Imperial College London

MATH97028 MATH97105

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Complex Analysis

Date: 20th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS <u>ONE PDF</u> TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. (a) For $-1 < x < 1$, find a closed form expression for

$$\fint_{-1}^{1} \frac{t}{t-x} \mathrm{d}t.$$

(4 marks)

(b) For $z \in \mathbb{C}$, find a closed form expression for

$$\int_{-1}^{1} x \log|x-z| \mathrm{d}x.$$

(4 marks)

(c) For $z \in \mathbb{C}$, find a closed form expression for

$$\int_{-1}^{1} \mathrm{e}^{x} \log|x-z| \mathrm{d}x.$$

in terms of

$$\mathrm{Ei}(z) := \int_{-\infty}^{z} \frac{\mathrm{e}^{\zeta}}{\zeta} \mathrm{d}\zeta,$$

where the path of integration can be chosen to be the union of two line segments: one segment from $-\infty$ to -1 and another segment from -1 to z. (6 marks)

(d) For $-1 < x < 1$, find u such that

$$\int_{-1}^{1} u(t) \log|t-x| \mathrm{d}t = \operatorname{atan} x$$

Hint: recall that $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{atan} x = 1/(x^2+1)$ and the inverse Hilbert transform formula:

$$u(t) = -\frac{1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{x-t} \mathrm{d}t + \frac{D}{\sqrt{1-x^2}}$$

for any constant D satisfies

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{x-t} \mathrm{d}t = f(x).$$

(6 marks)

(Total: 20 marks)

2. Consider computation of the matrix function $\sqrt{A}$ and recall the contour integral formula

$$\sqrt{A} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - A)^{-1} \mathrm{d}\zeta$$

which is valid for suitably chosen γ.

(a) (i) What are the conditions on a contour γ?

(3 marks)

(ii) Depict a suitable contour for the matrix

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

(3 marks)

(b) (i) Assume $A = V\Lambda V^{-1}$ is diagonalisable, that is $\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_d]$. Express $\sqrt{A}$ in terms of the eigenvalues λ_k.

(3 marks)

(ii) Show that this definition is consistent with the contour integral formula.

(5 marks)

(c) Recall for a Jordan block we have

$$\sqrt{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}} = \begin{pmatrix} \sqrt{\lambda} & 1/(2\sqrt{\lambda}) \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

Show this is consistent with the contour integral formula.

(6 marks)

(Total: 20 marks)

3. The Legendre polynomials

$$P_n(x) = \frac{2^n (1/2)_n}{n!} x^n + O(x^{n-1})$$

are orthogonal with respect to the weight 1 on $[-1, 1]$ and the ultraspherical polynomials

$$C_n(x) := C_n^{(3/2)}(x) = \frac{2^n (3/2)_n}{n!} x^n + O(x^{n-1})$$

are orthogonal with respect to the weight $(1-x^2)$ on $[-1, 1]$. Here $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. Denote the Cauchy transform of the weighted C_n polynomials as

$$Q_n(z) := \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{(1-x^2) C_n(x)}{x - z} \mathrm{d}x.$$

(a) Show that

(i) $\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) = C_{n-1}(x)$. (2 marks)

(ii) $\frac{\mathrm{d}}{\mathrm{d}x} \left[(1-x^2) C_n(x) \right] = -(n+1)(n+2) P_{n+1}(x)$. (3 marks)

(b) Find the three-term recurrence for $C_n(z)$. That is, give the coefficients c_{n-1}, a_n, and b_n so that

$$\begin{aligned} xC_0(x) &= a_0 C_0(x) + b_0 C_1(x) \\ xC_n(x) &= c_{n-1} C_{n-1}(x) + a_n C_n(x) + b_n C_{n+1}(x) \end{aligned}$$

You may use the three-term recurrence of $P_n(x)$,

$$xP_n(x) = \frac{n}{2n+1} P_{n-1}(x) + \frac{n+1}{2n+1} P_{n+1}(x)$$

and the relationships

$$3P_1(x) = C_1(x), \qquad (2n+1) P_n(x) = C_n(x) - C_{n-2}(x).$$

(4 marks)

(c) (i) What is $Q_0(z)$? (3 marks)

(ii) What 3-term recurrence does $Q_n(z)$ satisfy? (3 marks)

(d) For $z \in \mathbb{C}$ and $n > 0$, express the logarithmic singular integral

$$\frac{1}{\pi} \int_{-1}^{1} P_n(x) \log |x - z| \mathrm{d}x$$

in terms of $Q_n(z)$. (5 marks)

(Total: 20 marks)

4. Consider the Toeplitz operator, which we denote by

$$T[a] = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad \text{for} \qquad a(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

and the Wiener–Hopf factorisation if its symbol:

$$a(z) = \phi_+(z)\phi_-(z)$$

where

$$\phi_+(z) = \sum_{k=0}^{\infty} \ell_k z^k \qquad \text{and} \qquad \phi_-(z) = 1 + \sum_{k=1}^{\infty} u_k z^{-k}$$

analytic inside and outside the unit disk, respectively.

(a) Under what conditions is a non-degenerate, that is, it has a Wiener–Hopf factorisation?

(2 marks)

(b) Show that if $a(z)$ is non-degenerate that $T[a(z)] = T[\phi_-(z)]T[\phi_+(z)]$.

(5 marks)

(c) Consider the discrete Laplacian

$$\Delta = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

What is $(\Delta - 3I)^{-1}\mathbf{e}_0$, where $\mathbf{e}_0 = [1, 0, 0, \dots]^\top$? (5 marks)

(d) Consider $a(z) = \mathrm{e}^{z+2/z}$.

(i) Show that $a(z)$ is non-degenerate. (3 marks)

(ii) Find the Wiener–Hopf factorisation using an expression in terms of $\log a(z)$. (5 marks)

(Total: 20 marks)

5. The Cauchy transform of $f(x)$ over an interval (a,b) is denoted

$$\mathcal{C}_{(a,b)} f(z) := \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{f(x)}{x-z} \mathrm{d}x$$

(a) Define the two-sheeted Riemann surface analogue of the Cauchy transform of v on $[0,\infty)$ via the following two functions:

$$\mathcal{C}^1 v(z) := \frac{\mathcal{C}_{(0,\infty)} v(z) + \mathrm{i}\sqrt{-z}\,\mathcal{C}_{(0,\infty)} w(z)}{2}$$

and

$$\mathcal{C}^2 v(z) := \frac{\mathcal{C}_{(0,\infty)} v(z) - \mathrm{i}\sqrt{-z}\,\mathcal{C}_{(0,\infty)} w(z)}{2},$$

where $w(x) = \frac{v(x)}{\sqrt{x}}$. For f smooth and $v(x) = f(\sqrt{x})$, justify the relationship

$$\mathcal{C}_{(0,\infty)} f(z) = \begin{cases} \mathcal{C}^1 v(z^2) & \operatorname{Im} z > 0 \\ \mathcal{C}^2 v(z^2) & \operatorname{Im} z < 0 \end{cases}$$

using Plemelj's theorem. You may use without proof that the right-hand side has weaker-than-pole singularities.

(6 marks)

(b) Consider now a rational map

$$r(x) = \frac{x}{1-x^2}$$

(i) Find two pre-images $r_1^{-1}(z)$ and $r_2^{-1}(z)$ satisfying $r(r_j^{-1}(z)) = z$. Be precise on where the branch cuts of $r_j^{-1}(z)$ are.

(3 marks)

(ii) For $g(x) = f(r(x))$ defined on $[-1,1]$, the Cauchy transform satisfies

$$\mathcal{C}_{\mathbb{R}} f(z) = \mathcal{C}_{[-1,1]} g(r_1^{-1}(z)) + \mathcal{C}_{[-1,1]} g(r_2^{-1}(z)) + D$$

for some constant D (depending on f but not z). What is D?

(5 marks)

(iii) Justify the above result using Plemelj's theorem.

(6 marks)

(Total: 20 marks)

M3M6 2019/20 Solutions
Setter: Olver
Checker: Crowdy

1(a)

Plemelj guarantees that $\psi(z)$ is equal to the Cauchy transform of f

$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(t)}{t-z} \, \mathrm{d}t$$

if it satisfies the following conditions:

1. $\psi(\infty) = 0$,
2. $\psi(z)$ is analytic off $[-1, 1]$,
3. $\psi(z)$ has weaker than pole singularities everywhere and
4. $\psi(z)$ satisfies the jump

$$\psi_+(x) - \psi_-(x) = f(x) \qquad \text{for} \qquad -1 < x < 1.$$

We first find the Cauchy transform

$$\mathcal{C}f(z) := \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(t)}{t-z} \, \mathrm{d}t$$

by considering

$$\varphi(z) := z\mathcal{C}1(z) = z(\log(z-1) - \log(z+1))/(2\pi \mathrm{i}).$$

This satisfies conditions (2,3) and 4, since

$$\varphi^+(z) - \varphi^-(z) = x(\mathcal{C}^+ - \mathcal{C}^-)1(x) = x$$

We thus just need to rectify the behaviour at infinity. Using

$$\log(z-1) - \log(z+1) = \log(1 - 1/z) - \log(1 + 1/z) = -\frac{2}{z} + O(z^{-3})$$

we see that

$$\mathcal{C}[x](z) = z(\log(z-1) - \log(z+1))/(2\pi \mathrm{i}) + \frac{1}{\pi \mathrm{i}}$$

Then we have for the Hilbert transform

$$\mathcal{H}f(x) := \frac{1}{\pi} ⨍_{-1}^{1} \frac{f(t)}{x-t} \, \mathrm{d}t$$

that

$$\begin{aligned} \mathcal{H}f(x) &= -\mathrm{i}(\mathcal{C}^+ f + \mathcal{C}^- f) = -x \frac{\log_+(x-1) + \log_-(x-1) - 2\log(x+1)}{2\pi} - \frac{2}{\pi} \\ &= -x \frac{\log(1-x) - \log(x+1)}{\pi} - \frac{2}{\pi} \end{aligned}$$

Multiplying through by $-\pi$ gives the result

$$2 + x(\log(1-x) - \log(x+1)).$$

Seen similar

1(b) For the function

$$M(z) = \frac{1}{\pi}\int_{-1}^{1} f(x)\log(z-x)\,\mathrm{d}x$$

is equal to

$$M(z) = \frac{\log(z+1)}{\pi}\int_{-1}^{1} f(x)\,\mathrm{d}x + 2\mathrm{i}\mathcal{C}F(z)$$

for $F(z) = \int_x^1 f(x)\,\mathrm{d}x$. In this case we have $\int_{-1}^1 f(x)\,\mathrm{d}x = 0$ by symmetry and

$$F(x) = \frac{1}{2} - \frac{x^2}{2}.$$

To compute $\mathcal{C}F(z)$ we find as in 1(a) that

$$\mathcal{C}F(z) = (1/2 - z^2/2)\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} - \frac{z}{2\pi\mathrm{i}}$$

We thus get the answer (multiplying through by π)

$$\Re(2\pi\mathrm{i}\mathcal{C}F(z)) = \Re\left[(1/2 - z^2/2)(\log(z-1) - \log(z+1)) - z\right].$$

Seen similar

1(c) Using the same formula as above, it boils down to computing $\mathcal{C}F(z)$ for $F(x) = \int_x^1 \mathrm{e}^x\,\mathrm{d}x = \mathrm{e} - \mathrm{e}^x$. We note that for $x > 0$ we have for $\phi(z) = -\frac{\mathrm{e}^{-z}\mathrm{Ei}z}{2\pi\mathrm{i}}$ that

$$\phi_+(x) - \phi_-(x) = \mathrm{e}^{-x}$$

Thus we find that

$$\mathcal{C}F(z) = \mathrm{e}\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} - \mathrm{e}^{-1}\phi(-z-1) + \mathrm{e}\phi(1-z)$$

We thus get the answer using $\int_{-1}^1 \mathrm{e}^x\,\mathrm{d}x = (\mathrm{e} - \mathrm{e}^{-1})$,

$$\log|z+1|(\mathrm{e} - \mathrm{e}^{-1}) - 2\pi\Im\left[\mathrm{e}\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} - \mathrm{e}^{-1}\phi(-z-1) + \mathrm{e}\phi(1-z)\right].$$

Alternative solution: integrate by parts to get

$$\int_{-1}^1 \mathrm{e}^x \log(z-x)\,\mathrm{d}x = [\mathrm{e}^x\log(z-x)]_{-1}^1 - \int_{-1}^1 \frac{\mathrm{e}^x\,\mathrm{d}x}{x-z}$$

then changing variables $\zeta = x - z$ gives

$$\int_{-1}^1 \mathrm{e}^x \log(z-x)\,\mathrm{d}x = \mathrm{e}\log(z-1) - \mathrm{e}^{-1}\log(z+1) - \mathrm{e}^z(\mathrm{Ei}(1-z) - \mathrm{Ei}(-1-z))$$

leading to

$$\mathrm{e}\log|1-z| - \mathrm{e}^{-1}\log|1+z| - \Re\left[\mathrm{e}^{z}\left(\mathrm{Ei}(1-z) - \mathrm{Ei}(-1-z)\right)\right].$$

Unseen

1(d) Differentiating this is equivalent to solving

$$\mathcal{H}u(t) = \frac{1}{\pi(x^2+1)}$$

The inverse Hilbert formula tells us this is equal to

$$u(t) = -\frac{1}{\pi\sqrt{1-x^2}}\mathcal{H}[\sqrt{1-t^2}(1+t^2)^{-1}](x) + \frac{D}{\sqrt{1-x^2}}.$$

By considering $x = 0$ and using symmetry we determine that $D = 0$. We calculate the Hilbert transform by first calculating the Cauchy transform, that is,

$$\mathcal{C}[\sqrt{1-t^2}(1+t^2)^{-1}] = \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(z^2+1)} + \frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4(z-\mathrm{i})} - \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4(z+\mathrm{i})}$$

so that

$$\mathcal{H}[\sqrt{1-t^2}(1+t^2)^{-1}](x) = -\mathrm{i}(\mathcal{C}^+ + \mathcal{C}^-)[\sqrt{1-t^2}(1+t^2)^{-1}](x) = -\mathrm{i}\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{2(x-\mathrm{i})} + \mathrm{i}\frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{2(x+\mathrm{i})}.$$

Note this is real as its a complex number plus its complex conjugate. Therefore

$$u(x) = -\frac{1}{\pi\sqrt{1-x^2}}\left[-\mathrm{i}\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{2(x-\mathrm{i})} + \mathrm{i}\frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{2(x+\mathrm{i})}\right].$$

2(a)(i) γ must be a simply connected closed contour surrounding the spectrum and avoiding the branch cut of $\sqrt{z}$ on $(-\infty, 0]$.

Seen

2(a)(ii) The eigenvalues are $-1 \pm \mathrm{i}$. The contour should contain these two points while avoiding the branch cut.

Seen similar

2(b)(i)

$$\sqrt{A} = V\sqrt{\Lambda}V^{-1} = V \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_d} \end{pmatrix} V^{-1}$$

Seen

2(b)(ii)

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - A)^{-1}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - V\Lambda V^{-1})^{-1}\,\mathrm{d}\zeta = V\frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - \Lambda)^{-1}\,\mathrm{d}\zeta V^{-1}$$

Use Cauchy's integral formula to show

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - \Lambda)^{-1}\,\mathrm{d}\zeta = \sqrt{\Lambda}.$$

Seen

2(c) For A equal to the Jordan block, we use the fact that

$$(\zeta I - A)^{-1} = \begin{pmatrix} \zeta - \lambda & -1 \\ & \zeta - \lambda \end{pmatrix}^{-1} = \begin{pmatrix} (\zeta - \lambda)^{-1} & (\zeta - \lambda)^{-2} \\ & (\zeta - \lambda)^{-1} \end{pmatrix}$$

Then we have

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta I - A)^{-1}\,\mathrm{d}\zeta = \begin{pmatrix} \frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta - \lambda)^{-1}\,\mathrm{d}\zeta & \frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta - \lambda)^{-2}\,\mathrm{d}\zeta \\ & \frac{1}{2\pi\mathrm{i}} \oint_\gamma \sqrt{\zeta}(\zeta - \lambda)^{-1}\,\mathrm{d}\zeta \end{pmatrix}$$

The result follows using Cauchy's integral formula on each entry. **Unseen**

3(a)(i) We show each by showing that they have the right normalisation constant and are orthogonal.

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) = \frac{n 2^n (1/2)_n}{n!} x^{n-1} + O(x^{n-2}) = \frac{2^n (1/2)(3/2)_n}{(n-1)!} x^{n-1} + O(x^{n-2})$$

which matches the normalisation constant of C_{n-1}. Orthogonality follows from integration by parts: if f_m is a degree $m < n-1$ polynomial then

$$\int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x} P_n(x) f_m(x)(1-x^2)\,\mathrm{d}x = -\int_{-1}^{1} P_n(x) \frac{\mathrm{d}}{\mathrm{d}x}[f_m(x)(1-x^2)]\,\mathrm{d}x = 0$$

since the second term is of degree $m + 2 - 1 = m + 1 < n$.

Seen similar

3(a)(ii)

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)C_n(x) = -(n+2)\frac{2^n (3/2)_n}{n!} x^{n+1} + O(x^n) = -(n+2)(n+1)\frac{2^{n+1}(1/2)_n}{(n+1)!} x^{n+1} + O(x^n)$$

which matches the normalisation constant of P_{n+1}. Orthogonality follows from integration by parts: if f_m is a degree $m < n+1$ polynomial then

$$\int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)C_n(x)] f_m(x)\,\mathrm{d}x = -\int_{-1}^{1} (1-x^2) C_n(x) f_m'(x)\,\mathrm{d}x = 0$$

since the second term is of degree $m - 1 < n$.

Seen similar

3(b) We differentiate the 3-term recurrence for P_n: in particular we have

$$P_1(x) + x C_0(x) = \frac{\mathrm{d}}{dx}[x P_1(x)] = \frac{1}{3} P_0'(x) + \frac{2}{3} P_2'(x) = \frac{2}{3} C_1(x)$$

which shows that

$$x C_0(x) = \frac{1}{3} C_1(x)$$

Similarly

$$\begin{aligned} P_{n+1}(x) + x C_n(x) &= \frac{\mathrm{d}}{dx}[x P_{n+1}(x)] = \frac{n+1}{2n+3} P_n'(x) + \frac{n+2}{2n+3} P_{n+2}'(x) \\ &= \frac{n+1}{2n+3} C_{n-1}(x) + \frac{n+2}{2n+3} C_{n+1} \end{aligned}$$

which shows using the conversion relationship that

$$x C_n(x) = \frac{n+2}{2n+3} C_{n-1}(x) + \frac{n+1}{2n+3} C_{n+1}(x)$$

Seen similar

3(c) (i) We determine this via finding

$$(1-z^2)\frac{\log(z-1) - \log(z+1)}{2\pi \mathrm{i}} = (1-z^2)\left(\frac{\mathrm{i}}{\pi z} + O(z^3)\right) = -\frac{\mathrm{i}z}{\pi} + O(1/z)$$

Hence by Plemelj we have

$$(1-z^2)\frac{\log(z-1)-\log(z+1)}{2\pi\mathrm{i}}+\frac{\mathrm{i}z}{\pi}$$

Seen similar

3(c) (ii) Note that

$$\int_{-1}^{1}(1-x^2)\,\mathrm{d}x=\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\frac{4}{3}$$

We therefore find that

$$zQ_n(z)=-\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}(1-x^2)Q_n(x)\,\mathrm{d}x+\mathcal{C}[xQ_n]$$

The first integral is zero for $n\neq 0$ so we have

$$\begin{aligned} zQ_0(z) &= -\frac{2}{3\pi\mathrm{i}}+\frac{1}{3}Q_1(z)\\ zQ_n(z) &= \frac{n+2}{2n+3}Q_{n-1}(z)+\frac{n+1}{2n+3}Q_{n+1}(z)\end{aligned}$$

that is, the same 3-term recurrence as the polynomials themselves apart from the $n=0$ term.
3(d) For $n>0$ we use the fact that $\int_{-1}^{1}P_n(x)\,\mathrm{d}x=0$ and the formula

$$F_n(x):=\int_x^1 P_n(x)\,\mathrm{d}x=\frac{1}{n(n+1)}(1-x^2)C_{n-1}(x)$$

to find that

$$\frac{1}{\pi}\int_{-1}^{1}P_n(x)\log|x-z|\,\mathrm{d}x=-2\Im\mathcal{C}F_n(z)=-\frac{2}{n(n+1)}\Im Q_{n-1}(z)$$

Unseen

4(a) The winding number of a has to be zero, or equivalent, $\log a(z)$ has to be continuously-differentiable on the unit circle.

Seen

4(b) We do so by direct inspection. By multiplying out the Laurent series we determine

$$\begin{aligned} a(z) &= \phi_+(z)\phi_-(z) \\ &= \cdots + z^{-1}\left(\ell_0 u_1 + \ell_1 u_2 + \cdots\right) + \left(\ell_0 + \ell_1 u_1 + \ell_2 u_2 + \cdots\right) + z\left(\ell_1 + \ell_2 u_1 + \ell_2 u_2 + \cdots\right) + \cdots \end{aligned}$$

Multiplying out the Toeplitz operators we find for $k \geq j$

$$\boldsymbol{e}_k^\top T[\phi_-]T[\phi_+]\boldsymbol{e}_j = \ell_{k-j} + \ell_{k-j+1}u_1 + \cdots = a_{k-j}.$$

For $k < j$ we have

$$\boldsymbol{e}_k^\top T[\phi_-]T[\phi_+]\boldsymbol{e}_j = \ell_0 u_{j-k} + \ell_1 u_{j-k+1} + \cdots = a_{k-j}.$$

Seen

4(c) Note that

$$\Delta - 3I = T[z - 3 + z^{-1}]$$

We calculate the Wiener–Hopf factorisation by factorising the polynomial:

$$\begin{aligned} a(z) &= z - 3 + z^{-1} = z^{-1}(z^2 - 3z + 1) = z^{-1}(z - z_0)(z - z_1) \\ &= \underbrace{(z - z_0)}_{\phi_+}\underbrace{(1 - z^{-1}z_1)}_{\phi_-} \end{aligned}$$

where $z_0 = 3/2 + (\sqrt{5})/2$ and $z_1 = 3/2 - (\sqrt{5})/2$. We have that (using that $T[\phi_-^{-1}]^{-1}$ is upper triangular with a 1 in the top entry)

$$T[a]^{-1} = T[\phi_+^{-1}]T[\phi_-^{-1}]\boldsymbol{e}_0 = T[\phi_+^{-1}]\boldsymbol{e}_0 = -[z_0^{-1}, z_0^{-2}, z_0^{-3}, \ldots]^\top$$

since, using Taylor series,

$$\phi_+(z)^{-1} = \frac{1}{z - z_0} = -\frac{1}{z_0}\frac{1}{1 - z/z_0} = -\frac{1}{z_0} - \frac{z}{z_0^2} - \frac{z^2}{z_0^3} - \cdots .$$

Seen similar

4(d) (i) Here we use that

$$\Re a(z) = \mathrm{e}^{|z+2/z|} > 0$$

to show that the winding number is zero.

Unseen

4(d) (ii) Taking the logarithm we find

$$\log a(z) = z + 2/z$$

which implies that

$$\mathcal{C}[\log a](z) = \begin{cases} z & |z| < 1 \\ -2/z & |z| > 1 \end{cases}.$$

Therefore we have

$$\phi_+(z) = \mathrm{e}^{\mathcal{C}[\log a](z)} = \mathrm{e}^z$$

and

$$\phi_-(z) = \mathrm{e}^{-\mathcal{C}[\log a](z)} = \mathrm{e}^{2/z}.$$

Unseen

5(a) We check the 4 properties. (1) Decay is immediate as $z^2 \to \infty$ so that $\mathcal{C}^{1/2}v(z^2) \to 0$, and (2) weaker-than pole singularities is given. (3) Analyticity off the real line is immediate, and for $x < 0$ we have

$$\begin{aligned}2(\mathcal{C}^1 v)(x^2)_+ - 2(\mathcal{C}^2 v)(x^2)_- &= \mathcal{C}^-_{(0,\infty)} v(x^2) + \mathrm{i}(\sqrt{-x^2})_+ \mathcal{C}^-_{(0,\infty)} w(x^2) - \\ &\qquad \mathcal{C}^+_{(0,\infty)} v(x^2) + \mathrm{i}(\sqrt{-x^2})_- \mathcal{C}^+_{(0,\infty)} w(x^2) \\ &= -v(x^2) + |x| w(x^2) = 0\end{aligned}$$

hence it is continuous (and therefore analytic). Finally, (4) for $x > 0$ we have the right jump:

$$\begin{aligned}2(\mathcal{C}^1 v)(x^2)_+ - 2(\mathcal{C}^2 v)(x^2)_- &= \mathcal{C}^+_{(0,\infty)} v(x^2) + \mathrm{i}(\sqrt{-x^2})_- \mathcal{C}^+_{(0,\infty)} w(x^2) \\ &\qquad - \mathcal{C}^-_{(0,\infty)} v(x^2) + \mathrm{i}(\sqrt{-x^2})_+ \mathcal{C}^-_{(0,\infty)} w(x^2) \\ &= v(x^2) + x w(x^2) = 2v(x)\end{aligned}$$

Seen

5(b) (i)

We need to solve

$$r(\zeta) = \frac{\zeta}{1-\zeta^2} = z$$

That is,

$$z\zeta^2 + \zeta - z = 0$$

i.e. $r_1^{-1}(z) = \frac{-1+\sqrt{2z-\mathrm{i}}\sqrt{2z+\mathrm{i}}}{2z}$ and $r_2^{-1}(z) = \frac{-1-\sqrt{2z-\mathrm{i}}\sqrt{2z+\mathrm{i}}}{2}$. These have branch cuts from $(-\infty \pm \mathrm{i}/2, \pm\mathrm{i}/2)$.

Unseen

5(b) (ii) We need the right-hand side to decay at ∞. Thus we need to choose

$$D = -\mathcal{C}_{(-1,1)} g(r_1^{-1}(\infty)) - \mathcal{C}_{(-1,1)} g(r_2^{-1}(\infty)) = -\mathcal{C}_{(-1,1)} g(1) - \mathcal{C}_{(-1,1)} g(-1)$$

Unseen

5(b) (iii)

(1) Decay at infinity follows from the choice of constant. (2) To show analyticity we need to consider the branch cuts of r_j^{-1}. But on the branch cuts we have from the properties of square roots $(r_1^{-1})_\pm(z) = (r_2^{-1})_\mp(z)$ hence the jumps cancel and the right-hand side is continuous (and thence analytic). (3) Weaker-than-pole singularities follows as $r_j^{-1}(z)$ have at worst square-root singularities. Finally, (4) for $x \in \mathbb{R}$ we have only

$$r_1^{-1}(x) = \frac{-1+\sqrt{4x^2+1}}{2x}$$

approaching $[-1, 1]$ thus the difference is

$$\mathcal{C}^+ g(r_1^{-1}(x)) - \mathcal{C}^- g(r_1^{-1}(x)) = f(r(r_1^{-1}(x))) = f(x).$$

Unseen

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97028 MATH97105	1	Overall students performed well on parts a and b. Part c was intentionally the most challenging though gladly a number of students successfully completed. Part d was fine though many students missed the choice of constant.	
MATH97028 MATH97105	2	Students performed extremely well on this question, perhaps helped by the open book nature of the exam. On part a (ii), some students tried to use Gorshgorin despite it just being a 2x2 matrix.	
MATH97028 MATH97105	3	Several students forgot to show orthogonality in part (a), which is suprising as this was emphasised in lecture. Some students missed part (b) but part © and (d) were fine.	
MATH97028 MATH97105	4	Students performed well in parts (a) and (b), though some struggled with part (c), perhaps as this was the last topic they didn't study it as much. Some students messed up part (d) because they confused problems on the circle with that of the interval.	
MATH97028 MATH97105	5	Question 5 was the most challenging though most students did part (a) fine. One student managed to do most of the question so I believe it was appropriate in difficulty.	