IMPERIAL

MATH70058

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Manifolds

Date: Friday, May 31, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let X, Y be manifolds of dimension n and k respectively and let $F\colon X \to Y$ be a smooth function.

(i) Define what a regular value of F is. (2 marks)

(ii) Let $y \in Y$ be a regular value of F. Show that $F^{-1}(y)$ is a submanifold of X and compute its dimension. (4 marks)

(b) Determine if the set

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 + 2xz - 2yz = 1, 2x - y + z = 0\}$$

is a manifold (justify your answer). (4 marks)

(c) Let X be a manifold and let $Y \subset X$ be a connected submanifold. Consider the equivalence $\sim$ on X defined by, for all $x, y \in X$,

$$x \sim y \qquad \text{if and only if} \qquad x = y \text{ or } x, y \in Y.$$

Let Z be the quotient of X by this equivalence. Determine if Z is a manifold (justify your answer). (5 marks)

(d) Consider the topological space $X = [0, 1] \times \mathbb{R}$ and consider the equivalence $\sim$ on X defined by, for all $(t, x), (s, y) \in X$,

$$(t, x) \sim (s, y) \quad \text{if and only if} \quad (t, x) = (s, y) \text{ or } s = 0, t = 1, x = -y \text{ or } s = 1, t = 0, x = -y.$$

Let Z be the quotient of X by this equivalence. Determine if Z is a manifold (justify your answer). (5 marks)

(Total: 20 marks)

2. (a) Find a vector field on $\mathbb{P}^3_{\mathbb{R}}$ which vanishes exactly at one point. (6 marks)

(b) Let X be a manifold of positive dimension and let $x \in X$. Show that there exists a vector field v on X such that $v(x) \neq 0$. (6 marks)

(c) Let X be a manifold and let $x \in X$.

(i) Define a derivation of X at x. (2 marks)

(ii) Let V be a derivation on X at x and let $h \in C^\infty(X)$ be a function which is constant in a neighbourhood U of x. Show that $V(h) = 0$. (6 marks)

(Total: 20 marks)

3. (a) Let X be a manifold and let $\pi\colon E \to X$ be a vector bundle of rank r over X. Show that E is trivial if and only if there exist sections $s_1, \ldots, s_r\colon X \to E$ of E such that, for all $x \in X$, the elements

$$s_1(x), \ldots, s_r(x)$$

form a basis of $E_x = \pi^{-1}(x)$. (5 marks)

(b) Let X be a manifold, let $\pi\colon E \to X$ be a vector bundle of rank r over X and let $Z \subset X$ be a submanifold. Show that if we define $E_Z = \pi^{-1}(Z)$ and $\pi_Z\colon E_Z \to Z$ is the restriction of π to E_Z, then $\pi_Z\colon E_Z \to Z$ is a vector bundle of rank r over Z.

(5 marks)

(c) Let $X = \mathbb{R}$ and let $\pi\colon E \to X$ be a line bundle over X.

(i) Show that there exist an open neighbourhood $U \subset X$ of 0 and a smooth function $s\colon U \to \pi^{-1}(U)$ such that $\pi \circ s = \mathrm{id}_{\mathrm{U}}$ and $s(x) \neq 0$ for all $x \in U$. (2 marks)

(ii) Show that there exists a nowhere vanishing section s of E. (6 marks)

(iii) Show that E is a trivial line bundle. (2 marks)

(Total: 20 marks)

4. (a) Let X and Y be manifolds, let $F\colon X \to Y$ be a smooth function and let ω be a p-form on Y for some $p \geq 0$.

   (i) Show that if p is the dimension of Y then $d\omega = 0$. (2 marks)

   (ii) Show that $d(d\,\omega) = 0$. (4 marks)

   (iii) Show that $d(F^*\omega) = F^*(d\,\omega)$. (4 marks)

(b) Fix a positive integer n. Let $D = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and let ∂D be the boundary of D.

   (i) Show that there exists a $(n-1)$-form ω on ∂D such that

$$\int_{\partial D} \omega > 0.$$

(4 marks)

   (ii) Let $g\colon D \to \partial D$ be a smooth function.
   Show that there exists $x \in \partial D$ such that $g(x) \neq x$. (6 marks)

(Total: 20 marks)

5. (a) Let X be a manifold of dimension $n > 0$ and let $D \subset T_X$ be a distribution.

(i) Show that if D is of rank one then D is integrable. (4 marks)

(ii) Show that if D is of rank n then D is integrable. (4 marks)

(b) Let $X \subset \mathbb{R}^n$ be an open subset, let $x \in X$ and let $V : X \to TX = X \times \mathbb{R}^n$ be a vector field such that $V(x) \neq 0$.

Show that there exists an open neighbourhood U of x in X and a diffeomorphism $f : U \to \tilde{U} \subset \mathbb{R}^n$ such that, for all $y \in U$, we have

$$Df_y(V(y)) = (1, 0, \ldots, 0).$$

(5 marks)

(c) Let $X = \mathbb{R}^3$ and consider the vector fields

$$V, W : X \to TX = X \times \mathbb{R}^3 \qquad V(x, y, z) = (1, 0, y) \qquad W(x, y, z) = (x, 1, 0).$$

(i) Show that V and W define a distribution D on X. (2 marks)

(ii) Determine if D is integrable. (2 marks)

(iii) Determine all the integrable submanifolds $Y \subset X$ with respect to the distribution defined by V. (3 marks)

(Total: 20 marks)

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MATH70058

Manifolds (Solutions)

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1. (a) (i) A point $x \in X$ is called a regular point of F if the rank of F at x is equal to k or, alternatively, if the Jacobian $DF_x \colon T_xX \to T_{F(x)}Y$ is surjective. $\boxed{\text{seen} \Downarrow}$

A point $y \in Y$ is called a regular value of F if every point $x \in F^{-1}(y)$ is a regular point of F. $\boxed{2, \text{A}}$

(ii) Denote $Z_y = F^{-1}(y)$ and let $x \in Z_y$. $\boxed{\text{seen} \Downarrow}$

By continuity of F, there exist a chart (U, f) of X with $x \in U$ and a chart (V, g) of Y with $y \in V$ such that $F(U) \subset V$.

Consider the function

$$\tilde{F} = g \circ F \circ f^{-1} \colon f(U) \to g(V).$$

Then, since F is smooth, it follows that also $\tilde{F}$ is smooth and since y is a regular value of F, it follows that $g(y)$ is a regular value of $\tilde{F}$.

We have

$$\tilde{F}^{-1}(g(y)) = f(Z_y \cap U).$$

By the Implicit Function Theorem, it follows that $\tilde{F}^{-1}(g(y))$ is a submanifold of $\mathbb{R}^n$ of dimension $n-k$. Thus, there exist a $(n-k)$-dimensional affine subspace $A \subset \mathbb{R}^n$ and a chart (W, h) with $f(x) \in W \subset f(U)$ such that

$$h(W \cap \tilde{F}^{-1}(g(y))) = h(W) \cap A.$$

Hence, $(f^{-1}(W), h \circ f)$ is a chart on X such that

$$h \circ f(f^{-1}(W) \cap Z_y) = h(W) \cap A.$$

Thus, Z_y is a submanifold of X of dimension $n-k$. $\boxed{4, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(b) Consider the smooth function

$$F \colon \mathbb{R}^3 \to \mathbb{R}^2 \qquad F(x, y, z) = (x^2 - y^2 + 2xz - 2yz, 2x - y + z).$$

We want to show that the point $(1, 0)$ is a regular value.

We have that the Jacobian of F is given by

$$DF = \begin{pmatrix} 2(x+z) & -2(y+z) & 2(x-y) \\ 2 & -1 & 1 \end{pmatrix}.$$

This matrix has rank one if and only if $x - 2y - z = 0$. It is easy to show that the set $F^{-1}(1, 0)$ does not admit any such point.

Thus, $(1, 0)$ is a regular value and X is a manifold of dimension one. $\boxed{4, \text{A}}$

(c) In general, Z is not a manifold. For example, consider $X = \mathbb{R}^2$ and $Y = \{x = 0\} \subset X$. Assume by contradiction that Z is a topological manifold. Let p be the equivalence class of $(0,0)$. If $q \in Z$ is a different point, then Z is locally homeomorphic to an open subset of $\mathbb{R}^2$. Thus, Z has dimension two and there exists a chart (U,f) of Z such that $p \in U$ and $f(U)$ is a connected open subset of $\mathbb{R}^2$. On the other hand, $U \setminus \{p\}$ contains two connected components given by the equivalence classes of points (x,y) with $y > 0$ and $y < 0$ respectively. But $f(U) \setminus f(p)$ is connected, a contradiction.

unseen $\Downarrow$

5, B

unseen $\Downarrow$

(d) We show that Z is a manifold of dimension two.

Let $q\colon X \to Z$ be the quotient map. Define

$$\tilde{U} = ((0,1) \times \mathbb{R}) \qquad \text{and} \qquad W = ([0,1/2) \cup (1/2,1]) \times \mathbb{R}$$

and let

$$U = q(\tilde{U}) \qquad \text{and} \qquad V = q(W).$$

Then $q|_{\tilde{U}}\colon \tilde{U} \to U$ is a homeomorphism and we set $f = (q|_{\tilde{U}})^{-1}\colon U \to \tilde{U}$.

We also define $\tilde{V} = (1/2, 3/2) \times \mathbb{R}$ and

$$p\colon \tilde{V} \to V \qquad p(x,y) = \begin{cases} q(x,y) & \text{if } x \leq 1 \\ q(x-1,-y) & \text{if } x > 1. \end{cases}$$

Then p is a homeomorphism and we set $g = p^{-1}\colon V \to \tilde{V}$.

This easily implies that Z is Hausdorff and second countable. Thus, Z is a topological manifold of dimension two.

We now show that the atlas above is smooth. We only need to consider the transition function

$$\phi = f \circ g^{-1}\colon ((1/2,1) \cup (1,3/2)) \times \mathbb{R} \to ((0,1/2) \cup (1/2,1)) \times \mathbb{R}$$

given by

$$\phi(x,y) = \begin{cases} (x,y) & \text{if } x < 1 \\ (x-1,-y) & \text{if } x > 1. \end{cases}$$

Clearly, both ϕ and ϕ^{-1} are smooth functions. Thus, ϕ is a diffeomorphism and our claim follows.

5, D

2. (a) We first constrct a vector field on $\mathbb{P}^3_{\mathbb{R}}$ which is nowhere vanishing. There exists a natural smooth function unseen ⇓

$$\rho\colon S^3 \to \mathbb{P}^3_{\mathbb{R}} \qquad (x_0, x_1, x_2, x_3) \mapsto [x_0, x_1, x_2, x_3].$$

For each $x \in S^3$, the Jacobian of ρ at x is a linear map

$$D\rho|_x\colon T_xS^3 \to T_{\rho(x)}\mathbb{P}^3_{\mathbb{R}}.$$

Consider the smooth function

$$t\colon S^3 \to S^3 \qquad t(x) = -x.$$

We want to find a non vanishing vector field $V\colon S^3 \to TS^3$ on S^3 such that for all $x \in S^3$

$$V(t(x)) = Dt|_x \circ V(x).$$

Indeed, this would induce a non vanishing vector field W on $\mathbb{P}^3_{\mathbb{R}}$ such that

$$W(\rho(x)) = D\rho|_x \circ V(x) \in T_{\rho(x)}\mathbb{P}^3_{\mathbb{R}}.$$

If we consider the vector field V, given by

$$V(x_0, x_1, x_2, x_3) = (-x_1, x_0, -x_3, x_2),$$

then V is never vanishing and, since $Dt|_x = -\mathrm{Id}_{\mathbb{R}^4}$, for any $x \in S^3$, we have that

$$\begin{aligned} V(t(x)) &= V(-x_0, -x_1, -x_2, -x_3) \\ &= (x_1, -x_0, x_3, -x_2) \\ &= Dt|_x \circ V(x), \end{aligned}$$

as requested.

Now, fix a point $p \in \mathbb{P}^3_{\mathbb{R}}$. By using a bump function at p, we can construct a function $f \in C^\infty(\mathbb{P}^3_{\mathbb{R}})$ which is zero exactly at the point p. Thus, the vector field $W = f \cdot V$ is zero exactly at p. 6, C

meth seen ⇓

(b) Let n be the dimension of X. Let (U, f) be a chart on X such that $x \in U$. Let $\tilde{U} = f(U) \subset \mathbb{R}^n$ and let $\Delta_f\colon T_xX \to T_{f(x)}\tilde{U} \simeq \mathbb{R}^n$ be the induced isomorphism. Let $v \in \mathbb{R}^n$ be a non-zero vector and let V be the constant vector field so that $V(z) = v$ for all $z \in \tilde{U}$ and let $\rho\colon \tilde{U} \to [0,1]$ be a bump function on $\tilde{U}$ such that $\rho(f(x)) = 1$. Then the induced vector field $\xi = \Delta_f^{-1} \circ (\rho \cdot V) \circ f$ on U is such that $\xi(x) = v$ and it is zero, outside a neighbourhood of x. In particular, it can be extended to a smooth vector field v on X such that $v(x) = v$. 6, A

seen ⇓

(c) (i) A derivation of X at x is a linear map $\mathfrak{d}\colon C^\infty(X) \to \mathbb{R}$ which satisfies the Leibniz rule, i.e. for any $h_1, h_2 \in C^\infty(X)$, we have

$$\mathfrak{d}\,(h_1h_2) = h_1(x)\mathfrak{d}\,(h_2) + h_2(x)\mathfrak{d}\,(h_1)\,.$$

2, A

(ii) We first show that if h is a constant function then $V(h) = 0$. By linearity, we may assume that h is the constant function identically equal to 1. By the Leibniz rule, we have $\boxed{\text{meth seen} \Downarrow}$

$$V(1) = V(1 \cdot 1) = 1 \cdot V(1) + 1 \cdot V(1) = 2V(1).$$

Thus, $V(1) = 0$.

Now let $h \in C^\infty(X)$ be a function which is equal to a constant c in a neighbourhood U of x and let $g = h - c$. Then g is identically zero on the open subset U. Since $V(c) = 0$ and V is linear, it is enough to show that $V(g) = 0$.

Consider a bump function $\rho \in C^\infty(X)$ such that, there exist open sets

$$x \in W' \subset W \subset U$$

such that

$$\rho|_{W'} \equiv 1 \qquad \rho|_{X \setminus W} \equiv 0.$$

Let $\psi = 1 - \rho$. Then $\psi \cdot g = g$. Thus, the Leibniz rule implies

$$V(g) = V(\psi \cdot g) = g(x)V(\psi) + \psi(x)V(g) = 0,$$

as claimed. $\boxed{6, \text{B}}$

3. (a) Assume first that E is the trivial vector bundle. By definition, there exists an isomorphism of vector bundles $F\colon X \times \mathbb{R}^r \to E$. In particular, F is a diffeomorphism. Fix a basis $v_1, \dots, v_r \in \mathbb{R}^r$ and consider the functions

sim. seen ⇓

$$s_i\colon X \to E \qquad x \mapsto F(x, v_i) \qquad \text{for } i = 1, \dots, r.$$

It follows by construction that s_i is a smooth section of F. Since, for each $x \in X$, the induced map $F_x\colon E_x \to \mathbb{R}^r$ is an isomorphism of vector spaces and since $F^{-1}(s_i(x)) = v_i$, it follows that $s_1(x), \dots, s_r(x)$ form a basis of E_x.

Now assume that $s_1, \dots, s_r$ are sections of E such that, for all $x \in X$, we have that $s_1(x), \dots, s_r(x)$ form a basis of E_x.

Consider the function

$$F\colon X \times \mathbb{R}^r \to E \qquad (x, (v_1, \dots, v_r)) \mapsto \sum_{i=1}^{r} v_i s_i(x).$$

Then, for each $x \in X$, it induces a linear map

$$\mathbb{R}^r \to E_x \qquad (v_1, \dots, v_r) \mapsto \sum_{i=1}^{r} v_i s_i(x),$$

which is an isomorphism of vector spaces. Thus, F is a bijection.

We now show that F and F^{-1} are smooth functions. For each $x \in X$, there exist a chart (U, f) in X with $x \in X$ and a chart (V, g) on Y such that $V = \pi^{-1}(U)$ and $g(V) = f(U) \times \mathbb{R}^r$. By assumption, for each $i = 1, \dots, r$ the function

$$\tilde{s}_i := g \circ s_i \circ f^{-1}\colon f(U) \to f(U) \times \mathbb{R}^r = g(V)$$

is smooth. If $\pi_2\colon f(U) \times \mathbb{R}^r \to \mathbb{R}^r$ is the projection into the second factor, then

$$\tilde{F} := g \circ F \circ (f^{-1}, id_{\mathbb{R}^r})\colon f(U) \times \mathbb{R}^r \to f(U) \times \mathbb{R}^r \qquad (y, (v_1, \dots, v_r)) \mapsto (y, \sum v_i \cdot \pi_2 \circ \tilde{s}_i(y)).$$

It follows that both $\tilde{F}$ and $\tilde{F}^{-1}$ are smooth function.

Thus, F and F^{-1} are smooth functions, as claimed. 5, D

unseen ⇓

(b) By definition, we have that if n is the dimension of X then E is a manifold of dimension $n + r$, $\pi\colon E \to X$ is a smooth function and there exists an atlas $\{(U_i, f_i)\}_{i \in I}$ for X and an atlas $\{(V_i, g_i)\}_{i \in I}$ for E, satisfying the following properties:

1. $V_i = \pi^{-1}(U_i)$ for all $i \in I$;
2. $g_i(V_i) = f_i(U_i) \times \mathbb{R}^r \subset \mathbb{R}^{n+r}$, for each $i \in I$;
3. $\mathrm{pr}_1 \circ g_i(v) = f_i(\pi(v))$ for all $i \in I$ and $v \in V_i$; and
4. for each $x \in X$, the level set $E_x := \pi^{-1}(x)$ is a vector space of dimension r, such that for all $i \in I$, the map $g_i|_{E_x}\colon E_x \to \mathbb{R}^r$ is an isomorphism.

Let m be the dimension of Z. Since $Z \subset X$ is a submanifold, we may assume without loss of generality, that

$$f_i(U_i \cap Z) = f_i(U_i) \cap A \qquad \text{for all } i \in I \text{ such that } U_i \cap Z \neq \emptyset$$

where $A \subset \mathbb{R}^n$ is the standard m-dimensional affine sublinear space. Let $W_i = U_i \cap Z$ and let

$$W_i' = \pi_Z^{-1}(W_i) = E_Z \cap V_i.$$

Then W_i' is an open subset of E_Z. If

$$h_i = g_i|_{W_i'} \colon W_i' \to (f_i(U_i) \cap A) \times \mathbb{R}^r \subset A \times \mathbb{R}^r \simeq \mathbb{R}^{m+r},$$

then (W_i', h_i) is a co-ordinate chart of E_Z. For any $i, j \in I$ such that $U_i \cap Z, U_j \cap Z \neq \emptyset$, we have that the transition function $h_j \circ h_i^{-1}$ is the restriction of the transition function $g_j \circ g_i^{-1}$. Thus, it is smooth and, in particular, E_Z is a manifold.

Properties 3. and 4. above for E imply the same properties for E_Z. Thus, our claim follows. 5, B

meth seen ⇓

(c) (i) By definition of a line bundle, there exists a chart (U, f) of X such that $0 \in U$ and such that $\pi^{-1}(U) \simeq U \times \mathbb{R}$. In particular, as in the proof of Ex. (a) above, we can find a smooth function $s \colon U \to p^{-1}(U)$ such that $s(x)$ is a non-zero element in $\pi^{-1}(x) \simeq \mathbb{R}$ for all $x \in U$, as claimed. 2, A

unseen ⇓

(ii) Let U and $s \colon U \to \pi^{-1}(U)$ be as in Ex. (c)(i). We may assume that $U = (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Let $\epsilon' = \epsilon/2$ and let s' be the restriction of s to $(-\epsilon', \epsilon')$. We consider

$$T = \{x > 0 \mid \exists \tilde{s} \colon (-\epsilon, x) \to \pi^{-1}(-\epsilon, x) \text{ such that } \tilde{s} \text{ extends } s' \text{ and } \pi \circ \tilde{s} = \mathrm{id}\}.$$

Clearly, $\epsilon \in T$ and, therefore, T is not empty.
We now show that T is open in $\mathbb{R}_{>0}$. Let $x \in T$. We may assume that $x > \epsilon'$. Similarly as in i), we can find a chart (V, g) of X such that $x \in V$ and such that $\pi^{-1}(V) \simeq V \times \mathbb{R}$. We may assume that $V = (x - \delta, x + \delta)$ for some $\delta > 0$. By using a bump function, we can find a smooth function $t \colon (x - \delta, x + \delta) \to V \times \mathbb{R}$ which is nowhere zero and such that t extends the restriction of $\tilde{s}$ to $(x - \delta, x - \delta/2)$. Thus, $(x - \delta, x + \delta) \subset T$. It follows that J is open.
We now show that T is closed in $\mathbb{R}_{>0}$. Suppose it is not. Let $y = \inf\{x \in \mathbb{R}_{>0} \mid x \notin T\}$. In particular, $(0, y) \subset T$ and, similarly as above, we can extend s' to a neighbourhood of y, a contradiction.
It follows that $T = \mathbb{R}_{>0}$. Proceeding, similarly for $\mathbb{R}_{>0}$, our claim follows. 6, D

unseen ⇓

(iii) The claim follows immediately from Ex. (a) and Ex. (c)(ii).

2, A

4. (a) (i) By definition, $d\omega$ is a $(p+1)$-form on a manifold of dimension n. Since $\wedge^{p+1}T_y^*Y = 0$ for all $y \in Y$, it follows that $d\omega = 0$.

seen ⇓

2, A

(ii) Since the de Rham differential is defined locally around any point $y \in Y$, we may assume without loss of generality, that Y is an open subset of $\mathbb{R}^n$ where n is the dimension of Y. By linearity we may also assume that

seen ⇓

$$\omega = h dy_{i_1} \wedge \ldots \wedge dy_{i_p}$$

where $h \in C^\infty(Y)$.
Then

$$d(d\omega) = \sum_{m=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 h}{\partial y_m \partial y_j} dy_m \wedge dy_j \wedge dy_{i_1} \wedge \ldots \wedge dy_{i_p}$$

Since $h \in C^\infty(Y)$, we have that

$$\frac{\partial^2 h}{\partial y_m \partial x_j} dy_m \wedge dy_j = -\frac{\partial^2 h}{\partial y_j \partial y_m} dy_j \wedge dy_m.$$

Thus, $d(d\omega) = 0$.

4, A

(iii) Similarly as above, we may assume that $X = U \subset \mathbb{R}^n$ and $Y = V \subset \mathbb{R}^k$ are open subsets and we assume that

seen ⇓

$$\omega = h dy_{i_1} \wedge \ldots \wedge dy_{i_p}$$

where $h \in C^\infty(V)$.
Recall that if $\omega_1 \in \Omega^p(Y)$ and $\omega_2 \in \Omega^q(Y)$, then

$$F^*(\omega_1 \wedge \omega_2) = F^*(\omega_1) \wedge F^*(\omega_2).$$

We have

$$\omega = h dy_{i_1} \wedge \ldots \wedge dy_{i_p}$$

Thus,

$$F^\star \omega = h \circ F \cdot (F^\star dy_{i_1}) \wedge \ldots \wedge (F^\star dy_{i_p})$$

We also have that if $g \in C^\infty(Y)$ then

$$F^\star dg = d(g \circ F).$$

Thus,

$$d(F^\star dy_{i_j}) = d(d(y_{i_j} \circ F)) = 0.$$

We have

$$F^\star \omega = h \circ F \cdot (F^\star dy_{i_1}) \wedge \ldots \wedge (F^\star dy_{i_p})$$

and

$$d(F^\star dy_{i_j}) = 0.$$

Thus,

$$\begin{aligned} d(F^\star\omega) &= d(h\circ F)\wedge(F^\star dy_{i_1})\wedge\ldots\wedge(F^\star dy_{i_p}) \\ &= F^\star dh\wedge(F^\star dy_{i_1})\wedge\ldots\wedge(F^\star dy_{i_p}) \\ &= F^\star(dh\wedge dy_{i_1}\wedge\ldots\wedge dy_{i_p}) \\ &= F^* d\omega. \end{aligned}$$

The general claim follows again by linearity. $\boxed{4, \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(b) (i) ∂D is the $(n-1)$-dimensional sphere, which is orientable. Thus, it admits a volume form $\omega \in \Omega^{n-1}(\partial D)$, and

$$\int_{\partial D} \omega > 0.$$

$\boxed{4, \text{B}}$

$\boxed{\text{unseen} \Downarrow}$

(ii) Assume by contradiction that $g(x) = x$ for all $x \in \partial D$.
Let $\omega \in \Omega^{n-1}(\partial D)$ be as in Part (i). Then we have

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^*\omega = \int_D d\,(g^*\omega) = \int_D g^* d\omega$$

where the first equality comes from the fact that g is the identity on ∂D, the second from Stokes' theorem and the third follows by the property of the de Rham differential.
But $d\omega$ is a n-form on the $(n-1)$-dimensional manifold ∂D and, thus, it is zero.
Thus, we get a contradiction. $\boxed{6, \text{C}}$

5. (a) (i) We first show that D is involutive.

Being involutive is a local property. Thus, we can replace X by a smaller open subset of X and, in particular, we may assume that D is a trivial vector bundle. Thus, D admits a nowhere zero section s. Let V and W be vector fields on X such that $V(x), W(x) \in D_x$ for all $x \in X$. Since D has rank one, there exist smooth function f and g on X such that

$$V = f \cdot s \qquad \text{and} \qquad W = g \cdot s.$$

It follows that

$$[V, W] = [f \cdot s, g \cdot s] = (f \cdot s(g) - g \cdot s(f)) \cdot s.$$

Thus $[V, W]_x \in D_x$ and the claim follows.

By Frobenius Theorem, it follows that D is integrable. 4, M

(ii) Since D is of rank n, it follows that $D = T_X$ and, in particular, if V and W are vector fields on X such that $V(x), W(x) \in D_x$ for all $x \in X$, then $[V, W](x) \in T_x X = D_x$. Thus, D is involutive and again by Frobenius Theorem, it follows that D is integrable. 4, M

(b) After a suitable choice of co-ordinates, we may assume that $x = (0, \dots, 0) \in \mathbb{R}^n$ and that $V((0, \dots, 0)) = (1, 0, \dots, 0)$. After possibly shrinking $X \subset \mathbb{R}^n$ further, we may assume that there exists a flow $F\colon (-\epsilon, \epsilon) \times X \to \mathbb{R}^n$ for V such that if, for any $y \in X$ we define $\sigma_y(t) := F(t, y)$, then

$$[\sigma_y] = V(y) \in T_y X.$$

It follows that

$$\frac{\partial}{\partial t} F(0, (0, \dots, 0)) = (1, 0, \dots, 0). \tag{1}$$

Let $\pi_1 \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the last $n-1$ factors and let

$$W_0 := \pi_1(X \cap \{x_1 = 0\}).$$

Then $W_0 \subset \mathbb{R}^{n-1}$ is an open subset and we can consider the open subset

$$W := (-\epsilon, \epsilon) \times W_0 \subset \mathbb{R}^n$$

and the smooth function

$$g\colon W \to X \qquad (t, x_2, \dots, x_n) \mapsto F(t, (0, x_2, \dots, x_n)).$$

Since F is a flow, we have that $F(0, x) = x$ for all $x \in X$ and by (1) it follows that the Jacobian matrix

$$Dg|_{0,(0,\dots,0)}$$

is the identity matrix. Thus, by the inverse function theorem, after possibly shrinking W and X, we may assume that g is a diffeomorphism. Recall that, if we define $F_t(\cdot) := F(t, \cdot)$, then, by the definition of a flow, we have

$$F_{t+s} = F_t \circ F_s$$

and therefore, for all y, we have

$$F(t, y) = F_t(y) = F_0 \circ F_t(y) = F(0, \sigma_y(t)).$$

Thus, we also have

$$\begin{aligned}
Dg|_{(t,x_2,\ldots,x_n)}\left(\frac{\partial}{\partial t}\right) &= \frac{\partial}{\partial t} g(t,(x_2,\ldots,x_n)) \\
&= \frac{\partial}{\partial t} F(t,(0,x_2,\ldots,x_n)) \\
&= \frac{\partial}{\partial t} F(0,\sigma_{(0,x_2,\ldots,x_t)}(t)) = V(g(t,x_2,\ldots,x_n)).
\end{aligned}$$

Thus, if f is the inverse of g, then for all $(y_1,\ldots,y_n)$ we have

$$Df|_{(y_1,\ldots,y_n)}(V(y_1,\ldots,y_n)) = \frac{\partial}{\partial t}|_{(y_1,\ldots,y_n)}$$

and the claim follows. $\boxed{5, \mathrm{M}}$

(c) (i) For all $(x,y,z) \in X$, we have that $V(x,y,z)$ and $W(x,y,z)$ are linearly independent vectors. Thus D is a distribution of rank 2. $\boxed{2, \mathrm{M}}$

(ii) We have that

$$[V,W] = (\partial_x + y\partial_z)(x\partial_x + \partial_y) - (x\partial_x + \partial_y)(\partial_x + y\partial_z) = \partial_x - \partial_z$$

which is not a vector field in D. Thus D is not involutive and, therefore, it is not integrable. $\boxed{2, \mathrm{M}}$

(iii) The integrable submanifolds are given by the solutions of $\boxed{3, \mathrm{M}}$

$$z - xy = a \qquad y = b$$

for all $a, b \in \mathbb{R}$.

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70058 Manifolds

Question	Marker's comment
1	Overall good.nbsp;Few comments:1) The intersection of two submanifolds is not always a submanifold (ex. (b))2) You cannot write "it is a Mobius strip and therefore it is a manifold.". It needs a proof (ex. (d)).
2	Overall good. No particular comments.nbsp;
3	Ex. (c)(ii) was the hardest exercise of the exam. Several attempts only used a partition of the unity, which cannot work because every manifold admits a partition of the unity but it is not true that every line bundle is trivial.nbsp;
4	Overall good. If you use techniques from other classes to prove (b)(ii) then you should clearly explain all those techniques, which were not covered in this class.nbsp;
5	Overall good. No particular comments.nbsp;