

# Sheet 2 , question 6 (replacement)

---

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\exp(-n \tanh x)}{(1+x^2)} dx$$

and the trapezium rule approximation by  $I_h$  and  $I_h^{(N)}$ .

- Estimate the discretisation error (do not attempt to bound the integrand in the analytic strip).
- Estimate the truncation error.
- Determine the optimal meshwidth  $h$ .

## Solution

a) Consider  $f(z) = \frac{\exp(-z \tanh z)}{1+z^2}$  for  $z \in \mathbb{C}$ .

This has singularities at  $z = \pm i$ , but is analytic in the strip  $\{|Im\{z\}| < 1\}$ . So by  
 (see \* below for more detail)  
 Theorem 2.12 of the lecture notes<sup>1</sup> with  $a < 1$ ,  
 we have, for the discretization error,

$$|I - I_h| \leq \frac{2M}{e^{2\pi a/h} - 1}$$

$$\begin{aligned} e^{2\pi a/h} &\gg 1 \\ \text{as } h \rightarrow 0, \text{ so} \end{aligned}$$

$$\frac{2M}{e^{2\pi a/h} - 1} \approx \frac{2M}{e^{2\pi a/h}}$$

$$= O(e^{-2\pi a/h})$$

$$\text{as } h \rightarrow 0.$$

\* More detail:

Check the necessary conditions for Thm 2.12 as follows:

Note that for all  $a' \in (-a, a)$ ,

$$|F(x + ia')| = \left| \frac{\exp(-(x + ia') \tanh(x + ia'))}{1 + (x + ia')^2} \right|$$

And

$$\tanh z = \frac{e^{2z} - 1}{e^{2z} + 1}$$

$$= \frac{(e^{2z} - 1)(e^{2\bar{z}} + 1)}{(e^{2z} + 1)(e^{2\bar{z}} + 1)} \quad \text{with } z = x + iy$$

$$= \frac{e^{4x} - 1 + e^{2x}(e^{2iy} - e^{-2iy})}{e^{4x} + 1 + e^{2x}(e^{2iy} + e^{-2iy})}$$

$$= \frac{e^{4x} - 1 + 2ie^{2x} \sin y}{e^{4x} + 1 + 2e^{2x} \cos y}$$

$$\Rightarrow \operatorname{Re} \{ z \tanh z \} =$$

$$= \frac{n(e^{4n} - 1) - 2e^{2n} y \sin y}{e^{4n} + 1 + 2e^{2n} \cos y}$$

$\approx \pm n$  as  $n \rightarrow \pm \infty$  for finite  $y$

So

$$|\exp(-(n+ia') \tanh(n+ia'))|$$

$\approx \exp(\mp n)$  as  $n \rightarrow \pm \infty$

$$\Rightarrow |f(n+ia')| \approx \frac{\exp(\mp n)}{n^2} \text{ as } n \rightarrow \pm \infty$$

$$< \frac{1}{n^2} \text{ as } n \rightarrow \pm \infty$$

$$\Rightarrow |f(n+ia')| \rightarrow 0 \text{ as } n \rightarrow \pm \infty$$

and furthermore  $\int_{-\infty}^{\infty} |f(n+ia')| dn$  is bounded

(Since the contributions to this integral due the sections of the range of integration located to  $\pm\infty$  are zero, since

$$\int \frac{1}{n^2} dn = \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \pm\infty).$$

b) The truncation error is defined to be

$$\begin{aligned} I_h - I_h^{[N]} &= h \left( \sum_{n=-\infty}^{-N-1} + \sum_{n=N+1}^{\infty} f(x_n) \right). \\ &= 2h \sum_{n=N+1}^{\infty} f(nh) \quad \text{Since } f(n) \text{ is even.} \end{aligned}$$

$x_n = nh$

As shown in part (a),

$$|f(n)| \approx \frac{e^{-n}}{n^2} \quad \text{as } n \rightarrow +\infty$$

And, it follows Lemma 2.13 of the lecture notes with  $g(n) \leq n$ , that

$$h \sum_{n=N+1}^{\infty} e^{-nh} = O(e^{-Nh}) \quad \text{as } N \rightarrow \infty$$

(assuming that  $h \rightarrow 0$  with  $h \gg 1/N$ ). Then, since

$$|f(x_n)| \approx |f(n^h)| < e^{-nh} \quad \text{as } n \rightarrow \infty,$$

we can deduce from the above that

$$h \sum_{n=N+1}^{\infty} |f(x_n)| = O(e^{-Nh}) \quad \text{as } N \rightarrow \infty.$$

Thus, we have

$$\left| I_h - I_h^{[N]} \right| = O(e^{-Nh}).$$

- c) To determine the optimal mesh width  $h$  we equate the discretisation and truncation errors:

$$-\frac{2\pi a}{h} = -Nh$$

$$\Rightarrow h^2 = \frac{2\pi a}{N}$$

$$\Rightarrow h = \left(\frac{2\pi a}{N}\right)^{1/2} \text{ for any } a < 1$$

$$\Rightarrow \text{take } h = \left(\frac{2\pi}{N}\right)^{1/2}.$$