Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2020-2021

Partial Differential Equations in Action **MATH50008**

Midterm Exam

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You must turn in handwritten solutions written on paper and scanned. You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

IMPORTANT – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

1. **Total: 10 Marks** Consider the following initial boundary value problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{1}{2x}\frac{\partial u}{\partial x} = 0, \quad x > 0, t > 0 \\
&u(x,0) = 0, \\
&u(0,t) = \sin(t)
\end{aligned}
$$

   (a) Find the equation of the characteristics for this problem. Sketch the characteristics and draw in the (x,t)-plane the curve that separates regions where the solution is identically zero from regions where it is non-zero. 4 Marks

   (b) Using the method of characteristics, find the solution of this problem for all $x > 0,\ t > 0$. 3 Marks

   (c) In the region where the solution is non-zero, how many local maxima and minima does the solution exhibit at a given time t_0? Find how many local maxima and minima occur when $t_0 = 10$. 3 Marks

2. **Total: 10 Marks** We consider the one-dimensional flow of a fluid. It can be shown that the velocity $u(x,t)$ of the fluid in this problem is governed by the nonlinear partial differential equation

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0
$$

   Here, we assume that the initial velocity field is given by

$$
u(x,0) = \begin{cases} 0, & x < 0 \\ u_0, & x > 0 \end{cases}
$$

   where u_0 is a positive constant.

   (a) If you have to perform dimensional reduction, what are the relevant variables in this problem? 1 Mark

   (b) Use dimensional reduction and a similarity variable (that you will determine) to reduce this problem to a nonlinear ordinary differential equation with two boundary conditions. 4 Marks

   (c) Use your result in part (b) to obtain a continuous solution to this problem. 3 Marks

   (d) If the variable $u(x,t)$ now represents the displacement of the fluid instead of its velocity, explain why it is not possible for u to satisfy the original nonlinear partial differential equation. 2 Marks

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Partial Differential Equations in Action **MATH50008**

Solutions to Midterm Exam

1. **Total: 10 Marks** Here, we consider the following initial boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2x}\frac{\partial u}{\partial x} = 0, \quad x > 0, t > 0 \\ &u(x,0) = 0, \\ &u(0,t) = \sin(t) \end{aligned}$$

(a) The characteristics for this problem are solutions to the following equation

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = \frac{1}{2x}$$

So we can conclude that

$$u = \text{const.} \quad \text{on} \quad x^2 = t + c$$

where c is an arbitrary constant. We can see that the characteristic given by $c = 0$, i.e. $x(t) = \sqrt{t}$ divides the (x,t)-plane in two regions. For $x > \sqrt{t}$, $u = 0$ as $u(x,0) = 0$ and the characteristics in this region emanate from the $t = 0$ axis. In the region $x < \sqrt{x}$, u is not identically zero.

The characteristics in the region where $x > \sqrt{t}$ emanate from the $t = 0$ axis, they are subject to the parametrization $x = \xi$ when $t = 0$ leading to

$$x^2 = t + \xi^2$$

Conversely, in the region where $x < \sqrt{t}$, the characteristics emanate from the $x = 0$ axis, they are subject to the parametrization $t = \tau$ when $x = 0$ leading to

$$x^2 = t - \tau$$

2 Marks

A diagram of characteristics is provided in Fig. 1, where the dividing characteristic is shown in red. 2 Marks

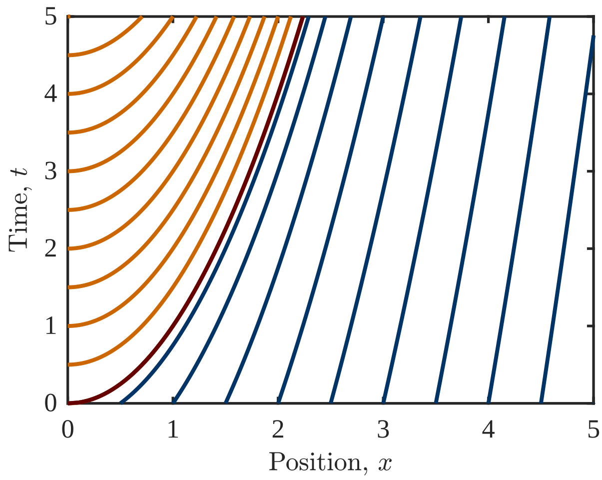

Figure 1: Diagram of characteristics for Q1.

(b) We have already seen that $u(x,t) = 0$ for $x > \sqrt{t}$. In the region $x < \sqrt{t}$, we need to use information from the boundary condition in $x = 0$, i.e. $u(0,t) = \sin t$. We can parametrize the characteristics emanating from this axis using τ as in saw in (a). The value of the solution on the $x = 0$ axis for a given t sets the value of the solution all along one of these characteristics emanating from the t-axis.

We find that

$$u = \sin\tau \quad \text{on} \quad x^2 = t - \tau$$

So the final solution is given by

$$u(x,t) = \begin{cases} \sin(t - x^2), & 0 < x \leq \sqrt{t} \\ 0, & x > \sqrt{t} \end{cases}$$

$\boxed{\text{3 Marks}}$

(c) If we fix $t = t_0$, then we know that the solution is given by

$$u(x,t_0) = \sin(t_0 - x^2)$$

for $x \leq \sqrt{t_0}$, which is only a function of x. Taking the derivative of u with respect to x for a given t_0, we obtain

$$\frac{du}{dx} = -2x\cos(t_0 - x^2)$$

you want to ask when this derivative is zero.

$$\frac{du}{dx} = 0 \Rightarrow x = 0 \quad \text{or} \quad \cos(t_0 - x^2) = 0$$

The first solution is not admissible as we were asked for local minima and maxima in the $x > 0$. The second solution leads to

$$x = \sqrt{t_0 - (2n+1)\frac{\pi}{2}}, \ n \in \{0, 1, 2, \cdots\}$$

Now for the roots to be real, we need

$$t_0 \geq (2n+1)\frac{\pi}{2}$$

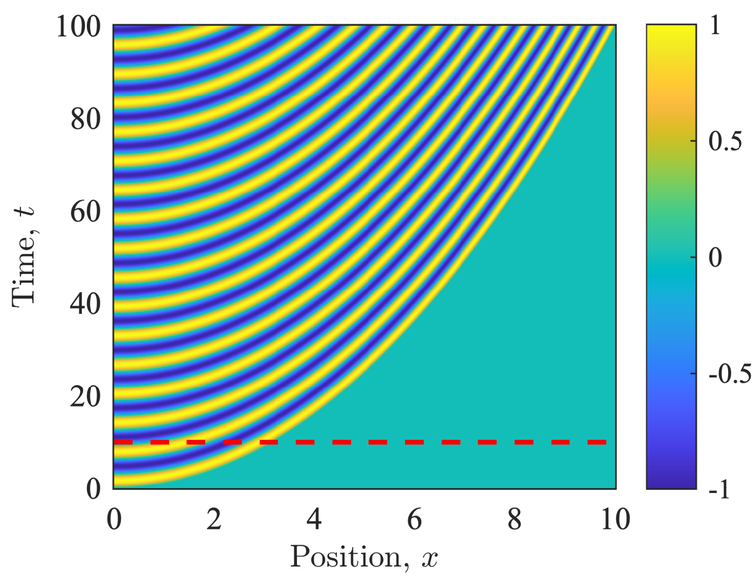

Figure 2: Solution for Q1 in the (x,t)-plane.

So the number of local minima/maxima in the given range is the largest value of n which satisfies this condition plus one as to include the $n = 0$ case. $\boxed{\text{2 Marks}}$

If $t_0 = 10$, then the number of local minima/maxima is given by

$$10 \geq (2N+1)\frac{\pi}{2} \Rightarrow N \leq \frac{10}{\pi} - 1 \Rightarrow N \leq 2$$

So the number of local maxima/minima is $N = 3$. $\boxed{\text{1 Mark}}$

2. $\boxed{\textbf{Total: 10 Marks}}$ We consider the one-dimensional flow of a fluid with no viscosity. It can be shown that the velocity $u(x,t)$ of the fluid in this problem is governed by the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Here, we assume that the initial velocity field is given by

$$u(x,0) = \begin{cases} 0, & x < 0 \\ u_0, & x > 0 \end{cases}$$

where u_0 is a positive constant.

(a) The relevant variables are u, u_0, x and t. $\boxed{\text{1 Mark}}$

(b) We will make the modelling assumption that

$$u = f(u_0, x, t)$$

and so look for numbers (a, b, c) such that

$$[u] = [u_0^a x^b t^c]$$

dimensionally this is equivalent to

$$LT^{-1} = L^a T^{-a} L^b T^c$$

which we can rewrite as the following system of linear equations

$$\begin{cases} L: & 1 = a + b \\ T: & -1 = -a + c \end{cases}$$

We can see that there are only two fundamental variables needed to describe this system $\{L, T\}$. A solution to this system of linear equation is given by $a = 1 - b$ and $c = -b$. This leads to

$$u = \alpha u_0 \left(\frac{x}{u_0 t}\right)^b$$

where α and b are arbitary constants and more generally

$$u = u_0 F(\eta)$$

with

$$\eta = \frac{x}{u_0 t}$$

which is dimensionless $[\eta] = 1$ and we will take as our similarity variable.

Substituting this result in the original PDE we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} &= u_0 F'(\eta)\frac{\partial \eta}{\partial t} = -\frac{x}{t^2}F'(\eta) \\ \frac{\partial u}{\partial x} &= u_0 F'(\eta)\frac{\partial \eta}{\partial x} = \frac{1}{t}F'(\eta) \end{aligned}$$

So

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \Rightarrow -\frac{x}{t^2}F'(\eta) + u_0 F(\eta)\frac{1}{t}F'(\eta)$$

We conclude that $F(\eta)$ is a solution of the following ODE

$$-\eta F'(\eta) + F(\eta)F'(\eta) = 0$$

Let us now take care of the boundary conditions.

- ☐ When $x > 0$, $t \to 0^+$, $\eta \to +\infty$, $u_0 = u_0 F(\infty) \Rightarrow F(\infty) = 1$;
- ☐ When $x < 0$, $t \to 0^+$, $\eta \to -\infty$, $0 = u_0 F(-\infty) \Rightarrow F(-\infty) = 0$.

4 Marks

(c) Let us rewrite the ODE as

$$(F(\eta) - \eta)F'(\eta) = 0$$

which is a nonlinear ODE. So we conclude that

$$\frac{dF}{d\eta} = 0 \quad \text{or} \quad F(\eta) = \eta$$

but F cannot be a constant in $\mathbb{R}$ as then the boundary conditions would not be fullfilled. So we find that on some interval $[a, b]$, $F(\eta) = \eta$. The only possibility for the solution to remain continuous and fulfill the boundary conditions is for the solution to this problem to be given by

$$F(\eta) = \begin{cases} 0, & \eta < 0 \\ \eta, & 0 \le \eta \le 1 \\ 1, & \eta > 1 \end{cases}$$

which one can easily check is a solution of the nonlinear ODE.
This translates eventually to a solution to our original problem written in the form

$$u(x,t) = \begin{cases} 0, & x < 0 \\ x/t, & 0 \le x \le u_0 t \\ u_0, & x > u_0 t \end{cases}$$

We have here obtained the rarefaction wave solution to the Riemann problem using a similarity solution. 3 Marks

(d) To answer this equation, let us have a look at the dimensions in this problem. In the original problem, u is said to be a velocity, i.e. $[u] = LT^{-1}$, so we know that

$$\left[\frac{\partial u}{\partial t}\right] = \frac{LT^{-1}}{T} = LT^{-2} \quad \text{and} \quad \left[u\frac{\partial u}{\partial x}\right] = LT^{-1}\frac{LT^{-1}}{L} = LT^{-2}$$

and so we confirm that the original PDE was dimensionally homogeneous. If now $[u] = L$, we would obtain

$$\left[\frac{\partial u}{\partial t}\right] = \frac{L}{T} = LT^{-1} \quad \text{and} \quad \left[u\frac{\partial u}{\partial x}\right] = L\frac{L}{L} = L$$

and so the equation would be dimensionally inhomogenous unless we introduced another dimensional parameter. So this equation cannot govern the fluid displacement. 2 Marks