Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Biological Fluid Mechanics

Date: Wednesday, 18 May 2011. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. The flow of blood down a curved artery of circular cross-section is modelled by the steady Dean equations,

$$\frac{K}{r}\left(\psi_\theta v_r - \psi_r v_\theta\right) = 4 + \nabla^2 v$$

$$\frac{K}{r}\left(\psi_\theta \Omega_r - \psi_r \Omega_\theta\right) = \nabla^2 \Omega - 2Kv\left(v_r \sin\theta + \frac{v_\theta}{r}\cos\theta\right)$$

$$\Omega = -\nabla^2\psi \equiv -\left(\psi_{rr} + \frac{1}{r}\psi_r + \frac{\psi_{\theta\theta}}{r^2}\right).$$

In terms of polar coordinates (r, θ), the down-pipe velocity is $v(r, \theta)$, while the cross-pipe flow has streamfunction $\psi(r, \theta)$.

(a) Explain the assumptions leading to these equations and identify the key terms. How does the Dean number K relate to the ordinary Reynolds number?

(b) When K is small an expansion is sought of the form

$$v = v_0 + Kv_1 + K^2 v_2 + \ldots \quad \psi = \psi_0 + K\psi_1 + \ldots \quad \Omega = \Omega_0 + K\Omega_1 + \ldots$$

Equating powers of K, find v_0, ψ_0, v_1 assuming no-slip boundary conditions on $r = 1$. (You may assume uniqueness, so that if a zero solution is possible, it is the only solution.)

(c) Show that Ω_1 satisfies the equation

$$\nabla^2\Omega_1 = 4(r^3 - r)\sin\theta,$$

and find its general solution by considering functions of the form $\Omega_1 \sim r^n \sin\theta$.

Hence find ψ_1. Sketch the contours of ψ_1, indicating with an arrow the direction of the flow.

(d) Obtain an equation which determines v_2 and describe the nature of the r and θ dependence of its solution. (You need not find v_2 exactly.)

2. Explain why it might benefit a spherical swimming organism to have its centre of mass displaced a distance h from its geometrical centre.

Such an organism is placed in a weak flow with local vorticity $\boldsymbol{\omega}$. Show that if the organism attempts to swim upwards (in the $\widehat{\mathbf{z}}$-direction) it actually swims in the approximate direction

$$\widehat{\mathbf{p}} = \widehat{\mathbf{z}} + B\,\widehat{\mathbf{z}} \wedge \boldsymbol{\omega},$$

where B is a parameter you should define.
(You may assume a sphere of small radius a in a fluid of viscosity μ feels a torque $4\pi\mu a^3\boldsymbol{\omega}$.)

A suspension of bacteria, with concentration c, swimming with uniform speed V_0, satisfies the equations

$$\begin{aligned}
\nabla\cdot\mathbf{u} &= 0\\
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla P + \alpha g c\,\widehat{\mathbf{z}} + \nu\nabla^2\mathbf{u}\\
c_t + \mathbf{u}\cdot\nabla c &= -V_0\nabla\cdot(c\widehat{\mathbf{p}}) + D\nabla^2 c.\\
\widehat{\mathbf{p}} &= \widehat{\mathbf{z}} + B\,\widehat{\mathbf{z}}\wedge(\nabla\wedge\mathbf{u})
\end{aligned}$$

Explain the origin of the significant terms in each equation.

The uniform state $c = c_0$, $\widehat{\mathbf{p}} = \widehat{\mathbf{z}}$ is perturbed with z-independent perturbations in the form

$$c = c_0 + \zeta c_1, \quad \mathbf{u} = \zeta\mathbf{u}_1, \quad P = \alpha g c_0 z + \zeta P_1, \quad \widehat{\mathbf{p}} = \widehat{\mathbf{z}} + \zeta\widehat{\mathbf{p}}_1$$

$$\text{where} \qquad \zeta = \varepsilon\exp(ikx + ily + st) \qquad \text{with} \quad 0 < \varepsilon \ll 1.$$

Show that the z-component of the momentum equation reduces to

$$(s + \nu\kappa^2)W = \alpha g c_1 \qquad \text{where} \quad \kappa^2 = k^2 + l^2,$$

while the bacterial equation requires

$$(s + D\kappa^2)c_1 = V_0 c_0 B\kappa^2 W,$$

where $W = \mathbf{u}_1\cdot\widehat{\mathbf{z}}$.
Deduce that instability occurs for all values of $B > 0$ provided that $\kappa < \kappa_c$, where κ_c is to be found.

3. (a) Explain why birds of a similar design have a maximum size. You may neglect the induced drag and assume that the metabolic rate of all animals scales as body mass to the $\frac{3}{4}$ power.

(b) Give a brief account of the generation of induced drag in steady flight.

(c) A bird at position $(X(t), Y(t), Z(t))$ flies with velocity (u, v, w) relative to the air. There is a wind with the vertical shear $(U(z), 0, 0)$. By considering the fictitious force acting when working in the rest frame of the bird, show that the bird is able to extract energy from the shear provided that on average uwU' is negative.

Sketch a typical efficient flight path for the bird, assuming $U > 0$ and $U' > 0$.

(d) Describe, with appropriate diagrams, the essence of the "clap and fling" mechanism of insect flight.

4. Describe the assumptions behind "Resistive Force Theory" and explain how it enables the propulsive force of a flagellum undergoing prescribed movement to be calculated.

A long thin needle-shaped organism of length l moves with speed U through fluid of viscosity μ at low Reynolds number. The drag is $2k\mu lU$ if it moves perpendicular to its axis and $k\mu lU$ if it moves parallel to its axis, where k is a dimensionless constant.

The needle is inclined at an angle α to the vertical and falls under gravity. Use the linearity of the Stokes equations to show that its velocity makes an angle β to the vertical where

$$\tan(\alpha - \beta) = \tfrac{1}{2}\tan\alpha \ .$$

(You may assume the organism does not rotate as it falls.) Show that the maximum possible value of β is given by $\tan\beta = 1/(2\sqrt{2})$ and that this occurs when $\tan\alpha = \sqrt{2}$. For this maximal value of β show that the speed of the organism is

$$U = \frac{mg}{\sqrt{2}k\mu l},$$

where mg is the weight of the organism, adjusted for buoyancy.

$$\left[\text{Recall that} \quad \cos x = \frac{1}{(1+\tan^2 x)^{1/2}} \quad \text{and} \quad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\ .\right]$$