Imperial College London

MATH97066 MATH97177

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Bifurcation Theory

Date: 2nd June 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. Consider a system of differential equations in R^4. How many stable equilibria and periodic orbits can be born at the following bifurcations:

   (i) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{2} \pm i, -2, 0$ and the Lyapunov coefficients $l_2 = l_3 = l_4 = 0$, $l_5 < 0$? (5 marks)

   (ii) An equilibrium state with the eigenvalues of the linearisation matrix equal to $\pm i, -\frac{1}{2} \pm i$ and the Lyapunov coefficients $L_1 = L_2 = L_3 = L_4 = 0$, $L_5 > 0$? (5 marks)

   (iii) A periodic orbit with the multipliers $1, -\frac{1}{2} \pm i$ and the Lyapunov coefficient $l_2 < 0$? (5 marks)

   (iv) A periodic orbit with the multipliers $-1, -\frac{1}{2}, \frac{1}{2}$ and the Lyapunov coefficients $L_1 = 0$, $L_2 > 0$? (5 marks)

(Total: 20 marks)

2. Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y + axy - y^2, \end{cases}$$

that depends on parameters ε and a.

   (i) Write the normal form, up to the terms of the third order, for this system near the equilibrium $(x, y) = (0, 0)$ at $\varepsilon = 0$. (8 marks)

   (ii) Deduce that the first Lyapunov coefficient equals to $L = -a/8$. Determine for which values of ε and a the equilibrium state at $(x, y) = (0, 0)$ is asymptotically stable. (6 marks)

   (iii) How many periodic orbits can exist near $(x, y) = (0, 0)$ for small $\varepsilon > 0$ at $a = 2$? At $a = -2$? At $a = 0$? (6 marks)

(Total: 20 marks)

3. Consider the following one-dimensional map:

$$x \mapsto \bar{x} = a - 2x^3,$$

where a is a parameter.

(i) Show that the map has no periodic points of period larger than 2. (4 marks)

(ii) Study bifurcations of the fixed point of the map. For which values of a is the fixed point stable? (8 marks)

(iii) Study bifurcations of points of period 2. Find the set of values of a for which stable orbits of period 2 exist. (8 marks)

(Total: 20 marks)

4. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω varies near a certain value $\omega_0 \in (0,\pi)$. Let the first Lyapunov coefficient be $L = -1$.

(i) In polar coordinates (r,ϕ), the normal form for this map near the fixed point can be written as

$$\bar{r} = (1+\mu)r\,(1 - r^2 + f(r,\phi,\mu,\omega)), \qquad \bar{\phi} = \omega + \phi + g(r,\phi,\mu,\omega),$$

where $f = O(r^3)$ and $g = O(r^2)$ are smooth functions. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$ and this curve attracts all orbits (except for the fixed point itself) from some neighbourhood of the fixed point independent of μ and ω. It is also known that this curve has an equation $r = h(\phi,\mu,\omega)$ where h is a smooth, positive, 2π-periodic function of ϕ. Show that

$$h = \sqrt{\mu} + O(\mu).$$

(12 marks)

(ii) Show that in the (μ,ω)-plane for any N there exist parameter values corresponding to the existence of periodic orbits of period larger than N. (8 marks)

(Total: 20 marks)

5. (Mastery Question) Consider a generic system of differential equations on a plane which has an equilibrium state with a double zero eigenvalue. By bringing the linear part of the system to the Jordan form, one can write the system in the following form

$$\dot{x} = u + f(x, u),$$
$$\dot{u} = g(x, u),$$

where the smooth functions f and g vanish at zero along with their first derivatives. Introducing a new variable y by the rule

$$y = u + f(x, u)$$

brings the system to the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= h(x, y), \end{aligned} \tag{1}$$

where h is a smooth function vanishing at zero along with its first derivatives.

(i) The transformation

$$x^{new} = x + \psi(x, y), \qquad y^{new} = y + \frac{\partial \psi}{\partial x} y + \frac{\partial \psi}{\partial y} h(x, y)$$

keeps system (1) in the same form, i.e., we have

$$\dot{x}^{new} = y^{new},$$
$$\dot{y}^{new} = H(x^{new}, y^{new})$$

for some function H. If $\psi = \alpha x^2 + \beta xy + \gamma y^2$ and $h = ax^2 + bxy + cy^2 + O(|x|^3 + |y|^3)$, what is the Taylor expansion, up to the quadratic terms, of the function $H(x^{new}, y^{new})$?

(8 marks)

(ii) Using the result of part (i), show that system (1) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = ax^2 + bxy + O(|x|^3 + |y|^3),$$

by a smooth coordinate transformation. (2 marks)

(iii) By induction in n, show that for every $n \geq 2$ system (1) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = P_n(x) + yQ_n(x) + O(|x|^{n+1} + |y|^{n+1}),$$

by a smooth coordinate transformation, where $P_n(x)$ is a polynomial of degree n and $Q_n(x)$ is a polynomial of degree $(n-1)$. (10 marks)

(Total: 20 marks)

Module: M3PA24/M4PA24/M5PA24
Setter: Turaev
Checker: Rasmussen
Editor: editor
External: external
Date: April 27, 2020
Version: Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

M3PA24/M4PA24/M5PA24 Bifurcation Theory

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1 - 20 marks; 3(i) 4 marks; 4(ii) 8 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2 - 20 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
4(i) 12 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
3(ii) 8 marks; 3(iii) 8 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Bifurcation Theory

Date: ??

Time: ??

Time Allowed: 2 Hours for M3PA24 paper; 2.5 Hours for M45PA24 papers

This paper has *4 Questions (M3PA24 version); 5 Questions (M45PA24 versions)*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. Consider a system of differential equations in R^4. How many stable equilibria and periodic orbits can be born at the following bifurcations:

(i) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{2} \pm i, -2, 0$ and the Lyapunov coefficients $l_2 = l_3 = l_4 = 0$, $l_5 < 0$? (5 marks)

(ii) An equilibrium state with the eigenvalues of the linearisation matrix equal to $\pm i, -\frac{1}{2} \pm i$ and the Lyapunov coefficients $L_1 = L_2 = L_3 = L_4 = 0$, $L_5 > 0$? (5 marks)

(iii) A periodic orbit with the multipliers $1, -\frac{1}{2} \pm i$ and the Lyapunov coefficient $l_2 < 0$? (5 marks)

(iv) A periodic orbit with the multipliers $-1, -\frac{1}{2}, \frac{1}{2}$ and the Lyapunov coefficients $L_1 = 0$, $L_2 > 0$? (5 marks)

(Total: 20 marks)

2. Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y + axy - y^2, \end{cases}$$

that depends on parameters ε and a.

(i) Write the normal form, up to the terms of the third order, for this system near the equilibrium $(x, y) = (0, 0)$ at $\varepsilon = 0$. (8 marks)

(ii) Deduce that the first Lyapunov coefficient equals to $L = -a/8$. Determine for which values of ε and a the equilibrium state at $(x, y) = (0, 0)$ is asymptotically stable. (6 marks)

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(Total: 20 marks)

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where a is a parameter.

(i) Show that the map has no periodic points of period larger than 2. (4 marks)

(ii) Study bifurcations of the fixed point of the map. For which values of a is the fixed point stable? (8 marks)

(iii) Study bifurcations of points of period 2. Find the set of values of a for which stable orbits of period 2 exist. (8 marks)

(Total: 20 marks)

4. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω varies near a certain value $\omega_0 \in (0, \pi)$. Let the first Lyapunov coefficient be $L = -1$.

(i) In polar coordinates (r, ϕ), the normal form for this map near the fixed point can be written as

$$\bar{r} = (1+\mu)r\,(1 - r^2 + f(r, \phi, \mu, \omega)), \qquad \bar{\phi} = \omega + \phi + g(r, \phi, \mu, \omega),$$

where $f = O(r^3)$ and $g = O(r^2)$ are smooth functions. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$ and this curve attracts all orbits (except for the fixed point itself) from some neighbourhood of the fixed point independent of μ and ω. It is also known that this curve has an equation $r = h(\phi, \mu, \omega)$ where h is a smooth, positive, 2π-periodic function of ϕ. Show that

$$h = \sqrt{\mu} + O(\mu).$$

(12 marks)

(ii) Show that in the (μ, ω)-plane for any N there exist parameter values corresponding to the existence of periodic orbits of period larger than N. (8 marks)

(Total: 20 marks)

5. (Mastery Question) Consider a generic system of differential equations on a plane which has an equilibrium state with a double zero eigenvalue. By bringing the linear part of the system to the Jordan form, one can write the system in the following form

$$\dot{x} = u + f(x,u),$$
$$\dot{u} = g(x,u),$$

where the smooth functions f and g vanish at zero along with their first derivatives. Introducing a new variable y by the rule

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$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= h(x,y), \end{aligned} \tag{1}$$

where h is a smooth function vanishing at zero along with its first derivatives.

(i) The transformation

$$x^{new} = x + \psi(x,y), \qquad y^{new} = y + \frac{\partial\psi}{\partial x}y + \frac{\partial\psi}{\partial y}h(x,y)$$

keeps system (1) in the same form, i.e., we have

$$\dot{x}^{new} = y^{new},$$
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for some function H. If $\psi = \alpha x^2 + \beta xy + \gamma y^2$ and $h = ax^2 + bxy + cy^2 + O(|x|^3 + |y|^3)$, what is the Taylor expansion, up to the quadratic terms, of the function $H(x^{new}, y^{new})$?

(8 marks)

(ii) Using the result of part (i), show that system (1) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = ax^2 + bxy + O(|x|^3 + |y|^3),$$

by a smooth coordinate transformation. (2 marks)

(iii) By induction in n, show that for every $n \geq 2$ system (1) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = P_n(x) + yQ_n(x) + O(|x|^{n+1} + |y|^{n+1}),$$

by a smooth coordinate transformation, where $P_n(x)$ is a polynomial of degree n and $Q_n(x)$ is a polynomial of degree $(n-1)$. (10 marks)

(Total: 20 marks)

Solutions.

1. Consider a system of differential equations in R^4. Consider a system of differential equations in R^4. How many stable equilibria and periodic orbits can be born at the following bifurcations:
(i) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{2} \pm i, -2, 0$ and the Lyapunov coefficients $l_2 = l_3 = l_4 = 0$, $l_5 < 0$?
(ii) An equilibrium state with the eigenvalues of the linearisation matrix equal to $\pm i, -\frac{1}{2} \pm i$ and the Lyapunov coefficients $L_1 = L_2 = L_3 = L_4 = 0$, $L_5 > 0$?
(iii) A periodic orbit with the multipliers $1, -\frac{1}{2} \pm i$ and the Lyapunov coefficient $l_2 < 0$?
(iv) A periodic orbit with the multipliers $-1, -\frac{1}{2}, \frac{1}{2}$ and the Lyapunov coefficients $L_1 = 0$, $L_2 > 0$?

Answers. (5 points each) (i) There can be no periodic orbits, as the centre manifold is 1-dimensional. Since there are no eigenvalues to the right of the imaginary axis, the equilibria are stable if and only if they are stable on the centre manifold. We have $l_5 \neq 0$, so up to 5 equilibria can be born, and three of them can be stable.

(ii) Up to 5 periodic orbits can be born here, 2 of them can be stable.

(iii) There are multipliers outside the unite circle, so no stable periodic orbits can be born.

(iv) We have $L_1 = 0$, $L_2 > 0$, so up to two orbits of double period can be born, only one of them will be stable.

2. Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y + axy - y^2, \end{cases}$$

that depends on parameters ε and a.

(i) Write the normal form, up to the terms of the third order, for this system near the equilibrium $(x, y) = (0, 0)$ at $\varepsilon = 0$. (8 points)

(ii) Deduce that the first Lyapunov coefficient equals to $L = -a/8$. Determine for which values of ε and a the equilibrium state at $(x, y) = (0, 0)$ is asymptotically stable. (6 points)

(iii) How many periodic orbits can exist near $(x, y) = (0, 0)$ for small $\varepsilon > 0$ at $a = 2$? At $a = -2$? At $a = 0$? (6 points)

Answers. (i) Let $z = x - iy$; the system at $\varepsilon = 0$ will take the form

$$\dot{z} = iz + \frac{a}{4}(z + z^*)(z - z^*) - \frac{i}{4}(z - z^*)^2 = iz + \frac{a - i}{4}z^2 + \frac{i}{2}zz^* - \frac{a + i}{4}(z^*)^2.$$

Let

$$z_{new} = z + \frac{1 + ia}{4}z^2 + \frac{1}{2}zz^* - \frac{a + i}{12i}(z^*)^2.$$

We get

$$\dot{z}_{new} - iz_{new} = \dot{z} + \frac{1 + ia}{2}z\dot{z} + \frac{1}{2}(z\dot{z}^* + z^*\dot{z}) - \frac{a + i}{6i}z^*\dot{z}^* - iz + \frac{a - i}{4}z^2 - \frac{i}{2}zz^* + \frac{a + i}{12}(z^*)^2 =$$

$$= \frac{-3a + i(a^2 + 2)}{24}z^2z^* + \ldots,$$

where the dots stand for 4th order terms and higher and the non-resonant third order terms. All the non-resonant third order terms can be killed by further normalising transformations, so the resulting normal form is

$$\dot{z}_{new} = iz_{new} - \frac{3a - i(a^2 + 2)}{24}z_{new}^2 z_{new}^* + \ldots$$

(ii) The first Lyapunov coefficient is, thus, $L = -a/8$. The origin is asymptotically stable at $\varepsilon < 0$ and unstable at $\varepsilon > 0$. The stability at $\varepsilon = 0$ is determined by the sign of L, so the origin is asymptotically stable, at $\varepsilon = 0$, for $a > 0$ and unstable for $a < 0$. At $\varepsilon = 0$, $a = 0$ the system is reversible, so the origin is stable, but not asymptotically stable. When $L \neq 0$, a single periodic orbit is born when $L\varepsilon < 0$. Thus, at $a = -2$ no periodic orbits can exist near the origin for small $\varepsilon > 0$ (as $L > 0$). At $a = 2$, we have one periodic orbit near the origin for small $\varepsilon > 0$. At $a = 0$, we have that the equilibrium is surrounded by a family of periodic orbits at $\varepsilon = 0$ (because the system is reversible), by Hopf theorem, no periodic orbits exists at $\varepsilon > 0$.

3. Consider the following one-dimensional map:

$$x \mapsto \bar{x} = a - 2x^3,$$

where a is a parameter.

(i) Show that the map has no periodic points of period larger than 2. (4 points)

(ii) Study bifurcations of the fixed point of the map. For which values of a is the fixed point stable? (8 points)

(iii) Study bifurcations of points of period 2. Find the set of values of a for which stable orbits of period 2 exist. (8 points)

Answers. (i) $\frac{d\bar{x}}{dx} \leq 0$, so this map is monotonically decreasing. Its second iteration is monotonically increasing, and such maps cannot have periodic points other than fixed points. Therefore the original map can have only points of period 2 (and one fixed point).

(ii) Bifurcations of the fixed point of the monotonically decreasing map correspond to a multiplier equal to (-1). This gives the following system for the bifurcating fixed point:

$$x = a - 2x^3, \qquad -1 = -6x^2.$$

The solutions are $x = \frac{1}{\sqrt{6}}$ at $a = \frac{4}{3\sqrt{6}}$ and $x = -\frac{1}{\sqrt{6}}$ at $a = -\frac{4}{3\sqrt{6}}$. The stability at the bifurcation moment is determined by the Schwartz derivative which equals here $12 - 216x^2 = -24 < 0$, so the fixed point is stable at the critical moments. Thus, the fixed point is stable for $|a| \leq \frac{4}{3\sqrt{6}}$.

(iii) Orbits of period 2 are born when the fixed point loses stability, i.e., when $|a|$ becomes larger than $\frac{4}{3\sqrt{6}}$. The only other bifurcation which may happen corresponds to the multiplier of the orbit of period 2 equal to 1. The equation for that is

$$x_2 = a - 2x_1^3, \qquad x_1 = a - 2x_2^3, \qquad 36x_1^2x_2^2 = 1, \qquad x_1 \neq x_2.$$

This gives us

$$x_2 - x_1 = 2(x_2^3 - x_1^3),$$

or

$$\frac{1}{2} = x_1^2 + x_1x_2 + x_2^2.$$

Along with the condition $x_1x_2 = \pm\frac{1}{6}$, this implies $2(x_1 - x_2)^2 = 1 \mp 1$. Since $x_1 \neq x_2$, we find

$$x_1x_2 = -\frac{1}{6}, \qquad x_1 - x_2 = \pm 1.$$

For certainty, assume $x_1 > x_2$, then $x_1 = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}, x_2 = -\frac{1}{2} \pm \frac{1}{2\sqrt{3}}$. The corresponding values of $a = x_2 + 2x_1^3$ are $a = \pm\frac{17}{18}\sqrt{3}$. The map has a single unstable orbit of period 2: $(x_1, x_2) = (1, -1)$ at $a = 0$. When $|a|$ gets larger than $\frac{4}{3\sqrt{6}}$ a stable orbit of period 2 emerges in the period-doubling bifurcation. These two orbits collide and disappear at $|a| = \frac{17}{18}\sqrt{3}$. Thus, stable orbits of period 2 exist when $\frac{4}{3\sqrt{6}} < |a| < \frac{17}{18}\sqrt{3}$.

4. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω varies near a certain value $\omega_0 \in (0,\pi)$. Let the first Lyapunov coefficient be $L = -1$.

(i) In polar coordinates (r,ϕ), the normal form for this map near the fixed point can be written as

$$\bar{r} = (1+\mu)r\,(1 - r^2 + f(r,\phi,\mu,\omega)), \qquad \bar{\phi} = \omega + \phi + g(r,\phi,\mu,\omega),$$

where $f = O(r^3)$ and $g = O(r^2)$ are smooth functions. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$ and this curve attracts all orbits (except for the fixed point itself) from some neighbourhood of the fixed point independent of μ and ω. It is also known that this curve has an equation $r = h(\phi,\mu,\omega)$ where h is a smooth, positive, 2π-periodic function of ϕ. Show that

$$h = \sqrt{\mu} + O(\mu).$$

(12 points)

(ii) Show that in the (μ,ω)-plane for any N there exist parameter values corresponding to the existence of periodic orbits of period larger than N. (8 points)

Answers (i) It is enough to show that there exists a constant K such that the annulus

$$|r - \sqrt{\mu}| \le K\mu$$

is forward invariant with respect to the map. By assumption, there exists a constant C such that

$$(1+\mu)r\,(1 - r^2 - Cr^3) < \bar{r} < (1+\mu)r\,(1 - r^2 + Cr^3).$$

Thus, the task is to check that for all small μ

$$\bar{r} > r \quad \text{if} \quad r = \sqrt{\mu} - K\mu$$

and

$$\bar{r} < r \quad \text{if} \quad r = \sqrt{\mu} + K\mu.$$

These inequalities can be rewritten as

$$1 < (1+\mu);(1 - (\sqrt{\mu} - K\mu)^2 - C(\sqrt{\mu} - K\mu)^3) = 1 + (2K - C)\mu\sqrt{\mu} + O(\mu^2)$$

and

$$1 > (1+\mu)\,(1 - (\sqrt{\mu} + K\mu)^2 + C(\sqrt{\mu} + K\mu)^3) = 1 - (2K - C)\mu\sqrt{\mu} + O(\mu^2),$$

so we have the required result if we choose $K > C/2$.

(ii) The map on the invariant curve is given by

$$\bar{\phi} = \omega + \phi + O(\mu).$$

At $\mu = 0$ the rotation number equals to $\frac{1}{2\pi}\omega$, so it changes in a certain non-empty interval as ω varies. By continuity, the same holds true for sufficiently small $\mu > 0$, hence there exist values of ω and $\mu > 0$ for which the rotation number equals to p/q with $q > N$ and p co-prime with q. This corresponds to an orbit of period q.

5. (Mastery question). Consider a generic system of differential equations on a plane which has an equilibrium state with a double zero eigenvalue. By bringing the linear part of the system to the Jordan form, one can write the system in the following form

$$\dot{x} = u + f(x,u),$$
$$\dot{u} = g(x,u),$$

where the smooth functions f and g vanish at zero along with their first derivatives. Introducing variable y by the rule

$$y = u + f(x,u)$$

brings the system to the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= h(x,y), \end{aligned} \tag{2}$$

where h is a smooth function vanishing at zero along with its first derivatives.

(i) (8 points) The transformation

$$x^{new} = x + \psi(x,y), \qquad y^{new} = y + \frac{\partial \psi}{\partial x} y + \frac{\partial \psi}{\partial y} h(x,y)$$

keeps system (2) in the same form, i.e., we have

$$\dot{x}^{new} = y^{new},$$
$$\dot{y}^{new} = H(x^{new}, y^{new})$$

for some function H. If $\psi = \alpha x^2 + \beta xy + \gamma y^2$ and $h = ax^2 + bxy + cy^2 + O(|x|^3 + |y|^3)$, what is the Taylor expansion, up to the quadratic terms, of the function $H(x^{new}, y^{new})$?

(ii) Using the result of part (i), show that system (2) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = ax^2 + bxy + O(|x|^3 + |y|^3),$$

by a smooth coordinate transformation. (2 points)

(iii) By induction in n, show that for every $n \geq 2$ system (2) can be brought to the form

$$\dot{x} = y,$$
$$\dot{y} = P_n(x) + yQ_n(x) + O(|x|^{n+1} + |y|^{n+1}),$$

by a smooth coordinate transformation, where $P_n(x)$ is a polynomial of degree n and $Q_n(x)$ is a polynomial of degree $(n-1)$. (10 points)

Answers. (i) We have

$$x^{new} = x + \alpha x^2 + \beta xy + \gamma y^2, \qquad y^{new} = y + 2\alpha xy + \beta y^2 + O(|x|^3 + |y|^3).$$

This gives

$$\dot{y}^{new} = h(x,y) + 2\alpha y^2 + O(|x|^3 + |y|^3),$$

So

$$H(x^{new}, y^{new}) = a(x^{new})^2 + bx^{new}y^{new} + (c + 2\alpha)(y^{new})^2 + O(|x^{new}|^3 + |y^{new}|^3).$$

(ii) Make the transformation described in (i) with $\alpha = -c/2$.

(iii) By induction assumption, the system can be brought to the form

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= P_n(x) + yQ_n(x) + O(|x|^{n+1} + |y|^{n+1}),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= P_{n+1}(x) + yQ_{n+1}(x) + cy^{n+1} + O(|x|^{n+2} + |y|^{n+2}),
\end{aligned}$$

where $P_{n+1}(x)$ is a polynomial of degree $(n+1)$, $Q_{n+1}(x)$ is a polynomial of degree n, and c is a constant. Now, the transformation

$$x^{new} = x - \frac{c}{2}x^2y^{n-1}, \qquad y^{new} = y - \frac{d}{dt}\left(\frac{c}{2}x^2y^{n-1}\right) = y - cxy^n + O(|x|^{n+2} + |y|^{n+2}),$$

brings the system to the desired form.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	**QuestionN**	**Comments for Students**	
PA24	1	the question done well	
PA24	2	the question done well	
PA24	3	this question is more difficult than others	
PA24	4	this question is more difficult than others	
PA24	5	the last part of the question is more difficult than others	