Mathematical Logic (MATH6/70132;P65)
Problem Class, week 6

[1] In 2.1.3 we introduced a language $\mathcal{L}$ which is appropriate for groups: with 2-ary relation symbol R (for equality), 2-ary function symbol m (for the group operation), 1-ary function symbol i (for inversion) and constant symbol e (for the identity element).

Let $\mathcal{A}, \mathcal{B}$ be the $\mathcal{L}$-structures $\langle \mathbb{Z}; =, +, -, 0\rangle$ and $\langle \mathbb{Q}; =, +, -, 0\rangle$ (respectively).

(a) True or false? Give reasons.

(i) Every $\mathcal{L}$-structure is a group.

(ii) $(\neg m(x_2, m(e, x_1)))$ is a formula of $\mathcal{L}$.

(iii) $R(e, m(x_1, i(x_1)))$ is a formula of $\mathcal{L}$.

(iv) $(\exists x_1)((\neg R(e, m(x_1, i(x_1)))))$ is an $\mathcal{L}$-formula.

(b) Suppose v is the valuation in $\mathcal{A}$ with $v(x_1) = 2$, $v(x_2) = 4$ and $v(x_j) = 0$ when $j \neq 1, 2$.

(i) Compute $v(m(x_2, m(e, x_1))$.

(ii) Find a term t with $v(t) = -6$.

(iii) Can you find a term t' with $v(t') = 7$?

(c) Find a closed $\mathcal{L}$-formula ϕ such that $\mathcal{A} \models \phi$ and $\mathcal{B} \not\models \phi$.

[2] It's difficult to do any reasoning in mathematics without using the equality symbol. In 1st-order logic, we use the following terminology to handle equality. We will say more about this during the lectures.

A first-order *language with equality* $\mathcal{L}^=$ is a 1st-order language with a distinguished 2-ary relation symbol $=$. An $\mathcal{L}^=$-structure $\mathcal{A}$ is *normal* if the symbol $=$ is interpreted as equality in $\mathcal{A}$.

We write the more usual '$x_1 = x_2$' instead of '$= (x_1, x_2)$' in $\mathcal{L}^=$-formulas.

Suppose $\mathcal{L}^=$ is a language with equality which also has a 2-ary relation symbol R.

(a) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula σ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \sigma_n$ iff the domain of $\mathcal{A}$ has at least n elements. [Hint: Think about the cases $n = 2, 3$ first; you can use $\wedge$ if you want.]

(b) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula τ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \tau_n$ iff the domain of $\mathcal{A}$ has exactly n elements.

(c) Find a closed $\mathcal{L}^=$ formula θ with the property that for every positive $n \in \mathbb{N}$:

there is a normal $\mathcal{L}^=$ structure $\mathcal{A}$ with n elements in its domain and $\mathcal{A} \models \theta$ if and only if n is even (your formula will need to use the symbol R).

Hint: (1) Say "R is an equivalence relation all class with exactly 2 elts."

(2) Another way,

Problem Class, Week 6.

1. (a)(i) F

(ii) F   no relation symbol

(iii) T

(iv) F   too many ( )

(b) (i) 6

(ii) $i(m(x_2, m(e, x_1)))$

(iii) No   v(term) is even here.

(c) $(\neg(\forall x)(\exists y)(\underbrace{x = y+y}_{R(x,\, m(y,y))}))$

2. $\mathcal{L}^=$

(a) $n=2$ $(\exists x_1)(\exists x_2)(x_1 \neq x_2)$ σ_2

σ_3: $n=3$ $(\exists x_1)(\exists x_2)(\exists x_3)((x_1 \neq x_2) \wedge (x_1 \neq x_2) \wedge \dots))$

σ_n $(\exists x_1)\cdots(\exists x_n) \bigwedge_{1 \le i < j \le n} (x_i \neq x_j)$

(b) $(\sigma_n \wedge (\neg \sigma_{n+1}))$.