Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Applied Probability

Date: Friday, 20 May 2011. Time: 10.00am. Time allowed: 2 hours.
This paper has FOUR questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the main book is full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (i) Let $\{N_t\}_{t\geq 0}$ be a counting process. Give a formal definition, if it is to be a time-homogeneous Poisson Process of rate $\lambda \in \mathbb{R}_+$.

(ii) Suppose $\{N_t\}_{t\geq 0}$ is a time-homogeneous Poisson process of rate $\lambda \in \mathbb{R}_+$ and $X_1, X_2, \ldots$ are i.i.d. and independent of $\{N_t\}$ such that the probability generating function, $G_X(s)$ is well-defined for each s. Show that the probability generating function of the compound Poisson process:

$$Y_t := \sum_{i=1}^{N_t} X_i$$

is

$$G_{Y_t}(s) = G_{N_t}(G_X(s)) = \exp\left\{(G_X(s)-1)\lambda t\right\}$$

for every $t \geq 0$.

(iii) Suppose that customers arrive at a shop according to an non-homogeneous Poisson-Process $\{N_t\}_{t\in[0,1]}$ of daily rate $\lambda(t) = 1/(t+1)$. Each customer will buy something with probability $p \in (0,1)$, otherwise nothing is bought. The shop is run by a single individual, who often has to leave before the end of the day; the time the shop is open is uniformly distributed between 0 and 1 day. Let Y be the total number of individuals that will buy something whilst the shop is open on a given day. Show that the probability generating function of Y is

$$G_Y(s) = \frac{1}{p(s-1)+1}\left[2^{p(s-1)+1} - 1\right].$$

Hence, or otherwise, find the expected number of individuals that will buy something.

2. (i) Let $\{N_t\}_{t\geq 0}$ be a counting process. Give a formal definition, if it is to be a birth process.

(ii) Let $p_n(t)$ be the probability of being in state $n \in \{0, 1, \dots\}$ at a time $t \geq 0$. For the birth process, derive the forward equations associated to $p_n(t)$:

$$\begin{aligned} p_n'(t) &= -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) \quad n \geq 1 \\ p_0'(t) &= -\lambda_0 p_0(t) \end{aligned}$$

where $\lambda_0, \lambda_1, \dots$ are the positive intensities.

(iii) A marine biologist is interested in the movements of dolphins. She is to navigate a boat until she observes a single dolphin, at which time she stops and waits there to observe the dolphins which pass her boat. As more dolphins pass her boat, the rate of arrival of dolphins increases linearly as a multiple of $\lambda \in \mathbb{R}_+$; e.g. the arrival rate with two dolphins is 2λ.

(a) Modelling the number of dolphins observed as a birth process $\{N_t\}_{t\geq 0}$ starting from when the boat is stopped ($N_0 = 1$), using the forward equations, show that

$$\frac{\partial G(s,t)}{\partial t} = \lambda s(s-1)\frac{\partial G(s,t)}{\partial s}$$

where

$$G(s,t) = \sum_{n=1}^{\infty} s^n p_n(t).$$

Hence **verify** that

$$G(s,t) = \frac{se^{-\lambda t}}{1-(1-e^{-\lambda t})s}.$$

(b) Calculate $\mathbb{P}(N_t \geq m)$ for $m \geq 2$ [Hint: consider a power-series expansion of $G(s,t)$] and hence show that the probability density function of T_m the first time the process takes the value m is

$$f(t_m) = (m-1)\lambda e^{-\lambda t}(1-e^{-\lambda t})^{m-2} \quad t \in \mathbb{R}_+.$$

3. (i) Let $\{X_n\}_{n\geq 0}$ be discrete-time Markov chain on a continuous state-space $E = \mathbb{R}$. Give a formal definition of a Markov kernel.

(ii) Suppose that for any $n \geq 1$, with $X_0 = x_0$ known, the Markov chain is defined by

$$X_n = X_{n-1} + \sigma Z_n \tag{1}$$

where $\sigma > 0$ is a known constant and Z_n are i.i.d. standard normal random variables (i.e. zero mean, unit variance).

(a) Derive, for any $n \geq 1$, the Markov transition kernel for X_n, given $X_{n-1} = x_{n-1}$.

(b) Show that for any $n \geq 1$ that

$$X_n | X_0 = x_0 \sim \mathcal{N}(x_0, n\sigma^2)$$

where $\mathcal{N}(\mu, \sigma^2)$ is a normal distribution of mean μ and variance σ^2. [It should be noted that you are **NOT** allowed to use the identity $X_n = x_0 + \sigma \sum_{i=1}^n Z_i$, NO MARKS will be awarded.] [You may use, without proof, that for any $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_+$

$$\phi(x; \mu, \sigma_1^2 + \sigma_2^2) = \int_{-\infty}^{\infty} \phi(x; y, \sigma_1^2)\phi(y; \mu, \sigma_2^2)dy$$

where

$$\phi(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left\{ -\frac{1}{2\sigma^2}(x-\mu)^2 \right\}.$$

Recall that the iterated kernel is, for $n \geq 2$:

$$K^n(x_0, x_n) = \int_{E^{n-1}} K^{n-1}(x_0, x_{n-1})K(x_{n-1}, x_n)dx_{1:n-1}.$$

]

(iii) A well known model for the **log** price of an asset, X_n, when time is measured in days, follows the process (1) on day n. The log price on day 0 (today) is known. An investor is given two choices. He can either buy the **asset** today, or he can enter into the following contract: if on a **given** day n the price is greater than S (a known number) he may purchase it for S, otherwise he does not. It is assumed that the value of money n days in the future is the same as today.

(a) Find the expected value of the asset **price** at time n.

(b) Show that the expected value of the contract, today, is:

$$e^{x_0+n\sigma^2/2}\Phi\left(\frac{x_0+n\sigma^2-\log(S)}{\sigma\sqrt{n}}\right) - S\Phi\left(\frac{x_0-\log(S)}{\sigma\sqrt{n}}\right)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \phi(u; 0, 1)du$$

and $1 - \Phi(x) = \Phi(-x)$ for any $x \in \mathbb{R}$.

4. (i) Let $X_0, X_1, X_2, \dots$ be a time-homogeneous Markov chain on a state-space $E = \{1, \dots, k\}$, with transition matrix P. Assuming it exists, give a definition of the stationary distribution

(ii) Let $n \geq 1$. The $n-$step transition matrix $P_n = (p_{ij}(n))$, of the chain in (i), is the matrix of $n-$step transition probabilities

$$p_{ij}(n) = \mathbb{P}(X_{m+n} = j | X_m = i)$$

with $m \geq 0$. Derive the Chapman-Kolmogorov equations: Let $m \geq 0$, $n \geq 1$, then we have

$$p_{ij}(m+n) = \sum_{l=1}^{k} p_{il}(m) p_{lj}(n).$$

(iii) (a) In a study of rainfall in Tel Aviv it was found that a two-state Markov chain gave a good description of the occurrence of wet and dry days during the rainy period of December-February. Let state 0 denote a dry day and state 1 a wet day. The Markov transition matrix is

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

for $\alpha, \beta \in (0,1)$. Show by induction, or otherwise that

$$P^n = \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta} \begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}$$

for any $n \geq 1$.

(b) The transition matrix was estimated as

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{pmatrix}$$

so $p_{00} = 3/4$. Compute the stationary distribution of P. The 10-step transition matrix is approximately

$$P^{10} = \begin{pmatrix} 12/21 & 9/21 \\ 12/21 & 9/21 \end{pmatrix}.$$

What do you notice? What consequence does this have?