Imperial College London
MATH 50004 Multivariable Calculus
Mid-Term Examination Date: 16th November 2022
Duration: 60 minutes

1. The function ϕ is given by

$$\phi = x^3 + x^2 + y^2 - 2x + \alpha y - z, \quad (\alpha \text{ constant}).$$

(a) [**4 marks**] At the point $(1, 0, -1)$ find a unit vector $\widehat{\mathbf{n}}$ in the direction in which ϕ experiences its greatest rate of change, and determine the magnitude of that rate of change.

(b) [**3 marks**] Find the rate of change of ϕ at $(1, 0, -1)$ in the direction towards the point $(3, 4, 5)$.

(c) [**4 marks**] Find the tangent plane to the surface $\phi = 1$ when $x = 1$ and $y = 0$.

(d) [**9 marks**] Verify the relations

$$\begin{array}{ll} \text{(i)} & \mathbf{u} \times \operatorname{curl} \mathbf{u} = \nabla \left(\tfrac{1}{2} |\mathbf{u}|^2 \right) - (\mathbf{u} \cdot \nabla)\mathbf{u}, \\ \text{(ii)} & \operatorname{curl}(\operatorname{curl} \mathbf{u}) = \nabla(\operatorname{div} \mathbf{u}) - \nabla^2 \mathbf{u}, \end{array}$$

for the vector field $\mathbf{u} = \nabla\phi$, with ϕ given above.

2. Consider the vector field

$$\mathbf{A} = (\beta x - y^3)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}, \quad (\beta \text{ constant}).$$

Let S be the hemisphere

$$x^2 + y^2 + z^2 = a^2, \quad z \geq 0,$$

and C be the circular boundary

$$x^2 + y^2 = a^2, \quad z = 0.$$

(a) [**4 marks**] Calculate $\operatorname{div} \mathbf{A}$ and $\operatorname{curl} \mathbf{A}$.

(b) [**3 marks**] Find the unit normal $\widehat{\mathbf{n}}$ to S such that $\widehat{\mathbf{n}} \cdot \mathbf{k} \geq 0$.

(c) [**6 marks**] Evaluate

$$\oint_C \mathbf{A} \cdot d\mathbf{r},$$

where C is traversed anti-clockwise, using a suitable parameterization of C.

You may assume

$$\int_0^{2\pi} \sin^4\theta \, d\theta = \frac{3\pi}{4}.$$

(d) [**7 marks**] Evaluate

$$\int_S (\operatorname{curl} \mathbf{A}) \cdot \widehat{\mathbf{n}} \, dS,$$

by projecting onto the $x - y$ plane and using plane polar coordinates (r, θ).

You may assume $dxdy = r\,drd\theta$.