Theorem 3.19: Algebra of limits

If $a_n \to a$ and $b_n \to b$ then:

1. $a_n + b_n \to a + b$,
2. $a_n b_n \to ab$,
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$.

Proof of 1. Fix any $\epsilon > 0$. Then

$$\exists N_a \in \mathbb{N}_{>0} \text{ such that } \forall n \geq N_a,\ |a_n - a| < \frac{\epsilon}{2},$$

$$\exists N_b \in \mathbb{N}_{>0} \text{ such that } \forall n \geq N_b,\ |b_n - b| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_a, N_b\}$, so $\forall n \geq N$,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□

Rough working for 2: First a bit of a trick,

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b - a_n(b - b_n)| \\ &\leq |a_n - a||b| + |a_n||b_n - b|. \end{aligned}$$

We can easily make $|a_n - a||b| < \frac{\epsilon}{2}$ if we take $|a_n - a| < \frac{\epsilon}{2|b|}$.

But we **cannot** deduce $|b_n - b| < \frac{\epsilon}{2|a_n|}$ from $b_n \to b$ because in the definition, ϵ has to be independent of n.

Instead we bound $|a_n| < A$ by Proposition 3.16; then we can take $|b_n - b| < \frac{\epsilon}{2A}$.

Proof of 2. $a_n \to a \implies \exists\, A > 0$ such that $|a_n| < A\ \forall n \in \mathbb{N}_{>0}$ by Proposition 3.16.

Fix $\epsilon > 0$. Then

$$\exists N_a \text{ such that } \forall n \geq N_a,\ |a_n - a| < \frac{\epsilon}{2(|b| + 1)},$$

$$\exists N_b \text{ such that } \forall n \geq N_b,\ |b_n - b| < \frac{\epsilon}{2A}.$$

(We added 1 to $2|b|$ to handle the case $|b| = 0$.)

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$,

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a||b| + |b_n - b||a_n| \\ &< \frac{\epsilon}{2}\frac{|b|}{|b|+1} + A\frac{\epsilon}{2A} \\ &< \epsilon/2 + \epsilon/2 \ = \ \epsilon. \end{aligned}$$

Alternative trick-less proof of 2: Write $a_n = a + e_n$ and $b_n = b + f_n$ so that (easy exercise!) $e_n, f_n \to 0$. Then

$$\begin{aligned} |a_n b_n - ab| &= |(a+e_n)(b+f_n) - ab| \ = \ |af_n + be_n + e_n f_n| \\ &\leq |a||f_n| + |b||e_n| + |e_n||f_n|. \qquad (*) \end{aligned}$$

Now the idea is that if we make $|e_n|, |f_n| < \epsilon$, the last term is $< \epsilon^2$ which *should be* even smaller. In fact this only works if $\epsilon \leq 1$ so we need to ensure this.

So now fix $\epsilon > 0$ and set $\epsilon' := \min(\epsilon, 1)/(|a| + |b| + 1)$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|e_n|,\ |f_n| \ < \ \epsilon' \ \overset{(*)}{\Longrightarrow} \ |a_n b_n - ab| \ < \ |a|\epsilon' + |b|\epsilon' + (\epsilon')^2.$$

Since $\epsilon' \leq 1$ we know $(\epsilon')^2 \leq \epsilon'$ so we get $|a_n b_n - ab| < \epsilon'(|a| + |b| + 1) \leq \epsilon$, so $a_n b_n \to ab$.

I deliberately missed out the rough working of how to choose ϵ'. Tonight **close your notes and write out your own proof of this result.** Do the rough working first, then write a concise, precise, logical proof. Don't be afraid to have several goes until the end result is undeniably a correct proof.

See exercise sheet for proof of (3). $\square$

Remark 3.20. Now it's easier to handle things like $a_n = \dfrac{n^2+5}{n^3-n+6}$.

Dividing by n^3, we get

$$a_n \ = \ \frac{1/n + 5/n^3}{1 - 1/n^2 + 6/n^3}$$

Using the fact that $1/n \to 0$ as $n \to \infty$

(Recall proof: $\forall \epsilon > 0$ choose $N_\epsilon > 1/\epsilon$ so that

$$n \geq N_\epsilon \implies n > 1/\epsilon \implies 1/n < \epsilon)$$

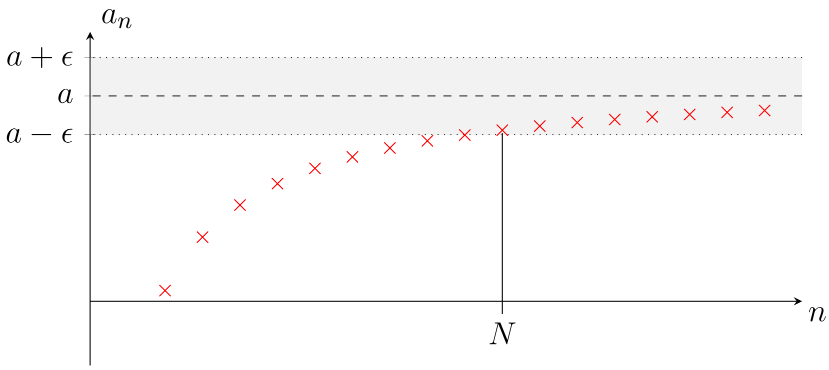
and the algebra of limits, we deduce

$$a_n \ \longrightarrow \ \frac{0 + 5 \times 0^3}{1 - 0^2 + 6 \times 0^3} \ = \ 0.$$

Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i : i \in \mathbb{N}_{>0}\}$. We write $a_n \uparrow a$.

Idea: Eventually we get in the ϵ-corridor around a (the shaded area) because $a - \epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}_{>0}\}$. We stay in there because a_n is monotonic and bounded above by a.



Proof. Set $a := \sup\{a_i : i \in \mathbb{N}_{>0}\}$ and fix $\epsilon > 0$. Now $a - \epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}\}$ (because a is the *smallest* upper bound), so $\exists N \in \mathbb{N}_{>0}$ such that $a_N > a - \epsilon$. Monotonic so $\forall n \geq N$ we have

$$a \geq a_n \geq a_N > a - \epsilon \implies |a_n - a| < \epsilon.$$

□

Example 3.22. Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \leq b_n$ $\forall n$ and $a_n \to a$, $b_n \to b$. Prove that $a \leq b$.

Draw a picture! It will eventually lead you to a proof along the following lines.

Suppose for a contradiction that $a > b$, then set $\epsilon = \frac{a-b}{2} > 0$. Then:

$$\exists\, N_a \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon \implies a_n > a - \epsilon = \frac{a+b}{2},$$

and $$\exists\, N_b \in \mathbb{N} \text{ such that } n \geq N \implies |b_n - b| < \epsilon \implies b_n < b + \epsilon = \frac{a+b}{2}.$$

So for $n \geq \max(N_a, N_b)$ we have $b_n < \frac{a+b}{2} < a_n$ which contradicts $a_n \leq b_n$.

Example 3.23. Prove that if

$$\left|\frac{a_{n+1}}{a_n}\right| \to L < 1$$

then $a_n \to 0$.

Idea: $a_n \approx c \cdot L^n$ for $n \gg 0$, $L < 1 \implies a_n \to 0$.

Since $\left|a_{n+1}/a_n\right|$ is not exactly L, to turn this in to a proof, we must instead estimate/bound it by $\left|a_{n+1}/a_n\right| < L'$ for some $L' < 1$. Though we cannot take $L' = L$ we can take $L' = L + \epsilon$ (because $\left|a_{n+1}/a_n\right| \to L$). So we need $L + \epsilon < 1$, so let's take $\epsilon = \frac{1-L}{2}$.

Proof. Fix $\epsilon = \frac{1-L}{2}$. Then $\epsilon > 0$ because $L < 1$, so $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n} - L\right| \ < \ \epsilon \implies \left|\frac{a_{n+1}}{a_n}\right| \ < \ L + \epsilon \ = \ \frac{1+L}{2} \ < \ 1.$$

Setting $L' := \frac{1+L}{2} < 1$ we find inductively that

$$\begin{aligned} |a_{N+k}| &\leq L'|a_{N+k-1}| \\ &\leq (L')^2|a_{N+k-2}| \\ &\leq \ldots \\ &\leq (L')^k|a_N|. \end{aligned} \tag{$*$}$$

[Exercise sheet: $\alpha^k \to 0$ as $k \to \infty$ if $|\alpha| < 1$.]

We apply this to $\alpha = L' < 1$. Fixing a new $\epsilon > 0$, $\exists M > 0$ such that $\forall k \geq M$,

$$(L')^k \ < \ \frac{\epsilon}{1 + |a_N|}. \tag{$**$}$$

(We wanted to write $\frac{\epsilon}{|a_N|}$ but we have to beware the case $|a_N| = 0$.)

So by $(*)$ and $(**)$ we have

$$|a_{N+k}| \ < \ \frac{\epsilon}{1+|a_N|}|a_N| \ < \ \epsilon \qquad \forall k \geq M.$$

Rewriting this:

$$\forall n \geq N + M, \quad |a_n| \ < \ \epsilon.$$

□

3.2 Cauchy Sequences

We're now world experts at proving a_n converges if we know what the limit is. Cauchy sequences gives us a way to prove convergence *without* knowing the limit.

Definition. $(a_n)_{n\geq 1}$ is called a *Cauchy* sequence if and only if

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0} \text{ such that } \forall n, m \geq N,\ |a_n - a_m| < \epsilon.$$

Remark 3.24. $m, n \geq N$ are arbitrary. It is not enough to say that $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See exercise sheet.

Proposition 3.25. *If $a_n \to a$ then (a_n) is Cauchy.*

Proof. $a_n \to a \implies \forall \epsilon > 0\ \exists N$ such that $n \geq N \implies |a_n - a| < \frac{\epsilon}{2}$. $(*)$

So $m \geq N \implies |a_m - a| < \frac{\epsilon}{2}$ $(\dagger)$.

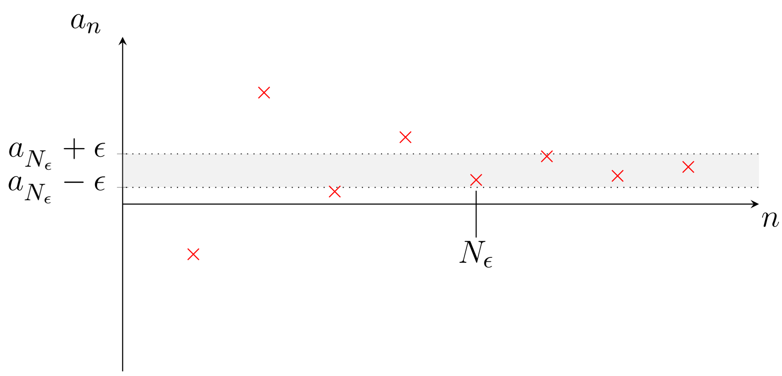
Combining these, for $m, n \geq N$ we have

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \underbrace{\epsilon/2}_{(*)} + \underbrace{\epsilon/2}_{(\dagger)} = \epsilon.$$

□

Next we want to prove the converse: Cauchy $\implies$ convergence.

We need a candidate for the limit a.



We will produce an auxiliary sequence which is *monotonic* (and bounded) $\implies$ convergent. Let $b_n := \sup\{a_i : i \geq n\}$. Then picture shows that $b_{N_\epsilon} \in (a_{N_\epsilon} - \epsilon, a_{N_\epsilon} + \epsilon]$ and b_ns are monotonically *decreasing* because $\{a_i : i \geq n+1\} \subseteq \{a_i : i \geq n\}$ so $b_{n+1} = \sup \leq \sup = b_n$.

So b_ns converge to $\inf\{b_n : n \in \mathbb{N}\}$. We will show that a_ns converge to same number, a, using the Cauchy condition.

Lemma 3.26. *(a_n) is Cauchy $\Longrightarrow$ (a_n) is bounded.*

Proof. Pick $\epsilon = 1$, then $\exists N$ such that $\forall n, m \geq N$, $|a_n - a_m| < 1$.
In particular, taking $m = N$ gives $|a_n| < 1 + |a_N|$ $\forall n \geq N$, so

$$|a_n| \ \leq \ \max\big\{|a_1|, |a_2|, \dots |a_{N-1}|, 1 + |a_N|\big\} \ \ \forall n \in \mathbb{N}. \qquad \square$$

Theorem 3.27

If (a_n) is a Cauchy sequence of real numbers then a_n converges.

Corollary 3.28. *(a_n) Cauchy $\Longleftrightarrow$ (a_n) convergent.*

> **Exercise 3.29.** Show this is not true in $\mathbb{Q}$: there exist Cauchy sequences (a_n) with $a_n \in \mathbb{Q}$ with no limit in $\mathbb{Q}$.

Proof. Since (a_n) is Cauchy, it is bounded by Lemma 3.26: $|a_n| \leq A$. So we can define $b_n := \sup\{a_i : i \geq n\}$.

Then $b_n \geq a_i$ $\forall i \geq n$ so $b_n \geq a_i$ $\forall i \geq n+1$ is an upper bound for $\{a_i : i \geq n+1\}$, so is $\geq \sup\{a_i : i \geq n+1\} = b_{n+1}$. So the sequence (b_n) is monotonically decreasing. And $b_n \geq a_n \geq -A$ shows it is also bounded below.

So we can define $a := \inf\{b_n : n \in \mathbb{N}\}$ and $b_n \downarrow a$. We claim that $a_n \to a$.

Fix $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \frac{\epsilon}{2} \quad \Longleftrightarrow \quad a_n - \frac{\epsilon}{2} < a_m < a_n + \frac{\epsilon}{2}.$$

Fix $i \geq N$ and take the supremum over all $m \geq i$:

$$\begin{aligned}
&\Longrightarrow \quad a_n - \frac{\epsilon}{2} \ < \ \underbrace{\sup\{a_m : m \geq i\}}_{= \, b_i} \ \leq \ a_n + \frac{\epsilon}{2} \\
&\Longrightarrow \quad a_n - \frac{\epsilon}{2} \ \leq \ \underbrace{\inf\{b_i : i \geq N\}}_{= \, a} \leq \ a_n + \frac{\epsilon}{2} \\
&\Longleftrightarrow \ |a - a_n| \ \leq \ \frac{\epsilon}{2} \ < \ \epsilon.
\end{aligned}$$

Since this holds for all $n \geq N$ it proves $a_n \to a$. $\square$

In the proof we twice used:

> **Exercise 3.30.** If $S \subseteq \mathbb{R}$ satisfies $x < M \ \ \forall x \in S$ then $\sup S \leq M$.

> **Example 3.31** (Decimals)**.** Suppose we didn't use decimals to construct $\mathbb{R}$ (e.g. if we used Dedekind cuts, or we just used the axioms without worrying about constructing the set).
>
> Then using Cauchy sequences we can now make sense of the decimal $a_0.a_1a_2a_3\ldots$ as follows. (Here we fix $a_0 \in \mathbb{Z}$ and $a_1, a_2, a_3, \dots \in \{0, 1, \dots, 9\}$.)
>
> Let $(A_n)_{n\geq 1}$ be the sequence of rational numbers defined by
>
> $$A_n \ := \ a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}\,.$$
>
> (A_n is the approximation to our decimal given by truncating at the nth place.)
>
> Exercise: for all $n, m \geq N$ we have $|A_n - A_m| < 10^{-N}$.
>
> Thus (A_n) is a Cauchy sequence: $\forall \epsilon > 0$ we can take $N > \epsilon^{-1}$ so that $10^N > \epsilon^{-1}$ so that $10^{-N} < \epsilon$.
>
> Thus it converges to a limit in $\mathbb{R}$. We call this limit $a_0.a_1a_2a_3\ldots$.