Math40003 Linear Algebra and Groups **Problem Sheet 3**
Seen A Solutions

Seen A

A.1. Exercise 3.3.7: Which of the following sets span $\mathbb{R}^3$?

While the most obvious way to show that a set spans $\mathbb{R}^3$ is to show that an arbitrary vector of the form (x, y, z) can be obtained by adding together multiples of the set elements, there is method which is usually a shortcut. If we know that set X is spanning, and every element of X can be written as a linear combination, then our original starting set must also be spanning. So in all of these, I'm going to investigate if

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

is in the span of the set in question.

(a)

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

$$\begin{aligned}\begin{pmatrix}1\\0\\0\end{pmatrix} &= \begin{pmatrix}1\\0\\0\end{pmatrix}\\ \begin{pmatrix}0\\1\\0\end{pmatrix} &= \begin{pmatrix}0\\1\\0\end{pmatrix}\\ \begin{pmatrix}0\\0\\1\end{pmatrix} &= \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\end{aligned}$$

(b)

$$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

There's no chance of representing $(1, 0, 0)$ as a linear combination of elements from this set, as in all these vectors, the first and third coordinates are equal.

(c)

$$\left\{\begin{pmatrix}3\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$$

$$\begin{aligned}\begin{pmatrix}1\\0\\0\end{pmatrix} &= \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\\ \begin{pmatrix}0\\0\\1\end{pmatrix} &= \frac{1}{2}\left(\begin{pmatrix}3\\0\\2\end{pmatrix} - 3\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\\ \begin{pmatrix}0\\1\\0\end{pmatrix} &= \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}\end{aligned}$$

(d)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The same observation as in (b) applies here. You might have also noticed that there are too few vectors, this is an idea that we'll get into more detail soon.

Definitions 3.3.1. and 3.3.5. are useful here.

A.2. For each of the following subsets of $\mathbb{R}^3$, work out if they are linearly independent or not.

(a) $\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right\}$

Definition 3.4.1., and Section 2.3, may be useful here.

One way to prove that a set is linearly independent is to check that if a linear combination of elements from the set is equal to zero, then all the coefficients are zero.

(a) Suppose that for some $\alpha, \beta \in \mathbb{R}$

$$\alpha \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the bottom line, we get that $\beta = 0$. Combining this with the middle line, we also get that $\alpha = 0$, and therefore the set is linearly independent.

(b) Suppose that for some $\alpha, \beta, \gamma \in \mathbb{R}$

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the top line, we get that $\alpha = \beta$, and from the middle line, we get that $\beta = \gamma$. The final line gives $\alpha = -\gamma$. If we put all this together, we get that $\alpha = -\alpha$, and therefore $\alpha = \beta = \gamma = 0$.

Therefore the set is linear independent.

(c)

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So the set is not linearly independent.

(d) This set is not linearly independent, it has too many vectors. In my opinion, the easiest way to find the actual α, β, γ and δ such that

$$\alpha \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is to assume that $\delta = -1$ and transform this problem into an augmented matrix.

$$\begin{pmatrix} 1 & 2 & 2 \\ -3 & -1 & -5 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ -3 & -1 & -5 & 2 \\ 2 & 1 & 4 & 5 \end{array}\right)$$

which reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 10 \end{array}\right)$$

Therefore

$$-17\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 10\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and the set is not linearly independent.

A.3. For which of the following values of $\bar{x}$ and $\bar{y}$ is the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \bar{x}, \bar{y} \right\} \subseteq \mathbb{R}^4$$

a basis?

(a)

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(b)

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

(c)

$$\bar{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

(d)

$$\bar{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 16 \\ 32 \\ 64 \\ 128 \end{pmatrix}$$

Definition 3.5.1., and hence Definitions 3.3.5. and 3.4.1., are the key ones for this question.

There are a lot of ways to show that something is not a basis. In (a), it will be easiest to see that the span of the set will not contain $(0, 0, 0, 1)^T$, so there's no way for the set to span $\mathbb{R}^4$, and hence it cannot be a basis. In (d), $2\bar{x} = \bar{y}$, and therefore the set cannot be linearly independent.

To show that something is a basis, we need to show that the set is both linearly independent and spans the vector space. Showing that the set from (b) spans the set is comparatively simple. To show that it spans $\mathbb{R}^4$, you can either show that an arbitrary element of $\mathbb{R}^4$ is contained in the span by explicitly calculating how, or if you show that a known basis is contained in the span of the set, then the set must span $\mathbb{R}^4$. (Why?)

In (b), it's clear that $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ are in the span of the set.

$$\bar{x} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{y} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and therefore the standard basis of $\mathbb{R}^4$ is contained in the span of the set in (b).

We can do the same for (c), but using more steps.

$$\bar{x} - 4\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} \quad \bar{y} - 2\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\2\end{pmatrix}$$

Then

$$-\frac{1}{3}\left(\begin{pmatrix}0\\0\\2\\1\end{pmatrix} - 2\begin{pmatrix}0\\0\\1\\2\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and

$$-\frac{1}{3}\left(-2\begin{pmatrix}0\\0\\2\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\2\end{pmatrix}\right) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

A.4. Extend the following set into a basis of $\mathbb{R}^4$.

$$\left\{\begin{pmatrix}1\\4\\-5\\2\end{pmatrix}, \begin{pmatrix}2\\-2\\5\\0\end{pmatrix}\right\}$$

Put together what you've learnt from the last 3 questions for this one!

Extending a basis is most easily done by adding elements of something you already know is a basis. In this case, it's comparatively easy to prove that

$$\left\{\begin{pmatrix}1\\4\\-5\\2\end{pmatrix}, \begin{pmatrix}2\\-2\\5\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$$

is a basis.

Seen B

B.1. For which a, b, c are the vectors $(1, 3, 1)$, $(2, 1, 1)$, (a, b, c) linearly dependent?

B.2. Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

(a) If $\{v_1, \ldots, v_n\}$ is a basis, for V, and $\{x_1, \ldots, x_r\}$ is a linearly independent subset of V with $r < n$, and if $v_i \notin Span\{x_1, \ldots, x_r\}$ for all $i = 1, \ldots, n$, then $\{x_1, \ldots, x_r, v_{r+1}, \ldots, v_n\}$ is a basis for V.

(b) If U is a subspace of V, then $U + U = U$.

(c) If U and W are subspaces of V, and $\dim U + \dim W = \dim V$, then $U \cap W = \{0_V\}$.

(d) If $\dim V = n$ and $v_1 \in V$, then there exist vectors $v_2, \ldots, v_n$ in V such that $\{v_1, \ldots, v_n\}$ spans V.

(e) If W is a subspace of V , then $\dim W \leq \dim V$ and $\dim W = \dim V$ if and only if $W = V$.

B.3. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$).

(a) Show that the functions

$$f_1(x) = 1, \quad f_2(x) = 1 + x + x^2, \quad f_3(x) = \sin x, \quad f_4(x) = \cos x$$

are linearly independent.

(b) Which of the following functions lie in $Span(f_1, f_2, f_3, f_4)$?

$$5 - 3x - 3x^2, \quad \tan x, \quad 10 - x - x^2 + \sin(x + \pi/3).$$

B.4. (a) Describe an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

(b) Find a basis for $W = Span(x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1) \leq \mathbb{R}[x]$.

B.5. Let V be the vector space of all 3×3 matrices over $\mathbb{R}$.

(a) Find a basis of V consisting of invertible matrices.

(b) Let $W = \{A \in V : A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

(c) Let $W \subset V$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.

Math40003 Linear Algebra and Groups — Problem Sheet 3
Unseen Questions

U.1. Prove that if V is a finite dimensional vector space, then V does not have an infinite basis.

U.2. Find a basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

U.3. Find a basis of the vector spaces introduced in C.2 on Problem Sheet 2.

U.4. Does every vector space have a basis?

WARNING: I'm an exceptionally cruel person to set this question with no guidance or advice. I'm willing to argue that this question is the most subtle piece of mathematics you'll see in your first and second year.