Intro to ODEs - Systems of ODEs

$$\vec{y}_{CF} = \vec{y}_H^{GS}(t; c_1, c_2) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

2nd step: Find any particular integral $\vec{y}_{PI}(t)$ (Quiz)

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$$\vec{y}_{PI} = \begin{bmatrix} a \\ b \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} 3 & 3 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -7/2 \\ 3 \end{bmatrix}$$

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When A has repeated eigenvalues

Case 1: A is still diagonalizable (it has n linearly independent eigenvectors). Then we can still use the method described. For example for $n = 2$ we have:

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When A has repeated eigenvalues

Case 2: A is not diagonalizable (it has less than n linearly indepenent eigenvectors). Then we will use the *Jordan normal form*. We first discuss an example and then see the general case.

Example:

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

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Eigenvectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \implies \quad \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix} = 2 \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix}$$

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We look for similarity transformation to a Jordan normal form. We look for a matrix of the form:

$$W = \begin{bmatrix} 1 & \alpha \\ -1 & \beta \end{bmatrix}$$

So that:

$$W^{-1}AW = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

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$$\begin{aligned} \alpha - \beta &= 1 + 2\alpha \\ \alpha + 3\beta &= -1 + 2\beta \end{aligned} \implies \alpha + \beta = -1 \implies \vec{w}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

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The Jordan normal form allows us to solve the non-diagonalizable systems of ODEs:

$$\frac{d\vec{y}}{dt} = A\vec{y} \quad \Longrightarrow \quad W^{-1}\frac{d\vec{y}}{dt} = \left[W^{-1}AW\right]W^{-1}\vec{y}$$

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$$
\begin{aligned}
\frac{dz_1}{dt} &= 2z_1 + z_2 \\
\frac{dz_2}{dt} &= 2z_2
\end{aligned}
$$

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$$\vec{Z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

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Now consider the case of a non-diagonalizable $A_{n \times n}$ with one repeated eigenvalue λ. Assume λ is associated with only a single eigenvector. We can use the Jordan normal form to obtain a solution to the associated systmes of linear ODEs.

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$$
\begin{aligned}
\frac{dz_n}{dt} &= \lambda z_n \\
\frac{dz_{n-1}}{dt} &= \lambda z_{n-1} + z_n \\
\frac{dz_{n-2}}{dt} &= \lambda z_{n-2} + z_{n-1} \\
&\vdots \\
\frac{dz_1}{dt} &= \lambda z_1 + z_2
\end{aligned}
$$

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Quiz: Find the general solution for this system of ODEs

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$