Imperial College London

MATH97041 MATH97150

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry 1: Algebraic Curves

Date: Wednesday, 19 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

You may use the results and exercises from the lecture notes, problem sheets and coursework, but make sure to clearly indicate what you are using in your answers. Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.

1. (a) Let X be a connected compact Riemann surface, and let p be a point of X. Let f be a meromorphic function on X with a pole of order 1 at p and no other poles. Let $\overline{f}$ be the morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ induced by f.

   (i) Compute the ramification degree of $\overline{f}$ at p and prove your answer. (5 marks)

   (ii) Show that $\overline{f}$ is an isomorphism of Riemann surfaces. (5 marks)

   (b) Let X and Y be connected compact Riemann surfaces and let $h\colon Y \to X$ be a morphism of degree 2. Use the Riemann-Hurwitz formula to show that the number of ramification points of h is even. (5 marks)

   (c) Let X be a Riemann surface and let C be a smooth projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$. We denote by U_0, U_1 and U_2 the affine charts on $\mathbb{P}^2_{\mathbb{C}}$. Consider a map of sets $h\colon X \to C$. Show that h is a morphism of Riemann surfaces if and only if all of the following properties are satisfied:

   * the set $V_0 = h^{-1}(U_0 \cap C)$ is open in X, and the functions $(x_1/x_0) \circ h$ and $(x_2/x_0) \circ h$ are holomorphic on V_0;
   * the set $V_1 = h^{-1}(U_1 \cap C)$ is open in X, and the functions $(x_0/x_1) \circ h$ and $(x_2/x_1) \circ h$ are holomorphic on V_1;
   * the set $V_2 = h^{-1}(U_2 \cap C)$ is open in X, and the functions $(x_0/x_2) \circ h$ and $(x_1/x_2) \circ h$ are holomorphic on V_2.

   (5 marks)

(Total: 20 marks)

2. (a) Let C be an affine plane curve in $\mathbb{C}^2$. We say that a line L in $\mathbb{C}^2$ is an *asymptote* of C if the projectivization of L is tangent to the projectivization of C at some point at infinity of C. This point at infinity may be singular; then we use the definition of tangent lines from problem sheet 3. Compute the asymptotes of the following affine plane curves in $\mathbb{C}$ and explain your answers:

   (i) the curve D defined by $y^2 = x^3 + 1$; (4 marks)

   (ii) the curve E defined by $y^2 = x^4 + xy^2$. (4 marks)

   (b) Show that an affine plane curve of degree d in $\mathbb{C}^2$ has at most d asymptotes (see part (a) for the definition of an asymptote). (6 marks)

   (c) Show that every projective plane curve C of degree 4 in $\mathbb{P}^2_{\mathbb{C}}$ with four singular points is reducible. *Hint: construct a conic D such that C and D have a common component.*

   (6 marks)

(Total: 20 marks)

3. (a) Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_0 = 0$, and let p be the point $[1, 0, 0]$. Let S be the set of effective divisors of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ whose support contains $L \cup \{p\}$. Show that S is a linear system. Compute its dimension and prove your answer. (6 marks)

(b) Let p_1, p_2, p_3, p_4 be distinct points in $\mathbb{P}^2_{\mathbb{C}}$ such that no three of them are collinear. Show that there exist two non-degenerate conics in $\mathbb{P}^2_{\mathbb{C}}$ whose intersection is equal to $\{p_1, p_2, p_3, p_4\}$. (6 marks)

(c) Let P be a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors. Let Q be the linear polynomial $ax_0 + bx_1 - x_2$, for some a and b in $\mathbb{C}$, and assume that Q does not divide P. Let $p = [p_0, p_1, p_2]$ be a zero of Q in $\mathbb{P}^2_{\mathbb{C}}$. Use the axiomatic characterization of intersection multiplicities to show that

$$\mathbf{I}(p, P, Q) = \mathrm{mult}_{(p_0,p_1)} P(x_0, x_1, ax_0 + bx_1).$$

Clearly indicate which axioms you are using. (8 marks)

(Total: 20 marks)

4. For every λ in $\mathbb{C} \setminus \{0, 1\}$, we denote by C_λ the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by the Legendre equation

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2).$$

(a) Let λ be an element in $\mathbb{C} \setminus \{0, 1\}$. Show that C_λ and C_μ are projectively equivalent for $\mu = 1 - \lambda$ and also for $\mu = 1/\lambda$. Deduce that C_λ is projectively equivalent to C_μ for every μ in

$$\left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.$$

(3 marks)

(b) Let C be a smooth non-degenerate cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and let p and q be two inflection points of C. Show that there exists a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(C) = C$ and $\Phi(p) = q$. (6 marks)

(c) Let λ be an element in $\mathbb{C} \setminus \{0, 1\}$. Determine the points p on C_λ different from $[0, 1, 0]$ such that the tangent line to C_λ at p passes through $[0, 1, 0]$. (3 marks)

(d) Let λ and μ be elements in $\mathbb{C} \setminus \{0, 1\}$. Show that C_λ and C_μ are projectively equivalent if and only if

$$\mu \in \left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.$$

(8 marks)

(Total: 20 marks)

5. In this question you may freely use that every connected compact Riemann surface satisfies the Riemann-Roch theorem from the mastery material. Let X be a connected compact Riemann surface of genus 1, and let p be a point of X. For every integer n, we consider the divisor np on X. We denote by $\ell(np)$ the dimension of the complex vector space $\mathcal{L}(np)$ of meromorphic functions f on X such that $f = 0$ or $f \neq 0$ and $(f) + np \geq 0$.

(a) Compute $\ell(np)$ for every integer n. (4 marks)

(b) Show that the complex vector space $\mathcal{L}(3p)$ has a basis of the form $\{1, f, g\}$ where f has a pole of order 2 at p and g has a pole of order 3 at p. (4 marks)

(c) Show that there exist complex numbers $a_0, a_1, \ldots, a_6$ such that a_0 is non-zero and

$$g^2 + a_1 fg + a_3 g = a_0 f^3 + a_2 f^2 + a_4 f + a_6.$$

(4 marks)

(d) Consider the homogeneous polynomial

$$P = x_1^2 x_2 + a_1 x_0 x_1 x_2 + a_3 x_1 x_2^2 - a_0 x_0^3 - a_2 x_0^2 x_2 - a_4 x_0 x_2^2 - a_6 x_2^3$$

in $\mathbb{C}[x_0, x_1, x_2]$, and let C be the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$. Show that there exists an isomorphism of Riemann surfaces $X \to C$ that maps p to $[0, 1, 0]$. You may use without proof that the projective plane curve C is smooth. (8 marks)

(Total: 20 marks)

You may use the results and exercises from the lecture notes, problem sheets and coursework, but make sure to clearly indicate what you are using in your answers. Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.

1. (a) Let X be a connected compact Riemann surface, and let p be a point of X. Let f be a meromorphic function on X with a pole of order 1 at p and no other poles. Let $\overline{f}$ be the morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ induced by f.

(i) Compute the ramification degree of $\overline{f}$ at p and prove your answer. (5 marks)

Solution: Seen similar, Category A. The ramification degree of $\overline{f}$ at p is equal to 1. **(1 mark)** To see this, note that $\overline{f}$ maps p to $[1, 0]$, the point at infinity of $\mathbb{P}^1_{\mathbb{C}}$. Consider the chart $(U_0, x_1/x_0)$ for $\mathbb{P}^1_{\mathbb{C}}$ and set $V = \overline{f}^{-1}(U_0)$. Then $(x_1/x_0) \circ \overline{f}$ is a holomorphic function on V, and it is equal to $1/f$ on $V \setminus p$. Since f has a pole of order 1 at p, the function $(x_1/x_0) \circ \overline{f}$ has a zero of order 1 at p. Therefore, $\overline{f}$ is unramified at p. **(4 marks)**

(ii) Show that $\overline{f}$ is an isomorphism of Riemann surfaces. (5 marks)

Solution: Unseen, Category A. The morphism $\overline{f}$ maps p to $[1, 0]$ and each other point of X to a point in $U_1 \subset \mathbb{P}^1_{\mathbb{C}}$. Since $\overline{f}$ is unramified at p, the degree of $\overline{f}$ is equal to 1, so that $\overline{f}$ is injective. **(3 marks)** The morphism $\overline{f}$ is also surjective because it is a non-constant morphism between connected compact Riemann surfaces. **(1 mark)** It follows that $\overline{f}$ is a bijective morphism of Riemann surfaces, and therefore an isomorphism. **(1 mark)**

(b) Let X and Y be connected compact Riemann surfaces and let $h\colon Y \to X$ be a morphism of degree 2. Use the Riemann-Hurwitz formula to show that the number of ramification points of h is even. (5 marks)

Solution: Unseen, Category A. Since h has degree 2, the ramification degree at each ramification point is equal to 2. If we denote by N the number of ramification points, then the Riemann-Hurwitz formula states that $2 - 2g(Y) = 4 - 4g(X) - N$, so that N is even.

(c) Let X be a Riemann surface and let C be a smooth projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$. We denote by U_0, U_1 and U_2 the affine charts on $\mathbb{P}^2_{\mathbb{C}}$. Consider a map of sets $h\colon X \to C$. Show that h is a morphism of Riemann surfaces if and only if all of the following properties are satisfied:

* the set $V_0 = h^{-1}(U_0 \cap C)$ is open in X, and the functions $(x_1/x_0) \circ h$ and $(x_2/x_0) \circ h$ are holomorphic on V_0;
* the set $V_1 = h^{-1}(U_1 \cap C)$ is open in X, and the functions $(x_0/x_1) \circ h$ and $(x_2/x_1) \circ h$ are holomorphic on V_1;
* the set $V_2 = h^{-1}(U_2 \cap C)$ is open in X, and the functions $(x_0/x_2) \circ h$ and $(x_1/x_2) \circ h$ are holomorphic on V_2.

(5 marks)

Solution: Seen similar. The conditions are necessary by the definition of a morphism of Riemann surfaces, because such morphisms are continuous and preserve holomorphy of functions under composition. **(Category A, 1 mark)** So let us show that they are also sufficient. Let p be a point on C. It suffices to find a chart (U, ψ) for C such that p lies in U, the set $V = h^{-1}(U)$ is open in X, and $\psi \circ h$ is holomorphic on V. Swapping the homogeneous coordinates on $\mathbb{P}^2_{\mathbb{C}}$ if necessary, we may assume that p lies in the affine chart U_2.

By the definition of the complex structure on C, we can find an open U around p in $C \cap U_2$ such that either x_0/x_2 or x_1/x_2 is a chart on U. Since $(x_0/x_2) \circ h$ and $(x_1/x_2) \circ h$ are holomorphic on the whole of V_2, we only need to show that the subset $V = h^{-1}(U)$ of V_2 is open. The map

$$\psi_2 \colon U_2 \to \mathbb{C}^2,\ [x_0, x_1, x_2] \mapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$$

is a homeomorphism, so that it suffices to show that $\psi_2 \circ h \colon V_2 \to \mathbb{C}^2$ is continuous; this follows from the fact that the component functions $(x_0/x_2) \circ h$ and $(x_1/x_2) \circ h$ are holomorphic, and therefore continuous, on V_2. **(Category B, 4 marks)**

(Total: 20 marks)

2. (a) Let C be an affine plane curve in $\mathbb{C}^2$. We say that a line L in $\mathbb{C}^2$ is an *asymptote* of C if the projectivization of L is tangent to the projectivization of C at some point at infinity of C. This point at infinity may be singular; then we use the definition of tangent lines from problem sheet 3. Compute the asymptotes of the following affine plane curves in $\mathbb{C}$ and explain your answers:

   (i) the curve D defined by $y^2 = x^3 + 1$; (4 marks)

   Solution: Seen similar, Category A. The projectivization $\overline{D}$ of D is defined by $x_1^2 x_2 = x_0^3 + x_2^3$. It has a unique point at infinity, namely, the point $[0, 1, 0]$. The tangent line of $\overline{D}$ at this point is the line at infinity defined by $x_2 = 0$. This line does not intersect the affine chart U_2, so that the affine plane curve D has no asymptotes.

   (ii) the curve E defined by $y^2 = x^4 + xy^2$. (4 marks)

   Solution: Seen similar, Category B. The projectivization $\overline{E}$ of E is defined by $x_1^2 x_2^2 = x_0^4 + x_0 x_1^2 x_2$. It again has $[0, 1, 0]$ as its unique point at infinity. The curve $\overline{E}$ is singular at $[0, 1, 0]$. To find the tangent lines, we pass to the affine chart U_1 containing $[0, 1, 0]$. The affine plane curve $\overline{E} \cap U_1$ is defined by

$$\left(\frac{x_2}{x_1}\right)^2 - \left(\frac{x_0}{x_1}\right)^4 - \left(\frac{x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right) = 0.$$

   The lowest degree part of this equation is

$$\left(\frac{x_2}{x_1}\right)^2 - \left(\frac{x_0}{x_1}\right)\left(\frac{x_2}{x_1}\right) = \frac{x_2}{x_1}\left(\frac{x_2}{x_1} - \frac{x_0}{x_1}\right)$$

   so that the affine tangent lines to $\overline{E} \cap U_1$ at $(0, 0)$ are given by $x_2/x_1 = 0$ and $x_2/x_1 = x_0/x_1$, and the projective tangent lines to $\overline{E}$ at $[0, 1, 0]$ are given by $x_2 = 0$ and $x_2 = x_0$. The first line does not meet U_2, while the second one is the projectivization of the line L in U_2 defined by $x = 1$. Thus L is the unique asymptote of E.

(b) Show that an affine plane curve of degree d in $\mathbb{C}^2$ has at most d asymptotes (see part (a) for the definition of an asymptote). (6 marks)

Solution: Unseen, Category A. The number of tangent lines at each point p of $\overline{C}$ is at most $\mathrm{mult}_p\overline{C}$. Therefore, if we denote by L_∞ the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_2 = 0$, then the number of asymptotes of C is at most

$$\sum_{p\in\overline{C}\cap L_\infty} \mathrm{mult}_p\overline{C} \leq \sum_{p\in\overline{C}\cap L_\infty} \mathbf{I}(p,\overline{C},L_\infty) = d$$

where the final equality follows from Bézout's theorem.

(c) Show that every projective plane curve C of degree 4 in $\mathbb{P}^2_{\mathbb{C}}$ with four singular points is reducible. *Hint: construct a conic D such that C and D have a common component.* (6 marks)

Solution: Seen similar, Category C. Let p_1, p_2, p_3, p_4 be distinct singular points of C, and let p_5 be a point of C different from $p_1, \ldots, p_4$. Let D be a conic through $p_1, \ldots, p_5$. Since the intersection multiplicity of C and D is at least 2 at each singular point of C, we have

$$\sum_{p\in C\cap D} \mathbf{I}(p,C,D) \geq 9 > \deg(C)\cdot\deg(D)$$

so that C and D have a common component by Bézout's theorem. This common component cannot be C itself because the degree of C is strictly larger than the degree of D. It follows that C is reducible.

(Total: 20 marks)

3. (a) Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_0 = 0$, and let p be the point $[1, 0, 0]$. Let S be the set of effective divisors of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ whose support contains $L \cup \{p\}$. Show that S is a linear system. Compute its dimension and prove your answer. (6 marks)

Solution: Seen similar, Category B. Let V be the set of degree 2 homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ that vanish at p. This is a sub-vector space of $\mathbb{C}[x_0, x_1, x_2]$ of dimension 5. The map

$$V \to \mathbb{C}[x_0, x_1, x_2],\ Q \mapsto x_0 Q$$

is an injective morphism of vector spaces, so that its image is a sub-vector space of $\mathbb{C}[x_0, x_1, x_2]$ of dimension 5 consisting of homogeneous polynomials of degree 3. The associated linear system is precisely the set S, so that S is a linear system of dimension $5 - 1 = 4$.

(b) Let p_1, p_2, p_3, p_4 be distinct points in $\mathbb{P}^2_{\mathbb{C}}$ such that no three of them are collinear. Show that there exist two non-degenerate conics in $\mathbb{P}^2_{\mathbb{C}}$ whose intersection is equal to $\{p_1, p_2, p_3, p_4\}$. (6 marks)

Solution: Seen similar, Category B. Since no three of the given points are collinear, there exist infinitely many non-degenerate conics containing p_1, p_2, p_3, p_4. Let C and D be two distinct non-degenerate conics through p_1, p_2, p_3, p_4. Then C and D have no common component, so that $C \cap D$ contains at most four points by Bézout's theorem. It follows that $C \cap D = \{p_1, p_2, p_3, p_4\}$.

Marking: Full marks for applying a projective transformation to reduce to the case where $\{p_1, p_2, p_3, p_4\}$ is the standard projective basis of $\mathbb{P}^2_{\mathbb{C}}$ and writing down explicit equations for two non-degenerate conics through these points.

(c) Let P be a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors. Let Q be the linear polynomial $ax_0 + bx_1 - x_2$, for some a and b in $\mathbb{C}$, and assume that Q does not divide P. Let $p = [p_0, p_1, p_2]$ be a zero of Q in $\mathbb{P}^2_{\mathbb{C}}$. Use the axiomatic characterization of intersection multiplicities to show that

$$\mathbf{I}(p, P, Q) = \mathrm{mult}_{(p_0, p_1)} P(x_0, x_1, ax_0 + bx_1).$$

Clearly indicate which axioms you are using. (8 marks)

Solution: Seen similar, Category D. We can view elements in $\mathbb{C}[x_0, x_1, x_2]$ as polynomials in the variable x_2 with coefficients in $\mathbb{C}[x_0, x_1]$. By Euclidean division, we can write $P = QR + S$ where R lies in $\mathbb{C}[x_0, x_1, x_2]$ and S lies in $\mathbb{C}[x_0, x_1]$. Evaluating at $(x_0, x_1, ax_0 + bx_1)$, we find that $S = P(x_0, x_1, ax_0 + bx_1)$ because $Q(x_0, x_1, ax_0 + bx_1)$ is the zero polynomial. By the symmetry and deformation axioms, we have

$$\mathbf{I}(p, P, Q) = \mathbf{I}(p, S, Q).$$

The polynomial S is a homogeneous polynomial in $\mathbb{C}[x_0, x_1]$. It is non-zero because Q does not divide P, and it is not constant because $S(0, 0) = P(0, 0, 0) = 0$. Therefore, we can write S as a product $S = L_1 \cdots L_r$ of non-zero linear homogeneous polynomials L_i in $\mathbb{C}[x_0, x_1]$. None of these polynomials L_i is divisible by Q, because Q does not divide P.

By the symmetry, additivity and transversality axioms, $\mathbf{I}(p, S, Q)$ is equal to the number of factors L_i that vanish at p. Each of these factors L_i has multiplicity 1 at (p_0, p_1), and the other factors L_j have multiplicity 0 at (p_0, p_1), so that

$$\mathbf{I}(p, S, Q) = \sum_{i=1}^{r} \mathrm{mult}_{(p_0,p_1)} L_i = \mathrm{mult}_{(p_0,p_1)} S.$$

(Total: 20 marks)

4. For every λ in $\mathbb{C} \setminus \{0, 1\}$, we denote by C_λ the cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by the Legendre equation

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2).$$

(a) Let λ be an element in $\mathbb{C} \setminus \{0, 1\}$. Show that C_λ and C_μ are projectively equivalent for $\mu = 1 - \lambda$ and also for $\mu = 1/\lambda$. Deduce that C_λ is projectively equivalent to C_μ for every μ in

$$\left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.$$

(3 marks)

Solution: Seen similar, Category A. If $\mu = 1 - \lambda$ then the projective transformation

$$\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},\ [x_0, x_1, x_2] \mapsto [x_2 - x_0, i x_1, x_2]$$

maps C_λ to C_μ. Similarly, if $\mu = 1/\lambda$ and ν is a square root of λ, then the projective transformation

$$\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},\ [x_0, x_1, x_2] \mapsto [x_0, \nu^{-1} x_1, \lambda x_2]$$

maps C_λ to C_μ. The remainder of the statement follows from the transitivity property of projective equivalence, because $\lambda(\lambda - 1)^{-1} = 1 - (1 - \lambda)^{-1}$.

(b) Let C be a smooth non-degenerate cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and let p and q be two inflection points of C. Show that there exists a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(C) = C$ and $\Phi(p) = q$. (6 marks)

Solution: Seen similar, Category C. By the solutions to Problem sheet 4, we can apply a projective transformation to reduce to the case where $p = [0, 1, 0]$, $q = [0, 0, 1]$, and C is defined by the homogeneous equation

$$a x_0^3 + x_1 x_2 (b x_0 + c x_1 + d x_2) = 0$$

for some a, b, c, d in $\mathbb{C}$ such that a, c, d are non-zero. Applying a further change of homogeneous coordinates to rescale the coordinates x_0, x_1 and x_2, we can reduce to the case where $a = c = d = 1$. Now the projective transformation

$$\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},\ [x_0, x_1, x_2] \mapsto [x_0, x_2, x_1]$$

satisfies the required properties.

The problem can also be solved using the Legendre form: let r be the third intersection point of the line L through p and q with C. By Problem sheet 4, the point r is again an inflection point of C. After a projective transformation, we may assume that $r = [0, 1, 0]$ and that C is the zero set of $x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)$ for some λ in $\mathbb{C} \setminus \{0, 1\}$. We have seen in the solutions to Problem sheet 4 that the projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ defined by $[x_0, x_1, x_2] \mapsto [x_0, -x_1, x_2]$ satisfies $\Phi(C) = C$ and $\Phi(p) = q$.

(c) Let λ be an element in $\mathbb{C} \setminus \{0, 1\}$. Determine the points p on C_λ different from $[0, 1, 0]$ such that the tangent line to C_λ at p passes through $[0, 1, 0]$. (3 marks)

Solution: Seen similar, Category A. Set $P = x_1^2 x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2)$ and let $p = [p_0, p_1, p_2]$ be a point on C_λ different from $[0, 1, 0]$. Since $[0, 1, 0]$ is the only point of C_λ where x_2 vanishes, we may assume that $p_2 = 1$. Now the tangent line to C_λ at p passes through $[0, 1, 0]$ if and only if $\partial_{x_1} P(p_0, p_1, 1) = 0$, that is, if and only if $p_1 = 0$. It follows that the desired points p are the elements of the set

$$\{[0, 0, 1],\ [1, 0, 1],\ [\lambda, 0, 1]\}.$$

(d) Let λ and μ be elements in $\mathbb{C} \setminus \{0, 1\}$. Show that C_λ and C_μ are projectively equivalent if and only if

$$\mu \in \left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.$$

(8 marks)

Solution: Unseen, Category D. The"if" implication follows from Part (a), so we only need to prove the "only if" implication. Assume that there exists a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(C_\lambda) = C_\mu$. Since Φ preserves inflection points, it maps $[0, 1, 0]$ to an inflection point p of C_μ. Composing Φ with a projective transformation that preserves C_μ and maps p to $[0, 1, 0]$ (whose existence is guaranteed by Part (b)), we may assume that $\Phi([0, 1, 0]) = [0, 1, 0]$. Then it follows from Part (c) that Φ maps the set $\{[0, 0, 1],\ [1, 0, 1],\ [\lambda, 0, 1]\}$ to the set $\{[0, 0, 1],\ [1, 0, 1],\ [\mu, 0, 1]\}$. In particular, it preserves the line L_1 defined by $x_1 = 0$. The projective transformation Φ must also map the tangent line of C_λ at $[0, 1, 0]$ to the tangent line of C_μ at $[0, 1, 0]$, which means that it preserves the line L_2 defined by $x_2 = 0$, and fixes the intersection point $[1, 0, 0]$ of L_1 and L_2.

We identify L_1 with $\mathbb{P}^1_{\mathbb{C}}$ by means of the isomorphism

$$\mathbb{P}^1_{\mathbb{C}} \to L_1,\ [x_0, x_1] \mapsto [x_0, 0, x_1].$$

Restricting Φ to L_1, we obtain a projective transformation of $\mathbb{P}^1_{\mathbb{C}}$ that fixes $[1, 0]$ and maps the set $\{[0, 1],\ [1, 1],\ [\lambda, 1]\}$ to the set $\{[0, 1],\ [1, 1],\ [\mu, 1]\}$. Since projective transformations of $\mathbb{P}^1_{\mathbb{C}}$ preserve cross-ratios, it follows that μ lies in

$$\left\{ \lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}.$$

(Total: 20 marks)

5. In this question you may freely use that every connected compact Riemann surface satisfies the Riemann-Roch theorem from the mastery material. Let X be a connected compact Riemann surface of genus 1, and let p be a point of X. For every integer n, we consider the divisor np on X. We denote by $\ell(np)$ the dimension of the complex vector space $\mathcal{L}(np)$ of meromorphic functions f on X such that $f = 0$ or $f \neq 0$ and $(f) + np \geq 0$.

(a) Compute $\ell(np)$ for every integer n. (4 marks)

Solution: Unseen. If n is negative, then $\ell(np) = 0$ because the divisor np has negative degree. If $n = 0$, then $\mathcal{L}(np)$ is the vector space of holomorphic functions on X, which has dimension $\ell(np) = 1$ because every holomorphic function on a connected compact Riemann surface is constant.

Finally, assume that n is positive, and set $D = np$. Let K be a canonical divisor on X. Then K has degree $2 - 2g(X) = 0$ so that $K - D$ has negative degree. Now it follows from the Riemann-Roch theorem that $\ell(D) = \deg(D) - g(X) + 1 = n$.

(b) Show that the complex vector space $\mathcal{L}(3p)$ has a basis of the form $\{1, f, g\}$ where f has a pole of order 2 at p and g has a pole of order 3 at p. (4 marks)

Solution: Unseen. By Part (i), all the functions in $\mathcal{L}(p)$ are constant, and $\mathcal{L}(2p)$ contains a non-constant function f, which must then have a pole of order 2 at p. Since $\mathcal{L}(3p)$ is strictly larger than $\mathcal{L}(2p)$, it contains a function g with a pole of order 3 at p. The functions 1, f and g are linearly independent, so that they form a basis of the three-dimensional vector space $\mathcal{L}(3p)$.

(c) Show that there exist complex numbers $a_0, a_1, \ldots, a_6$ such that a_0 is non-zero and

$$g^2 + a_1 fg + a_3 g = a_0 f^3 + a_2 f^2 + a_4 f + a_6.$$

(4 marks)

Solution: Unseen. The seven functions $1, f, g, f^2, fg, f^3, g^2$ all belong to $\mathcal{L}(6p)$, because they have a pole of order at most 6 at p and no other poles. The six functions $1, f, g, f^2, fg, f^3$ are linearly independent, because they each have a different order at p. Therefore, they form a basis of the six-dimensional vector space $\mathcal{L}(6p)$. It follows that g^2 is a complex linear combination of $1, f, g, f^2, fg, f^3$. The coefficient a_0 is non-zero because f^3 is the only basis element with a pole of order 6 at p.

(d) Consider the homogeneous polynomial

$$P = x_1^2 x_2 + a_1 x_0 x_1 x_2 + a_3 x_1 x_2^2 - a_0 x_0^3 - a_2 x_0^2 x_2 - a_4 x_0 x_2^2 - a_6 x_2^3$$

in $\mathbb{C}[x_0, x_1, x_2]$, and let C be the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$. Show that there exists an isomorphism of Riemann surfaces $X \to C$ that maps p to $[0, 1, 0]$. You may use without proof that the projective plane curve C is smooth. (8 marks)

Solution: Unseen. Consider the map

$$h\colon X \to C,\ q \mapsto \begin{cases} [0, 1, 0] & \text{if } q = p, \\ [f(q), g(q), 1] & \text{otherwise.} \end{cases}$$

We use Question 1(c) to check that this is a morphism of Riemann surfaces.

* The set $V_0 = h^{-1}(U_0)$ consists of all the points of $X \setminus \{p\}$ where f is non-zero. The complement of this set is finite, because f has finitely many zeros on $X \setminus \{p\}$ (it is meromorphic on X and not identically zero). Therefore, V_0 is open. We have $(x_1/x_0) \circ h = g/f$ on and $(x_2/x_0) \circ h = 1/f$ on V_0; these functions are holomorphic.
* The set $V_1 = h^{-1}(U_1)$ consists of p and all the points of $X \setminus \{p\}$ where g is non-zero. The complement of this set is again finite. Therefore, V_1 is open. We have $(x_0/x_1) \circ h = f/g$ on $V_1 \setminus \{p\}$ and $((x_0/x_1) \circ h)(p) = 0$. Since f/g has order 1 at p, it follows that $(x_0/x_1) \circ h$ is holomorphic on V_1. Similarly, $(x_2/x_1) \circ h$ is holomorphic on V_1 because $1/g$ has order 3 at p.
* We have $V_2 = h^{-1}(U_2) = X \setminus \{p\}$; this is an open subset of X, and on this set, $(x_0/x_2) \circ h$ and $(x_1/x_2) \circ h$ coincide with the holomorphic functions f and g, respectively.

Finally, we show that h is an isomorphism. Since X and Y are connected compact Riemann surfaces, it suffices to show that h has degree 1. Since p is the only point in the fiber over $[0, 1, 0]$, it is enough to prove that h is unramified at p. Locally around $[0, 1, 0]$, the function x_0/x_1 defines a chart for C, because $\partial_y P(x, 1, y)$ does not vanish at $(0, 0)$. The function $(x_0/x_1) \circ h = f/g$ has order 1 at p, so that h is unramified at p.

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Math96032/97041 /97150	1	In Question 1a many students were still struggling with the correct use of charts. Questions 1a and 1b were answered reasonably well. In Question 1c the standard mistake was to claim that the affine coordinate functions on the intersections of C with U_0, U_1 and U_2 form an atlas for C, which is false.
Math96032/97041 /97150	2	Question 2a was answered well overall, although the conversion back to the chart U_2 posed problems for some. Most students had the rough idea for question 2b but not everyone managed to write it down properly (invoking Bézout). Question 2c was answered well, the standard error was to take two lines through the 4 points and claim that their intersection point also lies on the quartic.
Math96032/97041 /97150	3	Questions 3a and 3b were answered well. Question 3c was discussed on the Piazza forum but only few students gave a complete solution; some ignored the assignment to use the axioms instead of resultants.
Math96032/97041 /97150	4	Questions 4a and 4c were answered well. Several students invoked the j-invariant for 4a; this was not the intended answer but received full marks. The answer to Question 4b was essentially the solution to question 2 on problem sheet 4, but relatively few students recognized this. Almost no-one made progress on Question 4(d), which was the hardest problem on the exam.
Math96032/97041 /97150	5	(mastery question) Questions 5a,b,c were answered well, but few students got far on question 5d.