Linear Algebra **Unseen Problem 2**

Recall that an $n \times n$ matrix A over a field F is *diagonalisable* if $\exists$ an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal.

This week the project is to investigate the probability that a random matrix in $M(n, F)$ (the $n \times n$ matrices over a field F) is diagonalisable. To make sense of this question for infinite fields F would need some measure theory, so we will focus on finite fields $\mathbb{F}_p$, for which the probability is just

$$\frac{|\{A \in M(n, \mathbb{F}_p) : A \text{ diagonalisable}\}|}{|M(n, \mathbb{F}_p)|}. \tag{1}$$

(A) Let's start with the case $n = 2$.

(i) What is $|M(2, \mathbb{F}_p)|$?

(ii) Let $D = \mathrm{diag}(\lambda_1, \lambda_2)$, the diagonal matrix with diagonal entries λ_1, λ_2. Show that if $\lambda_1 \neq \lambda_2$, then the only matrices in $M(2, \mathbb{F}_p)$ that commute with D are the diagonal ones.

(iii) By defn, a diagonalisable 2×2 matrix has the form $P^{-1}DP$, where $P \in GL(2, \mathbb{F}_p)$. Consider the map $\phi_D : GL(2, \mathbb{F}_p) \mapsto M(2, \mathbb{F}_p)$ that sends $P \mapsto P^{-1}DP$. Show that if $\lambda_1 \neq \lambda_2$ then

$$|\mathrm{Im}(\phi_D)| = \frac{|GL(2, \mathbb{F}_p)|}{(p-1)^2}.$$

Hence find the total number of diagonalisable matrices in $M(2, \mathbb{F}_p)$. (Recall from last week that $|GL(2, \mathbb{F}_p)| = (p^2 - 1)(p^2 - p)$.)

(iv) Now compute the probability (1) for $n = 2$. What is its limit as $p \to \infty$?

(B) Try to generalise – first to $n = 3$, then to arbitrary n.