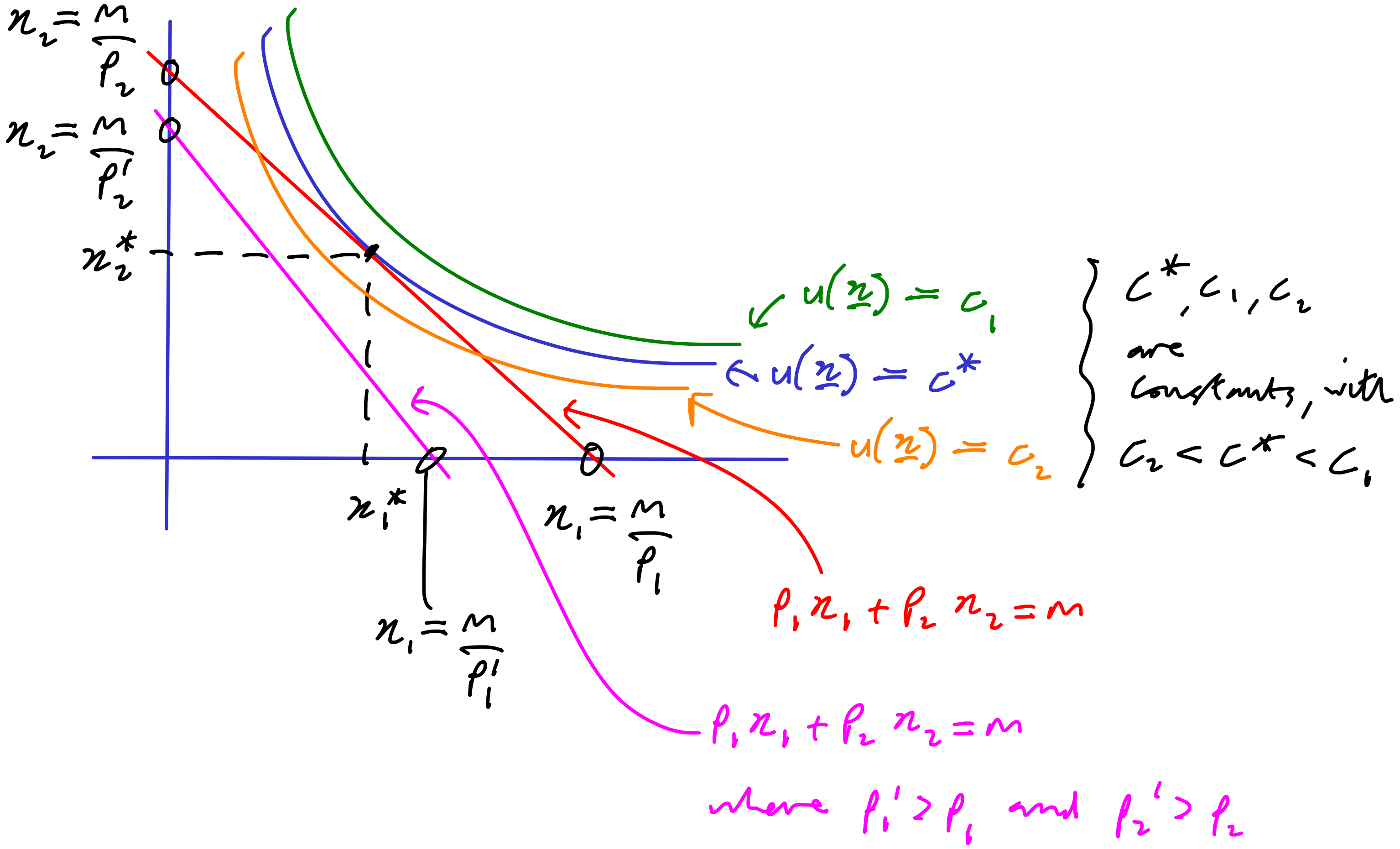
This indirect utility function is itself a quantity of interest, and we note some of its key properties here:

- Nonincreasing in $\underline{p}$:

$$\underline{p}' \geq \underline{p} \Rightarrow v\left(\underline{p}', m\right) \leq v\left(\underline{p}, m\right)$$

We won't provide a rigorous proof, but observe the following:



Recall that $\underline{p}(\underline{x}^*)^T \leq m$ (or $= m$ if assuming local non-satiation), i.e., $\underline{x}^* \in B_{\underline{p},m}$. As $\underline{p}$ increases, so $B_{\underline{p},m}$ recedes, i.e., for $\underline{p}' \geq \underline{p}$, $B_{\underline{p}',m} \subset B_{\underline{p},m}$. So $u(\underline{x}^*) \leq u(\underline{x})$. And intuitively, as $\underline{p}$ increases, (while m remains fixed) one would expect $\underline{x}^*$ and hence $u(\underline{x}^*)$ to not increase.

...and nondecreasing in m:

$$m' \geq m \Rightarrow v\left(\underline{p}, m'\right) \geq v\left(\underline{p}, m\right)$$

– this follows by arguments similar (but converse) to those used above.

- Homogeneous of degree 0 in $\left(\underline{p}, m\right)$:

$$v\left(t\underline{p}, tm\right) = v\left(\underline{p}, m\right) \quad \forall t > 0$$

(Makes sense intuitively – e.g. one would expect $\underline{x}^*$ and hence $u(\underline{x}^*)$ to be independent of a change in currency.)

- Quasi-convex in $\underline{p}$:

$$\left\{\underline{p} \in \mathbb{R}^n_{\geq 0} : v\left(\underline{p}, m\right) \leq k\right\} \text{ is a convex set for all } k, m \in \mathbb{R}.$$

i.e., $\forall t \in [0,1]$,

$$v(t\underline{p} + (1-t)\underline{p}', m) \leq \max\{v(\underline{p}, m), v(\underline{p}', m)\}$$

(or equivalently, if $v(\underline{p}, m) \leq k$ and $v(\underline{p}', m) \leq k$, then

$$v(t\underline{p} + (1-t)\underline{p}', m) \leq k.)$$

E.g., Consider sketch above: with $\underline{p}' \geq \underline{p}$, so

$\underline{p}' \geq t\underline{p} + (1-t)\underline{p}' \geq \underline{p}$.

- Continuous at all $\underline{p} \gg 0, m > 0$.

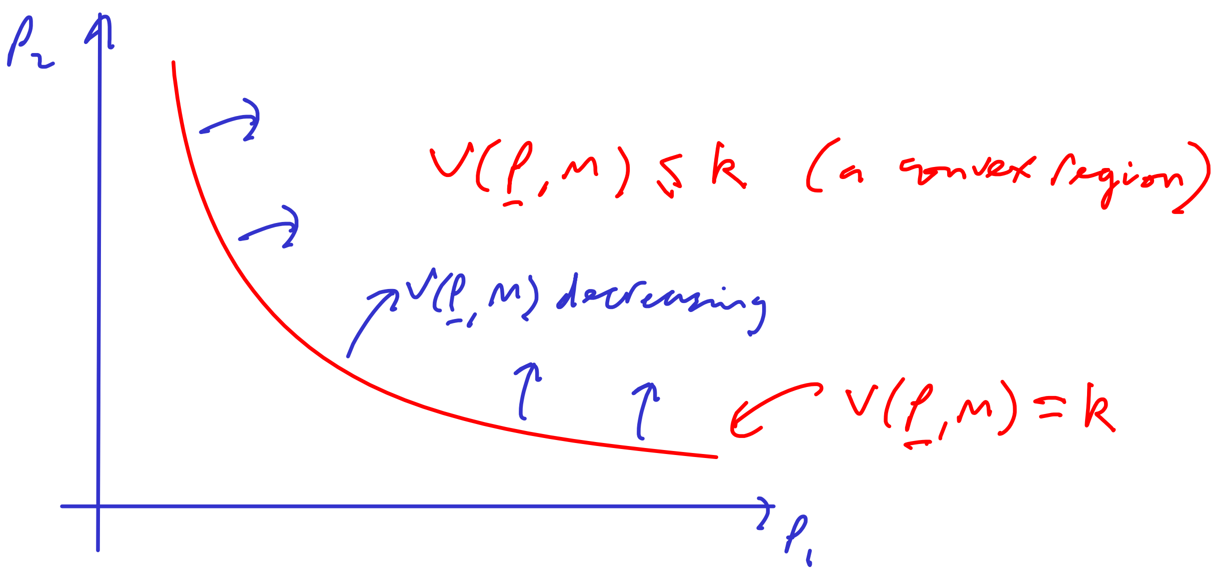
($\underline{p} \gg 0$ means $p_i > 0 \ \forall i = 1, \dots, n$.)

$\rightarrow$ $\underline{x}^*$ and hence $v(\underline{p}, m)$ might not exist unless $\underline{p} \gg 0$. E.g. consider above sketch with $p_1 = 0$.

The indirect utility function v is often illustrated using so-called price indifference curves. These are the level sets of the indirect utility function with a *fixed budget* m. That is

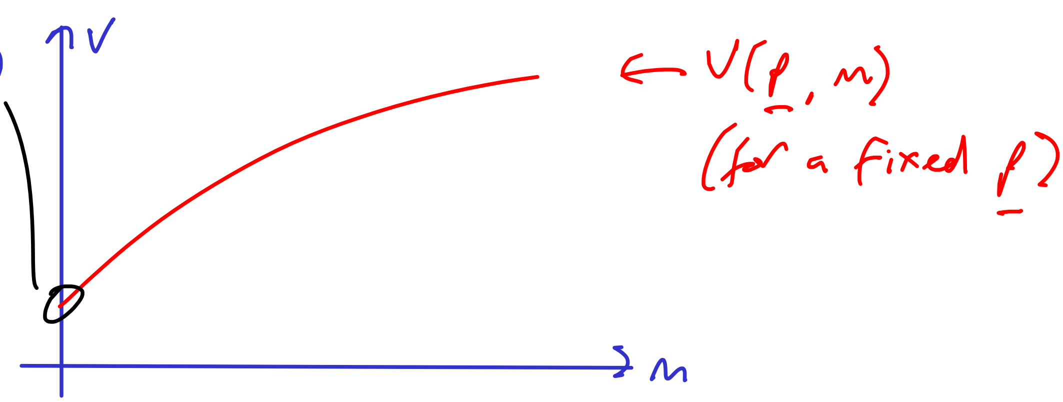
$$\{\underline{p} \in \mathbb{R}^n_{\geq 0} : v(\underline{p}, m) = k\}, \quad k \in \mathbb{R}, \ m \geq 0.$$

They are analogous to the indifference curves of the utility function:



A direct consequence of the local nonsatiation assumption of the underlying preferences is that for fixed $\underline{p}$, the indirect utility function $v(\underline{p}, \cdot)$ is strictly increasing in m:

Note that $v(\underline{p}, 0)$ is not necessarily 0 ($u(\underline{x})$ is an ordinal utility function).



Therefore, $v(\underline{p}, \cdot)$ is injective and can be inverted on its image. Denote this image with

$$U_p = \{v(\underline{p}, m) : m \geq 0\}.$$

Then we define the ***expenditure function*** $e(\underline{p}, u) : U_p \to [0, \infty)$,

$$u \mapsto e(\underline{p}, u) \quad \text{s.t.} \quad u = v(\underline{p}, e(\underline{p}, u))$$

(Note: here u denotes a fixed level of utility; not to be confused with $u(\underline{x})$.)

The expenditure function provides the minimum level of income required to obtain a given level of utility at prices $\underline{p}$. Note that $e(\underline{p}, u)$ can also be obtained as the solution to the optimisation problem

$$\text{Find} \quad \min_{\underline{x}} \underline{p}\,\underline{x}^T \quad \text{s.t.} \quad u(\underline{x}) \geq u.$$

(Note: If one could obtain a level of utility u with a level of income, say, m' with $m' < e(\underline{p}, u)$ we would have

$$u(\underline{x}') = u = v(\underline{p}, e(\underline{p}, u)) = u(\underline{x}^*(\underline{p}, e(\underline{p}, u)))$$

for some $\underline{x}'$ with $\underline{p}\,\underline{x}'^T \leq m' < e(\underline{p}, u)$, but then, due to the local non-satiation assumption, there would exist some $\underline{x}$ with both $\underline{p}\,\underline{x}^T < e(\underline{p}, u)$ and $u(\underline{x}) > u(\underline{x}^*(\underline{p}, e(\underline{p}, u)))$ (c.f., a similar argument used earlier), which is a contradiction.)

$e(\underline{p}, u)$ is simply the demand-side analogue of a firm's cost function $c^*(\underline{w}, y)$ - recall that this is given by

$$\min_{\underline{x}} \underline{w}\,\underline{x}^T \text{ s.t. } f(\underline{x}) \geq y.$$

They share some similar properties: (provided it exists) $e(\underline{p}, u)$ is:

- non-decreasing in $\underline{p}$
- homogeneous of degree 1 in $\underline{p}$
- concave in $\underline{p}$
- continuous in $\underline{p}$

The dual quantity to the expenditure function is the ***Hicksian demand*** (sometimes referred to as the *compensated demand*)

$$\underline{x}_H^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R}^n_{\geq 0}, \qquad (\underline{p}, u) \mapsto \underline{x}_H^*(\underline{p}, u) = \underset{\underline{x} \text{ s.t. } u(\underline{x}) \geq u}{\arg\min}\ \underline{p}\,\underline{x}^T$$

This is the analogue of the conditional factor demand function $\underline{x}^*(\underline{w}, y)$.

Recall that, on the firm side, for a specified level of output, the cost-minimising combination of production inputs can be found via Shephard's Lemma. We can also apply this result in the current scenario, yielding an expression for the expenditure-minimising consumption bundle in terms of prices and desired utility level:

$$x_{H,i}^{*}\left(\underline{p},u\right)=\frac{\partial e\left(\underline{p},u\right)}{\partial p_i}.$$

Recall that the Marshallian demand function is defined by

$$\underline{x}^{*}(\underline{p},m) = \underset{\underline{x} \text{ s.t. } \underline{p}\,\underline{x}^{T} = m}{\operatorname{argmax}} \; u(\underline{x})$$

assuming local non-satiation

Note that, when we refer to the demand function without qualification, it is assumed to be the Marshallian demand.

Unlike the Marshallian demand, the Hicksian demand function is not observable; indeed, it depends on utility, which is itself unobservable. Nonetheless, under some of the **usual regularity assumptions,** the Hicksian and Marshallian demands satisfy the following identities: for all $\underline{p} \gg 0$, $m > 0$

- $e\left(\underline{p},v\left(\underline{p},m\right)\right)= m$
- $v\left(\underline{p},e\left(\underline{p},u\right)\right)\equiv u$
- $x_{H,i}^{*}\left(\underline{p},v\left(\underline{p},m\right)\right)= x_i^{*}(\underline{p},m)$
- $x_i^{*}\left(\underline{p},e\left(\underline{p},u\right)\right)= x_{H,i}^{*}(\underline{p},u)$

The Slutsky equation

For economists, it is important to understand how consumers react to changes in the economic environment. For instance, we can consider how the optimal choice of consumption bundle $\underline{x}^*(\underline{p}, m)$ will change with respect to the price vector $\underline{p}$. *The* ***Slutsky equation*** states that the total effect of a change in demand of a good when the price of a good is changed can be decomposed into a ***substitution effect*** and an ***income effect.***

- the substitution effect is the change whilst keeping the level of utility fixed.
- the income effect is the change due to the consumer's increase/decrease in purchasing power.

The Slutsky equation:

Under certain regularity conditions (details omitted), for $\underline{p} \gg 0$ and $m > 0$ and $\forall i, j \in \{1, \dots, n\}$,

$$\boxed{\begin{aligned} &\partial_i x_j^*(\underline{p}, m) = \\ &= \partial_i x_{H,j}^*(\underline{p}, v(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_i^*(\underline{p}, m) \end{aligned}}$$

Note! Here, $\partial_i x_{H,j}^*(\underline{p}, v(\underline{p}, m))$ denotes

$$\left(\frac{\partial}{\partial p_i} x_{H,j}^*(\underline{p}, u)\right)\Bigg|_{u = v(\underline{p}, m)}$$

Proof:

For any $u \in U_p$, $\quad x_{H,j}^*(\underline{p}, u) = x_j^*(\underline{p}, e(\underline{p}, u))$

$$\Rightarrow \quad \frac{\partial}{\partial p_i} x^*_{H,j}(\underline{p},u) = \frac{\partial}{\partial p_i} x^*_j(\underline{p}, e(\underline{p},u))$$

$$= \left(\frac{\partial}{\partial p_i} x^*_j(\underline{p},m)\right)\Bigg|_{m=e(\underline{p},u)}$$

$$+ \left(\left(\frac{\partial}{\partial m} x^*_j(\underline{p},m)\right)\Bigg|_{m=e(\underline{p},u)}\right) \cdot \underbrace{\frac{\partial e(\underline{p},u)}{\partial p_i}}_{\substack{= x^*_{H,i}(\underline{p},u) \\ \text{by Shephard's Lemma}}}$$

Now set $u = v(\underline{p},m)$. Note that

$$e(\underline{p}, v(\underline{p},m)) = m.$$

So we get:

$$\left(\frac{\partial}{\partial p_i} x^*_{H,j}(\underline{p},u)\right)\Bigg|_{u=v(\underline{p},m)}$$

$$= \frac{\partial}{\partial p_i} x^*_j(\underline{p},m) + \left(\frac{\partial}{\partial m} x^*_j(\underline{p},m)\right) \cdot \overbrace{x^*_{H,i}(\underline{p}, v(\underline{p},m))}^{= x^*_i(\underline{p},m)}$$

which rearranges to give the result. //

$$\partial_i x_j^*(\underline{p}, m) =$$

$$= \partial_i x_{H,j}^*(\underline{p}, v(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_i^*(\underline{p}, m)$$

Corresponds to the substitution effect

Corresponds to the income effect