Exercise 8.1. Consider a discrete metric space $(X, \mathrm{d}_{\mathrm{disc}})$, that is $\mathrm{d}_{\mathrm{disc}}$ is a discrete metric on X. Show that $\mathrm{d}_{\mathrm{disc}}$ induces the discrete topology on X.

Hint: Identify the open sets in the discrete metric.

Exercise 8.2. Let (X, τ) be a topological space, $Y \subset X$, and

$$\tau_Y = \{U \cap Y \mid U \in \tau_X\}.$$

Show that τ_Y is a topology on Y.

Hint: you need to verify the three properties for the topology, and use basic relations for unions and intersections of sets.

Exercise 8.3. Let τ_{Eucl} be the Euclidean topology on $\mathbb{R}$, that is τ_{Eucl} is the collection of all open sets in $(\mathbb{R}, \mathrm{d}_1)$. Show that the collection

$$\{U \times V \mid U \in \tau_{\mathrm{Eucl}}, V \in \tau_{\mathrm{Eucl}}\}.$$

is not a topology on $\mathbb{R} \times \mathbb{R}$. Is condition T2 satisfied? How about condition T3?

Hint: Consider the union of two boxes.

Exercise 8.4. Let (A, τ) be a topological space, and let S and T be subsets of A. The following properties hold:

(i) if $S \subset T$ then $S^\circ \subset T^\circ$,

(ii) S is open in A if and only if $S = S^\circ$,

(iii)* S° is the largest open set contained in S.

Hint: Compare this to the corresponding exercise for the metric spaces, and see if those proofs can be adapted here.

Exercise 8.5. Let (X, d) be a metric space, and let τ be the topology on X induced from the metric d. Show that (X, τ) is a Hausdorff topological space.

Hint: For a pair of distinct points, consider the distance between those points, and use that to define balls around each of the two points, so that they do not intersect.

Exercise 8.6. Assume that the topological spaces (X, τ_X) and (Y, τ_Y) are topologically equivalent. Then, (X, τ_X) is Hausdorff if and only if (Y, τ_Y) is Hausdorff.

Hint: By the hypothesis, there is a homeomorphism from (X, τ_X) to (Y, τ_Y). Use this map to send pairs of distinct open sets to pairs of distinct open sets.

Unseen Exercise. (unseen) Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $f : X \to Y$ be a continuous and injective map. Then, if Y is Hausdorff, then X is Hausdorff.

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional