MATH50010: Probability for Statistics
Problem Sheet 8

1. A flea jumps randomly on vertices $\{1, 2, 3\}$ according to the transition probabilities shown in Figure 1. Let X_t be the position of the flea at time t ($t = 0, 1, ...$).

   (a) Write down the transition matrix $\boldsymbol{P}$.

   (b) Find $P(X_2 = 3|X_0 = 1)$.

   (c) Suppose that the flea is equally likely to start at any vertex at time 0. Find the probability distribution of X_1.

   (d) Suppose that the flea begins at vertex 1 at time 0. Find the probability distribution of X_2.

   (e) Suppose that the flea is equally likely to start on any vertex at time 0. Find the probability of obtaining the trajectory $(3, 2, 1, 1, 3)$.

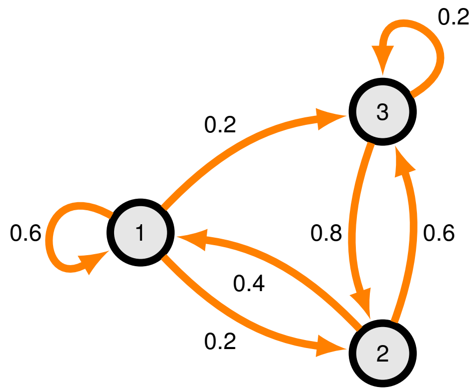

Figure 1: Transition diagram For Question 1

2. Suppose a gambler has \$1 initially. At each round, he either wins \$1 with probability p or loses \$1 with probability $q = 1 - p$. The game ends when the gambler obtains $\$N$. Find the probability that the gambler goes broke, i.e., that his capital reaches \$0. What is the fate of a gambler who faces an opponent who is infinitely rich? (A reasonable model for an individual playing against a casino, who will always take the gambler's bet.)

3. Consider the two Markov chains below and decide which are irreducible and which are periodic:

   (a) A random walk on a cycle with state space $\mathcal{E} = \{0, 1, \cdots, M - 1\}$. At each step the walk increases by 1 (mod M) with probability p and decreases by 1 (mod M) with probability $1 - p$. That is:

$$p_{ij} = \begin{cases} p & \text{if } j \equiv i + 1 \text{mod } M \\ 1 - p & \text{if } j \equiv i - 1 \text{mod } M \\ 0 & \text{otherwise} \end{cases}$$

(b) Simple symmetric random walk on $\mathbb{Z}^d$. At each step the walk moves from its current site to one of its $2d$ neighbours chosen uniformly at random. That is:

$$p_{ij} = \begin{cases} 1/2d & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $|i-j| = |i_1 - j_1| + \cdots + |i_d - j_d|$ for states $i = (i_1, ..., i_d)$, $j = (j_1, \cdots, j_d)$.

4. Consider the random walk on $\{0, 1, 2, \ldots\}$, where $p_{01} = 1$ and for $i > 0$,

$$p_{ij} = \begin{cases} q & j = i-1 \\ p & j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

where $p + q = 1$.

Let h_i be the probability of hitting 0 when the chain starts from $X_0 = i$.

(a) Explain why h_i satisfies

$$h_0 = 1 \qquad h_i = ph_{i+1} + qh_{i-1}, \quad i \geq 1.$$

(b) Show that if $u_i = h_{i-1} - h_i$, then $u_i = \left(\frac{q}{p}\right)^{i-1} u_1$.

(c) Hence determine h_i, distinguishing between the cases $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$.

5. Extend the idea of the previous question to the more general birth-death chain on $\{0, 1, 2, \ldots\}$ for which $p_{i\,i+1} = p_i$ and $p_{i\,i-1} = q_i = 1 - p_i$, with zero probability for all other transitions, and $p_i, q_i > 0$ for all $i \geq 1$.

(a) Show that $h_i = p_i h_{i+1} + q_i h_{i-1}$ and deduce that $u_i = \frac{q_i}{p_i} u_{i-1}$, for $u_i = h_{i-1} - h_i$.

(b) Write u_i in terms of $\gamma_i = \prod_{k=1}^{i} \frac{q_k}{p_k}$ and u_1.

(c) Determine u_1 and show that the chain is transient if and only if $\sum_{i=1}^{\infty} \gamma_i < \infty$.

6. Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Find π, the stationary distribution of P.

For discussion

7. Consider the random walk on $\mathbb{Z}$ with

$$p_{i,j} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \end{cases}$$

Show that all states are transient if $p \neq 1/2$ and recurrent if $p = 1/2$.

[Hint: You may use without proof Stirlings formula: $n! \sim \sqrt{2\pi n}(n/e)^n$ *as* $n \to \infty$ *where* $a_n \sim b_n$ *here means* $a_n/b_n \to 1$*]*