Classical Dynamics **Autumn Term 2024**

Answers to Problem Sheet 6

1. $H = xp$; this Hamiltonian was considered on Problem Sheet 4. Writing $S = W(x) - \alpha t$,
$$x\frac{\partial W}{\partial x} = \alpha,$$
so that $W = \alpha \log x$ (dropping an additive constant) and $S = \alpha \log x - \alpha t$. The new coordinate is
$$\beta = \frac{\partial S}{\partial \alpha} = \log x - t,$$
giving $x(t) = e^{\beta + t}$. This assumes that $x > 0$. What happens if $x < 0$?

2. The Hamilton-Jacobi equation is
$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + U(r) + f(r)g(\theta) + \frac{\partial S}{\partial t} = 0.$$
If separable $S = W_r(r) + W_\theta(\theta) - \alpha_1 t$, so that
$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + U(r) + f(r)g(\theta) - \alpha_1 = 0.$$
Multiply with r^2 to separate 2nd term
$$\frac{r^2}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + r^2 U(r) + r^2 f(r)g(\theta) - \alpha_1 r^2 = 0.$$
To separate third term take $f(r)$ proportional to r^{-2}. Without loss of generality take $f(r) = r^{-2}$ (as g can be rescaled). This works for any $g(\theta)$.
(ii) taking $f(r) = r^{-2}$ and $g(\theta)$ arbitrary the Hamilton-Jacobi equation can be written
$$-\frac{r^2}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 - r^2 U(r) + \alpha_1 r^2 = \frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + g(\theta) = \alpha_2,$$
which integrate to
$$W_\theta = \int \sqrt{2m(\alpha_2 - g(\theta))}\, d\theta, \quad W_r = \int \sqrt{2m\left(\alpha_1 - U(r) - \frac{\alpha_2}{r^2}\right)}\, dr.$$

3. Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 - cxt + \frac{\partial S}{\partial t} = 0.$$

Inserting $S = xf(t) + g(t)$ gives

$$\frac{1}{2}f^2 - ctx + xf'(t) + g'(t) = 0,$$

so that $f'(t) = ct$ and $g'(t) = -\frac{1}{2}(f(t))^2$.
Accordingly, $f(t) = \frac{1}{2}ct^2 + \alpha$ and

$$g'(t) = -\frac{1}{2}\left(\frac{c^2t^4}{4} + c\alpha t^2 + \alpha^2\right),$$

which integrates to

$$g(t) = -\frac{c^2t^5}{40} - \frac{c\alpha t^3}{6} - \frac{\alpha^2 t}{2},$$

ignoring an additive constant. A complete solution is

$$S = \left(\frac{1}{2}ct^2 + \alpha\right)x - \frac{c^2t^5}{40} - \frac{c\alpha t^3}{6} - \frac{\alpha^2 t}{2}.$$

The new coordinate is

$$\beta = \frac{\partial S}{\partial \alpha} = x - \frac{ct^3}{6} - \alpha t.$$

This can be written as

$$x = \frac{ct^3}{6} + \alpha t + \text{constant}.$$

This can (more easily) be obtained by directly integrating Hamilton's equations.

4. Hamilton's Principal function S can be understood as a type 2 generating function for which the new Hamiltonian is zero. Identifying S with a type 3 generating function where

$$q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}, \quad K = H + \frac{\partial F_3}{\partial t},$$

requiring the new Hamiltonian to vanish yields

$$H\left(-\frac{\partial S}{\partial p}, p, t\right) + \frac{\partial S}{\partial t} = 0.$$

Can you use a type 1 or type 4 generating function to obtain Hamilton-Jacobi equations?