Imperial College London

MATH70038

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Graph Theory

Date: 17 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Throughout this paper, all graphs are assumed to be finite and simple unless otherwise stated. The set $\mathbb{N}$ consists of the positive integers.

1. (a) Provide the examples requested below. For each example, give a brief justification of why it has the required properties.

   (i) Give an example of a connected graph G such that $\kappa(G) < \lambda(G)$. (Recall that $\kappa(G)$ is the *connectivity* of G and $\lambda(G)$ is the *edge connectivity* of G.) (2 marks)

   (ii) Give an example of a connected graph G such that $\kappa(G) = \lambda(G)$. (3 marks)

   (iii) Give an example of a graph whose automorphism group is trivial. (Recall that a group is trivial when it only consists of the identity element.) (1 mark)

   (iv) Find two graphs G and H such that $G \ncong H$ and H is a minor of G. (2 marks)

   (b) Let $G = (V, E, \varepsilon)$ be a graph and let $\sigma \in \mathrm{Aut}(G)$.

   (i) Show that for all $v \in V$, $\deg(\sigma(v)) = \deg(v)$. (2 marks)

   (ii) Let $n \in \mathbb{N}$ with $n > 1$. Let $G = P_n$ be the *path graph* on n vertices. Show that $\mathrm{Aut}(G)$ is a group of order 2. (3 marks)

   (c) Prove the following statement or disprove it by giving a counterexample.

   Let G and H be two graphs with the same order. Then $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$ implies that $G \cong H$. (3 marks)

   (d) Let G be a graph. Assume that G contains a cycle C and there are two vertices on C such that G contains a path of length at least k between them. Show that G contains a cycle of length at least $\sqrt{k}$. (Recall that we defined the length of a path to be the number of its edges.) (4 marks)

(Total: 20 marks)

2. (a) Answer the questions below. If you are asked to provide an example, give a brief justification of why it has the required properties.

   (i) Give an example of a bipartite graph that has a perfect matching. (1 mark)

   (ii) Give an example of a bipartite graph that does not have a perfect matching. (2 marks)

   (iii) Find a cover of the complete bipartite graph $K_{3,3}$. (Recall that $K_{3,3}$ is the bipartite graph with two parts A and B, each consisting of 3 vertices, and such that every vertex in A is connected to all vertices in B.) (2 marks)

   (iv) Let $G = (V, E, \varepsilon)$ be a graph and let $\sigma \in \mathrm{Aut}(G)$. Prove that there is a bijective map

$$\sigma_E : E \longrightarrow E,$$

   such that, for all $e \in E$, $\varepsilon(\sigma_E(e)) = \sigma(\varepsilon(e))$. (3 marks)

(b) Let $G = (V, E, \varepsilon)$ be a bipartite graph. Let $\sigma \in \mathrm{Aut}(G)$ and let σ_E be the map in (a) (iv).

   (i) Prove that if C is a cover of G, then $\sigma(G)$ is also a cover of G. (2 marks)

   (ii) Let M be a matching of G. Prove that $\sigma_E(M)$ is also a matching of G. (3 marks)

(c) Prove that a tree is planar. (Recall that we defined a tree to be a connected graph with no cycles.) (3 marks)

(d) Prove Hall's 1935 Marriage Theorem using Kőnig's Theorem. (You will have to show that if A and B is the partition of a bipartite graph G and $|N(S)| \geq |S|$ for all $S \subseteq A$, then G has a matching of A. The converse is easy and does not require Kőnig's Theorem, so you do not have to prove it here.) (4 marks)

(Total: 20 marks)

3. (a) For each of the following statements say if it is true or false. Justify your answer with a short proof or a counterexample. Credit will only be given for the part of your answer containing the justification.

   (i) All planar graphs are 3-colourable. (2 marks)

   (ii) Kőnig's Theorem remains true if we remove the requirement that the graph be bipartite. (3 marks)

   (iii) Let $n \in \mathbb{N}$. Then $\chi_C(K_n) = n$ (where $\chi_C(G)$ is the chromatic number of the graph G and K_n denotes the complete graph on n vertices). (3 marks)

(b) Let $n \in \mathbb{N}$, recall that $\overline{K_n}$ is the graph with n vertices and no edges between them.

   (i) Prove that $\overline{K_n}$ has chromatic polynomial $P_{\overline{K_n}}(x) = x^n$. (2 marks)

   (ii) Prove that K_n has chromatic polynomial $P_{K_n}(x) = \prod_{i=0}^{n-1}(x-i)$. (3 marks)

(c) Let $n \in \mathbb{N}$. Prove that a tree T with n vertices has chromatic polynomial $P_T(x) = x(x-1)^{n-1}$. (3 marks)

(d) Prove that if G is a planar graph with $\delta(G) \leq 4$, then G is 4-colourable. (Recall that $\delta(G)$ is the *minimum degree* of G and is defined as $\delta(G) = \min\{\deg(v) \mid v \in V_G\}$.) (4 marks)

(Total: 20 marks)

4. (a) Answer the questions below. If you are asked to provide an example, give a brief justification of why it has the required properties.

   (i) Let $k, \ell \in \mathbb{N}$. Define the *Ramsey number* $R(k, \ell)$. (2 marks)

   (ii) Let $n \in \mathbb{N}$. Prove that $R(1, n) = R(n, 1) = 1$ and that $R(2, n) = R(n, 2) = n$. (3 marks)

   (iii) Compute $R(3, 3)$. (For this question, you may use without proof the upper bound for $R(k, \ell)$ involving a binomial coefficient.) (3 marks)

   (b) Prove Mantel's theorem: if G is a graph such that $K_3 \not\subseteq G$ then $\|G\| \leq \frac{|G|^2}{4}$. (You may use Turán's Theorem in your solution. Recall that K_3 is the *complete graph* on 3 vertices.) (5 marks)

   (c) Let $n, k, r \in \mathbb{N}$. Show that if $k < r - 1$, then $\|T(n, k)\| \leq \|T(n, r-1)\|$. (Recall that $T(n, k)$ and $T(n, r-1)$ are Turán graphs.) (3 marks)

   (d) Use a version of Ramsey's Theorem to prove that any infinite sequence of real numbers contains an infinite subsequence which is either constant or strictly monotonic. (4 marks)

(Total: 20 marks)

5. (a) Let G be a graph and let $X, Y \subseteq V_G$ be disjoint.

(i) Define $\|X, Y\|$ and show it is equal to $\|Y, X\|$. (2 marks)

(ii) Define $d(X, Y)$ and show it is equal to $d(Y, X)$. (2 marks)

(iii) Prove that $d(X, Y)$ is a rational number between 0 and 1. (2 marks)

(b) State *Szemerédi's Regularity Lemma*. (2 marks)

(c) Prove the following statement.

Let G be a graph and let $X, Y, Z \subseteq V_G$ be disjoint. Suppose there is $\varepsilon \in \mathbb{R}_{>0}$ such that X, Y, Z are pairwise ε-regular and $d(X, Y), d(X, Z), d(Y, Z)$ are at least 2ε. Then the number of triples $(x, y, z) \in X \times Y \times Z$ such that x, y, z form a triangle in G is at least

$$(1 - 2\varepsilon)(d(X, Y) - \varepsilon)(d(X, Z) - \varepsilon)(d(Y, Z) - \varepsilon)|X||Y||Z|.$$

(*Hint: consider the set of vertices in X that have fewer than $(d(X, Y) - \varepsilon)|Y|$ neighbours in Y and use the regularity of (X, Y) to get an upper bound for its size.*)

(12 marks)

(Total: 20 marks)

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Graph Theory (Solutions)

Date: Wednesday, 17th May 2023
Time: 14:00
Time Allowed: 2 Hours for MATH60 paper; 2.5 Hours for MATH70 papers
This paper has *4 Questions (MATH60 version); 5 Questions (MATH70 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.
- **Throughout this paper, all graphs are assumed to be finite and simple unless otherwise stated. The set $\mathbb{N}$ consists of the positive integers.**

Solutions

1. (a) Below are some examples with justifications. Solutions that use less words are also accepted as long as the key points are there.

   i. The bowtie graph is one such example (the graph obtained by glueing two triangles K_3 on one vertex). Let v be the vertex with valency 4 in the bowtie graph. If v is deleted, the graph becomes disconnected, so $\kappa(G) = 1$.

   (1 mark) **Cat A** seen

   However, if we remove any edge the graph remains connected, which means that $\lambda(G) > 1$.

   (1 mark) **Cat A** seen

   ii. K_4 is an example of one such graph (K_3 is an example as well but in order to prove $\kappa(G) = 2$ you need to remember that $\kappa(G) \leq |G| - 1$ and not argue that removing two vertices makes the graph disconnected, because K_1 is connected).
   In fact, for all $n \in \mathbb{N}$, $\kappa(K_n) = \lambda(K_n) = n - 1$. By definition $\kappa(K_n) \leq n - 1 \geq \lambda(K_n)$, so it suffices to show that K_n remains connected if we remove less than $n-1$ vertices or less than $n-1$ edges. In the first case we have that the remaining graph is K_1, which is connected

   (1 mark) **Cat A** seen

   at this point one could remember that $\kappa(G) \leq \lambda(G)$ as proved in class or show that removing less than $n-1$ edges does not disconnect G because there are $n-1$ edges between any two points of G, so removing less than $n-1$ edges will leave at least an edge between any two points of G.

   (2 marks) **Cat A** seen

   iii. K_1 is one such graph because there is only one point in its vertex set.

   (1 mark) **Cat A** seen

   iv. Here we could take $G = P_5$ (the path graph with 5 vertices) and $H = P_4$ (the path graph with 5 vertices). Contracting an edge in P_5 gives P_4.

   (1 mark) **Cat A** seen

   Clearly P_4 has only 4 vertices but P_5 has 5 so they cannot be isomorphic.

   (1 mark) **Cat A** seen

   (b) i. Let $v \in V$ as in the question. Let $u \in V$, we prove that u is a neighbour of v if and only if $\sigma(u)$ is a neighbour of $\sigma(v)$.
   Suppose that uEv, then by definition of automorphism, it follows that $\sigma(u)E\sigma(v)$.

   (1 mark) **Cat B** seen

   Conversely, suppose $\sigma(u)E\sigma(v)$. The map σ^{-1} is also an automorphism, so we deduce that $(\sigma^{-1}\sigma(u))E(\sigma^{-1}\sigma(v))$, that is, uEv.

   (1 mark) **Cat B** seen

   ii. Let u, v be the two vertices of degree 1 in G. By the previous point an automorphism of G needs to send $\{u, v\}$ to itself (it either fixes both u and v or it swaps them). Let $\Gamma = \mathrm{Aut}(G)$. Clearly, $\iota = id_{V_G} \in \Gamma$ and, if σ is the reflection around the middle point of the path (for n odd) or the edge in the middle of G (n even), then $\sigma \in \Gamma$. We show that all automorphisms of G are either equal to ι or to σ. Note that σ is the only possible reflection as we will be using this fact.

(1 mark) **Cat C** unseen

Let $\tau \in \mathrm{Aut}(G)$. Suppose that $\tau(u) = u$. We prove that $\tau = \iota$ by induction on n. The base case is $n = 2$, for which the statement is trivially true. Let u' be the only neighbour of u. Since τ is an automorphism, we necessarily have that $\tau(u') = u'$. Hence, the restriction of τ to $V \setminus \{u\}$ is still fixing the endpoints of the path graph between u' and v. This shows, by induction, that τ restricted to $V \setminus \{u\}$ is the identity and, therefore, $\tau = \iota$.

(1 mark) **Cat C** unseen

It remains to show that if τ swaps u and v, then it is a reflection. This is done again by induction on n. The base cases are $n = 2$ and $n = 3$: the first one is obvious, while, in the second one, if τ swaps the two endpoints, then it must fix the remaining one; therefore, it is a reflection around the midpoint.
In general, again, let u' be the only neighbour of u, and let v' be the only neighbour of v. Since τ swaps u and v, it must also swap u' and v'. Therefore, τ restricts to a reflection on $G \setminus \{u, v\}$ by inductive hypothesis. It follows that τ is a reflection of the path G and hence $\tau = \sigma$.

(1 mark) **Cat C** unseen

There is also a shorter proof: we just note that the argument above shows that Γ_u, the stabiliser of u for the action of Γ, is trivial. Since the orbit of u for this action has size 2, it follows, by the orbit stabiliser theorem, that Γ has order 2.

(c) The statement is false in general. Two graphs forming a counterexample are $G = \overline{K_2}$ and $H = K_2$. Clearly the two graphs are not isomorphic because $\|G\| = 0$ and $\|H\| = 1$. To prove their automorphism groups are isomorphic, we identify their vertex set with $\{1, 2\}$ and we prove that $\mathrm{Aut}(G) = \mathrm{Aut}(H) = S_2$ (the symmetric group on the set $\{1, 2\}$). By definition, every automorphism is a permutation of the vertex set; therefore, it suffices to prove that the transposition (12) is in both automorphism groups. This is clear because (12) is only swapping the two vertices. Hence, for both graphs, there is an edge between 1 and 2 if and only if there is an edge between 2 and 1.

(3 marks) **Cat B** seen

Alternatively, let $n > 1$. Then the null graph with n vertices (G $= \overline{K_n}$) has automorphism group S_n, which is also the automorphism group of $H = K_n$. (Both things were proved in class). Indeed, every automorphism of a graph G consists of permutations of its vertices. For convenience, we identify the vertex set of the two graphs above with $[n] = \{1, \ldots, n\}$. By definition, permutation in $\sigma \in S_n$ is an automorphism of G if and only if for every pair of vertices $i, j \in [n]$, $iE_G j$ if and only if $\sigma(i)E_G\sigma(j)$. This is trivially true for all $\sigma \in S_n$, because there are no edges in G. The same is true for H, that is, if $\sigma \in S_n$ and $i, j \in [n]$, $iE_H j$ if and only if $\sigma(i)E_H\sigma(j)$, because in H there is an edge between any pair of vertices. Clearly the two graphs cannot be isomorphic because

$$\|G\| = 0, \quad \|H\| = \binom{n}{2}.$$

(d) Let P be the path with at least k edges between two points of C. Assume that C has less than $\sqrt{k}$ edges (otherwise the cycle with at least $\sqrt{k}$ edges is C).

(2 marks) **Cat D** unseen

This means that there is a portion $T = (u, \ldots, v)$ of P that is at least $\sqrt{k}$ and only intersects C in u and v. The concatenation of T with a portion of C between u and v forms a cycle of length at least $\sqrt{k}$.

(2 marks) **Cat D** unseen

(Total: 20 marks)

2. (a) i. K_2 is an example. It is bipartite and the only edge is a perfect matching.

(1 mark) $\mathbf{Cat\ A}$ seen

ii. A bipartite graph with partition $\{u_1, v_1\}$, $\{u_2\}$ and $u_1 E u_2$. Then v_1 cannot be matched, so there cannot be a perfect matching.

(2 marks) $\mathbf{Cat\ A}$ seen

iii. Let $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ be the partition of $K_{3,3}$. Then $\{u_1, u_2, u_3\}$ is a cover as it contains an endpoint of all the edges in $K_{3,3}$.

(2 marks) $\mathbf{Cat\ A}$ seen

iv. We define σ_E as follows. Let $e \in E$, and let $\{u, v\} = \varepsilon(e)$. Since σ is an automorphism, there is a unique $f \in E$ such that $\sigma_E(e) = f$. We define $\sigma_E(e) = f$.

(2 marks) $\mathbf{Cat\ A}$ seen

The map σ_E^{-1} is the inverse of σ_E, which, therefore, is a bijection.

(1 mark) $\mathbf{Cat\ A}$ seen

(b) i. Let $e \in E$. By the definition of a cover, there is $v \in C$ such that $v \in \varepsilon(\sigma_E^{-1}(e))$. Then $\sigma(v) \in \varepsilon(e)$. This shows that $\sigma(C)$ is a cover because it contains an endpoint of every edge in G.

(2 marks) $\mathbf{Cat\ B}$ unseen

ii. It suffices to show that for all $e, f \in M$, $\sigma_E(e)$ and $\sigma_E(f)$ are not incident. This follows from the definition of matching: if $e, f \in M$ and $e \neq f$, then e and f are not incident. Since $\varepsilon(\sigma_E(e)) = \sigma(\varepsilon(e))$, it follows that $\sigma_E(e)$ and $\sigma_E(f)$ are not incident.

(3 marks) $\mathbf{Cat\ B}$ unseen

(c) Induction on the order of the tree. The base case is K_1 which is clearly planar. Let $n = |T|$ and suppose that every tree of order less than n is planar. Let v be a leaf of T. Then $T \setminus v$ is also a tree and is planar by inductive hypothesis. Reattaching v to $T \setminus v$ can be done without crossing any edge (in a drawing of $T \setminus v$). It follows that T is also planar.

(3 marks) $\mathbf{Cat\ C}$ seen

(d) Suppose that G is bipartite with partitions A and B like in the statement of the marriage theorem. Suppose there is no matching of A. We show that there is $S \subseteq A$, $|N(S)| < |S|$. This will prove the contrapositive of the sufficiency statement in the marriage theorem.

Indeed, if A is not matched, the size of a maximal matching is less than $|A|$. Thus by Kőnig's Theorem, the size of a minimal cover C is less than the size of $|A|$. Suppose $C = C_A \cup C_B$ where $C_A \subseteq A$ and $C_B \subseteq B$. The inequality

$$|C| < |A|$$

implies that $|C_B| < |A \setminus C_A|$. Since C is a cover, there cannot be any edge between $A \setminus C_A$ and $B \setminus C_B$ (otherwise there would be some edges that do not have any endpoint in C). It means that the neighbours of $A \setminus C_A$ are all contained in C_B. Hence, $S = A \setminus C_A$ is the set we are looking for.

(4 marks) $\mathbf{Cat\ D}$ unseen

(Total: 20 marks)

3. (a) i. False. The complete graph on 4 vertices is planar but it is not 3-colourable, because its chromatic number $\chi_C(K_4)$ is 4. (2 marks) **Cat A** seen

ii. False. K_3 is a counterexample, because a non-empty matching can only consist of one edge (once we choose one edge, the other two are incident with that one, so they cannot be put in the same matching). On the other hand every vertex has degree 2, so every cover of K_3 needs to contain at least two vertices to cover all the 3 edges.

(3 marks) **Cat A** seen

iii. True. Let $G = K_n$. Let c be a function $V_G \to \{1, \dots, n-1\}$. Then, c is not injective and there are two vertices $u, v \in V_G$ such that $c(u) = c(v)$. The consequence is that c cannot be a colouring of G, because G is complete, and so uE_Gv. This shows that $\chi_C(G) \geq n$. However, it is easy to see that the function assigning a different colour to each vertex is an n-colouring of G; hence, $\chi_C(G) = n$. (3 marks) **Cat A** seen

(b) i. Let $k \in \mathbb{N}$. If we take the graph with n vertices and no edges we try to construct a k-colouring, then we have k choices of colour for each of the n vertices. This is because there are no edges between the vertices and therefore every vertex may be assigned any of the k colours without restriction. (2 marks) **Cat B** seen (Notes)

ii. Let $k \in \mathbb{N}$. By definition, $P_G(k)$ is the number of ways we may colour K_n using k different colours. A standard counting argument will help us determine how many of them there are. Let $k \geq n$. Suppose we choose a vertex $v \in V_G$, we may assign any of the k colours to it. Thus we have k possibilities and each choice of colour will determine a different colouring. The vertex v is adjacent to all other $n-1$ vertices, so if we now pick $w \in V_G \smallsetminus \{v\}$ we may use only $k-1$ colours for it. Continuing this way we find that there are

$$k(k-1)(k-2)\cdots(k-n+1)$$

distinct k-colourings of G. The polynomial that has this values for $k \geq n$ (so infinitely many times) is $\prod_{i=0}^{n-1}(x-i)$, which gives the right number of k-colourings even when $k < n$. (3 marks) **Cat B** seen (Problem Sheet 6)

(c) We proceed by induction on the number of vertices. Let $k \in \mathbb{N}$. The trivial tree K_1 has k k-colourings, which proves the base case of our induction.

(1 mark) **Cat C** unseen (similar to week 9 test)

Suppose that $n > 1$ and that the result is true for all trees of order less than n. In particular it is true if we remove a leaf v form T (note that a leaf always exists as proved in a problem sheet). Hence $T \smallsetminus v$ has chromatic polynomial

$$P_{T\smallsetminus v}(x) = x(x-1)^{n-2}.$$

Therefore $T \smallsetminus v$ admits $k(k-1)^{n-2}$ distinct k-colourings. Let c be one of them. All colouring of T are obtained by extending a colouring of $T \smallsetminus v$ to V_T. Let, therefore, c be a k-colouring of $T \smallsetminus v$, and let w be the only neighbour of v in T. If $c(w) = i \in \{1, \dots, k\}$, then every extension of c to V_T is obtained by assigning a colour in $\{1, \dots, k\} \smallsetminus \{i\}$ to v. Hence, there are $k-1$ different ways of extending c, which implies that T has

$$k(k-1)^{n-1}$$

distinct k-colourings. This shows that $P_T(x) = x(x-1)^{n-1}$.

(2 marks) **Cat C** unseen (similar to week 9 test)

(d) (What follows is inspired by the proof of the 5-colouring theorem I wrote in the notes). Induction on the number of vertices. The base case is K_1 for which the statement is trivially true. Let $n \in \mathbb{N}$ and suppose that every graph of order $n-1$ having minimum degree less than or equal to 4 is 4-colourable.

(1 mark)**Cat D** unseen

Let G be a graph of order n with $\delta(G) \leq 4$ and let v be a vertex with $\deg(v) = \delta(G)$. Certainly $G \smallsetminus v$ has minimum degree less than or equal to 4. Therefore, by the inductive hypothesis, there is a 4-colouring of $G \smallsetminus v$. Let c be one such colouring.

If v has less than 4 neighbours or c uses less than 4 colours on them, then there is $i \in \{1, \ldots, 4\}$ such that $c(u) \neq i$ for all u with uE_Gv. Therefore, it is possible to extend c to a 4-colouring of G by assigning the colour i to v.

(1 mark)**Cat D** unseen

Assume, then, that v has 4 neighbours that are coloured with 4 different colours. Let $x_1, \ldots, x_4$ be the neighbours of v labelled clockwise in a drawing of G. After permuting $\{1, \ldots, 4\}$ if necessary, we may assume that $c(x_i) = i$ for all $i \in \{1, \ldots, 4\}$. We show that it is possible to alter c in such a way that two neighbours of v have the same colour, thereby reducing to the previous case.

We define two auxiliary sets of vertices and induced subgraphs:

$$X(1,3) = \{u \in V_G \setminus \{v\} \mid c(u) \in \{1,3\}\}, \qquad G_{13} = G[X(1,3)];$$
$$X(2,4) = \{u \in V_G \setminus \{v\} \mid c(u) \in \{2,4\}\}, \qquad G_{24} = G[X(2,4)].$$

We show that either x_1 and x_3 lie in two different connected components of G_{13} or x_2 and x_4 lie in two different connected components of G_{24}. This will be enough to conclude the proof. Indeed, in the first case, let C be connected component of G_{13} containing x_1. Then we may swap the two colours on C to obtain a colouring of $G \smallsetminus v$ that has x_1 and x_3 painted with the same colour. In the second case, a similar argument shows how to alter the colouring c to obtain a colouring with less than 4 colours on the neighbours of v.

Assume that x_1 and x_3 are in the same connected component of G_{13}, then there is a path P joining them in G_{13}. In the drawing of G (the same in which we labelled the x_i's clockwise), x_2 and x_4 do not lie in the same plane region, if we consider those cut out by the cycle vPv. It follows that there cannot be a path from x_2 to x_4 all within G_{24} and therefore x_2 and x_4 belong to two different connected components of G_{24}. This finishes the proof.

(2 marks)**Cat D** similar to seen

(A less detailed argument can also be accepted, as long as the part on swapping the colours is outlined correctly.)

(Total: 20 marks)

4. (a) i. If it exists, we define $R(k,\ell) \in \mathbb{N}$ to be the smallest number such that if $|G| \geq R(k,\ell)$ then either $K_k \leq G$ or $\overline{K_\ell} \leq G$.

(2 marks) $\mathbf{Cat\ A}$ seen

ii. we note that $K_1 = \overline{K_1}$. Clearly every non-empty graph, is also every one element graph, and so every non-empty graph has K_1 and $\overline{K_1}$ as an induced subgraph, so $R(1,n) = R(n,1) = 1$.

(1 mark) $\mathbf{Cat\ A}$ seen

If a graph G does not contain K_2 as an induced subgraph, then it contains K_n if and only if it has order at least n.

(2 marks) $\mathbf{Cat\ A}$ seen

iii. Proposition 6.1.5 in the notes implies that $R(3,3) \leq \binom{4}{2} = 6$. The graph C_5 does not contain either K_3 or its complement as a subgraph, therefore $R(3,3) > 5$. This shows that $R(3,3) = 6$. (The precise reference to Proposition 6.1.5 is not required to get full marks.)

(3 marks) $\mathbf{Cat\ A}$ seen

(b) Suppose that G is K_3-free. Then $\|G\| \leq \|T(|G|,2)\|$ by Turán's theorem.

(1 mark) $\mathbf{Cat\ B}$ seen

Suppose that $|G| \geq 2$. If $|G|$ is even then $T(|G|,2)$ is the complete bipartite graph where both parts have $\frac{|G|}{2}$ vertices. Therefore $\|T(|G|,2)\| = |G|^2/4$.

(1 mark) $\mathbf{Cat\ B}$ seen

Suppose that $|G|$ is odd, say $|G| = 2n+1$ for some $n \in \mathbb{N}$. Then $T(|G|,2)$ is the complete bipartite graph where one part has n vertices, and the other has $(n+1)$. Therefore, $\|T(|G|,2)\| = n(n+1) \leq (4n^2+4n+1)/4 = |G|^2/4$.

(2 marks) $\mathbf{Cat\ B}$ seen

If $|G| = 1$ then $T(|G|,2) = K_1$ and the result is trivially true because there are no edges in K_1.

(1 mark) $\mathbf{Cat\ B}$ seen

(c) Since the definition of the Turán graph has two cases, there are three cases to analyse here. Suppose first that $n \leq k < r-1$. Then $T(n,k) = K_n$ and $T(n,r-1) = K_n$, so the result is obvious.

Assume that $k < n \leq r-1$. Then again the statement follows trivially because $k < r-1$ and $T(n,r-1)$ contains K_{r-1} as an induced subgraph (choose one vertex for each part of $T(n,r-1)$).

Finally, assume that $k < r-1 < n$. Let $V_1,\dots,V_{r-1}$ be a partition of $T(n,r-1)$. Then $T(n,k)$ may be obtained from $T(n,r-1)$ by redistributing the vertices in $V_{k+1},\dots,V_{r-1}$ to the first k parts. This is the same as some repeated edge deletions in $T(n,r-1)$: every time that we move a vertex v, from V_i to V_j (for $k < i \leq r-1$ and $j \leq k$) we delete the edges connecting v to the vertices in V_j and then repeat the process on the graph we just obtained until we see $T(n,k)$. Hence, it is clear that, if $k < r-1$, there are less edges in $T(n,k)$ than in $T(n,r-1)$.

(3 marks) $\mathbf{Cat\ C}$ seen (part of the proof of Turán's theorem but left as an exercise)

(d) Let $X = \{x_1, x_2, \dots\}$ be the sequence of real numbers. We define a colouring $c : \mathcal{P}_2(X) \to$

$\{1,2,3\}$: for all pairs $\{x_i, x_j\}$ such that $i < j$ we set

$$c(\{x_i, x_j\}) = \begin{cases} 1 & x_i < x_j \\ 2 & x_i > x_j \\ 3 & x_i = x_j. \end{cases}$$

The infinite Ramsey's Theorem ensures that there is a monochromatic subset $M \subseteq X$. This is a set such that $c(\{x_i, x_j\})$ is constant for all $x_i, x_j \in M$ with $i < j$, and, therefore, it forms the required subsequence.

(4 marks) **Cat D** unseen

(Total: 20 marks)

5. (a) i. $\|X, Y\|$ is the number of edges in G having one end in X and the other in Y, it is clear from the definition that this notation is symmetric. (2 marks) seen

ii. $d(X, Y) = \|X, Y\|/(|X||Y|)$ and the definition is again symmetric because $\|X, Y\| = \|Y, X\|$. (2 marks) seen

iii. Clearly $d(X, Y)$ is a rational number as it is defined as the quotient of two integers. Since our graphs are simple, each vertex in X has at most $\|Y\|$ neighbours. Hence, $\|X, Y\| \leq |X||Y|$. (2 marks) seen

(b) For every $\varepsilon \in \mathbb{R}_{>0}$ and every $m \in \mathbb{N}$ with $m > 1$, there is an integer M, such that every graph of order at least m admits an ε-regular partition $\{V_0, V_1, \ldots, V_k\}$ with $m \leq k \leq M$. (2 marks) seen

(c) First we note that $\varepsilon \leq 1/2$. Otherwise the densities cannot be at least 2ε. Define

$$A_{XY} = \{x \in X \mid |N(x) \cap Y| < (d(X, Y) - \varepsilon)|Y|\}.$$

We show that $|A_{XY}| < \varepsilon|X|$. Indeed,

$$d(A_{XY}, Y) < \frac{|A_{XY}|(d(X, Y) - \varepsilon)|Y|}{|A_{XY}||Y|} = d(X, Y) - \varepsilon.$$

Thus $d(X, Y) - d(A_{XY}, Y) > \varepsilon$, which implies that $|d(X, Y) - d(A_{XY}, Y)| = d(X, Y) - d(A_{XY}, Y) > \varepsilon$ (because $\varepsilon > 0$). Now, $\varepsilon \leq 1/2$; hence, $|Y| \geq \varepsilon|Y|$. Therefore, $|A_{XY}| < \varepsilon|X|$, because otherwise the pair (X, Y) would not be regular.

(3 marks) unseen, but related to Q5 May 2021

An analogous argument shows that $|A_{XZ}| < \varepsilon|X|$, where

$$A_{XZ} = \{x \in X \mid |N(x) \cap Z| < (d(X, Z) - \varepsilon)|Z|\}.$$

Let $A' = A \setminus (A_{XY} \cup A_{XZ})$. Then $|A'| \geq (1 - 2\varepsilon)|X|$ and A' has at least $(d(X, Y) - \varepsilon)|Y|$ neighbours in Y and at least $(d(X, Z) - \varepsilon)|Z|$ neighbours in Z.

(3 marks) unseen, but related to Q5 May 2021

Let $x \in X'$, and let

$$N_Y(x) = \{y \in Y \mid yEx\},$$
$$N_Z(x) = \{z \in Z \mid zEx\}.$$

Clearly the number of triangles that have x as one of their vertices (and the other two in Y and Z) is equal to $\|N_Y(x), N_Z(x)\|$ (our graphs are simple). By definition of density,

$$\|N_Y(x), N_Z(x)\| = d(N_Y(x), N_Z(x))|N_Y(x)||N_Z(x)|.$$

We have given a lower bound for the last two factors of this product and we now need a lower bound for the first one.

Note that $|N_Y(x)| \geq (d(X, Y) - \varepsilon)|Y| \geq \varepsilon|Y|$ because $d(X, Y) \geq 2\varepsilon$. Similarly $|N_Z(x)| \geq \varepsilon|Z|$. Hence, as (Y, Z) is an ε-regular pair,

$$|d(Y, Z) - d(N_Y(x), N_Z(x))| \leq \varepsilon.$$

In particular, since ε is positive, $d(Y,Z) - d(N_Y(x), N_Z(x)) \leq \varepsilon$; therefore,

$$d(N_Y(x), N_Z(x)) \geq d(Y,Z) - \varepsilon.$$

(3 marks) unseen

In short, there are at least $(1-2\varepsilon)|X|$ vertices $x \in X'$, and each of them is the vertex of at least

$$(d(X,Y) - \varepsilon)(d(X,Z) - \varepsilon)(d(Y,Z) - \varepsilon)|Y||Z|$$

distinct triangles with the desired property (one vertex X, one in Y, and one in Z). Hence, the total number of these triangles is at least

$$(1-2\varepsilon)(d(X,Y) - \varepsilon)(d(X,Z) - \varepsilon)(d(Y,Z) - \varepsilon)|X||Y||Z|,$$

as we wanted to prove.

(3 marks) unseen

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70052	1	Students did this rather well. The last part involves much algebra, and so students chose to describe the idea without working out all the algebraic detail.
MATH70052	2	Students had a decent tackle of this problem. Part b(ii) could be long if all terms in the equations are written out explicitly. In fact, it suffices only to retain the leading order terms. Some students skipped this part.
MATH70052	3	This question was quite well done.
MATH70052	4	tudents did not do as well as expected.
MATH70052	5	Except one, all did rather poorly. It was a long paper, and students appeared to struggle with time. The overall performance by students was all right. Given this is a long paper, the E and M will be set accordingly (to ensure a fair comparison with other modules).