

## Problem Sheet 1

1. Compute the residues of

- a).  $\frac{z^3 \sin z}{z^4 + a^4}$  at  $z = ae^{i\pi/4}$ , where  $a > 0$  is a non-zero real constant.
- b).  $\frac{z+1}{(z^2-1)^2}$  at  $z = 1$ .
- c).  $\frac{e^z}{z(z-a)^2}$  at  $z = a$ , where  $a$  is a non-zero complex constant.
- d).  $\frac{z^2 e^z}{z^3 - a^3}$  at  $z = a$ , where  $a$  is a non-zero complex constant.

2. Use contour integration to show that

- a).  $\int_0^{2\pi} \frac{1}{5 - 4 \cos \theta} d\theta = \frac{2\pi}{3}$
- b).  $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta = \frac{\pi}{6}$
- c).  $\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{6}$
- d).  $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$
- e).  $\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1} dx = -\frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}} \sin 1$
- f).  $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} dx = \frac{\pi}{2} e^{-2}$

Hint: For (a) and (b), use the change of variables  $z = e^{i\theta}$ .

3. Prove the *triangle inequality*, and the *negative triangle inequality* which states that for complex numbers  $z_1, z_2 \in \mathbb{C}$ ,

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

4. In a programming language of your choice, create a function  $s(x, \theta)$  which returns the value of the positive square root function, with the branch cut moved to  $\arg z = \theta$ . Note that the standard convention is  $\theta = \pi$ , and you will need to call this kind of default square root command from within your function. Test your function by plotting  $\arg s(x, \theta)$ , and checking for the right discontinuity.

5. Use contour integration to compute

$$\int_0^{\infty} \frac{x^{a-1}}{(x+1)^2} dx,$$

where  $0 < a < 2$ ,  $a \neq 1$ .

6. Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \pi(b - a),$$

where  $a, b > 0$ .

7. By integrating around a rectangular contour with vertices at  $\pm R$  and  $2\pi i \pm R$  (taking  $R \rightarrow \infty$ ) and with appropriate indentations, compute

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 - e^x} dx,$$

where  $0 < a < 1$ .

8. a). By integrating around a rectangular contour with vertices at  $\pm R$  and  $\pi i \pm R$  (taking  $R \rightarrow \infty$ ), show that

$$\int_{-\infty}^{\infty} e^{ikx} \operatorname{sech} x dx = \pi \operatorname{sech}(k\pi/2),$$

where  $k > 0$ .

- b). Now obtain the result from part (a) using a semi-circular contour (*Hint: When summing the residues, note a geometric series*).