Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Introduction to University Mathematics **MATH40001/MATH40009**

Part II – Problem Sheet 3

1. Show the following statements using only the axioms of the reals.
   (a) ✦ For all $a, b \in \mathbb{R}$, there exists exactly one $x \in \mathbb{R}$, such that $a + x = b$.
   (b) ✦✦ For all $a \in \mathbb{R}$, $-(-a) = a$ and $(-a) + (-b) = -(a + b)$.
2. Show the following statements using only the axioms of the reals.
   (a) ✦✦ For all $a, b \in \mathbb{R}$, $a \cdot b = 0$ if and only if $a = 0 \vee b = 0$.
   (b) ✦ The neutral elements 0 and 1 are uniquely defined.
3. We remind of the definition of the absolute value of a real number $a \in \mathbb{R}$.
$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$
   Show that
   (a) ✦ If $a \neq 0$, then $|a| > 0$.
   (b) ✦ $|a \cdot b| = |a| \cdot |b|$.
   (c) ✦✦ $|a + b| \leq |a| + |b|$.
   (d) ✦✦ $||a| - |b|| \leq |a - b|$.
4. (a) Show directly from the axioms:
      i. ✦✦ If $a \in \mathbb{R}$, $a > 0$, then $\frac{1}{a} > 0$
      ii. ✦✦✦ If $a, b > 0$, then $a < b$ if and only if $a^2 < b^2$
   (b) ✦ Show that for all $x \in \mathbb{R}$, such that $x > -1$, and for all $n \in \mathbb{N}$
$$(1 + x)^n \geq 1 + nx.$$
5. ✦✦✦
   (a) Prove that the set of natural numbers $\mathbb{N}$ is not bounded above.
   (b) Prove that for any $x, y \in \mathbb{R}$, $x > 0$, there exists an $n \in \mathbb{N}$, such that $nx > y$. As you should know from Video 16, this is called the Archimedean Property.
6. ✦✦ For each of the following sets find $\sup(S)$ and $\inf(S)$ if they exist.
   (a) $S = \{x \in \mathbb{R} | x^2 < 5\}$
   (b) $S = \{x \in \mathbb{R} | x^2 > 7\}$
   (c) $S = \{-\frac{1}{n} | n \in \mathbb{N}\}$
7. (a) ✦ Is the empty set bounded? Prove or disprove.
   (b) ✦✦ Show that for any $a, b \in \mathbb{R}$, $\inf[a, b] = \inf(a, b) = a$ and $\sup[a, b] = \sup(a, b) = b$.
8. ✦✦
   (a) Let A, B be bounded non-empty sets. Prove that if $A \subseteq B$, then
$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$
   (b) Let A be an non-empty set bounded above. Let $B := \{x | x \text{ is an upper bound for the set } A\}$. Suppose $\inf B$ and $\sup A$ exist. Prove $\inf B = \sup A$.