Problem sheet 2 – Q7c,d, solutions

c). We have (see equation (2.12) of the lecture notes – note that you will not need to memorize this equation for the exam):

$$I - I_h = \sum_{\pm} \int_{-\infty \pm ia'}^{\infty \pm ia'} \frac{f(w)}{1 - e^{\mp 2\pi i w/h}} \, dw \qquad (1)$$

where here, for complex w we define

$$f(w) = \frac{e^{-z^2 w^2}}{w^2 + 1}$$

(recall, $z \in \mathbb{R}$ – think of this as being fixed), and $a' \in (-1, 1)$. For convenience, also define

$$H_{\pm}(w) = \frac{f(w)}{1 - e^{\mp 2\pi i w/h}}.$$

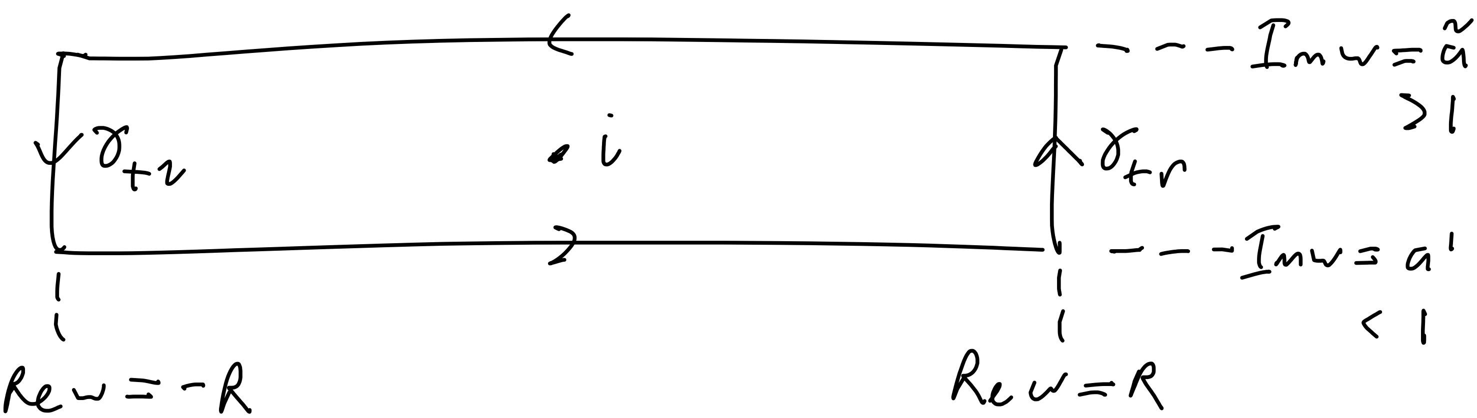
So (1) becomes

$$I - I_h = \sum_{\pm} \int_{-\infty \pm ia'}^{\infty \pm ia'} H_{\pm}(w) \, dw. \qquad (2)$$

Now consider

$$\lim_{R \to \infty} \oint_{\gamma_+ \circlearrowleft} H_+(w)\, dw$$

where γ_+ is the closed rectangular contour sketched here, and we integrate around it in the anticlockwise direction:



The horizontal sides of γ_+ lie along the lines $\operatorname{Im} w = a'$ and $\operatorname{Im} w = \tilde{a}$, for some $\hat{a} > 1$. The vertical sides $\gamma_{+l,r}$ of γ_+ lie along the lines $\operatorname{Re} w = \pm R$, for some $R > 0$, as shown.

$H_+(w)$ is analytic inside γ_+ except for a simple pole at $w = i$ with residue α_+, say. Writing

$$H_+(w) = \underbrace{\frac{e^{-z^2 w^2}}{(w+i)(w-i)}}_{= f(w)} \cdot \frac{1}{1 - e^{-2\pi i w/h}} \qquad (3)$$

we may identify

$$\alpha_+ = \left(\frac{e^{-z^2 w^2}}{w+i} \cdot \frac{1}{1 - e^{-2\pi i w/h}} \right) \Bigg|_{w=i}$$

$$= \frac{e^{z^2}}{2i(1 - e^{2\pi/h})}$$

By the residue theorem, we then have

$$\int_{\gamma_+} H_+(w)\, dw = 2\pi i \alpha_+$$

$$= \frac{\pi e^{z^2}}{1 - e^{2\pi/h}} \qquad (4)$$

Note that (4) holds for all $R > 0$, including as $R \to \infty$. By arguments used in the proof of Thm 2.12 (you will not need to reproduce such arguments in the exam) one can show that

$$\lim_{R\to\infty} \int_{\gamma_{+r}} H_+(w)\,dw = \lim_{R\to\infty} \int_{\gamma_{+l}} H_+(w)\,dw = 0$$

(5).

It then follows from (4) and (5) that

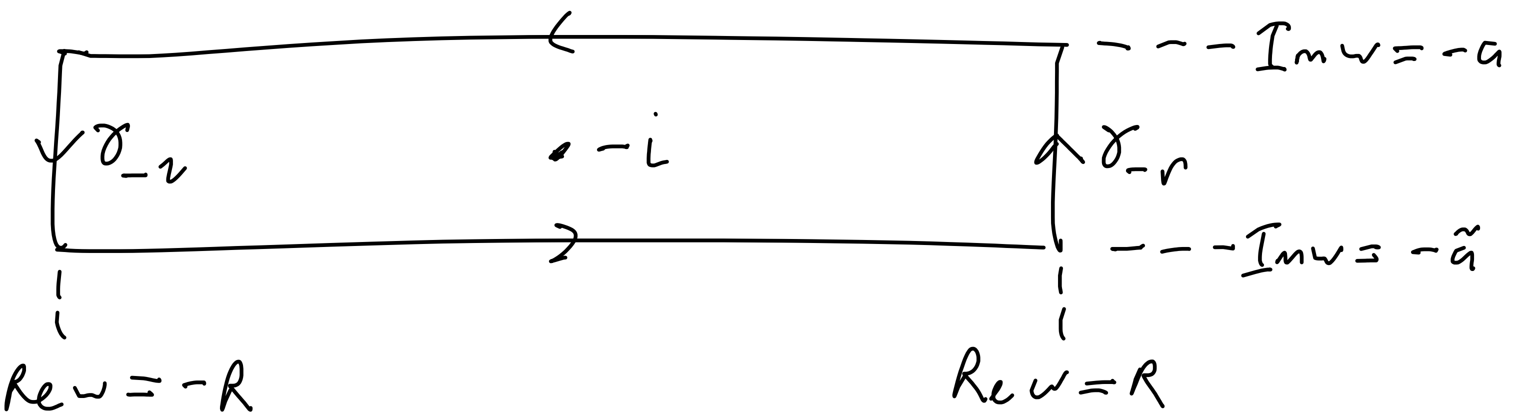
$$\int_{-\infty+ia'}^{\infty+ia'} H_+(w)\,dw + \int_{\infty+i\tilde{a}}^{-\infty+i\tilde{a}} H_+(w)\,dw = \frac{\pi e^{z^2}}{1-e^{2\pi/h}}$$

$$\Rightarrow \int_{-\infty+ia'}^{\infty+ia'} H_+(w)\,dw = \int_{-\infty+i\tilde{a}}^{\infty+i\tilde{a}} H_+(w)\,dw + \frac{\pi e^{z^2}}{1-e^{2\pi/h}}. \quad (6)$$

Next, consider

$$\lim_{R \to \infty} \int_{\underset{\circlearrowleft}{\gamma_-}} H_-(w)\, dw$$

where γ_- is the contour



Note that $H_-(w)$ is analytic inside γ_- except for a simple pole at $w = -i$ with residue:

$$\alpha_- = \left(\frac{e^{-z^2 w^2}}{w - i} \cdot \frac{1}{1 - e^{+2\pi i w / h}} \right) \Bigg|_{w=-i}$$

Note + sign!

$$= \frac{e^{z^2}}{-2i\left(1 - e^{2\pi/h}\right)} .$$

Then, following a similar approach to above, one may deduce that

$$\int_{\infty - ia'}^{-\infty - ia'} H_-(w)\,dw + \int_{-\infty - i\tilde{a}}^{\infty - i\tilde{a}} H_-(w)\,dw$$

$$= 2\pi i \alpha_-$$

$$= \frac{-\pi e^{z^2}}{1 - e^{2\pi/h}}$$

$$\Rightarrow \int_{-\infty - ia'}^{\infty - ia'} H_-(w)\,dw$$

$$= \int_{-\infty - i\tilde{a}}^{\infty - i\tilde{a}} H_-(w)\,dw + \frac{\pi e^{z^2}}{1 - e^{2\pi/h}} . \quad 7$$

Now, 2, 6 and 7 give

$$I - I_h = \frac{2\pi e^{z^2}}{1-e^{2\pi/h}} + \sum_{\pm} \left(\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} H_{\pm}(w)\, dw \right)$$

which we can rearrange into the form

$$\left(I_h + \frac{2\pi e^{z^2}}{1-e^{2\pi/h}} \right) - I = - \sum_{\pm} \left(\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} H_{\pm}(w)\, dw \right)$$

$\left(I_h + \frac{2\pi e^{z^2}}{1-e^{2\pi/h}} \right) = \tilde{I}_h$, say

$$\Rightarrow \quad |\tilde{I}_h - I| \leq \sum_{\pm} \left(\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} |H_{\pm}(w)|\, dw \right) \qquad (8)$$

But recall

$$H_{\pm}(w) = \frac{f(w)}{1-e^{\mp 2\pi i w/h}}$$

Hence
$$|H_{\pm}(w)| = \frac{|f(w)|}{|1-e^{\mp 2\pi i w/h}|}$$

writing $w = u + iv$

$$\leq \frac{|f(w)|}{\left|e^{\pm 2\pi v/h} - 1\right|}$$

$$= \frac{|f(w)|}{e^{2\pi \tilde{a}/h} - 1}$$ for $v = \pm\tilde{a}$, respectively, and as $h \to 0$ (so that $e^{2\pi\tilde{a}/h} > 1$)

So

$$\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} |H_{\pm}(w)|\, dw \leq \frac{1}{e^{2\pi\tilde{a}/h} - 1} \int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} |f(w)|\, dw$$

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And,

$$|f(w)| = \left|\frac{e^{-z^2 w^2}}{w^2+1}\right|$$

$$= \frac{e^{-z^2(u^2-v^2)}}{|w^2+1|}$$

$$= \frac{e^{(zv)^2} \cdot e^{-z^2u^2}}{|w^2+1|}$$

So

$$\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} |f(w)|\, dw = e^{(z\tilde{a})^2} \int_{-\infty}^{\infty} \frac{e^{-z^2u^2}}{|(u \pm i\tilde{a})^2 + 1|}\, du$$

But $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-z^2u^2}}{|(u \pm i\tilde{a})^2+1|}\, du$ is bounded for all $\tilde{a}$, including as $\tilde{a} \to \infty$ (due to the exponential decay of the integrand as $u \to \pm\infty$). Hence

$$\int_{-\infty \pm i\tilde{a}}^{\infty \pm i\tilde{a}} |f(w)|\, dw = O\left(e^{(z\tilde{a})^2}\right). \quad (10)$$

It then follows from (8), (9) and (10) that

$$|\tilde{I}_h - I| = O\left(\exp\left(-\frac{2\pi\tilde{a}}{h} + (z\tilde{a})^2\right)\right)$$

as $h \to 0$.

This is the discretisation error.

Now determine the value of $\tilde{a}$ that minimises the discretisation error. First let

$$\phi(\tilde{a}) = -\frac{2\pi\tilde{a}}{h} + (z\tilde{a})^2 .$$

Then

$$\phi'(\tilde{a}) = -\frac{2\pi}{h} + 2z^2\tilde{a}$$

so $\phi'(\tilde{a}) = 0$ if

$$\tilde{a} = \frac{\pi}{z^2 h}$$

$$= \tilde{a}^*, \quad \text{say}.$$

This value of $\tilde{a}$ maximises $\phi(a)$ ($\phi(a)$ is a quadratic in $\tilde{a}$ with positive coefficient of $\tilde{a}^2$).

And

$$\phi(\tilde{a}^*) = -\frac{2\pi}{h}\cdot\left(\frac{\pi}{z^2 h}\right) + \left(\frac{\pi}{zh}\right)^2$$

$$= -\left(\frac{\pi}{zh}\right)^2$$

$\Rightarrow$ The minimum discretisation error is

$$|\tilde{I}_h - I| = O\left(\exp\left(-\left(\frac{\pi}{2h}\right)^2\right)\right) \qquad 11$$

as $h \to 0$.

Notice that this is a 'faster' rate of convergence than that for $|I_h - I|$ which (as found in part (a)) is $O(\exp(-2\pi a/h))$ where $a<1$ (for this, there is only a factor of $\frac{1}{h}$ in the exponential rather than $\frac{1}{h^2}$).

Now consider the truncation error. It follows from Lemma 2.13 that, since $f(t) = e^{-\tau^2 t^2}/(t^2+1)$ $(t \in \mathbb{R})$ decays exponentially as $t \to \pm\infty$, then

$$|I_h - I_h^{[N]}| = O(f(hN))$$

exponential decay dominates algebraic

$$= O\left(\exp\left(-(zhN)^2\right)\right).$$

Balancing this with the discretisation error given by (1) gives

$$-(zhN)^2 = -\left(\frac{\pi}{zh}\right)^2$$

$$\Rightarrow \quad zhN = \frac{\pi}{zh}$$

$$\Rightarrow \quad h^2 = \frac{\pi}{z^2 N}$$

$$\Rightarrow \quad h = \frac{1}{z}\sqrt{\frac{\pi}{N}}$$

gives the optimal convergence rate, which is

$$|I - I_h^{[N]}| = O\left(\exp(-N\pi)\right) \text{ as } N \to \infty.$$