Problem Sheet 2, Geometry of Curves and Surfaces, 2021-2022

Problem 1. Let T be a real number. Consider the surface

$$H_T = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 3y^2 - z^2 = T\}.$$

Sketch the surface H_T in the three cases $T > 0$, $T = 0$, $T < 0$.

(a) Show that H_T is a regular surface if and only if T is non-zero.

(b) Find all points $P = (x, y, z)$ on H_T satisfying $x = y = 1$.

(c) For each such point P in part (b), find the equation of the tangent plane to H_T at P.

Problem 2. Let $S \subset \mathbb{R}^3$ be a regular level set of some smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, and let $G : \mathbb{R}^3 \to \mathbb{R}$ be another smooth function. We say that a point $p \in S$ is a critical point of the (restricted) map $G : S \to \mathbb{R}$ if the map $dG_p : T_pS \to \mathbb{R}$ is zero.

(a) Prove that p is a critical point of G if and only if $\nabla G(p)$ is a scalar multiple (possibly zero) of $\nabla F(p)$.

(b) Assume that S is the torus $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ in $\mathbb{R}^3$. How many critical points does the function $G(x, y, z) = y$ have on S?

Problem 3. Let $S \subset \mathbb{R}^3$ be a non-empty, compact, and connected surface. By the Jordan-Brouwer theorem (extension of the Jordan curve theorem in the plane), S divides $\mathbb{R}^3$ into two components, so that we can talk about inside and outside components of S. Let $N(p)$ be the outward normal vector at $p \in S$. Prove that the Gauss map $N : S \to \mathbb{S}^2$ is surjective. Are there any non-compact surfaces for which the Gauss map is not surjective?

Hint: for each $v \in \mathbb{S}^2$, consider a maximum of the function $x \mapsto \langle x, v \rangle$ on S.

Problem 4. Let K and H denote the Gaussian and mean curvatures of S at the point p, respectively. Prove that $H^2 \geq K$. At which points p does equality hold?

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Problem 5. Let S be a nonempty, compact, oriented regular surface in $\mathbb{R}^3$ and let $p \in S$ be a point which maximises the function $f : S \to \mathbb{R}$ defined by $f(x) = |x|^2$. Prove that the normal curvature of any curve $C \subset S$ passing thought p satisfies $|k_n| \geq 1/|p|$.

Conclude that the second fundamental form A_p at p is definite, and hence that $K(p) > 0$.