Applied Complex Analysis MATH60006/70006/97028

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Autumn Term 2022

Chapter 1: Review of Complex Analysis

In this first chapter we will review some of the fundamental results of complex analysis that will be relevant to our later chapters. Sometimes we will omit proofs from this chapter as we are interested in using the results to solve problems in applied mathematics/mathematical physics rather than studying how they are obtained.

For those in year 3/4 of their undergraduate studies at Imperial, it may help to refer back to your complex analysis notes from year 2 if you want to refresh your memory on any of the proofs of these results/see more examples. Similarly for those on Masters programmes it might be useful to look back at any complex analysis you did during your undergraduate degree. For those without access to any previous complex analysis notes, it may help to refer to any introductory text on complex analysis (see the reading list) to look up the proofs/see more examples.

That being said, these lecture notes are entirely self-contained in the sense that no external knowledge outside of these notes is necessary to study this course and for the exam/assessments. A knowledge of the basic results and properties of complex numbers as well as first/second year undergraduate knowledge in calculus and differential equations will be assumed throughout this course however.

1.1 Complex Numbers and their Properties

Let us begin by reviewing a few basic results about complex numbers before moving on to recap the complex derivative.

A complex number has the form $z = x + iy$, where $i = \sqrt{-1}$. We denote the **real** and **imaginary** parts respectively as $x = \mathrm{Re}\{z\}$ and $y = \mathrm{Im}\{z\}$. We may also think of z as a point in the complex plane, which in polar coordinates has **absolute value** $r = |z| = \sqrt{x^2 + y^2}$ and **argument** θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Note that θ is defined periodically with period 2π and may also be denoted $\theta = \arg\{z\}$.

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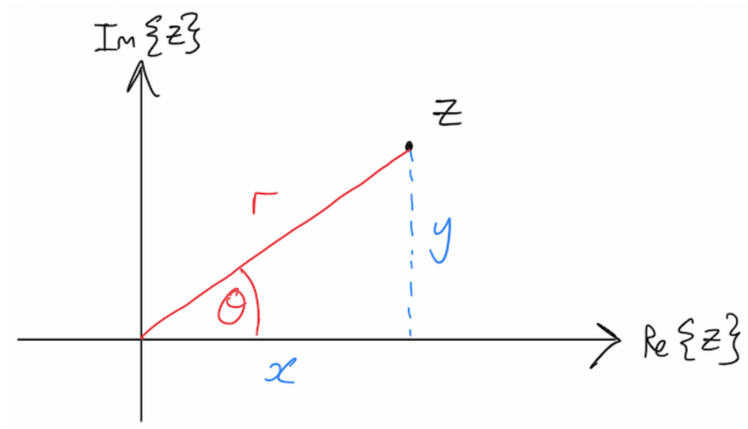


Figure 1: A complex number z visualised on the complex plane.

The complex number may then be represented in polar form $z = r(\cos\theta + i\sin\theta)$ or exponential form $z = re^{i\theta}$.

Complex numbers obey the following algebraic properties:

$$\begin{aligned} z_1 + z_2 =& (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 =& (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned}$$

i.e. they have the same algebra as the real numbers, with the convention $i^2 = -1$.

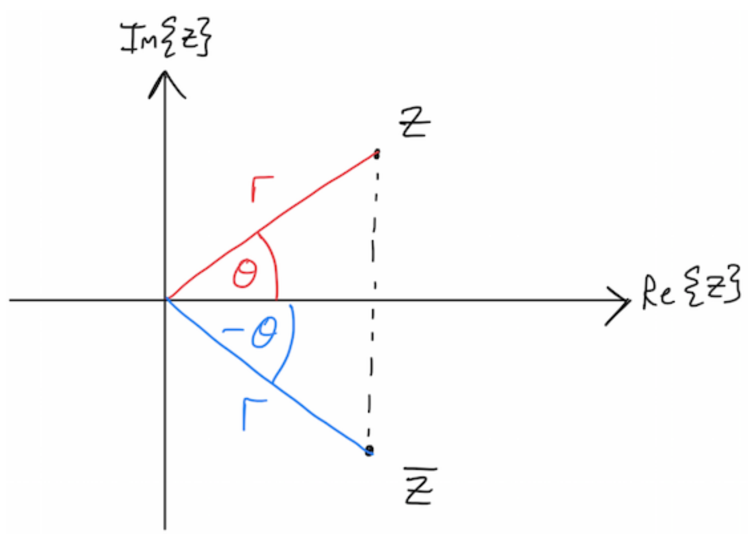


Figure 2: Complex conjugate $\overline{z} = x - iy$ is defined by reflecting in the x-axis.

If $z = x + iy$ then throughout this course the **complex conjugate** of z will be defined as $\overline{z} = x - iy$, which is the symmetric image of z with respect to the x-axis. Note the following properties of the complex conjugate:

$$\begin{aligned}|z| =& \sqrt{x^2+y^2} = |\overline{z}| \\ \arg\{z\} =& -\arg\{\overline{z}\} \\ z\overline{z} =& (x+iy)(x-iy) = x^2+y^2 = |z|^2\end{aligned}$$

This helps us to obtain a useful formula for the division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}.$$

We may also do multiplication in polar coordinates:

$$\begin{aligned}z_1z_2 =& (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) \\ =& r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\ =& r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right],\end{aligned}$$

where we have used the double-angle formulae for sin and cos. Then we see that

$$|z_1z_2| = |z_1|\,|z_2|$$

and

$$\arg\{z_1z_2\} = \arg\{z_1\} + \arg\{z_2\}.$$

Thus, we may take the nth power and, after that, nth root of z as:

$$\begin{aligned}z^n =& r^n(\cos(n\theta) + i\sin(n\theta)) \\ \sqrt[n]{z} =& \cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right) \qquad \text{for} \qquad k = 0, ..., n-1\end{aligned}$$

Each non-zero z has n different nth roots and these are related by rotations in the complex plane, see Figure 3.

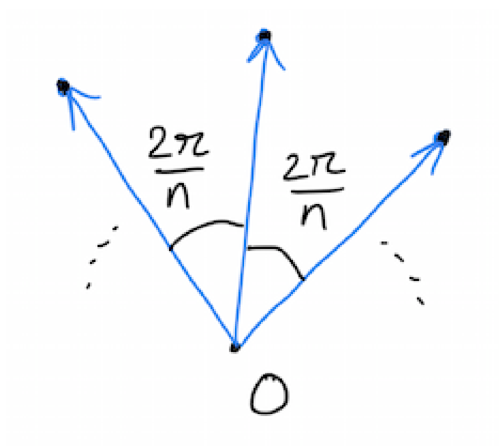


Figure 3: The nth roots are related by rotation through angle $2\pi/n$ in the complex plane.

The definition of the complex conjugate $\overline{z}$ also allows us to write

$$\mathrm{Re}\{z\} = \frac{z+\overline{z}}{2}, \quad \mathrm{Im}\{z\} = \frac{z-\overline{z}}{2i}.$$

We will use these results throughout the course.

The Triangle Inequality

A final result we will use throughout the course is the so called **triangle inequality**, which states for complex numbers z_1, $z_2 \in \mathbb{C}$, then

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

The proof of this statement is left as an exercise (see problem sheet 1).

1.2 Functions of Complex Variables

A standard notation we use for a complex function is

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real valued functions of x and y.

Examples

1. Some very simple complex functions are $f(z) = a$, where a is a constant and $f(z) = z$. By doing multiplication and addition with these, we obtain

2. Polynomials:

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

   Employing the division operation, we obtain

3. Rational functions:

$$R(z) = \frac{P(z)}{Q(z)}$$

   where P and Q are both polynomials.

4. Consider also the exponential function:

$$\exp\{z\} = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

   The coefficients of this series are exactly the same as for the real exponential function, so, because the algebraic operations on real and complex numbers obey the same rules, the exponential function defined on complex numbers obeys the same algebra as the exponential function on real numbers.

5. The formula $e^{i\theta} = \cos\theta + i\sin\theta$ gives $e^{-i\theta} = \cos\theta - i\sin\theta$ from which we can deduce

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

   One can show that all the regular rules for $\sin x$ and $\cos x$ apply for these complex valued functions too.

6. From the formula $e^{i\theta} = \cos\theta + i\sin\theta$, we see that $e^{i\theta}$ is 2π-periodic in θ:

$$e^{i\theta+2\pi ki} = e^{i\theta} \qquad \text{for} \qquad k \in \mathbb{Z}.$$

   The periodicity of the complex exponential function implies that its inverse function is **multi-valued**:

$$\log z = \log\left(re^{i\theta}\right) = \log|z| + i\arg\{z\} = \log r + i\left(\theta + 2k\pi\right) \qquad \text{for} \qquad k \in \mathbb{Z}.$$

   More on multi-valued functions later!

7. We may use the logarithm to define arbitrary complex powers of a complex number z:

$$z^{\alpha} = e^{\alpha \log z}$$

   This function is also multivalued (when α is not an integer).

The previous examples are 'good' functions, in the sense that they are analytic, as we will see soon. However, we may also define 'bad' functions of complex variables, such as:

$$\begin{aligned} f(z) &= \mathrm{Re}(z) \\ g(z) &= |z| \\ h(z) &= \overline{z} \end{aligned}$$

These functions are non-analytic (how do we know why at a glance?).

The analyticity is the main topic here. While a typical complex function is not analytic, many important functions are analytic and, as a consequence, possess non-trivial and useful properties. The analyticity can be defined in several equivalent ways. We start with the notion of the derivative.

1.3 The Complex Derivative

Definition 1.1. A complex function $f(z)$ is said to be **differentiable** at a point $z \in \mathbb{C}$ if the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}, \tag{1}$$

exists, and is **independent** of the path along which $h \to 0$ (note $h = h_1 + ih_2 \in \mathbb{C}$).

In particular, we may write $h = re^{i\theta}$, so the limit $h \to 0$ must be independent of θ, the direction along which h approaches zero.

Definition 1.2. A function $f(z)$ is **analytic** (or holomorphic) at the point z_0 if it has a derivative at all points close to z_0.

Note, for example, that $f(z) = |z|^2$ is not analytic at $z = 0$, even though it has a derivative at $z = 0$, because one can show that it does not have a derivative for any $z \neq 0$.

Definition 1.3. A function $f(z)$ is **analytic** in an open region D if the derivative $f'(z)$ exists at every point $z \in D$. A function that is analytic everywhere in $\mathbb{C}$ is called **entire**.

Examples

1. Let $f(z) = z^2$. Then at $z = 0$ we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{(0+h)^2 - 0^2}{h} \\ &= \lim_{h\to 0} h \to 0. \end{aligned}$$

   So the derivative at $z = 0$ exists.

2. Let $f(z) = \overline{z}$. Then at $z = 0$ we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{\overline{(0+h)} - \overline{0}}{h} \\ &= \lim_{h\to 0} \frac{\overline{h}}{h}. \end{aligned}$$

   Now if we let $h = h_1$ (so set $h_2 = 0$), then in the limit we find $f'(0) = 1$. On the other hand, if we let $h = ih_2$ (so $h_1 = 0$), then in the limit we find $f'(0) = -1$. So the limits are not equal and so the function is not differentiable at $z = 0$.

Theorem 1.4 (Cauchy-Riemann Conditions)**.** *Let $f(z) = u(x, y) + iv(x, y)$. The derivative $f'(z)$ exists at $z = x + iy$ if and only if u and v are differentiable and satisfy the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2b}$$

Proof. Suppose $f'(z)$ exists. Then the limit in eq. (1) exists and is independent of the direction in which the limit is taken. Therefore set $h = h_1$ and we see that:

$$f'(z) = \lim_{h_1\to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - u(x, y) + iv(x, y)}{h_1} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Similarly, if we take $h = ih_2$ then we find:

$$f'(z) = \lim_{h_2\to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - u(x, y) + iv(x, y)}{ih_2} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts gives eq. (2).

Suppose conversely that u and v satisfy eq. (2). We keep a general form for $h = h_1 + ih_2$ and then use Taylor expansions of functions of two-variables:

$$\begin{aligned} f(z+h) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) \\ u(x+h_1, y+h_2) &= u(x, y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + o\left(\sqrt{h_1^2 + h_2^2}\right) \\ v(x+h_1, y+h_2) &= v(x, y) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + o\left(\sqrt{h_1^2 + h_2^2}\right) \\ \implies \qquad f(z+h) - f(z) &= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2\right) + o\left(|h|\right). \end{aligned}$$

By eq. (2), the right-hand side of this formula is

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) h + o\left(|h|\right)$$

and therefore $f'(z) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)$ exists. $\square$

Properties of the Complex Derivative

The derivative satisfies the same rules as in the real case: for any analytic functions f and g, we have

1. $(f+g)' = f' + g'$,
2. $(fg)' = f'g + fg'$,
3. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$, $\quad (g \neq 0$ at these points),
4. (chain rule) for any z in the domain of g, $(f(g(z)))' = f'(g(z))g'(z)$.

Remark If g is the inverse of f, i.e. for any z in the domain of f, $g(f(z)) = z$, then $g'(f(z)) = \frac{1}{f'(z)}$, and $g'(z) = \frac{1}{f'(g(z))}$.

Some examples of how to derive the derivatives are given in what follows.

Some Examples

1). Suppose $f(z) = c$ for any $z \in \mathbb{C}$, where c is a constant complex number. Since for any $z, h \in \mathbb{C}$

$$\frac{f(z) + h - f(z)}{h} = \frac{c - c}{h} = 0,$$

the derivative of f is 0.

2). Suppose $f(z) = z$ for any $z \in \mathbb{C}$. Since for any $z, h \in \mathbb{C}$,

$$\frac{f(z) + h - f(z)}{h} = \frac{z + h - z}{h} = 1,$$

we get $f'(z) = 1$.

3). Using the product rule, for any $z \in \mathbb{C}$, we get

$$(z^n)' = nz^{n-1}$$

4). Now, for any $z \in \mathbb{C}$, letting $P(z) = a_0 + a_1 z + ... + a_n z^n$, we get the derivative of P as follows:

$$P'(z) = a_1 + ... + na_n z^{n-1}.$$

5). Since any rational function can be written as $\frac{P}{Q}$, where P and Q are polynomials, and since $(\frac{P}{Q})' = \frac{P'Q - PQ'}{Q^2}$, we obtain that rational functions are analytic everywhere except for the points where $Q(z) = 0$.

6). For $e^z = \sum_{i=0}^{\infty} \frac{z^n}{n!}$, its derivative is

$$(e^z)' = 1 + z + ... + \frac{nz^{n-1}}{n!} + ... = \sum_{i=0}^{\infty} \frac{z^n}{n!} = e^z.$$

The inverse function of e^z is $\log z$ (recall that $\log(re^{i\theta}) = \log r + i(\theta + 2k\pi)$). Using the rule for the derivative of the inverse function, we find

$$(\log z)' = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

7). For any $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$(z^\alpha)' = (e^{\alpha \log z})' = \frac{\alpha}{z} e^{\alpha \log z} = \alpha z^{\alpha - 1}.$$

Partial Complex Derivatives

Consider a complex function $f(z) = f(x, y)$, then using the chain rule for partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \\ &= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i}, \end{aligned}$$

where we have used the fact that $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$. So we have

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Using a similar derivation, or taking the complex conjugate, we find

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Theorem 1.5. *$f(z)$ is an analytic function if and only if*

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{3}$$

Proof.

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]. \end{aligned}$$

This expression is then equal to 0 if f is analytic (as f then satisifes the Cauchy-Riemann equations), and conversely, if this expression equals 0, f must satisfy the Cauchy-Riemann equations (as the Re and Im parts must both equal zero), so f is analytic. Hence

$$f(z) \text{ analytic } \Leftrightarrow \frac{\partial f}{\partial \overline{z}} = 0.$$

□

This final result provides another description of what an analytic function is; namely the analyticity of a complex function $f(z)$ depends solely on whether f is a function of z only. z good! $\overline{z}$ bad!

1.4 Harmonic Functions

Definition 1.6. Let $\phi = \phi(x, y)$ be a real function ($x, y \in \mathbb{R}$). ϕ is called **harmonic** if

$$\Delta\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

Theorem 1.7. *Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then u and v are harmonic.*

Proof. Indeed, using the Cauchy-Riemann equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

So $\nabla^2 u = 0$ (similarly for v). □

Definition 1.8. Let u be a harmonic function. Then we say that another harmonic function v is the **harmonic conjugate** of u if the complex function $f(z) = u + iv$ is analytic.

1.5 Integrals over paths in $\mathbb{C}$

Let's introduce the notion of an integral over a path in the complex plane.

Definition 1.9. Given a smooth curve $\gamma \in \mathbb{C}$ and a continuous function f defined on $\gamma = \{z = z(t) : t \in [a, b]\}$ ($a, b \in \mathbb{R}$), we define the integral of f over γ as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

The integral has the following basic properties.

1. If $\tilde{\gamma}$ is the same curve as γ, just with opposite orientation, then
$$\int_\gamma f(z)dz = -\int_{\tilde{\gamma}} f(z)dz.$$
2. Let the end point of γ_1 be the starting point of γ_2. Then
$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

We can also deduce the useful **ML-Inequality**:

Theorem 1.10 (ML-Inequality)**.** *For a continuous curve γ in the complex plane and a complex function $f(z)$, we have*

$$\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma}\{|f(z)|\} \times \mathit{length}(\gamma). \tag{4}$$

Proof. Let $z(t)$ be a parameterisation of γ with $t \in [a, b]$. Then

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq \max_{t\in[a,b]}\{|f(z(t))|\}\int_a^b |z'(t)|dt \\ &\leq \max_{z\in\gamma}\{|f(z)|\} \times \text{length}(\gamma),\end{aligned}$$

where we have used the fact that $\int_\gamma dz =$ length of the curve. □

1.6 Cauchy's Theorem and Cauchy's Integral Formula

Analytic functions have many wonderful, or even magical properties. For example, let $\mathcal{D}$ be a simply connected open region in $\mathbb{C}$, as in figure 4.

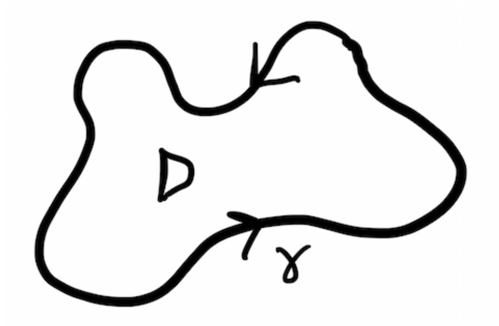


Figure 4: A simply connected region $\mathcal{D}$, bounded by a contour γ.

Then the following is true:

Theorem 1.11 (Cauchy's Theorem)**.** *If γ bounds a simply connected region $\mathcal{D}$, and f is analytic inside $\mathcal{D}$ and on γ, then*

$$\oint_\gamma f(z)dz = 0. \tag{5}$$

This theorem implies the following corollary: the path independence of integrals of analytic functions.
Corollary. *If $f(z)$ is analytic in a simply-connected domain $\mathcal{D}$, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

*for any curves γ_1 and γ_2 in $\mathcal{D}$ which have the same end points (i.e. the contour integral of $f(z)$ is **independent** of path in $\mathcal{D}$).*

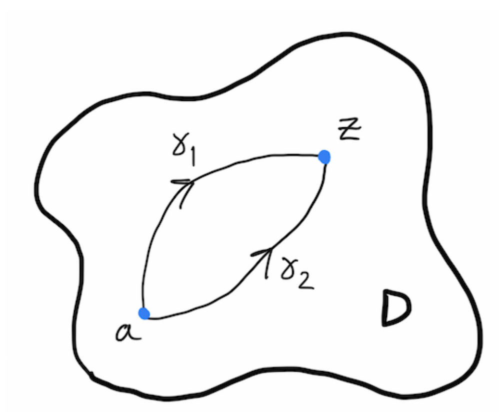


Figure 5: The integral of an analytic function in the simply-connected domain $\mathcal{D}$ can depend only on the end points of the path of integration, and does not depend on the path itself.

Proof. To prove this, consider the curve γ obtained by the concatenation of the paths γ_1 and γ_2 reversed ($\tilde{\gamma}_2$). We have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\tilde{\gamma}_2} f(z)dz \\ &= \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz.\end{aligned}$$

The path γ is closed, so $\int_\gamma f(z)dz = 0$ by Cauchy's theorem, hence

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

as claimed. ◻

We can also deduce the so called **deformation theorem**.

Theorem 1.12 (Deformation Theorem)**.** *If $f(z)$ is analytic in a region D bounded by γ_1 and γ_2, with γ_2 lying completely inside γ_1, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

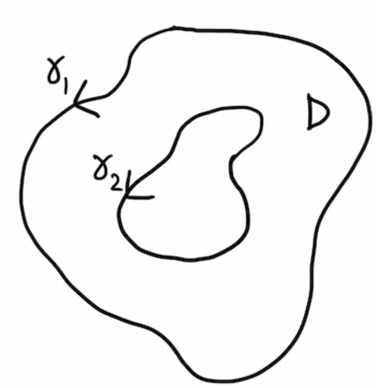


Figure 6: Curve γ_2 lying within curve γ_1.

Proof. To see this, we take a cut (shown by blue in figure 7) connecting the outer boundary γ_1 and the inner boundary γ_2.

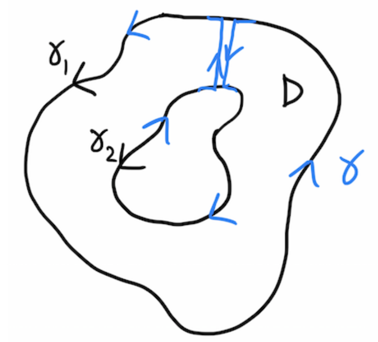


Figure 7: The integral over the outer boundary equals to the integral over the inner boundary.

The closed path starting with the end point of the blue cut, then going along γ_2 in reverse, then along the blue cut towards γ_1, then along γ_1, then back along the blue cut, bounds a simply-connected domain (the region D minus the blue cut). Therefore, the integral of $f(z)$ over this path is zero, according to Cauchy's Theorem. Since the blue cut is traversed twice in opposite directions, the contributions of the "blue parts" of the path cancel each other, which gives us that the integral over γ_1 plus the integral over $\tilde{\gamma}_2$ equals zero, which gives the result of the theorem. $\square$

Next comes one of the most central theorems to the topic of complex analysis.

Theorem 1.13 (Cauchy's Integral Formula)**.** *Let f be analytic inside and on a closed path γ bounding a simply-connected region $\mathcal{D}$. Then, at any point z interior to γ*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} d\xi. \tag{6}$$

Proof. Since f is analytic in $\mathcal{D}$, it follows that $\frac{f(\xi)}{\xi - z}$ is analytic in $\mathcal{D} \backslash \{\xi = z\}$. Therefore, by the Deformation theorem:

$$\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = \oint_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z} d\xi,$$

where γ_ε is a circle of small radius ε with centre at the point z, see figure 8.

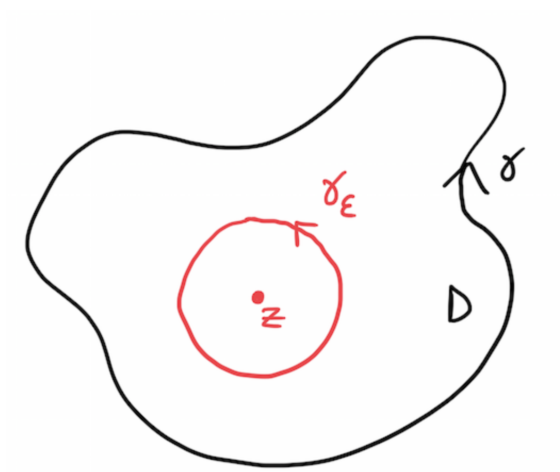


Figure 8: The integral over γ equals to the integral over a small circle around z.

This circle is given by equation $\xi = z + \varepsilon e^{i\theta}$, where $\theta \in [0, 2\pi]$. After making this substitution into the integral, we obtain

$$\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = \oint_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z} d\xi = \int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = i \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta = i \int_0^{2\pi} (f(z) + O(\varepsilon)) d\theta.$$

Now, taking the limit $\varepsilon \to 0$, we obtain

$$\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z),$$

which gives the required result upon re-arrangement. □

Note that this is indeed a magical formula: the values of the analytic function f at any point inside the domain $\mathcal{D}$ are determined by the values of f on the boundary of the domain only!

Magical formulas have magical consequences: using the Cauchy Integral formula, we will also show that f has derivatives of **all orders** and the Taylor series of f converges to f. Nothing similar happens for real functions of a real variable, they may have a first derivative, but not second derivative, or first and second derivatives, but no third derivative, etc; or derivatives of all orders may exist, but the Taylor series may diverge, or converge to a wrong function - all these complications disappear for analytic functions of complex variables.

Let us now show that an analytic function $f(z)$ has derivatives of all orders and that these are also analytic.

Theorem 1.14. *Let $f(z)$ be analytic inside and on a closed path γ bounding a simply-connected region D. Then for any z within D:*

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi. \tag{7}$$

Proof. We prove this via induction on n. The case $n = 0$ is the Cauchy Integral formula. When $n = k$, let $h \in \mathbb{C}$ such that $z + h$ lies inside γ. Then let

$$I = \frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} = \frac{(k-1)!}{2\pi i} \oint_\gamma \frac{1}{h} f(\xi) \left[\frac{1}{(\xi - (z+h))^k} - \frac{1}{(\xi - z)^k} \right] d\xi.$$

Now on the right hand side apply the formula:

$$A^k - B^k = (A - B)(A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1}), \quad \text{with:} \quad A = \frac{1}{(\xi - (z+h))}, \quad B = \frac{1}{\xi - z},$$

and

$$A - B = \frac{(\xi - z) - (\xi - (z+h))}{(\xi - (z+h))(\xi - z)} = \frac{h}{(\xi - (z+h))(\xi - z)},$$

giving

$$I = \frac{(k-1)!}{2\pi i} \oint_\gamma f(\xi) \left[\frac{1}{(\xi - (z+h))(\xi - z)} [A^{k-1} + \cdots + B^{k-1}] \right] d\xi.$$

Finally take the limit $h \to 0$ of both sides:

$$\lim_{h\to 0} I = \frac{d^k}{dz^k} f(z) \longrightarrow \frac{(k-1)!}{2\pi i} \oint_\gamma f(\xi) \left[\frac{1}{(\xi - z)^2}\right] \left(\frac{k}{(\xi - z)^{k-1}}\right) d\xi$$
$$\longrightarrow \frac{k!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi.$$

□

1.7 Estimates on the derivatives of analytic functions and Liouville's Theorem

Let us now derive some important results using the derivatives of the Cauchy Integral formula (7). In particular we will be interested in Liouville's theorem and its extension. Letting $\gamma = \{|\xi - z| = r\}$ in the Cauchy Integral formula (6) and its derivatives (7), we get $\xi = z + re^{i\theta}$ and $d\xi = ire^{i\theta} d\theta$, which gives

$$\begin{aligned} |f^{(n)}(z)| &= \left| \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right| \\ &\le \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(\xi)|}{r^{n+1}} r d\theta \\ &\le \frac{n!}{2\pi} \max_{\xi \in \gamma}(f(\xi)) \times \frac{2\pi}{r^n} \\ &= n! \frac{M}{r^n}, \end{aligned} \tag{8}$$

where M is the maximal value of $|f(z)|$ on γ. This provides estimates for the absolute value of the function $f(z)$ (the case $n = 0$) and its derivatives ($n \ge 1$).

Theorem 1.15 (Maximum Modulus Principle)**.** *The absolute value of an analytic function takes its maximum on the boundary of the analyticity domain.*

Proof. If we set $n = 0$ in the above, then we have $|f(z)| \le M$ for any r such that f is analytic in the disc of radius r around the point z. Thus the maximum value of f is M and indeed obtained on the boundary. □

Theorem 1.16 (Liouville's theorem)**.** *If an entire function $f(z)$ is bounded everywhere in the complex plane, including at ∞, it is a constant.*

Proof. By (8) with $n = 1$, we obtain

$$|f'(z)| \le \frac{M}{r}.$$

In the limit as $r \to \infty$ we have

$$f'(z) = 0 \Longrightarrow f(z) = \text{constant}.$$

□

This can be extended by allowing for algebraic growth at infinity.

Theorem 1.17 (Extension of Liouville's theorem)**.** *If an entire function $f(z)$ is bounded everywhere in the **finite** complex plane (the complex plane excluding complex infinity) and $|f(z)| \le M|z^k|$ ($M > 0, k > 0$) for $|z|$ large, then $f(z)$ is a polynomial of degree at most k.*

Proof. Following the same procedure as before, but now using $|f(z)| \leq M|z^k|$, when using the ML-inequality, we obtain

$$|f^{(n)}(z)| \leq n! \frac{Mr^k}{r^n}.$$

In the limit as $r \to \infty$ we have

$$f^{(n)}(z) = 0 \quad \text{for} \quad n > k.$$

Thus $f(z)$ is a polynomial of degree at most k. □

From Liouville's theorem we can also get the Fundamental theorem of Algebra.

Theorem 1.18 (The Fundamental Theorem of Algebra)**.** *Every non-constant polynomial has a root in the complex plane.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where $a_n \neq 0$. Indeed, suppose $p(z)$ has no zeros, then the function $\frac{1}{p(z)}$ would be analytic everywhere. Now

$$\left| \frac{1}{p(z)} \right| = \frac{1}{|z^n|} \left(\frac{1}{a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}} \right).$$

Then as $|z| \to \infty$, $|1/p(z)| \to 0$, so $1/p(z)$ is bounded. Therefore by Liouville's Theorem, $1/p(z) = \text{constant}$, but this contradicts our assumption that $p(z)$ was a non-constant polynomial. □

1.8 Taylor Series

Definition 1.19. Suppose $f(z)$ is analytic in $|z - z_0| \leq R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

for $|z - z_0| < R$, is called the **Taylor series** of $f(z)$ about z_0.

The existence of derivatives of all orders allows one to write the Taylor expansion of an analytic function $f(z)$ at any point z_0 of the analyticity domain. We know that for real functions of real variables the Taylor series can diverge, however it is not the case for functions of complex variables.

Theorem 1.20. *The Taylor series* $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ *of an analytic function* $f(z)$ *at any point* z_0 *converges to* $f(z)$ *for all* z *such that* $|z - z_0| < R$*, where the radius of convergence* R *equals the distance from* z_0 *to the nearest singularity point of* f*.*

Proof. Let $\gamma = \{\xi : |\xi - z_0| = R\}$. Then, using the Cauchy Integral formula (see (6)) we have:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0)} \left(\frac{1}{1 - (\frac{z-z_0}{\xi - z_0})} \right) d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}} \right) (z - z_0)^n \\
&= f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots ,
\end{aligned}$$

where we used formula (7) for the derivatives of $f(z)$. As we see, the Taylor series converges to f if $|z - z_0| < |\xi - z_0|$ for all $\xi \in \gamma$.
The path γ is taken to be as close as we want to the boundary of the analyticity domain of f, hence R cannot be smaller than the distance from z_0 to the nearest singularity. Of course, R cannot also be larger than this, since the sum of a convergent power series is analytic (non-singular) everywhere within the radius of convergence. Thus R equals the distance from z_0 to the closest singularity of f. ☐

1.9 Laurent Series

There is another type of series expansion that is widely used when dealing with complex functions, that is a Laurent series.

Definition 1.21. Suppose $f(z)$ is analytic in the annular region $r < |z - z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

Theorem 1.22 (Laurent expansion Theorem)**.** *Let $f(z)$ be analytic in the annular region $D = \{z : r < |z - z_0| < R\}$. Then $f(z)$ can be expressed in the form of a Laurent series:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n; \quad \textit{where} \quad a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \tag{9}$$

where γ is a closed curve in D that contains z_0 in its interior.

Proof. For simplicity of writing we set $z_0 = 0$ here. Then we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\text{blue}} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i}\oint_{\text{red}} \frac{f(\xi)}{\xi - z} d\xi \\
&= \underbrace{\frac{1}{2\pi i}\oint_{|\xi| = R'} \frac{f(\xi)}{\xi - z} d\xi}_{=I_1} + \underbrace{\frac{-1}{2\pi i}\oint_{|\xi| = r'} \frac{f(\xi)}{\xi - z} d\xi}_{=I_2},
\end{aligned}$$

where the blue and red paths and the radii r' and R' are as shown in figure 9.

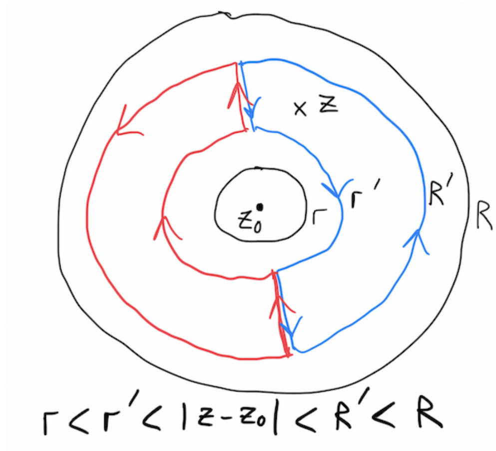


Figure 9: Diagram of the annular region $r < |z - z_0| < R$ and the blue and red contours.

Then

$$I_1 = \frac{1}{2\pi i} \oint_{|\xi|=R'} \frac{1}{\xi} \frac{f(\xi)}{1 - z/\xi} d\xi = \frac{1}{2\pi i} \oint_{|\xi|=R'} \sum_{n=0}^{\infty} \frac{f(\xi)}{\xi} \left(\frac{z}{\xi}\right)^n d\xi = \frac{1}{2\pi i} \sum_{n=0}^{\infty} z^n \oint_{|\xi|=R'} \frac{f(\xi)}{\xi^{n+1}} d\xi,$$

and

$$\begin{aligned} I_2 &= \frac{-1}{2\pi i} \oint_{|\xi|=r'} \frac{f(\xi)}{z(\xi/z - 1)} d\xi = \frac{1}{2\pi i} \oint_{|\xi|=r'} \frac{f(\xi)}{z(1 - \xi/z)} d\xi = \frac{1}{2\pi i} \oint_{|\xi|=r'} \frac{f(\xi)}{z} \sum_{n=0}^{\infty} \left(\frac{\xi}{z}\right)^n d\xi \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \oint_{|\xi|=r'} f(\xi)\xi^n d\xi = \frac{1}{2\pi i} \sum_{k=-\infty}^{-1} z^k \oint_{|\xi|=r'} \frac{f(\xi)}{\xi^{k+1}} d\xi, \end{aligned}$$

where we have used the substitution $n + 1 = -k$ in the last line. The sum of I_1 and I_2 is equal to $f(z)$ and gives the Laurent series expansion. □

The Taylor series is valid in a circle who's radius R grows outwards from an analytic point z_0 and is valid until the radius (of convergence) R reaches the first singularity/point of non-analyticity of f. The Laurent series coincides with the Taylor series if an analytic point z_0 is chosen, however the Laurent series's power is that it works when z_0 is not analytic - thus it gives us a series representation of our function f about points of non-analyticity. We'll talk about this more soon.

1.10 Zeros and Singularities of Complex Functions

Let's talk briefly about zeros and singularities of complex functions.

Definition 1.23. We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m - 1$ and $f^{(m)}(z_0) \neq 0$.

Theorem 1.24. *A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.*

Proof. Assume $f(z)$ has a zero of order m. Then Taylor expanding we have

$$\begin{aligned} f(z) &= 0 + 0 + \cdots + 0 + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots \\ &= (z-z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots \right] \\ &= (z-z_0)^m g(z). \end{aligned}$$

Conversely, if $f(z) = (z-z_0)^m g(z)$, with $g(z_0) \neq 0$, then clearly $f^{(k)}(z_0) = 0$ if $k < m$ and $f^{(m)}(z_0) = m! g(z_0) \neq 0$. □

Definition 1.25. A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.

Definition 1.26. A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Remark: Throughout this course we will only be interested in isolated singularities. An example of a non-isolated singularity would be a point which is a limit point of a sequence of other singularities of a function. For instance $f(z) = 1/\sin(1/z)$ has singularities at the points $z_n = 1/\pi n$ and the sequence $\{z_n\}$ of singular points has a limit point at $z = 0$. As far as I am aware there is no systematic theory of these points.

Definition 1.27. Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ for $0 < |z-z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Examples: $\frac{\sin z}{z}$, $\frac{1}{z^3(z+2)^2}$ and $e^{1/z}$ are an example of each type of singularity respectively.

Theorem 1.28. *A function $f(z)$ has a pole of order m at z_0 if and only if*

$$f(z) = \frac{g(z)}{(z-z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Proof. First let $f(z) = \frac{g(z)}{(z-z_0)^m}$. If $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$, then $g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$. Substituting this into our expression for $f(z)$ we find

$$f(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots ,$$

in other words, $f(z)$ has a pole of order m. On the other hand, if $f(z)$ has a pole of order m at z_0, then

$$\begin{aligned} f(z) &= \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots + a_0 + a_1(z-z_0) + \cdots \\ &= \frac{1}{(z-z_0)^m}\left[a_{-m} + a_{-(m-1)}(z-z_0) + \cdots \right] \\ &= \frac{g(z)}{(z-z_0)^m}, \end{aligned}$$

where $g(z)$ has the required properties. □

1.11 Residue Theory

Let's consider $f(z)$ represented by its Laurent series about z_0, that is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where $0 < |z - z_0| < R$. The coefficient a_{-1} in this expansion is special and has a name given to it.

Definition 1.29. The coefficient a_{-1} in the Laurent series expansion is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \mathrm{Res}(f, z_0).$$

Examples:

1. for $f = \frac{3}{z}$, we have $\mathrm{Res}(f, 0) = 3$;
2. for $f = \frac{1}{z^2}$, we have $\mathrm{Res}(f, 0) = 0$;
3. for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\mathrm{Res}(f, 0) = 0$;
4. for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\mathrm{Res}(f, 0) = 1$.

Computing Residues of poles

When the function $f(z)$ contains a pole of order m at z_0 there are some techniques we can use to find the residue of f at z_0. Suppose

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + \cdots ,$$

so that $f(z)$ has a pole of order m at z_0. Then:

- If $m = 1$: $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$,
- If $m = 2$: $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz} [(z - z_0)^2 f(z)]$,
- For a pole of order m: $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$.

There are a few more useful tricks to know: for instance, suppose $f(z)$ is of the form

$$f(z) = \frac{A(z)}{(z - z_0)^m},$$

where $A(z)$ is analytic at $z = z_0$ (and that $A(z_0) \neq 0$), then

$$\mathrm{Res}(f, z_0) = \frac{A^{(m-1)}(z_0)}{(m-1)!}. \tag{10}$$

Alternatively, if $f(z)$ contains a simple pole (pole of order $m = 1$) and $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0 and B has a simple zero at z_0 ($m = 1$), with $A(z_0) \neq 0$, then (**exercise:** derive this!)

$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}. \tag{11}$$

Example Let $f(z) = \frac{1}{z^4+1}$. Then the poles of f are the points such that $z^4 + 1 = 0$, which is equivalent to $z = e^{\frac{i\pi}{4} + \frac{i\pi k}{2}} = \pm\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$. By (11), the residue of f at $z = z_j$ is:

$$\text{Res}(f, z_j) = \frac{1}{4z_j^3} = \frac{z_j}{4z_j^4} = \frac{-z_j}{4}$$

(we use that $z_j^4 = -1$ here).

The next theorems explain why we are interested in the residues of a function, particularly at singularities.

Theorem 1.30. *Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (R is the radius of convergence), then*

$$Res(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z)dz.$$

Proof. Let $0 < r < R$ and $\gamma_r = \{z : |z - z_0| = r\}$. Then using the deformation theorem

$$\frac{1}{2\pi i} \oint_\gamma f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_r} f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_r} \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n dz,$$

upon substituting the Laurent series for $f(z)$. Now if we set $z - z_0 = re^{i\theta}$, then

$$\frac{1}{2\pi i} \oint_{\gamma_r} \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n dz = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{in\theta} ire^{i\theta} d\theta = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{n+1} a_n \int_0^{2\pi} e^{i(n+1)\theta} d\theta.$$

But

$$\int_0^{2\pi} e^{i(n+1)\theta} d\theta = \begin{cases} \left[\frac{1}{i(n+1)} e^{i(n+1)\theta}\right]_0^{2\pi} = 0, & \text{if } n \neq -1 \\ 2\pi, & \text{if } n = -1. \end{cases}$$

So putting everything together

$$\frac{1}{2\pi i} \oint_\gamma f(z)dz = \frac{1}{2\pi} a_{-1}(2\pi) = a_{-1} = \text{Res}(f, z_0).$$

□

Theorem 1.31 (Residue Theorem)**.** *Let $f(z)$ be a single-valued analytic function inside a domain $\mathcal{D}$ bounded by a closed path γ except at the isolated non-branching singularities (either essential singularities, or poles) $z_1, z_2, \cdots z_n$ which are lying inside γ. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f, z_j). \qquad (12)$$

Proof. Figure 10 shows a domain $\mathcal{D}$ and z_j are the isolated singularities of f.

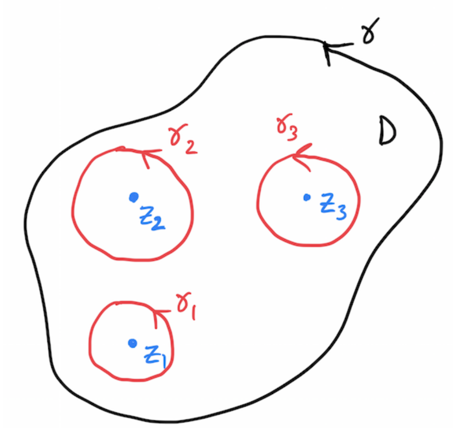


Figure 10: A domain $\mathcal{D}$, bounded by γ, with $\{z_j\}$ as singularities of f. The paths γ_j are small circles of radius r_j around z_j.

Consider a small circle around z_j, denoted by $\gamma_j = \{z : |z - z_j| = r_j\}$, where r_j is chosen sufficiently small so that γ_j lies inside γ and γ_j contains only z_j and no other singularities of $f(z)$. Then, using the deformation theorem

$$\begin{aligned}\oint_\gamma f(z)dz &= \sum_{j=1}^{n} \oint_{\gamma_j} f(z)dz \\ &= \sum_{j=1}^{n} (2\pi i \text{Res}(f, z_j)) \\ &= 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j),\end{aligned}$$

where we have used the previous theorem 1.30. □

Example Let $f(z) = \frac{1}{1+z^4}$. We already know that it has four poles: $z_j = \pm\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$ and the corresponding residues are

$$\text{Res}(f, z_j) = \frac{-z_j}{4}.$$

We want to compute the contour integral of f over different paths, for example, γ_1 and γ_2 as shown in figure 11; the points z_1 and z_4 are inside γ_1, and z_3 and z_4 are inside γ_2.

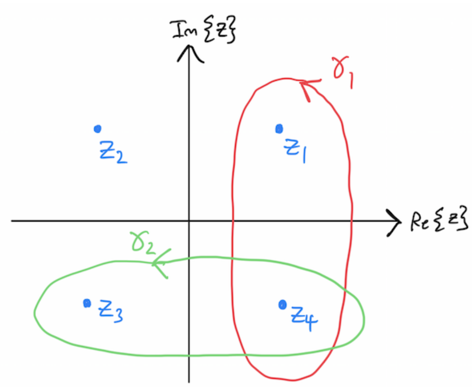


Figure 11: The poles $z_1, ..., z_4$ of f, and two paths γ_1 and γ_2.

We have:

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_1} \frac{1}{1+z^4}dz = 2\pi i \left[\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_4)\right] = -\frac{2\pi i}{4}(z_1 + z_4) = -\pi i \frac{\sqrt{2}}{2},$$

and

$$\oint_{\gamma_2} f(z)dz = \oint_{\gamma_2} \frac{1}{1+z^4}dz = 2\pi i \left[\mathrm{Res}(f, z_3) + \mathrm{Res}(f, z_4)\right] = -\frac{2\pi i}{4}(z_3 + z_4) = \frac{-\pi\sqrt{2}}{2}.$$

1.12 Analytic Continuation

We saw earlier that the values of an analytic function at any point z inside the analyticity domain are completely determined by its values on the boundary of the domain only (by the Cauchy Integral formula). Well, the somewhat opposite is also true: information about the local behaviour of an analytic function near any point inside the analyticity domain completely determines the function globally, i.e., everywhere in the domain.

Theorem 1.32 (Analytic Continuation)**.** *If f and g are analytic in a connected domain $\mathcal{D}$ and $f = g$ in some common region D' (this may be a line segment/contour or a patch or even a collection of points on a convergent sequence) within D, then $f \equiv g$ throughout $\mathcal{D}$.*

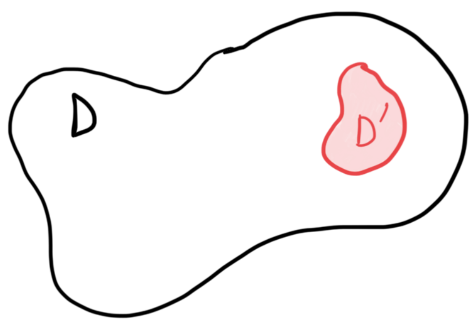


Figure 12: A region D with a patch D' inside indicated in red.

Example: Consider the function

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This is convergent for $|z| < 1$. So we have a representation for this analytic function everywhere inside the unit disc. However consider now the function

$$g(z) = \frac{1}{1-z}.$$

This function coincides with $f(z)$ inside the unit disc, and is analytic everywhere in the complex plane except at $z = 1$, hence $g(z)$ is an analytic continuation of $f(z)$ to all points outside and on the unit disc (except $z = 1$).

Consequence of this theorem

When an analytic function $f(z)$ in D' is analytically continued to a region D such that $D \cap D' \neq \emptyset$, this continuation is unique.

The main observation of the theory of complex analytic functions is that local data of an analytic function in a neighbourhood of any given point contains all the information about this function anywhere else in the complex plane, however far away from the initial point. In many situations this allows one to replace computations performed in one part of the complex plane by simpler computations at a different part. We will see examples of this approach in the following sections. The powerful idea is that whenever we have to work with a function of real variables which admit an analytic continuation to complex numbers, we continue the function to as large a domain as possible in the complex plane, and then perform computations anywhere in this domain in order to obtain (with luck) as much information as possible about the behaviour for real values of the variables.

1.13 Examples of Contour Integrals

Throughout this course we will be interested in evaluating many real integrals. A key technique to enable us to do this is to analytically continue the real integrand into the complex plane, where we will then close the contour (path of integration) and utilise the residue/Cauchy's theorem to enable us evaluate the original real integral. Let's look at some examples:

1). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

We introduce $f(z) = \frac{z^2}{z^4+1}$, and consider

$$\oint_\gamma \frac{z^2}{z^4+1} dz,$$

where $\gamma = \gamma_1 + \gamma_R$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$ and $\gamma_R = \{z : z = Re^{i\theta}, \theta \in [0, \pi]\}$ as shown in figure 13.

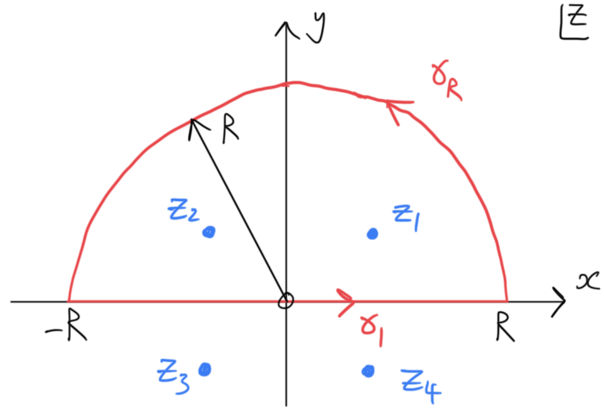


Figure 13: A semi-circular contour in the UHP encapsulating two of the four poles.

The function $f(z)$ has simple poles at $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$ and $e^{i7\pi/4}$, shown in blue in figure 13. Now, as $R \to \infty$, by the residue theorem

$$\begin{aligned}
\oint_\gamma f(z)dz &= 2\pi i\left[\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right] \\
&= 2\pi i\left(\left(\frac{z^2}{4z^3}\right)\Bigg|_{z_1} + \left(\frac{z^2}{4z^3}\right)\Bigg|_{z_2}\right) \\
&= \frac{\pi i}{2}\left[\frac{1}{z_1} + \frac{1}{z_2}\right] \\
&= \frac{\pi i}{2}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right) \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Now consider integrals around the separate components of γ as $R \to \infty$.
On γ_1:

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = I.$$

On γ_R: $z = Re^{i\theta}$:

$$\int_{\gamma_R} f(z)dz = \int_0^\pi \frac{R^2e^{2i\theta}}{R^4e^{4i\theta}+1}iRe^{i\theta}d\theta = \int_0^\pi \frac{iR^3e^{3i\theta}}{R^4e^{4i\theta}+1}d\theta.$$

As $R \to \infty$:

$$\begin{aligned}
\left|\int_{\gamma_R} f(z)dz\right| &\leq \max_{\theta\in\gamma}\left\{\frac{iR^3e^{3i\theta}}{R^4e^{4i\theta}+1}\right\} \times \pi \\
&\leq \frac{R^3}{R^4-1} \to 0 \text{ as } R \to \infty,
\end{aligned}$$

where we have used the ML-inequality in the first line and the fact that $|R^4e^{4i\theta}+1| \geq ||R^4e^{4i\theta}|-1| \geq R^4-1$ by the triangle inequality to get to the second line. Hence

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0.$$

So, putting everything together gives

$$\begin{aligned}
\lim_{R\to\infty}\oint_\gamma f(z)dz &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz + \lim_{R\to\infty}\int_{\gamma_1} f(z)dz \\
\Rightarrow \frac{\pi}{\sqrt{2}} &= 0 + I \\
\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx &= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Remark: We could have chosen to close γ_1 with a semicircular contour γ_R in the LHP (lower-half plane) if we had wished. Instead we would've picked up the residues of f at z_3 and z_4 in the residue theorem and, since we would be integrating clockwise now, would pick up a minus sign. All in all we should recover the same answer upon doing this (**exercise:** check this!).

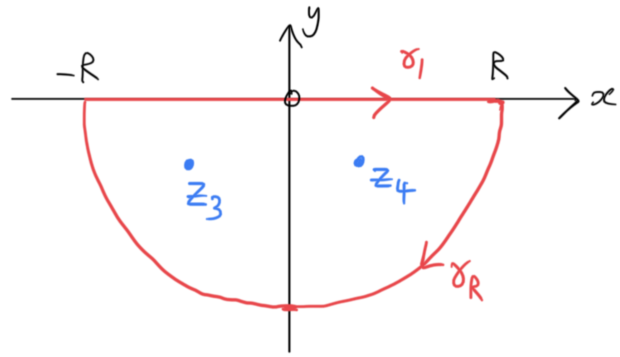


Figure 14: A semi-circular contour in the LHP.

2). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx, \quad a, k > 0.$$

Let $f(z) = \frac{e^{ikz}}{(z-ia)(z+ia)}$. Note we have an exponential in the integrand now - so we must take care with the sign in the exponential. Observe

$$e^{ikz} = e^{ik(x+iy)} = e^{ikx} e^{-ky}.$$

Now for $k > 0$, as was given, we need $-ky < 0$ giving $y > 0$ to have exponential decay of our integrand (and to avoid exponential growth). Thus we should choose to close our contour with γ_R as a semi-circle in the **upper-half plane**, see figure 15.

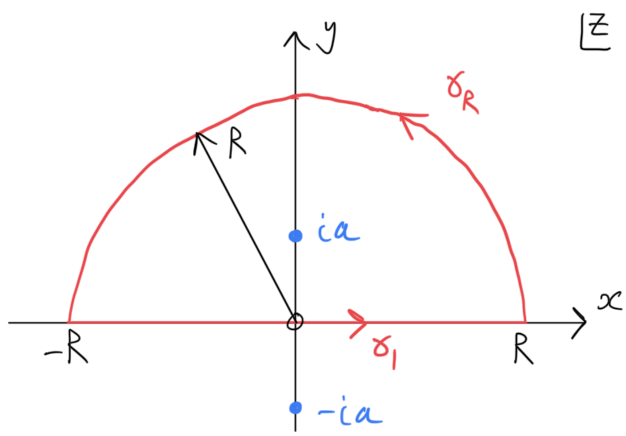


Figure 15: A semi-circular contour in the UHP containing the pole at ia.

Here $\gamma = \gamma_1 + \gamma_R$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$ and $\gamma_R = \{z : z = Re^{i\theta}, \theta \in [0, \pi]\}$. $f(z)$ has simple poles at $z = \pm ia$. By (11) the residue of $f(z)$ at $z = ia$ is $e^{ik(ia)}/(ia + ia) = e^{-ka}/(2ia)$. Thus by the residue theorem

$$\oint_\gamma \frac{e^{ikz}}{z^2 + a^2} dz = 2\pi i \left(\frac{e^{-ka}}{2ia} \right) = \frac{\pi}{a} e^{-ka}.$$

Now look at the integrals around the separate components of γ as $R \to \infty$. As before

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)dz = I.$$

On γ_R, we have $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$. As $R \to \infty$:

$$\begin{aligned}
\left| \int_{\gamma_R} f(z)dz \right| &= \left| \int_0^\pi \frac{e^{ikR(\cos\theta + i\sin\theta)}}{R^2 e^{2i\theta} + a^2} iRe^{i\theta} d\theta \right| \\
&\leq \int_0^\infty \left| \frac{e^{ikR\cos\theta} e^{-kR\sin\theta}}{R^2 e^{2i\theta} + a^2} R \right| d\theta \\
&\leq \frac{R}{R^2 - a^2} \to 0 \text{ as } R \to \infty,
\end{aligned}$$

since $k > 0$, and for $0 \leq \theta \leq \pi$; $\sin\theta \geq 0$. Thus

$$\int_{\gamma_R} f(z)dz = 0,$$

so we have $I = \frac{\pi}{a} e^{-ka}$.

3). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2} dx, \quad k > 0.$$

Note that $\cos kx = \mathrm{Re}\{e^{ikx}\}$. Hence, using the result from the last example

$$\begin{aligned}
I &= \mathrm{Re} \left\{ \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx \right\} \\
&= \frac{\pi}{a} e^{-ka}.
\end{aligned}$$

4). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx, \quad \text{where } 0 < a < 1.$$

Let $f(z) = e^{az}/(1 + e^z)$. Can we choose the contour with a semi-circle as before?

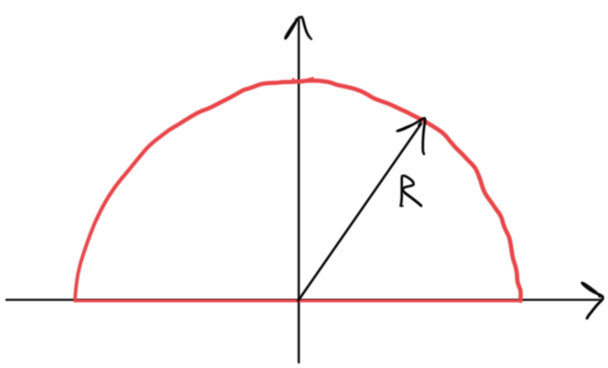


Figure 16: A semi-circular contour in the UHP.

On such a semi-circle $z = Re^{i\theta}$. Then

$$f(z) = \frac{e^{aR(\cos\theta + i\sin\theta)}}{1 + e^{R(\cos\theta + i\sin\theta)}} = \frac{e^{aRi\sin\theta}e^{aR\cos\theta}}{1 + e^{R(\cos\theta + i\sin\theta)}}.$$

Now the modulus of $e^{aRi\sin\theta}$ is 1, but look at the second term in the numerator, for $0 < \theta < \pi$ (or $-\pi < \theta < 0$ if we tried to close with a semi-circle below) the sign of $\cos\theta$ and hence of $aR\cos\theta$ **changes** over the range of θ values. So we can't use a semi-circle...somewhere we will experience exponential growth of the integrand!

Instead, consider

$$\oint_\gamma f(z)dz,$$

where $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$, $\gamma_2 = \{z : z = R + iy, y \in [0, 2\pi]\}$, $\gamma_3 = \{z : z = x + 2\pi i, x \in [R, -R]\}$ and $\gamma_4 = \{z : z = -R + iy, y \in [2\pi, 0]\}$.

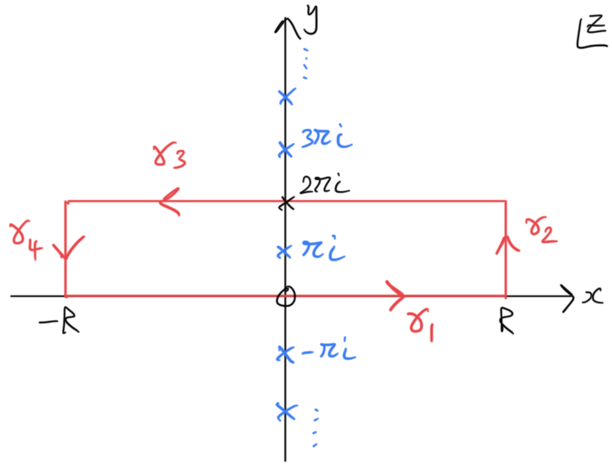


Figure 17: A rectangular contour containing one of the infinitely many poles.

Now $f(z) = e^{az}/(1 + e^z)$ has singularities where $e^z = -1$. Putting $z = x + iy$ into this leads to $z = (2k + 1)\pi i$, $k \in \mathbb{Z}$ (**exercise:** check this!). The singularities are shown in blue in figure 17. Only the singularity at πi is within γ. It turns out that these singularities are all simple poles - let's show this for the singularity at πi: let's look at $f(z)$ local to πi; let $z = \pi i + \varepsilon$, for small ε. Then

$$\begin{aligned} f(z) = f(\pi i + \varepsilon) &= \frac{e^{a(\pi i+\varepsilon)}}{1+e^{(\pi i+\varepsilon)}} \\ &= \frac{e^{\pi i a}e^{a\varepsilon}}{1-e^{\varepsilon}} \\ &= \frac{e^{\pi i a}(1+a\varepsilon+O(\varepsilon^2))}{1-(1+\varepsilon+O(\varepsilon^2))} \\ &= \frac{e^{\pi i a}(1+a\varepsilon+O(\varepsilon^2)}{-\varepsilon(1+O(\varepsilon))} \\ &= -\frac{e^{\pi i a}}{\varepsilon}(1+a\varepsilon+O(\varepsilon^2))(1+O(\varepsilon)) \\ &= -\frac{e^{\pi i a}}{\varepsilon}+O(1) \\ &= -\frac{e^{\pi i a}}{z-\pi i}+O(1), \end{aligned}$$

where we have used the expansions $e^x = 1 + x + x^2/2! + \cdots$ and $1/(1-z) = 1 + z + \cdots$ for $|z| < 1$ in the third and fifth lines respectively. Thus we see there is a simple pole at πi with residue $-e^{\pi i a}$. Hence by the residue theorem

$$\oint_\gamma f(z)dz = 2\pi i(-e^{\pi i a}) = -2\pi i e^{\pi i a}.$$

Consider now the integrals around separate components of γ.

On γ_2:

$$\begin{aligned} \left|\int_{\gamma_2} f(z)dz\right| &= \left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}} i dy\right| \\ &\le 2\pi \cdot \max_{y\in\gamma_2}\left\{\left|\frac{e^{aR}e^{aiy}}{1+e^Re^{iy}}\right|\right\} \\ &\le 2\pi\cdot\frac{e^{aR}}{e^R-1} \\ &\to 0 \text{ as } R\to\infty \text{ since } 0<a<1, \end{aligned}$$

where in the second line we have used the ML-inequality and in the third line have used $|1+e^R| \geq ||e^R|-1| = e^R - 1$ by the triangle inequality.

Similarly, one can show that

$$\lim_{R\to\infty}\int_{\gamma_4} f(z)dz = 0.$$

On γ_3:

$$\int_{\gamma_3} f(z)dz = \int_\infty^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{(x+2\pi i)}}dx = -\int_{-\infty}^{\infty}\frac{e^{2\pi i a}e^{ax}}{1+e^x}dx = -e^{2\pi i a}I.$$

Thus, putting everything together, we have

$$\oint_\gamma f(z)dz = -2\pi i e^{\pi i a} = (1-e^{2\pi i a})I.$$

$$
\begin{aligned}
\Rightarrow I &= \frac{-2\pi i e^{\pi i a}}{1-e^{2\pi i a}} \times \frac{e^{-\pi i a}}{e^{-\pi i a}} \\
&= \pi\left(\frac{2i}{e^{\pi i a}-e^{-\pi i a}}\right) \\
&= \frac{\pi}{\sin(\pi a)}.
\end{aligned}
$$

General Method to Evaluate $I = \int_a^b f(x)dx$:

1). Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.

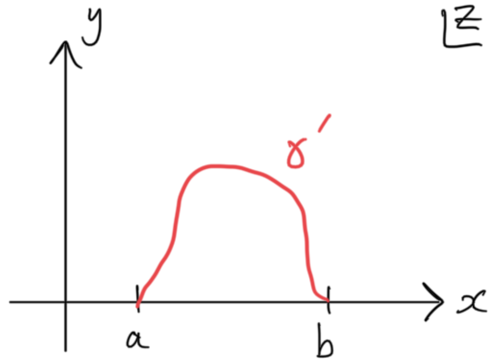


Figure 18: A contour γ' added to $[a, b]$.

2). Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$ (for example $\text{Re}\{g(x)\} = f(x)$ as in example 3). $g(z)$ is sometimes referred to as the **auxiliary** function.

3). Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.

4). If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

1.14 Branch Cuts for Multi-valued Functions

Throughout this course we'll encounter multi-valued functions such as non-integer powers of z (e.g $z^{1/2}$, $z^{1/3}$, etc) and logarithms.

Definition 1.33. A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., circling along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

A way to deal with multi-valued analytic functions is to decompose them into single-valued '**branches**', by removing from the complex plane the so-called '**branch cuts**'.

Definition 1.34. A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

A branch cut must pass through all branching points; often it has an arc which extends to infinity. There is much freedom in the choice of branch cuts and we can choose the cut to suit our needs.

Examples

1. Let's start off with the complex logarithm. We define

$$f(z) = \log z = \log|z| + i\arg\{z\}. \tag{13}$$

This is a multi-valued function, since the argument of z is not unique. We can add or take away any integer multiple of 2π and we have a different value for $\log z$ for the same point z.

The logarithm has an infinite number of **branches** (one corresponding to each integer multiple of 2π we can add to the argument). To enable us to work with this function, we need to make it single-valued. To do this, let's start by finding its branch points.

First, clearly $z = 0$ is a branch point, since on traversing a small circuit around $z = 0$, $\arg\{z\} = \theta$ changes by $2\pi \Rightarrow f(z) = \log z = \log|z| + i\theta$ changes its value by $2\pi i$. It is clear no other finite point can be a branch point for this function, but what about complex infinity? Well as it turns out $z = \infty$ (the point at infinity or complex infinity) is a branch point. To see this, introduce $w = 1/z$. Then $z = \infty$ corresponds to $w = 0$, and $f(z) = \log z = -\log w$. Hence on traversing a small circuit around $w = 0$, by the argument used for $z = 0$ above, $\log w$ changes value and hence $\log z$ changes value.

To construct a single-valued function from $\log z$ we introduce a branch cut joining together the branch points 0 and ∞. Figure 19 shows two possibilities.

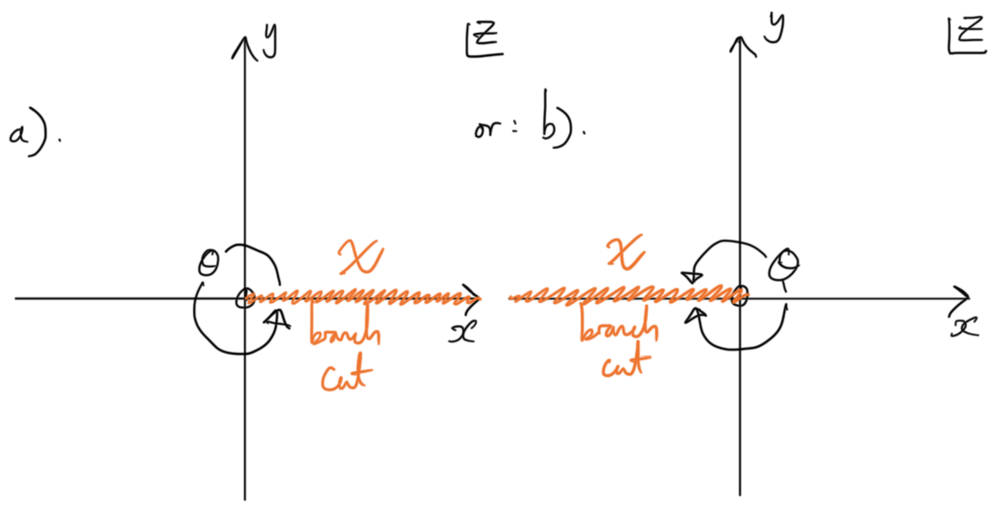


Figure 19: a). corresponds to restricting $0 \leq \theta \leq 2\pi$. b). corresponds to restricting $-\pi \leq \theta \leq \pi$.

Any line joining $z = 0$ to $z = \infty$ with θ appropriately restricted would work as a branch cut for this function, not just the cases a) and b) above.

One thing to note is that on either side of the branch cut, the value of $\arg\{z\} = \theta$ differ by 2π (they approach 0 and 2π in a). and they approach $-\pi$ and π in b).), so the branch cut is a discontinuity for the function $f(z)$.

2. Let's look at an example of a non-integer power of z. Let $f(z) = z^{1/2} = \sqrt{z}$. If we let $z = re^{i\theta}$, then $f(z) = (re^{i\theta})^{1/2} = \sqrt{r}e^{i\theta/2}$. We can now see that this function is multi-valued since if we let $\theta \mapsto \theta + 2\pi$, then $f(z) = \sqrt{r}e^{i\left(\frac{\theta+2\pi}{2}\right)} = -\sqrt{r}e^{i\theta/2}$.

   In fact we can see that this function has **two branches**, a usual way to describe each is to say one is where $f(z) \sim \sqrt{x}$ as $x \to \infty$ and the other where $f(x) \sim -\sqrt{x}$ as $x \to \infty$. There are only two branches, since a second circling of the origin takes us back to the same output as where we started (imagine adding 4π to theta in the calculation above and see what happens).

   $f(z) = \sqrt{z}$ also has branch points at $z = 0$ and $z = \infty$. To see $z = 0$ is a branch point, consider $z_0 = \varepsilon e^{i\theta}$, where ε is small. At this point $f(z_0) = \sqrt{\varepsilon}e^{i\theta_0/2}$.

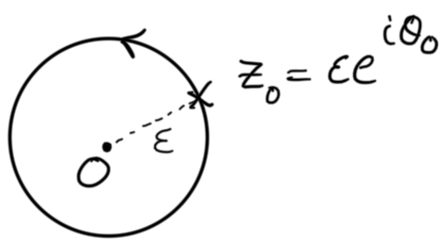
   

   Figure 20: A small circular path around $z = 0$.

   But after circling around $z = 0$ one time we have $f(z_0) = \sqrt{\varepsilon}e^{i\frac{(\theta_0+2\pi)}{2}} = -\sqrt{\varepsilon}e^{i\theta_0/2}$. i.e. $\sqrt{z_0}$ has changed its value $\Rightarrow z = 0$ is a branch point. To show $z = \infty$ is a branch point, in a similar manner to as in example 1, let $w = 1/z$ and then consider $f(w)$ upon making a small circle around $w = 0$.

   To construct a single-valued branch of this function we introduce a branch cut connecting $z = 0$ to $z = \infty$. As in example 1, figure 19, cases a) and b) would both work.

   Note in any case one can check which branch of the function we have implicitly chosen (the choice of restriction on our range of angles θ implicitly chooses the branch of the function we have). It turns out both cases a) and b) from figure 19 with the restrictions on θ as given select the branch where $f(z) \sim \sqrt{x}$ as $x \to \infty$. For example in case b) for $z = x \in \mathbb{R}_{>0}$, we have $\arg\{z\} = 0$, so $f(z) = \sqrt{z}e^{i0/2} = \sqrt{z} \sim \sqrt{x}$. To choose the branch which $\sim -\sqrt{x}$ as $x \to \infty$ the easiest thing to do is to add 2π to our range for θ (you can check this is what would happen).

3. More generally than example 2, regularly throughout this course we'll encounter functions of the form

   $$f(z) = \sqrt{(z-z_1)(z-z_2)} = ((z-z_1)(z-z_2))^{\frac{1}{2}}, \quad (z_1, z_2 \in \mathbb{C}).$$

   To analyze the behaviour of functions like this better, we introduce local coordinates: for $j = 1, 2$ let $r_j = |z - z_j|$ and $\theta_j = \arg\{z_j\}$, then $z - z_1 = r_1e^{i\theta_1}$ and $z - z_2 = r_2e^{i\theta_2}$, see figure 21.

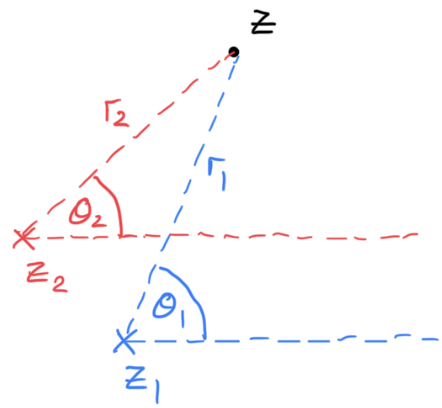


Figure 21: Schematic of the local coordinates.

Then

$$f(z) = \left[(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})\right]^{\frac{1}{2}} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{(\theta_1+\theta_2)}{2}} = (r_1 r_2)^{\frac{1}{2}} e^{i\Theta/2},$$

where $\Theta = \theta_1 + \theta_2$.

On traversing a **small** circuit around z_1 (small meaning it doesn't contain z_2), then θ_1 changes by 2π, but θ_2 remains unchanged $\Rightarrow \Theta$ changes by $2\pi \Rightarrow f(z)$ is multiplied by -1, i.e. $f(z)$ has changed, so z_1 is a branch point. Similarly, one can show that z_2 is a branch point.

Consider any other point, say z_3, in the finite plane (not ∞). On traversing a small circuit around z_3, neither θ_1 nor θ_2 changes $\Rightarrow f(z)$ doesn't change, so z_3 is **not** a branch point of $f(z)$.

Finally, consider the point at ∞. Looking at figure 22, as $R \to \infty$, the only point left **outside** of γ_R is the point at ∞.

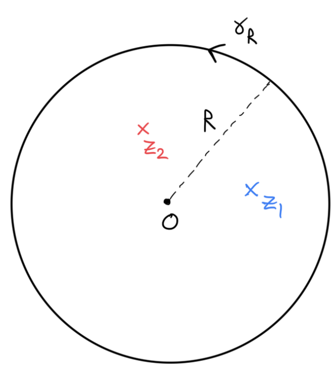


Figure 22: A circular path of radius $R \to \infty$ centered at the origin.

So we can think of moving around γ_R as moving around a small circuit about ∞. Since γ_R is large, it contains **both** z_1 and z_2, so on traversing it, both θ_1 and θ_2 change by $2\pi \Rightarrow \Theta$ changes by 4π, i.e.

$f(z)$ is multiplied by 1 and doesn't change, so ∞ is **not** a branch point.

We introduce a branch cut to make the function single-valued by connecting the branch points z_1 and z_2 together. There are two usual choices:

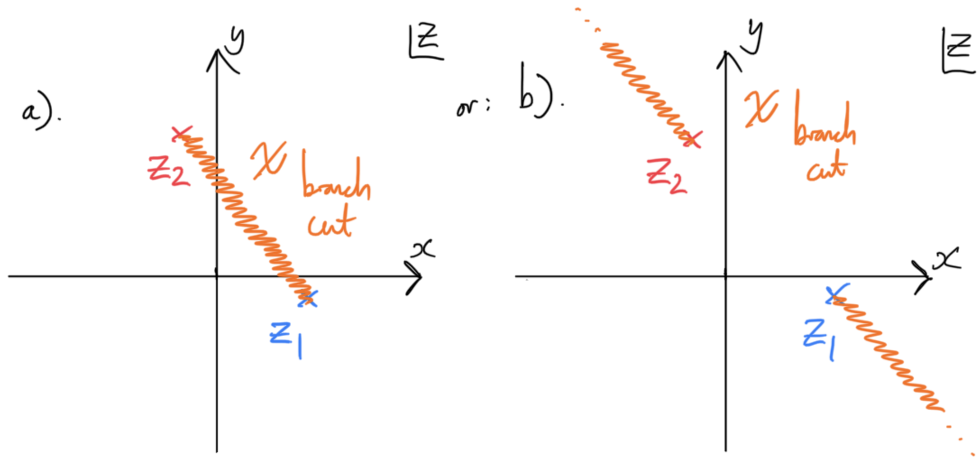


Figure 23: a). A branch cut joining z_1 and z_2 directly. b). A branch cut joining z_1 to z_2 by a straight line passing through the point at infinity.

Note that in case b) the branch cut passes through the point at ∞. Let's take a look at case a) in a little more detail. Consider the values of $f(z)$ at z_+ and z_-, two points just either side of the branch cut as shown in figure 24.

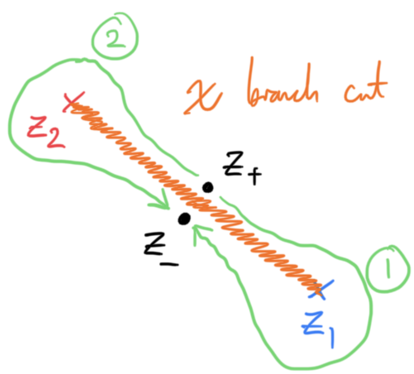


Figure 24: Two points on either side of the branch cut with two paths drawn to move from z_+ to z_-.

On moving from z_+ to z_- along the path labelled 1 in green, θ_1 changes by 2π but θ_2 stays the same (z_+ and z_- are **just** on either side of the cut). So Θ changes by $2\pi \Rightarrow f(z_+) = -f(z_-)$. Similarly the same thing happens with θ_2 changing if we move along the path labelled 2 in green. So $f(z)$ takes different values at points on opposite sides of the branch cut. This is a general feature of multi-valued functions and is crucial to their use in applied mathematics!

Cautionary Remark: In the next section and in other areas of the course we will be considering contour integration when branch cuts are present. So long as the branch cut remains **entirely outside** of the closed contour γ, the residue theorem and Cauchy's theorem remain valid and the same value of the integral will be found regardless of the branch cut chosen. What is not okay however is to let the contour under consideration cross over the branch cut or contain the entire branch cut in its inside.

1.15 Integrals involving multi-valued functions

When the real integrals we are interested in calculating contain multi-valued functions like $\sqrt{z}$ or $\log z$ we have to take a little more care using contour integration. Consider the following examples:

1). Evaluate

$$I = \int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \text{where } 0 < \alpha < 1.$$

Let

$$f(z) = \frac{z^{\alpha-1}}{z+1}.$$

This function is multi-valued (due to the power $\alpha - 1 \notin \mathbb{Z}$) with branch points at $z = 0$ and $z = \infty$. Let's take the branch cut connecting $z = 0$ to ∞ along the positive real axis, choosing $0 \leq \theta \leq 2\pi$. Note that $f(z)$ has a simple pole at $z = -1$. Consider

$$\oint_\gamma f(z)dz,$$

where $\gamma = \gamma_1 + \gamma_R + \gamma_2 + \gamma_\varepsilon$ given by $\gamma_1 = \{z : z = xe^{i0}; 0 < x \leq R\}$, $\gamma_R = \{z : z = Re^{i\theta}; 0 \leq \theta \leq 2\pi\}$, $\gamma_2 = \{z : z = xe^{2\pi i}; R \geq x > 0\}$ and $\gamma_\varepsilon = \{z : z = \varepsilon e^{i\theta}; 2\pi \geq \theta \geq 0\}$ as shown in figure 25 (note how I emphasize that the argument of the points in γ_1 and γ_2 are 0 and 2π respectively here).

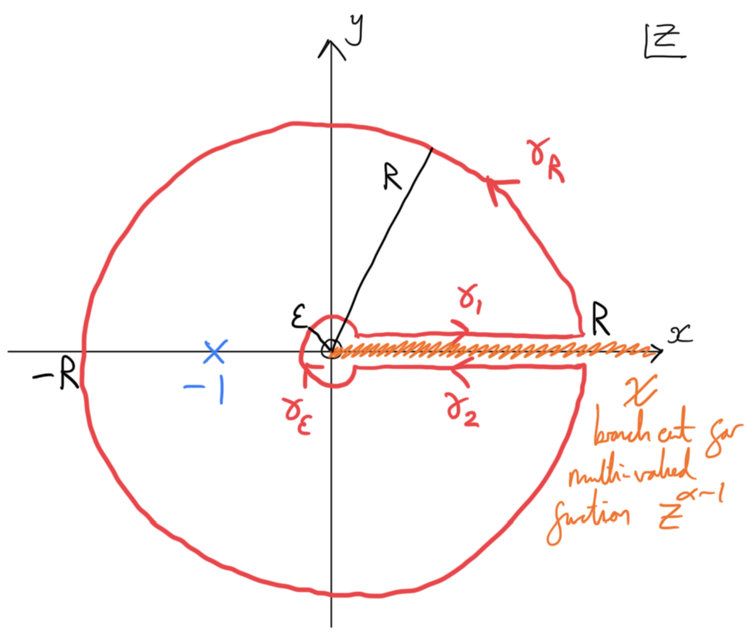


Figure 25: A 'keyhole' contour carefully avoiding the branch cut.

Then, by the residue theorem

$$\begin{aligned}\oint_\gamma f(z)dz &= 2\pi i \mathrm{Res}(f,-1)\\ &= 2\pi i(-1)^{\alpha-1}\\ &= 2\pi i(e^{\pi i})^{\alpha-1}\\ &= -2\pi i e^{\pi i\alpha},\end{aligned}$$

where we have used formula (10) with $m = 1$ in the second line to calculate the residue and the fact that $\arg\{-1\} = \pi$ based on our choice of branch cut $(0 \le \theta \le 2\pi)$ in the third line. Now let's consider the integrals along the separate components of γ.

On γ_R: $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$.

$$\begin{aligned}\Rightarrow \left|\int_{\gamma_R} f(z)dz\right| &\le \int_0^{2\pi}\left|\frac{(Re^{i\theta})^{\alpha-1}}{1+Re^{i\theta}}iRe^{i\theta}\right|d\theta\\ &= \int_0^{2\pi}\left|\frac{iR^\alpha e^{\alpha i\theta}}{1+Re^{i\theta}}\right|d\theta\\ &\le 2\pi\frac{R^\alpha}{R-1} \sim R^{\alpha-1} \to 0 \text{ as } R\to\infty,\end{aligned}$$

since $\alpha - 1 < 0$. Thus

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0.$$

On γ_ε: $z = \varepsilon e^{i\theta}$, $\theta \in [2\pi, 0]$.

$$\begin{aligned}\left|\int_{\gamma_\varepsilon} f(z)dz\right| &= \left|\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{\alpha-1}}{1+\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta\right|\\ &\le \int_0^{2\pi}\left|\frac{\varepsilon^\alpha e^{\alpha i\theta}}{1+\varepsilon e^{i\theta}}\right|d\theta\\ &\le 2\pi\frac{\varepsilon^\alpha}{-\varepsilon+1} \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)dz = 0.$$

On γ_1: $z = xe^{i0}$; $0 < x \le R$.

$$\begin{aligned}\int_{\gamma_1} f(z)dz &= \int_0^R \frac{(xe^{i0})^{\alpha-1}}{1+xe^{i0}}dx\\ &= \int_0^R \frac{x^{\alpha-1}}{1+x}dx,\end{aligned}$$

noting that $e^{i0(\alpha-1)} = e^{i0} = 1$ to reach the second line (this may seem over the top when the argument is 0, but when dealing with branch cuts it is very important to take care the correct values of the function are being calculated). Hence

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\gamma_1} f(z)dz = I.$$

On γ_2: $z = xe^{2\pi i}$; $R \geq x > 0$.

$$\begin{aligned}\int_{\gamma_2} f(z)dz &= \int_R^0 \frac{(xe^{2\pi i})^{\alpha-1}}{1+xe^{2\pi i}}dx \\ &= -\int_0^R \frac{e^{2\pi i\alpha}x^{\alpha-1}}{1+x}dx,\end{aligned}$$

noting that $e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha}e^{-2\pi i} = e^{2\pi i\alpha}$ to reach the second line (notice now how we would've missed this crucial extra factor along γ_2 had we not been careful and used 2π for the argument here). Hence

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\gamma_2} f(z)dz = -e^{2\pi i\alpha} I.$$

So, putting everything together:

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_R} + \int_{\gamma_\varepsilon} \right)$$

$$\begin{aligned}\Rightarrow -2\pi i e^{\pi i\alpha} &= (1-e^{2\pi i\alpha})I + 0 \\ \Rightarrow I &= \pi\left(\frac{2ie^{\pi i\alpha}}{e^{2\pi i\alpha}-1}\right) \times \frac{e^{-\pi i\alpha}}{e^{-\pi i\alpha}} \\ &= \pi\left(\frac{2i}{e^{\pi i\alpha}-e^{-\pi i\alpha}}\right) \\ &= \frac{\pi}{\sin(\alpha\pi)}.\end{aligned}$$

Remark on this example: Recall the example 4) from section 1.13, namely

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}dx, \quad 0 < \alpha < 1.$$

We found that

$$I = \frac{\pi}{\sin(\alpha\pi)}.$$

If we take $e^x = t$, then $e^{\alpha x} = t^\alpha$ and $dx = \frac{dt}{t}$. Also $x = -\infty \mapsto t = 0$ and $x = \infty \mapsto t = \infty$. Hence under this substitution we find

$$I = \int_0^\infty \frac{t^{\alpha-1}}{1+t}dt,$$

which is precisely the integral we just solved in the last example. So under this correspondence

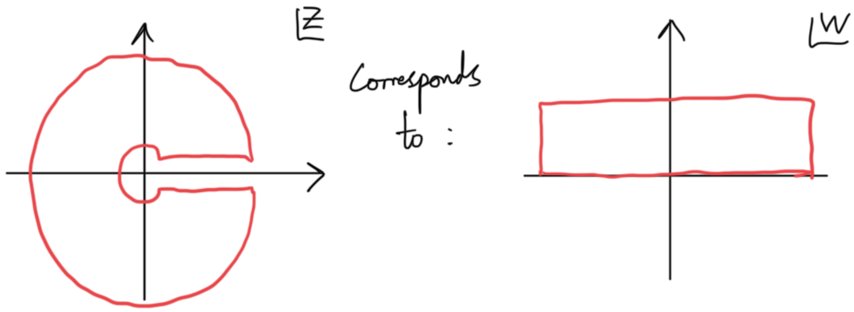


Figure 26: Correspondence between 'keyhole' and rectangle under the mapping $w = \log z$.

This can be seen from $w = \log z = \log |z| + i\theta$.

2). Evaluate

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}.$$

Here again we have a multi-valued integrand, with branch points at $x = \pm 1$. We consider

$$\oint_\gamma f(z)dz,$$

where

$$f(z) = \frac{1}{\sqrt{z^2-1}},$$

and $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_R$ as shown in figure 27 (note well here the slight difference between $f(z)$ and the function in the integrand of I).

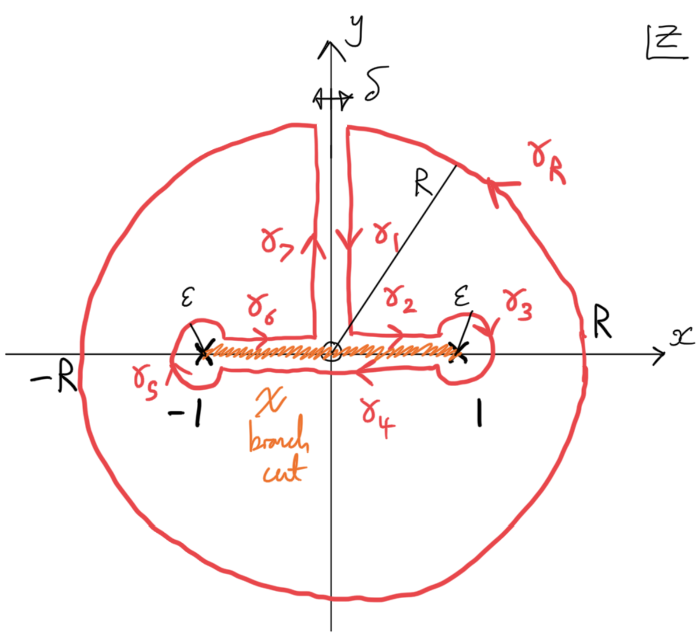


Figure 27: A contour with a cut that carefully excludes the branch cut along $[-1, 1]$.

The branch cut for $f(z)$ is chosen to connect the points $z = \pm 1$ by a straight line along the segment $[-1, 1]$ of the real axis. Our contour carefully excludes this.

With our choice of contour γ, $f(z)$ is analytic everywhere inside γ, hence by Cauchy's theorem

$$\oint_\gamma f(z)dz = 0.$$

Now to study the portions of γ near to the branch cut we introduce local coordinates: $r_1 = |z-1|$, $r_2 = |z+1|$ and lets choose $-\pi \leq \theta_1, \theta_2 \leq \pi$ (where $\theta_1 = \arg\{z-1\}$, $\theta_2 = \arg\{z+1\}$, see figure 28.

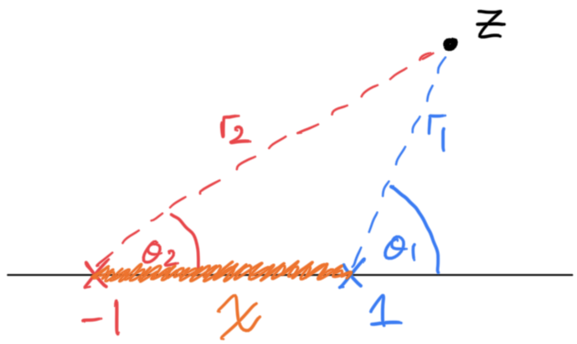


Figure 28: Local coordinates around the branch cut.

Note: The choices of the ranges for θ_1 and θ_2 are free to be chosen by us, however this choice will implicitly decide which branch of the multi-valued function we are taking. The choice I've made here corresponds to the branch of $f(z)$ that $\sim \sqrt{x^2-1}$ as $x \to \infty$ along the real axis - you'll see why when we consider the integral along γ_R soon.

Now let's examine the components of γ near the branch cut first. Let's start with γ_2.
On γ_2: $\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = |z-1| = 1-x$ and $r_2 = |z+1| = x+1$. See figure 29 for an illustration of where these values come from.

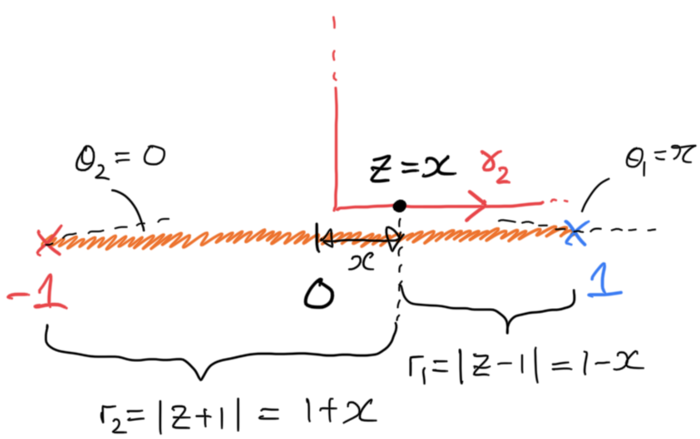


Figure 29: Values of the local coordinates when $z \in \gamma_2$.

Hence we find on γ_2:

$$\begin{aligned}\sqrt{z^2-1} &= (r_1r_2)^{1/2}e^{i\frac{(\theta_1+\theta_2)}{2}} \\ &= \sqrt{1-x^2}e^{i\pi/2} \\ &= i\sqrt{1-x^2}.\end{aligned}$$

On γ_4**:** $\theta_1=-\pi$, $\theta_2=0$, $r_1=1-x$ and $r_2=1+x$, so now we find

$$\begin{aligned}\sqrt{z^2-1} &= (r_1r_2)^{1/2}e^{i\frac{(\theta_1+\theta_2)}{2}} \\ &= \sqrt{1-x^2}e^{-i\pi/2} \\ &= -i\sqrt{1-x^2}.\end{aligned}$$

As expected with multi-valued functions, we have a jump in the function values on either side of the branch cut.
Similarly to γ_2, **on** γ_6**:** $\theta_1=\pi$, $\theta_2=0$, $r_1=1-x$ and $r_2=1+x$, so again $\sqrt{z^2-1}=i\sqrt{1-x^2}$ here.

Hence, as $R\to\infty$, $\varepsilon\to 0$ and $\delta\to 0$, we have

$$\begin{aligned}\int_{\gamma_2+\gamma_6} f(z)dz &= \int_{-1}^{1}\frac{1}{i\sqrt{1-x^2}}dx = -i\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx = -iI \\ \int_{\gamma_4} f(z)dz &= \int_{1}^{-1}\frac{1}{-i\sqrt{1-x^2}}dx = -i\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx = -iI\end{aligned}$$

What about the other components of γ?
First note that

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_7} f(z)dz = 0,$$

since along this portion of the plane there is no branch cut and hence the function varies continuously, so as $\delta\to 0$ this is equivalent to traversing one contour forwards and then backwards again giving no contribution.

On γ_3**:** $z=1+\varepsilon e^{i\theta}$; $\pi\geq\theta\geq-\pi$. Here

$$\begin{aligned}1-z^2 &= 1-(1+\varepsilon e^{i\theta})^2 \\ &= -2\varepsilon e^{i\theta}(1+O(\varepsilon)),\end{aligned}$$

hence

$$\begin{aligned}|f(z)| &= \left|\frac{1}{\sqrt{-2\varepsilon e^{i\theta}(1+O(\varepsilon))}}\right| \\ &= \left|\frac{1}{\sqrt{2\varepsilon}}\right|\left|\frac{1}{\sqrt{1+O(\varepsilon)}}\right| \\ &\approx \left|\frac{1}{\sqrt{2\varepsilon}}\right|(1+O(\varepsilon)),\end{aligned}$$

where to reach the final line the general binomial expansion ($(1+x)^n = 1 + nx + n(n-1)x^2/2! + \cdots$) was used giving $(1+O(\varepsilon))^{-1/2} = 1 + O(\varepsilon)$. This means that

$$\left|\int_{\gamma_3} f(z)dz\right| \leq \left|\frac{1}{\sqrt{2\varepsilon}}\right| (1+O(\varepsilon)) \times 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

$$\Rightarrow \lim_{\varepsilon\to 0} \int_{\gamma_3} f(z)dz = 0.$$

Similarly, one can show that $\int_{\gamma_5} f(z)dz = 0$.

Finally, consider γ_R. First we need to check which branch of our multi-valued function $f(z)$ we've chosen implicitly by our choice of restrictions for θ_1 and θ_2. Consider $x \in \mathbb{R}$, $x > 1$. Then $\theta_1 = \theta_2 = 0$, $r_1 = |x-1| = x-1$ and $r_2 = |x+1| = x+1$. So we get

$$\sqrt{z^2-1} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{(\theta_1+\theta_2)}{2}} = +\sqrt{x^2-1}.$$

Note: For a different range of θ_1, θ_2 we may have chosen the other branch and this red + sign would've been a $-$ sign; for instance setting $\pi \leq \theta_1 \leq 3\pi$ and keeping the range for θ_2 the same would've done this.

Hence, for z on γ_R: $z = Re^{i\theta}$; $\sqrt{z^2-1} \approx \sqrt{z^2} = +z$ (positive due to our choice of branch cut).

$$\begin{aligned}\Rightarrow \lim_{R\to\infty} \int_{\gamma_R} f(z)dz &= \int_0^{2\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= i\int_0^{2\pi} d\theta \\ &= 2\pi i.\end{aligned}$$

Hence, putting everything together

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0 \\ \delta\to 0}} \left(\int_\gamma = \sum_{j=1}^{7} \int_{\gamma_j} + \int_{\gamma_R} \right)$$

$$\Rightarrow 0 = -2iI + 2\pi i$$

$$\Rightarrow I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.$$

Remark: Indeed we could have arrived at this result in seconds using techniques of real calculus; setting $x = \sin\theta$ in I gives

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

Nevertheless this example teaches us how to deal with these types of functions so that when we do encounter integrals of this form that we can't do easily, we have a procedure to deal with them.

1.16 The residue at ∞

Suppose that an analytic function $f(z)$ is analytic at $z = \infty$, and further that $f(\infty) = 0$. Then we can write

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \frac{a_{-3}}{z^3} + \cdots$$

Definition 1.35. a_{-1} is called the residue of $f(z)$ at ∞. Denoted by $\mathrm{Res}(f, \infty) = a_{-1}$.

One can show that, in this case

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} f(z)dz &= \lim_{R\to\infty} \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{a_{-1}}{Re^{i\theta}} + \frac{a_{-2}}{R^2 e^{2i\theta}} + \cdots \right) iRe^{i\theta} d\theta \\ &= a_{-1},\end{aligned}$$

where γ_R is a circle centred at the origin of radius R on which $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$.

Example: In the previous example at the end of section 1.15, we could have computed the integral over γ_R as follows. First, expand $f(z)$ as

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{z^2 - 1}} \\ &= \frac{1}{\sqrt{z^2}} \frac{1}{\sqrt{1 - \frac{1}{z^2}}} \\ &= \frac{1}{z} \frac{1}{\sqrt{1 - \frac{1}{z^2}}} \\ &= \frac{1}{z} \left(1 + \frac{1}{2z^2} + O(1/z^4) \right) \\ &= \frac{1}{z} + \frac{1}{2z^3} + \cdots \end{aligned}$$

where in the third line we used the fact that $\sqrt{z^2} = +z$ by the choice of branch cut used and in the fourth line the general binomial expansion was used since on γ_R, $|z|$ is large so $1/z^2$ is small. Now we have $f(z)$ expanded in the form where we can use the above result. So we see that

$$\mathrm{Res}(f, \infty) = 1,$$

leading to

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = 2\pi i \mathrm{Res}(f, \infty) = 2\pi i,$$

as found earlier.

1.17 Principal Value Integrals

Suppose we wish to integrate a function $f(x)$ over the real interval $a < x < b$, but $f(x)$ has a singularity at say x_0, where $a < x_0 < b$ (for example, take $f(x) = 1/x$ between -1 and 1).

Definition 1.36. We define the **principal value integral** (denoted by $⨍$ - a dash through the integration symbol) over a singularity at x_0 as

$$⨍_a^b f(x)dx = \lim_{\varepsilon \to 0+} \left(\int_a^{x_0-\varepsilon} f(x)dx + \int_{x_0+\varepsilon}^b f(x)dx \right).$$

The idea here being that a small segment around x_0 is removed from the integration path such that x_0 is exactly in the middle. So long as the contributions as $\varepsilon \to 0$ on either side cancel out we are good and this is well-defined. This will become clearer in the following examples.

Examples: First let's do a baby example, then one we need complex analysis for!

1). Compute

$$⨍_{-1}^1 \frac{1}{x}dx.$$

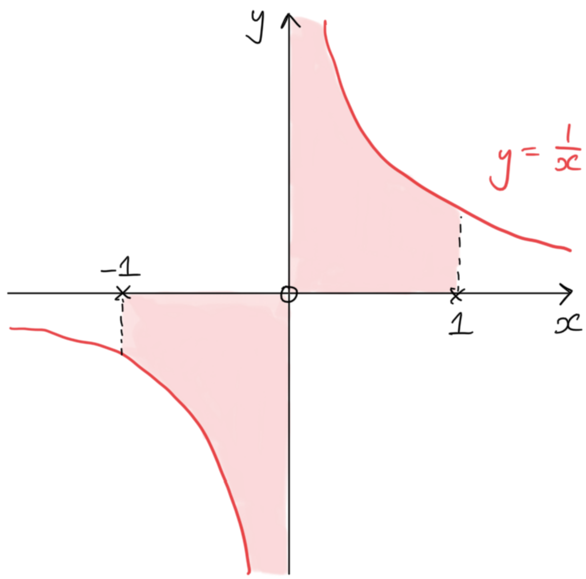


Figure 30: Shaded area 'beneath' the graph of $y = 1/x$ from -1 to 1.

Well:

$$\begin{aligned} ⨍_{-1}^1 \frac{1}{x}dx &= \lim_{\varepsilon \to 0+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x}dx + \int_{\varepsilon}^1 \frac{1}{x}dx \right] \\ &= \lim_{\varepsilon \to 0} \left([\log x]_{-1}^{-\varepsilon} + [\log x]_{\varepsilon}^1 \right) \\ &= \lim_{\varepsilon \to 0} (\log|-\varepsilon| - \log|-1| + \log|1| - \log|\varepsilon|) \\ &= \lim_{\varepsilon \to 0} (\log \varepsilon - \log \varepsilon) \\ &= 0, \end{aligned}$$

as expected on grounds of symmetry.

2). Compute

$$I = ⨍_0^\infty \frac{x^{\alpha-1}}{1-x} dx,$$

for some $0 < \alpha < 1$. This is a very similar example to example 1 in section 1.15, except this time the difference being the simple pole at $x = -1$ has been shifted to $x = 1$, along the path of integration! Hence the principal value notation.

In a similar manner, we take $f(z) = \frac{z^{\alpha-1}}{1-z}$. This function is multi-valued with branch points at $z = 0$ and ∞. Let's take the branch cut along the positive real axis (with $0 \geq \theta \geq 2\pi$). Then consider

$$\oint_\gamma f(z) dz,$$

where $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_\varepsilon + \gamma_R + \gamma_{\delta_+} + \gamma_{\delta_-}$ as shown in figure 31.

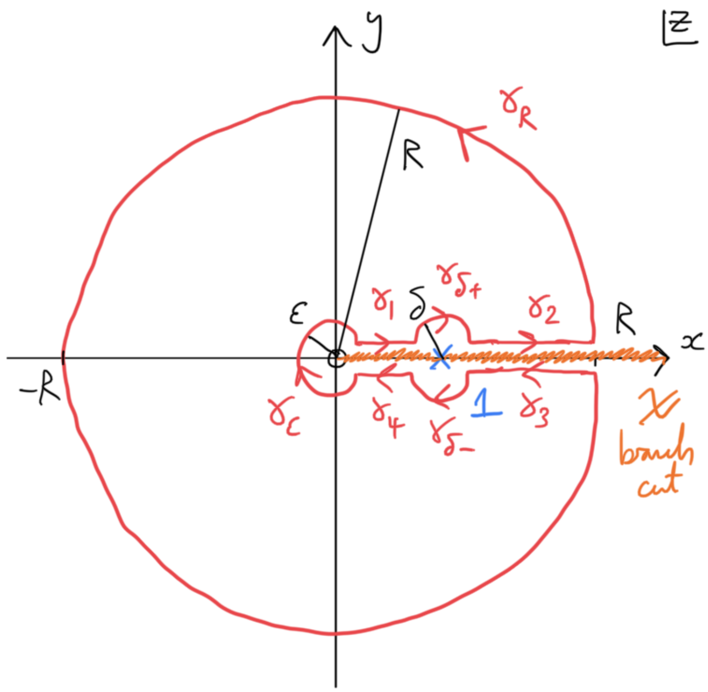


Figure 31: 'Keyhole' contour avoiding the branch cut with a 'bump' around the singular point at $z = 1$.

Consider $R \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$. $f(z)$ is analytic everywhere inside γ, hence by Cauchy's theorem

$$\oint_\gamma f(z) dz = 0.$$

Now consider the different sections of γ separately.
On γ_1, γ_2: $\theta = 0$, $z = x$, and one finds

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0 \\ \delta\to 0}} \int_{\gamma_1+\gamma_2} f(z) dz = ⨍_0^\infty \frac{x^{\alpha-1}}{1-x} dx = I.$$

On γ_3, γ_4: $\theta = 2\pi$, $z = xe^{2\pi i}$, and one finds

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0\\ \delta\to 0}} \int_{\gamma_3+\gamma_4} f(z)dz = \fint_{\infty}^{0} \frac{(xe^{2\pi i})^{\alpha-1}}{1-xe^{2\pi i}} dx = -\fint_{0}^{\infty} \frac{e^{2\pi i\alpha} x^{\alpha-1}}{1-x} dx = -e^{2\pi i\alpha} I.$$

On γ_ε: $z = \varepsilon e^{i\theta} \Rightarrow f(z) = \frac{\varepsilon^{\alpha-1} e^{i(\alpha-1)\theta}}{1-\varepsilon e^{i\theta}}$.

$$\begin{aligned} &\Rightarrow |f(z)| = \frac{|\varepsilon^{\alpha-1}|}{|1-\varepsilon e^{i\theta}|} \le \frac{|\varepsilon^{\alpha-1}|}{|1-\varepsilon|} \\ &\Rightarrow \left|\int_{\gamma_\varepsilon} f(z)dz\right| \le \frac{\varepsilon^{\alpha-1}}{1-\varepsilon} \times 2\pi\varepsilon = \frac{2\pi\varepsilon^\alpha}{1-\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

since $\alpha > 0$, where we have used the ML-inequality in the second line.

On γ_R: $z = Re^{i\theta} \Rightarrow f(z) = \frac{R^{\alpha-1} e^{i(\alpha-1)\theta}}{1-Re^{i\theta}}$.

$$\begin{aligned} &\Rightarrow |f(z)| = \frac{|R^{\alpha-1}|}{|1-Re^{i\theta}|} \le \frac{R^{\alpha-1}}{|R-1|} \\ &\Rightarrow \left|\int_{\gamma_R} f(z)dz\right| \le \frac{R^{\alpha-1}}{R-1} \times 2\pi R = \frac{2\pi R^\alpha}{R-1} \sim R^{\alpha-1} \to 0 \quad \text{as } R \to \infty, \end{aligned}$$

since $\alpha - 1 < 0$, where we have again used the ML-inequality in the second line.

On γ_{δ_+}: $z = 1 + \delta e^{i\theta}$; $\theta \in [\pi, 0]$. Local to $z = 1$ in the upper half plane upon Taylor expanding we have

$$f(z) = \frac{-1}{(z-1)} \left[z^{\alpha-1}|_{z=e^{i0}} + O(z-1) \right] = \frac{-1}{(z-1)} + O(1)$$

Thus

$$\begin{aligned} \lim_{\delta\to 0} \int_{\gamma_{\delta_+}} f(z)dz &= \lim_{\delta\to 0} \int_{\pi}^{0} \left(\frac{-1}{\delta e^{i\theta}} + O(1) \right) i\delta e^{i\theta} d\theta \\ &= \lim_{\delta\to 0} \int_{0}^{\pi} (i + O(\delta)) d\theta \\ &= \pi i. \end{aligned}$$

On γ_{δ_-}: $z = 1 + \delta e^{i\theta}$; $\theta \in [0, -\pi]$. Local to $z = 1$ in the lower half plane we have

$$f(z) = \frac{-1}{(z-1)} \left[z^{\alpha-1}|_{z=e^{2\pi i}} + O(z-1) \right] = \frac{-1}{(z-1)} (e^{2\pi\alpha i} + O(z-1))$$

Thus

$$\begin{aligned} \lim_{\delta\to 0} \int_{\gamma_{\delta_-}} f(z)dz &= \lim_{\delta\to 0} \int_{0}^{-\pi} \left(\frac{-e^{2\pi\alpha i}}{\delta e^{i\theta}} + O(1) \right) i\delta e^{i\theta} d\theta \\ &= \lim_{\delta\to 0} \int_{-\pi}^{0} (ie^{2\pi\alpha i} + O(\delta)) d\theta \\ &= i\pi e^{2\pi\alpha i}. \end{aligned}$$

So, putting everything together as usual

$$\begin{aligned}
\Rightarrow 0 &= (1 - e^{2\pi\alpha i})I + 0 + \pi i(1 + e^{2\pi\alpha i}) \\
\Rightarrow I &= -i\pi\left(\frac{1 + e^{2\pi\alpha i}}{1 - e^{2\pi\alpha i}}\right) \\
&= -i\pi\left(\frac{e^{-\pi\alpha i} + e^{\pi\alpha i}}{e^{-\pi\alpha i} - e^{\pi\alpha i}}\right) \\
&= \pi\cot(\pi\alpha).
\end{aligned}$$

Chapter 2: Singular Integral Equations

In this chapter we will learn how to solve equations of the form

$$\frac{1}{\pi}\int_a^b k(t-x)f(t)dt = g(x), \tag{14}$$

where $a < x < b$, a, b, finite, and $g(x)$ is given, for the unknown function $f(x)$. There is no explicit solution known for equations of this form for a general kernel function k, however, notable exceptions are when k is of the form

$$k(t-x) = \frac{1}{t-x}, \quad \text{or} \quad k(t-x) = \log(t-x).$$

Notice in these cases the associated integral equation (14) has a singularity along the integration path (when $t = x$), hence the name **singular integral equations**.

2.1 The Cauchy and Hilbert Transforms

In order to solve equations of the form (14) we will make use of the **Cauchy Transform** (of a function $f(\xi)$ over a path γ) (sometimes simply referred to as a Cauchy-type integral) defined by

$$C(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi, \tag{15}$$

where γ is a smooth contour and $f(\xi)$ is continuous for $\xi \in \gamma$. In general this transform is defined for complex valued functions $f(\xi)$, and γ can be any contour in the complex plane, but for simplicity in this course we will only consider the case when γ is an interval of the real line (in particular we will set $\gamma = [-1, 1]$). Since the transform depends on $f(\xi)$ and γ it is usually denoted by $C_{[f,\gamma]}(z)$ (or some other way illustrating the explicit f and γ dependence), but within our course it should be sufficiently clear what we mean so we drop this notation and simply use $C(z)$.

Proposition 2.37. *The Cauchy transform is* ***analytic*** *for all z, including as $z \to \infty$ (in fact $C(z) \sim O(\frac{1}{z})$ as $z \to \infty$),* ***except*** *for $z \in \gamma$.*

Proof. The analyticity extends from Cauchy's Integral Theorem. The proof of the decay can be found in 'Trogdon and Olver: Riemann-Hilbert Problems, Their Numerical Solution, and the Computation of Nonlinear Special Functions' on pages 34-35. □

For $z \in \gamma$, we define the **Hilbert Transform** (of a function $f(\xi)$ over a path γ) to be

$$H(z) = \frac{1}{2\pi i}⨍_\gamma \frac{f(\xi)}{\xi - z}d\xi, \tag{16}$$

where γ is a smooth contour and $f(\xi)$ is continuous for $\xi \in \gamma$. As before this transform is defined for arbitrary smooth curves $\gamma \in \mathbb{C}$, but we will restrict our attention to the case when γ is an interval of the real line. Similarly, the transform is usually denoted by $H_{[f,\gamma]}(z)$ (or some equivalent way), but for our purposes the notation $H(z)$ should be clear.

Remark: Note that for $z \in \gamma$, $H(z)$ has a singularity along the integration path, hence the use of the principal value integral sign.

Note: In other sources/previous versions of this course, the definition of $H(z)$ may have a different factor outside of the integral, like $-\frac{1}{\pi}$ for instance.

2.2 The Plemelj Formulae

Consider again the Cauchy transform (of a function $f(\xi)$ over a path γ) given by

$$C(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

Let's investigate what happens as $z \to z_0$, where z_0 is **any** point on γ (except for the end points of γ). To do this, we introduce the notation where, looking in the direction of integration along γ, we denote:

$C_+(z_0) =$ the limiting value of $C(z)$ as $z \to z_0$ from the **left** of γ.
$C_-(z_0) =$ the limiting value of $C(z)$ as $z \to z_0$ from the **right** of γ.

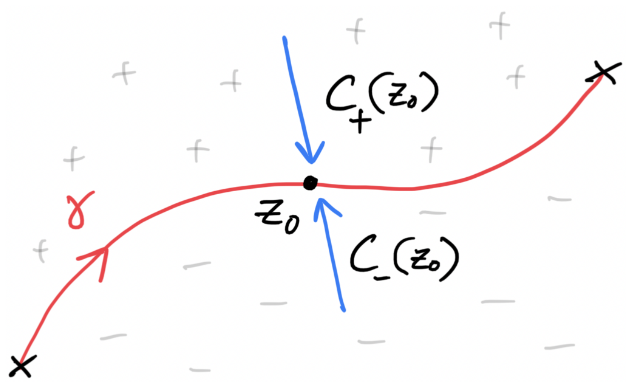


Figure 32

To examine $C_+(z_0)$, we consider the integral along the deformed contour $\gamma_\varepsilon + c_\varepsilon^+$ as shown in figure 33 and take the limit as $\varepsilon \to 0$.

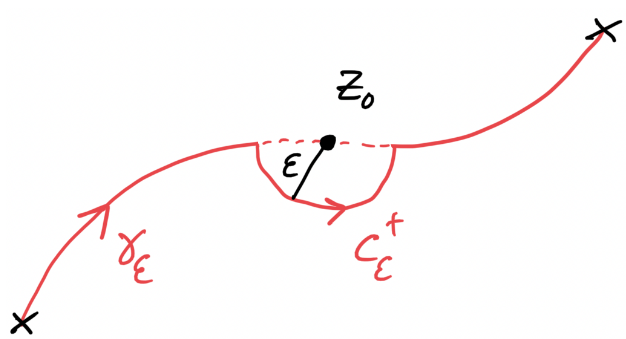


Figure 33

Here $\gamma_\varepsilon = \gamma$ with a section of length 2ε about z_0 removed and $c_\varepsilon^+ =$ semi-circle, centre z_0, radius ε. So, we have

$$C_+(z_0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon + c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi.$$

On c_ε^+ we have $z = z_0 + \varepsilon e^{i\theta}$, where $\theta_0 \leq \theta \leq \theta_0 + \pi$ (for some angle θ_0). Hence

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi &= \lim_{\varepsilon \to 0} \int_{\theta_0}^{\theta_0 + \pi} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta \\ &= i\pi f(z_0), \end{aligned}$$

where we have used the Taylor expansion of f about z_0 given by $f(z_0 + \varepsilon e^{i\theta}) = f(z_0) + O(\varepsilon)$ and then taken the limit $\varepsilon \to 0$ to reach the result. Hence we get

$$2\pi i C_+(z_0) = \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z_0} d\xi + \lim_{\varepsilon \to 0} \int_{c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi,$$

giving

$$C_+(z_0) = H(z_0) + \frac{1}{2} f(z_0), \tag{17}$$

where

$$H(z_0) = \frac{1}{2\pi i} ⨍_\gamma \frac{f(\xi)}{\xi - z_0} d\xi,$$

is the previously defined Hilbert transform. Now consider $C_-(z_0)$.

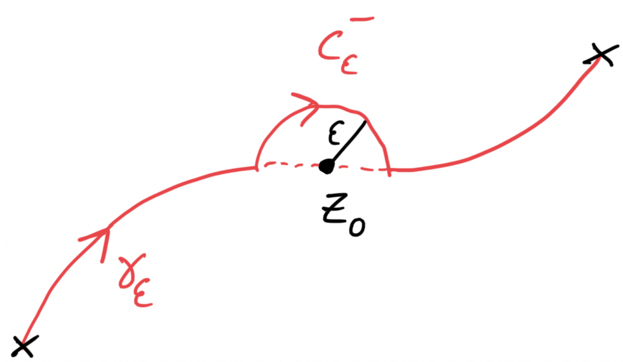


Figure 34

Here γ_ε is as before and c_ε^- is as c_ε^+ but the other half of the circle. Using similar analysis to before, but now noting $\theta_0 \geq \theta \geq \theta_0 - \pi$ (note this is $-\pi$ not $+\pi$ as we have gone around z_0 clockwise this time), we find (exercise):

$$\begin{aligned} C_-(z_0) &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon + c_\varepsilon^-} \frac{f(\xi)}{\xi - z_0} d\xi \\ &= H(z_0) - \frac{1}{2} f(z_0). \end{aligned} \tag{18}$$

Finally considering (17) + (18) and (17) - (18) we obtain:

$$C_+(z_0) + C_-(z_0) = 2H(z_0) \tag{19}$$
$$C_+(z_0) - C_-(z_0) = f(z_0). \tag{20}$$

These two equations are together known as the **Plemelj formulae**.

Example (to illustrate the Plemelj formulae in action):
Let's show that the Plemelj formulae hold for an example case. Let's take $f(x) = 1$ and take γ to be the segment of the real axis between -1 and 1: $-1 < \xi < 1$.

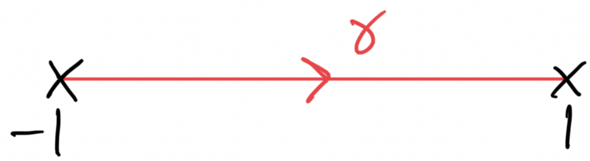


Figure 35: The path γ between $[-1, 1]$.

Thus, the **Cauchy transform** gives:

$$C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{1}{\xi - z} d\xi = \frac{1}{2\pi i} \log\left(\frac{z-1}{z+1}\right).$$

Next we want to consider $C_\pm(x)$, for $-1 < x < 1$. Note that $\log(\frac{z-1}{z+1})$ is a multi-valued function with branch points at ± 1. Taking a branch cut along γ gives a branch of this function analytic everywhere except for a jump discontinuity across γ. Introducing local coordinates:

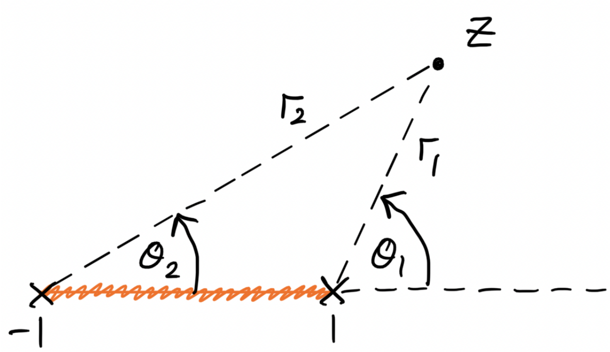


Figure 36

Here $r_1 = |z-1|$, $\theta_1 = \arg\{z-1\}$, $r_2 = |z+1|$, $\theta_2 = \arg\{z+1\}$ and let's take $-\pi < \theta_1, \theta_2 < \pi$. Then

$$C(z) = \frac{1}{2\pi i}\left(\log\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2)\right). \tag{21}$$

Now let $x^{\pm} = x \pm i\delta$ for $-1 < x < 1$ and δ small. At x^+: $\theta_1 = \pi$, $\theta_2 = 0$, so we get $\theta_1 - \theta_2 = \pi$. At x^-: $\theta_1 = -\pi$, $\theta_2 = 0$, so we get $\theta_1 - \theta_2 = -\pi$. At both $x^{\pm}$ we have $r_1 = 1 - x$ and $r_2 = 1 + x$. Then, using (21):

$$C_+(z) = \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right) + \frac{1}{2},$$
$$C_-(z) = \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right) - \frac{1}{2}.$$

Finally let's work out $H(x)$:

$$\begin{aligned} H(x) &= \frac{1}{2\pi i} ⨍_{-1}^{1} \frac{1}{\xi - x} d\xi \\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{1}{\xi - x}d\xi + \int_{x+\varepsilon}^{1}\frac{1}{\xi - x}d\xi\right] \\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\log|-\varepsilon| - \log|-1-x| + \log|1-x| - \log|\varepsilon|\right) \\ &= \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right). \end{aligned}$$

Then, for $x \in \gamma$, we have shown:

$$\begin{aligned} C_+(x) + C_-(x) = \frac{1}{\pi i}\log\left(\frac{1-x}{1+x}\right) &= 2H(x), \\ C_+(x) - C_-(x) = 1 &= f(x), \end{aligned}$$

both expected by Plemelj.

2.3 The Converse Problem

Our main goal of this chapter will be to learn how to solve equations of the form of (14). In order to motivate the solution scheme for this, let us for a moment consider the converse to this problem. That is, suppose, conversely, we are given a function $f(z)$ which is continuous along some smooth path γ, and our goal is to find a function, $G(z)$ say, which is analytic for **all** z **except** on γ, where for $z_0 \in \gamma$ it has the jump discontinuity: $G_+(z_0) - G_-(z_0) = f(z_0)$, and furthermore which vanishes at infinity.
The solution to this problem is **unique** and is given by

$$G(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi, \tag{22}$$

i.e the Cauchy transform of f on γ!

Why? Let's check this:

1). Analytic at all z except on γ:
$G(z)$ is the Cauchy transform of $f(z)$ which has this analyticity property.

2). $G_+(z_0) - G_-(z_0) = f(z_0)$, $z_0 \in \gamma$:
This is true by the Plemelj formulae.

3). $G(z)$ vanishes at infinity:
This is true since another property of the Cauchy transform is that $G(z) \sim O(1/z)$ as $z \to \infty$. So we have decay at infinity.

4). Uniqueness of solution:
Let's verify this. Suppose G_1 and G_2 are two solutions. Then the difference $G_1 - G_2$ is analytic for **all** z including on γ (since G_1 and G_2 have the same jump discontinuity across γ) and $G_1 - G_2$ vanishes as $z \to \infty$. Hence, by Liouville's theorem: $G_1 - G_2 = \text{constant} = 0$, by noting the behaviour at infinity. Thus $G_1 = G_2$, i.e we have uniqueness.

Remark: We required $G(z) \to 0$ as $z \to \infty$, but we can weaken this to allow for algebraic growth, i.e $G(z) \sim O(|z|^n)$ as $z \to \infty$, for some $n \in \mathbb{N}$.
In this case, we can check (exercise) using the extension of Liouville's theorem that $G(z)$ is determined up to a polynomial of degree n:

$$G(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi + p_n(z),$$

where $p_n(z)$ is an arbitrary polynomial of degree n (this may be determined by additional conditions if we have them).

2.4 The Inversion Problem: Cauchy Kernel

Let us return to our main focus of interest, namely problem (14). That is to find a function $f(x)$ satisfying

$$\frac{1}{\pi}⨍_a^b k(t-x)f(t)dt = g(x),$$

for $a < x < b$, where $k(t-x)$ and $g(x)$ are given functions. As mentioned earlier, for the kernel functions $k(t-x) = 1/(t-x)$ and $k(t-x) = \log(t-x)$ there are known techniques to solve the equation. Let's start with the so called **Cauchy kernel** $k(t-x) = 1/(t-x)$. We will also restrict ourselves to the case where $a = -1$, $b = 1$ throughout this course for simplicity (though in general this is not necessary). So, our aim is to find a function $f(x)$ satisfying

$$\frac{1}{\pi}⨍_{-1}^1 \frac{f(t)}{t-x}dt = g(x), \quad -1 < x < 1. \tag{23}$$

In terms of the Hilbert transform, we can rephrase this problem as

$$2iH(x) = g(x). \tag{24}$$

Let's solve for $f(x)$ as follows: first introduce the Cauchy transform:

$$C(z) = \frac{1}{2\pi i}\int_{-1}^1 \frac{f(t)}{t-z}dt.$$

This satisfies the Plemelj formulae

$$C_+(x) + C_-(x) = 2H(x), \tag{25}$$

$$C_+(x) - C_-(x) = f(x), \tag{26}$$

where

$$H(z) = \frac{1}{2\pi i} \fint_{-1}^{1} \frac{f(t)}{t-z} dt,$$

is the Hilbert transform of $f(x)$ on $[-1, 1]$. Now using (24), the first Plemelj equation (25) gives

$$C_+(x) + C_-(x) = -ig(x). \tag{27}$$

Now recall the 'converse problem' from section 2.3. We can solve problems of the form

$$G_+(x) - G_-(x) = \text{known function.} \tag{28}$$

So this motivates a new goal; can we convert (27) to an equation of the form of (28).
Suppose now we have some function $\phi(z)$, which is analytic everywhere except for a jump discontinuity across γ of the form

$$\phi_+(x) = -\phi_-(x), \tag{29}$$

for $-1 < x < 1$. Then consider $w(z) = \phi(z)C(z)$. For $-1 < x < 1$, we have

$$\begin{aligned} w_+(x) - w_-(x) &= \phi_+(x)C_+(x) - \phi_-(x)C_-(x) \\ &= \phi_+(x)(C_+(x) + C_-(x)) \\ &= -i\phi_+(x)g(x), \end{aligned} \tag{30}$$

where we have used (29) to reach the second line and (27) to reach the final line. So, provided we can find such a function $\phi(z)$, then (30) is of the form of the 'converse problem', whose solution is known.

Finding $\phi(z)$

For $\phi(z)$ we need a function analytic everywhere except for the jump discontinuity $\phi_+(x) = -\phi_-(x)$ across $-1 < x < 1$. We have already encountered a function satisfying this in the course already!
We take $\phi(z) = \sqrt{z^2 - 1}$, with a branch cut along $-1 < x < 1$.
Let's double check this gives the required jump in $\phi(z)$ across $[-1, 1]$. As usual, introduce local coordinates r_1, r_2, θ_1, θ_2 as shown.

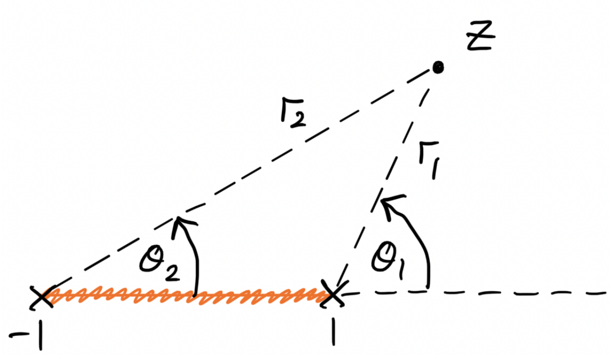


Figure 37: Usual local coordinates.

Here $r_1 = |z-1|$, $\theta_1 = \arg\{z-1\}$, $r_2 = |z+1|$, $\theta_2 = \arg\{z+1\}$ and let's take $-\pi \le \theta_1, \theta_2 \le \pi$ (recall that this choice corresponds to the branch that looks like $+z$ as $z \to \infty$).
Now $\phi(z) = \sqrt{z^2-1} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{\Theta}{2}}$, where $\Theta = \theta_1 + \theta_2$.
On the upperside of the branch cut, for $-1 < x < 1$, $y = \delta$ ($\delta > 0$ small), we have: $\theta_1 = \pi$, $\theta_2 = 0$, giving $\Theta = \pi$, resulting in $\phi(x) = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i\sqrt{1-x^2}$ (where we have used the usual facts that $r_1 = |z-1| = -(x-1) = 1-x$ and $r_2 = |z+1| = x+1$).
On the lowerside of the branch cut, for $-1 < x < 1$, $y = -\delta$, we have: $\theta_1 = -\pi$, $\theta_2 = 0$, giving $\Theta = -\pi$, resulting in $\phi(x) = (r_1 r_2)^{\frac{1}{2}} e^{-i\frac{\pi}{2}} = -i\sqrt{1-x^2}$.
Hence $\phi_+(x) = -\phi_-(x)$ for $-1 < x < 1$ as we required.

Returning to where we were

Let's return back to our solution scheme; we now have

$$w_\pm(x) = \phi_\pm(x) C_\pm(x) = \pm i\sqrt{1-x^2}\, C_\pm(x). \tag{31}$$

This means that (30) gives $w_+(x) - w_-(x) = -i\phi_+(x) g(x) = \sqrt{1-x^2}\, g(x)$. We know the solution to this problem as it is of the form of the 'converse problem'. The general solution is

$$w(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{t-z} dt + p_n(z), \tag{32}$$

where $p_n(z)$ is a polynomial. We can determine $p_n(z)$ as follows; as $z \to \infty$, $\phi(z) \sim O(z)$, $C(z) \sim O(1/z)$, so since $w(z) = \phi(z)C(z)$, then $w(z) \sim O(1)$. But in (32), the Cauchy-type integral $\sim O(1/z)$, hence it must be the case that we have $p_n(z) = \text{constant} = A_1$, say. So,

$$w(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{t-z} dt + A_1.$$

Now recall the other Plemelj equation (26) we haven't used yet

$$\begin{aligned} f(x) &= C_+(x) - C_-(x) \\ &= \frac{1}{i\sqrt{1-x^2}} [w_+(x) + w_-(x)], \end{aligned} \tag{33}$$

using (31). Now, using the Plemelj formulae for the function $w(z)$, we deduce that, for $-1 < x < 1$:

$$w_+(x) + w_-(x) = 2H_w(x),$$

where

$$H_w(x) = \frac{1}{2\pi i} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{t-x} dt + A_2,$$

for some constant A_2. Here I have used the subscript w to denote that this is the Hilbert transform corresponding to function $w(z)$ not $C(z)$. Hence, plugging this into (33) gives

$$f(x) = \frac{2}{i\sqrt{1-x^2}} H_x(x),$$

or

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, g(t)}{t-x} dt + \frac{A}{\sqrt{1-x^2}}, \tag{34}$$

for a constant $A \in \mathbb{R}$. (34) is known as the **Hilbert Inversion Formula**.

2.5 Example Problem

Let's do an example to illustrate how we use the inversion formula and solve for $f(x)$.
Example: Find a function $f(x)$ satisfying

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x} dt = 1, \quad -1 < x < 1.$$

i.e. the function $g(x) = 1$ here.
Solution: Applying the **Hilbert inversion formula** (34), we have

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} I(x) + \frac{A}{\sqrt{1-x^2}}, \tag{35}$$

where

$$I(x) = ⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} dt.$$

All that's left to do is to determine $I(x)$. Let's consider two different methods:

Method 1 (Plemelj formulae):

Introduce the Cauchy transform of $\sqrt{1-z^2}$ on $[-1,1]$:

$$C(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-z} dt.$$

Then, applying the **Plemelj formulae** (and noting that $I(x) = 2\pi i H(x)$ for $\sqrt{1-x^2}$ on $[-1,1]$):

$$I(x) = \pi i (C_+(x) + C_-(x)), \tag{36}$$

for $-1 < x < 1$. Now let's find $C_\pm(x)$ as follows: take $z = x \in \mathbb{R} > 1$. Let $t = \cos\theta$. Then

$$\begin{aligned}
C(x) &= \frac{1}{2\pi i} \int_{\pi}^{0} \frac{-\sin^2\theta}{\cos\theta - x} d\theta \\
&= \frac{1}{2\pi i} \int_{0}^{\pi} \frac{1-\cos^2\theta}{\cos\theta - x} d\theta \\
&= \frac{-1}{2\pi i} \int_{0}^{\pi} \frac{(x+\cos\theta)(x-\cos\theta) - (x^2-1)}{(x-\cos\theta)} d\theta \\
&= \frac{1}{2\pi i} \Bigg[\underbrace{-\int_{0}^{\pi} (x+\cos\theta) d\theta}_{=-\pi x} + (x^2-1) \underbrace{\int_{0}^{\pi} \frac{d\theta}{x-\cos\theta}}_{=\frac{\pi}{\sqrt{x^2-1}}} \Bigg],
\end{aligned}$$

where the second integral is left as an exercise in complex analysis and residue theory (see similar examples on problem sheet 1: consider $1/2\int_{-\pi}^{\pi}$ and convert to an integral around the unit circle). Hence we find

$$C(x) = \frac{1}{2i}(-x + \sqrt{x^2-1}),$$

for $x > 1$. Then, by analytic continuation, we must have

$$C(z) = \frac{1}{2i}(-z + \sqrt{z^2-1}),$$

for complex z. Then, recall that for $-1 < x < 1$, we know $\sqrt{z^2-1}|_{z=x\pm i\delta(\delta<<1)} = \pm i\sqrt{1-x^2}$. Hence for $-1 < x < 1$ we have

$$C_{\pm}(x) = \frac{1}{2i}(-x \pm i\sqrt{1-x^2}).$$

Thus (36) gives

$$\begin{aligned} I(x) &= \pi i(\frac{1}{2i}(-2x)) \\ &= -\pi x, \end{aligned} \tag{37}$$

which, on substitution into (35) gives

$$f(x) = \frac{(x+A)}{\sqrt{1-x^2}}.$$

To determine the constant A we need extra information given in the problem (for example the value of $f(x)$ at some point x_0.

Method 2 (Contour integration):

Consider the analytic continuation of the integrand of $I(x)$ into the complex z-plane given by

$$\frac{\sqrt{z^2-1}}{z-x}.$$

Note the difference inside the square root. Let's expand this function as $z \to \infty$:

$$\begin{aligned} \frac{\sqrt{z^2-1}}{z-x} &= \frac{\sqrt{1-\frac{1}{z^2}}}{1-\frac{x}{z}} \\ &= \left(1 - \frac{1}{2z^2} + O(\frac{1}{z^4})\right)\left(1 + \frac{x}{z} + O(\frac{1}{z^2})\right) \\ &= 1 + \frac{x}{z} + O(\frac{1}{z^2}), \end{aligned} \tag{38}$$

where we have used the expansions $(1-X)^{-1} = 1 + X + X^2 + \cdots$ and $(1+X)^n = 1 + nX + \frac{n(n-1)}{2!}X^2 + \cdots$ in the second line. Owing to the result (38) we consider the contour integral

$$\oint_{\gamma} \left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz,$$

where γ is the contour $\gamma = \gamma_1 + \gamma_2 + \gamma_+ + \gamma_- + \gamma_R + \gamma_{\varepsilon_1} + \gamma_{\varepsilon_2} + \gamma_{\varepsilon_+} + \gamma_{\varepsilon_-}$, as shown in figure 38.

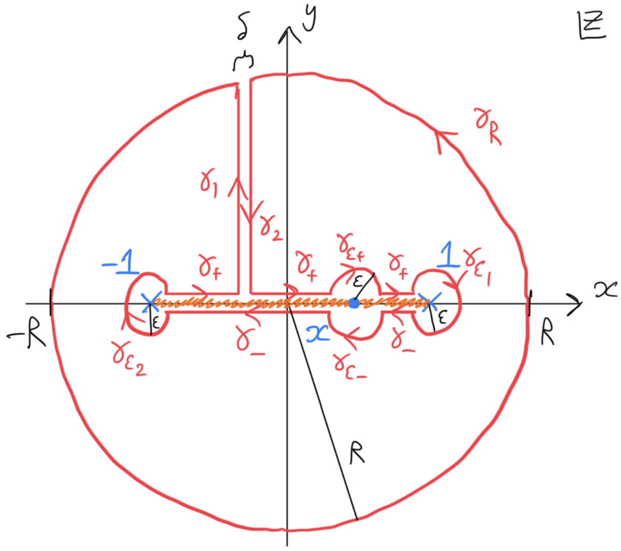


Figure 38: Contour path.

Here γ_R is a circle, centre 0, radius R. $\gamma_{\varepsilon_{1,2}}$ are circles, radius ε with centres on $1, -1$. $\gamma_{\varepsilon_\pm}$ are semi-circles, centre x, radius ε. $\gamma_\pm$ are straight lines that hug the top and bottom sides respectively of the branch cut for the multivalued function $\sqrt{z^2-1}$ taken along the real axis between $[-1,1]$ (we choose the branch that $\sim +z$ as $z \to \infty$). Now consider the limit as $\varepsilon \to 0$, $\delta \to 0$ and $R \to \infty$.
There are no singularities inside the contour, hence by Cauchy's theorem

$$\oint_\gamma \left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz = 0.$$

Now let's consider the integrals along the separate segments of γ. First note that

$$\int_{\gamma_1+\gamma_2} = 0,$$

since the integral is continuous between γ_1 and γ_2 and we integrate along them in opposite directions. Secondly, we can check that (exercise)

$$\lim_{\varepsilon\to 0}\int_{\varepsilon_{1,2}} = 0.$$

Now let's see what's happening on γ_R. Here $z = Re^{i\theta}$, where θ takes any range of 2π. We find

$$\begin{aligned}
\lim_{R\to\infty}\int_{\gamma_R}\left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz &= \lim_{R\to\infty}\int_0^{2\pi}\left[\left(1+\frac{x}{Re^{i\theta}}+O(\frac{1}{R^2})\right)-1\right] iRe^{i\theta}d\theta \\
&= \lim_{R\to\infty}\int_0^{2\pi}\left(ix + O(\frac{1}{R})\right)d\theta \\
&= 2\pi i x,
\end{aligned}$$

where we have used the fact that we know the integrand behaves as found in (38) as $z \to \infty$. Now recall that on the top and bottom sides of the branch cut we have $\sqrt{z^2-1}|_{\gamma_\pm,\gamma_{\varepsilon_\pm}} = \pm i\sqrt{1-x^2}$. Also note that on γ_{ε_+} we have $z = x + \varepsilon e^{i\theta}$, $0 \le \theta \le \pi$. Thus

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\gamma_+ + \gamma_{\varepsilon_+}} &= \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{i\sqrt{1-t^2}}{t-x}dt + \int_{x+\varepsilon}^{1}\frac{i\sqrt{1-t^2}}{t-x}dt + \int_{\pi}^{0}\frac{i\sqrt{1-x^2}}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta - \int_{-1}^{1}1dt\right]\\ &= i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt + \pi\sqrt{1-x^2} - 2.\end{aligned}$$

Similarly, on γ_-, γ_{ε_-} (recalling $\sqrt{z^2-1} = -i\sqrt{1-x^2}$ and with $z = x + \varepsilon e^{i\theta}$ where $-\pi \le \theta \le 0$ on γ_{ε_-}), it can be shown that (exercise):

$$\lim_{\varepsilon\to 0}\int_{\gamma_- + \gamma_{\varepsilon_-}} = i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt - \pi\sqrt{1-x^2} + 2.$$

Thus, summing up all the contributions and equating this to the result from Cauchy's theorem

$$2i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt + 2\pi i x = 0,$$

or in other words

$$I(x) = -\pi x, \tag{39}$$

as found via method 1. As before, substituting for $I(x)$ in (35) gives the unknown function $f(x) = \frac{x+A}{\sqrt{1-x^2}}$.

2.6 The Inversion Problem: Logarithmic Kernel

Once again let us return to problem (14), but now consider the **logarithmic kernel** given by $k(t-x) = \log(t-x)$ and once again consider the case where $a = -1$, $b = 1$ for simplicity. This results in us trying to find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = g(x), \quad -1 < x < 1. \tag{40}$$

Assuming $f(x)$ is 'nice' (e.g analytic), then in fact such an integral is non-singular even though the integrand itself is (roughly speaking this is because it is possible to show that the integral goes like $\varepsilon\log\varepsilon \to 0$ as $\varepsilon \to 0$). Such equations are sometimes referred to as **weakly** singular integral equations (this is why we don't use the $⨍$ symbol here).

Let's solve for $f(x)$ as follows

$$\begin{aligned}\int_{-1}^{1} f(t)\log|t-x|dt &= \int_{-1}^{x} f(t)\log(x-t)dt + \int_{x}^{1} f(t)\log(t-x)dt\\ &= \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon} f(t)\log(x-t)dt + \int_{x+\varepsilon}^{1} f(t)\log(t-x)dt\right],\end{aligned}$$

because the integral along the section of length 2ε about x tends to zero as $\varepsilon \to 0$ (again glossing over some details this is because $\varepsilon\log\varepsilon \to 0$ as $\varepsilon \to 0$). Now we will differentiate with respect to x under the integral sign. Note here however, that the variable x is also present in the limits of the integral, hence denoting $h(x,t) = f(t)\log(x-t)$, we find

$$\begin{aligned}\frac{d}{dx}\left\{\int_{-1}^{x-\varepsilon} f(t)\log(x-t)dt\right\} &= h(x,x-\varepsilon)\cdot\frac{d}{dx}(x-\varepsilon) - h(x,-1)\cdot\frac{d}{dx}(-1) + \int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt \\ &= \int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt + f(x-\varepsilon)\log\varepsilon,\end{aligned}$$

where in the first line the full formula for differentiating under the integral sign when limits are non-constant was used. Similarly, we can show

$$\frac{d}{dx}\left\{\int_{x+\varepsilon}^{1} f(t)\log(t-x)dt\right\} = \int_{x+\varepsilon}^{1}\frac{-f(t)}{t-x}dt - f(x+\varepsilon)\log\varepsilon.$$

Hence, we have

$$\begin{aligned}\frac{d}{dx}\left\{\int_{-1}^{1} f(t)\log|t-x|dt\right\} &= \lim_{\varepsilon\to 0}\left\{\int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt + \int_{x+\varepsilon}^{1}\frac{f(t)}{x-t}dt + (f(x-\varepsilon) - f(x+\varepsilon))\log\varepsilon\right\} \\ &= -⨍_{-1}^{1}\frac{f(t)}{t-x}dt,\end{aligned}$$

where in the first line the term $(f(x-\varepsilon) - f(x+\varepsilon))\log\varepsilon \to 0$ as $\varepsilon \to 0$ (again provided $f(x)$ is 'nice'). Thus we have a strategy to solve (40). Namely differentiating (40) with respect to x gives

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{f(t)}{t-x}dt = -g'(x), \quad -1 < x < 1. \tag{41}$$

This is now an equation of the form of (23) for which we can apply the Hilbert inversion formula to solve for $f(x)$.

2.7 Example Problems

Let's look at a couple of examples where the logarithmic kernel is used.

Example 1:

Find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = 3, \quad -1 < x < 1.$$

Solution: Differentiating wrt x we find

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{f(t)}{t-x}dt = 0, \quad -1 < x < 1.$$

Then, applying the Hilbert inversion formula (34) we get

$$f(x) = \frac{A}{\sqrt{1-x^2}},$$

where A is a constant. With logarithmic singular integral equations we can determine the value of A by substituting our form of $f(x)$ back into the original equation. Here we get

$$\frac{1}{\pi}\int_{-1}^{1}\frac{A\log|t-x|}{\sqrt{1-t^2}}dt = 3, \quad -1 < x < 1.$$

This equation holds for all x between -1 and 1. So, in particular, it holds for $x = 0$, which means

$$\frac{A}{\pi}\int_{-1}^{1}\frac{\log|t|}{\sqrt{1-t^2}}dt = 3,$$

or $A = \frac{3\pi}{I}$, where

$$I = \int_{-1}^{1}\frac{\log|t|}{\sqrt{1-t^2}}dt.$$

Now,

$$I = 2\underbrace{\int_{0}^{1}\frac{\log t}{\sqrt{1-t^2}}dt}_{=I_0} = 2I_0.$$

Let's compute I_0. To do this let $t = \sin\theta$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}\log(\sin\theta)d\theta \\ &= \frac{1}{2}\int_0^{\pi}\log(\sin\theta)d\theta, \end{aligned}$$

using symmetry arguments. Now put $\alpha = \theta/2$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}\log(\sin 2\alpha)d\alpha \\ &= \int_0^{\pi/2}\log(2\sin\alpha\cos\alpha)d\alpha \\ &= \int_0^{\pi/2}(\log 2 + \log(\sin\alpha) + \log(\cos\alpha))d\alpha. \end{aligned}$$

Now put $\beta = \pi/2 - \alpha$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}(\log 2 + 2\log(\sin\beta))d\beta \\ &= \frac{\pi}{2}\log 2 + 2I_0. \end{aligned}$$

Giving

$$I_0 = -\frac{\pi}{2}\log 2.$$

Thus we have

$$\begin{aligned} A &= \frac{3\pi}{2I_0} \\ &= -\frac{3}{\log 2}. \end{aligned}$$

Hence the solution to the original problem is given by

$$f(x) = \frac{-3}{\log 2}\cdot\frac{1}{\sqrt{1-x^2}}.$$

Example 2:

Find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = x, \quad -1 < x < 1.$$

Solution: Differentiating wrt x we find

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x}dt = -1, \quad -1 < x < 1.$$

Hence, applying the Hilbert inversion formula (34) we get

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

where A is a constant. But recall from earlier, in section 2.5 equations (37) and (39), we showed that

$$I(x) = ⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}dt = -\pi x.$$

Thus, we have

$$f(x) = \frac{A-x}{\sqrt{1-x^2}}.$$

Again, to determine A, substitute for $f(x)$ into the original equation

$$\frac{1}{\pi}\int_{-1}^{1} \left[\frac{-t}{\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}}\right]\log|t-x|dt = x, \quad -1 < x < 1.$$

This time, checking what happens for $x = 0$, we find

$$\frac{1}{\pi}\int_{-1}^{1} \left[\frac{-t}{\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}}\right]\log|t|dt = 0.$$

Now the first term here is an **odd** function, so it integrates to 0 over $[-1, 1]$, leaving us with

$$\frac{A}{\pi}\int_{-1}^{1} \frac{\log|t|}{\sqrt{1-t^2}}dt = 0,$$

and one can check that the integrand of this is < 0 for all values of t, hence this integral is **non-zero**. Thus we conclude

$$A = 0,$$

giving

$$f(x) = \frac{-x}{\sqrt{1-x^2}}.$$

2.8 Ideal Fluid Flow past a Flat Plate

Understanding branch cuts and Cauchy transforms allows us to solve problems which can be reduced to singular integral equations. A classic example of this is the Laplace equation for ideal fluid flow. We consider the case of uniform flow with angle α around an infinitesimally thin plate on $[-1, 1]$. This can be modelled as

$$\begin{aligned} \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= 0, \text{ everywhere off } [-1,1], \\ \psi(x, 0) &= 0, \text{ for } -1 < x < 1, \\ \psi(x, y) &\sim y\cos\alpha - x\sin\alpha, \text{ as } x^2 + y^2 \to \infty, \end{aligned} \tag{42}$$

where $\psi(x, y)$ is some real-valued function (called the streamfunction) which corresponds in some way to the fluid trajectories (for those taking fluids courses, or those simply interested, we have what are called streamlines, lines which the fluid follows over time, when $\psi(x, y) = \text{constant}$). Using the techniques we have developed over the last lectures, let's see how we can obtain a nice, closed-form expression for the solution $\psi(x, y)$ as the imaginary part of an analytic function.

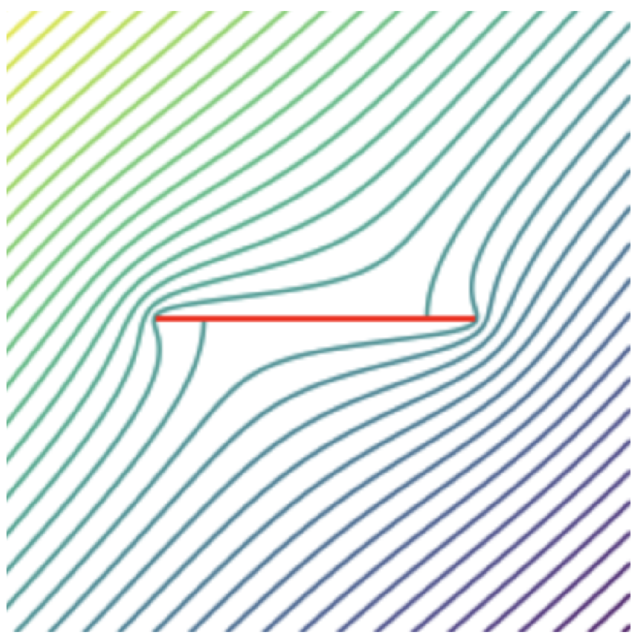


Figure 39: Ideal fluid flow at an angle of $\pi/4$ past a flat plate along $[-1, 1]$.

The figure above shows a plot of the solution we will find. The lines are called streamlines and represent how the fluid flows around the flat plate on $[-1, 1]$. In the plot $\alpha = \pi/4$ radians. We will solve this problem in stages:

1). Rephrasing the problem in $\psi(x, y)$ to a complex analytical problem in $w(z)$.

2). Reduction to a singular integral equation.

3). Calculating the inverse Hilbert transform using the inversion formula.

4). Calculating the remaining Cauchy transform.

Steps 3 and 4 involve the usual solution methods we have employed in our example problems so far.

Step 1: The real and imaginary parts of an analytic function satisfy Laplace's equation: that is if $w(z) = \phi(x,y) + i\psi(x,y)$, where ϕ and ψ are the real and imaginary parts, then

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

We proved this result in chapter 1. Therefore we can write the ideal fluid flow problem as a problem of calculating $w(z) = \phi(x,y) + i\psi(x,y)$ whose imaginary part, $\psi(x,y)$, is the solution to the ideal fluid flow pde (we are not interested in finding $\phi(x,y)$ here, although this can readily be found from $w(z)$ and does have meaning in a fluid mechanics context; it is known as the velocity potential and its x and y partial derivates give the horizontal and vertical velocity components of the fluid).

So, rephrased, we want to find an **analytic** function $w(z)$ satisfying

$$\begin{aligned} \operatorname{Im}\{w(x)\} &= 0, \quad \text{for } -1 < x < 1, \\ w(z) &\sim e^{-i\alpha} z, \quad \text{as } z \to \infty. \end{aligned}$$

(Exercise: check that $e^{-i\alpha} z$ gives the correct behaviour corresponding to (42)).

Step 2: Now $w(z)$ must be a function analytic everywhere off the plate between $[-1,1]$ with the correct far field and plate boundary conditions. Let's think about what the Cauchy transform does here... well, the Cauchy transform of some function, say $f(z)$, on $[-1,1]$ generates an analytic function everywhere off $[-1,1]$ that decays as $z \to \infty$. This is exactly what we need when paired with the correct far-field behaviour! So, let's make the ansatz

$$w(z) = \underbrace{e^{-i\alpha} z + c}_{\text{the behaviour at } \infty} + \underbrace{\frac{1}{2\pi i} \int_{-1}^{1} \frac{f(t)}{t - z} dt}_{= C(z)}, \tag{43}$$

where $c \in \mathbb{C}$ is a constant. Here $C(z)$ is as mentioned the Cauchy transform of some unknown function $f(z)$, which gives us a function analytic everywhere off the plate, which decays at ∞. It remains to satisfy the plate condition $\operatorname{Im}\{w(x)\} = 0$ on $-1 < x < 1$. Before we check this note that

$$\overline{C(z)} = \frac{-1}{2\pi i} \int_{-1}^{1} \frac{f(t)}{t - \overline{z}} dt = -C(\overline{z}),$$

so long as $f(x)$ is a real-valued function (for all z off $[-1,1]$). It then follows using this fact that, if we take $z = x + i\delta$, where $\delta << 1$ is real, and let $\delta \to 0$

$$\overline{C_+(x)} = \overline{C}_+(x) = -C_-(x). \tag{44}$$

Thus, we can find the imaginary part of the Cauchy transform, using the Plemelj formulae, by

$$\begin{aligned} 2H(x) = C_+(x) + C_-(x) &= C_+(x) - \overline{C_+(x)} = 2i\operatorname{Im}\{C_+(x)\} \\ &= C_-(x) - \overline{C_-(x)} = 2i\operatorname{Im}\{C_-(x)\}, \end{aligned}$$

where we have used (44) in each line to arrive at the two different forms for $2H(x)$. This means that we must have $\text{Im}\{C_+(x)\} = \text{Im}\{C_-(x)\} = \text{Im}\{C(x)\}$, giving

$$H(x) = i\text{Im}\{C(x)\}. \tag{45}$$

Now applying the plate condition on $z = x$ where $-1 < x < 1$:

$$\begin{aligned} 0 = \text{Im}\{w(x)\} &= \text{Im}\{(\cos\alpha - i\sin\alpha)(x)\} + \text{Im}\{c\} + \text{Im}\{C(x)\} \\ &= -x\sin\alpha + \text{Im}\{c\} - iH(x), \end{aligned}$$

upon using result (45). Hence we arrive at the **singular integral equation**

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(t)}{t-x}dt = -2x\sin\alpha + 2\text{Im}\{c\}, \quad -1 < x < 1,$$

which is an equation of the form (23) which we have learnt how to solve.

Step 3: For simplicity I'll suppose now $c = 0$, so $\text{Im}\{c\} = 0$, but in principle you could solve this equation with the methods we have learned with c arbitrary and be given extra information later on to determine it. Then we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(t)}{t-x}dt = -2x\sin\alpha, \quad -1 < x < 1.$$

Applying the Hilbert inversion formula, we find

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\int_{-1}^{1} \frac{\sqrt{1-t^2}(-2t\sin\alpha)}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

or

$$f(x) = \frac{2\sin\alpha}{\pi\sqrt{1-x^2}}I(x) + \frac{A}{\sqrt{1-x^2}},$$

where

$$I(x) = \int_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x}dt.$$

Exercise (Problem Sheet 2): show that $I(x) = \frac{\pi}{2} - \pi x^2$. Then

$$f(x) = \frac{\sin\alpha - 2\sin\alpha x^2 + A}{\sqrt{1-x^2}}.$$

Now from physical principles the solution should not blow-up. If $f(x)$ blows-up, then so does its Cauchy transform, and hence our solution. This means at the end points where $x = \pm 1$ we need to ensure $f(x)$ stays regular. Therefore, we choose $A = \sin\alpha$, so that

$$f(x) = \frac{2\sin\alpha(1-x^2)}{\sqrt{1-x^2}} = 2\sin\alpha\sqrt{1-x^2}, \tag{46}$$

which is regular at $x = \pm 1$.

Step 4: Now substituting back for $f(x)$ from (46) into our expression (43) for $w(z)$ gives

$$w(z) = e^{-i\alpha} z + \frac{1}{2\pi i} \int_{-1}^{1} \frac{2\sin\alpha\sqrt{1-t^2}}{t-z} dt.$$

But now recall from section 2.5 in Method 1, we showed

$$C(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-z} dt = \frac{1}{2i}(-z + \sqrt{z^2-1}).$$

Hence

$$w(z) = e^{-i\alpha} z - i\sin\alpha(-z + \sqrt{z^2-1}),$$

is the complex potential function we are looking for whose imaginary part gives the function $\psi(x, y)$ desired:

$$\psi(x, y) = \mathrm{Im}\{e^{-i\alpha} z - i\sin\alpha(-z + \sqrt{z^2-1})\}. \tag{47}$$

This is the solution for the streamfunction $\psi(x, y)$ for uniform flow at an angle α past a flat plate along the real axis between $[-1, 1]$. Plotting $\psi(x, y) =$ constant, for a range of different constants gives a plot like the one shown in figure 39.

2.9 Electrostatic Potential of a point charge near a flat plate

Another application of Laplace's equation is that of electrostatics. Suppose we are interested in determining the electric field surrounding a plate where a nearby point source is located. Let's suppose the point source is at $x = 2$, say, and the plate is located on $[-1, 1]$ and is being held at some constant potential, k, which we don't know. This problem can be modelled as

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \ \text{ for } z \text{ off } [-1, 1] \text{ and } x = 2, \\
V(z) &\sim \log|z-2| + O(1), \ \text{ for } z \to 2, \\
V(z) &\sim \log|z| + o(1), \ \text{ for } z \to \infty, \\
V(x) &= k, \ \text{ for } -1 < x < 1.
\end{aligned}$$

Here $V(z)$ is the electrostatic potential which we want to solve for. It turns out (we omit the details here) that we can represent the solution in the form

$$V(z) = \frac{1}{\pi} \int_{-1}^{1} f(t) \log|t-z| dt + \log|z-2|. \tag{48}$$

On $-1 < x < 1$, this satisfies

$$\frac{1}{\pi} \int_{-1}^{1} f(t) \log|t-x| dt = k - \log(2-x).$$

Differentiating wrt x

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{f(t)}{t-x} dt = -\frac{1}{2-x}.$$

Now applying the Hilbert inversion formula we find

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} \underbrace{⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-2)(t-x)} dt}_{=I(x)} + \frac{A}{\sqrt{1-x^2}}.$$

Exercise (Problem Sheet 2): Show that $I(x) = \frac{-\pi\sqrt{3}}{x-2}$. Thus

$$f(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{A}{\sqrt{1-x^2}}.$$

To determine the constant A we can look at the far-field behaviour of our solution. We know $V(z) \sim \log|z|$ as $z \to \infty$. It is also possible to show that (we omit the details here)

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-z|dt \sim \frac{1}{\pi}\int_{-1}^{1} f(t)dt \log|z|, \quad \text{as } z \to \infty.$$

Thus, since $\log|z-2| \sim \log|z|$ as $z \to \infty$, we must set

$$\frac{1}{\pi}\int_{-1}^{1} f(t)dt = 0, \tag{49}$$

since then in (48), $V(z) \sim 0 \times \log|z| + \log|z-2| \sim \log|z|$ as $z \to \infty$ as required.
Exercise (Problem Sheet 2): (49) gives

$$\frac{1}{\pi}\int_{-1}^{1} \left(\frac{\sqrt{3}}{(t-2)\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}} \right) dt = 0,$$

which leads to $A = 1$. Hence

$$f(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}},$$

meaning that upon back substitution into (48) we have

$$V(z) = \frac{1}{\pi}\int_{-1}^{1} \left[\frac{\sqrt{3}}{(t-2)\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} \right] \log|t-z|dt + \log|z-2|.$$

For any point z in the plane this integral can be calculated numerically and so we can plot the solution over a range of z values.

Chapter 3: Orthogonal Polynomials

3.1 Sturm-Liouville Problems

The Sturm-Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+r_1(x)y+\lambda r_2(x)y=0, \tag{50}$$

where $p(x)>0$, $r_2(x)\geq 0$ and p, r_1 and r_2 are continuous functions on the interval $[a,b]$, along with the boundary conditions

$$a_1y(a)+a_2p(a)y'(a)=0, \tag{51}$$

$$b_1y(b)+b_2p(b)y'(b)=0, \tag{52}$$

where $a_1^2+a_2^2\neq 0$ and $b_1^2+b_2^2\neq 0$, with the problem of finding a complex number μ (if any), such that (50)-(52) with $\lambda=\mu$ has a non-trivial solution is called a Sturm-Liouville eigenvalue problem. Such a value μ is called an eigenvalue and the corresponding non-trivial solutions $y(x)$ are called eigenfunctions.

In physics and applied mathematics many problems arise in the form of boundary value problems involving second-order ordinary differential equations; many of which can be reduced to the form of a Sturm-Liouville problem.

There is a rich theory of these equations and their properties and if interested in reading more about them then many classic texts contain more details (see the reading list for instance).

In this chapter we will be interested in a very similar ordinary differential equation, namely

$$\frac{d}{dx}[p(x)y'(x)]+q(x)y'(x)+\lambda y(x)=0,\quad a<x<b, \tag{53}$$

where $p(x)$, $q(x)$ are polynomials of degree 2 and 1 respectively and λ is a real constant. Note that for $q(x)=0$, this equation becomes a Sturm-Liouville problem and so there is some overlap between the theory we will study and that of Sturm-Liouville problems.

Why do we care about this particular equation? Well, as we will see shortly, this equation has some special polynomial solutions (orthogonal polynomials) that are of particular interest in computational mathematics and physics.

3.2 An Eigenvalue Problem

Let us now ask a question about our special ode (53). Namely, for what values of λ, say λ_n, does there exist a **polynomial** solution for $y(x)$ of degree n? i.e.

$$y(x)=\sum_{r=0}^{n}a_rx^r,\quad a_n\neq 0. \tag{54}$$

Solution: Substitute for $y(x)$ from (54) into the ode (53). First note that

$$y'(x)=\sum_{r=1}^{n}ra_rx^{r-1},$$

and writing

$$p(x) = p_2 x^2 + p_1 x + p_0,$$
$$q(x) = q_1 x + q_0,$$

gives

$$p(x)y'(x) = \sum_{r=1}^{n} (p_0 r a_r x^{r-1} + p_1 r a_r x^r + p_2 r a_r x^{r+1}),$$

so then

$$\frac{d}{dx}[p(x)y'(x)] = \sum_{r=2}^{n} (p_0 r(r-1) a_r x^{r-2}) + \sum_{r=1}^{n} (p_1 r^2 a_r x^{r-1} + p_2 r(r+1) a_r x^r).$$

Also we have

$$q(x)y'(x) = \sum_{r=1}^{n} (q_0 r a_r x^{r-1} + q_1 r a_r x^r).$$

Now re-labelling all the indices to start at $r = 0$ gives:

$$\begin{aligned}
\frac{d}{dx}[p(x)y'(x)] &= \sum_{r=0}^{n-2} (p_0 (r+2)(r+1) a_{r+2} x^r) + \sum_{r=0}^{n-1} p_1 (r+1)^2 a_{r+1} x^r + \sum_{r=0}^{n} p_2 (r+1) r a_r x^r, \\
q(x)y'(x) &= \sum_{r=0}^{n-1} q_0 (r+1) a_{r+1} x^r + \sum_{r=0}^{n} q_1 r a_r x^r, \\
\lambda y(x) &= \sum_{r=0}^{n} \lambda a_r x^r.
\end{aligned}$$

Substituting these back into (53) and setting the coefficients of x^r equal to zero gives:

a). For $0 \leq r \leq n-2$, we get

$$p_0 (r+2)(r+1) a_{r+2} + (p_1 (r+1)^2 + q_0 (r+1)) a_{r+1} + (p_2 (r+1) r + q_1 r + \lambda) a_r = 0. \tag{55}$$

This gives us a recurrence relation: if a_{r+1} and a_{r+2} are known, it gives us a_r.

b). For $r = n-1$, we get:

$$(p_2 n(n-1) + q_1 (n-1) + \lambda) a_{n-1} + (p_1 n^2 + q_0 n) a_n = 0. \tag{56}$$

c). For $r = n$, we get:

$$\lambda = \lambda_n = -(p_2 (n+1) + q_1) n. \tag{57}$$

This is the solution to the eigenvalue problem we asked.

The a_r's and hence the polynomial solutions for $y(x)$ to (53) can be constructed as:

1). Pick a value for $a_n \neq 0$ (freedom to do this as RHS of ode $= 0$).

2). Determine the value of λ_n if not given from (57).

3). Then use (56) to find the value of a_{n-1}.

4). Then use (55) to find a_{n-2}, a_{n-3}, $\cdots$, a_1, a_0.

Note: There are more efficient ways to construct such polynomial solutions as we will see soon.

Example: Legendre Polynomials:

Consider the Legendre equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \quad -1 < x < 1.$$

Written in standard form this is

$$\frac{d}{dx}[(1-x^2)y'] + \lambda y = 0,$$

i.e. $p(x) = 1 - x^2$, $q(x) = 0$, so $p_0 = 1$, $p_1 = 0$, $p_2 = -1$, $q_0 = 0$ and $q_1 = 0$. So, from (57), we have a polynomial solution of degree n to this equation if and only if $\lambda = n(n+1)$.
Let's construct a few:

$\mathbf{n = 0}$: Then $\lambda = 0$ giving $\frac{d}{dx}[(1-x^2)y'] = 0$, leading to $y(x) = a$, where a is a constant.

$\mathbf{n = 1}$: Then $\lambda = 2$ giving $(1-x^2)y'' - 2xy' + 2y = 0$. Let's seek a polynomial solution of degree $n = 1$: $y(x) = a_1 x + a_0$. Substituting this into the ode gives $-2a_1 x + 2(a_1 x + a_0) = 0$, or $a_0 = 0$. Thus we arrive at the polynomial solution $y(x) = Bx$, where B is a constant (**Exercise:** Check this is what relation (56) gives).

$\mathbf{n = 2}$: Then $\lambda = 6$ giving $(1-x^2)y'' - 2xy' + 6y = 0$. Let's seek a polynomial solution of degree $n = 2$: $y(x) = b_2 x^2 + b_1 x + b_0$. Substituting this into the ode gives $2b_2(1-x^2) - 2x(2b_2 x + b_1) + 6(b_2 x^2 + b_1 x + b_0) = 0$, or $b_1 = 0$, $b_0 = -\frac{1}{3}b_2$. Thus we arrive at the polynomial solution $y(x) = Cx^2 - \frac{C}{3}$, where C is a constant (**Exercise:** Check that this agrees with relations (55) and (56) from earlier).

3.3 Orthogonality of the polynomials

So far we have seen that polynomial solutions to (53) can exist. We have referred to these polynomial solutions as **orthogonal polynomials**. Let us now show what we mean by this.
Let's take two different polynomial solutions of the same equation (53), namely $y_n(x)$ and $y_m(x)$, where $n \neq m$. Then:

$$\frac{d}{dx}[p(x)y_n'(x)] + q(x)y_n'(x) + \lambda_n y_n(x) = 0, \tag{58}$$

and

$$\frac{d}{dx}[p(x)y_m'(x)] + q(x)y_m'(x) + \lambda_m y_m(x) = 0, \tag{59}$$

for $a < x < b$. Now let's multiply (58) by $y_m(x)w(x)$ and (59) by $y_n(x)w(x)$, where $w(x)$ (referred to as the weight function) is to be specified shortly, subtract and then integrate from a to b, giving

$$\int_a^b w(x)\left[y_n(py_m')' - y_m(py_n')' + q(y_ny_m' - y_my_n') + (\lambda_m - \lambda_n)y_ny_m\right]dx = 0,$$

where the argument of all the functions here is x so this has been suppressed for brevity. Integrating the first two terms by parts

$$\int_a^b w(x)\left[y_n(py_m')' - y_m(py_n')'\right]dx = \Big[w(x)\left[y_n(py_m') - y_m(py_n')\right]\Big]_a^b \\ - \int_a^b \left[w(y_n'py_m' - y_m'py_n') + w'(y_npy_m' - y_mpy_n')\right]dx.$$

Substituting back and re-arranging terms

$$\int_a^b \left[(wq - q'p)(y_ny_m' - y_my_n') + (\lambda_m - \lambda_n)wy_ny_m\right]dx + \Big[wp(y_ny_m' - y_my_n')\Big]_a^b = 0. \tag{60}$$

Now suppose that $w(x)$ is chosen so that $wq - w'p = 0$, i.e

$$\frac{w'}{w} = \frac{q}{p},$$

giving

$$w(x) = \exp\left\{\int^x \frac{q(u)}{p(u)}du\right\}. \tag{61}$$

The notation $\int^x$ here simply means to substitute in x at the end of the integration and ignore the lower limit (it cancels with the additive constant of integration). Further, suppose we also require that

$$w(a)p(a) = w(b)p(b) = 0 \tag{62}$$

(and we also choose $w(x) \geq 0$ on $x \in [a, b]$). Then, under these constraints, (60) gives

$$(\lambda_m - \lambda_n)\int_a^b w(x)y_n(x)y_m(x)dx = 0.$$

For $m \neq n$, we have $\lambda_m \neq \lambda_n$ (Exercise: why?), so, for $m \neq n$, we have

$$\int_a^b w(x)y_n(x)y_m(x)dx = 0. \tag{63}$$

These are known as the **orthogonality relations**. We often say that the polynomials y_n and y_m are **orthogonal with respect to the weight function** $w(x)$.

3.4 Classical Orthogonal Polynomials

Let's return to (62), namely our requirement that $w(a)p(a) = w(b)p(b) = 0$. We shall distinguish between three different cases for a, b:

1). a and b both finite (often taken to be $[-1,1]$).

In this case, we can satisfy (62) by requiring that $p(x)=0$ at $x=a,b$. Then $p(x)=-p_2(x-a)(b-x)$. This gives

$$\frac{q(x)}{p(x)}=\frac{q_0+q_1x}{-p_2(x-a)(b-x)}=\frac{\alpha}{x-a}-\frac{\beta}{b-x},$$

for some α, β which can be computed in terms of q_0, q_1 and p_2 (exercise). Then

$$\int^x \frac{q(u)}{p(u)}du=\alpha\log(x-a)+\beta\log(b-x)+\text{constant},$$

giving

$$w(x)=\text{constant}\cdot(x-a)^{\alpha}(b-x)^{\beta}.$$

Note α, $\beta>-1$ here so that $p(x)w(x)=0$ at $x=a,b$ holds still. These orthogonal polynomials are known as **Jacobi** polynomials, and sometimes are denoted by $P_n^{(\alpha,\beta)}(x)$.

There are special families of Jacobi weights with their own names, some famous ones are:

- Legendre: $(\alpha,\beta)=(0,0)$, $w(x)=1$, denoted $P_n(x)$.
- Chebyshev (1st kind): $(\alpha,\beta)=(-\frac{1}{2},-\frac{1}{2})$, $w(x)=\frac{1}{\sqrt{1-x^2}}$, denoted $T_n(x)$.

There are also Chebyshev (2nd kind), Ultraspherical polynomials and many more.

e.g. Recall the Legendre equation we saw earlier. For the Legendre polynomials $p(x)=1-x^2$, $q(x)=0$ giving $w(x)=e^0=1$. Hence the orthogonality relations are

$$\int_{-1}^{1} p_n(x)p_m(x)dx=0.$$

Figure 40 shows a plot of the first five Legendre polynomials.

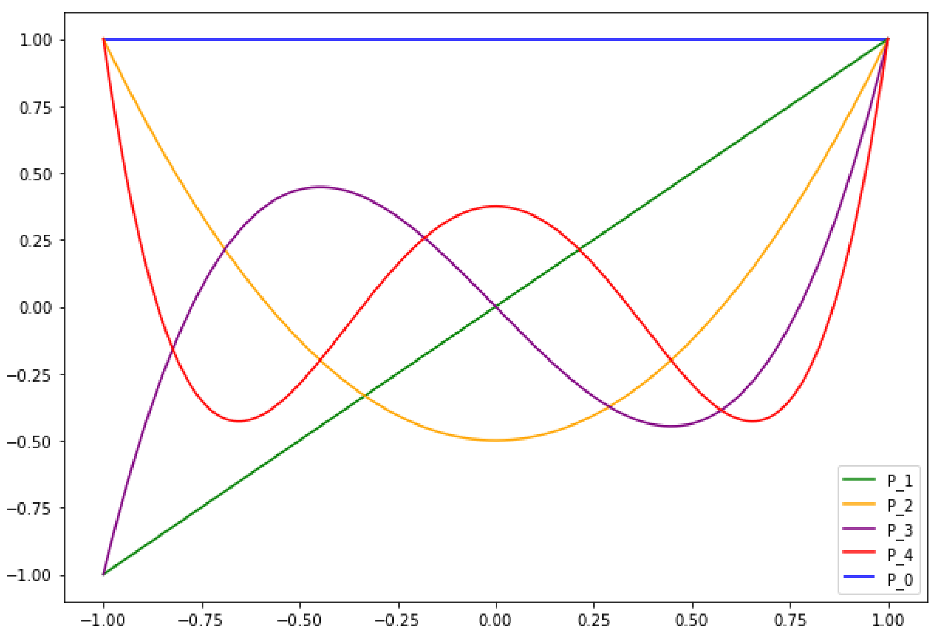


Figure 40: Plot of the first five Legendre polyomials: $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$ and $P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$.

2). $a \to -\infty$, $b \to \infty$. Take $p = p_0 > 0$ and $q(x) = -q_1 x$, $q_1 > 0$. Then

$$\int^x \frac{q(u)}{p(u)} du = -\frac{q_1 x^2}{2p_0},$$

giving

$$w(x) = e^{-\frac{q_1 x^2}{2p_0}}.$$

These are called **Hermitte** type and denoted $H_n(x)$.

3). a finite (without loss of generality take $a = 0$), and $b \to \infty$. In this case we can take $p(x) = p_1 x$, $p_1 > 0$. Then $p(0) = 0$. We must also have $w(x)q(x) \to 0$ as $x \to \infty$. If we have $q(x) = -q_1 x$, $q_1 > 0$. Then

$$w(x) = \exp\left\{ \int^x \frac{q(u)}{p(u)} du \right\} = e^{-\left(\frac{q_1}{p_1}\right)x}.$$

Then $w(x)q(x) \to 0$ as $x \to \infty$, so (62) is satisfied. These are called **Laguerre** polynomials, denoted $L_n(x)$ (It is possible to take $x^\alpha e^{-x}$ as the weight function, and this is done in many applications, these are then denoted $L_n^{(\alpha)}(x)$).

3.5 Uniqueness of Orthogonal Polynomials

We have so far shown that these polynomial solutions to (53) are orthogonal to one another with respect to $w(x)$ and have briefly looked at some of the classical examples. Let us now go further and show that these polynomials with $w(x)$ form a **unique** set (up to normalisation).

Proposition: Let $\{P_n(x)\}$ be a set of orthogonal polynomials on a given interval $[a, b]$ with a given weight function $w(x)$. Then this set is **unique** (up to normalisation).

Proof. First, note that, given **any** polynomial $F(x)$ of degree m, we can express $F(x)$ as

$$F(x) = \sum_{n=0}^{m} c_n p_n(x),$$

where the coefficients $\{c_n\}$ are unique, and $c_m \neq 0$. This is because x^m only appears in $p_m(x)$, so this fixes the coefficient c_m. Then x^{m-1} only subsequently appears in $p_{m-1}(x)$ after $p_m(x)$, so adding the appropriate multiple of $p_{m-1}(x)$ to match the coefficient of x^{m-1} fixes c_{m-1}, etc.
Now, suppose we have another set of orthogonal polynomials $\{Q_m(x)\}$ on $[a, b]$ with the same weight function $w(x)$. Consider $Q_m(x)$, for some m. By what we have argued previously

$$Q_m(x) = \sum_{n=0}^{m} c_n p_n(x),$$

for some unique set of coefficients $\{c_n\}$, with $c_m \neq 0$ (our aim will be to show that $c_n = 0$, for all $n < m$, thus giving $Q_m(x) = p_m(x)$).

Note that for $n \leq m$:

$$\int_a^b w(x)Q_m(x)p_n(x)dx = \sum_{r=0}^{m} c_r \int_a^b w(x)p_r(x)p_n(x)dx \qquad (64)$$
$$= c_n T_n,$$

where we have used the orthogonality relations to set the integrals equal to zero when $r \neq n$ and

$$T_n = \int_a^b w(x)p_n^2(x)dx \neq 0,$$

since $w(x) \geq 0$ for $x \in [a, b]$. But we can express $p_n(x)$ as

$$p_n(x) = \sum_{r=0}^{n} \tilde{c}_r Q_r(x),$$

for some unique set $\{\tilde{c}_r\}$, where $\tilde{c}_n \neq 0$. Hence for $n < m$:

$$\int_a^b w(x)Q_m(x)p_n(x)dx = \sum_{r=0}^{n} \tilde{c}_r \int_a^b w(x)Q_m(x)Q_r(x)dx = 0, \qquad (65)$$

since the integrals equal 0 for all $r < m$ by orthogonality. From setting (64) and (65) equal we get for $n < m$: $c_n T_n = 0$ meaning $c_n = 0$ (since $T_n \neq 0$). Thus

$$Q_m(x) = c_m p_m(x).$$

i.e. The set $\{p_n(x)\}$ orthogonal with respect to $w(x)$ on $[a, b]$ is unique up to normalisation (values of c_n in the proof). $\square$

Note: We can make the set $\{p_n(x)\}$ unique by requiring a normalisation. Some common examples of normalisations are:

(i). Set the coefficient of x^n in $p_n(x)$ to be 1 for all n, or setting

(ii). $\int_a^b w(x)p_n^2(x)dx = 1$ for all n.

3.6 The Three-Term Recurrence Relation

We now come to one of the central themes of orthogonal polynomials. That is, every family of orthogonal polynomials has a three-term recurrence relationship. If you know the three-term recurrence relationship, you can find the polynomials.

Theorem: Suppose $\{p_n(x)\}$ are a family of orthogonal polynomials on $[a, b]$ with respect to a weight $w(x)$. Then there exist constants A_n, B_n and C_n (i.e. constants depending on n) such that:

$$p_{n+1}(x) = (xA_n + B_n)p_n(x) + C_n p_{n-1}(x), \quad n \geq 1. \qquad (66)$$

Proof. Consider $xp_n(x)$: this is a polynomial of degree $n + 1$. We can express this in the form

$$xp_n(x) = \sum_{r=0}^{n+1} b_{n,r} p_r(x),$$

for some set of coefficients $\{b_{n,r}\}$, where $b_{n,n+1} \neq 0$. Then

$$xp_n(x) = b_{n,n+1}p_{n+1}(x) + \sum_{r=0}^{n} b_{n,r}p_r(x),$$

so

$$p_{n+1}(x) = \alpha_n x p_n(x) + \sum_{r=0}^{n} \alpha_{n,r}p_r(x), \tag{67}$$

where

$$\alpha_n = \frac{1}{b_{n,n+1}}, \quad \alpha_{n,r} = -\frac{b_{n,r}}{b_{n,n+1}}, \quad r \geq 0.$$

We will show that $\alpha_{n,n-2},\ \alpha_{n,n-3},\cdots,\ \alpha_{n,1},\ \alpha_{n,0}$ are all zero to obtain a recurrence relationship between p_{n+1}, p_n and p_{n-1}. To do this we make use of orthogonality properties. Let $s \leq n-2$. Multiply (67) by $w(x)p_s(x)$ and then integrate between a and b:

$$\int_a^b w(x)p_s(x)p_{n+1}(x)dx = \alpha_n \int_a^b w(x)xp_s(x)p_n(x)dx + \sum_{r=0}^{n} \alpha_{n,r} \int_a^b w(x)p_s(x)p_r(x)dx.$$

The LHS of this equation is equal to zero by orthogonality, since $s \leq n-2$. So, now writing

$$xp_s(x) = \sum_{r=1}^{s+1} b_{s,r}p_r(x),$$

we have:

$$0 = \alpha_n \sum_{r=0}^{s+1} b_{s,r} \int_a^b w(x)p_r(x)p_n(x)dx + \sum_{r=0}^{n} \alpha_{n,r} \int_a^b w(x)p_s(x)p_r(x)dx. \tag{68}$$

Assuming $n \geq 2$, consider $s = 0$. From (68) we get

$$0 = \alpha_n \sum_{r=0}^{1} b_{0,r} \int_a^b w(x)p_r(x)p_n(x)dx + \sum_{r=0}^{n} \alpha_{n,r} \int_a^b w(x)p_0(x)p_r(x)dx,$$

where every integral in the first term equates to 0 by orthogonality, since $n \geq 2$, and every integral in the second term, **except** the $r = 0$ term, also comes to 0 by orthogonality. So we find

$$0 = \alpha_{n,0} \int_a^b w(x)p_0^2(x)dx, \tag{69}$$

where this remaining integral is non-zero since $p_0^2(x) > 0$ (if $p_0 = 0$ then all other p's would be zero) and $w(x) > 0$. Thus we must have that

$$\alpha_{n,0} = 0.$$

In a similar fashion, the same process can be repeated for all s such that $0 \leq s \leq n-2$: from (68) we get:

$$0 = \alpha_n \sum_{r=0}^{s+1} b_{s,r} \int_a^b w(x)p_r(x)p_n(x)dx + \sum_{r=0}^{n} \alpha_{n,r} \int_a^b w(x)p_s(x)p_r(x)dx,$$

where all integrals in the first term are zero by orthogonality, since $s+1 \le n-1$, and all integrals in the second term equal 0 by orthogonality, **except** from the term when $r = s$. This gives

$$0 = \alpha_{n,s} \int_a^b w(x) p_s^2(x) dx,$$

where again, the integral here is non-zero for the same reason as in (69). Hence

$$\alpha_{n,s} = 0.$$

So we have shown that $\alpha_{n,s} = 0$ for $0 \le s \le n-2$. Hence (67) becomes

$$p_{n+1}(x) = \alpha_n x p_n(x) + \alpha_{n,n} p_n(x) + \alpha_{n,n-1} p_{n-1}(x),$$

which can be re-arranged and written as

$$p_{n+1}(x) = (xA_n + B_n) p_n(x) + C_n p_{n-1}(x), \quad n \ge 2.$$

To complete the proof the case when $n = 1$ needs to be checked. This is left as an exercise. □

3.7 Rodrigues Formula

We have seen that orthogonal polynomials possess a three-term recurrence relation. Later we will see some examples of how we might be able to derive this relationship. Now we focus on another way of generating orthogonal polynomials.

Theorem: Rodrigues Formula: For a weight function $w(x)$ and a polynomial $p(x)$ coming from the ordinary differential equation (53) we saw earlier, we can construct a family of orthogonal polynomials $\{p_n(x)\}$ up to a normalisation using the formula

$$p_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} [w(x) p^n(x)]. \tag{70}$$

This is called **Rodrigues Formula**.
The Formula works when the $p(x)$ and $w(x)$ for a set of orthogonal polynomials are known, and won't work for arbitrary $w(x)$ and $p(x)$.

For **Jacobi** polynomials: $p(x) = (x-a)(b-x)$.
For **Laguerre** polynomials: $p(x) = x$.
For **Hermitte** polynomials: $p(x) = 1$.

Proof. a). First we check that the polynomials generated by this formula are indeed orthogonal, i.e that

$$\int_a^b w(x) p_m(x) p_n(x) dx = 0, \quad \text{for } m < n,$$

or, equivalently (exercise: why?), that

$$I_m = \int_a^b w(x) x^m p_n(x) dx = 0, \quad \text{for } m < n.$$

Now Rodrigues formula gives

$$w(x)p_n(x) = \frac{d^n}{dx^n}[w(x)p^n(x)].$$

Thus, for $m < n$, using this in I_m gives

$$\begin{aligned} I_m &= \int_a^b x^m \frac{d^n}{dx^n}[w(x)p^n(x)]dx \\ &= \left[x^m \frac{d^{n-1}}{dx^{n-1}}(w(x)p^n(x))\right]_a^b - m\int_a^b x^{m-1}\frac{d^{n-1}}{dx^{n-1}}[w(x)p^n(x)]dx, \end{aligned}$$

upon an integration by parts. Now the boundary terms (terms inside the square bracket) evaluate to 0 following from the property that $w(x)p(x) = 0$ at $x = a, b$. Now we continue repeating integrating by parts, at each step we find that the boundary term equals 0, so, for $m < n$, we get

$$\begin{aligned} I_m &= (-1)^m m! \int_a^b \frac{d^{n-m}}{dx^{n-m}}(w(x)p^n(x))dx \\ &= \left[(-1)^m m! \frac{d^{n-m-1}}{dx^{n-m-1}}(w(x)p^n(x))\right]_a^b \\ &= 0, \end{aligned}$$

where again the boundary terms will equal zero when a, b are substituted in. This checks the orthogonality of the resulting polynomials.

b). Next we need to check the uniqueness. But now we have proved orthogonality this proof is identical to the proof for uniqueness we performed earlier in the chapter (section 3.5).

c). Finally we need to verify that Rodrigues formula indeed gives a polynomial of degree n. This needs to be checked on a case by case basis since $p(x)$ and $w(x)$ are of different degrees/$w(x)$ is non-polynomial for different families of orthogonal polynomials.

To illustrate this part of the proof we check this is true in the **Jacobi** case. For Jacobi polynomials $p(x) = (x-a)(b-x)$, $w(x) = \text{constant} \cdot (x-a)^\alpha (b-x)^\beta$. So, we have

$$w(x)p^n(x) = \text{constant} \cdot (x-a)^{n+\alpha}(b-x)^{n+\beta}.$$

Then, using Leibnitz rule

$$\begin{aligned} \frac{d^n}{dx^n}\left((x-a)^{n+\alpha}(b-x)^{n+\beta}\right) &= \sum_{r=0}^n \binom{n}{r} \frac{d^{n-r}}{dx^{n-r}}(x-a)^{n+\alpha}\frac{d^r}{dx^r}(b-x)^{n+\beta} \\ &= \sum_{r=0}^n c_r (x-a)^{r+\alpha}(b-x)^{n-r+\beta}, \end{aligned}$$

for some constants $\{c_r\}$. Then

$$\frac{1}{w(x)}\frac{d^n}{dx^n}\left((x-a)^{n+\alpha}(b-x)^{n+\beta}\right) = \sum_{r=0}^n c_r (x-a)^r (b-x)^{n-r},$$

which is indeed a polynomial of degree n (**Exercise:** check the coefficient of x^n in this is non-zero).

So, Rodrigues formula indeed generates $\{p_n(x)\}$. □

Some common Rodrigues formulae:

Laguerre: $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$

Hermitte: $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$

Notice that a constant is sometimes added outside the derivative in the Rodrigues formula. This is just for normalisation purposes (note that you will be given these formulae in any questions/have to derive them as part of the question. There is no need to memorise these specific formulae).

Example of Rodrigues formula in action:

Consider the Legendre equation

$$\frac{d}{dx}((1-x^2)y') + n(n+1)y = 0.$$

Here $p(x) = 1 - x^2$, $q(x) = 0$, leading to

$$\begin{aligned} w(x) &= \exp\left\{\int^x \frac{q(u)}{p(u)}du\right\} \\ &= e^0 \\ &= 1. \end{aligned}$$

So, Rodrigues formula gives

$$p_n(x) = \frac{d^n}{dx^n}[(1-x^2)^n].$$

From this we can calculate

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= \frac{d}{dx}(1-x^2) = -2x \\ p_2(x) &= \frac{d^2}{dx^2}((1-x^2)^2) = -4(1-3x^2). \end{aligned}$$

We can check that up to normalisation these agree with those we found previously (the example in section 3.2).

3.8 Generating Functions

Let us now examine another method of finding orthogonal polynomials, namely the so called **generating functions**. What are these? Well, for a set of orthogonal polynomials $\{p_n(x)\}$; we seek a $G(x, y)$ (called a **generating function**), such that

$$G(x, y) = \sum_{n=0}^{\infty} p_n(x)y^n, \tag{71}$$

(where we assume this sum converges for sufficiently small y) i.e the coefficients in the expansion of $G(x, y)$ for small y are all the orthogonal polynomials (up to normalisation).
We can construct such a $G(x, y)$ as follows:

- Introduce z, given implicitly by the relation

$$z = x + yp(z), \tag{72}$$

where $p(x)$ is the associated polynomial of degree 2 we are accustomed with.

- Note that (72) defines a quadratic equation in z. We choose the branch of z given by (72) which $\to x$ as $y \to 0$ (this will be clearer in examples).

- One can then show that

$$G(x, y) = \frac{w(z)}{w(x)} \cdot \frac{\partial z}{\partial x}, \tag{73}$$

or, noting that $\frac{\partial z}{\partial x} = 1 + yp'(z)\frac{\partial z}{\partial x}$, i.e that $\frac{\partial z}{\partial x} = \frac{1}{1-yp'(z)}$, then:

$$G(x, y) = \frac{w(z)}{w(x)} \cdot \frac{1}{1 - yp'(z)}. \tag{74}$$

Proof. Let

$$I_m = \int_a^b w(x)G(x, y)p_m(x)dx,$$

for $m \geq 0$. But from (71), we know that

$$\begin{aligned} I_m &= \int_a^b w(x) \left(\sum_{n=0}^{\infty} p_n(x) y^n \right) p_m(x) dx \\ &= T_m y^m, \end{aligned}$$

since by orthogonality all integrals equal zero unless $n = m$. Here

$$T_m = \int_a^b w(x)p_m^2(x)dx.$$

Now using (73) let's try to show we get the same thing:

$$I_m = \int_a^b w(x) \left(\frac{w(z)}{w(x)} \cdot \frac{\partial z}{\partial x} \right) p_m(x) dx.$$

Now, supposing y is small, we have $z \approx x$ (by (72)), so $x = a$, b correspond to $z = a$, b, and writing $\frac{\partial z}{\partial x}dx$ as dz, we get

$$\begin{aligned} I_m &= \int_{z=a}^b w(z)p_m(x)dz \\ &= \int_a^b w(z)p_m(z - yp(z))dz, \end{aligned}$$

where we have used (72) to substitute for x. Now with y taken sufficiently small we can expand $p_m(z-yp(z))$ about z, noting that p_m is a polynomial of degree m, this series will be finite:

$$p_m(z - yp(z)) = \sum_{r=0}^{m} \frac{\partial^r}{\partial z^r}(p_m(z)) \frac{(-yp(z))^r}{r!}.$$

This gives

$$I_m = \sum_{r=0}^{m} \frac{(-y)^r}{r!} \int_a^b w(z)(p(z))^r \frac{\partial^r}{\partial z^r}(p_m(z))dz.$$

Now considering each term in the sum separately: for $r = 0$ we have

$$\int_a^b w(z)p_m(z)dz = 0,$$

by orthogonality (with $p_0(z) = \text{constant}$). For $r \geq 1$ we integrate by parts

$$\begin{aligned} \int_a^b w(z)(p(z))^r \frac{\partial^r}{\partial z^r}(p_m(z))dz &= \left[w \cdot p^r \cdot \frac{\partial^{r-1}}{\partial z^{r-1}}(p_m) \right]_a^b - \int_a^b (w \cdot p^r)' \frac{\partial^{r-1}}{\partial z^{r-1}}(p_m)dz \\ &= -\int_a^b (w \cdot p^r)' \frac{\partial^{r-1}}{\partial z^{r-1}}(p_m)dz, \end{aligned}$$

since the boundary term equals zero as $w(z)p(z) = 0$ at $z = a$, b. Now let's integrate this by parts r times; at each integration the boundary terms will vanish (since $w(z)p(z) = 0$ at $z = a$, b). This gives

$$\begin{aligned} \int_a^b w(z)(p(z))^r \frac{\partial^r}{\partial z^r}(p_m(z))dz &= (-1)^r \int_a^b \frac{\partial^r}{\partial z^r}(w \cdot p^r) \cdot p_m dz \\ &= (-1)^r \int_a^b w(z)p_r(z)p_m(z)dz \\ &= \begin{cases} 0, & \text{for } r < m, \\ (-1)^r \cdot T_m, & \text{for } r = m, \end{cases} \end{aligned}$$

where to reach the second line we have used the fact that $\frac{\partial^r}{\partial z^r}(w \cdot p^r) = w(z)p_r(z)$, by Rodrigues formula. Thus

$$I_m = \frac{T_m}{m!} y^m,$$

as required. □

Examples involving generating functions:

1). Legendre polynomials: the Legendre equation is

$$\frac{d}{dx}[(1 - x^2)y'] + n(n+1)y = 0.$$

So $p(x) = 1 - x^2$, $q(x) = 0$, and as seen before this gives $w(x) = 1$. Then, introduce

$$z = x + y(1 - z^2),$$

which leads to

$$yz^2 + z - (x + y) = 0,$$

giving

$$z = \frac{-1 \pm \sqrt{1 + 4y(x+y)}}{2y}.$$

For small y, expand the square root using $(1+X)^n = 1 + nX + \frac{n(n-1)}{2!}X^2 + \cdots$, giving

$$\sqrt{1+4y(x+y)} = 1 + \frac{1}{2}(4y(x+y)) + \cdots = 1 + 2xy + O(y^2).$$

Now pick the square root so that, for small y;

$$z = \frac{-1 + (1 + 2xy + O(y^2))}{2y} = x + O(y),$$

as required (recall $z \sim x$ is the choice we want). If we had taken the -ve sign then we would have had $z \sim O(1/y)$ for small y. Then, from (74)

$$\begin{aligned} G(x,y) &= \frac{1}{1} \cdot \frac{1}{1 - y(-2z)} \\ &= \frac{1}{1+2yz} \\ &= \frac{1}{\sqrt{1+4y(x+y)}}. \end{aligned}$$

Hence, for small y, expanding $G(x,y)$ again using $(1+X)^n = 1 + nX + \frac{n(n-1)}{2!}X^2 + \cdots$, we find

$$\begin{aligned} G(x,y) &= 1 - \frac{1}{2}(4y(x+y)) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(4y(x+y))^2 + \cdots \\ &= 1 - 2xy - 2y^2 + 6y^2(x^2+y^2) + O(y^3) \\ &= 1 - 2xy + 2(3x^2-1)y^2 + O(y^3). \end{aligned}$$

Hence we can read off the values of

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_2(x) &= x^2 - \frac{1}{3}. \end{aligned}$$

You can check that these agree with earlier results up to normalisation.

2). Laguerre polynomials: consider the equation

$$xy'' + (1-x)y' + ny = 0, \quad 0 < x < \infty.$$

Written in standard form this is

$$\frac{d}{dx}(xy') - xy' + ny = 0.$$

So we can read off $p(x) = x$, $q(x) = -x$. Then

$$w(x) = \exp\left\{\int^x \frac{q(u)}{p(u)} du\right\} = e^{-x}.$$

Now introduce $z = x + yp(z) = x + yz \Rightarrow z = \frac{x}{1-y}$ (Note: $z \sim x$ for small y ✓). So,

$$\begin{aligned} G(x,y) &= \frac{w(z)}{w(x)} \cdot \frac{\partial z}{\partial x} \\ &= \frac{e^{-\frac{x}{1-y}}}{e^{-x}} \cdot \frac{1}{1-y} \\ &= \frac{e^{-\frac{xy}{1-y}}}{1-y}. \end{aligned}$$

Now let's expand $G(x,y)$ for small y

$$\begin{aligned} G(x,y) &= \left[1 - \frac{xy}{1-y} + \frac{1}{2}\left(\frac{xy}{1-y}\right)^2 + \cdots\right](1 + y + y^2 + \cdots) \\ &\approx [1 - xy(1 + y + O(y^2)) + \frac{1}{2}x^2y^2(1 + O(y)) + \cdots](1 + y + y^2 + \cdots) \\ &\approx 1 + (1-x)y + (1 - 2x + \frac{1}{2}x^2)y^2 + O(y^3). \end{aligned}$$

Hence we can read off the values (up to normalisation) of

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= 1 - x, \\ p_2(x) &= 1 - 2x + \frac{1}{2}x^2. \end{aligned}$$

Remark: The generating function $G(x,y)$ is **not** unique. e.g. For the Legendre polynomials,

$$G(x,y) = \frac{1}{\sqrt{1 - 2xy + y^2}},$$

is **also** a generating function (Exercise: expand this for small y and check this retrieves the same first three polynomials).

3.9 Examples of Deriving Recurrence Relations

Let's now look at some examples involving deriving recurrence relations using all of the techniques we have learnt and seen throughout the chapter so far.

1). Laguerre polynomials:

$$G(x,y) = \frac{e^{-\frac{xy}{1-y}}}{1-y} = \sum_{n=0}^{\infty} p_n(x)y^n.$$

Differentiate this above result with respect to y:

$$\frac{-x}{(1-y)^2} \cdot \frac{e^{-\frac{xy}{1-y}}}{1-y} + \frac{e^{-\frac{xy}{1-y}}}{(1-y)^2} = \sum_{n=1}^{\infty} np_n(x)y^{n-1},$$

where noticing that $\frac{e^{-\frac{xy}{1-y}}}{1-y} = \frac{e^x e^{-\frac{x}{1-y}}}{1-y}$ helps with the differentiation. Now substitute in the series for $G(x,y)$, giving

$$\frac{-x}{(1-y)^2}\sum_{n=0}^{\infty} p_n(x)y^n + \frac{1}{1-y}\sum_{n=0}^{\infty} p_n(x)y^n = \sum_{n=1}^{\infty} np_n(x)y^{n-1}.$$

Then multiplying by $(1-y)^2$ gives

$$-x\sum_{n=0}^{\infty} p_n(x)y^n + (1-y)\sum_{n=0}^{\infty} p_n(x)y^n = (1-y)^2\sum_{n=1}^{\infty} np_n(x)y^{n-1}.$$

Expanding out

$$(1-x)\sum_{n=0}^{\infty} p_n(x)y^n - \sum_{n=0}^{\infty} p_n(x)y^{n+1} = \sum_{n=1}^{\infty} np_n(x)y^{n-1} - 2\sum_{n=1}^{\infty} np_n(x)y^n + \sum_{n=1}^{\infty} np_n(x)y^{n+1}.$$

Now relabelling indices so that we just have y^n appearing gives

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty} p_n(x)y^n - &\sum_{n=1}^{\infty} p_{n-1}(x)y^n \\ &= \sum_{n=0}^{\infty}(n+1)p_{n+1}(x)y^n - 2\sum_{n=1}^{\infty} np_n(x)y^n + \sum_{n=2}^{\infty}(n-1)p_{n-1}(x)y^n.\end{aligned}$$

Comparing coefficients of y^n, we get

$$(1-x)p_n(x) - p_{n-1}(x) = (n+1)p_{n+1}(x) - 2np_n(x) + (n-1)p_{n-1}(x),$$

or upon rearranging

$$(n+1)p_{n+1}(x) = (2n+1-x)p_n(x) - np_{n-1}(x),$$

the **three-term recurrence relation** for the Laguerre polynomials. We can also derive recurrence relations involving the derivatives of these functions. For Laguerre polynomials for example we can show that (see problem sheet 3)

$$xp_n'(x) = np_n(x) - np_{n-1}(x).$$

2). Hermite polynomials:

$$y'' - 2xy' + 2ny = 0, \quad -\infty < x < \infty.$$

We associate $p(x) = 1$, $q(x) = -2x$, giving

$$\begin{aligned}w(x) &= \exp\left\{\int^x \frac{q(u)}{p(u)}du\right\} \\ &= e^{-x^2}.\end{aligned}$$

The Rodrigues formula gives

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}).$$

Then, using this definition, we have

$$
\begin{aligned}
H_{n+1}(x) &= (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) \\
&= (-1)^{n+1} e^{x^2} \frac{d^n}{dx^n} \left(\frac{d}{dx} (e^{-x^2}) \right) \\
&= (-1)^{n+1} e^{x^2} \frac{d^n}{dx^n} (-2x e^{-x^2}) \\
&= 2(-1)^n e^{x^2} \frac{d^n}{dx^n} (x e^{-x^2}) \\
&= 2(-1)^n e^{x^2} \left[x \frac{d^n}{dx^n} (e^{-x^2}) + n \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) \right],
\end{aligned}
$$

where we have applied Liebnitz rule in the final line. Hence, relating the final line back to Rodrigues formula we have

$$
H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x),
$$

the **three-term recurrence relation** for Hermite polynomials. A recurrence relation involving derivatives can be found as follows

$$
\begin{aligned}
\frac{d}{dx}(H_n(x)) &= (-1)^n 2x e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2}) \\
&= 2x H_n(x) - H_{n+1}(x),
\end{aligned}
$$

which, upon using the three-term recurrence relation leads to

$$
H_n'(x) = 2n H_{n-1}(x).
$$

3.10 Function Approximation with Orthogonal Polynomials

Let us finish this chapter by briefly discussing one of the main applications of orthogonal polynomials, that is function approximation.

A basic usage of orthogonal polynomials is for polynomial approximation. Suppose $f(x)$ is a degree n polynomial. Since $\{p_n(x)\}$ span all degree n polynomials, we know that

$$
f(x) = \sum_{k=0}^{n} f_k p_k(x), \tag{75}
$$

where

$$
f_k = \frac{\langle f(x), p_k(x) \rangle}{\langle p_k(x), p_k(x) \rangle},
$$

and we have used the inner product notation to represent

$$
\langle f(x), g(x) \rangle = \int_a^b f(x) g(x) w(x) dx.
$$

Example:

Let $f(x) = x^2 + 2x + 3$, and $[a, b] = [-1, 1]$. Let's show that we can build this polynomial using the Legendre polynomials, for which $w(x) = 1$ and $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 1/3$. Now, calculating the f_k we find

$$\begin{aligned} f_0 &= \frac{\langle f(x), p_0(x) \rangle}{\langle p_0(x), p_0(x) \rangle} \\ &= \frac{\int_{-1}^{1} (x^2 + 2x + 3) dx}{\int_{-1}^{1} dx} \\ &= \frac{10}{3}, \end{aligned}$$

$$\begin{aligned} f_1 &= \frac{\langle f(x), p_1(x) \rangle}{\langle p_1(x), p_1(x) \rangle} \\ &= \frac{\int_{-1}^{1} (x^2 + 2x + 3) x dx}{\int_{-1}^{1} x^2 dx} \\ &= 2, \end{aligned}$$

$$\begin{aligned} f_2 &= \frac{\langle f(x), p_2(x) \rangle}{\langle p_2(x), p_2(x) \rangle} \\ &= \frac{\int_{-1}^{1} (x^2 + 2x + 3)(x^2 - 1/3) dx}{\int_{-1}^{1} (x^2 - 1/3)^2 dx} \\ &= 1, \end{aligned}$$

so according to (75) we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{2} f_k p_k(x) \\ &= f_0 p_0(x) + f_1 p_1(x) + f_2 p_2(x) \\ &= \frac{10}{3} + 2x + x^2 - \frac{1}{3} \\ &= x^2 + 2x + 3, \end{aligned}$$

which is indeed $f(x)$.

We can also approximate other functions with orthogonal polynomials: provided the sum converges in x, we can represent an arbitrary function $f(x)$ by

$$f(x) = \sum_{k=0}^{\infty} f_k p_k(x).$$

In practice, smooth functions can then be approximated by a finite truncation

$$f(x) \approx \sum_{k=0}^{n} f_k p_k(x). \tag{76}$$

The accuracy of this approximation is typically dictated by the smoothness of f: the more times we can differentiate, the faster it converges. For analytic functions, it's dictated by the domain of analyticity, just like Laurent/Fourier series.

Example:

Let $f(x) = e^x$, and $[a, b] = [-1, 1]$. Let's show that we can closely approximate this function using just a few Legendre polynomials (again for which $w(x) = 1$ and $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 1/3$). Now, calculating the f_k this time we find

$$\begin{aligned} f_0 &= \frac{\langle f(x), p_0(x)\rangle}{\langle p_0(x), p_0(x)\rangle} \\ &= \frac{\int_{-1}^{1} e^x dx}{\int_{-1}^{1} dx} \\ &= \frac{1}{2}\left(e - \frac{1}{e}\right), \end{aligned}$$

$$\begin{aligned} f_1 &= \frac{\langle f(x), p_1(x)\rangle}{\langle p_1(x), p_1(x)\rangle} \\ &= \frac{\int_{-1}^{1} xe^x dx}{\int_{-1}^{1} x^2 dx} \\ &= \frac{3}{e}, \end{aligned}$$

$$\begin{aligned} f_2 &= \frac{\langle f(x), p_2(x)\rangle}{\langle p_2(x), p_2(x)\rangle} \\ &= \frac{\int_{-1}^{1} (x^2 - 1/3) e^x dx}{\int_{-1}^{1} (x^2 - 1/3)^2 dx} \\ &= \frac{15}{4}\left(e - \frac{7}{e}\right), \end{aligned}$$

so according to (76), truncating at $n = 2$ we have

$$\begin{aligned} f(x) &\approx \sum_{k=0}^{2} f_k p_k(x) \\ &\approx f_0 p_0(x) + f_1 p_1(x) + f_2 p_2(x) \\ &\approx \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}x + \frac{15}{4}\left(e - \frac{7}{e}\right)(x^2 - 1/3) \\ &\approx \frac{15}{4}\left(e - \frac{7}{e}\right)x^2 + \frac{3}{e}x + \frac{3}{4}\left(\frac{11}{e} - e\right). \end{aligned} \tag{77}$$

Figure 41 shows a plot of e^x, expansion (77) using the first three Legendre polynomials (denoted OP expansion) and the first three terms of the Taylor series for e^x (denoted Taylor expansion).

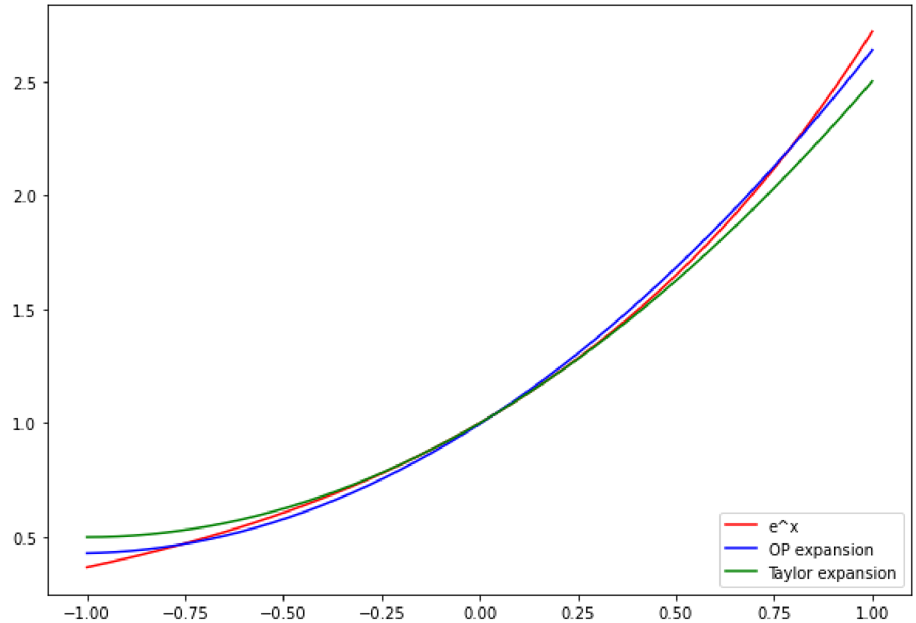


Figure 41: Plot of e^x, the expansion found using the first three Legendre polynomials (77) and the first three terms of the Taylor expansion of e^x, i.e $1 + x + \frac{x^2}{2}$.

e^x is entire and it turns out we get faster than exponential convergence using the orthogonal polynomial approximation (informally, one can see from the plot that the polynomial expansion is very close to the true curve and even performing better than the Taylor expansion of equivalent order for instance!).

Expansions in orthogonal polynomials can sometimes continue to work when other expansions will not, for example Taylor series. This is why expansions in orthogonal polynomials are used in problems where Taylor expansions would diverge.

On the real line, when dealing with the infinities, polynomial approximation is unnatural unless the function being approximated is a polynomial as the behaviour to the infinities is inconsistent (polynomials blow up and other functions may decay/blow up in stronger/slower ways). In cases like this it is common to also use the weight function $w(x)$ to aid in approximation (or $\sqrt{w(x)}$ or something similar). For example using Hermite polynomials we could express

$$f(x) \approx e^{-x^2} \sum_{k=0}^{n} f_k H_k(x).$$

There is a rich theory on the uses of orthogonal polynomials for function approximation and on which basis of polynomials with weight functions will work well for given situations, including methods for calculating the f_k to high accuracy.

3.11 Calculating Orthogonal Polynomials Numerically: Jacobi Operators

The three-term recurrence relation is the best way to compute successive orthogonal polynomials, even for cases where explicit formulae for the polynomials exist (e.g Chebyshev polynomials where using the recurrence relation avoids evaluating trigonometric functions).

The three-term recurrence relation given by (66)

$$p_{n+1}(x) = (xA_n + B_n)p_n(x) + C_n p_{n-1}(x), \quad n \geq 2,$$

can be written into the language of linear algebra as

$$x \begin{bmatrix} A_0 p_0(x) \\ A_1 p_1(x) \\ A_2 p_2(x) \\ \vdots \end{bmatrix} = J \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix},$$

where J is called a **Jacobi operator**; an infinite dimensional tri-diagonal matrix

$$J = \begin{bmatrix} -B_0 & 1 & 0 & \cdots & & \\ -C_1 & -B_1 & 1 & 0 & \cdots & \\ 0 & -C_2 & -B_2 & 1 & \ddots & \\ \vdots & 0 & -C_3 & -B_3 & 1 & \\ & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Suppose we are given that $p_0(x) = k_0$, constant (where $k_0 = 1$ can always be chosen as we have a degree of freedom with the uniqueness). Then we can construct the orthogonal polynomials as follows: noting

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix} = k_0,$$

we can combine this with the Jacobi operator to get

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & \\ -B_0 - A_0 x & 1 & 0 & \cdots \\ -C_1 & -B_1 - A_1 x & 1 & \ddots \\ 0 & -C_2 & -B_2 - A_2 x & \ddots \\ \vdots & 0 & -C_3 & \ddots \\ & \vdots & \ddots & \ddots \end{bmatrix}}_{=L_x} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} k_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

This is a lower-triangular system, so solving for the polynomials can be done via:

$$\begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{bmatrix} = L_x^{-1} \begin{bmatrix} k_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

or, it can be solved by forward re-currence

$$
\begin{aligned}
p_0(x) &= k_0, \\
p_1(x) &= (B_0 + A_0 x) p_0(x), \\
p_2(x) &= (B_1 + A_1 x) p_1(x) + C_1 p_0(x), \\
p_3(x) &= (B_2 + A_2 x) p_2(x) + C_2 p_1(x), \\
&\vdots
\end{aligned}
$$

Chapter 4: Conformal Mapping

A vast number of problems arising in fluid mechanics, electrostatics, heat conduction and many other physical applications can be formulated in terms of the 2D Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{78}$$

in a certain region D of the complex z-plane. The function $\phi(x, y)$, in addition to satisfying Laplace's equation, also satisfies specific boundary conditions on the boundary C of the region D.

Recalling from chapter 1 (section 1.4, p9) that the real and imaginary parts of an analytic function satisfy Laplace's equation, it follows that solving the above mentioned problem reduces to finding a function that is analytic in D and who's real/imaginary parts satisfy the corresponding boundary conditions on C. It turns out that finding this particular **analytic function** can be greatly simplified if the region D is something simple, such as the upper-half-plane or the interior of the unit disc.

This suggests that instead of solving Laplace's equation in D, we should first perform a change of variables from the complex variable z to the complex variable $\zeta = f(z)$, such that the region D of the z-plane is mapped to the upper-half plane or unit disc in the ζ-plane. In general, such transformations are called conformal and we study them in this chapter.

According to a theorem first discussed by Riemann, if D is a simply connected region (a region with no 'holes'), which is not the entire complex z-plane, then there exists an analytic function $f(z)$ such that $\zeta = f(z)$ transforms D onto the upper half ζ-plane. This theorem is famous in the field of complex analysis and is known as the Riemann-mapping theorem. Unfortunately (for its use in applied mathematics), the theorem does not provide a constructive approach for finding $f(z)$, it just proves the map exists whatever it is! Nevertheless, for certain simple domains it is possible to find an explicit formula for $f(z)$.

4.1 Conformal Transformations

Any complex function $\zeta = f(z)$ serves the purpose of defining the value of $\zeta = \xi + i\eta$ for a given value of the argument $z = x + iy$. Thus it can be thought of as a mapping of points in the z-plane to the corresponding points in the ζ-plane.

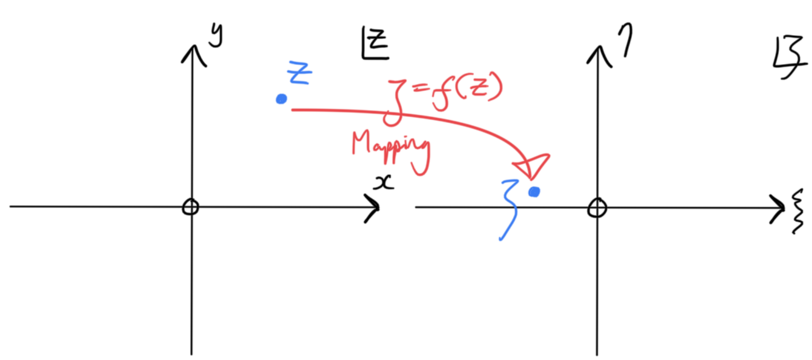


Figure 42: A mapping between the z and ζ planes.

To build up our understanding of these mappings, let's start by investigating mapping with a **linear function**. Let

$$\zeta = az + b, \qquad a,\, b \in \mathbb{C}. \tag{79}$$

Now if $a = 1$, then writing $b = b_r + ib_i$ we have $\zeta = z + b \Rightarrow \xi + i\eta = x + iy + b_r + ib_i \Rightarrow \xi = x + b_r,\ \eta = y + b_i$. So this represents a **parallel translation** of all the points in the complex-plane.

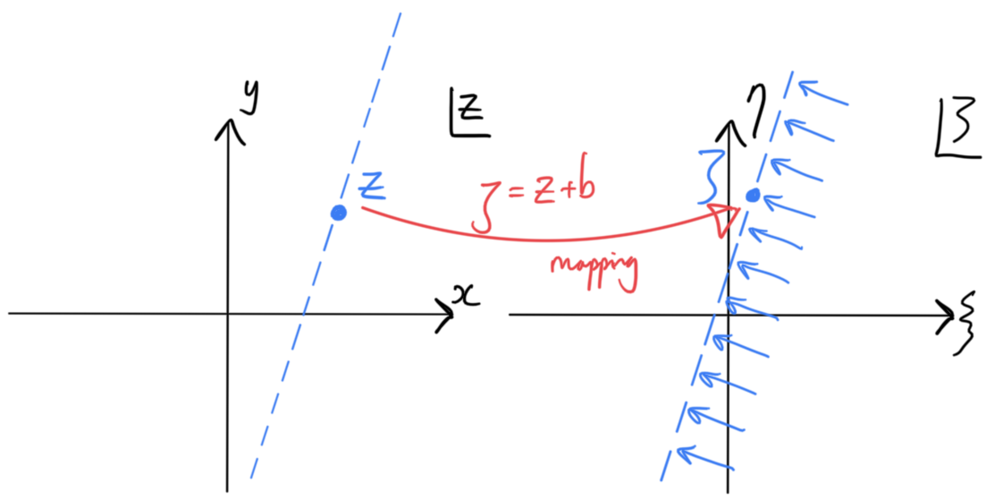


Figure 43: A parallel translation of points in the plane.

What if instead we set $b = 0,\ a \neq 0$. Then $\zeta = az$. Let $a = ke^{i\alpha}$ where $k = |a|$, $\alpha = \arg\{a\}$ and $z = re^{i\theta}$. Then: $\zeta = kre^{i(\theta+\alpha)}$, which is a stretching and rotating of the space.

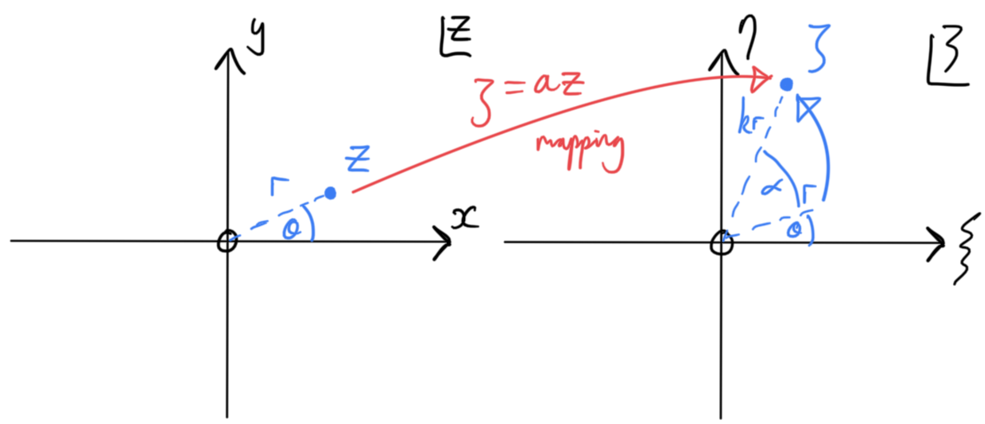


Figure 44: A stretching and rotating of the space.

Returning to the general case: $\zeta = az + b$. Consider two points z_0 and z_1 in the z-plane.

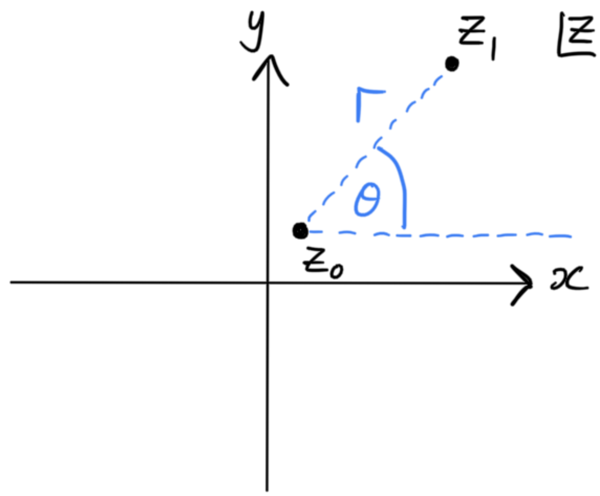


Figure 45: Points z_0 and z_1 in the z-plane.

Letting $\zeta_0 = az_0 + b$ and $\zeta_1 = az_1 + b$ gives $(\zeta_1 - \zeta_0) = a(z_1 - z_0)$. Writing $a = ke^{i\alpha}$, $z_1 - z_0 = re^{i\theta}$ gives $(\zeta_1 - \zeta_0) = kre^{i(\theta+\alpha)}$. So in other words, the line joining z_1 and z_0 in the z-plane is going to stretch and rotate.

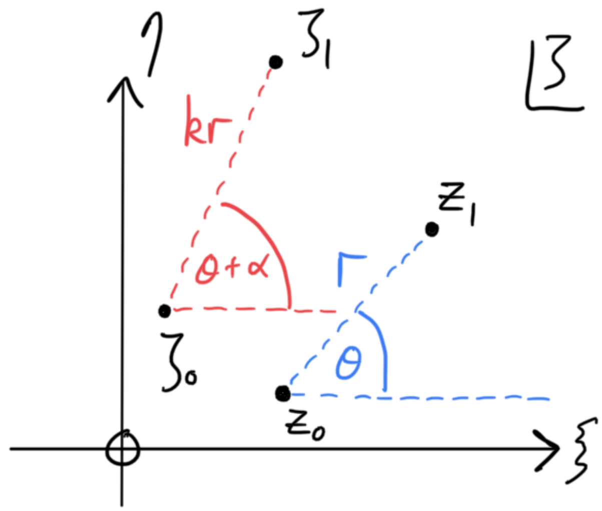


Figure 46: The line joining z_0 to z_1 is stretched and rotated.

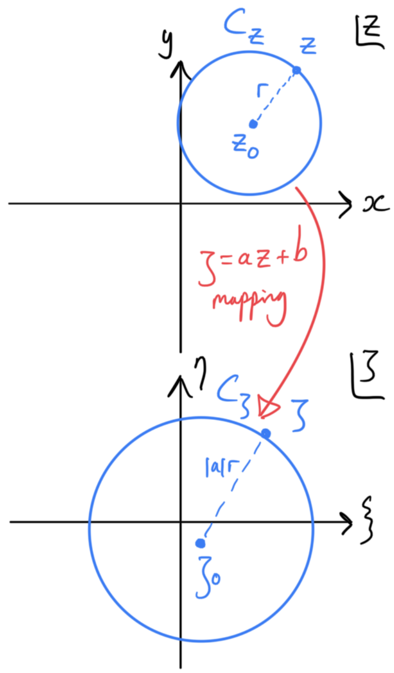
So, we notice the following facts about linear mapping:

(1) A straight line is mapped onto a straight line.

(2) The angle between two arbitrary lines will be preserved.

Let's also show the following result

(3) The linear function $\zeta = az + b$, with $a \neq 0$, maps any circle in the z-plane onto a circle in the ζ-plane.



Proof. Take a circle, C_z, in the z-plane of radius r, centre z_0. Then

$$|z - z_0| = r,$$

on C_z. Now take $\zeta = az + b$. Then let $\zeta_0 = az_0 + b$. Then

$$\begin{aligned} \zeta - \zeta_0 &= (az + b) - (az_0 + b) \\ &= a(z - z_0) \\ \Rightarrow |\zeta - \zeta_0| &= |a||z - z_0| \\ &= |a|r, \end{aligned}$$

which means that C_ζ is a circle. □

Figure 47: Linear mapping preserves circles.

So, linear mapping also preserves circles. Let us now define a conformal mapping:

Definition: The mapping $\zeta = f(z)$ is called **conformal** (meaning that it preserves the angle between two different arcs) at $z = z_0$ if $f(z)$ is **analytic** at z_0 and $f'(z_0) \neq 0$.

Let us justify why this is true. Indeed if $f(z)$ is analytic at $z = z_0$, then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

So, restricting ourselves to a small vicinity of z_0, i.e taking Δz small, see figure 48, gives

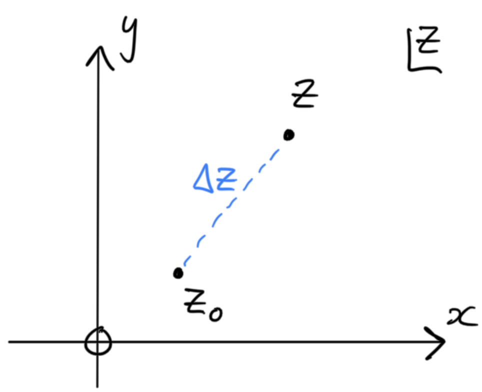


Figure 48: Points z in a small vicinity of z_0.

$$f(z_0 + \Delta z) - f(z_0) = \Delta z f'(z_0).$$

Now denoting $z = z_0 + \Delta z$, we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0)$$
$$\Rightarrow \zeta = f(z) = \underbrace{f'(z_0)}_{a}\, z + \underbrace{[f(z_0) - f'(z_0)z_0]}_{b}.$$

So, $\zeta = f(z)$ behaves locally like $az + b$, where $a = f'(z_0)$, and hence by property (2) of linear mapping preserves the angle between two arcs, so long as $a \neq 0$, i.e. $f'(z_0) \neq 0$.

Examples

1). Let D be the rectangular region in the z-plane bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 1$. Consider the transformation

$$\zeta = (1 + i)z + (1 + 2i).$$

The image of D under the transformation ζ is given by the rectangular region D' of the ζ-plane bounded by $\xi + \eta = 3$, $\xi - \eta = -1$, $\xi + \eta = 7$ and $\xi - \eta = -3$.

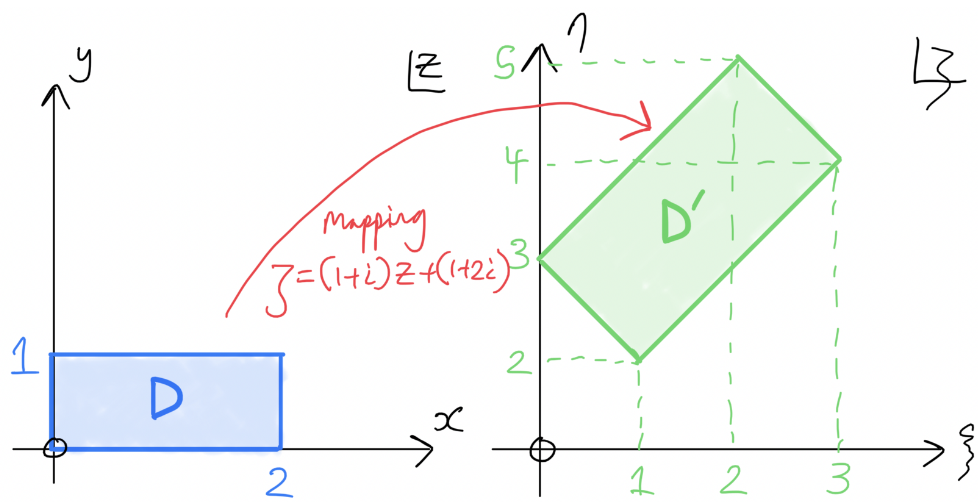


Figure 49: The rectangular regions D and D'.

If $\zeta = \xi + i\eta$, then $\xi + i\eta = (1 + i)(x + iy) + (1 + 2i)$, giving

$$\xi = x - y + 1, \qquad \eta = x + y + 2.$$

Hence the corners of D are mapped to $(1, 2)$, $(3, 4)$, $(2, 5)$ and $(0, 3)$. The line $x = 0$ is mapped to $\xi = 1 - y$, $\eta = y + 2 \Rightarrow \xi + \eta = 3$. Similar calculations can be done for the other sides of the rectangle.

The rectangle D is translated by $(1 + 2i)$, rotated by $\pi/4$ anticlockwise and dilated by a factor $\sqrt{2}$. In general a linear transformation $f(z) = az + b$ translates by b, rotates by $\arg\{a\}$ and stretches by $|a|$. For $a \neq 0$, as in this example, $f'(z) \neq 0$ so the linear transformation is conformal.

2). Let D be the triangular region bounded by $x = 1$, $y = 1$ and $x + y = 1$. Consider the transformation

$$\zeta = z^2.$$

The image of D under this transformation is given by the curvilinear triangle D'. One can check here $\xi = x^2 - y^2$ and $\eta = 2xy$, so for example, the line $x = 1$ is mapped to $\xi = 1 - y^2$, $\eta = 2y \Rightarrow \xi = 1 - \frac{\eta^2}{4}$. Similar calculations can be done for the other sides of the triangle.

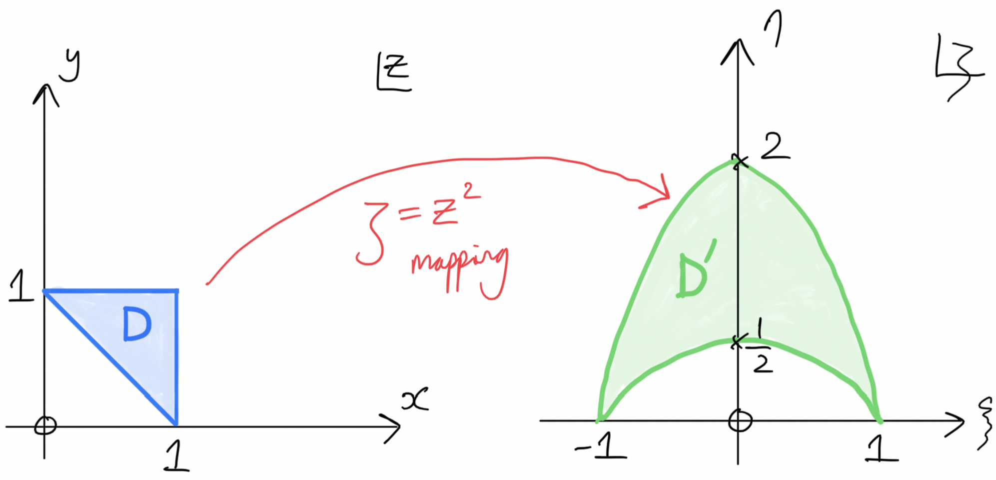


Figure 50: The triangular region D and the curvilinear triangular region D'.

Notice here since $f'(z) = 2z$, and the point $z = 0$ lies **outside** D, it follows that this mapping from D to D' is conformal. Hence the angles inside triangle D are equal to the respective angles in the curvilinear triangle D'.

4.2 Critical points and Inverse mappings

If $f'(z_0) = 0$, then the transformation $f(z)$ ceases to be conformal at $z = z_0$. Such a point is called a **critical point** of f. Let's find out what happens geometrically at a critical point. As before, let $\Delta z = z - z_0$, where z is a point near the critical point z_0. If the first non-vanishing derivative of $\zeta = f(z)$ at z_0 is of the nth order, then, representing $\Delta\zeta = \zeta - \zeta_0$, where $\zeta_0 = f(z_0)$, by a Taylor series, we find

$$\Delta\zeta = \frac{1}{n!} f^{(n)}(z_0)(\Delta z)^n + \frac{1}{(n+1)!} f^{(n+1)}(z_0)(\Delta z)^{n+1} + \cdots$$

Thus, as $\Delta z \to 0$, taking arguments of the above

$$\begin{aligned} \arg\{\Delta\zeta\} &\sim \arg\{f^{(n)}(z_0)(\Delta z)^n\} + \text{smaller terms} \\ &\sim \arg\{f^{(n)}(z_0)\} + \arg\{(\Delta z)^n\} \\ &\sim \arg\{f^{(n)}(z_0)\} + n \cdot \arg\{\Delta z\}. \end{aligned}$$

This means that the angle between any two infinitesimal line segments at the point z_0 is increased by the factor n.

Example

Let D be the triangular region bounded by $x = 0$, $y = 0$ and $x + y = 1$. The image of D under the transformation

$$\zeta = z^2,$$

is given by the curvilinear triangle D' as shown in figure 51 (exercise: check this!).

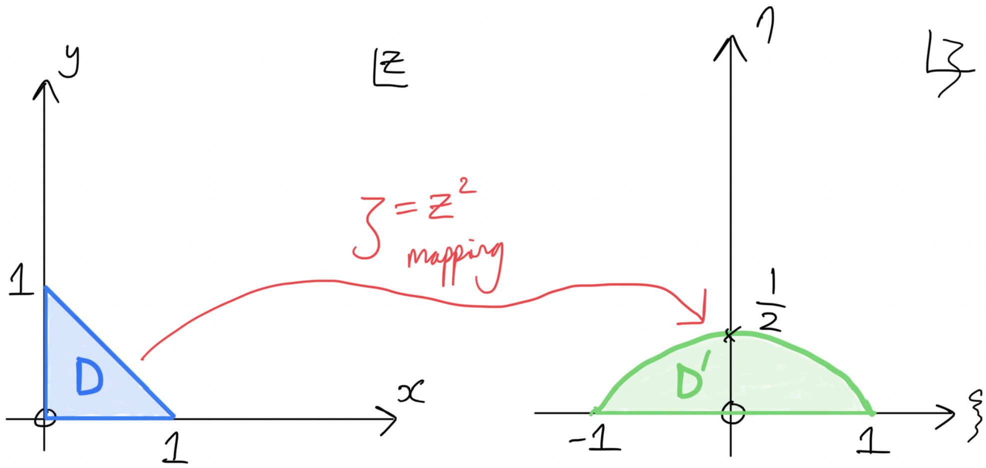


Figure 51: The triangular region D and the curvilinear triangular region D'.

The transformation $\zeta = z^2$ ceases to be conformal at $z = 0$. Because the second derivative at $z = 0$ is the first non-vanishing derivative there, it follows from our previous analysis that we should expect the angle at the point $z = 0$ (which is $\pi/2$) to be multiplied by a factor of 2. This is indeed the case, as the angle at the point corresponding to $z = 0$, i.e. $\zeta = 0^2 = 0$ is π in the ζ-plane.

Critical points can be important in determining whether the function $f(z)$ has an inverse. Finding the inverse of $\zeta = f(z)$ means solving this equation for z in terms of ζ.

As it turns out (we will not prove this here, the reader is directed to Ablowitz & Fokas p.342-343 for a proof) a conformal map $f(z)$ has a **unique** inverse that is also conformal. Furthermore, if f is analytic at z_0 but has $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, n-1$, then the mapping can still be **univalent** (one-to-one) near z_0 so that an inverse mapping exists. We will best see this by examples (this can involve introducing branch cuts to make functions single-valued).

Examples

1). The transformation $\xi = 4x^2 - 8y$, $\eta = 8x - 4y^2$ can be written in the form (exercise): $\zeta = (1+i)(z^2 + \overline{z}^2) + (2-2i)z\overline{z} + 8iz$. This map is **not conformal** since ζ is not an analytic function of z.

2). The transformation $\xi = x^3 - 3xy^2$, $\eta = 3x^2y - y^3$ can be written in the form (exercise): $\zeta = z^3$. This can be used to define a conformal mapping except at $z = 0$, where $f'(0) = 0$, $f''(0) = 0$ but $f'''(0) \neq 0$. The inverse mapping $z = \sqrt[3]{\zeta}$ is multi-valued so a branch cut would need to be introduced for this to give a single-valued inverse.

4.3 Power function mapping

Suppose that

$$\zeta = f(z) = z^{\alpha}, \tag{80}$$

where $\alpha \neq 0$ is real. This function is conformal everywhere, except at $z = 0, \infty$. To see this note that $f'(z) = \alpha z^{\alpha - 1}$, so for $\alpha > 1$, as $z \to 0$, $f'(z) \to 0$ and if $\alpha < 1$, $f'(z) \to \infty$ as $z \to 0$ (or $f'(z) \to 0$ as $z \to \infty$).

This suggests that the mapping $\zeta = z^{\alpha}$ preserves the angles at all finite points of the complex plane, but not at $z = 0$. Therefore it is used when one needs to change the angle between two lines passing through $z = 0$.

Example

Suppose we want to map from the wedge of angle θ in the z-plane to the upper half ζ-plane, as shown in figure 52.

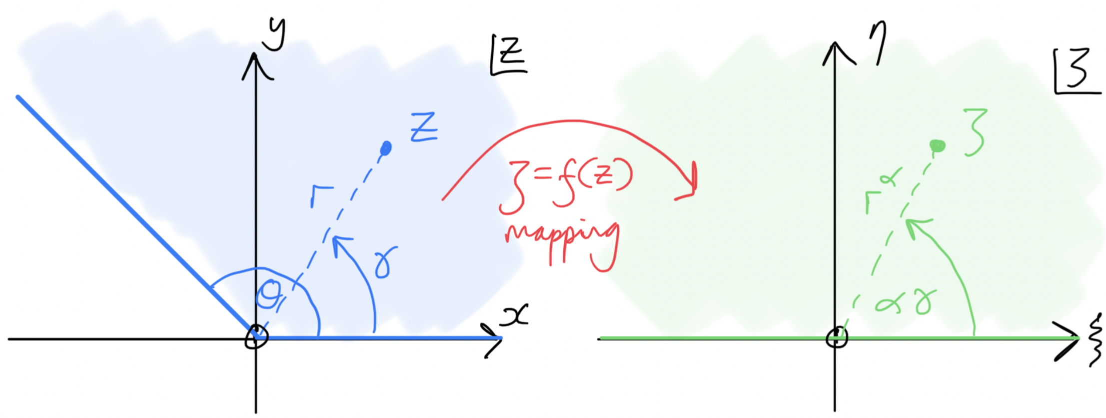


Figure 52: Power function mapping.

Then consider the mapping $\zeta = z^{\alpha}$, for some α to be determined. With $z = re^{i\gamma}$, we have $\zeta = r^{\alpha}e^{i\alpha\gamma}$. This keeps the positive real axis fixed, since $\gamma = 0$ there. We want the line at $\arg\{z\} = \theta$ to go to the negative real axis in the ζ-plane. So, we want $\gamma\alpha = \theta\alpha = \pi \Rightarrow \alpha = \pi/\theta$. So, the mapping

$$\zeta = z^{\frac{\pi}{\theta}},$$

achieves this. Notice at $z = 0$ we have

$$\frac{d\zeta}{dz} = \frac{\pi}{\theta} z^{\frac{\pi}{\theta} - 1},$$

which equals 0 for $\frac{\pi}{\theta} > 1$ and at $z \to \infty$ equals 0 for $\frac{\pi}{\theta} < 1$, as predicted by the theory.

4.4 Linear fractional transformation

An important class of mappings is given by the particular choice of $f(z)$:

$$\zeta = f(z) = \frac{az+b}{cz+d}, \tag{81}$$

where a, b, c, $d \in \mathbb{C}$ and $ad - bc \neq 0$. This transformation is called **linear fractional**, or **bilinear**, or commonly known as a **Möbius map**. Observe that, if $c = 0$, this becomes a linear transformation. For $c \neq 0$, we have

$$\begin{aligned}
\zeta &= \frac{1}{c}\left[\frac{az+b}{z+\frac{d}{c}}\right] \\
&= \frac{1}{c}\left[\frac{a(z+\frac{d}{c}) - \frac{ad}{c} + b}{z+\frac{d}{c}}\right] \\
&= \frac{1}{c}\left[a - \frac{\frac{ad}{c} - b}{z+\frac{d}{c}}\right] \\
&= \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d},
\end{aligned}$$

which is why we take $ad - bc \neq 0$ so the mapping is meaningful. Now multiply (81) by its denominator, giving

$$\begin{aligned}
cz\zeta + d\zeta &= az + b \\
\Rightarrow (c\zeta - a)z &= b - d\zeta \\
\Rightarrow z &= \frac{b-d\zeta}{c\zeta - a},
\end{aligned}$$

so the **inverse transformation** is also a linear fractional transformation. So, except for potential issues at the points $z = -\frac{d}{c}$ and $\zeta = \frac{a}{c}$, this is a one-one mapping each way. In order to take care of these points, we adopt the following conventions for the complex number ∞: (for $a \neq 0$)

- $\frac{a}{\infty} = 0$,
- $\frac{a}{0} = \infty$.

Then, we see that

$$\zeta = \frac{az+b}{cz+d} = \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c},$$

as $z \to \infty$ and further we have

$$\zeta = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d} = \infty,$$

as $z \to -\frac{d}{c}$. The linear fractional transformation can be decomposed into the following set of conformal mappings:

(1) $z_1 = cz + d$,

(2) $\zeta_1 = \frac{1}{z_1}$,

(3) $\zeta = \frac{a}{c} + \frac{bc-ad}{c}\zeta_1$.

Mappings (1) and (3) are linear mappings which we already know much about. (2) is a power function mapping with $\alpha = -1$. Writing $\zeta_1 = \frac{1}{z_1}$ as

$$\zeta_1 = \frac{1}{|z_1|}e^{-i\arg\{z_1\}},$$

we see the transformation reflects in the real axis and changes moduli to $\frac{1}{|z_1|}$.

4.5 Joukowski transformation

Suppose we wish to map the region exterior to the circular arc in the z-plane (as shown in figure 53) to the region exterior to the circle in the ζ-plane (as shown in figure 53). Let's find the mapping that does this!

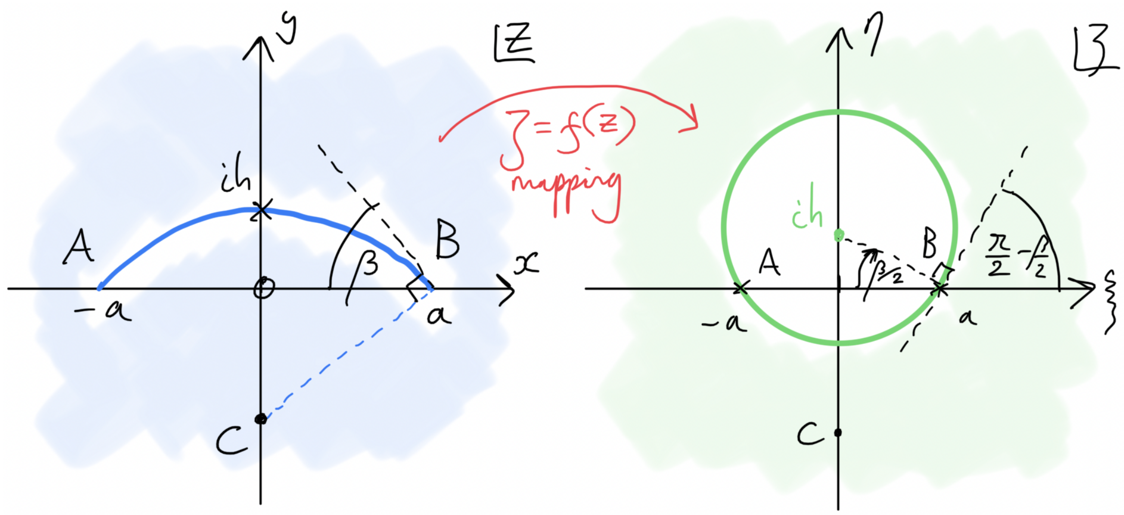


Figure 53: The Joukowski transformation.

(Exercise: show that the angle marked $\frac{\beta}{2}$ takes this value and hence the value of the angle marked $\frac{\pi}{2} - \frac{\beta}{2}$). To find the mapping we require, $\zeta = f(z)$, let's consider breaking down the problem into a series of maps.

First, let's apply the linear fractional transformation

$$z_1 = \frac{z-a}{z+a},$$

to the z-plane.

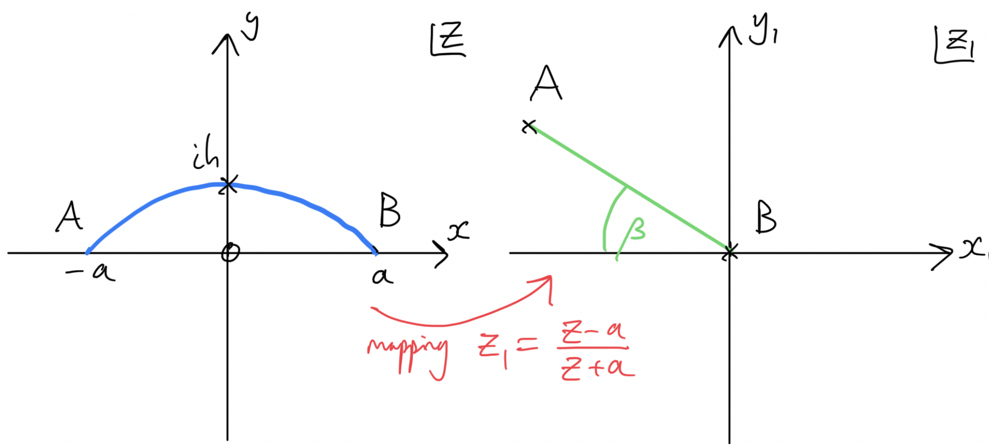


Figure 54: Möbius mapping from z to z_1 plane.

Notice that point B, where $z = a \to z_1 = 0$ and point A where $z = -a \to z_1 = \infty$ (so A goes off to infinity somewhere). To locate where let's take the derivative

$$\frac{dz_1}{dz} = \frac{z + a - (z - a)}{(z + a)^2} = \frac{2a}{(z + a)^2}.$$

Now at point B, $z = a \Rightarrow \frac{dz_1}{dz}|_B = \frac{1}{2a} \neq 0$, so angle β between the arc and the real-axis is preserved. The real axis for $-a < z < a$ corresponds to the negative real axis in the z_1-plane. Hence the angle between the arc and the negative real axis at B in the z_1-plane is β. Moreover the arc must now correspond to a straight line joining the point B to A, since A is at infinity and linear fractional transforms map circles to circles (which can be straight lines passing through ∞), see problem sheet 4. Hence A is located as drawn in figure 54.

Now let's apply the same transform to the circle in the ζ-plane. Introduce

$$\zeta_1 = \frac{\zeta - a}{\zeta + a}.$$

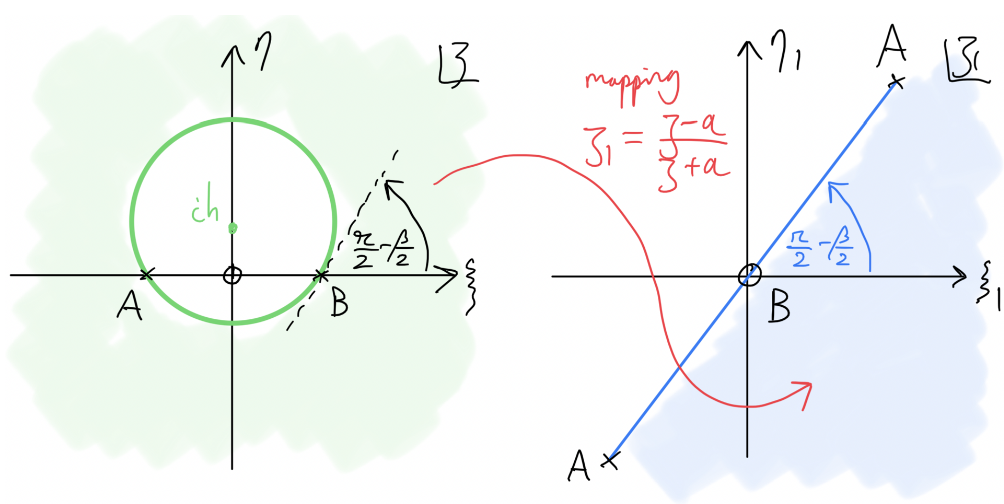


Figure 55: Möbius mapping from ζ to ζ_1 plane.

Then, point B at $\zeta = a \to \zeta_1 = 0$ and point A at $\zeta = -a \to \zeta_1 = \infty$. Now since A goes off to infinity and B to zero we will have a straight line again from A at ∞, through B, and back out to A at infinity (since now we are transforming an entire circle). This time we can check for ζ real and $\zeta > a$ we get a section of the real axis from 0 to 1 in the ζ_1-plane. Hence the angle $\frac{\pi}{2} - \frac{\beta}{2}$ between the real axis and the circle at B in the ζ-plane will be preserved between the arc (circle) and the segment of the real axis mentioned in the ζ_1-plane, as shown in figure 55.

Now we have transformed from both ends, it is clearer how to 'meet in the middle'. If we set

$$z_1 = \zeta_1^2,$$

then B becomes a point of non-conformality and the angles 'open' here, see figure 56.

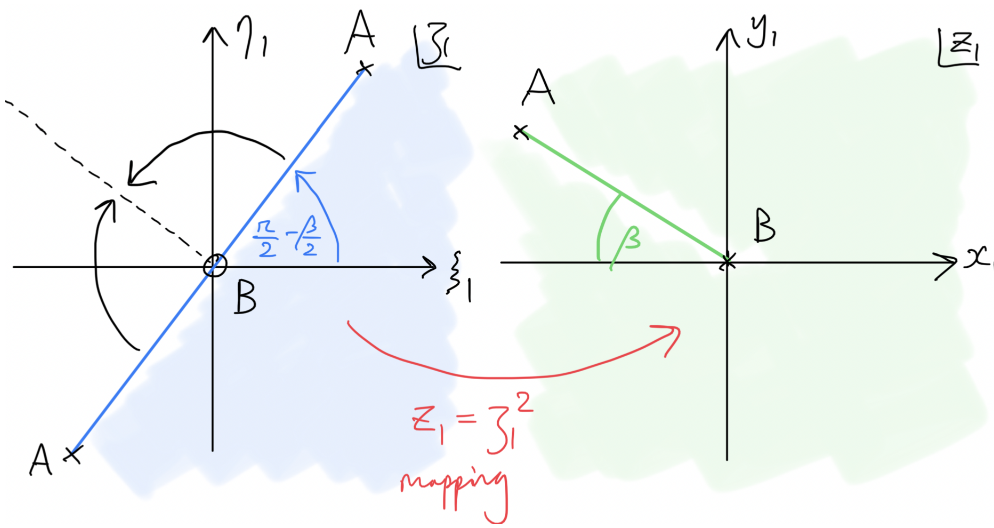


Figure 56: Power law mapping from ζ_1 to z_1 plane.

Putting the series of mappings together:

$$\begin{aligned}
\frac{z-a}{z+a} &= \frac{(\zeta-a)^2}{(\zeta+a)^2} \\
\Rightarrow \frac{z+a-2a}{z+a} &= \left(\frac{\zeta+a-2a}{\zeta+a}\right)^2 \\
\Rightarrow 1 - \frac{2a}{z+a} &= \left(1 - \frac{2a}{\zeta+a}\right)^2 = 1 - \frac{4a}{\zeta+a} + \frac{4a^2}{(\zeta+a)^2} \\
\Rightarrow \frac{1}{z+a} = \frac{2}{\zeta+a} - \frac{2a}{(\zeta+a)^2} &= \frac{2(\zeta+a)-2a}{(\zeta+a)^2} = \frac{2\zeta}{(\zeta+a)^2} \\
\Rightarrow z+a = \frac{(\zeta+a)^2}{2\zeta} &= \frac{\zeta^2+2a\zeta+a^2}{2\zeta} = \frac{\zeta}{2} + \frac{a^2}{2\zeta} + a
\end{aligned}$$

$$\Rightarrow z = \frac{1}{2}\left(\zeta + \frac{a^2}{\zeta}\right). \tag{82}$$

This is known as the **Joukowski transformation**. A conformal mapping taking the exterior of the circle centred at ih and passing through $\pm a$ (a real) in the ζ-plane to the exterior of the arc of the circle passing through $\pm a$ and ih in the z-plane. Inverting this mapping:

$$\begin{aligned} 2\zeta z &= \zeta^2 + a^2 \\ \iff \zeta^2 - 2z\zeta + a^2 &= 0 \\ \iff \zeta &= z \pm \sqrt{z^2 - a^2}. \end{aligned}$$

But do we take $+$ or $-$ here? Well, let's take the point $z = 2a$ for instance. Plugging this in gives $\zeta = 2a \pm \sqrt{4a^2 - a^2} = a(2 \pm \sqrt{3})$. So, we see that if we want the exterior of the circle mapped to the exterior of the arc, we must take the $+$ sign. So

$$\zeta = z + \sqrt{z^2 - a^2},$$

is the inverse we want.

4.6 The Schwarz-Christoffel transformation

Suppose that in the z-plane we have a polygon with n vertices, A_1, A_2, $\cdots$, A_n (see figure 57) and we are interested in mapping this polygon to the upper-half ζ-plane.

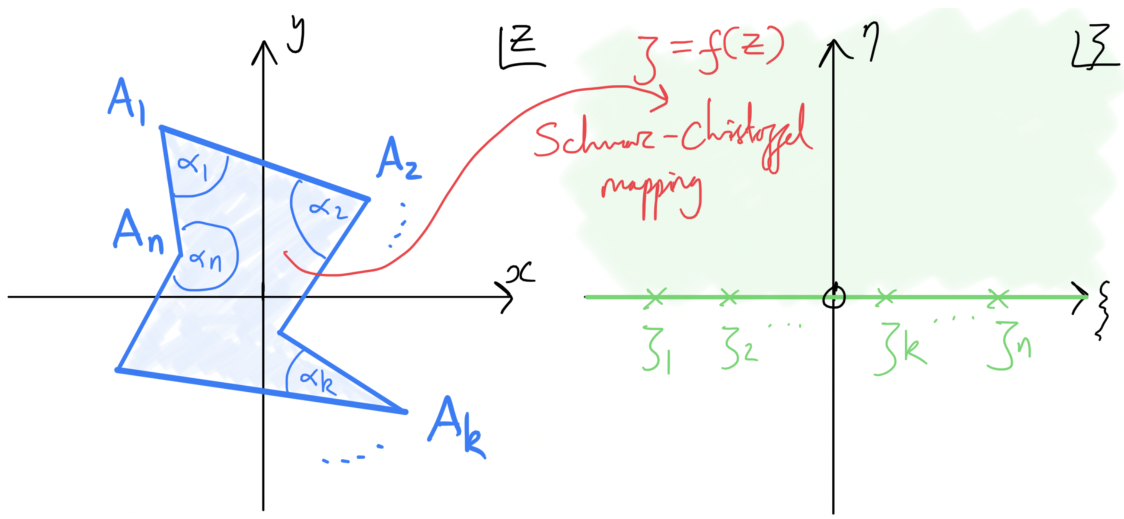


Figure 57: The Schwarz-Christoffel mapping.

This can be achieved by the transformation

$$z = c_1 \int (\zeta - \zeta_1)^{\frac{\alpha_1}{\pi} - 1} (\zeta - \zeta_2)^{\frac{\alpha_2}{\pi} - 1} \cdots (\zeta - \zeta_n)^{\frac{\alpha_n}{\pi} - 1} d\zeta + c_2, \tag{83}$$

where ζ_1, ζ_2, $\cdots$, ζ_n all lying on the real axis in the ζ-plane are the images of the vertices A_1, A_2, $\cdots$, A_n respectively, α_1, α_2, $\cdots$, α_n are the interior angles of the polygon at vertices A_1, A_2, $\cdots$, A_n respectively and c_1 and c_2 are two complex constants.

A key point about conformal transformations which is important here is that we have the freedom to decide on the image locations of exactly three points in the complex ζ-plane (i.e we can choose three of the ζ_i). After the first three choices, all successive points will now have a forced position as we have ran out of degrees of freedom in our mapping (see Ablowitz & Fokas for a proof). The complex constants c_1 and c_2 need to be found as part of the mapping and this can be done by substitution of values for z and ζ into the mapping formula once the integration is conducted.

When using the Schwarz-Christoffel transformation it is convenient to choose one of the boundary points in the ζ-plane, say ζ_1, to be at infinity. It turns out by doing this that the corresponding term $(\zeta - \zeta_1)^{\frac{\alpha_1}{\pi}-1}$ is omitted from (83) making the integration easier (not proved here). Another useful thing to note is that placing the free choices of ζ_i symmetrically about the imaginary axis in the ζ-plane often simplifies the resulting integral in (83) making the integration easier.

One further point to be aware of is that the polygon in the z-plane may extend to infinity (see figure 58), this can be dealt with straightforwardly (see the coming example).

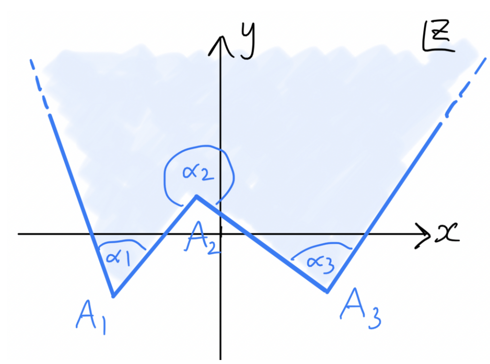


Figure 58: A polygon extending to infinity.

The proof of the Schwarz-Christoffel mapping result (83) is omitted here (see Ablowitz & Fokas p.348-352 if interested in details).

Example: Find a conformal map from the half-strip shown in figure 59 to the upper-half ζ-plane.

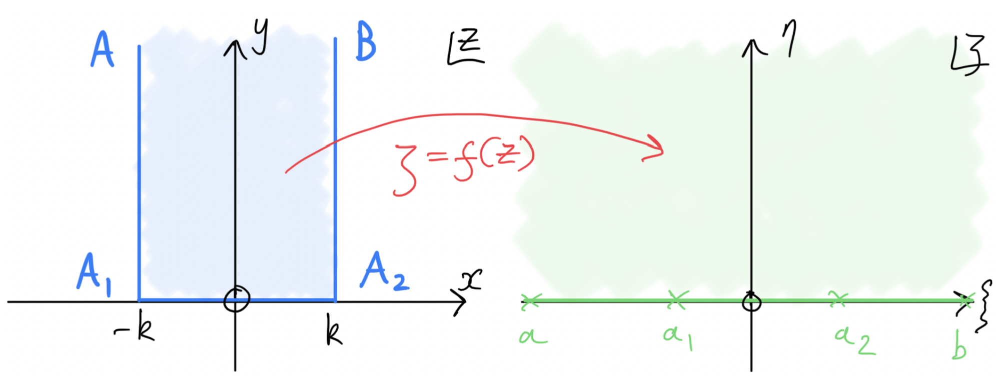


Figure 59: A vertical semi-infinite strip in the z-plane and the upper-half ζ-plane.

We associate point $A(z=\infty)$ with point $a(\zeta=\infty)$ (point B is the same point and is drawn to indicate the direction), $A_1(z=-k)$ with $a_1(\zeta=-1)$ and $A_2(z=k)$ with $a_2(\zeta=1)$. Thus the Schwarz-Christoffel formula (83) gives:

$$z = c_1 \int (\zeta+1)^{-\frac{1}{2}}(\zeta-1)^{-\frac{1}{2}} d\zeta + c_2,$$

where we have $\zeta_1=-1$, $\zeta_2=1$, $\alpha_1=\frac{\pi}{2}$, $\alpha_2=\frac{\pi}{2}$ and the point at infinity provides no contribution. Hence

$$\begin{aligned} z &= c_1 \int \frac{d\zeta}{\sqrt{\zeta^2-1}} + c_2 \\ &= ic_1 t + c_2 \\ &= ic_1 \sin^{-1}\zeta + c_2, \end{aligned}$$

where we have used the substitution $\zeta=\sin t$, giving $d\zeta=\cos t dt$ and $\sqrt{\zeta^2-1}=i\cos t$. Now we need to find c_1 and c_2. When $\zeta=-1$, $z=-k$ and when $\zeta=1$, $z=k$. Plugging these into what we've found gives

$$\begin{aligned} -k &= ic_1 \sin^{-1}(-1) + c_2, \\ k &= ic_1 \sin^{-1}(1) + c_2, \end{aligned}$$

which upon use of $\sin^{-1}(-1)=-\frac{\pi}{2}$ and $\sin^{-1}(1)=\frac{\pi}{2}$ give $c_1=-\frac{2ki}{\pi}$ and $c_2=0$. Hence a mapping is given by

$$z = \frac{2k}{\pi}\sin^{-1}(\zeta), \quad \text{or} \quad \zeta = \sin\left(\frac{\pi z}{2k}\right).$$

4.7 Complex Potentials

As discussed at the start of the chapter, and proved in chapter 1, the real and imaginary parts of an analytic function satisfy Laplace's equation. This means that if we are trying to solve the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

for some function ϕ, then we can instead seek an analytic function

$$w(z) = \phi(x,y) + i\psi(x,y),$$

known as the **complex potential**, who's real or imaginary part (it doesn't matter which we choose it to be) is the function we are looking for.

Boundary conditions: In all applications, regardless of the quantity in question, in order to find the solution of Laplace's equation, one needs to satisfy appropriate boundary conditions. There are two types of boundary-value problems that arise frequently in applications:

- **Dirichlet** type: where the value of the quantity of interest, say ϕ, is specified on the boundary.
- **Neumann** type: where the value of the normal derivative of ϕ is specified on the boundary.

(There are also mixed boundary conditions and in all cases one can show that the resulting solution to Laplace's equation is unique). Reformulated in terms of the complex potential, Dirichlet boundary conditions, say $\phi = c$, become of the form $\text{Re}\{w(z)\}$ (or $\text{Im}\{w(z)\}$) equal to c. To see how Neumann conditions reformulate, first recall from chapter 1 that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

then

$$\begin{aligned}\frac{\partial w}{\partial z} = \frac{dw}{dz} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(\phi + i\psi) \\ &= \frac{1}{2}\left[\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} - i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}\right] \\ &= \frac{1}{2}\left[2\frac{\partial \phi}{\partial x} - 2i\frac{\partial \phi}{\partial y}\right] \\ &= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y},\end{aligned}$$

where $\partial w/\partial z = dw/dz$ since $w(z)$ is analytic (a function of z only - no $\overline{z}$), and the Cauchy-Riemann equations were used to reach the third line. Note that if we were more interested in how dw/dz relates to ψ then we could use the Cauchy-Riemann equations to rewrite the final result in terms of ψ (giving $\partial\psi/\partial y + i\partial\psi/\partial x$). This shows that Neumann conditions, say $\partial\phi/\partial y = 0$ for instance, relate to the real and imaginary parts of dw/dz (here giving $\text{Im}\{dw/dz\} = 0$).

Next we outline some of the applications, although we omit the derivations and background as to why Laplace's equation models these.

Ideal fluid flow: As seen previously (section 2.8, p.61), ideal fluid flow can be modelled by finding a function $\psi(x, y)$ which satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

These problems can then be solved by finding an analytic function

$$w(z) = \phi(x, y) + i\psi(x, y),$$

known as the **complex potential**, who's imaginary part is the function we are looking for. A useful result to know is that

$$\frac{dw}{dz} = \overline{U} = u - iv,$$

where u and v are the horizontal and vertical velocity components of the fluid respectively. The function $\psi(x, y)$ is called the **streamfunction** and setting $\psi(x, y) =$ constant produces the streamlines of the flow (the paths the fluid particles follow).

Heat flow: The steady-state temperature $T(x, y)$ in a region satisfies Laplace's equation in that region. As such these problems can be solved by finding an analytic function

$$w(z) = T(x, y) + i\psi(x, y),$$

known as the **complex temperature**. Usually, as written, the real part is the temperature function we are looking for. The curves of the family $T(x, y) =$ constant are called isothermal lines (lines of constant temperature).

Electrostatics: As seen previously (section 2.9, p.64), Laplace's equation also appears in electrostatics applications. Problems can be reduced to finding the analytic function

$$w(z) = \phi(x, y) + i\psi(x, y),$$

called the **complex electrostatic potential**. Similarly to ideal fluid flow

$$\frac{dw}{dz} = -\overline{E} = -E_1 + iE_2,$$

(take note of the extra $-$ sign here though) where E_1 and E_2 are the horizontal and vertical components of the electric field. In this context the curves of $\phi(x, y) =$ constant and $\psi(x, y) =$ constant are called equipotential and flux lines respectively.

Let's now take a look at a couple of important complex potentials.

Uniform flow: Consider, for example, the two-terminal rectangular conductor of width L and height H as shown in figure 60.

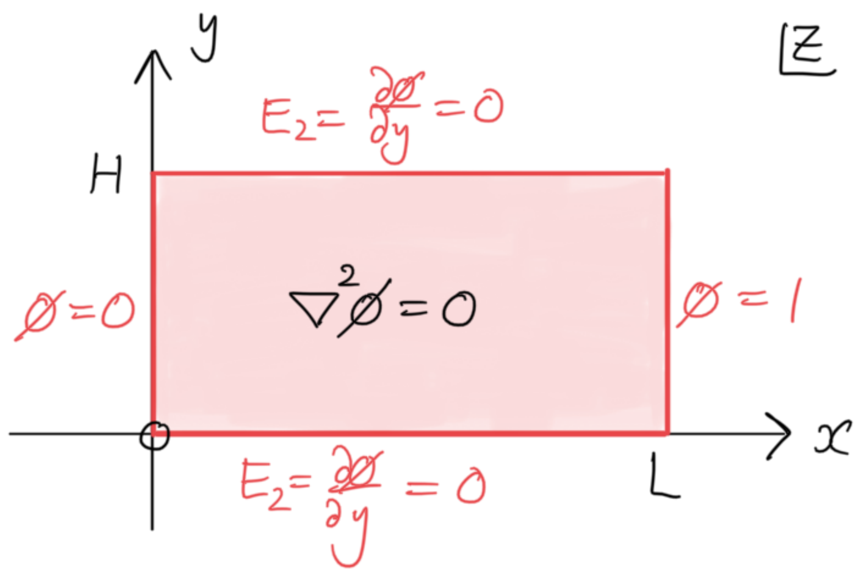


Figure 60: A rectangular conductor in the z-plane.

The left and right ends of the conductor are charged to voltages of $\phi = 0$ and $\phi = 1$ respectively and the upper and lower boundaries of the conductor are insulated $\partial\phi/\partial y = 0$. The voltage $\phi(x, y)$ inside the conductor satisfies Laplace's equation. Let's see if we can find the complex potential satisfying this problem!

So, let

$$w(z) = \phi(x, y) + i\psi(x, y),$$

and consider the ansatz

$$w(z) = kz,$$

for some complex constant $k = k_r + ik_i \in \mathbb{C}$. Well first observe that

$$\phi = \mathrm{Re}\{w(z)\} = \mathrm{Re}\{kz\} = k_r x - k_i y,$$

so when $x = 0$ and $\phi = 0$ we need to set $k_i = 0$ and when $x = L$ and $\phi = 1$ we need to set $k_r = 1/L$. So we find

$$w(z) = \frac{z}{L}, \tag{84}$$

but we have not checked that the two Neumann boundary conditions are satisfied. Indeed

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = \frac{1}{L},$$

which is real, so we must have

$$\frac{\partial \phi}{\partial y} = 0,$$

everywhere and hence the Neumann boundary conditions are also satisfied. The complex potential (84) therefore satisfies all the boundary conditions and is analytic, so is the solution to the problem we want. Taking it's real part

$$\phi = \mathrm{Re}\{w(z)\} = \frac{x}{L},$$

and we see the voltage changes linearly between the two terminals. Indeed the complex potential

$$w(z) = cz, \tag{85}$$

for some complex constant $c \in \mathbb{C}$ is known as the complex potential representing **uniform flow**. Whenever we see that a quantity must vary constantly and linearly over a domain we know this complex potential represents this.

If the domain is unbounded, see figure 61, then this complex potential still works. For instance, consider the problem of finding the complex potential representing uniform fluid flow at an angle of θ from the positive horizontal across the z-plane. Indeed if we consider

$$w(z) = e^{-i\theta} z,$$

then

$$\begin{aligned} \frac{dw}{dz} &= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \\ &= \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} \\ &= u - iv \\ &= \cos\theta - i\sin\theta, \end{aligned}$$

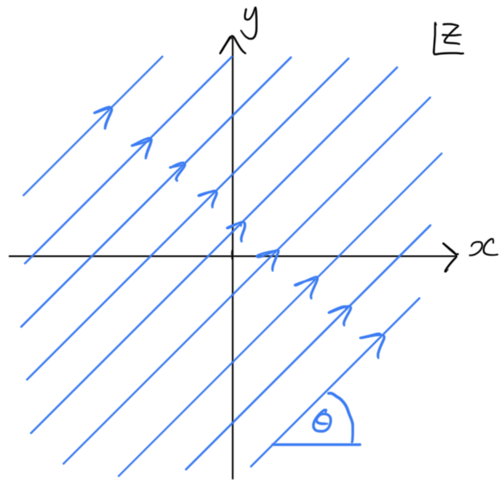


Figure 61: Uniform flow in the unbounded z-plane.

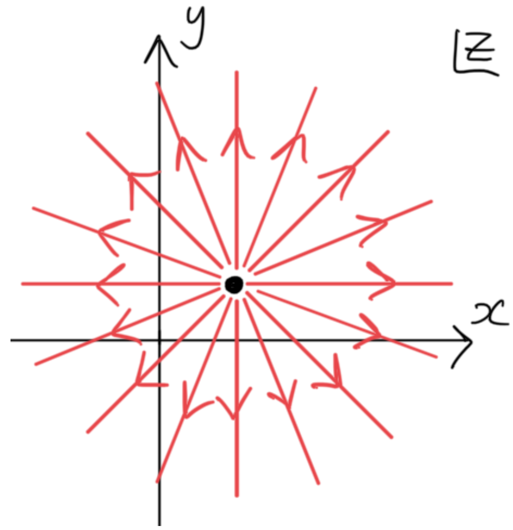


Figure 62: A point source in the z-plane.

so we see this gives fluid flow with velocities $u = \cos\theta$ and $v = \sin\theta$ everywhere.

A point source: Consider now the complex potential given by

$$w(z) = k\log(z - z_0),$$

where k, $z_0 \in \mathbb{C}$ are complex constants. This is known as the complex potential representing a **point source** centred at $z = z_0$. To understand this, let $z - z_0 = re^{i\theta}$, then

$$\begin{aligned} w(z) &= k\log(z - z_0) \\ &= k\log r + ik\theta \\ &= \phi(x, y) + i\psi(x, y), \end{aligned}$$

so then we see taking $k \in \mathbb{R}$ gives $\psi(x, y) = k\theta$, i.e lines of constant ψ (representing streamlines of fluid flow or lines of current flow etc.) emanate from the point z_0 either outwards (if $k > 0$ and this is known as a point **source**) or inwards (if $k < 0$ and this is then known as a a point **sink**), see figure 62. For applications purposes the constant k here is often written as $m/2\pi$ for some constant m known as the **strength** of the source.

Remark: Taking $k = k_i i$, i.e. taking k purely imaginary, leads to $w(z) = ik_i \log r - k_i\theta$, so then we have $\psi(x, y) = k_i \log r$, i.e the lines of constant ψ are now circles around the point z_0. For this reason the complex potential is often referred to as a **point vortex** in this case (much much more on these in the module on **Vortex Dynamics** next term).

Remark: There are many more complex potentials that we do not investigate here. What we have instead tried to do is to give some of the core building block potentials and develop your intuition as to why these solve the problems (this will become stronger from doing more examples), rather than give a library of different known potentials. Those taking courses in fluid mechanics in the department will see other complex potentials for things such as stagnation point flow, flows around cylinders, etc.

4.8 Conformal mapping to solve Laplace's equation

In all of the different applications of Laplace's equation we have discussed methods of complex analysis can be applied in order to solve problems, such as using Cauchy & Hilbert transforms to formulate singular integral equations (as we did in chapter 2). This now brings us to the focus of this chapter: obtaining solutions to Laplace's equation using complex potentials and the theory of conformal mapping.

This procedure involves the following steps:

(1) Use a conformal mapping to transform the region D of the z-plane in which the problem is set onto a simple region such as the unit disc or a half-plane of the ζ-plane.

(2) Solve the problem in the ζ-plane using our knowledge of simple complex potentials.

(3) Use this solution and the inverse mapping function to solve the original problem (recall that if $\zeta = f(z)$ is conformal then $f(z)$ has a unique inverse).

This procedure needs justification however... upon applying a conformal mapping, what if the resulting problem in the ζ-plane is no longer harmonic! Perhaps the change of variables completely changes the equation we need to solve. This is where the beauty of conformal mappings being applied to harmonic problems takes place!

Indeed, let $\phi(x, y)$ be the harmonic function satisfying Laplace's equation we wish to solve for in region D of the z-plane. Now assume the region D is mapped onto the region D' of the ζ-plane by a conformal mapping $\zeta = f(z)$, where $\zeta = \xi + i\eta$. Then $\phi(x, y) = \phi(x(\xi, \eta), y(\xi, \eta))$ remains harmonic in D'. Indeed, one can verify that (by differentiation and use of the Cauchy-Riemann equations)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left| \frac{df}{dz} \right|^2 \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right),$$

which, because $\frac{df}{dz} \neq 0$ since the mapping is conformal, proves the above assertion. Upon use of a conformal map, our problem remains harmonic! Let's now show how the procedure outlined above can be used to solve some example problems.

Examples

1). Solve Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

for the steady-state temperature distribution, T, inside the unit circle that takes the value T_1 for $0 \leq \theta < \pi$ (i.e. on the upper boundary) and takes the value T_2 for $\pi \leq \theta < 2\pi$ (i.e. on the lower boundary), by conformally mapping from the unit circle in a complex z-plane to the upper-half ζ-plane.

Solution: Let's draw the problem we want to solve in the z-plane.

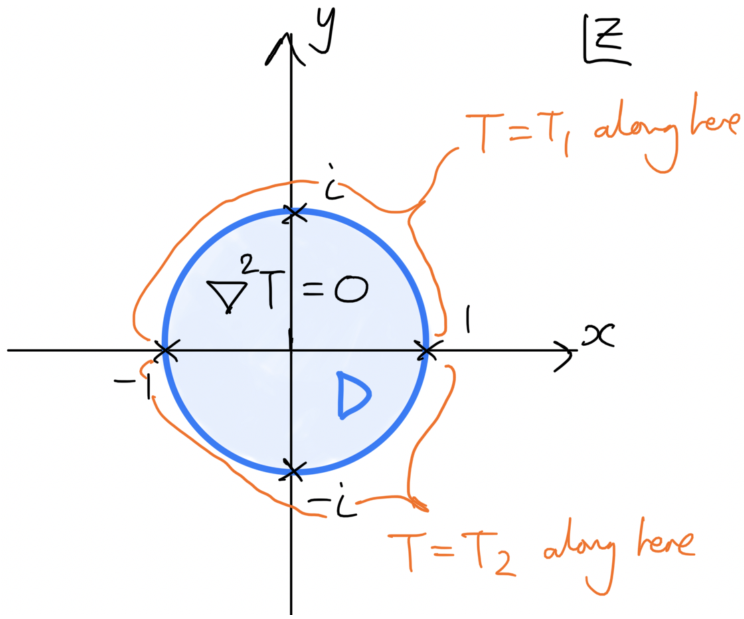


Figure 63: The physical problem in the z-plane.

We need to find a conformal map from the unit disc in the z-plane to the upper half ζ-plane. Let's try to work out the mapping we need by doing things in stages. First map from the z-plane to the z_1-plane by use of the map $z_1 = z + 1$. This simply translates the circle to the right by 1 unit.

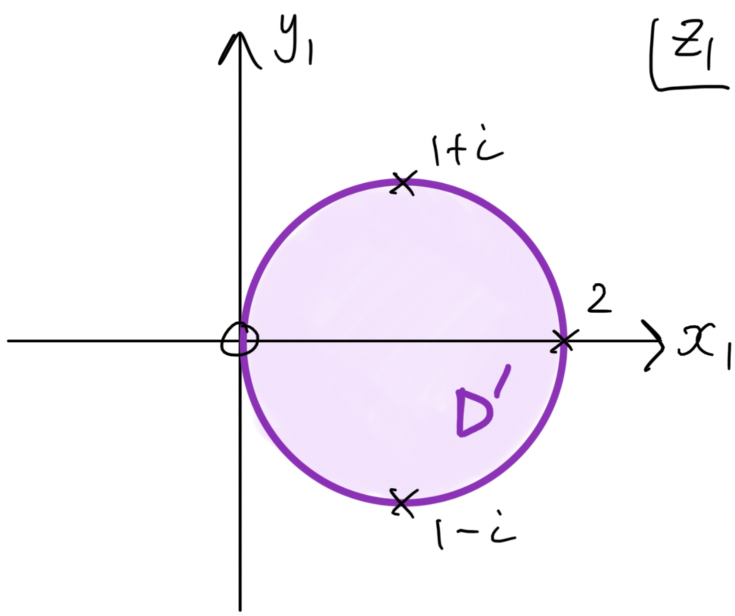


Figure 64: The domain in the z_1-plane.

Why did we do this? Well our goal is to obtain a half-plane (i.e. a circle with infinite radius), so if we can set one point of our unit circle on the point 0, then by using an inversion map (i.e. using

$1/z$ or equivalently the correct linear fractional transform if we knew it) we will get a line (a circle with infinite radius). We can then translate and rotate our line to obtain the half-plane we desire. As mentioned, let's now apply

$$z_2 = \frac{1}{z_1},$$

to our last region D'. Well in the z_1-plane our boundary is given by $|z_1 - 1| = 1$, or $z_1 = 1 + e^{i\alpha}$, $\alpha \in [0, 2\pi]$. So, we have

$$\begin{aligned} z_2 &= \frac{1}{1+e^{i\alpha}} \cdot \frac{1+e^{-i\alpha}}{1+e^{-i\alpha}} \\ &= \frac{1}{2}\frac{(1+\cos\alpha - i\sin\alpha)}{(1+\cos\alpha)} \\ &= \frac{1}{2} - \frac{1}{2}\frac{\sin\alpha}{1+\cos\alpha} i. \end{aligned}$$

i.e. the circle in the z_1-plane is mapped to the line $\text{Re}\{z_2\} = 1/2$ in the z_2-plane.

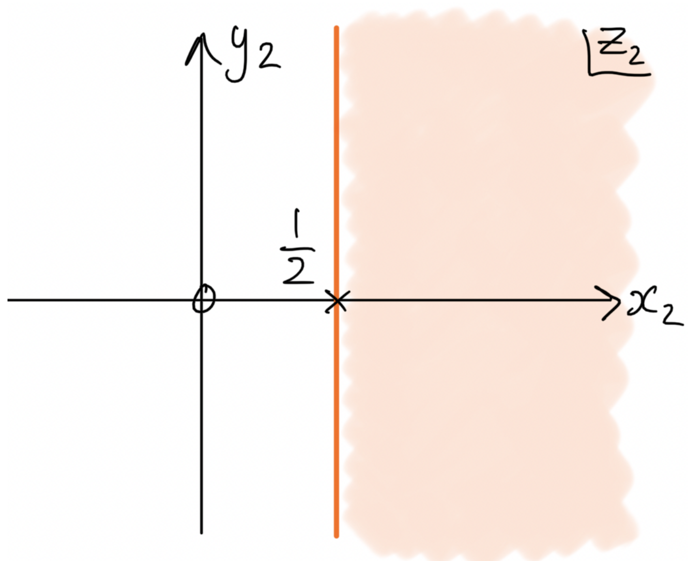


Figure 65: The domain in the z_2-plane.

In fact we map the interior of the circle in the z_1-plane to the right half-plane $\text{Re}\{z_2\} > 1/2$ in the z_2-plane. To see that this is the correct half-plane, take a point inside the circle in the z_1-plane (say $z_1 = 1$), and check which side of $\text{Re}\{z_2\} = 1/2$ this point is mapped to. Indeed, $z_1 = 1$ is mapped to $z_2 = 1/z_1 = 1$, so we have the right-half plane.

Finally, to map from the right-half plane we have to the upper-half plane apply a scaling by 2 and a rotation by $\pi/2$ anticlockwise, followed by a translation by $-i$. This corresponds to applying the mapping

$$\zeta = 2iz_2 - i.$$

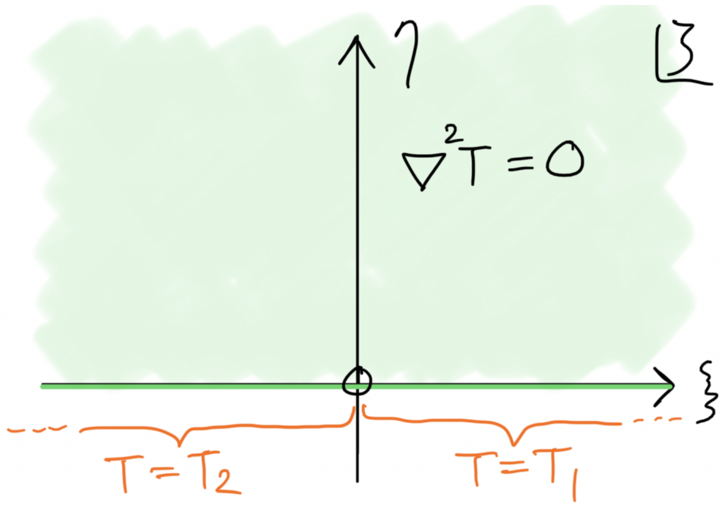


Figure 66: The upper-half ζ-plane.

One can check by carefully following where each portion of the circle maps to that the section where $T = T_1$ in the upper-half z-plane corresponds to the section $\xi \geq 0$ in the ζ-plane, and the section where $T = T_2$ in the lower-half z-plane corresponds to the line $\xi < 0$ in the ζ-plane as shown in figure 66. Putting

$$
\begin{aligned}
\zeta &= 2iz_2 - i, \\
z_2 &= \frac{1}{z_1}, \\
z_1 &= z + 1,
\end{aligned}
$$

together gives

$$
\zeta = i\left(\frac{1-z}{1+z}\right).
$$

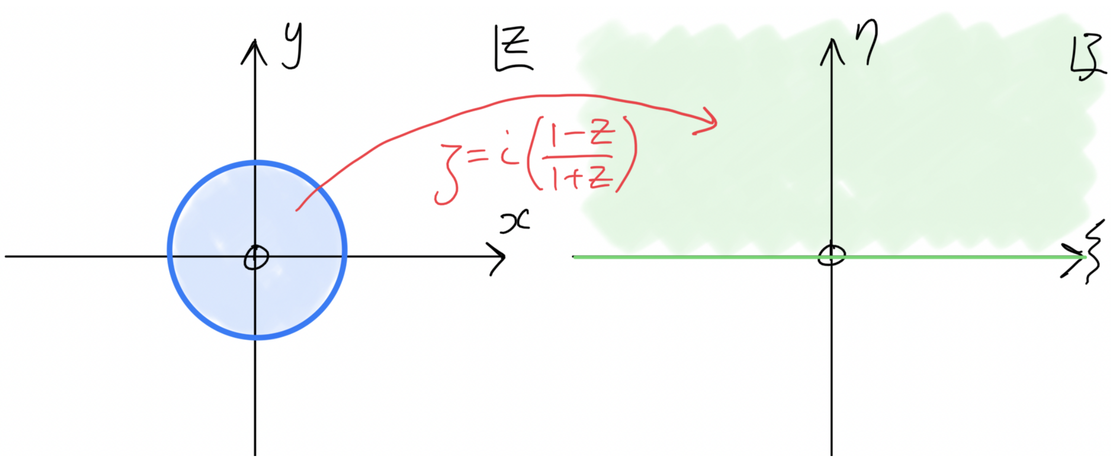


Figure 67: Mapping from unit disc to upper-half plane.

Now let's solve $\nabla^2 T = 0$ in the ζ-plane. We know $\nabla^2 T = 0$ everywhere in the upper-half plane with $T = T_1$ for $\xi \geq 0$ and $T = T_2$ for $\xi < 0$ (with $\eta = 0$ in both cases). For the solution to make sense physically the temperature should change continuously (except at this discontinuity at $\zeta = 0$ where our forcing conditions meet), so it makes sense to think of a ray emanating from 0 that rotates from the positive real ξ-axis around to the negative real ξ-axis as changing temperature from T_1 to T_2 continuously and linearly. Motivated by this, and thinking back to what we know about some of the simple complex potentials we know, this looks like the behaviour of a point source at $\zeta = 0$. Therefore, notice that the function

$$W(\zeta) = -ai \log \zeta + b,$$

where a, $b \in \mathbb{C}$ is indeed analytic in the UHP and has a logarithmic singularity at $\zeta = 0$. Further, by setting $\zeta = \rho e^{i\gamma}$, we see

$$\begin{aligned} W(\zeta) &= -ai(\log \rho + i\gamma) + b \\ &= (a\gamma + b) - a \log \rho i, \end{aligned}$$

so the real part of this varies **linearly** with γ, as we wanted due to physical considerations. Now since

$$W(\zeta) = T(\xi, \eta) + i\psi(\xi, \eta),$$

then we have

$$T(\xi, \eta) = a\gamma + b,$$

and we can allow this to satisfy the given boundary conditions by setting $b = T_1$ (so then when $\gamma = 0$, so we are on the positive real axis, $T = T_1$) and setting $a = (T_2 - T_1)/\pi$ (so when $\gamma = \pi$ and we are on the negative real axis, $T = T_2$). Hence the solution to the problem in the ζ-plane is given by

$$T(\xi, \eta) = \mathrm{Re}\{W(\zeta)\} = \mathrm{Re}\left\{ \frac{(T_2 - T_1)}{\pi} i \log \zeta + T_1 \right\} = \frac{(T_2 - T_1)}{\pi} \gamma + T_1,$$

where $\zeta = \xi + i\eta = \rho e^{i\gamma}$. So writing $\gamma = \tan^{-1}(\eta/\xi)$, this can be written as

$$T(\xi, \eta) = \frac{(T_2 - T_1)}{\pi} \tan^{-1}\left(\frac{\eta}{\xi}\right) + T_1,$$

where an appropriate branch cut would need to be taken due to the multivaluedness of the inverse tangent function present. It remains to find the solution for $T(x, y)$ in the z-plane. To do this, we need the inverse of the conformal map, or ξ and η in terms of x and y. Inverting the conformal map here gives

$$z = \frac{i - \zeta}{i + \zeta},$$

or

$$\xi = \frac{2y}{(1 + x)^2 + y^2}, \quad \eta = \frac{1 - (x^2 + y^2)}{(1 + x)^2 + y^2},$$

(exercise: check this!). Hence, the solution for the temperature in the z-plane can be written as

$$T(x,y) = \text{Re}\left\{ \frac{(T_2 - T_1)}{\pi} i \log\left(i\frac{(1-z)}{(1+z)} \right) + T_1 \right\},$$

or

$$T(x,y) = \frac{(T_2 - T_1)}{\pi} \tan^{-1}\left(\frac{1-(x^2+y^2)}{2y} \right) + T_1.$$

2). Find the streamfunction for uniform fluid flow, at an incidence of α radians, past a flat plate situated along the real axis between ± 1 in the z-plane. You may use the fact that for uniform fluid flow at an incidence of α radians past a unit cylinder centred at $\zeta = 0$ in the ζ-plane we know the complex potential is given by (if you are studying any fluids modules you will see this):

$$W(\zeta) = \frac{1}{2}\left(\zeta e^{-i\alpha} + \frac{e^{i\alpha}}{\zeta} \right).$$

Solution: Recall the Joukowski mapping (82) given by

$$z = \frac{1}{2}\left(\zeta + \frac{a^2}{\zeta} \right),$$

where $a \in \mathbb{R}$, which maps from the circle centred at ih and passing through $\pm a$ in the ζ-plane to the arc of the circle between $\pm a$ and passing through ih in the z-plane. Setting $a = 1$ and $h = 0$ gives us the flat plate in the z-plane that we wish to find the streamfunction for uniform flow past.

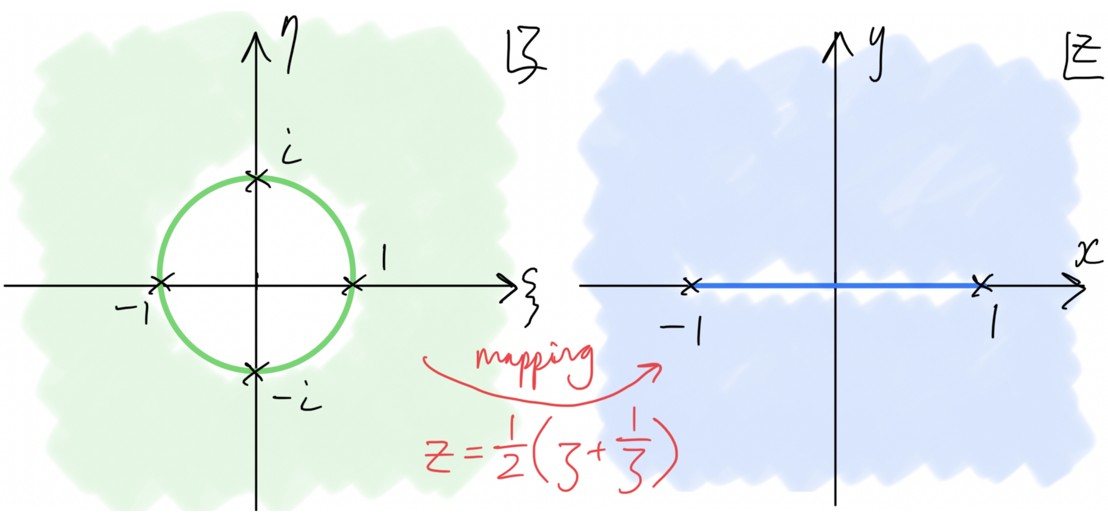


Figure 68: Joukowski mapping from exterior of unit circle to exterior of flat plate.

In the ζ-plane we know the complex potential is given by

$$W(\zeta) = \phi(\xi,\eta) + i\psi(\xi,\eta) = \frac{1}{2}\left(\zeta e^{-i\alpha} + \frac{e^{i\alpha}}{\zeta} \right),$$

so that, in the z-plane we must have

$$w(z) = \phi(x,y) + i\psi(x,y) = \frac{1}{2}\left(\zeta(z) e^{-i\alpha} + \frac{e^{i\alpha}}{\zeta(z)} \right).$$

The inverse mapping is given by

$$\zeta = z + \sqrt{z^2 - 1},$$

hence

$$\psi(x, y) = \frac{1}{2}\mathrm{Im}\left\{(z + \sqrt{z^2 - 1})e^{-i\alpha} + \frac{e^{i\alpha}}{z + \sqrt{z^2 - 1}}\right\}.$$

Simplifying this, using $1/\zeta = 2z - \zeta = z - \sqrt{z^2 - 1}$, one can show that (exercise):

$$\psi(x, y) = \mathrm{Im}\left\{e^{-i\alpha}z - i\sin\alpha(-z + \sqrt{z^2 - 1})\right\},$$

which is precisely the solution we found earlier in section 2.8, p.64, see equation (47), using singular integrals.

Chapter 5: The Wiener-Hopf Method

5.1 Riemann-Hilbert Problems

Definition 5.38. Given functions $f(z)$ and $g(z)$ defined on a smooth contour γ, a (scalar) **Riemann-Hilbert problem** consists of finding an analytic function $\phi(z)$ on $\mathbb{C}/\gamma$ which remains regular as $z \to \infty$ (i.e. $\lim_{z\to\infty} \phi(z) = \text{constant}$) such that the jump condition

$$\phi_+(t) - g(t)\phi_-(t) = f(t), \quad \text{for } t \in \gamma,$$

holds (where, as before, $\phi_+(t)$ and $\phi_-(t)$ represent the limiting values of ϕ as it approaches $t \in \gamma$ from the left or right).

Remark: We solved one of these for the function $\phi(z) = \sqrt{z^2 - 1}$ along the segment of the real line from $[-1, 1]$ way back in chapter 2 when deriving the Hilbert-inversion formula.

There are a huge number of applications to such problems (see for instance Trogdon & Olver 2015) and an entire course could be dedicated to their applications and solution methods. Some classical applications include:

(1) Solving integral-differential equations on half-lines via Wiener-Hopf factorisation (the focus of this chapter).

(2) Spectral analysis of Schrödinger operators.

(3) Problems in Ideal fluid flow.

More recently, non-classical applications have arisen from integrable systems:

(4) Solutions to Painlevé equations.

(5) Random matrix eigenvalue statistics.

(6) Asymptotics of orthogonal polynomials.

(7) Solving partial differential equations like the Korteweg de Vries (KdV) equation describing shallow water waves:

$$u_t + 6uu_x + u_{xxx} = 0.$$

In our course we will focus on one application and method of solution, namely the Wiener-Hopf method to solve a Riemann-Hilbert problem that arises from applying Fourier transforms to integral-differential equations.

Before we discuss the problems we will solve, let's recap some properties of the Fourier transform; a key tool in the Wiener-Hopf method.

5.2 The Fourier transform

Definition 5.39. Suppose $f(x)$ is defined for $-\infty < x < \infty$. For $s \in \mathbb{R}$, we define the **Fourier transform**, $F(s)$ (in many texts this is denoted $\hat{f}(s)$), of $f(x)$ to be

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx. \tag{86}$$

The inversion (**inverse Fourier transform**) of (86) is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds. \tag{87}$$

Note that these definitions hold for certain 'nice' functions (for example those which are piecewise differentiable and absolutely integrable). Note also the factor of $1/2\pi$ outside the integral in (87). In some definitions this factor is moved to (86) and sometimes the signs in the exponents of the exponential terms are also switched.

Let's do an example.

Example: Let

$$f(x) = \frac{1}{x^2 + a^2},$$

where $a > 0$ is real. Then we have

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2 + a^2}dx.$$

First, for $s = 0$:

$$F(s) = \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{1}{a^2}\int_{-\infty}^{\infty} \frac{dx}{1 + (x/a)^2} = \frac{1}{a}\int_{-\pi/2}^{\pi/2} dt = \frac{\pi}{a},$$

where the substitution $\tan t = x/a$ was used.

Now let $s > 0$; consider

$$g(z) = \frac{e^{isz}}{z^2 + a^2} = \frac{e^{isz}}{(z - ai)(z + ai)}.$$

Take γ to be $\gamma = \gamma_1 + \gamma_R$ as shown in figure 69. Then, by the residue theorem

$$\oint_{\gamma} g(z)dz = 2\pi i \text{Res}\{g, ai\} = \frac{\pi}{a}e^{-sa}.$$

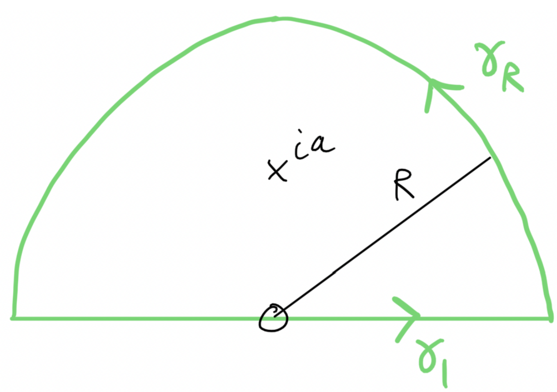


Figure 69: Contour γ.

Also, since $s > 0$, one can check that

$$\lim_{R\to\infty} \int_{\gamma_R} g(z)dz = 0.$$

Thus

$$F(s) = \frac{\pi}{a} e^{-sa}, \quad s > 0.$$

For $s < 0$, one can find that (by closing the contour in the LHP) $F(s) = \frac{\pi}{a} e^{sa}$, $s < 0$. Hence, putting everything together

$$F(s) = \frac{\pi}{a} e^{-a|s|}, \quad \forall s \in \mathbb{R}.$$

Let's check the inversion formula works:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{a} e^{-a|s|} e^{-isx} ds \\
&= \frac{1}{2a} \left(\int_{-\infty}^{0} e^{(a-ix)s} ds + \int_{0}^{\infty} e^{-(a+ix)s} ds \right) \\
&= \frac{1}{2a} \left[\frac{e^{(a-ix)s}}{a - ix} \right]_{-\infty}^{0} + \frac{1}{2a} \left[\frac{e^{-(a+ix)s}}{-(a+ix)} \right]_{0}^{\infty} \\
&= \frac{1}{2a} \left(\frac{1}{a - ix} + \frac{1}{a + ix} \right) \\
&= \frac{1}{x^2 + a^2},
\end{aligned}$$

as expected!

Fourier transforms of derivatives

Let $F_1(s)$ denote the Fourier transform of the derivative $f'(x)$ of some function $f(x)$. We have

$$\begin{aligned}
F_1(s) &= \int_{-\infty}^{\infty} f'(x) e^{isx} dx \\
&= \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} is f(x) e^{isx} dx \\
&= -isF(s),
\end{aligned}$$

provided we confine ourselves to functions which decay as $|x| \to \infty$ so that the boundary term from the integration by parts in line 2 tends to 0. In a similar manner, after conducting n integrations by parts, one can show that

$$F_n(s) = (-is)^n F(s). \tag{88}$$

This property allow one to convert differential equations for $f(x)$ to algebraic equations for $F(s)$. If we can then solve for $F(s)$, then we can retrieve $f(x)$ by the inversion formula.

Fourier transforms of convolution integrals

Let $F(s)$ and $G(s)$ be the Fourier transforms of $f(x)$ and $g(x)$ respectively. Then one can show that the Fourier transform of both

$$\int_{-\infty}^{\infty} g(y) f(x - y) dy \quad \text{and} \quad \int_{-\infty}^{\infty} f(y) g(x - y) dy,$$

is $F(s)G(s)$. That is, if

$$h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy \quad \Rightarrow \quad H(s) = F(s)G(s). \tag{89}$$

Proof.

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)dy \\ &= \int_{-\infty}^{\infty} f(y)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} G(s)e^{-is(x-y)}ds\right)dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)e^{isy}dy\right)G(s)e^{-isx}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)G(s)e^{-isx}ds, \end{aligned}$$

where in the second line we replaced $g(x-y)$ by its inverse Fourier transform, in the third line the order of integration was changed and in the final line we replaced $f(y)$ by its Fourier transform. Hence

$$H(s) = F(s)G(s),$$

as required. □

Using Fourier transforms to solve integral equations

In physical applications one often encounters integral equations of the form

$$\lambda f(x) + \int_{-\infty}^{\infty} k(x-y)f(y)dy = p(x), \quad -\infty < x < \infty, \tag{90}$$

where λ is a known parameter, $k(x)$ is known (often referred to as the kernel function), $p(x)$ is known (the 'driving' term) and we wish to solve for $f(x)$.

We can convert (90) to an algebraic equation in $F(s)$. Taking Fourier transforms of both sides gives

$$\lambda F(s) + K(s)F(s) = P(s),$$

which upon re-arranging for $F(s)$ gives

$$F(s) = \frac{P(s)}{\lambda + K(s)},$$

and then, assuming the Fourier transforms of $p(x)$ and $k(x)$, namely $P(s)$ and $K(s)$, can be found, applying the inversion gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{P(s)}{\lambda + K(s)}e^{-isx}ds.$$

5.3 Half-Fourier transforms

Motivation: First let's discuss a little motivation. Suppose equation (90) is given only for some restricted interval of the real line, say $x \geq 0$. Then is it somehow possible to replicate the method in the last subsection to solve for the unknown $f(x)$. Well, sort-of, but we need the concept of a Fourier transform which is not over the whole real line, but which instead works on half-lines: a **half-Fourier transform**.

The Wiener-Hopf method will allow us to solve such equations defined on half-lines. The method may also be applied to certain differential equations when domains are bounded by half-lines or lines have different boundary conditions on two segments. Half-Fourier transforms are a key tool in the Wiener-Hopf method and we study these now.

The Right-sided (+) Fourier transform

Introduce the function

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Definition 5.40. The **Right-sided Fourier transform**, $F_+(s)$, is given by

$$F_+(s) = \int_0^\infty f_+(x) e^{isx} dx, \tag{91}$$

where now $s = s_1 + is_2 \in \mathbb{C}$ is a complex variable.

Proposition 5.41. *If $|f_+(x)| < Ae^{\alpha x}$ as $x \to \infty$, where A, $\alpha \in \mathbb{R}$ with $A > 0$, α* ***any*** *sign, then the integral in* (91) *is convergent for all s such that $Im\{s\} > \alpha$. Furthermore, under these conditions, $F_+(s)$ is an* ***analytic*** *function of s.*

Proof. We outline some details of the proof. For convergence note

$$\begin{aligned} \left| \int_0^\infty f_+(x) e^{isx} dx - \int_0^t f_+(x) e^{isx} dx \right| &= \left| \int_t^\infty f_+(x) e^{isx} dx \right| \\ &\leq \int_t^\infty \left| f_+(x) e^{isx} \right| dx \\ &< \int_t^\infty A e^{(\alpha + si)x} dx \to 0, \quad \text{if } s_2 > \alpha. \end{aligned}$$

For analyticity see (Titchmarsh 'the theory of functions' p.100). The main idea is that the FT of a function is analytic, so then the half-FT can be shown to be analytic too. □

Asymptotic behaviour of $F_+(s)$

Observe that

$$\begin{aligned} F_+(s) &= \int_0^\infty f_+(x) e^{isx} dx \\ &= \left[\frac{1}{is} e^{isx} f_+(x) \right]_0^\infty - \frac{1}{is} \int_0^\infty f'_+(x) e^{isx} dx, \end{aligned}$$

after an integration by parts. Now assuming $|f_+(x) < Ae^{\alpha x}$ as $x \to \infty$, where $A > 0$, $\alpha \in \mathbb{R}$ is any sign, provided $\text{Im}\{s\} > \alpha$, we have

$$\begin{aligned} &= \left(0 - \frac{f_+(0)}{is} \right) - \frac{1}{is} \int_0^\infty f'_+(x) e^{isx} dx \\ &= -\frac{f_+(0)}{is} + \frac{1}{s^2} \left(\left[f'_+(x) e^{isx} \right]_0^\infty - \int_0^\infty f''_+(x) e^{isx} dx \right), \end{aligned}$$

after a second integration by parts. Now provided $f'_+(x)$ has sufficiently nice behaviour at infinity, we have

$$F_+(s) = \frac{if_+(0)}{s} - \frac{f'_+(0)}{s^2} + O\left(\frac{1}{s^3}\right), \tag{92}$$

i.e. $F_+(s)$ **decays as** $s \to \infty$, but stronger is the precise behaviour in (92) which we will use.

Inverting the right-sided Fourier transform

Note that

$$F_+(s) = \int_{-\infty}^{\infty} f_+(x)e^{isx}dx,$$

where we have lowered the bottom limit to $-\infty$, since $f_+(x) = 0$ for $x < 0$. Now writing $s = s_1 + is_2$, we find

$$F_+(s) = \int_{-\infty}^{\infty} \left(e^{-s_2x}f_+(x)\right)e^{is_1x}dx.$$

Therefore $F_+(s)$ can be regarded as the **ordinary** Fourier transform of $e^{-s_2x}f_+(x)$. Thus, provided $\mathrm{Im}\{s\} > \alpha$, we can apply the inverse formula as usual, giving

$$\begin{aligned} e^{-s_2x}f_+(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_+(s)e^{-s_1x}ds_1 \\ \Rightarrow f_+(x) &= \frac{e^{s_2x}}{2\pi}\int_{-\infty}^{\infty} F_+(s)e^{-s_1x}ds_1 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_+(s)e^{-isx}ds_1, \end{aligned}$$

where we substituted $s = s_1 + is_2$ in the final line.

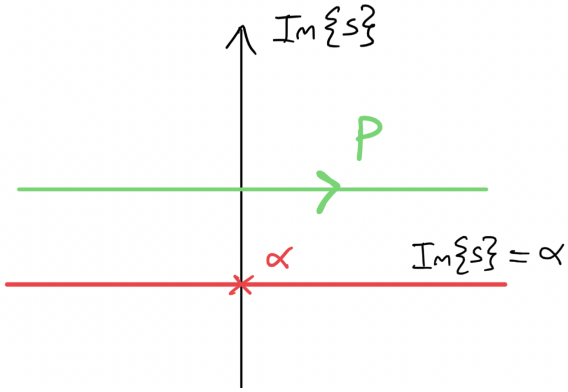


Figure 70: A suitable contour P.

Now for a constant s_2, $ds_1 = ds$, and we find

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds, \tag{93}$$

where P is a straight line along which $s_2 = \mathrm{Im}\{s\} =$ constant $> \alpha$.

Let's check this inversion formula (93) gives $f_+(x) = 0$ for $x < 0$. To evaluate this contour integral, let's close P with a semi-circle above, see figure 71.

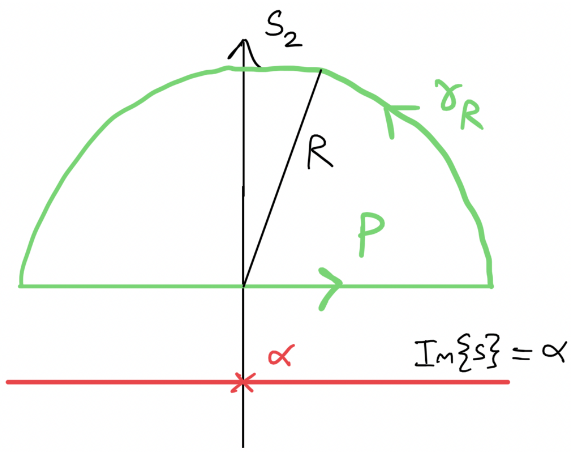


Figure 71: Close P in the half-plane above.

Let $\gamma = \gamma_R + P$. Since γ is contained in the region $\mathrm{Im}\{s\} > \alpha$, then $F_+(s)$ is analytic inside it. Also $e^{-isx} = e^{-is_1 x}e^{s_2 x}$, and this is also analytic in γ (as $|s| \to \infty$, $s_2 \to +\infty$, $x < 0$). Hence, by Cauchy's theorem

$$\oint_\gamma F_+(s)e^{-isx}ds = 0.$$

On γ_R, as $R \to \infty$, $s \approx Re^{i\theta}$; $\left|F_+(s)e^{-isx}\right| = e^{s_2 x}|F_+(s)| \le |F_+(s)|$, and $|F_+(s)| \to 0$ as $s \to 0$ by (92). Thus

$$\int_P F_+(s)e^{-isx}ds = 0,$$

as expected for $x < 0$.

Summary of key properties

$$f_+(x) = \begin{cases} f(x), & \text{for } x \ge 0, \\ 0, & \text{for } x < 0. \end{cases} \qquad F_+(s) = \int_0^\infty f_+(x)e^{isx}dx, \quad s \in \mathbb{C}.$$

If $|f_+(x)| < Ae^{\alpha x}$ as $x \to \infty$ (where $A, \alpha \in \mathbb{R}$ with $A > 0$, α **any** sign) then $F_+(s)$ exists and is an **analytic** function of s for $\mathrm{Im}\{s\} > \alpha$.

$$F_+(s) = \frac{if_+(0)}{s} - \frac{f'_+(0)}{s^2} + O\left(\frac{1}{s^3}\right), \quad \text{as } s \to \infty.$$

Inversion:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a straight line along which $s_2 = \mathrm{Im}\{s\} =$ constant $> \alpha$.

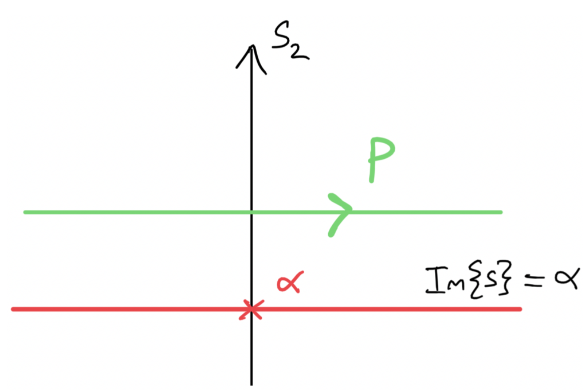


Figure 72: A suitable contour P.

The Left-sided $(-)$ Fourier transform

Now, similarly, for a function $g(x)$ specified for $x < 0$ (but not $x \ge 0$), we define

$$g_-(x) = \begin{cases} 0, & \text{for } x \ge 0, \\ g(x), & \text{for } x < 0. \end{cases}$$

Definition 5.42. The **Left-sided Fourier transform**, $G_-(s)$, is given by

$$G_-(s) = \int_{-\infty}^{0} g_-(x)e^{isx}dx, \quad s \in \mathbb{C}. \tag{94}$$

We then find the similar properties

- It can be shown that if $|g_-(x)| < Be^{\beta x}$ as $x \to -\infty$ (where B, $\beta \in \mathbb{R}$ with $B > 0$, β **any** sign) then $G_-(s)$ exists and is an **analytic** function of s for $\mathrm{Im}\{s\} < \beta$.
- $G_-(s)$ decays as $|s| \to \infty$.

Inversion:

$$g_-(x) = \frac{1}{2\pi}\int_Q G_-(s)e^{-isx}ds,$$

where Q is a straight line along which $s_2 = \mathrm{Im}\{s\} =$ constant $< \beta$.

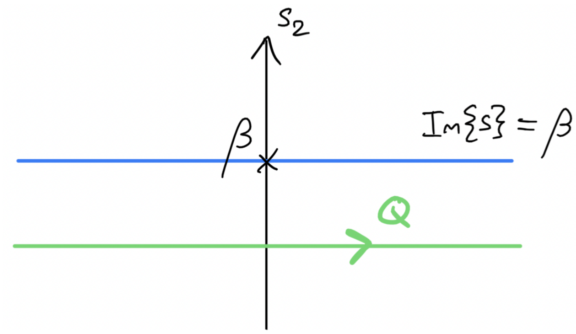


Figure 73: A suitable contour Q.

5.4 The Wiener-Hopf method

Let's return to the integral equation we want to be able to solve

$$\lambda f(x) = \int_0^\infty k(x-y)f(y)dy + p(x), \qquad x \geq 0. \tag{95}$$

Remark: Generally, one requires the kernel function $k(x)$ to have 'nice' (see shortly what we mean by this) behaviour for the Wiener-Hopf method to be applicable (in this course and in any questions given this will be the case).

Let's now outline the method by breaking it down into steps.

Step 1: Introduce unknown functions and take Fourier transforms to form the Riemann-Hilbert problem

We'd like to take a Fourier transform of (95), but we don't know how to do that directly. To circumvent this problem introduce the unknown functions

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ \int_0^\infty k(x-y)f(y)dy, & \text{for } x < 0. \end{cases}$$

Then the equation (95) can be written as

$$\int_0^\infty k(x-y)f(y)dy = \lambda f_+(x) - p_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty. \tag{96}$$

Now we have an equation (96) specified over the entire real line, so we can take its Fourier transform. On the RHS we get

$$\int_{-\infty}^{\infty} \left(\lambda f_+(x) - p_+(x) + g_-(x)\right) e^{isx} dx = \lambda F_+(s) - P_+(s) + G_-(s),$$

where $F_+(s)$ and $P_+(s)$ are the right-sided Fourier transforms of $f_+(x)$ and $p_+(x)$ respectively and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_0^\infty k(x-y)f(y)dy\right)e^{isx}dx &= \int_0^\infty f_+(y)\left(\int_{-\infty}^{\infty} k(x-y)e^{isx}dx\right)dy \\ &= \int_0^\infty f_+(y)\left(\int_{-\infty}^{\infty} k(t)e^{ist}e^{isy}dt\right)dy \\ &= \left(\int_0^\infty f_+(y)e^{isy}dy\right)\left(\int_{-\infty}^{\infty} k(t)e^{ist}dt\right) \\ &= F_+(s)\hat{K}(s),\end{aligned}$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. So, we have

$$F_+(s)\hat{K}(s) = \lambda F_+(s) - P_+(s) + G_-(s),$$

or

$$K(s)F_+(s) + G_-(s) = P_+(s), \tag{97}$$

where $K(s) = \lambda - \hat{K}(s)$.

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

If we seek solutions to the original integral equation such that $|f_+(x)| < Ae^{\alpha x}$ and $|g_-(x)| < Be^{\beta x}$ as $x \to \infty$, where A, B, α, $\beta \in \mathbb{R}$, then we know from the earlier proposition:

- $F_+(s)$ is analytic in $\text{Im}\{s\} > \alpha$,
- $G_-(s)$ is analytic in $\text{Im}\{s\} < \beta$.

Also, let's suppose $\alpha < \beta$ (**Note:** It turns out for exponentially decaying kernels, i.e $|k(x)| < Ce^{-\gamma|x|}$ as $x \to \infty$, that $\alpha < \beta$, i.e. a strip of analyticity exists. This is what we mean by 'nice' kernels making the method applicable.): then the regions of analyticity of $F_+(s)$ and $G_-(s)$ overlap in a strip $\Omega = \{s : \alpha < \text{Im}\{s\} < \beta\}$, see figure 74.

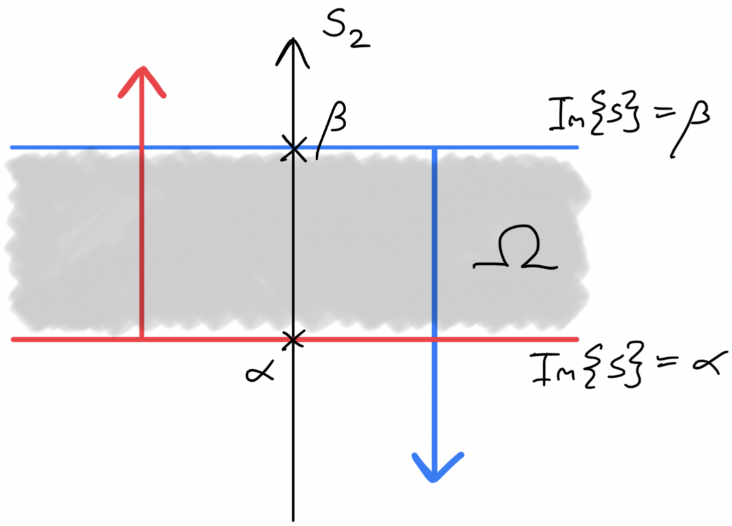


Figure 74: The strip of analyticity Ω.

Furthermore, suppose that $k(x)$ and $p(x)$ are such that $K(s)$ and $P_+(s)$ are:

(i) Analytic in Ω.

(ii) $K(s) \neq 0$ in Ω.

(iii) As $s \to \infty$ in Ω, $|K|$, $|P_+|$ are 'suitably' bounded.

Then, more generally, equation (97) and the conditions just outlined form the canonical Wiener-Hopf problem (note equation (97) and these conditions form a Riemann-Hilbert problem).

Step 3: The product decomposition

Find $K_+(s)$ and $K_-(s)$ where $K_+(s)$ is analytic in $\mathrm{Im}\{s\} > \alpha_1$, $K_-(s)$ is analytic in $\mathrm{Im}\{s\} < \beta_1$, $\alpha_1 < \beta_1$, where $\Omega_1 = \{s : \alpha_1 < \mathrm{Im}\{s\} < \beta_1\} \subseteq \Omega$, and $K(s) = K_+(s)K_-(s)$.

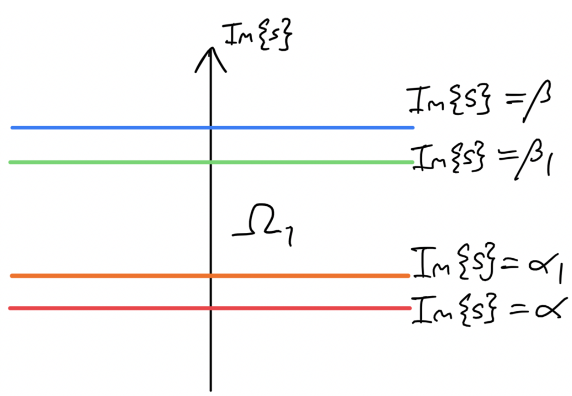


Figure 75: The new strip Ω_1.

After this, it follows that (97) can be rewritten as:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R(s), \tag{98}$$

for $s \in \Omega_1$, where

$$R(s) = \frac{P_+(s)}{K_-(s)}.$$

Step 4: The sum decomposition

Find $R_+(s)$ and $R_-(s)$ where $R_+(s)$ is analytic in $\text{Im}\{s\} > \alpha_2$, $R_-(s)$ is analytic in $\text{Im}\{s\} < \beta_2$, $\alpha_2 < \beta_2$, where $\Omega_2 = \{s : \alpha_2 < \text{Im}\{s\} < \beta_2\} \subseteq \Omega_1$, such that $R(s) = R_+(s) + R_-(s)$.

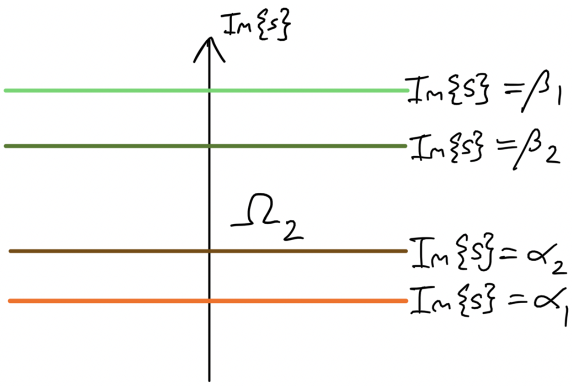


Figure 76: The new strip Ω_2.

After this, it follows that (98) can be written as:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R_+(s) + R_-(s), \tag{99}$$

for $s \in \Omega_2$.

Step 5: Analytic Continuation

Now re-arrange (99) to get

$$K_+(s)F_+(s) - R_+(s) = R_-(s) - \frac{G_-(s)}{K_-(s)},$$

for $s \in \Omega_2$.

Here, the LHS is analytic in $\text{Im}\{s\} > \alpha_2$ (call this region $+$) and the RHS is analytic in $\text{Im}\{s\} < \beta_2$ (call this region $-$).

Since $+$ and $-$ overlap in the strip Ω_2, then the RHS is the analytic continuation of the LHS into the $-$ region, and likewise the LHS is the analytic continuation of the RHS into the $+$ region.

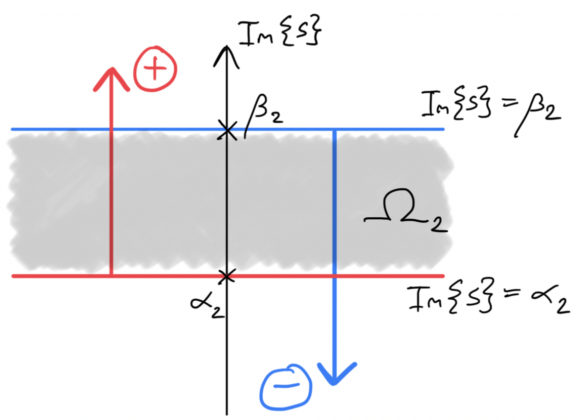


Figure 77: The regions of analyticity of the LHS and RHS, $+$ and $-$.

Hence, this means that the function

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in +, \\ R_-(s) - \frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

is **entire** (analytic everywhere in the complex s-plane).

Step 6: Behaviour at infinity

Since $K_+(s)$, $K_-(s)$, $R_+(s)$ and $R_-(s)$ are known, then so is their behaviour as $|s| \to \infty$. Furthermore, the behaviour of $F_+(s)$ and $G_-(s)$ is given by the asymptotic expansion of the relevant Fourier integral (formula (92) for $F_+(s)$). Hence we can determine the behaviour of $E(s)$ as $|s| \to \infty$. Then we can apply **Liouville's theorem**:

(a) If $E(s) \to$ constant, M, as $|s| \to \infty$, then $E(s) \equiv M$.

(b) If $E(s) \sim O(s^N)$, as $|s| \to \infty$, where $N \in \mathbb{N}$, then $E(s)$ is a polynomial of degree at most N.

Step 7: Invert

Having found $E(s)$, to find $f_+(x)$ (and also $g_-(x)$ if we wished), we invert:

$$F_+(s) = \frac{R_+(s) + E(s)}{K_+(s)} \qquad \text{and} \qquad G_-(s) = K_-(s)\left(R_-(s) - E(s)\right).$$

Then one determines $f_+(x)$ using the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a straight line contour ($\text{Im}\{s\} = \text{constant}$) within region $+$ (similarly for $g_-(x)$ if necessary). This final integral will often need to be closed in an appropriate way and evaluated using the residue/ Cauchy's theorem.

Step 8: Check!

In non-exam conditions, for example during research problems or when doing problem sheets, it can be good practice to substitute our solution function $f(x)$ back into the original integral equation to check if our answer is correct.

5.5 The Wiener-Hopf product and sum decompositions

The key steps in the Wiener-Hopf method are the ability to decompose $K(s)$ and $R(s)$ into product and sum decompositions with the appropriate analyticity properties. In this section we justify why this can always be done (for sufficiently 'nice' kernels) and the proof will provide us with a 'constructive' method for determining these decompositions.

Nevertheless, if the decompositions can be found by inspection this often proves faster and more convenient than using the formulae we will derive here. Let's start with the sum decomposition.

Proposition 5.43 (Wiener-Hopf sum decomposition)**.** *Suppose $R(s)$ is analytic in the strip $\Omega = \{s : \alpha < Im\{s\} < \beta\}$. Then we can write $R(s) = R_+(s) + R_-(s)$, where $R_+(s)$ is analytic in $Im\{s\} > \alpha_1$ and $R_-(s)$ is analytic in $Im\{s\} < \beta_1$, for some α_1 and β_1 where $\alpha < \alpha_1 < \beta_1 < \beta$.*

Proof. Here we will further assume $R(s) \to 0$ as $|s| \to \infty$ (this condition can be relaxed, but we assume it here to make the proof simpler).

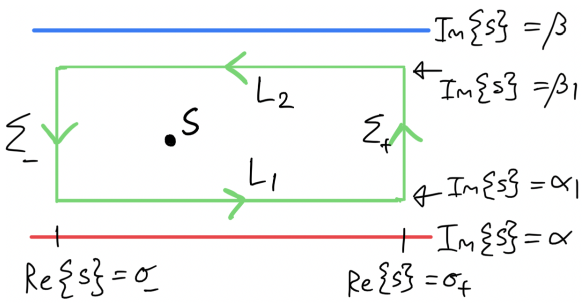


Figure 78: The region $\tilde{\Omega}$.

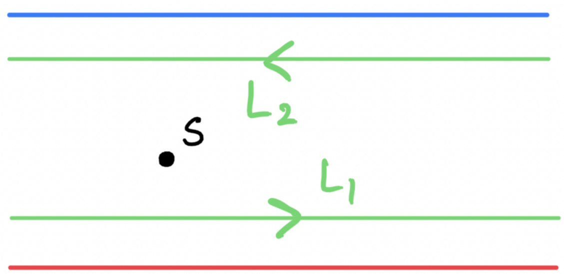


Figure 79: The region $\tilde{\Omega}$ with the sides sent to infinity.

Let $\gamma = L_1 + L_2 + \Sigma_- + \Sigma_+$, as shown in figure 78. Let $\tilde{\Omega}$ denote the interior of γ.

We know that $R(s)$ is analytic for $s \in \tilde{\Omega}$. Hence by Cauchy's integral formula

$$R(s) = \frac{1}{2\pi i} \oint_\gamma \frac{R(t)}{t-s} dt,$$

for $s \in \tilde{\Omega}$.

Now send $\sigma_- \to -\infty$ and $\sigma_+ \to \infty$. Since $R(t) \to 0$ as $t \to \infty$, one can show that

$$\lim_{\sigma_\pm \to \pm\infty} \int_{\Sigma_\pm} \frac{R(t)}{t-s} dt = 0.$$

Hence, for $s \in \tilde{\Omega}$ (which is the strip $\alpha_1 < \text{Im}\{s\} < \beta_1$), we have

$$R(s) = \underbrace{\frac{1}{2\pi i} \int_{L_1} \frac{R(t)}{t-s} dt}_{R_+(s)} + \underbrace{\frac{1}{2\pi i} \int_{L_2} \frac{R(t)}{t-s} dt}_{R_-(s)}. \tag{100}$$

$R_+(s)$ and $R_-(s)$ as defined above are **Cauchy-type** integrals (or Cauchy transforms; recall chapter 2) hence $R_+(s)$ is analytic in the UHP above L_1, $R_-(s)$ is analytic in the LHP below L_2 (and both analytic as $|s| \to \infty$ too). □

Proposition 5.44 (Wiener-Hopf product decomposition). *Suppose $K(s)$ is analytic and non-zero in the strip $\Omega = \{s : \alpha < Im\{s\} < \beta\}$. Then we can write $K(s) = K_+(s)K_-(s)$, where $K_+(s)$ is analytic and non-zero in $Im\{s\} > \alpha_1$ and $K_-(s)$ is analytic and non-zero in $Im\{s\} < \beta_1$, for some α_1 and β_1 where $\alpha < \alpha_1 < \beta_1 < \beta$.*

Proof. Our idea is that taking logarithms of both sides, this becomes a sum decomposition for $\log K(s)$. So, take $\hat{R}(s) = \log K(s)$. Then by the sum decomposition, we get

$$\hat{R}(s) = \hat{R}_+(s) + \hat{R}_-(s),$$

where $\hat{R}_+(s)$ is analytic in $\text{Im}\{s\} > \alpha_1$ and $\hat{R}_-(s)$ is analytic in $\text{Im}\{s\} < \beta_1$, and

$$\hat{R}_+(s) = \frac{1}{2\pi i} \int_{L_1} \frac{\hat{R}(t)}{t-s} dt,$$
$$\hat{R}_-(s) = \frac{1}{2\pi i} \int_{L_2} \frac{\hat{R}(t)}{t-s} dt,$$

where $L_1 = \{t : \text{Im}\{t\} = \alpha_1\}$ and $L_2 = \{t : \text{Im}\{t\} = \beta_1\}$. Hence

$$K(s) = \exp\left\{\hat{R}(s)\right\} = \exp\left\{\hat{R}_+(s)\right\} \cdot \exp\left\{\hat{R}_-(s)\right\},$$

so we identify

$$K_\pm(s) = \exp\left\{\hat{R}_\pm(s)\right\},$$

where $\hat{R}_{\pm}(s)$ are as defined above. □

Note: We refer to these as 'constructive' proofs, since the proofs give formulae that enable us to calculate $K_{\pm}(s)$ and $R_{\pm}(s)$ for a given problem if we cannot determine the decompositions by inspection.

5.6 Examples of finding product and sum decompositions

Example 1:

$$K(s) = \frac{1}{s^2+1} = \frac{1}{(s+i)(s-i)},$$

with Ω as shown in figure 80. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

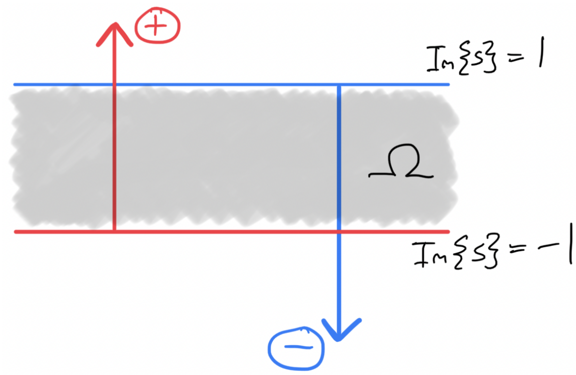


Figure 80: The strip Ω and regions + and −.

Example 2:

$$K(s) = \frac{s^2+2}{s^2+1} = \frac{(s+\sqrt{2}i)(s-\sqrt{2}i)}{(s+i)(s-i)},$$

with Ω as shown in figure 81. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{(s+\sqrt{2}i)}{s+i},$$

is **analytic** and **non-zero** in $\mathrm{Im}\{s\} > -1$, and

$$K_-(s) = \frac{(s-\sqrt{2}i)}{s-i},$$

is **analytic** and **non-zero** in $\mathrm{Im}\{s\} < 1$.

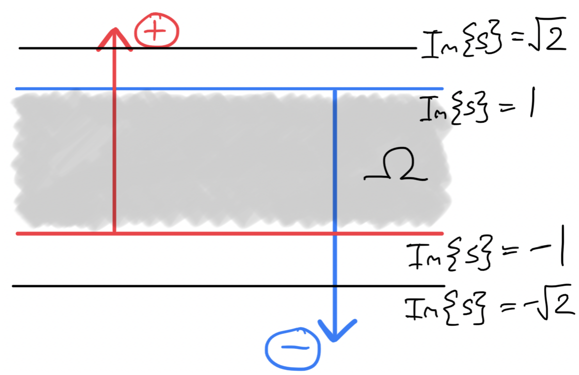


Figure 81: The strip Ω and regions + and −.

Example 3:

$$K(s) = \frac{s^2}{s^2+1} = \frac{s^2}{(s+i)(s-i)},$$

with Ω as shown in figure 82. Here $0 < \delta < 1$. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

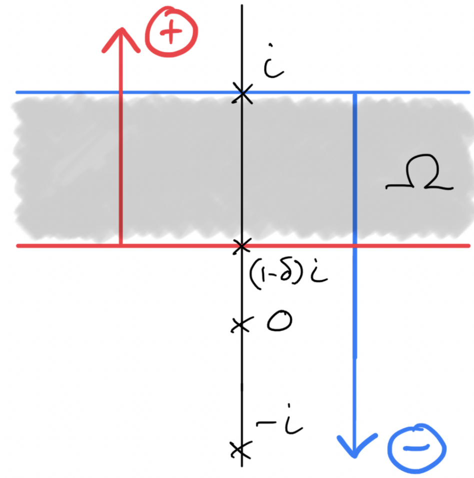


Figure 82: The strip Ω and regions $+$ and $-$.

Example 4:

$$R(s) = \frac{1}{s^2+1} = \frac{s^2}{(s+i)(s-i)},$$

with Ω as shown in figure 83. We can write $R(s) = R_+(s) + R_-(s)$, where

$$R_+(s) = \frac{i/2}{s+i}, \quad R_-(s) = \frac{-i/2}{s-i}.$$

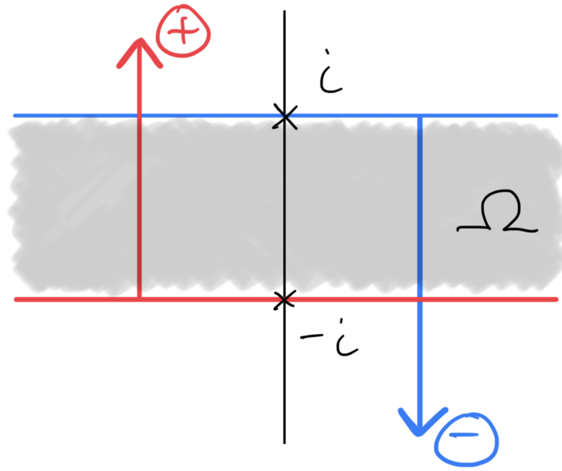


Figure 83: The strip Ω and regions $+$ and $-$.

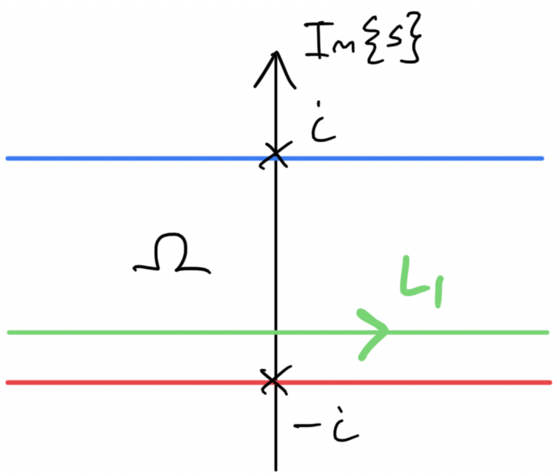


Figure 84: The contour L_1.

Let's show in this example that we can recover what we have found by inspection via the formula (100). Using this we find, for $R_+(s)$:

$$R_+(s) = \frac{1}{2\pi i}\int_{L_1} \frac{R(t)}{t-s}dt = \frac{1}{2\pi i}\int_{L_1} \frac{1}{t-s}\frac{1}{(t+i)(t-i)}dt.$$

To evaluate the integral we need to close L_1. Since the integrand $\to 0$ both above and below L_1, we can choose to close in either direction.

Let's close the contour below as shown in figure 85. As $R \to \infty$, one can check $\int_{\gamma_R} \to 0$. So, by the residue theorem

$$\begin{aligned} R_+(s) &= \frac{1}{2\pi i} \times -2\pi i \left(\frac{1}{-i-s} \left(\frac{-1}{2i} \right) \right) \\ &= \frac{i/2}{s+i}, \end{aligned}$$

as found before by inspection. Rather than then calculating $R_-(s)$ via formula (100), we would simply substitute back into $R(s) = R_+(s) + R_-(s)$ to find $R_-(s)$ more simply.

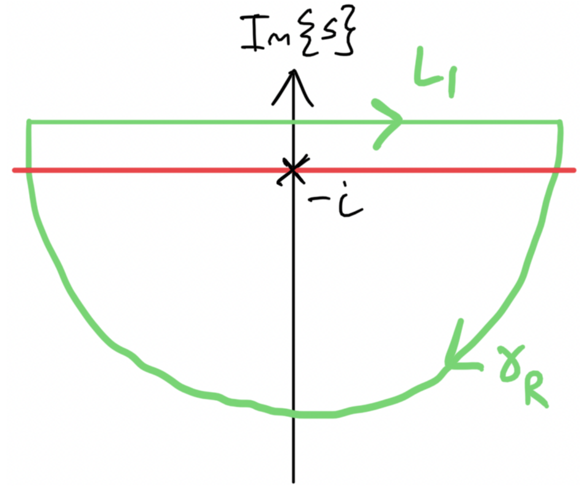


Figure 85: The closed contour γ.

5.7 Wiener-Hopf example problem 1

Let's finally do an example problem to see how everything works in practice. Let's start simple and set $p(x) = 0$. Solve

$$f(x) = \frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy, \qquad x \geq 0,$$

subject to the condition $f(0) = A_0$, constant.

Solution:

Step 1: Introduce functions & take Fourier transforms

Here $k(x) = \frac{1}{2} e^{-|x|}$. First, introduce

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ \frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy, & \text{for } x < 0. \end{cases}$$

Then we have the equation

$$\frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy = f_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty.$$

Now we can take Fourier transforms. On the RHS we get

$$\int_{-\infty}^\infty (f_+(x) + g_-(x)) e^{isx} dx = F_+(s) + G_-(s),$$

where $F_+(s)$ is the right-sided Fourier transforms of $f_+(x)$ and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\int_0^{\infty} e^{-|x-y|}f_+(y)dy\right)e^{isx}dx = F_+(s)\hat{K}(s),$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. Evaluating this, we find

$$\begin{aligned}
\hat{K}(s) &= \int_{-\infty}^{\infty}\frac{1}{2}e^{-|x|}e^{isx}dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{(1+is)x}dx + \frac{1}{2}\int_0^{\infty} e^{(-1+is)x}dx \\
&= \frac{1}{2}\int_0^{\infty} e^{-(1+is)x}dx + \frac{1}{2}\int_0^{\infty} e^{(-1+is)x}dx \\
&= \underbrace{\frac{1}{2}\left[\frac{e^{-(1+is)x}}{-(1+is)}\right]_0^{\infty}}_{\text{Need Im}\{s\}<1\text{ for convergence}} + \underbrace{\frac{1}{2}\left[\frac{e^{(-1+is)x}}{-1+is}\right]_0^{\infty}}_{\text{Need Im}\{s\}>-1\text{ for convergence}} \\
&= \frac{1}{2}\left(\frac{1}{1+is}+\frac{1}{1-is}\right) \\
&= \frac{1}{s^2+1}, \qquad \text{for } -1<\text{Im}\{s\}<1.
\end{aligned}$$

Plugging $s = s_1 + is_2$ into the integrands helps determine the regions of convergence for each. So, the Wiener-Hopf equation is

$$F_+(s)\hat{K}(s) = F_+(s) + G_-(s),$$

or

$$K(s)F_+(s) + G_-(s) = 0,$$

where

$$K(s) = 1 - \hat{K}(s) = \frac{s^2}{s^2+1}.$$

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

First let's consider $G_-(s)$. well

$$\begin{aligned}
g_-(x) &= \frac{1}{2}\int_0^{\infty} e^{-|x-y|}f(y)dy, \quad \text{for } x<0 \\
&= \frac{e^x}{2}\int_0^{\infty} e^{-y}f(y)dy,
\end{aligned}$$

since for $x<0$, $y>0$ we have $-|x-y| = x-y$. Thus

$$g_-(x) = Be^{1x},$$

where (provided this integral exists)

$$B = \text{constant} = \frac{1}{2}\int_0^{\infty} e^{-y}f(y)dy.$$

Hence $G_-(s)$ is analytic for $\text{Im}\{s\} < 1$.

Now consider $F_+(s)$. For the original equation to hold, the integral

$$\int_0^\infty e^{-|x-y|} f(y) dy,$$

should converge for all $x \geq 0$. Consider, as $t \to \infty$:

$$\left| \int_0^\infty e^{-|x-y|} f(y) dy - \int_0^t e^{-|x-y|} f(y) dy \right| = \left| \int_t^\infty e^{-|x-y|} f(y) dy \right|$$
$$\leq \int_t^\infty e^{-|x-y|} |f(y)| dy.$$

For a given x, as $t \to \infty$, $-|x-y| \sim -y$. So $e^{-|x-y|}|f(y)| \sim e^{-y} f(y)$. So, we have convergence if, as $y \to \infty$: $|f(y)| < Ae^{(1-\delta)y}$, for some $\delta > 0$, as then $e^{-y} f(y) \to 0$ as $y \to \infty$. Hence $F_+(s)$ is analytic for $\text{Im}\{s\} > 1-\delta$.

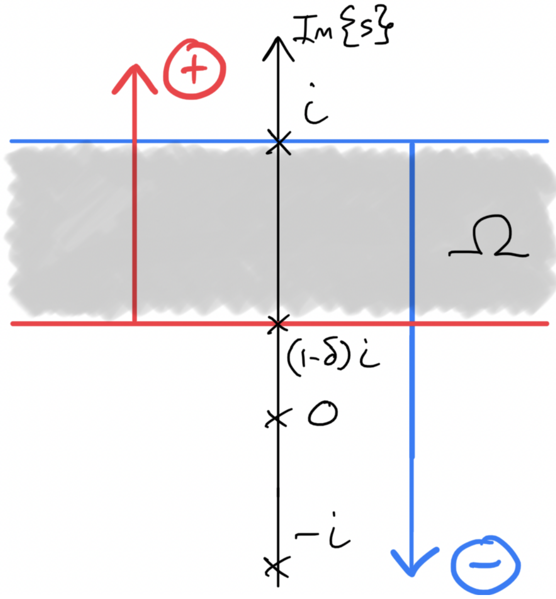


Figure 86: The strip of analyticity γ with the $+$ and $-$ regions indicated.

We found $K(s)$ is analytic for $-1 < \text{Im}\{s\} < 1$, but it also has a **zero** at $s = 0$, so we have ensured $s = 0$ is outside our choice of Ω.

Recall that we need to check $K(s)$ and $P_+(s)$ satisfy the analyticity and (for K) non-zero conditions to define our Ω.

Step 3: The product decomposition

We have

$$K(s) = \frac{s^2}{s^2+1} = \frac{s^2}{(s+i)(s-i)} = \underbrace{\frac{s^2}{s+i}}_{\substack{\text{analytic and} \\ \text{non-zero in } +}} \cdot \underbrace{\frac{1}{s-i}}_{\substack{\text{analytic and} \\ \text{non-zero in -}}} .$$

So we set

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

Now our equation becomes

$$K_+(s)F_+(s) = -\frac{G_-(s)}{K_-(s)}, \quad s \in \Omega.$$

Step 4: The sum decomposition

Since $p(x) = 0$, we avoid this step here: our equation is already separated into expressions analytic in +/- respectively.

Step 5: Analytic continuation

Thus

$$E(s) = \begin{cases} K_+(s)F_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

is **entire**.

Step 6: Behaviour at infinity

To determine $E(s)$, we consider its behaviour as $s \to \infty \in +$. Recall from (92), as $s \to \infty$:

$$F_+(s) = \frac{if_+(0)}{s} + O\left(\frac{1}{s^2}\right) = \frac{A_0 i}{s} + O\left(\frac{1}{s^2}\right).$$

Then

$$\begin{aligned} E(s) &= \frac{s^2}{s+i}F_+(s) \to \frac{s^2}{s+i}\left(\frac{A_0 i}{s} + O\left(\frac{1}{s^2}\right)\right), \quad \text{as } s \to \infty \\ &= A_0 i, \end{aligned}$$

a constant. Hence

$$E(s) \equiv A_0 i,$$

by Liouville's theorem.

Step 7: Invert

$$F_+(s) = \frac{A_0 i(s+i)}{s^2}, \quad G_-(s) = \frac{-A_0 i}{s-i}.$$

We obtain $f_+(x)$ from the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds = \frac{A_0 i}{2\pi}\int_P \frac{(s+i)e^{-isx}}{s^2}ds,$$

where P is some straight line contour in $+$. Recall, as shown earlier, this integral gives $f_+(x) = 0$ for $x < 0$ as expected.

For $x > 0$. Let $\gamma = P + \gamma_R$ as shown in figure (87). As $R \to \infty$ one can show $\int_{\gamma_R} \to 0$. Thus, by the residue theorem

$$f_+(x) = \frac{A_0 i}{2\pi} \times (-2\pi i) \times \text{Res}\{h(s), s = 0\},$$

where

$$h(s) = \frac{(s+i)e^{-isx}}{s^2} = \frac{(1 - isx + O(s^2))(s+i)}{s^2}$$
$$= \frac{i}{s^2} + \frac{x+1}{s} + O(1),$$

so that $\text{Res}\{h, s = 0\} = 1 + x$, giving

$$f_+(x) = A_0(1+x), \quad x \geq 0.$$

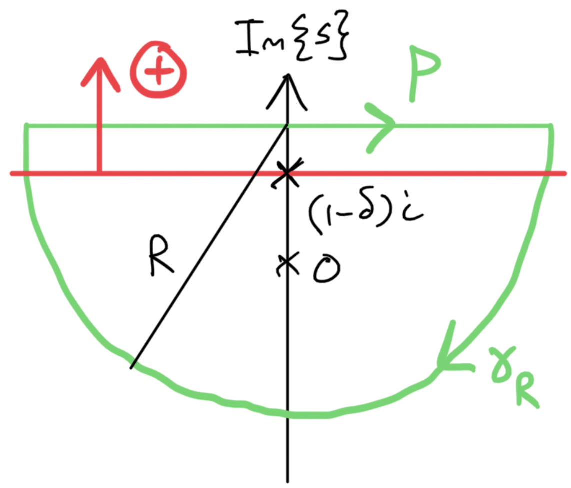


Figure 87: Closed contour γ below P.

Step 8: Check

One can check this is the correct solution by substituting back into the original equation (exercise).

Important remarks about this example

1). Let's check the behaviour of the solution as $x \to \infty$. Recall, we expect $|f(x)| < A_0 e^{(1-\delta)x}$, where $0 < \delta < 1$, i.e $1 - \delta > 0$. As $x \to \infty$, $f(x) \sim A_0 x < A_0 e^{(1-\delta)x}$, since $1 - \delta > 0$. ✓

2). Let's now take a detailed look at the $+$ and $-$ regions and decompositions. Recall, we took:

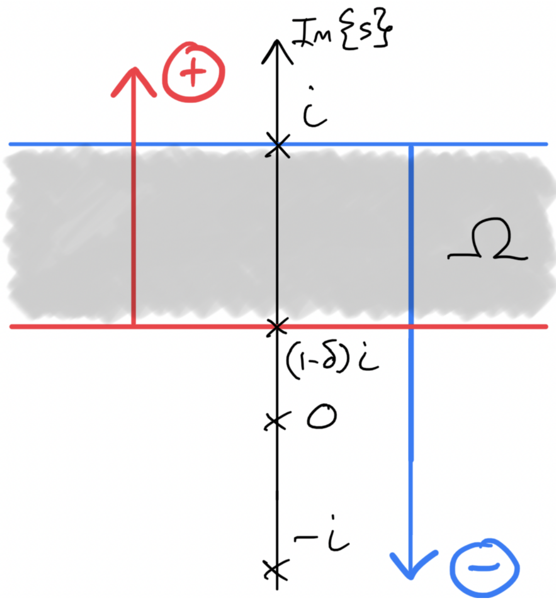


Figure 88: (a) The strip Ω we took during the problem.

The conditions we had to satisfy were:

(i) $G_-(s)$ analytic in $\text{Im}\{s\} < 1$.

(ii) $F_+(s)$ analytic in $\text{Im}\{s\} > 1 - \delta$ $(\delta > 0)$.

(iii) $K(s) = \frac{s^2}{(s+i)(s-i)}$ analytic and non-zero in Ω.

Question: What other choices for the region Ω are there? And what solutions do we find in these cases?

Well by (i), Ω must be contained in $\mathrm{Im}\{s\} < 1$. But (ii) does not add any restrictions, as we can take δ as large as we like. (iii) means we can't have $s = 0, \pm i$ within Ω. This means there are actually two more distinctive choices for Ω:

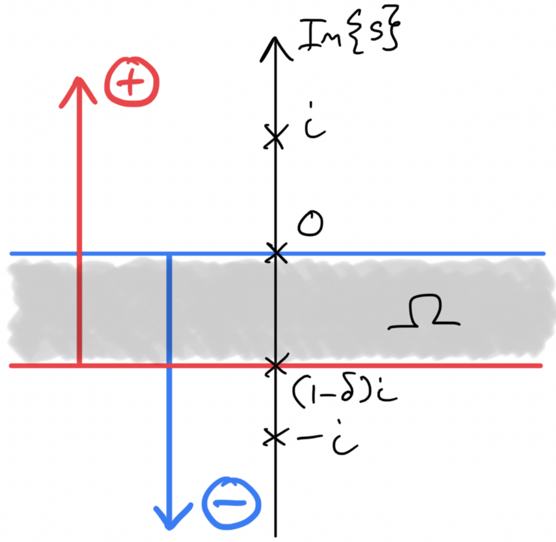


Figure 89: (b) Another possible choice for Ω.

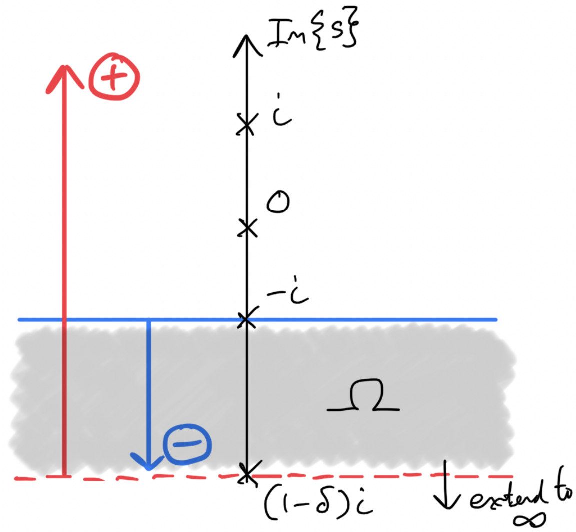


Figure 90: (c) Yet another possible choice for Ω.

(a) This was the choice we made: $\Omega = \{s : 1-\delta < \mathrm{Im}\{s\} < 1\}$, with $0 < \delta < 1$ (so that $1 - \delta > 0$), and

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

(b) In this case: $\Omega = \{s : 1-\delta < \mathrm{Im}\{s\} < 0\}$, with $1 < \delta < 2$, so $-1 < 1-\delta < 0$. In this case we get

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{s^2}{s-i}.$$

Recall

$$K_+(s)F_+(s) = -\frac{G_-(s)}{K_-(s)},$$

so we get

$$E(s) = \begin{cases} K_+(s)F_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -. \end{cases}$$

By analytic continuation $E(s)$ is an entire function. To determine $E(s)$ we look at the behaviour as $s \to \infty$. We get, as $s \to \infty$ (in $+$):

$$\begin{aligned} K_+(s)F_+(s) &\to \frac{1}{s} \cdot \frac{1}{s} \to 0 \\ \Rightarrow E(s) &\equiv 0 \\ \Rightarrow F_+(s) &= 0 \\ \Rightarrow f_+(x) &= 0. \end{aligned}$$

It is trivial to check that $f(x) = 0$ is a solution of the original equation.

So why does this region for Ω not retrieve the solution found in case (a)? Recall, $|f(x)| < Ae^{(1-\delta)x}$ as $x \to \infty$. In case (a) we had $1 - \delta > 0$ and so this allowed for solutions which became infinite as $x \to \infty$ (we found $f(x) = A_0(1+x)$). But in case (b), we're restricted so that $1 - \delta < 0$ and so any solutions we find must decay as $x \to \infty$. Here we've thus retrieved the solutions of (a) which decay as $x \to \infty$, i.e those for which $A_0 = 0$, i.e $f(x) = 0$.

(c) Case (c) is similar to case (b): exercise: find $K_+(s)$ and $K_-(s)$ here and show that you find only $f(x) = 0$ again.

Important remark on this: We want to pick the region Ω that allows for the largest amount of different solutions $\to$ (a) was the correct choice here.

5.8 Wiener-Hopf example problem 2

Solve

$$2\int_0^\infty e^{-|x-y|}f(y)dy = f(x) + 2xe^{-x}, \quad x \geq 0,$$

with $f(0) = A_0$, constant.

Solution:

Step 1: Introduce functions & take Fourier transforms

Denote $k(x) = 2e^{-|x|}$, $p(x) = 2xe^{-x}$ (in the general theory we had $p(x)$ on the other side of the equation - this will lead to a sign difference here). then introduce

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ 2\int_0^\infty e^{-|x-y|}f(y)dy, & \text{for } x < 0. \end{cases}$$

Then we have the equation

$$2\int_0^\infty e^{-|x-y|}f(y)dy = f_+(x) + p_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty.$$

Now we can take Fourier transforms. On the RHS we get $F_+(s) + P_+(s) + G_-(s)$, where $F_+(s)$ and $P_+(s)$ are the right-sided Fourier transforms of $f_+(x)$ and $p_+(x)$ respectively and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\int_{-\infty}^\infty \left(2\int_0^\infty e^{-|x-y|}f(y)dy\right)e^{isx}dx = \hat{K}(s)F_+(s),$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. Hence, we have

$$\hat{K}(s)F_+(s) = F_+(s) + P_+(s) + G_-(s).$$

Similar to the previous example (see $\hat{K}(s)$ in section 5.7), one can check that this time we find

$$\hat{K}(s) = \frac{4}{s^2+1}.$$

Now calculating $P_+(s)$:

$$\begin{aligned}
P_+(s) &= \int_0^\infty p_+(x)e^{isx}dx = 2\int_0^\infty xe^{(is-1)x}dx \\
&= 2\left[\frac{xe^{(is-1)x}}{is-1}\right]_0^\infty - 2\int_0^\infty \frac{e^{(is-1)x}}{(is-1)}dx \\
&= 0 - 2\left[\frac{e^{(is-1)x}}{(is-1)^2}\right]_0^\infty \\
&= \frac{-2}{(s+i)^2},
\end{aligned}$$

where one can check that the integrals converge provided $\text{Im}\{s\} > -1$. Hence our equation becomes

$$\left(\frac{4}{s^2+1}\right)F_+(s) = F_+(s) + G_-(s) - \frac{2}{(s+i)^2},$$

or

$$K(s)F_+(s) + G_-(s) = -P_+(s), \tag{101}$$

where

$$K(s) = 1 - \hat{K}(s) = \frac{s^2-3}{s^2+1} = \frac{(s+\sqrt{3})(s-\sqrt{3})}{(s+i)(s-i)}.$$

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

As in the first example (section 5.7), we require $|f_+(x)| < \tilde{A}e^{(1-\delta)x}$ as $x \to \infty$, for some $\delta > 0$. Thus $F_+(s)$ is analytic in $\{s : \text{Im}\{s\} > 1-\delta\}$.

Similarly for $G_-(s)$, as in the first example, we can show $g_-(x) = Be^x$, for some constant B. Thus $G_-(s)$ is analytic for $\{s : \text{Im}\{s\} < 1\}$.

As discussed after the previous example, there are again three choices to take for Ω:

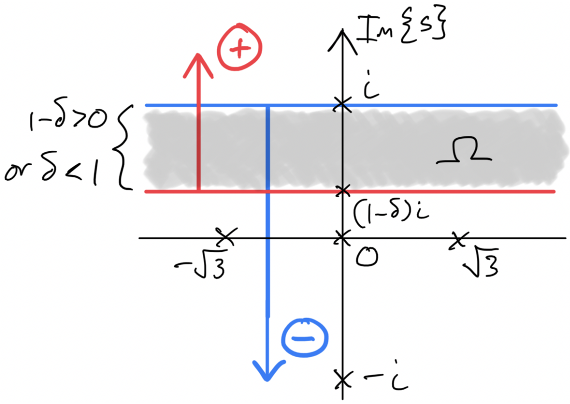


Figure 91: (a) One choice for Ω.

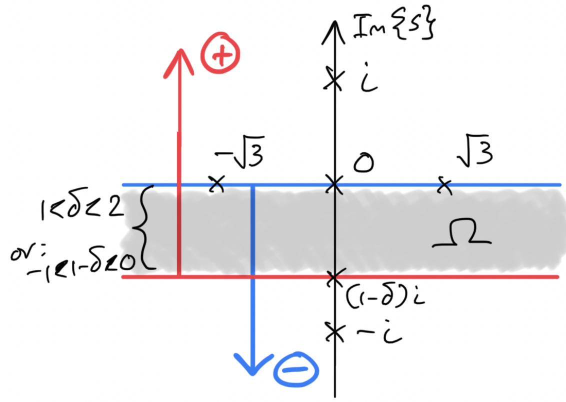


Figure 92: (b) Another possible choice for Ω.

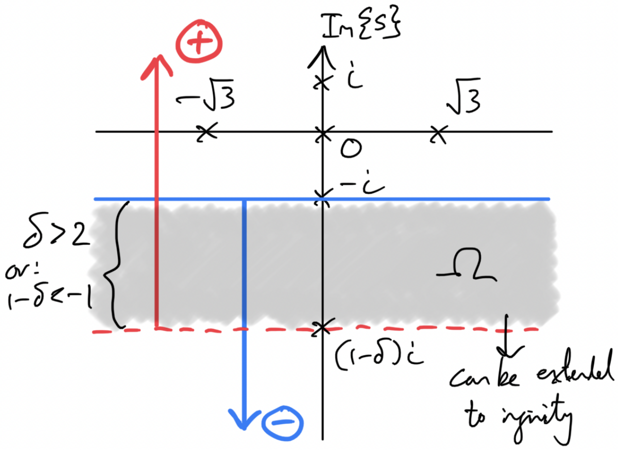


Figure 93: (c) A final possible choice for Ω.

Let's take case (a) (we'll look at regions (b) and (c) afterwards). Indeed, $K(s)$ is analytic provided $-1 < \text{Im}\{s\} < 1$ and non-zero provided $s \neq \pm\sqrt{3}$. ✓ $P_+(s)$ is analytic provided $\text{Im}\{s\} > -1$. ✓

Step 3: The product decomposition

Write $K(s) = K_+(s)K_-(s)$, where $K_+(s)$ is analytic and non-zero in $+$, $K_-(s)$ is analytic and non-zero in $-$.

$$K(s) = \frac{(s+\sqrt{3})(s-\sqrt{3})}{(s+i)(s-i)} = \underbrace{\left(\frac{(s+\sqrt{3})(s-\sqrt{3})}{s+i}\right)}_{K_+(s)} \underbrace{\left(\frac{1}{s-i}\right)}_{K_-(s)}.$$

Now we can write (101) as

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = -\frac{P_+(s)}{K_-(s)} = R(s) = \frac{2(s-i)}{(s+i)^2}. \tag{102}$$

Step 4: The sum decomposition

Writing $R(s) = R_+(s) + R_-(s)$, we have

$$R_+(s) = \frac{2(s-i)}{(s+i)^2}, \quad R_-(s) = 0,$$

since $R_+(s)$ is analytic in $+$. Unlike $K_\pm(s)$, it doesn't matter that one of $R_\pm(s)$ is zero. Thus (102) becomes

$$K_+(s)F_+(s) - R_+(s) = -\frac{G_-(s)}{K_-(s)}, \quad \text{for } s \in \Omega.$$

Step 5: Analytic continuation

Now the LHS is analytic in $+$ (except possibly infinite at ∞) and the RHS is analytic in $-$ (except possibly infinite at ∞). Since $+$ and $-$ overlap in Ω, and LHS = RHS in Ω then these are analytic continuations of one another, hence, if we define:

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

then $E(s)$ is analytic **everywhere** in the complex s-plane, except possibly at ∞.

Step 6: Behaviour at infinity

As $s \to \infty$ in $+$:

$$\begin{aligned} K_+(s)F_+(s) - R_+(s) &\sim (s + O(1))\left(\frac{if_+(0)}{s} + O\left(\frac{1}{s^2}\right)\right) - \left(\frac{2}{s} + O\left(\frac{1}{s^2}\right)\right) \\ &= A_0 i + O(1/s) \\ &= A_0 i, \end{aligned}$$

a constant. Hence, by Liouville's theorem $E(s) \equiv A_0 i$.

Step 7: Invert

Hence we find

$$\begin{aligned} F_+(s) &= \frac{A_0 i + R_+(s)}{K_+(s)}, \quad \text{for } s \in + \\ &= \frac{i(A_0 - 1)(s+i)^2 + i(s^2 - 3)}{(s+i)(s^2 - 3)}. \end{aligned}$$

Finally, to get $f_+(x)$, use the inversion formula

$$f_+(x) = \frac{1}{2\pi} \int_P F_+(s) e^{-isx} ds,$$

where P is a horizontal line extending to ∞ in $+$. So,

$$f_+(x) = \frac{1}{2\pi} \int_P \frac{(i(A_0 - 1)(s+i)^2 + i(s^2 - 3))e^{-isx}}{(s+i)(s+\sqrt{3})(s-\sqrt{3})} ds.$$

For $x < 0$, one can check that we have $f_+(x) = 0$ as expected. For $x > 0$, close P with a semi-circle γ_R in the LHP.

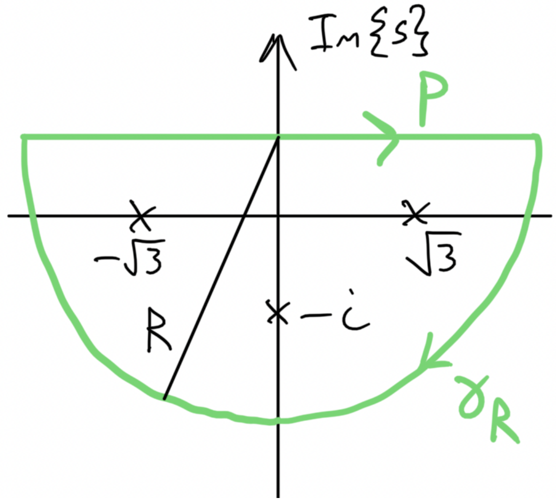


Figure 94: Closed contour γ below P.

Take $\gamma = P + \gamma_R$ as shown in figure 94. Look at the integral along γ_R: as $R \to \infty$:

$$|e^{-isx}F_+(s)| \cdot R = e^{s_2 x}|F_+(s)| \cdot R.$$

On the section of γ_R where $s_2 = \text{Im}\{s\} < 0$, then this expression tends to zero due to the exponential decay. On the sections of γ_R where $s_2 = \text{Im}\{s\} > 0$, since s_2 remains bounded there, and the lengths of these sections are finite, then since $F_+(s) \sim O(1/s)$ the contribution to the integral along these sections is also 0. Hence $\int_{\gamma_R} = 0$ as $R \to \infty$.

Hence by the residue theorem

$$f_+(x) = \frac{1}{2\pi}(-2\pi i)\left[\sum \text{residues of } e^{-isx}F_+(s) \text{ inside } \gamma\right].$$

Denoting $h(s) = e^{-isx}F_+(s)$, then $h(s)$ has poles inside γ at $s = -i, \pm\sqrt{3}$. We find

$$\text{Res}\{h, -i\} = ie^{-x},$$
$$\text{Res}\{h, -\sqrt{3}\} = i(A_0 - 1)\left(\frac{1 - \frac{i}{\sqrt{3}}}{2}\right)e^{i\sqrt{3}x},$$
$$\text{Res}\{h, \sqrt{3}\} = i(A_0 - 1)\left(\frac{1 + \frac{i}{\sqrt{3}}}{2}\right)e^{-i\sqrt{3}x}.$$

Hence we get

$$f(x) = e^{-x} + (A_0 - 1)\left(\cos\sqrt{3}x + \frac{1}{\sqrt{3}}\sin\sqrt{3}x\right).$$

Important remark about this example

Suppose we had opted for Ω as in (b). Then we would have

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{s^2 - 3}{s^2 - i}.$$

Then

$$R(s) = -\frac{P_+(s)}{K_-(s)} = \frac{2(s-i)}{(s+i)^2(s-\sqrt{3})(s+\sqrt{3})} = \frac{c_1}{s+i} + \frac{c_2}{(s+i)^2} + \frac{c_3}{s+\sqrt{3}} + \frac{c_4}{s-\sqrt{3}},$$

for some c_1, c_2, c_3 and c_4. So now:

$$R_+(s) = \frac{c_1}{s+i} + \frac{c_2}{(s+i)^2}, \quad R_-(s) = \frac{c_3}{s+\sqrt{3}} + \frac{c_4}{s-\sqrt{3}}.$$

Then we get

$$\frac{F_+(s)}{s+i} + \frac{(s-i)}{s^2-3}G_-(s) = R_+(s) + R_-(s),$$

leading to the function

$$E(s) = \begin{cases} \frac{F_+(s)}{s+i} - R_+(s), & s \in +, \\ -\frac{(s-i)}{s^2-3}G_-(s) + R_-(s), & s \in -, \end{cases}$$

is **entire**. As $s \to \infty$ we get

$$\frac{F_+(s)}{s+i} - R_+(s) \to 0,$$

hence $E(s) = 0$, by Liouville's theorem. Thus

$$F_+(s) = (s+i)R_+(s) = c_1 + \frac{c_2}{s+i},$$

and in fact one can check (exericse) that $c_1 = 0$ and $c_2 = i$, so we get

$$F_+(s) = \frac{i}{s+i}.$$

Finding $f_+(x)$ by applying the inversion formula, again closing in the LHP, one finds:

$$f(x) = e^{-x}.$$

Recall $|f(x)| < \tilde{A}e^{(1-\delta)x}$ as $x \to \infty$. We see that in case (b) we get the solutions from case (a) which decay as $x \to \infty$. This is because we now have $1-\delta < 0$ (remember in (a): $1-\delta > 0$, so it allowed for solutions which do **not** decay as $x \to \infty$ (we found the extra cosine and sine solution)).

Finally, in case (c), one can show that no solutions are found (this is because it requires solutions which decay faster than e^{-x} as $x \to \infty$).

5.9 Differential equations on a half-line

Suppose we want to solve

$$f''(x) - f'(x) - 2f(x) = 0, \quad x \geq 0, \tag{103}$$

with $f(0) = 1$, $\lim_{x\to\infty} f(x) = 0$ (all derivatives of f exist and are bounded as $x \to 0+$ and $x \to \infty$).

We know from our knowledge of how to solve odes that

$$f(x) = A_1e^{-x} + A_2e^{2x},$$

so we need to set $A_2 = 0$ and $A_1 = 1$ to satisfy our conditions. Now let's solve the equation using Fourier transforms (this is overkill for this toy example but will be useful for what's coming next)!

First, as usual, introduce

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Now we want to take the Fourier transform of (103), so we will need the Fourier transforms of the derivatives $f_+''(x)$ and $f_+'(x)$. Let's find them:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_+''(x)e^{isx}dx &= \int_0^\infty f_+''(x)e^{isx}dx \\
&= \underbrace{\left[f_+'(x)e^{isx}\right]_0^\infty}_{\substack{\text{provided Im}\{s\}>0\text{, then} \\ e^{isx}f_+'(x)\to 0 \text{ as } x\to\infty}} -is\int_0^\infty f_+'(x)e^{isx}dx \\
&= -f_+'(0) - is\int_0^\infty f_+'(x)e^{isx}dx \\
&= -f_+'(0) - is\left[f_+(x)e^{isx}\right]_0^\infty - s^2\int_0^\infty f_+(x)e^{isx}dx.
\end{aligned}$$

Hence we find

$$\int_{-\infty}^{\infty} f_+''(x)e^{isx}dx = -f_+'(0) + isf_+(0) - s^2F_+(s), \tag{104}$$

and also

$$\int_{-\infty}^{\infty} f_+'(x)e^{isx}dx = -f_+(0) - isF_+(s). \tag{105}$$

Higher derivatives can be calculated in a similar manner. So, rewriting (103) as

$$f_+''(x) - f_+'(x) - 2f_+(x) = 0, \quad -\infty < x < \infty,$$

and taking Fourier transforms, we find

$$-f_+(0) + isf_+(0) - s^2F_+(s) + f_+(0) + isF_+(s) - 2F_+(s) = 0,$$

now substituting for $f_+(0) = 1$ and letting $\varepsilon = f_+'(0)$ be an unknown constant

$$F_+(s) = \frac{is + 1 - \varepsilon}{s^2 - is + 2} = \frac{is + 1 - \varepsilon}{(s - 2i)(s + i)}.$$

This has an apparent singularity in the region $\text{Im}\{s\} > 0$ at $s = 2i$, where we require $F_+(s)$ to be analytic! It follows that we must take

$$(is + 1 - \varepsilon)|_{s=2i} = 0 \quad \Rightarrow \quad -1 - \varepsilon = 0 \quad \Rightarrow \quad \varepsilon = -1,$$

so that

$$F_+(s) = \frac{is + 2}{(s - 2i)(s + i)} = \frac{i(s - 2i)}{(s - 2i)(s + i)} = \frac{i}{s + i},$$

which is analytic in $\text{Im}\{s\} > 0$. Then finally, we can find $f_+(x)$ using the inversion formula

$$f_+(x) = \frac{1}{2\pi} \int_P F_+(s) e^{-isx} ds,$$

where P is a horizontal line extending to ∞ in $\text{Im}\{s\} > 0$. As usual, one can check that $f_+(x) = 0$ for $x < 0$. For $x > 0$, close P in the LHP with a semi-circular contour $\gamma = P + \gamma_R$ as shown in figure 95. One can check

$$\int_{\gamma_R} \to 0 \quad \text{as } R \to \infty.$$

So, by the residue theorem

$$f_+(x) = \frac{1}{2\pi}(-2\pi i) \underbrace{\text{Res}\{F_+(s)e^{-isx}, s = -i\}}_{=ie^{-x}}$$

$$\Rightarrow f_+(x) = e^{-x},$$

as expected.

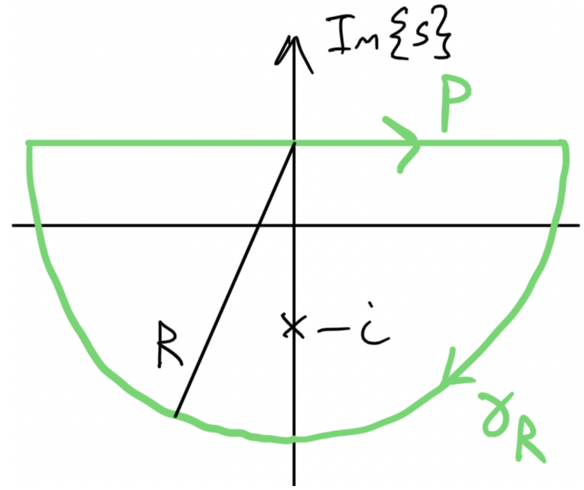


Figure 95: Closed contour γ below P.

5.10 Wiener-Hopf method to solve integral-differential equations

Now we come to the point of the 'toy' problem in the last section. What if the integral equations we have been dealing with throughout the chapter also contain derivatives of the unknown function $f(x)$ too. For example suppose

$$f''(x) + f'(x) + \lambda f(x) = \int_0^\infty k(x-y)f(y)dy, \quad \text{for } x \geq 0.$$

Well the Wiener-Hopf method can be employed in exactly the same way as we have been doing throughout the chapter, the only added difference is that we need to utilise the results (104) and (105) (or equivalent for higher derivatives) for taking Fourier transforms of derivatives when in step 1 of the method. The form of the resulting algebraic equation for $F_+(s)$ will be slightly different due to the added derivative terms (see problem sheet 5 for examples).

Chapter 6: Complex variable methods for the Biharmonic equation

6.1 Introduction and conversion to complex variables

The two-dimensional **biharmonic** equation is given by

$$\nabla^4 \psi = 0,$$

where $\psi(x, y)$ is a real-valued function that we are interested in. This equation has applications in **elasticity** theory, where ψ is known as the **Airy stress function** and in **Stokes flow** (viscous fluid flow occurring in the limit of the Reynolds number tending to 0), which will be the application we focus the analysis on in this chapter, where ψ represents the **streamfunction** of the fluid. This operator represents the Laplacian of the Laplacian, i.e

$$\begin{aligned}
\nabla^4 \psi &= \nabla^2(\nabla^2 \psi) \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \\
&= \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}.
\end{aligned}$$

This comes out a little messy, involving the sum of three separate derivatives. To make things easier, let's introduce complex variables

$$z = x + iy, \qquad \overline{z} = x - iy,$$

then we know that (from chapter 1):

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right], \\
\frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right].
\end{aligned}$$

This leads to

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

and hence

$$\nabla^4 = 16\frac{\partial^4}{\partial z^2 \partial \overline{z}^2}.$$

So in summary, upon changing to complex variables, the two dimensional biharmonic equation may be written as

$$\nabla^4 \psi = 16\frac{\partial^4 \psi}{\partial z^2 \partial \overline{z}^2} = 0$$

$$\Rightarrow \quad \frac{\partial^4 \psi}{\partial z^2 \partial \overline{z}^2} = 0. \tag{106}$$

6.2 Two complex potentials: The Goursat functions

The biharmonic equation (106) is now in a readily integrable form, so we integrate, twice in z, twice in $\overline{z}$, see problem sheet 6. Upon doing this, we find that the general solution for a real biharmonic function ψ can be written in terms of **two analytic functions**:

$$\psi = \mathrm{Im}\{\overline{z}f(z) + g(z)\}, \tag{107}$$

where $f(z)$ and $g(z)$, known as the **Goursat functions**, are functions that are analytic in the flow region where the biharmonic equation holds. Recall from chapter 4, in applications governed by Laplace's equation such as ideal fluid flow, we had that $\psi = \mathrm{Im}\{w(z)\}$, i.e. we needed just **one** complex potential $w(z)$.

So when dealing with problems governed by the biharmonic equation, we have to find **two** complex potentials, $f(z)$ and $g(z)$ to solve the problem.

Remark: The functions $f(z)$ and $g(z)$ do not themselves correspond to anything physical, however in certain combinations they do (see the coming sections 6.4 and 6.5 on Goursat functions). Although the Goursat functions must be analytic in the flow region, it is possible to endow these functions with point singularities in the flow field in order to model physical situations.

6.3 Physical quantities: The velocity field

We have seen that the solution for the streamfunction of a given Stokes flow results in finding the Goursat functions $f(z)$ and $g(z)$ for the flow. It is important to work out how to determine the physical quantities of interest, for example the horizontal and vertical velocity field u and v, from these two functions. Recall from chapter 4, for Laplace's equation we could find the velocity field via the relationship

$$\frac{dw}{dz} = u - iv,$$

so what are the equivalent relationships involving now $f(z)$ and $g(z)$ for the biharmonic equation?

Well, the velocity field is known to satisfy the relationship

$$u - iv = 2i\frac{\partial \psi}{\partial z}.$$

Now, by (107) we know that ψ can be written as

$$\psi = \frac{(\overline{z}f(z) + g(z)) - (z\overline{f(z)} + \overline{g(z)})}{2i}.$$

Thus plugging this into the previous formula gives

$$u - iv = -\overline{f(z)} + \overline{z}f'(z) + g'(z). \tag{108}$$

(**Remark:** It is also possible to calculate pressures, vorticities, components of the rate-of-strain tensor, forces and torques all in terms of the Goursat functions. Here we just concentrate on the velocity field.) Throughout this chapter we will assume **no-slip** boundary conditions, meaning that $u - iv = 0$ on any boundaries of the fluid.

6.4 Goursat functions for background flows

In this section we will find the Goursat functions for some types of classical fluid flow such as shear flow and stagnation point flow.

Note: Do not worry about memorising any Goursat functions presented in these next two sections, you will be given any Goursat functions needed in an exam question.

Linear shear flow

For linear shear flow we want the fluid to move as shown in figure 96. Here the velocity field is given by $(u, v) = (2\gamma y, 0)$, where γ is a real constant known as the shear strength and the streamfunction is given by $\psi = \gamma y^2$. From this information, let's try to work out the Goursat functions that represent this!

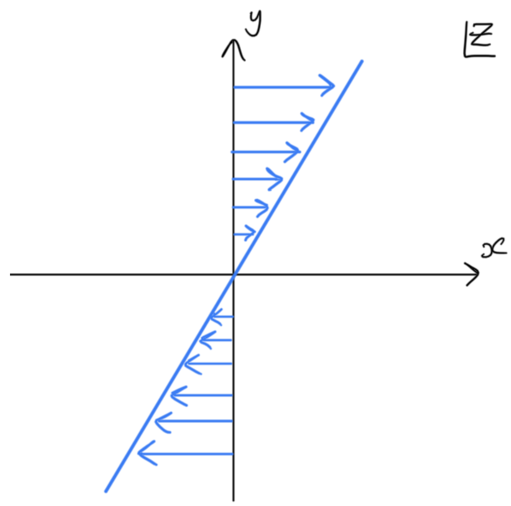


Figure 96: Schematic of linear shear flow.

We know

$$u - iv = -\overline{f(z)} + \overline{z} f'(z) + g'(z),$$

so plugging in for u and v gives

$$\begin{aligned} 2\gamma y - i(0) &= -\overline{f(z)} + \overline{z} f'(z) + g'(z) \\ \Rightarrow \gamma i \overline{z} - \gamma i z &= -\overline{f(z)} + \overline{z} f'(z) + g'(z). \end{aligned}$$

Notice now that on the LHS there are no functions of the form $\overline{z} z^n$, for $n \neq 0$. Therefore the only way the middle term on the RHS can make sense in the above equation is if

$$f'(z) = \text{constant} = A,$$

where $A \in \mathbb{C}$. Now given this, the first and second terms on the RHS are both functions of $\overline{z}$ only, and the third term is a function of z only. Comparing these with like terms on the LHS gives (comparing the terms in z):

$$g'(z) = -\gamma i z,$$

and (since $f'(z) = A \Rightarrow f(z) = Az + B$, where $B \in \mathbb{C}$ is a constant) (comparing the terms in $\overline{z}$):

$$-(\overline{A}\overline{z} + \overline{B}) + A\overline{z} = \gamma i \overline{z}$$
$$\Rightarrow (A - \overline{A})\overline{z} - \overline{B} = \gamma i \overline{z},$$

which means that we must have

$$\gamma i = A - \overline{A},$$
$$B = 0.$$

The first equation above yields $A = \frac{\gamma i}{2} + a$, where $a \in \mathbb{R}$ is a constant. It turns out that this constant makes no difference to the velocity field (if you're not convinced keep it and when you plug $f(z)$ and $g'(z)$ into $u - iv$ you'll see it indeed vanishes), so we can just set it to 0. This means the pair of Goursat functions

$$f(z) = \frac{\gamma i}{2} z, \qquad g'(z) = -\gamma i z,$$

correspond to a shear flow of strength γ.

Remark: Often the derivative of the Goursat function $g(z)$, namely $g'(z)$, is quoted along with $f(z)$ rather than $g(z)$. This is because in the equation for $u - iv$ just $g'(z)$ is required. When integrating this to find $g(z)$ the additive constant is arbitrary and changing its value merely shifts the value of the streamfunction by an additive constant (which we don't care about) and doesn't alter the velocity field of the fluid flow.

Stagnation point flow

These can come in different varieties, sometimes with the strength of flow in one direction being made stronger, though a typical stagnation point flow has velocity field and streamfunction given by something like $(u, v) = (kx, -ky)$, where k is a real constant and $\psi = kxy$. See figure 97 for a schematic of the fluid flow.

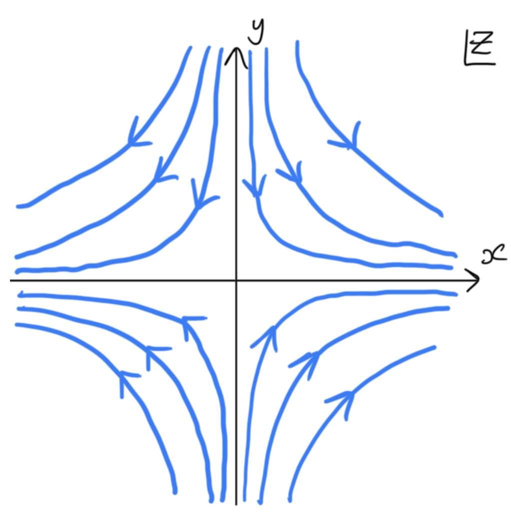


Figure 97: Schematic of a stagnation point flow.

Following a similar method to the last section we can show

$$f(z) = z, \qquad g'(z) = kz,$$

correspond to the stagnation point flow with the stagnation point at $z = 0$.

Uniform flow

What velocity field and streamfunction do the Goursat functions

$$f(z) = z, \qquad g'(z) = U,$$

where $U \in \mathbb{R}$ give? What flow does this correspond to?

6.5 Fundamental singularities of Stokes flow

In general $f(z)$ and $g(z)$ are analytic at all points in the flow, however in order to devise mathematical models for physical scenarios of interest, we can endow these functions with certain isolated (or point) singularities: recall the idea of a source/sink/point vortex in ideal fluid flow being modelled by endowing the complex potential $w(z)$ with a **logarithmic** singularity. Now let's look at a few point singularities in $f(z)$ and $g(z)$.

Stokeslet

The singularity combination

$$\begin{aligned} f(z) &= s \log(z - z_0), \\ g'(z) &= -\overline{s} \log(z - z_0) - \frac{s\overline{z_0}}{z - z_0}, \end{aligned} \tag{109}$$

where $s \in \mathbb{C}$ is called a **Stokeslet** of strength s at z_0. This acts like a point force in the fluid.

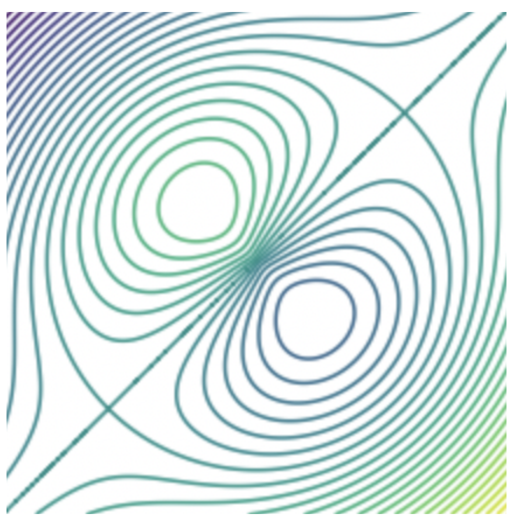


Figure 98: Plot of the streamlines produced by a stokeslet of strength $s = 1 + i$ at $z_0 = 0$.

Stresslet (force dipole)

The singularity combination

$$\begin{aligned} f(z) &= \frac{\mu}{z - z_0}, \\ g'(z) &= \frac{\mu \overline{z_0}}{(z - z_0)^2}, \end{aligned} \tag{110}$$

where $\mu \in \mathbb{C}$ is called a **stresslet** (or force dipole), of strength μ at z_0.

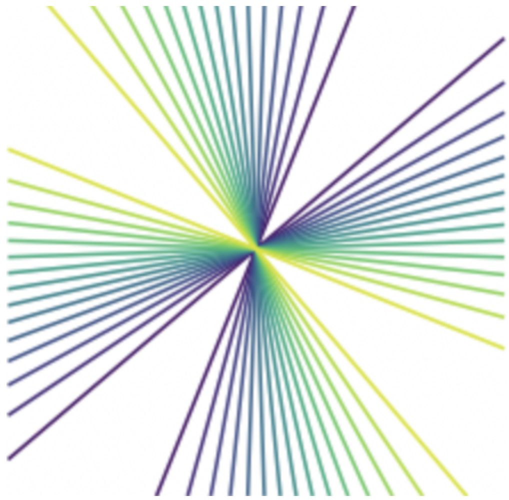


Figure 99: Plot of the streamlines produced by a stresslet of strength $\mu = 3 + i$ at $z_0 = 0$.

Rotlets and point sources/sinks

Suppose instead that $f(z)$ is analytic at z_0 but that $g(z)$ has a logarithmic singularity there of the form

$$g(z) = b \log(z - z_0),$$

where $b \in \mathbb{C}$. We call the singularity combination

$$\begin{aligned} f(z) &= 0, \\ g'(z) &= \frac{b}{z - z_0}, \end{aligned} \tag{111}$$

a **rotlet** of strength b at z_0 **if** $\text{Re}\{b\} = 0$ and a **source/sink** (depending on the sign of b) of strength b at z_0 **if** $\text{Im}\{b\} = 0$.

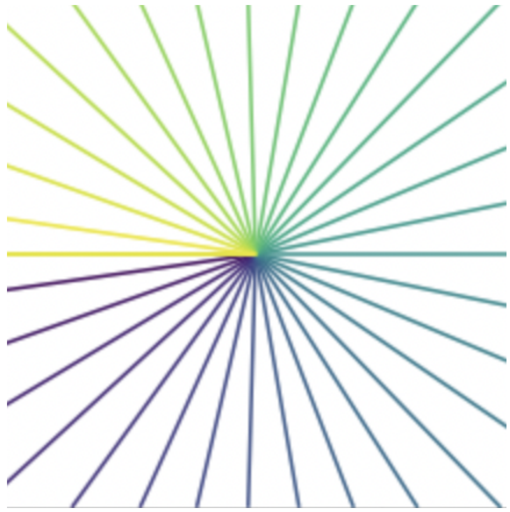


Figure 100: Plot of the streamlines produced by a source of strength $m = 1$ at $z_0 = 0$.

6.6 Stokes flows above a flat wall

In the last section we have looked at some Goursat functions describing flow across the entire plane. Let us now consider a Stokes flow generated by one (or more!) of these singularities positioned at a point z_0 above an infinite flat wall running along the real axis, see figure 101.

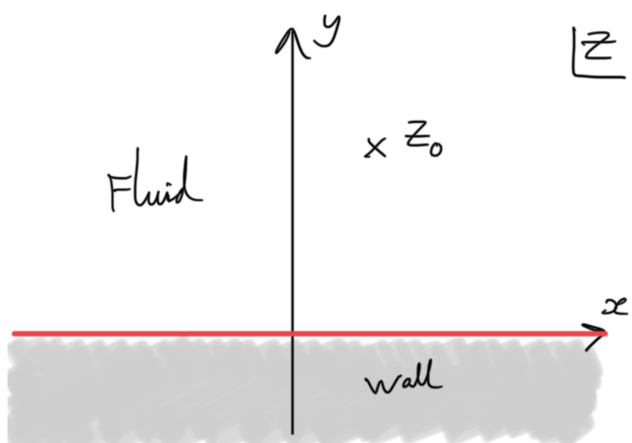


Figure 101: A point singularity at z_0 above a flat wall.

To find the correct Goursat functions satisfying such a set-up we must ensure they tend to the correct singularity structure at z_0, must tend to the right value in the far-field ($z \to \infty$) should there be any background flow introduced and they must also satisfy the boundary condition on the wall. Let's do an example:

Example: A point source near a flat wall

Suppose a point source, of strength $m \in \mathbb{R}$, is at a complex position z_0 above a straight infinite flat wall along the line $y = 0$, see figure 101. We assume a no-slip boundary condition on the wall (velocity is zero on the wall).

For z near to z_0, $f(z)$ must be analytic and $g(z)$ must take the local form of a source

$$g(z) \sim m \log(z - z_0).$$

This means that the Goursat functions for this flow will take the form

$$\begin{aligned} f(z) &= f_R(z), \\ g'(z) &= \frac{m}{z - z_0} + g'_R(z), \end{aligned} \tag{112}$$

where $f_R(z)$ and $g'_R(z)$ are functions analytic in the upper half plane (flow region) that need to be found. The no-slip boundary condition on the wall implies

$$u - iv = -\overline{f(z)} + \overline{z} f'(z) + g'(z) = 0, \quad \text{on } y = 0, \tag{113}$$

or, using the fact that $\overline{z} = z$ on the wall:

$$g'(z) = \overline{f}(z) - z f'(z), \tag{114}$$

where we define the analytic function

$$\overline{f}(z) \equiv \overline{f(\overline{z})}.$$

Remark: It is important to note that whilst equation (113) is only valid on the real axis, the equation (114) derived from it, which relates **analytic** functions, is valid everywhere in the flow region by analytic continuation.

Remark: Given some analytic function $f(z)$, the analytic function defined by

$$\overline{f}(z) \equiv \overline{f(\overline{z})}$$

is known as its **Schwarz conjugate function**.

Returning to the problem at hand, we have arrived at equation (114) which is valid everywhere in the flow region. The goal is now to use this equation to determine what the unknown analytic functions $f_R(z)$ and $g'_R(z)$ must be. If we substitute the forms we have so far from (112) into (114) we obtain

$$\frac{m}{z - z_0} + g'_R(z) = \overline{f}_R(z) - z f'_R(z). \tag{115}$$

Now comes a crucial observation: both $g'_R(z)$ and $f_R(z)$ are analytic at $z = z_0$, meaning neither of the terms in the above equation containing these functions can have a singularity there. However the simple pole on the LHS at $z = z_0$ has to appear somewhere on the RHS, meaning that the term $\overline{f}_R(z)$ must contribute this simple pole at $z = z_0$. Therefore, let

$$\overline{f}_R(z) = \frac{m}{z - z_0}, \tag{116}$$

then by (114) $g'(z)$ has the correct behaviour at $z = z_0$. This means that

$$f_R(z) = \frac{m}{z - \overline{z_0}}, \qquad f_R'(z) = -\frac{m}{(z - \overline{z_0})^2}.$$

And substituting these expressions into (115) then gives

$$\begin{aligned} g_R'(z) &= \frac{mz}{(z - \overline{z_0})^2} \\ &= \frac{m(z - \overline{z_0}) + m\overline{z_0}}{(z - \overline{z_0})^2} \\ &= \frac{m}{z - \overline{z_0}} + \frac{m\overline{z_0}}{(z - \overline{z_0})^2}. \end{aligned}$$

So putting everything back together, we have that the Goursat functions for the flow are given by

$$\begin{aligned} f(z) &= \frac{m}{z - \overline{z_0}}, \\ g'(z) &= \frac{m}{z - z_0} + \frac{m}{z - \overline{z_0}} + \frac{m\overline{z_0}}{(z - \overline{z_0})^2}. \end{aligned} \tag{117}$$

Inspection of these Goursat functions and comparing them with the different types of singularities from section 6.5 shows that for a point source of strength m at a position z_0 above a flat wall we require an image source of strength m, an image stresslet of strength m and another singularity not discussed here containing a double pole in $g'(z)$ with $f(z) = 0$ which is called a dipole of strength $m(\overline{z_0} - z_0)$ all positioned at the reflected point in the wall $\overline{z_0}$.

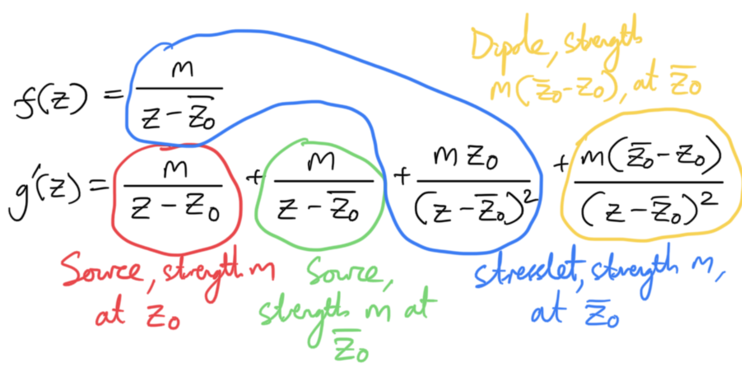


Figure 102: Breakdown of the 'image' singularities at $z = \overline{z_0}$ in the Goursat functions.

Question: The crucial part of the solution method depended on us deciding the form for the function $\overline{f}_R(z)$ in line (116). The pole at z_0 in this function was required to give the correct behaviour in $g'(z)$ as explained earlier. However, why was this **all** of $\overline{f}_R(z)$? Why couldn't there be any other higher order poles for example? Or other poles at other points inside the wall, like at $2\overline{z_0}$ for example?
Answer: Adding any other singularities inside the wall to $\overline{f}_R(z)$ would result in image singularities appearing **inside** the fluid region in $f(z)$ and $g'(z)$. We only want the dipole at z_0 of strength m to exist here and nothing else erroneous.

Question: A harder question to ask is why there were no other 'non-singular' functions of z added to $\overline{f}_R(z)$. For example, how did I know that a function like kz (k some complex constant) couldn't be added to $\overline{f}_R(z)$?
Answer: Adding any other analytic functions of z to $\overline{f}_R(z)$ would lead to some extra analytic functions added to $f(z)$ and $g'(z)$. As explored a little in section 6.4, it turns out this corresponds to adding some effects in the far-field which we don't want to do. We just want to see how the source drives the fluid.

Question: Related to the previous questions is the question of why did I not add a **constant** function to $\overline{f}_R(z)$. Let's investigate this a little more - in line (116) add a constant term ($+k$, k some complex number) to the function $\overline{f}_R(z)$ as well as the required pole. Follow through the rest of the working and determine the Goursat functions for the flow, you should find they are slightly different to those found in (117). Can you explain why this is not a problem?
Answer: Some new constant terms appear. These are no problem because they cancel out when we calculate the velocity field $u - iv$. They would change the values of ψ by a constant which as mentioned we aren't fussed about. Goursat functions are unique up to one additive constant.

Determining the flow field: Streamlines

Now we have the Goursat functions (117) for a point source above a flat wall we can plot the streamlines for the flow using either

$$u - iv = -\overline{f(z)} + \overline{z}f'(z) + g'(z),$$

to calculate the velocities or

$$\psi = \mathrm{Im}\{\overline{z}f(z) + g(z)\},$$

to get the streamfunction. Note that in using this second method $g'(z)$ needs to be integrated which introduces an arbitrary constant of integration. As mentioned earlier with the background flows, this constant simply changes the values of ψ on each streamline which we don't care about, so this can be safely set to 0.

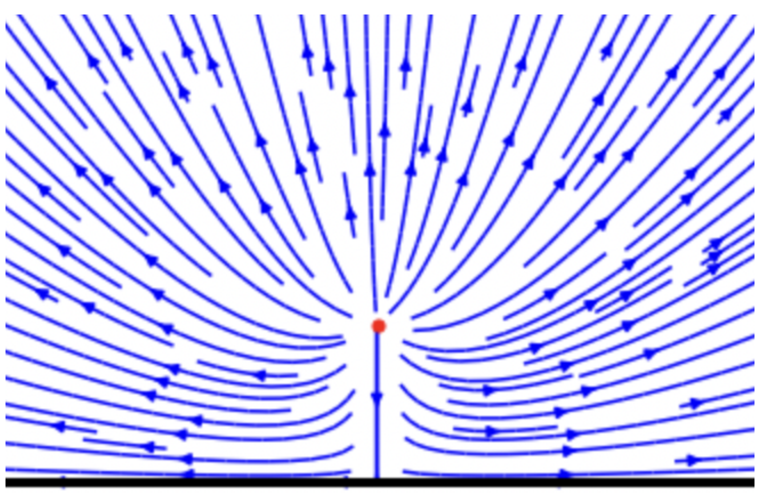


Figure 103: Plot of the streamlines produced by a source of strength $m = 1$ at position $z_0 = i$ above a flat wall.

6.7 Stokes flows near a sharp edge

Let's now go a little further and ask about what happens to Stokes flows near a sharp edge, i.e. a half-line situated along the positive real axis with fluid completely surrounding it, as shown in figure 104.

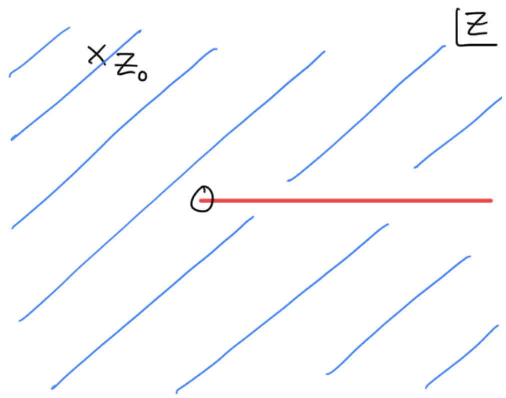


Figure 104: A point singularity at $z = z_0$ near to a half-line.

To aid us in studying this we will use a solution approach based on **conformal mapping**.

Remark: Note that for ideal flows governed by Laplace's equation the problems are **conformally invariant** (we showed this in Chapter 4). This however is **not** the case here with Stokes flow since the governing equation is not Laplace's equation (rather it's the biharmonic equation). Nevertheless conformal mappings can still be a useful tool as we see here.

Consider now the conformal mapping from the inside of the unit ζ-disc to the region exterior to the semi-infinite flat wall along the positive real axis in the z-plane, see figure 105.

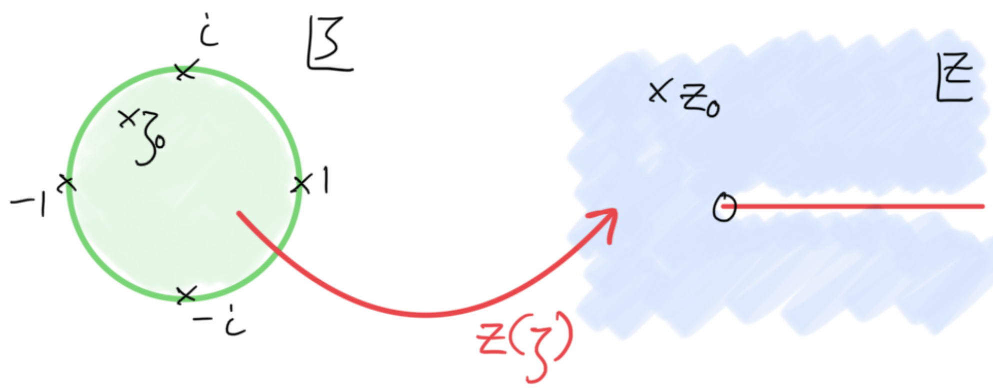


Figure 105: Schematic of the conformal mapping $z(\zeta)$ from the unit ζ-disc to the region surrounding the half-line in the z-plane.

This mapping is given by

$$z = -\left(\frac{1-\zeta}{1+\zeta}\right)^2,$$

see problem sheet 6. A point singularity (in $f(z)$ or $g'(z)$ or both) at z_0 will have pre-image point denoted by ζ_0, so that

$$z_0 = z(\zeta_0) = -\left(\frac{1-\zeta_0}{1+\zeta_0}\right)^2.$$

Now we introduce the functions

$$F(\zeta) = f(z(\zeta)),$$
$$G(\zeta) = g'(z(\zeta)),$$

where ζ is a point inside our unit disc and we will work with these in the ζ-plane.

The boundary condition on the wall

In the z-plane the wall should have the no-slip boundary condition $u - iv = 0$ along the wall, i.e

$$-\overline{f(z)} + \overline{z}f'(z) + g'(z) = 0.$$

In the ζ-plane, the wall corresponds to the boundary of the unit disc, where $|\zeta| = 1$, or in other words $\overline{\zeta} = \frac{1}{\zeta}$. So here the boundary condition becomes

$$-\overline{F(\zeta)} + \frac{\overline{z(\zeta)}}{z'(\zeta)}F'(\zeta) + G(\zeta) = 0,$$

where we have applied the chain rule in the middle term. Then using $\overline{\zeta} = \frac{1}{\zeta}$ we find

$$G(\zeta) = \overline{F}(1/\zeta) - \frac{z(\zeta)}{z'(\zeta)}F'(\zeta), \tag{118}$$

which is an equation valid on $|\zeta| = 1$, but also inside the fluid region **off** $|\zeta| = 1$ by analytic continuation.

Remark: The analysis here now takes a similar vein to that conducted in the last section: recall we needed to come up with the form for $f_R(z)$ based on the form of this equation (we chased where the poles could be). Given a particular example this is exactly what we will do, albeit now the analysis becomes more complicated due to the effect of the conformal mapping meaning this equation (118) is in terms of ζ, but our known singularities are in terms of z. To help facilitate this, we need to have an idea of what poles in $f(z)$ and $g'(z)$ transform into when undergoing the conformal map (i.e what does $1/(z - z_0)$ 'look like' in terms of ζ). Let's investigate this now.

Taylor expansions

Let's Taylor expand the conformal map $z(\zeta)$ about ζ_0. Well

$$\begin{aligned} z(\zeta) &= z(\zeta_0) + z'(\zeta_0)(\zeta-\zeta_0) + \frac{z''(\zeta_0)}{2!}(\zeta-\zeta_0)^2 + \cdots \\ \Rightarrow z - z_0 &= z'(\zeta_0)(\zeta-\zeta_0) + \frac{z''(\zeta_0)}{2!}(\zeta-\zeta_0)^2 + \cdots \\ \Rightarrow \frac{1}{\zeta-\zeta_0} &= \frac{z'(\zeta_0)}{z-z_0} + \frac{z''(\zeta_0)}{2!}\frac{(\zeta-\zeta_0)}{z-z_0} + \cdots \end{aligned}$$

Then using the expansion for $z - z_0$ in the second line inside the third line gives

$$\begin{aligned}
\frac{1}{\zeta - \zeta_0} &= \frac{z'(\zeta_0)}{z - z_0} + \frac{z''(\zeta_0)}{2!}\frac{(\zeta - \zeta_0)}{z'(\zeta_0)(\zeta - \zeta_0)[1 + O(\zeta - \zeta_0)]} + \cdots \\
&= \frac{z'(\zeta_0)}{z - z_0} + \frac{z''(\zeta_0)}{2z'(\zeta_0)}\frac{1}{[1 + O(\zeta - \zeta_0)]} + \cdots \\
&= \frac{z'(\zeta_0)}{z - z_0} + \frac{z''(\zeta_0)}{2z'(\zeta_0)} + O(\zeta - \zeta_0).
\end{aligned}$$

Calculating $z'(\zeta_0)$ and $\frac{z''(\zeta_0)}{2z'(\zeta_0)}$, we find

$$\frac{1}{\zeta - \zeta_0} = \frac{A}{z - z_0} + B + O(z - z_0), \tag{119}$$

where

$$\begin{aligned}
A &= z'(\zeta_0) = \frac{4(1 - \zeta_0)}{(1 + \zeta_0)^3}, \\
B &= \frac{z''(\zeta_0)}{2z'(\zeta_0)} = \frac{\zeta_0 - 2}{1 - \zeta_0^2}.
\end{aligned}$$

Thus we see that simple poles in z at z_0 correspond to simple poles in ζ at ζ_0 (with the extra analytic bits coming along but importantly we know how the singularities transform).

One can also compute

$$\frac{z(\zeta)}{z'(\zeta)} = -\frac{1}{4}(1 - \zeta^2),$$

so that our equation (118) becomes

$$G(\zeta) = \overline{F}(1/\zeta) + \frac{1}{4}(1 - \zeta^2)F'(\zeta). \tag{120}$$

Now we have all the information we need, so given some singularity at z_0 we are in a position to come up with the form of our ansatz for $F(\zeta)$. Let's do an example:

Example: A point source near a sharp edge

Consider a point source at $z = z_0$ near the half-line in figure 104. Due to this, from section 6.5, we know that local to z_0 the Goursat functions must take the form

$$\begin{aligned}
f(z) &= \text{analytic}, \\
g'(z) &= \frac{m}{z - z_0} + \text{analytic}.
\end{aligned} \tag{121}$$

Now $f(z)$ has **no** poles at z_0, thus $F(\zeta)$ must have **no** poles at ζ_0 (we saw through the mapping poles at z_0 correspond to poles at ζ_0). $g'(z)$ has a **simple** pole at z_0, but nothing stronger (double poles, triple poles etc), therefore $G(\zeta)$ will have a **simple** pole at ζ_0, but also nothing stronger. This means that the term on the RHS of (120) given by

$$\frac{1}{4}(1 - \zeta^2)F'(\zeta)$$

is **analytic** at $\zeta = \zeta_0$ (since $F'(\zeta)$ cannot have singularities in it at ζ_0 if $F(\zeta)$ can't). Thus, the pole at ζ_0 in $G(\zeta)$ on the LHS of (120) must be balanced by the term $\overline{F}(1/\zeta)$ on the RHS. So, we set

$$\overline{F}(1/\zeta) = \frac{a}{\zeta - \zeta_0},$$

giving

$$F(\zeta) = \frac{\overline{a}}{1/\zeta - \overline{\zeta_0}}, \tag{122}$$

where $a \in \mathbb{C}$ is a complex constant to be determined. From (122) by taking a ζ derivative we can get

$$F'(\zeta) = \frac{\overline{a}}{\zeta^2(1/\zeta - \overline{\zeta_0})^2}.$$

Then plugging this into (120) gives

$$G(\zeta) = \frac{a}{\zeta - \zeta_0} + \frac{1}{4}(1 - \zeta^2)\frac{\overline{a}}{\zeta^2(1/\zeta - \overline{\zeta_0})^2}. \tag{123}$$

Equations (122) and (123) then give the Goursat functions we desire, but we still need to determine the unknown value a. To do this, recall that

$$\frac{1}{\zeta - \zeta_0} = \frac{A}{z - z_0} + B + O(z - z_0),$$

so plugging this into (123) gives

$$G(\zeta) = \frac{aA}{z - z_0} + O(1),$$

and comparing this with (121) we see we need to set

$$a = \frac{m}{A}.$$

Equations (122) and (123) along with the value for a above now provide the Goursat functions we desire in terms of ζ, ζ_0 and m. Should we wish to write the solution back in terms of z (and z_0), we can use the inverse mapping given by

$$\zeta = \frac{1 - \sqrt{-z}}{1 + \sqrt{-z}}.$$

Below shows a plot of the streamlines generated by the point source when $m = 1$ and $z_0 = -1 + i/2$.

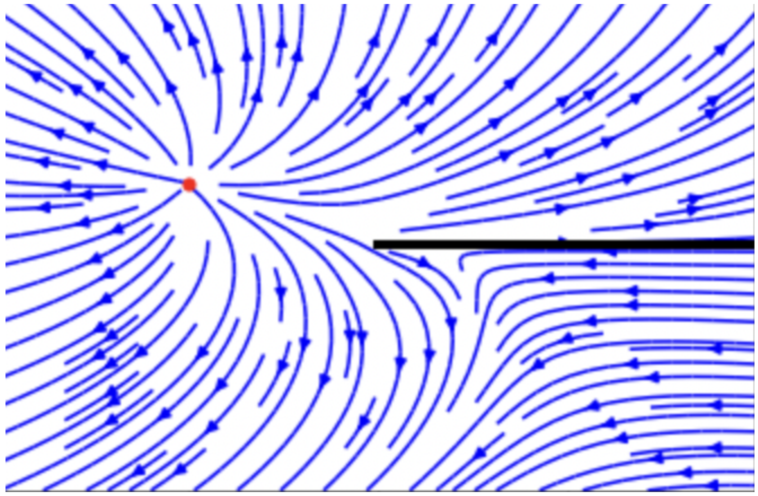


Figure 106: Plot of the streamlines produced by a source of strength $m = 1$ at $z_0 = -1 + i/2$ near to a sharp edge.