

**Mathematics Pre-arrival course**

**Solutions to Problem Sheet 3 – Linear Algebra, Sequences and Series**

1. The  $2 \times 2$  matrices  $A$  and  $B$  are defined by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ 3 & 2 \end{bmatrix}.$$

Calculate

- (a)  $AB$

Remember that  $(AB)_{i,j} = \sum_{k=1}^2 A_{i,k} B_{k,j}$ .

$$AB = \begin{bmatrix} 6 & -2 \\ -3 & -10 \end{bmatrix}$$

- (b)  $BA$

$$BA = \begin{bmatrix} -8 & 2 \\ 17 & 4 \end{bmatrix}$$

- (c)  $A^T$

$$A^T = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$$

- (d)  $B^T A^T$

$$B^T A^T = \begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -2 & -10 \end{bmatrix}$$

- (e)  $\det(A)$

$$\det(A) = 3 \times (-1) - 4 \times 2 = -11$$

- (f)  $A^{-1}$

Remember that for  $2 \times 2$  matrices, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then its inverse is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So, here we obtain

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}$$

One can easily check that  $AA^{-1} = I$ , where  $I$  is the identity matrix.

2. Which of the following statements are true for all square matrices  $A$ ,  $B$  and where  $I$  denotes the identity matrix:

$$I^2 = I, \quad A(B + I) = (B + I)A, \quad A(A + I) = (A + I)A.$$

- ☐  $I^2 = I$  is **true** by definition of the identity matrix;
- ☐  $A(B + I) - (B + I)A = AB + A - BA - A = AB - BA \neq 0$ , the second statement is **false** as matrices  $A$  and  $B$  do not necessarily commute;
- ☐  $A(A + I) - (A + I)A = AA + A - AA - A = 0$ , the third statement is **true**.

3. Describe the transformation that the following matrices represent:

(a)  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

This matrix is of the form  $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$  with  $\theta = \pi$ , it thus corresponds to a 2D rotation of  $180^\circ$  or  $\pi$  radians.

(b)  $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

This matrix is of the form  $\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$  with  $\theta = \pi/2$ , it thus corresponds to a 3D rotation of  $90^\circ$  or  $\pi/2$  radians about the  $y$ -axis.

(c)  $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Here, we recognize a shear matrix. In particular, this is a shear parallel to the  $y$ -axis mapping the vector  $(1, 0)$  to  $(1, 2)$ ; the transformation can be parametrized by  $x' = x$  and  $y' = y + 2x$ .

(d)  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

This one can be interpreted one of two ways:

- as a 2D rotation of  $90^\circ$  or  $\pi/2$  radians about the origin;
- as a combination of a reflection with respect to  $y = 0$  axis followed by a reflection with respect to axis  $y = x$ ; indeed reflection matrices are of the form

$$\text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

where  $\theta$  is the angle the reflection axis (going through the origin) makes with the  $x$ -axis and we can see that

$$\text{Ref}(\pi/4)\text{Ref}(\pi/2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

4. Find all of the invariant points under the following transformations

(a)  $\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$

Consider a point with position vector is given by  $(x, y)^T$ , with  $(x, y) \in \mathbb{R}^2$ , then this point is invariant under the transformation if and only if

$$\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{cases} 3x + y = x \\ 4x + 3y = y \end{cases} \iff y = -2x$$

(b)  $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Consider a point with position vector is given by  $(x, y)^T$ , with  $(x, y) \in \mathbb{R}^2$ , then this point is invariant under the transformation if and only if

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{cases} -y = x \\ -x = y \end{cases} \iff y = -x$$

This is consistent with the fact that this matrix represents a reflection with respect to  $y = -x$  axis, i.e. a reflection with  $\theta = -\pi/4$ .

(c)  $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$

Consider a point with position vector is given by  $(x, y)^T$ , with  $(x, y) \in \mathbb{R}^2$ , then this point is invariant under the transformation if and only if

$$\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{cases} 5x + 2y = x \\ x + 3y = y \end{cases} \iff \begin{cases} y = -2x \\ y = -x/2 \end{cases} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only the origin is invariant.

(d)  $\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$

Consider a point with position vector is given by  $(x, y)^T$ , with  $(x, y) \in \mathbb{R}^2$ , then this point is invariant under the transformation if and only if

$$\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{cases} 1x + 2y = x \\ 6x + 5y = y \end{cases} \iff \begin{cases} y = 0 \\ y = -3x/2 \end{cases} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only the origin is invariant.

NB: note that in all cases, we have only looked for invariant points, i.e. points which map onto themselves when the transformation is applied. In all cases, we can also find invariants lines; remember that the difference between invariant lines and points lies in the fact that points on an invariant line map onto points of the line under the transformation but needn't map onto themselves!

5. Given that  $A$  and  $B$  are invertible square matrices, prove that  $(AB)^{-1} = B^{-1}A^{-1}$ .

Since, by assumption,  $A$  and  $B$  are invertible square matrices, then  $A^{-1}$  and  $B^{-1}$  exist. We can write

$$B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Thus,  $B^{-1}A^{-1}$  is a left inverse of  $AB$ . Further, we have

$$ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

So  $B^{-1}A^{-1}$  is also a right inverse of  $AB$  and we conclude that  $(AB)^{-1} = B^{-1}A^{-1}$ .

6. Find all the values of  $k$  for which the matrix

$$\begin{bmatrix} k-1 & k-1 \\ k^2 & k+6 \end{bmatrix}$$

is singular.

The matrix is singular if its determinant is zero, so if

$$\begin{aligned} \begin{vmatrix} k-1 & k-1 \\ k^2 & k+6 \end{vmatrix} = 0 &\iff (k-1)(k+6) - k^2(k-1) = 0 \\ &\iff (k-1)[k^2 - k - 6] = 0 \\ &\iff (k-1)(k+2)(k-3) = 0 \end{aligned}$$

Thus, the matrix is singular if and only if  $k \in \{-2, 1, 3\}$ .

7. The  $3 \times 3$  matrix  $M$  is defined by

$$M = \begin{bmatrix} 3 & -2 & 2 \\ 2 & 3 & -1 \\ 1 & 3 & 1 \end{bmatrix}.$$

Calculate  $M^{-1}$  and solve the system of equations:

$$3x - 2y + 2z = -9$$

$$2x + 3y - z = 12$$

$$x + 3y + z = 3$$

To find the inverse of the matrix, we can use Gaussian elimination or we can remember that for a matrix  $M$ , its inverse (if it exists!) can be written as

$$M^{-1} = \frac{1}{\det(M)} \text{Adj}(M)$$

where  $\text{Adj}(M)$  is the adjoint of the square matrix  $M$ , i.e. the transpose of its cofactor matrix. So let's start with the determinant:

$$\det(M) = \begin{vmatrix} 3 & -2 & 2 \\ 2 & 3 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 3 \times 3 \times 1 + (-2) \times (-1) \times 1 + 2 \times 2 \times 3 - 1 \times 3 \times 2 - 3 \times (-1) \times 3 - 1 \times 2 \times (-2) = 30$$

where we have used here Sarrus' rule. Note that you can also use Leibniz's formula. Finally, we obtain that

$$M^{-1} = \frac{1}{30} \begin{bmatrix} + \begin{vmatrix} 3 & -1 \\ 3 & 1 \end{vmatrix} & - \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \\ - \begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} \\ + \begin{vmatrix} -2 & 2 \\ 3 & -1 \end{vmatrix} & - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} & + \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T = \frac{1}{30} \begin{bmatrix} 6 & -3 & 3 \\ 8 & 1 & -11 \\ -4 & 7 & 13 \end{bmatrix}^T = \frac{1}{30} \begin{bmatrix} 6 & 8 & -4 \\ -3 & 1 & 7 \\ 3 & -11 & 13 \end{bmatrix}$$

The system of equations can be written in matrix form as

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 12 \\ 3 \end{bmatrix}$$

So a solution is easily obtained by multiplying both sides on the left by  $M^{-1}$  leading to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1} \begin{bmatrix} -9 \\ 12 \\ 3 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 6 & 8 & -4 \\ -3 & 1 & 7 \\ 3 & -11 & 13 \end{bmatrix} \begin{bmatrix} -9 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

8. Find an expression in terms of  $n$  only for the following series:

(a)  $\sum_{r=1}^n (3r - 1)$

$$\sum_{r=1}^n (3r - 1) = 3 \sum_{r=1}^n r - \sum_{r=1}^n 1 = 3 \frac{n(n+1)}{2} - n = \frac{n(3n+1)}{2}$$

(b)  $\sum_{r=1}^n (2r^2 - r)$

$$\sum_{r=1}^n (2r^2 - r) = 2 \sum_{r=1}^n r^2 - \sum_{r=1}^n r = 2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)(4n-1)}{6}$$

(c)  $\sum_{r=1}^n r(r^2 - 2)$

$$\sum_{r=1}^n r(r^2 - 2) = \sum_{r=1}^n r^3 - 2 \sum_{r=1}^n r = \frac{n^2(n+1)^2}{4} - 2 \frac{n(n+1)}{2} = \frac{n(n+1)(n^2+n-4)}{4}$$

(d)  $\sum_{r=1}^n r^2(r+1)$

$$\sum_{r=1}^n r^2(r+1) = \sum_{r=1}^n r^3 + \sum_{r=1}^n r^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n+1)(n+2)}{12}$$

(e)  $\sum_{r=1}^n (2r^3 - 3r^2 + 5r - 3)$

$$\begin{aligned} \sum_{r=1}^n (2r^3 - 3r^2 + 5r - 3) &= 2 \sum_{r=1}^n r^3 - 3 \sum_{r=1}^n r^2 + 5 \sum_{r=1}^n r - 3 \sum_{r=1}^n 1 \\ &= 2 \frac{n^2(n+1)^2}{4} - 3 \frac{n(n+1)(2n+1)}{6} + 5 \frac{n(n+1)}{2} - 3n \\ &= \frac{1}{2} (n^4 + 2n^3 + n^2 - 2n^3 - 3n^2 - n + 5n^2 + 5n - 6n) \\ &= \frac{1}{2} (n^4 + 3n^2 - 2n) = \frac{1}{2} n (n^3 + 3n - 2) \end{aligned}$$

(f)  $\sum_{r=1}^n (r+2)(r-5) + \sum_{r=1}^n r(2r+1)(3r-2)$

$$\begin{aligned} \sum_{r=1}^n (r+2)(r-5) + \sum_{r=1}^n r(2r+1)(3r-2) &= \sum_{r=1}^n (6r^3 - 5r - 10) \\ &= 6 \sum_{r=1}^n r^3 - 5 \sum_{r=1}^n r - 10 \sum_{r=1}^n 1 \\ &= 6 \frac{n^2(n+1)^2}{4} - 5 \frac{n(n+1)}{2} - 10n \\ &= \frac{1}{2} [3(n^4 + 2n^3 + n^2) - 5(n^2 + n) - 20n] \\ &= \frac{1}{2} n [3n^3 + 6n^2 - 2n - 25] \end{aligned}$$

9. Use the method of differences to find a formula for the sums of the following series:

(a)  $\sum_{r=1}^n \frac{2}{(2r-1)(2r+1)}$

We have

$$\sum_{r=1}^n \frac{2}{(2r-1)(2r+1)} = \sum_{r=1}^n \frac{1}{(2r-1)} - \frac{1}{(2r+1)}$$

This sum is of the form  $\sum_{r=1}^n a_r - a_{r+1}$  with  $a_r = 2r - 1$ . Thus, we know that the sum will telescope as follows

$$\begin{aligned}\sum_{r=1}^n \frac{1}{(2r-1)} - \frac{1}{(2r+1)} &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= 1 \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \left(-\frac{1}{7} + \frac{1}{7}\right) + \cdots - \frac{1}{2n+1} \\ &= 1 - \frac{1}{2n+1} = a_1 - a_{n+1}\end{aligned}$$

and so we conclude that

$$\sum_{r=1}^n \frac{2}{(2r-1)(2r+1)} = \frac{2n}{2n+1}$$

(b)  $\sum_{r=1}^n \frac{1}{r(r+2)}$

We have

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{1}{2} \sum_{r=1}^n \frac{1}{r} - \frac{1}{r+2}$$

This sum is of the form  $\sum_{r=1}^n a_r - a_{r+2}$  with  $a_r = 1/r$ . Thus, we know that the sum will telescope as follows

$$\begin{aligned}\sum_{r=1}^n \frac{1}{r} - \frac{1}{r+2} &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \cdots - \frac{1}{n+1} - \frac{1}{n+2} \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = a_1 + a_2 - a_{n+1} - a_{n+2}\end{aligned}$$

and so we conclude that

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

(c)  $\sum_{r=1}^n \frac{r}{(r+2)(r+3)(r+4)}$

We have

$$\sum_{r=1}^n \frac{r}{(r+2)(r+3)(r+4)} = \sum_{r=1}^n -\frac{1}{r+2} + \frac{3}{r+3} - \frac{2}{r+4}$$

Let's write the first few terms

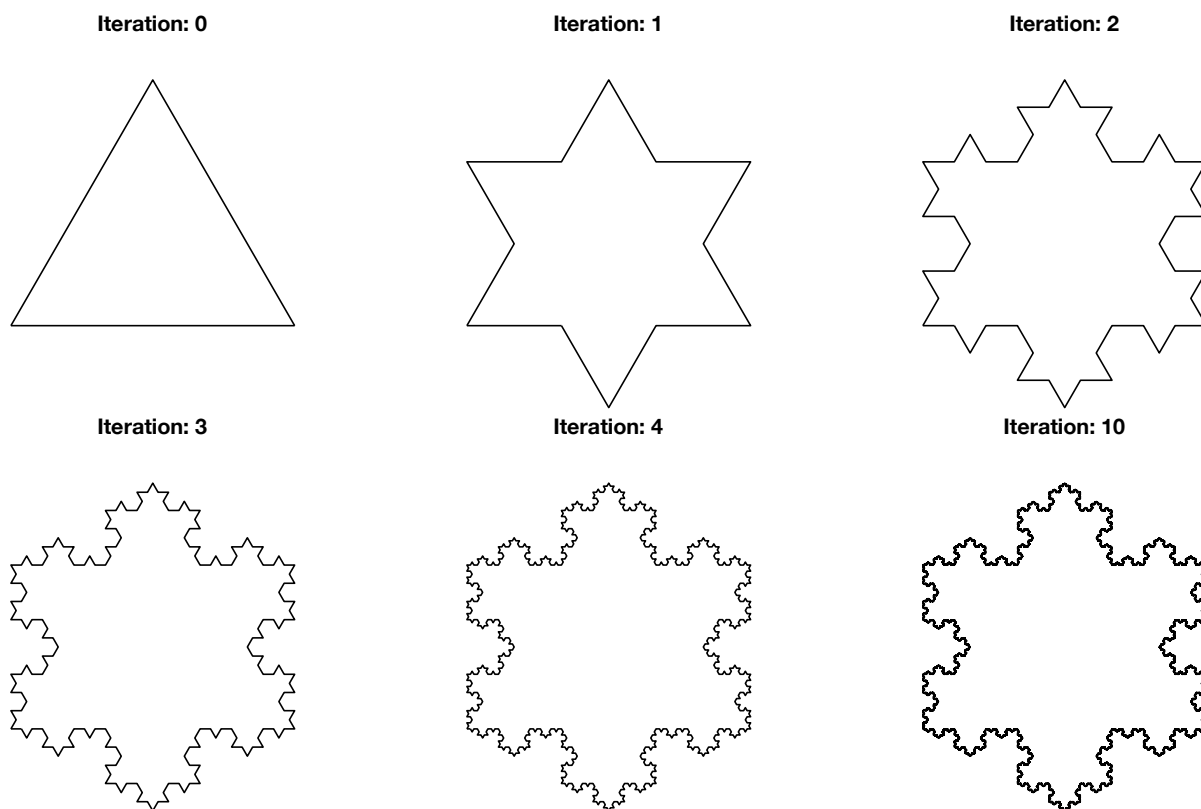
$$\begin{aligned}\sum_{r=1}^n \frac{r}{(r+2)(r+3)(r+4)} &= \left(-\frac{1}{3} + \frac{3}{4} - \frac{2}{5}\right) + \left(-\frac{1}{4} + \frac{3}{5} - \frac{2}{6}\right) + \left(-\frac{1}{5} + \frac{3}{6} - \frac{2}{7}\right) + \cdots \\ &= -\frac{1}{3} + \frac{3}{4} - \frac{1}{4} + \left(-\frac{1}{5} + \frac{3}{5} - \frac{2}{5}\right) + \left(-\frac{1}{6} + \frac{3}{6} - \frac{2}{6}\right) + \cdots \\ &= -\frac{1}{3} + \frac{3}{4} - \frac{1}{4} - \frac{2}{n+3} + \frac{3}{n+3} - \frac{2}{n+4}\end{aligned}$$

and so we conclude that

$$\sum_{r=1}^n \frac{r}{(r+2)(r+3)(r+4)} = \frac{n(n+1)}{6(n+3)(n+4)}$$

10. Consider the following iterative geometric construction. We start from an equilateral triangle with side length 1. At each subsequent steps, we first divide each side of the polygon in three equal parts, we construct an equilateral triangle on the middle part and we delete the middle part itself. The Koch snowflake is the piecewise continuous curve that results from repeating this process infinitely. For a given step  $n$  in the process, we denote  $S_n$  the number of sides of the polygon,  $L_n$  the length of a given side and  $P_n$  the total perimeter of the snowflake.

(a) Sketch the first few steps of the construction process.



(b) Devise formulas for  $S_n$ ,  $L_n$  and  $P_n$ .

Students should realize from the first few iterations they did by hand that the iteration process is such that at each step a given side is replaced by four shorter sides and each of these four new sides is of length  $1/3$  of the length at the preceding step in the construction. This, in turn, gives the following for a step  $n$  in the iteration process:

$$S_n = S_0 \cdot 4^n$$

$$L_n = \frac{L_0}{3^n}$$

$$P_n = S_n L_n$$

By construction we started with a triangle of side length 1. So we have:  $S_0 = 3$  and  $L_0 = 1$ .

$$S_n = 3 \cdot 4^n$$

$$L_n = \frac{1}{3^n}$$

$$P_n = 3 \cdot \left(\frac{4}{3}\right)^n$$

(c) What can you say about the perimeter of the snowflake curve?

We just showed that  $P_n = 3 \cdot (4/3)^n$ . In particular, as  $4/3 > 1$ , we know that  $P_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The length of the curve (the perimeter of the snowflake) diverges.

(d) What is the area enclosed by Koch's snowflake?

At any given step  $n$  in the iterative construction, the area of one of the small equilateral triangles added is  $1/9$  of the area of the triangle that was added at the previous step. If we denote  $a$  the area of the original triangle then, the area of one added triangle at step  $n$  is given by

$$a_n = \frac{a}{9^n}$$

At any step  $n$ , one of those small triangles is added on each of the sides of the polygon, thus it follows that the total area added at step  $n$  is given by

$$A_n = S_n \cdot a_n = 3 \cdot 4^{n-1} \cdot \frac{a}{9^n} = a \frac{4^{n-1}}{3^{2n-1}}$$

The total area enclosed by the curve is thus given by

$$A = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} a \frac{4^{n-1}}{3^{2n-1}} = a + \frac{a}{3} \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n = a + \frac{a/3}{1 - 4/9} = \frac{8a}{5}$$

We know that the original triangle was of side length 1. So we

$$a = \frac{1}{2} \cdot 1 \cdot \sin \frac{\pi}{3} = \sqrt{3}/4$$

Finally, we find that

$$A = \frac{2\sqrt{3}}{5}.$$

We have, in turn, just shown that Koch's snowflake is a curve of infinite length enclosing a very finite area!

This curve is called the Koch snowflake; it is one of the earliest examples of fractals which appeared in 1904 in a paper entitled "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire" (in english, "On a continuous curve without tangents, constructible from elementary geometry") by Swedish mathematician Helge von Koch.