Answers to Problem Sheet 7

1. Energy $E = \frac{1}{2}\dot{\theta}^2 - \cos\theta$ or

$$d\theta = \pm\sqrt{2(E + \cos\theta)}dt.$$

At the turning points (where $\dot{\theta} = 0$) $\cos\theta = -E$ so that $\theta = \pm\cos^{-1}(-E)$. Integrating the above equation from $\theta = 0$ to $\theta = \cos^{-1}(-E)$ yields a quarter of the period, T, as this motion represents a quarter of a full oscillation. That is

$$\int_0^{\cos^{-1}(-E)} \frac{d\theta}{\sqrt{2(E + \cos\theta)}} = \frac{T}{4}$$

as required.

(ii) Using $p = \sqrt{2(E + \cos\theta)}$

$$J = \oint p\, d\theta = 4\int_0^{\cos^{-1}(-E)} \sqrt{2(E + \cos\theta)}d\theta.$$

Now

$$T = \frac{1}{\nu} = \frac{dJ}{dE} = 2\sqrt{2}\int_0^{\cos^{-1}(-E)} \frac{d\theta}{\sqrt{E + \cos\theta}}.$$

(iii) If $E > 1$? the motion is rotation rather than libration. Here

$$J = \int_0^{2\pi} \sqrt{2(E + \cos\theta)}d\theta, \quad T = \frac{1}{\sqrt{2}}\int_0^{2\pi} \frac{d\theta}{\sqrt{E + \cos\theta}}$$

What happens if $E = 1$?

2. Hamilton's Characteristics Function $W = W_r(r) + \alpha_\theta\theta$ as θ is cyclic. The Hamilton-Jacobi equation yields

$$\frac{1}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2r^2} - \frac{k}{r} = \alpha_1.$$

Using $p_r = \partial W_r/\partial r$,

$$J_r = \oint p_r dr = 2\int_{r_1}^{r_2} \sqrt{2\alpha_1 + \frac{2k}{r} - \frac{a_\theta^2}{r^2}}\, dr$$

$$= 2 \int_{r_1}^{r_2} \sqrt{2\alpha_1 r^2 + 2kr - a_\theta^2} \, \frac{dr}{r} = 2\sqrt{-2\alpha_1} \int_{r_1}^{r_2} \sqrt{-r^2 - \frac{kr}{\alpha_1} + \frac{a_\theta^2}{2\alpha_1}} \, \frac{dr}{r}.$$

Here $\alpha_1 < 0$ for closed orbits[1]. r_1 and r_2 are the roots of the quadratic inside the square root and are the closest and furthest distance to the origin of the orbit. The integral of p_r from r_1 to r_2 represents one half of a full libration (hence the factor of 2 above).

Using the integral

$$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{x} \, dx = \frac{\pi}{2}\left(a + b - 2\sqrt{ab}\right) \qquad (b > a > 0).$$

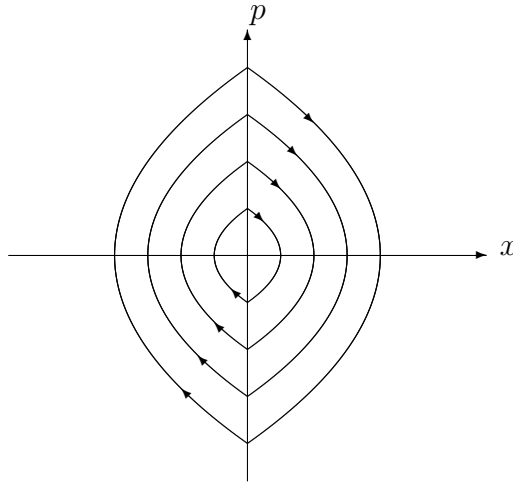with $a + b = -k/(\alpha_1)$ and $ab = -a_\theta^2/(2\alpha_1)$ yields

$$J_r = 2\sqrt{-2\alpha_1} \cdot \frac{\pi}{2}\left(-\frac{2k}{2\alpha_1} - 2\frac{\alpha_\theta}{\sqrt{-2\alpha_1}}\right) = \frac{2\pi k}{\sqrt{-2\alpha_1}} - J_\theta,$$

as $J_\theta = 2\pi\alpha_\theta$. As $H = \alpha_1$ we have

$$H(J_r, J_\theta) = -\frac{2\pi^2 k^2}{(J_r + J_\theta)^2}.$$

Remark: it is clear from this result that $\nu_r = \partial H/\partial J_r$ is the same as $\nu_\theta = \partial H/\partial J_\theta$.

3. (i) Closed orbits comprise parts of parabolas joined at p-axis.

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[1]The closed orbits are ellipses of the form $r = L/[1 + e\cos(\theta - \alpha)]$ with $0 \le e < 1$; see question 2 on Problem Sheet 5.

(ii)

$$J = \oint p\,dx = 4\int_0^{\frac{E}{\lambda}} \sqrt{2E - 2\lambda x}\,dx = -\,4\cdot\frac{1}{3\lambda}\left(2E - 2\lambda x\right)^{3/2}\Big|_0^{E/\lambda} = \frac{4}{3\lambda}(2E)^{3/2},$$

so that $2E = (\frac{3}{4})^{2/3}(\lambda J)^{2/3}$ giving

$$H = 2^{-7/3}3^{2/3}\lambda^{2/3}J^{2/3}.$$

(iii) Frequency

$$\nu = \frac{\partial H}{\partial J} = \frac{2}{3}2^{-7/3}3^{2/3}\lambda^{2/3}J^{-1/3}.$$

Under an adiabatic change of λ, J is unchanged. If λ is doubled (adiabatically) the frequency is increased by a factor $2^{2/3}$.

4. (i) H (total energy) and $p_1 + p_2$ (total momentum) are constants of the motion.

The Noether symmetry associated with conservation of energy is time translation $Q_1 = x_1 + s\{x_1, H\} = x_1 + sp_1/m_1$, $P_1 = p_1 + s\{p_1, H\} = p_1 - sU'(x_1 - x_2)$

$Q_2 = x_2 + s\{x_2, H\} = x_2 + sp_2/m_2$, $P_2 = p_2 + s\{p_2, H\} = p_2 + sU'(x_1 - x_2)$.

The Noether symmetry associated with conservation of momentum is translational invariance $Q_1 = x_1 + s$, $Q_2 = q_2 + s$, $P_1 = p_1$, $P_2 = p_2$.

(ii) Yes, the Hamiltonian is Liouville-integrable as $\{H, p_1 + p_2\} = 0$.

5. (i) $F_2 = \sum_{i=1}^{N} q_i P_i$.

(ii)

$$F_2 = \sum_{i=1}^{N} q_i P_i + s\alpha(q_1, ..., q_N, P_1, ..., P_N, t).$$

To verify that this generates the given deformation

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i + s\frac{\partial \alpha}{\partial q_i}.$$

Up to terms of order s, $\partial \alpha/\partial q_i = \{\alpha, p_i\}$ Therefore, $P_i = p_i + s\{p_i, \alpha\} + ...$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i + s\frac{\partial \alpha}{\partial P_i} + ... = q_i + s\{q_i, \alpha\},$$

as required.

(iii) As F_2 is a generating function

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_2}{\partial t} = H(q_i, p_i, t) + s\frac{\partial \alpha}{\partial t} + \ldots$$

Now

$$H(Q_i, P_i, t) = H(q_i + s\frac{\partial \alpha}{\partial p_i}, p_i - s\frac{\partial \alpha}{\partial q_i}, t) = H(q_i, p_i, t) + s\sum_{j=1}^{N} \frac{\partial H}{\partial q_j}\frac{\partial \alpha}{\partial p_j} - s\sum_{j=1}^{N} \frac{\partial H}{\partial p_j}\frac{\partial \alpha}{\partial q_j}$$

$$= H(q_i, p_i, t) + s\{H, \alpha\}.$$

If $K(Q_i, P_i, t) = H(Q_i, P_i, t)$, then

$$\frac{d\alpha}{dt} = \{\alpha, H\} + \frac{\partial \alpha}{\partial t} = 0.$$