**Exercise 7.1.** Let $(X, \mathrm{d})$ be a metric space, and $V$ be a subset of $X$. Show that the set $V$ is closed if and only if $\overline{V} = V$.

*Hint: use the definition of closed sets, and the definition of the closure of a set.*

**Exercise 7.2.** Let $V$ and $W$ be subsets of a metric space $(X, \mathrm{d})$. The following properties hold:

(i) if $V \subset W$, then $V^\circ \subset W^\circ$,

(ii) if $V \subset W$, then $\overline{V} \subset \overline{W}$,

**Exercise 7.3.** Let $V$ and $W$ be subsets of a metric space $(X, \mathrm{d})$. Prove that

$$\overline{V \cup W} = \overline{V} \cup \overline{W}.$$

Give an example of $(X, \mathrm{d})$, $V$ and $W$ such that

$$(V \cup W)^\circ \neq V^\circ \cup W^\circ.$$

**Exercise 7.4.** Let $(A_1, \mathrm{d}_1)$ and $(A_2, \mathrm{d}_2)$ be metric spaces. A map $f : A_1 \to A_2$ is continuous if and only if the pre-image of any closed set in $A_2$ is a closed set in $A_1$.

**Exercise 7.5.** Recall that the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ is denoted by $C([0, 1])$. We also defined the metrics $\mathrm{d}_1$, $\mathrm{d}_2$ and $\mathrm{d}_\infty$ on $C([0, 1])$. Consider the map

$$\Phi : C([0, 1]) \to \mathbb{R},$$

defined as

$$\Phi(f) = f(1/2).$$

(i) Is the map $\Phi$ from the metric space $(C([0, 1]), \mathrm{d}_\infty)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(ii) Is the map $\Phi$ from the metric space $(C([0, 1]), \mathrm{d}_1)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(iii) Is the map $\Phi$ from the metric space $(C([0, 1]), \mathrm{d}_2)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

*Hint: draw the graphs of few functions, and think about what it means for two functions in $C([0, 1])$ to be close together in each of those metrics.*

**Exercise 7.6.** Consider the metric spaces $X = (\mathbb{R}, \mathrm{d}_1)$ and $Y = (\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$. Show that the map $f(x) = x$ from $X$ to $Y$ is not continuous. Show that the map $g(x) = x$ from $Y$ to $X$ is continuous.

**Exercise 7.7.** Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, for $n \geq 1$, defined as

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be the constant map $f \equiv 0$.

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Questions marked with $*$ are optional

(i) Show that the sequence $(f_n)_{n\geq 1}$ in $C([0,1])$ converges to $f$ in the metric space $(C([0,1],\mathrm{d}_1)$.

(ii) Show that the sequence $(f_n)_{n\geq 1}$ in $C([0,1])$ does not converge to $f$ in the metric space $(C([0,1],\mathrm{d}_\infty)$.

(iii) Conclude that the identity map

$$\mathrm{id} : (C([0,1]),\mathrm{d}_1) \to (C([0,1]),\mathrm{d}_\infty)$$

is not continuous.

**Exercise 7.8.** Let $(X,\mathrm{d}_X)$ and $(Y,\mathrm{d}_Y)$ be metric spaces, and $f : X \to Y$ be a surjective map. Show that if $f$ is bi-Lipschitz, then it is a homeomorphisms.

**Unseen Exercise.** Let $(X,\mathrm{d})$ be a metric space, and $V$ be a subset of $X$. Prove that

(i) the set $V^\circ$ is open, and $V^\circ$ is the largest open set contained in $V$;

(ii) the set $\overline{V}$ is closed, and $\overline{V}$ is the smallest closed set which contains $V$.

*Hint: For the latter part of (i), you need to show that if $\Omega \subseteq V$ and $\Omega$ is an open set in $(X,\mathrm{d})$, then $\Omega \subseteq V^\circ$. For the latter part of (ii), you need to show that if $V \subseteq \Delta$ and $\Delta$ is a closed set in $(X,\mathrm{d})$, then $\overline{V} \subseteq \Delta$.*