

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebraic Topology

Date: 01 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD
WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS
ANSWERED AND PAGE NUMBERS PER QUESTION.**

Mathematical statements of an algebraic nature can be used without any justification (provided they are correct).

1. (a) Recall that the 3-sphere S^3 is the topological space defined by

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Let $n \geq 1$ be a positive integer and $p < n$ and $q < n$ two positive integers. Show that

$$k \longmapsto \left((z_1, z_2) \mapsto (e^{2\pi i \frac{kp}{n}} z_1, e^{2\pi i \frac{kq}{n}} z_2) \right)$$

defines an action of $\mathbb{Z}/n\mathbb{Z}$ on S^3 . (5 marks)

- (b) Show that if $\gcd(p, n) = \gcd(q, n) = 1$, this action is nice. (6 marks)
- (c) Let $\mathcal{L}_n(p, q)$ be the quotient of S^3 by the action thus defined, when $\gcd(p, n) = \gcd(q, n) = 1$. Assume that $\mathcal{L}_n(p, q)$ and $\mathcal{L}_{n'}(p', q')$ are homeomorphic. Show that $n = n'$. (5 marks)
- (d) (Recall that by definition of $\mathcal{L}_n(p, q)$, $\gcd(p, n) = \gcd(q, n) = 1$) Show that any $\mathcal{L}_n(p, q)$ is homeomorphic to $\mathcal{L}_n(1, l)$ for some $l > 0$. (4 marks)

(Total: 20 marks)

2. (a) Let $S = [0, 1] \times [0, 1]$ and let \sim the equivalence relation defined by $(x, 0) \sim (x, 1)$ for any $x \in [0, 1]$, $(0, y) \sim (1, 1 - y)$ for $y \in [0, 1]$. Let $\text{KB} := S/\sim$ be the quotient of S by this equivalence relation. Show that KB is a compact space. (6 marks)

- (b) Let $\tilde{p} = (\frac{1}{2}, \frac{1}{2}) \in S$ and let p be its projection in KB . Let \tilde{B} the ball of radius $\frac{1}{10}$ in S centred at \tilde{p} and B its projection in KB . Show that $\text{KB} \setminus B$ deformation retracts onto a subset homeomorphic to the wedge of two circle $S^1 \vee S^1$. (8 marks)
- (c) Give a presentation of $\pi_1(\text{KB}, x_0)$ for a point x_0 of your choosing. (6 marks)

(Total: 20 marks)

3. Recall that

$$S^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\}$$

$$\mathbb{D}_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\}$$

$$\mathbb{RP}^n := S^n /_{x \sim -x}.$$

For this question, proofs of the continuity of maps that you might define to establish homeomorphisms between various spaces can be omitted.

- (a) Let \sim be the equivalence relation on \mathbb{D}_n defined by $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ if and only if $(x_1, \dots, x_n) = (y_1, \dots, y_n)$ or $\sum_i x_i^2 = \sum_i y_i^2 = 1$. Show that $\mathbb{D}_n /_{\sim}$ is homeomorphic to S^n .

Hint : You may consider a map of the form

$$x = (x_1, \dots, x_n) \mapsto (1 - 2||x||, a(x)x_1, \dots, a(x)x_n)$$

for a well-chosen function a .

(8 marks)

- (b) Let \sim' be the equivalence relation on \mathbb{D}_n defined by $(x_1, \dots, x_n) \sim' (y_1, \dots, y_n)$ if and only if $(x_1, \dots, x_n) = (y_1, \dots, y_n)$ or $\sum_i x_i^2 = \sum_i y_i^2 = 1$ and $(x_1, \dots, x_n) = -(y_1, \dots, y_n)$. Show that $\mathbb{D}_n /_{\sim'}$ is homeomorphic to \mathbb{RP}^n .

Hint : You may consider a natural identification between \mathbb{D}_n and the northern hemisphere $\{x \in S^n \mid x_0 \geq 0\}$. (6 marks)

- (c) The Moebius strip M is the quotient of $[0, 1] \times [0, 1]$ by the equivalence relation where $(0, x)$ is associated to $(1, 1 - x)$.

- (i) Show that the image of $\{0\} \times [0, 1] \cup \{1\} \times [0, 1]$ is homeomorphic to S^1 .

- (ii) We consider A the copy of $\mathbb{RP}^1 \subset \mathbb{RP}^2$ defined by the projection of $S^1 := \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\} \subset S^2$ in \mathbb{RP}^2 . Show that a neighbourhood of $A \subset \mathbb{RP}^2$ is homeomorphic to the Moebius strip M

(6 marks)

(Total: 20 marks)

4. (a) Let P be a regular octagon in \mathbb{R}^2 with consecutive sides labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}$, following an arbitrarily chosen orientation of the boundary of the octagon. Σ the surface of genus 2 is the surface obtained by gluing a_i to a_i^{-1} and b_i to b_i^{-1} ($i = 1, 2$), linearly but reversing the orientation of ∂P_2 . Put a Δ -complex structure on Σ (a schematic picture together with a list of the simplices in each dimension will suffice). (8 marks)

(b) Give presentations of $\mathcal{C}_0^\Delta(\Sigma)$, $\mathcal{C}_1^\Delta(\Sigma)$ and $\mathcal{C}_2^\Delta(\Sigma)$.

(5 marks)

(c) Compute the homology groups of Σ .

(7 marks)

(Total: 20 marks)

5. You can freely use the following facts.

1. If $H \simeq \mathbb{Z}^d$ is a quotient of \mathbb{Z}^n by a subgroup Γ , $\text{rk}(\mathbb{Z}^n) = \text{rk}(H) + \text{rk}(\Gamma)$.
2. If $f : \mathbb{Z}^d \longrightarrow \mathbb{Z}^n$ is a group homomorphism, $\text{rk}(\mathbb{Z}^d) = \text{rk}(\text{Ker}(f)) + \text{rk}(\text{Im}(f))$.
3. Let Σ_g be the genus g surface, we have $H_0(\Sigma_g) = \mathbb{Z}$, $H_1(\Sigma_g) = \mathbb{Z}^{2g}$ and $H_2(\Sigma_g) = \mathbb{Z}$.

- (a) Let $C_0^\Delta(\Sigma_g)$, $C_1^\Delta(\Sigma_g)$ and $C_2^\Delta(\Sigma_g)$ be the chain groups associated with a Δ -complex structure on Σ_g . Show that

$$\text{rk}\left(C_0^\Delta(\Sigma_g)\right) - \text{rk}\left(C_1^\Delta(\Sigma_g)\right) + \text{rk}\left(C_2^\Delta(\Sigma_g)\right) = \text{rk}\left(H_0(\Sigma_g)\right) - \text{rk}\left(H_1(\Sigma_g)\right) + \text{rk}\left(H_2(\Sigma_g)\right)$$

where for an abelian group Γ isomorphic to \mathbb{Z}^d , $\text{rk}(\Gamma) = d$.

(6 marks)

- (b) Let $p : \Sigma_{g'} \longrightarrow \Sigma_g$ be a covering map of degree d . Show that any map $\varphi : \Delta_k \longrightarrow \Sigma_g$ that is injective on the interior of Δ_k lifts to d distinct maps $\tilde{\varphi} : \Delta_k \longrightarrow \Sigma_{g'}$ which are injective on the interior of Δ_k . Show furthermore that these lifts, restricted to the interior of Δ_k , have pairwise disjoint images. (7 marks)

- (c) Show that we have

$$g' = d(g - 1) + 1.$$

(7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60034

Algebraic Topology (Solutions)

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1. (a) **(unseen)**

Let

$$\varphi_k := (z_1, z_2) \mapsto (e^{2\pi i \frac{kp}{n}} z_1, e^{2\pi i \frac{kq}{n}} z_2).$$

Since $|e^{2\pi i \frac{kp}{n}} z_1|^2 + |e^{2\pi i \frac{kq}{n}} z_2|^2 = |z_1|^2 + |z_2|^2 = 1$, φ_k maps S^3 onto S^3 .

We have

$$\varphi_k \circ \varphi_{k'}(z_1, z_2) = (e^{2\pi i \frac{kp}{n}} e^{2\pi i \frac{k'p}{n}} z_1, e^{2\pi i \frac{kq}{n}} e^{2\pi i \frac{k'q}{n}} z_2) = (e^{2\pi i \frac{(k+k')p}{n}} z_1, e^{2\pi i \frac{(k+k')q}{n}} z_2)$$

which shows

$$\varphi_k \circ \varphi_{k'} = \varphi_{k+k'}.$$

Furthermore, $\varphi_0 := \text{Id}$ which implies that

$$A : k \mapsto \varphi_k$$

defines a group homomorphism $\mathbb{Z} \rightarrow \text{Homeo}(S^3)$. Finally, since $e^{2\pi i \frac{kq}{n}} = 1$ for $k \in n\mathbb{Z}$, $n\mathbb{Z} < \text{Ker}(A)$. A induces thus a group homomorphism

$$\alpha : \mathbb{Z}/n\mathbb{Z} \longrightarrow \text{Homeo}(S^3)$$

which is, by definition, an action of $\mathbb{Z}/n\mathbb{Z}$ on S^3 .

(b) **(unseen)**

For any $z = (z_1, z_2)$, let U_z be the ball of radius ϵ centred at z (for the Euclidean distance) intersected with S^3 for an ϵ to be specified. The map

$$\varphi_k := (z_1, z_2) \mapsto (e^{2\pi i \frac{kp}{n}} z_1, e^{2\pi i \frac{kq}{n}} z_2)$$

preserves the Euclidean distance. Thus the image of U_z by φ_k is the ball of radius ϵ centred at $\varphi_k(z)$ intersected with S^3 .

If we have $\gcd(p, n) = \gcd(q, n) = 1$, for any $k \notin n\mathbb{Z}$, both $e^{2\pi i \frac{kp}{n}}$ and $e^{2\pi i \frac{kq}{n}}$ are different from 1. For $z \in S^3$ at least z_1 or z_2 is not 0, thus $\varphi_k(z) \neq z$. If $\epsilon < \min_{1 \leq k < n} \frac{\|\varphi_k(z) - z\|}{2}$, we have $\varphi_k(U_z) \cap U_z = \emptyset$. The action of $\mathbb{Z}/n\mathbb{Z}$ is nice.

(c) **(unseen)**

Recall that S^3 is simply-connected. $\mathcal{L}_n(p, q)$ is the quotient of a simply-connected space by a nice action of $\mathbb{Z}/n\mathbb{Z}$, its fundamental based at any point is thus isomorphic to $\mathbb{Z}/n\mathbb{Z}$. But since two spaces that are homeomorphic have isomorphic fundamental groups, and that $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n'\mathbb{Z}$ if and only $n = n'$ (for positive integers), if $\mathcal{L}_n(p, q)$ and $\mathcal{L}_{n'}(p', q')$ are isomorphic, n must be equal to n' .

(d) **(unseen)** Since $\gcd(p, n) = 1$, there exists $k > 0$ such that $pk = 1 + mn$ for some $m > 0$ (Bézout theorem). Thus φ_k is of the form

$$(z_1, z_2) \mapsto (e^{2\pi i \frac{1}{n}} z_1, e^{2\pi i \frac{kq}{n}} z_2).$$

We set $l = kq$. φ_k generates the same group as φ_1 , so the quotients of S^3 by the actions of $\langle \varphi_1 \rangle$ and $\langle \varphi_k \rangle$ are the same space. By definition, the first one is $\mathcal{L}(p, q)$ and the second is $\mathcal{L}_n(1, l)$, which terminates the proof.

2. (a) **(unseen)**

Let $\mathcal{U} = \bigcup_{i \in I} U_i$ a cover of KB by open sets. Let $\pi : S \rightarrow \text{KB}$ the natural projection. By definition of the quotient topology, for all $i \in I$, $\pi^{-1}(U_i)$ is open, thus $\bigcup_{i \in I} \pi^{-1}(U_i)$ is open covering of S . Since S is compact, there is a finite collection $\pi^{-1}(U_{i_1}), \dots, \pi^{-1}(U_{i_k})$ that covers S . Since π is surjective, the $\pi(\pi^{-1}(U_{i_j})) = U_{i_j}$'s cover KB. We have thus extracted a finite subcover of \mathcal{U} which covers KB. This proves that KB is compact.

(b) **(seen)**

First note that the boundary of S maps to KB to a subset homeomorphic to $S^1 \vee S^1$. For any $p \in S \setminus B$, consider the line through $(\frac{1}{2}, \frac{1}{2})$ passing through p . It does intersect the boundary of S in two points. Let $\varphi(p)$ be the one of these two points such that $[p, \varphi(p)]$ does not intersect B . Note that

- * φ is continuous;
- * φ restricted to the boundary of S is the identity.

We therefore define $H(t, p) = (1 - t)p + t\varphi(p)$. $H(t, \cdot)$ is the identity for any t restricted to the boundary of S . H therefore induces a deformation retract of $\text{KB} \setminus \pi(B)$ onto $S^1 \vee S^1$.

(c) **(method seen)**

Let x_0 the projection of the point $(0, 0)$ in KB. Let C a neighbourhood of $\pi(B)$ in KB as represented in the picture below. Let $A = \text{KB} \setminus \pi(B)$. We have the following facts.

- * $\pi_1(A, x_0) = \pi_1(S^1 \vee S^1, x_0)$ as A deformation retracts onto $S^1 \vee S^1$. Thus $\pi_1(A, x_0)$ is the free group generated by a and b .
- * $A \cap C$ is path-connected and $\pi_1(A \cap C, x_0)$ is isomorphic to \mathbb{Z} as $A \cap C$ is homeomorphic to $S^1 \times (0, 1)$.
- * $A \cup C = \text{KB}$.

The generator of δ of $\pi_1(A \cap C, x_0)$ as in the picture is homotopic to $abab^{-1}$ in A . Van Kampen theorem this gives

$$\pi_1(\text{KB}) = \langle a, b \mid abab^{-1} = 1 \rangle.$$

3. (a) **(method seen)**

Consider the map

$$x = (x_1, \dots, x_n) \mapsto (1 - 2\|x\|, 2\sqrt{(\frac{1}{\|x\|} - 1)}x_1, \dots, 2\sqrt{(\frac{1}{\|x\|} - 1)}x_n).$$

It extends by continuity to $x = 0$ to $(1, 0, \dots, 0)$ and maps \mathbb{D}_n onto S^n and maps $\partial\mathbb{D}_n$ onto $(-1, 0, \dots, 0)$. It thus induces a continuous map $\mathbb{D}_n/\partial\mathbb{D}_n$ that is bijective. Since $\mathbb{D}_n/\partial\mathbb{D}_n$ is compact (as the quotient of a compact space is compact) and S^n is Hausdorff, it is a homeomorphism.

(b) **(method seen)** Consider the map

$$\begin{aligned} \mathbb{D}_n &\longrightarrow S^n \\ x &\mapsto (1 - \|x\|^2, (\sqrt{\frac{2}{\|x\|}} - 1)x) \end{aligned}$$

Post composing by the projection $S^n \rightarrow \mathbb{RP}^n$, it defines a continuous map

$$\varphi : \mathbb{D}_n \longrightarrow \mathbb{RP}^n$$

for which $\varphi(x) = \varphi(y)$ if and only if $x = y$ or $\|x\| = \|y\| = 1$ and $x = -y$. It thus induces a continuous bijection

$$\psi : \mathbb{D}_n/\sim' \longrightarrow \mathbb{RP}^n$$

which by the same arguments as the previous question is a homeomorphism.

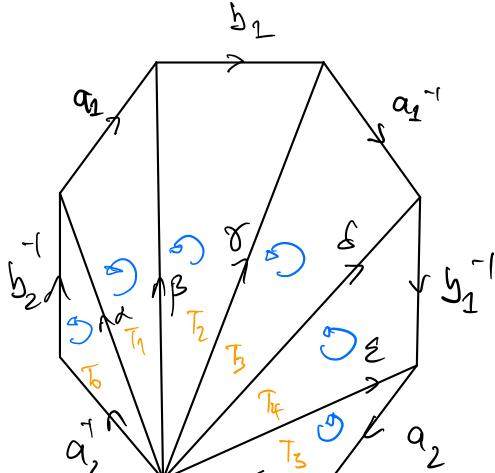
(c) **(method seen)** The map $f : [0, 2] \rightarrow S$ defined by $x \mapsto (0, x)$ for $x \leq 1$ and $x \mapsto (1, x-1)$ for $x-1$ induces a continuous map $g : [0, 2] \rightarrow S$ when projected onto M . Since $g(0) = g(2)$, g induces a continuous, bijective map $S^1 = [0, 2]/_{0 \sim 2} \rightarrow M$ which by the same argument as in the two previous questions is a homeomorphism.

(d) **(unseen)** Consider the following subspace Q in S^2 : $\{(x, y, z) \in S^2 \mid |z| \leq \epsilon, x \geq 0\}$ for ϵ sufficiently small. We note the two following points :

- * Q is isomorphic to $[0, 1]$;
- * the projection of Q in \mathbb{RP}^2 maps homeomorphically to a Moebius band M .

The boundary circle of this Moebius band coincides with the projection of the circle defined in S^2 by the equation $z = \epsilon$. The complement in \mathbb{RP}^2 of M is homeomorphic to the subset of S^2 defined by $z \geq \epsilon$ is homeomorphic to \mathbb{D}_2 , and \mathbb{RP}^2 is thus homeomorphic to \mathbb{D}_2 glued along $\{z = \epsilon\}$ to M along its boundary circle.

4. (a) (seen)



(b) (method seen)

$$\mathcal{C}_2^\Delta(\Sigma) = \bigoplus_{i=0}^5 \mathbb{Z} \cdot [T_i]$$

$$\mathcal{C}_1^\Delta(\Sigma) = \bigoplus_{i=1,1} \mathbb{Z} \cdot [a_i] \oplus \mathbb{Z} \cdot [b_i] \oplus \mathbb{Z} \cdot [\alpha] \oplus \mathbb{Z} \cdot [\beta] \oplus \mathbb{Z} \cdot [\gamma] \oplus \mathbb{Z} \cdot [\delta] \oplus \mathbb{Z} \cdot [\epsilon]$$

$$\mathcal{C}_0^\Delta(\Sigma) = \mathbb{Z} \cdot [p]$$

(c) (method seen) We have

- * $\partial_2[T_0] = [a] + [a_2] + [b_2];$
- * $\partial_2[T_1] = [\beta] - [a_1] - [\alpha];$
- * $\partial_2[T_2] = [\gamma] - [b_1] - [\beta];$
- * $\partial_2[T_3] = [\delta] + [a_1] - [\gamma];$
- * $\partial_2[T_4] = [\epsilon] + [b_1] - [\delta];$
- * $\partial_2[T_5] = -[a_2] - [b_2] - [\epsilon].$

Furthermore, since all the 1-simplices start and end in p , $\partial_1 \equiv 0$. We get from these calculations that

- * $\text{Ker}(\delta_2) = \mathbb{Z} \cdot (\sum_{i=0}^5 [T_i]);$
- * $\text{Ker}(\delta_1) = \mathcal{C}_1^\Delta(\Sigma).$

In view of the relations above, we get that the projection of the subgroup of $\mathcal{C}_1^\Delta(\Sigma)$ generated by $[a_1], [a_2], [b_1], [b_2]$ is an isomorphism onto $\mathcal{C}_1^\Delta(\Sigma)/\text{Im}(\partial_2) = H_1(\Sigma)$. Since $H_2(\Sigma) = \text{Ker}(\partial_2)$ and $H_2(\Sigma) = \text{Ker}(\partial_0) = \mathcal{C}_0^\Delta(\Sigma)$ we obtain

- * $H_2(\Sigma) = \mathbb{Z};$
- * $H_1(\Sigma) = \mathbb{Z}^4;$
- * $H_0(\Sigma) = \mathbb{Z}.$

5. (a) **(unseen)**

Use that

$$\text{rk}(\mathcal{C}_i^\Delta(\Sigma_g)) = \text{rk}(\text{Ker}(\partial_i)) + \text{rk}(\text{Im}(\partial_i))$$

and the fact that since $H_i(\Sigma_g) = \text{Ker}(\partial_i)/\text{Im}(\partial_{i+1})$

$$\text{rk}(\text{Ker}(\partial_i)) = \text{rk}(\text{Im}(\partial_{i+1})) + \text{rk}(H_i(\Sigma_g)).$$

Putting this equality together for $i = 0, 1$ and 2 yields the desired equality.

- (b) **(method seen)** Fix an arbitrary point x_0 in Δ_k . For y_0 any lift of $\varphi(x_0)$ to $\Sigma_{g'}$, there is a lift $\tilde{\varphi} : \Delta_k \longrightarrow \Sigma_{g'}$.

Such a lift is injective on the interior of Δ_k , for otherwise if there is $x \neq z$ such that $\tilde{\varphi}(x) = \tilde{\varphi}(z)$, $\varphi(x) = \varphi(z)$ contradicting the injectivity of φ . Take two different such lifts $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, we claim that the images of their interiors do not intersect. Otherwise, by injectivity of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, there exists x such that $\tilde{\varphi}_1(x) = \tilde{\varphi}_2(x)$. By uniqueness of the lift mapping x to a certain point, $\tilde{\varphi}_1 = \tilde{\varphi}_2$.

There are as many such lifts as there are pre-images of x_0 by p , that is by definition d the degree of p .

- (c) **(unseen)** By the previous question, lifts of maps φ defining the Δ -complex structure of Σ_g define a Δ -structure on Σ'_g . This Δ -complex structure has exactly d times the number of k -simplices of that Σ_g for all k . In particular,

$$\text{rk}(\mathcal{C}_k^\Delta(\Sigma_{g'})) = d \cdot \text{rk}(\mathcal{C}_k^\Delta(\Sigma_g)).$$

Thus

$$\text{rk}(C_0^\Delta(\Sigma_g)) - \text{rk}(C_1^\Delta(\Sigma_g)) + \text{rk}(C_2^\Delta(\Sigma_g)) = \text{rk}(H_0(\Sigma_g)) - \text{rk}(H_1(\Sigma_g)) + \text{rk}(H_2(\Sigma_g))$$

$$\text{rk}(C_0^\Delta(\Sigma_{g'})) - \text{rk}(C_1^\Delta(\Sigma_{g'})) + \text{rk}(C_2^\Delta(\Sigma_{g'})) = d(\text{rk}(C_0^\Delta(\Sigma_g)) - \text{rk}(C_1^\Delta(\Sigma_g)) + \text{rk}(C_2^\Delta(\Sigma_g)))$$

By the result of a,

$$\text{rk}(C_0^\Delta(\Sigma_g)) - \text{rk}(C_1^\Delta(\Sigma_g)) + \text{rk}(C_2^\Delta(\Sigma_g)) = 2 - 2g$$

thus we get

$$2 - 2g' = d(2 - 2g)$$

from which we derive

$$g' = d(g - 1) + 1.$$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60034, MATH97042, MATH70034	1	Students did reasonably well on average. Most students forgot to check that in the definition of group action, maps have to be homeomorphisms.
MATH60034, MATH97042, MATH70034	2	The delicate issue of dealing with the base point when applying van Kampen's theorem was missed by most students.
MATH60034, MATH97042, MATH70034	3	No comment.
MATH60034, MATH97042, MATH70034	4	No student dealt with the linear algebra to prove that the first homology group is indeed \mathbb{Z}^4 well.
MATH97042, MATH70034	5	No comment.