Imperial College London

MATH97013 MATH70051

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Vortex Dynamics

Date: 09 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

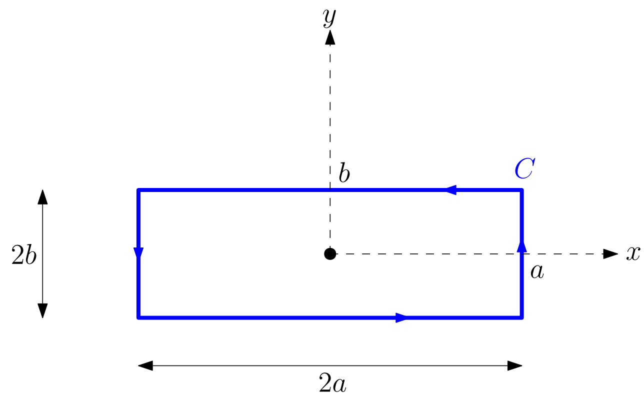
1. Consider the two dimensional incompressible flow of an ideal fluid in a Cartesian $\mathbf{x} = (x, y)$ plane having velocity field

$$\mathbf{u} = (u, v) = (x^2 - 2y^2, -2xy).$$

(a) Find the circulation

$$\oint_C \mathbf{u}.\mathbf{dx},$$

where C is the boundary of a rectangle shown in the figure:



(4 marks)

(b) Near the point $(1, 1)$ the velocity $\mathbf{u}$ can be written locally as

$$\mathbf{u} = (-1, -2) + \mathbf{u}_{\text{SBR}} + \mathbf{u}_{\text{str}} + \text{terms quadratic in } (x-1) \text{ and } (y-1),$$

where $\mathbf{u}_{\text{SBR}}$ is a solid body rotation about the point $(1, 1)$ and $\mathbf{u}_{\text{str}}$ is a linear irrotational extensional flow about the point $(1, 1)$. Find $\mathbf{u}_{\text{SBR}}$ and $\mathbf{u}_{\text{str}}$ explicitly. (6 marks)

(c) Find the principal axes of the extensional flow $\mathbf{u}_{\text{str}}$ and sketch the streamlines associated with it near the point $(1, 1)$.

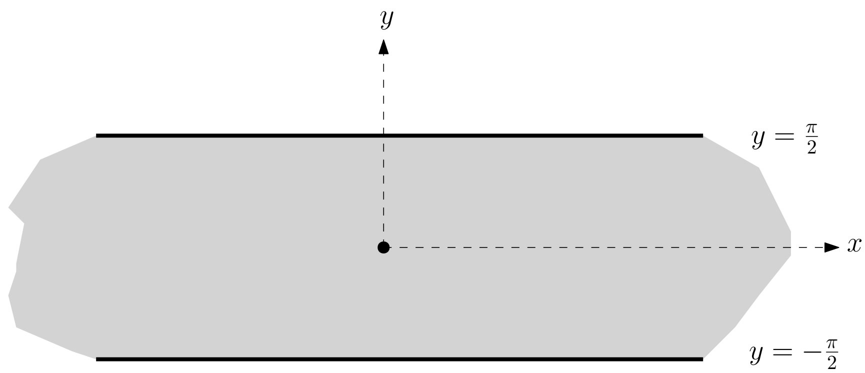
(5 marks)

(d) Suppose, at some instant, two point vortices of unit circulation are located at $(\pm 1, 0)$ in the background flow given by $\mathbf{u}$. Find the instantaneous velocity of each point vortex.

(5 marks)

(Total: 20 marks)

2. Consider the two dimensional incompressible flow (u, v) in a channel in the (x, y) plane as shown in the figure:



The two walls of the channel at $y = \pm\pi/2$ are impenetrable. The channel is of infinite width in the x direction.

(a) Find a conformal mapping $z = f(\zeta)$ that transplants the interior of a unit disc $|\zeta| < 1$ in a complex ζ plane to the fluid in the channel and where the unit circle $|\zeta| = 1$ is transplanted to the channel walls.

(4 marks)

(b) A point vortex of circulation Γ is now placed at $(0, 0)$ but the flow in the channel is otherwise irrotational. Using your result from part (a) or otherwise, find, *as a function of* z, the complex potential $w(z)$ associated with this flow scenario.

(6 marks)

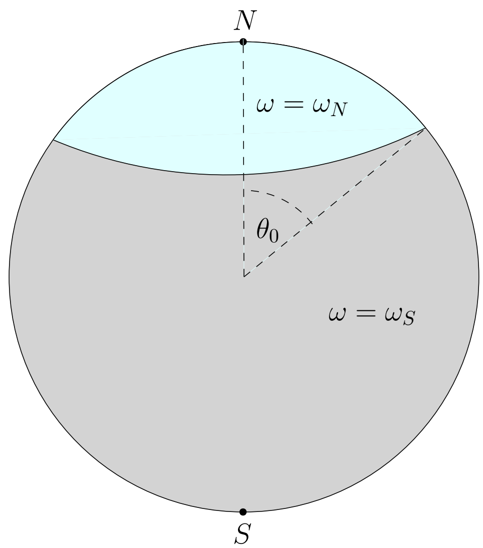
(c) Suppose that the point vortex of circulation Γ at $(0, 0)$ is now situated in a vorticity field that is uniform everywhere and of strength ω_0 (a non-zero constant). Find the streamfunction $\psi(x, y)$ associated with this modified flow scenario. (6 marks)

(d) Find the instantaneous velocity of the point vortex in part (c).

(4 marks)

(Total: 20 marks)

3. Consider motion of an ideal fluid of constant density on a spherical surface of unit radius. Let (r, θ, ϕ) be the usual spherical polar angles where θ is the latitudinal angle measured from an axis through the north and south poles of the sphere. A spherical cap containing the North pole N and extending down the sphere to the spherical polar angle θ_0 $(0 < \theta_0 < \pi)$ is filled with fluid of uniform vorticity ω_N (a constant) while the remainder of the sphere containing the south pole S is filled with fluid of uniform vorticity ω_S (a constant), as indicated in the figure:



(a) For this flow to be an equilibrium configuration what conditions must hold at the vortex jump on the latitude circle $\theta = \theta_0$? (2 marks)

(b) Using the complex-valued coordinate

$$\zeta = \cot\left(\frac{\theta}{2}\right) e^{\mathrm{i}\phi}$$

representing a point in a plane of stereographic projection, find the relationship between ω_N and ω_S required for this flow to be an equilibrium. Find an expression for the streamfunction $\psi(\zeta, \overline{\zeta})$ associated with this equilibrium flow.

(8 marks)

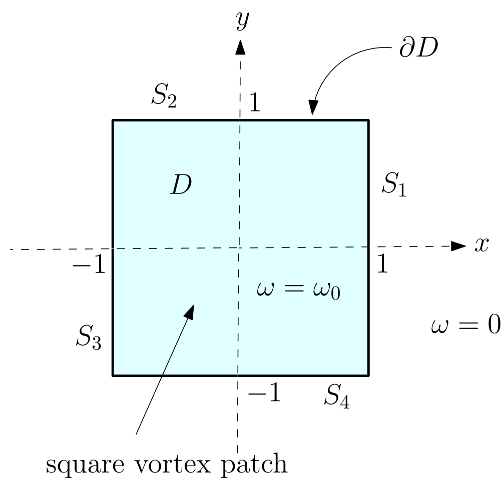
(c) A point vortex of circulation Γ is now added at the south pole S in this equilibrium flow. Find the modified streamfunction and show that this vortex configuration is also an equilibrium for any value of Γ. (4 marks)

(d) Determine the value of Γ that makes the fluid at the vortex jump on the latitude circle $\theta = \theta_0$ stationary.

(6 marks)

(Total: 20 marks)

4. At some instant a vortex patch D of uniform vorticity $\omega = \omega_0$ sits in an unbounded irrotational flow taking place in a two-dimensional (x, y) plane and has the shape of a square with 4 sides $\{S_j | j = 1, \ldots 4\}$ of length 2 as shown in the figure:



The flow everywhere is incompressible. The boundary of the patch is denoted by ∂D and is made up of the 4 sides of the square denoted by S_1, S_2, S_3 and S_4 as shown in the figure.

(a) Let $z = x + \mathrm{i}y$. As a function of z and $\overline{z}$, find an expression for the complex velocity field $u - \mathrm{i}v$ associated with this instantaneous flow in terms of Cauchy-type integrals around the boundary ∂D.

(4 marks)

(b) Find an expression for $\overline{z}$ as a function of z on <u>each</u> of the 4 sides $S_j, j = 1, 2, 3, 4$.

(4 marks)

(c) Using your results from parts (a) and (b) to show that the irrotational flow exterior to the vortex patch can be written as

$$u - \mathrm{i}v = \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} I\left(\frac{z}{\Omega_j}\right),$$

where $\Omega_j = e^{\pi \mathrm{i}(j-1)/2}$ and the function $I(z)$ is defined by

$$I(z) \equiv (2 - z) \log\left(\frac{z - (1 + \mathrm{i})}{z + (1 - \mathrm{i})}\right).$$

(8 marks)

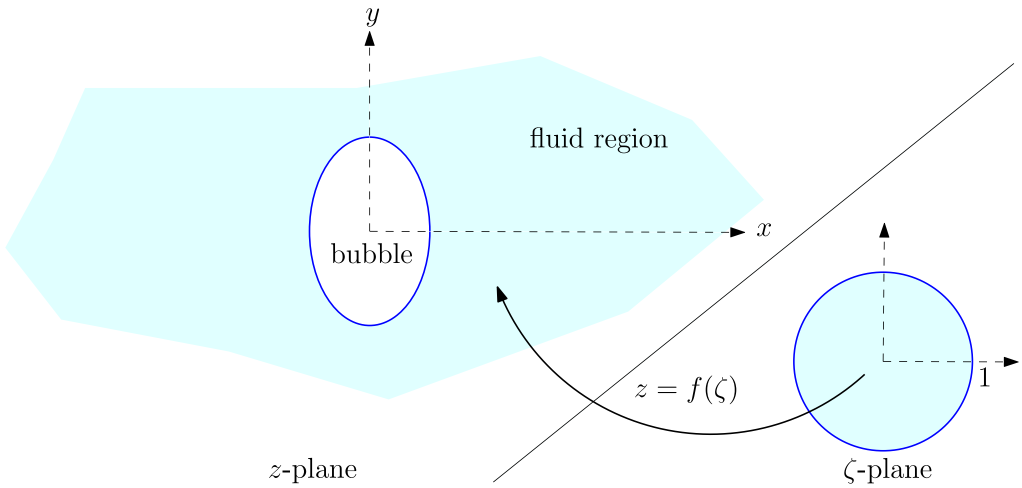
(d) Find the constant c such that

$$u - \mathrm{i}v \to \frac{c}{z}, \qquad \text{as } |z| \to \infty.$$

(4 marks)

(Total: 20 marks)

5. A constant-pressure gas bubble in an incompressible irrotational two-dimensional flow has circulation Γ around it and rotates steadily about the origin with angular velocity Ω without changing its shape. The figure shows the *steady* bubble in an (x, y)-plane corotating with the bubble:



The fluid domain occupied by the fluid exterior to the bubble in this corotating frame is described by a one-to-one conformal mapping

$$z = x + \mathrm{i}y = f(\zeta) = -\frac{1}{\zeta}\left(1 + \frac{8\zeta^2}{\zeta^2 - a^2}\right), \qquad a \in \mathbb{R}, \qquad a > 3,$$

from the interior of a parametric unit ζ-disc, $|\zeta| < 1$; the bubble boundary is the image of the circle $|\zeta| = 1$ under this conformal mapping. Owing to the fact that the bubble is at constant pressure it can be argued (using Bernoulli's theorem) that the fluid speed on the bubble boundary in this corotating frame must equal a constant q everywhere on the bubble boundary.

(a) Given the facts just described, show that, on the boundary of the bubble,

$$\overline{\left(\frac{dw}{dz}\right)} - \mathrm{i}\Omega z = q\frac{dz}{ds},$$

where $ds = \sqrt{dx^2 + dy^2}$ denotes an infinitesimal element of arclength along the boundary with s increasing as the bubble boundary is traversed in an anticlockwise direction.

(6 marks)

(b) Show that

$$\frac{df}{d\zeta} = [R(\zeta)]^2,$$

where $R(\zeta)$ is a rational function of ζ that you must determine.

(6 marks)

Question continues on the next page

(c) The chain rule can be used to show that, in terms of the variable ζ,

$$\frac{dz}{ds} = -\frac{\mathrm{i}\zeta f'(\zeta)}{|f'(\zeta)|}, \qquad \text{where } f'(\zeta) = \frac{df}{d\zeta}.$$

Use this result, together with your results from parts (a) and (b) to deduce that, as a function of the variable ζ, the derivative of the complex potential dw/dz is

$$\frac{dw}{dz} = \frac{\mathrm{i}q}{\zeta}\frac{R(1/\zeta)}{R(\zeta)} + \mathrm{i}\Omega\zeta\left(\frac{\zeta^2 - 9/a^2}{\zeta^2 - 1/a^2}\right).$$

(4 marks)

(d) Hence deduce that the boundary speed q must be related to Ω and a via

$$q = \frac{2\Omega(a^4+3)}{a^4-1}.$$

(4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

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XXX (Solutions)

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1. (a) The vorticity field is

sim. seen ⇓

$$\omega(x,y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2y - (-4y) = 2y.$$

By Stokes theorem the circulation Γ is the area integral of this vorticity field:

$$\Gamma = \int_{x=-a}^{x=a}\int_{y=-b}^{y=b} 2y\ dy\ dx = \int_{x=-a}^{x=a} \left[y^2\right]_{y=-b}^{y=b} dx = 0.$$

4, A

(b) We can write

meth seen ⇓

$$(u,v) = (x^2-2y^2, -2xy) = ((x-1+1)^2-2(y-1+1)^2, -2(x-1+1)(y-1+1))$$

Now let

$$X = x-1, \qquad Y = y-1$$

then

$$\begin{aligned}(u,v) &= ((X+1)^2 - 2(Y+1)^2, -2(X+1)(Y+1)) \\ &\approx (2X-4Y-1, -2X-2Y-2) + \text{quadratic terms in } X,Y.\end{aligned}$$

Now we know the vorticity field is $\omega(x,y) = 2y$ so $\omega(1,1) = 2$ hence, factoring off the solid body rotation (SBR) with angular velocity $\Omega = \omega(1,1)/2 = 1$ gives

$$\begin{aligned}(u,v) &\approx \underbrace{(-Y,X)}_{\text{SBR}} + (2X-3Y-1, -3X-2Y-2) + \text{quadratic in } X,Y \\ &= (-1,-2) + \underbrace{(-Y,X)}_{\text{SBR}} + \underbrace{(2X-3Y, -3X-2Y)}_{\text{irrotational strain}} + \text{quadratic in } X,Y\end{aligned}$$

6, B

(c) We can write the irrotational strain at $(1,1)$ the symmetric matrix form

sim. seen ⇓

$$\begin{pmatrix} 2 & -3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

To identify the principal axes of strain requires calculation of the eigenvectors of this 2-by-2 matrix. The eigenvalues λ satisfy

$$\det\begin{pmatrix} 2-\lambda & -3 \\ -3 & -2-\lambda \end{pmatrix} = 0$$

implying

$$\lambda = \pm\sqrt{13}.$$

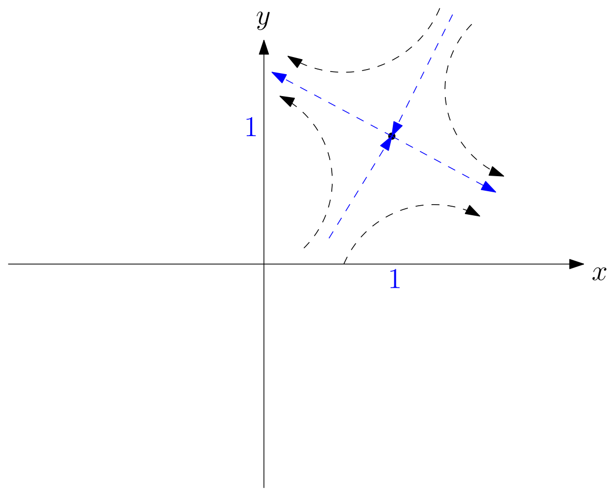
The corresponding eigenvectors are

$$\begin{pmatrix} 3 \\ 2-\sqrt{13} \end{pmatrix}, \qquad \lambda = \sqrt{13},$$

$$\begin{pmatrix} 3 \\ 2+\sqrt{13} \end{pmatrix}, \qquad \lambda = -\sqrt{13},$$

The local form of the strain therefore looks as shown in the figure:

5, A



$\boxed{\text{sim. seen} \Downarrow}$

(d) The additional flow due to the point vortices can be described by the complex potential

$$w(z) = -\frac{\mathrm{i}}{2\pi}\log(z-1) - \frac{\mathrm{i}}{2\pi}\log(z+1)$$

with corresponding complex velocity field

$$u - \mathrm{i}v = \frac{dw}{dz} = -\frac{\mathrm{i}}{2\pi}\frac{1}{z-1} - \frac{\mathrm{i}}{2\pi}\frac{1}{z+1}.$$

By the usual rules for point vortex motion, the point vortex at $z = 1$ will move with the velocity (U_1, V_1) (due to the combined effect of the background flow stated in the question and the additional contribution due to the other point vortex) namely,

$$U_1 - \mathrm{i}V_1 = \left(x^2 - 2y^2 + \mathrm{i}2xy - \frac{\mathrm{i}}{2\pi}\frac{1}{z+1}\right)\bigg|_{z=1} = 1 - \frac{\mathrm{i}}{4\pi}$$

so that

$$\boxed{(U_1, V_1) = \left(1, +\frac{1}{4\pi}\right)}$$

Similarly, the point vortex at $z = -1$ will move with the velocity (U_2, V_2) (due to the combined effect of the background flow stated in the question and the additional contribution due to the other point vortex) namely,

$$U_2 - \mathrm{i}V_2 = \left(x^2 - 2y^2 + \mathrm{i}2xy - \frac{\mathrm{i}}{2\pi}\frac{1}{z-1}\right)\bigg|_{z=-1} = 1 + \frac{\mathrm{i}}{4\pi}$$

so that

$$\boxed{(U_2, V_2) = \left(1, -\frac{1}{4\pi}\right)}$$

$\boxed{5, \mathrm{A}}$

2. (a) The self-inverse Cayley map seen ⇓

$$\zeta \mapsto \eta = \frac{1-\zeta}{1+\zeta}$$

transplants the unit disc $|\zeta| < 1$ to the right half η plane. Then the logarithmic map

$$\eta \mapsto z = \log \eta$$

maps the right half plane to the channel region of interest in a z plane. The required conformal mapping is therefore

$$z = f(\zeta) = \log \frac{1-\zeta}{1+\zeta}.$$

4, A

(b) On exponentiation of the result of part (a), meth seen ⇓

$$e^z = \frac{1-\zeta}{1+\zeta}.$$

Now using the fact that the Cayley map is self-inverse we conclude

$$\zeta = \frac{1-e^z}{1+e^z} = -\tanh\left(\frac{z}{2}\right).$$

Note that $\zeta = 0$ corresponds to $z = 0$. Now we know that the complex potential for a point vortex at the centre of the unit ζ disc is

$$-\frac{\mathrm{i}\Gamma}{2\pi}\log\zeta$$

hence, by the conformal invariance of the boundary value problem, the required complex potential $w(z)$ for a point vortex at $z = 0$ is

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\zeta(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(-\tanh\left(\frac{z}{2}\right)\right).$$

6, C

(c) We need to add a solution with constant vorticity ω_0 that *also* satisfies the boundary conditions that the walls of the channel are streamlines. From the simplicity of this geometry it should be clear that a simple shear will suffice, namely, unseen ⇓

$$(\hat{u}, \hat{v}) = (-\omega_0 y, 0),$$

since this satisfies the boundary condition that $\hat{v} = 0$ on $y = \pm\pi/2$ and, moreover, the associated vorticity is the required constant:

$$\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} = \omega_0.$$

The associated streamfunction $\hat{\psi}$ satisfies

$$\hat{u} = -\omega_0 y = \frac{\partial \hat{\psi}}{\partial y}, \qquad \text{implying} \qquad \hat{\psi} = -\frac{\omega_0 y^2}{2}.$$

Hence the total streamfunction asked for in the question is

$$\psi = \hat{\psi} + \mathrm{Im}[w(z)] = -\frac{\omega_0 y^2}{2} - \frac{\Gamma}{2\pi}\log\left|\tanh\left(\frac{z}{2}\right)\right|.$$

6, C

$\boxed{\text{sim. seen} \Downarrow}$

(d) The point vortex will move with the local "non-self-induced" velocity. It is clear that the simple shear $(-\omega_0 y, 0)$ will produce no motion of the vortex which sits on $y = 0$. The only possible contribution is from the non-self-induced contribution $\hat{u} - \mathrm{i}\hat{v} = d\hat{w}/dz|_{z=0}$ where $\hat{w}(z)$ is defined by

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log z + \hat{w}(z).$$

But

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(-\tanh\left(\frac{z}{2}\right)\right)$$

and since the Taylor expansion of the odd function $\tanh(z/2) = z/2 + \mathcal{O}(z^3)$ then

$$\hat{w}(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(-\frac{1}{2} + \mathcal{O}(z^2)\right)$$

which implies that $d\hat{w}/dz = \mathcal{O}(z)$ as $z \to 0$ implying the non-self-induced velocity from this term also vanishes. Therefore the instantaneous velocity of the vortex is zero, and it is in steady equilibrium. $\boxed{4, \text{B}}$

3. (a) For equilibrium, a vortex jump must be (i) a streamline and (ii) the fluid velocity must be continuous at the vortex jump. seen ⇓

2, A

(b) The vorticity ω is given in terms of the streamfunction as meth seen ⇓

$$\nabla^2_\Sigma \psi = -\omega$$

where the Laplace-Beltrami operator can be expressed in terms of ζ and its conjugate as

$$\nabla^2_\Sigma \psi = (1+\zeta\overline{\zeta})^2 \frac{\partial^2 \psi}{\partial \zeta \partial \overline{\zeta}}$$

If the vorticity is uniform, say equal to ω_S, then

$$(1+\zeta\overline{\zeta})^2 \frac{\partial^2 \psi}{\partial \zeta \partial \overline{\zeta}} = -\omega_S.$$

or

$$\frac{\partial^2 \psi}{\partial \zeta \partial \overline{\zeta}} = -\frac{\omega_S}{(1+\zeta\overline{\zeta})^2}.$$

This can be integrated to give

$$\frac{\partial \psi}{\partial \zeta} = \frac{\omega_S}{\zeta(1+\zeta\overline{\zeta})} + g(\zeta),$$

where $g(\zeta)$ is an analytic function of ζ. In order to avoid a singularity at the south pole at $\zeta = 0$ we must pick

$$g(\zeta) = -\frac{\omega_S}{\zeta}.$$

Then

$$\frac{\partial \psi}{\partial \zeta} = -\frac{\omega_S \overline{\zeta}}{(1+\zeta\overline{\zeta})}$$

which can be integrated again to give

$$\psi = -\omega_S \log(1+\zeta\overline{\zeta}).$$

On the other hand, the same steps for $\omega = \omega_N$ lead to

$$\frac{\partial \psi}{\partial \zeta} = \frac{\omega_N}{\zeta(1+\zeta\overline{\zeta})} + h(\zeta),$$

where $h(\zeta)$ is an analytic function of ζ. For ζ away from the South pole (i.e. $\zeta \neq 0$) the integrated function is no longer singular so we pick $h(\zeta) = 0$ (any other constant can be shown to lead to unbounded velocities as $\zeta \to \infty$) leading to

$$\frac{\partial \psi}{\partial \zeta} = \frac{\omega_N}{\zeta(1+\zeta\overline{\zeta})} = \frac{\omega_N}{\zeta} - \frac{\omega_N \overline{\zeta}}{(1+\zeta\overline{\zeta})}$$

which can be integrated to give

$$\psi = \omega_N \log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right) + c,$$

where c is some constant.

Therefore the streamfunction is given by

$$\psi = \begin{cases} \omega_N \log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right) + c, & 0 \le \theta \le \theta_0, \\ -\omega_S \log(1+\zeta\overline{\zeta}), & \theta_0 \le \theta < \pi. \end{cases}$$

The complex velocity $u_\phi - \mathrm{i}u_\theta$, where u_ϕ and u_θ are the zonal and meridional velocity components respectively, is related to $\partial\psi/\partial\zeta$:

$$u_\phi - \mathrm{i}u_\theta = \frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}$$

so that continuity of velocity at $\theta = \theta_0$ is equivalent to continuity of $\partial\psi/\partial\zeta$ and requires

$$-\frac{\omega_S\overline{\zeta}}{(1+\zeta\overline{\zeta})} = \frac{\omega_N}{\zeta} - \frac{\omega_N\overline{\zeta}}{(1+\zeta\overline{\zeta})} \qquad \text{on} \qquad |\zeta| = \cot(\theta_0/2),$$

where we have used the fact that $|\zeta| = \cot(\theta_0/2)$ at the vortex jump, or

$$-\frac{\omega_S\overline{\zeta}}{(1+\zeta\overline{\zeta})} = \frac{\omega_N}{\zeta(1+\zeta\overline{\zeta})} \qquad \text{on} \qquad |\zeta| = \cot(\theta_0/2)$$

which implies

$$\boxed{\omega_S = -\tan^2(\theta_0/2)\omega_N}$$

Note: It is also clear that by picking c such that

$$-\omega_S \log(1+\cot^2(\theta_0/2)) = \omega_N \log\left(\frac{\cot^2(\theta_0/2)}{1+\cot^2(\theta_0/2)}\right) + c,$$

then the streamfunction can be made continuous, and constant on $\theta = \theta_0$. 8, D

meth seen ⇓

(c) The streamfunction $\hat{\psi}$, say, with an additional point vortex at the South Pole is given by

$$\hat{\psi} = \psi - \frac{\Gamma}{4\pi}\log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right) = \begin{cases} \omega_N \log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right) - \frac{\Gamma}{4\pi}\log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right) + c, & 0 \le \theta \le \theta_0, \\ \\ -\omega_S \log(1+\zeta\overline{\zeta}) - \frac{\Gamma}{4\pi}\log\left(\frac{\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right), & \theta_0 \le \theta < \pi \end{cases}$$

Clearly the velocity field associated with $\hat{\psi}$ remains continuous at $\theta = \theta_0$ if ψ is continuous there. Moreover since the streamfunction added to ψ to get $\hat{\psi}$ is constant on $\theta = \theta_0$, namely equal to

$$-\frac{\Gamma}{4\pi}\log\left(\frac{\cot^2(\theta_0/2)}{1+\cot^2(\theta_0/2)}\right),$$

then the latitude circle $\theta = \theta_0$ remains a streamline. Therefore all the equilibrium conditions on the vortex jump are satisfied.

However it is also required to show that the point vortex at the South Pole will remain stationary: this requires that the non-self-induced flow (that caused by the equilibrium streamfunction ψ at the South Pole) induces zero velocity at $\zeta = 0$. However, it is easy to see that the velocity near the South Pole induced by ψ is proportional to

$$\frac{\partial\psi}{\partial\zeta} = -\omega_S\left[\frac{\overline{\zeta}}{1+\zeta\overline{\zeta}}\right], \qquad \theta_0 \le \theta \le \pi$$

which vanishes as $\zeta \to 0$. Therefore the point vortex is also steady, and we still have a global equilibrium for any value of Γ. 4, A

$\boxed{\text{meth seen} \Downarrow}$

(d) In order that the velocity on $\theta = \theta_0$ vanishes we must have

$$\left.\frac{\partial \hat{\psi}}{\partial \zeta}\right|_{|\zeta|=\cot(\theta_0/2)} = 0$$

or

$$-\frac{\omega_S \bar{\zeta}}{(1+\zeta\bar{\zeta})} - \frac{\Gamma}{4\pi}\left[\frac{1}{\zeta} - \frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}\right] = 0, \qquad \text{when} \qquad |\zeta| = \cot(\theta_0/2).$$

But since this can be written

$$\frac{1}{\zeta}\left[\frac{\omega_S \zeta\bar{\zeta}}{(1+\zeta\bar{\zeta})} + \frac{\Gamma}{4\pi}\left[1 - \frac{\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right]\right] = 0, \qquad \text{when} \qquad |\zeta| = \cot(\theta_0/2)$$

it can be satisfied provided that constants are chosen so that

$$\frac{\omega_S \zeta\bar{\zeta}}{(1+\zeta\bar{\zeta})} = -\frac{\Gamma}{4\pi}\left[\frac{1}{1+\zeta\bar{\zeta}}\right], \qquad \text{when} \qquad |\zeta| = \cot(\theta_0/2).$$

Rearrangement leads to

$$\frac{\Gamma}{4\pi} = -\omega_S |\zeta|^2 \Big|_{|\zeta|=\cot(\theta_0/2)}$$

implying the choice

$$\boxed{\Gamma = -4\pi\omega_S \cot^2(\theta_0/2) = 4\pi\omega_N}$$

$\boxed{6,\ \text{B}}$

4. (a) For a uniform vortex patch the streamfunction $\psi(z,\overline{z})$ satisfies $\boxed{\text{seen} \Downarrow}$

$$\nabla^2\psi = -\omega = \begin{cases} -\omega_0 & (x,y) \in D, \\ 0, & (x,y) \notin D, \end{cases}$$

or in variables $(z,\overline{z})$,

$$\frac{\partial^2\psi}{\partial z \partial \overline{z}} = \begin{cases} -\dfrac{\omega_0}{4}, & z \in D, \\ \\ 0, & z \notin D. \end{cases}$$

On integration with respect to $\overline{z}$:

$$\frac{\partial \psi}{\partial z} = \begin{cases} -\dfrac{\omega_0}{4}(\overline{z} - C_i(z)), & z \in D, \\ \\ \dfrac{\omega_0}{4} C_o(z), & z \notin D, \end{cases}$$

where $C_i(z)$ is analytic inside D and $C_o(z)$ is analytic outside D. Now

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial \psi}{\partial z}$$

so continuity of velocity on ∂D, the boundary of D, requires

$$-\frac{\omega_0}{4}(\overline{z} - C_i(z)) = \frac{\omega_0}{4} C_o(z), \qquad z \in \partial D.$$

This implies

$$\overline{z} = C_i(z) - C_o(z), \qquad \text{on } \partial D.$$

This is a scalar Riemann-Hilbert problem whose solution is well known to be given by

$$C_i(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{\overline{z}' dz'}{z' - z}, \qquad C_o(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{\overline{z}' dz'}{z' - z}.$$

With these expressions, the velocity field is

$$u - \mathrm{i}v = \begin{cases} -\dfrac{\mathrm{i}\omega_0}{2}(\overline{z} - C_i(z)), & z \in D, \\ \\ \dfrac{\mathrm{i}\omega_0}{2} C_o(z), & z \notin D. \end{cases}$$

$\boxed{4,\ \text{A}}$

(b) On S_1 we can use the parametrization $\boxed{\text{meth seen} \Downarrow}$

$$z = 1 + \mathrm{i}y, \qquad -1 \le y \le +1$$

so that

$$\overline{z} = 1 - \mathrm{i}y = 1 - (z-1) = 2 - z, \qquad \boxed{\overline{z} = 2 - z}$$

Similarly, on S_2 we can write

$$z = x + \mathrm{i} \qquad -1 \le x \le +1$$

so that

$$\overline{z} = x - \mathrm{i} = z - 2\mathrm{i}, \qquad \boxed{\overline{z} = z - 2\mathrm{i}}$$

Using similar arguments, on S_3 we find

$$\boxed{\overline{z} = -2 - z}$$

and on S_4,

$$\boxed{\overline{z} = z + 2\mathrm{i}}$$

$\boxed{4,\ \text{A}}$

(c) From part (a) the complex velocity exterior to the patch is

unseen ⇓

$$u - \mathrm{i}v = \frac{\mathrm{i}\omega_0}{2} C_o(z) = \frac{\omega_0}{4\pi} \oint_{\partial D} \frac{\overline{z}' dz'}{z' - z} = \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \int_{S_j} \frac{\overline{z}' dz'}{z' - z},$$

where we have split the boundary integral into the 4 contributions from each side of the square patch. Now notice that side S_j for $j = 2, 3, 4$ is just a rotation of side S_1 by $\Omega_j = e^{\mathrm{i}\pi(j-1)/2}$ so that in the integral

$$\int_{S_j} \frac{\overline{z}' dz'}{z' - z}$$

we can make the substitution

$$z' = \Omega_j u, \qquad u \in S_1.$$

Hence

$$\int_{S_j} \frac{\overline{z}' dz'}{z' - z} = \int_{S_1} \frac{\overline{u} du}{\Omega_j u - z} = \frac{1}{\Omega_j} \int_{S_1} \frac{\overline{u} du}{u - z/\Omega_j} := \frac{1}{\Omega_j} \tilde{I}(z/\Omega_j),$$

where

$$\tilde{I}(z) \equiv \int_{S_1} \frac{\overline{u} du}{u - z}.$$

Since on S_1 we know, from part (b), that $\overline{u} = 2 - u$,

$$\begin{aligned}
\tilde{I}(z) &= \int_{S_1} \frac{(2-u)du}{u - z} = \int_{1-\mathrm{i}}^{1+\mathrm{i}} \frac{(2 - (u - z) - z)du}{u - z} \\
&= [(2 - z)\log(u - z) - u]_{1-\mathrm{i}}^{1+\mathrm{i}} \\
&= (2 - z)\log\left(\frac{1 + \mathrm{i} - z}{1 - \mathrm{i} - z}\right) - (1 + \mathrm{i} - (1 - \mathrm{i})) \\
&= (2 - z)\log\left(\frac{1 + \mathrm{i} - z}{1 - \mathrm{i} - z}\right) - 2\mathrm{i}.
\end{aligned}$$

Hence

$$u - \mathrm{i}v = \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} \tilde{I}(z/\Omega_j) = \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} I(z/\Omega_j),$$

where

$$I(z) \equiv (2 - z)\log\left(\frac{1 + \mathrm{i} - z}{1 - \mathrm{i} - z}\right),$$

and where we have used the fact that

$$\sum_{j=1}^{4} \frac{1}{\Omega_j} = 0.$$

8, D

(d) There are two ways to do this part. The easiest method is to use the fact that one expects the flow far away to resemble the flow due to a point vortex of circulation equal to that of the square patch. The complex velocity field for a point vortex of circulation Γ is

meth seen ⇓

$$u - \mathrm{i}v = -\frac{\mathrm{i}\Gamma}{2\pi z}.$$

The circulation of the square patch is

$$\omega_0 \times 4$$

where we have multiplied ω_0 by the area of the square (equal to 4). Hence with $\Gamma = 4\omega_0$,

$$u - \mathrm{i}v \to -\frac{\mathrm{i}(4\omega_0)}{2\pi z} = -\frac{\mathrm{i}2\omega_0}{\pi z}$$

therefore we read off

$$c = -\frac{2\omega_0 \mathrm{i}}{\pi}.$$

Alternatively, the same answer can be obtained by looking at the large z behaviour of the answer to part (d). From part (d),

$$\begin{aligned} u - \mathrm{i}v &= \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} I(z/\Omega_j) \\ &= \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} \left((2 - z/\Omega_j) \log \left(\frac{1 + \mathrm{i} - z/\Omega_j}{1 - \mathrm{i} - z/\Omega_j} \right) \right) \\ &= \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} \left((2 - z/\Omega_j) \log \left(\frac{z - \Omega_j(1 + \mathrm{i})}{z - \Omega_j(1 - \mathrm{i})} \right) \right). \end{aligned}$$

Now using familiar expansions of the $\log$ function for large $|z|$:

$$\begin{aligned} \log \left(\frac{z - \Omega_j(1 + \mathrm{i})}{z - \Omega_j(1 - \mathrm{i})} \right) &= \log \left(\frac{1 - \Omega_j(1 + \mathrm{i})/z}{1 - \Omega_j(1 - \mathrm{i})/z} \right) \\ &\sim -\frac{2\mathrm{i}\Omega_j}{z} - \frac{2\mathrm{i}\Omega_j^2}{z^2} + \dots, \qquad \text{as } |z| \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} u - \mathrm{i}v &= \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \frac{1}{\Omega_j} \left((2 - z/\Omega_j) \log \left(\frac{z - \Omega_j(1 + \mathrm{i})}{z - \Omega_j(1 - \mathrm{i})} \right) \right) \\ &\sim \frac{\omega_0}{4\pi} \sum_{j=1}^{4} \left(\frac{2}{\Omega_j} - \frac{z}{\Omega_j^2} \right) \times \left(-\frac{2\mathrm{i}\Omega_j}{z} - \frac{2\mathrm{i}\Omega_j^2}{z^2} + \dots \right) \qquad \text{as } |z| \to \infty \\ &= \frac{\omega_0}{4\pi} \times \sum_{j=1}^{4} \left(-\frac{2\mathrm{i}}{z} \right) + \dots \\ &= -\frac{2\omega_0 \mathrm{i}}{\pi z} + \dots, \end{aligned}$$

as required, and where we have again used the fact that

$$\sum_{j=1}^{4} \frac{1}{\Omega_j} = 0.$$

$\boxed{\text{4, B}}$

5. (a) The complex velocity field for solid body rotation $u_{SBR} - \mathrm{i}v_{SBR}$ with angular velocity Ω is well known to be

unseen $\Downarrow$

$$u_{SBR} - \mathrm{i}v_{SBR} = -\mathrm{i}\Omega\overline{z}.$$

This was established in lectures. Outside the bubble there is an irrotational flow component described by dw/dz where $w(z)$ is the complex potential so that the total complex velocity field *in the corotating frame* is

$$u - \mathrm{i}v = \frac{dw}{dz} - (-\mathrm{i}\Omega\overline{z}) = \frac{dw}{dz} + \mathrm{i}\Omega\overline{z}$$

The complex unit tangent is

$$\frac{dz}{ds}$$

where ds denotes an element of arclength from which we infer, from the fact that the fluid speed on the boundary equals q everywhere in the corotating frame, that

$$u + \mathrm{i}v = q \times \frac{dz}{ds}$$

It follows from these arguments that, on the bubble boundary,

$$\overline{\left(\frac{dw}{dz}\right)} - \mathrm{i}\Omega z = q\frac{dz}{ds}.$$

6, M

(b) Note that we can write

unseen $\Downarrow$

$$f(\zeta) = -\frac{1}{\zeta}\left(\frac{9\zeta^2 - a^2}{\zeta^2 - a^2}\right) = -\frac{1}{\zeta}\left[9 + \frac{8a^2}{\zeta^2 - a^2}\right].$$

By the product rule,

$$\frac{df}{d\zeta} = \frac{1}{\zeta^2}\left(\frac{9\zeta^2 - a^2}{\zeta^2 - a^2}\right) - \frac{1}{\zeta}\left[-\frac{16\zeta a^2}{(\zeta^2 - a^2)^2}\right] = \left(\frac{1}{\zeta}\left(\frac{3\zeta^2 + a^2}{\zeta^2 - a^2}\right)\right)^2.$$

Therefore we read off

$$R(\zeta) = \frac{1}{\zeta}\left(\frac{3\zeta^2 + a^2}{\zeta^2 - a^2}\right).$$

6, M

(c) From part (a) we deduce, on taking a complex conjugate,

unseen $\Downarrow$

$$\frac{dw}{dz} = q\frac{d\overline{z}}{ds} - \mathrm{i}\Omega\overline{z}.$$

From part (b),

$$\frac{dz}{ds} = -\frac{\mathrm{i}\zeta f'(\zeta)}{|f'(\zeta)|} = -\frac{\mathrm{i}\zeta f'(\zeta)^{1/2}}{\overline{f'(\zeta)^{1/2}}} = -\frac{\mathrm{i}\zeta R(\zeta)}{R(1/\zeta)},$$

where we have used the fact that, on the bubble boundary where $\overline{\zeta} = 1/\zeta$,

$$\overline{R(\zeta)} = R(1/\zeta).$$

Also on $|\zeta| = 1$,

$$\overline{z} = \overline{f(\zeta)} = -\zeta\left(\frac{9 - a^2\zeta^2}{1 - a^2\zeta^2}\right) = -\zeta\left(\frac{\zeta^2 - 9/a^2}{\zeta^2 - 1/a^2}\right).$$

Therefore, on $|\zeta| = 1$,

$$\frac{dw}{dz} = \frac{\mathrm{i}q}{\zeta}\frac{R(1/\zeta)}{R(\zeta)} - \mathrm{i}\Omega\overline{z} = \frac{\mathrm{i}q}{\zeta}\frac{R(1/\zeta)}{R(\zeta)} + \mathrm{i}\Omega\zeta\left(\frac{\zeta^2 - 9/a^2}{\zeta^2 - 1/a^2}\right),$$

as required.

4, M

(d) The important point is that the previous expression for dw/dz is analytic in ζ and can therefore can analytically continued off $|\zeta| = 1$ to the interior of the unit ζ disc, which corresponds to the fluid domain exterior to the bubble under the conformal mapping correspondence. Since, from part (b), $\boxed{\text{unseen} \Downarrow}$

$$\frac{R(1/\zeta)}{R(\zeta)} = \zeta\left(\frac{3+a^2\zeta^2}{1-a^2\zeta^2}\right) \times \frac{\zeta(\zeta^2-a^2)}{(3\zeta^2+a^2)} = \zeta^2\left(\frac{3+a^2\zeta^2}{1-a^2\zeta^2}\right)\frac{(\zeta^2-a^2)}{(3\zeta^2+a^2)}$$

we find from part (c) that

$$\begin{aligned}\frac{dw}{dz} &= \mathrm{i}q\zeta\left(\frac{3+a^2\zeta^2}{1-a^2\zeta^2}\right)\frac{(\zeta^2-a^2)}{(3\zeta^2+a^2)} + \mathrm{i}\Omega\zeta\left(\frac{\zeta^2-9/a^2}{\zeta^2-1/a^2}\right)\\ &= -\frac{\mathrm{i}\zeta}{\zeta^2-1/a^2}\left[\frac{q}{a^2}\frac{(3+a^2\zeta^2)(\zeta^2-a^2)}{(3\zeta^2+a^2)} - \Omega(\zeta^2-9/a^2)\right].\end{aligned}$$

This rational function on the right hand side has poles when $\zeta^2 - 1/a^2 = 0$ and when $3\zeta^2 + a^2 = 0$. The possible poles at the roots of $3\zeta^2 + a^2 = 0$, i.e, at

$$\zeta = \pm\frac{\mathrm{i}a}{\sqrt{3}}$$

lie outside the unit disc because we are told $a > 3$. However, the roots of the denominator $\zeta^2 - 1/a^2$ at $\zeta = \pm 1/a$ lie inside the unit disc. These correspond to poles of the complex potential in the fluid region which are not allowed: they would represent flow singularities if admitted but there are no such singularities exterior to the bubble. Therefore the expression in square brackets must vanish when $\zeta^2 = 1/a^2$ in order to render dw/dz analytic in the fluid region as required:

$$\frac{q}{a^2} \times 4\frac{(1-a^4)}{(3+a^4)} + 8\frac{\Omega}{a^2} = 0.$$

On rearrangement we find the required

$$q = \frac{2\Omega(a^4+3)}{a^4-1}.$$

$\boxed{4, \text{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks