MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 8

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} \, dz.$$

Example.

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)}\, dz \\
&= \frac{1}{2\pi i}\, \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i} \right) dz \\
&= \frac{1}{2i} \left(e^{i} - e^{-i} \right) = \sin 1.
\end{aligned}$$

Theorem. (Generalised Cauchy's integral formula)

Let f be holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, for simple, closed, piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}} \, d\eta.$$

Proof. The proof is by induction on n. The case $n = 0$ is simply the Cauchy integral formula. Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n}} \, d\eta.$$

Let $h \in \mathbb{C}$ be small enough, so that $z+h$ is lying inside γ. Then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \oint_\gamma f(\eta)\,\frac{1}{h}\left(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n}\right) d\eta.$$

Recall

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

and apply it with $A = 1/(\eta - z - h)$ and $B = 1/(\eta - z)$. Then we obtain as $h \to 0$

$$\begin{aligned}
&\frac{1}{h}\left(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n}\right) \\
&\quad = \frac{1}{h}\,\frac{h}{(\eta - z - h)(\eta - z)}\,(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}) \\
&\quad\quad \to \frac{1}{(\eta - z)^2}\,\frac{n}{(\eta - z)^{n-1}}.
\end{aligned}$$

This implies

$$
\begin{aligned}
&\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\
&\qquad\qquad \to \frac{(n-1)!}{2\pi i} \oint_\gamma f(\eta) \frac{1}{(\eta - z)^2} \frac{n}{(\eta - z)^{n-1}}\, d\eta \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta.
\end{aligned}
$$

The proof is complete.

Corollary. If f is holomorphic in Ω, then all its defivatives $f', f'', \dots,$ are holomorphic.

Exercise:

Let f be continuous on a piecewise-smooth curve γ. At each point $z \notin \gamma$ define the value of a function F by

$$F(z) = \int_\gamma \frac{f(\eta)}{\eta - z}\, d\eta.$$

Show that F is holomorphic at $z \notin \gamma$ and

$$F'(z) = \int_\gamma \frac{f(\eta)}{(\eta - z)^2}\, d\eta.$$

Section: Applications of Cauchy's integral formulae.

Joseph Liouville

French 1809 - 1882

Corollary. (Liouville's theorem)

If an entire function is bounded, then it is constant.

Proof. Suppose that f is entire and bounded. Then there is a constant M such that

$$|f(z)| \leq M, \qquad \forall z \in \mathbb{C}.$$

Let $z_0 \in \mathbb{C}$ and let $\gamma_r = \{z : |z - z_0| = r\}$. Then

$$|f'(z_o)| = \left| \frac{1!}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^2} \, dz \right| \leq \frac{M}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

Therefore for any $z_0 \in \mathbb{C}$ we have $f'(z_0) = 0$ and thus f is constant.

Theorem. (Fundamental theorem of Algebra) Every polynomial of degree greater than zero with complex coefficients has at least one zero.

Proof. Assume that

$$p(z) = a_n\, z^n + a_{n-1}\, z^{n-1} \cdots + a_0 = 0.$$

has no zeros. Then $1/p(z)$ is entire. Clearly $|1/p(z)| \to 0$ as $|z| \to \infty$. Indeed, given $\varepsilon > 0$ there is R such that

$$\left|\frac{1}{p(z)}\right| < \varepsilon, \qquad \forall z : |z| > R.$$

Since $1/p(z)$ is entire it is also continuous and therefore there is a constant $M > 0$ such that

$$\left|\frac{1}{p(z)}\right| \leq M, \qquad z : |z| \leq R$$

and thus $|1/p(z)|$ is bounded in $\mathbb{C}$. This implies $1/p$ is constant and this contradicts the fact that $p(z)$ is a polynomial of degree greater than zero.

Corollary.
Every polynomial

$$P(z) = a_n z^n + \cdots + a_0$$

of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \ldots w_n$, then P can be factored as

$$P(z) = a_n (z - w_1)(z - w_2) \ldots (z - w_n).$$

Proof. We now know that P has at least one root, say w_1. Then writing $z = (z - w_1) + w_1$. Substituting this in $P(z) = a_n z^n + \ldots a_0$ and using the binomial formula we get

$$P(z) = b_n (z - w_1)^n + \cdots + b_1 (z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$ we have $b_0 = 0$ and thus

$$P(z) = (z - w_1) Q(z).$$

Repeating this we find

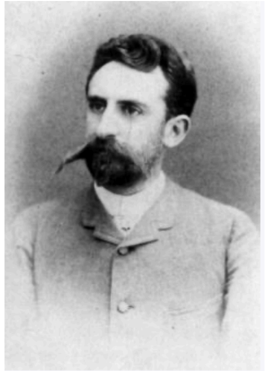
$$P(z) = a_n (z - w_1)(z - w_2) \ldots (z - w_n).$$

Theorem. (Morera's theorem)

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z)\,dz = 0,$$

then f is holomorphic.



Giacinto Morera

Italian, 1856 - 1909

Proof. We have proved before that f has a primitive F in D that satisfies $F' = f$. Then F is indefinitely complex differentiable, and therefore f is holomorphic.

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Quizzes

Question: What is the value of the integral $\int_\gamma \frac{e^{\pi z}}{z-i} dz$, where γ is the circle of radius $1/2$ centered at i, traversed in the direction such that its interior remains on the left.

Answers:

A. $2\pi i$

B. $-2\pi i$

C. 0

D. $4\pi i$

Thank you