Imperial College London

DEPARTMENT OF MATHEMATICS
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Introduction to University Mathematics **MATH40001/MATH40009**

Part I – Solutions to Problem Sheet 3: Binary relations

1. (a) This isn't a binary relation because if x and y are real numbers then $x - y$ is a real number, not a true-false statement.
   (b) This is a binary relation because if x and y are real numbers then $x^2 < y + 1$ is a true-false statement.
   (c) This is a ternary relation, not a binary relation.
   (d) This is not a well-defined binary relation on the reals, because if $y = 0$ then $x/y > 0$ involves division by zero so is not a well-defined object. Note that anyone who says $1/0 = \infty$ and $\infty > 0$ is talking nonsense: $1/0$ is *undefined*. It would have been a binary relation on the non-zero reals though.
   (e) This is a binary relation, because if x and y are real numbers then $x < y \implies y < x$ is a true-false statement, even though it's a slightly bizarre one. Note that it is true precisely when $x \geq y$.

2. (a) R is not reflexive because $R(2,2)$ is false. It is symmetric, because we just check all the cases; in all the three cases that $R(i,j)$ is true, $R(j,i)$ is also true. It is not antisymmetric, because $R(2,1)$ and $R(1,2)$ are both true, but $2 \neq 1$. Finally it is not transitive, because $R(2,1)$ and $R(1,2)$ are true, but $R(2,2)$ is not.
   (b) $R(3,3)$ is false because $3 \neq -3$, so R is not reflexive. If $a = -b$ then $b = -a$ so R is symmetric. We have $R(2,-2)$ true and $R(-2,2)$ true but $2 \neq -2$ so R is not antisymmetric. Finally $R(2,-2)$ is true and $R(-2,2)$ is true but $R(2,2)$ is not, so R is not transitive.
   (c) R is not reflexive because $R(37,37)$ is false. It is however symmetric, antisymmetric and transitive because each one of these statements is of the form "If $R(a,b)$ is true then...", and "`false` implies P" is always true whatever P is.
   (d) This binary relation is reflexive, symmetric, antisymmetric and transitive, because every one of these assertions is of the form "For all x in the empty set,…" so they're all true because there can be no counterexample.

3. (a) We need to prove that $\subseteq$ is reflexive, antisymmetric and transitive. Let $\mathbb{Z}$ denote the integers. Say $A \subseteq \mathbb{Z}$ is arbitrary. Then $A \subseteq A$ because $\forall z \in \mathbb{Z}$, $z \in A \implies z \in A$. This shows reflexivity. Now say that $B \subseteq \mathbb{Z}$ is also arbitrary. If $A \subseteq B$ and $B \subseteq A$ then $\forall z \in \mathbb{Z}$, $z \in A \implies z \in B$ and $z \in B \implies z \in A$, so $z \in A \iff z \in B$, which means that A and B have the same elements, so $A = B$.

   Finally say $C \in \mathbb{Z}$ is a third arbitrary subset, and let's assume $A \subseteq B$ and $B \subseteq C$. If $z \in \mathbb{Z}$ is arbitrary and we assume $z \in A$, then we deduce $z \in B$ (as $A \subseteq B$), and hence $z \in C$ (as $B \subseteq C$). We have just shown that for all $z \in \mathbb{Z}$, $z \in A \implies z \in C$. Hence $A \subseteq C$. Thus $\subseteq$ is transitive. We have thus checked all the axioms for a partial order, so $\subseteq$ is indeed a partial order on X.
   (b) It is not a total order. Indeed if $A = \{0\}$ and $B = \{37\}$ then both $A \subseteq B$ and $B \subseteq A$ are false.

4. We know $3 \sim 2$ and $2 \sim \pi$ so by transitivity we have $3 \sim \pi$. We know $1 \sim \pi$ so by symmetry $\pi \sim 1$. Applying transitivity to $3 \sim \pi$ and $\pi \sim 1$ we deduce $3 \sim 1$.

5. Assume $\leq$ is a total order on the complexes. Let (M) be the assertion that $0 \leq a$ and $0 \leq b$ implies $0 \leq ab$. Let (A) be the assertion that if $a \leq b$ then for all t, $t+a \leq t+b$. I will prove a contradiction assuming (M) and (A), and this solves the problem.

   Let a be any complex number. By totality of $\leq$ we know that either $0 \leq a$ or $a \leq 0$. If $0 \leq a$ then by (M) we have $0 \leq a^2$. However if $a \leq 0$ then by (A) we have $(-a)+a \leq (-a)+0$, and hence $0 \leq -a$, so $0 \leq (-a)^2 = a^2$. We conclude that for any complex number a we must have $0 \leq a^2$. However $-1 = i^2$ and hence $0 \leq -1$, and now adding 1 to both sides we deduce $1 \leq 0$. Also $1 = 1^2$ is a square so $0 \leq 1$. By antisymmetry we deduce $0 = 1$, a contradiction.

6. If R is transitive then I claim R^{op} is also transitive. Say $a, b, c \in X$ and assume $R^{op}(a,b)$ and $R^{op}(b,c)$; we want to deduce $R^{op}(a,c)$, and here's how we do it. By definition of "op" we can deduce $R(b,a)$ and $R(c,b)$. By transitivity of R we deduce $R(c,a)$, and by definition of "op" we deduce $R^{op}(a,c)$, which is what we wanted to prove.

   I claim that if R is antisymmetric then R^{op} is also antisymmetric. Say $a, b \in X$ and we know $R^{op}(a,b)$ and $R^{op}(b,a)$; we want to deduce $a = b$. By definition of op we deduce $R(b,a)$ and $R(a,b)$. By antisymmetry of R we deduce $a = b$ and we're done.

7. The error is when I write "choose $y \in X$ such that $R(x,y)$ is true". What if $R(x,y)$ is false for every y? The earlier question it contradicts is Q2c; this relation is symmetric and transitive, but not reflexive.

8. (a) This is true. Say $x \in X$. By definition $R(x,x) = S(f(x), f(x))$ which is true by reflexivity of S. But $x \in X$ was arbitrary, so R is reflexive.

   (b) This is true. Say $x, y \in X$ are arbitrary. We want to prove that $R(x,y) \implies R(y,x)$. So assume $R(x,y)$ is true. Then by definition of R we see that $S(f(x), f(y))$ is true. By symmetry of S we deduce $S(f(y), f(x))$. By definition of R we deduce $R(y,x)$. This was what we wanted.

   (c) This is not true. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate, so $f(x,y) = x$, and let $S(a,b)$ be the binary relation $a \leq b$ on the reals. Then S is antisymmetric. However $R((0,0),(0,1))$ and $R((0,1),(0,0))$ are both true, because they are both the proposition $0 \leq 0$, and yet $(0,0) \neq (0,1)$. Hence R is not antisymmetric.

   (d) This is true. If $R(x,y)$ and $R(y,z)$, then $S(f(x), f(y))$ and $S(f(y), f(z))$, so $S(f(x), f(z))$ by transitivity of S, so $R(x,z)$, which was what we wanted.

   (e) This is not true and the same counterexample as in (c) works, for the same reason.

   (f) This is not true and the same counterexample as in (c) works, for the same reason.

   (g) This is true, by parts (a), (b) and (d).

9. Let $X = \{1,2\}$ and let $Y = \{3,4\}$, and define $f : X \to Y$ by $f(1) = f(2) = 3$.

   (a) If R is any binary relation on X such that $R(1,1)$ is true and $R(2,2)$ is false (e.g. the binary relation defined letting $R(a,b)$ be $a = 1$), then the relations E and A are different, because $E(3,3)$ is true (set $x_1 = x_2 = 1$) but $A(3,3)$ is false (set $x_1 = x_2 = 2$).

   (b) Let R be the equivalence relation defined by equality – so $R(a,b)$ is the true-false statement $a = b$. Then $A(3,3)$ is false because we can set $x_1 = 1$ and $x_2 = 2$. Hence A is not reflexive and thus not an equivalence relation.

   (c) Let R be the same as in the previous part. Then $E(4,4)$ is false because we can't find $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = 4$ and $R(x_1, x_2)$ is true – indeed we can't find $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = 4$ at all, because 4 is not in the range of f.

10. This question is the "dual" to the friends question on the previous sheet, and shows that there is some kind of duality between subsets of a set, and equivalence relations on a set. Note also that this question is as long, if not longer, than the corresponding question on sheet 3!

The question asks us to prove an $\iff$ so we have to prove two things. First let's assume that f and g are pals, and let's prove that the equivalence relations defined by f and g are equal. So we need to prove that if x_1 and x_2 are arbitrary elements of X, then $f(x_1) = f(x_2) \iff g(x_1) = g(x_2)$. I guess we start by choosing a bijection $h : Y \to Z$ such that $g = h \circ f$.

Our goal, $f(x_1) = f(x_2) \iff g(x_1) = g(x_2)$, is *another* iff, so again we have really got two things to do. First let's prove $f(x_1) = f(x_2) \implies g(x_1) = g(x_2)$. Let's assume $f(x_1) = f(x_2)$. Applying h we deduce $h(f(x_1)) = h(f(x_2))$, or in other words, $(h \circ f)(x_1) = (h \circ f)(x_2)$. But $h \circ f = g$ so this is exactly what we wanted.

Next, in this first part of the argument, is to show that $g(x_1) = g(x_2) \implies f(x_1) = f(x_2)$. So let's assume $g(x_1) = g(x_2)$. We know that $g = h \circ f$ so what we know is that $h(f(x_1)) = h(f(x_2))$. But h is bijective and hence injective, and hence $f(x_1) = f(x_2)$.

Now let us start on the other implication. This way around we know that for all $x_1, x_2 \in X$, $f(x_1) = f(x_2) \iff g(x_1) = g(x_2)$, and we want to prove that there exists a bijection $h : Y \to Z$ such that $g \circ h = f$. So we're going to have to define this function h. Several people asked me whether they could just "define" h by saying $g = h \circ f$, but I don't think you can, we need to justify that such a function exists. The function h is a function from Y to Z so let's let $y \in Y$ be arbitrary and we need to come up with an element $z \in Z$ so we can define $h(y)$ to be this element. Here's how I want to define it. We know that f is surjective, so we can choose $x \in X$ such that $f(x) = y$. We can define $z = g(x)$ and then we can define $h(y) = z$. There is something a bit funny about this definition – it seems to depend on an auxiliary choice of x. We know f is surjective but it might not be injective, so there might be several possibilities for the choice of x. Does this affect our definition of h? Fortunately, it does not. For if x_1 and x_2 are two elments of X with $f(x_1) = f(x_2) = y$, then by our assumption $f(x_1) = f(x_2) \iff g(x_1) = g(x_2)$ we can deduce that $g(x_1) = g(x_2)$. This means that our definition of $h(y)$ as $g(x)$ does not depend on the choice of x.

Our job now is to prove that h does the job of showing that f and g are pals. First let's prove that $g = h \circ f$. Let $x \in X$ be arbitrary, and let's define $y = f(x)$. Recall our definition of $h(y)$ – it involved choosing some random element of X such that f of this element was y. But we showed above that our definition of $h(y)$ did not depend on this choice, so we may as well choose x. This means that $h(y) = g(x)$, and in particular $h(f(x)) = g(x)$. This is what we wanted to show.

Finally, we need to show that h is a bijection. Let's start with injectivity. Say y_1 and y_2 are elements of Y, with $h(y_1) = h(y_2)$. We want to show that $y_1 = y_2$. Recall that the definition of h involved choosing something in X, so let's choose x_1 and $x_2 \in X$ with $f(x_i) = y_i$. Then $h(y_1) = g(x_1)$ and $h(y_2) = g(x_2)$. Our assumption is hence that $g(x_1) = g(x_2)$. By our assumption ($f(x_1) = f(x_2) \iff g(x_1) = g(x_2)$), we can deduce that $f(x_1) = f(x_2)$. But this says exactly that $y_1 = y_2$, which is what we wanted to prove.

The last of the jobs we need to do is to prove that h is surjective. So say $z \in Z$, and our job is to find $y \in Y$ such that $h(y) = z$. Here's how to do it. We know that g is surjective, so there exists some $x \in X$ such that $g(x) = z$. Let's define $y = f(x)$. The definition of $h(y)$ involves choosing an element of X such that f of it is Y, but we checked that the definition did not depend on this choice, so we may as well choose x, and now by definition $h(y) = g(x)$, and $g(x) = z$, and this is the last of the many things which we need to check in order to completely solve this extremely long question.