Imperial College London
Department of Mathematics

Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2021/22

You have 1h. You should attempt all questions.

1. Let $V = \mathbb{R}^3$ and $T : V \to V$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_2 \\ -x_1 - x_2 + x_3 \end{pmatrix}.$$

Let $E = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Let $A = [T]_E$.

a. Write down A, explaining your answer. (2 marks)

b. Is the matrix A orthogonal? Explain your answer. (2 marks)

c. Find the eigenvectors of T. (6 marks)

d. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix. (1 mark)

e. Write e_1 as a linear combination of the eigenvectors of T, then express $T^n(e_1)$ as a linear combination of e_1, e_2, e_3 for all $n \in \mathbb{N}$. (T^0 is the identity transformation.) (4 marks)

f. Find a matrix $C \in M_3(\mathbb{R})$ such that $C^2 = A$. (5 marks)

(Total: 20 marks)

2. **a.** Let $n \in \mathbb{N} \setminus \{0\}$. Let $A_n = (a_{ij}) \in M_n(\mathbb{Q})$ be defined by

$$a_{ij} = \begin{cases} \binom{i-1}{j-1} & j \leq i \\ 0 & j > i. \end{cases}$$

(Recall that $\binom{r}{0} = 1$ for all $r \in \mathbb{N}$ (including $r = 0$)).

i. Write down A_n for $n = 1, 2, 3$. (2 marks)

ii. Compute $\det(A_n)$ for all $n \in \mathbb{N} \setminus \{0\}$. (2 marks)

iii. Find the eigenvectors of A_n for all $n \in \mathbb{N} \setminus \{0\}$. (3 marks)

iv. For which values of n is A_n diagonalisable? Explain your answer. (2 marks)

b. For each of the following statements, say whether it is true or false. If it is true, give a short proof; if it is false, give a counterexample.

i. Let $n \in \mathbb{N} \setminus \{0\}$ and let V be an n-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. Assume $\chi_T(X) = (X - \lambda)^n$ for some $\lambda \in F$. Then T is diagonalisable if and only if $T = \lambda \cdot id$, where $id : V \to V$ is the identity transformation. (3 marks)

ii. Let $n \in \mathbb{N} \setminus \{0\}$ and $A = (a_{ij}) \in M_n(\mathbb{Q})$ be an upper-triangular matrix (that is $a_{ij} = 0$ for $j < i$). Assume that A is invertible. Then A^{-1} is upper-triangular.

(3 marks)

iii. Let $n \in \mathbb{N} \setminus \{0\}$ and $A \in M_n(\mathbb{R})$. Then there is a real symmetric matrix $A_s \in M_n(\mathbb{R})$ and a matrix $A_a \in M_n(\mathbb{R})$, such that $A_a^T = -A_a$ and

$$A = A_s + A_a.$$

(2 marks)

iv. Let $n \in \mathbb{N} \setminus \{0\}$ and F be a field. Let $A, B \in M_n(F)$. If A and B are diagonalisable, then there is an invertible $P \in M_n(F)$ such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

(3 marks)

(Total: 20 marks)