

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Asymptotic Methods

Date: 24 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) (i) Define what it means for two non-zero functions $f(t)$ and $g(t)$ to satisfy the relation $f(t) = o(g(t))$ as $t \rightarrow t_0$. (1 mark)
- (ii) Define what it means for a function $f(t)$ to have the asymptotic expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n f_n(t) \quad \text{as } t \rightarrow t_0. \quad (3 \text{ marks})$$

- (iii) What relations must the asymptotic scale $f_0(t)$, $f_1(t)$, $f_2(t)$, ... satisfy? (1 mark)
- (b) Let $f(t) = \sqrt{t + e^{-t}}$ and $I(x) = \int_0^x f(t) dt$.

Calculate the first three (non-zero) terms in the asymptotic expansions of:

- (i) $f(t)$ as $t \searrow 0$, (3 marks)
 (ii) $f(t)$ as $t \nearrow +\infty$, (3 marks)
 (iii) $I(x)$ as $x \searrow 0$, (3 marks)
 (iv) $I(x)$ as $x \nearrow +\infty$. (6 marks)

[You should choose your asymptotic scales to be as simple as possible. Your answers for $I(x)$ may contain a definite integral that you are unable to calculate, provided that it is convergent and independent of x .]

(Total: 20 marks)

2. (a) Use Laplace's method to calculate the first two terms in the asymptotic expansion of

$$I(x) = \int_{-\pi}^{\pi} e^{x \cos t} e^{-t} dt \quad \text{as } x \nearrow +\infty,$$

and estimate the order of the correction.

[You may use $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, if needed.]

(12 marks)

- (b) Calculate the first term in the asymptotic expansion of

$$I(x) = \int_0^\pi e^{x \cos t} e^{-1/t} dt \quad \text{as } x \nearrow +\infty,$$

[Hint: The moving maximum cannot be solved for analytically, so you will need to calculate its leading-order approximation and an estimate of the error. You will also need to evaluate a function and its second derivative to a suitable order of accuracy. You do not need to justify your approximations in detail.] (8 marks)

(Total: 20 marks)

3. Suppose that $y(x)$ satisfies

$$\varepsilon y'' + 2(1+x)y' + y - y^{-1} = 0, \quad y(0) = 1, \quad y(1) = 2, \quad \varepsilon \searrow 0.$$

- (a) (i) Assuming that the boundary layer is at $x = 0$, determine the leading-order outer solution $y(x) = y_0(x) + \dots$. [Hint: Consider the derivative of $(1+x)y^2$.] (4 marks)
 - (ii) Determine the width of the boundary layer at $x = 0$ and an appropriate inner variable X , and find the leading-order inner solution $y(x) = Y_0(X) + \dots$. (5 marks)
 - (iii) Sketch the solution $y(x)$. (2 marks)
 - (iii) Explain why the boundary layer cannot be at $x = 1$ instead. (3 marks)
- (b) Now suppose that the boundary conditions instead are $y(-1) = 1$, $y(1) = 2$.
- (i) Assuming that the outer solution is the same as in (a), what is its leading-order behaviour as $x \searrow -1$? (1 mark)
 - (ii) Deduce the width of the boundary layer at $x = -1$ and the scaling of y , and determine appropriate inner variables X and Y . (2 marks)
 - (iii) Obtain the equation for the leading-order approximation $Y(X) = Y_0(X) + \dots$. What is the boundary condition for Y_0 at $X = 0$? What is the matching condition for Y_0 as $X \nearrow \infty$? [You do not need to solve the equation.] (3 marks)

(Total: 20 marks)

4. (a) Let F be a real constant, and consider the real solutions $x(t)$ of the equation

$$x'' + x = \varepsilon \left[-(x')^3 + 2F \cos(2t) x' \right], \quad \text{where } \varepsilon \searrow 0.$$

Use the method of multiple scales with a slow time $T = \varepsilon t$ to obtain the leading-order approximation $x_0 = A_0(T)e^{it} + A_0(T)^*e^{-it}$, valid up to $t = O(1/\varepsilon)$, and determine the evolution equation for $A_0(T)$. [You do not need to solve the equation.] (10 marks)

Find the non-zero constant solutions for A_0 . (2 marks)

- (b) Now consider the real solutions $x(t)$ and $y(t)$ of the coupled equations

$$x'' + x = \varepsilon \left[-(x')^3 + x'y \right], \quad y'' + 4y = \varepsilon \left[-(y')^3 - (x^2)y \right], \quad \text{where } \varepsilon \searrow 0.$$

Use the method of multiple scales to determine the leading-order solutions x_0 and y_0 with complex amplitudes A_0 and B_0 , respectively, and obtain the evolution equations for $A_0(T)$ and $B_0(T)$. (6 marks)

Show that the non-zero constant solutions satisfy $|A_0| = 1/\sqrt{18}$, $|B_0| = 1/6$. What do $\arg A_0$ and $\arg B_0$ satisfy? (2 marks)

(Total: 20 marks)

5. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Suppose that $D(\mathbf{x}, \mathbf{y}) > 0$ is given and is periodic in both y_1 and y_2 with period λ . Consider the partial differential equation

$$\nabla \cdot [D(\mathbf{x}, \mathbf{x}/\varepsilon) \nabla u] = f(\mathbf{x}),$$

for $u(\mathbf{x})$, as $\varepsilon \searrow 0$.

- (a) Use the method of multiple scales, assuming that u depends on \mathbf{x} and \mathbf{y} independently and is periodic in \mathbf{y} , to derive the homogenised equation

$$\nabla \cdot [\hat{\mathbf{D}}(\mathbf{x}) \nabla u_0] = f(\mathbf{x}), \quad \hat{\mathbf{D}} = \begin{pmatrix} \langle D \rangle + \langle D \partial_{y_1} a_1 \rangle & \langle D \partial_{y_1} a_2 \rangle \\ \langle D \partial_{y_2} a_1 \rangle & \langle D \rangle + \langle D \partial_{y_2} a_2 \rangle \end{pmatrix}$$

for the leading-order approximation $u \sim u_0(\mathbf{x})$. You should define the operator $\langle \cdot \rangle$ and specify the equations and conditions satisfied by $a_1(\mathbf{x}, \mathbf{y})$ and $a_2(\mathbf{x}, \mathbf{y})$.

[You may assume that if $k(\mathbf{y}) > 0$ and $w(\mathbf{y})$ are periodic with the same period, and satisfy $\nabla_y \cdot [k \nabla_y w] = 0$, then w must be independent of \mathbf{y} .]

(15 marks)

- (b) Now consider the equation with an additional term,

$$\nabla \cdot [\mathbf{v}(\mathbf{x}, \mathbf{x}/\varepsilon) u] + \nabla \cdot [D(\mathbf{x}, \mathbf{x}/\varepsilon) \nabla u] = f(\mathbf{x}),$$

where $\mathbf{v}(\mathbf{x}, \mathbf{y})$ is also periodic in y_1 and y_2 with period λ . Derive the homogenised equation

$$\nabla \cdot [\hat{\mathbf{v}}(\mathbf{x}) u_0] + \nabla \cdot [\hat{\mathbf{D}}(\mathbf{x}) \nabla u_0] = f(\mathbf{x}), \quad \hat{\mathbf{v}} = \langle \mathbf{v} \rangle + \langle D \nabla_y b \rangle,$$

where you should specify the equations and conditions satisfied by $b(\mathbf{x}, \mathbf{y})$.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60004/70004

Asymptotic Methods (Solutions)

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1. (ai) We say that $f(t) = o(g(t))$ as $t \rightarrow t_0$ if and only if $f/g \rightarrow 0$ as $t \rightarrow t_0$.

seen ↓

(aii) We say that $f(t) \sim \sum_{n=0}^{\infty} a_n f_n(t)$ as $t \rightarrow t_0$ if and only if, for all $N \geq 0$,

$$f(t) = \sum_{n=0}^N a_n f_n(t) + o(f_N(t)) \text{ as } t \rightarrow t_0. \text{ (Use of } O(f_{N+1}(t)) \text{ is also accepted.)}$$

3, A

(aiii) The scale must satisfy $f_1 = o(f_0)$, $f_2 = o(f_1)$, \dots as $t \rightarrow t_0$.

1, A

$$\begin{aligned} (\text{bi}) \quad \sqrt{t + e^{-t}} &= \sqrt{t + 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + O(t^4)} = [1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + O(t^4)]^{1/2} = \\ &= 1 + \frac{1}{2} [\frac{1}{2}t^2 - \frac{1}{6}t^3 + O(t^4)] + O(t^4) = 1 + \frac{1}{4}t^2 - \frac{1}{12}t^3 + O(t^4). \end{aligned}$$

meth seen ↓

3, A

$$\begin{aligned} (\text{bii}) \quad \sqrt{t + e^{-t}} &= t^{1/2} \left[1 + \frac{e^{-t}}{t} \right]^{1/2} = t^{1/2} \left[1 + \frac{1}{2} \frac{e^{-t}}{t} - \frac{1}{8} \frac{e^{-2t}}{t^2} + O(e^{-3t}/t^3) \right] = \\ &= t^{1/2} + \frac{e^{-t}}{2t^{1/2}} - \frac{e^{-2t}}{8t^{3/2}} + O(e^{-3t}/t^{-5/2}). \end{aligned}$$

3, B

(biii) For $x \searrow 0$, we can integrate the $t \searrow 0$ result term by term,

$$I(x) = \int_0^x 1 + \frac{1}{4}t^2 - \frac{1}{12}t^3 + O(t^4) dt = x + \frac{1}{12}x^3 - \frac{1}{48}x^4 + O(x^5).$$

3, A

(biv) For $x \nearrow +\infty$, the integral diverges, so we need to subtract the corresponding asymptotic behaviour of the integrand. Before we can integrate the large- t expansion term by term we also need to shift the limit of the integral to ∞ . Thus,

$$\begin{aligned} I(x) &= \int_0^x \sqrt{t} dt + \int_0^x \sqrt{t + e^{-t}} - \sqrt{t} dt = \\ &= \frac{2}{3}x^{3/2} + \int_0^\infty \sqrt{t + e^{-t}} - \sqrt{t} dt - \int_x^\infty \sqrt{t + e^{-t}} - \sqrt{t} dt. \end{aligned}$$

Using (bii), we obtain the leading term of the final integrand and integrate by parts,

$$\int_x^\infty \frac{e^{-t}}{2t^{1/2}} dt = \left[-\frac{e^{-t}}{2t^{1/2}} \right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^{3/2}} dt \sim \frac{e^{-x}}{2x^{1/2}},$$

making sure to verify that the remainder integral is negligible, for example by bounding it as

$$\left| \int_x^\infty \frac{e^{-t}}{t^{3/2}} dt \right| \leq \int_x^\infty \frac{e^{-t}}{x^{3/2}} dt = \frac{e^{-x}}{x^{3/2}}.$$

We conclude that

$$I(x) \sim \frac{2}{3}x^{3/2} + \left[\int_0^\infty \sqrt{t + e^{-t}} - \sqrt{t} dt \right] + \frac{e^{-x}}{2x^{1/2}},$$

where the integral is a finite constant.

6, C

(Total: 20 marks)

2. (a) In the interval $-\pi \leq t \leq \pi$, the function $\cos t$ attains its maximum at $t = 0$, so the exponentially dominant contribution to the integral comes from a small neighbourhood of that point. Since this is an internal quadratic maximum, we expect the correction from next power of t to cancel, so we expand to two more powers of t to obtain the second non-zero term.

meth seen ↓

We thus have

$$\begin{aligned}
I(x) &= \int_{-\varepsilon}^{\varepsilon} e^x \cos t e^{-t} dt + \text{EST} = \\
&= \int_{-\varepsilon}^{\varepsilon} e^{x[1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + O(t^6)]} \left[1 - t + \frac{t^2}{2} + O(t^3) \right] dt + \text{EST} = \\
&\stackrel{t=(2/x)^{1/2}s}{=} e^x \int_{-\varepsilon(x/2)^{1/2}}^{\varepsilon(x/2)^{1/2}} e^{-s^2} \left[1 + \frac{s^4}{6x} + O(x^{-2}) \right] \times \\
&\quad \left[1 - \frac{\sqrt{2}s}{x^{1/2}} + \frac{s^2}{x} + O(x^{-3/2}) \right] \left(\frac{2}{x} \right)^{1/2} ds + \text{EST} = \\
&= \left(\frac{2}{x} \right)^{1/2} e^x \int_{-\infty}^{\infty} e^{-s^2} \left[1 - \frac{\sqrt{2}s}{x^{1/2}} + \frac{\frac{1}{6}s^4 + s^2}{x} + O(x^{-3/2}) \right] ds + \text{EST}.
\end{aligned}$$

As expected, the $O(x^{-1/2})$ term is odd so its integral vanishes. We evaluate the remaining integrals using

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-s^2} s^{2n} ds &= 2 \int_0^{\infty} e^{-s^2} s^{2n} ds \stackrel{s=u^{1/2}}{=} \int_0^{\infty} e^{-u} u^{n-\frac{1}{2}} du = \Gamma(n + \frac{1}{2}), \\
\Gamma(\frac{1}{2}) &= \sqrt{\pi}, \quad \Gamma(1 + \frac{1}{2}) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma(2 + \frac{1}{2}) = \frac{3}{4}\sqrt{\pi},
\end{aligned}$$

and obtain

$$\begin{aligned}
I(x) &= \left(\frac{2\pi}{x} \right)^{1/2} e^x \left[1 + \frac{\frac{1}{6} \times \frac{3}{4} + \frac{1}{2}}{x} + O(x^{-3/2}) \right] = \\
&= \left(\frac{2\pi}{x} \right)^{1/2} e^x \left[1 + \frac{5}{8x} + O(x^{-3/2}) \right].
\end{aligned}$$

5, A
5, B

The contributions to the $x^{-3/2}$ term will also be integrals of odd functions, so the next term is expected to be $O(x^{-2})$ in the square brackets, or $O(e^x x^{-5/2})$ overall.

2, C

- (b) The standard Laplace's method fails because $e^{-1/t}$ decays to zero exponentially fast as $t \searrow 0$, where the maximum of $\cos t$ is attained. Instead, we need to expand about a moving maximum $t = t_*$ of the overall exponent $h(t) = x \cos t - t^{-1}$.

We find the maximum by requiring that the derivative of h vanishes, i.e.

$$0 = h'(t_*) = -x \sin t_* + t_*^{-2} \Rightarrow t_*^2 \sin t_* = \frac{1}{x}.$$

Since the relevant maximum of $\cos t$ is at $t = 0$, we expect the moving maximum t_* to be small as $x \nearrow +\infty$, so we expand the equation for small t_* and solve,

$$\begin{aligned} t_*^2 [t_* + O(t_*^3)] &= \frac{1}{x} \\ \Rightarrow t_* &= \frac{1}{x^{1/3}} [1 - O(t_*^2)]^{-1/3} = \frac{1}{x^{1/3}} [1 + O(t_*^2)] = \frac{1}{x^{1/3}} + O(x^{-1}). \end{aligned}$$

2, C

For Laplace's method, we'll need to expand $h(t)$ about t_* to the first varying term, which is the second derivative. We thus need to evaluate $h(t_*)$ to $\text{ord}(1)$, and find the leading-order behaviour of $h''(t_*)$:

$$\begin{aligned} h(t_*) &= x \cos(t_*) - t_*^{-1} = \\ &= x \left[1 - \frac{1}{2} \left(x^{-1/3} + O(x^{-1}) \right)^2 + O((x^{-1/3})^4) \right] - \frac{1}{x^{1/3} + O(x^{-1})} = \\ &= x \left[1 - \frac{1}{2} x^{-2/3} + O(x^{-4/3}) \right] - \frac{1}{x^{1/3}} \left[1 + O(x^{-2/3}) \right] \\ &= x - \frac{3}{2} x^{1/3} + O(x^{-1/3}), \\ h''(t_*) &= -x \cos(t_*) - 2t_*^{-3} = -x \left[1 + O(x^{-2/3}) \right] - 2x \left[1 + O(x^{-2/3}) \right] \\ &= -3x + O(x^{1/3}). \end{aligned}$$

4, D

The leading-order behaviour of the integral is then given by

$$\begin{aligned} I(x) &\sim \int_{-\varepsilon}^{\varepsilon} e^{h(t_*+s)} ds \sim \int_{-\varepsilon}^{\varepsilon} e^{h(t_*) + \frac{1}{2} h''(t_*) s^2} ds \\ &\sim e^{x - \frac{3}{2} x^{1/3}} \int_{-\infty}^{\infty} e^{-\frac{3}{2} x s^2} ds = \sqrt{\frac{2\pi}{3x}} e^{x - \frac{3}{2} x^{1/3}}. \end{aligned}$$

2, D

(Total: 20 marks)

3. We consider $\varepsilon y'' + 2(1+x)y' + y - y^{-1} = 0$, $y(0) = 1$, $y(1) = 2$, $\varepsilon \searrow 0$.

sim. seen \downarrow

- (ai) The leading-order outer solution $y \sim y_0(x)$ satisfies

$$\begin{aligned} 2(1+x)y'_0 + y_0 - y_0^{-1} &= 0 \Rightarrow 2(1+x)y_0y'_0 + y_0^2 - 1 = 0 \\ \Rightarrow [(1+x)y_0^2]' &= 1 \Rightarrow (1+x)y_0^2 = x + a_0 \Rightarrow y_0 = \pm \sqrt{\frac{x+a_0}{1+x}}. \end{aligned}$$

Assuming that the boundary layer is at $x = 0$, we only impose the boundary condition at $x = 1$ on the outer solution,

$$2 = y_0(1) = \pm \sqrt{\frac{1+a_0}{2}} \Rightarrow a_0 = 7, y_0 = +\sqrt{\frac{x+7}{x+1}}.$$

4, A

- (aii) For the inner solution near $x = 0$, we seek an inner variable $X = x/\varepsilon^\alpha$ with $y = Y(X) = \text{ord}(1)$, and obtain

$$\varepsilon^{1-2\alpha}Y'' + \varepsilon^{-\alpha}2(1+\varepsilon^\alpha X)Y' + Y - Y^{-1} = 0.$$

The only consistent balance involving the Y'' term is for $\alpha = 1$, so the boundary-layer width is $x = \text{ord}(\varepsilon)$ and the inner variable is $X = x/\varepsilon$.

2, A

The resulting leading-order equation is

$$Y_0'' + 2Y_0' = 0 \Rightarrow Y_0 = A_0 + B_0 e^{-2X}.$$

The boundary condition $y = 1$ at $x = 0$ yields

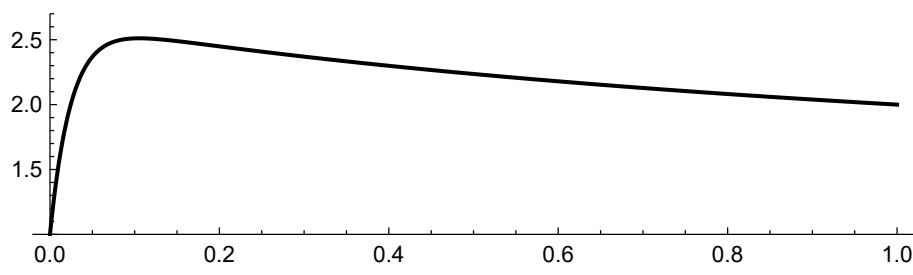
$$1 = Y_0(0) = A_0 + B_0 \Rightarrow Y_0 = A_0 + (1 - A_0)e^{-2X}.$$

The constant A_0 is determined by matching to the outer, i.e. equating

$$y_0(0) = \sqrt{7} \text{ and } Y_0(\infty) = A_0 \Rightarrow Y_0 = \sqrt{7} + (1 - \sqrt{7})e^{-2X}.$$

3, A

- (aiii) To sketch the solution, we note that as x decreases from 1 to 0, the outer approximation y_0 increases from 2 towards $\sqrt{7}$, and then the solution rapidly decreases to 1 in an $\text{ord}(\varepsilon)$ boundary layer.



2, A

- (aiv) For a boundary layer at $x = 1$, then the inner variable would be given by $x = 1 + \varepsilon X$, with the leading-order equation

$$Y_0'' + 4Y_0' = 0 \quad \Rightarrow \quad Y_0 = A_0 + B_0 e^{-4X}.$$

To match this solution with the outer, we take the limit $X \searrow -\infty$, and note that e^{-4X} diverges exponentially. This cannot match to the outer, so we have $B_0 = 0$ and the inner solution is constant $A_0 = 2$ at leading order, so the outer solution would still need to satisfy $y_0(1) = 2$.

- (bi) Near $x = -1$, we write $x = -1 + \Delta x$ and find that the previous outer solution satisfies

3, B

sim. seen ↓

1, A

- (bii) Based on the outer behaviour, we write $X = \Delta x/\varepsilon^\alpha = (x+1)/\varepsilon^\alpha$ and $y = \varepsilon^{-\alpha/2}Y$, and obtain

$$\varepsilon^{1-5\alpha/2}Y'' + 2\varepsilon^{-\alpha/2}XY' + \varepsilon^{-\alpha/2}Y - \varepsilon^{\alpha/2}Y^{-1} = 0.$$

The last term must be negligible as $\alpha > 0$, and we balance the remaining three terms using $\alpha = 1/2$. Thus, the boundary layer has width $\text{ord}(\varepsilon^{1/2})$ and the inner variables are given by

$$x = -1 + \varepsilon^{1/2}X, \quad y = \varepsilon^{-1/4}Y(X).$$

2, C

- (biii) The leading-order equation is

$$Y_0'' + 2XY_0' + Y_0 = 0.$$

The boundary condition at $x = -1$ becomes

$$1 = y(-1) = \varepsilon^{-1/4}Y(0) \quad \Rightarrow \quad Y(0) = \varepsilon^{1/4} \quad \Rightarrow \quad Y_0(0) = 0.$$

The matching condition to the outer solution is obtained from

$$y_0(x) \sim 6(x+1)^{-1/2} = 6\varepsilon^{-1/4}X^{-1/2} \quad \Rightarrow \quad Y_0(X) \sim 6X^{-1/2} \text{ as } X \nearrow \infty.$$

3, D

(Total: 20 marks)

4. (a) We introduce the slow time $T = \varepsilon t$ and treat it as independent of the fast time, so that $d/dt \rightarrow \partial/\partial t + \varepsilon\partial/\partial T$. The equation becomes

meth seen ↓

$$\partial_t^2 x + 2\varepsilon\partial_t\partial_T x + \varepsilon^2\partial_T^2 x + x = \varepsilon [-(x')^3 + F(e^{2it} + cc)x'] ,$$

and we substitute in $x = x_0 + \varepsilon x_1 + \dots$.

At $\text{ord}(\varepsilon^0)$, we obtain the leading-order equation

$$x_0'' + x_0 = 0 \quad \Rightarrow \quad x_0 = A_0(T)e^{it} + cc.$$

4, A

At $\text{ord}(\varepsilon^1)$, we obtain

$$\begin{aligned} x_1'' + x_1 &= -2\partial_t\partial_T x_0 - (x_0')^3 + (Fe^{2it} + cc)x_0' = \\ &= -2[iA_0'e^{it} + cc] - [iA_0e^{it} - iA_0^*e^{-it}]^3 + \\ &\quad + (Fe^{2it} + cc)(iA_0e^{it} - iA_0^*e^{-it}) = \\ &= [-2iA_0'e^{it} + cc] + [iA_0^3e^{3it} - 3i|A_0|^2A_0e^{it} + cc] + \\ &\quad + [iFA_0e^{3it} - iFA_0^*e^{it} + cc] = \\ &= i[A_0^3 + FA_0]e^{3it} + i[-2A_0' - 3|A_0|^2A_0 - FA_0^*]e^{it} + cc. \end{aligned}$$

4, B

The forcing proportional to e^{it} is secular, as it will generate a term proportional to te^{it} in x_1 . Hence we require that it vanishes, and obtain the evolution equation

$$A_0' = -\frac{3}{2}|A_0|^2A_0 - \frac{F}{2}A_0^*.$$

2, B

The non-zero steady solution is obtained from

$$0 = A_0' = -\frac{3}{2}A_0^2A_0^* - \frac{F}{2}A_0^* \quad \Rightarrow \quad A_0^2 = -\frac{F}{3} \quad \Rightarrow \quad A_0 = \pm i\sqrt{\frac{F}{3}}.$$

2, B

(b) The leading-order equations yield

meth seen ↓

$$x_0'' + x_0 = 0, \quad y_0'' + 4y_0 = 0 \quad \Rightarrow \quad x_0 = A_0 e^{it} + \text{cc}, \quad y_0 = B_0 e^{2it} + \text{cc}.$$

1, B

At next order, for the x_1 equation we can reuse the previous results, but replace the F term with

$$(B_0 e^{2it} + \text{cc})x' = iB_0 A_0 e^{3it} - iB_0 A_0^* e^{it} + \text{cc}.$$

This yields the equation and secularity condition

$$\begin{aligned} x_1'' + x_1 &= [\dots] e^{3it} + i[-2A'_0 - 3|A_0|^2 A_0 - B_0 A_0^*] e^{it} + \text{cc} \\ \Rightarrow \quad A'_0 &= -\frac{3}{2}|A_0|^2 A_0 - \frac{1}{2}B_0 A_0^*. \end{aligned}$$

2, D

For y_1 , we obtain the equation

$$\begin{aligned} y_1'' + 4y_1 &= -2\partial_t \partial_T y_0 - (y'_0)^3 - (x_0^2)' \\ &= [-4iB'_0 e^{2it} + \text{cc}] + [8iB_0^3 e^{6it} - 24i|B_0|^2 B_0 e^{2it} + \text{cc}] \\ &\quad - [2iA_0^2 e^{2it} - 2i(A_0^*)^2 e^{-2it}]. \end{aligned}$$

Requiring no secular term proportional to e^{2it} then yields

$$-4iB'_0 - 24i|B_0|^2 B_0 - 2iA_0^2 = 0 \quad \Rightarrow \quad B'_0 = -6|B_0|^2 B_0 - \frac{1}{2}A_0^2.$$

3, D

The non-trivial steady solutions satisfy

$$\begin{aligned} A_0^2 &= -\frac{1}{3}B_0, \quad |B_0|^2 B_0 = -\frac{1}{12}A_0^2 \quad \Rightarrow \quad |B_0|^2 B_0 = \frac{1}{36}B_0 \\ \Rightarrow \quad |B_0| &= \frac{1}{6}, \quad |A_0| = \sqrt{\frac{|B_0|}{3}} = \frac{1}{\sqrt{18}}. \end{aligned}$$

The A_0 equation also yields $2 \arg A_0 = \pi + \arg B_0$. There is no second restriction on the arguments of A_0 and B_0 (as the remaining one degree of freedom corresponds to an arbitrary shift in time).

2, D

(Total: 20 marks)

5. (a) We write $u = u(\mathbf{x}, \mathbf{y})$ where $\mathbf{y} = \mathbf{x}/\varepsilon$ is treated as a separate variable, so that $\nabla = \nabla_{\mathbf{x}} + \frac{1}{\varepsilon} \nabla_{\mathbf{y}}$. This yields, after multiplying through by ε^2 ,

$$(\nabla_{\mathbf{y}} + \varepsilon \nabla_{\mathbf{x}}) \cdot [D(\mathbf{x}, \mathbf{y}) (\nabla_{\mathbf{y}} + \varepsilon \nabla_{\mathbf{x}}) u] = \varepsilon^2 f(\mathbf{x}).$$

We also enforce that u is periodic in \mathbf{y} .

We expand $u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$, and solve the equation order by order. At $\text{ord}(\varepsilon^0)$,

$$\nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{y}} u_0] = 0.$$

The only \mathbf{y} -periodic solution u_0 is constant in \mathbf{y} , i.e. $u_0 = u_0(\mathbf{x})$.

At $\text{ord}(\varepsilon^1)$,

$$\begin{aligned} \nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{y}} u_1] + \nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{x}} u_0] + \nabla_{\mathbf{x}} \cdot [D \nabla_{\mathbf{y}} u_0] &= 0 \\ \Rightarrow \quad \nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{y}} u_1] &= -(\nabla_{\mathbf{y}} D) \cdot \nabla_{\mathbf{x}} u_0, \end{aligned}$$

where we have used the fact that u_0 is independent of \mathbf{y} .

We further use the independence of u_0 on \mathbf{y} to write

$$u_1 = a_1(\mathbf{x}, \mathbf{y}) \partial_{x_1} u_0 + a_2(\mathbf{x}, \mathbf{y}) \partial_{x_2} u_0 + c_1(\mathbf{x}) = \mathbf{a}(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} u_0(\mathbf{x}) + c_1(\mathbf{x}),$$

where a_1 and a_2 are solutions of the cell problem

$$\nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{y}} a_i] = -\partial_{y_i} D,$$

with the condition that $a_{1,2}$ are periodic. (As before, any function of \mathbf{x} alone is a homogeneous solution to the equations, so the $a_{1,2}$ are only determined up to addition of such a function.)

At $\text{ord}(\varepsilon^2)$,

$$f(\mathbf{x}) = \nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{y}} u_2] + \nabla_{\mathbf{y}} \cdot [D \nabla_{\mathbf{x}} u_1] + \nabla_{\mathbf{x}} \cdot [D \nabla_{\mathbf{y}} u_1] + \nabla_{\mathbf{x}} \cdot [D \nabla_{\mathbf{x}} u_0].$$

We take the average over a cell, defined by

$$\langle \dots \rangle = \frac{1}{\lambda^2} \int_0^\lambda \int_0^\lambda \dots dy_1 dy_2,$$

noting that the average of a divergence $\nabla_{\mathbf{y}} \cdot (\dots)$ vanishes due to the divergence theorem and periodicity in \mathbf{y} (which is satisfied by all quantities involved), and that any function of (or derivative by) \mathbf{x} alone can be factored out of the average. This yields

$$\begin{aligned} f &= 0 + 0 + \nabla_{\mathbf{x}} \cdot [\langle D \nabla_{\mathbf{y}} u_1 \rangle] + \nabla_{\mathbf{x}} \cdot [\langle D \rangle \nabla_{\mathbf{x}} u_0] = \\ &= \nabla_{\mathbf{x}} \cdot [\langle D(\nabla_{\mathbf{y}} a_1) \rangle \partial_{x_1} u_0 + \langle D(\nabla_{\mathbf{y}} a_2) \rangle \partial_{x_2} u_0 + \langle D \rangle \nabla_{\mathbf{x}} u_0] = \\ &= \nabla_{\mathbf{x}} \cdot [\hat{\mathbf{D}} \cdot \nabla_{\mathbf{x}} u_0], \quad \hat{\mathbf{D}} = \begin{pmatrix} \langle D \rangle + \langle D \partial_{y_1} a_1 \rangle & \langle D \partial_{y_1} a_2 \rangle \\ \langle D \partial_{y_2} a_1 \rangle & \langle D \rangle + \langle D \partial_{y_2} a_2 \rangle \end{pmatrix}. \end{aligned}$$

seen \downarrow

4, M

3, M

3, M

5, M

- (b) We repeat the calculation from (a) but with an extra term. The main equation becomes

unseen ↓

$$\varepsilon (\nabla_y + \varepsilon \nabla_x) \cdot [\mathbf{v}u] + (\nabla_y + \varepsilon \nabla_x) \cdot [D(\mathbf{x}, \mathbf{y}) (\nabla_y + \varepsilon \nabla_x) u] = \varepsilon^2 f(\mathbf{x}),$$

and the $\text{ord}(\varepsilon^0)$ result is unchanged:

$$\nabla_y \cdot [D \nabla_y u_0] = 0 \quad \Rightarrow \quad u_0 = u_0(\mathbf{x}).$$

1, M

At $\text{ord}(\varepsilon^1)$, we find

$$\begin{aligned} 0 &= \nabla_y \cdot [\mathbf{v}u_0] + \nabla_y \cdot [D \nabla_y u_1] + \nabla_y \cdot [D \nabla_x u_0] + \nabla_x \cdot [D \nabla_y u_0] = \\ &= (\nabla_y \cdot \mathbf{v}) u_0 + \nabla_y \cdot [D \nabla_y u_1] + (\nabla_y D) \cdot \nabla_x u_0 \\ &\Rightarrow \quad \nabla_y \cdot [D \nabla_y u_1] = -(\nabla_y \cdot \mathbf{v}) u_0 - (\nabla_y D) \cdot \nabla_x u_0, \end{aligned}$$

where we again have used the fact that u_0 is independent of y . The solution can then be written as

$$u_1 = b(\mathbf{x}, \mathbf{y}) u_0(\mathbf{x}) + \mathbf{a}(\mathbf{x}, \mathbf{y}) \cdot \nabla_x u_0(\mathbf{x}) + c_1(\mathbf{x}),$$

where the new term requires the y -periodic solution $b(\mathbf{x}, \mathbf{y})$ of the cell problem

$$\nabla_y \cdot [D \nabla_y b] = -\nabla_y \cdot \mathbf{v}.$$

2, M

At $\text{ord}(\varepsilon^2)$, we again take the average of the equation

$$\begin{aligned} f(\mathbf{x}) &= \nabla_y \cdot [\mathbf{v}u_1] + \nabla_x \cdot [\mathbf{v}u_0] + \\ &\quad + \nabla_y \cdot [D \nabla_y u_2] + \nabla_y \cdot [D \nabla_x u_1] + \nabla_x \cdot [D \nabla_y u_1] + \nabla_x \cdot [D \nabla_x u_0]. \end{aligned}$$

over a cell, discarding any divergence terms $\nabla_y \cdot (\dots)$, and obtain

$$\begin{aligned} f &= 0 + \nabla_x \cdot [\langle \mathbf{v} \rangle u_0] + 0 + 0 + \nabla_x \cdot \langle D \nabla_y u_1 \rangle + \nabla_x \cdot \langle D \rangle \nabla_x u_0 = \\ &= \nabla_x \cdot [\langle \mathbf{v} \rangle u_0 + \langle D(\nabla_y b) \rangle u_0 + \langle D(\nabla_y a_1) \rangle \partial_{x_1} u_0 + \langle D(\nabla_y a_2) \rangle \partial_{x_2} u_0 + \langle D \rangle \cdot \nabla_x u_0] \\ &= \nabla_x \cdot [\hat{\mathbf{v}} u_0] + \nabla_x \cdot [\hat{\mathbf{D}} \cdot \nabla_x u_0], \quad \hat{\mathbf{v}} = \langle \mathbf{v} \rangle + \langle D \nabla_y b \rangle. \end{aligned}$$

2, M

(Total: 20 marks)

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	QuestionNumber	Comments for Students
MATH60004/70004	1	Only half managed to give all the definitions correctly, with the definition of an asymptotic expansion proving particularly tricky. The basic expansions and the first integral went very well, but only a third managed the last part with the trickier integral, where you have to subtract a divergent term first.
MATH60004/70004	2	For the standard Laplace problem in (a), most managed to calculate the leading-order term, but only a few obtained both contributions to the next-order term correctly. A common mistake was unnecessarily treating the maximum as a moving one. In (b), the moving maximum is conceptually more complicated even though the calculations are fairly straightforward - around a third managed to figure out what to do.
MATH60004/70004	3	Part (a) is a fairly standard matched asymptotics question, and the vast majority did very well. Part (b) requires some more thought, but around a third managed to get the gist of it correct, although some marks were lost on not answering (i) properly (i.e. including the prefactor), or not answering all parts of (iii).
MATH60004/70004	4	Part (a) is a standard multiple-scales question, which went very well. In part (b) an additional frequency is introduced so that the secular terms are of a different form, which most students managed to handle fine.
MATH70004	5	Part (a) asked the students to reproduce the standard homogenisation procedure, which around half managed to do. Part (b) then added an extra term, but very few students did their algebra carefully enough to realise that this term comes in to the equations only at order epsilon, and none managed to complete the question.