Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 13

Cauchy Theorem $\iff$ BW - Theorem

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Proposition 3.39

If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.

Next, we are going to look at the relationship between these two theorems:

Theorem 3.27 - Cauchy Theorem

If (a_n) is a Cauchy sequence of real numbers then a_n converges.

Theorem 3.34 - Bolzano-Weierstrass

If (a_n) is a *bounded* sequence of real numbers then it has a *convergent subsequence*.

Bolzano-Weierstrass $\Rightarrow$ the Cauchy Theorem

Before continuing, a basic but useful trick:

Lemma 3.40

Fix $c > 0$. Then $a_n \to a$ if and only if

$$\forall \epsilon > 0 \; \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < c\epsilon \quad (\star)$$

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Proof: $\Rightarrow$. Assume $a_n \to a$, we prove $(\star)$. Fix $\epsilon > 0$ and let $\epsilon' := c\epsilon$. Then by the definition of convergence applied to $\epsilon' > 0$ we find

$$\exists N \in \mathbb{N}_{>0} \; : \; n \geq N \Rightarrow |a_n - a| < \epsilon',$$

which is $(\star)$.

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Warning!

Do not let c depend on ϵ (nor N or n)!

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Since $a_n \in [-R, R]\ \forall n$, then either

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Pick one of these intervals with infinite number of a_n s; call it $[A_1, B_1]$ of length R.

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Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ contains a_n for infinitely many n, with length $R/2$.

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the Cauchy Theorem $\Rightarrow$ Bolzano-Weierstrass continued

We get a sequence of intervals $[A_n, B_n]$ of length $2^{1-n}R$ which are nested – i.e. $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$ – with each containing a_n for infinitely many n.