Imperial College
London

MATH97013 MATH97092

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Vortex Dynamics

Date: Tuesday, 4 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the two dimensional flow of an ideal incompressible fluid with velocity field

$$\mathbf{u} = (u, v) = (Ax + By, Cx - Ay),$$

where A, B and C are real constants.

(a) Find the circulation

$$\oint_{\mathcal{C}} \mathbf{u}.\mathbf{dx},$$

where $\mathcal{C}$ is a single circuit of the curve defined by

$$x^2 + \frac{y^2}{4} = 1.$$

(4 marks)

Hint: the area of an ellipse is πab where a and b are the lengths of its semi-axes.

(b) Show that $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathbf{u}_{\mathrm{SBR}} + \mathbf{u}_{\mathrm{str}},$$

where $\mathbf{u}_{\mathrm{SBR}}$ is a solid body rotation and $\mathbf{u}_{\mathrm{str}}$ is an irrotational extensional flow. Find $\mathbf{u}_{\mathrm{SBR}}$ and $\mathbf{u}_{\mathrm{str}}$ explicitly. (6 marks)

(c) Now suppose that

$$A = \frac{B + C}{2}.$$

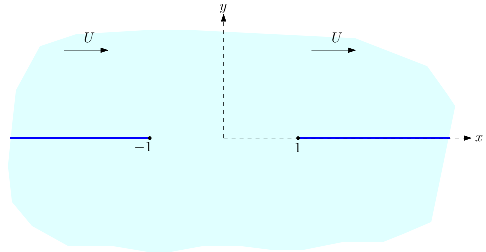
By first finding its principal axes of strain, sketch the streamlines associated with the extensional flow $\mathbf{u}_{\mathrm{str}}$.

(5 marks)

(d) Suppose, at some instant, two force-free point vortices of unit circulation are located at $(\pm 1, 0)$ in the background flow given by $\mathbf{u}$. Find the instantaneous velocity of each point vortex. (5 marks)

(Total: 20 marks)

2. Consider the two dimensional irrotational incompressible flow (u, v) in an (x, y) plane in the unbounded region exterior to a gap in a wall along the x-axis as shown in the Figure:



Impenetrable walls occupy the intervals $(-\infty, -1]$ and $[+1, +\infty)$. There is a uniform flow U as $y \to +\infty$ in the upper-half plane but the flow vanishes as $y \to -\infty$ in the lower half plane, i.e.,

$$(u, v) \to \begin{cases} (U, 0), & \text{as } y \to +\infty, \\ (0, 0), & \text{as } y \to -\infty. \end{cases}$$

(a) Find the conformal mapping $z = f(\zeta)$ that transplants the interior of a unit disc $|\zeta| < 1$ in a complex ζ plane to the fluid region exterior to the wall, with $\zeta = +\mathrm{i}$ mapping to the point at infinity in the upper-half plane $y > 0$, $\zeta = -\mathrm{i}$ mapping to the point at infinity in the lower-half plane $y < 0$, and the rest of the unit circle $|\zeta| = 1$ mapping to the wall.

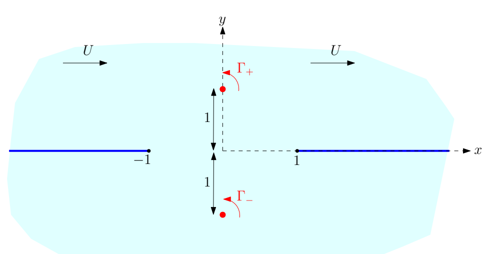
(4 marks)

(b) Show that the complex potential $w(z)$ describing the flow scenario shown in the Figure above can be written, as a function of ζ, as

$$W(\zeta) \equiv w(f(\zeta)) = \frac{U}{\zeta - \mathrm{i}}.$$

(6 marks)

(c) Suppose now that a point vortex of circulation Γ_+ is at $(0, +1)$ and a point vortex of circulation Γ_- is at $(0, -1)$; see the new Figure below. Find the new complex potential associated with this instantaneous flow as a function of ζ. (5 marks)

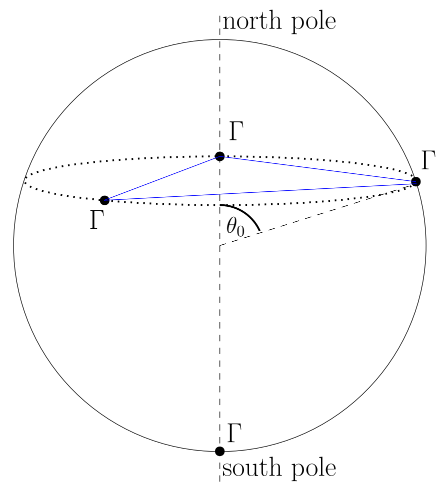


(d) Let the speed of the fluid at the point $(0, \sqrt{3})$ be (u_s, v_s). Write down an explicit formula for $u_s - \mathrm{i}v_s$ (simplification of your formula is not required, but your formula must be explicit, and all quantities appearing in it must be defined).

(5 marks)

(Total: 20 marks)

3. Four point vortices of equal circulation Γ are placed on the surface of a non-rotating sphere of unit radius on which an incompressible fluid of constant density flows. One point vortex is at the south pole. The three other point vortices are equally spaced around the latitude circle with spherical polar angle θ_0, relative to an axis through the north and south poles, as shown in the Figure:



(a) Let (r, θ, ϕ) be the usual spherical polar angles where θ is the latitudinal angle measured from an axis through the north and south poles of the sphere. Using a complex-valued coordinate

$$\zeta = \cot\left(\frac{\theta}{2}\right) e^{\mathrm{i}\phi}$$

representing a point in a plane of stereographic projection, find an expression for the stream function $\psi(\zeta, \overline{\zeta})$ associated with the instantaneous flow generated by these 4 point vortices. You must explain all your notation. (4 marks)

(b) Find the stream function $\psi_{SBR}(\zeta, \overline{\zeta})$ associated with solid body rotation of the fluid on the sphere with angular velocity Ω about an axis through the north and south poles. You may use the fact that if u_ϕ and u_θ are the zonal and meridional velocity components then

$$u_\phi - \mathrm{i}u_\theta = \sqrt{\frac{\zeta}{\overline{\zeta}}}(1 + \zeta\overline{\zeta})\frac{\partial \psi}{\partial \zeta}.$$

(4 marks)

(c) Use your results from parts (a) and (b) to determine the latitudinal angle θ_0 at which all the point vortices are in equilibrium (i.e. stationary). Explain your reasoning. (6 marks)

(d) Show that, in the equilibrium determined in part (c), the length of any straight line in $\mathbb{R}^3$ joining any pair of vortices is

$$\frac{2\sqrt{2}}{\sqrt{3}}.$$

(6 marks)

(Total: 20 marks)

4. At some instant a vortex patch of uniform vorticity ω_0 sits in an otherwise irrotational flow (u, v) that, at large distances, has the form

$$(u, v) \to (-2\dot{\gamma}xy, \dot{\gamma}(y^2 - x^2)),$$

where $\dot{\gamma}$ is a constant parameter. The flow is two-dimensional and incompressible. The shape of the vortex patch can be described by the following conformal map

$$z = x + \mathrm{i}y = f(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta^2, \qquad \alpha, \beta \in \mathbb{R}$$

from the interior of a unit disc $|\zeta| < 1$ in a parametric ζ plane to the fluid region *exterior* of the vortex patch. The values of α and β are such that this is a one-to-one mapping.

(a) Let $\psi(z, \overline{z})$ be the streamfunction associated with the flow. Show that, as $|z| \to \infty$,

$$\frac{\partial \psi}{\partial z} \to \frac{\dot{\gamma}z^2}{2} + \text{decaying terms.}$$

(4 marks)

(b) Find the velocity field everywhere in the (x, y) plane. (5 marks)

(c) Find the value of $\dot{\gamma}$ in terms of ω_0, α and β for which the flow inside the patch is purely in solid body rotation. (4 marks)

(d) Using your result from part (b) show that, as $|z| \to \infty$, the next term of the far-field flow given in part (a) is

$$\frac{\partial \psi}{\partial z} \to \frac{\dot{\gamma}z^2}{2} + \frac{\omega_0}{4}\left(2\beta^2 - \alpha^2\right)\frac{1}{z} + \text{faster decaying terms.}$$

(3 marks)

(e) Use your result in part (d) to deduce a formula for the area of the vortex patch as a function of α and β. (4 marks)

(Total: 20 marks)

5. (a) Show that in a two-dimensional incompressible flow of an ideal fluid in an (x, y) plane the vorticity ω is related to the streamfunction ψ by means of the relation

$$\omega = -\nabla^2 \psi, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Show also that any flow in which $\omega = h(\psi)$, where h is an arbitrary differentiable function of ψ, is a steady solution of the 2D vorticity equation. (2 marks)

(b) Let $z = x + \mathrm{i}y$ and choose $h(\psi) = -4e^{-2\psi}$. Verify that a general steady solution of the two-dimensional vorticity equation in a region D is given by

$$\psi(z, \bar{z}) = -\frac{1}{2} \log \left(\frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2} \right),$$

where $f(z)$ is an analytic function of z in D whose derivative $f'(z)$ has no zeros there.

(3 marks)

(c) For the stream function of part (b), we now make the choice

$$f(z) = -\mathrm{i}e^{\mathrm{i}z} \frac{z + 3\mathrm{i}}{z - \mathrm{i}}.$$

(i) Show that the flow becomes uniform as $y \to \pm\infty$ and find the speed of the uniform flow in each case. (4 marks)

(ii) Show that there is a point vortex at $(0, -1)$ and find its circulation. (3 marks)

(iii) Show that the point vortex considered in part (ii) is stationary. (5 marks)

(iv) Show that the point vortex considered in parts (ii) and (iii) is the only singularity in the flow and that, consequently, the stream function associated with this choice of $f(z)$ provides a global equilibrium of the 2D vorticity equation. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH97013/MATH97092

Vortex dynamics (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. Consider the two dimensional flow

$$\mathbf{u} = (u, v) = (Ax + By, Cx - Ay), \qquad A, B, C \in \mathbb{R},$$

where A, B and C are real constants.

(a) The easiest thing to do is to use Stokes theorem:

$$\oint_C \mathbf{u}.\mathbf{dx} = \int\int_{\substack{\text{Area}\\\text{enclosed}}} \nabla \wedge \mathbf{u}\, dA.$$

Then

$$\nabla \wedge \mathbf{u} = (0, 0, \omega), \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = C - B.$$

Since this is constant, the circulation is this value multiplied by the area of the ellipse which is $\pi \times 1 \times 2 = 2\pi$ (the semi-axis lengths can be read off the equation for the ellipse and are 1 and 2). The circulation is therefore

$$2\pi(C - B).$$

$\boxed{\text{meth seen} \Downarrow}$

$\boxed{4, \text{A}}$

(b) Recognizing the local angular velocity Ω of solid body rotation is

$$\Omega = \frac{\omega}{2} = \frac{C - B}{2}$$

the velocity can be rewritten as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx - Ay \end{pmatrix} = \underbrace{\frac{C - B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}}_{\text{solid body rotation}} + \underbrace{\begin{pmatrix} \left(\frac{B+C}{2}\right) y + Ax \\ \left(\frac{B+C}{2}\right) x - Ay \end{pmatrix}}_{\text{irrotational strain}}.$$

Hence

$$\mathbf{u}_{SBR} = \frac{C - B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}, \qquad \mathbf{u}_{\text{str}} = \begin{pmatrix} \left(\frac{B+C}{2}\right) y + Ax \\ \left(\frac{B+C}{2}\right) x - Ay \end{pmatrix}.$$

$\boxed{\text{meth seen} \Downarrow}$

$\boxed{6, \text{B}}$

(c) If

$$A = \left(\frac{B + C}{2}\right)$$

then

$$\mathbf{u}_{\text{str}} = \begin{pmatrix} Ay + Ax \\ Ax - Ay \end{pmatrix} = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

To find the principal axes of strain we need to find the eigenvectors and eigenvalues of this matrix. The eigenvalues are the solutions of

$$\begin{vmatrix} A - \lambda & A \\ A & -A - \lambda \end{vmatrix} = 0$$

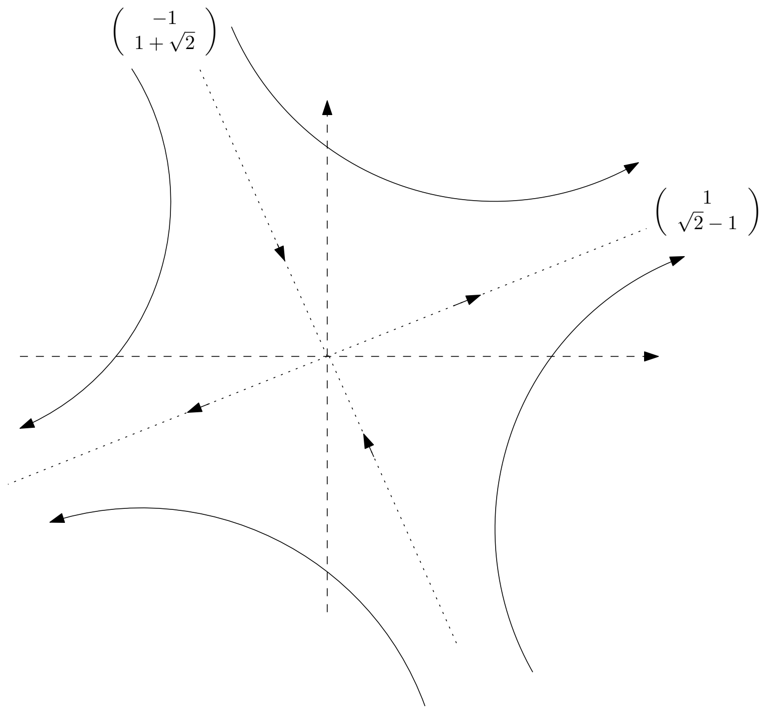
implying

$$\lambda^2 - A^2 - A^2 = 0, \qquad \lambda = \pm\sqrt{2}A.$$

The eigenvectors, which are the principal axes of strain (by definition), are

$$\begin{pmatrix} 1 \\ \pm\sqrt{2}-1 \end{pmatrix}.$$

A sketch of the streamlines associated with the extensional flow $\mathbf{u}_{\mathrm{str}}$ is therefore:



meth seen $\Downarrow$

5, A

(d) Since the vortices are force-free, they must move with the "non-self-induced velocity", or the velocity field produced by other vortices and any background flow effects. The easiest way to proceed is to compute the mutual effect of the two point vortices on each other and then add in the effect of the background flow *a posteriori*. The complex potential associated with the two point vortices of unit circulation is

$$w(z) = -\frac{\mathrm{i}}{2\pi}\log(z-1) - \frac{\mathrm{i}}{2\pi}\log(z+1)$$

hence the associated complex velocity field is

$$\tilde{u} - \mathrm{i}\tilde{v} = \frac{dw}{dz} = -\frac{\mathrm{i}}{2\pi}\frac{1}{z-1} - \frac{\mathrm{i}}{2\pi}\frac{1}{z+1}.$$

Therefore the velocity of the vortex at $(+1,0)$ or $z=1$ due to the other vortex is

$$\tilde{u}^+ - \mathrm{i}\tilde{v}^+ = -\frac{\mathrm{i}}{2\pi}\frac{1}{z+1}\bigg|_{z=1} = -\frac{\mathrm{i}}{2\pi}\frac{1}{2}.$$

Therefore adding in the effect of the background flow the velocity of this vortex is

$$\begin{pmatrix} A \\ C \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{4\pi} \end{pmatrix} = \begin{pmatrix} A \\ C + \dfrac{1}{4\pi} \end{pmatrix}.$$

The velocity of the vortex at $(-1,0)$ or $z=-1$ due to the other vortex is

$$\tilde{u}^- - \mathrm{i}\tilde{v}^- = -\frac{\mathrm{i}}{2\pi}\frac{1}{z-1}\bigg|_{z=-1} = +\frac{\mathrm{i}}{2\pi}\frac{1}{2}.$$

Therefore adding in the effect of the background flow the velocity of this vortex is

$$\begin{pmatrix} -A \\ -C \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{4\pi} \end{pmatrix} = \begin{pmatrix} -A \\ -C - \dfrac{1}{4\pi} \end{pmatrix}.$$

$\boxed{\text{meth seen} \Downarrow}$

$\boxed{5,\ \mathrm{B}}$

2. (a) The mapping is a composition of the Joukowski map from the disc to the exterior of the real interval $(-1,1)$

$$\eta = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{1}$$

seen ⇓

4, A

and an inversion

$$z = \frac{1}{\eta} = \frac{2\zeta}{1+\zeta^2} \equiv f(\zeta), \tag{2}$$

which transplants the finite interval $(-1,1)$ to the two semi-infinite wall portions.

(b) First note that if $\zeta = (1-\epsilon)\mathrm{i}$ then

$$z = \frac{2(1-\epsilon)\mathrm{i}}{1-(1-\epsilon)^2} \to \frac{\mathrm{i}}{\epsilon} \tag{3}$$

confirming that i is the preimage of the point at infinity in the upper half z plane. Since, as $\zeta \to \mathrm{i}$,

$$z = \frac{2\zeta}{(\zeta-\mathrm{i})(\zeta+\mathrm{i})} = \frac{1}{(\zeta-\mathrm{i})}\frac{2\zeta}{(\zeta+\mathrm{i})} \to \frac{1}{\zeta-\mathrm{i}}. \tag{4}$$

The function

$$W(\zeta) = \frac{U}{\zeta-\mathrm{i}} \to Uz, \tag{5}$$

as required. On the other hand, as $\zeta \to -\mathrm{i}$,

$$\frac{dw}{dz} = \frac{W'(\zeta)}{f'(\zeta)} \to 0 \tag{6}$$

since $W'(\zeta)$ tends to a constant and $f'(\zeta)$ tends to infinity. Therefore the velocity vanishes as $\zeta \to -\mathrm{i}$, or equivalently, as $y \to -\infty$.
Finally, on $|\zeta| = 1$, corresponding to the wall under the mapping,

$$\overline{W(\zeta)} = \frac{U}{\overline{\zeta}+\mathrm{i}} = \frac{U}{1/\zeta+\mathrm{i}} = \frac{U\zeta}{1+\mathrm{i}\zeta} = -\mathrm{i}U + \frac{U}{\zeta-\mathrm{i}} = W(\zeta) - \mathrm{i}U, \tag{7}$$

where we have used the fact that $\overline{\zeta} = 1/\zeta$ on $|\zeta| = 1$ and therefore confirming that

$$\mathrm{Im}[W(\zeta)] = \frac{W(\zeta) - \overline{W(\zeta)}}{2\mathrm{i}} \tag{8}$$

is constant on the wall. Since the imaginary part of the complex potential is the stream function, this confirms that the wall is a streamline, as required.

meth seen ⇓

6, C

(c) First note that inverting the map yields

$$\zeta = \frac{1-\sqrt{1-z^2}}{z}, \tag{9}$$

where we have chosen the sign of the square root to ensure that $\zeta = 0$ correponds to $z = 0$. Hence $z = \pm\mathrm{i}$ correspond to preimages

$$\zeta_\pm = \pm\mathrm{i}s, \qquad s = \sqrt{2} - 1. \tag{10}$$

Therefore the required complex potential is

$$\tilde{W}(\zeta) = \frac{U}{\zeta - \mathrm{i}} - \frac{\mathrm{i}\Gamma_+}{2\pi}\log\left[\frac{\zeta-\zeta_+}{|\zeta_+|(\zeta - 1/\overline{\zeta_+})}\right] - \frac{\mathrm{i}\Gamma_-}{2\pi}\log\left[\frac{\zeta-\zeta_-}{|\zeta_-|(\zeta - 1/\overline{\zeta_-})}\right], \quad (11)$$

where we have used the result, established in lectures, that the complex potential for a point vortex of circulation Γ at $\zeta = a$ in a unit disc (with impenetrable boundary) is

meth seen ⇓

5, B

$$-\frac{\mathrm{i}\Gamma}{2\pi}\log\left[\frac{\zeta - a}{|a|(\zeta - 1/\overline{a})}\right].$$

(d) First we need to find the preimage of the point $(0, \sqrt{3})$ or $z = \sqrt{3}\mathrm{i}$. Using the formula in part (c) this is

$$\zeta_s = \frac{1 - \sqrt{1+3}}{\sqrt{3}\mathrm{i}} = \frac{\mathrm{i}}{\sqrt{3}}.$$

Next we need to use the chain rule to find

$$\frac{d\tilde{w}}{dz} = \frac{\tilde{W}'(\zeta)}{f'(\zeta)},$$

where $\tilde{w}(z)$ is the complex potential associated with the flow. From part (c),

$$\tilde{W}'(\zeta) = -\frac{U}{(\zeta-\mathrm{i})^2} - \frac{\mathrm{i}\Gamma_+}{2\pi}\left[\frac{1}{\zeta-\zeta_+} - \frac{1}{\zeta - 1/\overline{\zeta_+}}\right] - \frac{\mathrm{i}\Gamma_-}{2\pi}\left[\frac{1}{\zeta-\zeta_-} - \frac{1}{\zeta - 1/\overline{\zeta_-}}\right].$$

By differentiation of the result of part (a) it also follows that

$$\frac{d}{d\zeta}\left(\frac{2\zeta}{1+\zeta^2}\right) = \frac{2}{1+\zeta^2} - \frac{4\zeta^2}{(1+\zeta^2)^2} = \frac{2(1-\zeta^2)}{(1+\zeta^2)^2}.$$

Therefore an explicit formula for the required speed (u_s, v_s) is given by

$$u_s - \mathrm{i}v_s = \left[-\frac{U}{(\zeta_s-\mathrm{i})^2} - \frac{\mathrm{i}\Gamma_+}{2\pi}\left[\frac{1}{\zeta_s-\zeta_+} - \frac{1}{\zeta_s - 1/\overline{\zeta_+}}\right] - \frac{\mathrm{i}\Gamma_-}{2\pi}\left[\frac{1}{\zeta_s-\zeta_-} - \frac{1}{\zeta_s - 1/\overline{\zeta_-}}\right]\right] \times \frac{(1+\zeta_s^2)^2}{2(1-\zeta_s^2)},$$

meth seen ⇓

where

5, D

$$\zeta_s = \frac{\mathrm{i}}{\sqrt{3}}, \qquad \zeta_\pm = \pm\mathrm{i}(\sqrt{2}-1).$$

Note: In part (d) it is important that students realize that the chain rule must be used, that they compute the derivatives correctly, and that all quantities in the given expression is defined explicitly.

3. (a) The stream function for a point vortex of circulation Γ at some projected point a is

$$\psi_{PV}(\zeta,\overline{\zeta}) = -\frac{\Gamma}{4\pi}\log\left[\frac{(\zeta-a)(\overline{\zeta}-\overline{a})}{(1+\zeta\overline{\zeta})(1+a\overline{a})}\right].$$

Let one of the point vortices on the latitude circle be at $(\theta,\phi)=(\theta_0,0)$ then, on setting

$$a = \cot(\theta_0/2)$$

the stream function for the configuration of 4 point vortices is

$$\psi(\zeta,\overline{\zeta}) = -\frac{\Gamma}{4\pi}\log\left[\frac{\zeta\overline{\zeta}(\zeta^3-a^3)(\overline{\zeta^3}-\overline{a^3})}{(1+\zeta\overline{\zeta})^4(1+a\overline{a})^4}\right].$$

meth seen ⇓

where we have used the fact that the stream function for a point vortex of circulation Γ at the south pole, corresponding to $\zeta=0$, is

4, A

$$\psi_{PV}(\zeta,\overline{\zeta}) = -\frac{\Gamma}{4\pi}\log\left[\frac{\zeta\overline{\zeta}}{(1+\zeta\overline{\zeta})}\right].$$

(b) For solid body rotation with angular velocity Ω a typical point at latitude θ has zonal velocity

$$u_\phi = \Omega\times\sin\theta.$$

On use of the given formula

$$u_\phi - \mathrm{i}u_\theta = 2\sqrt{\frac{\zeta}{\overline{\zeta}}}(1+\zeta\overline{\zeta})\frac{\partial\psi}{\partial\zeta}$$

we find

$$u_\phi - \mathrm{i}u_\theta = \sqrt{\frac{\zeta}{\overline{\zeta}}}(1+\zeta\overline{\zeta})\frac{\partial\psi_{SBR}}{\partial\zeta} = \Omega\times\sin\theta.$$

But it can be checked using simple trigonometry (this is quotable from lectures, or can be derived using the definition of ζ in terms of θ and ϕ) that

$$\sin\theta = \frac{2\sqrt{\zeta\overline{\zeta}}}{(1+\zeta\overline{\zeta})}$$

hence

$$\sqrt{\frac{\zeta}{\overline{\zeta}}}(1+\zeta\overline{\zeta})\frac{\partial\psi_{SBR}}{\partial\zeta} = \Omega\times\frac{2\sqrt{\zeta\overline{\zeta}}}{(1+\zeta\overline{\zeta})},$$

or

$$\frac{\partial\psi_{SBR}}{\partial\zeta} = \Omega\times\frac{2\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}$$

seen ⇓

implying, on integration with respect to ζ, that

4, A

$$\psi_{SBR} = -\frac{2\Omega}{1+\zeta\overline{\zeta}}. \tag{12}$$

No function of $\overline{\zeta}$ can be added since ψ_{SBR} is required to be real. A possible additive constant has been set to zero since it is inconsequential for the velocity field.

(c) We expect, from the symmetry of the arrangement, that the vortices on the latitude circle will rotate with constant angular velocity Ω, say, about the north-south axis. In a corotating frame with this angular velocity the stream function is

$$\psi(\zeta,\overline{\zeta}) = -\frac{\Gamma}{4\pi}\log\left[\frac{\zeta\overline{\zeta}(\zeta^3-a^3)(\overline{\zeta^3-a^3})}{(1+\zeta\overline{\zeta})^4(1+a\overline{a})^4}\right] + \frac{2\Omega}{1+\zeta\overline{\zeta}}.$$

This can be written as

$$\begin{aligned}\psi(\zeta,\overline{\zeta}) &= -\frac{\Gamma}{4\pi}\log\left[\frac{\zeta\overline{\zeta}(\zeta-a)\overline{(\zeta-a)}(\zeta^2+\zeta a+a^2)\overline{(\zeta^2+\zeta a+a^2)}}{(1+\zeta\overline{\zeta})^4(1+a\overline{a})^4}\right] + \frac{2\Omega}{1+\zeta\overline{\zeta}} \\ &= -\frac{\Gamma}{4\pi}\log\left[\frac{(\zeta-a)(\overline{\zeta}-\overline{a})}{(1+\zeta\overline{\zeta})(1+a\overline{a})}\right] \\ &\quad \underbrace{-\frac{\Gamma}{4\pi}\log\left[\frac{\zeta\overline{\zeta}(\zeta^2+\zeta a+a^2)\overline{(\zeta^2+\zeta a+a^2)}}{(1+\zeta\overline{\zeta})^3(1+a\overline{a})^3}\right] + \frac{2\Omega}{1+\zeta\overline{\zeta}}}_{\text{non self induced term}}.\end{aligned}$$

Since the velocity is proportional to the partial derivative of the stream function with respect to ζ, for the configuration to be stationary we therefore need this partial derivative of the non-self-induced contribution to the stream function to vanish at $\zeta = a$ (by symmetry, this will ensure stationarity at the other vortices):

$$\left\{-\frac{\Gamma}{4\pi}\left[\frac{1}{\zeta}+\frac{2\zeta+a}{\zeta^2+a\zeta+a^2}-\frac{3\overline{\zeta}}{1+\zeta\overline{\zeta}}\right]-\frac{2\Omega\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}\right\}\Bigg|_{\zeta=a} = 0.$$

After simplication this yields

$$\Omega = \frac{\Gamma(1+a^2)}{8\pi a^2}(a^2-2).$$

This clearly vanishes – and hence we have equilibrium – when

$$a = \sqrt{2} = \cot(\theta_0/2).$$

It follows that

$$\theta_0 = 2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

$\boxed{\text{meth seen } \Downarrow}$

(d) From the symmetry of this figure, there are **two** distances to consider: the distance between the polar vortex and any ring vortex; or the distance between any two ring vortices. These are shown by double arrows in the figure.

$\boxed{6, \text{ C}}$

The distance between the vortex at the south pole and any other vortex is clearly

$$2\sin(\chi/2),$$

where χ is the angle shown in the figure below. But, by simple trigonometry,

$$\cot(\chi/2) = \tan(\theta_0/2) = \frac{1}{\sqrt{2}}$$

from part(c), hence, again by simple trigonometry,

$$\sin(\chi/2) = \frac{\sqrt{2}}{\sqrt{3}}$$

and the required distance between vortices is

$$2 \times \sin(\chi/2) = \frac{2\sqrt{2}}{\sqrt{3}}.$$

On the other hand, the distance between any two of the latitudinal vortices is $2L$ where, from simple trigonometry (see right hand schematic below, "looking down" on the latitudinal ring):

$$\frac{L}{\sin\theta_0} = \frac{\sqrt{3}}{2}.$$

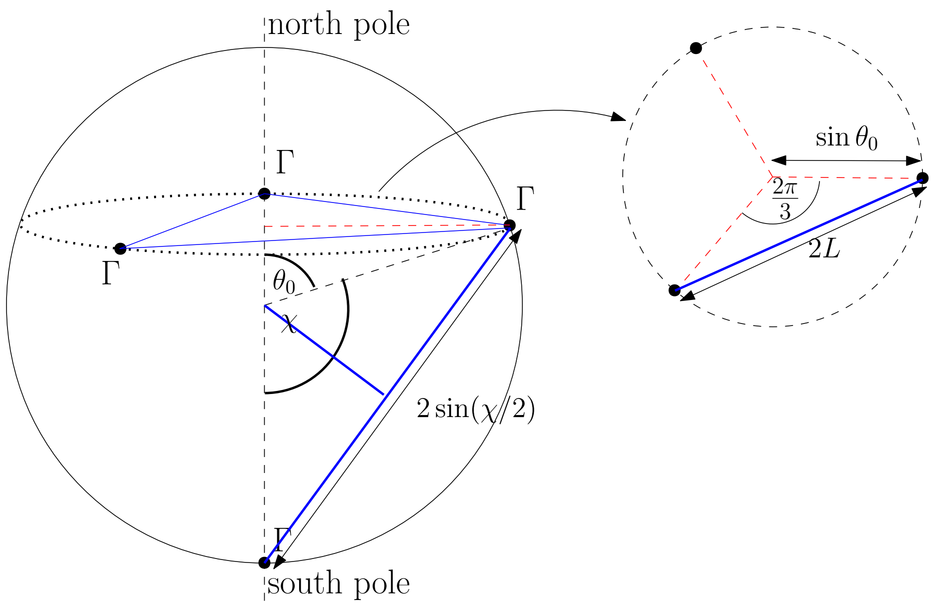
unseen ⇓

6, D

Therefore the distance between vortices is

$$2L = \sqrt{3}\sin\theta_0 = 2\sqrt{3}\sin(\theta_0/2)\cos(\theta_0/2) = 2\sqrt{3} \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}}.$$

These two distances are clearly the same, showing that all vortices are separated by the same distance at equilibrium.



Side note: the point vortices in this equilibrium lie at the vertices of a tetrahedron.

4. The map is

$$z = x + \mathrm{i}y = f(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta^2, \qquad \alpha, \beta \in \mathbb{R}$$

from the interior of a unit disc $|\zeta| < 1$ in a parametric ζ plane to the fluid region *exterior* of the vortex patch.

(a) Since

$$u - \mathrm{i}v = -2\dot{\gamma}xy + \mathrm{i}\dot{\gamma}(x^2 - y^2) = \mathrm{i}\dot{\gamma}z^2 \tag{13}$$

and since

$$2\mathrm{i}\frac{\partial\psi}{\partial z} = u - \mathrm{i}v \tag{14}$$

then, in the far-field,

$$\frac{\partial\psi}{\partial z} \to \frac{\dot{\gamma}z^2}{2} + \text{decaying terms.} \tag{15}$$

$\boxed{\text{meth seen} \Downarrow}$

$\boxed{4, \text{A}}$

(b) For a vortex patch D, say, the vorticity distribution is

$$\omega = -4\frac{\partial^2\psi}{\partial z\partial\overline{z}} = \begin{cases} \omega_0, & z \in D, \\ 0, & z \notin D. \end{cases} \tag{16}$$

On integration with respect to $\overline{z}$, and use of the boundary condition from part (a),

$$\frac{\partial\psi}{\partial z} = \begin{cases} -(\omega_0/4)(\overline{z} - C_i(z)), & z \in D, \\ (\omega_0/4)C_o(z) + (\dot{\gamma}/2)z^2, & z \notin D, \end{cases} \tag{17}$$

where $C_i(z)$ is analytic inside D and $C_o(z)$ is analytic outside D and decaying as $|z| \to \infty$. These analytic functions appear on integration with respect to $\overline{z}$. Continuity of velocity on the boundary of the patch ∂D implies

$$-(\omega_0/4)(\overline{z} - C_i(z)) = (\omega_0/4)C_o(z) + (\dot{\gamma}/2)z^2, \qquad z \in \partial D$$

or

$$-\overline{z} + C_i(z) = C_o(z) + (2\dot{\gamma}/\omega_0)z^2, \qquad z \in \partial D$$

We can rewrite this as a scalar Riemann-Hilbert problem,

$$\begin{aligned} C_i(z) - C_o(z) &= (2\dot{\gamma}/\omega_0)z^2 + \overline{z} \\ &= (2\dot{\gamma}/\omega_0)z^2 + \alpha\zeta + \frac{\beta}{\zeta^2}. \end{aligned} \tag{18}$$

The aim is to identify functions that are patently analytic inside the patch (e.g. polynomials in z) and analytic outside the patch (e.g. polynomials in ζ). Now, from the conformal mapping,

$$\frac{1}{\zeta} = \frac{z}{\alpha} - \frac{\beta}{\alpha}\zeta^2$$

hence we can write

$$\begin{aligned} C_i(z) - C_o(z) &= (2\dot{\gamma}/\omega_0)z^2 + \alpha\zeta + \beta\left(\frac{z}{\alpha} - \frac{\beta}{\alpha}\zeta^2\right)^2 \\ &= (2\dot{\gamma}/\omega_0)z^2 + \alpha\zeta + \frac{\beta z^2}{\alpha^2} - \frac{2\beta^2}{\alpha^2}(z\zeta^2) + \frac{\beta^3}{\alpha^2}\zeta^4. \end{aligned} \tag{19}$$

But, again from the conformal mapping,

$$z\zeta^2 = \alpha\zeta + \beta\zeta^4 \tag{20}$$

therefore

$$\begin{aligned} C_i(z) - C_o(z) &= (2\dot\gamma/\omega_0)z^2 + \alpha\zeta + \frac{\beta z^2}{\alpha^2} - \frac{2\beta^2}{\alpha^2}(\alpha\zeta + \beta\zeta^4) + \frac{\beta^3}{\alpha^2}\zeta^4 \\ &= (2\dot\gamma/\omega_0 + \beta/\alpha^2)z^2 + \left(\alpha - 2\beta^2/\alpha\right)\zeta - \frac{\beta^3}{\alpha^2}\zeta^4 \end{aligned} \tag{21}$$

From this form of the right hand side we can read off, based on the required analyticity and decay properties, that

$$C_i(z) = (2\dot\gamma/\omega_0 + \beta/\alpha^2)z^2, \qquad C_o(z) = \frac{\beta^3}{\alpha^2}\zeta^4 - \left(\alpha - 2\beta^2/\alpha\right)\zeta,$$

$\boxed{\text{meth seen} \Downarrow}$

where clearly $C_i(z)$ is analytic inside the patch, and $C_o(z)$ is analytic inside $|\zeta| < 1$, i.e., outside the patch, and decays as $|z| \to \infty$.

$\boxed{5, \text{D}}$

(c) It is clear that the interior flow inside the patch is pure solid body rotation, i.e. no irrotational strain component $C_i(z) = 0$, if we pick

$\boxed{\text{meth seen} \Downarrow}$

$$2\dot\gamma/\omega_0 = -\beta/\alpha^2. \tag{22}$$

$\boxed{4, \text{B}}$

(d) From part (b),

$$\begin{aligned} \frac{\partial\psi}{\partial z} &= (\omega_0/4)C_o(z) + \frac{\dot\gamma z^2}{2} \\ &= \frac{\dot\gamma z^2}{2} + \frac{\omega_0}{4}\left(-\left(\alpha - 2\beta^2/\alpha\right)\zeta + \frac{\beta^3}{\alpha^2}\zeta^4\right) \end{aligned} \tag{23}$$

$\boxed{\text{meth seen} \Downarrow}$

Hence as $|z| \to \infty$, since $\zeta \to \alpha/z$ to leading order,

$\boxed{3, \text{A}}$

$$\begin{aligned} \frac{\partial\psi}{\partial z} &\to \frac{\dot\gamma z^2}{2} + \frac{\omega_0}{4}\left(\left(2\beta^2/\alpha - \alpha\right)\frac{\alpha}{z}\right) + \text{faster decaying terms} \\ &= \frac{\dot\gamma z^2}{2} + \frac{\omega_0}{4}\left(\left(2\beta^2 - \alpha^2\right)\frac{1}{z}\right) + \text{faster decaying terms.} \end{aligned} \tag{24}$$

This is the required result.

(e) Since the circulation Γ of the vortex patch is

$$\Gamma = \omega_0\mathcal{A}, \tag{25}$$

where $\mathcal{A}$ is its area, then in the far field we expect the flow generated by it to resemble that due to a point vortex of the same circulation, i.e.

$$u - \mathrm{i}v \to -\frac{\mathrm{i}\Gamma}{2\pi}\frac{1}{z} + \text{faster decaying terms.} \tag{26}$$

Hence using the formula

$$2\mathrm{i}\frac{\partial\psi}{\partial z} = u - \mathrm{i}v$$

then we expect

$$\frac{\partial\psi}{\partial z} \to -\frac{\Gamma}{4\pi}\frac{1}{z} + \text{faster decaying terms.}$$

Equating the $1/z$ terms in this last equation with the result in part (d) gives

$$-\frac{\Gamma}{4\pi} = -\frac{\omega_0\mathcal{A}}{4\pi} = \frac{\omega_0}{4}\left(2\beta^2 - \alpha^2\right)$$

meth seen ⇓

from which we deduce that

4, A

$$\mathcal{A} = \pi(\alpha^2 - 2\beta^2).$$

5. (a) The two-dimensional vorticity equation for $\omega = \partial v/\partial x - \partial u/\partial y$ is

$$\frac{D\omega}{Dt} = 0,$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla,$$

and where $\mathbf{u} = (u, v, 0)$. Since

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

it is clear that

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi.$$

The steady vorticity equations, in two dimensions, is

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0.$$

In terms of the stream function this is

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = 0.$$

Hence, if $\omega = h(\psi)$ where h is a differentiable function, then

$$\frac{\partial \omega}{\partial x} = h'(\psi)\frac{\partial \psi}{\partial x}, \qquad \frac{\partial \omega}{\partial y} = h'(\psi)\frac{\partial \psi}{\partial y},$$

which, on substitution into the left hand side of the steady vorticity equation, gives

$$h'(\psi)\left(\frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}\right)$$

seen ⇓

which is identically zero. Hence $\omega = h(\psi)$ represents a general solution of the steady two-dimensional Euler equation. 2, M

(b) Suppose

$$\psi = -\frac{1}{2}\log\left[\frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2}\right]$$

then, on exponentiation,

$$e^{-2\psi} = \frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2}.$$

On differentiation of ψ with respect to z we get

$$\frac{\partial \psi}{\partial z} = -\frac{1}{2}\frac{f''(z)}{f'(z)} + \frac{\overline{f(z)}f'(z)}{(1 + f(z)\overline{f(z)})}.$$

A derivative with respect to $\overline{z}$ produces

$$\frac{\partial^2 \psi}{\partial z \partial \overline{z}} = \frac{\overline{f'(z)}f'(z)}{(1 + f(z)\overline{f(z)})} - \frac{f(z)\overline{f(z)}f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2} = \frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2}.$$

Therefore

$$\nabla^2 \psi = 4\frac{\partial^2 \psi}{\partial z \partial \overline{z}} = 4\frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2} = 4e^{-2\psi}$$

hence this is a solution of

$$\omega = -\nabla^2\psi = h(\psi)$$

where $h(\psi) = -4e^{-2\psi}$ and is a solution of the steady Euler equation in D if $f(z)$ is analytic there and its derivative $f'(z) \neq 0$ in D (it is clear the ψ will have logarithmic singularities at any zeros of $f'(z)$). seen $\Downarrow$

3, M

(c)(i) As $|z| \to \infty$ then

$$f(z) \to -\mathrm{i}e^{\mathrm{i}z}, \qquad f'(z) \to e^{\mathrm{i}z}.$$

Hence

$$\psi \to -\frac{1}{2}\log\left[\frac{e^{\mathrm{i}(z-\overline{z})}}{(1+e^{\mathrm{i}(z-\overline{z})})^2}\right].$$

But

$$\mathrm{i}(z-\overline{z}) = -2y,$$

so that as $|y| \to \infty$,

$$\begin{aligned}\psi \to -\frac{1}{2}\log\left[\frac{e^{-2y}}{(1+e^{-2y})^2}\right] &= \begin{cases} -\frac{1}{2}\log\left[e^{-2y}\right], & y \to +\infty, \\ -\frac{1}{2}\log\left[\frac{1}{e^{-2y}}\right], & y \to -\infty, \end{cases} \\ &= \begin{cases} y, & y \to +\infty, \\ -y, & y \to -\infty. \end{cases}\end{aligned}$$

Therefore the flow becomes uniform $(u,v) \to (U,0)$ as $y \to \pm\infty$ with

$$U = \begin{cases} 1, & y \to +\infty, \\ -1, & y \to -\infty. \end{cases}$$

unseen $\Downarrow$

4, M

(c)(ii) By differentiation, it can be shown that

$$f'(z) = e^{\mathrm{i}z}\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)^2. \tag{27}$$

On substitution into the expression for ψ it is clear that, locally near $z = -\mathrm{i}$,

$$\begin{aligned}\psi &= -\frac{1}{2}\log\left[|f'(z)|^2\right] - \frac{1}{2}\log\left[\frac{1}{(1+f(z)\overline{f(z)})^2}\right] \\ &= -\log|f'(z)| + \text{locally analytic function}, \\ &= -\log|z+\mathrm{i}|^2 + \text{locally analytic function}.\end{aligned}$$

Since we know that the stream function for a point vortex of circulation Γ at $z = z_0$ in the plane is

$$-\frac{\Gamma}{4\pi}\log|z-z_0|^2$$

then there is a point vortex at $z_0 = -\mathrm{i}$ of circulation $\Gamma = 4\pi$.

unseen $\Downarrow$

3, M

(c)(iii) From the expression above,

$$\frac{\partial\psi}{\partial z} = -\frac{1}{2}\frac{f''(z)}{f'(z)} + \frac{\overline{f(z)}f'(z)}{(1+f(z)\overline{f(z)})}$$

and

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial \psi}{\partial z}. \tag{28}$$

It is also easy to check, from (27), that

$$\frac{f''(z)}{f'(z)} = \mathrm{i} + \frac{2}{z+\mathrm{i}} - \frac{2}{z-\mathrm{i}}.$$

Therefore,

$$\begin{aligned} u - \mathrm{i}v &= 2\mathrm{i}\frac{\partial \psi}{\partial z} \\ &= 2\mathrm{i}\left\{ -\frac{1}{2}\left[\mathrm{i} + \frac{2}{z+\mathrm{i}} - \frac{2}{z-\mathrm{i}}\right] + \frac{\overline{f(z)}f'(z)}{(1+f(z)\overline{f(z)})} \right\} \\ &= 2\mathrm{i}\left\{ -\frac{1}{z+\mathrm{i}} \underbrace{-\frac{1}{2}\left[\mathrm{i} - \frac{2}{z-\mathrm{i}}\right] + \frac{\overline{f(z)}f'(z)}{(1+f(z)\overline{f(z)})}}_{\text{non}-\text{self}-\text{induced term}} \right\}. \end{aligned} \tag{29}$$

For the point vortex at $z = -\mathrm{i}$ to be stationary the non-self-induced term indicated must vanish. But the final term of this expression, proportional to $f'(z)$, clearly vanishes since $f'(z)$ itself vanishes at $z = -\mathrm{i}$. It therefore only remains to check that

$$-\frac{1}{2}\left[\mathrm{i} - \frac{2}{z-\mathrm{i}}\right]\Bigg|_{z=-\mathrm{i}} = 0. \tag{30}$$

But this readily verified. Hence this point vortex is in equilibrium.

unseen ⇓

5, M

(c)(iv) From the form of ψ its singularities can only occur as the zeros of $f'(z)$ and at the poles of $f(z)$ that are of order 2 or higher. If $f(z)$ has a simple pole (as it does in the given $f(z)$), i.e.,

$$f(z) = \frac{\mu}{z - z_0} + \text{locally analytic}$$

then

$$f'(z) = -\frac{\mu}{(z-z_0)^2} + \text{locally analytic}$$

so that the second order pole singularities at z_0 in both the numerator and denominator cancel out, meaning that ψ is, in fact, regular at z_0. Therefore the only singularity of ψ for the given $f(z)$ is the point vortex at $z = -\mathrm{i}$ and this has been shown to be stationary. Therefore ψ represents a global equilibrium of the 2d vorticity equation.

unseen ⇓

3, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam

ExamModuleCode	Question	Comments for Students
MATH97013/MATH97092	1	This question was done well: in the final part, some students failed to appreciate that the eigenvectors of a matrix can (and should) be used to plot the principal axes of strain.
	2	This question was done well too: the conformal mapping was familiar from discussions during the lectures. Part (b) was found to be the hardest, with many students failing to realize that they need to ensure the given function satisfies the condition of no-flow through the wall.
	3	This question was found to be the hardest of the non-Mastery questions. The final part showing that the vortices in equilibrium are equidistant was not done successfully by any candidates.
	4	Some excellent answers to this question. I was impressed by the students' overall understanding of these quite technical concepts.
	5	This question had some very good, but also some very poor, attempts. However, that is partly the point of mastery questions - to be more challenging. I was surprised to see a couple of students who failed even to appreciate what the steady vorticity equation is!