# Problem Sheet 1

We will discuss the solutions of this problem sheet in the problem class on Thursday, 18 January 2024.

1. **(Demand functions)** Students 1, 2 and 3 are interviewed about their coffee consumption. More specifically, they report their *reservation prices* for the quantity of cups of coffee they would be willing to purchase. The reservation price is the highest price per quantity such that they are willing to buy that specific quantity.

   **Student 1:**

   | Quantity | 1 | 2 | 3 |
   |---|---|---|---|
   | Price in £ | 3.00 | 2.00 | 1.00 |

   **Student 2:**

   | Quantity | 1 | 2 | 3 |
   |---|---|---|---|
   | Price in £ | 4.00 | 1.50 | 0.50 |

   **Student 3:**

   | Quantity | 1 | 2 | 3 |
   |---|---|---|---|
   | Price in £ | 1 | 0.00 | 0.00 |

   **a)** Calculate the respective individual demand functions $D_i$, $i \in \{1, 2, 3\}$ (as functions in the price) and the inverse demand functions $P_i$, $i \in \{1, 2, 3\}$ (as functions in the quantity demanded). If you properly consider the functions as maps $Q_i \colon A \to B$ and $P_i \colon C \to E$, what sets $A, B, C, E$ are most appropriate?

   **b)** Calculate the aggregate demand function $D$ and the corresponding inverse $P$ of the aggregate demand function.

   **c)** If the price for one cup of coffee is £ 0.75, how many cups will be sold in total?

2. **(Price elasticity of demand and supply)**

   **a)** Student A claims that since a linear demand function has constant slope, it also exhibits constant price elasticity. Under which conditions is Student A correct? Justify your answer.

   **b)** In case you agree with Student A, are there other possible demand functions that exhibit a constant price elasticity? If you disagree with Student A, are there alternatives to linear demand functions that exhibit a constant price elasticity?

   **c)** Now, Student B claims that since a linear supply function has constant slope, it also exhibits constant price elasticity. Under which conditions is Student B correct? Justify your answer.

   **d)** In case you agree with Student B, are there other possible supply functions that exhibit a constant price elasticity? If you disagree with Student B, are there alternatives to linear supply functions that exhibit a constant price elasticity?

3. **(Quasi-Concavity)** We have seen in the lecture that one of the assumptions of a production function is quasi-concavity. If $D \subseteq \mathbb{R}^n$ is a convex subset of $\mathbb{R}^n$, we say that a function $f \colon D \to \mathbb{R}$ is quasi-concave if

$$f((1 - \lambda)y + \lambda x) \geq \min\{f(x), f(y)\} \qquad \forall x, y \in D, \ \forall \lambda \in [0, 1].$$

   **a)** Show that for an interval $I \subseteq \mathbb{R}$ and $f \colon I \to \mathbb{R}$ is quasi-concave if and only if

   (i) $f$ is monotonically increasing; or
   (ii) $f$ is monotonically decreasing; or
   (iii) $f$ is monotonically increasing and then monotonically decreasing.

   **b)** Let $D \subseteq \mathbb{R}^n$ be convex. Show that if a function $f \colon D \to \mathbb{R}$ is concave, it is also quasi-concave. Show that the reverse implication does not hold by giving a counterexample.

   **c)** Let $D \subseteq \mathbb{R}^n$ be convex and $f \colon D \to \mathbb{R}$ a continuously differentiable function. Show that the following assertions are equivalent:

   (i) $f$ is quasi-concave.
   (ii) For all $y \in \mathbb{R}$ the set $f^{-1}([y, \infty))$ is convex.
   (iii) For all $x, y \in D$: If $f(y) \geq f(x)$, then $\nabla f(x)^\top (y - x) \geq 0$.

**d)** We say a function $g \colon D \to \mathbb{R}$ is quasi-convex if $f = -g$ is quasi-concave. State the equivalence in **c)** directly in terms of $g$.

**e)** Prove that the following production functions are increasing and quasi-concave.

- Leontief production function $f(x_1, x_2) = \min(ax_1, bx_2)$;
- linear production function $f(x_1, x_2) = ax_1 + bx_2$;
- Cobb-Douglas production function $f(x_1, x_2) = Ax_1^a x_2^b$,

where $A, a, b, x_1, x_2 \geq 0$.