(1) (2.4.7) Ex/Notation (Generalised soundness thm. for $K_{\mathcal{L}}$)

Suppose Σ is a set of $\mathcal{L}$-fmlas and ψ is an $\mathcal{L}$-fmla.

Notation: $\Sigma \models \psi$ means: if v is a valuation (in $\mathcal{A}$) with $v[\Sigma] = T$ (i.e. $v[\sigma] = T$ for all $\sigma \in \Sigma$) then $v[\psi] = T$.

Ex: If $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ then $\Sigma \models \psi$.

(Need to use the restriction in Gen to do this.).

(2.4.8) Theorem. (Deduction Thm.)

Let Σ be a set of $\mathcal{L}$-fmlas. and ϕ, ψ $\mathcal{L}$-fmlas.

If $\Sigma \cup \{\phi\} \vdash_{K_{\mathcal{L}}} \psi$, then $\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi)$.

Pf: Like the DT for L (1.2.5): By induction on the length of a deduction of ψ from $\Sigma \cup \{\phi\}$.

Base case. ψ is an axiom or in $\Sigma \cup \{\phi\}$: as in 1.2.5.

Ind. step. Suppose ψ is obtained from an earlier fmla. ②
by applying MP or Gen.

MP As in 1.2.5.

Gen Suppose ψ is obtained by using Gen. So ψ is $(\forall x_i)\theta$
and θ appears earlier in the deduction & uses only fmlas Δ from $\Sigma \cup \{\phi\}$ which do not have x_i as a free variable.

So ~~$\Delta \vdash \theta$~~ $\Delta \subseteq \Sigma \cup \{\phi\}$ & $\Delta \vdash \theta$.

Case 1. $\phi \notin \Delta$. So ~~Δ~~ $\Delta \subseteq \Sigma$ and $\Delta \vdash \underbrace{(\forall x_i)\theta}_{\psi}$ by Gen.

So $\Sigma \vdash \psi$. Use A1 $(\psi \to (\phi \to \psi))$
to obtain $\Sigma \vdash (\phi \to \psi)$.

Case 2 $\phi \in \Delta$. Let $\Delta' = \Delta \setminus \{\phi\} \subseteq \Sigma$. Note x_i not free in ϕ.

$\Delta' \cup \{\phi\} \vdash \theta$ with a shorter deduction.

By ind hyp. $\Delta' \vdash (\phi \to \theta)$. By Gen $\Delta' \vdash (\forall x_i)(\phi \to \theta)$

As $\Delta' \subseteq \Sigma$ we have $\Sigma \vdash (\forall x_i)(\phi \to \theta)$, x_i not free in ϕ.

By K2 $\vdash((\forall x_i)(\phi \to \theta) \to (\phi \to (\forall x_i)\theta))$. So $\Sigma \vdash (\phi \to \psi)$. //

(2.5) Gödel's Completeness Theorem.

(2.5.1) **Def.** Suppose Σ is a set of $\mathcal{L}$-fmlas.

1. Σ is consistent if there is ~~not~~ no $\mathcal{L}$-fmla ϕ with $\Sigma \vdash_{K_\mathcal{L}} \phi$ and $\Sigma \vdash_{K_\mathcal{L}} (\neg\phi)$.

2. Σ is complete if for every closed $\mathcal{L}$-fmla ϕ $\Sigma \vdash_{K_\mathcal{L}} \phi$ or $\Sigma \vdash_{K_\mathcal{L}} (\neg\phi)$.

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Assume that $\mathcal{L}$ is countable

ie. the variables are $x_0, x_1, x_2, \ldots$ and there are only countably many relation, function and constant symbols.

So we can list the $\mathcal{L}$-fmlas (or any subset thereof) as a list indexed by $\mathbb{N}$.

Eg enumerate the closed $\mathcal{L}$-fmlas as $\psi_0, \psi_1, \psi_2, \ldots$.

(2.5.2) Proposition Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas and ϕ a closed $\mathcal{L}$-fmla.

1. (Like 1.3.7) If $\Sigma \nvdash \phi$ then $\Sigma \cup \{(\neg\phi)\}$ is consistent.

2. (Lindenbaum Lemma) (like 1.3.8) There is a set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas which is consistent and complete.

Pf: ① As in 1.3.7

(uses DT &

$\vdash_{K_{\mathcal{L}}} (((\neg\phi) \to \phi) \to \phi)$).

② Use ① + enumeration

$(\psi_i : i \in \mathbb{N})$ as in (1.3.8).

#.

(2.5.3) Thm. (Model Existence Theorem).

Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas. Then there is an $\mathcal{L}$-structure $\mathcal{A}$ with $\mathcal{A} \models \Sigma$

(ie. $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$).

Pf: Next week. #.

(2.5.4) Theorem

(Generalised Completeness Thm.)

Suppose Σ is a set of closed $\mathcal{L}$-fmlas & ϕ is a closed $\mathcal{L}$-fmla.

Suppose that $\Sigma \models \phi$

(as Σ, ϕ consist of closed fmlas. this means: if $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \phi$.).

Then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$.

Pf: If Σ is inconsistent then $\Sigma \vdash \phi$ for any ϕ.

Suppose Σ is consistent and $\Sigma \nvdash \phi$. By 2.5.2 ①,

$\Sigma \cup \{(\neg\phi)\}$ is consistent.

By 2.5.3 there is an $\mathcal{L}$-str. $\mathcal{A}$ with $\mathcal{A} \models \Sigma$ and $\mathcal{A} \models (\neg\phi)$.
This contradicts $\Sigma \models \phi$.

#.

(2.5.5) <u>Thm</u>.
(Gödel Completeness thm).
If ϕ ~~is~~ is a logically valid $\mathcal{L}$-fmla, then $\vdash_{K_{\mathcal{L}}} \phi$.

<u>Pf</u>: 2.5.4 with $\Sigma = \emptyset$ gives this for closed ϕ.