

# MATH50001 Analysis II, Complex Analysis

## Lecture 20

## Section: Möbius Transformations.

### Definition.

A Möbius transformation (that is also called a bilinear transformation) is a map

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

*Special Möbius transformations.*

Let

$$f(z) = \frac{az + b}{cz + d}$$

and consider the following cases:

$$(M1) \quad z \mapsto az \quad (b = c = 0, d = 1);$$

if  $|a| = 1$ ,  $a = e^{i\theta}$ , then this is a rotation by  $\theta$ . If  $a > 0$  then  $f$  corresponds to a dilation and if  $a < 0$  the map consists of a dilation by  $|a|$  followed by a rotation of  $\pi$ .

$$(M2) \quad z \mapsto z + b \quad (a = d = 1, c = 0 - \text{translation by } b);$$

$$(M3) \quad z \mapsto \frac{1}{z} \quad (a = d = 0, b = c = 1 - \text{inversion}).$$

In (M1), if  $a = re^{i\theta}$ , the geometrical interpretation is an expansion by the factor  $r$  followed by a rotation anticlockwise by the angle  $\theta$ .

Theorem.

Every Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

is a composition of transformations of type (M1), (M2) and (M3).

*Proof.*

1. If  $c = 0$  and  $d \neq 0$ , then

$$f(z) = \frac{az + b}{d} = g_2 \circ g_1(z),$$

where

$$g_1(z) = \frac{a}{d} z, \quad g_2(z) = z + \frac{b}{d}.$$

2. If  $c \neq 0$ , then  $f(z) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z)$ , where

$$g_1(z) = cz, \quad g_2(z) = z + d, \quad g_3 = \frac{1}{z},$$

$$g_4(z) = \frac{1}{c}(bc - ad)z \quad g_5(z) = z + \frac{a}{c}.$$

Indeed,

$$g_1(z) = cz, \quad g_2 \circ g_1(z) = cz + d, \quad g_3 \circ g_2 \circ g_1(z) = \frac{1}{cz + d},$$

$$g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{bc - ad}{c(cz + d)},$$

$$g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{az + b}{cz + d} = f(z).$$

### Corollary.

A Möbius transformation transforms circles into circles, and interior points into interior points. (Here we mean that straight lines are also circles whose radius equal infinity).

*Proof.* Each of the transformations (M1), (M2) and (M3) transform circles into circles.

## Section: Cross-Ratios Möbius Transformation.

Theorem.

If  $w = f(z)$  is a Möbius transformation that maps the distinct points  $(z_1, z_2, z_3)$  into the distinct points  $(w_1, w_2, w_3)$  respectively, then

$$\left( \frac{z - z_1}{z - z_3} \right) \left( \frac{z_2 - z_3}{z_2 - z_1} \right) = \left( \frac{w - w_1}{w - w_3} \right) \left( \frac{w_2 - w_3}{w_2 - w_1} \right), \text{ for all } z.$$

*Proof.*

The Möbius transformation

$$g(z) = \left( \frac{z - z_1}{z - z_3} \right) \left( \frac{z_2 - z_3}{z_2 - z_1} \right)$$

maps  $z_1, z_2, z_3$  to  $0, 1, \infty$  respectively. Similarly the Möbius transformation

$$h(w) = \left( \frac{w - w_1}{w - w_3} \right) \left( \frac{w_2 - w_3}{w_2 - w_1} \right)$$

maps  $w_1, w_2, w_3$  to  $0, 1, \infty$  respectively. Therefore  $h^{-1} \circ g$  maps  $(z_1, z_2, z_3)$  into  $(w_1, w_2, w_3)$ .

**Example.** Find a Möbius transformation  $w = f(z)$  that maps the points  $1$ ,  $i$ , and  $-1$  on the unit circle  $|z| = 1$  onto the points  $-1$ ,  $0$ ,  $1$  on the real axis. Determine the image of the interior  $|z| < 1$  under this transformation.

*Proof.* Let  $z_1 = 1$ ,  $z_2 = i$ ,  $z_3 = -1$  and  $w_1 = -1$ ,  $w_2 = 0$ ,  $w_3 = 1$ . The mapping  $w = f(z)$  must satisfy the Cross-Ratios Möbius Transformation

$$\begin{aligned} \frac{z-1}{z-(-1)} \cdot \frac{i-(-1)}{i-1} &= \frac{w-(-1)}{w-1} \cdot \frac{0-1}{0-(-1)} \\ \implies \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} &= -\frac{w+1}{w-1} \implies \frac{z-1}{z+1}(-i) = -\frac{w+1}{w-1} \\ \implies (w-1)(z-1)i &= (w+1)(z+1) \\ \implies w((z-1)i - (z+1)) &= (z-1)i + (z+1) \\ \implies w = \frac{iz - i + z + 1}{zi - i - z - i - 1} &= \frac{z(1+i) + (1-i)}{iz(1+i) - (1+i)} = \frac{z - i}{iz - 1}. \end{aligned}$$

Note that if  $z = 0$  then  $f(0) = i$ .

**Example.** Find a linear fractional transformation  $w = f(z)$  that maps the points  $z_1 = -i$ ,  $z_2 = 1$ , and  $z_3 = \infty$  on the line  $y = x - 1$  onto the points  $w_1 = 1$ ,  $w_2 = i$ , and  $w_3 = -1$  on the unit circle  $|w| = 1$ .

*Proof.* Note that

$$\begin{aligned} \lim_{z_3 \rightarrow \infty} \frac{z+i}{z-z_3} \cdot \frac{1-z_3}{1+i} &= \lim_{t \rightarrow 0} \frac{z+i}{z-1/t} \cdot \frac{1-1/t}{1+i} \\ &= \lim_{t \rightarrow 0} \frac{z+i}{tz-1} \cdot \frac{t-1}{1+i} = \frac{z+i}{1+i}. \end{aligned}$$

Therefore in this case the cross-ratio could be written

$$\begin{aligned} \frac{z+i}{1+i} = \frac{w-1}{w+1} \cdot \frac{i+1}{i-1} &\implies \frac{z+i}{1+i} = -i \frac{w-1}{w+1} \\ &\implies w = \frac{-z-1}{z+2i-1}. \end{aligned}$$

## Section: Conformal mapping of a half-plane to the unit disc.

The upper half-plane can be mapped by a holomorphic bijection to the disc, and this is given by a Möbius transformation.

Let

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : \operatorname{Im} z = y > 0\}.$$

A remarkable surprising fact is that the unbounded set  $\mathbb{H}$  is conformally equivalent to the unit disc. Moreover, an explicit formula giving this equivalence exists. Indeed, let

$$w = f(z) = \frac{i - z}{i + z}, \quad g(w) = i \frac{1 - w}{1 + w}.$$

**Theorem.** Let  $\mathbb{D} = \{z : |z| < 1\}$ . Then the map  $f : \mathbb{H} \mapsto \mathbb{D}$  is a conformal map with inverse  $g : \mathbb{D} \mapsto \mathbb{H}$ .

*Proof.* Clearly both functions are holomorphic in their respective domains. If  $z = x + iy$ ,  $y > 0$ , then

$$\left| \frac{i-z}{i+z} \right|^2 = \left| \frac{x^2 + (y-1)^2}{x^2 + (y+1)^2} \right| < 1.$$

Let  $w = u + iv$ ,  $|w| < 1$ . Then

$$\begin{aligned} \operatorname{Im} g(w) &= \operatorname{Re} \left( \frac{1-u-iv}{1+u+iv} \right) = \operatorname{Re} \left( \frac{(1-u-iv)(1+u-iv)}{(1+u)^2 + v^2} \right) \\ &= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0. \end{aligned}$$

Finally

$$f \circ g(w) = \frac{i - i \frac{1-w}{1+w}}{i + i \frac{1-w}{1+w}} = \frac{1+w-1+w}{1+w+1-w} = w.$$

Similarly we also have  $g \circ f(z) = z$ .

Note that  $f$  is holomorphic in  $\mathbb{C} \setminus \{-i\}$  and, in particular, it is continuous on the boundary of  $\partial(\mathbb{H}) = \{z = x + i0 \in \mathbb{C}\}$ . Clearly

$$|f(z)|_{z=x+i0} = \left| \frac{i-x}{i+x} \right| = 1.$$

Thus  $f$  maps  $\mathbb{R}$  onto the boundary of the unit disc  $\partial\mathbb{D}$ . Moreover,

$$f(z) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i \frac{2x}{1+x^2}.$$

$$f(z) = \frac{i - x}{i + x} = \frac{1 - x^2}{1 + x^2} + i \frac{2x}{1 + x^2}.$$

Let  $x = \tan \theta$  with  $\theta \in (-\pi/2, \pi/2)$ . Since

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \quad \text{and} \quad \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

we obtain

$$f(z) = \cos 2\theta + i \sin 2\theta = e^{2i\theta}.$$

$$f(z) = \cos 2\theta + i \sin 2\theta = e^{2i\theta}, \quad \theta \in (-\pi/2, \pi/2).$$

Therefore the image of the real line is the arc consisting of the circle omitting the point  $-1$ . Moreover, if the value of  $x$  changes from  $-\infty$  to  $\infty$ ,  $f(x)$  changes along that arc starting from  $-1$  and first going through that part of the circle that lies in the lower half-plane. The point  $-1$  on the circle corresponds to “infinity” of the upper half-plane.

## Section: Riemann mapping theorem.

**Definition.** We say that  $\Omega \subset \mathbb{C}$  is *proper* if it is non-empty and not the whole of  $\mathbb{C}$ .

**Theorem.** Suppose  $\Omega$  is proper and simply connected. If  $z_0 \in \Omega$ , then there exists a unique conformal map  $f : \Omega \rightarrow \mathbb{D}$  such that

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0.$$

**Corollary** Any two proper simply connected open subsets in  $\mathbb{C}$  are conformally equivalent.

Thank you