IMPERIAL

MATH70130

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Stochastic Differential Equations in Financial Modelling

Date: Monday, April 28, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. *SDE.*

Consider the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2} dW_t,$$

where $X_0 = x_0$ is the deterministic initial condition and W is a standard Brownian motion.

(a) Prove that this SDE admits a unique global solution. (6 marks)

(b) Calculate the solution. *[Hint: transform in Stratonovich form and remember $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + C$ where $\sinh^{-1}$ is the inverse function of hyperbolic sine and C is an integration constant]* (8 marks)

(c) Calculate the expected value of the solution at a time $T > 0$. *[Hint: it may be easier obtaining and solving an ODE for the mean rather than attempting a direct calculation from the solution]* (6 marks)

(Total: 20 marks)

2. *Black-Scholes Option Pricing, Bull Call Spread.*

Given a stock with price S_t at time t, $t \geq 0$, consider a payoff Y that is the difference between a call option with smaller strike and a call option with a larger strike. The underlying asset S and the option maturities T are the same. In formula: if the two strikes are $K_1 > K_2$, then the bull call spread payoff is

$$Y = (S_T - K_2)^+ - (S_T - K_1)^+.$$

(a) Draw a plot of this payoff as a function of S_T. Explain what type of investor would be interested in buying this payoff and what views on the stock market this investor would have. (5 marks)

(b) Write a formula for the price of the bull call spread in a Black-Scholes model, where the stock price dynamics under the the physical measure P is

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$

with $S_0 = s_0$ the deterministic initial condition, and where interest rates r are constant and deterministic. Here W^P is a Brownian motion under the measure P. You can use the formula for a call option without deriving it if you remember it. (10 marks)

(c) Write a formula for the delta of the bull call spread, namely its sensitivity to the stock price at time 0. In other words, if V_0 is the price you computed in point b), write a formula for $\frac{\partial V_0}{\partial s_0}$. What does the sign of the delta allow you to conclude on the behaviour of the bull call spread with respect to the underlying stock and does this confirm your intuition on the payoff interpretation in point a)? (5 marks)

(Total: 20 marks)

3. *Option pricing Bachelier (butterfly spread).*

Consider a stock market where the stock price S follows the Bachelier dynamics, assuming the risk free interest rate is zero, namely $r = 0$,

$$dS_t = \sigma \; dW_t, \quad S_0 = s_0,$$

where W is a Brownian motion under the risk neutral measure Q, with deterministic initial stock price $s_0 > 0$. σ is a positive real volatility. Consider a butterfly spread option on the stock S with maturity T. This is a strategy based on buying one in-the-money call option with a low strike $L = S_0 - X$, selling two at-the-money call options with a middle strike $M = S_0$, and buying one out-of-the-money call option with a higher strike price $H = S_0 + X$, where X is a positive real constant. All options have maturity T. The payoff at maturity is

$$Y = (S_T - L)^+ - 2(S_T - M)^+ + (S_T - H)^+.$$

(a) Write a formula for the butterfly spread price at time 0 in the Bachelier model, call it V_0. Simplify the formula as much as possible. (10 marks)

(b) Write a formula for the delta of the butterfly spread in the Bachelier model, namely $\frac{\partial V_0}{\partial s_0}$, simplify it as much as possible and comment on the sign of the delta and on its significance. *[Hint: remember we ask for the sensitivity to the stock and not the joint sensitivity to stock and strikes]* (6 marks)

(c) Draw a plot of the butterfly spread final payoff Y as a function of S_T. Consider the limit situation when $X \downarrow 0$. Deduce intuitively the price of the butterfly by looking at the plot of the payoff and by thinking about what happens to the plot when $X \downarrow 0$. Check with a limit calculation for $X \downarrow 0$ from point a) whether the Bachelier price for the butterfly spread confirms this intuition. (4 marks)

(Total: 20 marks)

4. *VaR for bond and stock with different maturities in Black Scholes.*

Consider a portfolio with a first bond position in a notional N of zero-coupon bond with maturity U in an economy where we have a deterministic constant risk free interest rate r. Assume that in the same portfolio we are short a bond with maturity $T < U$ on the same notional, and that we hold an amount M of stock S, where the stock price follows the following dynamics under the measure P: $dS_t = \mu S_t dt + \sigma S_t dW_t,\ s_0$. We assume $M > 0, N > 0$.

(a) Define Value at Risk (VaR) for a time horizon T with confidence level α for a general portfolio. (6 marks)

(b) Compute the value at risk of this portfolio for a confidence level α at a risk horizon $h < T$. Simplify the formula as much as you can. (6 marks)

(c) Explain the drawback of VaR as a risk measure that is completely solved by expected shortfall. (4 marks)

(d) Is the equity dynamics you used for VaR the same you would have used to price an equity call option in the Black Scholes model? Discuss any possible differences. (4 marks)

(Total: 20 marks)

5. *Mastery question.*

The introduction of risk measures has been triggered by a number of crises and market incidents. Consider the Barings collapse.

(a) Write a short history of the Baring collapse, who caused it and how.

(5 marks)

(b) What option combination, in particular, turned out to create a huge loss? Explain the idea behind this option combination and draw the payoff as a function of the final stock price S_T.

(5 marks)

(c) In that particular historical moment, what was the point of setting up this position and what went wrong, leading to further huge losses that sank the bank? (5 marks)

(d) What risk measure has been introduced shortly after to limit the possibility that similar cases may happen again? Give a verbal definition of what this risk measure represents and what it may achieve. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

April 2025

This paper is also taken for the relevant examination for the Associateship.

Math60130/70130

Stochastic Differential Equations in Financial Modeling (Solutions)

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1. (a) A sufficient condition for existence and uniqueness of a global strong solution is that we have two conditions regarding Lipschitz continuity and linear growth. We know from the theory that for the SDE $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = Z$ with Z independent of $\sigma(\{W_t,\ t \le T\})$ and $E[Z^2] < +\infty$, and with $\mu : [0,T] \times R \to R$ (the drift) and $\sigma : [0,T] \times R \to R$ (the diffusion coefficient) being measurable, if we have global Lipschitz continuity

sim. seen $\Downarrow$

6, A

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y| \text{ for all } t \in [0,T] \text{ and all } x \in R$$

and linear growth

$$|\mu(t,x)| + |\sigma(t,x)| \le K'(1+|x|)| \text{ for all } t \in [0,T] \text{ and all } x \in R$$

for two constants K, K', then our SDE has a unique global solution X_t.

Let's check our conditions.

The initial condition has to be squared integrable, $E[X_0^2] < +\infty$, which is true in our case as $X_0 = x_0$ is a finite deterministic constant and $E[X_0^2] = x_0^2 < \infty$. Then we need to prove that the the drift and diffusion coefficient are measurable functions of X, t.

This is trivially true as the drift is a linear function of X and does not depend on t, $\mu(t, X) = X/2$. Also, the diffusion coefficient $\sigma(t, X) = \sqrt{1 + X_t^2}$ is trivially measurable as it is a continuous function of X and does not depend on t.

Next we need to check the Lipschitz continuity and linear growth condition. I will present a brute force proof of the Lipschitz condition but alternative proofs are ok and get full marks. For example, proving that $x \mapsto \sqrt{1+x^2}$ is 1-Lipschitz can be considerably shortened by noting that its derivative on the positive reals is bounded in modulus by 1 and appealing to the Mean Value Theorem. There are also other approaches that are shorter than the brute force approach below and that can get full marks.

Brute force proof. The Lipschitz condition reads

$$\begin{aligned} &|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| = \\ &= |x/2 - y/2| + |\sqrt{1+x^2} - \sqrt{1+y^2}| \le \dots \end{aligned} \tag{1}$$

At this point we need to deal with the second term before the inequality, as the first one is trivially $\frac{1}{2}|x-y|$. We claim that

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| \le ||x| - |y||.$$

To prove this, let us assume that $|x| \ge |y|$ (and hence also $x^2 > y^2$). If the opposite holds, we can swap the terms and proceed analogously, as they are inside an absolute value and swapping them does not affect the reasoning. With $|x| \ge |y|$, we can rewrite the above inequality as

$$\sqrt{1+x^2} - \sqrt{1+y^2} \le (|x| - |y|).$$

To check that this is true, and remembering that both sides are positive given $|x| \ge |y|$, we may square both sides and see if the inequality holds. Given positivity, if it holds for the squares it holds for the bases too. Squaring both sides, we get

$$(\sqrt{1+x^2} - \sqrt{1+y^2})^2 \le (|x| - |y|)^2 \iff$$

$$\iff 1 + x^2 + 1 + y^2 - 2\sqrt{(1+x^2)(1+y^2)} \le x^2 + y^2 - 2|x||y| \iff$$

$$\iff x^2 + y^2 - 2\sqrt{(1+x^2)(1+y^2)} \le x^2 + y^2 - 2|x||y| \iff$$

$$-2\sqrt{(1+x^2)(1+y^2)} \le -2|x||y| \iff 2\sqrt{(1+x^2)(1+y^2)} \ge 2|x||y|$$

which is true since $1 + x^2 \ge x^2$, $1 + y^2 \ge y^2$ and $\sqrt{\cdot}$ is an increasing function, so that $\sqrt{1+x^2} > \sqrt{x^2} = |x|$ and similarly for y. So we have clearly $-2\sqrt{(1+x^2)(1+y^2)} < -2|x||y|$ which, for the chain of $\iff$ above, is equivalent to

$$(\sqrt{1+x^2} - \sqrt{1+y^2})^2 \le (|x| - |y|)^2,$$

or

$$\sqrt{1+x^2} - \sqrt{1+y^2} \le |x| - |y|.$$

Substituting this in (1) we conclude

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le \frac{1}{2}|x-y| + ||x| - |y||. \tag{2}$$

This is not yet a Lipschitz condition. We need to show that

$$||x| - |y|| \le |x - y|.$$

To check this, let us look at all four cases.

a) $x \ge 0, y \ge 0$. Then the above inequality becomes an identity and we are done.

b) $x \ge 0, y \le 0$. Then the above inequality becomes

$$||x| - |y|| = |x - (-y)| = |x + y| \le |x - y|$$

where the inequality holds because of the signs of x and y, in particular with y being negative.

c) $x \le 0, y \ge 0$. Then the above inequality becomes

$$||x| - |y|| = |-x - y| \le |x - y|$$

as $-x$ is positive, so it offsets $-y$ leading to a smaller absolute value for the left hand side.

d) $x \le 0, y \le 0$. Then the above inequality becomes

$$||x| - |y|| = |-x - (-y)| = |-x + y| \le |x - y|$$

as $-x$ is positive and offsets the negative $+y$ on the left hand side, leading to a smaller absolute value.

So we have proven that $||x| - |y|| \le |x - y|$, and substituting in (2) we get

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le \frac{1}{2}|x-y| + ||x| - |y|| \le$$

$$\le \frac{1}{2}|x-y| + |x-y| = \frac{3}{2}|x-y|$$

which is Lipschitz continuity with constant $3/2$.

So we have proven Lipschitz continuity. For autonomous SDEs, Lipschitz implies linear growth so we can say that and stop. Or prove linear growth too as follows.

To prove linerar growth, we need to show that

$$|\mu(t,x)| + |\sigma(t,x)| \le K'(1+|x|)| \quad \text{for all } t \in [0,T] \text{ and all } x \in \mathbb{R}$$

We have

$$|\mu(t,x)| + |\sigma(t,x)| = |x| + \sqrt{1+x^2} \le \frac{1}{2}|x| + \sqrt{1} + \sqrt{x^2} = 1 + \frac{3}{2}|x| \le \frac{3}{2}(1+|x|)$$

$$\text{for all } t \in [0,T] \text{ and all } x \in \mathbb{R}$$

as in general, for two positive real number a and b, $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ (square both sides, that are positive, to convince yourself of this).

So we have a unique global solution.

sim. seen ⇓

8, C

(b) Calculation of the solution using Stratonovich calculus is based on transforming the Ito SDE in a Stratonovich SDE. We know that the transformation changes the drift into

$$\frac{1}{2}x \mapsto \frac{1}{2}x - \frac{1}{2}\sigma(t,x)\frac{\partial \sigma(t,x)}{\partial x}.$$

In our specific case,

$$\frac{1}{2}\sigma(t,x)\frac{\partial \sigma(t,x)}{\partial x} = \frac{1}{2}\sqrt{1+x^2}\frac{1}{2}\frac{2x}{\sqrt{1+x^2}} = -\frac{1}{2}x$$

so that the Stratonovich drift becomes

$$\frac{1}{2}x \mapsto \frac{1}{2}x - \frac{1}{2}x = 0.$$

The equivalent Stratonovich SDE with the same solution is therefore

$$dX_t = \sqrt{1+X_t^2} \circ dW_t, \quad x_0.$$

With Stratonovich, we can use formal rules of calculus. We can separate variables as in

$$\frac{dX_t}{\sqrt{1+X_t^2}} = \circ dW_t,$$

and integrate both sides

$$\int_{x_0}^{X_t} \frac{dX}{\sqrt{1+X^2}} = \int_0^t dW_t,$$

leading to

$$\sinh^{-1}(X)|_{x_0}^{X_t} = W_t$$

or

$$\sinh^{-1}(X_t) - \sinh^{-1}(x_0) = W_t$$

$$\sinh^{-1}(X_t) = \sinh^{-1}(x_0) + W_t$$

$$X_t = \sinh\left(\sinh^{-1}(x_0) + W_t\right).$$

sim. seen ⇓

6, A

(c) To calculate the expected value we could use the solution directly and proceed to a direct calculation but this is not convenient, as it involves a much more complicated calculation. Let us use the Ito SDE instead. Consider

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1+X_t^2}dW_t,$$

and write it in integral form

$$X_t = x_0 + \int_0^t \frac{1}{2}X_s ds + \int_0^t \sqrt{1+X_s^2}dW_s,$$

and now take the expected value conditional on information at time 0 on both sides. Recall the that expectation of an Ito integral is zero and that x_0 is deterministic, so we get

$$E_0[X_t] = x_0 + E_0[\int_0^t \frac{1}{2}X_s ds] + 0,$$

or

$$E_0[X_t] = x_0 + \int_0^t \frac{1}{2}E_0[X_s]ds + 0,$$

using Fubini's theorem. Let us now call $m_t = E_0[X_t]$ for all $t \geq 0$. The above equation can be written as

$$m_t = x_0 + \int_0^t \frac{1}{2}m_s \; ds.$$

Now differentiate both sides with respect to t, obtaining

$$\frac{dm_t}{dt} = \frac{1}{2}m_t,$$

with initial condition $m_0 = E_0[X_0] = E_0[x_0] = x_0$. The last differential equation is immediate to integrate. We have

$$\frac{dm}{m} = \frac{1}{2}dt$$

so that, integrating both sides

$$\ln(m)|_{m_0}^{m_t} = \frac{1}{2}t$$

or

$$\ln(m_t) - \ln(m_0) = \frac{1}{2}t$$

or, rearranging,

$$m_t = m_0 \exp\left(\frac{1}{2}t\right)$$

so that

$$E[X_t] = x_0 \exp\left(\frac{1}{2}t\right).$$

2. (a) The payoff can also be written as

meth seen ⇓

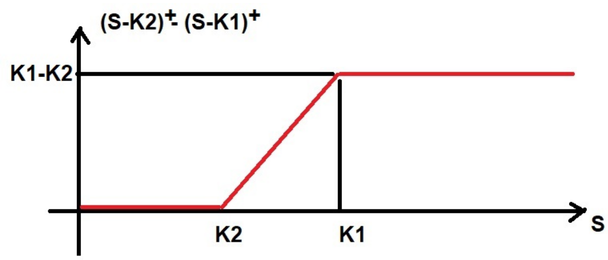
5, A

$$Y = (K_1 - K_2)1_{S_T > K_1} + (S - K_2)1_{K_2 < S_T \le K_1} + 0\ 1_{S \le K_2}.$$

This contingent claim consists of one long call with a lower strike price and one short call with a higher strike.

Note that the initial price of Y would be positive to us, since it is an in-the-money call minus an out-of-the-money call. This means that to purchase this payoff we need to pay.

As for the plot of the payoff of a bull call spread, excluding the initial payment needed to buy the product, the payoff looks like



What type of investor would buy this payoff? A bull call spread profits when the underlying stock rises in price, but profit is limited as the stock price rises above the strike price K_1, and the loss is also limited as the stock price falls below the strike price K_2. So differently from payoffs like risk reversals, the bull call spread allows for limited gain and limited losses, whichever the model (we know for example that under the Bachelier model a risk reversal can lead to an unlimited loss, potentially). Therefore, as a payoff it will be less risky than a bull or long risk reversal, but at the same time it will allow for less profit in case of strong positive performance of the stock at maturity.

Indeed, the loss is floored after the stock drops below K_2, but the potential profits are also capped as the stock rises above K_1. Hence this contract will be sought by a trader who does not want excessive risk and who is expecting the stock to increase.

The contract may be of interest to a trader with limited funds, because it is less expensive than a call option with strike K_2, as it reduces the price of the call by selling another call with a higher strike K_1. So one finances the purchase of a call with the sale of another call.

meth seen ⇓

10, B

(b) To price this payoff, we just need to price the two call options and substract. Indeed, we know from the Black Scholes formulas that the price of the payoff Y at time 0 is the Black-Scholes price of the call with strike K_2 minus the Black-Scholes price of the call with strike K_1, namely

$$V_0 = S_0\Phi(d_1(K_2)) - K_2 e^{-rT}\Phi(d_2(K_2)) - [S_0\Phi(d_1(K_1)) - K_1 e^{-rT}\Phi(d_2(K_1))]$$

where

$$d_{1,2}(K) = \frac{\ln(S_t/K) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

The sign of the price is positive, $V_0 > 0$. Indeed, the payoff itself is non-negative in every scenario, so its Q-expectation, leading to the price, will be positive. Another way to look at it is to consider that we are taking the difference of two call options with the same maturity, same stock and different strikes. The first option has a lower strike K_2 with $S_0 > K_2$, so the first call option is in-the-money. The call option we subtract or sell, has larger strike K_1 and $S_0 < K_1$, so the second call is out of the money. As a call that is in the money is more valuable than a call that is out of the money, everything else being equal, we deduce that the difference will be positive.

sim. seen $\Downarrow$

(c) To compute the delta, we need to take the partial derivative of V_0 with respect to s_0.

5, A

Recall that the delta at time 0 of a call options with stock S, strike K, maturity T, volatility σ and risk free rate r is given by

$$\Delta_{call}(K) = \Phi(d_1(K)), \quad d_1(K) = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Since our bull call spread (BuCS) is the difference of two call options with strikes K_2 and K_1 respectively, we have

$$\Delta_{BuCS} = \frac{\partial V_0}{\partial s_0} = \frac{\partial(\text{CallPrice}(K_2) - \text{CallPrice}(K_1))}{\partial s_0} =$$
$$= \frac{\partial \text{CallPrice}(K_2)}{\partial s_0} - \frac{\partial \text{CallPrice}(K_1)}{\partial s_0} = \Phi(d_1(K_2)) - \Phi(d_1(K_1)).$$

We can discuss the sign of the Delta to see the pattern of the BuCS price with respect to s_0. To do this, we wish to understand whether the delta of a call option, $\Phi(d_1(K))$, is increasing or decreasing in K, or neither. We know Φ is an increasing function as it is the normal CDF, so the question is whether $d_1(K)$ is increasing, decreasing or neither in K. We can write $d_1(K)$ as

$$d_1(K) = \frac{\ln(S_0) - \ln(K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

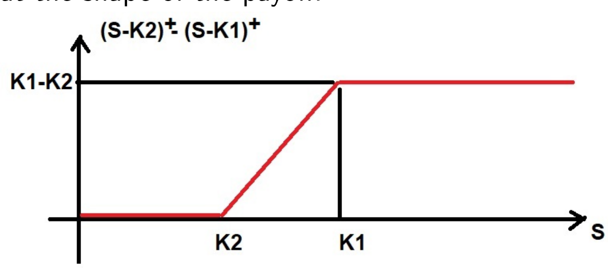
Then

$$\frac{\partial d_1(K)}{\partial K} = \frac{1}{\sigma\sqrt{T}} \frac{\partial(\ln(S_0) - \ln(K) + \left(r + \frac{1}{2}\sigma^2\right)T)}{\partial K} = \frac{1}{\sigma\sqrt{T}}\left(-\frac{1}{K}\right) < 0.$$

If K is positive, this is a negative numer. This means that $d_1(K)$ is decreasing in K. As the total delta is

$$\Delta_{BuCS} = \Phi(d_1(K_2)) - \Phi(d_1(K_1))$$

with $K_2 < K_1$, and the delta is decreasing in K, it follows that the difference is positive. Therefore the delta of a BuCS is positive. This is confirmed by looking at the shape of the payoff.



We see from the picture that if S increases, the value of the payoff increases.

3. (a) We recall the price of a call option in the Bachelier model: unseen $\Downarrow$

10, D

$$V^{Call}_{BaM}(0, s_0, K, T, \sigma) = (s_0 - K)\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right),$$

where ϕ is the pdf of a standard normal. From our previous points, the payoff Y of a butterfly satisfies

$$E^Q[e^{-rT}Y] = E^Q[e^{-rT}(S_T - L)^+] - 2E^Q[e^{-rT}(S_T - M)^+] + E^Q[e^{-rT}(S_T - H)^+]$$

or, if S follows the Bachelier model,

$$V^{Butter}_{BaM} = V^{Call}_{BaM}(K = L) - 2V^{Call}_{BaM}(K = M) + V^{Call}_{BaM}(K = H), \tag{3}$$

or, recalling $r = 0$ and $L = S_0 - X,\ M = S_0,\ H = S_0 + X$

$$V^{Butter}_{BaM} = X\Phi\left(\frac{X}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{X}{\sigma\sqrt{T}}\right)$$

$$-2\sigma\sqrt{T}\phi\,(0) +$$

$$-X\Phi\left(\frac{-X}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{-X}{\sigma\sqrt{T}}\right)$$

Note that $\phi(0) = 1/\sqrt{2\pi}$.
Recalling that $\Phi(-x) = 1 - \Phi(x)$ and that $\phi(-x) = \phi(x)$ we get

$$V^{Butter}_{BaM} = X\left(2\Phi\left(\frac{X}{\sigma\sqrt{T}}\right) - 1\right) - 2\sigma\sqrt{T}\left(\phi(0) - \phi\left(\frac{X}{\sigma\sqrt{T}}\right)\right) \tag{4}$$

Note that, as X is positive, $\Phi\left(\frac{X}{\sigma\sqrt{T}}\right)$ is calculated in a positive point and is therefore larger than $\frac{1}{2}$, since $\Phi(0) = 1/2$ and Φ is strictly increasing. It follows that $2\Phi\left(\frac{X}{\sigma\sqrt{T}}\right) - 1$ is larger than one, or that the first term in the V_{BaM} formula above is positive.

As concerns ϕ, the Gaussian standard pdf, this has a maximum in 0, so that the second term in the V_{BaM} formula above being subtracted is also positive. The price is thus the difference of two positive terms related to the Gaussian CDF and PDF respectively:

$$V^{Butter}_{BaM} = \underbrace{X\left(2\Phi\left(\frac{X}{\sigma\sqrt{T}}\right) - 1\right)}_{positive} - \underbrace{2\sigma\sqrt{T}\left(\phi(0) - \phi\left(\frac{X}{\sigma\sqrt{T}}\right)\right)}_{positive}.$$

unseen $\Downarrow$

6, D

(b) Careful here. You might be tempted to differentiate directly Formula (4) with respect to s_0 and conclude that the delta is 0, as Eq. (4) does not depend on s_0. However, as explained in the lectures for Black Scholes and as seen in some previous mock exams, if the strike depends on s_0, to calculate the delta you need to differentiate the price with a general K and susbstitute the specific K that is a function of s_0 after you differentiate, not before. This is because we want the sensitivity to s_0 as initial stock and not the joint sensitivity to s_0 and K. Thus, the equation you need to differentiate with respect to s_0 is (3), leaving the strikes L, M, H as generic and not as functions of s_0 yet.

We recall the delta for a call with strike K in the Bachelier model, if you don't remember it derive it as in the lecture notes.

$$\frac{\partial V_{BaM}(0)}{\partial s_0} = \frac{\partial\left((s_0-K)\Phi\left(\frac{s_0-K}{\sigma\sqrt{T}}\right)\right)}{\partial s_0} + \sigma\sqrt{T}\frac{\partial\, p_N\left(\frac{s_0-K}{\sigma\sqrt{T}}\right)}{\partial s_0} = \Phi\left(\frac{s_0-K}{\sigma\sqrt{T}}\right).$$

The butterfly spread payoff is call with strike L minus 2 calls with strike M plus a call with strike H. The delta is thus obtained by differentiating the three call prices in Eq (3) and is

$$\frac{\partial V_{BaM}^{Butter}}{\partial s_0} = \Phi\left(\frac{s_0-K}{\sigma\sqrt{T}}\right)|_{K=L} - 2\Phi\left(\frac{s_0-K}{\sigma\sqrt{T}}\right)|_{K=M} + \Phi\left(\frac{s_0-K}{\sigma\sqrt{T}}\right)|_{K=H} =$$

$$\Phi\left(\frac{s_0-L}{\sigma\sqrt{T}}\right) - 2\Phi\left(\frac{s_0-s_0}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{s_0-H}{\sigma\sqrt{T}}\right)$$

$$= \Phi\left(\frac{s_0-(s_0-X)}{\sigma\sqrt{T}}\right) - 2\Phi(0) + \Phi\left(\frac{s_0-(s_0+X)}{\sigma\sqrt{T}}\right)$$

$$= \Phi\left(\frac{X}{\sigma\sqrt{T}}\right) - 2\frac{1}{2} + \Phi\left(\frac{-X}{\sigma\sqrt{T}}\right)$$

$$= \Phi\left(\frac{X}{\sigma\sqrt{T}}\right) - 1 + 1 - \Phi\left(\frac{X}{\sigma\sqrt{T}}\right) = 0$$

where we used $\Phi(-x) = 1 - \Phi(x)$. So we conclude the delta is zero:

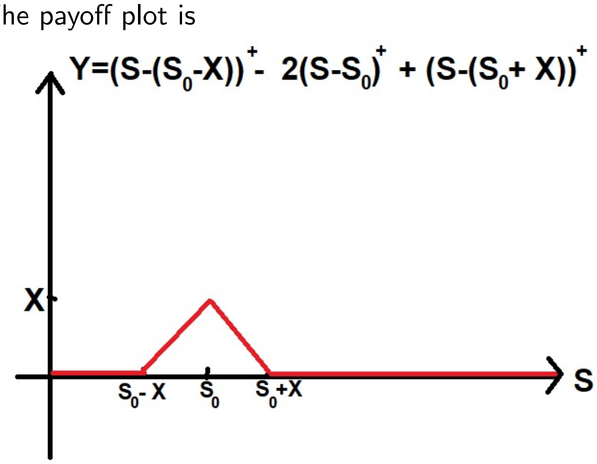
$$\frac{\partial}{\partial S_0} V_{BaM}^{Butter} = 0.$$

We obtained the same result as we would obtained with the wrong method, namely by differentiating Eq. (4), but this is a coincidence. You need to avoid differentiating Eq. (4) directly, even if in this case you would have obtained the same result.

Zero delta means that the butterfly has always the same price in Bachelier, regardless of the initial stock price S_0. This is due to the special dynamics of the model and to the specific choice of strikes $L = S_0 - X$ and $H = S_0 + X$.

$\boxed{\text{unseen} \Downarrow}$

(c) The payoff plot is $\boxed{4, \text{C}}$



Looking at the payoff we see that when $X \downarrow 0$ the payoff tends to be zero everywhere. As such, it will be worth 0 in terms of initial price.

The Bachelier prices are continuous in X around $X = 0$, so we can set $X = 0$ direcly in the formulas to compute the limit for $X \downarrow 0$.

$$\begin{aligned}
V_{BaM}^{Butter}(X=0) &= 0\left(2\Phi\left(\frac{0}{\sigma\sqrt{T}}\right)-1\right) - 2\sigma\sqrt{T}\left(\phi(0) - \phi\left(\frac{0}{\sigma\sqrt{T}}\right)\right) \\
&= (1-1) - 2\sigma\sqrt{T}(\phi(0)-\phi(0)) = 0
\end{aligned}$$

again, as expected from the payoff analysis.

4. (a) We define value at Risk (VaR). VaR is related to the potential loss on our portfolio over the time horizon h. Define this loss L_h as the difference between the value of the portfolio today (time 0) and in the future h. seen ⇓ 6, A

$$L_h = \text{Portfolio}_0 - \text{Portfolio}_h.$$

VaR with horizon h and confidence level α is defined as that number $q = q_{h,\alpha}$ such that

$$P[L_h < q] = \alpha$$

so that our loss at time h is smaller than q with P-probability α. In words, it's the level of loss over a period h that will not be exceeded with confidence α. meth seen ⇓

(b) Let us analyze the three positions and in particular their value at time h, where we have to assess VaR. A zero coupon bond with maturity U on a notional N promises to pay the notional N at time U. Its value at time $h < U$ is obtained by risk neutral pricing as 6, B

$$E_h^Q[e^{-r(U-h)}N] = e^{-r(U-h)}N$$

where we could take away the expectation as there is nothing random in the payoff or in the discount rate r.

A similar approach leads to the price $-e^{-r(T-h)}N$ for the short bond position, where the minus sign is due to the short position.

The value of the stock at time h is simply MS_h, namely the amount of stock we hold times the price of the stock at time h.

Putting all terms together the value of the portfolio is

The value of the portfolio at time h is

$$V_h = Ne^{-r(U-h)} - Ne^{-r(T-h)} + MS_h.$$

The value of the portfolio at time 0 is instead, trivially,

$$V_0 = Ne^{-rU} - Ne^{-rT} + MS_0.$$

The loss of the portfolio over the time h is

$$L_h = V_0 - V_h = Ne^{-rU} - Ne^{-rT} + MS_0 - Ne^{-r(U-h)} + Ne^{-r(T-h)} - MS_h.$$

We can set

$$K = Ne^{-rU} - Ne^{-rT} + MS_0 - Ne^{-r(U-h)} + Ne^{-r(T-h)}$$

and rewrite the loss as

$$L_h = K - MS_h.$$

The only random term here is S_h, which in the Black Scholes model is written, under the measure P, as

$$S_h = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)h + \sigma W_h\right)$$

where as usual we recall that $W_h \sim \mathcal{N}(0,h) \sim \sqrt{h}\mathcal{N}(0,1) \sim \sqrt{h}\mathcal{N}$ where we abbreviate $\mathcal{N} = \mathcal{N}(0,1)$. To compute $q_{h,\alpha} = VaR_{h,\alpha}$ we need to find the percentile such that

$$\mathbb{P}\{L_h < q_{h,\alpha}\} = \alpha.$$

Write $q = q_{h,\alpha}$ for brevity, and calculate

$$\mathbb{P}\{L_h < q\} = P\{K - MS_h < q\} = P\left\{S_h > \frac{K-q}{M}\right\}$$

$$= P\left\{S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)h + \sigma\sqrt{h}\mathcal{N}\right) > \frac{K-q}{M}\right\}$$

$$= P\left\{\mathcal{N} > \frac{\ln\frac{K-q}{MS_0} - \left(\mu - \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}}\right\}$$

$$= 1 - \Phi\left(\frac{\ln\frac{K-q}{MS_0} - \left(\mu - \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}}\right)$$

$$= \Phi\left(-\frac{\ln\frac{K-q}{MS_0} - \left(\mu - \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}}\right).$$

Thus the equation

$$\mathbb{P}\{L_h < q_{h,\alpha}\} = \alpha$$

becomes

$$\Phi\left(-\frac{\ln\frac{K-q}{MS_0} - \left(\mu - \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}}\right) = \alpha$$

or

$$-\frac{\ln\frac{K-q}{MS_0} - \left(\mu - \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}} = \Phi^{-1}(\alpha)$$

from which we can solve in q, obtaining

$$q = VaR_{h,\alpha} = K - MS_0 \exp\left(-\sigma\sqrt{h}\Phi^{-1}(\alpha) + \left(\mu - \frac{1}{2}\sigma^2\right)h\right)$$

$\boxed{\text{seen} \Downarrow}$

(c) VaR may violate subadditive behaviour, meaning that the VaR of the sum of two portfolios may happen to be larger than the sum of the single VaRs of each portfolio, violating the principle of diversifications. ES is a risk measure that is subadditive and respects the principle of diversification. $\boxed{4, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(d) No the dynamics is not the same, to price an option we need to use the risk neutral dynamics, where the drift parameter μ of S is replaced by the risk free rate $r = 0$ of the bank account. So to price an option we need to use the dynamics under Q, $dS_t = \sigma dW^Q$. To compute value at risk or expected shortfall the dynamics that is relevant up to the risk horizon is the dynamics under P, i.e. the dynamics with drift μ. $\boxed{4, \text{B}}$

5. (a) The Barings collapse was caused by a rogue trader named Leeson, operating from the Singapore Branch. Leason had been trading massively in futures and masking his losses using accounting tricks. At some point, to recoup on the losses, he sold a straddle, an option strategy that allowed him to cash in the option premium at time 0 and hoping the payoff at maturity would be very small, so that he would keep the initial sale profit without paying much at maturity. However, the market moved in a way he didn't expect and the straddle strategy backfired, causing further losses. $\boxed{5, \text{M}}$

$\boxed{5, \text{M}}$

(b) The option combination is a short straddle, consisting of selling a call and a put option on the same asset with the same maturity and the same strike. The plot is entirely negative and is a reversed V shaped plot with the vertex on the S_T axis at the point K.

The idea is that by selling this combination of options we cash in the price of the call and the price of the put at time 0 and hope the stock does not move. The more the stock moves, one way or the other, whether decreasing (activating the put payoff) or increasing (activating the call payoff), the more we have to pay at maturity, so we hope the stock doesn't move and we keep the cash made with the sale at time 0.

$\boxed{5, \text{M}}$

(c) An earthquake hit the Asian markets where the underlying asset was based, so the underlying asset moved a lot, causing the payoff to be paid to become very large and causing big losses in the straddle position. This compounded the crisis instead of helping, and the bank finally defaulted.

$\boxed{5, \text{M}}$

(d) Value at Risk was introduced to tackle situations like Barings. Every traded portfolio risk has to be measured with VaR. VaR is related to the potential loss on our portfolio over the time horizon h. Define this loss L_h as the difference between the value of the portfolio today (time 0) and in the future h.

$$L_h = \text{Portfolio}_0 - \text{Portfolio}_h.$$

VaR with horizon h and confidence level α is defined as that number $q = q_{h,\alpha}$ such that

$$P[L_h < q] = \alpha$$

so that our loss at time h is smaller than q with P-probability α. In words, it's the level of loss over a period h that will not be exceeded with confidence α. Leeson's positions VaR would have been to high to allow Leeson to take further positions and so would have avoided him reaching a level of losses that become impossible to recover.

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70130 Stochastic Differential Equations in Financial Modelling Markers Comments

Question 1	Most students did the problem quite well. Most common mistakes are: for the existence of the solution, forgetting that the initial condition must be independent of the Brownian filtration; not applying the triangular inequality in one of the proofs of Lipschitz continuity; Forgetting to say why the SDE coefficients are measurable. Almost everyone solved the SDE correctly and almost everyone made the correct calculation for the mean solution through the ODE, although at times mentioning a somewhat wrong reason for the zero mean of the Ito integral or omitting Fubini's theorem mention. A few students made a mistake in the very basic ODE solution. Overall it went quite well though.
Question 2	This question was answered very well by most students. Many students received full marks and most reductions in points were minor and due to imprecise statements / missing details in (a) and (c).
Question 3	This question was quite standard, and most candidates performed well overall. For parts (a) and (b), some candidates incorrectly recalled the formulae for the price and delta of the option or unnecessarily spent significant time deriving them from scratch instead of directly referring to the lecture notes. Additionally, a few students neglected to comment on the sign of the delta they computed, which was required for full marks. In part (c), a small number of students failed to explicitly 'deduce the price' as instructed by the question. Even though their analysis of the payoff behaviour was correct, omitting this step led to the loss of one mark.
Question 4	The question was generally answered well. There were a few cases of confusion regarding what subadditivity means and also relatively common arithmetic slips in the derivation of value at risk for the portfolio (which were understandable given the length of the computation, hence treated leniently). Beyond subadditivity, the inability to see the tail of the loss distribution beyond the confidence level was also accepted as a shortcoming of VaR that is solved by ES (although one can debate whether that issue is "completely" solved by ES).

Question 5 Most students gave a satisfactory answer ranging in the distinction interval (at or above 14). Few students noticed that the Leeson's straddle was at the money, $K=S_0$, to gamble on the market not moving, that is necessary. In the lecture notes this was hidden in the numerical example where both K and S_0 where the same, but it was not caught up by most students. Other common mistakes include not mentioning that VaR has also an holding period, but only he confidence level, and not mentioning that ES was introduced after VaR. Some students actually only talked about ES skipping VaR altogether, which is not the response the industry gave at the time. Some students went in more detail on the Barings history but most got the essentials right. Overall a good performance for this question with very few exceptions and an empty booklet.