Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 1

Ajay Chandra

Imperial College
London

Housekeeping

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- **Problem Sheets:** Our problem class is on Friday!
  - Before the program class, try to **attempt** all the problems on Problem Sheet 0.
  - You can also take a look at the problems on Problem Sheet 1 during the week, but aren't expected to have attempted them!

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Assessments

- Quiz 1 – 1% - released Friday, November 4th and due Tuesday, November 8th
- Quiz 2 – 1% - released Friday, November 11th and due Tuesday, November 15th
- Fall Midterm – 5%
- Quiz 3 – 1% - released Friday, November 25th and due Tuesday, November 29th
- Quiz 4 – 1% - released Friday, December 2nd and due Tuesday, December 6th
- Quiz 5 – 1% - released Friday, December 9th and due Tuesday, December 13th
- January Test – 10%
- Spring assignments and Midterm – 10%
- May Exam (covering both Fall and Spring material) – 70%.

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  - Sequences - what does it mean for an infinite list of numbers to converge?
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- **The first step: a solid grasp of mathematical logic is key!**

Exercise 1.1

At which stage does this argument go wrong?

Suppose $x = 2$.

1. $\Rightarrow x - 2 = 0$
2. $\Rightarrow x^2 - 2x = 0$
3. $\Rightarrow x(x - 2) = 0$
4. $\Rightarrow x = 0$ or $x = 2$.
5. Nowhere; the argument is correct.

$x = 2 \Rightarrow x = 0$ or $x = 2$

Exercise 1.2

"A unless B" is the same logical statement as

1. $A \iff B$
2. $\overline{A} \iff \overline{B}$
3. $A \Rightarrow B$
4. $A \Rightarrow \overline{B}$
5. $\overline{A} \Rightarrow B$ ✓
6. $\overline{A} \Rightarrow \overline{B}$
7. None of these; something else.
8. More than one of these.

Exercise 1.3

"Find two real numbers x which satisfy the equation $x^2 - 3x + 2 = 0$."

Student solution:

$$\begin{aligned} & x^2 - 3x + 2 = 0 \\ \Rightarrow \quad & (x-1)(x-2) = 0 \\ \Rightarrow \quad & x = 1 \text{ or } x = 2. \end{aligned}$$

How many marks would this get in an exam?

1. Two marks – completely solved the problem.
2. One mark – partially solved the problem.
3. No marks – failed to solve the problem.

$x^2 - 3x + 2 = 0 \Rightarrow x = 1$ or $x = 2$

Exercise 1.4

Is this a correct proof that $3|n^2 \Rightarrow 3|n$?
If $3|n$ then $n = 3m$ for some $m \in \mathbb{N}_{>0}$. ← $\{1,2,3,\ldots\}$
Therefore $n^2 = 3(3m^2)$ is divisible by 3.

1. Yes.
2. No.
3. Uh?

Proved
$3|n \Rightarrow 3|n^2$

Proving $3|n^2 \Rightarrow 3|n$

3 cases
$n = 3q$
$n = 3q+1$
$n = 3q+2$
$q \in \mathbb{Z}$

If $n = 3q+1$ or $3q+2$
then $n^2 = 3N+1$ for $N \in \mathbb{Z}$
so $3 \nmid n^2$
Therefore $n = 3q$

Exercise 1.5

What does $x \in \bigcup_{n=1}^{\infty} S_n$ mean?

1. $x \in S_n$ for some $n \in \mathbb{N}_{>0}$
2. Either $x \in S_n$ for some $n \in \mathbb{N}_{>0}$ or $x \in S_\infty$
3. Either $x \in S_n$ for some $n \in \mathbb{N}_{>0}$ or $x \in \lim_{n\to\infty} S_n$
4. Other

Defining $\mathbb{Q}$

Recall $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}_{>0} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with $+, \times, >$.

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Recall $\mathbb{Q} := \{(p, q) \in \mathbb{Z} \times \mathbb{N}_{>0}\}/\sim$, where $\sim$ is the equivalence relation

$$(p_1, q_1) \sim (p_2, q_2) \iff p_1 q_2 = p_2 q_1.$$

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We write the equivalence class of (p, q) as p/q or $\frac{p}{q}$.

Each equivalence class has a distinguished element (p', q') such that $\nexists\, n \in \mathbb{N}$ with $n > 1$ and $n | p'$, $n | q'$. We say $\frac{p'}{q'}$ is "in lowest terms".

Defining operations on $\mathbb{Q}$

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} \quad := \quad \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \, ,$$

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\frac{p_1}{q_1} + \frac{p_2}{q_2} \quad &:= \quad \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}\,, \\
\frac{p_1}{q_1} - \frac{p_2}{q_2} \quad &:= \quad \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}\,, \\
\frac{p_1}{q_1} \times \frac{p_2}{q_2} \quad &:= \quad \frac{p_1 p_2}{q_1 q_2}\,, \\
\frac{p_1}{q_1} \div \frac{p_2}{q_2} \quad &:= \quad \frac{p_1 q_2}{q_1 p_2}\,, \quad p_2 \neq 0, \\
\frac{p_1}{q_1} \leq \frac{p_2}{q_2} \quad &\iff \quad p_1 q_2 \leq p_2 q_1.
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These satisfy certain properties that we list next. They are sufficiently strong that you can deduce everything about $\mathbb{Q}$ just from these properties, i.e. you can treat them as axioms if you wish.

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3. $a + (b + c) = (a + b) + c$ (+ is *associative*)

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6. $\exists\, 0 \in \mathbb{Q} \colon a + 0 = a \quad \forall a \in \mathbb{Q}$
7. $\exists\, 1 \in \mathbb{Q} \colon 0 \neq 1, a \times 1 = a \quad \forall a \in \mathbb{Q}$

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8. $\forall a \in \mathbb{Q},\ \exists(-a) \in \mathbb{Q}$ such that $a + (-a) = 0$

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8. $\forall a \in \mathbb{Q},\ \exists (-a) \in \mathbb{Q}$ such that $a + (-a) = 0$
9. $\forall a \in \mathbb{Q} \setminus \{0\}\ \exists a^{-1} \in \mathbb{Q}$ such that $a \times (a^{-1}) = 1$

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10. for each $x \in \mathbb{Q}$ **precisely one** of (a), (b), (c) holds:

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12. $x > 0,\ y > 0 \implies xy > 0 \quad \forall x, y \in \mathbb{Q}$

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12. $x > 0,\ y > 0 \implies xy > 0 \quad \forall x, y \in \mathbb{Q}$
13. $\forall x \in \mathbb{Q}\ \exists\ n \in \mathbb{N}$ such that $n > x$ (Archimedean axiom)

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Notation: $a - b := a + (-b)$, and $a/b := a \times (b^{-1})$, while $a > b$ ($a < b$) is defined to mean $a - b > 0$ (respectively $-(a - b) > 0$).

Exercise 2.3

Prove that $x > y > z \Rightarrow x > z$.

Real numbers that are not rational: The real numbers $\mathbb{R}$ satisfy the exact same axioms, plus one more – the **completeness axiom** – designed to fix the problem that $\mathbb{Q}$ has holes. For instance,

Proposition 2.5

There is no $x \in \mathbb{Q}$ such that $x^2 = 3$.

Suppose, by contradiction, that $\exists x \in \mathbb{Q}$ w/ $x^2 = 3$

Let $x = p/q$ w/ $(p,q) \in \mathbb{Z} \times \mathbb{N}_{>0}$ in lowest terms

$$p^2 = 3q^2 \Rightarrow 3 \mid p^2 \underset{\text{Ex 1.4}}{\Longrightarrow} 3 \mid p$$

$$p = 3n \Rightarrow q^2 = 3n^2 \text{ so } 3 \mid q^2 \Rightarrow 3 \mid q \Rightarrow$$

(p,q) lowest terms $\#$