IMPERIAL

MATH70017

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Tensor Calculus & General Relativity

Date: Tuesday, April 29, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

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Formula Sheet

Equation of Parallel Transport:

$$\frac{dv^a}{d\lambda} + \Gamma^a_{bc} v^b \frac{dx^c}{d\lambda} = 0.$$

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b,$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad \mathcal{R} = g^{bd} R_{bd}.$$

Einstein tensor:

$$G^{ab} = R^{ab} - \frac{1}{2} \mathcal{R} g^{ab}.$$

Schwarzschild metric:

$$ds^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right),$$

where R is the Schwarzschild radius. In the equatorial plane, $\theta = \pi/2$:

$$ds^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 \; d\phi^2.$$

1. (a) Define four-velocity and four-acceleration. (3 marks)

(b) The trajectory of a uniformly accelerating particle is

$$x^0(\tau) = \frac{c^2}{A}\sinh(A\tau/c), \qquad x^1(\tau) = \frac{c^2}{A}\cosh(A\tau/c), \qquad x^2 = x^3 = 0,$$

where τ is the proper time, c is the speed of light, and A is a positive constant.

(i) Determine the four-velocity, $u^\mu(\tau)$, and four-acceleration, $a^\mu(\tau)$, of the particle.

(ii) Determine $a \cdot a$.

(7 marks)

(c) Consider the photon trajectory $x^0 = x^1 - L$ where L is a positive constant. For what τ does the particle considered in part (b) cross the photon trajectory? (5 marks)

(d) Suppose that the particle from part (b) reaches speed v at $x^0 = cT$. Determine A.

(5 marks)

(Total: 20 marks)

2. Cylindrical polar coordinates (ρ, θ, z) are defined through $x = \rho\cos\theta, \quad y = \rho\sin\theta$, and z is the same in cylindrical and cartesian coordinates. In cartesian coordinates the metric is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

(a) Show that the metric in cylindrical polar coordinates takes the form

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2.$$

(4 marks)

(b) The paraboloid $z = \frac{1}{2}\rho^2$ is embedded in the three dimensional space. Show that the metric on the paraboloid has the form

$$ds^2 = (1+\rho^2)d\rho^2 + \rho^2 d\theta^2.$$

(4 marks)

(c) Identify any quantities that are constant along geodesics on the paraboloid from part (b).

(4 marks)

(d) The non-zero Christoffel symbols for the paraboloid are

$$\Gamma^\rho_{\rho\rho} = \frac{\rho}{1+\rho^2}, \quad \Gamma^\rho_{\theta\theta} = -\frac{\rho}{1+\rho^2}, \quad \Gamma^\theta_{\rho\theta} = \Gamma^\theta_{\theta\rho} = \frac{1}{\rho}.$$

Obtain and solve the equation of parallel transport along the curve ρ=constant.

(8 marks)

(Total: 20 marks)

3. (a) Consider the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

where $y > 0$. The nonzero Christoffel symbols are

$$\Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^y_{yy} = -\frac{1}{y}, \quad \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}.$$

(i) Verify $\Gamma^y_{xx} = 1/y$.

(ii) Compute R^x_{yxy} and determine R_{yy}.

(10 marks)

(b) Starting with the Bianchi identity

$$\nabla_e R^a_{bcd} + \nabla_c R^a_{bde} + \nabla_d R^a_{bec} = 0,$$

obtain the contracted Bianchi identity

$$2\nabla_p R^p_q - \nabla_q \mathcal{R} = 0.$$

Comment briefly on the relevance of this result to General Relativity.

(10 marks)

(Total: 20 marks)

4. Null geodesics in the equatorial plane ($\theta = \pi/2$) of the Schwarzschild space-time satisfy the equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - Ru^3 = \text{constant}, \tag{1}$$

where $u = 1/r$ and R is the Schwarzschild radius.

(a) Show that null geodesics in the equatorial plane satisfy the orbit equation

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}Ru^2 = 0. \tag{2}$$

(4 marks)

(b) Consider the solution of (2) satisfying the boundary conditions

$$u(\phi = 0) = 0, \quad \frac{du}{d\phi}(\phi = 0) = \frac{1}{a},$$

where a is a positive constant.

(i) Give a physical interpretation of the boundary conditions (a diagram may be helpful).

(ii) Write the solution in the form

$$u = \frac{\sin\phi}{a} + \tilde{u}(\phi),$$

where $\tilde{u}$ satisfies the boundary conditions

$$\tilde{u}(0) = 0, \quad \tilde{u}'(0) = 0.$$

Suppose that $a \gg R$. Treating $\tilde{u}$ as small leads to the linear equation

$$\frac{d^2\tilde{u}}{d\phi^2} + \tilde{u} - \frac{3R}{2a^2}\sin^2\phi = 0.$$

Solve this equation subject to the given boundary conditions.

(iii) Explain how your solution from part (ii) leads to Einstein's approximate result for the deflection of light by a massive body

$$\alpha = \frac{2R}{a},$$

where α is the deflection angle, R is the Schwarzschild radius of the massive body and a is the impact parameter. Assume $a \gg R$.

(11 marks)

(c) Explain in principle how to obtain the deflection angle if $a \gg R$ does not hold.

Hints: use equation (1). You may assume that u has a maximum, u_0, which means that the photon trajectory does not cross the event horizon and end at the $r = 0$ singularity.

(5 marks)

(Total: 20 marks)

5. The Kerr metric is

$$ds^2 = c^2\left(1 - \frac{Rr}{\Sigma}\right)dt^2 + \frac{2Racr\sin^2\theta}{\Sigma}dt d\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$$

$$-\left(r^2 + a^2 + \frac{Ra^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - Rr + a^2$. Here a is a constant and R is a positive constant. If $R > 2|a|$, the space-time has horizons at

$$r_\pm = \frac{R \pm \sqrt{R^2 - 4a^2}}{2}.$$

(a) Show that g_{tt} is negative if

$$\frac{1}{2}R - \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta} < r < \frac{1}{2}R + \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta}.$$

Assume $R > 2|a|$. (7 marks)

(b) Consider the (non-geodesic) trajectory where r, θ and ϕ are constant. Show that that there is a region (the ergosphere) outside the outer horizon, $r = r_+$, where the r, θ, ϕ constant trajectory is space-like. Assume $R > 2|a|$. (6 marks)

(c) Consider the same trajectory from part (b). Can this be space-like if $R < 2|a|$? What happens if $R = 0$? (7 marks)

(Total: 20 marks)

Tensor Calculus and General Relativity **Summer Term 2025**

Answers to 2025 Examination

1. **(a)** Four-velocity and four-acceleration are defined through, $u^\mu = dx^\mu/d\tau$, and, $a^\mu = du^\mu/d\tau$, respectively. Here $x^\mu = x^\mu(\tau)$ is the trajectory of a particle as a function of the proper time τ.

(3 marks, bookwork, A)

(b) (i) Here

$$u^0 = \frac{dx^0}{d\tau} = c\cosh(A\tau/c), \quad u^1 = \frac{dx^1}{d\tau} = c\sinh(A\tau/c), \quad u^2 = u^3 = 0.$$

Differentiating again

$$a^0 = \frac{du^0}{d\tau} = A\sinh(A\tau/c), \quad a^1 = \frac{du^1}{d\tau} = A\cosh(A\tau/c), \quad a^2 = a^3 = 0.$$

(4 marks, seen, A)

(ii) $a \cdot a = (a^0)^2 - (a^1)^2 = A^2[\sinh^2(A\tau/c) - \cosh^2(A\tau/c)] = -A^2.$

(3 marks, seen, A)

(c) The particle and photon trajectories cross when

$$L = x^1 - x^0 = \frac{c^2}{A}\cosh(A\tau/c) - \frac{c^2}{A}\sinh(A\tau/c) = \frac{c^2}{A}e^{-A\tau/c},$$

so that

$$-A\tau/c = \log\frac{LA}{c^2},$$

or

$$\tau = \frac{c}{A}\log\frac{c^2}{LA}.$$

(5 marks, unseen, C)

(d) $x^0 = cT = (c^2/A)\sinh(A\tau/c)$. From part (b) $u^1 = c\sinh(A\tau/c)$ so that

$$A = \frac{c\sinh(A\tau/c)}{T} = \frac{u^1}{T} = \frac{v}{T\sqrt{1-\frac{v^2}{c^2}}}.$$

(5 marks, unseen, D)

2. **(a)** $x = \rho\cos\theta$, $dx = \cos\theta d\rho - \rho\sin\theta d\theta$

   $y = \rho\sin\theta$, $dy = \sin\theta d\rho + \rho\cos\theta d\theta$

   Hence $dx^2 + dy^2 = (\cos\theta d\rho - \rho\sin\theta d\theta)^2 + (\sin\theta d\rho + \rho\cos\theta d\theta)^2 = (\cos^2\theta + \sin^2\theta)d\rho^2 + \rho^2(\sin^2\theta + \cos^2\theta)d\theta^2 = d\rho^2 + \rho^2 d\theta^2$. Hence the result.

(4 marks, seen similar, A)

(b) $z = \frac{1}{2}\rho^2$, $dz = \rho d\rho$, $dz^2 = \rho^2 d\rho^2$ so that

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 = (1+\rho^2)d\rho^2 + \rho^2 d\theta^2.$$

(4 marks, seen similar, A)

(c) As the components of the metric do not depend on θ, $g_{\theta b}dx^b/ds = g_{\theta\theta}d\theta/s = \rho^2 d\theta/ds$ is constant along geodesics. A further constant is

$$g_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = (1+\rho^2)\left(\frac{d\rho}{ds}\right)^2 + \rho^2\left(\frac{d\theta}{ds}\right)^2 = 1.$$

(4 marks, seen similar, B)

(d) Using θ as a parameter the equation of parallel transport yields

$$\frac{dv^\rho}{d\theta} + \Gamma^\rho_{b\theta}v^b \cdot 1 = \frac{dv^\rho}{d\theta} + \Gamma^\rho_{\theta\theta}v^\theta = \frac{dv^\rho}{d\theta} - \frac{\rho}{1+\rho^2}v^\theta = 0,$$

$$\frac{dv^\theta}{d\theta} + \Gamma^\theta_{b\theta}v^b \cdot 1 = \frac{dv^\theta}{d\theta} + \Gamma^\theta_{\rho\theta}v^\rho = \frac{dv^\theta}{d\theta} + \frac{1}{\rho}v^\rho = 0.$$

Therefore

$$\frac{d^2v^\rho}{d\theta^2} - \frac{\rho}{1+\rho^2}\frac{dv^\theta}{d\theta} = \frac{d^2v^\rho}{d\theta^2} + \frac{1}{1+\rho^2}v^\rho = 0,$$

with general solution

$$v^\rho = A\cos\left[(1+\rho^2)^{-1/2}\theta + \beta\right],$$

which gives

$$v^\theta = \frac{1+\rho^2}{\rho}\frac{dv^\rho}{d\theta} = -\frac{A(1+\rho^2)^{1/2}}{\rho}\sin\left[(1+\rho^2)^{-1/2}\theta + \beta\right].$$

(8 marks, seen similar, C)

3. **(a)** (i)

$$\Gamma^y_{bc} = \frac{1}{2}g^{yd}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) = \frac{1}{2}g^{yy}\left(\partial_b g_{yc} + \partial_c g_{by} - \partial_y g_{bc}\right).$$

$b = c = x$, $\Gamma^y_{xx} = \frac{1}{2}y^2(0 + 0 - \partial_y y^{-2}) = 1/y$

(4 marks, seen similar, A)

(ii) $R^a_{bcd} = \partial_c\Gamma^a_{bd} + \Gamma^a_{ec}\Gamma^e_{bd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc}$

$R^x_{yxy} = \partial_x\Gamma^a_{yy} + \Gamma^x_{ex}\Gamma^e_{yy} - \partial_y\Gamma^x_{yx} - \Gamma^x_{ey}\Gamma^e_{yx}$

$= 0 + \Gamma^x_{yx}\Gamma^y_{yy} - \partial_y\frac{-1}{y} - \Gamma^x_{xy}\Gamma^x_{yx}$

$= 0 + \frac{1}{y^2} - \frac{1}{y^2} - \frac{1}{y^2} = -\frac{1}{y^2}.$

$R_{yy} = R^a_{yay} = R^x_{yxy} = -1/y^2$

(6 marks, seen similar, B)

(b) Contract a with d in the Bianchi identity and use the definition of the Ricci tensor which gives

$$-\nabla_e R_{bc} + \nabla_c R_{be} + \nabla_a R^a_{bec} = 0.$$

Multiplying by g^{be} gives (here $g^{be}R^a_{bec} = -R^a_c$ and the inverse metric commutes with the covariant derivative)

$$-\nabla_e R^e_c + \nabla_c\mathcal{R} - \nabla_a R^a_c = 0,$$

which is the contracted Bianchi identity (here e and a are dummy indices. (6 marks, bookwork, C)

Multiplying the Bianchi identity by g^{qr} yields $\nabla_p G^{pr}$ where G^{pr} is the Einstein tensor. This property is shared by the energy-momentum tensor motivating the identification of $T^{\mu\nu}$ and $G^{\mu\nu}$ in Einstein's equations.

(4 marks, bookwork, C)

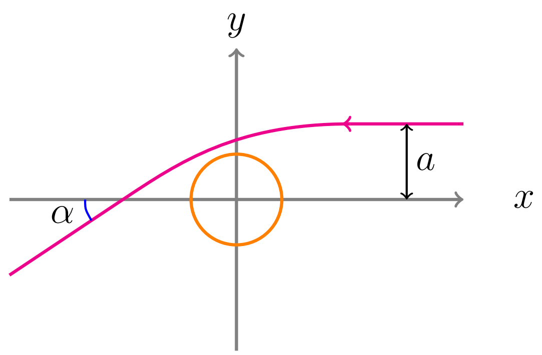
4. **(a)** Differentiating with respect to ϕ

$$2\frac{du}{d\phi} \cdot \frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} - 3Ru^2\frac{du}{d\phi} = 0.$$

Dividing by $du/d\phi$ gives the required orbit equation.

(4 marks, bookwork, A)

(b) (i) The boundary conditions for u which represents a light ray starting at $\phi = 0$ with impact parameter a. In other words for small ϕ it matches $u = a^{-1}\sin\phi$ or $y = a$ (a straight line).



(4 marks, bookwork, A)

(ii)

$$\frac{d^2\tilde{u}}{d\phi^2} + \tilde{u} = \frac{3R}{2a^2}\sin^2\phi = \frac{3R}{4a^2}(1 - \cos 2\phi).$$

A particular solution is

$$\tilde{u} = \frac{3R}{4a^2} + \frac{R}{4a^2}\cos 2\phi.$$

The general solution is

$$\tilde{u} = \frac{3R}{4a^2} + \frac{R}{4a^2}\cos 2\phi + A\cos\phi + B\sin\phi.$$

$\tilde{u}(0) = 0$ gives $A = -R/a^2$. $\tilde{u}'(0) = 0$ gives $B = 0$

$$\tilde{u} = \frac{3R}{4a^2} + \frac{R}{4a^2}\cos 2\phi - \frac{R}{a^2}\cos\phi.$$

(4 marks, seen similar, B)

(iii) The deflection angle α is defined so that $u(\pi + \alpha) = 0$. In this approximation α is small. We have

$$u = \frac{\sin\phi}{a} + \frac{3R}{4a^2} + \frac{R}{4a^2}\cos 2\phi - \frac{R}{a^2}\cos\phi$$

$$= -\frac{\sin(\phi - \pi)}{a} + \frac{3R}{4a^2} + \frac{R}{4a^2}\cos 2\phi - \frac{R}{a^2}\cos\phi \approx -\frac{\phi - \pi}{a} + \frac{2R}{a^2},$$

for ϕ near π. Accordingly, $u(\pi + \alpha) = 0$ has the approximate solution $\alpha = 2R/a$. (3 marks, seen similar, C)

(c) The boundary conditions $u(0) = 0$, $u'(0) = 1/a$ fix the constant in (1) as $1/a^2$ giving

$$\frac{du}{d\phi} = \pm\sqrt{\frac{1}{a^2} - u^2 + Ru^3},$$

or

$$d\phi = \pm\frac{du}{\sqrt{\frac{1}{a^2} - u^2 + Ru^3}}.$$

The photon trajectory starts at $\phi = 0$ and as ϕ increases to $\frac{1}{2}(\pi + \alpha)$, u increases to its maximum value u_0. As ϕ increases from $\frac{1}{2}(\pi + \alpha)$ to $\pi + \alpha$, u decreases from u_0 to $\pi + \alpha$. Considering the first part of this trajectory yields

$$\frac{\pi + \alpha}{2} = \int_0^{u_0} \frac{du}{\sqrt{\frac{1}{a^2} - u^2 + Ru^3}}.$$

u_0 is a root of

$$\frac{1}{a^2} - u^2 - Ru^3.$$

The deflection angle is defined through $u(\pi + \alpha) = 0$.

(5 marks, unseen, D)

5. **(a)**

$$g_{tt} = c^2\left(1 - \frac{Rr}{\Sigma}\right) = \frac{c^2}{\Sigma}(\Sigma - Rr).$$

As Σ is positive except at the ring singularity, g_{tt} is negative if

$$\Sigma - Rr = r^2 + a^2\cos^2 - Rr = (r - R/2)^2 - \frac{1}{4}R^2 + a^2\cos^2\theta < 0,$$

or

$$(r - R/2)^2 < \frac{1}{4}R^2 - a^2\cos^2\theta, \tag{1}$$

giving

$$-\frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta} < r - \frac{R}{2} < \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta},$$

hence the result.

(7 marks, seen similar)

(b) If r, θ and ϕ are constant $ds^2 = g_{tt}dt^2$ so the trajectory is space-like if g_{tt} is negative. From part (a) the region

$$r_+ < r < \frac{R}{2} + \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta}$$

is outside the outer event horizon with g_{tt} negative.

(6 marks, seen similar)

(c) Assume that $2|a| > R$. If the right hand side of (1) is negative the inequality has no solutions. This happens if

$$\cos^2\theta > \frac{R^2}{4a^2}.$$

For this range of θ values all trajectories are time-like. For other θ the trajectories are space-like for

$$\frac{1}{2}R - \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta} < r < \frac{1}{2}R + \frac{1}{2}\sqrt{R^2 - 4a^2\cos^2\theta}.$$

If $R = 0$ (the ring wormhole) the trajectory is time-like for all trajectories (except at the ring singularity).

(7 marks, unseen)

MATH70017 Tensor Calculus & General Relativity Markers Comments

Question 1	As expected (a) and (b) were well answered. There was more variability in the quality of answers to parts (c) and (d).
Question 2	This question was well answered overall. There was one minor issue. Most candidates only provided one constant in part (c).
Question 3	This question was split into two unrelated part worth 10 marks each. Part (a) was generally well answered. The answering of part (b) was more variable. Some candidates simply attempted to prove the contracted Bianchi identity without commenting on its relevance to General Relativity. Most candidates who did comment on the relevance did not mention the energy-momentum tensor!
Question 4	Overall the answering of this question was disappointing. As expected most candidates obtained the full 4 marks for part (a). The marks for part (b) were significantly lower than expected. This part, deriving Einstein's formula for the deflection of light by a massive body, could be described as a complex bookwork task. Although fairly complex the task was broken up into smaller parts and the key equations were included in the question. The marks for part (c) were also low but this was not wholly unexpected as this deflection problem is in the unseen category. Most students did not identify the constant in equation (1) with 1/a^2. Some candidates gave a correct but more complicated form of the constant using u_0.
Question 5	The Mastery question concerned the ergosphere in the Kerr space time. The whole question was generally well answered. One issue was that some students essentially repeated the analysis of part (a) when answering part (b).