Tensor Calculus and General Relativity **Spring Term 2023**

Problem Sheet 1

The standard Lorentz boost is (here K' is moving in the x direction with speed v relative to the frame K)

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z,$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

1. A particle in K' moves along the y'-axis at constant velocity w as $y' = wt'$, $x' = z' = 0$. What is the magnitude and direction of its velocity as measured in frame K?

2. The Lorentz boost given above can be written in the form

   $$ct' = ct\cosh\psi - x\sinh\psi, \quad x' = x\cosh\psi - ct\sinh\psi, \quad y' = y, \quad z' = z$$

   where $\tanh\psi = v/c$.

   Consider a third inertial frame K'' connected to K' through the boost

   $$ct'' = ct'\cosh\phi - x'\sinh\phi, \quad x'' = x'\cosh\phi - ct'\sinh\phi, \quad y'' = y', \quad z'' = z'$$

   where $\tanh\phi = w/c$. Find the boost connecting K'' and K. How fast is K'' moving relative to K?

3. In the twin paradox one twin remains on Earth while the other twin embarks on a high speed round trip. When the twins reunite the second twin has aged less than the first due to time dilation.

   This question is a variation of the famous problem. Here the second twin maintains a constant velocity relative to Earth so the twins never reunite. However, they remain in contact electronically.

   (i) Twin 2 starts her journey on the twins' 21st birthday. When twin 1 is 22 years old she transmits a video message to her sister. Twin 2 receives this message (which travels at the speed of light) on her 23rd birthday. What is the speed of the spacecraft?

   (ii) Twin 2 sends an immediate reply. How old is twin 1 when she receives the response?

   (iii) Include the events from parts (i) and (ii) in a space-time diagram.

4. Do any two Lorentz boosts commute?

5. A boost can be written in the form

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu}.$$

   Here $\Lambda^{\mu'}_{\nu}$ can be viewed as the elements of a 4×4 matrix Λ. Show that for the standard boost in the x direction $\Lambda = \exp(\psi K)$ where K is a 4×4 matrix and $\tanh \psi = v/c$ is as in question 2. What are the corresponding matrices for boosts in the y and z directions?

6. A particle is accelerating along the x^1-axis. The components of the four acceleration satisfy (ignoring the x^2 and x^3 directions)

$$ca^0 = Au^1, \qquad ca^1 = Au^0,$$

   where A is a constant. Suppose that the particle is at rest when $\tau = 0$. Determine $u^0(\tau)$ and $u^1(\tau)$. Find $x^0(\tau)$ and $x^1(\tau)$ assuming that $x^0 = x^1 = 0$ when $\tau = 0$. Sketch the worldline of the particle on a space-time diagram.