Imperial College London

MATH70022

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Finite Elements: Numerical Analysis and Implementation

Date: 15 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let $(K, P, \mathcal{N})$ be a finite element.

   (i) Provide a definition of the nodal variables $\mathcal{N}$. (3 marks)

   (ii) What does it mean for $\mathcal{N}$ to determine $\mathcal{P}$? (3 marks)

(b) Let $(K, P, \mathcal{N})$ be defined by:

* K is a triangle with vertices z_1, z_2 and z_3.
* P is $P_1 + \mathring{P}_3$, where P_1 are the linear polynomials, and $\mathring{P}_3$ is the subspace of the cubic polynomials P_3 that vanish on the boundary of K,
* $\mathcal{N} = (N_1, N_2, N_3, N_4)$, where $N_i[u] = u(z_i)$, $i = 1, 2, 3$, and $N_4[u] = u(z^*)$, where $z^* = z_1 + (z_2 - z_1)/3 + (z_3 - z_1)/3$.

   (i) Let $u \in P$. Show that u is linear when restricted to each of the edges of K. (1 mark)

   (ii) Show that $\mathcal{N}$ determines P. (4 marks)

(c) Find the nodal basis function ϕ_4. (3 marks)

(d) Let u solve $-\nabla^2 u = f$ in the unit square Ω with boundary conditions $u = 0$ on all sides of the square.

Let V be the finite element space constructed on a triangulation $\mathcal{T}$ of the unit square, using the finite element considered in Part (b).

Let $u_h \in \mathring{V}$ solve the finite element variational problem,

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}\, x = \int_\Omega f v \, \mathrm{d}\, x, \quad \forall v \in \mathring{V}, \tag{1}$$

where $\mathring{V}$ is the subspace of V containing functions that vanish on the boundary $\partial\Omega$ of Ω.

Let $b \in V$ be a function that vanishes everywhere except for one triangle K in $\mathcal{T}$. Further, let $b = \phi_4$ in K.

   (i) Show that

$$\int_K \nabla(u - u_h) \cdot \nabla b \, \mathrm{d}\, x = 0. \tag{2}$$

(3 marks)

   (ii) Hence, show that

$$\int_K u_h \, \mathrm{d}\, x = \int_K u \, \mathrm{d}\, x. \tag{3}$$

(3 marks)

(Total: 20 marks)

2. (a) Consider a finite element $(K, P, \mathcal{N})$, with nodal basis $\{\phi_i\}_{i=1}^n$.

(i) Provide a definition of local interpolation operator I_K. (2 marks)

(ii) Show that

$$N_i[I_K(v)] = N_i[v], \quad i = 1, \ldots, n. \tag{4}$$

(3 marks)

(iii) Show that I_K is the identity when restricted to P. (3 marks)

(b) For a nodal variable $N \in P'$, we define the norm $\|N\|_{C^l(K)'}$ by

$$\|N\|_{C^l(K)'} = \sup_{0<\|u\|_{C^l(K)}} \frac{|N[u]|}{\|u\|_{C^l(K)}}, \tag{5}$$

where

$$\|u\|_{C^l(K)} = \sup_{x \in K, r=0,\ldots,l} |D_r u(x)|, \tag{6}$$

and D_r is the rth derivative. Show that

$$\|I_K(u)\|_{H^k(K)} \leq C_1 \|u\|_{C^l(K)}, \tag{7}$$

where

$$C_1 = \sum_{i=1}^n \|\phi_i\|_{H^k(K_1)} \|N_i\|_{C^l(K)'}. \tag{8}$$

(5 marks)

(c) You may assume the Sobolev inequality for continuous functions $u \in C^l(K)$. This states that there exists a constant $0 < C_2 \leq \infty$, depending only on the shape and size of K, such that

$$\|u\|_{C^l(K)} \leq C_2 \|u\|_{H^k(K)}, \tag{9}$$

provided that $k > d/2 + l$, where d is the dimension of K.
Show that there exists $0 < C_3 \leq \infty$, such that

$$\|I_K(u)\|_{H^k(K)} \leq C_3 \|u\|_{H^k(K)}, \tag{10}$$

stating any assumptions that you make about k, l, and d. (3 marks)

(d) For each finite element below, consider $u \in H^2(K)$, and give all the values of k for which $\|I_K(u)\|_{H^k(K)}$ is finite, justifying your answer.

(i) The degree m Lagrange elements on the interval $[0, 1]$. (1 mark)

(ii) The degree m Lagrange elements on a triangle. (1 mark)

(iii) The degree m Lagrange elements on a tetrahedron. (1 mark)

(iv) The cubic Hermite elements (nodal variables are point evaluations on vertices plus cell centre, and derivative evaluations on vertices) on a triangle. (1 mark)

(Total: 20 marks)

3. (a) Let $(K, P, \mathcal{N})$ be a finite element.

   (i) Provide a definition of a local geometric decomposition for $(K, P, \mathcal{N})$. (4 marks)

   (ii) What does it mean for a local geometric decomposition to be C^0? (4 marks)

(b) Let $(K, P, \mathcal{N})$ be a finite element with a C^0 local geometric decomposition. Let $\mathcal{T}$ be a triangulation, and let V be a finite element space constructed on $\mathcal{T}$ using $(K, P, \mathcal{N})$ and the local geometric decomposition. Show that V is a C^0 finite element space. (5 marks)

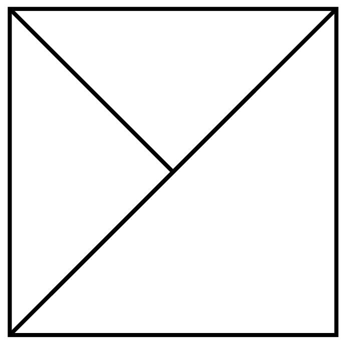

Figure 1: Diagram for parts 3(c-d)

(c) The diagram in Figure 1 shows a mesh which is not a triangulation. We can nevertheless proceed to define a finite element space V on this mesh, using linear Lagrange elements with the C^0 geometric decomposition assigning each nodal variable to its vertex.

   (i) Show that V is not a C^0 finite element space. (1 mark)

   (ii) Describe, with justification, a subspace of V containing only C^0 functions. (2 marks)

(d) We consider the finite element discretisation of the equation $u - \nabla^2 u = f$ on the unit square Ω with boundary conditions $\frac{\partial u}{\partial n} = 0$ on all edges. For a C^0 finite element space W, the finite element discretisation seeks $u_h \in W$ such that

$$\int_\Omega u_h v + \nabla u_h \cdot \nabla v \,\mathrm{d}\,x = \int_\Omega v f \,\mathrm{d}\,x, \quad \forall v \in W. \tag{11}$$

We say that a finite element discretisation is *consistent* if replacing u_h with the exact solution u to the strong form partial differential equation still gives equality for all test functions. Provide a modification to (11) with the two following properties:

1. The resulting finite element discretisation is consistent.
2. The modification vanishes when u_h vanishes on the interior edge going from bottom left to top right in the diagram in Figure 1.

(4 marks)

(Total: 20 marks)

4. For a convex polygonal domain Ω, we consider a variational problem on $H^1(\Omega)$, seeking $u \in H^1(\Omega)$ such that

$$a(u, v) = F[v], \quad \forall v \in H^1(\Omega), \tag{12}$$

where a and F are bilinear and linear forms on $H^1(\Omega)$, respectively. Further, we assume that a is symmetric.

(a) (i) State what it means for $a(\cdot, \cdot)$ to be coercive. (2 marks)

(ii) State what it means for $a(\cdot, \cdot)$ to be continuous. (2 marks)

(iii) State what it means for F to be continuous. (2 marks)

(iv) Write down a Galerkin finite element approximation to our variational problem, using a finite element space $V \subset H^1(\Omega)$. (4 marks)

(b) (i) Show that

$$a(u - u_h, v) = 0, \quad \forall v \in V, \tag{13}$$

where u_h is the finite element approximation to the solution u. (3 marks)

(ii) What does this tell us about the error in the solution? (3 marks)

(c) Using the earlier parts of this question, and assuming that a is continuous and coercive, show that

$$\|u - u_h\|_{H^1(\Omega)} \leq C \sup_{v \in V} \|v - u\|_{H^1(\Omega)}, \tag{14}$$

for some constant $0 < C \leq \infty$. (4 marks)

(Total: 20 marks)

5. In this question, we consider the Stokes equations, written in strong form as

$$-2\mu\nabla\cdot\epsilon(u) + \nabla p = f, \quad \nabla\cdot u = 0, \tag{15}$$

for (vector valued) velocity u and (scalar valued) pressure p, where

$$\epsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right). \tag{16}$$

We consider boundary conditions $u = 0$ on the boundary $\partial\Omega$ of the 3-dimensional domain Ω.

The variational formulation seeks $(u, p) \in (V, Q)$, where $V = (\mathring{H}^1)^3$ is the subspace of $(H^1)^3$ vanishing on the boundary, and $Q = \mathring{L}^2$ is the subspace of L^2 that integrates to zero, such that

$$c((u,p),\,(v,q)) = \int_\Omega f\cdot v\,\mathrm{d}\,x, \quad \forall (v,q) \in (\mathring{H}^1(\Omega))^3 \times \mathring{L}^2(\Omega), \tag{17}$$

where

$$c((u,p),\,(v,q)) = a(u,v) + b(v,p) + b(u,q), \tag{18}$$

$$a(u,v) = 2\mu\int_\Omega \epsilon(u) : \epsilon(v)\,\mathrm{d}\,x, \quad b(u,q) = \int_\Omega pq\,\mathrm{d}\,x. \tag{19}$$

(a) Show that if (u,p) solves the variational formulation, and further that $u \in H^2(\Omega)$ and $p \in H^1(\Omega)$, then (u,p) solves the strong form of the Stokes equations.

(8 marks)

(b) Show that the form $c((u,p),(v,q))$ is not coercive. (3 marks)

(c) Now we consider the discrete inf-sup condition, which requires that there exists $\beta_h > 0$ such that

$$\inf_{0\neq q\in Q_h}\sup_{0\neq v\in V_h}\frac{b(v,q)}{\|v\|_V\|q\|_Q} \geq \beta_h, \tag{20}$$

for finite element spaces $V_h \subset V$ and $Q_h \subset Q$. We define B^* as the map from Q to the dual space V', given by

$$(B^*q)[v] = b(v,p), \quad \forall q \in Q,\, v \in V. \tag{21}$$

We also define B_h^* as the map from Q_h to the dual space V_h', given by

$$(B_h^*q)[v] = b(v,p), \quad \forall q \in Q_h,\, v \in V_h. \tag{22}$$

Show that if B_h^* has a kernel, then the discrete inf-sup condition is not satisfied. (4 marks)

(d) Consider a mesh of squares, with each square subdivided into four triangles by the diagonals. When V_h is constructed from continuous linear elements (with the boundary condition subspace restriction) and Q_h is constructed from discontinuous constant elements (with the mean zero restriction), show that $\ker(B_h^*)$ is not empty. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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M70022

Finite Elements (Solutions)

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1. (a) $\boxed{\text{seen} \Downarrow}$

(i) The nodal variables $\mathcal{N}$ are a basis for the dual space P' of P. $\boxed{3, \text{A}}$

(ii) $\mathcal{N}$ determines P if $\mathcal{N}$ is indeed a basis for P'. $\boxed{3, \text{A}}$

(b) $\boxed{\text{unseen} \Downarrow}$

(i) If $v = P_1 + \mathring{P}_3$ then $v = v_1 + \mathring{v}_3$ where $v_1 \in P_1$ and $\mathring{v}_3 \in \mathring{P}_3$. On an edge of K, $\mathring{v}_3 = 0$, so $v = v_1$ there, *i.e* v is linear when restricted to the edge. $\boxed{1, \text{B}}$

(ii) $\boxed{\text{seen/sim.seen} \Downarrow}$

We define lines Π_1, Π_2, Π_3 intersecting z_1, z_2, z_2, z_3, and z_3, z_1 respectively. We choose nondegenerate linear functionals L_1, L_2, L_3 that vanish on Π_1, Π_2, Π_3 respectively. Now assume that $v \in P$ is such that $N_i[v] = 0$ for $i = 1, 2, 3, 4$. v restricted to Π_1 vanishes at two points, z_1 and z_2, so $v = 0$ on Π_1 by the fundamental theorem of algebra. Thus $v = L_1 q_1$ for a quadratic polynomial q_1 (since v is cubic). Similarly, $v = 0$ on Π_2. Therefore q_1 vanishes everywhere on Π_2 except potentially at z_2, but continuity requires that q_1 vanishes there too. Hence, $q_1 = L_2 q_3$, where q_3 is linear. Similarly, $v = 0$ on Π_3, hence $q_3 = cL_3$ for c constant. Finally, v vanishes at z^*, but none of L_1, L_2, L_3 vanish there, so we must have $c = 0$, i.e. $v \equiv 0$. $\boxed{4, \text{B}}$

(c) $\boxed{\text{unseen} \Downarrow}$ ϕ_4 vanishes on Π_1, Π_2, Π_3, so by similar arguments to above, $\phi_4 = cL_1L_2L_3$. We need $\phi_4(z^*) = 1$, so
$\phi_4(x) = L_1(x)L_2(x)L_3(x)/(L_1(z^*)L_2(z^*)L_3(z^*))$. $\boxed{3, \text{C}}$

(d) $\boxed{\text{unseen} \Downarrow}$ (i) u solves the same variational problem, but with $\mathring{V}$ replaced by $\mathring{H}^1$. Taking $v = b$ in both variational problems and subtracting gives

$$0 = \int_\Omega \nabla(u - u_h) \cdot \nabla b \, \mathrm{d}\, x = \int_K \nabla(u - u_h) \cdot \nabla b \, \mathrm{d}\, x, \tag{1}$$

since b is only supported in K. $\boxed{3, \text{D}}$

(ii) Integration by parts gives

$$-\int_K (u - u_h)\nabla^2 b \, \mathrm{d}\, x = 0 \tag{2}$$

where the boundary integral has vanished because $b = 0$ on ∂K. Since $-\nabla^2 b$ is constant and nonzero, we obtain the result by dividing by $-\nabla^2 b$. $\boxed{3, \text{D}}$

2. (a) $\boxed{\text{seen} \Downarrow}$

(i)

$$I_K[u](x) = \sum_{i=1}^{n} N_i[u]\phi_i(x). \tag{3}$$

$\boxed{2, \text{A}}$

(ii)

$$N_i[I_K[u]] = N_i[\sum_{j=1}^{n} N_j[u]\phi_j(x)] = \sum_{j=1}^{n} N_j[u]N_i[\phi_j(x)]] = \sum_{j=1}^{n} N_j[u]\delta_{ij} = N_i[u], \tag{4}$$

using linearity of N_i and the definition of the nodal basis. $\boxed{3, \text{A}}$

(iii) If $v \in P$, then we can write $v = \sum_i v_i\phi_i$. Then,

$$I_K(v) = \sum_i v_i I_K(\phi_i) = \sum_i v_i \sum_j N_j[\phi_i]\phi_j = \sum_i v_i \sum_j \delta_{ji}\phi_j = \sum_i v_i\phi_i = v, \tag{5}$$

as required. $\boxed{3, \text{A}}$

(b) $\boxed{\text{seen} \Downarrow}$

$$\|I_K(u)\|_{H^k(K)} = \left\| \sum_{i=1}^{n} N_i[u]\phi_i(x) \right\|_{H^k(K)} \tag{6}$$

$$\text{triangle inequality} \leq \sum_{i=1} |N_i[u]| \|\phi_i\|_{H^k(K)} \tag{7}$$

$$\text{definition of } C^l(K)' \text{ norm} \leq \underbrace{\sum_i \|\phi_i\|_{H^k(K)} \|N_i\|_{C^l(K)'}}_{=C_1} \|u\|_{C^l(K)}, \tag{8}$$

as required. $\boxed{5, \text{B}}$

(c) If $k > d/2 - l$, then we can use the Sobolev inequality to get

$$\|I_K(u)\|_{H^k(K)} \leq C_1\|u\|_{C^l(K)} \leq \underbrace{C_1C_2}_{=C_3} \|u\|_{H^k(K)}. \tag{9}$$

$\boxed{3, \text{C}}$

(d) (i) We have $l = 0$ because Lagrange elements involve function evaluation only. We have $d = 1$ because we are solving on an interval. Hence, we need $k > 1/2$. On the other hand, we only have $u \in H^2$, so we can take $k = 1$ or $k = 2$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{1, \text{D}}$

(ii) We have $l = 0$ for Lagrange, and $d = 2$, so we need $k > 1$. This means that only $k = 2$ is possible. $\boxed{1, \text{D}}$

(iii) We have $l = 0$ for Lagrange, and $d = 3$, so we need $k > 3/2$. This means that only $k = 2$ is possible. $\boxed{1, \text{D}}$

(iv) We have $l = 1$ for Hermite, and $d = 2$, so we need $k > 2$. This means that no values of k are possible. $\boxed{1, \text{D}}$

3. (a) (i) A local geometric decomposition for $(K, P, \mathcal{N})$ is an assignment of each nodal variable $N \in \mathcal{N}$ to a geometric entity of K. seen ⇓ 4, A

(ii) A local geometric decomposition for $(K, P, \mathcal{N})$ is C^0, if for each geometric entity w of K, there exists a subset $\mathcal{N}_w \subset \mathcal{N}$ containing only nodal variables that have been assigned to the closure of w, such that $(w, P|_w, \mathcal{N}_w)$ is a finite element, where $P|_w$ is the restriction of P to w. 4, A

(b) We need to show that $u \in V$ means that $u \in C^0$. To do this we need to check continuity of u across vertices, edges, and in 3D, faces. If V is constructed using elements with a C^0 geometric decomposition, then we can take any global entity of the triangulation (i.e., a vertex, edge, face, or cell), and the value of u should agree on w from any cell that contains w. If $(w, P|_w, \mathcal{N}_w)$ is a finite element, then since u in each cell shares those nodal variables, the value of u is completely determined on w in the same way from all cells. seen ⇓ 5, B

(c) (i) If we take the function which is zero on each vertex of the square, but equal to one in the middle, then there is a discontinuity because the function is zero in the entire bottom right triangle, but one in the middle. unseen ⇓ 1, C

(ii) The subspace requires that the value in the middle is the average of the values at the bottom left and top right vertices. Then, the function is linear along the entire diagonal, which matches the values in the bottom right triangle. 2, C

(d) Denote the diagonal edge as Γ. A consistent modification is to add a term unseen ⇓

$$-\int_\Gamma n^+ \cdot \nabla u_h^+ v_h^+ + n^- \cdot \nabla u_h^- v_h^- \, \mathrm{d}\, S \tag{10}$$

to the left hand side, where $+$ and $-$ indicate the values above and below Γ respectively. This is consistent since if we replace u_h with the exact solution u, we can separately integrate by parts in the regions above and below Γ, to obtain

$$\int_\Omega (u - \nabla^2 u - f) v \, \mathrm{d}\, x = 0, \tag{11}$$

as required. The modification vanishes according to property 2 because it only involves values of u_h on that boundary. 4, D

4. (a) (i) a is coercive if there exists a constant $\gamma > 0$ such that

seen $\Downarrow$

$$a(u,u) \geq \gamma \|u\|_{H^1}^2, \quad \forall u \in H^1. \tag{12}$$

2, A

(ii) a is continuous if there exists a constant $C > 0$ such that

$$a(u,v) \leq C\|u\|_{H^1}\|v\|_{H^1}, \quad \forall u, v \in H^1. \tag{13}$$

2, A

(iii) F is continuous if there exists a constant $C > 0$ such that

$$F[v] \leq C\|v\|_{H^1}, \quad \forall u, v \in H^1. \tag{14}$$

2, A

(iv) The Galerkin approximation seeks $u_h \in V$ such that

$$a(u_h, v) = F[v], \quad \forall v \in V. \tag{15}$$

4, A

(b) (i) We take $v \in V$ in the H^1 variational problem (possible since $V \subset H^1$, and substract the Galerkin approximation with the same v, using linearity,

seen $\Downarrow$

$$a(u - u_h, v) = a(u,v) - a(u_h, v) = F[v] - F[v] = 0, \tag{16}$$

as required.

3, B

(ii) The error is $u - u_h$. This tells us that the error is orthogonal to the whole of V, when using $a(\cdot,\cdot)$ as an inner product.

seen $\Downarrow$

(c) For arbitrary $v \in V$,

3, B

seen $\Downarrow$

$$\begin{aligned} \gamma\|u-u_h\|_{H^1(\Omega)} &\leq a(u-u_h, u-u_h) && (17)\\ &= b(u-u_h, u-v) + \underbrace{b(u-u_h, v-u_h)}_{=0} && (18)\\ &\leq C\|u-u_h\|_{H^1(\Omega)}\|u-v\|_{H^1(\Omega)}, && (19) \end{aligned}$$

and the result is obtained by dividing by $\|u-v\|_{H^1(\Omega)}$ and sup-ing over all $v \in V$.

4, C

5. (a) We first note that if u solves the weak form equation, then taking $q = \nabla \cdot u$, $v = 0$ gives $\|\nabla \cdot u\|_{L^2} = 0$, i,e. $\nabla \cdot u = 0$ in L^2. If $u \in H^2$ and $p \in H^1$, we may integrate by parts to get **seen/sim.seen ⇓**

$$\int_\Omega (-\nu\nabla^2 u - \nu\nabla \underbrace{(\nabla \cdot u)}_{=0} + \nabla p - f) v \,\mathrm{d}\, x, \quad \forall v \in V, \tag{20}$$

having dropped the boundary integral because v vanishes there. Then, we may choose as v a sequence of C_0^∞ functions converging to $-\mu\nabla^2 u + \nabla p - f$, and hence conclude that $-\mu\nabla^2 u + \nabla p = f$ in L^2. **8, M**

(b) If we take $u = 0$, then **seen ⇓**

$$c((u,p),(u,p)) = 0, \tag{21}$$

so c is not coercive. **5, M**

(c) We have **unseen ⇓**

$$\|B_h^* q\|_{V_h'} = \sup_{0 \neq v \in V_h} \frac{b(v,q)}{\|v\|_V}, \tag{22}$$

so the inf sup condition is equivalent to

$$\inf_{0 \neq q \in Q_h} \|B^* hq\|_{V_h'} \geq \gamma > 0. \tag{23}$$

If B_h^* has a kernel, then we can take q in the kernel and get zero, violating the inf sup condition. **3, M**

(d) We consider a function $q \in Q_h$ that is only supported in one square (subdivided into triangles). Inside the square, q is either -1 or 1, with the value alternating upon crossing the diagonal lines between triangles. We claim that $b(u,q) = 0$ for all $u \in V_h$. To check this, we just need to check it for each basis function supported in the square. The basis function equal to 1 at the square centre has constant divergence, so the q values cancel out and $b(u,q) = 0$. A basis function equal to 1 at a corner of the square also has constant divergence inside its support, and the same thing happens. Therefore, $q \in \ker B_h^*$. In fact there is one kernel function for each square. **unseen ⇓** **4, M**

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 21 of 20 marks

Total C marks: 13 of 12 marks

Total D marks: 14 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60022/70022	1	No Comments Received
MATH60022/70022	2	No Comments Received
MATH60022/70022	3	No Comments Received
MATH60022/70022	4	No Comments Received
MATH70022	5	No Comments Received