

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Quantum Mechanics 2

Date: Monday, May 12, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Formula Sheet

The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Connection to quantum information language: $X = \sigma_x$, $Y = \sigma_y$, $Z = \sigma_z$, $|\uparrow\rangle = |0\rangle$, $|\downarrow\rangle = |1\rangle$.

Useful relation:

$$e^{-i\mathbf{B}\cdot\boldsymbol{\sigma}} = \cos(B) - i \sin(B) \mathbf{e}_B \cdot \boldsymbol{\sigma}$$

where $\mathbf{e}_B = \mathbf{B}/|\mathbf{B}|$.

Results from non-degenerate perturbation theory:

$$E_n^{(1)} = \langle n | \hat{V} | n \rangle$$

$$E_n^{(2)} = \sum_{n' \neq n} \frac{|\langle n' | \hat{V} | n \rangle|^2}{\varepsilon_n - \varepsilon_{n'}}.$$

Result from time-dependent perturbation theory:

$$P_{n \rightarrow n'} = \frac{\lambda^2}{\hbar^2} \left| \int_0^t dt' \langle n' | \hat{V} | n \rangle e^{-i(\varepsilon_n - \varepsilon_{n'})t'/\hbar} \right|^2. \quad (1)$$

1. Symmetries in quantum mechanics

- (a) Free particle. A free particle is described by the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}$.

- (i) Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the translation operator. (2 marks)
- (ii) Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the parity operator. (2 marks)
- (iii) Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the time-reversal operator. (2 marks)

For each case, you are welcome to work within an explicit basis (like the position basis) or work without a basis.

- (b) Time-reversal symmetry and spin-orbit coupling.

- (i) Suppose a Hamiltonian is time-reversal invariant: $\hat{T}\hat{\mathcal{H}}\hat{T}^{-1} = \hat{\mathcal{H}}$. Further suppose that the time-reversal operator squares to minus one: $\hat{T}^2 = -1$. Show that this implies that all eigenstates of $\hat{\mathcal{H}}$ will be at least doubly-degenerate. (This is Kramer's theorem.) (4 marks)

- (ii) A (spin-half) electron moving in two dimensions is governed by the following Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \gamma \hat{\mathbf{p}} \cdot \hat{\mathbf{S}}$$

where \hat{p}_x, \hat{p}_y are the usual momentum operators, \mathbf{S} is a vector composed of the spin-half operators, $\hat{\mathbf{p}} \cdot \hat{\mathbf{S}} = \hat{p}_x \hat{S}_x + \hat{p}_y \hat{S}_y$, and γ is a positive real constant. The second term describes so-called spin-orbit coupling. Explain why this Hamiltonian is time-reversal invariant. (3 marks)

- (iii) Due to translational invariance, explain why the eigenstates of the Hamiltonian from (ii) (in the position and \hat{S}_z eigenbasis) have the form $\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \chi_{\mathbf{k}}$, where $\chi_{\mathbf{k}}$ is a spin-half spinor (or two-component vector). Determine the corresponding eigenenergies of $\hat{\mathcal{H}}$ for a given \mathbf{k} . (4 marks)

- (iv) The two eigenenergies you found in the previous part are not generally degenerate. Explain how Kramer's theorem is still satisfied for this system. (3 marks)

(Total: 20 marks)

2. Heisenberg equations of motion.

The following Hamiltonian describes ‘squeezing’ in quantum optics:

$$\hat{\mathcal{H}} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\gamma}{2}(\hat{a}\hat{a}e^{i\Omega t} + \hat{a}^\dagger\hat{a}^\dagger e^{-i\Omega t})$$

where ω , γ , and Ω are positive constants and \hat{a} is a bosonic (or ladder) operator.

- (a) Show that the Heisenberg equations of motion of \hat{a} are $i\frac{d}{dt}\hat{a}_H = \omega\hat{a}_H + \gamma e^{-i\Omega t}\hat{a}_H^\dagger$. (6 marks)
- (b) Let $\hat{b}_H = e^{i\Omega t/2}\hat{a}_H$. Show that \hat{b} satisfies bosonic commutation relations. Rewrite the equation of motion found in (a) in terms of the operator \hat{b} . (4 marks)
- (c) Taking the resonance condition $\Omega/2 = \omega$, solve the Heisenberg equations of motion. Suggestion: use the result of (b). (6 marks)
- (d) Suppose at $t = 0$, the system is in the vacuum state $|0\rangle$ where $\hat{a}|0\rangle = 0$. Find the expectation value of the number operator $\langle\psi(t)|\hat{a}^\dagger\hat{a}|\psi(t)\rangle$ at later times, again taking the system to be at resonance. (4 marks)

(Total: 20 marks)

3. Perturbation theory.

- (a) Consider the Hamiltonian

$$\hat{\mathcal{H}} = -w(|0\rangle\langle 0| - |1\rangle\langle 1|) + \lambda\gamma(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

Here, w and γ are positive real parameters and $\{|0\rangle, |1\rangle\}$ is an orthonormal basis spanning the Hilbert space. λ is the usual dimensionless perturbative parameter.

- (i) Using first-order perturbation theory, determine the ground state energy of this Hamiltonian to first order in λ . (3 marks)

- (ii) Find the λ^2 correction to the ground state energy using second order perturbation theory. (3 marks)

- (iii) Find the exact ground state energy of $\hat{\mathcal{H}}$. Show that this is consistent with the results you found in (i) and (ii). (4 marks)

- (b) Time-dependent perturbation theory. Consider the Hamiltonian

$$\hat{\mathcal{H}} = \hbar\omega\hat{a}^\dagger\hat{a} + \lambda\frac{\hbar\gamma}{2}(\hat{a}\hat{a}e^{i\Omega t} + \hat{a}^\dagger\hat{a}^\dagger e^{-i\Omega t})$$

where ω , γ , and Ω are positive constants and \hat{a} is a bosonic (or ladder) operator. λ is the usual dimensionless perturbative parameter. At $t = 0$, the system is in the vacuum state $|0\rangle$ where $\hat{a}|0\rangle = 0$. Using time-dependent perturbation theory, determine the probability that the second excited state is occupied at later time t to order λ^2 : $P_{0\rightarrow 2}$. Note: at this level of perturbation theory, this is the only non-vanishing transition. What is this transition probability $P_{0\rightarrow 2}$ at resonance: $\Omega = 2\omega$? Comment on the validity of perturbation theory at resonance.

(10 marks)

(Total: 20 marks)

4. Quantum gates and measurement.

(a) The CNOT and swap gates.

- (i) Recall our discussion of the CNOT gate from lectures. When acting on a two-qubit state in the computational basis, it does the following: $|n\rangle|m\rangle \rightarrow |n\rangle|m+n\rangle$ where integers inside kets and bras are taken to be mod 2. Let $|X_+\rangle$ be the +1 eigenstate of the Pauli \hat{X} operator. The two-qubit state $|X_+\rangle|X_+\rangle$ is sent through a CNOT gate. What is the resulting state in the computational basis?

(5 marks)

- (ii) The swap gate acts on an arbitrary two-qubit state as $|\phi\rangle|\psi\rangle \rightarrow |\psi\rangle|\phi\rangle$. Find a two-qubit eigenbasis for the swap gate.

(5 marks)

(b) Quantum measurement and probabilities.

- (i) Take the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. The first qubit is measured in the computational basis and it is determined to be in state $|0\rangle$. What is the two-qubit state after the measurement? Suppose instead that the first qubit of $|\psi\rangle$ is measured in the computational basis and it is determined to be in state $|1\rangle$. What is the two-qubit state after the measurement? Is $|\psi\rangle$ entangled?

(5 marks)

- (ii) A qubit is initially in the state $|0\rangle$. Then the operator $\hat{Z}\cos(\theta) + \hat{X}\sin(\theta)$ is measured (θ is an arbitrary angle). Finally \hat{Z} is measured. What is the probability that the state will be $|0\rangle$ after the final measurement?

(5 marks)

(Total: 20 marks)

5. Many particle systems

Consider the Hamiltonian that describes identical Fermions hopping between the vertices of an equilateral triangle:

$$\hat{\mathcal{H}}_0 = -w(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_3 + \hat{c}_3^\dagger \hat{c}_1) + \text{h.c.}$$

Here, “h.c.” means the Hermitian conjugate of the term preceding it (so that the full Hamiltonian is Hermitian) and \hat{c}_n are Fermionic operators (so that, for instance, $\{\hat{c}_2, \hat{c}_2^\dagger\} = 1$). We will also, in due course, consider adding the following interaction term

$$\hat{\mathcal{H}}_{\text{int}} = g\hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2.$$

This imposes a nearest-neighbor interaction between the first and second sites. The parameters w and g are real and positive. The full Hamiltonian is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}}$.

- (a) Describe how to write $\hat{\mathcal{H}}_0$ in diagonal form: $\hat{\mathcal{H}}_0 = \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k$ (using notation from lectures). For one particle, determine the eigenstates and eigenenergies of $\hat{\mathcal{H}}_0$. (6 marks)
- (b) Determine (or write down) an orthonormal basis for two-particle states of the system in terms of \hat{c}_1 , \hat{c}_2 , and \hat{c}_3 . (3 marks)
- (c) $\hat{\mathcal{H}}_0$ alone has both rotational and inversion symmetries. When $\hat{\mathcal{H}}_{\text{int}}$ is introduced, rotational symmetry is broken and one is left with only one inversion symmetry. The corresponding symmetry operator is $\hat{U} = e^{i\pi(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1)}$. This operator acts on the Fermionic operators as $\hat{U}\hat{c}_1\hat{U}^\dagger = \hat{c}_2$, $\hat{U}\hat{c}_2\hat{U}^\dagger = \hat{c}_1$, and $\hat{U}\hat{c}_3\hat{U}^\dagger = \hat{c}_3$. Show that \hat{U} commutes with $\hat{\mathcal{H}}$. (5 marks)
- (d) Determine a two-particle orthonormal eigenbasis of the \hat{U} operator. A good approach may be to act with \hat{U} on the basis you found in (b). Use this to determine an exact eigenstate of $\hat{\mathcal{H}}$ and corresponding eigenenergy. (6 marks)

(Total: 20 marks)

Solutions for Quantum Mechanics II Exam, 2025

1. Symmetries in quantum mechanics

- (a) Free particle. A free particle is described by the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}$.
- Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the translation operator.
Similar seen. Eigenstates of \hat{p} is also an eigenstates of the translation operator. So $\phi(x) = e^{ikx}$ works.
 - Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the parity operator.
Similar seen. The result of the previous part are not eigenstates of the parity operator. Taking linear combinations of degenerate states we see that $\phi(x) = \cos(kx)$ works. It has parity eigenvalue 1.
 - Find a simultaneous eigenstate of $\hat{\mathcal{H}}$ and the time-reversal operator.
Similar seen. The result of the previous part is real and so is an eigenstate of time-reversal. So $\phi(x) = \cos(kx)$ works here too.
- (b) Time-reversal symmetry and spin-orbit coupling.
- Suppose a Hamiltonian is time-reversal invariant: $\hat{T}\hat{\mathcal{H}}\hat{T}^{-1} = \hat{\mathcal{H}}$. Further suppose that the time-reversal operator squares to minus one: $\hat{T}^2 = -1$. Show that this implies that all eigenstates of $\hat{\mathcal{H}}$ will be at least doubly-degenerate. (This is Kramer's theorem.)
Seen. Proof given in module.
 - A (spin-half) electron moving in two dimensions is governed by the following Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \gamma \hat{\mathbf{p}} \cdot \hat{\mathbf{S}}$$

where \hat{p}_x, \hat{p}_y are the usual momentum operators, \mathbf{S} is a vector composed of the spin-half operators, $\hat{\mathbf{p}} \cdot \hat{\mathbf{S}} = \hat{p}_x \hat{S}_x + \hat{p}_y \hat{S}_y$, and γ is a positive real constant. The second term describes so-called spin-orbit coupling. Explain why this Hamiltonian is time-reversal invariant.

Unseen. It is important to recall that under time reversal momentum changes sign. Also spin changes sign. With this, it is pretty straightforward to show that $\hat{\mathcal{H}}$ is TRI.

- Due to translational invariance, explain why the eigenstates of the Hamiltonian from (ii) (in the position and \hat{S}_z eigenbasis) have the form $\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \chi_{\mathbf{k}}$, where $\chi_{\mathbf{k}}$ is a spin-half spinor (or two-component vector). Determine the corresponding eigenenergies of $\hat{\mathcal{H}}$ for a given \mathbf{k} .
Unseen. The momentum operators commute with the Hamiltonian. So a good approach is to simultaneously diagonalize \hat{p}_x, \hat{p}_y , and $\hat{\mathcal{H}}$. This directly leads to the form: $\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \chi_{\mathbf{k}}$. This is already an eigenstate of the momentum operators. Requiring it to be an eigenstate of $\hat{\mathcal{H}}$ determines the

spinor part. With some work, one sees that the eigenenergies are $E_{\mathbf{k},\pm} = \frac{\hbar k^2}{2m} \pm \frac{\hbar\gamma}{2}|\mathbf{k}|$.

- iv. The two eigenenergies you found in the previous part are not generally degenerate. Explain how Kramer's theorem is still satisfied for this system.

Unseen. The key is to consider different \mathbf{k} values. For the eigenstates, the spinor points either parallel or antiparallel to the \mathbf{k} vector. Therefore, states having \mathbf{k} are related to states with $-\mathbf{k}$ by the time-reversal operation. These form degenerate states protected by Kramer's theorem.

2. The following Hamiltonian describes ‘squeezing’ in quantum optics:

$$\hat{\mathcal{H}} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\gamma}{2}(\hat{a}\hat{a}e^{i\Omega t} + \hat{a}^\dagger\hat{a}^\dagger e^{-i\Omega t})$$

where ω , γ , and Ω are positive constants and \hat{a} is a bosonic (or ladder) operator.

- (a) Show that the Heisenberg equations of motion of \hat{a} are $i\frac{d}{dt}\hat{a}_H = \omega\hat{a}_H + \gamma e^{-i\Omega t}\hat{a}_H^\dagger$.

Similar seen. A direct computation using Heisenberg EOM formalism.

- (b) Let $\hat{b}_H = e^{i\Omega t/2}\hat{a}_H$. Show that \hat{b} satisfies bosonic commutation relations. Rewrite the equation of motion found in (a) in terms of the operator \hat{b} .

Unseen. If \hat{b}_H doesn't satisfy bosonic commutation relations, matters will be complicated so it is good to verify. Ans: $i\frac{d}{dt}\hat{b}_H = (\omega - \Omega/2)\hat{b}_H + \gamma\hat{b}_H^\dagger$. It removes the time dependence of the coefficients!

- (c) Taking the resonance condition $\Omega/2 = \omega$, solve the Heisenberg equations of motion. Suggestion: use the result of (b).

Unseen. The result from (b) simplifies to $i\frac{d}{dt}\hat{b}_H = \gamma\hat{b}_H^\dagger$. Solving this with appropriate initial conditions gives $\hat{b}_H(t) = \hat{b}(0)\cosh(\gamma t) - i\hat{b}_H^\dagger(0)\sinh(\gamma t)$. Finally, rewriting in terms of the original operator, $\hat{a}_H(t) = [\cosh(\gamma t)\hat{a} - i\sinh(\gamma t)\hat{a}^\dagger]e^{-i\Omega t/2}$.

- (d) Suppose at $t = 0$, the system is in the vacuum state $|0\rangle$ where $\hat{a}|0\rangle = 0$. Find the expectation value of the number operator $\langle\psi(t)|\hat{a}^\dagger\hat{a}|\psi(t)\rangle$ at later times, again taking the system to be at resonance.

Unseen. A key is to recognize that we can use our solution of the Heisenberg EOM to do this. That is, $\langle\psi(t)|\hat{a}^\dagger\hat{a}|\psi(t)\rangle = \langle 0|\hat{a}_H^\dagger\hat{a}_H|0\rangle$. Plugging our expression for \hat{a}_H and doing some manipulation, one finds $\langle\psi(t)|\hat{a}^\dagger\hat{a}|\psi(t)\rangle = \sinh^2(\gamma t)$.

3. Perturbation theory

- (a) Consider the Hamiltonian

$$\hat{\mathcal{H}} = -w(|0\rangle\langle 0| - |1\rangle\langle 1|) + \lambda\gamma(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

Here, w and γ are positive real parameters and $\{|0\rangle, |1\rangle\}$ is an orthonormal basis spanning the Hilbert space. λ is the usual dimensionless perturbative parameter.

- i. Using first-order perturbation theory, determine the ground state energy of this Hamiltonian to first order in λ .

Similar Seen. Ans: $E = -w + \lambda \cdot 0 = -w$.

- ii. Find the λ^2 correction to the ground state energy using second order perturbation theory.

Similar Seen. Ans: $E = -w + -\lambda^2 \frac{\gamma^2}{2w}$

- iii. Find the exact ground state energy of $\hat{\mathcal{H}}$. Show that this is consistent with the results you found in (i) and (ii).

Similar Seen. Ans: $E = -\sqrt{w^2 + \lambda^2 \gamma^2}$. Expanding, we get $E = -w\sqrt{1 + \lambda^2 \gamma^2/w^2} = -w(1 + \lambda^2 \gamma^2/(2w^2) + \dots)$ so consistent.

- (b) Time-dependent perturbation theory. Consider the Hamiltonian

$$\hat{\mathcal{H}} = \hbar\omega \hat{a}^\dagger \hat{a} + \lambda \frac{\hbar\gamma}{2} (\hat{a} \hat{a} e^{i\Omega t} + \hat{a}^\dagger \hat{a}^\dagger e^{-i\Omega t})$$

where ω , γ , and Ω are positive constants and \hat{a} is a bosonic (or ladder) operator. λ is the usual dimensionless perturbative parameter. At $t = 0$, the system is in the vacuum state $|0\rangle$ where $\hat{a}|0\rangle = 0$. Using time-dependent perturbation theory, determine the probability that the an excited state of the unperturbed Hamiltonian is occupied at later time t to order λ^2 . Hint: at this level of perturbation theory, the only non-vanishing transition is to the second excited state: $P_{0 \rightarrow 2}$. What is this transition probability $P_{0 \rightarrow 2}$ at resonance: $\Omega = 2\omega$? Comment on the validity of perturbation theory at resonance.

Unseen. Ans: $P_{0 \rightarrow 2} = \frac{\lambda^2 \gamma^2}{2(\omega - \Omega/2)^2} \sin^2((\omega - \Omega/2)t)$, all others vanish. At resonance, we have $P_{0 \rightarrow 2} = \frac{1}{2}\lambda^2 \gamma^2 t^2$. For this, perturbation theory will eventually (for sufficiently long time) break down.

4. Quantum gates and measurement.

- (a) The CNOT and swap gates.

- i. Recall our discussion of the CNOT gate from lectures. When acting on a two-qubit state in the computational basis, it does the following: $|n\rangle|m\rangle \rightarrow |n\rangle|m+n\rangle$ where integers inside kets and bras are taken to be mod 2. Let $|X_+\rangle$ be the $+1$ eigenstate of the Pauli \hat{X} operator. The two-qubit state $|X_+\rangle|X_+\rangle$ is sent through a CNOT gate. What is the resulting state in the computational basis?

Similar seen. After some computation one finds $|X_+\rangle|X_+\rangle \rightarrow \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle)$.

- ii. The swap gate acts on an arbitrary two qubit state as $|\phi\rangle|\psi\rangle \rightarrow |\psi\rangle|\phi\rangle$. Find a two-qubit eigenbasis for the swap gate.

Similar seen. Ans: $\{|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}$

- (b) Quantum measurement and probabilities

- i. Take the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. The first qubit is measured in the computational basis and it is determined to be in state $|0\rangle$. What is the state after the measurement? Now suppose the first qubit is measured in the computational basis and it is determined to be in state $|1\rangle$. What is the state after the measurement? Is $|\psi\rangle$ entangled?

Similar seen. Ans: For first case, $\frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle)$. For second case, $\frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle)$. (Need to show work.) It is not entangled as it can be written as a product state. The measurement of the first qubit does not influence the second, as seen in the calculation.

- ii. A qubit is initially in the state $|0\rangle$. Then the operator $\hat{Z}\cos(\theta) + \hat{X}\sin(\theta)$ is measured (θ is an arbitrary angle). Finally \hat{Z} is measured. What is the probability that the state will be $|0\rangle$ after the final measurement?

Unseen For first measurement: the probability of getting +1 is $(1+\cos(\theta))/2$ while the probability of getting -1 is $(1-\cos(\theta))/2$. Assuming +1 for the first measurement, the probability of getting +1 for the second is $(1+\cos(\theta))/2$ while the probability of getting -1 is $(1-\cos(\theta))/2$. Assuming -1 for the first measurement, the probability of getting +1 for the second is $(1-\cos(\theta))/2$ while the probability of getting -1 is $(1+\cos(\theta))/2$. Putting everything together, we find that the probability of getting +1 for the second measurement is: $P = (1+\cos(\theta))/2 \cdot (1+\cos(\theta))/2 + (1-\cos(\theta))/2 \cdot (1-\cos(\theta))/2 = (1+\cos^2(\theta))/2$.

5. Many particle systems

Consider the Hamiltonian that describes identical Fermions hopping between the vertices of an equilateral triangle:

$$\hat{\mathcal{H}}_0 = -w(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_3 + \hat{c}_3^\dagger \hat{c}_1) + \text{h.c.}$$

Here, “h.c.” means the Hermitian conjugate of the term preceding it (so that the full Hamiltonian is Hermitian) and \hat{c}_n are Fermionic operators (so that, for instance, $\{\hat{c}_2, \hat{c}_2^\dagger\} = 1$). We will also, in due course, consider adding the following interaction term

$$\hat{\mathcal{H}}_{\text{int}} = g\hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2.$$

This imposes a nearest-neighbor interaction between the first and second sites. The parameters w and g are real and positive. The full Hamiltonian is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}}$.

- (a) Describe how to write $\hat{\mathcal{H}}_0$ in diagonal form: $\hat{\mathcal{H}} = \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k$ (using notation from lectures). For one particle, determine the eigenstates and eigenenergies of $\hat{\mathcal{H}}_0$.

Seen. This is done by taking the Fourier transform of the \hat{c}_n operators: $\hat{c}_k = \frac{1}{\sqrt{N}} \sum_n e^{-ikn} \hat{c}_n$ where $N = 3$ and k takes on values $k = 0, 2\pi/3, 4\pi/3$. The single particle eigenstates are $\hat{c}_k^\dagger |0\rangle$ while the eigenenergies are $\varepsilon_k = -2w \cos(k)$ for these k values.

- (b) Determine (or write down) an orthonormal basis for two-particle states of the system in terms of \hat{c}_1 , \hat{c}_2 , and \hat{c}_3 .

Similar seen. Ans: $\hat{c}_1^\dagger \hat{c}_2^\dagger |0\rangle$, $\hat{c}_2^\dagger \hat{c}_3^\dagger |0\rangle$, $\hat{c}_3^\dagger \hat{c}_1^\dagger |0\rangle$.

- (c) $\hat{\mathcal{H}}_0$ alone has both rotational and inversion symmetries. When $\hat{\mathcal{H}}_{\text{int}}$ is introduced, rotational symmetry is broken and one is left with only one inversion symmetry. The corresponding symmetry operator is $\hat{U} = e^{i\pi(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1)}$. This operator acts on the Fermionic operators as $\hat{U}\hat{c}_1\hat{U}^\dagger = \hat{c}_2$, $\hat{U}\hat{c}_2\hat{U}^\dagger = \hat{c}_1$, and $\hat{U}\hat{c}_3\hat{U}^\dagger = \hat{c}_3$. Show that \hat{U} commutes with $\hat{\mathcal{H}}$.

Unseen. It is easiest to show that $\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger = \hat{\mathcal{H}}$, which is the same as $\hat{\mathcal{H}}$ and \hat{U} commuting. Key to realise this. Then clearly $\hat{U}\hat{\mathcal{H}}_0\hat{U}^\dagger = \hat{\mathcal{H}}_0$. For the other part, note $\hat{U}\hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2 \hat{U}^\dagger = \hat{c}_2^\dagger \hat{c}_2 \hat{c}_1^\dagger \hat{c}_1 = \hat{c}_1^\dagger \hat{c}_1 \hat{c}_2^\dagger \hat{c}_2$ (number operators commute).

- (d) Determine a two-particle orthonormal eigenbasis of the \hat{U} operator. A good approach may be to act with \hat{U} on the basis you found in (b). Use this to determine an exact eigenstate of $\hat{\mathcal{H}}$ and corresponding eigenenergy.

Unseen. Ans: $\frac{1}{\sqrt{2}}(\hat{c}_1^\dagger + \hat{c}_3^\dagger)\hat{c}_3^\dagger |0\rangle$, $\frac{1}{\sqrt{2}}(\hat{c}_1^\dagger - \hat{c}_2^\dagger)\hat{c}_3^\dagger |0\rangle$, $\hat{c}_1^\dagger \hat{c}_2^\dagger |0\rangle$. The first has inversion eigenvalue 1 while the last two have inversion eigenvalue -1. Since the first is a non-degenerate eigenstate of \hat{U} it will automatically be an eigenstate of $\hat{\mathcal{H}}$. Computing (with considerable work) its eigenenergy, one obtains $E = -w$.

MATH60018 Quantum Mechanics 2 Markers Comments

- Question 1 This question I believe was surprising for many students. In particular (a) perhaps was outside this box.
- Question 2 A mix. Many got full marks, some struggled. It was probably an expected question.
- Question 3 Probably another expected question. Combining time-independent and time-dependent perturbation theory in a single question might have made it longer than expected.
- Question 4 A quantum information question. I am always impressed with how well students learn quantum information. I thought (b, ii) was rather difficult but yet a number of students got full marks there.