Imperial College London

MATH97027 MATH97104

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Partial Differential Equations

Date: Wednesday, 2 June 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Answer the following two questions related to the method of characteristics:

(a) Solve the Cauchy problem:

$$\begin{cases} xu_x + u_y = y, & (x,y) \in \mathbb{R}^2, \\ u(x,0) = x^2, & x \in \mathbb{R}. \end{cases}$$

Give the maximal domain of existence of the classical solution $u(x,y)$. (8 marks)

(b) Consider the the global Cauchy problem

$$\begin{cases} u_t + 6(1-2u)u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 1/3, & x \leq 0, \\ 1/3 + 5x/12 & 0 < x < 1, \\ 3/4 & x \geq 1. \end{cases}$$

Draw the characteristic lines, compute the shocks and find the unique entropy solution of the problem.

(12 marks)

(Total: 20 marks)

2. Let $g \in C^1([0,\pi])$ such that $g(0) = g(\pi) = 0$. Consider the initial-boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi,\ t > 0, \\ u(0,x) = g(x), & 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, & t > 0. \end{cases} \tag{1}$$

(a) Solve the problem (1) by using separation of variables. (6 marks)

(b) Prove that the solution found in Part (a) is the unique solution to the Cauchy problem (1). (6 marks)

(c) Using the formula found in Part (a), prove that

$$\lim_{t\to\infty} \sup_{x\in[0,\pi]} |u(t,x)| = 0.$$

(4 marks)

(d) Assume that u_1 is the solution to the Cauchy problem (1) with initial datum $g_1(x)$, and u_2 is the solution to the Cauchy problem (1) with initial datum $g_2(x)$. Show that

$$\sup_{(t,x)\in[0,\infty)\times[0,\pi]} |u_1(t,x) - u_2(t,x)| \leq \sup_{x\in[0,\pi]} |g_1(x) - g_2(x)|$$

(4 marks)

(Total: 20 marks)

3. Consider the general one-dimensional wave equation with sources

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = F(t,x) & \text{in } t>0,\ x\in\mathbb{R}, \\ u(0,x) = g(x) & \text{for } x\in\mathbb{R}, \\ \partial_t u(0,x) = h(x) & \text{for } x\in\mathbb{R}, \end{cases} \tag{2}$$

with $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and F a continuously differentiable function on $\mathbb{R}$.

(a) Show the uniqueness of a classical solution, $u \in C^2((0,\infty)\times\mathbb{R})$, to the initial value problem (2). (6 marks)

(b) Show that

$$u(t,x) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\mathrm{d}y + \frac{1}{2c}\int_0^t\int_{x-ct+cs}^{x+ct-cs} F(s,y)\mathrm{d}y\mathrm{d}s,$$

is the classical solution to the Cauchy problem (2). (8 marks)

(c) Prove the stability of solutions with respect to data and sources for $x \in \mathbb{R}$ and $0 \le t \le T$ for any given $T > 0$; that is, show that for all $\epsilon > 0$, there exists $\delta > 0$ such that if

$$|g_1(x) - g_2(x)| \le \delta\,, \qquad |h_1(x) - h_2(x)| \le \delta\,, \qquad \text{and} \qquad |F_1(t,x) - F_2(t,x)| \le \delta$$

for all $x \in \mathbb{R}$ and $0 \le t \le T$, then

$$|u_1(t,x) - u_2(t,x)| \le \epsilon$$

for all $x \in \mathbb{R}$ and $0 \le t \le T$, being u_i, $i = 1,2$, the solutions to (2) with data and sources g_i, h_i, and F_i, $i = 1,2$, respectively. (6 marks)

(Total: 20 marks)

4. Recall that a function $u \in C^2(\mathbb{R}^2)$ is called harmonic if $\Delta u = 0$ in $\mathbb{R}^2$, while $w \in C^2(\mathbb{R}^2)$ is called subharmonic if $\Delta w \ge 0$ in $\mathbb{R}^2$.

(a) Let $w \in C^2(\mathbb{R}^2)$ be subharmonic. Show that for any $R > 0$ and any $\boldsymbol{x} \in \mathbb{R}^2$,

$$w(\boldsymbol{x}) \le \frac{1}{2\pi R}\int_{\partial B_R(\boldsymbol{x})} w(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}$$

and

$$w(\boldsymbol{x}) \le \frac{1}{\pi R^2}\int_{B_R(\boldsymbol{x})} w(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

where $B_R(\boldsymbol{x})$ is the open ball of radius R, centered at $\boldsymbol{x} \in \mathbb{R}^2$, and $\partial B_R(\boldsymbol{x})$ is its boundary. (8 marks)

(b) Show that if $u \in C^2(\mathbb{R}^2)$ is harmonic and $F \in C^2(\mathbb{R})$ is convex, then the function $w = F(u)$ is subharmonic. (6 marks)

(c) Assume that $u \in C^2(\mathbb{R}^2)$ is harmonic in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} |\nabla u(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = M < \infty.$$

Prove that u is constant. (6 marks)

(Total: 20 marks)

5. Let $a \geq 1$ be a fixed parameter, $\Omega \subset \mathbb{R}^2$ a smooth bounded domain, and $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field such that $\operatorname{div}(\boldsymbol{u}) = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the outward unit normal to the boundary of Ω. Consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = -\rho^a, & (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \\ \rho(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{3}$$

In what follows, assume that g is a continuously differentiable function on Ω.

(a) Assume that g is positive. Using the method of characteristics, find the solution ρ of the problem and deduce that ρ is positive, global in time, bounded above by $\sup_{\boldsymbol{x} \in \Omega} |g(\boldsymbol{x})|$, and $\lim_{t \to \infty} \sup_{\boldsymbol{x} \in \Omega} |\rho(t, \boldsymbol{x})| = 0$. How does the rate of convergence to 0 depend on a? (8 marks)

(b) Assume that g is positive. Use the energy method to prove that the solution found in Part (a) is unique. (8 marks)

(c) Find a value of $a \geq 1$ and an initial datum g for which the solution to (3) is not global in time. Find the maximum time of existence of this solution. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH96018/97027/97104

Introduction to Partial Differential Equations (Solutions)

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1. (a) The associated characteristic system is

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$$\frac{\mathrm{d}x}{\mathrm{d}s} = x, \qquad x(0) = \tau,$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = 1, \qquad y(0) = 0,$$
$$\frac{\mathrm{d}z}{\mathrm{d}s} = y, \qquad z(0) = \tau^2.$$

From the first two equations we obtain

$$x = X(s,\tau) = \tau \mathrm{e}^s, \qquad y = Y(s,\tau) = s,$$

so that

$$u(X(s,\tau), Y(s,\tau)) = Z(s,\tau) = \tau^2 + \int_0^s \sigma \mathrm{d}\sigma = \tau^2 + s^2/2.$$

Then we find

$$s = S(x,y) = y, \qquad \tau = T(x,y) = x\mathrm{e}^{-y}.$$

Therefore

$$u(x,y) = x^2 \mathrm{e}^{-2y} + y^2/2$$

is the unique solution. The domain is all of $\mathbb{R}^2$. 8, A

(b) The characteristic emanating from the point $(\xi, 0)$ on the xt-plane, along which the solution is constant and equals $g(\xi)$, has equation sim. seen ⇓

$$x = \xi + 6(1 - 2g(\xi))t = \begin{cases} \xi + 2t, & \xi < 0, \\ \xi + (2 - 5\xi)t & 0 \le \xi \le 1, \\ \xi - 3t & \xi > 1. \end{cases}$$

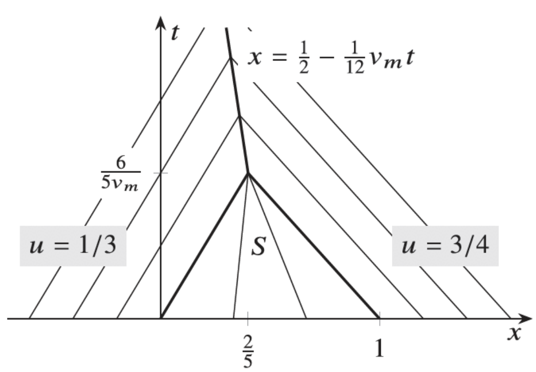

Figure 1: The solution u

We can see that the characteristics meet, creating a shock wave. The starting point of it is the point with smallest time coordinate, at which the characteristics intersect for $0 < \xi < 1$. In this case the characteristics form a pencil depending on the parameter ξ, and the pencil base point, where all characteristics with $0 < \xi < 1$ meet, is $(2/5, 1/5)$. See Figure 1.

We divide now in two cases. If $t < 1/5$, then it is clear that $u = 1/3$ for $x < 2t$ and $u = 3/4$ for $x > 1 - 3t$. In between the regions (the region S in the picture), we get

$$\xi = \frac{x - 2t}{1 - 5t} \qquad \Rightarrow \qquad u = g(x-6(1-2u)t) \qquad \Rightarrow \qquad u = \frac{1}{3}+\frac{5}{12}(x-6(1-2u)t).$$

Solving for u gives

$$u = \frac{4 + 5x - 30t}{12(1 - 5t)} \qquad 2t < x < 1 - 3t.$$

We now compute the equation of the shock $x = s(t)$ satisfying the Rankine-Hugoniot condition. Since here $u_- = 1/3$ and $u_+ = 3/4$ we have

$$s' = -\frac{1}{2}, \qquad s(1/5) = 2/5 \qquad \Rightarrow \qquad s(t) = \frac{1}{2} - \frac{1}{2}t.$$

Thus in the end we get

$$u(t,x) = \begin{cases} 1/3, & x \le \min\{2t, 1/2 - t/2\}, \\ \frac{4+5x-30t}{12(1-5t)} & 2t < x < 1 - 3t, \\ 3/4 & x \ge \max\{1 - 3t, 1/2 - t/2\}. \end{cases}$$

$\boxed{12, \text{C}}$

2. (a) We look for solutions of the type $u(t,x) = w(t)v(x)$. Substituting into the equation gives

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$$v(x)w'(t) - v''(x)w(t) = 0 \qquad \Rightarrow \qquad \frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)},$$

so that each side must both be equal to some constant $\lambda \in \mathbb{R}$. For w we obtain the equation

$$w'(t) - \lambda D w(t) = 0, \qquad \Rightarrow \qquad w(t) = C\mathrm{e}^{\lambda t},\ C \in \mathbb{R}.$$

For v we have the eigenvalue problem

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v'(0) = v'(\pi) = 0. \end{cases}$$

In the case $\lambda \geq 0$, we obtain that the only possible solution is $v = 0$. If $\lambda = -\mu^2 < 0$ we have

6, A

$$v(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x), \qquad v(0) = v(\pi) = 0.$$

Imposing the boundary conditions we find that $C_1 = 0$, C_2 arbitrary and $\lambda = -k^2$, for $k \in \mathbb{N}$. Together we the expression for w, we find that all functions

$$\varphi_k(t,x) = \mathrm{e}^{-k^2 t} \sin(kx)$$

solve the heat equation with Dirichlet boundary conditions. Setting

$$u(t,x) = \sum_{k=1}^{\infty} c_k \mathrm{e}^{-k^2 t} \sin(kx),$$

we therefore need to impose that

$$u(0,x) = \sum_{k=1}^{\infty} c_k \sin(kx) = g(x).$$

By expanding g in sine-Fourier series,

$$g(x) = \sum_{k=1}^{\infty} g_k \sin(kx), \qquad g_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(kx) \mathrm{d}x.$$

Thus we need $c_k = g_k$, giving

$$u(t,x) = \sum_{k=1}^{\infty} g_k \mathrm{e}^{-k^2 t} \sin(kx).$$

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(b) The solution is unique. If v is another solution, let $w = u - v$ and set

$$E(t) = \int_0^{\pi} |w(t,x)|^2 \mathrm{d}x.$$

Then

$$E'(t) = -2 \int_0^{\pi} |w_x(t,x)|^2 \mathrm{d}x \leq 0.$$

From this,

6, B

$$E(t) \leq E(0) = 0.$$

From the definition of $E(t)$, this implies

$$\int_0^{\pi} \left(u(t,x) - v(t,x)\right)^2 \mathrm{d}x = 0,$$

so that $u = v$.

$\boxed{\text{meth seen} \Downarrow}$

(c) From

$$u(t,x) = \sum_{k=1}^{\infty} g_k \mathrm{e}^{-k^2 t} \sin(kx),$$

we first notice that

$$\left| g_k \mathrm{e}^{-k^2 t} \sin(kx) \right| \le \mathrm{e}^{-t} |g_k|.$$

Since $\sum_{k \ge 1} |g_k| = S < \infty$, we obtain $\boxed{4, \mathrm{A}}$

$$\sup_{x \in [0,\pi]} |u(t,x)| \le S\mathrm{e}^{-t} \to 0 \quad \text{as } t \to \infty,$$

proving the claim. $\boxed{\text{unseen} \Downarrow}$

(d) This follows from the maximum principle. Since $v = u_1 - u_2$ solves

$$\begin{cases} v_t - v_{xx} = 0, & 0 < x < \pi,\ t > 0, \\ v(0,x) = g_1(x) - g_2(x), & 0 \le x \le \pi, \\ v(t,0) = v(t,\pi) = 0, & t > 0, \end{cases}$$

the maximum principle tell us that $\boxed{4, \mathrm{D}}$

$$\begin{aligned} \sup_{(t,x) \in [0,\infty) \times [0,\pi]} |u_1(t,x) - u_2(t,x)| &= \sup_{(t,x) \in [0,\infty) \times [0,\pi]} |v(t,x)| \\ &\le \sup_{x \in [0,\pi]} |v(0,x)| = \sup_{x \in [0,\pi]} |g_1(x) - g_2(x)|. \end{aligned}$$

3. (a) Given $u_1, u_2 \in C^2((0,\infty)\times\mathbb{R})$ two classical solutions of the initial value problem for the wave equation with source $F(t,x)$ and initial data $g(x)$ and $h(x)$, then $w = u_1 - u_2$ by linearity satisfies $\boxed{\text{seen} \Downarrow}$

$$\begin{cases} \partial_{tt}w - c^2\partial_{xx}w = 0 & \text{in } t>0,\ x\in\mathbb{R}, \\ w(0,x) = 0 & \text{for } x\in\mathbb{R}, \\ \partial_t w(0,x) = 0 & \text{for } x\in\mathbb{R}, \end{cases}$$

Using the D'Alembert representation formula for the solution of the initial value problem, we obtain that $w = 0$, and hence $u_1 = u_2$ as desired. $\boxed{6, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(b) By linearity and uniqueness of classical solution, we know that $u = u_1 + u_2$, where u_1 and u_2 satisfy the initial value problems:

$$\begin{cases} \partial_{tt}u_1 - c^2\partial_{xx}u_1 = 0 & \text{in } t>0,\ x\in\mathbb{R}, \\ u_1(0,x) = g(x) & \text{for } x\in\mathbb{R}, \\ \partial_t u_1(0,x) = h(x) & \text{for } x\in\mathbb{R}, \end{cases}$$

and

$$\begin{cases} \partial_{tt}u_2 - c^2\partial_{xx}u_2 = F(t,x) & \text{in } t>0,\ x\in\mathbb{R}, \\ u_2(0,x) = 0 & \text{for } x\in\mathbb{R}, \\ \partial_t u_2(0,x) = 0 & \text{for } x\in\mathbb{R}, \end{cases} \tag{1}$$

respectively. Using D'Alembert's formula for the initial value problem of the wave equation, we obtain that

$$u_1(t,x) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\mathrm{d}y.$$

We are reduced to show that the function

$$w(t,x) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} F(s,y)\mathrm{d}y\mathrm{d}s$$

is a classical solution of the second initial value problem, and by uniqueness $u_2 = w$. Since F is continuous, we can take t and x derivatives in w to obtain

$$\partial_t w = \frac{1}{2}\int_0^t (F(s,x+ct-cs) + F(s,x-ct+cs))\mathrm{d}s$$
$$\partial_x w = \frac{1}{2c}\int_0^t (F(s,x+ct-cs) - F(s,x-ct+cs))\mathrm{d}s$$

Here, we used the following derivative formula

$$\frac{d}{dt}\int_0^{\alpha(t)} G(t,x)\mathrm{d}x = G(t,\alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t G(t,x)\mathrm{d}x$$

for given differentiable functions α and G. Taking further derivatives and using that F and F_x are continuous, we get $\boxed{8, \text{B}}$

$$\partial_{tt}w = F(t,x) + \frac{c}{2}\int_0^t (F_x(s,x+ct-cs) - F_x(s,x-ct+cs))\,ds$$
$$\partial_{xx}w = \frac{1}{2c}\int_0^t (F_x(s,x+ct-cs) - F_x(s,x-ct+cs))\,ds$$
$$\partial_{xt}w = \frac{1}{2}\int_0^t (F_x(s,x+ct-cs) + F_x(s,x-ct+cs))\,ds$$

that are all continuous functions. Therefore, $w \in C^2((0,\infty)\times\mathbb{R})$, w satisfies the wave equation with source since $\partial_{tt}w - c^2\partial_{xx}w = F(t,x)$ and $w(0,x) = \partial_t w(0,x) = 0$. So it is a classical solution of the initial value problem (1). unseen ⇓

(c) Taking the representation formula for each of the solutions and taking the difference, then $w = u_1 - u_2$ is given by

$$w(t,x) = \frac{\bar{g}(x+ct)+\bar{g}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\bar{h}(y)\mathrm{d}y + \frac{1}{2c}\int_0^t\int_{x-ct+cs}^{x+ct-cs}\bar{F}(s,y)\mathrm{d}y\mathrm{d}s,$$

with $\bar{g} = g_1 - g_2$, $\bar{h} = h_1 - h_2$, and $\bar{F} = F_1 - F_2$. By assumption we know that

$$|\bar{g}(x)| \le \delta\,, \qquad |\bar{h}(x)| \le \delta\,, \qquad \text{and} \qquad |\bar{F}(t,x)| \le \delta$$

for all $x \in \mathbb{R}$ and $0 \le t \le T$. Estimating directly in the expression above, we obtain

$$|w(t,x)| \le \delta + t\delta + \frac{1}{2}t^2\delta \le \left(1 + T + \frac{1}{2}T^2\right)\delta$$

for all $x \in \mathbb{R}$ and $0 \le t \le T$. Given $\epsilon > 0$, one chooses $\delta(T)$ such that $(1 + T + \frac{1}{2}T^2)\delta = \epsilon$ and we conclude the desired estimate. 6, D

4. Recall that a function $u \in C^2(\mathbb{R}^2)$ is called harmonic if $\Delta u = 0$ in $\mathbb{R}^2$, while $w \in C^2(\mathbb{R}^2)$ is called subharmonic if $\Delta w \geq 0$ in $\mathbb{R}^2$. $\boxed{\text{seen} \Downarrow}$

(a) For $r < R$, define

$$g(r) = \frac{1}{2\pi r}\int_{\partial B_r(\boldsymbol{x})} w(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = \frac{1}{2\pi}\int_{\partial B_1(0)} w(\boldsymbol{x} + r\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}$$

Then, using the divergence theorem,

$$g'(r) = \frac{1}{2\pi}\int_{\partial B_1(0)} \nabla w(\boldsymbol{x} + r\boldsymbol{\sigma})\cdot\boldsymbol{\sigma}\mathrm{d}\boldsymbol{\sigma} = \frac{r}{2\pi}\int_{B_1(0)} \Delta w(\boldsymbol{x} + r\boldsymbol{y})\mathrm{d}\boldsymbol{y} \geq 0.$$

Thus g is increasing and $g(r) \geq g(0) = w(\boldsymbol{x})$, so we obtain the first inequality $\boxed{8, \text{A}}$

$$w(\boldsymbol{x}) \leq \frac{1}{2\pi r}\int_{\partial B_r(\boldsymbol{x})} w(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}.$$

For the second inequality, take the above inequality, multiply by r and integrate between 0 and R. Then

$$\frac{R^2}{2}w(\boldsymbol{x}) \leq \frac{1}{2\pi}\int_0^R \mathrm{d}r\int_{\partial B_r(\boldsymbol{x})} w(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = \frac{1}{2\pi}\int_{B_R(\boldsymbol{x})} w(\boldsymbol{y})\mathrm{d}\boldsymbol{y},$$

which proves the second inequality. $\boxed{\text{meth seen} \Downarrow}$

(b) By direct computation $\boxed{6, \text{B}}$

$$\Delta w = F''(u)|\nabla u|^2 + F'(u)\Delta u = F''(u)|\nabla u|^2 \geq 0,$$

since F is convex and u is harmonic. $\boxed{\text{unseen} \Downarrow}$

(c) Observe that $w(\boldsymbol{x}) = |\nabla u(\boldsymbol{x})|^2$ is subharmonic, as it is the sum of square of harmonic functions (using part (b)). Then for any $\boldsymbol{x} \in \mathbb{R}^2$,

$$\begin{aligned}|\nabla u(\boldsymbol{x})|^2 = w(\boldsymbol{x}) &\leq \frac{1}{|B_R(\boldsymbol{x})|}\int_{B_R(\boldsymbol{x})} w(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{|B_R(\boldsymbol{x})|}\int_{B_R(\boldsymbol{x})} |\nabla u(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y} \leq \frac{M}{|B_R(\boldsymbol{x})|} \to 0 \quad \text{as } R \to \infty,\end{aligned}$$

and therefore $\nabla u(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in \mathbb{R}^2$. Hence, u is constant. $\boxed{6, \text{D}}$

5. (a) The characteristics solve the system

unseen ⇓

$$\partial_t \boldsymbol{X}(t,\boldsymbol{b}) = \boldsymbol{u}(\boldsymbol{X}(t,\boldsymbol{b})), \qquad \boldsymbol{X}(0,\boldsymbol{b}) = \boldsymbol{b} \in \Omega.$$

8, M

Since $\boldsymbol{u}$ has bounded derivatives, the solution of the above ODE is global. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,\boldsymbol{X}(t,\boldsymbol{b})) = -(\rho(t,\boldsymbol{X}(t,\boldsymbol{b}))^a.$$

For $a=1$, we find

$$\rho(t,\boldsymbol{X}(t,\boldsymbol{b})) = g(\boldsymbol{b})\mathrm{e}^{-t}.$$

For $a>1$, we get

$$\rho(t,\boldsymbol{X}(t,\boldsymbol{b})) = \frac{g(\boldsymbol{b})}{(1+(a-1)(g(\boldsymbol{b}))^{a-1}t)^{\frac{1}{a-1}}}.$$

In all cases, ρ is global, $\rho \geq 0$ and $\sup_{\boldsymbol{x}}|\rho(t,\boldsymbol{x})| \leq \sup_{\boldsymbol{x}}|g(\boldsymbol{x})|$ and $\rho \to 0$ as $t \to \infty$ uniformly in $\boldsymbol{x}$, since for $a=1$

$$\sup_{\boldsymbol{x}\in\Omega}|\rho(t,\boldsymbol{x})| \leq \sup_{\boldsymbol{x}\in\Omega}|g(\boldsymbol{x})|\mathrm{e}^{-t}$$

and for $a>1$ we have

$$\sup_{\boldsymbol{x}\in\Omega}|\rho(t,\boldsymbol{x})| \leq \frac{1}{(a-1)^{\frac{1}{a-1}}}\frac{1}{t^{\frac{1}{a-1}}}.$$

The rate of convergence to zero decreases as a increases.

meth seen ⇓

(b) Consider the difference of two solutions ρ_1 and ρ_2 with the same initial datum. Since $\boldsymbol{u}$ is divergence free, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} + 2\int_\Omega \boldsymbol{u}(\boldsymbol{x})\cdot\nabla(\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x}))(\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$
$$= -2\int_\Omega (\rho_1(t,\boldsymbol{x})^a - \rho_2(t,\boldsymbol{x})^a)(\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$

Since $\boldsymbol{u}$ is divergence-free and $\boldsymbol{u}\cdot\boldsymbol{n}=0$, we have that

$$2\int_\Omega \boldsymbol{u}\cdot\nabla(\rho_1-\rho_2)(\rho_1-\rho_2) = \int_\Omega \boldsymbol{u}\cdot\nabla|\rho_1-\rho_2|^2$$
$$= \int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{n}|\rho_1-\rho_2|^2 - \int_\Omega \mathrm{div}(\boldsymbol{u})|\rho_1-\rho_2|^2 = 0.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = -2\int_\Omega (\rho_1(t,\boldsymbol{x})^a-\rho_2(t,\boldsymbol{x})^a)(\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$

Notice that by the mean value theorem, if y_1, y_2 are positive real numbers, for some z between y_1 and y_2 we have that $(y_1^a - y_2^a)(y_1-y_2) = az^{a-1}(y_1-y_2)^2 \geq 0$, since z is positive and $a \geq 1$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} \leq 0,$$

implying

$$\int_\Omega |\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} \leq \int_\Omega |\rho_1(0,\boldsymbol{x})-\rho_2(0,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = 0.$$

Therefore, $\rho_1 \equiv \rho_2$, and the solution to the problem is unique.

8, M

(c) Let us take $a = 2$ and $g(\boldsymbol{x}) = -1$. From part (a), unseen $\Downarrow$

$$\rho(t, \boldsymbol{X}(t, \boldsymbol{b})) = -\frac{1}{(1-t)} \qquad \Rightarrow \qquad \rho(t, \boldsymbol{x}) = -\frac{1}{(1-t)}.$$

Hence the solution becomes unbounded as $t \to 1^-$. Notice that any even number a would work. 4, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96018 MATH97027 MATH97104	1	Exercise done relatively well, there is only one shock in part b
MATH96018 MATH97027 MATH97104	2	Exercise done relatively well, the last part was to be done by using the maximum principle, using Fourier series for the difference of solutions does not work
MATH96018 MATH97027 MATH97104	3	Many of the students made a sign mistake in the computation of the x-derivative of the second part of the solution
MATH96018 MATH97027 MATH97104	4	Some of the students did not realize that the partial derivatives of a harmonic function is harmonic, hence the |grad u|^2 is the some of subharmonic functions, which is subharmonic.
MATH96018 MATH97027 MATH97104	5	This was a nonlinear problem, so when taking the difference of the two problems one has to be careful when treating the nonlinear part (x^a-y^a is not (x-y)^a)