Applied Complex Analysis - Lecture Seven

Andrew Gibbs

January 2025

Using contour deformation to evaluate

$$\int_{-\infty}^{\infty} f(z)dz,$$

where f has poles.

Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx, \quad 0 < a < 1.$$

Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx, \quad 0 < a < 1.$$

Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1.$$

Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1.$$

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

Analytic Continuation

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Complex ∞

Complex ∞

The *Riemann Sphere* is the compactification of $\mathbb{C}$:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

- Without delving on the details, we can define an open set $D \subset \bar{\mathbb{C}}$ on the Riemann sphere, where $\infty \in D$ implies that there exists an R such that $\{z : |z| > R\} \subset D$.
- A function $f(z)$ defined on an open set $D \subset \bar{\mathbb{C}}$ such that $\infty \in D$ is *analytic at* ∞ if $f(z^{-1})$ is analytic at zero.
- A version of Cauchy's integral theorem exists for functions analytic in $D \subset \overline{\mathbb{C}}$ with $\infty \in D$, but we will not need it.

Complex ∞

The *Riemann Sphere* is the compactification of $\mathbb{C}$:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

- Without delving on the details, we can define an open set $D \subset \bar{\mathbb{C}}$ on the Riemann sphere, where $\infty \in D$ implies that there exists an R such that $\{z : |z| > R\} \subset D$.
- A function $f(z)$ defined on an open set $D \subset \bar{\mathbb{C}}$ such that $\infty \in D$ is *analytic at* ∞ if $f(z^{-1})$ is analytic at zero.
- A version of Cauchy's integral theorem exists for functions analytic in $D \subset \overline{\mathbb{C}}$ with $\infty \in D$, but we will not need it.

Complex ∞

The *Riemann Sphere* is the compactification of $\mathbb{C}$:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

- Without delving on the details, we can define an open set $D \subset \bar{\mathbb{C}}$ on the Riemann sphere, where $\infty \in D$ implies that there exists an R such that $\{z : |z| > R\} \subset D$.
- A function $f(z)$ defined on an open set $D \subset \bar{\mathbb{C}}$ such that $\infty \in D$ is *analytic at* ∞ if $f(z^{-1})$ is analytic at zero.
- A version of Cauchy's integral theorem exists for functions analytic in $D \subset \overline{\mathbb{C}}$ with $\infty \in D$, but we will not need it.

Complex ∞

The *Riemann Sphere* is the compactification of $\mathbb{C}$:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

- Without delving on the details, we can define an open set $D \subset \bar{\mathbb{C}}$ on the Riemann sphere, where $\infty \in D$ implies that there exists an R such that $\{z : |z| > R\} \subset D$.
- A function $f(z)$ defined on an open set $D \subset \bar{\mathbb{C}}$ such that $\infty \in D$ is *analytic at* ∞ if $f(z^{-1})$ is analytic at zero.
- A version of Cauchy's integral theorem exists for functions analytic in $D \subset \overline{\mathbb{C}}$ with $\infty \in D$, but we will not need it.

The residue at $z = \infty$

If f is analytic at $z = \infty$ and $f(\infty) = 0$, then

$$f(z) = \sum_{n=-\infty}^{-1} \frac{a_n}{z^n}$$

By similar arguments to before, we can define

$$\mathrm{Res}(f, \infty) := a_{-1} = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz.$$

This provides an alternative approach to the earlier example!

The residue at $z = \infty$

If f is analytic at $z = \infty$ and $f(\infty) = 0$, then

$$f(z) = \sum_{n=-\infty}^{-1} \frac{a_n}{z^n}$$

By similar arguments to before, we can define

$$\mathrm{Res}(f, \infty) := a_{-1} = \lim_{R \to \infty} \int_{\gamma_R} f(z)dz.$$

This provides an alternative approach to the earlier example!

The residue at $z = \infty$

If f is analytic at $z = \infty$ and $f(\infty) = 0$, then

$$f(z) = \sum_{n=-\infty}^{-1} \frac{a_n}{z^n}$$

By similar arguments to before, we can define

$$\mathrm{Res}(f, \infty) := a_{-1} = \lim_{R\to\infty} \int_{\gamma_R} f(z) dz.$$

This provides an alternative approach to the earlier example!

Branch points and branch cuts

Branch points and branch cuts

A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., analytically continuing along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

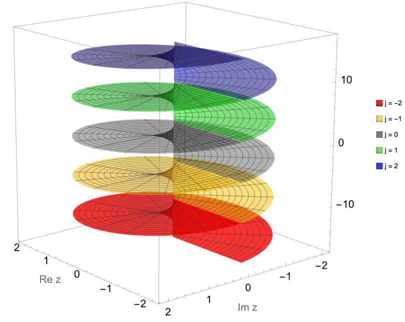
A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

Branch points and branch cuts

A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., analytically continuing along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

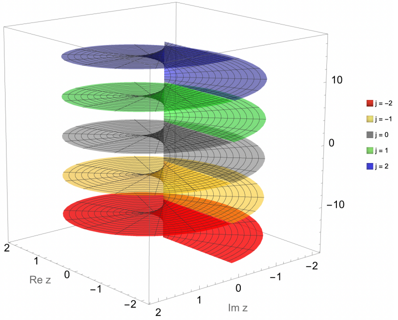
A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

Branch cut example: Complex logarithm



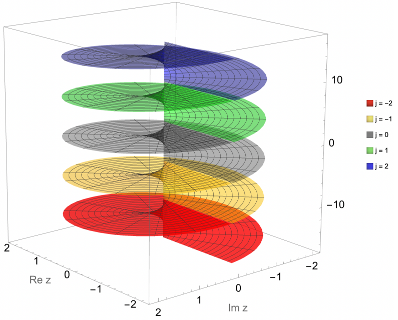
- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

Branch cut example: Complex logarithm



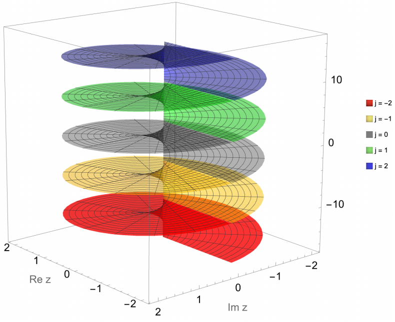
- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

Branch cut example: Complex logarithm



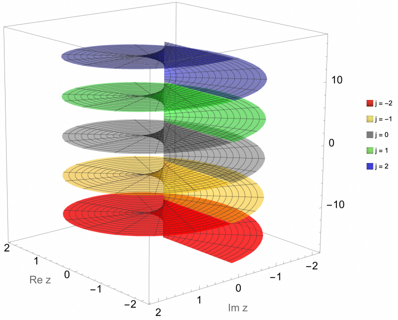
- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

Branch cut example: Complex logarithm



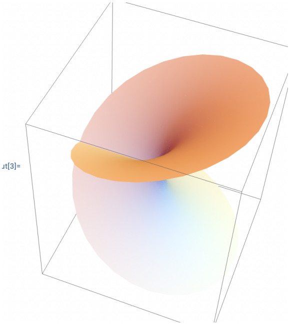
- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

Branch cut example: Complex logarithm



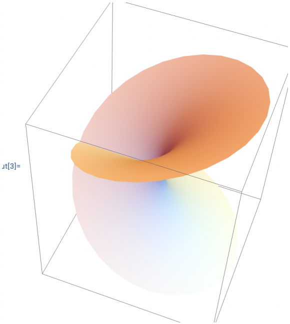
- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

Branch cut example: Square root



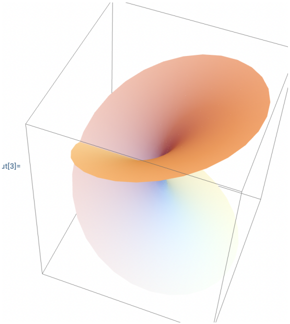
- Traversing a full circuit around 0 gives a different value
- Note that rotating by 4π brings us home - two branches
- Another branch point at complex infinity
- These are the familiar $\pm\sqrt{x}$
- Visualisation

Branch cut example: Square root



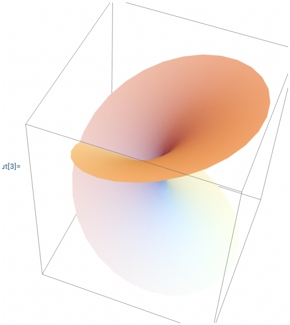
- Traversing a full circuit around 0 gives a different value
- Note that rotating by 4π brings us home - two branches
- Another branch point at complex infinity
- These are the familiar $\pm\sqrt{x}$
- Visualisation

Branch cut example: Square root



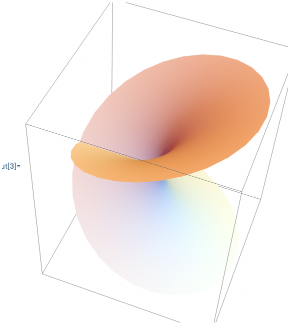
- Traversing a full circuit around 0 gives a different value
- Note that rotating by 4π brings us home - two branches
- Another branch point at complex infinity
- These are the familiar $\pm\sqrt{x}$
- Visualisation

Branch cut example: Square root



- Traversing a full circuit around 0 gives a different value
- Note that rotating by 4π brings us home - two branches
- Another branch point at complex infinity
- These are the familiar $\pm\sqrt{x}$
- Visualisation

Branch cut example: Square root



- Traversing a full circuit around 0 gives a different value
- Note that rotating by 4π brings us home - two branches
- Another branch point at complex infinity
- These are the familiar $\pm\sqrt{x}$
- Visualisation

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1+\theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i \frac{\theta_1 + \theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i \frac{\theta_1 + \theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1 + \theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1+\theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1+\theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

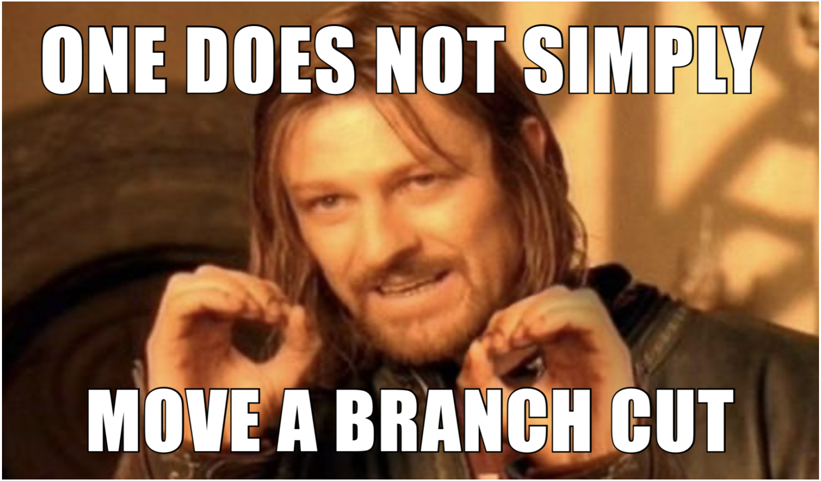
- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Moving branch cuts in mathematical software



Example problems

- $$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \alpha \in (0,1)$$

- $$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

Example problems

- $$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \alpha \in (0,1)$$
- $$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

Principal value integrals

- We say integral is *weakly singular* if $f(z) = O(|z|^\alpha)$ for some $\alpha > -1$ as $z \to 0$.
- Such integrals are absolutely convergent, similar in many ways to integrals of smooth functions.
- When $\alpha = -1$ (or worse), integrals are not absolutely convergent. But they may converge in a different sense.

$$\fint_a^b f(x)dx = \lim_{\epsilon \to 0+} \left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right).$$

Principal value integrals

- We say integral is *weakly singular* if $f(z) = O(|z|^\alpha)$ for some $\alpha > -1$ as $z \to 0$.
- Such integrals are absolutely convergent, similar in many ways to integrals of smooth functions.
- When $\alpha = -1$ (or worse), integrals are not absolutely convergent. But they may converge in a different sense.

$$\fint_a^b f(x)dx = \lim_{\epsilon \to 0+} \left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right).$$

Principal value integrals

- We say integral is *weakly singular* if $f(z) = O(|z|^{\alpha})$ for some $\alpha > -1$ as $z \to 0$.
- Such integrals are absolutely convergent, similar in many ways to integrals of smooth functions.
- When $\alpha = -1$ (or worse), integrals are not absolutely convergent. But they may converge in a different sense.

$$\fint_a^b f(x)dx = \lim_{\epsilon \to 0+} \left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right).$$

Examples

- $$\int_{-1}^{1} \frac{1}{x} dx$$
- $$\int_{-1}^{1} \frac{x^{\alpha-1}}{1-x}, \quad \alpha \in (0,1)$$