Math40002 Analysis 1 — Unseen 7

In this sheet, we define an alternative definition for a convergent series, and see its connection to the definitions we learned.

Definition 1. Let X be a set, and let $f : X \to \mathbb{R}$ be a function. (If $X = \mathbb{N}$ then this is a sequence.)

- For X finite, we define $\sum_{x\in X} f(x)$ simply to be the sum of the finite set $\{\, f(x) \mid x \in X \,\}$.

- For X infinite, we say the sum $\sum_{x\in X} f(x)$ is convergent to a real number L, if for all $\epsilon > 0$, there is some finite subset $I_0 \subseteq X$, such that for every finite set I, if $I_0 \subseteq I \subseteq X$, then $\left|\left(\sum_{x\in I} f(x)\right) - L\right| < \epsilon$.

1. Prove the limit defined above is unique, in the following sense: If $\sum_{x\in X} f(x)$ is convergent to both L_1 and L_2, then $L_1 = L_2$.
2. Prove that if $\sum_{n\in\mathbb{N}} a_n$ is convergent to L, then $\sum_{n=1}^{\infty} a_n$ is convergent to L.
3. Let X_1, X_2 be sets such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$. Let $f : X \to \mathbb{R}$. Prove that if $\sum_{x\in X_1} f(x)$ is convergent to L_1, $\sum_{x\in X_2} f(x)$ is convergent to L_2, then $\sum_{x\in X} f(x)$ is convergent to $L_1 + L_2$.

Definition 2. Let X be a set, and let $f : X \to \mathbb{R}$ be a function. We say the sum $\sum_{x\in X} f(x)$ is *Cauchy*, if for all $\epsilon > 0$, there is some finite subset $I_0 \subseteq X$, such that for every finite set $I \subseteq X$, if $I \subseteq (X \setminus I_0)$, then $\left|\left(\sum_{x\in I} f(x)\right)\right| < \epsilon$.

4. Prove that if $\sum_{x\in X} f(x)$ is convergent, then it is Cauchy.
5. Prove that if $\sum_{x\in X} f(x)$ is Cauchy, then it is convergent. (Hard!)
6. Deduce that if $\sum_{x\in X} f(x)$ is convergent and $X' \subseteq X$, then $\sum_{x\in X'} f(x)$ is convergent.
7. Prove that if $\sum_{n=1}^{\infty} |a_n|$ is convergent to L, then $\sum_{n\in\mathbb{N}} |a_n|$ is convergent to L.
8. Prove that $\sum_{n\in\mathbb{N}} a_n$ is convergent to L if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent to L.

   Can you see how this is connected to rearrangements of the series (a_n)?