Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Physics I: Quantum Mechanics

Date: Monday, 13 May 2013. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. A quantum wave function

Consider the wave function

$$\psi(x,t) = A\mathrm{e}^{-\lambda|x|}\mathrm{e}^{-\mathrm{i}\omega t},$$

where A, λ, ω are real positive constants.

(a) Find a value of A for which ψ is normalised to one.

(b) Calculate the expectation values $\langle x \rangle, \langle x^2 \rangle$ and the uncertainty Δx.
Sketch the probability distribution $|\psi(x,t)|^2$ as a function of x and mark the points $\langle x \rangle - \Delta x$ and $\langle x \rangle + \Delta x$.

(c) Calculate the probability to find the particle outside the region between $\langle x \rangle - \Delta x$ and $\langle x \rangle + \Delta x$.

2. **Dynamics of an harmonic oscillator**

Consider a simple harmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Assume that the system is initially in the state

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x),$$

where $\phi_0(x)$ and $\phi_1(x)$ are the normalised ground and first excited states of the system:

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-\frac{m\omega}{\hbar}x^2/2} \quad \text{and} \quad \phi_1(x) = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\,\mathrm{e}^{-\frac{m\omega}{\hbar}x^2/2}.$$

(a) Is the initial state normalised?

(b) Find the time dependent state $\psi(x, t)$.

(c) Calculate the expectation values of $\hat{x}$ when the system is in the states $\phi_0(x)$ and $\phi_1(x)$, and $\psi(x, t = 0)$, respectively.

(d) Use the result from (b) to calculate the time dependent expectation value of $\hat{x}$. (That is, calculate the expectation value of $\hat{x}$, when the system is in the state $\psi(x, t)$.)

(e) From the result in (d) and the Ehrenfest theorem, deduce the expectation value of the momentum in the initial state $\psi(x, t = 0)$.

Hint: You may use the fact that $\int_{-\infty}^{+\infty} x^2 \mathrm{e}^{-a^2x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2a^3}$, for $\mathrm{Re}(a) > 0$.

3. **Angular momentum eigenvalues.**

Consider the angular momentum operators $J_{x,y,z}$ fulfiling the commutation relations $[J_x, J_y] = \mathrm{i}\hbar J_z$, and cyclic permutations.

(a) Verify that the operators J_z and $J^2 = J_x^2 + J_y^2 + J_z^2$ commute. What can you conclude from this about their eigenvectors?

(b) Prove that the possible eigenvalues of $J^2 = J_x^2 + J_y^2 + J_z^2$ are given by $\hbar^2 j(j+1)$ with $2j \in \mathbb{N}$, and for each given value of j the eigenvalues of J_z are given by $\hbar m$ with m running in integer steps from $-j$ to j.

For this purpose it is useful to first show that:

(i) From $L^2|\beta, m\rangle = \hbar^2\beta|\beta, m\rangle$ and $L_z|\beta, m\rangle = \hbar m|\beta, m\rangle$ it follows that $m^2 \leq \beta$.

(ii) If $|\beta, m\rangle$ is an eigenvector of J_z corresponding to the eigenvalue $\hbar m$, then $J_+|\beta, m\rangle$ is either also an eigenvector of J_z corresponding to the eigenvalue $\hbar(m+1)$ or the zero vector, and $J_-|\beta, m\rangle$ is either an eigenvector of J_z corresponding to the eigenvalue $\hbar(m-1)$ or the zero vector.

Hint: You may find the following relations useful:

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_+, J_-] = 2\hbar J_z, \text{ and}$$

$$J_-J_+ = J^2 - J_z^2 - \hbar J_z.$$

4. **An asymmetric square well.**

Consider a potential of the form

$$V(x) = \begin{cases} \infty, & x \leq 0 \\ 0, & 0 < x \leq a \\ V_0, & x > a. \end{cases}$$

(a) Find a quantisation condition for the energies of the bound states for this potential.

(b) Use graphical methods to deduce the minimum value of V_0 for which there is a bound state.

(c) How many bound states are there for given values of V_0, a and m?

	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 1		Marks & seen/unseen

Parts

(a)

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx$$

$$= A^2 \left\{ \int_{-\infty}^{0} e^{2\lambda x} dx + \int_{0}^{\infty} e^{-2\lambda x} dx \right\}$$

$$= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx$$

$$= 2A^2 \left[-\frac{e^{-2\lambda x}}{2\lambda} \right]_0^{\infty} = \frac{A^2}{\lambda}$$

$$\Rightarrow \quad A = \sqrt{\lambda}$$

Seen in homework — 3

(b)

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} x |\psi(x,t)|^2 dx \cancel{= A^2}$$

$$= A^2 \int_{-\infty}^{\infty} \underbrace{x}_{\text{anti-symm}} \underbrace{e^{-2\lambda|x|}}_{\text{symm.}} dx$$

$$= \underline{\underline{0}}$$

unseen — 2

$$\langle \hat{x}^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx$$

$$= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx$$

2

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	EXAMINATION SOLUTIONS 2012-13	Course
Question 1		Marks & seen/unseen

Parts

Calculate

$$\int_0^\infty x^2 e^{-2\lambda x}\, dx :$$

$$= \frac{1}{4}\frac{d^2}{d\lambda^2}\int_0^\infty e^{-2\lambda x}\, dx$$ 4.

$$= \frac{1}{4}\frac{d^2}{d\lambda^2}\left(\frac{1}{2\lambda}\right) = \frac{1}{4\lambda^3}$$

$$\Rightarrow \langle \hat{x}^2 \rangle = \frac{1}{2\lambda^2}$$

$$\Delta x = \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2} = \frac{1}{\sqrt{2}\,\lambda}$$ 2

(*)

(c) Probability that particle is found outside region $[\langle \hat{x}\rangle - \Delta x, \langle \hat{x}\rangle + \Delta x]$:

seen similar in home-work

3

$$P = 1 - 2\int_0^{\langle x\rangle + \Delta x} |\psi(x,t)|^2 dx = 1 - 2\lambda \int_0^{\Delta x} e^{-2\lambda x} dx$$

$$= 1 - 2\lambda \left[-\frac{e^{-2\lambda x}}{2\lambda}\right]_0^{\Delta x}$$

$$= 1 - 2\lambda\left[\frac{-e^{-2\lambda \Delta x}}{2\lambda} + \frac{1}{2\lambda}\right]$$

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	EXAMINATION SOLUTIONS 2012-13	Course
Question 1		Marks & seen/unseen

Parts

(c)

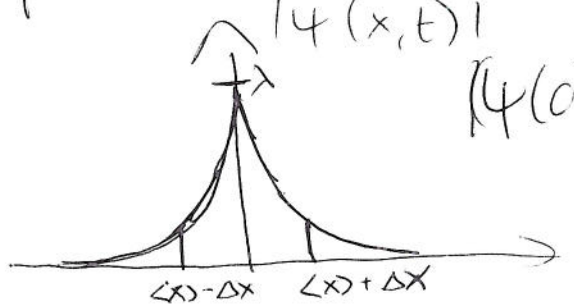
$$P = 1 - \left(1 - e^{-2\lambda\Delta x}\right)$$

$$= e^{-2\lambda\Delta x} = e^{-\sqrt{2}} \left(\simeq 0.24\right)$$

2

(*) part b:

unseen



$$|\psi(0)|^2 = \underline{\underline{\lambda}}$$

2

$$|\psi(\Delta x)|^2 = \lambda e^{-2\lambda \frac{1}{\sqrt{2}\lambda}}$$

$$= \lambda e^{-\sqrt{2}} \simeq \frac{\lambda}{4}$$

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 2		Marks & seen/unseen

Parts

(a) $\|\psi(x,t=0)\|^2 = \frac{1}{2}\|\phi_0\|^2 + \frac{1}{2}\|\phi_1\|^2 + \frac{1}{2}\int_{-\infty}^{\infty}\phi_1^*(x)\phi_2(x)dx + \frac{1}{2}\int_{-\infty}^{\infty}\phi_1(x)\phi_2^* dx$ — Unseen

$\phi_j(x)$ are orthonormal — 2

$\Rightarrow \|\psi(x,t=0)\|^2 = \frac{1}{2}+\frac{1}{2} = 1$

$\Leftrightarrow$ normalised

(b) $\psi(x,t) = \frac{1}{\sqrt{2}}\phi_0(x)e^{-iE_0t/\hbar} + \frac{1}{\sqrt{2}}\phi_1(x)e^{-iE_1t/\hbar}$ — 1

~~seen similar~~

harmonic oscillator: — Unseen

$E_j = \hbar\omega(j+1/2)$ — 1

$\psi(x,t) = \left(\frac{m\omega}{4\hbar\pi}\right)^{1/4} e^{-\frac{m\omega}{\hbar}x^2/2} e^{-i\frac{\omega}{2}t}$

$+ \left(\frac{m^3\omega^3}{4\hbar^3\pi}\right)^{1/4} x e^{-\frac{m\omega}{\hbar}x^2/2} e^{-i\frac{3}{2}\omega t}$ — 1

(c) $\langle x\rangle_{\phi_0} = \int_{-\infty}^{\infty} x|\phi_0(x)|^2 dx$ — Unseen

$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} \underset{\text{asymm}}{x}\, \underset{\text{symm}}{e^{-\frac{m\omega}{\hbar}x^2}} dx$ — 1

$= \underline{\underline{0}}$

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 2		Marks & seen/unseen

Parts

(c)

$$\langle x \rangle_{\phi_1} = \int_{-\infty}^{\infty} x |\phi_1(x)|^2 dx$$

$$= \left(\frac{4m^3\omega^3}{\hbar^3 \pi}\right)^{1/2} \int_{-\infty}^{\infty} \underset{\text{asymm}}{x^3}\, \underset{\text{symm}}{e^{-\frac{m\omega}{\hbar}x^2}} dx \qquad 1$$

$$= \underline{\underline{0}}$$

$$\langle x \rangle_{\psi_{t=0}} = \frac{1}{2}\langle x \rangle_{\phi_0} + \frac{1}{2}\langle x \rangle_{\phi_1}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} x\left(\phi_1^*(x)\phi_2(x)dx + \phi_1(x)\phi_2^*(x)dx\right)$$

$$= \int_{-\infty}^{\infty} x\, \phi_1(x)\phi_2(x) dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx \qquad 3$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{m\omega}{\hbar}\, \frac{\sqrt{\pi}}{2} \cdot \left(\frac{\hbar}{m\omega}\right)^{3/2}$$

$$= \underline{\underline{\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}}}$$

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 2		Marks & seen/unseen

Parts

(d) $\langle x\rangle_t = \int_{-\infty}^{\infty} x\,|\psi(x,t)|^2 dx$ — Unseen

$= \frac{1}{2}\langle x\rangle_{\phi_0} + \frac{1}{2}\langle x\rangle_{\phi_1}$

$+\frac{1}{2}\int_{-\infty}^{\infty} x\phi_1(x)\phi_2(x)dx \cdot \{e^{-i\frac{\omega}{2}t}e^{i\frac{3}{2}\omega t}$

$+ e^{-i\frac{3}{2}\omega t}e^{i\frac{\omega}{2}t}\}$ — 5

$= \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})\sqrt{\frac{2}{\pi}}\frac{m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$

$= \cos(\omega t)\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}$

(e) The Ehrenfest theorem states that: — Unseen

$\frac{d}{dt}\langle\hat{x}\rangle = \frac{\langle\hat{p}\rangle}{m}$ & $\frac{d}{dt}\langle\hat{p}\rangle = -\langle\frac{\partial V}{\partial x}\rangle$

here: $= -m\omega^2\langle\hat{x}\rangle$ — 2

$\Rightarrow \frac{d^2}{dt^2}\langle\hat{x}\rangle = -\omega^2\langle\hat{x}\rangle$ & $\langle\hat{p}\rangle = m\frac{d}{dt}\langle\hat{x}\rangle$

$\Rightarrow \langle\hat{x}(t)\rangle = \langle\hat{x}(0)\rangle\cos(\omega t) + \frac{\langle\hat{p}(0)\rangle}{m\omega}\sin(\omega t)$ — 2

Comparison with (d) yields

$\langle\hat{p}(0)\rangle = \underline{\underline{0}}$ — 1

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 3		Marks & seen/unseen

Parts

(a) seen in lecture

$$[J^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z]$$

$$= J_x[J_x, J_z] + [J_x, J_z]J_x + J_y[J_y, J_z] + [J_y, J_z]J_y$$ 2

$$= -i\hbar J_x J_y - i\hbar J_y J_x + i\hbar J_y J_x + i\hbar J_x J_y$$

$$= 0 \checkmark$$

$\Rightarrow$ they have a set of joint eigenvectors! 1

(b) (i) consider $\langle \beta, m | J^2 | \beta, m \rangle$:

$$\langle \beta, m | J^2 | \beta, m \rangle = \langle \beta, m | J_x^2 | \beta, m \rangle + \langle \beta, m | J_y^2 | \beta, m \rangle$$

$$+ \langle \beta, m | J_z^2 | \beta, m \rangle = \hbar^2 \beta \langle \beta, m | \beta, m \rangle$$

On the other hand since all

$\langle \beta, m | J_i^2 | \beta, m \rangle \geq 0$: 4

$$\langle \beta, m | J_x^2 | \beta, m \rangle + \langle \beta, m | J_y^2 | \beta, m \rangle$$

$$+ \langle \beta, m | J_z^2 | \beta, m \rangle \geq \langle \beta, m | J_z^2 | \beta, m \rangle$$

$$\parallel$$

$$\hbar^2 m^2 \langle \beta, m | \beta, m \rangle$$

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EXAMINATION SOLUTIONS 2012-13

Course: M3/4/5A4

Question: 3

Marks & seen/unseen

Parts (c)

$\Rightarrow \quad \hbar^2\beta \geq \hbar^2 m^2 \iff m^2 \leq \beta \quad \square$

(ii) $J_z J_+ |m,\beta\rangle = \left(J_+ J_z - [J_+, J_z] \right) |m,\beta\rangle$ — seen in lecture

$= (+\hbar J_+ + J_+ J_z)|\beta, m\rangle$

$= (\hbar J_+ + \hbar m J_+)|\beta, m\rangle$ — 2

$= \hbar(m+1) J_+ |\beta, m\rangle \quad \square$

$J_z J_- |\beta, m\rangle = (J_- J_z - [J_-, J_z])|\beta, m\rangle$

$= (J_- J_z - \hbar J_-)|\beta, m\rangle$ — 2

$= \hbar(m-1) J_- |\beta, m\rangle \quad \square$

these has to be a value m_{min} for which $J_-|\beta, m_{min}\rangle = 0$ and a value m_{max} for which $J_+|\beta, m_{max}\rangle = 0$ as m is bounded via $m^2 \leq \beta$. — 2

Now let $m_{min} := k$; $m_{max} := j$

Consider

$J_-|\beta, k\rangle = 0 \quad \& \quad J_+|\beta, j\rangle = 0$ — 1

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Course M3/4/5A4

Question 3

Marks & seen/unseen

Parts (c)

$$J_+J_-|\beta,k\rangle = (J_x+iJ_y)(J_x-iJ_y)|\beta,k\rangle$$

$$= (J_x^2+J_y^2+i[J_y,J_x])|\beta,k\rangle$$

$$= (J^2-J_z^2+\hbar J_z)|\beta,k\rangle$$

$$= (\hbar^2\beta-\hbar^2k^2+\hbar^2k)|\beta,k\rangle$$

$$= 0$$

seen in lecture

$$\Rightarrow \hbar^2\beta-\hbar^2k^2+\hbar^2k=0$$

2

$$\boxed{\beta+k-k^2=0}$$

$$\boxed{\beta=k(k-1)}$$

$$J_-J_+|\beta,j\rangle = (J_x-iJ_y)(J_x+iJ_y)|\beta,j\rangle$$

$$= (J_x^2+J_y^2+i[J_x,J_y])|\beta,j\rangle$$

$$= (J^2-J_z^2-\hbar J_z)|\beta,j\rangle$$

$$= (\hbar^2\beta-\hbar^2j^2-\hbar^2j)|\beta,j\rangle$$

$$= 0$$

2

$$\Rightarrow \boxed{\beta=j(j+1)}$$

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 3		Marks & seen/unseen
Parts (c)	$\Rightarrow \quad j(j+1) = k(k-1)$	seen in lecture
	2 solutions: (i) $k = j+1$	
	(ii) $k = -j$	2
	but $k < j \Rightarrow \boxed{k = -j}$	
	$\Rightarrow$ m runs from $-j$ to j in integer values and $\beta = j(j+1)$	4
	D	
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EXAMINATION SOLUTIONS 2012-13

Course: M3/4/5A4

Question: 4

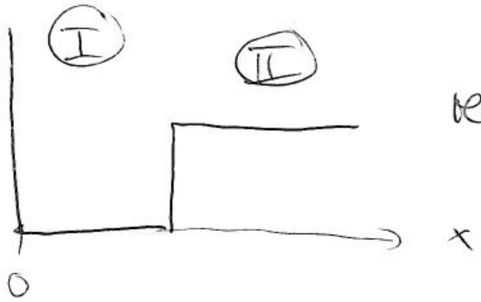
Marks & seen/unseen: seen in homework

Parts

(a)

$$V(x) = \begin{cases} \infty & x \le 0 \\ 0 & 0 < x \le a \\ V_0 & x > a \end{cases}$$

$E < V_0$:



Solution in region I:

$$\phi_I(x) = A\cos(kx) + B\sin(kx) \qquad 2$$

$$k = \sqrt{2mE}/\hbar$$

region II: $\phi_{II}(x) = Ce^{-\kappa x}$ 2

with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$

Continuity of ϕ and ϕ' at boundary between I & II and continuity of ϕ at $x=0$:

$\phi_I(0) = A = 0 \Rightarrow \phi_I(x) = B\sin(kx)$ 2

$\phi_I(a) = B\sin(ka) = Ce^{-\kappa a}$ 2

and $\phi_I'(a) = kB\cos(ka) = -\kappa Ce^{-\kappa a}$ 2

$\Rightarrow \boxed{k\cot(ka) = -\kappa}$ 2

quantisation condition!

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	EXAMINATION SOLUTIONS 2012-13	Course M3/4/5A4
Question 4		Marks & seen/unseen

Parts

$$\cot(ka) = -K/k$$

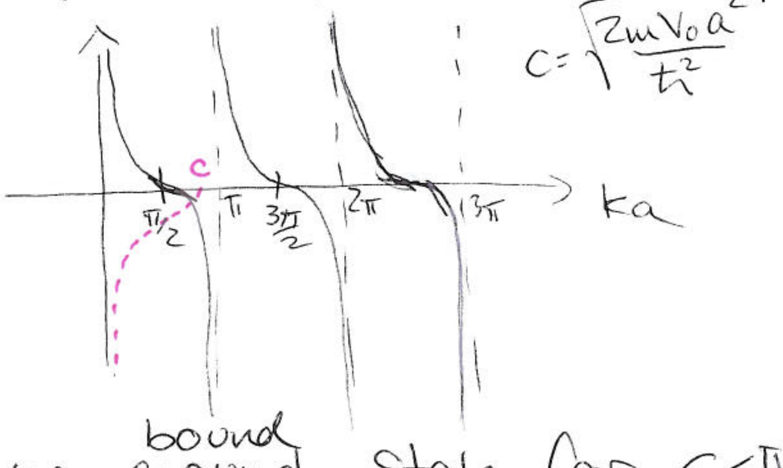
$$K/k = \sqrt{\frac{2m(V_0 - E)}{2mE}} = \sqrt{\frac{2mV_0a^2}{\hbar^2(ak)^2} - 1}$$

$$\boxed{\cot(ka) = -\sqrt{\frac{2mV_0a^2}{\hbar^2(ak)^2} - 1}}$$ 2

in energies: Substitute

$$k = \sqrt{2mE}/\hbar \ \ldots$$

(b) graphically: Seen Similar for Symmetric well



2

no ~~ground~~ bound state for $c < \pi/2$

minimum value of V_0 for bound state:

$$\sqrt{\frac{2mV_0a^2}{\hbar^2}} = \pi/2$$

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Parts

$$\Rightarrow V_{0\,min} = \frac{\pi^2}{8ma^2}\hbar^2$$ 2

(c) In dependence on a, m and V_0 these are

$$\left[\frac{c}{\pi} - \frac{1}{2}\right]_{>} \text{ bound states.}$$ 2