Imperial College London

MATH40007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: 15 May 2023

Time: 14:00 – 16:00 (BST)

Time Allowed: 2 hrs

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Figures 1–4 below show the first 4 generations, $n = 1, 2, 3, 4$, of a tree graph with a root node and $n \geq 1$ generations each considered as an electric circuit. The root node of each tree, shown in red, is at unit voltage, and all nodes in the lowest branch, or n-th generation, shown in black, are grounded. At all other nodes, shown in blue, Kirchhoff's current law holds. All the conductors between generation $(j-1)$ and generation j for $1 \leq j \leq n$ have conductance c_j, as indicated. Let $C_{\text{eff}}^{(n)}$ denote the effective conductance of such a tree circuit with n generations.

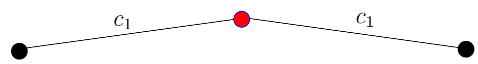


Figure 1: One generation: $n = 1$

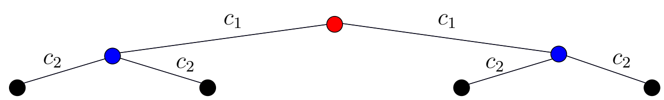


Figure 2: Two generations: $n = 2$

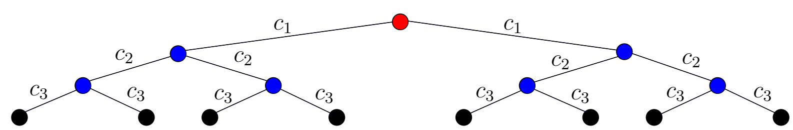


Figure 3: Three generations: $n = 3$

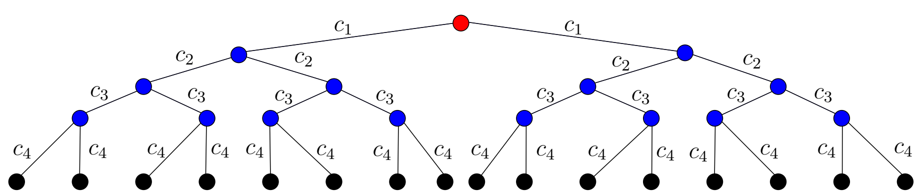


Figure 4: Four generations: $n = 4$

(a) Find the dimensions of the incidence matrix of a tree with $n \geq 1$ generations. (1 mark)

(b) Assume that $c_j = 1$ for $1 \leq j \leq n$.

  (i) Find $C_{\text{eff}}^{(n)}$. (4 marks)

  (ii) Find the voltages at all the nodes in a tree with n generations. (6 marks)

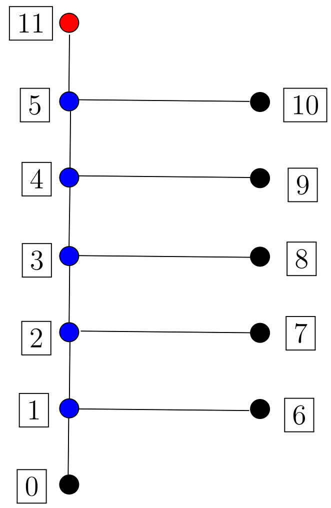
  (iii) Find $C_{\text{eff}}^{(\infty)}$. (1 mark)

(c) Let $c_j = 1/j$ for $j \geq 1$. Find $C_{\text{eff}}^{(\infty)}$. (4 marks)

(d) Let $c_j = j$ for $j \geq 1$. Find $C_{\text{eff}}^{(\infty)}$. (4 marks)

(Total: 20 marks)

2. Consider the electric circuit shown in the figure where all edges are conductors of unit conductance. Nodes $\boxed{0}$ and nodes $\boxed{6}$–$\boxed{10}$ are all grounded (shown black) and node $\boxed{11}$ is set to unit voltage. Kirchhoff's current law holds at nodes $\boxed{1}$–$\boxed{5}$. Let the j-th component of the 5-dimensional vector $\hat{\mathbf{x}}$ be the voltage at node $\boxed{j}$. The nodes should be ordered as follows: $\boxed{1}$, $\boxed{2}$, $\ldots$, $\boxed{10}$, $\boxed{0}$, $\boxed{11}$.



(a) If $\mathbf{I}_n$ denotes the n-by-n identity matrix, show that, with columns assigned to nodes in the order specified above, the Laplacian matrix $\mathbf{K}$ can be written

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_5 + \mathbf{I}_5 & -\mathbf{I}_5 & -\mathbf{P} \\ -\mathbf{I}_5 & \mathbf{I}_5 & \mathbf{0} \\ -\mathbf{P}^T & \mathbf{0} & \mathbf{I}_2 \end{pmatrix},$$

where $\mathbf{K}_5$ (a 5-by-5 matrix) and $\mathbf{P}$ (a 5-by-2 matrix) should be found explicitly. (2 marks)

(b) By writing

$$\hat{\mathbf{x}} = \sum_{m=1}^{5} a_m \mathbf{\Phi}_m,$$

where $\{\mathbf{\Phi}_m | m = 1, \ldots, 5\}$ is a convenient set of orthonormal eigenvectors with corresponding eigenvalues $\{\lambda_m | m = 1, \ldots, 5\}$, find formulas for the set of coefficients $\{a_m | m = 1, \ldots, 5\}$ expressed in terms of the eigenvalues $\{\lambda_m | m = 1, \ldots, 5\}$.

(4 marks)

(c) Show that the n-th element of $\hat{\mathbf{x}}$, for $n = 1, \ldots, 5$, can also be written as

$$\frac{\lambda_+^n - \lambda_-^n}{\lambda_+^6 - \lambda_-^6},$$

for suitable choices of the parameters λ_+ and λ_- which you should find. (6 marks)

(Question 2 continues on next page)

(d) Using your answers to (b) and (c), establish the formula

$$3\sqrt{5} = \sum_{m=1}^{5} \left(\frac{\lambda_+^6 - \lambda_-^6}{1 + \lambda_m} \right) (-1)^{m+1} \sin^2 \left(\frac{m\pi}{6} \right).$$

(4 marks)

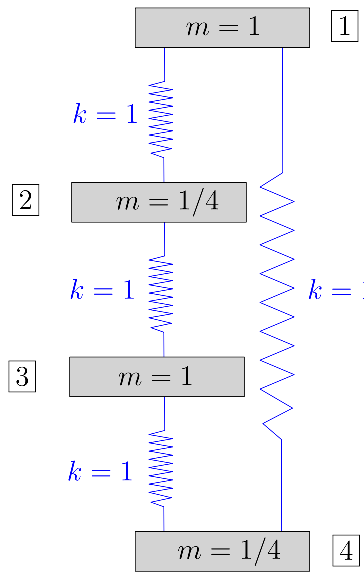
(e) Show that the effective conductance of the circuit is

$$1 - \frac{1}{3} \sum_{m=1}^{5} \frac{1}{(1 + \lambda_m)} \sin^2 \left(\frac{m\pi}{6} \right).$$

(4 marks)

(Total: 20 marks)

3. Consider the spring mass system shown in the figure: the masses labelled $\boxed{1}$ and $\boxed{3}$ have mass $m = 1$; the masses labelled $\boxed{2}$ and $\boxed{4}$ have mass $m = 1/4$. All four springs have a spring constant $k = 1$ as indicated. It is assumed that the masses can only be displaced in the direction parallel to the springs which are all aligned as shown in the figure.



Let $\mathbf{K}$ be the spring-constant-weighted Laplacian for this system considered as a graph with columns numbered according to the labels $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$. Let $\mathbf{M}$ be the corresponding diagonal matrix of masses. Introduce the 4-by-4 matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Calculate the matrix $\mathbf{S}^2$. (1 mark)

(b) Use your answer to part (a) to compute the eigenvalues and eigenvectors of $\mathbf{S}$. (4 marks)

(c) If ω denotes a natural frequency of free oscillation of this spring-mass system show that the possible values of ω^2 are the eigenvalues of the matrix $\hat{\mathbf{K}} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$. (3 marks)

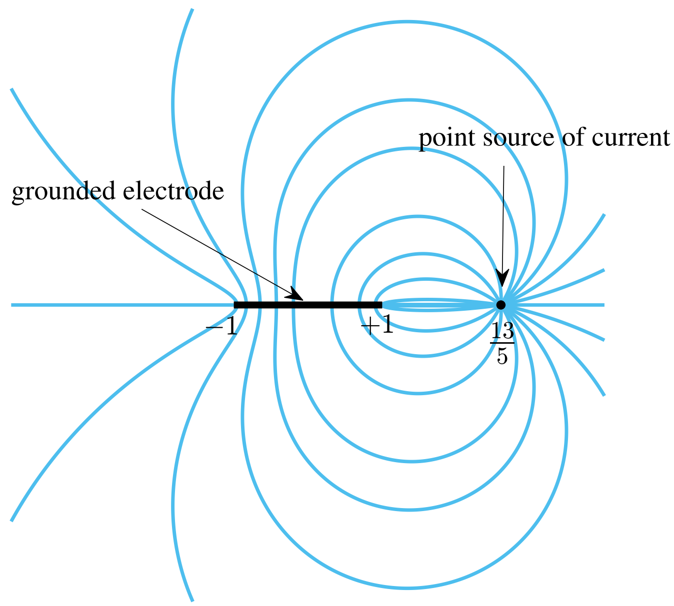
(d) Calculate the three matrices $\hat{\mathbf{K}}$, $\mathbf{S}\hat{\mathbf{K}}$ and $\hat{\mathbf{K}}\mathbf{S}$. (2 marks)

(e) Use your answer to part (d) to show that if $\mathbf{x}$ is an eigenvector of $\hat{\mathbf{K}}$ then $\mathbf{S}\mathbf{x}$ is also an eigenvector of $\hat{\mathbf{K}}$ with the same eigenvalue as $\mathbf{x}$. (2 marks)

(f) Using the result of part (e), or otherwise, find the possible values of ω^2. (8 marks)

(Total: 20 marks)

4. Consider a two-dimensional conductor of unit conductivity occupying the whole (x, y)-plane except for an electrode occupying the interval $x \in (-1, 1)$ on the x-axis. A voltage distribution $\phi(x, y)$ is caused by a point source of current of strength m at $(\frac{13}{5}, 0)$ with the electrode on the x-axis being grounded. The figure show typical current lines which are lines that are everywhere parallel to the current density vector $(J^{(x)}, J^{(y)})$.



The voltage distribution $\phi(x, y)$ at any point (x, y) in the conductor is given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad h(z) = -\frac{m}{2\pi} \log\left(\frac{5z - 1 - 5\sqrt{z^2 - 1}}{z - 5 - \sqrt{z^2 - 1}}\right), \qquad z = x + \mathrm{i}y,$$

where $\mathrm{i} = \sqrt{-1}$ and $\mathrm{Re}[.]$ denotes the real part of the complex-valued quantity in square brackets.

(a) Verify that the electrode is grounded, i.e., verify that $\phi(x, 0) = 0$ for $|x| < 1$. (6 marks)

(b) Verify, using this expression for $\phi(x, y)$, that this solution gives the required current source of strength m at $(\frac{13}{5}, 0)$. (6 marks)

(c) Find a formula for the complex current density $J^{(x)} - \mathrm{i}J^{(y)}$. (4 marks)

(d) Show that as $x \to 1^+$ (that is, as x tends to 1 from the right) the quantity $J^{(x)}$ has the singular behaviour

$$J^{(x)} \to \frac{c}{\sqrt{x - 1}}$$

and find the real constant c. (4 marks)

(Total: 20 marks)

Students may use any of the results below in the M40007 examination.

(I) Trigonometric addition formulas:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A, \qquad \cos(A+B) = \cos A\cos B - \sin B\sin A.$$

(II) The formula for an infinite geometric series is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

(III) The set of linearly independent vectors

$$\mathbf{x}_k = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/n}, \qquad k = 0, 1, \cdots, n-1,$$

are complex-valued eigenvectors of an n-by-n circulant matrix.

(IV) The matrix

$$\mathbf{K}_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 2 \end{pmatrix}, \qquad n \geq 2$$

has orthonormal eigenvectors

$$\boldsymbol{\Phi}_m = \sqrt{\frac{2}{n+1}}\begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \sin\left(\frac{2m\pi}{n+1}\right) \\ \cdot \\ \cdot \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}, \qquad m = 1, \cdots, n$$

with corresponding eigenvalues

$$\lambda_m = 2 - 2\cos\left(\frac{\pi m}{n+1}\right).$$

1. (a) The incidence matrix is m-by-n where m is the number of edges, n is the number of nodes. First count the nodes. For the tree with n generations there are

sim. seen ⇓

$$N = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1 \tag{1}$$

nodes where we have used a well-known result on the sum of a finite geometric progression. Next count the number of edges. For the tree with n generations there are

$$M = 2 + 2^2 + 2^2 + 2^3 + \cdots + 2^n = N - 1 = 2^{n+1} - 2. \tag{2}$$

The incidence matrix of a tree with n generations therefore is therefore

$$(2^{n+1} - 2) \text{ by } (2^{n+1} - 1). \tag{3}$$

1, A

(b) (i) By the symmetry of the graph and the chosen electrified nodes, it is clear that all nodes in the same generation k, with $1 \leq k \leq n$, of the tree will be at the same potential ϕ_k, with $\phi_0 = 1$ and $\phi_n = 0$. This means we can combine equipotential nodes at each generation, while preserving all edges, to produce the equivalent circuit shown on the left in this figure:

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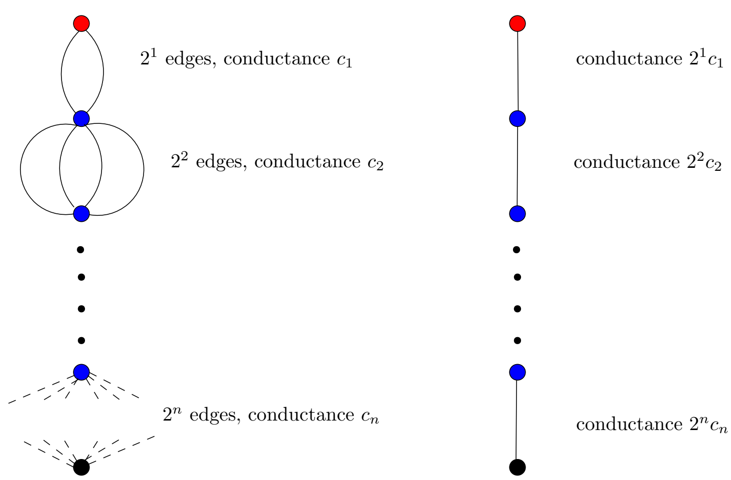

Figure 1: Equivalent circuits on combining equipotential nodes of-n-gen tree.

Then, because conductances in parallel are additive, this is equivalent to the series circuit shown on the right of the above figure. The effective conductance $C_{\text{eff}}^{(n)}$ of this equivalent set of n conductors in series is then known to satisfy

$$R_{\text{eff}}^{(n)} = \frac{1}{C_{\text{eff}}^{(n)}} = \sum_{k=1}^{n} \frac{1}{2^k c_k} \tag{4}$$

because resistances in series are additive. In the case when $c_k = 1$ for all $k = 1, \ldots, n$, then

$$\frac{1}{C_{\text{eff}}^{(n)}} = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = R_{\text{eff}}^{(n)}. \tag{5}$$

One can either quote and use the formula for a finite geometric series or, equivalently, on multiplying by $1/2$,

$$\frac{R_{\text{eff}}^{(n)}}{2} = \frac{1}{2^2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}. \tag{6}$$

Then subtracting (6) from (5) gives, after a telescoping sum,

$$\frac{R_{\text{eff}}^{(n)}}{2} = \frac{1}{2} - \frac{1}{2^{n+1}}, \qquad R_{\text{eff}}^{(n)} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}. \tag{7}$$

Thus,

$$C_{\text{eff}}^{(n)} = \frac{2^n}{2^n - 1}. \tag{8}$$

$\boxed{4, \text{A}}$

(ii) Labelling the root node as $\boxed{0}$, and using the fact that the potential is the conductance-weighted average of the neighbouring potentials in the series circuit on the right of the figure above,

$\boxed{\text{sim. seen} \Downarrow}$

$$\phi_k = \frac{2^k \phi_{k-1} + 2^{k+1}\phi_{k+1}}{2^k + 2^{k+1}}, \qquad 1 \le k \le n-1 \tag{9}$$

or

$$(2^k + 2^{k+1})\phi_k = 2^k \phi_{k-1} + 2^{k+1}\phi_{k+1}, \qquad 1 \le k \le n-1. \tag{10}$$

On dividing by 2^k, we arrive at the recurrence relation

$$(1+2)\phi_k = \phi_{k-1} + 2\phi_{k+1}, \qquad \text{or} \qquad 2\phi_{k+1} - 3\phi_k + \phi_{k-1} = 0 \tag{11}$$

for $1 \le k \le n-1$. There are two ways to solve this.
First – and this is the standard approach – we can seek solutions of the form $\phi_k = \lambda^k$,

$$2\lambda^2 - 3\lambda + 1 = 0, \qquad \text{or} \qquad (2\lambda - 1)(\lambda - 1) = 0. \tag{12}$$

so that

$$\phi_k = A + \frac{B}{2^k}, \qquad 1 \le k \le n-1 \tag{13}$$

for some constants A and B. For $k = 1$ the recurrence relation says

$$2\phi_2 - 3\phi_1 + 1 = 0, \qquad \text{or} \qquad 2\left(A + \frac{B}{2^2}\right) - 3\left(A + \frac{B}{2}\right) + 1 = 0 \tag{14}$$

or

$$A + B = 1 \tag{15}$$

With $\phi_n = 0$ the recurrence relation for $k = n-1$ says

$$-3\phi_{n-1} + \phi_{n-2} = 0, \qquad \text{or} \qquad -3\left(A + \frac{B}{2^{n-1}}\right) + A + \frac{B}{2^{n-2}} = 0 \tag{16}$$

giving

$$-2A + \frac{B}{2^{n-2}}\left(1 - \frac{3}{2}\right) = 0, \qquad \text{or} \qquad A + \frac{B}{2^n} = 0. \tag{17}$$

Thus, on subtracting (17) from (15),

$$B\left(1 - \frac{1}{2^n}\right) = 1, \qquad \text{or} \qquad B = \frac{2^n}{2^n - 1}, \qquad A = -\frac{1}{2^n - 1}. \tag{18}$$

Thus,

$$\boxed{\phi_k = \frac{1}{2^n - 1}\left(2^{n-k} - 1\right), \qquad 0 \le k \le n} \tag{19}$$

$\boxed{6, \text{C}}$

Alternative: A second method to solve

$$2\phi_{k+1} - 3\phi_k + \phi_{k-1} = 0 \tag{20}$$

for $1 \leq k \leq n-1$ is to notice it can be written as

$$2(\phi_{k+1} - \phi_k) - (\phi_k - \phi_{k-1}) = 0 \tag{21}$$

or, defining $p_k \equiv \phi_k - \phi_{k-1}$,

$$2p_{k+1} = p_k, \qquad \text{or} \qquad p_{k+1} = \frac{p_k}{2}. \tag{22}$$

It is clear that the solution is

$$p_k = \frac{p_1}{2^{k-1}}. \tag{23}$$

This implies that

$$\phi_k - \phi_{k-1} = \frac{p_1}{2^{k-1}} \tag{24}$$

implying,

$$\begin{aligned}
\phi_2 - \phi_1 &= \frac{p_1}{2}, \\
\phi_3 - \phi_2 &= \frac{p_1}{2^2}, \\
\phi_4 - \phi_3 &= \frac{p_1}{2^3} \\
&\dots \\
\phi_{k-1} - \phi_{k-2} &= \frac{p_1}{2^{k-2}} \\
\phi_k - \phi_{k-1} &= \frac{p_1}{2^{k-1}}
\end{aligned} \tag{25}$$

so that, noticing the telescoping sum,

$$\phi_k - \phi_1 = (\phi_1 - 1)\left[\frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^{k-1}}\right] = (\phi_1 - 1)\left(1 - \frac{1}{2^{k-1}}\right) \tag{26}$$

where we have used $p_1 = \phi_1 - \phi_0$ with $\phi_0 = 1$ and the fact that if

$$S = \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^{k-1}}, \qquad \frac{S}{2} = + \frac{1}{2^2} + \dots \frac{1}{2^k}, \tag{27}$$

then

$$\frac{S}{2} = \frac{1}{2} - \frac{1}{2^k}, \qquad \text{or} \qquad S = \left(1 - \frac{1}{2^{k-1}}\right). \tag{28}$$

Hence

$$\phi_k = \phi_1 + (\phi_1 - 1)\left(1 - \frac{1}{2^{k-1}}\right) \tag{29}$$

We do not yet know ϕ_1, but we know that $\phi_n = 0$, hence

$$0 = \phi_1 + (\phi_1 - 1)\left(1 - \frac{1}{2^{n-1}}\right) \tag{30}$$

or

$$0 = \phi_1\left(1 + 1 - \frac{1}{2^{n-1}}\right) - \left(1 - \frac{1}{2^{n-1}}\right) \tag{31}$$

which yields

$$\phi_1 = \frac{2^{n-1}-1}{2^n-1} \tag{32}$$

Now

$$\phi_1 - 1 = \frac{2^{n-1}-1-2^n+1}{2^n-1} = \frac{2^{n-1}-2^n}{2^n-1} = -\frac{2^{n-1}}{2^n-1} \tag{33}$$

Therefore, from (26),

$$\phi_k = \frac{2^{n-1}-1}{2^n-1} - \frac{2^{n-1}}{2^n-1}\left(1-\frac{1}{2^{k-1}}\right) = \frac{2^{n-k}-1}{2^n-1} \tag{34}$$

which is the same as the result found earlier.

Check: we can check this result by computing the current out of the root node which, by Ohm's law, is

$$\begin{aligned} 2\times(1-\phi_1) &= 2\left(1-\frac{1}{2^n-1}\left(2^{n-1}-1\right)\right) = 2\left(\frac{2^n-2^{n-1}}{2^n-1}\right) \\ &= 2^{n+1}\left(\frac{1-1/2}{2^n-1}\right) \\ &= \frac{2^n}{2^n-1} \end{aligned} \tag{35}$$

which agrees with the effective conductance computed in part (i).

sim. seen ⇓

(iii) It is clear from part (i) that

$$C_{\text{eff}}^{(\infty)} = \lim_{n\to\infty}\left(\frac{2^n}{2^n-1}\right) = 1. \tag{36}$$

1, A

(c) From (4), with $c_k = 1/k$, in this case

unseen ⇓

$$\frac{1}{C_{\text{eff}}^{(\infty)}} = \sum_{k=1}^{\infty}\frac{k}{2^k} = \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots = \frac{1}{2}\left(1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots\right). \tag{37}$$

But the well-known geometric series formula is

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots, \qquad |x|<1 \tag{38}$$

so, on differentiation,

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots, \qquad |x|<1. \tag{39}$$

Therefore, on recognizing the same sum in (37) with $x = 1/2$,

$$\frac{1}{C_{\text{eff}}^{(\infty)}} = \frac{1}{2}\times\left(\frac{1}{(1-1/2)^2}\right) = 2, \qquad \text{or} \qquad C_{\text{eff}}^{(\infty)} = \frac{1}{2}. \tag{40}$$

4, A

(d) From (4), with $c_k = k$, in this case

unseen ⇓

$$\frac{1}{C_{\text{eff}}^{(\infty)}} = \sum_{k=1}^{\infty}\frac{1}{k2^k} = \frac{1}{2}+\frac{1}{2}\frac{1}{2^2}+\frac{1}{3}\frac{1}{2^3}+\frac{1}{4}\frac{1}{2^4}+\ldots \tag{41}$$

But on integration of the geometric series formula (38),

$$-\log(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots, \qquad |x|<1. \tag{42}$$

Therefore, on recognizing the same sum in (41) with $x = 1/2$,

$$\frac{1}{C_{\text{eff}}^{(\infty)}} = -\log(1 - 1/2) = \log 2, \qquad \text{or} \qquad C_{\text{eff}}^{(\infty)} = \frac{1}{\log 2}. \tag{43}$$

$\boxed{4,\ \mathrm{D}}$

2. (a) It is convenient to proceed for general N where, in the given figure, the case $N = 5$ is shown. In each answer, we then set $N = 5$ as the final step.

sim. seen $\Downarrow$

If the nodes are labelled as specified in the question then the Laplacian has the sub-block structure:

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_N + \mathbf{I}_N & -\mathbf{I}_N & -\mathbf{P} \\ -\mathbf{I}_N & \mathbf{I}_N & \mathbf{0} \\ -\mathbf{P}^T & \mathbf{0} & \mathbf{I}_2 \end{pmatrix},$$

where $\mathbf{I}_j$ denotes the j-by-j identity matrix and $\mathbf{P}$ is the N-by-2 matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ . & . \\ . & . \\ . & . \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{44}$$

and

$$\mathbf{K}_N = \begin{pmatrix} 2 & -1 & 0 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 2 & -1 \\ 0 & 0 & 0 & 0 & . & . & -1 & 2 \end{pmatrix}, \tag{45}$$

is the matrix familiar from lectures and whose eigenvectors/eigenvalues are given on the examination handout.

For $N = 5$, clearly,

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{46}$$

and

$$\mathbf{K}_5 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{47}$$

2, A

(b) It is known from lectures, and the examination handout, that orthonormal

sim. seen $\Downarrow$

eigenvectors of $\mathbf{K}_N$ are

$$\mathbf{\Phi}_m = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin\left(\frac{m\pi}{N+1}\right) \\ \sin\left(\frac{2m\pi}{N+1}\right) \\ \cdot \\ \cdot \\ \sin\left(\frac{Nm\pi}{N+1}\right) \end{pmatrix}, \qquad m = 1, \cdots, N \tag{48}$$

with eigenvalues

$$\lambda_m = 2 - 2\cos\left(\frac{\pi m}{N+1}\right), \qquad m = 1, \cdots, N. \tag{49}$$

Therefore, for $N = 5$,

$$\mathbf{\Phi}_m = \sqrt{\frac{1}{3}} \begin{pmatrix} \sin\left(\frac{m\pi}{6}\right) \\ \sin\left(\frac{2m\pi}{6}\right) \\ \sin\left(\frac{3m\pi}{6}\right) \\ \sin\left(\frac{4m\pi}{6}\right) \\ \sin\left(\frac{5m\pi}{6}\right) \end{pmatrix}, \qquad m = 1, \cdots, 5. \tag{50}$$

To find the unknown voltages at the blue notes, collected in the N-dimensional vector $\hat{\mathbf{x}}$, we let

$$\mathbf{x} = \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \\ \mathbf{e} \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{51}$$

and where $\mathbf{0}$ denotes an N-dimensional zero vector. The system to solve is the usual $\mathbf{Kx} = \mathbf{f}$ which has the form

$$\begin{pmatrix} \mathbf{K}_N + \mathbf{I}_N & -\mathbf{I}_N & -\mathbf{P} \\ -\mathbf{I}_N & \mathbf{I}_N & \mathbf{0} \\ -\mathbf{P}^T & \mathbf{0} & \mathbf{I}_2 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}, \tag{52}$$

where the elements of $\mathbf{f}$ on the right hand side corresponding to the blue nodes are zero by Kirchhoff's current law. The subsystem to solve for $\hat{\mathbf{x}}$ is

$$(\mathbf{K}_N + \mathbf{I}_N)\hat{\mathbf{x}} = \mathbf{Pe} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}. \tag{53}$$

On letting

$$\hat{\mathbf{x}} = \sum_{j=1}^{N} a_j \mathbf{\Phi}_j \tag{54}$$

and substituting into (53), we find

$$(\mathbf{K}_N + \mathbf{I}_N)\hat{\mathbf{x}} = (\mathbf{K}_N + \mathbf{I}_N) \sum_{j=1}^{N} a_j \mathbf{\Phi}_j = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \tag{55}$$

or, using the properties of the eigenvectors,

$$\sum_{j=1}^{N}(\lambda_j+1)a_j\mathbf{\Phi}_j = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}. \tag{56}$$

We multiply this equation by $\mathbf{\Phi}_m^T$ and use the orthonormality of the eigenvectors:

$$\sum_{j=1}^{N}(\lambda_j+1)a_j\underbrace{\mathbf{\Phi}_m^T\mathbf{\Phi}_j}_{\delta_{mj}} = \mathbf{\Phi}_m^T\begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = \sqrt{\frac{2}{N+1}}\sin\left(\frac{Nm\pi}{N+1}\right). \tag{57}$$

We infer that

$$a_m = \sqrt{\frac{2}{N+1}}\frac{1}{(1+\lambda_m)}\sin\left(\frac{Nm\pi}{N+1}\right). \tag{58}$$

Hence, one representation of the solution is

$$\begin{aligned}\hat{\mathbf{x}} &= \sum_{m=1}^{N}\sqrt{\frac{2}{N+1}}\frac{1}{(1+\lambda_m)}\sin\left(\frac{Nm\pi}{N+1}\right)\mathbf{\Phi}_m \\ &= \sum_{m=1}^{N}\sqrt{\frac{2}{N+1}}\frac{(-1)^{m+1}}{(1+\lambda_m)}\sin\left(\frac{m\pi}{N+1}\right)\mathbf{\Phi}_m\end{aligned} \tag{59}$$

where we have used the trigonometric addition formula (also given on the handout)

$$\sin\left(\frac{Nm\pi}{N+1}\right) = (-1)^{m+1}\sin\left(\frac{m\pi}{N+1}\right) \tag{60}$$

If we now restrict to the special case $N=5$, the result is

$$\boxed{\hat{\mathbf{x}} = \sum_{m=1}^{5}\sqrt{\frac{1}{3}}\frac{1}{(1+\lambda_m)}\sin\left(\frac{5m\pi}{6}\right)\mathbf{\Phi}_m} \tag{61}$$

or

$$\boxed{\hat{\mathbf{x}} = \sum_{m=1}^{5}\sqrt{\frac{1}{3}}\frac{(-1)^{m+1}}{(1+\lambda_m)}\sin\left(\frac{m\pi}{6}\right)\mathbf{\Phi}_m} \tag{62}$$

$\boxed{\text{4, B}}$

(c) The solution can be found another way, using a recurrence relation. By the mean value property for harmonic potentials, the elements of $\hat{\mathbf{x}}$, denoted by $\boxed{\text{sim. seen} \Downarrow}$

$$\hat{\mathbf{x}} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_N \end{pmatrix} \tag{63}$$

satisfy

$$\phi_n = \frac{\phi_{n+1} + \phi_{n-1} + 0}{3} \tag{64}$$

or

$$\phi_{n+1} - 3\phi_n + \phi_{n-1} = 0. \tag{65}$$

This recurrence relation can be solved by seeking solutions of the form $\phi_n = \lambda^n$ which requires that

$$\lambda^2 - 3\lambda + 1 = 0, \qquad \text{or} \qquad \lambda_\pm = \frac{3 \pm \sqrt{5}}{2}. \tag{66}$$

The solution is also required to satisfy the boundary conditions

$$\phi_0 = 0, \qquad \phi_{N+1} = 1, \tag{67}$$

where we have extended our notation in a natural way so that ϕ_0 denotes the potential at node $\boxed{0}$ and ϕ_{N+1} denotes the potential at node $\boxed{2N+1}$. The required solution is therefore given by

$$\phi_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+^{N+1} - \lambda_-^{N+1}}, \qquad n = 1, \ldots, N \tag{68}$$

or, specializing now to $N = 5$,

$$\phi_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+^6 - \lambda_-^6}, \qquad n = 1, \ldots, 5. \tag{69}$$

$\boxed{6, C}$

(d) By the uniqueness theorem for harmonic potentials, the n-th element of the solution (62) must coincide to the value given by (69). It therefore follows that, for any $n = 1, \ldots, N$, $\boxed{\text{sim. seen} \Downarrow}$

$$\phi_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+^{N+1} - \lambda_-^{N+1}} = \sum_{m=1}^{N} \frac{2}{N+1} \frac{(-1)^{m+1}}{(1+\lambda_m)} \sin\left(\frac{m\pi}{N+1}\right) \sin\left(\frac{mn\pi}{N+1}\right), \tag{70}$$

for $n = 1, \ldots, N$.

If we now specialize to $N = 5$ and pick $n = 1$, and notice that

$$\lambda_+ - \lambda_- = \sqrt{5}, \tag{71}$$

then formula (70) gives the required result, namely,

$$3\sqrt{5} = \sum_{m=1}^{5} \frac{\lambda_+^6 - \lambda_-^6}{1 + \lambda_m} (-1)^{m+1} \sin^2\left(\frac{m\pi}{6}\right) \tag{72}$$

after multiplication by some numerical factors. $\boxed{4, B}$

(e) By definition, the effective conductance is the total current (into the circuit) from the unit-voltage node. By Ohm's law, and the fact that all edges have unit conductance, this is $\boxed{\text{sim. seen} \Downarrow}$

$$1 - \phi_5. \tag{73}$$

Now, by (70) with $n = 5$ and $N = 5$,

$$\phi_5 = \sum_{m=1}^{N} \frac{1}{3} \frac{(-1)^{m+1}}{(1+\lambda_m)} \sin\left(\frac{m\pi}{6}\right) \sin\left(\frac{5m\pi}{6}\right). \tag{74}$$

Given that

$$\sin\left(\frac{5m\pi}{6}\right) = \sin\left(\frac{(6-1)m\pi}{6}\right) = (-1)^{m+1}\sin\left(\frac{m\pi}{6}\right), \tag{75}$$

which follows from a trigonometric identity given on the handout, the effective conductance is

$$1 - \sum_{m=1}^{5} \frac{1}{3}\frac{1}{(1+\lambda_m)}\sin^2\left(\frac{m\pi}{6}\right). \tag{76}$$

$\boxed{4, \mathrm{B}}$

3. (a) A direct calculation shows that $\mathbf{S}^2 = \mathbf{I}_4$ where $\mathbf{I}_4$ denotes the 4-by-4 identity matrix. Alternatively, since

sim. seen ⇓

$$\mathbf{S}\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{pmatrix}$$

then the action of $\mathbf{S}$ on a typical vector $\mathbf{x}$ is to swap the first and third components of $\mathbf{x}$, as well as its second and fourth components. It is clear that doing this twice will leave the vector unchanged, implying again that $\mathbf{S}^2 = \mathbf{I}_4$.

1, A

(b) Suppose $\hat{\mathbf{x}}$ is an eigenvector of $\mathbf{S}$ with eigenvalue $\hat{\lambda}$, then

meth seen ⇓

$$\mathbf{S}\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}.$$

Operating on this again with $\mathbf{S}$ yields

$$\mathbf{S}^2\hat{\mathbf{x}} = \mathbf{S}(\hat{\lambda}\hat{\mathbf{x}}) = \hat{\lambda}\mathbf{S}\hat{\mathbf{x}} = \hat{\lambda}^2\mathbf{x} = \hat{\mathbf{x}}$$

where the last equality follows from part (a). Thus,

$$\hat{\lambda}^2 = 1, \qquad \text{or}, \qquad \hat{\lambda} = \pm 1.$$

These are the only possible eigenvalues. However, as noticed in the answer to part (a), the action of $\mathbf{S}$ on a typical vector $\hat{\mathbf{x}}$ is to swap the first and third components of $\hat{\mathbf{x}}$, as well as its second and fourth components. It is then clear that two linearly independent eigenvectors corresponding to $\hat{\lambda} = 1$ are

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

since these are invariant under the action of $\mathbf{S}$. Moreover, two linearly independent eigenvectors corresponding to $\hat{\lambda} = -1$ are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

since these swap sign under the action of $\mathbf{S}$.

4, A

(c) We know from lectures that Newton's second law, and the fact that the internal spring forces are given by $-\mathbf{K}\mathbf{x}$ where $\mathbf{x}$ is the vector of displacements, that

sim. seen ⇓

$$\mathbf{M}\frac{d^2\mathbf{x}}{dt^2} = -\mathbf{K}\mathbf{x} \tag{77}$$

Letting $\mathbf{x} = \mathbf{x}_0 e^{\pm i\omega t}$ yields

$$-\omega^2\mathbf{M}\mathbf{x}_0 = -\mathbf{K}\mathbf{x}_0 \tag{78}$$

or

$$\omega^2\mathbf{M}\mathbf{x}_0 = \mathbf{K}\mathbf{x}_0. \tag{79}$$

This is not a standard eigenvalue problem (although it is a generalized eigenvalue problem). On multiplying both sides by the diagonal matrix $\mathbf{M}^{-1/2}$ it can be written as

$$\omega^2 \mathbf{M}^{1/2}\mathbf{x}_0 = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{x}_0 \tag{80}$$

which itself can be rewritten as

$$\omega^2[\mathbf{M}^{1/2}\mathbf{x}_0] = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{M}^{1/2}\mathbf{x}_0 = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}[\mathbf{M}^{1/2}\mathbf{x}_0] \tag{81}$$

This shows that ω^2 are the eigenvalues of $\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$.

$\boxed{3,\ \mathrm{A}}$

(d) By the usual rules of construction of a weighted-Laplacian matrix, we find

$\boxed{\text{meth seen} \Downarrow}$

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

while

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}.$$

Hence

$$\mathbf{M}^{-1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Direct calculation gives

$$\hat{\mathbf{K}} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2} = \begin{pmatrix} 2 & -2 & 0 & -2 \\ -2 & 8 & -2 & 0 \\ 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & 8 \end{pmatrix}$$

and

$$\hat{\mathbf{K}}\mathbf{S} = \begin{pmatrix} 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & 8 \\ 2 & -2 & 0 & -2 \\ -2 & 8 & -2 & 0 \end{pmatrix}$$

and

$$\mathbf{S}\hat{\mathbf{K}} = \begin{pmatrix} 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & 8 \\ 2 & -2 & 0 & -2 \\ -2 & 8 & -2 & 0 \end{pmatrix}.$$

$\boxed{2,\ \mathrm{A}}$

(e) We note from part (d) that $\hat{\mathbf{K}}$ and $\mathbf{S}$ commute: $\hat{\mathbf{K}}\mathbf{S} = \mathbf{S}\hat{\mathbf{K}}$. Hence if $\mathbf{x}$ is an eigenvector of $\hat{\mathbf{K}}$ with eigenvalue λ then

$\boxed{\text{sim. seen} \Downarrow}$

$$\hat{\mathbf{K}}\mathbf{x} = \lambda\mathbf{x}.$$

Operating on this with $\mathbf{S}$:

$$\mathbf{S}\hat{\mathbf{K}}\mathbf{x} = \mathbf{S}(\lambda\mathbf{x}) = \lambda(\mathbf{S}\mathbf{x}) = \hat{\mathbf{K}}\mathbf{S}\mathbf{x}$$

where the last equality follows because $\hat{\mathbf{K}}$ and $\mathbf{S}$ commute. But the last equation, namely $\lambda(\mathbf{S}\mathbf{x}) = \hat{\mathbf{K}}(\mathbf{S}\mathbf{x})$, shows that $\mathbf{S}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{K}}$ with eigenvalue λ. 2, B

(f) First, we know from lectures that if $\hat{\mathbf{K}}$ and $\mathbf{S}$ commute then any eigenvectors of $\mathbf{S}$ that have eigenspace dimension (algebraic multiplicity) equal to 1 are also eigenvectors of $\hat{\mathbf{K}}$. However, this fact is of no direct use to us here because we know from part (b) that all eigenvalues of $\mathbf{S}$ have multiplicity 2. unseen $\Downarrow$

However, this statement also works with $\hat{\mathbf{K}}$ and $\mathbf{S}$ swapped and, although we do not know them yet, the eigenvalues of $\hat{\mathbf{K}}$ can *generically* be expected to have unit multiplicity. *If so* (and this assumption will be tested later for consistency) any such eigenvectors will *also* be eigenvectors of $\mathbf{S}$. This is consistent with the observation of part (e) and suggests that we should restrict consideration to seeking eigenvectors of $\hat{\mathbf{K}}$ of the form

$$\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix}. \tag{82}$$

This is because the first vector expresses a general eigenvector of $\mathbf{S}$ associated with its eigenvalue 1 and the second vector is a general eigenvector of $\mathbf{S}$ associated with its eigenvalue -1. With the first choice, and using $\hat{\mathbf{K}}$ as calculated in part (c), direct multiplication gives

$$\hat{\mathbf{K}}\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix} = \begin{pmatrix} 2a-4b \\ -4a+8b \\ 2a-4b \\ -4a+8b \end{pmatrix} = (2a-4b)\begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \end{pmatrix}$$

suggesting either that $2a - 4b = 0$, or $b = a/2$, yielding the eigenvector/eigenvalue pair

$$\begin{pmatrix} 1 \\ 1/2 \\ 1 \\ 1/2 \end{pmatrix}, \qquad \lambda = 2a - 4b = 0$$

or $a = 1, b = -2$ yielding the eigenvector/eigenvalue pair

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \end{pmatrix}, \qquad \lambda = 2a - 4b = 10.$$

With the second choice in (82), and using $\hat{\mathbf{K}}$ as calculated in part (c), direct multiplication gives

$$\hat{\mathbf{K}}\begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix} = \begin{pmatrix} 2a \\ 8b \\ -2a \\ -8b \end{pmatrix}.$$

Thus if $b = 0$ then an eigenvalue is $\lambda = 2$; if $a = 0$ the eigenvalue is $\lambda = 8$. These two possible solutions yield the eigenvector/eigenvalue pairs

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \lambda = 2$$

and

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \lambda = 8.$$

We know that the frequencies ω of free oscillation of the system satisfy $\omega^2 = \lambda$ so

$$\omega^2 = 0, 2, 8, 10$$

are the possible values of ω^2.

Note that these eigenvalues are all of unit (algebraic) multiplicity, which is consistent with our original assumption in deriving these values. $\boxed{8, \text{D}}$

4. (a) Let

meth seen ⇓

$$R(z) = \frac{5z - 1 - 5\sqrt{z^2-1}}{z - 5 - \sqrt{z^2-1}}$$

so that

$$\phi = \mathrm{Re}\left[-\frac{m}{2\pi}\log R(z)\right].$$

We aim to show that $|R(z)| = 1$ when $z \in [-1,1]$. For $z \in [-1,1]$, then $\sqrt{z^2-1} = \pm \mathrm{i}\sqrt{1-x^2}$ (the sign depends on whether you are on the "top" or "bottom" of the grounded electrode) and $z = x$, therefore

$$R(x) = \frac{5x - 1 \mp 5\mathrm{i}\sqrt{1-x^2}}{x - 5 \mp \mathrm{i}\sqrt{1-x^2}}.$$

Writing this in standard form gives:

$$\begin{aligned} R(x) &= \frac{5x - 1 \mp 5\mathrm{i}\sqrt{1-x^2}}{x - 5 \mp \mathrm{i}\sqrt{1-x^2}} \times \frac{x - 5 \pm \mathrm{i}\sqrt{1-x^2}}{x - 5 \pm \mathrm{i}\sqrt{1-x^2}} \\ &= \frac{(5x-1)(x-5) + 5(1-x^2) \mp \mathrm{i}(1-x^2)(5(x-5) - (5x-1))}{(x-5)^2 + 1 - x^2} \\ &= \frac{10 - 26x \pm 24\mathrm{i}\sqrt{1-x^2}}{26 - 10x}. \end{aligned}$$

We want to show this has unit modulus. But the squared modulus of the numerator is

$$\begin{aligned} (10-26x)^2 + 24^2(1-x^2) &= 10^2 - 2\times 10\times 26x + 26^2x^2 - 24^2x^2 + 24^2 \\ &= (26^2 - 24^4)x^2 - 2\times 10 \times 26x + (24^2 + 10^2) \\ &= (10x - 26)^2 \end{aligned}$$

since $26^2 - 24^4 = (26-24)(26+24) = 100$. Therefore $|R(z)| = 1$ when $z \in [-1,1]$ implying that this interval is grounded, i.e., $\phi = 0$ there.

6, A

(b) To ensure that there is a source, of strength m, at $z = 13/5$, we must check that the Taylor expansion of $R = R(z)$ about $z = 13/5$, i.e.,

meth seen ⇓

$$R(z) = R(13/5) + \left(z - \frac{13}{5}\right) R'(13/5) + \dots$$

is such that

$$R(13/5) = 0, \qquad R'(13/5) \neq 0.$$

(Note that if $R(13/5) = R'(13/5) = 0$ with $R''(13/5) \neq 0$ then the current source strength would be $2m$). When $z = 13/5$ the numerator of $R(z)$ is

$$5z - 1 - 5\sqrt{z^2-1} = 13 - 1 - 5\left(\frac{169}{25} - 1\right)^{1/2} = 12 - 5 \times \sqrt{\frac{144}{25}} = 0.$$

We should also check that the denominator does not vanish too:

$$z - 5 - \sqrt{z^2-1} = \frac{13}{5} - 5 - \left(\frac{169}{25} - 1\right)^{1/2} = \frac{13}{5} - \frac{25}{5} - \sqrt{\frac{144}{25}} = -\frac{24}{5} \neq 0.$$

Hence $R(13/5) = 0$ as required. By the quotient rule,

$$R'(z) = \frac{(z - 5 - \sqrt{z^2-1})(5 - (5z/\sqrt{z^2-1})) - (1 - (z/\sqrt{z^2-1}))(5z - 1 - 5\sqrt{z^2-1})}{(z - 5 - \sqrt{z^2-1})^2}$$

whose numerator evaluated at $z = 13/5$ is

$$-\frac{24}{5} \times \left(5 - 13/\sqrt{144/25}\right) \neq 0.$$

This confirms there is a logarithmic singularity, of strength $-m/(2\pi)$, at $z = 13/5$, which we know from lectures to be a point current source of strength m.

6, B

(c) We know from lectures that, for a conductor with unit conductivity,

meth seen ⇓

$$J^{(x)} - \mathrm{i}J^{(y)} = -h'(z) = \frac{m}{2\pi}\left[\frac{5 - (5z/\sqrt{z^2-1})}{5z - 1 - 5\sqrt{z^2-1}} - \frac{1 - (z/\sqrt{z^2-1})}{z - 5 - \sqrt{z^2-1}}\right].$$

This expression is acceptable. (It can, of course, be simplified, but this is not required).

4, A

(d) Examining the previous expression in part (c) for $x \to 1^+$ then $\sqrt{z^2-1} = \sqrt{x-1}\sqrt{x+1}$ and, isolating the most singular term as $x \to 1^+$,

meth seen ⇓

$$\begin{aligned} J^{(x)} - \mathrm{i}J^{(y)} = \frac{m}{2\pi}\Bigg[&- \frac{5x}{\sqrt{x-1}\sqrt{x+1}}\left(\frac{1}{5x - 1 - 5\sqrt{x^2-1}}\right) \\ &+ \frac{x}{\sqrt{x-1}\sqrt{x+1}}\left(\frac{1}{x - 5 - \sqrt{x^2-1}}\right) + \text{more regular terms}\Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} J^{(x)} - \mathrm{i}J^{(y)} &\approx \frac{m}{2\pi}\left[-\frac{5}{\sqrt{x-1}\sqrt{1+1}}\left(\frac{1}{5-1}\right) + \frac{1}{\sqrt{x-1}\sqrt{1+1}}\left(\frac{1}{1-5}\right) + \dots\right] \\ &= \frac{c}{\sqrt{x-1}} \end{aligned}$$

where

$$\boxed{c = -\frac{3m}{4\sqrt{2}\pi}}$$

4, D

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH40007	1	Question 1 appeared to be found quite difficult by the students, and the performance was mixed. Many students got the key idea of combining nodes at each generation, which leads to quite a straightforward question, except perhaps for the last part which requires noticing a connection with series expansions of functions.
MATH40007	2	This question required good knowledge of a piece of coursework as far as I understand. Part a was a matter of identifying the Laplacian matrix and most students did well here, except that some overlooked the suggested ordering of the elements. Part b had to be answered with the help of the formula sheet available at the exam. Many students got at least partial marks here, identifying the correct eigenvectors and -values, but often not determining the resulting expression for a_m correctly, frequently overlooking the identity I_5. Part c required a different approach that many students did not attempt. Nevertheless, a good number of students managed to determine \lambda_\pm. In Part d) and e) I gave marks for using the correct entry of the vector (1 and 5 respectively). Many students did not attempt these parts.
MATH40007	3	This question on whole was poorly done. part (f) (worth 8 marks) was not answered by majority of candidates - those that managed to score a few marks here, used the direct determinant route, as opposed to the approach outlined in the solution sheet by the lecturer; only a handful of candidates were able to derive the corresponding eigenvectors (and then only partially). Part b, those who attempted use of circulant matrix properties to compute eigenvalues/eigenvectors fared poorly - it was obvious that usage of circulants was poorly mastered; just stating answers in terms of power of omegas for eigenvectors/eigenvectors was marked down - full marks were awarded if candidates actually worked out these to be (1,0,1,0), (1,0,-1,0) etc. and the eigenvalue to be +/- 1.
MATH40007	4	This question on whole was very poorly done. Part a, the fact that sqrt(x*x-1) had to be considered as imaginary (for x<1), was missed by a large majority of candidates and thus resulted in poor scores. Part b, while most deduced that it had to be shown that R(13/5) =0, only a handful of candidates were able to answer this part fully, as what also had to be shown was that R'(13/5) was not zero and that the denominator of h(z) expression was analytic. Part c, was done well, but quite a large number here, assumed that a differentiation of a(z)/b(z) was required, when in fact they required differentiating a Log [a(z)/b(z)] - hence loosing marks. Part d, was found challenging by all, with only a handful of students making an attempt - but still failing to get the required result.