Analysis II, Complex Analysis
Assessed Coursework 2
Deadline the 25th of March, 2024, 13:00.

Q 1. [5]

a) Assume that u and v are harmonic in an open set Ω and that v is a harmonic conjugate to u in Ω. Define functions

$$U(x,y) = e^{u^2(x,y)-v^2(x,y)}\cos(2u(x,y)v(x,y))$$
$$V(x,y) = e^{u^2(x,y)-v^2(x,y)}\sin(2u(x,y)v(x,y)).$$

Show that U and V are harmonic in Ω and that V is a harmonic conjugate to U.

b) Let $u = u(x,y)$ be harmonic for all $(x,y) \in \mathbb{R}^2$. Show that if there is $M \in \mathbb{R}$ such that $u \leq M$, then $u = const$.

Q 2. [5]

a) Find the poles and their orders of the function

$$\frac{1}{e^z - 1} - \frac{1}{z}.$$

b) Find the Laurent expansion for the function

$$f(z) = \frac{1}{(z+2)^3} \quad \text{for} \quad |z| > 2.$$

Q 3. [5]

Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{16 + x^2}\, dx.$$

Q 4. [5]

a) Let $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ and let $f(z)$ be a continuous function on γ. Prove that

$$\overline{\oint_\gamma f(z)\, dz} = -\oint_\gamma \overline{f(z)}\, \frac{dz}{z^2}.$$

b) Prove that for a real $\lambda > 1$ there is a unique solution to the equation

$$ze^{\lambda - z} = 1$$

in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.