MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Repetition Material 3: Banach fixed point theorem

We review Banach's fixed point theorem (on contractions in metric spaces) here and explain how it is applied in the Differential Equations course (in the context of the Banach space of continuous functions on a compact interval).

Banach's fixed point theorem says that Lipschitz continuous functions on complete metric spaces with Lipschitz constant less than 1 have a unique fixed point. Let us first look at a one-dimensional special case to understand this better. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that there exists a constant $K \in (0,1)$ with $|f'(x)| \le K$ for all $x \in \mathbb{R}$. As seen in the lectures, it then follows that f is Lipschitz continuous with constant $K < 1$.

To see that f has a fixed point (i.e. there exists a $x^* \in \mathbb{R}$ such that $f(x^*) = x^*$), consider the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) := f(x) - x$, and note that x^* is a fixed point of f if and only if x^* is a zero of the function g. Now $g'(x) = f'(x) - 1 \le K - 1 < 0$, and thus, g is strictly monotonically decreasing. It follows that $\lim_{x\to\infty} g(x) = -\infty$ and $\lim_{x\to-\infty} g(x) = \infty$, and by the intermediate value theorem, there exists an $x^* \in \mathbb{R}$ with $g(x^*) = 0$, meaning that $f(x^*) = x^*$.

We now show that this fixed point is unique. Assume there is another fixed point $\bar{x}^* \in \mathbb{R}$. Then the mean value theorem implies that there exists a ξ between x^* and $\bar{x}^*$ with

$$x^* - \bar{x}^* = f(x^*) - f(\bar{x}^*) = f'(\xi)(x^* - \bar{x}^*)\,.$$

Hence, $f'(\xi) = 1$, which is a contradiction.

The following theorem deals with such a situation in the context of complete metric spaces.

Theorem 1 (Banach fixed point theorem)**.** *Let (X,d) be a complete metric space, and consider a* contraction *on X, that is a mapping $F : X \to X$ such that there exists a constant $K \in (0,1)$ with*

$$d\big(F(x), F(y)\big) \le K\, d(x,y) \quad \text{for all } x, y \in X\,. \tag{1}$$

Then F has a unique fixed point x^* in X, i.e. there exists exactly one $x^* \in X$ with

$$F(x^*) = x^*\,.$$

In addition, this fixed point x^* is limit of every sequence $\{x_n\}_{n\in\mathbb{N}_0}$, determined by a starting value $x_0 \in X$ and

$$x_{n+1} = F(x_n) \quad \text{for all } n \in \mathbb{N}\,. \tag{2}$$

Proof. Step 1. We show that the sequence $\{x_n\}_{n\in\mathbb{N}_0}$, as defined in (2) with an arbitrary starting value $x_0 \in X$, is a Cauchy sequence.
Firstly, we obtain

$$d(x_n, x_{n+1}) = d(F(x_{n-1}), F(x_n)) \le K\, d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}\,,$$

and an induction argument leads to

$$d(x_n, x_{n+1}) \le K^n\, d(x_0, x_1) \quad \text{for all } n \in \mathbb{N}_0\,.$$

Now consider two natural numbers $n < m$. By means of the triangle inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\le (K^n + K^{n+1} + \cdots + K^{m-1}) d(x_0, x_1)\,. \end{aligned}$$

With $(1-K)(K^n + \cdots + K^{m-1}) = K^n - K^m < K^n$, it follows that

$$d(x_n, x_m) \le \frac{K^n}{1-K}\, d(x_0, x_1) \quad \text{for all } \ m > n \ge 0\,. \tag{3}$$

The limit relation $\lim_{n\to\infty} K^n = 0$ implies that for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that the right hand side of (3) is smaller than ε for all $n > n_0$. It follows that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } \ m > n > n_0\,.$$

Due to symmetry of the metric d, this inequality also holds for $n > m > n_0$, and the case $n = m$ is trivial. Hence, the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence, which, due to the completeness of X, converges to a point $x^* \in X$.

Step 2. We show that x^* is a fixed point.
Consider

$$0 \le d(x_{n+1}, F(x^*)) = d(F(x_n), F(x^*)) \le K\, d(x_n, x^*) \quad \text{for all } \ n \in \mathbb{N}\,.$$

The right hand side of this relation converges to 0 as $n \to \infty$, and for this reason, also the left hand side converges to 0: $\lim_{n\to\infty} d(x_{n+1}, F(x^*))$, and hence,

$$\lim_{n\to\infty} x_{n+1} = F(x^*)\,.$$

It follows that $F(x^*) = x^*$.

Step 3. We show that x^* is the unique fixed point.
Assume there is another fixed point $y^* \ne x^*$, i.e. $F(y^*) = y^*$. We get

$$0 < d(x^*, y^*) = d(F(x^*), F(y^*)) \le K\, d(x^*, y^*) < d(x^*, y^*)\,,$$

which is a contradiction and finishes the proof of this theorem. □

In the Differential Equations course, we use the setting of the space of continuous functions $C^0(J, \mathbb{R}^d)$ on a compact interval J (see *Repetition Material 2*):

$$X = C^0(J, \mathbb{R}^d) := \{u : J \to \mathbb{R}^d : u \text{ is continuous}\}\,.$$

This space of functions is equipped with the supremum norm,

$$\|u\|_\infty := \sup_{t\in J} \|u(t)\| \quad \text{for all } u \in C^0(J, \mathbb{R}^d)\,,$$

which induces naturally a metric $d : X \times X \to \mathbb{R}_0^+$ on $X = C^0(J, \mathbb{R}^d)$ via

$$d(u_1, u_2) := \|u_1 - u_2\|_\infty\,.$$

In this setting, Banach's fixed point theorem is applied in the local (and global) version of the Picard–Lindelf theorem, see Theorems 2.11 and 2.13.