# Math40002 Analysis 1 Problem Sheet 1

1. Suppose the sets $S_n,\ n = 1, 2, 3, \ldots$ are all countable. Show that $S = \bigcup_{n=1}^{\infty} S_n$ is also countable. (Hint: recall the diagonal argument used in lectures.)

   **List the elements of the set $S_n = \{s_1^n, s_2^n, \ldots\}$. Then list the elements of $S = \bigcup_{n=1}^{\infty} S_n$ in an array with $s_1^1, s_2^1, \ldots$ on the first row, $s_1^2, s_2^2, \ldots$ on the second row, $s_1^n, s_2^n, \ldots$ on the $n$th row etc. Leave a gap if any of the $S_n$ are finite.**

   **Then draw the diagonals, where the $n$th diagonal is the list $s_n^1, s_{n-1}^2, \ldots, s_{n-i+1}^i, \ldots, s_1^n$ as in lectures.**

   **Now put all these finite diagonals end to end in a long linear list, as in lectures, throwing out any repeated elements. The result is a list of all the elements of $S$, which is therefore countable.**

   **Alternatively, supposing we have listed all the elements of $S_n$ $s_1^n, s_2^n, \ldots$ for each $n$, and given $s \in S$, fix $m(s) = \min\{j \in \mathbb{N}_{>0} : s \in S_j\}$ and fix $l(s) \in \mathbb{N}_{>0}$ by where $s$ sits in the listing of $S_{m(s)}$, that is $s_{l(s)}^{m(s)} = s$. Then define $f : S \to \mathbb{N}_{>0}$ by setting $f(s) = 2^{m(s)}3^{l(s)}$, one can check $f$ is an injection and since we have an injection of $S$ into a countable set, it must be countable.**

2. Suppose that $S$ and $T$ are countable. Show that $S \times T$ is countable. Hence show that $\bigcup_{n=1}^{\infty} S^n$ is countable, where $S^n := S \times \ldots \times S$ ($n$ times).

   **$S = \{s_1, s_2, \ldots\}$. $T = \{t_1, t_2, \ldots\}$. So write down the elements of $S \times T$ in an array with $(s_i, t_j)$ in the $i$th row and $j$th column. Then apply the usual diagonal argument to list these elements. One can similarly define a function $f$ from $S \times T \to \mathbb{N}_{>0}$ as in the previous answer.**

   **So setting $T = S$, we see that $S^2$ is countable. Setting $T = S^2$ we see that $S^3$ is countable. Inductively then, $S^n$ is countable. Therefore by Q1, $\bigcup_{n=1}^{\infty} S^n$ is countable.**

3. † Show the set of polynomials $p(x)$ with integer coefficients is countable. (Hint: use Q2.)

   A real number is called *algebraic* if it is a root of a polynomial with integer coefficients. Show that rational numbers $n/m$ and $n$th roots $\sqrt[n]{m}$ are algebraic. Show that the set of algebraic real numbers is countable.

   A real number is called *transcendental* if it is not algebraic. (Examples include $\pi$ and $e$, but this is hard to prove.) Prove that transcendental numbers exist, and that in fact there are uncountably many of them.

   **The set of polynomials of degree $n$ with integer coefficients is a subset of $\mathbb{Z}^{n+1}$ (each polynomial is equivalent to a list of $n+1$ integers, by taking the $n+1$ coefficients; we only get a subset because the first integer should be nonzero). Therefore the set of all polynomials with integer coefficients is a subset of $\bigcup_{n=1}^{\infty} \mathbb{Z}^n$, which is countable by Q2. And we showed in lectures that an infinite subset of a countable set is countable.**

   **$n/m$ is the root of $mx - n = 0$ while $\sqrt[n]{m}$ is the root of $x^n - m = 0$, so these are both algebraic.**

   **Each polynomial has a finite number of roots. So in the list of polynomials, replace each polynomial by its finite list of roots to get a list of all algebraic numbers.**

   **If the set of transcendental numbers were empty, finite or countable, then their union with the algebraic numbers would also be countable. Therefore $\mathbb{R}$ would be countable, but it is not. Therefore there are uncountably many transcendental numbers.**

4. Let $S^1 = \{s_1^1, s_2^1, s_3^1, \ldots\},\ S^2 = \{s_1^2, s_2^2, s_3^2, \ldots\},\ \ldots,\ S^n = \{s_1^n, s_2^n, s_3^n, \ldots\},\ \ldots$ be subsets of $\mathbb{N}$. Here the elements are ordered so that $s_i^n < s_{i+1}^n$ for all $i$ and $n$.

Define $t_n$ recursively to be strictly larger than $s_n^n$ and $t_{n-1}$ (e.g. set $t_n = \max\{s_n^n, t_{n-1}\} + 1$), or $t_{n-1} + 1$ if $\nexists\, s_n^n$ (i.e. if $S^n$ has $< n$ elements).

Show that $T = \{t_1, t_2, \ldots\} \subseteq \mathbb{N}$ is not equal to any $S^i$. Conclude that the set of subsets of $\mathbb{N}$ is *not* countable.

**Since we chose $t_n$ to be larger than $t_{n-1}$ then $T = \{t_1, t_2, t_3, \ldots\}$ is listed just like the $S^i$s, with the elements in ascending order. That is, $t_n$ is the $n$th smallest element of $T$.**

**By construction $t_n$ is *not* equal to the $n$th smallest element $s_n^n$ of $S_n$. (It was constructed to be larger.) Therefore $T \neq S_n$. But this is true for all $n$. Therefore $T$ is not in the list $S_1, S_2, S_3, \ldots$.**

**Therefore given any list of subsets of $\mathbb{N}$, there is a subset $T \subseteq \mathbb{N}$ not in that list. Therefore the set of subsets of $\mathbb{N}$ is not countable.**

5. Is the set of *finite* subsets of $\mathbb{N}$ countable?

**Yes. For each $n \in \mathbb{N}$ the set $\mathbb{N}^n$ is countable due to Q2. Every subset of $\mathbb{N}$ of size $n$ corresponds to a unique $n$-uple of $\mathbb{N}^n$ by arranging its elements in increasing order. Since this is an injection, it follows that the set of subsets of $\mathbb{N}$ with exactly $n$ elements, is countable. Finally, the union over $n \in \mathbb{N}$ of all these sets is also countable by Q1.**