Applied Complex Analysis - Lecture Eight

Andrew Gibbs

January 2025

Branch points and branch cuts

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A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

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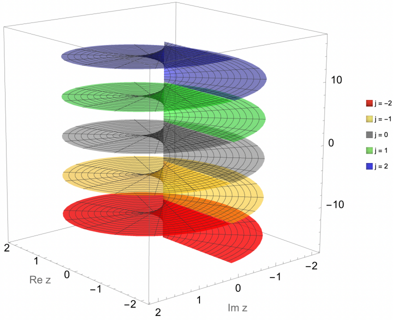
Branch cut example: Complex logarithm



- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

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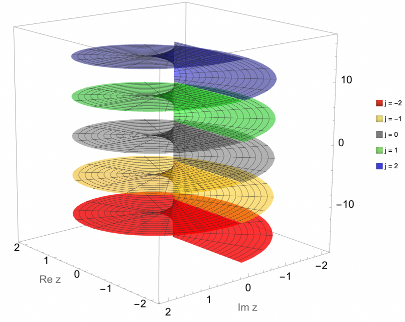

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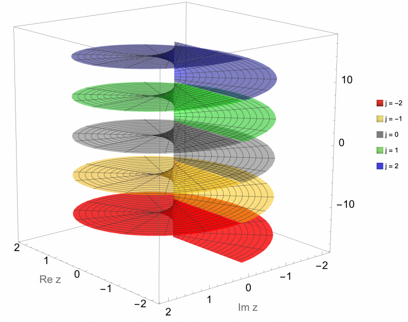

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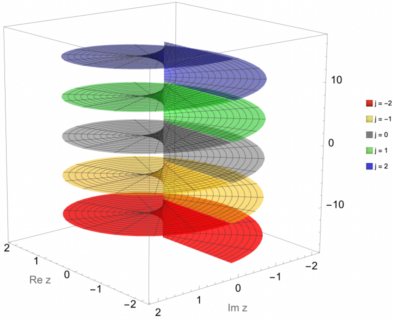

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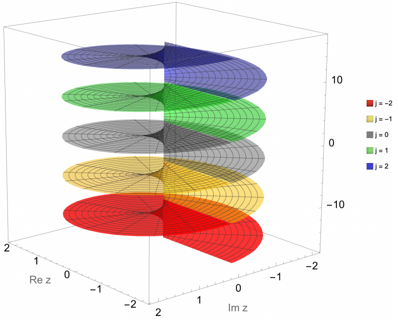

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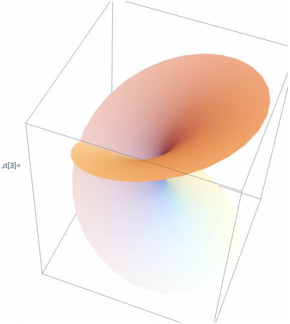

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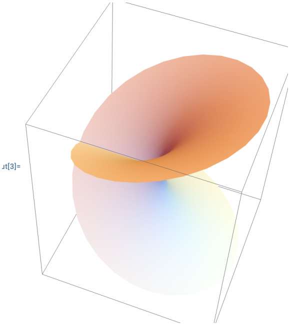

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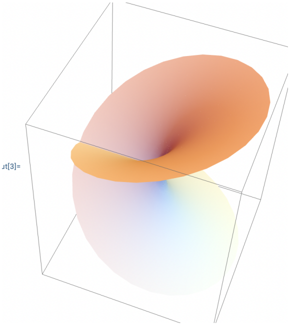

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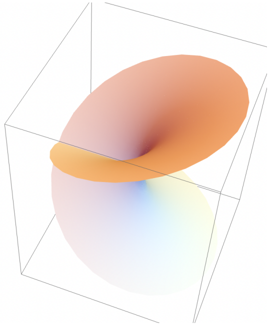

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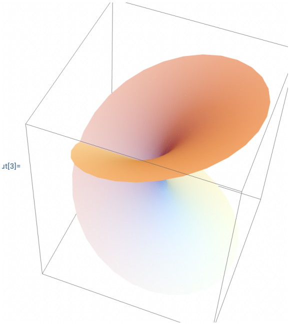

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Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1+\theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
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How many branches?

- z^n for $n \in \mathbb{Z}$ is single-valued, i.e. not a branch point.
- $z^{m/n}$ for $m, n \in \mathbb{Z}$ has n branches
- z^α for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ has infinitely many branches.
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Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

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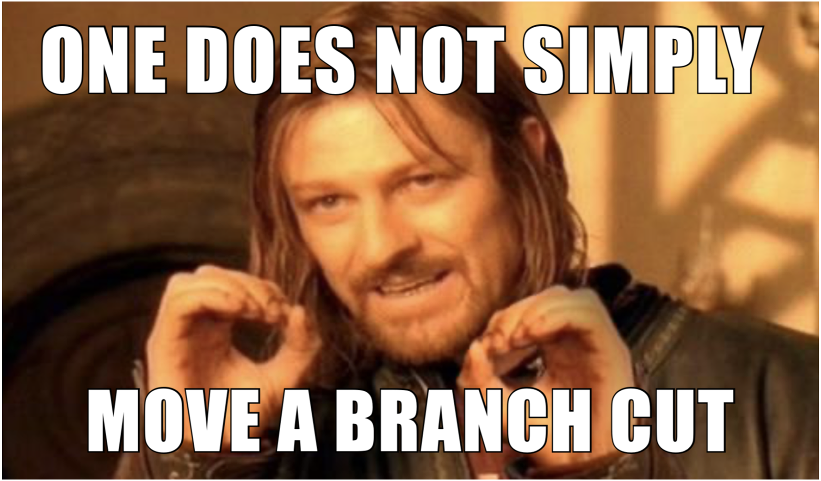
Moving branch cuts in mathematical software



Example problems

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Principal value integrals

- We say integral is *weakly singular* if $f(z) = O(|z|^\alpha)$ for some $\alpha > -1$ as $z \to 0$.
- Such integrals are absolutely convergent, similar in many ways to integrals of smooth functions.
- When $\alpha = -1$ (or worse), integrals are not absolutely convergent. But they may converge in a different sense.

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Examples

- $$\int_{-1}^{1} \frac{1}{x} dx$$

- $$\int_{-1}^{1} \frac{x^{\alpha-1}}{1-x}, \quad \alpha \in (0,1)$$