Network Science
Spring 2024
Problem sheet 3

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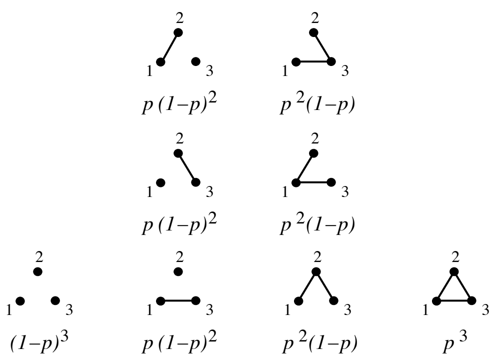
1. Consider the G_{Np} random graph model with $N = 3$ nodes.

   (a) Draw all 8 graphs that can be generated by the model and determine the realization probability of each graph.

   Solution: See the figure.

   $$N = 3,$$
   $$N' = \binom{N}{2} = 3.$$

   (b) What is the probability a graph has at most one edge?
   Solution: The left four of the above 8 graphs drawn above in (a) satisfy this property and thus we get $(1-p)^3 + 3p(1-p)^2$.

   (c) What is the probability that the graph is connected?
   Solution: The right four of the above 8 graphs drawn above in (a) satisfy this property and thus we get $3p^2(1-p) + p^3$.

   (d) What is the expected number of connected triples?
   Solution: A "connected triple" is three nodes a, b, c with edges $a - b$ and $b - c$. From the solution to (a), we see that the 2-link graphs each have 1 connected triple, while the 3-link graph has 3. The expected number of connected triples is then: $3p^2(1-p) + 3p^3 = 3p^2$

   (e) What is the expected clustering coefficient of node i, $\langle C_i \rangle$?
   Solution: The only graph where any nodes have non-zero clustering coefficient is the triangle, in which case the clustering of each node is one. So, $\langle C_i \rangle = p^3 * (1) = p^3$.

2. Now, consider the G_{Np} model with general N.

(a) Recall the degree distribution for the model. Explain what it means.
Solution: Let $k_i(G)$ be the degree of the graph G at node i. Then $P(k_i = k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$. This means that the probability that a graph G which is randomly generated from the model has k edges connected to node i is equal to $\binom{N-1}{k} p^k (1-p)^{N-1-k}$. Notice that k_i is a random variable which does not depend on i. It is also common to use the terminology $k_i \sim \text{Binomial}(N-1, p)$.

(b) Let $i \sim j$ be a randomly chosen link from a graph G generated by G_{Np}, and let k_i^j be the degree of node i attached to this link. Find the probability distribution for k_i^j. In other words, find an expression for $P(k_i^j = k)$. Hint: first consider $P(k_i^j = 1)$.
Solution: Since we know that node i has a link to node j, we have to consider the probability of node i linking to $k-1$ of the other $N-2$ nodes in the graph giving, $P(k_i^j = 0) = 0$ and for $k = 1, \ldots, N-2$ we have $P(k_i^j = k) = \binom{N-2}{k-1} p^{k-1} (1-p)^{N-1-k}$. In other words, k_i^j is distributed as $X + 1$ where $X \sim \text{Binomial}(N-2, p)$.

3. What code would you use to draw a G_{Np} random graph with 10 nodes and with probability p with NetworkX. Solution:

4. Show that if we let $p(N) = N^{-z}$ with $z > 2$ then $G \in G_{Np}$ w.h.p. has no two edges with a common vertex (or equivalently the degree at each node is at most one).
Solution: Let X_i be the degree of node i and $X = X_1 + \cdots + X_N$. Then $\langle X_i \rangle = p(N-1)$. So $\langle X \rangle = p(N-1)N$. It would be sufficient to show that $P(X \geq 1) \to 0$ because this implies $P(X = 0) \to 1$ as $N \to \infty$. By the Markov inequality we have $P(X \geq 1) \leq \langle X \rangle = p(N-1)N$ and this goes to zero when $p(N) = N^{-z}$ when $z > 2$. Notice that we actually proved a much stronger statement, namely that w.h.p. nodes are isolated.