon > 0$ *the function* $G_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$, $x \in \mathbb{R}$.] (6 marks)

Solution: We apply (i). The function G_ε is in $C^1(I)$ so by (i) we know that

$$\int (G_\varepsilon \circ u)\varphi' dx = -\int (G'_\varepsilon \circ u)u'\varphi dx.$$

Moreover, $G_\varepsilon(x) \to G(x) = |x|$ uniformly on compact sets as $\varepsilon \to 0$ so by similar arguments as in (i) it follows readily that the left-hand side converges to $\int (G \circ u)\varphi' dx = \int |u|\varphi' dx$. On the other hand

$$G'_\varepsilon(x) = \frac{x}{\sqrt{x^2 + \varepsilon^2}} \to \mathsf{sign}(x)$$

pointwise for $x \neq 0$, whence $G'_\varepsilon \circ u)u'\varphi \to \mathsf{sign}(u)u'\varphi$ a.e. In addition the integrand is bounded by $|u'||\varphi|$, which is in $L^1(I)$ so dominated convergence yields that

$$\int (G'_\varepsilon \circ u)u'\varphi dx \xrightarrow{\varepsilon \to 0} \int \mathsf{sign}(u)u'\varphi dx.$$

Overall, this yields that $|u|' = \mathsf{sign}(u)u'$.

(Total: 20 marks)

MATH70029 Functional Analysis Markers Comments

Question 1	Solved well for the most part.
Question 2	Some issues with Q2(c)(ii): need to use coercivity assumption!
Question 3	Solved mostly well.
Question 4	Solved well except 4c.
Question 5	Wide range in quality of solutions.