MATH50001/50017/50018 - Analysis II

Complex Analysis

Lecture 3

Cauchy-Riemann equations.

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f(z)/\partial z$.

Proof. Assuming $h = h_1 + ih_2$ we have

$$u(x+h_1, y+h_2) - u(x,y) = u'_x(x,y)\, h_1 + u'_y(x,y)\, h_2 + |h|\psi_1(h),$$

where $\psi_1(h) \to 0$ as $h \to 0$. Indeed,

$$\begin{aligned} u(x+h_1, y+h_2) - u(x,y) &= u(x+h_1, y+h_2) - u(x, y+h_2) + u(x, y+h_2) - u(x,y) \\ &= u'_x(x, y+h_2)h_1 + h_1\varphi_1(h) + u'_y(x,y)h_2 + h_2\varphi_2(h). \end{aligned}$$

Since $u'_x(x, y+h_2)$ is continuous we have

$$u'_x(x, y+h_2) - u'_x(x,y) = \varphi_3(h) \to 0 \quad \text{as} \quad h_2 \to 0$$

and thus

$$\begin{aligned} u(x+h_1, y+h_2) - u(x,y) &= u'_x(x,y)\, h_1 + u'_y(x,y)\, h_2 + h_1(\varphi_3(h) + \varphi_1(h)) + h_2\varphi_2(h) \\ &= u'_x(x,y)\, h_1 + u'_y(x,y)\, h_2 + |h|\psi_1(h), \end{aligned}$$

where $\psi_1(h) = |h|^{-1}(h_1(\varphi_3(h) + \varphi_1(h)) + h_2\varphi_2(h)) \to 0$, $h \to 0$.

Similarly

$$v(x+h_1, y+h_2) - v(x,y) = v'_x(x,y)\,h_1 + v'_y(x,y)\,h_2 + |h|\psi_2(h),$$

where $\psi_2(h) \to 0$ as $h \to 0$.

Using the Cauchy-Riemann equations $v'_x = -u'_y$ and $v'_y = u'_x$, we find

$$\begin{aligned} f(z+h) - f(z) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y) \\ &= u'_x(x,y)\,h_1 + u'_y(x,y)\,h_2 + i(v'_x(x,y)\,h_1 + v'_y(x,y)\,h_2) + |h|\psi(h) \\ &= u'_x(x,y)\,h_1 + u'_y(x,y)\,h_2 - iu'_y(x,y)h_1 + iu'_x h_2 + |h|\psi(h) \\ &= (u'_x - iu'_y)(h_1 + ih_2) + |h|\psi(h), \end{aligned}$$

where $\psi(h) = \psi_1(h) + i\psi_2(h) \to 0$, as $h \to 0$. Therefore f is holomorphic and

$$f'(z) = 2\,\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

Section: Cauchy-Riemann equations in polar coordinates

Usual Cauchy-Riemann equations for a holomorphic function $f = u + iv$ as they were defined before are:

$$u'_x = v'_y \qquad u'_y = -v'_x$$

Introduce polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan y/x.$$

Then

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\,(-1)\,\frac{y}{x^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{\cos\theta}{r}.$$

Therefore

$$u'_x = u'_r\,\cos\theta + u'_\theta\,\frac{-\sin\theta}{r}, \qquad v'_y = v'_r\,\sin\theta + v'_\theta\,\frac{\cos\theta}{r},$$

$$u'_y = u'_r\,\sin\theta + u'_\theta\,\frac{\cos\theta}{r}, \qquad v'_x = v'_r\,\cos\theta + v'_\theta\,\frac{-\sin\theta}{r}.$$

Multiplying u'_x by $\cos\theta$ and u'_y by $\sin\theta$ and adding the results we find

$$u'_r = u'_r\,\cos^2\theta + u'_r\,\sin^2\theta = u'_x\,\cos\theta + u'_y\,\sin\theta.$$

Using $u'_x = v'_y$ and $u'_y = -v'_x$ we conclude

$$\begin{aligned} u'_x \cos\theta + u'_y \sin\theta &= v'_y \cos\theta - v'_x \sin\theta \\ &= \left(v'_r \sin\theta + v'_\theta \frac{\cos\theta}{r}\right) \cos\theta - \left(v'_r \cos\theta - v'_\theta \frac{\sin\theta}{r}\right) \sin\theta = v'_\theta \frac{1}{r}. \end{aligned}$$

Then

$$u'_r = \frac{1}{r} v'_\theta \quad \text{and similarly} \quad v'_r = -\frac{1}{r} u'_\theta.$$

Example. Let

$$\begin{aligned} f(z) = u(x,y) + iv(x,y) = \ln(x^2+y^2) + 2i \arctan\frac{y}{x} \\ = \ln|z|^2 + 2i\mathrm{Arg}\,(z) = 2(\ln r + i\theta), \end{aligned}$$

where $z = r(\cos\theta + i\sin\theta)$. Then

$$u'_r = \frac{2}{r} = \frac{1}{r}\cdot 2 = \frac{1}{r} v'_\theta \quad \text{and} \quad 0 = v'_r = -\frac{1}{r} u'_\theta = 0.$$

Section: Power series

Definition. A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$.

The series is convergent at z if the partial sum $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has a limit

$$S(z) = \lim_{N\to\infty} S_N(z).$$

In this case we write $S(z) = \sum_{n=0}^{\infty} a_n z^n$.

For its absolute convergence we consider

$$\sum_{n=0}^{\infty} |a_n||z|^n.$$

Proposition. If $S(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\lim_{N\to\infty}(S(z) - S_N(z)) = 0$.

Example. The complex exponential function, which is defined by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for any $z \in \mathbb{C}$ and $R = \infty$.

Example. The geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

converges absolutely $|z| < 1$ and its radius of convergence $R = 1$.

Theorem. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:

(i) If $|z| < R$ the series converges absolutely.

(ii) If $|z| > R$ the series diverges.

Moreover, R is given by the formula

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}$$

The number R is called the *radius of convergence* of the power series, and the domain $|z| < R$ the *disc of convergence*.

Proof. Let $L = 1/R$ and suppose that $L \neq 0, \infty$. If $|z| < R$, choose $\varepsilon > 0$ so that

$$(L + \varepsilon)|z| = r < 1.$$

By the definition L, we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n, therefore

$$|a_n||z|^n \leq ((L + \varepsilon)|z|)^n = r^n$$

Comparison with the geometric series $\sum_{n=0}^{\infty} r^n$ shows that $\sum_{n=0}^{\infty} a_n z^n$ converges.

If $|z| > R$, then a similar argument proves that there exists a sequence of terms in the series whose absolute value goes to infinity, hence the series diverges.

Remark. Prove the above result for $R = 0$ and $R = \infty$ ($L = \infty$ and $L = 0$ respectively).

Remark. On the boundary of the disc of convergence, $|z| = R$, one can have either convergence or divergence.

Power series provide an important class of holomorphic functions.

Theorem. The power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Moreover, f' has the same radius of convergence as f.

Proof. Indeed, note that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^0 = 1.$$

Therefore

$$\sum_{n=1}^{\infty} n a_n z^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} a_n z^n$$

have the same radius of convergence.
It remains to show that $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ coincides with $f'(z)$.
Let R be the radius of convergence of f, $|z_0| < r < R$ and let

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then if h is chosen so that $|z_0 + h| < r$ we have

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = &\left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) \\ &+ \left(S_N'(z_0) - g(z_0)\right) + \left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right).\end{aligned}$$

We find that

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right|$$

$$\leq \sum_{n=N+1}^{\infty} |a_n|\, n\, r^{n-1} \to 0, \quad \text{as} \quad N \to \infty.$$

Given $\varepsilon > 0$ there is N_1 such that for any $N > N_1$ we have

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| < \varepsilon.$$

Since $\lim_{N\to\infty} S_N'(z_0) \to g(z_0)$ there is N_2 such that for any $N > N_2$ we have

$$|S_N'(z_0) - g(z_0)| < \varepsilon$$

Finally for any fixed $N > \max(N_1, N_2)$ we choose $\delta > 0$ such that if $|h| < \delta$

$$\left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right| < \varepsilon.$$

We now conclude

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - g_(z_0)\right| < 3\varepsilon, \quad |h| < \delta.$$

The proof is complete.

Corollary. A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwize differentiation.

Quizzes

Question 1: Let $f = u + iv$, be defined on an open set Ω of the complex plane $z = x + iy$. Which of the following conditions imply that f is holomorphic on Ω?

Answers:

A. The functions $u(x, y)$ and $v(x, y)$ are continuously differentiable on Ω

B. Partial derivatives u'_x, u'_y, v'_x, v'_y exist on Ω and satisfy Cauchy-Riemann equations there

C. The functions $u(x, y)$ and $v(x, y)$ are continuously differentiable on Ω and satisfy Cauchy-Riemann equations there

Question 2: What is the radius of convergence R of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$?

Answers:

A. $R = 1$

B. $R = 0$

C. $R = \infty$

Thank you