Introduction to Quantum Mechanics – Problem sheet 5

1. **Coherent states** - *This question is a good practice for working with the harmonic oscillator basis, and coherent states are an important tool in qantum optics and beyond, and we will encounter them again. Do have a go at parts (a), (b), and (c) for sure. part (d) is also good practice, but perhaps a little harder.*

   The so-called *coherent states* are superpositions of harmonic oscillator states of the form

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \quad \text{for} \quad z \in \mathbb{C} \tag{1}$$

   (a) Verify that the *overlap* $|\langle z_1|z_2\rangle|^2$ of two coherent states is given by $|\langle z_1|z_2\rangle|^2 = \mathrm{e}^{-|z_1-z_2|^2}$.

   (b) Verify that $|z\rangle$ are eigenvectors of the lowering operator $\hat{a}$ with eigenvalue z.

   (c) Verify that the expectation values of the position and momentum operators $\hat{q}$ and $\hat{p}$ in a coherent state $|z\rangle$ are given by

$$\langle \hat{q} \rangle = \sqrt{\frac{2\hbar}{m\omega}} \mathrm{Re}(z) \quad \text{and} \quad \langle \hat{p} \rangle = \sqrt{2m\omega\hbar}\, \mathrm{Im}(z).$$

   (d) Show that the raising operator does not have normalisable eigenstates.

2. **A two-dimensional harmonic oscillator** - *Part (a) can be simply viewed as an exercise in commutators, it is good to get some practice. Part (b) is short, but requires some deeper thought.*

   Let us consider two harmonic oscillators with annihilation and creation operators $\hat{a}_{1,2}$ and $\hat{a}^\dagger_{1,2}$, fulfilling the commutation relations

$$[\hat{a}_j, \hat{a}_k] = 0, \quad [\hat{a}^\dagger_j, \hat{a}^\dagger_k] = 0, \quad \text{and} \quad [\hat{a}_j, \hat{a}^\dagger_k] = \delta_{jk}.$$

   (a) Show that the operators defined as

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger_1 \hat{a}_1 - \hat{a}^\dagger_2 \hat{a}_2), \quad \hat{K}_+ = \hat{a}^\dagger_1 \hat{a}_2, \quad \text{and} \quad \hat{K}_- = \hat{a}^\dagger_2 \hat{a}_1$$

   fulfil the commutation relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad \text{and} \quad [\hat{K}_+, \hat{K}_-] = 2\hat{K}_0.$$

   (b) What can you conclude about the eigenvalues of $\hat{K}_0$ from what you know about the eigenvalues of the operator $\hat{N}$ of the harmonic oscillator?

3. **The harmonic oscillator method for numerical eigenvalue problems** - *This is a totally optional numerical application of the harmonic oscillator basis. Decide for yourself whether you think doing some playing on matlab helps you get more familiar with the content. It does help many people a lot, others not so much.*

   Download the matlab code for the numerical calculation for eigenvalues of one-dimensional potentials using the harmonic oscillator basis described in the lecture.

   (a) Modify it to obtain the first 10 eigenvalues of the Hamiltonian $\hat{H} = \hat{p}^2/2 + \hat{x}^4 - \lambda \hat{x}^2$ for different values of λ, such as, for example $\lambda = 1$, or $\lambda = 10$. What do you observe about the eigenvalues of these Hamiltonians?

   (b) Investigate the convergence of the obtained eigenvalues with respect to the matrix size N.