Introduction to Quantum Mechanics – Solutions to Problem sheet 5

1. Coherent states

(a) The *overlap* $|\langle z_1|z_2\rangle|^2$ of two coherent states is given by

$$\begin{aligned}
|\langle z_1|z_2\rangle|^2 &= \left| e^{-\frac{|z_1|^2}{2}} e^{-\frac{|z_2|^2}{2}} \sum_n \frac{z_1^{*n} z_2^n}{n!} \right|^2 \\
&= \left| e^{-\frac{|z_1|^2}{2}} e^{-\frac{|z_2|^2}{2}} e^{z_1^* z_2} \right|^2 \\
&= e^{-(|z_1|^2 + |z_2|^2 - 2|z_1^* z_2|)} \\
&= e^{-|z_1 - z_2|^2}.
\end{aligned}$$

(b) We directly apply the annihilation operator to find

$$\begin{aligned}
\hat{a}\,|z\rangle &= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \hat{a}\,|n\rangle \\
&= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \sqrt{n}\,|n-1\rangle
\end{aligned}$$

We relabel the sum with $m = n-1$ to find

$$\begin{aligned}
\hat{a}\,|z\rangle &= e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{z^{m+1}}{\sqrt{m!}}\,|m\rangle \\
&= z e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}}\,|m\rangle \\
&= z\,|z\rangle\,.
\end{aligned}$$

That is, $|z\rangle$ is indeed an eigenvector of $\hat{a}$ with eigenvalues $z \in \mathbb{C}$.

(c) Given that,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{\mathrm{i}}{\sqrt{2m\omega\hbar}}\hat{p},$$

for the expectation value of $\hat{a}$ in the state $|z\rangle$ we have

$$\sqrt{\frac{m\omega}{2\hbar}}\,\langle\hat{q}\rangle + \frac{\mathrm{i}}{\sqrt{2m\omega\hbar}}\,\langle\hat{p}\rangle = \langle\hat{a}\rangle = \frac{\langle z|\hat{a}|z\rangle}{\langle z|z\rangle} = z\frac{\langle z|z\rangle}{\langle z|z\rangle} = z.$$

Re-arranging and taking real and imaginary parts of z yields

$$\langle\hat{q}\rangle = \sqrt{\frac{2\hbar}{m\omega}}\mathrm{Re}(z) \quad \text{and} \quad \langle\hat{p}\rangle = \sqrt{2m\omega\hbar}\,\mathrm{Im}(z).$$

(d) To prove there are no normalisable eigenstates of the creation operator we assume that there exist a nonzero state $|\alpha\rangle$ with $\hat{a}^\dagger = \alpha\,|\alpha\rangle$ and derive a contradiction. Writing $|\alpha\rangle$ in the harmonic oscillator basis as $|\alpha\rangle = \sum_n c_n|n\rangle$ and applying $\hat{a}^\dagger$ yields

$$\begin{aligned}
\hat{a}^\dagger \left|\alpha\right\rangle &= \hat{a}^\dagger \sum_{n=0}^{\infty} c_n \left|n\right\rangle \\
&= \sum_{n=0}^{\infty} c_n \hat{a}^\dagger \left|n\right\rangle \\
&= \sum_{n=0}^{\infty} c_n \sqrt{n+1} \left|n+1\right\rangle \\
&= \sum_{m=1}^{\infty} c_{m-1} \sqrt{m} \left|m\right\rangle .
\end{aligned}$$

On the other hand we also demand $|\alpha\rangle$ to be an eigenvector of $\hat{a}^\dagger$ with eigenvalue α, that is,

$$\begin{aligned}
\hat{a}^\dagger \left|\alpha\right\rangle &= \alpha \left|\alpha\right\rangle \\
&= \sum_{n=0}^{\infty} \alpha c_n \left|n\right\rangle .
\end{aligned}$$

Taking the inner product with the harmonic oscillator eigenstate $|j\rangle$ we deduce

$$\alpha c_j = c_{j-1}\sqrt{j}.$$

That is for $j = 0$ we have $\alpha c_0 = 0$, and thus either we have $\alpha = 0$ or $c_0 = 0$. The latter would imply that $c_j = 0 \quad \forall j$ (i.e. $|\alpha\rangle$ is the zero vector), which is not an allowed eigenvector. This seems to allow the option that $\alpha = 0$ and $c_0 \neq 0$, but $c_0 = \alpha c_1$ and if $\alpha = 0$ that means that $c_0 = 0$ after all. Therefore the creation operator has no normalisable eigenstates.

2. A two-dimensional harmonic oscillator

(a) This is shown by making use of the commutation relations between $\hat{a}_{1,2}$ and $\hat{a}^\dagger_{1,2}$ and the properties of the commutator, in particular

$$\left[\hat{A}+\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right] + \left[\hat{B},\hat{C}\right],$$

$$\left[\hat{A}\hat{B},\hat{C}\right] = \hat{A}\left[\hat{B},\hat{C}\right] + \left[\hat{A},\hat{C}\right]\hat{B},$$

and

$$\left[\hat{A},\hat{B}\right] = \left[\hat{B},\hat{A}\right].$$

We first calculate,

$$\begin{aligned}
\left[\hat{K}_0,\hat{K}_+\right] &= \frac{1}{2}\left[\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2, \hat{a}^\dagger_1\hat{a}_2\right], \\
&= \frac{1}{2}\hat{a}^\dagger_1\left[\hat{a}_1,\hat{a}^\dagger_1\hat{a}_2\right] + \frac{1}{2}\left[\hat{a}^\dagger_1,\hat{a}^\dagger_1\hat{a}_2\right]\hat{a}_1 - \frac{1}{2}\hat{a}^\dagger_2\left[\hat{a}_2,\hat{a}^\dagger_1\hat{a}_2\right] - \frac{1}{2}\left[\hat{a}^\dagger_2,\hat{a}^\dagger_1\hat{a}_2\right]\hat{a}_2, \\
&= \frac{1}{2}\hat{a}^\dagger_1\hat{a}_2\left[\hat{a}_1,\hat{a}^\dagger_1\right] - \frac{1}{2}\hat{a}^\dagger_1\hat{a}_2\left[\hat{a}^\dagger_2,\hat{a}_2\right], \\
&= \hat{K}_+.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left[\hat{K}_0,\hat{K}_-\right] &= \frac{1}{2}\left[\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2, \hat{a}^\dagger_2\hat{a}_1\right], \\
&= \frac{1}{2}\hat{a}^\dagger_2\hat{a}_1\left[\hat{a}^\dagger_1,\hat{a}_1\right] - \frac{1}{2}\hat{a}^\dagger_2\hat{a}_1\left[\hat{a}_2,\hat{a}^\dagger_2\right], \\
&= -\hat{K}_-
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{K}_+,\hat{K}_-\right] &= \left[\hat{a}^\dagger_1\hat{a}_2, \hat{a}^\dagger_2\hat{a}_1\right], \\
&= \hat{a}^\dagger_1\left[\hat{a}_2,\hat{a}^\dagger_2\hat{a}_1\right] + \left[\hat{a}^\dagger_1,\hat{a}^\dagger_2\hat{a}_1\right]\hat{a}_2, \\
&= \hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2, \\
&= 2\hat{K}_0.
\end{aligned}$$

(b) Since $[\hat{N}_1,\hat{N}_2]$ commute, they have a complete set of joint eigenvectors $|n_1,n_2\rangle$, which are also the eigenvectors of $\hat{K}_0$, with eigenvalues $\frac{1}{2}(n_1-n_2)$, where $n_{1,2}$ are non-negative integers. Thus, the eigenvalues of $\hat{K}_0$ are either integers or half integers.

3. The harmonic oscillator method for numerical eigenvalue problems

(a) The modified program could look like this

```
clear all
N=100; %matrix size
lambda=10;

%matrices for position and momentum operators
n=1:N-1;
k=sqrt(n);

Q=sqrt(0.5)*(diag(k,1)+diag(k,-1));
P=i*sqrt(0.5)*(diag(k,-1)-diag(k,1));

%Hamiltonian
H=0.5*P^2+Q^4-lambda*Q^2;

%eigenvalues
EigSort=sort(eig(H));
EigSort(1:10)
```

We can observe that below the barrier the eigenvalues come in pairs.

(b) The convergence can be investigated, for example, with the following program.

```
clear all

Nn=[10:100]; %variable matrix size

for j=1:length(Nn)
    N=Nn(j);
%matrices for position and momentum operators
n=1:N-1;
k=sqrt(n);

Q=sqrt(0.5)*(diag(k,1)+diag(k,-1));
P=i*sqrt(0.5)*(diag(k,-1)-diag(k,1));

%Hamiltonian
H=0.5*P^2+Q^4-5*Q^2;

%eigenvalues
EigSort=sort(eig(H));
E(:,j)=EigSort(1:6);
end
plot(Nn,E,'.','markersize',10)
xlabel('matrix size')
ylabel('eigenvalues')
```

Producing a picture such as the one depicted in figure 1. The different coloured dots represent the first six eigenvalues for any matrix size. It can be seen that the eigenvalues quickly converge to constant values as functions of the matrix size N, and that the lowest two eigenvalues are almost degenerate.

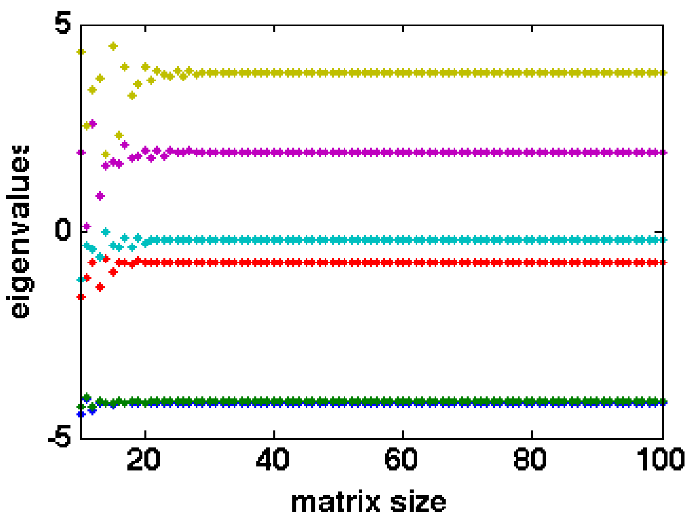

Figure 1: The different coloured dots show the first six eigenvalues of the finite matrix representation of the double well potential with $\lambda = 5$, in dependence on the matrix size.