# Quantum Mechanics II, Coursework 2

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1. Recall that the Pauli $Y$ operator corresponds to the Pauli matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

   This matrix has eigenvalues $1, -1$ which respectively correspond to the (normalised) eigenvectors

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

   The eigenstates of the Pauli $Y$ operator are therefore

$$|y_+\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}, \quad |y_-\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}.$$

   We can then express the state $|\psi\rangle = |0\rangle$ in terms of $|y_+\rangle, |y_-\rangle$ as

$$|0\rangle = \frac{1}{\sqrt{2}} \left(|y_+\rangle + |y_-\rangle\right),$$

   so the probability of measuring $+1$ is the square of the modulus of the coefficient of $|y_+\rangle$, i.e.,

$$\mathrm{P}(+1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{2}}.$$

2. Recall again that the Pauli matrices for $X$ and $Z$ are $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   Their $+1$ and $-1$ eigenstates can be easily calculated respectively as

$$|x_+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |z_+\rangle = |0\rangle, \quad |x_-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad |z_-\rangle = |1\rangle.$$

   We are given the operator $\hat{O} = \cos(\theta)\hat{Z} + \sin(\theta)\hat{X}$. By the properties of $\hat{Z}$ and $\hat{X}$, we have

$$\hat{Z}\,|x_+\rangle = |x_-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad \hat{Z}\,|z_+\rangle = |z_+\rangle = |0\rangle, \quad \hat{X}\,|x_+\rangle = |x_+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad \hat{X}\,|z_+\rangle = |z_-\rangle = |1\rangle,$$

   so that the expectations are

$$\langle x_+|\,\hat{Z}\,|x_+\rangle = 0, \quad \langle z_+|\,\hat{Z}\,|z_+\rangle = 1, \quad \langle x_+|\,\hat{X}\,|x_+\rangle = 1, \quad \langle z_+|\,\hat{X}\,|z_+\rangle = 0.$$

   Since $\hat{O}$ is a linear combination of $\hat{Z}$ and $\hat{X}$, we have

$$\langle x_+|\,\hat{O}\,|x_+\rangle = \sin\theta, \quad \langle z_+|\,\hat{O}\,|z_+\rangle = \cos\theta.$$

   To compute the probability of measuring $+1$, recall that

$$\langle\hat{O}\rangle = (+1)\mathrm{P}(+1) + (-1)\mathrm{P}(-1) = \mathrm{P}(+1) - \mathrm{P}(-1), \quad \mathrm{P}(+1) + \mathrm{P}(-1) = 1$$

   from the definition of expectation and probability, so we have $\mathrm{P}(+1) = \dfrac{1 + \langle\hat{O}\rangle}{2}$.

In this case, we have

$$\mathrm{P}_x(+1) = \frac{1 + \langle x_+|\,\hat{O}\,|x_+\rangle}{2} = \frac{1+\sin\theta}{2}, \quad \mathrm{P}_z(+1) = \frac{1 + \langle z_+|\,\hat{O}\,|z_+\rangle}{2} = \frac{1+\cos\theta}{2}.$$

Since we are given a bag of qubits with half in the $+1$ eigenstate of Pauli $X$ and half in the $+1$ eigenstate of Pauli $Y$, the total probability of measuring $+1$ is

$$\mathrm{P}_{\mathrm{tot}}(+1) = \frac{1}{2}\mathrm{P}_x(+1) + \frac{1}{2}\mathrm{P}_z(+1) = \boxed{\frac{2+\sin\theta+\cos\theta}{4}}.$$

3. To achieve maximum distinguishability between the two states of the qubit bag, we need to ensure that the probabilities $\mathrm{P}_x(+1)$ and $\mathrm{P}_z(+1)$ have the maximum difference, i.e., we maximise

$$\Delta\mathrm{P} = |\mathrm{P}_x(+1) - \mathrm{P}_z(+1)| = \left|\frac{1+\sin\theta}{2} - \frac{1+\cos\theta}{2}\right| = \left|\frac{\sin\theta - \cos\theta}{2}\right|.$$

Note that $\sin\theta - \cos\theta = \sqrt{2}\sin(\theta - \pi/4)$, which achieves its maximum $\sqrt{2}$ at $\theta = 2n\pi - \pi/4$ and its minimum $-\sqrt{2}$ at $\theta = 2n\pi + 3\pi/4$ for all $n \in \mathbb{Z}$.

After taking the modulus, we can then see that $\Delta\mathrm{P}$ is maximised when

$$\theta = n\pi - \frac{\pi}{4} = \boxed{\cdots, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \cdots}.$$

Without loss of generality, we may take $\theta = -\pi/4$. Then we have $\sin\theta = -1/\sqrt{2}$ and $\cos\theta = 1/\sqrt{2}$.

We calculate

$$\mathrm{P}_x(+1) = \frac{1+\sin\theta}{2} = \frac{1-1/\sqrt{2}}{2} = \frac{2-\sqrt{2}}{4}, \quad \mathrm{P}_z(+1) = \frac{1+\cos\theta}{2} = \frac{1+1/\sqrt{2}}{2} = \frac{2+\sqrt{2}}{4}.$$

Our strategy is to guess $|z_+\rangle$ when the outcome is $+1$ and to guess $|x_+\rangle$ when the outcome is $-1$, so the probability of success is

$$\mathrm{P}_{\mathrm{success}} = \frac{1}{2}\left(\mathrm{P}_z(+1) + (1 - \mathrm{P}_x(+1))\right) = \frac{1}{2}\left(\frac{2+\sqrt{2}}{4} + \left(1 - \frac{2-\sqrt{2}}{4}\right)\right) = \frac{2+\sqrt{2}}{4} \approx 85.4\%.$$

4. We consider again the operator $\hat{O} = \cos(\theta)\hat{Z} + \sin(\theta)\hat{X}$, represented in matrix form as

$$\sin\theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix},$$

which has eigenvalues $1, -1$. To find the corresponding eigenstates, we consider

$$\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\cos\theta + b\sin\theta\\ a\sin\theta - b\cos\theta\end{pmatrix} = \pm\begin{pmatrix}a\\ b\end{pmatrix},$$

which has solutions as normalised eigenvectors

$$\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, \begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix},$$

so the eigenstates can be expressed as

$$|+\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle, \quad |-\rangle = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}|1\rangle.$$

This can be rewritten as

$$|0\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle, \quad |1\rangle = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle,$$

since the inverse matrix of $\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$ is itself.

We are given the two-qubit state $|\psi\rangle$ which can be rewritten as

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}((|0\rangle)|0\rangle + (|1\rangle)|1\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\left(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\right)|0\rangle + \left(\sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle\right)|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\right)|+\rangle + \left(\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}|1\rangle\right)|-\rangle\right),
\end{aligned}$$

so if $|+\rangle$ is measured in the first qubit, the second qubit collapses to $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$, and the probability that the second qubit is measured to be in the $|1\rangle$ state is just $\boxed{\sin^2\frac{\theta}{2}}$.

5. We are given the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\gamma\cos(\Omega t)\hat{x} = \hat{\mathcal{H}}_0 + \lambda\hat{V},$$

where the unperturbed term $\hat{\mathcal{H}}_0$ and the perturbation term $\hat{V}$ are given by

$$\hat{\mathcal{H}}_0 := \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\gamma\cos(\Omega t)\hat{x}, \quad \hat{V} = \gamma\cos(\Omega t)\hat{x}.$$

We are given that at $t = 0$, the system is in the ground state, i.e., we start from $|0\rangle$.

Recall that the position operator is given by

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right),$$

where we have $\hat{a}|0\rangle = |0\rangle, \hat{a}^\dagger|0\rangle = |1\rangle$, so that

$$\langle n|\hat{a}|0\rangle = \langle n|0\rangle = 0, \quad \langle n|\hat{a}^\dagger|0\rangle = \langle n|1\rangle = \delta_{n1},$$

where $\delta_{n1}$ is the Kronecker delta. Combining these, we have

$$\begin{aligned}
\langle n|\hat{x}|0\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|\hat{a}|0\rangle + \langle n|\hat{a}^\dagger|0\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}\delta_{n1}, \\
\langle n|\hat{V}|0\rangle &= \gamma\cos(\Omega t)\langle n|\hat{x}|0\rangle = \gamma\cos(\Omega t)\sqrt{\frac{\hbar}{2m\omega}}\delta_{n1}.
\end{aligned}$$

This expression is equivalent to saying that $\langle n|\hat{V}|0\rangle = 0$ for all $n \neq 1$, so we can focus on the $n = 1$ case and neglect any terms higher than first order.

From expression (5.5) given in lecture notes, the probability of transitioning from 0 to 1 is given by

$$\mathrm{P}_{0\to 1} = \frac{\lambda^2}{\hbar^2}\left|\int_0^t dt' \langle 1|\hat{V}|0\rangle e^{-i\omega_{01}t'}\right|^2,$$

where by definition

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad \omega_{01} = \frac{\varepsilon_0 - \varepsilon_1}{\hbar} = \frac{\hbar\omega\left(0 + \frac{1}{2}\right) - \hbar\omega\left(1 + \frac{1}{2}\right)}{\hbar} = -\omega.$$

We can now compute the integral

$$\begin{aligned}\int_0^t dt' \langle 1|\hat{V}|0\rangle e^{-i\omega_{01}t'} &= \int_0^t \gamma\cos(\Omega t')\sqrt{\frac{\hbar}{2m\omega}}e^{i\omega t'}dt' \\ &= \frac{\gamma}{2}\sqrt{\frac{\hbar}{2m\omega}}\int_0^t \left(e^{i(\omega+\Omega)t'} + e^{i(\omega-\Omega)t'}\right)dt' \qquad \left(\text{by } \cos(\Omega t') = \frac{1}{2}\left(e^{i\Omega t'} + e^{-i\Omega t'}\right)\right) \\ &= \frac{\gamma}{2i}\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{e^{i(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{i(\omega-\Omega)t}-1}{\omega-\Omega}\right),\end{aligned}$$

and so the probability

$$\begin{aligned}\mathrm{P}_{0\to 1} &= \frac{\lambda^2}{\hbar^2}\left|\frac{\gamma}{2i}\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{e^{i(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{i(\omega-\Omega)t}-1}{\omega-\Omega}\right)\right|^2 \\ &= \boxed{\frac{\lambda^2\gamma^2}{8m\omega\hbar}\left|\frac{e^{i(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{i(\omega-\Omega)t}-1}{\omega-\Omega}\right|^2},\end{aligned}$$

which is the exact expression. However, since the question asked for approximation, we can consider the case near resonance ($\omega \approx \Omega$) and neglect the first term inside the modulus function, so that

$$\begin{aligned}\mathrm{P}_{0\to 1} &\approx \frac{\lambda^2\gamma^2}{8m\omega\hbar}\left|\frac{e^{i(\omega-\Omega)t}-1}{\omega-\Omega}\right|^2 \\ &= \frac{\lambda^2\gamma^2}{8m\omega\hbar}\left|e^{i(\omega-\Omega)t/2}\frac{e^{i(\omega-\Omega)t/2}-e^{-i(\omega-\Omega)t/2}}{\omega-\Omega}\right|^2 \\ &= \frac{\lambda^2\gamma^2}{8m\omega\hbar(\omega-\Omega)^2}\left|e^{i(\omega-\Omega)t/2}-e^{-i(\omega-\Omega)t/2}\right|^2 \\ &= \frac{\lambda^2\gamma^2}{8m\omega\hbar(\omega-\Omega)^2}\left|2i\sin\frac{(\omega-\Omega)t}{2}\right|^2 \\ &= \frac{1}{2m\omega\hbar}\left(\frac{\lambda\gamma}{\omega-\Omega}\sin\frac{(\omega-\Omega)t}{2}\right)^2.\end{aligned}$$

As a sanity check, for the case when we are right at resonance, our formula tells us

$$\mathrm{P}_{0\to 1} = \frac{\lambda^2\gamma^2}{8m\omega\hbar}t^2$$

which matches the usual form for the transition probability.

In conclusion, the approximate probability that at later time $t$, the system is in the $n^{\text{th}}$ excited state from the ground state is

$$\boxed{\mathrm{P}_{0\to n} = \begin{cases}\dfrac{1}{2m\omega\hbar}\left(\dfrac{\lambda\gamma}{\omega-\Omega}\sin\dfrac{(\omega-\Omega)t}{2}\right)^2, & n=1, \\ 0, \quad n \neq 1.\end{cases}}$$

6. We are given the Hamiltonian

$$\hat{\mathcal{H}} = \varepsilon\hat{S}_z + \lambda\gamma\hat{S}_x^2 = \hat{\mathcal{H}}_0 + \lambda\hat{V},$$

where the unperturbed term $\hat{\mathcal{H}}_0 := \varepsilon\hat{S}_z$ has eigenenergies

$$E_n^{(0)} = \langle n|\,\varepsilon\hat{S}_z\,|n\rangle = \varepsilon\,\langle n|\,(\hbar n\,|n\rangle) = \varepsilon\hbar n,$$

and we consider the perturbation term $\hat{V} := \lambda\gamma\hat{S}_x^2$.

Recall that the spin operators $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, so we have

$$\hat{S}_x^2 = \left(\frac{\hat{S}_+ + \hat{S}_-}{2}\right)^2 = \frac{\hat{S}_+^2 + \hat{S}_-^2 + \hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+}{4},$$

and we have already known that

$$\hat{S}_+ \left|s,m\right\rangle = \hbar\sqrt{s(s+1)-m(m+1)}\left|s,m+1\right\rangle, \quad \hat{S}_- \left|s,m\right\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\left|s,m-1\right\rangle,$$

so we have

$$\begin{aligned}
\hat{S}_+\hat{S}_- \left|s,m\right\rangle &= \hbar^2\left(s(s+1)-m(m-1)\right)\left|s,m\right\rangle, \\
\hat{S}_-\hat{S}_+ \left|s,m\right\rangle &= \hbar^2\left(s(s+1)-m(m+1)\right)\left|s,m\right\rangle, \\
\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right)\left|s,m\right\rangle &= \hbar^2\left(s(s+1)-m(m-1)+s(s+1)-m(m+1)\right)\left|s,m\right\rangle \\
&= 2\hbar^2\left(s(s+1)-m^2\right)\left|s,m\right\rangle.
\end{aligned}$$

Since we are only considering first-order perturbation energy, we neglect the $\hat{S}_+^2$ and $\hat{S}_-^2$ terms, and

$$\left\langle s,m\right|\hat{V}\left|s,m\right\rangle = \lambda\gamma\left\langle s,m\right|\hat{S}_x^2\left|s,m\right\rangle \approx \frac{\lambda\gamma}{4}\left\langle s,m\right|\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right)\left|s,m\right\rangle,$$

so the eigenenergies for the perturbation term takes the form

$$E_n^{(1)} \approx \frac{\lambda\gamma\hbar^2}{2}\left(s(s+1)-n^2\right).$$

Therefore, the total approximate eigenenergies of this Hamiltonian are

$$E_n = E_n^{(0)} + E_n^{(1)} \approx \boxed{\varepsilon\hbar n + \frac{\lambda\gamma\hbar^2}{2}\left(s(s+1)-n^2\right)}.$$