

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Function Spaces and Applications

Date: Thursday, 13 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD
INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION
NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. For every $n \in \mathbb{N}$, consider the function $g_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g_n(x) = \frac{n}{\pi(1 + n^2x^2)}.$$

- (a) Does the sequence $\{g_n\}_{n \in \mathbb{N}}$ converge pointwise almost everywhere in \mathbb{R} ? Does it converge in $L^1(\mathbb{R})$? (3 marks)
- (b) For every $\varepsilon > 0$, define the set $A_\varepsilon := \{x \in \mathbb{R} : |x| > \varepsilon\}$.
 - (i) Does the sequence $\{g_n\}_{n \in \mathbb{N}}$ converge uniformly in A_ε ? (4 marks)
 - (ii) Does the sequence $\{g_n\}_{n \in \mathbb{N}}$ converge in $L^1(A_\varepsilon)$? (5 marks)
- (c) Prove that for every continuous and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ it holds

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} g_n(x) f(x) dx = f(0).$$

(8 marks)

(Total: 20 marks)

2. Consider a measurable set $A \subset \mathbb{R}^d$ and a measurable function $f : A \rightarrow \mathbb{R}$. Assume that there exists $p_0 \in [1, +\infty)$ such that $f \in L^p(A)$ for every $p \in [p_0, +\infty)$.

- (a) Let $\alpha > 0$ and define the set $B_\alpha := \{x \in A : |f(x)| \geq \alpha\}$. We use the notation $|B_\alpha|$ for its Lebesgue measure. Prove that for every $p \in [p_0, +\infty)$

$$|B_\alpha| \leq \alpha^{-p} \|f\|_{L^p(A)}^p.$$

To prove this inequality, it might be useful to use the identity $|B_\alpha| = \int_{\mathbb{R}^d} \mathbf{1}_{B_\alpha}(x) dx$, where $\mathbf{1}_{B_\alpha}$ is the indicator function of the set B_α , and note that for $x \in B_\alpha$ we have $1 \leq \frac{|f(x)|}{\alpha}$.

(5 marks)

- (b) Use the inequality proven in the previous step to show that for every $\alpha < \|f\|_{L^\infty(A)}$

$$\liminf_{p \rightarrow \infty} \|f\|_{L^p(A)} \geq \alpha. \quad (1)$$

Why is the constraint $\alpha < \|f\|_{L^\infty(A)}$ needed in order to prove (1)? Infer from (1) that $\liminf_{p \rightarrow \infty} \|f\|_{L^p(A)} \geq \|f\|_{L^\infty(A)}$. Note that $\|f\|_{L^\infty(A)}$ may also be infinity. (5 marks)

- (c) Prove that for every $p > p_0$

$$\|f\|_{L^p(A)}^p \leq \|f\|_{L^\infty(A)}^{p-p_0} \|f\|_{L^{p_0}(A)}^{p_0} \quad (2)$$

and use this inequality to infer that

$$\limsup_{p \rightarrow \infty} \|f\|_{L^p(A)} \leq \|f\|_{L^\infty(A)}. \quad (3)$$

(6 marks)

- (d) Use the previous steps to conclude that

$$\lim_{p \rightarrow +\infty} \|f\|_{L^p(A)} = \|f\|_{L^\infty(A)}.$$

(4 marks)

(Total: 20 marks)

3. We recall that the notation \mathbb{N} stands for the natural numbers without 0. For any $p \in [1, +\infty]$ we denote by $(\ell_p, \|\cdot\|_{\ell_p})$ the Banach space consisting of sequences of real numbers $\{x_n\}_{n \in \mathbb{N}}$ with $\|x\|_{\ell_p} := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$, if $p \in [1, +\infty)$, and $\|x\|_{\ell_{\infty}} := \sup_{n \in \mathbb{N}} |x_n|$ when $p = +\infty$.

Let $k \in \mathbb{N}$, $k \geq 2$ and $p \in [1, +\infty]$. For every sequence of real numbers $x = \{x_n\}_{n \in \mathbb{N}}$ let us define the new sequence $T_k(x) = \{(T_k(x))_n\}_{n \in \mathbb{N}}$ with

$$(T_k(x))_n = \frac{x_n}{k^n}.$$

- (a) Show that, for every k and p as above, the operator $T_k : \ell_p \rightarrow \ell_1$ is linear and continuous. (6 marks)

- (b) Show that:

- (i) When $p = 1$ then

$$\|T_k\|_{\mathcal{L}(\ell_1, \ell_1)} = \frac{1}{k};$$

(4 marks)

- (ii) When $p = \infty$ then

$$\|T_k\|_{\mathcal{L}(\ell_{\infty}, \ell_1)} = \frac{1}{k-1}.$$

(4 marks)

- (c) Prove that the sequence $\{T_k\}_{k \in \mathbb{N}}$ converges in $\mathcal{L}(\ell_p, \ell_1)$ for every $p \in [1, +\infty]$. (6 marks)

(Total: 20 marks)

4. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. Consider the set of functions

$$K := \left\{ v \in L^1(\Omega) : |v(x, y)| \leq 1 \text{ almost everywhere in } \Omega \text{ and } \int_{\Omega} v(x, y) \, dx \, dy = 0 \right\}.$$

- (a) Prove that K is a convex set. (2 marks)
- (b) Prove that K is a closed subset of $L^2(\Omega)$. (4 marks)
- (c) Deduce that for every $f \in L^2(\Omega)$ there is a unique $u \in K$ such that

$$\int_{\Omega} (u(x, y) - f(x, y))(u(x, y) - v(x, y)) \, dx \, dy \leq 0 \quad \text{for every } v \in K.$$

(2 marks)

- (d) Find $u \in K$ defined in the previous step when

- (i) $f(x, y) = xy^{20}$ (4 marks)

- (ii) $f(x, y) = x^2$. (8 marks)

(Total: 20 marks)

5. Let $p \in [1; +\infty]$. For $f \in L^p(0, 1)$ define the function

$$T(f)(x) = \int_0^x f(z) dz, \quad x \in [0, 1]. \quad (4)$$

(a) For every $p \in [1, +\infty]$, prove that the map $T : L^p(0, 1) \rightarrow L^p(0, 1)$, $f \mapsto T(f)$ is linear and continuous.

(2 marks)

(b) For $\alpha \in (0, 1]$, we define the set $C^{0,\alpha}([0, 1])$ as the set of functions $g \in C^0([0, 1])$ such that

$$[g]_\alpha := \sup_{\substack{x, y \in [0, 1], \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|^\alpha} < +\infty.$$

You may assume that $C^{0,\alpha}([0, 1])$ is a Banach space with norm $\|\cdot\|_{0,\alpha} = \|\cdot\|_{C^0([0,1])} + [\cdot]_\alpha$. Here, $\|f\|_{C^0([0,1])} := \sup_{x \in [0,1]} |f(x)|$.

Let $p \in (1; +\infty]$ and T be as in (4). Prove that $T : L^p(0, 1) \rightarrow C^{0,1-\frac{1}{p}}([0, 1])$ is continuous. Here, we use the understanding that when $p = +\infty$, then $1 - \frac{1}{p} = 1$.

(6 marks)

(c) Use part (b) with $p = 2$ to prove that $T : L^2(0, 1) \rightarrow L^2(0, 1)$ is a compact operator.

(12 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96016 / MATH97025 / MATH97102

Function spaces and applications (Solutions)

Setter's signature

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1. (a) For every $x \neq 0$ fixed, we have that $g_n(x) \rightarrow 0$ when $n \rightarrow +\infty$. Hence, the sequence converges to 0 pointwise almost everywhere in \mathbb{R} .

seen \Downarrow

We note that for every $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |g_n(x)| dx = \int_{\mathbb{R}} g_n(x) dx = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1+y^2} dy = 1,$$

where in the first inequality we performed the change of variables $y = nx$. This implies that $\|g_n\|_{L^1(A)} = 1$ for every $n \in \mathbb{N}$ and, by uniqueness of the limit, that the sequence does not converge in $L^1(\mathbb{R})$.

3, A

- (b) (i) Let $\varepsilon > 0$ be fixed. We have that for every $n \in \mathbb{N}$ and $x \in A_\varepsilon$

sim. seen \Downarrow

$$|g_n(x)| \leq \frac{1}{x^2} \frac{1}{\pi n} \leq \frac{1}{\varepsilon^2} \frac{1}{\pi n}.$$

Since the right-hand side above provides a uniform bound in $x \in A_\varepsilon$, we conclude that $g_n \rightarrow 0$ uniformly in A_ε .

4, A

- (ii) Let $\varepsilon > 0$ be fixed. We have

meth seen \Downarrow

$$\int_{A_\varepsilon} |g_n(x)| dx = \frac{1}{\pi} \int_{|y| > \varepsilon n} \frac{1}{1+y^2}. \quad (1)$$

Since the function $h(y) = \frac{1}{1+y^2} \in L^1(\mathbb{R})$, we conclude that the right-hand side above vanishes in the limit $n \rightarrow \infty$. This implies that $\|g_n\|_{L^1(A_\varepsilon)} \rightarrow 0$ when $n \rightarrow \infty$, i.e. the sequence converges to zero in $L^1(A_\varepsilon)$.

5, B

- (c) We rewrite

meth seen \Downarrow

$$\int_{\mathbb{R}} g_n(x) f(x) dx = \frac{1}{\pi} \int_{\mathbb{R}} f\left(\frac{y}{n}\right) \frac{1}{1+y^2} dy. \quad (2)$$

Let us define the sequence $h_n(y) := f\left(\frac{y}{n}\right) \frac{1}{1+y^2}$. Since f is continuous, we have that when $n \rightarrow \infty$, then $h_n(y) \rightarrow f(0) \frac{1}{1+y^2}$ for every $y \in \mathbb{R}$ fixed. Moreover, since f is bounded on \mathbb{R} , for every $n \in \mathbb{N}$ we may bound

$$|h_n(x)| \leq \|f\|_{L^\infty(\mathbb{R})} \frac{1}{1+y^2},$$

with the right-hand side being in $L^1(\mathbb{R})$. We may thus apply the Dominated Convergence Theorem and conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n(x) f(x) dx = \frac{1}{\pi} f(0) \int_{\mathbb{R}} \frac{1}{1+y^2} dy = f(0). \quad (3)$$

8, D

2. (a) We write

sim. seen ↓

$$|B_\alpha| = \int_{\mathbb{R}^d} \mathbf{1}_{B_\alpha}(x) dx. \quad (4)$$

Since for $x \in B_\alpha$ it holds $1 \leq \frac{|f(x)|}{\alpha}$ and therefore also $1 \leq \alpha^{-p}|f(x)|^p$, we obtain

$$|B_\alpha| < \alpha^{-p} \int_{\mathbb{R}^d} |f(x)|^p \mathbf{1}_{B_\alpha}(x) dx \leq \alpha^{-p} \|f\|_{L^p(A)}^p. \quad (5)$$

5, A

(b) By definition of essential supremum, if $\alpha < \|f\|_{L^\infty(A)}$, then the set B_α has positive measure. This implies that

sim. seen ↓

$$\lim_{p \rightarrow \infty} |B_\alpha|^{\frac{1}{p}} = 1. \quad (6)$$

Using inequality of Part (a) (that is valid for any $p \in [1, +\infty)$), we have that

$$\liminf_{p \rightarrow \infty} \|f\|_{L^p(A)} \geq \alpha. \quad (7)$$

For $\alpha \geq \|f\|_{L^\infty(A)}$ the set B_α may be negligible and therefore limit (6) may vanish.

Since the inequality above is valid for every $\alpha < \|f\|_{L^\infty(A)}$ we may send $\alpha \uparrow \|f\|_{L^\infty(A)}$ and obtain that also

$$\liminf_{p \rightarrow \infty} \|f\|_{L^p(A)} \geq \|f\|_{L^\infty(A)}. \quad (8)$$

5, B

(c) Let $p > p_0$. Writing $|f|^p = |f|^{p-p_0} |f|^{p_0}$, we obtain

sim. seen ↓

$$\int_A |f(x)|^p dx \leq \|f\|_{L^\infty(A)}^{p-p_0} \int_A |f(x)|^{p_0} dx = \|f\|_{L^\infty(A)}^{p-p_0} \|f\|_{L^{p_0}(A)}^{p_0}.$$

This implies that for every $p > p_0$

$$\|f\|_{L^p(A)} \leq \|f\|_{L^\infty(A)}^{1-\frac{p_0}{p}} \|f\|_{L^{p_0}(A)}^{\frac{p_0}{p}}.$$

Taking the lim-sup in the above inequality and observing that the right-hand side converges to $\|f\|_{L^\infty(A)}$, we conclude

$$\limsup_{p \rightarrow +\infty} \|f\|_{L^p(A)} \leq \|f\|_{L^\infty(A)}.$$

6, A

(d) Using the previous parts, we have

seen ↓

$$\limsup_{p \rightarrow \infty} \|f\|_{L^p(A)} \leq \|f\|_{L^\infty(A)} \leq \liminf_{p \rightarrow \infty} \|f\|_{L^p(A)},$$

which yields that $\limsup_{p \rightarrow \infty} \|f\|_{L^p(A)} = \liminf_{p \rightarrow \infty} \|f\|_{L^p(A)}$ and thus that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p(A)} = \|f\|_{L^\infty(A)}.$$

4, A

3. (a) Let $k \in \mathbb{N}$, $k \geq 2$ be fixed. The linearity of the operator T_k is immediate. We thus only need to prove that it is bounded: We define the sequence $y^{(k)} = \{\frac{1}{k^n}\}_{n \in \mathbb{N}}$ so that we may write

seen \Downarrow

$$T_k(x) = \{y_n^{(k)} x_n\}_{n \in \mathbb{N}}.$$

If $x \in \ell_p$, then Hölder's inequality yields

$$\|T_k(x)\|_{\ell_1} \leq \|x\|_{\ell_p} \|y^{(k)}\|_{\ell_q},$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. This is equivalent to

$$\|T_k\|_{\mathcal{L}(\ell_p; \ell_1)} \leq \|y^{(k)}\|_{\ell_q}. \quad (9)$$

By the inclusion $\ell_p \subset \ell_q$ for $p \leq q$, it suffices to prove that $y^{(k)} \in \ell_1$: We have

$$\|y^{(k)}\|_{\ell_1} = \sum_{n \in \mathbb{N}} k^{-n} = \frac{1}{1 - k^{-1}} - 1 = \frac{1}{k - 1} < +\infty,$$

where we crucially used the assumption $k \geq 2$.

6, A

- (b) (i) If $p = 1$, then $q = +\infty$. Note that

meth seen \Downarrow

$$\|y^{(k)}\|_{\ell_\infty} = \frac{1}{k}.$$

We thus only need to prove that the upper bound in (9) is an equality. We do so by choosing $x \in \ell_1$ such that $x_1 = 1$ and $x_n = 0$ for every $n \geq 2$. Note that $\|x\|_{\ell_1} = 1$ and

$$\|T_k(x)\|_{\ell_1} = \frac{1}{k}.$$

- (ii) In this case, $q = 1$ and

4, C

$$\|y^{(k)}\|_{\ell_1} = \frac{1}{k - 1}.$$

meth seen \Downarrow

Arguing as in Part (b) (i) we only need to pick an element of ℓ_∞ that realizes the norm: We choose $x \in \ell_\infty$ with $x_n = 1$ for every $n \in \mathbb{N}$. This implies that

$$\|T_k(x)\|_{\ell_1} = \sum_{n \in \mathbb{N}} k^{-n} = \frac{1}{k - 1}.$$

4, C

- (c) We use the upper bound (9) obtained in part (a) to estimate

seen \Downarrow

$$\|T_k\|_{\mathcal{L}(\ell_p, \ell_1)} \leq \|y^{(k)}\|_{\ell_q} \stackrel{\ell_1 \subset \ell_q}{\leq} \|y^{(k)}\|_{\ell_1} = \frac{1}{k - 1} \rightarrow 0.$$

We conclude that T_k converges to 0 in $\mathcal{L}(\ell_p, \ell_1)$ for every $p \in [1, +\infty]$.

6, B

4. (a) In order to prove convexity, we need to check that for every two elements $f, g \in K$, we have that also the function $h_t := tf + (1-t)g \in K$ for every $t \in [0, 1]$. Note that for almost every $(x, y) \in \Omega$ we have that $|f(x, y)| \leq 1$ and $|g(x, y)| \leq 1$ so that also

sim. seen \Downarrow

$$|h_t(x, y)| \leq t|f(x, y)| + (1-t)|g(x, y)| \leq 1 \quad \text{for almost every } (x, y) \in \Omega.$$

Note that, by the linearity of the integral, the first inequality above also implies that $\|h_t\|_{L^1(\Omega)} \leq t\|f\|_{L^1(\Omega)} + (1-t)\|g\|_{L^1(\Omega)}$ and thus that also $h_t \in L^1(\Omega)$. Finally, again by the linearity of the integral, we conclude

$$\int_{\Omega} h_t = t \int_{\Omega} f + (1-t) \int_{\Omega} g = 0.$$

This implies that $h_t \in K$ and hence that K is convex.

2, A

- (b) We note that $K \subset L^2(\Omega)$: If $f \in K$, then $f \in L^\infty(\Omega)$ with $\|f\|_{L^\infty(\Omega)} \leq 1$. Since Ω is bounded, this yields that $f \in L^2(\Omega)$.

sim. seen \Downarrow

We now prove that K is closed in $L^2(\Omega)$: Let $\{f_n\} \subset K$ be a sequence such that $f_n \rightarrow f$ in $L^2(\Omega)$, for some $f \in L^2(\Omega)$. We need to prove that $f \in K$: We have

$$\left| \int_{\Omega} f_n - \int_{\Omega} f \right| \leq \int_{\Omega} |f_n - f| \leq |\Omega| \left(\int_{\Omega} |f_n - f|^2 \right)^{\frac{1}{2}}.$$

Since Ω is bounded and $f_n \rightarrow f$ in $L^2(\Omega)$, we conclude that

$$\int_{\Omega} f = \lim_{n \rightarrow \infty} \int_{\Omega} f_n = 0.$$

Alternatively, one could use weak convergence in $L^2(\Omega)$. It remains to prove $|f| \leq 1$ almost everywhere in Ω : This follows from the fact that we may extract a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ that converges to f pointwise almost everywhere in Ω .

4, B

- (c) Since $L^2(\Omega)$ is a Hilbert space with inner product $(f, g) = \int_{\Omega} fg$ and K is a convex and closed subset, we may apply the projection theorem and infer that a unique u exists for every $f \in L^2(\Omega)$.

seen \Downarrow

2, A

- (d) (i) Note that

$$\int_{\Omega} f = 0$$

meth seen \Downarrow

and that $|f(x, y)| \leq |x||y|^{20} \leq 1$ almost everywhere in Ω . Hence $f \in K$ and, by uniqueness, $u = f$.

4, C

- (ii) We note that $|f(x, y)| \leq 1$ almost everywhere in Ω . However

unseen \Downarrow

$$\int_{\Omega} f = \frac{4}{3}.$$

We claim that $\tilde{f}(x, y) = x^2 - \frac{1}{3} \in K$: Indeed, we have that also $|\tilde{f}(x, y)| \leq 1$ almost everywhere in Ω and that $\int_{\Omega} \tilde{f}(x, y) = 0$ by construction. In addition, we note that if $v \in K$ then

$$\int_{\Omega} (\tilde{f} - f)(\tilde{f} - v) = -\frac{1}{3} \int_{\Omega} (\tilde{f} - v) = 0.$$

By uniqueness of the projection, this implies that $\tilde{f} = u$.

8, D

5. (a) It is immediate to see that T is linear. Let $p \in [1; +\infty)$. We have

seen \Downarrow

$$\int_0^1 |T(f)(x)|^p dx = \int_0^1 \left| \int_0^x f(z) dz \right|^p dx$$

so that, by Hölder's inequality, we bound

$$\int_0^1 |T(f)(x)|^p dx \leq \int_0^1 \left(\int_0^x |f(z)|^p dz \right) |x|^{p-1} dx \leq \|f\|_{L^p(0,1)}^p.$$

If $p = \infty$, then we simply have

$$\|T(f)\|_{L^\infty(0,1)} \leq \int_0^1 |f(z)| dz \leq \|f\|_{L^\infty(0,1)}.$$

This implies that for every choice $p \in [1, \infty]$, the operator T is bounded (and hence continuous).

2, M

- (b) Let $f \in L^p(0, 1)$. We note that for every $x, y \in [0, 1]$

sim. seen \Downarrow

$$|T(f)(x) - T(f)(y)| = \left| \int_x^y f(z) dz \right| \leq \|f\|_{L^p(0,1)} |x - y|^{1-\frac{1}{p}}, \quad (10)$$

where in the last inequality we used Hölder's inequality.

The previous inequality implies that $T(f)$ is continuous in $[0, 1]$ and that $[T(f)]_{1-\frac{1}{p}} \leq \|f\|_{L^p(0,1)}$. Thus, $T(f) \in C^{0,1-\frac{1}{p}}([0, 1])$. Since $T(f)(0) = 0$, inequality (10) also implies that for every $x \in [0, 1]$ we have $|T(f)(x)| \leq \|f\|_{L^p(0,1)}$. This yields that for every $f \in L^p(0, 1)$ it holds $\|T(f)\|_{0,1-\frac{1}{p}} \leq 2\|f\|_{L^p(0,1)}$ and thus that $T : L^p(0, 1) \rightarrow C^{0,1-\frac{1}{p}}([0, 1])$ is a bounded and continuous map.

6, M

- (c) We prove that T is compact by showing that the image under T of the closed unit ball $\bar{B} := \{g \in L^2(0, 1), \|g\|_{L^2(0,1)} \leq 1\}$ is precompact in $L^2(0, 1)$. Let $\{g_n\} \subset \bar{B}$ and consider the corresponding sequence $\{T(g_n)\}_{n \in \mathbb{N}}$. By the previous step with $p = 2$, we know that $\{T(g_n)\}_{n \in \mathbb{N}} \subset C^{0,\frac{1}{2}}([0, 1])$. Since $C^{0,\frac{1}{2}}([0, 1]) \subset C^0([0, 1])$ (trivially $\|\cdot\|_{C^0([0,1])} \leq \|\cdot\|_{0,\frac{1}{2}}$), we thus aim at applying Ascoli-Arzelà's theorem to show that there exists a converging subsequence.

meth seen \Downarrow

We prove that $\{T(g_n)\}_{n \in \mathbb{N}}$ is a bounded sequence in $C^0([0, 1])$: Since by part (b) we have that $T : L^2(0, 1) \rightarrow C^{0,\frac{1}{2}}([0, 1])$ is bounded, we infer that $\{T(g_n)\}_{n \in \mathbb{N}}$ is bounded in $C^{0,\frac{1}{2}}([0, 1])$ and thus also in $C^0([0, 1])$.

We now prove that $\{T(g_n)\}_{n \in \mathbb{N}}$ is equicontinuous. This follows immediately by part (b) with $p = 2$: Since $\{g_n\}_{n \in \mathbb{N}} \subset \bar{B}$, for every $n \in \mathbb{N}$, we have that $[T(g_n)]_{\frac{1}{2}} \leq \|g_n\|_{L^2(0,1)} \leq 1$ or, equivalently, that for every $x, y \in [0, 1]$

$$|T(g_n)(x) - T(g_n)(y)| \leq |x - y|^{\frac{1}{2}}.$$

This implies that $\{T(g_n)\}_{n \in \mathbb{N}}$ is equicontinuous (note that it is also uniformly equicontinuous).

Since the conditions for Ascoli-Arzelà's theorem are satisfied, we conclude that there exists a $G \in C^0([0, 1])$ such that (up to a subsequence) $T(g_n) \rightarrow G$ in $C^0([0, 1])$. To conclude, we only need to show that $T(g_n) \rightarrow G$ also in $L^2(0, 1)$. This follows by

$$\|T(g_n) - G\|_{L^2(0,1)}^2 = \int_0^1 |T(g_n) - G|^2 \leq \|T(g_n) - G\|_{C^0([0,1])}^2.$$

We have thus shown that the set $T(\bar{B})$ (namely the image of \bar{B} under T) is precompact in $L^2(0, 1)$. Hence, the operator $T : L^2(0, 1) \rightarrow L^2(0, 1)$ is compact.

12, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96016, MATH97025, MATH97102	1	In part c) the most common mistake is to think that the dominated convergence theorem could be applied. Moreover, there is some confusion with the lebesgue integration over a ``point”: That is necessarily zero, since it is a negligible set.
MATH96016, MATH97025, MATH97102	2	The set $ B_\alpha $ has always finite measure. This follows from the inequality proven in point a).
MATH96016, MATH97025, MATH97102	3	In part b) the most common mistake is to only prove an upper bound for the norm of the functionals. In part d) it is already clear that the only possible limit for the functionals is zero: This follows from the inequalities proven in the previous steps.
MATH96016, MATH97025, MATH97102	4	In part b) there is still some confusion concerning the meaning of being closed in a metric space. In some case the convergence was assumed to be pointwise or L^1 : it needs to be in L^2 , which is the norm/topology that you are considering.
MATH96016, MATH97025, MATH97102	5	There is some confusion in part b). The continuity of T , since T is linear, needs to be proven by showing that the norm of the functional is bounded. In part c) there is some confusion between being compact and finite-rank.