Logic

20 lectures

- Propositional Logic
  - Semantics ('truth tables') — Tautologies
  - Syntax ('proof') — Theorems
- 1st order logic
  - Semantics ('1st order structures')
  - Formal deduction

Gödel's Completeness Thm.

Set theory

$$\Big(\big((p \to q) \wedge (q \to (\neg p))\big) \to (\neg p)\Big)$$

"If $\overbrace{\text{Mr Jones is happy}}^{p}$ then $\overbrace{\text{Mrs Jones is unhappy}}^{q}$
and if Mrs J. is unhappy then Mr J. is unhappy.
So Mr Jones is unhappy."

1. Propositional Logic.

1.1 Propositional Formulas.

'Proposition' = 'Statement'

either True (T)
or False (F).

Combine basic propositions using connectives.

(1.1.1) Connectives + truth table rules.

$p, q, \ldots$ statements.

Connectives

Negation $(\neg p)$

p	$(\neg p)$
T	F
F	T

Conjunction ('and')

$(p \wedge q)$ has value T $\Leftrightarrow$ p, q have value T.

Disjunction ('or')

$(p \vee q)$ has value T $\Leftrightarrow$ at least one of p, q has value T

Implication $(p \to q)$

$(p \to q)$ has value F only when p has value T and q has value F.

Biconditional $(p \leftrightarrow q)$

Value T precisely when p, q have the same value.

Summary:

p	q	$(p \wedge q)$	$(p \vee q)$	$(p \to q)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

(1.1.2) Def. A propositional formula is obtained from propositional variables $p_1, p_2, p_3, \ldots$ and connectives in the following way:

(i) Any prop. var. is a formula

(ii) If ϕ, ψ are formulas then $(\neg \phi)$ $(\phi \wedge \psi)$ $(\phi \to \psi)$ $(\phi \vee \psi)$ $(\phi \leftrightarrow \psi)$ are formulas

(iii) Any formula arises in this way.

Eg. Formulas

p_1 p_2 $(\neg p_2)$

$(p_1 \to (\neg p_2))$

$((p_1 \to (\neg p_2)) \to p_2)$

Not formulas $p_1 \wedge p_2$ (missing brackets)

$)(\neg p_9$

Remarks: (i) Because of the brackets, every formula is either a prop. variable or is built from 'shorter' formulas in a unique way.

② Any assignment of truth values to the variables in a formula ϕ determines the truth value of ϕ in a unique way, using (1.1.1)

Eg ϕ: $((p_1 \to (\neg p_2)) \to p_1)$

p_1	p_2	$\neg p_2$	$(p_1 \to (\neg p_2))$	ϕ
T	T	F	F	T
T	F	T	T	T
→ F	T	F	T	F
F	F	T	T	F

(1.1.3) Def. Let $n \in \mathbb{N}$

① A truth function of n variables is a function

$$f : \{T, F\}^n \longrightarrow \{T, F\}$$

(where $\{T, F\}^n = \{(x_1, \ldots, x_n) : \text{each } x_i \text{ is } T \text{ or } F\}$.

[Ex: How many?]

② Suppose ϕ is a formula whose variables are amongst $p_1, \ldots, p_n$. Obtain a truth function $F_\phi : \{T, F\}^n \longrightarrow \{T, F\}$ whose value at $(x_1, \ldots, x_n)$ is the truth value of ϕ when p_i has value x_i, when computed according to (1.1.1).

F_ϕ is the truth function of ϕ

[Eg. In previous $n = 2$ $F_\phi((F, T)) = F$.]

L2 + L3

⑥

$$\Big(\underbrace{\big((p \to q) \wedge (q \to (\neg p))\big)}_{\psi} \to \underbrace{(\neg p)}_{\chi}\Big) \quad : \phi$$

p	q	ψ	χ	ϕ
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(1.1.4) Def ① A formula ϕ is a <u>tautology</u> if its truth function F_ϕ always has value T.

② Say that formulas ψ, χ with variables amongst $p_1, \dots, p_n$ are <u>logically equivalent</u> (l.e.) if they have the same truth function, i.e. $F_\psi = F_\chi$ (as functions of n variables)

Eg. $((p \to q) \to (q \to (\neg p)))$

is l.e. to $(\neg p)$.

(1.1.5) Remark

1) ψ, χ are l.e. if and only if $(\psi \leftrightarrow \chi)$ is a tautology.

2) Suppose ϕ is a formula with variables $p_1, \dots, p_n$ and $\phi_1, \dots, \phi_n$ are fmlas with variables $q_1, \dots, q_m$. For each $i \leq n$ substitute ϕ_i for p_i in ϕ. Then

(i) the result is a formula θ &

(ii) if ϕ is a tautology, then so is θ.

(1.1.6)

Example Check

$$(((\neg p_2) \to (\neg p_1)) \to (p_1 \to p_2))$$

is a tautology. So if ϕ_1, ϕ_2 are any formulas, then

$$(((\neg \phi_2) \to (\neg \phi_1)) \to (\phi_1 \to \phi_2))$$

is a formula & in fact is a tautology.

Pf of 1.1.5(2).

(i) By induction on the number of connectives in ϕ.

(ii) Prove

$$F_\theta(q_1, \dots, q_m)$$

$$=$$

$$F_\phi\Big(F_{\phi_1}(q_1, \dots, q_m), \dots, F_{\phi_n}(q_1, \dots, q_m)\Big)$$

by induction on the number of connectives in ϕ. #.

Ex: $(p_1 \to (\neg p_1))$ not a tautology, but you can find ϕ_1 with $(\phi_1 \to (\neg \phi_1))$ being a tautology.

Examples of l.e. formulas.

1) $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. $((p_1 \wedge p_2) \wedge p_3)$

- usually omit brackets.

2) Similar with $\vee$

3) $(p_1 \vee (p_2 \wedge p_3))$ is l.e. $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

3') Similar with $\vee, \wedge$ interchanged

4) $(\neg(\neg p_1))$ is l.e. p_1

5) $(\neg(p_1 \wedge p_2))$ l.e. to $((\neg p_1) \vee (\neg p_2))$

5') ...

By 1.1.5 we obtain e.g. for formulas ϕ, ψ, χ $(\phi \wedge (\psi \wedge \chi))$ is l.e. to $((\phi \wedge \psi) \wedge \chi)$ etc.
[See P. sheet 1.]

(1.1.7) Lemma. There are 2^{2^n} truth functions of n variables.

Pf: A truth fn. is a fn.

$$G : \{T, F\}^n \to \{T, F\}.$$

$$|\{T, F\}^n|^* = 2^n$$

and each $G(\bar{v})$ for $\bar{v} \in \{T, F\}^n$ has two possible values. #

(1.1.8) Def. Say that a set of connectives is adequate if for every $n \geq 1$, every truth fn. of n variables is the truth fn. of some formula which involves only connectives from the set and variables $p_1, \dots, p_n$.

(1.1.9) Thm The set $\{\neg, \wedge, \vee\}$ is adequate.

Disjunctive normal form

Proof: Let $G : \{T, F\}^n \to \{T, F\}$.

Case 1 $G(\bar{v}) = F$ for all $\bar{v} \in \{T, F\}^n$.

Take ϕ to be $(p_1 \wedge (\neg p_1))$.

Case 2 List the $\bar{v}$ with $G(\bar{v}) = T$ as $\bar{v}_1, \ldots, \bar{v}_r$.

Write $\bar{v}_i = (v_{i1}, \ldots, v_{in})$ where each $v_{ij} \in \{T, F\}$.

Define

$$q_{ij} = \begin{cases} p_j & \text{if } v_{ij} = T \\ (\neg p_j) & \text{if } v_{ij} = F \end{cases}$$

Let ψ_i be $(q_{i1} \wedge q_{i2} \wedge \ldots \wedge q_{in})$

then $F_{\psi_i}(\bar{v}) = T$

$\iff \bar{v} = \bar{v}_i$.

$[\, F_{\psi_i}(\bar{v}) = T \iff$ each q_{ij} is T

$\iff \bar{v} = \bar{v}_i$. $]$

Now let ϕ be $\psi_1 \vee \psi_2 \vee \ldots \vee \psi_r$.

Then $F_\phi(\bar{v}) = T \iff$

$F_{\psi_i}(\bar{v}) = T$ for some $i \le r$

$\iff \bar{v} = \bar{v}_i$ for some $i \le r$.

Thus $F_\phi(\bar{v}) = G(\bar{v})$ for all $\bar{v}$. #

Example: $n = 3$: $\bar{v} = (T, F, F)$

$p_1 \wedge (\neg p_2) \wedge (\neg p_3)$ has value T only at $(p_1, p_2, p_3) = (T, F, F)$.

A formula ϕ as in case 2 is said to be in <u>disjunctive normal form</u>.

(1.1.10) <u>Cor</u>. Suppose χ is a formula whose truth fn. is not always F then χ is l.e. to a formula in d.n.f. //.

[Apply Case 2 to F_χ.]

<u>Eg</u> $\chi : ((p_1 \to p_2) \to (\neg p_2))$

$\underline{n=2}$ $F_\chi(\bar{v}) = T$

$\iff \bar{v} = (T,F)$ or (F,F)

dnf:

$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2)))$

(1.1.11) <u>Cor</u>. The following sets of connectives are adequate

1) $\{\neg, \vee\}$

2) $\{\neg, \wedge\}$

3) $\{\neg, \to\}$

<u>Pf</u>: 1) By (1.1.9) enough to show that we can express $\wedge$ in terms of $\neg, \vee$:

$(p_1 \wedge p_2)$ is l.e. to $(\neg((\neg p_1) \vee (\neg p_2)))$

2) Similar

3) Express $\vee$ using $\neg, \to$.

$(p \vee q)$ l.e. to

$$((\neg p) \to q) \quad // \#$$

(1.1.12) <u>Example</u> the following are <u>not</u> adequate:

(i) $\{\wedge, \vee\}$

(ii) $\{\neg, \leftrightarrow\}$

//

(1.1.13) <u>Example</u> NOR connective

$\downarrow$ has truth table

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

$\{\downarrow\}$ is <u>adequate</u>:

$(\neg p)$ is l.e. to $(p \downarrow p)$

$(p \wedge q)$ is l.e. to

$$((p \downarrow p) \downarrow (q \downarrow q)).$$

So as $\{\neg, \wedge\}$ is adequate, $\{\downarrow\}$ is also adequate. //

———.

(1.2) A formal system for propositional logic

Idea: Try to generate all tautologies from certain 'basis assumptions' (axioms) using appropriate deduction rules.

(1.2.2) Def. The formal system L for propositional logic has the following:

Alphabet: variables $p_1, p_2, p_3, \ldots$

connectives $\neg$ $\rightarrow$

punctuation) (

Formulas Finite sequences ('strings') of symbols from the alphabet as follows (as in 1.1.2)

(a) Any variable p_i is a formula;

(b) If ϕ, ψ are formulas then so are $(\neg\phi)$ $(\phi \rightarrow \psi)$

(c) Any formula arises in this way.

L-formulas

Axioms Suppose ϕ, ψ, χ are L-formulas. The following are *axioms* of L:

(A1) $(\phi \to (\psi \to \phi))$

(A2) $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

(A3) $(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$

Deduction rule Modus Ponens (MP)

From ϕ $(\phi \to \psi)$

Deduce ψ

A *proof* in L is a finite sequence of L-formulas $\phi_1, \phi_2, \ldots, \phi_n$ such that each ϕ_i is either an *axiom* or is obtained from earlier formulas in the sequence by applying the deduction rule MP.

The final formula in a proof is a *theorem* of L.

$\ldots\ldots \phi \ldots (\phi \to \psi) \ldots \psi \ldots$

applied MP.

Write $\vdash_L \phi$ to mean 'ϕ is a theorem of L'.

Note: ① Any axiom is a theorem of L.

② Every formula in a proof is a theorem of L.

(1.2.3) Example

Suppose ϕ is an L-formula. Then $\vdash_L (\phi \to \phi)$.

Here is a proof in L:

1. $\underbrace{(\phi \to ((\phi \to \phi) \to \phi))}_{\text{Call this } \chi}$ (A1)

2. $(\chi \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$ (A2)

3. $((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$ (1, 2 + MP)

4. $(\phi \to (\phi \to \phi))$ (A1)

5. $(\phi \to \phi)$ (by 3, 4 and MP).

Office hour (in Huxley 661)
1330–1400, 1415–1445.

(1.2.4) <u>Def.</u> Suppose Γ is a set of L-formulas. A <u>deduction</u> from Γ is a finite sequence of L-fmlas
$$\phi_1, \phi_2, \dots, \phi_n$$
such that each ϕ_i is either

an axiom,
a formula in Γ,

or is obtained from previous formulas $\phi_1, \dots, \phi_{i-1}$ using the deduction rule MP.

(n here is the <u>length</u> of the deduction).

Write $\Gamma \vdash_L \phi$ if there is a deduction from Γ ending with ϕ. Say that ϕ is a <u>consequence</u> of Γ.

<u>Note</u>: $\emptyset \vdash_L \phi$ is the same

empty set

as 'ϕ is a theorem of L'.
$$\vdash_L \phi.$$

(1.2.5) <u>Thm</u> (The Deduction Theorem)
Suppose Γ is a set of L-formulas and ϕ, ψ are L-fmlas. Suppose
$$\Gamma \cup \{\phi\} \vdash_L \psi$$
Then
$$\Gamma \vdash_L (\phi \to \psi).$$

(1.2.6) Cor. (Hypothetical Syllogism HS)

Suppose ϕ, ψ, χ are L-fmulas.

and $\vdash_L (\phi \to \psi)$

and $\vdash_L (\psi \to \chi)$.

Then $\vdash_L (\phi \to \chi)$.

Proof: Use DT with $\Gamma = \emptyset$.

Show $\{\phi\} \vdash_L \chi$

[Then by DT $\emptyset \vdash_L (\phi \to \chi)$

so $\vdash_L (\phi \to \chi)$.].

" Show there is a deduction of χ from $\{\phi\}$

1. $(\phi \to \psi)$ (theorem of L)
2. $(\psi \to \chi)$ (thm. of L)
3. ϕ (Deduction from ϕ)
4. ψ (1,3 + MP).
5. χ (2,4 + MP).

Thus: $\{\phi\} \vdash_L \chi$. #

(1.2.7) <u>Proposition</u> Suppose ϕ, ψ are L-formulas. Then:

(a) $\vdash_L ((\neg\psi) \to (\psi \to \phi))$

(b) $\{(\neg\psi), \psi\} \vdash_L \phi$

(c) $\vdash_L (((\neg\phi) \to \phi) \to \phi)$

<u>Pf</u>: (a) P. sheet 1.

(b) Deduction of ϕ from $\{(\neg\psi), \psi\}$:

use (a) & MP twice to obtain ϕ.

(c) Suppose χ is any formula. Then by (b) & MP

$\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L \chi$ ③

Let α be an axiom and let χ be $(\neg\alpha)$.

So $\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L (\neg\alpha)$ ... ①

By DT:

$\{((\neg\phi) \to \phi)\} \vdash_L ((\neg\phi) \to (\neg\alpha))$

Axiom A3:

$\vdash_L (((\neg\phi) \to (\neg\alpha)) \to (\alpha \to \phi))$ ... ②

①, ② + MP gives

$\{((\neg\phi) \to \phi)\} \vdash_L (\alpha \to \phi)$

As α is an axiom we get (by MP)

$\{((\neg\phi) \to \phi)\} \vdash_L \phi$.

Use DT.

#

Pf. of Deduction thm: Show:

if $\Gamma \cup \{\phi\} \vdash_L \psi$

then $\Gamma \vdash_L (\phi \to \psi)$.

Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ using a deduction of length n. Prove by induction on n that

$\Gamma \vdash_L (\phi \to \psi)$.

Base step $n=1$. In this case ψ is either an axiom or in Γ or it is ϕ.

In the first two cases

$$\Gamma \vdash_L \psi$$

Then use the A1 axiom

$$\vdash_L (\psi \to (\phi \to \psi))$$

& MP to get

$$\Gamma \vdash_L (\phi \to \psi).$$

If ψ is ϕ then

$$\Gamma \vdash_L (\phi \to \phi)$$

by 1.2.3. This does the base case.

Inductive step: Suppose the result holds for shorter deductions (i.e. length $< n$).

In our deduction $\Gamma \cup \{\phi\} \vdash_L \psi$ either:

(a) ψ is an axiom, or in Γ or is equal to ϕ

or (b) ψ is obtained from earlier formulas χ, $(\chi \to \psi)$ in the deduction using MP.

In case (a), we argue as in the base case to get

$$\Gamma \vdash_L (\phi \to \psi).$$

In case (b) we have

$$\Gamma \cup \{\phi\} \vdash_L \chi$$

and $\Gamma \cup \{\phi\} \vdash_L (\chi \to \psi)$

using shorter deductions

So by inductive hypothesis:

$$\Gamma \vdash (\phi \to \chi) \quad \ldots ①$$

& $\Gamma \vdash (\phi \to (\chi \to \psi))$ ... ②

Now use A2

$$\vdash_L ((\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi)))$$

to get using ② + MP:

$$\Gamma \vdash ((\phi \to \chi) \to (\phi \to \psi)).$$

Using ① + MP gives:

$$\Gamma \vdash (\phi \to \psi),$$

as required. ~~#~~.

(1) Week 4 Monday :

problem class .

(1.3.1) Theorem (Soundness of L)

Suppose ϕ is a theorem of L.

Then ϕ is a tautology .

(1.3.2) Notation

A (propositional) valuation v is an assignment of truth values to the propositional variables $p_1, p_2, \dots$

So $v(p_i) \in \{T, F\}$ (for $i \in \mathbb{N}$).

Using the truth table rules, this assigns a truth value $v(\phi) \in \{T, F\}$ to every L-fmla ϕ (satisfying

$v((\neg\phi)) \neq v(\phi)$

and $v((\phi \to \psi)) = F \iff v(\phi) = T$ & $v(\psi) = F$).

[Prob. sheet 2].

Pf of 1.3.1 : By induction on the length of a pf. of ϕ it is enough to show

(a) Every axiom is a tautotology ;

(b) MP preserves tautologies .

(a) Use truth tables, or argue as follows : Do A2 :

Suppose for a contradiction, v is a valuation with

$$v\Big(\big(\phi \to (\psi \to \chi)\big) \to \big((\phi \to \psi) \to (\phi \to \chi)\big)\Big) = F.$$

then:

$$v\big(\phi \to (\psi \to \chi)\big) = T \quad \ldots (1)$$

& $$v\big((\phi \to \psi) \to (\phi \to \chi)\big) = F \quad \ldots (2)$$

By (2) $v((\phi \to \psi)) = T$ .. (3)

& $v((\phi \to \chi)) = F$ .. (4)

By (4) $v(\phi) = T$ $v(\chi) = F$

By (3) $v(\psi) = T$

This contradicts (1). //

A1, A3 Ex.

(b) If ϕ $(\phi \to \psi)$ are tautologies and v is a valuation then

$v(\phi) = T$ $v(\phi \to \psi) = T$

so $v(\psi) = T$. #.

(1.3.3) Thm (Generalisation of Soundness)

Suppose Γ is a set of L-fmlas, ϕ is a fmla. and $\Gamma \vdash_L \phi$.

If v is a valuation with $v(\psi) = T$ for all $\psi \in \Gamma$ [write $v(\Gamma) = T$]

then $v(\phi) = T$.

Pf: As for 1.3.2.

(1.3.4) <u>Thm</u> (Completeness / Adequacy Thm. for L).

Suppose ϕ is a tautology. Then $\vdash_L \phi$.

//

<u>Steps in pf</u>:

1. Want to show: if $v(\phi) = T$ for all valuations v then $\vdash_L \phi$.

2. Generalisation:

Suppose that for every v with $v(\Gamma) = T$ we have $v(\phi) = T$.

Then $\Gamma \vdash_L \phi$.

[1. is the case $\Gamma = \emptyset$].

3. Equivalently:

If $\Gamma \nvdash_L \phi$ there is a v with $v(\Gamma) = T$ and $\underline{v(\phi) = F}$.

//

(1.3.6) <u>Def</u>: A set of L-fmlas Γ is <u>consistent</u> if there is no L-fmla ϕ with $\Gamma \vdash_L \phi$ and $\Gamma \vdash_L (\neg\phi)$.

<u>Rk</u>: By 1.3.1 (Soundness) there is <u>no</u> L-fmla ϕ with $\vdash_L \phi$ & $\vdash_L (\neg\phi)$

(Say that <u>L is consistent</u>.)

(1.3.7) <u>Proposition</u>

Suppose Γ is a consistent set of L-fmlas and $\Gamma \nvdash_L \phi$.

Then $\Gamma \cup \{(\neg\phi)\}$ is consistent.

Proof: Suppose not. So there is a fmla. ψ with

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \psi \quad \ldots (1)$$

and $\Gamma \cup \{(\neg\phi)\} \vdash_L (\neg\psi)$ ... (2)

Apply DT to (2):

$$\Gamma \vdash_L ((\neg\phi) \to (\neg\psi)).$$

By A3 & this $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi))$

$$\Gamma \vdash_L (\psi \to \phi) \quad \ldots (3)$$

By (1), (3) + MP obtain:

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \phi \quad \ldots (4)$$

Apply DT $\Gamma \vdash_L ((\neg\phi) \to \phi)$ ... (5)

By 1.2.7 (c)

$$\vdash_L (((\neg\phi) \to \phi) \to \phi)$$

By this and (5) + MP get $\Gamma \vdash_L \phi$.

Contradiction. #.

(1.3.8) **Prop.** (Lindenbaum Lemma)

Suppose Γ is a consistent set of L-fmlas. Then there is a <u>consistent</u> set of formulas $\Gamma^* \supseteq \Gamma$ such that for every L-fmla ϕ

either $\Gamma^* \vdash_L \phi$

or $\Gamma^* \vdash_L (\neg\phi)$.

[Sometimes say Γ^* is <u>complete</u>.]

<u>Pf</u>: The set of L-formulas is <u>countable</u>, so we can the L-formulas as $\phi_0, \phi_1, \phi_2, \ldots$

[Why countable? Alphabet

$\neg \; \to \;) \; (\; p_1 \; p_2 \ldots$

is countable.

Formulas are certain <u>finite sequences of symbols from the alphabet</u>. This is a countable set.

// Define inductively sets of L-fmlas

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

where $\Gamma_0 = \Gamma$

and $\Gamma^* = \bigcup_{i \in \mathbb{N}} \Gamma_i$.

Suppose Γ_n has been defined.

If $\Gamma_n \vdash_L \phi_n$ then

let $\Gamma_{n+1} = \Gamma_n$.

If $\Gamma_n \nvdash_L \phi_n$ then

let $\Gamma_{n+1} = \Gamma_n \cup \{(\neg\phi_n)\}$.

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An easy induction using Prop 1.3.7 shows that each Γ_n is <u>consistent</u>. Let $\Gamma^* = \bigcup_{n \in \mathbb{N}} \Gamma_n$

<u>Claim 1</u> Γ^* is consistent.

If $\Gamma^* \vdash \phi$

and $\Gamma^* \vdash (\neg \phi)$

then as deductions are <u>finite</u>

$\Gamma_n \vdash \phi$

and $\Gamma_n \vdash (\neg \phi)$ for some $n \in \mathbb{N}$. <u>Contradiction</u>

<u>Claim 2</u> Γ^* is complete.

Let ϕ be any formula.

We have $\phi = \phi_n$ for some $n \in \mathbb{N}$.

Then by construction

either $\Gamma_n \vdash \phi$

or $\Gamma_{n+1} \vdash (\neg \phi)$

In the first case

$\Gamma^* \vdash \phi$

in the second case

$\Gamma^* \vdash (\neg \phi)$.

$\#$

(1.3.9) *Lemma*. Suppose Γ^* is a set of L-fmlas which is consistent and complete.

Then there is a valuation v such that for every L-fmla ϕ

$$\boxed{v(\phi) = T \iff \Gamma^* \vdash_L \phi.}$$

Pf: ~~Eat~~ Each variable p_i is an L-formula. *So* by the properties of Γ^*

either $\Gamma^* \vdash_L p_i$

or $\Gamma^* \vdash_L (\neg p_i)$

(and only one of these is the case).

~~In the first case~~

Let v be the valuation with

$$v(p_i) = T \iff \Gamma^* \vdash_L p_i$$

(for each $i \in \mathbb{N}$).

Show this v has the required property.

Do this by induction on the length of ϕ: show

$$v(\phi) = T \iff \Gamma^* \vdash_L \phi$$

Base case: ϕ is a variable. ~~this~~ is the def. of ~~...~~ v.

Case 1: ϕ is $(\neg\psi)$

$\Rightarrow$: Suppose $v(\phi) = T$

So $v(\psi) = F$

(v is a valuation)

By induction $\Gamma^* \nvdash \psi$.

By completeness of Γ^*

obtain $\Gamma^* \vdash (\neg\psi)$

ie. $\Gamma^* \vdash \phi$. //

$\Leftarrow$: Conversely suppose

$\Gamma^* \vdash \phi$

ie. $\Gamma^* \vdash (\neg\psi)$

By consistency $\Gamma^* \nvdash \psi$

So by inductive assumption

$v(\psi) = F$, so

$v(\neg\psi) = T$ ie. $v(\phi) = T$.

//.

Case 2 ϕ is $(\psi \to \chi)$

$\Leftarrow$: Suppose $v(\phi) = F$

(Show $\Gamma^* \nvdash \phi$)

Then $v(\psi) = T$ & $v(\chi) = F$

By ind. hypothesis

$\Gamma^* \vdash \psi$ . . . (a)

& $\Gamma^* \nvdash \chi$.

If $\Gamma^* \vdash \phi$ then

$\Gamma^* \vdash (\psi \to \chi)$ . . . (b)

then (a), (b) give $\Gamma^* \vdash \chi$

a contradiction. So

$\Gamma^* \nvdash \phi$.

ϕ is $(\psi \to \chi)$

$\Rightarrow)$: Suppose

$\Gamma^* \nvdash \phi$.

(show $v(\phi) = F$)

So $\Gamma^* \nvdash (\psi \to \chi)$

then: $\Gamma^* \nvdash \chi$ ... (1)

(as $\vdash (\chi \to (\psi \to \chi))$)

Also $\Gamma^* \nvdash (\neg\psi)$ .. (2)

(as $\vdash_L ((\neg\psi) \to (\psi \to \chi))$ 1.2.7(a)).

By (1), (2) & ind. hyp

$v(\chi) = F$

and $v(\neg\psi) = F$

so $v(\psi) = T$.

Thus $v(\phi) = F$. // #

(1.3.10) Cor. Suppose Δ is a consistent set of L-formulas and $\Delta \nvdash_L \phi$.

Then there is a valuation v with $v(\Delta) = T$ and $v(\phi) = F$.

Pf: Let $\Gamma = \Delta \cup \{(\neg\phi)\}$.

By 1.3.7 Γ is also consistent.

Let 1.3.8 there is $\Gamma^* \supseteq \Gamma$ which is consistent and complete.

By 1.3.9 there is a valuation v with $v(\Gamma^*) = T$.

In particular $v(\Delta) = T$ and $v(\neg\phi) = T$

& so $v(\phi) = F$.
(as v is a valuation)

#.

(9)

(1.3.11) <u>Theorem</u>.

(Completeness / Adequacy Thm. for L).

<u>If</u> ϕ is an L-fmla and $v(\phi) = T$ for every valuation v

<u>then</u> $\vdash_L \phi$.

<u>Pf</u>: Suppose $\nvdash_L \phi$.

ok by soundness thm.

By 1.3.10 (with $\Delta = \emptyset$) there is a valuation v with $v(\phi) = F$. $\#$

//

Some informal notes on countabiliy.

Definition 0.1 Two sets A, B are said to be *equinumerous* if there exists a bijection $f : A \to B$. We denote this by $A \approx B$ We also say that under these circumstances A and B have the same *cardinality*, and write $|A| = |B|$.

So two sets are equinumerous if their elements can be 'paired off.' This seems reasonable, but it has the consequence that a set can be equinumerous to a proper subset of itself. For example, the succesor function $n \mapsto n + 1$ gives a bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$. (Exercise: find bijections between $\mathbb{N}$ and $2\mathbb{N}$, and between $\mathbb{N}$ and $\mathbb{Z}$.) Note however that we do have the following properties:

Lemma 0.2 *For sets A, B, C we have the following:*
(i) if $A \approx B$ then $B \approx A$;
(ii) $A \approx A$;
(iii) if $A \approx B$ and $B \approx C$ then $A \approx C$.

Proof: (i) The inverse of a bijection is a bijection.
(ii) Consider the identity function $A \to A$.
(iii) The composition of two bijections is a bijection. $\square$

Note that a set A is equinumerous with a subset of a set B iff there is an injective function $f : A \to B$.

Definition 0.3 A set is *countably infinite* (or *denumerable*) if it is equinumerous with $\mathbb{N}$. A set is *countable* if it is finite or countably infinite. A set which is not countable is called *uncountable*.

Uncountable sets exist. Thus there are 'different sizes of infinity.' This was first observed by Georg Cantor. Here is an example of an uncountable set: the argument used to show the uncountablilty is called Cantor's diagonal argument.

Example 0.4 Let S be the set of all sequences of zeros and 1's. So formally S is the set of all functions $s : \mathbb{N} \to \{0, 1\}$. Then S is uncountable. For suppose there were a bijection $g : \mathbb{N} \to S$. Then consider the sequence $s \in S$ given by

$$s(n) = \begin{cases} 0 & \text{if } g(n)(n) = 1 \\ 1 & \text{if } g(n)(n) = 0 \end{cases}$$

Note that $g(n)(n)$ is the n-th term in the sequence $g(n)$. So the sequence s differs from the n-th sequence $g(n)$ in the n-th place. In particular, for all $n \in \mathbb{N}$ we have $s \neq g(n)$. Thus g cannot be onto: contradiction.

We can use this to observe that the set of real numbers $\mathbb{R}$ is uncountable. There is an obvious bijection between S and a subset of $\mathbb{R}$. Send the sequence s to the real number with decimal expansion

$$s(0) \cdot s(1)s(2)s(3) \ldots .$$

Now applying the fact below that a subset of a countable set has to be countable, we see that $\mathbb{R}$ is uncountable.

Theorem 0.5 *(i) Every subset of $\mathbb{N}$ is countable.*
(ii) Every subset of a countable set is countable.

Proof: Clearly (ii) follows from (i) as a subset of a countable set is equinumerous with a subset of $\mathbb{N}$. To prove (i), suppose S is an infinite subset of $\mathbb{N}$. Then there exists a function $f : \mathbb{N} \to S$ given by:

$f(0)$ is the least element of S;

$f(n+1)$ is the least element of $S \setminus \{f(0), \ldots, f(n)\}$.

[We're using things which will only be formally justified later.]

Note also that by definition, f is injective. It is onto, because if $s \in S$ then $f(n) = s$ for some $n \leq s$. $\square$

Corollary 0.6 *A set S is countable if and only if there exists an injective function $g : S \to \mathbb{N}$.* $\square$

Theorem 0.7 *(i) Let A, B be countable sets. Then $A \times B$ is countable.*

(ii) Let B be a countable set and let S be the set of all finite sequences of elements of B. Then S is countable.

Proof: Recall that a natural number $n > 1$ is a prime number if the only natural numbers dividing it are 1 and itself. Recall also that any natural number $m > 1$ can be written in a unique way as a product of powers of prime numbers (this is the Fundamental Theorem of Arithmetic: see your first-year notes, or look it up in a basic text).

(i) Let $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ be bijections (actually, injectivity is enough). Define a function $h : A \times B \to \mathbb{N}$ by

$$h(a, b) = 2^{f(a)} 3^{g(b)}.$$

Then by FTA h is injective and so $A \times B$ is countable.

(ii) This is similar. Let $p_0, p_1, p_2, p_3, \ldots$ be the sequence of primes (in some order, usually taken to be increasing). Let $f : B \to \mathbb{N}$ be an injection. Define a function $h : S \to \mathbb{N}$ as follows. Let h send the empty sequnce to 0. For $s = s(0)s(1)\ldots s(n) \in S$ let

$$h(s) = p_0^{f(s(0))+1} p_1^{f(s(1))+1} \ldots p_n^{f(s(n))+1}.$$

Then FTA implies that h is injective and so S is countable. $\square$

Theorem 0.8 *(i) A non-empty set S is countable if and only if there exists a surjection $h : \mathbb{N} \to S$.*

(ii) A non-empty set S is countable if and only if there exists a surjection $g : T \to S$ for some countable set T.

(iii) If A is a countable set of countable sets then

$$\bigcup A = \{y : (\exists x \in A)(y \in x)\}$$

is countable.

Proof: (i) One direction is clear. So suppose there exists a surjection $h : \mathbb{N} \to S$. For $s \in S$ let $g(s)$ be the smallest element of $h^{-1}(s) = \{n \in \mathbb{N} : h(n) = s\}$ (this set is non-empty as h is surjective). Then $g = \{(s, g(s)) : s \in S\}$ is an injective function from S to $\mathbb{N}$. So S is countable by Corollary 0.6.

(ii) This follows trivially from (i).

(iii) Let $F : \mathbb{N} \to A$ be a surjection. So for each $n \in \mathbb{N}$, $F(n)$ is a countable set. So there exists a surjection $g_n : \mathbb{N} \to F(n)$. Then $h : \mathbb{N} \times \mathbb{N} \to \bigcup A$ given by

$$h(n, m) = g_n(m)$$

is a surjection. So the result follows from (ii) and countability of $\mathbb{N} \times \mathbb{N}$ (Theorem 0.7). $\Box$

(The proof of (iii) used implicitly the Axiom of Choice - we will come back to this.)

Exercise: Show that the following sets are countable (you may use any of the above results):

(a) The set of finite subsets of $\mathbb{N}$.

(b) The set of subsets of $\mathbb{N}$ with finite complement.

(c) The set of rational numbers.

(d) The set of real numbers which are roots of non-zero polynomial equations with rational coefficients.

(e) The set of those real numbers which can be described by sentences in English.

L8 Class rep? ✓.

2. First-order logic

~~Propositional~~ (Predicate logic).

Plans

1) Introduce 1^{st}-order structures — Semantics

2) 1^{st}-order languages

3) Describe a formal system — Syntax

4) Show that the theorems of the formal system are logically valid formulas i.e. ones true in all structures.

(1)

4: Gödel's completeness thm.

2.1 Structures.

(2.1.1) Def: Suppose A is a set and $n \geq 1$. An n-ary relation (on A) is a subset

$$\overline{R} \subseteq A^n$$

(where $A^n = \{(a_1, \dots, a_n) : a_i \in A\}$ ← n-tuple.

An n-ary function on A is a function $\overline{f} : A^n \to A$

Examples.

a) ordering $\leq$ on $\mathbb{R}$:

2-ary relation on $\mathbb{R}$

b) $+$ on $\mathbb{C}$: 2-ary function on $\mathbb{C}$

c) $\bar{P} \subseteq \mathbb{Z}$ $\quad \bar{P} = \{x \in \mathbb{Z} : x \text{ is even}\}$

1-ary relation on $\mathbb{Z}$

(sometimes see 'predicates' rather than 'relations').

Notation: $\bar{R} \subseteq A^n$ is an n-ary rel. on A and $(a_1, \dots, a_n) \in A^n$ write $\bar{R}(a_1, \dots, a_n)$ to mean $(a_1, \dots, a_n) \in \bar{R}$.

(2.1.2) Def. A first-order structure $\mathcal{A}$ consists of :

1) A non-empty set A (the domain of $\mathcal{A}$)

2) A set $\{\bar{R}_i : i \in I\}$ of relations on A, $\bar{R}_i \subseteq A^{n_i}$.

3) A set $\{\bar{f}_j : j \in J\}$ of functions on A, $\bar{f}_j : A^{m_j} \to A$

4) A set $\{\bar{c}_k : k \in K\}$ of constants : just elements of A.

The sets I, J, K indexing sets (can be empty). Usually, subsets of $\mathbb{N}$.

the information:

$$\left.\begin{array}{l} (n_i : i \in I) \\ (m_j : j \in J) \\ \text{the set } K \end{array}\right\} \text{ is called the } \underline{\text{signature}} \text{ of } \mathfrak{A}.$$

Might denote the structure by:

$$\mathfrak{A} = \langle \underbrace{A}_{\text{domain}} ; \underbrace{(\bar{R}_i : i \in I)}_{\text{relations}}, \underbrace{(\bar{f}_j : j \in J)}_{\text{fns.}}, \underbrace{(\bar{c}_k : k \in K)}_{\text{constants.}} \rangle$$

(2.1.3) **Examples.**

1. Orderings

$A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$

and $I = \{1\}$, $J, K = \emptyset$
$n_1 = 2$

$\bar{R}_1(a_1, a_2)$ to mean $a_1 < a_2$.

2. Groups Could use the signature:

$\bar{R}$ 2-ary relation of equality,

$\bar{m}$ 2-ary function (for multiplication)

$\bar{i}$ 1-ary function (for inversion)

$\bar{e}$ constant (for the identity element).

(3) **Rings** Signature :

$\bar{R}$ 2-ary relation for equality

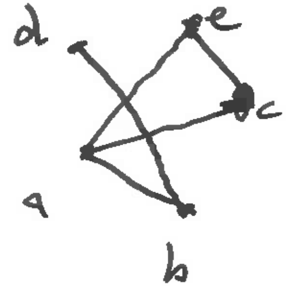
$\bar{m}$ 2-ary function for multiplication

$\bar{a}$ 2-ary function for addition

$\bar{n}$ 1-ary function $x \mapsto -x$

$\bar{0}, \bar{1}$ constants for zero and one.

(4) Graphs.



$\bar{R}$ 2-ary relation for equality

$\bar{E}$ 2-ary relation for 'adjacency'

eg. $\bar{E}(a, b)$

(2.2) First-order languages.

(4)

(2.2.1) Def. A first order language $\mathcal{L}$ has an alphabet of symbols :

variables : $x_0, x_1, x_2, \dots$

punctuation : ( ) ,

connectives : $\neg$ $\rightarrow$

quantifier : $\forall$

relation symbols : R_i $(i \in I)$

function symbols : f_j $(j \in J)$

constant symbols : c_k $(k \in K)$

Here I, J, K are indexing sets (could have J or K being $\emptyset$).

Each R_i comes with an <u>arity</u> n_i

Each f_j comes with an arity m_j.

The information $(n_i : i \in I)$, $(m_j : j \in J)$, K is called the <u>signature</u> of $\mathcal{L}$.

A first-order str. $\mathcal{A}$ with the same signature as $\mathcal{L}$ is called an <u>$\mathcal{L}$-structure</u>.

(2.2.2) <u>Def.</u> A <u>term</u> of $\mathcal{L}$ is defined as follows

i) any variable is a term;

ii) any constant symbol is a term;

iii) if f is an m-ary fn. symbol and $t_1, \ldots, t_m$ are terms then $f(t_1, \ldots, t_m)$ is a term.

iv) any term arises in this way.

<u>Example</u>: $\mathcal{L}$ has a 2-ary fn. symbol f and constant symbols c_1, c_2.

<u>Some terms</u>: c_1, c_2, x_1

$f(c_2, x_1)$; $f(f(c_2, x_1), x_2)$

<u>Not terms</u>: ffx_1x_2 $\quad g(x_1)$

not in $\mathcal{L}$

Typo in c/w qu. 2 : will repair.

(2.2.3) **Def.** (Formulas).

Definite $\mathcal{L}$-formulas inductively :

1. An <u>atomic formula</u> of $\mathcal{L}$ is of the form

$$R(t_1, \dots, t_m) \quad \text{with}$$

R an m-ary relation symbol and $t_1, \dots, t_m$ terms of $\mathcal{L}$.

2. i) Any atomic formula is an $\mathcal{L}$-formula

ii) if ϕ, ψ are $\mathcal{L}$-formulas then

$$(\neg \phi)$$
$$(\phi \to \psi)$$
$$(\forall x)\, \phi$$

are $\mathcal{L}$-formulas (where x is any variable)

<u>Example</u> Suppose $\mathcal{L}$ has

2-ary fn. symbol f
1-ary fn. symbol g
2-ary rel. symbol R
constant symbol c_1

Some terms

$$x_1,\ x_2,\ c_1,\ f(g(x_2), c_1)$$

Atomic formulas

$$R(x_1, f(g(x_2), c_1))$$
$$R(g(x_3), x_4)$$

$$(\forall x_4)\, R(g(x_3), x_4) \quad \dots$$

<u>Ex</u>: Take the signature for groups in 2.1.3 (2) write down some terms + atomic formulas.
Can you write the group axioms?

Eg. $(\forall x_1) R\left(m(i(x_1), x_1), e\right)$

(2.2.4) <u>Def</u>. (Short-hand)

Suppose ϕ, ψ are $\mathcal{L}$-fmlas

$(\exists x)\phi$ means
$$(\neg(\forall x)(\neg\phi))$$

$(\phi \vee \psi)$ means
$$((\neg\phi) \to \psi).$$

(2.2.5) <u>Def</u>. Suppose

$\mathcal{L}$ is as in 2.2.1 ~~...~~ and

$\mathcal{A} = \langle A; (\bar{R}_i : i \in I), (\bar{f}_j : j \in J), (\bar{c}_k : k \in K)\rangle$

is an $\mathcal{L}$-structure.

The correspondence between the relation, function + constant <u>symbols</u> in $\mathcal{L}$ and the actual relations, functions and contants of $\mathcal{A}$ (with matching arities) is called an <u>interpretation</u> of $\mathcal{L}$.
(or say $\mathcal{A}$ is an <u>interpretation</u> of $\mathcal{L}$).

(2.2.6) <u>Def</u>. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure. A <u>valuation</u> in $\mathcal{A}$ is a function v from the set of terms of $\mathcal{L}$ to A (the domain of $\mathcal{A}$) satisfying:

a) $v(c_k) = \bar{c}_k$

b) if $t_1, \ldots, t_m$ are terms and f is an m-ary fn. symbol (of $\mathcal{L}$) then

$$v\big(f(t_1, \ldots, t_m)\big) = \bar{f}\big(v(t_1), \ldots, v(t_m)\big)$$

(2.2.7) <u>Lemma</u>: Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and $a_0, a_1, \ldots \in A$. Then there is a unique valuation v (in $\mathcal{A}$) with

$$v(x_\ell) = a_\ell \quad (\text{for all } \ell \in \mathbb{N}).$$

[the variables are $x_0, x_1, \ldots$].

<u>Pf</u>: (Sketch) By induction on the length of terms: show that if we let

i) $v(x_\ell) = a_\ell$ (for $\ell \in \mathbb{N}$)

ii) $v(c_k) = \bar{c}_k$ ($k \in K$)

iii) $v\big(f(t_1, \ldots, t_m)\big) = \bar{f}\big(v(t_1), \ldots, v(t_m)\big)$

then this is a well-defined valuation.

#

<u>Example</u> Groups

$$\text{Signature} \quad \left.\begin{array}{l} R \\ m \\ i \\ e \end{array}\right\} \text{ as in } 2.1.3 .$$

Let $\mathcal{G}$ be a group

$$= \langle G ; \bar{R} ; \bar{m}, \bar{i}, \bar{e} \rangle$$

Let $g, h \in G$.

Let v be a valuation with

$$v(x_0) = g \;,\; v(x_1) = h \;, \ldots$$

$$v\Big(m\big(m(x_0, x_1), i(x_0)\big)\Big)$$

$$= g h g^{-1} .$$

$$v\big(m(x_0, x_1)\big)$$

$$= \bar{m}\big(v(x_0), v(x_1)\big)$$

$$= gh \qquad \text{etc.}$$

(2.2.9) Def.

Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and v is a valuation in $\mathcal{A}$. Define, inductively, for an $\mathcal{L}$-formula ϕ what's meant by

v satisfies ϕ (in $\mathcal{A}$)

(abbreviated as $v[\phi] = T$)

negation: $v[\phi] = F$ for v does not satisfy ϕ (in $\mathcal{A}$)).

(i) ~~St~~ Atomic formulas:

Suppose R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms (of $\mathcal{L}$). Then v satisfies the atomic formula $R(t_1, \dots, t_n)$

iff $\bar{R}(v(t_1), \dots, v(t_n))$

holds in $\mathcal{A}$.

(ii) Suppose ϕ, ψ are $\mathcal{L}$-formulas (& we already know about valuations satisfying ϕ, ψ).

(a) Say $v[(\neg\phi)] = T$

iff $v[\phi] = F$

(i.e. v satisfies $(\neg\phi)$ iff

v dns ϕ (in $\mathcal{A}$)).
does not satisfy

(b) Say $v[(\phi \to \psi)] = F$

iff $v[\phi] = T$ & $v[\psi] = F$.

(c)

(2.2.9) (ii) (c): v satisfies $(\forall x_\ell)\phi$ (in $\mathcal{A}$) iff whenever w is a valuation (in $\mathcal{A}$) which is <u>x_ℓ-equivalent</u> to v, then w satisfies ϕ.

// Uses:

(2.2.8) <u>Def</u>: Suppose v, w are valuations in an $\mathcal{L}$-str. $\mathcal{A}$ and x_ℓ is a variable. Say that v, w are <u>x_ℓ-equivalent</u> if $v(x_m) = w(x_m)$ when $m \neq \ell$.

[<u>Remark</u>: Def. 2.2.9 does not work if we allow the domain of $\mathcal{A}$ to be empty: there are no valuations.]

// If every valuation in $\mathcal{A}$ satisfies ϕ, say that ϕ is <u>true</u> in $\mathcal{A}$ or $\mathcal{A}$ is a <u>model</u> of ϕ & write $\mathcal{A} \models \phi$.

If $\mathcal{A} \models \phi$ for every $\mathcal{L}$-str. say that ϕ is <u>logically valid</u> and write $\models \phi$.

These are the analogues of the tautologies in prop. logic.

(2.2.10) Examples. (of using the terminology)

1) Suppose $\mathcal{L}$ has a 2-ary rel. symbol R. The $\mathcal{L}$-formula:

$$\left(R(x_1,x_2) \to ((R(x_2,x_3) \to R(x_1,x_3)))\right)$$

is true in

$$\mathcal{A} = \langle \mathbb{N} \,;\, < \rangle$$

domain

R interpreted as "$<$".

2) The same formula is not true in the str.

$$\mathcal{B} = \langle \mathbb{N} \,;\, \neq \rangle$$

~~(so R(x is interp)~~

(so $R(x_1,x_2)$ is interpreted as "$x_1 \neq x_2$")

Eg. let v be a valuation with $v(x_1) = 1 = v(x_3)$ and $v(x_2) = 2$.

Then (using 2.2.9)

$$v[R(x_1,x_2)] = T$$

$$v[R(x_2,x_3)] = T$$

$$v[R(x_1,x_3)] = F.$$

So $v[\text{formula}] = F$.

3) Recall $(\exists x_1)\phi$ is shorthand for

$$(\neg(\forall x_1)(\neg\phi))$$

<u>Lemma</u>. Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and v is a valuation in $\mathcal{A}$. Then v satisfies $(\exists x_1)\phi$ (in $\mathcal{A}$) <u>iff</u> there is a valuation w which is x_1-equivalent to v and where $w[\phi] = T$.

<u>Pf</u>: <u>$\Rightarrow$</u>: Suppose v satisfies $(\exists x_1)\phi$ (in $\mathcal{A}$)

ie v satisfies $(\neg(\forall x_1)(\neg\phi))$.

$\therefore$ By 2.2.9 we have

$$v[(\forall x_1)(\neg\phi)] = F.$$

Again (by 2.2.9(ii)(c)) there is a valuation w x_1-equiv. to v with

$$w[(\neg\phi)] = F.$$

But then, for this w,

$$w[\phi] = T,$$

as req^d. //.

<u>$\Leftarrow$</u>: Ex. $\#$

(2.2.11) <u>Example</u>:

$(\forall x_1)(\exists x_2) R(x_1, x_2)$ is true in $\langle \mathbb{Z}; < \rangle$ but not in $\langle \mathbb{N}; > \rangle$

(2.2.12) Suppose ϕ is any $\mathcal{L}$-fmla. Then

(1) $\left((\exists x_1)(\forall x_2) \phi \rightarrow (\forall x_2)(\exists x_1) \phi \right)$

is logically valid.

(2) $\left((\forall x_2)(\exists x_1) \phi \rightarrow (\exists x_1)(\forall x_2) \phi \right)$

is not necessarily logically valid.

//

(1) Ex. (using valuations).

(2) Give an example.

Some logically valid formulas.

Consider the ~~pr~~ propositional formula

$\chi \qquad (p_1 \rightarrow (p_2 \rightarrow p_1))$

Suppose $\mathcal{L}$ is 1st order language and ϕ_1, ϕ_2 are $\mathcal{L}$-fmlas.

Substitute $\left.\begin{array}{l} \phi_1 \text{ in place of } p_1 \\ \phi_2 \text{ in place of } p_2 \end{array}\right)$ in χ

The ~~res~~ resulting:

$\theta \qquad (\phi_1 \rightarrow (\phi_2 \rightarrow \phi_1))$

is an $\mathcal{L}$-formula and θ is logically valid.

Suppose v is a valuation in an $\mathcal{L}$-str. $\mathcal{A}$.

5

Suppose for a ↯ (contradiction)

that $v[\theta] = F$.

then $v[\phi_1] = T$ & $v[(\phi_2 \to \phi_1)] = F$

so $v[\phi_2] = T$

& $v[\phi_1] = F$.

Contradiction.

(2.2.13) Def. Suppose χ is an L-formula involving prop. vars. $p_1, \dots, p_n$. Suppose $\mathcal{L}$ is a 1^{st} order language and $\phi_1, \dots, \phi_n$ are $\mathcal{L}$-formulas.

A substitution instance of χ is obtained by replacing each p_i in χ by ϕ_i (for $i = 1, \dots, n$). Call the result θ.

(2.2.14) Thm (1) θ is an $\mathcal{L}$-formula.

(2) Let v be an valuation in an $\mathcal{L}$-str. $\mathcal{A}$. Let w be a prop. valuation with $w(p_i) = v[\phi_i]$ (for $i = 1, \dots, n$).

Then $\underbrace{v[\theta]}_{\mathcal{L}\text{-formula}} = \underbrace{w(\chi)}_{L\text{-formula}}$

(3) If χ is a tautology then θ is logically valid.

Pf: (Sketch) (1) Omit.

(3) follows from (2) (as $w(\chi) = T$ in this case).

(2) By induction on the number of connectives in χ.

Base case Ex.

Inductive step:

(a) χ is $(\neg\alpha)$

(b) χ is $(\alpha \to \beta)$

[for L-formulas α, β].

(a) Ex.

(b) θ is of the form

$$(\theta_1 \to \theta_2)$$

where θ_1 is obtained from α by substituting ϕ_i in place of p_i in α;

similarly for θ_2.

$w(\chi) = F$

$\iff w(\alpha) = T$ & $w(\beta) = F$

$\iff v[\theta_1] = T$ & $v[\theta_2] = F$

induction

$\iff v[(\theta_1 \to \theta_2)] = F$

$\iff v[\theta] = F$.

which does the ind. step in this case. //.

#.

Note: Not all logically valid formulas arise in this way

Eg. $((\exists x_2)(\forall x_1)\phi \to (\forall x_1)(\exists x_2)\phi)$

is logically valid, but not a subst. instance of a prop. tautology.

(2.3) Bound and free variables.

1) Eg: ψ_1

$$\Big(R_1(\overset{\text{free}}{x_1},\overset{\text{free}}{x_2}) \to (\forall \overset{\text{bound}}{x_3}) \underbrace{R_2(\overset{\text{free}}{x_1},\overset{\text{bound}}{x_3})}_{\text{scope of}}\Big)$$

(2.3.1) Def. Suppose ϕ, ψ are $\mathcal{L}$-formulas and $(\forall x_i)\phi$ occurs as a subformula of ψ

i.e. ψ is $\ldots\ldots(\forall x_i)\phi\ldots\ldots$

We say that ϕ is the scope of the quantifier $(\forall x_i)$ here.

A occurrence of a variable x_j in ψ is bound if it is in the scope of a quantifier $(\forall x_j)$ in ψ (or it is the x_j here).

Otherwise, it is a free occurrence of in ψ. Variables having a free occurrence in ψ are called free variables of ψ.

A formula with no free variables is called a closed formula (or an $\mathcal{L}$-sentence).

Examples

2) ψ_2:

$$\Big((\forall \overset{\text{bound}}{x_1}) \underbrace{R_1(\overset{\text{bound}}{x_1},\overset{\text{free}}{x_2})}_{\text{scope}} \to R_2(\overset{\text{free}}{x_1},\overset{\text{free}}{x_2})\Big)$$

Compare with:

bound

$$(\forall x_1)(\underbrace{R_1(x_1, x_2) \to R_2(x_1, x_2)}_{\text{scope}})$$

3) ψ_3:

free   bound free

$$((\exists x_3)\underbrace{R_1(x_1, x_2)}_{\text{scope}} \to (\forall x_2)\underbrace{R_2(x_2, x_3)}_{\text{scope}})$$

//

(2.3.2) <u>Notation</u>:

If ψ is an $\mathcal{L}$-formula with free variables amongst $x_1, \ldots, x_n$ write $\psi(x_1, \ldots, x_n)$ (instead of ψ).

If $t_1, \ldots, t_n$ are terms then by $\psi(t_1, \ldots, t_n)$ we mean the $\mathcal{L}$-formula obtained by replacing each <u>free</u> occurence of x_i in ψ by t_i (for $i = 1, \ldots, n$).

Eg

$\psi(x_1, x_2)$

$$\left(\left(\forall x_1\right) R_1(x_1, \overset{\text{free}}{x_2}) \to (\forall x_3) R_2(\overset{\text{free}}{x_1}, \overset{\text{free}}{x_2}, x_3)\right)$$

t_1 $\qquad f_1(x_1)$

t_2 $\qquad f_2(x_1, x_2)$

$\psi(t_1, t_2)$

$$\left((\forall x_1) R_1(x_1, f(x_1, x_2)) \to (\forall x_3) R_2\left(f_1(x_1), f(x_1, x_2), x_3\right)\right)$$

(2.3.3) **Theorem**

Suppose ϕ is a closed $\mathcal{L}$-formula and $\mathcal{A}$ is an $\mathcal{L}$-str. Then

either $\mathcal{A} \models \phi$

or $\mathcal{A} \models (\neg \phi)$

More generally, if ϕ has free variables amongst $x_1, \dots, x_n$ and v, w are valuations in $\mathcal{A}$ with

$v(x_i) = w(x_i)$ for $i = 1, \dots, n$

then $v[\phi] = T \iff w[\phi] = T$

(allow $n = 0$ here: no free vars.).

Pf: Note that the first statement follows from the general statement. If ϕ has no free variables then for any valuations v, w (in $\mathcal{A}$) they agree on the free variables, so

$v[\phi] = w[\phi]$.

//

Prove generalization by ind. on number of connectives + quantifiers in ϕ.

Base case: ϕ is atomic

$R(t_1, \dots, t_m)$ t_j terms.

The t_j only involve variables from $x_1, \dots, x_n$. So

$v(t_j) = w(t_j)$ for $j = 1, \dots, m$

(compare 2.2.6).

then (2.2.9 ?)

$v[R(t_1,\dots,t_m)] = T$

$\iff \bar{R}(v(t_1),\dots,v(t_m))$ holds in $\mathcal{A}$

$\iff \bar{R}(w(t_1),\dots,w(t_m))$ holds in $\mathcal{A}$

$\iff w[R(t_1,\dots,t_m)] = T$.

//

Ind. step. ϕ is $(\neg\psi)$, $(\psi \to \chi)$ or $(\forall x_i)\psi$.

First two cases: Ex.

Suppose ϕ is $(\forall x_i)\psi$

Suppose $v[\phi] = F$.

Want to show $w[\phi] = F$.

(By symmetry, this is enough.)

By Def 2.2.9 (ii)(c) there is a valuation v' x_i-equiv. to v with $v'[\psi] = F$.

The free variables of ψ are amongst $x_1,\dots,x_n, x_i$.

Let w' be the valuation x_i-equiv. to w with $w'(x_i) = v'(x_i)$.

Then v', w' agree on the free variables of ψ.

By ind. hyp. (on ψ)

$v'[\psi] = w'[\psi]$

so $w'[\psi] = F$. As w' is x_i-equiv. to w, we have $w[(\forall x_i)\psi] = F$, ie. $w[\phi] = F$.

//

#

Notation: If $\mathcal{A}$ is an $\mathcal{L}$-structure and $\psi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-fmla (whose free vars. are amongst $x_1, \ldots, x_n$) and $a_1, \ldots, a_n \in A$ (domain of $\mathcal{A}$)

then write

$$\mathcal{A} \models \psi(a_1, \ldots, a_n)$$

to mean $v[\psi] = T$ whenever v is a valuation with $v(x_i) = a_i$ for $i = 1, \ldots, n$.

(Note: By pf. of 2.3.3 this holds if $v[\psi] = T$ for some such v.)

(2.3.4) Warning Example

An example where

$$\mathcal{A} \models (\forall x_1)\, \phi(x_1)$$

but where we a term t_1 and a valuation v in $\mathcal{A}$ with $v[\phi(t_1)] = F$.

(?? Expect $v[\phi(t_1)] = T$)

But no.

$$\phi(x_1) : ((\forall x_2) R(\underset{\substack{\uparrow \\ \text{free}}}{x_1}, x_2) \to S(\underset{\substack{\uparrow \\ \text{free}}}{x_1})))$$

t_1 is the term x_2.

$$\phi(t_1) : ((\forall x_2) R(x_2, x_2) \to S(x_2)))$$

$\mathcal{A}$ Domain $\mathbb{N} = \{0, 1, 2, \dots\}$

$R(x_1, x_2)$ interpreted as '$x_1 \leq x_2$'

$S(x_1)$ interpreted as '$x_1 = 0$'

So $\mathcal{A} \models (\forall x_1) \phi(x_1)$

$(\mathcal{A} \models (\forall x_1)((\forall x_2) R(x_1, x_2) \to S(x_1))))$

But if $v(x_2) = 1$ then

$v[\phi(t_1)] = F$. //

(2.3.5) Def. Let ϕ be an $\mathcal{L}$-formula, x_i a variable and t a term of $\mathcal{L}$. We say that t is free for x_i in ϕ if there is no variable x_j in t such that x_i occurs free within the scope of a quantifier $(\forall x_j)$ in ϕ.

NOT free for x_i is ϕ:

$$\phi : \quad \dots\dots (\forall x_j) \underbrace{\dots \overset{\substack{\text{free} \\ \swarrow}}{x_i}}_{\text{scope}}$$

$$t \qquad \dots x_j \dots$$

In example: t_1 is not free for x_1 in ϕ.

(2.3.6) **Thm**

Suppose $\phi(x_1)$ is an $\mathcal{L}$-fmla (possibly with other free variables). Let t be a term free for x_1 in ϕ.

Then $\models ((\forall x_1)\phi(x_1) \to \phi(t))$.

In particular if $\mathcal{A}$ is a $\mathcal{L}$-str. with $\mathcal{A} \models (\forall x_1)\phi(x_1)$ then $\mathcal{A} \models \phi(t)$.

Eg: take $t = x_1$ here.

So if $\mathcal{A} \models (\forall x_1)\phi(x_1)$ then $\mathcal{A} \models \phi(x_1)$.

Follows from:

(2.3.7) **Lemma** Suppose v is a valuation in $\mathcal{A}$. Let v' be the val. in $\mathcal{A}$ which is x_1-equiv. to v with

$v'(x_1) = v(t)$.

Then $v'[\phi(x_1)] = T$

$\iff v[\phi(t)] = T$.

Lemma $\Rightarrow$ Thm:

Suppose v is a valuation with $v[\phi(t)] = F$. Show $v[(\forall x_1)\phi(x_1)] = F$.

Take v' as in the Lemma.

Then by Lemma $v'[\phi(x_1)] = F$

So $v[(\forall x_1)\phi(x_1)] = F$ as v' is x_1-equiv. to v. //

(1) 2.4 The Formal System $K_{\mathcal{L}}$

(2.4.1) Def. Suppose $\mathcal{L}$ is a 1st order language. The formal system $K_{\mathcal{L}}$ has formulas the $\mathcal{L}$-formulas and:

Axioms For ϕ, ψ, χ $\mathcal{L}$-formulas

A1 $(\phi \to (\psi \to \phi))$

A2 $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

A3 $(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$

K1 $((\forall x_i)\phi(x_i) \to \phi(t))$

where t is a term free for x_i in ϕ [ϕ can have other free variables]

K2 $((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi))$

if x_i is not free in ϕ.

Deduction rules: MP From ϕ and $(\phi \to \psi)$ deduce ψ

Gen (Generalisation) from ϕ deduce $(\forall x_i)\phi$.

A **proof** in $K_{\mathcal{L}}$ is a finite sequence of $\mathcal{L}$-fmlas, each of which is an axiom or deduced from previous fmlas. using a deduction rule. A **theorem** of $K_{\mathcal{L}}$ is the last fmla. in some proof.

Write $\vdash_{K_{\mathcal{L}}} \phi$ if ϕ is a theorem of $K_{\mathcal{L}}$

(or $\vdash \phi$).

(2.4.2) **Def**. Suppose Σ is a set of $\mathcal{L}$-fmlas. and ψ is an $\mathcal{L}$-fmla.

A **deduction** of ψ from Σ is a finite sequence of fmlas, ending with ψ, each of which is an axiom, an elt. of Σ or obtained from fmlas earlier in the deduction using MP or Gen with the restriction that when Gen is applied it does not involve a variable occurring freely in Σ.

Write $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ in this case.

(Say ψ is a consequence of Σ.)

(2.4.3) Remarks

(1) If Σ consists of closed fmlas, don't have to worry about the ~~ster~~ restriction.

(2) Without the restriction would have
" $\{\phi\} \vdash (\forall x_i)\phi$ "
– not sensible.

(3) Should have: if $\Sigma' \subseteq \Sigma$ and $\Sigma' \vdash \phi$ then $\Sigma \vdash \phi$.

So ought to modify the defn. to allow for this.

(2.4.4) Theorem. Suppose ϕ is an $\mathcal{L}$-fmla which is a substitution instance of a propositional tautology χ. Then $\vdash_{K_{\mathcal{L}}} \phi$.

Eg $((\neg(\neg\phi_1)) \to \phi_1)$ (for a $\mathcal{L}$-fmla ϕ_1) is a subst. instance of the prop. taut. $((\neg(\neg p_1)) \to p_1)$.

Pf: Let $p_1, \ldots, p_n$ be the prop. vars. in χ & we obtain ϕ by substituting $\psi_1, \ldots, \psi_n$ in place of $p_1, \ldots, p_n$ in χ.

By the Completeness Thm for L (1.3.11), there is a pf in L of χ: $\chi_1, \ldots, \chi_r$ where χ is χ_r.

If we substitute $\psi_1, \ldots, \psi_n$ in $\chi_1, \ldots, \chi_r$ in place of $p_1, \ldots, p_n$ we obtain a pf. of ϕ in $K_{\mathcal{L}}$. // . #.

(2.4.5) Thm. (Soundness of $K_{\mathcal{L}}$) 4

If $\vdash_{K_{\mathcal{L}}} \phi$ then

$\models \phi$ (ie. ϕ is logically valid).

Pf: Like the pf. for L (1.3.1)

- Show that the axioms are logically valid +
- the deduction rules preserve logical validity.

A1, A2, A3 are sub. instances of prop. tauts. so are logically valid by (2.2.14).

K1 is logically valid, by 2.3.6.

K2 $\left((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi) \right)$

(x_i not free in ϕ).

Let v be a valuation (in an $\mathcal{L}$-str. $\mathcal{A}$) with

$v[(\phi \to (\forall x_i)\psi] = F$.

then $v[\phi] = T$ & $v[(\forall x_i)\psi] = F$.

So there is v' x_i-equiv. to v with $v'[\psi] = F$.

x_i is not free in ϕ, so as v, v' are x_i-equiv. v, v' agree on the free variables in ϕ. thus

$v'[\phi] = v[\phi] = T$

2.3.3

So $v'[(\phi \to \psi)] = F$.

thus

$v[(\forall x_i)(\phi \to \psi)] = F$

(as v, v' are x_i-equiv.).

So $v[K2] = T$, as req'd. //.

MP: If $\models \phi$ and $\models (\phi \to \psi)$ then $\models \psi$. // Ex.

Gen: If $\models \phi$ then $\models (\forall x_i)\phi$. // Easy Ex.

#.

(i)

(2.4.6) **Cor.** (Consistency of $K_{\mathcal{L}}$)

There is no formula ϕ with $\vdash_{K_{\mathcal{L}}} \phi$ and $\vdash_{K_{\mathcal{L}}} (\neg\phi)$.

Pf: Follows from 2.4.5 as ϕ, $(\neg\phi)$ cannot both be logically valid. //. #.

(2.4.7) **Ex.** Suppose Σ is a set of $\mathcal{L}$-fmlas and ψ is an $\mathcal{L}$-fmla with $\Sigma \vdash_{K_{\mathcal{L}}} \psi$. Then for every valuation v with $v(\Sigma) = T$ (i.e. $v[\sigma] = T$ for all $\sigma \in \Sigma$) we have $v[\psi] = T$. //.

(2.4.8) **Thm.** (Deduction Thm.)

Suppose $\mathcal{L}$ is a 1st order language, Σ is a set of $\mathcal{L}$-fmlas and ϕ, ψ are $\mathcal{L}$-formulas. If

$$\Sigma \cup \{\phi\} \vdash_{K_{\mathcal{L}}} \psi$$

then

$$\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi).$$

Pf: Like the DT for L (1.2.5).

By induction on the length of the deduction of ψ from $\Sigma \cup \{\phi\}$.

Base case: If there is a one-step deduction, argue as in 1.2.5.

<u>Inductive step</u> Suppose ψ follows from earlier formulas in the deduction using MP or Gen

<u>MP</u> Exactly as in 1.2.5.

<u>Gen</u> Suppose ψ is obtained by using Gen. So ψ is $(\forall x_i)\,\theta$ and

$$\Sigma \cup \{\phi\} \vdash \theta$$

<u>and</u> x_i is not free in any formula in $\Sigma \cup \{\phi\}$.

By ind. hypothesis

$$\Sigma \vdash (\phi \to \theta)$$

Apply Gen. to this (x_i not free in Σ).

So $\Sigma \vdash (\forall x_i)(\phi \to \theta)$

Use axiom K2 (noting x_i not free in ϕ), MP to get

$$\Sigma \vdash (\phi \to (\forall x_i)\,\theta).$$

ie $\Sigma \vdash (\phi \to \psi)$ //.

this finishes the ind. step. #

---

(3)

2.5 Gödel's Completeness thm.

(2.5.1) **Def.** A set Σ of $\mathcal{L}$-formulas is consistent if there is no $\mathcal{L}$-fmla ϕ with $\Sigma \vdash_{K_\mathcal{L}} \phi$ and $\Sigma \vdash_{K_\mathcal{L}} (\neg\phi)$.

[2.4.6 : "ϕ is consistent" or $K_\mathcal{L}$ is consistent.].

Rk: If Σ is inconsistent then $\Sigma \vdash \chi$ for any $\mathcal{L}$-fmla χ.

[as with L.].

Recall: A closed $\mathcal{L}$-fmla ϕ is one with no free variables.

//

Show: If Σ is a consistent set of closed $\mathcal{L}$-fmlas then there is an $\mathcal{L}$-structure $\mathcal{A}$ with $\mathcal{A} \models \Sigma$ (i.e. $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$.).

//

Simplification: Assume $\mathcal{L}$ is countable i.e. the variables are $x_0, x_1, x_2, \ldots$ and there are countably many relation, function and constant symbols.

So we can list the $\mathcal{L}$-formulas (or any subset of the $\mathcal{L}$-fmlas) as a list indexed by $\mathbb{N}$.

Eg. enumerate the closed $\mathcal{L}$-fmlas as $\psi_0, \psi_1, \psi_2, \ldots$

($\{\psi_i : i \in \mathbb{N}\}$).

(2.5.2) Proposition Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas and ϕ a closed $\mathcal{L}$-fmla.

① (like 1.3.7) If

$$\Sigma \nvdash_{K_\mathcal{L}} \phi \quad \text{then}$$

$$\Sigma \cup \{(\neg\phi)\} \text{ is}$$

consistent.

② (Lindenbaum Lemma, (like 1.3.8)).

There is a consistent set

$\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas

such that for every closed $\mathcal{L}$-fmla ψ either

$$\Sigma^* \vdash \psi$$

$$\text{or} \quad \Sigma^* \vdash (\neg\psi).$$

Pf: ① As in 1.3.7.

(Use DT & $\vdash_{K_\mathcal{L}} ((\neg\phi) \to \phi) \to \phi$)

② Uses ①

and the enumeration ($\psi_i : i \in \mathbb{N}$)

of the closed $\mathcal{L}$-fmlas. #

(2.5.3) Thm. (Model Existence Thm.)
Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas. Then there is an $\mathcal{L}$-str. $\mathcal{A}$ with $\mathcal{A} \models \Sigma$.

Pf: Hard bit: later. #.

Notation: $\Sigma \models \phi$

means "if $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \phi$."

(2.5.4) Thm. Let Σ be a set of closed $\mathcal{L}$-fmlas and ϕ a closed $\mathcal{L}$-fmla. If $\Sigma \models \phi$, then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$.

5

Pf: If Σ is inconsistent then $\Sigma \vdash \psi$ for every ψ.

So assume Σ is consistent. Suppose $\Sigma \nvdash \phi$. By 2.5.2 1

$\Sigma \cup \{(\neg\phi)\}$ is consistent.

So by 2.5.3, there is an $\mathcal{L}$-str. $\mathcal{A}$ with

$$\mathcal{A} \models \Sigma \cup \{(\neg\phi)\}.$$

So $\mathcal{A} \models \Sigma$ and $\mathcal{A} \models (\neg\phi)$.

This contradicts $\Sigma \models \phi$.

#.

(2.5.5) <u>Thm.</u> (<u>$\mathcal{L}$ countable</u>) (Gödel Completeness Thm. for $K_{\mathcal{L}}$) If ϕ is an $\mathcal{L}$-fmla with $\models \phi$, then ϕ is a theorem of $K_{\mathcal{L}}$ (ie. $\vdash_{K_{\mathcal{L}}} \phi$).

<u>Pf</u>: If ϕ is closed, then this follows from 2.5.4 (by taking $\Sigma = \emptyset$).

Suppose ϕ has its free variables amongst $x_0, \ldots, x_n$ & ~~and~~ consider ψ:

$(\forall x_0) \ldots (\forall x_n) \phi$.

(this is closed).

6

As $\models \phi$

we have $\models \psi$:

So $\vdash_{K_{\mathcal{L}}} \psi$ (by the closed case!)

ie. $\vdash_{K_{\mathcal{L}}} (\forall x_0) \ldots (\forall x_n) \phi$ ... (*)

If θ is any fmla. then $((\forall x_i)\theta \to \theta)$ is an axiom of type K1. (with t being x_i)

Using (*), these axioms and MP we obtain $\vdash_{K_{\mathcal{L}}} \phi$.

#

1

(2.5.6) Cor. (Compactness Thm.)

Suppose $\mathcal{L}$ is a countable 1st order language, Σ is a set of closed $\mathcal{L}$-formulas and every finite subset of Σ has a model. Then Σ has a model.

Pf: Suppose Σ has no model. By 2.5.3, Σ is inconsistent. So there an $\mathcal{L}$-fmla ϕ with $\Sigma \vdash \phi$ and $\Sigma \vdash (\neg\phi)$.

Deductions only involve finitely many formulas in Σ.

So there is a finite subset $\Sigma_0 \subseteq \Sigma$ with

$$\Sigma_0 \vdash \phi \quad \text{and} \quad \Sigma_0 \vdash (\neg\phi).$$

So Σ_0 is a finite subset of Σ which is inconsistent. So Σ_0 has no model. ~~Contra~~ Contradiction. #

(2.5.3) Thm. (Model Existence Thm.)

~~The~~ Suppose $\mathcal{L}$ is a countable 1st order language and Σ is a set of closed $\mathcal{L}$-formulas which is consistent. Then there is a countable $\mathcal{L}$-str. $\mathcal{A}$ such that $\mathcal{A} \models \phi$ for every $\phi \in \Sigma$.

Sketch of proof:

Notation is cumulative.

Step 1 let $b_0, b_1, \ldots$ be new constant symbols. Form $\mathcal{L}^+$ by adding these symbols to those in $\mathcal{L}$. Regard Σ as a set of $\mathcal{L}^+$-formulas.

Check Σ is still consistent (in the formal system $K_{\mathcal{L}^+}$).

- See notes.

Note: $\mathcal{L}^+$ is a countable language.

Step 2 (Adding witnesses).

Lemma there is a consistent set of closed $\mathcal{L}^+$-formulas $\Sigma_\infty \supseteq \Sigma$ such that for every $\mathcal{L}^+$-fmla $\theta(x_i)$ with one free variable there is some b_j with

$$\Sigma_\infty \vdash_{K_{\mathcal{L}^+}} \left((\neg(\forall x_i)\theta(x_i)) \to (\neg\theta(b_j)) \right)$$

2

[Why?

- Think of $\theta(x_i)$ as $\neg\chi(x_i)$.

then this formula is

$$\big((\exists x_i)\chi(x_i) \to \chi(b_j)\big)$$

so b_j "witnesses" the existence of x_i satisfying $\chi(x_i)$.)]

<u>Step 3</u> By the Lindenbaum Lemma (2.5.2) there is <u>consistent</u> $\Sigma^* \supseteq \Sigma_\infty \supseteq \Sigma$ of closed $\mathcal{L}^+$-formulas s.t. for every closed $\mathcal{L}^+$-fmla ϕ either $\Sigma^* \vdash_{K_{\mathcal{L}^+}} \phi$ or $\Sigma^* \vdash_{K_{\mathcal{L}^+}} (\neg\phi)$.

(3)

<u>Step 4</u>. Let

$$A = \left\{ \bar{t} \;:\; \begin{array}{l} t \text{ is a closed term} \\ \text{of } \mathcal{L}^+ \end{array} \right\}$$

<u>Note</u>: (1) A term is closed if it only involves function + constant symbols of $\mathcal{L}^+$ (no variables).

(2) Use the $\bar{\ }$ to distinguish when we're thinking of a closed term as an elt. of A.

(3) As $\mathcal{L}^+$ is countable, A is countable.

=

Make A into an $\mathcal{L}^+$-structure:

(1) Each constant symbol c of $\mathcal{L}^+$ is interpreted as $\bar{c} \in A$.

(2) Suppose R is an n-ary relation symbol. Define the relation $\bar{R} \subseteq A^n$ by

$$(\bar{t}_1, \ldots, \bar{t}_n) \in \bar{R}$$

$$\iff \underbrace{\Sigma^* \vdash_{K_{\mathcal{L}^+}} R(t_1, \ldots, t_n)}_{\text{closed, atomic } \mathcal{L}^+\text{-formula}}.$$

where $t_1, \ldots, t_n$ are closed $\mathcal{L}^+$-~~fmlae~~ terms.

(3) Suppose f is an m-ary function symbol. Define a fn. $\bar{f} : A^m \to A$ by

$$\bar{f}(\bar{t}_1, \ldots, \bar{t}_m) = \overline{f(t_1, \ldots, t_m)}.$$

Call this structure $\mathcal{A}$.

Note: If v is a valuation in $\mathcal{A}$ & t is a closed term then

$$v(t) = \bar{t}. \quad (\text{by } (1) + (3))$$ (4)

Main Lemma: For every closed $\mathcal{L}^+$-formula ϕ

$$\Sigma^* \vdash_{K_{\mathcal{L}}} \phi \iff \mathcal{A} \models \phi$$ (*)

[It then follows that $\mathcal{A} \models \Sigma$.]

Pf: By induction on number of connectives + quantifiers in ϕ.

Base case ϕ is atomic i.e. $R(t_1, \ldots, t_n)$ for some ~~fm~~ $\mathcal{L}^+$-terms $t_1, \ldots, t_n$. The t_i are closed terms. This case follows by defn. of $\bar{R}$ in $\mathcal{A}$.

Inductive step Assume (*) holds for closed fmlas involving fewer connectives + quantifiers.

Case 1 ϕ is $(\neg\psi)$

Case 2 ϕ is $(\psi \to \chi)$

Case 3 ϕ is $(\forall x_i)\psi$.

In cases 1 + 2, ψ, χ are closed. So (*) holds for ψ, χ.

Case 1. ϕ is $(\neg\psi)$.

$\mathcal{A} \models \phi \iff \mathcal{A} \not\models \psi$

$\overset{\text{(by 2.3.3)}}{\iff}$

$\overset{\text{ind}}{\iff} \Sigma^* \not\vdash \psi$

$\overset{(*)}{\iff} \Sigma^* \vdash (\neg\psi)$

(as ψ is closed).

step 3 (of ind. step).

Case 2 Ex.

Case 3 ϕ is $(\forall x_i)\psi$

Case 3(a) x_i is not free in ψ

So ψ is closed + can use ind. hyp. .... Ex.

Case 3(b) x_i is free in ψ.

So $\psi(x_i)$ has a single free variable.

$\Longrightarrow$ See notes. //

6

$\Leftarrow$ in (*) Suppose for a contradiction that

$\mathcal{A} \models \phi$ and $\Sigma^* \nvdash \phi$.

By Step 3 $\Sigma^* \vdash (\neg \phi)$

By Step 2:

$$\Sigma^* \vdash \Big(\underbrace{\neg (\forall x_i) \psi(x_i)}_{\phi} \to (\neg \psi(b_j)) \Big)$$

for some constant symbol b_j.

i.e. $\Sigma^* \vdash ((\neg \phi) \to (\neg \psi(b_j)))$

Thus $\Sigma^* \vdash (\neg \psi(b_j))$.

$\neg \psi(b_j)$ is closed

so by Case 1, (*) applies.

We obtain $\mathcal{A} \models \neg \psi(b_j)$

(†)

this ~~contradiction~~ contradicts

$$\mathcal{A} \models (\forall x_i) \psi(x_i)$$

[take a valuation v in $\mathcal{A}$ with $v(x_i) = \bar{b}_j$; then v does not satisfy $\psi(x_i)$ (in $\mathcal{A}$) by † .].

Sketch.

(1) 2.6 <u>Equality</u>.

(2.6.1) <u>Def</u>. Suppose $\mathcal{L}^E$ is a 1^{st}-order language with a distinguished 2-ary relation symbol E.

(1) An $\mathcal{L}^E$-str. in which E is interpreted as equality $=$ is called a <u>normal</u> $\mathcal{L}^E$-structure.

(2) The following are the axioms of equality, Σ_E:

$$(\forall x_1) E(x_1, x_1)$$

$$(\forall x_2)(\forall x_1)(E(x_1, x_2) \to E(x_2, x_1))$$

$$(\forall x_1)(\forall x_2)(\forall x_3)(E(x_1, x_2) \to (E(x_2, x_3) \to E(x_1, x_3)))$$

For each n-ary rel. symbol R of $\mathcal{L}^E$:

$$(\forall x_1) \cdots (\forall x_n)(\forall y_1) \cdots (\forall y_n)$$
$$(R(x_1, \ldots, x_n) \wedge E(x_1, y_1) \wedge \cdots \wedge E(x_n, y_n) \longrightarrow R(y_1, \ldots, y_n))$$

For each m-ary function symbol f of $\mathcal{L}^E$:

$$(\forall x_1) \cdots (\forall x_m)(\forall y_1) \cdots (\forall y_m)$$
$$(E(x_1, y_1) \wedge \cdots \wedge E(x_m, y_m) \to E(f(x_1, \ldots, x_m), f(y_1, \ldots, y_m))).$$

(2.6.2) Remarks / Def.

① If $\mathcal{A}$ is a normal $\mathcal{L}^E$-str.

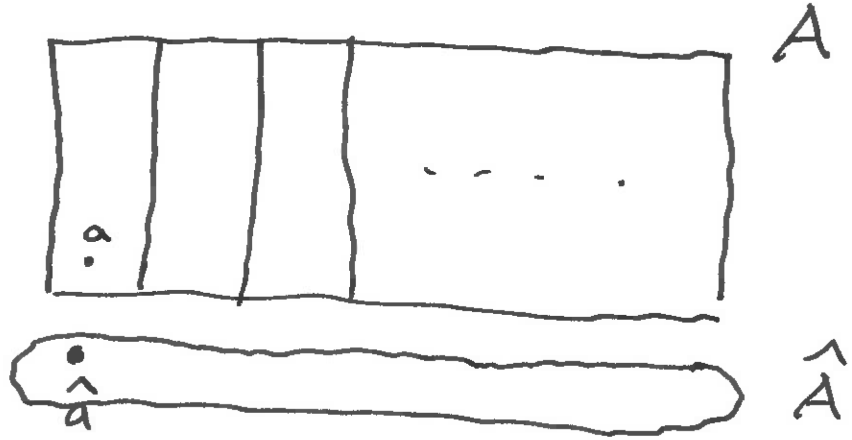
then $\mathcal{A} \models \Sigma_E$.

② Suppose

$\mathcal{A} = \langle A \,;\, \bar{E}, \ldots \rangle$

is a model of Σ_E.

Then $\bar{E}$ is an equivalence relation on A. Denote, for $a \in A$, $\hat{a} = \{ b \in A : \bar{E}(a,b) \text{ holds in } \mathcal{A} \}$.



Let $\hat{A} = \{ \hat{a} : a \in A \}$ ②

Make $\hat{A}$ into an $\mathcal{L}^E$-str. $\hat{\mathcal{A}}$ as follows:

If R is an n-ary rel.-symbol of $\mathcal{L}^E$ and $\hat{a}_1, \ldots, \hat{a}_n \in \hat{A}$, say that

$\bar{R}(\hat{a}_1, \ldots, \hat{a}_n)$ holds in $\hat{\mathcal{A}}$

iff $\bar{R}(a_1, \ldots, a_n)$ holds in $\mathcal{A}$.

This is well-defined, by $\mathcal{A} \models \Sigma_E$.

Similarly, if f is an m-ary function symbol and $\hat{a}_1, \ldots, \hat{a}_m \in \hat{A}$

let $\bar{f}(\hat{a}_1, \ldots, \hat{a}_m) = \widehat{\bar{f}(a_1, \ldots, a_m)}$.

This is well-defined as $\mathcal{A} \models \Sigma_E$.

If c is a constant symbol in $\mathcal{L}^E$ then interpret c in $\hat{\mathcal{A}}$ as

$\hat{\bar{c}}$ in $\hat{A}$ ($\bar{c}$ is the interpretation in A)

Note: in $\hat{A}$ we have

$\bar{E}(\hat{a}_1, \hat{a}_2) \iff$

$\bar{E}(a_1, a_2)$ holds in A

$\iff \hat{a}_1 = \hat{a}_2$.

So $\hat{A}$ is a normal $\mathcal{L}^E$-str.

(2.6.3) Lemma. Suppose A is an $\mathcal{L}^E$-str. with $A \models \Sigma_E$.

Let v be a valuation in A.

Let $\hat{A}$ be as given above.

Let $\hat{v}$ be the valuation in $\hat{A}$ with $\hat{v}(x_i) = \widehat{v(x_i)}$.

Then for every $\mathcal{L}^E$-formula ϕ

v satisfies ϕ in A

iff $\hat{v}$ satisfies ϕ in $\hat{A}$.

In particular, if ϕ is a closed $\mathcal{L}^E$-formula

$A \models \phi \iff \hat{A} \models \phi$.

Pf: By induction on the number of connectives + quantifiers in ϕ.

Note: if t is a term (of $\mathcal{L}^E$) then by defn. $\bar{f}, \bar{c}$ on $\hat{A}$

$\hat{v}(t) = \widehat{v(t)}$.

Using this & defn. of $\bar{R}$ ← n-ary gives the base step:

$\bar{R}(v(t_1), \dots, v(t_n))$ in A

$\iff \bar{R}(\hat{v}(t_1), \dots, \hat{v}(t_n))$ in $\hat{A}$.

Suppose every finite subset of Δ has a <u>normal</u> model. Then Δ has a <u>normal</u> model.

<u>Pf</u>: Every normal $\mathcal{L}^E$-str. is a model of Σ_E. So every finite subset of $\Delta \cup \Sigma_E$ has a model. By the Compactness Thm. (2.5.6) $\Delta \cup \Sigma_E$ has a model $\mathcal{A}$. Then by 2.6.4 $\hat{\mathcal{A}}$ is a normal model of Δ. #

<u>Notation</u>: From now on write $\mathcal{L}^{=}$ instead of $\mathcal{L}^E$ and "$x_1 = x_2$" instead of "$E(x_1, x_2)$" etc. when dealing with normal structures. Denote the axioms for equality as $\Sigma_=$. //

① Beginning Model Theory.

(2.6.6) <u>Theorem</u>.
(Countable Downward Löwenheim-Skolem Theorem.)

Suppose $\mathcal{L}^=$ is a countable 1st order language with equality and $\mathcal{B}$ is a normal $\mathcal{L}^=$-str. Then there is a <u>countable</u> <u>normal</u> $\mathcal{L}^=$-str. $\mathcal{A}$ such that for every closed $\mathcal{L}^=$-fmla ϕ

$$\mathcal{B} \models \phi \iff \mathcal{A} \models \phi.$$

<u>Example</u>:
$$\mathcal{B} = \langle \mathbb{R}; \overset{=}{L}, +, \cdot, 0, 1, \leq, \exp(\cdot) \rangle$$

$\mathcal{A} = \,??$

<u>Pf</u>: Let
$$\Sigma = \{ \text{closed } \mathcal{L}^=\text{-fmlas } \phi : \mathcal{B} \models \phi \}$$
(called the <u>$\mathcal{L}^=$-theory</u> $Th(\mathcal{B})$ of $\mathcal{B}$).

Then $\Sigma \supseteq \Sigma_=$ and (as Σ has a model) Σ is consistent.

By 2.6.4 ② Σ has a <u>countable</u> normal model $\mathcal{A}$.

So $Th(\mathcal{B}) = \Sigma \subseteq Th(\mathcal{A})$.

Conversely, if ϕ is closed and $\mathcal{B} \not\models \phi$ then (by 2.3.3) $\mathcal{B} \models (\neg\phi)$. Then $\mathcal{A} \models (\neg\phi)$.

So $\mathcal{A} \not\models \phi$. Thus $Th(\mathcal{A}) = Th(\mathcal{B})$ #

2

(2.7) <u>Example / Application</u>

<u>Linear orders</u>.

$\mathcal{L}^{=}$ is a 1^{st} order lang. with equality and a 2-ary relation symbol $\leq$.

(2.7.1) <u>Def</u>. A linear <u>order</u> $\mathcal{A} = \langle A \, ; \, \leq_A \rangle$ is a normal model of:

$\phi_1 : (\forall x_1)(\forall x_2)\big(((x_1 \leq x_2) \wedge (x_2 \leq x_1)) \leftrightarrow (x_1 = x_2)\big)$ &

$\phi_2 : (\forall x_1)(\forall x_2)(\forall x_3)\big((x_1 \leq x_2) \wedge (x_2 \leq x_3) \rightarrow (x_1 \leq x_3)\big)$

$\phi_3 : (\forall x_1)(\forall x_2)\big((x_1 \leq x_2) \vee (x_2 \leq x_1)\big)$

It is <u>dense</u> if also:

$\phi_4 : (\forall x_1)(\forall x_2)(\exists x_3)\big((x_1 < x_2) \rightarrow (x_1 < x_3) \wedge (x_3 < x_2)\big)$

[where '$x_1 < x_2$' is an abbreviation for $((x_1 \leq x_2) \wedge (x_1 \neq x_2))$]

It is <u>without endpoints</u> if

$\phi_5 : (\forall x_1)(\exists x_2)(x_1 < x_2)$

$\phi_6 : (\forall x_1)(\exists x_2)(x_2 < x_1)$.

Let $\Delta = \{\phi_1, \dots, \phi_6\}$.

<u>Ex</u>: $\Delta \vdash \Sigma_{=}$.

(3) Let

$\mathcal{Q} = \langle \mathbb{Q} ; \leq \rangle$ &

$\mathcal{R} = \langle \mathbb{R} ; \leq \rangle$ (usual ordering).

These are normal models of Δ.

(2.7.2) Thm.

For every closed $\mathcal{L}^{=}$-formula ϕ

$$\mathcal{Q} \models \phi \iff \mathcal{R} \models \phi$$

$$[\Leftarrow] \quad \Delta \vdash \phi$$

[In other words $Th(\mathcal{Q}) = Th(\mathcal{R})$ & this is axiomatized by Δ.]

(2.7.3) Def. / Fact (wk 6 p. class)

(1) Linear orders $\mathcal{A} = \langle A ; \leq_A \rangle$ and $\mathcal{B} = \langle B ; \leq_B \rangle$ are isomorphic if there is a bijection $\alpha : A \to B$ with, for all $a, a' \in A$

$$a \leq_A a' \iff \alpha(a) \leq_B \alpha(a').$$

(2) If $\mathcal{A}, \mathcal{B}$ are isomorphic and ϕ is closed then

$$\mathcal{A} \models \phi \iff \mathcal{B} \models \phi.$$

(2.7.4) Thm. (G. Cantor)

If $\mathcal{A}, \mathcal{B}$ are countable dense linear orders without endpoints, then $\mathcal{A}, \mathcal{B}$ are isomorphic. //

4

(2.7.5) __Lemma__ (Special case of Łos-Vaught Test).

Let $\Sigma = \Delta \cup \Sigma_=$

Then for every closed $\mathcal{L}^=$-fmla. ϕ we have either

$\Sigma \vdash \phi$ or $\Sigma \vdash (\neg\phi)$.

[Say that Σ is __complete__.]

__Pf__: Suppose not for some ϕ.

Then, as Σ is consistent, we have $\Sigma_1 = \Sigma \cup \{\neg\phi\}$ is consistent and $\Sigma_2 = \Sigma \cup \{\neg\neg\phi\}$ is consistent (by 2.5.2). So $\Sigma \cup \{\phi\}$ is consistent.

By 2.6.4 (2) Σ_1, Σ_2 have countable normal models $\mathcal{A}_1, \mathcal{A}_2$.

So $\mathcal{A}_1, \mathcal{A}_2$ are countable dense linear orderings without endpoints, so are isomorphic.

Therefore by 2.7.3 (2)

$Th(\mathcal{A}_1) = Th(\mathcal{A}_2)$.

__But__ $\mathcal{A}_1 \models \neg\phi$ and $\mathcal{A}_2 \models \phi$. Contradiction.

#

__Pf__ of 2.7.2. Show

$\mathcal{Q} \models \phi \iff \Sigma \vdash \phi$.

__$\Leftarrow$__: As $\mathcal{Q} \models \Sigma$ $\Sigma \vdash \phi$ implies $\mathcal{Q} \models \phi$ (by Soundness 2.4.6)

$\underline{\Rightarrow}$: If $\Sigma \nvdash \phi$

then by 2.7.5

$\Sigma \vdash (\neg\phi)$. Then

$\mathbb{Q} \models (\neg\phi)$, so $\mathbb{Q} \not\models \phi$.

But similarly

$\mathbb{R} \models \phi \iff \Sigma \vdash \phi$.

Done. #.

2.7.2 $\langle \mathbb{Q} ; \leq \rangle$ $\mathcal{L}^{=}$

$$\Sigma = \Sigma_E \cup \Delta$$

Σ_E: axiom for equality

Δ: d.l.o. without endpoints

Closed ϕ

$$\langle \mathbb{Q} ; \leq \rangle \models \phi \iff \Sigma \vdash \phi .$$

2.7.3 Thm There is an algorithm which decides, given a closed $\mathcal{L}^{=}$-formula, whether

$$\langle \mathbb{Q} ; \leq \rangle \models \theta$$

or $\langle \mathbb{Q} ; \leq \rangle \not\models \theta$

(by 2.3.3 the second case is equivalent to $\langle \mathbb{Q} ; \leq \rangle \models (\neg \theta)$).

Σ is a recursively enumerable set of formulas: we can write an algorithm which systematically generates all formulas in Σ.

The set of axioms for $K_{\mathcal{L}^{=}}$ is also recursively enumerable.

"So" the set of consequence of Σ is also recursively enumerable.

Method Run the method which generates all consequences of Σ. At some point we will see either θ or $(\neg \theta)$. At this point, the method stops. //

1) Depends on:

- the recursive axiomalization Σ_0 of $\mathrm{Th}(\langle \mathbb{Q} ; \leq \rangle)$

$\{ \phi \text{ closed} : \langle \mathbb{Q} ; \leq \rangle \models \phi \}$

- the completeness theorem for $K_{\mathcal{L}}^{=}$.

2) In this case, more practical methods exist.

3) Works for other structures.

4) No such algorithm for $\langle \mathbb{N} ; +, \cdot, 0, 1 \rangle$.

Gödel's Incompleteness thm.

Problem class

1/. For $n \in \mathbb{N}$ let σ_n be

$$(\exists x_1) \cdots (\exists x_n) \bigwedge_{1 \le i < j \le n} (x_i \ne x_j) .$$

Consider $\Sigma = \{\phi\} \cup \{\sigma_n : n \in \mathbb{N}\} \cup \Phi$,

Show Σ has a normal model. By CT enough to show every finite $\Sigma_0 \subseteq \Sigma$ has a normal model.

Any such Σ_0 is contained in $\Phi \cup \{\phi\} \cup \{\sigma_1, \ldots, \sigma_m\}$ for some $m \in \mathbb{N}$.

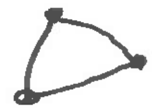
By assumption there is $p \ge m$ with $\mathbb{F}_p \models \phi$.

Then $\mathbb{F}_p \models \Sigma_0$. //

$$\underbrace{1 + \cdots + 1}_{n} \ne 0 \qquad \tau_n \qquad n \in \mathbb{N} .$$

Apply CT to $\Pi = \Sigma \cup \{\tau_n : n \ge 1\}$: get a field of char 0. //

3/. $\mu_n : (\exists x_1)\cdots(\exists x_n) \bigwedge_{1\le i<j\le n} R(x_i, x_j)$



$\lambda_n : (\exists x_1)\cdots(\exists x_n) \bigwedge_{1\le i<j\le n} \left(\neg R(x_i, x_j) \wedge (x_i \neq x_j)\right).$

(ii) Assume for a ↯ there is $n \in \mathbb{N}$ such that for for every m, there is a graph Γ_m with $\geq m$ vertices and $\Gamma_m \models (\neg\mu_n) \wedge (\neg\lambda_n)$.

Consider $\Sigma \cup \underbrace{\{\gamma\}}_{\text{graph axiom}} \cup \{\sigma_m : m \in \mathbb{N}\} \cup \{(\neg\mu_n) \wedge (\neg\lambda_n)\}$

Every finite subset of this has a normal model (by assumption). So by CT, Σ has a normal model. ↯ infinite Ramsey thm. #

3. Set Theory.

(3.0) Review of basic set theory.

(0) Extensionality Sets A, B are equal (written $A = B$) iff $(\forall x)((x \in A) \leftrightarrow (x \in B))$.

(1) Natural numbers.

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

think of

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} \\ 2 &= \{0, 1\} \\ &\vdots \\ n+1 &= \{0, \dots, n\}. \end{aligned}$$

Note: For $n, m \in \mathbb{N}$

$$m < n \iff m \in n.$$

(2) Power set. If A is any set, the power set $\mathcal{P}(A)$ is the set of subsets of A.

(3) Ordered pairs

The ordered pair (x, y) is the set $\{\{x\}, \{x, y\}\}$

Ex: For any x, y, z, w

$(x, y) = (z, w) \iff x = z$ and $y = w$

If A, B are sets then

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A^2 = A \times A, \quad A^3 = A^2 \times A, \dots$$

$$A^{n+1} = A^n \times A, \dots \quad A^0 = \{\emptyset\}.$$

The set of finite sequences of elts. of A is $\bigcup_{n \in \mathbb{N}} A^n$.

(4) Functions

Think of function $f: A \to B$ as a subset of $A \times B$.

$A = \mathrm{dom}(f)$ (Domain)

$B = \mathrm{ran}(f)$ (Range)

If $X \subseteq A$ then

$$f[X] = \{ f(a) : a \in X \} \subseteq B.$$

Set of functions from A to B

$$B^A \quad (\subseteq \mathcal{P}(A \times B)).$$

//

3.1 Cardinality

(3.1.1) Def. Sets A, B are equinumerous (or have the same cardinality) if there is a bijection $f: A \to B$.

Write $A \approx B$ or $|A| = |B|$.

(3.1.2) Def. A set A is finite if it is equinumerous with some elt. of $\mathbb{N}$. A set is countably infinite if it is equinumerous with $\mathbb{N}$.

Countable: finite or countably infinite.

(3.1.3) Basic facts. (Notes on BB).

(i) Every subset of a countable set is countable.

(ii) A set A is countable iff there is an injective fn. $f: A \to \mathbb{N}$.

(iii) If A, B are countable, then $A \times B$ is countable.

(iv)* If $A_0, A_1, \ldots$ are countable, then $\bigsqcup_{n \in \mathbb{N}} A_n$ is also countable.

(* requires Axiom of Choice.)

Ex: $\mathbb{R}$ is not countable (Cantor's diagonal argument.)

(3.1.4) Thm. (G. Cantor).

If X is a set, then there is no surjective function $f : X \to \mathcal{P}(X)$.

(In particular, $X \not\approx \mathcal{P}(X)$.)

Pf: Suppose f is such a function. Let $Y = \{ y \in X : y \notin f(y) \} \subseteq X$.

As f is surjective, there is $z \in X$ with $f(z) = Y$.

If $z \in Y$ then $z \notin f(z)$, i.e. $z \notin Y$. ↯.

If $z \notin Y$, then $z \notin f(z)$ so $z \in Y$. ↯.

So we have a contradiction. #.

(3.1.5) Def. For sets A, B write $|A| \leq |B|$ if there is an injective function $f : A \to B$. (So A is equinumerous with a subset of B.)

Ex: If $|A| \leq |B|$ & $|B| \leq |C|$ then $|A| \leq |C|$.

Note: $|X| \leq |\mathcal{P}(X)|$.

Use $x \longmapsto \{x\}$

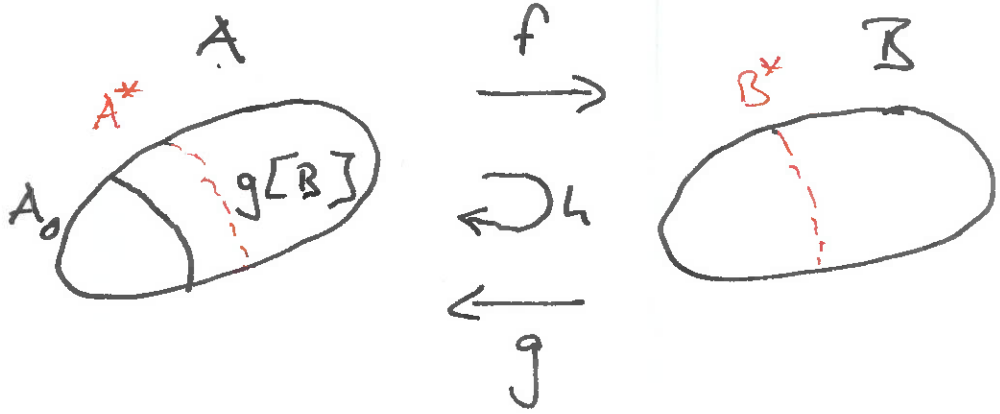
As $|X| \neq |\mathcal{P}(X)|$, obtain $|X| < |\mathcal{P}(X)|$.

Does $|A| \leq |B|$ and $|B| \leq |A|$ imply $|A| = |B|$?

Yes.

Thm. (Cantor - Schröder - Bernstein) Suppose A, B are sets and $f: A \to B$, $g: B \to A$ are injective functions. Then there is a bijective function $k: A \to B$.

Pf:



Let $h = g \circ f : A \to A$

Let $A_0 = A \setminus g[B]$

For $n > 0$ let $A_n = h[A_{n-1}]$

Let $A^* = \bigsqcup_{n \in \mathbb{N}} A_n$; $B^* = f[A^*]$.

Note: $h[A^*] \subseteq A^*$

(as $h[A_{n-1}] = A_n$)

So $g[B^*] = g[f[A^*]]$

$$= h[A^*] \subseteq A^*.$$

Claim: $g[B \setminus B^*] = A \setminus A^*$.

Once we have this

f gives a bijection $A^* \to B^*$

and g gives a bijection

$$B \setminus B^* \longrightarrow A \setminus A^*.$$

Then ~~let~~ define $k : A \to B$

by $k(a) = \begin{cases} f(a) & \text{if } a \in A^* \\ g^{-1}(a) & \text{if } a \notin A^* \end{cases}$.

Pf of Claim: (i) Let $a \in A \setminus A^*$. As $a \notin A_0$, there is $b \in B$ with $g(b) = a$.

Then $b \notin B^*$ as:

$$b \in B^* \Rightarrow b \in f[A^*]$$

$$\Rightarrow g(b) \in g[f[A^*]]$$

$$\Rightarrow g(b) \in h[A^*] \subseteq A^*$$

$$\Rightarrow a \in A^* \ ↯ . \quad //.$$

(ii) Let $b \in B$ & suppose ~~$g \in$~~ $g(b) \in A^*$.

Show $b \in B^*$ [So $g[B \setminus B^*] \subseteq A \setminus A^*$.]

$g(b) \notin A_0 = A \setminus g[B]$, so

$g(b) \in A_n$ for some $n > 0$.

Thus $g(b) = h(a)$ for some $a \in A_{n-1}$.

So $g(b) = g(f(a))$ $\therefore$

$b = f(a)$ for some $a \in A^*$.

Thus: $b \in f[A^*] = B^*$. #

3.1.7 Example.

The following sets are equinumerous:

1) S_1 = set of all sequences of 0's and 1's $= \{0,1\}^{\mathbb{N}}$

2) $S_2 = \mathbb{R}$

3) $S_3 = \mathcal{P}(\mathbb{N})$

4) $S_4 = \mathcal{P}(\mathbb{N} \times \mathbb{N})$

5) S_5 = set of sequences of natural numbers, $\mathbb{N}^{\mathbb{N}}$.

Pf: Find injective fns.

$$f_{i,j} : S_i \to S_j$$

$$(i,j \in \{1,\dots,5\})$$

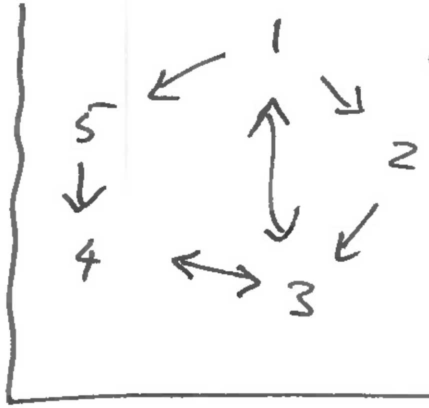
then use 3.1.6.

As

$$\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$$

we get

$$S_3 \approx S_4 .$$



$$S_1 \subseteq S_5 \subseteq S_4$$

= there's a bijection $f_{3,1} : \mathcal{P}(\mathbb{N}) \to S_1$

For $X \subseteq \mathbb{N}$ let

$$f_{3,1}(X) = (a_n)_{n \in \mathbb{N}}$$

$$a_n = \begin{cases} 0 & \text{if } n \notin X \\ 1 & \text{if } n \in X \end{cases} .$$

= Define

$f_{1,2} : S_1 \to \mathbb{R}$ by

$$(a_n)_{n \in \mathbb{N}} \mapsto 0 \cdot a_0 a_1 a_2 \dots$$

"decimal expansion". Injective

$\mathbb{Q} \approx \mathbb{N}$

So $\mathcal{P}(\mathbb{Q}) \approx \mathcal{P}(\mathbb{N})$

We have an injective fn.

$g : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$

$r \mapsto \{q \in \mathbb{Q} : q < r\}$.

Obtain $f_{2,3} : \mathbb{R} \to \mathcal{P}(\mathbb{N})$.

(3.2) Axioms for Set Theory.

Zermelo-Fraenkel Axioms (ZF)

Expressed in a 1^{st} order language (with $=$) using a single 2-ary relation symbol $\in$.

Say how we can build 'new sets from old ones'.

ZF axioms 1-6.

ZF1. (Extensionality)

$(\forall x)(\forall y)((x = y) \leftrightarrow (\forall z)((z \in x) \leftrightarrow (z \in y)))$

ZF2 (Empty set axiom)

$(\exists x)(\forall y)(y \notin x)$

ZF3 (Pairing axiom)

"Given sets x, y, we can form the set $z = \{x, y\}$"

$(\forall x)(\forall y)(\exists z)(\forall w)$
$((w \in z) \leftrightarrow ((w = x) \vee (w = y)))$.

Rk: 1) Using ZF1, ZF2 there is a unique set with the property in ZF2: the empty set $\emptyset$.

2) Using ZF3 can form

$1 = \{\phi\}$, $2 = \{\phi, 1\}$

3) Ex: Use ZF3 to form

$(x,y) = \{\{x\}, \{x,y\}\}$

ZF4 Union axiom

"For any set A there is a set $B = \bigcup A$".

i.e. $B = \bigcup\{z : z \in A\}$.

i.e.

$(\forall A)(\exists B)(\forall x)$

$\big((x \in B) \leftrightarrow (\exists z)((z \in A) \wedge (x \in z))\big)$

Eg If $A = \{x, y\}$

then $B = x \cup y$.

Eg $3 = \{0,1,2\} = \{0,1\} \cup \{2\}$

ZF5 Power set axiom

8

"If A is any set there is a set $P(A)$ whose elts. are the subsets of A".

Notation:

$z \subseteq A$ means

$(\forall y)((y \in z) \to (y \in A))$

Axiom:

$(\forall A)(\exists B)(\forall z)$

$((z \in B) \leftrightarrow (z \subseteq A))$.

ZF6 Axiom scheme of specification

Suppose $P(x, y_1, \ldots, y_r)$ is a formula in our 1^{st} order language. Then we have an axiom:

$$(\forall A)(\forall y_1)\cdots(\forall y_r)(\exists B)(\forall x)$$
$$(x \in B) \leftrightarrow ((x \in A) \wedge P(x, y_1, \ldots, y_r))$$

i.e. "given a set A and sets $y_1, \ldots, y_r$ we can form the set
$B = \{ x \in A : P(x, y_1, \ldots, y_r)$ holds $\}$."

L22.

Given A & $y_1, \dots, y_r$ can form

$$B = \{ x \in A : \underset{\substack{\uparrow \\ 1^{st} \text{ order}}}{P(x, y_1, \dots, y_r) \text{ holds}} \}$$

<u>Specification</u> (or <u>Comprehension</u>)

<u>Eg</u> 1) Let C be a non-empty set and $A \in C$.

then

" $\{ x : x \in B \text{ for all } B \in C \}$ "

$$\bigcap C = \{ x \in A : \underbrace{(\forall z)((z \in C) \to (x \in z))}_{P(x, C)} \}$$

(Doesn't depend on A)

2) $A \times B$

Recall, if $a \in A$ & $b \in B$ then

$$(a,b) = \{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

Using Specification, can 'construct'

$A \times B =$

$$\{ x \in \mathcal{P}(\mathcal{P}(A \cup B)) : \underbrace{(\exists a)(\exists b)((a \in A) \wedge (b \in B) \wedge (x = \{\{a\}, \{a,b\}\}))}_{P(x, A, B)} \}$$

<u>Ex</u>: Can form

$B^A \subseteq \mathcal{P}(A \times B)$ consisting of all functions $f : A \to B$.

ZF7 Axiom of infinity

(3.2.1) Def. 1) For a set a the successor of a is

$$a^+ = a \cup \{a\}$$

Eg

$$\phi^+ = \phi \cup \{\phi\} = \{\phi\} = 1$$

$$1^+ = \{\phi\} \cup \{1\} = \{0,1\} = 2$$

$$2^+ = \{0,1\} \cup \{2\} = \{0,1,2\} = 3.$$

etc

2) A set A is inductive if

$$(\phi \in A) \wedge ((\forall x)((x \in A) \rightarrow (x^+ \in A)))$$

The axiom of infinity ZF7 is

$$(\exists A)((\phi \in A) \wedge ((\forall x)((x \in A) \rightarrow (x^+ \in A))))$$

3.2.2 Def. Let A be any inductive set. Using Specification we can form

$$\mathbb{N} = \left\{ x \in A : \begin{array}{l} \text{if } B \text{ is an inductive set} \\ \text{then } x \in B \end{array} \right\}$$

Note: this doesn't depend on choice of A.

Notation Also denote $\mathbb{N}$ by ω or ω_0.

(3.2.3) Thm. 1) $\mathbb{N}$ is an inductive set. If B is any inductive set then $\mathbb{N} \subseteq B$.

2) "Proof by induction works for $\mathbb{N}$"
Suppose $P(x)$ is a 1st-order fmla such that

(i) $P(\emptyset)$ holds

(ii) for every $k \in \mathbb{N}$, if $P(k)$ holds then ~~$P(k+$~~ $P(k^+)$ holds.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Pf: (1) Ex.

(2) Consider $B \subseteq \mathbb{N}$

$B = \{ k \in \mathbb{N} : P(k) \text{ holds} \}$

By (i), (ii), B is an inductive set. So by (1), $B = \mathbb{N}$.

#

Could develop arithmetic in $\mathbb{N}$ (using n^+ as $n+1$) using this.

Hard ex: For $m, n \in \mathbb{N}$, write $m \leq n$ to mean ~~$\mathbb{x}$~~

$(m = n) \vee (m \in n)$

then this is a linear order on $\mathbb{N}$ & in fact is a well ordering.

3.3 Linear orderings

(3.3.1) Def. A linear ordering $(A; \leq)$ is a well ordering (or a w.o. set) if every not empty subset of A has a least element.

Examples (informal)

$(\mathbb{Z};\leq)$ is not a w.o. set.

$(\mathbb{N};\leq)$ is a w.o. set.

(3.3.2) Def. Suppose $\mathcal{A}_1 = (A_1;\leq_1)$ & $\mathcal{A}_2 = (A_2;\leq_2)$ are linear orderings. Say these are similar (or isomorphic) if there is bijection $\alpha: A_1 \to A_2$ with, $\forall a,b \in A_1$,

$$a \leq_1 b \iff \alpha(a) \leq_2 \alpha(b)$$

α is called a similarity between $\mathcal{A}_1, \mathcal{A}_2$; write $\mathcal{A}_1 \simeq \mathcal{A}_2$.

If α is injective &

$$a \leq_1 b \Rightarrow \alpha(a) \leq_2 \alpha(b)$$

say α is order preserving.

4

(3.3.3) Def.

(1) the reverse lexicographic product $\mathcal{A}_1 \times \mathcal{A}_2 = (A_1 \times A_2; \leq)$ is defined by

$(a_1, a_2) \leq (a_1', a_2')$

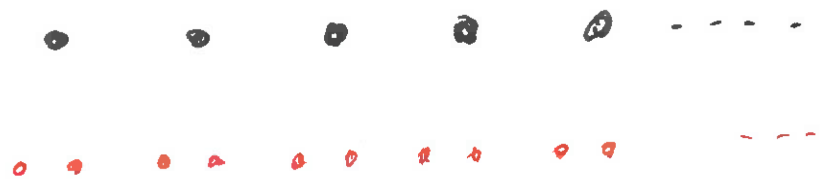
$\iff$ either $a_2 <_2 a_2'$

or $a_2 = a_2'$ and $a_1 \leq_1 a_1'$.

"– In $\mathcal{A}_2$, replace every elt. by a copy of $\mathcal{A}_1$":

Eg $\{0,1\} \times \mathbb{N}$

$\mathbb{N}$

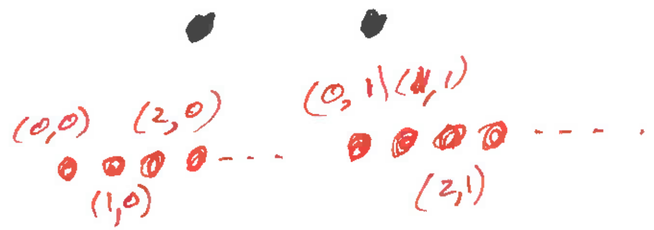


$\{0,1\} \times \mathbb{N} \simeq \mathbb{N}$

$\mathbb{N} \times \{0,1\}$

$\{0,1\}$

replace



$\mathbb{N} \times \{0,1\} \not\simeq \mathbb{N}$.

(1)

(3.3.3) (2) Sum

Given $\mathcal{A}_i = (A_i ; \leq_i)$ $(i=1,2)$

Regard A_1, A_2 as disjoint (or replace them by disjoint sets

$A_1 \times \{0\} = \{(a,0) : a \in A_1\}$
$A_2 \times \{1\} = \{(a,1) : a \in A_2\}$).

Define $\mathcal{A}_1 + \mathcal{A}_2 = (A_1 \cup A_2 ; \leq)$

where $a_1 \leq a_2$ for $a_1 \in A_1$ & $a_2 \in A_2$ & all other orderings as in $\mathcal{A}_1, \mathcal{A}_2$.

Eg $(\mathbb{N}, \leq) + (\mathbb{N}, \leq)$
$\cong \mathbb{N} \times \{0,1\}$

(3.3.4) Lemma. With this notation

(1) $\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_1 \times \mathcal{A}_2$ are linear orderings.

(2) If $\mathcal{A}_1, \mathcal{A}_2$ are w.o. sets then $\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_1 \times \mathcal{A}_2$ are w.o. sets.

[Eg $\mathbb{N} \times \mathbb{N}$ with reverse lex. ordering is a w.o. set.]

Pf: (1) Ex.

(2) Eg $\mathcal{A}_1 \times \mathcal{A}_2$.

Let $\emptyset \neq X \subseteq A_1 \times A_2$. Consider $Y = \{b \in A_2 : \exists a \in A_1$ with $(a,b) \in X\}$ $\subseteq A_2$. Let d be the least elt. of Y. Consider $Z = \{a \in A_1 : (a,d) \in X\}$; this has a least elt. c. Then $(c,d) \in X$ is the least elt. of X. #

(3.4) Ordinals.

(3.4.1) Def. (i) A set X is a transitive set if every element of X is also a subset of X (i.e. if $y \in x \in X$ then $y \in X$).

(ii) A set α is an ordinal if
(a) α is a transitive set
& (b) the relation $<$ on α given by (for $x, y \in \alpha$)
$x < y \iff x \in y$
is a (strict) well-ordering (on α).

$\in_\alpha$ also used

[Note: if α is an ordinal the by defn. $x \notin x$ for every $x \in \alpha$.].

Examples:

$0 = \emptyset$
$1 = 0^+ = \{\emptyset\} = \{0\}$
$2 = 1^+ = \{0, 1\}$
$3 = 2^+ = \{0, 1, 2\}$
$\vdots$

are ordinals.

[Non-example: $\{0, 1, 3\}$ is not a transitive set $3 \notin \{0, 1, 3\}$.]

(3.4.2) Lemma. If α is an ordinal then so is $\alpha^+ = \alpha \cup \{\alpha\}$.

Pf: If $\beta \in \alpha^+$ then either $\beta \in \alpha$ (in which case $\beta \subseteq \alpha \subseteq \alpha^+$) or $\beta = \alpha \subseteq \alpha^+$. So α^+ is a transitive set.
The ordering $\in$ on α^+ is that on α with an extra element α added as a greatest element

So this is a w.o. set $\#$ α

Prop. (3.4.3) (i) If $n \in \omega$ then n is an ordinal.
(ii) ω is a transitive set.

Pf: (i) $\emptyset$ is an ordinal.
So every $n \in \omega$ is an ordinal by induction (3.2.3) and 3.4.2.
(ii) Prove by induction on n that if $m \in n \in \omega$ then $m \in \omega$. (Ex). #.

(3.4.4) **Prop.** (i) If α is an ordinal then $\alpha \notin \alpha$.
(ii) If α is an ordinal & $\beta \in \alpha$ then β is an ordinal.
(iii) If α, β are ordinals and $\beta \subset \alpha$ then $\beta \in \alpha$.
(iv) If α is an ordinal then
$\alpha = \{ \beta : \beta$ is an ordinal and $\beta \in \alpha \}$.

Pf: (i) ✓.
(ii) Check defn.
(iii) Consider $\alpha \setminus \beta$: this is non-empty, so has a least elt. γ. Show $\gamma = \beta$. ~~Etc~~.
(iv) From (ii). #.

(3.4.5) **Def.** If α, β are ordinals, write $\alpha < \beta$ to mean $\alpha \in \beta$.
Write $\alpha \leq \beta$ to mean $\alpha < \beta$ or $\alpha = \beta$.

Note: $\alpha \leq \beta \iff \alpha \subseteq \beta$

($\Rightarrow$ defn. of ordinal
$\Leftarrow$ (iii).)

(3.4.6) <u>Prop.</u> Suppose α, β, γ are ordinals.

(i) If $\alpha < \beta$ & $\beta < \gamma$ then $\alpha < \gamma$.

(ii) If $\alpha \leq \beta$ & $\beta \leq \alpha$ then $\alpha = \beta$

(iii) Exactly one of $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$ holds

(iv) If X is a non-empty set of ordinals then X has a least element (equal to $\bigcap X$).

"The collection of ordinals is well-ordered (by $\leq$)"

<u>Pf:</u> (i) We have $\alpha \subset \beta \subset \gamma$ so $\alpha < \gamma$.

(ii) We have $\alpha \subseteq \beta$ & $\beta \subseteq \alpha$.

(iii) As $\alpha \notin \alpha$ it follows that at most one of these holds.

Show that if $\alpha \neq \beta$ then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

<u>Step 1</u> Consider $\alpha \cap \beta$. Show (using the Def.) that this is an ordinal.

<u>Step 2</u>. If $\alpha \not\subset \beta$ then $\alpha \cap \beta \subset \alpha$. As $\alpha \cap \beta$ is an ordinal 3.4.4 (iii) implies $\alpha \cap \beta \in \alpha$.

If $\beta \not\subset \alpha$ the same argument gives $\alpha \cap \beta \in \beta$. So $\alpha \cap \beta \in \alpha \cap \beta$. <u>Contradiction</u>

(iv) See notes (or Ex. comparing with (iii).) . #.

(3.4.7) Cor. (i) If X is a set of ordinals the $\bigcup X$ is an ordinal.

(ii) ω is an ordinal.

Pf: (i) $\bigcup X$ is a set of ordinals (3.4.4), so by 3.4.6 it is well-ordered by $\in$.
Easy to check that $\bigcup X$ is a transitive set. (Ex.).

(ii) ω is a set of ordinals (3.4.3), so use (i) and $\bigcup \omega = \omega$. #.

Now can form more 'infinite' ordinals

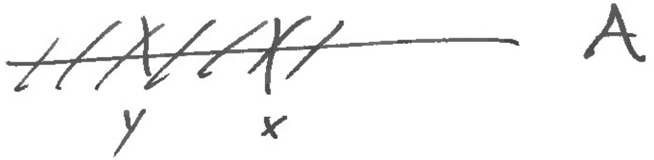
$$\omega^{+} = \{0, 1, \ldots, \omega\}$$
$$\omega^{++} = \{0, 1, \ldots, \omega, \omega^{+}\}$$
$$\vdots$$

(3.4.8) Thm. If $(A; \leq)$ is any w.o. set then there is a unique ordinal α which is 'similar' to $(A; \leq)$.

Eg there is an ordinal similar to $\mathbb{N} + \mathbb{N}$ etc.

(3.4.9) Def. Suppose $(A; \leq)$ is a w.o. set. Say $X \subseteq A$ is an initial segment of A if whenever $y < x \in X$ then $y \in X$.

[It is proper if $X \neq A$.]



Eg If α, β are ordinals & $\alpha < \beta$ then α is a proper initial segment of β

$$\alpha = \{ \delta \in \beta : \delta < \alpha \}$$

Notation If $z \in A$ write

$$A[z] = \{ a \in A : a < z \}$$

This is a proper initial segment of A.

(3.4.10) Lemma. Suppose $(A, \leq)$ is a w.o. set. If $X \subset A$ is a proper initial segment of A there is $z \in A$ with $X = A[z]$.

Pf: Let z be the least elt. of $A \setminus X$. Show $X = A[z]$.

Ex.

(3.4.11) Prop. Suppose $(A; \leq)$ is a w.o. set and $f : A \to A$ which is order-preserving and $f[A]$ is an initial segment of A. Then $f(x) = x$ for all $x \in A$.

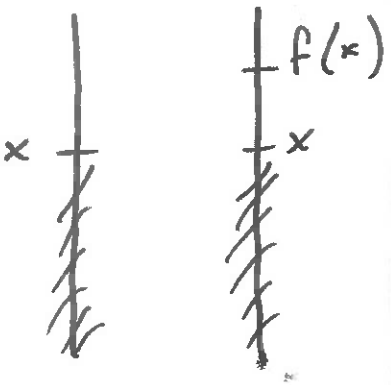
Pf: Suppose not. Let x be

the least elt of $\{ y \in A : f(y) \neq y \}$.

So if $z < x$ $f(z) = z$.

So $f | A[x]$ is the identity

$A \xrightarrow{f} A$ $\quad A[x] \longrightarrow A[x]$



As f is injective and $f(x) \neq x$, we have $f(x) > x$

So $x \notin f[A]$

Contradiction. #.

Cor (3.4.12) If $\alpha \neq \beta$ ~~then~~ are ordinals, then $\alpha \not\simeq \beta$.

Pf: May assume $\beta < \alpha$.

So β is a proper initial segment of α ($\beta = \{ \delta \in \alpha : \delta < \beta \}$)

Can't have α similar to a proper initial segment of itself (by 3.4.11). #.

Proof of 3.4.8 :

Given $(A ; \leq)$ w.o. set

Show there is a unique ordinal similar to $(A ; \leq)$.

Uniqueness Done.

Existence Consider

$$X = \{ x \in A : A[x] \text{ is similar to an ordinal} \}$$

By uniqueness, if $x \in X$ there is a unique ordinal α_x similar $A[x]$.

Let $S = \{ \alpha_x : x \in X \}$ (*)

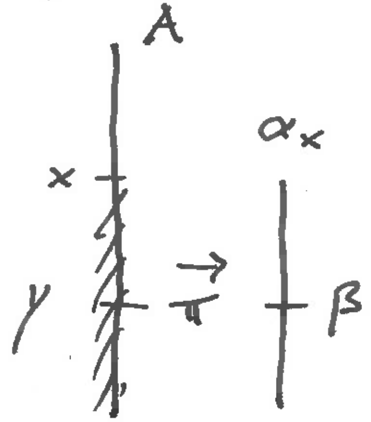
Claim: S is an ordinal.

Need to show: S is a transitive set (8).

i.e. $\beta \in \alpha_x \in S$ then $\beta \in S$.

Let $\pi : A[x] \to \alpha_x$ be a similarity.

Let $y = \pi^{-1}(\beta)$.



Then π restricted to $A[y]$ gives a similarity

$$A[y] \longrightarrow \underbrace{\{ \delta \in \alpha_x : \delta < \beta \}}_{\beta}$$

So $\beta = \alpha_y \in S$. #

Denote S by α.

If we know $X = A$ we are done as $x \longmapsto \alpha_x$ gives a similarity $A \to \alpha$.

By the above X is an initial segment of A.

If $X \neq A$ then there is $z \in A \setminus X$ with $X = A[z]$. (3.7.6)

We know $x \mapsto \alpha_x$ is a similarity $X \to \alpha$.

So then $z \in X$. $\lightning$.

So $X = A$ & we have a similarity $A \to \alpha$. #

3.4.8

Why is the set

$S = \{ \alpha_x : x \in X \}$ in (*)

$\uparrow$ ordinal

a set?

Needs ZF8 <u>Axiom of Replacement</u>.

(3.4.13) <u>Def</u>. Suppose

$F(x, y, z_1, \ldots, z_r)$ is

a formula (in our 1st order language) with the property:

whenever $s_1, \ldots, s_r$ are sets and b is a set, then there is a unique set a such that $F(a, b, s_1, \ldots, s_r)$ holds.

The $s_1, \ldots, s_r$ are referred to as <u>parameters</u>. With these fixed F gives a "function" $F(x, y, s_1, \ldots, s_r)$ on set $b \longmapsto a$.

F is called an <u>operation on sets</u> ($z_1, \ldots, z_r$ are the parameter variables).

<u>Examples</u>:

1) Without parameters

$F(a, b)$ says 'a is the power set of b'

2) $F(a, b, s_1)$

$\uparrow$ parameter

'a is the set of functions from b to s_1'.

Axiom of Replacement ZF8.

Suppose $F(x, y, z_1, \ldots, z_r)$ is an operation on sets and $s_1, \ldots, s_r$ are sets. Suppose B is a set. Then there is a set A such that

$$A = \{ a : F(a, b, s_1, \ldots, s_r) \text{ holds for some } b \in B \}.$$

(3.5) Transfinite Induction

(3.5.1) Thm. Suppose $P(x)$ is a 1st order property (of sets). Assume that for all ordinals α

(**) if $P(\beta)$ holds for all $\beta < \alpha$ then $P(\alpha)$ holds.

Then $P(\alpha)$ holds for all ordinals α.

Pf: Suppose for a contradiction that there is an ordinal γ s.t. $P(\gamma)$ does not hold. Consider

$$\{ \delta : \delta \text{ an ordinal } \delta \leq \gamma \text{ \& } P(\delta) \text{ does not hold} \}.$$

By 3.4.6 (iv) this ~~set~~ has a least element: call it α.

Then, for $\beta < \alpha$ $P(\beta)$ holds.

By (**) $P(\alpha)$ holds. ↯.

So $P(\gamma)$ holds for all ordinals γ.

#

(3.5.2) <u>Thm</u>. Suppose α is an infinite ordinal. Then

$$|\alpha| = |\alpha \times \alpha| .$$

(3.5.3) <u>Cor</u>. If $(A, \leq)$ is a w.o. set and A is infinite, then

$$|A| = |A \times A| .$$

<u>Pf</u>: By 3.4.8 there is an ordinal α $(A, \leq) \simeq \alpha$. $\boxed{\text{So in particular}}$ $|\alpha| = |A|$ & therefore $|A \times A| = |\alpha \times \alpha|$.

#

<u>Pf of 3.5.2</u>:

(0) Result holds with $\alpha = \omega$.

(1) <u>Assume</u> that if $\omega \leq \beta < \alpha$ then $|\beta| = |\beta \times \beta|$ and deduce that

$$|\alpha| = |\alpha \times \alpha| .$$

the result then follows by Transfinite Induction.

(2) May assume that α is uncountable (by the result for ω) and that if $\beta < \alpha$ then $|\beta| < |\alpha|$. Also then have

$$|\beta^+| < |\alpha| .$$

(3) Enough to show

$$|\alpha \times \alpha| \leq |\alpha| .$$

(by 3.1.6).

STEP 1 Suppose we have a well-ordering $\leq$ of $A = \alpha \times \alpha$ such that for all $x \in A$

$$|A[x]| < |\alpha|.$$

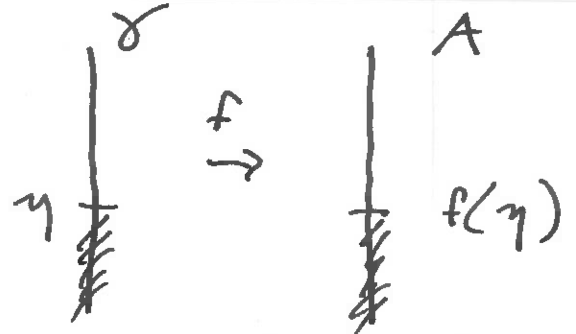
$\{y \in A : y < x\}$) Then $|\alpha \times \alpha| \leq |\alpha|$.

Pf: By 3.4.8 there is an ordinal γ which is similar to $(A, \leq)$. Let $f : \gamma \to A$ be the similarity.

Show $\gamma \subseteq \alpha$.

($\therefore$ $|\gamma| \leq |\alpha|$, so $|A| \leq |\alpha|$)

Let $\eta \in \gamma$; so $\eta < \gamma$.



As f is a similarity it gives a bijection

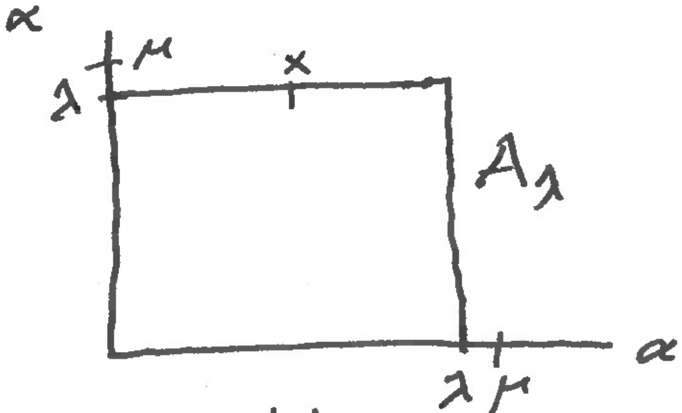
$$\eta = \{\delta \in \gamma : \delta < \eta\} \longrightarrow A[f(\eta)]$$

So $|\eta| = |A[f(\eta)]| < |\alpha|$.

Thus $\eta < \alpha$ (otherwise (by 3.4.6 (iii)) $\alpha \leq \eta$, so $\alpha \subseteq \eta$ and then $|\alpha| \leq |\eta|$). So $\eta \in \alpha$. Hence $\gamma \subseteq \alpha$. //.

L26. <u>STEP 2</u> Find a w. ordering $\leq$ on $A = \alpha \times \alpha$ st. $\forall x \in A$

$$|A[x]| < |\alpha| .$$



For $\lambda < \alpha$ let

$$A_\lambda = \{ (\theta, \zeta) \in \alpha \times \alpha : \max(\theta,\zeta) = \lambda \}$$

Define $\leq$ on A by

$(\theta', \zeta') < (\theta, \zeta) \iff$

$\max(\theta', \zeta') < \max(\theta, \zeta)$

<u>or</u> $\max(\theta', \zeta') = \lambda = \max(\theta, \zeta)$

and either $\zeta' < \zeta$ or

$\zeta' = \zeta = \lambda$ and $\theta' < \theta$.

(the ordering on A_λ is rev. lex.). ①

<u>Check</u>: $\leq$ is a well-ordering on A

(note: $A = \bigsqcup_{\lambda \in \alpha} A_\lambda$).

<u>Show</u> that if $x = (\theta, \zeta) \in A$

then $|A[x]| < |\alpha|$.

Let $\lambda = \max(\theta, \zeta)$; may assume

$\lambda \geq \omega$. Let $\mu = \lambda^+$.

So (by ②) $\mu < \alpha$ and $|\mu| < |\alpha|$.

By ind. hypothesis $|\mu \times \mu| = |\mu| < |\alpha|$

$A[x] = \{ y \in A : y < x \}$

$\subseteq \{ (\cancel{\zeta}\theta', \zeta') \in A : \max(\theta', \zeta') \leq \lambda \}$

$= \mu \times \mu$.

So $|A[x]| \leq |\mu \times \mu| < |\alpha|$.

As required. #

(3.6) Transfinite recursion.

Allows us to construct, for ordinals α, sets $G(\alpha)$ so that $G(\alpha)$ is obtained ~~for~~ from $G(\beta)$ with $\beta < \alpha$ by applying some operation F.

$\underbrace{}_{G \upharpoonright \alpha}$ ("G restricted to α")

$$G(0), G(1), \ldots, G(\beta), \ldots \quad (\beta < \alpha) \overset{F}{\rightsquigarrow} G(\alpha)$$

$G \upharpoonright \alpha : \alpha \longrightarrow \{G(\beta) : \beta < \alpha\}$

is the function $\beta \longmapsto G(\beta)$.

a set, by Axiom of Replacement

(3.6.1) Theorem (Transfinite Recursion) ②

Suppose F is an operation on sets. Then there is an operation G such that for all ordinals α we have $G(\alpha) = F(G \upharpoonright \alpha)$.

If G' is another such operation then $G'(\alpha) = G(\alpha)$ for all ordinals α.

Pf: Notes on BB. #.

In practice we usually do not write down F explicitly as a 1st order formula. //.

(3.6.2) Application.

Lindenbaum Lemma. (Compare 1.3.7 & 2.5.2)

Suppose $\mathcal{L}$ is a 1^{st} order lang. whose alphabet of symbols ~~[illegible]~~ A is well ordered.

Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas. Then there is a consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas. s.t. for every closed $\mathcal{L}$-fmla ψ either $\Sigma^* \vdash_{K_\mathcal{L}} \psi$

or $\Sigma^* \vdash_{K_\mathcal{L}} (\neg \psi)$.

[Σ^* is complete.].

As A is well ordered we can well order $S = \bigcup_{n \in \mathbb{N}} A^n$:

First order by length (the n), then by reverse lex. ordering for ~~[illegible]~~ sequences of the same length

Hence we obtain a well-ordering of the set of closed $\mathcal{L}$-fmlas.

Let λ be the unique ordinal similar to this w.o. set.

(Note this has no greatest elt.)

Then we can write the set of closed $\mathcal{L}$-fmlas as $\{\phi_\alpha : \alpha < \lambda\}$ - a set indexed by λ.

Construction of Σ^*: By Transfinite Recursion

Define for each ordinal α a set $G(\alpha) \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas.

$$G(\alpha) = \begin{cases} \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \cup \{\phi_\alpha\} & \text{if } \alpha<\lambda \text{ and } \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \vdash \phi_\alpha \\ \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \cup \{(\neg\phi_\alpha)\} & \text{if } \alpha<\lambda \text{ and } \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \nvdash \phi_\alpha \\ \Sigma \cup \bigcup_{\beta<\lambda} G(\beta) & \text{if } \alpha \geq \lambda \end{cases}$$

<u>Note</u>: If $\beta<\alpha$ then $G(\beta) \subseteq G(\alpha)$. Show (using transfinite induction and same argument as in countable case) that $G(\alpha)$ is consistent. Let $\Sigma^* = G(\lambda)$. Then for every closed $\mathcal{L}$-fmla ψ, $\psi = \phi_\alpha$ for some $\alpha<\lambda$. So $\psi \in \Sigma^*$ or $\neg\psi \in \Sigma^*$.

Can use similar arguments in other parts of the pf. of the Model Existence Thm. (2.5.3) to get Completeness Thm + Compactness thm. for $\mathcal{L}$.

3.7 Axiom of Regularity / Foundation.

ZF9.

$$(\forall x)((x \neq \phi) \to (\exists a)((a \in x) \wedge (a \cap x = \phi)))$$

In particular, there is no set b with $b \in b$.

Consider $x = \{b\}$

By ZF9 $b \cap \{b\} = \phi$

so $b \notin b$. #.

ZF 1-9

Zermelo - Fraenkel Set Theory

ZF.

Mathematical Logic (MATH6/70132; P65)
Problem Class, Lecture 27, Tuesday week 10
Supplementary problems.

[1] Some parts of the proofs in Section 3.4 were left as exercises. Do some of these, for example:
(i) Show that if α, β are ordinals then $\alpha \cap \beta$ is an ordinal.
(ii) Show that if X is a set of ordinals, then $\bigcup X$ is an ordinal.

[2] In question 5 of problem sheet 7 we consider $\beta = \{\gamma : \gamma \text{ is a countable ordinal}\}$. Justify carefully why this is a set. (Hint: Consider the set of all possible well-orderings on subsets of ω and use the Axiom of Replacement.)

The *Axiom of Choice* is the following statement:

(AC) Suppose A is a set of non-empty sets. Then there is a function $f : A \to \bigcup A$ such that $f(a) \in a$ for all $a \in A$.

(So the function f 'chooses' an element $f(a)$ from a, for every set $a \in A$.)

[3] Using AC, prove that if $f : C \to D$ is a surjective function, then there is an injective function $g : D \to C$ with $f(g(d)) = d$ for all $d \in D$. Does this statement imply AC (given the ZF axioms)?

[4] Suppose A, B are non-empty sets. Using AC, prove that $|A| \leq |B|$ iff there is a surjective function $f : B \to A$.

[5] Suppose A is any infinite set. The Fundamental Theorem of Cardinal Arithmetic (see 3.5.3) is a Theorem of ZFC (i.e. ZF plus AC) and says:

$$|A \times A| = |A|.$$

If B is any set, let B^A denote the set of functions from A to B. Show using FTCA that if $2 \leq |B| \leq |A|$, then $|B^A| = |\mathcal{P}(A)| = |2^A|$.

[Hint: Use the idea in Question 4(b) on Problem sheet 6.]

[6] Assume AC in this question. You may use the fact that that AC (together with the other ZF Axioms) implies that every set can be well-ordered.

Suppose A is any set. The *cardinality* of A is the least ordinal κ which is equinumerous with A. Temporarily denote this by $\mathrm{card}(A)$.

(i) Why is there such an ordinal?

(ii) If A, B are sets, prove that $|A| \leq |B|$ if and only if $\mathrm{card}(A) \leq \mathrm{card}(B)$.

(iii) Say why this justifies using the notation $|A|$ in place of $\mathrm{card}(A)$ for the cardinality of A.

(iv) Prove that an ordinal α is a cardinal (in the sense of qu 7 on problem sheet 7) iff $|\alpha| = \alpha$.

3) $f : C \to D$ surjective

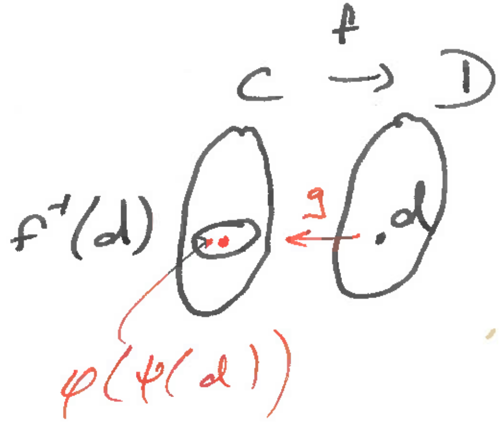
Prove (using AC) there is $g : C \to D$ with $f(g(d)) = d$ $\forall d \in D$.

Let $A = \{ \underbrace{f^{-1}(d)}_{\text{non-empty as } f \text{ surjective}} : d \in D \}$; $\bigcup A = C$

AC gives $\varphi : A \to C$ with $\varphi(f^{-1}(d)) \in f^{-1}(d)$

$\psi : D \to A$ $\quad d \mapsto f^{-1}(d)$

Let $g = \varphi \circ \psi : D \to C$. //



$\Leftarrow$ $\quad A$ set of non-empty sets

Let $C = \{ (b,a) : a \in A, \ b \in a \}$

$f \downarrow \uparrow$

$D = A$ $\quad f((b,a)) = a$.

$g(a) = (b, a)$ etc.

2) $\beta = \{ \gamma : \gamma \text{ is a countable ordinal} \}$ (Ex: β is smallest uncountable ordinal ω_1)

γ countable : there is a bijection between γ and a subset of ω.

Consider

$$\mathcal{S} = \{ (X, R) : X \subseteq \omega,\ R \subseteq X^2 \text{ is a well-ordering on } X \}$$

$\in \mathcal{P}(\omega) \times \ldots$

For each $(X, R) \in \mathcal{S}$ there is a unique ordinal $\gamma_{(X,R)}$ with $(X, R) \cong \gamma$.

$$S = \{ \gamma_{(X,R)} : (X,R) \in \mathcal{S} \} \quad \text{(using replacement)}$$

$$= \beta.$$

//.

(1)

4. Axiom of Choice + Consequences.

(4.1) Statement + WO principle

(4.1.1) Def. Axiom of Choice (AC)

Suppose A is a set of non-empty sets. Then there is a function

$$f : A \to \bigcup A$$

with $f(a) \in a$ for all $a \in A$.

Axioms ZF1-9 + AC :

ZFC.

(4.1.2) Example.

Suppose X is a non-empty set + let $A = \mathcal{P}(X) \setminus \{\emptyset\}$ (non-empty subsets of X).

By AC there is a function $f : A \to X$ such that

$f(Y) \in Y$ for every $\emptyset \neq Y \subseteq X$.

Such an f is called a choice function on X.

"Note: If $(X, \leq)$ is a w.o. set then we automatically have a choice fn. f: if $\emptyset \neq Y \subseteq X$ let

$$f(Y) = \min_{\leq}(Y)$$

(ie. least elt. of Y).

(4.1.3) Thm. (ZF) Suppose X is a non-empty set and $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ is a choice function. Then there is a well ordering $\leq$ of X i.e. a well-ordered set $(X, \leq)$.

Pf: Idea: Use transfinite recursion

$$\begin{matrix} \bullet & \bullet & \bullet & \cdots & \\ G(0) & G(1) & G(2) & & X \end{matrix}$$

$$G(\alpha) = f\Big(\underbrace{X \setminus \{G(\beta) : \beta < \alpha\}}_{\text{if this is } \neq \emptyset}\Big).$$

At stage α ~~G~~ $G \restriction \alpha$ is an injective fn. $\alpha \to X$.

Why does this 'terminate'?

(4.1.4) Hartogs' Lemma. (ZF) For any set X there is an ordinal α such that there is no injective function $h : \alpha \to X$.

Pf of 4.1.3 (given this):

Let α be as in Hartogs' Lemma, and be the least such ordinal.

Let ∞ be some set with $\infty \notin X$.

Consider $\tilde{X} = X \cup \{\infty\}$.

Using transfinite recursion, define an operation G :

For an ordinal γ, define

$$G(\gamma) = \begin{cases} f\left(X \setminus \{G(\beta) : \beta < \gamma\}\right) & \text{if } X \setminus \{G(\beta) : \beta < \gamma\} \text{ is } \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

<u>Note</u>: If $\infty \notin \operatorname{im}(G \restriction \gamma)$ then $G \restriction \gamma$ is an injective fn. $\gamma \longrightarrow X$.

By Hartogs' Lemma there is some ordinal α with $G(\alpha) = \infty$.

Take the least such α. So

$g = G \restriction \alpha \; : \; \alpha \longrightarrow X$ is an injective fn. which is surjective $\therefore$ ~~$g : X \to$~~ $g : \alpha \to X$ is a <u>bijection</u>.

Define $\leqslant$ on X by: $x_1 \leqslant x_2 \iff g^{-1}(x_1) \leqslant g^{-1}(x_2)$ (for $x_1, x_2 \in X$)

ordering on α. #

4.1.4 If A is any set, there is an ordinal α with no injective fn. $\alpha \to A$.

// Consider the set

$\mathcal{Y} = \{ (Y, \leq_Y) :$ $Y \subseteq A$ and $\leq_Y$ is a w.o. on $Y \}$

Let

$S = \{ \beta :$ β is an ordinal similar to some $(Y, \leq_Y) \in \mathcal{Y} \}$.

This is a set, by Axiom of Replacement.

$S = \{ \beta :$ β is an ordinal and there is an injective fn. $\beta \to A \}$.

Let $\sigma = \bigcup S$

This is an ordinal and $\beta \leq \sigma$ for all $\beta \in S$ (3.4.4 + 3.4.7).

Let $\alpha = \sigma^+$. Then α is an ordinal & for $\beta \in S$ we have $\beta \leq \sigma < \alpha$. So $\alpha \notin S$. $\#$.

(4.1.5) Cor. (Assuming ZF)

AC is equivalent to

WO (Well Ordering Principle):

if A is an set then there is $\leq_A \subseteq A \times A$ such that $(A, \leq_A)$ is a w.o. set.

[$ZF \vdash (AC \leftrightarrow WO)$]

Pf:

AC $\Rightarrow$ WO: AC gives us a choice fn. on A; then use 4.1.3.

WO $\Rightarrow$ AC: If A is a set of non-empty sets, let $B = \bigcup A$. By WO there is a w.o. $\leq_B$ on B. Define $f : A \to \bigcup A$ by $f(a) = \min_{\leq_B}(a)$. #

(4.1.6) Cor. (ZFC)

(i) If A is any set, there is an ordinal α with $\alpha \approx A$ (i.e. $|\alpha| = |A|$).

(ii) If A, B are sets then either $|A| \leq |B|$ or $|B| \leq |A|$.

(iii) (Fundamental Thm. of Cardinal Arithmatic)

If A is any infinite set then $|A \times A| = |A|$.

Pf: (i) By WO there is w.o. set $(A, \leq_A)$. This is similar to some ordinal α.

(ii) By (i) there are ordinals α, β with $A \approx \alpha$ & $B \approx \beta$. By 3.4.6 either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

(iii) By 3.5.3 & (i). #

(4.1.7) Lemma. (ZFC)

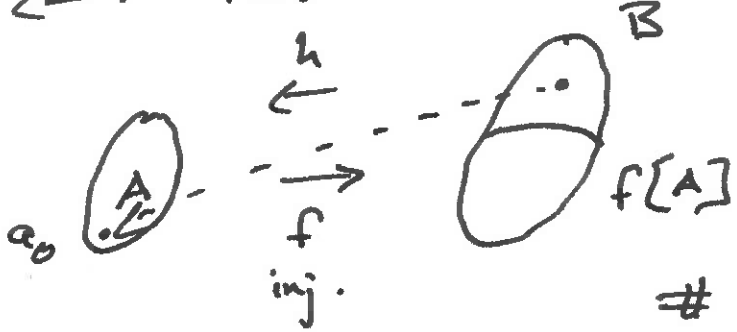
Suppose A, B are set with $A \neq \emptyset$.

Then

$|A| \leq |B| \iff$ there is a surjective fn. $h : B \to A$.

Pf (Ex.) $\overset{AC}{\Leftarrow}$: Problem class

$\Rightarrow$:



#.

(4.2) Cardinals and Cardinality.

Assume ZFC.

(4.2.1) Def. An ordinal α is a cardinal if it is not equinumerous with any $\beta < \alpha$.

Eg - if $n \in \omega$, then n is a cardinal.

- ω is a cardinal

- If ~~ξ~~ γ is any infinite ordinal then $\gamma \approx \gamma^+$ and so γ^+ is not a cardinal.

(4.2.2) Lemma. Suppose A is any set. Then there is a unique cardinal α with $\alpha \approx A$.

Pf: By 4.1.6 there is some ordinal α with $\alpha \approx A$. Take α to be the least such ordinal. Then α is a cardinal. #.

(4.2.3) Def. The unique cardinal equinumerous with A is called the cardinality of A. Denote it by card(A) or $|A|$.

<u>Ex</u>: there is an injective fn.
$f : A \to B \iff \text{card}(A) \leq \text{card}(B)$

(then it's consistent with previous notation to denote card(A) by $|A|$.).

<u>Eg</u>. 1) If A is a countably infinite set $|A| = \omega$

2) If α is any ordinal, then $|\alpha| = \alpha \iff \alpha$ is a cardinal.

3) Define the sequence of <u>alephs</u> as follows, by transfinite recursion

$\aleph_0 = \omega$

$\aleph_0 < \aleph_1 < \aleph_2 < \dots < \aleph_\alpha < \dots$

$\aleph_\alpha$ is the least cardinal which is $> \aleph_\beta$ for $\beta < \alpha$.

(4.2.4) <u>Def</u>. Suppose A, B are disjoint sets with $|A| = \kappa$ & $|B| = \lambda$

(so κ, λ are <u>cardinals</u>).

Let $\kappa + \lambda$ be $|A \cup B|$
and $\kappa \cdot \lambda$ be $|A \times B|$.

<u>Rk</u>: This doesn't depend on the choice of A & B here.

(4.2.5) Theorem. Suppose κ, λ are cardinals, $\kappa \leq \lambda$ and λ infinite.

Then (i) $\kappa + \lambda = \lambda$

(ii) $\kappa \cdot \lambda = \lambda$ (assuming $\kappa \neq 0$)

Pf: (ii) As $\kappa \leq \lambda$, we have $\kappa \subseteq \lambda$, so $|\kappa \times \lambda| \leq |\lambda \times \lambda|$.

Thus $|\kappa \times \lambda| \leq |\lambda \times \lambda| \underset{4.1.6}{=} |\lambda|$

as λ is a cardinal $\rightarrow$ $= \lambda$

As $\kappa \neq 0$ there is an injective fn

$$\lambda \longrightarrow \kappa \times \lambda$$
$$\beta \longmapsto (0, \beta).$$

So $|\lambda| \leq |\kappa \times \lambda|$.

Thus $\lambda = \kappa \cdot \lambda$, as reqd.

(i) $\lambda \leq \kappa + \lambda \leq \lambda + \lambda \approx \{0,1\} \times \lambda$

$= 2 \cdot \lambda \overset{(ii)}{=} \lambda$. #

(4.2.6) Thm. Suppose A is an infinite set of cardinality λ. Suppose that each elt. of A is a set of cardinality $\leq \kappa$. Then

$$\left| \bigcup A \right| \leq \kappa \cdot \lambda.$$

Eg. Suppose X is an infinite set & $|X| = \kappa \geq \omega$.

Let S be the set of finite sequences of elts. of X. So

$$S = \bigcup_{n \in \omega} X^n.$$

$$|X^n| = \underbrace{\kappa \cdot \kappa \cdot \ldots \cdot \kappa}_{n \text{ times}} \overset{4.2.5}{=} \kappa.$$

By 4.2.6 $|S| \leq \kappa \cdot \omega = \kappa$.

As $X \subseteq S$ $|S| \geq \kappa$. So $|S| = \kappa$. //

Pf of 4.2.6. May assume $\phi \notin A$. 4

For each $a \in A$ the set S_a of surjective functions $\kappa \to a$ is non-empty. (by 4.1.7 and $|a| \leq \kappa$.)

Assuming AC there is a fn. $F : A \to \bigcup_{a \in A} S_a$

with $F(a) \in S_a \quad \forall a \in A$

$\therefore \quad F(a) : \kappa \to a$ is a surjective fn.

Let $h : \lambda \to A$ be a bijection.

Define $g : \lambda \times \kappa \to \bigcup A$

by $g(\alpha, \beta) = F(h(\alpha))(\beta)$

(for $\alpha < \lambda$ & $\beta < \kappa$).

this is a surjective fn.

So by 4.1.7,

$|\lambda \times \kappa| \geq |\bigcup A|$

$\therefore \quad |\bigcup A| \leq \lambda \cdot \kappa$. //

Example: Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Suppose $X \subseteq \mathbb{R}$ spans $\mathbb{R}$ i.e. for each $r \in \mathbb{R}$ there are $q_1, \dots, q_s \in \mathbb{Q}$ and $x_1, \dots, x_s \in X$ with $r = \sum_{i=1}^{s} q_i x_i$.

Claim $|X| = |\mathbb{R}|$.

// Let S be the set of pairs $((q_1, \dots, q_n), (x_1, \dots, x_n))$ where $n \in \mathbb{N}$, $q_i \in \mathbb{Q}$, $x_i \in X$.

$$\subseteq \left(\bigcup_{n \in \mathbb{N}} \mathbb{Q}^n \right) \times \left(\bigcup_{n \in \mathbb{N}} X \right).$$

Thus
$$|S| \leq \left| \bigcup_{n \in \mathbb{N}} \mathbb{Q}^n \right| \cdot \left| \bigcup_{n \in \mathbb{N}} X^n \right| = \omega . |X| .$$

$|\mathbb{Q}|$

There is a surjection $S \to \mathbb{R}$: take the pair to $\sum_{i=1}^{n} q_i x_i \in \mathbb{R}$.

So $|\mathbb{R}| \leq |S|$.

Thus $|X| \leq |\mathbb{R}| \leq |S| \leq \omega . |X| = \max(\omega, |X|) = |X|$.

as $X \subseteq \mathbb{R}$

can't be ω as $|\mathbb{R}| > \omega$

So $|X| = |\mathbb{R}|$. //

4.3 Zorn's Lemma.

(1) A partially ordered set (poset) $(A, \leq)$ satisfies:

$$\forall x y z \in A \quad x \leq y \leq z \to x \leq z$$
$$\wedge \quad (x \leq y) \wedge (y \leq x) \to (x = y)$$
$$\wedge \quad (x \leq x).$$

Example: $A = \mathcal{P}(X)$, $\leq$ is $\subseteq$

(2) A chain C in a poset $(A, \leq)$ is a subset $C \subseteq A$ s.t.

$$\forall x, y \in C \quad (x \leq y) \vee (y \leq x).$$

(3) An upper bound of C in A is $a \in A$ such that $a \geq c \quad \forall c \in C$.

Example. If $C \subseteq \mathcal{P}(X)$ then $\bigcup C \in \mathcal{P}(X)$ is an upper bound for C in $(\mathcal{P}(X), \subseteq)$.

(4) An element $z \in A$ is a maximal element of A if

$$\forall x \in A \quad ((x \geq z) \to (x = z)).$$

(4.3.1) Def. Zorn's Lemma (ZL) is the statement

Suppose $(A, \leq)$ is a non-empty poset in which every chain in A has an upper bound in A. Then $(A, \leq)$ has a maximal element.

(4.3.2) Th.

(1) Assuming ZFC, ZL holds.
($ZFC \vdash ZL$).

(2) Assuming ZF + ZL then AC holds
($ZF \vdash (ZL \to AC)$).

(4.3.3) Example. (Assume ZFC)

Suppose V is a vector space over a field F. Then V has a basis.

Use ZL. Let A be the set of linearly independent subsets of V, ordered by $\subseteq$.

Claim: If $C \subseteq A$ is a chain then $\bigcup C \in A$.
(the can apply ZL).

Pf: Must show $\bigcup C$ is l.i.

ie. if $y_1, \ldots, y_n \in \bigcup C$ then $y_1, \ldots, y_n$ are l.i.. There are $C_1, \ldots, C_n \in C$ s.t. $y_i \in C_i$.

As C is a chain there is $j \leq n$ with $C_i \subseteq C_j$ for all $i \leq n$.

So $y_1, \ldots, y_n \in C_j$, and are therefore l.i.. // Claim.

By ZL there is a maximal element B of A. Show B is a basis of V.

B is a l.i. set. If $v \in V \setminus B$ then $B \cup \{v\}$ is not a l.i. set (as B is maxl. in A). This implies that v is a linear comb. of vectors in B (uses that B is l.i.)

$\therefore$ B spans V. //.

8)

Pf. of 4.3.2 (1)

AC $\Rightarrow$ ZL.

Given our poset $(A, \leq)$ (satisfying hypotheses of ZL).

Let $f : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ be a choice function.

Suppose, for a contradiction, that $(A, \leq)$ has no maximal element.

Let $C \subseteq A$ be a chain in A.

By assumption there is some $y \in A$ with $c \leq y$ for all $c \in C$.

As y is not maximal, so there is some $z \in A$ with $z > y \geq c$ for all $c \in C$.

Note: $C \cup \{z\}$ is also a chain.

Use transfinite recursion to define an operation G such that for all ordinals α

$$G(\alpha) \in A$$

$$G(\alpha) = f\left(\{z \in A : z > G(\beta) \text{ for all } \beta < \alpha\}\right)$$

So $G(0) < G(1) < \dots < G(\beta) < \cdots$

ie. for all ordinals $\beta < \alpha$ we have $G(\beta) < G(\alpha)$.

So for every α we have an injective function $G \restriction \alpha : \alpha \to A$.

this contradicts Hartogs' Lemma 4.1.4. # (ZL $\Rightarrow$ AC Q3 on P. sheet 8).

Postscript.

Assume ZFC

Gödel's Completeness Thm.

+ Compactness Thm.

hold for arbitrary 1st order languages.

(Also versions of Löwenheim-Skolem ...)

Exam

'18, '19, '20 papers on BB.

MATH60132

4 qns.

3 on Logic (different on '20)

1 on Set Th.

MATH70132

Mastery Material

Model Theory

Qu 5