Imperial College
London

MATH97022 MATH97100

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Special Relativity and Electromagnetism

Date: 28th May 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. The following questions are concerned with events in different frames of reference. The spatial origin of each frame is chosen so that $(0,0,0,0)$ refers to the same event in every frame. All axes are parallel at all time.

The origin of frame K' moves with velocity v relative to frame K along its x-axis.

At time t_0 measured in frame K, a flash of light is emitted from the origin of K. The flash is received at the origin of K' at its proper time t'_1.

(a) State the four-vector that describes in K the event of the emission of the flash. (2 marks)

(b) Transform the four-vector found in (a) to the four-vector that describes in K' the event of the emission of the flash. (3 marks)

(c) State the four-vector that describes in K' the event of the arrival of the flash. (2 marks)

(d) Transform the four-vector found in (c) to the four-vector that describes in K the event of the arrival of the flash. (3 marks)

(e) Determine the distances ℓ and ℓ' from emission to arrival that the flash has covered in K and K' respectively. (2 marks)

(f) Determine the times Δt and $\Delta t'$ that pass from emission to arrival of the flash in K and K' respectively. (2 marks)

(g) Show that $\ell/\Delta t = c$ results in $\ell'/\Delta t' = c$. (6 marks)

(Total: 20 marks)

2. A particle with proper mass M moves with velocity v along the x-axis of frame K. The particle decays spontaneously into two identical particles each with mass $m = (3/10)M$ that carry on moving along the x-axis of frame K.

(a) The frame of reference of the original particle of mass M is K'. Its axes are parallel with those in K and the two origins coincide at $t = 0 = t'$. Determine the two velocities v_1' and v_2' in K' of the particles that result from the spontaneous decay.
Hint: Use energy and momentum conservation. (5 marks)

The velocities v_1 and v_2 of the two particles in frame K are given by

$$\frac{v_i}{c} = \tanh(\psi_i) \text{ with } \psi_i = \phi + \psi_i'$$

where, correspondingly, $v/c = \tanh(\phi)$ and $v_i'/c = \tanh(\psi_i')$, so that, for example,

$$\frac{v/c}{\sqrt{1 - v^2/c^2}} = \sinh(\phi) \quad \text{and} \quad \frac{1}{\sqrt{1 - v^2/c^2}} = \cosh(\phi)$$

Hints: $\sinh(\phi + \psi) = \sinh(\phi)\cosh(\psi) + \cosh(\phi)\sinh(\psi)$ *and* $\cosh(\phi + \psi) = \cosh(\phi)\cosh(\psi) + \sinh(\phi)\sinh(\psi)$.

(b) Show that $M/m = \cosh(\psi_1') + \cosh(\psi_2')$. (5 marks)

(c) Show that momentum is conserved in K. (5 marks)

(d) Show that energy is conserved in K. (5 marks)

(Total: 20 marks)

3. The following questions are concerned with the four-potential $A^i = (\phi, \mathbf{A})$.

(a) Frame K' moves with velocity v relative to frame K along their common x-axis. The two origins coincide at time $t = 0 = t'$. All axes remain parallel throughout. Express the scalar potential ϕ' in frame K' in terms of ϕ and $\mathbf{A}$ in K. (3 marks)

The electric and magnetic fields are related to the potentials via

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad \text{and} \quad \mathbf{H} = \nabla \times \mathbf{A} \ .$$

In the following we will determine the y-component of the electric field E'_y in frame K' on the basis of the transformed potentials.

(b) Express the derivative with respect to (proper) time in K', $\partial/\partial t'$, in terms of derivatives with respect to coordinates in K. (3 marks)

(c) Using $A'_y = A_y$ and $\partial/\partial y' = \partial/\partial y$ as well as the results in (a) and (b), express E'_y in terms of components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in K. (8 marks)

The four-potential is determined by physical observables up to a gauge. One such gauge, known as the Lorenz-gauge, implements

$$\frac{1}{c}\partial_t \phi = \nabla \cdot \mathbf{A} \ .$$

(d) Show that the property $\frac{1}{c}\partial_t\phi = \nabla \cdot \mathbf{A}$ is Lorenz-invariant, $i.e.$ if it holds in one frame it holds in all frames. (3 marks)

(e) Show that $\partial_i \partial^i A^k = \frac{4\pi}{c} j^k$ with the Lorenz-gauge, where j^k is the four-current. *Hint:* $F^{ik} = \partial^i A^k - \partial^k A^i$ *and* $-\frac{4\pi}{c} j^k = \partial_k F^{ik}$. (3 marks)

(Total: 20 marks)

4. In the following we consider a particle with mass m moving in frame K parallel to its x-axis with speed v. In this frame, a magnetic field $\mathbf{H}$ constant in time and uniform in space and no electric field act on the particle.

(a) Determine explicitly all components of the Lorentz force

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H}$$

acting on the particle. (6 marks)

(b) Show that the magnitude of the momentum of the particle remains constant. (6 marks)

(c) An arbitrary contravariant vector A^i transforms according to

$$A'^i = \tilde{\mathcal{L}}^i{}_k A^k$$

with

$$\tilde{\mathcal{L}}^i{}_k = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\beta = v/c$. The electromagnetic field tensor is given by

$$F^{ij} = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} .$$

State the electric field in the rest-frame K' of the particle. (6 marks)

(d) Determine the Lorentz-force acting on the particle on the basis of the electric field in its rest-frame. (2 marks)

(Total: 20 marks)

5. (a) Derive the Poisson equation that relates the scalar potential ϕ to the charge density ρ from a Maxwell equation assuming that the vector potential is constant in time. (5 marks)

(b) Express the energy of an electric field

$$U = \frac{1}{8\pi} \int \mathrm{d}^3 r |\mathbf{E}|^2$$

in terms of the charge density and the scalar potential. (5 marks)

(c) Calculate the magnitude of the electric field as a function of the radial distance from the centre of a homogeneous, spherical charge density of radius R

$$\rho(\mathbf{r}) = \begin{cases} \frac{3Q}{4\pi R^3} & \text{for } |\mathbf{r}| < R \\ 0 & \text{otherwise} \end{cases}$$

using Gauss' law in integral form,

$$\int_{\partial V} \mathrm{d}\mathbf{n} \cdot \mathbf{E} = 4\pi \int_V \mathrm{d}^3 r \rho \ .$$

(5 marks)

(d) Use the result in (c) to determine the self-energy of a homogeneously charged sphere using the integral given in (b). (5 marks)

(Total: 20 marks)

Solutions of M345A6 2019/2020

1.a *(2 marks, very easy, A)*
This is essentially given, $\boxed{(ct_0, 0, 0, 0)_K}$.

1.b *(3 marks, easy, A)*
Using standard transform, $\boxed{(ct_0\gamma, -vt_0\gamma, 0, 0)_{K'}}$.

1.c *(2 marks, very easy, A)*
This is essentially given, $\boxed{(ct'_1, 0, 0, 0)_{K'}}$.

1.d *(3 marks, easy, A)*
Using standard transform, $\boxed{(ct'_1\gamma, vt'_1\gamma, 0, 0)_K}$.

1.e *(2 marks, easy, A)*
On the basis of (a) and (d), $\boxed{\ell = vt'_1\gamma}$ and on the basis of (b) and (c), $\boxed{\ell' = vt_0\gamma}$.

1.f *(2 marks, easy, A)*
On the basis of (a) and (d), $\boxed{\Delta t = t'_1\gamma - t_0}$ and on the basis of (b) and (c), $\boxed{\Delta t' = t'_1 - t_0\gamma}$.

1.g *(6 marks, medium, but seen, B)*
From $\ell/\Delta t = vt'_1/(t'_1 - t_0\sqrt{1 - v^2/c^2}) = c$ follows $t'_1 = t_0\sqrt{1 - v^2/c^2}/(1 - v/c)$. Using this in $\ell'/\Delta t' = vt_0\gamma/(t'_1 - t_0\gamma)$ produces

$$\boxed{\frac{\ell'}{\Delta t'} = \frac{vt_0\gamma}{t'_1 - t_0\gamma} = \frac{v}{\frac{1-v^2/c^2}{1-v/c} - 1} = c}$$

as desired. A signal that travels with c in K must travel with c in K'.

2.a *(5 marks, medium, similar, A)*
Momentum conservation immediately gives $\boxed{v'_1 = -v'_2}$. Energy conservation gives $Mc^2 = 2mc^2/\sqrt{1 - v'^2_1/c^2}$. Solving for v'_1 gives $\boxed{v'_1/c = 4/5}$.

2.b *(5 marks, medium, unseen (use of hyperbolic functions seen), B)*
This is essentially (a), as

$$\frac{mc^2}{\sqrt{1 - v'^2_1/c^2}} + \frac{mc^2}{\sqrt{1 - v'^2_2/c^2}} = Mc^2$$

implies $\boxed{\cosh(\psi'_1) + \cosh(\psi'_2) = M/m}$ after re-writing according to the question. As stated in (a), $\cosh(\psi'_1) = \cosh(\psi'_2)$ as $\psi'_1 = -\psi'_2$.

2.c *(5 marks, difficult, unseen (use of hyperbolic functions seen), C)*
Need to show

$$\frac{mv_1}{\sqrt{1 - v_1^2/c^2}} + \frac{mv_2}{\sqrt{1 - v_2^2/c^2}} = \frac{Mv}{\sqrt{1 - v^2/c^2}}$$

and thus

$$m\sinh(\psi_1) + m\sinh(\psi_2) = M\sinh(\phi) \ .$$

Using (b) this is equivalent to

$$\boxed{\begin{aligned} \sinh(\psi_1) + \sinh(\psi_2) &= \sinh(\phi)(\cosh(\psi_1') + \cosh(\psi_2')) \\ &\Leftrightarrow \\ \sinh(\psi_1' + \phi) + \sinh(\psi_2' + \phi) &= \sinh(\phi)(\cosh(\psi_1') + \cosh(\psi_2')) \\ &\Leftrightarrow \\ \sinh(\psi_1' + \phi) + \sinh(\psi_2' + \phi) &= 2\sinh(\phi)\cosh(\psi_1') \end{aligned}}$$

which is the hyperbolic trig identity.

2.d *(5 marks, difficult, unseen (use of hyperbolic functions seen), C)*
Need to show

$$\frac{mc^2}{\sqrt{1 - v_1^2/c^2}} + \frac{mc^2}{\sqrt{1 - v_2^2/c^2}} = \frac{Mc^2}{\sqrt{1 - v^2/c^2}}$$

and thus

$$m\cosh(\psi_1) + m\cosh(\psi_2) = M\cosh(\phi) \ .$$

Using (b) this is equivalent to

$$\boxed{\begin{aligned} \cosh(\psi_1) + \cosh(\psi_2) &= \qquad \cosh(\phi)(\cosh(\psi_1') + \cosh(\psi_2')) \\ &\Leftrightarrow \\ \cosh(\psi_1' + \phi) + \cosh(\psi_2' + \phi) &= \qquad \cosh(\phi)(\cosh(\psi_1') + \cosh(\psi_2')) \\ &\Leftrightarrow \\ \cosh(\psi_1' + \phi) + \cosh(\psi_2' + \phi) &= \qquad 2\cosh(\phi)\cosh(\psi_1') \end{aligned}}$$

which is based on the hyperbolic trig identity.

3.a *(3 marks, medium, seen similar, A)*
With $A^i = (\phi, \mathbf{A})$, the scalar potential ϕ transforms like any four-vector,

$$\boxed{\phi' = \frac{\phi - \frac{v}{c}A_x}{\sqrt{1 - v^2/c^2}} = (\phi - \beta A_x)\gamma}$$

where ϕ and A_x are taken in K.

3.b *(3 marks, medium, unseen, B)*
With

$$ct = \frac{ct' + \frac{v}{c}x'}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad x = \frac{x' + \frac{v}{c}ct'}{\sqrt{1 - v^2/c^2}}$$

or more compactly

$$ct = (ct' + \beta x')\gamma \quad \text{and} \quad x = (x' + \beta ct')\gamma$$

this is a matter of straight-forward multivariate calculus

$$\boxed{\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial t} + v\gamma\frac{\partial}{\partial x}}$$

3.c *(8 marks, difficult, unseen, D)*
The y-component of $\mathbf{E}'$ is given by

$$\boxed{\begin{aligned} E_y' = -\frac{1}{c}\frac{\partial}{\partial t'}A_y' - \frac{\partial}{\partial y'}\phi' &= -\frac{1}{c}\gamma\frac{\partial}{\partial t}A_y - \frac{v}{c}\gamma\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}(\phi - \beta A_x)\gamma \\ &= \gamma\left(-\frac{1}{c}\frac{\partial}{\partial t}A_y - \frac{\partial}{\partial y}\phi\right) - \beta\gamma\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) = \gamma\left(E_y - \beta H_z\right) \end{aligned}}$$

3.d *(3 marks, medium, unseen, C)*

The Lorenz-gauge can be written as $\boxed{\partial_i A^i = 0, \text{ which is a Lorentz-scalar}}$.

3.e *(3 marks, easy with hint, unseen, A)*

From the hint,

$$\boxed{\frac{4\pi}{c} j^k = -\partial_k F^{ik} = \partial_k \partial^k A^i - \partial_k \partial^i A^k = \partial_k \partial^k A^i}$$

as $\partial_k \partial^i A^k = 0$ from the gauge.

4.a *(6 marks, easy, seen, B)*

The Lorentz force is $\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H}$ so that with $\mathbf{v} = (v, 0, 0)$

$$\boxed{\dot{\mathbf{p}} = e\frac{v}{c}\begin{pmatrix} 0 \\ -H_z \\ H_y \end{pmatrix} .}$$

4.b *(6 marks, easy, seen, A)*

This follows from $\mathbf{p} = m\mathbf{v}\gamma \parallel \mathbf{v}$ and so

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} \cdot \mathbf{p} = 2\mathbf{p} \cdot \frac{e}{c}\mathbf{v} \times \mathbf{H} = 0}$$

4.c *(6 marks, difficult, seen, D)*

This question may be answered from memory or via

$$F'^{ij} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\beta\gamma E_x & -\gamma E_x & -\gamma E_y + \beta\gamma H_z & -\gamma E_z - \beta\gamma H_y \\ \gamma E_x & \beta\gamma E_x & \beta\gamma E_y - \gamma H_z & \beta\gamma E_z - \gamma H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\beta\gamma^2 E_x + \beta\gamma^2 E_x & \dots \\ \gamma^2 E_x - \beta^2\gamma^2 E_x & \dots \\ \gamma E_y - \beta\gamma H_z & \dots \\ \gamma E_z + \beta\gamma H_y & \dots \end{pmatrix}$$

Identifying the relevant terms and simplifying gives the transformed fields

$$\mathbf{E}' = \begin{pmatrix} E_x \\ \gamma(E_y - \beta H_z) \\ \gamma(E_z + \beta H_y) \end{pmatrix}$$

Since $\mathbf{E} = \mathbf{0}$, however,

$$\boxed{\mathbf{E}' = \begin{pmatrix} 0 \\ -\beta\gamma H_z \\ \beta\gamma H_y \end{pmatrix}}$$

4.d *(2 marks, medium, seen, A)*

The force is now only due to electric field,

$$\boxed{\dot{\mathbf{p}'} = e\mathbf{E}' = e\beta\gamma \begin{pmatrix} 0 \\ -H_z \\ H_y \end{pmatrix}}$$

which is the result in (a) multiplied by γ. *A neat aside: In other words, in the rest-frame of a moving charge the magnetic field appears as a electric field. One may think of a magnetic field as the relativistic effect of an electric field.*

5.a *(5 marks, medium, A)*
Gauss' law, $\nabla \cdot \mathbf{E} = 4\pi\rho$ together with $\mathbf{E} = -\nabla\phi$ as $\mathbf{A}$ is constant in time, gives

$$\boxed{\nabla^2\phi = -4\pi\rho}$$

5.b *(5 marks, medium, B)*
Write

$$\boxed{U = \frac{1}{8\pi}\int \mathrm{d}^3r\, \mathbf{E}\cdot\mathbf{E} = -\frac{1}{8\pi}\int \mathrm{d}^3r\, \mathbf{E}\cdot\nabla\phi = \frac{1}{8\pi}\int \mathrm{d}^3r\, \phi\cdot\nabla\cdot\mathbf{E} = \frac{1}{2}\int \mathrm{d}^3r\, \phi\rho}$$

using the divergence theorem and assuming that the surface integral vanishes (at infinity).

5.c *(5 marks, difficult, D)*
By symmetry the electric field points radially outwards. Integrating over a sphere with radius r gives, for $r < R$

$$\int_{\partial V} \mathrm{d}\mathbf{n}\cdot\mathbf{E} = 4\pi r^2 E(r) = 4\pi \int \mathrm{d}^3r\, \rho = 4\pi \frac{3Q}{4\pi R^3}\frac{4\pi}{3}r^3 = 4\pi\frac{Qr^3}{R^3}$$

and similarly for $r \geq R$,

$$4\pi r^2 E(r) = 4\pi Q$$

so that

$$\boxed{E(r) = \begin{cases} \frac{Qr}{R^3} & \text{for } r < R \\ \frac{Q}{r^2} & \text{otherwise} \end{cases}}$$

5.d *(5 marks, difficult, C)*
The self-energy is given by

$$\boxed{U = \frac{1}{8\pi}\int \mathrm{d}^3r\, |\mathbf{E}(\mathbf{r})|^2 = \frac{Q^2}{8\pi}\left(\int_0^R \mathrm{d}r\, 4\pi r^2\frac{r^2}{R^6} + \int_R^\infty \mathrm{d}r\, 4\pi r^2 \frac{1}{r^4}\right) = \frac{Q^2}{2}\left(\frac{1}{5R} + \frac{1}{R}\right) = \frac{3Q^2}{5R}}$$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97022 MATH97100	1	Generally done well, occasionally confusion about factors of c. The last part requires lots of algebra unless planned well, I was generous with marks in case of confusion.	
MATH97022 MATH97100	2	For many the hardest question. Some students not solving for velocity. Part c-d often done by comparing numerical values, which is not what I had in mind, but I gave full marks. Some used velocity addition theorem, which got very quickly very messy.	
MATH97022 MATH97100	3	This was a nice example of the transformation of fields. This worked well for most, but many missed that the gauge can be written elegantly in four-vector form and then, as a scalar, does not transform.	
MATH97022 MATH97100	4	This, I thought, was a pretty cool example of how magnetic fields in one frame become magnetic fields in another. This was generally done well, with few people making mistakes when transforming the electromagnetic field tensor.	
MATH97022 MATH97100	5	In general well done, often the two cases in c and d ($r>R$, $r<R$) were not considered separately.	