Imperial College London

MATH97049 MATH97160

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: 19th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

.

1. (a) Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular curve.

   (i) Define what is meant by the *length* and a parametrisation by *arc-length* of γ.

   (ii) Show that γ has a reparametrisation $\delta : [0, L] \to S$ by arc-length, where L is its *length*.

   (iii) Show that any reparametrisation of γ by arc-length is of the form $t \mapsto \delta(\pm t + c)$, where c is a constant.

(6 marks)

(b) Let $S \subset \mathbb{R}^3$ be a regular surface with orientation given by the map of unit normal vectors $N : S \to \mathbb{S}^2$ and let $\gamma : [0, L] \to S \subset \mathbb{R}^3$ be a regular curve parametrized by arc-length in S.

   (i) Define what is meant by the *curvature* κ and the *normal curvature* κ_n of γ.

   (ii) Define what is meant by the *geodesic curvature* κ_g of γ and a *geodesic* in S.

   (iii) Show the following identity $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

(7 marks)

(c) Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = 4\} \subset \mathbb{R}^3$ be an ellipsoid.

   (i) Show that S is a regular surface and compute its tangent plane T_pS at $p = (2, 0, 0) \in S$.

   (ii) Show that the intersection $S \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is a simple closed geodesic in S.

   (iii) Find a simple closed geodesic in S passing through the two points $(2, 0, 0), (1, 0, 1) \in S$.

(7 marks)

(Total: 20 marks)

2. (a) (i) Define what is meant by an *isometry* $f : S_1 \to S_2$ between two regular surfaces $S_1, S_2 \subset \mathbb{R}^3$ and show that an isometry $f : S_1 \to S_2$ preserves the arc-length parametrisation of curves in S_1 and S_2, respectively.

   (ii) Show that an isometry $f : S_1 \to S_2$ sends geodesics in S_1 to geodesics in S_2.

   (iii) Define what is meant by the *first fundamental form* or *metric* of a regular surface $S \subset \mathbb{R}^3$ and state *Gauss's Egregium Theorem*, carefully explaining all the terms in your statement. (You do not need to give the definition of Gaussian curvature.)

(9 marks)

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular plane curve without self-intersections and consider the set $S \subset \mathbb{R}^3$, given by $S = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \gamma(\mathbb{R})\}$.

   (i) Show that $S \subset \mathbb{R}^3$ is a regular surface that is homeomorphic to a plane.

   (ii) Find an isometry $\phi : \mathbb{R}^2 \to S$. What is the *Gaussian curvature* of S?

   (iii) Show that S is contained in a plane if and only if its *mean curvature* is zero.

(7 marks)

(c) Let γ and S be as in Part (b) and assume furthermore that γ is parametrised by arc-length and $\gamma(0) = (0,0)$ and $\gamma(2) = (1,0)$. Let $\pi : S \to \mathbb{R}^2$ denote the projection $\pi(x,y,z) = (x,y)$.

(i) Show that the subsets $\pi^{-1}(\gamma(t)) \subset S$ are geodesic lines of S for all $t \in \mathbb{R}$. Compute the minimum length of a curve in S from $(0,0,-1)$ to $(0,0,1)$.

(ii) What is the minimum length of a (regular) curve in S from $(0,0,-1)$ to $(1,0,0)$?

(4 marks)

(Total: 20 marks)

3. Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation given by the map of unit normal vectors $N : S \to \mathbb{S}^2$, and let $p \in S$ and $v \in T_pS$ a unit tangent vector to S at the point p.

(a) (i) Show that all regular curves $\gamma : (-\epsilon, \epsilon) \to S$ parametrised by arc-length and $\gamma(0) = p$, $\gamma'(0) = v$ have the same normal curvature at the point $p \in S$.

(ii) Define what is meant by the *principal curvatures* and *principal directions* of S at p.

(iii) Let $\gamma : (-\epsilon, \epsilon) \to S$ be a curve parametrised by arc-length. Show that $\gamma'(t) \in T_{\gamma(t)}S$ is a principal direction for all $t \in (-\epsilon, \epsilon)$ if and only if there exists a function $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ with $\frac{d}{dt}N(\gamma(t)) = -\lambda(t) \cdot \gamma'(t)$ for all $t \in (-\epsilon, \epsilon)$. In this case, show that, $\lambda(t)$ is the corresponding principal curvature at $\gamma(t) \in S$.

(iv) Let $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be the unit cylinder in $\mathbb{R}^3$. Describe all the regular curves on $S \in \mathbb{R}^3$ that satisfy the condition in (iii).

(8 marks)

(b) (i) Show that there is a regular chart $\phi : U \to S$ with $\phi((0,0)) = p$, where $U \subset \mathbb{R}^2$ an open neighbourhood of $(0,0)$, and a rigid motion (isometry) α of $\mathbb{R}^3$, such that the map $\psi : U \to \mathbb{R}^3$, given by $\psi(u,v) = \alpha(\phi(u,v))$, satisfies $\psi(u,v) = (u, v, \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2)) +$ (higher order terms), as $(u,v) \to (0,0)$, where λ_1, λ_2 are the principal curvatures at p.

(ii) Describe (e.g. by carefully drawing) a regular surface $S \subset \mathbb{R}^3$ and $p \in S$, with principal curvatures λ_1, λ_2 at $p \in S$ for each of the following four cases: (1) $\lambda_1 > 0, \lambda_2 > 0$, (2) $\lambda_1 > 0, \lambda_2 < 0$, (3) $\lambda_1 = 0, \lambda_2 > 0$ and (4) $\lambda_1 = 0, \lambda_2 = 0$.

(5 marks)

(c) (i) Define what is meant by the *Christoffel symbols* $\Gamma_{ij}^k : U \to \mathbb{R}$ for a given chart $\phi : U \to S$.

(ii) Using part b(i), or otherwise, show that there is a regular chart $\phi : U \to S \in \mathbb{R}^3$ for which all the Christoffel symbols $\Gamma_{ij}^k : U \to \mathbb{R}$ are zero. (Hint: you may want to choose ϕ so that the first fundmental form is the identity up to second order terms.)

(7 marks)

(Total: 20 marks)

4. (a) Let $\phi : [a,b] \to \mathbb{R}^3$ be a regular curve parametrised by arc-length and $\phi''(t) \neq 0$ for all $t \in [a,b]$.

(i) Define what is meant by the *Frenet frame* (T, N, B), *curvature* κ and *torsion* τ of ϕ.

(ii) Show that the Frenet frame (T, N, B) of ϕ satisfies the *Frenet equations*

$$(T', N', B') = (T, N, B) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

(ii) Let $\phi, \psi : [a,b] \to \mathbb{R}^3$ be curves parametrised by arc-length, with equal non-zero curvature and torsion for all $t \in [a,b]$. Show that there is a rigid motion A of $\mathbb{R}^3$ so that $\phi = A \circ \psi$.

(12 marks)

(b) (i) Let $\kappa : [-1,1] \to \mathbb{R}$ be a smooth function. Show that there exists a unique plane curve $\gamma : [-1,1] \to \mathbb{R}^2$, parametrised by arc-length and satisfying $\gamma(0) = (0,0)$, $\gamma'(0) = (1,0)$ and (signed) curvature equal to κ.

(ii) Is there a smooth *closed* curve $\gamma : [-1,1] \to \mathbb{R}^2$ parametrised by arc-length and (signed) curvature equal to $\kappa(t) = t^3 - t$ for all $t \in [-1,1]$?

(8 marks)

(Total: 20 marks)

5. Let $\gamma : [0, L] \to \mathbb{R}^3$ be a simple (i.e. γ is injective) closed curve, parametrised by arc-length and positive curvature $\kappa(s) > 0$ at all points $\gamma(s)$, $s \in [0, L]$.

(a) Using the Gauss-Bonnet theorem, or otherwise, show that if the curve γ is contained in a plane then its total curvature satisfies $\int_0^L \kappa(s)ds = 2\pi$. (4 marks)

(b) Let r be a positive constant and $\phi : [0, L] \times [0, 2\pi] \to \mathbb{R}^3$ denote the smooth map given by

$$\phi(s, t) = \gamma(s) + r(\cos(t)n(s) + \sin(t)b(s))$$

where $n(s)$ and $b(s)$ denote the unit normal and bi-normal vectors of γ at the point $\gamma(s)$.

(i) Show that for $r > 0$ sufficiently small, the image of ϕ is a regular surface $T \subset \mathbb{R}^3$ (called the *Tube* around γ) with a regular coordinate chart given by ϕ.

(ii) Show that the Gauss map of T is given by $N(s, t) = -(\cos(t)n(s) + \sin(t)b(s))$.

(iii) Show that the Gaussian curvature $K(p)$ at the point $p = \phi(s, t) \in T$ is equal to

$$K(p) = \frac{-k\cos(t)}{r(1 - r\kappa(s)\cos(t))}$$

Conclude that $K = 0$ if and only if the line through $b(s)$ is orthogonal to the tube.

(7 marks)

(c) Let $R = \{p \in T : K(p) \geq 0\} \subset T$ be a regular surface (with boundary) in $\mathbb{R}^3$.

(i) Show that $\int_R K dA = 2\int_0^L \kappa(s)ds$.

(ii) Show that the Gauss map $N|_R : R \to \mathbb{S}^2$ is surjective. (Hint: Given a unit vector $v \in \mathbb{S}^2$, consider the function $h : S \to \mathbb{R}$, given by $h(x, y, z) = (x, y, z) \cdot v$.)

(iii) Using parts (i) and (ii) above, or otherwise, show that the total curvature of γ satisfies $\int_0^L \kappa(s)ds \geq 2\pi$. (Hint: $\int_R K dA \geq 4\pi$.)

(9 marks)

(Total: 20 marks)

Solutions [category, marks]

1. (a) Let $\gamma : [a,b] \to \mathbb{R}^3$ be a regular curve.

   (i) Define what is meant by the *length* and a parametrisation by *arc-length* of γ.

   (ii) Show that γ has a reparametrisation $\delta : [0, L] \to S$ by arc-length, where L is its *length*.

   (iii) Show that any reparametrisation of γ by arc-length is of the form $t \mapsto \delta(\pm t + c)$, where c is a constant.

   (b) Let $S \subset \mathbb{R}^3$ be a regular surface with orientation given by the map of unit normal vectors $N : S \to \mathbb{S}^2$ and let $\gamma : [0, L] \to S \subset \mathbb{R}^3$ be a regular curve parametrized by arc-length in S.

   (i) Define what is meant by the *curvature* κ and the *normal curvature* κ_n of γ.

   (ii) Define what is meant by the *geodesic curvature* κ_g of γ and a *geodesic* in S.

   (iii) Show the following identity $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

   (c) Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = 4\} \subset \mathbb{R}^3$ be an ellipsoid.

   (i) Show that S is a regular surface and compute its tangent plane T_pS at $p = (2, 0, 0) \in S$.

   (ii) Show that the intersection $S \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is a simple closed geodesic in S.

   (iii) Find a simple closed geodesic in S passing through the two points $(2, 0, 0), (1, 0, 1) \in S$.

Solution 1.

(a) (i) [A, 2] The *length* of γ is $\int_a^b |\gamma'(t)|dt$, γ is parametrised by *arc-length* if $|\gamma'(t)| = 1$ for all t.

(ii) [A, 2] Consider $l(t) = \int_a^t |\gamma'(t)|dt$. Then l is smooth and $l'(t) > 0$ for all t, so it is a bijection $l : [a, b] \to [0, L]$. Let $\delta : [0, L] \to S$ defined by $\delta(s) = \gamma(l^{-1}(s))$; δ is smooth as composition of smooth function and $|\delta'(s)| = 1$ for all s, by the chain-rule and the inverse derivative rule.

(iii) [B, 2] Any other arc-length (re-)parametrisation of γ is of the form $\delta(\phi(t))$, where ϕ is a smooth bijection $\phi : [c, d] \to [0, L]$. By the chain-rule, we have $|\phi'(t)| = 1$ for all t; hence, $\phi'(t) = \pm 1$ for all t and ϕ is of the form $t \mapsto \pm t + c$, where c is a constant.

(b) (i) [A, 2] The *curvature* of γ is $\kappa(t) = |\gamma''(t)|$. The *normal curvature* of γ is $\kappa_n(t) = \kappa(t)\cos(\theta)$, where θ is the angle between the unit normals $n(t) = \frac{\gamma''(t)}{|\gamma''(t)|}$ and $N(\gamma(t))$ of γ and S.

(ii) [A, 2] The *geodesic curvature* of γ is $\kappa_g = \gamma''(t) \cdot (N(\gamma(t)) \times \gamma'(t))$. γ a *geodesic* if $\kappa_g = 0$.

(iii) [C, 3] The vectors $(\gamma'(t), N(\gamma(t)) \times \gamma'(t), N(\gamma(t)))$ form an orthonormal basis of $\mathbb{R}^3$. Since $|\gamma'(t)| = 1$ for all t, we have $\gamma'(t) \cdot \gamma''(t) = 0$; hence, $\gamma''(t) = \kappa_n N(\gamma(t)) + \kappa_g \gamma'(t) \times N(\gamma(t))$.

(c) (i) [B, 3] Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$ with $S = f^{-1}(4)$. It is clear that $\nabla f = (2x, 4y, 6z)$ is non-zero for $(x, y, z) \in S$. Hence S is a regular surface and $T_{(2,0,0)}S = \ker(\nabla f) = \{x = 0\}$.

(ii) [D, 3] The intersection $C = S \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is an ellipse in the plane $\{z = 0\}$. The unit normal to S is proportional to ∇f, which is contained in the plane $\{z = 0\}$ for all $(x, y, z) \in S \cap \{z = 0\}$. Hence the curvature vector of C is proportional to the unit normal of S along C, from which it follows the geodesic curvature of C is zero and C is a geodesic.

(iii) [D, 1] Arguing exactly as in (ii) above, the intersection $S \cap \{y = 0\}$ is a simple closed geodesic in S passing through the points $(2, 0, 0), (1, 0, 1) \in S$.

2. (a) (i) Define what is meant by an *isometry* $f : S_1 \to S_2$ between two regular surfaces $S_1, S_2 \subset \mathbb{R}^3$ and show that an isometry $f : S_1 \to S_2$ preserves the arc-length parametrisation of curves in S_1 and S_2, respectively.

(ii) Show that an isometry $f : S_1 \to S_2$ sends geodesics in S_1 to geodesics in S_2.

(iii) Define what is meant by the *first fundamental form* or *metric* of a regular surface $S \subset \mathbb{R}^3$ and state *Gauss's Egregium Theorem*, carefully explaining all the terms in your statement. (You do not need to give the definition of Gaussian curvature.)

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular plane curve without self-intersections and consider the set $S \subset \mathbb{R}^3$, given by $S = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \gamma(\mathbb{R})\}$.

(i) Show that $S \subset \mathbb{R}^3$ is a regular surface that is homeomorphic to a plane.

(ii) Find an isometry $\phi : \mathbb{R}^2 \to S$. What is the *Gaussian curvature* of S?

(iii) Show that S is contained in a plane if and only if its *mean curvature* is zero.

(c) Let γ and S be as in Part (b) and assume furthermore that γ is parametrised by arc-length and $\gamma(0) = (0, 0)$ and $\gamma(2) = (1, 0)$. Let $\pi : S \to \mathbb{R}^2$ denote the projection $\pi(x, y, z) = (x, y)$.

(i) Show that the subsets $\pi^{-1}(\gamma(t)) \subset S$ are geodesic lines of S for all $t \in \mathbb{R}$. Compute the minimum length of a curve in S from $(0, 0, -1)$ to $(0, 0, 1)$.

(ii) What is the minimum length of a (regular) curve in S from $(0, 0, -1)$ to $(1, 0, 0)$?

Solution 2.

(a) (i) [A, 3] A smooth map $f : S_1 \to S_2$ between two regular surfaces $S_1, S_2 \subset \mathbb{R}^3$ is an *isometry* if it is a bijection and $df_p(v) \cdot df_p(w) = v \cdot w$ for all $p \in S_1$ and $v, w \in T_pS_1$. Let $f : S_1 \to S_2$ be an isometry and $\gamma : [a, b] \to S_1$ be a curve parametrised by arc-length. Then $t \mapsto f(\gamma(t))$ is a curve in S_2 with tangent/velocity vector $df_{\gamma(t)}(\gamma'(t))$, whose length is equal to $|df_{\gamma(t)}(\gamma'(t))| = \sqrt{df_{\gamma(t)}(\gamma'(t)) \cdot df_{\gamma(t)}(\gamma'(t))} = \sqrt{\gamma'(t) \cdot \gamma'(t)} = |\gamma'(t)| = 1$ for all t. Hence the curve $f \circ \gamma : [a, b] \to S_2$ is paramatrised by arc-length as well.

(ii) [C, 3] Let $\phi : U \to S_1$ be a chart for S_1; then $\psi = f \circ \phi : U \to S_2$ is a chart for S_2. Assume that $\gamma : [a, b] \to S_1$ is a geodesic in S_1. This is equivalent to $\gamma''(t)$ is a multiple of $N(\gamma(t))$ for all t, which is equilavent to $\gamma''(t) \cdot \phi_u = 0$ and $\gamma''(t) \cdot \phi_v = 0$. Write $\gamma(t) = \phi(u(t), v(t))$, for smooth u, v, then $\gamma'(t) = \phi_u u' + \phi_v v'$ and $\gamma''(t) = (\phi_{uu}u' + \phi_{uv}v')u' + (\phi_{vu}u' + \phi_{vv}v')v'$. So $0 = \gamma''(t) \cdot \phi_u = (\phi_u \cdot \phi_{uu})(u')^2 + 2(\phi_u \cdot \phi_{uv})(u'v') + (\phi_u \cdot \phi_{vv})(v')^2 = 1/2\frac{\partial}{\partial u}(\phi_u \cdot \phi_u)(u')^2 + \frac{\partial}{\partial v}(\phi_u \cdot \phi_u)(u'v') + [\frac{\partial}{\partial v}(\phi_u \cdot \phi_v) - 1/2\frac{\partial}{\partial u}(\phi_v \cdot \phi_v)](v')^2 = 1/2(g_{11})_u(u')^2 + (g_{11})_v(u'v') + [(g_{12})_v - 1/2(g_{22})_v](v')^2$ and similarly for $0 = \gamma''(t) \cdot \phi_v$: these are determined by u, v and the first fundamental form $g = (g_{i,j})_{i,j=1}^2$ of S_1. Since f is an isometry, it preserves g with respect to the chart $\psi = f \circ \phi$, and $f(\gamma(t)) = \psi(u(t), v(t))$ implies that $(f \circ \gamma)'' \cdot \psi_u = \gamma'' \cdot \phi_u = 0$ and $(f \circ \gamma)'' \cdot \psi_v = \gamma'' \cdot \phi_v = 0$. Hence $f \circ \gamma : [a, b] \to S_2$ is a geodesic in S_2.

(iii) [A, 3]The *first fundamental form* or *metric* of a regular surface $S \subset \mathbb{R}^3$ is bilinear map $T_pS \times T_pS \to R$, given by $(v, w) \mapsto v \cdot w$, for all $p \in S$ and $v, w \in T_pS$. *Gauss's Egregium Theorem* states that the Gaussian curvature of a regular surface in $\mathbb{R}^3$ depends only on the first fundamental form of the surface; in particular it is a (local) isometric invariant.

(b) (i) [A, 2] Let $\phi : \mathbb{R}^2 \to S \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ given by $\phi(s, t) = (\gamma(s), t)$ for $(s, t) \in \mathbb{R}^2$. It is clear that ϕ is a regular chart for all S ($d\phi$ is injective at all points) and smooth bijection from $\mathbb{R}^2$ to S.

(ii) [B, 3] Let γ be (re-)parametrised by arc-length and let $\phi : \mathbb{R}^2 \to S$ given as in (i) above. It is clear that ϕ is an isometry (check the first fundemental form is the identity). Hence, by *Gauss's Egregium Theorem*, the *Gaussian curvature* of S is zero.

(iii) [B, 2] If S is contained in a plane, its Gauss map of unit normals is constant; hence, the second fundamental form is zero, from which it follows that the *mean curvature* is zero. Conversely, if the *mean curvature* of S is zero, since its Gaussian curvature is also zero, it follows that both of its principal curvatures are zero. We have seen in lectures such a surface is planar. (Alternatively, compute the mean curvature at $p \in S$ is $H(p) = \pm\frac{1}{2}\kappa \circ \pi(p)$, where κ is the curvature of γ. Hence, $H = 0$ iff $\kappa = 0$ iff γ contained in a line iff S contained in a plane.)

(c) (i) [D, 2] The subsets $\pi^{-1}(\gamma(t)) \subset S$ are straight lines; hence, $\kappa = 0$. Since $\kappa^2 = \kappa_n^2 + \kappa_g^2$, we conclude that $k_g = 0$; hence, the vertical lines are geodesics. Alternatively, since isometries preserve geodesics, $\pi^{-1}(\gamma(t)) = \phi(\{t\} \times \mathbb{R})$ are geodesics for each t. Since ϕ is a global isometry, it preserves length of curves; hence, the minimum length of a curve in S from $(0,0,-1)$ to $(0,0,1)$ is the minimum length of a curve in $\mathbb{R}^2$ from $(0,-1)$ to $(0,1)$, which is obviously 2.

(ii) [D, 2] Arguing as above, the minimum length of a curve in S from $(0,0,-1)$ to $(1,0,0)$ is the minimum length of a curve in $\mathbb{R}^2$ from $(0,-1)$ to $(2,0)$, which is obviously $\sqrt{5}$.

3. Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation given by the map of unit normal vectors $N : S \to \mathbb{S}^2$, and let $p \in S$ and $v \in T_pS$ a unit tangent vector to S at the point p.

(a) (i) Show that all regular curves $\gamma : (-\epsilon, \epsilon) \to S$ parametrised by arc-length and $\gamma(0) = p$, $\gamma'(0) = v$ have the same normal curvature at the point $p \in S$.

(ii) Define what is meant by the *principal curvatures* and *principal directions* of S at p.

(iii) Let $\gamma : (-\epsilon, \epsilon) \to S$ be a curve parametrised by arc-length. Show that $\gamma'(t) \in T_{\gamma(t)}S$ is a principal direction for all $t \in (-\epsilon, \epsilon)$ if and only if there exists a function $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ with $\frac{d}{dt}N(\gamma(t)) = -\lambda(t) \cdot \gamma'(t)$ for all $t \in (-\epsilon, \epsilon)$. In this case, show that, $\lambda(t)$ is the corresponding principal curvature at $\gamma(t) \in S$.

(iv) Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be the unit cylinder in $\mathbb{R}^3$. Describe all the regular curves on $S \in \mathbb{R}^3$ that satisfy the condition in (iii).

(b) (i) Show that there is a regular chart $\phi : U \to S$ with $\phi((0, 0)) = p$, where $U \subset \mathbb{R}^2$ an open neighbourhood of $(0, 0)$, and a rigid motion (isometry) α of $\mathbb{R}^3$, such that the map $\psi : U \to \mathbb{R}^3$, given by $\psi(u, v) = \alpha(\phi(u, v))$, satisfies $\psi(u, v) = (u, v, \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2)) +$ (higher order terms), as $(u, v) \to (0, 0)$, where λ_1, λ_2 are the principal curvatures at p.

(ii) Describe (e.g. by carefully drawing) a regular surface $S \subset \mathbb{R}^3$ and $p \in S$, with principal curvatures λ_1, λ_2 at $p \in S$ for each of the following four cases: (1) $\lambda_1 > 0, \lambda_2 > 0$, (2) $\lambda_1 > 0, \lambda_2 < 0$, (3) $\lambda_1 = 0, \lambda_2 > 0$ and (4) $\lambda_1 = 0, \lambda_2 = 0$.

(c) (i) Define what is meant by the *Christoffel symbols* $\Gamma^k_{ij} : U \to \mathbb{R}$ for a given chart $\phi : U \to S$.

(ii) Using part b(i), or otherwise, show that there is a regular chart $\phi : U \to S \in \mathbb{R}^3$ for which all the Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ are zero. (Hint: you may want to choose ϕ so that the first fundmental form is the identity up to second order terms).

Solution 3.

(a) (i) [A, 3] Since $\gamma'(t) \cdot N(\gamma(t)) = 0$ for all t, we have $\gamma''(t) \cdot N(\gamma(t)) + \gamma'(t) \cdot dN_{\gamma(t)}(\gamma'(t)) = 0$, where we used the chain rule $\frac{d}{dt}N(\gamma(t)) = dN_{\gamma(t)}(\gamma'(t))$. Hence, the normal curvature of γ at p is $-v \cdot dN_p(v)$.

(ii) [A, 2] Let $A : T_pS \times T_pS$ be the second fundamental form at $p \in S$, which is a symmetric bilinear form; hence, A can be diagonalised in an orthonormal basis v_1, v_2 of T_pS, the *principal directions*, and with real diagonal entries λ_1, λ_2, the *principal curvatures* of S at p, such that $A(v_1, v_1) = \lambda_1, A(v_1, v_2) = 0, A(v_2, v_1) = 0$ and $A(v_2, v_2) = \lambda_2$.

(iii) [B, 1] It follows immediately from the chain-rule $dN_{\gamma(t)}(\gamma'(t)) = \frac{d}{dt}N(\gamma(t))$.

(iv) [B, 2] Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be the unit cylinder in $\mathbb{R}^3$. It is clear that the vertical and horizontal directions are the principal directions of S. Hence the vertical straight lines and horizontal closed circles are all the curves in S that satisfy the condition in (iii).

(b) (i) [C, 3] We can arrange by a change of coordinates on U and rigid motion α on $\mathbb{R}^3$ that $\psi = \alpha \circ \phi : U \to \mathbb{R}^3$ satisfies $\psi(0, 0) = (0, 0, 0)$ and $\{\frac{\partial \psi}{\partial u}(0, 0) = (1, 0, 0), \frac{\partial \psi}{\partial v}(0, 0) = (0, 1, 0)\}$ are the principal directions $\{x_1, x_2\}$ at $\alpha(p)$. Note that $N(0, 0, 0) = (0, 0, 1)$ and $A(ux_1 + vx_2, ux_1 + vx_2) = \lambda_1 u^2 + \lambda_2 v^2$. Then we have a Taylor series: $\psi(u, v) =$

$(u, v, 0) + \frac{1}{2}(u^2 \frac{\partial^2 \psi}{\partial u^2} + 2uv \frac{\partial^2 \psi}{\partial u \partial v} + v^2 \frac{\partial^2 \psi}{\partial v^2}) + \ldots$, as $(u, v) \to (0, 0)$, and near $(0, 0)$ the quadratic term satisfies (quadratic term) $\cdot N(0, 0, 0) = A((u, v, 0), (u, v, 0))$, so we have $\psi(u, v) = (u, v, \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2)) +$ (higher order terms), as $(u, v) \to (0, 0)$, where λ_1, λ_2 are the principal curvatures at p.

(ii) [B, 2] (1) Draw an elliptic point for $\lambda_1 > 0, \lambda_2 > 0$, (2) a hyperbolic/saddle point for $\lambda_1 > 0, \lambda_2 < 0$, (3) a parabolic point (ruled surface) for $\lambda_1 = 0, \lambda_2 > 0$ and (4) a plane for $\lambda_1 = 0, \lambda_2 = 0$.

(c) (i) [A, 3] The Christoffel symbols are uniquely determined from the following equations:

$$\frac{\partial^2 \phi}{\partial u^2} = \Gamma^1_{11}\phi_u + \Gamma^2_{11}\phi_v + A(\phi_u, \phi_u) N \circ \phi$$

$$\frac{\partial^2 \phi}{\partial u \partial v} = \Gamma^1_{12}\phi_u + \Gamma^2_{12}\phi_v + A(\phi_u, \phi_v) N \circ \phi$$

$$\frac{\partial^2 \phi}{\partial v \partial u} = \Gamma^1_{21}\phi_u + \Gamma^2_{21}\phi_v + A(\phi_v, \phi_u) N \circ \phi$$

$$\frac{\partial^2 \phi}{\partial v^2} = \Gamma^1_{22}\phi_u + \Gamma^2_{22}\phi_v + A(\phi_v, \phi_v) N \circ \phi$$

(ii) [D, 4] Let $\psi : U \to S \in \mathbb{R}^3$ be the regular chart of S described in part (b)(i) above, with $\psi(u, v) \cdot (1, 0, 0) = u + a_1 u^2 + 2b_1 uv + c_1 v^2 +$ higher order terms and $\psi(u, v) \cdot (0, 1, 0) = v + a_2 u^2 + 2b_2 uv + c_2 v^2 +$ higher order terms, as $(u, v) \to (0, 0)$. Let $U = u + a_1 u^2 + 2b_1 uv + c_1 v^2$ and $V = v + a_2 u^2 + 2b_2 uv + c_2 v^2$; since $(u, v) \mapsto (U, V)$ has Jacobian equal to the identity at $(0, 0)$, it is a diffeomorphism is small neighbourhood of $(0, 0)$. It is clear that in (U, V) coordinates $\psi(U, V) = (U +$ (cubic terms)$, V +$ (cubic terms)$, \frac{1}{2}(\lambda_1 U^2 + \lambda_2 V^2) +$ (cubic terms)). It is clear that $\psi_U \cdot \psi_U = 1 +$ (quadratic terms), $\psi_U \cdot \psi_V =$ (quadratic terms) and $\psi_V \cdot \psi_V = 1 +$ (quadratic terms). Hence, $\frac{\partial^2 \psi}{\partial U^2} \cdot \psi_U = \frac{\partial^2 \psi}{\partial U^2} \cdot \psi_V = 0$ and similarly for the rest.

4. (a) Let $\phi : [a,b] \to \mathbb{R}^3$ be a regular curve parametrised by arc-length and $\phi''(t) \neq 0$ for all $t \in [a,b]$.

   (i) Define what is meant by the *Frenet frame* (T, N, B), *curvature* κ and *torsion* τ of ϕ.

   (ii) Show that the Frenet frame (T, N, B) of ϕ satisfies the *Frenet equations*

$$(T', N', B') = (T, N, B)\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

   (ii) Let $\phi, \psi : [a,b] \to \mathbb{R}^3$ be curves parametrised by arc-length, with equal non-zero curvature and torsion for all $t \in [a,b]$. Show that there is a rigid motion A of $\mathbb{R}^3$ so that $\phi = A \circ \psi$.

(b) (i) Let $\kappa : [-1,1] \to \mathbb{R}$ be a smooth function. Show that there exists a unique plane curve $\gamma : [-1,1] \to \mathbb{R}^2$, parametrised by arc-length and satisfying $\gamma(0) = (0,0)$, $\gamma'(0) = (1,0)$ and (signed) curvature equal to κ.

   (ii) Is there a smooth *closed* curve $\gamma : [-1,1] \to \mathbb{R}^2$ parametrised by arc-length and (signed) curvature equal to $\kappa(t) = t^3 - t$ for all $t \in [-1,1]$?

Solution 4.

(a) (i) [A, 4] $T(t) = \phi'(t)$, $N(t) = \frac{\phi''(t)}{|\phi''(t)|}$ and $B(t) = T(t) \times N(t)$. The *curvature* $\kappa(t) = |\phi''(t)|$ and the *torsion* $\tau(t)$ is defined by $B'(t) = -\tau(t)N(t)$ (or $\tau(t) = -B'(t) \cdot N(t)$).

(ii) [A, 4] It is clear that $T' = \kappa N$. Also, $|B| = 1$ implies $B' \cdot B = 0$, and $B \cdot T = 0$ implies $B' \cdot T + B \cdot (\kappa N) = 0$, hence $B' \cdot T = 0$ (since $B \cdot N = 0$). In other words, there is τ (with $\tau = -B' \cdot N$), such that $B' = -\tau N$. Finally, $|N| = 1$ implies $N' \cdot N = 0$; $T \cdot N = 0$ implies $N' \cdot T = -T' \cdot N = -\kappa$ and $N \cdot B = 0$ implies $N' \cdot B = -B' \cdot N = \tau$. Hence, $T' = \kappa N$, $N' = -\kappa T + \tau B$ and $B' = -\tau N$.

(iii) [B, 4] By post-composing with a rigid motion of $\mathbb{R}^3$, we may assume that $\phi(a) = \psi(a)$ and $(T_\phi(a), N_\phi(a), B_\phi(a)) = (T_\psi(a), N_\psi(a), B_\psi(a))$. Using the Frenet equations, we see that $\frac{d}{dt}(|T_\phi(t) - T_\psi(t)|^2 + |N_\phi(t) - N_\psi(t)|^2 + |B_\phi(t) - B_\psi(t)|^2) = 0$. Hence, $|T_\phi(t) - T_\psi(t)|^2 + |N_\phi(t) - N_\psi(t)|^2 + |B_\phi(t) - B_\psi(t)|^2$ is constant; since it is zero for $t = a$, it must be zero for all $t \in [a,b]$. Hence $\phi(t) = \phi(a) + \int_a^T T_\phi(t)dt = \psi(a) + \int_a^T T_\psi(t)dt = \psi(t)$ for all $t \in [a,b]$.

(b) (i) [C, 4] The Existence and Uniqueness Theorem for solution of linear differential equations guarantees that there exist unique smooth functions $T, N : [-1,1] \to \mathbb{R}^2$ with $T(0) = (1,0)$ and $N(0) = (0,1)$ such that

$$(T', N') = (T, N)\begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}$$

Let $M(t)$ be the 2×2 matrix with columns $T(t)$ and $N(t)$, and observe that

$$M'(t) = M(t)\begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}$$

We need to show that $M(t)^T M(t) = I$ for all $t \in [-1,1]$. This certainly holds when $t = 0$. But

$$\frac{d}{dt} M(t)^T M(t) = M'(t)^T M(t) + M(t)^T M'(t) = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}^T M(t)^T M(t) + M(t)^T M(t) \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}$$

and the linear system

$$\frac{d}{dt} A(t) = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}^T A(t) + A(t) \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}$$

with the initial condition $A(0) = I$ has a unique solution $A(t) = I$, $t \in [-1,1]$. Thus $M(t)^T M(t) = I$ for all $t \in [-1,1]$, and so $T(t), N(t)$ is an orthonormal frame for all $t \in [-1,1]$. $\gamma(t) = \int_0^t T(t)dt$ gives a curve $\gamma : [-1,1] \to \mathbb{R}^2$ with the properties claimed.

(ii) [D, 4] If there was such a curve, then $\int_{-1}^{1} \kappa(t)dt = 2\pi \text{Ind}(\gamma)$. But the LHS is a non-zero rational number, whereas the RHS is an integer multiple of π. Therefore there is no smooth *closed* curve $\gamma : [-1,1] \to \mathbb{R}^2$ parametrised by arc-length and (signed) curvature equal to $\kappa(t) = t^3 - t$ for all $t \in [-1,1]$.

5. Let $\gamma : [0, L] \to \mathbb{R}^3$ be a simple (i.e. γ is injective) closed curve, parametrised by arc-length and positive curvature $\kappa(s) > 0$ at all points $\gamma(s)$, $s \in [0, L]$.

(a) Using the Gauss-Bonnet theorem, or otherwise, show that if the curve γ is contained in a plane then its total curvature satisfies $\int_0^L \kappa(s)ds = 2\pi$.

(b) Let r be a positive constant and $\phi : [0, L] \times [0, 2\pi] \to \mathbb{R}^3$ denote the smooth map given by

$$\phi(s, t) = \gamma(s) + r(\cos(t)n(s) + \sin(t)b(s))$$

where $n(s)$ and $b(s)$ denote the unit normal and bi-normal vectors of γ at the point $\gamma(s)$.

(i) Show that for $r > 0$ sufficiently small, the image of ϕ is a regular surface $T \subset \mathbb{R}^3$ (called the *Tube* around γ) with a regular coordinate chart given by ϕ.

(ii) Show that the Gauss map of T is given by $N(s, t) = -(\cos(t)n(s) + \sin(t)b(s))$.

(iii) Show that the Gaussian curvature $K(p)$ at the point $p = \phi(s, t) \in T$ is equal to

$$K(p) = \frac{-k\cos(t)}{r(1 - r\kappa(s)\cos(t))}$$

Conclude that $K = 0$ if and only if the line through $b(s)$ is orthogonal to the tube.

(c) Let $R = \{p \in T : K(p) \geq 0\} \subset T$ be a regular surface (with boundary) in $\mathbb{R}^3$.

(i) Show that $\int_R K dA = 2\int_0^L \kappa(s)ds$.

(ii) Show that the Gauss map $N|_R : R \to \mathbb{S}^2$ is surjective. (Hint: Given a unit vector $v \in \mathbb{S}^2$, consider the function $h : S \to \mathbb{R}$, given by $h(s) = \gamma(s) \cdot v$.)

(iii) Using parts (i) and (ii) above, or otherwise, show that the total curvature of γ satisfies $\int_0^L \kappa(s)ds \geq 2\pi$. (Hint: $\int_R K dA \geq 4\pi$.)

Solution 5.

(a) [4] Let S be a regular surface in the plane with boundary γ. Since the normal curvature is zero (Gauss map is constant), the curvature of γ is equal to its geodesic curvature. Using the Gauss-Bonnet theorem and the fact that the Gaussian curvature of a planar surface is zero, we conclude that $\int_0^L \kappa(s)ds = \int_\gamma \kappa_g(s)ds = 2\pi\chi(S)$ and in particular $\chi(S) > 0$. Since S has a single boundary component, by the classification of surfaces, 'capping' up S with a disk we obtain a closed surface with positive Euler characteristic, so it is a sphere with χ equal to two; hence $\chi(S) = 1$, and $\int_0^L \kappa(s)ds = 2\pi$. (Alternatively, one can argue that the curve is convex hence its index is one).

(b) (i) [2] Let (t, n, b) be the Frenet frame of γ. Given s, if for all ϵ there is a solution $\phi(s, t_1) = \phi(s+\epsilon, t_2)$ for some t_1, t_2, using the Taylor series $\gamma(s+\epsilon) = \gamma(s)+\epsilon t(s)+$(quadratic terms) and the Frenet equation $n'(s) = -\kappa(s)t(s) + \tau(s)b(s)$, we conclude that $r \geq 1/\kappa(s)$ (by comparing the linear part of coefficients of $t(s)$ in both sides). We conclude that if $r \in (0, \inf_{s\in[0,L]} 1/\kappa(s))$ then ϕ is locally injective in $s \in [0, L]$. By compactness of $[0, L]$ and the triangle inequality (if $\phi(s_1, t) = \phi(s_2, t)$ then $|\gamma(s_1) - \gamma(s_2)| \leq 2r$), we conclude that if r is sufficiently small then ϕ is a regular chart for the tube T of γ. It is clear that γ gives a diffeomorhism from $[0, L]/(0 \sim L) \times [0, 2\pi]/(0 \sim 2\pi)$ to the regular surface $T \subset \mathbb{R}^3$.

(ii) [2] Compute $\phi_t = -r\sin(t)n(s) + r\cos(t)b(s)$ and $\phi_s = (1 - r\kappa(s)\cos(t))t(s) + \tau(s)\phi_t$. Hence $N(s,t) = \frac{\phi_s \times \phi_t}{|\phi_s \times \phi_t|} = -(\cos(t)n(s) + \sin(t)b(s))$ and $|\phi_s \times \phi_t| = r(1 - r\kappa(s)\cos(t))$.

(iii) [3] Note that $\phi(s,t) = \gamma(s) - rN(s,t)$, $N_t = \sin(t)n(s) - \cos(t)b(s)$ and $N_s = \kappa(s)\cos(t)t(s) - \tau(s)\cos(t)b(s) - \tau(s)\sin(t)n(s)$; hence, $K(\phi(s,t)) = \frac{(\phi_s \cdot N_s)(\phi_t \cdot N_t) - (\phi_s \cdot N_t)^2}{|\phi_s \times \phi_t|^2} = \frac{-\kappa(s)\cos(t)}{r(1 - r\kappa(s)\cos(t))}$.
We readily see $K = 0$ iff $\cos(t) = 0$ iff the line through $b(s)$ is orthogonal to the tube.

(c) (i) [1] Since $R = S \cap \{t \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$, we have $\int_R K dA = \int_0^L \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{-\kappa(s)\cos(t)}{r(1 - r\kappa(s)\cos(t))} dt ds = 2\int_0^L \kappa(s) ds$.

(ii) [4] Given a unit vector $v \in \mathbb{S}^2$, consider the function $h : S \to \mathbb{R}$, given by $h(x,y,z) = (x,y,z) \cdot v$. The maximum of this function must occur in some $p \in S$, with $N(p) = \nabla h = v$; the plane $\{h(x,y,z) = h(p)\}$ is tangent to the surface S and since h is maximized at p the surface must lie entirely on the half-space $\{h(x,y,z) \le h(p)\}$; hence, the curvature of S at p cannot be negative and therefore $p \in R$. In order words, $N|_R : R \to \mathbb{S}^2$ is surjective.

(iii) [4] From parts (i) and (ii), it suffices to show $\int_R K dA \ge 4\pi$. Let $\phi : U \to R$ be a chart around $p \in R$ and shrink U so that N restricts to a diffeomorphism of $\phi(U) \subset R$ onto its image. Then $\psi = N \circ \phi : U \to \mathbb{S}^2$ is a chart for $\mathbb{S}^2$ at $N(p)$, and we recognise that $K(\phi(u,v)) = \det dN_{\phi(u,v)}$, so from $\psi = N \circ \phi$, we get $\int_{\phi(F)} K dA = \int_F det(dN_\phi)|\phi_u \times \phi_v| dudv = \int_F |\psi_u \times \psi_v| dudv =$ area$(\psi(F))$ for any compact subset $F \subset U$. If we cover S by such compact sets $\phi(F)$, then some of them may overlap, but their images under N cover all of $\mathbb{S}^2$ and so we conclude that $\int_R K dA \ge$ area$(\mathbb{S}^2) = 4\pi$.

No comments provided