Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May – June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Design of Experiments & Surveys

Date: Friday, 03 June 2011. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows: book A - solutions to questions 1 & 2; book B - solutions to questions 3 & 4.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (i) Prove that a set of mutually orthogonal $n \times n$ latin squares contains at most $n-1$ elements.

(ii) Any observation made at the point x in the design region $[1, 2]$ has expected value

$$\beta_1 + \frac{\beta_2}{x}$$

where β_1 and β_2 are unknown parameters. All observations are independent and have common unknown variance $\sigma^2 > 0$. Show that the design measure ξ which places half the observations at $x = 1$ and half at $x = 2$ is G-optimal.

(iii) State the two most desirable features of a clinical trial to compare the effects of two treatments. Discuss briefly these features with reference to a clinical trial to compare the effect of giving a particular vaccination to children at 2 months of age with the effect of giving it to them at 9 months of age.

2. (i) State the two main reasons for using blocking in experimental design.

(ii) Define a balanced incomplete block design with parameters (t, b, r, k, λ).

(iii) The treatments of a block design D are the set of non-zero vectors

$$v = (v_1, v_2, v_3, v_4)$$

where $v_i \in \{0, 1\}$, the set of integers modulo 2. The blocks consist of all subsets of 3 treatments u, v, w satisfying

$$u + v + w = (0, 0, 0, 0)$$

where addition is normal vector addition with the results of all calculations being taken modulo 2. Prove that D is a balanced incomplete block design with 35 blocks and find its other parameters.

3. The factors in a 2^7 factorial experiment are A, B, C, D, E, F and G. However there are only enough experimental units to test 32 treatment combinations. It is decided to alias ABCDE and CDEFG with the mean to form a one-quarter fractional replicate containing the treatment combination in which every factor occurs at its low level.

(i) Show that no main effect is aliased with another main effect or with a two-factor interaction.

(ii) Which two-factor interactions are aliased with other two-factor interactions?

Suppose now that the 32 experimental units are arranged into blocks and that ABC and EFG are to be confounded with the block effect.

(iii) Show that no main effect is confounded with the block effect.

(iv) Which two-factor interactions are confounded with the block effect?

(v) Construct the principal block.

(vi) What advice would you give the experimenter about the labelling of the factors in this experiment?

4. (a) (i) Define a $t \times r$ Youden rectangle, where $r < t$.

(ii) Show that if a column of any $n \times n$ latin square ($n > 3$) is deleted then the result is a Youden rectangle.

(b) A sample of size two is taken (without replacement) from a population of size $N > 2$. Each possible sample is selected with probability proportional to the sum of the corresponding population values, all of which are strictly positive.

(i) If Y_i is the response from the i^{th} population member ($i = 1, 2, ..., N$) show that this individual is chosen with probability

$$\frac{Y_T + (N-2)Y_i}{(N-1)Y_T}$$

where Y_T is the population total.

(ii) Show that the expected value of the sample mean is

$$\frac{Y_T^2 + (N-2)\sum Y_i^2}{2(N-1)Y_T}.$$