Math40002 Analysis 1 — Problem Sheet 5

1. Fix $x > 0$. Prove $(1+x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$. Deduce that $(1+x)^{-n} \to 0$. Deduce that if $r \in (0,1)$ then $r^n \to 0$, and if $r \in (1,\infty)$ then $r^n \to \infty$.

   By the binomial theorem, $(1+x)^n = 1+nx+\ldots \geq 1+nx$ because $\ldots$ is all > 0 (or empty for $n = 0, 1$).

   Hence $|(1+x)^{-n} - 0| \leq 1/(1+nx)$. Now

   $$1/(1+nx) < \epsilon \iff n > (\epsilon^{-1} - 1)/x.$$

   So given any $\epsilon > 0$ we pick $N > (\epsilon^{-1} - 1)/x$ so that

   $$n \geq N \Rightarrow |(1+x)^{-n} - 0| \leq 1/(1+nx) < \epsilon.$$

   We can write $r = (1+x)^{-1}$ by setting $x := r^{-1} - 1 > 0$, then apply previous result.

   If $r \in (1,\infty)$ then fix any $R > 0$. Now use the first part of the question to see that $r^n \geq 1+n(r-1) \geq R$ for all $n \geq \frac{R-1}{r-1}$. That is, $r^n \to \infty$.

2. Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. In lectures we proved that if $L < 1$ then $a_n \to 0$.

   (a) Prove that if $L > 1$ then $|a_n| \to \infty$.

   (b) Give an example with $|a_{n+1}/a_n| < 1\ \forall n$ but $a_n \not\to 0$.

   Give (without proof) examples where $L = 1$ and

   (i) $a_n \to 0$, (iii) a_n divergent and bounded,

   (ii) $a_n \to a \neq 0$, (iv) $a_n \to \infty$.

   (a) If $L > 1$ then set $\epsilon = (L-1)/2 > 0$. Then $\exists N$ such that $\forall n \geq N$ we have $|a_{n+1}/a_n - L| < (L-1)/2$ and in particular $|a_{n+1}|/|a_n| > L - (L-1)/2 = (L+1)/2 > 1$.

   Let $\alpha := (L+1)/2 > 1$. Therefore we find inductively that $|a_{N+m}| > \alpha^m |a_N|$. But $\alpha^m \to \infty$ as $m \to \infty$ by previous question. In particular if we fix any $R > 0$ then $\exists M$ such that $\forall m \geq M$ we have $\alpha^m > R/|a_N|$.

   Putting it altogether we find that $\forall n \geq N + M$ we have $|a_n| > (R/|a_N|)|a_N| = R$. Thus $|a_n| \to \infty$ as $n \to \infty$.

   (b) Example: $a_n = 1 + 1/n$.

   (i) $a_n = 1/n$

   (ii) $a_n \equiv a$

   (iii) $a_n = (-1)^n$

   (iv) $a_n = n$

3. Let $(a_n)_{n\geq 1}$ be a sequence of *strictly positive* real numbers. Give an example such that $(1/a_n)_{n\geq 1}$ is unbounded. Suppose that $a_n \to a \neq 0$. Prove *from first principles* that $(1/a_n)_{n\geq 1}$ is bounded.

   Any example like $a_n = 1/n$ will do.

   Let $\epsilon = a/2 > 0$. Then $\exists N \in \mathbb{N}$ such that

   $$n \geq N \Rightarrow |a_n - a| < \epsilon \Rightarrow a_n > a - \epsilon = a/2 \Rightarrow 1/a_n < 2/a.$$

   Therefore $0 < 1/a_n \leq \max\left(a_1^{-1}, a_2^{-1}, \ldots, a_{N-1}^{-1}, 2/a\right)\ \forall n$ and so is bounded.

4.† Fix $r \in (0, 1/8)$. Define $(a_n)_{n\geq 1}$ by $a_1 := 1$ and $a_{n+1} = ra_n^2 + 1$.

   (a) Show that $a_{n+1} - a_n = r(a_n + a_{n-1})(a_n - a_{n-1})$.

   This is just $a_{n+1} - a_n = ra_n^2 - ra_{n-1}^2 = r(a_n + a_{n-1})(a_n - a_{n-1})$.

(b) Show that if $$0 < a_j < 2 \qquad \forall j \le n, \tag{1}$$ then $$|a_{n+1} - a_n| \; < \; (4r)^n/4. \tag{2}$$

Use $|a_{n+1} - a_n| < r(2+2)|a_n - a_{n-1}| = 4r|a_n - a_{n-1}| \le (4r)^2|a_{n-1} - a_{n-2}| \le \ldots \le (4r)^{n-1}|a_2 - a_1|$. But this equals $(4r)^{n-1}(r+1-1) = (4r)^n/4$, as required.

(c) Deduce that if (1) holds, then $a_{n+1} < r/(1-4r) + 1$.

By the triangle inequality, $a_{n+1} \le |a_{n+1} - a_n| + |a_n - a_{n-1}| + \ldots + |a_2 - a_1| + |a_1|$, which is $< \frac{1}{4}\Big((4r)^n + (4r)^{n-1} + \ldots + 4r\Big) + 1 \le r/(1-4r) + 1$ because $4r < 1$.

(d) Conclude that (1) holds for $j = n+1$ too, and so $\forall j$ by induction.

Since $r < 1/8$ we have $r/(1-4r) + 1 < 2$. (It is clear from the definition that $a_n > 0 \; \forall n$.)

(e) Using (2) deduce $|a_m - a_n| < (4r)^n/4(1-4r)$ for $m \ge n$.

By the same triangle inequality argument, $|a_m - a_n| < (4r)^{m-1}/4 + \ldots + (4r)^n/4$ which again is $\le (4r)^n/4(1-4r) \le (4r)^n/2$.

From Q1 $(4r)^n \to 0$ as $n \to \infty$ since $0 < 4r < 1$.

So $\forall \epsilon > 0 \; \exists N \in \mathbb{N}$ such that $n \ge N \;\Rightarrow\; (4r)^n < \epsilon \;\Rightarrow\; |a_m - a_n| < \epsilon/2$ for $m \ge n \ge N$. Thus a_m is Cauchy and so convergent.

(f) Deduce a_n is Cauchy. What does it converge to?

Let a be $\lim_{n\to\infty} a_n$. Taking limits in $a_{n+1} = ra_n^2 + 1$ gives $a = ra^2 + 1$ so that $a = \frac{1 \pm \sqrt{1-4r}}{2r}$. Then $\pm$ cannot be $+$ because we know from (1) that $a \in [0,2]$. So $a = \frac{1-\sqrt{1-4r}}{2r}$.

5.* Show that *any* sequence of real numbers $(a_n)_{n\ge0}$ has a subsequence which either converges, or tends to ∞, or tends to $-\infty$.

If (a_n) is bounded, it has a convergent subsequence by Bolzano-Weierstrass. Suppose instead it is unbounded above; we will show it has a subsequence tending to ∞ (unbounded below and $-\infty$ is similar).

We define a_{n_i} recursively such that $a_{n_i} > i$. Since 1 is not an upper bound, there is an $n_1 \in \mathbb{N}$ such that $a_{n_1} > 1$, so the recursion begins.

Assuming we've defined $n_1 < \ldots < n_i$ such that $a_{n_i} > i$, we need to define n_{i+1}. But $i+1$ is not an upper bound for the set $\{a_n : n > n_i\}$ (if it were then (a_n) would be bounded above by $\max(i+1, a_1, a_2, \ldots, a_{n_i})$.) So we can pick $a_{n_{i+1}} > i+1$ in it this set, as required.

Now given any $R \in \mathbb{R}$ pick $N \in \mathbb{N}$ with $N > R$. Then $\forall i \ge N$ we have $a_{n_i} > i \ge N > R$, which is the definition of $a_{n_i} \to \infty$.

6. At home Professor Papageorgiou has made a fully realistic mathematical model of a dart board. It is a copy of the unit interval $[0,1]$ in a frictionless vacuum. He throws a countably infinite number of darts at it, the nth landing at $a_n \in [0,1]$.

He then makes a small dot $(x - \epsilon_x, \, x + \epsilon_x)$ around each point $x \in [0,1]$ with his pen. Prove that however small he makes each dot, at least one of them will contain an infinite number of darts $a_n \in [0,1]$.

What if he only makes dots around each dart $a_n \in [0,1]$?

By Bolzano-Weierstrass there exists a subsequence b_n of the a_n which is convergent to some $b \in [0,1]$. Therefore consider any neighbourhood $(b - \epsilon_b, b + \epsilon_b)$ of the limit. There exists $N \in \mathbb{N}$ such that $b_n \in (b - \epsilon_b, b + \epsilon_b) \; \forall n \ge N$, so there are an *infinite* number of darts in this dot.

For some sequences (a_n) it is possible to find a neighbourhood of each dart with only finitely many darts in it. Eg if $a_n = 1/n$ then we can choose the neighbourhood $(1/(n+1), 1/(n-1))$ of a_n.

For some it is not; eg if $a_1 = 0$ and $a_n = 1/n$ for $n > 1$ – then any neighbourood of a_1 has infinitely many darts.

The general condition is that no point a_n of the sequence should be a limit of any subsequence.

7. Let $(a_n)_{n\geq 1}$ be the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$

   (i) Give (without proof) a subsequence of $(a_n)_{n\geq 1}$ which converges to $\ell = 0$, and one which converges to $\ell = 1$.

   (ii) Given any $\ell \in (0, 1)$, give (with proof) a subsequence convergent to ℓ.

(i) The subsequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ converges to $\ell = 0$.

The subsequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ converges to $\ell = 1$.

(ii) Let $\ell_n/10^n$ ($\ell_n \in \mathbb{N}$) be the decimal expansion of ℓ truncated at the nth decimal place. Since $\ell \neq 0$, the decimal expansion is nonzero so there is a k such that $\ell_k \neq 0$. Now take the subsequence of $(a_n)_{n\geq 1}$ given by

$$\frac{\ell_k}{10^k}, \frac{\ell_{k+1}}{10^{k+1}}, \ldots.$$

Notice we *do not* cancel the fractions into lower terms – the denominators must keep increasing so the ith term a_{n_i} satisfies that $n_i < n_{i+1}$ – i.e. subsequences always "move to the right" in the original sequence. By its definition, $|\ell_n/10^n - l| \leq 10^{-n}$. So given any $\epsilon > 0$, choose $N > 1/\epsilon$ and

$$\left| \frac{\ell_n}{10^n} - \ell \right| \leq 10^{-n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$. So the subsequence $\to \ell$, as required.

8. A student is learning about Cauchy sequences, and thinks they have a brilliant proof that allows them to precisely identify the limit of a Cauchy sequence straight from the Cauchy condition. The student gives their proof below, is it correct?

$$\forall \epsilon > 0 \; \exists \, N \in \mathbb{N} \text{ such that } n, m \geq N \;\Rightarrow\; |a_n - a_m| < \epsilon$$

$$\Rightarrow\; \forall n \geq N \quad |a_n - a_N| < \epsilon$$

$$\Rightarrow\; a_n \to a_N \quad \text{as} \quad n \to \infty.$$

The problem is that N can depend on ϵ; we only found N after fixing ϵ. So they only prove that $|a_n - a_N| < \epsilon$ for a fixed $\epsilon > 0$. To prove that $a_n \to a_N$ we need to prove $|a_n - a_N| < \epsilon$ for *any* $\epsilon > 0$, so we need to be able to change ϵ, but that may change N and so the "limit" a_N.