

## Topic: Probability and conditional probability

In today's problem class we will be reviewing the probability axioms and we will study problems involving conditional probabilities.

1. Given events  $E, F, G \subseteq \Omega$ , prove that

- (a)  $P(E^c \cap F) = P(F) - P(E \cap F)$
- (b)  $P(E \cup F) \leq P(E) + P(F)$
- (c)  $E \subseteq F, F \subseteq G \implies P(E) \leq P(G)$
- (d)  $P(E \cap F) \geq P(E) + P(F) - 1$

[(d) is known as *Bonferroni's Inequality*.]

2. Suppose that  $E$  and  $F$  are events such that  $P(E) = x$ ,  $P(F) = y$  and  $P(E \cap F) = z$ . Express the following terms in terms of  $x$ ,  $y$  and  $z$ :

- (a)  $P(E^c \cup F^c)$
- (b)  $P(E^c \cap F)$
- (c)  $P(E^c \cup F)$
- (d)  $P(E^c \cap F^c)$

3. A crime has been committed and a suspect is being held by police. He is either guilty,  $G$ , or not,  $G^c$ , and the probability of his being guilty on the basis of current evidence is  $P(G) = p$ , say. Forensic evidence is now produced which shows that the criminal must have a property,  $A$ , which occurs in a proportion,  $\pi$ , of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population, so that  $P(A|G^c) = \pi$ .

The suspect is now interrogated and found to have property A. Show that the odds on his guilt have now risen from  $\frac{P(G)}{P(G^c)} = p/(1-p)$  to  $\frac{P(G|A)}{P(G^c|A)} = \frac{P(G)}{\pi P(G^c)}$ .

*Note:* The odds on an event  $E$  are defined to be the ratio  $P(E)/P(E^c)$ , the odds-against  $E$  are  $P(E^c)/P(E)$ .

4. A shop sells fuses produced by three manufacturers; each manufacturer supplies a deluxe and a standard type of fuse. A mixed batch of 500 fuses is sold, and the number of faulty fuses of each type and for each manufacturer is recorded. By considering the following events;  $M_i \equiv$  “fuse produced by manufacturer  $i$ ” for  $i = 1, 2, 3$ ,  $D \equiv$  “Deluxe type of fuse” and  $F \equiv$  “Fuse Faulty”, a summary of the data can be presented as a 3-way table

	$M_1$		$M_2$		$M_3$	
	$D$	$D^c$	$D$	$D^c$	$D$	$D^c$
$F$	20	16	30	20	15	10
$F^c$	100	64	120	30	60	15

so that, for example, the number of deluxe fuses from manufacturer 1 that are faulty is 20, whereas the number of standard fuses from manufacturer 1 that are faulty is 16, etc.

- (a) A fuse is selected with equal probability from the 500. What is the probability that
  - i. it is faulty?
  - ii. it was produced by manufacturer 1?
- (b) Given that the selected fuse is faulty, what is the conditional probability that
  - i. it is a deluxe fuse?
  - ii. it is a fuse produced by manufacturer 1?
  - iii. it is a deluxe fuse produced by manufacturer 1?
- (c) Describe, evaluate, and comment on the following conditional probabilities:
  - i.  $P(F|M_1)$ ,  $P(F|M_2)$ ,  $P(F|M_3)$
  - ii.  $P(F|D)$ ,  $P(F|D^c)$
  - iii.  $P(F|M_1 \cap D)$ ,  $P(F|M_2 \cap D)$ ,  $P(F|M_3 \cap D)$ .
  - iv.  $P(F|M_1 \cap D^c)$ ,  $P(F|M_2 \cap D^c)$ ,  $P(F|M_3 \cap D^c)$ .