MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 5

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length}\,(\gamma) = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Theorem. Integration of continuous functions over curves satisfies the following properties:

- $$\int_\gamma (\alpha\, f(z) + \beta\, g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$
- If γ^- is γ with the reverse orientation, then
$$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz.$$
- (ML-inequality)
$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}\,(\gamma).$$

Proof. The first property follows from the definition and the linearity of the Riemann integral. The second property is left as an exercise. For the third one, we note that

$$\left|\int_\gamma f(z)\, dz\right| \;\leq\; \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\, dt \;=\; \sup_{z\in\gamma} |f(z)| \cdot \text{length}\,(\gamma).$$

Section: Primitive functions.

Definition. A primitive for f on $\Omega \subset \mathbb{C}$ is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem. If a continuous function f has a primitive F in an open set Ω, and γ is a curve in Ω that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$$

Proof. If γ is smooth, the proof is a simple application of the chain rule and the fundamental theorem of calculus. Indeed, if $z(t) : [a, b] \to \mathbb{C}$ is a parametrization for γ, then $z(a) = w_1$ and $z(b) = w_2$, and we have

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}F(z(t))\,dt = F(z(b)) - F(z(a)).$$

If γ is only piecewise-smooth then arguing the same as we did we have

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} (F(z(a_{k+1}) - F(z(a_k))$$
$$= F(z(a_n)) - F(z(a_0)) = F(z(b)) - F(z(a)).$$

Corollary. If γ is a closed curve in an open set Ω, f is continuous and has a primitive in Ω, then

$$\oint_{\gamma} f(z)\, dz = 0.$$

Proof. This is immediate since the end-points of a closed curve coincide.

For example, the function $f(z) = 1/z$ does not have a primitive in the open set $\mathbb{C} \setminus \{0\}$, since if C is the unit circle parametrized by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, we have

$$\oint_{C} f(z)\, dz = \int_{0}^{2\pi} \frac{i\, e^{it}}{e^{it}}\, dt = 2\pi i \neq 0.$$

Corollary. If f is holomorphic in an open connected set Ω and $f' = 0$, then f is constant.

Proof. Fix a point $w_0 \in \Omega$. It suffices to show that $f(w) = f(w_0)$ for all $w \in \Omega$. Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w. Since f is clearly a primitive for f', we have

$$\int_\gamma f'(z)\, dz = f(w) - f(w_0),$$

By assumption, $f' = 0$ so the integral on the left is 0, and we conclude that $f(w) = f(w_0)$ as desired.

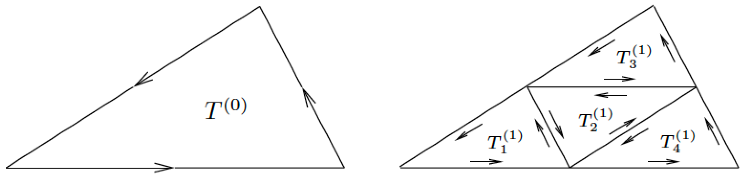
Section: Properties of holomorphic functions.

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω, then

$$\oint_T f(z)\, dz = 0,$$

whenever f is holomorphic in Ω.

Proof. Let $T^{(0)}$ be our original triangle (with a fixed orientation which we choose to be positive), and let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$, respectively. At the first step we find middle point of each side of $T^{(0)}$ and introduce four triangles $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, $T_4^{(1)}$ that are similar to the original triangle as follows:



Then

$$\oint_{T^{(0)}} f(z)\,dz = \oint_{T_1^{(1)}} f(z)\,dz + \oint_{T_2^{(1)}} f(z)\,dz + \oint_{T_3^{(1)}} f(z)\,dz + \oint_{T_4^{(1)}} f(z)\,dz.$$

There is some $j \in \{1, 2, 3, 4\}$ such that (WHY?)

$$\left|\oint_{T^{(0)}} f(z)\,dz\right| \leq 4\left|\oint_{T_j^{(1)}} f(z)\,dz\right|.$$

We choose a triangle that satisfies this inequality, and rename it $T^{(1)}$. Observe that if $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of $T^{(1)}$, respectively. Then

$$d^{(1)} = \frac{1}{2}\,d^{(0)} \quad \text{and} \quad p^{(1)} = \frac{1}{2}\,p^{(0)}.$$

We now repeat this process for the triangle $T^{(1)}$. Continuing this process, we obtain a sequence of triangles

$$T^{(1)}, T^{(1)}, T^{(2)}, \ldots, T^{(n)}, \ldots$$

with the properties that

$$\left|\oint_{T^{(0)}} f(z)\,dz\right| \leq 4^{n}\left|\oint_{T_j^{(n)}} f(z)\,dz\right|$$

and

$$d^{(n)} = 2^{-n}\,d^{(0)} \quad \text{and} \quad p^{(n)} = 2^{-n}\,p^{(0)},$$

where $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$.

Let $\Omega^{(n)}$ be the closed triangle such that $\partial\Omega^{(n)} = T^{(n)}$. Clearly we have a sequence of compact nested sets

$$\Omega^{(0)} \supset \Omega^{(1)} \supset \cdots \supset \Omega^{(n)} \supset \ldots,$$

whose diameter goes to 0. Then there exists a unique point z_0 that belongs to all triangles $\Omega^{(n)}$. Since f is holomorphic then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\, \psi(z),$$

where $\psi(z) \to 0$ as $z \to z_0$.

Since the constant $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, we can integrate the above equality over $T^{(n)}$ and obtain

$$\oint_{T^{(n)}} f(z)\, dz = \oint_{T^{(n)}} \psi(z)(z - z_0)\, dz.$$

Since z_0 belongs to all triangles we have $|z - z_0| \leq d^{(n)}$ and using the ML-inequality we arrive at

$$\left|\oint_{T^{(n)}} f(z)\, dz\right| \leq \varepsilon_n\, d^{(n)}\, p^{(n)},$$

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$. Therefore

$$\left|\oint_{T^{(n)}} f(z)\, dz\right| \leq \varepsilon_n\, 4^{-n} d^{(0)}\, p^{(0)},$$

and thus finally we obtain

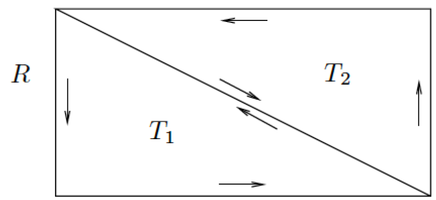
$$\left|\oint_{T^{(0)}} f(z)\, dz\right| \leq 4^n \left|\oint_{T_j^{(n)}} f(z)\, dz\right| \leq \varepsilon_n\, d^{(0)}\, p^{(0)} \to 0, \quad \text{as} \quad n \to \infty.$$

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z)\,dz = 0.$$

Proof. This immediately follows from the equality

$$\oint_R f(z)\,dz = \oint_{T_1} f(z)\,dz + \oint_{T_2} f(z)\,dz.$$



Quizzes

Question 1: What is the value of the integral $\int_\gamma \frac{1}{z^2} dz$, where γ is the unit circle $|z| = 1$ traversed in the direction such that its interior remains on the left.

Answers:

A. $2\pi i$

B. $-2\pi i$

C. 0

D. $4\pi i$

Question 2: What is the value of the integral $\int_\gamma \frac{1}{z} dz$, where γ is the circle of radius 1 centered at $2i$, traversed in the direction such that its interior remains on the left.

Answers:

A. $2\pi i$

B. $-2\pi i$

C. 0

D. $4\pi i$

Thank you