

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Introduction to Stochastic Differential Equations

Date: 18 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD
WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS
ANSWERED AND PAGE NUMBERS PER QUESTION.**

1.

- (a) Let W_t be a standard one-dimensional Brownian motion, i.e. a continuous-time Gaussian process with almost surely continuous paths, mean $\mathbb{E}W_t = 0$ and covariance $\mathbb{E}(W_t W_s) = \min(t, s)$.
- (i) Show that the function $R(t, s) = \min(t, s)$ is symmetric and nonnegative definite.
 - (ii) Show that the one dimensional stochastic process W_t with almost surely continuous paths is a standard one-dimensional Brownian motion if and only if $W_0 = 0$ a.s. and it has independent increments with $W_t - W_s \sim \mathcal{N}(0, t - s)$ for all $t \geq s \geq 0$.

(12 marks)

- (b) Show that the one-dimensional standard Brownian motion is a Markov process and give the formula of the corresponding Markov semigroup P_t . Show that for every $f \in C^1(\mathbb{R})$ we have

$$\frac{d}{dx} (P_t f) = P_t \left(\frac{df}{dx} \right).$$

(8 marks)

(Total: 20 marks)

2. Let W_t be a standard one dimensional Brownian motion and $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space and \mathcal{F}_t the natural filtration. Let $f(t, \omega) : [0, +\infty) \times \Omega \rightarrow \mathbb{R}$ such that f is $\mathcal{B} \times \mathcal{F}$ measurable, where \mathcal{B} denotes the Borel σ -algebra on $[0, +\infty)$, that $f(t, \omega)$ is \mathcal{F}_t -adapted, and that $\mathbb{E} \int_0^T f^2(t, \omega) dt < +\infty$.

- (a) Give the definition of the Stratonovich stochastic integral. (2 marks)
- (b) Calculate the Stratonovich stochastic integral

$$I_{Strat}(T) = \int_0^T W_t \circ dW_t.$$

(6 marks)

- (c) Assume that there exists a constant $C > 0$ and $\varepsilon > 0$ such that

$$\mathbb{E}|f(s, \cdot) - f(t, \cdot)|^2 \leq C|s - t|^{1+\varepsilon}, \quad s, t \in [0, T].$$

Prove that in this case the stochastic integral is independent of the choice of $t'_j \in [t_j, t_{j+1}]$ in the sense that

$$\int_0^T f(t, \omega) dW(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t'_j, \omega) \Delta W_j,$$

the convergence being in $L^1(\mathbb{P})$. (6 marks)

- (d) Show that $N_t = W_t^3 - 3tW_t$ is a martingale with respect to the filtration generated by W_t . (6 marks)

(Total: 20 marks)

3.

- (a) Let W_t denote a one dimensional Brownian motion and let $h(t) \in L^2[0, T]$, be deterministic. Define

$$Y(t, \omega) = \exp \left\{ \int_0^t h(s) dW_s(\omega) - \frac{1}{2} \int_0^t h^2(s) ds \right\}.$$

1. Obtain a stochastic differential equation for $Y(t, \omega)$.
2. Show that $Y(t, \omega)$ is a square integrable martingale with respect to the filtration generated by the Brownian motion W_t .

(6 marks)

- (b) Consider the Itô SDE

$$dX_t = \frac{1 - X_t}{1 - t} dt + dW_t, \quad X_0 = 0, \quad (1)$$

where W_t is a standard one dimensional Brownian motion and $t \in (0, 1)$.

1. Use an integrating factor or otherwise to show that the solution of (1) is

$$X_t = t + (1 - t) \int_0^t \frac{1}{1 - s} dW_s. \quad (2)$$

2. Calculate the mean, the variance and the autocorrelation function of X_t .

(14 marks)

(Total: 20 marks)

4.

- (a) Let X_t be a diffusion process on $[0, 1]^d$ with periodic boundary conditions. The drift vector is a periodic function $a(x)$ and the diffusion matrix is $2DI$, where $D > 0$ and I is the identity matrix.

1. Write down the SDE satisfied by X_t , the generator and the backward and forward (Fokker-Planck) Kolmogorov equations for X_t .
2. Assume that $a(x)$ is divergence-free ($\nabla \cdot a(x) = 0$). Show that X_t is an ergodic diffusion process and find the invariant distribution.
3. Assume that the initial distribution, $X_0 \sim \rho_0(x)$, satisfies $\rho_0 \in L^2([0, 1]^d)$. Show that the probability density function $\rho(x, t)$ converges exponentially fast to the invariant distribution in L^2 .

(12 marks)

- (b) Consider the SDE

$$dX_t = \left(-\alpha X_t + \frac{D}{X_t} \right) dt + \sqrt{2D} dW_t, \quad X_0 = x > 0,$$

where $\alpha, D > 0$.

1. Write down the generator the forward and backward Kolmogorov equations for X_t .
2. Show that this process is ergodic and find its invariant distribution.

(8 marks)

(Total: 20 marks)

5. Consider the Itô stochastic differential equation

$$dX_t = -(D\nabla V)(X_t) dt + (\nabla \cdot D)(X_t) dt + \sqrt{2}\Gamma(X_t) dW_t, \quad X_0 = x, \quad (3)$$

where $V \in C^\infty(\mathbb{R}^d)$, $D \in \mathbb{R}^{d \times d}$ is symmetric, $D = D^T$, with

$$\delta|\xi|^2 \leq \xi \cdot D(x)\xi \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad (4)$$

for some $\delta > 0$, uniformly in $x \in \mathbb{R}^d$. The matrix Γ satisfies $\Gamma\Gamma^T = D$, and W_t is a standard d -dimensional Brownian motion.

- (a) Write down the generator and the backward and forward (Fokker-Planck) Kolmogorov equations for the process X_t . (2 marks)

- (b) Assume that $Z := \int e^{-V} dx < +\infty$. Show that the process X_t is ergodic with respect to the probability measure

$$\mu(dx) = \frac{1}{Z}e^{-V} dx =: \rho_\infty(x) dx. \quad (5)$$

(3 marks)

- (c) Show that the generator of X_t is symmetric in the weighted L^2 space $L^2(\mathbb{R}^d; \rho_\infty)$ and calculate the Dirichlet form

$$D_{\mathcal{L}}(f) := \int_{\mathbb{R}^d} (-\mathcal{L}f)f \mu(dx), \quad (6)$$

for all smooth functions f for which $D_{\mathcal{L}}(f)$ is finite.

(3 marks)

- (d) Assume that the probability measure $\mu(dx)$ satisfies Poincaré's inequality

$$\text{Var}_\mu(f) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f|^2 \mu(dx), \quad (7)$$

with $\lambda > 0$ and for all functions $f \in L^2(\mathbb{R}^d; \rho_\infty)$ for which $\int |f|^2 \mu(dx)$ is finite. Assume that the initial condition X_0 is a random variable with probability density function $\rho_0 \in L^2(\mathbb{R}^d; \rho_\infty^{-1})$. Show that the law of the process X_t converges exponentially fast to the unique invariant measure in $L^2(\mathbb{R}^d; \rho_\infty^{-1})$. Calculate the rate of convergence in terms of the constants δ and λ in (4) and (7).

(6 marks)

- (e) Let $\rho(x, t)$ denote the law of the process (the solution of the Forward Kolmogorov equation) and define

$$H(t) = \int_{\mathbb{R}^d} \rho(x, t) \ln \left(\frac{\rho(x, t)}{\rho_\infty(x)} \right) dx. \quad (8)$$

Show that

$$\frac{dH(t)}{dt} \leq 0.$$

(6 marks)

(Total: 20 marks)

1. (a) (i) The function $R(t, s) = \min(t, s)$ is clearly symmetric. To show that it is positive definite, we calculate, using the notation $\mathbf{1}_A(s)$ for the characteristic function of the set A and for $a_1 \dots a_n \in \mathbb{R}$ and $t_1 \dots t_n \geq 0$:

$$\begin{aligned} \sum_{i,j=1}^n a_i a_j \min(t_i, t_j) &= \sum_{i,j=1}^n a_i a_j \int_0^\infty \mathbf{1}_{[0,t_i]}(s) \mathbf{1}_{[0,t_j]}(s) ds \\ &= \int_0^\infty \left(\sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}(s) \right)^2 ds. \end{aligned}$$

[4] MARKS – A

- (ii) We start by showing that the axioms defining Brownian motion imply the second characterisation of BM. First, since $\mathbb{E}W_0 = 0$, $\mathbb{E}(W_0^2) = 0$ and W_t is a Gaussian process, we deduce that $W_0 = 0$ a.s. Second, Let $0 = t_0 \leq t_1 \leq \dots t_n$ and set $\xi_n = W_{t_{n+1}} - W_{t_n}$, $n = 0, 1, \dots$. The random vector $\xi^n = (\xi_1, \dots, \xi_n)$ is a linear transformation of Gaussian random variables and therefore it is Gaussian. To see this, take ξ_k, ξ_m with $k, m \in \{1, 2, \dots, n\}$ and let $\lambda_1, \lambda_2 \in \mathbb{R}$ arbitrary. Then

$$\begin{aligned} \lambda_1 \xi_k + \lambda_2 \xi_m &= \lambda_1 (W_{t_{k+1}} - W_{t_k}) + \lambda_2 (W_{t_{m+1}} - W_{t_m}) \\ &= \lambda_1 W_{t_{k+1}} - \lambda_1 W_{t_k} + \lambda_2 W_{t_{m+1}} - \lambda_2 W_{t_m}, \end{aligned}$$

and similarly for an arbitrary number of intervals. Now we show that ξ_k, ξ_m with $k \neq m$ are uncorrelated (take $k \geq m + 1$ wlog):

$$\begin{aligned} \mathbb{E}(\xi_k \xi_m) &= \mathbb{E}\left((W_{t_{k+1}} - W_{t_k})(W_{t_{m+1}} - W_{t_m})\right) \\ &= \mathbb{E}(W_{t_{k+1}} W_{t_{m+1}}) - \mathbb{E}(W_{t_{k+1}} W_{t_m}) - \mathbb{E}(W_{t_k} W_{t_{m+1}}) + \mathbb{E}(W_{t_k} W_{t_m}) \\ &= t_{m+1} - t_m - t_{m+1} + t_m = 0. \end{aligned}$$

This shows that ξ_k, ξ_m are uncorrelated and, since they are Gaussian, they are also independent. It remains to calculate the variance of the Gaussian random variable ξ_k , $k = 1, \dots, n$. We calculate:

$$\begin{aligned} \mathbb{E}(\xi_k)^2 &= \mathbb{E}(W_{t_{k+1}} - W_{t_k})^2 \\ &= \mathbb{E}(W_{t_{k+1}})^2 - 2\mathbb{E}(W_{t_{k+1}} W_{t_k}) + \mathbb{E}(W_{t_k})^2 \\ &= t_{k+1} - t_k. \end{aligned}$$

This completes the proof of the first part.¹

Now we proceed with proving the converse statement. First, we show that W_t is a Gaussian

¹We could have also shown the independence property of the increments directly by calculating the characteristic function

$$\mathbb{E}e^{(i \sum_k \lambda_k \xi_k)} = e^{-\frac{1}{2} \sum_k \lambda_k^2 (t_{k+1} - t_k)} = \prod_k \mathbb{E}e^{i \lambda_k \xi_k}.$$

process:

$$\begin{aligned}
\sum_{k=1}^n \lambda_k W_{t_k} &= \lambda_n (W_{t_n} - W_{t_{n-1}}) + (\lambda_n + \lambda_{n-1}) W_{t_{n-1}} + \sum_{k=1}^{n-2} \lambda_k W_{t_k} \\
&= \dots \\
&= \sum_{k=1}^n \rho_k (W_{t_k} - W_{t_{k-1}}) = \sum_{k=1}^n \rho_k \xi_k,
\end{aligned}$$

for appropriately chosen constants ρ_k . Therefore, it is a linear combination of the Gaussian random variables ξ_k and, hence, it is Gaussian. Since the vector $(W_{t_1}, \dots, W_{t_n})$ is Gaussian for arbitrary n , it follows that W_t is a Gaussian process. Now we calculate the mean and covariance. Since $W_t - W_s \sim \mathcal{N}(0, t-s)$ and $W_0 = 0$ a.s. we have that $W_t \sim \mathcal{N}(0, t)$. We clearly have that $\mathbb{E}W_t = 0$. Furthermore, with $t \geq s$, and since the process has independent increments with variance $t-s$:

$$\begin{aligned}
\mathbb{E}(W_t W_s) &= \mathbb{E}((W_t - W_s + W_s) W_s) = \mathbb{E}((W_t - W_s) W_s) + \mathbb{E}(W_s^2) \\
&= \mathbb{E}((W_t - W_s)(W_s - W_0)) + \mathbb{E}(W_s - W_0)^2 \\
&= 0 + s = \min(t, s),
\end{aligned}$$

and the proof is complete. **[8] MARKS –A**

- (b) First we give the definition of a Markov process (students are not asked to do this). Let $\{\mathcal{F}_t^X\}_{t \in [0, +\infty)}$ denote the filtration generated by $\{X_t\}_{t \in [0, +\infty)}$. The process is Markov if, for all bounded Borel functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $t, h \geq 0$, we have

$$\mathbb{E}f((X_{t+h})|\mathcal{F}_h^X) = \mathbb{E}(f(X_{t+h})|X_h).$$

The Markov property of Brownian motion follows from the fact that Brownian motion has independent increments:

$$\mathbb{E}(f(W_{t+h})|\mathcal{F}_h) = \mathbb{E}(f(W_{t+h} - W_h + W_h)|\mathcal{F}_h) = \mathbb{E}(f(W_{t+h})|W_h).$$

The Markov semigroup, applied to a measurable function f is given by

$$(P_t f)(x) = \mathbb{E}(f(W_t)|W_0 = x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy$$

with $P_0 = I$.

To show that the heat semigroup and differentiation commute, we observe that $(P_t f)(x) = f \star \gamma$, where \star denotes the convolution and γ the standard gaussian. We then use properties of the convolution to calculate

$$\frac{d}{dx} (P_t f) = \frac{d}{dx} f \star \gamma = \frac{df}{dx} \star \gamma = \left(P_t \frac{df}{dx} \right).$$

[8] MARKS –A

2. (a) The Stratonovich stochastic integral is defined as

$$\int_0^T f(t, \omega) \circ dW(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t_j^*, \omega) \Delta W_j, \quad \text{where } t_j^* = \frac{1}{2}(t_j + t_{j+1}),$$

where the limit is in $L^2(\mathbb{P})$.

The Stratonovich stochastic integral is equivalently defined as the $L^2(\mathbb{P})$ -limit of the midpoint rule:

$$\int_0^T f(t, \omega) \circ dW_s = \lim_{\Delta t \rightarrow 0} \sum_j \frac{f(t_{j+1}, \omega) + f(t_j, \omega)}{2} \Delta W_j.$$

[2] MARKS -A

(b) We use the fact that the term on the right hand side is a telescopic sum:

$$\begin{aligned} \sum_j \frac{W_{j+1} + W_j}{2} \Delta W_j &= \frac{1}{2} \left(\sum_j W_{j+1} \Delta W_j + \sum_j W_j \Delta W_j \right) \\ &= \frac{1}{2} \sum_j (W_{j+1}^2 - W_{j+1}W_j + W_jW_{j+1} - W_j^2) \\ &= \frac{1}{2} \sum_j (W_{j+1}^2 - W_j^2) = \frac{1}{2} W_{j+1}^2, \end{aligned}$$

from which we deduce that

$$\int_0^t W_s \circ dW_s = \frac{1}{2} W_t^2.$$

Equivalently:

$$\begin{aligned} \sum_{n=0}^{N-1} W_{\frac{t_n+t_{n+1}}{2}} \Delta W_n &:= \sum_{n=0}^{N-1} \left[W_{n+\frac{1}{2}} (W_{n+1} - W_{n+\frac{1}{2}}) + W_{n+\frac{1}{2}} (W_{n+\frac{1}{2}} - W_n) \right] \\ &= \sum_{n=0}^{N-1} \left[-\frac{1}{2} (W_{n+\frac{1}{2}} - W_{n+1})^2 - \frac{1}{2} W_{n+\frac{1}{2}}^2 + \frac{1}{2} W_{n+1}^2 \right. \\ &\quad \left. + \frac{1}{2} (W_{n+\frac{1}{2}} - W_n)^2 - \frac{1}{2} W_n^2 + \frac{1}{2} W_{n+\frac{1}{2}}^2 \right] \\ &= \sum_{n=0}^{N-1} \left[-\frac{1}{2} (W_{n+\frac{1}{2}} - W_{n+1})^2 + \frac{1}{2} W_{n+1}^2 + \frac{1}{2} (W_{n+\frac{1}{2}} - W_n)^2 - \frac{1}{2} W_n^2 \right] \\ &= \sum_{n=0}^{N-1} \left[-\frac{1}{2} (W_{n+\frac{1}{2}} - W_{n+1})^2 + \frac{1}{2} (W_{n+\frac{1}{2}} - W_n)^2 \right] + \frac{1}{2} W_T^2 \xrightarrow{\Delta t \rightarrow 0} \frac{1}{2} W_T^2 \text{ in } L^2(\mathbb{P}), \\ \text{since } &\sum_{n=0}^{N-1} \left[-\frac{1}{2} (W_{n+\frac{1}{2}} - W_{n+1})^2 + \frac{1}{2} (W_{n+\frac{1}{2}} - W_n)^2 \right] \rightarrow 0, \text{ in } L^2(\mathbb{P}). \end{aligned}$$

[6] MARKS -B

(c) Let $t'_n = t_n + \delta t_n$, $\delta t_n \in [0, \Delta t]$.

We calculate:

$$\sum_{n=0}^{N-1} f(t'_n) \Delta W_{t_n} - \sum_{n=0}^{N-1} f(t_n) \Delta W_{t_n} = \sum_{n=0}^{N-1} (f(t'_n) - f(t_n)) \Delta W_{t_n}$$

We have:

$$\begin{aligned} \mathbb{E} \left| \sum_{n=0}^{N-1} (f(t'_n) - f(t_n)) \Delta W_{t_n} \right| &\leq \sum_{n=0}^{N-1} \mathbb{E} |(f(t'_n) - f(t_n)) \Delta W_{t_n}| \\ &\leq \sum_{n=0}^{N-1} (\mathbb{E}(f(t'_n) - f(t_n))^2)^{1/2} (\mathbb{E} \Delta W_{t_n}^2)^{1/2} \\ &\leq C \sum_{n=0}^{N-1} (t'_n - t_n)^{\frac{1+\varepsilon}{2}} (\Delta t_n)^{1/2} \\ &\leq C \sum_{n=0}^{N-1} (\Delta t)^{1+\varepsilon/2} \leq CN^{-\varepsilon/2} \rightarrow 0 \\ \implies \lim_{\Delta t \rightarrow 0} \sum_{n=0}^{N-1} f(t'_n) \Delta t_n &= \int_0^t f(s) dW_s \text{ in } L^1(\mathbb{P}) \text{ for all } \delta t_n \in [0, \Delta t]. \end{aligned}$$

[6] MARKS -B

(d) Brownian motion is a diffusion processes with generator $\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2}$. Let $f(x, t) = \frac{1}{3}x^3 - tx$. We apply Itô's formula to $Y_t = f(W_t, t)$ to obtain

$$\begin{aligned} df(W_t) &= (\partial_t f)(W_t, t) dt + (\partial_x f)(W_t, t) dW_t + \frac{1}{2} (\partial_x^2 f)(W_t, t) (dW_t)^2 \\ &= -W_t dt + (W_t^2 - t) dW_t + W_t dt. \end{aligned}$$

Consequently,

$$\frac{1}{3}W_t^3 - tW_t = \int_0^t (W_s^2 - s) dW_s.$$

We use the martingale representation theorem to deduce that $W_t^3 - 3tW_t$ is a martingale. The result then follows from the martingale representation theorem. Alternatively, we calculate, with $N_t = \frac{1}{3}W_t^3 - tW_t$,

$$\begin{aligned} \mathbb{E}[N_s | \mathcal{F}_t] &= \mathbb{E}[(W_s - W_t)^3 + 3W_s^2 W_t - 3W_s W_t^2 - W_t^3 | \mathcal{F}_t] - 3sW_t \\ &= 3W_t \mathbb{E}[W_s^2 | \mathcal{F}_t] - 3W_t^2 W_t + W_t^3 - 3sW_t \\ &= 3W_t \mathbb{E}[(W_s - W_t)^2 + 2W_s W_t - W_t^2 | \mathcal{F}_t] - 2W_t^3 - 3sW_t \\ &= 3W_t(s - t) + W_t^3 - 3sW_t \\ &= W_t^3 - 3tW_t = N_t. \end{aligned}$$

[6] MARKS -A

3. (a) Consider the Itô process

$$dZ_t = -\frac{1}{2}h^2(t)dt + h(t)dW_t.$$

The generator of this process is

$$\mathcal{L} = -\frac{1}{2}h^2(t)\frac{d}{dx} + \frac{1}{2}h^2(t)\frac{d^2}{dx^2}.$$

The process $Y(t, \omega)$ can be written in the form

$$Y(t, \omega) = e^{Z_t}$$

We use Itô's formula, noting that

$$\mathcal{L}e^x = -\frac{1}{2}h^2(t)\frac{d}{dx}e^x + \frac{1}{2}h^2(t)\frac{d^2}{dx^2}e^x = 0,$$

to calculate

$$\begin{aligned} dY_t &= (\mathcal{L}e^{Z_t})dt + \frac{d}{dx}e^{Z_t}h(t)dW_t \\ &= Y_t h(t)dW_t. \end{aligned}$$

Consequently,

$$dY_t = Y_t h(t)dW_t, \quad Y_0 = 1.$$

The result then follows from the martingale representation theorem.

[6] MARKS -B

(b) The SDE is

$$dX_t = \frac{1-X_t}{1-t}dt + dW_t, \quad X_0 = 0, \tag{1}$$

We rearrange terms:

$$dX_t - \frac{1-X_t}{1-t}dt = dW_t.$$

The integrating factor is

$$\exp\left(\int_0^t \frac{1}{1-s}ds\right) = \frac{1}{1-t}.$$

We multiply the equation through by $\frac{1}{1-t}$:

$$\frac{1}{1-t}dX_t - \frac{1-X_t}{(1-t)^2}dt = \frac{1}{1-t}dW_t.$$

The left hand side becomes an exact differential:

$$d\left(\frac{1}{1-t}X_t\right) = \frac{1}{(1-t)^2}dt + \frac{1}{1-t}dW_t.$$

We integrate from 0 to t to obtain

$$\frac{1}{1-t}X_t = \frac{1}{1-t} - 1 + \int_0^t \frac{1}{1-s} dW_s$$

We multiply through by $1-t$ and simplify the right hand side to obtain

$$X_t = t + (1-t) \int_0^t \frac{1}{1-s} dW_s. \quad (2)$$

[8] MARKS -D

- (c) We take the expectation in (??) to obtain

$$\mathbb{E}X_t = t.$$

we now calculate the variance, using Itô's formula:

$$\begin{aligned} \text{Var}(X_t) &= \mathbb{E}(X_t - \mathbb{E}X_t)^2 = \mathbb{E}\left((1-t) \int_0^t \frac{1}{1-s} dW_s\right)^2 \\ &= (1-t)^2 \int_0^t \left(\frac{1}{1-s}\right)^2 dt \\ &= (1-t)t. \end{aligned}$$

Finally, we calculate the autocorrelation function, assuming, without loss of generality, that $t \geq s$, and using Itô's formula:

$$\begin{aligned} C(X_t, X_s) &= \mathbb{E}((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)) \\ &= \mathbb{E}\left((1-t)(1-s) \int_0^t \frac{1}{1-\ell} dW_\ell \int_0^s \frac{1}{1-\rho} dW_\rho\right) \\ &= (1-t)(1-s) \int_0^s \frac{1}{(1-\rho)^2} d\rho \\ &= (1-t)s = s - ts. \end{aligned}$$

From symmetry, we deduce

$$C(X_t, X_s) = \min(s, t) - st.$$

[6] MARKS -C

4. (a) 1. The generator is

$$\mathcal{L} = a(x) \cdot \nabla + D\Delta.$$

The forward and backward Kolmogorov equations are

$$\frac{\partial f}{\partial t} = \mathcal{L}f$$

and

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^* \rho = -\nabla \cdot (a(x)\rho) + D\Delta\rho.$$

[2] MARKS -A

2. We need to find the solution of the stationary Fokker-Planck equation and to show that it is unique. Since $a(x)$ is divergence-free, we have that the Fokker-Planck operator is

$$\mathcal{L}^* = -a(x) \cdot \nabla + D\Delta.$$

The stationary Fokker-Planck equation becomes

$$-a(x) \cdot \nabla \rho + D\Delta \rho = 0$$

on $[0, 1]^d$ with periodic boundary conditions. Clearly, 1 is a normalized solution of this boundary value problem. To show that it is unique, we multiply the stationary FP equation by ρ , integrate over $[0, 1]^d$ and integrate by parts, using the fact that $\nabla \cdot a(x) = 0$ (which implies that $\int h a \cdot \nabla h dx = 0$) to obtain

$$D \int_{[0,1]^d} |\nabla \rho|^2 dx = 0,$$

from which we deduce that ρ is a constant. Hence, the diffusion process is ergodic and the invariant distribution is

$$\rho(x) = 1.$$

[2] MARKS -B

3. Let

$$h(x, t) = p(x, t) - 1.$$

It is a mean zero periodic function and, consequently, it satisfies Poincaré's inequality:

$$\int_{[0,1]^d} |h|^2 dx \leq C \int_{[0,1]^d} |\nabla h|^2 dx.$$

The function $h(x, t)$ satisfies the Fokker-Planck equation

$$\frac{\partial h}{\partial t} = -a(x) \cdot \nabla h + D\Delta h.$$

We multiply the equation by h integrate by parts on the left hand side, use the fact that $a(x)$ is divergence free and use Poincaré's inequality to obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{[0,1]^d} |h|^2 dx &= -D \int_{[0,1]^d} |\nabla h|^2 dx \\ &\leq -CD \int_{[0,1]^d} |h|^2 dx. \end{aligned}$$

Let

$$\eta(t) = \int_{[0,1]^d} |h(x, t)|^2 dx.$$

From the above calculation we obtain the inequality

$$\frac{d\eta}{dt} \leq -2CD\eta(t),$$

from which we deduce that

$$\eta(t) \leq \eta(0)e^{-2CDt}.$$

Consequently,

$$\int_{[0,1]^d} |p(x, t) - 1|^2 dx \leq e^{-2CDt} \int_{[0,1]^d} |p(x, 0) - 1|^2 dx,$$

which shows that the stochastic process X_t converges exponentially fast to its invariant distribution.

[8] MARKS – D

- (b) 1. The generator of X_t is

$$\mathcal{L} = \left(-ax + \frac{D}{x} \right) \partial_x + D\partial_x^2.$$

The forward and backward Kolmogorov equations are

$$\frac{\partial f}{\partial t} = \mathcal{L}f$$

and

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^* \rho = -\partial_x \left(\left(-ax + \frac{D}{x} \right) \rho \right) + D\Delta\rho.$$

[2] MARKS – A

2. The generator of the process X_t is of the form

$$\mathcal{L} = -\partial_x V(x) + D\partial_x^2$$

with $V(x) = \frac{ax^2}{2} - D \ln(x)$. The invariant distribution is the Gibbs distribution

$$\rho_s(x) = Z^{-1} e^{-V(x)/D} = Z^{-1} x e^{-\frac{ax^2}{2D}}.$$

The normalization constant is

$$Z = \int_0^\infty x e^{-\frac{ax^2}{2D}} dx = \frac{D}{a}.$$

The invariant distribution is

$$\rho_s(x) = \frac{a}{D} x e^{-\frac{ax^2}{2D}}.$$

[6] MARKS – C

5. (a) The generator is

$$\mathcal{L} = -D\nabla V \cdot \nabla + \nabla \cdot D \cdot \nabla + D \cdot \nabla \cdot \nabla.$$

Its adjoint is

$$\mathcal{L}^* \rho = \nabla \cdot (D\nabla V \rho - \nabla \cdot D\rho + \nabla \cdot (D\rho)) = \nabla \cdot (D\nabla V \rho + D\nabla \rho).$$

The forward and backward Kolmogorov equations are

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \quad u(x, 0) = f(x),$$

where $u(x, t) = \mathbb{E}(f(X_t) | X_0 = x)$ and

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^* \rho, \quad \rho(x, 0) = \rho_0(x),$$

where $X_t \sim \rho$.

[2] MARKS

- (b) We need to show that ρ_∞ is the unique normalizable solution of the stationary Fokker-Planck equation

$$\nabla \cdot (D(\nabla V \rho_\infty + \nabla \rho_\infty)) = 0. \quad (3)$$

We check that

$$\nabla V \rho_\infty + \nabla \rho_\infty = 0, \quad (4)$$

from which it follows that $\mathcal{L}^* \rho_\infty = 0$. Suppose now that there exists a second solution $\hat{\rho}$ of the stationary Fokker-Planck equation and write $\hat{\rho} = h\rho_\infty$. We have, using (??) and (??),

$$\mathcal{L}^* \hat{\rho} = 0 = \nabla \cdot (D\rho_\infty \nabla h).$$

We multiply by h , integrate over \mathbb{R}^d and integrate by parts to obtain

$$\int_{\mathbb{R}^d} \nabla \cdot (D\rho_\infty \nabla h) h \, dx = - \int_{\mathbb{R}^d} \langle D \cdot \nabla h, \nabla h \rangle \rho_\infty \, dx = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. From this equation and the fact that D is strictly positive definite we deduce that $h = \text{const} = 1$ (both ρ_∞ and $\hat{\rho}$ are normalized). Hence, ρ_∞ is unique.

Alternatively, we can use the coupling argument that was presented in the lectures.

[3] MARKS

- (c) Let $f, h \in L^2(\mathbb{R}^d; \rho_\infty)$ and assume that they are sufficiently smooth and decay sufficiently fast at infinity. From the calculation that we did in the previous part, the definition of the adjoint of an operator and the symmetry of D we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L} f h \rho_\infty \, dx &= \int_{\mathbb{R}^d} f \mathcal{L}^* (h \rho_\infty) \, dx \\ &= \int_{\mathbb{R}^d} f \nabla \cdot (D \rho_\infty \nabla h) \, dx \\ &= - \int_{\mathbb{R}^d} \langle D \nabla f, \nabla h \rangle \rho_\infty \, dx, \end{aligned}$$

from which it follows that \mathcal{L} is symmetric in $L^2(\mathbb{R}^d; \rho_\infty)$. From the calculation above it follows that

$$D_{\mathcal{L}}(f) = \int_{\mathbb{R}^d} \langle D\nabla f, \nabla f \rangle \rho_\infty dx.$$

[3] MARKS

- (d) Let ρ denote the law of the process X_t , i.e. the solution of the Fokker-Planck equation. Set $\rho = h\rho_\infty$. We have that $\mathcal{L}^*\rho = \nabla \cdot (D\nabla h\rho_\infty) = \rho_\infty \mathcal{L}h$. Furthermore, we have that

$$\mathbb{E}_\mu h = \int_{\mathbb{R}^d} h(x, t) \rho_\infty(x) dx = 1.$$

Set $\phi = h - 1$. We have that $\text{Var}_\mu(h) = \mathbb{E}_\mu \phi^2$. The Fokker-Planck equation can be now written for ϕ :

$$\frac{\partial \phi}{\partial t} = \mathcal{L}\phi, \quad \phi(x, 0) = \frac{\rho(x, 0)}{\rho_\infty(x)} - 1.$$

We multiply by $\phi\rho_\infty$, integrate over \mathbb{R}^d and use the formula the Dirichlet form, Poincaré's inequality and the fact that the matrix D is strictly positive definite to calculate:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\phi|^2 \rho_\infty dx &= \int_{\mathbb{R}^d} \phi \frac{\partial \phi}{\partial t} \rho_\infty dx \\ &= \int_{\mathbb{R}^d} \mathcal{L}\phi \phi \rho_\infty dx = - \int_{\mathbb{R}^d} \langle D\nabla \phi, \nabla \phi \rangle \rho_\infty dx \\ &\leq -\delta \int_{\mathbb{R}^d} |\nabla \phi|^2 \rho_\infty dx \\ &\leq -\lambda \delta \int_{\mathbb{R}^d} |\phi|^2 \rho_\infty dx, \end{aligned}$$

from which we deduce that

$$\int_{\mathbb{R}^d} |\phi(x, t)|^2 \rho_\infty(x) dx \leq e^{-2\lambda\delta t} \int_{\mathbb{R}^d} |\phi(x, 0)|^2 \rho_\infty(x) dx.$$

We use now that fact that $\phi = \rho/\rho_\infty - 1$ to rewrite the above inequality in the form

$$\int_{\mathbb{R}^d} |\rho(x, t) - \rho_\infty(x)|^2 \rho_\infty^{-1}(x) dx \leq e^{-2\lambda\delta t} \int_{\mathbb{R}^d} |\rho(x, 0) - \rho_\infty(x)|^2 \rho_\infty^{-1}(x) dx.$$

The rate of convergence in the weighted L^2 norm is $\lambda\delta$.

[6] MARKS

- (e) First we rewrite H in terms of $h = \rho/\rho_\infty$:

$$H(\rho(x, t)) = \int_{\mathbb{R}^d} \rho(x, t) \ln \left(\frac{\rho(x, t)}{\rho_\infty(x)} \right) dx = \int_{\mathbb{R}^d} h(x, t) \ln h(x, t) \rho_\infty(x) dx$$

We calculate the time derivative of $H(\rho(x, t))$, using the formula for the Dirichlet form and the fact that D is strictly positive definite:

$$\begin{aligned}
\frac{d}{dt} H(\rho(x, t)) &= \frac{d}{dt} \int_{\mathbb{R}^d} h(x, t) \ln h(x, t) \rho_\infty(x) dx \\
&= \int_{\mathbb{R}^d} \partial_t h(x, t) \ln h(x, t) \rho_\infty(x) dx + \int_{\mathbb{R}^d} h(x, t) \partial_t \ln h(x, t) \rho_\infty(x) dx \\
&= \int_{\mathbb{R}^d} (\mathcal{L}h) \ln h \rho_\infty dx + \int_{\mathbb{R}^d} h(\mathcal{L}h) \frac{1}{h} \rho_\infty dx \\
&= - \int_{\mathbb{R}^d} \langle D\nabla h, \nabla \ln h \rangle \rho_\infty dx + \int_{\mathbb{R}^d} \mathcal{L}h \rho_\infty dx \\
&\leq -\delta \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \rho_\infty dx + 0 \leq 0,
\end{aligned}$$

since the second integral on the right hand side vanishes because $\mathcal{L}^* \rho_\infty = 0$.

[6] MARKS

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

| ExamModuleCode | QuestionNumber | Comments for Students |
|---|----------------|---|
| Introduction to Stochastic Differential Equations_MATH97020 MATH70054 | 1 | Most students answered correctly this question. A few students were confused with how to differentiate the heat semigroup. |
| Introduction to Stochastic Differential Equations_MATH97020 MATH70054 | 2 | Most students answered correctly this question. There was some confusion about the type of convergence in Part c, the convergence is in L^1 and not L^2 . |
| Introduction to Stochastic Differential Equations_MATH97020 MATH70054 | 3 | Most students did well in this question. I was pleased to see that students could use correctly the integrating factor in the second part. |
| Introduction to Stochastic Differential Equations_MATH97020 MATH70054 | 4 | Most students did well. They could use Poincare's inequality on the torus to deduce exponentially fast convergence to equilibrium. |
| Introduction to Stochastic Differential Equations_MATH97020 MATH70054 | 5 | Very few students were able to answer this question. There was confusion on the role of the diffusion matrix D. Most students tried to repeat the calculation from the lecture notes, which corresponds to the case where D is the identity matrix. |