# Applied Complex Analysis - Lecture Two

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January 2025

Some questions:

- How will the mastery level be evaluated?
- How can I catch up on complex analysis?
- How much coding will there be in the coursework?

Summary

- In the previous lecture, we reviewed complex *numbers*
- We saw that many properties of standard arithmetic generalise in a natural way.
- Issues arise with complex argument, which carry over to radicals, multiplying numbers, etc
- This time, we will review complex *functions*

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## The triangle inequality

For $z = a + bi$:

$$||a| - |b|| \leq |z| \leq ||a| + |b||$$

For $z_1, z_2 \in \mathbb{C}$:

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq ||z_1| + |z_2||$$

Proof: exercise.

*Functions of complex variables*

# Functions of complex variables

Any function of a complex variable can be written as:

$$f(z) = u(x, y) + iv(x, y)$$

Elementary functions generalise to complex variables.

- Polynomials
- Rational functions
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# The exponential function

Familiar Taylor expansion holds for complex variables (more on this later):

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Conjugation (of Euler's identity) provides trig identities:

$$\cos\theta = \frac{e^{i\theta} + e^{i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}$$

Periodicity (not enjoyed by real-valued exp)

$$e^{i\theta + 2\pi k i} = e^{i\theta}, \quad k \in \mathbb{Z}$$

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# The complex logarithm

Due to periodicity of $\exp$, if defined as the inverse of $\exp$, $\log$ is multi-valued:

$$\log z = \log\left(|z|e^{i\arg z}\right) = \log|z| + i\arg z + 2\pi i k, \quad \text{for } k \in \mathbb{Z}.$$

- The inverse is an example of a *multi-valued* function.
- Mathematica demo
- This is distinct from $\exp(x)$ for $x \in \mathbb{R}$, which has a unique single-valued inverse $\log$.
- Actually, $\exp z$ has a single-valued inverse for $z$ in the complex strip $\mathrm{Im} z \in [-\pi, \pi)$.
- Due to $\arg$, not all real-valued properties of $\log$ generalise
- Can define $z^\alpha := \exp(\alpha \log z)$, also multi-valued for $\alpha \notin \mathbb{Z}$.

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# Analytic/Holomorphic functions

This is where the magic happens. All of our complex analysis applications require this property. There are multiple equivalent definitions.

- Holomorphic: Complex differentiable in a neighbourhood of a point
- Satisfying the Cauchy Riemann equations
- Analytic: Has a locally convergent Taylor expansion
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# The complex derivative

A function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z$ if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is independent of the path taken for the limit, noting $h \in \mathbb{C}$.

- We say $f$ is holomorphic (equiv. analytic) in $\Omega \subset \mathbb{C}$ if $f$ is complex differentiable for all $z \in \Omega$.
- The complex derivative satisfies all of the same rules as the real derivative, nice!
- Before exploring theoretical consequences, let's consider some examples...

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# Are these functions analytic?

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- $f(z) = z^2$
- $f(z) = \overline{z}$
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# Cauchy-Riemann equations

For $f(z) = u(x, y) + iv(x, y)$, the complex derivative $f'(z)$ exists if and only if the following hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

# Domain of analyticity

- We have seen friendly looking functions which are not analytic anywhere
- If a $f(z)$ is analytic for all $z \in \mathbb{C}$, we say $f$ is 'entire'
- Most functions we are interested are analytic in some non-trivial open $\Omega \subset \mathbb{C}$, called the 'domain of analyticity'.
- Conventionally, mathematicians say '$f$ is analytic' to mean '$f$ is analytic in some domain $\Omega$'. This can be ambiguous, so it is often best to clarify $\Omega$, if any confusion might arise.
- Confusingly - we will see later that for many $f$, we may have multiple choices for $\Omega$, but we cannot take the union of these choices!

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# Harmonic functions

For $\phi(x, y)$ real-valued, we say $\phi$ is *Harmonic* if:

$$\Delta\phi := \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

- There is a beautiful connection between *harmonic* functions and *analytic* functions: If $f(x, y) = u(x, y) + iv(x, y)$ is *analytic*, then $u$ and $v$ are *harmonic*.
- This means that complex analysis techniques can be used to solve Laplace problems, involving $\Delta\phi = 0$, such as ideal fluid flow and electrostatic potential. More on this later!

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# *Contour Integrals*

# Contour integrals

Given a curve $\gamma = \{z(t) : t \in [a, b]\}$, we can define the contour integral

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

- Direction matters! If $\tilde{\gamma}$ is parametrised in the opposite direction, then

$$\int_{\tilde{\gamma}} f(z)dz = -\int_\gamma f(z)dz$$

- If $\gamma$ is the disjoint union of $\gamma_1$ and $\gamma_2$, then

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

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# The ML principle

For a piecewise differentiable curve $\gamma$ in the complex plane and a complex function $f(z)$, we have

$$\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma}\{|f(z)|\} \times \text{length}(\gamma).$$

- Proof uses definition of length of a contour.
- Example: $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz$ for $|f(z)| = O(z^{-3/2})$ as $|z| \to \infty$.
- Example: $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz$ for $f(z) = g(z)e^{iz}$, $|g(z)| = O(z^{-1/2})$ as $|z| \to \infty$.

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# Jordan's Lemma

For a function of the form $f(z) = e^{iaz}g(z)$, defined on a contour $\gamma_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, such that $g(z) \leq M_R$ for $z \in \gamma_R$, we have

$$\left| \int_{\gamma_R} f(z) dz \right| \leq \frac{\pi}{a} M_R.$$

- Proof is a little more involved than ML.
- When applicable, this is a stronger result than the ML principle.
- Revisit previous example: $\lim_{R\to\infty} \int_{\gamma_R} f(z) dz$ for $f(z) = g(z)e^{iz}$, $|g(z)| = O(z^{-1/2})$ as $|z| \to \infty$.

# Cauchy's Integral Theorem

**Theorem**:
Suppose $f$ is analytic/holomorphic in some simply connected (no holes) open $D \subset \mathbb{C}$. For a closed contour $\gamma \subset D$, we have

$$\oint_{\gamma} f(z)dz = 0$$

(Proof omitted.)

**Corollary**:
If $\gamma_1$ and $\gamma_2$ have the same endpoints, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

- Proof
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**Theorem**:

Suppose $f$ is analytic/holomorphic in some simply connected (no holes) open $D \subset \mathbb{C}$. For a closed contour $\gamma \subset D$, we have

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(Proof omitted.)

**Corollary**:

If $\gamma_1$ and $\gamma_2$ have the same endpoints, then

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