Imperial College
London

Autumn Term, 2022

Analysis 1A

Lecture 19
Finishing rearrangements
Power series

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Theorem 4.34

$\sum a_n$ is absolutely convergent $\iff$ $(1) + (2) \Rightarrow (3) + (4)$, where

(1) $\sum_{a_n \geq 0} a_n$ is convergent (to A say),

(2) $\sum_{a_n < 0} a_n$ is convergent (to B say),

(3) $\sum a_n = A + B$,

(4) $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n).

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Let $p_1, p_2, p_3, \ldots$ be the nonnegative $a_n \geq 0$.
That is p_i is the i th nonnegative element of the sequence (a_n).
Similarly let $n_1, n_2, n_3, \ldots$ be the negative $a_n < 0$.

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Last lecture, we have showed

$$\text{Absolute convergence of } \sum a_n \Rightarrow (1) + (2) .$$

That is, $\sum_{i=1}^{n} p_i$ converges monotonically upwards to some $A \in \mathbb{R}$ and $\sum_{i=1}^{n} n_i$ converges monotonically ~~upwards~~ to some $B \in \mathbb{R}$.
downwards

Want to prove (1)+(2) $\Rightarrow$ (4)

Let $\varepsilon>0$, $\exists\, N_1$ s.t $\forall\, n\geq N_1$ $\quad A-\varepsilon < \sum_{i=1}^{n} p_i \leq A$ $\quad$ I

$\exists\, N_2$ s.t $\forall\, n\geq N_2$, $\quad B \leq \sum_{i=1}^{n} n_i < B+\varepsilon$ $\quad$ II

For any $I_1 \subset \{N_1+1, N_1+2, \ldots\ldots, \}$, $\quad 0 \leq \sum_{i\in I} p_i < \varepsilon$ $\quad$ III

$I_2 \subset \{N_2+1, N_2+2, \ldots, \ldots\}$ $\quad -\varepsilon < \sum_{i\in I_2} n_i \leq 0$ $\quad$ IV

Let b_n be a rearrangement of a_n

Want to show if $s_n = \sum_{j=1}^{n} b_j$, $s_n \to A+B$

$\exists N$ st $\{p_1, \dots, p_{N_1}\}$ and $\{n_1, n_2, \dots, n_{N_2}\}$ are in $\{b_1, \dots, b_N\}$

Then $\forall n \geq N$

$$|s_n - (A+B)| \leq \overset{\mathrm{I}}{\left|\sum_{i=1}^{N_1} p_i - A\right|} + \overset{\mathrm{II}}{\left|\sum_{i=1}^{N_2} n_i - B\right|} + \overset{\mathrm{III}}{\sum_{i \in I_1} p_i} + \overset{\mathrm{IV}}{\sum_{i \in I_2} |n_i|}$$

$$< \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon$$

So $s_n \to A+B$.

Now, just need to show (1)+(2) $\Rightarrow$ absolute convergence

Let N, N_1, N_2 be as above

$\forall n \geq N$

$N' \geq N_1$, $N'' \geq N_2$

$$\sum_{i=1}^{n} |a_i| = \underset{> A - \varepsilon}{\sum_{i=1}^{N'} p_i} - \underbrace{\sum_{i=1}^{N''} n_i}_{< B + \varepsilon} \leq A - B$$

Also know

$$\sum_{i=1}^{n} |a_i| > A - \varepsilon - (B + \varepsilon) = A - B - 2\varepsilon$$

So $\sum_{i=1}^{n} |a_i| \to A - B$

Turning to power series

Turning to power series - we introduce $[0, \infty] := [0, \infty) \cup \{+\infty\}$.

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Theorem 4.35 - Radius of Convergence

Fix a real or complex sequence (a_n) and consider the series $\sum a_n z^n$ for $z \in \mathbb{C}$. Then $\exists\ R \in [0,\infty]$ such that

- $|z| < R \implies \sum a_n z^n$ is absolutely convergent, and
- $|z| > R \implies \sum a_n z^n$ is divergent. ✓

Proof: Let $S = \{ |z| : a_n z^n \to 0 \} \subset [0,\infty)$

$$R = \begin{cases} \sup(S) & \text{if } S \text{ is bounded above} \\ \infty & \text{if } S \text{ is unbounded} \end{cases}$$ $\leftarrow$ Defines radius of convergence

Suppose $|z| > R$, then $|z| \notin S$, so $a_n z^n \not\to 0$ so $\sum a_n z^n$ must diverge

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Proof continued.

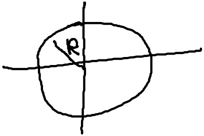
Let $|z| < R$, then $\exists\, w \in \mathbb{C}$ s.t. $|w| \in S$ and $|w| > |z|$

We know $a_n w^n \to 0$, so $a_n w^n$ is bounded, $|a_n w^n| \le A\ \forall n$

$\sum |a_n z^n| = \sum |a_n w^n| \cdot \left|\frac{z}{w}\right|^n$ converges by comparison with $\sum A \left|\frac{z}{w}\right|^n$

$\left|\frac{z}{w}\right| < 1$

Convergent geometric series.



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The R in Thm 4.35 is called the radius of convergence for $\sum a_n z^n$.

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The R in Thm 4.35 is called the radius of convergence for $\sum a_n z^n$. Note that Thm 4.35 doesn't tell us what happens when $|z| = R$.

Exercise 4.37

Consider the sequences

(a) $a_n = \frac{1}{n^2}$,

(b) $a_n = \frac{1}{n}$,

(c) $a_n = 1$.

Show their power series $\sum a_n z^n$ all have radius of convergence $R = 1$, and on $|z| = 1$ their behaviour is as follows,

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Show their power series $\sum a_n z^n$ all have radius of convergence $R = 1$, and on $|z| = 1$ their behaviour is as follows,

(a) convergent everywhere on $|z| = 1$,
$\big($Absolutely convergent because $\sum \frac{1}{n^2} < \infty$.$\big)$

(b) convergent somewhere,
$\big($Convergent at $z = -1$ by alternating series test, not convergent at $z = 1$.$\big)$

(c) convergent nowhere on $|z| = 1$.
$\big(a_n z^n \not\to 0$ as $n \to \infty$.$\big)$