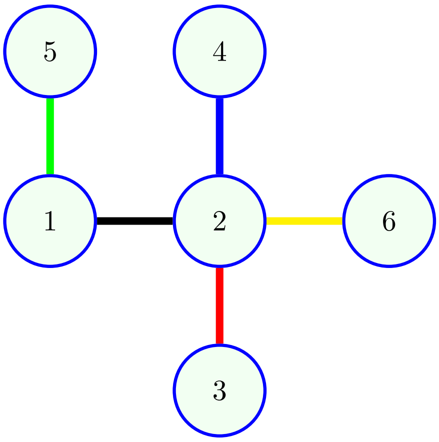
Network Science
Spring 2024
Problem sheet 6 Solutions



1. Consider the simple Barabasi-Albert model analyzed in lecture with $N_0 = 2$ and $m = 1$. A simulation of the model has generated the graph above at time $t = 4$, $G_1(t = 4)$.

   (a) What is the probability of generating this graph given the graph that existed at $t = 3$?
   Solution: The graph begins with nodes 1 and 2 and the black link at $t = 0$, and then one node and one link is added each iteration. At $t = 3$, we have the graph above with node 6 and the yellow link removed. We need to consider the probability of node 6 linking to node 2. At $t = 3$, node 2 had 3 of 8 stubs in the graph so the probability of generating $G_1(t = 4)$ is 3/8.

   (b) What is the probability of the simulation generating this graph?
   Solution: We need to think about the probability of node 3 connecting to node 2 at $t = 1$, the probability of node 4 connecting to node 2 at $t = 2$ and so on... The probability of node 3 connecting to node 2 at t=1 is 1/2. The probability of node 4 connecting to 2 at t=2 is 1/2. The probability of node 5 connecting to 1 at t=3 is 1/6 and the probability of node 6 connecting to 2 at t=4 is 3/8, so the probability of generating $G_1(t = 4)$ is $(1/2)(1/2)(1/6)(3/8) = 1/64$.

   (c) What is the expected degree of node 2 at time $t = 5$ given that we have $G_1(t = 4)$?
   Solution: The probability that a link will be added to node 2 is $4/10 = 2/5$ which is also the expected change in the degree. So, $\langle k_2(t = 5|G_1(t = 4)\rangle = 4 + 2/5 = 4.4$

2. Consider the following modification to the simple Barabasi-Albert model ($N_0 = 2, m = 1$): a new node connects to any existing node in the graph with equal probability.

   (a) How does the expected degree of a node at iteration $t + 1$, $\langle k_i(t + 1)\rangle$, depend on $\langle k_i(t)\rangle$ and t?
   Solution: We have $k_i(t + 1) = k_i(t) + \delta$ where δ is the number of links added to node i between iterations t and $t + 1$ (it will be either 0 or 1). Since, $P(\delta = 1) = \langle\delta\rangle = 1/N(t)$, we have $\langle k_i(t + 1)\rangle = \langle k_i(t)\rangle + 1/N(t)$, and $N(t) = 2 + t$.

(b) How does $\langle N_k(t+1)\rangle$, the expected number of nodes with degree k at iteration $t+1$ depend on $\langle N_k(t)\rangle$, $\langle N_{k-1}(t)\rangle$ and t? when $k > 1$?
Solution: As in lecture, we consider three cases. Case A: New node links to node with degree k and N_k decreases by 1; Case B: New node links to node with degree $k-1$ and N_k increases by 1; Case C: Neither A nor B and N_k remains the same, The probabilities of cases A and B are $P(A) = N_k/N$, $P(B) = N_{k-1}/N$, and $P(C) = 1 - P(A) - P(B)$. Using these expressions as in lecture,
$\langle N_k(t+1)\rangle =$
$\sum_{i=1}^{(t+1)!} P(G_i(t))\left[P(A)(N_k(t)-1) + P(B)(N_k(t)+1) + P(C)(N_k(t)\right]$
$= \sum_{i=1}^{(t+1)!} P(G_i(t)\left[N_k(t) + (N_{k-1}(t) - N_k(t))/N(t)\right]$, so:

$$\langle N_k(t+1)\rangle = \langle N_k(t)\rangle + \left(\langle N_{k-1}(t) - N_k(t)\rangle\right)/(2+t).$$

(c) How does $p_1(t+1)$, the expected fraction of nodes with degree 1 at iteration $t+1$ depend on $p_1(t)$ and t?
Solution: We consider two cases. Case A: New node links to node with degree 1 and N_1 decreases by 1 (ignoring the new node which is accounted for later); Case B: New node links to node with degree $k > 1$ and N_k is unchanged. The probabilities of cases A and B are $P(A) = N_1/N$, $P(B) = 1 - N_1/N$. Moreover, there is a new node with degree 1 each iteration. Using these results,

$$\begin{aligned}\langle N_1(t+1)\rangle &= \\ &\sum_{i=1}^{(t+1)!} P(G_i(t))\left[P(A)(N_1(t)-1) + P(B)(N_1(t)) + 1\right] \\ &= \sum_{i=1}^{(t+1)!} P(G_i(t)\left[N_1(t) - N_1(t)/N(t) + 1\right], \quad (1)\end{aligned}$$

so:

$$\langle N_1(t+1)\rangle = \langle N_1(t)\rangle - \langle N_1(t)\rangle/N(t) + 1.$$

Since $p_1(t) = \langle N_1(t)\rangle/N(t)$, and $N(t) = 2+t$, we find,

$$(3+t)p_1(t+1) = (2+t)p_1(t) - p_1(t) + 1.$$

Note: In the summations above, $P(A), P(B)$, and $N_1(t)$ are conditional on $G_i(t)$, so following the notation from lecture, I should have written $N_1(t|G_i(t))$