# M40007: Introduction to Applied Mathematics



Imperial College London

# 1 Spring-mass systems

Consider 3 masses connected by 2 springs as shown in Figure 1. The masses are labelled $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$. The springs are labelled ⓐ and ⓑ. In the upper row the masses are in their equilibrium positions; in the lower row the masses are in strained positions and non-zero tensions $T_a$ and $T_b$ exist in the springs joining the masses. For now, the **only** forces acting on the masses are due to the springs.

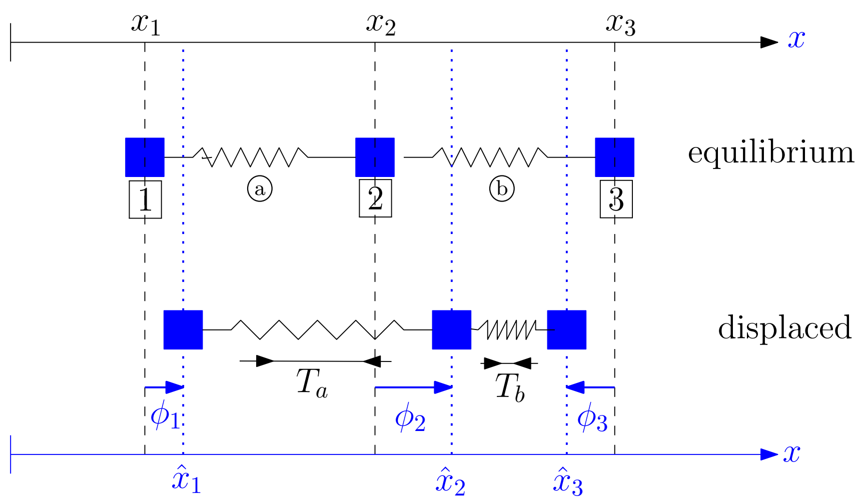


Figure 1: A spring-mass system. If any two masses connected by a spring are displaced from equilibrium a tension exists in the spring connecting them.

The equilibrium positions of the masses are

$$x_1, \qquad x_2, \qquad x_3 \tag{1}$$

and their positions once displaced are

$$\hat{x}_1, \qquad \hat{x}_2, \qquad \hat{x}_3. \tag{2}$$

The important quantities are the *displacements* from equilibrium which are then given by

$$\phi_1 = \hat{x}_1 - x_1, \qquad \phi_2 = \hat{x}_2 - x_2, \qquad \phi_3 = \hat{x}_3 - x_3. \tag{3}$$

**Hooke's law:** According to Hooke's law, the tension in spring ⓐ is

$$T_a = c_a(\phi_2 - \phi_1), \tag{4}$$

where $c_a > 0$ is the spring constant. The quantity $\phi_2 - \phi_1$ is the *extension* of the spring. As drawn in Figure 1 this tension will be positive since $\phi_2 > \phi_1$. This means that the masses will be pulled towards each other by spring ⓐ. Such a spring is sometimes called a *linear spring* because of the linear relationship between the force and the extension of the spring.

Similarly, Hooke's law says that the tension in spring (b) is

$$T_b = c_b(\phi_3 - \phi_2), \tag{5}$$

where $c_b > 0$ is its spring constant. As drawn in Figure 1 this tension will be negative since $\phi_3 < 0$ and $\phi_2 > 0$. This means that the masses will be pushed away from each other by spring (b).

## 2 Spring-mass system as a graph

Any spring-mass system can be modelled as a graph with the nodes being the masses and the *displacement* of each node being the node potential. The edges of the graph are the springs, each having a spring constant, and the *tension* in each spring being the edge variable.

Consider the spring-mass system shown in Figure 2.

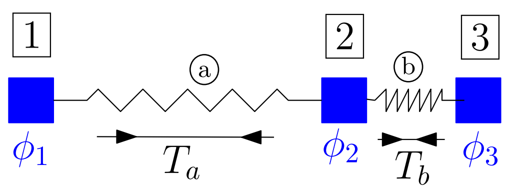


Figure 2: A typical spring-mass system.

There are $n = 3$ masses, or nodes to the graph, and $m = 2$ springs, or edges to the graph. In order to write down an incidence matrix for this graph, we need to make a choice of direction on each edge, i.e., pick the arrows. However, since the edges are aligned along an $x$-axis, say, the most natural choice is to pick all arrows to point to the right in Figure 2, along the positive $x$ direction. The incidence matrix of the graph is then

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{edge (a)} \\ \text{edge (b)} \end{array} \tag{6}$$

We will use

$$\mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{7}$$

to denote the vector of node potentials, i.e., the vector of the displacements of the masses. Then,

$$\mathbf{A\Phi} = \begin{pmatrix} \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \end{pmatrix}. \tag{8}$$

This is the vector of the extensions of the springs. If we introduce the diagonal matrix $\mathbf{C}$ where

$$\mathbf{C} = \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}, \tag{9}$$

where the positive spring constants are on the diagonal, then Hooke's law – encompassing (4) and (5) – can be written in matrix form as

$$\mathbf{T} = \begin{pmatrix} T_a \\ T_b \end{pmatrix} = \mathbf{CA\Phi}. \tag{10}$$

It is clear that Hooke's law for $\mathbf{T}$ in this mechanical spring-mass system is analogous to Ohm's law for the currents $\mathbf{w}$ in the electric circuit problem.

Also in analogy with the electric circuit problem, where

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w} \tag{11}$$

gives the divergence of the currents at the nodes, the quantity defined by

$$\mathbf{f}_I = -\mathbf{A}^T\mathbf{T} \tag{12}$$

now gives the divergence of the spring tensions at the masses. Physically, the divergence of the spring tensions at the masses gives the net *internal forces on the masses due to the tensions in the springs*; the subscript on $\mathbf{f}_I$ is added to emphasize that these are internal spring forces. To see this, note that

$$-\mathbf{A}^T\mathbf{T} = -\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_a \\ T_b \end{pmatrix} = \begin{pmatrix} T_a \\ T_b - T_a \\ -T_b \end{pmatrix}. \tag{13}$$

As is evident from Figure 2, the quantities in this vector are the total forces on each mass $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ in the positive $x$-direction due to the springs.

Putting the pieces together, from (12) and (10) we have

$$\mathbf{f}_I = -\mathbf{A}^T\mathbf{T} = -\mathbf{A}^T\mathbf{CA\Phi}. \tag{14}$$

Equivalently, the vector of total internal spring forces on each mass is given by

$$\mathbf{f}_I = -\mathbf{K\Phi}, \qquad \mathbf{K} = \mathbf{A}^T\mathbf{CA}. \tag{15}$$

Here $\mathbf{K}$ is the weighted Laplacian matrix now weighted by the set of (positive) spring constants.

What is the analogue of the Kirchhoff current law at the nodes in this spring-mass system? That depends on which physical problem is of interest. If we seek conditions for the masses to be *in equilibrium* – and therefore not moving – the required condition is that the total force on each mass must be zero. This is the condition of force balance in mechanical equilibrium.

For additional generality, in addition to the internal spring forces, we will also allow in our formulation for a set of *external forces* on each mass. These external forces will be collected in the vector

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{16}$$

Among the many possibilities, these external forces could be due to *gravitational forces* caused by a gravitational field, or due to *reaction forces* if any given mass is simply constrained not to move.

The constraint that the *total force* on each mass vanishes at mechanical equilibrium is then equivalent to the vector equation

$$\mathbf{f} + \mathbf{f}_I = 0, \tag{17}$$

where, to get the total force on each mass, we have added together the external forces and the internal forces due to the springs. For mechanical equilibrium, the total force on each mass must vanish.

On combining (15) and (17) we arrive at

$$\mathbf{f} = -\mathbf{f}_I = \mathbf{A}^T\mathbf{C}\mathbf{A}\boldsymbol{\Phi} = \mathbf{K}\boldsymbol{\Phi}. \tag{18}$$

Remarkably, this equation is exactly the same as that appearing in the electric circuit problem; it is just the physical interpretation of the vectors that changes. In the spring-mass application, the $\Phi$ is the vector of displacements of the masses at the nodes and $\mathbf{f}$ is the vector of external forces on the masses. The spring-constant-weighted Laplacian $\mathbf{K}$ encodes the influence of the springs.

# 3 Two masses hanging under gravity

Consider 2 masses attached to a ceiling, pulled down by the external force of gravity and connected by 2 springs as shown in Figure 3. It is convenient to think of the ceiling as node $\boxed{1}$ and the two masses, of mass $m_2$ and $m_3$, as nodes $\boxed{2}$ and $\boxed{3}$. The springs are labelled a and b. Their spring constants are $c_a$ and $c_b$ respectively.

The challenge is to determine the equilibrium displacements, $\phi_2$ and $\phi_3$, of the two masses, and also the reaction force $R$ at the ceiling.

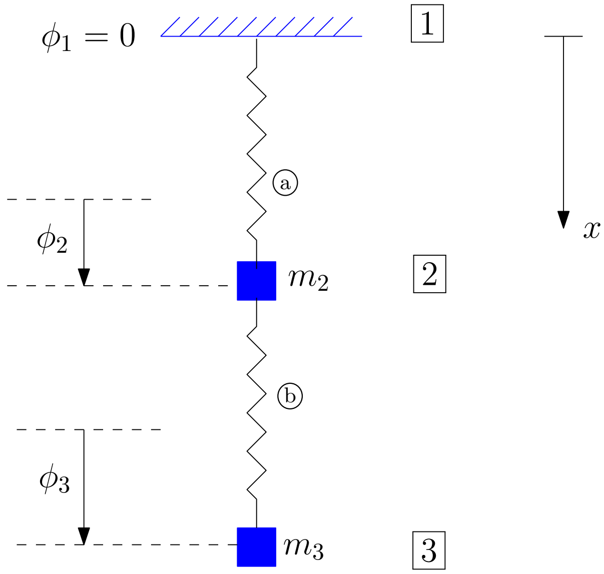


Figure 3: Two masses hanging under gravity. The ceiling is viewed as mass $\boxed{1}$ but has zero displacement.

By the usual constructions, it is easy to find the spring-constant-weighted Laplacian to be

$$\begin{array}{cc} & \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \mathbf{K} = & \begin{pmatrix} c_a & -c_a & 0 \\ -c_a & c_a + c_b & -c_b \\ 0 & -c_b & c_b \end{pmatrix} \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \end{array} \tag{19}$$

The external forces at the nodes are

$$\mathbf{f} = \begin{pmatrix} R \\ m_2 g \\ m_3 g \end{pmatrix}, \tag{20}$$

where $R$ is the reaction force at the ceiling. The force due to gravity, assumed to the in the positive $x$-direction, is given by the mass multiplied by the acceleration due to gravity $g$. The equations for equilibrium are therefore

$$\mathbf{f} = \begin{pmatrix} R \\ m_2 g \\ m_3 g \end{pmatrix} = \mathbf{K} \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} c_a & -c_a & 0 \\ -c_a & c_a + c_b & -c_b \\ 0 & -c_b & c_b \end{pmatrix} \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \tag{21}$$

This can be readily solved –either directly, or, for example, using Schur complements – to find

$$\phi_2 = \left(\frac{m_2 + m_3}{c_a}\right) g, \qquad \phi_3 = \frac{m_3 g}{c_b} + \left(\frac{m_2 + m_3}{c_a}\right) g \tag{22}$$

and

$$R = -(m_2 + m_3)g. \tag{23}$$

Since the displacement of node $\boxed{1}$ has been fixed, which is akin to fixing a unit voltage at the + node in a 2-point source/sink electric circuit problem, the reaction force $R$ can be thought of as akin to the effective conductance in the circuit problem.

**Remark:** In this course the masses in a spring-mass network should **only** be taken to be subject to the external force of gravity if this is stated explicitly.

# 4 Newton's second law of motion

The structure of the equations for *equilibrium* of a spring-mass system has been seen to fit neatly into the same mathematical framework that worked successfully for the electric circuit problem. However, the spring-mass system allows for an interesting generalization to the *non-equilibrium* setting.

The physical law needed to study this is *Newton's second law* of motion. It is one of the most basic laws of mechanics. Suppose a mass $m$ has position $\mathbf{x}(t)$ where we now allow this position to vary with time $t$. The velocity $\mathbf{v}$ of this mass is defined as the vector

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \tag{24}$$

and its acceleration $\mathbf{a}$ as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}. \tag{25}$$

If $\mathbf{F}$ denotes the total force acting on the mass, then Newton's second law of motion dictates that

$$\mathbf{F} = m\mathbf{a}. \tag{26}$$

The idea now is to allow the masses in the spring-mass systems just considered to be displaced from equilibrium with their subsequent positions determined according to Newton's second law of motion. In what follows there is an implicit physical assumption that the displacements remain sufficiently small that Hooke's law for linear springs remains valid.

Consider a spring-mass system with $n$ *movable* masses. The total force on any mass is

$$\mathbf{F} = \mathbf{f} + \mathbf{f}_I = \mathbf{f} - \mathbf{K}\mathbf{\Phi}, \tag{27}$$

where $\mathbf{K}$ is the $n$-by-$n$ Laplacian weighted by any spring constants exerting forces on the masses and $\mathbf{f}$ denotes any external forces. The vector of accelerations of the masses is

$$\frac{d^2\mathbf{\Phi}}{dt^2}. \tag{28}$$

Newton's second law (26) therefore implies

$$\mathbf{F} = \mathbf{f} - \mathbf{K}\mathbf{\Phi} = \mathbf{M}\frac{d^2\mathbf{\Phi}}{dt^2}, \tag{29}$$

where we have introduced the $n$-by-$n$ diagonal matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & m_{n-1} & 0 \\ 0 & 0 & 0 & 0 & m_n \end{pmatrix}. \tag{30}$$

Since all masses are positive then, provided they are non-zero, $\mathbf{M}$ is a positive-definite matrix.

# 5 Two masses between fixed walls

Consider the situation shown in Figure 4 where two masses, each of unit mass, are connected by three springs labelled ⓐ, ⓑ and ⓒ between 2 fixed walls. Let all spring constants be unity.

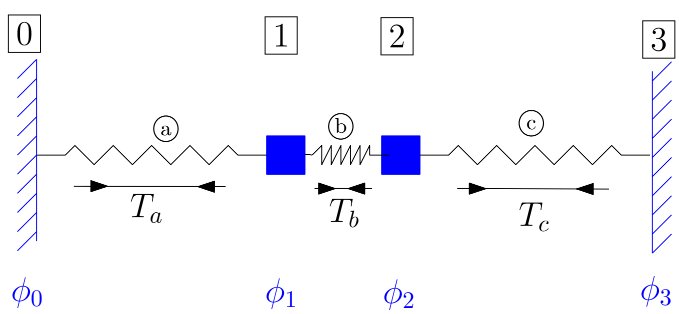


Figure 4: Two masses, three springs, between two fixed walls.

Let the walls and the masses each constitute a node in the graph and label them

$\boxed{0}$, $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ as shown in Figure 4. Let the vector of displacements be

$$\mathbf{\Phi} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{31}$$

The Laplacian matrix in this case is easy to find using the standard construction:

$$\mathbf{K} = \begin{array}{c} \begin{array}{cccc} \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \boxed{0} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \tag{32}$$

The walls are fixed implying that

$$\phi_0 = \phi_3 = 0. \tag{33}$$

It is therefore convenient to consider the reduced Laplacian

$$\hat{\mathbf{K}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{34}$$

since masses $\boxed{1}$ and $\boxed{2}$ move according to Newton's second law, which says that

$$\hat{\mathbf{f}} - \hat{\mathbf{K}}\hat{\mathbf{\Phi}} = \mathbf{I}\frac{d^2\hat{\mathbf{\Phi}}}{dt^2}, \tag{35}$$

where $\mathbf{I}$ is the 2-by-2 identity matrix – this is the matrix $\mathbf{M}$ when the masses are both unity – and

$$\hat{\mathbf{\Phi}} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \hat{\mathbf{f}} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{36}$$

where $f_i$ is the external force on mass $i$ for $i = 1, 2$.

**Free oscillations:** First let us assume there are no external forces on the masses so that they are in *free oscillation* under the influence of the internal spring forces. This means that

$$f_1 = f_2 = 0. \tag{37}$$

Equation (35) simplifies to

$$-\hat{\mathbf{K}}\hat{\mathbf{\Phi}} = \mathbf{I}\frac{d^2\hat{\mathbf{\Phi}}}{dt^2}. \tag{38}$$

This is a system of linear second order differential equations, with constant coefficients, for the displacements. It is natural to seek solutions of the form

$$\mathbf{\Phi} = \mathbf{\Phi}_0 e^{\mathrm{i}\omega t}, \tag{39}$$

where $\omega \in \mathbb{R}$ is a real constant and i is the square root of $-1$. The vector $\mathbf{\Phi}_0$ will be complex-valued. On substitution of (39) into (38) we find

$$-\hat{\mathbf{K}}\hat{\Phi}_0 e^{\mathrm{i}\omega t} = -\omega^2 \hat{\Phi}_0 e^{\mathrm{i}\omega t} \tag{40}$$

which simplifies to

$$\hat{\mathbf{K}}\hat{\Phi}_0 = \omega^2 \hat{\Phi}_0. \tag{41}$$

This is an *eigenvalue problem* for $\hat{\mathbf{K}}$. The vector $\hat{\Phi}_0$ is an eigenvector and $\omega^2$ is the corresponding eigenvalue.

The matrix $\hat{\mathbf{K}}$ is real and symmetric which means that it has 2 eigenvalues and they are both real. So too are the corresponding eigenvectors. The mathematical approach based on complex numbers used here appears to generate complex eigenvectors. However, the required real eigenvectors can be constructed on making the following three observations.

**Note 1:** Since both $\hat{\mathbf{K}}$ and $\omega^2$ are real then, on taking a complex conjugate of (41),

$$\hat{\mathbf{K}}\overline{\hat{\Phi}_0} = \omega^2 \overline{\hat{\Phi}_0} \tag{42}$$

which shows that $\overline{\hat{\Phi}_0}$ is *also* an eigenvector of $\hat{\mathbf{K}}$ and having the same eigenvalue as $\hat{\Phi}_0$.

**Note 2:** Since (38) is invariant if $\omega \mapsto -\omega$ then if $\mathbf{\Phi}_0 e^{\mathrm{i}\omega t}$ is a solution of (38) then so too is $\mathbf{\Phi}_0 e^{-\mathrm{i}\omega t}$.

**Note 3:** Notes 1 and 2 mean that if $\mathbf{\Phi}_0 e^{\mathrm{i}\omega t}$ is a solution of (38) then so too are the linear combinations

$$\mathrm{Re}\left[\mathbf{\Phi}_0 e^{\mathrm{i}\omega t}\right] = \frac{1}{2}\left[\mathbf{\Phi}_0 e^{\mathrm{i}\omega t} + \overline{\mathbf{\Phi}_0} e^{-\mathrm{i}\omega t}\right] \tag{43}$$

and

$$\mathrm{Im}\left[\mathbf{\Phi}_0 e^{\mathrm{i}\omega t}\right] = \frac{1}{2\mathrm{i}}\left[\mathbf{\Phi}_0 e^{\mathrm{i}\omega t} - \overline{\mathbf{\Phi}_0} e^{-\mathrm{i}\omega t}\right]. \tag{44}$$

Recall that the eigenspace of a matrix is a vector space so eigenvectors with the same eigenvalue can be added together to form other eigenvectors with the same eigenvalue.

Let us find the eigenvalues and eigenvectors for the 2-mass problem shown in Figure 4. It is convenient to set

$$\lambda = \omega^2 \tag{45}$$

and to solve the standard eigenvalue problem

$$\hat{\mathbf{K}}\hat{\Phi}_0 = \lambda \hat{\Phi}_0. \tag{46}$$

The existence of a non-trivial solution requires the following zero determinant condition to hold:

$$\text{der} \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = 0 \tag{47}$$

or

$$(2-\lambda)^2 - 1 = 0. \tag{48}$$

This leads to

$$\lambda = 1, 3. \tag{49}$$

The corresponding values of $\omega$ are

$$\omega = \pm 1, \pm\sqrt{3}. \tag{50}$$

It is common to call the values of $\omega$ found above to be the *natural frequencies of oscillation* of the system. The corresponding eigenvectors – also known as the *natural modes of oscillation* – are easy to find:

$$\lambda = 1, \qquad \hat{\Phi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{51}$$

$$\lambda = 3, \qquad \hat{\Phi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{52}$$

It is easy to check that these two eigenvectors are orthogonal, i.e.,

$$\hat{\Phi}_1^T \hat{\Phi}_2 = 0, \tag{53}$$

as expected from linear algebra results on the eigenvectors of real symmetric matrices.

The general solution for the time-dependent vector $\mathbf{\Phi}$ is therefore

$$\mathbf{\Phi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (c_1 e^{\mathrm{i}t} + \overline{c_1} e^{-\mathrm{i}t}) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (c_2 e^{\sqrt{3}it} + \overline{c_2} e^{-\sqrt{3}it}), \tag{54}$$

where $c_1, c_2 \in \mathbb{C}$ are two complex constants. This general solution therefore depends on 4 real constants; the real and imaginary parts of $c_1$ and $c_2$. Physically, those 4 constants would be determined by the initial positions and initial velocities of the 2 masses.

What are the reaction forces at the walls? These are the external forces at nodes

$\boxed{0}$ and $\boxed{3}$. Let the vector of external forces at the nodes be

$$\mathbf{f} = \begin{pmatrix} R_0 \\ 0 \\ 0 \\ R_3 \end{pmatrix}. \tag{55}$$

Now we can invoke Newton's second law in the form (29) which implies

$$\mathbf{f} - \mathbf{K\Phi} = \begin{pmatrix} R_0 \\ 0 \\ 0 \\ R_3 \end{pmatrix} - \mathbf{K} \begin{pmatrix} 0 \\ \phi_1(t) \\ \phi_2(t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2\phi_1(t) \\ -\omega^2\phi_2(t) \\ 0 \end{pmatrix} \tag{56}$$

to deduce that

$$R_0 = -\phi_1(t), \qquad R_3 = -\phi_2(t). \tag{57}$$

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**Exercise:** Verify that if

$$c_1 = \frac{1}{2}(A - \mathrm{i}B), \qquad c_2 = \frac{1}{2}(C - \mathrm{i}D) \tag{58}$$

then the solution (54) can be written in the alternative form

$$\mathbf{\Phi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A\cos t + B\sin t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C\cos\sqrt{3}t + D\sin\sqrt{3}t). \tag{59}$$

# 6 Three masses between fixed walls

Now consider the situation shown in Figure 5 where three masses, each of unit mass, are connected by four springs labelled a, b, c and d between 2 fixed walls. Let all the spring constants be unity.

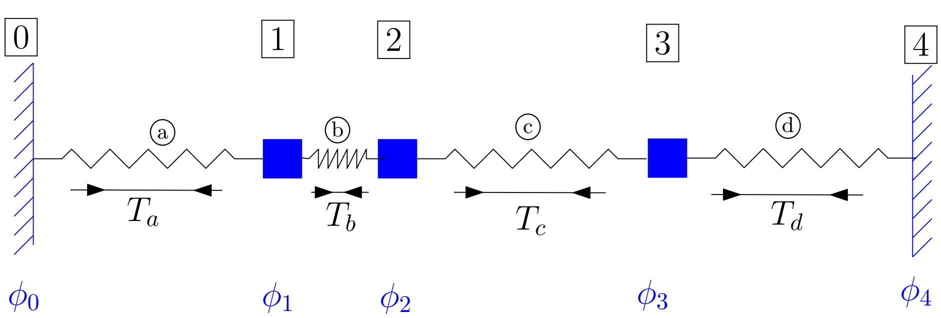


Figure 5: Two masses, three springs, between two fixed walls.

The walls and the masses constitute nodes in a graph; label them $\boxed{0}$, $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ as shown in Figure 5. Let the vector of displacements be

$$\mathbf{\Phi} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \tag{60}$$

The Laplacian matrix in this case is easy to find using the standard construction:

$$\mathbf{K} = \begin{array}{c} \begin{array}{ccccc} \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \boxed{0} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array} \tag{61}$$

The walls are fixed implying zero displacements:

$$\phi_0 = \phi_4 = 0. \tag{62}$$

It is therefore convenient to consider the reduced Laplacian

$$\hat{\mathbf{K}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{63}$$

and to consider only the motion of masses $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ under Newton's second law. This law says that

$$\hat{\mathbf{f}} - \hat{\mathbf{K}}\hat{\mathbf{\Phi}} = \mathbf{I}\frac{d^2\hat{\mathbf{\Phi}}}{dt^2}, \tag{64}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix – this is the matrix $\mathbf{M}$ when the masses are all unity – and

$$\hat{\mathbf{\Phi}} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \qquad \hat{\mathbf{f}} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{65}$$

where $f_i$ is the external force on mass $i$ for $i = 1, 2, 3$.

**Free oscillations:** First we assume there are no external forces on the masses so that they are in free oscillation under the influence of the internal spring forces. This means that

$$f_1 = f_2 = f_3 = 0, \qquad \hat{\mathbf{f}} = 0. \tag{66}$$

Equation (64) simplifies to

$$-\hat{\mathbf{K}}\hat{\Phi} = \mathbf{I}\frac{d^2\hat{\Phi}}{dt^2}. \tag{67}$$

As before we seek solutions of this system of linear second order differential equations of the form

$$\mathbf{\Phi} = \mathbf{\Phi}_0 e^{\mathrm{i}\omega t}, \qquad \omega \in \mathbb{R}. \tag{68}$$

On substitution of (68) into (67) we find

$$-\hat{\mathbf{K}}\hat{\Phi}_0 e^{\mathrm{i}\omega t} = -\omega^2\hat{\Phi}_0 e^{\mathrm{i}\omega t} \tag{69}$$

which simplifies, on cancellation of common terms, to the eigenvalue problem

$$\hat{\mathbf{K}}\hat{\Phi}_0 = \lambda\hat{\Phi}_0, \qquad \lambda = \omega^2. \tag{70}$$

The existence of non-trivial solutions requires the zero determinant condition to hold:

$$\det\begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = 0. \tag{71}$$

The solution of this characteristic equation leads to

$$\lambda = 2, 2 \pm \sqrt{2} \tag{72}$$

leading to the natural frequencies

$$\omega = \pm\sqrt{2}, \pm\sqrt{2 \pm \sqrt{2}}. \tag{73}$$

The corresponding eigenvectors are easy to find:

$$\lambda = 2, \qquad \hat{\Phi}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \tag{74}$$

$$\lambda = 2 + \sqrt{2}, \qquad \hat{\Phi}_2 = \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}, \tag{75}$$

$$\lambda = 2 - \sqrt{2}, \qquad \hat{\Phi}_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{76}$$

It can be verified that these eigenvectors are mutually orthogonal, as expected from results in linear algebra.

The general solution for $\mathbf{\Phi}$ can therefore be written as

$$
\begin{aligned}
\mathbf{\Phi} = & \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (c_1 e^{\sqrt{2}\mathrm{i}t} + \overline{c_1} e^{-\sqrt{2}\mathrm{i}t}) + \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} (c_2 e^{\sqrt{2+\sqrt{2}}\mathrm{i}t} + \overline{c_2} e^{-\sqrt{2+\sqrt{2}}\mathrm{i}t}) \\
& + \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} (c_3 e^{\sqrt{2-\sqrt{2}}\mathrm{i}t} + \overline{c_3} e^{-\sqrt{2-\sqrt{2}}\mathrm{i}t}),
\end{aligned} \tag{77}
$$

where $c_1, c_2$ and $c_3 \in \mathbb{C}$ are complex constants. This general solution depends on 6 real constants determined by the initial positions and initial velocities of the 3 masses.

The structure of this type of problem for the free oscillation of masses connected to springs between two fixed walls should be clear from these two examples. Adding more masses and springs leads to higher dimensional eigenvalue problems. It becomes increasingly challenging, however, to find the associated natural frequencies and natural modes of oscillation, at least, using this direct solution scheme based on evaluating determinants. Some deeper mathematical understanding of the structure of the Laplacian matrices underlying these oscillatory systems is needed.