Solutions to Question Sheet 8 - Probl. Class week 11

MATH40003 Linear Algebra and Groups   Term 2, 2022/23

This is the final problem sheet for this module, released on Monday of week 10. Solutions will be released on Monday of week 11. Question 5 is related to the material on Dihedral Groups covered in Lecture 19 in the notes.

Question 1 (a) Write down all of the cycle shapes of the elements of S_5. For each cycle shape, calculate how many elements there are with that shape. (Check that your answers add up to $|S_5| = 120$.)

  (b) How many elements of S_5 have order 2?

  (c) How many subgroups of size 3 are there in the group S_5?

Solution:

  (a)

	Shape	Example	Formula	Number
1	(1^5)	id		1
2	$(2^1 1^3)$	$(1\,2)$	$\binom{5}{2}$	10
3	$(3^1 1^2)$	$(1\,2\,3)$	$2\binom{5}{3}$	20
4	$(2^2 1^1)$	$(1\,2)(3\,4)$	$\frac{1}{2}\binom{5}{2}\binom{3}{2}$	15
5	$(4^1 1^1)$	$(1\,2\,3\,4)$	$3!\binom{5}{4}$	30
6	$(3^1 2^1)$	$(1\,2)(3\,4\,5)$	$2\binom{5}{3}$	20
7	(5^1)	$(1\,2\,3\,4\,5)$	$4!$	24

  (b) Cycle shapes 2 and 4 in the table give elements of order 2, so there are $10 + 15 = 25$ of them.

  (c) A subgroup of H order 3 must be cyclic, since if $g \in H \setminus \{e\}$ then $H = \{e, g, g^2\}$. The elements of order 3 are those with cycle shape 3 in the table. There are 20 of these, and each cyclic subgroup of order 3 contains two of them. Moreover, two distinct subgroups of order 3 intersect only in the trivial group. So there are 10 subgroups of order 3.

Question 2 What is the largest order of an element of S_8?

Solution: Consider the possible cycle shapes of an element of S_8. The one giving the largest order is the shape $5^1 3^1$. An element of this cycle shape has order 15.

Question 3 Let G be a group, and let S be a subset of G. Recall that we say that S *generates* G if every element in G can be written as a product of elements of S and their inverses.

   (i) Let $2 \leq k \leq n$. Show that a k-cycle $(a_1 \ldots, a_k)$ in S_n can be written as a product of $k-1$ distinct cycles of length 2. Deduce that the set of 2-cycles in S_n generates S_n.

   (ii) (Harder) Let α be the n-cycle $(1234\ldots n)$ and β the 2-cycle (12). Prove that $\langle \alpha, \beta \rangle = S_n$.

   [Hint: $\alpha\beta\alpha^{-1} = (23)$. Use tricks like this.]

Solution: (i) We have

$$(a_1\, a_2\, \cdots\, a_k) = (a_1\, a_2)(a_2\, a_3) \cdots (a_{k-1}\, a_k).$$

So any cycle is a product of 2-cycles. Since every element of S_n is a product of cycles, we see that every element is a product of 2-cycles. So the set of 2-cycles generates S_n.

(ii) By (i), it suffices to show that every 2-cycle is in $\langle \alpha, \beta \rangle$. We may assume $n \geq 3$. Note that $\alpha\beta\alpha^{-1} = (23)$ and similarly $\alpha(23)\alpha^{-1} = (34)$, etc. So using this repeatedly, we obtain $(12), (23), \ldots, (n-1, n), (n1) \in \langle \alpha, \beta \rangle$. But then note that $(13) = (23)(12)(23)$, $(14) = (34)(13)(34)$, etc . Repeating this we obtain $(1k) \in \langle \alpha, \beta \rangle$ for all $2 \leq k \leq n$. But then for $j \neq k$ we have $(1j)(1k)(1j) = (jk) \in \langle \alpha, \beta \rangle$.

Question 4 (a) Use the inclusion - exclusion principle to give a formula for the number of permutations in S_n which have no fixed points. Prove that the proportion of such permutations in S_n tends to $1/e$ as $n \to \infty$.

(b) Give a formula for the number of permutation in S_n which have one fixed point.

(c) A standard deck of 52 cards is shuffled at random. What (approximately) is the probability that at least one card is still in the same place after the shuffle?

Solution: (a) Perhaps you did the inclusion - exclusion principle in the Introductory or Probability and Statsistics module. If not, you should have looked on the internet (eg. at the Wikipedia article), or looked in a book. Suppose $A_1, \ldots, A_n$ are subsets of a set S. If $I \subseteq \{1, \ldots, n\}$ is non-empty let $A_I = \bigcap_{i \in I} A_i$. Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = - \sum_{I} (-1)^{|I|} |A_I|,$$

where the sum is over all non-empty subsets I of $\{1, \ldots, n\}$.

Let $S = S_n$ and for $i = 1, \ldots, n$ let A_i be the set of permutations in S_n fixing i. Note that $\bigcup_{i=1}^{n} A_i$ is the set of permutations fixing at least one point, which is the complement in S_n of the set we are interested in. Moreover, A_I is the set of permutations fixing all points in I: so this has size $(n - |I|)!$ It follows that the set of permutations in S_n which fix no point has size

$$d(n) = \left(n! + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)! \right) = n! \left(1 + \sum_{k=1}^{n} (-1)^k \frac{1}{k!} \right)$$

The familiar Taylor series for e^x then shows that $d(n)/n! \to e^{-1}$ as $n \to \infty$.

(b) A permutation in S_n fixes exactly the point i if and only if it fixes i and gives a fixed point-free permutation of the remaining $n-1$ points. So there are $d(n-1)$ such permutations and therefore exactly $nd(n-1)$ permutations in S_n fixing exactly one point.

(c) The probability that no card is in the same place is $d(52)/52!$ which is approximately $1/e$. So the probability that at least one card is still in the same place is approximately $1 - 1/e$.

Question 5 Suppose G is a group and $a, b \in G$ are of order 2. Let $c = ab$ and suppose that c has finite order $m \geq 3$.

(a) Prove that $aca = c^{-1}$ and deduce that for all $n \in \mathbb{N}$ we have $ac^n a = c^{-n}$.

(b) Show that $H = \{a^s c^t : s = 0, 1 \text{ and } 0 \leq t < m\}$ is a subgroup of G of order $2m$.

Solution: (a) Note that $a^{-1} = a$ and $b^{-1} = b$. Then

$$aca^{-1} = aca = aaba = ba = (ab)^{-1} = c^{-1}.$$

It follows that $c^{-n} = (aca^{-1})^n = ac^n a$, as required.

(b) To show that it is a subgroup note that $(a^s c^t)(a^\sigma c^\tau)$ is equal to $a^s c^{t+\tau}$ if $\sigma = 0$ and (by (b)) $a^{s+\sigma} c^{t-\tau}$ if $\sigma = 1$. This is in H. We can similarly show that H is closed under taking inverses.

Clearly H has at most $2m$ elements. We need to show that there are exaclty $2m$. If $(a^s c^t) = (a^\sigma c^\tau)$ for $\sigma, s = 0, 1$ and $0 \leq \tau, t < m$, then we show $s = \sigma$ and $t = \tau$. If $s \neq \sigma$ then a is a power of c and so commutes with c. It follows from (a) that $m = 2$, a contradiction. So $s = \sigma$ and as c has order m, it then follows that $t = \tau$.

[Remark: It now follows that H is isomorphic to the dihedral group D_{2m}.]

Question 6

(a) Describe the group G of rotational symmetries of a cube, saying what the possible axes of rotation are and what the possible angles of rotation are. Hence show that there are 24 such rotational symmetries.

(b) Use Question 8 from Problem Sheet 7 to give a different proof that $|G| = 24$, by thinking of G as a group of permutations on the faces of the cube.

(c) Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries of the cube which send this pair of faces to itself forms a subgroup of G of order 8.

Solution: (a) You should draw some pictures for this.

The possibilities are:

(i) Axis of rotation through centres of a pair of opposite faces; angle of rotation π. There are 3 symmetries of this type.

(ii) Axis of rotation through centres of a pair of opposite faces; angle of rotation $\pm\pi/2$. There are 6 symmetries of this type.

(iii) Axis of rotation through mid-points of a pair of (diametrically) opposite edges; angle of rotation π. There are 12 edges, so 6 such pairs of edges, and therefore 6 symmetries of this type.

(iv) Axis of rotation through mid-points of (diametrically) opposite vertices; angle of rotation $\pm 2\pi/3$. There are 4 such axes, so 8 symmetries of this type.

Adding in the identity element, we get $1 + 3 + 6 + 6 + 8 = 24$ rotational symmetries.

(b) Let X be the set of 6 faces, labelled $1, \ldots, 6$. In the notation of Qu 8 of sheet 7, let a be the face 1. Note that any face can be moved to any other face by a suitable rotation, so $Y = \{1, \ldots, 6\}$. Let $H = G_1$, the subgroup of rotations fixing face 1. Clearly this has order 4. By Qu8 on Sheet 7, there are 6 left cosets of H in G, so it follows that $|G| = 6.4 = 24$.

[If we were being more careful, we ought to justify why we can think of G as a subgroup of $\text{Sym}(X)$.]

(c) Note that, by inspection, the rotations which send the pair of faces to itself are the rotations about an axis through the centres of the faces (4 of these, type (i), (ii) above);

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rotations through π about axis through centres of pairs of opposite edges between the faces (2 of these, type (iii)); rotation through π about an axis through one of the other pairs of opposite faces (2 of these, type (i) above). This gives 8 symmetries in total.