MATH60006/70006 Applied Complex Analysis Dr Sam Brzezicki

Problem Sheet 5

1). Calculate the ordinary Fourier transform of

$$k(x) = ae^{-b|x|},$$

where a and b are real constants with $b > 0$. Give the region where this Fourier transform is analytic. [**Note:** This is a good Fourier transform to commit to memory to speed up exam questions (you can just quote its result rather than calculate it each time)]

2). Decompose the following into Wiener-Hopf **sums** for the given strips of analyticity Ω:

(i). $R(s) = \frac{1}{s^2+a^2}$, where $a \in \mathbb{R}$ is constant, with strip $\Omega = \{s : -a < \text{Im}\{s\} < a\}$.

(ii). $R(s) = \frac{1}{s^3-is^2-4s+4i}$, with strip $\Omega = \{s : 1/4 < \text{Im}\{s\} < 1/2\}$.

3). The function $f(x)$ satisfies the integral equation

$$f(x) = \lambda \int_0^\infty e^{-4|x-y|} f(y)dy,$$

for $x \geqslant 0$, where λ is a real constant. Use the Wiener-Hopf method to find $f(x)$ in the cases where:

(i). $\lambda = \frac{15}{8}$,

(ii). $\lambda = 2$,

(iii). $\lambda = \frac{17}{8}$.

4). The function $f(x)$ is bounded by a polynomial for $x \geqslant 0$ and satisfies

$$1 + \alpha x = f(x) + \frac{3}{2}\int_0^\infty f(y)e^{-|x-y|}dy$$

for $x \geqslant 0$, where α is a positive constant. Use the Wiener-Hopf method to find $f(x)$.

5). The function $f(x)$ satisfies the integral-differential equation

$$f'(x) + 6f(x) = 12\int_0^\infty f(y)e^{-4|x-y|}dy,$$

for $x \geqslant 0$ with $f(0) = 2$. Working in the analyticity strip:

(i). $\Omega_1 = \{s : 2 < \text{Im}\{s\} < 4\}$, (ii). $\Omega_2 = \{s : 0 < \mathrm{Im}\{s\} < 2\}$,

use the Wiener-Hopf method to find $f(x)$. What do you notice about your two answers?

6). The function $f(x)$ satisfies the integral-differential equation

$$f''(x) + f(x) = e^{-x} + \int_0^\infty f(y)e^{-|x-y|}dy,$$

for $x \geqslant 0$, with $f'(0) = 0$. Working in the analyticity strip $\Omega = \{s : 0 < \mathrm{Im}\{s\} < 1/\sqrt{2}\}$, use the Wiener-Hopf method to show that $f(0) = -\sqrt{2}$ and find $f(x)$.