Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 8 with Solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. Consider the function $u = \arctan(y/x)$. Show that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

and that $u(x,y)$ is also a solution of the Laplace equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We have:

$$\frac{\partial u}{\partial x} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right), \quad \frac{\partial u}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right).$$

by plugging in we see that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Now, by calculating the second partial derivatives we have:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{1}{(1+(\frac{y}{x})^2)^2}\left(\frac{2y}{x^3}\right),$$

So, we have $u(x,y)$ being the solution os the Laplace equation as requested.

2. Consider the function $u = x\ln(x^2+y^2) - 2y\arctan(y/x)$. Show that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + 2x$$

We have:

$$\frac{\partial u}{\partial x} = \ln(x^2+y^2) + 2\frac{x^2}{x^2+y^2} + 2\frac{y^2}{x^2+y^2} = \ln(x^2+y^2) + 2,$$

$$\frac{\partial u}{\partial y} = 2\frac{x^2}{x^2+y^2} - 2\arctan(\frac{y}{x}) - 2\frac{x^2}{x^2+y^2} = -2\arctan(\frac{y}{x}).$$

by plugging in we see that:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + 2x.$$

3. * Consider the function $u(x, y, z) = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$. Show that this function is a solution of the Laplace equation in three dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

We have $u = (x^2 + y^2 + z^2)^{1/2}$, so:

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{-1}{r^3} + \frac{3x^2}{r^5}.$$

One obtains simularly:

$$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{r^3} + \frac{3y^2}{r^5}, \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = \frac{-1}{r^3} + \frac{3z^2}{r^5}.$$

So, we can see by plugging these results that u is a solution of the Laplace equation.

4. Consider the following change of variables:

$$s = \frac{x}{x^2 + y^2}$$

$$t = \frac{y}{x^2 + y^2}.$$

(a) Obtain the Jacobian matrix associated with this change of variables.

We have from total differentiation:

$$ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy,$$

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy.$$

So,

$$J = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{bmatrix} = \begin{bmatrix} t^2 - s^2 & -2st \\ -2st & s^2 - t^2 \end{bmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} y^2 - x^2 & -2xy \\ -2xy & x^2 - y^2 \end{bmatrix}.$$

(b) Consider a function $u(s, t)$ that obeys the partial differential equation:

$$\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 = 0$$

Find the partial differential equation that this function obeys when it is expressed in terms of the variables x, y.

From the lecture notes we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial s}(t^2 - s^2) + \frac{\partial u}{\partial t}(-2ts),$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial u}{\partial s}(-2ts) + \frac{\partial u}{\partial t}(s^2 - t^2).$$

From this we have:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)(s^2 + t^2)^2 = 0.$$

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5. Solve the differential equations below. First, determine if they are exact or not. If exact, solve directly. If not, solve them by multiplying by a suitable integrating factor.

(a) $(3x - 2y)dy + (3x^2 + 3y)dx = 0$

Let $F = 3x^2 + 3y$ and $G = 3x - 2y$. We see that the condition of integratibility is satisfied:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = 3,$$

So the equation is exact. We look for $u(x, y)$ such that $\frac{\partial u}{\partial x} = F$ and $\frac{\partial u}{\partial y} = G$. From the first equation, we obtain:

$$u(x, y) = x^3 + 3xy + h(y),$$

where, $h(y)$ is constant of integration. From the second equation we then obtain:

$$u(x, y) = x^3 + 3xy - y^2 + c,$$

Hence the soluion of the ODE in implicit form is:

$$x^3 + 3xy - y^2 + c = 0.$$

(b) $e^y \sin x \frac{dy}{dx} + (1 + e^y) \cos x = 0$

Let $F = (1 + e^y) \cos x$ and $G = e^y \sin x$. We see that the condition of integratibility is satisfied:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = e^y \cos x,$$

So the equation is exact. We look for $u(x, y)$ such that $\frac{\partial u}{\partial x} = F$ and $\frac{\partial u}{\partial y} = G$. We find the soluion of the ODE in implicit form is:

$$(1 + e^y) \sin x = c.$$

(c) $(y^2 - x^2)\frac{dy}{dx} + 2xy = 0$

Let $F = 2xy$ and $G = y^2 - x^2$. We see that the condition of integratibility is not satisfied:

$$\frac{\partial F}{\partial y} = -\frac{\partial G}{\partial x} = 2x,$$

So the equation is not exact. We look for an integrating factor I to make the ODE exact. If we try $I(x)$ we will find out that does not work. So, let's try $I(y)$. We have:

$$2xI(y) + 2xy\frac{dI}{dy} = 0 \quad \Rightarrow \quad I(y) = \frac{1}{y^2}.$$

We look for $u(x, y)$ such that $\frac{\partial u}{\partial x} = FI$ and $\frac{\partial u}{\partial y} = GI$. We obtain

$$u(x, y) = \frac{x^2}{y} + y + c,$$

so, we find the soluion of the ODE in implicit form is:

$$\frac{x^2}{y} + y + c = 0.$$

(d) $(y^4 + 2y^2 - x^3 + 5x^2y - 21xy^2)\frac{dy}{dx} + (x^3 - 3x^2y + 5xy^2 - 7y^3) = 0$

Let $F = x^3 - 3x^2y + 5xy^2 - 7y^3$ and $G = y^4 + 2y^2 - x^3 + 5x^2y - 21xy^2$. We see that the condition of integratibility is satisfied:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = -3x^2 + 10xy - 21y^2,$$

So the equation is exact. We look for $u(x,y)$ such that $\frac{\partial u}{\partial x} = F$ and $\frac{\partial u}{\partial y} = G$. We find the soluion of the ODE in implicit form is:

$$\frac{x^4}{4} - x^3y + \frac{5}{2}x^2y^2 - 7xy^3 + \frac{y^5}{5} + \frac{2}{3}y^3 = c.$$

(e) $(x^3 - 2xy)\frac{dy}{dx} + (x + 2y^2) = 0$

Let $F = x + 2y^2$ and $G = x^3 - 2xy$. We see that the condition of integratibility is not satisfied:

$$\frac{\partial F}{\partial y} = 4y \quad \text{and} \quad \frac{\partial G}{\partial x} = 3x^2 - 2y,$$

So the equation is not exact. We look for an integrating factor I to make the ODE exact. We can try $I(x) = x^k$, where k is to be found (alternatively we can find an ODE for $I(x)$). Condition of integratibility is satisfied if:

$$\frac{\partial(FI)}{\partial y} = \frac{\partial(GI)}{\partial x} \quad \Rightarrow \quad 4x^ky = (k+3)x^{k+2} - 2(k+1)x^ky,$$

This is satisfied for $k = -3$, so $I(x) = x^{-3}$. We now look for $u(x,y)$ such that $\frac{\partial u}{\partial x} = FI$ and $\frac{\partial u}{\partial y} = GI$. We obtain

$$u(x,y) = -\frac{1}{x} - \frac{y^2}{x^2} + y + c,$$

so, we find the soluion of the ODE in implicit form is:

$$-\frac{1}{x} - \frac{y^2}{x^2} + y + c = 0.$$

(f) $(e^y + ye^x)dx + (e^x + xe^y + 1)dy = 0$

Let $F = e^y + ye^x$ and $G = e^x + xe^y + 1$. We see that the condition of integratibility is satisfied:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = e^y + e^x,$$

So the equation is exact. We look for $u(x,y)$ such that $\frac{\partial u}{\partial x} = F$ and $\frac{\partial u}{\partial y} = G$. We find the soluion of the ODE in implicit form is:

$$xe^y + ye^x + y = c.$$

6. A few examples to practise sketching of functions in two variables, as well as finding and classifying their extrema:

(a) Find the stationary points (extrema) of

$$f(x, y) = x^3 + y^2 - 3x$$

and determine their character (i.e, whether they are maxima, minima or saddle points).

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y = 0,$$

So we have two stationay points: $P_1 = (1, 0)$ and $P_2 = (-1, 0)$. We evaluate the Hessian matrix at these points to decide the character of these extrema.

$$H = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}.$$

At P_1 the determinant of H is positive ($\Delta = 12$), and the trace of H is also positive ($\tau = 8$), so P_1 is a minimum. At P_2 the determinant of H is negative ($\Delta = -12$), so P_2 is a sadde point.

(b) Consider the function

$$f(x, y) = xy(x + y - 1)$$

Find the extrema and determine their character. Sketch the relevant contour lines of the function.

$$\frac{\partial f}{\partial x} = 2xy + y^2 - y = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 2xy - x = 0,$$

So we have four stationay points: $P_1 = (0, 0)$, $P_2 = (0, 1)$, $P_3 = (1, 0)$ and $P_4 = (1/3, 1/3)$. We evaluate the Hessian matrix at these points to decide the character of these extrema.

$$H = \begin{bmatrix} 2y & 2x + 2y - 1 \\ 2x + 2y - 1 & 2x \end{bmatrix}.$$

So for P_1, P_2 and P_3 we have $\Delta < 0$, so they are saddle points. For P_4 we have $\Delta, \tau > 0$, so it is a minimum.

The $f = 0$ gives us the zero level set (contour), which is the 3 lines of $x = 0$, $y = 0$ and $x + y = 1$. We can also obtain the sign of the function f in each of the regions. Figure 1 shows an sketch of function f in the (x, y) plane.

(c) * Consider the function

$$f(x, y) = x(y - 2)^2 + x^2 - x$$

Find the extrema and determine their character. Sketch the relevant contour lines of the function.

$$\frac{\partial f}{\partial x} = (y - 2)^2 + 2x - 1 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x(y - 2) = 0,$$

So we have three stationay points: $P_1 = (0, 1)$, $P_2 = (0, 3)$ and $P_3 = (1/2, 2)$. We evaluate the Hessian matrix at these points to decide the character of these extrema.

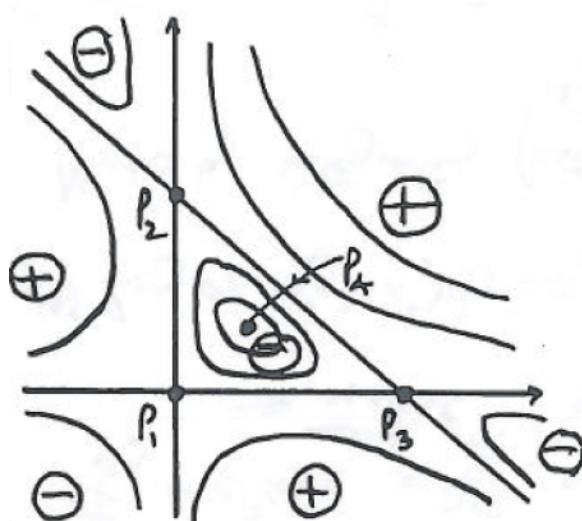$$H = \begin{bmatrix} 2 & 2(y - 2) \\ 2(y - 2) & 2x \end{bmatrix}.$$
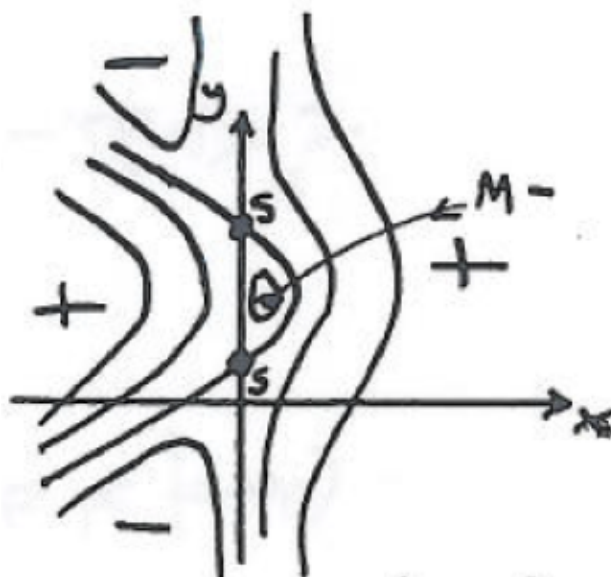
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Figure 1: Sketch for problem 6 (b).

Figure 2: Sketch for problem 6 (c).

So for P_1 and P_2 we have $\Delta < 0$, so they are saddle points. For P_3 we have $\Delta, \tau > 0$, so it is a minimum.

The $f = 0$ gives us the zero level set (contour), which is the one line of $x = 0$ and the parabola $x - 1 = -(y - 2)^2$. We can also obtain the sign of the function f in each of the regions. Figure 2 shows an sketch of function f in the (x, y) plane.