Imperial College
London

M4/5M9

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Functional Analysis

Date: Monday 22 May 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Let Ω be an open set of $\mathbb{R}^d$ and $h : \Omega \to [0, \infty)$ a measurable function. We consider the set

$$K = \{u \in L^2(\Omega) \mid |u(x)| \leq h(x), \text{ a.e. } x \text{ in } \Omega\},$$

where a.e. stands for "almost every" or "almost everywhere".

(a) Show that K is a non-empty closed convex subset of $L^2(\Omega)$.

We recall that a convex set K is such that for any given points v, w in K, $[v, w] \subset K$ where

$$[v, w] = \{(1 - t)v + tw \mid t \in [0, 1]\}.$$

To show that K is closed, it will be useful to remember that if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ that converges to u in $L^2(\Omega)$ (i.e. $\|u_n - u\|_2 \to 0$, where $\|u\|_2$ stands for the L^2-norm of u), then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges almost everywhere in Ω.

(b) We denote by P_K the projection of $L^2(\Omega)$ onto K. Let $u \in L^2(\Omega)$. Show that $P_K u$ is given for a.e. $x \in \Omega$ by:

$$P_K u(x) = \begin{cases} h(x) & \text{if } u(x) > h(x), \\ u(x) & \text{if } |u(x)| \leq h(x), \\ -h(x) & \text{if } u(x) < -h(x). \end{cases}$$

We recall that if C is a non-empty closed convex subset of a Hilbert space H with inner product $(\cdot, \cdot)$, then for any $w \in H$, $\xi = P_C w$ is characterized by

$$\begin{cases} \xi \in C, \\ (w - \xi, \eta - \xi) \leq 0, \quad \forall \eta \in C. \end{cases}$$

The proof involves computing integrals over the domain Ω. It will be useful to decompose them onto the following three subsets (forming a partition of Ω):

$$\begin{aligned} \Omega_+ &= \{x \in \Omega \mid u(x) > h(x)\}, \\ \Omega_- &= \{x \in \Omega \mid u(x) < -h(x)\}, \\ \Omega_0 &= \{x \in \Omega \mid |u(x)| \leq h(x)\}. \end{aligned}$$

2. Let Ω be a bounded open set of $\mathbb{R}^d$ (with $d \in \mathbb{N}$, $d \geq 1$). We recall that $C^0(\bar{\Omega})$ is the space of continuous functions from $\bar{\Omega}$ into $\mathbb{R}$. We consider a sequence $(f_n)_{n\geq 1}$ of elements of $C^0(\bar{\Omega})$ and a function $f \in C^0(\bar{\Omega})$ such that the following conditions hold:

(i) $(f_n)_{n\geq 1}$ is increasing meaning that $f_n(x) \leq f_{n+1}(x)$, for all $x \in \bar{\Omega}$;

(ii) $(f_n)_{n\geq 1}$ converges pointwise to f, meaning that $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in \bar{\Omega}$.

(a) Let $g_n(x) = f(x) - f_n(x)$. Show that g_n is continuous, that

$$g_n(x) \geq g_{n+1}(x) \geq 0, \quad \forall x \in \bar{\Omega}, \quad \forall n \in \mathbb{N}, \quad n \geq 1,$$

and that $g_n \to 0$ pointwise as $n \to \infty$.

(b) For all $\varepsilon > 0$ and for all $n \in \mathbb{N}$, $n \geq 1$, define:

$$V_n(\varepsilon) = \{x \in \bar{\Omega} \text{ such that } g_n(x) < \varepsilon\}.$$

Show that $V_n(\varepsilon)$ is an open subset of $\bar{\Omega}$ and that $V_n(\varepsilon) \subset V_{n+1}(\varepsilon)$, $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$, $n \geq 1$.

(c) Show that

$$\bigcup_{n\geq 1} V_n(\varepsilon) = \bar{\Omega}.$$

(d) Show that for all $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that $\bar{\Omega} = V_{N(\varepsilon)}(\varepsilon)$. To show this, it will be useful to remark that, Ω being bounded, $\bar{\Omega}$ is compact, and to remember that compact sets are characterized by the Borel-Lebesgue property: X is a compact metric space if and only if for any covering of X by a family of open sets $(U_i)_{i\in I}$ (i.e. $X = \cup_{i\in I} U_i$), there exists a finite subfamily $(U_i)_{i\in J}$ with J finite and $J \subset I$ which is still a covering of X (i.e. $X = \cup_{i\in J} U_i$).

(e) Deduce that $g_n \to 0$ uniformly on $\bar{\Omega}$ and that $f_n \to f$ uniformly on $\bar{\Omega}$.

3. We recall the following notations and definitions.
We denote by $|x|$ the euclidean norm of a vector $x \in \mathbb{R}^d$.
For any $x \in \mathbb{R}^d$ and $\rho > 0$, $B(x,\rho)$ (respectively $\bar{B}(x,\rho)$) stands for the open (respectively closed) ball centered at x and of radius ρ, i.e.

$$B(x,\rho) = \{y \in \mathbb{R}^d \mid |x-y| < \rho\}, \quad \bar{B}(x,\rho) = \{y \in \mathbb{R}^d \mid |x-y| \le \rho\}.$$

For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, Supp$f$ is the closed set defined by

$$x \notin \text{Supp}f \iff \exists \rho > 0, \text{ such that } f = 0, \text{a.e. in } B(x,\rho),$$

and a.e. stands for "almost everywhere" or "almost every".
For any subset A of $\mathbb{R}^d$, χ_A denotes the indicator function of A i.e. $\chi_A(x) = 1$ if and only if $x \in A$ and $\chi_A(x) = 0$ otherwise.
For a measurable subset A of $\mathbb{R}^d$, we denote by $\mu(A) \in [0,\infty]$ the Lebesgue measure of A.
Let $p \in [1,\infty)$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. We recall that f belongs to $L^p(\mathbb{R}^d)$ if and only if $|f|^p$ is Lebesgue integrable on $\mathbb{R}^d$ and we denote by $\|f\|_p = (\int_{\mathbb{R}^d} |f(x)|^p \, dx)^{1/p}$.

Throughout this question, f will denote a function in $L^p(\mathbb{R}^d)$ with $p \in [1,\infty)$.

(a) Let $n \in \mathbb{N}$, $n \ge 1$ and let $\chi_n = \chi_{B(0,n)}$ be the indicator function of $B(0,n)$. Let also $T_n : \mathbb{R} \to \mathbb{R}$ be such that

$$T_n(s) = \begin{cases} s & \text{if } \ |s| \le n, \\ n\,\text{Sign}(s) & \text{if } \ |s| > n, \end{cases}$$

where, for $s \in \mathbb{R} \setminus \{0\}$, Sign$(s)$ denotes the sign of s. For all $n \in \mathbb{N}$, $n \ge 1$, we define

$$f_n(x) = \chi_n(x)\, T_n\big(f(x)\big), \text{ a.e. } x \in \mathbb{R}^d. \tag{1}$$

Show that $f_n \in L^\infty(\mathbb{R}^d)$ and Suppf_n is compact.

(b) By Definition (1), for all $n \in \mathbb{N}$, $n \ge 1$, there exists a measurable subset A_n of $\mathbb{R}^d$ such that $\mu(A_n) = 0$ and

$$f_n(x) = \chi_n(x)\, T_n\big(f(x)\big), \quad \forall x \in \mathbb{R}^d \setminus A_n. \tag{2}$$

Show that there exists a measurable subset A of $\mathbb{R}^d$ such that $\mu(A) = 0$ and

$$f_n(x) = \chi_n(x)\, T_n\big(f(x)\big), \quad \forall x \in \mathbb{R}^d \setminus A, \quad \forall n \ge 1. \tag{3}$$

(c) Show that $\mu(B) = 0$ where $B = \{x \in \mathbb{R}^d \mid |f(x)| = +\infty\}$. To show this, it will be useful to remember a result from the course: any Lebesgue integrable function is finite almost everywhere.

(d) Show that $f_n \to f$ pointwise in $\mathbb{R}^d \setminus (A \cup B)$. Deduce that $f_n \to f$ almost everywhere.

(e) Show that $|f_n(x)| \le |f(x)|$ a.e. x in $\mathbb{R}^d$. Deduce that $|f(x) - f_n(x)|^p \le 2^p\,|f(x)|^p$ a.e. x in $\mathbb{R}^d$.

(f) Show that $\|f - f_n\|_p \to 0$ as $n \to \infty$.

4. We recall the following notations and definitions.
We denote by $|x|$ the euclidean norm of a vector $x \in \mathbb{R}^d$.
For any $x \in \mathbb{R}^d$ and $\rho > 0$, $B(x,\rho)$ (respectively $\bar{B}(x,\rho)$) stands for the open (respectively closed) ball centered at x and of radius ρ, i.e.

$$B(x,\rho) = \{y \in \mathbb{R}^d \,\big|\, |x-y| < \rho\}, \quad \bar{B}(x,\rho) = \{y \in \mathbb{R}^d \,\big|\, |x-y| \leq \rho\}.$$

For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, Supp$f$ is the closed set defined by

$$x \notin \text{Supp} f \iff \exists \rho > 0, \text{ such that } f = 0, \text{a.e. in } B(x,\rho),$$

and a.e. stands for "almost everywhere" or "almost every".
For a measurable subset A of $\mathbb{R}^d$, we denote by $\mu(A) \in [0,\infty]$ the Lebesgue measure of A.
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. We recall that f belongs to $L^1(\mathbb{R}^d)$ if and only if f is Lebesgue integrable on $\mathbb{R}^d$ and we denote by $\|f\|_1 = \int_{\mathbb{R}^d} |f(x)|\, dx$.

Throughout this question f will denote a function in the space $L_c^\infty(\mathbb{R}^d)$ of L^∞-functions with compact support. We recall that this implies that $f \in L^1(\mathbb{R}^d)$ and $\|f\|_1 \leq \|f\|_\infty \, \mu(\text{Supp} f) < \infty$, where $\mu(\text{Supp} f)$ denotes the Lebesgue measure of Supp f, which is finite since Supp f is compact.

(a) For any non-negative real number m, we define the function $T_m : \mathbb{R} \to \mathbb{R}$ such that

$$T_m(s) = \begin{cases} s & \text{if } |s| \leq m, \\ m\,\text{Sign}(s) & \text{if } |s| > m, \end{cases}$$

where, for $s \in \mathbb{R} \setminus \{0\}$, Sign$(s)$ denotes the sign of s. We recall a result of the course: the space $C_c^0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ with compact support is dense in $L^1(\mathbb{R}^d)$. Let $\varepsilon > 0$ be fixed. Thanks to this density property, there exists $g \in C_c^0(\mathbb{R}^d)$ such that $\|f - g\|_1 \leq \varepsilon$. Let $m = \|f\|_\infty$ and $h = T_m \circ g$. Show that $h \in C_c^0(\mathbb{R}^d)$ and that

$$\|g - h\|_1 = \int_{\{x \,|\, g(x) > \|f\|_\infty\}} (g(x) - \|f\|_\infty)\, dx \; - \int_{\{x \,|\, g(x) < -\|f\|_\infty\}} (g(x) + \|f\|_\infty)\, dx.$$

(b) Show that on the set $\{x \,|\, g(x) > \|f\|_\infty\}$, we have $0 < g(x) - \|f\|_\infty \leq g(x) - f(x)$, a.e. Similarly, show that on the set $\{x \,|\, g(x) < -\|f\|_\infty\}$, we have $0 < -(g(x) + \|f\|_\infty) \leq f(x) - g(x)$, a.e.

(c) Using Part (b), show that

$$\int_{\{x \,|\, g(x) > \|f\|_\infty\}} (g(x) - \|f\|_\infty)\, dx \leq \|g - f\|_1.$$

Using Part (a), deduce that $\|g - h\|_1 \leq 2\|g - f\|_1$. Finally deduce that $\|f - h\|_1 \leq 3\varepsilon$.

(d) Using the definition of T_m show that $\|h\|_\infty \leq \|f\|_\infty$. Deduce that for all $f \in L_c^\infty(\mathbb{R}^d)$ and all $\varepsilon > 0$ there exists a function $\tilde{f} \in C_c^0(\mathbb{R}^d)$ such that $\|\tilde{f} - f\|_1 \leq \varepsilon$ and $\|\tilde{f}\|_\infty \leq \|f\|_\infty$.

5. Mastery question. This question involves the use of weak convergence concepts, which are the subject of the assigned extra-reading. We define the space ℓ^p, for $1 \leq p < \infty$ as the space of sequences $x = (x_k)_{k\geq 1}$, $x_k \in \mathbb{R}$, $\forall k \geq 1$, such that

$$\|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p \Big)^{1/p} < \infty.$$

The space ℓ^p endowed with the norm $\|\cdot\|_p$ is a Banach space. We recall the Hölder inequality: for $x = (x_k)_{k\geq 1} \in \ell^p$, $y = (y_k)_{k\geq 1} \in \ell^q$ with q the conjugate exponent of p defined by $\frac{1}{p} + \frac{1}{q} = 1$, then $xy = (x_k y_k)_{k\geq 1}$ belongs to ℓ^1 and

$$\|xy\|_1 \leq \|x\|_p \, \|y\|_q.$$

We also recall that ℓ^p is separable i.e. ℓ^p admits a dense countable subset.

For $1 < p < \infty$, the dual space of ℓ^p is $(\ell^p)' = \ell^q$: any element $\varphi \in (\ell^p)'$ can be represented by an element $\xi = (\xi_k)_{k\geq 1} \in \ell^q$ such that

$$\langle \varphi, x \rangle_{(\ell^p)', \ell^p} = \sum_{k=1}^{\infty} \xi_k \, x_k, \quad \forall x = (x_k)_{k\geq 1} \in \ell^p.$$

Thus, ℓ^p is a reflexive space since $((\ell^p)')' = (\ell^q)' = \ell^p$. We recall that in a reflexive separable Banach space E, closed balls $\bar{B}(0,R) = \{x \in E \mid \|x\| \leq R\}$ are metrizable and compact for the weak topology $\sigma(E,E')$. This implies that if $(x^n)_{n\geq 1}$ is a bounded sequence in E, there exists a subsequence $(x^{n_r})_{r\geq 1}$ and $x \in E$ such that $x^{n_r} \rightharpoonup x$ as $r \to \infty$ for the weak topology $\sigma(E,E')$.

(a) Let $(x^n)_{n\geq 1}$ be a sequence in ℓ^p for $1 < p < \infty$, i.e. $x^n = \Big((x^n)_k\Big)_{k\geq 1} \in \ell^p$. Suppose that $x^n \rightharpoonup x$ for the weak topology $\sigma(\ell^p, \ell^q)$. Show that the following two statements hold:

(i) The sequence $(x^n)_{n\geq 1}$ is bounded in ℓ^p.

(ii) $(x^n)_k \to x_k$ as $n \to \infty$ in $\mathbb{R}$, $\forall k \geq 1$.

(b) Let us now suppose that $(x^n)_{n\geq 1}$ is a sequence in ℓ^p for $1 < p < \infty$, such that criteria (i) and (ii) of Part (a) hold for a given $x = (x_k)_{k\geq 1}$. Show that $x \in \ell^p$. For this, show that there exists a subsequence $(x^{n_r})_{r\geq 1}$ and $z \in \ell^p$ such that $x^{n_r} \rightharpoonup z$ as $r \to \infty$ in the weak topology $\sigma(\ell^p, \ell^q)$. Then, show that $z = x$.

(c) Under the same assumptions as in Part (b), show that $x^n \rightharpoonup x$ for the weak topology $\sigma(\ell^p, \ell^q)$. Proceed by contradiction and show that if this is not the case, there exists a subsequence $(x^{n_r})_{r\geq 1}$ and $\varepsilon > 0$ such that $d(x^{n_r}, x) > \varepsilon$, $\forall r \geq 1$. Here, d is a distance which metrizes the weak topology $\sigma(\ell^p, \ell^q)$ of the ball $\bar{B}(0,R)$ where R is such that $\sup_{n\geq 1} \|x^n\|_p \leq R$. Show that this implies the existence of $z \in \ell^p$, $z \neq x$ and of a subsequence $(x^{n_{r'}})_{r'\geq 1}$ such that $x^{n_{r'}} \rightharpoonup z$ as $r' \to \infty$ in the weak topology $\sigma(\ell^p, \ell^q)$. Using Part (a), show that this leads to a contradiction.

	EXAMINATION SOLUTIONS 2016-17	Course M345M9
Question 1		Marks & seen/unseen
Parts (a)	K is non empty because 0 (i.e. the function $x \in \Omega \to 0 \in \mathbb{R}$) belongs to K.	2 seen similar
	K is convex: indeed take two functions u_1 and u_2 in K and $t \in [0,1]$. Then, $\lvert u_1(x)\rvert \leq h(x)$ a.e. and $\lvert u_2(x)\rvert \leq h(x)$ a.e., so $\lvert(1-t)u_1(x) + tu_2(x)\rvert \leq (1-t)\lvert u_1(x)\rvert + t\lvert u_2(x)\rvert$ $\leq (1-t)h(x) + th(x) \leq h(x)$, a.e., showing that $(1-t)u_1 + tu_2 \in K$.	3 seen similar
	Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of K that converges in $L^2(\Omega)$ to u. Then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that converges almost everywhere to u on Ω. From $u_{n_k} \in K$, we have $\lvert u_{n_k}(x)\rvert \leq h(x)$, a.e. And since $u_{n_k}(x) \to u(x)$ a.e., it follows that $\lvert u(x)\rvert \leq h(x)$ a.e. and therefore, $u \in K$, showing that K is closed.	5 seen similar
(b)	Let $u \in L^2(\Omega)$ and temporarily denote by v the function proposed as a candidate for $P_K u$ in Part (b). We have $\lvert v\rvert \leq \lvert u\rvert$ a.e. Since $u \in L^2(\Omega)$, then $v \in L^2(\Omega)$ (with $\lVert v\rVert_2 \leq \lVert u\rVert_2$). Obviously $\lvert v\rvert \leq h$ so $v \in K$, which is the first item to be proved.	3 seen similar
	We now show that $(u-v, w-v)_2 \leq 0$, $\forall w \in K$, where $(\cdot,\cdot)_2$ is the inner product in $L^2(\Omega)$. We compute $(u-v, w-v)_2 = \int_\Omega (u-v)(x)\,(w-v)(x)\,dx$ and decompose the integral over the three sets as suggested in the question. On Ω_0, $v = u$, a.e. and the corresponding integral is equal to 0. Thus $(u-v, w-v)_2 = \int_{\Omega_+} (u-h)(x)\,(w-h)(x)\,dx$ $+ \int_{\Omega_-} (u+h)(x)\,(w+h)(x)\,dx$. Since $w \in K$, $-h \leq w \leq h$, so $w - h \leq 0$ and $w + h \geq 0$. On Ω_+, $u - h \geq 0$ so $(u-h)\,(w-h) \leq 0$ and the integral over Ω_+ is non positive. Similarly, on Ω_-, $u + h \leq 0$ so $(u+h)\,(w+h) \leq 0$ and the integral over Ω_- is non positive as well. In the end, we get $(u-v, w-v)_2 \leq 0$, which, thanks to the characterization recalled in the question, shows that $v = P_K u$.	7 unseen
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	EXAMINATION SOLUTIONS 2016-17	Course M345M9
Question 2		Marks & seen/unseen
Parts (a)	g_n is continuous as the difference of two continuous functions. As $(f_n)_n$ is an increasing sequence we have $f_n \leq f$. So $g_n \geq 0$ and $(g_n)_n$ is decreasing. Since $f_n \to f$ pointwise, $g_n \to 0$ pointwise.	3 seen
(b)	$V_n(\varepsilon)$ is an open set as the preimage of an open set of $\mathbb{R}$ (the open set $(-\infty, \varepsilon)$) by the continuous function g_n. If $x \in V_n(\varepsilon)$, then $g_{n+1}(x) \leq g_n(x) < \varepsilon$ showing that $x \in V_{n+1}(\varepsilon)$. Therefore, $V_n(\varepsilon) \subset V_{n+1}(\varepsilon)$.	4 seen
(c)	Let $\varepsilon > 0$ and $x \in \bar{\Omega}$ be given. Since $g_n(x) \to 0$, there exists $n = n(x) \in \mathbb{N}$ such that $g_n(x) < \varepsilon$, showing that $x \in V_n(\varepsilon)$. Therefore, $x \in \bigcup_{n \geq 1} V_n(\varepsilon)$.	3 seen
(d)	Since Ω is bounded, $\bar{\Omega}$ is compact. For a fixed $\varepsilon > 0$, the family $(V_n(\varepsilon))_n$ is an open covering of $\bar{\Omega}$ by open sets, from (b) and (c). Therefore, there exists a finite subcovering of $\bar{\Omega}$, i.e. there exists an integer k and k integer numbers $n_1 < n_2 < \ldots < n_k$ such that $\bar{\Omega} = \bigcup_{j=1}^{k} V_{n_j}(\varepsilon)$. But from (b) we have $V_{n_1}(\varepsilon) \subset V_{n_2}(\varepsilon) \subset \ldots \subset V_{n_k}(\varepsilon)$. So, $\bigcup_{j=1}^{k} V_{n_j}(\varepsilon) = V_{n_k}(\varepsilon)$. Denote $N(\varepsilon) = n_k$. We have $\bar{\Omega} = V_{N(\varepsilon)}(\varepsilon)$.	6 unseen
(e)	Let $\varepsilon > 0$ be fixed and $N(\varepsilon)$ be given by (d). Then, by (b), for all $n \geq N(\varepsilon)$, we have $V_n(\varepsilon) = \bar{\Omega}$. This means that $g_n(x) < \varepsilon$ for all $x \in \bar{\Omega}$ and $n \geq N(\varepsilon)$ which is precisely saying that $g_n \to 0$ uniformly on $\bar{\Omega}$. Now since $g_n = f - f_n$, we get that $f_n \to f$ uniformly on $\bar{\Omega}$.	4 seen similar
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	EXAMINATION SOLUTIONS 2016-17	Course M345M9
Question 3		Marks & seen/unseen
Parts (a)	Since Supp $\chi_n = \bar{B}(0,n)$ we have Supp$f_n \subset \bar{B}(0,n)$ and so, Suppf_n is compact.	1 seen similar
	Since $\lvert T_n(s)\rvert \le n$, $\forall s \in \mathbb{R}$, we have $\lvert T_n(f(x))\rvert \le n$, a.e. $x \in \mathbb{R}^d$. Then, with $\lvert \chi_n(x)\rvert \le 1$, $\forall x \in \mathbb{R}^d$, we obtain that $\lvert f_n(x)\rvert \le n$, a.e. $x \in \mathbb{R}^d$, and consequently, f_n belongs to $L^\infty(\mathbb{R}^d)$.	2 seen similar
(b)	Take $A = \cup_{n\ge1} A_n$. Then, $\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n) = 0$, since $\mu(A_n) = 0$, $\forall n \ge 1$. Then, since $\mathbb{R}^d \setminus A_n \supset \mathbb{R}^d \setminus (\cup_{n\ge1} A_n) = \mathbb{R}^d \setminus A$, (3) follows from (2).	3 seen similar
(c)	Since $f \in L^p(\mathbb{R}^d)$, then $\lvert f\rvert^p \in L^1(\mathbb{R}^d)$. From the result of the course recalled, we have $\lvert f(x)\rvert^p < \infty$, a.e. and consequently $\lvert f(x)\rvert < \infty$, a.e., which implies that $\mu(B) = 0$.	3 seen similar
(d)	Let $x \in \mathbb{R}^d \setminus (A \cup B)$. Let $N \in \mathbb{N}$, $N > \max(\lvert x\rvert, \lvert f(x)\rvert)$. N is finite because $x \notin B$ and $\lvert f(x)\rvert < \infty$. Then $\forall n \ge N$, $\chi_n(x) = 1$ and $T_n(f(x)) = f(x)$. So, $\forall n \ge N$, $f_n(x) = f(x)$. This implies that $f_n(x) \to f(x)$, and consequently that $f_n \to f$ pointwise in $\mathbb{R}^d \setminus (A \cup B)$. since $\mu(A \cup B) \le \mu(A) + \mu(B) = 0$, this implies that $f_n \to f$ almost everywhere.	4 unseen
(e)	Since $\lvert \chi_n(x)\rvert \le 1$, $\forall x \in \mathbb{R}^d$ and $\lvert T_n(s)\rvert \le \lvert s\rvert$, $\forall s \in \mathbb{R}$, we get that $\lvert f_n(x)\rvert \le \lvert f(x)\rvert$ a.e. x in $\mathbb{R}^d$. We deduce that $\lvert f(x) - f_n(x)\rvert \le 2\lvert f(x)\rvert$, and consequently $\lvert f(x) - f_n(x)\rvert^p \le 2^p \lvert f(x)\rvert^p$ a.e. x in $\mathbb{R}^d$.	3 seen similar
(f)	We have $\lVert f - f_n\rVert_p^p = \int_{\mathbb{R}^d} \lvert f(x) - f_n(x)\rvert^p\,dx$. But from (d), $\lvert f(x) - f_n(x)\rvert \to 0$ a.e. and consequently $\lvert f(x) - f_n(x)\rvert^p \to 0$ a.e. From (e), we get that $\lvert f(x) - f_n(x)\rvert^p$ is bounded by $2^p\lvert f(x)\rvert^p$ which is an integrable function independent of n. The Lebesgue dominated convergence theorem applies and we have $\lVert f - f_n\rVert_p^p = \int_{\mathbb{R}^d} \lvert f(x) - f_n(x)\rvert^p\,dx \to 0$ as $n \to \infty$, showing that $\lVert f - f_n\rVert_p \to 0$.	4 seen similar
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	EXAMINATION SOLUTIONS 2016-17	Course M345M9
Question 4		Marks & seen/unseen
Parts (a)	T_m is a continuous function and so is g, so h is continuous as the composition of two continuous functions. Furthermore, $g(x) = 0 \Rightarrow h(x) = 0$, which shows that Supp $h \subset$ Supp g. Since Supp g is compact, so is Supp h. Thus $h \in C_c^0(\mathbb{R}^d)$.	3 seen similar
	We have $\|g-h\|_1 = \int_{\mathbb{R}^d} \|g(x) - h(x)\| \, dx$. But for all x such that $\|g(x)\| \leq \|f\|_\infty$, $h(x) = g(x)$, so $\|g(x) - h(x)\| = 0$. Therefore, the integral is only non zero either when $g(x) > \|f\|_\infty$ or when $g(x) < -\|f\|_\infty$. In the first instance, $\|g(x) - h(x)\| = g(x) - \|f\|_\infty$. In the second one, $\|g(x) - h(x)\| = -(g(x) + \|f\|_\infty)$. Collecting these, we get the requested formula.	5 unseen
(b)	On the set $\{x \mid g(x) > \|f\|_\infty\}$, since $\|f\|_\infty > f(x)$, a.e. we have $0 < g(x) - \|f\|_\infty \leq g(x) - f(x)$ a.e. On the set $\{x \mid g(x) < -\|f\|_\infty\}$, since $-\|f\|_\infty < f(x)$, a.e. we have $0 < -(g(x) + \|f\|_\infty) \leq -g(x) + f(x)$ a.e.	3 seen similar
(c)	We have using (b): $\int_{\{x \mid g(x) > \|f\|_\infty\}} (g(x) - \|f\|_\infty) \, dx \leq \int_{\{x \mid g(x) > \|f\|_\infty\}} (g(x) - f(x)) \, dx$ $= \int_{\{x \mid g(x) > \|f\|_\infty\}} \|g(x) - f(x)\| \, dx \leq \int_{\mathbb{R}^d} \|g(x) - f(x)\| \, dx = \|g - f\|_1$. and similarly : $- \int_{\{x \mid g(x) < -\|f\|_\infty\}} (g(x) + \|f\|_\infty) \, dx \leq \int_{\{x \mid g(x) < -\|f\|_\infty\}} (-g(x) + f(x)) \, dx$ $= \int_{\{x \mid g(x) < -\|f\|_\infty\}} \|g(x) - f(x)\| \, dx \leq \int_{\mathbb{R}^d} \|g(x) - f(x)\| \, dx = \|g - f\|_1$. So, thanks to (a), $\|g-h\|_1 \leq 2\|g-f\|_1$. Then $\|f-h\|_1 \leq \|f-g\|_1 + \|g-h\|_1 \leq 3\|g-f\|_1$.	6 unseen
(d)	Since $\|T_m(s)\| \leq m$, we have $\|h\| \leq m = \|f\|_\infty$. Let $f \in L_c^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. Take $g \in C_c^0(\mathbb{R}^d)$ such that $\|f - g\|_1 \leq \eta := \varepsilon/3$. Then, the previous proof shows that $\tilde{f} := T_m \circ g$ for $m = \|f\|_\infty$ is such that $\|f - \tilde{f}\|_1 \leq 3\eta = \varepsilon$, which shows the requested result.	3 seen similar
	Setter's initials PD Checker's initials JB	Page number 4

<table>
<tr><td></td><td>EXAMINATION SOLUTIONS 2016-17</td><td>Course
M345M9</td></tr>
<tr><td>Question
5</td><td></td><td>Marks &
seen/unseen</td></tr>
<tr><td>Parts
(a)</td><td>Weakly convergent sequences are bounded so $(x^n)_{n\geq 1}$ is bounded in ℓ^p.</td><td>1
unseen</td></tr>
<tr><td></td><td>$x^n \rightharpoonup x$ means that $\forall y = (y_k)_{k\geq 1} \in \ell^q$, $\sum_{k\geq 1}(x^n)_k\, y_k \to \sum_{k\geq 1} x_k\, y_k$ in $\mathbb{R}$ as $n \to \infty$. Taking $y = e_i$ with $(e_i)_k = \delta_{ik}$, δ_{ik}being the Kronecker delta, we get: $(x^n)_i \to x_i$.</td><td>2
unseen</td></tr>
<tr><td>(b)</td><td>If $(x^n)_{n\geq 1}$ is bounded and since ℓ^p is reflexive for $1 < p < \infty$, we can use the theorem recalled in the question. This leads to the existence of a subsequence $(x^{n_r})_{r\geq 1}$ and $z \in \ell^p$ such that $x^{n_r} \rightharpoonup z$ as $r \to \infty$ in the weak topology $\sigma(\ell^p, \ell^q)$.</td><td>4
unseen</td></tr>
<tr><td></td><td>From the previous part, this leads to
$(x^{n_r})_i \to z_i$ as $r \to \infty$ in $\mathbb{R}$, $\forall i \geq 1$. But we know that $(x^n)_i \to x_i$ as $n \to \infty$. In particular, this implies that $(x^{n_r})_i \to x_i$ as $r \to \infty$
in $\mathbb{R}$, $\forall i \geq 1$. Therefore, $z_i = x_i$, $\forall i \geq 1$ and so $x = z \in \ell^p$</td><td>3
unseen</td></tr>
<tr><td>(c)</td><td>Now, suppose that $(x^n)_{n\geq 1}$ does not weakly converge to x in $\sigma(\ell^p, \ell^q)$. Since $(x^n)_{n\geq 1}$ is bounded, it is imbedded in a closed ball $\bar{B}(0, R)$ of ℓ^p. Since in the closed ball $\bar{B}(0, R)$ the weak topology $\sigma(\ell^p, \ell^q)$ is metrizable with a distance denoted by d, there exists a subsequence $(x^{n_r})_{r\geq 1}$ and $\varepsilon > 0$ such $d(x^{n_r}, x) > \varepsilon$, $\forall r \geq 1$.</td><td>4
unseen</td></tr>
<tr><td></td><td>Now, since $(x^{n_r})_{r\geq 1}$
is bounded, there exists a subsequence, $(x^{n_{r'}})_{r'\geq 1}$ and
$z \in \ell^p$ such that $x^{n_{r'}} \rightharpoonup z$ as $r' \to \infty$ in the weak topology $\sigma(\ell^p, \ell^q)$ and we have $d(z, x) \geq \varepsilon$. In particular, this implies $z \neq x$.</td><td>3
unseen</td></tr>
<tr><td></td><td>But from
Part (a), this implies that $(x^{n_{r'}})_i \to z_i$, $\forall i \geq 1$. And since $z \neq x$, there exists $i_0 \geq 1$ such that $z_{i_0} \neq x_{i_0}$ So, $(x^{n_{r'}})_{i_0} \not\to x_{i_0}$ as $r' \to \infty$, which contradicts the assumption that $(x^n)_{i_0} \to x_{i_0}$ as $n \to \infty$.
This shows by contradiction that $x^n \rightharpoonup x$ as $n \to \infty$ in $\sigma(\ell^p, \ell^q)$.</td><td>3
unseen</td></tr>
<tr><td></td><td>Setter's initials   Checker's initials
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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M345 M9 **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The average is 14.4/20 among 3rd year students and 11/20 among 4th and Msc Students (*). Part (a) was generally well done, but to most, except a few exceptions, part (b) was difficult. I did not expect it and knowing it, I would have put more steps. The idea was to take the function suggested in the question and show that it satisfied the condition for being the projection on the closed convex set K. To many, it was just difficult to set up the right methodology.

(*) before addition of bonus marks

Marker: Pierre DEGOND

Signature: **Date:** 26/05/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M345M9 **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The average is 15.8/20 for 3rd year students and 18/20 to fourth year and MSc (before addition of bonus marks). This was the most successful question of the exam. I am satisfied to see that almost all students could apply the Borel-Lebesgue definition of a compact set. On the other hand, I am surprised that so many students did not remember that the subtraction of two continuous functions is a continuous function and reproved it. It is a shame. Some precious time could have been gained.

Marker: Pierre DEGOND

Signature: Date: 26/05/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M345 M9 **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The average is 16/20 for 3rd year students and 15/20 for fourth year and MSc students (before addition of bonus marks).
This question was also generally well-done, although to a lesser extent than question 2. I am satisfied to see that the students are proficient at manipulating the concept of Lebesgue negligible set and Lebesgue integration concepts generally, in particular the Lebesgue dominated convergence theorem.

Marker: Pierre DEGOND

Signature:

Date: 26/05/2017

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Department of Mathematics

Examiner's Comments

Exam: M345M9 **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The average is 11.8/20 among 3rd year student and 15/20 among 4th year and MSc students (before addition of bonus marks).
The lower marks for 3rd year student reflects that some did not have the time to attempt it. The similar average as for the 3rd question for 4th year and MSc students reflects the fact that the two marks are correlated: those who were proficient in Lebesgue integration were also proficient in approximation of integrable functions by continuous functions. I am satisfied to see that these difficult concepts were well mastered by the students.

Marker: Pierre DEGOND

Signature: ______ **Date:** 26/05/2017

Please return with exam marks (one report per marker)

Exam M45H9 Session 2016-2017

Mastery Question

The average to this question is 6.3/20 which shows that most students did not have the time to seriously attempt it. There are noticeable exceptions: one 20/20 (who scored 20/20 on all questions of the exam: congratulations!) and two 14/20.
In general, I could see that the concepts of weak topology were not sufficiently mastered. This may be a hint that this should be reviewed at a revision class before the exam.

Pierre DEGOND 28/05/2017