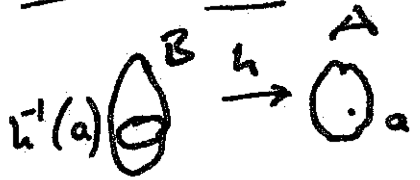
(4.1.7) **Lemma.** (ZFC)

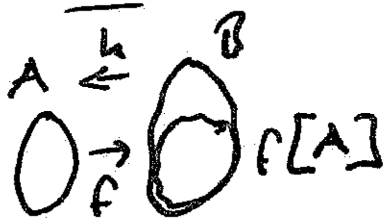
Suppose A, B are nonempty sets. Then

$|A| \leq |B| \iff$ there is a surjective function $B \to A$.

Pf: $\overset{AC}{\Leftarrow}$: Problem Sheet 8.



$\Rightarrow$: If $f : A \to B$ is injective. Let $a_0 \in A$. ~~Take~~ Define $h : B \to A$



by, for $b \in B$

$$h(b) = \begin{cases} a \text{ with } f(a) = b & \text{if } b \in f[A] \\ a_0 & \text{if } b \notin f[A] \end{cases}$$

#.

(4.2) Cardinals + Cardinality

Assume ZFC

(4.2.1) **Def.** An ordinal α is a cardinal if it is not equinumerous with any ordinal $\beta < \alpha$.

Eg - If $n \in \omega$ then n is a cardinal

- ω is a cardinal
- If γ any infinite ordinal then $\gamma \approx \gamma^+$ ($\approx$ equinumerous) so γ^+ is not a cardinal. (P. sheets 7 + 8).

(4.2.2) **Lemma.** Suppose A is any set. Then there is a unique cardinal α with $\alpha \approx A$.

(4.2.3) **Def:** Here α is called the cardinality of A, ~~dete~~ denoted by $\mathrm{card}(A)$ or $|A|$.

(Ex: there is an injective fn. $f : A \to B$ iff $\mathrm{card}(A) \leq \mathrm{card}(B)$.

So using $|A|$ for $\mathrm{card}(A)$ is ~~constst~~ consistent with previous.)

Pf of 4.2.2: By 4.1.6 there is some ordinal $\alpha \approx A$.
Take α to be the least such ordinal. Then α is a cardinal.

#.

Eg. (1) If A is a countably infinite set then $|A| = \omega$

(2) If α is any ordinal then $|\alpha| = \alpha \iff \alpha$ is a cardinal.

(3) Define the 'sequence of alephs' $\aleph_\alpha$ α ordinal is defined using transfinition recursion as:

Each $\aleph_\alpha$ is a cardinal,

$\aleph_0 = \omega$

$\aleph_0 < \aleph_1 < \cdots < \aleph_\alpha < \cdots$

$\aleph_\alpha$ is the least cardinal which is $> \aleph_\beta$ for all $\beta < \alpha$.

[Exists: by Cantor's thm:
$$\left|\mathcal{P}\left(\bigcup_{\beta<\alpha} \aleph_\beta\right)\right| > \aleph_\beta \quad \forall \beta < \alpha]$$

(4.2.4) **Def.** (Cardinal Arithmetic)

Suppose A, B are disjoint sets with $|A| = \kappa$ & $|B| = \lambda$ (so κ, λ are cardinals).

Let $\kappa + \lambda$ be $|A \cup B|$

$\kappa \cdot \lambda$ be $|A \times B|$

and κ^λ be $|A^B|$

Rk: Doesn't depend on choice of A, B.

(4.2.5) **Theorem**. Suppose κ, λ are cardinals with $\kappa \leq \lambda$ and λ infinite. Then

(i) $\kappa + \lambda = \lambda$

(ii) $\kappa \cdot \lambda = \lambda$ (if $\kappa \neq 0$)

(iii) If $2 \leq \kappa \leq \lambda$ then $2^\lambda = \kappa^\lambda$.

Pf: (ii) As $\kappa \leq \lambda$ we have $\kappa \subseteq \lambda$, so

$$\kappa \cdot \lambda = |\kappa \times \lambda| \leq |\lambda \times \lambda| \underset{4.1.6}{=} |\lambda| = \lambda.$$

As $\kappa \neq 0$, $0 \in \kappa$ so there is an injective fn. $\lambda \to \kappa \times \lambda$

$$\beta \mapsto (0, \beta)$$

so $\lambda = |\lambda| \leq |\kappa \times \lambda| = \kappa \cdot \lambda$.

Thus $\lambda = \kappa \cdot \lambda$. //

(i) $\lambda \leq \kappa + \lambda \leq \lambda + \lambda$

$\approx \{0,1\} \times \lambda$

$$|\{0,1\} \times \lambda| = 2 \cdot \lambda \overset{(ii)}{=} \lambda.$$

So $\lambda = \kappa + \lambda$. (~~#~~, (i))

(iii) Problem sheet 8. //

(4.2.6) <u>Thm</u>. Suppose A is an infinite set of cardinality λ. Suppose each elt. of A is a (non-empty) set of cardinality $\leq \kappa$. Then

$$\left| \bigcup A \right| \leq \kappa \cdot \lambda .$$

<u>Pf of 4.2.6</u>.

For $a \in A$ the set S_a of surjective functions $\kappa \to a$ is non-empty (by $|a| < \kappa$ & 4.1.7).

Assuming AC there is a function $F : A \to \bigcup S_a$ which $F(a) \in S_a \ \forall a \in A$.

So for all $a \in A$, $F(a) : \kappa \to a$ is surjective.

Let $h : \lambda \to A$ be a bijection.

Define $g : \lambda \times \kappa \to \bigcup A$

by $g(\alpha, \beta) = F(h(\alpha))(\beta)$

$\therefore$ (for $\alpha < \lambda$ & $\beta < \kappa$).

This is a surjective function.

So by 4.1.7

$$|\lambda \times \kappa| \geq \left| \bigcup A \right|$$

ie. $\therefore$ $\left| \bigcup A \right| \leq \lambda \cdot \kappa$. #.

$\underset{a}{h(a)} \in A$ $\qquad$ $F(h(\alpha)) : \underset{\beta}{\kappa} \to h(\alpha)$