

# MATH50004/50015/50019 Differential Equations

## Spring Term 2023/24

### Repetition Material 1: Higher-order differential equations

In this course, we focus on first-order differential equation, i.e. the right hand side depends only on the value of the solution, but not on its derivatives. In many applications, differential equations with higher-order derivatives appear naturally, and an example is given by the harmonic oscillator  $\dot{x} = -x$ , which is solved by the sin and cos function, and linear combinations of these two functions (this works, because this is a linear equation). Remember also that you have encountered higher-order differential equations already in Year 1, and you have learned how to solve (autonomous) linear higher-order differential equations.

In this short note, we show that higher-order differential equations can be equivalently rewritten as first-order differential equation. For this reason, all results in this course are formulated for first-order differential equations. Please note that this document is not required to understand any material of the course, and it is not examinable.

The formal definition of a higher-order differential equation is given as follows.

**Definition 1** (Higher-order differential equation). Consider  $n, d \in \mathbb{N}$ , an open set  $D \subset \mathbb{R} \times \mathbb{R}^{nd}$ , and a function  $f : D \rightarrow \mathbb{R}^d$ . An equation of the form

$$x^{(n)} = f(t, x, \dot{x}, \ddot{x}, \dots, x^{(n-1)}) \quad (1)$$

is called a  $d$ -dimensional *ordinary differential equation* of order  $n$ . An  $n$ -times differentiable function  $\lambda : I \rightarrow \mathbb{R}^d$  on an interval  $I \subset \mathbb{R}$  is called a *solution* to the differential equation (1) if  $(t, \lambda(t), \dot{\lambda}(t), \ddot{\lambda}(t), \dots, \lambda^{(n-1)}(t)) \in D$  and

$$\lambda^{(n)}(t) = f(t, \lambda(t), \dot{\lambda}(t), \ddot{\lambda}(t), \dots, \lambda^{(n-1)}(t)) \quad \text{for all } t \in I, \quad (2)$$

where  $\dot{\lambda}(t) := \frac{d\lambda}{dt}(t)$ ,  $\ddot{\lambda}(t) := \frac{d^2\lambda}{dt^2}(t)$  and  $\lambda^{(k)}(t) := \frac{d^k\lambda}{dt^k}(t)$  for  $k \in \mathbb{N}$  and  $t \in I$ .

Note that this definition generalises Definition 1.2. We also consider initial value problems for higher-order differential equations, and here we need to fix not only  $x$ -value of a solution at a particular time  $t_0$ , but also (some of) its derivatives.

**Definition 2** (Initial value problem). Consider  $n, d \in \mathbb{N}$ , an open set  $D \subset \mathbb{R} \times \mathbb{R}^{nd}$ , and a function  $f : D \rightarrow \mathbb{R}^d$ . The combination of the ordinary differential equation

$$x^{(n)} = f(t, x, \dot{x}, \dots, x^{(n-1)})$$

with an initial condition of the form

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1, \quad \dots, \quad x^{(n-1)}(t_0) = x_{n-1}, \quad (3)$$

where  $(t_0, x_0, x_1, \dots, x_{n-1}) \in D$ , is called an *initial value problem*, and (3) is called *initial condition*. A solution to the above initial value problem is a solution  $\lambda : I \rightarrow \mathbb{R}^d$  to the differential equation such that  $t_0$  is in the interior of  $I$  and

$$\lambda(t_0) = x_0, \quad \dot{\lambda}(t_0) = x_1, \quad \dots, \quad \lambda^{(n-1)}(t_0) = x_{n-1}.$$

We now demonstrate how differential equations of higher order can be transformed into first-order differential equations.

**Proposition 3** (Reduction to first-order systems). *Given a set  $D \subset \mathbb{R}^{1+nd}$  and a function  $f : D \rightarrow \mathbb{R}^d$ . Then the  $d$ -dimensional ordinary differential equation of order  $n$*

$$x^{(n)} = f(t, x, \dot{x}, \dots, x^{(n-1)}) \quad (4)$$

*is equivalent to the  $nd$ -dimensional first-order differential equation*

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_{n-1} &= y_n \\ \dot{y}_n &= f(t, y_1, y_2, \dots, y_n) \end{aligned} \quad (5)$$

*with  $y_1, y_2, \dots, y_n \in \mathbb{R}^d$  and  $(t, y_1, y_2, \dots, y_n) \in D$ . The equivalence is given by the following two statements:*

(i) *If  $\lambda$  is a solution to (4), then  $t \mapsto (\lambda(t), \dot{\lambda}(t), \dots, \lambda^{(n-1)}(t))$  is a solution to (5).*

(ii) *If  $t \mapsto (\mu_1(t), \mu_2(t), \dots, \mu_n(t))$  is a solution to (5), then  $\mu_1$  is a solution to (4).*

*In addition, if for a given  $(t_0, x_0, x_1, \dots, x_{n-1}) \in D$ , a solution to (4) satisfies the initial condition*

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1, \quad \dots, \quad x^{(n-1)}(t_0) = x_{n-1},$$

*then the corresponding solution to (5) satisfies the initial condition*

$$y_1(t_0) = x_0, \quad y_2(t_0) = x_1, \quad \dots, \quad y_n(t_0) = x_{n-1}.$$

*Proof.* (i) If  $\lambda$  is a solution to (4), then we have

$$\lambda^{(n)}(t) = f(t, \lambda(t), \dots, \lambda^{(n-1)}(t)).$$

Define  $(\mu_1(t), \dots, \mu_n(t)) := (\lambda(t), \dots, \lambda^{(n-1)}(t))$ . Then we get

$$\dot{\mu}_1(t) = \mu_2(t), \quad \dot{\mu}_2(t) = \mu_3(t), \quad \dots, \quad \dot{\mu}_{n-1}(t) = \mu_n(t)$$

and

$$\dot{\mu}_n(t) = f(t, \lambda(t), \dots, \lambda^{(n-1)}(t)) = f(t, \mu_1(t), \dots, \mu_n(t)).$$

This implies that  $\mu$  solves (5).

(ii) Let  $t \mapsto (\mu_1(t), \dots, \mu_n(t))$  be a solution of (5). This means that

$$\dot{\mu}_1(t) = \mu_2(t), \quad \dot{\mu}_2(t) = \mu_3(t), \quad \dots, \quad \dot{\mu}_n(t) = f(t, \mu_1(t), \dots, \mu_n(t)).$$

It follows that

$$\dot{\mu}_1(t) = \mu_2(t), \quad \ddot{\mu}_1(t) = \dot{\mu}_2(t) = \mu_3(t), \quad \dots, \quad \mu_1^{(n-1)}(t) = \mu_n(t),$$

and this implies

$$\mu_1^{(n)}(t) = \dot{\mu}_n(t) = f(t, \mu_1(t), \dots, \mu_n(t)) = f(t, \mu_1(t), \dot{\mu}_1(t), \dots, \mu_1^{(n-1)}(t)).$$

Consequently,  $t \mapsto \mu_1(t)$  is a solution of (4).

The statement regarding the initial conditions is clear. □