Question: Which of the following sets is **not** open in $\mathbb{R}^2$?

A. $\{(x, y) \in \mathbb{R}^2 \mid xy < 0\}$

B. $\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \in (0, 1)\}$

C. $\{(x, y) \in \mathbb{R}^2 \mid \text{there exists } k \in \mathbb{N} \text{ such that } x \in [1/k, +\infty)\}$

D. $\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y = 0\}$

Solution: (The answer is D)

- The set (in) A is open. To see that let (x, y) be an arbitrary point in A. We define
$$r = \min\{|x|, |y|\}$$
and note that since $(x, y) \in A$, $r > 0$. We need to show that $B_r((x, y)) \subset A$. Let $(x', y') \in B_r(x, y)$ be an arbitrary point. We have
$$|x - x'| \leq \|(x, y) - (x', y')\| < r \leq |x|,$$
which implies that x and x' have the same sign. Also,
$$|y - y'| \leq \|(x, y) - (x', y')\| < r \leq |y|,$$
which implies that$y$ and y' have the same sign. Therefore, $(x', y') \in A$. As (x', y') in $B_r((x, y))$ was arbitrary, we conclude that $B_r((x, y)) \subset A$.

- The set B is open. Let x be an arbitrary point in B. We define
$$r = \min\{\|x\| - 1, 2 - \|x\|\}.$$
Since $x \in B$, $r > 0$. As in the example in the lecture notes, one can see that $B_r(x) \subset B$.

- First note that the set in C is equal to the set $\{(x, y) \in \mathbb{R}^2) \mid x > 0\}$. One can see that this set is open in $\mathbb{R}^2$ by setting $r = x/2$.

- The set in item D is not open. For example, the point $(1/2, 0) \in D$, but there is no $r > 0$ such that $B_r((1/2, 0)) \subset D$. That is because for every $r > 0$, the point $(1/2, r/2) \in B_r((1/2, 0))$ and $(1/2, r/2) \notin D$.

Note that here it is important that we are asking if the set in item D is open in $\mathbb{R}^2$ or not. We know that the set $\mathbb{R}^1$ is open in $\mathbb{R}^1$, but when we consider the set in D in $\mathbb{R}^2$, it is not open.

Question: Which of the following statements is **not true**.

A. If the maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ are continuous, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^l$ is continuous.

B. If the maps $f : \mathbb{R}^1 \to \mathbb{R}^n$ and $g : \mathbb{R}^1 \to \mathbb{R}^n$ are continuous, then the map

$$x \mapsto \langle f(x), g(x) \rangle$$

is continuous on $\mathbb{R}^1$.

C. If the maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g \circ f : \mathbb{R}^n \to \mathbb{R}^l$ are continuous, then $g : \mathbb{R}^m \to \mathbb{R}^l$ is continuous.

D. If the maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ are continuous, then the map

$$x \mapsto \|f(x)\| \cdot \|g(x)\|$$

is continuous on $\mathbb{R}^n$.

Solution: (The answer is C)

- Part A is the chain rule for continuous maps, which has been stated in the lecture notes.
- The statement in part B is true, and can be proved using the basic properties of the inner product.
- The statement in part C is not true. For example, if f is a constant function, and g is any discontinuous function.
- The statement in part D is true and can be proved using the chain rule for the continuous maps.

Question: Let Ω be an open set in $\mathbb{R}^5$, $p \in \Omega$, and $f : \Omega \to \mathbb{R}$ be a map. Which of the following statements in true?

A. If f is differentiable at p, then it is continuous at points sufficiently close to p.

B. If f is differentiable at p, then its partial derivatives exist at all points sufficiently close to p.

C. If f is differentiable at p, then its partial derivatives at all directions exist at p.

D. All of the above statements are false.

Solution: (The answer is C)

One can build counter examples for items A and B. The statement in C is discussed in the lecture notes.

Question: Consider the maps $f : \mathbb{R}^1 \to \mathbb{R}^1$ and $g : \mathbb{R}^3 \to \mathbb{R}^1$ defined as

$$f(t) = e^{(t^3-1)}, \qquad g(x, y, z) = \sin(x - y) - \sin(y - z) + \sin(z - x) + 1$$

You may assume that f and g are differentiable on their domains of definition. What is the Jacobian of $f \circ g$ at $(0, 0, 0)$?

A. $\begin{bmatrix} 0 & -6 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 6 & -6 \end{bmatrix}$

C. $\begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$

Solution: (The answer is A) We note that

$$g(0, 0, 0) = 1,$$

and

$$\operatorname{Jac} g(x, y, z) = \begin{pmatrix} \cos(x - y) - \cos(z - x) & -\cos(x - y) - \cos(y - z) & \cos(y - z) + \cos(z - x) \end{pmatrix}$$

which implies that

$$\operatorname{Jac} g(0, 0, 0) = \begin{bmatrix} 0 & -2 & 2 \end{bmatrix}.$$

Moreover, $f'(t) = 3t^2 e^{(t^3-1)}$, and hence $f'(1) = 3e^0 = 3$. By the chain rule, we obtain

$$\operatorname{Jac}(f \circ g)(0, 0, 0) = \operatorname{Jac} f(g(0, 0, 0)) \cdot \operatorname{Jac} g(0, 0, 0) = 3 \cdot \begin{bmatrix} 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 6 \end{bmatrix}$$

Question: Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and consider the map $h : \mathbb{R}^n \to \mathbb{R}$ defined as
$$h(x) = \|Bx\|^2 .$$
Which of the following statements is true about the map h?

A. h is not differentiable at $(0, 0, \ldots, 0) \in \mathbb{R}^n$, but it is differentiable at all other points $p \neq (0, 0, \ldots, 0)$.

B. The partial derivatives of h exist at all points and all directions, but they are not continuous.

C. h is differentiable at all points $p \in \mathbb{R}^n$, and its derivative is $Dh(p)[v] = 2\langle Bp, v\rangle$.

D. h is differentiable at all points $p \in \mathbb{R}^n$, and its derivative is $Dh(p)[v] = 2\langle Bp, Bv\rangle$.

Solution h is the composition of the linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $g(x) = Bx$, and the map $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|^2$. We saw in lectures that g is everywhere differentiable with differential:
$$Dg(p)[v] = Bv$$
and that f is everywhere differentiable with differential:
$$Df(q)[v] = 2\,\langle q, v\rangle .$$
By the chain rule, h is differentiable and has differential:
$$Dh(p)[v] = Df(g(p))\,[Dg(p)[v]] = Df(Bp)[Bv] = 2\,\langle Bp, Bv\rangle .$$