Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Function Spaces and Applications

Date: 06 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the
top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

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WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS
ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. Let $p \in (1; +\infty)$ be fixed. Here and throughout the paper, $\mathbb{N}$ denotes the set of strictly positive integers. Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) := n^{\frac{1}{p}} e^{-n|x|}$.

   (a) Does the sequence $\{f_n\}_{n\in\mathbb{N}}$ converge pointwise almost everywhere in $\mathbb{R}$? If so, find the limit function. Does the sequence converge uniformly? (4 marks)

   (b) For which values $q \in [1; +\infty]$ are the functions $f_n \in L^q(\mathbb{R})$? For which values q is the sequence $\{f_n\}_{n\in\mathbb{N}}$ bounded in $L^q(\mathbb{R})$? (6 marks)

   (c) For which values $q \in [1; +\infty]$ is $\{f_n\}_{n\in\mathbb{N}}$ (strongly) convergent in $L^q(\mathbb{R})$? (4 marks)

   (d) Let $q \in [1; +\infty)$. We denote by q' the conjugate exponent to q, namely such that $\frac{1}{q} + \frac{1}{q'} = 1$. We recall that a sequence of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^q(\mathbb{R})$ converges weakly to $g \in L^q(\mathbb{R})$ if for every $\phi \in L^{q'}(\mathbb{R})$ we have that

   $$\int_{\mathbb{R}} g_n(x)\phi(x)\, dx \to \int_{\mathbb{R}} g(x)\phi(x)\, dx.$$

   For which values $q \in [1; +\infty)$ does the sequence $\{f_n\}_{n\in\mathbb{N}}$ converge weakly in $L^q(\mathbb{R})$? (6 marks)

   (Total: 20 marks)

2. For $p \in [1; +\infty]$, consider the Banach spaces of real sequences ℓ_p with the norm $\|\cdot\|_{\ell_p}$ seen in class. We define c_0 as the set of (real) decaying sequences, namely

$$c_0 := \left\{ \{x_n\}_{n \in \mathbb{N}}, \ x_n \in \mathbb{R} : \ \lim_{n \to \infty} x_n = 0 \right\}.$$

The main goal of this question is to show that, if $(c_0)'$ denotes the dual space of c_0, then $(c_0)' = \ell_1$.

(a) Prove that c_0 is a closed subspace of ℓ_∞. (6 marks)

(b) For every $y \in \ell_1$, consider the map $\phi_y : c_0 \to \mathbb{R}$ such that, for every $x \in c_0$, we have that $\phi_y(x) = \sum_{n \in \mathbb{N}} x_n y_n$. Prove that $\phi_y \in (c_0)'$.

(2 marks)

(c) Consider the map $\phi : \ell_1 \to (c_0)'$ such that, for every $y \in \ell_1$, the image ϕ_y is the map defined as in Step (b). Prove that ϕ is well-defined, injective and satisfies

$$\|\phi_y\|_{(c_0)'} = \|y\|_{\ell_1}$$

for every $y \in \ell_1$.

(4 marks)

(d) Let $T \in (c_0)'$ be fixed. For every $n \in \mathbb{N}$, let the sequence $e^{(n)} = \{e_k^{(n)}\}_{k \in \mathbb{N}}$ be defined as

$$e_k^{(n)} = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

Consider the sequence $y = \{y_n\}_{n \in \mathbb{N}}$ defined by $y_n := T(e^{(n)})$.

(i) Prove that

$$T(x) = \sum_{n \in \mathbb{N}} y_n x_n, \quad \text{for every } x \in c_0.$$

(4 marks)

(ii) Conclude that the map ϕ defined in Step (c) is an isomorphism and that $(c_0)' = \ell_1$.

(4 marks)

(Total: 20 marks)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f \in L^1(\mathbb{R})$. The goal of this exercise is to prove that

$$\lim_{\alpha \to \pm\infty} \int_{\mathbb{R}} f(t)e^{i\alpha t}\,dt = 0. \tag{1}$$

Recall that, whenever $\alpha, t \in \mathbb{R}$, the complex-valued function $e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t)$ satisfies $|e^{i\alpha t}| \le 1$ (where $|\cdot|$ stands for the modulus of a complex number).

(a) Let $(a, b) \subset \mathbb{R}$ be a bounded interval. Let $\chi_{(a,b)}$ be the characteristic function of the set (a, b). Prove that for every $\alpha \neq 0$

$$\int_{\mathbb{R}} \chi_{(a,b)}(t)e^{i\alpha t}\,dt = i\frac{e^{i\alpha a} - e^{i\alpha b}}{\alpha}.$$

(5 marks)

(b) Using part (a), prove (1) whenever f is the characteristic function of a bounded interval. Extend this result to any step function f having compact support, i.e. f such that there exist $N \in \mathbb{N}$, $\{c_j\}_{j=1}^N \subset \mathbb{R}$ and $(a_j, b_j) \subset \mathbb{R}$ bounded intervals, for every $j = 1, \cdots, N$, for which

$$f(t) = \sum_{j=1}^N c_j\chi_{(a_j,b_j)}.$$

(5 marks)

(c) You may assume that the following holds: For every $g \in C_c^0(\mathbb{R})$ such that $g \ge 0$, there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of step functions with compact support (as the ones in (b)) such that

$$\rho_n \le g \quad \text{for all } n \in \mathbb{N}, \qquad \rho_n \to g \quad \text{pointwise in } \mathbb{R}.$$

Prove that (1) holds for $f \in C_c^0(\mathbb{R})$ such that $f \ge 0$. (8 marks)

(d) Prove (1) for any $f \in C_c^0(\mathbb{R})$ (independently from their sign) and, in turn, for any $f \in L^1(\mathbb{R})$.

(2 marks)

(Total: 20 marks)

4. Let $p \in [1; +\infty]$, $\alpha \in \mathbb{R}$, and $n \in \mathbb{N}$. For every $x \in \ell_p$, we define the sequence $T_n(x) = \{T_n(x)_k\}_{k \in \mathbb{N}}$ as

$$
T_n(x)_k = \begin{cases} |\alpha|^j x_{2j+1} & \text{if } k = 2j+1 \text{ for some } j \in \mathbb{N} \text{ and } k \leq 2n+1 \\ 0 & \text{otherwise} \end{cases} \tag{2}
$$

or, equivalently, as

$$
T_n(x) = (x_1, 0, |\alpha|x_3, 0, |\alpha|^2 x_5, \cdots, 0, |\alpha|^n x_{2n+1}, 0, 0, 0, \cdots).
$$

Note that the sequence $T_n(x)$ is always zero after the $(2n+1)$-th term.

(a) Prove that $T_n : \ell_p \to \ell_1$ is a well-defined, linear and continuous map (i.e. $T_n \in \mathcal{L}(\ell_p, \ell_1)$).

(4 marks)

(b) Prove that, if $p \in (1; +\infty]$ and q denotes the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\|T_n\|_{\mathcal{L}(\ell_p, \ell_1)} = \left(\sum_{k=0}^{n} |\alpha|^{kq} \right)^{\frac{1}{q}}.
$$

(8 marks)

(c) Prove that if $p = 1$, then

$$
\|T_n\|_{\mathcal{L}(\ell_1, \ell_1)} = \max_{k=0,\cdots,n} |\alpha|^k.
$$

(4 marks)

(d) Prove that, if $|\alpha| < 1$, then the sequence $\{T_n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(\ell_p, \ell_1)$. (4 marks)

(Total: 20 marks)

5. This question is related to the additional content included in the Mastery reading. Let $(X, \|\cdot\|_X)$ be a Banach space. We recall that X' stands for the dual space. Consider a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

(a) Assume that $\{x_n\}_{n \in \mathbb{N}} \subset X$ is bounded. Is it true that there exists $x \in X$ such that $x_n \rightharpoonup x$?

(4 marks)

(b) Prove that if $\{x_n\}_{n \in \mathbb{N}}$ is bounded in X and there exists $x \in X$ and a dense subset $D \subset X'$ such that for every $f \in D$ it holds

$$f(x_n) \to f(x),$$

then $x_n \rightharpoonup x$.

(6 marks)

(c) Prove that the viceversa of part (b) is also true, namely that if $x_n \rightharpoonup x$ then the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded and there is a dense $D \subset X'$ such that $f(x_n) \to f(x)$ for every $f \in X'$. To do so, you may assume that if $x_n \rightharpoonup x$ in X then the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded.

(4 marks)

(d) Recall if $x \in X$, then
$$\|x\|_X = \sup_{\substack{f \in X' \\ \|f\|_{X'}=1}} |f(x)|.$$

Use this to prove that, if $x_n \rightharpoonup x$, then

$$\|x\|_X \leq \liminf_{n \to \infty} \|x_n\|_X.$$

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH 60020/70020/97025

Function Spaces and Applications (Solutions)

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1. (a) For every $x \neq 0$, we have that $f_n(x) \to 0$ when $n \to +\infty$. Hence, the sequence $f_n \to 0$ pointwise almost everywhere in $\mathbb{R}$. Since for every $n \in \mathbb{N}$ we have that $\sup_{x\in\mathbb{R}} f_n(x) = n^{\frac{1}{p}}$, the sequence cannot converge uniformly to 0 in $\mathbb{R}$.

(b) For every $n \in \mathbb{N}$ fixed, the function $f_n \in L^\infty(\mathbb{R})$ (see previous step). Moreover, for any $q \in [1, +\infty)$, we simply have

$$\int_{\mathbb{R}} |f_n|^q = n^{\frac{q}{p}} \int_{\mathbb{R}} e^{-nq|x|}\, dx < +\infty,$$

where the integral is finite since the function decays exponentially at infinity.

To find the values q such that the sequence is uniformly bounded, we note that

$$\int_{\mathbb{R}} |f_n|^q \overset{y=nx}{=} n^{\frac{q}{p}-1} \int_{\mathbb{R}} e^{-q|y|}\, dy$$

and that this is uniformly bounded in n whenever $q \leq p$.

(c) By Step (a) and the uniqueness of the limit, if $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^q(\mathbb{R})$, then it must converge to 0. From the previous step, we note that $\|f_n\|_{L^q(\mathbb{R})} \to 0$ whenever $q < p$. Hence, $f_n \to 0$ in $L^q(\mathbb{R})$ if $q \in [1, p)$. Note that the case $q = p$ is not included since, in this case, the functions have a constant, positive, L^p-norm (i.e. $\|f_n\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} e^{-p|y|}\, dy$ for all $n \in \mathbb{N}$). In the case $q > p$, the sequence is not even bounded in $L^q(\mathbb{R})$.

(d) Since weak convergence implies boundedness of the sequence, we immediately restrict the range of possible values of q to the interval $[1, p]$. Since strong convergence implies weak convergence, we already know that $f_n \rightharpoonup 0$ in $L^q(\mathbb{R})$ when $q \in [1, p)$. Thus, we only need to check whether the borderline case $q = p$ is included. We claim that $f_n \rightharpoonup 0$ also in $L^p(\mathbb{R})$: Let $\phi \in L^{p'}(\mathbb{R})$ be fixed. Since $p \in (1, +\infty)$, we have that the conjugate exponent $p' \in (1, +\infty)$. For every $\varepsilon > 0$, we may find a function $\tilde{\phi} \in C_0^\infty(\mathbb{R})$ such that $\|\tilde{\phi} - \phi\|_{L^{p'}(\mathbb{R})} < \varepsilon$. This is possible since the set $C_0^\infty(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$ whenever $p' \in [1, +\infty)$. We thus have:

$$\left| \int_{\mathbb{R}} \phi f_n \right| \leq \left| \int_{\mathbb{R}} (\phi - \tilde{\phi}) f_n \right| + \left| \int_{\mathbb{R}} \tilde{\phi} f_n \right|.$$

and, by Hölder's inequality

$$\left| \int_{\mathbb{R}} \phi f_n \right| \leq \|\phi - \tilde{\phi}\|_{L^{p'}(\mathbb{R})} \|f_n\|_{L^p(\mathbb{R})} + \|\tilde{\phi}\|_{L^\infty(\mathbb{R})} \|f_n\|_{L^1(\mathbb{R})}$$

Using Step (b), we have that $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\mathbb{R})} \leq C$. Using Step (c), we know that $\|f_n\|_{L^1(\mathbb{R})} \to 0$. Hence, the previous inequality implies that

$$\limsup_{n\to+\infty} \left| \int_{\mathbb{R}} \phi f_n \right| \leq C \|\tilde{\phi} - \phi\|_{L^{p'}(\mathbb{R})}.$$

Thanks to our choice of $\tilde{\phi}$, the right-hand side is smaller than $C\varepsilon$. Since ε is arbitrary (and C independent from ε and n), we conclude that

$$\int_{\mathbb{R}} \phi f_n \to 0.$$

Since $\phi \in L^{p'}(\mathbb{R})$ is arbitrary as well, we conclude that $f_n \rightharpoonup 0$ in $L^p(\mathbb{R})$.

2. (a) We begin by arguing the set inclusion $c_0 \subset \ell_\infty$. Let $x \in c_0$. Since the components
satisfy $x_n \to 0$, we may find $N \in \mathbb{N}$ such that $\sup_{n \geq N} |x_n| \leq 1$. Hence,

$$\sup_{n \in \mathbb{N}} |x_n| \leq \max_{n \leq N} |x_n| + 1,$$

i.e. $x \in \ell_\infty$. To argue that c_0 is also a subspace, it remains to note that, for every $x, y \in c_0$ and $\alpha, \beta \in \mathbb{R}$ the linear combination $\alpha x + \beta y \in c_0$. Here, the product and the sum are component-wise.

We finally argue that c_0 is closed: To do this, let us assume that $\{x^{(m)}\}_{m \in \mathbb{N}} \subset c_0$ is such that $x^{(m)} \to x$ in ℓ_∞. We need to show that $x \in c_0$ or, equivalently, that the components $x_k \to 0$. Let $\varepsilon > 0$: By the assumption on the convergence of the sequence, we may pick $x^{(n)}$ such that $\|x^{(n)} - x\|_{\ell_\infty} \leq \frac{\varepsilon}{2}$. Hence:

$$|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}| \leq \|x^{(n)} - x\|_{\ell_\infty} + |x_k^{(n)}| \leq \frac{\varepsilon}{2} + |x_k^{(n)}|.$$

Hence, it remains to pick $k \in \mathbb{N}$ big enough such that also $|x_k^{(n)}| \leq \frac{\varepsilon}{2}$. This is possible thanks to the assumption $\{x^{(n)}\}_{n \in \mathbb{N}} \subset c_0$.

(b) For every $y \in \ell_1$, consider the linear map $\phi_y : c_0 \to \mathbb{R}$ such that $\phi_y(x) = \sum_{n \in \mathbb{N}} y_n x_n$. We note that, since $c_0 \subset \ell_\infty$, this map is well-defined and, by Hölder's inequality, we have that

$$|\phi_y(x)| \leq \|y\|_{\ell_1} \|x\|_{\ell_\infty},$$

i.e. $\phi_y \in (c_0)'$ and $\|\phi_y\|_{(c_0)'} \leq \|y\|_{\ell_1}$.

(c) Thanks to Step (b), the map ϕ is well-defined. We begin with injectivity: Let
$y, \tilde{y} \in \ell_1$ be two distinct elements. Then, there exists $N \in \mathbb{N}$ such that $y_N \neq \tilde{y}_N$. Let $x \in c_0$ be defined by $x_k = 0$ if $k \neq N$ and $x_k = 1$ if $k = N$. Then, by construction, $\phi_y(x) \neq \phi_{\tilde{y}}(x)$, i.e. $\phi_y \neq \phi_{\tilde{y}}$ as elements in $(c_0)'$. We now need to prove that $\|\phi_y\|_{(c_0)'} = \|y\|_{\ell_1}$: Let $x^{(n)} = \{x_k^{(n)}\}_{k \in \mathbb{N}}$ be such that $x_k^{(n)} = \text{sign}(y_k)$ for every $k \leq n$ and $x_k^{(n)} = 0$. Note that $x^{(n)} \in c_0$ and $\|x^{(n)}\|_{\ell_\infty} = 1$ for every $n \in \mathbb{N}$. Then, we have

$$\phi_y(x^{(n)}) = \sum_{k \leq n} |y_k|, \quad \lim_{n \to +\infty} \phi_y(x^{(n)}) = \|y\|_{\ell_1}.$$

This and Step (b) imply that $\|\phi_y\|_{(c_0)'} = \|y\|_{\ell_1}$.

(d (i)) Let $x \in c_0$. For every $N \in \mathbb{N}$, we define the truncations $x^{(N)} = \sum_{n \leq N} x_n e^{(n)}$. We
note that $x^{(N)} \to x$ in c_0: We indeed have that

$$\sup_{n \in \mathbb{N}} |x_n^{(N)} - x_n| = \sup_{n \geq N} |x_n|$$

and that the right-hand side vanishes in the limit $N \to \infty$ by the assumption $x \in c_0$. Hence, since T is continuous, we have that $T(x^N) \to T(x)$. On the other hand, by linearity of T and the definition of y we also infer that

$$T(x^{(N)}) = \sum_{n \leq N} y_n x_n.$$

This yields the desired identity.

(d (ii))   We start by showing that $y \in \ell_1$ and we use an argument similar to the one of the previous step. Let $z^{(N)} = \sum_{n \leq N} \text{sign}(y_n) e^{(n)}$. Then $z^{(N)} \in c_0$ and $\|z^{(N)}\|_{\ell_\infty} = 1$ (here, we assume that $T \neq 0$ and, hence, that also $y \neq 0$. The case $T = 0$ is trivial). This implies that

$$\sum_{n \leq N} |y_n| = T(z^{(N)}) \leq \|T\|_{(c_0)'} \|z^{(N)}\|_{\ell_\infty} \leq C.$$

Sending $N \to +\infty$, we infer that $y \in \ell_1$.

This and Step (d)(i) imply that for every $T \in (c_0)'$ there exists $y \in \ell_1$ such that $T = \phi_y$. Hence, the map of Step (b) is also surjective. This, together with Step (b) and (c), yield that ϕ is an isomorphism and, therefore, that $\ell_1 = (c_0)'$.

4, D

3. (a) Assume $\alpha \neq 0$. We write

$$\int_{\mathbb{R}} \chi_{(a,b)}(t) e^{i\alpha t} \, dt = \int_a^b e^{i\alpha t} \, dt = \frac{1}{i\alpha} \int_a^b \frac{d}{dt}(e^{i\alpha t}) \, dt = i\frac{e^{i\alpha a} - e^{i\alpha b}}{\alpha},$$

where in the last identity wee used integration by parts.

5, A

(b) From Step (a) we have that

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$$\left| \int \chi_{(a,b)}(t) e^{i\alpha t} \, dt \right| \leq \frac{2}{\alpha},$$

where we used that $|e^{i\alpha t}| \leq 1$ for every $\alpha, t \in \mathbb{R}$. Sending $\alpha \to \pm\infty$ yields that the integral on the left-hand side above vanishes in the limit. This immediately extends to any step function f with compact support by linearity and the triangle inequality.

5, A

(c) Let $f \in C_c^0(\mathbb{R})$ be non-negative and let $\{\rho_n\}_{n \in \mathbb{N}}$ be a sequence of step functions such that $\rho_n \leq f$ and $\rho_n \to f$ pointwise. Consider the sequence $g_n := |f - \rho_n|$. Note that $g_n \to 0$ pointwise in $\mathbb{R}$ and that $g_n \leq 2f$. Since $f \in C_c^0(\mathbb{R}) \subset L^1(\mathbb{R})$, the Dominated convergence theorem yields that

meth seen ⇓

$$\int_{\mathbb{R}} |g_n| \to 0.$$

Hence, we have for any $n \in \mathbb{N}$

$$\left| \int_{\mathbb{R}} e^{i\alpha t} f(t) \right| \leq \left| \int_{\mathbb{R}} (f(t) - \rho_n(t)) e^{i\alpha t} \, dt \right| + \left| \int_{\mathbb{R}} \rho_n(t) e^{i\alpha t} \, dt \right| \leq \int_{\mathbb{R}} g_n + \left| \int_{\mathbb{R}} \rho_n(t) e^{i\alpha t} \, dt \right|.$$

From Step (b), we know that inequality (1) holds for every element of the sequence ρ_n. This implies that

$$\limsup_{\alpha \to \pm\infty} \left| \int_{\mathbb{R}} e^{i\alpha t} f(t) \right| \leq \int_{\mathbb{R}} g_n$$

and yields the desired inequality if we send $n \to +\infty$.

8, D

(d) Let $f \in C_c^0(\mathbb{R})$. Then we may write $f = f_+ - f_-$. Since both f_+ and f_- are defined as minima of two continuous functions, they are both continuous and non-negative. Step (c) thus applies to each of them. By the triangle inequality, we immediately infer (1) also for f. To extend (1) to any $f \in L^1(\mathbb{R})$, it remains to argue by density: Since $C_c^0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, for every $\varepsilon > 0$, we may find $\tilde{f} \in C_c^0(\mathbb{R})$ such that $\|f - \tilde{f}\|_{L^1(\mathbb{R})} \leq \varepsilon$. The triangle inequality thus implies that

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$$\left| \int_{\mathbb{R}} e^{i\alpha t} f(t) \right| \leq \|f - \tilde{f}\|_{L^1(\mathbb{R})} + \left| \int_{\mathbb{R}} e^{i\alpha t} \tilde{f}(t) \right|.$$

Sending $\alpha \to \pm\infty$ implies that

$$\limsup_{\alpha \to \pm\infty} \left| \int_{\mathbb{R}} e^{i\alpha t} f(t) \right| \leq \|f - \tilde{f}\|_{L^1(\mathbb{R})} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude the desired inequality.

2, C

4. (a) It is easy to see that $T_n(x)$ is linear. We have that

$$\|T_n(x)\|_{\ell_1} = \sum_{j \le 2n+1} |T_n(x)_j| = \sum_{j=0}^{n} |\alpha|^j |x_{2j+1}|.$$

If $p \in (1; +\infty]$, Hölder's inequality yields

$$\|T_n(x)\|_{\ell_1} \le \left(\sum_{j=0}^{n} |\alpha|^{jq}\right)^{\frac{1}{q}} \|x\|_{\ell_p},$$

i.e. $\|T\|_{\mathcal{L}(\ell_p,\ell_1)} \le \left(\sum_{j=0}^{n} |\alpha|^{jq}\right)^{\frac{1}{q}}$.
In the case $p = 1$, instead, we have

$$\|T_n(x)\|_{\ell_1} \le \left(\max_{j=0,\cdots,n} |\alpha|^j\right) \|x\|_{\ell_1},$$

i.e. $\|T\|_{\mathcal{L}(\ell_p,\ell_1)} \le \max_{j=0,\cdots,n} |\alpha|^j$.

(b) Since $p \in (1, +\infty]$, then the conjugate exponent $q \in [1, +\infty)$. Let $x \in \ell_p$ be defined as

$$x_k = \begin{cases} |\alpha|^{j(q-1)} & \text{whenever } k = 2j+1, \ j \in \mathbb{N} \text{ and } j \le n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|x\|_{\ell_p}^p = \sum_{j=0}^{n} |\alpha|^{jq}$. Hence, we have

$$\frac{\|T_n(x)\|_{\ell_1}}{\|x\|_{\ell_p}^p} = \left(\sum_{j=0}^{n} |\alpha|^{jq}\right)^{\frac{1}{q}},$$

i.e. $\|T\|_{\mathcal{L}(\ell_p,\ell_1)} = \left(\sum_{j=0}^{n} |\alpha|^{jq}\right)^{\frac{1}{q}}$.

(c) Let $J \in \{0, \cdots, n\}$ be such that $|\alpha|^J = \max_{j=0,\cdots,n} |\alpha|^j$ (this will depend on whether $|\alpha| < 1$ or $|\alpha| > 1$). We pick as element $x \in \ell_p$ the one such that $x_k = 0$ if $k \ne 2J+1$ and $x_{2J+1} = 1$. Note that $\|x\|_{\ell_p} = 1$ and that

$$|T_n(x)| = |\alpha|^J = \max_{j=0,\cdots,n} |\alpha|^j.$$

This implies $\|T\|_{\mathcal{L}(\ell_p,\ell_1)} = \max_{j=0,\cdots,n} |\alpha|^j$.

(d) Let $n, m \in \mathbb{N}$, $m \le n$ be fixed. For every $x \in \ell_p$ we have that

$$\|T_n(x) - T_m(x)\|_{\ell_1} = \sum_{j=m}^{n} |\alpha|^j |x_{2j+1}|.$$

Arguing as in Step (a) and using that $|\alpha| < 1$, we have that

$$\|T_n - T_m\|_{\mathcal{L}(\ell_p,\ell_1)} \le \begin{cases} \left(\sum_{j=m}^{+\infty} |\alpha|^{jq}\right)^{\frac{1}{q}} & \text{if } p \in (1; +\infty] \\ |\alpha|^m & \text{if } p = 1. \end{cases}$$

In both cases, the right-hand side vanishes in the limit $m \to +\infty$. The sequence $\{T_n\}_{n \in \mathbb{N}}$ is therefore a Cauchy sequence and, since $\mathcal{L}(\ell_p, \ell_1)$ is complete, it is convergent.

5. (a) By Banach-Alaoglu's theorem we only know that, up to a subsequence, $\{x_n\}_{n\in\mathbb{N}}$ converges in the weak* topology. If X is also reflexive, we know that the weak and weak* topologies do coincide and thus that $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence that is weakly convergent.

> 4, M

(b) We know that there is a dense set $D \subset X'$ such that for every $f \in D$ it holds $f(x_n) \to f(x)$. Recalling the definition of weak convergence, we need to prove that the previous limit is true for every $f \in X'$. Let $f \in X'$ be fixed. Since D is dense in X', we may find $\{f_k\}_{k\in\mathbb{N}} \subset D$ such that $f_k \to f$ in X'. Hence

$$|f(x_n) - f(x)| \leq |f(x_n) - f_k(x_n)| + |f_k(x_n) - f_k(x)| + |f_k(x) - f(x)|$$

and so

$$|f(x_n) - f(x)| \leq \|f - f_k\|_{X'}(\sup_{n\in\mathbb{N}} \|x_n\|_X + \|x\|_X) + |f_k(x_n) - f_k(x)|$$
$$\leq C\|f - f_k\|_{X'} + |f_k(x_n) - f_k(x)|,$$

where we used that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is assumed to be bounded. Since $f_k \in D$, we may send $n \to \infty$ and infer that

$$\limsup_{n\to+\infty} |f(x_n) - f(x)| \leq C\|f - f_k\|_{X'}.$$

It thus remains to send $k \to +\infty$.

> 6, M

(c) This direction is easier: If $x_n \rightharpoonup x$ and we know that the sequence is bounded, we may simply choose $D = X$ and have that for every $f \in D$ it holds $f(x_n) \to f(x)$ by definition of weak convergence.

> 4, M

(d) For every $f \in X'$ such that $\|f\|_{X'} = 1$, we write

$$|f(x_n)| \leq \|f\|_{X'}\|x_n\|_X = \|x_n\|_X.$$

Taking the inferior limit to both sides and using that, by weak convergence $f(x_n) \to f(x)$ we infer that

$$|f(x)| \leq \liminf_{n\to+\infty} \|x_n\|_X.$$

We now take the supremum over $f \in X'$ with $\|f\|_{X'} = 1$ and conclude that

$$\|x\|_X = \sup_{\substack{f\in X' \\ \|f\|_{X'}=1}} |f(x)| \leq \liminf_{n\to+\infty} \|x_n\|_X.$$

> 6, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.		

ExamModuleCode	QuestionNumber	Comments for Students
	1	The first parts have been done pretty well. Most of the students struggled with the last step. This was harder than the other parts, but a very similar problem was contained already in one of the problem sheets that I gave and fully corrected during the term.
	2	Well done overall
	3	This has been answered partially well. The most common mistakes are in part c and d which require to motivate why one can exchange the order of two limits. This has been overlooked by many students.
	4	Well done overall
	5	Well done overall