MATH50004 Differential Equations
Spring Term 2020/21
Mid-term exam on 22 February 2021

Question 1 (total: 12 points)

Consider an initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let J be an interval containing t_0 in its interior, and consider the Picard iterates $\{\lambda_n : J \to \mathbb{R}^d\}_{n \in \mathbb{N}_0}$ corresponding to this initial value problem.

(i) Suppose that there exists an $n \in \mathbb{N}_0$ such that $\lambda_{n+1} = \lambda_n$. Explain why then λ_n solves the above initial value problem. [3 points]

(ii) Give an example of an initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$, for which all Picard iterates are constant, i.e. $\lambda_{n+1} = \lambda_n$ for all $n \in \mathbb{N}_0$. [3 points]

(iii) Compute λ_0, λ_1 and λ_2 for the one-dimensional initial value problem $\dot{x} = 1 - x$ with $x(0) = 0$. [6 points]

Question 2 (total: 12 points)

Consider the one-dimensional differential equation

$$\dot{x} = \frac{e^{t^2}}{x^2},$$

whose right hand side $f : D \to \mathbb{R}$, $f(t, x) = \frac{e^{t^2}}{x^2}$, is defined on $D = \mathbb{R} \times (0, \infty)$.

(i) Prove that the conditions on the right hand side f of the local version of the Picard–Lindelöf theorem are satisfied for this differential equation. [4 points]

Hence, for any initial pair $(t_0, x_0) \in D$, the maximal solution $\lambda_{max} : (I_-(t_0, x_0), I_+(t_0, x_0)) \to \mathbb{R}$ exists that satisfies the initial condition $x(t_0) = x_0$.

(ii) Show that $\lambda_{max}(t) \in (0, x_0]$ for all $t \in (I_-(t_0, x_0), t_0]$. [4 points]

(iii) Assume without proof that $I_-(t_0, x_0)$ is finite. What is $\lim_{t \to I_-(t_0, x_0)} \lambda_{max}(t)$? Justify your answer using results from the course. [4 points]

Question 3 (total: 6 points)

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and locally Lipschitz continuous (with respect to x) and consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

where $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$. Suppose that f is bounded on all sets of the form $J \times \mathbb{R}$, where $J \subset \mathbb{R}$ is a compact interval. Show that then the maximal solution $\lambda_{max} : (I_-(t_0, x_0), I_+(t_0, x_0)) \to \mathbb{R}$ to the above initial value problem exists globally forward in time, i.e. $I_+(t_0, x_0) = \infty$.

<u>Hint.</u> You may want to prove this by contradiction.