MATH60005/70005/97405: Optimisation (Spring 21-22)

Instructions: read this first!

- This coursework has a total of 20 marks and accounts for 10% of the module.
- Students who want to take the final exam (90%) must submit this coursework.
- **Submission deadline:** Monday February 28th, 13:00 UK time, via Blackboard drop box.
- Submit a single file, handwritten answers (whenever possible) are allowed but readability is essential and part of the assessment.
- **Marking criteria:** Full marks will be awarded for work that 1) is mathematically correct, 2) shows an understanding of material presented in lectures, 3) gives details of all calculations and reasoning, and 4) is presented in a logical and clear manner.
- Do not discuss your answers publicly via our forum. If you have any queries regarding your interpretation of the questions, please contact the lecturer at dkaliseb@imperial.ac.uk
- Beware of plagiarism regulations. This is an **individual assessment**.

Questions

1. **(6 marks)** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 - x_1^2 x_2^2 + x_2^4 \,.$$

   Find all the stationary points of f and classify them. Justify your answer.

2. Let $\mathbf{z} \in \mathbb{R}^n$ be a signal that is related to the input $\mathbf{u} \in \mathbb{R}^n$ by

$$z_k = \sum_{j=1}^{k} w_j u_{k-j+1}, \quad k = 1, 2, \ldots, n \,,$$

   where $\mathbf{w} \in \mathbb{R}^n$ is a given signal. The goal of this problem is to find an optimal input $\mathbf{u}^*$ to achieve different objectives:

   i) Tracking a reference signal $\bar{\mathbf{z}} \in \mathbb{R}^n$. The tracking error is defined as

$$f_t := \sum_{k=1}^{n} (z_k - \bar{z}_k)^2 \,.$$

ii) Minimizing the energy of the input signal, expressed as

$$f_e := \sum_{k=1}^{n} (u_k)^2 \,.$$

iii) A second-order regularizer of the input, formulated as

$$f_s := \sum_{k=2}^{n-1} (u_{k+1} - 2u_k + u_{k-1})^2 \,.$$

2.1 **(5 marks)** Formulate the joint minimization of f_t, f_e, and f_s as a regularized least squares problem for $\mathbf{u}$ of the form

$$\min_{\mathbf{u} \in \mathbb{R}^n} \left\{ \mathcal{J}(\mathbf{u}) := \|\mathbf{T}\mathbf{u} - \mathbf{b}\|^2 + \delta \|\mathbf{E}\mathbf{u}\|^2 + \eta \|\mathbf{S}\mathbf{u}\|^2 \right\} \,,$$

giving precise definitions for $\mathbf{T}$, $\mathbf{b}$, $\mathbf{E}$, and $\mathbf{S}$. Here $\mathbf{E}$ and $\mathbf{S}$ represent matrices related to f_e and f_s, respectively. The regularisation parameters δ and η are non-negative. Determine an explicit expression for the solution to this problem, and discuss its existence and uniqueness.

2.2 Taking $n = 200$, $\mathbf{w}$ given by

$$w_k = \begin{cases} 1 & k \leq 100 \,, \\ 0 & \text{otherwise} \end{cases}$$

and the reference $\bar{\mathbf{z}} \in \mathbb{R}^{200}$ given by

$$\bar{z}_k = \begin{cases} -1 & \text{for } 1 \leq k \leq 50 \,, \\ 0 & \text{for } 51 \leq k \leq 90 \,, \\ 1 & \text{for } 91 \leq k \leq 140 \,, \\ 0 & \text{for } 141 \leq k \leq 200 \,, \end{cases}$$

analyse the effect of the regularization parameters δ and η. For this, show and discuss[1] your observations for each of the following:

2.2.1 **(2 marks)** A plot of $\mathcal{J}(\mathbf{u}^*)$ versus δ from 0 to 10^5, taking $\eta = 0$. What is the asymptotic behaviour of $\mathbf{u}^*$ and $\mathcal{J}(\mathbf{u}^*)$ as δ grows?

2.2.2 **(2 marks)** A single plot illustrating the reference signal $\bar{\mathbf{z}}$, and reconstructed signals $\mathbf{z}^*$ for $\delta = 0$ and $\eta = 0, 1, 10, 100, 1000$. What is the effect of the regularizer? Explain why the reconstructed signal for $(\delta, \eta) = (0, 0)$ coincides exactly with the reference signal.

2.3 **(5 marks)** Create the noisy signal $\hat{\mathbf{y}} = \bar{\mathbf{z}} + \mathcal{N}$, where $\mathcal{N}$ is the noise sequence in the companion file `noise.mat`. Formulate the denoising problem for computing a denoised

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[1]discuss here means a brief analysis of 2-4 lines, explaining relevant observations, as in the caption of a figure. It is not an essay, but it should go beyond naming what's in the axes.

signal $\mathbf{z}$ using a term of the form f_s as a regularizer, and study the differences with the regularization given by

$$f_{tv} := \sum_{k=1}^{n-1} (z_{k+1} - z_k)^2 \,.$$

Compare your solutions against the original noisy signal for both regularizers separately, when the regularization parameter λ is set to $\lambda = 1$ and $\lambda = 100$. Derive an explicit gradient iteration for solving the regularisation problem if an exact line search is used, expressing your results depending on λ and a regularization matrix $\mathbf{R}$.