M40007: Introduction to Applied Mathematics

Imperial College London

1 A "doubled" circulant matrix

Consider the Laplacian matrix associated with the graph of $2n+2$ nodes as shown in Figure 1.

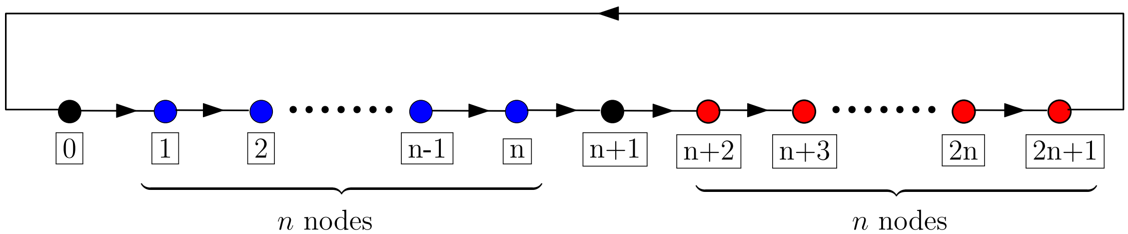

Figure 1: A graph with $2n+2$ nodes and $2n+2$ edges. Nodes $\boxed{0}$ and $\boxed{\text{n+1}}$ are marked out for special treatment in the ordering when the Laplacian is constructed.

The Laplacian for this graph, with the natural assignment where column j corresponds to node $\boxed{j-1}$ for $j = 1, \cdots, 2n+2$, is

$$
\mathbf{K} = \begin{array}{cc}
\begin{array}{cccccccccccc} \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & . & . & . & . & \boxed{\text{2n-1}} & \boxed{\text{2n}} & \boxed{\text{2n+1}} \end{array} & \\
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & . & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
&
\begin{array}{c} \boxed{0} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \\ . \\ . \\ . \\ . \\ \boxed{\text{2n-1}} \\ \boxed{\text{2n}} \\ \boxed{\text{2n+1}} \end{array}
\end{array}
\tag{1}
$$

It will be convenient, however, to reorder the assignment of nodes to the columns and rows of the matrix. The two nodes labelled $\boxed{0}$ and $\boxed{\text{n+1}}$ will be taken to correspond to the first and second column/row of the Laplacian matrix; the set of n blue nodes $\boxed{1}$-$\boxed{\text{n}}$ will correspond to the next n columns and rows, with the set of n red nodes $\boxed{n+2}$-$\boxed{2n+1}$ corresponding to the remaining columns and rows.

The reordered matrix, which will be denoted by $\tilde{\mathbf{K}}$, is

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
\boxed{0} & \boxed{\text{n+1}} & \boxed{1} & \boxed{2} & . & . & \boxed{\text{n-1}} & \boxed{\text{n}} & \boxed{\text{n+2}} & \boxed{\text{n+3}} & . & . & \boxed{\text{2n}} & \boxed{\text{2n+1}}
\end{array} \\
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & . & 0 & -1 \\
0 & 2 & 0 & 0 & 0 & . & 0 & -1 & -1 & 0 & 0 & . & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & . & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & . & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & . & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & . & 2 & -1 & 0 & 0 & 0 & . & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & . & -1 & 2 & 0 & 0 & 0 & . & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & . & 0 & 0 & 2 & -1 & 0 & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & . & 0 & 0 & -1 & 2 & -1 & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 & 2 & . & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & . & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & . & -1 & 2
\end{pmatrix}
\end{array}
\tag{2}
$$

The reason for the reordering now becomes apparent: the reordered $(2n+2)$-by-$(2n+2)$ matrix $\tilde{\mathbf{K}}$ above clearly has the sub-block decomposition

$$
\tilde{\mathbf{K}} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}_1^T & \mathbf{Q}_2^T \\ \mathbf{Q}_1 & \mathbf{K}_n & \mathbf{0} \\ \mathbf{Q}_2 & \mathbf{0} & \mathbf{K}_n \end{pmatrix}, \tag{3}
$$

where the n-by-n matrix,

$$
\mathbf{K}_n = \begin{pmatrix}
2 & -1 & 0 & 0 & . & . & 0 & 0 \\
-1 & 2 & -1 & 0 & . & . & 0 & 0 \\
0 & -1 & 2 & -1 & . & . & 0 & 0 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & . & 2 & -1 \\
0 & 0 & 0 & 0 & . & . & -1 & 2
\end{pmatrix}, \tag{4}
$$

appears *twice* as a sub-block in the matrix $\tilde{\mathbf{K}}$. Special cases of $\mathbf{K}_n$, for $n = 2$ and 3 are

$$
\begin{aligned}
n = 2, &\qquad \mathbf{K}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \\
n = 3, &\qquad \mathbf{K}_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.
\end{aligned}
\tag{5}
$$

We recognize these matrices as the ones that arose earlier when considering 2 and 3 masses, respectively, freely oscillating between two fixed walls.

It will come as no surprise that the matrix $\mathbf{K}_n$ will be seen in §2 to be the one relevant to the problem of n masses freely oscillating between two fixed walls. Since we wish to study this n-mass problem for any value of $n \geq 2$ the idea now is to deduce some general properties of the eigenvectors and eigenvalues of $\mathbf{K}_n$ by exploiting what we know about the eigenvectors/eigenvalues of the doubled circulant matrix just introduced. Note that the matrix $\mathbf{K}_n$ is *not* itself a circulant matrix.

The eigenvectors of the $(2n+2)$-dimensional circulant matrix $\mathbf{K}$ given in (1) are known, from earlier results, to be

$$
\mathbf{x}_m = \begin{pmatrix} 1 \\ \omega^m \\ \omega^{2m} \\ \omega^{3m} \\ . \\ . \\ . \\ \omega^{(2n+1)m} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/(2n+2)} = e^{\pi \mathrm{i}/(n+1)}, \qquad m = 0, 1, 2, \ldots, 2n+1
\tag{6}
$$

with corresponding eigenvalues

$$
\lambda_m = 2 - 2\cos\left(\frac{2\pi m}{2(n+1)}\right) = 2 - 2\cos\left(\frac{\pi m}{n+1}\right), \qquad m = 0, 1, 2, \ldots, 2n+1.
\tag{7}
$$

Let us rewrite this eigenvector with the modified ordering used to construct $\tilde{\mathbf{K}}$ as given in (2):

$$\tilde{\mathbf{x}}_m = \begin{pmatrix} 1 \\ \omega^{(n+1)m} \\ \omega^m \\ \omega^{2m} \\ . \\ . \\ . \\ \omega^{nm} \\ \omega^{(n+2)m} \\ . \\ . \\ . \\ \omega^{(2n+1)m} \end{pmatrix}, \qquad m = 0, 1, 2, \ldots, 2n+1. \tag{8}$$

These will be the eigenvectors of the reordered matrix $\tilde{\mathbf{K}}$. Now consider the vector

$$\mathrm{Im}[\tilde{\mathbf{x}}_m] = \begin{pmatrix} 0 \\ 0 \\ \mathbf{\Phi}_m \\ \mathbf{\Psi}_m \end{pmatrix}, \tag{9}$$

where we have used the fact that $\omega^{(n+1)m} = (-1)^m$, which is always real, and we have introduced the two n-dimensional vectors

$$\mathbf{\Phi}_m = \begin{pmatrix} \mathrm{Im}[\omega^m] \\ \mathrm{Im}[\omega^{2m}] \\ . \\ . \\ \mathrm{Im}[\omega^{nm}] \end{pmatrix}, \qquad \mathbf{\Psi}_m = \begin{pmatrix} \mathrm{Im}[\omega^{(n+2)m}] \\ \mathrm{Im}[\omega^{(n+3)m}] \\ . \\ . \\ \mathrm{Im}[\omega^{(2n+1)m}] \end{pmatrix}. \tag{10}$$

Since $\tilde{\mathbf{x}}_m$ is an eigenvector of the real matrix $\tilde{\mathbf{K}}$ with real eigenvalue λ_m as given in (7) then it is easily checked that $\mathrm{Im}[\tilde{\mathbf{x}}_m]$ is an eigenvector too, and with the same eigenvalue. Hence

$$\tilde{\mathbf{K}} \begin{pmatrix} 0 \\ \mathbf{\Phi}_m \\ \mathbf{\Psi}_m \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}_1^T & \mathbf{Q}_2^T \\ \mathbf{Q}_1 & \mathbf{K}_n & \mathbf{0} \\ \mathbf{Q}_2 & \mathbf{0} & \mathbf{K}_n \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{\Phi}_m \\ \mathbf{\Psi}_m \end{pmatrix} = \lambda_m \begin{pmatrix} 0 \\ \mathbf{\Phi}_m \\ \mathbf{\Psi}_m \end{pmatrix}. \tag{11}$$

On decomposing this into sub-blocks it is found that, for $m = 0, 1, \ldots, 2n+1$,

$$\begin{aligned} \mathbf{Q}_1^T \mathbf{\Phi}_m + \mathbf{Q}_2^T \mathbf{\Psi}_m &= 0, \\ \mathbf{K}_n \mathbf{\Phi}_m &= \lambda_m \mathbf{\Phi}_m, \\ \mathbf{K}_n \mathbf{\Psi}_m &= \lambda_m \mathbf{\Psi}_m. \end{aligned} \tag{12}$$

It is easy to show, using the fact that $\omega^{n+1} = -1$, that

$$\mathbf{\Psi}_m = (-1)^m \mathbf{\Phi}_m \tag{13}$$

so the last two equations in (12) are equivalent so it is enough to consider just

$$\mathbf{K}_n \mathbf{\Phi}_m = \lambda_m \mathbf{\Phi}_m, \qquad m = 0, 1, \ldots, 2n+1. \tag{14}$$

Since there are $2n+2$ possible values for m it appears, at first sight, that we have generated too many eigenvectors of $\mathbf{K}_n$. However, it is easy to verify that

$$\mathbf{\Phi}_{(2n+2)-m} = -\mathbf{\Phi}_m, \qquad m = 1, \ldots, n \tag{15}$$

and that

$$\mathbf{\Phi}_m = 0, \qquad m = 0, n+1. \tag{16}$$

Therefore only the n vectors $\mathbf{\Phi}_m$ for $m = 1, \cdots, n$ are linearly independent non-zero vectors. These are precisely the n eigenvectors of $\mathbf{K}_n$ we seek. In summary, we have shown that

$$\mathbf{\Phi}_m = A_m \begin{pmatrix} \mathrm{Im}[\omega^m] \\ \mathrm{Im}[\omega^{2m}] \\ . \\ . \\ \mathrm{Im}[\omega^{nm}] \end{pmatrix} = A_m \begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \sin\left(\frac{2m\pi}{n+1}\right) \\ . \\ . \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}, \qquad m = 1, \ldots, n \tag{17}$$

is an eigenvector of $\mathbf{K}_n$ with eigenvalue

$$\lambda_m = 2 - 2\cos\left(\frac{\pi m}{n+1}\right). \tag{18}$$

Distrinct eigenvectors are mutually orthogonal; the normalization constants A_m can be chosen to ensure that they are *orthonormal*, i.e.,

$$\mathbf{\Phi}_m^T \mathbf{\Phi}_p = \delta_{mp}. \tag{19}$$

A simple exercise reveals that

$$A_m = \sqrt{\frac{2}{n+1}}. \tag{20}$$

Notice that these normalization constants are the same for each $m = 1, \cdots, n$.

Exercise: Verify the first equation in (12), namely that

$$\mathbf{Q}_1^T \mathbf{\Phi}_m + \mathbf{Q}_2^T \mathbf{\Psi}_m = 0. \tag{21}$$

Exercise: Check that the general result (17) and (18) gives the known eigenvectors and eigenvalues in the case $n = 2$ and $n = 3$. Those results were derived earlier by direct methods based on determinants and solving the characteristic equation.

2 $n + 1$ masses pulled from a fixed wall

Consider $n + 1$ masses, each of unit mass, connected in line by $n + 1$ springs each with unit spring constant, with a fixed wall at one end and being pulled by an external force f_0 on the other as shown in Figure 2.

There is no upper restriction on n. This is useful since we will later consider the case when n gets very large $n \to \infty$.

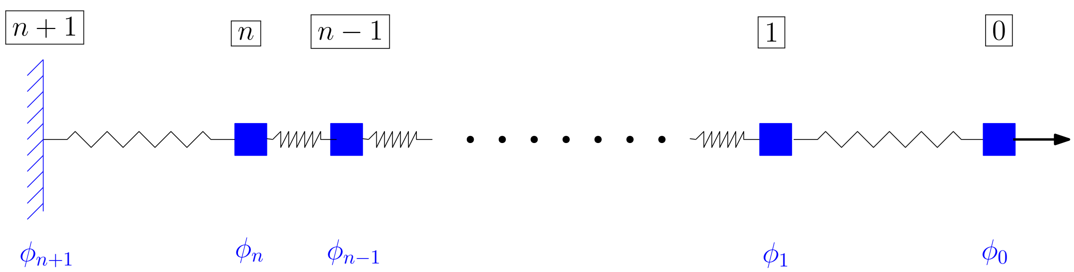

Figure 2: A graph with $2n + 2$ nodes and $2n + 2$ edges. Nodes $\boxed{0}$ and $\boxed{\text{n+1}}$ are marked out for special treatment in the ordering when the Laplacian is constructed.

As usual we will think of this system as a graph with $n + 2$ nodes with node $\boxed{0}$ being the end mass and node $\boxed{n+1}$ being the fixed wall. Let ϕ_j denote the displacement of node $\boxed{j}$.

It will be assumed that the fixed wall does not displace,

$$\phi_{n+1} = 0 \tag{22}$$

and that the external force f_0 on the mass at node $\boxed{0}$ is such that it causes unit displacement of the mass,

$$\phi_0 = 1. \tag{23}$$

The value of f_0 will be determined as part of the solution. The n masses at nodes $\boxed{1}$ - $\boxed{n}$ will be taken to be free of external forces.

Let the vector of potentials – or displacements – be given by

$$\mathbf{X} = \begin{pmatrix} \phi_0 \\ \phi_{n+1} \\ \phi_1 \\ . \\ . \\ \phi_n \end{pmatrix}, \tag{24}$$

where the ordering is such that the nodes $\boxed{0}$ and $\boxed{\text{n+1}}$ having *known* displacements are placed in the first two elements of the vector. The usual considerations imply that the corresponding Laplacian for this graph is

$$\mathbf{K} = \begin{array}{cc} \begin{array}{ccccccccc} \boxed{0} & \boxed{\text{n+1}} & \boxed{1} & \boxed{2} & \boxed{3} & . & . & \boxed{\text{n-1}} & \boxed{\text{n}} \end{array} & \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & . & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & 2 & -1 \\ 0 & -1 & 0 & 0 & 0 & . & . & -1 & 2 \end{pmatrix} & \begin{array}{c} \boxed{0} \\ \boxed{\text{n+1}} \\ \boxed{1} \\ \boxed{2} \\ . \\ . \\ \boxed{\text{n-1}} \\ \boxed{\text{n}} \end{array} \end{array} \tag{25}$$

This can be written in the block form

$$\mathbf{K} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{K}_n \end{pmatrix}, \tag{26}$$

where $\mathbf{I}_2$ is the 2-by-2 identity matrix,

$$\mathbf{Q} = \left.\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ . & . \\ . & . \\ . & . \\ 0 & 0 \\ 0 & -1 \end{pmatrix}\right\} \text{n columns} \tag{27}$$

and $\mathbf{K}_n$ is the matrix that we just studied using the "doubled circulant" matrix, namely,

$$\mathbf{K}_n = \begin{pmatrix} 2 & -1 & 0 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 2 & -1 \\ 0 & 0 & 0 & 0 & . & . & -1 & 2 \end{pmatrix}. \tag{28}$$

The force balance condition is

$$\mathbf{K}\mathbf{X} = \mathbf{f} = \begin{pmatrix} f_0 \\ -f_0 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \tag{29}$$

We can write this as

$$\begin{pmatrix} \mathbf{I}_2 & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{K}_n \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}} \\ \hat{\mathbf{X}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}} \\ \mathbf{0} \end{pmatrix}, \tag{30}$$

where

$$\hat{\mathbf{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{f}} = \begin{pmatrix} f_0 \\ -f_0 \end{pmatrix} \tag{31}$$

and

$$\hat{\mathbf{X}} = \begin{pmatrix} \phi_1 \\ . \\ . \\ \phi_n \end{pmatrix}. \tag{32}$$

The linear system (30) is equivalent to

$$\begin{aligned} \mathbf{I}_2\hat{\mathbf{e}} + \mathbf{Q}^T\hat{\mathbf{X}} &= \hat{\mathbf{f}}, \\ \mathbf{Q}\hat{\mathbf{e}} + \mathbf{K}_n\hat{\mathbf{X}} &= \mathbf{0} \end{aligned} \tag{33}$$

The second of these equations implies

$$\mathbf{K}_n\hat{\mathbf{X}} = -\mathbf{Q}\hat{\mathbf{e}} = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}. \tag{34}$$

This is the equation to solve for the unknown displacements (32) of the masses at nodes $\boxed{1}$ - $\boxed{\text{n}}$.

Let us solve equation (34) using the known eigenvectors of $\mathbf{K}_n$, namely,

$$\mathbf{K}_n \mathbf{\Phi}_j = \lambda_j \mathbf{\Phi}_j, \qquad j = 1, \ldots, n, \tag{35}$$

where

$$\mathbf{\Phi}_j = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ . \\ . \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}, \qquad j = 1, \ldots, n \tag{36}$$

which has corresponding eigenvalue

$$\lambda_j = 2 - 2\cos\left(\frac{\pi j}{n+1}\right), \qquad j = 1, \ldots, n. \tag{37}$$

This orthonormal set of vectors can be used as a basis of the solution space. Let

$$\hat{\mathbf{X}} = \sum_{j=1}^{n} a_j \mathbf{\Phi}_j \tag{38}$$

for some set of coefficients $\{a_j | j = 1, \ldots, n\}$ to be determined. The system (34) now tells us that

$$\mathbf{K}_n \hat{\mathbf{X}} = \mathbf{K}_n \left(\sum_{j=1}^{n} a_j \mathbf{\Phi}_j \right) = \left(\sum_{j=1}^{n} a_j \mathbf{K}_n \mathbf{\Phi}_j \right) = \left(\sum_{j=1}^{n} a_j \lambda_j \mathbf{\Phi}_j \right) = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}. \tag{39}$$

The orthonormality of the eigenvectors can be exploited to find the coefficients $\{a_j | j = 1, \ldots, n\}$. To see this, note that on multiplying (39) by $\mathbf{\Phi}_m^T$, it follows that

$$\left(\sum_{j=1}^{n} a_j \lambda_j \mathbf{\Phi}_m^T \mathbf{\Phi}_j \right) = \mathbf{\Phi}_m^T \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{m\pi}{n+1}\right), \tag{40}$$

where in the second equality we have picked off the first component of $\mathbf{\Phi}_m$. But, by the orthonormality of the eigenvectors,

$$\mathbf{\Phi}_m^T \mathbf{\Phi}_j = \delta_{mj}. \tag{41}$$

Thus (40) implies

$$a_m \lambda_m = \sqrt{\frac{2}{n+1}} \sin\left(\frac{m\pi}{n+1}\right). \tag{42}$$

Using the formula (37) for the eigenvalue λ_m we conclude

$$a_m = \sqrt{\frac{2}{n+1}} \times \frac{\sin\left(\frac{m\pi}{n+1}\right)}{2(1-\cos\left(\frac{m\pi}{n+1}\right))}, \qquad m = 1, \ldots, n. \tag{43}$$

The required solution is then

$$\hat{\mathbf{X}} = \sum_{j=1}^{n} \sqrt{\frac{2}{n+1}} \times \frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left(1-\cos\left(\frac{j\pi}{n+1}\right)\right)} \mathbf{\Phi}_j, \tag{44}$$

where we have substituted (43) into (38). If we now use (36) we find

$$\begin{aligned} \hat{\mathbf{X}} &= \sum_{j=1}^{n} \sqrt{\frac{2}{n+1}} \times \frac{\sin\left(\frac{j\pi}{n+1}\right)}{2\left(1-\cos\left(\frac{j\pi}{n+1}\right)\right)} \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \cdot \\ \cdot \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix} \\ &= \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin\left(\frac{j\pi}{n+1}\right)}{1-\cos\left(\frac{j\pi}{n+1}\right)} \begin{pmatrix} \sin\left(\frac{j\pi}{n+1}\right) \\ \sin\left(\frac{2j\pi}{n+1}\right) \\ \cdot \\ \cdot \\ \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}. \end{aligned} \tag{45}$$

This is the solution for the displacements. We can also write it element-wise. Picking off the individual elements of (45), and using m as the element label, we have found the n displacements to be

$$\phi_m = \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin\left(\frac{j\pi}{n+1}\right)}{1-\cos\left(\frac{j\pi}{n+1}\right)} \sin\left(\frac{mj\pi}{n+1}\right), \qquad m = 1, \ldots, n, \tag{46}$$

where we have used (32).

Equation (46) expresses the required solution. **But we already know the answer!** It should be clear from the way we have built up the mathematical framework that the problem for the displacements of the springs with the given boundary conditions (22)–(23) is equivalent to the problem of a random walker on a line with

p_m being the probability of reaching node $\boxed{0}$ before reaching node $\boxed{\text{n+1}}$ so that

$$p_0 = 1, \qquad p_{n+1} = 0. \tag{47}$$

We found the solution to this random walker problem earlier: it is

$$p_m = 1 - \frac{m}{n+1}. \tag{48}$$

We also understand that both p_m and ϕ_m can be understood as harmonic potentials on this graph. Therefore, by the uniqueness principle for harmonic potentials on this graph, we must have

$$p_m = \phi_m. \tag{49}$$

Using (46) and (48) this says that

$$1 - \frac{m}{n+1} = \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin\left(\frac{j\pi}{n+1}\right)}{1 - \cos\left(\frac{j\pi}{n+1}\right)} \sin\left(\frac{mj\pi}{n+1}\right), \qquad m = 1, \ldots, n. \tag{50}$$

This is quite a remarkable identity.

The result just obtained is valid for any integer $n \geq 1$. We can ask: what happens as $n \to \infty$?

Let us introduce a new variable

$$x = \frac{\pi m}{n+1}. \tag{51}$$

This replaces the integer index m with a non-integer variable x taking a discrete set of values $0 < x < \pi$ since $m = 1, \ldots, n$. Notice that, even as $n \to \infty$, this new variable x remains between 0 and π but never quite attains either value. We can now rewrite (50) in terms of this new variable:

$$1 - \frac{x}{\pi} = \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin\left(\frac{j\pi}{n+1}\right)}{1 - \cos\left(\frac{j\pi}{n+1}\right)} \sin(jx), \qquad x = \frac{\pi m}{n+1}, \qquad m = 1, \ldots, n. \tag{52}$$

If we now take the limit $n \to \infty$ and use the Taylor series

$$\begin{aligned} \sin\left(\frac{j\pi}{n+1}\right) &\approx \frac{\pi j}{n+1} + \ldots, \\ 1 - \cos\left(\frac{j\pi}{n+1}\right) &\approx 1 - \left(1 - \frac{1}{2!}\left(\frac{j\pi}{n+1}\right)^2 + \ldots\right) = \frac{1}{2}\left(\frac{j\pi}{n+1}\right)^2 + \ldots, \end{aligned} \tag{53}$$

we find that (52) becomes

$$1-\frac{x}{\pi}=\sum_{j=1}^{\infty}\frac{2}{j\pi}\sin jx, \qquad 0<x<\pi. \tag{54}$$

It can be checked that the left hand side of this equation is precisely the *Fourier sine series* of the function on the right hand side.

The Fourier sine series therefore emerges as the $n\to\infty$ of the solution of a discrete problem, involving $n+1$ masses attached to springs and pulled away from a wall, when one seeks to solve the problem of equilibrium using the eigenvectors of the Laplacian matrix as a basis for the solution.