Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3P22/M4P22

Enumerative Combinatorics

Date: Friday, 21 May 2010 Time: 2.00 pm – 4.00 pm

DO NOT OPEN THIS PAPER
UNTIL THE INVIGILATOR TELLS YOU TO.

This paper has FOUR questions.
Candidates should write their solutions in a single answer book.
Supplementary answer books should be used as necessary.

Affix one of the labels provided to each answer book that you use.
DO NOT use the label with your name on it.

Answer all the questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. (a) Give the definition of the Möbius function of a partially ordered set P. You can assume its existence and uniqueness. State the generalized Möbius inversion theorem.

(b) Let $X = \{1, 2, \dots, k\}$. Let P be the partially ordered set of all subsets of X (including $\emptyset$) with partial order $Y \leq Z$ if and only if $Y \subseteq Z$.

Prove that the Möbius function of P has values $\mu(Y, Z) = (-1)^{|Z|-|Y|}$ for all $Y, Z \in P$ with $Y \leq Z$. Deduce the inclusion-exclusion principle from the Möbius inversion theorem.

(c) Let $n \geq 1$ be an integer. Prove that

$$\sum_{j=0}^{n}(-1)^j \binom{n}{j} (n-j)^{n+1} = n!\frac{n(n+1)}{2}.$$

2. (a) Define the Stirling numbers of the second kind $S(n,k)$. Prove that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ and that the exponential generating function of the sequence $\{S(n,k)\}_{n=k}^{\infty}$ is $(e^x - 1)^k/k!$.

(b) For a positive integer n let c_n be the number of ways to seat n participants on several circular tables so that there are at least 3 participants on every table. Two seating plans are the same if and only if each person has the same right hand neighbour and also the same left hand neighbour. Stating carefully which results from the course you use prove that the exponential generating function of the sequence $\{c_n\}_{n=0}^{\infty}$ is

$$\frac{e^{-x-x^2/2}}{1-x}.$$

3. (a) Define the Ramsey numbers $R(p,q)$ and show that $R(p,q) \leq R(p,q-1) + R(p-1,q)$ for $p, q \geq 3$.

(b) Prove that $R(3,3) = 6$ and that $R(3,4) = 9$. Deduce that $R(4,4) \leq 18$.

(c) Let $n \geq 2$ and put $N = R(n,n)$. Show that for any sequence of distinct real numbers $a_1, \dots, a_N$ there exists a subsequence $a_{i_1}, a_{i_2}, \dots a_{i_n}$ with $i_1 < i_2 < \dots < i_n$ which is monotonic (either strictly increasing or decreasing).

4. Let p_n be the number of partitions of the positive integer n.

(a) Show that $p_n \leq p_{n-1}+p_{n-2}$ and deduce that $p_n < \mu^n$ for all $n > 0$, where $\mu = (1+\sqrt{5})/2$.

(b) State and prove Euler's identity.

(c) Let e_n be the number of those partitions of the integer n such that each part size appears at least twice. Put $e_0 = 1$. Prove that

$$\sum_{i=0}^{\infty} e_i x^i = \prod_{k=1}^{\infty} (1 + x^{2k} + x^{3k} + x^{4k} + \cdots)$$

and deduce that e_n is equal to the number of partitions of n such that no part is congruent to ± 1 modulo 6.