Topic: Elementary set theory and the sample space

In today's problem class we will be reviewing concepts from elementary set theory and we will link them to the concept of a sample space in probability.

1. Let A, B and C be three arbitrary events. Using only the operations of union, intersection and complement, write down expressions for the following events:
   (a) Only A occurs.
   (b) Both A and B, but not C occurs.
   (c) All three events occur.
   (d) At least one of A, B and C occurs.
   (e) At least two of A, B and C occur.
   (f) Precisely one of A, B and C occurs.
   (g) Precisely two of A, B and C occur.
   (h) None of A, B and C occurs.
   (i) Not more than two of A, B and C occur.

Solution:

(a) $A \cap B^c \cap C^c$

(b) $A \cap B \cap C^c$

(c) $A \cap B \cap C$

(d) $A \cup B \cup C$

(e) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$

(f) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

(g) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

(h) $A^c \cap B^c \cap C^c$

(i) $(A \cap B \cap C)^c \equiv A^c \cup B^c \cup C^c$

2. A football match contained exactly two penalties. Let $S_i, i = 1, 2$ denote the event that penalty i was scored and $M_i, i = 1, 2$ denote the event that penalty i was missed. We write e.g. $M_1 S_2$ for the outcome that the first penalty was missed and the second penalty scored.
   (a) Find the set which has as its elements all possible combinations of the outcomes of the two penalties (i.e. what is Ω, the sample space).
   (b) Let A denote the event that both penalties were missed, B denote the event that both were scored and C denote the event that at least one was scored.
   List the elements of A, B, C, $A \cap B$, $A \cup B$, $A \cup C$, $A \cap C$, $B \cup C$ and $B^c \cap C$.

Solution:

(a) $\Omega = \{M_1 M_2, M_1 S_2, S_1 M_2, S_1 S_2\}$

(b) $A = \{M_1M_2\}, \quad B = \{S_1S_2\}, \quad C = \{S_1M_1, M_1S_2, S_1S_2\}$

$$\begin{aligned} A \cap B &= \emptyset \\ A \cup B &= \{M_1M_2, S_1S_2\} \\ A \cup C &= \{M_1M_2, M_1S_2, S_1M_2, S_1S_2\} = \Omega \\ A \cap C &= \emptyset \\ B \cup C &= \{S_1S_2, M_1S_2, S_1M_2\} \\ B^c \cap C &= \{M_1S_2, S_1M_2\}. \end{aligned}$$

3. Two dice are thrown; let Ω be the sample space of possible outcomes, which correspond to pairs of values (e.g. (2,3), (6,1), (4,4)) indicating the scores on the first and second die respectively. Let A denote the subset of Ω containing outcomes in which the score on the second die is even, B denote the subset of outcomes for which the sum of scores on the two dice is even, and let C denote the subset of outcomes for which at least one of the scores is odd.
Write in terms of A, B and C (using union, intersection and complement) the following events:
   (a) Both scores are even.
   (b) The first score is odd and the second score is even.
   (c) Both scores are odd.
   (d) The second score is odd.

Solution: We write $\Omega = \{(i, j) : i, j \in \{1, \dots, 6\}\}$. Then

$$\begin{aligned} A &= \{(i,j) \in \Omega : i \in \{1,2,3,4,5,6\}, j \in \{2,4,6\}\} \quad \text{second score even} \\ B &= \{(i,j) \in \Omega : i + j \in \{2,4,6,8,10,12\}\} \quad \text{sum even} \\ C &= \{(i,j) \in \Omega : i \in \{1,3,5\} \text{ or } j \in \{1,3,5\}\} \quad \text{first or second score odd,} \\ C^c &= \{(i,j) \in \Omega : i \in \{2,4,6\} \text{ and } j \in \{2,4,6\}\} \quad \text{first and second score even.} \end{aligned}$$

(a) Both even: C^c, or $A \cap B$

(b) First odd, second even: $A \cap B^c$

(c) Both odd: $A^c \cap B$

(d) Second odd: A^c

4. Prove that $E \subseteq F$ is equivalent to $E \cup F = F$.

Solution: First, show that $E \cup F = F \Rightarrow E \subseteq F$.
Let $x \in E \Rightarrow x \in E \cup F = F \Rightarrow x \in F \Rightarrow E \subseteq F$.

Now show $E \subseteq F \Rightarrow E \cup F = F$ using double inclusion, i.e. show $E \cup F \subseteq F$ and $F \subseteq E \cup F$:
Let $x \in E \cup F \Rightarrow x \in E$ or $x \in F$, we know $x \in E \Rightarrow x \in F$ (as $E \subseteq F$) $\Rightarrow x \in F$.
If we let $x \in F \Rightarrow x \in E \cup F$, as required.

5. Can you use the result from Question 4 to show that if $E \subseteq F$ then $E \cup G \subseteq F \cup G$?

Solution: From Question 4 we know that $A \subseteq B \Leftrightarrow A \cup B = B$.

Let $A = (E \cup G)$ and $B = F \cup G$, so $(E \cup G) \subseteq (F \cup G) \Leftrightarrow (E \cup G) \cup (F \cup G) = (F \cup G)$ $\Leftrightarrow (E \cup F \cup G) = (F \cup G)$.

We need to show that $E \cup F \cup G = F \cup G$ if $E \subseteq F$ (*).

We know $E \subseteq F \Leftrightarrow E \cup F = F$ from Question 4. But from the LHS of (*) $(E \cup F \cup G) = (E \cup F) \cup G = F \cup G$ = RHS as required.