Introduction to Quantum Mechanics – Problem sheet 7

1. **An asymmetric square well. (Part of an exam question from 2013)** - *Part a) is a basic application of the ideas we employed for the finite box in the lecture notes. Part b) asks you to deduce the minimum value of V_0 for which there is a bound state, using an argument that we will only discuss in the lecture for the finite box. That doesn't mean you cannot have a go at it before the lecture already.*

   Consider a potential of the form

$$V(x) = \begin{cases} \infty, & x \leq 0 \\ 0, & 0 < x \leq L \\ V_0, & x > L. \end{cases}$$

   (a) Find a quantisation condition for the energies of the bound states for this potential.

   (b) Use graphical methods to deduce the minimum value of V_0 for which there is a bound state.

2. **The variational method for the approximate calculation of eigenfunctions** - *This exercise introduces a method to approximate the eigenfunction of the ground state of a potential, using an ansatz for the functional form of the wavefunction and optimising the parameters. This is not part of the core material, but a "widening your horizon" question.*

   The fact that the energy expectation value is bounded from below by the lowest eigenenergy can be used to provide an approximation for the lowest energy state of a given Hamiltonian by minimising the expectation value for a trial wave function depending on a set of parameters. Apply this idea to approximate the ground state energy and wave function for the Hamiltonian

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^4,$$

   using the normalised trial function

$$\phi(x) = \left(\frac{\lambda}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\lambda x^2/2},$$

   where λ is a real positive constant. Here we use rescaled units such that $\hbar = 1 = m$. Compare the approximate ground state energy with the numerically exact value $E_0 = 0.530\ldots$.

   Hint: The following integrals might be useful:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}dx = \sqrt{\frac{\pi}{a}}, \qquad \int_{-\infty}^{\infty} x^2\mathrm{e}^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}, \qquad \int_{-\infty}^{\infty} x^4\mathrm{e}^{-ax^2}dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}},$$

   for $\mathrm{Re}(a) > 0$.

3. **Number of bound states in a finite square well potential** - *This is just a little exercise on connecting the results from the finite box to some "real life" systems, as a practice to work with actual physical constants, which you need to look up for this exercise. It uses a formula that we will derive in the lecture.*

   There is a finite number N of bound states in a square well potential, given by $N = \left[\frac{\sqrt{2mV_0}L}{\pi\hbar}\right]_< + 1$, where $[X]_<$ denotes the largest integer smaller than X. How many bound states are there for an electron in a square well potential with $V_0 = 2\mathrm{eV}$ and width $L = 20\mathrm{nm}$? How many are there for a proton?