MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 7

Exercise 31.

The first four parts (i), (ii), (iii) and (iv) are elementary. To solve (v), it is essential to realise that Exercise 16 (ii) implies that if a solution stays in one of the four regions forward in time and is bounded forward in time, then it must converge to an equilibrium. To see this, note that in each region, each component of the solutions are monotone (e.g. in SW, first and second component decreasing), and if a solution is bounded forward in time, then it must converge, since both components must converge. Then Exercise 16 (ii) yields that it converges to an equilibrium. Proceed then with analysing how the flow can move between the different areas.

Exercise 32.

After proving the given hint, use the definition of stability (with ε and δ) to transfer stability from x^* to y^*. Note that the size of δ for y^* is related to the size of the δ for x^*, and here the action of the linear mapping T on the δ-neighbourhood needs to be considered. Then take an arbitrary point x in $W^s(x^*)$, and note that forward in time (after a finite time), the flow carries this point x in the above δ-neighbourhood. Establish that this is true not only for x but for a whole neighbourhood of x and finalise the proof.

Exercise 33.

Establish first that there exists a δ-ball around the equilibrium x^*, so that all solutions starting in this ball converge to x^* forward in time (this uses the definition of attractivity).

Exercise 34.

Let $x \in \partial M$ and assume, there exists a $\tau \in J_{max}(x)$ such that $\varphi(\tau, x) \notin \partial M$. Since ∂M is closed (in the open set D), there exists an $\varepsilon > 0$ such that either

$$B_\varepsilon(\varphi(\tau, x)) \subset \operatorname{int} M \quad \text{or} \quad B_\varepsilon(\varphi(\tau, x)) \subset D \setminus \overline{M}\,,$$

where int denotes the interior of M (i.e. the set of inner points of M). Proceed discussing these two cases to obtain a contradiction in each case.

Exercise 35.

Use Proposition 2.1 to show that the maximal solution λ_{max} to the initial value problem satisfies

$$\|\lambda_{max}(t)\| \le \alpha(t) + \int_{t_0}^{t} \beta(s)\|\lambda_{max}(s)\| \,\mathrm{d}s\,,$$

with continuous functions α and β. Assume then that the solution does not exist globally, find upper bounds for α and β, and apply Gronwall's lemma.