Imperial College London

## Department of Mathematics

## M3H History of Mathematics

| Question | Examiner's Comments |
| --- | --- |
| Q 1 | See longhand comments |
| Q 14 | Long Q: Set theory and foundations |

# M3H Exam 2018

Overall.

The unusual feature about this set of scripts (25 of them) is that there were considerably more than the number of students I ever saw. This started at around 20 (for reconnaissance purposes), soon shrank to 17, then down to well into single figures. This showed up in the results in several ways:

several failures (unusual for M3H), and several bare passes;

the number of good/very good scripts about the number of hard-core attenders;

a strong tendency for the weaker scripts to focus on the earlier questions (the question numbering reflected the chronological order).

Nearly everyone learned something useful. But, M3H is not a `half-course’, or a `soft course’: it requires a proper commitment from the student, like any other course.

Imperial College
London

M3H

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**History of Mathematics**

Date: Tuesday, 08 May 2018

Time: 10:00 AM - 12:00 PM

Time Allowed: 2 hours

**There are two sections in the paper.**

**Answer 5 questions out of 10 in section A.**

**Answer 2 questions out of 4 in section B.**

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

**Section A: answer 5 questions out of 10; 10 marks each.**

Give brief historical accounts of the following

(A1) Pythagoras and the Pythagoreans.

(A2) Cardano.

(A3) The abacus and its successors.

(A4) Centres of learning.

(A5) The development of decimals.

(A6) Mechanics up to and including Newton.

(A7) The wave equation.

(A8) Algebraic, transcendental and computable numbers.

(A9) Hilbert.

(A10) The Prime Number Theorem.

**Section B: answer 2 questions out of 4; 25 marks each.**

Give historical accounts of the following

(B1) Archimedes.

(B2) Geometry post-Renaissance.

(B3) The development of group theory.

(B4) Set theory, logic and foundations.

# M3H HISTORY OF MATHEMATICS: EXAMINATION SOLUTIONS, 2017-18.

The specimen solutions below are lightly edited from the relevant course teaching material. So they include full detail on dates etc. Students are not expected to memorise dates! The main points will suffice, along the lines of 'Who did what, (roughly) when'.

A1. *Pythagoras of Samos* (c. 580 – 500 BC; Samos: Dodecanese island)

Like Thales, Pythagoras travelled to Egypt and Mesopotamia, then returned to found a semi-monastic school – part academy, part secret society – at Croton ('instep of Italy'). His personal influence is difficult to separate from that of his school. The terms philosophy and mathematics are attributed to him. He believed in the transmigration of souls (Shakespeare: *Merchant of Venice* 4.1, 129-132; *Twelfth Night*, 4.2, 51-60).

As such quotations show, he had an impact for beyond mathematics proper: 'Never before or since has mathematics played so large a part in life and religion as it did among the Pythagoreans' (Boyer). But one could argue the converse: the number mysticism of the Pythagoreans ('All is number') imported into mathematics elements of superstition alien to it. **[3]**

*Pythagoras' Theorem*: In a right-angled triangle, the square on the hypotenuse [the 'long' side, opposite the right angle] is equal to the sum of the squares on the other two sides.

There is strong evidence that this *result* was known to the Babylonians (presumably without proof). Also, Pythagoras is known to have drawn mathematical inspiration from Babylon, and his number-superstition smacks too of Babylonian influence. We do not know whether or not Pythagoras had a *proof* of this theorem. **[2]**

*The pentagram; the golden section:* This was known to the Pythagoreans. [1]

*Cosmology*. The Pythagorean picture of cosmology was one in which the earth and planets (and sun) orbited about a 'central fire'. This sounds very modern nearly two millennia before Copernicus – but alas, it is based on supposed mystical properties of the number ten. **[1]**

*Harmonics*. Pythagoras is reputed to have noticed that dividing a vibrating string in simple arithmetic ratios produces harmonious overtones. 'Here we have perhaps the earliest quantitative law of acoustics – possibly the oldest of all quantitative physical laws' (Boyer). **[1]**

*Pythogorean triples*. These are triples of positive integers which can form the sides of a right-angled triangle: (3,4,5), (5,12,13) etc. **[1]**

*The end of the Pythagoreans*. After Pythagoras died (c. 500 BC) his school continued at Croton, but was later violently suppressed. The survivors scattered to spread their views in other parts of the Greek world. **[1]**

[Seen – lectures]

A2. *Girolamo Cardano* (1501-76): *Ars Magna (AM)*, 1545.

Cardano's *solution of the cubic* (published here in 1545) marks 'the beginning of the modern period in mathematics'. Scipione del Ferro (c. 1465-1526), Professor of Mathematics at Bologna, solved the cubic but did not publish his results. Niccolo Tartaglia (c. 1500-1557) (b. Niccolo Fontana; Tartaglia = stammerer), knowing of del Ferro's solution, found one himself, by 1541. Predictably, this led to a priority dispute with Cardano. [2]

*Complex numbers.* Though not properly assimilated into mathematics till much later (e.g. Argand diagram), these enter the stage with AM. In Ch. 37 of AM, Cardano solves

$$x(10-x) = 40,$$

obtaining roots $5 \pm \sqrt{-15}$, and notes that

$$(5+\sqrt{-15})(5-\sqrt{-15}) = 25-(-15) = 40.$$

Thus (Kline, 253) 'Without having fully overcome their difficulties with irrational and negative numbers, the Europeans added to their problems by blundering into what we now call complex numbers.'

As Boyer notes (p.322), 'whenever the three roots of a cubic are real and non-zero, the Cardan-Tartaglia formula leads inevitably to square roots of negative numbers'. For a detailed account of this 'irreducible case' of the cubic, and of the quartic, see e.g. W. L. Ferrar, *Higher algebra*, Ch. XXI. [3]

*Quintics and polynomials of higher degree*

The solution of cubics and quartics prompted an attack on the quintic and polynomials of degree at least 5. What was sought was a 'solution by radicals': a *formula* for expressing (or an algorithm for finding) the roots in terms of the coefficients. We now know that:

(i) Although a polynomial of degree $n$ does indeed have $n$ roots (possibly complex, counted according to multiplicity) (Fundamental Theorem of Algebra: Complex Analysis, 19th C.), (ii) If $n \geq 5$, no solution by radicals exists (Abel, Galois, Algebra – Field Theory; 19th C.).

The quintic here joins the classical Greek problems (circle-squaring, cube-duplicating, angle-trisecting and the proof of the parallel postulate) among the Holy Grails of mathematics. None of these problems is soluble, but, the search for a solution led to other things. [3]

Cardano (c. 1526) wrote a book *De Ludo Aleae* (On Dice Games), published posthumously in 1663. This was the first book written (though not the first book published) on Probability Theory. [2]

[Seen, lectures]

A3. *The abacus and its successors.*

*The Abacus* [4]

Herodotus (of Halicarnassus, Asia Minor, c. 484 – c. 425 BC), the Greek historian regarded as the 'father of history', records the use an abacus to solve arithmetical problems.

Early counting was done by means of small stones or pebbles (calculus = pebble, Latin; hence calcaria = limestone; chalk; calcium). Calculations were done by arranging pebbles on a flat surface (abax = flat surface, Greek). Boards divided up into squares, as with chess, were used.

The Greeks excelled in trade, and were adept at the calculations needed for commerce. By contact with Greek culture, Greek use of calculating techniques and devices passed to the Romans.

The upshot of all this is that calculating devices needed for the purposes of trade and administration were widely available, and widely used, throughout the advanced civilisations of the ancient world.

The term 'abacus' today refers to a frame containing parallel wires on which beads can move. Such abaci survived to modern times in China, Japan etc. They were made obsolete for calculation by the widespread availability of pocket calculators from the mid-1970s, and for commercial use by automatic tills with liquid-crystal display (LCD), etc.

*John Napier* (1550-1617) and *logarithms.* [3]

*Mirifici logarithmorum canonis descriptio* (1614),

*Mirifici logarithmorum canonis constructio* (1619, posth.)

Napier, a Scottish baron, was led to his discovery of logarithms (logos = ratio + arithmos = number) through being told (by Craig) of the use of prosthaphaeresis (by Tycho Brahe). His logarithms were an early form of modern logarithms to base 10. Logs to base 10 were used in schools (together with slide rules) until the pocket calculator arrived in the early 1970s.

*The computer.* [3]

The modern computer emerged from the demands of WWII (e.g., Bletchley Park; Turing; Enigma machines). After the war, Turing went from Bletchley Park, where he headed the mathematical group, to Manchester, where he worked on the first stored-programme electronic computer. The change from thermionic valves to microchips depended on later advances in the physics of semi-conductors, as does LCD etc. The change from mainframe computers to PCs has come in the last twenty years or so, as has e-mail and the Internet. [Seen – lectures]

A4. *Centres of learning.*

*Ancient academies*

The earliest important centres of learning are from the ancient Greek world: the *Pythagorean school* at Croton ('Italian instep'), 6th C. BC; *Plato's academy* at Athens (c. 400 BC; Emperor Justinian closed the Academy at Athens 529 AD), and the *Museum of Alexandria* (from its foundation by Alexander the Great, c. 330 BC). These were succeeded by the *House of Wisdom* in Baghdad (Caliph al-Mamun, 809-833) (Baghdad, then under the Abbasid Caliphate, fell to the Mongols in 1258). **[2]**

Spain under the Moors was an important centre of learning. Here scholars from Arab, Jewish and Christian backgrounds lived together in Cordoba, Toledo and elsewhere; important work was done on translation. **[1]**

*Early universities*

The academies (Pythagorean, Athenian, Alexandrian) of the ancient world played the role of universities in their day, as did their Arab counterparts in Baghdad, Cordoba etc. By the 12th C., the modern concept of a university as an autonomous academic institution awarding degrees, and as a centre of learning, teaching and research, began to emerge. This was a gradual process. The earliest continental universities are Bologna (founded 1088; Royal Charter 1130), Salamanca (founded 1134, from a Cathedral School, 1130; Royal Charter 1218) and Paris (mid-12th C.). In Britain, Oxford and Cambridge simply describe themselves as 'founded in the 12th C.' and 'founded in the 13th C.' respectively. Scotland has four ancient universities: St. Andrews, 1410; Glasgow, 1451; Aberdeen, 1495; Edinburgh, 1583. **[2]**

*Modern academies*

The *Royal Society of London (RS)* was founded in 1660 (Charter 1663).

The French *Académie des sciences* was founded in 1666.

The Royal Prussian Academy of Sciences was founded in Berlin in 1700.

The St. Petersburg Academy was founded by Catherine I, Peter the Great's widow, in 1725. **[2]**

*Modern universities*

In the UK, the major civic universities were founded around 1900 (Birmingham, Manchester etc.). Universities did not all admit women as full members; public funding was so scarce that private means were often necessary to study at university, etc. This gradually improved. The next wave of new universities (York, Sussex etc.) came in the 1960s. The former polytechnics attained university status in 1992. **[3]**

[Seen in lectures, except for the last part – general knowledge]

A5: *The emergence of the decimal system.*
*Aryabhata*, author of Aryabhatiya (499 AD) [2]

Here we find decimal-place notation: 'from place to place each is ten times the preceding'. The modern decimal numerals 1,2,3,4,5,6,7,8,9 are loosely called Arabic in English, but are called Hindu in Arabic; perhaps 'Hindu-Arabic' would be better. These evolved gradually; the key recognition that by use of place notation the same symbol could be used for three as for thirty, etc., had taken place by 595 AD (Indian source: date 346 in decimal notation), and in Western sources by 662 (Sebokt of Syria).

The zero symbol 0 came later. It had emerged by 876 in India (on an inscription in Gwalior), with the modern 0 for zero. The key components of (i) decimal base, (ii) positional notation, (iii) symbols for 0,1,2,...,9 were thus all in place. It seems that the Hindus did not invent any of them, but they did integrate them into (essentially) their modern form.

*Fibonacci, Leonardo of Pisa* (c.1180-1250) [2]

Leonardo of Pisa, son of Bonaccio (hence 'Fibonacci') wrote the *Liber Abaci* (Book of the Abacus) in 1202. This was the most influential European mathematical work before the Renaissance, and was the first such book to stress the value of the (Hindu-)Arabic numerals (Fibonacci had studied in the Muslim world and travelled widely in it).

*Nicholas Chuquet* (fl. c. 1500) [2]

*Triparty en la science des nombres*, 1484: the most important European mathematical text since the *Liber Abaci*.

Part I: Hindu-Arabic numerals; addition, subtraction; multiplication; division; Part II: Surds; Part III: Algebra; laws of exponents; solution of equations.

*Simon Stevin* (1548-1620) of Bruges (Brugge), Flemish military engineer. [2]
*Die Thiende, La disme* ('the tenth'), 1585.

Stevin's elementary book did more than any other to popularise the use of decimal fractions (decimals), and to spread awareness of their computational value and superiority.

*Napoleon Bonaparte* (1769-1821). [2]

Napoleon's main contribution to science was that his conquests spread the new metric system, in universal use today.

[Seen – lectures]

A6. *Mechanics up to Newton.*
*Archimedes* (of Syracuse, c.287-212 BC).

Archimedes constructed pulleys, by which he could move ships in Syracuse harbour single-handed. He was able to use this practically, in defending Syracuse against the Romans in the Second Punic War. He also worked on statics (law of the lever, moments, fulcrum etc.), and hydrostatics (Archimedes' Principle: a floating body displaces its own weight of water). [2]

*The Method.* In 1899 J. L. Heiberg found a palimpsest (a parchment which has been re-used) in the Library of the Monastery of the Holy Sepulchre in Jerusalem. Beneath the mediaeval liturgical text, he was able to decipher a previously unknown text by Archimedes, *The Method of Mechanical Theorems for Eratosthenes* (briefly, *The Method*). Here Archimedes describes the method which led him to, e.g., his quadrature of the parabola: an application of his 'principle of the lever' via a balancing argument. [2]

*Galileo Galilei* (1564-1642).
*The Two New Sciences* (1638): on mechanics. He showed that
(i) a body falling under gravity does so with constant acceleration;
(ii) the trajectory of a projectile is a parabola. [2]

*Sir Isaac Newton* (1642-1727); *Principia*, 1687
*Philosophiae naturalis principia mathematica*, 1687/1713/1726.

Newton's *Principia* is the most famous mathematical book ever published, and rightly so. It triggered the Scientific Revolution, and so helped to usher in the modern world. In the Preface, one finds *Newton's Laws of Motion*:
Law I (inertia): A body continues in its state of rest or uniform motion in a straight line unless force is applied to change it.
Law II: Force = Mass × acceleration; $F = ma$.
Law III: To every action there is an equal and opposite reaction. [2]

*Dynamics; Celestial Mechanics*

The great challenge of the new astronomy was to explain Kepler's Laws, arrived at empirically. It was suspected that an inverse square law of attraction was the key by, e.g., Hooke and Halley (Edmond Halley (1656-1742); 2nd Astronomer Royal). In 1684, Halley went to Cambridge, and asked Newton what the orbit of a body was under the inverse square law. Newton replied that it was an ellipse. Asked how he knew, he replied that he had calculated it (long before), but could not find the proof. Halley put the fear of being beaten by others (perhaps Hooke) into Newton's mind. This provoked Newton into writing the *Principia*, published in 1687 at Halley's expense. [2]
[Seen – lectures]

A7. *The wave equation.*
*Jean D'Alembert* (1717-1783) in 1746).

By considering the forces acting on a small segment of a string under tension and with small displacements, d'Alembert obtained

$$\partial^2 y/\partial x^2 = \frac{1}{c^2}\partial^2 y/\partial t^2,$$

the one-dimensional *wave equation*. Here $c$ has the dimensions of *velocity* $(L/T)$, and has the interpretation of the velocity of a *wave*. [2]

If $f$ is an arbitrary (smooth enough – twice continuously differentiable) function, differentiation shows that $f(x + ct)$ is a solution of the wave equation. Similarly, if $g$ is another arbitrary function, $g(x - ct)$ is also a solution. So by linearity, $f(x + ct) + g(x - ct)$ is also a solution to the wave equation. Think of $f$ as the profile of a wave. Then $f(x + ct)$ represents the wave travelling *left* with velocity $c$. Similarly, $g(x - ct)$ represents a wave with profile $g$ travelling to the *right* with velocity $c$. [2]
*Higher dimensions.* In two or three dimensions, the wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \qquad [1]$$

Useful general methods of solution include *separation of Variables*. [1]
*Boundary conditions (BCs).* If the string has length $\ell$, and is fixed at the ends $x = 0$ and $x = \ell$, one gets an infinity of solutions, $\sin k\ell = 0$ for $k\ell = n\pi$, $n$ integer, $k = n\pi/\ell$. By linearity of the wave equation, superpositions of these are solutions. This leads directly to *Fourier series* (*Joseph Fourier* (1768-1830); *Théorie Analytique de la Chaleur* (1822)). [2]
*The electromagnetic theory of light* (*James Clerk Maxwell* (1831-1879), *A treatise on electricity and magnetism*, Vol. 1, 2, OUP, 1891).
*Maxwell's Equations.* If $E$, $H$ are the electric intensity and the magnetic field in ES (electrostatic) units, $cE$ in EM (electromagnetic) units, *Maxwell's equations* (in a vacuum) lead to $\nabla^2 E = c^{-2}\partial^2 E/\partial t^2$, $\nabla^2 E = c^{-2}\partial^2 E/\partial t^2$. This is the *wave equation*, for propagation of $E$, $H$ with velocity $c$, the ratio of EM to ES units. This was known experimentally (c. $3 \times 10^{10}$ cm/sec., c. 186,000 miles/sec.) to be (approx.) the *speed of light*. Thus, *electromagnetic forces are propagated with the speed of light.* This suggested to Maxwell that *light waves are electromagnetic.* [2]
[Seen – lectures]

A8. *Algebraic, transcendental and computable numbers*

The irrational $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, and similarly for other 'surd-like' irrationals. Call a real number *algebraic* if it is a root of a polynomial with integer coefficients, *transcendental* otherwise. [1]

*Theorem* (Cantor, 1874). The algebraics are countable (and so the transcendentals are uncountable).

*Proof.* The algebraics are $\cup_{n=1}^{\infty} A_n$, where $A_n$ is the set of roots of polynomials of degree $n$ with integer coefficients. There are only countably many such polynomials $p(x) = a_0 x^n + \dots + a_n$; each has (at most) $n$ real roots (Fundamental Theorem of Algebra), so each $A_n$ is countable, so their union is countable. [2]

*Corollary.* Transcendental numbers exist! [Most reals are transcendental.]

This is a very fine example of a *non-constructive existence proof* – valuable, because it is *easy*. The 'obvious' – or constructive – way to show that transcendentals exist is to exhibit an example, but this is *much harder*. Joseph Liouville (1809-1882) showed in 1844 that decimals of the form $\sum a_n/10^{n!}$ are transcendental. Charles Hermite (1822-1901) showed in 1873 that $e$ is transcendental. Ferdinand Lindemann (1852-1939) showed in 1882 that $\pi$ is transcendental. As a corollary, *it is impossible to square the circle* (constructible numbers are algebraic; if the circle could be squared, $\sqrt{\pi}$ would be algebraic, so $\pi$ would be algebraic). [2]

*Note.* 1. This finally settles the ancient Greek problem (Anaxagoras). [1]

2. This convincingly shows the power of Cantor's new set theory. [1]

3. The same argument shows that the set of *computable numbers* is countable – so, most numbers are non-computable. For (though this is a 20th C. concept, due to Turing), a number is *computable* if it could in principle be the print-out from running a programme. But a programme is a finite string of symbols drawn from a finite alphabet; there can be at most countably many such (not finitely many – no upper bound on programme length), so there are only countably many computable numbers. Of course, $\pi$ is computable (from e.g. Brouncker's continued fraction), even though transcendental. [2]

*Generic reals.* So the typical (generic) real is not: rational; algebraic; computable. So: how do we get at it? Conclusion: the typical real is inaccessible in practical terms, even in principle! [1]

[Seen – lectures]

A9. *Hilbert*

*David Hilbert* (1862-1943), Professor of Mathematics at Göttingen, 1895. Hilbert came from Königsberg, where he was a pupil of Heinrich Weber (1842-1913); he succeeded Weber in Göttingen. His doctoral thesis of 1885 is on invariants. He proved the Hilbert Basis Theorem in 1888, and the Hilbert Nullstellensatz. **[2]**

*Zahlbericht* (1897) [Report on Numbers]. This book may be taken as the starting-point for the modern subject of Algebraic Number Theory; a key contributors to this was Hilbert's pupil E. Hecke (1887-1947). [1]

*Grundlagen der Geometrie* (1899) [foundations of Geometry]. Hilbert's book was an attempt to begin geometry axiomatically from scratch, by modern standards of rigour and with modern knowledge (such as non-Euclidean geometry) – "to bring Euclid's Elements up to date". Hilbert took the modern view of mathematics as *deductive reasoning from axioms*, and emphasised the importance of the *axiomatic method*, which we take for granted today. However, Hilbert's views on the foundations of mathematics were later shown to be too naive. [1]

*The Hilbert Problems.*

Hilbert proposed a famous list of (23) problems to the International Congress of Mathematicians in Paris in 1900. These have been extremely influential. The solution of a Hilbert problem is a major achievement in mathematics; for details of progress, see e.g. *Math. developments arising from Hilbert problems*, Proc. Symp. Pure Math. XXVIII, AMS, 1976. [2]

*The beginnings of functional analysis.*

Hilbert's pupil Erhard Schmidt (1876-1959) took his PhD in 1905. He worked on integral equations in *Hilbert space*, which he named – basically, the extension of Euclidean space to infinitely many dimensions. *Gram-Schmidt orthogonalisation* followed in 1907 and *Hilbert-Schmidt* operators in 1908. [2]

Hilbert's pupil Richard Courant (1888-1972) took his PhD in 1910 (Dirichlet's principle and conformal mapping). From Hilbert's lecture notes, Courant wrote 'Courant and Hilbert':

R. COURANT & D. HILBERT, *Methoden der mathematischen Physik*, I (1st ed. 1924, 2nd ed. 1931), II (1937).

This classic book arrived just in time to serve the needs of the new subject of Quantum Mechanics. **[2]**

[Seen – lectures]

A10. *The Prime Number Theorem (PNT).*

PNT states that

$$\pi(x) := \sum\nolimits_{p\leq x} 1 \sim li(x) := \int_2^x dt/\log t \sim x/\log x \qquad (x \to \infty) \quad (PNT)$$

This was conjectured on numerical grounds by GAUSS (c. 1799; letter of 1848) and A. M. LEGENDRE (1752-1833; in 1798, *Essai sur la Théorie des Nombres*). [2]

PNT was proved independently in 1896 by J. HADAMARD (1865-1963, French) and Ch. de la Vallée Poussin (1866-1962, Belgian). Both used Complex Analysis and $\zeta$. [2]

In 1737 L. EULER (1707-1783) found his Euler product, linking the primes to $\sum_{n=1}^{\infty} 1/n^{\sigma}$ for real $\sigma$ (later the Riemann zeta function). [1]

In 1859 B. RIEMANN (1626-66) studied

$$\zeta(s) := \sum\nolimits_{n=1}^{\infty} 1/n^s \qquad (s \in \mathbb{C})$$

using Complex Analysis (M2PM3), then still fairly new, developed by A. L. CAUCHY (1789-1857), 1825-29. He showed the critical relevance of the *zeros of* $\zeta(s)$ to the *distribution of primes*. One can show that:

(i) $\zeta$ can be continued analytically from $Re\ s > 1$ to the whole complex plane $\mathbb{C}$, where it is holomorphic except for a simple pole at 1 of residue 1 (III.3);

(ii) The only zeros of $\zeta$ outside the *critical strip* $0 < \sigma = Re\ s < 1$ are the so-called *trivial zeros* $-2, -4, \ldots, -2n, \ldots$ (trivial in that they follow from the *functional equation* for $\zeta$;

(iii) PNT is closely linked to non-vanishing of $\zeta$ on the 1-line: $\zeta(1+it) \neq 0$.

The *Riemann Hypothesis* (RH) of 1859 is that the only zeros of $\zeta$ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$. RH is still open, and is the most famous and important open question in Mathematics. Its resolution would have vast consequences for prime-number theory (especially error terms in PNT). [4]

Since counting primes relates to $\mathbb{N}$ ($\subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$), it seemed strange and unaesthetic to use complex methods. Great efforts were make to provide an *elementary proof*, over half a century.This was done in 1948 by A. SELBERG (1917-2009) and P. ERDÖS (1913-1996) (published independently in 1949). [1]

[Seen in lectures, in less detail. The above is taken from my Analytic Number Theory notes, Lecture 15.]

B1. *Archimedes* (of Syracuse, c.287-212 BC)

Archimedes lived in Syracuse, SE Sicily, then part of the Hellenistic world. He may have studied at Alexandria, and was in contact with mathematicians there. Syracuse sided with Carthage in the Punic Wars, and fell to Rome in 212 BC. Archimedes played a leading part in the defence of Syracuse during the long siege 214-212 BC, inventing engines of war: catapults to hurl stones, cranes of pull ships from the water and drop them, etc. (the siege is described in details by Plutarch, Livy and Polybius). The Roman general Marcellus admired Archimedes and ordered his life to be spared. Despite this, he was (as legend has it) killed by a Roman soldier who interrupted him at work drawing diagrams in the sand. Cicero relates that, on coming to Sicily as quaestor in 75 BC, he found 'at the gate of Achradina' an overgrown grave bearing the figure of a sphere with a cylinder circumscribed – the inscription Archimedes chose, to commemorate his favourite theorem (below).

Archimedes is without doubt the great mathematician of the ancient world, just as Newton is for the early modern period and Gauss is for the later modern period. These are the greatest three mathematicians in history. It is impossible to make sensible comparisons between them, as their times were so completely different. [4]

*Mechanics.* Archimedes constructed pulleys, by which he was able to move ships in Syracuse harbour single-handed. [1]

*Statics.* Law of the lever (moments, fulcrum etc.). [1]

*Hydrostatics.* Archimedes is known to the man in the street as having discovered Archimedes' Principle (that a floating body displaces its own weight of water), and for then (reputedly) having run down the street naked from his bath shouting 'Eureka' ('I have found it'). He wrote a treatise 'On floating bodies', for which he has been called the 'father of mathematical physics' – and which is still of value for, e.g., ship design. Other contributions include the Archimedes (helical) screw, still used for irrigation, and (reputedly) testing precious metals for purity by immersing them in water. [2]

*The Sand-reckoner.* Perhaps because of the transition from Attic to Ionian numerals around this time, Archimedes did not scorn calculation. He contributed to the debate (involving Aristarchus) about heliocentric universe, constructing 'astronomically large' numbers for this purpose (and touching on the law of exponents). [2]

*Calculation of $\pi$.* By inscribing and circumscribing $n$-gons to a circle for large $n$, Archimedes obtained the estimate $3\frac{10}{71} < \pi < 3\frac{10}{70}$ – superior to those of the Egyptians and Mesopotamians. [2]

*On Spirals.* In this book, Archimedes used the 'Archimedean spiral', the curve $r = c\theta$ (in modern polar coordinates), to trisect the angle, calculate areas, etc. [1]
*Quadrature of the Parabola.* Here Archimedes used the method of exhaustion to calculate the area $A_1$ of a parabolic segment (region bounded by a parabola and line) as

$$A_1 = \frac{4}{3}A_2,$$

where $A_2$ is the area of the triangle with the same base and 'height'. This first quadrature of a conic had considerable impact at the time. [2]
*On Conics and Spheroids.* Here Archimedes found the area of an ellipse with semi-axes $a, b$ (in particular the area of a circle of radius $r$) as respectively

$$A = \pi ab, \qquad A = \pi r^2. \qquad [2]$$

*On the Sphere and Cylinder.* Here Archimedes compares the volume of a sphere and circumscribing cylinder of the same radius $r$, and also their areas (his favourite theorem, which inspired the choice of the sphere-cylinder inscription on his grave). Hence,

$$V = \frac{4}{3}\pi r^3, \qquad S = 4\pi r^2. \qquad [2]$$

*The Method.* In 1899 J. L. Heiberg (Handout) found a palimpsest (a re-used parchment) in the Library of the Monastery of the Holy Sepulchre in Jerusalem. Beneath the mediaeval liturgical text, he was able to decipher a previously unknown text by Archimedes, *The Method of Mechanical Theorems for Eratosthenes* (briefly, *The Method*). Here Archimedes describes the method which led him to, e.g., his quadrature of the parabola: an application of his 'principle of the lever' via a balancing argument. It also led Archimedes to his favourite (sphere-cylinder) theorem. [2]

Archimedes' work shows strikingly how the 'pure' and 'applied', the theoretical and practical aspects of mathematics, need not be at variance but rather can reinforce and enrich each other. It is no accident that of the three greatest mathematicians of all time, Archimedes, Newton and Gauss, all three made distinguished contributions to science as well as to mathematics: the greatest mathematicians tend to be polymaths. [2]
*Archimedean Solids.* The five Platonic solids have regular polygonal faces *all of the same kind.* If several kinds of face are allowed, thirteen more

'semi-regular' solids are possible, and two infinite families, the 'prisms' and 'antiprisms' (B, 8.12). We know from Pappus (B 11.7) that Archimedes found the complete list (according to Heron, Archimedes ascribed one, the cuboctahedron, to Plato). The most notable one is the truncated icosahedron ('soccer ball'). The shape is familiar to those who watch football (typically, soccer balls have 20 white hexagonal and 12 black pentagonal faces), and occurs in a new form of carbon, $C_{60}$, doscovered in 1985 ny (Sir Harry) Kroto et al.. This form of carbon (in addition to diamond and graphite) is called *fullerene*, from the 'geodesic dome' of the architect Buckminster Fuller. It is found in outer space (it has even been suggested that life on earth originated from this source). [2]

B2. *Geometry post-Renaissance*

*René Descartes* (1596-1650) (B 17.2-10)

*The Method* (1637) (*Discours de la méthode* ...; Appendix, *Le Géométrie*. Book I Geometric solution of quadratics; Book II, Conics: conics as curves having equations of the second degree.

Modern analytical geometry, or 'Cartesian geometry', is loosely derived from Descartes' *La Géométrie*. The strengths of his book lie in its demonstration of the power of algebraic methods in geometry. Its weaknesses include failure to exploit graphical methods (curve-tracing, etc.), reluctance to use negative coordinates, and absence of standard rectangular ('Cartesian') axes (Descartes used oblique axes, to be found in Apollonius). [4]

*Girard Desargues* (1591-1661) and *Projective Geometry* (B 17.21, 22)

Desargues' first important book was *La Perspective* (1636). This led him on to his introduction of projective geometry in his *Brouillon projet* ... (1639) (Rough draft ...). As background, recall:

(i) Many results in geometry concern only *incidence properties* (whether lines meet, point lie on a line, etc.); these are preserved under projection.

(ii) Often one has to qualify statements because of exceptional cases involving 'infinity' – e.g., two lines in a plane meet in a point (unless parallel). Recall (Renaissance) *perspective* (vanishing point = 'point at infinity').

All this led to Projective Geometry, in which it is incidence properties rather than metrical ones that count. Here one works *projectively*, using *homogeneous coordinates* (in which point in a plane has three coordinates rather than two, determined up to a constant multiple). Projective methods are powerful, but need a thorough-going change of viewpoint. Together with analytic (= coordinate) and synthetic (= classical) methods, they complete the main tools needed to treat the geometric problems studied till then.

*Conics*. Projective methods allow a simple interpretation of conics as sections of circular cones by planes: conics are projection of circles.

Projective geometry is of great practical importance: it is the basis of computer graphics, hence of virtual reality etc. [4]

*Blaise Pascal* (1623-1662) (B 17.23-25)

*Essay pour les coniques* (1640) (one page!) Pascal's theorem (on hexagons inscribed in a conic), inspired by Desargues' work. [2]

NON-EUCLIDEAN GEOMETRY

*Nikolai Ivanovich Lobachevski* (1793-1856), Professor of Mathematics at Kazan.

Lobachevski worked in the 1820s on the Parallel Postulate, and became convinced – rightly – that it could not be deduced from the other Euclidean postulates. He then took the logical but epoch-making step of substituting for it an alternative postulate (through any point $C$ not on line $AB$ can be drawn *more than one* line in the plane $ABC$ not meeting $AB$. He did this in two Russian papers (of 1829 and 1835/37, both now lost), and *J. für Math.* **17** (1937), 295-320; *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840); *Pangéométrie* (1855). [3]

*Janos Bolyai* (1802-1860), son of Farkas (Wolfgang) Bolyai (1775-1856).

Gauss wrote to Farkas Bolyai on 17.12.1799, explaining his thinking on the Parallel Postulate. He worked on non-Euclidean geometry in the first two decades of the 19th C., but, as usual, published nothing on the subject.

*The science of absolute space* (1832). Janos Bolyai published this as a 26-page Appendix to his father's book *Tentamen* ....

Farkas had despaired of proving the parallel postulate, and warned his son against taking up the subject. But Janos announced his successful construction of (a) non-Euclidean geometry in letter to his father of 3.11.1823. His father then advised him to publish it, in case others did so. The credit for non-Euclidean geometry is thus shared between Lobachevski and Bolyai, who had the courage of their convictions and published their work (independently, and roughly simultaneously). [3]

*Eugenio Beltrami* (1835-1899). In 1868, Beltrami constructed a non-Euclidean geometry on a *pseudosphere*, a space of constant negative curvature, the surface of revolution of a tractrix (the locus of a mass dragged by a string whose end moves along a line). The more usual model for non-Euclidean geometry nowadays is Poincaré's (later), on a half-plane or disc. [2]

PROJECTIVE GEOMETRY

Desargues' projective geometry of the Brouillon projet (1639) was brought to modern form in the 19th C., largely by the French school:

*Gaspard Monge* (1746-1818), Professor at the Polytéchnique;

*Charles Jules Brianchon* (1785-1864); *Victor Poncelet* (1788-1867).

Poncelet (MS c. 1812, publ. 1862-4, Works I, II) emphasised *duality* in Projective Geometry: in two dimensions, one may interchange the words 'point' and 'line'; in three dimensions one may interchange 'point' and 'plane', leaving 'line' the same. Poncelet wrote his projective geometry in two columns, the right obtained from the left by duality. A prime example of duality is *Pascal's theorem* on hexagons inscribed to conics, and *Brianchon's theorem* on hexagons circumscribed about conics – *discovered* by duality.

*The Platonic solids.* The cube and octahedron are dual; the dodecahedron and icosahedron are dual; the tetrahedron is self-dual. [2]

GROUPS AND GEOMETRY

*Felix Klein* (1849-1925), Professor of Mathematics at Erlangen (1872).

Klein's inaugural lecture at Erlangen set out his *Erlanger Programm*, studying geometry in terms of invariance under groups of transformations. To Klein: Euclidean geometry is the study of properties invariant under the action of the Euclidean geroup (of rigid motions, or change of coordinate system – rotations and translations); similarly for 'projective', and 'affine'.

Klein also studied non-Euclidean geometry. He introduced the terms *elliptic geometry* (positive curvature – extending spherical geometry) and *hyperbolic geometry* (negative curvature: cf. Beltrami's pseudosphere).

*Sophus Lie* (1842-1889), a Norwegian mathematician, worked on continuous transformation groups (initially with Klein, though they quarreled later); *Theorie der Transformationsgruppen*, Vol 1-3 (1888-93, with F. Engel).

Ricci and Levi-Civita revolutionised differential geometry by introducing tensor methods in the early 20th C, providing the tools needed for Einstein's General Theory of Relativity.

H. F. Baker (1866-1956) of Cambridge continued and developed these ideas; he was succeeded in 1936 by W. V. D. Hodge (1903-75). Geometry and topology are now all but inseparable. Maxwell's equations have an interpretation in terms of Hodge's theory of harmonic forms. **[5]**

B3. *The development of Group Theory*

*Pre-history.* The first theorem in group theory pre-dates the group concept! *J.-L. Lagrange* (1736-1813) proved 'Lagrange's theorem' – the order of a subgroup divides the order of the group – in 1770; he here also conjectured the insolubility of the quintic. This was also conjectured by *Paolo Ruffini* (1765-1822), who attempted a proof in 1799. **[2]**

*Abel and Galois.*

The group concept emerged with Abel and Galois:
*Niels Henrik Abel* (1802-1829), a Norwegian mathematician, proved the insolubility of the quintic in 1824.
*Evariste Galois* (1811-1832) introduced the term *group* in 1830. He studied *Galois groups* – groups of automorphisms arising in field extensions [as one extends a field, it becomes easier to factorise polynomials: thus $x^2+1$ cannot be factorised over $\mathbb{R}$, but splits as $(x+i)(x-i)$ over $\mathbb{C}$]. **[3]**

*19th C.*

Abstract group theory emerged gradually, through the work of, e.g.., Cayley, Sylvester and Hamilton [in particular, Hamilton's *icosian game*, giving the icosahedral group in terms of generators and relations].
*Camille Jordan* (1838-1922), *Traité des substitutions et des équations algébriques* (1870). This textbook is the first major treatise incorporating group theory. In particular, it introduces the term 'abelian group' for a commutative group, and contaions the first textbook treatment of Galois Theory. The *Sylow theorems* date from 1872 (L. Sylow (1832-1918)), the *Jordan-Hölder theorem* from 1872 and 1889. **[3]**

*Geometry.*

*Felix Klein* (1849-1925), in his famous *Erlanger Programm* (1872) [Inaugural Lecture, Universit of Erlangen] brought group theory into geometry. To Klein, a particular geometry (Euclidean, affine, projective, ...) is the study of properties left invariant by the corresponding group – a view which survives. **[3]**

*DEs.* *Sophus Lie* (1842-1889), a Norwegian who studied with Klein at Göttingen, wrote a three-volume work (1888-1893) on continuous transformation groups – now called *Lie groups.* These are the prototypes of continuous groups, and arose out of the study of differential equations. The fundamental theorems of Lie theory relate Lie groups to *Lie algebras*, which express the local or differential aspects. [2]

*Crystallography.*

The subject of crystallography hinges on *symmetry groups*. Remarkably, progress here predates group theory. Beginning with Hessel in 1830 and Bravais in 1848, the classification of mineral crystals [or crystallographic space groups in 3 dimensions] was completed by Fedorov in 1890. In 2 dimensions, there are 17 crystallographic groups, occurring in wallpaper patterns [e.g., wall paintings in the Alhambra, Toledo, and the artistic work of Escher]. [3]

*Group representations.*

These represent group composition by homomorphic images as composition of linear transformstions, or matrix multiplication – developed in the years 1900-1920, in the work of *G. Frobenius* (1849-1917, German), *W. Burnside* (1852-1927, English) and *I. Schur* (1875-1941, German). This gives a powerful way of reducing group theory to matrix theory and linear algebra. [3]

*20th C.*

Group theory has been extensively developed, particularly in the UK, most notably by *Philip Hall* (1904-1982) and his school. The most dramatic result is the *classification of finite simple groups* [several well-understood infinite families, plus the *sporadic simple groups* culminating in the Fisher-Griess Monster] by *D. Gorenstein*, *D. Aschbacker* and others c. 1980 (outside our self-imposed time-frame: we stop at 1950). [3]

*Quantum Mechanics (QM).*

Group representations and the classification of Lie algebras have permeated *quantum mechanics* and *elementary particle physics*, beginning with *Weyl* (1928), *von Neumann* (1932) and *Wigner* (1959). This invasion made heavy mathematical demands of physicists in the early days of QM (recall that when Heisenberg gave his version of QM, known as matrix mechanics, he did not know what a matrix is!). The term *Gruppenpest* – 'group plague' – was coined to describe this. [3]
[Seen – lectures]

B4. *Set Theory, Logic and Foundations*

*George Boole* (1815-1864), Prof. Math., Queen's College, Cork, 1849.
*The mathematical analysis of logic* (1847);
*Investigation of the laws of thought* (1854).

Boole's work marks the beginning of mathematica logic proper. It has led to the subject of *Boolean algebra*, in algebra and computer programming. [2]

*Augustus De Morgan* (1806-1871), first Professor of Mathematics, UCL, 1828; first President LMS (1865-9).

De Morgan is remembered for *De Morgan's laws* in Set Theory. The LMS Headquarters in Russell Square is named De Morgan House. [2]

*John Venn* (1834-1923)
*The logic of chance* (1866); *Symbolic logic* (1881).

Venn taught Moral Science at Cambridge from 1862. He is remembered for *Venn diagrams* in Set Theory. [2]

*Georg Cantor* (1845-1918).

Cantor wrote *Grundlagen einer allgemeine Mannigfältigkeitslehre* (1883) [Foundations of a general theory of manifolds]. Here, and in a series of papers in *Mathematische Annalen* between 1879 and 1884, Cantor set up a general theory of sets, and in particular of infinite ('transfinite') numbers, of two kinds, *cardinal*, to do with *size* (one, two, ...) and *ordinal*, to do with *order* (first, second, ...). We return to set theory in the 20th C. below. We note here that Cantor's theory was violently attacked by Leopold Kronecker (1823-91), whose dictum was 'God gave us the integers; all the rest is the work of man'. But Cantor's views eventually prevailed (rightly); in particular, he was memorably defended by David Hilbert (1862-1943), who said 'No one shall expel us from the paradise that Cantor has created for us'.

Following Cantor's set theory, and the construction of $\mathbb{R}$ (by Cantor and Dedekind, in 1872), the question arose of setting up the *foundations of mathematics* – in particular, $\mathbb{Z}$ – 'from scratch', and by the new standards of rigour. Dedekind gave one approach (above). Another, more usual nowadays, was givenin 1889 by Giuseppe Peano (1858-1932) (who introduced the standard set-theoretic notation we use today. See the classic ('what the mathematician in the street should know about set theory and foundations')
P. R. HALMOS, *Naive set theory*, Van Nostrand, 1960. [6]

*Russell and Whitehead.*

Following Cantor's introduction of set theory in the late 19th C., it was soon noticed (by Burali-Forte and others) that one can be led into contradiction if one proceeds naively. An example is the 'paradox of the liar'. To see one form of this, write 'The statement on the other side of this piece of paper is false' on both sides of a piece of paper. Now read one side, and ask yourself whether what you read is true or false (it can be neither). For another form, the 'barber's paradox', see Kline, 1183.

This has become known as *Russell's paradox*, after the work of Bertrand Russell (1872-1970). Russell developed his 'theory of types' to avoid this. [See e.g. Halmos' *Naive set theory*.] Russell set out his theory in *Principles of Mathematics* (1903), and later (with A. N. Whitehead (1861-1947)) *Principia Mathematics* (3 volumes, 1910-13). [3]

*Axioms.*

The axioms of 19th C. mathematics needed to be augmented for technical reasons. One way to do this is via the *Axiom of Choice (AC)*, introduced by Ernst Zermelo (1871-1953). It is more usually employed in an equivalent formulation, *Zorn's Lemma* (Max Zorn (1906-93)) (cf. Hilbert's first problem). AC is needed for the Hahn-Banach theorem, so in Functional Analysis, etc.

It emerged that Hilbert's views on the nature of mathematics and its foundations were too naive. Kurt Gödel (1906-78) showed in 1931 that mathematics is *incomplete*: any mathematical theory rich enough to contain the set $\mathbb{N}$ of natural numbers must contain statements which can be neither proved nor disproved. As a corollary, the consistency question raised by Hilbert is undecidable. Gödel also showed in 1940 that AC is *consistent* with the other axioms of set theory. Alfred Tarski (1902-83) was a Polish Jew who was in the US at the start of WWII, and stayed there; he was a logician, working on e.g. model theory (also the "*Banach-Tarski paradox*" in 1924).

The 'other axioms of set theory' above are the *Zermelo-Fraenkel* axioms, ZF. These emerged from the work of Zermelo (above) and A. A. Fraenkel (1891-1965); ZF reached its modern form following work of Thoralf Skolem (1887-1963) in 1922 and 1928. Note that, by Gödel's incompleteness theorem, we do not (indeed, we *cannot*) know that ZF is consistent! But, we assume this whenever we do ordinary mathematics! – the 'elephant in the room'.

It was shown (by Cohen, in 1963) that AC is independent of ZF (that is, both AC and its negation are onsistent with ZF) (but this is outside our time-frame of up to 1950). [6]

*Computability.*

Alan M. Turing (1913-54) showed in 1937 that Hilbert's question on decidability has a negative answer. His work led to the development of *computable numbers*, and later to the development of the *computer*. See e.g.
A. HODGES, *Alan Turing – the enigma of intelligence*, 1983. [4]