Answers to Test 1

1. (i) (a) equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}\dot{r} - r\dot{\theta}^2 + \alpha,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(r^2\dot{\theta}) = 0.$$

(4 marks)

(b) inserting $r =$ constant into the second equation yields $\dot{\theta} =$ constant. $\dot{r} = 0$, $\ddot{r} = 0$ so that $r\dot{\theta}^2 = \alpha$ or $\dot{\theta} = \pm\sqrt{\alpha/r}$. Period $T = 2\pi/|\dot{\theta}| = 2\pi\sqrt{r/\alpha}$.

(5 marks)

(ii) Potential energy $V = mgy = mg\cosh x$.

$y = \cosh x$ so $\dot{y} = \dot{x}\sinh x$. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\cosh^2 x\ \dot{x}^2$. A Lagrangian is

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\cosh^2 x\ \dot{x}^2 - mg\cosh x.$$

(6 marks)

(iii) The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{d}{dt}\left(e^{\gamma t}\dot{y}\right) + ge^{\gamma t} = 0,$$

(this can also be written $\ddot{y} + \gamma\dot{y} = -g$). Integrating with respect to t

$$e^{\gamma t}\dot{y} = -\frac{g}{\gamma}e^{\gamma t} + c,$$

or $\dot{y} = -g/\gamma + ce^{-\gamma t}$. This integrates to $y = -gt/\gamma + Ce^{-\gamma t} + D$ where C and D are constants. For large t, $\dot{y} \approx -g/\gamma$ so rain hits the ground with speed g/γ.

(10 marks)

(Total: 25 marks)