Imperial College London

M3S4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Probability

Date: **Tuesday 24th May 2016**

Time: **09.30 – 11.30**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, \cdots, 7\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/8 & 0 & 7/8 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 0 & 3/4 \\ 0 & 1/9 & 7/9 & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(1) Draw the transition diagram.

(2) Determine the communicating classes.

(3) For each class, specify whether the class is transient or recurrent. *(Justify your answer)*

(4) Is the chain irreducible? *(Justify your answer)*

(5) Find $\mathbb{P}_3(X_2 = 6)$ and $\mathbb{P}_1(X_2 = 7)$.
([Note]. $\mathbb{P}_i(X_n = j)$ is the probability of X_n reaches state j starting from i)

2. Let $N = (N_t)_{t\geq 0}$ denote a Poisson process of rate $\lambda > 0$. Define the stochastic process $X = (X_t)_{t\geq 0}$ by

$$X_t = N_t - \lambda t.$$

(1) Prove that, for $s \leq t$, $X_s = \mathbb{E}(X_t|\mathcal{F}_s)$ where $\mathcal{F}_s = \sigma(N_u,\ u \leq s)$.

(2) Find the Laplace transform ϕ of X where $\phi(u) = \mathbb{E}(e^{-uX_t})$.

(3) Denote $S_t = \sum_{i=1}^{N_t} Y_i$ where $(Y_i)_{i\in\mathbb{N}_0}$ is a sequence of independent and identically random variables, independent of N.

(i) Prove that the Laplace transform of S_t is

$$\psi(u) = \mathbb{E}(e^{-uS_t}) = \exp(-\lambda t + \lambda t G(u)),$$

where G is the Laplace transform of Y.

(ii) Using (i), compute $\mathbb{E}(S_t)$ and $\mathrm{Var}(S_t)$.

3. (1) Define the generator of a continuous-time Markov chain.

(2) Suppose the generator of a continuous-time Markov chain with state space $E = \{1, 2, 3, 4\}$ is given by

$$\mathbf{G} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

What are the transition probabilities of the corresponding jump-chain?

(3) Consider a birth-death process with birth rates $\lambda_n = (n+1)\lambda$ and death rates $\mu_n = \mu n^2$, for $n \in \mathbb{N}_0$ and $0 < \lambda < \mu$.

(i) Write down the generator.

(ii) Find the stationary distribution.

4. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion.

(1) For $0 \leq s \leq t$, give the law of the random variable $B_t - B_s$.

(2) Prove that, for $t, s \geq 0$, $\mathrm{Cov}(B_t, B_s) = \min(s, t)$.

(3) Let $a > 0$ be a deterministic constant.
Prove that $(W_t)_{t \geq 0}$ with $W_t = aB_{t/a^2}$ is a standard Brownian motion.

(4) Let Z be a standard normal random variable [i.e. $Z \sim N(0,1)$]. For $t \geq 0$, denote, $X_t = \sqrt{t}Z$.
Show that X is not a Brownian motion.

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Applied Probability

Date: **Tuesday 24th May 2016**

Time: **09.30 – 12.00**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

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Statistical tables will not be provided.

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Mastery Question

5. Let W and $\tilde{W}$ be two independent standard Brownian motions. For $t \geq 0$, denote $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$.

(1) Prove that X is a Brownian motion.

(2) Denote $X_t = \exp(\sigma W_t - \frac{\sigma^2}{2}t)$ where $\sigma > 0$. Show that $\mathbb{E}(X_t) = 1$.