MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Problem Sheet 6

Exercise 26 (Inhomogeneous linear systems).
Consider the linear inhomogeneous differential equation

$$\dot{x} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} x - \begin{pmatrix} 2 \\ 2 \end{pmatrix} .$$

(i) Show that this differential equation has a constant solution.

(ii) Compute the flow of this differential equation.

Exercise 27 (Stability for one-dimensional differential equations).
Consider the one-dimensional differential equation

$$\dot{x} = f(x) ,$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, defined on an open set $D \subset \mathbb{R}$. We assume that $x^* \in D$ is an equilibrium.

(i) Show that if x^* is attractive, then x^* is stable.

(ii) Assume now that f is n-times differentiable. Analyse the stability of this equilibrium, in dependence on $n \in \mathbb{N}$ and the sign of $f^{(n)}(x^*)$, when

$$f^{(k)}(x^*) = 0 \quad \text{for all } \; k \in \{1, \dots, n-1\} , \quad \text{and} \quad f^{(n)}(x^*) \neq 0 .$$

Exercise 28 (Global attractivity).
Show that the trivial equilibrium of the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= -x + y - x^3 , \\ \dot{y} &= -x - y - y^3 \end{aligned}$$

is *globally attractive*, i.e. if $\varphi(t, x, y)$ denotes the flow of this differential equation, then we have

$$\lim_{t \to \infty} \varphi(t, x, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } \; (x, y) \in \mathbb{R}^2 .$$

Hint. Use polar coordinates (see *Quiz 1*) to analyse the time evolution $t \mapsto \|\varphi(t, x, y)\|$ of the radial component of the solution.

Remark. Note that the theorem on linearised stability (Theorem 4.10) can be applied here, but it will only give information on the local stability behaviour. Proving global stability results is in general much harder, but there are useful tools, and in the next weeks, you will learn much more about it.

Exercise 29 (Exponential stability of linear systems).

Consider the autonomous linear system

$$\dot{x} = Ax\,,$$

where $A \in \mathbb{R}^{d\times d}$. Show that then the trivial equilibrium $x^* = 0$ of this system is exponentially stable if and only if $\operatorname{Re}\rho < 0$ for all eigenvalues ρ of A.

Hint. This is the second part of Theorem 4.5. It is possible to prove this similarly to the first part.

Exercise 30 (Optional challenging question).

Let $g : \mathbb{R} \to \mathbb{R}^2$ be continuous and bounded, and consider a matrix $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $a \pm ib$, where $a < 0$ and $b \in \mathbb{R}$.

(i) Show that the linear inhomogeneous differential equation

$$\dot{x} = Ax + g(t)$$

has a bounded solution $\mu : \mathbb{R} \to \mathbb{R}^2$.

Hint. Analyse $t \mapsto \int_{-\infty}^{t} e^{A(t-s)} g(s)\,\mathrm{d}s$.

(ii) Show that the solution μ from (i) is the only bounded solution of the linear inhomogeneous differential equation.

(iii) Show that the solution μ from (i) is attractive in the sense that

$$\lim_{t\to\infty} \|\lambda(t, t_0, x_0) - \mu(t)\| = 0 \quad \text{for all } (t_0, x_0) \in \mathbb{R}^2\,,$$

where $\lambda : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the general solution of the linear inhomogeneous differential equation.

Comments on importance and difficulty of the exercises. Exercise 26 is an elementary computational exercise to find the flow of an inhomogeneous linear differential equation. Exercise 27 is very elementary, but requires confident use of typical one-dimensional arguments; these were trained in particular on Problem Sheet 4. Part (i) shows that in the special case of one-dimensional systems, attractivity implies stability. (ii) gives a simply criterion for stability in the one-dimensional case; additional explanations can be found in Example 4.8. Exercise 28 is an elementary analysis of a two-dimensional system using polar coordinates. Exercise 29 is similar to the first part of the proof of Theorem 4.5. Exercise 30 uses an integration technique that is quite important in the theory of differential equation: the Lyapunov–Perron integral given in the hint of (i). This does not play a role in the remainder of the course, but certain proofs of the stable and unstable manifold theorem (discussed in Subsection 1.5 of Chapter 4) make use of this integral.