Imperial College London

MATH97012 MATH70052

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Hydrodynamic Stability

Date: 11 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the modified Eckhaus equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial x^2} = \frac{1}{R}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial x^4} - \frac{1}{R}b(y),$$

which may be taken as a model for studying fluid motion and its stability. The equation is defined for $-\infty < x < \infty$ and $-1 \le y \le 1$, where a and $R > 0$ are real parameters, while $b(y)$ is a given function of y. The boundary conditions for the sought function $u(x, y, t)$ are

$$u = 0 \text{ at } y = \pm 1.$$

(a) Consider the basic state and perturbation for a general function $b(y)$.

(i) Find the equation satisfied by the steady basic state $u = U(y)$.
The basic state is perturbed by a small-amplitude disturbance $u'(x, y, t)$, i.e. $u = U(y) + u'(x, y, t)$. Write down the linearised equation and boundary conditions satisfied by u'.

(3 marks)

(ii) Seek temporal normal-mode solutions of the form

$$u'(x, y, t) = \bar{u}(y)e^{\sigma t + i\alpha x} + c.c.,$$

where $c.c.$ stands for the complex conjugate. Derive the equation and boundary conditions satisfied by $\bar{u}(y)$. Hence, or otherwise, show that $U(y)$ is stable when $b(y) \le 0$ for $-1 \le y \le 1$, and that $\sigma_i = -a\alpha$.

(4 marks)

(b) For the case where $b(y) = b > 0$ is a constant, investigate the eigenvalue problem formulated in Part (a)(ii) above.

(i) Show that the eigenvalues σ are

$$\sigma = -ia\alpha + b\,\alpha^2 - \alpha^4 - \frac{n^2\pi^2}{4R},$$

where $n = 1, 2, 3, \ldots$, and find the corresponding eigenfunctions.
For the case of $n = 1$, show that the basic state becomes unstable for

$$R > R_c = \pi^2/b^2.$$

(7 marks)

(ii) For $n = 1$, identify the upper and lower branches of the neutral curve in the limit $R \gg 1$. Sketch the neutral curve in the (α, R) parameter plane.

(3 marks)

(iii) Suppose that an initial perturbation of the form

$$u'(x, y, 0) = g(y)e^{i\alpha x} + c.c.$$

is introduced at $t = 0$, where $g(y)$ is an even function of y satisfying $g = 0$ at $y = \pm 1$. Find the solution for $u'(x, y, t)$ for $t > 0$.

(3 marks)

(Total: 20 marks)

2. Consider convection in a horizontal layer of fluid of depth d, where the temperature at the bottom and top is $\hat{\theta}_0$ and $\hat{\theta}_1 < \hat{\theta}_0$ respectively. The layer is bounded by a circular sidewall of radius r_0^*. The characteristic length, time scale and velocity are taken to be d, d^2/κ and κ/d, respectively, where κ is the heat diffusivity coefficient, and the non-dimensional coordinates $\mathbf{x} = (x, y, z)$, time variable t, velocity $\mathbf{u}$, pressure p and temperature θ are introduced as follows

$$\mathbf{x} = \hat{\mathbf{x}}/d, \quad t = \hat{t}/(d^2/\kappa); \quad \mathbf{u} = \hat{\mathbf{u}}/(\kappa/d), \quad p = \hat{p}/(\hat{\rho}_0 \kappa^2/d^2), \quad \theta = \hat{\theta}/(\hat{\theta}_0 - \hat{\theta}_1),$$

where the variables with a hat are dimensional, and $\hat{\rho}_0$ denotes the reference density. The layer is between $z = 0$ and 1, and the sidewall is assumed to be perfectly insulated.

When the basic state of pure conduction is perturbed by a small-amplitude disturbance $(u', v', w', p', \theta')$, the latter is governed by the linearised equations

$$\frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + Pr \nabla^2 \mathbf{u}' + Ra Pr\, \theta' \, \mathbf{k}, \quad \nabla \cdot \mathbf{u}' = 0, \quad \frac{\partial \theta'}{\partial t} - w' = \nabla^2 \theta',$$

under the Boussinesq approximation, where Ra and Pr denote the Rayleigh and Prandtl numbers, respectively, and $\mathbf{k}$ is the unit vector in the vertical (i.e. z) direction.

(i) Deduce from the momentum equations that

$$\frac{\partial}{\partial t} \nabla^2 \mathbf{u}' = Pr \nabla^4 \mathbf{u}' + Ra Pr \Big[\nabla^2 \theta' \, \mathbf{k} - \nabla \big(\frac{\partial \theta'}{\partial z} \big) \Big].$$

[*Hint: take the curl of the momentum equations for* $\mathbf{u}'$ *to obtain the equations for the vorticity* $\boldsymbol{\omega}' = \nabla \times \mathbf{u}'$*, and then take the curl of the resulting equations. Use the relation* $\nabla \times \boldsymbol{\omega}' = -\nabla^2 \mathbf{u}'$ *and the vector identities given on the next page.*]

(5 marks)

(ii) Let $(w', \theta') = \big(\tilde{w}(z), \tilde{\theta}(z)\big) f(x, y) e^{\sigma t}$. Derive the equations governing $\tilde{w}(z)$ and $\tilde{\theta}(z)$ as well as the equation satisfied by $f(x, y)$.

Suppose that both the top and bottom of the fluid layer are 'free'. Specify the boundary conditions on $\tilde{w}$ and $\tilde{\theta}$, and solve the eigenvalue problem to show that

$$(a^2 + n^2\pi^2)(a^2 + n^2\pi^2 + \sigma)(a^2 + n^2\pi^2 + \sigma/Pr) = a^2 Ra.$$

where a is a constant.

(5 marks)

(iii) Assume that the perturbation is axisymmetric such that f is a function of the radial coordinate $r \equiv (x^2 + y^2)^{1/2}$ only. Solve the equation for f to find $f(r)$ in terms of a Bessel function, and determine w' and θ'.

Find the radial velocity component u'_r of the perturbation from the continuity equation.

[*Hint: See the Laplace operator in polar coordinates* (r, ϕ)*, the Bessel function and the continuity equation given on the next page.*]

(5 marks)

(iv) Discuss how the impermeability condition $u'_r = 0$ and perfect insulation condition $d\theta'/dr = 0$ could be satisfied on the sidewall at $r = r_0 = r_0^*/d$? Discuss how the no-slip condition may be satisfied in the cases of $r_0 \gg 1$ and $r_0 = O(1)$.

(5 marks)

Question continues on the next page.

Vector identities Let ψ denote a scalar function, and $\mathbf{A}$ and $\mathbf{B}$ denote vectors functions. The following identities hold.

$$\nabla \times (\psi \mathbf{A}) = (\nabla \psi) \times \mathbf{A} + \psi \nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

The Laplace operator in polar coordinates (r, ϕ) is given by

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

The solution to the equation

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + f = 0,$$

that is finite at $r = 0$ is given by $f(r) = J_0(r)$, with $J_0(r)$ being the zeroth-order Bessel function of the first kind.

The continuity equation in the coordinates (r, ϕ, z) reads,

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r') + \frac{1}{r}\frac{\partial u_\phi'}{\partial \phi} + \frac{\partial w'}{\partial z} = 0,$$

where u_r' and u_ϕ' denote the radial and azimuthal velocity components, respectively.

(Total: 20 marks)

3. Linear stability of an exactly parallel shear flow with velocity profile $U(y)$ is studied by introducing small perturbations of normal-mode form: $\big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\big)e^{i(\alpha x-\omega t)} + c.c.$ It follows that $\big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\big)$ satisfies the equations,

$$i\alpha\bar{u} + \frac{dv}{dy} = 0, \quad i\alpha(U-c)\bar{u} + \frac{dU}{dy}\bar{v} = -i\alpha\bar{p} + \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \alpha^2\Big)\bar{u}, \tag{1}$$

$$i\alpha(U-c)\bar{v} = -\frac{d\bar{p}}{dy} + \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \alpha^2\Big)\bar{v}, \tag{2}$$

where $c = \omega/\alpha$ is the phase speed and Re the Reynolds number.

These equations will be applied to a parallel shear flow above an infinitely large rigid flat plate located at $y = 0$ with the velocity profile $U(y)$ being given by

$$U(y) = \begin{cases} 1 & y > 1, \\ y & 0 < y < 1. \end{cases}$$

(a) In the limit $Re \gg 1$, the equations can be reduced to the Rayleigh equation

$$(U-c)\Big(\frac{d^2}{dy^2} - \alpha^2\Big)\bar{v} - \frac{d^2U}{dy^2}\bar{v} = 0.$$

(i) Solve the Rayleigh equation for the $U(y)$ given above to find the solution for $\bar{v}$ in the different ranges of y: $0 < y < 1$ and $y > 1$. Impose appropriate boundary and far-field conditions as well as the jump conditions at $y = 1$:

$$\Big[(U-c)\frac{d\bar{v}}{dy} - \frac{dU}{dy}\bar{v}\Big]_{1^-}^{1^+} = 0, \quad \Big[\frac{\bar{v}}{(U-c)}\Big]_{1^-}^{1^+} = 0,$$

where $[\,\cdot\,]_{1^-}^{1^+}$ stand for the jump of the quantity across $y = 1$.

(5 marks)

(ii) Derive the dispersion relation satisfied by the phase speed c and wavenumber α, and determine whether the flow is inviscidly stable. Find the asymptote of $c(\alpha)$ for $\alpha \ll 1$.

(3 marks)

(iii) Find the pressure and streamwise velocity of the perturbation at $y = 0$, $\bar{p}_w = \bar{p}(0)$ and $\bar{u}(0)$. Explain why it is necessary to introduce a viscous Stokes layer.

(3 marks)

(b) Now analyse the disturbance in the Stokes layer by considering the continuity equation and the momentum equations with the viscosity term included, which are given in (1)–(2).

(i) Deduce that the thickness of the Stokes layer is $O\big((\omega Re)^{-1/2}\big)$, and hence introduce the local coordinate $\tilde{Y} = y/(\omega Re)^{-1/2}$, and deduce that $\bar{u}$ and $\bar{v}$ expand as

$$\bar{u} = \tilde{U}(\tilde{Y}) + \ldots, \quad \bar{v} = \alpha(\omega Re)^{-1/2}\tilde{V}(\tilde{Y}) + \ldots.$$

Show that

$$i\,\tilde{U} + \frac{d\tilde{V}}{d\tilde{Y}} = 0, \quad -i\omega\tilde{U} = -i\alpha\bar{p}_w + \omega\frac{d^2\tilde{U}}{d\tilde{Y}^2}.$$

Specify the appropriate boundary conditions, and find the solution for $\tilde{U}$ and $\tilde{V}$.

(4 marks)

Question continues on the next page.

(ii) Show that

$$\tilde{V} \to -(i\alpha/\omega)\bar{p}_w \tilde{Y} + \tilde{V}_\infty \quad \text{as} \quad \tilde{Y} \to \infty,$$

and determine the expression for $\tilde{V}_\infty$, which is independent of $\tilde{Y}$. Explain how $\tilde{V}_\infty$ affects the invisicid solution in the main layer.

(3 marks)

(iii) By examining balances in the streamwise momentum equation in the limit $\alpha \ll 1$, show that the Stokes layer solution in (ii) above becomes invalid when

$$\alpha = O(Re^{-1/4}).$$

(2 marks)

(Total: 20 marks)

4. Consider linear stability of a general steady uni-directional base flow with velocity ($\mathbf{U} = (U(y,z),0,0)$ and pressure $P(x)$ in the Cartesian coordinates $\mathbf{x} = (x,y,z)$. The flow is perturbed by a small-amplitude disturbance $(\mathbf{u}', p')$.

(i) Starting from the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u},$$

derive the *linearised* equations satisfied by $\mathbf{u}' = (u', v', w')$ and p'.

Seek solutions of the normal-mode form,

$$(\mathbf{u}', p') = \Big(\tilde{\mathbf{u}}(y,z), \tilde{p}(y,z)\Big)e^{i(\alpha x - \omega t)} + c.c.,$$

where $c.c.$ denotes the complex conjugate, and write down the equations governing $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ and $\tilde{p}$.

(4 marks)

(ii) Suppose now that $Re \gg 1$ so that the viscous terms in the momentum equations can be neglected. Show that the pressure $\tilde{p}$ satisfies the equation

$$\frac{\partial}{\partial y}\Big[\frac{\frac{\partial \tilde{p}}{\partial y}}{(U-c)^2}\Big] + \frac{\partial}{\partial z}\Big[\frac{\frac{\partial \tilde{p}}{\partial z}}{(U-c)^2}\Big] - \frac{\alpha^2 \tilde{p}}{(U-c)^2} = 0, \tag{4}$$

where $c = \omega/\alpha$ is the phase speed and is complex valued, i.e. $c = c_r + ic_i$.

Specify the boundary conditions in terms of $\tilde{p}$ and/or its derivatives for the two cases, where

(a) the flow is through a rectangular channel, bounded by two parallel rigid lower and upper surfaces at $y = 0$, 1 and two parallel rigid side surfaces at $z = 0$ and ℓ;

(b) the flow is through a rigid cylinder of a general cross section, which is a curved boundary in the (y,z) plane.

(7 marks)

(iii) By multiplying equation (4) by $\tilde{p}^*$, the complex conjugate of $\tilde{p}$, followed by integration by parts over the rectangular region, $0 < y < 1$ and $0 < z < \ell$, show that if the flow is unstable, then c_r lies in the range

$$U_{\min} < c_r < U_{\max},$$

where $U_{\max}$ and $U_{\min}$ denote the maximum and minimum of $U(y,z)$ respectively.

Prove further that

$$\Big[c_r - \tfrac{1}{2}\big(U_{\max} + U_{\min}\big)\Big]^2 + c_i^2 \le \Big[\tfrac{1}{2}\big(U_{\max} - U_{\min}\big)\Big]^2.$$

[*Hint: For the last inequality, follow the argument leading to Howard's semicircle theorem for the special case where $U = U(y)$ depends on y only.*]

(6 marks)

(iv) Are the results in Part (iii) valid for the case (b) in Part (ii)? Why?

(3 marks)

(Total: 20 marks)

5. When a boundary layer is perturbed by a two-dimensional disturbance, the perturbed flow field is written as

$$(u, v, p) = \Big(U(x, Y),\ Re^{-1/2}V(x, Y),\ P\Big) + \epsilon(u',\ v',\ p')$$

in the Cartesian coordinate system (x, y), where x and y are non-dimensionalised by L, the distance to the leading edge, $Y = Re^{1/2}y$ and the Reynolds number $Re = \hat{V}_\infty L/\nu$ with $\hat{V}_\infty$ being the reference velocity and ν the kinematic viscosity. The parameter $\epsilon \ll 1$ measures the magnitude of the disturbance. The boundary layer of interest is the so-called 'marginal separation' type, and its velocity has the near-wall and far-field behaviours:

$$U \to \tfrac{1}{2}\lambda_2 Y^2 \quad \text{as} \quad Y \to 0; \quad U \to 1 \quad \text{as} \quad Y \to \infty,$$

where λ_2 is a function of x. The flow field $(\mathbf{u}, p) = (u, v, p)$ is governed by the two-dimensional Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}.$$

(i) Derive the linearised equations governing the perturbation u', v' and p', which are functions of x, Y and t, e.g. $u' = u'(x, Y, t)$. Indicate the terms which represent the non-parallel-flow effect, and explain Prandtl's parallel-flow approximation that leads to the Orr-Sommerfeld equation.

(3 marks)

(ii) Suppose that in the main layer (deck) where $Y = O(1)$, the solution expands as

$$(u',\ v',\ p') = \big(\bar{u}(x, Y),\ Re^{-\frac{3}{14}}\bar{v}(x, Y),\ Re^{-\frac{3}{14}}\bar{p}(x, Y)\big)E + c.c.,$$

where $c.c.$ stands for the complex conjugate, and

$$E = e^{i(Re^{\frac{2}{7}}\alpha x - Re^{\frac{1}{7}}\omega t)}.$$

Derive the equations governing $\bar{u}$, $\bar{v}$ and $\bar{p}$, and verify that they have the solution

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{v} = -i\alpha A U, \quad \bar{p} = P \ \ \text{(constant)},$$

where A is a constant. Explain why it is necessary to introduce an upper layer.

(4 marks)

(iii) Deduce that the width of the upper layer is of $O(Re^{-2/7}L)$ and that $(u',\ v',\ p')$ expands as

$$(u',\ v',\ p') = Re^{-\frac{3}{14}}(u^\dagger, v^\dagger, p^\dagger)E + c.c.$$

Introduce the variable $\bar{y} = Re^{2/7}y = Re^{-3/14}Y$, and derive the equation satisfied by $p^\dagger$ as well as the boundary condition as required by matching with the main-layer solution. Solve for $p^\dagger$ to obtain the relation between P and A.

(4 marks)

Question continues on the next page.

(iv) Explain why a viscous lower deck is still required despite the fact that the main-layer solution satisfies the no-slip condition. Deduce that this layer has a width of $O(Re^{-\frac{4}{7}}L)$, and hence introduce $\tilde{y} = Re^{\frac{4}{7}}y = Re^{1/14}Y$ in the lower deck. Deduce that the solution expands as

$$(u',\, v',\, p') = \left(Re^{-\frac{1}{14}}\tilde{u}(x, \tilde{y}),\, Re^{-\frac{5}{14}}\tilde{v}(x, \tilde{y}),\, Re^{-\frac{3}{14}}\tilde{p}(x, \tilde{y})\right)E + c.c.$$

Comment on the orders of magnitude of the viscous effect and the non-parallelism.

(6 marks)

(v) Derive the equations governing $\tilde{u}$ and $\tilde{v}$. Specify the boundary and matching conditions.

(3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH70052, MATH97012

Hydrodynamic Stability (Solutions)

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1. (a) (i) The basic state satisfies the equation: $\dfrac{d^2U}{dy^2} = b(y)$, which is integrated to give

sim. seen ⇓

$$U(y) = \int_{-1}^{y} (y-\eta)b(\eta)d\eta + \frac{1}{2}\Big[\int_{-1}^{1}(\eta-1)b(\eta)d\eta\Big]\,(y+1).$$

The precise expression for $U(y)$ is however not needed.
The linearised equation for the disturbance:

$$\frac{\partial u'}{\partial t} + a\frac{\partial u'}{\partial x} + \frac{d^2U}{dy^2}\frac{\partial^2 u'}{\partial x^2} = \frac{1}{R}\frac{\partial^2 u'}{\partial y^2} - \frac{\partial^4 u'}{\partial x^4}, \tag{1}$$

where $d^2U/dy^2 = b(y)$. The boundary conditions are

$$u' = 0 \quad \text{at} \quad y = \pm 1.$$

3, A

(ii) For the assumed normal-mode form, we have $\dfrac{\partial}{\partial t} \to \sigma, \quad \dfrac{\partial}{\partial x} \to i\alpha.$

unseen ⇓

Substitution of the normal mode perturbation into equation (1) yields

$$\frac{d^2\bar{u}}{dy^2} + R\big[b(y)\,\alpha^2 - \alpha^4 - ia\alpha - \sigma\big]\bar{u} = 0. \tag{2}$$

The boundary conditions are

$$\bar{u} = 0 \quad \text{at} \quad y = \pm 1.$$

Multiplying (2) by u^*, the complex conjugate of $\bar{u}$, and integrating (by parts) over $[-1,1]$, we obtain

$$\Big[\bar{u}^*\frac{d\bar{u}}{dy}\Big]_{-1}^{1} - \int_{-1}^{1}\Big|\frac{d\bar{u}}{dy}\Big|^2 dy + R\int_{-1}^{1}\big[b\,\alpha^2 - \alpha^4 - ia\alpha - \sigma\big]|\bar{u}|^2 dy = 0.$$

The contribution from the end points $y = -1, 1$ vanishes due to the boundary conditions. Separating the real and imaginary parts gives

$$-\int_{-1}^{1}\Big|\frac{d\bar{u}}{dy}\Big|^2 dy + (R\alpha^2)\int_{-1}^{1} b(y)|\bar{u}|^2 dy - (R\alpha^4)J = R\sigma_r J, \qquad -a\alpha = \sigma_i,$$

where

$$J = \int_{-1}^{1}|\bar{u}|^2 dy > 0.$$

Clearly, if $b(y) < 0$ then $\sigma_r < 0$ and $U(y)$ is stable. Furthermore, $\sigma_i = -a\alpha$ as required. [The conclusion of stability may also be obtained by considering the equation for 'disturbance energy' $|u'|^2$. Due credit will be given for that.]

4, C

sim. seen ⇓

(b) (i) When $b(y)$ is a constant, the coefficient of equation (2) are all constant. The general solution is

$$\bar{u} = Ae^{\lambda y} + Be^{-\lambda y}.$$

with

$$\lambda^2 + R(b\,\alpha^2 - \alpha^4 - ia\,\alpha - \sigma) = 0. \tag{3}$$

The boundary conditions,

$$\bar{u}(-1) = Ae^{-\lambda} + Be^{\lambda} = 0, \quad \bar{u}(1) = Ae^{\lambda} + Be^{-\lambda} = 0,$$

can be satisfied only when $e^{-2\lambda} = e^{2\lambda}$, i.e. $e^{4\lambda} = 1 = e^{2n\pi i}$. Hence $\lambda = \frac{1}{2}n\pi i$ with $n = 1, 2, \ldots$. It follows from (3) that the eigenvalues are given by

$$\sigma(n) = b\,\alpha^2 - \alpha^4 - ia\,\alpha - (n\pi/2)^2/R. \tag{4}$$

Note that $A = -Be^{2\lambda} = -e^{n\pi i}B = (-1)^{(n+1)}B$. Hence the corresponding eigenfunctions are

$$\bar{u} = \cos(n\pi y/2) \quad \text{for odd } n; \quad \bar{u} = \sin(n\pi y/2) \quad \text{for even } n. \tag{5}$$

The growth rates are given by the real part of σ,

$$\sigma_r(n) = b\,\alpha^2 - \alpha^4 - (n\pi/2)^2/R. \tag{6}$$

The neutral curve corresponds to $\sigma_r = 0$, which gives

$$R = \frac{(n\pi/2)^2}{b\alpha^2 - \alpha^4}.$$

For each n, the minimum critical value of R occurs when $2b\,\alpha - 4\alpha^3 = 0$, i.e. at $\alpha = \alpha_c = \sqrt{b/2}$, and for $n = 1$, the critical Reynolds number is $R_c = \pi^2/b^2$.

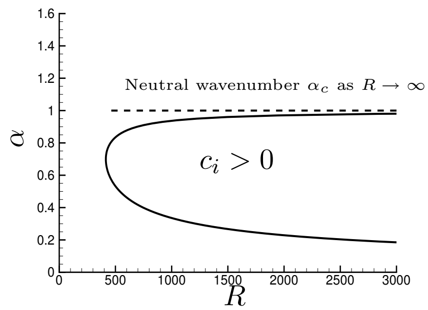

Figure 1: Sketch of the neutral curve (for illustrating qualitative features).

7, A

(ii) When $R \gg 1$, the balance of the first terms in (6) gives,

$$\alpha \to \sqrt{b} \quad \text{(the postive sign taken without losing generality)},$$

which is the asymptote of the upper branch. The balance of the first and third terms leads to

$$\alpha \to \pi/(2\sqrt{b})R^{-1/2}.$$

The neutral curve is shown in the sketch. 3, B

(iii) Since the set of the eigenfunctions (5) is complete, we can expand the transverse distribution of the initial condition, $g(y)$, as unseen ⇓

$$g(y) = \sum_{n\,\text{odd}} C_n \cos(n\pi y/2) + \sum_{n\,\text{even}} S_n \sin(n\pi y/2),$$

where $S_n = 0$ since g is even, and

$$C_n = \int_{-1}^{1} g(y)\cos(n\pi y/2)dy.$$

Each component evolves as a normal mode and so for $t > 0$,

$$u'(x, y, t) = e^{i\alpha(x-at)} \sum_{n\,\text{odd}} C_n \cos(n\pi y/2)e^{\sigma_r(n)t} + c.c.,$$

where $\sigma_r(n)$ is given by (6). 3, D

2. (i) Taking the curl of the momentum equations, unseen ⇓

$$\frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + Pr\nabla^2\mathbf{u}' + RaPr\,\theta'\,\mathbf{k},$$

we obtain the equations for the vorticity $\boldsymbol{\omega}' = \nabla \times \mathbf{u}'$,

$$\frac{\partial \boldsymbol{\omega}'}{\partial t} = Pr\nabla^2\boldsymbol{\omega}' + RaPr\nabla\theta' \times \mathbf{k},$$

where use has been made of the first of the given vector identities (with $\mathbf{A} = \mathbf{k}$ so that $\nabla \times \mathbf{k} = 0$). Taking the curl of the above equations and noting that $\nabla \times \boldsymbol{\omega}' = -\nabla^2\mathbf{u}'$, and using the second of the given identities, which implies

$$\nabla \times (\nabla\theta' \times \mathbf{k}) = (\nabla \cdot \mathbf{k})\nabla\theta' - (\nabla \cdot \nabla\theta')\mathbf{k} + (\mathbf{k} \cdot \nabla)\nabla\theta' - (\nabla\theta' \cdot \nabla)\mathbf{k}$$
$$= -\nabla^2\theta'\mathbf{k} + \nabla(\frac{\partial\theta'}{\partial z}),$$

we arrive at the required equation

$$\frac{\partial}{\partial t}\nabla^2\mathbf{u}' = Pr\nabla^4\mathbf{u}' + RaPr\Big[\nabla^2\theta'\,\mathbf{k} - \nabla(\frac{\partial\theta'}{\partial z})\Big]. \tag{7}$$

5, A

(ii) The third component in (7) is seen ⇓

$$\frac{\partial}{\partial t}\nabla^2 w' = Pr\nabla^4 w' + RaPr\nabla_1^2\theta', \tag{8}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Substitution of the assumed form of solutions for w' and θ' into the temperature equation, $\dfrac{\partial\theta'}{\partial t} - w' = \nabla^2\theta'$ (given in the question), leads to

$$(D^2 + \nabla_1^2 - \sigma)f\tilde{\theta} = -f\tilde{w}, \tag{9}$$

where the differential operator $D = \dfrac{d}{dz}$. Equation (9) can be rearranged into the 'variable separation' form,

$$\Big[(D^2 - \sigma)\tilde{\theta} + \tilde{w}\Big]/\tilde{\theta} = -(\nabla_1^2 f)/f.$$

Since the left-hand side is a function of z only while the right-hand side depends only on x and y, both sides should be a constant, a^2 say. Then

$$(D^2 - a^2 - \sigma)\tilde{\theta} = -\tilde{w}, \tag{10}$$

and

$$\nabla_1^2 f + a^2 f = 0. \tag{11}$$

Similarly, substituting the assumed form into a slightly re-arranged form of the equation for w', equation (8), we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma/Pr)\tilde{w} = a^2 Ra\,\tilde{\theta} \tag{12}$$

after use is made of (11). The required equations are (10) and (12).

The boundary conditions at $z = 0, 1$ (free surface) are

$$\tilde{w} = D^2\tilde{w} = 0, \quad \tilde{\theta} = 0.$$

The system of coupled equations (10) and (12) together with these boundary conditions defines the eigenvalue problem.

By inspection, the four boundary conditions on $\tilde{w}$ can be satisfied for

$$\tilde{w} = \sin(n\pi z),$$

and it follows from (10) that

$$\tilde{\theta} = \left[(n\pi)^2 + a^2 + \sigma\right]^{-1} \sin(n\pi z).$$

Substitution of $\tilde{w}$ and $\tilde{\theta}$ into equation (12) yields the required relation

$$(a^2 + n^2\pi^2)(a^2 + n^2\pi^2 + \sigma)(a^2 + n^2\pi^2 + \sigma/Pr) = a^2 Ra. \tag{13}$$

5, A

unseen ⇓

(iii) For axisymmetric disturbances, equation (11) reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + a^2\right] f = \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + a^2\right] f = 0.$$

The substitution $ar \to r$ converts it into the standard Bessel equation. The solution that is bounded at $r = 0$ is thus found as $f(r) = J_0(ar)$, where J_0 denotes the zero-th order Bessel function.

Inserting $f(r) = J_0(ar)$ along with $\tilde{w}$ and $\tilde{\theta}$ into w' and θ', we have

$$w' = J_0(ar)\sin(n\pi z)e^{\sigma t} + c.c.$$

$$\theta' = [a^2 + (n\pi)^2 + \sigma]^{-1} J_0(ar)\sin(n\pi z)e^{\sigma t} + c.c.$$

The radial velocity can be found from the continuity equation, which reduces, for axisymmetric disturbances, to

$$\frac{du_r'}{dr} + \frac{u_r'}{r} = -\frac{\partial w'}{\partial z} = -(n\pi)\cos(n\pi z)J_0(ar)e^{\sigma t} + c.c.$$

Observing the Bessel equation, we infer that

$$u_r' = a^{-1}(n\pi)\cos(n\pi z)J_0'(ar)e^{\sigma t} + c.c.$$

[Recall that the vertical component of the vorticity, $\omega_z' \equiv \dfrac{1}{r}\dfrac{\partial}{\partial r}(ru_\phi') - \dfrac{1}{r}\dfrac{\partial u_r'}{\partial \phi}$, must be zero. This is the case when the azimuthal velocity $u_\phi' = 0$.]

5, B

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(iv) For each a, the impermeability condition, $u_r'(r_0) = 0$, and perfect insulation condition, $\left.\dfrac{d\theta'}{dr}\right|_{r=r_0} = 0$, can be satisfied simultaneously by setting

$$J_0'(ar_0) = 0,$$

which fixes $a = j_0/r_0$, where j_0 is a root of $J_0' = 0$. However, the no-slip condition $w' = 0$ is not satisfied. When $r_0 \gg 1$, a thin boundary layer may be introduced to reduce the slip velocity to zero. The 'distant' sidewall is not expected to play a significant role.

For $r_0 = O(1)$, the sidewall may play a major role. In this case, to account for its effect, we note that the dispersion relation (13), a cubic polynomial of a^2, has 3 roots, denoted as a_j^2 $(j = 1, 2, 3)$, for which we can find $J_0(a_j r)$. The solution for w', u'_r and θ' can be constructed as superposition of three terms,

$$(w', u'_r) = \sum_j \Big(A_j J_0(a_j r) \sin(n\pi z),\ (n\pi) a_j^{-1} A_j J'_0(a_j r) \cos(n\pi z) \Big) e^{\sigma t} + c.c.,$$

$$\theta' = \sum_j \big[a_j^2 + (n\pi)^2 + \sigma \big]^{-1} A_j J_0(a_j r) \sin(n\pi z) e^{\sigma t} + c.c.$$

Now setting $w' = u'_r = d\theta'/dr = 0$ at $r = r_0$ gives

$$\sum_j J_0(a_j r_0) A_j = 0, \ \ \sum_j a_j^{-1} J'_0(a_j r_0) A_j = 0,$$

$$\sum_j \big[a_j^2 + (n\pi)^2 + \sigma \big]^{-1} a_j J'_0(a_j r_0) A_j = 0,$$

which is a system of three simultaneous equations for A_j. For non-zero solutions, the determinant of the coefficient matrix must vanish. This defines as a relation $\Delta(a_1, a_2, a_3, \sigma) = 0$, which in turn determines the eigenvalue, $\sigma = \sigma(Ra)$, since $a_j = a_j(Ra, \sigma)$. The critical Rayleigh number is found by setting $\sigma_r(Ra) = 0$. $\boxed{5, \text{D}}$

3. (a) (i) For the given profile, $U'' = 0$ and hence the Rayleigh equation reduces to sim. seen $\Downarrow$

$$\frac{d^2\bar{v}}{dy^2} - \alpha^2\bar{v} = 0.$$

The solution can be written as

$$\bar{v} = \begin{cases} C^+e^{-\alpha y} & y > 1, \\ C_1e^{-\alpha y} + C_2e^{\alpha y} & 0 < y < 1. \end{cases}$$

Now apply the jump conditions across the discontinuity $y = 1$. First the continuity of $(U - c)\bar{v}' - U'\bar{v}$ implies

$$-\alpha(1-c)C^+e^{-\alpha} = [-\alpha(1-c) - 1]C_1e^{-\alpha} + [\alpha(1-c) - 1]C_2e^{\alpha}, \qquad (14)$$

while the continuity of $\bar{v}/(U - c)$ gives

$$C^+e^{-\alpha} = C_1e^{-\alpha} + C_2e^{\alpha}, \qquad (15)$$

where we have used the fact that $U' = 1$ at $y = 1^-$, and U is continuous. The boundary condition $\bar{v} = 0$ at $y = 0$ gives $C_1 + C_2 = 0$, i.e.

$$C_1 = -C_2.$$

5, A

(ii) Use of the relation $C_1 = -C_2$ in (14) and (15) leads to meth seen $\Downarrow$

$$-\alpha(1-c)C^+e^{-\alpha} = \Big[2\alpha(1-c)\cosh\alpha - 2\sinh\alpha\Big]C_2,$$

$$C^+e^{-\alpha} = 2\sinh\alpha\, C_2,$$

from which we find

$$-\alpha(1-c)\sinh\alpha = \alpha(1-c)\cosh\alpha - \sinh\alpha.$$

Thus

$$c = 1 - \frac{1 - e^{-2\alpha}}{2\alpha},$$

which is real for all α, i.e. all modes are neutral ($\omega_i \equiv 0$). Thus the flow is inviscidly stable!

As $\alpha \to 0$,

$$c \to 1 - \frac{2\alpha - (2\alpha)^2/2 + O(\alpha^3)}{2\alpha} = \alpha + O(\alpha^2).$$

3, A

(iii) For $0 \le y < 1$, unseen $\Downarrow$

$$\bar{v} = (e^{\alpha y} - e^{-\alpha y})C_2.$$

From the continuity equation, the streamwise velocity is found as

$$\bar{u} = i(e^{\alpha y} + e^{-\alpha y})C_2.$$

The pressure follows from the streamwise momentum equation,

$$\begin{aligned} \bar{p} &= -(U - c)\bar{u} + i\alpha^{-1}U'\bar{v} \\ &= -i(y - c)(e^{\alpha y} + e^{-\alpha y})C_2 + i\alpha^{-1}(e^{\alpha y} - e^{-\alpha y})C_2. \end{aligned}$$

We have

$$\bar{u}(0) = 2iC_2, \quad \bar{p}_w = \bar{p}(0) = 2ic\,C_2,$$

indicating that $\bar{p}_w = c\bar{u}(0)$ as expected. Clearly, $\bar{u}(0) \neq 0$, i.e. the no-slip condition is not satisfied and so a viscous (Stokes) layer is required. 3, B meth seen $\Downarrow$

(b) (i) In the viscous layer, the viscous diffusion balances the unsteadiness in the streamwise momentum equation

$$i\alpha(U-c)\bar{u}+\frac{dU}{dy}\bar{v}=-i\alpha\bar{p}+\frac{1}{Re}(\frac{d^2}{dy^2}-\alpha^2)\bar{u}.$$

Let the thickness be of $O(\ell_s)$, which is now deduced. The viscous diffusion term and unsteadiness are

$$\frac{1}{Re}\frac{\partial^2\bar{u}}{\partial y^2}=O(\frac{\bar{u}}{Re\,\ell_s^2}),\qquad -i(\alpha c)\bar{u}=O(\omega\bar{u}).$$

The balance of the two, $\dfrac{\bar{u}}{Re\,\ell_s^2}=O(\omega\bar{u})$, gives

$$\ell_s=O\big((\omega Re)^{-1/2}\big).$$

In the Stokes layer, $\bar{u}$ is expected to be of $O(C_2)$, taken to be $O(1)$. The balance in the continuity equation, $i\alpha\bar{u}=O(\bar{v}/\ell_s)$, indicates that

$$\bar{v}=O\big(\alpha(\omega Re)^{-1/2}\big),$$

which is much smaller than $\bar{u}$.
Introduce $\tilde{Y}=y/(\omega Re)^{-1/2}$, which implies that $\partial/\partial y=(\omega Re)^{1/2}\partial/\partial\tilde{Y}$. With the expansion, the continuity equation becomes

$$i\tilde{U}+\frac{d\tilde{V}}{d\tilde{Y}}=0.$$

Note that two more terms, $\alpha U\bar{u}$ and $(dU/dy)\bar{v}$, are both of $O\big(\alpha(\omega Re)^{-1/2}\big)$, which is much smaller than $\omega\bar{u}$, provided that

$$\alpha(\omega Re)^{-1/2}\ll\omega. \tag{16}$$

The pressure gradient term $-i\alpha p=-i\alpha\bar{p}_w=-i\alpha c\bar{u}(0)=O(\omega)$.
Therefore the streamwise momentum equation reduces to

$$-i\omega\tilde{U}=-i\alpha\bar{p}_w+\omega\frac{d^2\tilde{U}}{d\tilde{Y}^2}.$$

The boundary and conditions are:

$$\tilde{U}=\tilde{V}=0\quad\text{at}\quad\tilde{Y}=0;\quad\tilde{U}\to\bar{u}(0)\quad\text{as}\quad\tilde{Y}\to\infty.$$

4, B

unseen $\Downarrow$

(ii) The solution for $\tilde{U}$ is found as

$$\tilde{U}=(\alpha/\omega)\bar{p}_w\Big[1-\exp\{-(-i)^{1/2}\tilde{Y}\}\Big].$$

The matching condition is satisfied since $\bar{p}_w=c\bar{u}(0)$.
Integrating the continuity equation, we obtain

$$\tilde{V}=-i(\alpha/\omega)\bar{p}_w\Big[\tilde{Y}-\int_0^{\tilde{Y}}\exp\{-(-i)^{1/2}\tilde{Y}\}d\tilde{Y}\Big]$$

$$\to -i(\alpha/\omega)\bar{p}_w\tilde{Y}-(\alpha/\omega)\bar{p}_w(-i)^{1/2}\quad\text{as}\quad\tilde{Y}\to\infty.$$

Thus $\tilde{V}_\infty = -(\alpha/\omega)\bar{p}_w(-i)^{1/2}$, which represents the displacement effect induced by the viscous motion. The unrescaled 'transpiration velocity' is $\alpha(\omega Re)^{-1/2}\tilde{V}_\infty = -(\alpha^2/\omega^{3/2})e^{-\pi i/4}\bar{p}_w Re^{-1/2}$, and it acts at the 'bottom' of the main layer. Its impact on the inviscid solution can be accounted for by replacing the boundary condition $\bar{v}(0) = 0$ by

$$\bar{v}(0) = -(\alpha^2/\omega^{3/2})e^{-\pi i/4}Re^{-1/2}\bar{p}(0),$$

where the coefficient of $\bar{p}(0)$ may be interpreted as an 'impedance'. This produces a small correction (actually growth rate) of $O(Re^{-1/2})$ when $\alpha = O(1)$. $\boxed{3, \text{D}}$

(iii) When $\alpha \ll 1$, $c = O(\alpha)$ as was noted in Part a (ii), and so $\omega = O(\alpha^2)$. The two terms in the streamwise momentum equation, $\alpha U\bar{u}$ and $(dU/dy)\bar{v}$, can no longer be neglected when $\boxed{\text{unseen} \Downarrow}$

$$\alpha(\omega Re)^{-1/2} = O(\omega), \quad \text{i.e.} \quad \alpha(\alpha^2 Re)^{-1/2} = O(\alpha^2).$$

Thus a new regime arises when $\alpha = O(Re^{-1/4}))$, which is the well-known triple-deck scale. (The 'impedance' coefficient increases to $O(Re^{-1/4})$.) $\boxed{2, \text{D}}$

4. (i) Substitution of the perturbed flow into the Navier-Stokes equations and followed by linearisation, the equations for the perturbation are found as

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$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} + w'\frac{\partial U}{\partial z} = -\frac{\partial p'}{\partial x} + \frac{1}{Re}\nabla^2 u',$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\nabla^2 v',$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\nabla^2 w'.$$

When the perturbation takes the normal-mode form, the equations for $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p})$ are the same as above, provided that

$$\frac{\partial}{\partial t} \to -i\omega, \quad \frac{\partial}{\partial x} \to i\alpha, \quad \nabla^2 \to \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2.$$

4, A

(ii) In the inviscid approximation, the above set of equations reduces to

sim. seen ⇓

$$i\alpha\tilde{u} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{17}$$

$$-i\omega\tilde{u} + i\alpha U\tilde{u} + \tilde{v}\frac{\partial U}{\partial y} + \tilde{w}\frac{\partial U}{\partial z} = -i\alpha\tilde{p}, \tag{18}$$

$$-i\omega\tilde{v} + i\alpha U\tilde{v} = -\frac{\partial \tilde{p}}{\partial y}, \tag{19}$$

$$-i\omega\tilde{w} + i\alpha U\tilde{w} = -\frac{\partial \tilde{p}}{\partial z}. \tag{20}$$

Eliminating $\tilde{u}$ from the first two equations, we obtain

$$-(U-c)\Big[\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z}\Big] + \Big[\tilde{v}\frac{\partial U}{\partial y} + \tilde{w}\frac{\partial U}{\partial z}\Big] = -i\alpha\tilde{p}. \tag{21}$$

Equations (19) and (20) give

$$\tilde{v} = -\frac{\frac{\partial \tilde{p}}{\partial y}}{i\alpha(U-c)}, \quad \tilde{w} = -\frac{\frac{\partial \tilde{p}}{\partial z}}{i\alpha(U-c)}. \tag{22}$$

substitution of which into (21) leads to an equation for $\tilde{p}$,

$$(U-c)\left\{\frac{\partial}{\partial y}\Big[\frac{\frac{\partial \tilde{p}}{\partial y}}{U-c}\Big] + \frac{\partial}{\partial z}\Big[\frac{\frac{\partial \tilde{p}}{\partial z}}{U-c}\Big]\right\} - \left\{\frac{\frac{\partial U}{\partial y}\frac{\partial \tilde{p}}{\partial y}}{U-c} + \frac{\frac{\partial U}{\partial z}\frac{\partial \tilde{p}}{\partial z}}{U-c}\right\} = \alpha^2\tilde{p}.$$

This can be rewritten into the required form

$$\frac{\partial}{\partial y}\Big[\frac{\frac{\partial \tilde{p}}{\partial y}}{(U-c)^2}\Big] + \frac{\partial}{\partial z}\Big[\frac{\frac{\partial \tilde{p}}{\partial z}}{(U-c)^2}\Big] = \frac{\alpha^2\tilde{p}}{(U-c)^2}, \tag{23}$$

as can be verified by using the product rule.

For case (a), the boundary condition at $y = 0, 1$, the impermeability condition $\tilde{v} = 0$, is satisfied when

$$\frac{\partial \tilde{p}}{\partial y} = 0.$$

The boundary condition at $z = 0, \ell$, the impermeability condition $\tilde{w} = 0$, is satisfied when

$$\frac{\partial \tilde{p}}{\partial z} = 0.$$

5, B

For case (b), the impermeability condition is expressed as $(\tilde{v}, \tilde{w}) \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal vector of the boundary. Use of (22) in $(\tilde{v}, \tilde{w}) \cdot \mathbf{n} = 0$ gives

unseen ⇓

$$\nabla \tilde{p} \cdot \mathbf{n} = 0 \quad \text{i.e.} \quad \frac{\partial \tilde{p}}{\partial n} = 0, \tag{24}$$

2, C

where ∇ is the gradient operator in the (y, z) plane.

(iii) Multiplying equation (23) by $\tilde{p}^*$, the complex conjugate of $\tilde{p}$, and integrating over the rectangular region, $0 < y < 1$ and $0 < z < \ell$, we have

$$\int_0^1 \int_0^\ell \tilde{p}^* \frac{\partial}{\partial y}\left[\frac{\frac{\partial \tilde{p}}{\partial y}}{(U-c)^2}\right] dydz + \int_0^1 \int_0^\ell \tilde{p}^* \frac{\partial}{\partial z}\left[\frac{\frac{\partial \tilde{p}}{\partial z}}{(U-c)^2}\right] dydz = \int_0^1 \int_0^\ell \frac{\alpha^2 |\tilde{p}|^2}{(U-c)^2} dydz.$$

Perform integration by parts with respect to y and z in the first and second integrals on the left-hand side, respectively:

$$-\int_0^1 \int_0^\ell \frac{\left|\frac{\partial \tilde{p}}{\partial y}\right|^2}{(U-c)^2} dydz - \int_0^1 \int_0^\ell \frac{\left|\frac{\partial \tilde{p}}{\partial z}\right|^2}{(U-c)^2} dydz = \int_0^1 \int_0^\ell \frac{\alpha^2 |\tilde{p}|^2}{(U-c)^2} dydz,$$

where the contributions from the boundaries are zero due to the boundary conditions. The key result can be written as

$$\int_0^1 \int_0^\ell \frac{Q(y,z)}{(U-c)^2} dydz = 0, \tag{25}$$

where

$$Q(y,z) = \left|\frac{\partial \tilde{p}}{\partial y}\right|^2 + \left|\frac{\partial \tilde{p}}{\partial z}\right|^2 + \alpha^2 |\tilde{p}|^2 \geq 0.$$

Noting that $(U-c)^{-2} = \left[(U-c_r)^2 - c_i^2 + 2c_i(U-c_r)i\right] / \left[(U-c_r)^2 + c_i^2\right]$ and split the integrand into the real and imaginary parts, equation (25) can be written as

$$\int_0^1 \int_0^\ell \tilde{Q}(y,z)\left[(U-c_r)^2 - c_i^2\right] dydz = 0; \tag{26}$$

$$2c_i \int_0^1 \int_0^\ell \tilde{Q}(y,z)(U-c_r) dydz = 0, \tag{27}$$

where we have put

$$\tilde{Q}(y,z) = \frac{Q(y,z)}{(U-c_r)^2 + c_i^2} \geq 0.$$

Since $c_i > 0$, the relation (27) indicates that $(U - c_r)$ must change its sign in the domain, i.e.

$$U_{\min} < c_r < U_{\max}, \tag{28}$$

that is, c_r lies in the range of $U(y,z)$.

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The argument leading to the required estimates (the semicircle theorem) is similar to that in the one-dimensional case where $U = U(y)$. The relation (26) can be written as

$$\int_0^1 \int_0^\ell \tilde{Q}\left[(U^2 + c_r^2 - 2c_r(U-c_r) - 2c_r^2 - c_i^2\right] dydz = 0,$$

from which and equation (27) follows

$$\int_0^1\int_0^\ell U^2\tilde{Q}\,dydz = \int_0^1\int_0^\ell (c_r^2+c_i^2)\,\tilde{Q}\,dydz. \tag{29}$$

In order to prove the required estimate, which is stricter than (28), we observe that

$$0 \geq (U-U_{\min})(U-U_{\max}),$$

and hence

$$\begin{aligned}
0 &\geq \int_0^1\int_0^\ell (U-U_{\min})(U-U_{\max})\tilde{Q}dydz \\
&= \int_0^1\int_0^\ell \Big[U^2-(U_{\min}+U_{\max})U+U_{\min}U_{\max}\Big]\tilde{Q}dydz. \\
&= \int_0^1\int_0^\ell \Big[c_r^2+c_i^2-(U_{\min}+U_{\max})c_r+U_{\min}U_{\max}\Big]\tilde{Q}dydz
\end{aligned}$$

where in the last step use has been made of (29) and (27). It follows that

$$c_r^2+c_i^2-(U_{\min}+U_{\max})c_r+U_{\min}U_{\max} \leq 0,$$

from which the semicircle theorem follows. $\boxed{6,\ \mathrm{C}}$

$\boxed{\text{unseen} \Downarrow}$

(iv) The results remain valid for a general closed domain, denoted as S say. In this case, multiplication of $\tilde{p}^*$ to (23) and integration over S yields

$$\int_S\left\{\tilde{p}^*\frac{\partial}{\partial y}\Big[\frac{\frac{\partial\tilde{p}}{\partial y}}{(U-c)^2}\Big]+\tilde{p}^*\frac{\partial}{\partial z}\Big[\frac{\frac{\partial\tilde{p}}{\partial z}}{(U-c)^2}\Big]\right\}dydz = \int_S\frac{\alpha^2|\tilde{p}|^2}{(U-c)^2}dydz.$$

After splitting the integrand into two parts, the left-hand side can be rewritten as

$$\int_S\left\{\frac{\partial}{\partial y}\Big[\frac{\tilde{p}^*\frac{\partial\tilde{p}}{\partial y}}{(U-c)^2}\Big]+\frac{\partial}{\partial z}\Big[\frac{\tilde{p}^*\frac{\partial\tilde{p}}{\partial z}}{(U-c)^2}\Big]-\frac{\left|\frac{\partial\tilde{p}}{\partial y}\right|^2}{(U-c)^2}-\frac{\left|\frac{\partial\tilde{p}}{\partial z}\right|^2}{(U-c)^2}\right\}dydz \equiv I_1-I_2.$$

By using Green's theorem, the first integral, I_1, can be converted into a contour integral along the boundary of S, denoted as C say,

$$I_1 = \int_C\left\{-\Big[\frac{\tilde{p}^*\frac{\partial\tilde{p}}{\partial z}}{(U-c)^2}\Big]dy+\Big[\frac{\tilde{p}^*\frac{\partial\tilde{p}}{\partial y}}{(U-c)^2}\Big]dz\right\} = \int_C\frac{\tilde{p}^*}{(U-c)^2}\frac{\partial\tilde{p}}{\partial n}ds = 0,$$

where the boundary condition (24) is used in the last step. As a result, the crucial identity (25) holds for a general domain. The arguments leading to the required estimate for c_r and the semicircle theorem remain the same. $\boxed{3,\ \mathrm{D}}$

5. (i) Substituting the expression for the perturbed flow into the Navier-Stokes equations, and neglecting all nonlinear terms, we obtain the linearized equations for the disturbances,

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$$\frac{\partial u'}{\partial x} + Re^{1/2}\frac{\partial v'}{\partial Y} = 0, \tag{30}$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \underline{\frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial U}{\partial Y}v' = -\frac{\partial p'}{\partial x} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\frac{\partial^2}{\partial x^2}\Big]u', \tag{31}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + \underline{Re^{-\frac{1}{2}}\frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial Y} + \frac{\partial V}{\partial Y}v'} = -Re^{\frac{1}{2}}\frac{\partial p'}{\partial Y} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\frac{\partial^2}{\partial x^2}\Big]v', \tag{32}$$

The underlined terms represent the non-parallel-flow effects, caused by the streamwise variation of the streamwise velocity U and the presence of a normal velocity $Re^{-1/2}V$, which is associated as well with the streamwise variation of U.

Parallel-flow approximation neglects the underlined terms in (31)-(32), and treats the variation of U with x as being parametric; the latter means that at each location, the profile is 'frozen' so that normal-mode solutions may be sought. The viscous terms are retained and so the analysis leads to the usual Orr-Sommerfeld equation.

[3 marks]

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(ii) Substituting the normal-mode form into (30)-(32), and noting that the operators

$$\frac{\partial}{\partial t} \to -iRe^{1/7}\omega, \quad \frac{\partial}{\partial x} \to iRe^{2/7}\alpha, \tag{33}$$

when acting on the perturbation, we obtain the equations

$$i\alpha\bar{u} + \frac{\partial \bar{v}}{\partial Y} = 0, \quad i\alpha U\bar{u} + \frac{\partial U}{\partial Y}\bar{v} = 0, \quad 0 = -\frac{\partial \bar{p}}{\partial Y}, \tag{34}$$

with $\bar{v}$ having to satisfy the boundary condition $\bar{v} = 0$ at $Y = 0$. Elimination of $\bar{u}$ between the first two equations in (34) gives

$$U\frac{\partial \bar{v}}{\partial Y} - \frac{\partial U}{\partial Y}\bar{v} = 0,$$

which is a first-order ordinary differential equation for $\bar{v}$ and has the solution

$$\bar{v} = -i\alpha AU,$$

where A is a constant yet to be determined, and the pre-factor $-i\alpha$ is introduced for convenience. It follows that

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{p} = P \ \text{(constant)}. \tag{35}$$

Note that $\bar{v} \to -i\alpha A \neq 0$ as $Y \to \infty$ and so an upper layer is required, where the perturbation attenuates to zero in the far field.

[4 marks]

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(iii) Let the width of the upper layer be $y = O(\ell)$ equivalent to $Y = O(Re^{1/2}\ell)$. Matching with the main-layer solution suggests that $v' = O(Re^{-3/14})$. We now deduce ℓ along with the orders of magnitude of u' and p'. The continuity equation implies

$$Re^{2/7}u' \sim v'/\ell. \tag{36}$$

Note that the background flow outside the boundary layer is $U = 1$, and so the momentum equations reduce to

$$\frac{\partial u'}{\partial x} = -\frac{\partial p'}{\partial x}, \quad \frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y},$$

which give

$$Re^{2/7}u' \sim Re^{2/7}p', \quad Re^{2/7}v' \sim p'/\ell.$$

Thus

$$p' \sim u' \sim Re^{2/7}\ell\, v', \tag{37}$$

which is combined with (36) to give $Re^{4/7}\ell v' \sim v'/\ell$, and hence

$$\ell = O(Re^{-2/7}).$$

It follows from (37) that

$$u' \sim p' \sim v' = O(Re^{-3/14}),$$

which explains the upper-layer expansion given in the question.
Substituting the expansion and $\bar{y} = Re^{2/7}y$ into the perturbation equations, we obtain

$$i\alpha u^\dagger + \frac{\partial v^\dagger}{\partial \bar{y}} = 0, \quad i\alpha u^\dagger = -i\alpha p^\dagger, \quad i\alpha v^\dagger = -\frac{\partial p^\dagger}{\partial \bar{y}}.$$

From these we find the equation for $p^\dagger$:

$$\frac{\partial^2 p^\dagger}{\partial \bar{y}^2} - \alpha^2 p^\dagger = 0.$$

The normal momentum equation and matching imply that $\left.\frac{\partial p^\dagger}{\partial \bar{y}}\right|_{\bar{y}=0} = -\alpha^2 A$. The solution for $p^\dagger$, which attenuates as $\bar{y} \to \infty$, is found as $p^\dagger = \alpha A e^{-\alpha\bar{y}}$. Matching the pressure in the main layer gives

$$P = \alpha A,$$

which is the pressure-displacement relation.

[4 marks]

unseen $\Downarrow$

(iv) As $Y \to 0$, the main layer solution behaves as

$$\bar{u} \to \lambda_2 Y, \quad \bar{v} \to -\tfrac{1}{2}\lambda_2(i\alpha A)Y^2. \tag{38}$$

indicating that the main-deck solution happens to satisfy the no-slip condition. Yet, a viscous layer (lower deck) still has to be introduced because neither of unsteadiness and the pressure gradient appears at leading order in the main layer, but they both will appear at leading order in the lower deck.

Let $Y = O(d)$ with $d \ll 1$ in the lower deck, where $U = \frac{1}{2}\lambda_2 Y^2 = O(d^2)$. It follows that the inertia term $U\dfrac{\partial u'}{\partial x} \sim O(d^2 Re^{2/7} u')$, while the viscous diffusion $\dfrac{\partial^2 u'}{\partial Y^2} \sim O(u'/d^2)$. The balance between the two,

$$d^2 Re^{2/7} u' \sim u'/d^2,$$

suggests that $d = O(Re^{-1/14})$, which corresponds to $y = Re^{-1/2}Y = O(Re^{-4/7})$.

The asymptotic behaviour of the main-deck solution, (38), suggests that in the lower deck $u' = O(d) = O(Re^{-1/14})$ and $v' = O(Re^{-3/14}d^2) = O(Re^{-5/14})$ as can be deduced by the matching principle. Similarly, $p' = O(Re^{-3/14})$ as in the main layer.

Note that the unsteadiness term $\partial u'/\partial t \sim Re^{1/7}u' = O(Re^{1/14})$, and the pressure gradient $\partial p'/\partial x \sim Re^{2/7}p' = O(Re^{1/14})$, both comparable with the viscous term, which is $u'/d^2 = O(Re^{1/14})$; these all appear at leading order. In contrast, the nonparallel-flow term $u'\partial U/\partial x \sim d^2 Re^{-1/14} = O(Re^{-3/14})$. It follows that non-parallelism contributes a small correction of $O(Re^{-2/7})$, which is the ratio of the wavelength of the mode to L, the length scale over which the base flow evolves.

Therefore, in the lower deck, the solution should expand as

$$(u',\, v',\, p') = (Re^{-1/14}\tilde{u},\, Re^{-\frac{5}{14}}\tilde{v},\, Re^{-\frac{3}{14}}\tilde{p})E + c.c.$$

[6 marks]

unseen $\Downarrow$

(v) Substituting this into (30)-(32) and using the fact that $U = Re^{-1/7}(\frac{1}{2}\lambda_2\tilde{y})$ as well as the relations in (33), we obtain

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\tilde{y}} = 0, \tag{39}$$

$$-i\omega\tilde{u} + \tfrac{1}{2}i\alpha\lambda_2\tilde{y}^2\tilde{u} + \lambda_2\tilde{y}\tilde{v} = -i\alpha\tilde{p} + \frac{d^2\tilde{u}}{d\tilde{y}^2}, \tag{40}$$

plus $d\tilde{p}/d\tilde{y} = 0$ so that $\tilde{p}$ is a constant and $\tilde{p} = P$.
The boundary conditions are

$$\tilde{u} = \tilde{v} = 0 \quad \text{at} \quad \tilde{y} = 0,$$

and the matching condition is

$$\tilde{u} \to \lambda_2 A\tilde{y} \quad \text{as} \quad \tilde{y} \to \infty.$$

[3 marks]

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Hydrodynamic Stability_MATH97012 MATH70052	1	All had decent or good attempts. Given that the question has mamy familiar elements, I was expecting higher marks.
Hydrodynamic Stability_MATH97012 MATH70052	2	All did rather well except that one seemed underprepared. The very last (and small) part of the question was hard for everyone, as had been expected.
Hydrodynamic Stability_MATH97012 MATH70052	3	All had decent or good attempts except one who seemed underprepared. Given that the question has mamy familiar elements, I was expecting higher marks.
Hydrodynamic Stability_MATH97012 MATH70052	4	All did rather well except that one seemed underprepared. The very last (and small) part of the question was hard for everyone, as had been expected.
Hydrodynamic Stability_MATH97012 MATH70052	5	The majority only managed to do the first half of the question. The second half is unfamilar and requires a more thorough understanding of the topic. Also the paper was rather long. These will be taken into account wjhen setting the PEM thresholds.