ii) $f(\underline{x}) = \sqrt{1 + \underline{x}^T Q \underline{x}}$ is convex, given $Q \succcurlyeq 0$

In previous weeks, we have shown that

$h(y) = \sqrt{1 + y^2}$ is convex. ✓

$$h'(y) = \frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{1+y^2}} > 0 \text{ for } y > 0 \rightarrow \text{non-decreasing}$$

$$h''(y) = \frac{\sqrt{1+y^2} - \frac{y^2}{\sqrt{1+y^2}}}{(1+y^2)} = \frac{1}{(1+y^2)^{3/2}} > 0$$

$$f(\underline{x}) = \sqrt{1 + \underline{x}^T Q \underline{x}} = \sqrt{1 + y^2}, \text{ where } y^2 = x^T Q x$$

$$\Rightarrow y = g(\underline{x}) = \sqrt{x^T Q x}$$

but $Q \succcurlyeq 0$, and we have seen in Week 2

$$Q = U^T D U \qquad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \qquad d_i \geq 0$$

$D \rightarrow$ Diagonal

$$Q = U^T \sqrt{D} \sqrt{D} U \implies \underline{x}^T U^T \sqrt{D} \sqrt{D} U \underline{x}$$

$$\underline{x}^T Q \underline{x} = \sqrt{\underline{x}^T U^T \sqrt{D} \sqrt{D} U \underline{x}}$$

$$= \| L x \| \quad \text{where } L = \sqrt{D} U$$

But we know that $\| \cdot \|$ is convex function and

$x \longrightarrow Lx$ is a linear transformation

$\implies \| Lx \|$ is convex in $\underline{x}$

$g(\underline{x}) : \mathbb{R}^M \longrightarrow \mathbb{R}^+ \quad g := \| Lx \|$ is convex

$$f(\underline{x}) = \sqrt{\underline{x}^T Q \underline{x} + 1} = \underbrace{h}_{\sqrt{1+\underline{y}^2}} \circ \underbrace{g}_{\|L x\|}(\underline{x}) \quad \text{convex}$$

because of composing $h \circ g$ w/ g convex and h convex and non-decreasing.

$$\min \; - x_1 x_2 x_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 3x_2 + 6x_3 \leq 48 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & -x_3 \leq 0 \end{aligned}$$