Tensor Calculus and General Relativity **Spring Term 2023**

Problem Sheet 2

1. Lorentz transformations may be defined through 4×4 matrices, Λ, with the property $\Lambda^T \eta \Lambda = \eta$ where $\eta = \text{diag}(1, -1, -1, -1)$. Show that the Lorentz transformations form a group under matrix multiplication.

2. An atom of mass m at rest absorbs a photon of energy E and recoils. What is the 4-momentum of (a) the atom at rest, (b) the photon and (c) the recoiling atom. Show that the rest mass m^* of the atom after absorbing the photon is
$$m^* = m\sqrt{1 + \frac{2E}{mc^2}}.$$

3. In Chapter 1 of the notes it is shown that $u = \gamma(c, \mathbf{v})$. Obtain a similar expression for the 4-acceleration a in terms of
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt}.$$

4. A tensor of type $(2, 0)$ has 16 components. A symmetric tensor has 10 independent components and an anti-symmetric tensor has 6 independent components.

   A tensor of type $(3, 0)$ has 64 components. How many independent components does a *totally symmetric* tensor of type $(3, 0)$ have? Totally symmetric means that the tensor is unchanged on the interchange of any two indices. How many independent components does a totally anti-symmetric tensor of type $(3, 0)$ have?

5. The energy-momentum tensor of a perfect fluid is given by
$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p\eta^{\mu\nu},$$
   where ρ is the proper density, p is the pressure and u^μ is the four-velocity of the fluid.

   Assuming that $u^i u^i \ll c^2$ and $p \ll \rho c^2$ (the non-relativistic limit)
$$c^{-2}T^{00} = \rho, \quad c^{-1}T^{i0} = c^{-1}T^{0i} = \rho u^i, \quad T^{ij} = \rho u^i u^j + p\delta^{ij}$$
   where $i = 1,\ 2,\ 3$.

   Extract the 4 PDEs contained in the tensor equation $\partial_\nu T^{\mu\nu} = 0$ (convert derivatives with respect to x^0 into derivatives with respect to time t).

6. The energy-momentum tensor for a particle of mass m at rest is

   $T^{00} = mc^2\delta^3(\mathbf{r})$ and all other components of $T^{\mu\nu}$ zero. Determine the energy-momentum tensor of a particle with constant velocity (simplify your answer if possible).

   Hint: $\delta(au) = \delta(u)/|a|$ $(a \neq 0)$.