Mathematical Logic (MATH6/70132;P65)
Solutions to Problem Sheet 7

[1] (a) Find subsets of $\mathbb{Q}$ which (with their induced orderings from $\mathbb{Q}$) are similar to:
(i) $\mathbb{N}+\mathbb{N}+\mathbb{N}$;
(ii) $\mathbb{N}\times\mathbb{Z}$;
(iii) $\mathbb{N}+\mathbb{N}^*$ (where $\mathbb{N}^*$ is the reverse ordering on $\mathbb{N}$).
You do not need to write down the similarities involved here.

(b) Suppose $\mathcal{A}=(A;\leq)$ is any (non-empty) linearly ordered set. Prove that $\mathbb{Q}\times\mathcal{A}$ is a dense linear ordering without endpoints.

Solution: (a) (i) $\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,1,1\frac{1}{2},1\frac{2}{3},1\frac{3}{4},\ldots,2,2\frac{1}{2},2\frac{2}{3},2\frac{3}{4},\ldots\}$.

(ii) $\{m+\frac{n}{n+1}:m\in\mathbb{Z},n\in\mathbb{N}\}$.

(iii) $\{-\frac{1}{n+1}:n\in\mathbb{N}\}\cup\{\frac{1}{n+1}:n\in\mathbb{N}\}$.

(b) Let $\mathcal{B}=\mathbb{Q}\times\mathcal{A}$. We already know that this is a linearly ordered set. To see that it has no least element, suppose $(q,a)\in\mathbb{Q}\times A$. There is $q'\in\mathbb{Q}$ with $q'<q$, and so $(q',a)<(q,a)$ in $\mathcal{B}$. Similarly $\mathcal{B}$ has no greatest element. For denseness, suppose $(q',a')<(q,a)$ (in $\mathcal{B}$). If $a'=a$ then $q'<q$ (in $\mathbb{Q}$) and there is $q''\in\mathbb{Q}$ with $q'<q''<q$. Then $(q',a)<(q'',a)<(q,a)$. If $a'<a$, let $q''<q$. Then $(q',a')<(q'',a)<(q,a)$.

[2] We say that a set is *finite* if and only if it is equinumerous with some natural number $n\in\omega$. Otherwise it is *infinite*.
(i) Suppose β is an infinite ordinal. Using results from 3.4 of the notes, prove that $\omega\leq\beta$. Deduce that $|\beta^\dagger|=|\beta|$. (ii) Prove that if $m,n\in\omega$ are equinumerous then $m=n$.
(iii) Suppose X is a non-empty finite set of ordinals. Prove that X has a largest element.
(iv) Suppose α is a finite ordinal. Prove that $\alpha\in\omega$.
(v) Suppose $x\subseteq n\in\omega$. Then x is finite.

Solution: (i) Let $n\in\omega$. By assumption, β is an ordinal and $\beta\neq n$. As n is an ordinal (3.4.3), we can use 3.4.6 to get $n<\beta$ (otherwise $\beta<n$, which implies $\beta\in\omega$ contradicting that β is infinite). Thus $n\in\beta$. It follows that $\omega\subseteq\beta$.

For the last part consider the function $f:\beta\to\beta^\dagger$ given by $f(0)=\beta$, $f(n^\dagger)=n$ if $n\in\omega$ and $f(\alpha)=\alpha$ if $\omega\leq\alpha\in\beta$. This is a bijection.

(ii) Note that if $m\neq n$ then either $m\subset n$ or $n\subset m$. So it will suffice to prove that if $n\in\omega$ and $x\subseteq n$ is equinumerous with n, then $x=n$. We prove this by induction on n. The case $n=\emptyset$ is trivial. Suppose we have the result for n: we deduce the result for $n^\dagger=n\cup\{n\}$. Let $x\subseteq n\cup\{n\}$ and $f:x\to n^\dagger$ a bijection.

Suppose first that $x\subseteq n$. Let $y=x\setminus\{f^{-1}(n)\}$. So $y\subset n$. Then f restricted to y is a bijection with n and so by inductive assumption $y=n$, contradicting $y\subset n$.

So $n\in x$. By composing with some other bijection (sending $f(n)$ to n), we may assume that $f(n)=n$. Then f restricted to $x\cap n$ gives a bijection between $x\cap n$ and n. By induction assumption, $x\cap n=n$ and therefore $x=n^\dagger$, completing the inductive step.

(iii) By assumption there is a bijection $f:n\to X$ for some $n\in\omega$. Prove by induction on n that X has a maximal element.

(iv) Suppose α is equinumerous with $n\in\omega$. We show $\alpha=n$. If not, by 3.4.6, then $\alpha\subset n$ or $n\subset\alpha$. In the first case, n is then equinumerous to a proper subset of itself, which contradicts (i). In the second case, let m be the least element of $\alpha\setminus n$. Then $m\in\omega$ (otherwise $\omega\subseteq n$ which is impossible) and $n\subseteq m\subset m^\dagger\subseteq\alpha$. There is then an injective function from $m^\dagger$ to n which contradicts the result in the proof of (i).

(v) Prove this by induction on n.

[3] Suppose X is a non-empty set of ordinals. From the notes, you know that $\bigcup X$ and $\bigcap X$ are ordinals and $\bigcap X \leq \alpha \leq \bigcup X$ for all $\alpha \in X$.
(i) Show that if β is an ordinal with $\alpha \leq \beta$ for all $\alpha \in X$, then $\bigcup X \leq \beta$.
(ii) Formulate and prove a similar statement about $\bigcap X$.

Solution: (i) For all $\alpha \in X$ we have $\alpha \leq \beta$ so $\alpha \subseteq \beta$ (by results in 3.4). Then $\bigcup X \subseteq \beta$. As $\bigcup X$ is an ordinal this implies $\bigcup X \leq \beta$.

(ii) The result says that if we have an ordinal δ with $\delta \leq \alpha$ for all $\alpha \in X$ then $\delta \leq \bigcap X$. From the notes (3.4.6) $\bigcap X \in X$, so this is immediate.

[4] Suppose α and β are ordinals with α similar to $\omega + \omega$ and β similar to $\omega \times \omega$ (with the orderings as defined in 3.3.3). Which of $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$ holds?

Solutions: We show that α is similar to a proper initial segment of β. It then follows that $\alpha < \beta$, as the other two alternatives would then imply that β is similar to a proper initial segment of itself, which is impossible (3.4.11). It is easy to see that $\omega + \omega$ is similar to the induced ordering on

$$A = \{(n, 0), (n, 1) : n \in \omega\} \subseteq \omega \times \omega,$$

and that A is a proper initial segment of $\omega \times \omega$.

[5] Let β be the set of all countable ordinals.
(i) Show that β is an ordinal.
(ii) Show that β is uncountable (- suppose not and obtain a contradiction).
(iii) Show that if γ is an uncountable ordinal then $\beta \leq \gamma$. (Again, try a proof by contradiction.)

Solution: (i) First, show that β is a transitive set. Suppose $\gamma \in \beta$ and $\delta \in \gamma$. Then γ is countable, and using results in 3.4 of the notes, δ is a subset of γ and δ is an ordinal. So δ is a countable ordinal, and therefore $\delta \in \beta$.

Also, β is a set of ordinals and it is therefore well-ordered by $\in$. So β is an ordinal.

(ii) Suppose not. Then β is a countable ordinal, whence $\beta \in \beta$ which is impossible, by definition of being an ordinal.

(iii) If γ is an uncountable ordinal then $\gamma \notin \beta$, i.e. $\gamma \not< \beta$. Thus as any two ordinals are comparable (Theorem 3.4.6), we have $\beta \leq \gamma$.

[6] (i) Suppose α is a ordinal and $X \subset \alpha$ is a proper initial segment of α. Prove that there is $\beta \in \alpha$ with $X = \beta$.
(ii) Suppose that $\gamma \neq \delta$ are ordinals. Prove that γ and δ are not similar.

Solution: (i) By 3.4.10 there is $\beta \in \alpha$ with $X = \{\delta \in \alpha : \delta < \beta\}$. But this set is just β (3.4.4).

(ii) By 3.4.6 we may, without loss, assume $\gamma < \beta$. So γ is a proper initial segment of δ and the result then follows from 3.4.11.

[7] A *cardinal* is an ordinal α with the property that for all ordinals $\beta < \alpha$ we have that α and β are not equinumerous.
(i) Prove that every natural number is a cardinal and ω is a cardinal.
(ii) Prove that the ordinal β in question 5 is a cardinal.
(iii) Show that if γ is any ordinal, there is a unique cardinal α which is equinumerous with γ.

Solution: (i) If $m, n \in \omega$ and $m < n$ then m, n are not equinumerous, by Qu2(i). So n is a cardinal. If $n \in \omega$ then n is finite and ω is infinite (by Qu 2) so they are not equinumerous. This ω is a cardinal.

(ii) If α is an ordinal and $\alpha < \beta$ then $\alpha \in \beta$, so α is countable. As β is uncountable, it is therefore not equinumerous with α. So β is a cardinal.

(iii) Consider

$$X = \{\delta \leq \gamma : \delta \text{ is an ordinal and } \delta \approx \gamma\} \subseteq \gamma.$$

As $\gamma \in X$, X is a non-empty set of ordinals, so has a least element α. If β is an ordinal with $\beta < \alpha$ then $\beta \leq \gamma$ and $\beta \notin X$ so β is not equinumerous with γ and therefore not equinumerous with α. So α is a cardinal. For the uniqueness part suppose α and α' are cardinals equinumerous with γ. So they are equinumerous. If $\alpha \neq \alpha'$ we have $\alpha < \alpha'$ or $\alpha' < \alpha$ (by 3.4.6) both of which are impossible by the definition of being a cardinal.