Network Science
Spring 2024
Problem Class 2

Problem class 2 exercises

1. Lecture 1 claimed that the adjacency matrix for a 2-dimensional rectangular lattice has a particular block-diagonal structure. However, the structure of the adjacency matrix depends on how the nodes are numbered. Provide a numbering of the nodes that results in the stated block-diagonal structure
   Solution: Say that the number of nodes is $N = m^2$. Number the nodes in the top row as $1, 2, 3, ..., m$. Number the nodes in the first column as $1, 1+m, 1+2m, ...,$ And finally the nodes in each row should be numbered in increasing order $j, j+1, ..., j+m-1$.

2. Use the results from problem sheet 2, question 3 to argue that $\lambda = 1$ is larger in magnitude than all other eigenvalues and explain why this is important for the computation of the pagerank centrality.
   Solution: The exercise shows that all elements of the eigenvector of $\mathbf{G}$ corresponding to eigenvalue $\lambda = 1$ have the same sign. The P-F theorem states that there is only one such eigenvector (neglecting multiplies of this eigenvector), and this eigenvector corresponds to a simple positive eigenvalue larger in magnitude than all other eigenvalues. The power method will find the eigenvector corresponding to the eigenvalue larger in magnitude than all others.

3. Question 3(b) from problem sheet 3 asks you to find $P(k_i^j = k)$, the degree distribution for a node connected to a randomly selected link in a G_{Np} graph. Write this distribution in terms of p_k (the degree distribution for G_{Np}, the expected degree $\langle k \rangle$, and k. Provide an interpretation of your result.
   Solution: We start from $P(k_i^j = k) = \binom{N-2}{k-1} p^{k-1}(1-p)^{(N-1-k)}$. We know that $p_k = \binom{N-1}{k} p^k (1-p)^{(N-1-k)}$, so we can rearrange $P(k_i^j = k)$ as, $\frac{k}{(N-1)p} p_k$. Recognizing that $\langle k \rangle = (N-1)p$, we arrive at our desired result, $P(D_i^j = k) = \frac{k}{\langle k \rangle} p_k$. We see that $P(k_i^j = k)$ and p_k are not generally the same, and when N is large, that you are more likely to find a node with degree larger than $\langle k \rangle$ by following a randomly selected link than you are by choosing an arbitrary node. Most nodes in a G_{Np} graph have degree close to $\langle k \rangle$, so the practical importance of this observation is limited.