Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2014

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3/4/5 S4

Applied Probability

Date: Wednesday, 14th May 2014 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Let $\mathbf{P}$ denote a $K \times K$–dimensional stochastic matrix for a $K \in \mathbb{N}$. Show that for all $n \in \mathbb{N}$ the matrix $\mathbf{P}^n$ is also a stochastic matrix.

(b) Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1,2,3,4,5,6,7,8\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(ii) Find all possible stationary distributions.

Please note that you need to justify your answers in (i)-(ii).

(c) Suppose that the weather in London can be modelled by a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$, where the state space consists of three states $E = \{1,2,3\}$, where the three states can be interpreted as follows: State 1: Rain; state 2: sun; state 3: fog. We assume the following:

– There are never two sunny days in a row. If it is a sunny day, then it is equally likely that the next day will be rainy or foggy.

– If it is rainy or foggy, then there is a 50% chance of having the same weather condition the next day.

– If the weather changes from rain or fog, only half of the time is this a change to a sunny day.

Write down the transition matrix $\mathbf{P}$. Which information would you need in order to fully specify the dynamics of this Markov chain?

2. (a) Consider a discrete-time homogeneous Markov chain on a countable state space. Show that the concept of communication is an equivalence relation.

   (b) Show that a Poisson process satisfies the Markov property.

   (c) Let $N = (N_t)_{t \geq 0}$ denote a Poisson process with rate $\lambda > 0$. Let $(Y_i)_{i \in \mathbb{N}}$ denote a sequence of i.i.d. random variables with cumulative distribution function given by

$$F_Y(y) := \mathbb{P}(Y_1 \leq y) = \begin{cases} 0, & \text{for } y < 0, \\ y^{\alpha}, & \text{for } 0 \leq y < 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

   where $\alpha > 0$. Suppose N and $(Y_i)_{i \in \mathbb{N}}$ are independent. Find

$$\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y | N_t > 0), \text{ for } t > 0, 0 < y < 1.$$

3. (a) Let $\lambda > 0$. Consider a continuous-time counting process $\{N_t\}_{t\geq 0}$ with values in $\mathbb{N}_0$ defined by the following three axioms: (1) $N_0 = 0$. (2) N has independent and stationary increments. (3) There is a 'single arrival', i.e. for any $t \geq 0$ and $\delta > 0$:

$$\begin{aligned}\mathbb{P}(N_{t+\delta} - N_t = 1) &= \lambda\delta + o(\delta)\\ \mathbb{P}(N_{t+\delta} - N_t \geq 2) &= o(\delta)\end{aligned}$$

Let $p_n(t) = \mathbb{P}(N_t = n)$ for $t \geq 0, n \in \mathbb{N}_0$.

(i) Using these axioms, show that the forward equations are given by

$$\begin{aligned}\frac{dp_0(t)}{dt} &= -\lambda p_0(t),\\ \frac{dp_n(t)}{dt} &= -\lambda p_n(t) + \lambda p_{n-1}(t), \qquad \text{for } n \geq 1.\end{aligned}$$

(ii) State the solution to the forward equations.

(iii) Prove by induction that your solution specified in (ii) is indeed a solution to the forward equations.
Hint: Recall that a one-dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t)f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g and initial condition $f(0) = C$ has solution

$$f(t) = \frac{\int_0^t g(u)M(u)du + C}{M(t)},$$

where M is the integrating factor $M(t) = \exp\left\{\int_0^t \alpha(u)du\right\}$.

(b) (i) Define a compound Poisson process.

(ii) Let $X = (X_t)_{t\geq 0}$ denote a compound Poisson process. For $0 < s < t$, find the characteristic function of the increment $X_t - X_s$. Does X have stationary increments?

4. (a) Customers arrive at a post office according to a Poisson process of rate 5. Three post office clerks serve the customers at three separate service desks. Their service times are independent and exponentially distributed with rate parameter 2. If an arriving customer finds all clerks busy, he immediately leaves without receiving any service.

   (i) For $t \geq 0$, let X_t denote the number of post office clerks who are busy (serving a customer) at time t. Suppose that $X = (X_t)_{t \geq 0}$ is a continuous-time Markov chain with state space $E = \{0, 1, 2, 3\}$. Find the generator associated with this Markov chain.

   (ii) Find the transition matrix of the jump chain associated with this Markov chain.

(b) Define a standard Brownian motion.

(c) Let $B = (B_t)_{t \geq 0}$ denote a standard Brownian motion. Let $a > 0$ denote a deterministic constant. Show that $W = (W_t)_{t \geq 0}$ with $W_t = aB_{t/a^2}$ is a standard Brownian motion.

(d) State the reflection principle for Brownian motion.

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2014

This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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1. (a) Recall that the stochastic matrix $\mathbf{P}$ must have nonnegative entries and needs to satisfy $\sum_{j=1}^{K} p_{ij} = 1$ for all $i = 1, \ldots, K$. We can write the latter condition in matrix notation: seen ⇓

$$\mathbf{P}\mathbf{1} = \mathbf{1}, \tag{1}$$

where $\mathbf{1}$ is the column vector of 1's.

We use induction to show that $\mathbf{P}^n$ is a stochastic matrix. We already know that this is true for $n = 1$. Then for $n+1$ we have

$$\mathbf{P}^{n+1}\mathbf{1} = \mathbf{P}\underbrace{\mathbf{P}^n\mathbf{1}}_{=\mathbf{1}} = \mathbf{1},$$

where we used that according to the induction hypothesis $\mathbf{P}^n$ is a stochastic matrix (which satisfies (1)). In addition, since all elements of $\mathbf{P}$ are nonnegative, this is also true for the elements of $\mathbf{P}^n$ (by induction hypothesis) and the product $\mathbf{P}\mathbf{P}^n$ stays non-negative, which concludes the proof. 4

(b) (i) We have a finite state space which can be divided into six communicating classes: The classes $T_1 = \{1\}, T_2 = \{2\}, T_3 = \{3\}, T_4 = \{4\}$ are not closed and hence transient. sim. seen ⇓ 2

The classes $C_1 = \{5, 6\}$ and $C_2 = \{7, 8\}$ are finite and closed and hence positive recurrent. 2

(ii) Note that we do not have a unique stationary distribution since we have two closed (essential) communicating classes.

Let π denote the vector of all stationary distributions. According to lectures, we know that $\pi_i = 0$ for all transient states i. I.e. $\pi_1 = \pi_2 = \pi_3 = \pi_4$. 2

We determine the remaining components by solving two systems of equations: We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}, \qquad \mathbf{P}(C_2) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We need to solve $(\pi_5, \pi_6)\mathbf{P}(C_1) = (\pi_5, \pi_6)$ which results in $\pi_5 = 0.2\pi_6$. 2

Also, solving $(\pi_7, \pi_8)\mathbf{P}(C_2) = (\pi_7, \pi_8)$ leads to $\pi_7 = \pi_8$. 2

Then all possible stationary distributions are given by $\pi := (0, 0, 0, 0, 0.2\pi_6, \pi_6, \pi_7, \pi_7)$ for constants $\pi_6, \pi_7 \geq 0$ such that $1.2\pi_6 + 2\pi_7 = 1$ since $\pi_i \geq 0$ for $i = 1, \ldots, 8$ and $\sum_{i=1}^{8} \pi_i = 1$. Also $\pi = \pi\mathbf{P}$. 2

(c) The transition matrix is given by unseen ⇓

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$

3

We would need to know the initial distribution $\nu^{(0)} := (\mathbb{P}(X_0 = 1), \mathbb{P}(X_0 = 2), \mathbb{P}(X_0 = 3))$ in order to specify the dynamics of the Markov chain fully. 1

2. (a) First we introduce the notation we use in the proof. Let E denote the state space of the Markov chain and let $p_{ij}(n)$ denote the n-step transition probability to go from state i to state j in n steps (for $i, j \in E$ and $n \in \mathbb{N}$). Recall: We say that state j is *accessible* from state i, written $i \to j$, if the chain may ever visit state j, with positive probability, starting from i. In other words, $i \to j$ if there exist $m \geq 0$ such that $p_{ij}(m) > 0$. Also, i and j *communicate* if $i \to j$ and $j \to i$, written $i \leftrightarrow j$. seen ⇓ 1

1. Reflexivity ($i \leftrightarrow i$): Note that we have $p_{ii}(0) = 1$ since

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

1

2. Symmetry (if $i \leftrightarrow j$, then $j \leftrightarrow i$): This follows directly from the definition. 1
3. Transitivity (if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$): $i \leftrightarrow j$ and $j \leftrightarrow k$ imply that there exist integers $n, m \geq 0$ such that $p_{ij}(n) > 0$ and $p_{jk}(m) > 0$. Hence (using the Chapman-Kolmogorov equations and the positivity of the transition probabilities)

$$p_{ik}(n+m) = \sum_{l \in E} p_{il}(n) p_{lk}(m) \geq p_{ij}(n) p_{jk}(m) > 0 \quad \Rightarrow i \to k.$$

Similarly, one can show that $i \leftarrow k$. 3

(b) For the Poisson process the state space is given by $E = \mathbb{N}_0$. Now we choose any $n \in \mathbb{N}$ and any sequence $0 \leq t_1 < \cdots < t_n < \infty$ and any states $j, i_1, \ldots, i_{n-1} \in E$ (such that the following conditional probability is well-defined). Then sim. seen ⇓ 1

$$A := \mathbb{P}(N_{t_n} = j | N_{t_1} = i_1, \ldots, N_{t_{n-1}} = i_{n-1}) = \frac{\mathbb{P}(N_{t_n} = j, N_{t_1} = i_1, \ldots, N_{t_{n-1}} = i_{n-1})}{\mathbb{P}(N_{t_1} = i_1, \ldots, N_{t_{n-1}} = i_{n-1})}.$$

Note that the set of equalities

$$N_{t_1} = i_1, \; \ldots, \; N_{t_{n-1}} = i_{n-1}, \; N_{t_n} = j$$

is equivalent to

$$N_{t_1} = i_1, N_{t_2} - N_{t_1} = i_2 - i_1 \ldots, N_{t_n} - N_{t_{n-1}} = j - i_{n-1}.$$

Hence, we get

$$\begin{aligned} A &= \frac{\mathbb{P}(N_{t_1} = i_1, N_{t_2} - N_{t_1} = i_2 - i_1 \ldots, N_{t_n} - N_{t_{n-1}} = j - i_{n-1})}{\mathbb{P}(N_{t_1} = i_1, N_{t_2} - N_{t_1} = i_2 - i_1 \ldots, N_{t_{n-1}} - N_{t_{n-2}} = i_{n-1} - i_{n-2})} \\ &= \frac{\mathbb{P}(N_{t_1} = i_1)\mathbb{P}(N_{t_2} - N_{t_1} = i_2 - i_1) \ldots \mathbb{P}(N_{t_n} - N_{t_{n-1}} = j - i_{n-1})}{\mathbb{P}(N_{t_1} = i_1)\mathbb{P}(N_{t_2} - N_{t_1} = i_2 - i_1) \ldots \mathbb{P}(N_{t_{n-1}} - N_{t_{n-2}} = i_{n-1} - i_{n-2})} \end{aligned}$$

where we used the fact that the increments are independent. Most of the terms cancel, and we obtain

$$A = \mathbb{P}(N_{t_n} - N_{t_{n-1}} = j - i_{n-1}).$$

Also, we have 3

$$\begin{aligned} B := \mathbb{P}(N_{t_n} = j | N_{t_{n-1}} = i_{n-1}) &= \frac{\mathbb{P}(N_{t_n} = j, N_{t_{n-1}} = i_{n-1})}{\mathbb{P}(N_{t_{n-1}} = i_{n-1})} \\ &= \frac{\mathbb{P}(N_{t_n} - N_{t_{n-1}} = j - i_{n-1}, N_{t_{n-1}} = i_{n-1})}{\mathbb{P}(N_{t_{n-1}} = i_{n-1})} \\ &= \frac{\mathbb{P}(N_{t_n} - N_{t_{n-1}} = j - i_{n-1})\mathbb{P}(N_{t_{n-1}} = i_{n-1})}{\mathbb{P}(N_{t_{n-1}} = i_{n-1})} = \mathbb{P}(N_{t_n} - N_{t_{n-1}} = j - i_{n-1}). \end{aligned}$$

where we used the independent increment property again. Hence $A = B$, so the Markov property holds. $\boxed{3}$

Note that the conditional probabilities in A is well-defined for states $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-1}$ in E. $\boxed{\text{unseen} \Downarrow}$

(c) Let $t > 0, 0 < y < 1$. Then

$$\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y | N(t) > 0) = \frac{\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y, N(t) > 0)}{\mathbb{P}(N(t) > 0)}.$$

Note that

$$\mathbb{P}(N(t) > 0) = 1 - e^{-\lambda t}.$$

Also, using the law of total probability, we have

$$\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y, N(t) > 0) = \sum_{n=1}^{\infty} \mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y, N(t) = n)$$
$$= \sum_{n=1}^{\infty} \mathbb{P}(\min(Y_1, Y_2, \ldots, Y_n) > y, N(t) = n) = \sum_{n=1}^{\infty} \mathbb{P}(\min(Y_1, Y_2, \ldots, Y_n) > y)\mathbb{P}(N(t) = n),$$

where we used the independence of N and $(Y_i)_{i \in \mathbb{N}}$. Also, since the $(Y_i)_{i \in \mathbb{N}}$ are i.i.d. we have

$$\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_n) > y) = (\mathbb{P}(Y_1 > y))^n = (1 - y^{\alpha})^n.$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{P}(\min(Y_1, Y_2, \ldots, Y_n) > y)\mathbb{P}(N(t) = n) = \sum_{n=1}^{\infty} (1 - y^{\alpha})^n \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$
$$= (\exp((1 - y^{\alpha})\lambda t) - 1)\exp(-\lambda t) = \exp(-y^{\alpha}\lambda t) - \exp(-\lambda t).$$

Altogether, we have

$$\mathbb{P}(\min(Y_1, Y_2, \ldots, Y_{N_t}) > y | N(t) > 0) = \frac{\exp(-y^{\alpha}\lambda t) - \exp(-\lambda t)}{1 - \exp(-\lambda t)}.$$

$\boxed{7}$

3. (a) (i) For $n = 0$, we have seen ⇓

$$\begin{aligned}p_0(t+\delta) &= \mathbb{P}(N_{t+\delta} = 0) = \mathbb{P}(\text{no event in}[0, t+\delta])\\ &= \mathbb{P}(\text{no event in}[0,t] \text{ and no event in}(t, t+\delta])\\ &= \mathbb{P}(\text{no event in}[0,t])\mathbb{P}(\text{no event in}(t,t+\delta])\\ &= p_0(t)[1-\lambda\delta + o(\delta)],\end{aligned}$$

where we used the independent increments property. Then we have

$$\frac{p_0(t+\delta) - p_0(t)}{\delta} = -\lambda p_0(t) + \frac{o(\delta)}{\delta}$$

letting $\delta \downarrow 0$ we get

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t).$$

2

For $n \geq 1$ (i.e. $n \in \mathbb{N}$) we have

$$\begin{aligned}p_n(t+\delta) &= \mathbb{P}(N_{t+\delta} = n)\\ &= \sum_{k=0}^{\infty} \mathbb{P}(N_{t+\delta} = n | N_t = k)\mathbb{P}(N_t = k) \quad \text{(Law of total probability)}\\ &= \sum_{k=0}^{\infty} \mathbb{P}((n-k) \text{ events in } (t, t+\delta])\mathbb{P}(N_t = k)\\ &= \mathbb{P}(1 \text{ event in } (t,t+\delta])\mathbb{P}(N_t = n-1) + \mathbb{P}(0 \text{ events in } (t,t+\delta])\mathbb{P}(N_t = n) + o(\delta)\\ &= p_{n-1}(t)\lambda\delta + p_n(t)(1-\lambda\delta) + o(\delta) = p_n(t)(1-\lambda\delta) + p_{n-1}(t)\lambda\delta + o(\delta).\end{aligned}$$

Re-arranging and letting $\delta \downarrow 0$ we have

$$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t).$$

3

(ii) The solution to the forward equations is given by seen ⇓

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \text{ for } n \in \mathbb{N}_0.$$

1

(iii) Proof by induction: Case $n = 0$. From the first axiom, we get that $p_0(0) = \mathbb{P}(N_0 = 0) = 1$. Solving the first forward equation leads to $p_0(t) = e^{-\lambda t} = \frac{(\lambda t)^0}{0!} e^{-\lambda t}$. sim. seen ⇓

1

Induction step from n to $n+1$: By the induction hypothesis we have that $p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. In order to find p_{n+1}, we need to solve

$$\frac{dp_{n+1}(t)}{dt} = -\lambda p_{n+1}(t) + \lambda \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Using the hint, the integrating factor is given by $M(t) = \exp(\lambda t)$ and $p_{n+1}(0) = 0$ (by the first axiom). Then

$$p_{n+1}(t) = e^{-\lambda t} \int_0^t \lambda \frac{(\lambda u)^n}{n!} e^{-\lambda u} e^{\lambda u} du = \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t},$$

which completes the proof.

3

(b) (i) Let $N = (N_t)_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$. In addition, let $Y_1, Y_2, \dots$ be a sequence of independent and identically distributed random variables, that are independent of N. Then the process $X = (X_t)_{t\geq 0}$ with

$$X_t = \sum_{k=1}^{N_t} Y_k$$

is a compound Poisson process. [seen ⇓] [2]

(ii) Let $\theta \in \mathbb{R}$. We use the same notation as in (i). Then the characteristic function of the increment is given by [unseen ⇓]

$$\begin{aligned}
\mathbb{E}(\exp(i\theta(X_t - X_s)) &= \mathbb{E}\left(\exp\left(i\theta\left(\sum_{k=N_s}^{N_t} Y_k\right)\right)\right)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\mathbb{E}\left(\exp\left(i\theta\sum_{k=N_s}^{N_t} Y_k\middle|\{N_s = n, N_t = m\}\right)\right)\mathbb{P}(N_s = n, N_t = m)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\mathbb{E}\left(\exp\left(i\theta\sum_{k=n}^{m} Y_k\right)\right)\mathbb{P}(N_s = n, N_t = m)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left[\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\right]^{m-n}\mathbb{P}(N_s = n)\mathbb{P}(N_t - N_s = m-n)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left[\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\right]^{m-n}\frac{e^{-\lambda s}(\lambda s)^n}{n!}\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{(m-n)}}{(m-n)!}\\
&= e^{-\lambda t}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left[\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\right]^{m-n}\frac{(\lambda s)^n}{n!}\frac{(\lambda(t-s))^{(m-n)}}{(m-n)!}\frac{m!}{m!}\\
&= e^{-\lambda t}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{m}\binom{m}{n}\left[\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\right]^{m-n}(\lambda(t-s))^{(m-n)}(\lambda s)^n\\
&= e^{-\lambda t}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\lambda(t-s) + \lambda s\right]^m\\
&= \exp\left(-\lambda t + \mathbb{E}\left(\exp\left(i\theta Y_1\right)\right)\lambda(t-s) + \lambda s\right) = \exp\left(\lambda(t-s)(\mathbb{E}\left(\exp\left(i\theta Y_1\right)\right) - 1\right),
\end{aligned}$$

where we used the definition of the conditional expectation, the independence of N and (Y_i), the independent and stationary increment property of N and the fact that the (Y_i) are i.i.d.. [6]

Since the characteristic function depends on t and s only through the difference $t-s$, we can conclude that X has stationary increments. [2]

4. (a) (i) This Markov chain is a birth-death process. The birth rates are determined by the arrival rate of the Poisson process. I.e. $\lambda_0 = \lambda_1 = \lambda_2 = 5$, when all three clerks are busy, there cannot be another "birth", hence $\lambda_3 = 0$. Since the service times are independent of each other and follow an exponential clock of rate 2, we get the following death rates $\mu_i = 2i$ for $i = 0, 1, 2, 3$. According to the lectures, the generator is then given by

sim. seen ⇓

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) \end{pmatrix} = \begin{pmatrix} -5 & 5 & 0 & 0 \\ 2 & -7 & 5 & 0 \\ 0 & 4 & -9 & 5 \\ 0 & 0 & 6 & -6 \end{pmatrix}.$$

6

(ii) Recall that the transition probabilities, denoted by p_{ij} for $i, j \in E$, of the corresponding jump chain are given by $p_{ij} = g_{ij}/(-g_{ii})$ for $i \neq j$ for $-g_{ii} > 0$. Noting that the row elements in the transition matrix have to sum up to one, we get that the transition matrix of the corresponding jump chain is given by

sim. seen ⇓

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/7 & 0 & 5/7 & 0 \\ 0 & 4/9 & 0 & 5/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

4

(b) A real-valued stochastic process $B = (B_t)_{t\geq 0}$ is a *standard Brownian motion* if (1) $B_0 = 0$ almost surely; (2) B has independent increments; (3) B has stationary increments; (4) The increments are Gaussian: For $0 \leq s < t$: $B_t - B_s \sim N(0, (t-s))$; (5) The sample paths are almost surely continuous, i.e. the function $t \to B_t$ is almost surely continuous in t.

seen ⇓

3

(c)
1. $W_0 = aB_{0/a^2} = aB_0 = 0$, since $B_0 = 0$.
2. Choose any $n \in \mathbb{N}$ and any $0 \leq t_1 < t_2 < \cdots < t_n$. Set $s_i = t_i/a^2$, $i = 1, \ldots, n$. Then $0 \leq s_1 < s_2 < \cdots < s_n$. Since B has independent increments, we know that $B_{s_1}, B_{s_2} - B_{s_1}, \ldots, B_{s_n} - B_{s_{n-1}}$ are independent. Multiplying the increments by the deterministic constant a, we get that $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$, are independent.
3. For $0 \leq s < t$, we have that $W_t - W_s = a(B_{t/a^2} - B_{s/a^2}) \overset{\text{law}}{=} aB_{(t-s)/a^2} = W_{t-s}$, since B has stationary increments.
4. For $0 \leq s < t$, we have that $W_t - W_s \overset{\text{law}}{=} aB_{(t-s)/a^2} \sim N(0, t-s)$, since $B_{(t-s)/a^2} \sim N(0, (t-s)/a^2)$.
5. The continuity of the sample paths $t \mapsto W_t$ is a direct consequence of the continuity of the paths of B.

sim. seen ⇓

5

(d) Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion. Let $x > 0$ and $\tau := \min\{s : B_s \geq x\}$. Then the reflection principle says that the stochastic process $B'' = (B''_t)_{t\geq 0}$ defined by

seen ⇓

$$B''_t = \begin{cases} B_t & \text{if} \quad t \leq \tau \\ x - (B_t - x) & \text{if} \quad t > \tau \end{cases}$$

is a (standard) Brownian motion.

2