Primer on the Calculus of Variations*

References

A readable introduction to the Calculus of Variations is Chapter 9 of 'Mathematical Methods in the Physical Sciences', 3rd edition, by M. L. Boas. See also section 2.2 of 'Classical Mechanics', 3rd edition, by Goldstein, Safko and Poole.

From elementary calculus the length, ℓ, of the curve $y = f(x)$ for $a \leq x \leq b$ is given by the integral

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1}$$

Now suppose that $y(a)$ and $y(b)$ are fixed. A simple problem in the Calculus of Variations is what curve joining $(a, y(a))$ and $(b, y(b))$ minimises the length ℓ? Clearly the answer is

$$y(x) = y(a) + \frac{\big[y(b) - y(a)\big](x - a)}{b - a},$$

which is a line segment joining the end points. Here we have an integral that depends on a function and seek the function that minimises the integral.

In the Brachistochrone problem a bead moves without friction on a wire under gravity. Assuming the bead is released from rest from a fixed starting point, what shape should the wire be to reach a fixed end point in the shortest time? Although a straight wire minimises the distance it does not minimise the travel time. The shape of the wire can be represented by the graph $y = y(x)$ as in the graph below (with the y axis pointing down). The bead starts at the origin $\mathcal{O}$ and reaches the end point (b, h) in time T. For a small segment of the wire $\delta t = \delta\ell/v$ where v is the speed of the bead and $\delta\ell$ is the length of the segment. The speed can by fixed using conservation of energy. The potential energy of the bead is $-mgy$ so the total energy is

$$E = \frac{1}{2}mv^2 - mgy.$$

$E = 0$ since the bead is released from rest ($v = 0$ for $y = 0$) giving $v = \sqrt{2gy}$. Accordingly,

$$\delta t = \frac{\delta\ell}{\sqrt{2gy}} = \sqrt{\frac{1 + y'^2}{2gy}}\,\delta x.$$
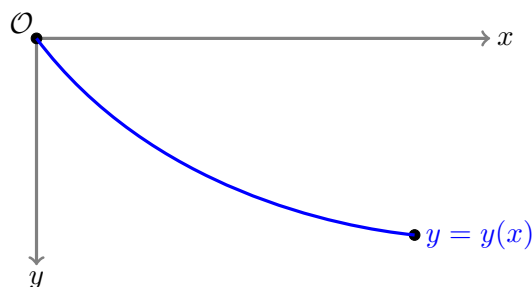

The total time is

$$t = \frac{1}{\sqrt{2g}} \int_0^b \sqrt{\frac{1 + [y'(x)]^2}{y}} \, dx.$$

We need to find the y which minimizes the integral

$$\int_0^b \sqrt{\frac{1 + [y'(x)]^2}{y}} \, dx,$$

with $y(0) =$ and $y(b) = h$.

Now consider more general integrals of the form

$$S = \int_a^b L(y, y', x) \, dx,$$

where the integrand or *Lagangian*, L, is a function of y, y' and x. The integral S is stationary with respect to variations in y, assuming $y(a)$ and $y(b)$ to be fixed, if the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0, \tag{2}$$

holds. This is the key equation of this module. When computing the partial derivative $\partial L/\partial y$ and $\partial L/\partial y$, the Lagrangian is treated as a function of three variables y, y' and x.
Lagrangians for the shortest curve and Brachistochrone problems are

$$L = \sqrt{1 + y'^2} \text{ and } L = \sqrt{\frac{1 + y'^2}{y}},$$

respectively.

For the shortest curve Lagrangian

$$\frac{\partial L}{\partial y} = 0, \quad \text{and} \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}},$$

giving the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0,$$

so that $y'/\sqrt{1+y'^2}$ and hence y' is constant, as expected for a straight line.

The Euler-Lagrange equation for the Brachistochrone problem is more complicated

$$\frac{d}{dx}\left(\frac{y^{-1/2}\,y'}{\sqrt{1+y'^2}}\right) + \frac{1}{2}y^{-3/2}\sqrt{1+y'^2} = 0.$$

With a little effort this can be written in the simpler form $2yy'' + 1 + y'^2 = 0$, which can be derived from the first order ODE

$$y(1 + y'^2) = \text{constant}, \tag{3}$$

The solution of this equation represents a cycloid. This can be verified using the parametric form

$$x = R(t - \sin t) \ \text{ and } \ y = R(1 - \cos t),$$

which represents a cycloid with radius R.

We now consider in detail why the Euler Lagrange equation (2) is a stationary condition for the integral
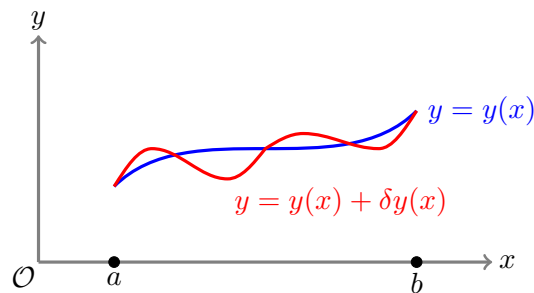
$$S = \int_a^b L(y, y', x)\,dx,$$

assuming fixed values of $y(a)$ and $y(b)$. It is instructive to recall the properties of stationary points of a function of one variable f. Consider a small change (or variation) in the argument

$$f(x + \delta x) = f(x) + f'(x)\delta x + \cdots.$$

If x is a stationary point, $f'(x) = 0$, and so at a stationary point a small variation in the argument has no effect.

With this in mind, consider some small variation in the function $y = y(x)$, say $\delta y(x)$, with $\delta y(a) = \delta y(b) = 0$.

Then the change in the integral S is given by

$$\delta S = \int_a^b L\left(y(x) + \delta y(x), y'(x) + \frac{d}{dx}\delta y(x), x\right) dx - \int_a^b L\big(y(x), y'(x), x\big) dx$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\big(y(x), y'(x), x\big)\delta y(x) + \frac{\partial L}{\partial y'}\big(y(x), y'(x), x\big)\frac{d}{dx}\delta y(x)\right] dx,$$

where the first of these integrals has been Taylor expanded about variations in the first and second arguments using

$$f(x + \delta x, y + \delta y, z) = f(x, y, z) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \dots$$

Applying integration by parts to the second term yields

$$\delta S = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y \; dx + \frac{\partial L}{\partial y'}\delta y\bigg|_a^b$$

$$= \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y \; dx,$$

as a consequence of the boundary conditions. If the Euler-Lagrange equation holds this is zero for arbitrary small variations δy satisfying the end point conditions $\delta y(a) = \delta y(b) = 0$.

In general, the Euler-Lagrange equation is a non-linear 2nd order differential equation. However, provided L has no explicit x-dependence, the equation can be integrated using the *Beltrami Formula* to yield a first order differential equation. If L has no explicit x-dependence and $y = y(x)$ is a stationary 'point' of S, then

$$H = y'\frac{\partial L}{\partial y'} - L = \text{constant}. \tag{4}$$

The proof is left as an exercise.

Returning to the Brachistochrone problem with Lagrangian $L = y^{-1/2}(1 + y'^2)^{1/2}$. This clearly has no explicit x-dependence, and so it follows from the Beltrami formula that

$$H = y'\frac{\partial L}{\partial y'} - L$$

$$= y' \cdot y^{-1/2} \cdot \frac{1}{2}(1 + y'^2)^{-1/2} \cdot 2y' - y^{-1/2}(1 + y'^2)^{1/2}$$

$$= -\frac{1}{y^{1/2}(1 + y'^2)^{1/2}},$$

is constant, which is equivalent to (3). The point is that the Beltrami result is an explicit formula for a constant quantity. When applied to mechanics Beltrami's formula is the conservation of energy.

Finding stationary points subject to constraints is a standard problem in multi-variable calculus. Here one can either (a) solve the constraint thereby reducing the number of variables *or* (b) add additional dummy variables (Lagrange multipliers) to implement the constraints. A simple example is: What is the minimum distance between the origin and the parabola $y = 1 - x^2$? This is essentially asking us to minimise the distance squared $Q = x^2 + y^2$, subject to the constraint $y = 1 - x^2$. Two

4

solutions are

(a) Using the constraint $Q = 1 - y + y^2$ with minimum value $\frac{3}{4}$.

(b) Consider $f(x, y, \lambda) = x^2 + y^2 + \lambda(y - 1 + x^2)$ including the Lagrange multiplier λ. The function f has stationary points at $(x, y, \lambda) = (1/\sqrt{2}, \frac{1}{2}, -1)$ and $(x, y, \lambda) = (0, 1, -2)$. The former yields a minimum value of $\frac{3}{4}$.

In the Calculus of Variations, constraints can be solved or implemented through Lagrange multipliers. An example of using a Lagrange multiplier in the Calculus of Variations is to show that the shape of a hanging rope is a catenary. As in previous examples the shape of the rope is represented by a function $y = y(x)$ for $a \le x \le b$ with $y(a)$ and $y(b)$ fixed. We now express the potential energy, V, of the rope as an integral. The potential energy of a small segment is $\delta V = \delta m \ gy$ where δm is the mass of the segment. Now $\delta m = \mu \delta \ell = \mu \sqrt{1 + y'^2} \delta x$, where μ is the mass per unit length of the rope (assumed to be constant). The total potential energy of the rope is

$$V = \mu g \int_a^b y\sqrt{1 + y'^2} \ dx.$$

We would like to minimise V for fixed $y(a)$ and $y(b)$ subject to the constraint that the length of the rope,

$$\ell = \int_a^b \sqrt{1 + y'^2} \ dx = \text{constant}.$$

Consider

$$S = \int_a^b y\sqrt{1 + y'^2} \ dx + \lambda \left[\int_a^b \sqrt{1 + y'^2} \ dx - \ell \right] = \int_a^b L \ dx,$$

where the Lagrangian

$$L = (y + \lambda)\sqrt{1 + y'^2} - \frac{\lambda \ell}{b - a},$$

includes the Lagrange multiplier λ. The constant term drops out of the Euler-Lagrange equation so one can use the Lagrangian $L = (y + \lambda)\sqrt{1 + y'^2}$. Using the Beltrami formula (or directly solving the Euler-Lagrange equation) it can be shown that the solution is a catenary

$$y + \lambda = \frac{1}{p}\cosh(px + \alpha),$$

Note that this includes *three* constants p, α and the Lagrange multiplier. These are fixed by the end point conditions and the length of the rope.

In the hanging rope problem the Lagrange multiplier, λ, is a constant. We will see that Lagrange multipliers may also be functions. Constant Lagrange multipliers in the Calculus of Variations are useful in Isoperimetric problems. For example, what is the largest area that can be enclosed by a closed curve of fixed length? For a related question see the exercise 3 below.

Exercises

1. Show that if the integral

$$S = \int_a^b L(y(x), y'(x))dx$$

is stationary for fixed $y(a)$ and $y(b)$ then

$$H = y'\frac{\partial L}{\partial y'} - L,$$

is a constant.

Remark: the result is not true if L has explicit x-dependence, i.e. if it is a function of y, y' and x.

2. The area of a surface of revolution obtained by rotating the curve $y = y(x)$ around the x-axis for $a \leq x \leq b$ is

$$A = 2\pi \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx.$$

(i) Take the end points to be $-L/2$ and $L/2$. A is to be minimised with $y(-L/2) = y(L/2) = R$ where R is a positive constant. Write down the Euler-Lagrange equation for $y(x)$. Show that $y(x)$ has the form $y(x) = p^{-1}\cosh(px)$ where p is constant. This is a catenary.

(ii) Show that if R/L is too small the solution from part (i) cannot be matched to the boundary conditions (a graphical plot may help here).

3. A curve of fixed length has fixed endpoints on a line. Show that the area enclosed by the curve and the line is maximised if the curve is circular.

Hint: Take the line to be the x-axis and the endpoints to be $(a, 0)$ and $(b, 0)$. Maximize the integral

$$\int_a^b y(x)dx \quad \text{with} \quad y(a) = y(b) = 0,$$

subject to the constraint that the length of the curve is fixed.