Imperial College London

MATH97008 MATH97088

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fluid Dynamics 1

Date: Thursday, 20 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) A general viscous incompressible flow is described in a Cartesian coordinate system (x_1, x_2, x_3), in which the velocity field is denoted by $\mathbf{V} = (V_1, V_2, V_3)$, where each component V_i depends on x_j $(j = 1, 2, 3)$ and the time variable t. Application of the mass momentum conservation law leads to the equations

$$\rho\Big[\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j}\Big] = \frac{\partial p_{ij}}{\partial x_j},$$

where ρ is the fluid density, and the Einstein summation convention is assumed. For a Newtonian fluid, the stress tensor p_{ij} is expressed, through the constitutive relation, as

$$p_{ij} = -p\delta_{ij} + \mu\Big(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\Big),$$

where p is the pressure, and δ_{ij} the Kronecker delta. The dynamic viscosity μ is a function of the temperature T, which depends on (x_1, x_2, x_3), i.e. $T = T(x_1, x_2, x_3)$. Assume that the continuity equation remains as

$$\nabla \cdot \mathbf{V} = 0.$$

Use the equations above to derive the momentum equations in the Navier-Stokes equations.

(3 marks)

(b) Let $(x, y, z) = (x_1, x_2, x_3)$ and apply the momentum equations derived in (a) to a steady and uni-directional flow, where the only non-zero velocity component u is in the x-direction. The temperature T is assumed to be independent of x. Show that $u = u(y, z)$ is independent of x and satisfies the equation,

$$\frac{\partial}{\partial y}\Big(\mu\frac{\partial u}{\partial y}\Big) + \frac{\partial}{\partial z}\Big(\mu\frac{\partial u}{\partial z}\Big) = \frac{\partial p}{\partial x},$$

and explain why the pressure gradient $\frac{\partial p}{\partial x}$ must be a constant. (4 marks)

(c) Apply the result in Part (b) to a steady uni-directional flow through a channel between two infinitely large parallel plates, which are located at $y = 0$ and $y = h$. Assuming that the plates are both fixed, and the flow is driven by a constant pressure gradient $\frac{\partial p}{\partial x} < 0$, and that

$$\mu = \mu_0 + \mu_1 T, \quad T = y,$$

where μ_0 and μ_1 are both constants.

(i) Deduce the equation for u and solve it to show that the velocity distribution across the channel is

$$u(y) = \frac{1}{\mu_1}\frac{\partial p}{\partial x}\left[y - \frac{h\ln\big[1 + (\mu_1/\mu_0)y\big]}{\ln(1 + \mu_1 h/\mu_0)}\right].$$

(6 marks)

(ii) Consider the pressure and viscous (frictional) forces acting on the fluid in the region: $0 \le y \le h$, $0 \le x \le L$ and $0 \le z \le 1$. Calculate the pressure forces acting on the cross sections at $x = 0$ and $x = L$, and the viscous stresses acting on the fluid by the plates. Show that the total viscous and pressure forces are in balance.

(7 marks)

(Total: 20 marks)

2. A thin spherical shell of radius R_0 is filled with gas and introduced to water, which occupies the entire three-dimensional space outside the shell. At the initial instant $t = 0$, the shell is destroyed and the gas comes in contact with the water. The fluid velocity at this moment is zero, and the subsequent motion will remain spherical and may be treated as being inviscid and irrotational. The water has a constant density ρ, and the pressure far from the gas bubble is p_∞ (see Figure 1). The gravity force is neglected. The pressure within the gas bubble, p_b, is uniform and but depends on $R(t)$, the radius of the bubble at time t, via the relation

$$p_b = k/\big(R(t)\big)^3.$$

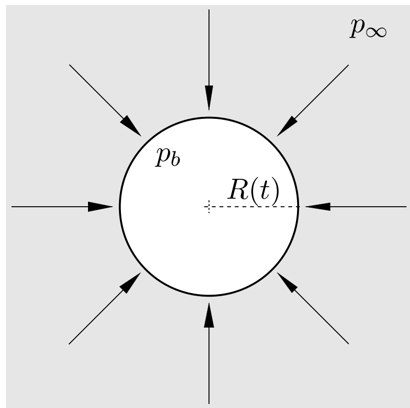

Figure 1: Gas bubble in water.

The motion of the fluid and the interface can be studied in spherical polar coordinates (r, ϑ, ϕ) by using the Laplace equation for the velocity potential φ,

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\varphi}{\partial r}\Big) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\Big(\sin\vartheta\frac{\partial\varphi}{\partial\vartheta}\Big) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\varphi}{\partial\phi^2} = 0,$$

and Bernoulli's equation

$$\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}|\mathbf{V}|^2 + \frac{p}{\rho} = F(t),$$

where the velocity $\mathbf{V} = (V_r, V_\vartheta, V_\phi) = \nabla\varphi$. *Hint: You may use without proof the fact that the gradient of any scalar function $f(r, \vartheta, \phi)$ is expressed, in spherical polar coordinates, as*

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial f}{\partial\vartheta}\mathbf{e}_\vartheta,$$

with $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\phi$ denoting the unit vectors in the radial, azimuthal and meridional directions, respectively.

(i) Show that the velocity potential φ and the radial velocity V_r have the expressions

$$\varphi = -\frac{f(t)}{r}, \qquad V_r = \frac{f(t)}{r^2} \quad \text{with} \quad f(t) = R^2\frac{dR}{dt}.$$

(3 marks)

Question continues on the next page.

(ii) Show that f satisfies the equation

$$-\frac{1}{R(t)}\frac{df}{dt}+\frac{1}{2}\frac{\big[f(t)\big]^2}{\big[R(t)\big]^4}=\frac{p_\infty-k/\big[R(t)\big]^3}{\rho}. \qquad (1)$$

(4 marks)

(iii) Show that

$$\frac{df}{dt}=R^{-2}f\frac{df}{dR},$$

and hence or otherwise, show that equation (1) can be rewritten as

$$\frac{dg}{dR}-\frac{1}{R}g=-\frac{2p_\infty}{\rho}R^3+\frac{2k}{\rho},$$

for $g=f^2$. Solve this equation to show that

$$R^2\frac{dR}{dt}=\pm\sqrt{R\Big[\frac{2p_\infty}{3\rho}(R_0^3-R^3)+\frac{2k}{\rho}\ln(R/R_0)\Big]}.$$

(6 marks)

(iv) Assuming that

$$R_0<R_c\equiv\left(\frac{k}{p_\infty}\right)^{1/3},$$

show that there exists the maximum radius, $R_{\max}>R_c$. Describe the characteristics of the motion of the bubble. (7 marks)

(Total: 20 marks)

3. (a) In a spherical polar coordinate system (r, ϑ, ϕ), the velocity field of an incompressible flow is denoted by $(V_r, V_\vartheta, V_\phi)$. For an axisymmetric flow, the meridional velocity $V_\phi = 0$ while the radial and azimuthal components, V_r and V_ϑ, are independent of ϕ. For a potential flow, there exists a velocity potential φ such that the velocity field $(V_r, V_\vartheta, V_\phi) = \nabla\varphi$.

Let the (stream) function $\psi(r, \vartheta)$ be defined through the relations

$$\frac{\partial \psi}{\partial r} = -\sin\vartheta \frac{\partial \varphi}{\partial \vartheta}, \qquad \frac{\partial \psi}{\partial \vartheta} = r^2 \sin\vartheta \frac{\partial \varphi}{\partial r}. \tag{2}$$

Show that contours of the function,

$$\psi = \text{constant},$$

correspond to streamlines of the flow. (4 marks)

Hint: You may use without proof the fact that the gradient of any scalar function $f(r, \vartheta, \phi)$ is expressed, in spherical polar coordinates, as

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{e}_\vartheta,$$

with $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\phi$ denoting the unit vectors in the radial, azimuthal and meridional directions, respectively.

(b) Consider a steady axisymmetric flow field corresponding to a superposition of a uniform flow V_∞, a point source $q > 0$ and a dipole m, both located at the origin in three-dimensional space. The velocity potential φ is then written as

$$\varphi = V_\infty r \cos\vartheta - \frac{q}{4\pi r} + \frac{m\cos\vartheta}{4\pi r^2}.$$

(i) Find the velocity field of this flow, and use equations (2) to find the corresponding stream function ψ. (6 marks)

(ii) Show that for suitable values of m and q, which you are required to determine, φ may represent the inviscid axisymmetric potential flow around a sphere located at the origin with radius a and a *porous surface*, through which suction is imposed, i.e. the fluid may come out of the sphere at a constant velocity $V_s > 0$ in the radial direction. (4 marks)

(iii) For the values of m and q determined in (ii), find the stagnation points. (2 marks)

The velocity potential φ may alternatively be viewed as representing the flow past an axi-symmetric body with a *rigid surface* and blunt leading edge. Again for the values of m and q determined in (ii), determine the shape of the body. Let $y = r\sin\vartheta$. Show that as $\vartheta \to 0$ on this body contour, y approaches a constant, which you are expected to find. (4 marks)

(Total: 20 marks)

4. (a) The velocity potential $\varphi(x, y)$ of a two-dimensional potential flow above a rigid wall at $y = h$ is given by

$$\varphi(x, y) = b\big[x^2 - (y - h)^2\big],$$

where b is a constant.

(i) Find the corresponding stream function $\psi(x, y)$, and construct the complex potential $w(z)$, where $z = x + iy$. (3 marks)

(ii) Suppose that a source of strength q is added at $z = iH$ (with $H > h$) to the flow described above. Write down the complex potential $w(z)$ with an appropriate image source being introduced such that $y = h$ may act as a rigid boundary. (3 marks)

(b) Consider the two-dimensional symmetrical potential flow due to a uniform oncoming stream with velocity V_∞ past a parabola

$$y = \pm 2a\sqrt{x},$$

and moreover a source of strength q is located at $(x, y) = (-3a^2, 0)$; see Figure 2.

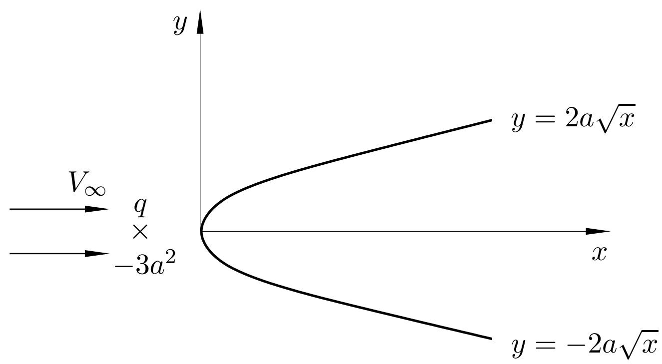

Figure 2: Symmetrical flow past a parabola.

(i) Show that the conformal mapping, $z = \zeta^2 + d$, maps the region $\Im\{\zeta\} > h$ in the auxiliary ζ-plane onto the exterior of the parabola in the physical z-plane provided that h and d are suitably chosen; determine h and d in terms of a. (3 marks)

(ii) Find the corresponding location of the source in the ζ-plane, and construct the appropriate complex potential $W(\zeta)$ in the ζ-plane that gives the flow in the physical z-plane. *Hint: Use the result in Part (a) (ii).* (4 marks)

(iii) For the case of $V_\infty = 0$, determine the modulus of the velocity, and using Bernoulli's equation calculate the pressure p along the parabola in terms of y. (4 marks)

(iv) Calculate the drag on the parabola (per unit length in the spanwise direction),

$$D = 2\int_0^\infty (p - p_\infty)dy,$$

where p is the pressure on the surface found in (iii), and p_∞ is the pressure at large distances from the parabola. (3 marks)

(Total: 20 marks)

5. Consider an incompressible inviscid irrotational flow past a (slitted) circular arc, which has a depth h and a chord length $2a$. A Cartesian coordinate system is used such that the chord is aligned with the x-axis and bisected by the y-axis as is illustrated in Figure 3(a). The modulus of the velocity of the oncoming flow far from the arc is V_∞, and the angle of attack is α.

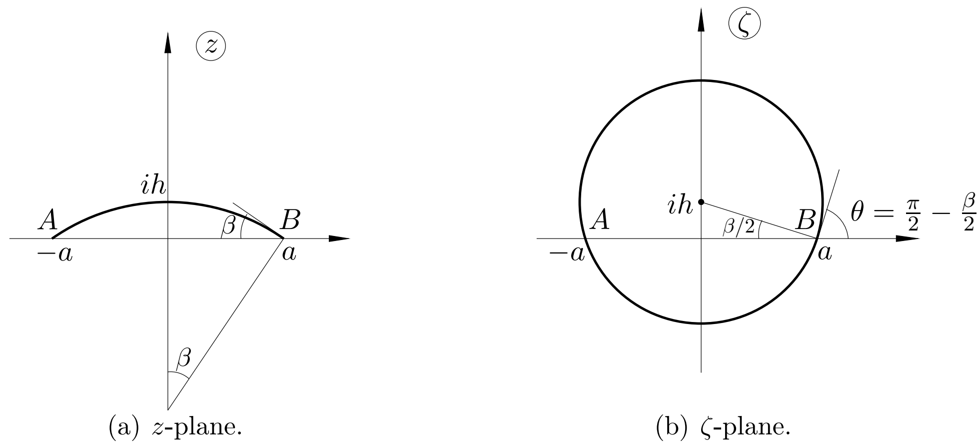

Figure 3: Joukowskii transformation.

It is known that the Joukowskii transformation,

$$z = \frac{1}{2}\left(\zeta + \frac{a^2}{\zeta}\right),$$

with its inversion being $\zeta = z + \sqrt{z^2 - a^2}$, maps the exterior of the circular arc onto the exterior of the circle centred at $\zeta = ih$ on the ζ-plane, as is shown in Figure 3(b).

(i) Find the radius and centre of the circle on which the arc sits, expressing your results in terms of a and h. Verify that the centre in the z-plane is mapped to $\zeta = ih$ in the ζ-plane. (3 marks)

(ii) Write down the appropriate form of the complex potential $W(\zeta)$ in the auxiliary ζ-plane. You may use without proof the fact that the complex potential of the flow past a circular cylinder of radius R centred at $z = 0$ is given by

$$w(z) = \tilde{V}_\infty\left(ze^{-i\alpha} + \frac{R^2}{ze^{-i\alpha}}\right) + \frac{\Gamma}{2\pi i}\ln z,$$

where α is the angle of attack. (3 marks)

(iii) Deduce the complex conjugate velocity $\overline{V}(z)$ in the z-plane with the free-stream velocity $\tilde{V}_\infty$ in the ζ-plane being determined in terms of V_∞. (3 marks)

(iv) Use the Joukowskii-Kutta condition to determine the circulation Γ in terms of α and β. (5 marks)

(v) Calculate the velocity at the point $z = ih$ on the *upper* side of the arc. Calculate the velocity at $z = ih$ but on the *lower* side of the arc. Which side has a larger velocity? (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96002, MATH97008, MATH97088

Fluid Dynamics 1(Solutions)

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1. (a) Substituting the constitutive relation into the momentum equations and differentiating using the Chain Rule, we obtain

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$$\rho\Big[\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}\Big]=-\frac{\partial p_{ij}}{\partial x_j}=-\frac{\partial p}{\partial x_j}\delta_{ij}+\frac{\partial}{\partial x_j}(\mu\frac{\partial V_i}{\partial x_j})+\frac{d\mu}{dT}\frac{\partial T}{\partial x_j}\frac{\partial V_j}{\partial x_i}+\mu\frac{\partial^2 V_j}{\partial x_j\partial x_i}.$$

The last term vanishes since $\frac{\partial^2 V_j}{\partial x_j\partial x_i}=\frac{\partial}{\partial x_i}(\frac{\partial V_j}{\partial x_j})$ and the continuity equation $\frac{\partial V_j}{\partial x_j}=0$ holds. 3, A

sim. seen ⇓

(b) For a steady flow, $\frac{\partial V_i}{\partial t}=0$. Since $u=u(x,y,z)$ is the only non-zero velocity component, it follows immediately from the continuity equation that

$$\frac{\partial u}{\partial x}=0,$$

implying that u is independent of x, while the y- and z-momentum equations reduce to

$$-\frac{\partial p}{\partial y}+\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{\partial u}{\partial y}=0,\qquad -\frac{\partial p}{\partial z}+\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{\partial u}{\partial z}=0.$$

Since $\partial T/\partial x=0$, we have $\partial p/\partial y=0$ and $\partial p/\partial z=0$, implying that the pressure p could only be a function of x.

The momentum equation in the x-direction reduces to

$$-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})+\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{\partial u}{\partial x}=0.$$

Since the last term on the left-hand side vanishes, the above equation is the required equation,

$$\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})=\frac{\partial p}{\partial x}.$$

The left-hand side depends only on y and z whereas the right-hand side depends on x only. It may therefore be deduced that both sides and hence the pressure gradient $\frac{\partial p}{\partial x}$ must be a constant. 4, A

unseen ⇓

(c) (i) As the plates are infinitely large, u must be independent of z. Noting also that T is a function of y only, the equation for u reduces to the ordinary differential equation,

$$\frac{\partial}{\partial y}((\mu_0+\mu_1 y)\frac{\partial u}{\partial y})=\frac{\partial p}{\partial x}.$$

Integrating once, we obtain

$$\frac{\partial u}{\partial y}=\frac{\partial p}{\partial x}\frac{y+C_1}{(\mu_0+\mu_1 y)}=\frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{y+C_1}{(y+\mu_0/\mu_1)},$$

where the right-hand side is re-arranged to aid the next integration. Integrating once more and using the boundary condition $u=0$ at $y=0$, we obtain

$$u=\frac{1}{\mu_1}\frac{\partial p}{\partial x}\int_0^y\frac{y+C_1}{(y+\mu_0/\mu_1)}dy=\frac{1}{\mu_1}\frac{\partial p}{\partial x}\Big[y+(C_1-\mu_0/\mu_1)\ln(\frac{y+\mu_0/\mu_1}{\mu_0/\mu_1})\Big].$$

Now determine the integration constant C_1 using the boundary condition $u=0$ at $y=h$:

$$\Big[h+(C_1-\mu_0/\mu_1)\ln(\frac{h+\mu_0/\mu_1}{\mu_0/\mu_1})\Big]=0,$$

from which we find that

$$C_1 - \mu_0/\mu_1 = -\frac{h}{\ln(1 + h\mu_1/\mu_0)}.$$

The solution is found as

$$u = \frac{1}{\mu_1}\frac{\partial p}{\partial x}\Big[y - h\frac{\ln(1 + y\mu_1/\mu_0)}{\ln(1 + h\mu_1/\mu_0)}\Big].$$

$\boxed{6, \text{C}}$

(ii) The pressure force acting on the cross sections at $x = 0$ and at $x = L$ are $p\big|_{x=0}h$ and $-p\big|_{x=L}h$ respectively, the sum of which is $\boxed{\text{unseen} \Downarrow}$

$$hp\big|_{x=0} - hp\big|_{x=L} > 0. \tag{1}$$

$\boxed{2, \text{A}}$

The stress tensor at the lower plate has components:

$$p_{11} = p_{22} = p_{33} = -p, \quad p_{13} = p_{23} = 0;$$

$$p_{12}\big|_{y=0} = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\mu_0}{\mu_1}\frac{\partial p}{\partial x}\Big[1 - \frac{h\mu_1/\mu_0}{\ln(1 + h\mu_1/\mu_0)}\Big].$$

The lower surface at $y = 0$ has unit normal direction $\mathbf{n} = (0, 1, 0)$. The shear stress of the fluid acting on the lower plate is given by $p_{ij}n_j$, which is

$$(p_{12}\big|_{y=0}, -p, 0). \tag{2}$$

The stress tensor at the upper plate at $y = h$ has components:

$$p_{11} = p_{22} = p_{33} = -p, \quad p_{13} = p_{23} = 0;$$

$$p_{12}\big|_{y=h} = \mu\frac{\partial u}{\partial y}\Big|_{y=h} = \frac{(\mu_0 + \mu_1 h)}{\mu_1}\frac{\partial p}{\partial x}\Big[1 - \frac{h\mu_1/\mu_0}{\ln(1 + h\mu_1/\mu_0)(1 + h\mu_1/\mu_0)}\Big].$$

The upper surface at $y = h$ has unit normal direction $\mathbf{n} = (0, -1, 0)$. The shear stress acting on the upper plate is given by $p_{ij}n_j$, which is

$$(-p_{12}\big|_{y=h}, p, 0). \tag{3}$$

The stresses of the lower and upper plates acting on the fluid are of opposite sign to (2) and (3) respectively. The total viscous force in the x-direction is

$$\Big(-p_{12}\big|_{y=0} + p_{12}\big|_{y=h}\Big)L = \frac{\partial p}{\partial x}hL = (p\big|_{x=L} - p\big|_{x=0})h < 0, \tag{4}$$

which has the same magnitude as, but of opposite the sign to, the total pressure force (1). The two are therefore in balance. $\boxed{5, \text{D}}$

2. (i) Since the flow remains spherical, the Laplace equation for $\varphi(r)$ reduces to sim. seen ⇓

$$\frac{\partial}{\partial r}\big(r^2\frac{\partial\varphi}{\partial r}\big) = 0.$$

It follows that

$$r^2\frac{\partial\varphi}{\partial r} = f(t), \qquad \text{i.e.} \qquad \frac{\partial\varphi}{\partial r} = \frac{f(t)}{r^2},$$

which is integrated to give

$$\varphi = -\frac{f(t)}{r}.$$

The radial velocity is, according to the given formula, given by

$$V_r = \frac{\partial\varphi}{\partial r} = \frac{f(t)}{r^2}.$$

At the interface (bubble surface), $r = R(t)$, $V_r = \frac{dR}{dt}$ by the kinematic condition. Hence

$$\frac{f(t)}{R^2} = \frac{dR}{dt},$$

which gives

$$f(t) = R^2\frac{dR}{dt}, \tag{5}$$

as required. 3, A

(ii) Applying Bernoulli's equation at $r \leq R(t)$ and $r \to \infty$ (where the fluid velocity is zero), and noting that meth seen ⇓

$$\frac{\partial\varphi}{\partial t} = -\frac{1}{r}\frac{df}{dt}, \quad |\mathbf{V}|^2 = |V_r|^2 = \frac{f^2}{r^4},$$

we have

$$-\frac{1}{r}\frac{df}{dt} + \frac{1}{2}\frac{f^2}{r^4} + \frac{p}{\rho} = \frac{p_\infty}{\rho}.$$

Now taking the limit $r \to R(t)$ and using the fact that $p \to p_b = k/R^3$, we obtain

$$-\frac{1}{R}\frac{df}{dt} + \frac{1}{2}\frac{f^2}{R^4} = \frac{p_\infty - k/R^3}{\rho}. \tag{6}$$

4, A

(iii) Note that sim. seen ⇓

$$\frac{df}{dt} = \frac{df}{dR}\frac{dR}{dt} = R^{-2}f\frac{df}{dR}.$$

Use of this relation in (6) converts the latter to

$$-\frac{ff'(R)}{R^3} + \frac{1}{2}\frac{f^2}{R^4} = -\frac{k}{\rho R^3} + \frac{p_\infty}{\rho}. \tag{7}$$

In terms of $g = f^2$, the equation can be rewritten as

$$g'(R) - \frac{1}{R}g = -\frac{2p_\infty}{\rho}R^3 + \frac{2k}{\rho}. \tag{8}$$

This is a linear first-order ordinary differential equation for g, and so can be solved by using the method of integration factor, which is found as

$$\exp\{-\int^R \frac{1}{R}dR\} = 1/R.$$

Multiplication of this to both sides of (8) leads to

$$\frac{d}{dR}\Big[\frac{g}{R}\Big] = -\frac{2p_\infty}{\rho}R^2 + \frac{2k}{\rho}\frac{1}{R}.$$

Integrating with respect to R, from R_0 where $g = f^2 = 0$, we obtain

$$\frac{g}{R} = -\frac{2p_\infty}{3\rho}(R^3 - R_0^3) + \frac{2k}{\rho}\ln(R/R_0),$$

and so

$$f^2 = R\Big[\frac{2p_\infty}{3\rho}(R_0^3 - R^3) + \frac{2k}{\rho}\ln(R/R_0)\Big]. \qquad (9)$$

Using (5), we can further write (9) as

$$R^2\frac{dR}{dt} = \pm\sqrt{R\Big[\frac{2p_\infty}{3\rho}[R_0^3 - R^3] + \frac{2k}{\rho}\ln(R/R_0)\Big]}. \qquad (10)$$

6, B

(iv) Let

unseen ⇓

$$G(R) = \frac{2p_\infty}{3\rho}[R_0^3 - R^3] + \frac{2k}{\rho}\ln(R/R_0).$$

Note that $G(R_0) = 0$, and

$$G'(R_0) = -\frac{2p_\infty}{\rho}R_0^2 + \frac{2k}{\rho}R_0^{-1}.$$

If $R_0 < R_c$, i.e.

$$\frac{2p_\infty R_0^2}{\rho} < \frac{2k}{\rho}R_0^{-1},$$

$G(R)$ would be negative for R less than R_0 since

$$G'(R) = -\frac{2p_\infty}{\rho}R^2 + \frac{2k}{\rho}R^{-1} > -\frac{2p_\infty}{\rho}R_0^2 + \frac{2k}{\rho}R_0^{-1} > 0,$$

but positive for R slightly greater than R_0. The bubble would then expand and the positive sign must be taken in (10). The bubble reaches its maximum, $R_{\text{max}} > R_0$, when $\frac{dR}{dt} = 0$, with R_{max} being determined by

$$G(R_{\text{max}}) = \frac{2p_\infty}{3\rho}(R_0^3 - R_{\text{max}}{}^3) + \frac{2k}{\rho}\ln(R_{\text{max}}/R_0) = 0.$$

In order to show that there exists a unique maximum, we note that

$$G'(R) = 0$$

at

$$R = R_c = (\frac{k}{p_\infty})^{1/3}.$$

It follows that $G > 0$ and $G' > 0$ for $R_0 < R < R_c$, but $G' < 0$ for $R > R_c$. Since $G \to -\infty$ as $R \to \infty$, there must exist a unique root $R > R_c$.

After reaching its maximum size, the bubble shrinks to R_0 and then expands again. The motion is oscillatory.

7, D

3. (a) A streamline is defined as a line at each point of which the velocity of the fluid is in the tangent direction. Therefore in order to show a contour $\psi =$ constant is a streamline, we need to show that its normal direction $\mathbf{n}$ and the velocity $\mathbf{V}$ are perpendicular, i.e. satisfy $\mathbf{V} \cdot \mathbf{n} = 0$. Since $\mathbf{V} = \nabla\varphi$, what is required is to show that $\boxed{\text{sim. seen} \Downarrow}$

$$\nabla\varphi \cdot \mathbf{n} = 0.$$

The normal direction of the contour is $\mathbf{n} = \nabla\psi$. Using the given formula,

$$\mathbf{n} = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\mathbf{e}_\vartheta.$$

The velocity at each point is

$$\mathbf{V} = \nabla\varphi = \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}\mathbf{e}_\vartheta.$$

It follows that

$$\mathbf{V} \cdot \mathbf{n} = \frac{\partial\varphi}{\partial r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\vartheta}\frac{\partial\psi}{\partial\vartheta} = \frac{\partial\varphi}{\partial r}\big(-\sin\vartheta\frac{\partial\varphi}{\partial\vartheta}\big) + \frac{1}{r^2}\frac{\partial\varphi}{\partial\vartheta}\big(r^2\sin\vartheta\frac{\partial\varphi}{\partial r}\big) = 0,$$

where use has been made of the given relations between ψ and φ in the last step. $\boxed{4, \text{B}}$

(b) (i) For the given velocity potential φ, $\boxed{\text{meth seen} \Downarrow}$

$$\frac{\partial\varphi}{\partial r} = V_\infty\cos\vartheta + \frac{q}{4\pi r^2} - \frac{m\cos\vartheta}{2\pi r^3}, \quad \frac{\partial\varphi}{\partial\vartheta} = -V_\infty r\sin\vartheta - \frac{m\sin\vartheta}{4\pi r^2}.$$

Substitution into the given relations in the question yields

$$\frac{\partial\psi}{\partial r} = -\sin\vartheta\frac{\partial\varphi}{\partial\vartheta} = \big(V_\infty r + \frac{m}{4\pi r^2}\big)\sin^2\vartheta, \tag{11}$$

$$\frac{\partial\psi}{\partial\vartheta} = r^2\sin\vartheta\frac{\partial\varphi}{\partial r} = r^2\big(V_\infty\cos\vartheta + \frac{q}{4\pi r^2} - \frac{m\cos\vartheta}{2\pi r^3}\big)\sin\vartheta. \tag{12}$$

Integrating (11) with respect to r, we find

$$\psi = \Big(\frac{1}{2}V_\infty r^2 - \frac{m}{4\pi r}\Big)\sin^2\vartheta + g(\vartheta),$$

where the unknown function $g(\vartheta)$ can be determined by inserting ψ into (12), which gives

$$g'(\vartheta) = \frac{q}{4\pi}\sin\vartheta,$$

and hence

$$g(\vartheta) = -\frac{q}{4\pi}\cos\vartheta$$

with an arbitrary constant being taken to be zero without affecting the flow field. Hence the (stream) function ψ is found as

$$\psi = \Big(\frac{1}{2}V_\infty r^2 - \frac{m}{4\pi r}\Big)\sin^2\vartheta - \frac{q}{4\pi}\cos\vartheta.$$

$\boxed{6, \text{A}}$

(ii) In terms of the given velocity potential φ, the velocity can be written as

unseen $\Downarrow$

$$V_r = \frac{\partial \varphi}{\partial r} = V_\infty \cos\vartheta + \frac{q}{4\pi r^2} - \frac{m\cos\vartheta}{2\pi r^3},$$

$$V_\vartheta = \frac{1}{r}\frac{\partial \varphi}{\partial \vartheta} = -V_\infty \sin\vartheta - \frac{m\sin\vartheta}{4\pi r^3}.$$

On the surface of the sphere, $r = a$, the normal velocity $V_r = V_s$, that is

$$V_r = \frac{\partial \varphi}{\partial r} = V_\infty \cos\vartheta + \frac{q}{4\pi a^2} - \frac{m\cos\vartheta}{2\pi a^3} = V_s$$

for any ϑ. This requires us to take

$$m = 2\pi a^3 V_\infty, \quad q = 4\pi a^2 V_s,$$

for which

$$\psi = \frac{1}{2}V_\infty\Big(r^2 - \frac{a^3}{r}\Big)\sin^2\vartheta - a^2 V_s \cos\vartheta.$$

4, B

sim. seen $\Downarrow$

(iii) Stagnation points are given by

$$V_r = V_\infty\Big(1 - \frac{a^3}{r^3}\Big)\cos\vartheta + \frac{a^2 V_s}{r^2} = 0, \quad V_\vartheta = -V_\infty\big(1 + \frac{a^3}{2r^3}\big)\sin\vartheta = 0.$$

From the second equation, we find

$$\vartheta_s = 0, \quad \vartheta_s = \pi.$$

The corresponding r_s are determined by

$$V_\infty\Big(1 - \frac{a^3}{r_s^3}\Big) \pm \frac{a^2 V_s}{r_s^2} = 0,$$

which can be rewritten as

$$(r_s/a)^3 \pm (V_s/V_\infty)(r_s/a) - 1 = 0,$$

where the plus and minus signs correspond to $\vartheta_s = 0$ and π respectively. There is only one positive root in each case (and can be found by using the standard formula for a cubic polynomial).

2, C

Of the two stagnation points, $\vartheta_s = \pi$ may represent the front stagnation point of a blunt leading edge of a rigid body. The streamline through this point is

unseen $\Downarrow$

$$\psi(r,\vartheta) = \psi(r_s, \vartheta_s) = a^2 V_s,$$

that is

$$\frac{1}{2}V_\infty\Big(r^2 - \frac{a^3}{r}\Big)\sin^2\vartheta = a^2 V_s(1 + \cos\vartheta).$$

This represents the body contour. In terms of $y = r\sin\vartheta$, it may be written as

$$y^2 - \frac{a^3\sin^3\vartheta}{y} = 2a^2(V_s/V_\infty)(1 + \cos\vartheta).$$

As $\vartheta \to 0$,

$$y \to 2\sqrt{\frac{V_s}{V_\infty}}\,a.$$

4, D

4. (a) (i) By Cauchy-Riemann equations, we have $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} = 2b(y-h), \quad \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x} = 2bx,$$

Integration of the first equation gives

$$\psi = 2bx(y-h) + g(y),$$

where the function $g(y)$ is determined by inserting ψ into the second equation,

$$2bx + g'(y) = 2bx,$$

which shows that $g(y)$ is a constant, whose value does not affect the flow. Thus

$$\psi = 2bx(y-h).$$

The complex potential is

$$w(z) = \varphi + i\psi = b\Big(x^2 - (y-h)^2\Big) + 2ibx(y-h) = b\Big[x + i(y-h)\Big]^2 = b(z-ih)^2.$$

Obviously, $y = h$ is one of the streamlines and may act as a rigid boundary. $\boxed{3, \text{A}}$

(ii) When a source q is added at $z = iH$ to the flow, an image source of equal strength must be added to $z = i\big(h-(H-h)\big) = (2h-H)i$, which is symmetric about $y = h$ in order to satisfy the boundary condition at $y = h$. Therefore the complex potential is $\boxed{\text{unseen} \Downarrow}$

$$w(z) = b(z-ih)^2 + \frac{q}{2\pi}\ln(z-iH) + \frac{q}{2\pi}\ln\Big[z - i(2h-H)\Big].$$

$\boxed{3, \text{B}}$

(b) (i) Let $\zeta = \xi + ih$ in the ζ-plane. The corresponding image in the physical z-plane is given by $z = x + iy$. Substitution into the mapping gives $\boxed{\text{meth seen} \Downarrow}$

$$x + iy = (\xi + ih)^2 + d = \xi^2 - h^2 + d + 2ih\xi,$$

from which we obtain

$$x = \xi^2 - h^2 + d, \quad y = 2h\xi.$$

Elimination of ξ leads to

$$x = \frac{y^2}{4h^2} - h^2 + d.$$

For the above to be identical to the parabola, $y = \pm 2a\sqrt{x}$, which is $x = y^2/(2a)^2$, we need to set

$$h = a, \quad d = h^2 = a^2.$$

$\boxed{3, \text{A}}$

(ii) The location of the source, $z = -3a^2$, is mapped to ζ, which satisfies $\boxed{\text{unseen} \Downarrow}$

$$-3a^2 = \zeta^2 + a^2,$$

and hence $\zeta = 2ai$.

In the ζ-plane, the complex potential is taken to be a supposition of a stagnation flow above a wall at $\zeta = h = a$ and a source at $\zeta = 2ai$ plus its image at $\zeta = 0$, and so

$$W(\zeta) = b\big(\zeta - ih\big)^2 + \frac{q}{2\pi}\ln(\zeta - 2ia) + \frac{q}{2\pi}\ln\zeta.$$

The composition of $W(\zeta)$ with the mapping gives $w(z)$. The complex conjugate velocity in the z-plane is calculated by

$$\overline{V}(z) = \frac{dW}{d\zeta}\frac{d\zeta}{dz} = \frac{dW}{d\zeta}\frac{1}{dz/d\zeta} = \Big[2b(\zeta - ih) + \frac{q}{2\pi(\zeta - 2ai)} + \frac{q}{2\pi\zeta}\Big]\frac{1}{2\zeta}. \quad (13)$$

The above indicates that as $\zeta \to \infty$ (which corresponds to $z \to \infty$),

$$\overline{V} \to b.$$

For the complex potential to represent the physical flow, we need to take

$$b = V_\infty.$$

As $\zeta \to 2ia$, we set $\zeta - 2ia = \tilde{\zeta}$ and $z + 3a^2 = \tilde{z}$, substitution of which into the mapping, $\zeta^2 + a^2 = z$, implies that

$$4ai\tilde{\zeta} \to \tilde{z}.$$

Use of the above in (13) confirms that as $\zeta \to 2ia$ (which corresponds to $z \to -3a^2$),

$$\overline{V} \to \frac{q}{2\pi\tilde{z}} = \frac{q}{2\pi}\frac{1}{\big(z - (-3a^2)\big)}.$$

Hence

$$\overline{V}(z) = \frac{2V_\infty}{\zeta(\zeta - ih)} + \frac{q}{4\pi}\frac{1}{\zeta}\Big[\frac{1}{\zeta - 2ai} + \frac{1}{\zeta}\Big].$$

$\boxed{4,\ \mathrm{C}}$

(iii) The parabola surface corresponds to $\zeta = \xi + ai$. When $V_\infty = 0$, the complex conjugate velocity along the parabola is given by $\boxed{\text{meth seen} \Downarrow}$

$$\overline{V} = \frac{q}{4\pi}\frac{1}{\xi + ia}\Big[\frac{1}{\xi - ia} + \frac{1}{\xi + ia}\Big] = \frac{q}{2\pi}\frac{\xi}{(\xi + ia)(\xi^2 + a^2)}.$$

It follows that

$$|\overline{V}| = \frac{q}{2\pi}\frac{\xi}{(\xi^2 + a^2)^{3/2}},$$

which can, after using the relation $\xi = y/(2a)$, be rewritten as

$$|\overline{V}| = \frac{q}{2\pi}\frac{4a^2 y}{(y^2 + 4a^4)^{3/2}}.$$

The pressure on the parabola surface is calculated by using the Bernoulli equation, which gives

$$p - p_\infty = -\frac{1}{2}\rho|\overline{V}|^2 = -\rho\big(\frac{q}{2\pi}\big)^2\frac{8a^4y^2}{(y^2 + 4a^4)^3}.$$

$\boxed{4,\ \mathrm{A}}$

The drag is calculated as

$$D = 2\int_0^\infty (p - p_\infty)dy = -\rho\big(\frac{q}{\pi}\big)^2(4a^4)\int_0^\infty \frac{y^2}{(y^2+4a^4)^3}dy.$$

The integral is evaluated by using the substitution

$$y = 2a^2\tan\vartheta,$$

and so

$$\int_0^\infty \frac{y^2}{(y^2+4a^4)^3}dy = (2a^2)^{-3}\int_0^{\frac{\pi}{2}} \tan^2\vartheta\cos^4\vartheta\, d\vartheta$$

$$= \frac{1}{4}(2a^2)^{-3}\int_0^{\frac{\pi}{2}} \sin^2(2\vartheta)\, d\vartheta = \frac{\pi}{16}(2a^2)^{-3}.$$

Substitution back into D gives

$$D = -\frac{\pi}{16}(2a^2)^{-1}\rho\big(\frac{q}{\pi}\big)^2.$$

$\boxed{3, \mathrm{B}}$

5. (i) Let the distance of the centre on the z-plane to the chord be denoted as d. The radius of the circle is $d+h$. The geometry indicates that

seen $\Downarrow$

$$d^2 + a^2 = (d+h)^2,$$

which gives

$$d = (a^2 - h^2)/(2h).$$

The radius is $d + h = (a^2 + h^2)/(2h)$.
The centre is at $z = -id = -i(a^2 - h^2)/(2h)$. It is mapped to

$$\zeta = z + \sqrt{z^2 - a^2} = -i(a^2 - h^2)/(2h) + \sqrt{-(a^2 - h^2)^2/(2h)^2 - a^2}$$
$$= -i(a^2 - h^2)/(2h) + i(a^2 + h^2)/(2h) = ih.$$

[3 marks]

sim. seen $\Downarrow$

(ii) Since the cylinder is centred at ih and its radius is $\sqrt{a^2 + h^2}$, the complex potential is

$$W(\zeta) = \tilde{V}_\infty\left[(\zeta - ih)e^{-i\alpha} + \frac{a^2 + h^2}{(\zeta - ih)e^{-i\alpha}}\right] + \frac{\Gamma}{2\pi i}\ln(\zeta - ih). \qquad (14)$$

[3 marks]

sim. seen $\Downarrow$

(iii) The complex conjugate velocity in the z-plane is calculated as

$$\overline{V} = \frac{dW}{d\zeta}\Big/\frac{dz}{d\zeta},$$

which is

$$\overline{V} = \left\{\tilde{V}_\infty\left[e^{-i\alpha} - \frac{a^2 + h^2}{(\zeta - ih)^2 e^{-i\alpha}}\right] + \frac{\Gamma}{2\pi i(\zeta - ih)}\right\}\frac{2}{1 - a^2/\zeta^2}.$$

Taking the limit $\zeta \to \infty$ (which corresponds to $z \to \infty$), we have

$$\overline{V} \to 2\tilde{V}_\infty e^{-i\alpha}.$$

In order to represent the physical flow, we take

$$\tilde{V}_\infty = \frac{1}{2}V_\infty.$$

[3 marks]

sim. seen $\Downarrow$

(iv) The trailing edge of the arc is $z = a$, which is mapped to $\zeta = a$. The figure shows that for $\zeta = a$,

$$\zeta - ih = \sqrt{a^2 + h^2}e^{-i\beta/2}. \qquad (15)$$

In order to satisfy the Joukowskii-Kutta condition, i.e. to keep the velocity at the trailing edge finite, we set

$$\frac{1}{2}V_\infty\left[e^{-i\alpha} - \frac{a^2 + h^2}{(\zeta - ih)^2 e^{-i\alpha}}\right] + \frac{\Gamma}{2\pi i(\zeta - ih)} = 0,$$

which may, after using (15), be written as

$$\frac{1}{2}V_\infty\Big[e^{-i\alpha}-e^{i(\alpha+\beta)}\Big]+\frac{\Gamma}{2\pi i}\frac{e^{i\beta/2}}{\sqrt{a^2+h^2}}=0.$$

We find that

$$\Gamma=-2\pi V_\infty\sqrt{a^2+h^2}\sin(\alpha+\beta/2), \tag{16}$$

[5 marks]

unseen $\Downarrow$

(v) Substitution of (16) into $W(\zeta)$ gives

$$\frac{dW}{d\zeta}=\frac{1}{2}V_\infty\Big[e^{-i\alpha}-\frac{a^2+h^2}{(\zeta-ih)^2e^{-i\alpha}}\Big]-\frac{V_\infty\sqrt{a^2+h^2}\sin(\alpha+\beta/2)}{i(\zeta-ih)}, \tag{17}$$

and

$$\overline{V}=\frac{dW}{d\zeta}\frac{2\zeta^2}{\zeta^2-a^2}. \tag{18}$$

Calculate the velocity at $z=ih$ on the *upper side* of the arc. This point is mapped to

$$\zeta=ih+i\sqrt{a^2+h^2},$$

Inserting this into (17) gives

$$\frac{dW}{d\zeta}=\frac{1}{2}V_\infty\Big[e^{-i\alpha}+e^{i\alpha}+2\sin(\alpha+\beta/2)\Big]=V_\infty\Big[\cos\alpha+\sin(\alpha+\beta/2)\Big],$$

and further we have

$$\zeta^2=-(2h^2+a^2+2h\sqrt{a^2+h^2}),$$
$$\zeta^2-a^2=-\Big[2a^2+2h^2+2h\sqrt{a^2+h^2}\Big].$$

Use of the above results in (18) gives

$$\overline{V}=\frac{\cos\alpha+\sin(\alpha+\beta/2)}{a^2+h^2+h\sqrt{a^2+h^2}}(2h^2+a^2+2h\sqrt{a^2+h^2})V_\infty,$$

or

$$\overline{V}=\big(\cos\alpha+\sin(\alpha+\beta/2)\big)\Big[2-\frac{a^2}{h^2+a^2+h\sqrt{a^2+h^2}}\Big]V_\infty. \tag{19}$$

Similarly, calculate the velocity at $z=ih$ but on *lower* side of the arc. This point is mapped to

$$\zeta=ih-i\sqrt{a^2+h^2}.$$

This is used in (17) to calculate

$$\frac{dW}{d\zeta}=\frac{1}{2}V_\infty\Big[e^{-i\alpha}+e^{i\alpha}-2\sin(\alpha+\beta/2)\Big]=V_\infty\Big[\cos\alpha-\sin(\alpha+\beta/2)\Big].$$

Also we have

$$\zeta^2=-(2h^2+a^2-2h\sqrt{a^2+h^2}),$$
$$\zeta^2-a^2=-\Big[2a^2+2h^2-2h\sqrt{a^2+h^2}\Big].$$

Inserting the above expressions into (18), we obtain

$$\overline{V} = \big(\cos\alpha - \sin(\alpha + \beta/2)\big)\Big[2 - \frac{a^2}{h^2 + a^2 - h\sqrt{a^2 + h^2}}\Big]V_\infty. \qquad (20)$$

Comparison between (19) and (20) indicates clearly that the speed on the upper side is greater than that on the lower side.

[6 marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96002 MATH97008 MATH97008	1	Although the overall performance on the paper was good, question 1 turned out to be most troublesome. Many took the viscosity coefficient to be constant, and were unable to obtain the solution for u. Even the best students could not tackle the last part, the verification of force balance.
MATH96002 MATH97008 MATH97008	2	The majority did well on Question 2. The question is set such that the Laplace and Bernoulli equations are to be solved, but quite a few chose to start from the continuity and momentum equations, a route that was accepted.
MATH96002 MATH97008 MATH97008	3	Question 3 was well done. A few mistakenly imposed no-slip condition to determine the dipole strength. The very last part was tackled only by a few (which was expected since it is catered to test and reward the most devoted).
MATH96002 MATH97008 MATH97008	4	Most did well on Question 4. A common oversight of quite a few was that the complex potential constructed from the streamfunction and velocity potential was not written in terms of complex position z. Again the very last part was done by only a few.
MATH96002 MATH97008 MATH97008	5	All except a few did well on Question 5. The amount of work of Msci and MSc students managed to produce in 2.5 hours was rather impressive.