MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Repetition Material 2: Uniform convergence and the space of continuous functions on a compact interval

In order to obtain a solution to an initial value problem of the form

$$\dot{x} = f(t,x)\,, \quad x(t_0) = x_0\,,$$

we have considered Picard iterates $\{\lambda_n : J \to \mathbb{R}^d\}_{n\in\mathbb{N}_0}$ in Definition 2.2, and we noticed in Exercise 10 that uniform convergence of the Picard iterates to a limit function $\lambda_\infty : J \to \mathbb{R}^d$ is of utmost importance. Recall that uniform convergence means that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|\lambda_n(t) - \lambda_\infty(t)\| < \varepsilon \quad \text{for all } t \in J \text{ and } n \geq N\,.$$

The crucial point here is that N only depends on ε and not on $t \in J$ (meaning that convergence is uniform in t).

We assume that J to be a compact interval, and we will see that uniform convergence is equivalent with convergence in the space $C^0(J,\mathbb{R}^d)$, which is the space of continuous functions $u : J \to \mathbb{R}^d$ defined on a compact interval $J \subset \mathbb{R}$:

$$C^0(J,\mathbb{R}^d) := \{u : J \to \mathbb{R}^d : u \text{ is continuous}\}\,.$$

You already know from the analysis course last term that this space is a normed vector space, where addition of two functions $u_1, u_2 : J \to \mathbb{R}^d$ is defined by

$$(u_1 + u_2)(t) := u_1(t) + u_2(t) \quad \text{for all } \ t \in J\,,$$

and scalar multiplication with $\alpha \in \mathbb{R}$ of a function $u : J \to \mathbb{R}^d$ is given by

$$(\alpha u)(t) := \alpha u(t) \quad \text{for all } \ t \in J\,.$$

You have seen already in the analysis course last term that there are norms that make C^0 a normed vector space. We will concentrate on one particular norm in this course, given by the *supremum norm*

$$\|u\|_\infty := \sup_{t\in J} \|u(t)\|\,.$$

It turns out that this norm is tailor-made for us, since it corresponds one-to-one to uniform convergence. More precisely, a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ in C^0 converges uniformly to a function $u_\infty : J \to \mathbb{R}^d$ if and only if

$$\lim_{n\to\infty} \|u_n - u_\infty\|_\infty = 0\,.$$

It is important for us that the normed vector space $(C^0, \|\cdot\|_\infty)$ is complete, and this is formulated in the following theorem. Note that completeness means that all Cauchy sequences converge.

Theorem 1 (C^0 is a Banach space)**.** *Let $J \subset \mathbb{R}$ be a compact interval. Then the space of continuous functions $C^0(J,\mathbb{R}^d)$, equipped with the supremum norm, is a Banach space.*

Note that not all norms on $C^0(J,R^d)$ make this space a Banach space, and we refer to Exercise 9 for an example. For the proof of this theorem, we refer to the Analysis course in Autumn term.