**Math40003 Linear Algebra and Groups** **Problem Sheet 0**

1. Find the following:

(a) $\begin{pmatrix} 1 & 2 \\ -4 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 8 \\ 18 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & -1 \\ 8 & \frac{3}{2} & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ \frac{1}{5} \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ (d) $\begin{pmatrix} 7 & -1 & 1 & 3 \\ 1 & 0 & -1 & \frac{1}{2} \\ 2 & 0 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ -3 \\ 4 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 \\ -4 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 & 0 \\ -4 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2. Consider the matrices

$$\mathsf{P} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \mathsf{Q} = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix} \qquad \text{and} \qquad \mathsf{R} = \begin{pmatrix} 6 & 3 & 1 \end{pmatrix}$$

Determine which of the following matrix products may be defined, and find (by hand) those which can.

(i) $\mathsf{PQ}$

(ii) $\mathsf{QP}$

(iii) $\mathsf{PR}$

(iv) $\mathsf{RP}$

3. Let $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Show that $\mathsf{A}^2 = 4\mathsf{A} + \mathsf{I}_2$ where $\mathsf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4. * Let $\mathsf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Find $\mathsf{A}^n$ for all positive integers $n$.

5. Let $\mathsf{A}$ be a $2 \times 2$ matrix which commutes with every other $2 \times 2$ matrix, i.e. $\mathsf{AB} = \mathsf{BA}$ for any $2 \times 2$ matrix $\mathsf{B}$. Show that $\mathsf{A}$ must be of the form $k\mathsf{I}_2$, for some $k \in \mathbb{R}$.