MATH50010 - Probability for Statistics
Unseen Problem 7

Suppose a flea hops randomly on the vertices of a triangle, with all jumps equally likely (and 0 probability of staying in the same place). Find the probability that after n hops, the flea is back where it started.

Let $p_{11}^{(n)} = \Pr(X_n = 1|X_0 = 1)$ *be the probability of interest. Then, conditioning on* X_1*, since* $\Pr(X_1 = 2|X_0 = 1) = \Pr(X_1 = 3|X_0 = 1) = 1/2$*,*

$$\begin{aligned} p_{11}^{(n)} &= \frac{1}{2}\Pr(X_n = 1|X_1 = 2, X_0 = 1) + \frac{1}{2}\Pr(X_n = 1|X_1 = 3, X_0 = 1) \\ &= \frac{1}{2}\Pr(X_n = 1|X_1 = 2) + \frac{1}{2}\Pr(X_n = 1|X_1 = 3) = \frac{1}{2}p_{21}^{(n-1)} + \frac{1}{2}p_{31}^{(n-1)} \end{aligned}$$

Now by symmetry (since probability of going from 2 to 1 in n steps is equal to probability of going from 3 to 1 in n steps), $p_{21}^{(n-1)} = p_{31}^{(n-1)}$*, so that the above simplifies to*

$$p_{11}^{(n)} = p_{21}^{(n-1)}.$$

Moreover, by the law of total probability

$$p_{21}^{(n-1)} + p_{22}^{(n-1)} + p_{23}^{(n-1)} = 1$$

and again by symmetry this gives

$$2p_{21}^{(n-1)} + p_{22}^{(n-1)} = 1.$$

This then gives

$$p_{11}^{(n)} = \frac{1}{2}(1 - p_{22}^{(n-1)}) = \frac{1}{2}(1 - p_{11}^{(n-1)}),$$

since symmetry gives $p_{11}^{(n-1)} = p_{22}^{(n-1)}$*. We can clearly see that* $p_{11}^{(n)} \to \frac{1}{3}$ *as* $n \to \infty$ *because the states are symmetrical. Another way to see this is to set* $p_{11}^{(n)} = p_{11}^{(n-1)} = p$ *and solve the resulting equation* $p = \frac{1}{2}(1 - p)$ *to get* $p = \frac{1}{3}$*.*
Then note that

$$\frac{1}{3} - p_{11}^{(n)} = \frac{1}{3} - \frac{1}{2}(1 - p_{11}^{(n-1)}) = \frac{1}{2}\left(p_{11}^{(n-1)} - \frac{1}{3}\right).$$

Iterating this argument gives

$$\frac{1}{3} - p_{11}^{(n)} = \left(\frac{-1}{2}\right)^n \left(1 - \frac{1}{3}\right)$$

since $p_{11}^{(0)} = 1$*. This then gives*

$$p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3}\left(\frac{-1}{2}\right)^n \to \frac{1}{3}.$$

Check: $p_{11}^0 = 1$ *and* $p_{11}^1 = 0$ *as it should be.*
Alternatively, we seek the diagonal entries of powers of the transition matrix.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

This is not too bad if we use symmetry: we know that all the diagonal entries of P^n must be equal: clearly $p_{11}^{(n)} = p_{22}^{(n)} = p_{33}^{(n)}$. This means all we need to do is take the average of the trace of P^n (recall the trace is the sum of the diagonal entries).

Now the trace of P^n is also the sum of its eigenvalues. And the eigenvalues of P^n are all of the form λ^n where λ ranges over the eigenvalues of P.

So we compute the eigenvalues of P. For ease of notation we'll work with $Q = 2P$

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$\det(Q - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = \det \begin{pmatrix} 0 & 1+\lambda & 1-\lambda^2 \\ 0 & -1-\lambda & 1+\lambda \\ 1 & 1 & -\lambda \end{pmatrix}$$

Evaluating this determinant then gives the characteristic polynomial

$$(1+\lambda)^2 + (1-\lambda^2)(1+\lambda) = (1+\lambda)\left(2+\lambda-\lambda^2\right) = (1+\lambda)(1+\lambda)(2-\lambda),$$

giving the eigenvalues of Q as 2 and -1 (twice). The eigenvalues of P are then 1 and $\frac{-1}{2}$ (twice). The probability we seek is then the average of the trace of P^n, which is exactly

$$\frac{1}{3}\left(1 + 2\left(\frac{-1}{2}\right)^n\right).$$

Happily this agrees with the first method.