Classical Dynamics **Autumn Term 2024**

Problem Sheet 2

1. The Lagrangian for a free particle of mass m in the plane is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

where x and y are cartesian coordinates.

Parabolic coordinates are defined through

$$x = uv, \quad y = \frac{v^2 - u^2}{2}.$$

Find the Lagrangian for a free particle in parabolic coordinates.

2. What is the holonomic constraint in the pendulum on a trolley problem discussed in the notes?

3. A bead of mass m moves without friction on a helical wire. The helix can be parametrized as follows

$$x = R\cos u, \quad y = R\sin u \quad z = \alpha u,$$

where u is a real parameter. Here α and R are constants. The acceleration due to gravity is the constant g (the gravitational force is pointing in the negative z direction). Obtain a Lagrangian $L(z, \dot{z})$ for the bead and solve the equation of motion.

4. The motion of a marble of mass m moving in a spherical bowl with radius R is described by the Lagrangian

$$L = \frac{mR^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgR\cos\theta.$$

Here θ and ϕ are spherical polar angles where the angle θ is measured with respect to the negative z-axis. Obtain the equations of motion. Use the equations of motion to show there are circular trajectories of the form $\theta =$constant for $0 < \theta < \pi/2$. Obtain the orbital period of these trajectories (this depends on θ).

5. Consider the Lagrangian

$$L = e^{\gamma t}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right],$$

where γ is a constant. Show that this Lagrangian governs the motion of a particle subject to a non-conservative force. Comment on the nature of the force for (a) $\gamma > 0$ and (b) $\gamma < 0$.

6. *Rotating Pendulum Problem*

A simple pendulum is mounted on a rotating turntable with constant angular velocity Ω. The pivot is on the axis of rotation.

(i) Show that the kinetic energy of the pendulum bob is

$$T = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2 \sin^2\theta\right).$$

Here m is the mass of the bob, l is the length of the rod (assumed to be massless) and θ is the angle between the rod and the vertical. The potential energy is the same as for a non-rotating pendulum, ie. $V = -mgl\cos\theta$.

Hint: what is the kinetic energy of a particle in spherical polar coordinates?

(ii) Obtain the equation of motion.

(iii) Find all solutions of the form $\theta =$constant.

(iv) Find a conserved quantity and show that $T + V$ is not a constant of the motion (unless θ is constant).

(v) Determine the frequency of small oscillations about the constant θ solutions from part (iii).