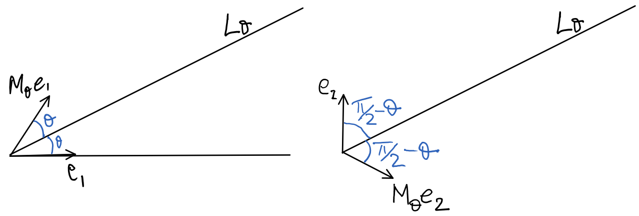
B.1. (a) Let M_θ be the reflection in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 \tan\theta\}$. Using any school geometry or trigonometry you like, show that the matrix representing M_θ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Drawing it, you see that e_1 gets reflected to the unit vector making an angle 2θ with the x_1-axis, i.e. $\binom{\cos 2\theta}{\sin 2\theta}$.



Similarly e_2 makes an angle $\pi/2 - \theta$ anticlockwise from L_θ, so gets reflected to a unit vector whose angle is $\pi/2 - \theta$ *clockwise* from L_θ. Thus it makes an angle $\theta - (\pi/2 - \theta) = 2\theta - \pi/2$ with the x_1-axis, so is the unit vector $\binom{\cos(2\theta-\pi/2)}{\sin(2\theta-\pi/2)} = \binom{\sin 2\theta}{-\cos 2\theta}$.

Thus the matrix is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ as claimed.

(b) Let R_α be a rotation about the origin, and let M_β be the reflection in a line through the origin. Prove that $M_\beta R_\alpha$ is a reflection.

We compute the product

$$\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos(2\beta-\alpha) & \sin(2\beta-\alpha) \\ \sin(2\beta-\alpha) & -\cos(2\beta-\alpha) \end{pmatrix}.$$

By (c) this is reflecton in $L_{\beta-\alpha/2}$

(c) Let M_α and M_β be reflections in straight lines through the origin. Prove that $M_\alpha M_\beta$ is a rotation.

We compute

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} = \begin{pmatrix} \cos 2(\alpha-\beta) & -\sin 2(\alpha-\beta) \\ \sin 2(\alpha-\beta) & \cos 2(\alpha-\beta) \end{pmatrix},$$

which is the rotation $R_{2(\alpha-\beta)}$.

B.2. Let $\mathbb{R}[x]$ be the set of all polynomials with variable x and real coefficients, with standard addition and scalar multiplication. Show that this is a vector space over $\mathbb{R}$.

A1 Follows from associativity of $\mathbb{R}$ and definition of polynomial addition. i.e. let $f(x), g(x), h(x) \in \mathbb{R}[x]$ then we have:

$$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^m a_i x^i \\ g(x) &= \textstyle\sum_{i=1}^n b_i x^i \\ h(x) &= \textstyle\sum_{i=1}^s c_i x^i \end{aligned}$$

Let $t = max\{m, n, s\}$ then define $a_i = 0$ for $m \le i \le t$, similarly define $b_i = 0$ for $n \le i \le t$, and define $c_i = 0$ for $s \le i \le t$. So we get:

$$\begin{aligned} f(x) &= \textstyle\sum_{i=1}^t a_i x^i \\ g(x) &= \textstyle\sum_{i=1}^t b_i x^i \\ h(x) &= \textstyle\sum_{i=1}^t c_i x^i \end{aligned}$$

Now

$$\begin{aligned} f(x)+(g(x)+h(x)) &= \sum_{i=1}^{t} a_i x^i + (\sum_{i=1}^{t} b_i x^i + \sum_{i=1}^{t} c_i x^i) \\ &= \sum_{i=1}^{t} (a_i+(b_i+c_i))x^i \\ &= \sum_{i=1}^{t} ((a_i+b_i)+c_i)x^i \\ &= (\sum_{i=1}^{t} a_i x^i + \sum_{i=1}^{t} b_i x^i) + \sum_{i=1}^{t} c_i x^i \\ &= (f(x)+g(x))+h(x) \end{aligned}$$

A2 Follows from commutitivity of $\mathbb{R}$ and definition of polynomial addition.

A3 0_V here is the polynomial 0.

A4 The inverse of $f(x) = \sum_{i=1}^{m} a_i x^i$ is $-f(x) = \sum_{i=1}^{m} -a_i x^i$ clearly we get $f(x) + (-f(x)) = 0_V$.

A5 Follows from distributivity of $\times$ over $+$ in $\mathbb{R}$ and definition of polynomial addition.

A6 Follows from distributivity of $+$ over $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.

A7 Follows from commutativity of $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.

A8 Follows from definition of multiplying a polynomial by a constant/scalar.

B.3. For each of the following sets and operations, decided whether it is a vector space over $\mathbb{R}$ or not.

(a) The set $\mathbb{R}^2$ with standard addition, but with scalar multiplication defined by

$$r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ry \\ rx \end{pmatrix}$$

Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{aligned} r \odot (s \odot \mathbf{v}) &= r \odot \left(s \odot \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= r \odot \begin{pmatrix} sy \\ sx \end{pmatrix} \\ &= \begin{pmatrix} rsx \\ rsy \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (rs) \odot \mathbf{v} &= (rs) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rsy \\ rsx \end{pmatrix} \end{aligned}$$

Because $r \odot (s \odot \mathbf{v}) \neq (rs) \odot \mathbf{v}$, A7 is not satisfied.

(b) The set $\mathbb{R}^2$ with standard scalar multiplication, but with addition defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b+y \\ a+x \end{pmatrix}$$

All of the axioms A5 - A8 are satisfied since the usual definition of scalar multiplication is used.

Consider the axiom A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then

$$\begin{aligned} (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} y_1+y_2 \\ x_1+x_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1+x_2+y_3 \\ y_1+y_2+x_3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
\mathbf{u} \oplus (\mathbf{v} + \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} y_2 + y_3 \\ x_2 + x_3 \end{pmatrix} \\
&= \begin{pmatrix} y_1 + x_2 + x_3 \\ x_1 + y_2 + y_3 \end{pmatrix}
\end{aligned}$$

Since $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ then A1 is not satisfied.

(c) The set $\mathbb{R}^2$ with addition and scalar multiplication defined by

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a + x + 1 \\ b + y \end{pmatrix} \\
r \odot \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} r(a+1) - 1 \\ rb \end{pmatrix}
\end{aligned}$$

Consider A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then

$$\begin{aligned}
(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 + x_3 + 2 \\ y_1 + y_2 + y_3 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 + x_3 + 1 \\ y_2 + y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 + x_3 + 2 \\ y_1 + y_2 + y_3 \end{pmatrix}
\end{aligned}$$

Because $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$, A1 is satisfied.

Consider A2 and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{v} \oplus \mathbf{w} &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{w} \oplus \mathbf{v} &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix}
\end{aligned}$$

Because $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$, A2 is satisfied.

Consider A3, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then

$$\begin{aligned} \mathbf{e} \oplus \mathbf{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x - 1 + 1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathbf{v} \end{aligned}$$

Therefore, A3 is satisfied.

Consider A4, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned} \mathbf{v} \oplus (-1 \odot \mathbf{v}) &= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \left(-1 \odot \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 1 - 1 \\ -y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 2 \\ -y \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ &= \mathbf{e} \end{aligned}$$

Therefore, A4 is satisfied.

Consider A5 and let $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\begin{aligned} r \odot (\mathbf{v} \oplus \mathbf{w}) &= r \odot \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (r \odot \mathbf{v}) \oplus (r \odot \mathbf{w}) &= r \odot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus r \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} rx_1 + r - 1 \\ ry_1 \end{pmatrix} \oplus \begin{pmatrix} rx_2 + r - 1 \\ ry_2 \end{pmatrix} \\ &= \begin{pmatrix} rx_1 + r - 1 + rx_2 + r - 1 + 1 \\ ry_1 + ry_2 \end{pmatrix} \\ &= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix} \end{aligned}$$

Because $r \odot (\mathbf{v} \oplus \mathbf{w}) = (r \odot \mathbf{v}) \oplus (r \odot \mathbf{w})$, A5 is satisfied.

Consider A6, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned} (r + s) \odot \mathbf{v} &= (r + s) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} (r + s)x + (r + s) - 1 \\ (r + s)y \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}(r \odot \mathbf{v}) \oplus (s \odot \mathbf{v}) &= r \odot \begin{pmatrix} x \\ y \end{pmatrix} \oplus s \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix} \oplus \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\ &= \begin{pmatrix} rx + r - 1 + sx + s - 1 + 1 \\ ry + sy \end{pmatrix} \\ &= \begin{pmatrix} (r+s)x + (r+s) - 1 \\ (r+s)y \end{pmatrix}\end{aligned}$$

Because $(r+s) \odot \mathbf{v} = (r \odot \mathbf{v}) \oplus (s \odot \mathbf{v})$, A6 is satisfied.

Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned}r \odot (s \odot \mathbf{v}) &= r \odot \left(s \odot \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= r \odot \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\ &= \begin{pmatrix} r(sx + s - 1) + r - 1 \\ rsy \end{pmatrix} \\ &= \begin{pmatrix} r(sx + s) - 1 \\ rsy \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}(rs) \odot \mathbf{v} &= rs \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rsx + rs - 1 \\ rsy \end{pmatrix}\end{aligned}$$

Because $r \odot (s \odot \mathbf{v}) = (rs) \odot \mathbf{v}$, then A7 is satisfied.

Consider A8, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned}1 \odot \mathbf{v} &= 1 \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x + 1 - 1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathbf{v}\end{aligned}$$

Therefore, A8 is satisfied.

Therefore, the set $\mathbb{R}^2$ with addition described by $\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a + 1 \\ y + b \end{pmatrix}$ and scalar multiplication described by $r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix}$ is a vector space.

B.4. Let F be a field. Show every F-vector space V with additive identity 0_V has the following properties:

(a) The vector 0_V is the unique vector satisfying the equation $0_V \oplus v = v$ for all vectors v in V.

(b) Let 0 be the additive identity in F. Then $0 \odot v = 0_V$ for all vectors v in V.

(a) Suppose that there are two vectors 0_v and $0'_V$ that satisfy

$$0_V \oplus \mathbf{v} = \mathbf{v} \quad 0'_V \oplus \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in V.$$

Let $\mathbf{v} = 0'_V$ in the first equation and $\mathbf{v} = 0_V$ in the second equation gives

$$0_V \oplus 0'_V = 0'_V \quad 0'_V \oplus 0_V = 0_V$$

By the commutative law, A2, $0_V \oplus 0'_V = 0'_V \oplus 0_V$, therefore, $0_V = 0'_V$ and hence the zero vector is unique.

(b) Using the distributive law, A6, for any $\mathbf{v} \in V$,

$$0 \odot \mathbf{v} = (0+0) \odot \mathbf{v} = (0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v})$$

By the additive identity axiom, A3, and the commutative law, A2,

$$0 \odot \mathbf{v} = 0 \odot \mathbf{v} \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$$

Therefore, $(0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$, and therefore, $0 \odot \mathbf{v} = 0_V$.

B.5. Describe all subspaces of $\mathbb{R}^3$.

(a) The set containing the zero vector, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, i.e. the zero subspace,

(b) the set describing any straight line going through the origin (including the x, y, and z axes),

(c) the set describing any plane that goes through the origin,

(d) and $\mathbb{R}^3$.

B.6. Let U and W be subspaces of vector space V over F. Prove that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

($\Rightarrow$) Suppose $U \cup W$ is a subspace. Suppose, for contradiction, we have $w \in W \setminus U$ and $u \in U \setminus W$. Then $u + w \notin U \cup W$:

Suppose $u + w \in U$ then $w = (u + w) + (-u) \in U$ which contradicts $w \in W \setminus U$, so $u + w \notin U$. Similarly we get $u + w \notin W$.

This contradicts SS2 for $U \cup W$, so either $W \setminus U = \emptyset$ (so $W \subseteq U$) or $U \setminus W = \emptyset$ (so $U \subseteq W$).

($\Leftarrow$) Clear as either $U \cup W = U$ or $U \cup W = W$.