IMPERIAL

MATH70030

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fourier Analysis & Theory of Distributions

Date: Friday, May 16, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Let $f \in L_1[-\pi, \pi]$ have Fourier series $\sum_{-\infty}^{\infty} c_k e^{ikx}$.

   (a) (i) Find the Fourier coefficients of the function $f(x)e^{imx}$ with an integer m. (3 marks)

   (ii) Let $n \geq 2$ be a fixed positive integer. Can the Fourier coefficients of $f \in L_2[-\pi, \pi]$ satisfy $|c_k| \geq 1/\sqrt{|k| \log |k|}$ for all k such that $|k| > n$? Justify your answer. (7 marks)

   (b) (i) Assume that $c_0 = 0$. Continue f by periodicity to the whole real line. Show that $F(t) = \int_0^t f(x)dx$ is integrable on finite intervals, periodic, and its Fourier coefficients are $c_k/(ik)$, $k \neq 0$. Justify your calculations. (5 marks)

   (ii) Show that if $f(x)$ is absolutely continuous, we have $nc_n \to 0$, $|n| \to \infty$. (5 marks)

2. (a) (i) State the definition of the Fourier transform of a function in $L_1[-\infty, \infty]$. Does it belong to $L_1[-\infty, \infty]$? (give a short justification for your answer) (5 marks)

   (ii) State the definition of the Fourier transform of a function in $L_2[-\infty, \infty]$. Does it belong to $L_2[-\infty, \infty]$? (no justification needed) (5 marks)

   (b) Find the Fourier transform g of the function $f(x) = e^{-x^2/2}$ in 2 different ways:

   (i) by contour integration; (5 marks)

   (ii) by deriving a differential equation for g. (5 marks)

3. (a) State what it means for a functional on a normed linear space to be continuous. (5 marks)

   (b) Let $C_\infty[-\pi, \pi]$ be the space of continuous functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$ equipped with the sup-norm. Let C^* be its adjoint (the space of linear continuous functionals on $C_\infty[-\pi, \pi]$). Using results from the lectures, do the following.

   (i) Show that the Fejér sum $\sigma_n(0)$ evaluated at $x = 0$ is an element of C^* for any $n = 1, 2, \ldots$ ($\sigma_n(0) : C \to \mathbb{C}$, $f \mapsto \sigma_n(f, 0)$). Find its norm. (8 marks)

   (ii) Show that the sequence $\sigma_n(0)$ converges in weak* sense and find its limit. (Make sure to demonstrate that the limit is indeed in C^*.) (7 marks)

4. (a) Let E be a topological linear space. Show that the convergence of a sequence of functionals in E^* in the weak* topology is equivalent to the convergence of this sequence on all elements of E. (10 marks)

   (b) (i) State the definitions of the space of Schwarz functions and the space of tempered distributions. (5 marks)

   (ii) State the definition of the Fourier transform of a tempered distribution. (2 marks)

   (iii) Find the Fourier transform of 1. (3 marks)

5. Prove the existence of a function $f \in L_1[-\pi, \pi]$ such that its Fourier partial sums S_n, $n = 1, 2, \ldots$ do not converge to f in the metric of $L_1[-\pi, \pi]$.

You may use the following theorem without proof: S_n converge to f for any $f \in L_1[-\pi, \pi]$ if and only if S_n are bounded in the operator norm uniformly in n. (20 marks)

Fourier analysis and distributions.
Exam 2025 solutions

1ai $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{imx}e^{-inx}\,dx = c_{n-m}$, $n\in\mathbb{Z}$. [3 marks]

1aii If $f\in L_2[-\pi,\pi]$ the Fourier coefficients must satisfy $\sum_{-\infty}^{\infty}|c_k|^2<\infty$. But $\sum_{2}^{\infty}\frac{1}{k\log k}$ diverge by the integral test. Therefore the answer to the question is <u>no</u> [7 marks]

1bi $F(t)$ is absolutely continuous and therefore integrable on finite intervals.

$$F(t+2\pi)-F(t)=\int_t^{t+2\pi} f(x)\,dx=\int_0^{2\pi} f(x)\,dx=$$
$$=2\pi c_0=0,\quad \forall t\in\mathbb{R}.$$

Integrating by parts, using absolute continuity of F, we obtain for $k\neq 0$

$$\frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-ikt}\,dt=\frac{1}{ik}\int_0^{2\pi} f(t)e^{-ikt}\,dt=\frac{c_k}{ik}$$

[5 marks]

1bii. If f is absolutely continuous, it has a representation

$$f(x) = \int_c^x g(u)\,du \quad , \quad g \in L_1 , \quad g(x) = f'(x).$$

By 1bi, $c_n = \frac{\gamma_n}{in}$, where γ_n are
$n \neq 0$

Fourier coefficients of $g(x)$. Now by Riemann-Lebesgue theorem, $\gamma_n \to 0, |n| \to \infty$

[5 marks]

2ai. Let $f \in L_1(-\infty, \infty)$

The Fourier transform $g(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt$.

$g(\lambda)$ does not in general belong to $L_1(-\infty,\infty)$, for example if f is an indicator function of a finite interval.

[5 marks]

2aii. Let $f \in L_2(-\infty,\infty)$

Then by Plancherel thm,

$\int_{-N}^{N} f(x) e^{-i\lambda x} dx$ converges as $N \to \infty$

in L_2 and its limit $g(\lambda)$ is called

the Fourier transform of f.

$g \in L_2(-\infty\infty)$ [5 marks]

2bi $g(\lambda) = \int_{-\infty}^{\infty} e^{-x^2/2 - i\lambda x} dx$, $\lambda \in \mathbb{R}$.

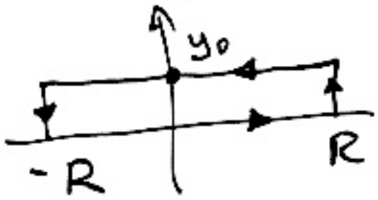
~~Ch~~ Fix $\epsilon > 0$ and choose R s.t.

$$g(\lambda) = \int_{-R}^{R} e^{-x^2/2 - i\lambda x} + \delta_R, \quad |\delta_R| \le \varepsilon.$$

[Note that $\delta_R \to 0$ as $R \to \infty$]

By analyticity, the integral of $e^{-z^2/2 - i\lambda z}$

along the rectangular contour is 0.



Moreover, the integrals along the vertical lines

$$\left| \int_{\pm R}^{\pm R + iy_0} e^{-z^2/2 - i\lambda z} \, i\,dy \right| \le C_{\lambda, y_0} e^{-R^2/2} \xrightarrow[R \to \infty]{} 0$$

Therefore

$$g(\lambda) = +\int_{-R+iy_0}^{R+iy_0} e^{-\frac{1}{2}z^2 - i\lambda z}\,dz + \delta_R'$$

$$= \int_{-R}^{R} e^{-\frac{1}{2}(x+iy_0)^2 - i\lambda(x+iy_0)}\,dx + \delta_R'$$

$$= \int_{-R}^{R} e^{-\frac{x^2}{2} - ixy_0 + y_0^2/2 - i\lambda x + \lambda y_0}\,dx + \delta_R'$$

Now set $y_0 = -\lambda$. Then

$$g(\lambda) = \int_{-R}^{R} e^{-\frac{x^2}{2} + \lambda^2/2 - \lambda^2}\,dx + \delta_R' =$$

$$= e^{-\lambda^2/2}\int_{-R}^{R} e^{-x^2/2}\,dx + \delta_R' \to e^{-\lambda^2/2}\sqrt{2\pi}, \quad R \to \infty.$$

[5 marks]

2bii

$$g'(\lambda) = F(-ix\,e^{-x^2/2}) = i\,F((e^{-x^2/2})')$$

$$= -\lambda\, g(\lambda) \Rightarrow g(\lambda) = c\,e^{-\lambda^2/2}$$

Here $c = g(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$

[5 marks]

3 a A functional f on E is called continuous if for any $x_0 \in E$, $\varepsilon > 0$, there is a neighbourhood U_0 of x_0 s.t. $|f(x) - f(x_0)| < \varepsilon$, $x \in U$

[5 marks]

3b i

$$\sigma_n(f, x) = \int_{-\pi}^{\pi} f(x+z)\,\varphi_n(z)\,dz\,, \quad n = 1, 2 \ldots$$

where Φ_n is Fejér kernel, $\int_{-\pi}^{\pi} \Phi_n(z)\,dz = 1$,

$\Phi_n(z) \geq 0$.

σ_n is clearly linear and

$$|\sigma_n(f, \otimes)| \leq \|f\| \int_{-\pi}^{\pi} \varphi_n(z)\,dz = \|f\|\,,$$

equality reached on $f \equiv 1$.

Thus $\sigma_n \in C^*$ and $\|\sigma_n\| = 1 \quad \forall n$

[8 marks]

3b ii. Since $\delta_n(f,0) \to f(0)$

by a result from lectures,

we have $\delta_n \xrightarrow{w^*} \delta$,

where $\delta(f) = f(0)$.

δ is a linear functional and

$|\delta(f)| \leq \|f\|$, so $\delta \in C^*$

[7 marks]

4a Weak-* topology is generated by the system of neighbourhoods of zero $U_{\varepsilon,A} = \{ f : |f(x)| < \varepsilon,\ x \in A,\ A\text{-finite} \}$

By linearity, we can assume $f_n \to 0$.

1) Let $f_n \to 0$ in the topology. Then, $\forall \varepsilon > 0$, in particular, $f_n \in U_{\varepsilon, x}$, $n \geq N_0(\varepsilon)$,

i.e. $f_n(x) \to f(x) = 0$

2) Let $f_n(x) \to 0$ $\forall x$. Let $A = \{x_1, x_2 \dots x_k\}$

For a fixed $\varepsilon > 0$ choose N_0 s.t. $|f_n(x_j)| < \varepsilon$, $j = 1, \dots, k$. Then $f_n \in U_{\varepsilon, A}$, $n \geq N_0$

$n \geq N_0$

[10 marks]

So $f_n \to 0$ in the topology
(ε, A - arbitrary).

4bi S_∞ is the set of functions on $\mathbb{R}$ which are infinitely differentiable and such that for any $p, q = 0, 1 \ldots$ there are constants $C(p,q,f)$ s.t.

$$|x^p f^{(q)}(x)| < C(p,q,f) \quad \forall x \in \mathbb{R}$$

The space of tempered distributions is $(S_\infty)^*$ [5 marks]

4bii g - Fourier transform of $f \in (S_\infty)^*$ is given by $(g, \varphi) = (f, F(\varphi))$, $\varphi \in S_\infty$

[2 marks]

4biii $(F[1], \varphi) = (1, F[\varphi]) =$

$$= \int_{-\infty}^{\infty} F[\varphi]\, dx = 2\pi \varphi(0). \text{ Thus}$$

$$F[1] = 2\pi\delta$$ [3 marks]

5. By the theorem, it is sufficient to show that the operator norms $\| S_n \|$ are not uniformly bounded.

We have
$S_n(f,x) = \int_{-\pi}^{\pi} f(x+z)\, D_n(z)\, dz$, where $D_n(z)$ is the Dirichlet kernel.

Now, for any fixed f, by Fubini thm,

$$\| S_n(f,x) \|_{L_1} \leq \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dz\, |f(x+z)|\, |D_n(z)|$$

$$\leq \| f \|_{L_1} \| D_n \|_{L_1},$$ so the operator norm $\| S_n \| \leq \| D_n \|_{L_1}$.

On the other hand, for the Fejèr kernel
$\| S_n \| \geq \| S_n(\Phi_N) \|_{L_1}$ (since $\| \Phi_N \|_{L_1} = 1$).

But $\| S_n(\Phi_N) \|_{L_1} = \| \mathcal{B}_N(D_n) \|_{L_1}$ and $\sigma_N(D_n) \to D_n$, $N \to \infty$. Therefore

$$\| S_n \| = \| D_n \|_{L_1}$$

But $\|D_n\|_{L_1} = \int_{-\pi}^{\pi} |D_n(t)|\,dt \to \infty$,

$n \to \infty$ so $\|S_n\|$ are not uniformly bounded.

[20 marks]

MATH70030 Fourier Analysis & Theory of Distributions Markers Comments

Question 1	Most issues with 1aii and 1bii
Question 2	In 2bi often details were lacking.
Question 3	In 3bi it is important to use the fact that the integral of Fejer kernel is 1.
Question 4	Most difficulties were in 4a.
Question 5	To determine the operator norm of S_n(f) one uses, in particular, Fejer kernel.