Imperial College London

MATH97223 MATH97238

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Classical Dynamics

Date: 19th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. (a) The area of the surface of revolution obtained by rotating the curve $y = y(x) > 0$ around the x-axis for $a \le x \le b$ is

$$A = 2\pi \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx.$$

The area is to be minimised with $y(a)$ and $y(b)$ fixed. Write down the Euler-Lagrange equation for this problem and verify that

$$y(x) = \frac{1}{p}\cosh(px + c),$$

satisfies the equation (p and c are constants). (7 marks)

(b) A bead of mass m moves without friction or gravity on a spiral-shaped wire described in polar coordinates by the equation $r = \exp(\theta)$ for $-\infty < \theta < \infty$. Find a Lagrangian for this system and obtain the equation of motion (simplify your answer if possible). Solve the equation of motion.

Hint: treat the equation of motion as a first order ODE for $\omega(t) = \dot{\theta}(t)$.

(7 marks)

(c) Define Poisson brackets. Show that if $A(q_i, p_i, t)$ is a function on phase space then

$$\frac{d}{dt}A = \{A, H\} + \frac{\partial A}{\partial t},$$

where H is the Hamiltonian function. (6 marks)

(Total: 20 marks)

2. The motion of a particle of unit mass in a plane is governed by the Lagrangian

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + br^2\dot{\theta} - \frac{3}{2}b^2r^2,$$

where r and θ are polar coordinates and b is a positive constant.

(a) Show that θ is a cyclic variable and obtain the equations of motion. (5 marks)

(b) Show that there are circular orbits of the form $r =$ constant. What are the periods of these orbits?

Hint: distinguish clockwise and anti-clockwise orbits.

(5 marks)

(c) Show that the corresponding Hamiltonian is

$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + 2b^2r^2 - bp_\theta.$$

(5 marks)

(d) Describe the solutions of Hamilton's equation where $p_\theta = 0$.

(5 marks)

(Total: 20 marks)

3. The time evolution of a physical system is determined by the Hamiltonian

$$H(q,p) = p^2 + e^q.$$

(a) Write down Hamilton's equations for this system. (3 marks)

(b) The canonical transformation

$$Q = -2\sinh^{-1}(pe^{-q/2}), \quad P = \sqrt{p^2 + e^q},$$

maps the above Hamiltonian $H(q,p)$ to the new Hamiltonian $K(Q,P) = P^2$.

Solve the equations of motion for the new Hamiltonian to obtain $Q(t)$, $P(t)$. Use your result to find $q(t)$, $p(t)$ thereby solving the equations of motion considered in part (a).

Hint: show that $\tanh(Q/2) = -p/P$. (8 marks)

(c) Outline briefly what is meant by a type 1 generating function for a canonical transformation (proofs are not required). (4 marks)

(d) Obtain the type 1 generating function for the time-independent canonical transformation quoted in part (b). (5 marks)

(Total: 20 marks)

4. (a) The Lagrangian for a symmetric top fixed at one point is

$$L = T - V = \frac{I_1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta,$$

where ϕ, θ and ψ are Euler angles, M is the total mass and l is the distance between the fixed point and the centre of mass.

Determine p_ϕ and p_ψ and explain why they are constants of the motion. Express the energy

$$E = T + V = \frac{I_1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + Mgl\cos\theta,$$

in the form

$$E = \frac{I_1}{2}\dot{\theta}^2 + \frac{p_\psi^2}{2I_3} + U(\theta),$$

where $U(\theta)$ is an effective potential depending on θ and the constants p_ϕ and p_ψ.

(10 marks)

(b) Solve the Hamilton-Jacobi equation for the Hamiltonian

$$H = q^2p^2 + \log q,$$

and use the result to determine $q(t)$. (10 marks)

(Total: 20 marks)

5. Consider the Hamiltonian

$$H = \frac{p^2}{2} + \lambda |x|.$$

(a) Sketch the trajectories in the xp plane (take $\lambda = 1$). (6 marks)

(b) Treating λ as a positive constant, write H as a function of λ and the action variable

$$J = \oint p \, dx.$$

(8 marks)

(c) Suppose that λ is slowly (adiabatically) increased until it is twice its initial value. What is the effect of this change on the frequency of oscillation? (6 marks)

(Total: 20 marks)

Answers to May 2020 Examination

1. (a) The Euler-Lagrange equation is

$$\frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}} = \sqrt{1+y'^2},$$

on taking the integrand $y\sqrt{1+y'^2}$ as the Lagrangian (dropping the factor of 2π).

For the proposed soliution $y = p^{-1}\cosh(px+c)$, $\sqrt{1+y'^2} = \sqrt{1+\sinh^2(px+c)} = \cosh(px+c)$ so the left hand side of the equation is

$$\frac{d}{dx}\frac{p^{-1}\cosh(px+c)\sinh(px+c)}{\cosh(px+c)} = \frac{1}{p}\frac{d}{dx}\sinh(px+c) = \cosh(px+c),$$

which agrees with the right hand side.

(7 marks, seen similar, A)

(b) In polar coordinates $T = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$. On the wire $r = e^\theta$, $\dot{r} = e^\theta\dot{\theta}$ so that $L = T = me^{2\theta}\dot{\theta}^2$. The Euler-Lagrange equation is

$$\frac{d}{dt}2e^{2\theta}\dot{\theta} - 2e^{2\theta}\dot{\theta}^2 = 2e^{2\theta}(\ddot{\theta}+\dot{\theta}^2) = 0,$$

so that $\ddot{\theta} = -\dot{\theta}^2$. Treating this as a first order ODE for $\omega = \dot{\theta}$, $d\omega/\omega^2 = -dt$ so that $\omega^{-1} = t + c$ or $\omega = (t+c)^{-1}$ which integrates to $\theta = \log(t+c) + d$.

(7 marks, seen similar, B)

(c) Let $A(q_i, p_i, t)$ and $B(q_i, p_i, t)$ be functions defined on phase space. The Poisson bracket of A and B is defined by

$$\{A, B\} = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

Using the chain rule

$$\frac{dA}{dt} = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i\right) + \frac{\partial A}{\partial t}$$

$$= \sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial A}{\partial t} = \{A, H\} + \frac{\partial A}{\partial t},$$

using Hamilton's equations.

(6 marks, bookwork, A)

2. (a)

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + br^2\dot{\theta} - \frac{3}{2}b^2r^2.$$

θ is cyclic as $\partial L/\partial\theta = 0$. The equations of motion are

$$\frac{d}{dt}\dot{r} - r\dot{\theta}^2 - 2br\dot{\theta} + 3b^2r = 0,$$

$$\frac{d}{dt}(r^2\dot{\theta} + r^2b) = 0.$$

(5 marks, seen similar, A)

(b) Inserting the trial solution $r =$ constant in the first equation gives $r = 0$ (particle stuck at origin) or $\dot{\theta}^2 + 2b\dot{\theta} - 3b^2 = 0$. Therefore $(\dot{\theta}+3b)(\dot{\theta}-b) = 0$ giving $\dot{\theta} = b$ (anti-clockwise) or $\dot{\theta} = -3b$ (clockwise). The periods are $T = 2\pi/b$ (anti-clockwise) or $T = \frac{2}{3}\pi/b$ (clockwise).

(5 marks, seen similar, B)

(c) $p_r = \partial L/\partial\dot{r} = \dot{r}$, $p_\theta = \partial L/\partial\dot{\theta} = r^2(\dot{\theta} + b)$ so that

$$H = p_r\dot{r}+p_\theta\dot{\theta}-L = p_r^2+p_\theta\left(\frac{p_\theta}{r^2} - b\right)-\frac{p_r^2}{2}-\frac{r^2}{2}\left(\frac{p_\theta}{r^2} - b\right)^2-br^2\left(\frac{p_\theta}{r^2} - b\right)+\frac{3}{2}b^2r^2$$

$$= \frac{p_r^2}{2} + \frac{r^2}{2}\left(\frac{p_\theta}{r^2} - b\right)^2 + \frac{3}{2}b^2r^2 = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + 2b^2r^2 - bp_\theta.$$

(5 marks, seen similar, B)

(d) Here

$$\dot{r} = \frac{\partial H}{\partial p_r} = p_r, \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{r^3} - 4b^2r.$$

Setting $p_\theta = 0$ gives $\ddot{r} = -4b^2r$, the equation of a simple harmonic oscillator with angular frequency $2b$. The general solution is $r(t) = A\cos(2bt + \beta)$. $p_\theta = 0$ gives $\dot{\theta} = -2b$ so that $bt = -\theta +$ constant and $r(t) = A\cos(2bt + \beta) = A\cos(2\theta + \gamma)$.

(5 marks, unseen, D)

3. (a) $\dot{q} = \partial H/\partial p = 2p, \quad \dot{p} = -\partial H/\partial q = -e^q.$

(3 marks, seen similar, A)

(b) $\dot{Q} = \partial K/\partial P = 2P$, $\dot{P} = -\partial K/\partial Q = 0$. Therefore P is constant and $Q = 2Pt + c$ with c constant.

$-\sinh(Q/2) = pe^{-q/2}$ and

$$\tanh(Q/2) = \frac{\sinh(Q/2)}{\sqrt{1+\sinh^2(Q/2)}} = -\frac{pe^{-q/2}}{1+p^2e^{-q}} = -\frac{p}{P}.$$

Accordingly,

$$p = -P\tanh(Q/2), \quad e^{-q/2} = -\frac{\sinh(Q/2)}{p} = \frac{\cosh(Q/2)}{P},$$

or

$$p = -P\tanh(Pt+b), \quad q = 2\log\left(\frac{P}{\cosh(Pt+b)}\right).$$

(8 marks, seen similar, C)

(c) The old and new momenta, p and P, can be written as follows

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q},$$

where the generating function F is a function of q, Q and t.

(4 marks, bookwork, B)

(d) $-\sinh(Q/2) = pe^{-q/2}$ so that $p = -e^{q/2}\sinh(Q/2) = \partial F/\partial q$ so that $F = -2e^{q/2}\sinh(Q/2)$.

(5 marks, seen similar, C)

4. (a) $p_\phi = \partial L/\partial\dot{\phi} = I_1 \sin^2\theta\dot{\phi} + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$,

$p_\psi = \partial L/\partial\dot{\psi} = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$.

These momenta are constants as ϕ and ψ are cyclic.

The formula for p_ϕ can be written as $p_\phi = I_1\sin^2\theta\dot{\phi} + p_\psi\cos\theta$ so that

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta},$$

hence $E = \frac{1}{2}I_1\dot{\theta}^2 + {p_\psi}^2/(2I_3) + U$, with

$$U = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgl\cos\theta.$$

(10 marks, seen similar, A)

(b) $S = W - \alpha t$ where

$$q^2\left(\frac{\partial W}{\partial q}\right)^2 + \log q = \alpha,$$

so that

$$\frac{\partial W}{\partial q} = \frac{1}{q}\sqrt{\alpha - \log q}.$$

which integrates to

$$W = -\frac{2}{3}(\alpha - \log q)^{3/2}.$$

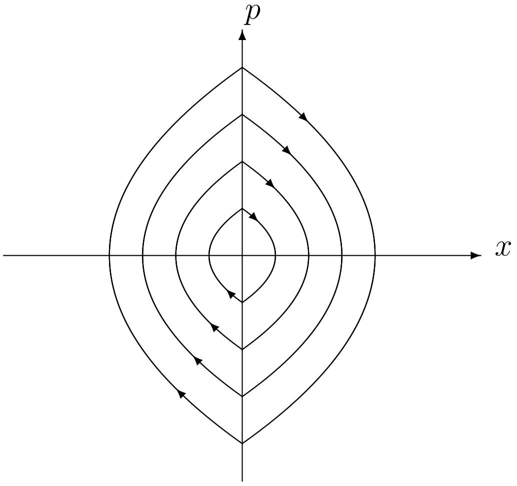
The new coordinate is

$$\beta = \frac{\partial S}{\partial\alpha} = -(\alpha - \log q)^{1/2} - t.$$

$$(\beta + t)^2 = \alpha - \log q \quad \text{or} \quad q(t) = \exp\left[\alpha - (t+\beta)^2\right].$$

(10 marks, unseen, D)

5. (a) Closed orbits comprise parts of parabolas joined at p-axis.



(6 marks, seen similar)

(b)

$$J = \oint p dx = 4 \int_0^{\frac{E}{\lambda}} \sqrt{2E - 2\lambda x} dx = -4 \cdot \frac{1}{3\lambda} (2E - 2\lambda x)^{3/2} \Big|_0^{E/\lambda} = \frac{4}{3\lambda} (2E)^{3/2},$$

so that $2E = (\frac{3}{4})^{2/3} (\lambda J)^{2/3}$ giving

$$H = 2^{-7/3} 3^{2/3} \lambda^{2/3} J^{2/3}.$$

(8 marks, seen similar)

(c) Frequency

$$\nu = \frac{\partial H}{\partial J} = \frac{2}{3} 2^{-7/3} 3^{2/3} \lambda^{2/3} J^{-1/3}.$$

Under an adiabatic change of λ, J is unchanged. If λ is doubled (adiabatically) the frequency is increased by a factor $2^{2/3}$.

(6 marks, unseen)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97223 MATH97238	1	Mostly well answered. Some students used the Beltrami formula in part (a) which was not necessary.	
MATH97223 MATH97238	2	Mostly well answered. A few students got stuck in part (b) as they tried to use the equation of motion for theta rather than r.	
MATH97223 MATH97238	3	This question is a variation of a problem sheet question. Some students were too closely following the line of the problem sheet question (e.g. the problem sheet used a type 4 generating function as opposed to the type 1 used in this question).	
MATH97223 MATH97238	4	Part (b) was generally answered well. Some students missed the point of part (a) even though this is fairly straightforward.	
MATH97223 MATH97238	5	Mostly well done. The only complaint is that there were quite a few poor sketches for part (a). This was surprising given that a generous 6 marks had been allocated to this part!	