IMPERIAL

MATH70057

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Riemannian Geometry

Date: Monday, May 12, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Given a Riemannian manifold (M, g) define the Levi-Civita connection associated to (M, g). (5 marks)

(b) Define what it means for a map $f : M \to N$ between Riemannian manifolds (M, g) and (N, h) to be a local isometry. (5 marks)

(c) Let M be a submanifold of $\mathbb{R}^n$. Define, without proof, the metric and connection induced on M by the ambient $\mathbb{R}^n$, endowed with the canonical Euclidean metric. In each case, write down the formula of the induced object in terms of the ambient one. (4 marks)

(d) Let (M, g) be the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the metric inherited from $(\mathbb{R}^3, g_{can})$, and $(N, h) = (\mathbb{R}^2, h_{can})$, where g_{can} and h_{can} represent the canonical Euclidean metric on each case. Prove there exists a map $f : N \to M$ which is a local isometry. (6 marks)

(Total: 20 marks)

2. Let (M, g) be a connected Riemannian manifold and $\gamma : [0, a] \to M$ be a regular curve.

(a) Define the length $\ell(\gamma)$ of γ.

(3 marks)

(b) Given $p, q \in M$, define $\mathrm{dist}_M(p, q)$.

(3 marks)

(c) Prove (M, dist_M) is a metric space. (Note: you can use facts about geodesic balls proved during the lectures).

(5 marks)

(d) Assume γ is parametrised by arc length and $f(s, t) : (-\varepsilon, \varepsilon) \times [a, b] \to M$ a smooth map such that $f(0, t) = \gamma(t)$, $f(s, a) = \gamma(a)$ and $f(s, b) = \gamma(b)$, for all s and t. Deduce, using direct differentiation and the properties of the connection, the formula

$$\frac{d}{ds}\ell(\gamma_s)\bigg|_{s=0} = -\int_0^a \left\langle \frac{D}{dt}\gamma'(t), \frac{\partial f}{\partial s}(0, t) \right\rangle dt.$$

(7 marks)

(e) Show carefully, using the formula of the previous item, that if γ realises the distance between its end points, then $\frac{D}{dt}\gamma'(t) = 0$, for all t.

(2 marks)

(Total: 20 marks)

3. (a) Define what it means for a Riemannian manifold to be geodesically complete. (6 marks)

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Assume that $0 \in \mathbb{R}$ is a regular value of f. Show that $M = f^{-1}(0)$ is a geodesically complete Riemannian manifold with respect to the metric inherited from $\mathbb{R}^n$.

   (6 marks)

   (c) Let (M, g) be a non-compact complete Riemannian manifold. Show that there exists a geodesic parametrised by arc length $\gamma : [0, +\infty) \to M$ that realises the distance between any of its end points, i.e. for all $0 < s < t$, we have $\mathrm{dist}_M(\gamma(s), \gamma(t)) = t - s$. (Hint: begin fixing $p \in M$ and argue that there exist a sequence $x_n \in M$ such that $\mathrm{dist}_M(p, x_n) = \infty$.) (8 marks)

(Total: 20 marks)

4. Let (M, g) be a Riemannian manifold.

   (a) Define the Riemann curvature tensor of M and write its main symmetries. (5 marks)

   (b) Given $p \in M$ and $\sigma \subset T_pM$ a two dimensional plane, define the sectional curvature $K(\sigma)$. (5 marks)

   (c) Write down, without proof, the simplified expression of the Riemannian curvature tensor in the case of a manifold with constant sectional curvature. (5 marks)

   (d) Let $R > 0$. Prove that the sphere $S^n(R) = \{x \in \mathbb{R}^{n+1} : |x| = R\}$ has constant sectional curvature and compute its curvature. (5 marks)

(Total: 20 marks)

5. Let (M, g) be a compact Riemannian manifold and $X \in \mathfrak{X}(M)$ a tangent vector field such that the flow $\phi : (-\varepsilon, \varepsilon) \times M \to M$, is a flow by isometries, i.e. $\phi_t(\cdot) = \phi(t, \cdot) : M \to M$ is an isometry of M for each fixed t.

(a) Given $p \in M$ and $c : (-\delta, \delta) \to M$ a curve such that $c(0) = p$ and $c'(0) = v$, let $f(s,t) = \phi_t(c(s))$. Prove the following two formulas:

(i) $\frac{\partial f}{\partial s}(t, 0) = d\phi_t(v)$.

(2 marks)

(ii) $\frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \nabla_v X$.

(3 marks)

(b) Use the previous formulas to conclude X satisfies the equation $0 = \langle \nabla_V X, W\rangle + \langle \nabla_W X, V\rangle$.

(5 marks)

(c) Show that if $X(p) = 0$, then X is tangent to the geodesic spheres centered at p. (5 marks)

(d) Show that if $X(p) = 0$ and $\gamma : [0, a] \to M$ is a geodesic such that $\gamma(0) = p$, then the restriction of X to γ can be written as

$$X(\gamma(t)) = \frac{d}{ds} \exp_p(t(v + sw))\Big|_{s=0}$$

where $v, w \in T_pM$ are such that $\langle v, w\rangle = 0$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70057

Riemannian Geometry (Solutions)

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1. (a) A connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying:

   1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
   2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
   3. $\nabla_X(fY) = df(X)Y + f\nabla_X Y$

   for all X, Y and Z tangent vector fields and f smooth real function. In addition, the Levi-Civita connection associated to a Riemannian metric g over M, is characterised by being the only connection satisfying

   1. $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
   2. $\nabla_X Y - \nabla_Y X = [X, Y]$. $\boxed{5, \text{A}}$

   (b) We say that f is a local isometry if:

   1. for all $p \in M$ and $v, w \in T_pM$ we have $h(df_p(v), df_p(w)) = g(v, w)$
   2. for each $p \in M$ there exists a neighbourhood $U \subset M$ of p, such that the restriction of f to U is a diffeomorphism onto its image. $\boxed{5, \text{A}}$

   (c) The induced metric g of M is the restriction to TM of the ambient Euclidean metric $\langle \cdot, \cdot \rangle$, i.e. given $p \in M$ and $v, w \in T_pM$, we use the natural inclusion $T_pM \subset \mathbb{R}^n$ to define $g(v, w) := \langle v, w \rangle$.

   The connection on (M, g) is simply the Levi-Civita connection associated to g as defined above. In terms of the ambient metric, it is defined by the formula $\nabla_X Y = (\bar{\nabla}_X Y)^T$ where $\bar{\nabla}$ is the connection in Euclidean space and T is the projection onto TM. $\boxed{4, \text{A}}$

   (d) Let $f(s, t) = (\cos(s), \sin(s), t)$. First, note that

$$df(e_1)(s, t) = \partial_s f(s, t) = (-\sin(s), \cos(s), 0)$$

   and

$$df(e_2)(s, t) = \partial_t f(s, t) = (0, 0, 1).$$

   It follows that $|df(e_1)|^2 = |df(e_2)|^2 = 1$ and $\langle df(e_1), df(e_2) \rangle = 0$. We claim that this implies the result. First, the connection on C is the restriction of the ambient connection, so the formulas above imply the first item of the definition. Second, since C is a 2-dimensional smooth manifold, e.g. regular level set of the function $F(x, y, z) = x^2 + y^2$, the Inverse Function theorem implies the local diffeomorphism property. $\boxed{6, \text{B}}$

2. (a) $\ell(\gamma)$ is given by the formula $\ell(\gamma) = \int_0^a g(\gamma'(t), \gamma'(t))^{1/2} dt$. $\boxed{3, \text{A}}$

(b) Let $\mathcal{C}(p,q) = \{c : [0,1] \to M : c$ is piecewise smooth, $c(0) = p$ and $c(1) = q\}$. We define $\mathrm{dist}_M(p,q) = \inf_{c \in \mathcal{C}(p,q)} \ell(c)$. $\boxed{3, \text{A}}$

(c) For simplicity we write $d = \mathrm{dist}_M$. We need to check

1. $d(p,q) \geq 0$
2. $d(p,q) = 0$ if and only if $p = q$.
3. $d(p,q) = d(q,p)$
4. $d(p,q) \leq d(p,x) + d(x,q)$.

For (1), note that $d(p,q)$ is defined as the infimum of a set of distances, which are all non-negative.

For (2) we have to prove each direction. First, if $p = q$, we can use the constant curve $c(t) = p$ on $t \in [0,1]$. In particular $c'(t) = 0$, for all t, which implies $0 \leq d(p,q) \leq \ell(c) = 0$. For the other direction, let us assume that $d(p,q) = 0$. We showed during the lectures, that, given $p \in M$, there exists $\delta > 0$ such that $\exp_p$ is a diffeomorphism on $B(t)$ the ball of radius $t > 0$ centered at $0 \in T_pM$, for all $t \in (0, \delta)$. Moreover, we used the Gauss lemma to show $\exp_p(B(t)) = \{x \in M : d(p,x) < t\}$. This implies that $q \in \cap_{t>0} \exp_p(B(t)) = \{p\}$.

For (3), note that, given $c \in \mathcal{C}(p,q)$ defining $\bar{c}(t) := c(1-t)$ on $t \in [0,1]$, we have $\bar{c} \in \mathcal{C}(q,p)$ and $\ell(c) = \ell(\bar{c})$. Taking the infimum on the formula we obtain (3).

Finally, for (4), given $a \in C(p,x)$ and $b \in C(x,q)$, we define the curve $(a*b)(t) = a(2t)$ for $t \in [0,1/2]$ and $c(t) = b(2(t-1/2))$ on $t \in [1/2,1]$. We have $\ell(a*b) = \ell(a) + \ell(b)$ and $a*b \in C(p,q)$. Therefore, $d(p,q) \leq \ell(a*b) = \ell(a) + \ell(b)$, for all $a \in C(p,x)$ and $b \in C(x,q)$. Taking the infimum of ℓ with respect to both a and b, we obtain (4). $\boxed{5, \text{C}}$

(d) By assumption $|\gamma'(t)| = 1$, for all t.

$$
\begin{aligned}
\frac{d}{ds}\ell(\gamma_s)\bigg|_{s=0} &= \frac{d}{ds}\bigg|_{s=0} \int_a^b \left|\frac{\partial f}{\partial t}(s,t)\right| dt \\
&= \int_a^b \frac{d}{ds}\bigg|_{s=0} \left|\frac{\partial f}{\partial t}(s,t)\right| dt \\
&= \int_a^b \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}(0,t), \frac{\partial f}{\partial t}(0,t) \right\rangle dt
\end{aligned}
$$

where we used the compatibility of the connection with the metric. From the first Symmetry lemma we have $\frac{D}{ds}\frac{\partial f}{\partial t}(0,t) = \frac{D}{dt}\frac{\partial f}{\partial s}(0,t)$. Therefore,

$$
\begin{aligned}
\frac{d}{ds}\ell(\gamma_s)\bigg|_{s=0} &= \int_a^b \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}(0,t), \frac{\partial f}{\partial t}(0,t) \right\rangle dt \\
&= \int_a^b \frac{D}{dt}\left\langle \frac{\partial f}{\partial s}(0,t), \frac{\partial f}{\partial t}(0,t) \right\rangle - \left\langle \frac{\partial f}{\partial s}(0,t), \frac{D}{dt}\frac{\partial f}{\partial t}(0,t) \right\rangle dt \\
&= \left\langle \frac{\partial f}{\partial s}(0,b), \frac{\partial f}{\partial t}(0,b) \right\rangle - \left\langle \frac{\partial f}{\partial s}(0,a), \frac{\partial f}{\partial t}(0,a) \right\rangle - \int_a^b \left\langle \frac{\partial f}{\partial s}(0,t), \frac{D}{dt}\frac{\partial f}{\partial t}(0,t) \right\rangle dt
\end{aligned}
$$

which implies the desired formula. $\boxed{7, \text{C}}$

(e) If γ minimises the distance between its end points, then the derivative of its length with respect to any variation is zero. Consider the variation $f(s,t) = \exp_{\gamma(t)}(s\rho(t)\frac{D}{dt}\gamma'(t))$, where ρ is a smooth function such that $\rho(a) = \rho(b) = 0$ and $\rho(t) > 0$ for $t \in (a,b)$. From the previous item we get

$$0 = \int_a^b \rho(t) \left| \frac{D}{dt}\gamma'(t) \right|^2 dt.$$

Since $t \mapsto \rho(t)|\frac{D}{dt}\gamma'(t)|^2$ is continuous and non-negative, we conclude it must be identically zero, but we chose $\rho(t) > 0$ in the interior. Therefore, $\frac{D}{dt}\gamma'(t) = 0$ for all t. $\boxed{2, \mathrm{A}}$

3. (a) We say that (M,g) is geodesically complete if geodesics on M exist for all past and future times, i.e. for all $p \in M$ the exponential map exp_p is defined in all of T_pM. $\boxed{6, \mathrm{B}}$

(b) Because M is a regular level set of a smooth function f, we know that M is an $(n-1)$-dimensional closed submanifold of $\mathbb{R}^n$. By Hopf-Rinow, being geodesically complete is equivalent to being complete with respect to the induced distance on M. Therefore it is enough to check that. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to dist_M. First, we claim that such a sequence is also a Cauchy sequence with respect to the ambient distance. In fact, note that given $p, q \in M \subset \mathbb{R}^n$, the ambient distance $|p-q|$ is the infimum of all the lengths of curves joining p and q, while $\mathrm{dist}_M(p,q)$ is the infimum with respect to curves fully contained in M. This implies

$$|p - q| \leq \mathrm{dist}_M(p,q),$$

which proves the claim. Finally, since M is a closed subset of a complete metric space, then it is itself complete as a metric space. In particular, $\lim_{n\to\infty} x_n = x$ exists and belong to M. $\boxed{6, \mathrm{C}}$

(c) Given $p \in M$, since M is not compact, by Hopf-Rinow, there exist a sequence $x_n \in M$ such that $\ell_n = \operatorname{dist}_M(p, x_n) \to +\infty$. Also by Hopf-Rinow, there are geodesics parametrised by arc length $\gamma_n : [0, \ell_n] \to M$ such that $\gamma(0) = p$ and $\gamma(\ell_n) = x_n$. Each one of this geodesics is characterised by its initial condition $v_n = \gamma_n'(0) \in T_pM$. Since $|v_n| = 1$, for all n, we can assume, after passing to a subsequence, that $\lim_{n\to\infty} v_n = v$, where $v \in T_pM$ and $|v| = 1$. We claim that the ray we are looking for is the geodesic $\gamma : [0, +\infty) \to M$, determined by the initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$. We want to show, that for all $0 < s < t$, we have $\operatorname{dist}_M(\gamma(s), \gamma(t)) = t - s$. First, note that this equality is true for the geodesics γ_n for n large enough, i.e. such that $\ell_n > t$. In other words, $\operatorname{dist}_M(\gamma_n(s), \gamma_n(t)) = t - s$. Otherwise, $\operatorname{dist}_M(\gamma_n(s), \gamma_n(t)) < \ell(\gamma_n|_{[s,t]}) = t - s$, which means there exist a curve $c : [s, t] \to M$, with $c(s) = \gamma_n(s)$ and $c(t) = \gamma_n(t)$, and such that $\ell(c|_{[s,t]}) < t - s$. It follows that the curve $\alpha = \gamma_n|_{[0,s]} * c|_{[s,t]} * \gamma_n|_{[t,\ell_n]}$ satisfies $\ell(\alpha) < \ell_n = \operatorname{dist}_M(p, x_n)$, which contradicts the minimising property of γ_n. Finally, by the smooth dependence of solutions to ODEs on their initial conditions, we know that $\gamma_n \to \gamma$ uniformly in compact sets. In particular, the curves $\theta \mapsto (\gamma_n(\theta), \gamma_n'(\theta))$ converge uniformly to the curve $\theta \mapsto (\gamma(\theta), \gamma'(\theta))$ in TM, for θ on any fixed compact set. It follows that $\operatorname{dist}_M(\gamma(s), \gamma(t)) = \lim_{n\to\infty} \operatorname{dist}_M(\gamma_n(s), \gamma_n(t)) = t - s$, as we wanted to show. $\boxed{8, \text{D}}$

4. (a) Let ∇ be the Levi-Civita connection. The Riemann curvature tensor of M is given by the formula $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Its main symmetries are:

   1. $R(X,Y)Z = -R(Y,X)Z$
   2. $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$
   3. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.

$\boxed{5, \text{A}}$

(b) Let $x, y \in T_pM$ be an orthonormal basis of T_pM. The curvature of M at p with respect to σ is given by the formula $K(\sigma) = g(R(x,y)y,x)$. $\boxed{5, \text{A}}$

(c) If (M,g) has constant sectional curvature κ, the Riemann tensor can be expressed as

$$g(R(X,Y)Z,W) = \kappa(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)).$$

$\boxed{5, \text{B}}$

(d) That $S^n(R)$ has constant curvature follows from the fact that for any $p, q \in S^n(R)$ and $\sigma \subset T_pM$ and $\eta \subset T_qM$, 2-dimensional planes, there is an isometry of $\mathbb{R}^{n+1}$ sending S^n into itself, p to q and σ to η. However, to compute the actual value of the curvature we need to perform one of several possible computations. The quickest way using the tools from the lectures, would be to use Gauss' formula $K(\sigma) - \bar{K}(\sigma) = B(X,X)B(Y,Y) - |B(X,Y)|^2$, where $\bar{K}$ is the ambient sectional curvature (which in our case is 0), B is the second fundamental form of $S^n(R)$ as submanifold of $\mathbb{R}^{n+1}$ and X, Y is an orthonormal basis of T_pM. Since $S^n(R)$ is a hypersurface it is enough to do this using a normal vector. After rotating if necessary we can assume $p = (R, 0, \dots, 0)$. Consider the curve $c(t) = (R\cos(t), R\sin(t), 0, \dots, 0)$ and the corresponding unitary normal vector $N(t) = (\cos(t), \sin(t), 0, \dots, 0)$. Then $R \cdot \nabla_v N(p) = \frac{D}{dt} N(0) = v$ where $v = (0, 1, 0, \dots, 0)$. It follows $\langle \nabla_v N(p), v \rangle = 1/R$. Given that v is arbitrary, up to rotations, and the relation between B and the derivatives of the normal, we get $B(X,X)B(Y,Y) = 1/R^2$. We conclude that the round sphere of radius R $K(\sigma) = 1/R^2$. $\boxed{5, \text{D}}$

5. (a) (i) Note that,for fixed t, $\frac{d}{ds}f(t,s)|_{s=0} = \frac{d}{ds}\phi_t(c(s))|_{s=0}$. This is exactly the definition of $d\phi_t(c'(0))$. Since $c'(0) = v$ we obtain the formula. $\boxed{2, \text{M}}$

(ii) Note that, for s fixed, $\frac{\partial}{\partial t}f(t,s)|_{t=0} = \frac{\partial}{\partial t}\phi_t(c(s))|_{t=0} = X(c(s))$, as this is the defining property of the flow ϕ_t. Then $\frac{D}{ds}X(c(s))|_{s=0} = \nabla_{c'(0)}X = \nabla_v X$ by the relation between the covariant derivative and the connection. $\boxed{3, \text{M}}$

(b) Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = w \in T_pM$. We denote by $h(s,t) = \phi_t(\gamma(s))$.

Since ϕ_t are all isometries, it follows $0 = \frac{d}{dt}\langle v, w\rangle = \frac{d}{dt}\langle d\phi_t(v), d\phi_t(w)\rangle$. By item (i) of the previous question we can then write $0 = \frac{d}{dt}\langle \frac{\partial f}{\partial s}(t,0), \frac{\partial h}{\partial s}(t,0)\rangle$. By the compatibility of the Levi-Civita connection with the metric and the first symmetry lemma, we obtain

$$0 = \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}(t,0), \frac{\partial h}{\partial s}(t,0)\right\rangle + \left\langle \frac{\partial f}{\partial s}(t,0), \frac{D}{dt}\frac{\partial h}{\partial s}(t,0)\right\rangle$$

$$= \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}(t,0), \frac{\partial h}{\partial s}(t,0)\right\rangle + \left\langle \frac{\partial f}{\partial s}(t,0), \frac{D}{ds}\frac{\partial h}{\partial t}(t,0)\right\rangle$$

which by item (ii) of the previous question and the definitions of f and h, gives us $0 = \langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle$. $\boxed{5, \text{M}}$

(c) Since $X(p) = 0$ it follows that $\phi_t(p) = p$ for all t. Moreover, since ϕ_t are all isometries, given $x \in M$, we also have $\mathrm{dist}_M(p, \phi_t(x)) = \mathrm{dist}_M(p, x) = r$, is a fixed constant. It follows that if x belongs to the geodesic sphere $S_r(p)$, then $\phi_t(x) \in S_r(p)$, for all t. In particular, $X(x) = \frac{d}{dt}\phi_t(x) \in T_xS_r(p)$. $\boxed{5, \text{M}}$

(d) Since ϕ_t are all isometries, then the image $\phi_t(\gamma(s))$ of a geodesic $\gamma(s)$, is also a geodesic. In other words, the variation $f(s,t) = \phi_t(\gamma(s))$ is a variation by geodesics. We proved in the lectures that the variational vector field, which in this case is $\frac{\partial f}{\partial t}(0,s) = X(\gamma(s))$. is a Jacobi vector field. It follows that if $\gamma(0) = p$ then X is a Jacobi vector field along γ with $X(0) = 0$. By uniqueness of ODEs on its initial conditions, it follows that all these vector fields are of the form $\frac{d}{dt}\exp_p(s(v + tw))|_{t=0}$, where, $w = \frac{d}{ds}X(s) = \nabla_v X(p)$. Finally, we saw on the last item that X must be tangent to the geodesic spheres centered at p. From the Gauss lemma we know that $\gamma'(s)$ is orthogonal to such geodesics, this proves $\langle X(\gamma(s)), \gamma'(s)\rangle = 0$. Differentiating with respect to s and using the formula from part (b) we get $\langle \nabla_{\gamma'(s)}X(\gamma(s)), \gamma'(s)\rangle = 0$, which at $s = 0$ gives us $\langle v, w\rangle = 0$. $\boxed{5, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 17 of 20 marks

Total C marks: 18 of 12 marks

Total D marks: 13 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70057 Riemannian Geometry Markers Comments

Question 1	Very good overall. Some people forgot to ask for (local) diffeomorphism in the definition of (local) isometry and just asked for the preservation of the inner product.
Question 2	The most common mistake was regarding the proof of d(p,q)=0 = p=q in part (c). Some people did not talk about geodesic balls in that context. Over all, the cohort did well on this question.
Question 3	In part (b) some people used the metric space statement "a closed subset of a complete metric space is also complete". However, note that in that statement the distance on the subset is meant to be the restriction of the ambient distance. While in the case of Riemannian manifolds the distance of the submanifold is very different from the ambient one.
Question 4	Good overall. Very varied responses. Part (d) was the hardest, but students were able to tackle it with different methods. Well done!
Question 5	The answers to parts (c) and (d) were usually very long. For example, in (c) it was not necessary to do any computations. It was enough to simply explain that geodesic balls are invariant under isometries that fix their centre. This is a simple fact that follows from isometries preserving the distance.