Mathematical Logic (MATH60132 and MATH70132)
2024-25, Coursework 2

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 10 March 2025. The coursework is marked out of 20 and the marks per question are indicated below.
The work which you submit should be your own, unaided work. Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet, generative AI agent or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1] (5 marks) Suppose $\mathcal{L}$ is a 1st-order language, Δ is a set of $\mathcal{L}$-formulas and ϕ an $\mathcal{L}$-formula. We write $\Delta \models \phi$ to mean that for every valuation v (in an $\mathcal{L}$-structure $\mathcal{A}$), if $v[\Delta] = T$, then $v[\phi] = T$.

(i) Suppose that the variable x_i is not free in any formula in Δ. Show that if $\Delta \models \phi$, then $\Delta \models (\forall x_i)\phi$.

(ii) Give (with justification) an example where $\Delta \models \phi$ and $\Delta \not\models (\forall x_i)\phi$.

[Note that (i) is used in the proof of the Generalised Soundness Theorem (2.4.7 in the notes), so your solution to (i) should not quote that result.]

[2] (11 marks) Let $\mathcal{L}^=$ be the 1st-order language (with equality) for groups, having a 2-ary function symbol $\cdot$ for the group operation and a constant symbol e for the identity element. For each natural number $n \geq 1$, let $\mathcal{C}_n$ denote the cyclic group of order 2^n, considered as a normal $\mathcal{L}^=$-structure. Let Φ be the set consisting of all closed $\mathcal{L}^=$-formulas ϕ having the property that:

$$\text{there are only finitely many } n \geq 1 \text{ with } \mathcal{C}_n \models (\neg\phi).$$

(i) Show that Φ is consistent.

(ii) Show that if χ is a closed $\mathcal{L}^=$-formula and $\Phi \vdash \chi$, then $\chi \in \Phi$.

(iii) Suppose $\mathcal{A}$ is a normal model of Φ (and denote its domain by A). Show that:

(a) $\mathcal{A}$ is an infinite abelian group.

(b) If $g \in A$ has finite order k, then k is a power of 2.

(c) For every odd number $m \in \mathbb{N}$ we have $\{a^m : a \in A\} = A$.

(d) $\mathcal{A}$ has exactly one subgroup H_n of order 2^n, for each $n \in \mathbb{N}$.

(e) If H_n is as in (d), then $H_n \subseteq H_{n+1}$.

[Hint: You may use that: if $G = \langle g \rangle$ is a cyclic group of finite order k and $m \in \mathbb{N}$, then g^m has order $k/\gcd(m,k)$; and if d divides k, then there is a unique subgroup of G of order d.]

(iv) Show that there is a normal model of Φ with an element of infinite order.

[Hint: Consider a language $\mathcal{L}_c^=$ consisting of $\mathcal{L}^=$ together with an extra constant symbol c and use the Compactness Theorem.]

[3] (4 marks) Let $\mathcal{L}^E$ be a 1st-order language with equality and $\mathcal{A}$ a normal $\mathcal{L}^E$-structure. Prove that there is an $\mathcal{L}^E$-structure $\mathcal{B}$ which satisfies the axioms for equality Σ_E, is NOT a normal $\mathcal{L}^E$-structure and has $\hat{\mathcal{B}}$ isomorphic to $\mathcal{A}$ (in the notation of Section 2.6 of the notes).