Math40003 Linear Algebra and Groups **Problem Sheet 1**
Seen A Solutions

Section A

A.1. Consider the following two real matrices:

$$A := \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$$

Calculate $A^2 - B^2$ and $(A+B)(A-B)$.

The relevant parts of the notes for this question are Definitions 2.2.1, 2.2.2, and 2.2.3.

Personally, I find calculating multiplication involving 2×2 or 3×3 matrices by hand to be simpler than opening up MATLAB, but if you're not averse to computer calculation then you should use that. Matrix multiplication involves a lot of smaller calculations, so there's plenty of opportunity for arithmetic errors to creep in. The more intermediate steps you write out, the easier it is to find those mistakes when things go wrong.

Here are the calculations used for this question.

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1.1+5.-1 & 1.5+5.1 \\ -1.1+1.-1 & -1.5+1.1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 10 \\ -2 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A+B &= \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B^2 &= \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2.2+3.2 & 2.3+3.4 \\ 2.2+4.2 & 2.3+4.4 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 18 \\ 12 & 22 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A-B &= \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 3 & -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 - B^2 &= \begin{pmatrix} -4 & 10 \\ -2 & -4 \end{pmatrix} - \begin{pmatrix} 10 & 18 \\ 12 & 22 \end{pmatrix} \\ &= \begin{pmatrix} -14 & -8 \\ -14 & -26 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (A+B)(A-B) &= \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -27 & -18 \\ -16 & -13 \end{pmatrix} \end{aligned}$$

A.2. Solve the following system of simultaneous linear equations by finding the augmented matrix and applying row operations.

(a)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 8 \\ 3x_1 - 6x_2 + 2x_3 &= 18 \\ x_3 - 2x_4 &= 5 \\ 2x_1 - 2x_2 + 3x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ 3x_1 - 8x_2 + 2x_3 &= 5 \\ 2x_1 - 5x_2 + x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_3 + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 0 \\ 4x_1 - 3x_2 - x_3 - 7x_4 &= 4 \end{aligned}$$

(d)
$$\begin{aligned} -x_2 + x_3 - 3x_4 &= 0 \\ x_1 + 3x_2 + x_3 - x_4 &= 0 \\ 2x_1 + 5x_2 + 3x_3 - 5x_4 &= 0 \end{aligned}$$

If you're having trouble, you may wish to review Example 2.3.4 from the notes.

If you're feeling particularly thorough, you can opt to rewrite the simultaneous equations so that the variables are aligned in columns. For (a), this would look like:

$$\begin{array}{rrrrcr} 1.x_1+ & -2.x_2+ & 1.x_3+ & -1.x_4 & = & 8 \\ 3.x_1+ & -6.x_2+ & 2.x_3+ & 0.x_4 & = & 18 \\ 0.x_1+ & 0.x_2+ & 1.x_3+ & -2.x_4 & = & 5 \\ 2.x_1+ & -2.x_2+ & 0.x_3+ & 3.x_4 & = & 4 \end{array}$$

It's a little easier to derive the augmented matrix from the equations when they're written out this way, although it's usually unnecessary. It's good for situations where the variables have very complicated names, so it's less obvious what order they come in. I tend to only do this step if I've tried to solve the equations once already, and I need to start over because something has gone so wrong that the first attempt is unfixable.

For work that really matters, e.g. anything assessed, if you have to preform a Gaussian elimination in detail, you should check the result against a calculator. If you find that your result is incorrect, you're left with the difficult task of trying to find where you went wrong. When I'm in this position (e.g. when I'm marking your Gaussian eliminations), I tend to work through them backwards. The later row operations are simpler, as the matrix has already been simplified when they are applied, so it's easier to spot mistakes. On the other hand, when the operations were written out, the mathematician doing the calculation was more tired and had more numbers in their head when they did the later calculations, so the errors are more likely to be towards the end than the beginning.

(a)

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 3 & -6 & 2 & 0 & 18 \\ 0 & 0 & 1 & -2 & 5 \\ 2 & -2 & 0 & 3 & 4 \end{array}\right) \xrightarrow[R4\to R4-2R1]{R2\to R2-3R1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 0 & -1 & 3 & -6 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 2 & -2 & 5 & -12 \end{array}\right)$$

$$\xrightarrow[R2\leftrightarrow R4]{} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 2 & -2 & 5 & -12 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & -1 & 3 & -6 \end{array}\right)$$

$$\xrightarrow[R4\to R4+R3]{R2\to 0.5R2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 2.5 & -6 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R3\to R3+2R4]{} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 2.5 & -6 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R2\to R2+R3-2.5R4]{} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R1\to R1+2R2-R3+R4]{} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Therefore $x_1 = 3$, $x_2 = -\frac{1}{2}$, $x_3 = 3$, and $x_4 = -1$.

(b)

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 3 & -8 & 2 & 5 \\ 2 & -5 & 1 & 1 \end{array}\right) \xrightarrow[R3\to R3-2R1]{R2\to R2-3R1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

At this point, we can see that $x_2 - x_3$ needs to equal to both -1 and -3 for x_2 and x_3 to be part of a solution to the equations. This is impossible, so there is no solution to these equations. If we keep going with the elimination, we'll see another way for us to show that there are no solutions.

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \end{array}\right) \xrightarrow[R3\to R3-R2]{R1\to R1+3R2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

From this row, we can see that $0 = 0x_1 + 0x_2 + 0x_3 = -2$, which also shows that these equations are inconsistent.

(c)

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 2 & -1 & 1 & -3 & 0 \\ 4 & -3 & -1 & -7 & 4 \end{array}\right) \xrightarrow[R3\to R3-4R1]{R2\to R2-2R1} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 5 & -5 & 0 \\ 0 & -3 & 7 & -11 & 4 \end{array}\right)$$

$$\xrightarrow[R3\to R3-3R2]{R2\to -R2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -5 & 5 & 0 \\ 0 & 0 & -8 & 4 & 4 \end{array}\right)$$

$$\xrightarrow[R3\to -0.5R3]{} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -5 & 5 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 \end{array}\right)$$

$$\xrightarrow[R2\to R2+5R3]{R1\to R_1+2R3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2.5 & -2.5 \\ 0 & 0 & 1 & -0.5 & -0.5 \end{array}\right)$$

From the first row, we see that $x_1 = -1$, as well as $x_2 + 2.5x_4 = -2.5$ and $x_3 - 0.5x_4 = -0.5$. If we set $x_4 = \lambda$ for any $\lambda \in \mathbb{R}$, we get that $x_1 = -1$, $x_2 = -2.5(1+\lambda)$, $x_3 = 0.5(\lambda - 1)$, and $x_4 = \lambda$.

(d)

$$\left(\begin{array}{cccc|c} 0 & -1 & 1 & -3 & 0 \\ 1 & 3 & 1 & -1 & 0 \\ 2 & 5 & 3 & -5 & 0 \end{array}\right) \xrightarrow[R1\leftrightarrow R2]{R3\to R3-2R2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & -1 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[R3\to R3-R2]{R2\to -R2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R1\to R1-3R2]{} \left(\begin{array}{cccc|c} 1 & 0 & 4 & -10 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So there are infinitely many solutions. Let λ and μ be real numbers, and we'll set $x_3 = \lambda$ and $x_4 = \mu$. Then $x_2 = \lambda - 3\mu$ and $x_1 = 10\mu - 4\lambda$.

A.3. Which of these matrices A is invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

If you're having trouble, you may wish to review Example 2.3.4 from the notes.

There are lots of way of solving this question. You can find an inverse through trial and error, e.g.

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} -9 & 7 \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This only works for very simple matrices, such as 2×2 matrices. Preforming Gaussian elimination on the augmented matrix is a much more reliable method.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ -1 & 12 & -7 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3\to R3+R1]{R2\to R2-2R2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -7 & 4 & -2 & 1 & 0 \\ 0 & 14 & -8 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R3\to R3+2R2]{} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -7 & 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 2 & 1 \end{array}\right)$$

At this point, we can see that the matrix is not invertible, as we will never be able to get the identity matrix on the right hand side.

For matrices with parameters in them, we're still able to perform Gaussian elimination, as long as we're willing to multiply rows by numbers that depend on that parameter. We need to be careful not to divide by zero.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ a & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R2\to R2-aR1]{} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1-a & -1 & -a & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

At this point, I'd like to divide by $1-a$, so I need to assume that $1-a \neq 0$.

$$\begin{aligned} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1-a & -1 & -a & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) &\xrightarrow[R2\to\frac{1}{1-a}R2]{} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{1-a} & \frac{-a}{1-a} & \frac{1}{1-a} & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \\ &\xrightarrow[R3\to R3+R2]{R1\to R1-R2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{1-a} & 1+\frac{a}{1-a} & \frac{-1}{1-a} & 0 \\ 0 & 1 & \frac{-1}{1-a} & \frac{-a}{1-a} & \frac{1}{1-a} & 0 \\ 0 & 0 & 2-\frac{1}{1-a} & \frac{-a}{1-a} & \frac{1}{1-a} & 1 \end{array}\right) \end{aligned}$$

If $2-\frac{1}{1-a}$ is non-zero, then we'll be able to finish the elimination, and the matrix will be invertible. If $2-\frac{1}{1-a}=0$ then there will not be an inverse.

We now need to go back and handle the case where $1-a=0$.

$$\begin{aligned} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1-a & -1 & -a & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) &= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \\ &\xrightarrow[R2\leftrightarrow R3]{} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \end{aligned}$$

We'll be able to complete the Gaussian elimination here.

A.4. Prove Theorem 2.4.4. The key definition is 2.4.1.

Theorem 2.4.4:

Let A be an $m \times n$ matrix, and let E be an elementary $m \times m$ matrix. The matrix multiplication EA applies the same elementary row operation on A that was preformed on I to obtain E.

Proof:

Let $A = (a_{ij})$ and let $E = (e_{ij})$. Then

$$\begin{aligned} EA &= (e_{ij})(a_{ij}) \\ &= (\textstyle\sum_{k=1}^{m} e_{ik}a_{kj}) \end{aligned}$$

According to Definition 2.4.1, there are three types of elementary matrix, so if we investigate these three types separately, we will have a greater handle on what E actually is.

First, let's assume that $E = E_r(\alpha)$. Then the elementary operation that was preformed on I to obtain $E_r(\alpha)$ was $R_r \mapsto \alpha R_r$, and e_{ij} equals 0 if $i \neq j$, equals 1 if $i = j \neq r$, and equals α if $i = r = j$. Then

$$\sum_{k=1}^{m} e_{ik}a_{kj} = \begin{array}{rl} a_{ij} & i \neq r \\ \alpha a_{ij} & i = r \end{array}$$

And therefore EA is the matrix obtained by multiplying the r row of A by α.

Second, let's assume that $E = E_{rs}(\alpha)$. Then the elementary operation that was preformed on I to obtain $E_{rs}(\alpha)$ was $R_r \mapsto R_r + \alpha R_s$, and e_{ij} equals 1 if $i = j$, equals α if $i = r$ and $j = s$, and equals 0 otherwise. Then

$$\sum_{k=1}^{m} e_{ik}a_{kj} = \begin{array}{rl} a_{ij} & i \neq r \\ a_{ij} + \alpha a_{sj} & i = r \end{array}$$

And therefore EA is the matrix obtained by adding αR_s to the r row of A.

Third, and finally, let's assume that $E = E_{rs}$. Then the elementary operation that was preformed on I to obtain E_{ij} is swapping rows r and s, and e_{ij} equals 1 if $s \neq i = j \neq r$, equals 1 if $i = r$ and $j = s$, equals 1 if $i = s$ and $j = r$, and equals 0 otherwise. Then

$$\sum_{k=1}^{m} e_{ik}a_{kj} = \begin{array}{ll} a_{ij} & i \neq r, s \\ a_{sj} & i = r \\ a_{rj} & i = s \end{array}$$

And therefore EA is the matrix obtained by swapping R_s and R_r.

A.5. Exercise 2.5.9. Theorem 2.5.8 will be useful here.

(a)

$$\begin{aligned} A^{-1}B^{-1}(BA) &= A^{-1}(B^{-1}B)A \\ &= A^{-1}IA \\ &= A^{-1}A \\ &= I \end{aligned}$$

(b)

$$\begin{aligned} B^{-1}A^{-1}(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I \end{aligned}$$

(c)

$$BAA^{-1}B^{-1}B^{-1}A^{-1}(ABBAA^{-1}B^{-1}) = I$$

Since inverses are unique (Theorem 2.5.8.), we have found the inverses of all these matrices.