Exercise 2.1. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^1 > 0\}$?

(d) $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^i \in [0, 1)\}$?

(e) $\mathbb{Q}^n := \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Exercise 2.2. Let $(x_i)_{i=0}^{\infty}$ be a sequence of vectors $x_i \in \mathbb{R}^n$ with $x_i \to x$. Suppose that the x_i satisfy $\|x_i\| < r$ for all i and some $r > 0$. Show that:

$$\|x\| \leq r.$$

Exercise 2.3. (a) Show that if U_1, U_2 are open in $\mathbb{R}^n$, then so are the sets

$$i) \;\; U_1 \cup U_2 \qquad\qquad ii) \;\; U_1 \cap U_2$$

(b) Suppose U_α, for α in an index set I, is a collection of open sets in $\mathbb{R}^n$.

(i) Show that $\bigcup_{\alpha \in I} U_\alpha$ is open in $\mathbb{R}^n$.

(ii) Give an example showing that $\bigcap_{\alpha \in I} U_\alpha$ need not be open.

Exercise 2.4. Suppose $A \subset \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}^m$. Show that $\lim_{x \to p} f(x) = F$ if and only if for any sequence $(x_i)_{i=0}^{\infty}$ in $A \setminus \{p\}$ which converges to p we have

$$f(x_i) \to F, \quad \text{as } i \to \infty.$$

Exercise 2.5. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \dots, 0)$ is continuous on $\mathbb{R}$.

(b) Let $A \subset \mathbb{R}^n$ and suppose we are given a map $f : A \to \mathbb{R}^m$ where

$$f(x^1, \dots, x^n) \mapsto \left(f^1\big((x^1, \dots, x^n)\big), \dots, f^m\big((x^1, \dots, x^n)\big)\right).$$

Show that f is continuous at $p \in A$ if and only if each map $f^k : A \to \mathbb{R}$ is continuous at p, for $k = 1, \dots, m$.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f\big((x^1, x^2, \dots, x^n)\big) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$, [1].

Exercise 2.6.*

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

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Questions marked with $*$ are optional

[1] Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

Exercise 2.7. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f(x) = x.$$

Show that f is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \mathrm{id},$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Exercise 2.8. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = (2\xi \quad 2\eta)$$

Exercise 2.9. One might hope that the differential can be calculated by finding

$$\lim_{x \to p} \frac{f(x) - f(p)}{\|x - p\|}.$$

By considering the example of Exercise 2.7 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Exercise 2.10. Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at p and

$$Dh(p) = Df(p) + Dg(p)$$