ExamModuleCode	Question Number	Comments for Students
M3A4	1	This question was fairly straightforward for most, as intended, and there were quite a few full marks
M3A4	2	Most people were fine with parts a and b (of course similar questions had been seen). I was very picky in part a if calculations were missing. In part c) I often saw answers to a different question (that was similar to questions from previous years)! Only a few got part c right. A few people tried to use classical probability arguments in part c, that failed.
M3A4	3	Parts a and b were straight-forward, in part c some people tried to use H|phi_0>=hbsr*omegs/2|phi_0> without success. In part d (i) many didn't think about using the normalisation condition.
M3A4	4	This question was perceived as quite a bit harder than it was intended. Most of this question had been seen in class... Perhaps this was partly due to lack of time? I was very generous when marking this question.
M45A4	5	The mastery question seems to have been perceived as much harder than intended. I can only guess that this is a combination of a lack of time and a psychological effect, that made the material more intimidating, since most of the actual content was based on calculations involving 2x2 matrices and vectors with 2 complex components, which should not have been too hard...

Imperial College
London

M3A4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Physics 1: Quantum Mechanics

Date: Tuesday 21 May 2019
Time: 10.00 - 12.00
Time Allowed: 2 Hours
This paper has 4 Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College London

M4/5A4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Physics 1: Quantum Mechanics

Date: Tuesday 21 May 2019
Time: 10.00 - 12.30
Time Allowed: 2 Hours 30 Minutes
This paper has 5 Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
All required additional material will be provided.

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1. **A quantum wave function in a harmonic potential well**

Consider a quantum particle in a harmonic potential described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2,$$

where ω is a real valued frequency. Assume that the particle is in a state with the wave function

$$\phi(x) = N\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2},$$

where N is a real positive constant.

(a) Calculate the value of N for which $\phi(x)$ is normalised to one.

(b) Calculate the expectation value of the position of the particle.

(c) Calculate the expectation value of the energy.

(d) What would the allowed region for a classical particle of the same energy be?

(e) Calculate the probability to find the quantum particle in the classically allowed region, and the probability to find it in the forbidden region to the right of the classically allowed region.

Hint: The following integrals might be useful:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \qquad \int_{-\infty}^{\infty} x^2 \mathrm{e}^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}},$$

for $\mathrm{Re}(a) > 0$, and

$$\frac{1}{\sqrt{\pi}}\int_{-z}^{z} \mathrm{e}^{-x^2} dx = \mathrm{erf}(z), \quad \text{and} \quad \mathrm{erf}(1) \approx 0.8427.$$

2. **The principles of quantum mechanics.**

Consider a system on the Hilbert space $\mathbb{C}^3$ and a Hamiltonian $\hat{H}$ represented by the matrix

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & E & 0 \\ E & 0 & E \\ 0 & E & 0 \end{pmatrix},$$

with $E \in \mathbb{R}$. Let another observable $\hat{A}$ be described by the matrix

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $a \in \mathbb{R}$.

(a) Calculate the eigenvalues and a set of normalised eigenvectors of $\hat{H}$.

(b) Assume that at some specific time the system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix}.$$

Give the probabilities for each possible outcome from a measurement of the observable A. Calculate the expectation value of $\hat{A}$.

(c) Assume that an energy measurement at time zero yields the outcome E, and a measurement of the observable A performed immediately after this yields the result a. If we measure the energy immediately after this measurement of A again, with which probability do we obtain which outcome?

3. **Harmonic oscillator and coherent states.**

Consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right), \tag{1}$$

with the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, fulfilling $[\hat{a}, \hat{a}^\dagger] = 1$, and the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. The eigenvectors $|\phi_n\rangle$ of $\hat{H}$ form a complete orthonormal basis. The action of the ladder operators on the basis states is given by

$$\hat{a}|\phi_n\rangle = \sqrt{n}|\phi_{n-1}\rangle, \quad \hat{a}^\dagger|\phi_n\rangle = \sqrt{n+1}|\phi_{n+1}\rangle. \tag{2}$$

(a) Verify that the $|\phi_n\rangle$ are indeed eigenvectors of $\hat{H}$ and deduce the corresponding eigenvalues.

(b) Verify that if $|\phi_n\rangle$ is normalised to one, so is $|\phi_{n+1}\rangle$.

(c) Remembering that $\hat{a}$ in terms of position and momentum reads

$$\hat{a} := \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{q} + \mathrm{i}\sqrt{\frac{1}{m\omega\hbar}}\hat{p}\right),$$

find the position representation of the ground state of the harmonic oscillator (up to a normalisation constant).

(d) The so-called coherent states $|z\rangle$ are eigenstates of the (non-Hermitian) lowering operator $\hat{a}$.

(i) Use the eigenvalue equation

$$\hat{a}|z\rangle = \lambda_z|z\rangle,$$

to show that in the basis $|\phi_n\rangle$ the normalised coherent states have the form

$$|z\rangle = \mathrm{e}^{-\frac{|\lambda_z|^2}{2}}\sum_{n=0}^{\infty}\frac{\lambda_z^n}{\sqrt{n!}}|\phi_n\rangle.$$

(ii) Use the method of stationary states to show that under time-evolution with the harmonic oscillator Hamiltonian (1) an initially coherent state is mapped into another coherent state with time-dependent $\lambda_z(t)$, which you should give explicitly.

4. **Free Gaussian wave packet and uncertainty relations.**

Consider the time evolution of a Gaussian wave packet of the form

$$\psi(x,t) = \exp\left(-\alpha_t(x-x_t)^2 + \frac{\mathrm{i}}{\hbar}p_t(x-x_t) + \frac{\mathrm{i}}{\hbar}\gamma_t\right), \tag{3}$$

with the (possibly time-dependent) parameters $x_t, p_t \in \mathbb{R}$ and $\alpha_t, \gamma_t \in \mathbb{C}$.

(a) Calculate the expectation values of the position $\langle\hat{q}\rangle$ and the momentum $\langle\hat{p}\rangle$.

(b) Prove that the product of the uncertainties of two observables $\hat{A}$ and $\hat{B}$ is bounded from below according to the uncertainty principle

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|, \tag{4}$$

by considering the norm of the state

$$|\chi\rangle = (\hat{A} - \langle\hat{A}\rangle - \mathrm{i}\lambda(\hat{B} - \langle\hat{B}\rangle))|\psi\rangle$$

for real $\lambda \in \mathbb{R}$ and an arbitrary normalised state $|\psi\rangle$.

(c) Use the position representation of the position and momentum operators to deduce the uncertainty relation for $\hat{p}$ and $\hat{q}$.

(d) Use the fact that for a minimum uncertainty state $|\psi\rangle$ the state $|\chi\rangle$ in part (a) is the zero state to deduce under which condition (3) is a minimum uncertainty wave packet in position and momentum.

(e) If the wave packet (3) has minimum uncertainty at time zero and evolves under a Hamiltonian with vanishing potential, $\hat{H} = \frac{\hat{p}^2}{2m}$, does the uncertainty stay minimal over time? Give a reason for your answer.

5. **Mastery Question - Biorthogonal quantum mechanics.**

Consider the matrix operator

$$\hat{K} = \begin{pmatrix} -\mathrm{i}\gamma & 1 \\ 1 & \mathrm{i}\gamma \end{pmatrix}, \tag{5}$$

with real parameter $\gamma \in \mathbb{R}$.

(a) Calculate the eigenvalues of $\hat{K}$, for which parameters are the eigenvalues real?

(b) In the parameter region in which the eigenvalues are real, calculate the eigevectors of $\hat{K}$ and the eigenvectors of $\hat{K}^\dagger$. Show that they form a biorthogonal set, and normalise them such that their bi-orthogonal norm is one.

(c) Assume that a quantum system is described by the state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

at time zero.

(i) Expand the state $|\psi(0)\rangle$ in the eigenbasis of $\hat{K}$.

(ii) Assume the system evolves under a time-evolution generated by the Hamiltonian $\hat{K}$. Calculate the state at time t.

(iii) Verify that the bi-orthogonal norm of the time-dependent state remains constant if the eigenvalues of $\hat{K}$ are real. How does the conventional norm behave as a function of time?

Quantum Mechanics 2018/19 Exam Solutions

1. A quantum wave function in a harmonic potential well

(Parts (a) seen, part (b) and (c) seen similar, part (d) and (e) unseen but straightforward standard material)

(a) The norm of $\phi(x)$ in dependence on N is given by

$$\int_{-\infty}^{+\infty} |\phi(x)|^2 \mathrm{d}x = N^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{m\omega}{\hbar}x^2} \mathrm{d}x = N^2 \sqrt{\frac{\pi\hbar}{m\omega}},$$

thus, for the wave function to be normalised to one we have

$$N = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}.$$

(3 points)

(b) The expectation value of the position is given by

$$\langle \hat{x} \rangle = \int_{-\infty}^{+\infty} x|\phi(x)|^2 \mathrm{d}x = N^2 \int_{-\infty}^{+\infty} x\mathrm{e}^{-\frac{m\omega}{\hbar}x^2} \mathrm{d}x = 0,$$

since the integrand is an odd function. One could also argue that $\langle \hat{x} \rangle = 0$ due to the symmetry of the problem.

(3 points)

(c) The expectation value of the energy is given by

$$\langle \hat{H} \rangle = \int_{-\infty}^{+\infty} \phi^*(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right) \phi(x) \mathrm{d}x.$$

We first calculate

$$\frac{\partial^2}{\partial x^2}\phi(x) = \left(\frac{m^2\omega^2}{\hbar^2}x^2 - \frac{m\omega}{\hbar}\right)\phi(x),$$

and thus

$$\begin{aligned}
\langle \hat{H} \rangle &= N^2 \int_{-\infty}^{+\infty} \left(\frac{\hbar\omega}{2} - \frac{m\omega^2}{2}x^2 + \frac{m\omega^2}{2}x^2\right) \mathrm{e}^{-\frac{m\omega}{\hbar}x^2} \mathrm{d}x \\
&= \frac{\hbar\omega}{2} N^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{m\omega}{\hbar}x^2} \mathrm{d}x \\
&= \frac{\hbar\omega}{2}.
\end{aligned}$$

(5 points)

(d) The allowed region for a classical particle is given by the region where the total energy is larger than the potential energy. That is, it is bounded at the points where the total energy equals the potential energy (i.e., the turning points, where the momentum is zero). At the turning points it holds

$$\frac{1}{2}m\omega^2 x^2 = E = \frac{\hbar\omega}{2},$$

and thus the classically allowed region is the interval $[x_-, x_+]$, with

$$x_\pm = \pm\sqrt{\frac{\hbar}{m\omega}}$$

(3 points)

(e) The probability to find the particle in the classically allowed region is given by

$$\begin{aligned} P_{\text{allowed}} &= \int_{x_-}^{x_+} |\phi(x)|^2 \mathrm{d}x \\ &= N^2 \int_{-\sqrt{\frac{\hbar}{m\omega}}}^{+\sqrt{\frac{\hbar}{m\omega}}} \mathrm{e}^{-\frac{m\omega}{\hbar}x^2} \mathrm{d}x \\ &= N^2 \sqrt{\frac{\hbar}{m\omega}} \int_{-1}^{+1} \mathrm{e}^{-u^2} \mathrm{d}u, \end{aligned}$$

where we have made the substitution $u = \sqrt{\frac{m\omega}{\hbar}}x$. Thus we find

$$P_{\text{allowed}} = erf(1) \approx 0.8427.$$

(4 points)

The probability to find the particle to the right of the allowed region is half of the probability to find it outside the allowed region, due to the symmetry, that is,

$$P_{\text{right}} = \frac{1}{2}(1 - P_{\text{allowed}}) \approx 0.0787.$$

(2 points)

2. The principles of quantum mechanics.

(Part (a) same as a question in a previous exam, part (b): seen similar, but only with non-degenerate eigenvalues, part (c) unseen.)

(a) From the characteristic polynomial of $\hat{H}$ we find for the eigenvalues λ of $\hat{H}$

$$-\lambda^3 + E^2\lambda = 0,$$

and thus $\lambda_0 = 0$ and $\lambda_\pm = \pm E$. For the components of the eigenvector ϕ_0 we find from $\hat{H}|\phi_0\rangle = 0$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & E & 0 \\ E & 0 & E \\ 0 & E & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is,

$$y_0 = 0, \text{ and } x_0 = -z_0.$$

Together with the normalisation condition that yields

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

up to an arbitrary phase factor $e^{i\varphi}$. Similarly we find the remaining eigenvectors

$$\phi_+ = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \text{and} \quad \phi_- = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

(5 points - for results *and* calculations)

(b) The probability to measure an eigenvalue of an observable is given by the modulus square of the coefficient of the wave function in the basis of the eigenvectors of the observable. Since the eigenvectors of $\hat{A}$ are just the basis vectors, we can directly read off the probabilities for the different eigenvalues from the wave function as

$$P(a) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad P(0) = \frac{1}{3}.$$

(4 points)

The expectation value can be either deduced from $\langle\psi|\hat{A}|\psi\rangle$ via vector and matrix multiplications, or we calculate

$$\langle\hat{A}\rangle = \sum_j P(\lambda_j)\lambda_j = aP(a) = \frac{2}{3}a.$$

(3 points)

(c) A measurement of the value E at time $t = 0$ projects the system onto the corresponding eigenstate of $\hat{H}$, that is, before the measurement of $\hat{A}$ we have

$$|\psi\rangle = |\phi_+\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

(2 points)

Now performing the measurement of $\hat{A}$ with the outcome a projects the state onto the subspace belonging to the degenerate eigenvalue a, that is the space spanned by the first two basis vectors, and, thus, taking the normalisation into account, we have

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix}. \tag{1}$$

(3 points)

The probabilities to measure the different outcomes in an energy measurement are now given by the overlap with the three eigenvectors of $\hat{H}$, that is,

$$P(E) = \frac{3}{4}, \quad , P(-E) = \frac{1}{12}, \quad P(0) = \frac{1}{6}. \tag{2}$$

(3 points - for results *and* calculations)

3. Harmonic oscillator and coherent states.

(Parts (a), (b), (c) seen, part (d) (i) similar to coursework question from a previous year, which was available as extra material, part (d)(ii) unseen.)

(a) Applying $\hat{H}$ to the basis vector $|\phi_n\rangle$ we find

$$\begin{aligned}\hat{H}|\phi_n\rangle &= \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|\phi_n\rangle\\ &= \hbar\omega\left(\hat{a}^\dagger\sqrt{n}|\phi_{n-1}\rangle+\frac{1}{2}|\phi_n\rangle\right)\\ &= \hbar\omega\left(n|\phi_n\rangle+\frac{1}{2}|\phi_n\rangle\right)\\ &= \hbar\omega\left(n+\frac{1}{2}\right)|\phi_n\rangle,\end{aligned}$$

that is, the $|\phi_n\rangle$ are eigenvectors of $\hat{H}$ corresponding to the eigenvalues $\hbar\omega(n+\frac{1}{2})$.

(3 points)

(b) We have by definition

$$|\phi_{n+1}\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^\dagger|\phi_n\rangle,$$

and thus

$$\langle\phi_{n+1}| = \frac{1}{\sqrt{n+1}}\langle\phi_n|\hat{a}.$$

That is

$$\langle\phi_{n+1}|\phi_{n+1}\rangle = \frac{1}{n+1}\langle\phi_n|\hat{a}\hat{a}^\dagger|\phi_n\rangle.$$

Using the commutation relation $[\hat{a},\hat{a}^\dagger]=1$ and the fact that $\hat{N}|\phi_n\rangle = n|\phi_n\rangle$ we thus have

$$\langle\phi_{n+1}|\phi_{n+1}\rangle = \langle\phi_n|\phi_n\rangle.$$

That is, if $|\phi_n\rangle$ is normalised to one, so is $|\phi_{n+1}\rangle$.

(4 points)

(c) We recall the expressions for $\hat{q}$ and $\hat{p}$ in position representation,

$$\langle x|\hat{q}|\phi\rangle = x\phi(x), \quad \text{and} \quad \langle x|\hat{p}|\phi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial x}\phi(x),$$

to find

$$\langle x|\hat{a}|\phi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x+\sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right)\phi(x).$$

Thus, the ground-state wave function fulfils the condition

$$\frac{\partial}{\partial x}\phi_0(x) = -\frac{m\omega}{\hbar}x\phi_0(x).$$

Thus, we find

$$\phi_0(x) = \phi_0(x) = c\,\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}, \qquad c\in\mathbb{C}.$$

(4 points)

(d) (i) We have from the definition of $\hat{a}$ that

$$\begin{aligned}\hat{a}|z\rangle &= \sum_{n=0}^{\infty} z_n \hat{a}|\phi_n\rangle \\ &= \sum_{n=1}^{\infty} z_n \sqrt{n}|\phi_{n-1}\rangle \\ &= \sum_{m=0}^{\infty} z_{m+1}\sqrt{m+1}|\phi_m\rangle\end{aligned}$$

If $|z\rangle$ is an eigenvector of $\hat{a}$ that has to be equal to $\lambda_z|z\rangle$, that is

$$\sum_{m=0}^{\infty} z_{m+1}\sqrt{m+1}|\phi_m\rangle = \lambda_z \sum_{n=0}^{\infty} z_n|\phi_n\rangle$$

Comparing the coefficients yields

$$z_{n+1} = \frac{\lambda_z}{\sqrt{n+1}} z_n,$$

that is

$$z_n = \frac{\lambda_z^n}{\sqrt{n!}} z_0,$$

For the state $|z\rangle$ to be normalised it has to hold

$$\sum_n |z_n|^2 = 1.$$

Inserting the expression for z_n obtained above yields

$$\sum_n \frac{|\lambda_z|^{2n}}{n!}|z_0|^2 = \mathrm{e}^{|\lambda_z|^2}|z_0|^2 = 1,$$

and thus

$$|z_0|^2 = \mathrm{e}^{-|\lambda_z|^2}.$$

Choosing the arbitrary phase $z_0 = \mathrm{e}^{-\frac{|\lambda_z|^2}{2}}$ that yields

$$z_n = \mathrm{e}^{-\frac{|\lambda_z|^2}{2}} \frac{\lambda_z^n}{\sqrt{n!}}.$$

(5 points)

(ii) According to the method of stationary states we have

$$|z(t)\rangle = \mathrm{e}^{-\frac{|\lambda_z(0)|^2}{2}} \sum_{n=0}^{\infty} \frac{\lambda_z(0)^n}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}E_n t/\hbar}|\phi_n\rangle.$$

using that $E_n = \hbar\omega(n+\frac{1}{2})$ we thus find

$$|z(t)\rangle = \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \mathrm{e}^{-\frac{|\lambda_z(0)|^2}{2}} \sum_{n=0}^{\infty} \frac{(\lambda_z(0)\mathrm{e}^{-\mathrm{i}\omega t})^n}{\sqrt{n!}}|\phi_n\rangle.$$

That is, up to a phase this is a coherent state with

$$\lambda_z(t) = \lambda_z(0)\mathrm{e}^{-\mathrm{i}\omega t}.$$

(4 points)

4. Free Gaussian wave packet and uncertainty relations.

(Parts (a),(b), and (c) seen (in lecture and in lecture notes), part (d) seen in lecture, part (e) unseen)

(a) Straightforward calculations yield

$$\langle \hat{q} \rangle = x_t$$
$$\langle \hat{p} \rangle = p_t.$$

(4 points)

(b) For simplicity we consider the shifted operators

$$\hat{a} := \hat{A} - \langle \hat{A} \rangle, \quad \text{and} \quad \hat{b} := \hat{B} - \langle \hat{B} \rangle, \tag{3}$$

with

$$[\hat{a}, \hat{b}] = [\hat{A}, \hat{B}] =: -\mathrm{i}\hat{C}. \tag{4}$$

We have

$$(\Delta A)^2 = \langle \hat{a}^2 \rangle \quad \text{and} \quad (\Delta B)^2 = \langle \hat{b}^2 \rangle. \tag{5}$$

Let us consider the norm of the state $|\chi\rangle = (\hat{a} - \mathrm{i}\lambda\hat{b})|\psi\rangle$ for $\lambda \in \mathbb{R}$ and an arbitrary normalised state $|\psi\rangle$. We have

$$\begin{aligned}
\langle \chi|\chi \rangle &= \langle \psi|(\hat{a} + \mathrm{i}\lambda\hat{b})(\hat{a} - \mathrm{i}\lambda\hat{b})|\psi\rangle \\
&= \langle (\hat{a} + \mathrm{i}\lambda\hat{b})(\hat{a} - \mathrm{i}\lambda\hat{b}) \rangle \\
&= \langle \hat{a}^2 + \lambda^2\hat{b}^2 + \mathrm{i}\lambda(\hat{b}\hat{a} - \hat{a}\hat{b}) \rangle \\
&= \langle \hat{a}^2 \rangle + \lambda^2 \langle \hat{b}^2 \rangle + \mathrm{i}\lambda \langle \mathrm{i}\hat{C} \rangle \\
&= \langle \hat{a}^2 \rangle + \lambda^2 \langle \hat{b}^2 \rangle - \lambda \langle \hat{C} \rangle
\end{aligned} \tag{6}$$

On the other hand, we know that $\langle \chi|\chi \rangle \geq 0$. That is, we have

$$\langle \hat{a}^2 \rangle + \lambda^2 \langle \hat{b}^2 \rangle - \lambda \langle \hat{C} \rangle \geq 0, \tag{7}$$

for all values of $\lambda \in \mathbb{R}$. The location of the minimum in dependence in λ is found where

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \langle \chi|\chi \rangle = 2\langle \hat{b}^2 \rangle \lambda - \langle \hat{C} \rangle = 0, \tag{8}$$

that is

$$\lambda = \frac{\langle \hat{C} \rangle}{2\langle \hat{b}^2 \rangle}. \tag{9}$$

Reinserting this into the condition $\langle \chi|\chi \rangle_{min} \geq 0$ yields

$$\begin{aligned}
\langle \chi|\chi \rangle_{min} &= \langle \hat{a}^2 \rangle + \frac{\langle \hat{C} \rangle^2}{4\langle \hat{b}^2 \rangle^2} \langle \hat{b}^2 \rangle - \frac{\langle \hat{C} \rangle^2}{2\langle \hat{b}^2 \rangle} \\
&= \langle \hat{a}^2 \rangle - \frac{\langle \hat{C} \rangle^2}{4\langle \hat{b}^2 \rangle} \geq 0,
\end{aligned} \tag{10}$$

that is

$$(\Delta A)^2 \geq \frac{\langle \hat{C} \rangle^2}{4\langle \hat{b}^2 \rangle}, \tag{11}$$

or

$$(\Delta A)(\Delta B) \geq \frac{|\langle \hat{C} \rangle|}{2}. \qquad \square$$

(6 points)

(c)
$$\langle x|\hat{q}|\phi\rangle = x\phi(x), \quad \text{and} \quad \langle x|\hat{p}|\phi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial x}\phi(x),$$

we find when applying $[\hat{q},\hat{p}]$ to a test function $\phi(x)$

$$\langle x|\hat{q}\hat{p} - \hat{p}\hat{q}|\phi\rangle = -\mathrm{i}\hbar x\frac{\partial\phi(x)}{\partial x} + \mathrm{i}\hbar\frac{\partial x\phi(x)}{\partial x} = \mathrm{i}\hbar\phi(x).$$

That is, we find

$$[\hat{q},\hat{p}] = \mathrm{i}\hbar\hat{I}.$$

(3 points)

(d) For the state $|\psi\rangle$ to saturate the lower bound of the uncertainty relation it needs to hold

$$(\hat{A} - \langle\hat{A}\rangle - \mathrm{i}\lambda(\hat{B} - \langle\hat{B}\rangle))|\psi\rangle = 0,$$

for some $\lambda \in \mathbb{R}$, that is

$$\hat{A}|\psi\rangle = (\langle\hat{A}\rangle + \mathrm{i}\lambda(\hat{B} - \langle\hat{B}\rangle))|\psi\rangle.$$

In particular, for $\hat{p}$ and $\hat{q}$ in position representation that means

$$-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = (\langle\hat{p}\rangle - \mathrm{i}\lambda\langle\hat{q}\rangle)\psi(x) + \mathrm{i}\lambda x\psi(x).$$

Separating variables and integrating that is

$$\begin{aligned} -\mathrm{i}\hbar\int\frac{\mathrm{d}\psi(x)}{\psi(x)} &= \int(\langle\hat{p}\rangle - \mathrm{i}\lambda\langle\hat{q}\rangle + \mathrm{i}\lambda x)\mathrm{d}x \\ -\mathrm{i}\hbar\ln(\psi(x)) &= (\langle\hat{p}\rangle - \mathrm{i}\lambda\langle\hat{q}\rangle)x + \mathrm{i}\frac{\lambda}{2}x^2. \end{aligned}$$

That is

$$\psi(x) = \exp\left(-\frac{\lambda}{2\hbar}x^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle x + \frac{\lambda}{\hbar}\langle\hat{q}\rangle x\right)$$

Rewriting this slightly leads to

$$\psi(x) = \exp\left(-\frac{\lambda}{2\hbar}(x - \langle\hat{q}\rangle)^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(x - \langle\hat{q}\rangle) + \frac{\lambda}{2\hbar}\langle\hat{q}\rangle^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle\langle\hat{q}\rangle\right),$$

up to a phase, for some $\lambda \in \mathbb{R}$, with the additional condition of $\lambda > 0$ to guarantee that $|\psi\rangle$ is normalisable.

Comparing this to the Gaussian wave packet in the question we see that this is a minimum uncertainty wave packet if $\alpha \in \mathbb{R}$.

(5 points)

(e) According to Ehrenfest's theorem the quantum free particle moves in the same way as the classical one. If there is an initial uncertainty in the momentum this leads to a spread in position over time, i.e. an increase in position uncertainty. That is, even for an initial minimum uncertainty wave packet the uncertainty does not stay minimal over time.

(2 points)

5. Mastery question - Biorthogonal quantum mechanics.

This question is based on reading material arxiv:1308.2609. The students were instructed to read Sections 1, 2, 3, 4, and 8.

(a) From the characteristic polynomial

$$(-\mathrm{i}\gamma - \lambda)(\mathrm{i}\gamma - \lambda) - 1 = 0$$

we find the eigenvalues

$$\lambda_\pm = \pm\sqrt{1-\gamma^2},$$

which are real for $|\gamma| \leq 1$.

(2 points)

(b) We find the components of the eigenvectors from $\hat{K}|\phi_\pm\rangle = \lambda_\pm|\phi_\pm\rangle$,

$$\begin{pmatrix} -\mathrm{i}\gamma & 1 \\ 1 & \mathrm{i}\gamma \end{pmatrix}\begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} = \begin{pmatrix} \lambda_\pm x_\pm \\ \lambda_\pm y_\pm \end{pmatrix},$$

as

$$|\phi_\pm\rangle = \begin{pmatrix} 1 \\ \mathrm{i}\gamma \pm \sqrt{1-\gamma^2} \end{pmatrix},$$

up to a normalisation factor.

(2 points)

Using that the eigenvalues of $\hat{K}^\dagger$ are the complex conjugates of those of $\hat{K}$, and assuming we are in the region where $|\gamma| \leq 1$ we find the eigenvectors of $\hat{K}^\dagger$ from $\hat{K}^\dagger|\chi_\pm\rangle = \lambda_\pm|\chi_\pm\rangle$,

$$\begin{pmatrix} \mathrm{i}\gamma & 1 \\ 1 & -\mathrm{i}\gamma \end{pmatrix}\begin{pmatrix} u_\pm \\ v_\pm \end{pmatrix} = \begin{pmatrix} \lambda_\pm u_\pm \\ \lambda_\pm v_\pm \end{pmatrix},$$

as

$$|\chi_\pm\rangle = \begin{pmatrix} 1 \\ -\mathrm{i}\gamma \pm \sqrt{1-\gamma^2} \end{pmatrix},$$

again, up to a normalisation factor.

(2 points)

To verify that they indeed form a bi-orthogonal set we calculate

$$\langle\chi_+|\phi_-\rangle = 1 + (\mathrm{i}\gamma + \sqrt{1-\gamma^2})(\mathrm{i}\gamma - \sqrt{1-\gamma^2}) = 1 - 1 = 0$$

and

$$\langle\chi_-|\phi_+\rangle = 1 + (\mathrm{i}\gamma - \sqrt{1-\gamma^2})(\mathrm{i}\gamma + \sqrt{1-\gamma^2}) = 1 - 1 = 0$$

(2 points)

For these states to be bi-orthogonally normalised (i.e. to fulfil $\langle\chi_\pm|\phi_\pm\rangle = 1$) we replace

$$|\phi_\pm\rangle \to n_\pm|\phi_\pm\rangle,$$

and

$$|\chi_\pm\rangle \to n_\mp|\chi_\pm\rangle,$$

with

$$n_\pm = \frac{1}{\sqrt{\langle\chi_\pm|\phi_\pm\rangle}}$$
$$= \frac{1}{\sqrt{2(1-\gamma^2 \pm \mathrm{i}\gamma\sqrt{1-\gamma^2})}}$$
$$= \sqrt{\frac{1}{2}\left(1 \mp \mathrm{i}\frac{\gamma}{\sqrt{1-\gamma^2}}\right)}$$

(2 points)

(c) (i) Using that the $|\phi\pm\rangle$ are normalised with respect to the bi-orthogonal norm, i.e. $\langle\chi\pm|\phi\pm\rangle = 1$ we have

$$|\psi(0)\rangle = \langle\chi_+|\psi(0)\rangle|\phi_+\rangle + \langle\chi_-|\psi(0)\rangle|\phi_-\rangle.$$

Calculating the coefficients explicitly thus yields

$$|\psi(0)\rangle = n_-^*|\phi_+\rangle + n_+^*|\phi_-\rangle.$$

(3 points)

(ii) The time evolved state is simply given by the application of $\mathrm{e}^{-\mathrm{i}\hat{K}t/\hbar}$ onto $|\psi(0)\rangle$, that is

$$|\psi(t)\rangle = n_-^*\mathrm{e}^{-\mathrm{i}\lambda_+ t/\hbar}|\phi_+\rangle + n_+^*\mathrm{e}^{-\mathrm{i}\lambda_- t/\hbar}|\phi_-\rangle$$
$$= n_-^*\mathrm{e}^{-\mathrm{i}\sqrt{1-\gamma^2}\,t/\hbar}|\phi_+\rangle + n_+^*\mathrm{e}^{\mathrm{i}\sqrt{1-\gamma^2}\,t/\hbar}|\phi_-\rangle.$$

(2 points)

(iii) The bi-orthogonal norm is given by $\langle\tilde{\psi}(t)|\psi(t)\rangle$ with

$$\langle\tilde{\psi}(t)| = n_-^*\mathrm{e}^{-\mathrm{i}\lambda_+ t/\hbar}\langle\chi_+| + n_+^*\mathrm{e}^{-\mathrm{i}\lambda_- t/\hbar}\langle\chi_-|.$$

That is,

$$\langle\tilde{\psi}(t)|\psi(t)\rangle = |n_+|^2\langle\chi_+|\phi_+\rangle + |n_-|^2\langle\chi_-|\phi_-\rangle$$
$$= |n_+|^2 + |n_-|^2$$
$$= \frac{1}{2}\left(1 + \mathrm{i}\frac{\gamma}{\sqrt{1-\gamma^2}}\right) + \frac{1}{2}\left(1 - \mathrm{i}\frac{\gamma}{\sqrt{1-\gamma^2}}\right) = 1.$$

(3 points)

The conventional norm on the other hand fulfils

$$\langle\psi(t)|\psi(t)\rangle = |n_+|^2\langle\phi_+|\phi_+\rangle + |n_-|^2\langle\phi_-|\phi_-\rangle + 2\mathrm{Re}\left(n_+^* n_- \mathrm{e}^{-\mathrm{i}(\lambda_- - \lambda_+)t/\hbar}\langle\phi_-|\phi_+\rangle\right)$$

That is, it oscillates in time around the initial value $\langle\psi(0)|\psi(0)\rangle = |n_+|^2\langle\phi_+|\phi_+\rangle + |n_-|^2\langle\phi_-|\phi_-\rangle$.

(2 points)