Applied Probability Midterm

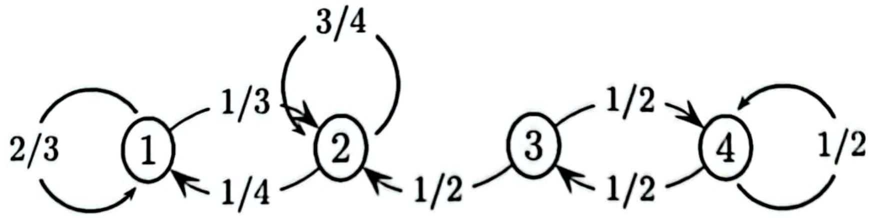
11 November 2022

Question 1: Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain on a countable state space E with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$.

(a) **(2 points)** State the Markov condition.

(b) **(2 points)** Describe in about 3-6 sentences, using results (without proofs!) from lectures, why it makes sense to say that "recurrence is a class property".

Question 2: Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain with state space $E = \{1,2,3,4\}$ and transition diagram given by



(a) **(2 points)** Find the transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$.

(b) **(2 points)** Determine the communicating classes.

(c) **(2 points)** For each class, specify whether the class is transient, positive recurrent or null recurrent and justify your answer.

(d) **(1 point)** Does this Markov chain have a unique stationary distribution? Justify your answer.

(e) Suppose that $\mathrm{P}(X_0 = 3) = 1$. Find the following probabilities:

i. (1 point) $\mathrm{P}(X_2 = 1)$.

ii. (1 point) $\mathrm{P}(X_2 = 2)$.

iii. (1 point) $\mathrm{P}(X_2 = 3)$.

iv. (1 point) $\mathrm{P}(X_2 = 4)$.

v. (1 point) $\mathrm{P}(X_0 = 3, X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 3)$.

Question 3: (4 points) Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain with state space $E = \{1,2\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Find

$$\lim_{n\to\infty} \mathrm{E}\left(X_n^2\right).$$