L8/.

①

2. First-order logic.

(Predicate logic).

Plan:

1) 1^{st} order structures
2) 1^{st} order languages / formulas
) 'semantics'

3) A formal system for 1^{st}-order logic "syntax"

4) Theorems of the formal system are the "logically valid formulas"

Gödel's Completeness Theorem

(2.1) Structures.

(2.1.1) Def. Suppose A is a set and $n \geq 1$. An n-ary relation on A is a subset $\bar{R} \subseteq A^n$

(where $A^n = \{(a_1, \ldots, a_n) : a_i \in A\}$
↑ n-tuple.

An n-ary function on A is a function $\bar{f} : A^n \to A$.

Examples

a) $=$ on any set A is a 2-ary relation on A.

b) $\leq$ ordering on $\mathbb{R}$ 2-ary relation on $\mathbb{R}$.

c) $+$ on $\mathbb{C}$ 2-ary function on $\mathbb{C}$

d) $\bar{P} \subseteq \mathbb{Z}$ $\bar{P} = \{x \in \mathbb{Z} : x \text{ is even}\}$

['Predicate' : 'Relation'].

Notation If $\bar{R} \subseteq A^n$ & $(a_1, \ldots, a_n) \in A^n$ write $\bar{R}(a_1, \ldots, a_n)$ (or say this holds) if $(a_1, \ldots, a_n) \in \bar{R}$. //

(2.1.2) **Def** A first-order structure $\mathcal{A}$ consists of:

1) A non-empty set A (the domain of $\mathcal{A}$)

2) A set $\{\bar{R}_i : i \in I\}$ of relations on A, $\bar{R}_i \subseteq A^{n_i}$ $(i \in I)$

3) A set $\{\bar{f}_j : j \in J\}$ of functions $\bar{f}_j : A^{m_j} \to A$ (for $j \in J$)

4) A set $\{\bar{c}_k : k \in K\} \subseteq A$ of constants: just elements of A.

- the sets I, J, K are indexing sets (can be empty). Usually subsets of $\mathbb{N}$.

- The information

$$\left.\begin{array}{l}(n_i : i \in I)\\(m_j : j \in J)\\K\end{array}\right\}$$

is called the signature of $\mathcal{A}$.

Denote:

$$\mathcal{A} = \langle \underset{\text{domain}}{A} ; \underset{\text{relations}}{(\bar{R}_i : i \in I)}, \underset{\text{functions}}{(\bar{f}_j : j \in J)}, \underset{\text{constants}}{(\bar{c}_k : k \in K)} \rangle$$

(2.1.3) Examples.

(1) Orderings

$A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$

Might take $I = \{1\}$

$n_1 = 2$

$J = K = \emptyset$

$\bar{R}_1(a_1, a_2)$ to mean "$a_1 \le a_2$".

(2) Groups.

Groups could use the signature

$\bar{R}$ 2-ary relation for $=$

$\bar{m}$ 2-ary function for group operation

$\bar{i}$ 1-ary function for inverse.

$\bar{e}$ constant (for identity elt.).

(3) Rings

$\bar{R}$ 2-ary rel. for $=$

$\bar{m}$ 2-ary function for multiplication

$\bar{a}$ 2-ary function for addition

$\bar{n}$ 1-ary function for $x \mapsto -x$

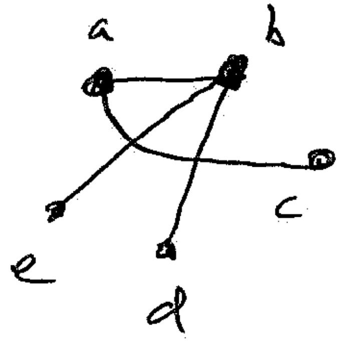
$\bar{0}, \bar{1}$ constants (for zero and one).

(4) Arithmetic

$$\langle \mathbb{N} ; \underbrace{\bar{R}, \bar{a}, \bar{m}}_{\text{as in (3)}}, \bar{s}, \bar{0} \rangle$$

$\bar{s}$ function $x \mapsto x+1$

(5) Graphs. (simple, loopless)



$\bar{E}$ 2-ary relation for edge relation (adjacency).

$\bar{R}$ equality.

(2.2) First-order languages.

(2.2.1) Def. A 1^{st} order language $\mathcal{L}$ has an alphabet of symbols:

variables $\quad x_1 \; x_2 \; x_3 \; \ldots$

punctuation $\quad (\quad) \quad ,$

connectives $\quad \neg \quad \rightarrow$

quantifier $\quad \forall$

relation symbols: $\quad R_i \quad (i \in I)$

function symbols: $\quad f_j \quad (j \in J)$

constant symbols: $\quad c_k \quad (k \in K)$

I, J, K are indexing sets (could have J, K being $\emptyset$)

Each R_i has an arity n_i

Each f_j has an arity m_j.

The information

$(n_i : i \in I), (m_j : j \in J), K$

is called the signature of $\mathcal{L}$.

A 1^{st} order structure $\mathcal{A}$ with the same signature as $\mathcal{L}$ is called an $\mathcal{L}$-structure.

The correspondence between the rel., fn. and constant symbols in $\mathcal{L}$ and the relations, fns. + constants in $\mathcal{A}$ ($R_i \rightsquigarrow \bar{R}_i$) etc.) is called an interpretation of $\mathcal{L}$.

(2.2.2) **Def.** A term of $\mathcal{L}$ is defined as follows:

i) Any variable is a term;

ii) Any constant symbol is a term;

iii) If f is an m-ary ~~[illegible]~~ function symbol and $t_1, \ldots, t_m$ are terms, then $f(t_1, \ldots, t_m)$ is a term.

Example If $\mathcal{L}$ has a 2-ary function symbol f and constant symbols c_0, c_1 (and ...)

Some terms: c_0, c_1, x_1

$f(c_0, x_1)$ $\quad$ $f(x_2, f(c_0, x_1))$