

MATH60046/MATH70046

Time Series Analysis

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Use Blackboard to obtain all course resources

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Chapter 1

Introduction

1.1 Module admin and structure

Pre-requisites

Probability for Statistics

Statistical Modelling I (preferable but not essential).

Course materials

Lecture notes

Figures booklets

30 hours of teaching (approx 25 lectures/5 problems class)

Non-assessed quiz questions

5 non-assessed problem sheets

Assessment

90% Exam

10% Coursework

1.2 What are times series?

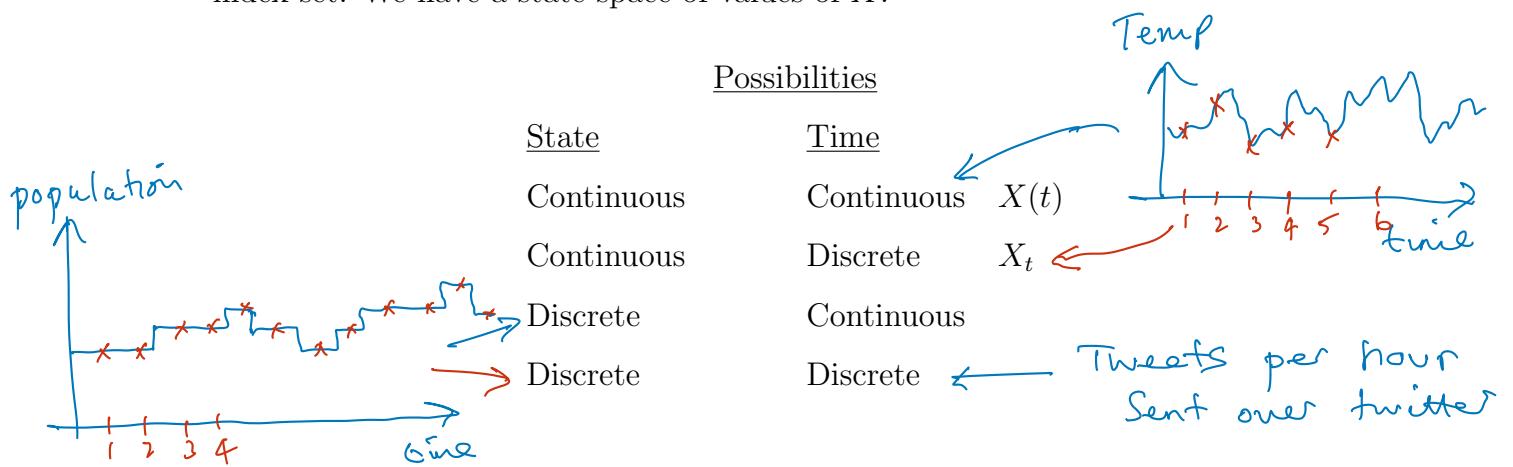
A time series is a series of data points indexed (or listed) in time order [wikipedia].

Any metric that is measured over time is a time series.

Times series analysis (TSA) could be described as a branch of applied stochastic processes. We start with an indexed family of real-valued random variables

Stochastic (random process, t assumed to run from $-\infty$ to ∞) $\{X_t : t \in T\}$ *typically \mathbb{Z}*

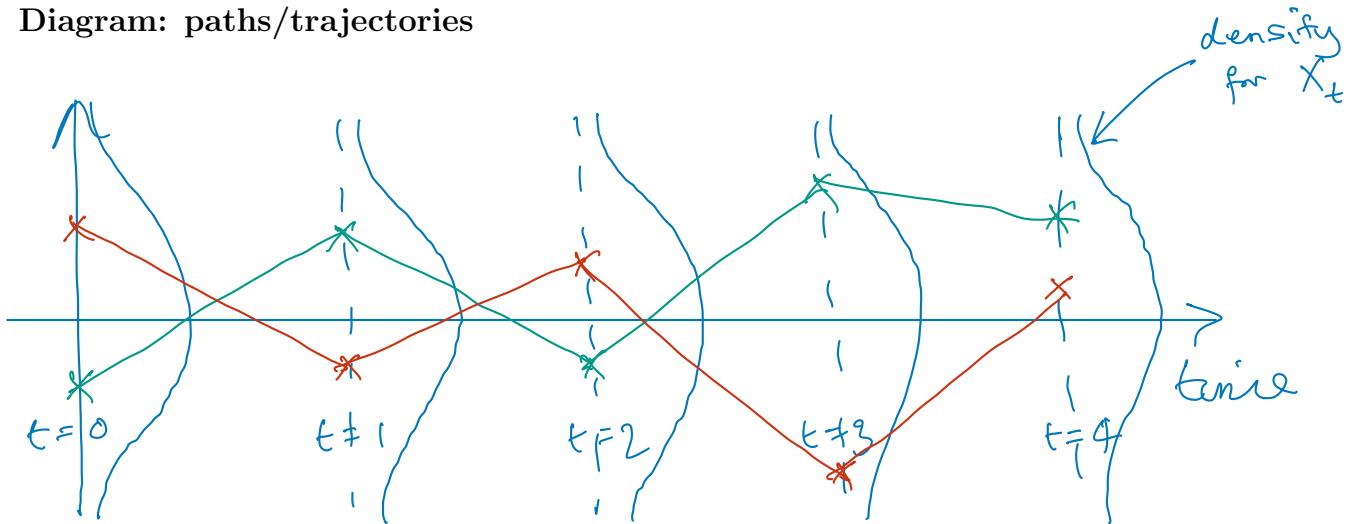
where t is the index, here taken to be time (but it could be space). T is called the index set. We have a state space of values of X .



In addition X could be univariate or multivariate. We shall concentrate on discrete time. Samples are taken at equal intervals.

We wish to use TSA to characterize time series and understand structure. Our job is to make inference on the underlying stochastic process from a single realisation - the observed time series.

Diagram: paths/trajectories



Examples: Figures 1–4, in all cases points are joined for clarity.

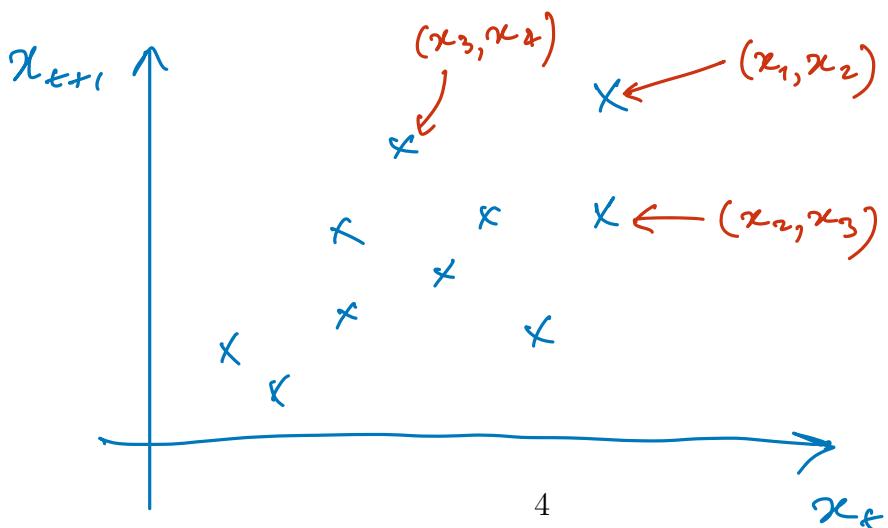
- [1] wind speed in a certain direction at a location, measured every 0.025s.
- [2] monthly average measurements of the flow of water in the Willamette River at Salem, Oregon.
- [3] the daily record of the change in average daily frequency that tells us how well an atomic clock keeps time on a day to day basis.
- [4] the change in the level of ambient noise in the ocean from one second to the next.
- [5] part of the Epstein-Barr Virus DNA sequence (the entire sequence consists of approximately 172,000 base pairs).
- [6] Daily US Dollar/Sterling exchange rate and the corresponding returns from 1981 to 1985.

The visual appearances of these datasets are quite different. For example, consider the wind speed and atomic clock data,

<u>Wind speed</u>	<u>Atomic clock</u>
Adjacent points are close in value	Positive values tend to be followed by negative values

For the numerical data, we can illustrate this using lag 1 scatter plots.

Diagram: lag 1 scatter plots



(See Figures 4a and 5). Realizations of the series denoted x_1, \dots, x_N . So plot x_t versus x_{t+1} as t varies from 1 to $N - 1$.

From these scatter plots we note the following:

- [1] for the wind speed and US dollar series, the values are positively correlated.
- [2] Willamette river data is similar, but points are more spread out.
- [3] for the atomic clock data, the values are negatively correlated.
- [4] for the ocean noise data and the US dollar returns series there is no clear clustering tendency.

We could similarly create lag τ scatter plots by plotting x_t versus $x_{t+\tau}$ for integer τ , but they would be unwieldy to deal with and interpret.

A better approach is to realize that the series x_1, \dots, x_N can be regarded as a realization of the corresponding random variables X_1, \dots, X_N , and we will proceed by studying the covariance relationships between these random variables.

1.3 A brief aside - covariance and correlation

The concept of covariance and correlation will be crucial in this module, therefore we BRIEFLY recap some of the key ideas.

Covariance

Covariance is a measure of joint variability of two random variables, X and Y say.

Defined as

$$\text{cov}(X, Y) \equiv E\{(X - E\{X\})(Y - E\{Y\})\} = E\{XY\} - E\{X\}E\{Y\}.$$

$$\begin{aligned} E\{XY - XE\{Y\} - E\{X\}Y + E\{X\}E\{Y\}\} &= E\{XY\} - E\{X\}E\{Y\} - E\{X\}E\{Y\} + E\{X\}E\{Y\} \\ &= E\{XY\} - E\{X\}E\{Y\}. \end{aligned}$$

- Positive covariance \Rightarrow when X is above its mean then Y also tends to be.
- Negative covariance \Rightarrow when X is above its mean then Y tends to be below its mean.
- Zero covariance means there is no relationship of this type and $E\{XY\} = E\{X\}E\{Y\}$

Therefore, this gives a measure of linear dependency between 2 random variables.

NOTE: $\text{cov}(X, X) = \text{var}(X)$

All variance and covariance terms (known as the joint second moments), can be summarized in the *variance-covariance matrix* (also commonly known as just the *covariance matrix*).

Define vector $\mathbf{X} = (X, Y)^T$ with mean $E\{\mathbf{X}\} = \underline{\mu} = (\mu_X, \mu_Y)^T$ and variances σ_X^2 and σ_Y^2 , respectively.

$$\Sigma \equiv E\{(\mathbf{X} - \mu)(\mathbf{X} - \mu)^T\} = \begin{pmatrix} \sigma_X^2 & \text{cov}(X, Y) \\ \text{cov}(X, Y) & \sigma_Y^2 \end{pmatrix}$$

Variances

This can be extended to higher dimensions. For a random vector $\mathbf{X} = (X_1, X_2, \dots, X_m)$, we have a $m \times m$ covariance matrix $\Sigma = (\sigma_{ij})$ where $\sigma_{ij} = \text{cov}(X_i, X_j)$.

Σ is a symmetric positive semi-definite matrix.

Correlation

Correlation is a normalized measure of covariance. It is useful because covariance is proportional to variance so can be misleading.

Covariance has all the properties of an inner-product, $\text{cov}(\cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$, namely

- Bilinear: $\text{cov}(aX + bY, Z) = a\text{cov}(X, Z) + b\text{cov}(Y, Z)$

$$\begin{aligned} & E\{(aX + bY)Z\} - E\{aX + bY\}E\{Z\} = E\{aXZ + bYZ\} - E\{aX\}E\{Z\} - E\{bY\}E\{Z\} \\ & = a(E\{XZ\} - E\{X\}E\{Z\}) + b(E\{YZ\} - E\{Y\}E\{Z\}) \end{aligned}$$

- Symmetric: $\text{cov}(X, Y) = \text{cov}(Y, X)$

- Positive ^{Semi}definite: $\text{cov}(X, X) \geq 0$ for all X .

This means we can invoke the Cauchy-Swartz inequality:

$$\begin{aligned} |\langle x, y \rangle| &\leq \sqrt{\|x\|^2 \|y\|^2} \\ |\text{cov}(X, Y)| &\leq \sqrt{\text{var}(X)\text{var}(Y)} \end{aligned}$$

We therefore define correlation ρ as

$$\rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

CS inequality means $-1 \leq \rho \leq 1$.

Equality holds when there's a perfect linear relationship, i.e. $Y = aX + b$,

$$\begin{aligned} \text{cov}(X, Y) &= \text{cov}(X, aX + b) = a\text{cov}(X, X) = a\sigma_X^2 \\ \text{var}(Y) &= \text{cov}(aX + b, aX + b) = \text{cov}(aX, aX) = a^2\text{var}(X), \end{aligned}$$

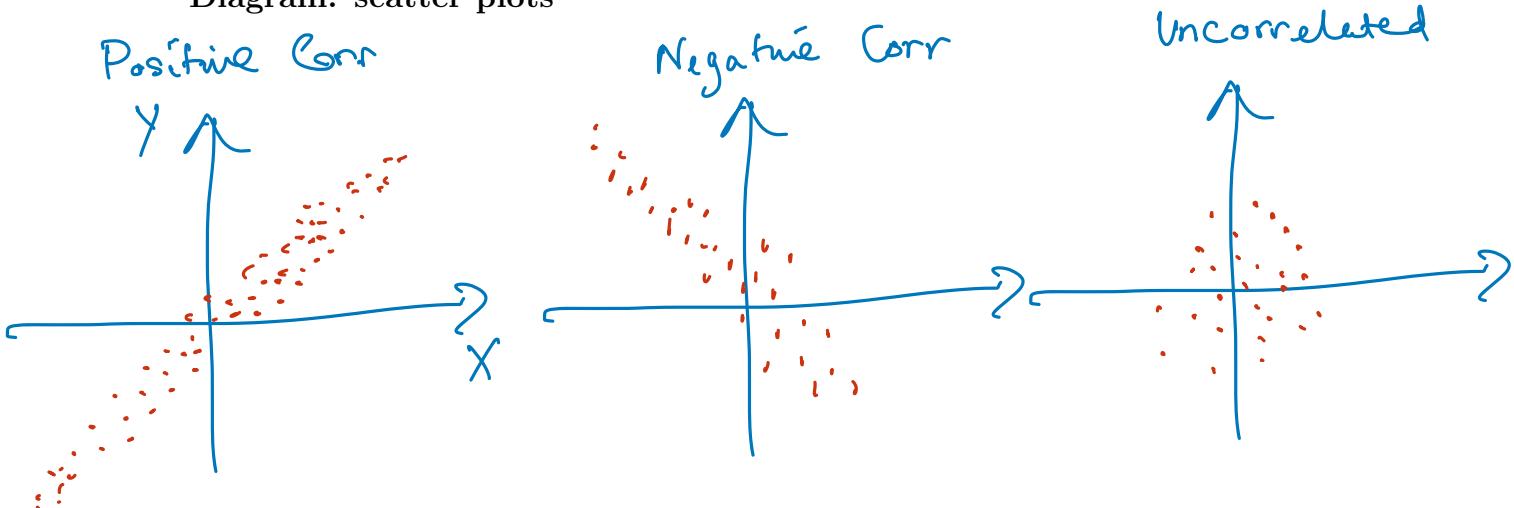
Therefore

$$\rho = \frac{a\sigma_X^2}{\sqrt{a^2\sigma_X^2\sigma_X^2}} = \frac{a\sigma_X^2}{|a|\sigma_X^2} = \begin{cases} 1, & \text{when } a > 0 \\ -1, & \text{when } a < 0. \end{cases}$$

In this circumstance we say the random variables are perfectly positively/negatively correlated.

When $\text{cov}(X, Y) = 0$ we say random variables are uncorrelated

Diagram: scatter plots



Formally,

$$\text{cov}(X, Y) = \int \int xyf(x, y)dxdy - \int xf(x)dx \int yf(y)dy$$

$E\{XY\}$

$E\{X^2\}, E\{Y^2\}$

Therefore, if X and Y are independent, then $f(x, y) = f(x)f(y)$ which implies

$$\text{cov}(X, Y) = \int xf(x)dx \int yf(y)dy - \int xf(x)dx \int yf(y)dy = 0.$$

So,

independence \Rightarrow uncorrelated

uncorrelated $\not\Rightarrow$ independence, in general

BUT, uncorrelated \Rightarrow independence for joint normal distribution

Its context in time series analysis

In time series analysis we will typically be interested in looking at the covariance or correlation within the random process $\{X_t\}$ at two time points, i.e. the covariance or correlation between random variables X_{t_1} and X_{t_2} .

Chapter 2

Real-Valued discrete time stationary processes

Video 4

2.1 Stationarity

Video 5

2.1.1 Joint distribution $\{X_t, t \in \mathbb{Z}\}$

Denote the process by $\{X_t\}$. For fixed t , X_t is a random variable (r.v.), and hence there is an associated cumulative probability distribution function (cdf):

$$F_t(a) = P(X_t \leq a),$$

and

index of random variable *pdf for X_t*

$$\begin{aligned} E\{X_t\} &= \int_{-\infty}^{\infty} x dF_t(x) = \int_{-\infty}^{\infty} x f_t(x) dx \\ \text{var}\{X_t\} &= \int_{-\infty}^{\infty} (x - \mu_t)^2 dF_t(x). \end{aligned}$$

But we are interested in the relationships between the various r.v.s that form the process. For example, for any t_1 and $t_2 \in T$,

$$F_{t_1, t_2}(a_1, a_2) = P(X_{t_1} \leq a_1, X_{t_2} \leq a_2)$$

gives the bivariate cdf. More generally for any $t_1, t_2, \dots, t_n \in T$,

$$F_{t_1, t_2, \dots, t_n}(a_1, a_2, \dots, a_n) = P(X_{t_1} \leq a_1, \dots, X_{t_n} \leq a_n)$$

The class of all stochastic processes is too large to work with in practice. We consider only the subclass of stationary processes.

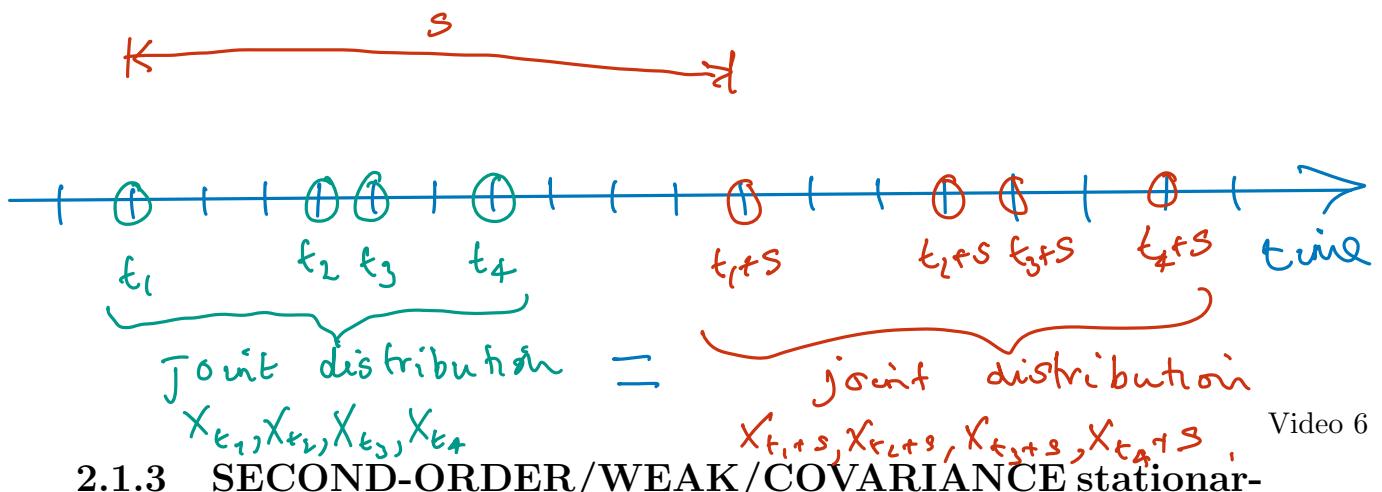
2.1.2 COMPLETE/STRONG/STRICT stationarity

$\{X_t\}$ is said to be completely stationary if, for all $n \geq 1$, for any $t_1, t_2, \dots, t_n \in T$, and for any s such that $t_1+s, t_2+s, \dots, t_n+s \in T$ are also contained in the index set, the joint cdf of $\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$ is the same as that of $\{X_{t_1+s}, X_{t_2+s}, \dots, X_{t_n+s}\}$ i.e.,

$$F_{t_1, t_2, \dots, t_n}(a_1, a_2, \dots, a_n) = F_{t_1+s, t_2+s, \dots, t_n+s}(a_1, a_2, \dots, a_n),$$

so that the probabilistic structure of a completely stationary process is invariant under a shift in time.

Diagram: strict stationarity

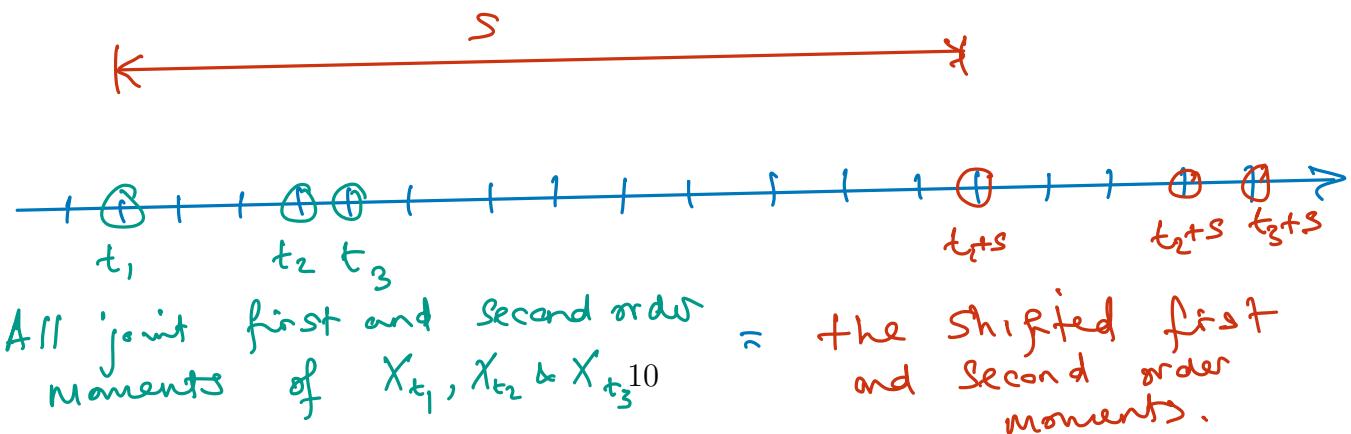


2.1.3 SECOND-ORDER/WEAK/COVARIANCE stationarity

Definition 1

$\{X_t\}$ is said to be second-order stationary if, for all $n \geq 1$, for any $t_1, t_2, \dots, t_n \in T$, and for any s such that $t_1+s, t_2+s, \dots, t_n+s \in T$ are also contained in the index set, all the joint moments of orders 1 and 2 of $\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$ exist, are finite, and equal to the corresponding joint moments of $\{X_{t_1+s}, X_{t_2+s}, \dots, X_{t_n+s}\}$.

Diagram: second-order stationarity



$$E\{X_{t_1}\} = E\{X_{t_1+s}\}$$

$$E\{X_{t_1}^2\} = E\{X_{t_1+s}^2\}$$

$$E\{X_{t_1}\} = E\{X_{t_1+s}\}$$

$$E\{X_{t_2}\} = E\{X_{t_2+s}\}$$

$$E\{X_{t_1}^2\} = E\{X_{t_1+s}^2\},$$

$$E\{X_{t_2}^2\} = E\{X_{t_2+s}^2\},$$

$$E\{X_{t_1}X_{t_2}\} = E\{X_{t_1+s}X_{t_2+s}\}.$$

It is immediate from this definition that,

$$E\{X_{t_1}\} = E\{X_{t_1+s}\}$$

for all t_1 and all s .

$$E\{X_t\} \equiv \mu \quad ; \quad \text{var}\{X_t\} \equiv \sigma^2 \quad (= E\{X_t^2\} - \mu^2),$$

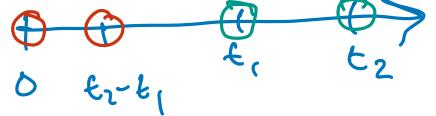
are constants independent of t . If we let $s = -t_1$,

$$E\{X_{t_1}X_{t_2}\} = E\{X_{t_1+s}X_{t_2+s}\}$$

$$E\{X_{t_1}^2\} = E\{X_{t_1+s}^2\}$$

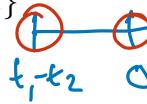
for all t_1 and all s .

$$\begin{aligned} E\{X_{t_1}X_{t_2}\} &= E\{X_{t_1+s}X_{t_2+s}\} \\ &= E\{X_0X_{t_2-t_1}\}, \end{aligned}$$



and with $s = -t_2$,

$$\begin{aligned} E\{X_{t_1}X_{t_2}\} &= E\{X_{t_1+s}X_{t_2+s}\} \\ &= E\{X_{t_1-t_2}X_0\}. \end{aligned}$$



Hence, $E\{X_{t_1}X_{t_2}\}$ is a function of the absolute difference $|t_2 - t_1|$ only, similarly, for the covariance between X_{t_1} & X_{t_2} :

$$\text{cov}\{X_{t_1}, X_{t_2}\} = E\{(X_{t_1} - \mu)(X_{t_2} - \mu)\} = E\{X_{t_1}X_{t_2}\} - \mu^2.$$

For a discrete time second-order stationary process $\{X_t\}$ we define the autocovariance sequence (acvs) by

$$s_\tau \equiv \text{cov}\{X_t, X_{t+\tau}\} = \text{cov}\{X_0, X_\tau\}.$$

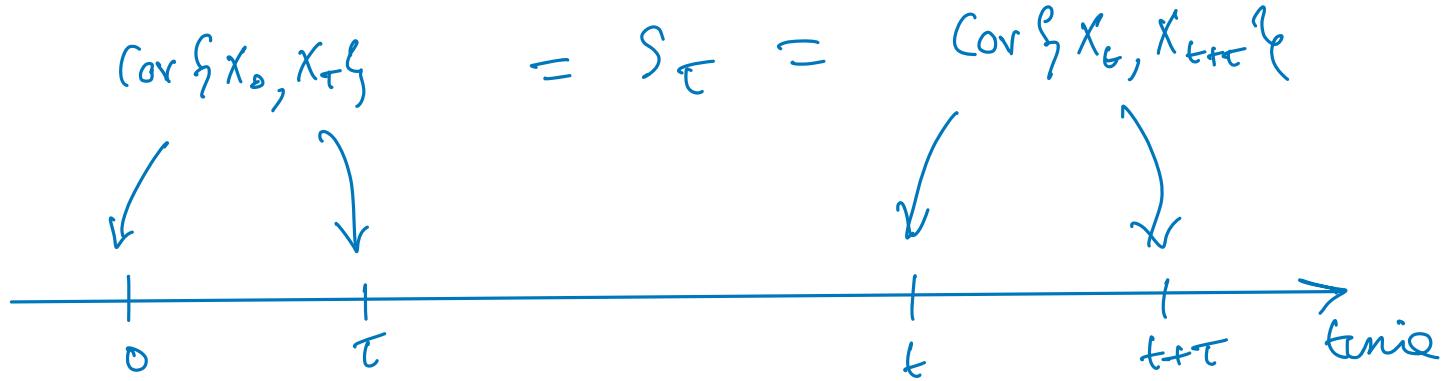
Furthermore, suppose a stochastic process $\{X_t\}$ is not stationary, then by the Definition 1 there must exist some index set $t_1, \dots, t_m \in T$ and some s such that not all joint moments of order 1 and 2 are constant. In the case of order 1 moments, that implies $\{X_t\}$ does not have a constant mean. In the case of order 2 moments, there must exist $t_i, t_j \in \{t_1, \dots, t_m\}$ such that $E\{X_{t_i}X_{t_j}\} \neq E\{X_{t_i+s}X_{t_j+s}\}$, and therefore $\text{cov}\{X_{t_i}, X_{t_j}\}$ does not depend only on the absolute difference $|t_2 - t_1|$.

An equivalent definition of second-order stationary is therefore as follows.

Definition 2

$\{X_t\}$ is said to be second-order stationary if for all $t \in T$ and all τ such that $t+\tau \in T$, $E\{X_t\}$ is finite and constant for all t , and $s_\tau \equiv \text{cov}\{X_t, X_{t+\tau}\}$ is finite and depends only on τ and not on t .

Diagram: autocovariance



Video 7

2.1.4 Properties and Notation

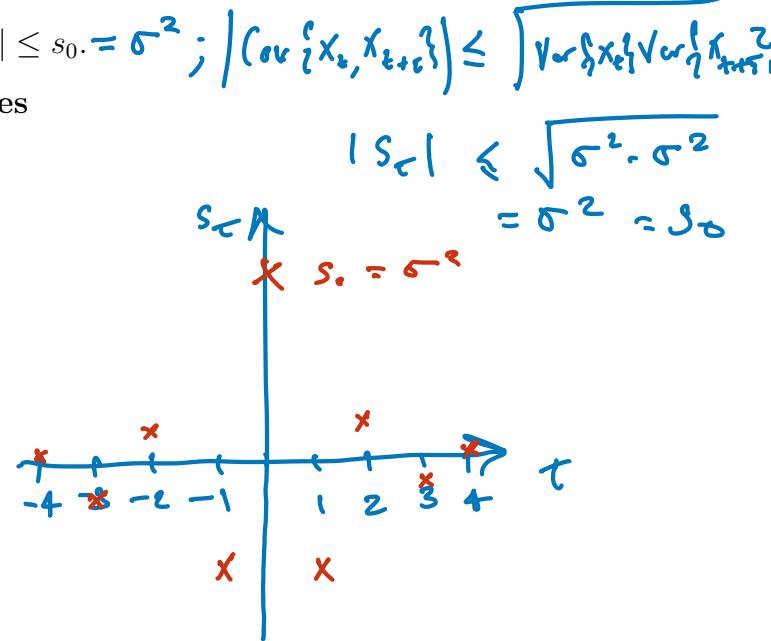
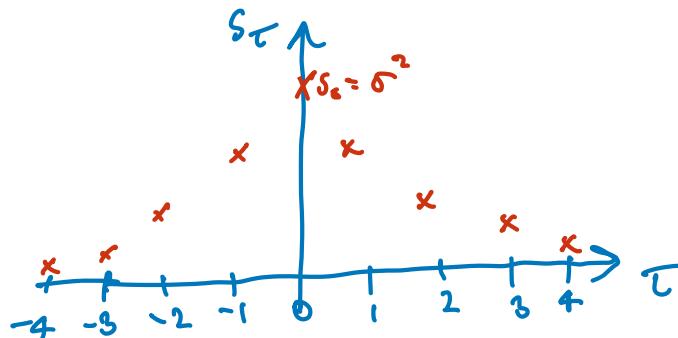
1. τ is called the lag.
2. $s_0 = \sigma^2$ and $s_{-\tau} = s_\tau$.
3. The autocorrelation sequence (acs) is given by

$$\rho_\tau = \frac{s_\tau}{s_0} = \frac{\text{cov}\{X_t, X_{t+\tau}\}}{\sigma^2}.$$

The sample or estimated autocorrelation sequence (acs), $\{\hat{\rho}_\tau\}$, for each of our time series are given in Figs. 6 and 7. [We shall see how to compute these in Chapter 4.] Note e.g., that for the Willamette river data X_t and X_{t+6} seem to be negatively correlated, while X_t and X_{t+12} seem positively correlated (consistent with the river flow varying with a period of roughly 12 months).

4. Since ρ_τ is a correlation coefficient, $|\rho_\tau| \leq 1$; $|\text{cov}\{X_t, X_{t+\tau}\}| \leq \sqrt{\text{Var}\{X_t\} \text{Var}\{X_{t+\tau}\}}$

Diagram: autocovariance sequences



Worked example: variogram

For the stationary process $\{X_t\}$ with mean μ , acvs $\{s_\tau\}$ and variance $s_0 = \sigma^2$

$$v_\tau = E\{(X_{t+\tau} - X_t)^2\}/2 \quad \begin{matrix} \text{Variance of the difference} \\ \text{between two points} \\ \text{separated by } \tau \end{matrix}$$

Show

(used in geostatistics) has a maximum of $2\sigma^2$.

$$\begin{aligned} v_\tau &= E\left\{\left[(X_{t+\tau} - \mu) - (X_t - \mu)\right]^2\right\}/2 \\ &= [E\{(X_{t+\tau} - \mu)^2\} + E\{(X_t - \mu)^2\} - 2E\{(X_{t+\tau} - \mu)(X_t - \mu)\}]/2 \\ &= [\sigma^2 + \sigma^2 - 2\text{Cov}\{X_t, X_{t+\tau}\}]/2 \\ &= [2\sigma^2 - 2s_\tau]/2 = \sigma^2 - s_\tau \end{aligned}$$

We know $-s_0 \leq s_\tau \leq s_0$, \therefore Maximum Value it can take is $v_\tau = \sigma^2 - (-s_0)$
 $= 2\sigma^2$

5. The sequence $\{s_\tau\}$ is positive semidefinite, i.e., for all $n \geq 1$, for any t_1, t_2, \dots, t_n contained in the index set, and for any set of nonzero real numbers a_1, a_2, \dots, a_n

$$\sum_{j=1}^n \sum_{k=1}^n s_{t_j-t_k} a_j a_k \geq 0.$$

Proof

Let

$$\Sigma_{ij} = \text{Cov}\{X_{t_i}, X_{t_j}\} \quad \mathbf{a} = (a_1, a_2, \dots, a_n)^T, \quad \mathbf{V} = (X_{t_1}, X_{t_2}, \dots, X_{t_n})^T$$

and let Σ be the variance-covariance matrix of \mathbf{V} . Its j, k -th element is given by $s_{t_j-t_k} = E\{(X_{t_j} - \mu)(X_{t_k} - \mu)\}$. Define the r.v.

$$\Sigma = \begin{pmatrix} s_{t_1-t_1} & s_{t_1-t_2} & \cdots & s_{t_1-t_n} \\ s_{t_2-t_1} & \ddots & & \\ \vdots & & \ddots & \\ s_{t_n-t_1} & \cdots & \cdots & s_{t_n-t_n} \end{pmatrix} \quad w = \sum_{j=1}^n a_j X_{t_j} = \mathbf{a}^T \mathbf{V}.$$

Then

$$0 \leq \text{var}\{w\} = \text{var}\{\mathbf{a}^T \mathbf{V}\} = \mathbf{a}^T \Sigma \mathbf{a} = \sum_{j=1}^n \sum_{k=1}^n s_{t_j - t_k} a_j a_k.$$

$\underline{\mu} = \mathbf{E}\{\underline{v}\}$

$$\begin{aligned} \text{var}\{\underline{a}^T \underline{v}\} &= E\left\{(\underline{a}^T \underline{v} - \underline{a}^T \underline{\mu})^2\right\} = E\left\{(\underline{a}^T (\underline{v} - \underline{\mu}))^2\right\} \\ &= E\left\{\underline{a}^T (\underline{v} - \underline{\mu})(\underline{v} - \underline{\mu})^T \underline{a}\right\} = \underline{a}^T E\left\{(\underline{v} - \underline{\mu})(\underline{v} - \underline{\mu})^T\right\} \underline{a} \\ &= \underline{a}^T \Sigma \underline{a}. \end{aligned}$$

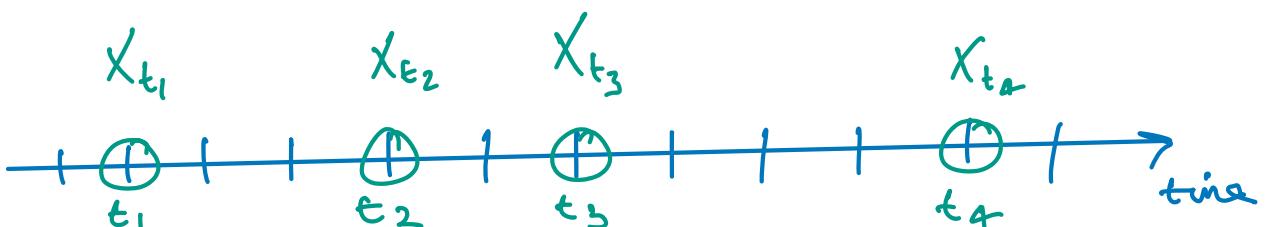
6. The variance-covariance matrix of equispaced X 's, $(X_1, X_2, \dots, X_N)^T$ has the form

$$\begin{bmatrix} s_0 & s_1 & \dots & s_{N-2} & s_{N-1} \\ s_1 & s_0 & \dots & s_{N-3} & s_{N-2} \\ \vdots & & \ddots & & \\ s_{N-2} & s_{N-3} & \dots & s_0 & s_1 \\ s_{N-1} & s_{N-2} & \dots & s_1 & s_0 \end{bmatrix}$$

which is known as a symmetric Toeplitz matrix – all elements on a diagonal are the same. Note the matrix has only N unique elements, s_0, s_1, \dots, s_{N-1} .

7. A stochastic process $\{X_t\}$ is called Gaussian if, for all $n \geq 1$ and for any t_1, t_2, \dots, t_n contained in the index set, the joint cdf of $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ is multivariate Gaussian.

Diagram: jointly Gaussian *Jointly Gaussian/normal*



- 2nd-order stationary Gaussian \Rightarrow complete stationarity (since MVN completely characterized by 1st and 2nd moments). It is not true in general that 2nd-order stationary \Rightarrow complete stationarity.
- Complete stationarity \Rightarrow 2nd-order stationary in general.

8. The simple term “stationary” will be taken to mean second-order stationary unless stated otherwise.

2.1.5 Examples of discrete stationary processes

[1] White noise process

Also known as a purely random process. Let $\{X_t\}$ be a sequence of uncorrelated r.v.s such that

$$\mathbb{E}\{X_t\} = \mu, \quad \text{var}\{X_t\} = \sigma^2 \quad \forall t$$

and

$$s_\tau = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \quad \text{or} \quad \rho_\tau = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

forms a basic building block in time series analysis. Very different realizations of white noise can be obtained for different distributions of $\{X_t\}$. Examples are given in Figures 8 and 9 for processes with (a) Gaussian, (b) exponential, (c) uniform and (d) truncated Cauchy distributions.

[2] q -th order moving average process MA(q)

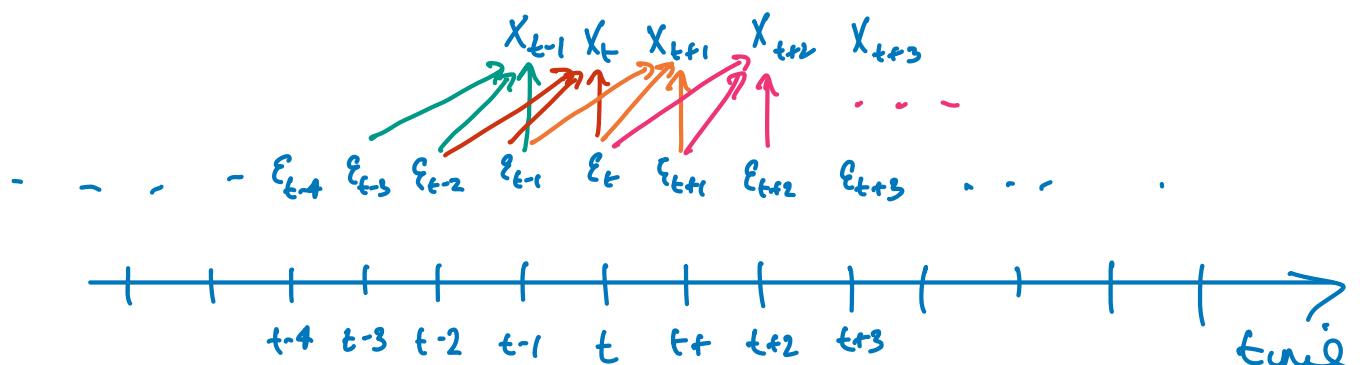
X_t can be expressed in the form

$$\begin{aligned} X_t &= \mu - \theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q} \\ &= \mu - \sum_{j=0}^q \theta_{j,q}\epsilon_{t-j}, \end{aligned}$$

where μ and $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$), and $\{\epsilon_t\}$ is a zero-mean white noise process with variance σ_ϵ^2 .

Diagram: moving average process

MA(2).



W.l.o.g. assume $E\{X_t\} = \mu = 0$.

Then $\text{cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\}$.

Recall: $\text{cov}(X, Y) = E\{(X - E\{X\})(Y - E\{Y\})\} = E(XY) - E(X)E(Y)$

Since $E\{\epsilon_t \epsilon_{t+\tau}\} = 0 \quad \forall \tau \neq 0$ we have for $\tau \geq 0$.

$$\begin{aligned} \text{cov}\{X_t, X_{t+\tau}\} &= \sum_{j=0}^q \sum_{k=0}^q \theta_{j,q} \theta_{k,q} E\{\epsilon_{t-j} \epsilon_{t+\tau-k}\} \\ &= \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q} \theta_{j+\tau,q} \quad (k = j + \tau) \\ &\equiv s_\tau, \end{aligned}$$

$$X_t = -\sum_{j=0}^q \theta_{j,q} \epsilon_{t-j} \quad \text{when } E(X) = E(Y) = 0$$

$$X_{t+\tau} = -\sum_{j=0}^q \theta_{j,q} \epsilon_{t+\tau-j}$$

which does not depend on t . Since $s_\tau = s_{-\tau}$, $\{X_t\}$ is a stationary process with acvs given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q} \theta_{j+|\tau|,q} & |\tau| \leq q \\ 0 & |\tau| > q \end{cases}$$

N.B. No restrictions were placed on the $\theta_{j,q}$'s to ensure stationarity, though obviously, $|\theta_{j,q}| < \infty, j = 1, \dots, q$.

Examples (see Figures 10 and 11)

$$X_t = \epsilon_t - \theta_{1,1} \epsilon_{t-1} \quad \text{MA(1)}$$

[acvs:]

$$s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{1-|\tau|} \theta_{j,1} \theta_{j+|\tau|,1} \quad |\tau| \leq 1,$$

$$s_\tau = 0 \quad |\tau| > 1.$$

so,

$$\begin{aligned} s_0 &= \sigma_\epsilon^2 (\theta_{0,1} \theta_{0,1} + \theta_{1,1} \theta_{1,1}) \\ &= \sigma_\epsilon^2 (1 + \theta_{1,1}^2); \end{aligned}$$

and,

$$\begin{aligned} s_1 &= \sigma_\epsilon^2 \theta_{0,1} \theta_{1,1} \\ &= -\sigma_\epsilon^2 \theta_{1,1}. \end{aligned}$$

[acs:]

$$\rho_\tau = \frac{s_\tau}{s_0}.$$

$$\rho_0 = 1.0; \quad \rho_1 = \frac{-\theta_{1,1}}{1 + \theta_{1,1}^2}; \quad \rho_2 = \rho_3 = \dots = 0.$$

$$X_{t+1} = \epsilon_{t+1} - \epsilon_t$$

(a) $\theta_{1,1} = 1.0, \sigma_\epsilon^2 = 1.0,$

we have,

$$s_0 = 2.0; s_1 = -1.0; s_2 = s_3 = \dots = 0.0,$$

giving,

$$\rho_0 = 1.0; \rho_1 = -0.5; \rho_2 = \rho_3 = \dots = 0.0.$$

(b) $\theta_{1,1} = -1.0, \sigma_\epsilon^2 = 1.0,$

we have,

$$s_0 = 2.0; s_1 = 1.0; s_2 = s_3 = \dots = 0.0,$$

giving,

$$\rho_0 = 1.0; \rho_1 = 0.5; \rho_2 = \rho_3 = \dots = 0.0.$$

Note: if we replace $\theta_{1,1}$ by $\theta_{1,1}^{-1}$ the model becomes

$$X_t = \epsilon_t - \frac{1}{\theta_{1,1}} \epsilon_{t-1}$$

and the autocorrelation becomes

$$\rho_1 = \frac{-\frac{1}{\theta_{1,1}}}{1 + \left(\frac{1}{\theta_{1,1}}\right)^2} = \frac{-\theta_{1,1}}{\theta_{1,1}^2 + 1},$$

i.e., is unchanged!!!

We cannot identify the MA(1) process uniquely from its autocorrelation!

model $(\theta_{1,1}) \rightarrow$ same
model $(1/\theta_{1,1}) \rightarrow$ ρ_1
many to one

[3] p-th order autoregressive process AR(p)

Video 9

$\{X_t\}$ is expressed in the form

response \downarrow *Predictors* \downarrow *noise/error term* -

$$X_t = \phi_{1,p} X_{t-1} + \phi_{2,p} X_{t-2} + \dots + \phi_{p,p} X_{t-p} + \epsilon_t,$$

where $\phi_{1,p}, \phi_{2,p}, \dots, \phi_{p,p}$ are constants ($\phi_{p,p} \neq 0$) and $\{\epsilon_t\}$ is a zero mean white noise process with variance σ_ϵ^2 . In contrast to the parameters of an MA(q) process, the $\{\phi_{k,p}\}$ must satisfy certain conditions for $\{X_t\}$ to be a stationary

process – i.e., not all AR(p) processes are stationary (more later).

Examples (Figures 12 and 13) $P(X_{t+1}|X_t, X_{t-1}, X_{t-2}, \dots) = P(X_{t+1}|X_t)$

$$X_t = \phi_{1,1}X_{t-1} + \epsilon_t \quad \text{AR(1) - Markov process} \quad (2.1)$$

$$= \phi_{1,1}\{\phi_{1,1}X_{t-2} + \epsilon_{t-1}\} + \epsilon_t$$

$$= \phi_{1,1}^2X_{t-2} + \phi_{1,1}\epsilon_{t-1} + \epsilon_t$$

$$= \phi_{1,1}^3X_{t-3} + \phi_{1,1}^2\epsilon_{t-2} + \phi_{1,1}\epsilon_{t-1} + \epsilon_t$$

:

$$= \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k}.$$

$X_{t-2} = \phi_{1,1}X_{t-3} + \epsilon_{t-2}$
 $\text{MA}(\infty)$ more on this later.

Here we take the initial condition $X_{-N} = 0$ and let $N \rightarrow \infty$.

Note $E\{X_t\} = 0$.

$$\text{var}\{X_t\} = E \left\{ \left(\sum_{k=0}^{\infty} \phi^k \epsilon_{t-k} \right)^2 \right\} = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \phi^k \phi^{k'} E\{\epsilon_{t-k} \epsilon_{t-k'}\} = \sum_{k=0}^{\infty} \phi^{2k} \sigma_{\epsilon}^2.$$

For $\text{var}\{X_t\} < \infty$ we must have $|\phi_{1,1}| < 1$, in which case

$$\text{var}\{X_t\} = \frac{\sigma_{\epsilon}^2}{1 - \phi_{1,1}^2}. \quad = S_0$$

To find the form of the acvs, we notice that for $\tau > 0$, $X_{t-\tau}$ is a linear function of $\epsilon_{t-\tau}, \epsilon_{t-\tau-1}, \dots$ and is therefore uncorrelated with ϵ_t . Hence

$$E\{\epsilon_t X_{t-\tau}\} = 0,$$

$$X_{t-\tau} = \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-\tau-k}$$

$$E\{\epsilon_t X_{t-\tau}\} = \sum_{k=0}^{\infty} \phi_{1,1}^k E\{\epsilon_t \epsilon_{t-\tau-k}\}$$

" 0

so, assuming stationarity and multiplying the defining equation (2.1) by $X_{t-\tau}$:

$$X_t X_{t-\tau} = \phi_{1,1} X_{t-1} X_{t-\tau} + \epsilon_t X_{t-\tau}$$

$$\Rightarrow E\{X_t X_{t-\tau}\} = \phi_{1,1} E\{X_{t-1} X_{t-\tau}\} \quad s_{\tau-1}$$

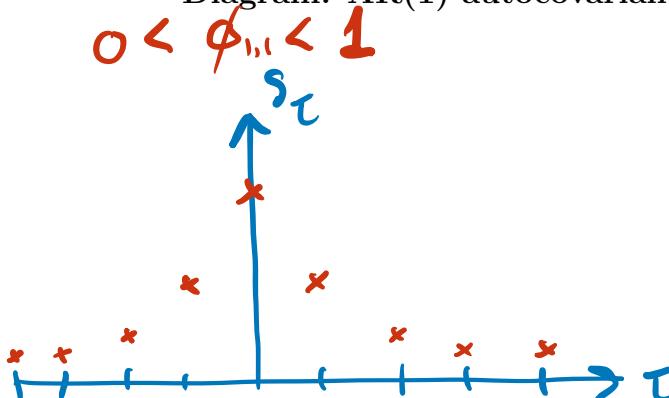
$$\text{i.e., } s_{\tau} = \phi_{1,1} s_{\tau-1} = \phi_{1,1}^2 s_{\tau-2} = \dots = \phi_{1,1}^{\tau} s_0$$

$$\Rightarrow \rho_{\tau} = \frac{s_{\tau}}{s_0} = \phi_{1,1}^{\tau} \quad s_{\tau-1} = \phi_{1,1} s_{\tau-2}.$$

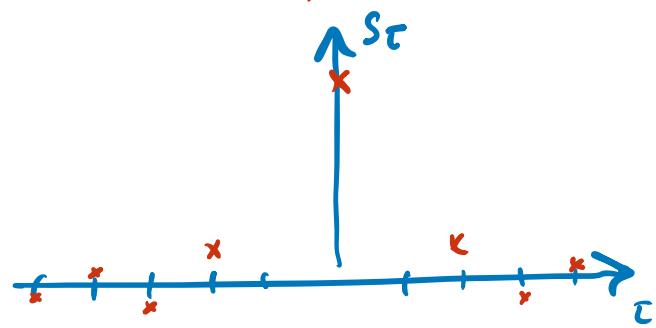
But ρ_{τ} is an even function of τ , so we obtain an exponentially decaying sequence

$$\rho_{\tau} = \phi_{1,1}^{|\tau|} \quad \tau = 0, \pm 1, \pm 2, \dots$$

Diagram: AR(1) autocovariance sequence



$-1 < \phi < 0$



[4] (p, q) 'th order autoregressive-moving average process

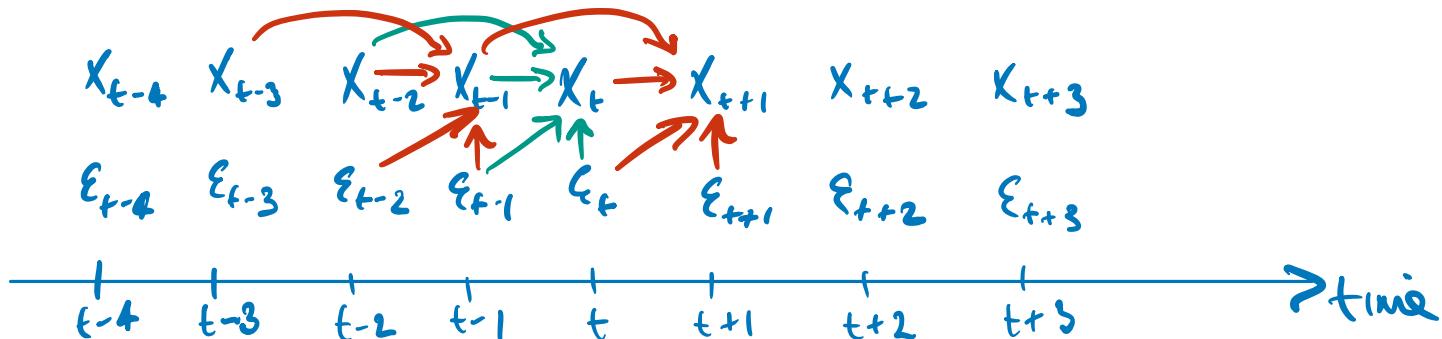
Here $\{X_t\}$ is expressed as

$$X_t = \underbrace{\phi_{1,p} X_{t-1} + \dots + \phi_{p,p} X_{t-p}}_{AR(p)} + \underbrace{\epsilon_t - \theta_{1,q} \epsilon_{t-1} - \dots - \theta_{q,q} \epsilon_{t-q}}_{MA(q)}$$

Allows for noise/shock terms to correlate

where the $\phi_{j,p}$'s and the $\theta_{j,q}$'s are all constants ($\phi_{p,p} \neq 0; \theta_{q,q} \neq 0$) and again $\{\epsilon_t\}$ is a zero mean white noise process with variance σ_ϵ^2 .

Diagram: ARMA(2,1) dependencies



The ARMA class is important as many data sets may be approximated in a more parsimonious way (meaning fewer parameters are needed) by a mixed ARMA model than by a pure AR or MA process.

Very flexible models, is prone to overfitting

$\begin{matrix} p+q+1 \\ \downarrow \quad \downarrow \\ AR \quad MA \end{matrix} \quad \sigma_\epsilon^2 \quad \text{parameters}$

Worked example: ARMA(p, q)

Suppose that $\{Y_t\}$ is a zero mean stationary $AR(p)$ process:

$$Y_t = \sum_{j=1}^p \phi_{j,p} Y_{t-j} + \epsilon_t.$$

Show that the process $\{X_t\}$ given by

$$X_t = \sum_{k=0}^q \beta_k Y_{t-k}; \quad \beta_0 = 1$$

is an ARMA(p, q) process (i.e. a moving average of an AR(p) process is ARMA).

$$\begin{aligned}
 X_t &= \sum_{k=0}^q \beta_k Y_{t-k} \\
 &= \sum_{k=0}^q \beta_k \left(\sum_{j=1}^p \phi_{j,p} Y_{t-k-j} + \epsilon_{t-k} \right) \\
 &= \sum_{j=1}^p \phi_{j,p} \left(\sum_{k=0}^q \beta_k Y_{t-k-j} \right) + \sum_{k=0}^q \beta_k \epsilon_{t-k} \\
 &= \underbrace{\sum_{j=1}^p \phi_{j,p} X_{t-j}}_{AR(p)} + \underbrace{\sum_{k=0}^q \beta_k \epsilon_{t-k}}_{MA(q)} \\
 &\text{Switch sum double}
 \end{aligned}$$

ARMA(p, q)

[5] p 'th order autoregressive conditionally heteroscedastic

model $ARCH(p)$ Used for modelling Volatility Clustering Video 10

Standard linear models were found to be inappropriate for describing the dependence of financial log-return series of stock indices, share prices, exchange rates etc. New multiplicative noise models were developed. One such is the $ARCH(p)$ model.

Assume we have a time series $\{X_t\}$ that has a variance (volatility) that changes through time,

$$X_t = \sigma_t \varepsilon_t \tag{2.2}$$

where here $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (iid) r.v.s with zero mean and unit variance. (This is stronger than simply uncorrelated). Here, σ_t represents the local conditional standard deviation of the process.

zero mean

$\gamma \sim \text{Bernoulli}(0.5)$ $X_t\}$ is ARCH(p) if it satisfies equation (2.2) and
 $X_t \sim \begin{cases} N(0,1) & \text{if } \gamma=0 \\ N(0,2) & \text{if } \gamma=1 \end{cases}$ $\sigma_t^2 = E\{X_t^2 | X_{t-1}, \dots, X_{t-p}\}$
 $\sigma_t^2 = \alpha + \beta_{1,p}X_{t-1}^2 + \dots + \beta_{p,p}X_{t-p}^2$, (2.3)

where $\alpha > 0$ and $\beta_{j,p} \geq 0, j = 1, \dots, p$ (to ensure σ_t^2 is positive).

$V\mu(x) = E(x^2)$ Example: ARCH(1)

$$\begin{aligned} &= 0.5 \cdot 1 + 0.5 \cdot 2 \\ &= 1.5 \end{aligned}$$

Define,

$$V\mu(x|\gamma=0) = 1$$

$$V\mu(x|\gamma=1) = 2$$

So $X_t^2 = \sigma_t^2 + v_t$ and the model can be written as

$$X_t^2 = \alpha + \beta_{1,1}X_{t-1}^2 + v_t,$$

$$\sigma_t^2 = \alpha + \beta_{1,1}X_{t-1}^2$$

$$E\{X_t^2 | X_{t-1}\}$$

$$v_t = X_t^2 - \sigma_t^2, \Rightarrow \sigma_t^2 = X_t^2 - v_t.$$

when stationary

$$\begin{aligned} E\{X_t^2\} &= E\{E\{X_t^2 | X_{t-1}\}\} \\ &= \alpha + \beta_{1,1}E\{X_{t-1}^2\}. \end{aligned}$$

$$E\{X_t^2\} = \frac{\alpha}{1 - \beta_{1,1}}$$

i.e., as an AR(1) model for $\{X_t^2\}$. The errors, $\{v_t\}$, have zero mean, but as

$v_t = \sigma_t^2(\epsilon_t^2 - 1)$ the errors are heteroscedastic.

$$v_t = \sigma_t^2 \epsilon_t^2 - \sigma_t^2 \quad E\{v_t\} = E\{\sigma_t^2\} \in \{\sigma_t^2 - 1\}.$$

[6] Harmonic with random amplitude (see Figures 14 and 14a)

Video 11

Here $\{X_t\}$ is expressed as

$$X_t = \epsilon_t \cos(2\pi f_0 t + \phi)$$



f_0 is a fixed frequency and $\{\epsilon_t\}$ is zero mean white noise with variance σ_ϵ^2 .

Case (a) ϕ is constant.

$$\begin{aligned} E\{X_t\} &= E\{\epsilon_t \cos(2\pi f_0 t + \phi)\} \\ &= E\{\epsilon_t\} \cos(2\pi f_0 t + \phi) = 0. \end{aligned}$$

$$\begin{aligned} \text{var}\{X_t\} &= E\{X_t^2\} \\ &= E\{\epsilon_t^2\} \cos^2(2\pi f_0 t + \phi) \\ &= \sigma_\epsilon^2 \cos^2(2\pi f_0 t + \phi). \end{aligned}$$

So the variance depends on t and the process is nonstationary.

Case (b) $\phi \sim U[-\pi, \pi]$ and indep. of $\{\epsilon_t\}$.

$$\text{random} \quad \text{random} \quad \text{independence}$$

$$E\{X_t\} = E\{\epsilon_t \cos(2\pi f_0 t + \phi)\} = E\{\epsilon_t\} E\{\cos(2\pi f_0 t + \phi)\} = 0.$$

" 0 "

$$\begin{aligned} \text{Independence of } \epsilon_t \text{ and } \phi & \quad \text{cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\} \\ & = E\{\epsilon_t \epsilon_{t+\tau}\} E\{\cos(2\pi f_0 t + \phi) \cos(2\pi f_0(t + \tau) + \phi)\} \end{aligned}$$

Since $\{\epsilon_t\}$ is white noise we have,

$$E\{\epsilon_t \epsilon_{t+\tau}\} = \begin{cases} \sigma_\epsilon^2 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0, \end{cases}$$

So, for $\tau \neq 0$, $\text{cov}\{X_t, X_{t+\tau}\} = 0$, while for $\tau = 0$,

$$\text{cov}\{X_t, X_t\} = s_0 = \sigma_\epsilon^2 E\{\cos^2(2\pi f_0 t + \phi)\}.$$

Now,

$$E\{g(Y)\} = \int g(y) f_Y(y) dy$$

$$\begin{aligned} E\{\cos^2(2\pi f_0 t + \phi)\} &= \int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \phi) \frac{1}{2\pi} d\phi \quad \text{pdf of } U[-\pi, \pi] \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [1 + \cos(4\pi f_0 t + 2\phi)] \frac{1}{2\pi} d\phi \\ &= \frac{1}{2}. \end{aligned}$$

$(\cos(a+b)) = (\cos(a))(\cos(b)) - \sin(a)\sin(b)$

So,

$$s_0 = \sigma_\epsilon^2 / 2,$$

and the process is stationary.

The random phase idea is illustrated in Figure 14a: the random point at which data collection started corresponds to breaking-in to the ‘sinusoidal-like’ behaviour at a random point, which equates to a random phase.

2.2 Trend removal and seasonal adjustment

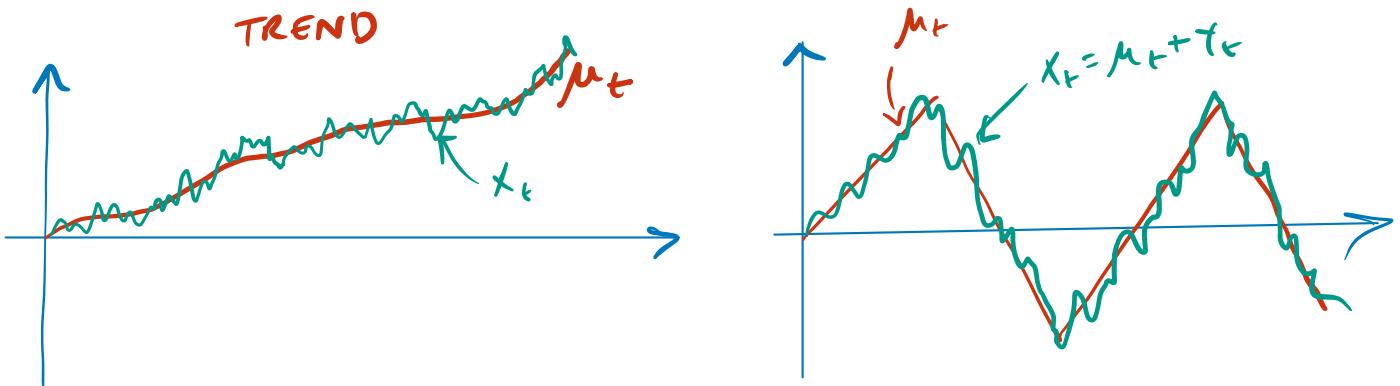
There are certain, quite common, situations where the observations exhibit a trend – a tendency to increase or decrease slowly steadily over time – or may fluctuate in a periodic/seasonal manner. The model is modified to

$$X_t = \mu_t + Y_t$$

μ_t = time dependent mean.

Y_t = zero mean stationary process.

Diagram: trend and seasonal model



Example CO₂ data

X_t = monthly atmospheric CO₂ concentrations expressed in parts per million (ppm) derived from in situ air samples collected at Mauna Loa observatory, Hawaii. Monthly data from May 1988 – December 1998, giving $N = 128$.

The data is plotted in Figure 15. We can see both a trend and periodic/seasonal effects.

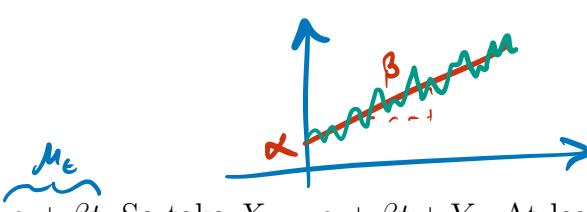
2.2.1 Trend adjustment

Represent a simple linear trend by $\alpha + \beta t$. So take $X_t = \alpha + \beta t + Y_t$. At least two possible approaches:

- (a) Estimate α and β by least squares, and work with the residuals

$$\hat{Y}_t = X_t - \hat{\alpha} - \hat{\beta}t.$$

For the CO₂ data these are shown in the middle plot of figure 15.



Benefits :
Attempts to recover
 Y_t directly
Drawbacks :
its only an
estimate of Y_t
So it will contain
errors

first differencing

(b) Take first differences:

$$X_t^{(1)} = X_t - X_{t-1} = \underbrace{X_t}_{\alpha + \beta t + Y_t} - \underbrace{X_{t-1}}_{\alpha + \beta(t-1) + Y_{t-1}} = \beta + Y_t - Y_{t-1} = \beta + Y_t^{(1)}$$

For the CO₂ data these are shown in the bottom plot of figure 15.

Note: if $\{Y_t\}$ is stationary so is $\{Y_t^{(1)}\}$ $E\{Y_t^{(1)}\} = E\{Y_t\} - E\{Y_{t-1}\} = 0$

$$\begin{aligned} \text{Cov}\{Y_t^{(1)}, Y_{t+\tau}^{(1)}\} &= E\{Y_t^{(1)}Y_{t+\tau}^{(1)}\} = E\{(Y_t - Y_{t-1})(Y_{t+\tau} - Y_{t+\tau-1})\} \\ &= S_\tau - S_{\tau+1} - S_{\tau-1} + S_\tau \quad \{S_\tau\} \text{ is a vs} \\ &= 2S_\tau - S_{\tau+1} - S_{\tau-1} \quad \text{of } \{Y_t\} \end{aligned}$$

In the case of linear trend, if we difference again:

$$\begin{aligned} X_t^{(2)} &= X_t^{(1)} - X_{t-1}^{(1)} = (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\ &= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\ &= Y_t - 2Y_{t-1} + Y_{t-2}, \quad (\equiv Y_t^{(1)} - Y_{t-1}^{(1)} = Y_t^{(2)}), \end{aligned}$$

so that the effect of $\mu_t (= \alpha + \beta t)$ has been completely removed.

Consider a quadratic trend $X_t = \alpha + \beta t + \gamma t^2 + Y_t$

$$\begin{aligned} X_t^{(1)} &= \alpha + \beta t + \gamma t^2 + Y_t - (\alpha + \beta[t-1] + \gamma[t-1]^2 + Y_{t-1}) \\ &= \beta + Y_t - Y_{t-1} + 2t\gamma - \gamma = \beta + Y_t^{(1)} + 2t\gamma - \gamma \end{aligned}$$

$$X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)} = Y_t^{(1)} - Y_{t-1}^{(1)} - 2\gamma = Y_t^{(2)} - 2\gamma$$

$$X_t^{(3)} = X_t^{(2)} - X_{t-1}^{(2)} = Y_t^{(2)} - Y_{t-1}^{(2)} = Y_t^{(3)}$$

$$\begin{aligned} X_t^{(3)} &= X_t^{(2)} - X_{t-1}^{(2)} = Y_t^{(2)} - Y_{t-1}^{(2)} = Y_t^{(3)} \\ &= X_t^{(1)} - X_{t-1}^{(1)} - (X_{t-1}^{(1)} - X_{t-2}^{(1)}) \\ &\vdots \\ &= X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3} \end{aligned}$$

If μ_t is a polynomial of degree $(d-1)$ in t , then d th differences of μ_t will be zero ($d = 2$ for linear trend). Further,

$$\begin{aligned} X_t^{(d)} &= \sum_{k=0}^d \binom{d}{k} (-1)^k X_{t-k} \\ &= \sum_{k=0}^d \binom{d}{k} (-1)^k Y_{t-k}. \end{aligned}$$

There are other ways of writing this. Define the difference operator

$$\Delta = (1 - B)$$

$$\begin{aligned}\Delta X_t &= (1 - B)X_t = X_t - BX_t \\ &= X_t - X_{t-1} \\ &= X_t^{(1)}\end{aligned}$$

where $BX_t = X_{t-1}$ is the *backward shift operator* (sometimes known as the *lag operator* L – especially in econometrics). Then,

$$X_t^{(d)} = \Delta^d X_t = \Delta^d Y_t.$$

$$\begin{aligned}\Delta^2 X_t &= \Delta \Delta X_t \\ &= \Delta X_t^{(1)} \\ &= X_t^{(1)} - X_{t-1}^{(1)} \\ &= X_t^{(2)}\end{aligned}$$

For example, for $d = 2$:

used for polynomials, 1st order aka linear

$$\begin{aligned}X_t^{(2)} &= (1 - B)^2 X_t = (1 - B)(X_t - X_{t-1}) \\ &= (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\ &= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\ &= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) \\ &= (1 - B)^2 Y_t = \Delta^2 Y_t.\end{aligned}$$

This notation can be incorporated into the ARMA set up. Recall if $\{X_t\}$ is ARMA(p, q),

$$X_t = \underbrace{\phi_{1,p}X_{t-1} + \dots + \phi_{p,p}X_{t-p}}_{AR(p)} + \epsilon_t - \underbrace{\theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q}}_{MA(q)},$$

$$\begin{aligned}AR &\quad X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q} \quad MA \\ (1 - \phi_{1,p}B - \phi_{2,p}B^2 - \dots - \phi_{p,p}B^p)X_t &= (1 - \theta_{1,q}B - \theta_{2,q}B^2 - \dots - \theta_{q,q}B^q)\epsilon_t \\ \Phi(B)X_t &= \Theta(B)\epsilon_t.\end{aligned}$$

Here

$$\begin{aligned}AR \quad \Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - \dots - \phi_{p,p}B^p \\ MA \quad \text{and} \quad \Theta(B) &= 1 - \theta_{1,q}B - \theta_{2,q}B^2 - \dots - \theta_{q,q}B^q\end{aligned}$$

are known as the *associated or characteristic polynomials*.

Further, we can generalize the class of ARMA models to include differencing to account for certain types of non-stationarity, namely, X_t is called ARIMA(p, d, q) if

$$\Phi(B)(1 - B)^d X_t = \Theta(B)\epsilon_t,$$

$$\text{or } \Phi(B)\underbrace{\Delta^d X_t}_{d^{\text{th}} \text{ differenced}} = \Theta(B)\epsilon_t.$$

25

AR
MA
The d^{th} process is difference ARMA(p, q)

2.2.2 Seasonal adjustment

The model is

$$X_t = \nu_t + Y_t$$

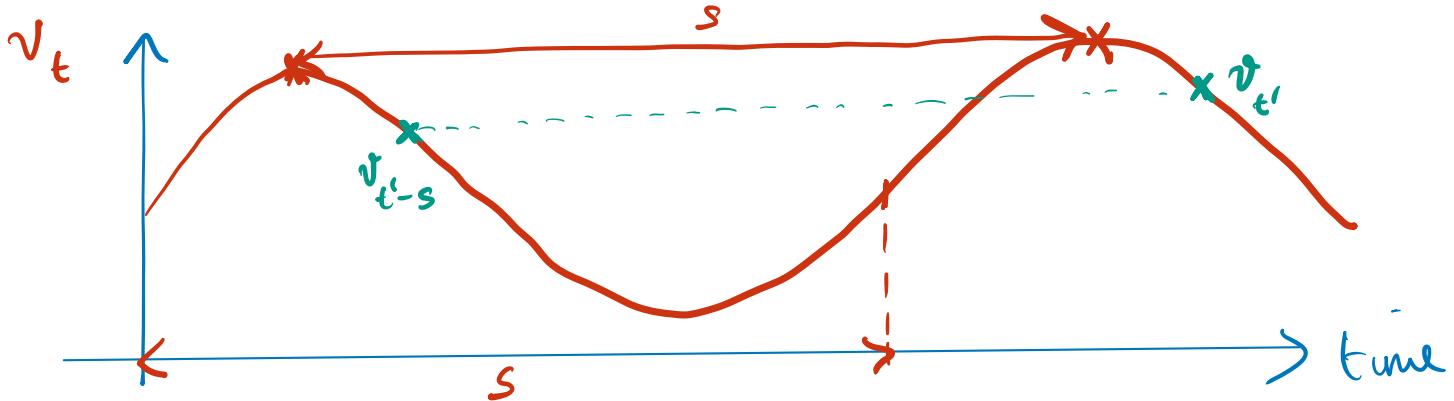
where

ν_t = seasonal component,

Y_t = zero mean stationary process.

Presuming that the seasonal component maintains a constant pattern over time with period s , there are again several approaches to removing ν_t .

Diagram: seasonal behaviour



A popular approach used by Box & Jenkins is to use the operator $(1 - B^s)$:

$$\begin{aligned} X_t^{(s)} &= (1 - B^s)X_t = X_t - X_{t-s} \\ &= (\nu_t + Y_t) - (\nu_{t-s} + Y_{t-s}) \\ &= Y_t - Y_{t-s} \end{aligned}$$

since ν_t has period s (and so $\nu_{t-s} = \nu_t$).

Figure 16 shows this technique applied to the CO₂ data – most of the seasonal structure and trend has been removed by applying the seasonal operator after the differencing operator:

$$(1 - B^{12})(1 - B)X_t.$$

Exercise

Worked example: trend and seasonality

Consider the model

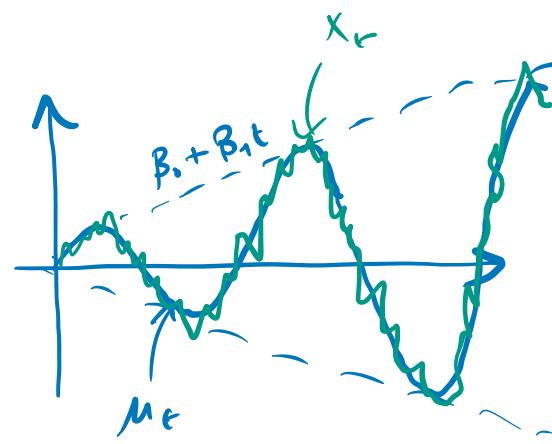
$$X_t = (\beta_0 + \beta_1 t) \nu_t + Y_t$$

where Y_t is a zero-mean stationary process. Show that

ν_t has period 12

$$W_t = (1 - B^{12})^2 X_t$$

only involves $\{Y_t\}$.



$$\begin{aligned}
 (1 - B^{12})^2 &= (1 - B^{12})(1 - B^{12}) X_t \\
 &= (1 - B^{12})(X_t - X_{t-12}) \\
 &= (1 - B^{12}) \left[(\beta_0 + \beta_1 t) \nu_t + Y_t - (\beta_0 + \beta_1 [t-12]) \nu_{t-12} \right. \\
 &\quad \left. - Y_{t-12} \right] \\
 &= (1 - B^{12}) [Y_t + 12 \beta_1 \nu_t - Y_{t-12}] \\
 &= Y_t + 12 \beta_1 \nu_t - Y_{t-12} - Y_{t-12} - 12 \beta_1 \nu_{t-12} + Y_{t-24} \\
 &= Y_t - 2 Y_{t-12} + Y_{t-24}
 \end{aligned}$$

Which only involves $\{Y_t\}$

2.3 The General Linear Process and the stationarity and invertability of AR/MA/ARMA processes

Video 14

2.3.1 General Linear Process

Consider a process of the form

$$X_t = \sum_{k=-\infty}^{\infty} g_k \epsilon_{t-k},$$

where $\{\epsilon_t\}$ is a purely random white noise process, and $\{g_k\}$ is a given sequence of real-valued constants satisfying $\sum_{k=-\infty}^{\infty} g_k^2 < \infty$, which ensures that $\{X_t\}$ has finite variance.

Note: this processes breaks causality with the $\{X_t\}$ depending on future values of $\{\epsilon_t\}$. We therefore restrict ourselves to

$$g_{-1}, g_{-2}, \dots = 0,$$

and we obtain what is called the General Linear Process (GLP)

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k},$$

where X_t depends only on present and past values $\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \dots$ of the purely random process.

The GLP is stationary because

$$\begin{aligned} E\{X_t\} &= \sum_{k=0}^{\infty} g_k E\{\epsilon_{t-k}\} = 0 \\ \text{cov}\{X_t, X_{t+\tau}\} &= E\{X_t X_{t+\tau}\} \\ &= E\left\{ \sum_{k=0}^{\infty} g_k \epsilon_{t-k} \sum_{k'=0}^{\infty} g_{k'} \epsilon_{t+\tau-k'} \right\} \\ &= \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} g_k g_{k'} E\{\epsilon_{t-k} \epsilon_{t+\tau-k'}\} \\ &= \sigma_{\epsilon}^2 \sum_{k=0}^{\infty} g_k g_{k+\tau}. \end{aligned}$$

$X_{t+\tau} = \sum_{k=0}^{\infty} g_k \epsilon_{t+\tau-k}$
 $t-k = t+\tau-k'$
 $k' = k-\tau$

This clearly depends only on τ . Furthermore, we know $|\rho_{\tau}| \leq 1$, so

$$|s_{\tau}| = |\text{cov}\{X_t, X_{t+\tau}\}| \leq \sigma_X^2 = \sigma_{\epsilon}^2 \sum_k g_k^2 < \infty,$$

so the covariance is bounded also. Hence, a GLP is stationary.

If we can write an AR or ARMA process in GLP form then we know it is stationary. NOTE: we have already shown MA processes are stationary, moreover, a MA process is already in GLP form.

To do this, we introduce the “z-polynomial”

$$\text{MA}(q) \quad X_t = -\sum_{k=0}^{\infty} \theta_{k,q} \epsilon_{t-k}$$

$$G(z) = \sum_{k=0}^{\infty} g_k z^k,$$

$$\text{where } z \in \mathbb{C}. \text{ Note } X_t = G(B)\epsilon_t = \sum_{k=0}^{\infty} g_k B^k \epsilon_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k} \quad g_k = -\theta_{k,q} \quad 0 \leq k \leq q$$

$$g_k = 0 \quad k > q$$

We will be dealing with z-polynomials of the form

$$\text{poles of } G(z) \text{ are where } G(z) \text{ is infinite} \quad G(z) = \frac{G_1(z)}{G_2(z)}, \text{ say.}$$

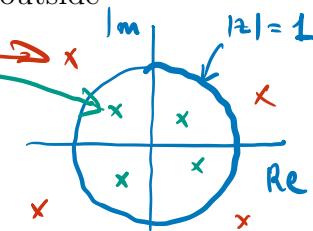
$$\text{roots of } G(z) = 0 \quad G_2(z) = (z-z_1)(z-z_2)\dots(z-z_p)$$

Call the roots of $G_2(z)$ (the “poles” of $G(z)$) in the complex plane z_1, z_2, \dots, z_p , where the zeros are ordered so that z_1, \dots, z_k are inside and z_{k+1}, \dots, z_p are outside the unit circle $|z| = 1$. Then,

$$\frac{1}{G_2(z)} = \sum_{j=1}^p \frac{A_j}{z - z_j} = \sum_{j=1}^k \frac{A_j}{z} \times \frac{1}{(1 - \frac{z_j}{z})} + \sum_{j=k+1}^p \frac{A_j}{z_j} \times \frac{-1}{(1 - \frac{z}{z_j})}$$

partial fractions

$$= \sum_{j=1}^k \frac{A_j}{z} \sum_{l=0}^{\infty} \left(\frac{z_j}{z}\right)^l - \sum_{j=k+1}^p \frac{A_j}{z_j} \sum_{l=0}^{\infty} \left(\frac{z}{z_j}\right)^l$$



which is uniformly convergent for $|z| = 1$. Replace z by the backshift operator B and apply to $\{\epsilon_t\}$:

$$X_t = \frac{g_1(B)}{g_2(B)} \epsilon_t \quad \left\{ \frac{1}{G_2(B)} \right\} \epsilon_t = \left\{ \sum_{j=1}^k A_j B^{-1} \sum_{l=0}^{\infty} z_j^l B^{-l} - \sum_{j=k+1}^p A_j z_j^{-1} \sum_{l=0}^{\infty} z_j^{-l} B^l \right\} \epsilon_t$$

$$= \underbrace{\sum_{j=1}^k A_j \sum_{l=0}^{\infty} z_j^l \epsilon_{t+l+1}}_{\text{Roots of } g_2(z) \text{ inside unit circle, correspond to future terms of } \{\epsilon_t\}} - \underbrace{\sum_{j=k+1}^p A_j \sum_{l=0}^{\infty} z_j^{-l-1} \epsilon_{t-l}}_{\text{Roots of } g_2(z) \text{ outside unit circle, correspond to past and present terms.}}$$

Consider B to have unit norm.

$\text{Roots of } g_2(z) \text{ inside unit circle, correspond to future terms of } \{\epsilon_t\}$

$\text{Roots of } g_2(z) \text{ outside unit circle, correspond to past and present terms.}$

Hence, if all the roots of $G_2(z)$ are outside the unit circle (i.e. all the poles of $G(z)$ are outside the unit circle) only past and present values of $\{\epsilon_t\}$ are involved and the GLP exists.

Another way of stating this is that

$$G(z) < \infty \quad |z| \leq 1 \quad \Leftrightarrow$$

i.e., $G(z)$ is analytic inside and on the unit circle.

So, all the

$$\begin{cases} \text{poles of } G(z) \text{ lie outside the unit circle} \\ \text{roots (zeros) of } G^{-1}(z) \text{ lie outside the unit circle} \end{cases}$$

No roots of $G_2(z)$
inside or on the
unit circle.
GLP exists

2.3.2 Stationarity

For the AR(p) process

$$\begin{aligned} X_t &= \phi_{1,p} X_{t-1} + \dots + \phi_{p,p} X_{t-p} + \epsilon_t \\ X_t - \phi_{1,p} X_{t-1} - \dots - \phi_{p,p} X_{t-p} &= \epsilon_t \\ \Phi(B)X_t &= \epsilon_t \\ \Rightarrow X_t &= \Phi^{-1}(B)\epsilon_t = G(B)\epsilon_t, \end{aligned}$$

$\Phi(B) = 1 - \phi_{1,p}B - \dots - \phi_{p,p}B^p$
where $G(B) = \frac{1}{\Phi(B)}$

so that $G(z) = \Phi^{-1}(z)$. Thus the model is stationary if

$$G(z) < \infty, \quad |z| \leq 1.$$

\Rightarrow All the poles of $G(z)$ are outside the unit circle.

Hence the requirement for stationarity is that all the roots of $\Phi(z)$ must lie outside the unit circle.

For the ARMA(p, q) process

$$\begin{aligned} \Phi(B)X_t &= \Theta(B)\epsilon_t \\ \Rightarrow X_t &= \frac{\Theta(B)}{\Phi(B)}\epsilon_t = G(B)\epsilon_t, \end{aligned}$$

$$G(B) = \frac{\Theta(B)}{\Phi(B)}.$$

so that $G(z) = \frac{\Theta(z)}{\Phi(z)}$. Thus the model is stationary if

$$G(z) < \infty, \quad |z| \leq 1.$$

\Rightarrow All the poles of $G(z)$ are outside the unit circle.

Hence the requirement for stationarity is that all the roots of $\Phi(z)$ must lie outside the unit circle.

For the MA(q) process

$$X_t = \Theta(B)\epsilon_t = G(B)\epsilon_t$$

and since $G(B) = \Theta(B)$ is a polynomial of finite order $G(z) < \infty$, $|z| \leq 1$, automatically.

Video 16

2.3.3 Invertibility

We say a process is invertible if it can be written in AR form (AR processes are therefore invertible by definition).

Consider inverting the GLP into autoregressive form

$$\begin{aligned} X_t &= \sum_{k=0}^{\infty} g_k \epsilon_{t-k} \\ &= \sum_{k=0}^{\infty} g_k B^k \epsilon_t \\ X_t &= G(B)\epsilon_t \end{aligned}$$

non-negative powers of B

↓

$$\text{AR: } \Phi(B)X_t = \epsilon_t \Rightarrow G^{-1}(B)X_t = \epsilon_t$$

The expansion of $G^{-1}(B)$ in powers of B gives the required autoregressive form provided $G^{-1}(B)$ admits a causal power series expansion

$$G^{-1}(z) = \sum_{k=0}^{\infty} h_k z^k.$$

Our requirement now is that $G^{-1}(z)$ is analytic for $|z| \leq 1$. Thus the model is invertible if

$$G^{-1}(z) < \infty, \quad |z| \leq 1.$$

⇒ All the poles of $G^{-1}(z)$ are outside the unit circle. Equivalently, all roots of $G(z)$ are outside the unit circle.

Consider the MA(q) model

$$X_t = \Theta(B)\epsilon_t, \quad \Theta(B) = 1 - \theta_{1,q} B - \theta_{2,q} B^2 - \dots - \theta_{q,q} B^q$$

then,

$$\Theta^{-1}(B)X_t = \epsilon_t$$

and in general, the expansion of $\Theta^{-1}(B)$ is a polynomial of infinite order. Hence, $G^{-1}(z) = \Theta^{-1}(z)$, and so the invertibility condition is that $\Theta(z)$ has no roots inside or on the unit circle; i.e. all the roots of $\Theta(z)$ lie outside the unit circle.

For the ARMA(p, q) process

$$\begin{aligned}\Phi(B)X_t &= \Theta(B)\epsilon_t \\ \Rightarrow \frac{\Phi(B)}{\Theta(B)}X_t &= \epsilon_t\end{aligned}$$

so that $G^{-1}(z) = \frac{\Phi(z)}{\Theta(z)}$. Thus the model is invertible if

$$G^{-1}(z) < \infty, \quad |z| \leq 1.$$

\Rightarrow All the poles of $G^{-1}(z)$ are outside the unit circle.

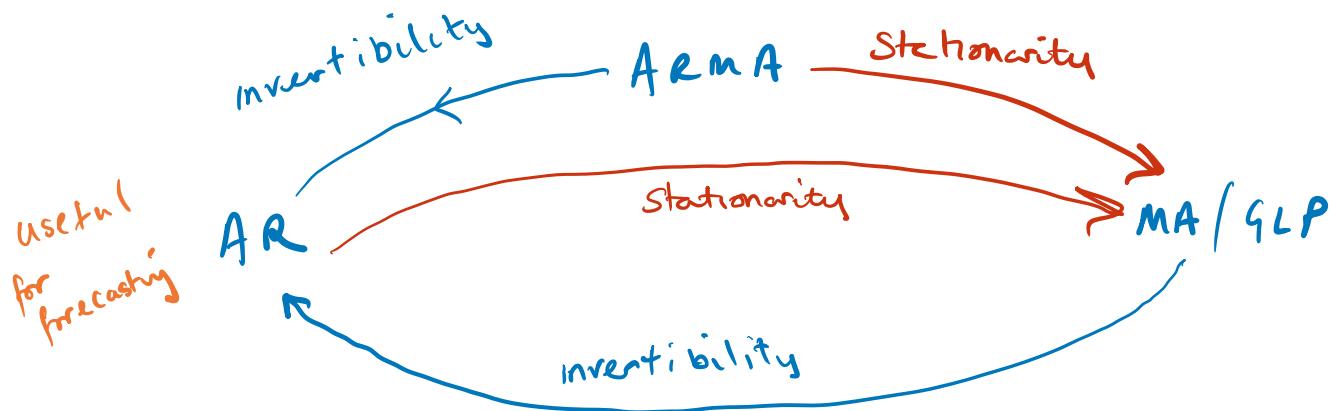
Hence the requirement for invertibility is that all the roots of $\Theta(z)$ must lie outside the unit circle.

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

2.3.4 Summary and examples

	AR(p)	MA(q)	ARMA(p, q)
Stationarity	Roots of $\Phi(z)$ outside $ z \leq 1$	Always stationary	Roots of $\Phi(z)$ outside $ z \leq 1$
Invertibility	Always invertible	Roots of $\Theta(z)$ outside $ z \leq 1$	Roots of $\Theta(z)$ outside $ z \leq 1$

Diagram: moving between different processes



Worked example: stationarity Determine whether the following AR process is stationary

$$X_t = \frac{5}{2}X_{t-1} - X_{t-2} + \epsilon_t.$$

$$\Phi(B)X_t = \epsilon_t$$

$$X_t - \frac{5}{2}X_{t-1} + X_{t-2} = \epsilon_t$$

$$(1 - \frac{5}{2}B + B^2)X_t = \epsilon_t$$

$$\Phi(B)X_t = \epsilon_t \quad \text{where} \quad \Phi(z) = 1 - \frac{5}{2}z + z^2$$

$$1 - \frac{5}{2}z + z^2 = 0 \Rightarrow 2 - 5z + 2z^2 = 0$$

$$(2z-1)(z-2) = 0$$

$$\Rightarrow \text{Roots are } z=2 \text{ and } z=\frac{1}{2}$$

\therefore Not stationary

Worked example: invertability

Determine whether the following MA process is invertible

$$X_t = \epsilon_t - 1.3\epsilon_{t-1} + 0.4\epsilon_{t-2}.$$

$$X_t = (1 - 1.3B + 0.4B^2)\epsilon_t$$

$$X_t = \Theta(B)\epsilon_t \quad \text{where} \quad \Theta(z) = 1 - 1.3z + 0.4z^2$$

$$1 - 1.3z + 0.4z^2 = 0$$

$$4z^2 - 13z + 10 = 0$$

$$(4z-5)(z-2) = 0 \Rightarrow \text{Roots at } z = \frac{5}{4}, z = 2.$$

Both are outside unit circle
 \therefore Invertible.

Worked example: stationarity and invertability

Determine whether the following model is stationary and/or invertible,

$$X_t = 1.3X_{t-1} - 0.4X_{t-2} + \epsilon_t - 1.5\epsilon_{t-1}. \quad \text{ARMA}(2,1)$$

$$(1 - 1.3B + 0.4B^2)X_t = (1 - 1.5B)\epsilon_t$$

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

So $\Phi(z) = 1 - 1.3z + 0.4z^2$
has roots outside of unit circle \Rightarrow stationary

$\Theta(z) = 1 - 1.5z$, which has root of $2/3$
hence not invertible.

Worked example: stationarity, invertability and autocovariance

Consider the ARMA(1,1) model

$$X_t = \phi X_{t-1} + \epsilon_t - \theta \epsilon_{t-1}$$

1. State conditions on ϕ and θ for $\{X_t\}$ to be both stationary and invertible.
2. In the case where it is stationary, express $\{X_t\}$ as a GLP.
3. Determine the autocovariance sequence $\{s_\tau\}$ of $\{X_t\}$.

$$1) X_t - \phi X_{t-1} = \epsilon_t - \theta \epsilon_{t-1} \Rightarrow \Phi(z) = 1 - \phi z \\ \Theta(z) = 1 - \theta z$$

\therefore Root of Φ is $z = 1/\phi \Rightarrow |\phi| < 1$ for stationarity
 Similarly $|\theta| < 1$ for invertibility

$$2) GLP \text{ form is } X_t = q(B) \epsilon_t \\ (1 - \phi B) X_t = (1 - \theta B) \epsilon_t \\ X_t = \frac{(1 - \theta B)}{(1 - \phi B)} \epsilon_t$$

$$\begin{aligned} \text{So } q(z) &= \frac{1 - \theta z}{1 - \phi z} = (1 - \theta z)(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \dots) \\ &= 1 + \phi z + \phi^2 z^2 + \dots - \theta z - \theta \phi z^2 - \theta \phi^2 z^3 + \dots \\ &= \sum_{k=0}^{\infty} \phi^k z^k - \theta \sum_{k=0}^{\infty} \phi^k z^{k+1} \\ &= 1 + (\phi - \theta) \sum_{k=1}^{\infty} \phi^{k-1} z^k \end{aligned}$$

$$x_t = \left(1 + (\phi - \theta) \sum_{k=1}^{\infty} \phi^{k-1} g_k \right) \varepsilon_t$$

$$= \varepsilon_t + (\phi - \theta) \sum_{k=1}^{\infty} \phi^{k-1} \varepsilon_{t-k}$$

This is in GLP form $g_0 = 1$ and $g_k = (\phi - \theta) \phi^{k-1}$ ($k \geq 1$).

$$\begin{aligned} 3). \quad s_0 &= \sigma_\varepsilon^2 \sum_{k=0}^{\infty} g_k^2 = \sigma_\varepsilon^2 \left(1 + (\phi - \theta)^2 + (\phi - \theta)^2 \phi^2 + (\phi - \theta)^2 \phi^4 + \dots \right) \\ &= \sigma_\varepsilon^2 \left(1 + (\phi - \theta)^2 \sum_{k=0}^{\infty} \phi^{2k} \right) \\ &= \sigma_\varepsilon^2 \left(1 + \frac{(\phi - \theta)^2}{1 - \phi^2} \right) \end{aligned}$$

$$\begin{aligned} \text{if } \tau \neq 0 \quad s_\tau &= \sigma_\varepsilon^2 \sum_{k=0}^{\infty} g_k g_{k+\tau} \\ &= \sigma_\varepsilon^2 \left(1 \cdot (\phi - \theta) \phi^{\tau-1} + (\phi - \theta) \cdot (\phi - \theta) \phi^\tau \right. \\ &\quad \left. + (\phi - \theta) \phi (\phi - \theta) \phi^{\tau+1} + \dots \right) \end{aligned}$$

$$= \sigma_\varepsilon^2 \left((\phi - \theta) \phi^{\tau-1} + (\phi - \theta)^2 \phi^\tau \sum_{k=0}^{\infty} \phi^{2k} \right).$$

$$= \sigma_\varepsilon^2 \left((\phi - \theta) \phi^{\tau-1} + (\phi - \theta)^2 \frac{\phi^\tau}{1 - \phi^2} \right)$$

Chapter 3

Video 18

Spectral analysis of discrete time stationary processes

Spectral analysis is a study of the frequency domain characteristics of a process, and describes the contribution of each frequency to the variance of the process.

3.1 Spectral representation for discrete time stationary processes

Video 19

3.1.1 The spectral representation theorem

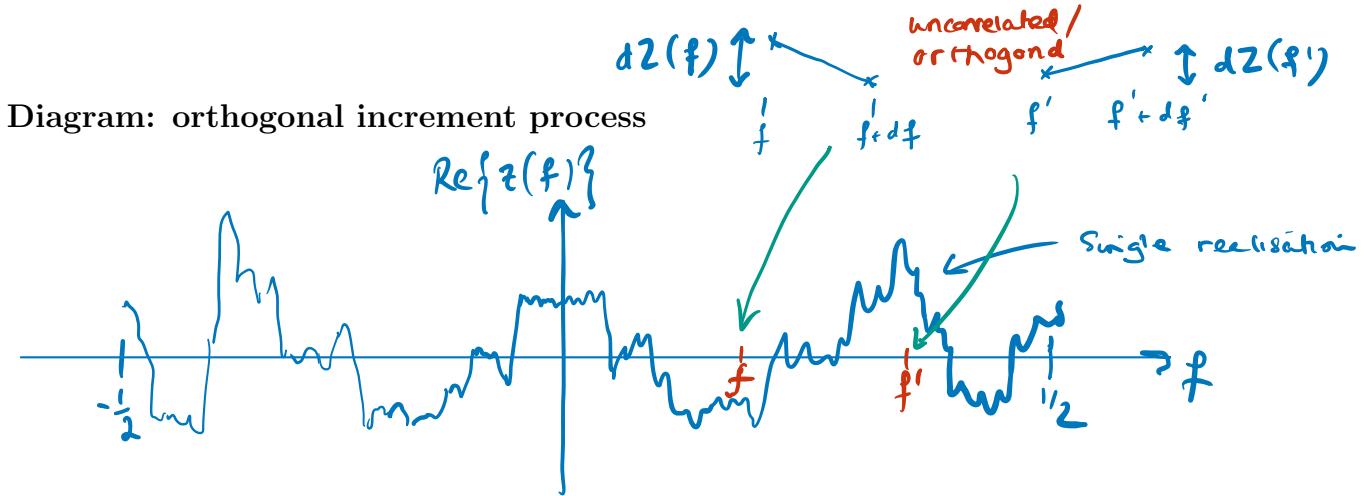
Let us define a complex-valued “jump” process $\{Z(f)\}$ on $[-1/2, 1/2]$

$$dZ(f) \equiv \begin{cases} Z(f + df) - Z(f), & 0 \leq f < 1/2; \\ 0, & f = 1/2; \\ dZ^*(-f), & -1/2 \leq f < 0, \end{cases}$$

where df is a small positive increment. If the intervals $[f, f + df]$ and $[f', f' + df']$ are non-intersecting subintervals of $[-1/2, 1/2]$, then the r.v.’s $dZ(f)$ and $dZ(f')$ are uncorrelated. We say that the process has *orthogonal increments*, and the process itself is called an *orthogonal process* – this orthogonality result is very important.

orthogonality = uncorrelated

See the inner product interpretation
of Covariance



Let $\{X_t\}$ be a real-valued discrete time stationary process, with zero mean.

The *spectral representation theorem* states that there exists an orthogonal process $\{Z(f)\}$, defined on $[-1/2, 1/2]$, such that

Stochastic version of
 the Fourier transform
 (Fourier 1822
 Heat equations)

for all integers t . The process $\{Z(f)\}$ has the following properties:

[1] $E\{dZ(f)\} = 0, \forall |f| \leq 1/2$.

[2] $E\{|dZ(f)|^2\} \equiv dS^{(I)}(f)$, say, $\forall |f| \leq 1/2$, where $S^{(I)}(f)$ is called the integrated spectrum of $\{X_t\}$.

[3] For any two distinct frequencies f and $f' \in (-1/2, 1/2]$

$$\text{cov}\{dZ(f'), dZ(f)\} = E\{dZ^*(f')dZ(f)\} = 0.$$

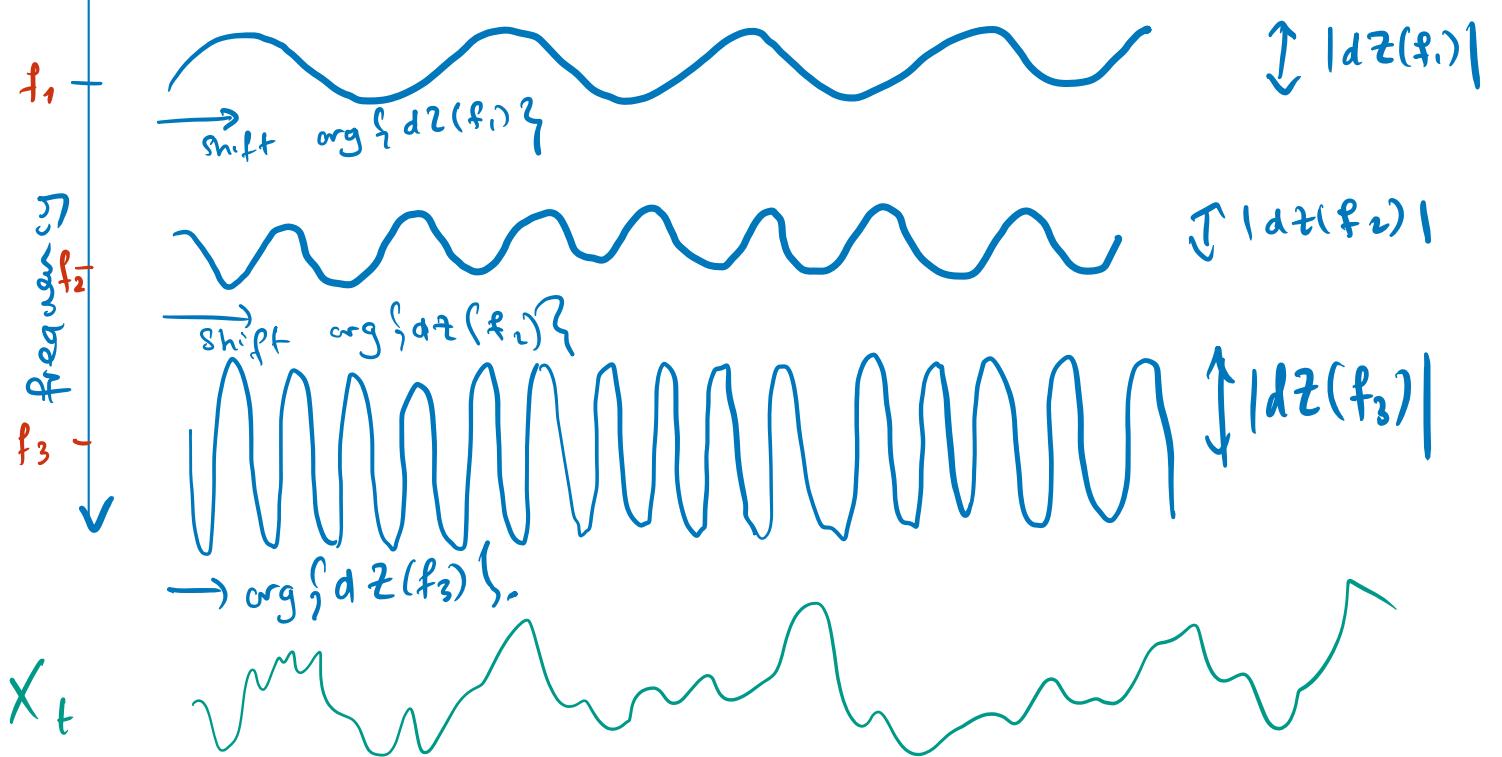
For complex r.v.s $\text{Cov}\{Wz\} = E\{W^*z\} - E\{W^*\}E\{z\}$.

The spectral representation

$$z = re^{i\theta} \quad X_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ(f) = \int_{-1/2}^{1/2} e^{i2\pi ft} |dZ(f)| e^{i\arg\{dZ(f)\}},$$

means that we can represent any discrete stationary process as an “infinite” sum of complex exponentials at frequencies f with associated random amplitudes $|dZ(f)|$ and random phases $\arg\{dZ(f)\}$.

Diagram: spectral representation



The orthogonal increments property can be used to define the relationship between the autocovariance sequence $\{s_\tau\}$ and the integrated spectrum $S^I(f)$:

$$s_\tau = E\{X_t X_{t+\tau}\} = E\{X_t^* X_{t+\tau}\} = E \int_{-1/2}^{1/2} e^{-i2\pi f' t} dZ^*(f') \int_{-1/2}^{1/2} e^{i2\pi f(t+\tau)} dZ(f)$$

$$E\{X_t X_{t+\tau}\} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{i2\pi(f-f')t} e^{i2\pi f\tau} E\{dZ^*(f') dZ(f)\}.$$

Because of the orthogonal increments property,

$$E\{dZ^*(f') dZ(f)\} = \begin{cases} dS^{(I)}(f) & f = f' \\ 0 & f \neq f' \end{cases}$$

so

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f\tau} dS^{(I)}(f),$$

which shows that the integrated spectrum determines the acvs for a stationary process. If in fact $S^{(I)}(f)$ is differentiable everywhere with a derivative denoted by $S(f)$ we have

$$E\{|dZ(f)|^2\} = dS^{(I)}(f) = S(f) df.$$

The function $S(\cdot)$ is called the spectral density function (sdf). Hence

$$s_\tau = \int_{-1/2}^{1/2} S(f) e^{i2\pi f\tau} df.$$

But, from standard Fourier theory, a square summable deterministic sequence $\{g_t\}$ say has the Fourier representation

$$g_t = \int_{-1/2}^{1/2} G(f) e^{i2\pi f t} df,$$

where

$$G(f) = \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi f t}.$$

If we assume that $S(f)$ is square integrable, then $S(f)$ is the Fourier transform (FT) of $\{s_\tau\}$,

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f \tau}.$$

Hence,

$$\{s_\tau\} \longleftrightarrow S(f),$$

i.e., $\{s_\tau\}$ and $S(f)$ are a FT pair.

1-1 mapping between
 $S(f)$ and s_τ
preservation of information

Video 20

3.1.2 Spectral Density Function

Subject to its existence, $S(\cdot)$ has the following interpretation: $S(f) df$ is the average contribution (over all realizations) to the power from components with frequencies in a small interval about f . The power – or variance – is

$$S_0 = \int_{-1/2}^{1/2} S(f) e^{i2\pi f \cdot 0} df = \int_{-1/2}^{1/2} S(f) df \quad \sigma^2 = \text{var}\{X_t\} = \int_{-1/2}^{1/2} S(f) df.$$

Hence, $S(f)$ is often called the power spectral density function or just power spectrum.

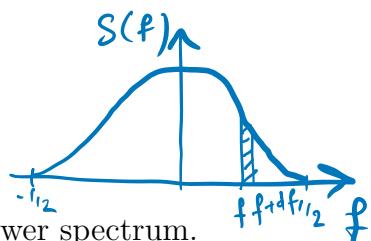
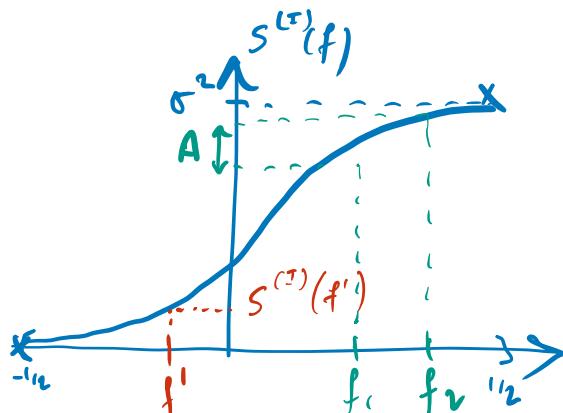
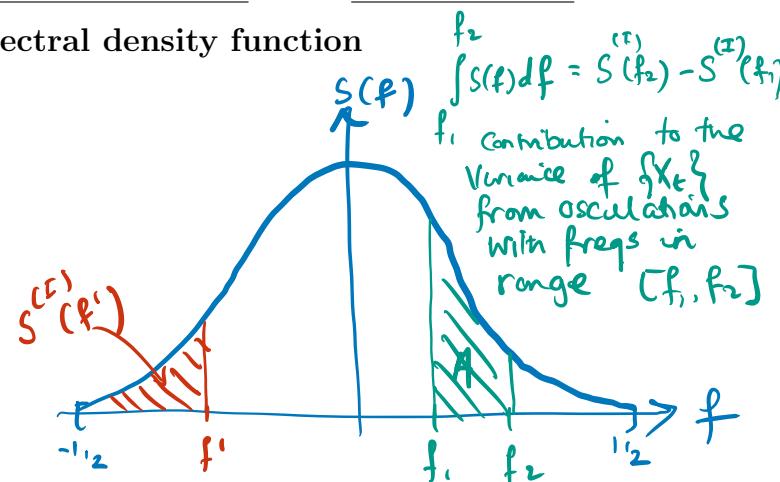


Diagram: integrated spectrum and spectral density function



Similar to CDF



Similar to a pdf

Properties: (assuming existence)

[1] $S^{(I)}(f) = \int_{-1/2}^f S(f') df'$. Fundamental Thm of Calculus

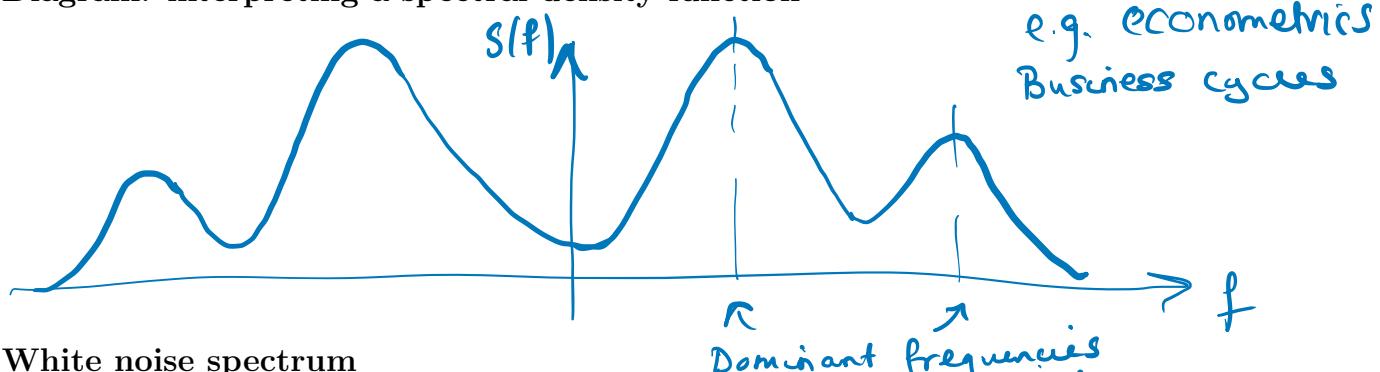
[2] $0 \leq S^{(I)}(f) \leq \sigma^2; \quad S(f) \geq 0$.

[3] $S^{(I)}(-1/2) = 0; \quad S^{(I)}(1/2) = \sigma^2; \quad \int_{-1/2}^{1/2} S(f) df = \sigma^2$.

[4] $f < f' \Rightarrow S^{(I)}(f) \leq S^{(I)}(f');$ $S(-f) = S(f)$. Hint: s_τ is symmetric exercise to show

Except, basically, for the scaling factor σ^2 , $S^{(I)}(f)$ has all the properties of a probability distribution function, and hence is sometimes called a spectral distribution function.

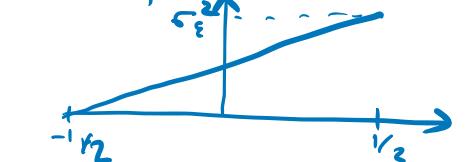
Diagram: interpreting a spectral density function



White noise spectrum

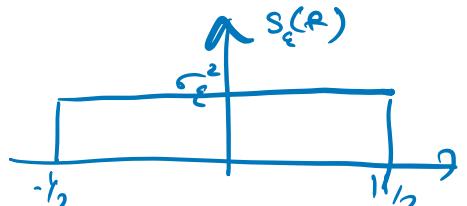
Recall that a white noise process $\{\epsilon_t\}$ has acvs:

$$s_\tau = \begin{cases} \sigma_\epsilon^2 & \tau = 0 \\ 0 & \text{otherwise.} \end{cases}$$



Therefore, the spectrum of a white noise process is given by:

$$S_\epsilon(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f \tau} = s_0 = \sigma_\epsilon^2,$$



i.e., white noise has a constant spectrum.

Video 21

3.1.3 Classification of Spectra

For most practical purposes any integrated spectrum, $S^{(I)}(f)$ can be written as

$$S^{(I)}(f) = S_1^{(I)}(f) + S_2^{(I)}(f)$$

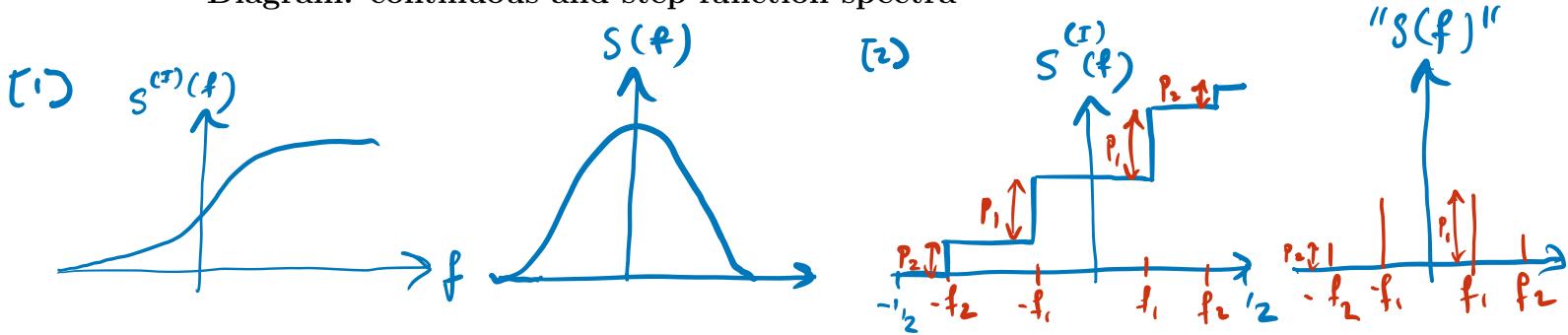
where the $S_j^{(I)}(f)$'s are nonnegative, nondecreasing functions with $S_j^{(I)}(-1/2) = 0$ and are of the following types:

- [1] $S_1^{(I)}(\cdot)$ is absolutely continuous, i.e., its derivative exists for almost all f and is equal almost everywhere to an sdf $S(\cdot)$ such that

$$S_1^{(I)}(f) = \int_{-1/2}^f S(f') df'.$$

- [2] $S_2^{(I)}(\cdot)$ is a step function with jumps of size $\{p_l : l = 1, 2, \dots\}$ at the points $\{f_l : l = 1, 2, \dots\}$.

Diagram: continuous and step function spectra



We consider the integrated spectrum for two ‘pure’ forms :

case (a): $S_1^{(I)}(f) \geq 0; S_2^{(I)}(f) = 0$.

$\{X_t\}$ is said to have a purely continuous spectrum and $S(f)$ is absolutely integrable, with

$$\int_{-1/2}^{1/2} S(f) \cos(2\pi f\tau) df \quad \text{and} \quad \int_{-1/2}^{1/2} S(f) \sin(2\pi f\tau) df \rightarrow 0,$$

as $|\tau| \rightarrow \infty$. [This is known as the Riemann-Lebesgue theorem]. But,

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f\tau} S(f) df = \int_{-1/2}^{1/2} S(f) \cos(2\pi f\tau) df + i \int_{-1/2}^{1/2} S(f) \sin(2\pi f\tau) df.$$

Hence $s_\tau \rightarrow 0$ as $|\tau| \rightarrow \infty$. In other words, the acvs diminishes to zero (called “mixing condition”).

case (b): $S_1^{(I)}(f) = 0; S_2^{(I)}(f) \geq 0$.

Here the integrated spectrum consists entirely of a step function, and the $\{X_t\}$ is said to have a purely discrete spectrum or a line spectrum. The acvs for a process with a line spectrum never damps down to 0.

Examples

case (a): white noise, ARMA process.

case (b): harmonic process.

$$X_t = A \cos(2\pi f_0 t + \phi)$$

fixed
random

Figs. 18 and 19 give examples of processes with purely continuous and purely discrete spectra. Note that other combinations are possible:

	$S_1^{(I)}(\cdot)$	$S_2^{(I)}(\cdot)$
Purely cts.	≥ 0	$= 0$
Purely disc.	$= 0$	≥ 0
Mixed	Non-white	≥ 0
Discrete	white	≥ 0

ARMA, MA harmonic

An example of a process with a discrete spectrum is a harmonic with additive white noise.

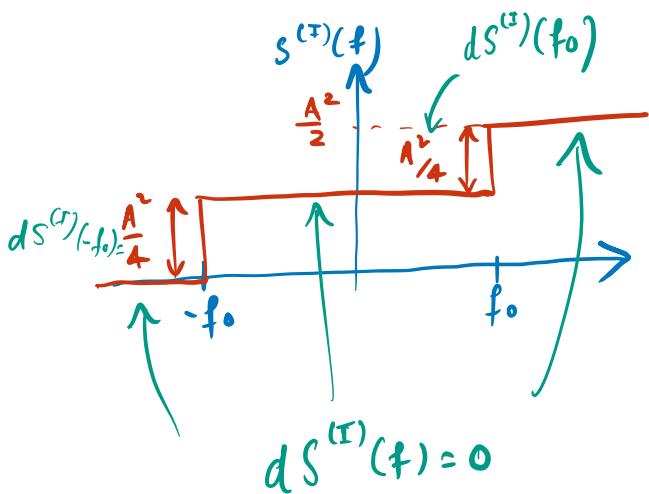
Diagram: example Purely discrete Spectrum

$$X_t = \underbrace{A \cos(2\pi f_0 t + \phi)}_{\text{oscillates at freq } f_0, -f_0} \quad \text{where } \phi \sim \mathcal{U}[-\pi, \pi]$$

$$E\{X_t\} = A \int_{-\pi}^{\pi} \cos(2\pi f_0 t + \phi) \cdot \frac{1}{2\pi} d\phi = 0$$

$$\text{Var}\{X_t\} = A^2 \int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \phi) \cdot \frac{1}{2\pi} d\phi = \frac{A^2}{2}$$

$$\text{Cov}\{X_t, X_{t+\tau}\} = \frac{A^2}{2} \cos(2\pi f_0 \tau) \quad (\text{see problem sheet 1}).$$



$$\begin{aligned}
 S_t &= \int_{-\pi/2}^{\pi/2} e^{i2\pi f_0 \tau} dS^{(I)}(f) \\
 &= e^{-i2\pi f_0 \tau} \frac{A^2}{4} + e^{i2\pi f_0 \tau} \frac{A^2}{4} \\
 &= \frac{A^2}{4} \cdot 2 \cos(2\pi f_0 \tau) \\
 &= \frac{A^2}{2} \cos(2\pi f_0 \tau).
 \end{aligned}$$

$$X_t = A \cos(2\pi f_0 t + \phi) + \xi_t$$

$$S^{(I)}(f) = \underbrace{\dots}_{\text{white noise}} + \underbrace{\dots}_{\text{harmonic}}$$



3.1.4 Spectral density function vs. autocovariance function

The sdf and acvs contain the same amount of information in that if we know one of them, we can calculate the other. However, they are often not equally informative. The sdf usually proves to be the more sensitive and interpretable diagnostic or exploratory tool.

Figure 20 show the sdf and acvs for two different processes - one with two spectral peaks and one with three. The sdf is able to distinguish between the processes while the acvs's are not noticeably different. [NB: dB = $10 \log_{10}(\text{power})$].

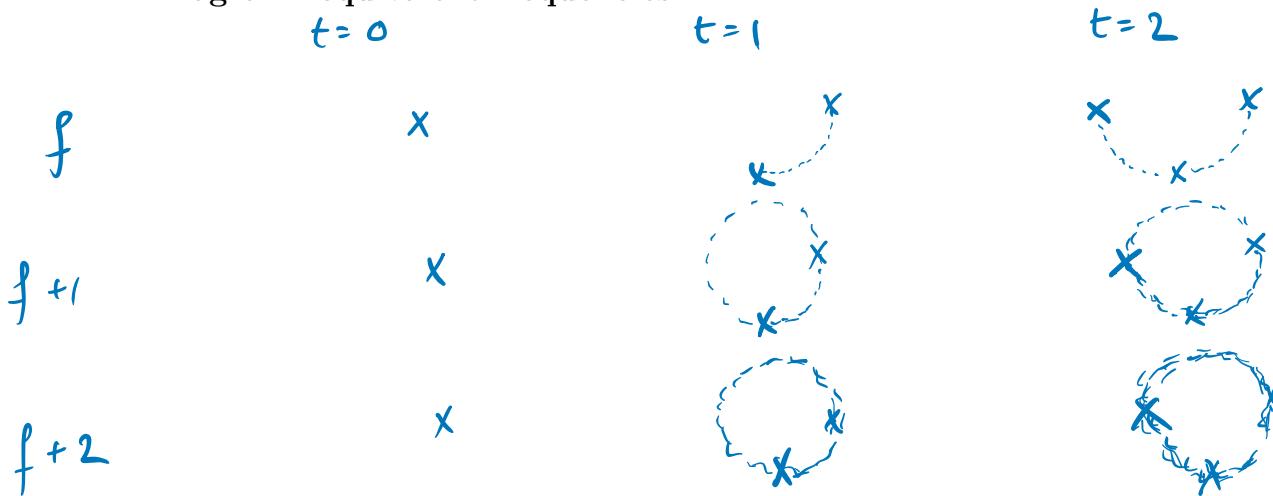
$$e^{i2\pi(f \pm k)t} = e^{i2\pi ft} e^{\pm i2\pi kt} \underbrace{= 1}_{k \in \mathbb{Z}}$$

Video 22

3.2 Sampling and Aliasing

Why does the spectral representation of a stationary process only include frequencies in $[-1/2, 1/2]$? This is because the values of $\exp(i2\pi ft)$ and $\exp(i2\pi(f \pm k)t)$, $k \in \mathbb{Z}$, are identical, hence it is only necessary to have frequencies in the range $[-1/2, 1/2]$; all other frequencies are redundant.

Diagram: equivalent frequencies



Note:

$$S(f+1) = \sum_{\tau=-\infty}^{\infty} s_{\tau} e^{i2\pi(f+1)\tau} = \sum_{\tau=-\infty}^{\infty} s_{\tau} e^{i2\pi f\tau} e^{i2\pi\tau} = S(f).$$

So far we have only been looking at unit-less discrete time. Suppose we had ob-



servations every second ($\Delta t = 1s$) then the interval is $[-1/2, 1/2]\text{Hz}$. Observations every minute ($\Delta t = 1\text{min}$) and it becomes $[-1/2, 1/2]\text{min}^{-1}$. But observing every minute is the same as observing every 60 seconds ($\Delta t = 60s$) and $[-1/2, 1/2]\text{min}^{-1}$ is the same as $[-1/120, 1/120]\text{Hz}$. The general version of the spectral representation is

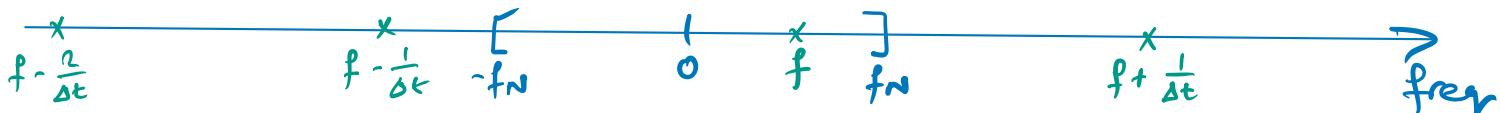
$$X_t = \int_{-f_N}^{f_N} e^{i2\pi ft\Delta t} dZ(f)$$

$$\text{Sampling rate} = \frac{1}{\Delta t}$$

where $f_N \equiv 1/(2\Delta t)$ is called the Nyquist frequency.

Diagram: equivalent frequencies

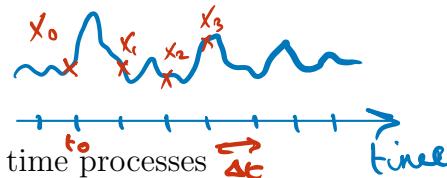
At a Sampling rate of Δt , the frequencies are indistinguishable, because, $e^{i2\pi f\Delta t} = e^{i2\pi(f \pm \frac{k}{\Delta t})\Delta t}$ $k \in \mathbb{Z}$



We have also so far we have only looked at discrete time series $\{X_t\}$. However, such a process is usually obtained by sampling a continuous time process at equal intervals Δt , i.e., for a sampling interval $\Delta t > 0$ and an arbitrary time offset t_0 , we can define a discrete time process through

$$X_t \equiv X(t_0 + t\Delta t), \quad t = 0, \pm 1, \pm 2, \dots$$

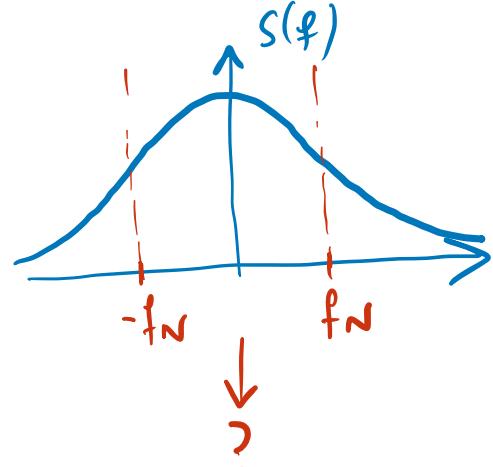
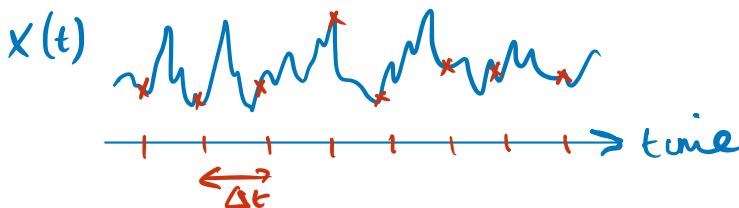
"zero" tone
Sampling interval



There also exists a spectral representation theorem for continuous time processes

$$X(t) = \int_{-\infty}^{\infty} e^{i2\pi ft\Delta t} dZ(f)$$

Diagram: aliasing



If $\{X(t)\}$ is a stationary process with, say, sdf $S_{X(t)}(\cdot)$ and acvf $s(\tau)$, then $\{X_t\}$ is also a stationary process with, say, sdf $S_{X_t}(\cdot)$ and acvs $\{s_\tau\}$. It can be shown that

$$S_{X_t}(f) = \sum_{k=-\infty}^{\infty} S_{X(t)}\left(f + \frac{k}{\Delta t}\right) \quad \text{for } |f| \leq \frac{1}{2\Delta t}.$$

↑
Sdf discrete time (observed)
↑
44
Sdf for discrete time (hidden) process

Thus, the discrete time sdf at f is the sum of the continuous time sdf at frequencies $f \pm \frac{k}{\Delta t}$, $k = 0, 1, 2, \dots$

This formula can be interpreted as “fold $S_{X(t)}(f)$ about the Nyquist frequency, and add” (see Figure 21a). One translation of the English term “aliasing” in German is “faltung” meaning folding.

If $S_{X(t)}$ is essentially zero for $|f| > 1/(2\Delta t)$ we can expect good correspondence between $S_{X_t}(f)$ and $S_{X(t)}(f)$ for $|f| \leq 1/(2\Delta t)$ (since $S_{X(t)}(f \pm k/(2\Delta t)) \approx 0$ for $k = 1, 2, \dots$). If $S_{X(t)}$ is large for some $|f| > 1/(2\Delta t)$, the correspondence can be quite poor, and an estimate of S_{X_t} will not tell us much about $S_{X(t)}$.

Figure 21 illustrates how sampling at a particular rate can't differentiate between different frequency waves. However, if you sample at a rate commensurate with the highest frequency wave present (the bottom one in the plot) — i.e., take samples frequently enough that you are sampling within single oscillations of that highest frequency wave — then you can distinguish the cases.

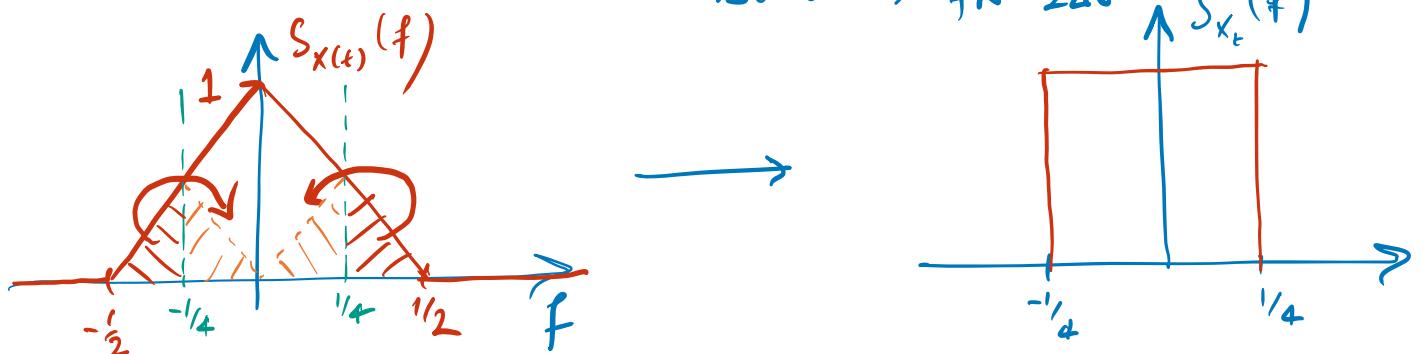
Worked example: aliasing

A continuous time process $X(t)$, t in seconds, has sdf

$$S_{X(t)}(f) = \begin{cases} 1 - 2|f| & |f| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

with f in Hz. It is sampled with a sample interval $\Delta t = 2$ s to produce $\{X_t\}$. What is the sdf of $\{X_t\}$?

$$\Delta t = 2 \Rightarrow f_N = \frac{1}{2\Delta t} = \frac{1}{4}$$

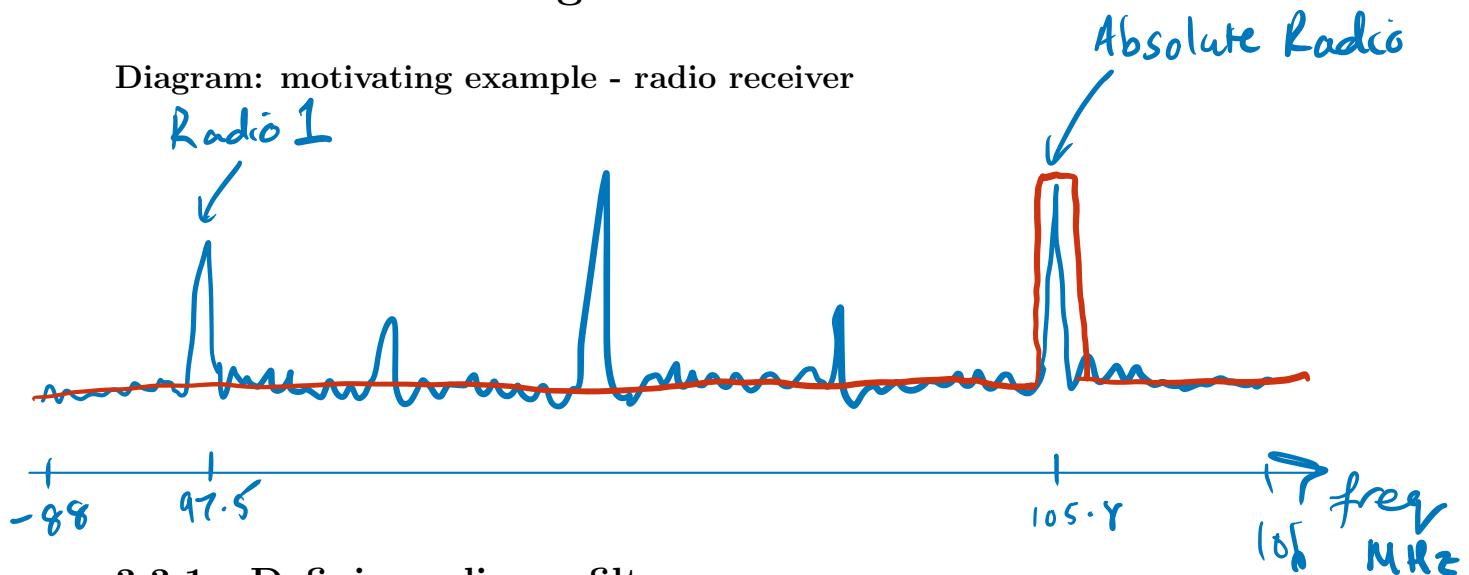


Formula: W log , take $0 \leq f \leq \frac{1}{4}$

$$\begin{aligned} S_{X_t}(f) &= \sum_{k=-\infty}^{\infty} S_{X(t)}\left(f + \frac{k}{\Delta t}\right) \\ &= \dots + S_{X(t)}\left(f - \frac{2}{2}\right) + S_{X(t)}\left(f - \frac{1}{2}\right) + S_{X(t)}(f) + S_{X(t)}\left(f + \frac{1}{2}\right) + S_{X(t)}\left(f + \frac{2}{2}\right) + \dots \\ &= 0 + 45 - 2\left|f - \frac{1}{2}\right| + 1 - 2|f| + 0 + 0 + \dots \\ &= 1 \end{aligned}$$

3.3 Linear filtering

Diagram: motivating example - radio receiver



3.3.1 Defining a linear filter

A digital filter maps a sequence to another sequence. A digital filter L that transforms an input sequence $\{x_t\}$ into an output sequence $\{y_t\}$ is called a linear time-invariant (LTI) digital filter if it has the following three properties:

[1] Scale-preservation:

$$L\{\{\alpha x_t\}\} = \alpha L\{\{x_t\}\}.$$

[2] Superposition:

$$L\{\{x_{t,1} + x_{t,2}\}\} = L\{\{x_{t,1}\}\} + L\{\{x_{t,2}\}\}.$$

[3] Time invariance:

If

$$L\{\{x_t\}\} = \{y_t\}, \quad \text{then} \quad L\{\{x_{t+\tau}\}\} = \{y_{t+\tau}\}.$$

Here τ is integer-valued, and the notation $\{x_{t+\tau}\}$ refers to the sequence whose t -th element is $x_{t+\tau}$.

From now on we shall drop the double brackets and the input and output ~~long~~ sequences will be implicitly understood, i.e., we will use $L\{x_t\} = y_t$ as shorthand for $L\{\{x_t\}\} = \{y_t\}$.

Suppose we use a sequence with t -th element $\exp(i2\pi ft)$ as the input to a LTI digital filter: Let $\xi_{f,t} = e^{i2\pi ft}$, and let $y_{f,t}$ denote the output:

$$y_{f,t} = L\{\xi_{f,t}\}.$$

The filter's response
to a single frequency
 f .

By properties [1] and [3]:

$$\xi_{f,t+\tau} = e^{i2\pi f(t+\tau)} = e^{i2\pi f\tau} e^{i2\pi ft} = e^{i2\pi f\tau} \xi_{f,t}$$

$$y_{f,t+\tau} = L\{\xi_{f,t+\tau}\} = L\{e^{i2\pi f\tau} \xi_{f,t}\} = e^{i2\pi f\tau} L\{\xi_{f,t}\} = e^{i2\pi f\tau} y_{f,t}.$$

[3] [1]

In particular, for $t = 0$:

$$y_{f,\tau} = e^{i2\pi f\tau} y_{f,0}.$$

Now set $\tau = t$:

$$\xi_{f,t} = e^{i2\pi ft} y_{f,0}.$$

Thus, when $\xi_{f,t}$ is input to the LTI digital filter, the output is the same sequence multiplied by some constant, $y_{f,0}$, which is independent of time but will depend on f . Let $G(f) = y_{f,0}$. Then

$$L\{\xi_{f,t}\} = \xi_{f,t}G(f).$$

$G(f)$ is called the transfer function or frequency response function of L . We can write

$$G: \mathbb{R} \rightarrow \mathbb{C}$$

$$G(f) = |G(f)|e^{i\theta(f)}$$

$$z = re^{i\phi}$$

where,

$$|G(f)| \quad \text{gain}$$

$$\theta(f) = \arg\{G(f)\} \quad \text{phase}$$

Any LTI digital filter can be expressed in the form:

$$L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u} \equiv Y_t,$$

discrete time convolution

where $\{g_u\}$ is a real-valued deterministic sequence called the impulse response sequence. Note,

$$L\{e^{i2\pi ft}\} = \sum_{u=-\infty}^{\infty} g_u e^{i2\pi f(t-u)} = e^{i2\pi f t} G(f),$$

with

$$G(f) = \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi f u} \quad \text{for } |f| \leq \frac{1}{2},$$

where

$$\{g_u\} \longleftrightarrow G(f) \quad (\text{FT pair}).$$

Now

$$Y_t = \sum_u g_u X_{t-u}$$

{ X_t } second order
stationary process.

and we recall from the spectral representation theorem that

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_X(f) \quad Y_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f),$$

$$\Rightarrow \underbrace{\int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f)}_{\int_{-1/2}^{1/2} e^{i2\pi ft} \sum_u g_u e^{-i2\pi fu} dZ_X(f)} = \sum_u g_u \int_{-1/2}^{1/2} e^{i2\pi f(t-u)} dZ_X(f)$$

so that, by the 1:1 property of the FT, $dZ_Y(f) = G(f) dZ_X(f)$, and

Stochastic version of $\int e^{i2\pi ft} F(f) df = \int e^{i2\pi ft} H(f) df$

$$\Leftrightarrow F(f) = H(f)$$

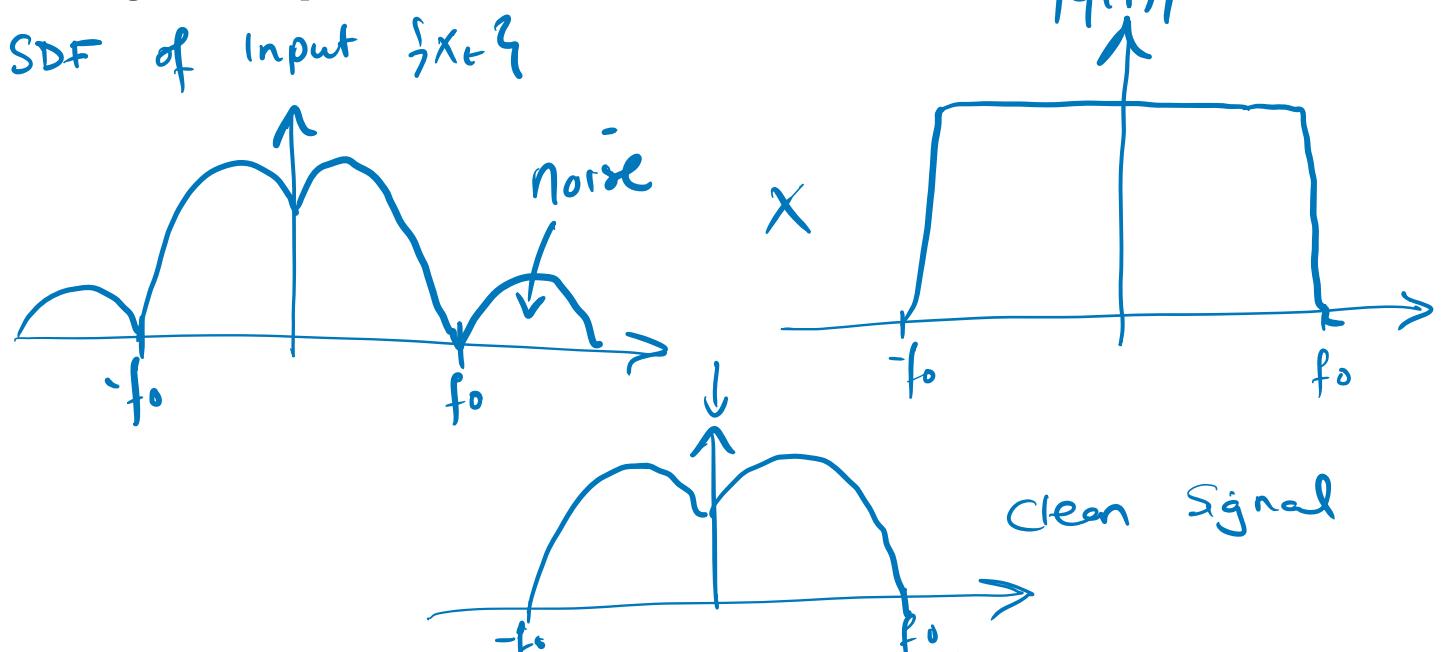
$$E\{|dZ_Y(f)|^2\} = |G(f)|^2 E\{|dZ_X(f)|^2\},$$

$$S_Y(f) df = |g(f)|^2 S_X(f) df$$

and if the spectral densities exist

$$S_Y(f) = |G(f)|^2 S_X(f).$$

Diagram: low-pass band filter



This relationship can also be used to determine the sdf's of discrete parameter stationary processes.

3.3.2 Determination of sdf's by LTI digital filtering

MA processes

q -th order moving average: $\text{MA}(q)$,

$$X_t = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q},$$

with usual assumptions (mean zero). Define

$$\begin{aligned} g_0 &= 1, \quad g_1 = -\theta_{1,q}, \quad g_2 = -\theta_{2,q}, \dots \\ g_q &= -\theta_{q,q} \end{aligned}$$

$$L\{\epsilon_t\} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q},$$

so that $X_t = L\{\epsilon_t\}$. To determine $G(f)$, input $e^{i2\pi ft}$:

$$\begin{aligned} L\{e^{i2\pi ft}\} &= e^{i2\pi ft} - \theta_{1,q}e^{i2\pi f(t-1)} - \dots - \theta_{q,q}e^{i2\pi f(t-q)} \\ &= e^{i2\pi ft} \underbrace{\left[1 - \theta_{1,q}e^{-i2\pi f} - \dots - \theta_{q,q}e^{-i2\pi fq} \right]}, \end{aligned}$$

so that

$$G(f) = \underbrace{(1 + \sum_{k=1}^q \theta_k e^{-ik2\pi f})}_{Q(z)} \text{ evaluated at } z = e^{-i2\pi f}$$

$$G_\theta(f) = 1 - \theta_{1,q}e^{-i2\pi f} - \dots - \theta_{q,q}e^{-i2\pi fq}.$$

Since,

$$S_X(f) = |G_\theta(f)|^2 S_\epsilon(f) \quad \text{and} \quad S_\epsilon(f) = \sigma_\epsilon^2,$$

we have

$$S_X(f) = \sigma_\epsilon^2 |1 - \theta_{1,q}e^{-i2\pi f} - \dots - \theta_{q,q}e^{-i2\pi fq}|^2.$$

Example sdfs for MA(1) processes can be found in Figure 22.

Worked example: MA(1) spectrum Derive the spectral density function of the MA(1) process $X_t = \epsilon_t - \theta\epsilon_{t-1}$, and from it compute s_1 .

$$\begin{aligned} \text{From formula above, } S_X(f) &= \sigma_\epsilon^2 |1 - \theta e^{-i2\pi f}|^2 \\ S_X(f) &= \sigma_\epsilon^2 (1 - \theta e^{-i2\pi f})(1 - \theta e^{i2\pi f}) \\ &= \sigma_\epsilon^2 (1 - 2\theta \cos(2\pi f) + \theta^2) \end{aligned}$$

$$\begin{aligned} \text{Then: } s_1 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{i2\pi f \tau} df \Rightarrow s_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{i2\pi f \tau} df \end{aligned}$$

$$= \sigma_\epsilon^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + \theta^2 - 2\theta \cos(2\pi f)] e^{i2\pi f \tau} df$$

$$= \sigma_\epsilon^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + \theta^2 - 2\theta \cos(2\pi f)] \cos(2\pi f \tau) df.$$

$$\text{exercise} \quad = -\sigma_\epsilon^2 \theta \quad \text{as before!}$$

AR processes

p -th order autoregressive process: AR(p),

$$X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p} = \epsilon_t$$

Define

$$L\{X_t\} = X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p},$$

so that $L\{X_t\} = \epsilon_t$. By analogy to MA(q)

$L\{e^{i2\pi f t}\}$

$$G_\phi(f) = \underbrace{1 - \phi_{1,p}e^{-i2\pi f} - \dots - \phi_{p,p}e^{-i2\pi fp}}_{\Phi(z) \text{ evaluated at } z = e^{-i2\pi f}}.$$

Since,

$$|G_\phi(f)|^2 S_X(f) = S_\epsilon(f) \quad \text{and} \quad S_\epsilon(f) = \sigma_\epsilon^2,$$

we have

$$S_X(f) = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \dots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

Worked example: AR(1) spectrum

Derive the spectrum of the AR(1) process $X_t = \phi X_{t-1} + \epsilon_t$, ($|\phi| < 1$).

$$\begin{aligned} X_t - \phi X_{t-1} &= \epsilon_t \\ L\{X_t\} &= \epsilon_t \quad L\{X_t\} = X_t - \phi X_{t-1} \\ \therefore G_\phi(f) &= 1 - \phi e^{-i2\pi f} \\ \Rightarrow S_X(f) &= \frac{\sigma_\epsilon^2}{|1 - \phi e^{-i2\pi f}|^2} = \frac{\sigma_\epsilon^2}{1 + \phi^2 - 2\phi \cos(2\pi f)} \end{aligned}$$

Example sdfs for AR(1) processes are given in Figure 23.

Interpretation of AR spectra

Video 25

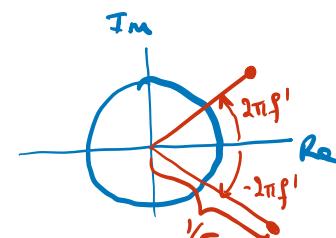
Recall that for an AR process we have characteristic equation

$$1 - \phi_{1,p}z - \phi_{2,p}z^2 - \dots - \phi_{p,p}z^p$$

and the process is stationary if the roots of this equation lie outside the unit circle.

Consider an AR(2) process with complex characteristic roots, the roots forming a complex conjugate pair:

$$z_1 = \frac{1}{r}e^{-i2\pi f'}, \quad z_2 = \frac{1}{r}e^{i2\pi f'}.$$



Now

$$1 - \phi_{1,2}z - \phi_{2,2}z^2 = (1 - az)(1 - bz) = 1 - (a + b)z + abz^2$$

so the roots are $z_1 = 1/a$ and $z_2 = 1/b$ and $\phi_{1,2} = (a + b)$, $\phi_{2,2} = -ab$. Then $a = re^{i2\pi f'}$ and $b = re^{-i2\pi f'}$ and $\phi_{1,2} = 2r \cos(2\pi f')$ and $\phi_{2,2} = -r^2$. The AR process can be written

$$X_t - 2r \cos(2\pi f') X_{t-1} + r^2 X_{t-2} = \epsilon_t.$$

The spectrum can be written in terms of the complex roots, by substituting $z = e^{-i2\pi f}$ in the characteristic equation.

$$\begin{aligned} S_X(f) &= \frac{\sigma_\epsilon^2}{|1 - az|^2 |1 - bz|^2} \Big|_{z=e^{-i2\pi f}} \\ &= \frac{\sigma_\epsilon^2}{|1 - \underbrace{re^{i2\pi f'} e^{-i2\pi f}}_a|^2 |1 - \underbrace{re^{-i2\pi f'} e^{-i2\pi f}}_b|^2} \end{aligned}$$

$S_x(f) \sim \frac{\sigma_\epsilon^2}{|\Phi(z)|^2} \Big|_{z=e^{i2\pi f}}$

Now,

$$|1 - re^{i2\pi f'} e^{-i2\pi f}|^2 = 1 - 2r \cos(2\pi(f' - f)) + r^2.$$

Similarly,

$$|1 - re^{-i2\pi f'} e^{-i2\pi f}|^2 = 1 - 2r \cos(2\pi(f' + f)) + r^2.$$

So,

$$S_X(f) = \frac{\sigma_\epsilon^2}{(1 - 2r \cos(2\pi(f' + f)) + r^2)(1 - 2r \cos(2\pi(f' - f)) + r^2)}.$$

when $f = f'$
 $(1-r)^2$

The spectrum will be at its largest when the denominator is at its smallest - when r is close to 1 this occurs when $f \approx \pm f'$. Also notice that at $f = \pm f'$ as $r \rightarrow 1$ (from below as $0 < r < 1$ since the root is outside the unit circle) so the spectrum becomes larger.

Generally speaking complex roots will induce a peak in the spectrum, indicating a tendency towards a cycle at frequency f' . Also, the larger the value of r the more dominant the cycle. This may be termed *pseudo-cyclical* behaviour (recall that a deterministic cycle will show up at a sharp spike - i.e., a line spectrum).

Example AR(2) spectra for real and complex-valued roots are given in Figures 24 and 25, respectively.

ARMA processes

(p, q) -th order autoregressive, moving average process: ARMA(p, q),

$$X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q}$$

$$L\phi\{X_t\} = L\theta\{\epsilon_t\}$$

If we write this as

$$X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p} = Y_t;$$

$$Y_t = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \dots - \theta_{q,q}\epsilon_{t-q},$$

then we have

$$\left. \begin{aligned} |G_\phi(f)|^2 S_X(f) &= S_Y(f), \\ S_Y(f) &= |G_\theta(f)|^2 S_\epsilon(f), \end{aligned} \right\} |G_\phi(f)|^2 S_X(f) = |G_\theta(f)|^2 S_\epsilon(f)^2$$

and

$$S_Y(f) = |G_\theta(f)|^2 S_\epsilon(f),$$

so that

$$\begin{aligned} S_X(f) &= S_\epsilon(f) \frac{|G_\theta(f)|^2}{|G_\phi(f)|^2} \\ &= \sigma_\epsilon^2 \frac{|1 - \theta_{1,q}e^{-i2\pi f} - \dots - \theta_{q,q}e^{-i2\pi fq}|^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \dots - \phi_{p,p}e^{-i2\pi fp}|^2} \end{aligned}$$

Differencing

See trend / Seasonal removal

Let $\{X_t\}$ be a stationary process with sdf $S_X(f)$. Let $Y_t = \underbrace{X_t - X_{t-1}}_{\Delta X_t}$. Then

$$\begin{aligned} L\{e^{i2\pi ft}\} &= e^{i2\pi ft} - e^{i2\pi f(t-1)} \\ &= e^{i2\pi ft} (1 - e^{-i2\pi f}) \\ &= e^{i2\pi ft} G(f) \end{aligned}$$

$$\begin{aligned} g_0 &= 1 \\ g_1 &= -1 \\ g_k &= 0 \text{ otherwise} \end{aligned}$$

so

$$\begin{aligned} |G(f)|^2 &= |1 - e^{-i2\pi f}|^2 = |e^{-i\pi f}(e^{i\pi f} - e^{-i\pi f})|^2 \\ &= |e^{-i\pi f} 2i \sin(\pi f)|^2 = 4 \sin^2(\pi f). \end{aligned}$$

$$\therefore S_Y(f) = S_{\Delta X}(f) = 4 \sin^2(\pi f) S_X(f)$$

Chapter 4

Video 27

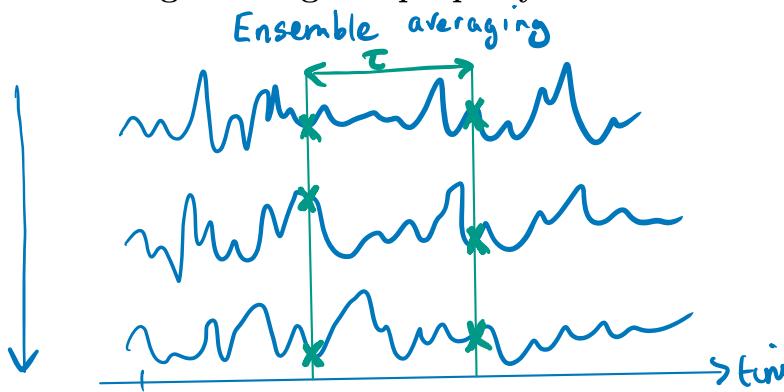
Estimation

4.1 Estimation of mean and autocovariance function

Ergodic property

Methods we shall look at for estimating quantities such as the autocovariance function will use observations from a single realization. Such methods are based on the strategy of replacing ensemble averages by their corresponding time averages.

Diagram: ergodic property



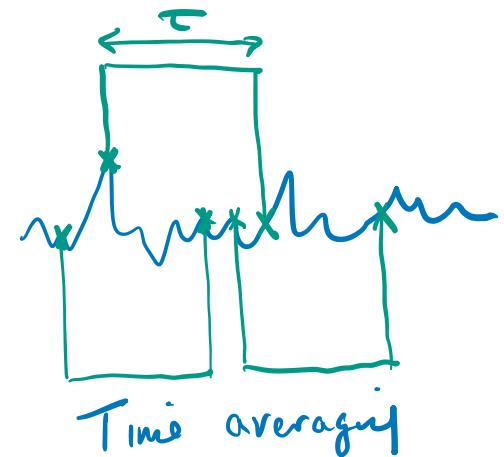
Reminder: bias, variance and mean square error

Parameter θ which I estimate with $\hat{\theta}$

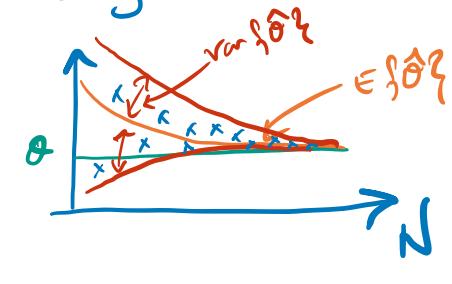
$$\begin{aligned} \text{MSE}(\hat{\theta}) &= E\{(\hat{\theta} - \theta)^2\} = E\{\hat{\theta}^2 + 2\hat{\theta}\theta + \theta^2\} \\ &= E\{\hat{\theta}^2\} - 2\theta E\{\hat{\theta}\} + \theta^2 \\ &= E\{\hat{\theta}^2\} - E^2\{\hat{\theta}\} + E^2\{\hat{\theta}\} - 2\theta E\{\hat{\theta}\} + \theta^2 \\ &= E\{\hat{\theta}^2\} - E^2\{\hat{\theta}\} + (E\{\hat{\theta}\} - \theta)^2 \end{aligned}$$

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$\text{Var}(\hat{\theta})$ $\text{Bias}^2(\hat{\theta})$



To converge in MSE, we need both the variance and bias to go to zero as $N \rightarrow \infty$



4.1.1 Sample mean

Given a stationary time series X_1, X_2, \dots, X_N . Let

$$\bar{X} = \frac{1}{N} \sum_{t=1}^N X_t.$$

Then,

$$E\{\bar{X}\} = \frac{1}{N} \sum_{t=1}^N E\{X_t\} = \frac{1}{N} N\mu = \mu$$

so \bar{X} is an unbiased estimator of μ . Hence, \bar{X} converges to μ in mean square if

$$\lim_{N \rightarrow \infty} \text{var}\{\bar{X}\} = 0.$$

Now,

$$\begin{aligned} \text{var}\{\bar{X}\} &= E\{(\bar{X} - \mu)^2\} \\ &= E\left\{\left(\frac{1}{N} \sum_{i=1}^N (X_i - \mu)\right)^2\right\} \\ &= \frac{1}{N^2} \sum_{t=1}^N \sum_{u=1}^N E\{(X_t - \mu)(X_u - \mu)\} \\ &= \frac{1}{N^2} \sum_{t=1}^N \sum_{u=1}^N s_{u-t} \quad \text{row sums} \\ &= \frac{1}{N^2} \sum_{\tau=-(N-1)}^{N-1} \sum_{k=1}^{N-|\tau|} s_\tau \quad \text{diagonal sums}_\tau \\ &= \frac{1}{N^2} \sum_{\tau=-(N-1)}^{N-1} (N - |\tau|) s_\tau \\ &= \frac{1}{N} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau. \end{aligned}$$

The summation interchange merely swaps row sums for diagonal sums.

$\sum_{\tau=-\infty}^{\infty} |s_\tau| < \infty$ To make further progress we need the condition $\sum_{\tau=-\infty}^{\infty} |s_\tau| < \infty$. By the Cesàro summability theorem, if $\sum_{\tau=-(N-1)}^{N-1} s_\tau$ converges to a limit as $N \rightarrow \infty$,

$$\Rightarrow \sum_{\tau=-\infty}^{\infty} |s_\tau|^2 < \infty \quad \left[\text{it must since} \quad \left| \sum_{\tau=-(N-1)}^{N-1} s_\tau \right| \leq \sum_{\tau=-(N-1)}^{N-1} |s_\tau| < \infty \quad \forall N \right]$$

\Rightarrow F.T exists
S(f) then $\sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau$ converges to the same limit.

We can thus conclude that,

$$\lim_{N \rightarrow \infty} N \text{var}\{\bar{X}\} = \lim_{N \rightarrow \infty} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau$$

$$= \lim_{N \rightarrow \infty} \sum_{\tau=-(N-1)}^{N-1} s_\tau = \sum_{\tau=-\infty}^{\infty} s_\tau. = S(0)$$

The assumption of absolute summability of $\{s_\tau\}$ also implies that $\{X_t\}$ has a purely continuous spectrum with sdf

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau},$$

so that

$$S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau.$$

Thus

$$\lim_{N \rightarrow \infty} N \text{var}\{\bar{X}\} = S(0),$$

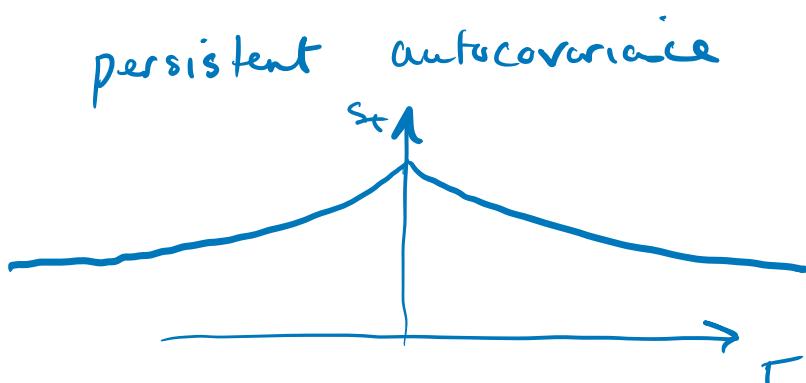
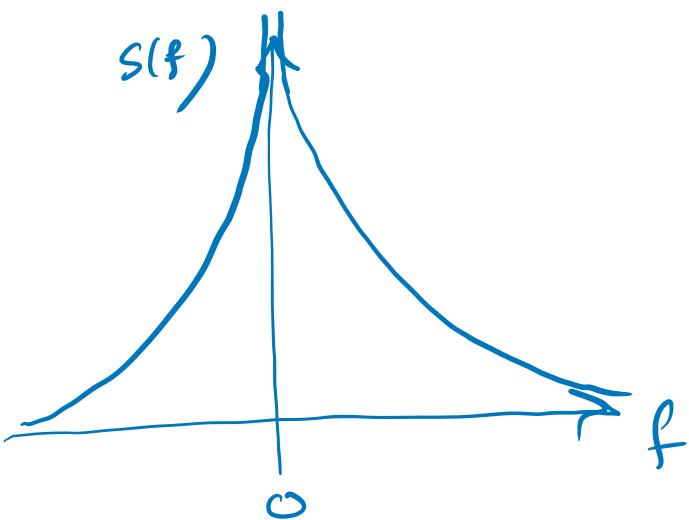
i.e.,

$$\text{var}\{\bar{X}\} \approx \frac{S(0)}{N} \text{ for large } N,$$

and therefore, $\text{var}\{\bar{X}\} \rightarrow 0$. Note (i) that the convergence of \bar{X} depends only on the spectrum at $S(0)$, i.e. at $f = 0$, and (ii) \bar{X} is a consistent estimator for μ .

Brief aside: long memory

"Long Memory" process is defined as a process $\sum_{\tau=-\infty}^{\infty} s_\tau$ does not converge. They have a spectrum that asymptotes at $f=0$.



4.1.2 Autocovariance Sequence

Now,

$$s_\tau = E\{(X_t - \mu)(X_{t+\tau} - \mu)\}$$

so that a natural estimator for the acvs is

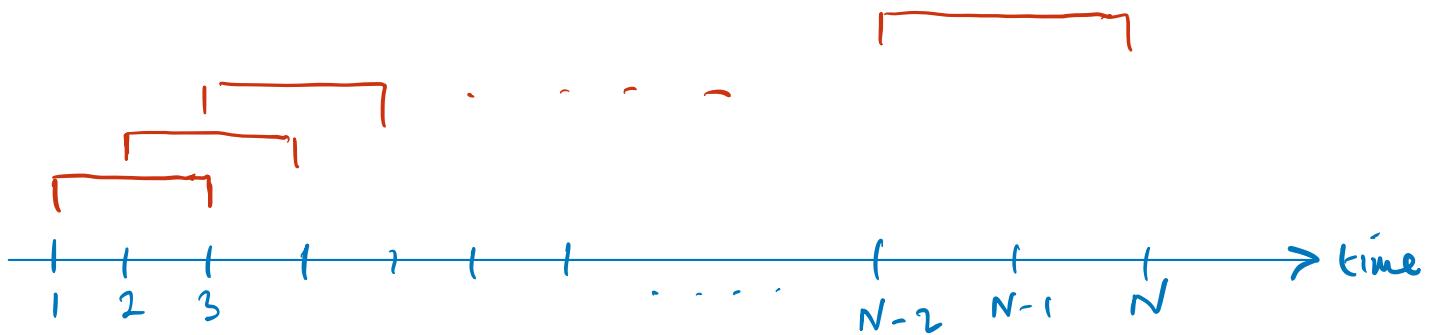
$$\hat{s}_\tau^{(u)} = \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \quad \tau = 0, \pm 1, \dots, \pm(N-1).$$

Note $\hat{s}_{-\tau}^{(u)} = \hat{s}_\tau^{(u)}$ as it should.

Diagram: autocovariance estimator

e.g. $\tau = 2$

$N - |\tau|$ pairs



If we replace \bar{X} by μ :

$$\begin{aligned} E\{\hat{s}_\tau^{(u)}\} &= \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} E\{(X_t - \mu)(X_{t+|\tau|} - \mu)\} \\ &= \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} s_\tau = s_\tau, \quad \tau = 0, \pm 1, \dots, \pm(N-1). \end{aligned}$$

Thus, $\hat{s}_\tau^{(u)}$ is an unbiased estimator of s_τ when μ is known. (Hence the (u) – for unbiased). Most texts refer to $\hat{s}_\tau^{(u)}$ as unbiased – however, if μ is estimated by \bar{X} , $\hat{s}_\tau^{(u)}$ is typically a biased estimator of s_τ !!!

A second estimator of s_τ is typically preferred:

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \quad \tau = 0, \pm 1, \dots, \pm(N-1).$$

With \bar{X} replaced by μ :

$$E\{\hat{s}_\tau^{(p)}\} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} s_\tau = \left(1 - \frac{|\tau|}{N}\right) s_\tau,$$

so that $\hat{s}_\tau^{(p)}$ is a biased estimator, and the magnitude of its bias increases as $|\tau|$ increases. Most texts refer to $\hat{s}_\tau^{(p)}$ as biased.

Why should we prefer the “biased” estimator $\hat{s}_\tau^{(p)}$ to the “unbiased” estimator $\hat{s}_\tau^{(u)}$?

- [1] For many stationary processes of practical interest

$$\text{mse}\{\hat{s}_\tau^{(p)}\} < \text{mse}\{\hat{s}_\tau^{(u)}\}.$$

- [2] If $\{X_t\}$ has a purely continuous spectrum we know that $s_\tau \rightarrow 0$ as $|\tau| \rightarrow \infty$. It therefore makes sense to choose an estimator that decreases nicely as $|\tau| \rightarrow N - 1$ (i.e. choose $\hat{s}_\tau^{(p)}$).

- [3] We know that the acvs must be positive semidefinite, the sequence $\{\hat{s}_\tau^{(p)}\}$ has this property, whereas the sequence $\{\hat{s}_\tau^{(u)}\}$ may not.

4.2 Spectral estimation

4.2.1 A naïve non-parametric spectral estimator – the periodogram

Suppose the zero mean discrete stationary process $\{X_t\}$ has a purely continuous spectrum with sdf $S(f)$. We have,

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_{\tau} e^{-i2\pi f\tau} \quad |f| \leq \frac{1}{2}.$$

With $\mu = 0$, we can use the biased estimator of s_{τ} :

$$\hat{s}_{\tau}^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}$$

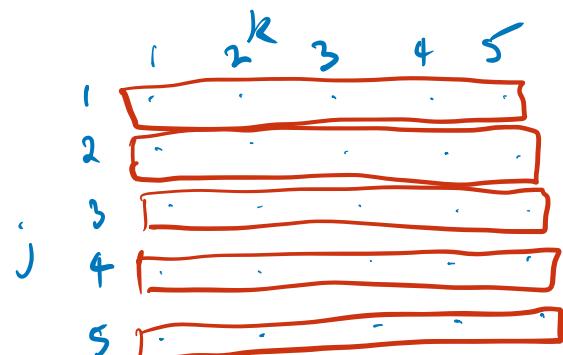
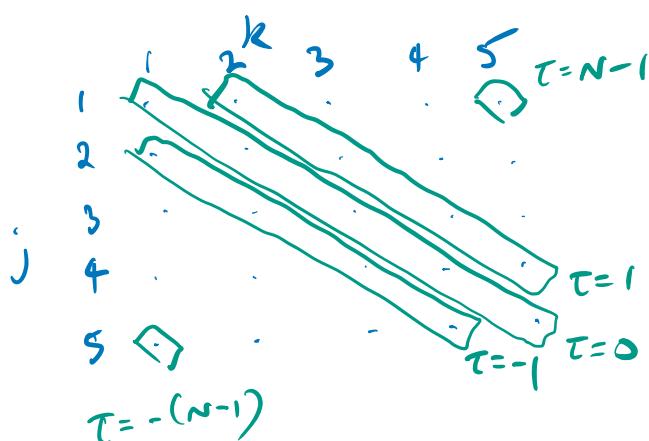
for $|\tau| \leq N-1$, but not for $|\tau| \geq N$. Hence we could replace s_{τ} by $\hat{s}_{\tau}^{(p)}$ for $|\tau| \leq N-1$ and assume $s_{\tau} = 0$ for $|\tau| \geq N$. Then a spectrum estimate could be

$$\begin{aligned} \hat{S}^{(p)}(f) &= \sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_{\tau}^{(p)} e^{-i2\pi f\tau} = \frac{1}{N} \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|} e^{-i2\pi f\tau} \\ &= \frac{1}{N} \sum_{j=1}^N \sum_{k=1}^N X_j X_k e^{-i2\pi f(k-j)} \quad \text{row sum} \quad \frac{1}{N} \left(\sum_{j=1}^N X_j e^{i2\pi f j} \right) \cdot \left(\sum_{k=1}^N X_k e^{-i2\pi f k} \right) \\ &= \frac{1}{N} \left| \sum_{t=1}^N X_t e^{-i2\pi f t} \right|^2, \end{aligned}$$

*Fourier transform
of data X_1, \dots, X_N
perform by F.F.T
algorithm.*

where the summation interchange has merely swapped diagonal sums for row sums.

Diagram: summation interchange
matrix $(j, k)^{\text{th}}$ element $X_j X_k e^{-i2\pi f(k-j)}$



$\hat{S}^{(p)}(f)$ defined above is known as the periodogram, and is defined over $[-1/2, 1/2]$.

Note that $\{\hat{s}_\tau^{(p)}\}$ and $\hat{S}^{(p)}(f)$ are a FT pair:

$$\{\hat{s}_\tau^{(p)}\} \longleftrightarrow \hat{S}^{(p)}(f)$$

(hence the (p) for periodogram), just like the population quantities

$$\{s_\tau\} \longleftrightarrow S(f).$$

Hence, $\{s_\tau^{(p)}\}$ can be written as

Inverse F.T.

$$s_\tau^{(p)} = \int_{-1/2}^{1/2} \hat{S}^{(p)}(f) e^{i2\pi f\tau} df \quad |\tau| \leq N - 1.$$

If $\hat{S}^{(p)}(f)$ were an ideal estimator of $S(f)$ we would have

[1] $E\{\hat{S}^{(p)}(f)\} \approx S(f) \quad \forall f.$

[2] $\text{var}\{\hat{S}^{(p)}(f)\} \rightarrow 0 \text{ as } N \rightarrow \infty \text{ and,}$

[3] $\text{cov}\{\hat{S}^{(p)}(f), \hat{S}^{(p)}(f')\} \approx 0 \text{ for } f \neq f'.$

We find that

[1] is a good approximation for some processes, but very bad for others.

[2] is blatantly false (see Figure 26),

[3] holds if f and f' are certain distinct frequencies, namely, the Fourier frequencies

$$f_k = k/N \quad (\Delta t = 1).$$

We firstly look at the expectation in [1] (assuming $\mu = 0$). Video 31

$$\begin{aligned} E\{\hat{S}^{(p)}(f)\} &= \sum_{\tau=-(N-1)}^{(N-1)} E\{s_\tau^{(p)}\} e^{-i2\pi f\tau} \\ &= \sum_{\tau=-(N-1)}^{(N-1)} \left(1 - \frac{|\tau|}{N}\right) s_\tau e^{-i2\pi f\tau}. \end{aligned}$$

Hence, if we know the acvs $\{s_\tau\}$ we can work out from this what $E\{\hat{S}^{(p)}(f)\}$ will be. We can obtain much more insight by considering:

$$E\{|J(f)|^2\} \quad \text{where} \quad J(f) = \frac{1}{\sqrt{N}} \sum_{t=1}^N X_t e^{-i2\pi f t}, \quad |f| \leq \frac{1}{2}.$$

Note

" $\hat{S}^{(p)}(f)$ "

$\hat{S}^{(p)}(f) = |J(f)|^2$

We know from the spectral representation theorem that,

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f't} dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^N \left(\int_{-1/2}^{1/2} \frac{1}{\sqrt{N}} e^{i2\pi f't} dZ(f') \right) e^{-i2\pi ft} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^N \frac{1}{\sqrt{N}} e^{-i2\pi(f-f')t} dZ(f') \end{aligned}$$

Then

$$\begin{aligned} E\{\hat{S}^{(p)}(f)\} &= E\{|J(f)|^2\} = E\{J^*(f)J(f)\} \\ &= E\left\{ \int_{-1/2}^{1/2} \sum_{t=1}^N \frac{1}{\sqrt{N}} e^{i2\pi(f-f')t} dZ^*(f') \int_{-1/2}^{1/2} \sum_{s=1}^N \frac{1}{\sqrt{N}} e^{-i2\pi(f-f'')s} dZ(f'') \right\} \\ &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{t=1}^N \frac{1}{\sqrt{N}} e^{i2\pi(f-f')t} \sum_{s=1}^N \frac{1}{\sqrt{N}} e^{-i2\pi(f-f'')s} E\{dZ^*(f') dZ(f'')\} \\ &= \int_{-1/2}^{1/2} \mathcal{F}(f-f') S(f') df', \end{aligned}$$

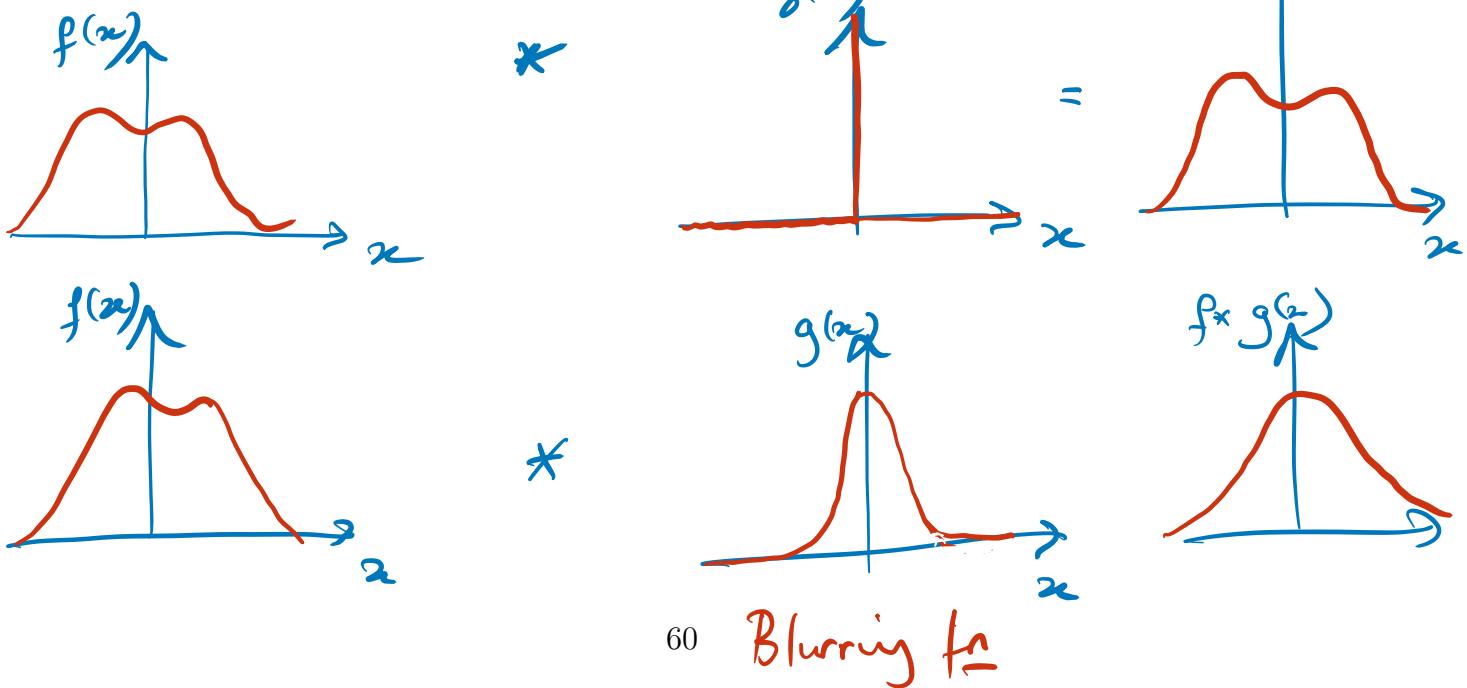
$\left\{ \begin{array}{l} S(f') df' \\ 0 \end{array} \right. \quad f' = f'' \\ o/w \quad o/\omega$

where \mathcal{F} is Féjer's kernel defined by

$$\mathcal{F}(f) = \left| \sum_{t=1}^N \frac{1}{\sqrt{N}} e^{-i2\pi ft} \right|^2 = \frac{\sin^2(N\pi f)}{N \sin^2(\pi f)}.$$

This result tells us that the expected value of $\hat{S}^{(p)}(f)$ is the true spectrum convolved with Féjer's kernel.

Diagram: effects of convolution



To understand the implications of this we need to know the properties of Féjer's kernel:

[1] For all integers $N \geq 1$, $\mathcal{F}(f) \rightarrow N$ as $f \rightarrow 0$.

[2] For $N \geq 1$, $f \in [-1/2, 1/2]$ and $f \neq 0$, $\mathcal{F}(f) < \mathcal{F}(0)$.

[3] For $f \in [-1/2, 1/2]$, $f \neq 0$, $\mathcal{F}(f) \rightarrow 0$ as $N \rightarrow \infty$.
pointwise

[4] For any integer $k \neq 0$ such that $f_k = k/N \in [-1/2, 1/2]$, $\mathcal{F}(f_k) = 0$.

[5] $\int_{-1/2}^{1/2} \mathcal{F}(f) df = 1$.

Fourier frequencies

Figure 27 shows Féjer's kernel on a $10 \log_{10}$ scale (dBs) for $N = 8, 32$ and 128 . $\mathcal{F}(f)$ is symmetric about the origin and consists of a broad central peak ("lobe") and $N - 2$ sidelobes which decrease as $|f|$ increases. From [1], [3] and [5] it follows that as $N \rightarrow \infty$, $\mathcal{F}(f)$ acts as a Dirac δ function with an infinite spike at $f = 0$.

So

$$\lim_{N \rightarrow \infty} E\{\hat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \delta(f - f') S(f') df' = S(f),$$

i.e., $\hat{S}^{(p)}(f)$ is *asymptotically unbiased* as an estimator of $S(f)$.

$E[\hat{S}^{(p)}(f)] \rightarrow \infty$

For a process with large dynamic range, defined as

$E[\hat{S}^{(p)}(f)] \rightarrow 0$

$$10 \log_{10} \left(\frac{\max_f S(f)}{\min_f S(f)} \right),$$

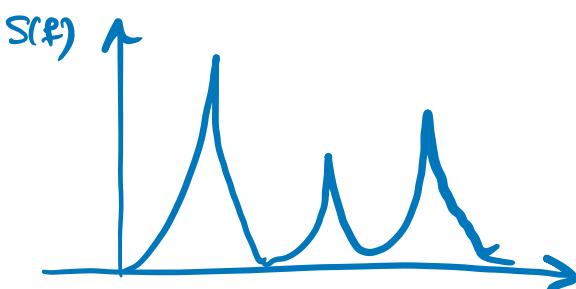
$f \neq 0$

[5] $\int_{-1/2}^{1/2} S(f) df = 1$.

since the expected value of the periodogram is a convolution of Féjer's kernel and the true spectrum, power from parts of the spectrum where $S(f)$ is large can "leak" via the sidelobes to other frequencies where $S(f)$ is small.

Diagram: side-lobe leakage

$E[\hat{S}^{(p)}(f)] \rightarrow \infty$



$S * F(f)$

→



Figures 28 and 29 demonstrate this "sidelobe leakage" for two processes, the first with a dynamic range of 14 dB, the second with a dynamic range of 65 dB.

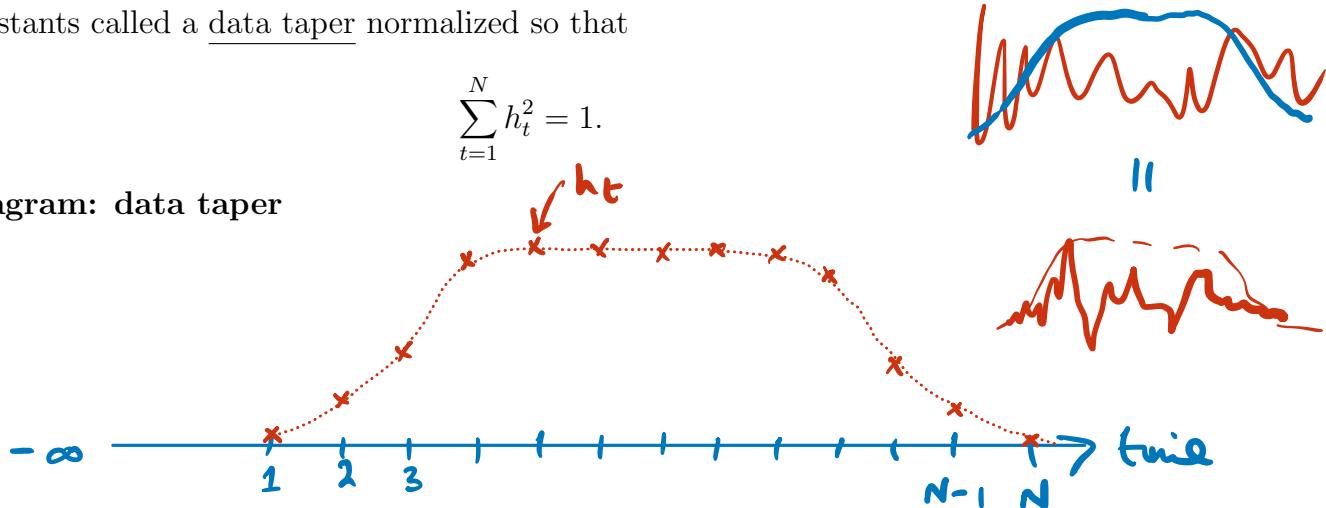
4.2.2 Bias reduction – Tapering

Much of the bias in the periodogram can be attributed to sidelobe leakage due to the presence of Féjer's kernel. Tapering is a technique which reduces the sidelobes associated with Féjer's kernel.

Let X_1, X_2, \dots, X_N be a portion of length N of a zero mean stationary process with sdf $S(f)$. We form the product $\{h_t X_t\}$ where $\{h_t\}$ is a sequence of real-valued constants called a data taper normalized so that

$$\sum_{t=1}^N h_t^2 = 1.$$

Diagram: data taper



Define

$$J(f) = \sum_{t=1}^N h_t X_t e^{-i2\pi f t} \quad |f| \leq 1/2.$$

By the spectral representation theorem,

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f' t} dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^N h_t \left(\int_{-1/2}^{1/2} e^{i2\pi f' t} dZ(f') \right) e^{-i2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^N h_t e^{-i2\pi(f-f')t} dZ(f') \\ &= \int_{-1/2}^{1/2} H(f - f') dZ(f'), \end{aligned}$$

where,

$$H(f) = \sum_{t=1}^N h_t e^{-i2\pi f t} \quad \text{i.e., } \{h_t\} \longleftrightarrow H(f).$$

Let,

$$\hat{S}^{(d)}(f) = |J(f)|^2 = \left| \sum_{t=1}^N h_t X_t e^{-i2\pi f t} \right|^2.$$

Then,

$$|J(f)|^2 = J^*(f)J(f) = \int_{-1/2}^{1/2} H^*(f - f') dZ^*(f') \int_{-1/2}^{1/2} H(f - f'') dZ(f''),$$

and hence,

$$\begin{aligned} E\{\hat{S}^{(d)}(f)\} &= E\{|J(f)|^2\} \\ &= \int_{-1/2}^{1/2} |H(f - f')|^2 S(f') df' \\ &= \int_{-1/2}^{1/2} \mathcal{H}(f - f') S(f') df', \end{aligned}$$

$\mathcal{E}\{\mathrm{d}Z^*(f')\mathrm{d}Z(f'')\}$
 $= \int_S(f') df' f' = f''$
 ? 0 0/w

where $\mathcal{H}(f) = |H(f)|^2$, i.e.,

$$\mathcal{H}(f) = \left| \sum_{t=1}^N h_t e^{-i2\pi ft} \right|^2.$$

A spectral estimator of the form of $\hat{S}^{(d)}(f)$ is called a direct spectral estimator (hence the (d)). Note, if $h_t = \frac{1}{\sqrt{N}}$ for $1 \leq t \leq N$, then

$$\hat{S}^{(d)}(f) = \hat{S}^{(p)}(f) \quad \text{and} \quad \mathcal{H}(f) = \mathcal{F}(f),$$

i.e., $\hat{S}^{(d)}(f)$ is the same as the periodogram,

$\mathcal{H}(f)$ is the same as Fejer's kernel.

The key idea behind tapering is to select $\{h_t\}$ so that $\mathcal{H}(f)$ has much lower sidelobes than $\mathcal{F}(f)$. Recall that $\mathcal{F}(f)$ corresponds to a rectangular taper

$$h_t = \begin{cases} \frac{1}{\sqrt{N}} & \text{for } 1 \leq t \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

There is thus a sharp discontinuity between where the taper is “ON” ($1 \leq t \leq N$) and where it is “OFF” ($N < t < 0$). Tapering effectively creates a smooth transition at the ends of the data.

Diagram: aims of tapering

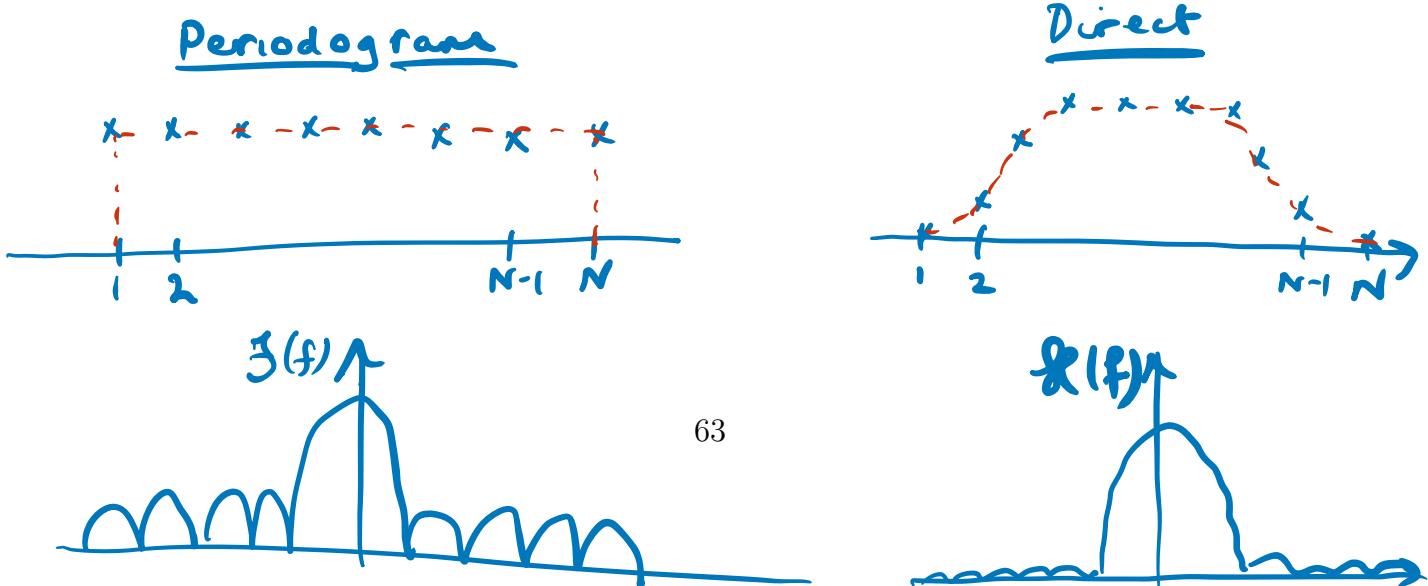


Figure 30 shows the effect of tapering on the shape of the spectral window $\mathcal{H}(f)$.

The $p \times 100\%$ cosine taper is defined by

$$h_t = \begin{cases} \frac{C}{2} \left[1 - \cos \left(\frac{2\pi t}{\lfloor pN \rfloor + 1} \right) \right], & 1 \leq t \leq \frac{\lfloor pN \rfloor}{2}; \\ C, & \frac{\lfloor pN \rfloor}{2} < t < N + 1 - \frac{\lfloor pN \rfloor}{2}; \\ \frac{C}{2} \left[1 - \cos \left(\frac{2\pi(N+1-t)}{\lfloor pN \rfloor + 1} \right) \right], & N + 1 - \frac{\lfloor pN \rfloor}{2} \leq t \leq N, \end{cases}$$

where C is a normalizing constant that forces $\sum_{t=1}^N h_t^2 = 1$.

As we perform more tapering, the main lobe of $\mathcal{H}(f)$ gets wider, but the sidelobes get lower. This means that the more tapering we perform:

Resolution of the spectrum DECREASES (bad!)

Sidelobe leakage DECREASES (good!).

Figure 31 demonstrates how the modification of the spectral window inherent in tapering reduces the sidelobe leakage at the expense of widening the main lobe (this results in smoothing bias) for the AR(4) process with high dynamic range.

Chapter 5

Video 33

Parametric model fitting (for autoregressive processes)

Here we concentrate on zero-mean models of the form

$$X_t - \phi_{1,p}X_{t-1} - \dots - \phi_{p,p}X_{t-p} = \epsilon_t.$$

As we have seen the corresponding sdf is

$$S(f) = \frac{\sigma_\epsilon^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \dots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

This class of models is appealing to use for time series analysis for several reasons:

- [1] Any time series with a purely continuous sdf can be approximated well by an AR(p) model if p is large enough.
- [2] There exist efficient algorithms for fitting AR(p) models to time series.
- [3] Quite a few physical phenomena are reverberant and hence an AR model is naturally appropriate.

5.1 Yule-Walker method

Video 34

We start by multiplying the defining equation by X_{t-k} :

$$X_t X_{t-k} = \sum_{j=1}^p \phi_{j,p} X_{t-j} X_{t-k} + \epsilon_t X_{t-k}.$$

$$X_t = \sum_{j=1}^p \phi_{j,p} X_{t-j} + \epsilon_t$$

$$X_{t-k} = \sum_{\ell=0}^{\infty} g_{\ell} \epsilon_{t-\ell-k} \quad (44)$$

$$E\{\epsilon_t X_{t-k}\} = \sum_{\ell=0}^{\infty} g_{\ell} E\{\epsilon_t \epsilon_{t-\ell-k}\}$$

Taking expectations, for $k > 0$:

$$s_k = \sum_{j=1}^p \phi_{j,p} s_{k-j}.$$

Let $k = 1, 2, \dots, p$ and recall that $s_{-\tau} = s_{\tau}$ to obtain

$$\begin{aligned} s_1 &= \phi_{1,p} s_0 + \phi_{2,p} s_1 + \dots + \phi_{p,p} s_{p-1} \\ s_2 &= \phi_{1,p} s_1 + \phi_{2,p} s_0 + \dots + \phi_{p,p} s_{p-2} \\ &\vdots \quad \vdots \\ s_p &= \phi_{1,p} s_{p-1} + \phi_{2,p} s_{p-2} + \dots + \phi_{p,p} s_0 \end{aligned}$$

$\left. \begin{array}{l} \text{P equations} \\ \text{P unknowns} \\ (\phi_{1,p}, \dots, \phi_{p,p}) \end{array} \right\}$

or in matrix notation,

$$\gamma_p = \Gamma_p \phi_p,$$

where $\gamma_p = [s_1, s_2, \dots, s_p]^T$; $\phi_p = [\phi_{1,p}, \phi_{2,p}, \dots, \phi_{p,p}]^T$ and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \dots & s_{p-1} \\ s_1 & s_0 & \dots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \dots & s_0 \end{bmatrix}$$

Note: this is a symmetric Toeplitz matrix which we have met already. All elements on a diagonal are the same.

Suppose we don't know the $\{s_{\tau}\}$, but the mean is zero, then take

$$\hat{s}_{\tau} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}, \quad \leftarrow \hat{s}_{\tau}^{(\tau)}$$

and substitute these for the s_{τ} 's in γ_p and Γ_p to obtain $\hat{\gamma}_p, \hat{\Gamma}_p$, from which we estimate ϕ_p as $\hat{\phi}_p$:

$$\hat{\phi}_p = \hat{\Gamma}^{-1} \hat{\gamma}_p.$$

Finally, we need to estimate σ_{ϵ}^2 . To do so, we multiply the defining equation by X_t and take expectations to obtain

$$\begin{aligned} s_0 &= \sum_{j=1}^p \phi_{j,p} s_j + \overbrace{E\{\epsilon_t X_t\}} \\ &= \sum_{j=1}^p \phi_{j,p} s_j + \sigma_{\epsilon}^2, \end{aligned}$$

$$\begin{aligned} E\{\epsilon_t X_t\} &= \sum_{\ell=0}^{\infty} g_{\ell} E\{\epsilon_t \epsilon_{t-\ell}\} \\ &= \sigma_{\epsilon}^2 \end{aligned}$$

YW estimators

so that as an estimator for σ_ϵ^2 we take

$$\hat{\sigma}_\epsilon^2 = \hat{s}_o - \sum_{j=1}^p \hat{\phi}_{j,p} \hat{s}_j.$$

The estimators $\hat{\phi}_p$ and $\hat{\sigma}_\epsilon^2$ are called the Yule-Walker estimators of the AR(p) parameters.

The estimate of the sdf resulting is

$$\hat{S}(f) = \frac{\hat{\sigma}_\epsilon^2}{|1 - \sum_{j=1}^p \hat{\phi}_{j,p} e^{-i2\pi f j}|^2}.$$

There are two important modifications which we can make to this approach:

- [1] We could use for $\{\hat{s}_\tau\}$ a modified autocovariance incorporating tapering:

$$\hat{s}_\tau = \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|}.$$

- [2] To invert $\hat{\Gamma}_p$ by brute force matrix inversion requires $O(p^3)$ operations. Fortunately, there is an algorithm due to Levinson and Durbin which takes advantage of the highly structured nature of the Toeplitz matrix, and carries out the estimation in $O(p^2)$ or fewer operations.

Worked example: Yule-Walker

Yule-Walker Calculations for an AR(2) process
 Given $\hat{s}_0 = 3, \hat{s}_1 = 2, \hat{s}_2 = 1$

$$\hat{\Gamma}_2 = \begin{bmatrix} \hat{s}_0 & \hat{s}_1 \\ \hat{s}_1 & \hat{s}_0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \hat{\gamma}_2 = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\hat{\phi}_2 = \hat{\Gamma}_2^{-1} \hat{\gamma}_2 = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -1/5 \end{bmatrix}$$

$$\text{And } \hat{\sigma}_\epsilon^2 = \hat{s}_0 - \sum_{j=1}^2 \hat{\phi}_{j,2} \hat{s}_j = 3 - \left(\frac{4}{5} \cdot 2 - \frac{1}{5} \cdot 1\right) = 8/5$$

So the estimated Model is

$$X_t - \frac{4}{5} X_{t-1} + \frac{1}{5} X_{t-2} = \varepsilon_t$$

where $\{\varepsilon_t\}$ is zero mean white noise
with estimated Variance $\hat{\sigma}_\varepsilon^2 = 8/5$

$$\therefore \Phi(z) = 1 - \frac{4}{5}z + \frac{1}{5}z^2$$

Roots satisfy $1 - \frac{4}{5}z + \frac{1}{5}z^2 = 0$
 $5 - 4z + z^2 = 0$

$$z = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

Examples: The AR(4) process again.

- Figure 32: Shows simulations from the AR(4) process defined by,

$$X_t = 2.7607X_{t-1} - 3.8106X_{t-2} + 2.6535X_{t-3} - 0.9258X_{t-4} + \epsilon_t$$

- Figure 33: Shows AR(4) processes fitted to the AR(4) data using Yule-Walker method and

$$\hat{s}_\tau = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}.$$

Very poor, even for $N = 1024$.

- Figure 34: Shows AR(8) processes fitted to the AR(4) data using Yule-Walker method and

$$\hat{s}_\tau = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}.$$

Although the process fitted is not the correct one, the extra parameters have improved the fit.

- Figure 35: Shows AR(4) process fitted to the AR(4) data, using Yule-Walker, but with the 50% split cosine bell taper used:

$$\hat{s}_\tau = \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|}.$$

The improvement over the other Yule-Walker estimates is dramatic.

The parameter estimates for the fitted AR(4) models when $N=1024$ are:

	true	Yule-Walker	tapered Y-W
$\phi_{1,4}$	2.7607	1.8459	2.7636
$\phi_{2,4}$	-3.8106	-1.7138	-3.8108
$\phi_{3,4}$	2.6535	0.6200	2.6502
$\phi_{4,4}$	-0.9258	-0.1437	-0.9211
σ_ϵ^2	1.0	14.9758	1.0841

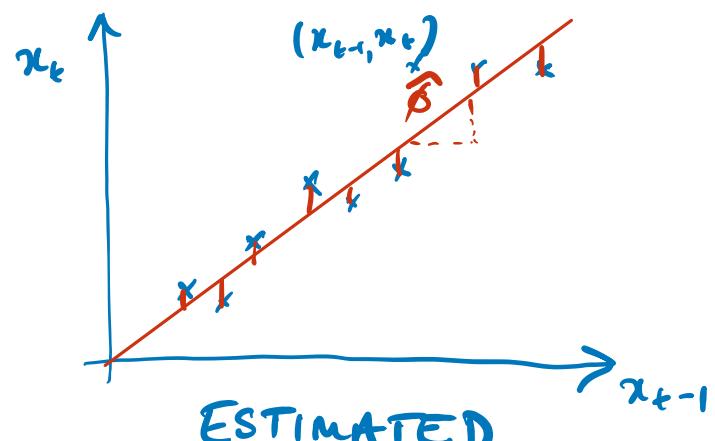
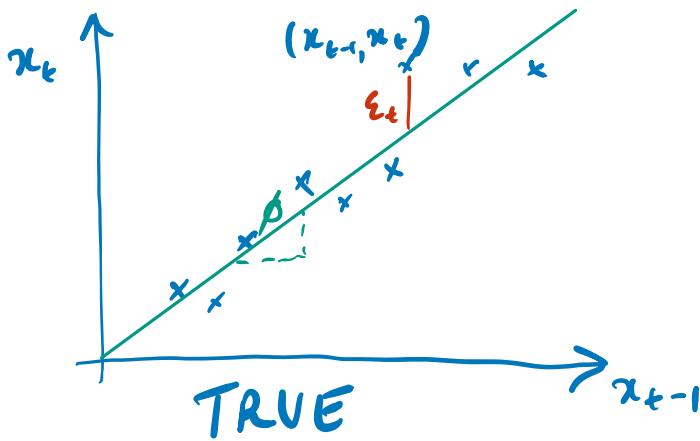
5.2 Least squares

Let $\{X_t\}$ be a zero-mean AR(p) process, i.e.,

$$X_t - \phi_{1,p}X_{t-1} - \phi_{2,p}X_{t-2} - \dots - \phi_{p,p}X_{t-p} = \epsilon_t.$$

Diagram: Least squares formulation

Eg AR(1) $X_t = \phi X_{t-1} + \epsilon_t$



We can formulate an appropriate least squares model in terms of data X_1, X_2, \dots, X_N as follows:

$$\mathbf{X} = F\phi + \epsilon,$$

$$\begin{aligned} X_{p+1} &= \phi_{1,p} X_p + \dots + \phi_{p,p} X_1 + \epsilon_{p+1} \\ X_{p+2} &= \phi_{1,p} X_{p+1} + \dots + \phi_{p,p} X_2 + \epsilon_{p+2} \\ &\vdots \\ X_N &= \phi_{1,p} X_{N-1} + \dots + \phi_{p,p} X_{N-p} + \epsilon_N \end{aligned}$$

where,

$$F = \begin{bmatrix} X_p & X_{p-1} & \dots & X_1 \\ X_{p+1} & X_p & \dots & X_2 \\ \vdots & & & \vdots \\ X_{N-1} & X_{N-2} & \dots & X_{N-p} \end{bmatrix}$$

and,

$$\mathbf{X} = \begin{bmatrix} X_{p+1} \\ X_{p+2} \\ \vdots \\ X_N \end{bmatrix}; \quad \phi = \begin{bmatrix} \phi_{1,p} \\ \phi_{2,p} \\ \vdots \\ \phi_{p,p} \end{bmatrix}; \quad \epsilon = \begin{bmatrix} \epsilon_{p+1} \\ \epsilon_{p+2} \\ \vdots \\ \epsilon_N \end{bmatrix}.$$

We can thus estimate ϕ by finding that ϕ such that

$$\begin{aligned} \text{SS}(\phi) &= \sum_{t=p+1}^N \left(X_t - \sum_{k=1}^p \phi_{k,p} X_{t-k} \right)^2 \quad \left[= \sum_{t=p+1}^N \epsilon_t^2 \right] \\ &= (\mathbf{X} - F\phi)^T (\mathbf{X} - F\phi) \end{aligned}$$

$$X_t = \sum_{k=1}^p \phi_{k,p} X_{t-k} + \epsilon_t$$

is minimized. If we denote the vector that minimizes the above as $\hat{\phi}$, standard least squares theory tells us that it is given by

$$\hat{\phi} = (F^T F)^{-1} F^T \mathbf{X}.$$

$$\begin{aligned} \frac{\partial}{\partial \phi} (\underline{X} - F\underline{\phi})^T (\underline{X} - F\underline{\phi}) &= 2(-F)^T (\underline{X} - F\underline{\phi}) \\ &= 2(F^T F \underline{\phi} - F^T \underline{X}) \\ \text{Set } = 0 : \quad F^T F \underline{\phi} &= F^T \underline{X} \Rightarrow \hat{\underline{\phi}} = (F^T F)^{-1} F^T \underline{X} \end{aligned}$$

We can estimate the innovations variance σ_ϵ^2 by the usual estimator of the residual variation, namely

$$\hat{\sigma}^2 = \frac{(\mathbf{X} - F\hat{\phi})^T (\mathbf{X} - F\hat{\phi})}{(N - 2p)}.$$

(Note: there are $N - p$ effective observations, and p parameters are estimated).

The estimator $\hat{\phi}$ is known as the forward least squares estimator of ϕ .

Figure 36 shows the AR(4) spectra corresponding to the least squares estimates of ϕ and tapered Yule-Walker estimates for comparison.

5.3 Maximum likelihood

Video 36

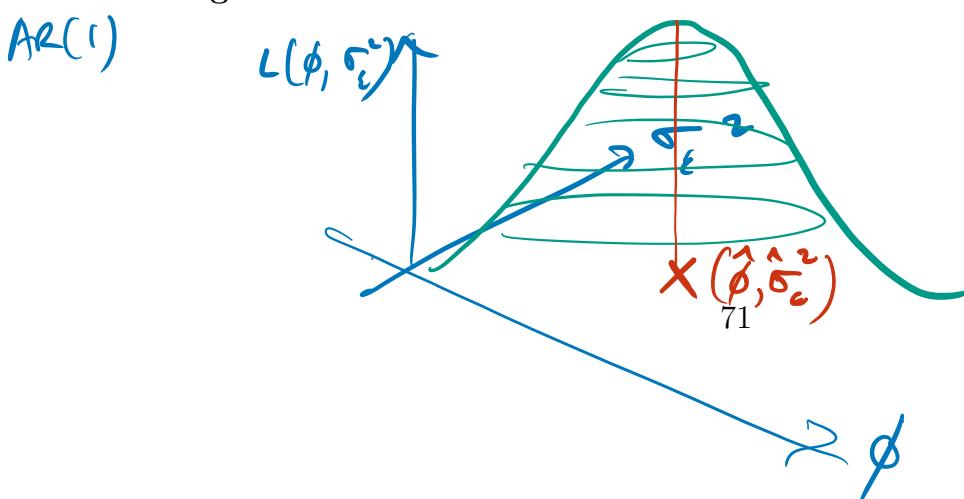
For a portion X_1, \dots, X_N from an AR(p) process, the likelihood function for the parameters ϕ and σ_ϵ^2 is

$$L(\phi, \sigma_\epsilon^2) = f(X_1, \dots, X_N | \phi, \sigma_\epsilon^2)$$

where f is the joint density function of X_1, \dots, X_N for an AR(p) process. The maximum likelihood estimates are the values of ϕ, σ_ϵ^2 that maximise L , namely

$$\hat{\phi}, \hat{\sigma}_\epsilon^2 = \operatorname{argmax} L(\phi, \sigma_\epsilon^2).$$

Diagram: maximum likelihood



To define f , we need to make an assumption on the distribution of the data.

Here, we will assume this is a Gaussian process.

$$P(A, B) = P(B) P(A|B)$$

We can write

$$\begin{aligned} f(X_1, \dots, X_N | \phi, \sigma_\epsilon^2) &= f(\underbrace{X_1, \dots, X_{N-1}}_B | \phi, \sigma_\epsilon^2) f(X_N | X_1, \dots, X_{N-1}, \phi, \sigma_\epsilon^2) \\ &= f(\underbrace{X_1, \dots, X_{N-1}}_B | \phi, \sigma_\epsilon^2) f(X_N | X_{N-1}, \dots, X_{N-p}, \phi, \sigma_\epsilon^2). \end{aligned}$$

because only explicitly depends on X_{N-1}, \dots, X_{N-p}
"pth order Markov property"

Applying the same argument gives

$$\begin{aligned} f(X_1, \dots, X_{N-1} | \phi, \sigma_\epsilon^2) &= f(X_1, \dots, X_{N-2} | \phi, \sigma_\epsilon^2) f(X_{N-1} | X_1, \dots, X_{N-2}, \phi, \sigma_\epsilon^2) \\ &= f(X_1, \dots, X_{N-2} | \phi, \sigma_\epsilon^2) f(X_{N-1} | X_{N-2}, \dots, X_{N-p-1}, \phi, \sigma_\epsilon^2). \end{aligned}$$

Iterating gives

$$f(X_1, \dots, X_N | \phi, \sigma_\epsilon^2) = f(X_1, \dots, X_{N-2} | \phi, \sigma_\epsilon^2) f(X_{N-1} | X_{N-2}, \dots, X_{N-p}, \phi, \sigma_\epsilon^2) \\ \cdot f(X_N | X_{N-1}, \dots, X_{N-p}, \phi, \sigma_\epsilon^2)$$

$$f(X_1, \dots, X_N | \phi, \sigma_\epsilon^2) = f(X_1, \dots, X_p) \prod_{t=p+1}^N f(X_t | X_{t-1}, \dots, X_{t-p}, \phi, \sigma_\epsilon^2).$$

With

$$X_t = \epsilon_t + \sum_{j=1}^p \phi_{j,p} X_{t-j},$$

$$E\{X_t\} = 0 + \sum_{j=1}^p \phi_{j,p} X_{t-j}$$

under the Gaussian assumption we have $X_t | X_{t-1}, \dots, X_{t-p} \sim N\left(\sum_{j=1}^p \phi_{j,p} X_{t-j}, \sigma_\epsilon^2\right)$,

and therefore

Gaussian pdf

$$f(X_t | X_{t-1}, \dots, X_{t-p}, \phi, \sigma_\epsilon^2) = \frac{1}{\sqrt{2\pi}\sigma_\epsilon^2} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(X_t - \sum_{j=1}^p \phi_{j,p} X_{t-j})^2\right).$$

Method 1: it is possible to obtain a closed form approximate solution by considering the X_1, \dots, X_p to be deterministic. Then

$$L(\phi, \sigma_\epsilon^2) \propto \prod_{t=p+1}^N f(X_t | X_{t-1}, \dots, X_{t-p}, \phi, \sigma_\epsilon^2).$$

Maximising $L(\phi, \sigma_\epsilon^2)$ is equivalent to maximising the log-likelihood $\ell(\phi, \sigma_\epsilon^2) = \ln(L(\phi, \sigma_\epsilon^2))$.

We therefore have

$$\begin{aligned} \ell(\phi, \sigma_\epsilon^2) &= C + \sum_{t=p+1}^N \ln f(X_t | X_{t-1}, \dots, X_{t-p}, \phi, \sigma_\epsilon^2) \\ &= C' - (N-p) \ln(\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2} \sum_{t=p+1}^N (X_t - \sum_{j=1}^p \phi_{j,p} X_{t-j})^2 \\ &= C' - (N-p) \ln(\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2} SS(\phi) \end{aligned}$$

We notice that the ϕ that maximises this expression is the Least Squares estimator, namely

$$\hat{\phi} = (F^T F)^{-1} F^T \mathbf{X}.$$

This gives

$$\ell(\hat{\phi}, \sigma_\epsilon^2) = C' - (N - p) \ln(\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2} (\mathbf{X} - F\hat{\phi})^T (\mathbf{X} - F\hat{\phi}).$$

To find the maximum likelihood estimator of σ_ϵ^2 , we now differentiate this wrt σ_ϵ^2 and set to zero. It is straight forward to show

$$\hat{\sigma}_\epsilon^2 = \frac{(\mathbf{X} - F\hat{\phi})^T (\mathbf{X} - F\hat{\phi})}{(N - p)}.$$

However, this is a biased estimator and therefore the unbiased estimator presented for Least Squares estimation is often preferred.

Method 2: we consider maximising the complete likelihood

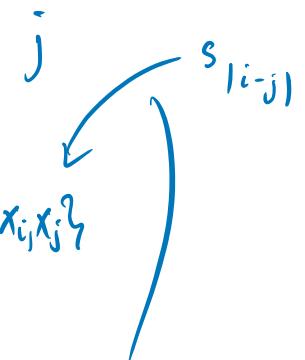
$$L(\phi, \sigma_\epsilon^2) = f(X_1, \dots, X_p) \prod_{t=p+1}^N f(X_t | X_{t-1}, \dots, X_{t-p}, \phi, \sigma_\epsilon^2).$$

However, this is a complicated function of the model parameters because

$$f(X_1, \dots, X_p) = (2\pi)^{-p/2} \det(\Sigma)^{-1/2} \exp(-\frac{1}{2} \mathbf{X}_p^T \Sigma^{-1} \mathbf{X}_p),$$

where $\mathbf{X}_p = (X_1, \dots, X_p)^T$ and Σ is the covariance matrix of \mathbf{X}_p , namely

$$\Sigma = \begin{pmatrix} s_0 & s_1 & \cdots & s_{p-1} \\ s_1 & s_0 & & \\ \vdots & & \ddots & \\ s_{p-1} & & & s_0 \end{pmatrix}.$$

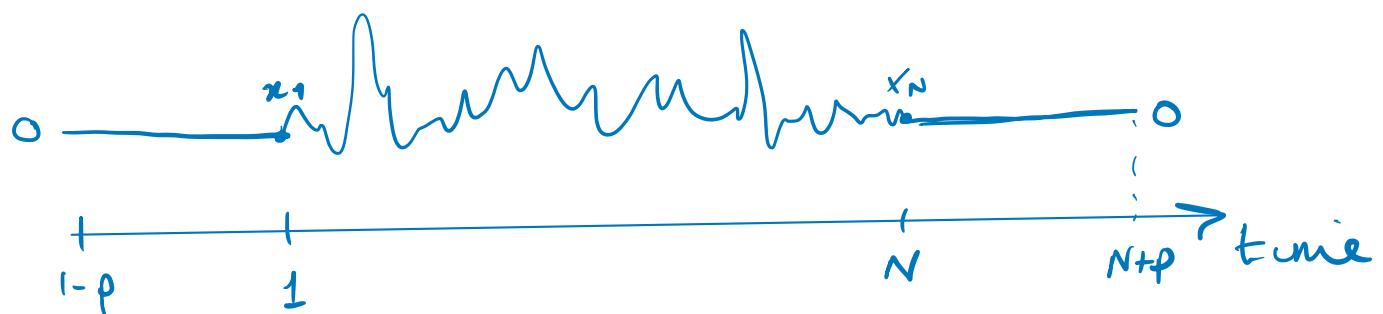


We have seen that s_τ can be a complicated function of the model parameters. There is no closed form solution to this but instead it can be optimised by numerical procedures (e.g. Nelder-Mead simplex).

5.4 Notes

Video 37

- [1] Least squares and maximum likelihood methods produce estimated models which need not be stationary. This may be a concern for prediction, however, for spectral estimation, the parameter values will still produce a valid sdf (i.e., nonnegative everywhere, symmetric about the origin and integrates to a finite number).
- [2] The Yule-Walker can be considered to be a matching of moments method. Furthermore, estimates can be formulated as a least squares problem. Consider adding zeros to our observations X_1, X_2, \dots, X_N , both at the beginning and end of the data.



$$x_1 = \phi_{1,p} x_0 + \cdots + \phi_{r,p} x_{1-p} + \epsilon_1$$

⋮ ⋮ ⋮

$$x_{N+p} = \phi_{1,p} x_{N+p-1} + \cdots + \phi_{p,p} x_N + \epsilon_{N+p}$$

⋮ ⋮ ⋮

This gives

$$\mathbf{X}_{YW} = W\boldsymbol{\phi} + \boldsymbol{\epsilon}_{YW},$$

where,

$$W = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ X_1 & 0 & 0 & \dots & \dots & 0 \\ X_2 & X_1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ X_{p-1} & \vdots & & & & 0 \\ X_p & X_{p-1} & \dots & \dots & \dots & X_1 \\ \vdots & \vdots & & & & \vdots \\ X_N & X_{N-1} & \dots & \dots & \dots & X_{N-p+1} \\ 0 & X_N & & & & X_{N-p+2} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & X_N \end{bmatrix}$$

and,

$$\mathbf{X}_{YW} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_{YW} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note that,

$$\frac{1}{N} W^T W = \begin{bmatrix} \hat{s}_0^{(p)} & \hat{s}_1^{(p)} & \dots & \hat{s}_{p-1}^{(p)} \\ \hat{s}_1^{(p)} & \ddots & & \\ \vdots & \ddots & \ddots & \\ \hat{s}_{p-1}^{(p)} & \dots & \dots & \hat{s}_0^{(p)} \end{bmatrix} = \hat{\Gamma}_p$$

and

$$\frac{1}{N} W^T \mathbf{X}_{YW} = \begin{bmatrix} \hat{s}_1^{(p)} \\ \vdots \\ \hat{s}_p^{(p)} \end{bmatrix} = \hat{\gamma}_p,$$

so that

$$(W^T W)^{-1} W^T \mathbf{X}_{YW} = (\hat{\Gamma}_p)^{-1} \hat{\gamma}_p.$$

which is identical to the Yule-Walker estimate.

- [3] Maximum likelihood method is in some ways equivalent to Least Squares when assuming Gaussianity. Other distributions can be assumed, however, if the distribution is chosen incorrectly it will give poor estimates.

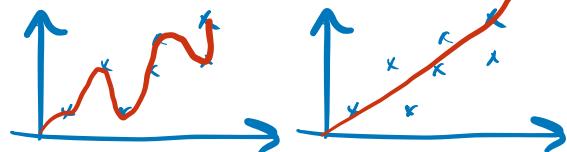
$$X_t = \phi_{1,p} X_{t-1} + \dots + \phi_{p,p} X_{t-p} + \epsilon_t$$

5.5 Model selection $X_t - \hat{\phi}_{1,p} X_{t-1} - \dots - \hat{\phi}_{p,p} X_{t-p} \approx \epsilon_t$ Video 38

The above estimation methods for the parameters of an AR(p) process assume we know p when fitting the model. Often this is not the case so we wish to fit the model with several different values of p and selecting the “best” model.

Typically, this is achieved via the Akaike information criterion (AIC), which is defined as

$$\text{AIC} = 2k - 2\ell(\hat{\phi}, \hat{\sigma}_\epsilon^2).$$



We choose the model which gives *lowest* AIC score. Here, k is the number of free parameters to be estimated when fitting the model, so for an AR(p) process, $k = p+1$ (the p coefficients plus the variance term). The AIC looks to reward goodness of fit but penalise model complexity.

The procedure is as follows

1. Choose a set of values for p that you wish to try, e.g. $p = 1, 2, 3, 4, 5, \dots, 10$.
2. For each value of p , estimate the model parameters via maximum likelihood, compute the value of the likelihood function at these estimates, and compute the AIC.
3. Choose the value of p and its associated parameter estimates that gave the lowest AIC.

Chapter 6

Forecasting

6.1 Formulation

Suppose we wish to predict the value of X_{t+l} of a process, given $X_t, X_{t-1}, X_{t-2}, \dots$

Let the appropriate model for $\{X_t\}$ be an ARMA(p, q) process:

$$\Phi(B)X_t = \Theta(B)\epsilon_t.$$

Consider a forecast $X_t(l)$ of X_{t+l} (an l -step ahead forecast) which is a linear combination of $X_t, X_{t-1}, X_{t-2}, \dots$:

$$X_t(l) = \sum_{k=0}^{\infty} \pi_k X_{t-k}.$$

Note: this assumes a semi-infinite realization of $\{X_t\}$. Let us now assume that $\{X_t\}$ can be written as a one-sided linear process, so that

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t,$$

$$\underline{\Psi}(B) = \underline{\psi}_0 + \underline{\psi}_1 B + \underline{\psi}_2 B^2 + \dots$$

and

what we want to predict using $X_t(l)$ $\rightarrow X_{t+l} = \sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k} = \Psi(B)\epsilon_{t+l}.$

Hence,

$$\begin{aligned} T\Psi(B)X_t &= X_t(l) = \sum_{k=0}^{\infty} \pi_k X_{t-k} \\ T\Psi(B) &= \pi_0 + \pi_1 B + \pi_2 B^2 + \dots \\ &= \sum_{k=0}^{\infty} \pi_k \Psi(B)\epsilon_{t-k} \\ &= \Pi(B)\Psi(B)\epsilon_t. \end{aligned}$$

these are the things I want to learn.

GLP form for X_{t-k}

object of interest



Let $\delta(B) = \Pi(B)\Psi(B)$ so that,

$$\delta(B) = \delta_0 + \delta_1 B + \delta_2 B^2 + \dots$$

$$\begin{aligned} X_t(l) &= \delta(B)\epsilon_t \\ &= \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k}. \end{aligned}$$

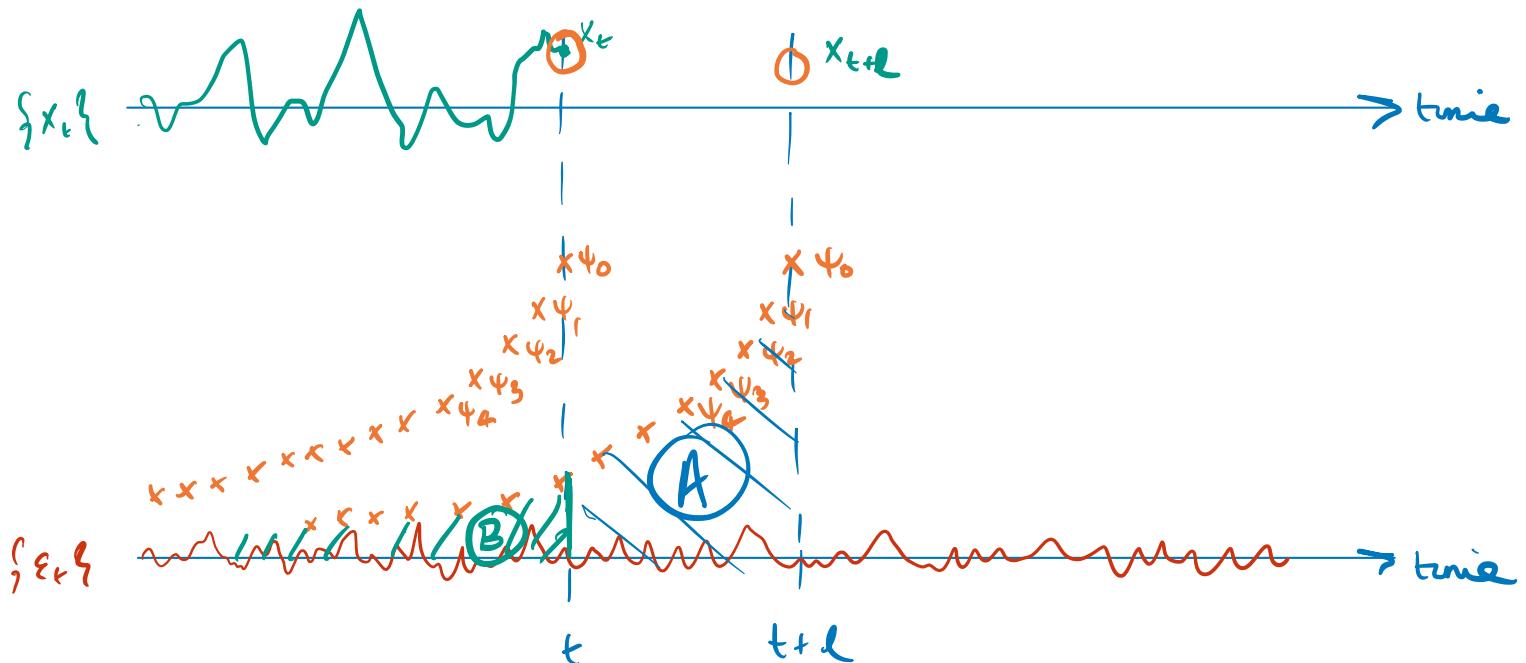
Now,

$$\begin{aligned} X_{t+l} &= \sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k} \\ &= \sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} + \sum_{k=l}^{\infty} \psi_k \epsilon_{t+l-k} \end{aligned} \quad \begin{matrix} (A) \\ (B) \end{matrix}$$

(A) Involves future ϵ_t s, and so represents the “unpredictable” part of X_{t+l} .

(B) Depends only on past and present values of ϵ_t , thus representing the “predictable” part of X_{t+l} .

Diagram: predictable and unpredictable parts



Hence we would expect,

$$\begin{aligned} X_t(l) &= \sum_{k=l}^{\infty} \psi_k \epsilon_{t+l-k} \\ &= \sum_{j=0}^{\infty} \psi_{j+l} \epsilon_{t-j}, \quad j = k - l. \end{aligned}$$

so that $\delta_k \equiv \psi_{k+l}$.

This can be readily proved. For linear least squares, we want to minimize,

$$\begin{aligned}
 E\{(X_{t+l} - X_t(l))^2\} &= E \left\{ \underbrace{\left(\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} + \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k} - \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} \right)^2}_{X_{t+l}} \right\} \\
 &= E \left\{ \underbrace{\left(\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} \right)}_{\text{future}} + \underbrace{\left(\sum_{k=0}^{\infty} [\psi_{k+l} - \delta_k] \epsilon_{t-k} \right)}_{\text{past and present}} \right\}^2 \\
 &= \sigma_{\epsilon}^2 \left\{ \left(\sum_{k=0}^{l-1} \psi_k^2 \right) + \sum_{k=0}^{\infty} (\psi_{k+l} - \delta_k)^2 \right\}.
 \end{aligned}$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}$, $k = 0, 1, 2, \dots$ as expected. With this choice of $\{\delta_k\}$ the second term vanishes, and we have,

$$\begin{aligned}
 \sigma^2(l) &= E\{(X_{t+l} - X_t(l))^2\} \\
 &= \sigma_{\epsilon}^2 \sum_{k=0}^{l-1} \psi_k^2, \quad = \text{Var}\{\textcircled{A}\}.
 \end{aligned}$$

which is known as the l -step prediction variance.

When $l = 1$, $\delta_k = \psi_{k+1}$,

$$\begin{aligned}
 X_t(1) &= \delta_0 \epsilon_t + \delta_1 \epsilon_{t-1} + \delta_2 \epsilon_{t-2} + \dots \\
 &= \psi_1 \epsilon_t + \psi_2 \epsilon_{t-1} + \psi_3 \epsilon_{t-2} + \dots \\
 X_{t+1} &= \psi_0 \epsilon_{t+1} + \psi_1 \epsilon_t + \psi_2 \epsilon_{t-1} + \dots
 \end{aligned}$$

so that,

$$X_{t+1} - X_t(1) = \psi_0 \epsilon_{t+1} = \epsilon_{t+1} \quad \text{since } \psi_0 = 1.$$

Hence ϵ_{t+1} can be thought of as the “one step prediction error”. Also of course,

$$X_{t+1} = X_t(1) + \epsilon_{t+1}$$

so that ϵ_{t+1} is the essentially “new” part of X_{t+1} which is not linearly dependent on past observations. The sequence $\{\epsilon_t\}$ is often called the innovations process of $\{X_t\}$, and, as used here, σ_{ϵ}^2 is called the innovations variance.

If we wish to write $X_t(l)$ explicitly as a function of X_t, X_{t-1}, \dots then we note first that,

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k}, \quad \text{say}$$

so that,

$$X_t(l) = \Psi^{(l)}(B) \epsilon_t, \quad \text{say}$$

where,

$$\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k.$$

Assuming that $\Psi(z)$ is analytic in and on the unit circle (stationary and invertible) then we can write

$$X_t = \Psi(B) \epsilon_t \quad \text{and} \quad \epsilon_t = \Psi^{-1}(B) X_t,$$

and thus

$$\begin{aligned} X_t(l) &= \Psi^{(l)}(B) \epsilon_t = \Psi^{(l)}(B) \Psi^{-1}(B) X_t \\ &= G^{(l)}(B) X_t, \quad \text{say} \end{aligned}$$

$\underbrace{\Psi^{-1}(B) \Psi^{(l)}(B)}_{T(B)}$

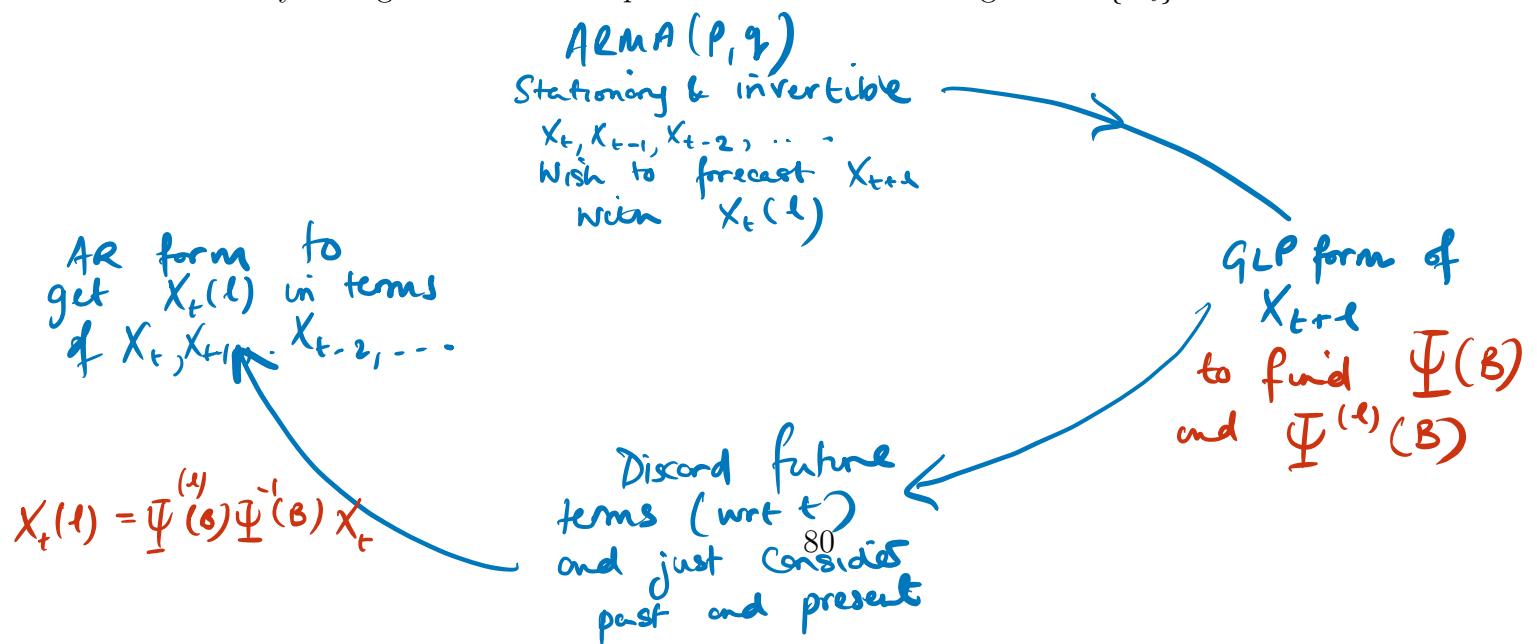
with,

$$G^{(l)}(z) = \Psi^{(l)}(z) \Psi^{-1}(z).$$

If we consider the sequence of predictors $X_t(l)$ for different values of t (with l fixed) then this forms a new process, which since

$$X_t(l) = G^{(l)}(B) X_t,$$

may be regarded as the output of a linear filter acting on the $\{X_t\}$.



6.2 Examples

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6.2.1 AR(1)

$$X_t - \phi_{1,1} X_{t-1} = \epsilon_t \quad |\phi_{1,1}| < 1.$$

Then

$$\Psi(z) = 1 - \phi_{1,1}z$$

$$X_t = (1 - \phi_{1,1}B)^{-1}\epsilon_t.$$

So,

$$\begin{aligned} \Psi(z) &= 1 + \phi_{1,1}z + \phi_{1,1}^2 z^2 + \dots \\ &= \psi_0 + \psi_1 z + \psi_2 z^2 + \dots \end{aligned}$$

i.e., $\psi_k = \phi_{1,1}^k$. Hence,

$$\begin{aligned} X_t(l) &= \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k} \\ &= \sum_{k=0}^{\infty} \phi_{1,1}^{k+l} \epsilon_{t-k} = \phi_{1,1}^l \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k} \\ &= \phi_{1,1}^l X_t. \end{aligned}$$

$\Psi(B)\epsilon_t = X_t$

The l -step prediction variance is $\text{Var}(X_t)$

$$\sigma^2(l) = \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \psi_k^2 \right) = \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \phi_{1,1}^{2k} \right) = \sigma_\epsilon^2 \frac{(1 - \phi_{1,1}^{2l})}{(1 - \phi_{1,1}^2)}.$$

Alternatively,

$$X_t(l) = G^{(l)}(B)X_t,$$

Variance of the forecast error

with $G^{(l)}(z) = \Psi^{(l)}(z)\Psi^{-1}(z)$. But,

$$\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k = \sum_{k=0}^{\infty} \phi_{1,1}^{k+l} z^k,$$

and,

$$\Psi^{-1}(z) = 1 - \phi_{1,1}z,$$

so that

$$\begin{aligned} G^{(l)}(z) &= (\phi_{1,1}^l + \phi_{1,1}^{l+1}z + \phi_{1,1}^{l+2}z^2 + \dots)(1 - \phi_{1,1}z) \\ &= \phi_{1,1}^l, \end{aligned}$$

i.e., $X_t(l) = \phi_{1,1}^l X_t$ as before.

We have demonstrated that for the AR(1) model the linear least squares predictor of X_{t+l} depends only on the most recent observation, X_t , and does not involve X_{t-1}, X_{t-2}, \dots , which is what we would expect bearing in mind the Markov nature of the AR(1) model. As $l \rightarrow \infty$, $X_t(l) \rightarrow 0$, since $X_t(l) = \phi_{1,1}^l X_t$ and $|\phi_{1,1}| < 1$. Also, the l -step prediction variance,

$$\sigma^2(l) \rightarrow \frac{\sigma_\epsilon^2}{(1 - \phi_{1,1}^2)} = \text{var}\{X_t\}.$$

In fact the solution to the forecasting problem for the AR(1) model can be derived directly from the difference equation,

$$X_t - \phi_{1,1} X_{t-1} = \epsilon_t.$$

by setting future innovations ϵ_t to be zero:

$$X_t(1) = \phi_{1,1} X_t + 0$$

$$X_t(2) = \phi_{1,1} X_t(1) + 0$$

⋮

$$X_t(l) = \phi_{1,1} X_t(l-1) + 0$$

so that,

$$X_t(l) = \phi_{1,1}^l X_t.$$

For general AR(p) processes it turns out that $X_t(l)$ depends only on the last p observed values of $\{X_t\}$, and may be obtained by solving the AR(p) difference equation with the future $\{\epsilon_t\}$ set to zero. For example for an AR(p) process and $l = 1$,

$$X_t(1) = \phi_{1,p} X_t + \dots + \phi_{p,p} X_{t-p+1} + 0$$

$$X_t(2) = \phi_{1,p} X_t(1) + \phi_{2,p} X_t + \dots + \phi_{p,p} X_{t-p+2} + 0$$

$$X_t(3) = \phi_{1,p} X_t(2) + \phi_{2,p} X_t(1) + \dots + \phi_{p,p} X_{t-p+3} + 0$$

6.2.2 ARMA(1,1)

$$(1 - \phi_{1,1}B)X_t = (1 - \theta_{1,1}B)\epsilon_t.$$

Take $\phi_{1,1} = \phi$ and $\theta_{1,1} = \theta$,

$$X_t = \frac{(1 - \theta B)}{(1 - \phi B)}\epsilon_t = \Psi(B)\epsilon_t.$$

So,

$$\begin{aligned}\Psi(z) &= (1 - \theta z)(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \dots) \\ &= 1 + (\phi - \theta)z + \phi(\phi - \theta)z^2 + \dots + \phi^{l-1}(\phi - \theta)z^l + \dots \\ &= \psi_0 + \psi_1 z + \psi_2 z^2 + \dots\end{aligned}$$

QLP form

$$\Psi(z) = \frac{(1 - \theta z)}{(1 - \phi z)}$$

$$\Psi^{-1}(z) = \frac{(1 - \phi z)}{(1 - \theta z)}$$

So,

$$\psi_l = \begin{cases} 1 & l = 0 \\ \phi^{l-1}(\phi - \theta) & l \geq 1 \end{cases}$$

The l -step prediction variance is

$$\begin{aligned}\text{Var(A)} \quad \sigma^2(l) &= \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \psi_k^2 \right) = \sigma_\epsilon^2 \left(1 + \sum_{k=1}^{l-1} \psi_k^2 \right) \\ &= \sigma_\epsilon^2 \left(1 + (\phi - \theta)^2 \sum_{k=1}^{l-1} \phi^{2k-2} \right) \\ &= \sigma_\epsilon^2 \left(1 + (\phi - \theta)^2 \frac{(1 - \phi^{2l-2})}{(1 - \phi^2)} \right).\end{aligned}$$

Now,

$$\begin{aligned}\Psi^{(l)}(z) &= \sum_{k=0}^{\infty} \psi_{k+l} z^k \\ &= \phi^{l-1}(\phi - \theta) \sum_{k=0}^{\infty} \phi^k z^k \\ &= \phi^{l-1}(\phi - \theta)(1 - \phi z)^{-1},\end{aligned}$$

and,

$$\Psi^{-1}(z) = \frac{(1 - \phi z)}{(1 - \theta z)}.$$

So,

$$\begin{aligned}G^{(l)}(z) &= \Psi^{(l)}(z)\Psi^{-1}(z) \\ &= \phi^{l-1}(\phi - \theta)(1 - \theta z)^{-1},\end{aligned}$$

$$= \phi^{l-1} \frac{(\phi - \theta)}{1 - \phi z} \cdot \frac{(1 - \phi z)}{1 - \theta z}$$

and,

$$\begin{aligned} X_t(l) &= G^{(l)}(B)X_t \\ &= \phi^{l-1}(\phi - \theta)(1 - \theta B)^{-1}X_t. \end{aligned}$$

Consider $l = 1$,

$$\begin{aligned} X_t(1) &= (\phi - \theta)(1 - \theta B)^{-1}X_t \\ &= (\phi - \theta)(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \dots)X_t \\ &= (\phi - \theta)X_t + \theta(\phi - \theta)X_{t-1} + \theta^2(\phi - \theta)X_{t-2} + \dots \\ &= \phi X_t - \theta [X_t - (\phi - \theta)X_{t-1} - \theta(\phi - \theta)X_{t-2} - \dots - \theta^{k-1}(\phi - \theta)X_{t-k} - \dots] \end{aligned}$$

1 step ahead forecast in terms
 of $X_t, X_{t-1}, X_{t-2}, \dots$

But consider,

$$\begin{aligned} \epsilon_t &= \Psi^{-1}(B)X_t = (1 - \phi B)(1 - \theta B)^{-1}X_t \\ &= (1 - \phi B)(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \dots)X_t \\ &= X_t - (\phi - \theta)X_{t-1} - \theta(\phi - \theta)X_{t-2} - \dots - \theta^{k-1}(\phi - \theta)X_{t-k} - \dots \end{aligned}$$

Therefore,

$$X_t(1) = \phi X_t - \theta \epsilon_t.$$

So can again be derived directly from the difference equation,

$$\begin{aligned} X_t &= \phi X_{t-1} - \theta \epsilon_{t-1} + \epsilon_t, \quad \text{ARMA(1,1)} \\ X_{t+1} &= \phi X_t - \theta \epsilon_t + \epsilon_{t+1} \\ \text{by setting } \underline{\text{future}} \text{ innovations } \epsilon_t \text{ to zero.} &\quad \leftarrow \text{set equal to zero to get } X_t(1) \end{aligned}$$

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6.2.3 MA(1) (invertible)

$$X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1} \quad |\theta_{1,1}| < 1.$$

So,

$$X_t = \Psi(B)\epsilon_t$$

$$\begin{aligned} \Psi(z) &= \psi_0 + \psi_1 z + \psi_2 z^2 + \dots \\ &= 1 - \theta_{1,1}z \quad \text{"0"} \end{aligned}$$

Hence, $\psi_0 = 1$; $\psi_1 = -\theta_{1,1}$; $\psi_k = 0$, $k \geq 2$.

$$\begin{aligned} X_t(l) &= \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k} = \Psi^{(l)}(B) \epsilon_t \\ &= \psi_l \epsilon_t + \psi_{l+1} \epsilon_{t-1} + \dots \end{aligned}$$

So,

$$\begin{aligned} \Psi^{(l)}(z) &= \sum_{k=0}^{\infty} \psi_{k+l} z^k = \psi_l z^0 + \psi_{l+1} z^1 + \dots \\ &= \begin{cases} -\theta_{1,1} & l = 1 \\ 0 & l \geq 2. \end{cases} \quad \text{Handwritten note: } \begin{array}{l} \psi_1 z^0 + \psi_2 z^1 + \psi_3 z^2 + \dots = -\theta_{1,1} \\ \psi_2 z^0 + \psi_3 z^1 + \dots = 0 \end{array} \end{aligned}$$

Hence,

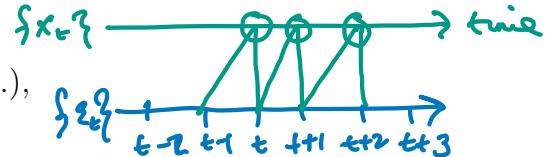
$$G^{(l)}(z) = \Psi^{(l)}(z) \Psi^{-1}(z) = \begin{cases} -\theta_{1,1}(1 - \theta_{1,1}z)^{-1} & l = 1 \\ 0 & l \geq 2. \end{cases}$$

Thus, for $l = 1$,

$$G^{(1)}(z) = -\theta_{1,1}(1 + \theta_{1,1}z + \theta_{1,1}^2 z^2 + \dots),$$

and hence,

$$\begin{aligned} X_t(1) &= G^{(1)}(B) X_t \\ &= -\sum_{k=0}^{\infty} \theta_{1,1}^{k+1} X_{t-k} \end{aligned}$$



$x_t(2), x_t(3), x_t(4), \dots$
are all zero

6.3 Forecast errors and updating

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We have seen that when $\delta_k = \psi_{k+l}$ the forecast error is $\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k}$.

Let,

$$\begin{aligned} e_t(l) &= X_{t+l} - X_t(l) \\ &= \sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k}. \quad \text{Handwritten note: } \textcircled{A} \end{aligned}$$

Then,

$$e_t(l+m) = \sum_{j=0}^{l+m-1} \psi_j \epsilon_{t+l+m-j}.$$

Clearly,

$$\mathbb{E}\{e_t(l)\} = \mathbb{E}\{e_t(l+m)\} = 0.$$

$$\begin{aligned}
& E \left\{ \left(\sum_{j=0}^{l-1} \psi_j \epsilon_{t+j} \right) \left(\sum_{k=0}^{l+m-1} \psi_k \epsilon_{t+k} \right) \right\} \\
& = \sum_{j=0}^{l-1} \sum_{k=0}^{l+m-1} \psi_j \psi_k E \left\{ \epsilon_{t+j} \epsilon_{t+k} \right\} \\
& \quad \text{except } -j = m-k
\end{aligned}$$

Hence,

$$\text{cov}\{e_t(l), e_t(l+m)\} = E\{e_t(l)e_t(l+m)\} = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k \psi_{k+m} \quad (j = k - m),$$

and

$$\text{var}\{e_t(l)\} = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k^2 = \sigma^2(l).$$

E.g.,

$$\text{cov}\{e_t(1), e_t(2)\} = \sigma_\epsilon^2 \psi_1.$$

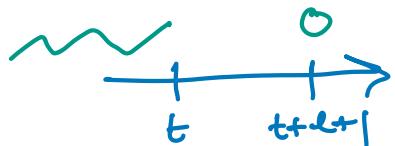
This could be quite large – should the forecast for a series wander off target, it is possible for it to remain there in the short run since forecast errors can be quite highly correlated. Hence, when X_{t+1} becomes available we should update the forecast.

$$\begin{aligned}
X_{t+1}(l) &= \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t+1-k} \\
&= \psi_l \epsilon_{t+1} + \psi_{l+1} \epsilon_t + \psi_{l+2} \epsilon_{t-1} + \dots,
\end{aligned}$$



but,

$$\begin{aligned}
X_t(l+1) &= \sum_{k=0}^{\infty} \psi_{k+l+1} \epsilon_{t-k} \\
&= \psi_{l+1} \epsilon_t + \psi_{l+2} \epsilon_{t-1} + \psi_{l+3} \epsilon_{t-2} + \dots,
\end{aligned}$$



and,

$$\begin{aligned}
X_{t+1}(l) &= X_t(l+1) + \psi_l \epsilon_{t+1} \\
&= X_t(l+1) + \psi_l (X_{t+1} - X_t(1)).
\end{aligned}$$

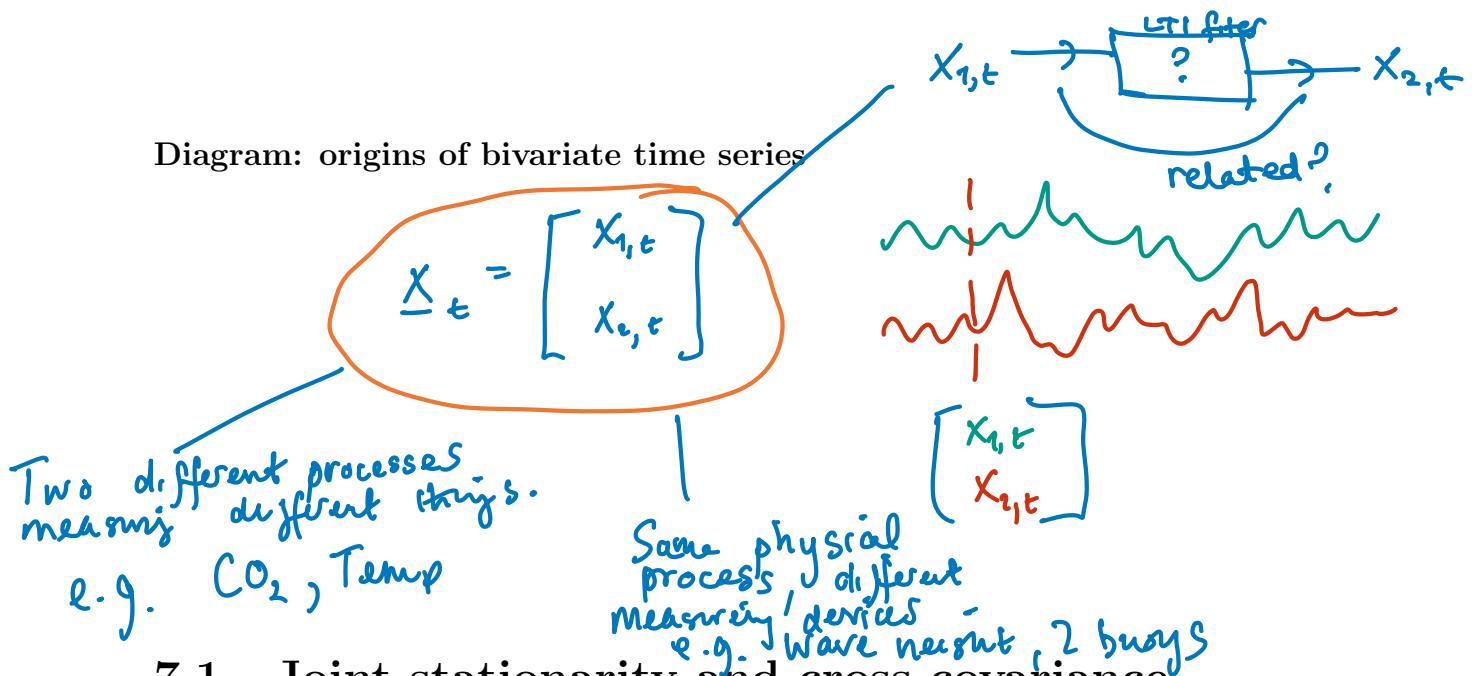
new forecast *new data point with* *update previous forecast* *continue to update as data comes in, and store it in memory*

Hence, to forecast X_{t+l+1} we can modify the $l+1$ -step ahead forecast at time t by producing an l -step ahead forecast at time $t+1$ using X_{t+1} as it becomes available.

Chapter 7

Bivariate Time Series

Diagram: origins of bivariate time series



7.1 Joint stationarity and cross-covariance

7.1.1 Joint stationarity

The two real-valued discrete time stochastic processes $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are said to be jointly stationary stochastic processes if $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are each, separately, second-order stationary processes, and $\text{cov}\{X_{1,t}, X_{2,t+\tau}\}$ is a function of τ only. Then $\{X_{1,t}; X_{2,t}\}$ forms a stationary bivariate process.

7.1.2 Cross-covariance

The acvs are

$$s_{X_{1,\tau}} = E\{[X_{1,t} - \mu_{X_1}][X_{1,t+\tau} - \mu_{X_1}]\}$$

$$S_{1,\tau}$$

$$s_{X_2,\tau} = E\{[X_{2,t} - \mu_{X_2}][X_{2,t+\tau} - \mu_{X_2}]\}$$

$s_{2/\tau}$

so that,

$$s_{X_1,0} = \text{var}\{X_{1,t}\} = \sigma_{X_1}^2$$

$$s_{X_2,0} = \text{var}\{X_{2,t}\} = \sigma_{X_2}^2.$$

The cross-covariance sequence (ccvs) is given by

$$\begin{aligned} s_{X_1 X_2, \tau} &= \text{cov}\{X_{1,t}, X_{2,t+\tau}\} \\ &= E\{[X_{1,t} - \mu_{X_1}][X_{2,t+\tau} - \mu_{X_2}]\}. \end{aligned}$$

$\tau > 0$, $X_{1,t}$ is lagging behind $X_{2,t+\tau}$

The cross-correlation sequence (ccs) is

$$\rho_{X_1 X_2, \tau} = \frac{s_{X_1 X_2, \tau}}{\sqrt{s_{X_1,0} s_{X_2,0}}} = \frac{s_{X_1 X_2, \tau}}{\sigma_{X_1} \sigma_{X_2}}.$$

Note that,

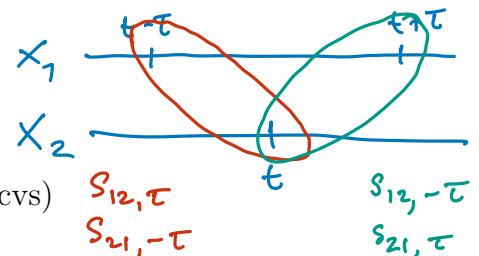
$$\begin{aligned} s_{X_2 X_1, \tau} &= \text{cov}\{X_{2,t}, X_{1,t+\tau}\} \\ &= E\{[X_{2,t} - \mu_{X_2}][X_{1,t+\tau} - \mu_{X_1}]\}. \end{aligned}$$

$$\begin{aligned} s_{X_2 X_1, \tau} &= E\{(X_{2,t} - \mu_{X_2})(X_{1,t+\tau} - \mu_{X_1})\} \\ &= E\{(X_{2,t+\tau} - \mu_{X_2})(X_{1,t} - \mu_{X_1})\} \\ &= s_{X_1 X_2, -\tau} \end{aligned}$$

Hence,

$$s_{X_1 X_2, \tau} = s_{X_2 X_1, -\tau} \quad \text{but}$$

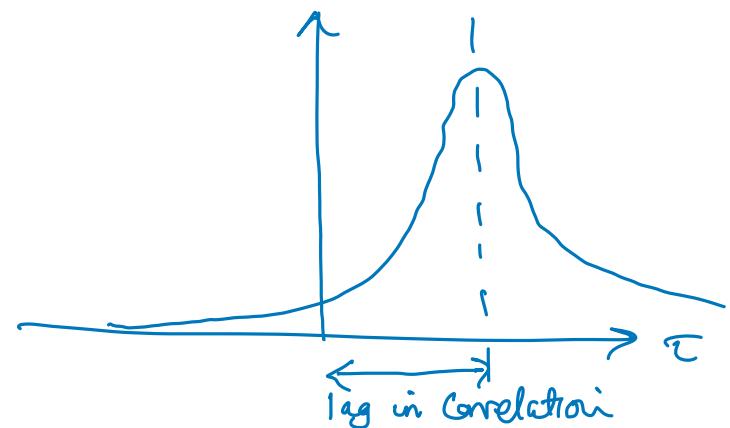
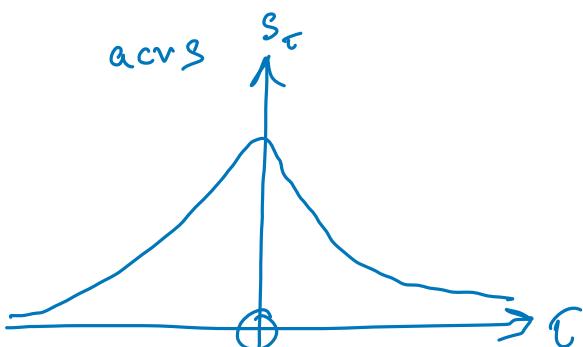
$$s_{X_1 X_2, \tau} \neq s_{X_1 X_2, -\tau} \quad (\text{unlike acvs})$$



The ccvs is generally quite asymmetric.

Diagram: cross-covariance sequence

ccvs



Estimation

Given

$$X_{1,1}, X_{1,2}, \dots, X_{1,N}$$

$$X_{2,1}, X_{2,2}, \dots, X_{2,N}$$

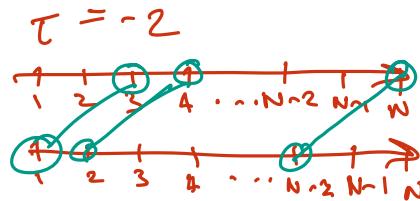
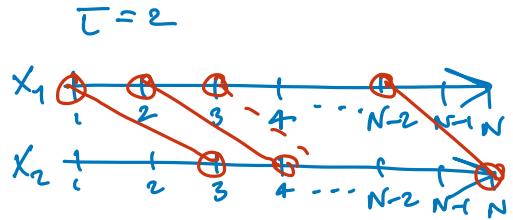
a natural estimator for the ccvs is

$$\hat{s}_{X_1 X_2, \tau} = \begin{cases} \frac{1}{N} \sum_{t=1}^{N-\tau} (X_{1,t} - \bar{X}_1)(X_{2,t+\tau} - \bar{X}_2) & \tau = 0, 1, 2, \dots, N-1 \\ \frac{1}{N} \sum_{t=1-\tau}^N (X_{1,t} - \bar{X}_1)(X_{2,t+\tau} - \bar{X}_2) & \tau = -1, -2, \dots, -(N-1), \end{cases}$$

so that the estimated ccs is

$$\hat{\rho}_{X_1 X_2, \tau} = \frac{\hat{s}_{X_1 X_2, \tau}}{\hat{\sigma}_{X_1} \hat{\sigma}_{X_2}}.$$

Linear filtering with noise

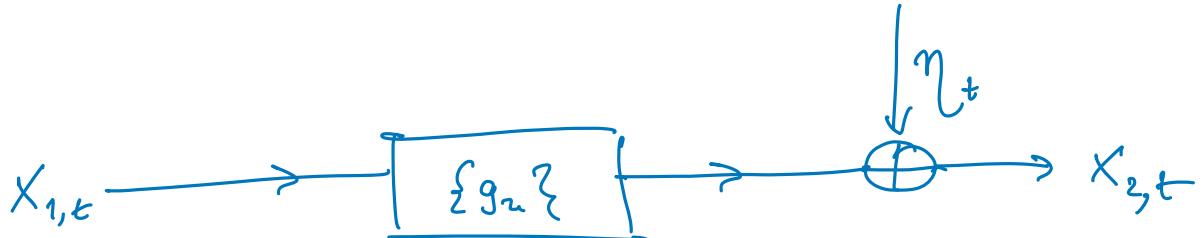


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$$X_{2,t} = \sum_{u=-k}^k g_u X_{1,t-u} + \eta_t$$

where $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are zero mean stationary processes, $\{\eta_t\}$ is a zero mean (possible coloured) noise with variance σ_η^2 , uncorrelated with $\{X_{1,t}\}$.

Diagram: linear filter with noise model



Then,

$$\begin{aligned}
 s_{X_1 X_2, \tau} &= \text{cov}\{X_{1,t}, X_{2,t+\tau}\} \\
 &= E\{X_{1,t} X_{2,t+\tau}\} \\
 &= E\left\{X_{1,t} \left[\sum_{u=-k}^k g_u X_{1,t+\tau-u} + \eta_{t+\tau} \right]\right\} \\
 &= \sum_{u=-k}^k g_u E\{X_{1,t}, X_{1,t+\tau-u}\} + E\{X_{1,t} \eta_{t+\tau}\} \\
 &= \sum_{u=-k}^k g_u s_{X_1, \tau-u}.
 \end{aligned}$$

Since,

$$\begin{aligned}
 \sigma_{X_2}^2 &= \text{var}\{X_{2,t}\} = \mathbb{E}\{X_{2,t}^2\} \\
 &= \mathbb{E}\left\{\left(\sum_{u=-k}^k g_u X_{1,t-u} + \eta_t\right)^2\right\} \\
 &= \mathbb{E}\left\{\left(\sum_{u=-k}^k g_u X_{1,t-u}\right)^2\right\} + \mathbb{E}\{\eta_t^2\} + 2 \cdot \mathbb{E}\left\{\sum_{u=-k}^k g_u X_{1,t-u} \eta_t\right\} \\
 &= \mathbb{E}\left\{\sum_{u=-k}^k g_u X_{1,t-u} \sum_{v=-k}^k g_v X_{1,t-v}\right\} + \sigma_\eta^2 \\
 &= \sum_{u=-k}^k \sum_{v=-k}^k g_u g_v \mathbb{E}\{X_{1,t-u} X_{1,t-v}\} + \sigma_\eta^2 \\
 &= \sum_{u=-k}^k \sum_{v=-k}^k g_u g_v s_{X_1,u-v} + \sigma_\eta^2
 \end{aligned}$$

the ccs is

$$\rho_{X_1 X_2, \tau} = \frac{\sum_{u=-k}^k g_u s_{X_1, \tau-u}}{\sigma_{X_1} \sqrt{\sum_{u=-k}^k \sum_{v=-k}^k g_u g_v s_{X_1, u-v} + \sigma_\eta^2}}$$

noise Could be zero
 Only depends on the input, and the Variance of the noise.

Consider

$g_u =$	$\begin{cases} 1 & u=5 \\ 0 & \text{o/w} \end{cases}$
$\sigma_\eta^2 = 0$	
$X_{2,t} = X_{1,t-5}$	

$$\ell_{X_1, X_2, \tau} = \frac{s_{X_1, \tau-5}}{\sigma_{X_1}^2} = \ell_{X_1, \tau-5}$$



7.2 Cross-Spectra

Consider frequency domain characterization of the real-valued bivariate process $\{X_{1,t}; X_{2,t}\}$. Assume that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are both zero mean processes with spectral density functions

$$S_{X_j}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_j, \tau} e^{-i2\pi f \tau}; \quad |f| \leq 1/2, \quad j = 1, 2.$$

Then the cross spectra are

$$S_{X_j X_k}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_j X_k, \tau} e^{-i2\pi f \tau}; \quad |f| \leq 1/2, \quad j \neq k = 1, 2,$$

Complex Valued because ccs is no longer symmetric, So imaginary parts do not cancel

$$\begin{aligned}
 S_{X_j X_k}^*(f) &= \sum_{\tau=-\infty}^{\infty} S_{X_j X_k, \tau} e^{i 2 \pi f \tau} \\
 &= \sum_{\tau=-\infty}^{\infty} S_{X_j X_k, \tau} e^{-i 2 \pi (-f) \tau} \\
 &= S_{X_j X_k}(-f)
 \end{aligned}$$

assuming the ccovs is square summable.

Note that for real processes $S_{X_j X_k}^*(f) = S_{X_j X_k}(-f)$.

Inverse Fourier transformation gives

$$S_{X_j X_k, \tau} = \int_{-1/2}^{1/2} S_{X_j X_k}(f) e^{i 2 \pi f \tau} df.$$

Now write

$$X_{j,t} = \int_{-1/2}^{1/2} e^{i 2 \pi f t} dZ_{X_j}(f); \quad X_{k,t} = \int_{-1/2}^{1/2} e^{i 2 \pi f' t} dZ_{X_k}(f'),$$

so that,

$$\begin{aligned}
 S_{X_j X_k, \tau} &= \text{cov}\{X_{j,t}, X_{k,t+\tau}\} \\
 &= E\{X_{j,t} X_{k,t+\tau}\} \\
 &= E\{X_{j,t}^* X_{k,t+\tau}\} \\
 &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{-i 2 \pi f t} e^{i 2 \pi f'(t+\tau)} E\{dZ_{X_j}^*(f) dZ_{X_k}(f')\}.
 \end{aligned}$$

But this must be a function of τ only, so that $E\{dZ_{X_j}^*(f) dZ_{X_k}(f')\} = 0$ for $f \neq f'$, i.e., $dZ_{X_j}^*$ and dZ_{X_k} are cross-orthogonal as well as individually orthogonal.

Hence,

$$\begin{aligned}
 S_{X_j X_k, \tau} &= \int_{-1/2}^{1/2} e^{i 2 \pi f \tau} E\{dZ_{X_j}^*(f) dZ_{X_k}(f)\} \\
 \Rightarrow S_{X_j X_k}(f) df &= E\{dZ_{X_j}^*(f) dZ_{X_k}(f)\} \\
 \Rightarrow S_{X_k X_j}^*(f) &= S_{X_j X_k}(f).
 \end{aligned}$$

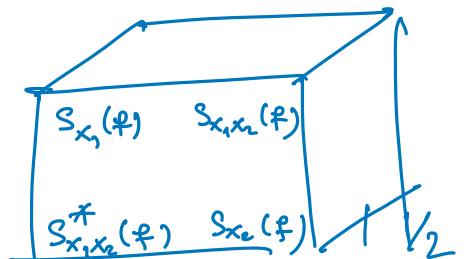
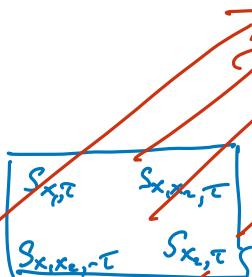
$$S_{X_k X_j}^*(f) df = E\{dZ_{X_k}^*(f) dZ_{X_j}(f)\}$$

The complete spectral properties are given by the spectral matrix

$$S(f) = \begin{pmatrix} S_{X_1}(f) & S_{X_1 X_2}(f) \\ S_{X_2 X_1}(f) & S_{X_2}(f) \end{pmatrix}.$$

$$\begin{aligned}
 S_{X_n X_3}^*(f) df &= \left(E\{dZ_{X_n}^*(f) dZ_{X_3}(f)\} \right) \\
 &= E\{dZ_{X_n}^*(f) dZ_{X_3}^*(f)\} \\
 &= S_{X_3 X_n}(f).
 \end{aligned}$$

$$\text{i.e. } S(f) = S^H(f)$$



$-l_1 l_2 \theta$
Nice to estimate

Since $S_{X_j X_k}(f)$ is a complex quantity we can write it as

$$S_{X_j X_k}(f) = |S_{X_j X_k}(f)| e^{i\theta_{X_j X_k}(f)}, \quad re^{i\theta}$$

where $|S_{X_j X_k}(f)|$ is the cross-amplitude spectrum

$\theta_{X_j X_k}(f)$ is the phase spectrum.

$\theta_{X_j X_k}(f)$ is defined only up to an integer multiple of 2π (since $e^{i2\pi} = e^{i4\pi} = \dots = 1$).

Worked example: group delay

$$x_{2,t} = c x_{1,t-d} + \eta_t \quad \text{in linear filtering with noise} \\ g_d = c, \quad g_n = 0 \text{ o/w.}$$

$$\text{we've seen that } S_{X_1 X_2, \tau} = \sum g_u S_{X_1, \tau-u}$$

$$S_{X_1 X_2, \tau} = c S_{X_1, \tau-d}$$

$$\text{Cross-spectrum is } S_{X_1 X_2}(f) = c \sum_{\tau=-\infty}^{\infty} S_{X_1, \tau-d} e^{-i2\pi f \tau} \\ = c e^{-i2\pi f d} \sum_{\tau'=-\infty}^{\infty} S_{X_1, \tau-d} e^{-i2\pi f (\tau-d)} \quad \tau' = \tau - d$$

$$= \underbrace{c e^{-i2\pi f d}}_{\text{lag is in phase information}} S_{X_1}(f)$$

of the cross-spectrum,

$$S_{X_1 X_2}(f) = \underbrace{c S_{X_1}(f)}_{|S_{X_1 X_2}(f)|} e^{i \underbrace{(-2\pi f d)}_{\theta_{X_1 X_2}(f)}}$$

$$\theta_{X_1 X_2}(f) = -2\pi f d$$

$$-\underbrace{\frac{1}{2\pi} \frac{d}{df} \theta_{X_1 X_2}(f)}_{\text{'group delay'}} = d.$$

Coherence

The quantity

$$\gamma_{X_j X_k}^2(f) = \frac{|S_{X_j X_k}(f)|^2}{S_{X_j}(f)S_{X_k}(f)},$$

is called the magnitude squared coherence at f .

$$\begin{aligned} S_{X_1 X_2}(f) df &= E \left\{ dZ_{X_1}^*(f) dZ_{X_2}(f) \right\} \\ S_{X_1}(f) df &= E \left\{ |dZ_{X_1}(f)|^2 \right\} \\ S_{X_2}(f) df &= E \left\{ |dZ_{X_2}(f)|^2 \right\}. \\ \gamma_{X_1 X_2}^2(f) &= \frac{|E \left\{ dZ_{X_1}^*(f) dZ_{X_2}(f) \right\}|^2}{E \left\{ |dZ_{X_1}(f)|^2 \right\} E \left\{ |dZ_{X_2}(f)|^2 \right\}} \\ &= \frac{\left| \text{Cov} \left\{ dZ_{X_1}(f), dZ_{X_2}(f) \right\} \right|^2}{\text{Var} \left\{ dZ_{X_1}(f) \right\} \text{Var} \left\{ dZ_{X_2}(f) \right\}} \end{aligned}$$

It is a real valued coefficient such that

$$0 \leq \gamma_{X_j X_k}^2(f) \leq 1.$$

It measures the linear correlation between the components of $\{X_{j,t}\}$ and $\{X_{k,t}\}$ at frequency f in the same sense as the coefficient of determination R^2 does in ordinary regression.

7.2.1 Example

Figure 37 shows measurements of ocean waves versus time recorded by two different instruments. Figure 38 shows the elements of the estimated spectral matrix for all frequencies. The Nyquist frequency is $1/(2 \times (4/30)) = 3.75\text{Hz}$ but the frequency axis has been truncated at 0.5Hz . Figure 39 shows the estimated cross-amplitude and phase spectra. Figure 40 shows the estimated coherence. The coherence between the datasets is highest around $0.1\text{-}0.2\text{Hz}$, so this is the frequency range where the instruments behave most similarly (since they are measuring the same waves).

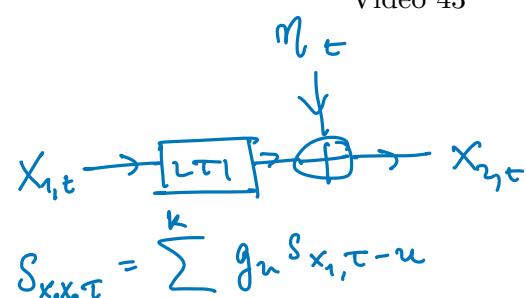
7.3 Linear filtering with noise

The model is

$$X_{2,t} = \sum_{u=-k}^k g_u X_{1,t-u} + \eta_t.$$

Then

$$\begin{aligned} S_{X_1 X_2}(f) &= \sum_{\tau=-\infty}^{\infty} s_{X_1 X_2, \tau} e^{-i2\pi f \tau} \\ &= \sum_{u=-k}^k g_u \sum_{\tau=-\infty}^{\infty} s_{X_1, \tau-u} e^{-i2\pi f \tau} \\ &= \sum_{u=-k}^k g_u e^{-i2\pi f u} \sum_{\tau=-\infty}^{\infty} s_{X_1, \tau-u} e^{-i2\pi f(\tau-u)} \\ &= G(f) S_{X_1}(f). \end{aligned}$$



$$\tau' = \tau - u$$

We can write the model as:

$$\underbrace{\int_{-1/2}^{1/2} e^{i2\pi f t} dZ_{X_2}(f)}_{X_{2,t}} = \sum_{u=-k}^k g_u \underbrace{\int_{-1/2}^{1/2} e^{i2\pi f(t-u)} dZ_{X_1}(f)}_{X_{1,t}} + \underbrace{\int_{-1/2}^{1/2} e^{i2\pi f t} dZ_{\eta}(f)}_{\eta_t}.$$

Hence,

$$dZ_{X_2}(f) = \sum_{u=-k}^k g_u e^{-i2\pi f u} dZ_{X_1}(f) + dZ_{\eta}(f).$$

Because:

$$\int_{-1/2}^{1/2} e^{i2\pi f t} dZ_{X_2}(f) = \int_{-1/2}^{1/2} e^{i2\pi f t} \left[\left(\sum_u g_u e^{-i2\pi f u} \right) dZ_{X_1}(f) + dZ_{\eta}(f) \right] df$$

Thus,

$$E\{|dZ_{X_2}(f)|^2\} = \sum_{u=-k}^k g_u e^{-i2\pi f u} \sum_{v=-k}^k g_v e^{i2\pi f v} E\{|dZ_{X_1}(f)|^2\} + E\{|dZ_{\eta}(f)|^2\}$$

$$E\{dZ_{X_2}^*(f) dZ_{X_2}(f)\}$$

since cross-products have expectation zero.

Hence,

$$S_{X_2}(f) = |G(f)|^2 S_{X_1}(f) + S_{\eta}(f).$$

Then,

$$\gamma_{X_1 X_2}^2(f) = \frac{|S_{X_1 X_2}(f)|^2}{S_{X_1}(f) S_{X_2}(f)}$$

$$\begin{aligned} \gamma_{X_1 X_2}^2(f) &= \frac{|G(f)|^2 S_{X_1}^2(f)}{S_{X_1}(f)[|G(f)|^2 S_{X_1}(f) + S_{\eta}(f)]} \\ &= \left[1 + \frac{S_{\eta}(f)}{|G(f)|^2 S_{X_1}(f)} \right]^{-1}. \end{aligned}$$

Now,

$$\begin{aligned} S_\eta(f) &= S_{X_2}(f) - |G(f)|^2 S_{X_1}(f) \\ &= S_{X_2}(f) \left[1 - \frac{|G(f)|^2}{S_{X_2}(f)} S_{X_1}(f) \right]. \end{aligned}$$

But,

$$\gamma_{X_1 X_2}^2(f) = \frac{|G(f)|^2 S_{X_1}^2(f)}{S_{X_1}(f) S_{X_2}(f)} = \frac{|G(f)|^2 S_{X_1}(f)}{S_{X_2}(f)},$$

so,

$$\begin{aligned} S_\eta(f) &= S_{X_2}(f)[1 - \gamma_{X_1 X_2}^2(f)] \\ \text{“noise”} &\quad \text{“total times unexplained proportion”} \end{aligned}$$

Video 44

7.4 Bivariate autoregressive processes

A bivariate model arises as an extension to the univariate AR(p) process. Let

$$\mathbf{X}_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_t = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

The VAR(p) model can be expressed as

$$\begin{aligned} \mathbf{X}_t &= \phi_{1,p} \mathbf{X}_{t-1} + \dots + \phi_{p,p} \mathbf{X}_{t-p} + \boldsymbol{\epsilon}_t, \\ \Phi(B) \mathbf{X}_t &= \boldsymbol{\epsilon}_t \end{aligned}$$

where,

$$\Phi(B) = \mathbf{I} - \phi_{1,p} B - \phi_{2,p} B^2 - \dots - \phi_{p,p} B^p,$$

where \mathbf{I} is the (2×2) identity matrix, and now $\{\phi_{i,p}\}$ are (2×2) matrices of parameters.

ϵ_t is a bivariate white noise process, such that

$$E\{\epsilon_t\} = 0$$

$$\Sigma = \begin{pmatrix} \text{Var}\{\epsilon_{1,t}\} & \text{Corr}\{\epsilon_{1,t}, \epsilon_{2,t}\} \\ \text{Corr}\{\epsilon_{2,t}, \epsilon_{1,t}\} & \text{Var}\{\epsilon_{2,t}\} \end{pmatrix}$$

and

$$E\{\epsilon_s \epsilon_t^T\} = \begin{cases} \Sigma, & t = s \\ 0 & \text{otherwise} \end{cases}$$

and Σ is a (2×2) covariance matrix. Thus the elements of ϵ_t may be correlated.

FACT : The condition for stationarity is that the roots of the *determinantal polynomial*, $|\Phi(z)|$, lie outside the unit circle.

Worked example: bivariate VAR(1)

Determine whether the bivariate system

$$\begin{aligned} X_{1,t} &= \frac{1}{2}X_{1,t-1} + \frac{1}{10}X_{2,t-1} + \epsilon_{1,t} \\ X_{2,t} &= \frac{1}{2}X_{1,t-1} + \frac{1}{2}X_{2,t-1} + \epsilon_{2,t} \end{aligned}$$

is stationary.

We can write this system as

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/10 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

$$(I - \phi B) \underline{X}_t = \underline{\epsilon}_t \quad \text{where}$$

$$\phi = \begin{pmatrix} \frac{1}{2} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Phi(z) := I - \phi z$$

$$\det(\Phi(z)) = \det \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/10 \\ 1/2 & 1/2 \end{pmatrix} z \right\} = \det \begin{pmatrix} 1 - \frac{z}{2} & -\frac{z}{10} \\ -\frac{z}{2} & 1 - \frac{z}{2} \end{pmatrix}$$

$$= \left(1 - \frac{z}{2}\right)^2 - \frac{z}{2} \cdot \frac{z}{10} = 1 - z + z^2/5$$

$$\text{Roots are } \frac{5}{2} \pm \frac{5}{2} \sqrt{\frac{1}{5}}$$

Both > 1 \Rightarrow Stationary.