Imperial College London

MATH97021 MATH70018

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 2

Date: 27 May 2022

Time: 12:00 – 14:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. This question is composed of several shorter problems touching on material covered throughout the module. If any computations are required to complete these subproblems, they should be fairly short. Unless stated otherwise, assume these subproblems are unrelated to each other.

(a) Evaluate and simplify the following expressions: $[\hat{p}, \hat{x}^2]$ and $[\hat{a}^\dagger, \hat{a}\hat{a}]$. (3 marks)

(b) Consider the matrix Hamiltonian $\mathcal{H} = \sigma_z + \lambda(\sigma_x + \sigma_z)$. Use perturbation theory to determine the first order (in λ) correction to the ground state energy. (4 marks)

(c) Suppose that $|\psi\rangle$ is time-reversal invariant. Show that $\langle\psi|(\hat{x}\hat{p} + \hat{p}\hat{x})|\psi\rangle = 0$. (3 marks)

(d) True or false: A single-qubit (unitary) quantum gate can transform $|0\rangle \to |0\rangle$ and $|1\rangle \to |0\rangle$. Explain. (3 marks)

(e) Consider the matrix Hamiltonian $\hat{\mathcal{H}} = \alpha\hat{S}^2$, where $\hat{\mathbf{S}}$ gives the spin components of a spin-s system. Show that the expectation value of $\hat{S}_z$ will be time-independent for motion governed by this Hamiltonian. (3 marks)

(f) Alice and Bob each hold a single qubit from an entangled pair corresponding to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The first entry in each ket corresponds to Alice's qubit and the second to Bob's. Alice measures her qubit in the X-Pauli basis and determines it is in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What then is the state of Bob's qubit? (4 marks)

(Total: 20 marks)

2. In this problem, we will consider Hamiltonians and dynamics involving the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ which satisfy the following commutation relations: $[\hat{A}, \hat{B}] = i\hat{C}$, $[\hat{B}, \hat{C}] = 0$, and $[\hat{C}, \hat{A}] = i\hat{B}$.

(a) Suppose that $\hat{A}$ and $\hat{B}$ are Hermitian operators. Explain why this then means that $\hat{C}$ is also Hermitian. In the remainder of the problem we will take all three operators to be Hermitian. (4 marks)

(b) Consider dynamics governed by the Hamiltonian $\hat{\mathcal{H}} = \hbar\Omega\hat{A}$ where Ω is a positive constant.

(i) Determine the Heisenberg equations of motion for $\hat{B}_{\mathrm{H}}(t)$ and $\hat{C}_{\mathrm{H}}(t)$. (4 marks)

(ii) Suppose at $t = 0$, the expectation value of $\hat{B}$ and $\hat{C}$ are $\bar{B}$ and $\bar{C}$ respectively. Find the expectation value of $\hat{B}$ and $\hat{C}$ for later times. (5 marks)

(c) Consider the operator $\hat{D} = \hat{B}^2 + \hat{C}^2$.

(i) This operator commutes with $\hat{A}$, $\hat{B}$, and $\hat{C}$. Explicitly show that $[\hat{A}, \hat{D}] = 0$. (3 marks)

(ii) Let $|\phi\rangle$ be an eigenstate of $\hat{D}$. Suppose that $\hat{C}|\phi\rangle \neq 0$. Show that this eigenstate of $\hat{D}$ is degenerate (i.e. there will be another linearly independent eigenstate of $\hat{D}$ having the same eigenvalue). (4 marks)

(Total: 20 marks)

3. Consider a two-particle system in one-dimension that interact through a Harmonic potential described by the Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega^2(\hat{x}_1 - \hat{x}_2)^2$. The position and momentum operators here satisfy the usual commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.

(a) Show that this Hamiltonian commutes with the total momentum $\hat{P} = \hat{p}_1 + \hat{p}_2$. This means that the system is translationally invariant. (4 marks)

(b) Introduce the new operators $\hat{X} = \frac{1}{2}(\hat{x}_1 + \hat{x}_2)$, $\hat{x} = \hat{x}_1 - \hat{x}_2$, $\hat{p} = \frac{1}{2}(\hat{p}_1 - \hat{p}_2)$. These operators also satisfy canonical commutation relations: $[\hat{X}, \hat{P}] = i\hbar$, $[\hat{x}, \hat{p}] = i\hbar$, $[\hat{X}, \hat{p}] = [\hat{x}, \hat{P}] = 0$. Verify explicitly that $[\hat{x}, \hat{p}] = i\hbar$. Rewrite the Hamiltonian in terms of these new operators ($\hat{x}$, $\hat{p}$, $\hat{X}$, $\hat{P}$).

(5 marks)

(c) Determine the eigenstates and eigenenergies of $\hat{\mathcal{H}}$. You may cite without derivation any results from lecture or the notes. Assume the particles are distinguishable.

(5 marks)

(d) Now let's take the two particles to be indistinguishable. How do the operators $\hat{x}$, $\hat{p}$, $\hat{X}$, and $\hat{P}$ transform under the operation of particle exchange? Which of the eigenstates found in part (c) are allowable for the cases where (i) the two particles are identical Bosons (ii) the two particles are identical Fermions?

(6 marks)

(Total: 20 marks)

4. In this question we will consider various quantum circuits.

(a) Recall that the (single-qubit) X-gate transforms states in the computational basis as: $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$. Suppose the X-gate is applied to the normalised state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. What is the resulting state? Suppose that a measurement is applied in the computational basis to this state after the X-gate is applied. What is the probability that the experimental outcome corresponds to the state $|0\rangle$? (5 marks)

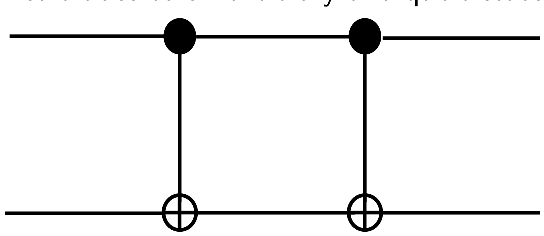
(b) Show that there is no unitary gate which serves to make a copy of a qubit onto a target qubit as $|\phi\rangle|0\rangle \to |\phi\rangle|\phi\rangle$ for arbitrary single-qubit state $|\phi\rangle$. (5 marks)

A suggested approach:

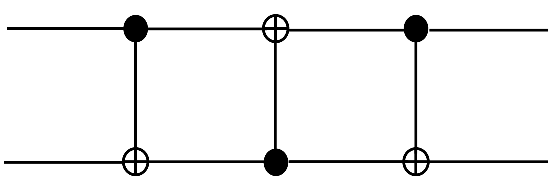
(i) Suppose such a unitary transformation exists. Determine how this transformation acts on the states $|0\rangle|0\rangle$ and $|0\rangle|1\rangle$.

(ii) Act with the unitary on a superposition of the states from (i) to reach a contradiction.

(c) The circuit diagram below shows two CNOT gates applied in succession. Describe what this circuit does to an arbitrary two-qubit state and hence simplify the circuit. (4 marks)



(d) Three CNOT gates are applied in succession as in the circuit below. Note that the middle CNOT gate sends $|n, m\rangle \to |n + m, m\rangle$ (binary arithmetic applies to quantities inside kets, and states are in the computational basis). Show that this full circuit serves as an exchange operator. That is, for arbitrary single-qubit states $|\phi\rangle$ and $|\phi'\rangle$ it performs the operation $|\phi\rangle\,|\phi'\rangle \to |\phi'\rangle\,|\phi\rangle$. (6 marks)



(Total: 20 marks)

5. A Hamiltonian that arises in the BCS theory of superconductivity is the following.

$$\hat{\mathcal{H}} = \varepsilon(\hat{c}^\dagger \hat{c} + \hat{d}^\dagger \hat{d}) + \Delta(\hat{c}\hat{d} + \hat{d}^\dagger \hat{c}^\dagger).$$

Here, ε and Δ are taken to be real positive parameters and $\hat{c}$ and $\hat{d}$ are Fermion operators satisfying the usual anticommutation relations, e.g. $\{\hat{c}, \hat{c}^\dagger\} = \{\hat{d}, \hat{d}^\dagger\} = 1$, $\{\hat{c}, \hat{d}^\dagger\} = 0$. This problem has a four-dimensional Hilbert space.

(a) Determine an orthonormal basis spanning the Hilbert space of this problem. (3 marks)

(b) Let $\hat{N} = \hat{c}^\dagger \hat{c} + \hat{d}^\dagger \hat{d}$, which is the (total) particle number operator. Show that $[\hat{N}, \hat{\mathcal{H}}] \neq 0$. Therefore the Hamiltonian of this problem does not conserve particle number. (5 marks)

(c) Let $\hat{U} = e^{i\pi \hat{N}}$. Explain why $\hat{U}$ is both unitary and Hermitian. Show that $[\hat{U}, \hat{\mathcal{H}}] = 0$. Therefore $\hat{U}$ and $\hat{\mathcal{H}}$ can be simultaneously diagonalised. (6 marks)

(d) Determine the ground state energy of $\hat{\mathcal{H}}$. Hint: by using the symmetry you found, you only need to work with 2×2 matrices. (6 marks)

(Total: 20 marks)

Solutions for Quantum Mechanics II Exam, 2022

1. This question involves several sub-problems covering the material appearing throughout the module. If any computations are required to complete these subproblems, they should be fairly short. Unless stated otherwise, assume these subproblems are unrelated to each other.

   (a) Evaluate and simplify $[\hat{p}, \hat{x}^2]$.

   Similar seen. $[\hat{p}, \hat{x}^2] = -i2\hbar\hat{x}$. $[\hat{a}^\dagger, \hat{a}\hat{a}] = -2\hat{a}$.

   (b) Consider the matrix Hamiltonian $\mathcal{H} = \sigma_z + \lambda(\sigma_x + \sigma_z)$. Use perturbation theory to determine the first order (in λ) correction to the ground state energy.

   Similar seen. This is a direct application of first order perturbation theory: $E_{(1)} = \langle\downarrow| (\sigma_z + \sigma_x) |\downarrow\rangle = -1 + 0 = -1$. So the ground state energy to first order is $E = -1 - \lambda$

   (c) Suppose that $|\psi\rangle$ is time-reversal invariant. Show that $\langle\psi|(\hat{x}\hat{p} + \hat{p}\hat{x}) |\psi\rangle = 0$.

   Unseen. Let $|\phi\rangle = \hat{p}|\psi\rangle$ and $|\xi\rangle = \hat{x}|\psi\rangle$. Also let $|\phi'\rangle = \hat{T}|\phi\rangle$, $|\xi'\rangle = \hat{T}|\xi\rangle$. Then it can be worked out that $|\phi'\rangle = -|\phi\rangle$ and $|\xi'\rangle = |\xi\rangle$. Since time reversal is antiunitary we have $\langle\xi|\phi\rangle = \langle\phi'|\xi'\rangle$. This then tells us that $\langle\xi|\phi\rangle = -\langle\phi|\xi\rangle$ which means that $\langle\psi|\hat{x}\hat{p}|\psi\rangle = -\langle\psi|\hat{p}\hat{x}|\psi\rangle$.

   (d) True or false: A single-qubit (unitary) quantum gate can transform $|0\rangle \to |0\rangle$ and $|1\rangle \to |0\rangle$.

   Unseen. False. For instance, this transformation is not invertible. On the other hand, unitary transformations are invertible.

   (e) Consider the matrix Hamiltonian $\hat{\mathcal{H}} = \alpha\hat{S}^2$, where $\hat{\mathbf{S}}$ gives the spin components of a spin-s system. Show that the expectation value of $\hat{S}_z$ will be time-independent for motion governed by this Hamiltonian.

   Unseen. Here we can apply the Heisenberg equations of motion to see that $\frac{d}{dt}(\hat{S}_z)_H = 0$. This is because $\hat{S}_z$ commutes with $\hat{S}^2$ (why)? Therefore the expectation value will be time independant.

   (f) Alice and Bob each hold a single qubit from an entangled pair corresponding to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The first entry in each ket corresponds to Alice's qubit while the second to Bob's. Alice measures her qubit in the X-Pauli basis and determines it is in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What then is the state of Bob's qubit?

   Unseen. The key is to realise that the state can be rewritten as $\frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$ where $|+\rangle$ and $|-\rangle$ are eigenstates of the Pauli-X operator. From this we see that Bob's qubit after the measurement will be $|+\rangle$.

2. In this problem, we will consider Hamiltonians and dynamics involving the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ which satisfy the following commutation relations: $[\hat{A}, \hat{B}] = i\hat{C}$, $[\hat{B}, \hat{C}] = 0$, and $[\hat{C}, \hat{A}] = i\hat{B}$.

(a) Suppose that $\hat{A}$ and $\hat{B}$ are Hermitian operators. Explain why this then means that $\hat{C}$ is also Hermitian. In the remainder of the problem we will take all three operators to be Hermitian.

Similar seen. Taking the adjoint of one of the given relations we have $-i\hat{C}^\dagger = -[\hat{A}^\dagger, \hat{B}^\dagger] = -[\hat{A}, \hat{B}]$. The RHS is $-i\hat{C}$. So it is Hermitian.

(b) Consider dynamics governed by the Hamiltonian $\hat{\mathcal{H}} = \hbar\Omega\hat{A}$ where Ω is a positive constant.

i. Determine the Heisenberg equations of motion for $\hat{B}_{\mathrm{H}}(t)$ and $\hat{C}_{\mathrm{H}}(t)$.
Similar seen. This is a matter of using expression from the Heisenberg EOM and the given commutation relations. After some work one finds $\frac{d}{dt}\hat{B}_H = -\Omega\hat{C}_H$ and $\frac{d}{dt}\hat{C}_H = \Omega\hat{B}_H$.

ii. Suppose at $t = 0$, the expectation value of $\hat{B}$ and $\hat{C}$ are $\bar{B}$ and $\bar{C}$ respectively. Find the expectation value of $\hat{B}$ and $\hat{C}$ for later times.
Unseen. For this, we need to solve the Heisenberg equations of motion. One way is to take another time derivative of the equations from (b). Then one finds $\frac{d^2}{dt^2}\hat{B}_H = -\Omega^2\hat{B}_H$ and $\frac{d^2}{dt^2}\hat{C}_H = -\Omega^2\hat{C}_H$. One finds $\hat{B}_H = \hat{B}\cos(\Omega t) - \hat{C}\sin(\Omega t)$ and $\hat{C}_H = \hat{C}\cos(\Omega t) + \hat{B}\sin(\Omega t)$. Taking the expectation value of these expressions immediately gives the result.

(c) Consider the operator $\hat{D} = \hat{B}^2 + \hat{C}^2$.

i. This operator commutes with $\hat{A}$, $\hat{B}$, and $\hat{C}$. Explicitly show that $[\hat{A}, \hat{D}] = 0$.
Unseen. Direct computation: $[\hat{A}, \hat{D}] = [\hat{A}, \hat{B}]\hat{B} + \hat{B}[\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]\hat{C} + \hat{C}[\hat{A}, \hat{C}] = i\hat{C}\hat{B} + i\hat{B}\hat{C} - i\hat{C}\hat{B} - i\hat{B}\hat{C} = 0$.

ii. Let $|\phi\rangle$ be an eigenstate of $\hat{D}$. Suppose that $\hat{C}|\phi\rangle \neq 0$. Show that this eigenstate of $\hat{D}$ is degenerate (i.e. there will be another linearly independent eigenstate of $\hat{D}$ having the same eigenvalue).
Unseen. Suppose that $|\phi\rangle$ is a non-degenerate eigenstate of $\hat{D}$. Then, since $\hat{A}$ and $\hat{B}$ commute with $\hat{D}$, $|\phi\rangle$ will also be an eigenstate of $\hat{A}$ and $\hat{B}$ (OK to cite notes or show explicitly): $\hat{A}\,|\phi\rangle = a\,|\phi\rangle\,, \hat{B}\,|\phi\rangle = b\,|\phi\rangle$. Then applying $|\phi\rangle$ to both sides of the equation $i\hat{C} = [\hat{A}, \hat{B}]$ gives $\hat{C}\,|\phi\rangle = 0$ which is a contradiction.

3. Consider a two-particle system in one-dimension that interact through a Harmonic potential described by the Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega^2(\hat{x}_1 - \hat{x}_2)^2$. The position and momentum operators here satisfy the usual commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.

(a) Show that this Hamiltonian commutes with the total momentum $\hat{P} = \hat{p}_1 + \hat{p}_2$. This means that the system is translationally invariant.

Similar seen. It is clear that the total momentum commutes with the kinetic energy. Also note that $[\hat{P}, \hat{x}_1 - \hat{x}_2] = -i\hbar + i\hbar = 0$. So it commutes with the Hamiltonian.

(b) Introduce the new operators $\hat{X} = \frac{1}{2}(\hat{x}_1 + \hat{x}_2)$, $\hat{x} = \hat{x}_1 - \hat{x}_2$, $\hat{p} = \frac{1}{2}(\hat{p}_1 - \hat{p}_2)$. These operators also satisfy canonical commutation relations: $[\hat{X}, \hat{P}] = i\hbar$, $[\hat{x}, \hat{p}] = i\hbar$, $[\hat{X}, \hat{p}] = [\hat{x}, \hat{P}] = 0$.

Unseen. $[\hat{x}, \hat{p}] = \frac{1}{2}[\hat{x}_1, \hat{p}_1] + \frac{1}{2}[\hat{x}_2, \hat{p}_2] = i\hbar$. The equations can be inverted to express the quantities entering the Hamiltonian in terms of the new operators. Then direct substitution gives $\hat{\mathcal{H}} = \frac{\hat{P}^2}{4m} + \frac{1}{m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$.

(c) Determine the eigenstates and eigenenergies of $\hat{\mathcal{H}}$. You may cite without derivation any results from lecture or the notes. Assume the particles are distinguishable.

Unseen. With a change of variables $\mu = m/2$, $\Omega = \sqrt{2}\omega$, $M = 2m$ we find that the Hamiltonian becomes $\hat{\mathcal{H}} = \frac{\hat{P}^2}{2M} + \frac{1}{2\mu}\hat{p}^2 + \frac{1}{2}\mu\Omega^2\hat{x}^2$. This Hamiltonian describes a free particle and a harmonic oscillator. So we see that the eigenstates will be $\Phi_{k,n}(X, x) = e^{ikX}\phi_n(x)$ where $\phi_n(x)$ are eigenstates of the harmonic oscillator (see notes). The eigenenergies are $E_{k,n} = \frac{k^2}{2M} + \hbar\Omega(n + 1/2)$.

(d) Now let's take the two particles to be indistinguishable. How do the operators $\hat{x}$, $\hat{p}$, $\hat{X}$, and $\hat{P}$ transform under the operation of particle exchange? Which of the eigenstates found in part (c) are allowable for the cases where (i) the two particles are identical Bosons (ii) the two particles are identical Fermions?

Unseen. Under particle exchange, $\hat{X}$ and $\hat{P}$ are unchanged while $\hat{x}$ and $\hat{p}$ obtain minus signs. Recall that states of the Harmonic oscillator are even/odd when n is even/odd. From this we see that the states with n odd will be acceptable Fermion states while states with even n will be acceptable Boson states. Note that all states are eigenstates of the exchange operator $\hat{\mathscr{P}}$.

4. In this question we will consider various quantum circuits.

(a) Recall that the (single-qubit) X-gate transforms states in the computational basis as: $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$. Suppose the X-gate is applied to the normalised state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. What is the resulting state? Suppose that a measurement is applied in the computational basis to this state after the X-gate is applied. What is the probability that the experimental outcome corresponds to the state $|0\rangle$?

Similar seen. Because gates operate linearly, we have $|\psi\rangle \to \alpha|1\rangle + \beta|0\rangle$. Using postulates of QM, we see that the probability that the state is in $|0\rangle$ is $|\beta|^2$.

(b) Show that there is no unitary gate which serves to make a copy of a qubit onto a target qubit as $|\phi\rangle|0\rangle \to |\phi\rangle|\phi\rangle$ for any single-qubit state $|\phi\rangle$.

Unseen. Follow the suggestion. $\hat{U}\,|00\rangle = |00\rangle$, $\hat{U}\,|10\rangle = |11\rangle$. Now use linearity and act with this gate on $|\phi\rangle\,|0\rangle$ where $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$ to obtain $|\phi\rangle\,|0\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. However, if $\hat{U}$ copies as it is assumed it does then $\hat{U}\,|\phi\rangle\,|0\rangle = |\phi\rangle\,|\phi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |01\rangle + |10\rangle)$ which clearly differs from the result of the previous sentence. So a contradiction is reached.

(c) The circuit diagram below shows two CNOT gates applied in succession. Describe what this circuit does to an arbitrary two-qubit state and hence simplify the circuit.

Unseen. Applying the two CNOT gates to a state in the computational basis we get $|n, m\rangle \to |n, n+m\rangle \to |n, 2n+m\rangle$. Then noting that binary arithmetic applies to quantities inside kets we find that the final state is $|n, m\rangle$. So the gate leaves states in the computational basis unchanged. Since the computational basis is complete, any state is unchanged. So the circuit does nothing.

(d) Three CNOT gates are applied in succession as in the circuit below. Note that the middle CNOT gate sends $|n, m\rangle \to |n+m, m\rangle$ (binary arithmetic applies to quantities inside kets, and states are in the computational basis). Show that this full circuit serves as an exchange operator. That is, for arbitrary single-qubit states $|\phi\rangle$ and $|\phi'\rangle$ it performs the operation $|\phi\rangle|\phi'\rangle \to |\phi'\rangle|\phi\rangle$.

Unseen. Applying the CNOT gates to a state in the computational basis: $|n, m\rangle \to |n, n+m\rangle \to |2n+m, n+m\rangle = |m, n+m\rangle \to |m, n+2m\rangle = |m, n\rangle$. The the circuit sends $|n, m\rangle$ to $|m, n\rangle$. So $|\phi\rangle|\phi'\rangle = \sum \phi_n \phi'_m |n, m\rangle \to \sum \phi_n \phi'_m |m, n\rangle = |\phi'\rangle|\phi\rangle$.

5. A Hamiltonian that arises in the BCS theory of superconductivity is the following.

$$\hat{\mathcal{H}} = \varepsilon(\hat{c}^\dagger \hat{c} + \hat{d}^\dagger \hat{d}) + \Delta(\hat{c}\hat{d} + \hat{d}^\dagger \hat{c}^\dagger).$$

Here, ε and Δ are taken to be real positive parameters and $\hat{c}$ and $\hat{d}$ are Fermion operators satisfying the usual anticommutation relations, e.g. $\{\hat{c}, \hat{c}^\dagger\} = \{\hat{d}, \hat{d}^\dagger\} = 1$, $\{\hat{c}, \hat{d}^\dagger\} = 0$. This problem has a four-dimensional Hilbert space.

(a) Determine (or write down) an orthonormal basis spanning this Hilbert space.

Similar seen. $|0\rangle$, $\hat{c}^\dagger|0\rangle$, $\hat{d}^\dagger|0\rangle$, $\hat{c}^\dagger\hat{d}^\dagger|0\rangle$.

(b) Let $\hat{N} = \hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d}$, which is the (total) particle number operator. Show that $[\hat{N}, \hat{\mathcal{H}}] \neq 0$. Therefore the Hamiltonian of this problem does not conserve particle number.

Unseen. A computation, making regular use of commutator identities, gives $[\hat{N}, \hat{\mathcal{H}}] = 2\Delta(\hat{d}\hat{c} + \hat{c}^\dagger\hat{d}^\dagger)$ which is nonzero.

(c) Let $\hat{U} = e^{i\pi\hat{N}}$. Explain why $\hat{U}$ is both unitary and Hermitian. Show that $[\hat{U}, \hat{\mathcal{H}}] = 0$. Therefore $\hat{U}$ and $\hat{\mathcal{H}}$ can be simultaneously diagonalised.

Unseen. Note that $\hat{U}^\dagger = e^{-i\pi\hat{N}}$. So $\hat{U}^\dagger\hat{U} = 1$ which means it is unitary. Next note that $\hat{N}$ has integer eigenvalues.This means that $\hat{U} = \hat{U}^\dagger$ since $e^{i\pi n} = e^{-i\pi n}$ for any integer n. So it is also Hermitian.

It can be worked out that $\hat{U}\hat{c}\hat{U}^\dagger = -\hat{c}$ and $\hat{U}\hat{d}\hat{U}^\dagger = -\hat{d}$. This means that the Hamiltonian is unchanged under the unitary transformation.

(d) Determine the ground state energy of $\hat{\mathcal{H}}$. Hint: by using the symmetry you found, you only need to work with 2×2 matrices.

Unseen. We can diagonalise within subspaces having even and odd particle numbers. Using the basis elements found in (a) the two matrices one finds are

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -\Delta \\ -\Delta & 2\varepsilon. \end{pmatrix}$$

The eigenvalues are ε (degenerate) and $\varepsilon \pm \sqrt{\varepsilon^2 + \Delta^2}$. So the ground state energy is $\varepsilon - \sqrt{\varepsilon^2 + \Delta^2}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60018 MATH97021 MATH70018	1	This question was of varying difficulty, ranging from very straightforward to challenging. I think I will need to spend more time explaining TRS.
MATH60018 MATH97021 MATH70018	2	I was pleased with a variety of approaches used to solve part c. A lot of creativity was demonstrated.
MATH60018 MATH97021 MATH70018	3	This question at first glance might have seemed unfamiliar. However, many found that it demonstrated familiar concepts.
MATH60018 MATH97021 MATH70018	4	I was very pleased with many high marks on this question. It focused on new material, introduced in 2022.
MATH97021	5	This was a more challenging Mastery question. Some did very well on it, many struggled.