Problem Sheet 5, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let $S \subset \mathbb{R}^3$ be a compact, connected surface without boundary which is not diffeomorphic to a sphere. Prove that S contains points where the Gaussian curvature is negative, zero, and positive.

Solution: We have already seen that points of positive curvature exist on compact surfaces; let p be such a point. Then, by continuity, we have $K > 0$ on an open neighbourhood $V \subset S$ of p. If the curvature is nonnegative at all points of S, then by Gauss-Bonnet we have

$$2\pi\chi(S) = \int_S K dA \geq \int_V K dA > 0.$$

Thus, $\chi(S) > 0$, which implies that S is diffeomorphic to a sphere. But this is a contradiction, so there is $q \in S$ such that $K(q) < 0$. Since S is connected, it is also path connected. Let us choose a path from p to q in S. Since K is continuous along that path, the intermediate value theorem implies that there is a point on the path where K becomes 0.

Problem 2. Fix constants $a, b > 0$ and consider the ellipsoid

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \right\}.$$

(a) Compute the Gaussian curvature of S at each point (no need to calculate them at the north and south poles) using the map

$$\phi(u, v) = (a\cos(u)\cos(v), a\sin(u)\cos(v), b\sin(v)).$$

(b) Apply the Gauss-Bonnet theorem to S, and conclude that

$$\int_0^1 \frac{dx}{\left(b^2 + (a^2 - b^2)\, x^2\right)^{3/2}} = \frac{1}{ab^2}.$$

Solution: (a) It is easy to see that the restriction of ϕ to sufficiently small open sets is a chart for S. We have

$$\begin{aligned}
\phi_u(u, v) &= (-a\sin(u)\cos(v), a\cos(u)\cos(v), 0),\\
\phi_v(u, v) &= (-a\cos(u)\sin(v), -a\sin(u)\sin(v), b\cos(v)).
\end{aligned}$$

Thus, the unit normal to S is (choose either of the signs)

$$\begin{aligned}
N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} &= \frac{\left(ab\cos u\cos^2 v, ab\sin u\cos^2 v, a^2\cos v\sin v\right)}{\left(a^2b^2\cos^2 u\cos^4 v + a^2b^2\sin^2 u\cos^4 v + a^4\cos^2 v\sin^2 v\right)^{1/2}}\\
&= \frac{(a\cos v)\,(b\cos u\cos v, b\sin u\cos v, a\sin v)}{(\pm a\cos v)\left(b^2\cos^2 v + a^2\sin^2 v\right)^{1/2}}\\
&= \pm\frac{(b\cos u\cos v, b\sin u\cos v, a\sin v)}{\left(b^2\cos^2 v + a^2\sin^2 v\right)^{1/2}}.
\end{aligned}$$

The first and second fundamental forms are given by

$$g = \begin{pmatrix} \langle \phi_u(u,v), \phi_u(u,v) \rangle & \langle \phi_u(u,v), \phi_v(u,v) \rangle \\ \langle \phi_v(u,v), \phi_u(u,v) \rangle & \langle \phi_v(u,v), \phi_v(u,v) \rangle \end{pmatrix} = \begin{pmatrix} a^2\cos^2(v) & 0 \\ 0 & a^2\sin^2(v) + b^2\cos^2(v) \end{pmatrix}$$

and

$$\begin{aligned} A &= \begin{pmatrix} \langle N(\phi(u,v)), \phi_{uu}(u,v) \rangle & \langle N(\phi(u,v)), \phi_{uv}(u,v) \rangle \\ \langle N(\phi(u,v)), \phi_{vu}(u,v) \rangle & \langle N(\phi(u,v)), \phi_{vv}(u,v) \rangle \end{pmatrix} \\ &= \frac{1}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^{1/2}} \begin{pmatrix} -ab\cos^2(v) & 0 \\ 0 & -ab \end{pmatrix} \end{aligned}$$

Therefore, at $\phi(u,v) \in S$, the Gaussian curvature is

$$K = \frac{\det(A)}{\det(g)} = \frac{\left(a^2b^2\cos^2(v)\right) / \left(a^2\sin^2(v) + b^2\cos^2(v)\right)}{\left(a^2\cos^2(v)\right)\left(a^2\sin^2(v) + b^2\cos^2(v)\right)} = \frac{b^2}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^2}.$$

(b) We see that S is diffeomorphic to the unit sphere S^2 by the map

$$(x,y,z) \in S \quad \mapsto \quad (x/a, y/a, z/b) \in \mathbb{S}^2.$$

Therefore, $\chi(S) = \chi(\mathbb{S}^2) = 2$, and hence the Gauss-Bonnet theorem gives us

$$\int_S K\,dA = 2\pi\chi(S) = 4\pi.$$

We can evaluate the total curvature of S using the parametrisation from (a), which covers all of S in the range $0 < u < 2\pi, 0 < v < \pi$ except for a regular curve. That is,

$$\begin{aligned} \int_S K\,dA &= \int_0^{\pi}\int_0^{2\pi} K(\phi(u,v))|\phi_u(u,v) \times \phi_v(u,v)|\,dudv \\ &= \int_0^{\pi}\int_0^{2\pi} \frac{b^2}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^2} a|\cos(v)|\sqrt{a^2\sin^2(v) + b^2\cos^2(v)}\,dudv \\ &= 2\pi\int_0^{\pi} \frac{ab^2|\cos(v)|}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^{3/2}} dv \\ &= 4\pi\int_0^{\pi/2} \frac{ab^2\cos(v)}{\left(a^2\sin^2(v) + b^2\cos^2(v)\right)^{3/2}} dv \end{aligned}$$

where the last step uses the fact that the integrand is symmetric about $v = \frac{\pi}{2}$. At this point we substitute $x = \sin(v), dx = \cos(v)dv$ to conclude that

$$4\pi = 4\pi\int_0^1 \frac{ab^2}{\left(a^2x^2 + b^2\left(1 - x^2\right)\right)^{3/2}} dx,$$

which implies that

$$\int_0^1 \frac{dx}{\left(b^2 + \left(a^2 - b^2\right)x^2\right)^{3/2}} = \frac{1}{ab^2}.$$

Problem 3. Let $S \subset \mathbb{R}^3$ be a regular surface, and assume that $\gamma_1 : [0, t_1] \to S$ and $\gamma_2 : [0, t_2] \to S$ are geodesics parametrised by arc length, and assume that these are not part of a single common geodesic on S. Prove that there are only finitely many pairs (τ_1, τ_2) such that $\gamma_1(\tau_1) = \gamma_2(\tau_2)$.

Solution: Consider the set

$$A = \left\{(t, t') \in [0, t_1] \times [0, t_2] \mid \gamma_1(t) = \gamma_2(t')\right\}.$$

This is a closed set in $[0, t_1] \times [0, t_2]$. That is because A is the pre-image of the diagonal $\{(x, y) \in \mathbb{R}^3 \mid x = y\}$ under the continuous map $(t, t') \mapsto (\gamma_1(t), \gamma_2(t'))$. Obviously, the diagonal is a closed set. Since, any closed set in a compact set is compact, and $[0, t_1] \times [0, t_2]$ is compact, A is compact.

Assume in the contrary that A is infinite. Since A is compact, there must be a sequence of pairs $(\tau_i, \tau_i')_{i=0}^{\infty}$ in A which converges to some $(\tau, \tau') \in [0, t_1] \times [0, t_2]$. By the continuity of γ_1 and γ_2, and $\gamma_1(\tau_i) = \gamma_2(\tau_i')$, we conclude that $\gamma_1(\tau) = \gamma_2(\tau')$.

On the other hand, we have

$$\gamma_1'(\tau) = \lim_{i\to\infty} \frac{\gamma_1(\tau) - \gamma_1(\tau_i)}{\tau - \tau_i} = \lim_{i\to\infty} \frac{\gamma_2(\tau') - \gamma_2(\tau_i')}{\tau' - \tau_i'} \left(\frac{\tau' - \tau_i'}{\tau - \tau_i}\right) = \gamma_2'(\tau') \lim_{i\to\infty} \frac{\tau' - \tau_i'}{\tau - \tau_i}.$$

Since $\gamma_1'(\tau)$ and $\gamma_2'(\tau')$ exist and are unit vectors, the limit of the ratio on the right hand side exists, and must be ± 1. In particular, $\gamma_1'(\tau) = \pm\gamma_2'(\tau')$. Since γ_1 and γ_2 (up to reversing direction) have the same position and unit tangent vector at times τ and τ' respectively, the geodesic equations say that they determine the same geodesic on S. This contradicts the assumptions in the problem.

Problem 4. Let p be a point on a regular surface S, and let $T \subset S$ be a curvilinear triangle whose sides are geodesics, and p belongs to the interior of T. Prove that if α_1, α_2, and α_3 denote the interior angles of T, then

$$\lim_{T\to p} \frac{\left(\sum_{i=1}^{3} \alpha_i\right) - \pi}{\text{area}(T)} \to K(p)$$

where the limit is taken over any sequence of such curvilinear triangles T which converge to p. Explain how this gives another proof of Gauss's Theorema Egregium.

Solution: Let θ_1, θ_2, and θ_3 denote the exterior angles of T, that is $\alpha_i = \pi - \theta_i$ for each i. Applying the Gauss-Bonnet theorem to T, we have

$$2\pi = \int_{\partial T} k_g \, ds + \sum_{i=1}^{3} \theta_i + \int_T K \, dA = \sum_{i=1}^{3} (\pi - \alpha_i) + \int_T K \, dA.$$

Thus,

$$\int_T K \, dA = \left(\sum_{i=1}^{3} \alpha_i\right) - \pi.$$

In particular, the ratio

$$\frac{(\sum \alpha_i) - \pi}{\text{area}(T)}$$

is the average value of K on the interior of the triangle T. Since K is continuous, this average approaches $K(p)$ as $T \to p$, as claimed.

To see why this proves the Theorema Egregium, this limit characterizes the curvature at p in terms of geodesic triangles. We have seen that the notion of a geodesic is intrinsic, as it is characterized by the geodesic equations, which depend only on the metric; so are the angles θ_i and the area of T (defined as an integral of $\sqrt{\det(g)}$). Thus every term in the limit is intrinsic, and we conclude that the curvature $K(p)$ is as well.

Problem 5. Let $S \subset \mathbb{R}^3$ be a regular surface with curvature $K \leq 0$. Assume that S is diffeomorphic to a plane, and $\gamma : (a, b) \to S$ is a geodesic parametrised by arc length. Prove that γ is injective. Give a counterexample when S is not diffeomorphic to a plane.

Solution: Suppose $\gamma(t)$ is not injective, so that (up to shifting t by a constant, and changing the direction) we have $\gamma(0) = \gamma(L)$, for some $L > 0$. The set $\{t \in [0, L] \mid \gamma(t) = \gamma(0)\}$ is closed. It contains 0 as an isolated point, since $\gamma'(0) \neq 0$, and it contains L. Thus, there is the smallest positive element t_0 in that set. This implies that $\gamma([0, t_0])$ is a simple closed geodesic (it may not be regular). Since S is diffeomorphic to a plane, by the Jordan curve theorem, $\gamma([0, t_0])$ bounds a region, say D, which is homeomorohic to an open ball in $\mathbb{R}^2$. Applying Gauss-Bonnet theorem to D, we get

$$\int_{\gamma([0,t_0])} k_g \, ds + \int_D K \, dA + \Theta = 2\pi,$$

where Θ is the exterior angle between $\gamma'(t_0)$ and $\gamma'(0)$. The integral over $\gamma([0, t_0])$ is zero, since the curve is a geodesic. The integral over K is non-positive, since $K \leq 0$ by assumption. We also know that $-\pi\Theta \leq \pi$ by our convention. Thus, the left hand side of the above equation is bonded from above by π. This is a contradiction.

For the latter part of the problem, consider the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. The curve $\gamma(t) = (\cos(t), \sin(t), 0)$ is a geodesic, as we saw in an example in the lectures. This curve does not bound a disk or indeed any other compact surface in S, and we cannot apply the Gauss-Bonnet theorem.