Imperial College London

MATH70058

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Manifolds

Date: 18 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let X be a manifold of dimension n and let $Y \subset X$ be a submanifold of dimension $m < n$.

   (i) Show that Y is a manifold of dimension m. (4 marks)

   (ii) Show that if $Z \subset Y$ is a submanifold of Y then Z is a submanifold of X. (4 marks)

   (iii) Assume that Y is compact. Show that there exists a smooth function

$$F \colon X \to \mathbb{R}$$

   such that $F(x) \geq 0$ for all $x \in X$ and $F(y) = 0$ if and only if $y \in Y$. (4 marks)

(b) Recall that the n-dimensional projective space $\mathbb{P}^n_{\mathbb{R}}$ is the quotient of

$$W = \mathbb{R}^{n+1} \setminus \{\underline{0}\}$$

by the equvalence relation on W given by:

$$x, y \in W \quad x \sim y \qquad \text{if and only if} \quad x = \lambda y \quad \text{for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

We denote by $[x_0, \dots, x_n] \in X$ the equivalence class of $(x_0, \dots, x_n) \in W$ in X.

   (i) Let $X = \mathbb{P}^3_{\mathbb{R}}$ be the three-dimensional projective space and let

$$Y = \left\{ [x_0, x_1, x_2, x_3] \in X \mid x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 \right\}.$$

   Show that Y is a submanifold of X. (4 marks)

   (ii) Show that $\mathbb{P}^1_{\mathbb{R}}$ is diffeomorphic to the circle S^1. (4 marks)

(Total: 20 marks)

2. (a) (i) Find a vector field on the circle S^1 which vanishes at exactly one point. (3 marks)

(ii) Find a vector field on the two-dimensional sphere S^2 which vanishes at exactly one point. (3 marks)

(iii) Find a non-vanishing vector field on the one-dimensional projective space $\mathbb{P}^1_{\mathbb{R}}$. (2 marks)

(iv) Find a non-vanishing vector field on the three-dimensional projective space $\mathbb{P}^3_{\mathbb{R}}$. (4 marks)

(b) Let X be a manifold and let $x \in X$.

(i) Recall that

$$R_x(X) = \{h \in C^\infty(X) \mid \text{ the rank of } h \text{ at } x \text{ is zero } \}.$$

Show that if $V\colon C^\infty(X) \to \mathbb{R}$ is a linear map such that

$$V(h) = 0 \qquad \text{for all } h \in R_x(X)$$

then V is a derivation at $x \in X$. (4 marks)

(ii) Let $V\colon C^\infty(X) \to \mathbb{R}$ be a derivation on X at x, let U be an open neighbourhood of x in X and let $h_1, h_2 \in C^\infty(X)$ such that

$$h_1(y) = h_2(y) \qquad \text{for all } y \in U.$$

Show that

$$V(h_1) = V(h_2).$$

(4 marks)

(Total: 20 marks)

3. (a) Let X be a manifold. Let $p\colon E \to X$ and $q\colon F \to X$ be vector bundles on X of rank r_1 and r_2 respectively. Show that the direct sum of E and F, defined as the set

$$E \oplus F = \{(v, w) \in E \times F \mid p(v) = q(w)\}$$

and equipped with the function

$$r\colon E \oplus F \to X \qquad (v, w) \mapsto p(v),$$

defines a vector bundle of rank $r_1 + r_2$ on X. (8 marks)

(b) Let X and Y be manifolds.

(i) Let $x \in X$ and $y \in Y$ show that there exists an isomorphism of vector spaces

$$T_{(x,y)}X \times Y \to T_xX \oplus T_yY.$$

(3 marks)

(ii) Assume that the tangent bundles of X and Y are trivial. Show that the tangent bundle of the product $X \times Y$ is also trivial. (4 marks)

(c) Let X be a manifold of dimension n and let $\pi\colon E \to X$ be a vector bundle of rank r. Consider the function $s\colon X \to E$, defined by $s(x) = 0 \in E_x$ for all $x \in X$.

(i) Show that s is a section of E. (2 marks)

(ii) Show shat the image of s is a submanifold of E. (3 marks)

(Total: 20 marks)

4. (a) Let X be a compact orientable manifold of dimension n without boundary and let ω be an $(n-1)$-form on X. Show that $d\omega$ must vanish at some point. (4 marks)

(b) Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the 1-form on X given by

$$\omega(x,y) = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

(i) Show that $d\omega = 0$. (4 marks)

(ii) Let $S^1 \subset X$ be the unit circle. Compute

$$\int_{S^1} \omega.$$

(4 marks)

(iii) Show that there does not exist any function $f \in C^\infty(X)$ such that $df = \omega$. (4 marks)

(c) Show that the sphere S^2 is orientable. (4 marks)

(Total: 20 marks)

5. (a) Let X be a manifold and let V_1, V_2 be derivations on X.

(i) Define the Lie bracket $[V_1, V_2]$ of V_1 and V_2. (2 marks)

(ii) Show that $[V_1, V_2]$ is a derivation on X. (3 marks)

(iii) Show that if $f \in C^\infty(X)$ then

$$[V_1, fV_2] = V_1(f)V_2 + f[V_1, V_2].$$

(3 marks)

(b) Let $X = \mathbb{R}^3$ and consider the derivation on X given by

$$V(x, y, z) = e^{x-y}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

(i) Show that

$$V = \left[\frac{\partial}{\partial x}, V\right] = \left[V, \frac{\partial}{\partial y}\right].$$

(3 marks)

(ii) Show that V and $\frac{\partial}{\partial x}$ span a distribution D on X. (3 marks)

(iii) Show that D is integrable. (3 marks)

(iv) Find all the integrable submanifolds of X. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH700058

Manifolds (Solutions)

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1. (a) (i) First note that Y is a topological space which is Hausdorff and secound countable, since these two properties are preserved by taking subspaces. sim. seen $\Downarrow$

We now show that Y is a topological manifold. By definition, for any point $y \in Y$ there exists an open subset $U \subset X$ such that $y \in U$ and a chart (U, g) of X such that

$$g(U \cap Y) = g(U) \cap A$$

where $A \subset \mathbb{R}^n$ is an m-dimensional affine subspace. After possibly composing g by a translation and a linear isomorphism, we may assume that A is the standard linear subspace

$$A_m = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}.$$

Thus, $A \simeq \mathbb{R}^m$ and if we take $V = U \cap Y$ and $f = g|_V \colon V \to A$, then (V, f) is a chart for Y. Therefore, Y is a topological manifold of dimension m.

We now show that the charts defined above define a smooth atlas on Y. Let $y \in Y$ and let (U_1, g_1) and (U_2, g_2) be two charts on X such that $y \in U_1 \cap U_2$ and such that

$$g_i(U_i \cap Y) = g_i(U_i) \cap A_i \qquad \text{for } i = 1, 2$$

where A_1 and A_2 are m-dimensional affine subspaces of $\mathbb{R}^n$. As above, we may assume that A_i coincides with the standard linear subspace A_m. As above, let $V_i = U_i \cap Y$ and $f_i = g_i|_{V_i}$. Since X is a manifold, we have that the transition function

$$\phi = g_2 \circ g_1^{-1} \colon g_1(U_1 \cap U_2) \to g_2(U_1 \cap U_2)$$

is a diffeomorphism. It follows that also the function

$$\psi = f_2 \circ f_1^{-1} = \phi|_{f_1(V_1 \cap V_2)}$$

is a diffeomorphism and, therefore, the claim follows. 4, A

(ii) Let $z \in Z$. In particular, $z \in Y$, and there exists an open subset $U \subset X$ such that $y \in U$ and a chart (U, g) of X such that unseen $\Downarrow$

$$g(U \cap Y) = g(U) \cap A$$

where $A \subset \mathbb{R}^n$ is an m-dimensional affine subspace. As in the previous exercise, we may assume that A is the standard linear subspace

$$A_m = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}.$$

Since g is a diffeomorphism, we may assume that X is an open subset of $\mathbb{R}^n$ and $Y = X \cap A$. Since $Z \cap Y$ is a submanifold, it follows that there exists a chart (W, h) of Y such that $z \in W$ and $h(W \cap Z) = h(W) \cap A'$ where A' is an affine subspace of $\mathbb{R}^m$ of dimension k. Let $W' \subset \mathbb{R}^n$ be an open set such that $W' \cap Y = W$. We consider the function $\tilde{h} \colon W' \to \mathbb{R}^n$ defined by

$$(x_1, \dots, x_n) \mapsto (h_1(x_1, \dots, x_m), \dots, h_m(x_1, \dots, x_m), x_{m+1}, \dots, x_n)$$

where $h = (h_1, \dots, h_m)$. Then $(W', \tilde{h})$ is a chart of X such that $z \in W'$ and

$$\tilde{h}(Z \cap W') = \tilde{h}(W') \cap A.$$

Thus, Z is a submanifold of X. 4, C

(iii) Let $\{(U_i, g_i)\}$ be charts on X, for $1 = 1, \ldots, \ell$, such that $Y \subset U_1 \cup \cdots \cup U_\ell$ and such that, for each i, we have that

unseen ⇓

$$g_i(U_i \cap Y) = g_i(U_i) \cap A$$

where, as in the previous exercises, we may assume that A is the standard linear subspace

$$A_m = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}.$$

Consider the function

$$h_i \colon g_i(U_i) \to \mathbb{R} \qquad (x_1, \ldots, x_n) \mapsto \sum_{j=m+1}^{n} x_j^2.$$

Let $U_{\ell+1} = X \setminus Y$ and let $\{r_i \colon U_i \to [0,1]\}_{i=1,\ldots,\ell+1}$ be a partition of the unity with respect to the cover $\{U_1, \ldots, U_{\ell+1}\}$. Note that, for any $i = 1, \ldots, \ell$, we can extend the function $r_i \cdot (h_i \circ g_i) \colon U_i \to \mathbb{R}$ as a smooth funciton on the whole manifold X by setting it equal to zero outside U_i. Call $f_i \colon X \to \mathbb{R}$ the extension. Set $f_{\ell+1} = r_{\ell+1}$. Consider the function

$$F \colon X \to \mathbb{R} \qquad x \mapsto \sum_{i=1}^{\ell+1} f_i(x).$$

Then F admits all the desired properties.

4, D

(b) (i) Let $U_0 = \{[x_0, x_1, x_2, x_3] \in X \mid x_0 \neq 0\}$. Recall that if we define $g_1 : U_1 \to \mathbb{R}^3$ by

unseen ⇓

$$[x_0, x_1, x_2, x_3] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right),$$

then (U_0, g_0) is a chart of X. We have that

$$g_0(U_0 \cap Y) = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 - b^2 - c^2 = 1\}.$$

Since 1 is a regular value for the function $\mathbb{R}^3 \to \mathbb{R}$ defined by

$$(a, b, c) \mapsto a^2 - b^2 - c^2,$$

it follows that $g_0(U_0 \cap Y)$ is a submanifold of $\mathbb{R}^3$. Since g_0 is a diffeomorphism, we can conclude that $U_0 \cap Y$ is a submanifold of U_0.
A similar calculation holds for the other three standard charts of X, i.e. U_1, U_2 and U_3. Thus, Y is a submanifold of X.

4, A

(ii) Consider the function

unseen ⇓

$$F \colon \mathbb{P}^1_{\mathbb{R}} \to S^1$$

defined by

$$[t_0, t_1] \mapsto \left(\frac{2t_0 t_1}{t_0^2 + t_1^2}, \frac{t_0^2 - t_1^2}{t_0^2 + t_1^2} \right).$$

It is easy to check that F is well-defined.
We now show that F is smooth. Consider the chart (U_0, g_0) given by the open subset $U_0 = \{[t_0, t_1] \mid t_0 \neq 0\} \subset \mathbb{P}^1_{\mathbb{R}}$ and the homeomorphism $g_0 \colon U_0 \to \mathbb{R}$ given by $[t_0, t_1] \mapsto \frac{t_1}{t_0}$. Then

$$F \circ g_0^{-1}(s) = \left(\frac{2s}{1 + s^2}, \frac{1 - s^2}{1 + s^2} \right).$$

In particular, $F \circ g_0$ is a smooth function, which implies that $F|_{U_0}$ is smooth. Moreover, $F \circ g_0$ is invertible and its inverse is given by

$$(x, y) \mapsto \frac{x}{1+y}$$

It follows that $F|_{U_0} \colon U_0 \to S^1 \setminus \{(0, -1)\}$ is a diffeomorphism. Using the second chart, (U_1, g_1), it is easy to conclude that F is a diffeomorphism. $\boxed{4, \text{C}}$

2. (a) (i) Let $X = S^1$ be the unit circle and let $P = (1, 0) \in X$. Consider a smooth function $g\colon S^1 \to \mathbb{R}$ such that $g(x, y) = 0$ if and only if $(x, y) = P$. For example, we can choose $g(x, y) = (x-1)^2 + y^2$. Let unseen ⇓

$$V(x, y) = (y \cdot g(x, y), -xg(x, y)).$$

Then, V is a smooth function and, for all $(x, y) \in S^1$, we have that

$$(x, y) \cdot V(x, y) = 0.$$

Thus, V defines a vector field on X. It is easy to check that V only vanishes at P. 3, B

(ii) Let us consider S^2 as the subset seen ⇓

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

Let $S = (0, 0, -1)$ and let $N = (0, 0, 1)$. In order to construct a vector field on S^2, it is enough to construct a vector field v_1 on $U = S^2 \setminus \{N\}$ and a vector field v_2 on $V = S^2 \setminus \{S\}$ so that v_1 and v_2 coincide on $U \cap V$.
Recall that the stereographic projection defines a diffeomorphism $g\colon V \to \mathbb{R}^2$ given by

$$(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

and with inverse

$$(s, t) \mapsto \left(\frac{2s}{1+s^2+t^2}, \frac{2t}{1+s^2+t^2}, \frac{1-s^2-t^2}{1+s^2+t^2}\right).$$

In particular, if (U, f) is the chart which is also defined by the stereographic projection then the transition function is given by

$$\phi = f \circ g^{-1}\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\} \qquad (s, t) \mapsto \left(\frac{s}{s^2+t^2}, \frac{t}{s^2+t^2}\right).$$

The inverse of ϕ is

$$\phi^{-1}\colon (S, T) \mapsto \left(\frac{S}{S^2+T^2}, \frac{T}{S^2+T^2}\right).$$

In order to define a vector field on V, it is enough to define a vector field on $\mathbb{R}^2$ and since the tangent bundle of $\mathbb{R}^2$ coincides with $\mathbb{R}^2 \times \mathbb{R}^2$, it is enough to define a smooth function $\tilde{v}_2\colon \mathbb{R}^2 \to \mathbb{R}^2$. Similarly, a vector field on U is given by a smooth function $\tilde{v}_1\colon \mathbb{R}^2 \to \mathbb{R}^2$. The two vector fields coincide on $U \cap V$ if we have

$$D\phi \circ \tilde{v}_2 = \tilde{v}_1$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$.
Consider the constant function

$$\tilde{v}_2\colon g(V) = \mathbb{R}^2 \to \mathbb{R}^2 \qquad (s, t) \to (1, 0)\,.$$

Then $\tilde{v}_2$ is smooth and nowhere zero.
We have

$$D\phi = \begin{pmatrix} \frac{t^2-s^2}{(s^2+t^2)^2} & \frac{-2st}{(s^2+t^2)^2} \\ \frac{-2st}{(s^2+t^2)^2} & \frac{s^2-t^2}{(s^2+t^2)^2} \end{pmatrix}$$

We define $\tilde{v}_1 = D\phi \circ \tilde{v}_2 \circ \phi^{-1}$. We have

$$\tilde{v}_1 = \begin{pmatrix} T^2 - S^2 & -2ST \\ -2ST & S^2 - T^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (T^2 - S^2, -2ST).$$

Note that $\tilde{v}_1$ can be extended as a smooth function on $\mathbb{R}^2$ with exactly one zero at $(0,0)$. Thus, $\tilde{v}_1$ and $\tilde{v}_2$ define a vector field with a single zero at N. $\boxed{3, \text{B}}$

(iii) From Exercise 1. (b)(ii), we know that $\mathbb{P}^1_{\mathbb{R}}$ is diffeomorphic to the circle S^1. Thus, it is enough to find a non-vanishing vector field on S^1. An example is given by the smooth function $(x, y) \in S^1 \to (-y, x)$. $\boxed{\text{unseen} \Downarrow}$
Alternatively, the same method used in Ex. (iv) would produce a non-vanishing vector field on $\mathbb{P}^1_{\mathbb{R}}$. $\boxed{2, \text{B}}$

(iv) There exists a natural smooth function $\boxed{\text{unseen} \Downarrow}$

$$\rho \colon S^3 \to \mathbb{P}^3_{\mathbb{R}} \qquad (x_0, x_1, x_2, x_3) \mapsto [x_0, x_1, x_2, x_3].$$

For each $x \in S^3$, the Jacobian of ρ at x is a linear map

$$D\rho|_x \colon T_x S^3 \to T_{\rho(x)} \mathbb{P}^3_{\mathbb{R}}.$$

Consider the smooth function

$$t \colon S^3 \to S^3 \qquad t(x) = -x.$$

We want to find a non vanishing vector field $V \colon S^3 \to TS^3$ on S^3 such that for all $x \in S^3$

$$V(t(x)) = Dt|_x \circ V(x).$$

Indeed, this would induce a non vanishing vector field W on $\mathbb{P}^3_{\mathbb{R}}$ such that

$$W(\rho(x)) = D\rho|_x \circ V(x) \in T_{\rho(x)} \mathbb{P}^3_{\mathbb{R}}.$$

If we consider the vector field V, given by

$$V(x_0, x_1, x_2, x_3) = (-x_1, x_0, -x_3, x_2),$$

then V is never vanishing and, since $Dt|_x = -\mathrm{Id}_{\mathbb{R}^4}$, for any $x \in S^3$, we have that

$$\begin{aligned} V(t(x)) &= V(-x_0, -x_1, -x_2, -x_3) \\ &= (x_1, -x_0, x_3, -x_2) \\ &= Dt|_x \circ V(x), \end{aligned}$$

as requested. $\boxed{4, \text{D}}$

(b) (i) Assume that $V \colon C^\infty(X) \to \mathbb{R}$ is a linear map such that $\boxed{\text{seen} \Downarrow}$

$$V(h) = 0 \qquad \text{for all} \quad h \in R_x(X).$$

Let $h_1, h_2 \in C^\infty(X)$ and define

$$h(y) := h_1(y) \cdot h_2(y) - h_1(x) \cdot h_2(y) - h_2(x) \cdot h_1(y).$$

We have that h has rank 0 at x, i.e. $h \in R_x(X)$ and, thus,

$$0 = V(h) = V(h_1 \cdot h_2) - h_1(x) V h_2 - h_2(x) V h_1.$$

It follows that V is a derivation. $\boxed{4, \text{A}}$

(ii) Let $h = h_1 - h_2 \in C^\infty(X)$. We want to show that $V(h) = 0$. There exists a bump function $\rho \in C^\infty(X)$ such that, there exist open sets $\boxed{\text{sim. seen} \Downarrow}$

$$x \in V \subset W \subset U$$

such that

$$\rho|_V \equiv 1 \qquad \rho|_{X\setminus W} \equiv 0.$$

Let $\psi = 1 - \rho$. Then $\psi \cdot h = h$.
Thus, the Leibniz rule implies

$$V(h) = V(\psi \cdot h) = h(x)V(\psi) + \psi(x)V(h) = 0,$$

as claimed. $\boxed{4, \text{A}}$

3. (a) Since E and F are vector bundles on X, there exist atlases $\{(U_i, f_i)\}_{i\in I}$ and $\{(U'_j, f'_j)\}_{j\in I'}$ and homeomorphisms unseen ⇓

$$g_i\colon p^{-1}(U_i) \to f_i(U_i) \times \mathbb{R}^{r_1}$$

for all $i \in I$ and

$$h_i\colon q^{-1}(U'_j) \to f'_j(U'_j) \times \mathbb{R}^{r_2}$$

for all $j \in I'$. After replacing both these atlases, by the atlas given by considering all the intersections $U_i \cap U'_j$ for all $i \in I$ and $j \in I'$, we may assume that the two atlases coincide.

For all $i \in I$, we define

$$V_i = \{(v, w) \in E \times V \mid p(v) = q(w) \in U_i\} = r^{-1}(U_i) \subset E \oplus F$$

and the bijection

$$\ell_i\colon V_i \to f_i(U_i) \times \mathbb{R}^{r_1+r_2} \qquad (v, w) \mapsto (f_i(v), \mathrm{pr}_2(g_i(v)) \oplus \mathrm{pr}_2(h_i(w))),$$

for all $i \in I$. In particular, ℓ_i defines a topology on V_i such that ℓ_i is a homeomorpism for each $i \in I$. We can therefore define a topology on $E \oplus F$ by requiring that a subset of $E \oplus F$ is open, if its intersection with V_i is open in V_i for all $i \in I$. It is easy to check that $E \oplus F$ is Hausdorff and second countable. Moreover, for each $i \in I$, (V_i, ℓ_i) is a chart and, therefore, $E \oplus F$ is a topological manifold of dimension $r_1 + r_2$.

The transition functions on $E \oplus F$ with respect to such an atlas are given by the direct sum of the transition functions of E and F and, in particular, they are diffeomorphisms. Therefore, $E \oplus F$ is a manifold and the function $r\colon E \oplus F \to X$ is a smooth function. Thus, $E \oplus F$ is a vector bundle of rank $r_1 + r_2$. 8, D

(b) (i) Let n and m be the dimension of X and Y respectively. Let (U, f) be a chart on X such that $x \in U$ and let (V, g) be a chart on Y such that $y \in V$. By construction, $(U \times V, (f, g))$ is a chart on $X \times Y$ containing (x, y). Thus, we have isomorphisms of vector spaces seen ⇓

$$\Delta_{(f,g)}\colon T_{(x,y)}(X \times Y) \to \mathbb{R}^{n+m} \qquad \Delta_f\colon T_xX \to \mathbb{R}^n \qquad \Delta_g\colon T_yY \to \mathbb{R}^m.$$

The claim follows from the fact that $\mathbb{R}^{n+m}$ is isomorphic to $\mathbb{R}^n \oplus \mathbb{R}^m$.

Alternatively, it is possible to show that the following map is an isomorphism

$$T_xX \oplus T_yY \to T_{(x,y)}(X \times Y) \qquad [\sigma_1, \sigma_2] \mapsto [(\sigma_1, \sigma_2)]$$

where if $\sigma_1\colon (-\epsilon, \epsilon) \to X$ and $\sigma_2\colon (-\epsilon, \epsilon) \to Y$ are two smooth curves then

$$(\sigma_1, \sigma_2)\colon (-\epsilon, \epsilon) \to X \times Y \qquad t \mapsto (\sigma_1(t), \sigma_2(t))$$

is the induced smooth curve. 3, A

(ii) Let $V\colon X \to TX$ be a vector field on X and let $W\colon Y \to TY$ be a vector field on Y. By the previous exercise, we have that, for all $(x, y) \in X \times Y$, there exists an isomorphism of vector spaces unseen ⇓

$$T_{(x,y)}X \times Y \to T_xX \oplus T_yY.$$

Thus, it follows easily that the function

$$V \oplus W \colon X \times Y \to T(X \times Y) \qquad (x, y) \mapsto V(x) \oplus W(y)$$

define a vector field on $X \times Y$.

Let n be the dimension of X and let m be the dimension of Y. Since, by assumption X and Y have trivial tangent bundles, we know that there exist vector fields $V_1, \dots, V_n$ of X and $W_1, \dots, W_m$ of Y such that, at each point $x \in X$ $V_1(x) \dots, V_n(x)$ are linearly independent in T_xX and, for each $y \in Y$, $W_1(y), \dots, W_m(y)$ are linearly independent in T_yY. Consider the vector fields $M_{i,j} = V_i \oplus W_j$ of $X \times Y$, for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Then, for each $(x, y) \in X \times Y$, we have that $\{M_{i,j}(x, y)\}_{i,j}$ are linearly independent. Thus, it follows that $T(X \times Y)$ is a trivial vector bundle. $\boxed{4, \text{C}}$

(c) (i) By definition, we have that $\pi \circ s(x) = x$ for all $x \in X$. Thus, it is enough to show that s is a smooth function. $\boxed{\text{unseen} \Downarrow}$

Since E is a vector bundle on X, there exist an atlas $\{(U_i, f_i)\}_{i \in I}$ and homeomorphisms

$$g_i \colon \pi^{-1}(U_i) \to f_i(U_i) \times \mathbb{R}^{r_1}$$

for all $i \in I$. We have that the function

$$\tilde{s}_i = g_i \circ s_i \circ f_i^{-1} \colon f_i(U_i) \to f_i(U_i) \times \mathbb{R}^r$$

is given by $\tilde{s}_i(x) = (x, 0)$ for all i and it is, therefore, smooth.

It follows that s is smooth, as claimed. $\boxed{2, \text{A}}$

(ii) Using the same notation as in the previous exercise, it is enough to show that $s(X) \cap \pi^{-1}(U_i)$ is a submanifold of $\pi^{-1}(U_i)$ for all $i \in I$. The homeomorphism g_i defines a chart on $\pi_{-1}(U_i)$ such that $\boxed{\text{unseen} \Downarrow}$

$$g_i(s(X) \cap \pi^{-1}(U_i) = f_i(U_i) \times \{0\}.$$

Note that $f_i(U_i) \times \{0\} = g_i(\pi^{-1}(U_i)) \cap A$ where A is the standard linear subspace of $\mathbb{R}^{n+r}$. Thus, $s(X) \cap \pi^{-1}(U_i)$ is a submanifold of $\pi^{-1}(U_i)$ for all $i \in I$, as claimed. $\boxed{3, \text{A}}$

4. (a) Assume that $d\omega$ is never zero. Then, $d\omega$ is a volume form on X and, in particular, by Stokes' Theorem we have that $\boxed{\text{seen} \Downarrow}$

$$0 \neq \int_X d\omega = \int_{\partial X} \omega$$

contradicting the fact that X is a manifold without boundary. $\boxed{4, \text{A}}$

(b) (i) We have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} d\omega &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx \\ &= \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} dx \wedge dy - \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} dy \wedge dx = 0 \end{aligned}.$$

$\boxed{4, \text{B}}$

(ii) Let $\boxed{\text{unseen} \Downarrow}$

$$\gamma \colon [0, 2\pi] \to X \qquad \gamma(\theta) = (\cos\theta, \sin\theta) \in X.$$

Then

$$\int_{S^1} \omega = \int_{[0,2\pi]} \gamma^* \omega = \int_0^{2\pi} \frac{\cos^2\theta - \sin\theta(-\sin\theta)}{\cos^2\theta + \sin^2\theta} d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

$\boxed{4, \text{B}}$

(iii) Assume by contradiction that there exists a a smooth function, f on X such that $\omega = df$. $\boxed{\text{unseen} \Downarrow}$
Since S^1 has no boundary, Stokes' theorem imply

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = 0.$$

Hence, we get a contradiction. $\boxed{4, \text{B}}$

(c) Consider the inclusion $i \colon S^2 \to \mathbb{R}^3$ as the sphere of radius one. Let $\boxed{\text{seen} \Downarrow}$

$$\omega = i^*(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2).$$

We want to show that ω is a volume form on S^2. Consider the vector fields:

$$V_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \qquad V_2 = -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3}.$$

Then for any $x = (x_1, x_2, x_3) \in S^2$, we have

$$\omega(V_1, V_2)(x) = x_3^3 + x_2^2 x_3 + x_3 x_1^2 = x_3.$$

Thus, ω is non-zero if $x_3 \neq 0$. By symmetry, it follows easily that ω is no-where zero and therefore it is a volume form. In particular, S^2 is orientable.

Alternatively, it is easy to show that the charts defined by the stereographic projection have transition functions whose Jacobian has positive determinant. $\boxed{4, \text{A}}$

5. (a) (i) We define $[V_1, V_2]$ as the function

seen ⇓

$$[V_1, V_2] \colon C^\infty(X) \to C^\infty(X) \qquad [V_1, V_2](g) = V_1(V_2(f)) - V_2(V_1(g)).$$

2, M

(ii) Since V_1 and V_2 are $\mathbb{R}$-linear, it is immediate to check that also $[V_1, V_2]$ is $\mathbb{R}$-linear. We now show that $[V_1, V_2]$ satisfies the Leibniz rule. For any $f, g \in C^\infty(X)$, we have

seen ⇓

$$\begin{aligned}
[V_1, V_2](f \cdot g) &= V_1(V_2(fg)) - V_2(V_1(fg)) \\
&= V_1(fV_2(g) + gV_2(f)) - V_2(fV_1(g) + gV_1(f)) \\
&= V_1(f)V_2(g) + fV_1(V_2(g)) + V_1(g)V_2(f) + gV_1(V_2(f)) \\
&\qquad - (V_2(f)V_1(g) + fV_2(V_1(g)) + V_2(g)V_1(f) + gV_2(V_1(f))) \\
&= fV_1(V_2(g)) + gV_1(V_2(f)) - fV_2(V_1(g)) - gV_2(V_1(f)) \\
&= f[V_1, V_2](g) + g[V_1, V_2](f).
\end{aligned}$$

Thus, the Leibniz rule holds

3, M

(ii) For any $g \in C^\infty(X)$, we have

sim. seen ⇓

$$\begin{aligned}
[V_1, fV_2](g) &= V_1(fV_2(g)) - fV_2(V_1(g)) \\
&= V_1(f)V_2(g) + fV_1(V_2(g)) - fV_2(V_1(g)) \\
&= V_1(f)V_2(g) + f[V_1, V_2](g).
\end{aligned}$$

Thus, the equality holds.

3, M

(b) (i) For any $g \in C^\infty(X)$, we have

unseen ⇓

$$\begin{aligned}
\left[\frac{\partial}{\partial x}, V\right](g) &= \frac{\partial}{\partial x}\left(e^{x-y}\left(\frac{\partial}{\partial x}g + \frac{\partial}{\partial y}g\right)\right) - e^{x-y}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}g + \frac{\partial}{\partial y}\frac{\partial}{\partial x}g\right) \\
&= e^{x-y}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}g + \frac{\partial}{\partial y}g\right) + e^{x-y}\left(\frac{\partial}{\partial x}g + \frac{\partial}{\partial y}g\right) - e^{x-y}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}g + \frac{\partial}{\partial y}\frac{\partial}{\partial x}g\right) \\
&= V(g).
\end{aligned}$$

The second equality is obtained by a similar calculation.

3, M

(ii) For any $(x, y, z) \in X$, we have that $V(x, y, z)$ and $\frac{\partial}{\partial x}$ are linearly independent. Thus, they define a distribution of rank two on X.

unseen ⇓

3, M

(iii) Note that D is the distribution spanned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, which is integrable since

unseen ⇓

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0.$$

Alternatively, the result follows from the identity in Ex. (i) above.

3, M

(iv) Since D is the distribution spanned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, it follows that all the integrable submanifolds of D are given by

unseen ⇓

$$\{(x, y, z) \in X \mid z = c\}$$

for some $c \in \mathbb{R}$.

3, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70058	1	Most of the students got Question a) right. A common mistake for b) was to write a function from P^3 -> R given by the defining equation of Y. This is not a well defined function .
MATH70058	2	Most of the students got most of the material correctly.
MATH70058	3	A common mistake: the zero section is not a constant function because the zero element depends on the fibre of the vector bundle that we are considering. Thus, it is not completely obvious that the zero section is a smooth function but it needs to be proven.
MATH70058	4	Most of the students got this question right.
MATH70058	5	Most of the students got this question right, except for the last problem. Please check the solutions.