

MATH60142/70142 The Mathematics of Business and Economics
Class test

Friday 28th February 2025
Duration: 40 minutes

Question (20 marks in total)

Suppose that a firm's production process requires the input of two different goods and produces a single output. For $i = 1, 2$, let $x_i \geq 0$ denote the quantity of the i -th good that the firm inputs, and $w_i > 0$ denote the (fixed) cost to the firm of each unit of this good. Furthermore, let y denote the quantity of its output that it produces. Suppose also that the firm's production function is given by

$$f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$$

(where these are positive square roots).

The firm wishes to determine the values x_1^* and x_2^* of x_1 and x_2 (respectively) that yield a level of output y for the minimum cost.

(a) **(2 marks)** Write down the Lagrangian for this minimisation problem.

(b) **(4 marks)** Derive a set of necessary conditions for x_1^* and x_2^* .

(c) **(5 marks)** Solve these conditions to determine x_1^* and x_2^* as functions of w_1 , w_2 and y . (You do not need to show that these solutions provide a minimum rather than a maximum.)

(d) **(2 marks)** Suppose that the firm is inputting these optimal quantities $x_1 = x_1^*$ and $x_2 = x_2^*$. What should the rate of change of x_2 be at this point if the firm is to vary x_1 while maintaining the same level of output y ? Express your answer in terms of w_1 and w_2 and provide a justification for it.

Suppose now that the firm sells each unit of its output for a (fixed) price p .

(e) **(6 marks)** If the firm produces a *non-zero* level of output y , what should that level be in order to maximise its profits with minimised costs? Note that you should provide proof that your solution for y provides a maximum rather than a minimum.

(f) **(1 mark)** Would the most profitable level of output for the firm actually be $y = 0$? Justify your answer.