Problem Sheet 1

Problem 1. Let $\phi : [-\epsilon, +\epsilon] \to \mathbb{R}^3$ be a regular curve, with tangent vector T and principle normal vector N. Let

$$\psi : [-\epsilon, +\epsilon] \to \{aT(0) + bN(0) \mid a, b \in \mathbb{R}\}$$

be the orthogonal projection of ϕ onto the plane spanned by $T(0)$ and $N(0)$. Prove that ϕ and ψ have the same curvature at time $t = 0$.

Problem 2. Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}^3$ be regular curves parametrized by arc length. Suppose that their curvatures k_1, k_2 and torsions τ_1, τ_2 are positive everywhere, and that their binormal vectors are identical, $B_1(t) = B_2(t)$ for all $t \in [a, b]$. Prove that there is a constant vector $\vec{v} \in \mathbb{R}^3$ such that $\phi_2(t) = \phi_1(t) + \vec{v}$.

Problem 3. For each $n \in \mathbb{Z}$, draw (construct, explain) a closed regular plane curve ϕ with $\mathrm{Ind}(\phi) = n$.

Problem 4. Let $\phi : [a, b] \to \mathbb{R}^2$ be a regular curve which is parametrized by arc length, and let $v \in \mathbb{R}^2$. Consider the function $f_v : [a, b] \to \mathbb{R}$ defined as

$$f_v(t) = |\phi(t) - v|^2.$$

a) Show that there is $t_0 \in (a, b)$ satisfying $f_v'(t_0) = 0$ if and only if the circle C of radius $\sqrt{f_v(t_0)}$ centred at v is tangent to ϕ at $\phi(t_0)$.

b) Assume that the curvature $k(t_0) \neq 0$ for some $t_0 \in (a, b)$. Determine, in terms of $k(t_0)$, the unique value of R such that there is $v \in \mathbb{R}^2$ satisfying $f_v(t_0) = R^2$, $f_v'(t_0) = 0$ and $f_v''(t_0) = 0$.

Remark: The above problem characterises $|k(t)|$ in terms of the radius of the circle which “best” approximates ϕ at $\phi(t)$ (that is, it is a tangent of order 2 to the curve).

Problem 5. Let $\phi : [-\epsilon, +\epsilon] \to \mathbb{R}^3$ be a regular curve parametrized by arc length. Assume that $\phi(0) = (0, 0, 0)$ and the Frenet frame at time $t = 0$ is

$$T(0) = (1, 0, 0), \quad N(0) = (0, 1, 0), \quad B(0) = (0, 0, 1).$$

Writing $\phi(t) = (x(t), y(t), z(t))$ and assuming that $k(0) \neq 0$, determine the leading nonzero terms of the Taylor series for each of the coordinates x, y, z at $t = 0$ in terms of the curvature $k_0 = k(0)$ and the torsion $\tau_0 = \tau(0)$.