

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics 1

Date: 22 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider incompressible flows governed by the Navier-Stokes equations

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V}, \quad \nabla \cdot \mathbf{V} = 0, \quad (1)$$

where the density ρ and the kinematic viscosity ν are constants.

- (a) (i) Let $\mathbf{V} = (V_1, V_2, V_3)$ in a Cartesian coordinate system (x_1, x_2, x_3) , and define $E = V^2/2 = (V_1^2 + V_2^2 + V_3^2)/2$. Show that

$$\frac{\partial E}{\partial t} + V_j \frac{\partial E}{\partial x_j} = -\frac{1}{\rho} \frac{\partial}{\partial x_i} (p V_i) + \nu \frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) - \nu \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_j},$$

where Einstein's summation convention is assumed.

(4 marks)

- (ii) Suppose that the flow is in a volume \mathcal{D} enclosed by a rigid surface S , on which the no-slip condition, $\mathbf{V} = 0$, is applied. Show that

$$\frac{d}{dt} \iiint_{\mathcal{D}} E d\tau = -\nu \iiint_{\mathcal{D}} \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_j} d\tau.$$

[Hint: Use the Divergence Theorem, which states that for any vector function \mathbf{G} ,

$$\iiint_{\mathcal{D}} (\nabla \cdot \mathbf{G}) d\tau = \iint_{\mathcal{S}} \frac{\partial G_j}{\partial x_j} d\tau = \iint_{\mathcal{S}} (\mathbf{G} \cdot \mathbf{n}) ds. \quad]$$

(3 marks)

- (b) (i) From the Navier-Stokes equations (1), derive the equation for vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{V}$,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{V} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}, \quad (2)$$

and interpret the physical meaning of each term on the right-hand side.

[You may use without proof the identities: (a) $(\mathbf{V} \cdot \nabla) \mathbf{V} = \boldsymbol{\omega} \times \mathbf{V} + \nabla(V^2/2)$, and (b) for any two vector functions, \mathbf{A} and \mathbf{B} ,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B}) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}. \quad]$$

(4 marks)

- (ii) Consider a velocity field,

$$\mathbf{V} = (u(y), -\alpha y, \beta z), \quad (3)$$

where α and β are constants with $\alpha > 0$.

Find the constraint on α and β such that the continuity equation is satisfied, and calculate the rate of strain.

Show that the vorticity has only one nonzero component, which is in the z -direction, i.e. $\boldsymbol{\omega} = (0, 0, \omega_3)$, and deduce from equation (2), or otherwise, the equation for ω_3 . Solve the equation to obtain ω_3 , assuming that ω_3 vanishes exponentially as $y \rightarrow \pm\infty$.

(9 marks)

(Total: 20 marks)

2. A cylindrical tank of radius R is filled with water and placed on a horizontal disk. The disk, and along with it the tank, are then brought to a steady rotation with angular velocity Ω about the axis of symmetry through the centre of the tank (see Figure 1). The density of the water, ρ , is assumed to be constant.

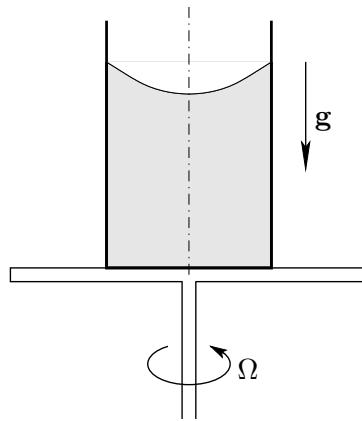


Figure 1: Rotating cylindrical water tank.

Assume that after an initial “transient” period the water inside the tank assumes a steady rotation as a rigid body so that the velocity field is given by

$$(V_r, V_\phi, V_z) = (0, \Omega r, 0)$$

in the cylindric coordinate system (r, ϕ, z) with its origin being at the centre of the tank bottom, where z is aligned with the axis of symmetry and points upward, while r measures the distance to the axis. The flow is governed by the Navier-Stokes equations given in (4) (see the next page), in which $f_z = -g$ due to the Earth’s gravitational field, while $f_r = f_\phi = 0$.

- (i) Deduce the pressure distribution, $p(r, z)$, inside the water, and show that the height of the water surface, z , depends on r as

$$z(r) = \Omega^2 r^2 / (2g) + (C - p_a) / (\rho g),$$

where p_a is the atmospheric pressure, and C is a constant, whose value depends on the amount of water in the tank.

(8 marks)

- (ii) Given that the mass of the water in the tank, M , is calculated as

$$M = \rho \int_0^R z \cdot (2\pi r) dr,$$

determine the constant C in terms of M . Find the critical value Ω_c of the angular velocity Ω above which a dry patch forms at the bottom of the tank.

(8 marks)

- (iii) Determine the radius of the dry patch, which forms when $\Omega > \Omega_c$.

(4 marks)

The governing Navier-Stokes equations in the cylindrical polar coordinates are given on the next page.

(Total: 20 marks)

You can use without proof the Navier-Stokes equations in the cylindrical polar coordinates (figure 2) given below.

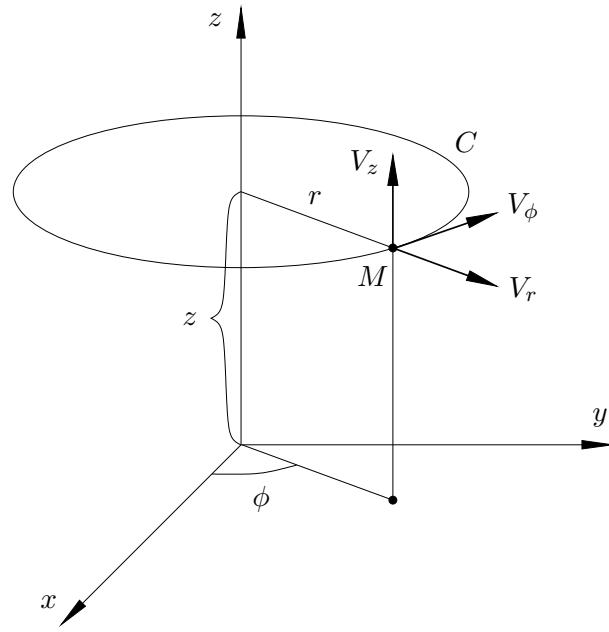


Figure 2: Cylindrical coordinates.

$$\begin{aligned} \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\phi}{r} \frac{\partial V_r}{\partial \phi} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\phi^2}{r} = f_r - \frac{1}{\rho} \frac{\partial p}{\partial r} + \\ + \nu \left(\frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \phi^2} - \frac{2}{r^2} \frac{\partial V_\phi}{\partial \phi} + \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} \right), \end{aligned} \quad (4a)$$

$$\begin{aligned} \frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + \frac{V_\phi}{r} \frac{\partial V_\phi}{\partial \phi} + V_z \frac{\partial V_\phi}{\partial z} + \frac{V_r V_\phi}{r} = f_\phi - \frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \\ + \nu \left(\frac{\partial^2 V_\phi}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 V_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial V_r}{\partial \phi} + \frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r^2} \right), \end{aligned} \quad (4b)$$

$$\begin{aligned} \frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\phi}{r} \frac{\partial V_z}{\partial \phi} + V_z \frac{\partial V_z}{\partial z} = f_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \\ + \nu \left(\frac{\partial^2 V_z}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \phi^2} + \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right), \end{aligned} \quad (4c)$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r} \frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r} = 0. \quad (4d)$$

3. Consider the steady incompressible flow of a fluid with density ρ and kinematic viscosity ν between two infinitely large parallel porous plates at $y = 0$ and $y = h$. The lower plate is fixed while the upper plate moves at a constant speed $U > 0$ in its own plane. Meanwhile, a steady spatially uniform velocity V normal to the plates is imposed at both plates as is illustrated in figure 3. Let x be the coordinate parallel to the plates, and u and v denote the velocity components in the x and y directions respectively. There is no pressure gradient in the x or z direction.

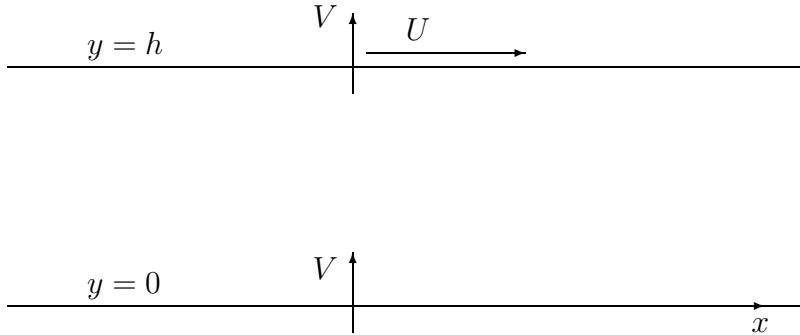


Figure 3: Sketch of the flow configuration.

- (i) Show that the Navier-Stokes equations admit an exact solution of the form

$$(u, v) = (u(y), V),$$

and derive the equation satisfied by $u(y)$. Show that the appropriate solution is

$$u(y) = \frac{e^{Vy/\nu} - 1}{e^{Vh/\nu} - 1} U. \quad (8 \text{ marks})$$

- (ii) Simplify the solution $u(y)$ in each of the following limiting cases:

$$(a) Vh/\nu \ll 1; \quad (b) Vh/\nu \gg 1 \text{ with } V > 0; \quad (c) |Vh/\nu| \gg 1 \text{ with } V < 0.$$

Comment briefly on the nature of the flow in each case. (6 marks)

- (iii) Calculate the stress acting on the upper plate, and investigate how the drag is influenced by V , which may be positive or negative.

[You may use without proof the inequality $e^x > 1 + x$, valid for an arbitrary $x \neq 0$.]

(6 marks)

(Total: 20 marks)

4. Consider a two-dimensional inviscid potential flow represented by the complex potential

$$w(z) = V_\infty z + \frac{q}{2\pi} \ln z, \quad (5)$$

which is a superposition of a uniform flow with velocity $V_\infty > 0$ and a two-dimensional (line) source or sink of strength q , corresponding to $q > 0$ or $q < 0$, respectively.

- (i) Introduce polar coordinates (r, ϑ) by writing $z = r e^{i\vartheta}$ and find the velocity potential $\varphi(r, \vartheta)$ and stream function $\psi(r, \vartheta)$.

(2 marks)

- (ii) For each case of $q > 0$ and $q < 0$, find the stagnation point of the flow and the contour of the stream function passing through the stagnation point. Explain why the complex potential (5) can represent a flow past/over a rigid body/surface of a certain shape, which you are expected to determine; illustrate the body/surface shape with a sketch.

[Hint: You may use without proof the result that the radial and azimuthal velocities, V_r and V_ϑ , are given by

$$V_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \vartheta}, \quad V_\vartheta = \frac{1}{r} \frac{\partial \varphi}{\partial \vartheta} = -\frac{\partial \psi}{\partial r}. \quad]$$

(8 marks)

- (iii) The same complex potential $w(z)$ given by (5) may alternatively be viewed as representing the inviscid two-dimensional potential flow past a circular cylinder (centred at the origin) with radius a and a porous surface, across which the fluid may be injected into, or sucked from, the flow field at velocity $V_s(\vartheta)$ in the normal (radial) direction.

Determine the required distribution of the injection/suction velocity $V_s(\vartheta)$. Calculate the pressure distribution $p(\vartheta)$ on the surface of the cylinder ($r = a$), assuming that the pressure at infinity is p_∞ .

Show that the x -component of the force per unit axial length on the cylinder, F_X , is given by the integral

$$F_X = -a \int_0^{2\pi} p(\vartheta) \cos \vartheta d\vartheta.$$

Evaluate F_X using the pressure distribution $p(\vartheta)$ obtained above.

(7 marks)

For the case of $q < 0$, discuss the implication of the result when the flow is viewed as due to the cylinder moving to the left at a constant velocity V_∞ in an otherwise motionless fluid.

Discuss whether F_X could be calculated by using the Blasius-Chaplygin formula,

$$F_X - iF_Y = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz} \right)^2 dz,$$

where $w(z)$ is the complex potential and ρ the density of the fluid, and C denotes the body contour.

(3 marks)

(Total: 20 marks)

5. Consider an inviscid incompressible flow above flat ground, on which a thin fence of height h is installed; see figure 4. The ground coincides with the x -axis and the fence occupies an interval $[0, h]$ of the y -axis. Far upstream of the fence, the flow velocity is V_∞ and the pressure is p_∞ .

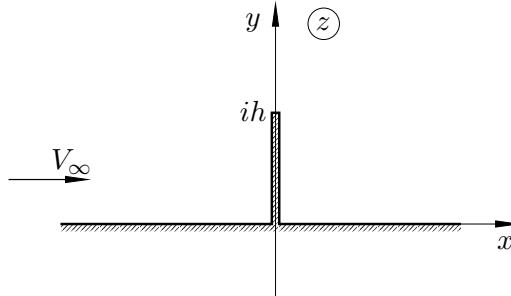


Figure 4: The flow over a fence on flat ground.

The flow is assumed to be irrotational so that the solution for it may be obtained in the form of a complex potential by using the method of conformal mapping.

- (a) (i) Show that the domain in the physical plane is mapped onto the upper half of the auxiliary ζ -plane by the conformal mapping

$$\zeta = \sqrt{z^2 + h^2}.$$

Identify the images of the front and rear sides of the fence. (6 marks)

- (ii) Assuming that in the auxiliary plane the complex potential is given by

$$W(\zeta) = \tilde{V}_\infty \zeta,$$

find the value of \tilde{V}_∞ such that in the physical plane the far-field velocity is V_∞ .

Calculate the velocity distribution on the front and rear sides of the fence. (4 marks)

- (iii) Find the pressure distribution along the ground ($z = x$) by using Bernoulli's equation, and evaluate the integral pressure force,

$$F = \int_{-\infty}^{\infty} (p - p_\infty) dx,$$

that is exerted on the ground. (5 marks)

- (b) The solution may alternatively be obtained by noting that the present flow is equivalent to that due to a uniform flow with velocity V_∞ approaching a flat plate of finite length $2h$ perpendicularly.

Deduce the complex potential based on the equivalence noted above and the information in the *hints* given below, and demonstrate that the complex potential in the physical plane, $w(z)$, is the same as that obtained in part (a). (5 marks)

[*Hints: For a flow past a flat plate of length $2a$ at an angle of attack α , as is shown in Figure 5, the Joukovskii transformation,*

$$\zeta = z + \sqrt{z^2 - a^2},$$

Question continues on the next page.

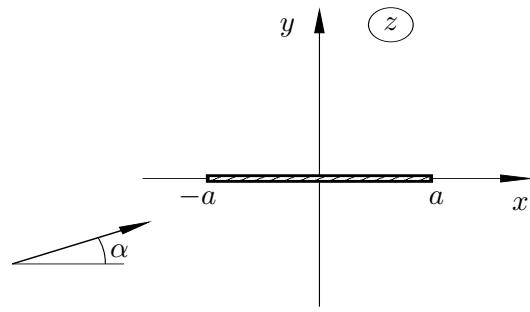


Figure 5: Flow past a flat plate at an angle of attack.

maps the domain exterior to the slit occupied by the plate onto the exterior of a circle in the auxiliary ζ -plane, where the complex potential is given by

$$W(\zeta) = \frac{1}{2}V_\infty \left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}} \right) + \frac{\Gamma}{2\pi i} \ln \zeta. \quad]$$

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH60001, MATH70001, MATH97008

Fluid Dynamics 1 (Solutions)

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1. (a) (i) The Navier-Stokes equations can be rewritten as

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$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j}.$$

Multiplying V_i to the above equation, which amounts to multiplying each of V_i ($i = 1, 2$ and 3) to each of the three momentum equations and taking the sum of them, we have

$$V_i \frac{\partial V_i}{\partial t} + V_i V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho} V_i \frac{\partial p}{\partial x_i} + \nu V_i \frac{\partial^2 V_j}{\partial x_j \partial x_j}. \quad (1)$$

Noting that

$$V_i \frac{\partial V_i}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} V_i^2 \right) = \frac{\partial E}{\partial t}, \quad V_i V_j \frac{\partial V_i}{\partial x_j} = V_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} V_i^2 \right) = V_j \frac{\partial E}{\partial x_j},$$

$$V_i \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i} (p V_i), \quad V_i \frac{\partial^2 V_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) - \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_j},$$

where the continuity equation, $\frac{\partial V_j}{\partial x_j} = 0$, is used. Use of this in (1) gives the required equation.

4, B

(ii) Note further that $V_j \frac{\partial E}{\partial x_j} = \frac{\partial}{\partial x_j} (V_j E)$, use of which leads to

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} (V_j E) = -\frac{1}{\rho} \frac{\partial}{\partial x_i} (p V_i) + \nu \frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) - \nu \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_j}.$$

Integrate the equation over the domain D :

$$\begin{aligned} & \frac{d}{dt} \iiint_D E d\tau + \iiint_D \frac{\partial}{\partial x_j} (V_j E) d\tau \\ &= -\frac{1}{\rho} \iiint_D \frac{\partial}{\partial x_i} (p V_i) d\tau + \nu \iiint_D \frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) d\tau - \nu \iiint_D \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_j} d\tau. \end{aligned}$$

Here we note that D is fixed in space, and so the integration and $\partial/\partial t$ commute. Furthermore, the integrands of the second term on the left-hand side, and the first and second terms on the right-hand side, are of the divergence form, $\nabla \cdot \mathbf{G} = \frac{\partial G_j}{\partial x_j}$, with $\mathbf{G} = E\mathbf{V}$, $p\mathbf{V}$ and ∇E . They can be converted to the integrals of $\mathbf{G} \cdot \mathbf{n}$ on the surface S , which are zero since $\mathbf{G} = 0$ on S due to the no-slip condition. The required identity follows.

3, B

(b) (i) Taking the curl of the momentum equations with $(\mathbf{V} \cdot \nabla)\mathbf{V}$ being replaced by $\boldsymbol{\omega} \times \mathbf{V} + \nabla(V^2/2)$, and noting that $\nabla \times (\nabla p) = 0$ and $\nabla \times \nabla(V^2/2) = 0$, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{V}) = \nu \nabla^2 \boldsymbol{\omega}.$$

Using the vector identify (given in the question), we have

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{V}) = (\nabla \cdot \mathbf{V})\boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega})\mathbf{V} + (\mathbf{V} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{V}. \quad (2)$$

seen ↓

Since $\nabla \cdot \mathbf{V} = 0$ (the continuity equation) and $\nabla \cdot \boldsymbol{\omega} = 0$ (a vector identity), there follows the required equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{V} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}.$$

The first term on the right-hand side represents stretching of the vorticity by the rate of the strain. More precisely, this term (a vector) may be projected to the direction of $\boldsymbol{\omega}$ and that perpendicular to it, with the former representing the stretching and the latter the 'tilting'. The second term on the right-hand side represents the diffusion of the vorticity.

- (ii) The continuity equation requires that $-\alpha + \beta = 0$, and so $\beta = \alpha$. The vorticity is given by

$$\boldsymbol{\omega} = (0, 0, -\frac{\partial u}{\partial y}).$$

The rate of strain is calculated as

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -\alpha, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = \beta = \alpha; \\ \varepsilon_{xy} &= \varepsilon_{yx} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \frac{\partial u}{\partial y}, \quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0, \\ \varepsilon_{xz} &= \varepsilon_{zx} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0.\end{aligned}$$

4, A

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It follows from (2) and the given v and w that $\omega_3 = -\frac{\partial u}{\partial y}$ satisfies the equation

$$-\alpha y \frac{\partial \omega_3}{\partial y} = \beta \omega_3 + \nu \frac{\partial^2 \omega_3}{\partial y^2} \quad \text{i.e.} \quad -\alpha \frac{\partial}{\partial y} (y \omega_3) = \nu \frac{\partial^2 \omega_3}{\partial y^2},$$

since $\beta = \alpha$. The above equation is integrated to give

$$-\alpha y \omega_3 = \nu \frac{\partial \omega_3}{\partial y} + C_1,$$

which can be written as

$$\frac{d\omega_3}{dy} + (\alpha/\nu)y\omega_3 = \hat{C}_1.$$

The equation may be solved using the method of integration factor:

$$\frac{d}{dy} (\omega_3 e^{(\alpha/\nu)y^2/2}) = \hat{C}_1 e^{(\alpha/\nu)y^2/2},$$

which is integrated to give

$$\omega_3 = \hat{C}_2 e^{-(\alpha/\nu)y^2/2} + \hat{C}_1 e^{-(\alpha/\nu)y^2/2} \int_0^y e^{(\alpha/\nu)y'^2/2} dy.$$

For ω_3 to vanish exponentially as $y \rightarrow \pm\infty$, we have to set $\hat{C}_1 = 0$, leaving us with

$$\omega_3 = \hat{C}_2 e^{-(\alpha/\nu)y^2/2}.$$

5, C

2. (i) With the three components of the body force,

$$f_r = f_\phi = 0, \quad f_z = -g,$$

and the velocity components

$$V_r = 0, \quad V_\phi = \Omega r, \quad V_z = 0$$

corresponding to 'solid-body' rotation about the axis, the radial and azimuthal momentum equations reduce to

$$-\frac{V_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad \text{or} \quad \frac{\partial p}{\partial r} = \rho \frac{V_\phi^2}{r} = \rho \Omega^2 r, \quad (3)$$

$$\frac{\partial p}{\partial \phi} = 0,$$

which means that p is independent of ϕ (as was expected).

The vertical momentum equation yields

$$0 = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} \quad \text{or} \quad \frac{\partial p}{\partial z} = -\rho g. \quad (4)$$

Integrating equation (3), we have

$$p = \frac{1}{2} \rho \Omega^2 r^2 + \Phi(z), \quad (5)$$

where $\Phi(z)$ is a function of z . Substitution of (5) into (4) gives

$$\Phi' = -\rho g,$$

and hence

$$\Phi = -\rho g z + C, \quad (6)$$

where C is a constant. Substituting (6) back into (5), we obtain the pressure distribution in the water,

$$p = \frac{1}{2} \rho \Omega^2 r^2 - \rho g z + C.$$

Since the fluid is in solid rotation, there is no deformation (i.e. the rate-of-strain tensor is a null tensor), and hence on the upper surface of the water in the tank, the stress is simply the pressure, which should be the same as the atmospheric pressure p_a . Therefore, the equation for the upper surface is written as

$$z = \frac{\Omega^2}{2g} r^2 + \frac{C - p_a}{\rho g} \quad (7)$$

which indicates that the shape of the interface is of parabolic form.

8, A

- (ii) The constant of integration, C , depends on the amount of water contained in the cylinder. The mass of the water is given by the integral

$$M = \rho \int_0^R z \cdot 2\pi r \cdot dr. \quad (8)$$

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Substituting (7) into (8) and performing the integration, we find

$$M = 2\pi\rho \int_0^R \left(\frac{\Omega^2}{2g} r^3 + \frac{C - p_a}{\rho g} r \right) dr = 2\pi\rho \left(\frac{\Omega^2}{8g} R^4 + \frac{C - p_a}{2\rho g} R^2 \right), \quad (9)$$

from which we have

$$C = p_a + \frac{2\rho g}{R^2} [M/(2\pi\rho) - \frac{\Omega^2}{8g} R^4].$$

Since z increases with r , the water surface would “touch” the cylinder bottom first when $z = 0$ at $r = 0$, and this happens, according to (7), when $C - p_a$ becomes zero. This reduces equation (9) to

$$M = \pi\rho\Omega^2 R^4 / (4g), \quad (10)$$

which determines the critical value Ω_c of the angular velocity,

$$\Omega = \frac{2}{R^2} \sqrt{\frac{gM}{\pi\rho}} \equiv \Omega_c.$$

8, B

- (iii) The radius of the dry patch, R_c , is defined by $z = 0$, i.e.

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$$0 = \frac{\Omega^2}{2g} R_c^2 + \frac{C - p_a}{\rho g}. \quad (11)$$

The mass is now calculated as

$$M = \rho \int_{R_c}^R z \cdot 2\pi r \cdot dr = 2\pi\rho \left[\frac{\Omega^2}{8g} (R^4 - R_c^4) + \frac{C - p_a}{2\rho g} (R^2 - R_c^2) \right]. \quad (12)$$

Eliminating $(C - p_a)/(\rho g)$ from (11) and (12), we obtain

$$M = \pi\rho \frac{\Omega^2}{4g} \left[(R^4 - R_c^4) - 2R_c^2(R^2 - R_c^2) \right],$$

which is arranged into

$$R_c^4 - 2R^2 R_c^2 + (R^4 - \frac{4gM}{\pi\rho\Omega^2}) = 0 \quad \text{i.e.} \quad R_c^4 - 2R^2 R_c^2 + R^4 \left(1 - \frac{\Omega_c^2}{\Omega^2}\right) = 0,$$

from which we find that

$$R_c = \sqrt{1 - \Omega_c/\Omega} R. \quad (13)$$

where the root with $R_c > R$ is rejected.

4, D

3. (i) Based on the information given in the question, the flow is two dimensional, and its velocity field $(u, v) = (u(y), v(y))$ is independent of x and z since the plates are infinitely large.

unseen ↓

The continuity equation simplifies to $\frac{\partial v}{\partial y} = 0$, implying that v is independent of y . Hence $v \equiv V$ for $0 \leq y \leq 1$.

The x -momentum equation yields:

$$V \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (14)$$

while the y -momentum equation implies $\frac{\partial p}{\partial y} = 0$.

Solving (14) for $\frac{\partial u}{\partial y}$ first using the method of integration factor, we obtain

$$\frac{\partial u}{\partial y} = \hat{C} e^{Vy/\nu} \quad (\hat{C} \text{ is a constant}),$$

which is integrated to give

$$u = C e^{Vy/\nu} + D.$$

Apply the boundary conditions:

$$0 = C + D, \quad U = C e^{Vh/\nu} + D.$$

and hence

$$C = -D = \frac{U}{e^{Vh/\nu} - 1}$$

so that the solution is

$$u(y) = \frac{e^{Vy/\nu} - 1}{e^{Vh/\nu} - 1} U.$$

8, A

- (ii) (a) For $Vh/\nu \ll 1$, Vy/ν is small, and Taylor expansion of the exponentials yields

$$u = \frac{Vy/\nu + \frac{1}{2}(Vy/\nu)^2 + \dots}{Vh/\nu + \frac{1}{2}(Vh/\nu)^2 + \dots} U = \frac{Uy}{h} \left[1 - \frac{1}{2} \frac{Vh}{\nu} (1 - y/h) + \dots \right].$$

unseen ↓

This is the plane Couette flow to leading-order accuracy.

(b) For $Vh/\nu \gg 1$ and $V > 0$, rewrite u as

$$u = \frac{e^{V(y-h)/\nu} - e^{-Vh/\nu}}{1 - e^{-Vh/\nu}} U.$$

Since $e^{-Vh/\nu} \ll 1$ is negligible,

$$u \approx U e^{V(y-h)/\nu},$$

which shows that the flow velocity concentrates in the region where $y - h = O(\nu/V) \ll O(h)$, a thin layer in the vicinity of the upper plate. Outside this thin layer of $O(\nu/V)$ width, the velocity is exponentially small. In this case, the plate at $y = 0$ plays no role at leading order.

(c) For $|Vh/\nu| \gg 1$ and $V < 0$, $e^{Vh/\nu} \ll 1$ is negligible, and so

$$u \approx U(1 - e^{Vh/\nu}).$$

The velocity varies primarily in the region where $y = O(\nu/V) \ll O(h)$, a thin layer in the vicinity of the lower plate. Outside this thin layer of $O(\nu/V)$ width, the velocity appears uniform. In this case, the plate at $y = h$ plays no role at leading order.

6, B

unseen ↓

- (iii) As the flow is two-dimensional, it suffices to calculate the stress tensor $\mathcal{P} = (p_{ij})_{2 \times 2}$ at the upper surface ($y = h$):

$$p_{11} = -p, \quad p_{12} = p_{21} = \mu \frac{\partial u}{\partial y} = (\rho UV) \frac{e^{Vh/\nu}}{e^{Vh/\nu} - 1} = \frac{\rho UV}{1 - e^{-Vh/\nu}}, \quad p_{22} = -p.$$

The unit normal direction of the upper plate is $\mathbf{n} = (0, -1)$, and so the stress is

$$\mathbf{R} = \mathcal{P} \cdot \mathbf{n} = (-p_{12}, p).$$

The x -component of the stress is $R_1 = -p_{12} < 0$, with the negative sign indicating a drag (opposite to the direction of the plate motion); here $p_{12} > 0$ whether $V > 0$ or $V < 0$.

To determine how the suction/blowing affects the drag, we consider

$$\begin{aligned} \frac{\partial |R_1|}{\partial V} &= (\rho U) \frac{1 - e^{-Vh/\nu} - (Vh/\nu)e^{-Vh/\nu}}{(1 - e^{-Vh/\nu})^2} \\ &= (\rho U)e^{-Vh/\nu} \frac{e^{Vh/\nu} - (Vh/\nu) - 1}{(1 - e^{-Vh/\nu})^2} > 0. \end{aligned}$$

The RHS is positive according to the inequality given (with $x = Vh/\nu$), indicating that $|R_1|$ is an increasing function of V . This implies that suction through the upper plate ($V > 0$) enhances the drag, while blowing through the upper plate ($V < 0$) reduces the drag.

6, D

4. (i) Substituting $z = r e^{i\vartheta}$ into $w(z)$, we have

sim. seen ↓

$$w = V_\infty r(\cos \vartheta + i \sin \vartheta) + \frac{q}{2\pi}(\ln r + i \vartheta).$$

The real and imaginary parts of $w(z)$ are the velocity potential and stream function, respectively:

$$\varphi(r, \vartheta) = V_\infty r \cos \vartheta + \frac{q}{2\pi} \ln r, \quad \psi(r, \vartheta) = V_\infty r \sin \vartheta + \frac{q}{2\pi} \vartheta.$$

2, A

(ii) Using the relations between the velocity (V_r, V_ϑ) and φ or ψ , we obtain

$$V_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \vartheta} = V_\infty \cos \vartheta + \frac{q}{2\pi} r^{-1}, \quad V_\vartheta = \frac{1}{r} \frac{\partial \varphi}{\partial \vartheta} = -\frac{\partial \psi}{\partial r} = -V_\infty \sin \vartheta.$$

sim. seen ↓

A stagnation point corresponds to

$$V_r = V_\infty \cos \vartheta + \frac{q}{2\pi} r^{-1} = 0, \quad V_\vartheta = -V_\infty \sin \vartheta = 0. \quad (15)$$

It follows from the second equation that $\vartheta = 0$ and $\vartheta = \pi$, and correspondingly from the first equation of (15) that

$$\vartheta_s = 0, \quad r_s = -q/(2\pi V_\infty), \quad (16)$$

and

$$\vartheta_s = \pi, \quad r_s = q/(2\pi V_\infty). \quad (17)$$

Since r must be positive, in each case there is only one stagnation point. For $q > 0$, the stagnation point is given by (17), and the stream function contour going through it is determined by $\psi(r, \vartheta) = \psi(r_s, \pi) = q/2$, i.e.

$$V_\infty r \sin \vartheta + \frac{q}{2\pi} \vartheta = \frac{q}{2},$$

which can also be written as

$$y = \frac{q}{2V_\infty} \left(1 - \frac{\vartheta}{\pi}\right).$$

Clearly, y monotonically increases from $y = 0$ to $d \equiv q/(2V_\infty)$ as ϑ decreases from π to 0, while x increases monotonically from $-q/(2\pi V_\infty)$ to $+\infty$. The contour is symmetric about the x -axis, and is shown in figure 1. Since contours of stream

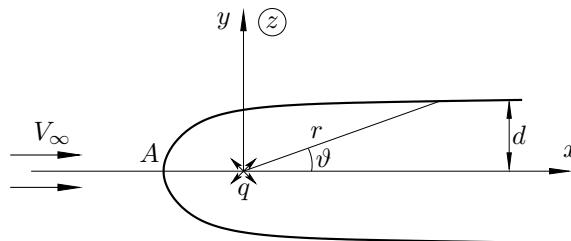


Figure 1: The contour of the body.

function are streamlines, the velocity is in the tangent direction of the contour so that the impermeability condition is satisfied. Furthermore, the domain exterior to the contour contains no singularity, and thus the complex potential represents the flow past a body with the shape given by the contour.

5, A

unseen ↓

For $q < 0$, the stagnation point is given by (16), and the stream function contour going through it is determined by $\psi(r, \vartheta) = \psi(r_s, 0) = 0$, i.e.

$$V_\infty r \sin \vartheta + \frac{q}{2\pi} \vartheta = 0,$$

which can be written as

$$y = -\frac{q}{2\pi V_\infty} \vartheta \rightarrow -\frac{q}{2V_\infty} \text{ as } \vartheta \rightarrow \pi.$$

The contour shape is the reflection of that shown in figure 1 about the y -axis. Again, the domain exterior to the contour contains no singularity. However, the body extends to infinity to the left. On noting the symmetry about the x -axis, the complex potential represents the flow over the upper surface of the body up to $x = -q/(2\pi V_\infty)$ and then over $y = 0$.

3, D

unseen ↓

- (iii) If the complex potential is viewed as representing the flow past a circular cylinder with radius a , then at $r = a$,

$$V_r = V_\infty \cos \vartheta + \frac{q}{2\pi a} \equiv V_s(\vartheta), \quad V_\vartheta = -V_\infty \sin \vartheta,$$

where $V_s(\vartheta)$ is the required distribution of the suction/injection velocity.

The pressure on the cylinder, $p(\vartheta)$, is calculated using the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2} [(V_\infty \cos \vartheta + \frac{q}{2\pi a})^2 + (-V_\infty \sin \vartheta)^2] = \frac{p_\infty}{\rho} + \frac{1}{2} V_\infty^2.$$

It follows that

$$p(\vartheta) = p_\infty - \frac{1}{2} \rho (\frac{q}{2\pi a})^2 - \frac{\rho q V_\infty}{2\pi a} \cos \vartheta.$$

The stress on the surface is in the opposite direction of the outward normal. The force on a surface element $ds = 1 \cdot ad\vartheta$ is $(-pad\vartheta)$, the projection of which to the x -direction gives $dF_X = (-pad\vartheta) \cos \vartheta$. Integration along the circle leads to the required formula, substitution of $p(\vartheta)$ into which yields

$$F_X = -a \int_0^{2\pi} \left[p_\infty - \frac{1}{2} \rho (\frac{q}{2\pi a})^2 - \frac{\rho q V_\infty}{2\pi a} \cos \vartheta \right] \cos \vartheta d\vartheta = \frac{\rho q V_\infty}{2\pi}.$$

7, C

We note that $F_X < 0$ when $q < 0$, which corresponds to a *propulsion* when the flow is viewed as due to the cylinder moving to the left. (This result is due to the fact that $V_s < 0$ on the front side ($\pi/2 < \vartheta < 3\pi/2$), i.e. the body is 'swallowing in' the fluid, thereby generating propulsion.)

The Blasius-Chaplygin formula is not applicable because the formula was derived under the condition that the body surface is a streamline, but this is not the case any longer in the presence of suction/injection. Indeed, if the Blasius-Chaplygin formula were used, we would end up with

$$\begin{aligned} F_X - iF_Y &= \frac{1}{2} i\rho \oint_C (V_\infty + \frac{q}{2\pi z})^2 dz = \frac{1}{2} i\rho \oint_C (V_\infty^2 + \frac{qV_\infty}{\pi z} + \frac{q^2}{4\pi^2 z^2}) dz \\ &= \frac{1}{2} i\rho \int_0^{2\pi} (V_\infty^2 + \frac{qV_\infty}{\pi a e^{i\vartheta}} + \frac{q^2}{4\pi^2 a^2 e^{2i\vartheta}}) a e^{i\vartheta} (id\vartheta) = -\rho q V_\infty, \end{aligned}$$

that is, $F_X = -\rho q V_\infty$, which is different from that obtained earlier and wrong.

3, D

5. (a) (i) In order to show that the transformation, $\zeta = \sqrt{z^2 + h^2}$, maps the physical domain in the z -plane onto the upper half of the auxiliary ζ -plane with the rigid body surface coinciding with the real ξ -axis as is shown in figure 2, we treat $\zeta = \sqrt{z^2 + h^2}$ as a composition of a sequence of mappings defined by elementary functions.

sim. seen \Downarrow

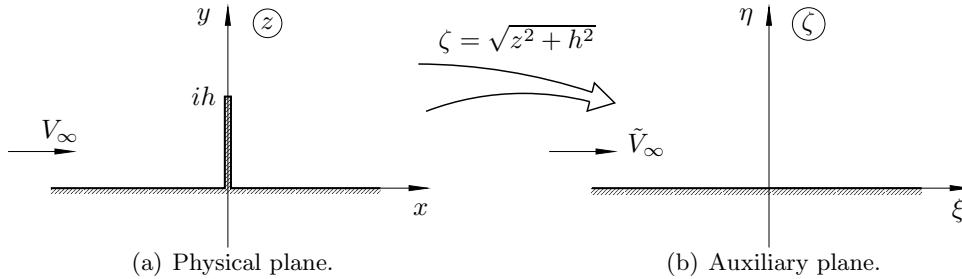


Figure 2: Required mapping.

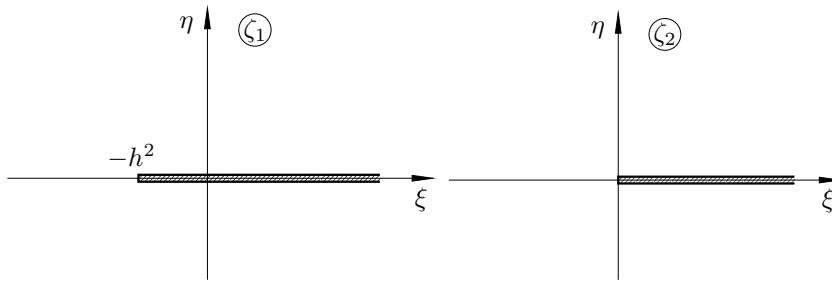


Figure 3: ζ_1 -plane (left) and ζ_2 -plane (right).

First the mapping via a power function,

$$\zeta_1 = z^2,$$

transforms the rigid body surface (the flat ground together with the fence) onto a slit along the real ξ -axis in the ζ_1 -plane with the tip $z = ih$ being mapped into the point $\zeta_1 = (ih)^2 = -h^2$ (see the left plot of figure 3).

Then the transformation

$$\zeta_2 = \zeta_1 + h^2$$

shifts the slit to the right along the real axis by a distance h^2 so that the tip of the slit is translated to $\zeta_2 = 0$ as is shown in right plot of figure 3.

Finally, the 'folded' upper and lower surfaces of the slit along the positive real axis is 'opened up' using the power function

$$\zeta = \sqrt{\zeta_2}.$$

The image is the required upper half plane.

The composition of these three elementary functions is precisely the mapping:

$$\zeta = \sqrt{\zeta_2} = \sqrt{\zeta_1 + h^2} = \sqrt{z^2 + h^2}. \quad (18)$$

The rear/front side is mapped to the upper/lower side of the interval $(0, h^2)$ on the real axis on the ζ_2 -plane. Under the final mapping, $\zeta = \sqrt{\zeta_2}$, the upper and lower sides are mapped to $(0, h)$ and $(-h, 0)$, respectively.

[6 marks]

seen ↓

- (ii) Substituting (18) into the complex potential $W(\zeta) = \tilde{V}_\infty \zeta$, we arrive at the sought complex potential in the z -plane,

$$w(z) = \tilde{V}_\infty \sqrt{z^2 + h^2}. \quad (19)$$

The complex-conjugate velocity at any point in the z -plane is

$$\bar{V} = dw/dz = \tilde{V}_\infty z / \sqrt{z^2 + h^2}. \quad (20)$$

In particular, in the oncoming flow ($z \rightarrow \infty$) $\bar{V} \rightarrow \tilde{V}_\infty$, which means that $\tilde{V}_\infty = V_\infty$, and we can write equations (19) and (20) as

$$w(z) = V_\infty \sqrt{z^2 + h^2}, \quad u - iv \equiv \bar{V} = V_\infty z / \sqrt{z^2 + h^2}. \quad (21)$$

unseen ↓

Note that the mapping of the front and rear sides implies that $\sqrt{z^2 + h^2} = \sqrt{h^2 - y^2}$ when $z = 0^+ + iy$ is on the rear side of the fence, while $\sqrt{z^2 + h^2} = -\sqrt{h^2 - y^2}$ when $z = 0^- + iy$ is on the front side. Thus on the rear side, $v = -V_\infty y / \sqrt{h^2 - y^2} \leq 0$, and on the front side $v = V_\infty y / \sqrt{h^2 - y^2} \geq 0$.

[4 marks]

- (iii) On the ground surface, where $z = x$, we have $\bar{V} = V_\infty x / \sqrt{x^2 + h^2}$. Using the Bernoulli equation, $\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}$, we find that

$$p = p_\infty + \frac{\rho}{2}(V_\infty^2 - V^2) = p_\infty + \frac{\rho}{2}V_\infty^2 \left(1 - \frac{x^2}{x^2 + h^2}\right) = p_\infty + \frac{1}{2}\rho V_\infty^2 \frac{h^2}{x^2 + h^2},$$

and the integral for the pressure force may be written as

$$F = 2 \int_0^\infty (p - p_\infty) dx = \rho V_\infty^2 h^2 \int_0^\infty \frac{dx}{x^2 + h^2} = \rho V_\infty^2 h \arctan t \Big|_0^\infty = \frac{\pi}{2} \rho V_\infty^2 h,$$

where the evaluation of the integral is aided by the substitution: $x = ht$.

[5 marks]

unseen ↓

- (b) In order to use the results given, the vertically oriented plate is mapped to a horizontal one by the mapping: $z_1 = iz$. Then z_1 plays the role of z in the Joukovskii transformation, leading to

$$\zeta = iz + \sqrt{(iz)^2 - h^2} = i(z + \sqrt{z^2 + h^2}), \quad (22)$$

where we note that $a = h$.

In the complex potential, the angle of attack $\alpha = \pi/2$ since the oncoming flow is perpendicular to the plate. In order to render the flow symmetric about the line bisecting the plate as is required by the geometric configuration, we have to set the circulation $\Gamma = 0$. With these the complex potential becomes

$$W(\zeta) = \frac{1}{2}V_\infty \left(-i\zeta + \frac{h^2}{-i\zeta}\right) = -\frac{1}{2}iV_\infty \left(\zeta - \frac{h^2}{\zeta}\right) = -\frac{1}{2}iV_\infty (\zeta^2 - h^2)/\zeta.$$

Substituting (22) into the above equation, we have

$$w(z) = -\frac{1}{2}iV_\infty \frac{-(z + \sqrt{z^2 + h^2})^2 - h^2}{i(z + \sqrt{z^2 + h^2})} = V_\infty \sqrt{z^2 + h^2},$$

which is the same as that obtained in part (a).

[5 marks]

Review of mark distribution:

Total A marks: 31 of 32 marks

Total B marks: 21 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	QuestionNumber	Comments for Students
MATH60001/70001	1	The majority did well on the entire question except on a small portion that is unfamiliar.
MATH60001/70001	2	Almost all did well.
MATH60001/70001	3	The average mark on this Question is the lowest among 4 questions. Many found Part (ii) and Part (iii) challenging. These two parts required physical interpretations in conjunction with the mathematical derivation.
MATH60001/70001	4	The overall performance was slightly below what I expected, but the distribution of the marks is good.
MATH70001	5	There were some good marks, but a few was somewhat underprepared for Part b, or struggled with time. The overall performance on the entire paper turned out to be more or less as expected. There is a good distribution of marks.