Imperial College London

MATH97051 MATH97162

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Manifolds

Date: Wednesday, 2 June 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism, which is not necessarily smooth.

   (i) Show that if $X = \mathbb{R}^n$ and $\mathcal{A} = \{(X, f)\}$, then $\mathcal{A}$ is a smooth atlas on X. (3 marks)

   (ii) Let X be the manifold defined in (i). Determine if X is diffeomorphic to $\mathbb{R}^n$ with the standard smooth structure (justify your answer). (4 marks)

(b) Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with quotient map $q\colon \mathbb{R}^2 \to X$. Determine for which $\alpha \in \mathbb{R}$, the following subset is a submanifold of X (justify your answer):

$$Y_\alpha := \{q(x, \alpha x) \mid x \in \mathbb{R}\} \subset X.$$

(4 marks)

(c) Determine which of the following subsets are submanifolds of $\mathbb{R}^3$ (justify your answer):

   (i) $\{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 = 1 \quad \text{and} \quad x = yz\} \subset \mathbb{R}^3$. (3 marks)

   (ii) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 = y^2 + z^2\} \subset \mathbb{R}^3$. (3 marks)

   (iii) $\{(x, y, z) \in \mathbb{R}^3 \mid x^3 - y^3 = 0\} \subset \mathbb{R}^3$. (3 marks)

(Total: 20 marks)

2. (a) (i) Let X be a manifold, let D be a derivation on X and let Y be a submanifold of X. Show that there exists a unique derivation $D'\colon C^\infty(Y) \to C^\infty(Y)$ such that

$$D(g)|_Y = D'(g|_Y)$$

for all smooth functions $g \in C^\infty(X)$ if and only if $D(f)|_Y = 0$ for all smooth functions $f \in C^\infty(X)$ such that $f|_Y = 0$. (6 marks)

(ii) Let $U \subset \mathbb{R}^2$ be an open subset and let D be a derivation on U.
Determine if there exists always a derivation D' on $\mathbb{R}^2$ such that

$$D'(g)|_U = D(g|_U)$$

for all smooth functions $g \in C^\infty(\mathbb{R}^2)$ (justify your answer). (4 marks)

(b) Let X be a manifold and let $D\colon C^\infty(X) \to C^\infty(X)$ be a derivation.

(i) Show that for all smooth functions f on X and for every positive integer k, we have

$$D(f^k) = kf^{k-1}D(f).$$

(4 marks)

(ii) Determine if $D^2 = D \circ D\colon C^\infty(X) \to C^\infty(X)$ is also a derivation on X (justify your answer).

(3 marks)

(c) Let X be a manifold and let $i\colon Y \hookrightarrow X$ be a submanifold. Let $V\colon Y \to TY$ be a vector field on Y. Show that there exists a vector field $W\colon X \to TX$ on X such that

$$Di|_x(V(x)) = W(x)$$

for all $x \in Y$. (3 marks)

(Total: 20 marks)

3. (a) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and let $X = S^2 \times S^2$. Show that the tangent bundle of X is not trivial. (4 marks)

(b) Recall that $X = \mathrm{SL}(2, \mathbb{R})$ is the manifold defined by the 2×2 matrices with determinant equal to one.

Determine if there exists a nowhere zero vector field on X (justify your answer). (4 marks)

(c) Show that the tangent bundle of the three-dimensional sphere $S^3 \subset \mathbb{R}^4$ is trivial. (4 marks)

(d) Let X be a manifold and let $\pi\colon T^*X \to X$ be its cotangent bundle. Let $s\colon X \to T^*X$ be a function such that $\pi \circ s = Id_X$. For all $x \in X$, denote by s_x the element in $\mathrm{Hom}(T_xX, \mathbb{R})$ defined by $s(x)$.

Show that s is a smooth 1-form if and only if, for all vector fields V on X, the function $f\colon X \to \mathbb{R}$ defined by $f(x) = s_x(V(x))$ is a smooth function. (4 marks)

(e) Let X be a compact manifold and let $f \in C^\infty(X)$ be a smooth function. Show that df vanishes in at least one point of X. (4 marks)

(Total: 20 marks)

4. (a) Let $X = S^2 \subset \mathbb{R}^3$ be the unit sphere and let $\omega = xdx \wedge dy + zdx \wedge dz \in \Omega^2(\mathbb{R}^3)$. Compute $\int_X \omega$. (4 marks)

(b) Show that the circle S^1 is orientable. (3 marks)

(c) Write down explicitly a volume form on S^2. (6 marks)

(d) Let $p \geq 1$ be an integer and let X be a non-compact manifold. Let ω be an exact p-form on X with compact support. Determine if there exists a $(p-1)$-form η with compact support and such that $\omega = d\eta$ (justify your answer). (5 marks)

(e) Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $i\colon S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion. Let ω be a closed 1-form on X with compact support. Show that

$$\int_{S^1} i^*\omega = 0.$$

(2 marks)

(Total: 20 marks)

5. (a) Let X be a manifold. Let $\epsilon > 0$ and let

$$F, G \colon (-\epsilon, \epsilon) \times X \to X$$

be two flows on X with associated vector fields V_1 and V_2 respectively.
Show that $[V_1, V_2] = 0$ if and only if $F_s \circ G_t = G_t \circ F_s$ for all $s, t \in (-\epsilon, \epsilon)$.

(4 marks)

(b) Let $F \colon X \to Y$ be a submersion. Show that there exists a distribution D on X such that for every $x \in X$, the fibre $Z_x := F^{-1}(F(x))$ is an integral submanifold of D such that $x \in Z_x$.

(3 marks)

(c) Let $X = \mathbb{R}^3$ and consider the vector fields

$$V = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} \qquad \text{and} \quad W = \frac{\partial}{\partial z}.$$

(i) Show that V and W span a distribution on X. (2 marks)

(ii) Find an integral submanifold of D passing through $(0, 0, 0)$. (3 marks)

(iii) Determine if D is involutive (justify your answer). (3 marks)

(d) (i) Let X be an n-dimensional manifold and let D be an involutive distribution of rank $n-1$ on X. Let ω be a 1-form such that $\omega(V) = 0$ for all the sections V of D. Show that $\omega \wedge d\omega = 0$. (3 marks)

(ii) Find a 1-form ω on a manifold X such that $\omega \wedge d\omega$ is not identically zero. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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M97051

Manifolds (Solutions)

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1. (a) (i) Since f is a homeomorphism, it follows that (X, f) is a chart on X, which clearly covers X. The only transition function on X is the identity, which is a smooth function. Thus $\mathcal{A}$ is an atlas. $\boxed{3, \text{A}}$

(ii) Let $Y = \mathbb{R}^n$ be the manifold with the standard smooth structure, i.e. it is defined by the chart (Y, g) where $g = id_Y$. We want to show that the function $F := f\colon X \to Y$ is a diffeomorphism with respect to the smooth structures on X and Y.

First note that

$$g \circ F \circ f^{-1} = id_{\mathbb{R}^n}$$

is a smooth function. Thus F is a smooth function. Similarly,

$$f \circ F^{-1} \circ g^{-1} = id_{\mathbb{R}^n}$$

is also a smooth function and therefore, F^{-1} is a smooth function. Thus F is a diffeomorphism. $\boxed{4, \text{B}}$

(b) Note that we may replace α by its fractional part (i.e. $\alpha - \lfloor \alpha \rfloor$) and we may assume that $\alpha \in [0, 1)$.

Assume that α is a rational number. We want to show that Y_α is a submanifold of X. Let $(x, y) \in [0, 1]^2$ such that $[(x, y)] \in Y_\alpha$. It is enough to show that Y_α is a submanifold of X around $[(x, y)]$. After translating, we may assume without loss of generality that $(x, y) \in (0, 1)^2$. Note that the segment $[0, 1] \times \{y\}$ intersects $q^{-1}(Y_\alpha)$ in finitely many points. Thus, there exists an open subset $V \in (0, 1)^2$ containing (x, y) such that $q^{-1}(Y_\alpha) \cap V = V \cap A$ where A is an affine subspace of $\mathbb{R}^2$. Thus, the claim follow.

Assume now that $\alpha \notin \mathbb{Q}$. We want to show that Y_α is not a submanifold of X. Note that if $m, n \in \mathbb{Z}$ then $\alpha m - \alpha n \in \mathbb{Z}$ if and only if $m = n$. Thus the set $Y_\alpha \cap q(\{0\} \times [0, 1])$ is a finite countable set. In particular, it admits an accumulation point $p \in Y_\alpha$. It follows that no open subset of X containing p intersects Y_α in a set homeomorphic to an open subset of $\mathbb{R}$. Thus, our claim follows. $\boxed{4, \text{D}}$

(c) (i) Consider the function

$$F\colon \mathbb{R}^3 \to \mathbb{R}^2 \qquad (x, y, z) \mapsto (x^4 + y^4 + z^4 - 1, x - yz).$$

We want to show that $(0, 0)$ is a regular value and therefore the subset is a submanifold of $\mathbb{R}^3$.

We have that

$$DF = \begin{pmatrix} 3x^3 & 3y^3 & 3z^3 \\ 1 & z & y \end{pmatrix}$$

Note that if $y^4 \neq z^4$ then the minor containing the second and third column has non zero determinant. Thus, we may assume that $y = \pm z$. In particular, if $x = yz$ then $x = \pm y^2$, which easily implies that if the rank of DF is not two, then $x = y = z = 0$, and in particular $x^4 + y^4 + z^4 - 1 \neq 0$. Thus, our claim follows. $\boxed{3, \text{A}}$

(ii) If the subset is a submanifold, then it is a manifold of dimension two and in particular there exists a chart (U, f) containing the origin $(0, 0, 0)$ where U is connected and $f\colon U \to f(U) \subset \mathbb{R}^2$ is a homeomorphism. Note that $U \setminus \{(0, 0, 0)\}$ is a disconnected set. Indeed $U \cap \{x > 0\}$ and $U \cap \{x < 0\}$ are the two connected components. But there is no connected open subset in $\mathbb{R}^2$ which can be disconnected by removing a point. Thus, it is not a submanifold. $\boxed{3, \mathrm{C}}$

(iii) The subset coincides with $\{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\}$ which is an hyperplane and, therefore, a submanifold of $\mathbb{R}^3$. $\boxed{3, \mathrm{A}}$

2. (a) (i) First assume that such a D' exists. Then if f is a smooth function on X such that $f|_Y = 0$, we have

$$D(f)|_Y = D'(f|_Y) = D'(0) = 0.$$

Now assume that $D(f)|_Y = 0$ for all f as above. First, we want to show that if $h \in C^\infty(Y)$ then there exists $\tilde{h} \in C^\infty(X)$ such that $\tilde{h}|_Y = h$. We want to show that for all $p \in Y$, there exists an open subset $U_p \subset X$ containing p such that if h_p is a smooth function on $U_p \cap Y$ then there exists a smooth function $\tilde{h}_p$ on U_p which restricts to h_p. Indeed, by taking a chart, we may assume that U is an open subset of $\mathbb{R}^n$ and $Y = A \cap U$ where $A = \{x_1 = \cdots = x_k = 0\}$ is the standard linear subspace. In this case, the function $h_p(x_{k+1}, \ldots, x_n)$ can be extended as $\tilde{h}_p(x_1, \ldots, x_n) = h_p(x_{k+1}, \ldots, x_n)$. By considering a partition of the unity relative to an open covering induced by the open subsets U_p, it follows easily that we can define the function $\tilde{h}$ as above.

Now, we can define

$$D'(h) = D(\tilde{h}).$$

Note that, by assumption, $D'(h)$ does not depend on the choice of the extension and therefore D' is well-defined. Moreover, it follows easily that D' is linear over $\mathbb{R}$ and it satisfies the Leibniz rule. Thus, it is a derivation. Furthermore, if g is a smooth function on X, then g is an extension of $g|_Y$ and therefore

$$D(g)|_Y = D'(g|_Y)$$

as requested.

Finally, if D'' is also a derivation satisfying the required properties, then we have

$$D''(h) = D''(\tilde{h}|_Y) = D(\tilde{h})|_Y = D'(h).$$

Thus, $D'' = D'$. $\boxed{6, \mathrm{D}}$

(ii) Let $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ and consider the derivation

$$D = \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x}.$$

Assume that there exists D' as claimed and let $g = x$. Then

$$D'(x)|_U = D(x|_U) = \frac{1}{x^2 + y^2}$$

but $D'(x)$ is a smooth function on $\mathbb{R}^2$, a contradiction. $\boxed{4, \mathrm{B}}$

(b) (i) We may proceed by induction on k. The claim is easily shown to be true for $k = 1$. Let $k > 1$. By the Leibniz rule, we have

$$D(f^k) = f^{k-1}D(f)+fD(f^{k-1}) = f^{k-1}D(f)+(k-1)f\cdot f^{k-2}D(f) = kf^{k-1}D(f).$$

Thus, the claim follows. $\boxed{4, \text{A}}$

(ii) In general, $D \circ D$ is not a derivation. For example let $X = \mathbb{R}$ and let $D = \frac{\partial}{\partial x}$. Then $D^2(x) = D(1) = 0$ but $D^2(x^2) = D(x) = 1$. Thus (i) above does not hold. $\boxed{3, \text{B}}$

(c) We first prove the result assuming that X is an open subset of $\mathbb{R}^n$ and $Y = A \cap X$ where $A = \{x_1 = \cdots = x_k = 0\}$ is the standard affine subspace. In this case, TX is the trivial bundle $X \times \mathbb{R}^n$ and similarly $TA = A \times \mathbb{R}^{n-k}$. Thus, the vector field V is defined by a smooth function $\tilde{V}\colon A \to \mathbb{R}^{n-k}$ so that

$$V(x_{k+1},\ldots,x_n) = ((x_{k+1},\ldots,x_n), \tilde{V}(x_{k+1},\ldots,x_n)).$$

Let $\tilde{V}_1,\ldots,\tilde{V}_{n-k}$ be the components of $\tilde{V}$. Then, it is enough to define

$$W(x_1,\ldots,x_n) = ((x_1,\ldots,x_n),(0,\ldots,0,\tilde{V}_1(x_{k+1},\ldots,x_n),\ldots,\tilde{V}_{n-k}(x_{k+1},\ldots,x_n))$$

and the claim holds.

We now prove the general case. Let $\{(U_i, f_i)\}$ be an atlas of X such that, for each i, we have that $f_i(U_i \cap Y) = f_i(U_i) \cap A$ where A is the affine subspace defined above. Then using the construction above, we may find vector fields W_i on U_i extending $V|_{U_i \cap Y}$. Let $h_i\colon X \to [0, 1]$ be a partition of the unity with respect to the open covering $\{U_i\}$. Then $h_i W_i$ extends to a smooth vector field $\tilde{W}_i$ on X which is zero outside U_i and such that $W := \sum \tilde{W}_i$ is a smooth vector field on X with the desired property. $\boxed{3, \text{C}}$

3. (a) Assume that the tangent bundle of X is trivial. Then we may find a vector field V on X which is nowhere zero. We showed that there exists a natural diffeomorphism $TX = TS^2 \times TS^2$. Let $\pi\colon TX \to TS^2$ be the projection to the first factor and pick $z \in S^2$. Then

$$x \in S^2 \mapsto \pi \circ V(x, z) \in TS^2$$

defines a vector field on S^2 which is never zero, a contradiction. $\boxed{4, \text{A}}$

(b) The manifold X can be considered as the submanifold of $\mathbb{R}^4$ defined by

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid xw - yz - 1 = 0\}.$$

Thus, the tangent space of X at the point (a, b, c, d) coincides with the linear subspace

$$\{(x, y, z, w) \in \mathbb{R}^4 \mid dx - cy - bz + aw = 0\}.$$

In particular the function

$$(a, b, c, d) \mapsto (a, b, -c, -d)$$

defines a nowhere zero vector field on X. $\boxed{4, \text{C}}$

(c) The sphere

$$X = S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2-1=0\}$$

has tangent space at the point (a,b,c,d) given by the linear subspace

$$\{(x,y,z,w) \in \mathbb{R}^4 \mid ax+by+cz+dw=0\}.$$

Consider the vector fields V_1, V_2, V_3 on X given by the functions

$$(a,b,c,d) \mapsto (b,-a,d,-c)$$

$$(a,b,c,d) \mapsto (c,-d,-a,b)$$

$$(a,b,c,d) \mapsto (d,c,-b,-a)$$

We want to show that at every point $(a,b,c,d) \in S^3$ the three vector fields are linearly independent. Assume first that $d \neq 0$. Then the determinant of the minor

$$\begin{pmatrix} b & -a & d \\ c & d & -a \\ d & c & -b \end{pmatrix}$$

is $d(a^2+b^2+c^2+d^2) = d \neq 0$. Similarly, it follows that if $a \neq 0$, $b \neq 0$ or $c \neq 0$ then the matrix above has rank three on every point of the sphere. Thus, the tangent bundle of S^3 is trivial. $\boxed{4, \mathrm{A}}$

(d) Since smoothness is a local property, it is enough to solve the problem in a neighbourhood U of x, for any point $x \in X$. Thus, by taking a chart around x we may assume that X is an open subset of $\mathbb{R}^n$. Assume that s is smooth. In local co-ordinates, we may write

$$V = \sum v_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad s = \sum g_i dx_i$$

where $v_1, \dots, v_n, g_1, \dots, g_n$ are smooth on X. Then we have

$$f(x) = \sum_{i=1}^{n} v_i g_i$$

which is also a smooth function on X. Thus, the claim follows.

Assume now that for any vector field V on X, the function $f \colon X \to \mathbb{R}$ defined by $f(x) = s_x(V(x))$ is a smooth function. Similarly as above, we may write $s = \sum g_i dx_i$ where $g_1, \dots, g_n$ are functions. We want to show that each g_i is smooth. Consider the vector field $V = \frac{\partial}{\partial x_i}$. Then, by assumption, $g_i(x) = s_x(V(x))$ is a smooth function, and the claim follows. $\boxed{4, \mathrm{A}}$

(e) Since X is compact, the function f admits a minimum at a point $x \in X$. Let (U,g) be a chart containing x. Then the function $\tilde{f} := f \circ g^{-1} = 0$ admits a minimum at the point $g(x)$ and in particular the rank of the function $\tilde{f}$ is zero at the point $g(x)$. Thus, the rank of f at x is zero, i.e. df vanishes at that point. $\boxed{4, \mathrm{B}}$

4. (a) We may write

$$\omega = d\left(\frac{1}{2}x^2dy - \frac{1}{2}z^2dx\right).$$

Thus ω is exact and since S^2 is a compact manifold without border, Stokes' Theorem implies that $\int_{S^2}\omega = 0$. $\boxed{4, \text{A}}$

(b) Consider the the inclusion $i\colon S^1 \to \mathbb{R}^2$ as the circle of radius one. Let

$$\omega = i^*(x_2dx_1 - x_1dx_2).$$

We want to show that ω is a volume form. Consider the vector field $V = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$. Then for all $x = (x_1, x_2) \in S^1$, we have $\omega(V)(x) = x_2^2 + x_1^2 = 1$. Thus ω is no-where zero and, therefore, it is a volume form. It follows that S^1 is orientable. $\boxed{3, \text{A}}$

(c) Consider the inclusion $i\colon S^2 \to \mathbb{R}^3$ as the sphere of radius one. Let

$$\omega = i^*(x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2).$$

We want to show that ω is a volume form on S^2. Consider the vector fields:

$$V_1 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} \qquad V_2 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3}.$$

Then for any $x = (x_1, x_2, x_3) \in S^2$, we have

$$\omega(V_1, V_2)(x) = x_3^3 + x_2^2x_3 + x_3x_1^2 = x_3.$$

Thus, ω is non-zero if $x_3 \neq 0$. By symmetry, it follows easily that ω is no-where zero and therefore it is a volume form. $\boxed{6, \text{D}}$

(d) In general, there is no such η. For example, let $X = \mathbb{R}$ and let f be a smooth function such that $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x > 1$. Let $\omega = df$. Then ω is an exact 1-form which has support contained in the interval $[0, 1]$. In particular, ω has compact support. Assume that there exists η with compact support and such that $\omega = d\eta$. Let $g = f - \eta$. Then $dg = 0$ and, in particular, g is a constant function, such that $g(x) = 0$ for x sufficiently negative and $g(x) = 1$ for x sufficiently large, a contradiction. $\boxed{5, \text{B}}$

(e) Let C be a circle inside X with centre at the origin and with radius sufficiently large so that ω is zero along C. Consider the manifold $Y \subset X$ whose boundary is $C \cup S^1$. Then, by Stokes' Theorem, we have

$$\int_{S^1}\omega = \int_{S^1}\omega - \int_C\omega = -\int_{\partial Y}\omega = -\int_Y d\omega = 0,$$

where the sign is due to the choice of the orientation on ∂Y. Thus, the result follows. $\boxed{2, \text{C}}$

5. (a) Assume that $F_s \circ G_t = G_t \circ F_s$ for all $s, t \in (-\epsilon, \epsilon)$. Then, for all smooth function f on X we have

$$\begin{aligned} V_1(V_2(f))(x) &= \frac{\partial}{\partial s}(V_2(f) \circ F(s,x))|_{s=0} \\ &= \frac{\partial}{\partial s}(\frac{\partial}{\partial t}(f \circ G_t \circ F_s(x))|_{t=0})|_{s=0} \\ &= \frac{\partial}{\partial t}(\frac{\partial}{\partial s}(f \circ F_s \circ G_t(x))|_{s=0})|_{t=0} \\ &= \frac{\partial}{\partial t}(V_1(f) \circ G(t,x))|_{t=0} = V_2(V_1(f))(x). \end{aligned}$$

Thus, $V_1(V_2(f)) - V_2(V_1(f)) = 0$ and, in particular, $[V_1, V_2] = 0$.

Assume now that $[V_1, V_2] = 0$. Then, as above, it follows that

$$\frac{\partial}{\partial s}(\frac{\partial}{\partial t}(f \circ G_t \circ F_s(x))|_{t=0})|_{s=0} = \frac{\partial}{\partial t}(\frac{\partial}{\partial s}(f \circ F_s \circ G_t(x))|_{s=0})|_{t=0}$$

for all smooth functions f on X. Since $F_0(x) = G_0(x) = x$, it follows that $F_s \circ G_t = G_t \circ F_s$ for all $s, t \in (-\epsilon, \epsilon)$. $\boxed{4, \mathrm{M}}$

(b) By assumption, for all $x \in X$, we have that $F(x)$ is a regular value for F and, in particular, $Z_x := F^{-1}(F(x))$ is a submanifold of X whose tangent space at x is

$$T_x Z_x = \mathrm{Ker}[DF|_x \colon T_x X \to T_{F(x)} Y].$$

Thus,

$$D := \bigcup_{x \in X} \mathrm{Ker}[DF|_x \colon T_x X \to T_{F(x)} Y]$$

is a distribution on X which satisfies the required properties. $\boxed{3, \mathrm{M}}$

(c) (i) For any $(x, y, z) \in \mathbb{R}^3$, we have that $V(x, y, z)$ and $W(x, y, z)$ are linearly independent vectors in $T_{(x,y,z)}\mathbb{R}^3$. Thus, they define a subbundle D of $T\mathbb{R}^3$ or rank two, i.e. a distribution. $\boxed{2, \mathrm{M}}$

(ii) Let $Z = \{(x, 0, z) \mid x, z \in \mathbb{R}\} \subset \mathbb{R}^3$. Then Z is an integral submanifold of D passing through $(0, 0, 0)$. $\boxed{3, \mathrm{M}}$

(iii) Let $f(x, y, z) = y$. Then

$$[V, W](f) = V(W(f)) - W(V(f)) = 0 - y = -y.$$

Thus D is not involutive. $\boxed{3, \mathrm{M}}$

(d) (i) It is enough to prove the claim locally around any point of $x \in X$. Thus, we may assume that X is an open subset of $\mathbb{R}^n$ and, by Frobenius Theorem, we may assume that D is integral, i.e there exist, co-ordinates $x_1, \dots, x_n$ such that D is spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}$. In particular, we must have $\omega = f dx_n$ where f is a smooth function.

Thus, we have

$$\omega \wedge d\omega = f dx_n \wedge (\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} dx_i \wedge dx_n) = 0.$$

(ii) Let $X = \mathbb{R}^3$ and let $\omega = x dy + dz$. Then $\boxed{3, \mathrm{M}}$

$$\omega \wedge d\omega = (x dy + dz) \wedge (dx \wedge dy) = -dx \wedge dy \wedge dz.$$

$\boxed{2, \mathrm{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97051MATH97162	1	Many students got this question right. On the other hand, very often, not many details were given in the solution of question b). It is not enough to say that if alpha is rational then the subset is a manifold because it looks like a line.
MATH97051MATH97162	2	This was a technical questions (mainly part a) and c)). A common mistake was to define a derivation or a vector field only along the subvariety Y of X. Instead it is important to show that such an object can be extended to the whole manifold X.
MATH97051MATH97162	3	A common mistake here was to consider the integral of the differential of a funtion defined on a manifold. This is a one-form. Thus, unless the manifold is of dimension one, such an integral is not well defined. We can only integrate top forms. Similarly, it is not enough to show that the integral of a top form is not zero to conclude that the form is non-vanishing.
MATH97051MATH97162	4	Many students got this question right. Students were asked to write down a volume form on the sphere and many got it right, but not everyone was able to justify their answer, maybe because of a lack of time.
MATH97051MATH97162	5	This was probably the hardest questions, but most of the students got at least part of it right. I did not see any particular common mistake.