Imperial College London

MATH97027 MATH70019

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Theory of Partial Differential Equations

Date: 12 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

5. Consider the following initial boundary value problem

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = u^3, & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T) \\ u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), & x \in [0,1], \end{cases} \tag{11}$$

with $g, h \in C^\infty([0,1])$. Assume that $u \in C^2([0,T) \times (0,1))$ with u solution to (11). Define

$$E(t) = \frac{1}{2}\int_0^1 \left(|\partial_t u|^2 + |\partial_x u|^2 - \frac{1}{2}u^4 \right)(t,x)dx, \qquad I(t) = \frac{1}{2}\int_0^1 |u(t,x)|^2 dx. \tag{12}$$

(a) Show that $E(t) = E(0)$ and (8 marks)

$$I''(t) = \int_0^1 \left(3|\partial_t u|^2 + |\partial_x u|^2 \right)(t,x)dx - 4E(t) \tag{13}$$

(b) In the following, assume that $E(0) < 0$ and $\int_0^1 g(x)h(x)dx > 0$.

(i) Prove that $I(t) > 0$, $I'(t) > 0$, $I''(t) > 0$ for all $t \in [0,T)$. (4 marks)

(ii) Assume the Cauchy-Schwartz inequality (4 marks)

$$\left(\int_0^1 v(x)w(x)dx \right)^2 \leq \left(\int_0^1 |v(x)|^2 dx \right) \left(\int_0^1 |w(x)|^2 dx \right).$$

First show

$$I(t)I''(t) \geq \frac{3}{2}(I'(t))^2, \tag{14}$$

then deduce that

$$\frac{d}{dt}\log(I'(t)) \geq \frac{d}{dt}\log(I^{\frac{3}{2}}(t)). \tag{15}$$

(iii) Prove that there exists a *finite* $t_\star > 0$ such that (4 marks)

$$\lim_{t \to t_\star^-} I(t) = +\infty. \tag{16}$$

(Total: 20 marks)

1. Consider the Cauchy problem for the Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & (t,x) \in (0,+\infty) \times \mathbb{R} \\ \rho(0,x) = g(x), & x \in \mathbb{R} \end{cases} \qquad g(x) = \begin{cases} 0 & x \leq 0, \\ 1 & 0 < x \leq 1, \\ 0 & x > 1. \end{cases} \tag{1}$$

(a) (i) Solve the characteristic system associated to the problem (1) and draw the characteristic lines. (4 marks)

(ii) Assume that $t \leq 2$. Compute the shock curve, find the unique entropy solution and draw the characteristic lines. (12 marks)

(b) For $t > 2$, compute the shock curve, find the unique entropy solution and draw the characteristic lines. (4 marks)

(Total: 20 marks)

2. Let $\kappa > 0$, $\alpha \geq 0$ and $N \in \mathbb{N}$ be given constants. Consider the following initial boundary value problem

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0, & (t,x) \in (0,+\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,+\infty) \\ u(0,x) = \alpha \sin(\pi N x), & x \in [0,1]. \end{cases} \tag{2}$$

(a) (i) Using the method of separation of variables, find a solution to (2). (8 marks)

(ii) Prove that the solution found is the unique one. (8 marks)

(b) Determine a time $T^\star \geq 0$, which can depend on α, κ, N, such that (4 marks)

$$|u(t,x)| \leq e^{-10}, \qquad \text{for all } t \geq T^\star,\ x \in [0,1]. \tag{3}$$

Is it true that $T^\star \to 0$ as $N \to \infty$?

(Total: 20 marks)

3. Let $c > 0$. Consider the one-dimensional wave equation

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

(a) Assume that $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Show how to obtain the d'Alembert formula, given by

$$u(t,x) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy. \tag{5}$$

(10 marks)

(b) Let $c = 1$ and consider the following initial data

$$g(x) = 0, \qquad h(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases} \tag{6}$$

(i) Appealing to (5), determine the solution u to (4). Is it a classical solution to (4)? (6 marks)

(ii) Plot $u(t,x)$ as function of x, at times $t = 0,\ t = 1/2,\ t = 1,\ t = 2$. Report on the figures all the relevant values of x and $u(t,x)$. Compute (4 marks)

$$\lim_{t \to +\infty} u(t,x).$$

(Total: 20 marks)

4. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Consider the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{7}$$

with $f \in C^2(\Omega),\ h \in C^2(\partial\Omega)$. Assume that a solution u to (7) is such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

(a) For any $w \in C^2(\Omega) \cap C(\overline{\Omega})$, define

$$E[w] = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 dx + \int_\Omega w(x)f(x)dx. \tag{8}$$

(i) Show that if u solves (7), then (10 marks)

$$E[u] \le E[w], \qquad \text{for all } w \in C^2(\Omega) \cup C(\overline{\Omega}), \text{ such that } w = h \text{ on } \partial\Omega. \tag{9}$$

(ii) If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with $u = h$ on $\partial\Omega$ satisfies (9), prove that u solves (7). (6 marks)

(b) Consider $f = 0$ and assume that $u \in C^3(\Omega) \cap C(\overline{\Omega})$. Let $B_R(x)$ be the open ball of radius R centered in $x \in \Omega$ such that the closed ball $\overline{B}_R(x)$ is contained in Ω, i.e. B_R is compactly contained in Ω. Prove that

$$|\nabla u(x)| \le \frac{C}{R}\sup_{\overline{B}_R(x)} |u|, \tag{10}$$

for a constant $C > 0$ which does not depend on R. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH60019/70019/97027/97104

Theory of Partial Differential Equations (Solutions)

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1. (a) (i) The ODEs to solve for the characteristic system is

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$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = z, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = 0, \tag{1}$$

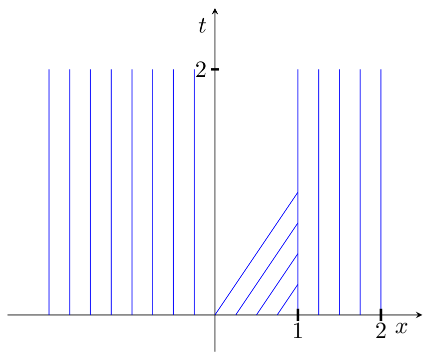
with initial conditions

$$t(0) = 0, \quad x(0) = \tau, \quad z(0) = g(\tau). \tag{2}$$

Solving the system, we obtain the formula

$$x = \tau + g(\tau)t = \begin{cases} \tau, & \tau \leq 0, \\ \tau + t, & 0 < \tau \leq 1, \\ \tau, & \tau > 1. \end{cases} \tag{3}$$

The characteristic lines are shown in the figure below



(ii) The region

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$$S := \{(t, x) : 0 < x \leq t\}$$

is not covered by the characteristics. We then connect the states 0 and 1 through a rarefaction wave, that is a solution of the form $\rho(t,x) = x/t$ in the region S. (We could have chosen to form a shock but since we are passing from 0 to 1 this shock wouldn't be entropic).

On the other hand, from (3), we see that the characteristic lines intersect at time $t = 0$ and $x = 1$. At $x = 1$ the initial datum g has a decreasing discontinuity, we can therefore expect the formation of a shock discontinuity. We compute the shock curve $(t, \sigma(t))$ emanating from $(0, 1)$ appealing to the Rankine-Hugoniot condition. In particular, for the Burgers equation we know that σ satisfies

$$\sigma'(t) = \frac{1}{2}\frac{\rho_+(t,\sigma(t))^2 - \rho_-(t,\sigma(t))^2}{\rho_+(t,\sigma(t)) - \rho_-(t,\sigma(t))} = \frac{1}{2}(\rho_+(t,\sigma(t)) + \rho_-(t,\sigma(t))) \tag{4}$$

$$\sigma(0) = 1. \tag{5}$$

In addition, the shock is entropic if

$$\rho_+(t,\sigma(t)) < \sigma'(t) < \rho_-(t,\sigma(t)). \tag{6}$$

The value at the right of the shock will always be $\rho_+ \equiv 0$, meaning that an entropic shock is such that $\sigma'(t) > 0$ for all $t \geq 0$. To determine ρ_-, we observe that on the left of the shock we have the region where $\rho \equiv 1$ and

$\rho(t,x) = x/t$ on S. But since $\sigma'(t) > 0$, the shock curve does not enter in the region S at least for some time $t^* > 0$. Consequently, we have that

$$\rho_-(t,\sigma(t)) \equiv 1, \qquad \text{for } t \le t^*. \tag{7}$$

With this, we find $\sigma'(t) = 1/2$ for $t \le t^*$. Therefore

$$\sigma(t) = t/2 + 1, \qquad t \le t^*,$$

so that the shock curve is the straight line $(t, t/2+1)$. The shock enters in the region S if

$$t = \frac{t}{2} + 1 \quad \Longrightarrow \quad t = 2. \tag{8}$$

In particular, we can take $t^* \le 2$.

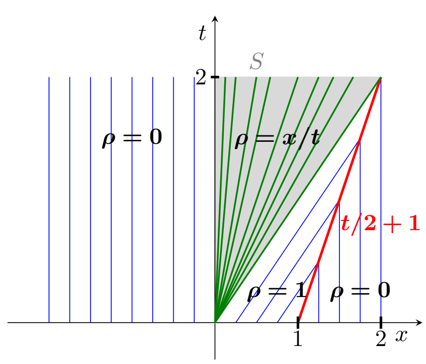
For $t \le 2$, we can define the solution as

$$\rho(t,x) = \begin{cases} 0, & x \le 0, \\ x/t, & 0 < x \le t, \\ 1, & t < x \le t/2+1, \\ 0, & x > t/2+1, \end{cases} \qquad \text{for all } 0 \le t \le 2. \tag{9}$$

Notice that this is an entropic solution since

$$0 = \rho_+(t,\sigma(t)) < \sigma'(t) = 1/2 < \rho_-(t,\sigma(t)) = 1, \qquad \text{for all } 0 \le t \le 2.$$

Thanks to Theorem 3.7 in Chapter 2 of the lecture notes, we know that (9) is the unique entropy solution since also g is bounded. In the picture below you find the characteristics with the rarefaction wave and the shock line for the solution (9).



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(b) When $t = 2$, the straight line $(t, t/2+1)$ enters in the region $S = \{(t,x) : 0 < x < t\}$. This implies that we need to modify the shock curve since we will not see anymore the value $\rho \equiv 1$ as in the previous case. In particular, on the left of the shock we now have

$$\rho_-(t,\sigma(t)) = \frac{\sigma(t)}{t}, \qquad t > 2. \tag{10}$$

Since $\sigma(2) = 2$, the ODE we need to solve is given by

$$\sigma'(t) = \frac{\sigma(t)}{2t}, \qquad \text{for } t > 2 \tag{11}$$

$$\sigma(2) = 2. \tag{12}$$

The solution is

$$\sigma(t) = \sqrt{2t}.$$

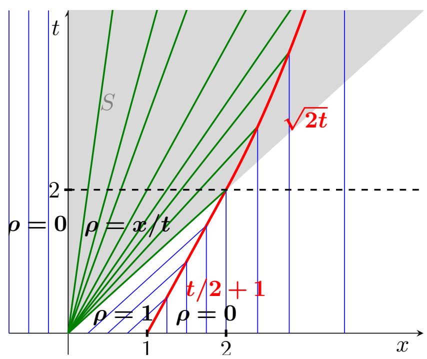
We only have to check that this shock is entropic, namely

$$0 = \rho_+(t, \sigma(t)) < \sigma'(t) = \frac{1}{\sqrt{2t}} < \rho_+(t, \sigma(t)) = \frac{\sqrt{2}}{\sqrt{t}}. \tag{13}$$

Therefore, the unique entropy solution for $t > 2$ is

$$\rho(t, x) = \begin{cases} 0, & x \leq 0, \\ x/t, & 0 < x \leq \sqrt{2t}, \\ 0, & x > \sqrt{2t}, \end{cases} \qquad \text{for all } t > 2, \tag{14}$$

where we also use that $\rho(2, x)$ is bounded to apply Theorem 3.7 in Chapter 2 of the lecture notes. In the picture below you find the characteristics with the rarefaction wave and the shock line for the solution $\rho(t, x)$ for all $t \geq 0$.



2. (a) (i) We look for a solution of the form $u(t,x) = w(t)v(x)$. Plugging this ansatz into the equation, we get

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$$w'(t)v(x) - \kappa w(t)v''(x) = 0 \qquad \Longrightarrow \qquad \frac{1}{\kappa}\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)}. \tag{15}$$

But the last identity is only possible if there is a constant $\lambda \in \mathbb{R}$ such that

$$\begin{cases} v'' = \lambda v, & \text{for } x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{16}$$

and

$$\begin{cases} w' = \lambda w, & \text{for } t > 0, \\ w(0) = C, \end{cases} \tag{17}$$

where C is an arbitrary constant that we will fix later. To solve (16) and (17), we distinguish three different cases below.

Case $\lambda = 0$: here we get $v(x) = A + Bx$ for some constants A, B. From the boundary conditions, we find $0 = v(0) = A$ and thus $0 = v(1) = B$, so that $v \equiv 0$. But we know that $u(0,x) \neq 0$ and therefore this solution is not admissible.

Case $\lambda > 0$: being λ positive, we can say that $\lambda = \mu^2 > 0$. Therefore, the solution to the ODE in (16) is given by

$$v(x) = Ae^{-\mu x} + Be^{\mu x}$$

for some constants A, B. The boundary conditions now imply

$$0 = v(0) = A + B, \quad \Longrightarrow \quad A = -B, \tag{18}$$

$$0 = v(1) = A(e^{-\mu} - e^{\mu}), \quad \Longrightarrow \quad A = 0. \tag{19}$$

The fact that $A = 0$ follows because we are assuming $\mu > 0$. Also in this case we obtain $u \equiv 0$ which is not possible.

Case $\lambda < 0$: We denote $\lambda = -\mu^2 < 0$ and the solution to (16) is now given by

$$v(x) = A\cos(\mu x) + B\sin(\mu x). \tag{20}$$

The boundary conditions imply

$$v(0) = A = 0, \qquad v(1) = A\cos(\mu) + B\sin(\mu) = 0. \tag{21}$$

We deduce that B can be arbitrary whereas $\mu = \mu_n = n\pi$ for some $n \in \mathbb{N}$ so that

$$v(x) = B\sin(\mu_n x), \qquad B = \text{arbitrary}. \tag{22}$$

With $\lambda = -\mu_n^2$, the solution to (17) is

$$w(t) = Ce^{-\kappa\mu^2 t}, \qquad C = \text{arbitrary}. \tag{23}$$

From (22) and (23), we obtain that a possible solution is of the form

$$u(t,x) = w(t)v(x) = BCe^{-\kappa n^2\pi^2 t}\sin(\pi n x), \qquad B,\, C \text{ arbitrary },\, n \in \mathbb{N}. \tag{24}$$

Finally, from the initial condition for our problem we get

$$\alpha\sin(\pi N x) = u(0,x) = BC\sin(\pi n x), \quad \Longrightarrow BC = \alpha,\, n = N, \tag{25}$$

meaning that

$$u(t,x) = \alpha e^{-\kappa N^2\pi^2 t}\sin(\pi N x), \tag{26}$$

is a solution to the problem under consideration.

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8, B

(ii) To prove the uniqueness of the solution (26), we are going to use the energy method. Assume that u_1, u_2 are two solutions of the same problem. By the linearity of the PDE, we infer that their difference $U = u_1 - u_2$ satisfy

$$\begin{cases} \partial_t U - \kappa\partial_{xx}U = 0, & (t,x) \in (0,+\infty)\times(0,1), \\ U(t,0) = U(t,1) = 0, & t \in (0,+\infty) \\ U(0,x) = 0, & x \in [0,1]. \end{cases} \tag{27}$$

Multiplying the equation by U and integrating in space we have

$$\int_0^1 (U\partial_t U)(t,x)\mathrm{d}x = \kappa\int_0^1 (U\partial_{xx}U)(t,x)\mathrm{d}x. \tag{28}$$

Since $U\partial_t U = \partial_t(U^2)/2$, integrating by parts on the right-hand side we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |U(t,x)|^2\mathrm{d}x = -\kappa\int_0^1 |\partial_x U|^2(t,x)\mathrm{d}x \le 0, \tag{29}$$

where we also used that $U(t,0) = U(t,1) = 0$. Integrating in time the inequality above we finally obtain that

$$\int_0^1 |U(t,x)|^2\mathrm{d}x \le \int_0^1 |U(0,x)|^2\mathrm{d}x = 0, \tag{30}$$

meaning that $U \equiv 0$. Therefore, the solution (26) is the unique one.

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(b) Using (26), we must find a time $T^\star$ such that

$$\alpha e^{-\kappa N^2\pi^2 t}|\sin(\pi N x)| \le e^{-10}, \qquad \text{for all } t \ge T^\star,\ x \in [0,1]. \tag{31}$$

If $\alpha = 0$, there is nothing to prove. In the following we consider $\alpha \neq 0$. Then, for $\bar{x} = 1/2N$ we have $|\sin(\pi N\bar{x})| = 1$. Consequently, we just need to verify that

$$\alpha e^{-\kappa N^2\pi^2 t} \le e^{-10}, \qquad \text{for all } t \ge T^\star. \tag{32}$$

We rewrite the inequality above as

$$e^{\kappa N^2\pi^2 t - 10} \ge \alpha \quad \Longrightarrow \quad \kappa N^2\pi^2 t \ge 10 + \log\alpha \tag{33}$$

If $10 + \log(\alpha) \le 0$ we can define $T^\star = 0$, otherwise we choose

$$T^\star = \frac{1}{\kappa N^2\pi^2}(10 + \log(\alpha)). \tag{34}$$

From the formula we see that for κ, α fixed then $T^\star \to 0$ as $N \to +\infty$.

3. (a) To derive the d'Alembert formula, by adding and subtracting the term $c\partial_{tx}u$ we get

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$$0 = \partial_{tt}u - c^2\partial_{xx}u = \partial_t(\partial_t u + c\partial_x u) - c\partial_{tx}u - c^2\partial_{xx}u. \tag{35}$$

Analogously, if we add and subtract $c\partial_{xt}u$ we infer

$$0 = \partial_t(\partial_t u + c\partial_x u) - c\partial_{tx}u - c\partial_x(\partial_t u + c\partial_x u) + c\partial_{xt}u, \tag{36}$$

or equivalently

$$(\partial_t - c\partial_x)(\partial_t u + c\partial_x u) = 0. \tag{37}$$

Defining

$$v = \partial_t u + c\partial_x u, \tag{38}$$

by (37) we know that v solves the linear transport equation

$$\partial_t v - c\partial_x v = 0 \qquad \Longrightarrow \qquad v(t,x) = \psi(x+ct), \tag{39}$$

where ψ is an arbitrary function to be chosen later. From (38) we have

$$\partial_t u + c\partial_x u = \psi(x+ct). \tag{40}$$

To solve this problem, we can compute u along the characteristics. Namely, observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t,\tau+ct)) = (\partial_t u + c\partial_x u)(t,\tau+ct) = \psi(\tau+2ct). \tag{41}$$

Let $u(0,x) = \varphi(x)$ with φ to be determined. Integrate (41) between $(0,t)$ to get

$$u(t,\tau+ct) = \varphi(\tau) + \int_0^t \psi(\tau+2cs)\mathrm{d}s. \tag{42}$$

Choosing now $x = \tau + ct$ we obtain

$$u(t,x) = \varphi(x-ct) + \int_0^t \psi(x-ct+2cs)\mathrm{d}s. \tag{43}$$

Doing the change of coordinates $y = x - ct + 2cs$, we find

$$u(t,x) = \varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y. \tag{44}$$

We can determine φ and ψ by imposing the initial conditions, so that

$$g(x) = u(0,x) = \varphi(x) \tag{45}$$

$$h(x) = \partial_t u(0,x) = \psi(x) - c\varphi'(x) \qquad \Longrightarrow \qquad \psi(x) = h(x) + cg'(x). \tag{46}$$

Inserting (45) and (46) into (44) we get

$$u(t,x) = g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left[h(y) + cg'(y)\right]\mathrm{d}y. \tag{47}$$

Integrating the term involving g', we finally get the d'Alembert formula

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\mathrm{d}y. \tag{48}$$

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6, C

(b) (i) Recall that we are considering $c = 1$ from now on. From the d'Alembert formula we deduce that

$$u(t,x) = \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y. \tag{49}$$

An immediate solution is

$$u(t,x) = \frac{1}{2}\text{length}\left([-1,1]\cap[x-t,x+t]\right) \tag{50}$$

where with length$(\dots)$ we denote the length of the set inside the brackets, e.g. length$([a,b]) = b-a$.
Another possibility is to observe that

$$H(s) := \int_{-\infty}^{s} h(y)\mathrm{d}y = \begin{cases} 0, & x \leq -1, \\ x+1 & -1 < x \leq 1, \\ 2 & x > 1. \end{cases} \tag{51}$$

Then the solution u can be written as

$$u(t,x) = \frac{1}{2}(H(x+t) - H(x-t)). \tag{52}$$

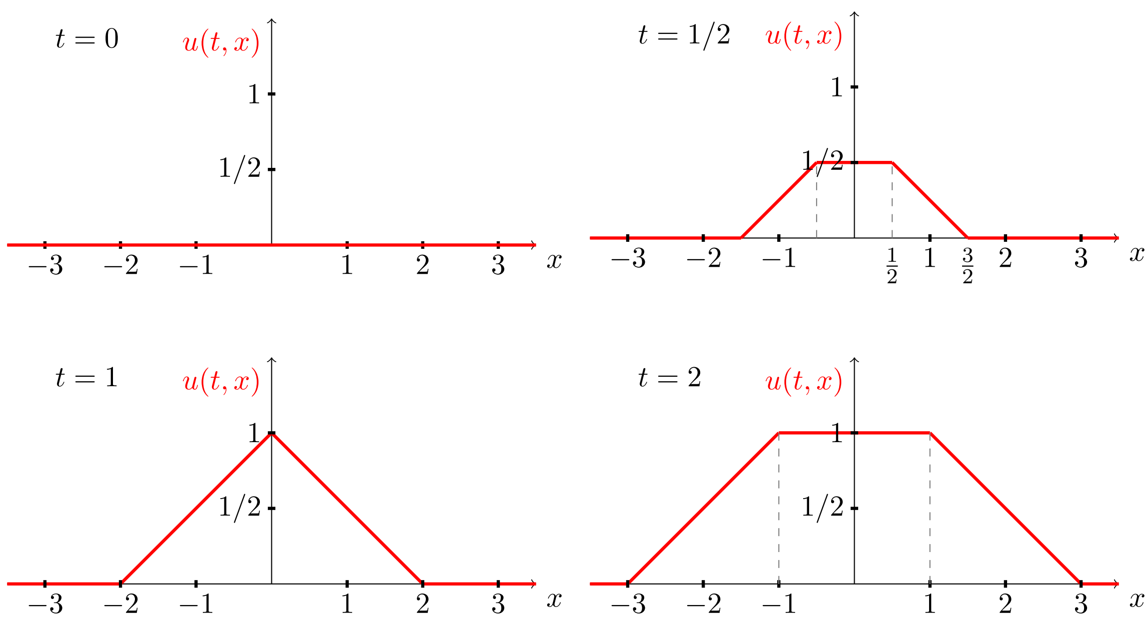
Otherwise, with a more direct computation we can determine the value of

$$F(a,b) = \int_a^b h(y)\mathrm{d}y, \tag{53}$$

with $a < b$, by considering different cases.
Case $\mathbf{a < -1, b > 1}$: since $h(y) = 0$ if $|y| \leq 1$, we get $F(a,b) = \int_{-1}^{1}\mathrm{d}y = 2$.
Case $\mathbf{-1 \leq a < 1, b > 1}$: here we get $F(a,b) = \int_a^1 \mathrm{d}y = 1-a$.
Case $\mathbf{1 \leq a < b, b > 1}$: in this case we never encounter the region $|y| \leq 1$ so that $F(a,b) = 0$.
Case $\mathbf{a < -1, -1 < b < 1}$: for this interval $F(a,b) = \int_{-1}^{b}\mathrm{d}y = b+1$.
Case $\mathbf{-1 \leq a < b, -1 < b < 1}$: here $[a,b] \subset [-1,1]$, meaning that $F(a,b) = \int_a^b \mathrm{d}y = b-a$.
Case $\mathbf{a < b, b < -1}$: this is the last case and we have $F(a,b) = \int_a^b \mathrm{d}y = 0$.

The solution with this initial datum is not classical since it fails to be $C^2(\mathbb{R})$. This is because h has a jump discontinuity.

(ii) To plot the function $u(t,x)$, one can first observe that $u(t,x)$ is either a constant or linear in x. One can thus determine where $u(t,x)$ is constant and then connect these intervals by straight lines, since we know that u is also a continuous function. The plots of the function $u(t,x)$ at the requested times are the following. unseen ⇓ 4, D



By the formula (50), we have that

$$\lim_{t \to +\infty} u(t,x) = \frac{1}{2} \int_{-\infty}^{+\infty} h(y) \mathrm{d}y = 1 \tag{54}$$

4. (a) (i) We first assume that u is a solution to the Poisson equation $\Delta u = f$ in Ω with boundary conditions $u = h$ on $\partial\Omega$. Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that $w = h$ on $\partial\Omega$. Then we can rewrite $E[w]$ as

seen/sim.seen ⇓

10, A

$$E[w] = \frac{1}{2}\int_{\Omega} |\nabla w(x) - \nabla u(x) + \nabla u(x)|^2 \mathrm{d}x + \int_{\Omega} w(x) f(x) \mathrm{d}x. \tag{55}$$

The first integral is

$$\frac{1}{2}\int_{\Omega} |\nabla w(x) - \nabla u(x) + \nabla u(x)|^2 \mathrm{d}x = \frac{1}{2}\int_{\Omega} \left(|\nabla w(x) - \nabla u(x)|^2 + |\nabla u(x)|^2\right) \mathrm{d}x \tag{56}$$

$$+ \int_{\Omega} (\nabla w(x) - \nabla u(x)) \cdot \nabla u(x) \mathrm{d}x. \tag{57}$$

Integrating by parts, we rewrite the (57) as

$$\int_{\Omega} (\nabla w(x) - \nabla u(x)) \cdot \nabla u(x) \mathrm{d}x = -\int_{\Omega} (w(x) - u(x)) \Delta u(x) \mathrm{d}x \tag{58}$$

$$+ \int_{\partial\Omega} (w(\sigma) - u(\sigma)) \partial_{\mathbf{n}} u(\sigma) \mathrm{d}\sigma. \tag{59}$$

But $w = u = h$ on $\partial\Omega$, meaning that the term in (59) is identically zero. While since $\Delta u = f$, we can rewrite $E[w]$ as

$$E[w] = \frac{1}{2}\int_{\Omega} \left(|\nabla w(x) - \nabla u(x)|^2 + |\nabla u(x)|^2\right) \mathrm{d}x + \int_{\Omega} w(x) f(x) \mathrm{d}x \tag{60}$$

$$- \int_{\Omega} (w(x) - u(x)) f(x) \mathrm{d}x, \tag{61}$$

$$= E[u] + \frac{1}{2}\int_{\Omega} |\nabla w(x) - \nabla u(x)|^2 \geq E[u]. \tag{62}$$

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(ii) We now have to prove that if $u \in C^2(\Omega) \cup C(\overline{\Omega})$ and $u = h$ on $\partial\Omega$ is such that

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$$E[u] \leq E[w], \qquad \text{for all } w \in C^2(\Omega) \cup C(\overline{\Omega}), \text{ with } w = h \text{ on } \partial\Omega, \tag{63}$$

then u solves the Poisson equation. Define the function $\Phi = \Phi(t) : [-1, 1] \to [0, \infty)$

$$\Phi(t) = E[u + tv] \tag{64}$$

$$= E[u] + t\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \mathrm{d}x + t^2 \int_{\Omega} |\nabla v(x)|^2 \mathrm{d}x + t\int_{\Omega} v(x) f(x) \mathrm{d}x. \tag{65}$$

Consider now $v \in C^2(\Omega) \cup C^2(\overline{\Omega})$ such that $v = 0$ on $\partial\Omega$. Then $w = u + tv \in C^2(\Omega) \cup C^2(\overline{\Omega})$ and $w = h$ on $\partial\Omega$. But since u is a minimizer of the energy E, Φ attains its minimum at $t = 0$, where $\Phi'(0) = 0$. We compute that

$$\Phi'(t) = \int_{\Omega} \nabla u(x) \cdot \nabla w(x) \mathrm{d}x + \int_{\Omega} v(x) f(x) + 2t\int_{\Omega} \left(|\nabla v(x)|^2 + v(x) f(x)\right) \mathrm{d}x \tag{66}$$

Using that $\Phi'(0) = 0$ we get

$$0 = \Phi'(0) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \mathrm{d}x + \int_\Omega v(x) f(x) \mathrm{d}x. \tag{67}$$

Now, since $v = 0$ on $\partial\Omega$, an integration by parts gives

$$0 = \int_\Omega \nabla u(x) \cdot \nabla v(x) \mathrm{d}x + \int_\Omega v(x) f(x) \mathrm{d}x = \int_\Omega (f(x) - \Delta u(x)) v(x) \mathrm{d}x. \tag{68}$$

Since the above holds for every v, we get that $\Delta u = f$ in Ω.

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(b) If $f = 0$ then $\Delta u = 0$ on Ω, hence u is harmonic on Ω. Since we are assuming that $u \in C^3(\Omega)$, we also have that $\Delta \partial_{x_i} u = 0$ for $i = 1, 2$. Namely also $\partial_{x_i} u$ is harmonic on Ω. Let us consider first $\partial_{x_1} u$. By the mean value formula for harmonic functions given in Theorem 2.1 of Chapter 5 of the lecture notes, we know that

$$(\partial_{x_1} u)(x) = \frac{1}{\pi R^2} \int_{B_R(x)} (\partial_{xy_1} u)(y) \mathrm{d}y \tag{69}$$

But we can see $\partial_{x_1} u = \nabla \cdot (u, 0)$, therefore, from the divergence theorem we have

$$(\partial_{x_1} u)(x) = \frac{1}{\pi R^2} \int_{\partial B_R(x)} n_1 u(y) \mathrm{d}\sigma, \tag{70}$$

where $\boldsymbol{n} = (n_1, n_2)$ is the normal vector to $\partial B_R(x)$. Taking absolute values in (70) we get

$$|(\partial_{x_1} u)(x)| = \frac{1}{\pi R^2} \left| \int_{\partial B_R(x)} n_1 u(y) \mathrm{d}\sigma \right| \leq \frac{1}{\pi R^2} \int_{\partial B_R(x)} |u(y)| \mathrm{d}\sigma \tag{71}$$

$$\leq \frac{2\pi R}{\pi R^2} \sup_{\overline{B}_R(x)} |u| = \frac{2}{R} \sup_{\overline{B}_R(x)} |u|. \tag{72}$$

Arguing analogously for $\partial_{x_2} u$, we know that

$$|\nabla u(x)| = \sqrt{|\partial_{x_1} u(x)|^2 + |\partial_{x_2} u(x)|^2} \leq \frac{2\sqrt{2}}{R} \sup_{\overline{B}_R(x)} |u|. \tag{73}$$

5. (a) Computing the time derivative of E and using the PDE, we find that

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$$E'(t) = \int_0^1 (\partial_t u(\partial_{tt}u - u^3) + \partial_{tx}u\partial_x u)(t,x)\mathrm{d}x. \tag{74}$$

Integrating by parts the last term, using that $u(t,0) = u(t,1) = 0$, we get

$$E'(t) = \int_0^1 (\partial_t u(\partial_{tt}u - \partial_{xx}u - u^3))(t,x)\mathrm{d}x = 0, \tag{75}$$

where the last identity follows because u is a solution of the PDE under consideration.

For what concerns $I(t)$, first we find

$$I'(t) = \int_0^1 (u\partial_t u)(t,x)\mathrm{d}x. \tag{76}$$

Then

$$I''(t) = \int_0^1 (u\partial_{tt}u)(t,x)\mathrm{d}x + \int_0^1 |\partial_t u|^2(t,x)\mathrm{d}x. \tag{77}$$

Using again the PDE, we rewrite the first integral above as

$$\int_0^1 (u\partial_{tt}u)(t,x)\mathrm{d}x = \int_0^1 (u\partial_{xx}u)(t,x)\mathrm{d}x + \int_0^1 |u(t,x)|^4\mathrm{d}x \tag{78}$$

$$= -\int_0^1 |\partial_x u|^2(t,x)\mathrm{d}x + \int_0^1 |u(t,x)|^4\mathrm{d}x, \tag{79}$$

where in the last identity we again integrated by parts. Finally, since

$$\int_0^1 |u(t,x)|^4\mathrm{d}x = -4E(t) + 2\int_0^1 (|\partial_t u|^2 + |\partial_x u|^2)(t,x)\mathrm{d}x, \tag{80}$$

combining the identity above with (79) and (77) we prove

$$I''(t) = \int_0^1 (3|\partial_t u|^2 + |\partial_x u|^2)(t,x)\mathrm{d}x - 4E(t). \tag{81}$$

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(b) (i) From (81), the conservation of the energy $E(t) = E(0)$ and the hypothesis $E(0) < 0$ we get

$$I''(t) \geq -4E(0) > 0. \tag{82}$$

Then, we have $I(t) \geq 0$. If $I(t) = 0$ for some t, then $u(t) \equiv 0$. However, if $u(t) \equiv 0$ then $E(t) = 0$. But since $E(t) = E(0) < 0$ we know that $u(t) \equiv 0$ is not possible. Therefore $I(t) > 0$. Finally, first observe that

$$I'(0) = \int_0^1 u(0,x)\partial_t u(0,x)\mathrm{d}x = \int_0^1 g(x)h(x) > 0, \tag{83}$$

where the last inequality follows by hypothesis. Then, by a Taylor expansion of I' around $t = 0$ we know that there exists $t^\star \in [0,t]$ such that

$$I'(t) = I'(0)t + \frac{I''(t^\star)}{2}t^2 > 0, \tag{84}$$

where we used (82) and (83).

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(ii) Multiplying (81) by $I(t)$, since $E(t) < 0$ and $I(t) \geq 0$ we find

$$I(t)I''(t) \geq \frac{3}{2}\left(\int_0^1 |u(t,x)|^2\right)\left(\int_0^1 |\partial_t u|^2(t,x)\right). \tag{85}$$

Using the Cauchy-Schwartz inequality for the terms on the right-hand side, we obtain

$$I(t)I''(t) \geq \frac{3}{2}\left(\int_0^1 (u\partial_t u)(t,x)\right)^2 = \frac{3}{2}(I'(t))^2. \tag{86}$$

Since we know that $I(t) > 0$ and $I'(t) > 0$, if we divide the inequality above by $I(t)I'(t)$ we get

$$\frac{I''(t)}{I'(t)} \geq \frac{3}{2}\frac{I'(t)}{I(t)} \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\log(I'(t)) \geq \frac{\mathrm{d}}{\mathrm{d}t}\log(I^{\frac{3}{2}}(t)) \tag{87}$$

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(iii) Integrating in time the inequality in (87) we deduce that

$$\log\left(\frac{I'(t)}{I^{\frac{3}{2}}(t)}\right) \geq \log\left(\frac{I'(0)}{I^{\frac{3}{2}}(0)}\right). \tag{88}$$

Notice that we are using again that $I'(0) > 0$ and $I(0) > 0$, since otherwise the term on the right-hand side might be not defined. Calling $\alpha = I'(0)/I^{\frac{3}{2}}(0)$, we can rewrite the previous inequality as

$$I'(t) \geq \alpha I^{\frac{3}{2}}(t). \tag{89}$$

This implies that

$$\int_{I(0)}^{I(t)} \frac{\mathrm{d}I}{I^{\frac{3}{2}}} \geq \alpha \int_0^t \mathrm{d}s \qquad \Longrightarrow \qquad 2\left(\frac{1}{\sqrt{I(0)}} - \frac{1}{\sqrt{I(t)}}\right) \geq \alpha t. \tag{90}$$

Manipulating the last inequality we infer

$$\sqrt{I(t)} \geq \frac{2}{\alpha t - 2\sqrt{I(0)}}. \tag{91}$$

Therefore, defining

$$t_\star = 2\frac{\sqrt{I(0)}}{\alpha} = 2\frac{I(0)^2}{I'(0)}$$

we prove

$$\lim_{t \to t_\star^-} I(t) = +\infty. \tag{92}$$

In particular, this means that the solution u blows up in finite time.

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

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ExamModuleCode	**QuestionNumber**	**Comments for Students**
Theory of Partial Differential Equations_MATH60019 MATH97027 MATH70019	1	When finding "the unique entropy solution", it is important to check that the entropy conditions are satisfied.
Theory of Partial Differential Equations_MATH60019 MATH97027 MATH70019	2	no particular comments
Theory of Partial Differential Equations_MATH60019 MATH97027 MATH70019	3	part b) one could have answered about the fact that the solution is not classical and the limit at infinite times without computing explicitly the solution.
Theory of Partial Differential Equations_MATH60019 MATH97027 MATH70019	4	the last exercise was challenging and non-standard
Theory of Partial Differential Equations_MATH60019 MATH97027 MATH70019	5	