Imperial College
London

MATH97042 MATH97151

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometry 2: Algebraic Topology

Date: Friday, 7 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Draw a connected CW complex with fundamental group $\mathbb{Z}/6\mathbb{Z}$.
(No proof necessary, but the construction must be illustrated clearly.) (4 marks)

(b) Compute for each integer $n \geq 1$ the fundamental group of the $n \times n$ grid

$$G_n = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \leq x \leq n, \quad 0 \leq y \leq n, \\ \text{at least one of } x, y \text{ is an integer} \end{array} \right\}.$$

(5 marks)

(c) Let $X = D^2 \cup_f T^2$, where the map $f : \partial D^2 \to T^2$, expressed as a map $S^1 \to S^1 \times S^1$, sends x to $(x, 1)$. Compute $\pi_1(X)$. (5 marks)

(d) Let X be the Klein bottle with a point removed.

(i) Prove that $\pi_1(X) \cong F_2$ is the free group on two generators. (2 marks)

(ii) Show that there is no path-connected space Y such that X is homotopy equivalent to $Y \times Y$. (4 marks)

(Total: 20 marks)

2. In all parts of this question, $p : X \to Y$ will be a covering map of some finite degree $d \geq 1$.

(a) Let Σ_g be the closed, orientable surface of genus g. Draw explicit pictures of covers $p : \Sigma_5 \to \Sigma_3$ and $p : \Sigma_5 \to \Sigma_2$ and a nontrivial deck transformation for each. (4 marks)

(b) If Y is a finite CW complex, show that X admits the structure of a finite CW complex with n-skeleton $X^n = p^{-1}(Y^n)$ for all $n \geq 0$. (4 marks)

(c) If Y is a finite CW complex, prove that $\chi(X) = d \cdot \chi(Y)$, where χ is the Euler characteristic. (4 marks)

(d) Prove that if Y is a finite CW complex and $X = \mathbb{RP}^{2k}$ for some integer k, then $Y \cong \mathbb{RP}^{2k}$ as well. (4 marks)

(e) Construct a covering $p : \mathbb{RP}^3 \to Y$ where Y is not homeomorphic to $\mathbb{RP}^3$. (4 marks)

(Total: 20 marks)

3. (a) Let $p : (X, x) \to (Y, y)$ be a covering map of degree 2.

   (i) Prove that p is regular. (3 marks)

   (ii) Prove that if $H_1(Y)$ is finite then it has even order. (4 marks)

(b) Let $p : (X, x) \to (Y, y)$ be a covering map, and suppose that there is a continuous map $s : (Y, y) \to (X, x)$ such that $p \circ s = \mathrm{Id}_Y$. Prove that p is a homeomorphism. (5 marks)

(c) Let $\mathrm{Sym}^2(\mathbb{C}) = \dfrac{\mathbb{C}^2}{(a, b) \sim (b, a)}$ be the space of *unordered* pairs of complex numbers, and let $Q = \{z^2 + pz + q \mid p, q \in \mathbb{C}\}$ be the space of monic quadratic polynomials with complex coefficients. These come with the quotient map $\pi : \mathbb{C}^2 \to \mathrm{Sym}^2(\mathbb{C})$ and a homeomorphism

$$r : \mathrm{Sym}^2(\mathbb{C}) \xrightarrow{\cong} Q$$
$$[(a, b)] \mapsto (z - a)(z - b).$$

   (i) State an explicit formula (no proof necessary) for $r^{-1}(z^2 + cz + d)$. (1 mark)

   (ii) Define $\Delta \subset \mathbb{C}^2$ by $\Delta = \{(a, a) \mid a \in \mathbb{C}\}$. Prove that $\mathbb{C}^2 \setminus \Delta$ is path-connected, and that

$$\pi|_{\mathbb{C}^2 \setminus \Delta} : \mathbb{C}^2 \setminus \Delta \longrightarrow \mathrm{Sym}^2(\mathbb{C}) \setminus \pi(\Delta)$$

   is a covering map. (4 marks)

   (iii) A *quadratic formula* is a continuous map $q : Q \to \mathbb{C}$ such that $q(z^2 + cz + d)$ is a root of $z^2 + cz + d$, or equivalently such that

$$(q \circ r)\Big([(a, b)]\Big) \text{ is equal to either } a \text{ or } b.$$

   Prove that a quadratic formula does not exist. (3 marks)

(Total: 20 marks)

4. (a) Let $X_n = \mathbb{R}^3 \setminus \{(0, 0, k) \mid k = 1, 2, \ldots, n\}$ be the complement of n points in $\mathbb{R}^3$ for some $n \geq 1$. Compute $H_*(X_n)$. (6 marks)

(b) Let X be a path-connected space and $a, b \in X$ points with disjoint open neighborhoods U and V such that

$$U \cong V \cong \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

Compute the singular homology of $\dfrac{X}{\{a, b\}}$ in terms of the homology of X. (6 marks)

(c) Let Y denote the k-skeleton of a d-simplex Δ^d for some $k \leq d$, meaning the union of all sub-simplices of dimension at most k.

   (i) Show that $H_n(Y) = 0$ for all $n \geq k + 1$. (2 marks)

   (ii) Show that $\tilde{H}_n(Y) \cong H_{n+1}(\Delta^d / Y)$ for all $n \geq 0$. (3 marks)

   (iii) Compute $H_n(\Delta^d / Y)$ for $1 \leq n \leq k$. (3 marks)

(Total: 20 marks)

5. Let X be a path-connected, finite CW complex. In this problem we will suppose that X is a *topological group*: it has a point $e \in X$ and continuous "multiplication" and "inverse" maps

$$m : X \times X \to X, \qquad i : X \to X$$

such that the binary operation $x \cdot y = m(x, y)$ turns X into a group, with identity e and inverses $x^{-1} = i(x)$ for all $x \in X$.

(a) Prove that for all $x \in X$, the "left multiplication by x" map $\ell_x : X \to X$ defined by $\ell_x(y) = m(x, y)$ is homotopic to the identity map on X. (5 marks)

(b) Compute the Lefschetz number $\tau(\ell_x)$ for all $x \in X$. Your answer should be expressed in terms of topological invariants of X. (4 marks)

(c) Prove that if X is not the trivial group then X has Euler characteristic zero. (3 marks)

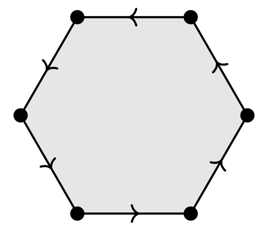
(d) Prove that S^{2n} is not a topological group for any $n \geq 1$. (4 marks)

(e) Let $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be an invertible linear map, representable by an $(n + 1) \times (n + 1)$ complex matrix. Use A to construct a continuous map $\mathbb{CP}^n \to \mathbb{CP}^n$ with nonzero Lefschetz number, and deduce that A has an eigenvector. You may assume that the group $GL(n+1, \mathbb{C})$ is a path-connected, finite CW complex. (4 marks)

(Total: 20 marks)

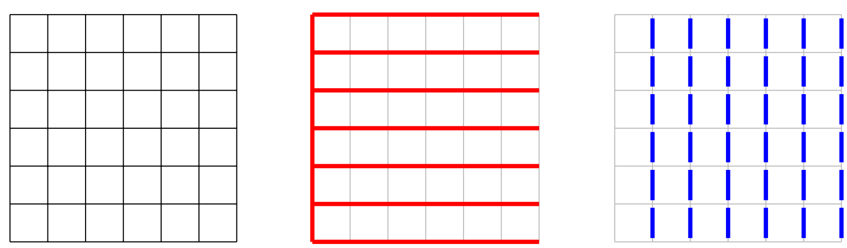
Solutions

1. (a) We draw a complex with one cell in each of dimensions 0, 1, and 2:



The boundary of the 2-cell wraps around the 1-cell six times. (4 marks)

(b) We view G_n as a graph whose vertices are the points $(x, y) \in G_n$ with $x, y \in \mathbb{Z}$, and whose edges are vertical and horizontal line segments of length one connecting the vertices. We then draw a maximal tree $T \subset G_n$ as the union of the vertical line from $(0, 0)$ to $(0, n)$ and the horizontal lines from $(0, i)$ to (n, i), $i = 0, 1, 2, \ldots, n$. For example, when $n = 6$ we have:



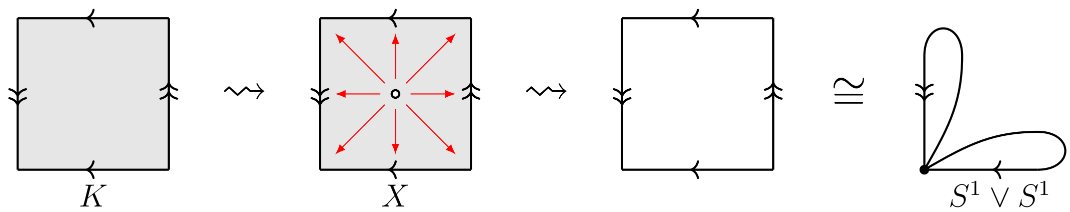
Then fixing a vertex $x \in G_n$, we know that $\pi_1(G_n, x)$ is isomorphic to the free group with one generator for every edge in $G_n \setminus T$. These are precisely the vertical edges from $(i, j-1)$ to (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq n$, so $\pi_1(G_n, x) \cong F_{n^2}$. (5 marks)

(c) Viewing T^2 as $S^1 \times S^1$, the group $\pi_1(T^2, (1, 1)) \cong \mathbb{Z}^2$ is generated by loops $\gamma, \eta : [0, 1] \to T^2$ defined by $\gamma(t) = (e^{2\pi i t}, 1)$ and $\eta(t) = (1, e^{2\pi i t})$. We construct X by attaching a 2-cell to T^2 along γ, so we know (by a computation involving the Seifert–van Kampen theorem, though this need not be invoked) that

$$\pi_1(X) = \frac{\pi_1(T^2, (1, 1))}{\langle\!\langle [\gamma] \rangle\!\rangle} \cong \frac{\langle \gamma, \eta \mid \gamma \cdot \eta = \eta \cdot \gamma \rangle}{\langle\!\langle \gamma \rangle\!\rangle} \cong \langle \eta \rangle \cong \mathbb{Z}.$$

(5 marks)

(d) (i) We draw a CW complex for the Klein bottle K, and then remove a point from the center of the 2-cell to get X. We then deformation retract X onto its 1-skeleton $S^1 \vee S^1$ by pushing the rest of the 2-cell radially outward, as shown below:

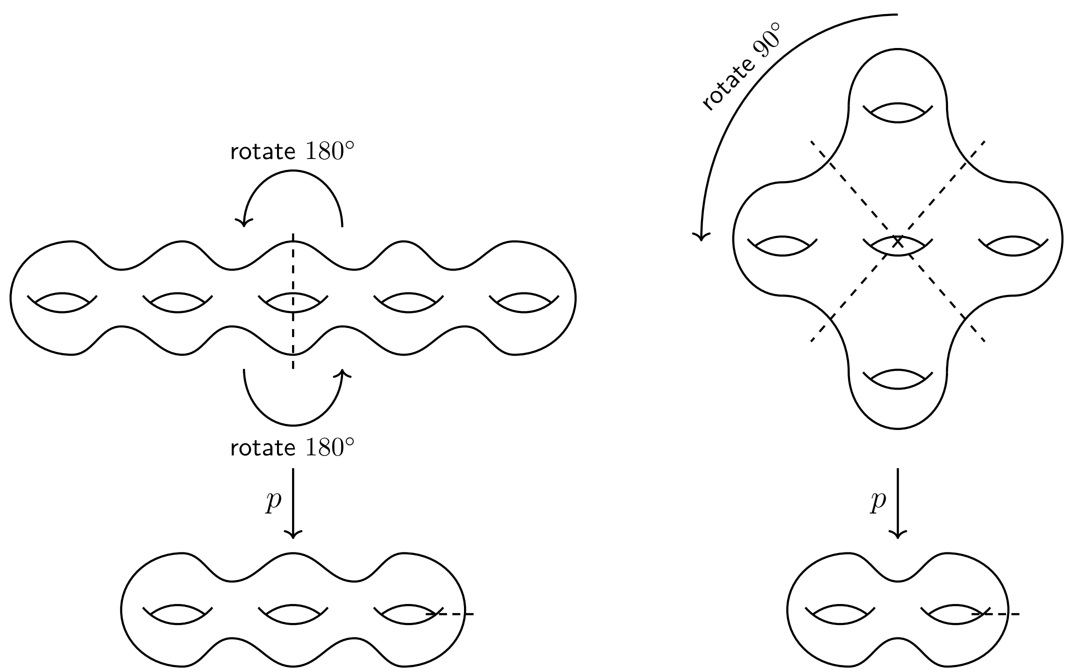


So $\pi_1(X) \cong \pi_1(S^1 \vee S^1) \cong F_2$ is the free group on two generators. (2 marks)

(ii) If $X \simeq Y \times Y$ then $\pi_1(X, x) \cong \pi_1(Y, y) \times \pi_1(Y, y)$ for appropriate base points x and y, so if $G = \pi_1(X, x)$ then we have $F_2 \cong G \times G$; note that G must be nontrivial since F_2 is. For any nontrivial element $g \neq 1$ in G, the subgroup $H = \{(g^i, g^j) \mid i, j \in \mathbb{Z}\}$ is an abelian subgroup of $G \times G \cong F_2$, and it is isomorphic to either $\mathbb{Z}^2$ or $(\mathbb{Z}/n\mathbb{Z})^2$ for some $n \geq 2$ by a map $(g^i, g^j) \mapsto (i, j)$, depending on whether g has infinite order or some finite order n. But any subgroup of a free group is free, and the only non-trivial abelian group which is free is $\mathbb{Z}$, so we must also have $H \cong \mathbb{Z}$, contradiction. (4 marks)

(Total: 20 marks)

2. (a)



(4 marks)

(b) We start by letting $X^0 = p^{-1}(Y^0)$. Now if we have built X^{n-1} for some $n \geq 1$, then the n-cells of Y are attached by maps

$$\sigma_\alpha : \partial D^n_\alpha \to Y^{n-1}.$$

We fix a point $z \in \partial D^n_\alpha$ and let $y_\alpha = \sigma_\alpha(z) \in Y^{n-1}$, and then we let $x^1_\alpha, \ldots, x^d_\alpha$ denote the points of $p^{-1}(x_\alpha) \subset X^{n-1}$. Then each map $p : (X^{n-1}, x^i_\alpha) \to (Y^{n-1}, y_\alpha)$ is a covering map and D^n is simply connected, so:

$$\begin{tikzcd} & X^{n-1} \arrow[d, "p"] \\ \partial D^n \arrow[ur, dashed, "\sigma^i_\alpha"] \arrow[r, "\sigma_\alpha"] & Y^{n-1} \end{tikzcd}$$

there is a unique lift $\sigma^i_\alpha : \partial D^n \to X^{n-1}$ with $\sigma^i_\alpha(z) = x^i_\alpha$. We let

$$X^n = \left(\bigsqcup_\alpha \bigsqcup_{i=1}^n D^n_{\alpha,i} \right) \cup_{\sqcup_{\alpha,i} \sigma^i_\alpha} X^{n-1}.$$

Now the preimage of each cell D^n_α in Y^n is precisely the union of the d cells $D^n_{\alpha,i}$ in X^n, so p extends to a covering $X^n \to Y^n$. Since Y is finite, we have $Y = Y^N$ for some $N > 0$ and then $X = p^{-1}(Y) = p^{-1}(Y^N) = X^N$, so we have realized X as a CW complex. (4 marks)

(c) Give X the CW complex structure from the previous part. If Y has a_i i-cells for each i, then X has da_i i-cells, so

$$\chi(X) = \sum_{i=0}^N (-1)^i \cdot da_i = d \left(\sum_{i=0}^N (-1)^i a_i \right) = d \cdot \chi(Y).$$

(4 marks)

(d) Since $\mathbb{RP}^{2k}$ can be built out of one i cell for each $i = 0, 1, 2, \ldots, 2k$, we have

$$\chi(\mathbb{RP}^{2k}) = \sum_{i=0}^{2k} (-1)^i = 1 - 1 + 1 - 1 + \cdots + 1 - 1 + 1 = 1.$$

Then $d \cdot \chi(Y) = \chi(\mathbb{RP}^{2k}) = 1$, and both d and $\chi(Y)$ are integers with $d \geq 1$, so we must have $d = 1$. A covering map of degree 1 is a homeomorphism, so $Y \cong \mathbb{RP}^{2k}$. (4 marks)

(e) We write $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, and then the group $\mathbb{Z}/4\mathbb{Z} = \langle a \mid a^4 = 1 \rangle$ acts freely on S^3 by

$$a^k \cdot (z_1, z_2) = (i^k z_1, i^k z_2).$$

Then $a^2(z_1, z_2) = (-z_1, -z_2)$, so we have

$$\mathbb{RP}^3 \cong \frac{S^3}{\langle a^2 \rangle} = \frac{S^3}{x \sim a^2 x},$$

and if we let $Y = S^3/\langle a \rangle$ then the map $p : \mathbb{RP}^3 \to Y$ which sends the equivalence classes

$$\{x, a^2x\}, \{ax, a^3x\} \mapsto \{x, ax, a^2x, a^3x\}$$

is a cover of degree two. (4 marks)

(Total: 20 marks)

3. (a) (i) Since $\deg(p) = 2$, we know that $p_*\pi_1(X,x)$ is an index-2 subgroup of $\pi_1(Y,y)$, hence a normal subgroup. But $p_*\pi_1(X,x)$ is normal iff p is regular, so p is regular. (3 marks)

(ii) The subgroup $p_*\pi_1(X,x)$ is a normal, index-2 subgroup of $\pi_1(Y,y)$, so it must be the kernel of a map from $\pi_1(Y,y)$ onto a group of order 2. The image of this map $f : \pi_1(Y,y) \to \mathbb{Z}/2\mathbb{Z}$ is abelian, so f factors through the abelianization of $\pi_1(Y,y)$:

$$\begin{tikzcd} \pi_1(Y,y) \arrow[r, "\mathrm{ab}"] \arrow[dr, "f"'] & H_1(Y) \arrow[d, dashed, "\tilde{f}"] \\ & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

and the resulting $\tilde{f} : H_1(Y) \to \mathbb{Z}/2\mathbb{Z}$ is surjective since f is, so $\ker(\tilde{f})$ is an index-2 subgroup of $H_1(Y)$, hence $H_1(Y)$ must have even order. (4 marks)

(b) If s exists then we have the commutative diagram at right, showing that s is a lift of Id_Y to (X,x). By the lifting criterion, we must therefore have

$$\begin{tikzcd} & (X,x) \arrow[d, "p"] \\ (Y,y) \arrow[ur, "s"] \arrow[r, "\mathrm{Id}_Y"'] & (Y,y) \end{tikzcd}$$

$$(\mathrm{Id}_Y)_*\pi_1(Y,y) \subset p_*\pi_1(X,x),$$

so $\pi_1(Y,y) = (\mathrm{id}_Y)_*\pi_1(Y,y)$ is a subgroup of $p_*\pi_1(X,x)$, which can only happen if $p_*\pi_1(X,x) = \pi_1(Y,y)$. But then the fundamental theorem of covering spaces tells us that $p : (X,x) \to (Y,y)$ is isomorphic as a covering map to $\mathrm{Id}_Y : (Y,y) \to (Y,y)$, so p is a homeomorphism. (5 marks)

(c) (i) We have $r^{-1}(z^2+cz+d) = \left[\left(\dfrac{-c+\sqrt{c^2-4d}}{2}, \dfrac{-c-\sqrt{c^2-4d}}{2}\right)\right]$. (1 mark)

(ii) The homeomorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f(w,z) = (w, z-w)$ sends Δ to $\{(a,0) \mid a \in \mathbb{C}\} = \mathbb{C} \times \{0\}$, so then it identifies $\mathbb{C}^2 \setminus \Delta$ with $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$, which is clearly path-connected since both factors are. (2 marks)
Given a point $[(a,b)] \in \mathrm{Sym}^2(\mathbb{C}) \setminus \pi(\Delta)$, we have $a \neq b$, so we take disjoint open neighborhoods $U, V \subset \mathbb{C}$ of a and b respectively, and then $[U \times V] \subset \mathrm{Sym}^2(\mathbb{C}) \setminus \Delta$ is an open neighborhood of $[(a,b)]$. Letting $p = \pi|_{\mathbb{C}^2 \setminus \Delta}$, we have

$$p^{-1}\Big([U \times V]\Big) = (U \times V) \sqcup (V \times U),$$

and p restricts to a homeomorphism on each of $U \times V$ and $V \times U$. We can do this for any $[(a,b)] \in \mathrm{Sym}^2(\mathbb{C}) \setminus \Delta$, so p must be a covering map of degree 2. (2 marks)

(iii) If q exists, then we can define a continuous map $s : \mathrm{Sym}^2(\mathbb{C}) \setminus \pi(\Delta) \to \mathbb{C}^2 \setminus \Delta$ by

$$s\Big([(a,b)]\Big) = \Big((q \circ r)\Big([(a,b)]\Big),\ \ a+b-(q\circ r)\Big([(a,b)]\Big)\Big)$$

and verify that $s\Big([(a,b)]\Big)$ is either (a,b) or (b,a), so $p \circ s = \mathrm{Id}$. By the previous part this requires p to be a homeomorphism, but $\deg(p) = 2$, contradiction. (3 marks)

(Total: 20 marks)

4. (a) We write $\mathbb{R}^3 = X_n \cup Y$, where $Y \cong \bigsqcup_{i=1}^{n} D^3$ is the union of n disjoint open disks, each of radius $\frac{1}{3}$, centered at $(0,0,k)$ for $k = 1, 2, \ldots, n$. Then the Mayer–Vietoris sequence for this union has the form

$$\cdots \to \underbrace{H_{k+1}(\mathbb{R}^3)}_{\cong 0 \text{ if } k \geq 0} \to H_k(X_n \cap Y) \to H_k(X_n) \oplus \underbrace{H_k(Y)}_{\cong 0 \text{ if } k \geq 1} \to \underbrace{H_k(\mathbb{R}^3)}_{\cong 0 \text{ if } k \geq 1} \to \cdots,$$

and so by exactness we have $H_k(X_n) \cong H_k(X_n \cap Y)$ for all $k \geq 1$. Now $X_n \cap Y$ is a disjoint union of n punctured open disks, each homeomorphic to

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid 0 < (x^2+y^2+z^2)^{1/2} < \tfrac{1}{3}\right\},$$

and these deformation retract onto a 2-sphere, say of radius $\frac{1}{6}$, so $H_k(X_n \cap Y) \cong \bigoplus_{i=1}^{n} H_k(S^2)$ is isomorphic to $\mathbb{Z}^n$ if $k = 2$ and zero otherwise. Finally, we have $H_0(X_n) \cong \mathbb{Z}$ since X_n is path-connected, so

$$H_k(X_n) \cong \begin{cases} \mathbb{Z}^n, & k = 2 \\ \mathbb{Z}, & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(6 marks)

(b) Let $A = \{a, b\}$. Then (X, A) is a good pair since $U \cup V$ deformation retracts onto A, so $\tilde{H}_n(X/A) \cong H_n(X, A)$ for all n; if $n \geq 1$ then this gives $H_n(X/A) \cong H_n(X, A)$, so we compute the latter. We examine the long exact sequence of a pair

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \cdots.$$

When $n \geq 2$, both $H_n(A)$ and $\tilde{H}_{n-1}(A)$ are zero, so $H_n(X) \cong H_n(X, A)$ and the latter is isomorphic to $\tilde{H}_n(X/A) \cong H_n(X/A)$. When $n = 1$ we have $H_1(A) = 0$ and $\tilde{H}_0(X) = 0$, while $\tilde{H}_0(A) \cong \mathbb{Z}$ since A has two path components, so this becomes

$$0 \to H_1(X) \xrightarrow{\pi} H_1(X, A) \xrightarrow{\delta} \underbrace{\tilde{H}_0(A)}_{\cong \mathbb{Z}} \to 0.$$

Since the last group is free abelian, we deduce that $H_1(X, A) \cong H_1(X) \oplus \tilde{H}_0(A)$. (This can be stated without proof.) And $H_0(X/A) \cong \mathbb{Z} \cong H_0(X)$ since both are path-connected, so

$$H_n(X/A) \cong \begin{cases} H_n(X), & n \neq 1 \\ H_n(X) \oplus \mathbb{Z}, & n = 1. \end{cases}$$

(6 marks)

(c) (i) Viewing Y as a sub-complex of the natural Δ-complex structure on Δ^d, we have $C_n^\Delta(Y) = 0$ for all $n \geq k+1$ and so $H_n(Y) \cong H_n^\Delta(Y) = 0$ for all $n \geq k+1$. (2 marks)

(ii) For each $n \geq 0$, we apply the long exact sequence of a pair to (Δ^d, Y) to get

$$\cdots \to H_{n+1}(\Delta^d) \to H_{n+1}(\Delta^d, Y) \to \tilde{H}_n(Y) \to \tilde{H}^n(\Delta^d) \to \cdots ,$$

in which the first and last terms are zero because Δ^d is contractible, so $\tilde{H}_n(Y) \cong H_{n+1}(\Delta^d, Y)$ by exactness. Since (Δ^d, Y) is a good pair and $n+1 > 0$, we also know that $H_{n+1}(\Delta^d, Y) \cong \tilde{H}_{n+1}(\Delta^d/Y) \cong H_{n+1}(\Delta^d/Y)$ and hence $\tilde{H}_n(Y) \cong H_{n+1}(\Delta^d/Y)$. (3 marks)

(iii) We note that Δ^d/Y is naturally a CW complex built by attaching cells of dimensions $k+1, k+2, \ldots, d$ to the point $[Y]$. Thus its cellular chain complex has the form

$$\cdots \to C_{k+1}^{\mathrm{CW}}(\Delta^d/Y) \to 0 \to \cdots \to 0 \to C_0^{\mathrm{CW}}(\Delta^d/Y) \to 0$$

and so $H_n(\Delta^d/Y) \cong H_n^{\mathrm{CW}}(\Delta^d/Y) = 0$ for $n = 1, 2, \ldots, k$. (3 marks)

(Total: 20 marks)

5. (a) Let $\gamma : I \to X$ be a path from $\gamma(0) = e$ to $\gamma(0) = x$. Then $H(y,t) = m(\gamma(t), y)$ is a homotopy from $h_0(y) = m(\gamma(0), y) = m(e, y) = y$ to $h_1(y) = m(\gamma(1), y) = m(x, y)$, and we have $h_0 = \mathrm{Id}_X$ and $h_1 = \ell_x$. (5 marks)

(b) Since the Lefschetz number is homotopy invariant, we have $\tau(\ell_x) = \tau(\mathrm{Id}_X)$, and the identity map has Lefschetz number $\chi(X)$, so $\tau(\ell_x) = \chi(X)$. (4 marks)

(c) Let $x \in X$ be a point different from e. Then $m(x,y) \neq y$ for all $y \in X$, so $\ell_x : X \to X$ is a continuous map without fixed points. The Lefschetz fixed theorem says that $\tau(\ell_x) = 0$, but we have also computed that $\tau(\ell_x) = \chi(X)$, so in fact $\chi(X) = 0$. (3 marks)

(d) We know that S^{2n} is a path-connected, finite CW complex, and it can be formed by attaching a single $2n$-cell to a single 0-cell, so $\chi(S^{2n}) = 1^0 + 1^{2n} = 2$. Since this is nonzero we conclude that S^{2n} cannot be a topological group. (4 marks)

(e) Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the quotient map. Then for any nonzero $v \in \mathbb{C}^{n+1}$ we have $Av \neq 0$, so $\pi(Av) \in \mathbb{CP}^n$; and then the map

$$\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{A} \mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{CP}^n$$

satisfies $\pi(A(v)) = \pi(A(\lambda v))$ for all nonzero $\lambda \in \mathbb{C}$, so by the universal property of quotients we get a continuous map $\tilde{A} : \mathbb{CP}^n \to \mathbb{CP}^n$ defined by $\tilde{A}([v]) = [Av]$.

If we choose a path of matrices $A_t \in GL(n+1, \mathbb{C})$ with $A_0 = \mathrm{Id}$ and $A_1 = A$, then $\tilde{A}_t$ defines a homotopy from $\tilde{A}_0 = \mathrm{Id}_{\mathbb{CP}^n}$ to $\tilde{A}_1 = \tilde{A}$, so the Lefschetz numbers satisfy

$$\tau(\tilde{A}) = \tau(\mathrm{Id}_{\mathbb{CP}^n}) = \chi(\mathbb{CP}^n) = n+1.$$

This is nonzero, so $\tilde{A}$ has a fixed point $[v]$, which means that $[Av] = [v]$, hence there is some nonzero $\lambda \in \mathbb{C}$ such that $Av = \lambda v$. (4 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96033 MATH97042 MATH97151	1	People generally did well on this problem. Very few people got full marks on 1d(ii), because they did not give a complete proof that a group $G \times G$ cannot be isomorphic to the free group F_2.
MATH96033 MATH97042 MATH97151	2	I tried to interpret the pictures in (a) generously, but many were still not covers because e.g. they collapsed a whole curve to a point. On (d), RP^n is not the only complex with one k-cell for each of $k=0,1,\ldots,n$; instead, use $\deg(p)=1$ to claim that p is a homeomorphism. Nobody answered part (e) correctly -- too hard!
MATH96033 MATH97042 MATH97151	3	Most people answered 3a(ii) incorrectly. If G has an index-2 subgroup H and G^{ab} is finite, then G need not be finite (consider $G=O(3)$, $H=SO(3)$, $G^{ab}=\{1\}$), and H^{ab} may not have index 2 in G^{ab} (let G be a dihedral group and H the subgroup of rotations; then $\lvert H^{ab}\rvert=\lvert H\rvert=\lvert G\rvert/2$ is usually bigger than $\lvert G^{ab}\rvert$).
MATH96033 MATH97042 MATH97151	4	In (a) and (b), some people computed the homology of a complement instead of a quotient, or vice versa. Parts c(i) and c(ii) were usually done well, but c(iii) was substantially harder.
MATH96033 MATH97042 MATH97151	5	Parts (a)-(d) were mostly well done, but most people did not solve (e).