

Problem Sheet 1

1. Compute the residues of

- a). $\frac{z^3 \sin z}{z^4 + a^4}$ at $z = ae^{i\pi/4}$, where $a > 0$ is a non-zero real constant.
- b). $\frac{z+1}{(z^2-1)^2}$ at $z = 1$.
- c). $\frac{e^z}{z(z-a)^2}$ at $z = a$, where a is a non-zero complex constant.
- d). $\frac{z^2 e^z}{z^3 - a^3}$ at $z = a$, where a is a non-zero complex constant.

2. Use contour integration to show that

$$\begin{aligned} \text{a). } & \int_0^{2\pi} \frac{1}{5 - 4 \cos \theta} d\theta = \frac{2\pi}{3} \\ \text{b). } & \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta = \frac{\pi}{6} \\ \text{c). } & \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{6} \\ \text{d). } & \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12} \\ \text{e). } & \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1} dx = -\frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}} \sin 1 \\ \text{f). } & \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} dx = \frac{\pi}{2} e^{-2} \end{aligned}$$

Hint: For (a) and (b), use the change of variables $z = e^{i\theta}$.

3. Prove the *triangle inequality*, and the *negative triangle inequality* which states that for complex numbers $z_1, z_2 \in \mathbb{C}$,

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

4. In a programming language of your choice, create a function $s(x, \theta)$ which returns the value of the positive square root function, with the branch cut moved to $\arg z = \theta$. Note that the standard convention is $\theta = \pi$, and you will need to call this kind of default square root command from within your function. Test your function by plotting $\arg s(x, \theta)$, and checking for the right discontinuity.

5. Use contour integration to compute

$$\int_0^\infty \frac{x^{a-1}}{(x+1)^2} dx,$$

where $0 < a < 2$, $a \neq 1$.

6. Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \pi(b - a),$$

where $a, b > 0$.

7. By integrating around a rectangular contour with vertices at $\pm R$ and $2\pi i \pm R$ (taking $R \rightarrow \infty$) and with appropriate indentations, compute

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 - e^x} dx,$$

where $0 < a < 1$.

8. a). By integrating around a rectangular contour with vertices at $\pm R$ and $\pi i \pm R$ (taking $R \rightarrow \infty$), show that

$$\int_{-\infty}^{\infty} e^{ikx} \operatorname{sech} x dx = \pi \operatorname{sech}(k\pi/2),$$

where $k > 0$.

- b). Now obtain the result from part (a) using a semi-circular contour (*Hint: When summing the residues, note a geometric series*).