Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

This paper is also taken for the relevant examination for the Associateship.

M3S4/M4S4

Applied Probability

Date: Wednesday, 2 June 2010 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (i) Let $\{N_t\}_{t \geq 0}$ be a counting process. Give a formal definition, if it is to be a time-homogeneous Poisson Process of rate $\lambda \in \mathbb{R}_+$.

(ii) Suppose males and females arrive at a queue at a major department store according to independent Poisson processes $\{N_t^1\}_{t \geq 0}$ and $\{N_t^2\}_{t \geq 0}$ of rates $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$ respectively. Let $\bar{N}_t = N_t^1 + N_t^2$ be the total number of individuals arriving at the queue. Show that $\{\bar{N}_t\}_{t \geq 0}$ is a Poisson process of rate $\alpha + \beta$. Note that you may use, without proof, that for a Poisson random variable N of parameter α, $\mathbb{E}[e^{-sN}] = \exp\{\alpha(e^{-s}-1)\}$.

(iii) As part of an internal assessment, the department store wishes to predict the average total time individuals spend queuing. Suppose that each individual waits in the queue for a time that is independently and exponentially distributed with parameter $\rho \in \mathbb{R}_+$. Let Y_t denote the total-time spent queuing. Show that the characteristic function $\phi_{Y_t}(s)$ is equal to

$$\phi_{Y_t}(s) = \exp\left\{\left(\frac{is}{\rho - is}\right)(\alpha + \beta)t\right\}$$

and hence find $\mathbb{E}[Y_t|\alpha, \beta]$. Recall that for a random variable Z, the characteristic function is defined as

$$\phi_Z(s) = \mathbb{E}\left[\exp\{isZ\}\right]$$

with $i = \sqrt{-1}$.

(iv) To complete the prediction, the department store assumes that α and β are independent exponential 1 random variables. Calculate $\mathbb{E}[Y_t]$. What do you notice as ρ increases?

2. (i) Let $\{N_t\}_{t\geq 0}$ be a counting process. Give a formal definition, if it is to be a birth process.

(ii) Let $p_n(t)$ be the probability of being in state $n \in \{0, 1, \dots\}$ at a time $t \geq 0$. For the birth process, derive the forward equations associated to $p_n(t)$:

$$\begin{aligned} p_n'(t) &= -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) \quad n \geq 1 \\ p_0'(t) &= -\lambda_0 p_0(t). \end{aligned}$$

(iii) Suppose that the counts of emitted particles from a given radio-active source are recorded and that there are I particles emitted when we start at time 0. At subsequent time-points, when $n \geq I$ particles have been emitted, the probability that a particle is emitted, in $(t, t+\delta)$, is $n\lambda\delta + o(\delta)$, $(\lambda, t, \delta > 0)$. Let N_t denote the number of emitted particles at time t. Using the forward equations show that

$$p_n(t) = \binom{n-1}{I-1} e^{-\lambda I t}(1 - e^{-\lambda t})^{n-I}$$

for $n \geq I$. [Hint: start by considering $n = I$ and build up an inductive proof. You may use the fact that:

$$\int_0^t (1 - e^{-\lambda s})^{n-I} e^{\lambda s(n+1-I)} ds = \frac{(1 - e^{-\lambda t})^{n-I+1} e^{\lambda t(n+1-I)}}{\lambda(n - I + 1)}.$$

]

(iv) Using the result for $p_n(t)$, deduce that:

$$\mathbb{E}[N_t] = I e^{\lambda t}.$$

Note that you may quote, without proof, any distributional results that you know about $p_n(t)$.[Hint: Setting $p = e^{-\lambda t}$, consider differentiating, on both sides, the equation

$$\sum_{k=I}^{\infty} p_n(t) = 1$$

w.r.t. p and solving for the associated expectation.]

3. (i) Let $X_0, X_1, X_2, \ldots$ be a sequence of discrete-valued random variables on a state-space E. When is the process $\{X_n\}_{n\geq 0}$ a Markov chain?

(ii) Now, suppose $\{X_n\}$ is an irreducible, positive recurrent Markov chain with a doubly infinite time index n.
(a) Let $Y_n = X_{-n}$; show that $\{Y_n\}$ is a Markov chain.
(b) What does it mean for the chain $\{X_n\}$ to be time-reversible?
(c) Show that $\{X_n\}$ is time-reversible, if and only if:

$$\pi_i p_{ij} = \pi_j p_{ji}$$

for every $i, j \in E$. Here π is the stationary distribution of the transition matrix $P = (p_{ij})$ associated to $\{X_n\}$.

(iii) Two containers, A and B are placed adjacent to each other and gas is allowed to pass through a small aperture joining them. A total of $m > 1$ gas molecules is distributed between the containers. We assume that at each epoch of time, one molecule picked uniformly at random from the m available, passes through this aperture. Let $\{X_n\}$ be the number of molecules in container A after n units of time.
(a) Why is $\{X_n\}$ a Markov chain? Write down the transition probabilities.
(b) Assume that the chain is time-reversible. Without solving $\pi_{0:m} = \pi_{0:m}P$, show that the stationary distribution is:

$$\pi_i = \binom{m}{i}\left(\frac{1}{2}\right)^m \quad 0 \leq i \leq m.$$

No marks will be awarded for solving $\pi_{0:m} = \pi_{0:m}P$. You may use the fact that:

$$\sum_{i=0}^{m}\binom{m}{i} = 2^m.$$

[Hint: Consider solving $\pi_i p_{i,i+1} = \pi_{i+1}p_{i+1,i}$ $(0 \leq i \leq m-1)$ obtaining π_i, $1 \leq i \leq m$ in terms of π_0.]

4. (i) Let $\{X_t\}_{t\geq 0}$ be a discrete-valued stochastic process. Give a formal definition, if it is to be a birth-death process, of birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$.

(ii) What is the generator for the birth-death process? Assuming that

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} < +\infty$$

(with the first term defined to be 1) use the generator and strong induction (or otherwise), to show that the stationary distribution is, for any $n \geq 1$

$$\pi_n = \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} \pi_0.$$

(iii) An investment fund allows n investors. If there are $0 \leq k \leq n-1$ investors in the fund, a new investor is allowed, in $(t, t+\delta)$ with probability $\lambda \exp\{k+1\}\delta + o(\delta)$ $(\lambda, \delta, t > 0)$. If there are $k \geq 1$ investors, one might leave in $(t, t+\delta)$ with probability $\mu e^k \delta + o(\delta)$.
a) Suppose that $\lambda < \mu$. Find the probability, in equilibrium, that there are no investors allowed.
b) Given that no investors are allowed, how long would you expect to wait until an investment is allowed?