Imperial College London

MATH70016

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Special Relativity and Electromagnetism

Date: 12 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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1. In the following the spatial origins of frames K and K' are chosen so that $(0,0,0,0)$ refers to the same event in both frames. All axes are parallel at all time. The origin of frame K' moves with velocity V relative to frame K along its x-axis. Dashed variables refer to measurements in the dashed frame K', undashed variables to measurements in the undashed frame K.

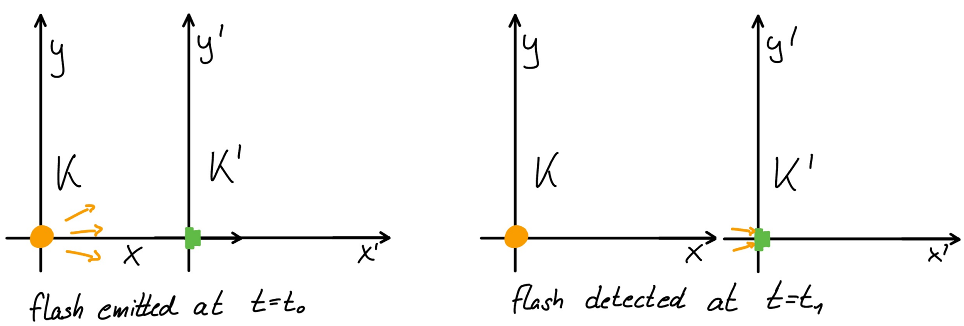
To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$.

(a) Lorentz transform and simplify the following four-vectors and tensors from frame K' to frame K.

(i) $x'^i = (ct', Vt', 0, 0)$. (2 marks)

(ii) $A'^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (2 marks)

(b)



A flash of light of a source at rest in K departs from the origin of K at time t_0. It is observed to arrive at the origin of K' at time t_1 measured in K.

(i) State the four-vector d^i measured in K of the event of the flash's departure from the origin of K. (2 marks)

(ii) State the four-vector a^i measured in K of the event of the flash's arrival in the origin of K'. (2 marks)

(iii) Transform the d^i to the four-vector d'^i measured in K'. (2 marks)

(iv) Transform the a^i to the four-vector a'^i measured in K'. (2 marks)

(v) Calculate t_1 as a function of t_0, V and c, given that the flash travels with the speed of light. (2 marks)

(vi) Determine the distance ℓ' the flash has travelled in K' using d'^i and a'^i. (2 marks)

(vii) Determine the time τ' the flash has travelled in K' using d'^i and a'^i. (2 marks)

(viii) Determine the speed the flash has travelled with as observed in K' using ℓ' and τ'. (2 marks)

(Total: 20 marks)

2. In the following the spatial origins of frames K and K' are chosen so that $(0,0,0,0)$ refers to the same event in both frames. All axes are parallel at all time. The origin of frame K' moves with velocity V relative to frame K along its x-axis. Dashed variables refer to measurements in the dashed frame K', undashed variables to measurements in the undashed frame K.

To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$.

(a) State the general form of the energy-momentum four-vector p^i in terms of a particle's rest mass m, its velocity $\mathbf{V}$, where $|\mathbf{V}| = V$, and the speed of light c.

(4 marks)

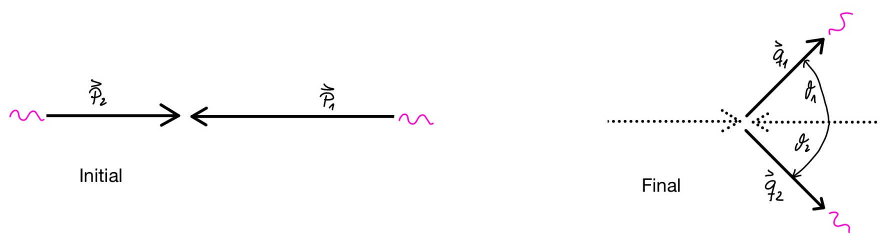
(b) Express $p_i p^i$ in terms of the particle's rest mass m and the speed of light c.

(4 marks)

(c) A photon is a particle with positive (kinetic) energy $\mathcal{E} = |\mathbf{p}|c$ and momentum $\mathbf{p}$, but no rest mass, so described by the energy-momentum four-vector $p^i_\gamma = (|\mathbf{p}|, \mathbf{p})$. Show that $p^i_\gamma p_{\gamma\, i} = 0$.

(4 marks)

(d)



As illustrated above, two photons with momentum $\mathbf{p}_1$ and $\mathbf{p}_2$ antiparallel, annihilate and produce two photons with momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ in the process. These photons are identical, which means that $|\mathbf{q}_1| = |\mathbf{q}_2| = q$.

(i) Show that $\theta_1 = -\theta_2 = \theta$.
Hint: Use momentum conservation and orient the coordinate system such that $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel to the x-axis.

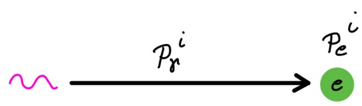
(4 marks)

(ii) Determine θ as a function of q, $\mathbf{p}_1$ and $\mathbf{p}_2$.
Hint: Use momentum conservation in the x-direction only.

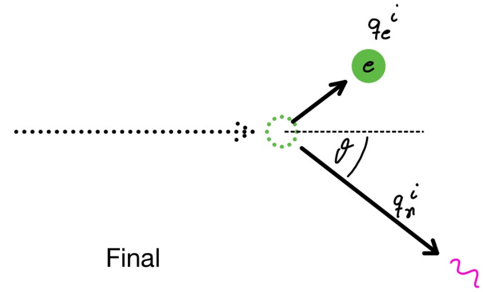
(4 marks)

(Total: 20 marks)

3.



Initial



Final

In the following we consider so-called "Compton-scattering", that is the collision of an electron and a photon but with changed momenta, resulting in an electron and a photon, as illustrated above. The momentum four-vector of the electron is denoted by p_e^i before and q_e^i after the collision. The momentum four-vector of the photon is denoted by $p_\gamma^i = (p_\gamma, p_\gamma \mathbf{v})$ before and $q_\gamma^i = (q_\gamma, q_\gamma \mathbf{w})$ after the collision with $p_\gamma, q_\gamma > 0$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ normalised, so that $|\mathbf{v}| = |\mathbf{w}| = 1$.

We study the collision in the frame of reference where the electron originally is at rest, so that $p_e^i = (mc, \mathbf{0})$ with m the rest mass of the electron.

(a) Writing $q_e^i = (\mathcal{E}/c, \mathbf{q}_e)$ show that $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{q}_e$ are in the same plane.

(5 marks)

(b) Given the result in (a), we can orient our coordinate system such that $\mathbf{v} = (1, 0, 0)^\mathsf{T}$ and $\mathbf{w} = (\cos\theta, \sin\theta, 0)^\mathsf{T}$ as suggested in the figure.

(i) By physical reasoning or otherwise, determine the minimum energy $\mathcal{E}$ of the electron after the collision and the corresponding angle θ.

(5 marks)

(ii) The energy of the electron after the collision, $\mathcal{E}$, varies with the deflection angle of the photon, θ. The maximum of $\mathcal{E}$ is obtained when $\theta = \pi$ and $\mathbf{v} = -\mathbf{w}$ so that $\mathbf{q}_e = q_e \mathbf{v}$. For this special case write $\mathcal{E}$ in terms of m, p_γ and c.
Hint: Momentum conservation produces q_e in terms of p_γ and q_γ. Energy conservation produces q_γ in terms of m, $\mathcal{E}$ and c. The squared (kinetic) energy of a particle is $\mathcal{E}^2 = m^2c^4 + q_e^2 c^2$.

(5 marks)

(iii) For the general case of arbitrary deflection angle θ express $\mathcal{E}$ as a function of m, p_γ, θ and c.
Hint: Write an expression for q_e^i from energy-momentum conservation. Extract $\mathcal{E}$ in terms of m, p_γ, q_γ and c. Then square q_e^i and solve for q_γ.

(5 marks)

(Total: 20 marks)

4. In four-vector form the inhomogeneous Maxwell equations are written as

$$\partial_\ell F^{i\ell} = -\frac{4\pi}{c} j^i$$

where $F^{i\ell}$ is the electromagnetic field tensor and j^i is the current four-vector. The covariant $\partial_\ell = \frac{\partial}{\partial x^\ell}$ denotes the partial derivative with respect to the contravariant x^ℓ.

(a) Using the definition of the electromagnetic field tensor

$$F^{i\ell} = \partial^i A^\ell - \partial^\ell A^i$$

and the Lorenz gauge

$$\partial_i A^i = 0$$

derive the wave-equation for A^i,

$$\partial_\ell \partial^\ell A^i = 0$$

in four-vector form. (5 marks)

(b) Determine the relation between ω and $\mathbf{k}$ in $k^i = (\omega/c, \mathbf{k})$, such that

$$A^i = B^i \exp(\imath k^j x_j)$$

with B^i constant solves the wave-equation for A^i. (5 marks)

(c) Electric and magnetic fields can be written in terms of the vector potential $\mathbf{A}$ and the scalar potential ϕ as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi \quad \text{and} \quad \mathbf{H} = \nabla \times \mathbf{A} \ .$$

Writing A^i in terms of the scalar and vector potentials, show that if $A^i = B^i \exp(\imath k^j x_j)$ with $k^i = (\omega/c, \mathbf{k})$ solves the wave equation, then with the Lorenz gauge it follows that

$$\mathbf{k} \cdot \mathbf{H} = 0 \quad \text{and} \quad \mathbf{k} \cdot \mathbf{E} = 0 \ .$$

Hint: For $\mathbf{k} \cdot \mathbf{H}$, show that $\mathbf{H} = -\imath \mathbf{k} \times \mathbf{A}$. For $\mathbf{k} \cdot \mathbf{E}$, use the Lorenz gauge to show $\mathbf{k} \cdot \mathbf{A} = \omega\phi/c$ and use that to remove ϕ from $\mathbf{k} \cdot \mathbf{E}$. (10 marks)

(Total: 20 marks)

5. Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \cdot \mathbf{H} = 0$$
$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}$$
$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

(a) From Maxwell's equations, express $\partial\rho/\partial t$ in terms of the current. (5 marks)

(b) The potential $\phi(\mathbf{r})$ at $\mathbf{r}$ of a point charge e_a at $\mathbf{r}_a$ is $\phi = e_a/|\mathbf{r} - \mathbf{r}_a|$.

(i) Consider the potential $\phi(r)$ due to N charges e_a with $a = \{1, 2, \dots, N\}$ at positions $\mathbf{r}_a$. Show that at a large distance r away from these charges, where $r \gg |\mathbf{r}_a|$ for all a, the leading order term in the potential goes like $1/|\mathbf{r}|$, i.e. that of a point charge with total charge $\sum_a^N e_a$ at the origin. (5 marks)

(ii) Determine the potential $\phi(\mathbf{r})$ of a line of charges of length L along the z-direction centred around the origin. It has density

$$\rho(\mathbf{r}) = \alpha\delta(x)\delta(y)\theta(L/2 - z)\theta(L/2 + z)$$

where $\mathbf{r} = (x, y, z)$ is the position in space, α is the charge density along the line and $\theta(z)$ is the Heaviside step-function, defined by

$$\theta(z) = \begin{cases} 1 & \text{for } z > 0 \\ 0 & \text{otherwise} \end{cases} .$$

The total charge over a length ℓ is therefore $\alpha\ell$.
Hint :

$$\int dx \, \frac{1}{\sqrt{1 + x^2}} = \sinh^{-1}(x) + C$$

where $\sinh^{-1}$ is the inverse hyperbolic sine function and C denotes the constant of integration. (5 marks)

(iii) We now consider the potential $\phi(\mathbf{r})$ at large distances from the charged line, where $R = \sqrt{x^2 + y^2}$ is large compared to z and L, *i.e.* $R \gg z$ and $R \gg L$. State $\phi(\mathbf{r})$ to leading order in $1/R$.
Hint :

$$\sinh^{-1}(x) = x + \mathcal{O}(x^3)$$

(5 marks)

(Total: 20 marks)

1. (a.i) Elementary calculation, $\boxed{x^i = \big((ct + \beta V t)\gamma, (Vt + \beta ct)\gamma, 0, 0\big)}$. seen ⇓

2, A

(a.ii) This is best done in matrix form, sim. seen ⇓

$$\boxed{\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\boxed{= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

2, A

(b.i) This is a straight-forward observation, $\boxed{d^i = (ct_0, 0, 0, 0)}$. seen ⇓

2, A

(b.ii) This is a straight-forward observation, $\boxed{a^i = (ct_1, Vt_1, 0, 0)}$. seen ⇓

2, A

(b.iii) Elementary calculation $\boxed{d'^i = (ct_0\gamma, -ct_0\beta\gamma, 0, 0)}$. sim. seen ⇓

2, B

(b.iv) Elementary calculation $\boxed{a'^i = ((ct_1 - \beta V t_1)\gamma, (Vt_1 - \beta ct_1)\gamma, 0, 0) = (ct_1/\gamma, 0, 0, 0)}$. sim. seen ⇓

2, B

(b.v) The flash travels distance Vt_1 in time $t_1 - t_0$, so $Vt_1/(t_1 - t_0) = c$ and therefore $\boxed{t_1 = t_0/(1-\beta)}$. sim. seen ⇓

2, C

(b.vi) The distance is given by the spatial elements of d'^i and a'^i, $\boxed{\ell' = ct_0\beta\gamma = t_0 V\gamma}$. sim. seen ⇓

2, C

sim. seen ⇓

(b.vii) The time is given by the time elements of d'^i and a'^i, $\boxed{c\tau' = ct_1/\gamma - ct_0\gamma}$. Using t_1 from b.v then gives $\boxed{\tau' = t_0(1/((1-\beta)\gamma) - \gamma) = t_0\beta\gamma}$.

2, C

sim. seen ⇓

(b.viii) The speed of the flash is $\boxed{\ell'/\tau' = V/\beta = c}$ as expected as the speed of light is invariant across frames. This

2, C

2. (a) As seen many times, $\boxed{p^i = (mc\gamma, m\mathbf{V}\gamma)}$, or, literally according to the question asking for things in terms of m, $\mathbf{V}$ and c, seen ⇓

$$\boxed{p^i = \left(\frac{mc}{\sqrt{1-\frac{|\mathbf{V}|^2}{c^2}}}, \frac{m\mathbf{V}}{\sqrt{1-\frac{|\mathbf{V}|^2}{c^2}}}\right)}$$

4, A

seen ⇓

(b) Either by elementary calculation or from memory $\boxed{p^i p_i = m^2c^2\gamma^2 - m^2V^2\gamma = m^2c^2}$. 4, A

seen ⇓

(c) Either by elementary calculation or from memory $\boxed{p^i_\gamma p_{\gamma\, i} = |\mathbf{p}||\mathbf{p}| - \mathbf{p}\cdot\mathbf{p} = 0}$. 4, A

sim. seen ⇓

(d.i) With $\mathbf{p}_1$ and $\mathbf{p}_2$ parallel to the x-axis, their four-momenta are $p^i_1 = (|p_1|, p_1, 0, 0)$ and $p^i_2 = (|p_2|, p_2, 0, 0)$. Correspondingly, $q^i_1 = (q, q\cos\theta_1, q\sin\theta_1, 0)$ and $q^i_2 = (q, q\cos\theta_2, q\sin\theta_2, 0)$. Energy-momentum conservation gives $p^i_1 + p^i_2 = q^i_1 + q^i_2$ and thus $\boxed{0 + 0 = q\sin\theta_1 + q\sin\theta_2}$ so that indeed $\theta_1 = -\theta_2$. 4, B

unseen ⇓

(d.ii) By momentum conservation $p_1 + p_2 = 2q\cos\theta$, so $\boxed{\theta = \cos^{-1}((p_1+p_2)/(2q))}$. 4, B

3. (a) From four-momentum conservation, $\boxed{p_e^i + {p_\gamma}^i = (mc + p_\gamma, p_\gamma \mathbf{v}) = q_e^i + {q_\gamma}^i = (\mathcal{E}/c + q_\gamma, q_\gamma \mathbf{w} + \mathbf{q}_e)}$ or momentum conservation directly, it follows that $\boxed{p_\gamma \mathbf{v} = q_\gamma \mathbf{w} + \mathbf{q}_e}$ and as $p_\gamma > 0$ it immediately follows that $\mathbf{v}$ is a linear combination of $\mathbf{w}$ and $\mathbf{q}_e/p_\gamma$, hence all three are in the same plane.

sim. seen ⇓

5, A

unseen ⇓

(b.i) The minimum energy of an electron at rest is obtained when it remains at rest, so $\boxed{\mathcal{E} = mc^2}$ and there is no collision and hence no momentum transfer, which implies that $\boxed{\theta = 0}$.

5, B

(b.ii) When $\mathbf{q}_e = q_e \mathbf{v}$ momentum conservation gives $p_\gamma = -q_\gamma + q_e$, so that $q_e = p_\gamma + q_\gamma$. Energy conservation further gives $mc + p_\gamma = \mathcal{E}/c + q_\gamma$, so that $q_\gamma = mc + p_\gamma - \mathcal{E}/c$. The squared energy of the particle is

unseen ⇓

$$\mathcal{E}^2 \;=\; m^2c^4 + q_e^2c^2 \;=\; m^2c^4 + (p_\gamma + q_\gamma)^2c^2 \;=\; m^2c^4 + (mc^2 + 2p_\gamma c - \mathcal{E})^2$$

After simplifying, this gives $2\mathcal{E}(mc^2 + 2p_\gamma c) = m^2c^4 + (mc^2 + 2p_\gamma c)^2$ and therefore

$$\boxed{\mathcal{E} = \frac{m^2c^3 + 2{p_\gamma}^2 c + 2mp_\gamma c^2}{mc + 2p_\gamma} = mc^2 + \frac{2{p_\gamma}^2 c}{mc + 2p_\gamma}}$$

5, D

(b.iii) As suggested by the hint, from energy momentum conservation $p_e^i + {p_\gamma}^i - {q_\gamma}^i = q_e^i$ which gives $\mathcal{E}/c = mc + p_\gamma - q_\gamma$. Squaring the expression for q_e^i gives $\boxed{mc^2 = mc^2 + 0 + 0 + 2p_e^i p_{\gamma_i} - 2p_e^i q_{\gamma_i} - 2{p_\gamma}^i q_{\gamma_i}}$. Calculating the cross-terms individually, $p_e^i p_{\gamma_i} = p_\gamma mc$, $p_e^i q_{\gamma_i} = q_\gamma mc$ and ${p_\gamma}^i q_{\gamma_i} = p_\gamma q_\gamma - p_\gamma q_\gamma cos\theta$, gives $0 = p_\gamma mc - q_\gamma mc - p_\gamma q_\gamma(1 - cos\theta)$ and thus $q_\gamma = p_\gamma mc/(mc + p_\gamma(1 - cos\theta)$. Using this in the expression for $\mathcal{E}$ finally produces

unseen ⇓

$$\boxed{\mathcal{E} = mc^2 + p_\gamma c - \frac{p_\gamma mc^2}{mc + p_\gamma(1 - cos\theta)} = mc^2 + \frac{{p_\gamma}^2 c(1 - \cos\theta)}{mc + p_\gamma(1 - cos\theta)}\;,}$$

which is consistent with (c.ii), the result for $\theta = \pi$, when $\cos\theta = -1$.

5, D

4. (a) In vacuum $j^i = 0$, so that $\partial_\ell F^{i\ell} = 0$ from the inhomogeneous Maxwell equations. From the definition of the electromagnetic field tensor it follows that $\boxed{0 = \partial_\ell \partial^i A^{i\ell} - \partial_\ell \partial^\ell A^i}$, but the Lorenz gauge implies that the first term on the RHS vanishes anyway, so $\boxed{0 = \partial_\ell \partial^\ell A^i}$. .

unseen ⇓

5, C

(b) First, we note that $\partial_\ell A^i = \mathring{\imath} k_\ell A^i$, so that $\partial_\ell \partial^\ell A^i = -k_\ell k^\ell A^i$. The wave equation requires this to vanishes, which means that $\boxed{k_\ell k^\ell = 0}$, implying $\boxed{\omega^2/c^2 - \mathbf{k}\cdot\mathbf{k} = 0}$. This is the well-known dispersion relation in vacuum.

sim. seen ⇓

5, A

(c) Firstly, $A^i = (\phi, \mathbf{A})$, so that

unseen ⇓

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\phi = -\mathring{\imath}\frac{\omega}{c}\begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} + \mathring{\imath}\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} A^0$$

using $\boxed{\partial_\ell A^i = \mathring{\imath} k_\ell A^i}$, where the covariant k_ℓ has elements $(\omega/c, -k_x, -k_y, -k_z)$. Multiplying the expression for $\mathbf{E}$ above by $\mathbf{k}$ gives

$$\boxed{\mathbf{k}\cdot\mathbf{E} = -\mathring{\imath}\frac{\omega}{c}\mathbf{k}\cdot\mathbf{A} + \mathring{\imath} k^2 \phi}$$

The Lorenz-gauge on the other hand means $0 = \partial_\ell A^\ell$, so that

$$\boxed{0 = \mathring{\imath}\left(\frac{\omega}{c}\phi - \mathbf{k}\cdot\mathbf{A}\right) \ .}$$

This can be used to re-write $\mathbf{k}\cdot\mathbf{E}$ as $\mathbf{k}\cdot\mathbf{A}$ in the expression above can be replaced by $\omega\phi/c$,

$$\boxed{\mathbf{k}\cdot\mathbf{E} = -\mathring{\imath}\frac{\omega^2}{c^2}\phi + \mathring{\imath} k^2 \phi = -\mathring{\imath}\left(\frac{\omega^2}{c^2} - k^2\right)\phi = 0}$$

using the dispersion relation $k^2 = \omega^2/c^2$.

Also allow for $\nabla\cdot\mathbf{E} = -\mathring{\imath}\mathbf{k}\cdot\mathbf{E}$ and assuming vacuum, so $\nabla\cdot\mathbf{E} = 0$. 5, D

Showing a similar property for $\mathbf{H}$ turns out to be easier, as

$$\boxed{\mathbf{H} = \nabla\times\mathbf{A} = -\mathring{\imath}\mathbf{k}\times\mathbf{A}}$$

which immediately gives $\boxed{\mathbf{k}\cdot\mathbf{H} = -\mathring{\imath}\mathbf{k}\cdot(\mathbf{k}\times\mathbf{A}) = 0}$. 5, B

5. (a) Level: a $\boxed{\text{sim. seen} \Downarrow}$

From Gauss' law, the last of the Maxwell equations given, $4\pi\dot{\rho} = \nabla \cdot \dot{\mathbf{E}}$. According to Ampère's law, the second from last, $\dot{\mathbf{E}} = c\nabla \times \mathbf{H} - 4\pi\mathbf{j}$, so that $\boxed{4\pi\dot{\rho} = \nabla \cdot \left(c\nabla \times \mathbf{H} - 4\pi\mathbf{j}\right) = -4\pi\nabla \cdot \mathbf{j}}$, as the gradient of a curl vanishes. It follows that $\boxed{\dot{\rho} = -\nabla \cdot \mathbf{j}}$, the continuity equation. $\boxed{5,\ \mathrm{M}}$

(b.i) Level: b $\boxed{\text{unseen} \Downarrow}$

The total potential of N charges is simply the superposition of each individual potential,

$$\phi(\mathbf{r}) = \sum_a^N \frac{e_a}{|\mathbf{r} - \mathbf{r}_a|} = \frac{1}{r}\sum_a^N \frac{e_a}{|\mathbf{r}/r - \mathbf{r}_a/r|} \ .$$

Using now that r is large compared to all $|\mathbf{r}_a|$ then gives

$$\boxed{\phi(\mathbf{r}) = \frac{1}{r}\sum_a^N e_a + \ldots}$$

which is the potential of a total charge $\sum_a^N e_a$ at the origin. There is no need to write the modulus as a square root and expand. $\boxed{5,\ \mathrm{M}}$

(b.ii) Level: d $\boxed{\text{unseen} \Downarrow}$

Using $\phi(\mathbf{r}) = \int \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\rho(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ gives

$$\boxed{\phi(\mathbf{r}) = \int_{-L/2}^{L/2} \mathrm{d}z' \frac{\alpha}{|\mathbf{r} - (0,0,z')|}}$$

Writing the modulus as a square root and taking out a factor $1/R$ with $R = \sqrt{x^2 + y^2}$, the radial distance, then produces

$$\phi(\mathbf{r}) = \frac{1}{R}\int_{-L/2}^{L/2} \mathrm{d}z' \frac{\alpha}{\sqrt{1 + (z - z')^2/R^2}} \ .$$

After substituting $q = (z' - z)/R$ the integral simplifies to the one given in the hint,

$$\boxed{\phi(\mathbf{r}) = \frac{\alpha}{R}\int_{-(L/2+z)/R}^{(L/2-z)/R} \frac{\mathrm{d}q}{\sqrt{1+q^2}} = \alpha\left(\sinh^{-1}\left(\frac{L/2 - z}{R}\right) - \sinh^{-1}\left(-\frac{L/2 + z}{R}\right)\right)}$$

$\boxed{5,\ \mathrm{M}}$

(b.iii) Level: c $\boxed{\text{unseen} \Downarrow}$

At large radial distance, the potential should be that of a point charge, $\boxed{\phi(\mathbf{r}) = L\alpha/R}$, as found in (5.a).

This is confirmed when considering large R. Expanding the result above in (5.b.ii) using the hint, indeed gives

$$\boxed{\phi(\mathbf{r}) = \alpha\left(\frac{L/2 - z}{R} + \frac{L/2 + z}{R}\right) + \ldots = \alpha\frac{L}{R}}$$

The more complicated case of large z is explicitly not considered. $\boxed{5,\ \mathrm{M}}$

Review of mark distribution:
Total A marks: 30 of 32 marks
Total B marks: 22 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 100 of 100 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60016/70016	1	Straight forward question which can be mastered by knowing how to Lorentz-transform. Some confusion about basics in b.ii and b.v, which I have compensated by not deducting marks from follow-on errors. I like the extra comments by students realising they should obtain c in the last question.
MATH60016/70016	2	This question was very easy. I was generous with confusion about the sign of the momentum, as the figure suggests that a positive momentum for a particle results in movement to the right for one particle and to the left for the other. A very large number of students got full marks. Marks where lost mostly by simple mistakes in rearranging terms.
MATH60016/70016	3	This question was challenging, in particular the last two parts of it. With a solid working understanding of four-vector arithmetic and using the hints it was quite doable. Still, I was impressed how quickly many students worked out the answer, drawing on the hints and previous results at the right time. When marking, I spent a considerable amount of time re-arranging what students had obtained to match the model answers. Most students did well in this question. A common mistake was to abandon four-vector notation too early in b.ii and b.iii. It was also somewhat common to overlook the simple physical reality that the minimal momentum transfer occurs when the photon passes the electron without interacting, i.e. \theta=0, so that the electron stays put, E=mc^2. Similarly, an educated guess suggests \theta=\pi for the maximum momentum transfer. The general result in part b.iii can then be checked by comparison to b.i and b.ii.
MATH60016/70016	4	This was a question about electromagnetic waves, covering basics in the first part and more advanced material in the later parts.The question was generally answered well. The first question required the knowledge of the requirement of vacuum to use that the four-current vanishes. The dispersion relation was generally derived easily. The last part was made easier by the hints, although some students struggled to put them together, in particular how the Lorenz-gauge enters into kE vanishing.
MATH70016	5	This was a question about charge densities and currents. The first part of the question about the continuity equation was answered well in general. The second and the last part was generally answered well, by many students either expanding explicitly or drawing on physical intuition. I made some allowances for the latter, if it involved some mathematical reasoning. Question b.ii was clearly the most difficult, at least at first sight, but easy once the Dirac \delta functions are used up, reducing what is left to the integral in the hint.