IMPERIAL

MATH50007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Network Science

Date: Wednesday, May 14, 2025

Time: Start time 10:00 – End time 12:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

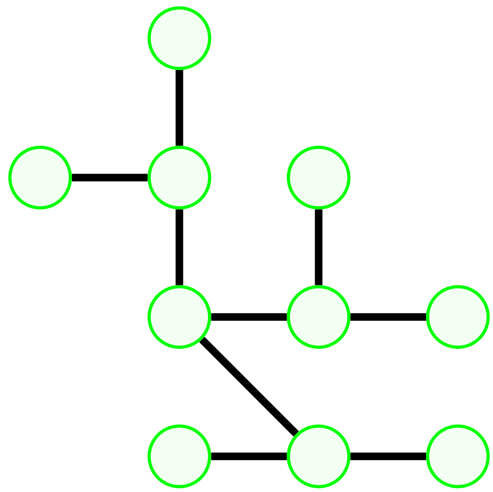
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the k-regular Cayley tree graph G drawn below. The nodes with degree 1 are termed *leaf nodes*, and the *root node* is the node that is equidistant from each leaf node. At iteration zero only the root node exists.



(i) Give k for G. (2 marks)

(ii) Draw the graph at iteration 1, give the corresponding adjacency matrix A and the Laplacian matrix L. (6 marks)

(ii) Give the diameter of G. (2 marks)

(b) Consider a k-regular Cayley tree graph after two iterations, where iteration zero is the *root node*. Assume that $k > 2$. Suppose the graph is divided into k disjoint sets by removing $k - 1$ links at the root node. Find the modularity M of this partition in terms of k and L, where L is the total number of links. (6 marks)

(c) For a simple, undirected, connected graph of N nodes, define a *loop* to be a path of length r that starts and ends at the same node i. The total number of *loops* of length r, l_r, is given by

$$l_r = \mathrm{Tr}(\mathrm{A}^r),$$

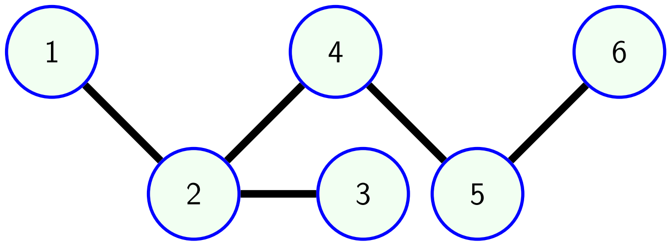
where Tr is the trace of a matrix and A^r is the adjacency matrix of the graph to the power of r. Show that

$$l_r = \sum_{i=1}^{N} \lambda_i^r,$$

where λ_i is the i^{th} eigenvalue of A. (4 marks)

(Total: 20 marks)

2. Consider the graph G_1 drawn below.



(a) Suppose G_1 was generated with the configuration model. Give the corresponding degree sequence.

(2 marks)

(b) Consider the following modification to the Barabasi-Albert model. Suppose at $t = 0$, we start with three connected nodes $i \sim j \sim k$, with two links in total, where the end nodes i, k have degree one, and the centre node j has degree two. We will call this a $triple$. Suppose at each iteration t another $triple$ is added, where a link is added from an end node of the $triple$ to a node in the existing graph. The node in the existing graph is selected using preferential attachment.

(i) Give $N(t)$ and $L(t)$ after t iterations, where $N(t)$ is the number of nodes and $L(t)$ the number of links after t iterations. (2 marks)

(ii) G_1 is one graph realisation of the model described at iteration $t = 1$, where nodes 4, 5 and 6 were introduced at $t = 1$. What is the probability of instead generating a path graph at $t = 1$ using the modified Barabasi-Albert model described? (4 marks)

(iii) Derive an equation relating the expected number of nodes with degree 2 at iteration $t + 1$, $\langle N_2(t + 1)\rangle$, to one or more of: $\langle N_1(t)\rangle$ and $\langle N_2(t)\rangle$. (5 marks)

(iv) Assume a stationary degree distribution $p_{k\infty}$ exists, where $p_{k\infty} = \lim_{t\to\infty}(p_k(t))$ and $p_k(t)$ is the degree distribution at t. Determine the stationary degree distribution when $k = 2$, by imposing the condition that $\lim_{t\to\infty}(t(p_2(t + 1) - p_2(t))) = 0$. Write your solution in terms of a recurrence relation for $p_{1\infty}$ and $p_{2\infty}$, the stationary degree distribution as $t \to \infty$ for $k = 1$ and $k = 2$, respectively. (7 marks)

(Total: 20 marks)

3. Consider a simple, connected, N-node graph with adjacency matrix A and Laplacian matrix L. Consider the following coupled system of ODEs for the graph:

$$\frac{dx_i}{dt} = x_i^3 - \alpha x_i^2 - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} \mathrm{L}_{ij} \mathrm{L}_{jl} x_l, \quad i = 1, ..., N. \tag{1}$$

The initial condition for each node is $x_i(t = 0) = x_i^0$, where $x_i^0 > 0$ and is not an equilibrium solution.

(a) Show that $\boldsymbol{z}$, the N-element vector of ones, is an eigenvector of L and find the corresponding eigenvalue. (3 marks)

(b) Show that $x_i = \alpha$ is a non-trivial equilibrium solution of (1). (3 marks)

(c) Consider Equation (1) and $x_i(t) = \alpha + \epsilon y_i(t)$ for all i, where $\epsilon \ll 1$. Show that in the limit $\epsilon \to 0$ the N-element vector $\boldsymbol{y}$ satisfies the following system of ODEs

$$\frac{dy_i}{dt} = \alpha^2 y_i - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} \mathrm{L}_{ij} \mathrm{L}_{jl} y_l. \tag{2}$$

(4 marks)

(d) Explain how to construct a matrix B so that the transformation $\boldsymbol{w} = \mathrm{B}\boldsymbol{y}$ allows the system of coupled ODEs given by Equation (2), to be written as

$$\frac{d\boldsymbol{w}}{dt} = \mathrm{M}\boldsymbol{w},$$

a decoupled system of ODEs, where $\boldsymbol{w}$ is a $N \times 1$ column vector, M is a diagonal $N \times N$ matrix with $M_{ii} = \alpha^2 - \beta\lambda_i^2$, and λ_i is the i^{th} eigenvalue of L. (5 marks)

(e) Assume that $y_i(t = 0) = y_i^0$, where $y_i^0 > 0$ and a known constant. Find the condition for synchronisation of non-trivial solutions of Equation (2), such that $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for distinct i, j. Write the condition in terms of α, β and $\rho(\mathrm{L})$, where $\rho(\mathrm{L})$ is the spectral radius of L. (5 marks)

(Total: 20 marks)

4. Consider a modification to the network-SI model, where upon infection, a person can recover with temporary immunity, and could become susceptible again. The spread of a disease on a N node simple graph G, with adjacency matrix A, has three state variables $s_i(t)$, $x_i(t)$, and $r_i(t)$, where $i = 1, 2, ..., N$, and each variable can be 0 or 1. A node is either susceptible $s_i = 1$, infected $x_i = 1$ or recovered $r_i = 1$. The variables are governed by the coupled ODEs

$$\frac{d\langle s_i \rangle}{dt} = -\beta \langle s_i \rangle \sum_{j=1}^{N} A_{ij} \langle x_j \rangle + \alpha \langle r_i \rangle, \tag{3}$$

$$\frac{d\langle x_i \rangle}{dt} = \beta \langle s_i \rangle \sum_{j=1}^{N} A_{ij} \langle x_j \rangle - \gamma \langle x_i \rangle, \tag{4}$$

$$\frac{d\langle r_i \rangle}{dt} = \gamma \langle x_i \rangle - \alpha \langle r_i \rangle, \tag{5}$$

where the states of nodes are assumed to be statistically independent. We require $s_i(t) + x_i(t) + r_i(t) = 1$ for all nodes. For a time-span of length Δt, let $\beta \Delta t$ be the probability that a susceptible node is infected, $\gamma \Delta t$ be the probability that an infected node recovers, and $\alpha \Delta t$ be the probability that a recovered node loses immunity to become susceptible again.

(a) The master equation for $x_i(t)$ is

$$P(x_i(t + \Delta t) = 1) = (1 - \gamma \Delta t) P(x_i(t) = 1) + \beta \Delta t \sum_{j=1}^{N} A_{ij} P(s_i(t) = 1, x_j(t) = 1) + O(\Delta t^2).$$

Show how the master equation for $x_i(t)$ leads to Equation (4) when $\Delta t \to 0$. (5 marks)

(b) Derive the master equation for $r_i(t)$ and $s_i(t)$. (8 marks)

(c) Consider the expression

$$\langle s_i x_j \rangle - \langle s_i x_j s_l \rangle - \langle s_i x_j x_l \rangle, \tag{6}$$

where the states of nodes are no longer assumed to be statistically independent.

(i) Show that the expression in Equation (6) is equal to $P(s_i = 1, x_j = 1, r_l = 1)$. (3 marks)

(ii) By assuming the states of nodes i and l are statistically independent, re-state expression (6) as an expression containing only first and second moments. (4 marks)

(Total: 20 marks)

Information sheet to be provided with Final Exam
Network Science 2025

There may be material on the exam which is not included here. There may be material here which is not needed for the exam.

Adjacency matrix: A_{ij} is the number of links from node j to node i.

Simple graphs: A graph is *simple* if it is undirected, unweighted, and does not have multiedges or self-loops.

Complete graph: A simple graph where each distinct pair of nodes is linked.

Graph Laplacian (also called the *Laplacian matrix*): $\mathrm{L} = \mathrm{D} - \mathrm{A}$ where D is the *diagonal degree matrix* for the graph, $\mathrm{D}_{ii} = k_i$, k_i is the degree of node i, and $\mathrm{L}_{ij} = \delta_{ij} k_j - \mathrm{A}_{ij}$.

Centralities and similarities

- *eigenvector centrality*: $x_i = \alpha \sum_{j=1}^{N} \mathrm{A}_{ij} x_j$, where α is a proportionality constant.
- *Katz centrality*: $x_i = \alpha \sum_{j=1}^{N} (\mathrm{A}_{ij} x_j) + 1$
- *PageRank centrality*: $x_i = \sum_{j=1}^{N} \left[(1-m) \frac{\mathrm{A}_{ij}}{\max(k_j^{out}, 1)} x_j + m \frac{x_j}{N} \right]$, where m is constant such that $0 < m < 1$ and k_j^{out} is the outgoing degree of node j.
- *cosine similarity*: $\sigma_{ij} = \frac{n_{ij}}{\sqrt{k_i k_j}}$, where n_{ij} is the number of common neighbours of node i and j.
- *Jaccard similarity*: $\sigma_{ij} = \frac{n_{ij}}{k_i + k_j - n_{ij}}$

Matrix resolvent: $R(\mathrm{M}; \mu)$ is the *resolvent* for a square N x N matrix, M. The resolvent is defined as $R = (\mu \mathrm{I} - \mathrm{M})^{-1}$ for μ where $\mu \neq \lambda_i,\ i = 1, 2, ..., N$, and λ_i is the i^{th} eigenvalue of M.

- $\rho(\mathrm{M})$ is the *spectral radius* of M: $\rho(\mathrm{M}) = \max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_N|\}$
- If $|\mu| > \rho(\mathrm{M})$, then $R(\mathrm{M}; \mu) = \sum_{l=0}^{\infty} \frac{\mathrm{M}^l}{\mu^{l+1}}$

Perron-Frobenius theorem: We have applied the Perron-Frobenius (P-F) theorem to three different classes of real square matrices:

1. *Positive matrices* where each element of the matrix is positive, $\mathrm{B}_{ij} > 0$. Then, the theorem tells us that there is a real positive eigenvalue λ where:
   - $\lambda = \rho(\mathrm{B}) > 0$, and all other eigenvalues are smaller in magnitude.
   - This eigenvalue is simple, all elements of the corresponding eigenvector have the same sign, and there are no other eigenvectors where all elements have the same sign. Note that a simple eigenvalue has algebraic multiplicity equal to 1.
2. *Irreducible matrices* Let $\mathrm{B}_{ij} > 0$ if there is a link in a graph from node i to node j with $\mathrm{B}_{ij} = 0$ otherwise. Then B is irreducible if and only if the corresponding graph is strongly connected (i.e. every node is reachable from every other node). For irreducible matrices, there is a real, positive eigenvalue λ where:
   - $\lambda = \rho(\mathrm{B}) > 0$, and this eigenvalue is simple.
   - All elements of the corresponding eigenvector have the same sign, and there are no other eigenvectors where all elements have the same sign
     - There may be other eigenvalues equal in magnitude to λ
3. *Non-negative matrices* where each element of the matrix is non-negative: $\mathrm{B}_{ij} \geq 0$. There is a real, non-negative eigenvalue λ where:

- $\lambda = \rho(\mathrm{B}) \geq 0$, and there may be other eigenvalues equal in value or equal in magnitude
- All non-zero elements of the corresponding eigenvector will have the same sign, and there may be other eigenvectors with the same property
- Note: This version of the P-F theorem is considerably weaker than the other 2

Markov's inequality: Let X be a random variable that assumes only non-negative values. Then, for all $a > 0$, $P(X \geq a) \leq \frac{\langle X \rangle}{a}$.

Chebyshev's inequality: Let X be a random variable with finite expected value, μ, and finite non-zero variance, σ^2. Then, for any $a > 0$, $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$.

The **configuration model** requires the specification of an N-element *degree sequence*: $\{k_1, k_2, ..., k_N\}$ with $k_i > 0$ for all i. The probability of two stubs being linked is $\frac{1}{K-1}$,
$K = \sum_{i=1}^{N} k_i$ is the **total degree**.

Average degree: $\overline{k} = \frac{1}{N}\sum_{i=1}^{N} k_i$.

Second moment of the degree distribution: $\overline{k^2} = \frac{1}{N}\sum_{i=1}^{N} k_i^2 = \sum_{k=1}^{k_{max}} p_k k^2$, where p_k is the probability that a node has degree k.

Preferential attachment: $\rho_i(G_a(t))$: probability that node i in graph $G_a(t)$ receives a link at time t. For linear preferential attachment: $\rho_i(G_a(t)) = \frac{k_i(G_a(t))}{K(t)}$
Graph diffusion equation: $\frac{d\langle \mathbf{n} \rangle}{dt} = -\alpha \mathrm{L} \langle \mathbf{n} \rangle$.

Fick's law on graphs: $\langle j_{ab} \rangle = -\alpha(\langle n_a \rangle - \langle n_b \rangle)$; n_a is the number of particles on node a, j_{ab} the net flux of particles per unit time from node b to node a.

Orthogonal diagonalization: A square real matrix, M, is orthogonally diagonalizable if and only if M is symmetric. Then, $\mathrm{M} = \mathrm{V}\Lambda\mathrm{V}^T$.

Rayleigh quotient: For a symmetric matrix M, $r(\mathrm{M}, \boldsymbol{x}) = \frac{\boldsymbol{x}^T \mathrm{M} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$ is maximized when $\boldsymbol{x} = \boldsymbol{v_1}$ in which case $r = \lambda_1$. ($\mathrm{M}\boldsymbol{v_1} = \lambda_1 \boldsymbol{v_1}$, $\lambda_1 \geq \lambda_2 \geq ...$)

Gershgorin's theorem: Let $\mathrm{B} \in \mathbb{C}^{N \times N}$ and suppose that $\mathrm{X}^{-1}\mathrm{B}\mathrm{X} = \mathrm{H} + \mathrm{F}$, where H is diagonal and F has zeros on its main diagonal. Then the eigenvalues of B lie on the union of the discs $\Delta_1, \Delta_2, \ldots, \Delta_N$, where $\Delta_i = \{l \in \mathbb{C} : |l - \mathrm{H}_{ii}| \leq \sum_{j=1}^{N} |\mathrm{F}_{ij}|\}$.

Random walks on graphs: $\mathrm{T}_{ij} = \frac{\mathrm{A}_{ij}}{k_i}$; T is the *transition matrix*, and T_{ij} is the probability that a walker takes a step from node i to node j on a simple graph.

Network-SI model: $\frac{d\langle x_i \rangle}{dt} = \beta \sum_{j=1}^{N} \mathrm{A}_{ij} \langle (1 - x_i) x_j \rangle$, where $x_i(t)$ is a random variable that indicates the state of node i at time t.

Degree-based approximation: $\frac{d\phi_k}{dt} = k\beta(1 - \phi_k) \sum_{k'=1}^{k'_{max}} \theta(k, k')\phi_{k'-1}$, where ϕ_k is the probability that a node with degree k is infectious and $\theta(k, k')$ the function giving the probability of a link on a node with degree k being connected to a node with degree k'.

Second-moment equation: $\frac{d\langle s_i x_j \rangle}{dt} = \beta \sum_{l=1}^{N} (\mathrm{A}_{jl} \langle s_i s_j x_l \rangle - \mathrm{A}_{il} \langle s_i x_j x_l \rangle)$, where $s_i = 1 - x_i$.

Modularity: The modularity of a set of nodes, S_a, is $M_a = \frac{1}{2L}\sum_{i \in S_a}\sum_{j \in S_a}\left(\mathrm{A}_{ij} - \frac{k_i k_j}{2L}\right)$

The *modularity matrix* B is defined using, $\mathrm{B}_{ij} = \mathrm{A}_{ij} - \frac{k_i k_j}{2L}$.

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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M50007

Network Science (Solutions)

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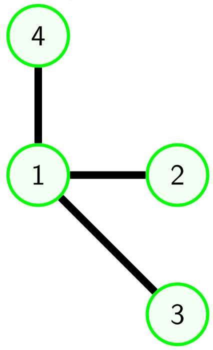
1. (a) (i) $k = 3$ sim. seen ⇓

(ii) The graph after one iteration is 2, A



Then

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The Laplacian matrix L is given by $\mathrm{L} = \mathrm{D} - \mathrm{A}$. D is a diagonal matrix with $D_{ii} = k_i$, where the degree of node i is k_i. Here,

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore, subtracting A, gives

$$\mathrm{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

6, A

(iii) This is a tree graph, and the longest shortest path is a path from a leaf node, through the root node, to another leaf node. The diameter is 4. 2, A

(b) Now, $M = M_1 + M_2 + ... + M_k$, where M_i is the modularity of each set n_i. Define $i = 1, ..., k-1$ to be the sets that do not contain the root node, and M_k to contain the root node. Firstly, notice that $M_1 = M_2 = ... = M_{k-1}$ since the graph is symmetric about the root node. Therefore, we need only determine M_1 and M_k. meth seen ⇓

$$M_i = \frac{1}{2L}\left(2L_i - \frac{K_i^2}{2L}\right),$$

where L_i is the number of links in set i, and K_i the number of stubs in set i. Now, $L_1 = L/k - 1$ and $L_k = L/k$. $K_1 = K/k - 1$, $K_k = K/k + k - 1$ and $K = 2L$. Therefore,

$$M_1 = M_2 = ... = M_{k-1} = \frac{1}{2L}\left(2\left(\frac{L}{k} - 1\right) - \frac{(\frac{2L}{k} - 1)^2}{2L}\right)$$

and

$$M_k = \frac{1}{2L}\left(\frac{2L}{k} - \frac{(\frac{2L}{k} - 1 + k)^2}{2L}\right).$$

So

$$M = \frac{k-1}{2L}\left(2\left(\frac{L}{k}-1\right) - \frac{(\frac{2L}{k}-1)^2}{2L}\right) + \frac{1}{2L}\left(\frac{2L}{k} - \frac{(\frac{2L}{k}-1+k)^2}{2L}\right).$$

For two iterations $L/k = k$ and this simplifies to

$$M = \frac{k-1}{2L}\left(2\left(k-1\right) - \frac{(2k-1)^2}{2L}\right) + \frac{1}{2L}\left(2k - \frac{(3k-1)^2}{2L}\right).$$

$\boxed{6, \text{C}}$

$\boxed{\text{unseen} \Downarrow}$

(c) Since A is symmetric, it has N real non-negative eigenvalues and is diagonalisable as $\mathrm{A} = \mathrm{V}\Lambda\mathrm{V}^{\mathrm{T}}$, where V is the orthogonal matrix of eigenvectors and Λ the diagonal matrix of eigenvalues. Then

$$\mathrm{A}^{\mathrm{r}} = (\mathrm{V}\Lambda\mathrm{V}^{\mathrm{T}})^{\mathrm{r}} = \mathrm{V}\Lambda^{\mathrm{r}}\mathrm{V}^{\mathrm{T}}.$$

since $\mathrm{V}\mathrm{V}^{\mathrm{T}} = \mathrm{I}$. Now,

$$l_r = \mathrm{Tr}(\mathrm{V}\Lambda^{\mathrm{r}}\mathrm{V}^{\mathrm{T}}) = \mathrm{Tr}(\mathrm{V}^{\mathrm{T}}\mathrm{V}\Lambda^{\mathrm{r}}) = \mathrm{Tr}(\Lambda^{\mathrm{r}}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \lambda_{\mathrm{i}}^{\mathrm{r}},$$

since the trace of a matrix product is invariant under cyclic permutations of the product.

$\boxed{4, \text{D}}$

2. (a) Node 1 has 1 stub, node 2 has 3 stubs, node 3 has one stub, node 4 has two stubs, node 5 has 2 stubs, node 6 has one stub. This gives $d = [1, 3, 1, 2, 2, 1]$. sim. seen ⇓

2, A

(b) (i)

$$N(t) = 3 + 3t \quad L(t) = 2 + 3t.$$

sim. seen ⇓

2, A

(ii) In G_1 node 4 is the end node. A path graph is created if node 4 connects to node 1 or node 3. These are the only other realisations at $t = 1$. The node that connects to node 4 is selected by preferential attachment. Node 2, has probability $= 1/2$ of being selected, and therefore the probability of generating G_1 is $1/2$. Node 1 and node 3 have probability $= 1/4$ of being selected. Therefore, a path graph is created with probability $1/4 + 1/4 = 1/2$. meth seen ⇓

4, A

(iii) First, note that at each time step, two nodes with degree two and one node with degree one will be added. Define the number of nodes with degree 2 as N_2. We will consider three cases. Case A: a node in the existing graph with degree 1 receives a link, in which case N_2 increases by three. Case B: a node in the existing graph with degree 2 receives a link and therefore N_2 increases by one. Case C: neither A or B occur and N_2 increases by two. Using the formulation from the lecture, sim. seen ⇓

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[P(A)(N_2(G_m(t)) + 3) + P(B)(N_2(G_m(t)) + 1)$$
$$+(1 - P(A) - P(B))(N_2(G_m(t)) + 2)\Big].$$

This simplifies to

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[(N_2(G_m(t)) + P(A) - P(B) + 2\Big].$$

Then the linear preferential attachment models tells us $P(A) = \frac{N_1}{2L}$, $P(B) = \frac{2N_2}{2L}$, and therefore

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[(N_2(G_m(t)) + \frac{N_1(G_m(t))}{2L} - \frac{2N_2(G_m(t))}{2L} + 2\Big].$$

Substituting

$$\langle N_k(t)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)N_k(G_m(t))$$

gives

$$\langle N_2(t+1)\rangle = \langle N_2(t)\rangle\left(1 - \frac{2}{2L}\right) + \frac{\langle N_1(t)\rangle}{2L} + 2,$$

and substituting for $L(t)$ gives

$$\langle N_2(t+1)\rangle = \langle N_2(t)\rangle\left(1 - \frac{1}{(2+3t)}\right) + \frac{\langle N_1(t)\rangle}{2(2+3t)} + 2. \qquad (1)$$

5, B

(iv) Substituting $\langle N_k(t)\rangle = p_k N(t)$ into Equation (1) we get

$\boxed{\text{sim. seen} \Downarrow}$

$$N(t+1)p_2(t+1) = N(t)p_2(t)\left(1 - \frac{1}{2+3t}\right) + N(t)p_1(t)\frac{1}{4+6t} + 2.$$

Substituting for $N(t)$ gives

$$(6+3t)p_2(t+1) = p_2(t)\left(3 + 3t - \frac{3+3t}{2+3t}\right) + p_1(t)\frac{3+3t}{4+6t} + 2.$$

Re-arranging gives

$$6p_2(t+1) - 3p_2(t) = 3t(p_2(t) - p_2(t+1)) - p_2(t)\frac{3+3t}{2+3t} + p_1(t)\frac{3+3t}{4+6t} + 2.$$

Taking the limit as $t \to \infty$ and imposing the condition $t(p_2(t+1) - p_2(t)) \to 0$ gives

$$3p_{2\infty} = -p_{2\infty} + \frac{1}{2}p_{1\infty} + 2.$$

Therefore,

$$p_{2\infty} = \frac{1}{8}p_{1\infty} + \frac{1}{2}.$$

$\boxed{7, \text{D}}$

3. (a) $\mathrm{L}\boldsymbol{z} = \mathrm{D}\boldsymbol{z} - \mathrm{A}\boldsymbol{z}$. Now, $\mathrm{D}\boldsymbol{z} = \mathrm{A}\boldsymbol{z} = \boldsymbol{k}$ where $\boldsymbol{k}$ is a column vector containing the degree of each node in the graph. Therefore $\mathrm{L}\boldsymbol{z} = \mathbf{0}$, so $\boldsymbol{z}$ is an eigenvector with corresponding eigenvalue $\lambda = 0$. seen ⇓

3, A

(b) Re-write the equation as sim. seen ⇓

$$\frac{dx_i}{dt} = -x_i^2(\alpha - x_i) - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} \mathrm{L}_{ij}\mathrm{L}_{jl}x_l, \tag{2}$$

When $x_i = \alpha$ the first term is zero by inspection. Now, the summation in vector form is $\mathrm{L}^2\boldsymbol{x}$ and for $\boldsymbol{x} = \alpha\boldsymbol{z}$, $\mathrm{L}\boldsymbol{x} = 0$ since $\lambda = 0$ is an eigenvalue of L with eigenvector $\boldsymbol{z}$. Therefore, $dx_i/dt = 0$ for all i, and we have an equilibrium.

3, A

(c) Substituting $x_i(t) = \alpha + \epsilon y_i(t)$ in Equation (2) gives sim. seen ⇓

$$\epsilon\frac{dy_i}{dt} = \epsilon y_i(\alpha + \epsilon y_i)(\alpha + \epsilon y_i) - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} \mathrm{L}_{ij}\mathrm{L}_{jl}(\alpha + \epsilon y_l), \quad i = 1, 2, ..., N. \tag{3}$$

We know from (b) that $\sum_{j=1}^{N} \sum_{l=1}^{N} L_{ij}L_{jl}\alpha = 0$. Then dividing both sides by ϵ gives

$$\frac{dy_i}{dt} = y_i(\alpha^2 + 2\epsilon\alpha y_i) - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} L_{ij}L_{jl}y_l + O(\epsilon^2), \tag{4}$$

and taking the limit as $\epsilon \to 0$ gives

$$\frac{dy_i}{dt} = \alpha^2 y_i - \beta \sum_{j=1}^{N} \sum_{l=1}^{N} L_{ij}L_{jl}y_l. \tag{5}$$

3, A

1, B

(d) In matrix form Equation (5) becomes sim. seen ⇓

$$\frac{d\boldsymbol{y}}{dt} = \alpha^2\boldsymbol{y} - \beta\mathrm{L}^2\boldsymbol{y}. \tag{6}$$

We can orthogonally diagonalize the Laplacian as $\mathrm{L} = \mathrm{V}\Lambda\mathrm{V}^T$, where the columns of V contain the eigenvectors of L, normalised to have length one, and Λ is a diagonal matrix where Λ_{ii} is the eigenvalue corresponding to the eigenvector stored in the i^{th} column of V. Therefore $\mathrm{L}^2 = \mathrm{V}\Lambda\mathrm{V}^\mathrm{T}\mathrm{V}\Lambda\mathrm{V}^\mathrm{T} = \mathrm{V}\Lambda^2\mathrm{V}^\mathrm{T}$, since $\mathrm{V}\mathrm{V}^\mathrm{T} = \mathrm{I}$. Equation (6) can be written as

$$\frac{d\boldsymbol{y}}{dt} = \left(\alpha^2\mathrm{I} - \beta\mathrm{V}\Lambda^2\mathrm{V}^T\right)\boldsymbol{y}. \tag{7}$$

Writing $\alpha^2\boldsymbol{y} = \alpha^2\mathrm{V}\mathrm{I}\mathrm{V}^T\boldsymbol{y}$ we get

$$\frac{d\boldsymbol{y}}{dt} = \left(\alpha^2\mathrm{V}\mathrm{I}\mathrm{V}^T - \beta\mathrm{V}\Lambda^2\mathrm{V}^T\right)\boldsymbol{y}. \tag{8}$$

Define a new matrix $\mathrm{M} = \alpha^2\mathrm{I} - \beta\Lambda^2$ to give

$$\frac{d\boldsymbol{y}}{dt} = \mathrm{V}\mathrm{M}\mathrm{V}^T\boldsymbol{y}. \tag{9}$$

Then define $\boldsymbol{w} = \mathrm{V}^T \boldsymbol{y}$ results in

$$\mathrm{V}\frac{d\boldsymbol{w}}{dt} = \mathrm{VM}\boldsymbol{w}, \tag{10}$$

since $\boldsymbol{y} = \mathrm{V}\boldsymbol{w}$. Multiplying by V^T results in

$$\frac{dw_i}{dt} = M_{ii}w_i, \quad i = 1, 2, ..., N, \tag{11}$$

where $M_{ii} = \alpha^2 - \beta\lambda_i^2$ and λ_i is the i^{th} eigenvalue of L. $\boxed{3, \mathrm{B}}$

$\boxed{2, \mathrm{C}}$

(e) The solution of Equation (11) is $w_i = w_i^0 e^{M_{ii}t}$. Then $\boldsymbol{y} = \mathrm{V}\boldsymbol{w}$, so $\boxed{\text{sim. seen} \Downarrow}$

$$\boldsymbol{y} = w_1^0 e^{M_{11}t}\boldsymbol{v_1} + w_2^0 e^{M_{22}t}\boldsymbol{v_2} + ... + w_N^0 e^{M_{NN}t}\boldsymbol{v_N}.$$

w_i^0 is determined by the initial condition for $y_i(t = 0)$, and $w_i^0 > 0$ since $\boldsymbol{v_i}$ are linearly independent and $y_i^0 > 0$. Now, suppose we have ordered the eigenvectors so that $\lambda_1 = 0$ corresponds to $\boldsymbol{v}_1 = \boldsymbol{z}/\sqrt{N}$. Then $\lambda_i > 0$ for $i > 1$ by the properties of L. Since $M_{ii} = \alpha^2 - \beta\lambda_i^2$, $M_{11} = \alpha^2$ and for $i > 1$, $M_{ii} < 0$ if $\beta > \alpha^2/\lambda_i^2$. λ_i are real, since G is simple and connected, and α^2/λ_i^2 is greatest for λ_2, the smallest postive eigenvalue of L. It follows that, if $\beta > \alpha^2/\lambda_2^2$ then when $i > 1$, $M_{ii} < 0$. Thus, as $t \to \infty$,

$$\boldsymbol{y} \approx w_1^0 e^{\alpha^2 t}\boldsymbol{v_1}.$$

Since $\boldsymbol{v_1} = \boldsymbol{z}/\sqrt{N}$, every element of $\boldsymbol{y}$ is the same and synchronisation occurs. $\boxed{5, \mathrm{D}}$

4. (a) Re-write as

sim. seen ⇓

$$\frac{P(x_i(t+\Delta t)=1)-P(x_i(t)=1)}{\Delta t} = -\gamma P(x_i(t)=1)+\beta\sum_{j=1}^{N} A_{ij}P(s_i(t)=1,x_j(t)=1)+O(\Delta t)$$

where we have divided through by Δt. Now $\langle x_i\rangle = P(x_i=1)$ so the LHS becomes

$$\frac{\langle x_i(t+\Delta t)\rangle - \langle x_i(t)\rangle}{\Delta t}.$$

Then $P(s_i(t)=1, x_j(t)=1) = P(s_ix_j=1) = \langle s_ix_j\rangle$, since $s_ix_j=0$ or $s_ix_j=1$ only. Letting $\Delta t\to 0$ gives

$$\frac{d\langle x_i\rangle}{dt} = \beta\sum_{j=1}^{N} A_{ij}\langle s_ix_j\rangle - \gamma\langle x_i\rangle \tag{12}$$

Assuming the states of nodes are all statistically independent we can write $\langle s_ix_j\rangle \approx \langle s_i\rangle\langle x_j\rangle$ and we get the required equation.

5, A

(b) First, the master equation for r_i. A node is recovered at $t+\Delta t$ if it is recovered at t and does not lose immunity, or an infected node recovers during the step. Other events occur with probability of $O(\Delta t^2)$. So the master equation for r_i is

sim. seen ⇓

$$P(r_i(t+\Delta t)=1) = P(r_i(t)=1, r_i\nrightarrow 0) + P(x_i(t)=1, r_i\to 1) + O(\Delta t^2).$$

Now,

$$P(r_i(t)=1, r_i\nrightarrow 0) = P(r_i\nrightarrow 0|r_i(t)=1)P(r_i(t)=1).$$

The probability of a recovered node losing immunity in the time step Δt is $\alpha\Delta t$ and therefore the probability of this event not happening is $1-\alpha\Delta t$. Consequently,

$$P(r_i(t)=1, r_i\nrightarrow 0) = (1-\alpha\Delta t)P(r_i(t)=1).$$

Next,

$$P(x_i(t)=1, r_i\to 1) = P(r_i\to 1|x_i(t)=1)P(x_i(t)=1).$$

The probability of a node recovering in the time step is Δt is $\gamma\Delta t$ and so the master equation for r_i is

$$P(r_i(t+\Delta t)=1) = (1-\alpha\Delta t)P(r_i(t)=1) + \gamma\Delta t P(x_i(t)=1) + O(\Delta t^2).$$

Now s_i. We know that $P(s_i(t)=1)+P(x_i(t)=1)+P(r_i(t)=1)=1$ since the three events are disjoint and no other cases exist. Then, $P(s_i(t+\Delta t)) = 1-P(x_i(t+\Delta t))-P(r_i(t+\Delta t)$. Using the given master equation for x_i and the derived equation for r_i, we find:

$$P(s_i(t+\Delta t)=1) = P(s_i(t)=1)-\beta\Delta t\sum_{j=1}^{N} A_{ij}P(s_i(t)=1,x_j(t)=1)+\alpha\Delta t P(r_i(t)=1)+O(\Delta t^2).$$

8, B

sim. seen ⇓

(c) (i) The linearity of expectation allows the expression to be written as $\langle s_i x_j(1 - s_l - x_l)\rangle$. Now $s_i + x_i + r_i = 1$, so $\langle s_i x_j(1 - s_l - x_l)\rangle = \langle s_i x_j r_l\rangle$. This is equal to one when all variables are one, or is zero otherwise. So by definition $P(s_i = 1, x_j = 1, r_l = 1) = \langle s_i x_j r_l\rangle$.

3, B

sim. seen ⇓

(ii) In part (a) we wrote the expression as

$$\langle s_i x_j\rangle - \langle s_i x_j s_l\rangle - \langle s_i x_j x_l\rangle = P(s_i = 1, x_j = 1, r_l = 1).$$

We can write the joint probability as

$$P(s_i = 1, x_j = 1, r_l = 1) = P(s_i = 1, r_l = 1|x_j = 1)P(x_j = 1).$$

Since nodes i and l are statistically independent

$$P(s_i = 1, r_l = 1|x_j = 1) = P(s_i = 1|x_j = 1)P(r_l = 1|x_j = 1).$$

Therefore,

$$\langle s_i x_j\rangle - \langle s_i x_j s_l\rangle - \langle s_i x_j x_l\rangle = P(s_i = 1|x_j = 1)P(r_l = 1|x_j = 1)P(x_j = 1).$$

Then, using

$$P(s_i = 1|x_j = 1) = \frac{P(s_i = 1, x_j = 1)}{P(x_j = 1)} \quad \textit{and} \quad P(r_l = 1|x_j = 1) = \frac{P(r_l = 1, x_j = 1)}{P(x_j = 1)},$$

gives

$$P(s_i = 1|x_j = 1)P(r_l = 1|x_j = 1)P(x_j = 1) = \frac{P(s_i = 1, x_j = 1)P(r_l = 1, x_j = 1)}{P(x_j = 1)},$$

and

$$\frac{P(s_i = 1, x_j = 1)P(r_l = 1, x_j = 1)}{P(x_j = 1)} = \frac{\langle s_i x_j\rangle\langle r_l x_j\rangle}{\langle x_j\rangle}.$$

4, C

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks

MATH50007 Network Science Markers Comments

Question 1	Students did very well on part (a), and most students achieved full marks here. In part (b), often students incorrectly identified the degree in each set, leading to an incorrect term in the modularity of the set, and sometimes students incorrectly partitioned the graph. Most students correctly identified the method they should use. In part (c), marks were commonly lost when students failed to write/use that the eigenvectors were orthogonal, or that the trace is commutable.
Question 2	Nearly every student answered part (a) and (b)(i) correctly. Part (b)(ii) students often mistook the question for asking the probability of generating G_1, rather than a path graph, or students did not state the use of preferential attachment to generate probabilities. In this model, the end node of the new triple is attached to the graph, not the centre node, and sometimes students did not invoke this. In part (b)(iii), often students gave the correct final equation, but missed out vital steps in their calculations. These included stating each event and its probability, and summing over all possible generations of the graph at time t. Part (b)(iv) was the hardest part of the question, and indeed, students often struggled with taking the limit of their equation. Often, students had combined terms as one fraction, and this caused errors when they took the limit.
Question 3	Students generally did well on parts (a), (b), (c), and (d) though some were affected by time pressure. Results for part (e) were more mixed. Several students were able to solve the decoupled ODEs to obtain w(t), but then there were difficulties using V and the first eigenvector in V to obtain the solution for y(t) and the condition for synchronization.
Question 4	It seemed like many students were significantly affected by time pressure leading to basic errors and omissions. For example, little or no explanation was often given when moving from indicator variables to probabilities (or back), and O(Delta t) terms were frequently not handled correctly. Part(b) was a particular struggle. Since the master equations could easily be inferred from the given ODEs, it was important to provide a careful derivation. Often correct (or nearly correct) master equations were given with little or no explanation of how to derive them. Several students took the ODEs as a starting point for their derivation. This was allowed (even though the ODEs are derived from the master equations), however then a careful, rigorous treatment of the time derivative was essential.