Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 20
Finishing Power series

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Where we left off

Theorem 4.35 - Radius of Convergence

Fix a real or complex sequence (a_n) and consider the series $\sum a_n z^n$ for $z \in \mathbb{C}$. Then $\exists\, R \in [0, \infty]$ such that

- $|z| < R \implies \sum a_n z^n$ is absolutely convergent, and
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The R in Thm 4.35 is called the radius of convergence for $\sum a_n z^n$. Note that Thm 4.35 doesn't tell us what happens when $|z| = R$.

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Here is an exercise that I strongly encourage you to do if you haven't before:

Exercise 4.38

Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to a \in [0, \infty]$ as $n \to \infty$.
Then $R = \frac{1}{a}$ is the radius of convergence of $\sum a_n z^n$.

$$\rightarrow \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| \rightarrow a \cdot |z| < 1 \rightarrow |z| < \frac{1}{a}$$

guarantees absolute convergence

If $a = 0 \Rightarrow R = \infty$

$a = \infty \Rightarrow R = 0$

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$$R = \sup\{r : a_n r^n \to 0\}$$

Note that the converse is not true, that is if $\sum a_n z^n$ has a radius of convergence R, it is possible for the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ to not exist!

Ratio test useless

$a_n = \begin{cases} 2^n & \text{for } n \text{ even} \\ 3^n & \text{for } n \text{ odd} \end{cases}$ $\quad \frac{3^{n+1}}{2^n}, \quad \frac{2^{n+1}}{3^n}$ $\quad R = \frac{1}{3}$

Products of Power Series

Products of Power Series

Consider

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n \;&=\; (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\
\text{``}=\text{''}\; & a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \\
=\; & \sum_{n=0}^{\infty} c_n z^n,
\end{aligned}
$$

$$\underbrace{\left(\sum_{j=0}^{n} \overset{z^j}{\overset{\downarrow}{a_j}}\, \overset{z^{n-j}}{\overset{\downarrow}{b_{n-j}}} \right)}_{c_n} z^n$$

where $c_0 = a_0 b_0,\ c_1 = a_0 b_1 + a_1 b + 0, \dots$

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

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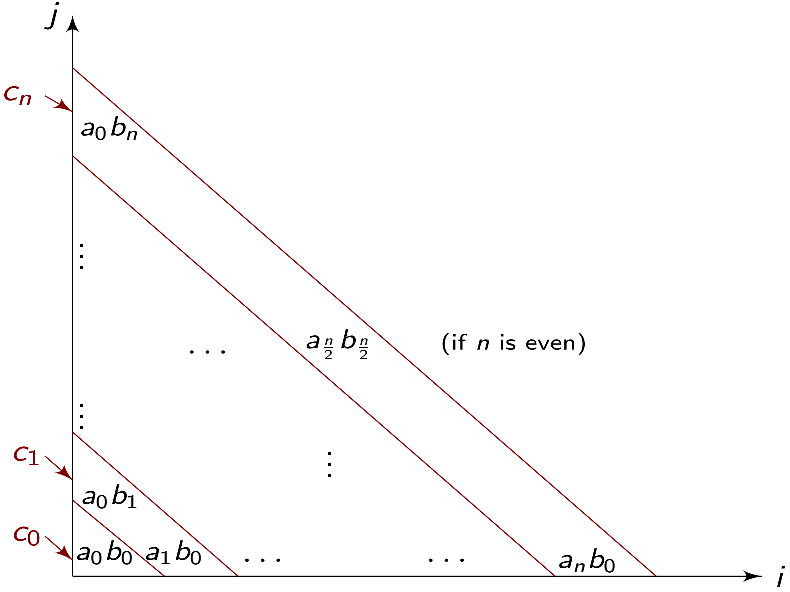
where $c_0 = a_0 b_0,\ c_1 = a_0 b_1 + a_1 b + 0, \dots$

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i=0}^{n} a_i b_{n-i}\ .$$


j
i
c_n
c_1
c_0
$a_0\,b_n$
$a_{\frac{n}{2}}\,b_{\frac{n}{2}}$
(if n is even)
$a_0\,b_1$
$a_0\,b_0$
$a_1\,b_0$
$a_n\,b_0$

So we set $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ and ask when is the product $\sum a_n z^n \sum b_n z^n$ equal to $\sum c_n z^n$?

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We can also do this without the z^n s:

Definition

Given series $\sum a_n$, $\sum b_n$ their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^{n} a_i b_{n-i}$.

Notice we used power series to motivate this definition.

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Notice we used power series to motivate this definition.

It is not the only way we could collect all the terms $a_i b_j$ to turn $\sum a_i \sum b_j$ into a single sum. This is why we give it the specific name *Cauchy product*.

$\sum_{i=1}^{\infty} a_i b_i$ "dot product"

The next theorem says that the Cauchy Product preserves the value of products of absolutely convergent series.

Theorem 4.39 - Cauchy Product

If $\sum a_n, \sum b_n$ are absolutely convergent, then their Cauchy product $\sum c_n$ is absolutely convergent to $\left(\sum a_n\right) \cdot \left(\sum b_n\right)$.

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Corollary 4.40

If $\sum a_n z^n$ and $\sum b_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.

Pf First we show that $\forall z \in \mathbb{C}$, with $|z| < \min(R_A, R_B)$

$|z| < R_A$
$|z| < R_B$

$\sum c_n z^n$ is absolutely convergent.
This follows from Thm 4.39 since, for such z, $\sum a_n z^n$ and $\sum b_n z^n$ are absolutely convergent.

By Thm on Power series, $|z| \leq R_C$

That is $|z| < \min(R_A, R_B) \Rightarrow |z| \leq R_C$.

Then we are done by the next exercise.

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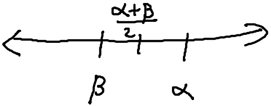
Question: Can we prove $R_C = \min(R_A, R_B)$

Exercise 4.41

Fix $\alpha, \beta \in \mathbb{R}$. Prove that if $[x < \alpha \Rightarrow x \leq \beta]$ then $\alpha \leq \beta$.

Pf Suppose, by contradiction, that $\beta < \alpha$



Set $x = \frac{\alpha+\beta}{2}$

Then $x < \alpha$, but $x > \beta$ $\#$.

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Example 4.42

$\sum z^n$ has $R_A = 1$. $\leftarrow \frac{1}{1-z}$

$1 - z$ has $R_B = \infty$.

So their Cauchy product $\underbrace{\sum c_n z^n}_{=1}$ has $R_C \geq 1$.

$b_0 = 1$

$b_1 = -1$

$b_n = 0$ for $n > 1$

$R_C = \infty$

$c_0 = a_0 b_0 = 1$

$c_1 = a_0 b_1 + b_0 a_1 = 0$

Prove $c_n = 0$ for $n > 0$

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If $\sum a_n z^n$ and $\sum b_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.

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$\sum z^n$ has $R_A = 1$.
$1 - z$ has $R_B = \infty$.
So their Cauchy product $\sum c_n z^n$ has $R_C \geq 1$.

End of the examinable material for **Term 1**!