Coursework

06 March 2023 19:00

1 (a) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(1-\lambda)(-3-\lambda)+6=0 \Rightarrow \lambda^2+2\lambda+3=0$$

$$\lambda_{1,2} = -1 \pm i\sqrt{2}$$ 2

$$\lambda_1 = -1+i\sqrt{2} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1-\frac{i}{\sqrt{2}} \end{pmatrix}$$ 2

$$\lambda_2 = -1-i\sqrt{2} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1+\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(-1+i\sqrt{2})t}\begin{pmatrix} 1 \\ 1-\frac{i}{\sqrt{2}} \end{pmatrix} + c_2 e^{(-1-i\sqrt{2})t}\begin{pmatrix} 1 \\ 1+\frac{i}{\sqrt{2}} \end{pmatrix}$$ 2

$$x = e^{-t}\left[c_1\cos\sqrt{2}t + ic_1\sin\sqrt{2}t + c_2\cos\sqrt{2}t - ic_2\sin\sqrt{2}t\right]$$

$$= e^{-t}\left[\underbrace{A}_{\frac{c_1+c_2}{2}}\cos\sqrt{2}t + \underbrace{B}_{\frac{i(c_1-c_2)}{2}}\sin\sqrt{2}t\right]$$

$$y = \frac{1}{2}e^{-t}\left[c_1\left(1-\frac{i}{\sqrt{2}}\right)\cos\sqrt{2}t + ic_1\left(1-\frac{i}{\sqrt{2}}\right)\sin\sqrt{2}t + c_2\left(1+\frac{i}{\sqrt{2}}\right)\cos\sqrt{2}t - ic_2\left(1+\frac{i}{\sqrt{2}}\right)\sin\sqrt{2}t\right]$$

$$= e^{-t}\left[\left(A-\frac{B}{\sqrt{2}}\right)\cos\sqrt{2}t + \left(B+\frac{A}{\sqrt{2}}\right)\sin\sqrt{2}t\right]$$

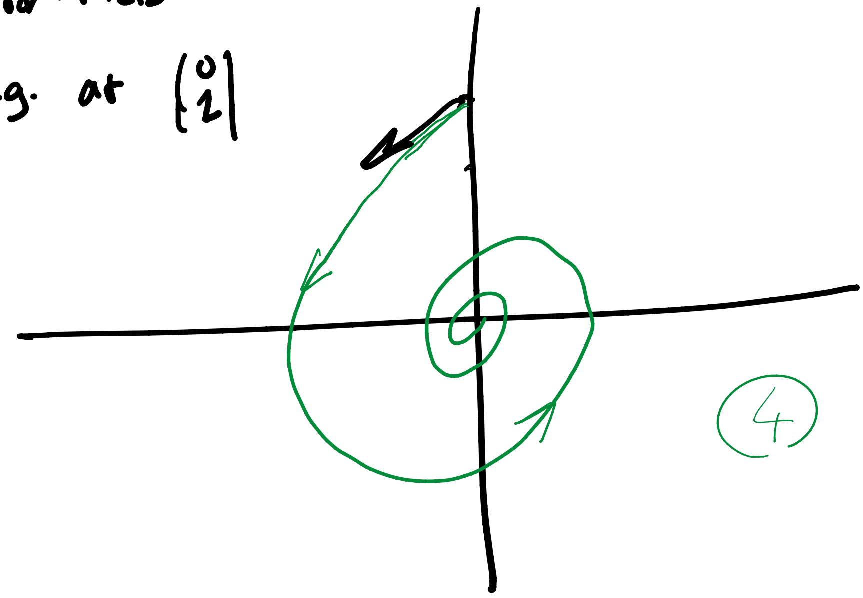
$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = e^{-t}\begin{bmatrix} A \\ A-\frac{B}{\sqrt{2}} \end{bmatrix}\cos\sqrt{2}t + e^{-t}\begin{bmatrix} B \\ B+\frac{A}{\sqrt{2}} \end{bmatrix}\sin\sqrt{2}t}$$ 4

b) Attracting (stable) spiral approach $\begin{bmatrix}0\\0\end{bmatrix}$ in an oscillatory manner as $t \to \infty$ (3)

Calculate the vector field for direction e.g. at $\begin{bmatrix}0\\1\end{bmatrix}$

$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix}$$

(3)



(4)

c) The system in (1) has $\tau = -2$ and $\Delta = 3$. (2)

This second order ODE is equivalent to a system of 2D ODEs (defining $u = \frac{dx}{dt}$):

$$\frac{d}{dt}\begin{pmatrix}x\\u\end{pmatrix} = \begin{pmatrix}0 & 1\\ -w^2 & -\gamma\end{pmatrix}\begin{pmatrix}x\\u\end{pmatrix} \Rightarrow \begin{matrix}\tau = -\gamma\\ \Delta = w^2\end{matrix}$$

(4) (2)

to have the same dynamics we need to have the same τ and Δ. So: $\boxed{\begin{matrix}\gamma = 2\\ W = \pm\sqrt{3}\end{matrix}}$ (2)

Also accepted if condition for an attracting spiral given.

(only $+\sqrt{3}$ could get full mark)

d) $$A = \begin{pmatrix}1 & -2\\ 3 & -3+\varepsilon\end{pmatrix} \Rightarrow \begin{matrix}\tau = -2+\varepsilon\\ \Delta = (-3+\varepsilon)+6 = 3+\varepsilon\end{matrix}$$

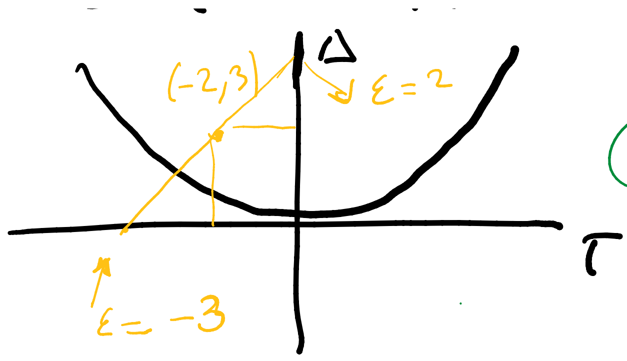
(2)

Bifurcations when

$\tau = 0 \Rightarrow \varepsilon = 2$ (2)

or

$\Delta = 0 \Rightarrow \varepsilon = -3$ (2)



(4)

e) ansatz : $\vec{y}_{PI} = \begin{pmatrix} a e^{-t} \\ b e^{-t} \end{pmatrix}$ (3)

$$-\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a = a - 2b + 1 \\ -b = 3a - 3b \end{cases} \Rightarrow$$

$$b = 2a + 1 \Rightarrow \quad -2a - 1 = 3a - 6a - 3 \Rightarrow \boxed{\begin{matrix} a = 1 \\ b = 3/2 \end{matrix}}$$

$$\begin{bmatrix} x_{GS} \\ y_{GS} \end{bmatrix} = e^{-t}\begin{bmatrix} A \\ A - \frac{B}{\sqrt{2}} \end{bmatrix}\cos\sqrt{2}t + e^{-t}\begin{bmatrix} B \\ B + \frac{A}{\sqrt{2}} \end{bmatrix}\sin\sqrt{2}t + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} e^{-t}$$ (2) (2)

$$x_{GS}(0) = 1 \Rightarrow \quad A + 1 = 1 \Rightarrow A = 0$$

$$y_{GS}(0) = 0 \Rightarrow \quad A - \frac{B}{\sqrt{2}} + 3/2 = 0 \Rightarrow B = \frac{3}{2}\sqrt{2}$$

$$\boxed{\begin{bmatrix} x_{GS} \\ y_{GS} \end{bmatrix} = e^{-t}\begin{bmatrix} 0 \\ -3/2 \end{bmatrix}\cos\sqrt{2}t + e^{-t}\begin{bmatrix} \frac{3}{2}\sqrt{2} \\ \frac{3}{2}\sqrt{2} \end{bmatrix}\sin\sqrt{2}t + \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} e^{-t}}$$ (3)