IMPERIAL

MATH70141

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Introduction to Game Theory

Date: Wednesday, May 21, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Throughout this question consider an N-player ($N \geq 2$), finite, simultaneous move game, G.

(a) Define what it means to have a **pure strategy equilibrium** of such a game. [Note: If your definition uses the terminology '**best response**' then you should define this] (4 marks)

(b) Define what it means for a pure strategy of a player to be **strictly dominated** by another pure strategy of that player. (2 marks)

(c) Prove that if we delete a strictly dominated pure strategy from G, then the game, say G', with this pure strategy removed has the same pure strategy equilibria as G. (5 marks)

At the university of ImperfectSweetial each member of a class of n ($n \geq 3$) students taking a course in Game Theory takes part in a simultaneous move game where they have two options. A student may either take an imperfect sweet of type a, or they may focus their effort towards creating the perfect sweet.

If at least m students, where $2 \leq m < n$, act in pursuit of the perfect sweet then it will be discovered and each student who pursues it will receive a perfect sweet. If less than m students pursue the perfect sweet then it is not discovered, but, feeling guilty about the lack of success, the lecturer offers each student who pursues the perfect sweet an imperfect sweet of type b.

(d) Prove that the strategy profile in which **all** students pursue the perfect sweet is the **unique equilibrium** of the game if and only if the students prefer imperfect sweets of type b to those of type a. You may assume that all students equally enjoy sweets (i.e student i enjoys sweets as much as student j, for all $1 \leq i, j \leq n$) and that the perfect sweet is preferred by students to any type of imperfect sweet. (9 marks)

(Total: 20 marks)

2. Throughout this question consider a finite, two-player, zero-sum game played between players A and B.

(a) In such a game:

(i) Define what it means for a strategy, α^*, of player A to be an **equaliser strategy**.

(2 marks)

(ii) Prove that if α^* and β^* are equaliser strategies for players A and B respectively, then these strategies are max-min and min-max and form an equilibrium of the game.

(5 marks)

In a conflict between two armies, army A has 2 regiments and army B has 3 regiments. The armies are fighting over two battlefields X and Y where battlefield X is worth k $(k > 1)$ times as much as battlefield Y to both armies. Each army must decide how to allocate their regiments to the two battlefields.

Army A can send both regiments to battlefield X (pure strategy a_1), or they can send one regiment to each of X and Y (pure strategy a_2), or they can send both regiments to battlefield Y (pure strategy a_3). Similarly army B can decide to send any number of its regiments to X and any remaining regiments to Y.

At each battlefield a battle takes place. The victor of this battle is the army with the most regiments at that battlefield. They gain the value of the battlefield and the loser loses the value of the battlefield. If both send the same number of regiments to the battlefield then neither wins and both receive no gain or loss for this battle. The overall payoff to each army is the sum of their payoffs at the two battlefields.

(b) Modelling this situation as a zero-sum game. Show that:

(i) In seeking a solution to the game, we can rule out the possibility of army B placing all their regiments at Y; (4 marks)

(ii) In the subgame which excludes this pure strategy for B, A has an equaliser strategy in which their pure strategies (a_1, a_2, a_3) are played with probabilities

$$\frac{1}{k^2+1}, \quad \frac{k-1}{k^2+1}, \quad \frac{k^2-k+1}{k^2+1},$$

respectively. (3 marks)

(c) Find a solution and the value of the game. (6 marks)

(Total: 20 marks)

3. The normal (strategic) form representation of a two-player, **cooperative** game being played between players A and B is given below. The first entry in each ordered pair gives the payoff to player A and the second the payoff to player B.

A \ B	b_1	b_2
a_1	$0, 0$	λ, μ
a_2	$1, 0$	$0, 1$

Here $0 < \lambda, \mu < 1$ and $\lambda + \mu < 1$.

(a) (i) Determine the threat point for the game. (4 marks)

(ii) Sketch the payoff set and identify the bargaining set. Indicate the pareto-optimal frontier of this set. (4 marks)

(iii) Find the Nash bargaining solution for the game and show how the players can implement this solution. (7 marks)

Suppose now that the same game is to be played **non-cooperatively** between the players and that $\mu = 0$.

(b) Determine all equilibria in this new game. (5 marks)

(Total: 20 marks)

4. [Throughout this question you may assume any results about impartial games and the game of Nim unless you are asked to prove them.]

(a) Given impartial games G and H, define:

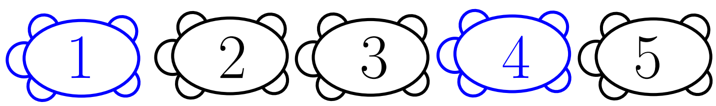
(i) What it means to call G **losing**; [Note: If your definition uses the terminology '**winning**' then you should define this] (2 marks)

(ii) The **game sum** of G and H; (2 marks)

(iii) What it means for G to be termed **equivalent** to H. (2 marks)

(b) Given an impartial game G. Prove that G is a losing game if and only if G is equivalent to the game with no options. (6 marks)

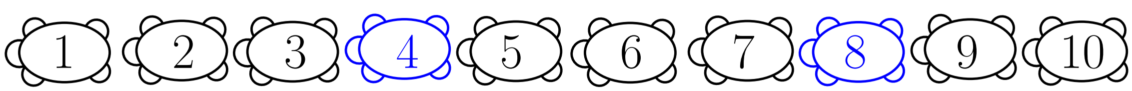
The impartial game of **Turning Turtles** consists of a line of n turtles which can either be upright or upside down. The figure below shows a turning turtles game with 5 turtles in which the first and fourth turtles are upside down. Upside down turtles have been coloured blue to distinguish them from upright turtles which are coloured black.



At each move a player **must** turn one upright (black) turtle upside down (blue) and then **may** also turn over any single turtle to the left of this turtle. This second turtle, unlike the first, may be turned either onto its back or onto its front (from black to blue or from blue to black). Players take turns to make moves.

The game follows normal play rules in which the player who turns the last turtle onto its back, leaving a row of upside down turtles (a row of blue turtles), wins (i.e. the player with no remaining moves loses).

(c) Consider the turning turtles game shown below involving ten turtles with turtles four and eight upside down.



Determine whether the game is winning or losing. If winning determine all possible winning moves from this position. (8 marks)

[No turtles were harmed in the making of this question]

(Total: 20 marks)

5. Arbitrated by the Mad Hatter over a decadent tea party, Alice and the Cheshire Cat play a game of **Hatter Poker**. This game uses a deck of **three** cards: a card of value 1, a card of value 2 and a card of value 3. Each of Alice and the Cheshire Cat is dealt a card at random with all possible deals being equally likely. A player knows the value of their own card, but does not know the value of their opponent's card.

After seeing her card, Alice has the option to gamble (G) or fold (F). When she folds, she has to give one of her biscuits to the Cheshire Cat. When she gambles, the Cheshire Cat has the option to see (S) or pass (P). When the Cheshire Cat chooses pass, he has to give one of his biscuits to Alice. When the Cheshire Cat chooses see, the players reveal their cards and the player with the lower value card must give two of their biscuits to the player who had the higher value card.

(a) Draw an extensive form (a game tree) representation of the game including any information sets and payoffs. (6 marks)

(b) Simplify this game by assuming that at an information set where a player's move is always at least as good as their other possible move, no matter what the other player's move or chance moves were or will be, then the player will always choose that move (i.e. simplify the game by eliminating any weakly/strictly dominated strategies). You can either reproduce a simplified game tree to illustrate this or explain which strategies you are removing. (5 marks)

(c) Produce the strategic form of the simplified game from part (b) and find an equilibrium of that game. (5 marks)

(d) (i) What are the behaviour properties of your equilibrium? (i.e. describe how the players should play the game according to your equilibrium strategies.) (2 marks)

(ii) Is your equilibrium subgame-perfect? Why/why not? (1 mark)

(iii) Should Alice agree to play this game? Why/why not? (1 mark)

(Total: 20 marks)

Parts

(a). In an N-player ($N \geq 2$) game we say that pure strategy $s_i \in S_i$ (pure strategy set of player i) is a <u>best response</u> to the strategies $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N$ for each of the other players if

$$g_i(s_1, s_2, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_N) \geq g_i(s_1, s_2, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_N),$$

for all $s_i' \in S_i$.

A <u>pure strategy equilibrium</u> in such a game is a profile of strategies such that each one is a <u>best response</u> to the others.

4 $\boxed{A}$ seen definition

(b). A strategy $s_i \in S_i$ for player i is <u>strictly dominated</u> by another strategy $s_i' \in S_i$ if

$$g_i(s_1, s_2, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_N) < g_i(s_1, s_2, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_N),$$

for all $s_j \in S_j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, N$.

2 $\boxed{A}$ seen definition

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 1

Marks & seen/unseen

Parts

(c). Suppose s' strictly dominates s for player i. This means that s is never part of an equilibrium since player i could always deviate to s' to do better.

Denoting the original game by G and the game with s removed from S_i by G', we can say that any equilibrium in G doesn't contain s, so therefore it is a possible set of strategies in G' where it is also an equilibrium. 2 A seen proof

On the other hand, any equilibrium in G' is a possible set of strategies in G where it is either an equilibrium, or, if not, then player i must be able to benefit by deviating to strategy s. But if this were the case then they could deviate to s' to do better which is a possible strategy in G', violating the fact we started with an equilibrium of G'. 3 B Seen proof

$\square$

Setter's initials: SJB

Checker's initials:

	EXAMINATION SOLUTIONS 2024-25	Course Intro to Game Theory
Question 1		Marks & seen/unseen
Parts		

(d). Let the pure strategies of the n students be:

s_i: take imperfect sweet type a

s_p: pursue perfect sweet

Let the payoffs for each sweet type be:

a : imperfect sweet type a

b : imperfect sweet type b

p : perfect sweet $\begin{pmatrix} a<p \\ b<p \end{pmatrix}$

We want to prove:

$\underbrace{(s_p, \cdots, s_p)}_{n}$ unique equilibrium $\Leftarrow\!\Rightarrow$ $a < b$

<u>($\Leftarrow$)</u>: First assume that $a < b < p$. Then pure strategy s_p <u>strictly dominates</u> s_i. This is because, if m or more students are currently choosing s_p, a payoff of p is obtained, but if less than m are choosing s_p, a payoff of b is obtained. Pure strategy s_i guarantees a payoff of a which is less than the worst case in playing s_p.

Thus, applying part (c), all students may delete strategy s_i from the game and we arrive at a dominance solvable game with a unique strategy profile (and hence unique equilibrium) of $(s_p, \cdots, s_p)$. (all n students playing s_p).

Seen similar game in class

C 1 define strats + payoffs

seen similar

2 B dominance

seen similar

1 C

seen similar

Setter's initials SJB	Checker's initials	Page number 3

($\Rightarrow$):
We need to show:

$$\underbrace{(s_p, \dots, s_p)}_{n} \text{ unique equilibrium} \Rightarrow a < b.$$

Take the contrapositive of this statement:

$$a \geq b \Rightarrow \underbrace{(s_p, \dots, s_p)}_{n} \text{ \underline{not} unique equilibrium}.$$

$\boxed{D}$ 1 unseen

We will prove this. First taking $a \geq b$ means that s_p does not strictly dominate s_i. Indeed, consider a profile of strategies given by:

$\boxed{C}$ 1 seen similar

$$\underbrace{(s_i, s_i, \dots, s_i)}_{n} \qquad (*)$$

We ask if $(*)$ is an equilibrium. All players currently recieve a payoff of a. If any single player were to switch to s_p, they would obtain a payoff of b (since at least m, which has minimum value $2\frac{3}{3}$, players are needed to play s_p to get payoff p). But $a \geq b$, so s_i is a best response for this player and hence also for all other players, so $(*)$ gives an equilibrium and hence we have shown $\underbrace{(s_p, \dots, s_p)}_{n}$ is not the unique equilibrium as we have found another. □

3 $\boxed{D}$ unseen

Q1: Total: 20

Marks & seen/unseen

Parts

(a).

(i). α^* is an <u>equaliser strategy</u> for A if and only if

$$g(\alpha^*, b) = \text{constant}, \ \forall b \in B_S.$$

A 2 seen definition

(ii). Assume α^*, β^* are ES for A and B respectively. Then:

$$\inf_\beta \{g(\alpha^*, \beta)\} = g(\alpha^*, \beta^*), \quad \because \alpha^* \text{ ES}$$
$$= g(\alpha, \beta^*), \quad \because \beta^* \text{ ES}$$
$$\geq \inf_\beta \{g(\alpha, \beta)\}, \ \forall \alpha \in A_S.$$

Hence:

$$\inf_\beta \{g(\alpha^*, \beta)\} \geq \inf_\beta \{g(\alpha, \beta)\}, \ \forall \alpha,$$

so α^* is <u>max-min</u>.

Similarly β^* is min-max.

B 3 seen proof

Further, let $g(\alpha^*, b) = c, \ \forall b \in B_S$.

Then $g(\alpha^*, \beta^*) = c$ since β^* is just a weighted mixture of the b's. This means that (α^*, β^*) are in equilibrium since B cannot change to any other mixture β of b's to do better. Similarly for A. □

A 2 seen proof

Setter's initials: SJB

Checker's initials:

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question: 2

Marks & seen/unseen

Parts: (b).

(unseen game)

(i). Let battlefield Y be worth 1, so X is worth k. We draw a normal form of the game:

payoffs

Payoffs for army A:

A \ B	b_1: XXX	b_2: XXY	b_3: XYY	b_4: YYY
a_1: XX	$-k$	-1	$k-1$	$k-1$
a_2: XY	$1-k$	$-k$	-1	$k-1$
a_3: YY	$1-k$	$1-k$	$-k$	-1

$\boxed{A}$ 2 normal form seen similar

To make the algebra easier, add k to all payoffs:

A \ B	b_1: XXX	b_2: XXY	b_3: XYY	~~b_4: YYY~~
a_1: XX	0	$k-1$	$2k-1$	$\cancel{2k-1}$
a_2: XY	1	0	$k-1$	$\cancel{2k-1}$
a_3: YY	1	1	0	$\cancel{k-1}$

weak dominance

Since $k>1$, we see that b_4: YYY is <u>weakly dominated by b_3: XYY</u> for B, so we can safely delete b_4 from the game in attempting to seek a solution.

$\boxed{B}$ 2 seen similar

Setter's initials: SJB

Checker's initials:

Parts

(b).

(ii). For the subgame with b_4 removed we check that

$$\alpha^* = \left(\frac{1}{k^2+1}, \frac{k-1}{k^2+1}, \frac{k^2-k+1}{k^2+1}\right)$$ gives an ES for player A:

$$g(\alpha^*, b_1) = \frac{1}{k^2+1}(k-1+k^2-k+1) = \frac{k^2}{k^2+1}$$

$$g(\alpha^*, b_2) = \frac{1}{k^2+1}(k-1+k^2-k+1) = \frac{k^2}{k^2+1}$$

$$g(\alpha^*, b_3) = \frac{1}{k^2+1}(2k-1+(k-1)^2) = \frac{k^2}{k^2+1},$$

So indeed α^* is an ES for A.

$\boxed{A}$ 2 seen similar

understand conclusion $\boxed{C}$ 1 seen similar

(c). We seek an ES for player B in this subgame. Let $\beta^* = (p, q, 1-p-q)$. Then:

$$g(a_1, \beta^*) = (k-1)q + (2k-1)(1-p-q)$$
$$= (1-2k)p - kq + (2k-1) \quad ①$$

$$g(a_2, \beta^*) = p + (k-1)(1-p-q)$$
$$= (2-k)p + (1-k)q + (k-1) \quad ②$$

$$g(a_3, \beta^*) = p + q \quad ③$$

$$① = ③: \quad 2kp + (k-1)q = (2k-1) \quad ④$$

$$② = ③: \quad (k-1)p + kq = (k-1) \quad ⑤$$

seek ES for B $\boxed{C}$ 2 seen similar

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 2

Marks & seen/unseen

Parts (c). (continued).

Solving (4), (5) simultaneously for p, q gives:

$$p = \frac{k^2 - k + 1}{k^2+1}, \quad q = \frac{k-1}{k^2+1}.$$

Hence we find:

$$\beta^* = \left(\frac{k^2-k+1}{k^2+1}, \frac{k-1}{k^2+1}, \frac{1}{k^2+1}\right)$$ is an ES for B

This is a valid randomised strategy.

D 2 seen similar

By part (a)(ii), (α^*, β^*) is an equilibrium of the subgame, so extending this to the full game gives α^* and β' where

$$\beta' = \left(\frac{k^2-k+1}{k^2+1}, \frac{k-1}{k^2+1}, \frac{1}{k^2+1}, 0\right),$$ as an equilibrium of the game.

D 1 seen similar

The value of the game is $g(\alpha^*, \beta') - k$ (remembering to $-k$ since we added k to all payoffs)

~~$g(\alpha^*, \beta')$ ...~~

$$v = -k + \frac{k^2}{k^2+1} = \frac{k(k-k^2-1)}{k^2+1}.$$

D 1 seen similar

Q2: Total: 20

Setter's initials: SJB

Checker's initials

	EXAMINATION SOLUTIONS 2024-25	Course Intro to Game Theory
Question 3		Marks & seen/unseen

Parts

(a).

(i). A's payoffs are:

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & \boxed{0} & \lambda \\ a_2 & 1 & \boxed{0} \end{array}$$

In this game we find A's max-min strategy to be $\hat{\alpha} = \left(\frac{1}{\lambda+1}, \frac{\lambda}{\lambda+1}\right)$ giving A's threat level as $\underline{\underline{t_A = \frac{\lambda}{\lambda+1}}}$.

$\boxed{A}$ 2 seen similar

B's payoffs are:

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & \boxed{0} & \boxed{\mu} \\ a_2 & \boxed{0} & 1 \end{array}$$

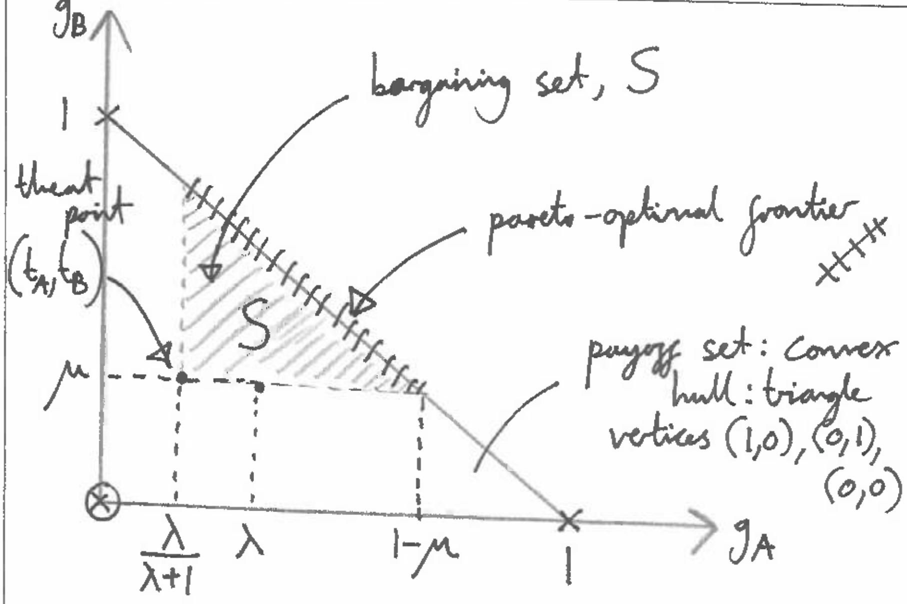
With B as maximiser this game has a pure strategy equilibrium at (a_1, b_2) giving B's threat level as $\underline{\underline{t_B = \mu}}$.

$\boxed{B}$ 2 seen similar

Setter's initials	Checker's initials	Page number
SJB		9

	EXAMINATION SOLUTIONS 2024-25	Course Intro to Game Theory
Question 3		Marks & seen/unseen

Parts (a). (ii).



seen similar [A] 3 (1: payoff set 1: pareto-frontier 1: bargaining set)

[B] 1 Threat point correct place

(iii). We maximise the Nash product $(x-t_A)(y-t_B)$ over the pareto-optimal frontier, which has equation $x+y=1$:

$$\left(x-\frac{\lambda}{\lambda+1}\right)(y-\mu)=\left(x-\frac{\lambda}{\lambda+1}\right)(-x+1-\mu)$$

$$=-x^2+\left(1-\mu+\frac{\lambda}{\lambda+1}\right)x-(1-\mu)\frac{\lambda}{\lambda+1},$$

which is maximised when:

$$x^*=\frac{1}{2}\left(1-\mu+\frac{\lambda}{\lambda+1}\right)$$

$$y^*=\frac{1}{2}\left(1+\mu-\frac{\lambda}{\lambda+1}\right).$$

[A] 3 seen similar

Setter's initials SJB | Checker's initials | Page number 10

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 3

Marks & seen/unseen

Parts (a). (iii). (continued).

We need to check if $(x^*, y^*) \in S$. Well it must be the case that $1-\mu > \frac{\lambda}{\lambda+1}$, because otherwise we would have $\mu\lambda + \mu \geq 1$, which contradicts the fact that $\lambda + \mu < 1$. This means that

$$\frac{\lambda}{\lambda+1} < x^* < 1-\mu,$$

So indeed x^* is within the limits of the bargaining set S and since (x^*, y^*) lies on the line $x+y=1$ it is the Nash bargaining solution of the game.

Seen concept, unseen problem

D

3 unseen

One way to implement this solution is for player A to play a_2 and for player B to play

$$\beta = \left(\frac{1}{2}\left(1-\mu+\frac{\lambda}{\lambda+1}\right), \frac{1}{2}\left(1+\mu-\frac{\lambda}{\lambda+1}\right)\right).$$

B

1 seen similar

Setter's initials: SJB

Checker's initials

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 3

Marks & seen/unseen

Parts (b).

		b_1	b_2
A	a_1	$0, \boxed{0}$	$\lambda, \boxed{0}$
	a_2	$1, 0$	$0, \boxed{1}$

(Column player: B)

seen similar

The pure strategy a_1 for A has two pure best responses (b_1 and b_2) for B. Hence the game is degenerate owing to B's weakly dominated strategy. $\boxed{D}$ 1 spot degeneracy

(a_1, b_2) forms a pure strategy equilibrium of the game, however because B is indifferent between $\boxed{B}$ 1 pure equilibrium

b_1 and b_2 when A chooses a_1, B can mix. This defines an equilibrium as long as a_1 remains a best response for A to B's mixing. This happens if and only if

seen similar

$$g_A(a_1, \beta) \geq g_A(a_2, \beta),$$

for player B's mixed strategy $\beta = (q, 1-q)$. This gives:

$\boxed{C}$ 3

$$\lambda(1-q) \geq q \iff \underline{\underline{q \leq \frac{\lambda}{\lambda+1}}}.$$

seen similar

Therefore: $(a_1, (q, 1-q))$, where $\underline{\underline{0 \leq q \leq \frac{\lambda}{\lambda+1}}}$

form an infinite set of all equilibria of the game.

Q3: Total: 20

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 4

Marks & seen/unseen

Parts

(a).

(i). G is <u>winning</u> if and only if at least one of its options is <u>losing</u>. G is <u>losing</u> if and only if all its options are <u>winning</u>. These definitions are recursive not cyclic due to the ending condition.

Note: Allow a definition for losing without mention to winning.

A

2 seen definition

(ii). Let the options of G be $G_1, \cdots, G_n$ and the options of H be $H_1, \cdots, H_m$. The <u>game sum</u> of G and H, $G+H$, is the game with options

$$G_1+H,\ G_2+H,\ \cdots,\ G_n+H$$
$$G+H_1,\ G+H_2,\ \cdots,\ G+H_m.$$

A

2 seen definition

(iii). G is <u>equivalent</u> to H, $G \equiv H$ if and only if for any other game J, $G+J$ is losing if and only if $H+J$ is losing.

A

2 seen definition

(b). We want to prove that:

$$G \text{ losing} \iff G \equiv 0.$$

<u>($\Leftarrow$):</u>

First assume $G \equiv 0$. Then use $J=0$ in our definition of equivalence in part (a)(iii) giving $G+0$ losing $\iff$ $0+0$ losing, i.e. G losing $\iff$ 0 losing, but this is the definition of a losing game so indeed G is losing.

A

2 seen proof

Setter's initials: SJB

Checker's initials

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 4

Marks & seen/unseen

Parts (b). (continued).

($\Rightarrow$): We need to show $G \equiv 0$, i.e. for any other game J, $G+J$ losing $\Leftrightarrow$ $0+J=J$ losing. i.e we need to prove: ① G, J losing $\Rightarrow$ $G+J$ losing. ② $G+J$ losing $\Rightarrow$ J losing.

①: G, J losing. We need to show every option of $G+J$ is winning. These options are either of form $G'+J$ where G' is an option of G, but G' is winning since G is losing, so there is an option of G', say G'' that is losing. This is simpler than G, so we appeal to top-down induction assuming that G'', J losing $\Rightarrow$ $G''+J$ losing by inductive hypothesis. So $G'+J$ winning since it has a losing option, $G''+J$. Or, options of $G+J$ could be of form $G+J'$, but these are also all winning by a similar argument to the above. So indeed all options of $G+J$ are winning so $G+J$ is losing. B 2 seen proof

②: Assume for a contradiction that $G+J$ is losing but J is winning. So there is a move to an option J' of J which is losing. Now using ①, since G, J' are losing, $G+J'$ is losing, but $G+J'$ is an option of $G+J$, which would mean $G+J$ is winning. ※ B 1 seen proof

Thus $G+J$ is losing. C 1 seen proof

$\square$

Setter's initials: SJB

Checker's initials

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EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 4

Marks & seen/unseen

Parts (c). We seek an equivalence between turning turtles and Nim to enable us to use the theory of Nim values and game sums. (unseen game)

Indeed, consider a single upright turtle of value n. On a move this turtle must be turned upside down (reducing n) and ~~can~~ we can either end our turn (corresponding to removing a Nim pile of size n) or can turn upright a turtle of some value $m < n$ (corresponding to reducing a Nim pile of size n to one of size $m < n$). B 2 seen similar

Consider now two upright turtles of sizes n, m where $m < n$. If we turn over the turtle of size n, then as before we can reduce this to any value of $k < n$, except for $k = m$, since this already has an upright turtle. However, by turning over this turtle, i.e. turning over both the turtle of size n and size m, we are not changing the Nim value of the game from what it would be if we could leave two upright turtles of size m, since $*m + *m \equiv 0$, i.e. we have an equivalent game. D 2 unseen

So we can analyse ~~turning~~ turtles as a sum of Nim games, with the caveat that when reducing a Nim pile down to the size of some other pile in

Setter's initials: SJB

Checker's initials:

Parts (c). (continued).

the game, we essentially also remove those two piles of equal size (since they together are equivalent to a lost game, this doesn't change the winning/losing analysis using Nim values).

Thus, the Nim value of the turning turtles game shown is:

$$1 \oplus 2 \oplus 3 \oplus 5 \oplus 6 \oplus 7 \oplus 9 \oplus 10$$

$$\begin{aligned} = {} & 1 \oplus 2 \\ & \oplus 1 \oplus 2 \\ & \oplus 1 \qquad \oplus 4 \\ & \qquad \oplus 2 \oplus 4 \\ & \oplus 1 \oplus 2 \oplus 4 \\ & \oplus 1 \qquad\qquad \oplus 8 \\ & \qquad \oplus 2 \qquad \oplus 8 \quad = \underline{\underline{7}}, \end{aligned}$$

[C] 2 seen similar

So the game is <u>winning</u>. A winning move reduces the Nim value to 0. We can do this by:

- Turning over the turtle of value 7,
- Turning over the turtles of values 6 and 1,
- Turning over the turtles of values 5 and 2.

[D] 2 unseen

(If only the 7 found, award 0 marks)

Q4: Total: 20

EXAMINATION SOLUTIONS 2024-25

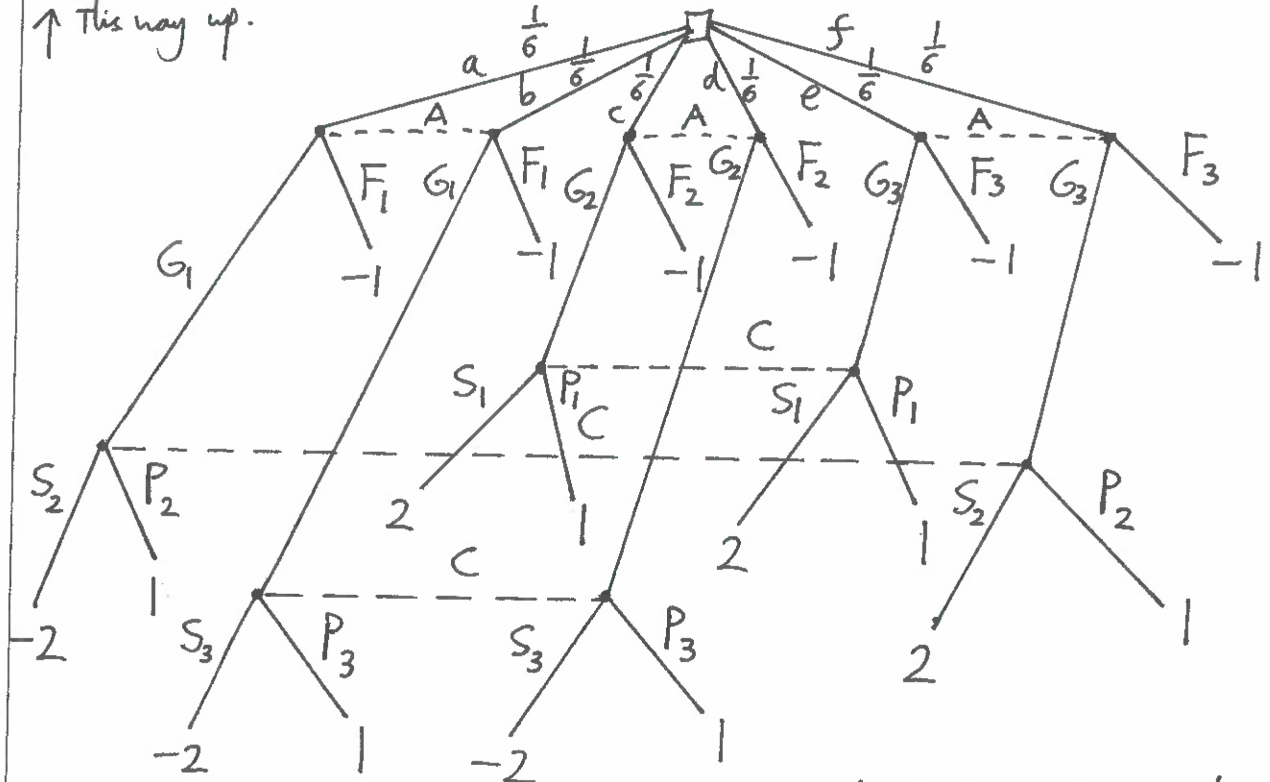
Course: Intro to Game Theory

Question 5

Marks & seen/unseen

Parts (a).

↑ This way up.



Let G_i = Alice gambles when she has card i.
F_i = Alice folds when she has card i.
S_i = Cat sees when he has card i.
P_i = Cat passes when he has card i.
For $i = 1,2,3$.

6 possible deals: a: 1,2, b: 1,3, c: 2,1, d: 2,3, e: 3,1 and f: 3,2, where x,y means Alice gets x, Cat gets y. Zero-sum game; just payoffs to Alice shown.

[unseen game] } 6

1: moves/strategies

1: 6 deals

2: info. sets of Cat

1: info. sets of Alice

1: correct tree

[NB: trees may start with chance node for A only $\frac{1}{3}$ each, then after her singleton info sets introduce chance nodes for Cat $\frac{1}{2}$ each.]

Seen similar trees

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Checker's initials

	EXAMINATION SOLUTIONS 2024-25	Course Intro to Game Theory
Question 5		Marks & seen/unseen
Parts (b).	Each player has 3 information sets with two moves each, so a strategy for Alice is e.g. $G_1F_2G_3$ (meaning gamble with 1, fold with 2, gamble with 3), a strategy for Cat is e.g. $P_1P_2S_3$ (pass with 1, pass with 2, see with 3).	1 discussion on strategies
	By looking at the game tree, we see that, for the Cheshire Cat, P_1 is always better than S_1, so we can delete strategies S_1** for the Cat. Similarly, S_3 is always better than P_3, so we can delete strategies $**P_3$ for the Cat. (Intuitively, the Cat should never see when he has 1, and never pass when he has 3). This leaves the Cat with <u>2 strategies</u>: <u>$P_1P_2S_3$</u> and <u>$P_1S_2S_3$</u>.	Seen similar 2 remove Cat's strategies
	Alice should never fold with the 3, so we delete strategies $**F_3$ for Alice. Now when Alice has the 2, we can see that if she gambles the Cat will <u>always</u> make a certain play: Cat sees with 3 and passes with 1. These each happen with conditional probability $\frac{1}{2}$, so we can do an expected value calculation: By playing G_2 Alice will get $\frac{1}{2}\times 1 + \frac{1}{2}\times(-2) = -\frac{1}{2}$.	Seen similar
	Setter's initials SJB	Checker's initials	Page number 18

	EXAMINATION SOLUTIONS 2024-25	Course Intro to Game Theory
Question 5		Marks & seen/unseen

Parts

(b). (continued). But $-\frac{1}{2} > -1$, which Alice gets by playing F_2, hence Alice should never play F_2 and we can delete strategies $*F_2*$. This leaves Alice with 2 strategies: $G_1G_2G_3$ and $F_1G_2G_3$.

remove Alice strategies. 2

Note: if some are missed in (b), but retrospectively found + removed in (c), award the missing marks from (b).

(c).

		C	
		$P_1S_2S_3$	$P_1P_2S_3$
A	$G_1G_2G_3$	$\boxed{-\frac{1}{3}}$	0
	$F_1G_2G_3$	0	$\boxed{-\frac{1}{6}}$

3 normal form

No pure strategy equilibria. We seek a mixed strategy equilibrium. This occurs when Alice plays $(G_1G_2G_3, F_1G_2G_3) = \left(\frac{1}{3}, \frac{2}{3}\right)$ and the Cat plays $(P_1S_2S_3, P_1P_2S_3) = \left(\frac{1}{3}, \frac{2}{3}\right)$.

seen similar

We could extend this solution to this subgame back and it is a solution of the whole game as we have only removed dominated/weak dominated strategies.

2 equilibrium

seen similar

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTIONS 2024-25

Course: Intro to Game Theory

Question 5

Marks & seen/unseen

Parts

(d).

(i). Alice should always gamble when she has the 2 or the 3, but should 'bluff' and gamble with the 1 card $\frac{1}{3}$ of the time when she has it, folding it the other $\frac{2}{3}$ of the time.

The Cheshire Cat should always pass his 1 and always see when he has the 3, but when he has the 2 should see just $\frac{1}{3}$ of the time, passing the other $\frac{2}{3}$ of the time.

1 seen similar

1

(ii). Yes, the equilibrium found is subgame-perfect as there are no subgames, other than the entire game, in this game, so the equilibrium is trivially an equilibrium in all subgames.

1 seen similar

(iii). The value of the game is $g(\alpha,\beta)$, where α and β are the player's equilibrium strategies. This is $V = -\frac{1}{9}$, and so the game yields a slight advantage to the Cheshire Cat.

1 unseen

It would be wise for Alice to resist the Cat's cheeky grin and not play the Mad Hatter's game if she values her biscuits.

QS: Total: 20

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MATH70141 Introduction to Game Theory Markers Comments

Question 1	(a) This is the most important concept in the course, but several students always manage to find ways to drop points on this definition! In the question the game was described to have N players where N could be greater than 2, yet a significant proportion of students gave a definition that spoke about two players A and B. This cost students a point as it conveyed a lack of awareness of how this definition extends for more than 2 players. Other common things to miss were the fact that a best response holds for all the own players strategies against a fixed set of strategies for the other players. (b) This was done very well, almost all students scoring full points. (c) Some students failed to show this held in both directions. To show G and G' have the same equilibria we can't just check any equilibrium in G' is an equilibrium in G, because what if there are some extra ones in G? So, assuming this and showing these also had to be in G' was important. (d) This is a similar problem to those seen in class and at the end of problem set 2, and, for the most part, students did well on this problem, correctly aiming to prove both directions and using good proof techniques – exhaustion, contradiction and direct proof were all used well. Some student's arguments became loose – arguing that certain things would happen based on iterated play of the game is a little airy and stronger arguments used strict domination to eliminate strategies rather than argue they wouldn't be played after many iterations of the game. Students that took a pause before diving in, taking the time to define strategies and payoffs for the players always seemed to perform well and this is a very good way to tackle an unseen game theoretic situation – first set up the game explicitly and clearly, then jump in, you'll find you and we readers get confused far less often!
Question 2	(a)(i) This was done very well with (according to my memory) every student defining this correctly, however there were some differing levels of quality in the definition – firstly, the game is a zero-sum two player game, so asked to define this in this context we would drop the subscript notation on the payoff function, which was a very rare thing for students to do. Of course, it is not incorrect to use a payoff function for one of the players here, just less tidy and organic. Secondly although it is also correct that an equaliser strategy equalises payoff vs any of the opponent's mixed strategies, this is not usually how we define them. We usually define it to equalise payoffs vs the opponent's pure strategies. (ii) This part was usually either not attempted by students, or completed to 3, 4 or full marks. This is an important result from the lecture notes so it is a shame to see that students don't put together

the definitions of equilibria and equaliser strategies to deduce equilibrium must hold for 3 easy marks. The final points were for showing the strategies were then max-min and min-max in the context of a zero-sum game which follows from their definitions so students who knew those usually scored these extra points.

(b) (i) This part also contained 2 points for modelling the game using a normal form which was pleasing to see every (as far as I can remember) student try to do with most correctly formulating the payoffs. The remaining two marks were slightly less well done, some students correctly using weak dominance by one of the other pure strategies or strict dominance by a suitable mixed strategy, but many students argued the pure strategy with all regiments at X strictly dominated all regiments at Y arguing 1-k -1 which is not necessarily true as this leads to k2 where all we know is k1.

(ii) This part was done in two different ways – either students set alpha = (p,q,1-p-q) as a mixed strategy for A and then set g(alpha,b_1) = g(alpha,b_2) = g(alpha_b_3) and solved the system for p and q, or the students simply set alpha = (1/(k^2+1), ...etc) as given and calculated the three quantities above showing that they were all equal. This second method saved valuable time and showed nice understanding into what an equaliser strategy meant once it had been given, but both worked! Of course, given that the ES was given in the question, the system found using the first method needed to be solved for full points or checked that the given ES satisfied it.

(c) This part was usually attempted but often had mistakes made in the algebra. Students that set about seeking an ES for B scored a handful of the points, now this took considerable algebra but could be found, however it was often found incorrectly. A handful of students used the examiners intended technique (well done!) to spot that the reduced game was symmetric up to a relabelling of the strategies and so the same ES (with the strategies sometimes needing to be reordered) worked for B! Students that spotted this earned themselves 4 points in a line of work! To complete the problem, we should extend our ES for B into the full game with their weak dominated strategy included, several students missed including this zero probability into their solution to the game and dropped a point. The value of the game was calculated correctly by many students.

Question 3 The first comment on this problem is a naughty examiner comment – I realised all too late in the middle of the exam that the statement "0lambda, mu 1 and ..." is extremely poorly written. I think originally, I had "here 0 lambda, mu 1." Which albeit still poorly written, is harder to read as "0 lambda and mu 1 and ..." as the full stop directly after means the English wouldn't be read naturally in the second way. Even so this was a error on my part and even after looking over several times seemed to escape all of us who looked at it! As such, if students read this statement as 0 mu 1 and 0 lambda 1 (as intended) or as 0 lambda and mu 1 (so a harder, more flexible case of the intended), full marks could still be awarded – i.e. I allowed the correct solutions to either of these cases to gain the marks. As it turned out the second unintended case did add in a little more work, but this was only minor as the case where mu 0 causes the payoff set to change, but not really the pareto-frontier which still lies along x + y = 1, so part (iii) does not grow unintentionally long.

(a) (i) This was generally done well, t_B = mu was intended but for the case when mu 0 then t_B = 0 was introduced. Students were awarded full marks for either t_B = 0 or t_B = mu (and well done if they discussed both) due to the unintended flexibility in the question.

(ii) Here comes the largest mistake by the students on the entire problem which caused so many of them so much heartache! The condition lambda + mu 1 means that the point (lambda, mu) must lie to the left (as we look at the line with the y axis pointing up) side of the line x+y = 1. So the payoff set was a triangle with the pareto-frontier along part of the line x + y = 1. Almost all students either placed (lambda, mu) somewhere to the right of the line, giving them a pareto-frontier that was two line segments rather than one (and these had messy equations leading to horrendous algebra in part (iii)), or, even when they placed (lambda, mu) correctly, proceeded to join up the four points of payoffs and called that the payoff set rather than the whole of the convex hull of these points, which, as we are examining a cooperative game, since (0,1) and (1,0) are two points, the whole of the line joining them is in the convex hull and so this forms the pareto-frontier! Again this second possible error made part (iii) far more complicated! Only a very small number of students produced completely correct diagrams here and these generally set them up well for the rest of the question.

(iii) Students generally performed the basic part of this problem well – maximising their Nash product using their threat points across their pareto-frontiers (as mentioned few of these ended up being correct, but full marks were still awarded here when student performed the correct calculation using their determined threat points and their pareto-frontiers). However 3 points out of the 7 here were for checking that the obtained solution indeed lay within the bargaining set determined in part (ii). Due to the parameters lambda and mu varying slightly this is non trivial (in most cases) and very few

students checked this – in fact few students even made the comment that their found point was in their set (likely meaning they did not think to check this). This is important as in many cases students found points that lay outside their bargaining sets. This part of the question was almost universally omitted and so although there were many scores of 15/16/17 very few students broke into the 18/19/20 range on this question.
(b) This part was done very well. Most students realised the degeneracy and correctly identified all the equilibria. Occasionally a mistake was made in allowing A to mix rather than B, but these were few. A small number of students failed to notice the degeneracy and usually ended up with just 1 point for finding the pure equilibrium (and sometimes the mixed equilibrium at the other end of the range of the infinite family).

Question 4

a) Was in general well succeeded, with some occasional omission of the winning definition for (i). Some mistakes were due to using Nim sum in (ii, not including another game J for (iii) or not mentioning that G+J and H+J must have same losing condition.
(b) The side = was in general well done, but the other side was problematic for a lot of submissions with partial proofs based on induction.
(c) There was some partial attempts at defining an equivalence to 3 Nim piles, hence getting a wrong Nim sum of 2 and wrong winning states. This was well done only by a handful of submissions who got the correct Nim value and equivalence but no-one found all the winning positions.

Question 5

(a) 3 marks on this tree were done well – students generally put together close to correct tree diagrams with appropriate payoffs and probabilities on cards, however the information sets for Alice and for the Cheshire Cat as well as the possible card of the Cheshire Cat were often missed out/incomplete/incorrect. Recall that there are in essence 6 possible deals here, each of 1/2/3 for Alice with each time one of the other two options for the Cheshire Cat, so good attempts at drawing a tree usually had 6 edges leaving one chance node at the start representing these 6 possibilities! Of course it is also correct to have 3 edges at the start representing Alice's card, followed by chance nodes later if Alice gambles representing the possibilities for the Cat's card. However, many students omitted these second chance nodes for the Cat, this will leave your model of the game without the information of the Cat's card being taken into account anywhere and as such incomplete. Another common mistake was to join up all of the Cat's decision nodes, or occasionally all of Alice's decision nodes to form a single information set. An information set should take into account the knowledge of that player at that time – so Alice's nodes should not all be joined as each node (or pair of nodes if you did the 6 edges at the start) represents her knowledge that she has a 1/2/3 – joining them all would be akin to her having imperfect recall – her looking at her card, then completely forgetting what card she had when she gambles/passes (we'll assume she wants to play optimally and uses the information of her card to make decisions!). Similarly for the Cheshire Cat who also has 3 information sets – the one where he has a 1/2/3 and each time Alice has gambled (these were hard to draw nicely, but good candidates found creative ways to make sure it was clear these cases lay in separate information sets).

(b) Several candidates noticed that there were some clear strictly/weakly dominated strategies – strategies involving the Cat passing with a 3, or seeing with a 1, or Alice folding with the 3. However many students incorrectly assumed Alice gambling with the 1 as dominated too – this is Hatter Poker, the Cat may have the 2 and as such Alice may be able to bluff the Cat into thinking she has the 3, so this is not always a strict dominated strategy as it turns out. It is also at this point were very good candidates first introduced the strategies and reduced strategies of the game! As mentioned in question 1, it is nearly always very good practice to define your strategies and payoffs at the start of modelling a game. Good candidates had very clear representations for their strategies – calling things a_1 or a_2 here doesn't help, but a nice clear pure strategy like G_1G_2P_3 representing that Alice will gamble if she gets the 1, gamble with the 2 and pass with the 3 etc helped keep things clear and made it easier to spot the dominated strategies! Saying things like "Alice passing with the 3" is a strictly dominated strategy was condoned, but take care because this isn't a strategy in itself. We can say any strategy where Alice passes when she has the 3 is weakly dominated by one when she gambles with the 3, because her strategy must also include information about what she does with the 1 or the 2.

(c) The strategic game form reduces to a 2x2 game as there are some more dominated strategies here. Generally many candidates arrived at this point with the wrong games due to an error in one of the earlier parts – as such as many marks as possible were still awarded here for correctly solving the games that students had formulated.

(d) Part (i) was almost 0 marks to everyone as nobody had sufficiently been able to describe what their equilibrium strategies meant in terms of playing the game. It is incorrect to say something like "Alice should gamble with the 3 and the Cat passes with the 2"…what if Alice gets the 1 then? Your strategy must describe every action a player needs to take in every possibility that can occur. Few students were able to do this. Many students correctly identified that their game tree had no other subgames in it, and as such, their equilibria would be subgame perfect. Finally many students correctly found the value of their game and argued that if positive Alice should play and if negative she shouldn't. If the student's game ended up with a 0 value – I accepted either yes or no so long as the student said the value was 0 – it's the value of the game that informs this decision. These marks in (d) were awarded for correct answers based on the game the student had formulated in (c) so these were still available if mistakes were made in the modelling parts of (a) and (b).