$$\min f(\underline{x}) \,, \quad \underline{x} \in \Delta_m = \left\{ \underline{x} \in \mathbb{R}^m : \sum_{i=1}^{m}{}' x_i = 1, \quad x_i \geq 0 \right\}$$

We need to show that stationarity is equiv. to

$$(*) \quad \frac{\partial f}{\partial x_j}(\underline{x}^*) = \begin{cases} = \mu \,, & x_i^* > 0 \\ \geq \mu \,, & x_i^* = 0 \end{cases} \,, \text{ for some } \mu \in \mathbb{R}.$$

1) $(*) \overset{?}{\Longrightarrow}$ Stationarity

We assume $\exists\ \underline{x}^* \in \Delta_m$ and $\mu$ such that $(*)$ holds.

We want to show stationarity, that is

$$\nabla f(\underline{x}^*)^T (\underline{x} - \underline{x}^*) \geq 0 \quad \forall x \in \Delta_m .$$

Because of $(*)$, $\nabla f(\underline{x}^*)$ is a vector $\sum\limits_{j,\, x_j^* > 0}' \mu \underline{e}_j + \sum\limits_{j,\, x_j^* = 0}' (\mu + \delta_j) \underline{e}_j$ with $\delta_j \geq 0$.

canonical vector $\begin{array}{r} \\ \\ \\ j\text{-coord.} \\ \\ \end{array} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$

Now, $\nabla f(\underline{x}^*)^T \underline{x} = \sum\limits_{j,\, x_j^* > 0}' \mu x_j + \sum\limits_{j,\, x_j^* = 0}' (\mu + \delta_j) x_j \geq \sum\limits_{j=1}^{m}{}' \mu \cdot x_j \underset{x \in \Delta_m}{=} \mu \sum\limits_{j=1}^{m} x_j = \mu.$

Then, $\nabla f(\underline{x}^*)^T (\underline{x} - \underline{x}^*) \geq \mu - \underbrace{\nabla f(\underline{x}^*)^T (\underline{x}^*)}_{\shortparallel}$

$$\mu - \mu \cdot \sum_{j,\, x_j^* > 0}' x_j^* \underset{\underline{x}^* \in \Delta_m}{=} \mu - \mu = 0 .$$

2) Stationarity $\Longrightarrow (*)$

Assume $\underline{x}^*$ stationary and pick $i \neq j$ such that $x_i^* > 0$ and $x_j^* > 0$. For sufficiently small $\delta$, take

$$\left.\begin{aligned} \underline{x}^+ &= \underline{x}^* + \delta \underline{e}_i - \delta \underline{e}_j \\ \underline{x}^- &= \underline{x}^* - \delta \underline{e}_i + \delta \underline{e}_j \end{aligned}\right\} \ \underline{x}^+ \text{ and } \underline{x}^- \in \Delta_m$$

Using the def. of stationarity with $\underline{x}^+$ and $\underline{x}^-$

$$\nabla f(\underline{x}^*)^T(\underline{x}^+ - \underline{x}^*) \geq 0$$

$$\frac{\partial f}{\partial x_i}(\underline{x}^*) \geq \frac{\partial f}{\partial x_j}(\underline{x}^*)$$

and

$$\nabla f(\underline{x}^*)^T(\underline{x}^- - \underline{x}^*) \geq 0$$

$$\frac{\partial f}{\partial x_j}(\underline{x}^*) \geq \frac{\partial f}{\partial x_i}(\underline{x}^*)$$

Implying that

$$\frac{\partial f}{\partial x_i}(\underline{x}^*) = \frac{\partial f}{\partial x_j}(\underline{x}^*)$$

for all positive coordinates

Now, we assume that $x_i^* > 0$ and $x_j^* = 0$.

The stationarity condition with $\underline{x}^-$ reads

$$\nabla f(\underline{x}^*)^T(\underline{x}^- - \underline{x}^*) = \nabla f(\underline{x}^*)^T(-\delta \underline{e}_i + \delta \underline{e}_j) \geq 0$$

$$= \delta\left(\frac{\partial f}{\partial x_j}(\underline{x}^*) - \frac{\partial f}{\partial x_i}(\underline{x}^*)\right) \geq 0$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x_j}(\underline{x}^*) \geq \frac{\partial f}{\partial x_i}(\underline{x}^*), \text{ whenever } x_j^* = 0.$$

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