We further note that if the short-run fixed factors $\underline{x}_F$ are fixed at their long-run conditional factor demand for a given output, y^* (i.e. $\underline{x}_F = \underline{x}_F^*(\underline{w}, y^*)$), then

$$LMC(y^*) = SMC(y^*)$$

Proof

The argument is as follows:

$$c^*(\underline{w}, y) \equiv c_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y) \quad \forall y > 0$$

That means, we can take the total derivative on both sides. That is

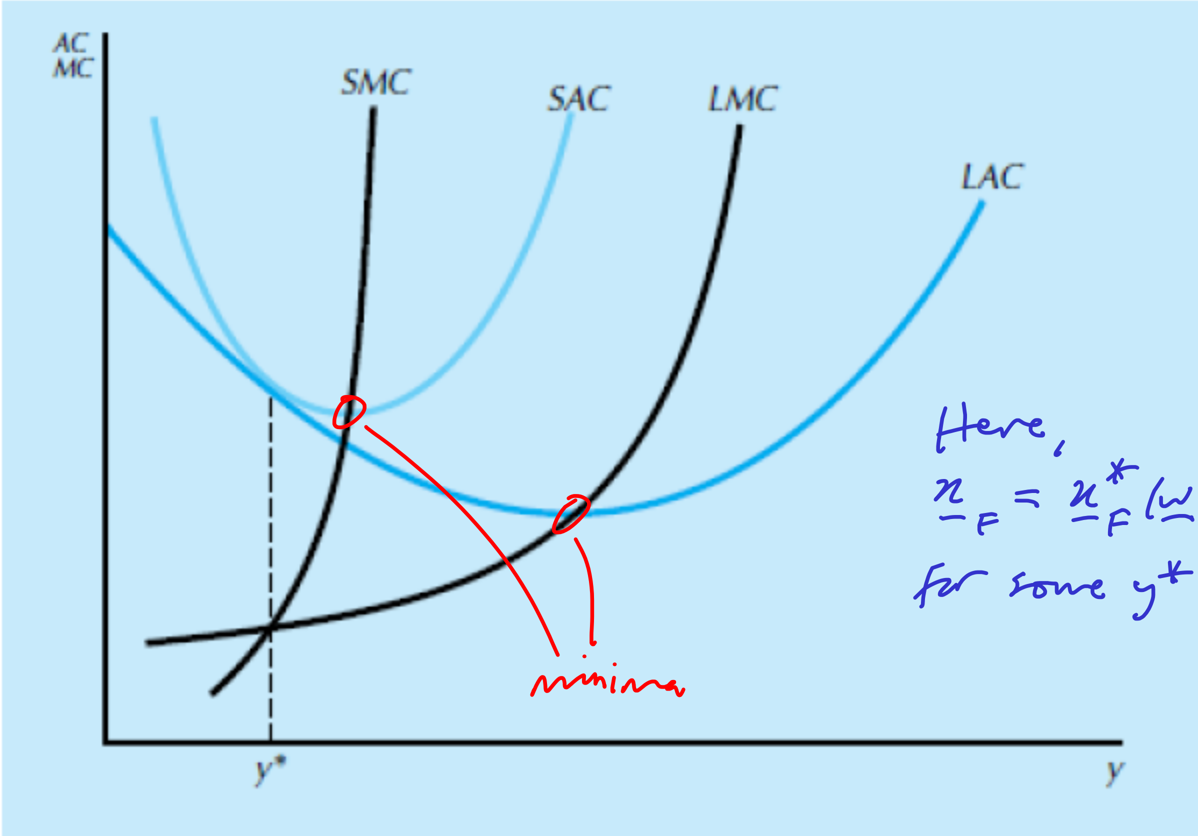
$$\frac{d c^*(\underline{w}, y)}{dy} = \frac{d}{dy} c_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)$$

$$= \underbrace{\sum_{j=1}^{n} \frac{\partial c_s^*(\underline{w}, \underline{x}_F, y)}{\partial (\underline{x}_F)_j} \Bigg|_{\underline{x}_F = \underline{x}_F^*(\underline{w}, y)} \cdot \frac{d(\underline{x}_F^*(\underline{w}, y))}{dy}}_{=\,0 \text{ since } \underline{x}_F = \underline{x}_F^*(\underline{w}, y) \text{ minimises } c_s^*(\underline{w}, \underline{x}_F, y)} + \frac{\partial c_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= \frac{\partial c_s^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= SMC(y).$$

But $\frac{d c^*(\underline{w}, y)}{dy} \equiv \frac{\partial c^*(\underline{w}, y)}{\partial y}$ since $\underline{w}$ is constant. //



Here, $\underline{x}_F = \underline{x}_F^*(\underline{w}, y^*)$ for some y^*

So, if $\underline{x}_F = \underline{x}_F^*(\underline{w}, y^*)$, then when $y = y^*$ the additional cost per additional unit of output (i.e. the marginal cost) is the same in the short-run as in the long-run (i.e., whether or not any of the inputs are constrained).

Profit maximisation given minimised costs

We have now considered the choices that a firm must make to minimise its costs, given knowledge of its factor prices $\underline{w}$ and a given level of output y. As mentioned previously, we now consider how a firm should subsequently choose an optimal level of output y in order to maximise profits conditional on minimised costs.

To set up the profit-maximisation question in this conditional framework, we initially maintain the assumption of perfect competition; we also assume to begin with that we are operating in the short-run.

Recall that previously, profit maximisation was framed as a question of how much input to use, and that the output of the firm was specified by the production function f. Now, all of the firm's technical constraints are implicitly specified by the cost function.

We therefore reformulate our profit maximisation problem:

We wish to solve

$$\underset{y \geq 0}{\arg\max} \left\{ py - c_s^*(\underline{w}, \underline{x}_F, y) \right\}$$

for y. Note that in the short-run, $\underline{x}_F$ is fixed. And p and $\underline{w}$ are also fixed (by the assumption of perfect competition). Hence $py - c_s^*(\underline{w}, \underline{x}_F, y)$ is simply a function of y.

First- and second-order conditions for the optimal level of output given minimised costs are given by:

$$\frac{\partial}{\partial y}\left(py - c_s^*(\underline{w}, \underline{x}_F, y)\right) = 0 \Rightarrow p = SMC(y)$$

$$\frac{\partial^2}{\partial y^2}\left(py - c_s^*(\underline{w}, \underline{x}_F, y)\right) \leq 0 \Rightarrow \frac{\partial^2 c_s^*(\underline{w}, \underline{x}_F, y)}{\partial y^2} \geq 0$$

i.e., $\frac{\partial SMC(y)}{\partial y} \geq 0$.

These conditions suggest that, in order to maximise profits, the output should be such that the corresponding short-run marginal cost is increasing and equal to the output price p.

For a cost-minimising competitive firm, this specifies a relationship between the market-defined output price p and the quantity of output that the firm should provide.

Example 1:

Suppose that a firm's short-run cost function for a good is specified as

$$C_S^*(w_1, w_2, \underline{x}_F, y) = 2\sqrt{w_1 w_2}\, y^2 + \underbrace{FC(\underline{w}_F, \underline{x}_F)}_{\text{fixed costs}}$$

If the market price for the good is £16 $(= p)$ and each input costs the firm £4 $(= w_1, w_2)$, how many units of the good should the firm produce in the short run, and what is their maximised profit if fixed costs are £12?

Solution:

$$SMC(y) = \frac{\partial C_S^*(w_1, w_2, \underline{x}_F, y)}{\partial y} = 4\sqrt{w_1 w_2}\, y$$ (a linear function of y)

Then

F.O.C.: $SMC(y) = p \Rightarrow y = \frac{p}{4\sqrt{w_1 w_2}} = 1, = \hat{y}$, say

S.O.C.: $\frac{\partial SMC(y)}{\partial y} = 4\sqrt{w_1 w_2} = 16 \geq 0 \quad \forall y$

So $\hat{y} = 1$ gives the maximum profit. And this is:

maximum profit $= p\hat{y} - c_s^*(w_1, w_2, \underline{x}_F, \hat{y})$

$$= 16.1 - \left(2\sqrt{w_1 w_2} \cdot (1)^2 + 12\right)$$

$$= -4$$

One may deduce that...

In the short-run, i.e. when there are fixed costs, the most profitable position for a firm may be one that returns negative profit .

Example 2:

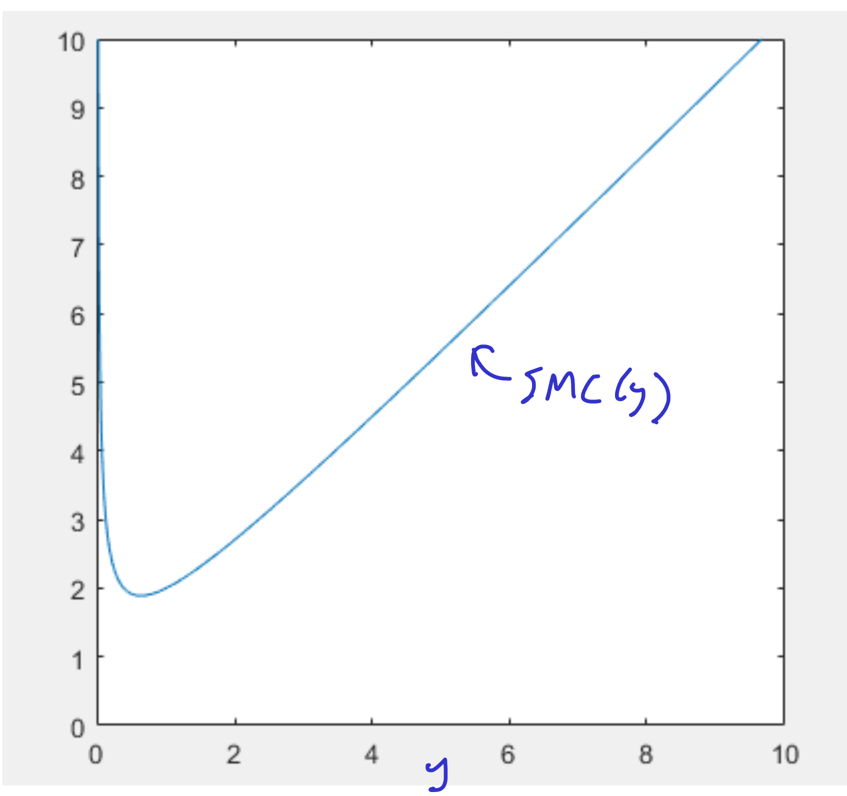
Consider the cost function

$$c_s^*(\underline{w}, \underline{x}_F, y) = w_1 y^{1/2} + w_2 y^2 + FC(\underline{w}_F, \underline{x}_F).$$

What is the maximised profit here, when $w_1 = 2$, $w_2 = \frac{1}{2}$, and $p = 2$?

Solution

$$SMC(y) = \frac{\partial c_s^*}{\partial y}(\underline{w}, \underline{x}_F, y) = \frac{w_1}{2} y^{-1/2} + 2w_2 y = y^{-1/2} + y$$



So. F.O.C.: $SMC(y) = p \Rightarrow y^{-1/2} + y = 2$

$$\Rightarrow \frac{1}{y} = (2-y)^2$$

$$\Rightarrow y^3 - 4y^2 + 4y - 1 = 0$$

$$\Rightarrow (y-1)(y^2 - 3y + 1) = 0$$

$$\Rightarrow y = 1, \frac{3 \pm \sqrt{5}}{2}$$

But note (check!), $\frac{3+\sqrt{5}}{2}$ doesn't give $SMC(y) = 2$, so discard it. Next,

S.O.C.: $$\frac{\partial SMC(y)}{\partial y} = -\frac{y^{-3/2}}{2} + 1$$

$$= \begin{cases} \frac{1}{2} & \text{for } y = 1 \\ <0 & \text{for } y = \frac{3-\sqrt{5}}{2} \end{cases}$$

$\Rightarrow$ $y = 1 = \hat{y}$, say, maximises the profit (for $y>0$).

$$\Rightarrow \text{max. profit} = p\hat{y} - c_s^*(\underline{w}, \underline{x}_F, \hat{y})$$

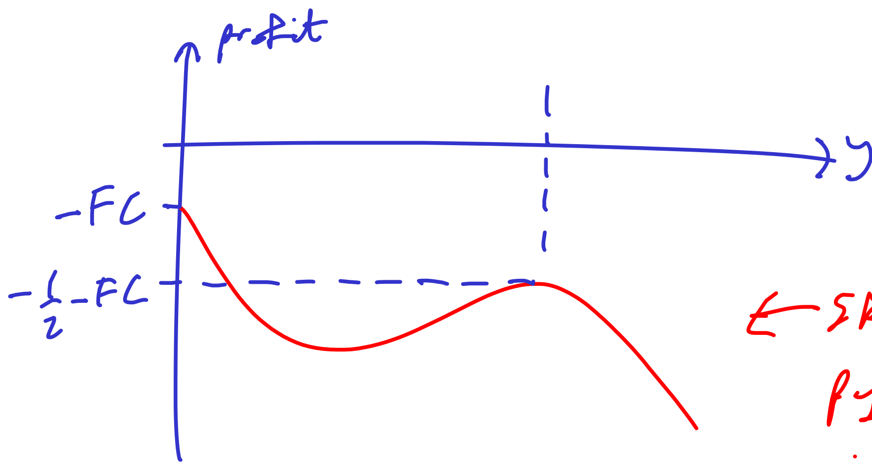
$$= 2 - \left(2 + \frac{1}{2} + FC\right)$$

$$= -\frac{1}{2} - FC$$

$< -FC$

" profit if $y=0$.

Our earlier analysis didn't pick up the global maximum of the profit at $y=0$, since the F.O.C. is not satisfied there:



← Sketch of profit curve $py - c_s^*(\underline{w}, \underline{x}_F, y)$ in this case.

(Note that in Example 1, the profit function $py - c_s^*(\underline{w}, \underline{x}_F, y)$ is a quadratic in y, and its value at $\hat{y} = 1$ is greater than at $y=0$).

So, as illustrated in Example 2, in some circumstances it may be preferable for a firm to go out of business rather than provide $y > 0$.

(that is, produce no output, i.e., set $y=0$)

(i.e., produce no output)

Indeed, we can generalise: it will be preferable to go out of business when

the profit for $y=0$ exceeds $py - c_s^*(\underline{w}, \underline{x}_F, y)$ $\forall y > 0$, i.e., when

$$\underbrace{-\underline{w}_F \underline{x}_F^T}_{\text{profit for } y=0\ (\underline{x}_S^*(\underline{w}, \underline{x}_F, 0) = 0)} > py - \left(\underline{w}_F \underline{x}_F^T + \underline{w}_V \underline{x}_S^*(\underline{w}, \underline{x}_F, y)\right) \quad \forall y > 0$$

$$\iff \frac{\underline{w}_V \underline{x}_S^*(\underline{w}, \underline{x}_F, y)}{y} > p \quad \forall y > 0$$

$$\iff SAVC(y) > p \quad \forall y > 0$$

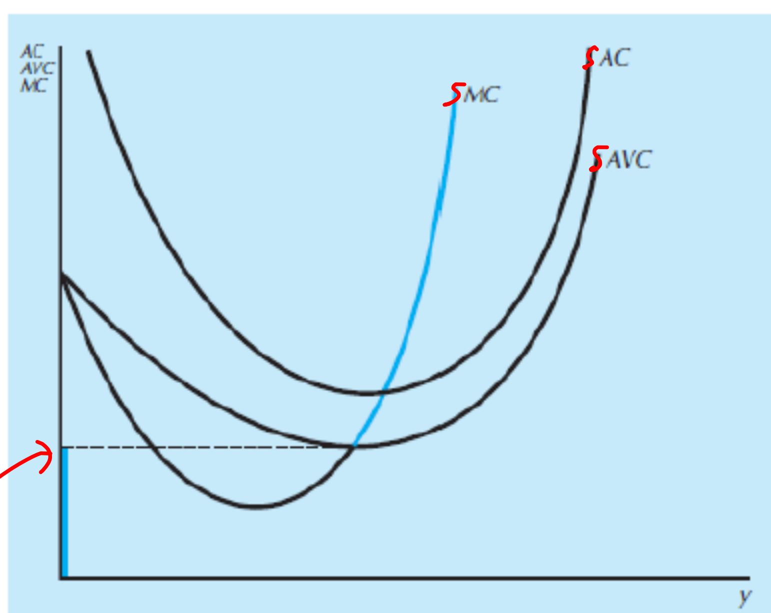
This is known as the ***shutdown condition***; when satisfied, it is preferable for the firm to go out of business (i.e., produce nothing).

So we must refine our definition of the firm's chosen short-run supply. The competitive cost-minimising firm should choose a positive level of output y such that:

- $SMC(y) = p$;
- $SMC(y)$ is increasing in y;
- and $SAVC(y) \leq p$.

If no such $y > 0$ exists for the given p, then the firm should set $y = 0$.

These conditions are satisfied by the portion of the SMC curve that is increasing in y and that lies on or above the $SAVC$ curve:



If p is less than this value then the firm should shut down production (i.e., produce nothing) (since in this case there exists no $y > 0$ such that $SMC(y) = p \geq SAVC(y)$).

In the long run, we have a very similar story. Neither the first- nor second-order conditions above explicitly require the costs to be dependent on fixed factors of production; these translate to the long-run scenario as would be expected. The long-run profit-maximising supply for a cost-minimising firm is given by y such that

- $LMC(y) = p$
- $LMC(y)$ must be increasing in y.
- $LAC(y) \leq p$

One arrives at the above by noting that in the long-run, the profit is given by

$$py - c^*(\underline{w}, y)$$

which is maximised for y s.t.

$$\underbrace{\frac{\partial c^*(\underline{w}, y)}{\partial y}}_{= LMC(y)} = p \quad \text{and} \quad \frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} \geq 0$$

and $$py - c^*(\underline{w}, y) \geq \underbrace{-c^*(\underline{w}, 0)}_{\text{profit for } y=0,\ =0\ (\underline{x}^*(\underline{w},0)=\underline{0})}$$

i.e., $$\underbrace{\frac{c^*(\underline{w}, y)}{y}}_{= LAC(y)} \leq p$$

Once more, if no such $y > 0$ exists for the given p, then the firm should choose to go out of business (i.e., produce nothing).

Profit maximisation for a noncompetitive firm

To contrast, we consider the profit maximisation problem for a cost-minimising monopolist. Whilst monopolists have more control over output prices than in a competitive market, they cannot choose price and output independently of one another; they must respect the market demand for their product. We therefore assume that the monopolist chooses the amount of output to provide, y, and the output price is determined according to the market demand for this output, i.e. as a function of y, $p(y)$.

The function $p(y)$ is the inverse of the market's demand function and is referred to as the inverse demand function "facing the firm"; we note that it may be dependent on other determinants, but assume these to be held constant in our analysis.

i.e., $p(y) = D^{-1}(y)$, or, $y = D(p(y))$.

To maximise profits, we therefore seek :

$$\underset{y \geq 0}{\arg\max} \left\{ p(y)y - c_S^*(\underline{w}, \underline{x}_F, y) \right\}.$$

First- and second-order conditions for finding a profit-maximising position for a monopolist facing an inverse demand function are therefore given by

$$\frac{\partial}{\partial y}\left(p(y)y - c_S^*(\underline{w}, y)\right) = 0 \quad \Rightarrow \quad \frac{\partial p(y)}{\partial y} y + p(y) = SMC(y) \qquad (FOC)$$

$$\frac{\partial^2}{\partial y^2}\left(p(y)y - c_S^*(\underline{w}, y)\right) \leq 0 \quad \Rightarrow \quad \frac{\partial^2 c_S^*(\underline{w}, \underline{x}_F, y)}{(\partial y)^2} \geq \frac{\partial^2 p(y)}{(\partial y)^2} y + 2\frac{\partial p(y)}{\partial y} \quad (SOC)$$

We can rearrange the FOC as follows:

$$p(y)\left[1 + \frac{1}{\varepsilon_D(y)}\right] = SMC(y) \qquad (*)$$

where $\varepsilon_D(y) = \dfrac{\partial y}{\partial p(y)} \cdot \dfrac{p(y)}{y}$ is the price elasticity of demand.

But with $y = D(p(y))$ we have (by differentiating wrt y):

$$1 = D'(p(y)) \cdot p'(y)$$

$$\Rightarrow \quad p'(y) = \frac{1}{D'(p(y))}$$

Now note that $\varepsilon_D(y) < 0$ (demand y decreases with increasing price p), and $SMC(y) \geq 0$ ($SMC(y) = \frac{\partial c_S^*(\underline{w}, \underline{x}_F, y)}{\partial y}$ and $c_S^*(\underline{w}, \underline{x}_F, y)$ increases with increasing output y).

Then, it follows from $*$ that a necessary condition for the firm to maximize profit is that $|\varepsilon_D| \geq 1$ (so that the LHS of $*$ is also ≥ 0), i.e., it should face elastic demand.

Example:

Consider the monopolist faced with a linear inverse demand

$$p(y) = a_1 - a_2 y \qquad a_1, a_2 > 0$$

and ~~Cobb-Douglas~~ variable costs in the short term :

$$c_S^*(\underline{w}, \underline{x}_F, y) = 2\sqrt{w_1 w_2}\, y^2 + FC(\underline{w}_F, \underline{x}_F).$$

What is the maximum profit that this monopolist can achieve?

Solution

First,

$$\frac{1}{\varepsilon_D(y)} = \frac{\partial p(y)}{\partial y} \cdot \frac{y}{p(y)} = \frac{-a_2 y}{a_1 - a_2 y}$$

(note this is ≤ 0 since $a_1, a_2 > 0$, $0 \leq y \leq a_1/a_2$)

Then $|\varepsilon_D| \geq 1$ provided

$$|a_2 y| \leq |a_1 - a_2 y|$$

$\Downarrow$ since $a_1, a_2 > 0$, $0 \leq y \leq a_1/a_2$

i.e. provided $y \leq \dfrac{a_1}{2a_2}$.

So any profit-maximising level of output must be <u>below</u> $\frac{a_1}{2a_2}$.

Now solve the FOC:

$$p(\hat{y})\left[1 + \frac{1}{\varepsilon_D(\hat{y})}\right] = SMC(\hat{y}) \left(= \frac{\partial}{\partial \hat{y}} c_S^*(\underline{w}, \underline{x}_F, \hat{y})\right)$$

$$\Rightarrow \quad a_1 - 2a_2\hat{y} = 4\sqrt{w_1 w_2}\,\hat{y}$$

$$\Rightarrow \quad \hat{y} = \frac{a_1}{2a_2 + 4\sqrt{w_1 w_2}} \leq \frac{a_1}{2a_2}$$ as required.

Evaluating the SOC verifies that this is a maximum (exercise: check).

Then $p(\hat{y}) = a_1 - a_2\hat{y}$

and the maximum profit is

$$p(\hat{y})\cdot\hat{y} - c_S^*(\underline{w}, \underline{x}_F, \hat{y}) =$$

$$= (a_1 - a_2\hat{y})\hat{y} - \left(2\sqrt{w_1 w_2}\,\hat{y}^2 + FC(\underline{w}_F, \underline{x}_F)\right)$$

$$\Rightarrow \ldots$$

$$= \frac{a_1^2}{4(a_2 + 2\sqrt{w_1 w_2})} - FC(\underline{w}_F, \underline{x}_F).$$

We can see from this example that it is also possible for profit-maximising monopolists to experience losses in the short-run; this is not a phenomenon unique to competitive markets.

(i.e., the above maximum profit could be < 0 if FC is large enough).

The above optimisation assumes that $y > 0$. Just as for competitive firms, however, we note that the profit-maximising (loss-minimising) position for a monopolist may be to go out of business, i.e. to set $y = 0$. This happens when the losses incurred by setting output according to the above first- and second-order conditions are greater than the fixed costs, i.e. when

$$SAVC(y) > p(y) \quad \forall y > 0.$$

One can see this as follows.

$$\text{profit} = p(y).y - c_s^*(\underline{w}, \underline{x}_F, y)$$

For $y=0$, this reduces to minus the fixed costs. So it is best to set $y=0$ if these fixed costs are greater than $p(y).y - c_s^*(\underline{w}, \underline{x}_F, y) \ \forall y > 0$, i.e., if

$$0 > p(y).y - \underline{x}_V \underline{x}_s^*(\underline{w}, \underline{x}_F, y) \quad \forall y > 0$$

i.e., if

$$\underbrace{\frac{\underline{x}_V \underline{x}_s^*(\underline{w}, \underline{x}_F, y)}{y}}_{= SAVC(y)} > p(y) \quad \forall y > 0.$$

So, in summary, for a cost-minimising monopolist, the short run profit-maximising output y will satisfy the following conditions:

$\cdot)\ p(y)\left[1 + \frac{1}{\varepsilon_D(y)}\right] = SMC(y)$

$\cdot)\ \frac{\partial^2 c^*(\underline{w}, \underline{x}_F, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{\partial y^2} y + 2\frac{\partial p(y)}{\partial y}$

$\cdot)\ SAVC(y) \leq p(y)$

(If no such $y > 0$ exists, then the firm should set $y = 0$.

We also note that, as for competitive firms, the extension to the long-run is trivial. For a cost-minimising monopolist, the long-run profit-maximising output y will satisfy the following conditions:

$\cdot)\ p(y)\left[1 + \frac{1}{\varepsilon_D(y)}\right] = LMC(y)$

$\cdot)\ \frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{\partial y^2} y + 2\frac{\partial p(y)}{\partial y}$

$\cdot)\ LAC(y) \leq p(y)$

Theory of the Consumer

We now focus on the theory of the consumer, where we will formalise the notion of consumer preferences and show how optimal behaviour of the consumer with respect to their preferences will lead to a specification of the demand function.
In the course of our analysis, we will see a lot of similarities and analogies to the Theory of the Firm.

Preferences & Utility

We start by considering the goods consumed by a consumer.

Define the ***consumption bundle*** for a particular consumer to be the quantities of a collection of goods that the consumer is willing to consume:

$$\underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_{\geq 0}.$$

The set of possible consumption bundles is referred to as the ***consumption set***; this is usually taken to be some *closed and convex* set

$$X \subseteq \mathbb{R}^n_{\geq 0}.$$

Consumers are assumed to have preferences between bundles $\underline{x}, \underline{x}' \in X$:

- $\underline{x} \preccurlyeq \underline{x}'$ means that the consumer has a preference for bundle $\underline{x}'$ over bundle $\underline{x}$.

  i.e., the consumer wants $\underline{x}'$ at least as much as they want $\underline{x}$

- $\underline{x} \prec \underline{x}'$ means that the consumer has a *strict* preference for $\underline{x}'$ over $\underline{x}$.

  i.e., the consumer wants $\underline{x}'$ more than they want $\underline{x}$

  (so $\underline{x} \prec \underline{x}' \Leftrightarrow (\underline{x} \preccurlyeq \underline{x}' \wedge \underline{x}' \not\preccurlyeq \underline{x})$)

- $\underline{x} \sim \underline{x}'$ denotes indifference between $\underline{x}$ and $\underline{x}'$.

  (so $\underline{x} \sim \underline{x}' \Leftrightarrow (\underline{x} \preccurlyeq \underline{x}' \wedge \underline{x}' \preccurlyeq \underline{x})$)

We are working under the condition that the preference relation satisfies the three axioms of a **complete weak order** on X. That is

- Completeness: $\forall \underline{x}, \underline{x}' \in X$, $\underline{x} \precsim \underline{x}'$ or $\underline{x}' \precsim \underline{x}$
  (i.e., any two bundles can be compared for preference.)
- Reflexivity: $\forall \underline{x} \in X$, $\underline{x} \precsim \underline{x}$
- Transitivity: $\forall \underline{x}, \underline{x}', \underline{x}'' \in X$, if $\underline{x} \precsim \underline{x}'$ and $\underline{x}' \precsim \underline{x}''$
  then $\underline{x} \precsim \underline{x}''$

Beware that reflexivity actually follows from completeness.

In addition, the following assumptions are *useful* but not necessary: (Axioms of consumer preferences)

Continuity

$\forall \underline{x} \in X$, the sets $\{\underline{x}' \in X : \underline{x} \precsim \underline{x}'\}$ and $\{\underline{x}' \in X : \underline{x}' \precsim \underline{x}\}$ are both closed. (One definition of a closed set is that any sequence of points in the set that converges, converges to a point in the set.) Roughly speaking, if bundles $\underline{x}'$ and $\underline{x}''$ are very similar, and $\underline{x}'$ is preferred to $\underline{x}$, then so should $\underline{x}'$ be. Or, if $\underline{x}$ is preferred to $\underline{x}'$, it should also be preferred to $\underline{x}''$.

Weak / Strong Monotonicity ('More is preferable to less')

$$\underline{x} \le \underline{x}' \quad \Longrightarrow \quad \underline{x} \preccurlyeq \underline{x}' \quad \text{(weak)}$$

$$\underline{x} \le \underline{x}' \text{ and } \underline{x} \ne \underline{x}' \quad \Longrightarrow \quad \underline{x} \prec \underline{x}' \quad \text{(strong)}$$

Local nonsatiation

$\forall \underline{x} \in X$ and $\forall \varepsilon > 0$, $\exists\, \underline{x}' \in X$ with $\|\underline{x} - \underline{x}'\| < \varepsilon$ and $\underline{x} \prec \underline{x}'$.

i.e., for any bundle $\underline{x}$, there is always another bundle $\underline{x}'$ arbitrarily close to $\underline{x}$ that is strictly preferred to it.

(Strict) Convexity

Convexity:

$\forall \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \precsim \underline{x}'$ and $\underline{x} \precsim \underline{x}''$

$$\underline{x} \precsim t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in [0,1].$$

Strict convexity:

$\forall \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \precsim \underline{x}'$ and $\underline{x} \precsim \underline{x}''$ and $\underline{x}' \neq \underline{x}''$

$$\underline{x} \prec t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in (0,1).$$

Note – we have not yet used the symbols $\succcurlyeq$ or $\succ$; we can use this as would be expected, i.e.

$$\underline{x} \precsim \underline{x}' \iff \underline{x}' \succsim \underline{x}$$

but it is no more than a notational convenience.

How does a consumer decide between bundles in some subset of X? How do we judge the suitability, or usefulness, of a consumption bundle $\underline{x}$? More to the point, how can we, as economists, model the unobserved preference allocation of consumers?

It is useful to model consumer preferences by a ***utility function***, which we define to be a real mapping $u: X \to \mathbb{R}$.
We say that u ***represents the preference relation*** $\preccurlyeq$ if

$$\forall \underline{x}, \underline{x}' \in X: \quad u(\underline{x}') \leq u(\underline{x}) \iff \underline{x}' \precsim \underline{x}$$

- If only the ordering imposed by a utility function is relevant, one speaks of an **ordinal utility**. If u is an ordinal utility, any strictly increasing transformation of u represents the same preferences.

That is, if one is only interested in whether a consumer prefers $\underline{x}$ to $\underline{x}'$ and not by $\underline{\text{how much}}$ the consumer prefers $\underline{x}$ to $\underline{x}'$, then one considers an ordinal utility.

Eg. The preferences $\underline{x} \prec \underline{x}' \prec \underline{x}''$ can be represented by the utility function

$$u(\underline{x}) = 1, \quad u(\underline{x}') = 3, \quad u(\underline{x}'') = 8$$

or by

$$v(\underline{x}) = 2, \quad v(\underline{x}') = 5, \quad v(\underline{x}'') = 10$$

The functions u and v are said to be ordinally equivalent. And if $g(u)$ is a strictly increasing transformation of u, then it, too, will be ordinally equivalent to u. Note that we will only consider ordinal utilities.

- If one wants to compare different utility differences, say $|u(\underline{x}) - u(\underline{x}')|$, i.e., if one is interested in by $\underline{\text{how much}}$ a consumer prefers, say, $\underline{x}$ to $\underline{x}'$,

 one speaks of a **cardinal utility.** Cardinal utilities are in general only preserved by affine and increasing transformations. (eg, $u \mapsto 2u+1$, but not $u \mapsto -2u+1$).
 (so as to preserve order)

Existence of an (ordinal) utility function: (Debreu's Theorem, 1954)

Suppose a consumption set X is imbued with a preference relation that is complete, transitive, continuous and strongly monotonic. Then there exists a continuous utility function $u : X \to \mathbb{R}$ that represents this preference relation.

Note – the assumption of strong monotonicity can be dropped, though the proof is more complex.

Proof:

Outline:

- We will consider bundles of goods that contain the same amount of each good, i.e. 'homogeneous' bundles;
- We will show that if, for every $\underline{x} \in X$, there exists a homogeneous bundle to which the consumer is indifferent, then the level of the homogeneous bundle can be taken as an appropriate utility function, i.e. one that preserves the ordering of $\succcurlyeq$;
- We will then show that such a homogeneous bundle exists and is unique.

Details:

Let $\underline{e} = (1, \dots, 1)$ be a length n vector of 1's. ($\underline{e}$ is a homogeneous bundle of level 1.)

Suppose that for any consumption bundle $\underline{x} \in X$ there exists $u(\underline{x}) \in \mathbb{R}$ such that $u(\underline{x}) . \underline{e} \sim \underline{x}$ (†).

We will now show that $u(\underline{x})$ represents the preference relation $\succcurlyeq$.

Indeed, for any $\underline{x}, \underline{x}' \in X$ with $\underline{x} \neq \underline{x}'$

$$u(\underline{x}) > u(\underline{x}') \Rightarrow u(\underline{x}).\underline{e} > u(\underline{x}').\underline{e} \quad \text{(component-wise)}$$

$\downarrow$ strong monotonicity

$$\Rightarrow u(\underline{x}).\underline{e} \succ u(\underline{x}').\underline{e}$$

$\downarrow$ by transitivity and (†)

$$\Rightarrow \quad \underline{x} \succ \underline{x}' \quad ①$$

Similarly, one can show that $u(\underline{x}) \leq u(\underline{x}') \Rightarrow \underline{x} \precsim \underline{x}'$ ②

It follows from ① and ② that $u(\underline{x}') \leq u(\underline{x}) \Leftrightarrow \underline{x}' \precsim \underline{x}$.

So $u(\underline{x})$ represents the preference relation $\precsim$.

Now, to prove the existence of $u(\underline{x})$, let $\underline{x} \in X \subseteq \mathbb{R}^n_{\geq 0}$.

Define

$$B = \{t \in \mathbb{R} \mid t\underline{e} \succsim \underline{x}\} \text{ and } W = \{t \in \mathbb{R} \mid t\underline{e} \precsim \underline{x}\}.$$

Note that

$$\left(\max_i x_i\right) \cdot \underline{e} \geq \underline{x} \Rightarrow \left(\max_i x_i\right) \cdot \underline{e} \succsim \underline{x}$$

$$\Rightarrow \left(\max_i x_i\right) \in B$$

So B is non-empty.

And $0 \cdot \underline{e} \precsim \underline{x}$, so W is also non-empty.

Also, by the continuity of $\precsim$, B and W are both <u>closed</u>.

Then, since B is non-empty and closed it has upper and lower bounds which are also contained in B. Similarly for W. Set

$$t^* = \inf B \in B \quad (\text{a lower bound of } B)$$

and let

$$t_n = t^* - \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Then $t_n < t^* \Rightarrow t_n \notin B$

$$\Rightarrow t_n \cdot \underline{e} \prec \underline{x}$$

$$\Rightarrow t_n \in W$$

Furthermore, $t_n \to t^*$ as $n \to \infty$, and W is closed, so $t^* \in W$. But then since $t^* \in B$ and $t^* \in W$ then $t^* \underline{e} \sim \underline{x}$, so $u(\underline{x}) = t^*$ exists.

Finally, we can prove the uniqueness of $u(\underline{x})$ as follows. Suppose $\underline{x} \sim u_1(\underline{x})\underline{e}$ and also that $\underline{x} \sim u_2(\underline{x})\underline{e}$.

Then,

$$u_1(\underline{x})\underline{e} \succsim \underline{x} \succsim u_2(\underline{x})\underline{e}$$

by transitivity

$$\Rightarrow \quad u_1(\underline{x})\underline{e} \succsim u_2(\underline{x})\underline{e}$$

by monotonicity, for $a, b \in \mathbb{R}$,

$$a\underline{e} \geq b\underline{e} \Rightarrow a\underline{e} \succsim b\underline{e}$$

and $b\underline{e} > a\underline{e} \Rightarrow b\underline{e} \succ a\underline{e}$

So in fact

$$a\underline{e} \succsim b\underline{e} \iff a\underline{e} \geq b\underline{e}$$
$$\iff a \geq b$$

$$\Rightarrow \quad u_1(\underset{\sim}{x}) \geq u_2(\underset{\sim}{x})$$

But similarly, one can show that $u_2(\underset{\sim}{x}) \geq u_1(\underset{\sim}{x})$

Hence $u_1(\underset{\sim}{x}) = u_2(\underset{\sim}{x})$. //

Proof of the continuity of $u(\underline{x})$ of the preceding theorem — beyond the scope of this course.

Some additional notes:

1) $\underline{x} \prec \underline{x}' \iff \underline{x} \precsim \underline{x}'$ and $\underline{x}' \not\precsim \underline{x}$

$\iff u(\underline{x}) \leq u(\underline{x}')$ and $u(\underline{x}') \nleq u(\underline{x})$

$\iff u(\underline{x}) < u(\underline{x}')$

---

2) 'Non-satiation' means that the consumer is never satisfied in the sense that no matter what bundle of goods they have, there is another very similar bundle that they (strictly) prefer.

Claim: Let $\succsim$ be a strong monotonic preference relation over $\mathbb{R}^n_{\geq 0}$. Then $\succsim$ is locally nonsatiated.

Proof: Fix some $\varepsilon > 0$. Consider arbitrary $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and let $\underline{e} = (1, \dots, 1)$ $(\in \mathbb{R}^n_{\geq 0})$. For any $\lambda > 0$, we also have $\underline{x} + \lambda \underline{e} \in \mathbb{R}^n_{\geq 0}$. Since clearly $\underline{x} + \lambda \underline{e} > \underline{x}$ then $\underline{x} + \lambda \underline{e} \succ \underline{x}$ by (strong) monotonicity. Now consider the following metric:

$$d(\underline{x} + \lambda \underline{e}, \underline{x}) = \|\underline{x} + \lambda \underline{e} - \underline{x}\| = \lambda \|\underline{e}\| = \lambda \sqrt{n}.$$

Then for $\lambda < \frac{\varepsilon}{\sqrt{n}}$, $d(\underline{x} + \lambda \underline{e}, \underline{x}) < \varepsilon$ yet $\underline{x} + \lambda \underline{e} \succ \underline{x}$. //

However, the converse is not always true.

Properties of a utility function

If the underlying preferences are complete, transitive, continuous and (strictly) monotone, the corresponding utility function will be continuous and (strictly) monotone.

Check:

Consider a preference relation with utility function u.

Weak monotonicity $\Rightarrow$

$$\underline{x} \leq \underline{x}' \Rightarrow \underline{x} \precsim \underline{x}' \Rightarrow u(\underline{x}) \leq u(\underline{x}')$$

Strong monotonicity $\Rightarrow$

$$\underline{x} \leq \underline{x}' \text{ and } \underline{x} \neq \underline{x}' \Rightarrow \underline{x} \prec \underline{x}' \Rightarrow u(\underline{x}) < u(\underline{x}')$$

Some sources also discuss the strict monotonicity of preference relations, which they define as:

$$\underline{x} \leq \underline{x}' \Rightarrow \underline{x} \precsim \underline{x}', \text{ while } \underline{x} \ll \underline{x}' \Rightarrow \underline{x} \prec \underline{x}'.$$

If the preferences are (strictly) convex, the utility function is (strictly) quasi-concave.

Recall the definition of convexity for a preference relation:

$$\forall \underline{x}, \underline{x}', \underline{x}'' \in X \text{ with } \underline{x} \precsim \underline{x}' \text{ and } \underline{x} \precsim \underline{x}'',$$

$$\underline{x} \precsim t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in [0,1]. \quad (*)$$

Then, if $\precsim$ is convex and u is its associated utility function, then $\forall \underline{x}', \underline{x}'' \in X$, assuming without loss of generality (by completeness) that $\underline{x}' \precsim \underline{x}''$, then $\forall t \in [0,1]$,

$$\underline{x}' \precsim t\underline{x}' + (1-t)\underline{x}'' \quad (\text{take } \underline{x} = \underline{x}' \text{ in } ⊛)$$

$$\Rightarrow \quad u(t\underline{x}' + (1-t)\underline{x}'') \geq u(\underline{x}') \geq \min\{u(\underline{x}'), u(\underline{x}'')\}$$

$\Rightarrow$ u is quasi-concave

Similarly for strict convexity/quasi-concavity.

Substitution in demand

Suppose the availability of good i drops, such that x_i must decrease. In order to preserve the same level of utility in their overall consumption bundle, consumers will want to compensate by replacing with a separate good. By how much should the consumer alter x_j such that the utility remains constant?

This is analogous to the problem of technical substitution.

Indeed, we define the ***marginal rate of substitution*** (MRS) to be the rate of change of good j with respect to the change in good i:

$$MRS_{i,j}(\underline{x}) = \frac{-\partial u(\underline{x})/\partial x_i}{\partial u(\underline{x})/\partial x_j}$$

where we also define

$$MU_i(\underline{x}) = \frac{\partial u(\underline{x})}{\partial x_i}$$

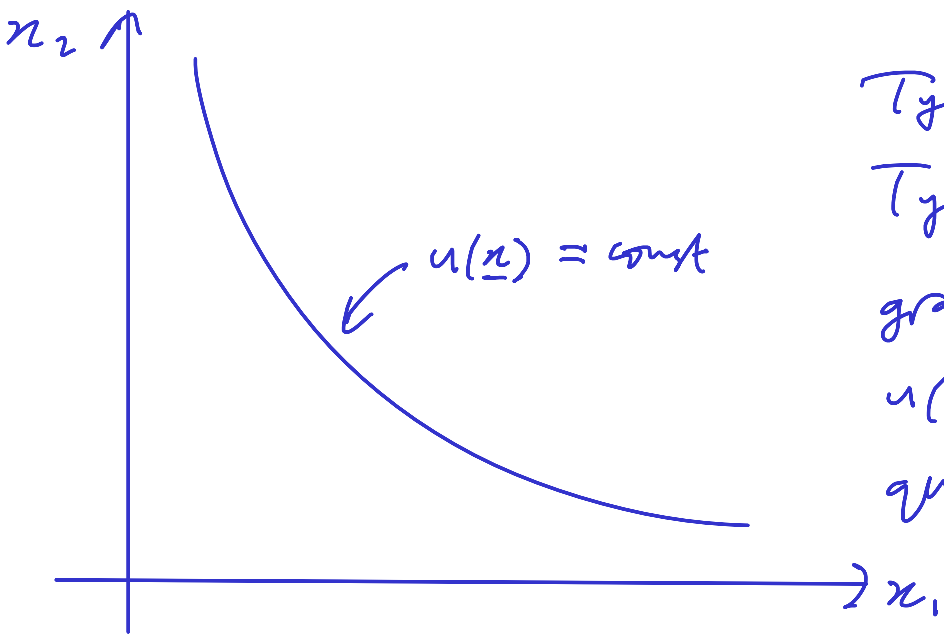
to be the ***marginal utility*** with respect to good i.

The MRS is, of course, the consumer-side analogue to the MRTS (marginal rate of technical substitution). One can check that the $MRS_{i,j}$ is indeed invariant under a strictly monotonic transformation of the utilities.

$$\frac{\partial g(u(\underline{x}))}{\partial x_i} = g'(u(\underline{x})) \cdot \frac{\partial u(\underline{x})}{\partial x_i}$$

So $MRS_{i,j}$ is independent of the choice of utility function.
($g'(u(\underline{x}))$ is cancelled from $(\partial g(u(\underline{x}))/\partial x_1)/((\partial g(u(\underline{x}))/\partial x_2))$.)
Note that g should be a strictly increasing function so that $g(u(\underline{x}))$ is indeed a utility function.

Just as it is often useful to consider a graphical representation of a firm's economic and technological capabilities, it can be useful to graphically represent consumer preferences. As a demand-side analogue to the isoquant, we define the ***indifference curve*** to be a level set of the utility function:



Typical indifference curve. Typically, it has negative gradient (MRS), and $u(\underline{x})$ is monotonic and quasi-concave.

Budget Restraints, Utility Maximisation and Demand

In practice, consumers can't simply pick their most preferred bundle – ∃ budget restraints. A fundamental assumption underlying consumer-side economic analysis is that the consumer will choose to purchase the most preferred consumption bundle from the set of all *affordable* bundles.

Represent the set of all affordable bundles by the **budget set**:

$$B \equiv B_{\underline{p},m} = \{\underline{x} \in X : \underline{p}\,\underline{x}^T \leq m\}$$

where p_i is the price per unit of the i^{th} good that the consumer must pay, and m is their budget.

At the heart of consumer choice, then, is the problem of finding the most preferred bundle $\underline{x} \in B$.

This is the problem of finding

$$\underset{\underline{x} \in B_{\underline{p},m}}{\operatorname{argmax}} \{u(\underline{x})\}$$

A solution to this problem will exist so long as u is continuous and B is closed and bounded...

and provided $p_i > 0 \ \forall i$ (c.f. existence of cost-minimising bundle $\underline{x}(\underline{w}, y)$) – details omitted.

Denote the constrained utility-maximising bundle $\underline{x}^* \in B$:

- $\underline{x}^*$ will be independent under a strictly increasing transformation of utility function.
- $\underline{x}^*$ will, in general, be dependant both on prices $\underline{p}$ and on the budget m.
- $\underline{x}^*$ is homogeneous of degree zero *jointly* in prices and budget.

i.e. $\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \ \forall t > 0$.

(This is to be expected – $\underline{p}$ and m are being multiplied by the <u>same</u> non-zero factor.)

How can we find $\underline{x}^*$?

Kuhn-Tucker conditions – these are necessary conditions on $\underline{x}^*$ to be a solution of this constrained optimisation problem which has <u>inequality</u> constraints. These generalise Lagrange's conditions for the solution of an optimisation problem with equality constraints.

If we make some reasonable regularity assumptions about the consumer's preference ordering $\preccurlyeq$, we can simplify our constrained optimisation problem.

Assume *local nonsatiation* and suppose $\underline{x}^* = \underset{\underline{x} \in B}{\operatorname{argmax}}\, u(\underline{x})$:

- If $\underline{p}\,\underline{x}^{*\top} < m$, that means $\underline{x}^*$ is in the interior of B, then there would exist some $\underline{x}$, close enough to $\underline{x}^*$, such that both $\underline{p}\,\underline{x}^\top < m$ and (by nonsatiation) $\underline{x} \succ \underline{x}^*$.
- This would imply that $\underline{x}^*$ did not maximize $u(x)$, and so we have a contradiction.

$$\Rightarrow \underline{p}\,\underline{x}^{*\top} \nless m$$
$$\Rightarrow \underline{p}\,\underline{x}^{*\top} = m.$$

So under these assumption, $\underline{x}^*$ costs the consumer all of their budget. But it means that we need only seek $\underset{\underline{x} \in \partial B}{\arg\max}\{u(\underline{x})\}$, which allows us to use the Lagrangian approach again.

Some economists call the fact that utilities are maximised only if people spend all their money **Walras' Law.**

Example: Consider the consumer with utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \quad , \quad 0 < \alpha < 1$$

(a Cobb-Douglas utility function. This corresponds to a strongly monotonic and hence locally non-satiated preference:

$$\underline{x} \geq \underline{x}' \text{ and } \underline{x} \neq \underline{x}' \Rightarrow u(\underline{x}) > u(\underline{x}') \Rightarrow \underline{x} \succ \underline{x}')$$

We seek $\underset{\underline{x} \in X}{\arg\max}\, u(\underline{x})$ such that $\underline{p}\,\underline{x}^\top = m$.

Introduce Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1 x_1 + p_2 x_2 - m)$$

F.O.C.: $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow p_1 x_1 + p_2 x_2 = m$ (1)

$\dfrac{\partial \mathcal{L}}{\partial x_i} = 0, \; i = 1, 2$

$\Rightarrow \quad \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1$ (2)

$(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2$ (3)

(2) ÷ (3) $\Rightarrow \quad \dfrac{\alpha}{1-\alpha} \dfrac{x_2}{x_1} = \dfrac{p_1}{p_2}$

$\Rightarrow \quad x_2 = \dfrac{(1-\alpha)}{\alpha} \left(\dfrac{p_1}{p_2}\right) x_1$

Then (1) $\Rightarrow \quad \left(1 + \dfrac{(1-\alpha)}{\alpha}\right) p_1 x_1 = m$

$\Rightarrow \quad x_1^*(p_1, p_2, m) = \alpha \dfrac{m}{p_1}$

$x_2^*(p_1, p_2, m) = (1-\alpha) \dfrac{m}{p_2}$

Notice $\displaystyle MRS = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2}$

$$= -\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}$$

$$= -\frac{\alpha}{1-\alpha}\,\frac{x_2}{x_1}$$

$$= -\frac{p_1}{p_2} \quad \text{at } (x_1^*, x_2^*)$$

gradient of $p_1 x_1 + p_2 x_2 = \text{const.}$

Check:

- $\underline{x}^*$ is independent of a strictly increasing transformation of $u(\underline{x})$, since such a transformation will not change the MRS. In particular, one might consider

$$\hat{u}(\underline{x}) = \log(u(\underline{x})) = \alpha \log x_1 + (1-\alpha)\log x_2$$

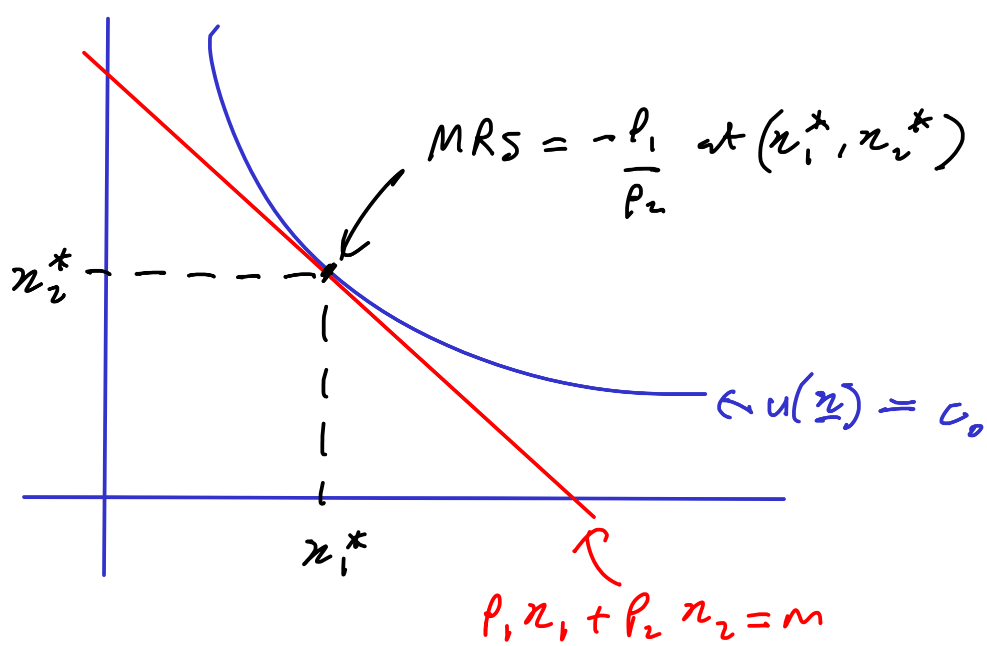
- $\underline{x}^*$ is homogeneous of degree 0 jointly in $\underline{p}$ and m:

$$\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \quad \forall t > 0.$$

Note also that, for $i = 1, 2$, $x_i^*(\underline{p}, m)$ increases as m increases (while p_i is held fixed), and decreases as p_i increases (while m is held fixed), as one might expect.

A note on second-order conditions for the Lagrangian:

Can we derive a condition on $\underline{x}^*(\underline{p}, m)$ to guarantee it gives a maximum of $u(\underline{x})$ rather than a minimum?



Consider the change in $u(\underline{x})$ as $\underline{x}$ moves along the line $p_1 x_1 + p_2 x_2 = m$. Let $\underline{x}(t)$ denote a point on this line, Take $t = x_1$. Now, considering u as a function of the single variable t, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt}$$

$$\Downarrow \quad x_2 = \frac{m - p_1 x_1}{p_2}$$

$$= \left(\frac{\partial}{\partial x_1} - \frac{p_1}{p_2}\frac{\partial}{\partial x_2}\right) u$$

So

$$\frac{d^2 u}{dt^2} = \left(\frac{\partial}{\partial x_1} - \frac{p_1}{p_2}\frac{\partial}{\partial x_2}\right)^2 u$$

$$= \underbrace{\frac{\partial^2 u}{\partial x_1^2} + \frac{p_1^2}{p_2^2}\frac{\partial^2 u}{\partial x_2^2} - 2\frac{p_1}{p_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}}_{= \; (t), \text{ say}}$$

Now recall that

$$L(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m)$$

$$\Rightarrow \quad \frac{\partial^2 L}{\partial \lambda^2} = 0, \quad \frac{\partial L}{\partial \lambda \partial x_i} = -p_i \quad \text{for } i = 1, 2$$

$$\frac{\partial^2 L}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}, \quad \frac{\partial^2 L}{\partial x_1 x_2} = \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

The Hessian matrix of L is therefore

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 \\ -p_2 & \partial^2 u/\partial x_1 \partial x_2 & \partial^2 u/\partial x_2^2 \end{pmatrix} = H_L(\underline{x}, \underline{p}),$$

say.

And the determinant of this is

$$|H_L(\underline{x}, \underline{p})| = -p_1\left(-p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_1 \frac{\partial^2 u}{\partial x_2^2}\right)$$

$$-p_2\left(-p_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_2 \frac{\partial^2 u}{\partial x_1^2}\right)$$

$$= 2p_1 p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - p_1^2 \frac{\partial^2 u}{\partial x_2^2} - p_2^2 \frac{\partial^2 u}{\partial x_1^2}$$

$$= -p_2^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{p_1^2}{p_2^2}\frac{\partial^2 u}{\partial x_2^2} - 2\frac{p_1}{p_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)$$

$$= -p_2^2 \cdot \dagger$$

So a necessary condition for $\underline{x}^*$ to give a maximum of $u(\underline{x})$ (subject to the constraint $\underline{p}\,\underline{x}^T = m$) is that

$$|H_L(\underline{x}, \underline{p})| \geq 0.$$

Now note that with $\hat{\underline{h}} = (p_2, -p_1)$ (note that this is parallel to the line $p_1 x_1 + p_2 x_2 = m$) and

$$(D^2 u)_{i,j} \equiv \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i,j = 1,2$$

we have

$$\hat{\underline{h}} \, D^2 u(\underline{x}) \, \hat{\underline{h}}^T$$

$$= (p_2 \ \ -p_1) \begin{pmatrix} p_2 \dfrac{\partial^2 u}{\partial x_1^2} - p_1 \dfrac{\partial^2 u}{\partial x_1 \partial x_2} \\ p_2 \dfrac{\partial^2 u}{\partial x_1 \partial x_2} - p_1 \dfrac{\partial^2 u}{\partial x_2^2} \end{pmatrix}$$

$$= p_2^2 \frac{\partial^2 u}{\partial x_1^2} - 2 p_1 p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_1^2 \frac{\partial^2 u}{\partial x_2^2}$$

$$= -|H_u(\underline{x}, \underline{p})|$$

So we can rephrase the above as follows. A necessary condition for $\underline{x}^*$ to maximize $u(\underline{x})$ subject to the constraint $\underline{p}\,\underline{x}^* = m$ is that

$$\underline{h} \, D^2 u(\underline{x}^*) \, \underline{h}^T \leq 0$$

for all vectors $\underline{h} \in \mathbb{R}^n$ such that

$$\nabla \underline{u}(\underline{x}^*) \cdot \underline{h}^T = 0 .$$

(i.e., all vectors $\underline{h}$ parallel to $\hat{\underline{h}} = (p_2, -p_1)$; the latter is equivalent to our F.O.C. on $\underline{x}^*$).

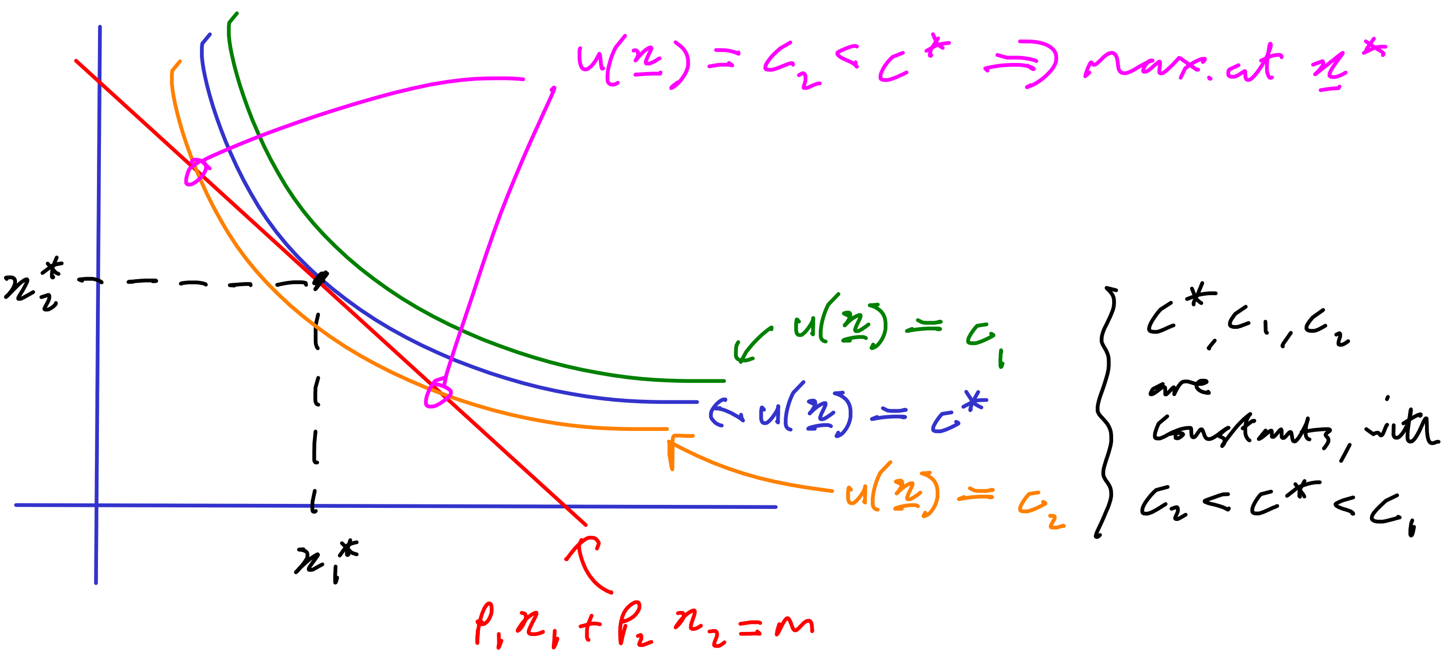
That is, $\nabla^2 \underline{u}(\underline{x}^*)$ should be negative semi-definite with respect to all such $\underline{h}$. A sufficient condition is that $\nabla^2 \underline{u}(\underline{x}^*)$ should be negative definite with respect to all such non-zero $\underline{h}$. (Compare this to the conditions for a maximum of $u(\underline{x})$ subject to <u>no</u> constraints on $\underline{x}$. Recall that a Taylor series expansion of $u(\underline{x})$ about $\underline{x}^*$ gives, for $\underline{x}$ local to $\underline{x}^*$:

$$u(\underline{x}) - u(\underline{x}^*) = \frac{1}{2}(\underline{x} - \underline{x}^*) \nabla^2 u(\underline{x}^*)(\underline{x} - \underline{x}^*)^T .)$$

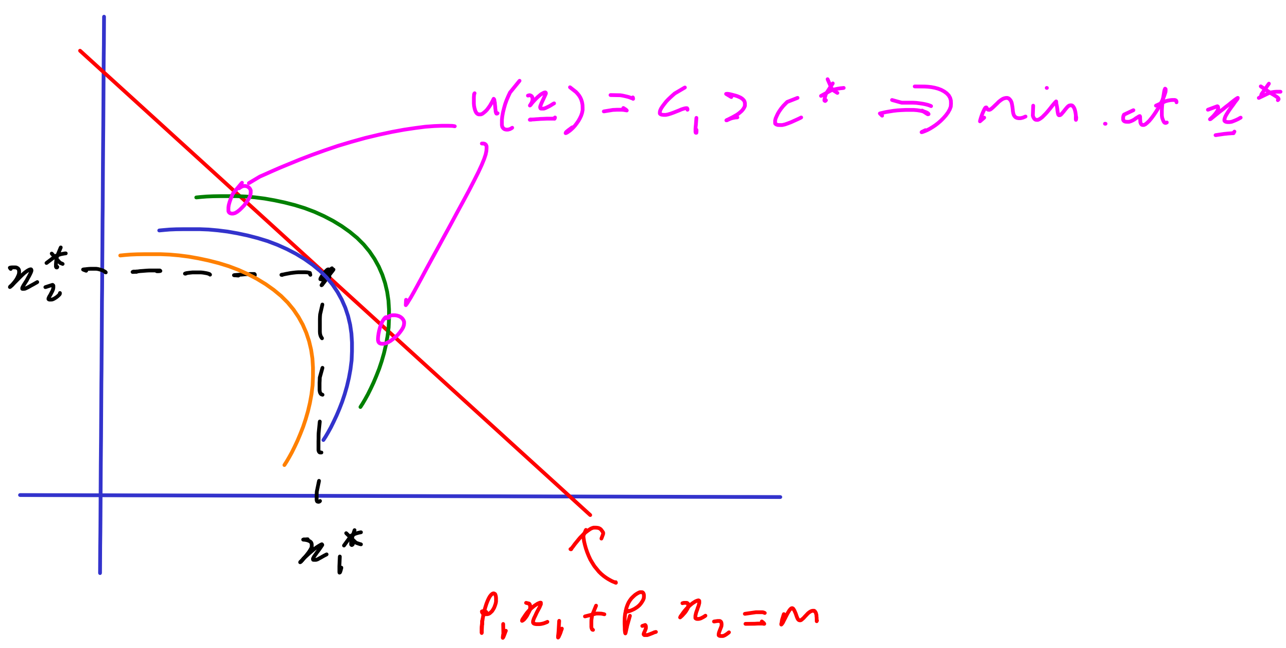
Note that $H_L(\underline{x}, \underline{p})$ is referred to as the <u>bordered</u> Hessian (here the Hessian refers to $\nabla^2 u(\underline{x})$).

For $u(\underline{x})$ monotonic (as is usually the case) the above condition on $\underline{x}^*$ is equivalent to $u(\underline{x})$ being quasi-concave at least local to $\underline{x}^*$:

i.e.,



rather than



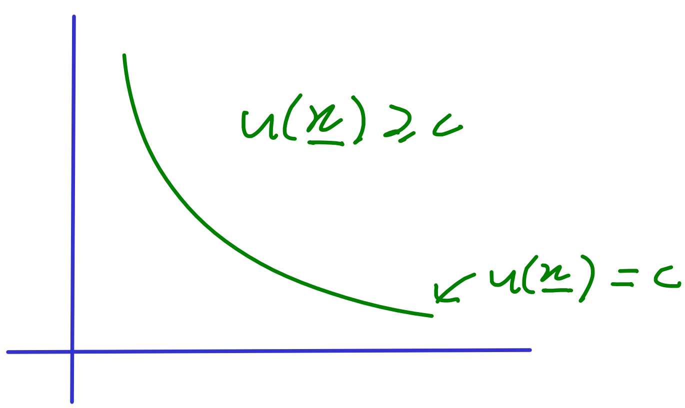
Details omitted here.

The choice of the consumption bundle that maximises the consumer's constrained utility function will be exactly the bundle that the consumer demands; this is unsurprisingly referred to as the ***demanded bundle*** or ***demand function***,

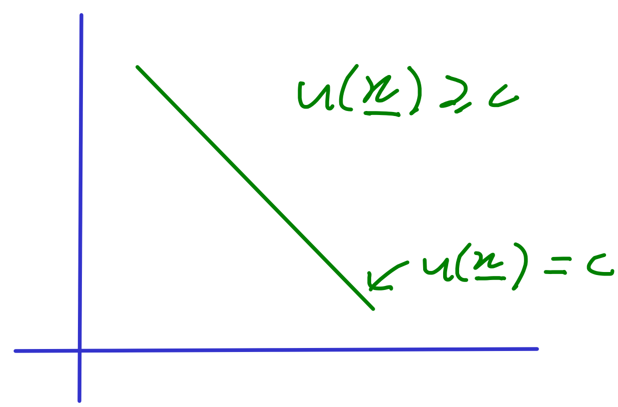
$$\underline{x}^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \qquad \underline{x}^*\left(\underline{p}, m\right) = \underset{\underline{x} \in \partial B}{\operatorname{argmax}}\, u\left(\underline{x}\right).$$

We call this the Marshallian demand function (or, uncompensated demand function).

Note that we have discussed the existence of the argmax. However, it is *per se* not clear whether the argmax is unique, that means, whether the maximum is attained at a single point over ∂B. This can be guaranteed if the underlying preferences are strictly convex (and prices are strictly positive), i.e. if $u(\underline{x})$ is strictly quasi-concave, or equivalently, the sets $\{\underline{x} \mid u(\underline{x}) \geq c\}$ for constant c, are strictly convex (so their boundaries do not contain any straight sections). Eg.



not



Moreover, the function $\underline{x}^*(\underline{p}, m)$ is homogeneous of degree 0 in $(\underline{p}, m)$.

ie. $\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \quad \forall t > 0.$

We also note that, faced with a set of goods with prices $\underline{p}$, the maximum utility achievable with a given budget m is known as the ***indirect utility function***:

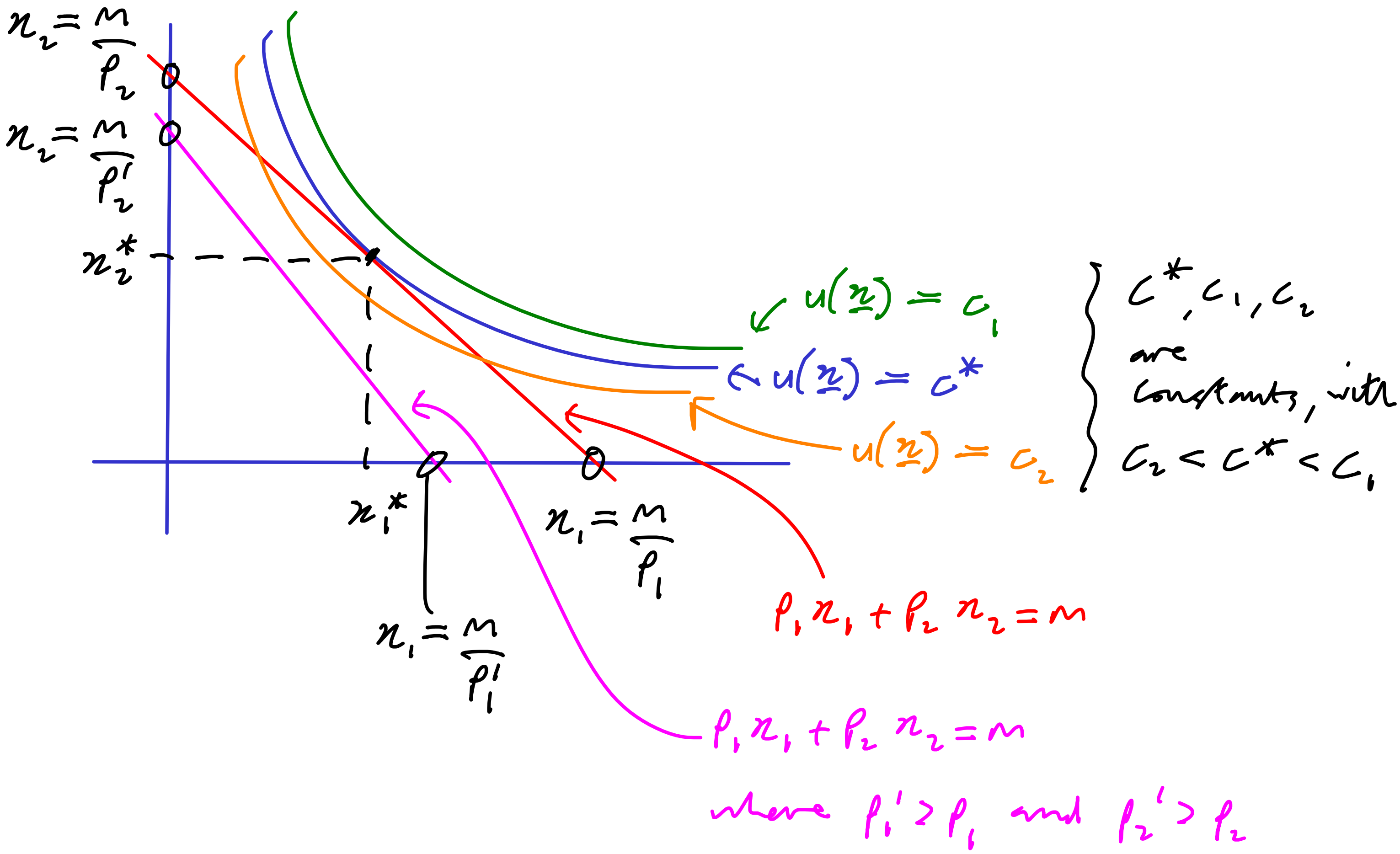
$$v : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad v\left(\underline{p}, m\right) = u\left(\underline{x}^*\left(\underline{p}, m\right)\right)$$

This indirect utility function is itself a quantity of interest, and we note some of its key properties here:

- Nonincreasing in $\underline{p}$:

$$\underline{p}' \geq \underline{p} \Rightarrow v\left(\underline{p}', m\right) \leq v\left(\underline{p}, m\right)$$

We won't provide a rigorous proof, but observe the following:



Recall that $\underline{p}(\underline{x}^*)^T \leq m$ (or $= m$ if assuming local non-satiation), i.e., $\underline{x}^* \in B_{\underline{p},m}$. As $\underline{p}$ increases, so $B_{\underline{p},m}$ recedes, i.e., for $\underline{p}' \geq \underline{p}$, $B_{\underline{p}',m} \subset B_{\underline{p},m}$. So $u(\underline{x}^*) \leq u(\underline{x})$. And intuitively, as $\underline{p}$ increases, (while m remains fixed) one would expect $\underline{x}^*$ and hence $u(\underline{x}^*)$ to not increase.

...and nondecreasing in m:

$$m' \geq m \Rightarrow v\left(\underline{p}, m'\right) \geq v\left(\underline{p}, m\right)$$

– this follows by arguments similar (but converse) to those used above.

- Homogeneous of degree 0 in $\left(\underline{p}, m\right)$:

$$v\left(t\underline{p}, tm\right) = v\left(\underline{p}, m\right) \quad \forall t > 0$$

(Makes sense intuitively – e.g. one would expect $\underline{x}^*$ and hence $u(\underline{x}^*)$ to be independent of a change in currency.)

- Quasi-convex in $\underline{p}$:

$$\left\{\underline{p} \in \mathbb{R}^n_{\geq 0} : v\left(\underline{p}, m\right) \leq k\right\} \text{ is a convex set for all } k, m \in \mathbb{R}.$$

i.e., $\forall t \in [0,1]$,

$$v(t\underline{p} + (1-t)\underline{p}', m) \leq \max\{v(\underline{p}, m), v(\underline{p}', m)\}$$

(or equivalently, if $v(\underline{p}, m) \leq k$ and $v(\underline{p}', m) \leq k$, then

$$v(t\underline{p} + (1-t)\underline{p}', m) \leq k.)$$

E.g., Consider sketch above: with $\underline{p}' \geq \underline{p}$, so

$\underline{p}' \geq t\underline{p} + (1-t)\underline{p}' \geq \underline{p}$.

- Continuous at all $\underline{p} \gg 0, m > 0$.

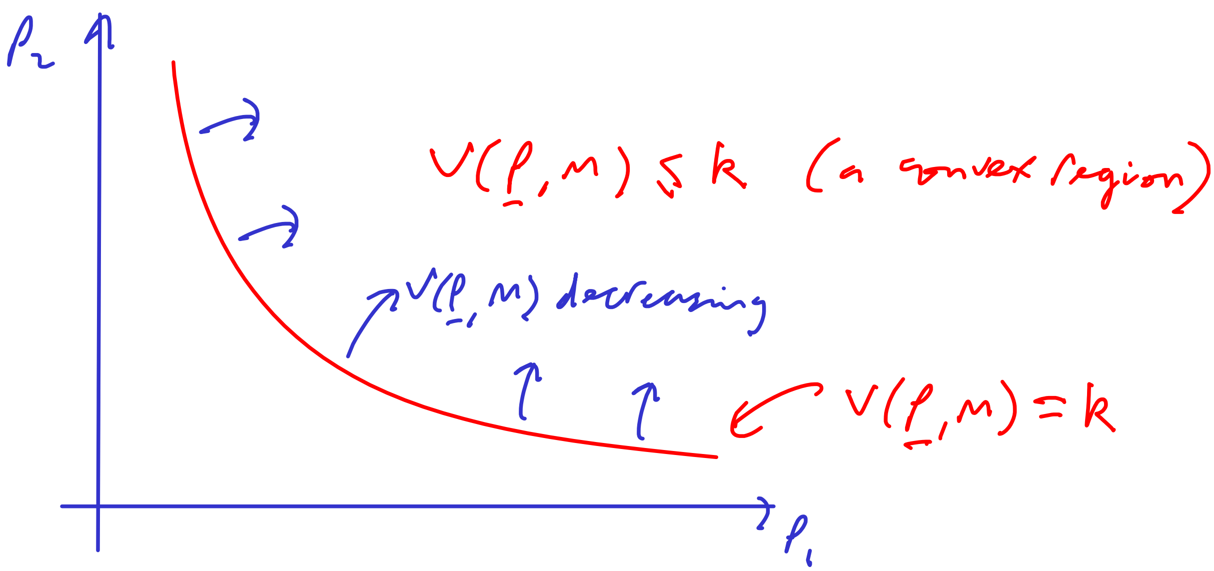
($\underline{p} \gg 0$ means $p_i > 0 \;\forall i = 1, \dots, n$.)

– $\underline{x}^*$ and hence $v(\underline{p}, m)$ might not exist unless $\underline{p} \gg 0$. E.g. consider above sketch with $p_1 = 0$.

The indirect utility function v is often illustrated using so-called price indifference curves. These are the level sets of the indirect utility function with a *fixed budget* m. That is

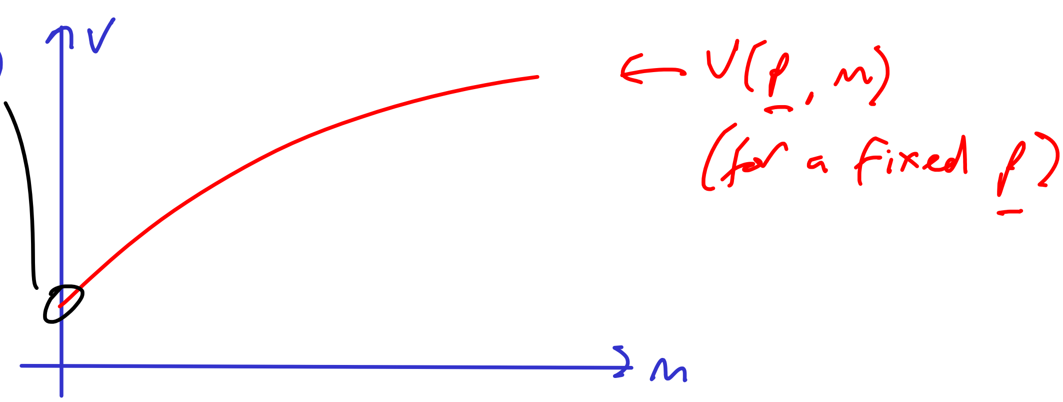
$$\{\underline{p} \in \mathbb{R}^n_{\geq 0} : v(\underline{p}, m) = k\}, \; k \in \mathbb{R}, \; m \geq 0.$$

They are analogous to the indifference curves of the utility function:



A direct consequence of the local nonsatiation assumption of the underlying preferences is that for fixed $\underline{p}$, the indirect utility function $v(\underline{p}, \cdot)$ is strictly increasing in m:

Note that $\underline{v}(\underline{p}, 0)$ is not necessarily 0 ($u(\underline{x})$ is an ordinal utility function).



Therefore, $v(\underline{p}, \cdot)$ is injective and can be inverted on its image. Denote this image with

$$U_p = \{v(\underline{p}, m) : m \geq 0\}.$$

Then we define the ***expenditure function*** $e(\underline{p}, u) : U_p \to [0, \infty)$,

$$u \mapsto e(\underline{p}, u) \text{ s.t. } u = v(\underline{p}, e(\underline{p}, u))$$

(Note: here u denotes a fixed level of utility; not to be confused with $u(\underline{x})$.)

The expenditure function provides the minimum level of income required to obtain a given level of utility at prices $\underline{p}$. Note that $e(\underline{p}, u)$ can also be obtained as the solution to the optimisation problem

$$\text{Find } \min_{\underline{x}} \underline{p}\,\underline{x}^T \text{ s.t. } u(\underline{x}) \geq u.$$

(Note: If one could obtain a level of utility u with a level of income, say, m' with $m' < e(\underline{p}, u)$ we would have

$$u(\underline{x}') = u = v(\underline{p}, e(\underline{p}, u)) = u(\underline{x}^*(\underline{p}, e(\underline{p}, u)))$$

for some $\underline{x}'$ with $\underline{p}\,\underline{x}'^T \leq m' < e(\underline{p}, u)$, but then, due to the local non-satiation assumption, there would exist some $\underline{x}$ with both $\underline{p}\,\underline{x}^T < e(\underline{p}, u)$ and $u(\underline{x}) > u(\underline{x}^*(\underline{p}, e(\underline{p}, u)))$ (c.f., a similar argument used earlier), which is a contradiction.)

$e(\underline{p}, u)$ is simply the demand-side analogue of a firm's cost function $c^*(\underline{w}, y)$ - recall that this is given by

$$\min_{\underline{x}} \underline{w}\,\underline{x}^T \quad \text{s.t.} \quad f(\underline{x}) \geq y.$$

They share some similar properties: (provided it exists) $e(\underline{p}, u)$ is:

- non-decreasing in $\underline{p}$
- homogeneous of degree 1 in $\underline{p}$
- concave in $\underline{p}$
- continuous in $\underline{p}$

The dual quantity to the expenditure function is the ***Hicksian demand*** (sometimes referred to as the *compensated demand*)

$$\underline{x}_H^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R}^n_{\geq 0}, \qquad (\underline{p}, u) \mapsto \underline{x}_H^*(\underline{p}, u) = \operatorname*{argmin}_{\substack{\underline{x} \text{ s.t.} \\ u(\underline{x}) \geq u}} \underline{p}\,\underline{x}^T$$

This is the analogue of the conditional factor demand function $\underline{x}^*(\underline{w}, y)$.

Recall that, on the firm side, for a specified level of output, the cost-minimising combination of production inputs can be found via Shephard's Lemma. We can also apply this result in the current scenario, yielding an expression for the expenditure-minimising consumption bundle in terms of prices and desired utility level:

$$x_{H,i}^*\left(\underline{p},u\right)=\frac{\partial e\left(\underline{p},u\right)}{\partial p_i}.$$

Recall that the Marshallian demand function is defined by

$$\underline{x}^*(\underline{p},m) = \underset{\underline{x} \text{ s.t. } \underline{p}\,\underline{x}^T = m}{\operatorname{argmax}} u(\underline{x})$$

assuming local non-satiation

Note that, when we refer to the demand function without qualification, it is assumed to be the Marshallian demand.

Unlike the Marshallian demand, the Hicksian demand function is not observable; indeed, it depends on utility, which is itself unobservable. Nonetheless, under some of the **usual regularity assumptions,** the Hicksian and Marshallian demands satisfy the following identities: for all $\underline{p} >> 0$, $m > 0$

- $e\left(\underline{p},v\left(\underline{p},m\right)\right)= m$
- $v\left(\underline{p},e\left(\underline{p},u\right)\right)\equiv u$
- $x_{H,i}^*\left(\underline{p},v\left(\underline{p},m\right)\right)= x_i^*(\underline{p},m)$
- $x_i^*\left(\underline{p},e\left(\underline{p},u\right)\right)= x^*_{H,i}(\underline{p},u)$

The Slutsky equation

For economists, it is important to understand how consumers react to changes in the economic environment. For instance, we can consider how the optimal choice of consumption bundle $\underline{x}^*(\underline{p}, m)$ will change with respect to the price vector $\underline{p}$. *The* ***Slutsky equation*** states that the total effect of a change in demand of a good when the price of a good is changed can be decomposed into a ***substitution effect*** and an ***income effect.***

- the substitution effect is the change whilst keeping the level of utility fixed.
- the income effect is the change due to the consumer's increase/decrease in purchasing power.

The Slutsky equation:

Under certain regularity conditions (details omitted), for $\underline{p} \gg 0$ and $m > 0$ and $\forall i,j \in \{1,\dots,n\}$,

$$\boxed{\begin{aligned}&\partial_i x_j^*(\underline{p}, m) = \\ &= \partial_i x_{H,j}^*(\underline{p}, V(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_i^*(\underline{p}, m)\end{aligned}}$$

Note! Here, $\partial_i x_{H,j}^*(\underline{p}, V(\underline{p}, m))$ denotes

$$\left(\frac{\partial x_{H,j}^*(\underline{p}, u)}{\partial p_i}\right)\Bigg|_{u = V(\underline{p}, m)}$$

Proof:

For any $u \in U_p$, $x_{H,j}^*(\underline{p}, u) = x_j^*(\underline{p}, e(\underline{p}, u))$

$$\Rightarrow \quad \frac{\partial}{\partial p_i} x^*_{H,j}(\underline{p}, u) = \frac{\partial}{\partial p_i} x^*_j(\underline{p}, e(\underline{p}, u))$$

$$= \left(\frac{\partial}{\partial p_i} x^*_j(\underline{p}, m)\right)\Bigg|_{m = e(\underline{p}, u)}$$

$$+ \left(\left(\frac{\partial}{\partial m} x^*_j(\underline{p}, m)\right)\Bigg|_{m = e(\underline{p}, u)}\right) \cdot \underbrace{\frac{\partial e(\underline{p}, u)}{\partial p_i}}_{= x^*_{H,i}(\underline{p}, u)}$$

by Shephard's Lemma

Now set $u = v(\underline{p}, m)$. Note that

$$e(\underline{p}, v(\underline{p}, m)) = m.$$

So we get:

$$\left(\frac{\partial}{\partial p_i} x^*_{H,j}(\underline{p}, u)\right)\Bigg|_{u = v(\underline{p}, m)}$$

$$= \frac{\partial}{\partial p_i} x^*_j(\underline{p}, m) + \left(\frac{\partial}{\partial m} x^*_j(\underline{p}, m)\right) \cdot \overbrace{x^*_{H,i}(\underline{p}, v(\underline{p}, m))}^{= x^*_i(\underline{p}, m)}$$

which rearranges to give the result. //

$$\partial_i x_j^*(\underline{p}, m) =$$

$$= \partial_i x_{H,j}^*(\underline{p}, v(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_i^*(\underline{p}, m)$$

Corresponds to the substitution effect

Corresponds to the income effect

It is clear from the Slutsky equation that the income effect plays a major part in determining how the demand for a set of goods will react to changes in their prices. For firms, consumers and economists alike, then, it is important to ascertain how the demand of certain goods will react to changes in consumer budget.

Indeed, economists class goods according to the manner in which they react to changes in consumer income:

- For ***normal goods***, an increase in income will result in an increase in demand;

$$\frac{\partial}{\partial m} x_j^*(\underline{p}, m) \geq 0 .$$

- For ***inferior goods***, an increase in income will result in a decrease in demand.

$$\frac{\partial}{\partial m} x_j^*(\underline{p}, m) < 0 .$$

It is also worth noting the different subclasses of normal goods: suppose we have an increase in consumer income m...

- ...for ***luxury goods***, demand will increase more than proportionally to income;

i.e., $$\underbrace{\frac{\partial x_j^*(\underline{p}, m)}{\partial m} \cdot \frac{m}{x_j^*(\underline{p}, m)}}_{\text{Income elasticity of demand (IED)}} > 1$$

(roughly speaking, $\frac{\delta x_j^*}{x_j^*} > \frac{\delta m}{m}$)

Consumers are likely to already be buying non-luxury goods, so any increase in their income is likely to result in a greater (relative) increase in their demand for luxury goods than for non-luxury goods.

- ...for ***necessary goods***, demand will increase less than proportionally $(0 \leq IED < 1)$
- ...and if demand increases proportionally to income, the consumer is said to have ***homothetic preferences*** for the set of goods under consideration. $(IED = 1)$

Finally, we note that goods can also be classified according to how changes in price impact their consumer demand:

- For ***ordinary goods***, a decrease in price will lead to an increase in their demand;

  i.e., $\frac{\partial x_j^*(\underline{p}, m)}{\partial p_j} \leq 0$

- For ***Giffen goods***, a decrease in price will lead to a decrease in demand

  i.e., $\frac{\partial x_j^*(\underline{p}, m)}{\partial p_j} > 0$

That means our previously stated law of demand only holds for ordinary goods, but not for Giffen goods.

What is an example for a Giffen good? Some theoretical considerations can help us finding necessary conditions for Giffen goods. In particular, the Slutsky equation helps to establish a relation between ordinary and normal goods on the one hand side, as well as Giffen and inferior goods on the other side.

Recall that from Slutsky's Equation with $i = j$:

$$\partial_j x_j^*(\underline{p}, m) =$$

$$= \partial_j x_{H,j}^*(\underline{p}, V(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_j^*(\underline{p}, m)$$

But $$x^*_{H,j}(\underline{p}, V(\underline{p}, m)) = \partial_j e(\underline{p}, V(\underline{p}, m))$$

$$\Rightarrow \quad \partial_j x^*_{H,j}(\underline{p}, V(\underline{p}, m)) = \partial_j^2 e(\underline{p}, V(\underline{p}, m))$$

Recall now that $e(\underline{p}, u)$ is concave in $\underline{p}$.

Aside: Let $f(\underline{x})$ be a function defined on a convex subset U of $\mathbb{R}^n$. If f is concave, then its restriction to every line segment in U is a concave function of a single variable.

Proof: For $\underline{x}$ and $\underline{x}'$ in U, define

$$g(t) = f(t\underline{x} + (1-t)\underline{x}') \quad \forall t \in [0,1]$$

i.e., $g(t)$ is the restriction of f to the line segment that joins $\underline{x}$ and $\underline{x}'$ (note that this is wholly contained in U as U is convex).

Then, for t, t' and s in $[0,1]$,

$$g(st + (1-s)t')$$

$$= f((st + (1-s)t')\underline{x} + (1-(st + (1-s)t'))\underline{x}')$$

$$= f\left(st\underline{x} + (1-s)t'\underline{x} + \underline{x}' - st\underline{x}' - (1-s)t'\underline{x}'\right)$$

$$= f\left(st(\underline{x}-\underline{x}') + (1-s)t'(\underline{x}-\underline{x}') + \underline{x}'\right)$$

$$= f\left(s\left(t\underline{x} + (1-t)\underline{x}'\right) - s\underline{x}' + \right.$$

$$\left. + (1-s)t'(\underline{x}-\underline{x}') + \underline{x}'\right)$$

$$= f\left(s\left(t\underline{x} + (1-t)\underline{x}'\right) + (1-s)\left(t'\underline{x} + (1-t')\underline{x}'\right)\right)$$

$\downarrow$ f is concave means

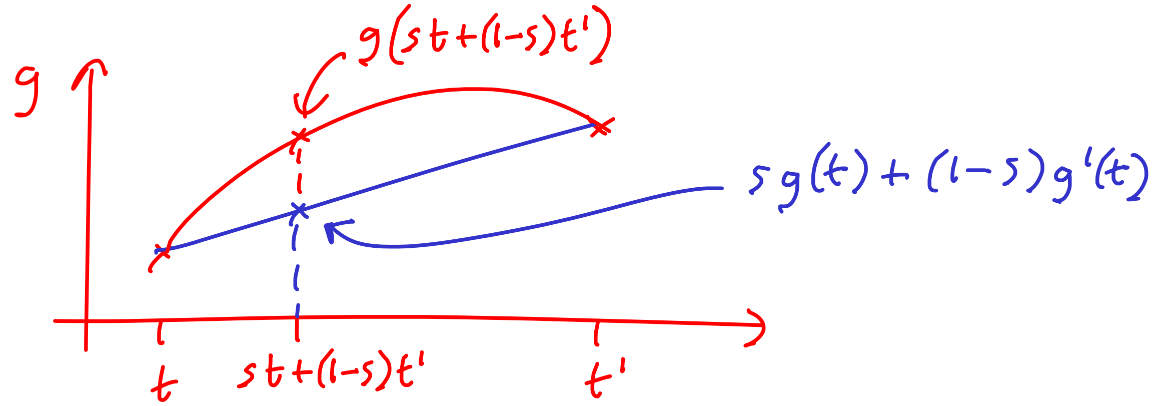
$$f(s\underline{w} + (1-s)\underline{w}') \geq sf(\underline{w}) + (1-s)f(\underline{w}') \quad \forall s \in [0,1]$$

$$\geq sf\left(t\underline{x} + (1-t)\underline{x}'\right) + (1-s)f\left(t'\underline{x} + (1-t')\underline{x}'\right)$$

$$\geq s\,g(t) + (1-s)\,g(t').$$

$\Rightarrow$ g is concave in t. //

Ex



So $g''(t) \leq 0 \quad \forall t \in [0,1]$.

$\Rightarrow$ $e(\underline{p}, u)$ is a concave function of p_j if all other p_i's and u are fixed

$\Rightarrow \quad \partial_j^2 e(\underline{p}, u) \leq 0$

$\Rightarrow \quad \partial_j x^*_{H,j}(\underline{p}, V(\underline{p}, m)) \leq 0$

So, recalling

$$\partial_j x_j^*(\underline{p}, m) =$$

$$= \partial_j x^*_{H,j}(\underline{p}, V(\underline{p}, m)) - \left(\partial_{n+1} x_j^*(\underline{p}, m)\right) x_j^*(\underline{p}, m)$$

and the fact that $x_j^*(\underline{p}, m) \geq 0$, one may deduce that

$$\frac{\partial x_j^*(\underline{p}, m)}{\partial m} \geq 0 \quad \Rightarrow \quad \frac{\partial x_j^*(\underline{p}, m)}{\partial p_j} \leq 0$$

So a normal good is always an ordinary good.

On the other hand

$$\frac{\partial x_j^*(\underline{p},m)}{\partial p_j} > 0 \Rightarrow \frac{\partial x_j^*(\underline{p},m)}{\partial m} < 0$$

So a Giffen good is always an inferior good.

Example of Giffen goods: inferior quality staple foods. If a consumer bases most – but not all of – their diet on these, supplemented by smaller quantities of some better quality foodstuffs, then if the prices of these staple foods rises, the consumer may rely on them so much that they must forgo the better quality foods and replace these with <u>more</u> of the (now more expensive) staple foods. (Note, this assumes that there are no cheaper substitute goods.)

Part 2 – Markets and Competition

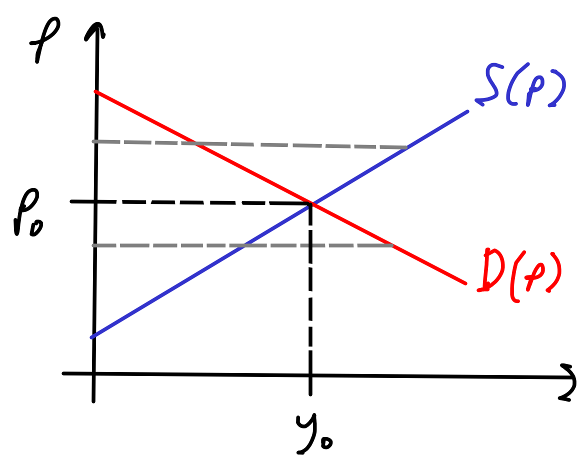
Markets – Demand, Supply, and Equilibrium

We define a market for a good or service to simply be the union of the individuals and firms that operate on both the supply and demand sides of a potential transaction. There are many different types of market that we should be aware of, some of which form rich areas of study themselves. We will continue to focus on markets for goods and services, however notable other markets include:

- stock market,
- labour market,
- capital market

The intentions and wishes of each side of a market are, of course, specified through the demand and supply curves, and we recall from the start of the course that (for a competitive market) the prices at which goods are sold is settled through the price mechanism.

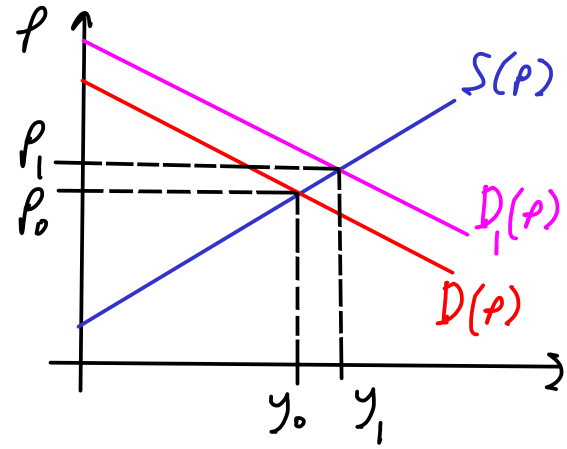
Recall that the price at which supply equals demand is referred to as the **equilibrium price**. Suppose that a market is settled at an equilibrium price p_0:



If $p < p_0$, then $D(p) > S(p)$ so p increases. Conversely, if $p > p_0$, then $D(p) < S(p)$ so p decreases

- If the demand for a good changes for some reason, then the demand curve will shift; demand and supply will no longer be equal at p_0.

- An increase in demand, or a decrease in supply, will lead to an ***excess in demand***.
- A decrease in demand, or an increase in supply, will lead to an ***excess in supply***.
- These excesses invoke the price mechanism, changing the equilibrium price. The speed at which this happens will vary between markets.



Excesses in demand or supply can be measured as long as we can express supply and demand in terms of the good's price.

- For an individual consumer, demand is measurable through the Marshallian demand function, under the assumption that they are maximising their utility. i.e., $x^*(\underline{p}, m)$, where $\underline{p}$ gives the prices of the goods and m is the consumer's budget.

- For an individual firm, we have the (short-run or long-run) supply curve (under profit-maximising assumptions), denoted by $y^*(\underline{p})$.

We want the **market demand** and **market supply** (also referred to as the industry demand and industry supply) i.e., the total demand for a good across the market and the total quantity of the good supplied. These are simply the sums of the individual demands/quantities, respectively.

Suppose a market for a single good contains I consumers and J firms. Further, suppose that consumer i has demand given by

$$x_i^*(\underline{p}, m_i) \quad , i = 1, \dots, I$$

and that the supply curve for firm j is specified by

$$y_j^*(\underline{p}) \quad , \quad j = 1, \dots, J$$

The market demand for the good is then defined as

$$X^*(\underline{p}, \underline{m}) = \sum_{i=1}^{I} x_i^*(\underline{p}, m_i)$$

and the corresponding market supply is defined as

$$Y^*(\underline{p}) = \sum_{j=1}^{J} y_j^*(\underline{p})$$

At equilibrium, $X^*(\underline{p}, \underline{m}) = Y^*(\underline{p})$

Example:

Suppose that the market for bananas contains 1000 utility-maximising consumers with demand functions

$$x_i^*(\underline{p}, m_i) = m_i \frac{p_B}{p_B^2 + p_A^2} \quad , \quad i = 1, \dots, 1000$$

which is dependent on the price p_B of bananas as well as the price p_A of apples.

Further, suppose the banana market comprises two suppliers, with supply curves

$$y_j^*(\underline{p}) = \frac{p_B}{2j} \quad , j = 1, 2.$$

What is the equilibrium price for bananas?

$$X^*(\underline{p},\underline{m}) = \sum_{i=1}^{1000} x_i^*(\underline{p},m_i)$$

$$= M \frac{p_B}{p_B^2 + p_A^2} \quad \text{where } M = \sum_{i=1}^{1000} m_i$$

and $$Y^*(\underline{p}) = y_1^*(\underline{p}) + y_2^*(\underline{p}) = \frac{p_B}{2} + \frac{p_B}{4} = \frac{3p_B}{4}$$

Then the equilibrium price of bananas is the value of p_B such that

$$X^*(\underline{p},\underline{m}) = Y^*(\underline{p})$$

i.e.,

$$M \frac{p_B}{p_B^2 + p_A^2} = \frac{3p_B}{4}$$

$$\Rightarrow \quad p_B = \sqrt{\frac{4M}{3} - p_A^2}$$

Consumers' and Producers' Surplus – Social Welfare

To analyse the consequences of a change in prices or income – or more generally, a change in policy – it is useful to have a measure of social welfare. We will see that a handy such measure is the sum of ***consumers'*** and ***producers' surplus.*** It also gives rise to another characterization of the equilibrium price and equilibrium quantity, maximising this social welfare measure.

We put ourselves into the general framework of utility maximising consumers and profit maximising firms where the utility and production functions satisfy our usual assumptions. Suppose we have J firms with cost functions $c_j^*(\cdot)$, $j \in \{1, \dots, J\}$, and I consumers with respective utility functions $u_i(\cdot)$, $i \in \{1, \dots, I\}$, and corresponding quantities (i.e. indirect utility function v_i, expenditure function e_i, Marshallian demand x_i^*, and Hicksian demand $x_{H,i}^*$ as well as profit-maximising output y_j^*)

Consider fixed income levels $m_1, \dots, m_I$ and a price change from $\underline{p}^{(1)} \in \mathbb{R}^n_{\geq 0}$ to $\underline{p}^{(2)} \in \mathbb{R}^n_{\geq 0}$. Suppose that the price change affects only one single product and that w.l.o.g. the product gets more expensive. To save notation, we will only explicitly denote the variable with a price change, suppressing all the other ones. So in the specific good, we will consider a price change from $p^{(1)} > 0$ to $p^{(2)} > 0$ where we assume without loss of generality that $p^{(1)} < p^{(2)}$.

With $\Pi_j^*(p) = p\, y_j^*(p) - c_j^*(y_j^*(p))$

(Note, we are effectively considering the case of a single product – just the one whose price is changing. Those products whose prices don't change will <u>not</u> affect $d\Pi^*/dp$ – see below. Recall that here, the subscript j indicates the firm.)

we have

$$\sum_{j=1}^{J} \left(\Pi_j^*(p^{(2)}) - \Pi_j^*(p^{(1)}) \right) = \sum_{j=1}^{J} \int_{p^{(1)}}^{p^{(2)}} \frac{d\Pi_j^*(p)}{dp}\, dp$$

where

$$\frac{d\Pi_j^*(p)}{dp} = y_j^*(p) + p\, y_j^{*\prime}(p) - c_j^{*\prime}(y_j^*(p)) \cdot y_j^{*\prime}(p)$$

$$= y_j^*(p) + y_j^{*\prime}(p)\left(p - c_j^{*\prime}(y_j^*(p))\right)$$

$$\left.\frac{d\left(p y_j - c_j^*(y_j)\right)}{d y_j}\right|_{y_j = y_j^*(p)} = 0$$

$$= y_j^*(p)$$

Thus,

$$\sum_{j=1}^{J}\left(\pi_j^*(p^{(2)}) - \pi_j^*(p^{(1)})\right) = \sum_{j=1}^{J}\int_{p^{(1)}}^{p^{(2)}} y_j^*(p)\,dp$$

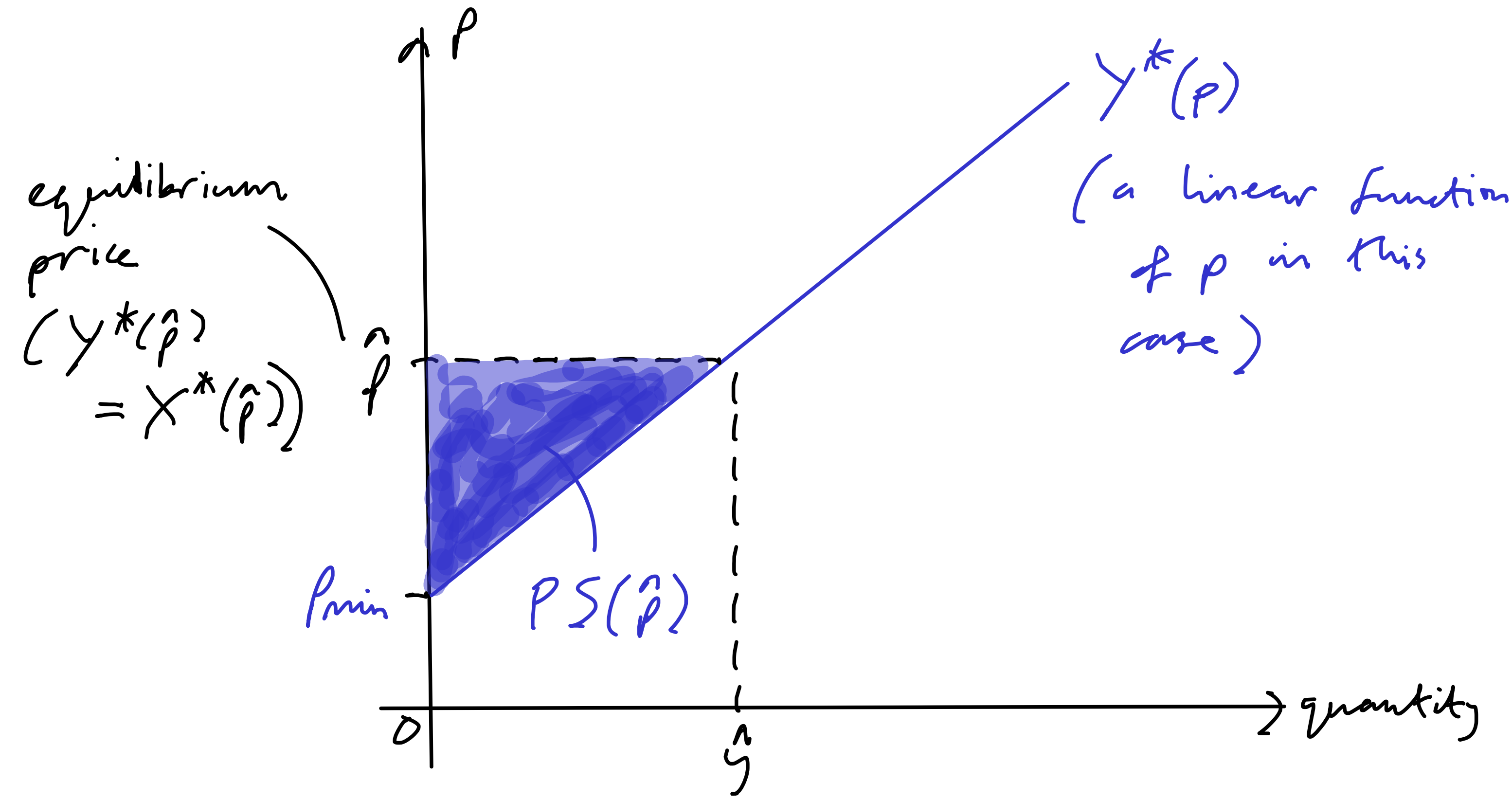
$$= \int_{p^{(1)}}^{p^{(2)}} Y^*(p)\,dp,$$

i.e., the area bounded by the market supply curve $Y^*(p)$ for $p^{(1)} < p < p^{(2)}$

Consequently, we introduce the ***producers' surplus at price*** $\hat{\mathbf{p}}$ as one part of the measure for social welfare measure:

$$PS(\hat{p}) = \int_0^{\hat{p}} Y^*(p)\,dp$$

This is the additional (aggregated) profit made by the producers by selling the product at the (equilibrium) market price $\hat{p}$ rather than at the minimum price p_{min}, say, at which they would be prepared to sell it. E.g.,



Note:

$$PS(\hat{p}) = \int_0^{\hat{p}} y^*(p)\,dp = \int_{p_{min}}^{\hat{p}} y^*(p)\,dp$$

The consumer side is a bit trickier. Following the utility maximisation rationale of the lecture, each individual consumer cares about the difference in their individual indirect utility, that is

$$V_i(p^{(2)}, m_i) - V_i(p^{(1)}, m_i)$$

However, this approach is problematic since

- it doesn't make sense to aggregate <u>ordinal</u> utilities,
- we would like to compare the effect on consumers to that on producers.

$\longrightarrow$ We instead seek a monetary measure.

A natural possibility is to consider the difference of the individual expenditure functions, keeping the initial (indirect) utility fixed. This quantity is known as ***compensating variation*** :

$$CV_i(p^{(1)}, p^{(2)}, m_i)$$

$$= e_i(p^{(2)}, V_i(p^{(1)}, m_i)) - \underbrace{e_i(p^{(1)}, V_i(p^{(1)}, m_i))}_{= m_i}$$

$$= \int_{p^{(1)}}^{p^{(2)}} \frac{d}{dp} e_i(p, V_i(p^{(1)}, m_i))\, dp$$

$$= \int_{p^{(1)}}^{p^{(2)}} \frac{d}{dp} e_i(p, V_i(p^{(1)}, m_i))\, dp$$

by Shephard's Lemma

$$= \int_{p^{(1)}}^{p^{(2)}} x^{*}_{H,i}\left(p, v_i(p^{(1)}, m_i)\right) dp$$

Note that here, this subscript i indicates the consumer rather than a product.

But this is problematic for the reason that Hicksian demand is not observable – only Marshallian demand is observable.

Furthermore it is somewhat inelegant that compensated variation is in general not anti-symmetric, i.e.,

$$CV_i(p^{(2)}, p^{(1)}, m_i) \neq - CV_i(p^{(1)}, p^{(2)}, m_i),$$

whereas the change in profit on the producer side is:

$$\sum_{j=1}^{J}\left(\Pi_j^*(p^{(2)}) - \Pi_j^*(p^{(1)})\right) = - \sum_{j=1}^{J}\left(\Pi_j^*(p^{(1)}) - \Pi_j^*(p^{(2)})\right).$$

We prefer to work with a quantity that relies on Marshallian demand only and that is anti-symmetric. Thus, we consider:

$$\int_{p^{(1)}}^{p^{(2)}} x_i^*(p, m_i)\,dp$$

Recall,

$$CV_i(p^{(1)}, p^{(2)}, m_i) = \int_{p^{(1)}}^{p^{(2)}} x_{H,i}^*(p, v_i(p^{(1)}, m_i))\,dp$$

And $\quad x_{H,i}^*(p, v(p,m)) = x_i^*(p,m)$

so $\quad x_{H,i}^*(p^{(1)}, v_i(p^{(1)}, m_i)) = x_i^*(p^{(1)}, m_i)$

and $\quad x_{H,i}^*(p^{(2)}, v_i(p^{(2)}, m_i)) = x_i^*(p^{(2)}, m_i)$.

And Slutsky's equation gives:

$$\frac{\partial x_i^*(p, m_i)}{\partial p} = \partial_1 x_{H,i}^*(p, v_i(p, m_i)) - \left(\frac{\partial x_i^*(p, m_i)}{\partial m_i}\right) \cdot x_i^*(p, m_i)$$

$$\geq \partial_1 x_{H,i}^*(p, v_i(p, m_i)) \qquad (1)$$

Note that here, as above, the subscript i indicates the consumer.

if we assume the good is normal $\left(\text{i.e., } \frac{\partial x_i^*(p, m_i)}{\partial m_i} \geq 0\right)$.

But

$$\partial_1 x^*_{H,i}(p, v_i(p, m_i)) = \left(\frac{\partial}{\partial p} x^*_{H,i}(p, u)\right)\Bigg|_{u = v_i(p, m_i)}$$

$$= \frac{\partial}{\partial p} x^*_{H,i}(p, v_i(p^{(1)}, m_i)) \quad \text{at} \quad p = p^{(1)}$$

So, recalling (1), we have

$$\frac{\partial x_i^*(p, m_i)}{\partial p} \geq \frac{\partial}{\partial p} x^*_{H,i}(p, v_i(p^{(1)}, m_i)) \quad \text{at} \quad p = p^{(1)}.$$

So at $p = p^{(1)}$, the gradient of the $x_i^*(p, m_i)$ curve is greater than that of the $x^*_{H,i}(p, v_i(p^{(1)}, m_i))$ curve.

Similarly, one can show that

$$\frac{\partial x_i^*(p, m_i)}{\partial p} \geq \frac{\partial}{\partial p} x^*_{H,i}(p, v_i(p^{(2)}, m_i)) \quad \text{at} \quad p = p^{(2)}.$$

And note that since $p^{(1)} < p^{(2)}$ then

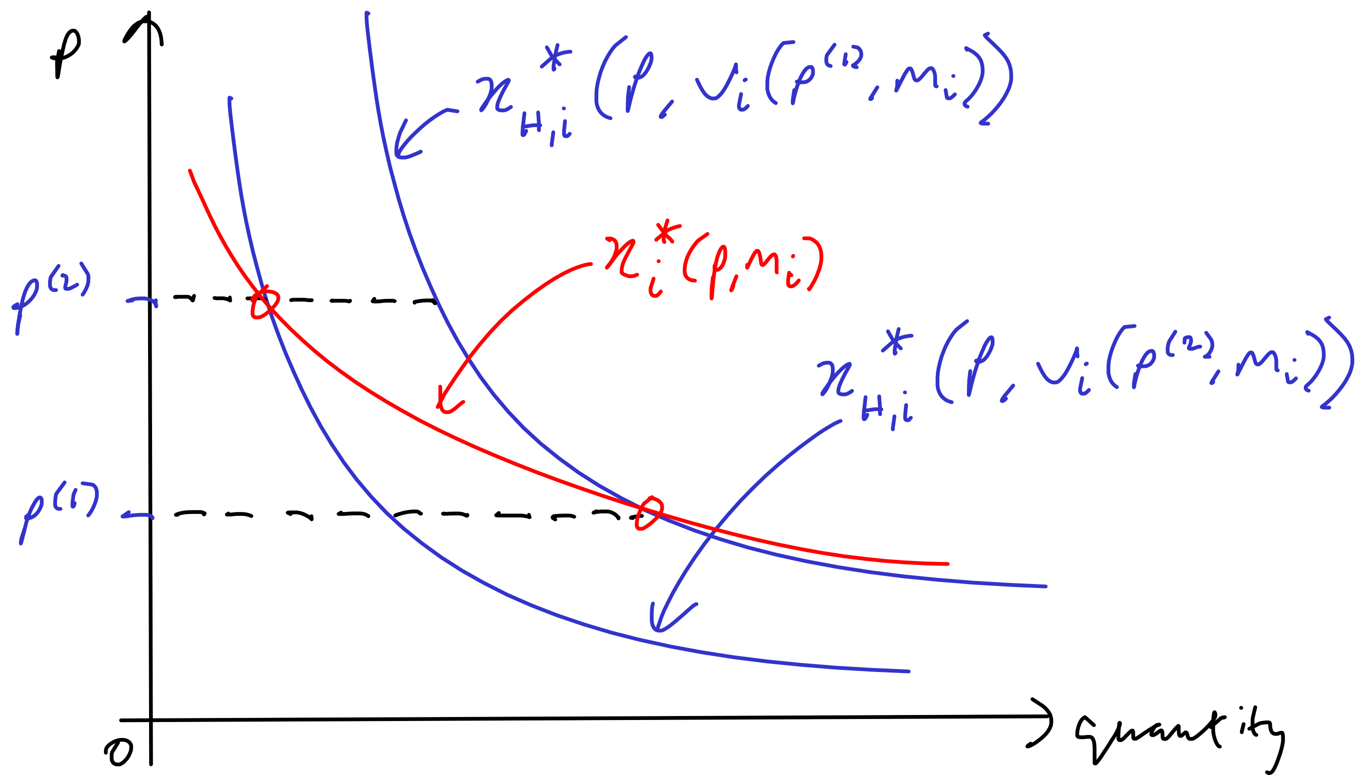
$$v_i(p^{(1)}, m_i) > v_i(p^{(2)}, m_i)$$

and so

$$x^*_{H,i}(p, v_i(p^{(1)}, m_i)) > x^*_{H,i}(p, v_i(p^{(2)}, m_i))$$

And as p increases, so both $x^*_{H,i}(p, v_i(p^{(1)}, m_i))$ and $x^*_{H,i}(p, v_i(p^{(2)}, m_i))$ decrease.

So we have



So

$$-CV_i(p^{(2)}, p^{(1)}, m_i) \le \int_{p^{(1)}}^{p^{(2)}} x_i^*(p, m_i)\,dp \le CV_i(p^{(1)}, p^{(2)}, m_i) = \int_{p^{(1)}}^{p^{(2)}} x^*_{H,i}(p, v_i(p^{(1)}, m_i))\,dp$$

(Compare areas in sketch above.)

We define the consumers' surplus at price $\hat{p}$ by:

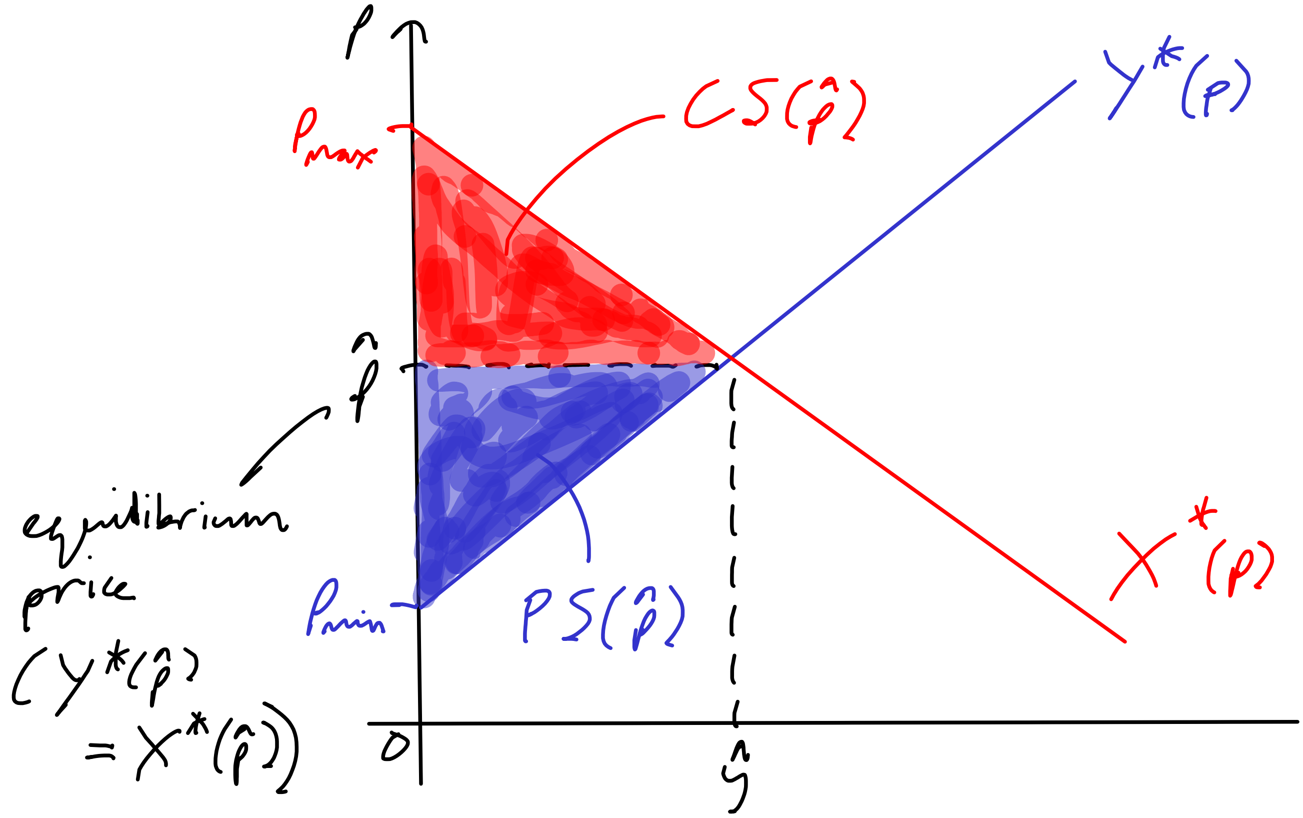
$$CS(\hat{p}) = \int_{\hat{p}}^{\infty} \sum_{i=1}^{I} x_i^*(p, m_i)\, dp$$

$$= \int_{\hat{p}}^{\infty} X^*(p, \underline{m})\, dp$$

$$= \int_{\hat{p}}^{p_{max}} X^*(p, \underline{m})\, dp$$

where p_{max} is the maximum price the consumers would be prepared to pay for the product.

So we have:



Finally, the sum of consumers' surplus and producers' surplus, the ***community surplus***, can be considered as a measure of social welfare.

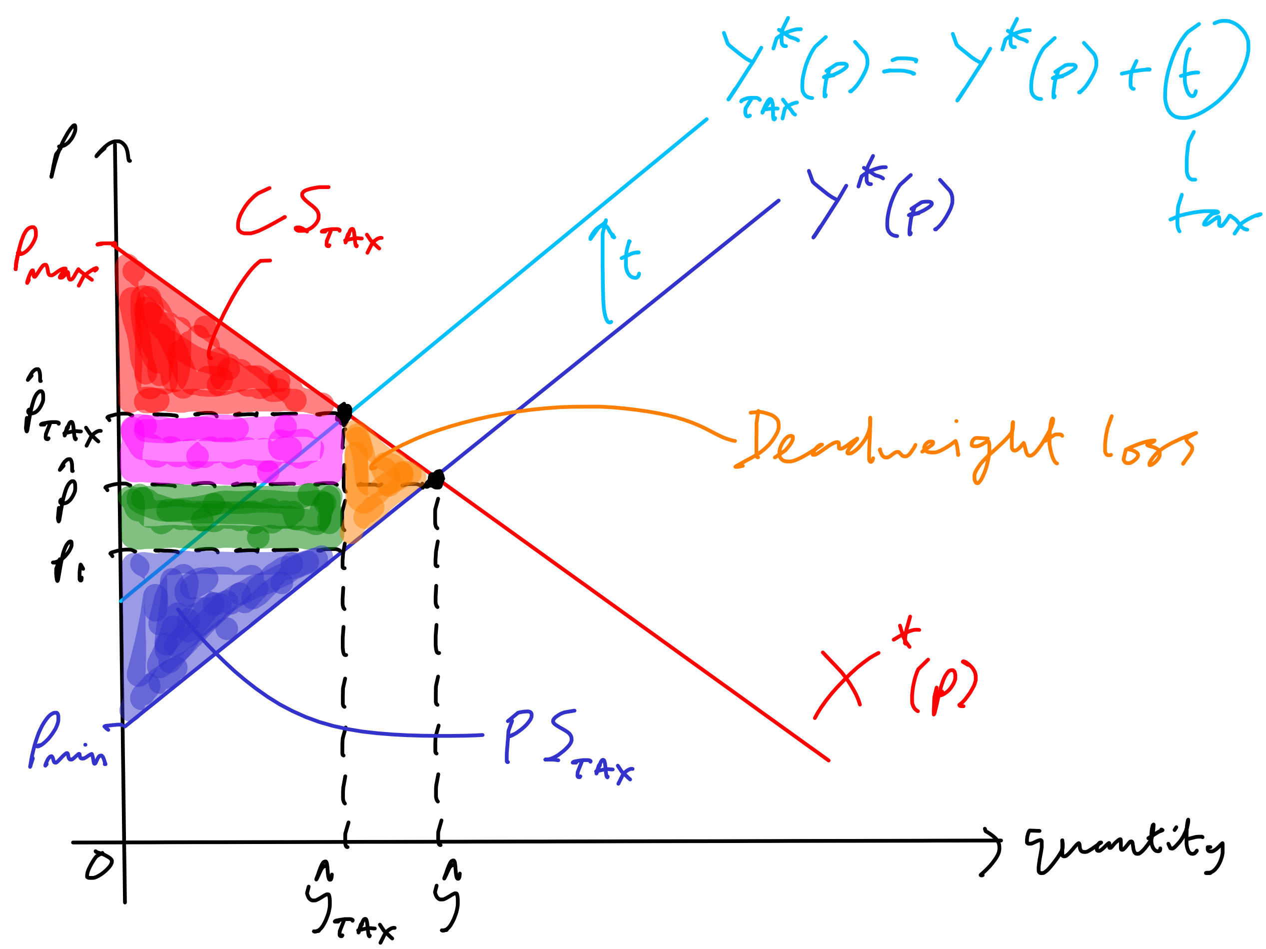
i.e., $CS(\hat{p}) + PS(\hat{p})$

Changes in the market demand or industry supply of a good will lead to a change in its equilibrium price. Taxes and subsidies are an interesting example of factors that lead to such a change.

<u>Indirect Taxes and Equilibrium:</u>

An indirect tax is one that can be passed on to another party. In the context of providing goods and services, an indirect tax on producers is one that is passed on to consumers. In general, a tax that is dependent on the quantity of good being produced can be treated as an indirect tax.

How much of an indirect tax is passed on to consumers?



The new supply function $Y^*_{TAX}(p)$ accounts for a tax that the producer passes on to the consumer. The new equilibrium price and quantity traded are $\hat{p}_{TAX}$ and $\hat{y}_{TAX}$, respectively. Now, because of this tax, the consumer pays an additional

$$(\hat{p}_{TAX} - \hat{p})\hat{y}_{TAX} \qquad (1)$$

for the amount $\hat{y}_{TAX}$ of the product while the producer earns

$$(\hat{p} - p_1)\hat{y}_{TAX} \qquad (2)$$

less on selling that amount (the producer is now selling an amount $\hat{y}_{TAX}$ at a price $\hat{p}_{TAX}$, but after deducting tax, only earns $p_1 \hat{y}_{TAX}$ from the sale). (1) and (2) are the cost of the tax to the consumer and the producer, respectively (also referred to as the tax incidence or burden on them).

After accounting for these tax costs, the consumer and producer are left with surpluses of CS_{TAX} and PS_{TAX} as indicated in sketch the above.

Finally, the 'deadweight loss' is the difference

$$(CS + PS) - (CS_{TAX} + PS_{TAX})$$

ie., the loss in community surplus due to the tax.