(1)

$$\mathcal{A}_1 = (A_1 ; \le_1)$$
$$\mathcal{A}_2 = (A_2 ; \le_2)$$
$$\mathcal{A}_1 \times \mathcal{A}_2 = ((A_1 \times A_2) ; \le)$$

If $\mathcal{A}_1, \mathcal{A}_2$ are w.o. sets then so is $\mathcal{A}_1 \times \mathcal{A}_2$:

Let $\emptyset \ne Z \subseteq A_1 \times A_2$

Consider
$$Y = \{ b \in A_2 : \exists a \in A_1, (a,b) \in Z \}$$

This has a least elt. d.

Let $X = \{ a \in A_1 : (a,d) \in Z \}$

This has a least elt c.

Now show: (c,d) is the least elt. of Z. #

3.4 Ordinals.

(3.4.1) **Def.** 1) A set X is **transitive** if every element of X is also a subset of X (i.e. if $y \in x \in X$ then $y \in X$).

2) A set α is an **ordinal** if

(a) α is a transitive set

(b) the relation $<$ on α given by: for $x, y \in \alpha$

$$x < y \iff x \in y$$

is a strict well ordering on α.

Eg. 1

$$3 = \{0, 1, 2\}$$
$$\cup|$$
$$2 = \{0, 1\}$$

2 $\{0, 2\}$ is not a transitive set.

Note: (i) By definition if α is an ordinal we have $\alpha \notin \alpha$: Suppose $\alpha \in \alpha$ then $\alpha < \alpha$ (in α): contradicts strictness.

(2) **Notation**: use $\alpha, \beta, \cdots$ for ordinals. Sometimes use $\in_\alpha$ for the ordering $\in$ on α.

(3.4.1) **Lemma.** If α is an ordinal, then so is $\alpha^+ = \alpha \cup \{\alpha\}$.

Pf: Trasitive: Suppose $\beta \in \alpha^+$

either: $\beta \in \alpha$: as α is an ordinal $\beta \subseteq \alpha$.

or $\beta = \alpha$: $\beta \subseteq \alpha$. #.

Well ordered by $\in$:

Ordering on α^+ puts a greatest element (α) 'above' all elts. in α.

α^+ : $\underbrace{\rule{4cm}{0.4pt}}_{\alpha}$ $\bullet$

this is still a well ordering.

Strict : By Note ①. #.

Examples:

$\emptyset$ is an ordinal

By Lemma $\underset{1}{\emptyset^+}$, $\underset{2}{(\emptyset^+)^+}$, ...

are ordinals. More properly:

(3.4.3) Prop. ① If $n \in \omega$ then n is an ordinal.

② ω is a transitive set.

Pf: ① By induction (3.2.3) ③ it's enough to show that $\emptyset$ is an ordinal and if n is an ordinal then n^+ is an ordinal (for $n \in \omega$)

– By 3.4.2. #①.

② Prove by induction on $n \in \omega$ that if $m \in n \in \omega$ then $m \in \omega$. #②

3.4.4 Prop.

1) If α is an ordinal then $\alpha \notin \alpha$.

2) If α is an ordinal and $\beta \in \alpha$ then β is an ordinal

3) If α, β are ordinals and $\beta \subset \alpha$ (i.e. $\beta \subseteq \alpha$ and $\beta \neq \alpha$) then $\beta \in \alpha$.

4) If α is an ordinal then
$$\alpha = \{\beta : \beta \text{ is an ordinal and } \beta \in \alpha\}$$

Pf: 1) Done.

2) Check the def.

4) By (2).

3) Note that $\alpha \setminus \beta \neq \emptyset$, so has a least element γ (with respect to the ordering $\in$ on α) **(4)**

Show: $\gamma = \beta$.

- Show: $\gamma \subseteq \beta$.

If $\delta \in \gamma$ then $\delta \in \gamma \subseteq \alpha$ (α is transitive & $\gamma \in \alpha$.)

So $\delta \in \alpha$. As $\delta < \gamma$ in the ordering on α and γ is the least elt. of α not in β, we obtain $\delta \in \beta$. //

- Show: $\beta \subseteq \gamma$. Let $\delta \in \beta$

We have $\delta, \gamma \in \alpha$. As $\in$ is a l.o. on α have:

$\delta \in \gamma$ — what we want.

or $\delta = \gamma$]

or $\gamma \in \delta$] as $\delta \in \beta$ ordinal get $\gamma \in \beta$. Contradicts def. of γ. #

3.4.5 **Def.** If α, β are ordinals write $\alpha < \beta$ to mean $\alpha \in \beta$ and $\alpha \le \beta$ to mean $(\alpha < \beta)$ or $(\alpha = \beta)$.

Ex: $\alpha \le \beta$ iff $\alpha \subseteq \beta$

[$\Rightarrow$ def. ; $\Leftarrow$: 3.4.4 (3)].

3.4.6 Proposition.

Suppose α, β, γ are ordinals.

1) If $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.
2) If $\alpha \le \beta$ and $\beta \le \alpha$ then $\alpha = \beta$.
3) Exactly one of $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$ holds.
4) If X is a non-empty set of ordinals, then X has a least element.

" The collection of ordinals is well-ordered by $\le$ ".

Pf: (1), (2) by Ex.

(3) Show that if $\alpha \neq \beta$ then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

Step 1. Show that $\alpha \cap \beta$ is an ordinal.

Step 2. If $\alpha \not\subseteq \beta$ then $\alpha \cap \beta \subset \alpha$. As $\alpha \cap \beta$ is an ordinal 3.4.4 (3) gives $\alpha \cap \beta \in \alpha$. Similarly if $\beta \not\subseteq \alpha$ then $\alpha \cap \beta \in \beta$. This gives $\alpha \cap \beta \in \alpha \cap \beta$. Contradiction # (3)

(4) Let $\delta \in X$.

Let β = least elt. of

$$\{\gamma \in X : \gamma \leq \delta\}$$

$$= \{\delta\} \cup (\delta \cap X).$$

$$\subseteq \delta^+.$$

Show: β is least

elt. of X. #.