MATH50010 – Autumn 2022 – Midterm

You should state carefully any results from lectures that are used.

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \Pr)$.

(a) (2 marks) State two equivalent conditions for the function $X : \Omega \to \mathbf{R}$ to be a random variable with respect to $\mathcal{F}$.

(b) (4 marks) Let $\Omega = \{1, 2, 3, 4\}$. Give an example of a function $X : \Omega \to \mathbf{R}$ and two sigma algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ such that X is a random variable with respect to $\mathcal{F}_1$ but not $\mathcal{F}_2$.

(c) (3 marks) Show that if F_X is the cumulative distribution function of a random variable X, then $\lim_{x\to+\infty} F_X(x) = 1$.

In the remainder of the question, let X be an absolutely continuous Beta$(1, \beta)$ random variable with probability density function given by

$$f_X(x) = \beta(1 - x)^{\beta-1}, \quad \text{for} \quad 0 < x < 1, \tag{1}$$

and zero otherwise, where $\beta \in \{1, 2, \ldots\}$.

(d) (2 marks) Write down the cumulative distribution function of X.

(e) (1 mark) Write down the value $\Pr(X = 0.75)$ for $\beta = 2$.

(f) (2 marks) Determine the probability density function of the random variable $Y = 1 + 5X$.

(g) (3 marks) Let $U \sim \textsc{Uniform}(0, 1)$, find a function H such that $H(U)$ has the same distribution as X. Explain how this can be used to draw a random sample from the distribution of X, assuming that we can sample uniform random variables easily.

(h) (3 marks) Assume that $X_1, \ldots, X_n$ are i.i.d. random variables with density function f_X for a fixed $\beta \in \{1, 2, \ldots\}$ as in (1). Calculate the distribution of $Y = \min\{X_1, \ldots, X_n\}$. Comment on the form of this distribution.