Problem Sheet 5, Geometry of Curves and Surfaces, 2020-2021

Problem 1. Let $S \subset \mathbb{R}^3$ be a compact, connected surface without boundary which is not diffeomorphic to a sphere. Prove that S contains points where the Gaussian curvature is negative, zero, and positive.

Problem 2. Fix constants $a, b > 0$ and consider the ellipsoid

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \right\}.$$

(a) Compute the Gaussian curvature of S at each point (no need to calculate them at the north and south poles)

$$\phi(u, v) = (a\cos(u)\cos(v), a\sin(u)\cos(v), b\sin(v)).$$

(b) Apply the Gauss-Bonnet theorem to S, and conclude that

$$\int_0^1 \frac{dx}{\left(b^2 + (a^2 - b^2)\, x^2\right)^{3/2}} = \frac{1}{ab^2}.$$

Problem 3. Let $S \subset \mathbb{R}^3$ be a regular surface, and assume that $\gamma_1 : [0, t_1] \to S$ and $\gamma_2 : [0, t_2] \to S$ are geodesics parametrised by arc length, and assume that these are not part of a single common geodesic on S. Prove that there are only finitely many pairs (τ_1, τ_2) such that $\gamma_1(\tau_1) = \gamma_2(\tau_2)$.

Problem 4. Let p be a point on a regular surface S, and let $T \subset S$ be a curvilinear triangle whose sides are geodesics, and p belongs to the interior of T. Prove that if $\theta_1, \theta_2, \theta_3$ denote the interior angles of T, then

$$\lim_{T \to p} \frac{\left(\sum_{i=1}^{3} \theta_i\right) - \pi}{\text{area}(T)} \to K(p)$$

where the limit is taken over any sequence of such curvilinear triangles T which converge to p. Explain how this gives another proof of Gauss's Theorema Egregium.

Problem 5. Let $S \subset \mathbb{R}^3$ be a regular surface with curvature $K \leq 0$. Assume that S is diffeomorphic to a plane, and $\gamma : (a, b) \to S$ is a geodesic parametrised by arc length. Prove that γ is injective. Give a counterexample when S is not diffeomorphic to a plane.