Test 1

1. The motion of a particle of mass m in the $(x-y)$-plane is governed by the Lagrangian

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy\,.$$

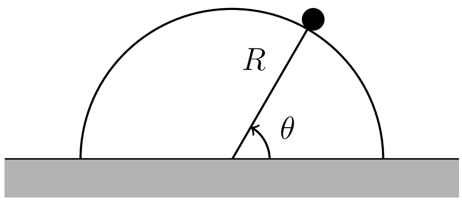
   (a) i. Identify the kinetic energy and the potential energy of the particle.
   ii. What is the force acting on the particle? Is it a conservative force?

[4 marks]

   (b) i. Obtain the equations of motion.
   ii. Write the Lagrangian in polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$, and obtain the Euler-Lagrange equations for this Lagrangian.
   iii. Explicitly show that the equations obtained in parts i. and ii. are equivalent. *Hint: Compute $\ddot{x}$ and $\ddot{y}$ in polar coordinates and use the result from (b)ii.*

[9 marks]

   (c) Now assume that the particle sits on a hemisphere of radius R, and that there are no frictional forces.



   i. Derive the equation of motion for θ which governs the dynamics of the particle whilst it remains on the hemisphere.
   ii. Derive the normal constraint force imparted on the particle by the hemisphere, in terms of θ.
   iii. Discuss the relationship between Newton's second law and the answers to parts i. and ii..

[12 marks]

[Total: 25 marks]

Answers to Test 1

1. (a) i.

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \,, \quad \text{and} \quad V = mgy \,.$$

[2 marks]

ii. The force is calculated as $\mathbf{F} = -\nabla V$, to give

$$\mathbf{F} = -mg\mathbf{j} \,,$$

where $\mathbf{j}$ is the unit vector in the y-direction. This is a conservative force since it has the form $\mathbf{F} = -\nabla V$.

[2 marks]

[1(a): 4 marks in total]

(b) i. The Euler-Lagrange equations give

$$\frac{d}{dt}(m\dot{x}) = 0 \,, \quad \text{and} \quad \frac{d}{dt}(m\dot{y}) = -mg \,.$$

The equation of motion is therefore a particle in free fall

$$\ddot{x} = 0 \,, \quad \text{and} \quad \ddot{y} = -g \,.$$

[2 marks]

ii. The Lagrangian in polar coordinates is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta \,.$$

The corresponding Euler-Lagrange equations are, for r and θ respectively, given by

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - mg\sin\theta \,, \quad \text{and} \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mgr\cos\theta \,.$$

Simplifying gives

$$\ddot{r} = r\dot{\theta}^2 - g\sin\theta \,, \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\cos\theta \,.$$

[3 marks]

iii. Since we have $(x, y) = (r\cos\theta, r\sin\theta)$, it follows that

$$\begin{aligned}
\ddot{x} &= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \,, \\
\ddot{y} &= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \,.
\end{aligned}$$

Now, using the Euler-Lagrange equation, $\ddot{r} = r\dot{\theta}^2 - g\sin\theta$, in each of the above equations gives

$$\begin{aligned}
\ddot{x} &= -g\sin\theta\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta \,, \\
\ddot{y} &= -g\sin^2\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta \,.
\end{aligned}$$

Now using the Euler-Lagrange equation $r\ddot{\theta}+2\dot{r}\dot{\theta} = -g\cos\theta$ in these equations gives

$$\ddot{x} = 0\,,$$
$$\ddot{y} = -g(\cos^2\theta + \sin^2\theta) = -g\,,$$

as required.

[4 marks]

[1(b): 9 marks in total]

(c) i. METHOD 1: Take the Lagrangian from (b) ii. and substitute in $r = R$ to give

$$L = \frac{mR^2}{2}\dot{\theta}^2 - mgR\sin\theta\,.$$

The Euler-Lagrange equation for θ is

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) = -mgR\cos\theta\,, \quad \text{or} \quad \ddot{\theta} = -\frac{g}{R}\cos\theta\,.$$

METHOD 2: Form a constrained Lagrangian as

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta + \lambda(r-R)\,.$$

The Euler-Lagrange equation for θ is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mgr\cos\theta\,.$$

Using the variation in λ to conclude that $r = R$, the result follows. It is also valid to use the constraint $r^2 - R^2 = 0$.

[4 marks]

ii. Determining the full Euler-Lagrange equations from the constrained Lagrangian above gives

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - mg\sin\theta + \lambda\,,$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mgr\cos\theta\,.,$$
$$r = R\,.$$

The constraint force is then given by evaluating the first of these at $r = R$ as

$$\lambda = -mR\dot{\theta}^2 + mg\sin\theta\,.$$

[6 marks]

iii. The answer to part i. is Newton's second law tangentially to the hemisphere, and the radial Euler-Lagrange equation computed in part ii. is Newton's law understood in the radial direction.

[2 marks]

[1(c): 12 marks in total]