IMPERIAL

MATH70079

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Statistical Finance

Date: Friday, May 3, 2024

Time: 14:00 – 15:30 (BST)

Time Allowed: 1.5 hours

This paper has 3 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

The open-book material allowed during the examinations consists of any material provided by the lecturers and annotated by the students, i.e. annotated lecture notes, annotated slides, and annotated problem class sheets. Books and electronic devices are not allowed.

1. Let $\{Y_t\}_{t\in\mathbb{Z}}$ be a strictly stationary GARCH(1,1) process, with natural filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathbb{Z}}$.

(a) Define Y_t and derive the conditional expectations $\mathbf{E}(Y_t|\mathcal{F}_{t-1})$ and $\mathbf{E}(Y_t^2|\mathcal{F}_{t-1})$ and the unconditional expectations $\mathbf{E}(Y_t)$ and $\mathbf{E}(Y_t^2)$, by specifying any conditions for their existence.

Derive analogous expectations $\mathbf{E}(X_t|\mathcal{G}_{t-1})$, $\mathbf{E}(X_t^2|\mathcal{G}_{t-1})$, $\mathbf{E}(X_t)$ and $\mathbf{E}(X_t^2)$, for the weakly stationary AR(1) process $\{X_t\}_{t\in\mathbb{Z}}$, with natural filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t\in\mathbb{Z}}$, defined as $X_t = \phi X_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim \mathrm{WN}(0, \varphi^2)$ and $-1 < \phi < 1$.

By comparing the results, argue that a linear process like X_t cannot properly describe the time varying volatility that characterise time series of returns.

(b) Discuss the conditions for weak stationarity of the GARCH(1,1) process $\{Y_t\}_{t\in\mathbb{Z}}$ and derive its autocovariance function. What does the latter suggest about the data generating process of $\{Y_t\}_{t\in\mathbb{Z}}$?

(c) Consider now the ARCH(1) process obtained as a particular case of the GARCH(1,1) process that you have defined in (a), with intercept α_0 and parameter $0 < \alpha_1 < 1$. Show that

$$\mathbf{E}\left[\sigma_{t+s}^2|\mathcal{F}_{t-1}\right] = \frac{1-\alpha_1^s}{1-\alpha_1}\alpha_0 + \alpha_1^s\sigma_t^2$$

and interpret the result. *Hint:* Proceed by induction.

[Total 15 marks]

2. Let L be a random variable representing a loss, following the convention that losses are counted positively.

(a) Define value-at-risk $\mathrm{VaR}_\alpha(L)$ and expected shortfall $\mathrm{ES}_\alpha(L)$ for any $\alpha \in (0,1)$ and explain how these definitions are related. If any conditions are needed to ensure that these risk measures are well defined, remember to state them.

(b) Assume that L has continuous cumulative distribution function F_L and that $\mathbf{E}[L^+] < \infty$, where $x^+ = \max\{x, 0\}$, for all $x \in \mathbb{R}$. Show that, for any $\alpha \in (0,1)$,

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\mathbf{E}\left[L\mathbf{1}_{\{L \geq \mathrm{VaR}_\alpha(L)\}}\right] = \mathbf{E}[L|L \geq \mathrm{VaR}_\alpha(L)]$$

and interpret the result. *Hint:* use a change of variable.

(c) Suppose that $L \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. (Recall that then $\mathbf{P}[L \leq l] = 1 - \exp(-\lambda l)$, $l \geq 0$.) Compute $\mathrm{VaR}_\alpha(L)$ and $\mathrm{ES}_\alpha(L)$ for any $\alpha \in (0,1)$. *Hint:* Note that $x \log x - x$ is an antiderivative of $\log x$, where $x \log x \to 0$ when $x \to \infty$.

[Total 15 marks]

3. Let $T \in \mathbb{N}$ and let $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t=0}^T$ be a portfolio invested in $d \in \mathbb{N}$ risky assets with price process $\boldsymbol{S} = (\boldsymbol{S}_t)_{t=0}^T$ (in $\mathbb{R}_{>0}^d$) and a risk-free asset with deterministic price process $B = (B_t)_{t=0}^T$ (in $\mathbb{R}_{>0}$).

(a) Define what it means for $\boldsymbol{\pi}$ to be an arbitrage. Also, financially interpret the conditions that define an arbitrage.

(b) Consider a specific model where $d = 1$ and $T = 1$. Suppose that $B_t = 1$ for $t = 0, 1$, $S_0 = 2$ and

$$\mathbf{P}[S_1 = 3] = \frac{1}{2}, \quad \mathbf{P}[S_1 = 2] = \frac{2}{6}, \quad \mathbf{P}[S_1 = 1] = \frac{1}{6}.$$

Define an equivalent martingale measure (EMM) and verify which of the following is an EMM:

$$\begin{aligned}
&\mathbf{Q_1}[S_1 = 3] = \frac{1}{2}, \quad \mathbf{Q_1}[S_1 = 2] = \frac{3}{8}, \mathbf{Q_1}[S_1 = 1] = \frac{1}{8} \\
&\mathbf{Q_2}[S_1 = 3] = \frac{3}{4}, \quad \mathbf{Q_2}[S_1 = 2] = \frac{1}{8}, \mathbf{Q_2}[S_1 = 1] = \frac{1}{8} \\
&\mathbf{Q_3}[S_1 = 3] = \frac{1}{8}, \quad \mathbf{Q_3}[S_1 = 2] = \frac{3}{4}, \mathbf{Q_3}[S_1 = 1] = \frac{1}{8}
\end{aligned}$$

(c) Is the price process S arbitrage-free? Motivate your answer.

(d) Show that the payoff $X = \min\{S_0, S_1\}$ is unattainable in this model and discuss a strategy to price the derivative.

[Total 15 marks]

Imperial College London
Department of Mathematics
MSc in Statistics
Academic year 2023–2024, Spring term

MATH70079 Introduction to Statistical Finance

Examination, Solutions

(Below, marks are indicated as **[#marks]**.)

1. (15 marks)

(a) A strictly stationary GARCH(1,1) process $Y = \{Y_t\}_{t\in\mathbb{Z}}$ is defined as follows,

$$\begin{aligned} Y_t &= \sigma_t Z_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad t \in \mathbb{Z}, \end{aligned} \tag{1}$$

with parameters $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\beta_1 \geq 0$, and $(Z_t)_{t\in\mathbb{Z}} \sim \mathrm{IID}(0,1)$. **[1]**

The process $(\sigma_t)_{t\in\mathbb{Z}}$ is adapted to $\mathcal{F}$ by construction, as each variable σ_t^2 is defined as a measurable function of (past) variables $Y_{t-1}, Y_{t-2}, \ldots$, and thus is $\mathcal{F}_{t-1}-$measurable.

$$\mathbf{E}(Y_t|\mathcal{F}_{t-1}) \overset{\text{def}}{=} \mathbf{E}(\sigma_t Z_t|\mathcal{F}_{t-1}) \overset{\sigma_t \mathcal{F}_{t-1}-\text{meas}}{=} \sigma_t \mathbf{E}(Z_t|\mathcal{F}_{t-1}) \overset{Z_t \text{IID}}{=} \sigma_t \mathbf{E}(Z_t) = 0 \quad \textbf{[0.5]}$$

$$\mathbf{E}(Y_t^2|\mathcal{F}_{t-1}) \overset{\text{def}}{=} \mathbf{E}(\sigma_t^2 Z_t^2|\mathcal{F}_{t-1}) \overset{\sigma_t^2 \mathcal{F}_{t-1}-\text{meas}}{=} \sigma_t^2 \mathbf{E}(Z_t^2|\mathcal{F}_{t-1}) \overset{Z_t \text{IID}}{=} \sigma_t^2 \mathbf{E}(Z_t^2) = \sigma_t^2. \quad \textbf{[0.5]}$$

By the tower property, denoting $\mathbf{E}_{t-1}(Y_t) = \mathbf{E}(Y_t|\mathcal{F}_{t-1})$ and reminding that $\mathbf{E}_{t-1}(Y_t) = 0$, one has $\mathbf{E}(Y_t) = \mathbf{E}_0 \ldots \mathbf{E}_{t-1}(Y_t) = 0$. **[1]**

Using the representation implied by equation (FTS.16) in the lecture notes, i.e. $\sigma_t^2 = \alpha_0 \left((1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} (\alpha_1 Z_{t-j}^2 + \beta_1) \right)$, one has

$$\begin{aligned} \mathbf{E}[Y_t^2] = \mathbf{E}[Z_t^2]\mathbf{E}[\sigma_t^2] &= \mathbf{E}[Z_t^2]\alpha_0 \left(1 + \sum_{i=1}^{\infty} \mathbf{E}\left[\prod_{j=1}^{i} (\alpha_1 Z_{t-j}^2 + \beta_1) \right] \right) \\ &= \alpha_0 \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} (\alpha_1 \underbrace{\mathbf{E}[Z_{t-j}^2]}_{=1} + \beta_1) \right) \\ &= \alpha_0 \left(1 + \sum_{i=1}^{\infty} (\alpha_1 + \beta_1)^i \right) \\ &= \frac{\alpha_0}{1 - (\alpha_1 + \beta_1)}, \quad \textbf{[1]} \end{aligned}$$

provided that $\alpha_1 + \beta_1 < 1$, to ensure convergence of the geometric series and existence of the conditional variance, and where we have used the mutual independence between Y_t and Z_t, the mutual independence of the variables in Z, implying that $\mathbf{E}(Z_t^2 Z_s^2) = 0 \ \forall s \neq t,\ s,t \in \mathbb{Z}$, and the fact that expectation and summation can always be swapped since the summands are non-negative. **[1] (seen)**

As far as the AR(1) process is concerned,

$$\mathbf{E}(X_t|\mathcal{G}_{t-1}) = \mathbf{E}(\phi X_t + \varepsilon_t|\mathcal{G}_{t-1}) = \phi X_{t-1}$$

by linearity of conditional expectation and by the same arguments used to derive the conditional expectation of the GARCH(1,1) process **[1]**. Similarly, as ε_t is uncorrelated with X_{t-1},

$$\mathbf{E}(X_t^2|\mathcal{G}_{t-1}) = \phi^2 X_{t-1}^2 + \sigma^2. \quad [\mathbf{1}]$$

To compute the unconditional moments of X_t, we exploit the assumption of weak stationarity, that implies that $\mu = \mathbf{E}(X_t)$ and $\gamma_0 = \mathbf{E}(X_t^2) - \mu^2$, $\forall t \in \mathbb{Z}$. Hence,

$$\mu = \mathbf{E}(X_t) = \mathbf{E}(\phi X_{t-1} + \varepsilon_t) = \phi \mathbf{E}(X_{t-1}) = \phi\mu$$

that holds for $\phi \in (-1, 1)$ if

$$\mu = 0. \quad [\mathbf{1}]$$

Similarly,

$$\gamma_0 = \mathbf{E}(X_t^2) = \frac{\sigma^2}{1-\phi^2}. \quad [\mathbf{1}]$$

The GARCH(1,1) process Y and the AR(1) process X have both mean equal to zero (conditional and unconditional) and constant unconditional variance. However, while the conditional variance of a GARCH(1,1) depends on past variables and thus is time varying, the conditional variance of an AR(1), $\mathbf{V}(X_t|\mathcal{G}_{t-1}) = \sigma^2$, is constant with respect to time, meaning that past information does reduce the unconditional variance of the process but does not imply any time variation in the conditional variance. Thus, a linear process, whose one-step-ahead prediction error variance is always equal to the noise variance, is not capable to capture the volatility clustering typical of financial time series. **[2]** **(seen similar)**

(b) Weak stationarity is ensured by existence of the first and second moments of the process proved in point (a) and by the covariance function of the process, $\gamma_k = \mathbf{E}(Y_t Y_{t-k})$ as the process has mean equal to zero, being a function of k and not of t, for $k \in \mathbb{Z}$. The condition $\mathbf{E}(Y_t|\mathcal{F}_{t-1}) = 0$, that defines a martingale difference sequence, implies (that $\mathbf{E}(Y_t) = 0$ and) that Y_t is uncorrelated with any $\mathcal{F}_{t-1}$-measurable variable so that $\gamma_{t,t-k} = 0$ for $k > 0$. Under stationarity, $\gamma_k = 0$ for $k > 0$. As the process is stationary, $\gamma_{-k} = \gamma_k = 0$ i.e. the autocovariance function of Y_t is equal to zero at each lag $k \neq 0$. **[0.5]** The process is then the realisation of a white noise process with zero mean and variance equal to $\sigma^2 = 1/(1-\alpha-\beta)$. **[0.5]** **(seen)** The process Y is uncorrelated but not independent.

(c) Consider the GARCH(1,1) process in (a) with $\beta_1 = 0$.

Using the hint, for $s = 1$ the equality is true as:

$$\mathbf{E}\left[\sigma_{t+1}^2|\mathcal{F}_{t-1}\right] = \mathbf{E}\left[\alpha_0 + \alpha_1 Y_t^2|\mathcal{F}_{t-1}\right] = \alpha_0 + \alpha_1 \mathbf{E}[Y_t^2|\mathcal{F}_{t-1}] = \alpha_0 + \alpha_1\sigma_t^2. \quad [\mathbf{1}]$$

Let us assume that the equality is true for $s > 1$ and compute:

$$\begin{aligned}
\mathbf{E}\left[\sigma_{t+s+1}^2 | \mathcal{F}_{t-1}\right] &= \mathbf{E}\left[\alpha_0 + \alpha_1 Y_{t+s}^2 | \mathcal{F}_{t-1}\right] \\
&= \alpha_0 + \alpha_1 \mathbf{E}\left[Y_{t+s}^2 | \mathcal{F}_{t-1}\right] \\
&= \alpha_0 + \alpha_1 \mathbf{E}_{t-1}\left[Y_{t+s}^2\right] \\
&= \alpha_0 + \alpha_1 \mathbf{E}_{t-1}\mathbf{E}_{t+s-1}\left[Y_{t+s}^2\right] \\
&= \alpha_0 + \alpha_1 \mathbf{E}_{t-1}\mathbf{E}_{t+s-1}\left[\sigma_{t+s}^2 Z_{t+s}^2\right] \\
&= \alpha_0 + \alpha_1 \mathbf{E}\left[\sigma_{t+s}^2 | \mathcal{F}_{t-1}\right] \\
&= \alpha_0 + \alpha_1 \left(\frac{1-\alpha_1^s}{1-\alpha_1}\alpha_0 + \alpha_1^s \sigma_t^2\right) \\
&= \alpha_0 \left(1 + \alpha_1 \frac{1-\alpha_1^s}{1-\alpha_1} + \alpha_1^{s+1}\sigma_t^2\right) \\
&= \alpha_0 \left(\frac{1-\alpha_1^{s+1}}{1-\alpha_1} + \alpha_1^{s+1}\sigma_t^2\right). \quad [1]
\end{aligned}$$

where in the fourth row we have applied the tower property and in the seventh row we have used the inductive hypothesis. As the statement is true for $s = 1$ and, if true for s then holds for $s + 1$, we conclude that it is true for all $s \in \mathbb{N}$.

Interpretation: in an ARCH process, the best (in minimum mean square error) s-step-ahead prediction of the conditional variance that we can make given the information at time $t - 1$ is scaled function of the volatility at time t. The scaling factor α^s implies that any shock in the volatility persists at a geometric rate α. This persistence is coherent with the stylised fact that commonly defined as volatility clustering. [1] (**unseen**)

2. (15 marks)

(a) Let $\alpha \in (0, 1)$. The definitions are

$$\begin{aligned}
\mathrm{VaR}_\alpha(L) &:= \inf\{l \in \mathbb{R} : \mathbf{P}[L > l] \leq 1 - \alpha\}, \\
\mathrm{ES}_\alpha(L) &:= \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\mathrm{d}u,
\end{aligned}$$

where $q_u(L) := \inf\{l \in \mathbb{R} : \mathbf{P}[L \leq l] \geq u\}$. [2] Since $\mathbf{P}[L \leq l] = 1 - \mathbf{P}[L > l]$, we have

$$q_u(L) = \inf\{l \in \mathbb{R} : \mathbf{P}[L > l] \leq 1 - u\} = \mathrm{VaR}_u(L). \quad [1]$$

The value-at-risk $\mathrm{VaR}_\alpha(L)$ is well-defined for any random variable L, whereas the expected shortfall $\mathrm{ES}_\alpha(L)$ requires $\mathbf{E}[L^+] < \infty$. [1] (**seen**)

(b) The proof is based on the proof of Lemma 2.16 in McNeil, Frey, Embrecths (2005). Let $\alpha \in (0, 1)$. The expected shortfall (ES) of loss L at confidence level α is

$$\mathrm{ES}_\alpha(L) := \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\mathrm{d}u. \quad [1]$$

As the cdf F is continuous by assumption, it is also strictly increasing, so

$$\mathrm{ES}_\alpha(L) := \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(u)\mathrm{d}u \quad [1]$$

(seen)

The result is obtained by a change of variable. Let u be the realisation of a Uniform r.v. $U = F(L)$, taking values in (0,1), so that $u = F_L(l)$. U is the inverse transformation of $L = F^{-1}(U)$ having cdf F and taking values in $\mathbb{R}$. As $\mathrm{d}u = \mathrm{d}F_L$ and $F_L^{-1}(\alpha) = VaR_\alpha$ [2] we have

$$\mathrm{ES}_\alpha(L) := \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(u)\mathrm{d}u = \frac{1}{1-\alpha}\int_{VaR_\alpha(L)}^\infty l\mathrm{d}F_L = \frac{1}{1-\alpha}\mathbf{E}\big[L\mathbf{1}_{\{L\geq \mathrm{VaR}_\alpha(L)\}}\big]. \quad [1]$$

The equality

$$\frac{1}{1-\alpha}\mathbf{E}\big[L\mathbf{1}_{\{L\geq \mathrm{VaR}_\alpha(L)\}}\big] = \mathbf{E}\big[L|L\geq \mathrm{VaR}_\alpha(L)\big]$$

follows by the definition of conditional expectation and by the fact that for a continuous cdf $1-\alpha = P(L\geq q_\alpha(L))$. The relation allows us to interpret ES as the expected loss, given that VaR is breached. [1] **(unseen)**

(c) The cdf $F_L(l) = 1-\exp(-\lambda l)$, $l\geq 0$, of L is strictly increasing and continuous on $[0,\infty)$ with $F_L(0) = 0$. Therefore,

$$\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha), \quad \alpha\in(0,1), \quad [1]$$

where we work out $F_L^{-1}(\alpha)$ as follows:

$$\alpha = F_L(l) \quad\Longleftrightarrow\quad \alpha = 1-\exp(-\lambda l) \quad\Longleftrightarrow\quad l = -\frac{\log(1-\alpha)}{\lambda}.$$

Thus,

$$\mathrm{VaR}_\alpha(L) = -\frac{\log(1-\alpha)}{\lambda}. \quad [2] \tag{2}$$

The first moment of the exponential distribution is known to be finite, so that $\mathbf{E}[L^+] = \mathbf{E}[L] < \infty$. Then, by (2), we get

$$\begin{aligned}\mathrm{ES}_\alpha(L) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\mathrm{d}u = -\frac{1}{1-\alpha}\int_\alpha^1 \frac{\log(1-u)}{\lambda}\mathrm{d}u \\ &= -\frac{1}{(1-\alpha)\lambda}\int_0^{1-\alpha}\log u\,\mathrm{d}u = -\frac{1}{(1-\alpha)\lambda}[u\log u - u]_0^{1-\alpha} \\ &= -\frac{\log(1-\alpha)}{\lambda} + \frac{1}{\lambda}\end{aligned}$$

using the hint for the final two equalities. [2] **(seen similar)**

3. **(15 marks)**

(a) An adapted, self-financing [1] portfolio $\boldsymbol{\pi}$ is an arbitrage if

(i) $V_0(\boldsymbol{\pi}) = 0$, **Interpretation:** The portfolio requires no initial wealth. [1]

(ii) $\mathbf{P}[V_T(\boldsymbol{\pi}) \geq 0] = 1$, **Interpretation:** Losses are impossible. [1]

(iii) $\mathbf{P}[V_T(\boldsymbol{\pi}) > 0] > 0$. **Interpretation:** There is a positive probability of profit. [1]

(**seen**)

(b) For $\mathbf{Q}$ to be an equivalent martingale measure, firstly, $\mathbf{Q}[A] > 0$ if and only if $\mathbf{P}[A] > 0$ and, secondly, the discounted price process $(S_t/B_t)_{t=0}^1$ should be a martingale with respect to the market filtration $(\mathcal{F}_t^S)_{t=0}^1$ under $\mathbf{Q}$. [2]

The first requirement is satisfied by all the three measures as $\mathbf{Q_i}[S_1 = 3] > 0$, $\mathbf{Q_i}[S_1 = 1] > 0$ for $\mathbf{i} = 1, 2, 3$, and their sum is smaller than one. For the second requirement, as S_0 is non-random (hence $\mathcal{F}_0^S = \{\varnothing, \Omega\}$) and $S_1/B_1 = S_1$, we verify that:

$$\mathbf{E}^{\mathbf{Q_3}}\left[\frac{S_1}{B_1}\middle|\mathcal{F}_0^S\right] = \mathbf{E}^{\mathbf{Q_3}}[S_1] = 3\mathbf{Q_3}[S_1 = 3] + 2\mathbf{Q_3}[S_1 = 2] + \mathbf{Q_3}[S_1 = 1] = \frac{3}{8} + \frac{3}{2} + \frac{1}{8} = 2 = S_0.$$

which is not the case for the other two measures. [2] (**unseen**)

(c) The price process is arbitrage-free by the first fundamental theorem of asset pricing, which states that if (and only if) there exists an EMM then S is arbitrage free. [1] (**seen**)

(d) To show that the payoff X is unattainable, we prove that no replicating portfolio exists that matches the payoff. [1]

We assume that X is attainable, so there exists an adapted, self-financing portfolio $\boldsymbol{\pi}$ with $V_1(\boldsymbol{\pi}) = X$, and argue for a contradiction. Then,

$$V_1(\boldsymbol{\pi}) = \pi_{0,0}B_1 + \pi_{0,1}S_1 = \begin{cases} \pi_{0,0} + 3\pi_{0,1} & \text{on event } \{S_1 = 3\} \\ \pi_{0,0} + 2\pi_{0,1} & \text{on event } \{S_1 = 2\} \\ \pi_{0,0} + 1\pi_{0,1} & \text{on event } \{S_1 = 1\} \end{cases} \qquad [1]$$

and note that,

$$X = \min\{S_0, S_1\} = \begin{cases} 2 & \text{on event } \{S_1 = 3\} \\ 2 & \text{on event } \{S_1 = 2\} \\ 3 & \text{on event } \{S_1 = 1\}. \end{cases} \qquad [1]$$

Therefore, for X to be attainable, we need to find $\pi_{0,0}$ and $\pi_{0,1}$ such that

$$\begin{cases} \pi_{0,0} + 3\pi_{0,1} = 2 \\ \pi_{0,0} + 2\pi_{0,1} = 2 \\ \pi_{0,0} + 2\pi_{0,1} = 3. \end{cases}$$

Note that the last two equations can only jointly hold for $\pi_{0,0} = 2, \pi_{0,1} = 0$. However, in this case, the left hand side of the first equation becomes $2 + 2 \times 0 = 2 \neq 3$ so we have a contradiction. Therefore, there exists no such portfolio and X is not attainable. **(seen similar)** **[1]**

Since X is not attainable in this model, we cannot use Theorem P.39 to find an arbitrage-free price process of X by replication. **[1]** On the other hand, we can use Theorem P.45 to find a risk neutral price since we know that there is an EMM $\mathbf{Q}$ and $E^{\mathbf{Q}}[|X|] \leq S_0 = 3 < \infty$. Therefore we would typically choose to use risk neutral pricing. **[1]** **(seen)**