

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Classical Dynamics

Date: Thursday, May 30, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

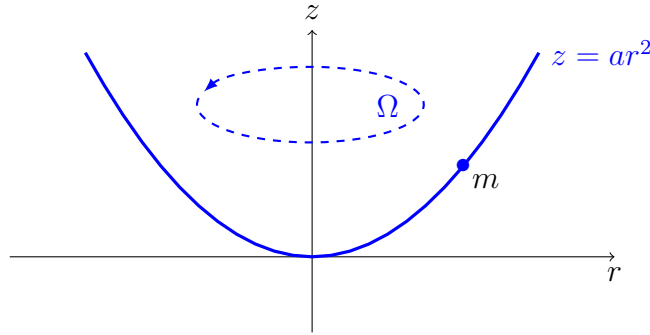
Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

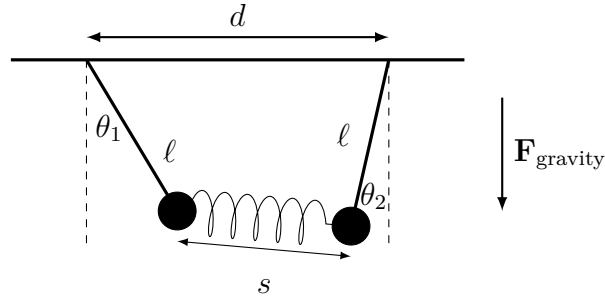
1. **Lagrangian Mechanics I** A point particle of mass m slides without friction along a wire that is bent upwards in the shape of a parabola. The wire is rotated at its minimum around the z -axis at constant angular velocity Ω (see sketch). Gravity acts in direction of $-\hat{e}_z$. We analyse the problem in cylindrical coordinates (r, φ, z) . The wire's shape is then given by $z = ar^2$, where a is an inverse length indicating the curvature of the wire.



- Formulate the two constraint functions ϕ_1, ϕ_2 ensuring the bead's rotation and attachment to the wire, and specify whether they are rheonomic or scleronomic. (4 marks)
- Explain whether or not the choice of ϕ_1, ϕ_2 is unique. (2 marks)
- State the Equations of Motion in r, φ, z using Lagrange multipliers.
(In a cylindrical coordinate system, the acceleration is given by $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$ and the gradient by $\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}$.) (6 marks)
- Find the equilibrium point(s) in dependence of Ω . The stability depends on whether the rotation frequency Ω is below or above some critical Ω_c . Find Ω_c and physically describe the three regimes $\Omega^2 < \Omega_c^2$, $\Omega^2 > \Omega_c^2$, and $\Omega^2 = \Omega_c^2$. (8 marks)

(Total: 20 marks)

2. Two identical pendulums of equal mass and length m, ℓ are suspended at equal height and horizontal distance d . The two masses are further connected by a spring of spring strength k . The distance between the two masses, and hence the elongation of the spring, is denoted by s . The spring's rest length, i.e. its elongation when no forces act upon it, is also d . The system's state is described by the two angle coordinates θ_1 and θ_2 which denote the angular displacement of the two pendulums with respect to the direction of gravity (see sketch).



- (a) State the Lagrangian $L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$ of the system. To keep the expression simple, you should use s without expressing it in terms of θ_1, θ_2 . (*Remember the spring's rest length.*) (5 marks)
- (b) We now assume that the angles are very small, $\theta_1, \theta_2 \approx 0$. To leading order in small angles the *linearised* Lagrangian is then given by

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{m\ell^2}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mg\ell}{2} (\theta_1^2 + \theta_2^2) - \frac{k\ell^2}{2} (\theta_1 - \theta_2)^2.$$

- (i) Derive the Equations of Motion from the given Lagrangian. (3 marks)
- (ii) State a conserved quantity and explain your choice. (2 marks)
- (iii) The EoM found in (i) can be written in a matrix form, i.e.

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \mathbf{M} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \quad (1)$$

The matrix's two eigenvectors are $v_1 = (1, 1)^T$ and $v_2 = (1, -1)^T$. Provide two different general oscillating solutions ($\theta_j = A_j e^{i\omega t}$ for $j = 1, 2$) stating their respective frequencies. Assume in both cases the maximal amplitude of either mass to be θ_0 .

Hint: It may be helpful to introduce $\omega_0^2 = g/\ell$ and $\omega_1^2 = k/m$. (6 marks)

- (iv) Now imagine the right mass would be externally forced to follow a small oscillation $\theta_2 = \epsilon \cos(\Omega t)$. State the linearised equation of motion for θ_1 . For which choice of Ω do you expect the left mass to swing the strongest (resonance)? (4 marks)

(Total: 20 marks)

3. We consider the Kepler problem of two masses m_1 and m_2 at positions $\mathbf{r}_1, \mathbf{r}_2$ which are interacting via gravity. Define $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The trajectory of \mathbf{r} then takes place in a plane where it is characterised by $r = |\mathbf{r}|$ and ϕ . One can show that ϕ depends on r as

$$\phi(r) = \arccos\left(\frac{\ell/r - \frac{\mu\alpha}{\ell}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{\ell^2}}}\right) + \phi_0,$$

where $\alpha = Gm_1m_2$.

- (a) (i) The two terms E and ℓ featured in the formula are constant. What physical quantities do they define? (2 marks)
- (ii) State a third physical quantity that is conserved in the Kepler problem. (1 mark)
- (iii) What is the torque \mathbf{K} in this system? Explain your result. (2 marks)
- (b) Define the parameter and excentricity

$$p = \frac{\ell^2}{\mu\alpha} \quad \epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu\alpha^2}}.$$

- (i) Setting $\phi_0 = 0$ (and thus fixing the coordinate system), show that the trajectory in ϕ and r is alternatively given by (3 marks)

$$\frac{p}{r} = 1 + \epsilon \cos \phi.$$

- (ii) State and sketch all possible geometrical shapes of the trajectory in dependence of ϵ . (You do not need to calculate any trajectory.) (4 marks)
- (c) (i) Sketch and label a planetary orbit, the position and linear momentum of the planet relative to the Sun, and two conserved vectorial quantities. (5 marks)
- (ii) Using your sketch, state and explain Kepler's three laws of planetary motion. (3 marks)

(Total: 20 marks)

4. (a) (i) Given a Lagrangian $L(q, \dot{q}, t)$, what is the associated Hamiltonian $H(q, p, t)$? What is this mathematical transform called? (2 marks)
- (ii) A particle in a homogeneous gravity field is described by the Lagrangian

$$L(z, \dot{z}, t) = \frac{m}{2} \dot{z}^2 - mgz.$$

Write down the corresponding Hamiltonian $H(z, p)$ of the particle, and state its Equations of Motion. (3 marks).

- (b) (i) State the Hamilton Jacobi equation for the free-falling particle using Hamilton's principal function $S(z, t)$. (2 marks)
- (ii) Why can we use a separation ansatz $S(z, t) = W(z) - \alpha t$ for this Hamiltonian? (1 mark)
- (iii) Show that a solution to the Hamilton's characteristic function is given by

$$W(z) = \int_{z_0}^z dz' \sqrt{2m(\alpha - mgz')}.$$

(3 marks)

- (iv) How do you interpret α physically? (1 mark)
- (v) Define a new variable $\beta = \frac{\partial S}{\partial \alpha}$ and show that the trajectory is given by

$$z(t) = z_0 + \sqrt{\frac{2(\alpha - mgz_0)}{m}}(\beta + t) - \frac{g}{2}(\beta + t)^2.$$

(4 marks)

- (vi) Since $mgz_0 < \alpha$, the prefactor of the linear term in t (velocity) is always positive. Since the particle could also start with a negative initial velocity, there seems to be a mistake here. Explain where this apparent contradiction emerges within the Hamilton-Jacobi framework, and how it can be resolved. (4 marks)

(Total: 20 marks)

5. We consider a *anharmonic oscillator*. This describes the motion of a particle in a potential of the shape

$$U(x) = \frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4 \quad (k, \lambda > 0).$$

- (a) (i) Write down the Equation of Motion of a particle of mass m using the rescaled quantities $\omega^2 = k/m$ as well as $\epsilon = \lambda/m$. (2 marks)
- (ii) For some fixed total energy $E > 0$, calculate the two turning points $\pm X$ at which the particle has zero velocity. (3 marks)

- (b) Assume $\epsilon \ll \omega^2$. We solve the EoM perturbatively assuming

$$x(t) = x_0(t) + \epsilon x_1(t), \quad (2)$$

neglecting higher-order terms in ϵ .

- (i) Set $\epsilon = 0$. State the harmonic solution $x_0(t)$ to the EoM with initial conditions $x(0) = 1, \dot{x}(0) = 0$. (2 marks)
- (ii) Inserting the perturbative ansatz (Eq. (2)) into the Equation of Motion, and ignoring higher order terms, show that $x_1(t)$ needs to satisfy

$$\ddot{x}_1(t) + \omega^2 x_1(t) = -\cos^3(\omega t). \quad (3)$$

(5 marks)

- (iii) What physical system does this EoM correspond to? (1 mark)
- (iv) The general solution to the ODE in Eq. (3) is given by

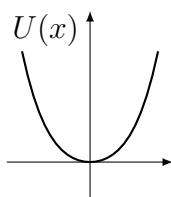
$$x_1(t) = A \cos(\omega t) + \left(B - \frac{3}{8\omega}t\right) \sin(\omega t) + \frac{1}{32\omega^2} \cos(3\omega t).$$

For which choice of A and B is $x_0 + \epsilon x_1(t)$ a valid solution to the initial condition $x(0) = 1, \dot{x}(0) = 0$? State the full perturbative solution to first order in ϵ . (3 marks)

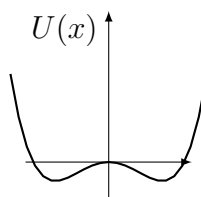
- (v) Is this solution physically feasible at all times? *Hint: See 5 (a) (ii)* (1 mark)
- (c) For $k < 0, \lambda > 0$ the particle shows a qualitatively different behaviour (see sketch). Explain how the perturbative scheme, and in particular $x_0(t)$, would need to be adapted in order to describe a small oscillation in this potential. (3 marks)

(Total: 20 marks)

$k > 0, \lambda > 0$



$k < 0, \lambda > 0$



Solutions to Exam I

1. Particle on parabolic wire

(a) The two constraints are given by

$$\phi_1(r, \varphi, z) = \varphi - \Omega t \stackrel{!}{=} 0 \quad (1)$$

which encodes the rotation of the wire with constant angular frequency, and

$$\phi_2(r, \varphi, z) = z - ar^2 \stackrel{!}{=} 0 \quad (2)$$

which encodes the constraint of the wire. The constraint function is not unique so any other commensurate formulation is fine.

The first constraint is *rheonomic* as it explicitly depends on time. The second constraint is *scleronomic* as it does not explicitly depend on time.

[2 marks for each constraint, 2 marks for correct classification. This is a category A question (and a slight variation of a problem covered in the lecture notes).]

(b) The choice is not unique. Any function that vanishes when the constraint is met is valid. The only physically unique combination is $\lambda \nabla \phi$, the constraint force.

(c) The EoM follow from

$$m\ddot{\mathbf{r}} = \mathbf{F} + \lambda_1(t)\nabla\phi_1 + \lambda_2(t)\nabla\phi_2. \quad (3)$$

The only external force acting on the mass is gravity

$$\mathbf{F} = -mg\hat{\mathbf{e}}_z$$

The first constraint then follows from the gradient (provided as a hint)

$$\lambda_1(t)\nabla\phi_1 = -\lambda_1\frac{1}{r}\Omega\hat{\mathbf{e}}_\varphi \quad (4)$$

The second constraint is

$$\lambda_2(t)\nabla\phi_2 = \lambda_2(t)(\hat{\mathbf{e}}_z - 2ar\hat{\mathbf{e}}_r). \quad (5)$$

Putting things together in Eq. (3) (the acceleration is provided as a hint), and writing down the individual three coordinates, one obtains three EoM

$$m(\ddot{r} - r\dot{\phi}^2) = -2\lambda_2(t)ar \quad (6)$$

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{1}{r}\lambda_1(t) \quad (7)$$

$$m\ddot{z} = \lambda_2(t) - mg \quad (8)$$

[1 point for the gravity force, 2 points for each constraint force (i.e. correct right hand side), 3 points for stating the EoM (i.e. correct left hand side).]

- (d) We begin by considering the equilibrium points at which \ddot{r}, \ddot{z} vanish. After substituting $\dot{\phi} = \Omega$, Eqs. (6), (8) give

$$m\ddot{r} = -2ar\lambda_2(t) + mr\Omega^2 \stackrel{!}{=} 0 \quad (9)$$

$$m\ddot{z} = \lambda_2(t) - mg \stackrel{!}{=} 0 \quad (10)$$

From this readily follows that at equilibrium,

$$\ddot{r} = -2arg + r\Omega^2 = (\Omega^2 - 2ag)r = 0.$$

For general Ω , the only solution to this linear equation is $r = 0$.

One reads off that for $\Omega_c^2 = 2ag$, *any* r is an equilibrium point; In this special case, the centrifugal forces and gravity cancel each other out at any point.

For $\Omega^2 < \Omega_c^2$, the rotation is slow and the equilibrium point at $r = 0$ is stable. Gravity dominates and the mass is drawn inwards as $r \rightarrow 0$. For $\Omega^2 > \Omega_c^2$, the rotation is fast and centrifugal forces push the mass outwards to $r \rightarrow \infty$.

[4 points for setting EoM to zero and finding Equilibrium point. 1 point for recognising $\Omega_c = 2ag$, 1 point each for every regime explained.]

2. (a) The kinetic energy is given by

$$T = \frac{m}{2}(\ell\dot{\theta}_1)^2 + \frac{m}{2}(\ell\dot{\theta}_2)^2 = \frac{m\ell^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2).$$

The potential energy due to gravity is (up to arbitrary constant)

$$U_{\text{grav}} = -mg\ell(\cos\theta_1 + \cos\theta_2).$$

The potential energy stored in the spring is

$$U_{\text{spring}} = \frac{k}{2} (s - d)^2,$$

i.e., it is minimal for $s = d$. The Lagrangian is given by $L = T - U_{\text{grav}} - U_{\text{spring}}$,

$$L = \frac{m\ell^2}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2) + mg\ell (\cos \theta_1 + \cos \theta_2) - \frac{k}{2} (s - d)^2$$

[1 mark each for kinetic energy and gravity potential energy, 2 marks for spring energy. 1 mark for combining into Lagrangian.]

(b) A conserved quantity is the total energy

$$E = \frac{m\ell^2}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg\ell (\cos \theta_1 + \cos \theta_2) + \frac{k}{2} (s - d)^2$$

This is a consequence of the time-independence of the Lagrangian and Noether's theorem.

[1 mark for stating, 1 mark for defining the energy.]

(c) i. Applying Euler-Lagrange Equations,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_j} = \frac{\partial L}{\partial \theta_j} \quad j = 1, 2,$$

and dividing L by $m\ell^2$ (with no effect on the EoM), one finds

$$\begin{aligned} \ddot{\theta}_1 &= -\frac{g}{\ell} \theta_1 - \frac{k}{m} (\theta_1 - \theta_2) \\ \ddot{\theta}_2 &= -\frac{g}{\ell} \theta_2 + \frac{k}{m} (\theta_1 - \theta_2) \end{aligned}$$

[1 point for Euler-Lagrange equation, 1 point each for EoM of θ_1 and θ_2 .]

ii. The EoM in matrix form read

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -g/\ell - k/m & k/m \\ k/m & -g/\ell - k/m \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Identifying $\omega_0^2 = g/\ell$, $\omega_1^2 = k/m$, one finds

$$\mathbf{M} = \begin{pmatrix} -\omega_0^2 - \omega_1^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 \end{pmatrix}$$

Here, ω_0 is the angular frequency with which the simple pendulum oscillates in absence of the spring; ω_1 is the angular frequency with which the spring oscillates in absence of the pendulums.

- iii. The eigenvectors are provided in the exam. We compute the associated eigenvalues

$$\mathbf{M} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{M} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -(\omega_0^2 + 2\omega_1^2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and hence

$$\lambda_1 = -\omega_0^2 \quad \lambda_2 = -(\omega_0^2 + 2\omega_1^2).$$

Assume as an ansatz, as given in the exam,

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}.$$

Inserting into the EoM in Matrix form gives

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} = \mathbf{M} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$$

For this solution to be valid, $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ needs to be an eigenvector and $-\omega^2$ an associated eigenvalue. The two such possibilities are given above. One finds therefore either

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \theta_0 \end{pmatrix} e^{i\omega t}$$

with angular frequency

$$\omega = \omega_0,$$

or

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \theta_0 \\ -\theta_0 \end{pmatrix} e^{i\omega t}$$

with angular frequency

$$\omega = \sqrt{\omega_0^2 + 2\omega_1^2},$$

The first solution describes synchronous oscillation in phase of the two masses. This occurs with the natural frequency of the pendulums. The second solution describes the two masses swinging inwards and outwards, in opposite phase. This occurs with a higher

frequency that depends on the natural frequency of the pendulum and the spring. The two solutions are called the “modes” of the system.

[2 marks for computing eigenvalues. 2 marks for inserting ansatz and recognising frequencies. 1 mark each for stating solution with θ_0 and correct frequency.]

iv. We replace $\theta_2 = \epsilon \cos(\Omega t)$ in the EoM for θ_1 to obtain

$$\ddot{\theta}_1 = -\left(\frac{g}{\ell} + \frac{k}{m}\right)\theta_1 + \frac{k\epsilon}{m}\cos(\Omega t).$$

This is a driven harmonic oscillator with eigenfrequency

$$\omega^2 = \frac{g}{\ell} + \frac{k}{m} = \omega_0^2 + \omega_1^2.$$

Resonance occurs at

$$\Omega = \sqrt{\omega_0^2 + \omega_1^2}.$$

3. (a) i. E defines the total energy of the system, ℓ defines the magnitude of the angular momentum. *[1 mark for each stated result]*
- ii. The centre of mass or the Laplace-Runge-Lenz vector are two other (not independent) conserved quantities. *[1 mark for any correct quantity]*
- iii. Since angular momentum is conserved, the torque vanishes. This is because $\mathbf{K} = \dot{\mathbf{L}} = 0$.
[1 mark for stating torque is zero, 1 mark for correct explanation.]
- (b) i. Set $\phi_0 = 0$. Invert $\arccos(x)$ to obtain

$$\cos(\phi) = \frac{\ell/r - \frac{\mu\alpha}{\ell}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{\ell^2}}}$$

Then reduce

$$\frac{\ell/r - \frac{\mu\alpha}{\ell}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{\ell^2}}} = \frac{\ell/r - \frac{\mu\alpha}{\ell}}{\frac{\mu\alpha}{\ell}\sqrt{\frac{2\ell E}{\mu\alpha^2} + 1}} = \frac{1}{\sqrt{\frac{2\ell E}{\mu\alpha^2} + 1}} \left(\frac{\ell^2}{\mu\alpha r} - 1 \right)$$

Using the constants as given in the exam, one then identifies

$$\cos \phi = \frac{1}{\epsilon} \left(\frac{p}{r} - 1 \right)$$

from which follows the desired result

$$\frac{p}{r} = 1 + \epsilon \cos \phi.$$

[1 point for inverting formula, 1 point for substituting p and ϵ , 1 point for correctly reproducing result.]

- ii. Depending on ϵ (and hence E), the geometric shape is
- Circular, if $\epsilon = 0$, as then $p/r = 1$ and hence $r = p$ constant.
 - Elliptic, if $0 < \epsilon < 1$.
 - Parabolic, if $\epsilon = 1$.
 - Hyperbolic, if $\epsilon > 1$.

[1 mark each]

- (c) Sketch: 1 point each for correctly drawing: An ellipse with the sun in a focus; a planet on the ellipse; linear momentum tangentially to the trajectory; angular momentum perpendicular to the drawing plane and commensurate with the linear momentum; the Laplace Runge Lenz vector pointing towards $\phi = 0$.

- (d) 1 point each for

- The planet moves on an ellipse with the sun in a focus.
- The swept area per unit time is constant
- The period T of a planet grows with the semi-major axis a as $T^2 \sim a^3$.

4. (a) i. The associated Hamiltonian is $H(q, p, t) = \dot{q}p - L(q, \dot{q}, t)$ where \dot{q} is eliminated using $p = \frac{\partial L}{\partial \dot{q}}$. Equivalently, one may define

$$H(q, p, t) = \sup_{\dot{q}} [\dot{q}p - L(q, \dot{q}, t)].$$

This transform is the Legendre transform. *[1 mark for stating the definition, 1 mark for naming Legendre transform]*

- ii. The corresponding Hamiltonian is

$$H(z, p) = \frac{p^2}{2m} + mgz$$

The EoM are

$$\dot{z} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \dot{p} = -\frac{\partial H}{\partial z} = -mg.$$

[1 mark for Hamiltonian, 1 mark each for a EoM]

- (b) i. The HJE are

$$\frac{1}{2m} \left(\frac{\partial S}{\partial z} \right)^2 + mgz = -\frac{\partial S}{\partial t}.$$

[2 marks if correct, 1 mark if Hamiltonian in 4 (a)(ii) was wrong, but correctly inserted $S(z, t)$.]

- ii. Since the Hamiltonian is not explicitly time-dependent, we may use a separation ansatz. *[1 mark]*
 iii. Inserting the separated ansatz, one finds

$$\frac{1}{2m} \left(\frac{dW}{dz} \right)^2 + mgz = \alpha$$

Algebraic manipulation gives

$$W' = \pm \sqrt{2m(\alpha - mgz)}$$

We decide for the positive branch to obtain

$$W(z) = \int_{z_0}^z dz' \sqrt{2m(\alpha - mgz')}$$

[1 mark for inserting ansatz, 1 mark for resolving to square root, 1 mark for integrating.]

- iv. The constant α is the total (conserved) energy of the system (kinetic + potential). *[1 mark]*
 v. Defining

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{1}{2} \int_{z_0}^z dz' \frac{\sqrt{2m}}{\sqrt{\alpha - mgz'}} - t = \sqrt{\frac{1}{2g}} \int_{z_0}^z dz' \frac{1}{\sqrt{\frac{\alpha}{mg} - z'}} - t$$

Integrating, one finds

$$\beta = -\sqrt{\frac{2}{g}} \sqrt{\frac{\alpha}{mg} - z'} \Big|_{z_0}^z - t.$$

From there follows

$$\beta + t = \frac{2}{g} \left[\sqrt{\frac{\alpha}{mg} - z_0} - \sqrt{\frac{\alpha}{mg} - z(t)} \right],$$

such that

$$\sqrt{\frac{\alpha}{mg} - z(t)} = \sqrt{\frac{\alpha}{mg} - z_0} - \sqrt{\frac{g}{2}}(\beta + t)$$

Squaring both sides, and rearranging, one finds

$$z(t) = z_0 + \sqrt{\frac{2(\alpha - mgz_0)}{m}}(\beta + t) - \frac{g}{2}(\beta + t)^2$$

[1 mark for $\frac{\partial S}{\partial \alpha}$, 1 mark for integrating, 2 marks for evaluating and inverting expression.]

vi. In resolving

$$(W')^2 = 2m(\alpha - mgz)$$

we chose the positive branch of the square root. However, choosing the negative branch would lead to

$$W = -\sqrt{2m(\alpha - mgz)},$$

and hence

$$\beta = \sqrt{\frac{2}{g}} \left[\sqrt{\frac{\alpha}{mg} - z'} \right]_{z_0}^z - t$$

from where the cross term in the inversion would change sign.
[2 marks for recognising square root, 2 marks for showing how crossterm would be negative]

5. (a) i. One finds

$$m\ddot{x} = -\frac{dU}{dx} = -kx - \lambda x^3,$$

which after substitution gives

$$\ddot{x} = -\omega^2 x - \epsilon x^3.$$

[1 mark for EoM, 1 mark for substituting.]

ii. Equating

$$U(X) = E$$

one finds

$$\frac{k}{2}X^2 + \frac{\lambda}{4}X^4 = E$$

and hence ($z = X^2$)

$$z^2 + \frac{2k}{\lambda}z - \frac{4E}{\lambda} = 0$$

and

$$z = -\frac{k}{\lambda} + \sqrt{\left(\frac{k}{\lambda}\right)^2 + \frac{4E}{\lambda}} = \frac{k}{\lambda} \left(\sqrt{1 + \frac{4E\lambda}{k^2}} - 1 \right)$$

such that

$$X = \sqrt{\frac{k}{\lambda} \left(\sqrt{1 + \frac{4E\lambda}{k^2}} - 1 \right)}.$$

[1 mark for stating condition, 2 marks for solving quartic equation]

- (b) i. One states the standard result of the harmonic oscillator

$$x_0(t) = \cos(\omega t)$$

which satisfies the required initial conditions.

[1 mark]

- ii. The EoM are

$$\ddot{x} = -\omega^2 x - \epsilon x^3$$

Inserting the ansatz, one finds

$$(\ddot{x}_0 + \epsilon \ddot{x}_1) = -\omega^2 (x_0 + \epsilon x_1) - \epsilon (x_0 + \epsilon x_1)^3$$

On the one hand, this simplifies since $\ddot{x}_0 = -\omega^2 x_0$ by construction such that one is left with

$$\epsilon \ddot{x}_1 = -\omega^2 \epsilon x_1 - \epsilon (x_0 + \epsilon x_1)^3.$$

On the other hand, ignoring higher-order terms and dividing by ϵ , one is left with

$$\ddot{x}_1 + \omega^2 x_1 = -\cos(\omega t)^3$$

- iii. This corresponds to a harmonic oscillator of frequency ω that is driven by an external force $F = \cos^3(\omega t)$.
 iv. The perturbative correction x_1 needs to satisfy $x_1(0) = 0, \dot{x}_1(0) = 0$ for the whole ansatz to be compatible with the initial conditions. This means that

$$x_1(0) = A + \frac{1}{32\omega^2} \stackrel{!}{=} 0 \quad B = 0$$

and so

$$x(t) = \cos(\omega t) + \epsilon \left[\frac{1}{32\omega^2} (\cos(3\omega t) - \cos(\omega t)) - \frac{3}{8\omega} t \sin(\omega t) \right]$$

- v. The solution contains an unbounded term $t \sin(\omega t)$ which is unphysical as it eventually surpasses any finite energy barrier.

(c) For $k < 0, \lambda > 0$, the EoM for the particle is

$$m\ddot{x} = |k|x - \lambda x^3.$$

The potential is bistable with two minima at $x = \pm\sqrt{\frac{-k}{\lambda}}$. The second derivative of the potential is

$$U''(x) = k + 3\lambda x^2$$

and hence at the minima takes the value

$$U''(x) = k + 3|k| = 2|k|.$$

Hence, a small oscillation around either of the equilibrium points is described to lowest order by

$$x_0(t) = \pm\sqrt{\frac{-k}{\lambda}} + A \cos(2\omega t + \varphi_0)$$

where $\omega = k/m$ is an angular frequency and A, φ_0 some amplitude and phase to match initial conditions.

[The important points are 1 mark for recognising bistability, 1 mark for recognising shifted equilibrium points, 1 mark for doubled frequency.]