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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 6

1. In this problem, we use the method of separation of variables to solve Laplace's equation on a rectangular bounded domain. First, it is useful to note that the boundary conditions impose $u(0, y) = u(2, y) = 0$, so we require solutions that are periodic in x.

   Let $u(x, y) = X(x)Y(y)$, a substitution in Laplace's equation leads to

$$X''Y + XY'' = 0$$

   Separating the variables, we find that there exists a constant λ such that

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

   where the sign of the constant was guided by the fact that we are looking for periodic solutions in x. The general solutions to these two ODEs are given by

$$\begin{aligned} X(x) &= A\cos\lambda x + B\sin\lambda x \\ Y(y) &= C\cosh\lambda y + D\sinh\lambda y \end{aligned}$$

   but the boundary conditions impose

$$\begin{aligned} &X(0) = 0 \Rightarrow A = 0 \\ &X(2) = 0 \Rightarrow \sin 2\lambda = 0 \Rightarrow \lambda = \frac{n\pi}{2} \end{aligned}$$

   and as well as $Y(0) = 0 \Rightarrow C = 0$. So in both cases (a) and (b), we obtain that

$$u = \sum_{n=1}^{\infty} B_n \sin\lambda_n x \sinh\lambda_n y, \quad \text{with } \lambda_n = \frac{n\pi}{2}$$

   (a) In this first case, we have $u = 1$ on $y = 4$, this leads to

$$1 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)\sinh(2n\pi) \qquad (0 < x < 2)$$

   This is a half-range Fourier sine series; thus, we know that

$$\begin{aligned} B_n\sinh(2\pi n) &= \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{2}{n\pi}\left[1 - (-1)^n\right] \\ &= \begin{cases} 0, & \text{if } n \text{ even} \\ 4/n\pi, & \text{if } n \text{ even} \end{cases} \end{aligned}$$

   So we conclude that the solution is given by

$$u = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x/2)\sinh((2m+1)\pi y)}{(2m+1)\sinh(2\pi(2m+1))}$$

(b) Here, we have $u = \sin(\pi x/2)$ on $y = 4$, which imposes

$$\sin\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} B_n \sinh(2n\pi) \sin\left(\frac{n\pi x}{2}\right)$$

One can easily compute the Fourier coefficients via the integral as we did above but here the easiest is to equate the coefficients of $\sin(n\pi x/2)$. We find that

$$B_1 = \frac{1}{\sinh(2\pi)} \quad \text{and} \quad B_n = 0,\ n > 1$$

so we finally obtain

$$u(x, y) = \frac{\sin(\pi x/2)\sinh(n\pi y/2)}{\sinh(2\pi)}$$

2. Consider Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

on the half-space $-\infty < x < \infty$, $y > 0$ with boundary condition $\phi(x, 0) = p(x)$. Taking a Fourier transform in x, we get

$$\frac{\partial^2 \widehat{\phi}}{\partial y^2} - \omega^2 \widehat{\phi} = 0$$

which has for general solution

$$\widehat{\phi}(\omega, y) = A\exp(\omega y) + B\exp(-\omega y)$$

As we require that $\phi(x, y)$ be bounded when $x \to \pm\infty$, the same applies to $\widehat{\phi}(\omega, y)$ as $\omega \to \pm\infty$. Therefore, using the boundary conditions, we find that the appropriate solution is given by

$$\widehat{\phi}(\omega, y) = \widehat{p}(\omega)\exp(-|\omega| y)$$

where $\widehat{p}(\omega)$ is the Fourier transform of $p(x)$. We thus have that

$$\phi(x, y) = \mathcal{F}^{-1}\left\{\widehat{\phi}(\omega, y)\right\} = p(x) * \mathcal{F}^{-1}\left\{\exp(-|\omega| y)\right\}$$

where we have used the convolution theorem. So if we can find the inverse Fourier transform of $\exp(-|\omega| y)$, then we are done! Let's write the inversion formula

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\exp(-|\omega| y)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-|\omega| y)\exp(i\omega x) d\omega \\
&= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} \exp((ix + y)\omega) d\omega + \int_{0}^{+\infty} \exp((ix - y)\omega) d\omega\right\} \\
&= \frac{1}{2\pi}\left\{\frac{1}{ix + y} - \frac{1}{ix - y}\right\} = \frac{y}{\pi(x^2 + y^2)}
\end{aligned}$$

So using the convolution theorem, we finally obtained that

$$\phi(x, y) = \int_{-\infty}^{+\infty} \frac{y}{\pi} \frac{p(\xi)}{(x - \xi)^2 + y^2} d\xi$$

3. Now, we are trying to find a bounded solution to the Neumann problem, i.e. a solution ϕ such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial \phi}{\partial y}(x, 0) = q(x), \quad \phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty$$

in the half-space $-\infty < x < \infty$ and $y > 0$.

Let $\psi = \partial\phi/\partial y$, we find that

$$\psi_{xx} + \psi_{yy} = \phi_{yxx} + \phi_{yyy} = \frac{\partial}{\partial y}\left[\phi_{xx} + \phi_{yy}\right] = 0$$

i.e. that ψ is solution to Laplace's equation with the Dirichlet boundary conditions, $\psi(x,0) = q(x)$. From Q2, we know that a solution to this problem is written

$$\psi(x,y) = \int_{-\infty}^{+\infty} \frac{y}{\pi}\frac{q(\xi)}{(x-\xi)^2 + y^2} d\xi$$

By integration over y, we thus find

$$\begin{aligned}
\phi(x,y) &= \int \psi(x,y) dy \\
&= \int dy \int_{-\infty}^{+\infty} d\xi \frac{y}{\pi}\frac{q(\xi)}{(x-\xi)^2 + y^2} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\xi q(\xi)\left[\int dy \frac{y}{(x-\xi)^2 + y^2}\right] \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\xi q(\xi)\left[\frac{1}{2}\ln\left((x-\xi)^2 + y^2\right)\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln\left((x-\xi)^2 + y^2\right) q(\xi) d\xi
\end{aligned}$$

4. Following the lecture notes, we suppose that we have found two solutions ϕ_1 and ϕ_2 to the interior Neumann problem in volume V bounded by the surface S_0 with N holes with boundaries S_i, $i = 1, \dots, N$.

   Let $\psi \equiv \phi_1 - \phi_2$, then we have

   $$\nabla^2\psi = 0 \quad \text{in } V, \qquad \frac{\partial\psi}{\partial n} = 0 \quad \text{on all boundaries } S_0, \dots, S_N$$

   Using Green's first identity, we write that

   $$\sum_{i=0}^{N} \int_{S_i} \psi\frac{\partial\psi}{\partial n} dS = \int_V \left\{\psi\nabla^2\psi + (\nabla\psi)\cdot(\nabla\psi)\right\} dV$$

   Now, as $\partial\psi/\partial n$ vanishes on all boundaries, the LHS of this equation is zero. Similarly, as ψ is solution to Laplace's equation, the first term on the RHS fo the equation also vanishes and we are left with

   $$\int_V |\nabla\psi|^2 dV = 0 \Rightarrow \nabla\psi = 0$$

   So we finally find that $\psi = \text{constant}$ and so we conclude that ϕ_1 and ϕ_2 differ at most by a constant.

5. In this problem, we try to solve

   $$\nabla^2\phi = f(\mathbf{r}) \quad \text{in } V, \qquad \frac{\partial\phi}{\partial n} = q(\mathbf{r}) \quad \text{on } S$$

   Consider the Green's function that satisfies

   $$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{in } V, \qquad \frac{\partial G}{\partial n} = 0 \quad \text{on } S,$$

   If we apply Green's second identity to the functions ϕ and G, we obtain

   $$\int_V \left(\phi\nabla^2 G - G\nabla^2\phi\right) dV = \int_S \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n}\right) dS$$

Substituting for $\nabla^2 G$, $\nabla^2\phi$ and applying Neumann boundary conditions, we find

$$\int_V \phi\delta(\mathbf{r}-\mathbf{r}_0)dV - \int_V Gf(\mathbf{r})dV = -\int_S Gq(\mathbf{r})dS$$

Using the sifting property of the delta function, we finally obtain

$$\phi(\mathbf{r}_0) = -\int_S Gq(\mathbf{r})dS + \int_V Gf(\mathbf{r})dV + \text{constant}$$

where the constant arises because the solution for G is only unique up to an additive constant.

6. To start, we need to verify that G as given in the problem satisfies

$$\nabla^2 G = \delta(\mathbf{r}-\mathbf{r}_0) \quad \text{for } z>0 \text{ with } \frac{\partial G}{\partial z} = 0 \quad \text{on } z=0$$

First, we need to verify that $\nabla^2 G = 0$, for all $\mathbf{r} \neq \mathbf{r}_0$. By definition of G, we have

$$\nabla G = -\frac{1}{4\pi}\nabla\cdot\left[\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{1}{|\mathbf{r}-\mathbf{r}_0'|}\right]$$

Let us define

$$g(x,y,z) = \left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{-1/2} + \left[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2\right]^{-1/2}$$

Then, we have

$$\nabla G = -\frac{1}{4\pi}\left[\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right]$$

By definition of g, we write

$$\frac{\partial g}{\partial z} = -\frac{z-z_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} - \frac{z+z_0}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}}$$

We obtain similar expressions for $\partial g/\partial x$ and $\partial g/\partial y$. Assembling all these terms, we finally obtain

$$\nabla G = \frac{1}{4\pi}\left[\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\mathbf{r}-\mathbf{r}_0'}{|\mathbf{r}-\mathbf{r}_0'|^3}\right]$$

Further, we know that

$$\nabla^2 G = \nabla\cdot\nabla G = -\frac{1}{4\pi}\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right]$$

In particular, we have

$$\begin{aligned}
\frac{\partial g^2}{\partial x^2} =& -\frac{\partial}{\partial x}\left[\frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} + \frac{x+x_0}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}}\right] \\
=& -\frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} - \frac{1}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} \\
&+ \frac{3(x-x_0)^2}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}} + \frac{3(x+x_0)^2}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{5/2}}
\end{aligned}$$

We proceed similarly for $\partial^2 g/\partial y^2$ and $\partial^2 g/\partial z^2$. Adding all these terms, we find that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

so we conclude that

$$\forall \mathbf{r} \neq \mathbf{r}_0,\ \nabla^2 G = 0$$

What about $\mathbf{r} = \mathbf{r}_0$? We want to show that then we need a delta function in $\mathbf{r} = \mathbf{r}_0$. To do this, we integrate the equation for G over the whole half-space and write

$$\int_V \nabla^2 G dV = \int_V \delta(\mathbf{r} - \mathbf{r}_0) dV = 1$$

We need to check that this holds for our choice of G! By the divergence theorem, we have

$$\begin{aligned}
\int_V \nabla^2 G dV &= \int_S \nabla G \cdot \widehat{\mathbf{n}}\, dS \\
&= \frac{1}{4\pi} \int_S \left[\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} + \frac{\mathbf{r} - \mathbf{r}'_0}{|\mathbf{r} - \mathbf{r}'_0|^3} \right] \cdot \widehat{\mathbf{n}}\, dS \\
&= \frac{1}{4\pi} \left[\int_S \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \widehat{\mathbf{n}}\, dS + \int_S \frac{\mathbf{r} - \mathbf{r}'_0}{|\mathbf{r} - \mathbf{r}'_0|^3} \cdot \widehat{\mathbf{n}}\, dS \right]
\end{aligned}$$

Now as we work on the upper half-space, we know that either $\mathbf{r}_0$ or $\mathbf{r}'_0$ are enclosed in the surface S but not both; so making use of Gauss theorem, we obtain that

$$\int_V \nabla^2 G dV = 1$$

Finally, there is one more thing to check... the boundary conditions! We need $\partial G / \partial z = 0$ on $z = 0$. From what we have just written, we conclude directly that

$$\frac{\partial G}{\partial z} = -\frac{1}{4\pi} \frac{\partial g}{\partial z} = 0$$

Next, we use Green's second identity with G and ϕ to write

$$\int_V \left(\phi \nabla^2 G - G \nabla^2 \phi \right) dV = \int_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) dS$$

The surface S is here composed of the plane $z = 0$ and a surface at infinity on which there is no contribution as we assume that ϕ decays in the far-field. So we obtain

$$\int_V \phi \delta(\mathbf{r} - \mathbf{r}_0) dV = \int_{z=0} G(\mathbf{r}, \mathbf{r}_0) q(x, y) dx dy$$

where we have used the fact that $\partial / \partial n = -\partial / \partial z$ (since $\widehat{\mathbf{n}}$ points outside the surface). We finally obtain

$$\phi(\mathbf{r}_0) = \int_{z=0} G(\mathbf{r}, \mathbf{r}_0) q(x, y) dx dy$$

Finally, if q is such that

$$q(x, y) = \begin{cases} q_0, & x^2 + y^2 \leq R^2 \\ 0, & x^2 + y^2 > R^2 \end{cases}$$

Then, we write

$$\phi(\mathbf{r}_0) = \int_{x^2 + y^2 \leq R^2} G(\mathbf{r}, \mathbf{r}_0) q(x, y) dx dy$$

and setting $x_0 = y_0 = 0$ and $z = 0$, we obtain

$$\phi(0, 0, z_0) = -\frac{q_0}{2\pi} \int_{x^2 + y^2 \leq R^2} \frac{1}{\sqrt{x^2 + y^2 + z_0^2}} dx dy$$

Using plane polar coordinates (r, θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$. So this integral reads

$$\phi(0,0,z_0) = -\frac{q_0}{2\pi}\int_0^{2\pi}\int_0^R \frac{r}{\sqrt{r^2+z_0^2}}drd\theta$$
$$= -q_0\int_0^R \frac{r}{\sqrt{r^2+z_0^2}}dr = -q_0\left[\sqrt{r^2+z_0^2}\right]_0^R$$

and we finally conclude that

$$\phi(0,0,z) = -q_0\left[\sqrt{R^2+z^2} - z\right]$$

7. First, we look for the Green's function satisfying $\nabla^2 G = \delta(\mathbf{r})$. The problem is radially symmetry, so we look for solutions of the form $G \equiv G(r)$, with r the radial plane polar coordinate. The Laplacian of G is then given by

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dG}{dr}\right] = 0, \quad \text{for } r \neq 0$$

This equation can easily be integrated in

$$G(r) = A\ln r + B$$

How can we find the integration constants? To do so, we apply the condition that

$$\int_{x^2+y^2<\infty} \nabla^2 G dxdy = 1$$

Now since $\nabla^2 G = 0$ expect at the origin, we can write that

$$\int_{x^2+y^2<\infty} \nabla^2 G dxdy = \lim_{\varepsilon\to 0}\int_{x^2+y^2<\varepsilon^2} \nabla^2 G dxdy$$

The divergence theorem then tells us that

$$\lim_{\varepsilon\to 0}\int_{x^2+y^2<\varepsilon^2} \nabla^2 G dxdy = \lim_{\varepsilon\to 0}\oint_{r=\varepsilon}\frac{dG}{dr}ds = \lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{A}{\varepsilon}\varepsilon d\theta = \lim_{\varepsilon\to 0}2\pi\frac{A}{\varepsilon}\varepsilon = 2\pi A$$

Hence, we require $A = (2\pi)^{-1}$. Now as the Green's function solution is always unique up to an additive constant, we can take $B = 0$ for convenience.

By a shift of origin, the corresponding solution to $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$ is given by

$$G = \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_0|$$

Now this function does not satisfy the boundary condition giving that $G = 0$ on $y = 0$. To achieve this, we consider a singularity of equal and opposite strength located at a distance y_0 below the $y = 0$ axis. The new Green's function reads

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_0| - \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_0'|$$

with $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{r}_0' = (x_0, -y_0)$. It is easy to confirm that in this case $G = 0$ on $y = 0$. Let us put this Green's function to good use! We consider the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{for } y > 0, \quad \phi(x, 0) = p(x)$$

As we did in the lectures, we can use Green's second identity in 2D which gives us here

$$\phi(\mathbf{r}_0) = \int_C p(x) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_0) dx dy$$

In this case, the boundary C is simply the x-axis $y = 0$ and the outward normal points in the negative y direction so that $\partial/\partial n = -\partial/\partial y$. So we write

$$\phi(x_0, y_0) = -\int_{-\infty}^{+\infty} p(x) \left. \frac{\partial G}{\partial y} \right|_{y=0} dx$$

Note that the Green's function reads

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi} \left\{ \ln\left[(x - x_0)^2 + (y - y_0)^2\right] - \ln\left[(x - x_0)^2 + (y + y_0)^2\right] \right\}$$

and by differentiation, we obtain

$$\frac{\partial G}{\partial y} = \frac{1}{4\pi} \left\{ \frac{2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{2(y + y_0)}{(x - x_0)^2 + (y + y_0)^2} \right\}$$

and setting $y = 0$ gives us

$$\frac{\partial G}{\partial y} = -\frac{1}{\pi} \frac{y_0}{(x - x_0)^2 + y_0^2}$$

So we finally obtain that

$$\phi(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{+\infty} \frac{p(x)}{(x - x_0)^2 + y_0^2} dx$$

which is the result we obtained via a different method in Q2.

8. In this problem, we will solve Laplace's equation

$$\nabla^2 u(\mathbf{r}) = 0$$

in plane polar coordinates (r, θ). Before we start, recall that the Laplacian in plane polar coordinates is written

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(a) Here, we use the method of separation of variables and write $u(r, \theta) = R(r)\Theta(\theta)$. Substituting this in the Laplace equation, we obtain:

$$\frac{\Theta}{r}\frac{d}{dr}\left[r\frac{dR}{dr}\right] + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Dividing through by $R\Theta$ and multiplying by r^2, we obtain

$$\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

So we conclude that there exists a (separation) constant n^2 such that

$$\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = n^2 \quad \text{and} \quad \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2$$

where we have taken the constant to be written as n^2 for later convenience.
First, we consider the case where $n \neq 0$. The ODE for Θ is easy to solve and we get

$$\Theta(\theta) = A\exp(in\theta) + B\exp(-in\theta)$$

Now the equation for R can be rearranged to read

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0$$

To solve this equation, we make the following substitution $r = \exp(t)$. The derivatives transform according to

$$\begin{aligned}\frac{dR}{dr} &= \frac{dR}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{dR}{dt} \\ \frac{d^2R}{dr^2} &= \frac{d}{dt}\left[\frac{1}{r}\frac{dR}{dt}\right] = -\frac{1}{r^2}\frac{dR}{dt} + \frac{1}{r^2}\frac{d^2R}{dt^2}\end{aligned}$$

So the equation can be rewritten

$$\frac{d^2R}{dt^2} - n^2R = 0$$

which is a second-order ODE with constant coefficients which integrates into

$$R(t) = C\exp(nt) + D\exp(-nt)$$

or

$$R(r) = Cr^n + Dr^{-n}$$

Now looking more closely at the solution for Θ, it is clear that we need this function to be 2π periodic. Otherwise when θ increases by 2π, the function would not be single-valued! So we require n to be an integer.
We conclude that, for $n \neq 0$, we get the following family of solutions for Laplace's equation

$$u_n(r,\theta) = (A_n\cos(n\theta) + B_n\sin(n\theta))\,(C_n\exp(nt) + D_n\exp(-nt))$$

where A_n, B_n, C_n, D_n are arbitrary constants.
We need now to deal with the case $n = 0$. In this case, we obtain

$$\begin{aligned}\frac{d^2\Theta}{d\theta^2} &= 0 \Rightarrow \Theta(\theta) = A\theta + B \\ \frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) &= 0 \Rightarrow R(r) = C\ln r + D\end{aligned}$$

Once again, by periodicity of Θ, we require that $A = 0$ and so we obtain the following particular solution

$$u_0(r,\theta) = C_0\ln r + D_0$$

By superposition principle, the general solution to this problem reads

$$u(r,\theta) = C_0\ln r + D_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta))\,(C_n\exp(nt) + D_n\exp(-nt))$$

or

$$u(r,\theta) = C_0\ln r + D_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta))\,\left(C_nr^n + D_nr^{-n}\right)$$

where we have omitted negative values of n as we can include them in the sum by a redefinition of the coefficients.

(b) We know that in the most general case the transverse displacements of a stretched membrane are governed by a two-dimensional wave equation. In this case, however, we are looking for the steady-state solution and so $u(r,\theta)$ will solve Laplace's equation subject to the imposed Dirichlet boundary conditions.
Using the result obtained in (b), we can conclude that:

- ☐ Since we are looking for a solution which is finite everywhere inside the disc of radius a, then we necessarily have $C_0 = 0$ (otherwise the solution would diverge in $r = 0$).
- ☐ For the same reason, we conclude that $D_n = 0$, for $n > 0$.
- ☐ Now the boundary condition at the rim imposes that

$$u(a,\theta) = D_0 + \sum_{n=1}^{\infty} C_n a^n (A_n \cos n\theta + B_n \sin n\theta) = \varepsilon(\sin\theta + 2\sin 2\theta)$$

  By equating the coefficients in front of the $sinn\theta$ and $\cos n\phi$, we conclude that $D_0 = 0$ and $A_n = 0$, for all n. Finally, we must have $C_1 B_1 a = \varepsilon$ and $C_2 B_2 a^2 = 2\varepsilon$ and $B_n = 0$ for $n > 2$.

We conclude that the appropriate shape for the drumskin (valid over the whole skin) is given by

$$u(r,\theta) = \frac{\varepsilon r}{a}\sin\theta + \frac{2\varepsilon r^2}{a^2}\sin 2\theta$$

We show the shape of the drumskin in Fig. 1.

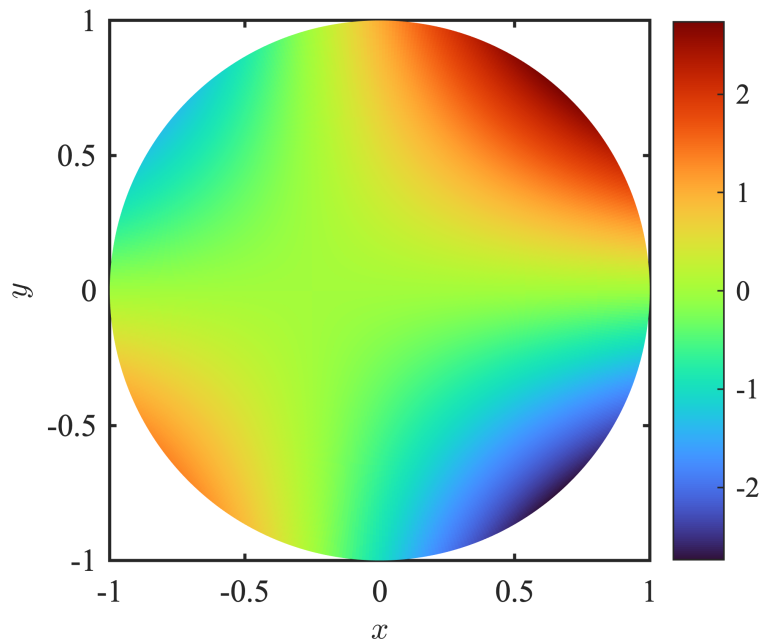

Figure 1: Steady-state vertical displacement of the deformed drumhead with $a = 1$ and $\varepsilon = 1$.

9. A circular disc of radius a is heated in such a way that its perimeter $r = a$ has a steady temperature distribution given by $A + B\cos^2\theta$, where r and θ are plane polar coordinates and A and B are real constants. The evolution of the temperature in the disc obeys the two dimensional heat equation but once again, this is a steady-state problem for which the heat equation becomes the Laplace equation. The most general solution to the Laplace equation in plane polar coordinates was obtained in Q8 and reads

$$T(r,\theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (A_n\cos(n\theta) + B_n \sin(n\theta)) \left(C_n r^n + D_n r^{-n}\right)$$

Now the region $r < a$ contains the point $r = 0$; since both $\ln r$ and r^{-n} diverge at that point, we require that $C_0 = 0$ and $D_n = 0$ for all n to have a physically admissible solution (temperature can not become infinite!).

Now on $r = a$, we have

$$T(a, \theta) = A + B\cos^2\theta = A + \frac{1}{2}B(\cos 2\theta + 1)$$

Equating the coefficients of $\cos n\theta$ and $\sin n\theta$ including $n = 0$, gives that

$$A + \frac{1}{2}B = D_0, \quad A_2 C_2 a^2 = \frac{1}{2}B, \quad A_n C_n = 0, \ \forall n \neq 2 \quad \text{and} \quad B_n = 0, \ \forall n$$

The solution everywhere in the disc is thus given by

$$T(r, \theta) = A + \frac{B}{2} + \frac{Br^2}{2a^2}\cos 2\theta$$

Recall that we could equate the coefficients to find our constants because the sinusoidal functions in the sum form a mutually orthogonal set over the range $0 \leq \theta \leq 2\pi$. We show in Fig. 2 the temperature distribution in the disc.

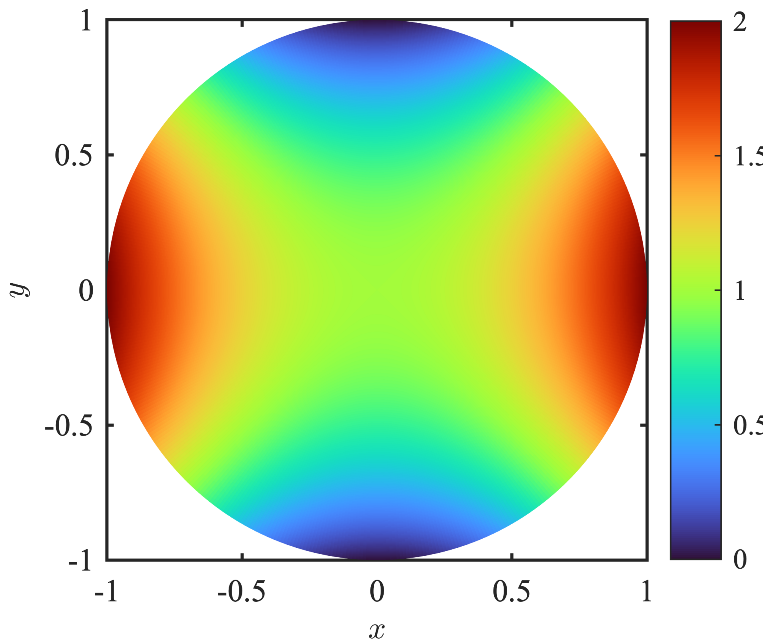

Figure 2: Steady-state temperature distribution in a disc of radius $a = 1$ with $A = 0$ and $B = 2$.