

**Imperial College  
London**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

**Integrable Systems**

Date: Friday, 29 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) A planet of mass  $m$  orbits in the plane about a fixed gravitating mass  $M$ . The constant of gravitation is  $G$ . Write down the Hamiltonian for the system in plane polar coordinates, with origin at the position of  $M$ .
- (b) Explain why this system is integrable.
- (c) Write down the condition for the radial motion to be bounded, and find its turning points  $r_+$  and  $r_-$ .
- (d) Write down the Hamilton-Jacobi equation for this system, and show that it is separable in these coordinates.
- (e) Construct the action integrals  $J_\theta$  and  $J_r$  corresponding to the angular and radial motions respectively.
- (f) By considering the contour integral

$$I = \frac{1}{2\pi i} \oint_C \sqrt{-Em - \frac{2Gm^2M}{\rho} + \frac{L^2}{\rho^2}} d\rho,$$

where  $C$  is a large circle enclosing the origin in the complex plane, and the branch cut of the integrand is taken between  $r_\pm$ , find the relation between the energy  $E$ , and the two action integrals.

[Hint: Evaluate the integral first by calculating the residue of the integrand at infinity; then shrink the contour to two loops, one enclosing the origin and the other enclosing the branch cut.]

- (g) Hence show that the motion of the planet in the plane is periodic.

2. (a) A set  $G$  of  $3 \times 3$  matrices  $M$  acts on 3-component column vectors as follows,

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = M \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

satisfying the condition that

$$(y^1)^2 + (y^2)^2 - (y^3)^2 = (x^1)^2 + (x^2)^2 - (x^3)^2.$$

Find the condition any such  $M$  must satisfy, and hence show that  $G$  is a Lie group.

- (b) Verify that the Lie algebra  $\mathfrak{g}$  of  $G$  is spanned by the basis:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the structure constants of  $\mathfrak{g}$  with respect to this basis.

- (c) A mechanical system has Lagrangian

$$L = \frac{1}{2}((\dot{x}^1)^2 + (\dot{x}^2)^2 - (\dot{x}^3)^2) - V((x^1)^2 + (x^2)^2 - (x^3)^2).$$

Write down its Hamiltonian.

- (d) Construct three Hamiltonian vector fields  $X_{e_1}$ ,  $X_{e_2}$  and  $X_{e_3}$ , such that their  $x^i$  components give the  $\mathfrak{g}$  action of  $e_i$  corresponding to the  $G$ -action described above. Explain clearly how these vector fields act on the momentum components  $p_i$ .

- (e) Construct the three components  $J_i$  of the momentum map, and write down the Poisson brackets between them and with  $H$ . Construct a function of these components which Poisson commutes with each of the  $J_i$ , and hence show that the mechanical system is integrable.

3. Let  $Q$  and  $P$  be  $N \times N$  Hermitian matrices, with Poisson brackets between their components

$$\{Q^{ij}, P_{k\ell}\} = \delta_k^j \delta_\ell^i,$$

$$\{Q^{ij}, Q^{kl}\} = 0,$$

$$\{P_{ij}, P_{k\ell}\} = 0.$$

- (a) Write down the symplectic form corresponding to the Poisson bracket.
- (b) Find the equations of motion for the Hamiltonian  $H = \frac{1}{2}\text{Tr}(Q^2 + P^2)$ , and write down their solutions.
- (c) Show that the Hamiltonian and the Poisson bracket are invariant under the action of the unitary group  $U(N)$ :

$$\tilde{Q} = UQU^{-1},$$

$$\tilde{P} = UPU^{-1}.$$

Find the corresponding Lie algebra action, and hence construct the momentum map  $J$ .

- (d) Taking a particular value for this momentum map  $J = \mu$ ,

$$\mu_{jk} = i(1 - \delta_{jk}),$$

and assuming without proof that the isotropy group of  $\mu$  is sufficiently large, reduce the phase space of the system, choosing  $U$  so as to diagonalise the matrix  $Q$ , with eigenvalues  $x^i$ ,  $i = 1 \dots N$ . Find the corresponding form of the matrix  $P$ , and hence show that the Hamiltonian in the reduced  $N$ -dimensional phase space is

$$\frac{1}{2} \sum_{i=1}^N (p_i)^2 + (x^i)^2 + \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{1}{(x^i - x^j)^2},$$

where the new coordinate variables are the eigenvalues  $x^i$  of  $Q$ .

4. (a) Let  $\psi$  be a two-component column vector satisfying both:

$$\partial_x \psi = L\psi,$$

and

$$\partial_t \psi = M\psi,$$

where  $L, M$  are  $sl(2, \mathbb{C})$ -valued matrix functions of  $(x, t)$ , polynomial in a spectral parameter  $\zeta$ . By cross-differentiating these two equations, find their consistency condition.

- (b) Taking

$$L = \begin{pmatrix} \zeta & v \\ v & -\zeta \end{pmatrix},$$

and

$$M = \begin{pmatrix} \zeta^3 + a_2\zeta^2 + \zeta a_1 + a_0 & \zeta^2 b_2 + \zeta b_1 + b_0 \\ \zeta^2 c_2 + \zeta c_1 + c_0 & -\zeta^3 - \zeta^2 a_2 - \zeta a_1 - a_0 \end{pmatrix},$$

and matching coefficients of  $\zeta^n$  in the consistency condition, find the coefficients  $(a_2, b_2, c_2), (a_1, b_1, c_1), (a_0, b_0, c_0)$ , as functions of  $v$  and its derivatives. For definiteness, you should assume that these functions contain no constant term.

- (c) Hence find the partial differential equation satisfied by  $v$ .

