Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 3

1. In this problem, we consider the following kinematic wave equation

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0$$

where k is a constant. We have

$$u = f(x-(k+u)t)$$

where f is an arbitrary function of a single variable. Differentiating partially with respect to t gives

$$\frac{\partial u}{\partial t} = f'(x-(k+u)t)\left(-t\frac{\partial u}{\partial t} - (u+k)\right)$$

which implies upon rearrangement that

$$\frac{\partial u}{\partial t} = -\frac{(u+k)f'(x-(k+u)t)}{1+tf'(x-(k+u)t)}$$

Similary, we obtain that

$$\frac{\partial u}{\partial x} = -\frac{f'(x-(k+u)t)}{1+tf'(x-(k+u)t)}$$

It is therefore clear that

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0$$

2. Here, we consider the same governing equation as in Q1. We want to solve this equation for $t>0$ in $-\infty < x < \infty$, given the following initial condition

$$u(x,0) = \begin{cases} 1, & x < -1 \\ (1-x)/2, & |x| < 1 \\ 0, & x > 1 \end{cases}$$

(a) The characteristics satisfy the following equation

$$\frac{dx}{dt} = u + k$$

which are straight lines as we know that u is constant on the characteristics, whose slope depends on where they cross the x-axis, i.e. the slope is determined by the initial condition. We denote $x(0)=\xi$. Given the initial condition, we find that the equation of the characteristics is given by

$$x = \begin{cases} \xi + (k+1)t, & \xi < -1 \\ \xi + ((1-\xi)/2 + k)t, & |\xi| < 1 \\ \xi + kt & \xi > 1 \end{cases}$$

We give in Fig. 1 a plot of these characteristics for $k = 0.75$.

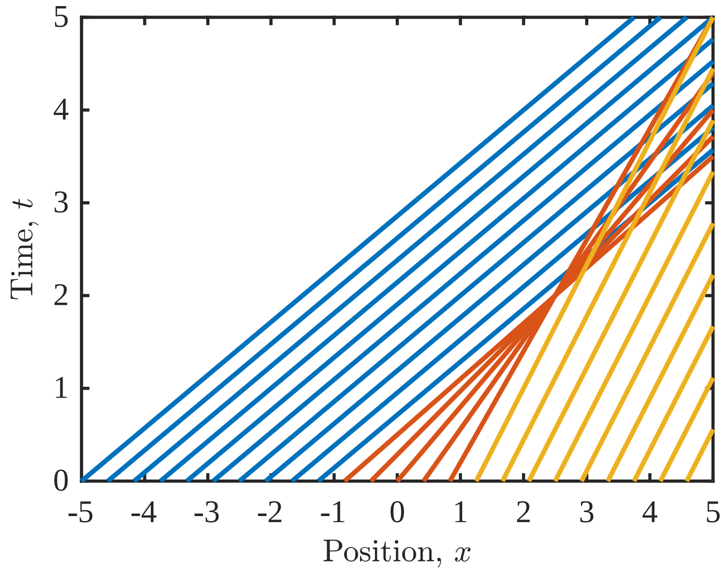

Figure 1: Diagram of characteristics for Q2 with $k = 0.75$.

(b) It is clear from our diagram of characteristics that all the characteristics emanating from the interval $[-1, 1]$ cross at $t = 2$. From the figure, we also see that this is the earliest time that any two characteristics cross. At the time of crossing of all these characteristics, we have

$$\frac{\partial x}{\partial \xi} = 0 \Rightarrow \frac{\partial}{\partial \xi}\left[\xi + ((1-\xi)/2 + k)t\right] = 0 \Rightarrow t = 2$$

Therefore, we can conclude that $t_s = 2$ is the shock formation time. Reinjecting this in the equation of the characteristics for $\xi = 0$, we find that

$$x_s = x(t_s) = (1/2 + k)t_s = 2\,(k + 1/2) = 1 + 2k$$

(c) We have found in Q1 that

$$u(x,t) = f(x - (u+k)t)$$

Thus, using the initial condition, we find that at $t = 0$ we have

$$u(x,0) = f(x) = \begin{cases} 1 & x < -1 \\ (1-x)/2 & |x| \leq 1 \\ 0 & x > 1 \end{cases}$$

Therefore, we must have for $t > 0$

$$u(x,t) = \begin{cases} 1 & x - (u+k)t < -1 \\ (1 - [x - (u+k)t])/2 & |x - (u+k)t| \leq 1 \\ 0 & x - (u+k)t > 1 \end{cases}$$

Let's rewrite this explicitly. For $|x - (u+k)t| \leq 1$, we can rearrange the expression to find

$$u = \frac{1 - x + kt}{2 - t}$$

By substituting this expression in the implicit solution yields, after some algebra, the following explicit formula

$$u(x,t) = \begin{cases} 1 & x < -1 + t + kt \\ \frac{1-x+kt}{2-t} & -1 + t + kt \leq x \leq 1 + kt \\ 0 & x > 1 + kt \end{cases}$$

In particular, it is clear that this solution profile u when plotted against x shows an infinite slope when $t = 2$ (i.e. the time when we expect shock formation).

3. In this problem, we consider the following initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x, 0) = -x, \quad x \in \mathbb{R}$$

The characteristic with label ξ is given by

$$u = -\xi \quad \text{on} \quad x = -\xi t + \xi$$

Those are straight lines. At $t = 1$, all characteristics irrespective of ξ cross at the point $(0, 1)$ in the (x, t)-plane. A diagram of characteristics is shown in Fig. 2. We can isolate ξ in the equation of the characteristics to find that for $t < 1$, the solution is given by

$$u(x, t) = -\frac{x}{1 - t}$$

We can see that the initial profile $u(x, 0) = -x$ steepens as t increases and at $t = 1$, it is vertical!

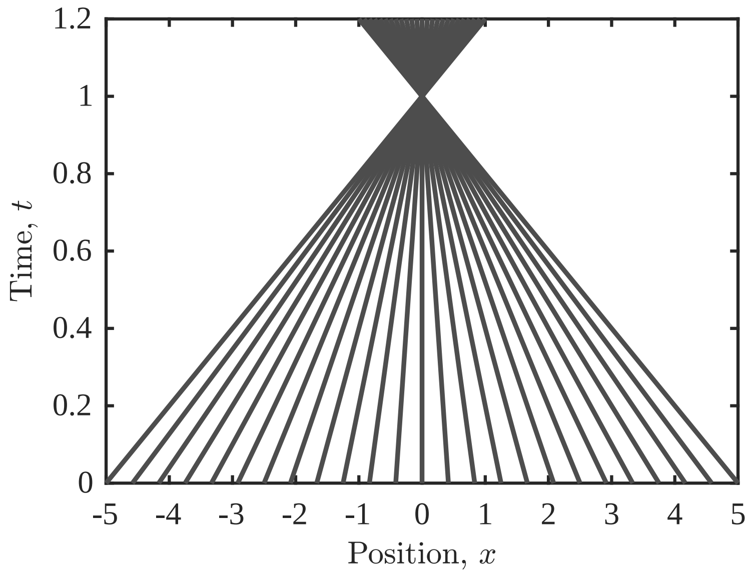

Figure 2: Diagram of characteristics for Q3.

4. Consider the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0$$

subject to the following initial conditions

$$u(x, 0) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

(a) The solution can be found using characteristics

$$\begin{cases} \text{I: } \xi < 0 & u = 1 \quad \text{on} \quad x = t + \xi \\ \text{II: } 0 < \xi < 1 & u = 1 - \xi \quad \text{on} \quad x = (1 - \xi)t + \xi \\ \text{III: } \xi > 1 & u = 0 \quad \text{on} \quad x = \xi \end{cases}$$

Using these characteristics, we can solve the problem explicitly in each region, indeed

- ☐ In region I, we have $\xi = x - t < 0$, hence $u(x,t) = 1$ for $x < t$;
- ☐ In region II, we can invert the characteristic equation to find $\xi = (x-t)/(1-t)$, hence $u(x,t) = 1 - (x-t)/(1-t) = (1-x)/(1-t)$ in the interval $0 < (x-t)/(1-t) < 1$, i.e. $t < x < 1$;
- ☐ In region II, we simply obtain that $u(x,t) = 0$ for $x > 1$.

(b) At $t = 1$, characteristics cross and a shock clearly forms. Consider that for $t > 1$, a shock form at position $x = s(t)$. Behind the shock the solution is given by $u_- = 1$and in front of the shock, it is given by $u_+ = 0$. The Rankine-Hugoniot condition gives us the speed of propagation of the shock

$$\frac{ds}{dt} = \frac{1}{2},$$

which is an ODE subject to the initial condition $s(1) = 1$. So the shock path is given by

$$s(t) = \frac{1}{2}t + \frac{1}{2}$$

Beyond $t = 1$, the solution is thus given by

$$u(x,t) = \begin{cases} 1 & x < (t+1)/2 \\ 0 & x > (t+1)/2 \end{cases}$$

(c) In Fig. 3, we give the required snapshots of the solution for $t = 0$, $t = 1/2$, $t = 1$ and $t = 3/2$.

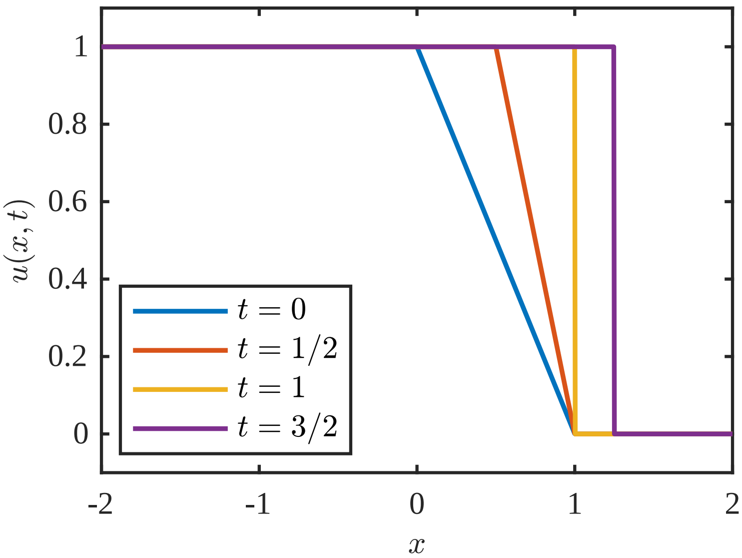

Figure 3: Solution snapshot for Q4.

5. The density of cars in a traffic flow problem is governed by the following conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

where $q(\rho)$ is the traffic flux whose functional form is given by

$$q(\rho) = \alpha \rho \ln(\rho_m/\rho)$$

where α and ρ_m are positive constants.

(a) The initial condition that we are given read

$$\rho(x,0) = \begin{cases} \rho_m, & x < -x_0 \\ -\rho_m x/x_0, & -x_0 < x < 0 \\ 0, & x > 0 \end{cases}$$

They are sketched on Fig. 4.

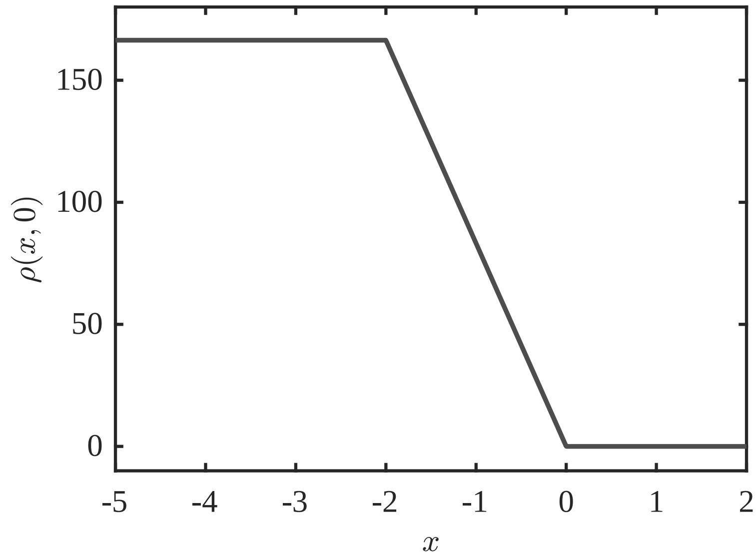

Figure 4: Initial conditions for Q5 with $\rho_m = 166.4$ cars/mile and $x_0 = 2$ miles.

(b) First, let us rewrite the equation as

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

where

$$c(\rho) = \alpha\left[\ln\left(\frac{\rho_m}{\rho}\right) - 1\right]$$

We will use the method of characteristics to solve this problem. The characteristics are given as follows:

- ☐ The characteristics with label $\xi < -x_0$ are solutions to $dx/dt = -a$, i.e. $x = -\alpha t + \xi$. Those are straight lines with slope $-1/\alpha$ in the (x,t)-plane;
- ☐ The characteristics with label $-x_0 < \xi < 0$ are given by

$$\frac{dx}{dt} = \alpha\left[\ln\left(\frac{-x_0}{\xi}\right) - 1\right]$$

as the initial condition in this interval is $\rho_0(\xi) = -\rho_m \xi / x_0$. The equation of the characteristics is then

$$x = \alpha\left[\ln\left(\frac{-x_0}{\xi}\right) - 1\right]t + \xi$$

- ☐ The characteristics with label $\xi > 0$ are lines with slope 0 in the (x,t)-plane.

Now realize that, when $\xi = -x_0/e$, we have $dx/dt = 0$. For $-x_0 < \xi < -x_0/e$, the slopes of the characteristics are negative while they are positive for $-x_0/e < \xi < 0$. So the characteristics fan out and there is no shock formation.

Finally, to find the position on the road where $\rho = \rho_m/2$ after 2 hours, we need to find the label ξ_0 where $\rho(\xi_0, 0) = \rho_m/2$. This is clearly given by $\xi_0 = -x_0/2$. We know that ρ is constant along the characteristics so the position on the road we are after will be given by

$$x^* = 2\alpha(\ln(2) - 1) - \frac{x_0}{2} < -\frac{x_0}{2}$$

6. We go back to the kinematic wave equation from Q1 and set $k = 0$ to obtain

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0$$

(a) It is clear that $\rho = \rho_0 = \text{constant}$ is a solution of the governing equation. Further, we write that

$$\frac{\partial}{\partial t}\left(\frac{x}{t}\right) = -\frac{x}{t^2} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}$$

So we finally obtain

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0$$

So $\rho = x/t$ is also a solution. This solution is represented in Fig. 5. It is a triangular shaped wave with a jump discontinuity at the point $x = s(t)$, which for now is arbitrary.

(b) The jump condition gives us that

$$\frac{ds}{dt} = \frac{[q]_-^+}{[\rho]_-^+}$$

where $q(\rho) = \rho^2/2$. For the solution $\tilde{\rho}$, we find that

$$\frac{ds}{dt} = \frac{\rho_0^2/2 - (s/t)^2/2}{\rho_0 - s/t} = \frac{1}{2}\left(\rho_0 + \frac{s}{t}\right)$$

(c) Finally, this ODE can be rewritten

$$\frac{ds}{dt} - \frac{1}{2}\frac{s}{t} = \frac{\rho_0}{2}$$

We can solve this linear first-order ODE with an integrating factor. The integrating factor is $1/t^{1/2}$ so that

$$\frac{d}{dt}\left(\frac{s}{t^{1/2}}\right) = \frac{\rho_0}{2t^{1/2}}$$

which gives after integration

$$\frac{s}{t^{1/2}} = \rho_0 t^{1/2} + C \Rightarrow s(t) = \rho_0 t + C t^{1/2}$$

How can we determine the value of C? We know that the total area below the curve is necessarily the same as for the initial condition; this will help us obtain C. Indeed, the area of the triangle (above the level ρ_0) must be equal to A; this gives the following relation

$$A = \frac{1}{2} C t^{1/2} C t^{-1/2} = \frac{C^2}{2} \Rightarrow C = \sqrt{2A}$$

Hence, we finally find that

$$s(t) = \rho_0 t + \sqrt{2At}$$

7. In this problem, we consider the viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

subject to the following boundary conditions

$$u \to u_1, \quad \text{as } x \to -\infty$$
$$u \to u_2, \quad \text{as } x \to +\infty$$

with $u_1 > u_2$.

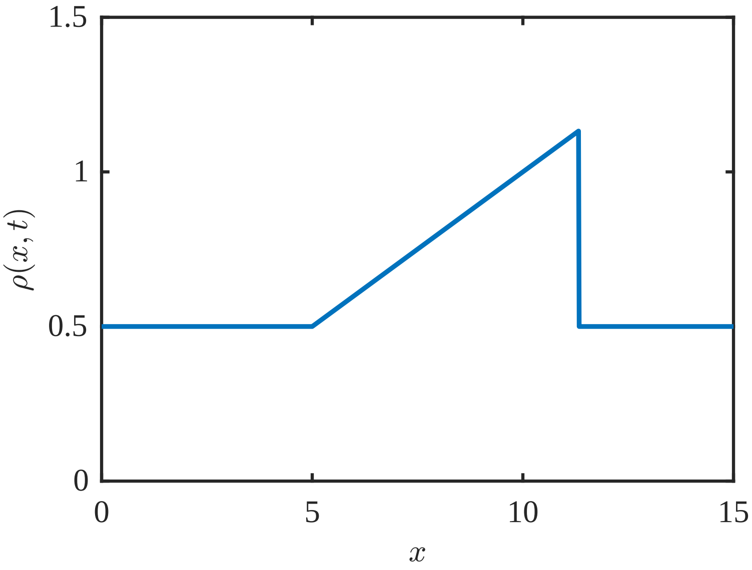

Figure 5: Solution profile for Q6 with $A = 2$, $t = 10$, $\rho_0 = 0.5$.

First assume that a travelling wave solution exists, i.e. a wave with a permanent form

$$u(x,t) \equiv u(x-ct)$$

Then transform the independent variables $(x,t) \to \eta = x - ct$. In this case, the partial derivatives become

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d}{d\eta} = -c\frac{d}{d\eta} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{d}{d\eta} = \frac{d}{d\eta}$$

As a consequence, the viscous Burgers equation can be rewritten

$$-cu' + uu' = \nu u''$$

where the prime denotes a derivative with respect to η. If we integrate once, we obtain

$$-cu + \frac{1}{2}u^2 = \nu u' + K$$

where K is an integration constant. We can evaluate this constant using the boundary conditions. Indeed, we know that in the limit where $\eta \to \pm\infty$ (i.e. in the limit of $x \to \pm\infty$), we have $u'(\eta) \to 0$. So we can write

$$-cu_1 + \frac{1}{2}u_1^2 = K = -cu_2 + \frac{1}{2}u_2^2$$

In particular, this allows us to write that the wave speed is given by

$$c = \frac{1}{2}\frac{u_1^2 - u_2^2}{u_1 - u_2} = \frac{\left[\frac{1}{2}u^2\right]_{+\infty}^{-\infty}}{[u]_{+\infty}^{-\infty}}$$

which simplifies to

$$c = \frac{1}{2}(u_1 + u_2)$$

Provided the expression for c, we can now obtain that

$$K = -\frac{1}{2}(u_1 + u_2)u_1 + \frac{1}{2}u_1^2 = -\frac{1}{2}u_1 u_2$$

We finally obtain the following ODE for u

$$\nu\frac{du}{d\eta} = \frac{1}{2}u^2 - \frac{1}{2}(u_1 + u_2)u + \frac{1}{2}u_1 u_2$$

This ODE is separable and we can write it

$$\frac{du}{u^2 - \frac{1}{2}(u_1 + u_2)u + \frac{1}{2}u_1 u_2} = \frac{d\eta}{\nu}$$

Now the denominator on the LHS can be factorized

$$u^2 - \frac{1}{2}(u_1 + u_2)u + \frac{1}{2}u_1 u_2 = \frac{1}{2}(u - u_1)(u - u_2)$$

So we can rewrite the LHS fraction as

$$\frac{1}{u^2 - \frac{1}{2}(u_1 + u_2)u + \frac{1}{2}u_1 u_2} = \frac{2}{u_1 - u_2}\left[\frac{1}{u - u_1} - \frac{1}{u - u_2}\right]$$

So we can integrate the ODE as follows

$$\int \frac{du}{u - u_1} - \int \frac{du}{u - u_2} = \int \frac{u_1 - u_2}{2\nu} d\eta \Rightarrow \ln \frac{|u - u_1|}{|u - u_2|} = \frac{u_1 - u_2}{2\nu}\eta$$

Now, as $u_1 < u < u_2$, the solution reads

$$\frac{u_1 - u}{u - u_2} = \exp\left[\frac{u_1 - u_2}{2\nu}\eta\right]$$

Rearranging the terms, we find that

$$u(x, t) = \frac{u_1 + u_2 e^{\frac{u_1 - u_2}{2\nu}(x - ct)}}{1 + e^{\frac{u_1 - u_2}{2\nu}(x - ct)}}$$

or equivalently

$$u(x, t) = \frac{u_2 + u_1 e^{-\frac{u_1 - u_2}{2\nu}(x - ct)}}{1 + e^{-\frac{u_1 - u_2}{2\nu}(x - ct)}}$$