Mock Exam 2: Mastery Question (Q5)

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We consider a set of N robots with state (position,velocity) $(x_i(t), v_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ interacting under second-order dynamics of the form

$$\frac{dx_i}{dt} = v_i \,, \tag{1}$$

$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) + u_i(t)\,, \tag{2}$$

$$x_i(0) = x_0\,, \quad v_i(0) = v_0\,, \qquad i = 1, \dots, N\,, \tag{3}$$

where $u_i(t) \in \{u : \mathbb{R}_+ \to \mathbb{R}^2\}$ correspond to control signals for each robot, and $a(r)$ is a communication kernel of the type

$$a(r) = \frac{1}{(1 + r^2)}\,.$$

Our goal is to drive the system to consensus, that is, to converge towards a configuration in which

$$v_i = \bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j \qquad \text{for all } i.$$

For this, we write a finite horizon control problem of the form

$$\min_{\mathbf{u}(\cdot)} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \left(\|\bar{v} - v_j\|^2 + \gamma\|u_j\|^2\right)\, dt\,,$$

with $\gamma > 0$, subject to the dynamics (1)-(3).

i Write the necessary optimality conditions for this problem, giving an explicit expression of the optimal control as a function of the adjoint variable.

ii The result in (a) gives the optimal control for a given initial condition. If an optimal feedback was sought instead, explain what is the practical difficulty associated to its synthesis, and how could we circumvent it.

Answer

i While existence of a minimiser $\mathbf{u}^*$ follows from the smoothness and convexity properties of the system dynamics and the cost, the Pontryagin Minimum Principle yields first-order necessary conditions for the optimal control. Let $(p_i(t), q_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ be adjoint variables associated to (x_i, v_i), then the optimality system consists of a solution $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{p}^*, \mathbf{q}^*)$ satisfying the system dynamics along with the adjoint equations

$$-\frac{dp_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \frac{a'\left(\|x_j - x_i\|\right)}{\|x_j - x_i\|} \left\langle q_j - q_i, v_j - v_i \right\rangle \left(x_j - x_i\right) , \tag{4}$$

$$-\frac{dq_i}{dt} = p_i + \frac{1}{N}\sum_{j=1}^{N} a\left(\|x_j - x_i\|\right)\left(q_j - q_i\right) - \frac{2}{N}\left(\bar{v} - v_i\right), \tag{5}$$

$$p_i(T) = 0\,, \quad q_i(T) = 0\,, \qquad i = 1, \dots, N\,, \tag{6}$$

and the optimality condition

$$\mathbf{u}(t) = \underset{\mathbf{w}\in\mathbb{R}^{2N}}{argmin} \sum_{j=1}^{N} \left(\left\langle q_j, \frac{dv_j}{dt} \right\rangle + \frac{\gamma}{N}\|w_j\|^2 \right) = -\frac{N}{2\gamma}\mathbf{q}^t\,. \tag{7}$$

ii For an optimal feedback law we must follow a dynamic approach, characterizing the value function of this problem as the solution of a HJB PDE. The main bottleneck of this approach is the curse of dimensionality. In this case, we expect N to be large, and with 4 states per robot, we need to solve the HJB PDE in $4N$ dimensions, which is not tractable with conventional grid-based schemes. As discussed in the lecture notes, an alternative to this is the use of deep neural networks to approximate the value function in high-dimensional spaces.