Math40002 Analysis 1 — Problem Sheet 0

1. What is the biggest element of the set $\{x \in \mathbb{R} \colon x < 1\}$? Give a careful proof.

It does not exist. Suppose it did, call it $m < 1$. Let $n = (m+1)/2$. Then $m = (m+m)/2 < (m+1)/2 < (1+1)/2 = 1$ shows that $m < n < 1$, so n is a larger element of the set: a contradiction.

2. Prove that for every positive integer $n \neq 3$, the number $\sqrt{n} - \sqrt{3}$ is irrational.

Suppose $\sqrt{n} - \sqrt{3} = r$ is rational.
Write as $\sqrt{n} = r + \sqrt{3}$ and square to give $n = r^2 + 2r\sqrt{3} + 3$.

So either $r = 0$ (impossible; $n \neq 3$) or $\sqrt{3} = \frac{n-r^2-3}{2r}$. But this is rational, a contradiction.

3.* Show that any positive *eventually periodic* decimal expansion is rational, and in fact can be written as the fraction

$$p \Big/ 99\ldots 9900\ldots 00 \quad (m \text{ 9s and } n \text{ 0s})$$

for some integers $p, m, n \geq 0$.
Deduce that any integer divides some number of the form $99\ldots 9900\ldots 00$.

Let the decimal expansion be $x = a_0.a_1a_2\ldots a_n(\overline{b_1\ldots b_m})$, where $\overline{\cdot}$ denotes recurring periodically. Then

$$\begin{aligned} x &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}(10^{-m} + 10^{-2m} + 10^{-3m} + \ldots) \\ &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}\frac{1}{10^m - 1} \\ &= \frac{(10^m-1)a_0a_1\ldots a_n + b_1\ldots b_m}{(10^m-1)10^n}, \end{aligned}$$

which is of the form claimed, with $p = (10^m - 1)a_0a_1\ldots a_n + b_1\ldots b_m$.

As proved in lectures, $x = 1/q$ has periodic decimal expansion since it is rational. Therefore we get $1/q = p/99\ldots 9900\ldots 00$ for some intreger p, and thus $99\ldots 9900\ldots 00/q = p$ as required.

4. Kevin tries to show $\sqrt{12} - \sqrt{3}$ is rational, by the following argument.

$$\begin{aligned} & \sqrt{12} - \sqrt{3} = p/q, \quad p, q \in \mathbb{N}, \\ \Rightarrow\quad & 12 - 2\sqrt{12}\sqrt{3} + 3 = p^2/q^2, \\ \Rightarrow\quad & 15 - 2\sqrt{36} = p^2/q^2. \end{aligned}$$

Since $\sqrt{36} = 6$ is indeed rational, this looks good to him. Can you help him by pointing out three ways in which he's gone wrong? Be kind to him!

1. Firstly, $\sqrt{12} - \sqrt{3} = \sqrt{3}$ is *not* rational.

2. But if he is trying to show that $\sqrt{12} - \sqrt{3}$ is rational then he needs to end up with something implying $\sqrt{12} - \sqrt{3} = p/q$; it's no use to have $\sqrt{12} - \sqrt{3} = p/q$ implying something else. He's assumed the result he's trying to prove.

3. However, his argument *is* the start of a good contradiction to be obtained from assuming that $\sqrt{12} - \sqrt{3} \in \mathbb{Q}$. He just needs to carry on from the last line to get $3q^2 = p^2$, therefore p is divisible by 3, therefore q is divisible by 3, etc...the usual contradiction one gets from assuming that $\sqrt{3} \in \mathbb{Q}$.