IMPERIAL

MATH70033

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebraic Curves

Date: Wednesday, May 14, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

You may reference results from the lecture notes; however, ensure you clearly indicate which results you are using in your answers.
Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.
Throughout all problems, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.

1. (a) (i) Define an *affine algebraic set* in $\mathbb{K}^n$. (2 marks)

   (ii) Prove that any affine algebraic subset of $\mathbb{K}$ is either $\mathbb{K}$ or a finite set. (3 marks)

   (iii) Prove that a finite union of projective plane curves $C_i \subset \mathbb{P}^2(\mathbb{K})$ for $1 \leq i \leq n$ is a projective plane curve. (4 marks)

   (iv) Let $C \subset \mathbb{P}^2(\mathbb{K})$ be a projective plane curve. Prove that there exists a finite subset Z of points on C such that $C \setminus Z$ is an affine plane curve over $\mathbb{K}$. (4 marks)

   (b) Let $C \subset \mathbb{P}^2(\mathbb{K})$ be a smooth projective plane curve. The dual curve $\hat{C} \subset \mathbb{P}^2(\mathbb{K})$ is defined as the Zariski closure of the set

$$\{[a,b,c] \in \mathbb{P}^2(\mathbb{K}) \mid T_pC = \{ax+by+cz=0\} \text{ for some point } p \in C\}.$$

   It is known that $\hat{C}$ is also a projective plane curve in $\mathbb{P}^2(\mathbb{K})$ (you do not need to prove this). Prove that the dual curve of $C = \{\frac{1}{2}x_0x_2^2 + \frac{1}{3}x_1^3 = 0\} \subset \mathbb{P}^2(\mathbb{K})$ is the curve

$$\hat{C} = \left\{ y_1^3 - \frac{9}{2}y_0y_2^2 = 0 \right\} \subset \mathbb{P}^2(\mathbb{K}).$$

   (7 marks)

(Total: 20 marks)

2. (a) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{K})$ be a projective plane curve, where F has no repeated factors.

   (i) Define what it means for C to be *singular* at a point $p = [a,b,c] \in C$. (2 marks)

   (ii) Prove that if C is irreducible, then it has only finitely many singular points. (4 marks)

   (iii) Suppose C is of degree 3. Prove that if C is irreducible, then it has at most one singular point. (5 marks)

   (b) Is there a line $L \subset \mathbb{P}^2(\mathbb{K})$ and a conic $C \subset \mathbb{P}^2(\mathbb{K})$ that intersect at a single point p such that $I_p(L,C) = 3$? Is there one with $I_p(L,C) = 2$? For each case, either provide an example or prove non-existence. (4 marks)

   (c) Let $f(x) = \prod_{i=1}^n (x - \alpha_i) \in \mathbb{K}[x]$. Prove that the resultant of f and f' is equal to

$$\mathcal{R}_{f,f'} = (-1)^{\frac{1}{2}n(n-1)} \left(\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j) \right)^2.$$

   (5 marks)

(Total: 20 marks)

3. (a) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{K})$ be a smooth projective plane curve.

   (i) Define what it means for a point $p \in C$ to be an *inflection point*. (2 marks)

   (ii) Suppose C is of degree 3, and L is a line passing through a point $p \in C$. Prove that $I_p(L,C) = 3$ if and only if p is an inflection point of C and L is the tangent line to C at p. (State clearly all the results used.) (5 marks)

   (b) Consider the cubic curve $C = \{F = x^2y + 2xy^2 + xz^2 + yz^2 = 0\} \subset \mathbb{P}^2(\mathbb{K})$.

   (i) Show that C is a smooth curve. (4 marks)

   (ii) Find one inflection point p of C. (4 marks)

   (c) Let $p_1, \dots, p_5$ be five distinct points in $\mathbb{P}^2$ such that no three of them are collinear. Show that there exists a unique conic passing through $p_1, \dots, p_5$. (5 marks)

(Total: 20 marks)

4. (a) Let X be a connected compact Riemann surface.

   (i) Define what it means for a function $f : X \to \mathbb{C} \cup \{\infty\}$ to be *meromorphic*. (2 marks)

   (ii) Suppose the meromorphic function $f : X \to \mathbb{C} \cup \{\infty\}$ has a pole at $p \in X$ of order one and no other poles. Let $\overline{f} : X \to \mathbb{P}^1(\mathbb{C})$ be the holomorphic map induced by f. Prove that the ramification index of $\overline{f}$ at p is equal to one. (4 marks)

   (iii) Prove that the map $\overline{f} : X \to \mathbb{P}^1(\mathbb{C})$ in part (ii) is bijective. (5 marks)

   (b) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of degree $d \geq 2$ which does not contain the point $[0,0,1]$. Let $f : C \to \mathbb{P}^1(\mathbb{C})$ be the map defined by $[x,y,z] \mapsto [x,y]$.

   (i) Prove that the preimage $f^{-1}([a,b])$ for a point $[a,b] \in \mathbb{P}^1(\mathbb{C})$ is the set of intersection points of C with the line $L_{[a,b]} = \{bx - ay = 0\}$. (1 mark)

   (ii) Prove that the map f has degree d. (4 marks)

   (c) Consider the cubic curve $C = \{F = x_0x_1x_2 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^2(\mathbb{C})$. Find a non-constant holomorphic map $f \colon \mathbb{P}^1(\mathbb{C}) \to C$.
   [Hint: Consider lines connecting the unique singular point of C to other points on the curve C.] (4 marks)

(Total: 20 marks)

5. (a) Let $C_1, C_2 \subset \mathbb{P}^2(\mathbb{C})$ be smooth projective plane curves of degrees d_1 and d_2, respectively, and let $f : C_1 \to C_2$ be a non-constant holomorphic map.

(i) What is the Euler characteristic of C_1 in terms of d_1? (2 marks)

(ii) If $d_1, d_2 \geq 2$, prove that $d_1 \geq d_2$. (4 marks)

(iii) If the map f is of degree 2, prove that the number of ramification points of f is even. (5 marks)

(iv) If the map f is biholomorphic, prove that either $d_1 = d_2$, or $\{d_1, d_2\} = \{1, 2\}$. (4 marks)

(b) Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of genus 1. Find a holomorphic map $f : C \to \mathbb{P}^1$ of degree 3 with at most 6 ramification points. (5 marks)

(Total: 20 marks)

You may reference results from the lecture notes; however, ensure you clearly indicate which results you are using in your answers.
Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.
Throughout all problems, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.

1. (a) (i) Define an *affine algebraic set* in $\mathbb{K}^n$. (2 marks)

 (ii) Prove that any affine algebraic subset of $\mathbb{K}$ is either $\mathbb{K}$ or a finite set. (3 marks)

 (iii) Prove that a finite union of projective plane curves $C_i \subset \mathbb{P}^2(\mathbb{K})$ for $1 \leq i \leq n$ is a projective plane curve. (4 marks)

 (iv) Let $C \subset \mathbb{P}^2(\mathbb{K})$ be a projective plane curve. Prove that there exists a finite subset Z of points on C such that $C \setminus Z$ is an affine plane curve over $\mathbb{K}$. (4 marks)

 (b) Let $C \subset \mathbb{P}^2(\mathbb{K})$ be a smooth projective plane curve. The dual curve $\hat{C} \subset \mathbb{P}^2(\mathbb{K})$ is defined as the Zariski closure of the set

$$\{[a,b,c] \in \mathbb{P}^2(\mathbb{K}) \mid T_pC = \{ax+by+cz=0\} \text{ for some point } p \in C\}.$$

It is known that $\hat{C}$ is also a projective plane curve in $\mathbb{P}^2(\mathbb{K})$ (you do not need to prove this). Prove that the dual curve of $C = \{\frac{1}{2}x_0x_2^2 + \frac{1}{3}x_1^3 = 0\} \subset \mathbb{P}^2(\mathbb{K})$ is the curve

$$\hat{C} = \left\{ y_1^3 - \frac{9}{2}y_0y_2^2 = 0 \right\} \subset \mathbb{P}^2(\mathbb{K}).$$

(7 marks)

(Total: 20 marks)

2. (a) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{K})$ be a projective plane curve, where F has no repeated factors.

 (i) Define what it means for C to be *singular* at a point $p = [a,b,c] \in C$. (2 marks)

 (ii) Prove that if C is irreducible, then it has only finitely many singular points. (4 marks)

 (iii) Suppose C is of degree 3. Prove that if C is irreducible, then it has at most one singular point. (5 marks)

 (b) Is there a line $L \subset \mathbb{P}^2(\mathbb{K})$ and a conic $C \subset \mathbb{P}^2(\mathbb{K})$ that intersect at a single point p such that $I_p(L,C) = 3$? Is there one with $I_p(L,C) = 2$? For each case, either provide an example or prove non-existence. (4 marks)

 (c) Let $f(x) = \prod_{i=1}^n (x - \alpha_i) \in \mathbb{K}[x]$. Prove that the resultant of f and f' is equal to

$$\mathcal{R}_{f,f'} = (-1)^{\frac{1}{2}n(n-1)} \left(\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j) \right)^2.$$

(5 marks)

(Total: 20 marks)

3. (a) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{K})$ be a smooth projective plane curve.

  (i) Define what it means for a point $p \in C$ to be an *inflection point*. (2 marks)

  (ii) Suppose C is of degree 3, and L is a line passing through a point $p \in C$. Prove that $I_p(L, C) = 3$ if and only if p is an inflection point of C and L is the tangent line to C at p. (State clearly all the results used.) (5 marks)

 (b) Consider the cubic curve $C = \{F = x^2y + 2xy^2 + xz^2 + yz^2 = 0\} \subset \mathbb{P}^2(\mathbb{K})$.

  (i) Show that C is a smooth curve. (4 marks)

  (ii) Find one inflection point p of C. (4 marks)

 (c) Let $p_1, \ldots, p_5$ be five distinct points in $\mathbb{P}^2$ such that no three of them are collinear. Show that there exists a unique conic passing through $p_1, \ldots, p_5$. (5 marks)

(Total: 20 marks)

4. (a) Let X be a connected compact Riemann surface.

  (i) Define what it means for a function $f : X \to \mathbb{C} \cup \{\infty\}$ to be *meromorphic*. (2 marks)

  (ii) Suppose the meromorphic function $f : X \to \mathbb{C} \cup \{\infty\}$ has a pole at $p \in X$ of order one and no other poles. Let $\overline{f} : X \to \mathbb{P}^1(\mathbb{C})$ be the holomorphic map induced by f. Prove that the ramification index of $\overline{f}$ at p is equal to one. (4 marks)

  (iii) Prove that the map $\overline{f} : X \to \mathbb{P}^1(\mathbb{C})$ in part (ii) is bijective. (5 marks)

 (b) Let $C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of degree $d \geq 2$ which does not contain the point $[0,0,1]$. Let $f : C \to \mathbb{P}^1(\mathbb{C})$ be the map defined by $[x,y,z] \mapsto [x,y]$.

  (i) Prove that the preimage $f^{-1}([a,b])$ for a point $[a,b] \in \mathbb{P}^1(\mathbb{C})$ is the set of intersection points of C with the line $L_{[a,b]} = \{bx - ay = 0\}$. (1 mark)

  (ii) Prove that the map f has degree d. (4 marks)

 (c) Consider the cubic curve $C = \{F = x_0x_1x_2 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^2(\mathbb{C})$. Find a non-constant holomorphic map $f\colon \mathbb{P}^1(\mathbb{C}) \to C$.
 [Hint: Consider lines connecting the unique singular point of C to other points on the curve C.] (4 marks)

(Total: 20 marks)

5. (a) Let $C_1, C_2 \subset \mathbb{P}^2(\mathbb{C})$ be smooth projective plane curves of degrees d_1 and d_2, respectively, and let $f : C_1 \to C_2$ be a non-constant holomorphic map.

(i) What is the Euler characteristic of C_1 in terms of d_1? (2 marks)

(ii) If $d_1, d_2 \geq 2$, prove that $d_1 \geq d_2$. (4 marks)

(iii) If the map f is of degree 2, prove that the number of ramification points of f is even. (5 marks)

(iv) If the map f is biholomorphic, prove that either $d_1 = d_2$, or $\{d_1, d_2\} = \{1, 2\}$. (4 marks)

(b) Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of genus 1. Find a holomorphic map $f : C \to \mathbb{P}^1$ of degree 3 with at most 6 ramification points. (5 marks)

(Total: 20 marks)

MATH70033 Algebraic Curves Markers Comments

Question 1	Part (a) was handled very well overall. For part (b), there was a correction to the exam question: the definition of C^{\hat} was updated to consider only the smooth points of C (if C is singular), and then take the Zariski closure. The reverse inclusion — that every point of C^{\hat} except [0,0,1] is of the form TpC for some point p \in C— was largely missed in student answers, and most students lost marks for that part.
Question 2	This question was mostly handled well.
Question 3	Part (a)(ii) is a direct consequence of combining several statements from the lecture notes. In part (b)(ii), the inflection points can be found easily by applying the result of part (a)(ii), rather than computing them directly.
Question 4	The surjectivity of \bar{f} follows from the fact that a non-constant holomorphic map between compact connected Riemann surfaces is surjective — a point that most students missed. The key was to show precisely that, for all but finitely many [a,b], the line L[a,b] intersects C with multiplicity 1. However, most students missed the detailed reasoning required for this argument. The question asked for an explicit equation for the holomorphic map.
Question 5	To construct the holomorphic map f in part (b), one could simply apply the map from Question 4(b) and verify that it satisfies the required conditions.