Math40003 Linear Algebra and Groups   Problem Sheet 0

1. Find the following:

(a) $\begin{pmatrix} 1 & 2 \\ -4 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 8 \\ 18 \end{pmatrix} = \begin{pmatrix} 44 \\ -23 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & -1 \\ 8 & \frac{3}{2} & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 4\frac{4}{5} \\ 12 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 3b \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -1 & 1 & 3 \\ 1 & 0 & -1 & \frac{1}{2} \\ 2 & 0 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ a \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 16-a \\ 6 \\ 17 \\ -6-a \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 \\ -4 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ -4x+\frac{1}{2}y \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 2 & 0 \\ -4 & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ -4x+\frac{1}{2}y+z \end{pmatrix}$

2. Consider the matrices

$$\mathsf{P} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \mathsf{Q} = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix} \qquad \text{and} \qquad \mathsf{R} = \begin{pmatrix} 6 & 3 & 1 \end{pmatrix}$$

Determine which of the following matrix products may be defined, and find (by hand) those which can.

(i) PQ
(ii) QP
(iii) PR
(iv) RP

P is 3×2, Q is 2×2 and R is 1×3.

(i) PQ : $(3 \times 2)(2 \times 2)$. **Yes.**
(ii) QP : $(2 \times 2)(3 \times 2)$. **No.**
(iii) PR : $(3 \times 2)(1 \times 3)$. **No.**
(iv) RP : $(1 \times 3)(3 \times 2)$. **Yes.**

$$\begin{aligned} \mathsf{PQ} &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 7 \\ 10 & 9 \\ 10 & 16 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathsf{RP} &= \begin{pmatrix} 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 15 \end{pmatrix} \end{aligned}$$

3. Let $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Show that $\mathsf{A}^2 = 4\mathsf{A} + \mathsf{I}_2$ where $\mathsf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \mathsf{A}^2 &= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 4\mathsf{A} + \mathsf{I}_2 &= 4\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 8 \\ 8 & 12 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \end{aligned}$$

4. * Let $\mathsf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Find A^n for all positive integers n.

$$\begin{aligned} \mathsf{A}^2 &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} \\ 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} \\ 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} & 3 \times \frac{1}{3} \times \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Claim: For all $n \in \mathbb{N}$, $\mathsf{A}^n = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

Prove claim by induction:

Base case: **For** $n = 1$ $\mathsf{A}^1 = \mathsf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$**.**

Inductive hypothesis: **Suppose result is true for** $n = k$**.**

Inductive step:

$$\mathsf{A}^{n+1} = \mathsf{A}\mathsf{A}^n = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Conclusion: **By the principle of mathematical induction** $A^n = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ **for all** $n \in \mathbb{N}$**.**

5. Let A be a 2×2 matrix which commutes with every other 2×2 matrix, i.e. $\mathsf{AB} = \mathsf{BA}$ for any 2×2 matrix B. Show that A must be of the form $k\mathsf{I}_2$, for some $k \in \mathbb{R}$.

   Let

   $$\mathsf{A} = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \qquad \textbf{and} \qquad \mathsf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \qquad \textbf{for } k, l, m, n, p, q, r, s \in \mathbb{R}$$

   Then

   $$\mathsf{AB} = \begin{pmatrix} kp + lr & kq + ls \\ mp + nr & mq + ns \end{pmatrix}$$

   and

   $$\mathsf{BA} = \begin{pmatrix} kp + mq & lp + nq \\ kr + ms & lr + ns \end{pmatrix}$$

   If $\mathsf{AB} = \mathsf{BA}$**, then**

   $$\begin{aligned} kp + lr &= kp + mq \\ \therefore lr &= mq \qquad \forall r, q \in \mathbb{R} \end{aligned}$$

   If we let, $q = 0$ **and** $r = 1$ **this implies that** $l = 0$**. Similarly if we let,** $r = 0$ **and** $q = 1$ **this implies that** $m = 0$**..**

   So,

   $$l = m = 0 \tag{1}$$

   Also,

   $$mp + nr = ms + kr \qquad \forall r, s \in \mathbb{R}$$

But, $m = 0$**, so**

$$\begin{aligned} nr &= kr \qquad \forall r \in \mathbb{R} \\ n &= k \end{aligned} \tag{2}$$

Equation 1 and Equation 2 imply that

$$\begin{aligned} \mathsf{A} &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ &= k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= k\mathsf{I}_2 \qquad \textbf{as required} \end{aligned}$$