MATH70020

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Function Spaces and Applications

Date: 12 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(a) Show that, for the series $\sum_{n=1}^{\infty} |a_n u_n|$ to converge for all $(u_n)_{n\in\mathbb{N}} \in \ell^\infty$, a necessary and sufficient condition is that $(a_n) \in \ell^1$. (5 marks)

(b) We assume from now on that $(a_n) \in \ell^1$. Define

$$\|(u_n)\| = \sum_{n=1}^{\infty} |a_n u_n|.$$

Find a necessary and sufficient condition for $\|\cdot\|$ to be a norm on ℓ^∞. We assume from now on that this necessary and sufficient condition holds. (5 marks)

(c) Show that the norms $\|\cdot\|$ and $\|\cdot\|_{\ell^\infty}$ are not equivalent. (4 marks)

(d) Choosing $a_n = \frac{1}{n^2}$, show that ℓ^∞ endowed with the norm $\|\cdot\|$ is not a Banach space. (6 marks)

(Total: 20 marks)

2. Consider the real Hilbert space $L^2(\mathbb{R})$ endowed with the scalar product

$$\langle f\,,\,g\rangle = \int f(x)g(x)\,dx$$

and the induced classical L^2 norm.

For $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, let the subset $C \subset L^2(\mathbb{R})$ be given by

$$C = \Big\{ f \in L^2(\mathbb{R}),\ \alpha \leq f(x) \leq \beta \text{ almost everywhere} \Big\}.$$

(a) Show that C is non-empty if and only if $\alpha \leq 0$ and $\beta \geq 0$. (5 marks)

(b) Show that C is convex and closed. (5 marks)

(c) By a theorem seen in class, there exists an orthogonal projection $P : L^2(\mathbb{R}) \to C$. Which identity does this orthogonal projection satisfy? (4 marks)

(d) Show that

$$Pf(x) = \begin{cases} f(x) & \text{if } \alpha \leq f(x) \leq \beta \\ \alpha & \text{if } f(x) \leq \alpha \\ \beta & \text{if } f(x) \geq \beta. \end{cases}$$

(6 marks)

(Total: 20 marks)

3. Consider the map

$$S \,:\, L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$f \mapsto \left[t \mapsto e^{-t^2} f(t-1)\right].$$

(a) Show that the map S is bounded, and compute its operator norm. (5 marks)

(b) Show that S is injective, but not surjective. (5 marks)

(c) Show that the image of S, $S(L^2(\mathbb{R}))$, is dense in $L^2(\mathbb{R})$. (5 marks)

(d) Show that the operator norm of $S^2 = S \circ S$ is strictly smaller than 1. (5 marks)

(Total: 20 marks)

4. The complex Hilbert space $L^2((-1,1))$ is endowed with the scalar product

$$\langle f\,,\,g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt,$$

and the induced classical L^2 norm.

Consider the operator

$$T\,:\,f \mapsto \left[t \mapsto \int_{-1}^{1} e^{itx} f(x)\,dx\right].$$

(a) Show that T maps $L^2((-1,1))$ to $\mathcal{C}((-1,1))$ boundedly (here, L^2 is endowed with the L^2 norm, and $\mathcal{C}$ with the uniform (or sup) norm). (4 marks)

(b) Show that T is a compact operator from $L^2((-1,1))$ to $\mathcal{C}((-1,1))$. (4 marks)

(c) Deduce that T is a compact operator from $L^2((-1,1))$ to itself. We now focus on this operator. (4 marks)

(d) Compute the adjoint of T. (4 marks)

(e) Show that TT^* is compact, self adjoint, and state what the spectral theorem for compact self-adjoint operators asserts. (4 marks).

(Total: 20 marks)

5. [Mastery question] Let H be a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For a sequence (x_n) in H, and an element x of H, we write

- $x_n \to x$ as $n \to \infty$ if the sequence (x_n) converges to x (for the norm $\| \cdot \|$).
- $x_n \rightharpoonup x$ as $n \to \infty$ if the sequence (x_n) converges weakly to x.

(a) Let $\mathcal{D}$ be a dense subset of H. Show that $x_n \rightharpoonup x$ as $n \to \infty$ if the following assertions are verified:

(i) (x_n) is a bounded sequence

(ii) <u>and</u> $\langle x_n , y \rangle \to \langle x , y \rangle$ as $n \to \infty$ for any y in $\mathcal{D}$.

(6 marks)

(b) Show (by finding a counterexample) that assuming only $(a)(ii)$ does not imply that $x_n \rightharpoonup x$ as $n \to \infty$. (7 marks)

(c) Show that $x_n \to x$ as $n \to \infty$ if and only if

(i) $x_n \rightharpoonup x$ as $n \to \infty$

(ii) <u>and</u> $\|x_n\| \to \|x\|$.

(7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH 60020 - 70020

Function Spaces and Applications (Solutions)

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1. (a) It is possible to bound $|a_n u_n|$ by $|a_n| \|(u_n)\|_{\ell^\infty}$. Therefore, $\sum |a_n u_n|$ converges if $(u_n) \in \ell^\infty$ and $(a_n) \in \ell^1$.

Conversely, if $\sum |a_n u_n|$ converges for all $(u_n) \in \ell^\infty$, it is also the case for u_n identically equal to 1. But then $\sum |a_n u_n| = \sum |a_n|$ converges, hence $(a_n) \in \ell^1$. $\boxed{5, \mathrm{A}}$

(b) It is immediate to check that the triangle inequality $\|(u_n + v_n)\| \leq \|u_n\| + \|v_n\|$ and positive homogeneity $\|(\lambda u_n)\| = |\lambda| \|(u_n)\|$ hold for $\| \cdot \|$. There remains to see whether $\| \cdot \|$ is definite, or in other words whether $\|(u_n)\| = 0$ if and only if $(u_n) = 0$.

We claim that definiteness holds if and only if $a_n \neq 0$ for all n. Indeed, if $a_n \neq 0$ for all n, then $\sum |a_n u_n| = 0$ implies that $(u_n) = 0$. Conversely, if $a_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then the sequence $u_n = \delta_{n,n_0}$ is nonzero, but satisfies $\sum |a_n u_n| = 0$. $\boxed{5, \mathrm{B}}$

(c) Indeed, choosing $u_n = \delta_{n,n_0}$, we see that $\|(u_n)\|_{\ell^\infty} = 1$ while $\|(u_n)\| = |a_{n_0}|$. Since $a_n \to 0$ as $n \to \infty$, there cannot hold

$$\|(u_n)\|_{\ell^\infty} \leq C \|(u_n)\|$$

for a constant C. $\boxed{4, \mathrm{C}}$

(d) For $k \in \mathbb{N}$, define the sequence

$$(u_n^k) = \begin{cases} \sqrt{n} & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

It is a Cauchy sequence for $\| \cdot \|$, as is easily checked, but it does not converge in ℓ^∞. Indeed, the only possible limit would be $u_n = \sqrt{n}$, which does not belong to ℓ^∞. $\boxed{6, \mathrm{D}}$

2. (a) If $\alpha \leq 0$ and $\beta \geq 0$, then the function identically equal to zero belongs to C. Conversely, if $\alpha > 0$, then any function f which is a.e. $\geq \alpha$ is such that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx \geq \int_{\mathbb{R}} |\alpha|^2 \, dx = \infty,$$

so that f cannot belong to L^2. The same argument applies if $\beta < 0$. $\boxed{5, \mathrm{A}}$

(b) To see that C is convex, pick f, g two functions in C and consider $h = tf + (1-t)g$, for $t \in [0,1]$. The function h belongs to L^2, since L^2 is a vector space; and furthermore $h(x) \geq \alpha$ since

$$\alpha = t\alpha + (1-t)\alpha \leq tf(x) + (1-t)g(x) = h(x).$$

The same argument applies to show that $h(x) \leq \beta$ almost everywhere. $\boxed{5, \mathrm{B}}$

(c) It is characterized by the following identity: for any $f, g \in L^2(\mathbb{R})$,

$$\|f - Pf\| = \inf_{g \in C} \|f - g\|.$$

$\boxed{4, \mathrm{A}}$

(d) The function Pf obviously belongs to C, so there remains to check that

$$\|f - Pf\| = \inf_{g \in C} \|f - g\|.$$

In order to prove this identity, we split $\mathbb{R}$ into three domains D, E, F where $f(x)$ takes values in $[\alpha, \beta]$, $(-\infty, \alpha)$ and (β, ∞) respectively. On the domain D, $f - Pf = 0$, so that

$$\int_D |f - Pf|^2 \, dx = 0 = \inf_{g \in C} \int_D |f - g|^2 \, dx = 0.$$

On the domain E, $f - Pf = f - \alpha$, and $|f - \alpha| \geq |f - g|$ for $g \in C$ almost everywhere so that

$$\int_E |f - Pf|^2 \, dx = \int_E |f - \alpha|^2 \, dx = \inf_{g \in C} \int_E |f - g|^2 \, dx.$$

Similarly,

$$\int_F |f - Pf|^2 \, dx = \int_F |f - \beta|^2 \, dx = \inf_{g \in C} \int_F |f - g|^2 \, dx.$$

Summing up the three above equalities gives

$$\int_{\mathbb{R}} |f - Pf|^2 \, dx = \inf_{g \in C} \int_{\mathbb{R}} |f - g|^2 \, dx,$$

which is the desired identity. $\boxed{6, \mathrm{D}}$

3. (a) The map S is bounded by the following estimate, which uses that $\|e^{-t^2}\|_{L^\infty} = 1$:

$$\|Sf\|_{L^2} = \|e^{-t^2} f(t-1)\|_{L^2} \leq \|f(t-1)\|_{L^2} = \|f\|_{L^2}.$$

This shows that S is bounded on L^2, with operator norm ≤ 1. To show that the operator norm of S is one, choose $f(t) = \mathbf{1}_{[-1-\delta,-1+\delta]}(t)$, and observe that $\frac{\|Sf\|_{L^2}}{\|f\|_{L^2}} \to 1$ as $\delta \to 0$. $\boxed{5,\text{ A}}$

(b) S is injective since $Sf = 0$ is equivalent to $f = 0$. It is not surjective: if $g \in L^2$, $Sf = g$ implies that

$$f(t) = g(t+1)e^{(t+1)^2},$$

which is not an L^2 function for all $g \in L^2$. $\boxed{1,\text{ B}}$

(c) The image of S is dense since $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Indeed, choose $g \in L^2$, and (g_n) a sequence in $\mathcal{C}_0^\infty$ converging to g in the L^2 topology. Then, letting $\boxed{4,\text{ C}}$

$$f_n(t) = g_n(t+1)e^{(t+1)^2},$$

it is a sequence in L^2 functions such that

$$Sf_n = g_n \to g \qquad \text{in } L^2 \text{ as } n \to \infty.$$

$\boxed{5,\text{ B}}$

(d) A small computation shows that

$$[S^2 f] = e^{-t^2-(t-1)^2} f(t-2).$$

Therefore, using that $\|e^{-t^2-(t-1)^2}\|_{L^\infty} = e^{-1/2}$,

$$\|S^2 f\|_{L^2} = \|e^{-t^2-(t-1)^2} f(t-2)\|_{L^2} \leq \|e^{-t^2-(t-1)^2}\|_{L^\infty} \|f(t-2)\|_{L^2} = e^{-1/2}\|f\|_{L^2}.$$

$\boxed{5,\text{ A}}$

4. (a) If $f \in L^2$, Tf is a continuous function, by a basic property of the Lebesgue integral, and since f is locally L^1. Furthemore, the sup norm of Tf can be bounded thanks to the Cauchy-Schwarz inequality

$$\|Tf\|_\infty \leq \|f\|_{L^2(-1,1)}\|1\|_{L^2(-1,1)} = \sqrt{2}\|f\|_{L^2(-1,1)}.$$

$\boxed{4, \text{B}}$

(b) By the Arzela-Ascoli theorem, it suffices to show that the functions Tf are equicontinuous if f is in the unit ball of L^2. This can be proved as follows: if $s, t \in (-1, 1)$,

$$\begin{aligned}|Tf(t) - Tf(s)| &= \left|\int_{-1}^1 e^{itx} f(x)\,dx - \int_{-1}^1 e^{isx} f(x)\,dx\right| = \left|\int_{-1}^1 [e^{itx} - e^{isx}] f(x)\,dx\right| \\ &\leq |t-s| \left|\int_{-1}^1 |x f(x)|\,dx\right| \leq |t-s|\|x\|_{L^2(-1,1)}\|f\|_{L^2(-1,1)} \leq |t-s|\|f\|_{L^2},\end{aligned}$$

where we used successively the inequality $|e^{itx} - e^{isx}| \leq |x||t-s|$, and the Cauchy-Schwarz inequality.

$\boxed{4, \text{C}}$

(c) On $(-1, 1)$, the L^2 norm is controlled by the uniform topology:

$$\|f\|_{L^2(-1,1)} = \left[\int_{-1}^1 |f(s)|^2\,ds\right]^{1/2} \leq \left[\int_{-1}^1 \|f\|_\infty^2\,ds\right]^{1/2} = \sqrt{2}\|f\|_{L^\infty}.$$

Therefore, the operator $T : L^2 \to L^2$ is compact since the operator $T : L^2 \to \mathcal{C}_0$ is compact.

$\boxed{4, \text{A}}$

(d) To compute the adjoint operator, we will use the formula

$$\langle Tf\,,\, g\rangle = \langle f\,,\, T^*g\rangle.$$

By definition of T, and with the help of the Fubini theorem,

$$\langle Tf\,,\, g\rangle = \int_{-1}^1 \int_{-1}^1 e^{itx} f(x)\,dx\,\overline{g(t)}\,dt = \int_{-1}^1 f(x)\overline{\int_{-1}^1 g(t)e^{-itx}\,dt}\,dx.$$

We can read off the formula for T^* from the above:

$$T^*f(x) = \int_{-1}^1 g(t)e^{-itx}\,dt.$$

$\boxed{4, \text{D}}$

(e)

The operator T^* is compact (Schauder's theorem), therefore, TT^* is compact. Furthermore, the adjoint operator of TT^* is itself. By the spectral theorem for compact self-adjoint operators, there exists a Hilbertian basis (ψ_n) of L^2, and eigenvalues (μ_n) such that

$$(TT^*)\psi_n = \mu_n\psi_n.$$

$\boxed{4, \text{A}}$

5. (a) Assume that (x_n) is bounded: $\|x_n\| \leq R$, and $\langle x_n , y\rangle \to \langle x , y\rangle$ for any $y \in \mathcal{D}$. Choose $z \in H$, and $\epsilon > 0$. By density of $\mathcal{D}$, there exists $z \in \mathcal{D}$ such that $\|y - z\| < \epsilon$. Then

$$\langle x_n , y\rangle - \langle x , y\rangle = \langle x_n - x , z\rangle + \langle x_n - x , y - z\rangle.$$

On the one hand, the Cauchy-Schwarz inequality gives $|\langle x_n - x , y - z\rangle| \leq 2R\epsilon$. On the other hand, we know by assumption that $\langle x_n - x , z\rangle \to 0$. Therefore,

$$\limsup |\langle x_n , y\rangle - \langle x , y\rangle| \leq \epsilon.$$

Since this is true for any $\epsilon > 0$, this implies that

$$|\langle x_n , y\rangle - \langle x , y\rangle| \to 0 \quad \text{as } n \to \infty,$$

or in other words that x_n converges weakly to x. $\boxed{6, \mathrm{M}}$

(b) The counterexample is as follows: choose a Hilbertian basis $(e_n)_{n\in\mathbb{N}}$, and let on the one hand $x_n = ne_n$, and on the other hand $\mathcal{D} = \mathrm{Span}\,\{e_n,\ n \in \mathbb{N}\}$.
For any $k \in \mathbb{N}$,

$$\langle x_n , e_k\rangle \to 0 \quad \text{as } n \to \infty,$$

so that for any $y \in \mathcal{D}$,

$$\langle x_n , y\rangle \to 0 \quad \text{as } n \to \infty$$

However, letting

$$w = \sum_{k\geq 0} \frac{1}{k} e_k,$$

it is an element of H such that

$$\langle w , e_n\rangle = 1 \quad \text{for any } k, n \in \mathbb{N}.$$

$\boxed{7, \mathrm{M}}$

(c) It follows from basic properties of the norm and scalar product clear that, if $x_n \to x$, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.
Conversely, assume that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$. Then, expanding,

$$\|x_n - x\|^2 = \langle x_n - x , x_n - x\rangle = \|x_n\|^2 + \|x\|^2 - 2\langle x , x_n\rangle.$$

By assumption, $\|x_n\|^2 \to \|x\|^2$ and $\langle x , x_n\rangle \to \|x\|^2$. Therefore, $\|x_n - x\| \to 0$, which is the desired result! $\boxed{7, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60020/70020	1	No Comments Received
MATH60020/70020	2	No Comments Received
MATH60020/70020	3	No Comments Received
MATH60020/70020	4	No Comments Received
MATH70020	5	No Comments Received