

MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 13

Section: Poles of holomorphic functions.

Definition. Suppose a holomorphic function f has an isolated singularity at z_0 and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

- If $a_n = 0$ for all $n < 0$, z_0 is called a removable singularity
- If $a_n = 0$ for $n < -m$ where m a fix positive integer, but $a_{-m} \neq 0$, z_0 is called a pole of order m .
- If $a_n \neq 0$ for infinitely many negative n 's, z_0 is called an essential singularity.

Example.

$$f(z) = \frac{\sin z}{z}; \quad f(z) = e^{1/z}; \quad f(z) = \frac{1}{z^3(z+2)^2}.$$

Theorem. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. If g is holomorphic at z_0 and $g(z_0) \neq 0$ then for some $R > 0$

$$g(z) = a_0 + a_1(z - z_0) + \dots, \quad |z - z_0| < R,$$

where $a_0 = g(z_0) \neq 0$. Then

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \dots, \quad 0 < |z - z_0| < R.$$

This implies that z_0 is a pole of order m .

Conversely, if f has a pole of order m at z_0 , then the Laurent expansion of f about z_0 equals

$$\begin{aligned} f(z) &= \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots \\ &\quad + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots \\ &= \frac{1}{(z - z_0)^m} (a_{-m} + a_{-m+1}(z - z_0) + \dots). \end{aligned}$$

L'Hopitals Rule.

Theorem. Let f and g be holomorphic in an open set Ω and $z_0 \in \Omega$. Suppose $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}.$$

Proof. We have

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} \left(\frac{z - z_0}{z - z_0} \right) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \left(\frac{z - z_0}{z - z_0} \right) \\ &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \rightarrow z_0} \frac{z - z_0}{g(z) - g(z_0)} \\ &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \left(\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} \right)^{-1} = \frac{f'(z_0)}{g'(z_0)}. \end{aligned}$$

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0 . The residue of f at z_0 is

$$\text{Res}[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0 . Then

$$\text{Res}[f, z_0] = \frac{1}{2\pi i} \oint_{\gamma} f(z) dz.$$

Proof. Let $0 < r < R$. By using the Deformation theorem we obtain

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma} f(z) dz &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z) dz \\ &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n dz \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{in\theta} i r e^{i\theta} d\theta = a_{-1}. \end{aligned}$$

Theorem. Let f be holomorphic function inside and on a simple, closed, piecewise-smooth curve γ except at the singularities z_1, \dots, z_n in its interior. Then

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^n \text{Res}[f, z_j].$$

Proof. Let $\gamma_j = \{z : |z - z_j| = r_j\} \subset \Omega$. Then by using the Deformation theorem we find

$$\oint_{\gamma} f(z) dz = \sum_{j=1}^n \oint_{\gamma_j} f(z) dz.$$

Example. Evaluate $\oint_{|z|=1} e^{1/z} dz$.

Clearly

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

Therefore

$$\oint_{|z|=1} e^{1/z} dz = 2\pi i.$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

m = 1. Then $g(z) = a_{-1} + a_0(z - z_0) + \dots$ and therefore

$$\text{Res}[f, z_0] = a_{-1} = \lim_{z \rightarrow z_0} g(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

m = 2. Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\text{Res}[f, z_0] = a_{-1} = \frac{d}{dz} g(z) \Big|_{z=z_0} = \lim_{z \rightarrow z_0} \frac{d}{dz} ((z - z_0)^2 f(z)).$$

m.

$$\text{Res}[f, z_0] = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_0)^m f(z)).$$

Quizzes

Thank you

