Imperial College London

MATH40004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: 16 May 2023

Time: 14:00 – 17:00 (BST)

Time Allowed: 3hrs

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) (i) Let $h(x) = (f \circ g)(x)$ denote the usual composition function. Calculate $\frac{dh}{dx}$ and $\frac{d^2h}{dx^2}$ giving your answers in terms of derivatives of f and g. (4 marks)

(ii) If $f(x) = e^{-x}$ and $g(x) = x^3$ find all local maxima/minima and points of inflection of $h(x)$ in this case. (4 marks)

(iii) Sketch the function $h(x)$ defined in part (ii) above. (2 marks)

(b) (i) State, without proof, Taylor's theorem giving the remainder in its local form, i.e. not as an integral. (3 marks)

(ii) Find the first two non-zero terms of the Taylor expansion of $f(x) = x \log x$ about $x = 1$, and give the remainder stating clearly any variables that you use. (4 marks)

(iii) Use the expansion found above to determine a rational approximation $1.1 \log 1.1$ and give a bound for the error made in using this approximation. (3 marks)

(Total: 20 marks)

2. (a) Consider the parabola $y = x^2$, for $x \geq 0$.

   (i) The parabola is revolved about the x and y axes, and the resulting surface areas of revolution are denoted by S_x and S_y, respectively. Find expressions for these for the segment $x \in [0, L]$ for some $L > 0$, and order them according to size as L varies. [Note: Avoid calculating integrals explicitly if you can.] (4 marks)

   (ii) As in part (i), the interval $x \in [0, L]$, $L > 0$ of the parabola, is revolved about the x and y axes, and the volumes of the resulting solids swept out by the area between the parabola and the $x-$axis, are denoted by V_x and V_y, respectively.
   Calculate V_x and V_y and order them according to the value of L, and determine whether it is possible to find a $L = L_0$ such that $V_x(L_0) = V_y(L_0)$. (4 marks)
   Explain your findings geometrically when $L < L_0$ and $L > L_0$. (2 marks)

(b) Given a function $f(t)$, $t \geq 0$, a new function $F(s)$ is defined by the linear transformation

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \qquad s > 0. \tag{1}$$

   (i) Find $F(s)$ for each of the functions $f_1(t) = t$ and $f_2(t) = \sin \omega t$, where the frequency ω is real. (5 marks)

   (ii) For a function $g(t)$, $t \geq 0$, it is given that its corresponding $F(s)$ according to the definition (1) above, is

$$F(s) = \frac{1}{s^2(s^2 + 4)}.$$

   Find $g(t)$ using the results in part 2(b)(i) above. (5 marks)

(Total: 20 marks)

3. (a) Given the series $\sum_{n=1}^{N} \alpha_n$ and $\sum_{n=1}^{N} \beta_n$, use induction to show that

$$\left(\sum_{n=1}^{N} \alpha_n\right)\left(\sum_{n=1}^{N} \beta_n\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j.$$

(4 marks)

(b) Now consider the series $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ defined for $-\pi \le x \le \pi$.

(i) Use the result in part (a) to show that

$$\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right)^2 dx = \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(4 marks)

(ii) Hence, if the function $f(x)$, defined on $-\pi \le x \le \pi$, has Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, derive Parseval's theorem

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(4 marks)

(c) Find the Fourier series of the function

$$f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ \pi + x & -\pi \le x \le 0 \end{cases},$$

extended periodically over the whole real line. (4 marks)

Use Parseval's theorem to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

(4 marks)

(Total: 20 marks)

4. (a) (i) Obtain the Fourier transform of $f(x) = e^{-a|x|}$ (a>0). (3 marks)

(ii) Obtain the inverse Fourier transfrom of the following function.

$$\hat{g}(\omega) = \frac{1}{(4+\omega^2)^2}.$$

(5 marks)

(b) Consider the following second order linear ODE of the Euler-Cauchy type (x>1):

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{x}{\ln x} \tag{1}$$

(i) Find the general solution of the associated *homogeneous* problem in terms of the functions of an appropriate functional basis using the change of independent variable $z = \ln x$. (6 marks)

(ii) Using the method of variation of parameters (that uses an appropriate ansatz that contains an unknown function to be determined), find a particular integral and hence the general solution of the full problem given by Eq. (1). (6 marks)

(Total: 20 marks)

5. (a) Consider the following system of coupled linear differential equations:

$$\frac{dx}{dt} = 5x + 8y$$
$$\frac{dy}{dt} = -6x - 9y$$

(i) Find the general solution of the system in terms of the eigenvectors and eigenvalues. (3 marks)

(ii) Sketch the phase portrait for this system in the phase plane and describe the asymptotic behavior. (4 marks)

(iii) Particularize the solution when $x(0) = 2$ and $y(0) = -2$ and draw the corresponding trajectory on the phase plane. (3 marks)

(b) Consider the following first order nonlinear ODE:

$$\frac{dx}{dt} = (r+2)x - x^2 - (r+1) \tag{2}$$

where $r \in \mathbb{R}$ is a parameter.

(i) Find the fixed points of the system x^* and determine their stability for representative values of r. (4 marks)

(ii) Sketch the possible values of x^* against r and classify any bifurcation found. (3 marks)

(iii) Indicate the set of initial conditions that lead to divergence of $x(t)$ as $t \to \infty$ depending on r. (3 marks)

(Total: 20 marks)

6. (a) Obtain the general solution for the following ODE by checking if it is exact and if not turn it into an exact ODE:

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)\frac{dy}{dx} = 0.$$

(7 marks)

(b) Consider the function

$$u(x, y) = (x + y - 4)(xy^2 - y^2).$$

(i) Find the set of points (x_*, y_*) that form the zero contour, i.e., $u(x_*, y_*) = 0$. (2 marks)

(ii) Find the stationary points of $u(x, y)$ and classify them stating clearly the conditions used for the classification. You should use the Hessian of $u(x, y)$ at the stationary points to classify them. Note, you do not need to classify any stationary point for which the Hessian is zero. (6 marks)

(iii) Sketch the contour plot of $u(x, y)$ indicating the zero contour ($u = 0$) and a few other representative contours. Indicate in your sketch the positions of the stationary points and the regions of positive and negative u. Using your sketch, argue the consistency of the classifications obtained in part (ii) for the stationary points. (5 marks)

(Total: 20 marks)

1. (a) (i) $h = f(g(x))$, seen ⇓

$$\frac{dh}{dx} = f'(g)\,g'(x), \qquad \frac{d^2h}{dx^2} = f''(g)\,(g')^2 + f'(g)\,g''(x)$$

4, A

(ii) Now $f'(g) = -e^{-g}$, $f''(g) = e^{-g}$, $g'(x) = 3x^2$, $g''(x) = 6x$, hence sim. seen ⇓

$$h' = -e^{-x^3}\,3x^2, \qquad h'' = e^{-x^3}\,9x^4 - e^{-x^3}\,6x$$

$h'(x) = 0$ implies $x = 0$, but this is an inflection point, not a local max/min - see below also. There are no local maxima or minima. 2, B

$h''(x) = 0$ when $x = 0$ or $x^3 = 2/3$. Hence two inflection points $(0, 1)$ and $((2/3)^{1/3}, e^{-2/3}))$. 2, A

(iii) $h = e^{-x^3}$ and tends to 0 as $x \to \infty$ and tends to $+\infty$ as $x \to -\infty$. Also monotonic decreasing since we have shown that $h' \leq 0$, and inflection points found already. Sketch in the figure. unseen ⇓

2, B

seen ⇓

(b) (i) For a function that is $n + 1$ times differentiable we have

3, A

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{n!} f^{(n+1)}(\xi),$$

where ξ is a number between x_0 and x.

sim. seen ⇓

(ii) $f(1) = \log(1) = 0$. Also, $f' = 1 + \log(x)$, $f'' = 1/x$, $f''' = -1/x^2$, and so we have the Taylor series 2, A

2, B

$$x \log x = (x - 1) + \frac{(x - 1)^2}{2!} + \frac{(x - 1)^3}{3!}(-1/\xi^2),$$

where ξ is a number between 1 and x.

sim. seen ⇓

(iii) Using the result above and keeping 2 terms gives $x \log x \approx 0.1 + 0.1^2/2 = 21/200$. The error is E with 3, D

$$|E| = \frac{0.1^3}{6\xi^2} \leq \frac{10^{-3}}{6},$$

since $1 \leq \xi \leq 1.1$.

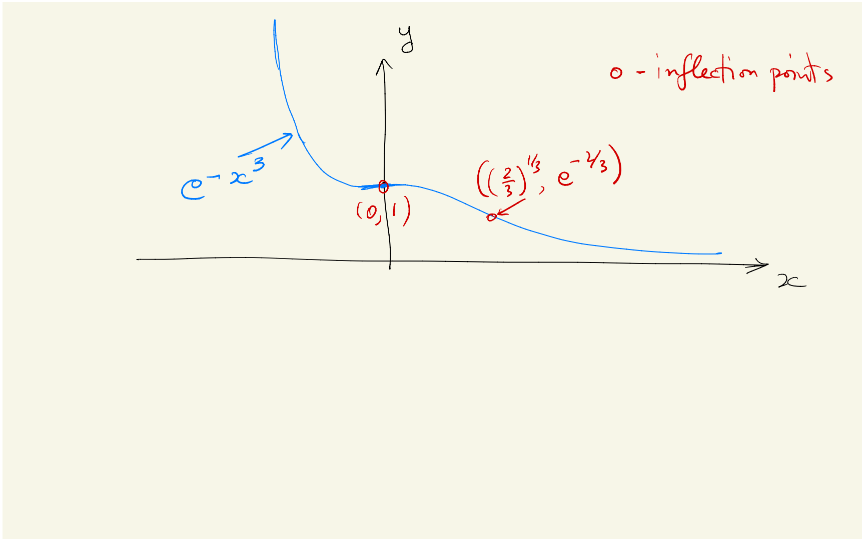

Figure 1: Figure for problem 1(a)iii.

2. (a) (i) Know from class notes

seen ⇓

$$S_x = 2\pi \int_0^L f[1+(f')^2]^{1/2} dx, \qquad S_y = 2\pi \int_0^L x[1+(f')^2]^{1/2} \frac{dy}{dx} dx$$

seen/sim.seen ⇓

2, A

hence

$$S_x = 2\pi \int_0^L x^2(1+4x^2)^{1/2} dx, \qquad S_y = 2\pi \int_0^L x(1+4x^2)^{1/2} 2x dx$$

No need to carry out the integrals, $S_y = 2S_x$ and so always $S_x < S_y$. 2, A

(ii) The required volumes are

sim. seen ⇓

unseen ⇓

$$V_x = \pi \int_0^L y^2 dx = \pi \int_0^L x^4 dx = \frac{\pi L^5}{5},$$

2, B

$$V_y = 2\pi \int_0^L xy dx = 2\pi \int_0^L x^3 dx = \frac{\pi L^4}{2}$$

These are equal when $L = L_0 = 5/2$, and $V_x < V_y$ for $L < L_0$, while $V_x > V_y$ when $L > L_0$. 2, C

Geometrically, when L is small enough, the distance from the $x-$axis to a point on the curve is smaller than the distance of the point to the $y-$axis, hence the swept volume V_x is smaller than that for V_y. This ordering changes for L big enough, and V_x overtakes V_y as seen through the L^5 power. 2, D

unseen ⇓

(b) (i) Calculate (integration by parts)

$$F_1(s) = \int_0^\infty te^{-st} dt = \left[\frac{e^{-st}}{-s} t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt = \left[\frac{e^{-st}}{-s^2}\right]_0^\infty = \frac{1}{s^2}$$

2, B

$$F_2(s) = \int_0^\infty \sin\omega t\, e^{-st} dt = \left[\sin\omega t\, \frac{e^{-st}}{-s}\right]_0^\infty + \int_0^\infty \frac{e^{st}}{s} \omega \cos\omega t\, dt$$

$$= \left[\frac{e^{-st}}{-s^2} \omega \cos\omega t\right]_0^\infty - \frac{\omega^2}{s^2} \int_0^\infty \sin\omega t\, e^{-st} dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} F_2(s)$$

Hence, $F_2(s) = \frac{\omega}{s^2+\omega^2}$. 3, C

unseen ⇓

(ii) Use partial fractions to write $F(s) = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$. 3, A

We can recognise these immediately as $(1/4)/s^2$ coming from $(1/4)t$, and $-\frac{1}{8}\frac{2}{s^2+2^2}$ coming from $-(1/8)\sin 2t$. Hence 2, D

$$g(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

unseen ⇓

3. (a) For $N = 1$ we clearly have $\left(\sum_{n=1}^{1}\alpha_n\right)\left(\sum_{n=1}^{1}\beta_n\right) = \alpha_1\beta_1 = \sum_{i=1}^{1}\sum_{j=1}^{1}\alpha_i\beta_j$, hence holds for $N = 1$. 4, D

Assume it holds for general N. Need to prove that the formula holds for $N+1$. Calculate

$$\left(\sum_{n=1}^{N+1}\alpha_n\right)\left(\sum_{n=1}^{N+1}\beta_n\right) = \left[\left(\sum_{n=1}^{N}\alpha_n\right) + \alpha_{N+1}\right]\left[\left(\sum_{n=1}^{N}\beta_n\right) + \beta_{N+1}\right]$$

$$= \left(\sum_{n=1}^{N}\alpha_n\right)\left(\sum_{n=1}^{N}\beta_n\right) + \beta_{N+1}\left(\sum_{n=1}^{N}\alpha_n\right) + \alpha_{N+1}\left(\sum_{n=1}^{N}\beta_n\right) + \alpha_{N+1}\beta_{N+1}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j + \beta_{N+1}\left(\sum_{i=1}^{N}\alpha_i\right) + \alpha_{N+1}\left(\sum_{j=1}^{N}\beta_j\right) + \alpha_{N+1}\beta_{N+1}$$

$$= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\alpha_i\beta_j$$

meth seen ⇓

(b) (i) Begin with the calculation 4, C

$$\left(\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)^2$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[a_i\cos(ix) + b_i\sin(ix)\right]\left[a_j\cos(jx) + b_j\sin(jx)\right]$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[a_ia_j\cos(ix)\cos(jx) + a_ib_j\cos(ix)\sin(jx) + a_jb_i\sin(ix)\cos(jx) + b_ib_j\sin(ix)\sin(jx)\right]$$

On integration over $[-\pi,\pi]$ and using orthogonality, the only terms that survive are $i = j$ in the first and last terms, i.e.

$$\int_{-\pi}^{\pi}\sum_{i=1}^{\infty}\left(a_i^2\cos^2 ix + b_i^2\sin^2 ix\right)dx = \sum_{i=1}^{\infty}\int_{-\pi}^{\pi}\left(\frac{a_i^2}{2}(1+\cos 2ix) + \frac{b_i^2}{2}(1-\cos 2ix)\right)dx$$

$$= \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(ii) From the definition of $f(x)$ we have

seen/sim.seen ⇓

3, B

1, D

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{a_0^2}{4} + a_0\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) + \left(\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)^2\right]$$

The second term is zero, and the third has been worked out above. Hence

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(c) The function as given is even about $x = 0$, hence we need the cosine series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, where

seen ⇓

meth seen ⇓

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\, dx \qquad \Rightarrow \qquad a_0 = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\left(\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n}dx\right) = \frac{2}{\pi}\left[-\frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2(1-\cos n\pi)}{\pi n^2}$$

Hence, $a_0 = \pi$, $a_{2n} = 0$, $a_{2n-1} = \frac{4}{(2n-1)^2\pi}$, for $n \geq 1$.

4, A

Using Parseval's theorem shown above, and noting that $f(x)$ is even, gives

$$\frac{2}{\pi}\int_0^{\pi}(\pi - x)^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2, \qquad i.e.$$

$$\frac{2}{\pi}\frac{\pi^3}{3} = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_{2n-1}^2 \quad \Rightarrow \quad \frac{2}{3}\pi^2 = \frac{1}{2}\pi^2 + \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} \quad \Rightarrow$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

2, A

2, B

Review of mark distribution:

Total A marks: 24 of 32 marks
Total B marks: 15 of 20 marks
Total C marks: 9 of 12 marks
Total D marks: 12 of 16 marks
Total marks: 60 of 80 marks
Total Mastery marks: 0 of 20 marks

4.(a(i)) Using the definition of the Fourier transform we have: seen ⇓

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x}, dx = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}.$$

3, A

(a(ii)) Form last parts results we note that using convolution theorem we have meth seen ⇓

$$\mathcal{F}\{f(x) * f(x)\} = \hat{f}(\omega)^2 = \frac{4a^2}{(a^2 + \omega^2)^2}$$

By definition of convolution we have:

$$f(x) * f(x) = \int_{-\infty}^{\infty} e^{-a|x-u|} e^{-a|u|} du.$$

To obtain the integral we consider the $x > 0$ and $x < 0$ separately and combine the two results using absolute values as above. To illustrate this for $x > 0$ we have:

$$\int_{-\infty}^{\infty} e^{-a|x-u|} e^{-a|u|} du = \int_{-\infty}^{0} e^{-a(x-u)} e^{au} du + \int_{0}^{x} e^{-a(x-u)} e^{-au} du + \int_{x}^{\infty} e^{a(x-u)} e^{-au} du$$

$$= e^{-ax} \left[\frac{e^{2au}}{2a} \right]_{-\infty}^{0} + e^{-ax} \left[u \right]_0^x + e^{ax} \left[\frac{e^{-2au}}{-2a} \right]_x^{\infty}$$

$$= (\frac{1}{a} + x) e^{-ax}.$$

Similarly, we obtain the result for $x < 0$. We then combine the two results using absolute values to obtain:

$$f(x) * f(x) = (\frac{1}{a} + |x|) e^{-a|x|}.$$

So setting $a = 2$ we have:

$$\mathcal{F}^{-1}\{\frac{1}{(4 + \omega^2)^2}\} = \frac{1}{16} (\frac{1}{2} + |x|) e^{-2|x|}.$$

5, B

(b(i)) We have (from the lectures, or can be obtained): meth seen ⇓

$$x \frac{dy}{dx} = \frac{dy}{dz},$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

So, we obtain the following ODE in terms of $y(z)$:

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = \frac{1}{z} e^z.$$

3, C

The characteristic equation of the corresponding homogeneous ODE is:

$$\lambda^2 - 2\lambda + 1 = 0,$$

which has a repeated root of $\lambda = 1$. So, we have from the lectures the following two functions as the basis for the solution vector space:

$$y_1 = e^z, \quad y_2 = z e^z.$$

So the complementary function, the general solution of the homogeneous problem is:

$$y_{CF} = c_1 e^z + c_2 z e^z = c_1 x + c_2 x \ln x$$

.

3, A

(b(ii)) We will stay with the ODE in terms of independent variable z, we try the ansatz $y_{PI}(z) = A(z)e^z$, where $A(z)$ is a function to be determined. By plugging the ansatz in the ODE, we have: $\boxed{\text{unseen} \Downarrow}$

$$\left[\frac{d^2A}{dz^2}e^z + 2\frac{dA}{dz}e^z + Ae^z\right] - 2\left[\frac{dA}{dz}e^z + Ae^z\right] + Ae^z = \frac{d^2A}{dz^2}e^z = \frac{1}{z}e^z$$

So, we get the following second order ODE for $A(z)$ that after two integration we obtain:

$$\frac{d^2A}{dz^2} = \frac{1}{z} \quad \Rightarrow \quad A = z\ln z + a_1 z + a_2,$$

with a_1 and a_2 as constants of integration that can be absorbed into the complementary function. So we have for the general solution: $\boxed{4,\ \text{D}}$

$$y_{GS} = y_{CF} + y_{PI} = c_1e^z + c_2ze^z + z\ln ze^z = c_1x + c_2x\ln x + x\ln x\ln(\ln x)$$

. $\boxed{2,\ \text{A}}$

5.(a(i)) We have for matrix A: meth seen ⇓

$$A = \begin{pmatrix} 5 & 8 \\ -6 & -9 \end{pmatrix}.$$

We calculate eigenvalues and eigenvectors of the matrix A:

$$\lambda_1 = -3 \quad \Rightarrow \quad \vec{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\lambda_2 = -1 \quad \Rightarrow \quad \vec{v_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

Therefore, we have for the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

4, A

(a(ii)) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a stable fixed point and the solutions asymptotically approach this fixed point along $\vec{v_2}$. Figure 1 shows the phase portrait for this system. The faster decay along $\vec{v_1}$ that corresponds to a smaller eigenvalue should be visible in the portrait for full marks. meth seen ⇓

3, C

(a(iii)) Using the initial conditions we have $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. So given the general solution in part (i) we obtain $c_1 = 2$ and $c_2 = 0$. So, we have: sim. seen ⇓

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This trajectory that is along $\vec{v_1}$ is highlighted on Figure 1. 3, B

(b(i)) The fixed points of the system are obtained by putting the right hand side to zero. We have

$$(x-1)(r+1-x) = 0 \quad \Rightarrow \quad x*_1 = 1, \quad x*_2 = r+1$$

So, we have two fixed points (except for r=0, for which we have only one fixed point). We consider the two cases of $r < 0$ and $r > 0$. In Figure 2a we have determined the stability of these fixed points for the different values of r. Note that at $r = 0$ we have one fixed point of $x* = 1$, which is a half-stable fixed point.

4, A

(b(ii)) The bifurcation diagram for this system is sketched in Figure 2b. There is a transcritical bifurcation at $r_c = 0$. 2, B

1, D

(b(iii)) Based on the bifurcation diagram, we see that for $r > 0$, all initial values of $x < 1$ diverge as $t \to \infty$. For $r \leq 0$, all initial values of $x < r + 1$ diverges as $t \to \infty$. 3, D

6. (a) Let $F(x,y) = 3x^2y + 2xy + y^3$ and $G(x,y) = x^2 + y^2$, so we obtain: seen/sim.seen ⇓

$$\frac{\partial F}{\partial y} = 3x^2 + 2x + 3y^2,$$

$$\frac{\partial G}{\partial x} = 2x.$$

So, the condition of integratibility is not satisfied. 2, A

Now, we look for an integrating factor. It turns out that we can find $\lambda(x)$ that does not depend on y. We have:

$$\lambda\frac{\partial F}{\partial y} = \lambda\frac{\partial G}{\partial x} + \frac{d\lambda}{dx}G \quad \Rightarrow \quad \frac{d\lambda}{dx} = 3\lambda \quad \Rightarrow \quad \lambda = e^{3x}.$$

2, A

For the solution $u(x,y) = 0$, we have:

$$u(x,y) = \int G(x,y)e^{3x}\,dy = e^{3x}(x^2y + \frac{1}{3}y^3) + g(x).$$

So we have:

$$\frac{\partial u}{\partial x} = F(x,y)e^{3x} \Rightarrow \frac{dg}{dx} = 0 \Rightarrow g(x) = c,$$

where c is constant of integration. So we have:

$$u(x,y) = 0 \Rightarrow e^{3x}(x^2y + \frac{1}{3}y^3) + c = 0.$$

3, B

meth seen ⇓

(b(i))

$$u(x,y) = (x+y-4)y^2(x-1).$$

Zero contour $(u=0)$:

$$x_* + y_* = 4, \quad x_* = 1, \quad y_* = 0.$$

2, A

(b(ii) We need to evaluate the gradient of u and set it to zero. We have: unseen ⇓

$$\frac{\partial u}{\partial x} = y^2(y+2x-5) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y + 2x - 5 = 0,$$

$$\frac{\partial u}{\partial y} = (x-1)y(2x+3y-8) = 0 \quad \Rightarrow \quad x = 1 \quad \text{or} \quad y = 0 \quad \text{or} \quad x + 2y - 4 = 0.$$

Given the above, the whole line of $y = 0$ is stationary. In addition we have the following stationary points:

$$P_1 = (1,3), \quad P_2 = (\frac{7}{4}, \frac{3}{2}).$$

For the character of the stationary points we calculate the Hessian. We have: 2, B

$$H(x,y) = \begin{bmatrix} 2y^2 & 3y^2 + 4yx - 10y \\ 3y^2 + 4yx - 10y & 2(x-1)(x+3y-4) \end{bmatrix}.$$

Now, we can evalute Hessain at each stationay point and based on the value of trace and determinant of the matrix, we can decide if the stationary point is a maximum, minimum or saddle point.

$$H_{P_1} = \begin{bmatrix} 18 & 9 \\ 9 & 0 \end{bmatrix} \quad \Rightarrow \quad \tau = 18, \quad \Delta = -81 < 0 \quad \Rightarrow \quad P_1 \text{ is a saddle point.}$$

$$H_{P_2} = \begin{bmatrix} \frac{9}{2} & \frac{9}{4} \\ \frac{9}{4} & \frac{27}{8} \end{bmatrix} \quad \Rightarrow \quad \tau = \frac{63}{8} > 0, \quad \Delta = \frac{81}{8} \quad \Rightarrow \quad P_2 \text{ is a minimum.}$$

$$H_{y=0} = \begin{bmatrix} 0 & 0 \\ 0 & 2(x-1)(x-4) \end{bmatrix} \quad \Rightarrow \quad \tau = 2(x-1)(x-4), \quad \Delta = 0 \quad \Rightarrow \quad y = 0 \text{ is a minimum or maximur}$$

Given the form of τ above, for $1 < x < 4$ line $y = 0$ is a maximum. For $x < 1$ and $x > 4$ $\tau > 0$ so $y = 0$ is a minimum. Note for $x = 1$ and $x = 4$ the Hessian is zero, so we need higher order derivatives to classify such points, which is out of the scope of this course. $\boxed{4, \text{D}}$

$\boxed{\text{unseen} \Downarrow}$

(b(iii)) Figure 3 shows the sketch of the level sets of u, the stationary points and the sign of the function u. $\boxed{3, \text{C}}$

For the first point, which based on the Hessian, we obtained to be a saddle point, we see that the value of the function u in the vicinity of the point becomes negative (decreases from 0) in certain directions and it becomes positive (increases from 0) in other directions, so this is consistent with the points P_1 being a saddle-point. We also have $u(P_2) < 0$, so this is also consistent with P_2 being a minimum. Finally, we note that $y = 0$ line is a zero contour, and we see part of this line which is a minimum is between postive regions and the middle part of this line that is a maximum is between negative regions, which also is consistent. $\boxed{2, \text{A}}$

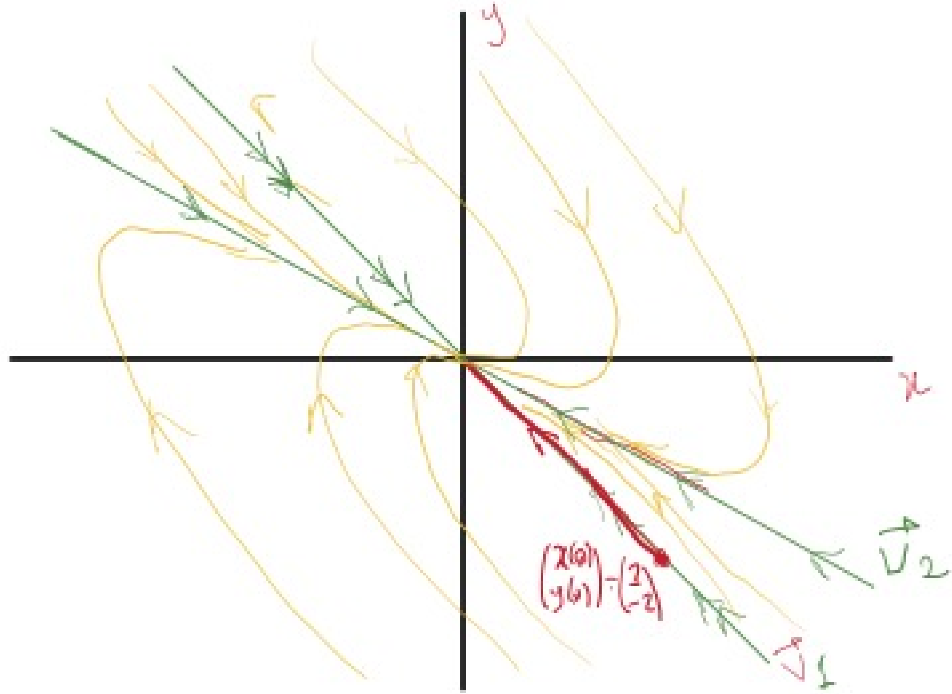

Figure 1: Phase portrait regarding problem 5a

Review of mark distribution:
Total A marks: 24 of 32 marks
Total B marks: 15 of 20 marks
Total C marks: 9 of 12 marks
Total D marks: 12 of 16 marks
Total marks: 60 of 80 marks
Total Mastery marks: 0 of 20 marks

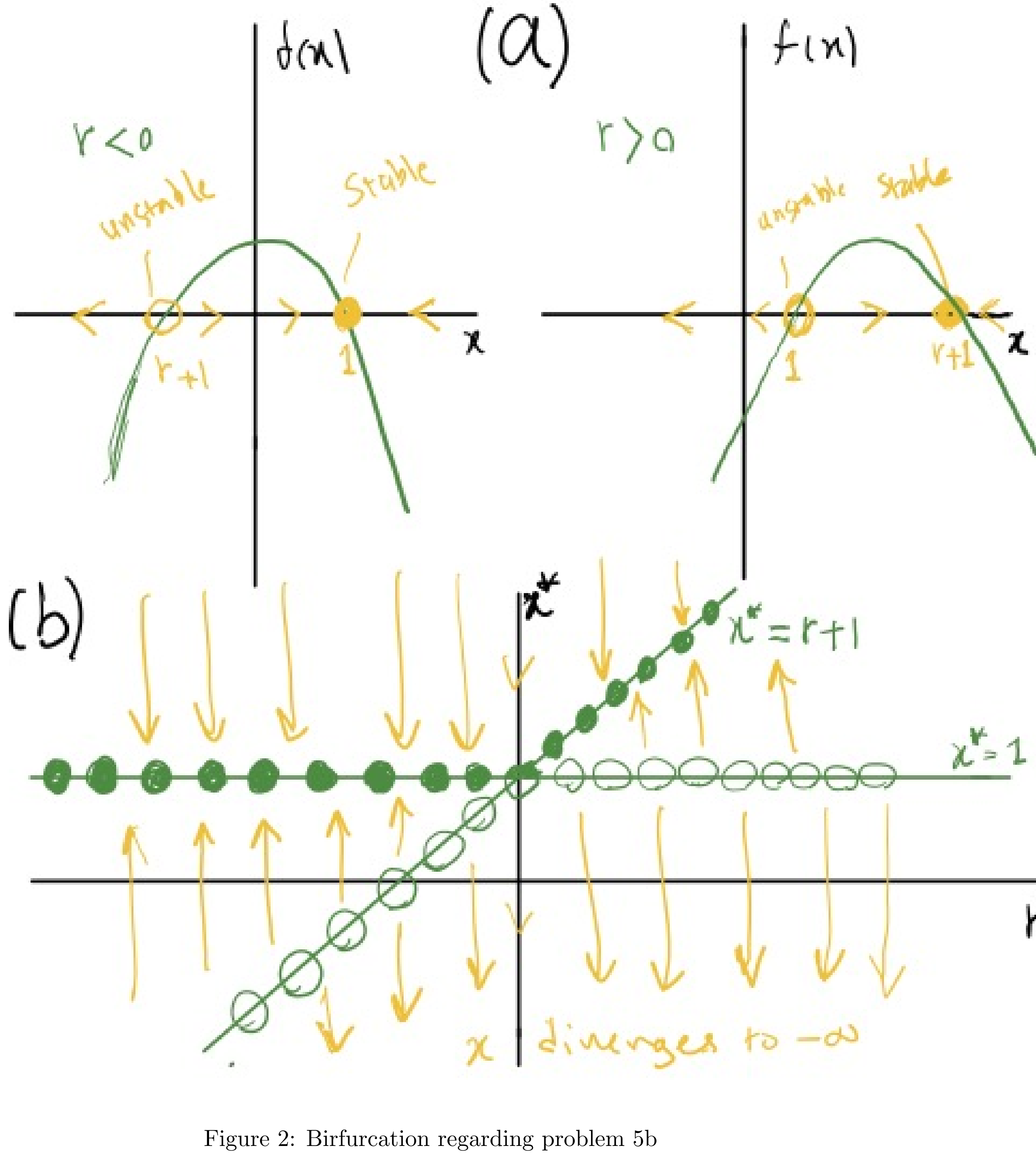

Figure 2: Birfurcation regarding problem 5b

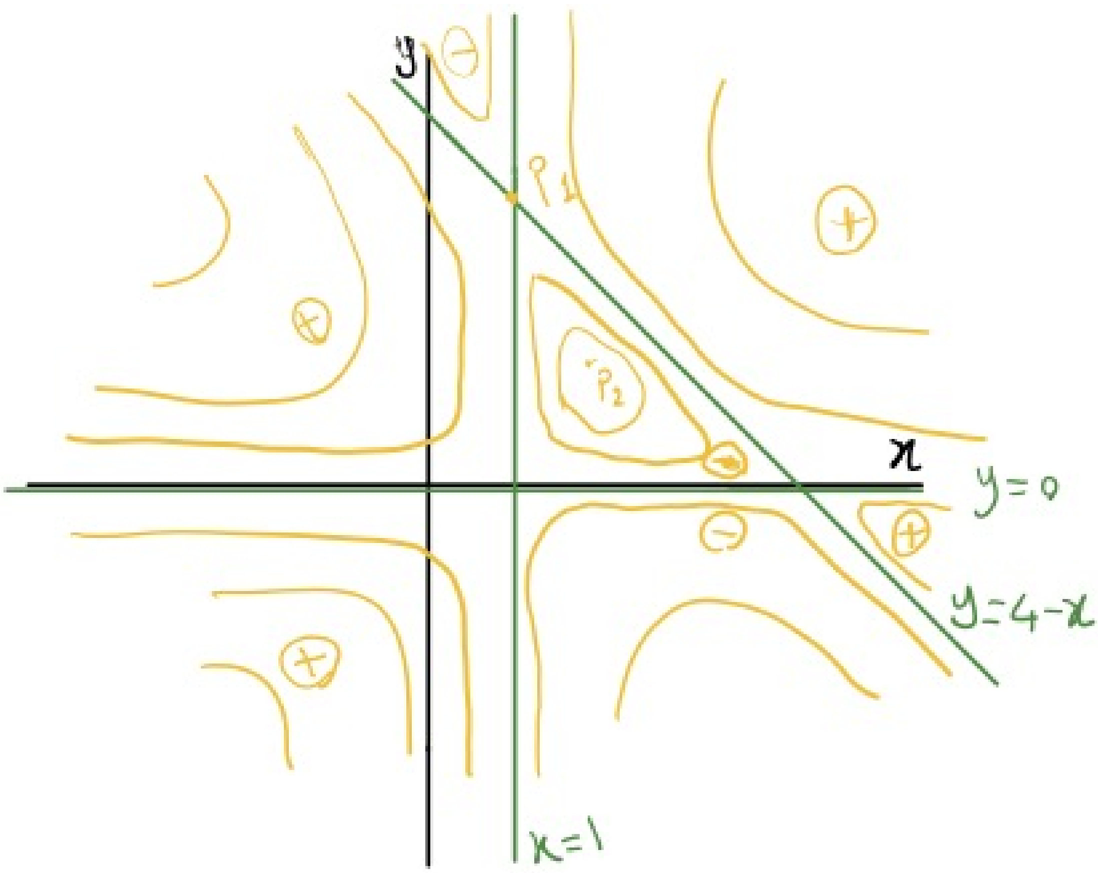

Figure 3: Plot regarding problem 6b

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH40004	1	I think most students found this question fairly straightforward and the many answers were very good. One thing I would recommend is to present solutions more clearly, especially sketches of graphs (label them, point out important points, etc.)Overall, this was answered well.
MATH40004	2	Students did rather well on this questions. The last part,(inversion of the Laplace transform) turned out to be challenging for many, who got lost, or made serious mistakes. There was only one or two full marks, as almost all fell on the geometric interpretation of the nature of the volumes.
MATH40004	3	No Comments Received
MATH40004	4	4ai) Most got this correct; some however solely referred to the lecture notes and did not show calculations; remember to enter calculations. 4aii) Only a few students got this one right. You had to use the convolution theorem. Some only showed the intergral, but did not actually compute the convolutions (which you had to do using the right split of the integral). 4bi) Most got this one right; some made some typo's in the sign in front of dy/dz, so that the characteristic equation & final expression was not fully correct. 4bii) Not everyone used the correct ansatz and when using the wrong one, the right solution wasn't attainable.
MATH40004	5	No Comments Received

MATH40004	6	This question was on the whole done badly by the majority of students, which is disappointing given that good methodology was generally seen, but let down by small errors or possibly a lack of experience.Part (a) required students to test an ode for exactness - this was done well and a majority of students correctly identified the ode was not exact.A good number of students then proceeded with the correct idea - to seek a factor to make the ode exact, however it was finding this correctly which proved difficult for most. In most cases methodology was good but overcomplicated causing students to formulate more difficult problems than necessary... it's a good idea to try and keep things simple here. Students that successfully found an appropriate factor then often went on to correctly find the solution to the ode.Part (b) caused many more problems than necessary. The first part was often done well and students correctly found the zero contours setting the function value to 0.Part (ii) however was a mess - most students started with the correct method - setting the partial derivatives in x and in y to 0, however solving this resulting system of equations was generally done poorly... it might be good advice here for future students to note that keeping things nice, tidy and factorised here is good practice... students that expanded everything upon differentiating very often struggled to spot all the solutions to the system, whereas those that kept everything nice and factored did better - this is in a much easier form to spot the roots. Generally those students that found all the roots characterised them well using the Hessian. Finally producing a good visualisation of the surface was done poorly. Most students correctly drew on the zero contours, though placing their analysis from the second part onto this diagram was often performed badly... of course where the analysis went badly wrong in the second part then it is natural to have confusion on to how to add this correctly to the image, but attempts to do this were generally poor and should have been better.Very few students even commented on their diagrams... the question asked for comment so it was a shame that most students failed to do this when explaining their diagram correctly, even if the surface picture was incorrect, would have scored them some easy points. When asked to comment - comment - and explain why. Bad comments said things akin to "it agrees with my analysis"...good comments simply said things like "I see all around the point P_1 the surface has negative values, this agrees with my analysis of P_1 being a minimum" - saying what you see shows you know why!