MATH50010: Probability for Statistics
Problem Sheet 7

Questions marked with (†) build on material from Tuesdays lecture.

1. Suppose $(\Omega, \mathcal{F}\,\Pr)$ is a probability space and $A, B, C \in \mathcal{F}$. Show that

$$\Pr(A \cap B|C) = \Pr(A|B \cap C)\Pr(B|C).$$

Using the definition of conditional probability,

$$\begin{aligned}\Pr(A \cap B \mid C) &= \frac{Pr(A \cap B \cap C)}{Pr(C)} \\ &= \frac{\Pr(A \mid B \cap C)\Pr(B \cap C)}{\Pr(C)} \\ &= \Pr(A \mid B \cap C)\Pr(B \mid C).\end{aligned}$$

Note: we will appeal to this result repeatedly when working with Markov chains - it will eventually become second nature!

2. (a) (†) Let $\mathbf{P}$ be the transition matrix of a discrete Markov chain $(X_n)_{n\geq 0}$. Show by induction that the n-step transition matrix satisfies $\mathbf{P}_n = \mathbf{P}^n$.

   (b) (†) Show that a *stochastic matrix* has at least one eigenvalue equal to 1. Hence show that if $\mathbf{P}$ is a stochastic matrix, then so is $\mathbf{P}^n$ for all $n \in \mathbb{N}$.

   (a) For $n = 0, 1$, the result holds as $P_0 = I$, the identity matrix, and $P_1 = P$ by definition. Assume the result is true for $n = k$, so $P_k = P^k$ and consider the (i, j) entry of P^{k+1}. Then,

$$\begin{aligned}P^{k+1}_{ij} &= (P^k P)_{ij} \\ &= \sum_{l\in\mathcal{E}} P^k_{il} P_{lj} \\ &= \sum_{l\in\mathcal{E}} P_{il}(k) P_{lj}(1)\end{aligned}$$

   by induction. Using the Chapman-Kolmogorov equations, we can write this as

$$P^{k+1}_{ij} = P_{ij}(k+1)$$

   which is the (i, j) entry of P_{k+1}. The result follows as the entry (i, j) was arbitrary.

   (b) Recall that a stochastic matrix P has non-negative entries and rows that sum to one. Let $\mathbf{1}$ be a column vector of ones. Then, as all the rows of P sum to one

$$P\mathbf{1} = \mathbf{1}$$

   and so $\mathbf{1}$ is an eigenvector of any stochastic matrix with eigenvalue equal to 1. Further, if P is a stochastic matrix, then all entries of P^n must also be non-negative. As the rows of P sum to one,

$$P^n\mathbf{1} = P^{n-1}(P\mathbf{1}) = P^{n-1}\mathbf{1} = \cdots = \mathbf{1}.$$

   Hence, the rows of P^n also sum to one and P^n is a stochastic matrix.

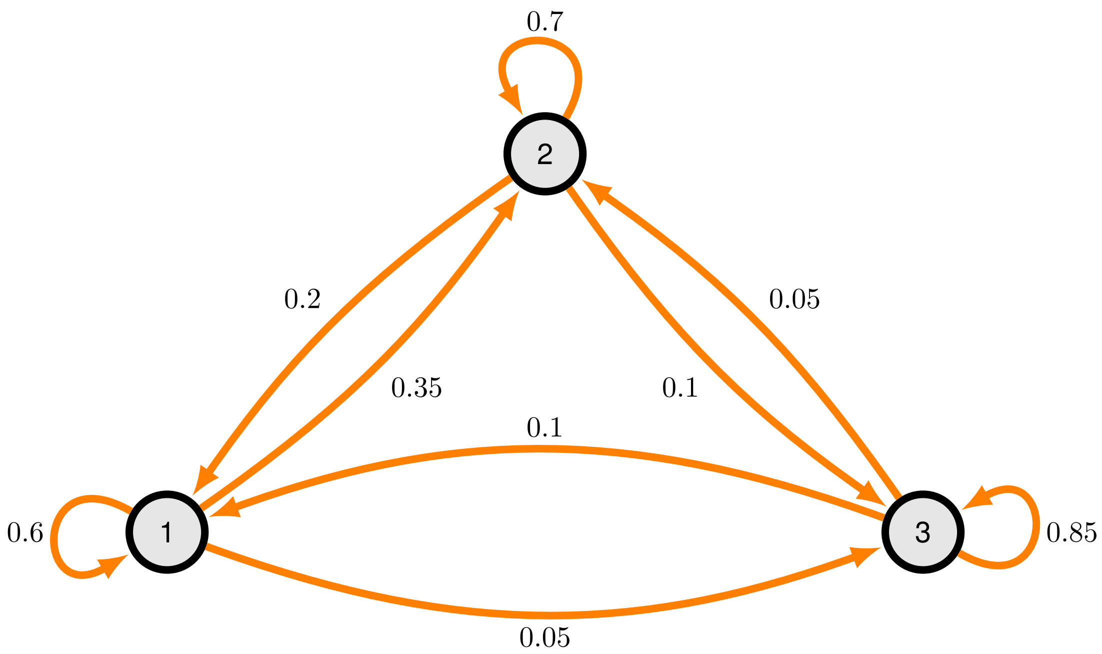

Figure 1: Question 3 (b): Transition matrix

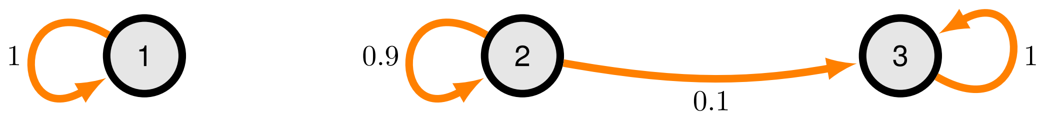

Figure 2: 3(c): Transition matrix

3. For each matrix, decide whether it is stochastic. If it is, draw the corresponding transition diagram (assuming the state space is given by $E = \{1, 2, 3\}$).

$$\text{(a)} \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & -1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0.6 & 0.35 & 0.05 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.05 & 0.85 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Not stochastic as it has negative entries.

(b) Stochastic matrix as it has positive entries with rows that sum to one.

(c) Stochastic matrix as it has positive entries with rows that sum to one.

4. Suppose we use a Markov chain to model the autumn weather in Bretagne (France), from day to day, as follows. If it is raining today, there is a probability 0.3 that it will rain again tomorrow. Similarly, if it is raining today, the probability of cloudy weather (with no rain) the next day is 0.5, and the probability of sunshine the next day is 0.2. Taking the states in order as rainy, cloudy, sunny, the full transition matrix for the Markov chain is:

$$\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

Suppose today is a Friday in Bretagne, in mid-October, and it is raining.

(a) Calculate the probability that both the next two days, Saturday and Sunday, will be sunny.

(b) Calculate the probability of rain on Sunday.

(c) Suggest aspects of this model that are unrealistic. How might the model be improved?

Let X_0, X_1, X_2 be the state of the weather on Friday, Saturday and Sunday respectively. Denote the three possible states of the chain as $\{R, C, S\}$ representing rainy, cloudy and sunny.

(a) The probability is

$$\begin{aligned}
\Pr(X_2 = S,\, X_1 = S \mid X_0 = R) &= \Pr(X_2 = S \mid X_1 = S, X_0 = R)\Pr(X_1 = S \mid X_0 = R) \\
&= \Pr(X_2 = S \mid X_1 = S)\Pr(X_1 = S \mid X_0 = R) \\
&= \Pr(X_1 = S \mid X_0 = S)\Pr(X_1 = S \mid X_0 = R) \\
&= 0.3 \times 0.2 = 0.06
\end{aligned}$$

using the markov property and time homogeneity.

(b) The probability of rain on Sunday is

$$\begin{aligned}
\Pr(X_2 = R \mid X_0 = R) &= \sum_{j \in \{R,C,S\}} \Pr(X_2 = R, X_1 = j \mid X_0 = R) \\
&= \sum_{j \in \{R,C,S\}} \Pr(X_2 = R \mid X_1 = j, X_0 = R)\Pr(X_1 = j \mid X_0 = R) \\
&= \sum_{j \in \{R,C,S\}} \Pr(X_2 = R \mid X_1 = j)\Pr(X_1 = j \mid X_0 = R) \\
&= 0.3 \times 0.3 + 0.5 \times 0.3 + 0.2 \times 0.2 \\
&= 0.28.
\end{aligned}$$

(c) We are assuming i) the Markov property and ii) time homogeneity. Neither of these will be good assumptions in general. Moreover, we are summarizing a complex and heterogeneous system by three states.

Many improvements to the model are possible, including

- *Modelling weather variations within the days.*
- *Enlarging the state space.*
- *Including seasonal variations. To incorporate this correctly we would need to use an inhomogeneous Markov chain.*
- *Allowing higher order Markov behaviour, i.e. given X_{n-1}, allow for conditional dependency between X_n and X_{n-k} for $k > 1$.*

5. Let X_n be the maximum reading obtained in the first n rolls of a fair die. Show that $\{X_n\}$ is a Markov chain, and give the transition probabilities.

Let D_n be the score on the nth roll of a fair die. Then,

$$X_n = \max\{D_1, \ldots, D_n\} = \max\{X_{n-1}, D_n\}.$$

As D_n is independent of $X_1, \ldots, X_{n-1}$, the random variable X_n depends on $X_1, \ldots, X_{n-1}$ only through X_{n-1} so it is a Markov chain. Consider the transition probabilities. Note that X_n will *remain the same as X_{n-1} unless a higher number is rolled. So, for all $i < j$, since it is impossible for the maximum to decrease,*

$$\Pr(X_n = i \mid X_{n-1} = j) = 0.$$

When $i > j$,

$$\Pr(X_n = i \mid X_{n-1} = j) = 1/6,$$

since if $i > j$, $\Pr(X_n = i|X_{n-1} = j) = \Pr(D_n = i) = 1/6$ For $i = j$, a number less than or equal to j must be rolled. Thus,

$$\Pr(X_n = j \mid X_{n-1} = j) = j/6.$$

Combining these probabilities, the transition matrix is

$$P = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix}.$$

6. Let (S_n) be a symmetrical random walk on the state space $\mathbb{Z}$, so that if $S_{n-1} = k$, then $S_n = k-1$ with probability $\frac{1}{2}$ and $S_n = k+1$ with probability $\frac{1}{2}$.

   Determine whether each of these processes below is a time homogeneous Markov chain and, if so, find the transition matrix.

   (a) $A = (S_n)_{n \geq 0}$,
   (b) $B = (S_n + n)_{n \geq 0}$,
   (c) $C = (S_n + n^2)_{n \geq 0}$,
   (d) $D = (S_n + (-1)^n)_{n \geq 0}$,
   (e) $E = (|S_n|)_{n \geq 0}$,
   (f) $F = (S_n^2 - n)_{n \geq 0}$,
   (g) $G = (S_{2n})_{n \geq 0}$.

   For each chain, let X_n be the state of the chain at state n and let P be the transition matrix (where it exists).

   (a) True. The transition matrix is $P_{ij} = 1/2$ if $|i - j| = 1$ and 0 otherwise.

   (b) True. The transition probabilities are given by

   $$\Pr(X_n = i \mid X_{n-1} = j) = \Pr(S_n = i - n \mid S_{n-1} = j - n + 1).$$

   These are equal to $1/2$ when $i = j$ or $i = j + 2$ and 0 otherwise.

   (c) False. If the chain was a time homogenous Markov chain then

   $$\begin{aligned} \Pr(X_2 = 0, X_1 = 0 \mid X_0 = 0) &= \Pr(X_2 = 0 \mid X_1 = 0) \Pr(X_1 = 0 \mid X_0 = 0) \\ &= \Pr(X_1 = 0 \mid X_0 = 0)^2. \end{aligned}$$

   However,

   $$\Pr(X_1 = 0 \mid X_0 = 0) = \Pr(S_1 = -1 \mid S_0 = 0) = 1/2,$$

   and

   $$\Pr(X_2 = 0, X_1 = 0 \mid X_0 = 0) = \Pr(S_2 = -4, S_1 = -1 \mid S_0 = 0) = 0,$$

   by properties of a random walk.

(d) *False (not time homogenous). The transition probabilities are given by*

$$\Pr(X_n = i \mid X_{n-1} = j) = \Pr(S_n = i - (-1)^n \mid S_{n-1} = j - (-1)^{n-1}).$$

For n even: $P_{ij} = 1/2$ if $i = j + 1$ or $i = j + 3$ and 0 otherwise. For n odd: $P_{ij} = 1/2$ if $i = j - 1$ or $i = j - 3$ and 0 otherwise.

(e) *True. The transition probabilities satisfy $P_{0,1} = 1$ and $P_{0,j} = 0$ for $j \neq 1$. For $i \geq 1$ we have $P_{ij} = 1/2$ if $|i - j| = 1$ and 0 otherwise.*

(f) *False. If it was a time homogeneous Markov chain then we would have*

$$\Pr(X_5 = 0 \mid X_4 = 0) = \Pr(X_1 = 0 \mid X_0 = 0).$$

However,

$$\Pr(X_1 = 0 \mid X_0 = 0) = \Pr(S_1^2 = 1 \mid S_0^2 = 0) = 1$$

and

$$\Pr(X_5 = 0 \mid X_4 = 0) \quad = \quad \Pr(S_5^2 = 5 \mid S_4^2 = 4) = 0.$$

(g) *True. Using the Chapman-Kolmogorov equations on the original Markov chain $(S_n)_{n \geq 0}$,*

$$\Pr(S_{2n} = i \mid S_{2(n-1)} = j) = \sum_k \Pr(S_{2n} = i \mid S_{2n-1} = k) \Pr(S_{2n-1} = k \mid S_{2n-2} = j).$$

Thus, $P_{ii} = 1/2$, $P_{ij} = 1/4$ if $|i - j| = 2$ and 0 otherwise.

For discussion

7. (†) Define the sequence $Y_1, Y_3, Y_5, \ldots$ of independent and identically distributed random variables by

$$\Pr(Y_{2k+1} = -1) = \Pr(Y_{2k+1} = 1) = \frac{1}{2}, \qquad k \geq 0.$$

Further, define $Y_{2k} = Y_{2k-1} Y_{2k+1}$

(a) Show that $(Y_{2k})_{k \geq 0}$ is a sequence of independent, identically distributed random variables, with the same distribution as the odd Ys.

(b) Show that $Y_1, Y_2, \ldots$ is a sequence of pairwise independent random variables.

(c) Show further that $p_{ij}(n) = \Pr(Y_n = j | Y_0 = i)$ satisfies the Chapman-Kolmogorov equations.

(d) Explain why (Y_k) is not a Markov chain (hence the C-K equations are necessary but not sufficient for a stochastic process to be Markov).

(e) Show that $Z_n = (Y_n, Y_{n+1})$ is a (non-homogeneous) Markov chain.

(a) First consider the distribution of Y_{2k}. By independence of Y_{2k-1} and Y_{2k+1},

$$\begin{aligned}
\Pr(Y_{2k} = -1) &= \Pr(Y_{2k-1} = 1,\, Y_{2k+1} = -1) + \Pr(Y_{2k-1} = -1,\, Y_{2k+1} = 1) \\
&= \Pr(Y_{2k-1} = 1) \Pr(Y_{2k+1} = -1) + \Pr(Y_{2k-1} = -1) \Pr(Y_{2k+1} = 1) \\
&= 1/2.
\end{aligned}$$

As $Y_{2k} \in \{-1, 1\}$, we have $\Pr(Y_{2k} = 1) = 1 - \Pr(Y_{2k} = -1) = 1/2$. Now, we show that Y_{2i} and Y_{2j} are independent for $i \neq j$. For $|j - i| > 1$, note that Y_{2i-1}, Y_{2i+1}, Y_{2j-1} and Y_{2j+1} are all independent and so, as functions of independent random variables, Y_{2i} and Y_{2j} are also independent. When $j = i + 1$,

$$\begin{aligned}
\Pr(Y_{2i} = 1, Y_{2(i+1)} = 1) &= \Pr(Y_{2i-1} = Y_{2i+1},\, Y_{2i+1} = Y_{2i+3}) \\
&= \Pr(Y_{2i-1} = Y_{2i+1} = Y_{2i+3} = 1) \\
&\quad + \Pr(Y_{2i-1} = Y_{2i+1} = Y_{2i+3} = -1) \\
&= (1/2)^3 + (1/2)^3 \\
&= 1/4.
\end{aligned}$$

Further, $\Pr(Y_{2i} = 1) \Pr(Y_{2(i+1)} = 1) = 1/4$ *so*

$$\Pr(Y_{2i} = 1, Y_{2(i+1)} = 1) = \Pr(Y_{2i} = 1) \Pr(Y_{2(i+1)} = 1).$$

Similarly,

$$\begin{aligned}
\Pr(Y_{2i} = 1, Y_{2(i+1)} = -1) &= \Pr(Y_{2i-1} = Y_{2i+1},\, Y_{2i+1} \neq Y_{2i+3}) \\
&= \Pr(Y_{2i-1} = Y_{2i+1} = 1,\, Y_{2i+3} = -1) \\
&\quad + \Pr(Y_{2i-1} = Y_{2i+1} = -1,\, Y_{2i+3} = 1) \\
&= 1/4 \\
&= \Pr(Y_{2i} = 1) \Pr(Y_{2(i+1)} = -1).
\end{aligned}$$

Following similar arguments, it is possible to show that for $a, b \in \{-1, 1\}$ and $j = i \pm 1$

$$\Pr(Y_{2i} = a, Y_{2j} = b) = \Pr(Y_{2i} = a) \Pr(Y_{2j} = b).$$

Thus, we have shown that $(Y_{2k})_{k \geq 0}$ is independent and identically distributed with the same distribution as the odd Ys.

(b) *As we have already shown pairwise independence among the odd and even indices, it is enough to consider independence between* $Y_{2k}Y_{2l+1}$ *for all* k, j. *Writing* $Y_{2k} = Y_{2k-1}Y_{2k+1}$, *independence follows easily unless* $k = l$ *or* $k = l + 1$. *It is sufficient to consider the case* $k = l$. *For* $i, j \in \{-1, 1\}$

$$\Pr(Y_{2k} = i, Y_{2k+1} = j) = \begin{cases} \Pr(Y_{2k+1} = j, Y_{2k-1} = j) & \text{if } i = j \\ \Pr(Y_{2k+1} = j, Y_{2k-1} = i) & \text{if } i \neq j \end{cases} = \begin{cases} 1/4 & \text{if } i = j \\ 1/4 & \text{if } i \neq j \end{cases} = 1/4$$

(c) *As the random variables are pairwise independent,*

$$\Pr(Y_{m+n} = j \mid Y_0 = i) = \Pr(Y_{m+n} = j).$$

Further, using pairwise independence again

$$\begin{aligned} \sum_l \Pr(Y_{m+n} = j \mid Y_m = l) \Pr(Y_m = l \mid Y_0 = i) &= \Pr(Y_{m+n} = j) \sum_l \Pr(Y_m = l) \\ &= \Pr(Y_{m+n} = j). \end{aligned}$$

Hence, the Chapman Kolmogorov equations hold:

$$\Pr(Y_{m+n} = j \mid Y_0 = i) = \sum_l \Pr(Y_{m+n} = j \mid Y_m = l) \Pr(Y_m = l \mid Y_0 = i).$$

(d) *Consider*

$$\Pr(Y_3 = 1 \mid Y_2 = 1, Y_1 = 1) = \Pr(Y_3 = 1 \mid Y_1Y_3 = 1, Y_1 = 1) = 1.$$

However,

$$\begin{aligned} \Pr(Y_3 = 1 \mid Y_2 = 1) &= \Pr(Y_3 = 1 \mid Y_3Y_1 = 1) \\ &= \frac{\Pr(Y_3 = 1, Y_3Y_1 = 1)}{\Pr(Y_3Y_1 = 1)} \\ &= \frac{1/4}{1/2} \\ &= 1/2. \end{aligned}$$

Thus, $\Pr(Y_3 = 1 \mid Y_2 = 1, Y_1 = 1) \neq \Pr(Y_3 = 1 \mid Y_2 = 1)$ *so the chain cannot be a Markov chain.*

(e) *For* n *even,*

$$\begin{aligned} \Pr(Z_n = z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_0 = z_0) &= \Pr(Y_{n+1} = y_{n+1}, Y_n = y_n \mid Y_n = y_n, \ldots, Y_0 = y_0) \\ &= \Pr(Y_n/Y_{n-1} = y_{n+1} \mid Y_n = y_n, \ldots, Y_0 = y_0) \\ &= \begin{cases} 1 & y_{n+1} = y_n/y_{n-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is precisely $\Pr(Z_n = z_n \mid Z_{n-1} = z_{n-1})$. *For* n *odd,*

$$\begin{aligned} \Pr(Z_n = z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_0 = z_0) &= \Pr(Y_{n+1} = y_{n+1} \mid Y_n = y_n, \ldots, Y_0 = y_0) \\ &= \Pr(Y_{n+2}Y_n = y_{n+1} \mid Y_n = y_n, \ldots, Y_0 = y_0) \\ &= \Pr(Y_{n+2} = y_{n+1}/y_n \mid Y_n = y_n, \ldots, Y_0 = y_0) \\ &= 1/2 \end{aligned}$$

which is also the value of $\Pr(Z_n = z_n \mid Z_{n-1} = z_{n-1})$ *so* Z_n *is a Markov chain.*