Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Group Coursework

Instructions: The deadline to submit this coursework is on **Friday 14 October at 1200 (UK Time)**. The **neatness, completeness and clarity of the answers** will contribute to the final mark. You can turn in handwritten or typed solutions (for instance, using LaTeX). You should upload your answers to this coursework as a single PDF via the Turnitin Assignment called **Group coursework** which you will find in the *Assessments* folder of our Blackboard site. While it is not mandatory, we encourage you to work in groups of **up to 3 students**. On the front page of your submission, you must indicate the firstname, lastname and CID of *all* the students who contributed to the solutions. Your submission filename must have the following format: `Coursework_FIRSTNAME_LASTNAME.pdf`, where FIRSTNAME and LASTNAME are the first and last names of the submitter. **Each group should nominate a submitter for the assignment and only submit one file; the submissions will be marked and all members of the group will receive the same final mark for this assignment.**

Maths students must attempt all three questions, JMC students will only attempt the first two questions.
In this coursework, you may assume any results from the course notes, lecture notes or old videos, as long as you state them correctly.

1. **Total: 20 Marks**

   (a) Prove or disprove: if P, Q and R are propositions, then $(P \wedge Q) \vee R$ is logically equivalent to $P \wedge (Q \vee R)$. 2 Marks

   (b) Let P_1, P_2, P_3, $\ldots$, P_{37} be 37 propositions. Can it ever be true that $P_n \implies \neg P_{n+1}$ for $1 \leq n \leq 36$, and $P_{37} \implies \neg P_1$? 3 Marks

   (c) Using truth tables, prove that $P \implies Q$ and $\neg Q \implies \neg P$ always have the same truth value. 2 Marks

   (d) For Y a set of real numbers, consider the proposition $P(Y)$, defined as $\exists x \in \{1, 2, 3\}, \forall y \in Y, y < x$. For each of the following choices of Y, either prove or disprove the proposition $P(Y)$.

      i. $Y = \{1, 2, 3\}$; 2 Marks

      ii. $Y = \{y \in \mathbb{R} \,|\, y < 3\}$; 2 Marks

      iii. $Y = \emptyset$. 2 Marks

   (e) Now for A and B sets of real numbers, consider the proposition $Q(A, B)$ defined as $\exists x \in A, \forall y \in B, y < x$ and the proposition $R(A, B)$ defined as $\forall y \in B, \exists x \in A, y < x$.

      i. Find an explicit example of sets A and B such that $Q(A, B)$ is false and $R(A, B)$ is true. 2 Marks

      ii. Now find an explicit example of sets A and B such that both $Q(A, B)$ and $R(A, B)$ are false. 2 Marks

      iii. Is it possible to find an example of sets A and B such that $Q(A, B)$ is true and $R(A, B)$ is false? 3 Marks

2. **Total: 20 Marks**

   (a) Say $f : X \to Y$ is injective but not surjective, and $g : Y \to X$ is surjective but not injective. Prove that $f \circ g$ cannot be the identity function. 3 Marks

(b) Give examples of sets X and Y, and functions $f : X \to Y$ and $g : Y \to X$, such that $f \circ g$ is the identity function but $g \circ f$ is not. $\boxed{\text{3 Marks}}$

(c) Suppose that X, Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions. Suppose also that Y is the empty set. Prove that $g \circ f$ is injective. $\boxed{\text{3 Marks}}$

(d) Say A, B, C, D are sets and $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions. Assume that f is bijective and g, h are injective. Prove that $h \circ g \circ f$ is injective. $\boxed{\text{2 Marks}}$

(e) Say $f : X \to Y$ is a function. If $S \subseteq X$ is a subset of X, we define $f(S)$ to be the *image* of S in Y, that is, $f(S) = \{y \in Y \mid \exists x \in S, f(x) = y\}$. Now let S and T be subsets of X. For each of the following statements, give a proof or a counterexample.

i. $S \subseteq T \implies f(S) \subseteq f(T)$; $\boxed{\text{1 Mark}}$

ii. $f(S) \subseteq f(T) \implies S \subseteq T$; $\boxed{\text{2 Marks}}$

iii. $f(S \cup T) = f(S) \cup f(T)$; $\boxed{\text{3 Marks}}$

iv. $f(S \backslash T) = f(S) \backslash f(T)$. $\boxed{\text{3 Marks}}$

3. **Total: 20 Marks**

(a) The curve C has polar equation

$$r = \tan\theta, \quad 0 \leq \theta < \frac{\pi}{2}.$$

Find a cartesian equation of C in the form $y = f(x)$ and give a sketch of the curve. $\boxed{\text{5 Marks}}$

(b) Let ABC be a triangle in the Euclidean plane with corners having position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$. A line passing through one of the corners of the triangle and perpendicular to the opposite side is called an **altitude** of the triangle.

i. Show that the altitude through corner A can be defined as the set of points X with position vectors $\mathbf{x}$ satisfying

$$\mathbf{x} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}).$$

$\boxed{\text{2 Marks}}$

ii. Prove that the three altitudes of a triangle meet at a common point. $\boxed{\text{2 Marks}}$

iii. Find this point when $A(1, 2)$, $B(2, -1)$ and $C(0, 3)$. $\boxed{\text{2 Marks}}$

iv. Determine for which triangles this point lies on the boundary of the triangle. $\boxed{\text{2 Marks}}$

(c) In this problem we will consider the **extended** complex plane. This is the usual complex plane $\mathbb{C}$, with a single point ∞ added to it representing the point at infinity. We denote this set by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Now consider the following transformation

$$f : \begin{cases} \overline{\mathbb{C}} \to \overline{\mathbb{C}} \\ z \mapsto \frac{az+b}{cz+d}, \ z \neq -\frac{d}{c}, \infty \\ -\frac{d}{c} \mapsto \infty \\ \infty \mapsto \frac{a}{c} \end{cases}$$

where here a, b, c, $d \in \mathbb{C}$ and $ad - bc \neq 0$.

i. Prove or disprove that f is injective. $\boxed{\text{3 Marks}}$

ii. Prove or disprove that f is surjective. $\boxed{\text{3 Marks}}$

iii. What happens if we were to let $ad - bc = 0$? $\boxed{\text{1 Mark}}$