Problem Sheet 1

Problem 1. Let $\phi : [-\epsilon, +\epsilon] \to \mathbb{R}^3$ be a regular curve, with tangent vector T and principle normal vector N. Let

$$\psi : [-\epsilon, +\epsilon] \to \{aT(0) + bN(0) \mid a, b \in \mathbb{R}\}$$

be the orthogonal projection of ϕ onto the plane spanned by $T(0)$ and $N(0)$. Prove that ϕ and ψ have the same curvature at time $t = 0$.

Solution: Regardless of how the curve ϕ is parametrised, the projected map ψ is the same on the plane spanned by $T(0)$ and $N(0)$. So, without loss of generality, we may assume that ϕ is parametrised by arc-length.

As $T(0)$, $N(0)$ and $B(0)$ form a basis for $\mathbb{R}^3$, we may write ϕ as

$$\phi(t) = a(t)T(0) + b(t)N(0) + c(t)B(0), \quad t \in [a, b].$$

Differentiating the above equation with respect to t, at $t = 0$, we obtain

$$\phi'(0) = a'(0)T(0) + b'(0)N(0) + c'(0)B(0).$$

On the other hand, by definition, we have $\phi'(0) = T(0)$. Since $T(0)$, $N(0)$ and $B(0)$ are linearly independent, we conclude that $a'(0) = 1$ while $b'(0) = c'(0) = 0$.

Differentiating the above equation two times at $t = 0$, we obtain

$$\phi''(0) = a''(0)T(0) + b''(0)N(0) + c''(0)B(0).$$

By definition we have $\phi''(0) = k_\phi(0)N(0)$. This implies that $k_\phi(0) = b''(0)$, and $a''(0) = c''(0) = 0$. Therefore, $k_\phi(0) = b''(0)$.

The curve ψ is given in coordinates in the plane with orthonormal basis $T(0)$ and $N(0)$ as

$$\psi(t) = a(t)T(0) + b(t)N(0).$$

However, this curve is not necessarily parametrised by arc length. By a proposition in the lecture notes, the signed curvature is given by the formula

$$\kappa_\psi(0) = \frac{\langle \psi''(0), N_\psi(0)\rangle}{|\psi'(0)|^2}$$

We have

$$\psi''(0) = a''(0)T(0) + b''(0)N(0) = b''(0)N(0),$$
$$|\psi'(0)| = |a'(0)T(0) + b'(0)N(0)| = |a'(0)T(0)| = |a'(0)| = 1.$$

Moreover, $N_\psi(0) = \pm N(0)$, since $N(0)$ belongs to the plane generated by $T(0)$ and $N(0)$, so one of $(T(0), +N(0))$ and $(T(0), -N(0))$, forms a positively oriented basis for that plane. Therefore,

$$\kappa_\psi(0) = \pm b''(0).$$

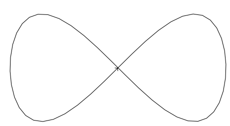
Now since, $k_\psi(0) = |\kappa(0)|$ and $b''(0) > 0$, we conclude that the two curves have the same curvature.

Problem 2. Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}^3$ be regular curves parametrised by arc length. Suppose that their curvatures k_1, k_2 and torsions τ_1, τ_2 are positive everywhere, and that their binormal vectors are identical, $B_1(t) = B_2(t)$ for all $t \in [a, b]$. Prove that there is a constant vector $\vec{v} \in \mathbb{R}^3$ such that $\phi_2(t) = \phi_1(t) + \vec{v}$.

Solution: Let (T_i, N_i, B_i) be the Frenet frame for ϕ_i , $i = 1, 2$, respectively. Since $B_1 = B_2$, we have $B_1' = B_2'$, and hence by the Frenet equations, we obtain $-\tau_1 N_1 = -\tau_2 N_2$. Both N_1 and N_2 are unit vectors, so upon taking lengths we get $\tau_1 = \tau_2$, using the fact that these are both positive. Dividing by the non-zero values $-\tau_1 = -\tau_2$, we obtain $N_1 = N_2$ as well. Since $N_1 = N_2$ and $B_1 = B_2$, it follows that $T_1 = T_2$ as well (note that $T_i = N_i \times B_i$). This implies that $(\phi_1 - \phi_2)' = T_1 - T_2 = 0$, and hence $\phi_1 - \phi_2$ is a constant.

Problem 3. For each $n \in \mathbb{Z}$, draw (construct, explain) a closed regular plane curve ϕ with $\mathrm{Ind}(\phi) = n$.

Solution: For $n > 0$, take ϕ to be a curve which travels counterclockwise along a unit circle n times. For $n < 0$, travel n times clockwise along the same circle. For $n = 0$, draw a figure 8 as follows



For explicit examples, one may use the curves

$$t \mapsto (\cos(nt), \sin(nt)), \quad 0 \le t \le 2\pi,$$

$$t \mapsto (\cos(nt), -\sin(|n|t)), \quad 0 \le t \le 2\pi,$$

and

$$t \mapsto (\sin(t), \sin(2t)), \quad 0 \le t \le 2\pi,$$

respectively.

Problem 4. Let $\phi : [a, b] \to \mathbb{R}^2$ be a regular curve which is parametrised by arc length, and let $v \in \mathbb{R}^2$. Consider the function $f_v : [a, b] \to \mathbb{R}$ defined as

$$f_v(t) = |\phi(t) - v|^2.$$

a) Show that there is $t_0 \in (a, b)$ satisfying $f_v'(t_0) = 0$ if and only if the circle C of radius $\sqrt{f_v(t_0)}$ centred at v is tangent to ϕ at $\phi(t_0)$.

b) Assume that the curvature $k(t_0) \neq 0$ for some $t_0 \in (a, b)$. Determine, in terms of $k(t_0)$, the unique value of R such that there is $v \in \mathbb{R}^2$ satisfying $f_v(t_0) = R^2$, $f_v'(t_0) = 0$ and $f_v''(t_0) = 0$.

Remark: The above problem characterises $|k(t)|$ in terms of the radius of the circle which "best" approximates ϕ at $\phi(t)$ (that is, it is a tangent of order 2 to the curve).

Solution: a) Obviously, $|\phi(t_0) - v| = \sqrt{f_v(t_0)}$, which shows that the point $\phi(t_0)$ lies on the circle C. We note that

$$f_v'(t) = \frac{d}{dt}\langle \phi(t) - v, \phi(t) - v\rangle = 2\langle \phi(t) - v, \phi'(t)\rangle.$$

Therefore, $f_v'(t_0) = 0$ if and only if the tangent vector $\phi'(t_0)$ is orthogonal to the radius $\phi(t_0) - v$ of C. Since the tangent vector to C at $\phi(t_0)$ is also orthogonal to this radius, it follows that the tangent vectors to C and ϕ at $\phi(t_0)$ are proportional, and so ϕ is tangent to C.

b) By condition $f_v'(t_0) = 0$ and part (a), the point v must be on the the perpendicular line to the curve ϕ at $\phi(t_0)$. Let v be an arbitrary point on that line. We will identify v using condition $f_v''(t_0) = 0$.

We can calculate the second derivative, as follows

$$\begin{aligned} f_v''(t_0) &= 2\langle \phi'(t_0), \phi'(t_0)\rangle + 2\langle \phi(t_0) - v, \phi''(t_0)\rangle \\ &= 2|\phi'(t_0)|^2 + 2\langle \pm\sqrt{f_v(t_0)}N(t_0), k(t_0)N(t_0)\rangle \\ &= 2 \pm 2\sqrt{f_v(t_0)}k(t_0). \end{aligned}$$

The sign in the above equation depends on whether the center v is located on the perpendicular line.

By the above equation, $f_v''(t_0) = 0$ if and only if $1 \pm \sqrt{f_v(t_0)}k(t_0) = 0$, and since $\sqrt{f_v(t_0)}$ and $k(t_0)$ are both nonnegative, we must have

$$\sqrt{f_v(t_0)} = \frac{1}{k(t_0)}.$$

We can choose $R = \sqrt{f_v(t_0)}$, and we have $f_v(t_0) = R^2$.

Problem 5. Let $\phi : [-\epsilon, +\epsilon] \to \mathbb{R}^3$ be a regular curve parametrised by arc length. Assume that $\phi(0) = (0, 0, 0)$ and the Frenet frame at time $t = 0$ is

$$T(0) = (1, 0, 0), \quad N(0) = (0, 1, 0), \quad B(0) = (0, 0, 1).$$

Writing $\phi(t) = (x(t), y(t), z(t))$ and assuming that $k(0) \neq 0$ and $\tau(0) \neq 0$, determine the leading nonzero terms of the Taylor series for each of the coordinates x, y, z at $t = 0$ in terms of the curvature $k_0 = k(0)$ and the torsion $\tau_0 = \tau(0)$.

Solution: For convenience let us introduce the notation $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

Since ϕ is smooth, it has Taylor series (as a formal power series which may not necessarily converge)

$$\sum_{i=0}^{\infty} \frac{\phi^{(i)}(0)}{i!} t^i.$$

We can compute the first few derivatives of ϕ (using the Frenet equations):

$$\phi'(t) = T(t), \quad \phi''(t) = k(t)N(t), \quad \phi'''(t) = k'(t)N(t) + k(t)(-k(t)T(t) + \tau(t)B(t)).$$

Therefore, at $t = 0$, we have

$$\phi(0) = (0, 0, 0), \quad \phi'(0) = e_1, \quad \phi''(0) = k_0 e_2, \quad \phi'''(0) = k'(0)e_2 - k_0^2 e_1 + k_0\tau_0 e_3.$$

This gives us

$$\phi(t) = te_1 + \frac{t^2}{2}k_0 e_2 + \frac{t^3}{6}(-k_0^2 e_1 + k'(0)e_2 + k_0\tau_0 e_3) = (t - \frac{k_0^2 t^3}{6}, \frac{k_0 t^2}{2} + \frac{k'(0)t^3}{6}, \frac{k_0\tau_0 t^3}{6}) + O(t^4).$$

We conclude that the degree 3 Taylor polynomial of ϕ is

$$(t, \frac{k_0}{2}t^2, \frac{k_0\tau_0}{6}t^3).$$