ExamModuleCode	Question Number	Comments for Students
M45A52	1	Note: comments are relevant for bothh M3 and M45 students. This question consisted mostly of (relatively) non-technical parts. It was clear that most of you revised thoroughly for the exam and could answer correctly (a)-(e). Part (f) was the most difficult part, but a few of you did this correctly.
M45A52	2	Most did well on this question which was an application to the Heisenberg equations of motion to an unfamiliar example. Some of you became stuck on evaluating the necessary fermionic commutation relation.
M45A52	3	This was an application of perturbation theory to a single spin. Most did this first parts correctly, but some of you had trouble working the problem for arbitrary spin and so resorted to the spin-half case (for which some partial credit was given). Nobody received full marks on the last part, but some of you knew that a spin rotation was needed.
M45A52	4	This problem received slightly lower marks than the others. Evaluating the necessary integral gave some candidates trouble. Some made the last part (on adiabaticity) overly complicated and found results having incorrect units.
M45A52	5	I was pleasantly surprised with how well many of you did on the mastery problem. It was a difficult problem to work in an exam setting, and suspect many of you benefited from working closely through the optional coursework 1 problem.

Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics II

Date: Tuesday 07 May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.

- Calculators may not be used.

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics II

Date: Tuesday 07 May 2019

Time: 14.00 - 16.30

Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

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Formula Sheet

Relations satisfied by the spin operators:

$$\hat{S}^2 \ket{s,m} = \hbar^2 s(s+1) \ket{s,m}$$
$$\hat{S}_z \ket{s,m} = \hbar m \ket{s,m}$$
$$\hat{S}_+ \ket{s,m} = \hbar\sqrt{s(s+1) - m(m+1)} \ket{s,m+1}$$
$$\hat{S}_- \ket{s,m} = \hbar\sqrt{s(s+1) - m(m-1)} \ket{s,m-1}.$$

Results from non-degenerate perturbation theory:

$$E_n^{(1)} = \bra{n} \hat{V} \ket{n}$$
$$E_n^{(2)} = \sum_{n' \neq n} \frac{|\bra{n'} \hat{V} \ket{n}|^2}{\varepsilon_n - \varepsilon_{n'}}.$$

Result from time-dependent perturbation theory:

$$P_{n \to n'} = \frac{\lambda^2}{\hbar^2} \left| \int_{t_i}^{t_f} dt \, \bra{n'} \hat{V}(t) \ket{n} \, e^{-i(\varepsilon_n - \varepsilon_{n'})t/\hbar} \right|^2 \tag{1}$$

where t_i and t_f denote the initial and final times and $n \neq n'$.

1. This problem consists of a number of short sub-problems covering material throughout the module. Throughout this problem, notation used in lectures and the notes is adhered to. Minimal calculation should be needed for this problem.

   (a) Evaluate and simplify $[\hat{a}, (\hat{a}^\dagger)^2]$ where $\hat{a}$ is a bosonic annihilation operator satisfying $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = 0$.

   (b) Evaluate and simplify $[\hat{c}, \hat{c}^\dagger \hat{c}]$ where $\hat{c}$ is a fermionic annihilation operator satisfying $\{\hat{c}, \hat{c}^\dagger\} = 1$ and $\{\hat{c}, \hat{c}\} = 0$.

   (c) State the Pauli exclusion principle. Explain why this principle is automatically enforced in the (fermionic) second quantised language.

   (d) State the spin-statistics theorem.

   (e) In a paragraph, compare and contrast orbital angular momentum with the spin of a fundamental particle.

   (f) Let $\hat{S}_a$ be the spin operators for a spin 7/2 particle. Consider the Hamiltonian $\hat{\mathcal{H}} = J\left[3(\hat{S}_x)^2 + 2\hat{S}_z\hat{S}_y + 2\hat{S}_y\hat{S}_z + 7(\hat{S}_z)^4\right]$ where J is a real constant having dimensions of energy. True or false: $\hat{\mathcal{H}}$ has a non-degenerate ground state. Explain.

2. In this problem, we consider the Hamiltonian

$$\hat{\mathcal{H}} = \Delta(\hat{c}\hat{d} + \hat{d}^\dagger\hat{c}^\dagger)$$

where $\hat{c}$ and $\hat{d}$ are fermionic annihilation operators. That is, letting $\hat{c} = \hat{c}_1$ and $\hat{d} = \hat{c}_2$ they satisfy the relations $\{\hat{c}_n, \hat{c}_{n'}^\dagger\} = \delta_{nn'}$ and $\{\hat{c}_n, \hat{c}_{n'}\} = 0$ where curly brackets denote the anticommutator. The quantity Δ in the above is a constant parameter having dimensions of energy. This Hamiltonian contains essential elements of "Cooper pairing" in superconductors.

(a) What restriction does the requirement that $\hat{\mathcal{H}}$ is Hermitian place on Δ? Evaluate and simplify the commutator $[\hat{c}, \hat{c}^\dagger\hat{d}^\dagger]$.

(b) Show that the Heisenberg equations of motion for the fermionic operators can be written as

$$i\hbar\frac{d}{dt}\hat{c}_H = -\Delta\hat{d}_H^\dagger$$
$$i\hbar\frac{d}{dt}\hat{d}_H^\dagger = -\Delta\hat{c}_H.$$

(c) Solve the above Heisenberg equations of motion to thereby express $\hat{c}_H$ and $\hat{d}_H$ in terms of $\hat{c}$ and $\hat{d}$ with time-dependent coefficients.

(d) Suppose at time $t = 0$, the initial state is the vacuum $|0\rangle$ (where $\hat{c}|0\rangle = \hat{d}|0\rangle = 0$). Evaluate the expectation value of the total particle number operator $\hat{N} = \hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d}$ for later times.

3. Consider a general spin-s particle governed by the Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{V}$ where $\hat{\mathcal{H}}_0 = A\hat{S}_z$ and $\hat{V} = B\hat{S}_x$. Here, A and B are positive constants having dimensions of frequency and $\lambda \in \mathbb{R}$ is our usual perturbative parameter.

(a) What are the eigenenergies of $\hat{\mathcal{H}}_0$? What is the ground state of $\hat{\mathcal{H}}_0$?

(b) Can $\hat{\mathcal{H}}_0$ and $\hat{V}$ be simultaneously diagonalised?

(c) Using first-order perturbation theory, determine the first-order correction (in λ) to the ground state energy.

(d) Using second-order perturbation theory, determine the second-order correction (in λ) to the ground state energy.

(e) Find the exact ground state energy of $\hat{\mathcal{H}}$. Show that this is consistent with the results from (c) and (d).

4. Consider the quantum harmonic oscillator $\hat{\mathcal{H}}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$ under the time-dependent perturbation $\hat{V} = \hat{x}f(t)$. Here, $f(t) = e^{-t^2/\tau^2}f_0$ where f_0 and τ are real positive constants having dimensions of force and time respectively. The full Hamiltonian for the system is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{V}$ where $\lambda \in \mathbb{R}$ is our usual perturbative parameter. Suppose at $t = -\infty$, the system is in the harmonic oscillator ground state.

(a) Using the ladder operators, $\hat{a}$ and $\hat{a}^\dagger$ where $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$, show that the Hamiltonian can be written as

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda f(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger).$$

(b) Using time-dependent perturbation theory, show that to lowest non-trivial order in λ, the probability for the system to transition to the first excited state of the harmonic oscillator at $t = \infty$ is

$$P_{0\to1} = \frac{\pi\lambda^2 f_0^2\tau^2}{2m\hbar\omega}e^{-\omega^2\tau^2/2}$$

(c) Determine the probability for the system to remain in the harmonic oscillator ground state at $t = \infty$ to second order in λ.

(d) Give the condition on the parameter τ for the system to be in the adiabatic regime. Is your result from (c) consistent with your expectations in the adiabatic regime?

5. (Mastery) Time-dependent unitary transformations.

(a) Suppose $|\psi\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar\partial_t |\psi\rangle = \hat{\mathcal{H}} |\psi\rangle$. Suppose $|\psi'\rangle$ is related to $|\psi\rangle$ by a time-dependent unitary transformation: $|\psi'\rangle = \hat{U}(t) |\psi\rangle$ where $\hat{U}(t)$ is a time-dependent unitary operator. Show that $|\psi'\rangle$ also satisfies a time-dependent Schrödinger equation $i\hbar\partial_t |\psi'\rangle = \hat{\mathcal{H}}' |\psi'\rangle$. What is $\hat{\mathcal{H}}'$?

(b) For the remainder of this problem, let us restrict to unitary transformation corresponding to the operator

$$\hat{U}(t) = e^{i\hat{p}\xi/\hbar} e^{-i\hat{x}m\dot{\xi}/\hbar} e^{i\alpha}$$

where ξ is a real time-dependent quantity having dimensions of length and $\dot{\xi} = d\xi/dt$. Also $\alpha(t)$ is an unspecified real function of t.

Show that under this unitary transformation, $\hat{x}' \equiv \hat{U}(t)\hat{x}\hat{U}^\dagger(t) = \hat{x}+\xi$ and $\hat{p}' \equiv \hat{U}(t)\hat{p}\hat{U}^\dagger(t) = \hat{p} + m\dot{\xi}$.

(c) Next consider the time-dependent Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2(\hat{x} - f(t))^2$$

where $f(t)$ is an unspecified real function of t having dimensions of length. This Hamiltonian corresponds to a driven harmonic oscillator. Taking ξ to satisfy Newton's classical equation of motion

$$\ddot{\xi} + \omega^2\xi = \omega^2 f(t),$$

show that with a suitable choice of $\alpha(t)$, the transformed Hamiltonian is:

$$\hat{\mathcal{H}}' = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

(you do not need to attempt to solve the differential equation you find for $\alpha(t)$ or $\xi(t)$).

(d) Consider the ground-state solution to the Schrödinger equation in the primed frame: $|\psi'\rangle = e^{-i\omega t/2} |\phi_0\rangle$. Determine the expectation value of $\hat{x}$ in the original frame $\langle\psi| \hat{x} |\psi\rangle$ corresponding to this solution. Comment on your result within the context of Ehrenfest's theorem.

Solutions for MA52 Exam, 2019

1. This problem consists of a number of short sub-problems covering material throughout the module. Throughout this problem, notation used in lecture and the notes is adhered to. Minimal calculation should be needed for this problem.

   (a) Evaluate and simplify $[\hat{a}, (\hat{a}^\dagger)^2]$ where $\hat{a}$ is a bosonic annihilation operator satisfying $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = 0$.

   Seen.

   3 Marks (Level 1)

   The result follows from the application of the identity $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$ and the bosonic commutation relations stated in the problem. Using this, one finds $[\hat{a}, (\hat{a}^\dagger)^2] = 2\hat{a}^\dagger$.

   (b) Evaluate and simplify $[\hat{c}, \hat{c}^\dagger\hat{c}]$ where $\hat{c}$ is a fermionic annihilation operator satisfying $\{\hat{c}, \hat{c}^\dagger\} = 1$ and $\{\hat{c}, \hat{c}\} = 0$.

   Seen.

   3 Marks (Level 1)

   The result follows from the application of the identity $[\hat{A}, \hat{B}\hat{C}] = \{\hat{A}, \hat{B}\}\hat{C} - \hat{B}\{\hat{A}, \hat{C}\}$ and the fermionic commutation relations stated in the problem. Using this, one finds $[\hat{c}, \hat{c}^\dagger\hat{c}] = \hat{c}$.

   (c) State the Pauli exclusion principle. Explain why this principle is automatically enforced in the (fermionic) second quantised language.

   Seen.

   4 Marks (Level 1)

   The PEP states that two identical Fermions cannot occupy the same single-particle state. This is enforced in the second-quantised language because fermionic creation operators anticommute. Suppose that $\hat{c}_n^\dagger$ creates a fermion in the nth single-particle state. Since $\hat{c}_n^\dagger\hat{c}_{n'}^\dagger = -\hat{c}_{n'}^\dagger\hat{c}_n^\dagger$ we have $(\hat{c}_n^\dagger)^2 = 0$.

   (d) State the spin-statistics theorem.

   Seen.

   2 Marks (Level 1)

   Bosons have integer spin while Fermions have half-integer spin.

   (e) In a paragraph, compare and contrast orbital angular momentum with the spin of a fundamental particle.

   Seen.

   4 Marks (Level 1)

   Marks awarded for accuracy and clarity.

   Sample paragraph:

The spin and orbital angular momentum operators satisfy the same algebra: $[\hat{L}_a, \hat{L}_b] = i\hbar\varepsilon_{abc}\hat{L}_c$ and $[\hat{S}_a, \hat{S}_b] = i\hbar\varepsilon_{abc}\hat{S}_c$. Important differences between orbital angular momentum are revealed by the quantum numbers ℓ and s arising from the possible eigenvalues of $\hat{L}^2$ and $\hat{S}^2$. For orbital angular momentum, ℓ is restricted to non-negative integer values. Additionally, a single quantum mechanical system with orbital angular momentum can transition between different values of ℓ (e.g. consider exciting into higher angular momentum states). On the other hand, for a fundamental particle, s takes on only a fixed value. Also, unlike orbital angular momentum, s for certain particles (fermions) can take on a half-integer value. For instance, an electron always has $s = 1/2$.

(f) Let $\hat{S}_a$ be the spin operators for a spin 7/2 particle. Consider the Hamiltonian $\hat{\mathcal{H}} = J\left[3(\hat{S}_x)^2 + 2\hat{S}_z\hat{S}_y + 2\hat{S}_y\hat{S}_z + 7(\hat{S}_z)^4\right]$ where J is a real constant having dimensions of energy. True or false: $\hat{\mathcal{H}}$ has a non-degenerate ground state. Explain.

Unseen

4 Marks (Level 4)

This Hamiltonian is too complicated to diagonalise in an exam setting! The key is to notice that the Hamiltonian is invariant under the operation of time reversal. All of the coefficients of the spin operators in the Hamiltonian are real. Additionally, under time reversal, $\hat{S}_a \to -\hat{S}_a$. Therefore the whole Hamiltonian is unchanged under time reversal. Since we are working with a half-integer spin system, from a result we showed in lecture $\hat{T}^2 = -1$. Therefore Kramer's theorem is applicable which tells us that there are no non-degenerate eigenstates. So the answer is false: the ground state cannot be non-degenerate.

2. In this problem, we consider the Hamiltonian

$$\hat{\mathcal{H}} = \Delta(\hat{c}\hat{d} + \hat{d}^\dagger\hat{c}^\dagger)$$

where $\hat{c}$ and $\hat{d}$ are fermionic annihilation operators. That is, letting $\hat{c} = \hat{c}_1$ and $\hat{d} = \hat{c}_2$ they satisfy the relations $\{\hat{c}_n, \hat{c}_{n'}^\dagger\} = \delta_{nn'}$ and $\{\hat{c}_n, \hat{c}_{n'}\} = 0$ where curly brackets denote the anticommutator. The quantity Δ in the above is a constant parameter having dimensions of energy. This Hamiltonian contains essential elements of "Cooper pairing" in superconductors.

(a) What restriction does the requirement that $\hat{\mathcal{H}}$ is Hermitian place on Δ? Evaluate and simplify the commutator $[\hat{c}, \hat{c}^\dagger\hat{d}^\dagger]$.

Unseen

4 Marks (Level 1)

By setting $\hat{\mathcal{H}}^\dagger = \hat{\mathcal{H}}$, one finds the requirement that Δ is real.

By using the identity stated in 1(b), we have $[\hat{c}, \hat{c}^\dagger\hat{d}^\dagger] = \{\hat{c}, \hat{c}^\dagger\}\hat{d}^\dagger - \hat{c}^\dagger\{\hat{c}, \hat{d}^\dagger\} = \hat{d}^\dagger$.

(b) Show that the Heisenberg equations of motion for the fermionic operators can be written as

$$i\hbar \frac{d}{dt}\hat{c}_H = -\Delta \hat{d}_H^\dagger$$

$$i\hbar \frac{d}{dt}\hat{d}_H^\dagger = -\Delta \hat{c}_H.$$

Unseen

6 Marks (3 Level 1 + 3 Level 2)

Using expressions derived in lecture for the Heisenberg equations of motion, we have

$$i\hbar \frac{d}{dt}\hat{c}_H = \mathcal{U}^\dagger[\hat{c}, \hat{\mathcal{H}}]\mathcal{U}$$

$$i\hbar \frac{d}{dt}\hat{d}_H = \mathcal{U}^\dagger[\hat{d}, \hat{\mathcal{H}}]\mathcal{U}$$

The result follows from correctly evaluating the above commutation relations.

(c) Solve the above Heisenberg equations of motion to thereby express $\hat{c}_H$ and $\hat{d}_H$ in terms of $\hat{c}$ and $\hat{d}$ with time-dependent coefficients.

Unseen

5 Marks (Level 2)

The result follows from solving the operator differential equations subject to the initial conditions $\hat{c}_H(t = 0) = \hat{c}$ and $\hat{d}_H(t = 0) = \hat{d}$. With some work, one finds

$$\hat{c}_H = \cos(\Delta t/\hbar)\hat{c} + i\sin(\Delta t/\hbar)\hat{d}^\dagger$$

$$\hat{d}_H = \cos(\Delta t/\hbar)\hat{d} - i\sin(\Delta t/\hbar)\hat{c}^\dagger.$$

(d) Suppose at time $t = 0$, the initial state is the vacuum $|0\rangle$ (where $\hat{c}|0\rangle = \hat{d}|0\rangle = 0$). Evaluate the expectation value of the total particle number operator $\hat{N} = \hat{c}^\dagger\hat{c} + \hat{d}^\dagger\hat{d}$ for later times.

Unseen

5 Marks (2 Level 3 + 3 Level 4)

The key here is to realise that you do not need to solve the Heisenberg equations of motion for the particle number operator. Instead, one can recognise that $\hat{N}_H = \hat{c}_H^\dagger\hat{c}_H + \hat{d}_H^\dagger\hat{d}_H$. This is true because the time-evolution operator is unitary. For example, $\mathcal{U}^\dagger\hat{c}^\dagger\hat{c}\mathcal{U} = \mathcal{U}^\dagger\hat{c}^\dagger\mathcal{U}\mathcal{U}^\dagger\hat{c}\mathcal{U} = \hat{c}_H^\dagger\hat{c}_H$.

Then using the result of (c) and the properties of the vacuum, one finds that

$$\langle\hat{N}\rangle = 2\sin^2(\Delta t/\hbar).$$

3. Consider a general spin-s particle governed by the Hamiltonian. $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{V}$ where $\hat{\mathcal{H}}_0 = A\hat{S}_z$ and $\hat{V} = B\hat{S}_x$. Here, A and B are positive constants having dimensions of frequncy and λ is our usual perturbative parameter.

(a) What are the eigenenergies of $\hat{\mathcal{H}}_0$? What is the ground state of $\hat{\mathcal{H}}_0$?

Seen

4 Marks (Level 1)

The eigenenergies of $\hat{\mathcal{H}}_0$ are $\varepsilon_m = A\hbar m$ where m takes on values from $-s$ to s in integer steps. Since A is positive, the ground state is $|s, -s\rangle$.

(b) Can $\hat{\mathcal{H}}_0$ and $\hat{V}$ be simultaneously diagonalised?

Unseen

4 Marks (Level 2)

$\hat{\mathcal{H}}_0$ and $\hat{V}$ cannot be simultaneously diagonalised. This is because $\hat{S}_x$ and $\hat{S}_z$ do not commute with one another (they are incompatible observables).

(c) Using first-order perturbation theory, determine the the first-order correction (in λ) to the ground state energy.

Similar seen

4 Marks (Level 2)

Since s is fixed, denote $|s, m\rangle \equiv |m\rangle$.

The key is to use the spin raising and lowering operators $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ to express $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$. It then follows that $\langle m| \hat{S}_x |m\rangle = 0$. So there is no perturbative correction at linear order in λ for any of the eigenenergies.

(d) Using second-order perturbation theory, determine the second-order correction (in λ) to the ground state energy.

Unseen

5 Marks (3 Level 3 + 2 Level 4)

Consider the following matrix element $\langle m| \hat{S}_x |-s\rangle$. Since the ket in the previous expression is an eigenstate of $\hat{S}_z$ with the smallest possible eigenvalue, $\langle m| \hat{S}_x |-s\rangle = \frac{1}{2}\langle m| \hat{S}_+ |-s\rangle$. Using the properties of the spin raising operator, we then have $\langle m| \hat{S}_x |-s\rangle = \frac{\hbar}{2}\sqrt{s(s+1) - (-s)((-s)+1)}\delta_{m,-s+1} = \hbar\sqrt{s/2}\delta_{m,-s+1}$. Therefore, only one term contributes to the second-order perturbation theory result:

$$E_{-s}^{(2)} = \frac{|\langle -s+1| \hat{V} |-s\rangle|^2}{\varepsilon_{-s} - \varepsilon_{-s+1}} = -\hbar\frac{B^2 s}{2A}.$$

(e) Find the exact ground state energy of $\hat{\mathcal{H}}$. Show that this is consistent with the results from (c) and (d).

Unseen

4

Instead of trying to diagonalise $\hat{\mathcal{H}}$ directly, it is best to apply a unitary spin rotation. The following relation is useful $e^{-i\hat{S}_y\alpha/\hbar}\hat{S}_z e^{i\hat{S}_y\alpha/\hbar} = \cos(\alpha)\hat{S}_z + \sin(\alpha)\hat{S}_x$ which follows from the commutation relations of the spin operators (okay to derive this result or state from memory). Taking $\tan(\alpha) = \lambda B/A$, we then find

$$\hat{\mathcal{H}} = \sqrt{A^2 + \lambda^2 B^2}\, e^{-i\hat{S}_y\alpha/\hbar}\hat{S}_z e^{i\hat{S}_y\alpha/\hbar}.$$

The ground state is then $e^{-i\hat{S}_y\alpha/\hbar}\left|-s\right\rangle$ which has the corresponding eigenenergy $E = -\sqrt{A^2 + \lambda^2 B^2}\,\hbar s$. Taylor expanding this result in λ to second order, we have

$$E = -A\sqrt{1 + \lambda^2 B^2/A^2}\,\hbar s \approx -\hbar s A - \hbar s \frac{B^2}{2A}\lambda^2$$

which is consistent with the previous results.

4. Consider the quantum harmonic oscillator $\hat{\mathcal{H}}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$ under the time-dependent perturbation $\hat{V} = \hat{x}f(t)$. Here, $f(t) = e^{-t^2/\tau^2}f_0$ where f_0 and τ are real positive constants having dimensions of force and time respectively. The full Hamiltonian for the system is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{V}$ where λ is our usual perturbative parameter. Suppose at $t = -\infty$, the system is in the harmonic oscillator ground state.

(a) Using the ladder operators, $\hat{a}$ and $\hat{a}^\dagger$ where $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}$, show that the Hamiltonian can be written as

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda f(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger).$$

The result follows from expressing $\hat{x}$ and $\hat{p}$ in terms of the ladder operators, and utilising the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$.

(b) Using time-dependent perturbation theory, show that to lowest non-trivial order in λ, the probability for the system to transition to the first excited sate of the harmonic oscillator at $t = \infty$ is

$$P_{0\to 1} = \frac{\pi\lambda^2 f_0^2\tau^2}{2m\hbar\omega}e^{-\omega^2\tau^2/2}$$

The result follows from adapting the result we derived in lecture to a different initial condition $t = -\infty$. A potentially tricky part of this problem is evaluating the integral.

(c) Determine the probability for the system to remain in the harmonic oscillator ground state at $t = \infty$ to second order in λ.

Unseen

5 Marks (2 Level 3 + 3 Level 4)

The result we derived in lecture for $P_{n \to n'}$ only applies when $n \neq n'$. However, the transition probabilities must satisfy the sum rule $\sum_{n'} P_{n \to n'} = 1$. That is, the probability of the system transitioning to some final state is unity. From this, and adapting to the problem at hand, we have

$$P_{0 \to 0} = 1 - \sum_{n \neq 0} P_{0 \to n}.$$

Next we can recognize that $P_{0 \to n} = 0$ when n is not 1 or 0 at the level of second order perturbation theory. This is clear from the matrix elements $\langle n | \hat{x} | 0 \rangle$. Therefore only one term in the above sum contributes and we have

$$P_{0 \to 0} = 1 - P_{0 \to 1}$$

and we have already computed $P_{0 \to 1}$ in part (b).

(d) Give the condition on the parameter τ for the system to be in the adiabatic regime. Is your result from (c) consistent with your expectations in the adiabatic regime?

Unseen

4 Marks (4 Level 3)

The pulse is slowly varying when τ is a large parameter. For the adiabatic regime, we need for the rate of variation to be slow compared to the energy level-spacing of the Hamiltonian (divided by $\hbar$). In particular, we need $\hbar/\tau \ll \hbar\omega$ or $\tau\omega \gg 1$. As $\tau\omega \to \infty$, we see that $P_{0 \to 0} \to 1$. This is consistent with what we have learned about adiabatic theory. That is, to lowest order in adiabatic perturbation theory, the time-dependent state of this problem will always be the instantaneous ground state of the Hamiltonian.

5. Time-dependent unitary transformations.

(a) Suppose $|\psi\rangle$ satisfies the time-dependent Schrodinger equation $i\hbar\partial_t |\psi\rangle = \hat{\mathcal{H}} |\psi\rangle$. Suppose $|\psi'\rangle$ is related to $|\psi\rangle$ by a time-dependent unitary transformation: $|\psi'\rangle = \hat{U}(t) |\psi\rangle$ where $\hat{U}(t)$ is a time-dependent unitary operator. Show that $|\psi'\rangle$ also satisfies a time-dependent Schrodinger equation $i\hbar\partial_t |\psi'\rangle = \hat{\mathcal{H}}' |\psi'\rangle$. What is $\hat{\mathcal{H}}'$?

Seen

5 Marks (Level 1)

Inserting the expression for $|\psi\rangle = \hat{U}^\dagger |\psi'\rangle$ into the TDSE gives $i\hbar\partial_t\hat{U}^\dagger |\psi'\rangle + i\hbar\hat{U}^\dagger\partial_t |\psi'\rangle = \hat{\mathcal{H}}\hat{U}^\dagger |\psi'\rangle$. Multiplying through by $\hat{U}$ and rearranging then gives $\hat{\mathcal{H}}' = \hat{U}\hat{\mathcal{H}}\hat{U}^\dagger - i\hbar\hat{U}\partial_t\hat{U}^\dagger$.

(b) For the remainder of this problem, let us restrict to unitary transformation corresponding to the operator

$$\hat{U}(t) = e^{i\hat{p}\xi/\hbar}e^{-i\hat{x}m\dot{\xi}/\hbar}e^{i\alpha}$$

where ξ is a real time-dependent quantity having dimensions of length and $\dot{\xi} = d\xi/dt$. Also $\alpha(t)$ is an unspecified real function of t.

Show that under this unitary transformation, $\hat{x}' \equiv \hat{U}(t)\hat{x}\hat{U}^\dagger(t) = \hat{x} + \xi$ and $\hat{p}' \equiv \hat{U}(t)\hat{p}\hat{U}^\dagger(t) = \hat{p} + m\dot{\xi}$.

Seen

5 Marks (Level 3)

The result follows from recognising that $\hat{U}^\dagger = e^{i\hat{x}m\dot{\xi}/\hbar}e^{-i\hat{p}\xi/\hbar}e^{i\alpha}$ and the properties of position and momentum translation operators (okay to state these from memory or derive them from first principles).

(c) Next consider the time-dependent Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2(\hat{x} - f(t))^2$$

where $f(t)$ is an unspecified real function of t having dimensions of length. This Hamiltonian corresponds to a driven harmonic oscillator. Taking ξ to satisfy Newton's classical equation of motion

$$\ddot{\xi} + \omega^2\xi = \omega^2 f(t),$$

show that with a suitable choice of $\alpha(t)$, the transformed Hamiltonian is:

$$\hat{\mathcal{H}}' = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

(you do not need to attempt to solve the differential equation you find for $\alpha(t)$ or $\xi(t)$).

Unseen

6 Marks (Level 4) For the expression that was derived in (a), we need to know $i\hbar\hat{U}\partial_t\hat{U}^\dagger$. Differentiating each factor in $\hat{U}^\dagger$ and being careful with commutation relations we have

$$i\hbar\hat{U}\partial_t\hat{U}^\dagger = \hat{U}\left(-\hat{x}m\ddot{\xi}\hat{U}^\dagger + \hat{U}^\dagger\hat{p}\dot{\xi} - \hat{U}^\dagger\hbar\dot{\alpha}\right) = -\hat{U}\hat{x}\hat{U}^\dagger m\ddot{\xi} + \hat{p}\dot{\xi} - \hbar\dot{\alpha}.$$

Then using the result of (b), we have

$$i\hbar\hat{U}\partial_t\hat{U}^\dagger = -(\hat{x} + \xi)m\ddot{\xi} + \hat{p}\dot{\xi} - \hbar\dot{\alpha}.$$

Then, with the boosted momentum and position operators and the above result, after some work one finds that the transformed Hamiltonian is

$$\hat{\mathcal{H}}' = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \hat{x}m(\omega^2(\xi - f) + \ddot{\xi}) + \left(\frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}m\omega^2(\xi - f)^2 + m\xi\ddot{\xi} - \hbar\dot{\alpha}\right).$$

Therefore, taking ξ to satisfy Newton's equation and α to satisfy

$$\hbar\dot{\alpha} = \frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}m\omega^2(\xi - f)^2 + m\xi\ddot{\xi}$$

we have that $\hat{\mathcal{H}}'$ takes on the simple form.

(d) Consider the ground-state solution to the Schrodinger equation in the primed frame: $|\psi'\rangle = e^{-i\omega t/2}|\phi_0\rangle$. Determine the expectation value of $\hat{x}$ in the original frame $\langle\psi|\hat{x}|\psi\rangle$ corresponding to this solution. Comment on your result within the context of Ehrenfest's theorem.

Unseen

4 Marks (Level 4)

We note that

$$\langle\psi|\hat{x}|\psi\rangle = \langle\psi'|\hat{U}\hat{x}\hat{U}^\dagger|\psi'\rangle = \langle\phi_0|\hat{U}\hat{x}\hat{U}^\dagger|\phi_0\rangle.$$

Then using our relation for the boosted position operator,

$$\langle\psi|\hat{x}|\psi\rangle = \langle\phi_0|\hat{x}|\phi_0\rangle + \xi = \xi.$$

So the expectation value of $\hat{x}$ satisfies Newton's classical equation of motion. Noting that the potential we are working with is quadratic, this is required by Ehrenfest's theorem.