

## MATH50001 - Problems Sheet 6

1. Compute the integral

$$\int_0^{2\pi} \frac{\sin^2 \theta}{2 + \cos \theta} d\theta.$$

2. Let  $\text{Log}(z)$  be the principle value of  $\log(z)$  and let

$$f(z) = -\text{Log}(z) + \int_1^z \frac{e^\eta}{\eta} d\eta.$$

a) Prove that  $f(z)$  has holomorphic continuation to the whole complex plane  $\mathbb{C}$ .

b) Find Taylor series for  $f(z)$  at  $z_0 = 0$ .

3. Let  $f$  be continuous on  $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ . Prove

$$\overline{\oint_{\gamma} f(z) dz} = -\oint_{\gamma} \frac{\overline{f(z)}}{z^2} dz.$$

4. Compute

$$f(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z(z-w)}$$

for all  $w : |w| \neq 1$ , where

$$\gamma = \{z \in \mathbb{C} : |z| = 1\}.$$

5. Let  $A = \{z : r \leq |z| \leq R\}$ , where  $0 < r < R < \infty$ . Show that there is a positive number  $\varepsilon$  such that for an arbitrary polynomial  $p$

$$\max_{z \in A} |p(z) - z^{-1}| > \varepsilon.$$

6. How many roots has the polynomial

$$w(z) = z^3 + 5z + 1 \quad \text{if } |z| > 1?$$

7. Show that the polynomial  $z^5 + 15z + 1$  has precisely four zeros in the annular region  $\{z : 3/2 < |z| < 2\}$ .

8. Let  $w(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$ . How many zeros (counting multiplicities) does  $w$  has in the unit disc.

2

**9.** How many zeros has the complex polynomial  $3z^9 + 8z^6 + z^5 + 2z^3 + 1$  in the annulus  $\{z : 1 < |z| < 2\}$ ?

**10.** Let  $a > e$ . Show that the equation  $e^z = az^n$  has  $n$  roots inside the circle  $|z| < 1$ .