MATH60142/70142 The Mathematics of Business and Economics
Class test - solutions

Friday 28th February 2025
Duration: 40 minutes

Question (20 marks in total) - solutions in red

Suppose that a firm's production process requires the input of two different goods and produces a single output. For $i = 1, 2$, let $x_i \geq 0$ denote the quantity of the i-th good that the firm inputs, and $w_i > 0$ denote the (fixed) cost to the firm of each unit of this good. Furthermore, let y denote the quantity of its output that it produces. Suppose also that the firm's production function is given by

$$f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$$

(where these are positive square roots).

The firm wishes to determine the values x_1^* and x_2^* of x_1 and x_2 (respectively) that yield a level of output y for the minimum cost.

(a) **(2 marks)** Write down the Lagrangian for this minimisation problem.

The Lagrangian is

$$L(x_1, x_2, \lambda) = (w_1 x_1 + w_2 x_2) - \lambda(f(x_1, x_2) - y), \tag{1}$$

where $f(x_1, x_2)$ is as in the question statement. It is also acceptable to write $(w_1 x_1 + w_2 x_2)$ as $\underline{w}\underline{x}^T$ (where $\underline{w} = (w_1, w_2)$ and $\underline{x} = (x_1, x_2)$), or any other equivalent.

(b) **(4 marks)** Derive a set of necessary conditions for x_1^* and x_2^*.

The necessary conditions are

$$\frac{\partial}{\partial x_i} L(x_1, x_2, \lambda) = 0 \quad \text{for } i = 1, 2, \tag{2}$$

and

$$\frac{\partial}{\partial \lambda} L(x_1, x_2, \lambda) = 0, \tag{3}$$

which are equivalent to

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \lambda) = \frac{w_i}{\lambda} \quad \text{for } i = 1, 2, \tag{4}$$

and

$$f(x_1, x_2) = y. \tag{5}$$

(c) **(5 marks)** Solve these conditions to determine x_1^* and x_2^* as functions of w_1, w_2 and y. (You do not need to show that these solutions provide a minimum rather than a maximum.)

First, (4) gives

$$\frac{1}{2\sqrt{x_1}} + \sqrt{x_2} = \frac{w_1}{\lambda} \quad \text{and} \quad \sqrt{x_1} + \frac{1}{2\sqrt{x_2}} = \frac{w_2}{\lambda}. \tag{6}$$

The ratio of these gives

$$\sqrt{\frac{x_2}{x_1}} = \frac{w_1}{w_2}, \tag{7}$$

which can be rearranged to give

$$\sqrt{x_2} = \frac{w_1}{w_2}\sqrt{x_1}. \tag{8}$$

Then (5) gives

$$y = \sqrt{x_1} + \sqrt{x_2} = \left(1 + \frac{w_1}{w_2}\right)\sqrt{x_1} = \left(\frac{w_2}{w_1} + 1\right)\sqrt{x_2}, \tag{9}$$

from which it follows that

$$x_1^*(w_1, w_2, y) = \left(\frac{w_2 y}{w_1 + w_2}\right)^2 \quad \text{and} \quad x_2^*(w_1, w_2, y) = \left(\frac{w_1 y}{w_1 + w_2}\right)^2. \tag{10}$$

Note that the student may have included some of this working in their answer to part (b).]

(d) **(2 marks)** Suppose that the firm is inputting these optimal quantities $x_1 = x_1^*$ and $x_2 = x_2^*$. What should the rate of change of x_2 be at this point if the firm is to vary x_1 while maintaining the same level of output y? Express your answer in terms of w_1 and w_2 and provide a justification for it.

The correct rate of change is $-w_1/w_2$. One could justify this by stating that $-w_1/w_2$ is the gradient of every isocost in the (x_1, x_2)-plane and the isoquant $f(x_1, x_2) = y$ is tangential to an isocost at the point (x_1^*, x_2^*). Alternatively, one might compute it as $MRTS(x_1^*, x_2^*)$ using the formula $MRTS(x_1, x_2) = -(\partial f(x_1, x_2)/\partial x_1)/(\partial f(x_1, x_2)/\partial x_2)$ and substituting for x_1^* and x_2^* using (10).

Suppose now that the firm sells each unit of its output for a (fixed) price p.

(e) **(6 marks)** If the firm produces a *non-zero* level of output y, what should that level be in order to maximise its profits with minimised costs? Note that you should provide proof that your solution for y provides a maximum rather than a minimum.

The profit generated by the firm when producing a level of output y with minimised costs is given by

$$\pi(y) = py - (w_1 x_1^* + w_2 x_2^*), \tag{11}$$

where $x_1^* = x_1^*(w_1, w_2, y)$ and $x_2^* = x_2^*(w_1, w_2, y)$ are given by (10). For this to be maximised, a necessary condition is that

$$\frac{\mathrm{d}\pi}{\mathrm{d}y} = 0, \tag{12}$$

which by (11) becomes

$$p - \left(w_1 \frac{\partial}{\partial y} x_1^*(w_1, w_2, y) + w_2 \frac{\partial}{\partial y} x_2^*(w_1, w_2, y)\right) = 0, \tag{13}$$

which with (10) gives

$$p - 2w_1\left(\frac{w_2}{w_1 + w_2}\right)^2 y - 2w_2\left(\frac{w_1}{w_1 + w_2}\right)^2 y = 0, \tag{14}$$

which implies that

$$y = \frac{p(w_1 + w_2)}{2w_1 w_2} = y^*, \text{ say.} \tag{15}$$

To check this gives a maximum profit rather than a minimum, note that $\pi(y)$ as given by (11) with $x_1^* = x_1^*(w_1, w_2, y)$ and $x_2^* = x_2^*(w_1, w_2, y)$ as given by (10), is simply a quadratic function of y with a negative coefficient of y^2, and so its only stationary point is a maximum. Alternatively, one may show that $\mathrm{d}^2\pi(y)/\mathrm{d}y^2 = -2w_1 w_2/(w_1 + w_2)$ which is negative for all y.

(f) **(1 mark)** Would the most profitable level of output for the firm actually be $y = 0$? Justify your answer. No. Again, this is because $\pi(y)$ is a quadratic in y with a negative coefficient of y^2 and so its maximum at $y = y^*$ as given by (15) is also its global maximum and so certainly greater than its value at $y = 0$. Alternatively, one might explicitly compute $\pi(y^*)$ and compare it with the value of $\pi(0)$ which is 0.