Imperial College London

MATH50007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Network Science

Date: 17 May 2023

Time: 10:00 – 12:00 (BST)

Time Allowed: 2 hrs

This paper has 4 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1.

(a) Consider a "square graph", a simple connected graph with four nodes and four links where each node has degree two. Number the nodes from 1 to 4, and assume that nodes 1 and 4 are not linked.

(i) What is the adjacency matrix for the graph? (4 marks)

(ii) Consider a partition of the graph into two parts defined by the vector, $\mathbf{s} = [1, 1, -1, -1]^T$, where nodes 1 and 2 are placed in one set, and nodes 3 and 4 are placed in the other. Laplacian graph partitioning is based on the observation that the cut size, c, can be computed using $\mathbf{s}$ and the Laplacian matrix, $\mathbf{L}$. Use $\mathbf{L}$ and the given $\mathbf{s}$ to compute the cut size for the stated partition.
(6 marks)

(b) Let G be a simple, connected N-node graph. Consider the partition of G into N 1-node communities. Derive an expression for the modularity of this partition. Give your answer in terms of N, the average degree, $\overline{k}$, and the 2nd-moment of the degree distribution, $\overline{k^2}$. (5 marks)

(c) The Katz centrality is generally considered to be superior to the eigenvector centrality for directed graphs. Provide a 3-node directed graph (with no multiedges or self-loops) which illustrates this superiority, and provide a brief explanation of how your graph shows that the Katz centrality is better than the eigenvector centrality for directed graphs. (5 marks)

(Total: 20 marks)

2. Consider the following model for two diseases spreading on a simple connected N-node network with adjacency matrix, $\mathbf{A}$. There are two diseases, X and Y. If node i has disease X at time t, then $x_i(t) = 1$ and $x_i(t) = 0$ otherwise. If node i has disease Y at time t, then $y_i(t) = 1$ and $y_i(t) = 0$ otherwise. If $x_i = y_i = 0$, the node is *healthy*, and a node can have both diseases where $x_i(t) = y_i(t) = 1$.

For a timespan of length Δt: let $\beta\Delta t$ be the probability that a healthy node acquires X via a link to a node infected with X, and let $\gamma\Delta t$ be the probability that a node infected only with Y acquires X via a link to a node infected with X.

(a) Show that $P(x_i = 0, y_i = 0, x_j = 1) = \langle (1 - x_i)(1 - y_i)x_j \rangle$ (4 marks)

(b) The master equation for x_i takes the following form:

$$P(x_i(t + \Delta t) = 1) = P(x_i(t) = 1) + \Delta t \sum_{j=1}^{N} A_{ij} \left[\beta T_1 + \gamma T_2\right] + \mathcal{O}(\Delta t^2),\ i = 1, 2, ..., N. \quad (1)$$

Provide a clear, concise explanation of why $T_1 = P(x_i(t) = 0, y_i(t) = 0, x_j(t) = 1)$, and determine what T_2 should be. (6 marks)

(c)

(i) Apply the limit $\Delta t \to 0$ to (1) to derive the following equation:

$$\frac{d\langle x_i \rangle}{dt} = \sum_{j=1}^{N} A_{ij} \left(\beta \langle T_3 \rangle\ + \gamma \langle T_4 \rangle\right), \quad (2)$$

and explain what T_3 and T_4 should be in terms of one or more of x_i, x_j, y_i, and y_j. (7 marks)

(ii) Using your solution to (i), show how (2) simplifies when $\beta = \gamma$, and provide an interpretation of this simplified result based on the model description. (3 marks)

(Total: 20 marks)

3. Consider the following system of N differential equations:

$$\frac{dx_i}{dt} = -\sum_{j=1}^{N} \frac{k_j}{k_i} L_{ij} x_j, \quad i = 1, 2, ..., N, \tag{3}$$

where x_i is the i^{th} element of $\mathbf{x} \in \mathbb{R}^N$, L_{ij} corresponds to the Laplacian matrix for a simple connected graph with N nodes, and k_i is the degree of the i^{th} node in the graph. The initial condition is $\mathbf{x}(t = 0) = \mathbf{x}_0$. You should assume that a complete set of mutually orthogonal eigenvectors for the Laplacian matrix, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, along with their corresponding eigenvalues are given.

(a) Show that for a complete graph, (3) simplifies to the graph diffusion equation with diffusivity, $\alpha = 1$. (3 marks)

(b) Show that $x_i = \frac{c}{k_i}$, $i = 1, 2, ..., N$ is an equilibrium solution to (3). Here, c is an arbitrary non-zero constant. (5 marks)

(c) Consider solutions to (3) for an initial condition which is *not* an equilibrium solution and whose elements are all non-negative but is otherwise arbitrary.

(i) Explain how to construct a matrix $\mathbf{M}$ so that the transformation $\mathbf{u} = \mathbf{M}\mathbf{x}$ results in a system of ODEs of the form,

$$\frac{d\mathbf{u}}{dt} = -\mathbf{\Lambda}\mathbf{u}.$$

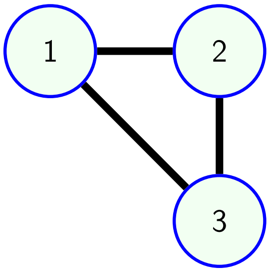
where $\mathbf{\Lambda}$ is a diagonal matrix. (4 marks)

(ii) Show that in the limit $t \to \infty$, $\mathbf{x} \to \mathbf{x}_{eq}$ where $\mathbf{x}_{eq}$ is an equilibrium solution to (3) with at least one non-zero element.

(8 marks)

(Total: 20 marks)

4. The triangle graph shown is referenced in the questions below.



(a) Consider the degree sequence, $d = \{2, 2, 2\}$. What is the probability of the triangle graph being generated by the configuration model with degree sequence, d? (5 marks)

(b) Consider the following modified version of the Barabasi-Albert model. The initial graph is a 3-node triangle graph. Each iteration, a new triangle is introduced, and a link is added connecting an arbitrary node in the new triangle with a node in the existing graph. The node in the existing graph is selected using the usual linear preferential attachment model. After t iterations, a graph will have $N(t) = 3t + 3$ nodes and $L(t) = 4t + 3$ links. Derive an equation relating the expected number of nodes with degree 2 at iteration $t+1$, $\langle N_2(t+1)\rangle$, to one or more of: $\langle N_1(t)\rangle$, $\langle N_2(t)\rangle$, and $L(t)$.

(7 marks)

(c) Let $\mathbf{A}$ be the adjacency matrix of an N-node simple graph, and let λ_i be the ith eigenvalue of $\mathbf{A}$ with $i \in \{1, 2, ..., N\}$. Provide a careful derivation of an expression relating the sum $S = \sum_{i=1}^{N} \lambda_i^3$ to the number of triangles in the graph. Note that the triangle graph shown above contains one triangle. (8 marks)

(Total: 20 marks)

Information sheet to be provided with Final Exam
Network Science 2023

Adjacency matrix: A_{ij} is the number of links from node j to node i

Simple graphs: A graph is *simple* if it is undirected, unweighted, and does not have multiedges or self-loops.

Complete graph: A simple graph where each distinct pair of nodes is linked.

Graph Laplacian (also called the *Laplacian matrix*): $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where $\mathbf{D}$ is the *diagonal degree matrix* for the graph, $D_{ii} = k_i$, k_i is the degree of node i, and $L_{ij} = \delta_{ij} k_j - A_{ij}$.

Centralities and similarities

- *eigenvector centrality*: $x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j$
- *Katz centrality*: $x_i = \alpha \sum_{j=1}^{N} (A_{ij} x_j) + 1$
- *PageRank centrality*: $x_i = \sum_{j=1}^{N} \left[(1-m) \frac{A_{ij}}{\max(k_j^{out},1)} x_j + m \frac{x_j}{N} \right]$
- *cosine similarity*: $\sigma_{ij} = \frac{n_{ij}}{\sqrt{k_i k_j}}$
- *Jaccard similarity*: $\sigma_{ij} = \frac{n_{ij}}{k_i + k_j - n_{ij}}$

Matrix resolvent: $R(\mathbf{M}; \mu)$ is the *resolvent* for a square N x N matrix, $\mathbf{M}$. The resolvent is defined as $R = (\mu \mathbf{I} - \mathbf{M})^{-1}$ for μ where $\mu \neq \lambda_i,\ i = 1, 2, ..., N$. Here λ_i is the i^{th} eigenvalue of $\mathbf{M}$

- If $|\mu| > \rho(\mathbf{M})$, then $R(\mathbf{M}; \mu) = \sum_{l=0}^{\infty} \frac{\mathbf{M}^l}{\mu^{l+1}}$
- $\rho(\mathbf{M})$ is the *spectral radius* of $\mathbf{M}$: $\rho(\mathbf{M}) = \max \{|\lambda_1|, |\lambda_2|, ..., |\lambda_N|\}$

Perron-Frobenius theorem: We have applied the Perron-Frobenius (P-F) theorem to three different classes of real square matrices:

1. *Positive matrices* where each element of the matrix is positive, $\mathbf{B} > 0$. Then, the theorem tells us that there is a real positive eigenvalue λ where:
   - $\lambda = \rho(\mathbf{B}) > 0$, and all other eigenvalues are smaller in magnitude.
   - This eigenvalue is simple, all elements of the corresponding eigenvector have the same sign, and there are no other eigenvectors where all elements have the same sign
2. *Irreducible matrices* Let $B_{ij} > 0$ if there is a link in a graph from node i to node j with $B_{ij} = 0$ otherwise. Then $\mathbf{B}$ is irreducible if and only if the corresponding graph is strongly connected (i.e. every node is reachable from every other node). For irreducible matrices, there is a real, positive eigenvalue λ where:
   - $\lambda = \rho(\mathbf{B}) > 0$, and this eigenvalue is simple
   - All elements of the corresponding eigenvector have the same sign, and there are no other eigenvectors where all elements have the same sign
     - There may be other eigenvalues equal in magnitude to λ
3. *Non-negative matrices* where each element of the matrix is non-negative: $\mathbf{B} \geq 0$. There is a real, non-negative eigenvalue λ where:
   - $\lambda = \rho(\mathbf{B}) \geq 0$, and there may be other eigenvalues equal in value or equal in magnitude
   - All non-zero elements of the corresponding eigenvector will have the same sign, and there may be other eigenvectors with the same property
   - Note: This version of the P-F theorem is considerably weaker than the other 2

Markov's inequality: Let X be a random variable that assumes only non-negative values. Then, for all $a > 0$, $P(X \geq a) \leq \frac{\langle X \rangle}{a}$.

Chebyshev's inequality: Let X be a random variable with finite expected value, μ, and finite non-zero variance, σ^2. Then, for any $a > 0$, $P\left(|X - \mu| \geq a\right) \leq \frac{\sigma^2}{a^2}$.

The **configuration model** requires the specification of an N-element *degree sequence*: $\{k_1, k_2, ..., k_N\}$ with $k_i > 0$ for all i. The probability of two stubs being linked is $\frac{1}{K-1}$,
$K = \sum_{i=1}^{N} k_i$ is the **total degree**.

Average degree: $\overline{k} = \frac{1}{N} \sum_{i=1}^{N} k_i$.

Second moment of the degree distribution: $\overline{k^2} = \frac{1}{N} \sum_{i=1}^{N} k_i^2 = \sum_{k=1}^{k_{max}} p_k k^2$.

Preferential attachment: $\rho_i(G_a(t))$: probability node i in graph $G_a(t)$ receives a link. For linear preferential attachment: $\rho_i(G_a(t)) = \frac{k_i(G_a(t))}{K(t)}$

Graph diffusion equation: $\frac{d\langle \mathbf{n} \rangle}{dt} = -\alpha \mathbf{L} \langle \mathbf{n} \rangle$.

Fick's law on graphs: $\langle j_{ab} \rangle = -\alpha(\langle n_a \rangle - \langle n_b \rangle)$; n_a is the number of particles on node a.

Orthogonal diagonalization: A square real matrix, $\mathbf{M}$, is orthogonally diagonalizable if and only if $\mathbf{M}$ is symmetric. Then, $\mathbf{M} = \mathbf{V} \boldsymbol{\Lambda} \mathbf{V}^T$.

Rayleigh quotient: For a symmetric matrix $\mathbf{M}$, $r(\mathbf{M}, \mathbf{x}) = \frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$ is maximized when $\mathbf{x} = \mathbf{v}_1$ in which case $r = \lambda_1$. ($\mathbf{M}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, $\lambda_1 \geq \lambda_2 \geq ...$)

Gershgorin's theorem: Let $\mathbf{B} \in \mathbb{C}^{N \times N}$ and suppose that $\mathbf{X}^{-1}\mathbf{B}\mathbf{X} = \mathbf{H} + \mathbf{F}$, where $\mathbf{H}$ is diagonal and $\mathbf{F}$ has zeros on its main diagonal. Then the eigenvalues of $\mathbf{B}$ lie on the union of the discs $\Delta_1, \Delta_2, \ldots, \Delta_N$, where $\Delta_i = \{l \in \mathbb{C} : |l - H_{ii}| \leq \sum_{j=1}^{N} |F_{ij}|\}$.

Random walks on graphs: $T_{ij} = \frac{A_{ij}}{k_i}$; $\mathbf{T}$ is the *transition matrix*, and T_{ij} is the probability that a walker takes a step from node i to node j on a simple graph.

Network-SI model: $\frac{d\langle x_i \rangle}{dt} = \beta \sum_{j=1}^{N} A_{ij} \left\langle (1 - x_i)\, x_j \right\rangle$.

Degree-based approximation: $\frac{d\phi_k}{dt} = k\beta\,(1 - \phi_k) \sum_{k'=1}^{k'_{max}} \theta(k, k') \phi_{k'-1}$

Second-moment equation: $\frac{d\langle s_i x_j \rangle}{dt} = \beta \sum_{l=1}^{N} \left(A_{jl} \langle s_i s_j x_l \rangle - A_{il} \langle s_i x_j x_l \rangle\right)$.

Modularity: The modularity of a set of nodes, S_a, is $M_a = \frac{1}{2L} \sum_{i \in S_a} \sum_{j \in S_a} \left(A_{ij} - \frac{k_i k_j}{2L}\right)$

The *modularity matrix* $\mathbf{B}$ is defined using, $B_{ij} = A_{ij} - \frac{k_i k_j}{2L}$.

Spectral modularity maximization: Find $\tilde{\mathbf{s}}$ such that $\tilde{\mathbf{s}}^T \mathbf{B} \tilde{\mathbf{s}}$ is maximized with $|\tilde{\mathbf{s}}|^2 = N$.

Cut size: For a partition that breaks a simple connected graph into two disjoint groups of nodes, the cut size, c, is the number of links crossing from one group to another, and $c = \frac{1}{4} \mathbf{s}^T \mathbf{L} \mathbf{s}$.

Laplacian graph partitioning: Find $\tilde{\mathbf{s}}$ such that $\tilde{c} = \tilde{\mathbf{s}}^T \mathbf{L} \tilde{\mathbf{s}}$ is minimized with $|\tilde{\mathbf{s}}|^2 = N$.

Weighted graphs: $\mathbf{W}$ is the *weight matrix*; $W_{ij} \neq 0$ if there is a link from j to i and is 0 otherwise.

Weighted normalized Laplacian: $\hat{\mathbf{L}} = \hat{\mathbf{D}}^{-1/2} (\hat{\mathbf{D}} - \mathbf{W}) \hat{\mathbf{D}}^{-1/2}$. $\hat{\mathbf{D}}$ is a diagonal matrix with, $\hat{D}_{ii} = \sum_{j=1}^{N} W_{ij}$

Weighted normalized cut size: $\hat{\xi} = \frac{1}{4} \left(\frac{1}{\hat{K}_a} + \frac{1}{\hat{K}_b}\right) \sum_{i=1}^{N} \sum_{j=1}^{N} W_{ij}\,(1 - s_i s_j)$, $\hat{K}_a = \sum_{i \in S_a} \sum_{j=1}^{N} W_{ij}$.

Image segmentation: Find $\tilde{\mathbf{s}} \in \mathbb{R}^N$ such that $\frac{1}{4} \left(\frac{1}{\hat{K}_a} + \frac{1}{\hat{K}_b}\right) \sum_{i=1}^{N} \sum_{j=1}^{N} W_{ij}\,(1 - \tilde{s}_i \tilde{s}_j)$ is minimized with $|\tilde{\mathbf{s}}|^2 = N$.

Spectral clustering requires consideration of the eigenvalues and eigenvectors of $\hat{\mathbf{A}} = \hat{\mathbf{D}}^{-1/2} \mathbf{W} \hat{\mathbf{D}}^{-1/2}$

1. (a) Consider a "square graph", a simple connected graph with four nodes and four links where each node has degree two. Number the nodes from 1 to 4, and assume that nodes 1 and 4 are not linked. sim. seen ⇓

(i) What is the adjacency matrix for the graph? 4, A

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

There is a link between two nodes when $A_{ij} = 1$. This is the only possible graph which satisfies the question requirements.

(ii) Consider a partition of the graph into two parts defined by the vector, $\mathbf{s} = [1, 1, -1, -1]^T$, where nodes 1 and 2 are placed in one set, and nodes 3 and 4 are placed in the other. Laplacian graph partitioning is based on the observation that the cut size, c, can be computed using $\mathbf{s}$ and the Laplacian matrix, $\mathbf{L}$. Use $\mathbf{L}$ and the given $\mathbf{s}$ to compute the cut size for the stated partition. 6, A

Solution: $c = \frac{\mathbf{s}^T\mathbf{L}\mathbf{s}}{4}$, and $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where $\mathbf{D}$ is the diagonal degree matrix for the graph ($\mathbf{D} = 2\mathbf{I}$ for this graph). $\mathbf{A}\mathbf{s} = [0, 0, 0, 0]^T$, so $\mathbf{s}^T\mathbf{A}\mathbf{s} = 0$. $\mathbf{D}\mathbf{s} = [2, 2, -2, -2]^T$, and $\mathbf{s}^T\mathbf{D}\mathbf{s} = 8$, so $c = (8 - 0)/4 = 2$ (which can be easily verified by inspection of the graph).

sim. seen ⇓

(b) Let G be a simple, connected N-node graph. Consider the partition of G into N 1-node communities. Derive an expression for the modularity of this partition. Give your answer in terms of N, the average degree, $\overline{k}$, and the 2nd-moment of the degree distribution, $\overline{k^2}$. 5, B

Solution: Note that $\overline{k} = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{K}{N}$ and $\overline{k^2} = \frac{1}{N}\sum_{i=1}^{N} k_i^2$. The modularity of set i is $M_i = \frac{1}{2L}\left(2L_i + \frac{K_i^2}{2L}\right)$ where L is the total number of links in the graph, L_i is the number of links connecting nodes within the set and K_i is the number of stubs attached to nodes within the set. Here, $L_i = 0$, and $K_i = k_i$, the degree of node i. Then, $M_i = -\frac{k_i^2}{4L^2}$, and the modularity of the partition is,

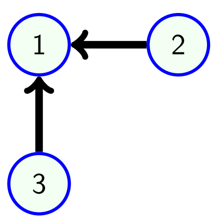
$$M = \sum_{i=1}^{N} M_i = -\frac{1}{4L^2}\sum_{i=1}^{N} k_i^2 = -\frac{N}{4L^2}\overline{k^2}.$$

Now, $K = 2L$, so $M = -N\frac{\overline{k^2}}{K^2} = -\frac{\overline{k^2}}{N\overline{k}^2}$.

sim. seen ⇓

(c) The Katz centrality is generally considered to be superior to the eigenvector centrality for directed graphs. Provide a 3-node directed graph (with no multiedges or self-loops) which illustrates this superiority, and provide a brief explanation of how your graph shows that the Katz centrality is better than the eigenvector centrality for directed graphs. 5, C

Solution:



The eigenvector centrality of node i is $x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j$ where $A_{ij} = 1$ if there is a link from node j to node i and is zero otherwise. Nodes 3 and 2 do not have any in-links, so $x_2 = x_3 = 0$. The centralities of node 1's neighbors are both zero, so $x_1 = 0$ despite node 1 having two in-links. With the Katz centrality, $x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j + 1$, and $x_2 = x_3 = 1$. Then, $x_1 = 2\alpha + 1$. Node 1 now has a larger centrality than nodes 2 and 3 (provided $\alpha > 0$) as expected. (Other examples are possible.)

2. Consider the following model for two diseases spreading on a simple connected N-node network with adjacency matrix, $\mathbf{A}$. There are two diseases, X and Y. If node i has disease X at time t, then $x_i(t) = 1$ and $x_i(t) = 0$ otherwise. If node i has disease Y at time t, then $y_i(t) = 1$ and $y_i(t) = 0$ otherwise. If $x_i = y_i = 0$, the node is *healthy*, and a node can have both diseases where $x_i(t) = y_i(t) = 1$.

For a timespan of length Δt, let $\beta\Delta t$ be the probability that a healthy node acquires X via a link to a node infected with X and let $\gamma\Delta t$ be the probability that a node infected only with Y acquires X via a link to a node infected with X.

sim. seen ⇓

(a) Show that $P(x_i = 0, y_i = 0, x_j = 1) = \langle(1 - x_i)(1 - y_i)x_j\rangle$ 4, A

Solution: Let $T = (1 - x_i)(1 - y_i)x_j$. $(1 - x_i)$, $(1 - y_i)$, and x_j can each be either 0 or 1. So, T can also only be 0 or 1. Using the definition of the expectation, $\langle T\rangle = P(T = 1) * 1 + P(T = 0) * 0 = P(T = 1)$. The only way T can be equal to one is if $(1 - x_i) = 1$ and $(1 - y_i) = 1$ and $x_j = 1$, so $\langle(1 - x_i)(1 - y_i)x_j\rangle = P(x_i = 0, y_i = 0, x_j = 1)$.

sim. seen ⇓

(b) The master equation for x_i takes the following form:

$$P(x_i(t + \Delta t) = 1) = P(x_i(t) = 1) + \Delta t \sum_{j=1}^{N} A_{ij} [\beta T_1 + \gamma T_2] + \mathcal{O}(\Delta t^2), \;\; i = 1, 2, ..., N. \tag{1}$$

Provide a clear, concise explanation of why $T_1 = P(x_i(t) = 0, y_i(t) = 0, x_j(t) = 1)$, and determine what T_2 should be. 3, A

3, B

Solution: The product $\beta\Delta t T_1$ should be:

sim. seen ⇓

$$\text{(probability of transmission via the link } (i, j) \text{ given } i \text{ is healthy and } j \text{ has X)} *$$
$$\text{(probability } i \text{ is healthy and } j \text{ has X)}\,.$$

The first term in the product is $\beta\Delta t$, and the the second term is $P(x_i = 0, y_i = 0, x_j = 1)$ based on the definition of the state variables. Using the same reasoning and the definition of γ, T_2 should be:

$$\text{(probability } i \text{ has Y but not X and } j \text{ has X)}\,.$$

So, $T_2 = P(x_i = 0, y_i = 1, x_j = 1)$. (All quantities here are evaluated at time t).
(3 marks for explanation of T_1 and 3 marks for derivation of T_2; 6 points in total for (b).)

sim. seen ⇓

(c) (i) Apply the limit $\Delta t \to 0$ to (1) to derive the following equation:

$$\frac{d\langle x_i\rangle}{dt} = \sum_{j=1}^{N} A_{ij} \left(\beta\langle T_3\rangle + \gamma\langle T_4\rangle\right), \tag{2}$$

and explain what T_3 and T_4 should be in terms of one or more of x_i, x_j, y_i, and y_j.

7, A

Solution: Subtracting $P(x_i(t) = 1)$ from both sides of (1), and dividing by Δt gives,

$$\frac{P(x_i(t + \Delta t) = 1) - P(x_i(t) = 1)}{\Delta t} = \sum_{j=1}^{N} A_{ij} [\beta T_1 + \gamma T_2] + \mathcal{O}(\Delta t).$$

Now, $\langle x_i \rangle = P(x_i = 1)$, so the LHS becomes $\frac{\langle x_i(t+\Delta t)\rangle - \langle x_i(t)\rangle}{\Delta t}$. Then, letting $\Delta t \to 0$ gives:

$$\frac{d\langle x_i \rangle}{dt} = \sum_{j=1}^{N} A_{ij}\left[\beta T_1 + \gamma T_2\right],$$

From (a), we know that $T_1 = \langle (1-x_i)(1-y_i)x_j \rangle$, so $T_3 = (1-x_i)(1-y_i)x_j$. Now, $T_2 = P(x_i = 0, y_i = 1, x_j = 1) = \langle (1-x_i)y_i x_j \rangle$, so $T_4 = (1-x_i)y_i x_j$

unseen ⇓

(ii) Using your solution to (i), show how (2) simplifies when $\beta = \gamma$, and provide an interpretation of this simplified result based on the model description. 3, B

Solution: Using linearity of expectation, $\langle T_3 \rangle + \langle T_4 \rangle = \langle T_3 + T_4 \rangle = \langle (1-x_i)x_j \rangle$, and the model equation becomes,

$$\frac{d\langle x_i \rangle}{dt} = \sum_{j=1}^{N} A_{ij}\left(\beta \langle (1-x_i)x_j \rangle\right).$$

This is the network-SI model. When $\beta = \gamma$, the presence of Y doesn't effect how X spreads through the network.

3. Consider the following system of N differential equations:

$$\frac{dx_i}{dt} = -\sum_{j=1}^{N} \frac{k_j}{k_i} L_{ij} x_j, \quad i = 1, 2, ..., N, \tag{3}$$

where x_i is the i^{th} element of $\mathbf{x} \in \mathbb{R}^N$, L_{ij} corresponds to the Laplacian matrix for a simple connected graph with N nodes, and k_i is the degree of the i^{th} node in the graph. The initial condition is $\mathbf{x}(t = 0) = \mathbf{x}_0$. You should assume that a complete set of mutually orthogonal eigenvectors for the Laplacian matrix, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, along with their corresponding eigenvalues are given.

sim. seen ⇓

(a) Show that for a complete graph, (3) simplifies to the graph diffusion equation with diffusivity, $\alpha = 1$. 3, A

Solution: For a complete graph, $k_i = N - 1$ for all i, so $k_j/k_i = 1$ for any i and any j. It follows that (3) simplifies to $\frac{dx_i}{dt} = -\sum_{j=1}^{N} L_{ij} x_j$. The graph diffusion equation is $\frac{d\mathbf{x}}{dt} = -\alpha \mathbf{L}\mathbf{x}$ which, when $\alpha = 1$, is equivalent to the simplified equation.

sim. seen ⇓

(b) Show that $x_i = \frac{c}{k_i}$, $i = 1, 2, ..., N$ is an equilibrium solution to (3). Here, c is an arbitrary non-zero constant. 5, A

Solution: Substituting the proposed solution into the RHS of (3) and then simplifying gives,

$$\sum_{j=1}^{N} \frac{k_j}{k_i} L_{ij} x_j = \sum_{j=1}^{N} \frac{k_j}{k_i} L_{ij} \left(\frac{c}{k_j}\right) = \frac{c}{k_i} \sum_{j=1}^{N} L_{ij}$$

Now, $L_{ij} = \delta_{ij} k_j - A_{ij}$, $\sum_{j=1}^{N} \delta_{ij} k_j = k_i$, and $\sum_{j=1}^{N} A_{ij} = k_i$, so $\sum_{j=1}^{N} L_{ij} = 0$, and when $x_i = \frac{c}{k_i}$, $\frac{dx_i}{dt} = 0$ for all i as required.

sim. seen ⇓

(c) Consider solutions to (3) for an initial condition which is *not* an equilibrium solution and whose elements are all non-negative but is otherwise arbitrary.

(i) Explain how to construct a matrix $\mathbf{M}$ so that the transformation $\mathbf{u} = \mathbf{M}\mathbf{x}$ results in a system of ODEs of the form,

$$\frac{d\mathbf{u}}{dt} = -\mathbf{\Lambda}\mathbf{u}.$$

where $\mathbf{\Lambda}$ is a diagonal matrix. 4, B

Solution: Equation (3) in matrix-vector form is:

$$\frac{d\mathbf{x}}{dt} = -\mathbf{D}^{-1}\mathbf{L}\mathbf{D}\mathbf{x},$$

where $\mathbf{D}$ is the diagonal degree matrix for the graph. Orthogonally diagonalizing $\mathbf{L}$ gives,

$$\frac{d\mathbf{x}}{dt} = -\mathbf{D}^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathbf{T}}\mathbf{D}\mathbf{x},$$

where $\mathbf{V}$ and $\mathbf{\Lambda}$ are the eigenvector and diagonal eigenvalue matrices for $\mathbf{L}$, respectively, and $\mathbf{V}^T\mathbf{V} = \mathbf{V}\mathbf{V}^T = \mathbf{I}$. Then, let $\mathbf{u} = \mathbf{V}^T\mathbf{D}\mathbf{x}$ and left-multiply both sides of the equation above with $\mathbf{V}^T\mathbf{D}$ to obtain,

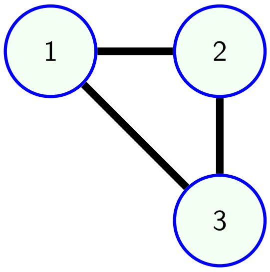
$$\frac{d\mathbf{u}}{dt} = -\mathbf{\Lambda}\mathbf{u}.$$

So, $\mathbf{M} = \mathbf{V}^T\mathbf{D}$ where $\mathbf{D}$ is the diagonal degree matrix for the graph, and the columns of $\mathbf{V}$ contain the eigenvectors of $\mathbf{L}$ normalized to have unit length.

(ii) Show that in the limit $t \to \infty$, $\mathbf{x} \to \mathbf{x}_{eq}$ where $\mathbf{x}_{eq}$ is an equilibrium solution to (3) with at least one non-zero element. $\boxed{8, \mathrm{D}}$

Solution: Starting from, $\frac{d\mathbf{u}}{dt} = -\mathbf{\Lambda u}$, and solving these uncoupled ODEs, the i^{th} element of $\mathbf{u}$ is $u_i = u_{0,i}e^{-\lambda_i t}$. Since the graph is connected, there will be one zero eigenvalue ($\lambda_1 = 0$) with eigenvector, $\mathbf{v}_1 = \frac{\mathbf{z}}{\sqrt{N}} = \frac{1}{\sqrt{N}}\{1, 1, ..., 1\}$ and all other eigenvalues will be positive. In the limit $t \to \infty$, we then have, $\mathbf{u} \to u_{0,1}\mathbf{e}_1$ where $\mathbf{e}_1 = \{1, 0, 0, ..., 0\}$. Now, $\mathbf{x} = \mathbf{D}^{-1}\mathbf{V}\mathbf{e}_1 u_{0,1}$, and $\mathbf{V}e_1 = \mathbf{v}_1$, so $\mathbf{x} \to u_{0,1}\mathbf{D}^{-1}\mathbf{v}_1$. The i^{th} element of this vector is $\frac{u_{0,1}}{\sqrt{N}k_i}$, so the vector is a nontrivial equilibrium solution provided that $u_{0,1} \neq 0$. Using the definition of $\mathbf{u}$, we have $u_{0,1} = \mathbf{v}_1^T\mathbf{D}\mathbf{x}_0$. Since the graphs is connected, each element on the diagonal of $\mathbf{D}$ is positive, and since the initial condition is not an equilibrium solution, it is non-trivial and at least one element of $\mathbf{x}_0 \neq 0$. It follows that $u_{0,1} \neq 0$.

4. The triangle graph shown is referenced in the questions below.



sim. seen ⇓

(a) Consider the degree sequence, $d = \{2, 2, 2\}$. What is the probability of the triangle graph being generated by the configuration model with degree sequence, d? 5, B

Solution: Let s_1 and s_2 be the two stubs on node 1, and $P(s_i \sim n_a)$ be the probability that stub i connects to node a. A triangle graph is formed if $s_1 \sim n_2$ and $s_2 \sim n_3$. For the configuration model, $P(s_1 \sim n_2, s_2 \sim n_3) = P(s_2 \sim n_3 | s_1 \sim n_2)P(s_1 \sim n_2)$, and $P(s_1 \sim n_2) = 2/(K-1) = 2/5, P(s_2 \sim n_3|s_1 \sim n_2) = 2/(K-3) = 2/3$, and the probability of generating this graph is $4/15$. A triangle is also generated if $s_1 \sim n_3$ and $s_2 \sim n_2$ which also has probability $4/15$, so the probability of generating a triangle graph is $8/15$.

sim. seen ⇓

(b) Consider the following modified version of the Barabasi-Albert model. The initial graph is a 3-node triangle graph. Each iteration, a new triangle is introduced and a link is added connecting an arbitrary node in the new triangle with a node in the existing graph. The node in the existing graph is selected using the usual linear preferential attachment model. After t iterations, a graph will have $N(t) = 3t+3$ nodes and $L(t) = 4t+3$ links. Derive an equation relating the expected number of nodes with degree 2 at iteration $t+1$, $\langle N_2(t+1)\rangle$, to one or more of: $\langle N_1(t)\rangle$, $\langle N_2(t)\rangle$, and $L(t)$. 7, C

Solution: We will consider three cases, A: a node in the existing graph with degree 1 receives a link in which case the number of nodes with degree 2 increases by 3 (two nodes with degree 2 are added each iteration via the new triangle), B: a node in the existing graph with degree 2 receives a link and N_2 increases by 1, and C: neither A nor B and N_k increases by 2. Using the formulation presented in lecture,

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)[P(A)\,(N_2(G_m(t))+3)+$$
$$P(B)\,(N_2(G_m(t))+1) + ((1-P(A)-P(B))\,(N_2(t)+2)].$$

The above equation simplifies to,

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m(t))[N_2(G_m(t)) + 2 + P(A) - P(B)].$$

The linear preferential attachment model gives, $P(A) = \frac{N_1}{2L}$ and $P(B) = \frac{2N_2}{2L}$, so,

$$\langle N_2(t+1)\rangle = 2 + \langle N_1(t)\rangle \frac{1}{2L(t)} + \langle N_2(t)\rangle \left(1 - \frac{1}{L(t)}\right).$$

Note that $N_1 = 0$ for all t based on the model definition. So, the answer simplifies to,

$$\langle N_2(t+1)\rangle = 2 + \langle N_2(t)\rangle \left(1 - \frac{1}{L(t)}\right).$$

unseen $\Downarrow$

(c) Let $\mathbf{A}$ be the adjacency matrix of an N-node simple graph, and let λ_i be the ith eigenvalue of $\mathbf{A}$ with $i \in \{1, 2, ..., N\}$. Derive an expression relating the sum $S = \sum_{i=1}^{N} \lambda_i^3$ to the number of triangles in the graph. Note: the triangle graph shown above contains 1 triangle. 8, D

Solution: Let $\mathbf{C} = \mathbf{A}^3$. Then, $C_{ii} = \sum_{l=1}^{N} \sum_{m=1}^{N} A_{il} A_{lm} A_{mi}$. The term inside the summation is 1 if and only if nodes (i, l, m) form a triangle and is zero otherwise. It follows that C_{ii} is twice the number of triangles that contain node i (since e.g. $l = 2, m = 3$ and $l = 3, m = 2$ are both considered in the double summation). The total number of distinct triangles is $\frac{\sum_{i=1}^{N} C_{ii}}{6} = \frac{trace(\mathbf{A}^3)}{6}$. The factor of 6 appears because a triangle involving nodes (i, l, m) is counted twice in C_{ii}, C_{ll}, and C_{mm}. Now, $\mathbf{Ax} = \lambda \mathbf{x}$, and $\mathbf{A}^3 \mathbf{x} = \lambda \mathbf{A}^2 \mathbf{x} = \lambda^2 \mathbf{Ax} = \lambda^3 x$, so if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}$, λ^3 will be an eigenvalue of $\mathbf{A}^3$ with eigenvector $\mathbf{Ax}$, and the number triangles is $\frac{trace(\mathbf{A}^3)}{6} = \frac{\lambda_1^3 + \lambda_2^3 + ... + \lambda_N^3}{6}$.

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH50007	1	The class did very well on this question.
MATH50007	2	Many students wrongly assumed that x_i, y_i and x_j are independent in part (a). Some students were not precise with the limit required in part (i) or (c), e.g., taking the limit of the left-hand side of the equality but not the right-hand side. Some dropped the O(Delta t) term without explanation. Many students did not distinguish between T_1 and T_3. T_3 is the expectation of T_1. (Same for T_2 and T_4.)
MATH50007	3	Overall, the class did well on parts (a) and (b) but struggled with part (c). A key aspect of part (c) was to rewrite the system of ODEs in matrix-vector form using the diagonal degree matrix.
MATH50007	4	Students did well on parts (a) and (b). Part (c) was more challenging, and a key point was to recognize that the elements of A³are related to the number of paths with length 3.