Imperial College London

MATH97065 MATH70008

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Dynamical Systems

Date: 09 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In all your answers, you may quote results derived in the course (lecture notes inclusive of exercises) without proof, but any such results must be carefully stated or referenced.

1. Consider the following two maps $f_{1,2} : [0,1] \to [0,1]$ of the interval $[0,1]$ to itself:

$$\begin{aligned} f_1(x) &:= x^2, \\ f_2(x) &:= \begin{cases} \frac{e^x - 1}{\sqrt{e} - 1} & \text{if } x \in [0, \frac{1}{2}], \\ \frac{e^{1-x} - 1}{\sqrt{e} - 1} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

For each of these maps:

(a) Determine all ω-limit sets that are attractors. (6 marks)

(b) Determine whether or not the map has sensitive dependence on initial conditions. Motivate your answer. (6 marks)

(c) Determine the topological entropy. Motivate your answer. (8 marks)

(Total: 20 marks)

2. Consider the map $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} 1-2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2x-1, & \text{if } x \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ 2-2x, & \text{if } x \in \left(\frac{3}{4}, 1\right], \end{cases} \tag{1}$$

see also the graph of g sketched in Fig. 1.

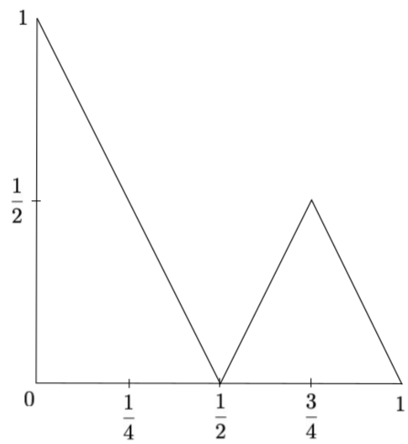

Figure 1: Graph of the map g (1).

Let $h : \Sigma_3^+ \to [0,1] \cup \{\emptyset\}$ be defined as

$$h(\omega_0\omega_1\omega_2\ldots) = \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} g^{-i}(I_{\omega_i})}. \tag{2}$$

where $I_0 := (0, \frac{1}{2})$, $I_1 := (\frac{1}{2}, \frac{3}{4})$ and $I_2 := (\frac{3}{4}, 1)$ and Σ_3^+ denotes the set of semi-infinite sequences $\omega_0\omega_1\omega_2\omega_3\ldots$ with $\omega_i \in \{0,1,2\}$ for all $i \in \mathbb{N}_0$.

(a) Determine the subset $\hat{\Sigma} := \{\omega \in \Sigma_3^+ \mid h(\omega) = \emptyset\}$. Motivate your answer. (6 marks)

(b) (i) Determine $h(\overline{02})$ and $h(\overline{12})$. (4 marks)

(ii) Determine $h^{-1}(\frac{1}{3})$ and $h^{-1}(\frac{3}{4})$. (4 marks)

(c) Prove that for any open set $U \subset [0,1]$, there exists $N \in \mathbb{N}$ such that $g^n(U) = [0,1]$ for all $n \geq N$. (6 marks)

(Total: 20 marks)

3. Consider the map g as defined in Equation (1) in Question 2. Let ρ be a probability measure on $\mathscr{B}([0,1])$ with Lebesgue density

$$p(x) := \begin{cases} \frac{4}{3} & \text{if } x \in [0, \frac{1}{2}], \\ \frac{2}{3} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

(a) Show that ρ is a g-invariant probability measure. (5 marks)

(b) Is ρ ergodic? Motivate your answer. (5 marks)

(c) Let $g' : [0,1] \setminus \{\frac{1}{2}, \frac{3}{4}\} \to \mathbb{R}$ denote the derivative of g. Explain why the expression

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(|g'(g^i(x))|)$$

is well-defined for all but a countable subset of $[0,1]$. Determine its value and discuss its meaning in terms of growth rates of derivatives. (5 marks)

(d) Show that there exists $f : [0,1] \to [0,1]$, topologically conjugate to g, for which the Lebesgue measure is an ergodic invariant measure. (5 marks)

(Total: 20 marks)

4. Consider the connectivity matrices

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

and let $\sigma_A : \Sigma^+_{3,A} \to \Sigma^+_{3,A}$ and $\sigma_B : \Sigma^+_{5,B} \to \Sigma^+_{5,B}$ be the (left) shifts on the associated topological Markov chains.

(a) Draw the Markov graphs associated with $\Sigma^+_{3,A}$ and $\Sigma^+_{5,B}$. (4 marks)

Let $\hat{h} : \Sigma^+_{3,A} \to \Sigma^+_{5,B}$ be defined as $\hat{h}(\omega_0\omega_1\omega_2\ldots) := \hat{h}(\omega_0\omega_1)\hat{h}(\omega_1\omega_2)\hat{h}(\omega_2\omega_3)\ldots$ with $\hat{h}(00) := 0$, $\hat{h}(01) := 1$, $\hat{h}(02) := 2$, $\hat{h}(10) := 3$, $\hat{h}(20) := 4$.
We consider $\Sigma^+_{3,A}$ and $\Sigma^+_{5,B}$ endowed with the usual (product) topology.

(b) (i) Show that $\sigma_B \circ \hat{h} = \hat{h} \circ \sigma_A$. (4 marks)

(ii) Show that $\hat{h}$ is invertible and continuous, with continuous inverse. (4 marks)

(c) Recall the maps g and h, as defined in Equation (1) in Question 2.
Let it be given that $h \circ \sigma_A = g \circ h$.

(i) Show that g is topologically semi-conjugate to σ_B. (4 marks)

(ii) Determine the Markov partition of $[0, 1]$ consisting of the labelling intervals for this topological semi-conjugacy. (4 marks)

(Total: 20 marks)

5. Consider the two-parameter family of maps $F_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ with

$$F_{\alpha,\beta}(x) := x + \alpha + \frac{1}{10\pi}\sin(2\pi\beta x).$$

and $\alpha, \beta \in \mathbb{R}$.

(a) For which values of $\alpha, \beta \in \mathbb{R}$ is $F_{\alpha,\beta}$ the lift of a circle homeomorphism $f_{\alpha,\beta} : S^1 \to S^1$, with

$$f_{\alpha,\beta}(x) := F_{\alpha,\beta}(x) \mod 1$$

for all $x \in [0,1) \simeq S^1$? (4 marks)

You may assume for the remainder of this Question that $f_{\alpha,3}$ is a circle homeomorphism. Consider the rotation number

$$\rho(f_{\alpha,3}) := \lim_{n\to\infty} \frac{1}{n}\Big(F^n_{\alpha,3}(x) - x\Big) \mod 1.$$

(b) (i) Show that $\rho(f_{\frac{2}{3},3}) = \frac{2}{3}$. (4 marks)

(ii) Determine whether $f_{\frac{2}{3},3}$ is topologically conjugate to the rigid rotation $x \to x + \frac{2}{3} \mod 1$. Motivate your answer. (4 marks)

(c) (i) Show that there exists a unique $\delta \in (\frac{1}{2}, \frac{2}{3})$ such that $\rho(f_{\delta,3}) = \frac{\pi}{6}$. (4 marks)

(ii) Determine whether $f_{\delta,3}$ is topologically conjugate to the rigid rotation $x \to x + \frac{\pi}{6} \mod 1$. Motivate your answer. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH96040/MATH97065/MATH97176/MATH97285

Dynamical Systems (Solutions)

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1. (a) f_1 has fixed points $\{0,1\}$ and $f(x) < x$ on $(0,1)$. Hence, the fixed points are the only possible forward limit points of orbits and thus are the only ω-limit sets: $\omega(x) = \{0\}$ if $x \in [0,1)$, $\omega(x) = \{1\}$ if $x = 1$. $\{0\}$ is an attractor, $\{1\}$ is not an attractor. sim. seen $\Downarrow$ 3, A

f_2 is a piecewise expanding full-branch map as $f_2([0,\frac{1}{2}]) = f_2([\frac{1}{2},1]) = [0,1]$ and on $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$ we have that $1 < |f_2'(x)|$. This implies among other things that f_2 has a dense orbit in $[0,1]$ which implies that it cannot have an attractor that is strictly smaller than $[0,1]$. $[0,1]$ is equal to the closure of the dense orbit and thus an ω-limit set. It is also a (trivial, by the definition) attractor. 3, B

(b) f_1 has no sensitive dependence on initial conditions since (for instance) f_1 is uniformly contracting on $[0,\frac{3}{7}]$ (as $f_1'(x) < 1$ for all $x \in [0,\frac{1}{2})$). unseen $\Downarrow$ 3, B

f_2 is a uniformly expanding full-branch map. It has been established in the course (in various ways) that such maps have sensitive dependence. seen $\Downarrow$ 3, A

(c) $h_{\mathrm{top}}(f_1) = 0$ since $f_1(1) = 1$ and for all $x \in [0,1)$, $\lim_{n\to\infty} f^n(x) = 0$. Namely, this implies that for any $\varepsilon < 1$, $\lim_{n\to\infty} \mathrm{sep}(n,\varepsilon,f_1) = 2$, and hence that $h_{\mathrm{top}}(f_1) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \ln \mathrm{sep}(n,\varepsilon,f_1) = 0$. unseen $\Downarrow$ 4, B

$h_{\mathrm{top}}(f_2) = \ln 2$ by a result in the course notes that asserts that piecewise expanding full-branch maps of the interval with n branches have topological entropy equal to $\ln n$. seen $\Downarrow$ 4, A

2. (a) $\{I_0, I_1, I_2\}$ is an expanding Markov partition for g. As $\overline{g(I_0)} = [0,1]$ and $\overline{g(I_1)} = \overline{g(I_2)} = \overline{I_0}$, results from the course yield that g is a factor of $\sigma_A : \Sigma^+_{3,A} \to \Sigma^+_{3,A}$ in the sense that $h \circ \sigma_A = g \circ h$, where A is the connectivity matrix featuring in Question 4 and $h : \Sigma^+_{3,A} \to [0,1]$ is surjective. The established action of g on the Markov partition (above) implies that $h(\omega) = \emptyset$ for all $\omega \in \Sigma^+_3 \setminus \Sigma^+_{3,A} =: \hat{\Sigma}$. unseen ⇓ 6, B

(b) (i) $h(\overline{02}) \in [0,1]$ since $\overline{02} \in \Sigma^+_{3,A}$. Let $x := h(\overline{02})$, then $2 - 2(1-2x) = x \Leftrightarrow x = 0$. Indeed $g(0) = 1 \in \overline{I_2}$ and $g(1) = 0 \in \overline{I_0}$. sim. seen ⇓
$h(\overline{12}) = \emptyset$ since $\overline{12} \in \hat{\Sigma}$. 4, B

(ii) $\frac{1}{3} \in I_0$ and $g(\frac{1}{3}) = \frac{1}{3}$ so $h^{-1}(\frac{1}{3}) = \{\overline{0}\}$. $h^{-1}(\frac{1}{3})$ is single-valued since all double-valued pre-images refer to boundaries of refinements of Markov partitions and these boundaries form a subset of $\{p/q \in \mathbb{Q} \mid p \in \mathbb{N},\ q = 2^{-n}$ for some $n \in \mathbb{N}\}$.
$\frac{3}{4} \in \overline{I_1} \cap \overline{I_2}$ and $g(\frac{3}{4}) = \frac{1}{2}$, $g(\frac{1}{2}) = 0$, $g(0) = 1$, etc (as above). We furthermore note that $g(I_1) = g(I_2) = I_0$, so that $h^{-1}(\frac{3}{4})$ starts with either 10 or 20 and then continuous as $\overline{02}$. Hence, $h^{-1}(\frac{3}{4}) := \{10\overline{02}, 20\overline{02}\}$. 4, C

(c) Since h is continuous, every open set U contains the image under h of a cylinder set $C_{\alpha_0 \ldots \alpha_{n-1}}$ for some $\Sigma^+_{3,A}$-admissible subsequence $\alpha_0 \ldots \alpha_{n-1}$ (and some $n \in \mathbb{N}$). Consider $h(C_{\alpha_0 \ldots \alpha_{n-1}}) \subset U$, then $g^n(h(C_{\alpha_0 \ldots \alpha_{n-1}})) = h(\sigma^n_A(C_{\alpha_0 \ldots \alpha_{n-1}})) = h(\Sigma^+_{3,A}) = [0,1]$. unseen ⇓ 6, D

3. (a) For every measurable $A \in \mathscr{B}([0,1])$, $\rho(A) = \rho(A_0) + \rho(A_1) + \rho(A_2)$ with $A_i := A \cap I_i$. Also, $g^{-1}(A_i) \cap g^{-1}(A_j) = \emptyset$ with $i,j \in \{0,1,2\}$, if $i \neq j$. Hence it suffices to check invariance of ρ for measurable subsets of I_0, I_1 and I_2. sim. seen ⇓

If $A \in I_0$ then, with λ denoting Lebesgue measure, we have $\rho(g^{-1}(A)) = \frac{1}{2} * \frac{4}{3}\lambda(A) + \frac{1}{2} * \frac{2}{3}\lambda(A) + \frac{1}{2} * \frac{2}{3}\lambda(A) = \frac{4}{3}\lambda(A) = \rho(A)$. Similarly, when $A \in I_1$ or $A \in I_2$, $\rho(g^{-1}(A)) = \frac{1}{2} * \frac{4}{3}\lambda(A) = \frac{2}{3}\lambda(A) = \rho(A)$. 5, A

(b) Yes. ρ is the push-forward of a Markov measure $\rho = h_*(\mu_{\nu,P})$ with sim. seen ⇓

$$\nu = \frac{1}{6}(4,1,1) \text{ and } P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

yielding (from lecture notes) a piecewise constant Lebesgue density on each I_i of $|\nu_i|/|I_i|$: $|\nu_0|/|I_0| = \frac{4}{6} * 2 = \frac{4}{3}$ and $|\nu_1|/|I_1| = |\nu_2|/|I_2| = \frac{1}{6} * 4 = \frac{2}{3}$. As the Markov measures are ergodic (lecture notes; observing that the Markov chain is irreducible) and ergodicity is preserved under measurable semi-conjugacy (lecture notes), so indeed ρ is ergodic. 5, A

(c) The subset of $[0,1]$ on which the argument in the sum is not defined ($\{x \in [0,1] \,|g^n(x) \in \{\frac{1}{2}, \frac{3}{4}\}$ for some $n \in \mathbb{N}\}$) is countable. When the sum exists, it is always equal to $\ln 2$ (as $|g'(x)| = 2$ for all $x \in [0,1] \setminus \{\frac{1}{2}, \frac{3}{4}\}$), so the value of the expression is $\ln 2$. It is the Lebesgue-a.s. asymptotic exponential growth rate of the absolute value of the derivative along orbits: unseen ⇓

$$\frac{1}{n}\ln(|(g^n)'(x)|) = \frac{1}{n}\ln(|\Pi_{i=0}^{n-1} g'(g^i(x))|) = \frac{1}{n}\sum_{i=0}^{n-1}\ln(|g'(g^i(x)|).$$

5, D

(d) If ρ is g-invariant then $T_*\rho$ is $T^{-1}\circ g\circ T$-invariant. T can be found such that $T_*\rho = \lambda$ by realizing that the piecewise constant density must be scaled up and down on the respective domains to achieve a constant density. The sought transformation is unseen ⇓

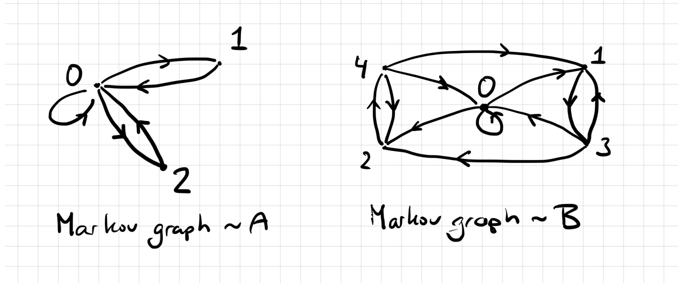
$$T(x) := \begin{cases} \frac{4}{3}x & \text{if } x \in [0, \frac{1}{2}], \\ \frac{2}{3}x + \frac{1}{3} & \text{if } x \in (\frac{1}{2}, 1], \end{cases} \qquad T^{-1}(x) := \begin{cases} \frac{3}{4}x & \text{if } x \in [0, \frac{2}{3}], \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in (\frac{2}{3}, 1]. \end{cases}$$

Indeed, writing for any $A \in \mathscr{B}([0,1])$, $A_0 := A \cap [0, \frac{2}{3}]$ and $A_1 := A \cap (\frac{2}{3}, 1]$, then

$$\begin{aligned} T_*\rho(A) &= T_*\rho(A_0) + T_*\rho(A_1) = \rho(T^{-1}(A_0)) + \rho(T^{-1}(A_1)) \\ &= \frac{4}{3} * \frac{3}{4}\lambda(A_0) + \frac{2}{3} * \frac{3}{2}\lambda(A_1) = \lambda(A_0) + \lambda(A_1) = \lambda(A). \end{aligned}$$

5, D

4. (a) sim. seen ⇓



4, A

(b) (i) For all $\omega \in \Sigma^+_{3,A}$, unseen ⇓

$$\begin{aligned}
\hat{h} \circ \sigma_A(\omega) &= \hat{h}(\omega_1\omega_2 \dots) = \hat{h}(\omega_1\omega_2)\hat{h}(\omega_2\omega_3)\dots, \\
\sigma_B \circ \hat{h}(\omega) &= \sigma_B(\hat{h}(\omega_0\omega_1)\hat{h}(\omega_1\omega_2)\hat{h}(\omega_2\omega_3)\dots) = \hat{h}(\omega_1\omega_2)\hat{h}(\omega_2\omega_3)\dots.
\end{aligned}$$

4, A

(ii) For all $\omega \in \Sigma^+_{3,A}$, $\hat{h}(\omega) \in \Sigma^+_{5,B}$ since iff $B_{ij} = 1$ then for $\alpha, \alpha' \in \{0,1,2\}$ such that $\hat{h}(\alpha\alpha') = i$, there exists $\beta \in \{0,1,2\}$ such that $\hat{h}(\alpha'\beta) = j$. Define for any $\tilde{\omega} \in \Sigma^+_{5,B}$, $\hat{h}^{-1}(\tilde{\omega}) := \hat{h}^{-1}(\tilde{\omega}_0)\hat{h}^{-1}(\tilde{\omega}_1)\hat{h}^{-1}(\tilde{\omega}_2)\dots$ with $\hat{h}^{-1}(0) := 0$, $\hat{h}^{-1}(1) := 0$, $\hat{h}^{-1}(2) := 0$, $\hat{h}^{-1}(3) := 1$, $\hat{h}^{-1}(4) := 2$. It follows that (by construction) $\hat{h}^{-1}(\tilde{\omega}) \in \Sigma^+_{3,A}$. One directly verifies unseen ⇓

$$\begin{aligned}
\hat{h}^{-1} \circ \hat{h}(\omega) &= \hat{h}^{-1}(\hat{h}(\omega_0\omega_1)\hat{h}(\omega_1\omega_2)\hat{h}(\omega_2\omega_3)\dots) = \omega_0\omega_1\omega_2\dots = \omega, \\
\hat{h} \circ \hat{h}^{-1}(\tilde{\omega}) &= \hat{h}(\hat{h}^{-1}(\tilde{\omega}_0)\hat{h}^{-1}(\tilde{\omega}_1)\hat{h}^{-1}(\tilde{\omega}_2)\dots) \\
&= \hat{h}(\hat{h}^{-1}(\tilde{\omega}_0)\hat{h}^{-1}(\tilde{\omega}_1))\hat{h}(\hat{h}^{-1}(\tilde{\omega}_1)\hat{h}^{-1}(\tilde{\omega}_2))\hat{h}(\hat{h}^{-1}(\tilde{\omega}_2)\hat{h}^{-1}(\tilde{\omega}_3))\dots \\
&= \tilde{\omega}_0\tilde{\omega}_1\tilde{\omega}_2\dots = \tilde{\omega}.
\end{aligned}$$

Continuity of $\hat{h}$ and its inverse is guaranteed by the fact that $\hat{h}$ maps cylinder sets of $\Sigma^+_{3,A}$ to cylinder sets of $\Sigma^+_{5,B}$ and $\hat{h}^{-1}$ maps cylinder sets of $\Sigma^+_{5,B}$ to cylinder sets of $\Sigma^+_{5,B}$, as the cylinder sets generate all open sets. 4, C

(c) (i) unseen ⇓

$$h \circ \sigma_A = g \circ h \;\;\Rightarrow\;\; h \circ \hat{h}^{-1} \circ \sigma_B \circ \hat{h} = g \circ h \;\;\Rightarrow\;\; (h \circ \hat{h}^{-1}) \circ \sigma_B = g \circ (h \circ \hat{h}^{-1}).$$

4, A

(ii) By the definition of $\hat{h}$, the labelling intervals for $\Sigma^+_{5,B}$ are precisely the elements of the first refinement of the Markov partition $\mathcal{R} := \{I_0, I_1, I_2\}$ introduced in Question 2, $\mathcal{R}_2 := \{I_{00}, I_{01}, I_{02}, I_{10}, I_{20}\}$ where $I_{02} := I_0 \cap g^{-1}(I_2) = (0, \frac{1}{8})$, $I_{01} := I_0 \cap g^{-1}(I_1) = (\frac{1}{8}, \frac{1}{4})$, $I_{00} := I_0 \cap g^{-1}(I_0) = (\frac{1}{4}, \frac{1}{2})$, $I_{10} := I_1 \cap g^{-1}(I_0) = I_1 = (\frac{1}{2}, \frac{3}{4})$, $I_{20} := I_2 \cap g^{-1}(I_0) = I_2 = (\frac{3}{4}, 1)$. unseen ⇓ 4, C

5. (a) $F_{\alpha,\beta}$ is continuous and continuously differentiable. In order for $F_{\alpha,\beta}$ to be a lift of a circle homeomorphism one needs to check that (1) $F_{\alpha,\beta}(x+1) = F_{\alpha,\beta}(x) + 1$ and (for invertibility) that (2) $F_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Ad (1): this implies that $\beta \in \mathbb{Z}$. Ad(2): from the fact that the derivative is given by $F'_{\alpha,\beta}(x) = 1 + \frac{\beta}{5}\cos(2\pi\beta x)$ it follows that iff $|\beta| \leq 5$, $F'_{\alpha,\beta}$ does not change sign and can only be equal to zero at isolated points, implying that $F_{\alpha,\beta}$ is invertible. $\boxed{4, \mathrm{M}}$

(b) (i) The point 0 has period 3 since $F_{\frac{2}{3},3}(0) = \frac{2}{3}$, $F_{\frac{2}{3},3}(\frac{2}{3}) = \frac{4}{3}$ and $F_{\frac{2}{3},3}(\frac{4}{3}) = 2$. As the rotation number is constant and can be deduced from any orbit, it follows that

$$\rho(f_{\frac{2}{3},3}) := \lim_{n\to\infty} \frac{F^n_{\frac{2}{3},3}(0)}{n} \mod 1 = \frac{2}{3}.$$

$\boxed{4, \mathrm{M}}$

(ii) Note that $(f^3_{\frac{2}{3},3})'(0) = f'_{\frac{2}{3},3}(0) * f'_{\frac{2}{3},3}(\frac{2}{3}) * f'_{\frac{2}{3},3}(\frac{4}{3}) = (1+\frac{3}{5})^3 > 1$. This implies that this periodic orbit is a repeller (or, equivalently, that this orbit is an attractor for $f^{-1}_{\frac{2}{3},1}$. As repellers (and attractors) are preserved under topological conjugacy and rigid rotations have none such, $f_{\frac{2}{3},3}$ cannot be topologically conjugated to a rigid rotation. $\boxed{4, \mathrm{M}}$

(c) (i) When $\alpha = \frac{1}{2}$, the point 0 has period 2 since $F_{\frac{1}{3},2}(0) = \frac{1}{2}$, $F_{\frac{1}{2},3}(\frac{1}{2}) = 0$ and hence $\rho(f_{\frac{1}{2},3}) = \frac{1}{2}$. We note that $\frac{1}{2} < \frac{\pi}{6} < \frac{2}{3}$.
For all $\alpha \geq \alpha'$, it holds that $F_{\alpha,3}(x) \geq F_{\alpha',3}(x)$ for all $x \in \mathbb{R}$ (strictly increasing family). This implies (using the definition of ρ for lifts, which is the formula in the question without the "mod 1") that $\rho(F_{\alpha,3}) \geq \rho(F_{\alpha',3})$. Moreover, since $F_{\alpha,3}$ is a continuous function of α, so is $\rho(F_{\alpha,3})$. Hence, since $\rho(F_{\frac{1}{2},3}) < \frac{\pi}{6} < \rho(F_{\frac{2}{3},3})$, there exists $\delta \in (\frac{1}{2},\frac{2}{3})$ such that $\rho(F_{\delta,3}) = \frac{\pi}{6}$. A result from the notes moreover establishes the fact that in a strictly increasing family of circle homeomorphisms (here $f_{\alpha,3}$), the rotation number is strictly increasing at any parameter value (here α) at which the rotation number is irrational. This implies that within the parameter interval $(\frac{1}{2},\frac{2}{3})$ the value of δ for which $\rho(f_{\delta,3}) = \frac{\pi}{6}$ is unique. $\boxed{4, \mathrm{M}}$

(ii) By Denjoy's theorem (from notes), it follows from the fact that $f_{\alpha,3}$ is C^2 (twice continuously differentiable), that whenever $\rho(f_{\alpha,3}) \notin \mathbb{Q}$ then $f_{\alpha,3}$ is topologically conjugate to a rigid rotation. As the rotation number is preserved under topological conjugacy, thus, $f_{\delta,3}$ is topologically conjugate to the rigid rotation $x \to x + \frac{\pi}{6}$. $\boxed{4, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total Mastery marks: 20 of 20 marks
Total marks: 100 of 100 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please

ExamModuleCode	QuestionNumber	Comments for Students
MATH60008/70008/97065	1	This question received a broad range of scores, from quite good to quite poor. There is not much in general to be commented on. As expected, easier parts were answered on average better than harder ones.
	2	This question received a broad range of scores, from quite good to quite poor. There is not much in general to be commented on. As expected, easier parts were answered on average better than harder ones.
	3	This question received a broad range of scores, from quite good to quite poor. There is not much in general to be commented on. As expected, easier parts were answered on average better than harder ones.
	4	This question received a broad range of scores, from quite good to quite poor. There is not much in general to be commented on. As expected, easier parts were answered on average better than harder ones.
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