This test is based on the material from lectures 1-14, problem sheets 1-5 (weeks 5 to 9).

1. The alphabet has 26 letters – 5 vowels and 21 consonants. How many 5-letter words are there with 3 distinct consonants and 2 distinct vowels? [Assume that the order of the letters matters, and "words" include imaginary words such as "athov".]

   Please write your answer as an integer.

   Solution:
   $$\binom{5}{2}\binom{21}{3}5! = 1596000$$

2. Let X denote a discrete random variable with Bern(0.7) distribution. Find $\mathrm{P}(X > 1)$.

   Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

   Solution: We note that $\mathrm{Im}X = \{0, 1\}$. Hence,
   $$\mathrm{P}(X > 1) = 0.$$

3. Multiple answer: Consider the sample space $\Omega = \{2, 4, 6\}$ and the event space $\mathcal{F} = \mathcal{P}(\Omega)$, i.e. the power sigma-algebra of Ω. Which of the following functions are discrete random variables on $(\Omega, \mathcal{F})$? **Please select all correct statements.**

   a) $X : \Omega \to \mathbb{R}$, such that $X(\omega) = -\sqrt{7}$
   b) $X : \Omega \to \mathbb{R}$, such that $X(\omega) = \exp(\omega) + 1$
   c) $X : \Omega \to \mathbb{R}$, such that $X(\omega) = 1/\omega^2$
   d) $X : \Omega \to \mathbb{R}$, such that $X(\omega) = \omega^\omega$
   e) None of the above.

   Solution: a), b), c), d) See Definition 7.3.1. in the notes.

4. Which of the following distributions is the most suitable one for modelling the number of volcanic eruptions in a region of the world with active volcanos.

   a) Poisson
   b) Geometric
   c) Binomial
   d) Gaussian
   e) Beta

   Solution: The Poisson distribution is widely used for modelling counts, in particular, counts of rare events.

5. Consider the discrete random variable X with

$$\mathrm{P}(X=1)=0.25, \qquad \mathrm{P}(X=3)=0.25, \qquad \mathrm{P}(X=10)=0.5,$$

and $\mathrm{P}(X=x)=0$ for $x \notin \{1,3,10\}$.

Define a new random variable by $Y=X^2$. Find $\mathrm{P}(Y \leq 10)$.

Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

Solution:

$$\mathrm{P}(Y \leq 10)=\mathrm{P}(X^2 \leq 10)=\mathrm{P}(-\sqrt{10} \leq X \leq \sqrt{10})=\mathrm{P}(X=1)+\mathrm{P}(X=3)=0.5.$$

6. Consider the discrete random variable X with

$$\mathrm{P}(X=1)=0.25, \qquad \mathrm{P}(X=3)=0.25, \qquad \mathrm{P}(X=10)=0.5,$$

and $\mathrm{P}(X=x)=0$ for $x \notin \{1,3,10\}$.

Define a new random variable by $Y=X^2$. Find $\mathrm{E}(Y)$.

Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

Solution: Using LOTUS, we get

$$\mathrm{E}(Y)=1^2 \times 0.25+3^2 \times 0.25+10^2 \times 0.5=52.5.$$

7. Multiple answer: **Please select all correct statements.**

   a) $\bigcup_{n=1}^{\infty}[-5-1/n, 5+1/n] = [-6, 6]$
   b) $\bigcap_{n=1}^{\infty}[-5-1/n, 5+1/n] = [-5, 5]$
   c) $\bigcap_{n=1}^{\infty}[-5-1/n, 5+1/n] = (-5, 5)$
   d) $\bigcap_{n=1}^{\infty}[-5+1/n, 5+1/n] = (-5, 5)$
   e) None of the above.

Solution: a), b)
For d) we have: $\bigcap_{n=1}^{\infty}[-5+1/n, 5+1/n] = (-5, 5]$

8. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } 0 \leq x \leq a, \\ 0, & \text{otherwise} \end{cases}$$

for $a \geq 0$.

For which a is f a valid probability density function?

Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

Solution: f is clearly nonnegative. In addition, we need that

$$1 \stackrel{!}{=} \int_0^a x^2 dx = \frac{1}{3}a^3 \iff a = 3^{1/3} \approx 1.44.$$

9. Consider a continuous random variable $X \sim \mathrm{Exp}(2)$. Find $\mathrm{E}(\sqrt{\exp(X)})$.

Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

Solution: By the law of the unconscious statistician, we note that

$$\mathrm{E}(\sqrt{\exp(X)}) = \int_{-\infty}^{\infty} e^{x/2} f_X(x) dx = \int_0^{\infty} e^{x/2} 2e^{-2x} dx = 2\int_0^{\infty} e^{-3x/2} dx = \frac{4}{3} \approx 1.33.$$

10. Suppose that one Spaghetti (a long, thin pasta) is 30cm long. In order to fit it into your small pot, you break it into two pieces. Suppose that the break point is uniformly chosen over the interval $[0, 30]$. What is the expected length of the longer piece?

    Please write your answer in decimals, i.e. if your answer is 1/2, please write 0.5. Please round to two decimal places if needed.

Solution: Let $X \sim U([0, 30])$ denote the position where the Spaghetti was broken. Let $Y = g(X)$ denote the longer piece when the Spaghetti was broken at X, i.e.

$$Y = g(X) := \max\{X, 30 - X\} = \begin{cases} 30 - X, & 0 \leq X < 15, \\ X, & 15 \leq X \leq 30. \end{cases}$$

Since X follows a continuous uniform distribution on $[0, 30]$, its density is given by $f(x) = 1/30$ if $x \in [0, 30]$ and 0 otherwise. Using LOTUS, we get that

$$\begin{aligned} \mathrm{E}(Y) = \mathrm{E}(g(X)) &= \int_0^{15} (30 - x)\frac{1}{30}dx + \int_{15}^{30} x\frac{1}{30}dx = x - \frac{1}{2}x^2\frac{1}{30}\Big|_0^{15} + \frac{1}{2}x^2\frac{1}{30}\Big|_{15}^{30} \\ &= 15 - \frac{15^2}{60} - 0 + \frac{30^2}{60} - \frac{15^2}{60} = 15 - \frac{15}{4} + 15 - \frac{15}{4} = 30 - \frac{30}{4} = \frac{3}{4} \times 30 = 22.5. \end{aligned}$$

Alternatively, we can derive the cdf as follows. For $y < 15$, we have $F_Y(y) = \mathrm{P}(Y \leq y) = 0$ and for $y > 30$, we have $F_Y(y) = \mathrm{P}(Y \leq y) = 1$. For $15 \leq y \leq 30$, we have

$$\begin{aligned} F_Y(y) &= \mathrm{P}(Y \leq y) = \mathrm{P}(\max\{X, 30 - X\} \leq y) = \mathrm{P}(X \leq y, 30 - X \leq y) = \mathrm{P}(X \leq y, 30 - y \leq X) \\ &= \mathrm{P}(30 - y \leq X \leq y) = F_X(y) - F_X(30 - y) = \frac{y - (30 - y)}{30} = \frac{2y - 30}{30} = \frac{y}{15} - 1. \end{aligned}$$

Hence, for $15 \leq y \leq 30$, the pdf of Y is given by $f_Y(y) = \frac{1}{15}$ and 0 otherwise. [This means that $Y \sim U([15, 30])$.] Then

$$\mathbb{E}(Y) = \int_{15}^{30} y f_Y(y)dy = \frac{1}{15}\frac{1}{2}y^2\Big|_{15}^{30} = 22.5.$$