Mathematical Logic (MATH6/70132;P65)
Problem Sheet 3

[1] The first-order language $\mathcal{L}$ has one unary function symbol f and one unary relation symbol P. Let ϕ be the formula $(\forall x_1)(P(x_1) \to P(f(x_1)))$. Give an interpretation of $\mathcal{L}$ in which ϕ is true, and one in which it is false.

[2] The language $\mathcal{L}$ has a binary relation symbol E, a binary function symbol m, a unary function symbol i and a constant symbol e. Let G be a group and consider G as an $\mathcal{L}$-structure by interpreting E as equality, m as multiplication, i as inversion, and e as the identity element of G. Let v be a valuation (of $\mathcal{L}$) in G and let

$$H = \{v(t) : t \text{ is a term of } \mathcal{L}\}.$$

(a) Show that H is a subgroup of G.

(b) Show that H is generated by $\{v(x_j) : x_j \text{ is a variable of } \mathcal{L}\}$.

(c) What is H if we omit the function symbol i from the language?

[3] Suppose F is a field. The language $\mathcal{L}_F$ appropriate for considering F-vector spaces V has a 2-ary relation symbol R (for equality); a 2-ary function symbol a (for addition in the vector space); a constant symbol 0 (for the zero vector) and, for every $\alpha \in F$, a 1-ary function symbol f_α (for scalar multiplication by α).

Convince yourself that it is possible to express the axioms for being an F-vector space as a set of formulas in this language.

[4] Let ϕ be a formula in a first-order language $\mathcal{L}$ and let v be a valuation (in some $\mathcal{L}$-structure $\mathcal{A}$). Suppose there is a valuation v' which is x_i-equivalent to v and satisfies ϕ. Show that v satisfies $(\exists x_i)\phi$.

[5] In each of the following formulas, indicate which of the occurrences of the variables x_1 and x_2 are bound and which are free:
(a) $(\forall x_2)(R_2(x_1, x_2) \to R_2(x_2, c_1))$;
(b) $(R_1(x_2) \to (\forall x_1)(\forall x_2)R_3(x_1, x_2, c_1))$;
(c) $((\forall x_1)R_1(f(x_1, x_2)) \to (\forall x_2)R_2(f(x_1, x_2), x_1))$.

Decide whether the term $f(x_1, x_1)$ is free for x_2 in each of the above formulas (explain briefly your answer).

[6] For the formula $\phi(x_2)$ given by $((\exists x_1)R(x_1, f(x_1, x_2)) \to (\forall x_1)R(x_1, x_2))$ (in a particular language $\mathcal{L}$) give an example of a term t which is not free for x_2 in $\phi(x_2)$. Find an $\mathcal{L}$-structure $\mathcal{A}$ in which $(\forall x_2)\phi(x_2)$ is true and a valuation v in $\mathcal{A}$ which does not satisfy $\phi(t)$.

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