Imperial College London

MATH40002

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
January 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 1

Date: 9 January 2023

Time: 14:00 – 15:00

Time Allowed: 1 hour

This paper has 2 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

January 2023 Examination
Analysis 1

1. For all the parts of this first problem, decide whether the statement is true or false. If it is true, give a proof. If it is false, provide a counter-example, and provide any needed justification that your counter-example is indeed a counter-example.

   (a) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that, for all $n \in \mathbb{N}_{>0}$,

   $$\lim_{n\to\infty} |a_{n+1} - a_n| = 0 \ .$$

   (True or False) The sequence a_n must be convergent.

   (b) (5 marks) Suppose that a_n is a non-negative sequence such that $\sum_{n=1}^{\infty} a_n$ is convergent. **(True or False)** The series $\sum_{n=1}^{\infty} a_{n+1}a_n$ must be convergent.

   (c) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that, for all $p \in \mathbb{N}$ with $p > 1$, the subsequence $(a_{pj})_{j=1}^{\infty}$ is convergent.

   (True or False) The sequence $(a_n)_{n=1}^{\infty}$ must be convergent.

   (d) (5 marks) Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent.

   (True or False) It must be the case that the sequence $(|a_n|^{1/n})_{n=1}^{\infty}$ is convergent with $|a_n|^{1/n} \to r \in [0, 1]$.

2. (a) (5 marks) Let a_n be a real sequence with $a_n \to a \neq 0$. Carefully prove from first principles (using the definition of convergence but not the algebra of limits) that $\frac{1}{a_n} \to \frac{1}{a}$.

   (b) (5 marks) Let S be a countably infinite set, and define $\mathscr{P}$ to be the collection of all subsets $A \subset S$ with the property that **both** A and $S \setminus A$ are infinite. Is $\mathscr{P}$ countably infinite or uncountably infinite? Prove your answer.

   (c) (5 marks) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of positive terms with

   $$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \text{ for all } n \ .$$

   Suppose that $\sum_{n=1}^{\infty} b_n$ converges, prove $\sum_{n=1}^{\infty} a_n$ converges.

   (d) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be the sequence defined by

   $$a_1 = 9 \ , \quad a_2 = 6 \ , \quad \text{and} \quad a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n} \quad \text{for} \quad n \geq 2 \ .$$

   Prove that the sequence a_n is convergent.

January 2023 Examination
Analysis 1

1. For all the parts of this first problem, decide whether the statement is true or false. If it is true, give a proof. If it is false, provide a counter-example, and provide any needed justification that your counter-example is indeed a counter-example.

   (a) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that, for all $n \in \mathbb{N}_{>0}$,

   $$\lim_{n\to\infty} |a_{n+1} - a_n| = 0 \ .$$

   (True or False) The sequence a_n must be convergent.

   Solution: False (1 mark). We can take $a_n = \sum_{j=1}^{n} \frac{1}{j}$. (2 marks for counterexample). We have shown in lectures that $a_n \uparrow \infty$, but we also have $|a_{n+1} - a_n| = \frac{1}{n+1} \to 0$. (2 marks for justifying counterexample, less important for marks if we have already discussed counterexample in lecture, like the harmonic series)

   (b) (5 marks) Suppose that a_n is a non-negative sequence such that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (True or False) The series $\sum_{n=1}^{\infty} a_{n+1}a_n$ must be convergent.

   Solution: True (1 mark). Note that since $\sum_{n=1}^{\infty} a_n$ is convergent, we must have $a_n \to 0$ so a_n is bounded, that is there exists M such that $|a_n| \leq M$ for all n. It follows that $|a_{n+1}a_n| \leq M|a_n|$. However, since $\sum_{n=1}^{\infty} M|a_n| = M\sum_{n=1}^{\infty}|a_n|$ is convergent, by comparison we know that $\sum_{n=1}^{\infty}|a_{n+1}a_n|$ is convergent - so $\sum_{n=1}^{\infty} a_{n+1}a_n$ is absolutely convergent. (4 marks for proof)

   (c) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that, for all $p \in \mathbb{N}$ with $p > 1$, the subsequence $(a_{pj})_{j=1}^{\infty}$ is convergent.

   (True or False) The sequence $(a_n)_{n=1}^{\infty}$ must be convergent.

   Solution: False (1 mark). Take (2 marks for counterexample that works)

   $$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{if } n \text{ is not prime.} \end{cases}$$

   Note that any subsequence $(a_{pj})_{j=1}^{\infty}$ converges to 0 as $j \to \infty$. However, if we choose $n(i)$ to be the i-th prime, then $a_{n(i)} \to 1$ as $i \to \infty$. Since we have subsequences of a_n converging to different limits, a_n cannot be convergent - recall that if $a_n \to a$, then every subsequence of a_n would have to converge to a. (2 marks for justification).

   (d) (5 marks) Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent.

   (True or False) It must be the case that the sequence $(|a_n|^{1/n})_{n=1}^{\infty}$ is convergent with $|a_n|^{1/n} \to r \in [0,1]$.

Solution: False (1 mark). The sequence $|a_n|^{1/n}$ need not be convergent, take $a_n = (1/2)^n$ for n even and $a_n = (1/3)^n$ for n odd (2 marks for counterexample). We have that $0 \leq a_n \leq (1/2)^n$, and since $\sum_{n=1}^{\infty}(1/2)^n$ is a convergent geometric series it follows that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. On the other hand, $|a_n|^{1/n} = 1/2$ for n even and $|a_n|^{1/n} = 1/3$ for n odd, so we can find two subsequences of $|a_n|^{1/n}$ with one converging to $1/2$ and the other converging to $1/3$. Therefore $|a_n|^{1/n}$ can't be convergent (2 marks for justification).

2. (a) (5 marks) Let a_n be a real sequence with $a_n \to a \neq 0$. Carefully prove from first principles (using the definition of convergence but not the algebra of limits) that $\frac{1}{a_n} \to \frac{1}{a}$.

Solution: Let $\epsilon = |a|/2$. Then there exists N_1 such that, for all $n \geq N_1$, $|a_n - a| < |a|/2$. Therefore, for $n \geq N_1$, we have

$$|a| \leq |a_n - a| + |a_n| < |a|/2 + |a_n| \quad \Rightarrow |a_n| > |a|/2 \, .$$

(2 marks for bounding the sequence away from 0).

Now, let $\epsilon > 0$ be arbitary. Then there exists N_2 such that for $n \geq N_2$, $|a_n - a| < |a|^2\epsilon/2$. It follows that if we set $N = \max(N_1, N_2)$, then we have

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a| \cdot |a_n|} < \frac{2}{|a|^2}|a_n - a| < \epsilon \, ,$$

where in the first inequality we used that $n \geq N_1$ and in the second we used that $n \geq N_2$. (3 marks for rest of the proof)

(b) (5 marks) Let S be a countably infinite set, and define $\mathscr{P}$ to be the collection of all subsets $A \subset S$ with the property that **both** A and $S \setminus A$ are infinite. Is $\mathscr{P}$ countably infinite or uncountably infinite? Prove your answer.

Solution: The set $\mathscr{P}$ is uncountably infinite (2 marks). On the problem sheets we showed that the set of infinite subsets of S is uncountably infinite and that the set of finite subsets of S is countable. The map $A \mapsto S \setminus A$ is a bijection on the set of subsets of S, in particular the fact that the set of finite subsets of S is countable tells us that the set $\{A \subset S : \ S \setminus A \text{ is finite }\}$ is also countable.

Now we note that

$$\begin{aligned}\{A \subset S : A \text{ infinite }\} = \{&A \subset S : A \text{ infinite and } S \setminus A \text{ finite }\} \\ &\sqcup \{A \subset S : A \text{ infinite and } S \setminus A \text{ infinite }\}\end{aligned}$$

We know that the first set on the right hand side is a subset of a countable set, so it is countable. We also know that the set on the left hand side is uncountable. Therefore the second set on the right hand side must be uncountably infinite.

(c) (5 marks) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of positive terms with

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \text{ for all } n \, .$$

Suppose that $\sum_{n=1}^{\infty} b_n$ converges, prove $\sum_{n=1}^{\infty} a_n$ converges.

Solution: Let $c_n = a_n/b_n$, then, we have

$$c_{n+1} = a_{n+1}/b_{n+1} \leq a_n/b_n = c_n$$

so that c_n is monotone decreasing. Since $c_n \geq 0$, it follows that c_n is convergent (3 marks for showing that ratios form a convergent subsequence).

It follows that c_n is bounded by some M - in particular $a_n = c_n b_n \leq M b_n$. We conclude that $\sum_{n=1}^{\infty} a_n$ must be convergent since $\sum_{n=1}^{\infty} M b_n$ is convergent (2 marks for finishing proof with comparison test).

(d) (5 marks) Let $(a_n)_{n=1}^{\infty}$ be the sequence defined by

$$a_1 = 9\ , \quad a_2 = 6\ , \quad \text{and} \quad a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n} \quad \text{for} \quad n \geq 2\ .$$

Prove that the sequence a_n is convergent.

Solution: It suffices to show that the a_n are monotone decreasing and bounded below (2 marks for identifying strategy that works).

Note that $a_1 \geq a_2 \geq a_3 > 0$ and we have

$$a_{n-1} \geq a_n \geq a_{n+1} > 0 \Rightarrow a_n \geq a_{n+1} \geq a_{n+2} > 0\ .$$

It follows by induction that the a_n are monotone decreasing (2 marks) and bounded below (1 mark).