Imperial College London

MATH97023 MATH70017

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Tensor Calculus and General Relativity

Date: 08 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Formula Sheet

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b.$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad R = g^{bd} R_{bd}.$$

1. (a) The standard Lorentz boost from an inertial frame K (with coordinates x^μ) to another inertial frame translating with speed v in the positive x^1-direction is

$$x^{0\prime} = \gamma\left(x^0 - \frac{v}{c}x^1\right), \quad x^{1\prime} = \gamma\left(x^1 - \frac{v}{c}x^0\right), \quad x^{2\prime} = x^2, \quad x^{3\prime} = x^3,$$

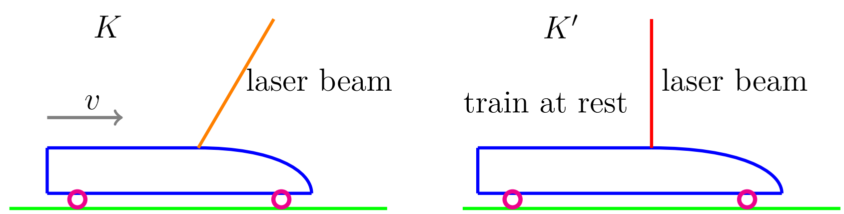
where $\gamma = (1 - v^2/c^2)^{-1/2}$. Write down the inverse transformation. (4 marks)

(b) Define four-momentum for massive and massless particles. Give the components of four-momentum for a photon of energy E moving in the x^3-direction.

(6 marks)

(c) A laser at rest generates photons of energy E. The laser is mounted on a train with constant speed v in the x^1-direction of an inertial frame K. The laser emits photons vertically (in the $x^{3\prime}$ direction of a frame K' where the train is at rest). What is the four-momentum of the photons in frame K?

Hint: Use the inverse Lorentz transformation.



(6 marks)

(d) How fast should the moving train from part (c) be so that the direction of the photons is at 30 degrees to the vertical?

(4 marks)

(Total: 20 marks)

2. The metric on a two-dimensional surface (a semi-infinite cigar) with coordinates ρ and θ is

$$ds^2 = d\rho^2 + \tanh^2 \rho \; d\theta^2.$$

Here $\rho \geq 0$ and $\theta \equiv \theta + 2\pi$ is an angle.

The non-zero Christoffel symbols are

$$\Gamma^{\theta}_{\rho\theta} = \Gamma^{\theta}_{\theta\rho} = \frac{1}{\sinh \rho \cosh \rho}, \qquad \Gamma^{\rho}_{\theta\theta} = -\frac{\sinh \rho}{\cosh^3 \rho}.$$

(a) (i) Write down the geodesic equation for this surface. (4 marks)

(ii) Show that the circles defined by $\rho = R$, where R is a positive constant, are not geodesics. (4 marks)

(b) (i) Let ϕ be a scalar field. Compute the Laplacian of ϕ on the semi-infinite cigar, i.e. determine

$$g^{ab} \nabla_a \nabla_b \phi.$$

(8 marks)

(ii) Evaluate $g^{ab} \nabla_a \nabla_b \phi$ where $\phi = \log(\sinh \rho)$. (4 marks)

(Total: 20 marks)

3. The metric on the unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta \; d\phi^2,$$

where θ and ϕ are spherical polar coordinates. The non-zero Christoffel symbols are

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

(a) (i) Write down the non-zero components of the metric and inverse metric. (3 marks)

(ii) Verify that $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$. (5 marks)

(b) Consider the family of geodesics (great circles)

$$\theta = s, \quad \phi = t.$$

Here s gives the distance along the geodesic and the parameter t fixes the geodesic. Consider

$$w^\theta = \frac{\partial\theta}{\partial t} = 0, \quad w^\phi = \frac{\partial\phi}{\partial t} = 1.$$

This satisfies the equation of geodesic deviation

$$\frac{D^2 w^a}{\partial s^2} = -R^a_{bcd} u_b w_c u_d,$$

where $u^a = \partial x^a/\partial s$ and $w^a = \partial x^a/\partial t$; here $u^\theta = 1$, $u^\phi = 0$. The above derivative is defined through

$$\frac{Dv^a}{\partial s} = \frac{\partial v^a}{\partial s} + \Gamma^a_{bc} v^b u^c.$$

(i) Determine

$$\frac{Dw^\phi}{\partial s},$$

for the family of geodesics. (5 marks)

(ii) Determine

$$\frac{D^2 w^\phi}{\partial s^2},$$

and use the result to obtain $R^\phi_{\theta\phi\theta}$. (7 marks)

(Total: 20 marks)

4. The Schwarzschild metric in the equatorial plane ($\theta = \pi/2$) is

$$ds^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \frac{dr^2}{1 - R/r} - r^2 d\phi^2,$$

where R is the Schwarzschild radius.

(a) Show that

$$k = \left(1 - \frac{R}{r}\right) \frac{dt}{ds} \quad \text{and} \quad h = r^2 \frac{d\phi}{ds},$$

are constant along (time-like) geodesics. Hence, show that geodesics in the equatorial plane satisfy

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{R}{r}\right) - \frac{R}{r} = c^2 k^2 - 1. \tag{1}$$

(8 marks)

(b) The equation corresponding to (1) for null geodesics is

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{\tilde{h}^2}{r^2}\left(1 - \frac{R}{r}\right) = c^2 \tilde{k}^2,$$

where λ is a parameter. Here

$$\tilde{k} = \left(1 - \frac{R}{r}\right) \frac{dt}{d\lambda} \quad \text{and} \quad \tilde{h} = r^2 \frac{d\phi}{d\lambda}$$

are constants along null geodesics. Show that radially infalling photons ($\tilde{h} = 0$) satisfy

$$\frac{dr}{dt} = -c\left(1 - \frac{R}{r}\right).$$

Solve this equation to show that

$$ct = -r - R\log\left(\frac{r}{R} - 1\right) + \text{constant}.$$

(7 marks)

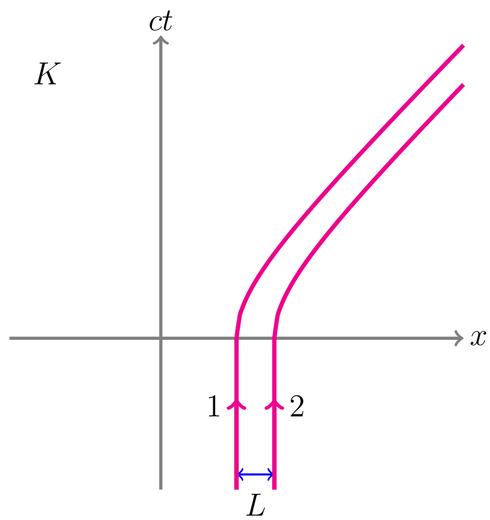
(c) Express ct as a function of r for a massive infalling ($h = 0$) particle assuming that $k = 1/c$.
Hint:

$$\int \frac{x^{3/2}\, dx}{x - 1} = \frac{2}{3}x^{3/2} + 2x^{1/2} - 2\tanh^{-1}\left(\frac{1}{\sqrt{x}}\right) + \text{constant}.$$

(5 marks)

(Total: 20 marks)

5. (a) In an inertial frame K with coordinates t and x two identical spacecraft are at rest. At $t = 0$ both spacecraft accelerate uniformly along the x-axis as shown in the space-time diagram below.



The trajectory of the first vehicle is

$$t = \frac{c}{A}\sinh(A\tau_1/c), \quad x = \frac{c^2}{A}\cosh(A\tau_1/c), \quad (\tau_1 \geq 0)$$

where τ_1 is the proper time and A is a positive constant.
The trajectory of the second vehicle is

$$t = \frac{c}{A}\sinh(A\tau_2/c), \quad x = \frac{c^2}{A}\cosh(A\tau_2/c) + L, \quad (\tau_2 \geq 0)$$

where L is a positive constant (the spatial separation of the spacecraft in K) and τ_2 is the proper time for the second vehicle. At what τ_1 is the first spacecraft unable to send a signal (travelling at the speed of light) to the second spacecraft? (10 marks)

(b) Write the Schwarzschild metric

$$ds^2 = c^2\left(1 - \frac{R}{r}\right)dt^2 - \frac{dr^2}{1 - R/r} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

in the coordinates (t, x, y, z) where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

and t is unchanged. Hint: $dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\ d\phi^2 = dx^2 + dy^2 + dz^2$.

(10 marks)

(Total: 20 marks)

Answers to June 2022 Examination

1. (a)

$$x^0 = \gamma\left(x^{0'} + \frac{v}{c}x^{1'}\right), \quad x^1 = \gamma\left(x^{1'} + \frac{v}{c}x^{0'}\right), \quad x^2 = x^{2'}, \quad x^3 = x^{3'},$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. **(4 marks, seen, A)**

(b) For a massive particle four-momentum is defined through $p^\mu = mu^\mu$ where $u^\mu = dx^\mu/d\tau$ is the four-velocity and m is the mass. For massless particles p^μ is null and p^0 is identified as E/c, i.e. $p = (E/c, \mathbf{p})$ where $|\mathbf{p}| = E/c$. For a photon of energy E moving in the x^3-direction $p = (E/c, 0, 0, E/c)$.

(6 marks, bookwork, A)

(c) In frame K' the components of the four-momentum are $p^{0'} = E/c, p^{1'} = p^{2'} = 0, p^{3'} = E/c$.

Using the inverse Lorentz boost from part (a) the components of the (contravariant) four-momentum in frame K are

$$p^0 = \gamma\left(p^{0'} + \frac{v}{c}p^{1'}\right) = \gamma E/c, \quad p^1 = \gamma\left(p^{1'} + \frac{v}{c}p^{0'}\right) = \gamma v E/c^2,$$
$$p^2 = p^{2'} = 0, \quad p^3 = p^{3'} = E/c.$$

(5 marks, seen similar, B)

(d)

$$\frac{1}{\sqrt{3}} = \tan 30^\circ = \frac{p^1}{p^3} = \frac{\gamma v}{c},$$

so that $\frac{1}{3} = v^2/(c^2 - v^2)$ or $v^2 = \frac{1}{4}c^2$, giving $v = c/2$ (half the speed of light).

(5 marks, unseen, C)

(Total: 20 marks)

2. (a) (i) geodesic equations

$$\frac{d^2\rho}{ds^2} - \frac{\sinh\rho}{\cosh^3\rho}\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{\sinh\rho\cosh\rho}\frac{d\rho}{ds}\frac{d\theta}{ds} = 0.$$

(4 marks, seen similar, A)

(ii) Inserting $\rho = R$=constant into the first geodesic equation gives $d\theta/ds = 0$ which is not compatible with $u \cdot u = (d\rho/ds)^2 + \tanh^2\rho(d\theta/ds)^2 = \tanh^2\rho(d\theta/ds)^2 = 1$.

(4 marks, seen similar, A)

(b) (i) $g^{ab}\nabla_a\nabla_b\phi = g^{ab}\nabla_a\partial_b\phi = g^{\rho\rho}\nabla_\rho\partial_\rho\phi + g^{\theta\theta}\nabla_\theta\partial_\theta\phi$

Using $\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc}v_a$

$$\nabla_\rho\partial_\rho\phi = \partial^2_\rho\phi - \Gamma^a_{\rho\rho}\partial_a\phi = \partial^2_\rho\phi.$$

Similarly,

$$\nabla_\theta\partial_\theta\phi = \partial^2_\theta\phi - \Gamma^a_{\theta\theta}\partial_a\phi = \partial^2_\theta\phi - \Gamma^\rho_{\theta\theta}\partial_\rho\phi = \partial^2_\theta\phi + \frac{\sinh\rho}{\cosh^3\rho}\partial_\rho\phi.$$

Accordingly,

$$g^{ab}\nabla_a\nabla_b\phi = \nabla_\rho\partial_\rho\phi + \coth^2\rho\nabla_\theta\partial_\theta\phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\sinh\rho\cosh\rho}\frac{\partial}{\partial\rho} + \coth^2\rho\frac{\partial^2}{\partial\theta^2}\right)\phi.$$

(8 marks, seen similar, C)

(ii) $\phi = \log(\sinh\rho)$, $\partial_\rho\phi = \coth\rho$, $\partial^2_\rho\phi = -1/\sinh^2\rho$.

$$g^{ab}\nabla_a\nabla_b\phi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\sinh\rho\cosh\rho}\frac{\partial}{\partial\rho}\right)\phi = -\frac{1}{\sinh^2\rho} + \frac{1}{\sinh\rho\cosh\rho}\coth\rho = 0.$$

(4 marks, unseen, D)

(Total: 20 marks)

3. (a) (i) The non-zero components of the metric and inverse metric are $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$, $g^{\theta\theta} = 1$, $g^{\phi\phi} = 1/\sin^2\theta$.

(3 marks, seen similar, A)

(ii) Using the formula for Γ^a_{bc},

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2}g^{\theta d}\left(\partial_\phi g_{\phi d} + \partial_\phi g_{d\phi} - \partial_d g_{\phi\phi}\right) = \frac{1}{2}g^{\theta\theta}(0 + 0 - \partial_\theta g_{\phi\phi}) = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta.$$

(5 marks, seen similar, A)

(b) (i)

$$\frac{Dw^\phi}{\partial s} = \frac{\partial w^\phi}{\partial s} + \Gamma^\phi_{bc}w^b u^c = 0 + \Gamma^\phi_{\phi\theta}w^\phi u^\theta = \Gamma^\phi_{\phi\theta} = \cot s.$$

(5 marks, unseen, B)

(ii)

$$\frac{D^2 w^\phi}{\partial s^2} = \frac{\partial}{\partial s}\frac{Dw^\phi}{\partial s} + \Gamma^\phi_{bc}\frac{Dw^b}{\partial s}u^c = \frac{\partial}{\partial s}\cot s + \Gamma^\phi_{\phi\theta}\frac{Dw^\phi}{\partial s}u^\theta = -\frac{1}{\sin^2 s} + \cot^2 s = -1.$$

By the equation of geodesic deviation this is $-R^\phi_{\theta\phi\theta}u^\theta w^\phi u^\theta = -R^\phi_{\theta\phi\theta}$ giving $R^\phi_{\theta\phi\theta} = 1$.

(7 marks, unseen, D)

(Total: 20 marks)

4. (a) As the components of the metric are independent of t, $g_{t\nu}dx^\nu/ds = g_{tt}dt/ds = c^2(1 - R/r)dt/ds = c^2 k$ is constant along geodesics.

Similarly, the components of the metric are independent of ϕ, and so $g_{\phi\nu}dx^\nu/ds = -r^2 d\phi/ds = -h$ is constant along geodesics.

Now

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2$$
$$= c^2k^2\left(1-\frac{R}{r}\right)^{-1} - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

Multiplying by $(1 - R/r)$ gives the required result. **(8 marks, seen, A)**

(b) Setting $\tilde{h} = 0$ in the ODE gives

$$\frac{dr}{d\lambda} = \pm c\tilde{k} = \pm c\left(1-\frac{R}{r}\right)\frac{dt}{d\lambda}.$$

Multiplying by $d\lambda/dt$ gives

$$\frac{dr}{dt} = -c\left(1-\frac{R}{r}\right).$$

The negative sign is taken as the photon is infalling.

Separating variables

$$cdt = -\frac{dr}{1-R/r} = -\frac{r\ dr}{r-R} = -dr - \frac{R\ dr}{r-R},$$

which integrates to the given result. **(7 marks, seen similar, B)**

(c) Setting $k = 1/c$ and $h = 0$ in (1) yields

$$\left(\frac{dr}{ds}\right)^2 = \frac{R}{r},$$

which for an infalling particle is

$$\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{dr}{dt}\frac{k}{1-R/r} = -\frac{R^{1/2}}{r^{1/2}}.$$

The negative sign is taken as the particle is infalling. Separating variables

$$c\ dt = -\frac{R^{-1/2}r^{1/2}\ dr}{1-R/r} = -\frac{R^{-1/2}r^{3/2}\ dr}{r-R} = -\frac{R\ x^{3/2}\ dx}{x-1},$$

where $x = r/R$. This integrates to

$$ct = -\frac{2}{3}R^{-1/2}r^{3/2} - 2R^{1/2}r^{1/2} + 2R\tanh^{-1}(\sqrt{R/r}) + \text{constant}.$$

(5 marks, unseen, D)

(Total: 20 marks)

5. (a) For large τ_2 the second vehicle approaches the light ray $ct = x - L$. This line intersects the world-line of the first vehicle when

$$\frac{c^2}{A}\sinh(A\tau_1/c) = \frac{c^2}{A}\cosh(A\tau_1/c) - L,$$

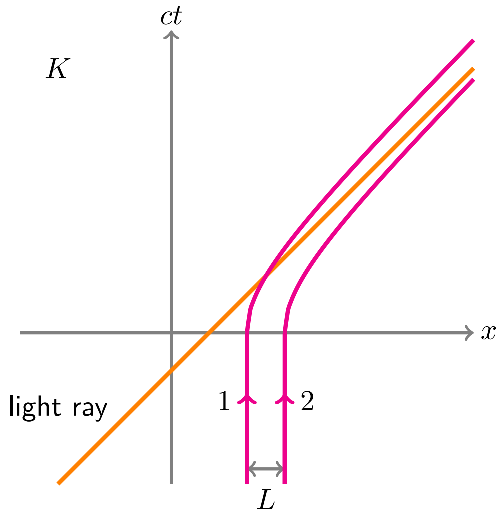
or $\frac{c^2}{A}e^{-A\tau_1/c} = L$ giving

$$\tau_1 = \frac{c}{A}\log\frac{c^2}{AL},$$

if $c^2 \geq AL$.

If $c^2 < AL$, the first vehicle's trajectory is $x = c^2/A$, $t = \tau_1$ so that

$$c\tau_1 = \frac{c^2}{A} - L.$$



(10 marks, unseen)

(b)

$$\frac{1}{1 - R/r} - 1 = \frac{R}{r - R}$$

so that

$$\begin{aligned} ds^2 &= c^2\left(1 - \frac{R}{r}\right)dt^2 - dr^2 - \frac{R}{r - R}dr^2 - r^2(d\theta^2 + \sin^2\theta\; d\phi^2) \\ &= c^2\left(1 - \frac{R}{r}\right)dt^2 - dx^2 - dy^2 - dz^2 - \frac{R}{r - R}dr^2. \end{aligned}$$

Differentiating $r = (x^2 + y^2 + z^2)^{1/2}$ yields

$$dr = \frac{x\; dx + y\; dy + z\; dz}{r}.$$

Accordingly,

$$ds^2 = c^2\left(1 - \frac{R}{r}\right)dt^2 - \left(1 + \frac{x^2 R}{r^2(r - R)}\right)dx^2 - \left(1 + \frac{y^2 R}{r^2(r - R)}\right)dy^2 - \left(1 + \frac{z^2 R}{r^2(r - R)}\right)dz^2$$

$$-\frac{2R}{r^2(r-R)}(xy\ dx\ dy + yz\ dy\ dz + zx\ dz\ dx).$$

(10 marks, unseen)

(Total: 20 marks)

Category A

1(a)(b) 10 marks, 2(a) 8 marks, 3(a) 8 marks, 4(a) 8 marks

Total: **34/80**

Category B

1(c) 5 marks, 3(b)(i) 5 marks, 4(b) 7 marks

Total: **17/80**

Category C

1(d) 5 marks, 2(b)(i) 8 marks

Total: **13/80**

Category D

2(b)(ii) 4 marks, 3(b)(ii) 7 marks, 4 (c) 5 marks

Total: **16/80**

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Tensor Calculus and General Relativity_MATH60017 MATH97023 MATH70017	1	This question was possibly too easy. The special relativity question from the 2021 exam proved to more challenging
Tensor Calculus and General Relativity_MATH60017 MATH97023 MATH70017	2	This question involved various calculations involving a semi-infinite cigar metric. Although this metric appeared in a test the calculations were different. Part (b) involving the Laplacian was generally well answered.
Tensor Calculus and General Relativity_MATH60017 MATH97023 MATH70017	3	This question includes an unseen part where the curvature on a sphere is calculated via geodesic deviation. There was an unfortunate typographical error in the statement of the equation of geodesic deviation. The marker did not penalise errors due to this issue.
Tensor Calculus and General Relativity_MATH60017 MATH97023 MATH70017	4	As in previous years Q4 was about the Schwarzschild space-time. The question was well answered.
Tensor Calculus and General Relativity_MATH60017 MATH97023 MATH70017	5	The Mastery question was split into two unseen parts; (a) dealing with accelerating spacecraft in Special Relativity and (b) recasting the Schwarzschild metric in cartesian-like coordinates. As expected the marks for part (b) were generally a few points lower than in the first four questions. However, part (a) proved much more difficult than expected with marks much lower than other parts of the examination. The identity cosh(A)-sinh(A)=exp(-A) is very useful in part (a)!