Computational PDEs MATH60025/70025, 2024-2025
Released : 17 March 2025
Upload Deadline : 1.00 pm, 31 March 2024
The project mark, will be weighted to comprise 35% of the overall Module.

You are required to investigate the Questions below and summarise your findings in form of a well written project report – on which you will be assessed.

Please name your files in following way:

- Technical report : **CPDES_Q3_yourCID.pdf** (limit your report to 16 pages or less (including plots). **Anything beyond the 16 page limit will NOT be marked!**
- All your code(s), label as follows :
  CPDES_Q3_of_X_yourCID.m (Matlab scripts example) or
  CPDES_Q3_of_X_yourCID.py (Python scripts).
  Zip all program files and call your zipped folder: **CPDES_Q3_programs_yourCID.zip**

Where in the above **CID** will be your College ID number.

Notes: *Important*

1. Marking will consider both the correctness of your code as well as the soundness of your analysis and clarity and legibility of the technical report.
2. Exam mark will primarily be based on contents of your written technical report. You are warned that if you ONLY submit the codes for the work with **NO technical report**, you can **NOT** expect a pass mark.
3. Do **NOT** include source code listing in your technical report.
4. All figures created by your code should be well-made and properly labelled in the technical pdf report.
5. The codes **must** be submitted.

Project 3: Hyperbolic Systems

Part A: (15 Marks)

The one-dimensional wave equation for $u(x,t)$ is given by

$$u_{tt} = c^2(x)\, u_{xx}, \tag{1.1}$$

where x represents a spatial coordinate and t the time. At $t = 0$

$$u(x,0) = \exp\left(-(2x-5)^2\right)\ ; \qquad \frac{\partial u}{\partial t}(x,0) = 0. \tag{1.2}$$

The wave speed $c(x) = 1$.

1. Investigate how the solution to Eqn. 1.1 evolves for $t > 0$. Discretise the equation based on your lecture notes, such that it is second-order accurate in time and space using the Leapfrog scheme from lectures:

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = c_j^2 \left(\frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{(\Delta x)^2} \right). \tag{1.3}$$

   At the computational end points, namely $x = \pm 10$, investigate the treatment of boundary conditions which satisfy the following:

   - Minimal numerical reflections off the left outer boundary; *i.e.* the waves pass through the boundary at $x = -10$ (also known as a transparent condition).
   - The solid wall condition on the right boundary of $\partial u / \partial x = 0$ at $x = 10$.

   Through appropriate numerical experiments investigate, discuss and demonstrate the accuracy of your numerical solution, and any dissipation and dispersive effects, as the Courant number varies, that you observe through appropriate plots, showing key features.

   Investigate how your results are affected as you vary the CFL parameter.

   Include a discussion on the "modified PDE" that Eqn.(1.4) represents, and discuss in relation to your numerical investigations.

2. Next, modify your code to set the wave speed such that it has the form

$$c(x) = (2+x)/2, \ \text{ in the region } \ -1 \le x \le 0,$$

   and $c(x) = 1$ elsewhere. Discuss and show by appropriate plots any new features you find.

Part B: (14 Marks)

The PML form of the wave equation is as follows

$$u_{tt} + 2\sigma u_t + \sigma^2 u = c^2(x) u_{xx},$$

with σ the PML absorption parameter.

1. Using a similar strategy as used earlier, namely the Leapfrog scheme, undertake a discrete dissipation-dispersion Fourier analysis to analyse the discretised equation, and investigate dissipation and dispersive behaviour. Are there any limits or bounds for σ which minimises dispersive and (or) maximises dissipative behaviour, but still provides a stable numerical scheme? Analyse and discuss for the case $c(x) = 1$.
2. Prescribing a PML layer for $x \leq -7$, devise a code to solve the above PML form of the wave equation; take $c(x) = 1$. The right boundary is prescribed to be a solid wall, satisfying the $u_x = 0$ condition at $x = 10$. At the left boundary devise an appropriate $\sigma(x)$ distribution, which allows minimal reflection off the PML interface at $x = -7$, as well as maximal dissipation of the wave prior to it reaching the left boundary at $x = -10$. Through appropriate plots show in a reasonably concise form your results, effectiveness of your PML and discuss your findings.

Part C: (6 Marks)

An implicit discretisation of the wave equation is as follows

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = c_j^2 \left(\frac{(\delta^2 U)_j^{k+1} + 2(\delta^2 U)_j^k + (\delta^2 U)_j^{k-1}}{4(\Delta x)^2} \right). \tag{1.4}$$

Here you may take $c_j = 1$.

Undertake a Fourier stability analysis to investigate stability of the scheme. Is there a CFL number bound on the numerical stability?