

M3/4/5P60, Geometric Complex Analysis,
 Coursework 1,
 Due: 20 February 2025, 1pm

Problem 1. *Prove the following version of the Schwarz lemma: Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic map which has a 0 of order $m \geq 1$ at 0. Then,*

- (a) *for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|^m$,*
- (b) *if $|f(z_0)| = |z_0|^m$ for some $z_0 \neq 0$, there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{2\pi i \theta} z^m$.*

Problem 2.

- (a) *Show that $\rho(z) = \frac{2}{1+|z|^2}$ is a conformal metric on \mathbb{C} .*
- (b) *Show that the length of the real interval $[-x, x]$, for $x > 0$, with respect to ρ is uniformly bounded from above.*
- (c) *Let $h(z) = \frac{z-1}{z+1}$. Show that for all $z \neq -1$,*

$$(h^* \rho)(z) = \rho(z).$$

Problem 3. *Let*

$$\Pi = \{w \in \mathbb{C} \mid \operatorname{Im} w > 0, -\pi/2 < \operatorname{Re} w < \pi/2\}$$

and consider the biholomorphism $\sin : \Pi \rightarrow \mathbb{H}$, and the conformal metric $\rho(z) = 1/\operatorname{Im} z$ on \mathbb{H} .

- (a) *Calculate $\eta = \sin^* \rho$, that is, the pull-back of ρ by \sin on Π .*
- (b) *Show that η is independent of the choice of the biholomorphism \sin , that is, if $g : \Pi \rightarrow \mathbb{H}$ is another biholomorphism, we have $g^* \rho = \eta$ on Π .*

Problem 4. *Let $U = \mathbb{C} \setminus (-\infty, 0]$, and for each $a \in (0, \infty)$, let $f_a : U \rightarrow \mathbb{C}$ be the holomorphic map*

$$f_a(z) = \frac{1}{1+az}.$$

Consider the family of maps

$$\mathcal{F} = \{f_a : U \rightarrow \mathbb{C} \mid a \in (0, \infty)\}.$$

For $\epsilon > 0$, let

$$U_\epsilon = \{z \in \mathbb{C} \mid \forall x \in (-\infty, 0], |z - x| > \epsilon\}.$$

- (a) *Show that for every $\epsilon > 0$ and every $a \in (0, \infty)$, on U_ϵ we have*

$$\left| \frac{1}{1+az} \right| \leq \frac{1}{\epsilon a}.$$

- (b) *Show that for every $\epsilon > 0$, the family \mathcal{F} is normal on U_ϵ .*
- (c) *Show that the family \mathcal{F} is normal on U .*
- (d) *Is the family \mathcal{F} compact? Justify your answer.*