Math40002 Analysis 1 **Problem Sheet 4**

1. Consider the following properties of a sequence of real numbers $(a_n)_{n\geq 0}$:
   (i) $a_n \to a$, or
   (ii) "a_n eventually equals a" – i.e. $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ a_n = a$, or
   (iii) "(a_n) is bounded" – i.e. $\exists\, R \in \mathbb{R}$ such that $|a_n| < R\ \ \forall n \in \mathbb{N}_{>0}$.

   For each statement (a-e) below, which of (i-iii) is it equivalent to? Proof?
   (a) $\exists\, N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ \forall \epsilon > 0,\ |a_n - a| < \epsilon$.
   (b) $\forall \epsilon > 0$ there are only finitely many $n \in \mathbb{N}_{>0}$ for which $|a_n - a| \geq \epsilon$.
   (c) $\forall N \in \mathbb{N}_{>0},\ \exists\, \epsilon > 0$ such that $n \geq N \ \Rightarrow\ |a_n - a| < \epsilon$.
   (d) $\exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}_{>0},\ |a_n - a| < \epsilon\ \forall n \geq N$.
   (e) $\forall R > 0\ \exists\, N \in \mathbb{N}_{>0}$ such that $n \geq N \ \Rightarrow\ a_n \in (a - \frac{1}{R}, a + \frac{1}{R})$.

2. Given a sequence $(a_n)_{n\geq 1}$ of *complex* numbers, define what $a_n \to a$ means. For $x, y \in \mathbb{R}$ and $z := x + iy \in \mathbb{C}$ show $\max(|x|, |y|) \ \leq\ |z| \ \leq\ \sqrt{2}\max(|x|, |y|)$, and
$$a_n \to a + ib \in \mathbb{C} \qquad \Longleftrightarrow \qquad \mathrm{Re}(a_n) \to a \quad \text{and} \quad \mathrm{Im}(a_n) \to b.$$

3. Suppose that $a_n \leq b_n \leq c_n\ \forall n$ and that $a_n \to a$ and $c_n \to a$. Prove that $b_n \to a$.

4. Suppose that $a_n \to 0$ and (b_n) is bounded. Prove that $a_n b_n \to 0$.

5. * Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \to a$ and $b_n \to b \neq 0$. Prove that the set $\{a_n : n \in \mathbb{N}_{>0}\}$ is bounded and that
$$\exists\, N \in \mathbb{N}_{>0} \quad \text{such that} \quad n \geq N \ \Rightarrow\ |b_n| > |b|/2.$$
   Therefore $(a_n/b_n)_{n\geq N}$ is a sequence of real numbers; prove it tends to a/b.

6. We call a sequence *sorta-Cauchy* if it satisfies the condition
$$\forall \epsilon > 0\ \exists\, N \in \mathbb{N}_{>0}\ \ n \geq N \ \Rightarrow\ |a_n - a_{n+1}| < \epsilon.$$
   Give an example of a sorta-Cauchy sequence which diverges to $+\infty$. Conclude that sorta-Cauchy is not as strong as Cauchy.

7. Give an example of a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

   In lectures we show that in $\mathbb{R}$, a sequence is Cauchy if and only if it is convergent. Show that it is impossible to prove this using only the arithmetic and order axioms of $\mathbb{R}$ (i.e. all the axioms except the completeness axioms – the one about the existence of least upper bounds).

8. Let $(a_n)_{n\in\mathbb{N}_{>0}}$ be a bounded sequence.

(a) For each $n \in \mathbb{N}_{>0}$, define the set $S_n = \{a_j : j \geq n\}$. Prove that, for every $n \in \mathbb{N}_{>0}$, there exists some $b_n \in \mathbb{R}$ such that $b_n = \sup(S_n)$.

(b) Let $B = \{b_n : n \in \mathbb{N}_{>0}\}$ where b_n is defined as above. Prove that there exists some $l \in \mathbb{R}$ such that $l = \inf(B)$. (Remark: l is called the limit supremum of the sequence $(a_n)_{n\in\mathbb{N}_{>0}}$, and the usual notation is $l = \limsup_{n\to\infty} a_n$).

(c) For each of the sequences below, find the value of $\limsup_{n\to\infty} a_n$ and give justification for your answer.

   i. $a_n = (-1)^n$

   ii. $a_n = \frac{(-1)^n}{n}$

Starred questions * *are good to prepare to discuss at your Problem Class.*