IMPERIAL

MATH70057

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Riemannian Geometry

Date: Friday, May 10, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Let (M, g) be a Riemannian manifold.

(a) Define the length $\ell(\gamma)$ of a piecewise smooth curve $\gamma : [a, b] \to M$.

(3 marks)

(b) Let Σ be a closed submanifold of M and $p \in M \setminus \Sigma$. Define the distance $\operatorname{dist}(p, \Sigma)$.

(3 marks)

(c) Show, using the first variation formula, that if γ is a piecewise smooth curve such that $\ell(\gamma) = \operatorname{dist}(\gamma(a), \Sigma)$ then γ must be a smooth geodesic orthogonal to Σ.

(6 marks)

(d) A geodesic $\gamma : [0, +\infty) \to M$ is called a *ray* if $\ell(\gamma|_{[a,b]}) = \operatorname{dist}(\gamma(a), \gamma(b))$, for all $0 \leq a \leq b$. Show that if (M, g) is complete and non-compact then it contains a ray.

(8 marks)

(Total: 20 marks)

2. Let $f : (-\varepsilon, \varepsilon) \times [0,a] \to M$ be a smooth map representing a variation of closed curves (i.e., $f(s,0) = f(s,a)$ for all s and $\frac{\partial f}{\partial t}(0,0) = \frac{\partial f}{\partial t}(0,a)$). Denote $\ell : [0,a] \to \mathbb{R}$ the function

$$\ell(s) = \int_0^a \left\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t) \right\rangle^{1/2} dt.$$

(a) Assume that $t \mapsto \frac{\partial f}{\partial t}(0,t)$ is a smooth closed geodesic parametrised by arc length and that the variation is orthogonal to the curve, i.e. $\frac{\partial f}{\partial s}(0,t) \perp \frac{\partial f}{\partial t}(0,t)$. Prove that

$$\frac{d^2}{ds^2}\ell(s)|_{s=0} = \int_0^a \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle + \left\langle R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt.$$

(10 marks)

(b) Use the formula, as well as the results seen in the lectures, to prove that a closed orientable surface with positive curvature must be a sphere.

(10 marks)

(Total: 20 marks)

3. (a) Define what it means for a map $f : (M, g) \to (M, g)$ to be an isometry of M.

(3 marks)

(b) What does it mean to say that a connection is symmetric and compatible with the metric?

(3 marks)

(c) Deduce Koszul's formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) = X\left(g(Y,Z)\right) + Y\left(g(X,Z)\right) - Z\left(g(X,Y)\right) \\ - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

Hint: you can save time by first finding the main relevant symmetry.

(6 marks)

(d) Let (M^n, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function. Let $\tilde{g}$ be a new Riemannian metric on M given by $\tilde{g} = e^{2f} g$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of (M, g) and $(M, \tilde{g})$, respectively. Show that, for $X, Y \in \chi(M)$, it holds:

$$\widetilde{\nabla}_X Y = \nabla_X Y + df(X)Y + df(Y)X - g(X,Y)\nabla f.$$

(4 marks)

(e) Show that if $\dim M$ is odd and f is an orientation preserving isometry with a fixed point, then M admits a geodesic of fixed points, i.e. $\gamma : [0, a] \to M$ such that $f(\gamma(t)) = \gamma(t)$, for all $t \in [0, a]$. Hint: remember that an orthogonal transformation of an odd dimensional vector space always has a fixed point.

(4 marks)

(Total: 20 marks)

4. (a) Let $U \subset M$ be a proper open subset of a complete Riemannian manifold (M, g), i.e. $M \setminus U \neq \emptyset$. Show that U is not geodesically complete with respect to the inherited metric.

(4 marks)

(b) State the definitions of Jacobi vector field and conjugate points.

(4 marks)

(c) Let J be a Jacobi vector field along a geodesic parametrised by arc length $\gamma : [0, a] \to M$.

(i) Show that $t \mapsto \langle J'(t), \gamma'(t) \rangle$ is a constant function.

(2 marks)

(ii) Assume $J(0) = 0$ and $J'(0) \perp \gamma'(0)$. Show that

$$\frac{d^2}{dt^2}|J(t)| = 2\{|J'(t)|^2 - K(\{J(t), \gamma'(t)\})|J(t)|^2\},$$

where $K(\{J(t), \gamma'(t)\})$ denotes the sectional curvature of the plane containing both $J(t)$ and $\gamma'(t)$.

(3 marks)

(d) Let (M, g) be a Riemannian manifold with negative sectional curvature and $p \in M$. Prove that p does not have conjugate points in M.

(3 marks)

(e) Let (M, g) be a complete Riemannian manifold with negative sectional curvature. Let $p \in M$ and $f = \exp_p : T_pM \to M$. Show that (T_pM, f^*g) is a complete Riemannian manifold.

(4 marks)

(Total: 20 marks)

5. Let $B^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $f : B^2 \to \mathbb{R}$ a radially symmetric function i.e. $f(x) = f(|x|)$. We define a new metric on B^2 by $\tilde{g}(v, w)_x = e^{2f(x)}\langle v, w\rangle$

(a) Denote by $X(r, \theta) = r(\cos\theta, \sin\theta)$ for $(r, \theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Denoting with subindices the derivatives with respect to r and θ, show that

$$X_{rr} = 0$$
$$X_{r\theta} = X_{\theta r} = \frac{X_\theta}{r}$$
$$X_{\theta\theta} = -rX_r.$$

(5 marks)

(b) Show that:

$$\widetilde{\nabla}_{X_\theta}\widetilde{\nabla}_{X_r}X_r = \left(\frac{f'}{r} + (f')^2\right)X_\theta$$
$$\widetilde{\nabla}_{X_r}\widetilde{\nabla}_{X_\theta}X_r = \left(f'' + \frac{2f'}{r} + (f')^2\right)X_\theta$$

Hint: you may use, without proof, the formula in (3)(d).

(5 marks)

(c) Show that the sectional curvature in the new metric is given by

$$\widetilde{K}(\{X_\theta, X_r\}) = -e^{-2f}\left(\frac{f'}{r} + f''\right).$$

(5 marks)

(d) Finally, conclude that if $f = \log(\frac{2}{1-r^2})$ then $\widetilde{K} \equiv -1$.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH70057

Riemannian Geometry (Solutions)

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1. (a) $\ell(\gamma) = \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \sqrt{g(\gamma'(t), \gamma'(t))} dt$, where $a = t_0 < t_1 \cdots < t_k = b$ are the singular points of γ.

(b) $\operatorname{dist}(p, \Sigma) = \inf\{\ell(\gamma) : \gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) \in \Sigma\}$.

(c) Let $f : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a smooth map such that $f(0,t) = \gamma(t)$, $\gamma(s,a) = \gamma(a)$, $\gamma(s,b) \in \Sigma$, for all $(s,t) \in (-\varepsilon, \varepsilon) \times [a,b]$. Without loss of generality we can assume that $|\frac{\partial f}{\partial t}(0,t)| = |\gamma'(t)| = 1$, for all t. Under these conditions, and because we are assuming γ is minimising, we have:

$$0 = \frac{d}{ds}\bigg|_{s=0} \ell(s) = \sum_{i=1}^{k} \langle V(t), \gamma(t_+) - \gamma(t_-)\rangle \bigg|_{t=t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \langle V(t), \nabla_{\gamma'(t)}\gamma'(t)\rangle dt,$$

where $V(t) = \frac{\partial f}{\partial s}(0,t)$. This formula holds for all vector fields V along γ such that $V(0) = 0$ and $V(a) \perp \Sigma$.

First we can consider $V(t) = \rho(t)\nabla_{\gamma'}\gamma'(t)$, where $\rho(t) > 0$ and smooth, except at the singular points of γ. In this case, we obtain $\int_{t_i}^{t_{i+1}} \rho(t)|\nabla_{\gamma'(t)}\gamma'(t)|^2 dt = 0$, which can only happens if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for all t. Therefore, the general first variation formula reduces to $0 = \sum_{i=1}^{k} \langle V(t), \gamma(t_+) - \gamma(t_-)\rangle \bigg|_{t=t_i}^{t_{i+1}}$. Since we can choose a variation with arbitrary values for all $V(t_i)$'s, we conclude that $\gamma'(t_i^+) = \gamma'(t_i^-)$, so γ must be smooth.

(d) Let $p \in M$. Since M is complete and non-compact there exists a sequence $x_n \in M$ such that $\operatorname{dist}(p, x_n) = T_n \to +\infty$. Moreover, such distance is realised by a geodesic $\gamma_n : [0, T_n] \to M$, which without loss of generality we assume to be parametrised by unit speed. Since $\gamma_n(0) = 0$ and $v_n = \gamma_n'(0) \in T_pM$ is a unitary vector, we can assume, after passing to a subsequence if necessary, that $v_n \to v \in T_pM$ where $|v| = 1$. Let γ be the geodesic with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$. By the smooth dependence on the initial conditions, it follows that $\gamma_n \to \gamma$ on any compact interval $[0,T]$ and the convergence is smooth. Let $q_n = \gamma_n(T)$ and $q = \gamma(T)$. Then, by the triangle inequality $\ell(\gamma_n|_{[0,T]}) = \operatorname{dist}(p, q_n) \le \operatorname{dist}(p,q) + \operatorname{dist}(q, q_n)$. From the smooth dependence we obtain $\ell(\gamma_n|_{[0,T]}) \to \ell(\gamma|_{[0,T]})$ and $\operatorname{dist}(q, q_n) \to 0$. We conclude

$$\ell(\gamma|_{[0,T]}) \le \operatorname{dist}(p,q),$$

which must be an equality by the definition of distance.

2. (a)

$$\begin{aligned}
\frac{d^2}{ds^2}E(s)\Big|_{s=0} &= \frac{d}{ds}\int_0^a \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle^{-1/2} \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle\, dt\Big|_{s=0} \\
&= \int_0^a -\frac{1}{2}\left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle^2 + \left\langle \frac{D}{ds}\frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle + \left|\frac{D}{ds}\frac{\partial f}{\partial t}\right|^2 dt \\
&= \int_0^a -\frac{1}{2}\left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle^2 + \left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left|\frac{D}{dt}\frac{\partial f}{\partial s}\right|^2 dt
\end{aligned}$$

Regarding the first term, note that $\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle = \langle V, \gamma'\rangle' - \langle V, \gamma''\rangle$ is zero. In fact, the first term vanishes because we are assuming $V \perp \gamma'$ and the second because γ is a geodesic.

For the second term we have

$$\begin{aligned}
\left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle &= \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \\
&= \frac{d}{dt}\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma'\right\rangle - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \gamma''\right\rangle + \langle R(V,\gamma')V, \gamma'\rangle.
\end{aligned}$$

Note that the first term in this expression vanishes after integration because of the fundamental theorem of calculus and the fact that the variation is cyclic. The second term is automatically zero since $\gamma'' = 0$.

Finally, we obtain the desired formula after replacing the terms in the computation above.

(b) If M is not a sphere we know $\pi_1(M) \neq 0$ and we saw in the lectures this implies the existence of a smooth closed minimising geodesic.

Let $\gamma : [0, a] \to M$ be a closed embedded geodesic, then $\gamma(0) = \gamma(a) = p$ and $\gamma'(0) = \gamma'(a) = v$. In this case we can consider a variation such that $f(s, 0) = f(s, a)$ and $\frac{\partial f}{\partial s}(0, t) = V(t)$ is a parallel vector field with $|V(t)| \equiv 1$ and $V(t) \perp \gamma'(t)$. Since the geodesic is minimising, then using the formula from the previous item we have

$$0 \leq \frac{d^2}{ds^2}\ell(0) = \int_0^a |V'(t)|^2 + \langle R(V,\gamma')V, \gamma'\rangle\, dt$$

Finally, $|V'(t)| = 0$ and $\langle R(V,\gamma')V,\gamma'\rangle = -K(\sigma) < 0$ is (minus) the sectional curvature of the plane $\sigma = \mathrm{span}\{V, \gamma'\}$. It follows, $0 \leq \frac{d^2}{ds^2}E(0) < 0$, which is a contradiction.

3. (a) We say that f is an isometry if f is a diffeomorphism such that $\langle v, w\rangle = \langle df_p(v), df_p(w)\rangle$, for all $p \in M$ and all $v, w \in T_pM$.

   (b) Symmetric means $\nabla_X Y - \nabla_Y X = [X, Y]$, for all $X, Y \in \chi(M)$ and compatible with the metric means $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

   (c) The relevant symmetry comes from combining both symmetry and compatibility with the metric, i.e. $\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle$. This induces a cyclic permutation of the roles of (X, Y, Z) while at the same time it changes the sign. So after three iterations we return to the same element with the right sign.

   (d) This follows directly from Koszul's formula. In fact, substracting the formulas for each connection we get:

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_X Y, Z) - 2e^{2f} g(\widetilde{\nabla}_X Y, Z) &= X(e^{2f})g(Y, Z) - Z(e^{2f})g(X, Y) + Y(e^{2f})g(Z, X) \\ &= 2e^{2f}\Big(X(f)g(Y, Z) - Z(f)g(X, Y) + Y(f)g(Z, X)\Big). \end{aligned}$$

   Dividing by $2e^{2f}$ and rearranging the terms we get

$$\begin{aligned} g(\widetilde{\nabla}_X - \widetilde{\nabla}_X Y, Z) &= g(df(X)Y, Z) - g(g(X, Y)\nabla f, Z) + g(df(Y)X, Z) \\ &= g(df(X)Y + df(Y)X - g(X, Y)\nabla f, Z). \end{aligned}$$

   Since this must hold for any Z we conclude the proof of the claim.

   (e) Let $p \in M$ be a fixed point of f, i.e. $f(p) = p$. Then $df_p : T_pM \to T_pM$ is an orientation preserving orthonormal transformation of T_pM onto itself. Since $\dim T_pM$ is odd, it follows that there exists $v \in T_pM \setminus \{0\}$ such that $df_p(v) = v$. Let $\gamma(t) = \exp_p(tv)$ be the geodesic starting at p with velocity v. Since f is an isometry, then $\tilde{\gamma}(t) = f(\gamma(t))$ is also a geodesic of M. Since, $\tilde{\gamma}(0) = f(\gamma(0)) = f(p) = p = \gamma(0)$ and $\tilde{\gamma}'(0) = df_p(\gamma'(0)) = df_p(v) = v = \gamma'(0)$, it follows that $f(\gamma(t)) = \tilde{\gamma}(t) = \gamma(t)$.

4. (a) Let $x \in \partial U$. Then, for all small $r > 0$, the geodesic ball $B_r(x)$ (with respect to the metric g on M) contains $p \in U$ and $q \in M \setminus U$. Let γ be a geodesic of M joining p and q, i.e. $\gamma(0) = p$ and $\gamma(1) = q$, which exists if $r > 0$ is small enough. Then, for a short time γ is a geodesic of (U, g). If this geodesic could be continued for all time in (U, g) then it would also be a geodesic of M, contradicting the fact that such geodesic leave U infinite time.

(b) A vector field $t \mapsto J(t)$ along a geodesic γ, is a Jacobi vector field if it satisfies $\frac{D}{dt}\frac{D}{dt}J + R(J, \gamma')\gamma' = 0$. We say that $p = \gamma(0)$ and $q = \gamma(a)$ are conjugate, if there exists a non-zero Jacobi vector field along γ such that $J(0) = 0$ and $J(a) = 0$.

(c) (i) Differentiating

$$\begin{aligned} \langle J', \gamma' \rangle' &= \langle J'', \gamma' \rangle + \langle J', \gamma'' \rangle \\ &= -\langle R(J, \gamma')\gamma', \gamma' \rangle + 0 \\ &= 0 \end{aligned}$$

where we have used the symmetries of R to show $\langle R(J, \gamma')\gamma', \gamma' \rangle = 0$ and the fact that γ is a geodesic to conclude $\gamma'' = 0$.

(c) (ii) By the previous item, it follows that $\langle J'(t), \gamma'(t) \rangle = \langle J'(0), \gamma'(0) \rangle = 0$. Therefore,

$$\begin{aligned} (|J(t)|^2)'' &= 2\{|J'(t)|^2 + \langle J''(t), J(t) \rangle\} \\ &= 2\{|J'(t)|^2 - \langle R(J(t), \gamma'(t))\gamma'(t), J(t) \rangle\} \end{aligned}$$

The claim follows because $J(t)/|J(t)|$ and γ' form an orthonormal basis.

(d) Assume J is a non-zero Jacobi vector field along γ, such that $J(0) = J(a) = 0$. Let t_0 be the time where $|J(t)| > 0$ is maximum. Using the formula from the previous item we have

$$0 \geq -K(\{J(t_0), \gamma'(t_0)\})|J(t_0)|^2 > 0,$$

which is a contradiction.

(e) From the previous item, we know that there are no conjugate points on M. In particular, $\exp_p$ is a local diffeomorphism. In this way, the pull-back f^*g is well-defined and $\exp_p : (T_pM, f^*g) \to (M, g)$ is a local isometry. The curves $t \mapsto f(tv)$ are geodesics on M which exist for all time, given that M is complete. Since f is a local isometry, it follows that $t \mapsto tv$ are geodesics in (T_pM, f^*g) which exist for all time. From Hopf-Rinow, it follows that (T_pM, f^*g) is complete.

5. (a) Differentiating the formula of $\phi(r,\theta)$ we get:

$$\begin{aligned} X_r &= (\cos\theta, \sin\theta) \\ X_\theta &= r(-\sin\theta, \cos\theta). \end{aligned}$$

Differentiating once more one obtains:

$$\begin{aligned} X_{rr} &= 0 \\ X_{r\theta} &= X_{\theta r} = \frac{X_\theta}{r} \\ X_{\theta\theta} &= -rX_r. \end{aligned}$$

Note also that $\nabla f = f' X_r$, $df(X_r) = f'(r)$ and $df(X_\theta) = 0$.

(b) Now we can apply the formula from Problem 4 to obtain

$$\begin{aligned} \widetilde{\nabla}_{X_r} X_r &= 0 + 2df(X_r)X_r - |X_r|^2 \nabla f \\ &= 2f'X_r - f'X_r \\ &= f'X_r \end{aligned}$$

$$\begin{aligned} \widetilde{\nabla}_{X_\theta} X_r &= \frac{X_\theta}{r} + 0 + df(X_r)X_\theta + 0 \\ &= \frac{X_\theta}{r} + f'X_\theta \\ &= \left(\frac{1}{r} + f'\right) X_\theta. \end{aligned}$$

$$\begin{aligned} \widetilde{\nabla}_{X_\theta} X_\theta &= -rX_r + 0 + 0 - r^2 f' X_r \\ &= -r^2 \left(\frac{1}{r} + f'\right) X_r \end{aligned}$$

Differentiating once more we get:

$$\begin{aligned} \widetilde{\nabla}_{X_\theta} \widetilde{\nabla}_{X_r} X_r &= \widetilde{\nabla}_{X_\theta} (f'X_r) \\ &= f' \widetilde{\nabla}_{X_\theta} X_r \\ &= \left(\frac{f'}{r} + (f')^2\right) X_\theta \end{aligned}$$

$$\begin{aligned} \widetilde{\nabla}_{X_r} \widetilde{\nabla}_{X_\theta} X_r &= \widetilde{\nabla}_{X_r} \left\{ \left(\frac{1}{r} + f'\right) X_\theta \right\} \\ &= \left(-\frac{1}{r^2} + f''\right) X_\theta + \left(\frac{1}{r} + f'\right)^2 X_\theta \\ &= \left(f'' + \frac{2f'}{r} + (f')^2\right) X_\theta \end{aligned}$$

(c) Subtracting the expressions, it follows

$$\widetilde{R}(X_\theta, X_r)X_r = -\left(\frac{f'}{r} - f''\right) X_\theta$$

and
$$\widetilde{R}(X_\theta, X_r, X_r, X_\theta) = -e^{2f}\left(rf' - r^2 f''\right).$$

Since $\tilde{g}(X_\theta, X_\theta) = e^{2f}r^2$ and $\tilde{g}(X_r, X_r) = e^{2f}$, we obtain

$$\widetilde{K}(\{X_\theta, X_r\}) = -e^{-2f}\left(\frac{f'}{r} + f''\right).$$

(d) $f = \log\left(\frac{2}{1-r^2}\right)$ differentiating we obtain

$$\begin{aligned} f' &= \frac{2r}{1-r^2} \\ f'' &= \frac{2}{1-r^2} + \frac{4r^2}{(1-r^2)^2} \\ e^{-2f} &= \frac{(1-r^2)^2}{4} \end{aligned}$$

which, substituting in the formula above, implies

$$\widetilde{K}(\{X_\theta, X_r\}) = -1.$$