3 Sequences

A sequence $(a_n)_{n\geq 1}$ of real (or complex, etc.) numbers is an infinite list of numbers $a_1, a_2, a_3, \ldots$ all in $\mathbb{R}$ (or $\mathbb{C}$, etc.) Formally:

Definition. A *sequence* is a function $a : \mathbb{N}_{>0} \to \mathbb{R}$.

Notation: We let $a_n \in \mathbb{R}$ denote $a(n)$ for $n \in \mathbb{N}_{>0}$. The data $(a_n)_{n=1,2,\ldots}$ is equivalent to the function $a : \mathbb{N}_{>0} \to \mathbb{R}$ because a function a is determined by its values a_n over all $n \in \mathbb{N}_{>0}$.
We will denote a by $a_1, a_2, a_3, \ldots$ or $(a_n)_{n\in\mathbb{N}_{>0}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark 3.1. a_is could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}_{>0}\} \subset \mathbb{R}$. E.g. $(a_n) = 1, 0, 1, 0, \ldots$ is different from $(b_n) = 1, 0, 0, 1, 0, 0, 1, \ldots$ This is why we use round brackets ( ) instead of { }.

We can describe a sequence in many ways,

- As a **list** $1, 0, 1, 0, \ldots,$
- Via a **closed formula**, like $a_n = \frac{1-(-1)^n}{2}$ for the sequence above,
- By a **recursion**, e.g. the Fibonacci sequence $F_1 = 1 = F_2$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ (so (F_n) is $1, 1, 2, 3, 5, 8, 13, \ldots$)
- By a summation, e.g. $a_n = \sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Such a sequence $a_n = \sum_{i=1}^n b_n$ is called a **series** and will be studied later in the course.

Notice a_n is **not** $\frac{1}{n}$.

Exercise 3.2. Show any sequence (a_n) can be written as a series $a_n = \sum_{i=1}^n b_i$ for an appropriate choice of sequence (b_n).

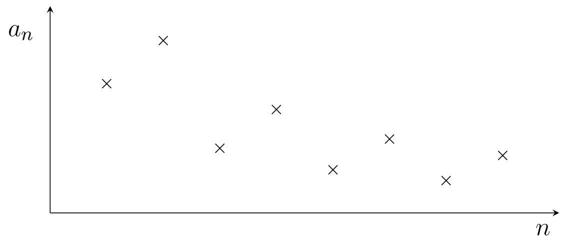
So why study series a_n? Sometimes the associated sequence b_n has nicer properties.

3.1 Convergence of Sequences

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)". We will spend a while exploring various formulations before we choose our definitive definition.

Idea 1: a_n should get closer and closer to a. Not necessarily monotonically, e.g. for:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0.$$



Idea 2: But notice that $\frac{1}{n}$ also gets closer and closer to -73.6! So we want to say instead that a_n gets "*as close as we like to* a" or "*arbitrarily close to* a". We will measure this with $\epsilon > 0$: we say a_n gets to within ϵ of a by

$$|a_n - a| \ < \ \epsilon \qquad \text{or} \qquad a_n \in \ (a - \epsilon, a + \epsilon).$$

We phrase "a_n gets *arbitrarily* close to a" by "*a_n gets to within ϵ of a for* ***any*** $\epsilon > 0$". This suggests the following definition.

Exercise 3.3. Dedekind tries to define $a_n \to a$ if and only if $\forall n$ sufficiently large, $|a_n - a|$ is *arbitrarily small.* When pushed they define a real number $b \in \mathbb{R}$ to be arbitrarily small if it is smaller than any $\epsilon > 0$ i.e. $\forall \epsilon > 0, \ |b| < \epsilon$.

Leaving aside what he means by "sufficiently large" for now, which of these sequences converges (to some $a \in \mathbb{R}$) according to their definition?

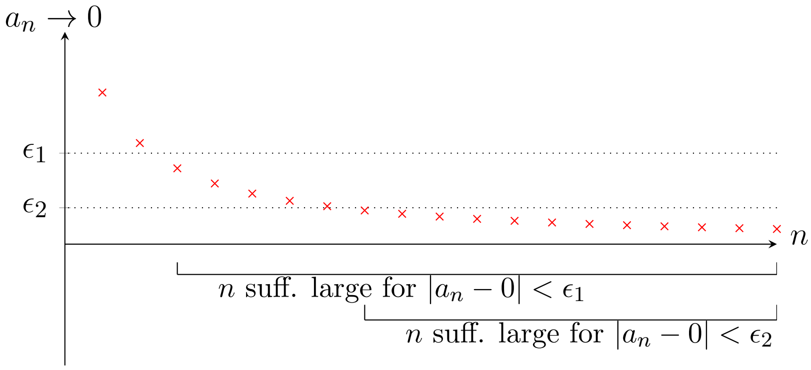
1. 0, 1, 0, 1, . . .
2. 1, 1, 1, 1, . . . ✓
3. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . .
4. $a_n = 2^{-n}$
5. More than one of these
6. None of these

Notice his definition of b being "arbitrarily small" means $b = 0$. (Proof: if $b \neq 0$ then $\exists\, \epsilon := \frac{|b|}{2} > 0$ such that $|b| \not< \epsilon$ so b is not arbitrarily small.)

So for Dedekind, $a_n \to a$ if and only if $a_n = a$ for all n sufficiently large.

Idea 3: Dedekind said that once n is large enough, $|a_n - a|$ is less than every $\epsilon > 0$, but that means it's zero, i.e. $a_n = a$. The problem they missed is that if we take smaller ϵ we will usually have to take bigger n to make $|a_n - a| < \epsilon$.

So we want to say that to get *arbitrarily close to the limit* a (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. If I change $\epsilon > 0$ to be smaller, I may have to go further down the sequence to get within ϵ of a.



Don't fall for the same trap as Dedekind - there will not be a "n sufficiently large" that works for all ϵ at once! (Unless $a_n \equiv a$ eventually.)

That is, we want to *reverse* the order of specifying n and ϵ: only once we've seen how small ϵ is do we know how big to take n. If we chose a smaller ϵ we can then choose a larger n.

For *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon > 0$" to "$\forall \epsilon > 0$, $\exists n$". *This allows n to depend on ϵ.*

Exercise 3.4. Dedekind takes your point, and modifies his definition of $a_n \to a$ to

$$\forall \epsilon > 0 \; \exists n \in \mathbb{N}_{>0} \text{ such that } |a_n - a| < \epsilon.$$

Which of these sequences converges to $a = 0$ according to his new definition?

1. 0, 1, 0, 1, ... ✓
2. 1, 1, 1, 1, ...
3. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... ✓
4. $a_n = 2^{-n}$ ✓
5. More than one of these ✓
6. None of these

Sequences 1, 3 and 4 all converge to 0 according to this definition, but we really don't want 1 to converge. We do want $|a_n - a| < \epsilon$ eventually, but we also want it to *stay there!*

Idea 4: So we measure "*eventually*" (or "sufficiently large") by a point $N \in \mathbb{N}_{>0}$ beyond which ("$\forall n \geq N$") a_n **stays** within ϵ of a. That is

Definition (Convergence)

We say that $a_n \to a$ as $n \to \infty$ if and only if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \; |a_n - a| < \epsilon.$$

Read this as follows:

However close ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}_{>0}$) beyond which ($n \geq N$) *all* a_n are indeed that close to the limit a ($|a_n - a| < \epsilon$).

Remark 3.5. N depends on ϵ! For a while we will sometimes denote it N_ϵ, as a reminder. We often write ($a_n \to a$ as $n \to \infty$) as just ($a_n \to a$) or ($\lim_{n\to\infty} a_n = a$).

Equivalently:

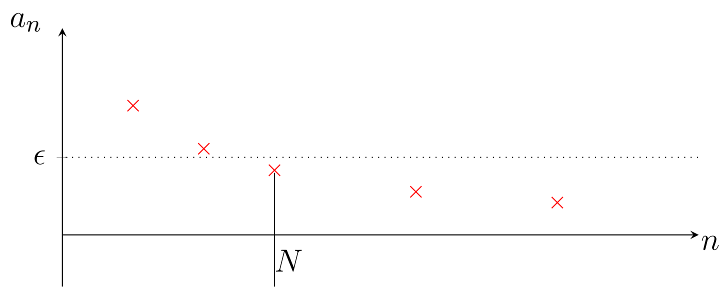
$$\forall \epsilon > 0 \; \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } \big[n \geq N_\epsilon \implies |a_n - a| < \epsilon \big]$$

or equivalently

$$\forall \epsilon > 0, \; \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } |a_n - a| < \epsilon \;\; \forall n \geq N_\epsilon.$$

Example 3.6. Prove $\frac{1}{n} \to 0$ as $n \to \infty$.

Rough working: Fix $\epsilon > 0$. I want to find $N_\epsilon \in \mathbb{N}_{>0}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N_\epsilon$.



Since this is equivalent to $n > \epsilon^{-1}$ then it is enough to take $N_\epsilon > \epsilon^{-1}$, which we know exists by the Archimedean axiom (e.g. $N_\epsilon = \lfloor \epsilon^{-1} \rfloor + 1$). So now the formal proof runs as follows:

Proof. Fix $\epsilon > 0$. Pick any $N_\epsilon \in \mathbb{N}_{>0}$ such that $N_\epsilon > \frac{1}{\epsilon}$. Then $n \geq N_\epsilon \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N_\epsilon} < \epsilon$. □

How to prove $a_n \to a$

$$\boxed{\forall \epsilon > 0 \; \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } |a_n - a| < \epsilon \; \forall n \geq N_\epsilon}$$

(I) Fix $\epsilon > 0$.

(II) Calculate $|a_n - a|$.

(II′) Find a good estimate $|a_n - a| \leq b_n$.

(III) Try to solve $b_n < \epsilon$. $(*)$

(IV) Find $N_\epsilon \in \mathbb{N}_{>0}$ such that $(*)$ holds whenever $n \geq N_\epsilon$.

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see examples below).

Notice you only have to do this for **one** $\epsilon > 0$, so long as it is arbitrary; that way you've done it for **any** $\epsilon > 0$.

The key point is to choose b_n so that solving $b_n < \epsilon$ is easier than solving $|a_n - a| < \epsilon$.

Example 3.7. Prove that $a_n = \dfrac{n+5}{n+1} \to 1$.

Point out the steps I-V in this example

Rough working:

$$|a_n - 1| \;=\; \left|\frac{n+5}{n+1} - 1\right| \;=\; \frac{4}{n+1} \;<\; \frac{4}{n}\,.$$

This is $< \epsilon \iff n > 4/\epsilon$, so take $N \geq 4/\epsilon$.

Proof. Fix $\epsilon > 0$. Pick N such that $N \geq 4/\epsilon$. Then $\forall n \geq N$,

$$|a_n - 1| \;=\; \frac{4}{n+1} \;\leq\; \frac{4}{N+1} \;<\; \frac{4}{N} \;\leq\; \epsilon.$$ □

Example 3.8. Prove that $a_n = \dfrac{n+2}{|n-2|} \to 1$.

You'll get $\implies$, $\impliedby$ and $<$, $>$ the wrong way round here

Rough working: We assume $n > 2$ so we can drop the absolute value, this is okay since we can always choose $N_\epsilon > 2$. We have

$$|a_n - 1| \;=\; \left|\frac{n+2}{n-2} - 1\right| \;=\; \frac{4}{n-2}\,.$$

We *want* $\frac{4}{n-2} < \epsilon$, so we want implications in the $\Longleftarrow$ direction
(i.e. $\frac{4}{n-2} < \epsilon \Longleftarrow n \geq N$)
not the $\Longrightarrow$ direction
(i.e. the fact that $\frac{4}{n-2} < \epsilon \Longrightarrow \frac{4}{n} < \epsilon$ *is of no use to us*).
[Notice the importance of the direction of implications!]

So we need something *bigger* than $\frac{4}{n-2}$, i.e. an estimate $\frac{4}{n-2} < b_n$ for which it is easier to solve $b_n < \epsilon$. So we make the denominator *smaller.*

To make $n-2$ smaller, make 2 bigger! e.g. $2 < \frac{n}{2}$ for $n > 4$. Then $\frac{4}{n-2} < \frac{4}{n-n/2} = \frac{8}{n}$.

As well as $n > 4$ we also want $b_n = \frac{8}{n} < \epsilon \Longleftrightarrow n > \frac{8}{\epsilon}$. So take $N_\epsilon > \max(4, 8/\epsilon)$.
(Notice using $2 < n$ here would ruined everything.)

Proof. Fix $\epsilon > 0$. Choose $N_\epsilon \in \mathbb{N}$ such that $N_\epsilon > \max(4, 8/\epsilon)$. Then $n \geq N_\epsilon \Longrightarrow n > \frac{8}{\epsilon}$ $(*)$ and $n > 4$ $(\dagger)$

$$\Longrightarrow \left|\frac{n+2}{n-2} - 1\right| \;=\; \frac{4}{n-2} \;\overset{(\dagger)}{<}\; \frac{4}{n-n/2} \;=\; \frac{8}{n} \;\overset{(*)}{<}\; \epsilon. \qquad \square$$

Definition. We say that a_n *converges* if and only if $\exists a \in \mathbb{R}$ such that $a_n \to a$, i.e.

$$\boxed{\exists a \text{ such that } \forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \Longrightarrow |a_n - a| < \epsilon.}$$

Negating the above statement gives the following

Definition. We say a_n *diverges* if and only if it does not converge (to any $a \in \mathbb{R}$), i.e.

$$\boxed{\forall a\ \exists \epsilon > 0 \text{ such that } \forall N \in \mathbb{N}_{>0},\ \exists n \geq N \text{ such that } |a_n - a| \geq \epsilon.}$$

Unpack this statement in words, one quantifier at a time

Remark 3.9. Notice *diverge* does not mean $\to \pm\infty$, for instance we will prove later that $a_n = (-1)^n$ diverges.

Exercise 3.10. Fix a sequence of real numbers $(a_n)_{n\geq 1}$. Consider

$$\boxed{\forall n \geq 1 \; \exists \epsilon > 0 \text{ such that } |a_n| < \epsilon}$$

This means?

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded
3. Precisely nothing ✓
4. More than one of these
5. None of these

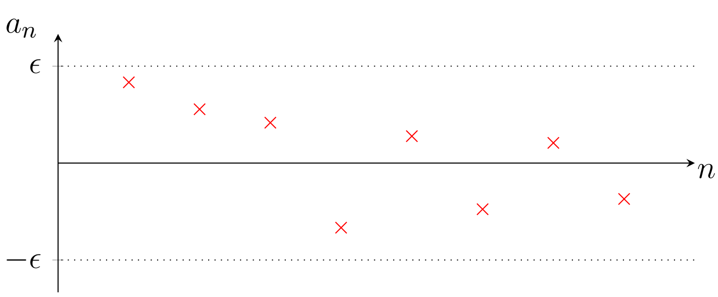
Proof. Fix any $n \in \mathbb{N}_{>0}$. Take $\epsilon = |a_n| + 1$. □

Order of $\exists$, $\forall$ very important!

Exercise 3.11. What about

$$\boxed{\exists \epsilon > 0 \text{ such that } \forall n \geq 1, \; |a_n| < \epsilon}?$$

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded ✓
3. Precisely nothing
4. More than one of these
5. None of these



It says $a_n \in (-\epsilon, \epsilon) \; \forall n \iff |a_n|$ is bounded by ϵ.

We can also define limits for *complex sequences*. Let $|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$.

Definition. $a_n \in \mathbb{C}$, $\forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ if and only if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n - a| < \epsilon.$$

This definition is equivalent to $(\operatorname{Re} a_n) \to \operatorname{Re} a$ *and* $(\operatorname{Im} a_n) \to \operatorname{Im} a$ (see problem sheet 4!).

Example 3.12. Prove $a_n = \dfrac{e^{in}}{n^3 - n^2 - 6} \to 0$ as $n \to \infty$.

Rough working:

$$|a_n - 0| \;=\; \left|\frac{e^{in}}{n^3 - n^2 - 6}\right| \;=\; \left|\frac{1}{n^3 - n^2 - 6}\right|$$

which we would like to be $< b_n = \frac{1}{c_n}$ for some more manageable c_n smaller than $n^3 - n^2 - 6$, but not too small! (I.e. we still want $c_n \to \infty$ so $b_n \to 0$.) So let $c_n = n^3 - (\,\text{something bigger than}\, n^2 + 6)$.

We use $\frac{n^3}{2}$ to make the c_n simple. For $n \geq 4$, we have $\frac{n^3}{2} > n^2 + 6$. So for $n \geq 4$

$$\left|\frac{1}{n^3 - n^2 - 6}\right| \;<\; \frac{1}{n^3 - n^3/2} \;=\; \frac{2}{n^3} \;\leq\; \frac{2}{n},$$

which is $< \epsilon$ for $n > \frac{2}{\epsilon}$.

Proof. $\forall \epsilon > 0$ choose $N \geq \max\big(4, 2/\epsilon\big)$. Then $\forall n \geq N$,

$$|a_n - 0| \;=\; \left|\frac{1}{n^3 - n^2 - 6}\right| \;<\; \frac{1}{n^3 - n^3/2} \;=\; \frac{2}{n^3} \;\leq\; \frac{2}{N^3} \;\leq\; \frac{2}{N} \;\leq\; \epsilon. \qquad \square$$

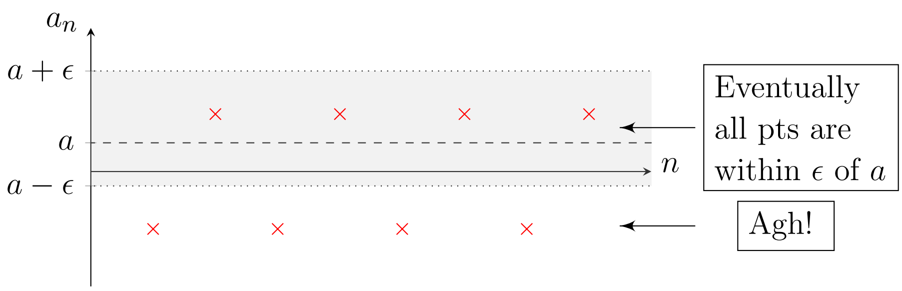
Once we've prepared right, the proof is only 2 lines

Example 3.13. Set $\delta = 10^{-1000000}$, $a_n = (-1)^n \delta$. Prove that a_n diverges, that is it does not converge (to any $a \in \mathbb{R}$).

Assume for contradiction that $a_n \to a$, i.e.

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n - a| < \epsilon.$$

Rough working: Draw a picture! But don't make δ small in your picture, as then you won't see the contradiction. Magnify it to be big.



For small enough $\epsilon > 0$ (the picture shows that any $\epsilon \le \delta$ will do), the fact that a is within ϵ of δ (a_{2n}) and $-\delta$ (a_{2n+1}) will be a contradiction.

Proof 1. Fix $a \in \mathbb{R}$. Take $\epsilon = \delta$.

Then if $\exists N$ such that $\forall n \ge N$, $|a_n - a| < \epsilon$ this implies

1. $|a_{2N} - a| < \epsilon \iff a \in (\delta - \epsilon, \delta + \epsilon) \implies a > \delta - \epsilon = 0$, and
2. $|a_{2N+1} - a| < \epsilon \iff a \in (-\delta - \epsilon, -\delta + \epsilon) \implies a < -\delta + \epsilon = 0$ ※

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge.

Or, *Proof 2*: Both $\pm\delta$ close to the limit a so must be close to each other by the triangle inequality:

$$|\delta - (-\delta)| \ \le\ |\delta - a| + |a - (-\delta)| \ <\ \epsilon + \epsilon \ \implies\ 2\delta < 2\epsilon = 2\delta \ ※$$

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge. □

An alternative approach to that question is provided by the following.

Ask them again what $\forall n,\ \exists \epsilon > 0$ such that $|a_n| < \epsilon$ means?

Theorem 3.14: Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$.

Idea: For n large, a_n is arbitrarily close to both a and b. So a arbitrarily close to $b \implies a = b$.

Proof 1.

1. $\forall \epsilon\ \exists N_a$ such that $\forall n \geq N_a,\ |a_n - a| < \epsilon$,
2. $\forall \epsilon\ \exists N_b$ such that $\forall n \geq N_b,\ |a_n - b| < \epsilon$.

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$, both 1 and 2 hold, so

$$|a-b| \ = \ |(a-a_n)+(a_n-b)| \ \leq \ |a-a_n|+|a_n-b| \ < \ 2\epsilon.$$

This is true $\forall\, \epsilon$, so in fact $|a-b| = 0$.

Proof of this last claim:
If not, set $\epsilon = \frac{1}{2}|a-b| > 0$ to get the contradiction $|a-b| < |a-b|$. □

Proof 2. By contradiction. Assume $a \neq b$ and again draw a *magnified* picture.

Eventually a_n is in *both* corridors. So if we choose ϵ sufficiently small so that the corridors don't overlap then we get a contradiction.

Set $\epsilon = \frac{|a-b|}{2} > 0$. Then $\exists N_a, N_b$ such that $\forall n \geq N_a, N_b$, we have

$$|a_n - a| \ < \ \epsilon \quad \text{and} \quad |a_n - b| \ < \ \epsilon.$$

Without loss of generality, $a > b$. Then $a_n > a - \epsilon$ and $a_n < b + \epsilon$

$$\implies b+\epsilon \ > \ a-\epsilon$$
$$\implies 2\epsilon \ > \ a-b = 2\epsilon \ ※$$ □

We throw away $a_n < a+\epsilon$, $b_n > b-\epsilon$: see diagram. Manipulating $|a_n - a| < \epsilon$ by algebra will not get you a proof.

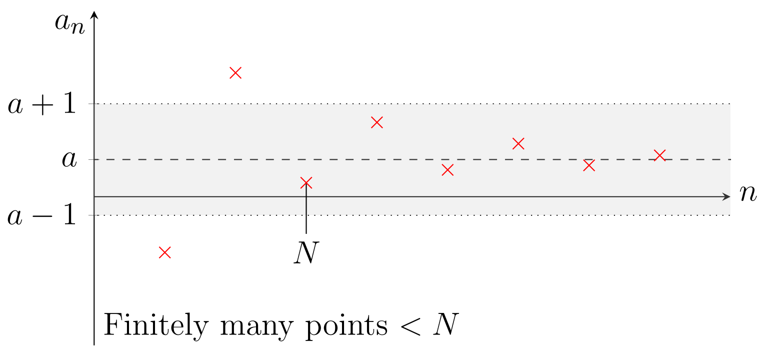
Exercise 3.15. Let a_n be defined by $a_1 = a_2 = 0$ and $a_n = \frac{1}{n-2}$ for $n > 2$. Show $a_n \to 0$.
Which step is incorrect in this student 's strategy?
Fix $\epsilon > 0$. We assume $n > 2$. Then

1. We want $|\frac{1}{n-2} - 0| = \frac{1}{n-2} < \epsilon$
2. $\implies n - 2 > 1/\epsilon$
3. $\implies n > 2 + 1/\epsilon$
4. $\implies n > 1/\epsilon \quad (*)$
5. So take $N > \max(1/\epsilon, 2)$, then
6. $\forall n \geq N$, $n > 1/\epsilon$ which is $(*)$
7. So $\frac{1}{n-2} \to 0$ ✓
8. More than one mistake
9. All correct

Although steps 2 and 4 cannot be reversed, they're not wrong (they're just not useful). But 7 IS wrong. It does not follow from 6 because $(*)$ does not imply the steps above it – it is implied by them. The implications are in the wrong direction.

Proposition 3.16. *If (a_n) is convergent, then it is bounded.*
$\big[$*I.e.* $a_n \to a \implies \exists A \in \mathbb{R}$ *such that* $|a_n| \leq A \; \forall n.\big]$

Proof. Fix $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < 1 \implies |a_n| < 1 + |a|$.



Then $|a_n|$ is bounded $\forall n$ by $\max\big\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 1\big\}$. $\square$

Notice $a_n = \frac{1}{n-7}$ is not a counterexample! It is not a well defined sequence of real numbers because a_7 is either not defined or not real. Instead we could take

$$a_n = \begin{cases} \frac{1}{n-7} & n \neq 7, \\ 0 & n = 7. \end{cases}$$

This is then indeed bounded as $\forall n \in \mathbb{N}_{>0}$ we have

$$-1 = a_6 \leq a_n \leq a_8 = 1.$$

Exercise 3.17. Give an example of a bounded sequence that is divergent.

Exercise 3.18. Let (a_n) be a bounded sequence. Let (b_n) be a sequence with $b_n = a_n$ for all $n \geq 100$. Prove that b_n is bounded.