Imperial College London

Department of Mathematics
Imperial College London
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Introduction to University Mathematics **MATH40001/MATH40009**

Part II – Problem Sheet 1

In your proofs of the following questions, you should use only Peano's axioms, and the other axioms and definitions from lectures, as well as results *proved* (not merely stated) in lectures. You may assume the results from the previous questions.

1. Let x, y be in $\mathbb{N}$. We call x a predecessor of y if $\nu(x) = y$.

   (a) ✦ Show that the number 0 does not have a predecessor in $\mathbb{N}$.

   (b) ✦✦✦ Show that every nonzero element in $\mathbb{N}$ has a unique predecessor in $\mathbb{N}$.

   (c) ✦ Define now the predecessor function $\pi : \mathbb{N} - \{0\} \to \mathbb{N}$, where $\pi(n)$ is the predecessor of n for $n \in \mathbb{N}$. Show that π is a bijection. What is its inverse function?

2. (a) ✦ Show that if $n \in \mathbb{N}$, then $n \neq \nu(n)$.

   (b) ✦✦ A set X is called Dedekind-infinite if there exists a bijection $X \to S$, where S is a proper subset of X (i.e. $S \neq X$). Show that $\mathbb{N}$ is Dedekind-infinite.

3. Show that, for all x, y in $\mathbb{N}$ (you can assume part c) to show a) and b)):

   (a) ✦ If $x + y = x$, then $y = 0$;

   (b) ✦✦ If $x + y = 0$, then $x = 0$ and $y = 0$;

   (c) ✦✦✦ $x + y = y + x$

   (d) ✦✦✦ $x \cdot y = y \cdot x$;

4. Show that, for all x, y in $\mathbb{N}$:

   (a) ✦ $1 \cdot x = x = x \cdot 1$;

   (b) ✦ $(x + y) \cdot z = x \cdot z + y \cdot z$.

   (c) ✦✦ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

5. ✦✦ The axiom of recursion says that there exists a unique function $R : \mathbb{N} \to \mathbb{N}$, such that $R(0) = n_0$ and $R(\nu(n)) = \nu(R(n))$ for a fixed natural number n_0. Show the uniqueness of R.

6. (a) ✦ Show that, for all x, y, z in $\mathbb{N}$, either $x \leq y$ or $y \leq x$.

   (b) Show that for all a, b in $\mathbb{N}$,

      i. ✦✦ $a \cdot b = 0$ implies $a = 0$ or $b = 0$.

      ii. ✦✦ $a \cdot b = a$ implies $a = 0$ or $b = 1$.

      iii. ✦✦ $a \cdot b = 1$ implies $a = b = 1$.

      iv. ✦✦✦ Show that divisibility is a partial order on $\mathbb{N}$. Recall this is the relation $x \mid y$, reading "x divides y". Is it a total order? Prove or disprove it.
      Hint: Use the previous parts i, ii and iii.

7. (a) ✦ Show that $8|n^2 - 1$ for any odd integer $n > 1$.
Hint: Prove first that $n^2 - 1$ is the product of two consecutive even natural numbers.

(b) ✦✦ Show that for all a, b in $\mathbb{N}$, if $a, b > 1$, then $ab > a$.

(c) ✦✦✦ Given a natural number $n > 1$, show that the smallest divisor d of n such that $d > 1$ is prime.

8. (a) ✦✦ Use the well-ordering principle to show that every amount of postage, that is more than one cent, can be formed using 2 cent and 3 cent stamps.

(b) ✦✦✦ Let $X \subseteq \mathbb{N}$ be a nonempty subset with the following properties: (1) $0 \notin X$; (2) if $a, b \in X$, then $a + b \in X$; and (3) if $a, b \in X$ and $a < b$, then $b - a \in X$. Prove that there exists a unique $d \in \mathbb{N}$ such that $X = \{d, 2d, 3d, \ldots\}$.