Imperial College London

Department of Mathematics
Imperial College London
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Partial Differential Equations in Action **MATH50008**

Problem Sheet 5

1. ✦ In this question, we consider the following wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < \pi$$

Use the method of separation of variables to solve the equation with Dirichlet boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0, \quad \text{for all } t > 0$$

and initial conditions

$$u(x,0) = \sin x + 2\sin 7x \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for all } 0 \le x \le \pi$$

2. ✦ If the function $y(x,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad t > 0, 0 < x < \infty$$

and is subject to the following initial and boundary conditions

$$\frac{\partial y}{\partial x}(0,t) = 0, \quad \text{for } t \ge 0,$$
$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad \text{for } 0 < x < \infty$$

Express the solution to this problem using an appropriate Fourier transform.

3. ✦✦ It can be shown that the small vertical vibrations of a uniform beam are governed by the following fourth-order PDE

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

Seek a separable solution of the form $u(x,t) = X(x)T(t)$ and show that

$$X(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$
$$T(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$$

where A, B, C, D, a, b, β are arbitrary constants. Using this result, find the solution subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(0,t) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \quad \text{for all } t > 0$$

and the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for } 0 < x < L$$

Find the dependence on n of the frequency for the n^{th} mode. How does this compare with that for the wave equation in Q1?

4. ✦✦ Use an energy argument to show that the solution to the 1D wave equation with homogeneous Dirichlet boundary conditions and arbitrary initial conditions is unique.

5. ✦✦✦ The wave equation describing the transverse vibrations $u(x, y, t)$ of a stretched membrane under tension τ and having a uniform surface density ρ is

$$c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}$$

with $c^2 = \tau/\rho$. Find a separable solution appropriate to a stretched membrane clamped on a frame of length L and width H. Further, show that the natural angular frequencies of such a membrane are given by

$$\omega_{nm}^2 = \frac{\pi^2 \tau}{\rho} \left(\frac{n^2}{L^2} + \frac{m^2}{H^2} \right)$$

where $(n, m) \in \mathbb{N}^*$.

[Hint: you should proceed to two separation of variables and determine two separation constants. You may want to first separate space and time variables.]

6. ✦✦ Shrödinger's equation for a free non-relativistic particle can be written

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = i\hbar \frac{\partial \psi}{\partial t}$$

where $\psi(\mathbf{r}, t)$ is the quantum mechanical wavefunction of a particle of mass m.

(a) Find a solution, separable in the four independent variables, that can be written in the form of a plane wave,

$$\psi(x, y, z, t) = A \exp\left[i(\mathbf{k} \cdot \mathbf{r} - \omega t) \right]$$

Using the relationships from de Broglie ($\mathbf{p} = \hbar \mathbf{k}$) and Einstein ($E = \hbar \omega$), show that the separation constants must be such that

$$p_x^2 + p_y^2 + p_z^2 = 2mE$$

where $\mathbf{p}$ is the particle momentum, m is its mass and E is its energy.

(b) Obtain a different separable solution describing a particle confined to a box of side a (i.e. ψ must vanish at the walls of the box). Show that the energy of the particle can then only take the quantised values

$$E = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2)$$

where n_x, n_y and n_z are integers.

7. ✦✦✦ Let's talk about the design of guitars! A guitar is a fretted musical instrument. Like most western fretted instruments, it is designed to play the so-called twelve-tone equal-tempered scale. In the twelve-tone equal-tempered scale, an octave (i.e. the interval between one musical pitch and another with double its frequency) is divided in twelve equally tempered intervals, which means twelve intervals equally spaced on a logarithmic scale. These intervals are called semitones or half-steps.

(a) How should you place the frets on a guitar?

[Hint: Think about the frequencies of the normal modes of the guitar string.]

(b) When playing the guitar, does it matter at which location on a string you pluck it?

[Hint: You should first transform this question in a PDE problem; think about what plucking a string means in terms of initial conditions.]