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# Problem Sheet 1 Solutions:

1).

a). We have $f(z) = \frac{A(z)}{B(z)}$ where $A(z) = z^3 \sin z$, $B(z) = z^4 + a^4$ which has a simple zero at $z_0 = ae^{i\frac{\pi}{4}}$. Hence, using formula (1.3) from lectures:

$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)} = \frac{\sin(ae^{i\frac{\pi}{4}})}{4}.$$

b). Write:

$$f(z) = \frac{z+1}{(z^2-1)^2} = \frac{z+1}{(z+1)^2(z-1)^2} = \frac{1}{(z+1)(z-1)^2}.$$

Then using formula (1.2) from lectures with $A(z) = \frac{1}{z+1}$ gives:

$$\text{Res}(f, 1) = \frac{A'(1)}{1!} = -\frac{1}{4}.$$

c). Again, using formula (1.2) and taking $A(z) = \frac{e^z}{z}$, gives:

$$\text{Res}\left(\frac{e^z}{z(z-a)^2}, a\right) = \frac{A'(a)}{1!} = \frac{e^a}{a}\left(1 - \frac{1}{a}\right)$$

d). We have $f(z) = \frac{A(z)}{B(z)}$, where $A(z) = z^2 e^z$ and $B(z) = z^3 - a^3$ which has a simple zero at $z_0 = a$. Hence, using formula (1.3) gives:

$$\text{Res}(f, a) = \frac{A(z_0)}{B'(z_0)} = \frac{e^a}{3}.$$

2).

a). Let $z = e^{i\theta}$. Then the integral becomes one taken anticlockwise around the unit circle $|z| = 1$. Also $\cos\theta = \frac{1}{2}(z + \bar{z})$ which becomes $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ for points on the circle $|z| = 1$ ($\bar{z} = \frac{1}{z}$ here). Hence the integral becomes

$$I = \int_{|z|=1} \frac{-iz^{-1}\,dz}{5 - 4\left(\frac{1}{2}(z + \frac{1}{z})\right)} = \frac{i}{2}\int_{|z|=1} \frac{dz}{(z-2)(z-\frac{1}{2})}$$

Only the simple pole at $z = \frac{1}{2}$ lies inside the contour, hence by the residue theorem:

$$I = 2\pi i \,\text{Res}\left(\frac{i}{2}\frac{1}{(z-2)(z-\frac{1}{2})}, \frac{1}{2}\right) = \frac{2\pi}{3}.$$

b) Apply the same method as in part (a), one finds:

$$I = -\frac{i}{4}\oint_{|z|=1} \frac{z^4+1}{z^2(z+\frac{1}{2})(z+2)}\,dz.$$

Now the integrand is analytic everywhere inside $|z|=1$ except for a double pole at $z=0$ and a simple pole at $z=-\frac{1}{2}$. Using the residue theorem:

$$I = 2\pi i\left(\frac{-i}{4}\right)\left[\frac{(-\frac{1}{2})^4+1}{(-\frac{1}{2})^2(\frac{3}{2})} + \frac{(-1)(\frac{5}{2})}{1}\right] = \frac{\pi}{2}\left(\frac{1}{3}\right) = \frac{\pi}{6}$$

c) Let $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$, and $\gamma$ be the contour consisting of the section of the real axis between $-R$ and $R$, for some $R>0$, together with the semi-circle $\gamma_R$ in the upper half plane with centre $0$ and radius $R$. Consider $R\to\infty$. In this limit the integral around $\gamma_R$ is $0$. Also, $f(z)$ is analytic everywhere inside $\gamma$ except for simple poles at $z=i$ and $z=2i$. The residues at these are respectively $-\frac{i}{6}$ and $\frac{i}{12}$. The result follows using the residue theorem:

$$I = 2\pi i\left[-\frac{i}{6}+\frac{i}{12}\right] = \frac{\pi}{6}$$

d) Let $f(z) = \dfrac{z^2-z+2}{z^4+10z^2+9}$, and $\gamma$ be the same contour as in part (c). $f(z)$ is analytic everywhere inside $\gamma$ except for simple poles at $z=i$ and $z=3i$. The residues at these are respectively, $-\frac{(1+i)}{16}$ and $\frac{3-7i}{48}$. The result follows from the residue theorem:

$$I = 2\pi i\left[\frac{-(1+i)}{16}+\frac{3-7i}{48}\right] = \frac{5\pi}{12}$$

e) Let $f(z) = \dfrac{e^{2iz}}{z^2+z+1}$ and once again $\gamma$ the same contour as in parts (c) and (d). $f(z)$ is analytic everywhere inside $\gamma$ except for a simple pole at $z=\frac{1}{2}(-1+\sqrt{3}i)$ with residue $\dfrac{e^{-(\sqrt{3}+i)}}{\sqrt{3}i}$. Again the integral over $\gamma_R$ decays and using the residue theorem, we find:

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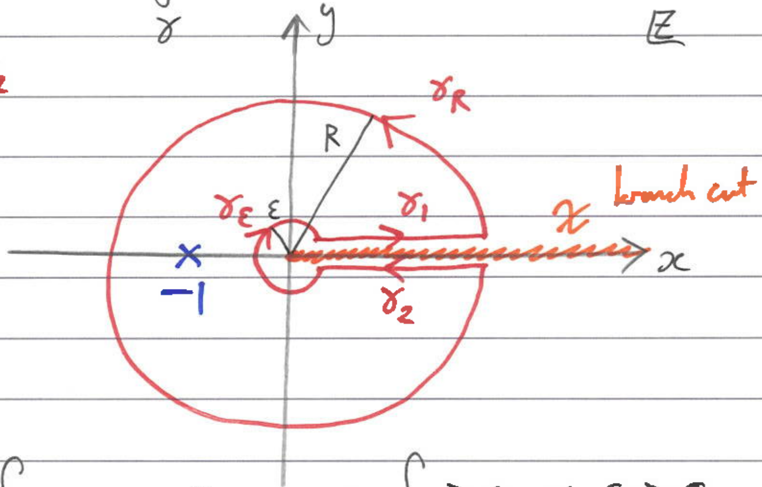
$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}}\left(\cos(1) - i\sin(1)\right).$$

Now noting that $e^{2ix} = \cos 2x + i\sin 2x$ and considering the imaginary part of the above gives the result.

f). Let $f(z) = \dfrac{e^{iz}}{z^2+4}$ and $\gamma$ again be the same contour from parts (c)–(e). $f(z)$ is analytic everywhere inside $\gamma$ except for a simple pole at $z = 2i$ with residue $\dfrac{-ie^{-2}}{4}$. Again the integral over $\gamma_R$ tends to zero and using the residue theorem and considering the real parts of the integral gives the result.

3). Let: $I = \displaystyle\int_0^{\infty} \frac{x^{a-1}}{(x+1)^2}\,dx$, $f(z) = \dfrac{z^{a-1}}{(z+1)^2}$. Take the branch cut of $f(z)$ along the positive real axis so then $0 \leq \theta \leq 2\pi$. Consider: $\displaystyle\oint_{\gamma} f(z)\,dz$, where:

$\gamma = \gamma_R + \gamma_\varepsilon + \gamma_1 + \gamma_2$




One can show $\displaystyle\int_{\gamma_R} \to 0$ as $R \to \infty$ and $\displaystyle\int_{\gamma_\varepsilon} \to 0$ as $\varepsilon \to 0$.

On $\gamma_1$: $z = xe^{i0} \Rightarrow \displaystyle\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\gamma_1} f(z)\,dz = \int_0^{\infty} \frac{x^{a-1}}{(x+1)^2}\,dx = I$

On $\gamma_2$: $z = xe^{2\pi i}$

$$\Rightarrow \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\gamma_2} f(z)\,dz = \int_{\infty}^{0} \frac{e^{2\pi a i}\, x^{a-1}}{(x+1)^2}\,dx = -e^{2\pi a i} I$$

By the residue Theorem:

$$\oint_{\gamma} f(z)\,dz = 2\pi i \operatorname{Res}(f, -1) = 2\pi i\left((a-1)(-1)^{a-2}\right)$$
$$= 2\pi i (a-1) e^{\pi a i}$$

So we have: $(1-e^{2\pi a i})I = 2\pi i(a-1)e^{\pi a i}$

$$\Rightarrow \dots \Rightarrow I = \frac{\pi(1-a)}{\sin(\pi a)}.$$

4). Let $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$.

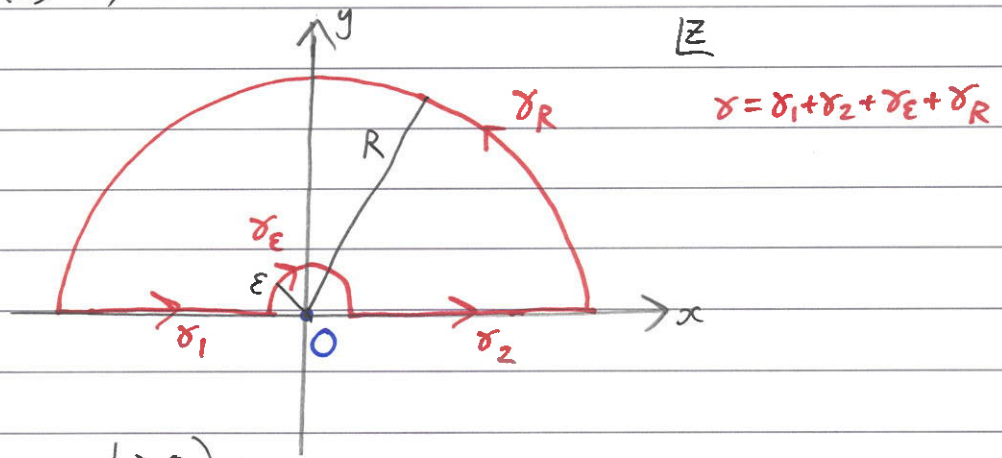
One can check for $z=x$, the real part of $f(x) = \dfrac{\cos(ax)-\cos(bx)}{x^2}$.

Note that for $z$ local to $z=0$ we have:

$$f(z) = \frac{1}{z^2}\left[\left(1+iaz-\frac{(az)^2}{2!}+O(z^3)\right)-\left(1+ibz-\frac{(bz)^2}{2!}+O(z^3)\right)\right]$$

$$= \frac{i(a-b)}{z} + O(1),$$ so $f(z)$ has a simple pole at $z=0$.

Consider: $\oint_\gamma f(z)\,dz$, where:




(close in UHP since $a,b>0$).

By Cauchy's Theorem: $\oint_\gamma f(z)\,dz = 0$.

One can check that $\lim_{R\to\infty}\int_{\gamma_R} \to 0$. Next consider the integral around $\gamma_\varepsilon$. Substituting $z=\varepsilon e^{i\theta}$, $0\leq\theta\leq\pi$ and taking $\varepsilon\to 0$ and using the above expansion for $f(z)$, we have:

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,dz = \lim_{\varepsilon\to 0}\int_\pi^0\left(\frac{i(a-b)}{\varepsilon e^{i\theta}}+O(1)\right)i\varepsilon e^{i\theta}\,d\theta = \lim_{\varepsilon\to 0}\int_0^\pi\left((a-b)+O(\varepsilon)\right)d\theta = \pi(a-b)$$

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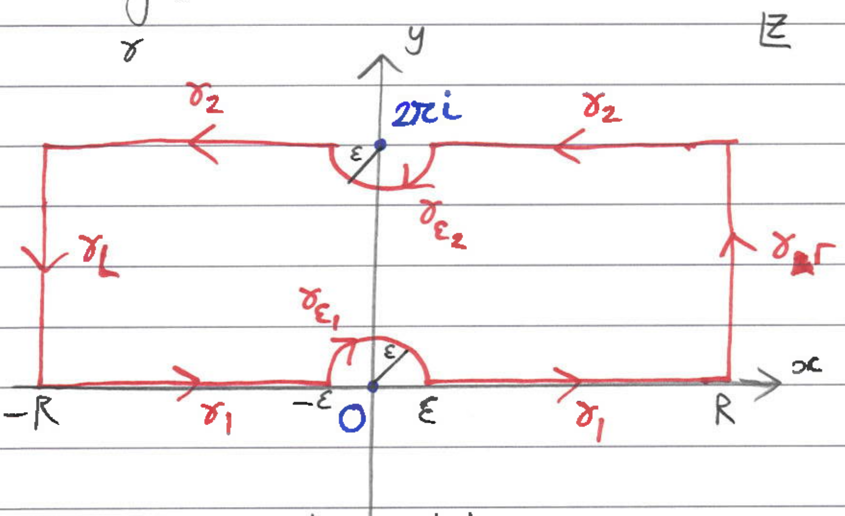
Taking the limit as $R \to \infty$, $\varepsilon \to 0$ we have:

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi(b-a)$$

Taking the real part of the above gives the result.

5). Denote $I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1-e^{x}}\,dx$. Introduce $f(z) = \dfrac{e^{az}}{1-e^{z}}$.

Note that $f(z)$ has singularities at the points where $e^{z} = 1$ i.e. at $z = 2n\pi i$, $n \in \mathbb{Z}$. Consider: $\displaystyle\oint_{\gamma} f(z)\,dz$, where: $\gamma = \gamma_1 + \gamma_2 + \gamma_L + \gamma_r + \gamma_{\varepsilon_1} + \gamma_{\varepsilon_2}$




It follows from Cauchy's Theorem that

$$\oint_{\gamma} f(z)\,dz = 0.$$

Let us now consider the integral of $f(z)$ around each section of $\gamma$ separately. First consider the sides of the rectangle. On $\gamma_r$: $z = R + iy$, where $0 \leq y \leq 2\pi$.

Thus:

$$\left|\int_{\gamma_r} f(z)\,dz\right| \leq 2\pi \frac{e^{aR}}{e^{R}-1} = 2\pi \frac{e^{(a-1)R}}{1-e^{-R}} \to 0 \text{ as } R \to \infty$$

(since $a-1<0$).

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_r} f(z)\,dz = 0.$$

Similarly: $\displaystyle\left|\int_{\gamma_L} f(z)\,dz\right| \leq 2\pi \frac{e^{-aR}}{1-e^{-R}} \to 0$ as $R \to \infty$ (since $a>0$)

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_L} f(z)\,dz = 0.$$

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Next consider contributions due to the integrals along the semi-circles.
Local to $z=0$, we have:

$$f(z) = \frac{1+O(z)}{1-(1+z+O(z^2))} = \frac{1+O(z)}{-z(1+O(z))} = -\frac{1}{z}(1+O(z))$$

Then, on $\gamma_{\varepsilon_1}$: $z=\varepsilon e^{i\theta}$, $0\leqslant\theta\leqslant\pi$:

$$\lim_{\varepsilon\to 0}\int_{\gamma_{\varepsilon_1}} f(z)\,dz = \lim_{\varepsilon\to 0}\int_{\pi}^{0} \frac{-1}{\varepsilon e^{i\theta}}(1+O(\varepsilon))\, i\varepsilon e^{i\theta}\,d\theta = i\pi.$$

Using similar arguments, one can show that in the limit as $\varepsilon\to 0$;

$$\lim_{\varepsilon\to 0}\int_{\gamma_{\varepsilon_2}} f(z)\,dz = i\pi e^{2\pi i a}.$$

Finally, on the top side of the rectangle $\gamma_2$: $z = x+2\pi i$

$$f(z) = \frac{e^{a(x+2\pi i)}}{1-e^{x+2\pi i}} = e^{2\pi a i} f(x).$$

Thus, putting everything together, in the limit as $R\to\infty$, $\varepsilon\to 0$:

$$(1-e^{2\pi a i})I + i\pi(1+e^{2\pi a i}) = 0$$

Thus, on re-arrangement, we get:

$$I = -i\pi\left(\frac{1+e^{2\pi i a}}{1-e^{2\pi i a}}\right) = \pi\cot(\pi a).$$

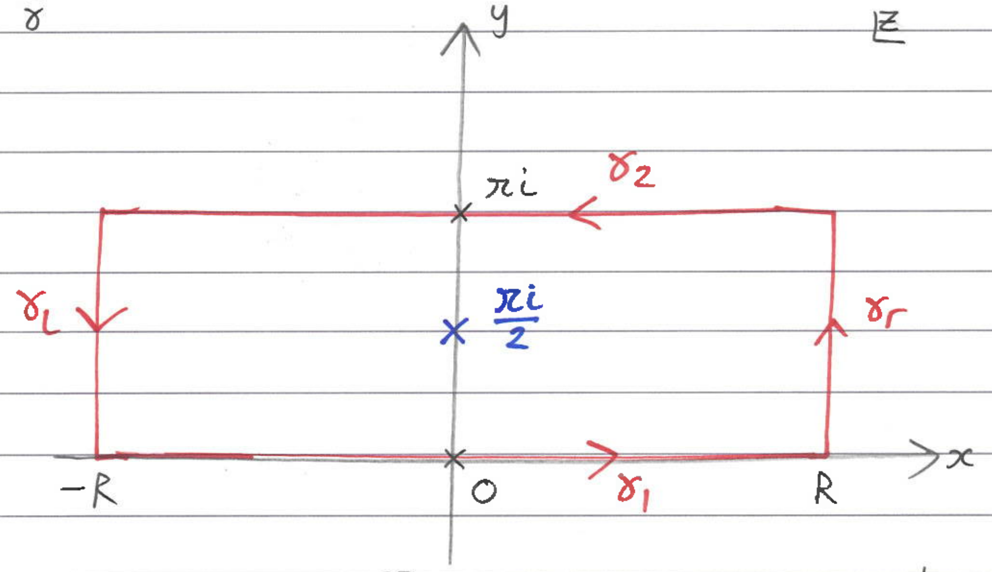
6).

a). Denote $I = \int_{-\infty}^{\infty} e^{ikx}\operatorname{sech} x\,dx$. We may write $e^{ikx}\operatorname{sech} x = \dfrac{2e^{(1+ik)x}}{e^{2x}+1}$

Hence introduce:

$$f(z) = \frac{2e^{(1+ik)z}}{e^{2z}+1}.$$

Consider $\oint_\gamma f(z)\,dz$, where: $\gamma = \gamma_1 + \gamma_2 + \gamma_l + \gamma_r$




$f(z)$ has singularities where $e^{2z}+1=0$, i.e. at $z=(n+\frac{1}{2})\pi i$, $n\in\mathbb{Z}$.
Thus $f(z)$ has a single singularity inside $\gamma$ at $z=\frac{\pi}{2}i$.

Local to $\frac{\pi}{2}i$ we have $z=\frac{\pi}{2}i+z_\varepsilon$, where $|z_\varepsilon|=\varepsilon$ is small. Then:

$$f(z) = \frac{2e^{(1+ik)(\frac{\pi}{2}i+z_\varepsilon)}}{e^{\pi i+2z_\varepsilon}+1} = \frac{2e^{\frac{\pi}{2}i}e^{-\frac{\pi}{2}k}e^{(1+ik)z_\varepsilon}}{e^{\pi i}e^{2z_\varepsilon}+1}$$

$$= \frac{2ie^{-\frac{\pi}{2}k}e^{(1+ik)z_\varepsilon}}{1-e^{2z_\varepsilon}} = \frac{2ie^{-\frac{\pi}{2}k}(1+O(\varepsilon))}{1-(1+2z_\varepsilon+O(\varepsilon^2))}$$

$$= \frac{-ie^{-\frac{\pi}{2}k}(1+O(\varepsilon))}{z_\varepsilon(1+O(\varepsilon))} = \frac{-ie^{-\frac{\pi}{2}k}}{z-\frac{\pi}{2}i} + O(1)$$

i.e. $f(z)$ has a simple pole at $z=\frac{\pi}{2}i$ with residue $-ie^{-\frac{\pi}{2}k}$. It follows from the residue theorem that

$$\oint_\gamma f(z)\,dz = 2\pi e^{-\frac{\pi}{2}k}.$$

Let us now consider the integral of $f(z)$ around each section of $\gamma$ separately.

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As in problem 5, one can check that $\int_{\gamma_r}$ and $\int_{\gamma_L} \to 0$ as $R \to \infty$.

As $R \to \infty$, it is clear that the integral along $\gamma_1$ is equal to $I$. Furthermore, on $\gamma_2$ we have $z = x + \pi i$ and hence one can find $f(z) = -e^{-k\pi} f(x)$. Note however that as we integrate along $\gamma_2$ we do so from right to left. It follows that as $R \to \infty$ the integral along $\gamma_2$ is $e^{-k\pi} I$.

Combining everything, we deduce that

$$(1 + e^{-k\pi}) I = 2\pi e^{-\frac{\pi}{2}k}$$

and hence

$$I = \pi \operatorname{sech}\left(\frac{\pi}{2}k\right).$$

b). Close in the real axis with a semi-circle $\gamma_R$ in the upper-half plane, of radius $R$ and centred at the origin. Label the closed contour consisting of the real axis and $\gamma_R$ as $\gamma$. Denote $I$ and introduce $f(z)$ as in part (a).

It follows from observations made in part (a), that as $R \to \infty$, $f(z)$ has singularities in the interior of $\gamma$ at the points $z_n = (n + \frac{1}{2})\pi i$, $n \in \mathbb{Z}_{\geq 0}$. Using similar analysis to that used in part (a), one can show that the residue of $f(z)$ at $z_n$ is given by

$$\operatorname{Res}(f, z_n) = i(-1)^{n+1} e^{-(n+\frac{1}{2})\pi k}.$$

Hence it follows from the residue theorem that:

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{n=0}^{\infty} \left( i(-1)^{n+1} e^{-(n+\frac{1}{2})\pi k} \right)$$

$$= 2\pi e^{-\frac{\pi}{2}k} \sum_{n=0}^{\infty} (-1)^n e^{-n\pi k}$$

$$= \pi \operatorname{sech}\left(\frac{\pi}{2}k\right),$$

where we have used the fact that, since $k>0$, then $e^{-\pi k}<1$ and so

$$\sum_{n=0}^{\infty} (-1)^n e^{-n\pi k} = \frac{1}{1+e^{-\pi k}} .$$

Now consider the integral around $\gamma_R$ : we have :

$$|f(z)| \leqslant \frac{2e^{x-ky}}{|e^{2x}-1|} = \frac{2e^{-ky}}{|e^{x}-e^{-x}|} \longrightarrow 0 \quad \text{as } y \to \infty. \quad x \to \pm\infty .$$

We may deduce that as $R \to \infty$: $\int_{\gamma_R} f(z)\,dz = 0$,

giving $I = \pi \operatorname{sech}\left(\frac{\pi}{2} k\right)$ as before.