Imperial College London

DEPARTMENT OF MATHEMATICS
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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 1

1. Laplace's equation is written $\Delta u = 0$, where Δ is the Laplacian operator. We work in Cartesian coordinates here, i.e.
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
Taking partial derivatives of the function $u(x,y) = e^{kx}\cos(ky)$, we find that
$$\frac{\partial u}{\partial x} = ke^{kx}\cos(ky) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(ke^{kx}\cos(ky)) = k^2 e^{kx}\cos(ky)$$
and
$$\frac{\partial u}{\partial y} = -ke^{kx}\sin(ky) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-ke^{kx}\sin(ky)) = -k^2 e^{kx}\cos(ky)$$
We thus conclude that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
and so that u is indeed solution of the Laplace equation.

2. The 1D diffusion equation is defined by
$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$
In this problem, we consider the profile of temperature in a rod of length L. We would like to obtain

   (a) its steady-state solution with the boundary conditions $u(0,t) = T_1$ and $u(L,t) = T_2$. A steady-state solution is independent of time, we can thus write $u = U(x)$ and we are looking for the solution of the following equation $U''(x) = 0$. The general solution of this equation is a linear function
$$U(x) = Ax + B$$
The boundary conditions at $x = 0$ gives us $B = T_1$ and the boundary condition at $x = L$ gives us $AL + B = T_2$, i.e. $A = (T_2 - T_1)/L$. We thus conclude that the steady-state solution we are after is given by
$$U(x) = T_1 + \frac{T_2 - T_1}{L}x$$
In this example, the boundary conditions correspond to maintaining the two ends of the rod at different temperatures T_1 and T_2; the steady-state that develops is thus linear temperature profile from temperature T_1 to temperature T_2.

   (b) We proceed just as above. The boundary condition at $x = 0$ still gives us $B = T_1$ but the boundary condition in $x = L$ gives us $U'(L) = 0$, i.e. $A = 0$. Thus, we find that the steady-state solution is given by
$$U(x) = T_1$$
In this second example, one end of the rod is maintained at a given temperature T_1 but the end of the rod in $x = L$ is not. Rather, the boundary condition applied correspond to

having the first-order derivative of the temperature in space to be 0. This second boundary condition corresponds to no heat flux at the boundary, i.e. the rod is thermally insulated in $x = L$. As the rod is insulated at one end, the only temperature profile that the rod can display in steady-state is a constant temperature set by the boundary condition in $x = 0$.

3. The propagation of sound in air in governed by the so-called wave equation. We consider here the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \sin t + x^3$$

Notice that in this problem we are asked for **a solution**, not the most general solution. Let us exploit the linearity of the wave equation! According to the superposition principle, we can split $u = v + w$, such that v and w are solutions of the following equations:

$$\frac{\partial^2 v}{\partial t^2} - 4\frac{\partial^2 v}{\partial x^2} = \sin t$$
$$\frac{\partial^2 w}{\partial t^2} - 4\frac{\partial^2 w}{\partial x^2} = x^3$$

By splitting this problem, we have simplified them and the solutions for each equation can be easily obtained. In particular, we can easily check that $v(x,t) = -\sin(t)$ is a solution of the first equation and $w(x,t) = -x^5/80$ is a solution of the second equation.

We finally write that **a solution** of the original problem is given by

$$u(x,t) = -\sin(t) - \frac{x^5}{80}$$

4. We consider the following equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} = 0$$

Let us check that this equation admits solutions of the form $u(x,y) = e^{\alpha x + \beta y}$. To do so, we reinject this functional form in the fourth-order PDE and obtain

$$\begin{aligned} 0 &= \frac{\partial^4}{\partial x^4}\left(e^{\alpha x+\beta y}\right) + \frac{\partial^4}{\partial y^4}\left(e^{\alpha x+\beta y}\right) + 2\frac{\partial^4}{\partial x^2 \partial y^2}\left(e^{\alpha x+\beta y}\right) \\ &= \alpha^4 e^{\alpha x+\beta y} + \beta^4 e^{\alpha x+\beta y} + 2\alpha^2\beta^2 e^{\alpha x+\beta y} \\ &= (\alpha^4 + \beta^4 + 2\alpha^2\beta^2)e^{\alpha x+\beta y} \end{aligned}$$

We conclude that this equation admits solutions of the form $u(x,y) = e^{\alpha x+\beta y}$ if and only if

$$\alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 0 \iff (\alpha^2+\beta^2)^2 = 0 \iff \alpha^2+\beta^2 = 0 \iff \alpha = 0 \text{ and } \beta = 0$$

5. In this problem, we consider the equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) Let us rewrite the equation in the coordinates $s = x$ and $t = x - y$. We substitute $v(s,t) = u(x,t)$ in the equation and use the chain rule to get

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial v}{\partial t} = \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial y}\frac{\partial v}{\partial t} = -\frac{\partial v}{\partial t}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right) = \frac{\partial s}{\partial x}\frac{\partial^2 v}{\partial s^2} + \frac{\partial t}{\partial x}\frac{\partial^2 v}{\partial s \partial t} + \frac{\partial s}{\partial x}\frac{\partial^2 v}{\partial t \partial s} + \frac{\partial t}{\partial x}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial t^2} + 2\frac{\partial^2 v}{\partial s \partial t}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial t}\right) = -\frac{\partial s}{\partial y}\frac{\partial v}{\partial t \partial s} + \frac{\partial t}{\partial y}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial t^2}$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right) = \frac{\partial s}{\partial y}\frac{\partial^2 v}{\partial s^2} + \frac{\partial t}{\partial y}\frac{\partial v}{\partial s \partial t} + \frac{\partial s}{\partial y}\frac{\partial v}{\partial t \partial s} + \frac{\partial t}{\partial y}\frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial s \partial t}$$

Therefore, we obtain that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial s^2}$$

So our equation reduces to

$$\frac{\partial^2 v}{\partial s^2} = 0$$

(b) The general solution for this equation is thus $v(s,t) = f(t) + sg(t)$ where f and g are arbitrary differentiable functions. Going back to the (x, y) coordinates, we obtain that the general solution to this PDE is given by

$$u(x, y) = f(x - y) + xg(x - y)$$

6. We assume that u represents a mass density. The dimension of u can be expressed in the fundamental dimensions $\{M, L, T\}$ and we write that

$$[u] = ML^{-3}$$

Further, we have that

(a) $\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial t}\right)\right] = \frac{1}{[x]}\left[u\frac{\partial u}{\partial t}\right] = \frac{[u]}{[x]}\left[\frac{\partial u}{\partial t}\right] = \frac{[u]^2}{[x][t]} = M^2L^{-7}T^{-1}$

(b) $\left[\frac{\partial^2}{\partial x^2}\left(u^3\frac{\partial u}{\partial t}\right)\right] = \frac{[u]^4}{[x]^2[t]} = M^4L^{-14}T^{-1}$

(b) $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}(u^7)\right) = \frac{[u]^7}{[x][t]} = M^7L^{-22}T^{-1}$

7. To solve this problem, we will use dimensional analysis. First, we need to make a list of the relevant variables in this problem. We are trying to determine the frequency of oscillation ω.

Physical intuition tells us that we expect the frequency to depend on:

- ☐ the mass of the cylinder m, after all in the spring-mass system the frequency of oscillation depends on the mass;
- ☐ the cylinder diameter D as this is the geometric quantity that determines how much fluid is displaced when the cylinder oscillates;
- ☐ the specific weight γ of the liquid (i.e., the force exerted by gravity on a unit volume of a fluid, $\gamma = \rho g$, with ρ the density of the fluid)

Our modelling assumption is written

$$\omega = f(m, D, \gamma)$$

We are thus asking the question of whether we can find (a, b, c) such that

$$[\omega] = \left[m^a D^b \gamma^c\right]$$

We then have a total of $n = 4$ physical quantities in our problem whose dimensions as given by

$$\begin{aligned}
[D] &= L \\
[\omega] &= T^{-1} \\
[m] &= M \\
[\gamma] &= ML^{-2}T^{-2}
\end{aligned}$$

We were able to express these dimensions using $m = 3$ fundamental dimensions, $\{M, L, T\}$. If our modelling assumption is correct, we should obtain

$$T^{-1} = M^a L^b \left(ML^{-2}T^{-2}\right)^c = M^{a+c} L^{b-2c} T^{-2c}$$

which leads to the following system:

$$\begin{cases} M: & a + c = 0 \\ L: & b - 2c = 0 \\ T: & -2c = -1 \end{cases}$$

which is solved by $c = 1/2$, $b = 1$ and $a = -1/2$. As we had 3 relevant variables and 3 fundamental dimensions, the system is fully determined and we do not have any dimensionless group. Finally, we find that

$$\omega = \alpha m^{-1/2} D^1 \gamma^{1/2} = \alpha D \sqrt{\frac{\gamma}{m}}$$

In conclusion, the frequency decreases when the mass of the cylinder increases but increases as the density of the fluid increases.

8. We consider that dominoes of width w and height h are placed standing in a straight line, evenly spaced at a distance d of each other. We want to use dimensional analysis to obtain information about the speed of the wave v that dominoes produce when toppling. We need a modelling assumption!

   Obviously, the distance between domino is relevant in this problem. The wave is created by dominoes toppling from their vertical position, the typical time a domino takes to hit its neighbor is most likely related to the height and width of the domino and the earth's gravity g which accelerates the domino. In conclusion, we are writing that

$$v = f(d, h, w, g)$$

   For this relation to be dimensionally homogeneous, we are trying to find the numbers $(\alpha, \beta, \gamma, \delta)$ such that

$$[v] = \left[d^\alpha h^\beta w^\gamma g^\delta\right]$$

   Expressing these in their fundamental dimensions, we obtain

$$LT^{-1} = L^\alpha L^\beta L^\gamma (L/T^2)^\delta = L^{\alpha+\beta+\gamma+\delta} T^{-2\delta}$$

   we have here 4 relevant variables but only two fundamental dimensions, we know that 2 exponents will not be determined and so we should define two dimensionless groups! We have the following system of equations

$$\begin{cases} L: & \alpha + \beta + \gamma + \delta = 0 \\ T: & -2\delta = -1 \end{cases}$$

Solving these equations we find that $\delta = 1/2$ and $\beta = 1/2 - \alpha - \beta$, which leads to

$$v = cd^{\alpha}h^{1/2-\alpha-\gamma}w^{\gamma}g^{1/2} = c\sqrt{gh}\left(\frac{d}{h}\right)^{\alpha}\left(\frac{w}{h}\right)^{\gamma} = c\sqrt{gh}\Pi_1^{\alpha}\Pi_2^{\gamma}$$

where c, α, γ are arbitrary numbers. So we conclude that

$$v = \sqrt{gh}F(\Pi_1, \Pi_2)$$

where the dimensionless groups measure the ratio of the distance between domino to their height and the ratio of the width of the domino to their height. Note that you could have picked another solution, now the form of the solution we provide in the problem should guide you in the choice of the relevant exponents here. It is quite common for dominoes to be thin and spaced out, so we can place ourselves in the limit where $w \ll h$, i.e. $\Pi_2 \to 0$ and write

$$v = \sqrt{gh}G(\Pi_1)$$

To determine, fully the speed, one would need to measure $G(\Pi_1)$ for a particular type of domino (fixing the height) and varying the inter-domino distance d.

9. To solve this problem, we will use dimensional analysis rather than solve the complex fluid dynamics problem! First, we need to make a list of the relevant variables in this problem.

   Granted what was provided to us in this problem, it sounds natural to consider that the volume flow rate Q depends on

   - ☐ the dynamics viscosity of the oil, ν, this quantifies how the fluid resists flow, so Q should be lower for a more viscous oil;
   - ☐ the cross-sectional area of the pipe A obviously and we expect Q to increase as we increase A;
   - ☐ the pressure drop per unit length, p, this quantifies how much we are "pushing" on the fluid and so if p increases, you'd expect Q to increase as well.

   Our modelling assumption is written

$$Q = f(A, p, \nu)$$

   We are thus asking the question of whether we can find (a, b, c) such that

$$[Q] = \left[A^a p^b \nu^c\right]$$

   We then have a total of $n = 4$ physical quantities in our problem whose dimensions as given by

$$\begin{aligned}
&[A] = L^2\\
&[p] = [\text{Force}][\text{Area}^{-1}][\text{Length}^{-1}] = (MLT^{-2})(L^{-2})(L^{-1}) = ML^{-2}T^{-2}\\
&[\nu] = ML^{-1}T^{-1}\\
&[Q] = L^3T^{-1}
\end{aligned} \tag{1}$$

   We were able to express these dimensions using $m = 3$ fundamental dimensions, $\{M, L, T\}$. If our modelling assumption is correct, we should obtain

$$L^3T^{-1} = M^{b+c}L^{2a-2b-c}T^{-2b-c}$$

   which leads to the following system:

$$\begin{cases} M: & b + c = 0\\ L: & 2a - 2b - c = 3\\ T: & -2b - c = -1 \end{cases}$$

which is solved by $a = 2$, $b = 1$ and $c = -1$. As we had 3 relevant variables and 3 fundamental dimensions, the system is fully determined and we do not have any dimensionless group. Finally, we find that

$$Q = \alpha \frac{pA^2}{\nu}$$

where α is an arbitrary constant which we will not need to determine. In conclusion, the volume flow rate scales as the square of the cross-section.

$$2 = \frac{Q_2}{Q_1} = \left(\frac{A_2}{A_1}\right)^2 \Rightarrow A_2 = \sqrt{2}A_1$$

To double the flow rate Q, one only needs to multiply the cross-section by a factor of $\sqrt{2}$.

10. In this problem, we put ourselves in the shoes of the great GI Taylor!

    (a) Once again, we use dimensional analysis. The modelling hypothesis is here given to us and reads:

    $$R = f(E, t, \rho)$$

    Now we can express all of these quantities in the fundamental dimensions $\{M, L, T\}$ and find

    $$\begin{aligned}
    [R] &= L \\
    [t] &= T \\
    [E] &= ML^2T^{-2} \\
    [\rho] &= ML^{-3}
    \end{aligned}$$

    We want to see if we can find numbers (a, b, c) such that

    $$[R] = \left[E^a t^b \rho^c\right]$$

    i.e.

    $$L = \left(ML^2T^{-2}\right)^a T^b \left(ML^{-3}\right)^c = M^{a+c}L^{2a-3c}T^{-2a+b}$$

    which to the following system

    $$\begin{cases} M: & a + c = 0 \\ L: & 2a - 3c = 1 \\ T: & -2a + b = 0 \end{cases}$$

    This system is solved by $a = 1/5$, $b = 2/5$ and $c = -1/5$. We conclude that the radius of the shock wave can be written:

    $$R = \alpha \left(\frac{Et^2}{\rho}\right)^{1/5}$$

    where α is an arbitrary number. Now, α can usually be determined by carrying out an experiment. Luckily (or unfortunately...), the Trinity experiment was carried out for us!

    (b) It was shown by G.I. Taylor that if $E = 1$ J and $\rho = 1$ kg/m^3, then $R = t^{2/5}$ m/s$^{2/5}$. All quantities are expressed in SI units, in particular $E = 1 \text{ J} = 1 \text{ kg.m}^2/\text{s}^2$, so we can conclude that $\alpha = 1$ and the radius of the blast is given by

    $$R = \left(\frac{Et^2}{\rho}\right)^{1/5}$$

    (c) We can now estimate the energy of the blast. From the photographs, we can estimate that $R = 130$ m at time $t = 0.025$ s; the density of the air (at $20°$C) being $\rho = 1.2 \text{ kg.m}^{-3}$, one can easily invert the relation we previously obtained and estimate the energy of the blast was $E = 7.13 \times 10^{13} \text{ J} \approx 17$ kilotons of TNT (the conversion from Joules to kilotons of TNT is such that a kiloton (of TNT) is equal to 4.184 terajoules (4.184×10^{12} J)).