Imperial College
London

MATH97225 MATH97226

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Graph Theory

Date: Wednesday, 26 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let G be a graph, with the vertices labelled as $v_0, v_1, \ldots$. The degree sequence of G is the sequence $(\deg(v_i))_{i=0}^{|G|}$.

(a) Define each of the following graphs (a diagram will suffice), label the vertices, and give its degree sequence:

(i) $T(5,3)$. (1 mark)

(ii) $K_{2,3}$. (1 mark)

(iii) $\mathbb{Z}$ where nEm iff $n = m \pm 1$. (1 mark)

(b) For each of the following sequences, give a graph and its labelling which has that sequence as a degree sequence:

(i) $(1, 2, 3, 2)$. (2 marks)

(ii) $(a_i)_{i=0}^{\infty}$ where $a_i = r + 1$ where r is the remainder obtained when dividing i by 4. (4 marks)

(c) (i) Let $S \subseteq \mathbb{N}$ be non-empty. Prove that every infinite sequence of natural numbers where every number in S occurs infinitely often, but no other numbers occur, is the degree sequence of some graph. A diagram of a graph by itself will not suffice as proof. (5 marks)

(ii) Find an infinite sequence $(x_n)_{n=1}^{\infty}$ which does not occur as the degree sequence of a graph. (2 marks)

(d) Find infinite G and H which have the same degree sequence but are not isomorphic. (4 marks)

(Total: 20 marks)

2. (a) Find graphs G and H and a homomorphism $\phi : G \to H$ such that all three of the following conditions hold:

   * there is a spanning tree $T_1 \subseteq G$ such that $\phi(T_1)$ is a tree;
   * there is a spanning tree $T_2 \subseteq G$ such that $\phi(T_2)$ is not a tree; and
   * if S is a spanning tree of H then $\phi^{-1}(S) = \{g \in G : \phi(g) \in S\}$, as an induced subgraph of G, is a tree. (4 marks)

(b) Suppose G is a finite, bipartite graph. Suppose also that the maximum degree of a vertex in G is r. Let X be a set of vertices such that all the elements of X belong to the same part and have degree r.

(i) Let $N(X) = \{v \in G : \exists x \in X \, xEv\}$. Prove that $|X| \leq |N(X)|$. (4 marks)

(ii) Prove that there is a matching that covers X. (1 mark)

(c) A perfect matching of a bipartite graph is a matching that matches every vertex of G.

(i) Prove that if G is bipartite and has a Hamiltonian cycle then G has a perfect matching. (3 marks)

(ii) Find a connected graph G such that G is bipartite and has a perfect matching, but does not have a Hamiltonian cycle. You must justify your answer, a diagram by itself will not suffice. (2 marks)

(d) Let G be a tripartite graph, with parts A, B, and C. We say that G is pair-wise perfectly matched if the three induced bipartite graphs $A \cup B$, $B \cup C$, and $A \cup C$ all have perfect matchings.

(i) Find a tripartite G which has a Hamiltonian cycle, but which is not pair-wise perfectly matched. You must justify your answer, a diagram will not suffice. (3 marks)

(ii) Find a connected tripartite G which is pair-wise perfectly matched, but does not have a Hamiltonian cycle. You must justify your answer, a diagram by itself will not suffice. (3 marks)

(Total: 20 marks)

3. (a) Let G and H be planar graphs with $G \subseteq H$. A given drawing of G *extends* to H if it is possible to construct a drawing of H by adding points and lines to the drawing of G.

   (i) Find planar graphs G and H, such that $G \subseteq H$, and there is a a drawing of G that does not extend to H. (2 marks)

   (ii) Prove that if G and H are planar graphs such that $G \subseteq H$ then there is a drawing of G that extends to H. (2 marks)

   (iii) Let G be a finite planar graph with two connected components, and let $x, y \in V_G$ lie in different connected components. Let G' be the graph obtained by adding an edge between x and y. Prove that there is a drawing of G that extends to G'. (2 marks)

   (b) Let $G \in \mathcal{G}(n,p)$ be a random graph for constant, non-zero p. Prove that $\mathbb{P}\{G \text{ is planar}\} \to 0$ as $n \to \infty$.

   (6 marks)

   (c) Let $R(n,m)$ be the Ramsey number for $n, m \in \mathbb{N}$.

   (i) Prove that if $|G| > lR(n,m)$ then either $K_{n+1} \leq G$, $\overline{K_l} \leq G$ or $K_{1,m} \leq G$. (4 marks)

   (iii) Prove that $R(\oplus_{i=1}^{m} K_2, K_n) \leq mn$. (4 marks)

(Total: 20 marks)

4. (a) Recall that $\mathrm{ex}(n,G)$ is the extremal value for n and G, i.e. the maximum number of edges a graph H such that $|H| = n$ can have while not containing a subgraph isomorphic to G.

   (i) Calculate $\mathrm{ex}(n, K_2 \oplus K_2)$ for all n. (4 marks)

   (ii) Prove that if $G \subseteq H$ and $n > |H|$ then $\mathrm{ex}(n,G) \leq \mathrm{ex}(n,H)$. (2 marks)

   (b) You may not use the Erdős-Stone Theorem or any of its corollaries in this question. Let $c(n,H) = \frac{2\mathrm{ex}(n,H)}{n(n-1)}$.

   (i) Prove that for all H, if $m < n$ then $c(n,H) \leq c(m,H)$. (12 marks)

   (ii) Prove that $\lim_{n\to\infty} c(n,H)$ exists. (2 marks)

(Total: 20 marks)

5. Let X and Y be disjoint set of vertices in graph G, and let $d \in (0, 1]$. You may find the defintions of the density $d(X, Y)$ and what it means for (X, Y) to be an ϵ-regular pair useful for this question.

Any results you use must be from the lecture notes, and must be stated explicitly.

(a) Let $\epsilon \in (0, \frac{d}{2})$. Suppose that (X, Y) is an ϵ-regular pair with density greater than or equal to d. Prove that if $|X| = |Y| = n$ then there are at most ϵn vertices in X with less than $\frac{dn}{2}$ neighbours in Y. (6 marks)

(b) Let $d > \frac{1}{2}$ and let $\epsilon \in (0, \frac{d}{4})$. Let $Z \subseteq V_G$ be disjoint from both X and Y, and suppose $|Z| = n$. Suppose that (X, Y), (Y, Z) and (X, Z) are all ϵ-regular with density at least d. Prove the induced subgraph of G with vertex set $X \cup Y \cup Z$ contains a subgraph isomorphic to K_3. (14 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96066/MATH97225/MATH97226

Graph Theory (Solutions)

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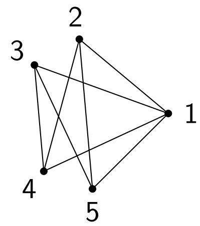
Robert Barham	....	....

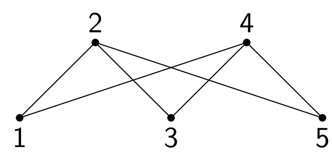
1. (a) Any labelling is acceptable, and so any permutation of the given sequences could be valid. unseen $\Downarrow$

(i) $(4, 3, 3, 3, 3)$ comes from:



1, A

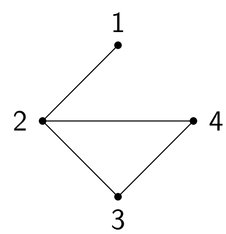
(ii) $(2, 3, 2, 3, 2)$ comes from:



1, A

(iii) Every vertex of this graph has the same degree, so any labelling of $\mathbb{Z}$, such as the standard function $f : \mathbb{N} \to \mathbb{Z}$ which starts as $0, 1, -1, 2, -2, \ldots$, will give the sequence $(2, \ldots)$. 1, A

(b) (i) Let $V_G = \{v_1, v_2, v_3, v_4\}$, where the only edges are: v_1Ev_2, v_2Ev_3, v_2Ev_4, v_3Ev_4 has degree sequence $(1, 2, 3, 4)$. unseen $\Downarrow$



2, A

(ii) Let $f : \mathbb{N} \setminus \{0\} \to \mathbb{Z}$ and $g : \mathbb{N} \setminus \{0\} \to V_G$ be bijections.
For all $n \in \mathbb{N}$, let $\{g(4n+1), g(4n+2), g(4n+3), g(4n+4)\}$ be isomorphic to the graph from (b)(i) as an induced subgraph of G.
The only other edges are between $g(4n)$ and $g(4m)$ where $f(n) = f(m) \pm 1$. 4, B

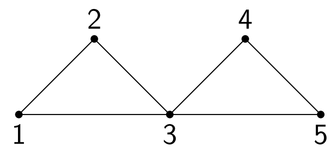
(c) (i) For every $n \in \mathbb{N}$, every vertex of K_{n+1} has degree n. Let $G_n = \oplus_{i=1}^{\infty} K_{n+1}$, i.e. infinitely many copies of K_{n+1} with no edges between them. Let $G_S = \oplus_{s \in S} G_s$. If the ith entry in the degree sequence is s then picking a vertex from G_s will result in a labelling of G_S with the desired degree sequence. unseen $\Downarrow$

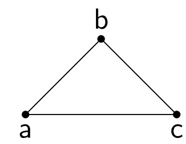
5, B

(ii) We may take infinitely many of the x_n's to be zero. If the subsequence of non-zero entries is not the degree sequence of any finite graph, then this sequence cannot be the degree sequence of an infinite graph.
e.g. $(1, 0, \ldots)$. 2, B

(d) Let G be G_2 from the (c)(i), and let H be the graph from (a)(iii). These both have the degree sequence $(2, \ldots)$, but one is connected while the other is not, so they cannot be isomorphic. unseen $\Downarrow$ 4, A

2. (a) There are lots of G, H, and ϕ with these three properties. e.g.: unseen ⇓

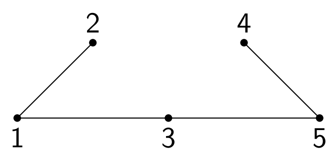
$G =$ 

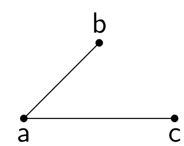
$H =$ 

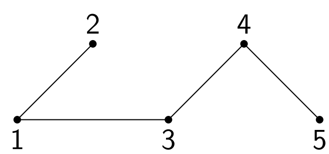
Let $\phi : G \to H$ be the map such that:

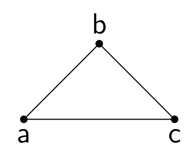
v	$\phi(v)$
1	a
2	b
3	c
4	b
5	a

This is a homomorphism. For T_1 and T_2:

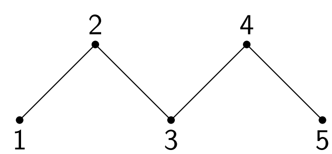
$T_1 =$ 

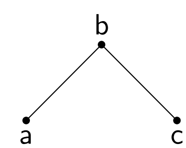
$\phi(T_1) =$ 

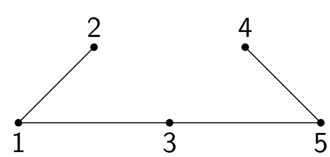
$T_2 =$ 

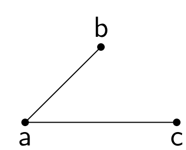
$\phi(T_2) =$ 

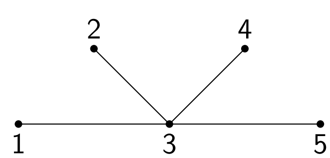
H has three spanning trees, and the preimage of each of them is also a tree.

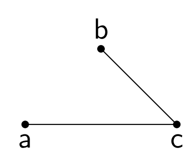
$\phi^{-1}(S_a) =$ 

$S_a =$ 

$\phi^{-1}(S_b) =$ 

$S_b =$ 

$\phi^{-1}(S_c) =$ 

$S_c =$ 

4, B

(b) (i) Every vertex of X has degree r, so there are $r|X|$ many edges between X and $N(X)$. If we count these vertices from the $N(X)$ side, we find that unseen ⇓

$$\begin{aligned} \textstyle\sum_{v \in N(X)} \deg(v) &= r|X| \\ \textstyle\sum_{v \in N(X)} \frac{\deg(v)}{r} &= |X| \end{aligned}$$

Since the degree of every vertex in G is at most r, we have $\frac{\deg(v)}{r} \leq 1$, and therefore $\sum_{v \in N(X)} \frac{\deg(v)}{r} = |N(X)|$.
Putting it all together, we get that $|N(X)| \geq |X|$. 4, C

(ii) Let G' be the induced subgraph of G on $X \cup N(X)$. Since $|N(X)| \geq |X|$, we have that X is a minimal cover of G'. Hence, by Kőnig's Theorem, there is a maximal matching M such that $|M| = |X|$, which implies that X is matched by M.

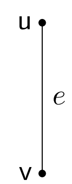
If we view M as a matching of G instead of G', it may be no longer maximal, but it still matches every vertex of X. $\boxed{1, \text{A}}$

(c) (i) Let G be a bipartite graph with a Hamiltonian cycle $(x_1, \ldots, x_n, x_1)$. If n is odd, then x_n belongs to the same part as x_1, which contradicts the fact that $x_n E x_1$, so n must be even. Suppose the Hamiltonian cycle uses edges $(e_1, \ldots, e_n)$ in order. Let $M = \{e_i : i \text{ is odd.}\}$. $\boxed{\text{unseen} \Downarrow}$

The endpoints of the odd-labelled edge e_{2i-1} are $\{v_{2i-1}, v_{2i}\}$, and thus if e_{2i-1} and e_{2j-1} share an endpoint then they are actually equal. Therefore M is a matching.

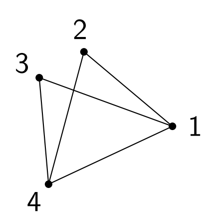
If i is odd then x_i is matched by e_i, and if i is even then x_i is matched by e_{i-1}, and therefore M matches every vertex in the cycle. Since this cycle is Hamiltonian, this means that M is a perfect matching. $\boxed{3, \text{A}}$

(ii) There are lots of graphs with this property. The simplest is:



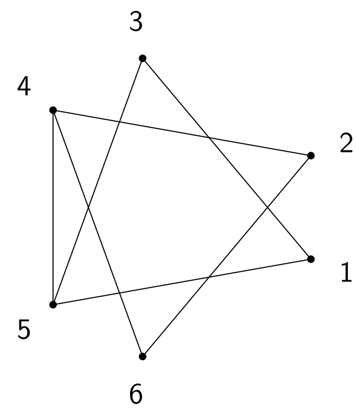
The set $\{e\}$ is a perfect matching, but since this graph is a tree it cannot contain any cycles, and hence does not contain a Hamiltonian cycle. $\boxed{1, \text{A}}$

(d) (i) Let G be a tripartite graph with parts X, Y and Z. If $|X|$, $|Y|$ and $|Z|$ are not all equal, then G can't be pair-wise perfectly matched. If we can find such a G with a Hamiltonian cycle then we're done. A minimal example is $T(4,3)$:



$T(4,3)$ has Hamiltonian cycle $(1,2,4,3,1)$ and tripartite structure $\{1\}$, $\{2,3\}$, $\{4\}$. Since $|\{1\}| \neq |\{2,3\}|$, this graph is not pair-wise perfectly matched. $\boxed{2, \text{A}}$

(ii) Again, lots of possible graphs. The one given here is minimal, more complicated graphs will require more sophisticated justifications. Let G be the following graph:



Edges are denoted by pairs of numbers, e.g. the edge between 1 and 4 is written as 14.

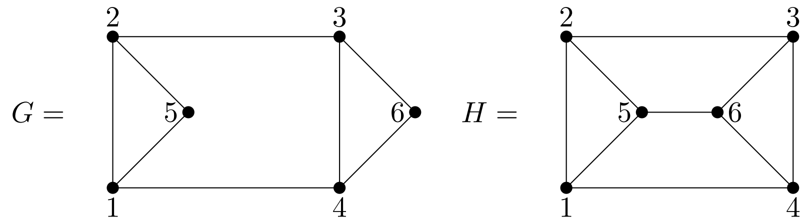
This graph is tripartite, with parts $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$. It's connected, as the cycles $(1,3,5,1)$ and $(2,4,6,2)$ cover every vertex, and it's possible to move between those two cycles via the edge 45.

It's pairwise perfectly matched thanks to the following perfect matchings: $\{13,24\}$, $\{35,46\}$, and $\{15,26\}$. $\boxed{2, \text{B}}$

There is no Hamiltonian cycle. If there was, we could start it at any vertex, so lets choose 1. There are only two edges with 1 as an endpoint, so we can choose which ever one we like, so lets use 13. There is only one edge not yet used with 3 as an endpoint, so we must now use edge 35. At 5, if we use edge 15 then the cycle has finished without using every vertex, so will not be Hamiltonian.

Therefore this Hamiltonian cycle must start $1,3,5,4$. None of the remaining vertices share an edge with 1, so there is no way to complete this cycle so the result is Hamiltonian. Therefore there is no Hamiltonian cycle. $\boxed{3, \text{B}}$

3. (a) (i) Let G and H be as follows: unseen $\Downarrow$



The labelling makes it clear that $G \subset H$, but there's no way of drawing an edge from 5 to 6 in the drawing of G. 2, A

(ii) Suppose H is a planar graph, and draw it. If $G \subseteq H$ then G can be constructed from H by deleting edges and vertices. By copying those deletions on the drawing of H, you arrive at a drawing of G. Reversing the deletions extends that drawing of G to a drawing of H. 2, A

(iii) Let X and Y be the connected components of G that contain x and y respectively. From the lecture notes, we know that there is a drawing of X such that x lies on the outer face, and there is a drawing of Y such that y lies of the outer face. Since both X and Y are finite, these drawings are bounded, and hence it is possible to draw them both at the same time with no edges overlapping. This drawing of G has both x and y on the same face, so it is possible to draw a line from x to y without crossing any edges. Therefore this drawing extends from G to G'. meth seen $\Downarrow$ 2, A

(b) Let $G \in \mathcal{G}(n, p)$. The probability that a given 5 vertex subset of G is isomorphic to K_5 is $p^{\binom{5}{2}} = p^{10}$. There are $\binom{n}{5}$ many 5 element subsets of G, so meth seen $\Downarrow$

$$\mathbb{P}[K_5 \subseteq G] = 1 - (1 - p^{10})^{\binom{n}{5}}.$$

The limit of this as $n \to \infty$ is 1.

Wagner's Theorem states that if K_5 or $K_{3,3}$ is a minor of G then G is not planar. If K_5 is a subgraph of G, then certainly K_5 is a minor, so if $K_5 \subseteq G$ then G is not planar, so $\mathbb{P}[K_5 \subseteq G] \leq \mathbb{P}[G \text{ is not planar.}]$.

Therefore $\mathbb{P}[G \text{ is planar.}] \to 0$ as $n \to \infty$. 6, A

(c) (i) Let G be a graph such that $|G| > lR(n, m)$. If G contains a vertex v with degree greater than or equal to $R(n, m)$ then consider unseen $\Downarrow$

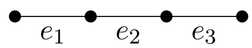
$$N(v) = \{u \in V_G : uEv\}.$$

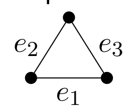
as an induced subgraph of G. By assumption, $|N(v)| \geq R(n, m)$, so either $K_n \subseteq N(v)$ or $\overline{K_m} \subseteq N(v)$.
If $K_n \subseteq N(v)$ then $K_n \cup \{v\} \cong K_{n+1}$.
If $\overline{K_m} \subseteq N(v)$ then $\overline{K_m} \cup \{v\} \cong K_{1,m}$.
Suppose that every vertex of G has degree less than $R(n, m)$. If we pick one vertex $v_1 \in V_G$ then there are at most $R(n, m)$ many vertices which share an edge with v_1, so there are at least $(l - 1)R(n, m)$ many vertices with no edge to v_1. Pick one of them, and label it v_2. Repeating this procedure results in $v_1, \ldots, v_l$, none of which have an edge between them. 4, C

(ii) Let G be a graph such that $|G| > mn$. If there is an $X \leq G$ such that $X \cong \overline{K_n}$ then $\overline{G}$ has K_n as a subgraph, so suppose that there is no such X.

Since $|G| > mn$ there are m many disjoint subsets of G of size n, called $A_1, \ldots, A_n$. By the last argument, each of the A_i must each contain at least 1 edge. Let x_i and y_i be the endpoints of one of the edges of A_i. Then, if we disregard any extraneous edges, $\{x_i, y_i : 1 \leq i \leq m\} \cong \oplus_{i=1}^{m} K_2$. $\boxed{4, \text{D}}$

4. (a) (i) Let G be a graph such that $|G| = n$ and $K_2 \oplus K_2 \not\subseteq G$. If e_1 and e_2 are edges from G then they must share an endpoint. If there are three edges e_1, e_2, and e_3 such that the endpoints that e_1 and e_3 share with e_2 are different, then the endpoints of e_1, e_2 and e_3 are either arranged as a path or a triangle. unseen ⇓





In the path case, e_1 and e_3 form a subgraph isomorphic to $K_2 \oplus K_2$. If we add a fourth edge to the triangle, then that fourth edge must not share an endpoint with one of e_1, e_2, or e_3, resulting in a subgraph isomorphic to $K_2 \oplus K_2$. Therefore if $\| G \| \geq 4$ then there must be a single vertex v such that every edge has v as an endpoint. There are $n-1$ vertices from G remaining, so if $K_2 \oplus K_2 \not\subseteq G$ then $\| G \| \leq n-1$.
These arguments can be put together to show that

$$\mathrm{ex}(n, K_2 \oplus K_2) = \begin{cases} 0 & n = 1 \\ 1 & n = 2 \\ 3 & n = 3 \\ n-1 & n \geq 4 \end{cases}$$

4, C

(ii) Suppose that K is a graph with $|K| > |H|$. If $||K|| \geq \mathrm{ex}(|K|, H)$ then there is a subgraph of K isomorphic to H. Since $G \subseteq H$, that means there is also a subgraph of K isomorphic to K. Therefore if $n \geq |H|$ then $\mathrm{ex}(n, G) \leq \mathrm{ex}(n, H)$. 2, A

(b) (i) Let G be a graph such that $|G| = n+1$ and $||G|| = \mathrm{ex}(n+1, H) - 1$. unseen ⇓

$$\begin{aligned} c(n+1, H) &= \tfrac{2\mathrm{ex}(n+1,H)}{(n+1)n} \\ &= \tfrac{2(||G||+1)}{(n+1)n} \\ &= \tfrac{2||G||}{(n+1)n} + \tfrac{2}{(n+1)n} \end{aligned}$$

4, D

There is a vertex $v \in G$ such that $\deg(v) \leq \mathrm{AvDeg}(G)$. Deleting v removes n many possible edges from G, but the proportion of those n possible edges which are actual edges is less than $\frac{\mathrm{AvDeg}(G)}{n}$. Therefore 2, D

$$\frac{||G||}{\binom{|G|}{2}} \leq \frac{||G \setminus \{v\}||}{\binom{|G|-1}{2}}$$

and hence

$$\frac{2||G||}{(n+1)n} \leq \frac{2||G \setminus \{v\}||}{n(n-1)}$$

Combining this with the fact that $\frac{2}{(n+1)n} < \frac{2}{n(n-1)}$ gives

$$\frac{2(||G||+1)}{(n+1)n} \leq \frac{2(||G \setminus \{v\}||+1)}{n(n-1)}$$

Since $||G|| = \mathrm{ex}(n+1, H) - 1$, $H \not\subseteq G$, and hence H is not a subgraph of any subgraph of G. This means that $||G \setminus \{v\}|| < \mathrm{ex}(n, H)$. 3, D

$$\begin{aligned} \tfrac{2(||G\setminus\{v\}||+1)}{n(n-1)} &\leq \tfrac{2\mathrm{ex}(n,H)}{n(n-1)} \\ &= c(n, H) \end{aligned}$$

Therefore $c(n+1, H) \leq c(n, H)$ for all H and for all n, so iterating this argument will show that if $m < n$ then $c(n, H) \leq c(m, H)$. $\boxed{3, \text{D}}$

(ii) $c(n, H) \geq 0$ for all n, and therefore the sequence $(c(n, H))_{n=1}^{\infty}$ is a bounded below, decreasing sequence of rational numbers, and hence is convergent. $\boxed{2, \text{A}}$

5. (a) Let $\epsilon \in (0, \frac{d}{2})$ and let (X, Y) be an ϵ-regular pair such that $d(X, Y) \geq d$. Suppose $|X| = |Y| = n$, and let $A = \{x \in X \,:\, ||\{x\}, Y|| < \frac{dn}{2}\}$. $||\{x\}, Y|| < \frac{dn}{2}$ for all $x \in A$, so $||A, Y|| < \frac{dn}{2}|A|$. Dividing both sides of this inequality by $|A||Y|$ gives $d(A, Y) < \frac{d}{2}$. We've assumed that $d(X, Y) > d$, so $d(A, Y) < d(X, Y)$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{3, \text{M}}$

We can get a lower bound for $|d(X, Y) - d(A, Y)|$ by using a lower bound for $d(X, Y)$ and an upper bound for $d(A, Y)$.

$$|d(X, Y) - d(A, Y)| \geq \left| d - \frac{d}{2} \right| = \frac{d}{2} > \epsilon$$

If there are more than $n\epsilon$ many vertices in X with less than $\frac{dn}{2}$ neighbours in Y then $|A| > \epsilon|X|$, so the ϵ-regularity of (X, Y) would imply that $|d(X, Y) - d(A, Y)| < \epsilon$. $\boxed{2, \text{M}}$
This gives a contradiction, so there must be less than $n\epsilon$ many such vertices. $\boxed{1, \text{M}}$

(b) Let $d > \frac{1}{2}$ and let $\epsilon \in (0, \frac{d}{4})$, and let X,Y, and Z be pairwise disjoint, pairwise ϵ-regular subsets of V_G such that $|X| = |Y| = |Z| = n$. Suppose that $d(X, Y), d(Y, Z), d(X, Z) > d$. Applying part (a) to the pairs (X, Y) and (X, Z) shows that there are at most ϵn many vertices in X with less than $\frac{dn}{2}$ neighbours in Y, and at most ϵn many vertices in X with less than $\frac{dn}{2}$ neighbours in Z. $\boxed{\text{unseen} \Downarrow}$ $\boxed{6, \text{M}}$

Therefore there are at least $1 - 2\epsilon$ many vertices of X with more than $\frac{dn}{2}$ neighbours in Y and more than $\frac{dn}{2}$ neighbours in Z. Call the set of such vertices A. Let $a \in A$. Let $N_Y(a) = \{y \in Y \,:\, yEa\}$. We chose a so that each vertex at least $\frac{dn}{2}$ many neighbours in Y, therefore $|N_Y(a)| > \frac{dn}{2}$. However, $\epsilon < \frac{d}{4}$, therefore $|N_Y(a)| > 2\epsilon n$. We define $N_Z(a)$ similarly, and discover that $|N_Z(a)| > 2\epsilon n$.

(Y, Z) is ϵ-uniform, so $|d(Y, Z) - d(N_Y(a), N_Z(a))| < \epsilon$. We assumed that $d(Y, Z) > d$ therefore

$$||N_Y(a), N_Z(a)|| > (d - \epsilon)(2\epsilon n)^2 > 0$$

$\boxed{7, \text{M}}$

Therefore there is at least one edge with one endpoint in $N_Y(a)$ and one in $N_Z(a)$. Let these endpoints be b and c respectively. Then aEb, bEc and aEc. Therefore $\{a, b, c\} \cong K_3$. $\boxed{1, \text{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH96066 MATH97225 MATH97226	1	Since the vertices had to be labelled with natural numbers, you could not label vertices with -1 in 1(a)(iii) While it is possible to create an entirely new graph for 1(b)(ii), using either (b)(i) or (c)(i) is simpler. If take infinitely many copies of the graph from 1(b)(i), and connect them together in the manner of the graph from 1(a)(iii) at a degree two vertex, you end up with this degree sequence.
MATH96066 MATH97225 MATH97226	2	The intended solution to 2(b)(i) was a proof by contradiction. If there are fewer neighbours of X than elements of X, then the degree of one of the neighbours must be greater than r. Kőnig's Theorem was the intended solution for 2(b)(ii), but Hall's Theorem, or a direct argument, were all possible.
MATH96066 MATH97225 MATH97226	3	(c) was challenging. The key to both parts was to decompose the graphs into parts of equal size. In (c)(i), if you have a graph with the required size, then you have l many pairwise disjoint subgraphs with R(n,m) many vertices. Considering those subgraphs connect with each other give the required subgraphs. In (c)(ii), you have m many pairwise disjoint subgraphs with n many vertices. If every one of those subgraphs contains an edge is one case, there being one with no edges was the second.

MATH96066 MATH97225 MATH97226	4	There is a more elegant solution to 4(b)(ii) than the one in the solutions, which is done by taking an extremal graph with n vertices and considering the m-vertex subgraphs.
MATH96066 MATH97225 MATH97226	5	5(a). If there are more than a small proportion of vertices of X with a small degree, then regularity means that the density between X and Y must also be small. 5(b). Since there is only a small proportion of vertices in X with small degree in the pairs (X,Y) and (X,Z), that means that there is a vertex v with large degree in X to both Y and Z. There are a lot of neighbours of v in Y and Z, enough to apply regularity to it. This lets us find an edge which completes the triangle.