Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Statistical Theory I

Date: Tuesday, 24 May 2011. Time: 2.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (i) Prove that a minimum variance unbiased estimator (MVUE) is unique: i.e. show that, if T_1 and T_2 are both MVUE for θ, then $T_1 = T_2$ with probability 1.

(ii) (a) What does it mean by saying that a statistic a is *ancillary*?

(b) What, in the context of ancillarity, is a *quasi-sufficient* (*conditionally sufficient*) statistic s given a?

(c) What property is needed for the independence of an ancillary statistic and its quasi-sufficient statistic?

(d) Explain why, if $x_1, x_2, \ldots, x_n$ is a random sample from a distribution having the cumulative distribution function $F(x-\theta)$ $(x \in \mathbb{R},\ \theta \in \mathbb{R})$, then the range $r = \max_i(x_i) - \min_i(x_i)$ is ancillary for θ.

(e) Illustrate (d) by finding the minimal sufficient statistic for a random sample from *Uniform* $(\theta - 1, \theta + 1)$, and identifying a quasi-sufficient statistic s.

(iii) A random sample $x_1, x_2, \ldots, x_n$ is taken from the *Delayed Exponential* distribution having probability density function (pdf)

$$f(x \mid \theta) = \begin{cases} \exp\{-(x-\theta)\} & (x \geq \theta), \\ 0 & \text{(otherwise)}, \end{cases}$$

where θ is an unknown parameter.

If the prior distribution of θ is *Uniform* $(0, 1)$, obtain the posterior pdf.

Show that the Bayes maximum likelihood estimate of θ is $\min\{\min_i(x_i),\, 1\}$.

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables from $N(\theta, \theta)$, where parameter $\theta > 0$ is unknown.

(i) Find the total efficient score $U_\bullet(\theta)$ and total Fisher information $I_\bullet(\theta)$.

(ii) Determine the maximum likelihood estimator (MLE) $\widehat{\theta}$ for θ, and its asymptotic distribution.

(iii) Obtain the asymptotic relative efficiency of the mean estimator $\overline{X}$ of θ and the MLE $\widehat{\theta}$.

(iv) Determine the MLE $\widehat{\psi}$ for $\psi = \ln(\theta)$, and its asymptotic distribution.

(v) Determine the MLE $\widehat{\xi}$ for $\xi = \Phi(\theta)$, and its asymptotic distribution, where Φ is the cumulative distribution function of $N(0, 1)$.

Give your reasoning throughout

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables from Poisson distributions, where $E(X_k) = k\theta \;\; (k = 1, 2, \ldots, n)$, and $\theta > 0$ is unknown.

(i) Show that $S = \sum_1^n X_k$ is a sufficient statistic for θ.

(ii) Explain without proof why S is complete for θ.

(iii) By considering the expectation of the estimator

$$T = \begin{cases} 1 & (X_1 + X_n = 0), \\ 0 & (X_1 + X_n \neq 0), \end{cases}$$

find the uniformly minimum variance unbiased estimator of $\xi = \exp\{-(n+1)\theta\}$.

Give your reasoning throughout

4. A random sample $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ is taken from a distribution having probability density function (pdf)

$$f(x \mid \theta) = \begin{cases} \dfrac{\theta}{(1+x)^{\theta+1}} & (x > 0), \\ 0 & (\text{otherwise}), \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(i) Show that the uniformly most powerful (UMP) test of the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta > \theta_0$ can be written as 'reject H_0 if statistic $z(\boldsymbol{x})$ exceeds value c', where you should find $z(\boldsymbol{x})$.

(ii) By considering the distribution of $\ln(1 + X)$, where X has the pdf above, obtain explicit expressions for the size α and the power function $\beta(\theta)$ of the test in (i).

Give your reasoning throughout