IMPERIAL

MATH70036

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Theory

Date: Monday, April 28, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Throughout the paper, you may use any results from the course that you require provided you state them clearly.

1. For each of the following statements, decide whether it is true or false. For those that you think are true, give a proof; for those you think are false, give a counterexample.

   Standard notation is used: G, H, N are groups; G' denotes the commutator subgroup of G; $\mathsf{Aut}(G)$ is the automorphism group of G; $GL_2(q)$, $SL_2(q)$ and $PSL_2(q)$ denote general linear, special linear and projective special linear groups over the field of q elements; C_n is a cyclic group of order n.

   (i) If N is a normal subgroup of G, then N' is also a normal subgroup of G. (2 marks)

   (ii) If N is a normal subgroup of G', then N is a normal subgroup of G. (3 marks)

   (iii) If $N \leq G$ and G' is a normal subgroup of N, then N is a normal subgroup of G. (3 marks)

   (iv) $\mathsf{Aut}(G \times H)$ has a subgroup that is isomorphic to $\mathsf{Aut}(G) \times \mathsf{Aut}(H)$. (3 marks)

   (v) If $G \ncong H$, then $\mathsf{Aut}(G \times H) \cong \mathsf{Aut}(G) \times \mathsf{Aut}(H)$. (3 marks)

   (vi) $SL_2(5) \cong C_2 \times PSL_2(5)$. (3 marks)

   (vii) $GL_2(4) \cong C_3 \times SL_2(4)$. (3 marks)

(Total: 20 marks)

2. (a) State the four *Sylow theorems*.
State also *Burnside's Transfer Theorem*.

(5 marks)

(b) Prove that any group of order 80 has a normal Sylow subgroup. (4 marks)

(c) Let G be a group of order $1764 = 2^2 3^2 7^2$.

(i) Prove that G is not simple.

(ii) Prove further that G has a normal Sylow subgroup.

(7 marks)

(d) Let G be a finite group of order $2^a m$ where m is odd, and suppose that G has a cyclic Sylow 2-subgroup. Prove that G has a normal subgroup of order m. (Proof required – it is not enough to quote a more general result from which this follows.) (4 marks)

(Total: 20 marks)

3. Let G be a finite group with a normal subgroup N, and let $H = G/N$, so that G is an extension of N by H. Recall that we say the extension *splits* if there is a subgroup H_0 of G such that $G = NH_0$ and $N \cap H_0 = 1$.

(a) Giving your reasoning, decide which of the following statements are true and which are false:

(i) The cyclic group C_8 is a split extension of C_4 by C_2.

(ii) The dihedral group D_8 of order 8 is a split extension of C_4 by C_2.

(iii) The quaternion group Q_8 is a split extension of C_4 by C_2.

(8 marks)

(b) Let G be an extension of N by H, let p be a prime, and suppose that

- N is a p-group,
- p does not divide $|H|$, and
- H is soluble.

Prove that the extension splits. (6 marks)

(c) Let P be a non-abelian p-group where p is prime, let $Z = Z(P)$ be the centre of P, and let $H = P/Z$, so that P is an extension of Z by H. Prove that this extension does not split.

(6 marks)

(Total: 20 marks)

4. (a) Let G be a finite group. Define what is meant by the statement that G is *nilpotent*. Define also the *Frattini subgroup* $\Phi(G)$. (4 marks)

(b) Let p be a prime, and let P be a finite p-group. Prove that P is nilpotent. (4 marks)

(c) Let p be an odd prime, and let P be the semidirect product $(C_p)^3 \rtimes_\iota C_p$, where $C_p = \langle x \rangle$ and $\iota : C_p \to \mathrm{Aut}((C_p)^3) \cong GL_3(p)$ is defined by

$$\iota(x) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So P has order p^4, and generators a, b, c, x, where $\langle a, b, c \rangle \cong C_p \times C_p \times C_p$, $x^p = 1$ and

$$xax^{-1} = a,\ xbx^{-1} = b,\ xcx^{-1} = ac.$$

(i) Let $A = \langle a \rangle$. Show that $A \triangleleft P$ and P/A is abelian.

(ii) Find (in terms of p) the order of the Frattini subgroup $\Phi(P)$. Give your reasoning.

(6 marks)

(d) Let G be a finite group, and suppose that $G/\Phi(G)$ is abelian.

(i) Prove that if M is a maximal subgroup of G, then $M \triangleleft G$.

(ii) Prove that G is nilpotent.

(6 marks)

(Total: 20 marks)

5. **(Mastery Question)** Let V be a finite-dimensional vector space over a field F.

(a) Define what is meant by a *non-degenerate alternating bilinear form* $\beta : V \times V \to F$.

(2 marks)

(b) Let $\beta : V \times V \to F$ be a non-degenerate alternating bilinear form. Prove that V has a basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ such that

$$\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \quad \beta(e_i, f_j) = \delta_{ij} \quad \text{for all } i, j$$

(where as usual, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise). (5 marks)

(c) Let β be as in (b), and recall that the *symplectic group* $Sp(V)$ is defined by

$$Sp(V) = \{g \in GL(V) \; : \; \beta(g(u), g(v)) = \beta(u, v) \text{ for all } u, v \in V\}.$$

(i) Show that if $\dim V = 2$, then $Sp(V) = SL(V)$.

(ii) Show that if $F = \mathbb{F}_q$, the finite field of q elements, and $\dim V = 2m$ as in (b), then

$$|Sp(V)| = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1).$$

(8 marks)

(d) Define $P(V)$ to be the set of 1-dimensional subspaces of V, and let $G = Sp(V)$ act on $P(V)$ in the natural way (i.e. $g \in G$ sends $\langle v \rangle \mapsto \langle g(v) \rangle$ for any $\langle v \rangle \in P(V)$). Prove the following facts about this action.

(i) $G = Sp(V)$ acts transitively on $P(V)$.

(ii) Assume that $\dim V \geq 4$. Then for $\langle v \rangle \in P(V)$, the stabilizer $G_{\langle v \rangle}$ has 2 orbits on the set $P(V) \setminus \{\langle v \rangle\}$.

(5 marks)

(Total: 20 marks)

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1. (i) True: N' char $N \triangleleft G \Rightarrow N' \triangleleft G$. **(2 marks)**

(ii) False: eg. $N = \langle (12)(34) \rangle V_4 = A_4' \triangleleft A_4$ but $N \not\triangleleft A_4$. **(3 marks)**

(iii) True: G/G' abelian $\Rightarrow N/G' \triangleleft G/G' \Rightarrow N \triangleleft G$. **(3 marks)**

(iv) True: for $\alpha \in \mathrm{Aut}(G)$, $\beta \in \mathrm{Aut}(H)$, define $\pi_{\alpha,\beta} \in \mathrm{Aut}(G \times H)$ to send $(g,h) \mapsto (g^\alpha, h^\beta)$ for $g \in G, h \in H$. Then the map $(\alpha,\beta) \mapsto \pi_{\alpha,\beta}$ is an injective homom from $\mathrm{Aut}(G) \times \mathrm{Aut}(H) \to \mathrm{Aut}(G \times H)$. **(3 marks)**

(v) False: eg. $G = C_2, H = C_2 \times C_2$: then $\mathrm{Aut}(G \times H) = \mathrm{Aut}(C_2^3) = GL_3(2)$ which is not isomorphic to $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$. **(3 marks)**

(vi) False: let $G = SL_2(5)$. If $G \cong C_2 \times PSL_2(5)$ then $G' \leq PSL_2(5)$, so $G' \neq G$; but by a result in lectures, $G' = G$, contradiction. **(3 marks)**

(vii) True: let $Z = \{\lambda I : \lambda \in \mathbb{F}_4^*\} \cong C_3$, and let $S = SL_2(4)$, $G = GL_2(4)$. Note that $|G| = 3|S| = |Z|\,|S|$. Also $Z \cap S = 1$, so $|ZS| = |Z|\,|S| = |G|$. Hence $G = ZS$. Also $Z \triangleleft G$ and $S \triangleleft G$. Now it follows that $G \cong Z \times S$. **(3 marks)**

Total $20 = 8A + 6B + 6C$

2. (a) The Sylow theorems:

Sylow I: Let $|G| = p^a m$, where p is prime and p does not divide m. Then G has a subgroup of order p^a.

Sylow II: If $n_p(G)$ denotes the number of Sylow p-subgroups of G, then $n_p(G) \equiv 1 \bmod p$.

Sylow III: Let Q be a p-subgroup of G. Then there exists $P \in Syl_p(G)$ such that $Q \leq P$.

Sylow IV: $Syl_p(G)$ is a single conjugacy class of subgroups of G; that is, for any $P, Q \in Syl_p(G)$, there exists $g \in G$ such that $Q = {}^gP$.

Burnside's transfer theorem: Let p be prime, $P \in Syl_p(G)$, and suppose that $P \leq Z(N_G(P))$. Then G has a normal p-complement (i.e. a normal subgroup N such that $G = PN$ and $P \cap N = 1$).

(5 marks, A)

(b) Let $|G| = 80 = 2^4 5$. Let n_5 be the number of Sylow 5-subgroups, so $n_5 \equiv 1 \bmod 5$ and n_5 divides 2^4. If G has a normal Sylow 5-subgroup, then $n_5 = 1$. If not, $n_5 = 16$, and so as $P \cap Q = 1$ for any two Sylow 5-subgroups P, Q, the number of elements of order 5 in G is $16 \times 4 = 64$. The remaining 16 elements of G must form a unique Sylow 2-subgroup, which is therfore normal in G. **(4 marks, similar seen, B)**

(c)(i) Let $|G| = 2^2 3^2 7^2$. Suppose G is simple. Then $n_7 > 1$ and is 1 mod 7 and divides 36, so $n_7 = 36$. So for $P \in Syl_7(G)$, $|G : N_G(P)| = 36 = |G : P|$. Hence $N_G(P) = P$. As $|P| = 7^2$, P is abelian, so $P = C_G(P) = N_G(P)$. By Burnside, G has a normal 7-complement, a contradiction as G is simple. **(3 marks, similar seen, B)**

(ii) Suppose G has no normal Sylow subgroups. As in (i) G has a normal 7-complement N. So $N \lhd G$ and $|N| = 36$. If $n_3(N) = 1$ then N has a unique, hence characteristic Sylow 3-subgroup P, and so $P \lhd G$, contradiction. Hence $n_3(N) = 4$. For $Q \in Syl_3(N)$ we then have $N_N(Q) = Q$ by the argument in (i), and so Burnside shows that N has a normal 3-complement R. Then $R \in Syl_2(N)$, so R char $N \lhd G$ and so $R \lhd G$, contradiction. **(4 marks, D)**

(d) Let $P \in Syl_2(G)$, so $P \cong C_{2^a}$ for some a. As P is abelian, $P \leq C_G(P)$. Hence $N_G(P)/C_G(P)$ has odd order. By a result in lectures, $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Now $\mathrm{Aut}(C_{2^a})$ has order $\phi(2^a) = 2^{a-1}$ (where ϕ is the Euler function). Hence $N_G(P)/C_G(P) = 1$. We now have $P = C_G(P) = N_G(P)$. So by Burnside, G has a normal 2-complement, as required. **(4 marks, C – special case of more general argument seen)**

3. (a) (i) False: $C_8 = \langle x \rangle$ has a unique element of order 2 (namely, x^4), and so any two subgroups C_4 and C_2 must both contain this element, so cannot intersect in 1. (**3 marks, A**)

(ii) True: let $D_8 = \langle x, y \rangle$, where $x^4 = y^2 = 1$ and $y^{-1}xy = x^{-1}$. Then $N = \langle x \rangle \triangleleft D_8$ and $D_8 = N\langle y \rangle$ with $N \cap \langle y \rangle = 1$. (**3 marks, A**)

(iii) False: Q_8 has a unique element of order 2, so the argument in part (i) applies. (**2 marks, A**)

(b) As N and H are soluble, so is G. By Hall's theorem, G has a Hall p'-subgroup H_0. Then $|G| = |N|\,|H| = |N|\,|H_0|$ and $N \cap H_0 = 1$. So $|NH_0| = |N|\,|H_0| = |G|$, and we have $G = NH_0$, $N \cap H_0 = 1$, showing the extension splits. (**6 marks, D**)

(c) Since P is a non-abelian p-group we have $1 < Z < P$. Suppose the extension splits, so P has a subgroup K such that $P = ZK$ and $Z \cap K = 1$. Since K is a nontrivial p-group, $Z_0 = Z(K) \neq 1$. Elements of Z_0 commute with K, and also commute with $Z = Z(P)$, and hence they commute with the whole of P. Hence $Z_0 \leq Z(P) = Z$, contradicting the fact that $Z_0 \cap Z \leq K \cap Z = 1$. Hence the extension does not split. (**Set as exercise, 6 marks, C**)

4. (a) There are quite a few correct definitions - here is the original one in lectures: G is nilpotent if there is a series of normal subgroups $1 = G_0 < G_1 < \cdots < G_r = G$ such that $G_{i+1}/G_i \leq Z(G/G_i)$ for all i. (**2 marks, A**)

The Frattini subgroup $\Phi(G)$ is the intersection of all the maximal subgroups of G. (**2 marks, A**)

(b) Let P be a finite p-group. Define a series of subgroups $1 = P_0 \leq P_1 \leq P_2 \leq \cdots$ by $P_0 = 1$, $P_1 = Z(P)$, $P_2/P_1 = Z(P/P_1)$, and inductively $P_{i+1}/P_i = Z(P/P_i)$. As these are all p-groups, and nontrivial p-groups have nontrivial centres, we have strict containments $P_0 < P_1 < P_2 < \cdots$, and the series terminates at $P_r = P$. Hence P is nilpotent. (**Seen, 4 marks, A**)

(c) (i) From the given relations we see that $a \in Z(P)$, so $A = \langle a \rangle \lhd P$. Also $P/A = \langle bA, cA, xA \rangle$ and these three generators commute, noting that $xcx^{-1}A = acA = cA$. So P/A is abelian. (**3 marks, B**)

(ii) The generators bA, cA, xA all have order p, so $P/A \cong C_p^3$. By lectures, $\Phi(P)$ is minimal subject to having elementary abelian quotient. Hence $\Phi(P) = A$, of order p. (**3 marks, C**)

(d) (i) Suppose $G/\Phi(G)$ abelian. Let M be a max subgroup of G. Then $\Phi(G) \leq M$ (by defn of $\Phi(G)$), and $M/\Phi(G)$ is normal in $G/\Phi(G)$ as this is abelian. Hence $M \lhd G$. **(4 marks, D)**

(ii) By (i), every max subgroup of G is normal, so G nilpotent by a standard result in lectures (Thm 7.5). **(2 marks, D)**

5. (a) Alternating: β bilinear and $\beta(v,v) = 0$ for all $v \in V$. Non-degenerate: $\beta(u,v) = 0$ for all $v \in V$ implies $u = 0$. (**2 marks**)

(b) Pick $e_1 \in V$ and f such that $\beta(e_1, f) = \lambda \neq 0$. Replacing f by $f_1 = \lambda^{-1} f$, we have $(e_1, f_1) = 1$.

Let $W = \langle e_1, f_1 \rangle$. If $v \in W \cap W^\perp$, then $v = ae_1 + bf_1$, and $\beta(v, e_1) = \beta(v, f_1) = 0$ implies $a = b = 0$. Hence $W \cap W^\perp = 0$, and so $V = W \oplus W^\perp$. Inductively, we can choose a symplectic basis e_i, f_i $(2 \leq i \leq m)$ of $W^\perp$, and adjoining $e_, f_1$ we get a symplectic basis e_i, f_i $(1 \leq i \leq m)$ of V. **(5 marks)**

(c) (i) Let $\dim V = 2$, so V has basis e_1, f_1 as in (b). The matrix of β wrt this basis is $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $A = (a_{ij}) \in Sp(V)$ iff $AJA^T = J$. The LHS works out as $\begin{pmatrix} 0 & |A| \\ -|A| & 0 \end{pmatrix}$, hence $A \in Sp(V)$ iff $|A| = 1$ iff $A \in SL(V)$. **(3 marks)**

(ii) The order of $Sp(V)$ is equal to the number of symplectic bases e_i, f_i of V (since for any two symplectic bases, there is a unique elemnt of $Sp(V)$ sending one to the other). To count these:

$$\text{no. of choices of } e_1 = q^{2m} - 1.$$

We then choose $f_1 \in V \setminus e_1^\perp$, normalized so that $\beta(e_1, f_1) = 1$. So

$$\text{no. of choices of } f_1 = (q^{2m} - q^{2m-1})/(q-1) = q^{2m-1}.$$

Given e_1, f_1, we then choose e_2, f_2 in the same way in the $2m-2$-dimensional space $\langle e_1, f_1 \rangle^\perp$. The number of choices for e_2 is $q^{2m-2} - 1$, and for f_2 is q^{2m-3}. Continuing (or inductively), we see that the total number of symplectic bases is

$$\prod_{i=1}^{m} (q^{2i} - 1) q^{2i-1} = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1).$$

(5 marks)

(d) Let

$$\Delta_1 = \{(v,w) : v, w \neq 0, \langle v \rangle \neq \langle w \rangle, \beta(v,w) = 0\}, \Delta_2 = \{(v,w) : \beta(v,w) = 1\}.$$

By Witt's theorem, $Sp(V)$ acts transitively on both Δ_1 and Δ_2.

For part (i), observe that for any $v \in V \setminus 0$, there exists w such that $(v,w) \in \Delta_2$, so by transitivity on Δ_2, G is transitive on $P(V)$.

For part (ii), it follows from transitivity on Δ_1, Δ_2 that the orbits of $G_{\langle v \rangle}$ are $\{\langle w \rangle : \beta(v,w) = 0\} \setminus \langle v \rangle$ and $\{\langle w \rangle : \beta(v,w) \neq 0\}$ – the $\dim V \geq 4$ assumption is needed to ensure that the first one is nonempty. **(5 marks)**

All the material for this solution can be found in the notes for the mastery material.

MATH70036 Group Theory Markers Comments

Question 1	Marks quite low for Q1. Parts (ii), (vi) and (vii) seem to have been particularly challenging.
Question 2	A lot of good attempts at Q2.
Question 3	Reasonable performance.
Question 4	A lot of good solutions to Q4.
Question 5	Very few good attempts at Q5.