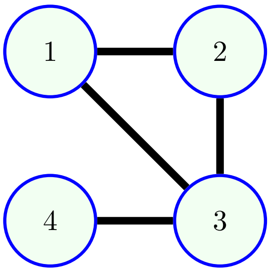
Network Science
Spring 2024
Problem sheet 10



1. The modularity matrix for the graph shown above has the following eigenvalues and eigenvectors: $\lambda_1 = -1.57, \lambda_2 = -1, \lambda_3 = 0.319, \lambda_4 = 0$,

$$\mathbf{v}_1 = [-0.2054671, -0.2054671, 0.85011161, -0.43917741]^T$$

$$\mathbf{v}_2 = [7.07106781e-01, -7.07106781e-01, 6.89664944e-16, -3.72946812e-16]^T$$

$$\mathbf{v}_3 = [-0.4558325, -0.4558325, 0.16525815, 0.74640686]^T$$

$$\mathbf{v}_4 = [0.5, 0.5, 0.5, 0.5]^T$$

Explain how the spectral community detection algorithm will partition the graph into two communities.

2. Consider a complete N-node weighted graph with weight matrix, $\mathbf{W}$, and diagonal degree matrix $\hat{\mathbf{D}}$ where $\hat{D}_{ii} = \hat{k}_i = \sum_{j=1}^{N} W_{ij}$.

   (a) Show that the normalized Laplacian, $\hat{\mathbf{L}} = \hat{\mathbf{D}}^{-1/2}(\hat{\mathbf{D}} - \mathbf{W})\hat{\mathbf{D}}^{-1/2}$, has a zero eigenvalue and find the corresponding eigenvector

   (b) Now consider a partition of the graph into two groups with $s_i = \pm 1$ indicating which group node i has been assigned to. Provide an expression for the normalized cut size in terms of $\mathbf{s}, \mathbf{z}, \mathbf{W}$, and $\hat{\mathbf{D}}$ where $\mathbf{z}$ is a column vector of N ones. The normalized cut size for a weighted graph is defined as: (wieghted cut size) $* (1/\hat{K}_a + 1/\hat{K}_b)$ where $\hat{K}_a$ and $\hat{K}_b$ are the total weighted degrees (computed from $\hat{k}_i$) of the two groups of nodes in the partition.

   (c) It can be shown that minimizing the normalized cut size is equivalent to finding $\mathbf{y}$ such that $\frac{\mathbf{y}^T\hat{\mathbf{L}}\mathbf{y}}{\mathbf{y}^T\hat{\mathbf{D}}\mathbf{y}}$ is minimized with $y_i \in \{1, -b\}$ and $\mathbf{y}^T\hat{\mathbf{D}}\mathbf{z} = 0$. Here $\hat{\mathbf{L}} = \hat{\mathbf{D}} - \mathbf{W}$, $\mathbf{y} = (\mathbf{z} + \mathbf{s}) - b(\mathbf{z} - \mathbf{s})$, and $b = \hat{K}_a/\hat{K}_b$. Find a transformation for $\mathbf{y}$ that results in the following equivalent minimization problem: Find $\mathbf{x}$ such that $\frac{\mathbf{x}^T\tilde{\mathbf{L}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is minimized. Typically the 1st constraint on $\mathbf{y}$ ($y_i \in \{1, -b\}$) is removed, leaving one constraint for both $\mathbf{x}$ and $\mathbf{y}$. What is this constraint for $\mathbf{x}$?

(d) Find the vector $\mathbf{x}$ that solves the minimization problem described in (c) with the constraint that you derived. Provide an interpretation of your result.

(e) The minimization problem in (c) is typically solved for image segmentation applications (e.g. problems related to artificial vision). The spectral clustering method is based on computing the eigenvectors corresponding to the *largest* eigenvalues of $\hat{\mathbf{A}} = \hat{\mathbf{D}}^{-1/2}\mathbf{W}\hat{\mathbf{D}}^{-1/2}$. How are the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$ and $\hat{\mathbf{L}}$ related?

3. Consider an undirected connected weighted graph with weight matrix $\mathbf{W}$ which has all elements non-negative.

(a) Propose a definition for a step of a random walk on this graph in terms of $\mathbf{W}$ and quantities derived from $\mathbf{W}$

(b) What is the stationary distribution for your model?

(c) Describe (give a sketch rather than a rigorous argument) how the locations of walkers at large times are related to non-trivial graph partitions obtained from the normalized minimum cut problem discussed in 2(c) (after the relaxation of the first constraint on $\mathbf{y}$).