Mathematics Year 1, Calculus and Applications I

List of Quizzes

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1. For the following functions construct specific $\varepsilon - \delta$ definitions of continuity at $x = 0$. In other words given a ε you need to find $\delta(\varepsilon)$.

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ x^2 & \text{for } x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & \text{for } x \geq 0 \\ |x|^{1/2} & \text{for } x < 0 \end{cases}$$

2. Consider the function

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

   (a) Is $f(x)$ a continuous function?

   (b) Show that $f'(0)$ exists and find its value.

   (c) Define $g(x) = f'(x)$, $x \neq 0$, and $g(0) = f'(0)$. Determine whether $g(x)$ is differentiable or not.

   (d) If instead of x^2 in the definition of $f(x)$ we had x^n where n is a positive integer. How many derivatives of $f(x)$ would exist in this case?

3. A spherical balloon is being blown up by injecting air into it at 1 liter per second. When its radius is 1 m, find the rate at which its area is increasing (pay attention to the units).

4. Sand is being piled onto a conical pile at a constant rate of $R\ cm^3/s$. As the pile grows, frictional forces between sand particles constrain the height of the pile to be equal to the radius of its base.

   (i) When the height equals 1 cm, find the rate at which it is increasing.

   (ii) If the height at time t is $h(t)$, find an explicit expression for it. What happens to its rate of change as t becomes large? Explain physically/intuitively.

5. (a) Determine the regions of increase and decrease of the function $f(x) = x^3 - 2x + 1$.

   (b) Sketch functions for which the intermediate value theorem holds and:

      (i) For a chosen y^* there are *at most two* values of x^*.

(ii) For a chosen y^* I can choose an interval $[a,b]$ to have as many x^* as I want. [Any guess as to what function this is?]

(iii) For a chosen y^* there does not exist a x^*, i.e. Theorem 6 does not hold.

6. Find $\tan^{-1}(\tan \frac{3\pi}{4})$ and $\arctan(\tan 2\pi)$. (No computers/calculators!)

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7. (a) Let $f(x) = x^a$ for $x > 0$ and a any real number. Prove that $f'(x)$ exists and is equal to ax^{a-1}. [Hint: Write as an exponential function.]

   (b) We know that $\frac{d}{dx}\log x = \frac{1}{x}$. Show that $\frac{d}{dx}\log_a x = \frac{1}{x\log a}$.
   [Hint: Let $y = \log_a x$ so that $x = a^y$.]

8. Find the equation of the tangent line at $(0, \log 3)$ to the graph of the curve defined implicitly by $e^y - 3 + \log(x+1)\cos y = 0$.

9. What is wrong with the following solution using L'Hôpital's rule:

$$\lim_{x\to 0}\frac{x^2+1}{x} = \lim_{x\to 0}\frac{2x}{1} = 0$$

10. Consider the step-ladder function $f(x) = 1 + [x]$ for $x \geq 0$, where $[x]$ denotes the integer part of x. (For example $[0.9] = 0$, $[2.1] = 2$, etc.)

    (a) Sketch $f(x)$.

    (b) Calculate and sketch $F(x) = \int_0^x f(t)dt$.

11. Need to calculate $\int_0^1 e^x dx$. (Of course the answer is $e-1$.)

    The Upper Riemann sum is $U_n = \sum_{i=1}^n e^{i/n}\frac{1}{n}$.

    Show that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n (e^{1/n})^i = e-1$.

    [Recall: Geometric series $\sum_{i=1}^n r^i = \frac{r-r^{n+1}}{1-r}$.]

12. Prove that

$$\int_0^{\log 2} e^{-x}\cos x^2\, dx \leq \frac{1}{2}.$$

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13. An inverted conical tank of base radius 6m and height 10m is full and is to be pumped empty (from the top) by a pump that has power output of 10^5 joules per hour (this is approximately 27.77 watts).

    (a) What is the water level at the end of 6 minutes of pumping?

    (b) How fast is the water level dropping this time?

14. Show that $\int_0^\infty \frac{dx}{\sqrt{1+x^8}}$ is convergent, by comparison with $1/x^4$.

15. (a) Calculate the integral $\int \frac{x^2}{(1+x^2)^{3/2}}dx$ using (i) a trigonometric substitution, (ii) integration by parts.

    (b) Calculate $\int_0^{\pi/2}\cos^7 x dx$.

16. An object moves from left to right along the curve $y = x^{3/2}$ at constant speed. If the object is at $(0,0)$ at noon and at $(1,1)$ at 1:00 pm, where is it at 1:30pm?

17. The region between the graphs of $\sin x$ and x over the interval $0 \leq x \leq \pi/2$, is revolved about the $y-$axis. Sketch the resulting solid and find its volume.

18. Find the area of the surface of the solid formed by rotating the curve $y = x^2$, $0 \leq x \leq 1$, about (i) the $x-$ axis, and (ii) the y-axis.

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19. (i) Consider the particular case of the centre of mass of a plate of area A that has unit density per unit area. Write down formulas for the centre of mass in terms of double integrals.

    (ii) Now take A to be the region $0 \leq x \leq 1$, $0 \leq y \leq 1$ (unit density also). Compute the double integrals to show that the centre of mass is $(1/2, 1/2)$ as expected by intuition.

    (iii) Now take A as in part (ii) but assume that the density in half the square between $0 \leq x \leq 1/2$ is 1, and the density in the remaining half of the square is ρ_0. Compute the centre of mass in this case.

    (iv) Check that your formula in (iii) agrees with your answer in (ii) when $\rho_0 = 1$. What happens as ρ_0 tends to 0 and ∞? Explain using physical intuition.

20. (a) Consider the functions $f(x) = x^m$ and $g(x) = x^n$ where $0 < m < n$. Find a condition satisfied by m and n that guarantees that the centre of mass of the region between the graphs $y = f(x)$ and $y = g(x)$ is *outside* the region.

    (b) We know from Example 2 from Recording 20 that $m = 1$, $n = 2$ has the centre of mass within the region. What are the smallest integer pair m, n for which the centre of mass is outside the region?

21. Use the results developed in Recording 21 to show that a circle of radius a has perimeter $2\pi a$ and area πa^2.

22. (a) For what real values of α (if any at all) does the series $\sum_{n=1}^{\infty} \frac{e^{\alpha n}}{n}$ converge?

    (b) Calculate $\sum_{n=100}^{\infty} \frac{1}{n^{1/100}}$.

    (c) Let $P = \sum_{n=1}^{K} 3^n$ and $Q = \sum_{n=1}^{K} (1/3)^n$. Find P, Q and PQ. Which ones converge as $K \to \infty$? Could you have anticipated this without calculations?

23. (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n^2}$ converge?

    (b) Does the series $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converge?

    (c) Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n}$ convergent? Is it absolutely convergent?

24. (a) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by using (i) the comparison test, (ii) the integral test.

    (b) Does the series $\sum_{n=1}^{\infty} \frac{3n+\sqrt{n}}{2n^{3/2}+2}$ converge or diverge?

    (c) For what values of $p > 0$ does the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge?

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25. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - \log n}$.

26. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

27. (a) Prove that $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty}(\sum_{m=0}^{n} a_m b_{n-m})x^n$.

    (b) Find the power series of $\frac{\log(1+x^2)}{1+x^2}$.

28. Use Taylor's theorem to find a second order accurate formula for the finite difference approximation of $f^{(2)}(x)$ using information from the points $f(x)$, $f(x \pm h)$.

29. Use Taylor series to compute $1/e$ to three decimal accuracy. Compare the number of terms needed with example 3 in Recording 29. Explain.

30. Find the first four non-zero terms in the Maclaurin expansion of $\log(1 + \sqrt{\sin x})$.

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31. The Chebyshev[1] polynomials of the first kind are defined by $T_n = \cos(n \cos^{-1} x)$, where $n = 0, 1, 2, \ldots$.

    (a) Show that

$$T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x$$

    (b) Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

    (c) Verify that $T_n(x)$, $n = 0, 1, 2, \ldots$, satisfy the Chebyshev differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0.$$

32. Find and sketch the periodic extensions of

    (a) $f(x) = x^2$, defined on $x \in [-\pi, \pi]$. Call this function $g_1(x)$.

    (b) $f(x) = e^x$, defined on $x \in [0, 1]$. Call this function $g_2(x)$.

    (c) Calculate $\int_0^{7\pi} g_1(x)dx$ and $\int_{1/2}^{21/2} g_2(x)dx$.

33. Express $f(x) = \sin(x + \pi/4) + \sin^2 x - \cos^2 x$ as a complex trigonometric series.

34. Give examples of functions $f(x)$ that are periodic of period 2π and are:

    (i) Smooth (i.e. infinitely differentiable).

    (ii) Piecewise continuous.

    (iii) Piecewise continuous with piecewise continuous first derivative.

    (iv) Piecewise continuous with piecewise continuous first derivative but not second derivative.

    (v) As nasty a function as you can think of whose Fourier series exists.

35. Find directly the Fourier series of $\phi(x) = x$, $-\pi < x < \pi$ without recourse to shifts and reflections used in class.

36. (a) For the function $f(x) = e^x$ defined on $[0, \pi]$, find the cosine series and the sine series. Call them $S_c(x)$ and $S_s(x)$.

[1]Pafnuti Chebyshev (1821-1894), Russian mathematician. Chebyshev polynomials are used widely in solutions of partial differential equations, including aerodynamics, biofluiddynamics, materials, waves, ...

(b) Discuss convergence of S_c and S_s for all values of $x \in [-\pi, \pi]$.

(c) Now consider $\frac{d}{dx}(S_c)$ and $\frac{d}{dx}(S_s)$. Which one (if any) converges and why?

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37. Sketch the function $f(x)$ given below and find its Fourier series:

$$f(x) = \begin{cases} 0 & -3 \leq x < -2 \\ -x-1 & -2 < x \leq 1 \\ 0 & -1 \leq x < 1 \\ 1-(x-1)^2 & 1 < x \leq 2 \\ 0 & 2 \leq x \leq 3 \end{cases}$$

38. Suppose $u(x)$ is a smooth $2\pi-$periodic function defined on $-\pi \leq x \leq \pi$. Use Parseval's theorem to find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_x^2(x)dx, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} u_{xx}^2(x)dx,$$

in terms of $\|u\|^2 := \frac{1}{2\pi}\int_{-\pi}^{\pi} u^2(x)dx$.

Hence show that if $0 < m < n$ are integers, then

$$\int_{-\pi}^{\pi}\left(\frac{d^m u}{dx^m}\right)^2 dx \leq \int_{-\pi}^{\pi}\left(\frac{d^n u}{dx^n}\right)^2 dx.$$
