

# Analysis 1A

Lecture 15

Series continued:

Absolute convergence

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#### Example 4.4

Show that  $\sum_{n=1}^{\infty} \frac{1}{n}$  is not convergent.

#### Example 4.5

Show that  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent.

## Proposition 4.6

Suppose  $a_n \geq 0 \ \forall n$  ( $\iff s_n = \sum_{i=1}^n a_i$  is monotonically increasing), Then the following two facts are true:

- 1  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $(s_n)$  is bounded above.
- 2 Similarly  $\sum_{n=1}^{\infty} a_n$  diverges to  $+\infty$  (i.e.  $\forall M > 0 \exists N \in \mathbb{N}$  such that  $s_n > M \ \forall n \geq N$ ) if and only if  $(s_n)$  is unbounded.

Proof ①  $s_n$  is monotonically increasing so

$s_n$  convergent  $\iff s_n$  bounded above

" $\sum a_n$  convergent"

$s_n$  not bounded  
 $\downarrow$  then

②  $s_n$  is unbounded  $\iff \forall M > 0, \exists N \in \mathbb{N}$  s.t.  $s_N > M$

$\iff \forall M > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, s_n \geq s_N > M$

$\iff s_n \rightarrow \infty$

" $\sum a_n$  diverges to  $\infty$ "



### Theorem 4.7 Comparison Tests

If  $0 \leq a_n \leq b_n$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.

Moreover,  $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ .

Proof Let  $A_n = \sum_{i=1}^n a_i$ ,  $B_n = \sum_{i=1}^n b_i$

$0 \leq A_n \leq B_n$  and  $B_n$  is bounded above (since  $\sum b_n$  is convergent)  
so  $B_n \leq M$

Therefore  $A_n$  is bounded above by  $M$ , and so

$\sum_{n=1}^{\infty} a_n$  convergent.



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### Exercise 4.8 - Converse of Comparison Test

If  $0 \leq a_n \leq b_n$  then

$\sum a_n$  diverges to  $+\infty \implies \sum b_n$  diverges to  $+\infty$ .

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### Remark 4.9

So from the fact that  $\sum \frac{1}{n^2}$  is convergent, we can now deduce that  $\sum \frac{1}{n^\alpha}$  convergent for  $\alpha \geq 2$  by the Comparison Test. In fact we can improve on this.

### Example 4.10

Let  $\alpha > 1$ , then show that  $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$  is convergent.

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$$\begin{aligned} 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \dots &= 1 + \left( \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} \right) + \left( \frac{1}{4^{\alpha}} + \dots + \frac{1}{7^{\alpha}} \right) \\ &\quad + \left( \frac{1}{8^{\alpha}} + \dots + \frac{1}{15^{\alpha}} \right) + \left( \frac{1}{16^{\alpha}} + \dots + \frac{1}{31^{\alpha}} \right) + \dots \end{aligned}$$

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*Rough work*

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Bound the  $k$ th bracketed term:

$$\left( \frac{1}{(2^k)^{\alpha}} + \dots + \frac{1}{(2^{k+1} - 1)^{\alpha}} \right) \leq \frac{1}{2^{k\alpha}} + \dots + \frac{1}{2^{k\alpha}} = \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}.$$

### Example 4.10

Let  $\alpha > 1$ , then show that  $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$  is convergent.

Example continued:

Proof Let  $s_n = \sum_{j=1}^n \frac{1}{j^\alpha}$

Since  $\left(\frac{1}{2^{k\alpha}} + \dots + \frac{1}{(2^{k+1}-1)^\alpha}\right) \leq \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}$

So  $s_4 \leq 1 + \frac{1}{2^{1\alpha}} + \frac{1}{2^{2\alpha}}$

For any  $n \leq 2^{k+1}-1$

$$s_n \leq \sum_{j=0}^k \frac{1}{2^{j\alpha}} \leq \sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$$

where  $r = 2^{-(1-\alpha)}$  ( $|r| < 1$ )

$s_n$  is bounded above, so convergent.  $\blacksquare$

### Theorem 4.11 - Algebra of Limits for Series

If  $\sum a_n$ ,  $\sum b_n$  are convergent then so is  $\sum(\lambda a_n + \mu b_n)$ , to

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

Proof

$$\sum_{j=1}^{\infty} \lambda a_j + \mu b_j = \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j \rightarrow \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j$$

by Algebra of limits for sequences.  $\blacksquare$

Next, we discuss a notion of convergence for real and complex series that is **stronger** than the convergence of the sequence of partial sums.

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### Definition - Absolute Convergence

For  $a_n \in \mathbb{R}$  or  $\mathbb{C}$ , we say the series  $\sum_{n=1}^{\infty} a_n$  is *absolutely convergent* if and only if the series  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

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### Remark 4.12

It is possible for a series to be convergent, but not absolutely convergent!

### Example 4.13

We note that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is *not* absolutely convergent (remember the harmonic series), but show that it is convergent.

*Rough Working:*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \dots$$

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with  $k$ -th bracket  $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$ .

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This is positive and  $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$ .

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We saw this is convergent in Example 4.5. So cancellation between consecutive terms is enough to make series converge by comparison with  $\sum \frac{1}{k(k-1)}$ .

### Example 4.13

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Proof We know  $\sum \frac{1}{2k(2k-1)}$  is convergent to  $L$ .

and we know  $\frac{(-1)^n}{n} \rightarrow 0$ ,  $\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = S_n$

For  $\epsilon > 0$  (i)  $\exists N_1$  st  $\forall n \geq N_1, |S_n - L| < \epsilon$

(ii)  $\exists N_2$  st  $\forall n \geq N_2, \left| \frac{(-1)^n}{n} \right| < \epsilon$

Let  $\sigma_n = \sum_{j=1}^n \frac{(-1)^{j+1}}{j}$ ,  $\sigma_n = S_{\lfloor n/2 \rfloor} + \delta_n$   $\delta_n = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

Let  $N = 2 \max(N_1, N_2) + 1$

Then for  $n \geq N$

$|\sigma_n - L| \leq |S_{\lfloor n/2 \rfloor} - L| + |\delta_n|$    
 $\quad \quad \quad \uparrow \quad \quad \quad \uparrow$    
 $\quad \quad \quad \lfloor n/2 \rfloor \geq N_1 \quad n \geq N_2$    
 $| \sigma_n - L | \leq \epsilon + \epsilon = 2\epsilon$ . So  $\sigma_n \rightarrow L$ .