MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Mid-term exam on 23 February 2024

Question 1 (total: 10 points)

Consider the one-dimensional differential equation

$$\dot{x} = |x| .$$

(i) Clarify the question whether all solutions to this differential equation are continuously differentiable (proof or counterexample). [3 points]

(ii) Fix the initial condition $x(0) = 1$, and let $(\lambda_n : \mathbb{R} \to \mathbb{R})_{n \in \mathbb{N}_0}$ be the corresponding Picard iterates. Compute the first three Picard iterates λ_0, λ_1, and λ_2. [7 points]

Question 2 (total: 14 points)

Consider the nonautonomous differential equation

$$\dot{x} = f(t, x) ,$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.

(i) State the global version of the Picard–Lindelöf theorem, in order obtain local solutions to the above differential equation satisfying an initial condition $x(t_0) = x_0$, where $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$. [2 points]

(ii) The proof of this theorem makes use of $P : C^0([t_0 - h, t_0 + h], \mathbb{R}^d) \to C^0([t_0 - h, t_0 + h], \mathbb{R}^d)$, given by

$$P(u)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))\, \mathrm{d}s \quad \text{for all } t \in [t_0 - h, t_0 + h] .$$

Show that there exists $h > 0$ such that P is a contraction when $C^0([t_0 - h, t_0 + h], \mathbb{R}^d)$ is equipped with the supremum norm $\| \cdot \|_\infty$. [8 points]

(iii) One can show that under the assumptions of the global version of the Picard–Lindelöf theorem, solutions to any initial value problem exist globally (this is not asked here). Can you give an example of a differential equation that does not satisfy the conditions of the global version of the Picard–Lindelöf theorem, but for which solutions to any initial value problem exist globally. Give a brief justification (no rigorous proof necessary). [4 points]

Question 3 (total: 6 points)

Consider an autonomous differential equation

$$\dot{x} = f(x) ,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous. Prove that for all $y_0 \in \mathbb{R}^d$, there exist $T > 0$ and $x_0 \in \mathbb{R}^d$ such that there exists a solution $\lambda : I \to \mathbb{R}^d$ to this differential equation with $\lambda(0) = x_0$ and $\lambda(T) = y_0$.