Imperial College London

MATH70056

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebraic Geometry

Date: 5 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Throughout this exam k denotes an algebraically closed field. In answering these questions you may use any results from the course as long as you state them clearly.

1. (a) For each of the following subsets X of affine space, either show that X is affine algebraic and find its ideal $I(X)$, or prove that X is not affine algebraic:

   (i) The subset $\{t^2 - 1, t^2 - t\}$ of $\mathbb{A}^2_k$. (5 marks)

   (ii) The subset $\{(x, y) : y = e^x\}$ of $\mathbb{A}^2_{\mathbb{C}}$. (5 marks)

   (b) Find the irreducible components of each of the following affine algebraic sets, and prove that each component you find is irreducible:

   (i) The subset $Z(y^2 - xz, z^2 - yw)$ of $\mathbb{A}^4_k$. (5 marks)

   (ii) The subset $Z(yz - xy, z^3 - y^2z - xy)$ of $\mathbb{A}^3_k$. (5 marks)

(Total: 20 marks)

2. (a) Define the notions of rational map and rational function on an irreducible variety. (2 marks)

   (b) Show that if X is a variety, then giving a rational map from X to $\mathbb{P}^1_k$ is equivalent to giving a rational function on X. (3 marks)

   (c) Let $f : \mathbb{P}^2_k \to \mathbb{P}^1_k$ be a rational map. Show that there exist homogeneous polynomials f_0, f_1 in $k[x, y, z]$, of the same degree, such that f is represented by the pair (U, g), where U is the complement of $\tilde{Z}(f_0, f_1)$ in $\mathbb{P}^2_k$, and g is the map $U \to \mathbb{P}^1_k$ defined by $g([x : y : z]) = [f_0(x, y, z) : f_1(x, y, z)]$. (5 marks)

   (d) Show that no nonconstant rational map $\mathbb{P}^2_k \to \mathbb{P}^1_k$ is regular. (5 marks)

   (e) Show that every regular map $\mathbb{P}^2_k \to \mathbb{P}^2_k$ is either constant or surjective. (5 marks)

(Total: 20 marks)

3. Identify the space of homogeneous polynomials of degree 2 in $k[x, y, z]$, up to rescaling, with $\mathbb{P}^5$, via:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz \mapsto [a : b : c : d : e : f].$$

Let X denote the subset of $\mathbb{P}^5$ corresponding to *reducible* polynomials of degree 2, and let Y be the subset of $\mathbb{P}^2$ corresponding to homogeneous polynomials of degree 2 that are (up to rescaling) squares of a linear homogeneous polynomial.

(a) Show that X and Y are irreducible closed subvarieties of $\mathbb{P}^5_k$, of dimensions 4 and 2, respectively. (5 marks)

(b) Show that there exists a single irreducible homogeneous polynomial f in $k[x_0, \ldots, x_5]$ such $X = \tilde{Z}(f)$. (You do not need to explicitly determine this polynomial.) (5 marks)

(c) Give, with proof, an explicit set S of homogeneous polynomials in $k[x_0, \ldots, x_5]$ such that $Y = \tilde{Z}(S)$. (You may assume k does not have characteristic 2.) (10 marks)

(Total: 20 marks)

4. An *automorphism* of a variety X is a regular isomorphism $f : X \to X$.

(a) Show that every automorphism of $\mathbb{A}^1_k$ is of the form $x \mapsto ax + b$, for $a \in k^\times$ and $b \in k$. (6 marks)

(b) Show that every automorphism of $\mathbb{P}^1_k$ is a projective transformation; i.e. of the form $[x : y] \mapsto [ax + by : cx + dy]$, for $a, b, c, d \in k$ such that $ad - bc \neq 0$. (7 marks)

(c) Determine the automorphisms of the curve $X = Z(y^2 - x^3)$ in $\mathbb{A}^2_k$. (7 marks)

(Total: 20 marks)

5. Identify the space of nonzero homogeneous polynomials of degree 2 in x, y, z, up to rescaling, with $\mathbb{P}^5_k$, by sending the polynomial $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ to the point $[a : b : c : d : e : f]$ of $\mathbb{P}^5_k$. Let $X \subset \mathbb{P}^5_k \times \mathbb{P}^5_k \times \mathbb{P}^5_k$ denote the set of triples of homogeneous polynomials (f, g, h) of degree 2 in $k[x, y, z]$ (up to rescaling), such that f, g, h have a common zero in $\mathbb{P}^2_k$.

(a) Let $\tilde{X}$ denote the subset of $\mathbb{P}^5_k \times \mathbb{P}^5_k \times \mathbb{P}^5_k \times \mathbb{P}^2_k$ consisting of tuples (P, Q, R, s), where P, Q, R are homogeneous polynomials of degree 2 and s is a point of $\mathbb{P}^2_k$ such that $P(s) = Q(s) = R(s) = 0$. Show that $\tilde{X}$ is a closed subset of $(\mathbb{P}^5_k)^3 \times \mathbb{P}^2_k$. (2 marks)

(b) By considering the projection of $\tilde{X}$ to $\mathbb{P}^2_k$, or otherwise, show that $\tilde{X}$ is irreducible of dimension 14. (7 marks)

(c) By considering the projection of $\tilde{X}$ to $(\mathbb{P}^5_k)^3$, or otherwise, show that X is a an irreducible closed subvariety of $(\mathbb{P}^5_k)^3$. (7 marks)

(d) Show further that the projection $\tilde{X} \to X$ is finite-to-one on an open subset U of X, and conclude that X has dimension 14. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH70056

Algebraic Geometry (Solutions)

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1. (a) (i) Setting $x = t^2 - 1, y = t^2 - t$, we find that $t = x - y + 1$. Substituting this into $x = t^2 - 1$ yields the equation $x - (x - y + 1)^2 + 1 = 0$, so the set in question is contained in $Z(x-(x-y+1)^2+1)$. Conversely, if (x, y) is a point on $Z(x - (x - y + 1)^2 + 1)$, then setting $t = x - y + 1$ we see that $x = t^2 - 1$, and $y = t^2 - t$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{5, \text{A}}$

(ii) Suppose this set were affine algebraic. Then its intersection with the closed subset $Z(y-1)$ would be a closed subset of the line $y = 1$. But this intersection consists of the points $(2\pi in, 1)$ for an integer n, and this is not closed in the affine line since any proper closed subset of the affine line is finite. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{5, \text{A}}$

(b) (i) Note that if $y = z = 0$, then x and w can be arbitrary; thus the set in question contains the irreducible plane $Z(y, z)$. On the other hand, if either y or z is nonzero then so is the other (for instance if y is nonzero then xz must be nonzero.) We then have $x = \frac{y^2}{z}$ and $w = \frac{z^2}{y}$ which implies that $xw = yz$. Thus the complement of $Z(y, z)$ in the set in question is contained in $Z(y^2-xz, z^2-yw, xw-yz)$. The latter is irreducible as well (it is the cone over the twisted cubic curve), or equivalently the image of $\mathbb{A}^2_k$ in $\mathbb{A}^4_k$ under the map $(t, u) \mapsto (t^3, t^2u, tu^2, u^3)$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{5, \text{B}}$

(ii) Note that since $xy-yz$ vanishes on this set, we either have $y = 0$ or $x = z$ for all points in this set. In the former case, the equation $z^3 - y^2z - xy$ becomes $z^3 = 0$, so any such point is on the line $y = z = 0$. In the latter case, the second equation becomes $x^3-y^2x-xy = 0$, which means that either $x = 0$ or $x^2 = y^2-y$. Our variety is thus the union of the varieties $Z(y, z)$, $Z(x, z)$, and $Z(x - z, x^2 - y^2 - y)$. Each of these varieties is irreducible: the ideals (y, z) and (x, z) are clearly prime, and $k[x, y, z]/\langle x - z, x^2 - y^2 - y\rangle$ is isomorphic to $k[x, y]/\langle x^2 - y^2 - y\rangle$ which is clearly a domain as $x^2 - y^2 - y$ is irreducible. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{5, \text{A}}$

2. (a) Let X and Y be irreducible varieties. A rational map $X \to Y$ is an equivalence class of pairs (U, f), where U is a nonempty open subset of X and $f : U \to Y$ is a regular map. We say (U, f) is equivalent to (V, g) if the restrictions of f and g to $U \cap V$ agree. A rational function on X is an equivalence class of pairs (U, f) where U is a nonempty open subset of X and f is a regular function on U, under the same equivalence relation. seen $\Downarrow$ 2, A

(b) Let $X \to \mathbb{P}^1_k$ be a rational map represented by a pair (U, f). Then $f^{-1}(\infty)$ is closed in U, and its complement V is an open subset of U. The restriction of f to V takes values in $\mathbb{A}^1_k$, and can thus be regarded as a regular function on V; then the pair (V, f) is a rational function on X. Conversely, given a rational function on X represented by a pair (V, f), we can regard f as a map $V \to \mathbb{A}^1_k$ and compose with the inclusion into $\mathbb{P}^1_k$ to obtain a rational map from X to $\mathbb{P}^1_k$. These two constructions are inverse to each other. sim. seen $\Downarrow$ 3, A

(c) By part (b), a rational map $\mathbb{P}^2 \to \mathbb{P}^1$ can be represented by a pair (V, f), where V is a distinguished open subset $D(h)$ of $\mathbb{A}^2_k \subset \mathbb{P}^2_k$ and f is a regular function on V. Then we can write $f = \frac{g}{h^r}$, where g and h lie in $k[x, y]$. Let f_0 be the homogenization of g with respect to z and f_1 be the homogenization of h^r; multiplying either f_0 or f_1 by a suitable power of z we can arrange for f_0 and f_1 to have the same degree. Then $\frac{f_0(x,y,1)}{f_1(x,y,1)} = \frac{g(x,y)}{h(x,y)^r} = f(x, y)$, so our rational function f is represented by the map $[x : y : z] \mapsto [f_0(x, y, z) : f_1(x, y, z)]$ on the complement of the common zero locus of f_0 and f_1. sim. seen $\Downarrow$ 5, B

(d) Following part c, we write such a map as $[f_0(x, y, z) : f_1(x, y, z)]$. As the map is nonconstant, f_0 and f_1 have positive degree. By the projective dimension theorem, they have a common zero q in $\mathbb{P}^k_2$. Now suppose this map were regular, and let $[x_0 : x_1]$ be the image of q under this map. Let $t = [t_0 : t_1]$ be any point in $\mathbb{P}^1_k$ other than $[x_0 : x_1]$. The preimage of t under this map is the locus $t_0 f_1(x, y, z) - t_1 f_0(x, y, z) = 0$; this in particular contains the point q. But this is impossible because we chose t to not be the image of q. unseen $\Downarrow$ 5, C

(e) Let $f : \mathbb{P}^2_k \to \mathbb{P}^2_k$ be a regular map that is not surjective, and let p be a point not in its image. Making a linear change of coordinates on $\mathbb{P}^2_k$ we can assume that $p = [0 : 0 : 1]$. Then the rational map $\pi : \mathbb{P}^2_k \to \mathbb{P}^1_k$ defined by $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ is defined everywhere except p, so the composition $\pi \circ f$ is a regular map $\mathbb{P}^2_k \to \mathbb{P}^1_k$. By part (d) this map is constant, so its image is a single point p, and the image of f is contained in the closure of the preimage of that point under π. But the latter is a line in $\mathbb{P}^2_k$, so applying part (d) again we see that f is constant. (One can also argue that if f is not surjective, then $\pi \circ f$ is finite-to-one, and use dimension arguments.) sim. seen $\Downarrow$ 5, C

3. (a) Identify the space of linear polynomials in x, y, z with $\mathbb{P}^2_k$, and consider the maps: $f : \mathbb{P}^2_k \times \mathbb{P}^2_k \to \mathbb{P}^5_k$ and $g : \mathbb{P}^2_k \to \mathbb{P}^5_k$ that take a pair of linear polynomials P, Q (resp. a single linear polynomial P) to the polynomials PQ and P^2, respectively. In coordinates we have sim. seen ⇓

$$f([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) = (a_0b_0 : a_1b_1 : a_2b_2 : a_0b_1{+}a_1b_0 : a_0b_2{+}a_2b_0 : a_1b_2{+}a_2b_1]$$

$$g([a_0 : a_1 : a_2]) = [a_0^2 : a_1^2 : a_2^2 : 4a_0a_1 : 4a_0a_2 : 4a_1a_2].$$

These are just regular maps, and their images are X and Y respectively. Since projective spaces and their products are complete, it follows that X and Y are closed. Moreover the maps are finite to one (as factorizations of polynomials are unique up to reordering), so X has dimension 4 and Y has dimension 2. 5, B

(b) We have seen that a codimension 1 subvariety of projective space is cut out by a single homogeneous polynomial. Since X has codimension 1 and is irreducible (it is the image of $\mathbb{P}^2_k \times \mathbb{P}^2_k$, which is irreducible,) the claim follows. meth seen ⇓ 5, A

(c) Our description of g above shows that the polynomials $4ab - d^2, 4ac - e^2$, and $4bc - f^2$ vanish on Y, as do the polynomials $de - 2af, df - 2be$, and $ef - 2cd$. It suffices to show that Y is the zero set of these polynomials. Let $p = [a : b : c : d : e : f]$ be a point of the image. Then one of a, b, c is nonzero (as otherwise d^2, e^2, f^2 all vanish. Without loss of generality (using an automorphism that permutes the variables) we can assume that a is nonzero. unseen ⇓

I now claim that $p = g([2a : d : e])$. Indeed, we have:

$$g([2a : d : e]) = [4a^2 : d^2 : e^2 : 8ad : 8ae : 4de]$$

and it suffices to show that this is equivalent to p. The relations $d^2 = 4ab$, $e^2 = 4ac$ and $de = 2af$ show that this is the point p, rescaled by $4a$. 10, D

4. (a) An automorphism of $\mathbb{A}^1_k$ is given by a regular map of the form $t \mapsto p(t)$ for t a polynomial. As this map must be bijective, $p(t) - c$ has exactly one root for and $c \in k^\times$; since k is algebraically closed, and in particular infinite, $p(t)$ must have degree one. Conversely, it is clear that any linear polynomial does give an isomorphism. sim. seen ⇓ 6, A

(b) We have seen that any linear change of variables of this form does indeed give an automorphism, and that this set of automorphisms is closed under composition and inverses. Now let g be an arbitrary automorphism, and choose a linear automorphism f such that f sends $g(\infty)$ to ∞. Then fg fixes infinity, so restricts to an automorphism of $\mathbb{A}^1_k$. Such an automorphism is given by a linear map by part (a), of the for $t \mapsto at + b$, but this extends to the linear automorphism $[x : y] \mapsto [ax + by : y]$ of $\mathbb{P}^1_k$. Thus f and fg are linear automorphisms, so g is as well. meth seen ⇓ 7, B

(c) It is clear that any map of the form $(x, y) \mapsto (c^2x, c^3y)$ for $c \in k^\times$ is an automorphism; we will show that every automorphism has this form. Let g be an arbitrary automorphism of X; then g induces an automorphism of $k[x, y]/\langle y^2 - x^3\rangle$, which we identify with the subring $k[t^2, t^3]$ of $k[t]$. Then g also induces an automorphism h of the field of fractions of this ring. Since $h(t)^2 = h(t^2)$, we have that $h(t)$ is the square root of a polynomial, and must thus itself be polynomial. Thus h induces a map $k[t] \to k[t]$. By arguing with g^{-1} in place of g we see that h^{-1} also maps $k[t]$ to $k[t]$, so h is an automorphism of $k[t]$ and hence of the form $t \mapsto at + b$ for some a, b. But this only preserves $k[t^2, t^3]$ when $b = 0$, so h has the form $t \mapsto at$. Then $g(X) = g(t^2) = h(t^2) = c^2t^2 = c^2X$, and similarly $g(Y) = c^3Y$. unseen ⇓ 7, D

5. (a) The condition $P(s) = 0$ can be written in polynomial terms as $a_P s_0^2 + b_P s_1^2 + c_P s_2^2 + d_P s_0 s_1 + E_P s_0 s_2 + f_P s_1 s_2$, where $a_P, b_P, \ldots$ are the coefficients of P, and s_0, s_1, s_2 are the coordinates of s. This is homogeneous of degree 1 in the P-coordinates and degree 2 in the s-coordinates. The other conditions are similar. $\boxed{2, \mathrm{M}}$

(b) For any given $s \in \mathbb{P}^2_k$, the fiber over s is a self-product of three linear codimension 1 subspaces of $\mathbb{P}^5_k$. Thus the projection to $\mathbb{P}^2_k$ is surjective with irreducible fibers of dimension 12. Since it is a map of projective varieties it is proper, so by a result in lectures $\tilde{X}$ is irreducible of dimension 14. $\boxed{7, \mathrm{M}}$

(c) Since $\tilde{X}$ is projective it is complete, so its image X under the first projection is closed; since $\tilde{X}$ is irreducible so is X. $\boxed{7, \mathrm{M}}$

(d) Let V be the closed subset of X consisting of tuples (P, Q, R) such that Q and R are both scalar multiples of P. On the complement of V the common zeros of P, Q, and R form a finite set by dimension considerations, so on the complement of this open set the map $\tilde{X} \to X$ is finite-to-one. In particular X and $\tilde{X}$ have the same dimension. $\boxed{4, \mathrm{M}}$

Review of mark distribution:
Total A marks: 31 of 32 marks
Total B marks: 22 of 20 marks
Total C marks: 10 of 12 marks
Total D marks: 17 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70056	1	A number of people struggled to find irreducible components in part b. A useful trick is to look for reducible polynomials in the ideal; factoring these will express V(I) as a union of proper closed subvarieties.
MATH70056	2	Many students attempted to use the fact that regular maps from P^m to P^n are given by m-tuples of homogeneous polynomials of the same degree in part c. We did not prove this for rational maps, however, and indeed this is the content of the question. I was pleased to see that many students remembered the "projection from a point" technique in part e.
MATH70056	3	I was glad to see that most students thought of the appropriate technique for showing X and Y were closed (exhibiting them as the image of a map from some projective variety).
MATH70056	4	Most people did well on parts a and b; part c was quite difficult.
MATH70056	5	I was very happy to see how many people understood the techniques involved here! One thing many people missed is that the theorem that says if f: X -> Y has irreducible fibers of the same dimension and Y is irreducible then X is irreducible requires f to be proper.