MATH70030

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fourier Analysis and Theory of Distributions

Date: 24 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Prove or disprove that

   (i) $L^2[-1,1]$ is a subset of $L^1[-1,1]$ (5 marks)

   (ii) $L^1[-1,1]$ is a subset of $L^2[-1,1]$ (5 marks)

   (b) Let $f(x) = 1/(|x|(\log|x/(2\pi)|)^2)$, $x \in [-\pi,\pi] \setminus 0$. Do the limits of Fourier coefficients a_k, b_k exist as $k \to \infty$? If yes, find the limits. Justify your answer. (10 marks)

2. (a) Let $f \in L^1[-\pi,\pi]$. State a sufficient condition for its Fourier series to converge at a given point $x \in (-\pi,\pi)$. (5 marks)

   (b) (i) Computing the integral, find the inverse Fourier transform of the function $F(\lambda) = \frac{2\sin(\lambda a)}{\lambda}$, where a parameter $a > 0$. (10 marks)

   (ii) Verify your answer in the previous question by computing the Fourier transform of the result. (5 marks)

3. (a) Show that if a Euclidean linear space is finite-dimensional, any linear functional on it is continuous. (10 marks)

   (b) Let $C_j[-1,1]$, $j = 1,2$, be the spaces of continuous functions on $[-1,1]$ with the norm $\|f\|_1 = \sup_{x\in[-1,1]}|f(x)|$ for $j = 1$, and the norm $\|f\|_2 = \int_{-1}^{1}|f(x)|dx$ for $j = 2$. Prove or disprove that $\delta(f) = f(0)$ is a linear continuous functional

   (i) on $C_1[-1,1]$ (5 marks)

   (ii) on $C_2[-1,1]$ (5 marks)

4. (a) Let H be a Hilbert space and let x_n be a sequence of its elements such that x_n converges to some $x \in H$ weakly. Suppose also that the sequence of norms $\|x_n\|$ converges to $\|x\|$. Show that x_n converges to x strongly. (10 marks)

   (b) Find the third distributional derivative of the function $f(x)$ equal to x^2 for $x \geq 0$ and $-x^2$ for $x < 0$. (10 marks)

5. Let $f(z)$ be holomorphic in the unit disc $0 \le |z| < 1$. Denote

$$h(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dt.$$

The space H^2 is a linear space of holomorphic functions in the unit disc such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} h(r) < \infty.$$

Show that

(a) The function $h(r)$ is a nondecreasing function of r. (10 marks)

(b) The space H^2 is complete. (10 marks)

Fourier analysis and distributions exam solutions 2023

1a.

(i) By Cauchy-Schwarz inequality

$$\|f\|_{L^1} = \int_{-1}^{1} |f|\,dt \leq \left(\int_{-1}^{1} 1\,dt\right)^{1/2} \left(\int_{-1}^{1} |f|^2 dt\right)^{1/2}$$

$$= \sqrt{2}\,\|f\|_{L_2}$$ [5 marks]

Therefore $L_2[-1,1] \subset L_1[-1,1]$

(ii) False:

e.g. $\frac{1}{\sqrt{|x|}} \in L_1[-1,1]$ but $\frac{1}{\sqrt{|x|}} \notin L_2[-1,1]$

[5 marks]

1b. $f(x) = f(-x)$ so consider only $x \geq 0$.

$$\int \frac{dx}{x \log^2 \frac{x}{2\pi}} = \int \frac{d \log \frac{x}{2\pi}}{\log^2 \frac{x}{2\pi}} = \frac{-1}{\log \frac{x}{2\pi}} + C$$

Therefore $\displaystyle\int_0^{\pi} \frac{dx}{x \log^2 \frac{x}{2\pi}} < \infty$

Thus, $f \in L_1[-\pi, \pi]$.

By Riemann-Lebesgue lemma,

$a_k, b_k \to 0, \quad k \to \infty$.

In fact $a_n = 0 \ \forall k$ as f is even.

[10 marks]

2a. Fourier series converge at x

if for some $\delta > 0$

$$\int_{-\delta}^{\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty .$$

[5 marks]

2Bi $\quad F(\lambda) = \dfrac{2 \sin(a\lambda)}{\lambda}, \quad a > 0$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin a\lambda}{\lambda} e^{i\lambda x} d\lambda =$$

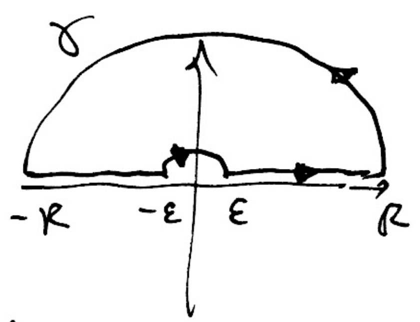
$$\doteq \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} (I_1 - I_2),$$

$$I_1 = \frac{1}{2\pi i} \int_{[-R,R]\setminus[-\varepsilon,\varepsilon]} \frac{1}{\lambda} e^{i\lambda(x+a)} \, d\lambda$$

$$I_2 = \frac{1}{2\pi i} \int_{[-R,R]\setminus[-\varepsilon,\varepsilon]} \frac{1}{\lambda} e^{i\lambda(x-a)} \, d\lambda$$

1) $x > a$

Choose contour



both for I_1, I_2.

Then $$0 = \int_\gamma \frac{1}{2\pi i \lambda} e^{i\lambda(x \pm a)} \, d\lambda$$

$$= I_1 - \tfrac{1}{2} + \underset{\varepsilon \to 0}{o(1)},$$

So $I_1 = \tfrac{1}{2} + o(1)$

Similarly, $I_2 = \tfrac{1}{2} + o(1)$

$\Rightarrow f(x) = \tfrac{1}{2} - \tfrac{1}{2} + \cancel{o(1)} = 0$.

2) $0 < x < a$

Choose contour  for I_1

and  for I_2

Then $I_1 = \frac{1}{2} + o(1)$,

$$I_2 = -\frac{1}{2} + o(1).$$

$$\Rightarrow \quad f(x) = \frac{1}{2} + \frac{1}{2} = 1$$

3) $f(x) = f(-x)$ by changing $\lambda \to -\lambda$ in the integral.

Thus $f = \begin{cases} 1, & -a < x < a \\ 0, & |x| > a. \end{cases}$

2Bii

$$F(\lambda) = \int_{-a}^{a} e^{-ix\lambda} dx = \frac{e^{-ia\lambda} - e^{ia\lambda}}{-i\lambda} =$$

$$= \frac{2\sin a\lambda}{\lambda}.$$

3a. Let E be finite-dimensional, $e_1, \ldots e_n$ its basis

$$f = \sum_{k=1}^{n} f_k e_k, \quad \|f\|^2 = \sum_{k=1}^{n} |f_k|^2 .$$

$$\varphi(f) = \sum_{k=1}^{n} f_k \varphi(e_k), \text{ so}$$

[10 marks]

$$|\varphi(f)| \leq \max_k |\varphi(e_k)| \cdot \sum_{k=1}^{n} |f_k|$$

$$\leq \max_k |\varphi(e_k)| \sqrt{n} \|f\|,$$

so φ is bounded and hence continuous.

3b. In both cases $\delta(f_1 + f_2) = (f_1 + f_2)(0) = f_1(0) + f_2(0)$ so δ is linear.

3bi $C_1[-1,1]$:

$$|\delta(f)| = |f(0)| \leq \|f\|_1, \text{ so}$$

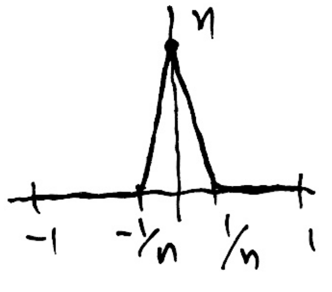
δ is bounded and hence continuous.

[5 marks]

3bii. $C_2[-1,1]$

Consider the sequence $f_n(x)$,

$f_n(0) = n$, $f_n(x) = 0$ outside $(-1/n, 1/n)$, f_n continuous.



Then $\|f_n\| = \int_{-1}^{1} |f(x)|dx \leq 2$

$\delta(f_n) = f_n(0) = n$

Thus δ is not bounded on the ball $\|f\| \leq 2$ and hence not continuous

[5 marks]

4a $x_n \xrightarrow{w} x$, $x_n, x \in H$,

$\|x_n\| \to \|x\|$,

We have

$\|x_n - x\|^2 = (x_n - x, x_n - x) =$

$= \|x_n\|^2 - (x, x_n) - \overline{(x, x_n)} + \|x\|^2$

$(x, x_n) \to (x, x) = \|x\|^2$ since

$f(x_n) = (x, x_n)$ is a continuous linear functional on H

Hence $\|x_n - x\|^2 \to 0$, $n \to \infty$.

[10 marks]

4B

$$f' = 2|x|$$

$$f'' = 2\,\mathrm{sgn}(x)$$

$$(f''', \varphi) = -(f'', \varphi') =$$

$$= -2\int_{-\infty}^{\infty} \mathrm{sgn}(x)\,\varphi'\,dx =$$

$$= 2\left(\int_{-\infty}^{0} \varphi'\,dx - \int_{0}^{\infty} \varphi'\,dx\right) =$$

$$= 4\varphi(0)$$

$$\Rightarrow \quad \underline{f''' = 4\delta}$$

[10 marks]

5a. Since $f(z)$ is holomorphic in the unit disc $|z|<1$, we have a representation

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \, , \quad |z| < 1 \, ,$$

so $f(re^{it}) = \sum_{n=0}^{\infty} c_n r^n e^{int}$, $r<1$

is in $L_2[-\pi, \pi]$.

[10 marks]

By Parseval equality,

$$h(r) = \sum_{n=0}^{\infty} c_n^2 r^{2n} \, , \quad r < 1 \, .$$

Therefore $h(r)$ is nondecreasing.

5b. Let $f \in H^2$. Using 5a, we can write $\lim_{r\to 1} h(r) = \sup_{0<r<1} h(r) = \|f\|^2 < \infty$

Thus $\sum_{n=0}^{N} c_n^2 r^{2n} < K$ for some K and all $r<1$, N. (sum of positive terms)

So $\sum_{n=0}^{N} c_n^2 \leq K \quad \forall N$

Thus $\sum_{n=0}^{\infty} c_n^2 < \infty$, and

we have by dominated convergence

$$\|f\|^2 = \sum_{n=0}^{\infty} c_n^2 .$$

Hence H^2 and $\ell^2(0,1,\dots)$ are isometric. Since ℓ^2 is complete, H^2 is complete.

[10 marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60030/70030	1	Q1b presented a difficulty, although conceptually easy, it required some computational habits.
MATH60030/70030	2	Q2b revealed some difficulties with complex integration
MATH60030/70030	3	Generally well done
MATH60030/70030	4	This question was in general well done
MATH70030	5	Q5 unfortunately was not done well.