Imperial College London

MATH97054 MATH97165

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Complex Manifolds

Date: Tuesday, 11 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Show that if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic then both the real part and the imaginary part of f are harmonic functions. (3 marks)

(b) Let $X = \mathbb{C}^n$ and let $Y \subset X$ be a connected compact submanifold. Show that Y is a point. (3 marks)

(c) Show that if $f\colon \mathbb{C}^n \to \mathbb{C}$ is a smooth function then f is holomorphic if and only if, at each point $x \in \mathbb{C}^n$, the differential $df_x\colon \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear. (4 marks)

(d) Prove the Inverse Function Theorem for holomorphic functions assuming the Implicit Function Theorem proved in class. (5 marks)

(e) Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$, with a complex structure $J \in \mathrm{End}(V_{\mathbb{R}})$ and a compatible Riemannian metric g. Prove that the decomposition $\wedge^k V_{\mathbb{C}} = \sum_{p+q=k} V^{p,q}$ is orthogonal with respect to the induced Hermitian inner product h on $\wedge^k V_{\mathbb{C}}$.

(5 marks)

(Total: 20 marks)

2. [The Fubiny-Study metric] Let $\mathcal{O}(-1) = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in L\}$.

(a) Show that the projection on the first coordinate $\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$ is a holomorphic line bundle. (4 marks)

(b) Let $\mathcal{O}(1)$ be the dual of $\mathcal{O}(-1)$ as a bundle over $\mathbb{CP}^n$.

(i) Show that there are natural hermitian metrics on $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ inherited from the canonical hermitian metric of $\mathbb{C}^{n+1}$. (3 marks)

(ii) Compute expressions for these metrics on the chart

$$U_0 = \{[z_0 : \cdots : z_n] : z_0 \neq 0\}.$$

(4 marks)

(iii) Show that $\mathcal{O}(-1)$ does not have non-trivial holomorphic global sections. (4 marks)

(c) Let D be the Chern connection of the natural hermitian metric on $\mathcal{O}(1)$ and F_D its curvature.

(i) Show that $F_D = \partial\bar{\partial} \log(1 + \sum |z_j|^2)$. (3 marks)

(ii) Show that $\omega_{FS} = \frac{i}{2\pi} F_D$ induces a hermitian metric on $\mathbb{CP}^n$. (2 marks)

(Total: 20 marks)

3. (a) Let $D \subset \mathbb{C}$ be a polydisk. Show that $H^{p,q}(D) = 0$, for $q \geq 1$. State clearly any results you are using. (4 marks)

(b) Show that $\overline{\partial \alpha} = \bar{\partial}\bar{\alpha}$. Conclude that a real (p,p)-form is ∂-closed (exact) if and only if it is $\bar{\partial}$-closed (exact). (5 marks)

(c) Use the $\bar{\partial}$-Poincaré Lemma to state and prove the ∂-Poincaré Lemma. (5 marks)

(d) (A $\partial\bar{\partial}$-Poincaré Lemma) Let $D \subset \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in A^{p,q}(D)$ satisfies $d\omega = 0$, with $p, q \geq 1$. Prove that there is $\psi \in A^{p-1,q-1}(D)$ such that $\partial\bar{\partial}\psi = \omega$ (Hint: use the three Poincaré Lemmas, i.e. for d, ∂ and $\bar{\partial}$). (6 marks)

(Total: 20 marks)

4. (a) Let (V, J) be a real vector space with an almost complex structure J. Show that $\dim V$ is even and that $iv := J(v)$ makes V a complex manifold. (4 marks)

(b) Let (M, g) be a compact oriented Riemannian manifold of dimension n. Prove that the volume form $\mathrm{vol}(g) \in A^n(M)$ is harmonic. (Hint: First show that $(\mathrm{vol}(g), d\psi) = \int_M d\psi$ for any $\psi \in A^{n-1}(M)$.) (4 marks)

(c) Let (V, g) be a vector space with a compatible almost complex structure J and fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$ and vol_W its volume form. Show that

$$\omega^m|_W \leq m! \, \mathrm{vol}_W,$$

with equality, if and only if, W is a complex subspace, i.e. $J(W) \subset W$. (4 marks)

(d) Let M be a Kahler manifold and ω its Kahler form. Show that ω^ℓ defines a non-zero homology class in $H_D^{\ell,\ell}(M)$, for all $1 \leq \ell \leq \dim M/2$. In particular, $\dim H_D^{\ell,\ell}(M) \neq 0$. (4 marks)

(e) Show that a complex submanifold of a Kahler manifold minimises the volume in its homology class. (4 marks)

(Total: 20 marks)

5. (a) The objective of the following items is to show a $\bar{\partial}$-Poincaré lemma for $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, i.e. *given a smooth function* $g : \mathbb{C}^* \to \mathbb{C}$, *there exists a smooth function* $f : \mathbb{C}^* \to \mathbb{C}$ *with* $\partial f / \partial \bar{z} = g$.

(i) Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of open annuli centered at 0 such that $\bar{A}_k \subset A_{k+1}$ and $\cup_k A_k = \mathbb{C}^*$. Show that for any $k \in \mathbb{N}$, there exists $h_k \in C^\infty(\mathbb{C})$ such that $\partial h_k/\partial \bar{z} = g$ on A_k.

(4 marks)

(ii) Assume that $f_{k-1} \in C^\infty(\mathbb{C})$ satisfies $\partial f_{k-1}/\partial \bar{z} = g$ on A_{k-1}. Show that for any $\varepsilon_k > 0$, there is a holomorphic function $\alpha_k \in C^\infty(\mathbb{C}^*)$, such that $|f_{k-1} - h_k - \alpha_k| < \varepsilon_k$ on A_{k-1}, where h_k is as in the previous item. (4 marks)

(iii) Define a $f_k \in C^\infty(\mathbb{C}^*)$ recursively by $f_1 = h_1$ and $f_k = h_k + \alpha_k$, with $\varepsilon_k = 2^{-k}$. Show that f_k converges uniformly in compacts of $\mathbb{C}^*$ to a f as in the statement. (4 marks)

(b) Conclude that the Dolbeault cohomology groups $H^{0,1}(\mathbb{C}^*)$ and $H^{1,1}(\mathbb{C}^*)$ are both zero.

(4 marks)

(c) Compute $H^{0,0}(\mathbb{C}^*)$ and $H^{1,0}(\mathbb{C}^*)$. (4 marks)

(Total: 20 marks)

1. (a) Show that if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic then both the real part and the imaginary part of f are harmonic functions. (3 marks)

(b) Let $X = \mathbb{C}^n$ and let $Y \subset X$ be a connected compact submanifold. Show that Y is a point. (3 marks)

(c) Show that if $f\colon \mathbb{C}^n \to \mathbb{C}$ is a smooth function then f is holomorphic if and only if, at each point $x \in \mathbb{C}^n$, the differential $df_x\colon \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear. (4 marks)

(d) Prove the Inverse Function Theorem for holomorphic functions assuming the Implicit Function Theorem proved in class. (5 marks)

(e) Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$, with a complex structure $J \in \mathrm{End}(V_{\mathbb{R}})$ and a compatible Riemannian metric g. Prove that the decomposition $\wedge^k V_{\mathbb{C}} = \sum_{p+q=k} V^{p,q}$ is orthogonal with respect to the induced Hermitian inner product h on $\wedge^k V_{\mathbb{C}}$. (5 marks)

(Total: 20 marks)

Solution.

(1)(a) Let $f(x,y) = u(x,y) + iv(x,y)$. By Cauchy-Riemann $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$. Therefore, $\Delta u = (\partial_x\partial_x + \partial_y\partial_y)u = \partial_x\partial_y v - \partial_y\partial_x v = 0$ and $\Delta v = (\partial_x\partial_x + \partial_y\partial_y)v = -\partial_x\partial_y u + \partial_y\partial_x u = 0$.

(1)(b) Let $f : Y \to \mathbb{C}$ be the projection onto the first coordinate and $p \in Y$, with $|f(p)| = \max|f|$. In holomorphic coordinates $z = (z_1, \ldots, z_n)$ around p, with $z(p) = 0$,$z_i \mapsto f(0, \ldots, z_i, \ldots, 0)$ is a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ on an open set of the complex plane attaining it maximum modulus at 0. By the maximum principle for complex functions of one variable, it follows that f is constant on each direction z_i. This implies that the Taylor expansion of f at p is constant. In particular f is constant on a neighborhood of p and by the identity principle it must be constant everywhere.

(1)(c) Since the map df is already $\mathbb{R}$-linear, being $\mathbb{C}$-linear is equivalent to commutation with the canonical almost complex structure of $\mathbb{C}^n$, which we denote by J. Writing the matrix of df in one canonical coordinate $z_i = x_i + iy_i$ at a time, denoting $v = (\cdots, a, b, \cdots)$, we obtain the expressions

$$\begin{pmatrix} & \vdots & \vdots & \\ \cdots & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \cdots \\ \cdots & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \cdots \\ & \vdots & \vdots & \end{pmatrix} v = \begin{pmatrix} \vdots \\ a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} \\ a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} \\ \vdots \end{pmatrix}.$$

Let $v = (\cdots, 1, 0 \cdots)$. Then $df \cdot J(v) = df(\cdots, 0, 1, \cdots) = (\cdots, \partial u/\partial y, \partial v/\partial y, \cdots)$ and $J \cdot df(v) = J(\cdots, \partial u/\partial x, \partial v/\partial x, \cdots) = (\cdots, -\partial v/\partial x, \partial u/\partial x, \cdots)$. It follows that J and df commute if and only if f satisfies the Cauchy-Riemann equations.

(1)(d) We show that given a holomorphic $f : D \subset \mathbb{C}^n \to \mathbb{C}^n$ with $f(0) = 0$ and $Jf(0)$ not singular, there is $r'' \in \mathbb{R}^n_+$ and $g : \Delta(0, r'') \to \mathbb{C}^n$ holomorphic, such that $f \circ g = id$ on $\Delta(0, r'')$. In fact, consider the function $\Phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, given by $\Phi(z, w) = f(z) - w$. We use the notation $r = (r', r'') \in \mathbb{R}^n \times \mathbb{R}^n$ and $J\Phi = (J'\Phi, J''\Phi)$. By assumption $J'\Phi(0, 0) = Jf(0)$ is not singular. Then, the Implicit Function Theorem implies the existence of a small positive $r = (r', r'')$ and a holomorphic map $g : \Delta(0, r'') \to \Delta(0, r')$ such that for $(z, w) \in \Delta(0, r')$, we have $\Phi(z, w) = f(z) - w = 0$ if and only if $z = g(w)$. In other words, $f \circ g(w) = w$.

(1)(e) Remember that g and J are compatible if $g(Jv, Jw) = (v, w)$ for any $v, w \in V$. This implies that J is skew-symmetric, i.e. $g(Jv, w) = g(J^2v, Jw) = g(v, -Jw)$. By the spectral theorem there is an orthonormal basis $\{x_i, y_i\}_{i=1}^n$ where J has diagonal block form, i.e $J(x_i) = y_i$ and $J(y_i) = -x_i$. Denote by $g_{\mathbb{C}}$ the hermitian extension of g on $V_{\mathbb{C}}$. The inner product on $\wedge^k V_{\mathbb{C}}$ is defined as follows: let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ be elements of $V_{\mathbb{C}}$, then $\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det\{g_{\mathbb{C}}(\alpha_i, \beta_j)\}$. Let $z_i = x_i - iy_i \in V_{\mathbb{C}}$. The space $V^{p,q} \subset \wedge^k V_{\mathbb{C}}$ is generated by the elements of the form $z_I \wedge z_J$, where $|I| = p$ and $|J| = q$. Therefore, to show that the spaces $\wedge^{p,q} V$ are orthogonal, it is enough to show that z_i is orthogonal to z_j, for all $j \neq i$ and to all $\bar{z}_j$. We check this now:

$$\begin{aligned}
g_{\mathbb{C}}(z_i, z_j) &= g_{\mathbb{C}}(x_i - iy_i, x_j - iy_j) = g_{\mathbb{C}}(x_i, x_j) - g_{\mathbb{C}}(x_i, iy_j) - g_{\mathbb{C}}(iy_i, x_j) + g_{\mathbb{C}}(iy_i, iy_j) \\
&= g(x_i, x_j) + ig(x_i, y_j) - ig(y_i, x_j) + g(y_i, y_j) \\
&= g(x_i, x_j) + ig(x_i, y_j) - ig(Jy_i, Jx_j) + g(Jy_i, Jy_j) \\
&= g(x_i, x_j) + ig(x_i, y_j) + ig(x_i, y_j) + g(x_i, x_j) \\
&= \delta_{ij} + 0 + 0 + \delta_{ij} \\
&= 2\delta_{ij}.
\end{aligned}$$

$$\begin{aligned}
g_{\mathbb{C}}(z_i, \bar{z}_j) &= g_{\mathbb{C}}(x_i - iy_i, x_j + iy_j) = g_{\mathbb{C}}(x_i, x_j) + g_{\mathbb{C}}(x_i, iy_j) - g_{\mathbb{C}}(iy_i, x_j) - g_{\mathbb{C}}(iy_i, iy_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) - ig(y_i, x_j) - g(y_i, y_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) - ig(Jy_i, Jx_j) - g(Jy_i, Jy_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) + ig(x_i, y_j) - g(x_i, x_j) \\
&= 0.
\end{aligned}$$

2. [The Fubiny-Study metric] Let $\mathcal{O}(-1) = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in L\}$.

(a) Show that the projection on the first coordinate $\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$ is a holomorphic line bundle. (4 marks)

(b) Let $\mathcal{O}(1)$ be the dual of $\mathcal{O}(-1)$ as a bundle over $\mathbb{CP}^n$.

(i) Show that there are natural hermitian metrics on $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ inherited from the canonical hermitian metric of $\mathbb{C}^{n+1}$. (3 marks)

(ii) Compute expressions for these metrics on the chart

$$U_0 = \{[z_0 : \cdots : z_n] : z_0 \neq 0\}.$$

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(iii) Show that $\mathcal{O}(-1)$ does not have non-trivial holomorphic global sections. (4 marks)

(c) Let D be the Chern connection of the natural hermitian metric on $\mathcal{O}(1)$ and F_D its curvature.

(i) Show that $F_D = \partial\bar{\partial}\log(1 + \sum |z_j|^2)$. (3 marks)

(ii) Show that $\omega_{FS} = \frac{i}{2\pi} F_D$ induces a hermitian metric on $\mathbb{CP}^n$. (2 marks)

(Total: 20 marks)

Solution.

(a) Let $[z] = [z_0, \dots, z_n]$ be homogeneous coordinates in $\mathbb{CP}^n$. Then

$$\mathcal{O}(-1) = \{([z], \lambda z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \lambda \in \mathbb{C}\}.$$

Remember the standard atlas for $\mathbb{CP}^n$ is given by $\mathcal{A} = \{U_i, \phi_i\}_{i=0}^n$, where $U_i = \{[z] \in \mathbb{CP}^n : z_i \neq 0\}$ and $\phi_i([z]) = z_i^{-1}(z_0, \dots, \widehat{z_i}, \dots, z_n)$ are homeomorphisms with inverse $\phi_i^{-1}(w_0, \dots, \widehat{w_i}, \dots, w_n) = [w_0, \dots, w_{i-1}, 1, w_i, \dots, w_n]$. Define $\tilde{\mathcal{A}} = \{\tilde{U}_i, \tilde{\phi}_i\}_{i=0}^n$, where $\tilde{U}_i = \{([z], \lambda z) \in U_i \times \mathbb{C}^{n+1} : \lambda \in \mathbb{C}\}$ and $\tilde{\phi}_i$ are constructed as the composition $\tilde{\phi}_i = (\phi_i \times id) \circ \psi_i$, where $\psi_i([z], \lambda z) = ([z], \lambda z_i)$. We now check that $\tilde{\mathcal{A}}$ is a holomorphic atlas for $\mathcal{O}(-1)$ and $\{\psi_i\}_{i=0}^n$ is a set of trivialisations that makes $\mathcal{O}(-1)$ a vector bundle of rank one.

In fact, in coordinates $\psi_{ji} = \psi_j \psi_i^{-1} : U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}$ is given by

$$\psi_j \psi_i^{-1}([z], a) = \psi_j([z], (a z_i^{-1}) z) = ([z], (z_j z_i^{-1}) a)$$

and $\tilde{\phi}_{ji} : \phi_i(U_{ij}) \times \mathbb{C} \to \phi_j(U_{ij}) \times \mathbb{C}$, i.e $\tilde{\phi}_j \tilde{\phi}_i^{-1} = (\phi_j \times id)\psi_{ji}(\phi_i^{-1} \times id)$, by

$$\begin{aligned}
\tilde{\phi}_{ji}(z_0, \dots, \widehat{z_i}, \dots, z_n, a) &= (\phi_j \times id)\psi_{ji}(\phi_i^{-1} \times id)(z_0, \dots, \widehat{z_i}, \dots, z_n, a) \\
&= (\phi_j \times id)\psi_{ji}([z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n], a) \\
&= (\phi_j \times id)([z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, w_n], z_j a) \\
&= (z_j^{-1}(z_0, \dots, \widehat{z_j}, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n), z_j a).
\end{aligned}$$

Since these are defined on the regions where $z_j \neq 0$, it follows that the maps are biholorphisms. Moreover, the formula shows that projection on the second variable of ψ_{ji} is the linear isomorphism $a \mapsto z_j a$ of $\mathbb{C}$.

(b)(i) As we saw above, the fibre of $\mathcal{O}(-1)$ over $[z] \in \mathbb{CP}^n$ is the set $L([z]) \in \{([z], \lambda z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \lambda \in \mathbb{C}\}$, which is a complex subspace of $\mathbb{C}^{n+1}$. Then the restriction of the canonical hermitian inner product of $\mathbb{C}^{n+1}$ is an hermitian inner product on $L([z])$, for each $[z]$. Since the fibres vary smoothly with $[z]$ these define a hermitian metric on $\mathcal{O}(-1)$. Finally, the hermitian inner products gives a canonical isomorphism between the fibers of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$, making it also a holomorphic hermitian vector bundle.

(b)(ii) Notice $h^{-1} = h^*$. So it is enough to compute it for $\mathcal{O}(-1)$. In coordinates, a hermitian metric on a vector bundle of rank one is determined by a scalar function h. We compute

$$\begin{aligned}\tilde{\phi}_0^{-1}(\widehat{z}_0, z_1, \ldots, z_n, 1) &= \psi_0^{-1}(\phi_0^{-1} \times id)(\widehat{z}_0, z_1, \ldots, z_n, 1)\\ &= \psi_0^{-1}([1, z_1, \ldots, z_n], 1)\\ &= ([1, z_1, \ldots, z_n], 1 \cdot (1, z_1, \ldots, z_n)).\end{aligned}$$

So $h = \langle (1, z_1, \ldots, z_n), (1, z_1, \ldots, z_n) \rangle_{\mathbb{C}^{n+1}} = 1 + |z_1|^2 + \cdots + |z_n|^2$.

(b)(iii) A section s assigns to each $[z] \in \mathbb{CP}^n$ and element $s([z]) \in \mathbb{C}^{n+1}$, i.e. $s : \mathbb{CP}^n \to \mathbb{C}^{n+1}$. Since $\mathbb{CP}^n$ is a compact complex manifold, if this map is holomorphic then it must be constant, i.e. $s = s_0 \in L$, for all $L \in \mathbb{CP}^n$, but the only vector that belongs to all lines is the origin. Therefore, $s_0 = 0$.

(c)(i) The metric on $\mathcal{O}(1)$ is $h^* = h^{-1} = (1+|z_1|^2+\cdots+|z_n|^2)^{-1}$. In class we saw that $F_D = \bar{\partial}\partial \log h^*$, then $F_D = \bar{\partial}\partial \log(1 + |z|^2)^{-1} = -\bar{\partial}\partial \log(1 + |z|^2) = \partial\bar{\partial} \log(1 + |z|^2)$, as claimed.

(c)(ii)

$$\begin{aligned}\partial\bar{\partial}\log(1+|z|^2) &= \partial\left(\frac{1}{1+|z|^2}\sum_{i=1}^n \bar{\partial}(z_i\bar{z}_i)\right)\\ &= \partial\left(\frac{1}{1+|z|^2}\sum_{i=1}^n z_i d\bar{z}_i\right)\\ &= -\frac{1}{(1+|z|^2)^2}\left(\sum_{i=1}^n \bar{z}_i dz_i\right) \wedge \left(\sum_{i=1}^n z_i d\bar{z}_i\right) + \frac{1}{1+|z|^2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i\\ &= -\frac{1}{(1+|z|^2)^2}\left(\sum_{i,j=1}^n z_j\bar{z}_i dz_i \wedge d\bar{z}_j\right) + \frac{1}{1+|z|^2}\sum_{i,j=1}^n \delta_{ij} dz_i \wedge d\bar{z}_j\\ &= \frac{1}{(1+|z|^2)^2}\sum_{i,j=1}^n (-z_j\bar{z}_i + (1+|z|^2)\delta_{ij}) dz_i \wedge d\bar{z}_j\end{aligned}$$

Since $\omega = \frac{i}{2}\sum_{ij} h_{ij} dz_i \wedge d\bar{z}_j$,it is enough to show that

$$h = h_{ij}(z) := (-z_j\bar{z}_i + (1+|z|^2)\delta_{ij})$$

is a positive definite hermitian matrix.

$$
\begin{aligned}
\langle h(z)w, w\rangle &= \sum_{ij} -z_j\bar{w}_j\bar{z}_i w_i + (1+|z|^2)\delta_{ij}w_i\bar{w}_j \\
&= -\langle z, w\rangle\langle w, z\rangle + (1+|z|^2)|w|^2 \\
&= -\langle z, w\rangle\overline{\langle z, w\rangle} + (1+|z|^2)|w|^2 \\
&= -|\langle z, w\rangle|^2 + (1+|z|^2)|w|^2.
\end{aligned}
$$

From Cauchy-Scharwz inequality it follows that $|z|^2|w|^2 \geq |\langle z, w\rangle|^2$. Therefore, $\langle h(z)w, w\rangle \geq |w|^2$, so h is positive definite.

3. (a) Let $D \subset \mathbb{C}$ be a polydisk. Show that $H^{p,q}(D) = 0$, for $q \geq 1$. State clearly any results you are using. (4 marks)

(b) Show that $\overline{\partial \alpha} = \bar{\partial}\bar{\alpha}$. Conclude that a real (p,p)-form is ∂-closed (exact) if and only if it is $\bar{\partial}$-closed (exact). (5 marks)

(c) Use the $\bar{\partial}$-Poincaré Lemma to state and prove the ∂-Poincaré Lemma. (5 marks)

(d) (A $\partial\bar{\partial}$-Poincaré Lemma) Let $D \subset \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in A^{p,q}(D)$ satisfies $d\omega = 0$, with $p, q \geq 1$. Prove that there is $\psi \in A^{p-1,q-1}(D)$ such that $\partial\bar{\partial}\psi = \omega$ (Hint: use the three Poincaré Lemmas, i.e. for d, ∂ and $\bar{\partial}$). (6 marks)

(Total: 20 marks)

Solutions.

(3)(a) Present the full statement of the $\bar{\partial}$-Poincaré lemma on a polydisk. It implies that $\bar{\partial}$-closed forms of type (p,q) on D, are exact for $q \geq 1$. It follows that the corresponding cohomomology is zero.

(3)(b) By linearity it is enough to show for a form $\alpha = f dz_I \wedge d\bar{z}_J$, with $|I| = p$ and $|J| = q$.

$$\overline{\partial \alpha} = \overline{\frac{\partial f}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J} = \frac{\partial \bar{f}}{\partial \bar{z}_i} d\bar{z}_i \wedge d\bar{z}_I \wedge d\ z_J = \bar{\partial}(\bar{f} d\bar{z}_I \wedge d\ z_J) = \bar{\partial}\bar{\alpha}.$$

If $\bar{\beta} = \beta$, on one hand we have $\overline{\partial\beta} = \bar{\partial}\bar{\beta} = \bar{\partial}\beta$. On the other hand, $\beta = \partial\alpha$ if and only if $\beta = \bar{\beta} = \bar{\partial}\bar{\alpha}$. These formulas show that β is ∂-closed (exact) if and only if it is $\bar{\partial}$-closed (exact).

(3)(c) Let $\alpha \in A^{p+1,q}(D)$ be a ∂-closed form. Then, by part (ii) the form $\bar{\alpha}$ is a $\bar{\partial}$-closed form in $A^{q,p+1}(D)$. From the $\bar{\partial}$-Poincaré lemma there is $\beta \in A^{q,p}(D)$ with $\bar{\alpha} = \bar{\partial}\beta$. Conjugating, $\alpha = \partial\bar{\beta}$, so α is a ∂-exact form in $A^{p,q}(D)$. In other words, we showed: Theorem: every ∂-closed form in $A^{p+1,q}(D)$ is ∂-exact.

(3)(d) By the standard Poincaré lemma, there is a form $\alpha \in A^{p+q-1}(D)$ with $d\alpha = \omega$. Since ω is a form of type (p,q) and $d = \partial + \bar{\partial}$, then $\alpha = \alpha' + \alpha'' + \eta$, with α' of type $(p-1,q)$, α'' of type $(p,q-1)$ and η a d-closed form. Without loss of generality we can assume that $\eta = 0$, i.e. $\alpha = \alpha' + \alpha''$. Moreover, $0 = d\alpha = (\partial + \bar{\partial})(\alpha' + \alpha'') = (\partial\alpha' + \bar{\partial}\alpha'') + \partial\alpha'' + \bar{\partial}\alpha'$, where the three terms have different types. We have $0 = \partial\alpha' + \bar{\partial}\alpha'' = \partial\alpha'' = \bar{\partial}\alpha'$. By the ∂- and $\bar{\partial}$-Poincaré lemmas, there are forms $\gamma'' \in A^{p-1,q-1}$ and $\gamma' \in A^{p-1,q-1}$, such that $\partial\gamma'' = \alpha'$ and $\bar{\partial}\gamma' = \alpha''$. Finally, $\partial\bar{\partial}(\gamma' - \gamma'') = \partial\alpha'' + \bar{\partial}\partial\gamma'' = \partial\alpha'' + \bar{\partial}\alpha' + (\partial\alpha' + \bar{\partial}\alpha'') = d(\alpha' + \alpha'') = \omega$.

4. (a) Let (V, J) be a real vector space with an almost complex structure J. Show that $\dim V$ is even and that $iv := J(v)$ makes V a complex manifold. (4 marks)

(b) Let (M, g) be a compact oriented Riemannian manifold of dimension n. Prove that the volume form $\mathrm{vol}(g) \in A^n(M)$ is harmonic. (Hint: First show that $(\mathrm{vol}(g), d\psi) = \int_M d\psi$ for any $\psi \in A^{n-1}(M)$.) (4 marks)

(c) Let (V, g) be a vector space with a compatible almost complex structure J and fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$ and vol_W its volume form. Show that

$$\omega^m|_W \leq m! \, \mathrm{vol}_W,$$

with equality, if and only if, W is a complex subspace, i.e. $J(W) \subset W$. (4 marks)

(d) Let M be a Kahler manifold and ω its Kahler form. Show that ω^ℓ defines a non-zero homology class in $H_D^{\ell,\ell}(M)$, for all $1 \leq \ell \leq \dim M/2$. In particular, $\dim H_D^{\ell,\ell}(M) \neq 0$. (4 marks)

(e) Show that a complex submanifold of a Kahler manifold minimises the volume in its homology class. (4 marks)

(Total: 20 marks)

Solutions.

(4)(a) That the dimension is even follows from $0 \leq \det(J)^2 = \det(J^2) = \det(-id) = (-1)^{\dim V}$. Since V is real vector space, to see that it is a complex vector space, we just need to check $i(v + w) = J(v + w) = J(v) + J(w) = iv + iw$ and $i^2 v = i(J(v)) = J^2(v) = -v$.

(4)(b) The Hodge Decomposition $\Omega^n(M) = \mathrm{im}\, d_{n-1} \oplus \mathrm{im}\, \delta_{n+1} \oplus \mathcal{H}^n(M)$ is an orthogonal decomposition. Since $\Omega^{n+1}(M) = 0$, we have $\mathrm{im}\, \delta_{n+1} = 0$. So, $\mathrm{vol}(g)$ is harmonic if it is orthogonal to all exact forms. Notice $(\mathrm{vol}(g), d\psi) = \int_M d\psi \wedge * \mathrm{vol}(g) = \int_M d\psi = 0$.

(4)(c) The restrictions of g and ω to W are also symmetric (non-degenerate) and skew-symmetric bilinear forms, respectively. By the spectral theorem, there is an orthonormal basis $\{v_i, w_i\}_{i=0}^m$ of W, for which $\omega|_W = \sum_{i=1}^m \lambda_i dv_i \wedge dw_i$. Let $\pi : V \to W$ be the orthogonal projection. Since $\omega|_W(\cdot, \cdot) = g(J\cdot, \cdot)|_W = g(\pi \circ J\cdot, \cdot)$, it follows that the basis v_i, w_i is giving diagonal blocks for the linear map $\pi \circ J : W \to W$. In particular $|\lambda_i| \leq 1$ with equality if and only if $J(v_i) = w_i$ and $J(w_i) = -v_i$. Finally, $\omega_W^m = m!(\lambda_1 \cdots \lambda_m) \sum_{i=1}^m dv_i \wedge dw_i = m!(\lambda_1 \cdots \lambda_m) \mathrm{vol}_W$, which proves the claim.

(4)(d) The fundamental form ω is of type $(1, 1)$ and is closed since M is Kahler. Therefore, for every ℓ, the forms ω^ℓ are of type (ℓ, ℓ) and closed, so they define an element of $H^{p,p}(M)$. However, by the previous exercise ω^n is a multiple of the volume form of M, in particular $\int_M \omega^n \neq 0$ and ω^n is not exact, i.e. $0 \neq [\omega^n] = [\omega]^n$.

(4)(e) Let S and Σ be $2m$-dimensional submanifolds of a complex manifold. Assume that they are homologous i.e. $S - \Sigma = \partial U$ as singular cycles. Then $\int_S \omega^m = \int_\Sigma \omega^m + \int_U d\omega^m$. If S is complex and M is Kahler (i.e. $d\omega = 0$), then by part (c) we have $m!\,\mathrm{vol}(S) = \int_S \omega^m = \int_\Sigma \omega^m + \int_U d\omega^m \leq m!\,\mathrm{vol}(\Sigma)$.

5. (a) The objective of the following items is to show a $\bar{\partial}$-Poincaré lemma for $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, i.e. *given a smooth function* $g : \mathbb{C}^* \to \mathbb{C}$, *there exists a smooth function* $f : \mathbb{C}^* \to \mathbb{C}$ *with* $\partial f/\partial \bar{z} = g$.

   (i) Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of open annuli centered at 0 such that $\bar{A}_k \subset A_{k+1}$ and $\cup_k A_k = \mathbb{C}^*$. Show that for any $k \in \mathbb{N}$, there exists $h_k \in C^\infty(\mathbb{C})$ such that $\partial h_k/\partial \bar{z} = g$ on A_k.

   (4 marks)

   (ii) Assume that $f_{k-1} \in C^\infty(\mathbb{C})$ satisfies $\partial f_{k-1}/\partial \bar{z} = g$ on A_{k-1}. Show that for any $\varepsilon_k > 0$, there is a holomorphic function $\alpha_k \in C^\infty(\mathbb{C}^*)$, such that $|f_{k-1} - h_k - \alpha_k| < \varepsilon_k$ on A_{k-1}, where h_k is as in the previous item. (4 marks)

   (iii) Define a $f_k \in C^\infty(\mathbb{C}^*)$ recursively by $f_1 = h_1$ and $f_k = h_k + \alpha_k$, with $\varepsilon_k = 2^{-k}$. Show that f_k converges uniformly in compacts of $\mathbb{C}^*$ to a f as in the statement. (4 marks)

   (b) Conclude that the Dolbeault cohomology groups $H^{0,1}(\mathbb{C}^*)$ and $H^{1,1}(\mathbb{C}^*)$ are both zero.

   (4 marks)

   (c) Compute $H^{0,0}(\mathbb{C}^*)$ and $H^{1,0}(\mathbb{C}^*)$. (4 marks)

(Total: 20 marks)

Solutions.

(a)(i) For each k, let $\rho_k \in C^\infty(\mathbb{C})$ such that $\rho_k = 0$ in a small ball around the origin and $\rho_k = 1$ on A_{k+2}. Then, the function $g_k = \rho_k g \in C^\infty(\mathbb{C}^*)$ extends smoothly to 0, by defining $g_k(0) = 0$. By the Poincaré Lemma in one variable, there exists a smooth function h_k such that $\partial h_k/\partial \bar{z} = g_k$. Since $g_k = g$ on $A_k \subset A_{k+2}$, this is the function we were looking for.

(a)(ii) On A_{k-1}, we have $\partial/\partial \bar{z}(f_{k-1} - h_k) = 0$. It follows that $f_{k-1} - h_k$ is holomorphic on this annulus A_{k-1} so it has a Laurent expansion converging on this set. Any truncation α of the expansion gives us a holomorphic function on $\mathbb{C}^*$. Since the Laurent series converges towards $f_{k-1} - h_k$ we can choose to cut the expansion at any point where the diference is uniformly smaller than ε.

(a)(iii) Let $K \subset \mathbb{C}^*$ be a compact set. Then, $K \subset A_m$ for some large m. The expression $f_n = f_m + \sum_{i=m+1}^{n}(f_i - f_{i-1})$ is the sum of the function f_m, that solves our problem i.e., $\partial f_m/\partial \bar{z} = g$ on K, plus a series of harmonic functions. Moreover, the series of harmonic functions $\sum_{i=m+1}^{\infty}(f_i - f_{i-1})$ satisfies the conditions of Weirstrass M-test by part (ii) and our choice of $\varepsilon_k = 2^{-k}$. It follows that it converges uniformly and absolutely. The limit of a uniformly convergent sequence of holomorphic functions is holomorphic so $(\partial/\partial \bar{z})(\sum_{i=m+1}^{\infty}(f_i - f_{i-1})) = 0$. Therefore, $f = \lim_{n\to\infty} f_n = f_m + \sum_{i=m+1}^{\infty}(f_i - f_{i-1})$ is well defined and solves our problem.

(b) Given $\alpha = g d\bar{z}$, let f be such that $\partial f/\partial \bar{z} = g$, then $\alpha = \bar{\partial} f$. It follows that every $(0,1)$-form is exact. Similarly, given $\beta = g dz \wedge d\bar{z}$, we have $\beta = \bar{\partial}(-f dz)$, so again every $(1,1)$-form is exact.

(c) From part (c) and Hodge Theorem it follows that $H^{1,0}(\mathbb{C}^*) \simeq H^1(\mathbb{C}^*)$ and $H^{0,0}(\mathbb{C}^*) = H^0(\mathbb{C}^*)$. Since the singular homology is a topological invariant and $\mathbb{C}^*$ retracts into S^1, it follows that $H^{1,0}(\mathbb{C}^*) \simeq \mathbb{R}$ and $H^{0,0}(\mathbb{C}^*) \simeq \mathbb{R}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH97054MATH97165	1	This question was well answered in general. As a minor point, in part (b) some students forgot to inlcude the hypothesis of connectedness in their argument.
MATH97054MATH97165	2	Most people did not check all the conditions of the definition of a holomorphic vector bundle in part (a), e.g. that transition functions are holomorphic, i.e. at the intersection of two trivialisations.
MATH97054MATH97165	3	(a) was intented to be a short question. It was only necessary to state de Poincare lemma and use it directly. Several people chose to present part of the proof of the Poincare lemma, but this was not necessary. A few people proved (c) assuming the form was real, even though the formula from (b) holds for any form.
MATH97054MATH97165	4	I was happy to see parts (c) and (e) were in general well written. In part (a) some people seemed to assume that the space had an inner product compatible with the almost complex structure, but this was not part of the assumptions.
MATH97054MATH97165	5	In (ii) some people talked about a Taylor series (polynomial) when in reality the correct expansion for annuli is a Laurent series (it has finitely many terms of negative degree). To take the limit in (iii) it is important to look at the difference of consecutives elements of the sequence, because these are harmonic functions, some people did not mention that subtle point.