Algebra III: Rings and Modules
Problem Sheet 4, Autumn Term 2022-23

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1. Let R be a commutative ring and let $I \subseteq R$ be an ideal.
   (i) Prove that I is a free R-module if and only if I is principal and is generated by an element which is not a zero divisor. [Optional: Find a non-commutative ring where this is false.]
   (ii) Deduce that a commutative ring R is a principal ideal domain if and only if every ideal $I \subseteq R$ is free as an R-module.

2. Let R be a ring and let M be a free R-module. Give a proof or counterexample to each of the following statements:
   (i) Every spanning set for M over R contains a basis for M.
   (ii) Every linearly independent subset of M over R can be extended to a basis for M.

3. Let R be a non-trivial commutative ring. Prove that R is a field if and only if every finitely generated R-module is free. [Optional: Prove this is also equivalent to every R-module being free. You will need to use the axiom of choice.]

4. Let R be a ring, let $S \subseteq R$ be a multiplicative submonoid and let $N \leq M$ be R-modules. Show that there is an isomorphism of $S^{-1}R$-modules $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.

5. Let R be a ring, M a right R-module and N a left R-module. The tensor product $M \otimes_R N$ is defined to be the abelian group

$$\begin{aligned} M \otimes_R N = \mathbb{Z}[M \times N] \,/\, (&(va, w) - (v, aw),\, (v, w) + (v', w) - (v + v', w), \\ &(v, w) + (v, w') - (v, w + w') \mid a \in R, v, v' \in M, w, w' \in N). \end{aligned}$$

   For left R-modules M and N, let $\mathrm{Hom}_R(M, N)$ denote the set of left R-module homomorphisms $f : M \to N$, which is an abelian group under pointwise addition.

   From now on, let R be a commutative ring.

   (i) Let M, N be left R-modules (which we can also view as right modules since R is commutative). Show that $M \otimes_R N$ is an R-module with action $a(v \otimes_R w) = av \otimes_R w$ for $a \in R$, $v \in M$ and $w \in N$.
   (ii) Let M, N be left R-modules. Show that $\mathrm{Hom}_R(M, N)$ is an R-module, with action: for $a \in R$ and $\varphi : M \to N$, define $a \cdot \varphi : M \to N$ by $(a \cdot \varphi)(b) = a\varphi(b)$ for $b \in M$.
   (iii) Show that, if M, N, and T are all R-modules, then $\mathrm{Hom}_R(M \otimes_R N, T)$ is identified with the set of R-*bilinear maps* $\varphi : M \times N \to T$, which means functions satisfying $\varphi(au, v) = a\varphi(u, v) = \varphi(u, av)$ and $\varphi(u + u', v) = \varphi(u, v) + \varphi(u', v)$ as well as $\varphi(u, v + v') + \varphi(u, v) + \varphi(u, v')$. Use this to give an alternative definition of tensor product.

6. Let R be a ring and let M be a left R-module. We say that R is a *ring with involution* (or a $*$-*ring*) if R is equipped with a map $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, $1^* = 1$ and $(x^*)^* = x$ for all $x, y \in R$, i.e. $*$ is an anti-homomorphism and an involution.

   (i) Show that $M^* = \mathrm{Hom}_R(M, R)$ is a right R-module with action: for $a \in R$ and $\varphi \in \mathrm{Hom}_R(M, R)$, define $\varphi \cdot a : M \to R$ by $(\varphi \cdot a)(b) = \varphi(b) \cdot_R a$ for $b \in M$. This is known as the *dual module*.

   (ii) Let R be a commutative ring. Show that R is a ring with involution. For a group G, show that $R[G]$ is a ring with involution.

   (iii) Let R be a ring with involution. Show that any right R-module M can be viewed as a left R-module with action: for $a \in R$ and $m \in M$, define $x \cdot m = m \cdot_M x^*$. Use this to define a left R-module structure on $\mathrm{Hom}_R(M, R)$. For left R-modules M and N, define a (sensible) left R-module structure on the tensor product of abelian groups $M \otimes_{\mathbb{Z}} N$. [Optional: How do these R-module structures compare to those defined in (5) in the commutative case?]

7. Let R be a ring and let M be an R-module and let $N \leq M$ be a submodule. Show that M is Noetherian if and only if N and M/N are Noetherian.

8. Let a, b be nonzero positive integers. Find the Smith normal form of the following matrices in their respective rings:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{Q}), \qquad \begin{pmatrix} X^2 - 5X + 6 & X - 3 \\ (X-2)^3 & X^2 - 5X + 6 \end{pmatrix} \in M_2(\mathbb{Q}[X]).$$

9. Let G be the abelian group given by generators a, b, c and the relations $6a + 10b = 0$, $6a + 15c = 0$, $10b + 15c = 0$ (i.e. G is the free abelian group generated by a, b, c quotiented by the subgroup $(6a + 10b, 6a + 15c, 10b + 15c)$). Determine the structure of G as a direct sum of cyclic groups.

10. A ring R has the *invariant basis number* property (IBN) if, for all positive integers m, n, $R^n \cong R^m$ as R-modules implies $m = n$.

    (i) For an ideal $I \subseteq R$ and an R-module M, we define an R-submodule $IM = \{am \in M : a \in I, m \in M\} \leq M$. Prove that M/IM is an R/I-module in a natural way.

    (ii) Prove that non-zero commutative rings have IBN. You may assume that every non-zero commutative ring has a maximal ideal. [This is equivalent to the axiom of choice.]

    (iii) Let S be a ring and M a free S-module with basis $\{x_i \mid i \geq 1\}$. Let $R = \mathrm{End}_S(M)$. Prove that R does not have IBN. [Hint: Note that $M \cong M_{\text{even}} \oplus M_{\text{odd}}$ where M_{even} and M_{odd} are the submodules generated by x_i for i even and odd respectively. Use this to show that $R \cong R^2$ as R-modules.]

+11. Let G be a finite group, let $N = \sum_{g \in G} g \in \mathbb{Z}[G]$ and let $r \in \mathbb{Z}$ be an integer with $(r, |G|) = 1$

    (i) Show that the ideal $(N, r) \subseteq \mathbb{Z}[G]$ is projective as a $\mathbb{Z}[G]$-module.

    (ii) Let $G = C_n$ be a finite cyclic group. Show that (N, r) is free as a $\mathbb{Z}[G]$-module.

    (iii) Let $G = Q_8$ be the quaternion group of order 8. Show that $(N, 3)$ is not free as a $\mathbb{Z}[G]$-module. Is it stably free as a $\mathbb{Z}[G]$-module?