Exercise 3.1. Suppose $\Omega, \Omega' \subset \mathbb{R}^n$ are open, $g : \Omega \to \Omega'$ and $f : \Omega' \to \Omega$ are functions such that g is differentiable at $p \in \Omega$ and f is differentiable at $g(p) \in \Omega'$ and moreover

$$f \circ g(x) = x, \qquad \forall\, x \in \Omega.$$
$$g \circ f(x) = x, \qquad \forall\, x \in \Omega'.$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

Exercise 3.2 (*). (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$P : (x, y) \mapsto xy$$

is differentiable at each point $p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^2$, with Jacobian:

$$DP(p) = \begin{pmatrix} \eta & \xi \end{pmatrix}.$$

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ given by:

$$Q : z \mapsto (f(z), g(z))$$

is differentiable at q and:

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

(c) Show that $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(z) = f(z)g(z)$ for all $z \in \mathbb{R}^n$ is differentiable at q, and:

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

[Hint: Note that $F = P \circ Q$.]

Exercise 3.3. (a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = \begin{pmatrix} x^2 + e^{x+y} \\ x - \log y \\ 2xy + 1 \end{pmatrix}.$$

Assuming f is differentiable at a point $\begin{pmatrix} x \\ y \end{pmatrix}$, what is its derivative?

(b) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by $g(x, y, z) = x + y + z$. Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.

Exercise 3.4. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable, and find the differential when:

(a) $f(x, y) = x^2 + y^2 - x - xy$,

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Questions marked with $*$ are optional

(b) $f(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x,y) = x^5y^2$.

Unseen Exercise. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} x^2 \sin(1/x) & \text{if } y = 0, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Is the map f differentiable at $(0,0) \in \mathbb{R}^2$? Justify your answer using the definition of the derivative.