

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
May 2023

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Stochastic Calculus with Applications to non-Linear Filtering**

Date: 11 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

**This paper has 5 Questions.**

**Please Answer All Questions in 1 Answer Booklet**

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

For the following questions, assume the set-up: Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $(\mathcal{F}_t)_{t \geq 0}$  be a filtration in  $\mathcal{F}$  and  $V$  be a standard one-dimensional  $\mathcal{F}_t$ -adapted Brownian motion under  $\mathbb{P}$ . Let  $f$  and  $\sigma$  be bounded Lipschitz continuous real valued functions and let  $X$  be the  $\mathcal{F}_t$ -adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s) ds + \int_0^t \sigma(X_s) dV_s. \quad (1)$$

Assume that  $X_0$  has distribution  $\pi_0$  at time 0, is independent of  $V$  and  $\mathbb{E}[(X_0)^2] < \infty$ . Let  $W$  be a standard  $\mathcal{F}_t$ -adapted one-dimensional Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$  independent of  $V$  and  $X_0$ , and  $Y$  be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \quad (2)$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded measurable function. The process  $Y = \{Y_t, t \geq 0\}$  is called the observation process. Let  $\{\mathcal{Y}_t, t \geq 0\}$  be the filtration associated with the process  $Y$ , that is  $\mathcal{Y}_t = \sigma(Y_s, s \in [0, t])$ . The filtering problem consists in determining the conditional distribution  $\pi_t$  of the signal  $X_t$  given  $\mathcal{Y}_t$ . That is,  $\pi_t(A) = \mathbb{E}[I_A(X_t) | \mathcal{Y}_t]$  for any Borel set  $A \in \mathcal{B}(\mathbb{R})$  ( $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -field on  $\mathbb{R}$  and  $I_A$  is the indicator function of the set  $A$  and  $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t]$  for any bounded Borel measurable function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ .

1. (a) Give the definition of a standard one-dimensional Brownian motion. (2 marks)
- (b) Let  $\theta \in [0, 2\pi]$ . Prove that the process  $B = \{B_t, t \geq 0\}$  defined as

$$B_t = \sin(\theta)V_t + \cos(\theta)W_t$$

is a Brownian motion. (8 marks)

- (c) For integers  $p, n > 0$  consider the sums

$$S_n^p(t) = \sum_{i=0}^{n-1} (W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^p.$$

- (i) Prove that there exists a constant  $c(t)$  independent of  $n$  such that

$$\mathbb{E}[(S_n^2(t) - t)^2] \leq \frac{c(t)}{n}$$

(4 marks)

- (ii) Deduce from part (c)(i), that  $\lim_{n \rightarrow \infty} S_{n^2}^2(t) = t$ ,  $P$ -almost surely. (2 marks)
- (iii) Prove that  $\lim_{n \rightarrow \infty} S_{n^2}^4(t) = 0$ ,  $P$ -almost surely. (2 marks)
- (iv) What does the result at part (c)(ii) tell us about the finite variation of a typical Brownian path. (2 marks)

(Total: 20 marks)

2. (a) Give the definition of a d-dimensional semimartingale. (2 marks)
- (b) State Itô's formula as applied to semimartingales [no proof required]. (3 marks)
- (c) Let  $R = \{R_t, t \geq 0\}$  be a continuous process satisfying the following identity

$$R_t = \exp \left( \int_0^t \sin(R_s) dV_s - \frac{1}{2} \int_0^t (\sin(R_s))^2 ds \right), \quad t \geq 0.$$

- (i) Use Itô's formula to prove that  $R$  satisfies the evolution equation:

$$R_t = 1 + \int_0^t \sin(R_s) R_s dV_s. \quad (3)$$

- (ii) State the Novikov condition. (2 marks)
- (iii) Using Novikov's condition prove that the process  $R$  is a martingale. (3 marks)
- (iv) Prove that  $\mathbb{E}[R_t^2] \leq e^t$ . (4 marks)

(Total: 20 marks)

3. Let  $a > 0$ . Assume that the signal  $X$  satisfies the equation

$$dX_t = -aX_t dt + 2dV_t, \quad (4)$$

and that  $X_0$  with a normal distribution with mean 0 and variance 1 .

- (a) Solve the equation (4). (3 marks)
- (b) Find the distribution  $p_t$  of  $X_t$ . (5 marks)
- (c) Prove that  $X$  is a stationary process if and only if  $a = 2$ . (4 marks)
- (d) Prove that

$$\int_0^t \mathbb{E}[|X_s|] ds < \infty.$$

(5 marks)

- (e) Compute

$$\lim_{t \rightarrow \infty} \int_0^t \mathbb{E}[|X_s|] ds.$$

(3 marks)

You may use any results given in the course without proof, provided that you make it clear which ones you are using.

(Total: 20 marks)

4. Let  $Z = \{Z_t, t \geq 0\}$  be the process defined by

$$Z_t = \exp \left( - \int_0^t h(X_s) dW_s - \frac{1}{2} \int_0^t h(X_s)^2 ds \right), \quad t \geq 0.$$

Let  $\tilde{\mathbb{P}}$  be a probability measure which is absolutely continuous with respect to  $\mathbb{P}$  and such that its Radon-Nikodym derivative with respect to  $\mathbb{P}$  is given by  $Z$ . That is,

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t, \quad \text{for any } t \geq 0.$$

(a) Prove that if  $\xi$  is an  $\mathcal{F}_s$ -measurable random variable, then

$$\mathbb{E}[\xi Z_t] = \mathbb{E}[\xi Z_s] \quad \text{for any } 0 \leq s \leq t.$$

Why is this property important ? (4 marks)

(b) Show that the two-dimensional process  $(V, Y)$  is a Brownian motion under  $\tilde{\mathbb{P}}$ . (6 marks)

(c) Prove that the process  $t \mapsto \frac{1}{Z_t}$  is a martingale under  $\tilde{\mathbb{P}}$ . (3 marks)

(d) Let  $\rho_t : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$  be the set function defined

$$\rho_t(A) = \tilde{\mathbb{E}} \left[ \frac{I_A(X_t)}{Z_t} \middle| \mathcal{Y}_t \right], \quad A \in \mathcal{B}(\mathbb{R}),$$

where  $\tilde{\mathbb{E}}[\cdot]$  denotes expectation with respect to  $\tilde{\mathbb{P}}$ . Prove the Kallianpur-Striebel's formula. That is, prove that

$$\pi_t(A) = \frac{\rho_t(A)}{\rho_t(\mathbb{R})}.$$

for any Borel set  $A \in \mathbb{R}$ . (7 marks)

(Total: 20 marks)

### Mastery Question

5. Let  $\xi = \{\xi_t, t \geq 0\}$  be a bounded  $\mathcal{F}_t$ -adapted process. Define  $a = \{a_t, t \geq 0\}$  and  $b = \{b_t, t \geq 0\}$  to be the following stochastic processes

$$\begin{aligned}a_t &= \int_0^t \xi_s ds + W_t, \\b_t &= a_t - \int_0^t \mathbb{E}[\xi_s | \sigma_s^a] ds,\end{aligned}$$

where  $\sigma^a = \{\sigma_t^a, t \geq 0\}$  is the filtration generated by the process  $a$ , that is  $\sigma_t^a = \sigma(a_s, s \in [0, t])$ .

- (a) (i) Give the definition of a square-integrable martingale. (2 marks)
- (ii) Define of the quadratic variation of a square-integrable martingale. (3 marks)
- (b) (i) Prove that the process  $b$  is  $\sigma_t^a$ -adapted. (2 marks)
- (ii) Prove that the process  $b$  is a square-integrable martingale. (7 marks)
- (c) State Lévy's characterization of a Brownian motion. (2 marks)
- (d) Prove that the process  $b = \{b_t, t \geq 0\}$  is a Brownian motion. (4 marks)

(Total: 20 marks)

## Marking Scheme

### Question 1. [20 marks]

(a) [2 marks] A real-valued stochastic process  $B = \{B_t, t \geq 0\}$  is a standard Brownian motion if the following properties are satisfied

1.  $B$  is continuous a.s. and  $B_0 = 0$ .
2.  $B$  has independent increments. That is, if  $t_1 < t_2 < \dots < t_n$ , then the  $n$  the random variables

$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \dots, B_{t_n} - B_{t_{n-1}}$$

are independent random variables.

3. For any  $s < t$  the random variable  $B_t - B_s$  is normally distributed with mean 0 and variance  $t - s$ .

(b) [8 marks] We verify the three properties:

1.  $B$  is a linear combination of two stochastic processes that are continuous almost surely and that start from 0, hence it will inherit the same properties.
2. There are many ways to check this, but perhaps the easiest is to do it via characteristic functions. We have that

$$\begin{aligned} \mathbb{E} \left[ e^{\sum_{j=1}^{n-1} i\lambda_j (B_{t_{j+1}} - B_{t_j})} \right] &= \mathbb{E} \left[ e^{\sum_{j=1}^{n-1} i\lambda_j (B_{t_{j+1}} - B_{t_j})} \right] \\ &= \mathbb{E} \left[ e^{\sum_{j=1}^{n-1} i\lambda_j \sin(\theta) (V_{t_{j+1}} - V_{t_j})} \right] \mathbb{E} \left[ e^{\sum_{j=1}^{n-1} i\lambda_j \cos(\theta) (W_{t_{j+1}} - W_{t_j})} \right] \\ &= \prod_{j=1}^{n-1} \mathbb{E} \left[ e^{i\lambda_j \sin(\theta) (V_{t_{j+1}} - V_{t_j})} \right] \prod_{j=1}^{n-1} \mathbb{E} \left[ e^{i\lambda_j \cos(\theta) (W_{t_{j+1}} - W_{t_j})} \right] \\ &= \prod_{j=1}^{n-1} \mathbb{E} \left[ e^{i\lambda_j (B_{t_{j+1}} - B_{t_j})} \right], \end{aligned} \tag{1}$$

where the first and the last identity follows from the independence of the Brownian motions  $V$  and  $W$  and the second identity follows from the independence of the increments of  $V$ , respectively,  $W$ .

3.  $B_t - B_s$  is a sum of two independent normally distributed random variables  $\sin(\theta)(V_t - V_s) \sim N(0, \sin(\theta)^2(t - s))$  and  $\cos(\theta)(W_t - W_s) \sim N(0, \cos(\theta)^2(t - s))$ , hence  $B_t - B_s$  is also normal with mean equal to the sum of the two means and variance the sum of the two variances which give the claim.

(c) (i) [4 marks, seen similar] Observe that

$$S_n^2(t) - t = \sum_{i=0}^{n-1} \left( (W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^2 - \frac{t}{n} \right) =: \sum_{i=0}^{n-1} X_i$$

with the obvious definition for the  $X_i$ 's, which makes them independent random variables, with mean 0. Also  $X_i = \frac{t}{n}Z_i$ , where the random variables  $Z_i$  are distributed as  $Z^2 - 1$  where  $Z$  is normal with mean 0 and variance 1. It follows that

$$\mathbb{E}[(S_n^2(t) - t)^2] = \sum_{i=0}^{n-1} \left(\frac{t}{n}\right)^2 \mathbb{E}[(Z^2 - 1)^2] = \frac{c(t)}{n}.$$

where  $c(t) = t^2 \mathbb{E}[(Z^2 - 1)^2]$ , hence the claim holds as an identity not an inequality (there may be alternative proofs, where an upper bound for the various terms is obtained).

(c) (ii) [2 marks, not seen] Observe

$$\mathbb{E} \left[ \sum_{n \geq 1} (S_{n^2}^2(t) - t)^2 \right] = \sum_{n \geq 1} \mathbb{E} [(S_{n^2}^2(t) - t)^2] = c(t) \sum_n \frac{1}{n^2} < \infty$$

Hence  $\sum_{n \geq 1} (S_{n^2}^2(t) - t)^2 < \infty$ ,  $P$ -almost surely and therefore

$$\lim_{n \rightarrow \infty} (S_{n^2}^2(t) - t)^2 = 0$$

(the infinite sum will not be finite otherwise).

(c) (iii) [2 marks, seen similar] We have that,  $P$ -almost surely,

$$0 \leq \lim_{n \rightarrow \infty} S_{n^2}^4(t) \leq \lim_{n \rightarrow \infty} \max_{i=0, \dots, n-1} (W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^2 \sum_{i=0}^{n-1} (W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^2 = 0 \times t,$$

where we used the almost sure continuity of the Brownian paths (and therefore the uniform continuity of the paths), and the result at (c) (ii).

(c) (iv) [2 marks, not seen] One can conclude that, almost surely, the Brownian sample paths have finite quadratic variation but infinite (first) variation.

**Question 2. [20 marks]**

(a) **[2 marks, seen]** Let  $X_t$  be an  $\{\mathcal{F}_t\}$ -adapted process with continuous paths. If  $X_t$  can be decomposed as  $X_t = M_t + V_t$ ,  $t \geq 0$ , where  $M_t$  is an  $\{\mathcal{F}_t\}$ -adapted martingale with continuous paths and the paths of  $V_t$  are of finite variation, then we call  $X_t$  a semimartingale.

(b) **[3 marks, seen]**

Let  $X_t^1, \dots, X_t^d$  be semi-martingales and  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  a function which is one time continuously differentiable with respect to  $t$  and twice with respect to  $x_i$ ,  $i = 1, 2, \dots, d$ . Then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X_s) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d[X^i, X^j]_s. \end{aligned}$$

(c) Let  $R = \{R_t, t > 0\}$  be a continuous process satisfying the following identity

$$R_t = \exp \left( \int_0^t \sin(R_s) dV_s - \frac{1}{2} \int_0^t (\sin(R_s))^2 ds \right), \quad t \geq 0.$$

(i) **[6 marks, not seen]** Let  $\nu = \{\nu_t, t > 0\}$  be the semimartingale defined by

$$\nu_t = \int_0^t \sin(R_s) dV_s - \frac{1}{2} \int_0^t (\sin(R_s))^2 ds, \quad t \geq 0.$$

Observe that

$$\langle \nu \rangle_t = \int_0^t (\sin(R_s))^2 ds$$

Then, by Itô's formula, we get that

$$\begin{aligned} R_t &= \exp(\nu_t) \\ &= 1 + \int_0^t \exp(\nu_s) d\nu_s + \frac{1}{2} \int_0^t \exp(\nu_s) d\langle \nu \rangle_s \\ &= 1 + \int_0^t R_s d\nu_s + \frac{1}{2} \int_0^t R_s d\langle \nu \rangle_s \\ &= 1 + \int_0^t R_s \left( \sin(R_s) dV_s - \frac{1}{2} (\sin(R_s))^2 ds \right) + \frac{1}{2} \int_0^t R_s (\sin(R_s))^2 ds \\ &= 1 + \int_0^t R_s \sin(R_s) dV_s \end{aligned}$$

Hence

$$R_t = 1 + \int_0^t \sigma(R_s) dV_s,$$

where  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is defined as  $\sigma(x) = x \sin(x)$ .

(ii) [2 marks, seen] Novikov's condition states that if  $u = \{u_t, t > 0\}$  is a process defined as  $u_t = \exp\left(M_t - \frac{1}{2}\langle M \rangle_t\right)$  for  $M$  a continuous local martingale, then a sufficient condition for  $u$  to be a martingale is that

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle M \rangle_t \right) \right] < \infty, \quad 0 \leq t < \infty.$$

(iii) [3 marks, seen similar] In this case the process  $t \rightarrow \int_0^t \sin(R_s) dV_s$  is a continuous local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by  $t \rightarrow \int_0^t (\sin(R_s))^2 ds$ . Moreover, since the process  $|\sin(R_s)| \leq 1$  is bounded, it follows that

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle M \rangle_t \right) \right] = \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^t (\sin(R_s))^2 ds \right) \right] \leq \exp \left( \frac{t}{2} \right) < \infty, \quad 0 \leq t < \infty.$$

Hence, by Novikov's condition, the process  $R = \{R_t, t > 0\}$  is a martingale under  $\mathbb{P}$ .

(iv) [4 marks, see similar] Note

$$\begin{aligned} R_t^2 &= \tilde{R}_t \exp \left( \int_0^t (\sin(R_s))^2 ds \right) \leq e^t \tilde{R}_t \\ \tilde{R}_t &= \exp \left( \int_0^t 2 \sin(R_s) dV_s - \frac{1}{2} \int_0^t (2 \sin(R_s))^2 ds \right), \quad t \geq 0. \end{aligned}$$

Then, similarly to the process  $R$ , the process  $\tilde{R}$  is also a martingale. Hence

$$\mathbb{E} [R_t^2] \leq e^t \mathbb{E} [\tilde{R}_t] = e^t \mathbb{E} [\tilde{R}_0] = e^t.$$

**Question 3. [20 marks]**

(a) [3 marks, seen similar] We have, by Itô's formula, that

$$e^{at} X_t = X_0 + \int_0^t X_s de^{as} + \int_0^t e^{as} dX_s = X_0 + \int_0^t 2e^{as} dV_s,$$

hence

$$X_t = X_0 e^{-at} + \int_0^t 2e^{a(s-t)} dV_s.$$

(b) [5 marks, seen similar] We use the fact that if  $f : [0, t] \rightarrow \mathbb{R}$  be a Borel measurable function such that  $v := \int_0^t f(s)^2 ds < \infty$ , then the random variable  $\int_0^t f(s) dV_s$  has a normal distribution with mean 0 and variance  $v$ . It follows that

$$\int_0^t 2e^{a(s-t)} dV_s \sim N\left(0, \frac{2}{a}(1 - e^{-2at})\right)$$

also

$$X_0 e^{-at} \sim N(0, e^{-2at})$$

Since  $X_0 e^{-at}$  and  $\int_0^t 2e^{a(s-t)} dV_s$  are independent normal random variables, we deduce that

$$p_t = \mathcal{L}(X_t) = N(0, v_t).$$

where  $v_t = e^{-2at} + \frac{2}{a}(1 - e^{-2at})$ .

(c) [4 marks, not seen]

$\implies$  Note that

$$v_t = \frac{2}{a} + e^{-2at} \left(1 - \frac{2}{a}\right).$$

It follows that  $v_t$  is either strictly decreasing or strictly increasing depending on whether  $a > 2$  or  $a < 2$ . Since  $X$  is stationary it follows that  $a = 2$ .

$\Leftarrow$  If  $a = 2$ , then  $p_t = \mathcal{L}(X_t) = N(0, 1)$ , hence  $X$  is stationary.

(d) [5 marks, not seen]

We have that

$$\begin{aligned} E[|X_s|] &= \frac{2}{\sqrt{2\pi v_s}} \int_0^\infty x e^{-\frac{x^2}{2v_s}} dx \\ &= -\frac{2}{\sqrt{2\pi v_s}} v_s e^{-\frac{x^2}{2v_s}} \Big|_0^\infty = \sqrt{\frac{2}{\pi}} v_s \leq \sqrt{\frac{2}{\pi} \left(\frac{2}{a} + 1\right)}, \end{aligned}$$

hence

$$\int_0^t E[|X_s|] ds \leq t \sqrt{\frac{2}{\pi} \left(\frac{2}{a} + 1\right)} < \infty.$$

(e) [**3 marks, not seen**] Observe that  $\lim_{s \rightarrow \infty} v_s = \frac{2}{a}$ , hence there exists a time  $t_0 > 0$  sufficiently large such that  $v_s > \frac{1}{a}$  for any  $s \geq t_0$ . It follows that

$$\int_0^t E[|X_s|] ds \geq \int_{t_0}^t \sqrt{\frac{2}{\pi}} v_s ds \geq \sqrt{\frac{2}{\pi a}} (t - t_0)$$

It follows that

$$\lim_{t \rightarrow \infty} \int_0^t E[|X_s|] ds \geq \lim_{t \rightarrow \infty} \sqrt{\frac{2}{\pi a}} (t - t_0) = \infty,$$

hence the required limit is  $\infty$ .

**Question 4. (20 marks)**

**(a) [4 marks, seen].** Since  $Z$  is a martingale and  $\xi$  is an  $\mathcal{F}_s$  measurable random variable, we have, by using the properties of the conditional expectation

$$\mathbb{E}[\xi Z_t] = \mathbb{E}[\mathbb{E}[\xi Z_t | \mathcal{F}_s]] = \mathbb{E}[\xi \mathbb{E}[Z_t | \mathcal{F}_s]] = \mathbb{E}[\xi Z_s] \quad \text{for any } 0 \leq s \leq t.$$

This property ensures that the definition of  $\tilde{\mathbb{P}}$  is consistent, as for  $A \in \mathcal{F}_s \subset \mathcal{F}_t$

$$\tilde{\mathbb{P}}(A) = \int_A Z_t d\mathbb{P} = \mathbb{E}[\xi Z_t]$$

and also

$$\tilde{\mathbb{P}}(A) = \int_A Z_s d\mathbb{P} = \mathbb{E}[\xi Z_s]$$

**(b) [6 marks, seen similar].** Let  $a \in \mathbb{R}$  be a constant and  $\hat{Z}^a = \{\hat{Z}_t^a, t \geq 0\}$  be the process defined by

$$\hat{Z}_t^a = \exp \left( - \int_0^t a dV_s + \int_0^t h(X_s) dW_s - \frac{1}{2} \int_0^t (a^2 + h(X_s)^2) ds \right), \quad t \geq 0.$$

Let  $\hat{\mathbb{P}}$  be a probability measure which is absolutely continuous with respect to  $\mathbb{P}$  and such that its Radon-Nikodym derivative with respect to  $\mathbb{P}$  is given by  $Z$ . That is,

$$\left. \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \hat{Z}_t^a, \quad \text{for any } t \geq 0.$$

By Girsanov's theorem, under the probability measure  $\hat{\mathbb{P}}$ , the process

$$t \rightarrow \left( V_t + \int_0^t a ds, W_t + \int_0^t h(X_s) ds \right), \quad t \geq 0$$

is a standard two dimensional Brownian motion. Choosing  $a = 0$ , we deduce that the process

$$t \rightarrow \left( V_t, Y_t = W_t + \int_0^t h(X_s) ds \right), \quad t \geq 0$$

is a Brownian motion under  $\hat{\mathbb{P}}$ . Since  $a = 0$ , it follows that  $\hat{Z}_t^0 = Z_t$  and therefore  $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$ . As a result the pair  $(V, Y)$  is indeed a two-dimensional Brownian motion.

**(c) [3 marks, seen similar].** Let  $\tilde{Z} = \{\tilde{Z}_t, t \geq 0\}$  be the process

$$\tilde{Z}_t = \frac{1}{Z_t} = \exp \left( \int_0^t h(X_s) dY_s - \frac{1}{2} \int_0^t h(X_s)^2 ds \right), \quad t \geq 0.$$

By Itô's formula, it follows that

$$\tilde{Z}_t = 1 + \int_0^t \tilde{Z}_s h(X_s) dY_s.$$

Hence  $\tilde{Z}$  is a local martingale under  $\tilde{\mathbb{P}}$ . It is a genuine martingale either by Novikov condition or by observing that it is a non-negative super-martingale with constant expectations as

$$\tilde{\mathbb{E}}[\tilde{Z}_t] = \mathbb{E}[\tilde{Z}_t Z_t] = 1. \quad \text{for any } t \geq 0.$$

**(d) [5 marks, seen].** It suffices to show that, for any Borel measurable function  $\varphi$ , we have

$$\pi_t(\varphi) \tilde{\mathbb{E}}[\tilde{Z}_t | \mathcal{Y}_t] = \tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t) | \mathcal{Y}_t] \quad \tilde{\mathbb{P}}\text{-a.s.}$$

As both the left and right hand sides of this equation are  $\mathcal{Y}_t$ -measurable, this is equivalent to showing that for any bounded  $\mathcal{Y}_t$ -measurable random variable  $b$ ,

$$\mathbb{E}[\pi_t(\varphi) \tilde{\mathbb{E}}[\tilde{Z}_t | \mathcal{Y}_t] b] = \mathbb{E}[\tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t) | \mathcal{Y}_t] b].$$

Since  $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t]$  we have

$$\mathbb{E}[\pi_t(\varphi) b] = \mathbb{E}[\varphi(X_t) b].$$

Writing this under the measure  $\tilde{\mathbb{P}}$ ,

$$\tilde{\mathbb{E}}[\pi_t(\varphi) b \tilde{Z}_t] = \tilde{\mathbb{E}}[\varphi(X_t) b \tilde{Z}_t].$$

By the tower property of the conditional expectation, since by assumption the function  $b$  is  $\mathcal{Y}_t$ -measurable

$$\tilde{\mathbb{E}}[\pi_t(\varphi) \tilde{\mathbb{E}}[\tilde{Z}_t | \mathcal{Y}_t] b] = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[\varphi(X_t) \tilde{Z}_t | \mathcal{Y}_t] b]$$

which proves that the result holds  $\tilde{\mathbb{P}}$ -a.s.

**Question 5. (20 marks)**

**(a)(i) [2 marks, seen]** Let  $M = \{M_t, t \geq 0\}$  be a stochastic process. The process  $M$  is an  $\mathcal{F}_t$ -adapted martingale if  $M_t$  is integrable for all  $t \geq 0$  and  $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ , for all  $s, t$  with  $s < t$ . A martingale  $M$  is called square integrable if  $\mathbb{E}[M_t^2] < \infty$  for all  $t \geq 0$ .

**(a)(ii) [3 marks, seen]** Let  $M = \{M_t, t \geq 0\}$  be a continuous square integrable martingale. Then there exists a unique (up to indistinguishability) adapted increasing process, denoted by  $[M]_t$  such that  $M_t^2 - [M]_t$  is a martingale. The process  $[M]_t$  is called the quadratic variation process of  $M$ .

**(b) (i) [2 marks, not seen]** The process  $b$  is  $\sigma_t^a$ -adapted as both  $a_t$  and  $\int_0^t \mathbb{E}[\xi_s | \sigma_s^a] ds$  are.

**(b) (ii) [7 marks, not seen]** Observe that

$$\begin{aligned} b_t &= a_t - \int_0^t \mathbb{E}[\xi_s | \sigma_s^a] ds \\ &= W_t + \int_0^t (\xi_s - \mathbb{E}[\xi_s | \sigma_s^a]) ds. \end{aligned} \tag{2}$$

By taking conditional expectation

$$\begin{aligned} \mathbb{E}[b_t | \sigma_s^a] - b_s &= \mathbb{E}\left[W_t + \int_0^t \xi_r dr \mid \mathcal{Y}_s\right] - \left(W_s + \int_0^s \xi_r dr\right) \\ &\quad - \mathbb{E}\left[\int_0^t \mathbb{E}[\xi_r | \sigma_r^a] dr \mid \mathcal{Y}_s\right] + \int_0^s \mathbb{E}[\xi_r | \sigma_r^a] dr \\ &= \mathbb{E}\left[W_t - W_s + \int_s^t \xi_r dr \mid \mathcal{Y}_s\right] \\ &\quad - \mathbb{E}\left[\int_0^t \mathbb{E}[\xi_r | \sigma_r^a] dr \mid \mathcal{Y}_s\right] + \int_0^s \mathbb{E}[\xi_r | \sigma_r^a] dr. \end{aligned}$$

Since  $a_t$  and  $\int_0^t \mathbb{E}[\xi_r | \sigma_r^a] dr$  are  $\mathcal{Y}_t$ -adapted,

$$\mathbb{E}[b_t | \mathcal{Y}_s] - b_s = \mathbb{E}[W_t - W_s | \sigma_s^a] + \int_s^t \mathbb{E}[\xi_r - \mathbb{E}[\xi_r | \sigma_r^a] | \sigma_s^a] dr = 0,$$

where we have used the fact that for  $r \geq s$ ,

$$\mathbb{E}[\mathbb{E}[\xi_r | \sigma_r^a] | \sigma_s^a] = \mathbb{E}[\xi_r | \sigma_s^a]$$

and

$$\mathbb{E}[W_t - W_s | \sigma_s^a] = \mathbb{E}[\mathbb{E}[W_t - W_s | \mathcal{F}_s] | \sigma_s^a] = 0.$$

Finally  $b_t$  is square integrable since both  $W_t$  and  $\int_0^t (\xi_s - \mathbb{E}[\xi_s | \sigma_s^a]) ds$  are square integrable (the second term is, in fact, bounded).

(c) [**2 marks, seen**] A stochastic process  $M$  is a Brownian motion if and only if  $M$  is a continuous martingale such that  $M_0 = 0$  and  $[M]_t = t$  for  $t \geq 0$ .

(d) [**4 marks, seen similar**] The cross-variation of  $b$  is the same as the cross-variation of  $W$  as the other term in (2) give zero cross-variation. So  $b$  is a continuous martingale and its cross-variation is given by

$$\langle I \rangle_t = \langle W \rangle_t = t. \quad (3)$$

Hence  $b$  is a Brownian motion by Lévy's characterisation of a Brownian motion.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	QuestionNumber	Comments for Students
MATH70055	1	A good question reasonably well answered by the students.
MATH70055	2	Very high marks for this question. Clearly the corresponding material was well understood.
MATH70055	3	Very high marks for this question. Clearly the corresponding material was well understood.
MATH70055	4	Students for this question most difficult, despite it being closest to the material taught in lectures.
MATH70055	5	A good question reasonably well answered by the students.