1. (a) Here, we consider the following initial value problem

$$\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} + u = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = \frac{1}{1+x^2}$$

(i) Find the equation of the characteristics for this PDE. Draw them in the (x,t)-plane. (4 marks)

(ii) Find an explicit solution $u(x,t)$ to this problem. (4 marks)

(b) In an infinitely long one-dimensional telecommunication cable, the voltage $u(x,t)$ is governed by the following partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda\frac{\partial u}{\partial t} + \lambda^2 u = c^2\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0$$

where λ and c are positive real constants. Assume that the initial conditions are given by

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

(i) What are the dimensions of λ and c? (2 marks)

(ii) Let $y(x,t) = e^{\lambda t}u(x,t)$ and show that $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

(4 marks)

(iii) Find the explicit solution to the original initial value problem. You will express your solution for $u(x,t)$ in terms of λ, c and $f(x)$. Hint: You may use d'Alembert's solution to the wave equation without rederiving it. (6 marks)

(Total: 20 marks)

2. In this question, we consider the following traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$\rho(x,0) = \rho_0(x)$$

where the wave velocity is given by Greenshield's law: $c(\rho) = v_m\,(1 - 2\rho/\rho_m)$, with $v_m = 2$ and $\rho_m = 4$.

Here, we assume that the initial linear density of cars is given by the following piecewise continuous function

$$\rho_0(x) = \begin{cases} 0, & x < -1 \\ 2(1+x), & -1 < x < 0 \\ 1, & x > 0 \end{cases}$$

(a) Find the equations of the characteristics for this problem. Draw them in the (x,t)-plane and point out any fan regions. Show that a shock forms at $t = 1/2$. (4 marks)

(b) Find an explicit solution valid for $0 < t < 1/2$. (4 marks)

(c) Find the explicit solution after the shock has formed. To do so, you will need to determine the explicit solution for the shock path. Draw an amended diagram of characteristics including the shock path. Until when is the solution you just obtained valid? Justify your reasoning. (7 marks)

(d) Sketch on the same graph the solution for: (i) $t = 0$, (ii) $t = 1/2$, (iii) $t = 1$, and (iv) $t = 3$. Clearly label your plots and important values your solutions take. (5 marks)

(Total: 20 marks)

3. (a) A perfectly elastic and flexible string of length L with linear density ρ under uniform tension τ is stretched horizontally and fixed at both ends. Here, we assume that the string is plucked, i.e. the string is vertically displaced and at $t = 0$, it is released with zero velocity.

(i) Show that the vertical displacement in the string is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left[\frac{n\pi x}{L}\right] \cos\left[\frac{n\pi ct}{L}\right]$$

where $c = \sqrt{\tau/\rho}$ is the wavespeed and c_n are real constants. (5 marks)

(ii) Determine the constants c_n in the case where the string is plucked at a third of its length by an amount A. The initial conditions of this problem are given by

$$u(x,0) = \begin{cases} 3Ax/L, & 0 < x < L/3 \\ 3A(L-x)/(2L), & L/3 < x < L \end{cases} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < L$$

(6 marks)

(iii) We now go back to the general solution to the vibrating string problem obtained in (a)(i). Suppose that the string is further constrained at its midpoint, such that $u(L/2,t) = 0$ for all t. What condition does this impose on the coefficients c_n? No work is needed but you should briefly justify your answer. (2 marks)

(b) We now consider a two dimensional square metal plate $0 < x < 1$, $0 < y < 1$. The steady-state temperature in the plate satisfies Laplace's equation. Here, we assume that: (1) the edge at $y = 0$ is maintained at a constant temperature $T > 0$, (2) the edge at $y = 1$ is maintained at zero temperature and (3) the other two edges (namely, in $x = 0$ and $x = 1$) are thermally insulated. Write down the PDE problem that you need to solve. What is the steady-state temperature $u(x,y)$ in the plate?

(7 marks)

(Total: 20 marks)

4. (a) First, consider that the linear density of a chemical is governed by the following diffusion problem

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0$$
$$u(x,0) = 0, \quad x > 0 \quad ; \quad u(x,t) \to 0, \quad x \to \infty, \quad t \geq 0 \quad ; \quad u(0,t) = u_0, \quad t \geq 0$$

We consider the transformation: $\{\tilde{x} = ax, \tilde{t} = a^\beta t, \tilde{u} = a^\gamma u\}$ where a is a positive real constant.

(i) For what value of β is the diffusion equation invariant under this transformation?

(3 marks)

(ii) If our PDE problem is invariant under the above transformation, we know that a similarity solution is of the form $u(x,t) = t^{\gamma/\beta} f(xt^{-1/\beta})$. Use the boundary conditions to find the value of γ which leaves the entire PDE problem invariant under the transformation.

(3 marks)

(iii) Show that this PDE problem can be reduced to solving the following ODE boundary value problem

$$f''(\eta) + \frac{\eta}{2D} f'(\eta) = 0$$
$$f(0) = u_0 \quad \text{and} \quad f(\eta) \to 0, \eta \to +\infty$$

where $\eta = x/\sqrt{t}$.

(3 marks)

(b) The evolution of the velocity field $v(r,t)$ in a two dimensional fluid vortex is governed by the following PDE in polar coordinates

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (rv)}{\partial r}\right]$$

where r is the radial distance to the center of the vortex. We will assume that the initial conditions are given by $v(r,0) = 1/r$. Here, we derive a bounded solution to this problem using similarity arguments.

(i) Show that this problem is invariant under the transformation

$$\tilde{r} = ar, \quad \tilde{t} = a^\beta t, \quad \tilde{v} = a^\gamma v$$

for any positive real constant a and for $\beta = 2$ and $\gamma = -1$.

(4 marks)

(ii) Show that a similarity solution to this problem can be written $v(r,t) = g(\xi)/r$ with $\xi = r^2/t$. Deduce that a bounded solution to this problem is given by

$$v(r,t) = \frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{4t}\right)\right]$$

(7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH50008

PDEs in Action (Solutions)

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1. (a) In this problem, we consider the following initial value problem

sim. seen ⇓

$$\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} + u = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = \frac{1}{1+x^2}$$

seen ⇓

(i) The method of characteristics tells us that

$$\frac{du}{dt} = -u \quad \text{on} \quad \frac{dx}{dt} = -2.$$

The characteristics for this PDE satisfy the following ODE

$$\frac{dx}{dt} = -2 \Rightarrow \int dx = \int -2dt \Rightarrow x = c_1 - 2t$$

where we have assumed that $x(0) = c_1$. So the characteristics of this problem form an array of straight lines with slope $-1/2$ each passing through the point $(c_1, 0)$ (see Fig. 1).

3, A

1, A

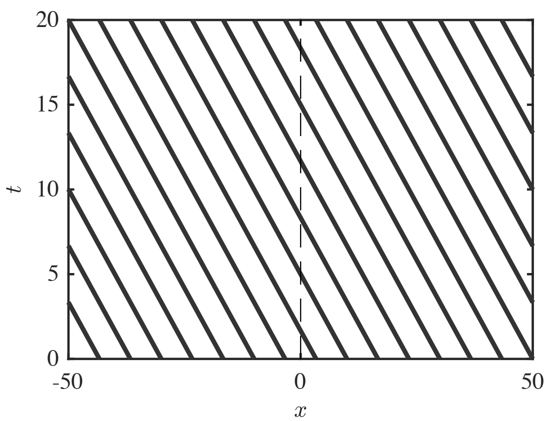

Figure 1: Characteristics associated with $u_t - 2u_x + u = 0$.

seen ⇓

(ii) To find an explicit solution to this problem, we integrate the following ODE to find the value of $u(x,t)$ along the characteristic lines

$$\frac{du}{dt} = -u \Rightarrow \int \frac{du}{u} = -\int dt \Rightarrow \log u = -t + c_2'$$

which means that $u = c_2 e^{-t}$, where c_2 is a constant. In particular, on the characteristics, we know that the constant are linked by $c_2 = f(c_1)$ so that we can then write the general solution as

$$u(x,t) = f(x+2t)e^{-t}$$

where f is an arbitrary differentiable function.
Now the ICs give us that $u(x,0) = 1/(1+x^2)$, which means that $f(s) = 1/(1+s^2)$. Therefore, the solution is

$$u(x,t) = \frac{1}{1+(x+2t)^2}e^{-t}$$

4, A

(b) (i) By dimensional homogeneity, we write that $\boxed{\text{meth seen} \Downarrow}$

$$\left[\frac{\partial^2 u}{\partial t^2}\right] = [\lambda^2 u] \Rightarrow [\lambda]^2[u] = [u]T^{-2} \Rightarrow [\lambda] = T^{-1}$$
$$\left[\frac{\partial^2 u}{\partial t^2}\right] = \left[c^2\frac{\partial^2 u}{\partial x^2}\right] \Rightarrow [c]^2[u]L^{-2} = [u]T^{-2} \Rightarrow [c] = LT^{-1}$$

$\boxed{2, \text{A}}$

(ii) Let $y(x,t) = u(x,t)e^{\lambda t}$, we can write that $\boxed{\text{unseen} \Downarrow}$

$$\frac{\partial y}{\partial t} = \frac{\partial u}{\partial t}e^{\lambda t} + \lambda e^{\lambda t}u$$
$$\frac{\partial^2 y}{\partial t^2} = e^{\lambda t}\left[\frac{\partial^2 u}{\partial t^2} + 2\lambda\frac{\partial u}{\partial t} + \lambda^2 u\right]$$

Similarly, we obtain

$$\frac{\partial^2 y}{\partial x^2} = e^{\lambda t}\frac{\partial^2 u}{\partial x^2}$$

leading to

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = e^{\lambda t}\left[\overbrace{\frac{\partial^2 u}{\partial t^2} + 2\lambda\frac{\partial u}{\partial t} + \lambda^2 u - c^2\frac{\partial^2 u}{\partial x^2}}^{0}\right] = 0$$

where we have used the fact that $u(x,t)$ satisfies the original PDE. So y indeed satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

$\boxed{4, \text{A}}$

(iii) Before we can write an explicit solution to this problem, we need to determine the initial conditions for $y(x,t)$. When $t = 0$, we write $\boxed{\text{meth seen} \Downarrow}$

$$y(x,0) = u(x,0)e^0 = f(x).$$

We also have to consider the time derivative of y which reads

$$\frac{\partial y}{\partial t} = \lambda e^{\lambda t}u + e^{\lambda t}\frac{\partial u}{\partial t}.$$

and gives in $t = 0$,

$$\frac{\partial y}{\partial t}(x,0) = \lambda f(x).$$

D'Alembert's solution for the wave equation thus gives us here:

$$y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{\lambda}{2c}\int_{x-ct}^{x+ct} f(s)ds$$

and we finally conclude that we can express the solution in terms of λ, c and $f(x)$ as

$$u(x,t) = \frac{e^{-\lambda t}}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{\lambda e^{-\lambda t}}{2c}\int_{x-ct}^{x+ct} f(s)ds$$

$\boxed{6, \text{B}}$

2. Recall that we consider the following traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$\rho(x,0) = \rho_0(x)$$

where the wave velocity is given by Greenshield's law: $c(\rho) = v_m\,(1 - 2\rho/\rho_m)$, with $v_m = 2$ and $\rho_m = 4$.

Here, we assume that the initial linear car density field is given by

$$\rho_0(x) = \begin{cases} 0, & x < -1 \\ 2(1+x), & -1 < x < 0 \\ 1, & x > 0 \end{cases}$$

meth seen ⇓

(a) The method of characteristics gives us that

$$\frac{d\rho}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = c(\rho), \quad x(0) = \xi$$

which means that

$$\rho = \rho_0(\xi) \quad \text{on} \quad x = c(\rho_0(\xi))t + \xi$$

So based on the initial conditions, we obtain the following equation for the characteristics

$$\begin{cases} \text{I} — \xi < -1: & x = 2\left[1 - \frac{0}{2}\right]t + \xi = 2t + \xi \\ \text{II} — \ -1 < \xi < 0: & x = 2\left[1 - \frac{2(1+\xi)}{2}\right]t + \xi = -2\xi t + \xi \\ \text{III} — \xi > 0: & x = 2\left[1 - \frac{1}{2}\right]t + \xi = t + \xi \end{cases}$$

2, A

The diagram of characteristics is given on Fig. 2.

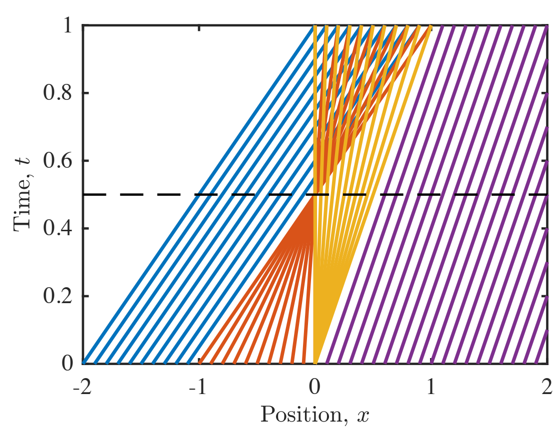

Figure 2: Diagram of characteristics with characteristics from region I (blue), region II (orange), region III (purple). We notice that there is an expansion fan between regions II and III (yellow). Characteristics from the region II all cross in $x = 0$ at $t = 1/2$, leading to shock formation.

1, A

To show that a shock is forming, you can notice that all the characteristics in region II cross in a single point. Their equation reads $x = -2\xi t + \xi$, so at $t = 1/2$, we realize that all characteristics are located in $x = -\xi + \xi = 0$. We conclude that a shock forms at $t = 1/2$ in $x = 0$. 1, A

(b) Here, we derive an explicit solution valid for $0 < t < 1/2$. Before the shock forms, we have $\boxed{\text{sim. seen} \Downarrow}$

- ☐ $-1 < \xi < 0$: In this region, we have

$$\rho = 2(1+\xi) \quad \text{on} \quad x = -2\xi t + \xi$$

In particular, we conclude that $\xi = x/(1-2t)$, which means that

$$\rho(x,t) = 2\left(\frac{x}{1-2t} + 1\right)$$

and this for

$$-1 < \xi < 0 \Rightarrow -1 < \frac{x}{1-2t} < 0 \Rightarrow 2t - 1 < x < 0$$

- ☐ $\xi < -1$: In this region, we have

$$\rho = 0 \quad \text{on} \quad x = 2t + \xi$$

In particular, we conclude that $\xi = x - 2t$, which means that

$$\rho(x,t) = 0$$

and this for

$$\xi < -1 \Rightarrow x - 2t < -1 \Rightarrow x < 2t - 1$$

- ☐ $\xi > 0$: In this region, we have

$$\rho = 1 \quad \text{on} \quad x = t + \xi$$

In particular, we conclude that $\xi = x - t$, which means that

$$\rho(x,t) = 1$$

and this for

$$\xi > 0 \Rightarrow x > t$$

- ☐ Finally, we need to deal with the fan region, $0 < x < t$. In the expansion fan, the solution varies linearly from $\rho = 2$ to $\rho = 1$, we thus know that

$$u(x,t) = 2 - \frac{x}{t} = 2\left[1 - \frac{x}{2t}\right]$$

We conclude that the explicit solution for $u(x,t)$ before the shock forms is given by

$$u(x,t) = \begin{cases} 0, & x < 2t-1 \\ 2[x/(1-2t)+1], & 2t-1 < x < 0 \\ 2[1 - x/(2t)], & 0 < x < t \\ 1, & x > t \end{cases}$$

$\boxed{4, \text{A}}$

(c) To find the explicit solution after the shock has formed, we need to proceed to shock fitting. If we denote $s(t)$ the position of the shock, we have

sim. seen ⇓

Before the shock: $\rho_- = 0$

After the shock: $\rho_+ = 2\left[1 - \dfrac{s}{2t}\right]$ (which comes from the fan region)

The Rankine-Hugoniot jump condition reads

$$\frac{ds}{dt} = \frac{[q(\rho)]_+^-}{[\rho]_+^-} \quad \text{with} \quad q(\rho) = 2\rho\left[1 - \frac{\rho}{4}\right]$$

subject to the initial condition $s(1/2) = 0$. Here, we have

$$q_- = 0 \quad \text{and} \quad q_+ = 2\left[1 - \frac{s}{2t}\right]\left[1 + \frac{s}{2t}\right]$$

We thus have to integrate the equation

$$\frac{ds}{dt} = 1 + \frac{s}{2t} \quad \text{which is of the form} \quad s' + P(t)s = Q(t), \quad \text{with } P(t) = -\frac{1}{2t},\ Q(t) = 1$$

This can easily be done with an integration factor and we find

$$s(t) = e^{-\int P(t)dt}\int Q(t)e^{\int P(t)dt} + Ce^{-\int P(t)dt} = 2t + C\sqrt{t}$$

where C is an integration constant to be determined. We use the initial condition above to write $C = -\sqrt{2}$ and so conclude that

2, C

$$s(t) = 2t - \sqrt{2t}$$

The explicit solution is thus given by

1, C

$$u(x,t) = \begin{cases} 0, & x < 2t - \sqrt{2t} \\ 2[1 - x/(2t)], & 2t - \sqrt{2t} < x < t \\ 1, & x > t \end{cases}$$

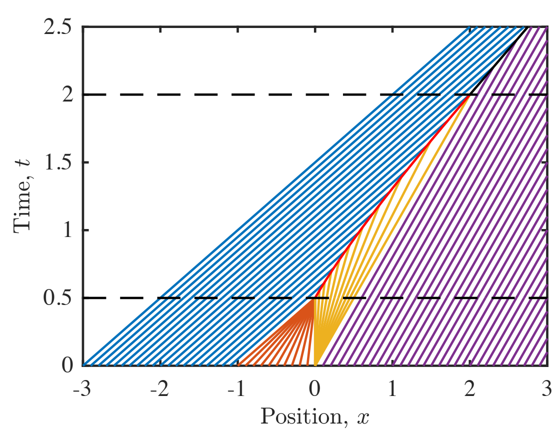

Figure 3: Amended diagram of characteristics with shock path up to $t = 2$ shown in red and for $t > 2$ shown in black.

The amended diagram of characteristics is given in Fig. 3. $\boxed{2, \mathrm{B}}$

From the diagram of characteristics, we see that the characteristics originally emanating from region III ($\xi > 0$) cross the shock path at some well defined time. The shock path is given by $s(t) = 2t - \sqrt{2t}$, while the equation of the characteristics in region III were given by $x(t) = t + \xi$. In particular, setting $\xi = 0$ (left most characteristic in the region), we find that $s(t) = x(t)$ when $t = \sqrt{2t} \Rightarrow t = 2$ which is consistent with the diagram drawn. We thus conclude that our explicit solution above is only valid in the range $1/2 < t < 2$. $\boxed{2, \mathrm{D}}$

$\boxed{\text{meth seen} \Downarrow}$

(d) First, we realize that to find the position of the shock at $t = 3$, we need to modify our jump condition. Indeed, we have just seen that our shock fitting procedure was only valid up to $t = 2$. For $t > 2$, we have the following

$$\text{Before the shock:} \quad \rho_- = 0 \Rightarrow q_- = 0$$

$$\text{After the shock:} \quad \rho_+ = 1 \Rightarrow q_+ = \frac{3}{2}$$

and the jump condition thus reads

$$\frac{ds}{dt} = \frac{3}{2} \quad \text{subject to} \quad s(2) = 2$$

and we conclude that for $t > 2$, the shock path is given by

$$s(t) = \frac{3t}{2} - 1$$

and in particular, $s(3) = 7/2$. $\boxed{2, \mathrm{B}}$

The solution for: (i) $t = 0$, (ii) $t = 1/2$, (iii) $t = 1$, and (iv) $t = 3$ is sketched on Fig. 4. $\boxed{\text{sim. seen} \Downarrow}$

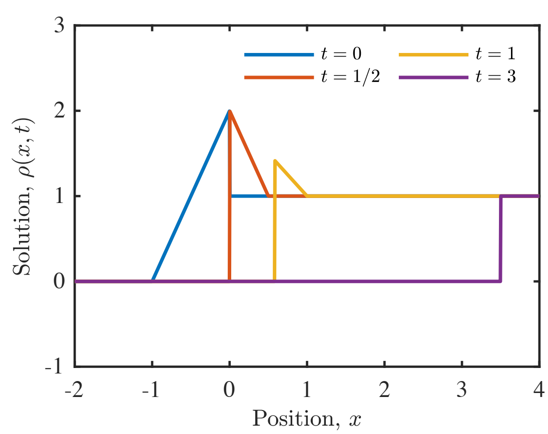

Figure 4: Sketch of the solution for various times.

$\boxed{3, \mathrm{A}}$

3. (a) (i) The vertical displacement of the string obeys the 1D wave equation $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

$\boxed{1, \text{A}}$

We seek separated solutions of the form $u(x,t) = X(x)T(t)$. By substituting this in the wave equation, we have

$$XT'' = c^2 X'' T$$

which means that there exists a separation constant λ^2 such that

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2$$

where the sign of the separation constant is dictated by the fact that we require solutions which are periodic in space.
So we obtain a set of two ODEs. Let us first turn our attention to that for $X(x)$: the general solution to $X'' + \lambda^2 X = 0$ is given by

$$X(x) = B\cos\lambda x + C\sin\lambda x$$

As we are looking for nontrivial solutions, the boundary conditions impose

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \Rightarrow B = 0$$
$$u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0 \Rightarrow \sin\lambda L = 0 \Rightarrow \lambda = n\pi/L$$

with n an integer.
Now back to the temporal equation $T'' + c^2\lambda^2 T = 0$, which has for general solution

$$T(t) = D\cos c\lambda t + E\sin c\lambda t$$

Since the initial conditions are such that the string is released without velocity $\frac{\partial u}{\partial t}(x,0) = 0$, we require that $T'(0) = 0$ and conclude that $E = 0$.
Combining these, we obtain the family of solutions

$$u_n(x,t) = C_n D_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

Finally, we conclude that the general solution of the original PDE is given by the following superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

as required. (Note that the sum only runs over the positive integers as the negative integers terms can be absorbed in the sum by defining $c_n = C_n D_n - C_{-n} D_{-n}$). $\boxed{4, \text{A}}$

(ii) To determine the coefficients c_n, we make use of the initial conditions, which are given by $\boxed{\text{meth seen} \Downarrow}$

$$u(x,0) = f(x) = \begin{cases} 3Ax/L, & 0 < x < L/3 \\ 3A(L-x)/(2L), & L/3 < x < L \end{cases} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < L$$

Imposing this initial condition, we get that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

This expression is the half-range Fourier sine series for $f(x)$ and so the coefficients are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$\boxed{2, \text{A}}$

We write
$$\frac{c_n L}{2} = \frac{3A}{L}\mathcal{I}(0, L/3) + \frac{3A}{2}\mathcal{J} - \frac{3A}{2L}\mathcal{I}(L/3, L)$$

with

$$\mathcal{I}(A,B) = \int_A^B x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n\pi}\left[-x\cos\left(\frac{n\pi x}{L}\right)\right]_A^B + \frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_A^B$$

leading to

$$\begin{aligned}
\mathcal{I}(0, L/3) &= -\frac{L^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) \\
\mathcal{I}(L/3, L) &= \frac{L^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right) - \frac{L^2}{n\pi}(-1)^n - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)
\end{aligned}$$

Similarly,

$$\mathcal{J} = \int_{L/3}^{L}\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{L}{n\pi}(-1)^n + \frac{L}{n\pi}\cos\left(\frac{n\pi}{3}\right)$$

and finally assembling all these terms, we write that

$$\begin{aligned}
\frac{c_n L}{2} =& -\frac{AL}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{3AL}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) + \frac{3AL}{2n\pi}\cos\left(\frac{n\pi}{3}\right) - \frac{3AL}{2n\pi}(-1)^n \\
&- \frac{AL}{2n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{3AL}{2n\pi}(-1)^n + \frac{3AL}{2n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)
\end{aligned}$$

So we conclude that
$$c_n = \frac{9A}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$\boxed{4, \text{C}}$

(iii) If we impose that the string is constrained at its midpoint, such that $u(L/2, t) = 0$ for all t. Then, we have $c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Indeed, all odd modes display a maximum amplitude in $x = L/2$ and they will be killed by the additional constraint. $\boxed{\text{unseen} \Downarrow}$

$\boxed{2, \text{D}}$

(b) The steady-state temperature of the metal plate is governed by the following Laplace problem $\boxed{\text{meth seen} \Downarrow}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1,\ 0 < y < 1$$

and the boundary conditions are given by

$$\begin{cases} \partial_x u(0, y) = 0 & u(x, 0) = T \\ \partial_x u(1, y) = 0 & u(x, 1) = 0 \end{cases}$$

$\boxed{1, \text{B}}$

We seek separated solutions of the form $u(x, y) = X(x)Y(y)$. Reinjecting this in the Laplace equation, we obtain

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

with λ a constant as the LHS only depends on x and the RHS only depends on y. To determine the separation constant, we consider the following boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(1, y) = 0$$

The only possibility (if we require non trivial solutions) is to have $\lambda < 0$ and we denote $\lambda = -\mu^2$. We thus have the following ODE problems

$$\begin{aligned} &X'' + \mu^2 X = 0, \quad X'(0) = 0, \quad X'(1) = 0 \\ &Y'' - \mu^2 Y = 0, \quad Y(0) = T, \quad Y(1) = 0 \end{aligned}$$

The general solution to the first problem is given by $\boxed{1, \text{B}}$

$$X(x) = A\cos(\mu x) + B\sin(\mu x) \Rightarrow X'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

which upon application of the BCs reduces to the following family of solutions

$$X_n(x) = A_n \cos(n\pi x)$$

We need to consider two cases: $\boxed{1, \text{B}}$

- ☐ For $n \neq 0$, we have $\mu_n \neq 0$ and the second ODE to solve is $Y'' - \mu^2 Y = 0$ whose general solution is $Y(y) = Ce^{\mu y} + De^{-\mu y}$. Upon imposing the BCs, we find that

  $$Y(1) = 0 \Rightarrow Ce^{\mu} + De^{-\mu} \Rightarrow D = -Ce^{2\mu}$$

  We conclude that for $n \neq 0$, we obtain the family of solutions

  $$u_n = c_n[e^{n\pi y} - e^{2n\pi}e^{-n\pi y}]\cos(n\pi x)$$

- ☐ For $n = 0$, we have $\mu_0 = 0$ and the second ODE reduces to $Y'' = 0 \Rightarrow Y(y) = Cy + D$, imposing $Y(1) = 0$, we conclude that

  $$u_0 = c_0(y - 1)$$

$\boxed{2, \text{B}}$

The general solution finally is given by the following superposition

$$u(x,y) = c_0(y-1) + \sum_{n=1}^{\infty} c_n \cos(n\pi x)\left(e^{n\pi y} - e^{2n\pi}e^{-n\pi y}\right)$$

where c_n are real constants. Finally, setting $y = 0$ in the solution, we get $\boxed{1, \mathrm{B}}$

$$u(x,0) = -c_0 + \sum_{n=1}^{\infty} c_n \left(1 - e^{2n\pi}\right)\cos(n\pi x) = T$$

equating the coefficients of the cosine terms on both sides (which we can do as $\{\cos(n\pi x)\}_{n\in\mathbb{N}}$ is an orthogonal family of functions), we find that $c_0 = -T$ and $c_n = 0$, $\forall n \geq 1$. The solution finally reads:

$$u(x,y) = (1-y)T$$

$\boxed{1, \mathrm{B}}$

4. (a) In this first part, we consider a diffusion problem. seen ⇓

(i) We consider the transformation: $\{\tilde{x} = ax, \tilde{t} = a^\beta t, \tilde{u} = a^\gamma u\}$ where a is a positive real constant. Then, we know that the derivative transform as follows: $\partial_t = a^\beta \partial_{\tilde{t}}$ and $\partial_x = a\partial_{\tilde{x}} \Rightarrow \partial_{xx} = a^2\partial_{\tilde{x}\tilde{x}}$ and so this leads to

$$\text{LHS} = \frac{\partial u}{\partial t} = a^{\beta-\gamma}\frac{\partial \tilde{u}}{\partial \tilde{t}}$$
$$\text{RHS} = D\frac{\partial^2 u}{\partial x^2} = a^{2-\gamma}D\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$$

This leads to

$$a^{\beta-2}\frac{\partial \tilde{u}}{\partial \tilde{t}} = D\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$$

It is thus clear that the diffusion equation is invariant under the above transformation if and only if $\beta = 2$, i.e. that provided $\beta = 2$, if u is solution of the original equation, then so is $\tilde{u}(\tilde{x}, \tilde{t}) = a^\gamma u(ax, a^\beta t)$. We do not obtain here any condition on γ as it cancels out. $\boxed{3, \text{B}}$

(ii) In this problem, the boundary conditions are given as follows sim. seen ⇓

$$u(x,0) = 0, \quad x > 0 \quad ; \quad u(x,t) \to 0, \quad x \to \infty, \quad t \geq 0 \quad ; \quad u(0,t) = u_0, \quad t \geq 0$$

Starting from the similarity ansatz $u(x,t) = t^{\gamma/\beta}f(\eta)$ with $\eta = xt^{-1/2}$, we know that

$$u(0,t) = u_0,\ t > 0 \Rightarrow t^{\gamma/\beta}f(0) = u_0,\ t > 0$$

which is only true for all times is $\gamma/\beta = 0$, which imposes $\gamma = 0$. We then conclude that the boundary conditions all translate to

$$f(0) = u_0 \quad \text{and} \quad f(\eta) \to 0,\ \eta \to +\infty$$

$\boxed{3, \text{D}}$

(iii) The similarity solution is thus here given by $u(x,t) = f(\eta)$ with $\eta = xt^{-1/2}$. In particular, we have meth seen ⇓

$$\frac{\partial u}{\partial t} = \frac{\partial \eta}{\partial t}f'(\eta) = -\frac{1}{2}\frac{x}{t^{3/2}}f'(\eta)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}f'(\eta)\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{t}}f'(\eta)\right) = \frac{1}{t}f''(\eta)$$

where the primes denote derivatives with respect to η. Combining these, we obtain

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \Rightarrow -\frac{1}{2}\frac{x}{t^{3/2}}f'(\eta) = \frac{D}{t}f''(\eta)$$

We thus conclude that the diffusion problem reduces to the following ODE problem

$$f''(\eta) + \frac{\eta}{2D}f'(\eta) = 0$$
$$f(0) = u_0, \quad f(\infty) = 0$$

$\boxed{3, \text{C}}$

(b) Here, we consider the equation governing the velocity field in a two-dimensional vortex:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (rv)}{\partial r}\right]$$

where r is the radial distance to the center of the vortex. We assume that the initial conditions are given by $v(r,0) = 1/r$.

unseen $\Downarrow$

(i) Once again, we consider the transformation: $\{\tilde{r} = ar, \tilde{t} = a^{\beta}t, \tilde{v} = a^{\gamma}v\}$ where a is a positive real constant. Then, we know that the derivative transform as follows: $\partial_t = a^{\beta}\partial_{\tilde{t}}$ and $\partial_r = a\partial_{\tilde{r}}$ and so this leads to

$$\begin{aligned}\text{LHS} &= \frac{\partial v}{\partial t} = a^{\beta-\gamma}\frac{\partial \tilde{v}}{\partial \tilde{t}} \\ \text{RHS} &= \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (rv)}{\partial r}\right] = a^{2-\gamma}\frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial (\tilde{r}\tilde{v})}{\partial \tilde{r}}\right]\end{aligned}$$

This leads to

$$a^{\beta-2}\frac{\partial \tilde{v}}{\partial \tilde{t}} = \frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial (\tilde{r}\tilde{v})}{\partial \tilde{r}}\right]$$

and so we conclude that this PDE is left unchanged by the above transformation if and only if $\beta = 2$. 2, C

To determine the value of γ which leaves the entire PDE problem invariant, we consult the initial conditions. In particular, we notice that

$$v(r,0) = \frac{1}{r} \Rightarrow a^{-\gamma}\tilde{v}(\tilde{r},0) = \frac{a}{\tilde{r}} \Rightarrow a^{-(\gamma+1)}\tilde{v}(\tilde{r},0) = \frac{1}{\tilde{r}}$$

and conclude that for the initial conditions to be left unchanged (i.e. they are written the same way as in the original problem but in the new variables), we require that $\gamma = -1$. 2, D

(ii) A similarity solution for this problem can thus be written unseen $\Downarrow$

$$v(r,t) = t^{-1/2}f(rt^{-1/2})$$

which we can rewrite as

$$v(r,t) = \frac{\eta}{r}f(\eta), \text{with } \eta = r/\sqrt{t} \Rightarrow v(r,t) = \frac{g(\xi)}{r} \quad \text{with } \xi = \eta^2 = r^2/t$$

The redefinition of the similarity variable is supported by the fact that $\eta = x/\sqrt{t}$ is itself invariant under the transformation $r \to r/L$ and $t \to t/L^2$. In particular, we have

$$\begin{aligned}&\frac{\partial v}{\partial t} = \frac{1}{r}\frac{\partial \xi}{\partial t}g'(\xi) = -\frac{r}{t^2}g'(\xi) \\ &\frac{\partial (rv)}{\partial r} = \frac{\partial \xi}{\partial r}g'(\xi) = \frac{2r}{t}g'(\xi) \Rightarrow \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv)}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{2}{t}g'(\xi)\right) = \frac{4r}{t^2}g''(\xi)\end{aligned}$$

We conclude that g is solution of the following ODE

$$4g''(\xi) + g'(\xi) = 0$$

whose general solution is given by $\boxed{4, \text{D}}$

$$g(\xi) = A + Be^{-\xi/4}$$

The initial conditions translate to $\lim_{\xi \to +\infty} g(\xi) = 1$ and we thus conclude that $A = 1$. As stated in the problem, we are looking for bounded solutions. Boundedness requires that $g(0) = 0$ (to prevent the divergence of $v(r,t) = g(\xi)/r$ in $r = 0$). This imposes that $A = -B \Rightarrow B = -1$. We finally conclude that the solution to this problem reads

$$v(r,t) = \frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{4t}\right)\right]$$

$\boxed{3, \text{D}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks