Imperial College London

MATH50007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Network Science

Date: 20 May 2022

Time: 09:00 – 11:00 (BST)

Time Allowed: 2:00 hours

Upload Time Allowed: 30 minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1.

(a) Consider the graph, G, which corresponds to the following adjacency matrix,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(i) Draw the graph G.

(2 marks)

(ii) What is the Laplacian matrix, L, for G? (3 marks)

(iii) Consider the scaled Laplacian defined as $\hat{\mathrm{L}} = \mathrm{M}^{-1}\mathrm{L}\mathrm{M}$ with $\mathrm{M} \in \mathbb{R}^{5\mathrm{x}5}$. Give a matrix M such that $\mathrm{v} = [1, 1, 1, 0, 0]^{\mathrm{T}}$ is an eigenvector of $\hat{\mathrm{L}}$, and include the reasoning used to construct the matrix. What is the eigenvalue that corresponds to v? (5 marks)

(b) Let G correspond to a simple graph with N nodes and L links.

(i) Consider a partition of the graph G where all N nodes are placed in a single set. Show that the modularity of this partition is zero. (4 marks)

(ii) Now assume that G consists of two connected components. Show that any partition of G which assigns each node in G to one of two disjoint sets cannot have a modularity greater than $\frac{1}{2}$.

(6 marks)

(Total: 20 marks)

2.

(a) Consider N-node graphs generated by the configuration model with degree sequence, $d = \{k_1, k_2, k_3, ..., k_N\}$, where the sum of the degrees, K, is even.

(i) Assume that $k_1 = 1$. What is the probability that node 1 will be linked with node N? (2 marks)

(ii) Show that the probability that a graph will have one or more self-loops is at most $\frac{\left(\overline{k^2} - \overline{k}\right)}{2\overline{k}} \left(\frac{K}{K-1}\right)$ (6 marks)

(b) Consider the following random graph model which progresses iteratively by removing one link at each iteration. Begin with a complete graph, G_0, with N nodes and $\binom{N}{2}$ links. At each iteration, choose a link uniformly at random and remove it.

(i) What will the degree distribution be after the first iteration? (2 marks)

(ii) Determine $\langle k_i(t = 1)\rangle$, the expected degree of node i after the first iteration (5 marks)

(iii) For a graph generated after the t^{th} iteration, let $N_k(t)$ be the number of nodes with degree k, and let $L_k(t)$ be the number of links connecting a node pair where both nodes have degree k. Given $N_k(t)$ and $L_k(t)$ for all k, compute $\langle N_0(t + 1)\rangle$, the expected number of nodes with degree zero after the $(t + 1)^{th}$ iteration. (5 marks)

(Total: 20 marks)

3. Consider the following system of N differential equations:

$$\frac{dx_i}{dt} = x_i(\mu - x_i) + \sum_{j=1}^{N} L_{ij}x_j + \sum_{j=1}^{N}\sum_{l=1}^{N} L_{ij}L_{jl}x_l \quad i = 1, 2, ..., N, \tag{1}$$

or in matrix-vector form:

$$\frac{d\mathrm{x}}{dt} = \mu\mathrm{x} - \mathrm{y} + \mathrm{L}\mathrm{x} + \mathrm{L}^2\mathrm{x}, \tag{2}$$

where x and y are N-element vectors, and the i^{th} element of y is $y_i = x_i^2$. The initial condition is $\mathrm{x}(t = 0) = \mathrm{x}_0$, L is the Laplacian matrix for a simple graph with N nodes, the parameter μ is a non-negative constant, and you should assume that a complete set of mutually orthogonal eigenvectors for L, $\{\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_N\}$, along with their corresponding eigenvalues are given.

(a) Say that at time $t = \tau$, $x_i = \mu$ for $i = 1, 2, ..., N$. Show that $\frac{dx_i}{dt} = 0$ at $t = \tau$ for all i.

(5 marks)

(b) Let $x_i = \mu + \epsilon u_i(t)$ for all i. Show that in the limit $\epsilon \to 0$, the N-element vector, u, satisfies the following system of ODEs:

$$\frac{d\mathrm{u}}{dt} = -\mu\mathrm{u} + \mathrm{L}\mathrm{u} + \mathrm{L}^2\mathrm{u}. \tag{3}$$

(4 marks)

(c) Let $\mathrm{w} = \mathrm{M}\mathrm{u}$ where u satisfies (3). Find a matrix M such that w satisfies a nontrivial decoupled system of ODEs,

$$\frac{dw_i}{dt} = a_i w_i, \ \ i = 1, 2, ..., N,$$

and carefully explain what a_i should be for all i. (6 marks)

(d) Let $\mathrm{u}(t)$ satisfy (3), let $E(t) = \mathrm{u}(t)^T\mathrm{u}(t)$, and assume that the initial condition $\mathrm{u}(t = 0) = \mathrm{u}_0$, is constrained to have length 1 ($\mathrm{u}_0^T\mathrm{u}_0 = 1$). Find the maximum possible value of $E(t = t_1)$ where $t_1 > 0$ in terms of t_1, μ and $\rho(\mathrm{L})$ where $\rho(\mathrm{L})$ is the spectral radius of L. Include a careful explanation of your reasoning. (5 marks)

(Total: 20 marks)

4. Consider the spread of an infectious disease on an N-node simple graph, G, with adjacency matrix, A. There are three state variables for each node, $s_i(t)$, $x_i(t)$, and $y_i(t)$, $i = 1, 2, ..., N$, and each variable can be 0 or 1. A node is either susceptible ($s_i = 1$), infected but not contagious ($y_i = 1$), or contagious ($x_i = 1$). The master equations are:

$$P\left(x_i\left(t+\Delta t\right)=1\right) = P\left(x_i(t)=1\right) + \beta\Delta t \sum_{j=1}^{N} A_{ij}\left[P\left(s_i\left(t\right)=1, x_j\left(t\right)=1\right)\right] + \mu\Delta t P\left(y_i\left(t\right)=1\right) + O(\Delta t^2), \tag{1a}$$

$$P\left(s_i\left(t+\Delta t\right)=1\right) = P\left(s_i(t)=1\right) - \Delta t \sum_{j=1}^{N} A_{ij}\left[\left(\beta+\gamma\right) P\left(s_i\left(t\right)=1, x_j\left(t\right)=1\right)\right] + O(\Delta t^2), \tag{1b}$$

and we require, $s_i(t) + x_i(t) + y_i(t) = 1$ for all nodes at all times. The parameters β, γ, and μ are non-negative constants.

(a) Show that $\langle x_i y_j \rangle = P(x_i = 1, y_j = 1)$ and $\langle s_i x_j \rangle - \langle s_i x_j s_l \rangle - \langle s_i x_j x_l \rangle = P(s_i = 1, x_j = 1, y_l = 1)$. Note that all quantities are evaluated at time, t. (5 marks)

(b) Apply the limit $\Delta t \to 0$ to this model, and derive an ODE for $\langle y_i \rangle$ of the form

$$\frac{d\left\langle y_i\right\rangle}{dt} = RHS,$$

where RHS consists of some or all of: first and second moments of the state variables (including mixed moments), the adjacency matrix, and the model parameters (β, γ, μ). (6 marks)

(c) Say that G is a complete graph, and let $\overline{x} = \frac{1}{N}\sum_{i=1}^{N}\left\langle x_i\right\rangle$ and $\overline{x^2} = \frac{1}{N}\sum_{i=1}^{N}\left(\left\langle x_i\right\rangle^2\right)$. Assume that $\frac{\overline{x^2}}{(\overline{x})^2} \ll N$. Show that,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left\langle x_i\right\rangle A_{ij}\left\langle x_j\right\rangle\right) \approx N^2\overline{x}^2.$$

(5 marks)

(d) Consider the third moment, $\langle y_i s_j x_l \rangle$. Assuming that the states of nodes i and l are statistically independent, carefully show that $\langle y_i s_j x_l \rangle$ can be restated as an expression containing only first and second moments (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH50007

Network Science (Solutions)

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1. (a) Consider the graph, G, which corresponds to the following adjacency matrix,

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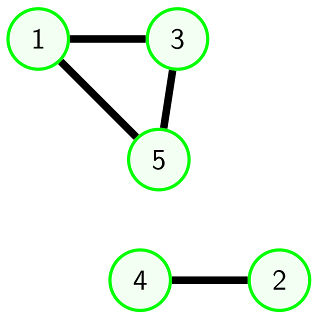
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

sim. seen ⇓

(i) Draw the graph G.

2, A



(other arrangements are possible; links should be placed between nodes i and j where $A_{ij} = 1$)

sim. seen ⇓

(ii) What is the Laplacian matrix, L, for G?

3, A

Solution: $\mathrm{L} = \mathrm{D} - \mathrm{A}$ where D is the diagonal degree matrix with $D_{ii} = k_i$. For G, the node degrees are, $\mathrm{k} = [2, 1, 2, 1, 2]$, so:

$$L = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{bmatrix}$$

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sim. seen ⇓

(iii) Consider the scaled Laplacian defined as $\hat{\mathrm{L}} = \mathrm{M}^{-1}\mathrm{L}\mathrm{M}$ with $\mathrm{M} \in \mathbb{R}^{5\mathrm{x}5}$. Give a matrix M such that $\mathrm{v} = [1, 1, 1, 0, 0]^{\mathrm{T}}$ is an eigenvector of $\hat{\mathrm{L}}$, and include the reasoning used to construct the matrix. What is the eigenvalue that corresponds to v?

5, B

Solution: If M is the permutation matrix where LM is L with its 2nd and 5th rows swapped, then M^{-1} will be a permutation matrix such that $\mathrm{M}^{-1}\mathrm{L}$ swaps the 2nd and 5th rows of L. $\mathrm{M}^{-1}\mathrm{L}\mathrm{M}$ is then the Laplacian matrix for a graph, G', which corresponds to G with nodes 2 and 5 swapped. For any connected component in a N-node simple graph, the vector which is 1 for all nodes in the component and zero otherwise is an eigenvector for the Laplacian with eigenvalue 0. The permutation matrix, M is the 5 x 5 identity matrix with the 2nd and 5th columns swapped:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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Note that other permutations are possible, and there are matrices that are not permutation matrices which are also acceptable.

(b) Let G correspond to a simple graph with N nodes and L links.

(i) Consider a partition of the graph G where all N nodes are placed in a single set. Show that the modularity of this partition is zero. $\boxed{4, \text{A}}$

Solution: The modularity will be $M = \frac{1}{2L}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(A_{ij} - \frac{k_i k_j}{2L}\right)$ where A is the adjacency matrix for the graph. Since $\sum_{j=1}^{N} A_{ij} = k_i$, and $\sum_{i=1}^{N} k_i = 2L$, we have $\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} = 2L$. Now,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{k_i k_j}{2L} = \frac{1}{2L}\sum_{i=1}^{N} k_i\left(\sum_{j=1}^{N} k_j\right) = 2L.$$

So,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(A_{ij} - \frac{k_i k_j}{2L}\right) = 2L - 2L,$$

and the modularity is zero.

$\boxed{\text{unseen} \Downarrow}$

(ii) Now assume that G consists of two connected components. Show that any partition of G which assigns each node in G to one of two disjoint sets cannot have a modularity greater than $\frac{1}{2}$. $\boxed{6, \text{D}}$

Solution: Let K_1 and K_2 be the total degrees for the nodes in components 1 and 2, and let S_a and S_b be the two sets defining the partition with total degrees, K_a and K_b. Let L_a be the total number of links between nodes in S_a with an analogous definition for L_b. The modularity of the partition is:

$$M = \frac{1}{K}\left[\left(2L_a - K_a^2/K\right) + \left(2L_b - K_b^2/K\right)\right],$$

where K is the total degree of the graph, and $K = K_a + K_b$. Note that $2L_a \leq K_a$ with equality occurring only if S_a contains all of the nodes in one component and none of the nodes in the other. Identical reasoning gives, $2L_b \leq K_b$. Using these inequalities, we have

$$M \leq \frac{1}{K}\left[K - \frac{K_a^2 + K_b^2}{K}\right] = \frac{2K_a K_b}{K^2} = \frac{2K_a(K - K_a)}{K^2}.$$

Now, $K_a(K - K_a)$ is maximized when $K_a = K/2$, so $M \leq \frac{2(K/2)(K-K/2)}{K^2} = \frac{1}{2}$ as required. The first part of the solution shows that the modularity for the graph is maximized for the partition where each set corresponds to a connected component (3 pts), and the 2nd part shows that the graph with the maximum modularity is the one where each connected component has half of the total stubs in the graph (3pts).

2. (a) Consider N-node graphs generated by the configuration model with degree sequence, $d = \{k_1, k_2, k_3, ..., k_N\}$, where the sum of the degrees, K, is even. $\boxed{\text{sim. seen} \Downarrow}$

(i) Assume that $k_1 = 1$. What is the probability that node 1 will be linked with node N? $\boxed{2, \text{A}}$

Solution: The probability that the stub on node 1 connects to any other stub is $\frac{1}{K-1}$, so the probability is, $\frac{k_N}{K-1}$.

$\boxed{\text{sim. seen} \Downarrow}$

(ii) Show that the probability that a graph will have one or more self-loops is at most $\frac{\left(\overline{k^2}-\overline{k}\right)}{2\overline{k}}\left(\frac{K}{K-1}\right)$ $\boxed{6, \text{C}}$

Solution: Let X_i be the number of self-loops on node i, and let $X = \sum_{i=1}^{N} X_i$ be the total number of self-loops in the graph. As shown in lecture, $\langle X_i \rangle = \frac{k_i(k_i-1)}{2(K-1)}$, and then, using linearity of expectation, $\langle X \rangle = \sum_{i=1}^{N} \frac{k_i(k_i-1)}{2(K-1)}$. Since $\overline{k} = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{K}{N}$, and $\overline{k^2} = \frac{1}{N}\sum_{i=1}^{N} k_i^2$:

$$\langle X \rangle = \frac{K}{2(K-1)}\frac{N}{K}\sum_{i=1}^{N}\frac{k_i^2 - k_i}{N} = \frac{K}{2(K-1)}\frac{1}{\overline{k}}\left(\overline{k^2} - \overline{k}\right),$$

and Markov's inequality tells us that $P(X \geq 1) \leq \langle X \rangle$, so the probability of the graph having one or more self-loops is at most

$$\frac{K}{2(K-1)}\frac{1}{\overline{k}}\left(\overline{k^2} - \overline{k}\right) = \frac{\left(\overline{k^2} - \overline{k}\right)}{2\overline{k}}\left(\frac{K}{K-1}\right).$$

(b) Consider the following random graph model which progresses iteratively by removing one link at each iteration. Begin with a complete graph, G_0, with N nodes and $\binom{N}{2}$ links. At each iteration, choose a link uniformly at random and remove it.

(i) What will the degree distribution be after the first iteration? $\boxed{2, \text{A}}$

Solution: Of the N nodes, the two that had been connected by the removed link will have degree $N-2$, and all other nodes will have degree $N-1$, so $p_{N-2} = \frac{2}{N}$, $p_{N-1} = \frac{N-2}{N}$, and $p_k = 0$ for $k = 0, 1, ..., N-3$. Note that this result can also be derived using the probability that node i has a link removed which is $\frac{k_i(t=0)}{L(t=0)}$ where $L(t=0) = N(N-1)/2$ and $k_i(t=0) = N-1$.

(ii) Determine $\langle k_i(t=1) \rangle$, the expected degree of node i after the first iteration $\boxed{5, \text{B}}$

Solution: There are $\binom{N}{2}$ graphs which can be generated with equal probability depending on which link is removed. In $N-1$ of these graphs, node i will have degree $N-2$, and in all others, it will have degree $N-1$. The expected degree is:

$$\langle k_i(t) \rangle = \frac{N-1}{\binom{N}{2}}(N-2) + \frac{\binom{N}{2} - (N-1)}{\binom{N}{2}}(N-1) = \frac{2(N-2)}{N} + \frac{(N-2)(N-1)}{N},$$

and the last expression simplifies to, $\langle k_i(t) \rangle = \frac{(N-2)(N+1)}{N}$. Note: this result can also be obtained directly from the distribution obtained in (i).

$\boxed{\text{sim. seen} \Downarrow}$

(iii) Let $N_k(t)$ be the number of nodes with degree k after the t^{th} iteration, and let $L_k(t)$ be the number of links connecting a node pair where both nodes have degree k. Given $N_k(t)$ and $L_k(t)$ for all k, compute $\langle N_0(t+1)\rangle$, the expected number of nodes with degree zero after the $(t+1)^{th}$ iteration. $\boxed{5, \text{D}}$

Solution: There are two cases to consider. A: A link connecting two degree-1 nodes is removed. Then, $\Delta N_0 = +2$, and $P(A) = L_1(t)/L(t)$. B: A link connecting a node with degree 1 with a node with some other degree is removed. Then, $\Delta N_0 = +1$, and $P(B) = \frac{N_1(t)-2L_1(t)}{L(t)}$. Putting these results together,

$$\langle N_0(t+1)\rangle = N_0(t) + 2P(A) + P(B) = N_0(t) + \frac{N_1(t)}{L(t)},$$

where $L(t) = \binom{N}{2} - t$. (1 point for correctly analyzing one of the two cases and 3 points for correctly analyzing both)

3. Consider the following system of N differential equations:

$$\frac{dx_i}{dt} = x_i\left(\mu - x_i\right) + \sum_{j=1}^{N} L_{ij}x_j + \sum_{j=1}^{N}\sum_{l=1}^{N} L_{ij}L_{jl}x_l \quad i = 1, 2, ..., N, \tag{1}$$

or in matrix-vector form:

$$\frac{d\mathrm{x}}{dt} = \mu\mathrm{x} - \mathrm{y} + \mathrm{L}\mathrm{x} + \mathrm{L}^2\mathrm{x}, \tag{2}$$

where x and y are N-element vectors, and the i^{th} element of y is $y_i = x_i^2$. The initial condition is $\mathrm{x}(t = 0) = \mathrm{x}_0$, L is the Laplacian matrix for a simple graph with N nodes, the parameter μ is a non-negative constant, and you should assume that a complete set of mutually orthogonal eigenvectors for L, $\{\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_N\}$, along with their corresponding eigenvalues are given.

sim. seen ⇓

(a) Say that at time $t = \tau$, $x_i = \mu$ for $i = 1, 2, ..., N$. Show that $\frac{dx_i}{dt} = 0$ at $t = \tau$ for all i.

Solution: By inspection, we see that the first term on the RHS of (1) will be zero. This leaves $\mathrm{L}\mathrm{x} + \mathrm{L}^2\mathrm{x}$. We know $\sum_{j=1}^{N} L_{ij}x_j = \sum_{j=1}^{N} A_{ij}\left(x_i - x_j\right) = 0$ if $x_i = x_j$ for all i and j. So, $\mathrm{L}\mathrm{x} = \mu\mathrm{L}\mathrm{z} = 0$ and $\mathrm{L}^2\mathrm{x} = \mathrm{L}(\mathrm{L}\mathrm{x}) = 0$, and

5, A

$$\mathrm{y} - \mu\mathrm{x} + \mathrm{L}\mathrm{x} + \mathrm{L}^2\mathrm{x} = \frac{d\mathrm{x}}{dt} = 0$$

if $\mathrm{x} = \mu\mathrm{z}$. Here, $\mathrm{z} = [1, 1, ..., 1]^T$, and this is the matrix-vector form of the desired equation.

sim. seen ⇓

(b) Let $x_i = \mu + \epsilon u_i(t)$ for all i. Show that in the limit $\epsilon \to 0$, the N-element vector, u, satisfies the following system of ODEs:

$$\frac{d\mathrm{u}}{dt} = -\mu\mathrm{u} + \mathrm{L}\mathrm{u} + \mathrm{L}^2\mathrm{u}. \tag{3}$$

4, A

Solution: Substituting the expansion into (3) gives,

$$\epsilon\frac{du_i}{dt} = -\left(\mu + \epsilon u_i\right)\left(\epsilon u_i\right) + \sum_{j=1}^{N} L_{ij}\left(\mu + \epsilon u_j\right) + \sum_{j=1}^{N}\sum_{l=1}^{N} L_{ij}L_{jl}\left(\mu + \epsilon u_l\right) \; i = 1, 2, ..., N,$$

From (a), we know that $\sum_{j=1}^{N} L_{ij}\mu = \sum_{j=1}^{N}\sum_{l=1}^{N} L_{ij}L_{jl}\mu = 0$. Then, dividing both sides of the equation by ϵ gives,

$$\frac{du_i}{dt} = -\left(\mu + \epsilon u_i\right)\left(u_i\right) + \sum_{j=1}^{N} L_{ij}u_j + \sum_{j=1}^{N}\sum_{l=1}^{N} L_{ij}L_{jl}u_l \; i = 1, 2, ..., N.$$

Finally, when $\epsilon \to 0$, $\left(\mu + \epsilon u_i\right) \to \mu$, and we have the desired result.

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(c) Let $\mathrm{w} = \mathrm{M}\mathrm{u}$ where u satisfies (3). Find a matrix M such that w satisfies a nontrivial decoupled system of ODEs,

$$\frac{dw_i}{dt} = a_i w_i, \; i = 1, 2, ..., N,$$

and carefully explain what a_i should be for all i.

6, B

Solution: Orthogonally diagonalize the Laplacian as $\mathrm{L} = \mathrm{V}\Lambda\mathrm{V}^T$ where the columns of V contain the eigenvectors of L normalized to have length one, and Λ is a diagonal matrix where Λ_{ii} is the eigenvalue corresponding to the eigenvector stored in the i^{th} column of V. Then $\mathrm{L}^2 = \mathrm{V}\Lambda\mathrm{V}^T\mathrm{V}\Lambda\mathrm{V}^T = \mathrm{V}\Lambda^2\mathrm{V}^T$ since the eigenvectors of L are mutually orthogonal and $\mathrm{V}^T\mathrm{V} = \mathrm{I}$. Equation (3) is then,

$$\frac{d\mathrm{u}}{dt} = \left(-\mu\mathrm{I} + \mathrm{V}\Lambda\mathrm{V}^T + \mathrm{V}\Lambda^2\mathrm{V}^T\right)\mathrm{u}.$$

Since $\mu\mathrm{I} = \mu\mathrm{V}\mathrm{I}\mathrm{V}^T$, we can simplify the ODEs to,

$$\frac{d\mathrm{u}}{dt} = \left[\mathrm{V}S\mathrm{V}^T\right]\mathrm{u},$$

where $\mathrm{S} = \Lambda + \Lambda^2 - \mu\mathrm{I}$. If we define $\mathrm{w} = \mathrm{V}^\mathrm{T}\mathrm{u}$, we have,

$$\mathrm{V}\frac{d\mathrm{w}}{dt} = \mathrm{VSw}$$

since $\mathrm{u} = \mathrm{Vw}$. It follows that,

$$\frac{dw_i}{dt} = S_{ii}w_i,\ i = 1, 2, ..., N,$$

where $S_{ii} = a_i = \lambda_i + \lambda_i^2 - \mu$ and λ_i is the i^{th} eigenvalue of the Laplacian. So, we can set $\mathrm{M} = \mathrm{V}^\mathrm{T}$ (two points for correctly finding M and four points for a_i.)

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(d) Let $\mathrm{u}(t)$ satisfy (3), let $E(t) = \mathrm{u}(t)^T\mathrm{u}(t)$, assume that the initial condition $\mathrm{u}(t=0) = \mathrm{u}_0$, has length 1 ($\mathrm{u}_0^T\mathrm{u}_0 = 1$) and also assume that u_0 has been chosen to maximize $E(t = t_1)$. Find this maximum $E(t_1)$ in terms of μ and $\rho(\mathrm{L})$ where $\rho(\mathrm{L})$ is the spectral radius of L. Include a careful explanation of your reasoning. 5, D

Solution: The solution to the system of ODEs from (c) is, $\mathrm{w} = e^{\mathrm{S}t}\mathrm{w}_0$. We know, $\mathrm{u} = \mathrm{Vw}$ and $\mathrm{V}^T\mathrm{V} = \mathrm{I}$, so $\mathrm{u} = \mathrm{V}e^{\mathrm{S}t}\mathrm{V}^T\mathrm{u}_0$, and $E = \mathrm{u}_0^T\mathrm{V}e^{2\mathrm{S}t}\mathrm{V}^T\mathrm{u}_0$ with $S_{ii} = \lambda_i + \lambda_i^2 - \mu$. E can be written as a Rayleigh quotient, $E = \frac{\mathrm{x}^Te^{2\mathrm{S}t}\mathrm{x}}{\mathrm{x}^T\mathrm{x}}$ with $\mathrm{x} = \mathrm{V}^T\mathrm{u}_0$ (note that $\mathrm{x}^T\mathrm{x} = 1$). The maximum $E(t)$ is then the largest eigenvalue of the diagonal matrix $e^{2\mathrm{S}t}$ which is determined by the largest element in S. Since the eigenvalues of L are non-negative, the most-positive eigenvalue is also the spectral radius. So, the maximum $E(t_1)$ will be, $exp\left\{2t_1\left[\rho(\mathrm{L}) + \rho(\mathrm{L})^2 - \mu\right]\right\}$.

4. Consider the spread of an infectious disease on an N-node simple graph, G, with adjacency matrix, A. There are three state variables for each node, $s_i(t)$, $x_i(t)$, and $y_i(t)$, $i = 1, 2, ..., N$, and each variable can be 0 or 1. A node is either susceptible ($s_i = 1$), infected but not contagious ($y_i = 1$), or contagious ($x_i = 1$). The master equations are:

$$P\left(x_i\left(t+\Delta t\right)=1\right) = P\left(x_i(t)=1\right) + \beta\Delta t\sum_{j=1}^{N} A_{ij}\left[P\left(s_i\left(t\right)=1, x_j\left(t\right)=1\right)\right] + \mu\Delta t P\left(y_i\left(t\right)=1\right) + O(\Delta t^2), \tag{1a}$$

$$P\left(s_i\left(t+\Delta t\right)=1\right) = P\left(s_i(t)=1\right) - \Delta t\sum_{j=1}^{N} A_{ij}\left[\left(\beta+\gamma\right)P\left(s_i\left(t\right)=1, x_j\left(t\right)=1\right)\right] + O(\Delta t^2), \tag{1b}$$

and we require, $s_i(t) + x_i(t) + y_i(t) = 1$ for all nodes. The parameters β, γ, and μ are non-negative constants.

sim. seen ⇓

(a) Show that $\langle x_i y_j\rangle = P(x_i = 1, y_j = 1)$ and $\langle s_i x_j\rangle - \langle s_i x_j s_l\rangle - \langle s_i x_j x_l\rangle = P(s_i = 1, x_j = 1, y_l = 1)$. Note that all quantities are evaluated at time, t. 5, A

Solution: The product $x_i y_j$ can be either zero or one. It is one only when both x_i and y_j are one. Then, by the definition of the expectation, $\langle x_i y_j\rangle = P(x_i = 1, y_j = 1) * 1 + P(\text{all other cases}) * 0 = P(x_i = 1, y_j = 1)$ as required. For the second expression, linearity of expectation allows the three moments to be combined as, $\langle s_i x_j(1 - s_l - x_l)\rangle$ which is equal to $\langle s_i x_j y_l\rangle$ since $s_l + x_l + y_l = 1$ implies $\langle s_l\rangle + \langle x_l\rangle + \langle y_l\rangle = 1$ (again due to linearity of expectation). Now, $s_i x_j y_l = 1$ when the three variables are all one and will be zero otherwise, so by the definition of expectation, $s_i x_j y_l = P(s_i = 1, x_j = 1, y_l = 1)$, and it follows that $\langle s_i x_j\rangle - \langle s_i x_j s_l\rangle - \langle s_i x_j x_l\rangle = P(s_i = 1, x_j = 1, y_l = 1)$. (2 points for the 1st part and 3 points for the 2nd)

(b) Apply the limit $\Delta t \to 0$ to this model, and derive an ODE for $\langle y_i\rangle$ of the form

$$\frac{d\left\langle y_i\right\rangle}{dt} = RHS,$$

where RHS consists of some or all of: first and second moments of the state variables (including mixed moments), the adjacency matrix, and the model parameters (β, γ, μ). 6, C

Solution: We know: $P(x_i(t) = 1) = \langle x_i\rangle$, $P(y_i(t) = 1) = \langle y_i\rangle$, $P(s_i(t) = 1) = \langle s_i\rangle$, $\langle y_i\rangle = 1 - \langle x_i\rangle - \langle s_i\rangle$, so subtracting equations (1a) and (1b) from 1 gives,

$$\left\langle y_i(t+\Delta t)\right\rangle = \left\langle y_i(t)\right\rangle + \gamma\Delta t\sum_{j=1}^{N}\left[A_{ij}\left\langle s_i(t)x_j(t)\right\rangle\right] - \mu\Delta t\left\langle y_i(t)\right\rangle + O(\Delta t^2),$$

Finally, move the first term on the RHS to the LHS, divide both sides by by Δt and let $\Delta t \to 0$. On the LHS, we will have $\frac{\langle y_i(t+\Delta t)\rangle - \langle y_i(t)\rangle}{\Delta t} \to \frac{d\langle y_i\rangle}{dt}$, the last term on the RHS becomes $O(\Delta t)$ and $\to 0$, and all other terms on the RHS have their Δt terms drop out prior to the application of the limit, so we are left with:

$$\frac{d\left\langle y_i\right\rangle}{dt} = \gamma\sum_{j=1}^{N} A_{ij}\left\langle s_i x_j\right\rangle - \mu\left\langle y_i\right\rangle.$$

sim. seen ⇓

(c) Say that G is a complete graph, and let $\overline{x} = \frac{1}{N}\sum_{i=1}^{N}\langle x_i\rangle$, and $\overline{x^2} = \frac{1}{N}\sum_{i=1}^{N}\left(\langle x_i\rangle^2\right)$. Assume that $\frac{\overline{x^2}}{\overline{x}^2} \ll N$. Show that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\langle x_i\rangle A_{ij}\langle x_j\rangle\right) \approx N^2\overline{x}^2$$

.

5, A

Solution: Since the graph is complete,

$$\sum_{j=1}^{N} A_{ij}\langle x_j\rangle = \sum_{j=1}^{N}\langle x_j\rangle - \langle x_i\rangle = N\overline{x} - \langle x_i\rangle.$$

Then,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\langle x_i\rangle A_{ij}\langle x_j\rangle\right) = \overline{x}N\sum_{i=1}^{N}\langle x_i\rangle - \sum_{i=1}^{N}\langle x_i\rangle^2 = N^2\overline{x}^2 - N\overline{x^2}$$

.

Finally,

$$N^2\overline{x}^2 - N\overline{x^2} = N^2\overline{x}^2\left(1 - \frac{\overline{x^2}}{N\overline{x}^2}\right) \approx N^2\overline{x}^2$$

since $\frac{\overline{x^2}}{N\overline{x}^2} \ll 1$.

sim. seen ⇓

(d) Consider the third moment, $\langle y_i s_j x_l\rangle$. Assuming that the states of nodes i and l are statistically independent, carefully show that $\langle y_i s_j x_l\rangle$ can be restated as an expression containing only first and second moments

4, B

Solution: $\langle y_i s_j x_l\rangle = P(y_i = 1, s_j = 1, x_l = 1) = P(y_i = 1, x_l = 1|s_j = 1)P(s_j = 1)$. Since nodes i and l are statistically independent, $P(y_i = 1, x_l = 1|s_j = 1) = P(y_i = 1|s_j = 1)P(x_l = 1|s_j = 1)$, so:

$$\langle y_i s_j x_l\rangle = P(y_i = 1|s_j = 1)P(x_l = 1|s_j = 1)P(s_j = 1).$$

Now, $P(y_i = 1|s_j = 1) = \frac{P(y_i=1,s_j=1)}{P(s_j=1)}$, and $P(x_l = 1|s_j = 1) = \frac{P(x_l=1,s_j=1)}{P(s_j=1)}$, which gives:

$$\langle y_i s_j x_l\rangle = \frac{P(y_i = 1, s_j = 1)P(x_l = 1, s_j = 1)}{P(s_j = 1)} = \frac{\langle y_i s_j\rangle\langle x_l s_j\rangle}{\langle s_j\rangle}.$$

Alternate solution:

$$P(y_i = 1, s_j = 1, x_l = 1) = P(y_i = 1|s_j = 1, x_l = 1)P(s_j = 1, x_l = 1),$$

and since nodes i and l are indpendent, $P(y_i = 1|s_j = 1, x_l = 1) = P(y_i = 1|s_j = 1)$. Then,

$$P(y_i = 1, s_j = 1, x_l = 1) = P(y_i = 1|s_j = 1)P(s_j = 1, x_l = 1) = \frac{P(y_i = 1, s_j = 1)}{P(s_j = 1)}P(s_j = 1, x_l = 1) = \frac{\langle y_i s_j\rangle\langle x_l s_j\rangle}{\langle s_j\rangle} \quad (5)$$

as before.

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionN	Comments for Students
Network Science_MATH50007	1	I didn't notice anything out-of-the-ordinary. The two more challenging parts related to 1) finding a transformation matrix for the Laplacian of the given graph and 2) finding an upper bound for the modularity for 2-community partitions. For 1), the idea was to use a permutation matrix to renumber the graph nodes as was done in a similar problem sheet exercise. For 2) many students showed that a particular partition had the desired upper bound, but not many showed that there were no other other partitions with larger modularities.
Network Science_MATH50007	2	Given the difficulty of the question, I think the class did well overall here. It is difficult to work on a "new" model in an exam setting, and the class handled this challenge well.
Network Science_MATH50007	3	The last part of the question was designed to be challenging, and for the most part, the class did fine on the other parts. For the last part, a some students found the needed upper bound for the energy, but few showed that this upper bound is tight. The key was to use the eigenvector corresponding to the largest eigenvalue as the initial condition.
Network Science_MATH50007	4	It seemed that time pressure was unfortunately a strong factor on this question, but where students were able to work through parts, they generally did well though there were some difficulties going back and forth between probabilities and expectations on part (a).