Imperial College London

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Analysis 1A

Lecture 12 - Subsequences and the Bolzano-Weierstrass Theorem

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$\varepsilon = \frac{1-L}{2} = |L'-L|$

First, an example we didn't get to last week:

$L' = \frac{1+L}{2}$

Example 3.23

Prove that if

$$\left|\frac{a_{n+1}}{a_n}\right| \leq L' < 1 \longleftarrow \left|\frac{a_{n+1}}{a_n}\right| \to L < 1$$

then $a_n \to 0$.

Idea: $\left|\frac{a_{n+1}}{a_n}\right| \approx L$, $\underset{\to 0}{|a_n|} \approx L^{n-1}|a_1| = \boxed{L^n \cdot c} \leftarrow$ some constant

Fun fact: if $|L|<1$ then $L^n \to 0$

Proof

Since $L<1$, $1-L>0$ so let $\varepsilon = \frac{1-L}{2} > 0$. Then $\exists N$ s.t. $\forall n \geq N$

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \varepsilon \implies \left|\frac{a_{n+1}}{a_n}\right| < L' = \frac{1+L}{2}$$

By induction, $|a_{N+k}| < L'|a_{N+k-1}| < (L')^2 |a_{N+k-2}| \cdots < \overbrace{(L')^k |a_N|}^{<\varepsilon}$

Let $\varepsilon > 0$. $\exists M$ s.t. ~~$\forall k \geq M$~~ $(L')^M < \frac{\varepsilon}{|a_N|+1}$ $\quad k \geq M,\ (L')^k \leq (L')^M$

Then for $n \geq N+M$, $|a_n| \leq (L')^{n-N} |a_N| < \varepsilon$. ∎ $a_n \to 0$

Definition

A *subsequence* of (a_n) is a new sequence $b_i = a_{n(i)}\ \forall i \in \mathbb{N}_{>0}$, where $n(1) < n(2) < \cdots < n(i) < \ldots\ \forall i \in \mathbb{N}_{>0}$.

$n(\cdot)$ is a function $\mathbb{N}_{>0} \to \mathbb{N}_{>0}$ sending $i \mapsto n(i)$ which is strictly monotonically increasing.

$j > i \Rightarrow n(j) > n(i)$

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Exercise 3.32

Prove, using induction, that our assumption on $n(\cdot)$ implies that $n(i) \geq i$ for all $i \in \mathbb{N}_{>0}$.

Example 3.33

Here are some subsequences of $a_n = (-1)^n$:

- $b_n = a_{2n}$

$b_1 = a_2$ $\quad n(i) = 2i$

$b_2 = a_4$

$b_3 = a_6$

Example 3.33

Here are some subsequences of $a_n = (-1)^n$:

- $b_n = a_{2n}$, so $b_n = 1 \ \forall n \implies b_n \to 1$.

$a_2 = 1$
$a_4 = 1$
$a_6 = 1$

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- $d_n = a_{3n}$, so $d_n = (-1)^n$ $(= a_n)$ doesn't converge.
- $e_n = a_{n+17}$, so $e_n = (-1)^{n+1}$ $(= -a_n)$ doesn't converge.

Next we work up to the following technical-sounding but vitally important theorem,:

Theorem 3.34 - Bolzano-Weierstrass

If (a_n) is a *bounded* sequence of real numbers then it has a *convergent subsequence*.

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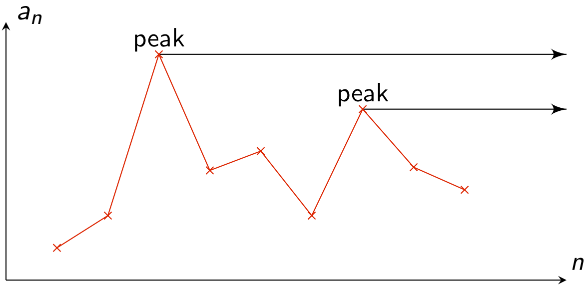
Remark 3.35

A bounded sequence will have *many* convergent subsequences, and they may converge to different limits; think of $a_n = (-1)^n$ for instance.

Idea for proof of Bolzano-Weierstrass

Use "peak points" of (a_n):

j is a peak if $\forall k > j$
$a_k < a_j$



Proposition 3.39

If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.