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Question 1	Topic BASIC MATERIAL	Marks& seen/unseen

Parts

(a). A pair of strategies, α^* for player A and β^* for player B, are said to be in equilibrium if:

$$g_A(\alpha^*,\beta^*) \geq g_A(\alpha,\beta^*), \quad \forall \alpha \in \mathbb{A}_S$$
$$g_B(\alpha^*,\beta^*) \geq g_B(\alpha^*,\beta), \quad \forall \beta \in \mathbb{B}_S.$$

A 3 Seen definition

(b). We have:

$$\max_{\alpha\in\mathbb{A}_S}\{g_A(\alpha,\beta^*)\} \geq \max_{a\in A_S}\{g_A(a,\beta^*)\}, \quad ①$$

since A_S is a subset of $\mathbb{A}_S$.

A 1

Conversely, writing $\alpha = (p_1, p_2, \cdots, p_n)$, $p_i \geq 0$, $\sum_i p_i = 1$, then:

Seen proof

$$g_A(\alpha,\beta^*) = \sum_i p_i g_A(a_i,\beta^*), \quad \text{by definition}$$
$$\leq \sum_i p_i \max_{a\in A_S}\{g_A(a,\beta^*)\}, \quad \because p_i \geq 0\ \forall i.$$
$$= \max_{a\in A_S}\{g_A(a,\beta^*)\}\cdot\sum_i p_i = \max_{a\in A_S}\{g_A(a,\beta^*)\}.$$

B 2 Seen proof

i.e. $\max_{\alpha\in\mathbb{A}_S}\{g_A(\alpha,\beta^*)\} \leq \max_{a\in A_S}\{g_A(a,\beta^*)\}$. ②

A 1

Taking ① and ② together proves equality. □

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Question 1 | Topic

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Parts

(c). A mixed strategy, α^*, for player A is called an equaliser strategy if:

$$g_B(\alpha^*, b) = c, \text{ constant}, \quad \forall b \in B_S .$$

1 $\boxed{A}$ constant $\forall b$

1 $\boxed{B}$ Subscript B

Seen definition

(d). We have:

$$g_A(a_i, \beta^*) = c_1 \ , \ \forall i,$$
$$g_B(\alpha^*, b_j) = c_2 \ , \ \forall j,$$

since α^* and β^* are equaliser strategies (c_1, c_2 constants).

1 $\boxed{A}$

seen proof

$$\Rightarrow g_A(\alpha^*, \beta^*) = \sum_i p_i\, g_A(a_i, \beta^*), \quad \text{writing } \alpha^* = \sum_i p_i a_i$$
$$= c_1 \sum_i p_i ,$$
$$= c_1 = g_A(a_i, \beta^*), \ \forall i .$$

Similarly $g_B(\alpha^*, \beta^*) = c_2 = g_B(\alpha^*, b_j), \ \forall j$.

2 $\boxed{B}$

seen proof

By part (b), there can be no alternative mixed strategies that would perform better than α^* for A and β^* for B, thus α^*, β^* are mutual best responses and are in equilibrium.

1 $\boxed{A}$

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Question 1

Topic

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Parts

(e). (i).

		B		
		b_1: $3n-1$	b_2: $3n$	b_3: $3n+1$
A	a_1 = even	2,5	6,1	4,3
	a_2 = odd	5,2	3,4	1,6

~~unseen~~ game.

2 $\boxed{A}$ seen similar

(ii). We can delete strategy $b_2 : 3n$ for player B since it is strictly dominated by strategy $b_3 : 3n+1$ for B. This leaves us with the game:

1 $\boxed{C}$ seen similar

		B	
		b_1	b_3
A	a_1	2, $\boxed{5}$	4,3
	a_2	5,2	1, $\boxed{6}$

No pure strategy equilibria.

Let $\alpha^* = (p, 1-p)$ and $\beta^* = (q, 1-q)$. We seek a pair of equaliser strategies for the players, which by part (d), form an equilibrium of the game.

For α^* to be an ES, we insist:

$$g_B(\alpha^*, b_1) = g_B(\alpha^*, b_2)$$

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Parts (e). (ii). (Contined.)

$$\Rightarrow \quad 5p + 2(1-p) = 3p + 6(1-p)$$

$$\Rightarrow \quad 3p + 2 = 6 - 3p$$

$$\Rightarrow \quad \underline{p = \frac{2}{3}}.$$

Similarly, for β^* to be in ES for B, we insist:

$$g_A(a_1, \beta^*) = g_A(a_2, \beta^*)$$

$$\Rightarrow \quad 2q + 4(1-q) = 5q + 1 - q$$

$$\Rightarrow \quad \underline{q = \frac{1}{2}}.$$

Thus the game has just one equilibrium, when the players play the strategies:

$$\underline{\left(\left(\tfrac{2}{3}, \tfrac{1}{3}\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right)\right)}, \text{ respectively.}$$

Seen Similar

$\boxed{C}$

3 Some Sensible method to find equilibria

extend to full game.

$\boxed{C}$

1

Seen Similar

Q1: Total: 20

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Question 2

Topic ZERO-SUM GAMES / CONGESTION GAMES

Marks& seen/unseen

Parts

(a).

(i). (a_1, b_1) is a pure strat. equilibrium if $x \geq 1$ and $x \leq 1$, i.e. $x = 1$.

(a_1, b_2) is a pure strat. equilibrium if $x \leq 1$ and $x \geq 3$, which is impossible.

(a_2, b_1) is a pure strat. equilibrium if $1 \leq x \leq 3$.

(a_2, b_2) is a pure strat. equilibrium if $x \geq 3$ and $x \leq 3$, i.e. $x = 3$.

$\Rightarrow$ The game has a pure strategy equilibrium $\Leftrightarrow$ $\underline{\underline{1 \leq x \leq 3}}$.

A 3 Seen Similar

(ii). If V is the value of the game, then there must exist at least one payoff larger than or equal to $V = \frac{11}{3} > 3$. Hence $x > 3$ for this to be the case.

B 1 unseen

(iii). Since $x > 3$ there is no pure strategy equilibrium by (i). Since this is a finite game we know by Nash's theorem there exists an equilibrium in mixed strategies. We seek a pair of equaliser strategies, (α^*, β^*), for the players. Denoting $\alpha^* = (p, 1-p) = \beta^*$ (from the symmetric payoff structure)

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Parts (a).(iii). (continued.)

Then:

$$g(a_1, \beta^*) = g(a_2, \beta^*)$$

$$\Rightarrow p + x(1-p) = px + 3(1-p)$$

$$\Rightarrow \cancel{} (1-x)p + x = (x-3)p + 3$$

$$\Rightarrow p = \frac{(x-3)}{2(x-2)}$$

(since $x > 3$ we have $0 < p < 1$ as required).

Some method to find equilibrium. $\boxed{B}$ 3 seen similar

Thus:

$$\alpha^* = \beta^* = \left(\frac{(x-3)}{2(x-2)}, \frac{(x-1)}{2(x-2)} \right)$$ are a

pair of ES and hence form an equilibrium of the game. Therefore, for example, $V = g(\alpha^*, b_1) = \frac{11}{3}$.

$$\Rightarrow V = 1 \cdot \frac{(x-3)}{2(x-2)} + x \cdot \left(\frac{(x-1)}{2(x-2)} \right) = \frac{11}{3}$$

$$\Leftrightarrow (x-3) + x^2 - x = \frac{11}{3} \cdot 2(x-2)$$

$$\Leftrightarrow 3x^2 - 22x + 35 = 0$$

$$\Leftrightarrow (3x-7)(x-5) = 0$$

$$\Leftrightarrow x = \frac{7}{3} \text{ or } x = 5, \text{ but } \frac{7}{3} < 3, \text{ so}$$

the only value of x giving $V = \frac{11}{3}$ is $x = 5$.

$\boxed{C}$ 2 unseen

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Question 2	Topic	Marks& seen/unseen
Parts (b).		

(i). In a congestion game with N users, the strategies $P_1, \dots, P_N$ (paths through the network from origin to destination) of all N users define an equilibrium if each strategy is a best response to the other strategies, i.e., that, for each user i:

$$\text{Cost}(P_i) \leqslant \text{Cost}(Q_i),$$

for all possible different strategies Q_i of user i.

3 seen definition

(ii). Suppose that x users take the route AB, y users $(y \leqslant x)$ take the route BC along the top path, $x-y$ users take BC through the middle and $10-x$ users take AC along the bottom.

Then:

$$C_{top} = x + y$$

$$C_{mid} = x + 2$$

$$C_{bot} = 20 - 2x$$

In equilibrium we must have $C_{top} = C_{mid}\ (\pm 1)$. This gives $y = 1, 2, 3$. Similarly, $C_{bot} = C_{mid}\ (\pm 1)$ giving $\underline{x = 6}$. We check each case to see which are in equilibrium, finding all equilibria to be:

seen similar

2

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Parts (b). (ii). (continued.)

- ~~3~~ users along the top route, cost = 7 each; 5 users along the middle route, cost = 8 each; 4 users along the bottom path, cost = 8 each.
- 2 users along the top route, cost = 8 each; 4 users along the middle route, cost = 8 each; 4 users along the bottom route, cost = 8 each.

B — 2 description of all equilibria — seen similar

(iii). Average cost per user

$$= \frac{1}{10}\left(x^2+y^2+2(x-y)+(10-x)(20-2x)\right)$$

$$= \frac{1}{10}\left(x^2+y^2+2x-2y+200-40x+2x^2\right)$$

$$= \frac{1}{10}\left(3x^2-38x+y^2-2y+200\right)$$

$$= \frac{1}{10}\left(3\left(x-\frac{19}{3}\right)^2-3\left(\frac{19}{3}\right)^2+(y-1)^2-1+200\right)$$

$$= \frac{3}{10}\left(x-\frac{19}{3}\right)^2+\frac{1}{10}(y-1)^2+\frac{1}{10}\left(199-\frac{19^2}{3}\right)$$

D — 2 — seen similar

This is minimised when $x=6$, $y=1$; so the social optimal flow is given by 1 user along the top, 5 users through the middle and 4 users along the bottom path.

D — 2 — seen similar

Q2 Total: 20

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Question 3

Topic COOPERATION / DEGENERACY

Marks& seen/unseen

Parts

(a).

(i). A's payoffs are:

$$\begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & \boxed{1} & 2 & 3 \\ a_2 & 3 & \boxed{1} & 2 \end{array}$$

There is no pure strategy equilibria. b_3 is dominated by b_2 for B in this game, so can be deleted. In the remaining 2×2 game we can seek an ES for A, finding that $\alpha^* = \left(\frac{2}{3}, \frac{1}{3}\right)$ is max-min for A, giving payoff $\frac{5}{3}$, so then: $\underline{\underline{t_A = \frac{5}{3}}}$.

B

3

seen Similar

B's payoffs are:

$$\begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & 2 & 2 & \boxed{2} \\ a_2 & \boxed{1} & \boxed{0} & 4 \end{array}$$

There is a pure strategy equilibrium at (a_1, b_3), so then b_3 is a max-min strategy for B giving payoff 2, so that: $\underline{\underline{t_B = 2}}$.

B

2

seen Similar

The threat point is thus $\left(\frac{5}{3}, 2\right)$.

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Course: Introduction to Game Theory

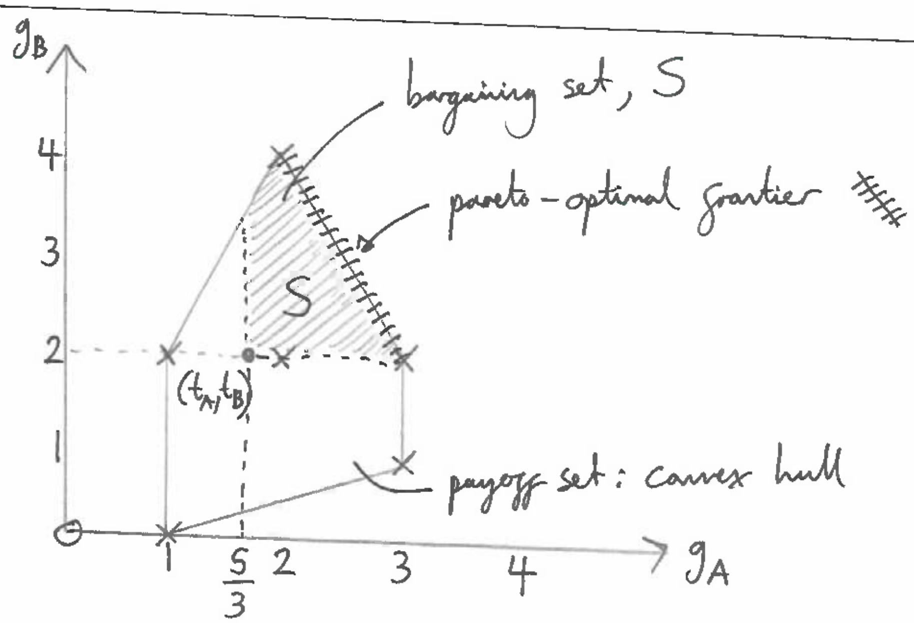
Question 3

Topic

Parts

Marks& seen/unseen

(ii).



$\boxed{A}$ 4 Seen Similar (1: pareto-f 2: S 1: convex hull)

(iii). We maximise the Nash product $(x-t_A)(y-t_B)$ over the pareto-optimal frontier; which has equation $y=-2x+8$:

$\boxed{A}$ 1 Nash-p

$$(x-\tfrac{5}{3})(y-2) = (x-\tfrac{5}{3})(6-2x)$$

$$= -2x^2+\tfrac{28}{3}x-10,$$

$\boxed{B}$ 1 line eq.

$\boxed{A}$ 1 quadratic.

which is maximised when : $\underline{x=\frac{7}{3},\ y=\frac{10}{3}}$.

$\boxed{C}$ 1 max point.

This is in the bargaining set S, hence it gives the Nash bargaining solution. To implement this one possibility is that B plays b_3 and A plays $(\frac{1}{3},\frac{2}{3})$. [or other joint strategies plausible].

$\boxed{C}$ 1 Seen Similar

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Question 3

Topic

Marks& seen/unseen

Parts

(b).

		B: b_1	b_2	b_3
A	a_1	$1, \boxed{2}$	$2, \boxed{2}$	$3, \boxed{2}$
	a_2	$3, 1$	$1, 0$	$2, \boxed{4}$

There are two pure strategy equilibria at (a_1, b_2) and (a_1, b_3).

Seen similar 1 $\boxed{D}$

if this isn't stated this is fine, this mark is awarded if solved correctly later.

Notice that the pure strategy a_1 for A has <u>three</u> pure strategy best responses (b_1, b_2 and b_3) for B. Hence the game is degenerate. If A were to assign <u>any</u> positive probability to a_2, then B plays ~~its~~ weakly dominant strategy b_3. Thus, in <u>any</u> equilibrium of the game, A must play a_1.

1 $\boxed{D}$

A plays a_1 reason.

Seen similar

unseen: B having 3 pure BR.

Denote $\beta = (q_1, q_2, 1-q_1-q_2)$ for B's strategy. (a_1, β) forms an equilibrium of the game so long as a_1 remains a best response for A against β.

This means we need to insist:

$$g_A(a_1, \beta) \geq g_A(a_2, \beta)$$

Seen similar 2 $\boxed{D}$

~~[illegible]~~ a_1 BR condition.

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Question 3

Topic

Parts (b). (continued.)

Marks& seen/unseen

$\Rightarrow \; q_1 + 2q_2 + 3(1-q_1-q_2) \geqslant 3q_1 + q_2 + 2(1-q_1-q_2)$

$\Rightarrow \; 3 - 2q_1 - q_2 \geqslant 2 + q_1 - q_2$

$\Rightarrow \; 1 \geqslant 3q_1$, or: $\underline{\underline{0 \leqslant q_1 \leqslant \frac{1}{3}}}$ (1), since $q_1 \geqslant 0$ to have a valid strategy β.

1 D Seen similar

Moreover, for β to be a valid strategy for B, we need $q_1 + q_2 \leqslant 1$, i.e.

$\underline{\underline{0 \leqslant q_2 \leqslant 1 - q_1}}$ (2), since $q_2 \geqslant 0$ to have a valid β.

1 D Seen similar

Hence all equilibria of the game are of form:

$(a_1, (q_1, q_2, 1-q_1-q_2))$,

with conditions (1) and (2) on q_1 and q_2.

Q3: Total: 20

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Question 4

Topic IMPARTIAL GAMES

Parts

Marks& seen/unseen

(a). If $G \equiv *m$ for an impartial game G, then we call m the Nim value of G. (here $*m$ represents a single Nim pile of size m).

2 A seen definition

(b). We use top-down induction. Assume that the proposition holds for all games simpler than G (refering to G as the impartial game in question).

1 A seen proof

Thus, the claim holds for all options of G, and hence if these are all winning, then G is losing because no matter which move is made in G we arrive at a winning option where a winning move can be made by the other player.

1 A seen proof

If not all options of G are winning, then one of them, let's say H, is losing, and since the claim holds true for H then by moving to H the player forces a win, hence G is winning.

1 A seen proof

To complete the inductive proof requires a base case, which we take as the game with no options; the simplest game.

1 A

□

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Question 4 | Topic

Parts | Marks & seen/unseen

unseen game

(c). (i). A split-Nim pile of size 1 cannot be split, so it is the same as an ordinary Nim pile of size 1 with Nim value $= \underline{\underline{1}}$.

1 A Seen similar arguments

A split-Nim pile of size 2 can be reduced to a split-Nim pile of size 1, or 0, or can be split into two split-Nim piles of size 1, which has Nim value $1 \oplus 1 = 0$, so is losing. Thus the Nim value of the split-Nim pile of size 2 is $\text{mex}(0,1,0) = \underline{\underline{2}}$.

1 A Seen similar arguments with mex rule

A split-Nim pile of size 3 has the additional option of being split into split-Nim piles of sizes 1 and 2. These have Nim values 1 and 2 (above), so this option has Nim value $1 \oplus 2 = 3$. Therefore the split-Nim pile of size 3 has Nim value $\text{mex}(0,1,2,3) = \underline{\underline{4}}$.

1 B Seen similar arguments with mex rule

A split-Nim pile of size 4 can be reduced to single split-Nim piles of sizes 0, 1, 2 or 3 which have Nim-values 0, 1, 2 and 4 or has two other options: being split into two split-Nim piles of sizes 1 and 3, with Nim value $1 \oplus 4 = 5$, or being split into two split-Nim piles of sizes

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Parts (c). (i). (continued.)	2 and 2, with Nim value $2 \oplus 2 = 0$. Thus the Nim value of the split-Nim pile of size 4 is $\text{mex}(0,1,2,4,5,0) = \underline{\underline{3}}$.	B 2 seen sim. arguments w. mex rule.
(ii).	The Nim value of the split-Nim position with three piles of sizes 1, 2 and 3 is $1 \oplus 2 \oplus 4 = 7$, so this is a <u>winning</u> position. values from (i)	C 1 seen-sim. calculation.
	A winning move must reduce the Nim value to 0, so must be made in the pile of size 4. We need to obtain a Nim-value of 3 from our move in the pile of size 3 (so that then the option we move to has Nim value $1 \oplus 2 \oplus 3 = 0$ and is thus losing). This is achieved in only one way, by splitting the pile into two piles of sizes 1 and 2 respectively (Nim-value 3 from part (i).).	C 1 move in pile size 4; D 2 unseen game seen sim. arguments.
	The Nim-value of the split-Nim position with three piles of sizes 1, 2 and 4 is $1 \oplus 2 \oplus 3 = 0$, so this is a <u>losing</u> position and no winning moves exist.	C 1 Seen sim. calculation.
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Question 4

Topic

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Parts (c). (iii).

The statement is true.

We prove this by contradiction: assume that there were two split-Nim piles that were equivalent, but have different sizes.

Since the two piles are equivalent, their game sum will have Nim-value 0 and thus be losing.

However, in such a game, the larger of the two split-Nim piles could be reduced to the same size of the smaller pile, creating a game sum of two identical games (the two equal split-Nim piles) which, by the copycat principle for impartial games, is always losing; ~~this~~ Thus, this pile equalising move, would constitute a winning move in the original game, we have a contradiction and ~~the~~ so our original assumption must be false, proving the desired conjecture. □

unseen

D

1

Sensible method/ attempt of proof.

unseen

D

3

proof

unseen

Q4: Total: 20

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EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic: MASTERY:

Parts

Marks & seen/unseen

(a). We assign payoff 0 to each player for a drawn game, +1 for a won game and −1 for a lost game. Hence all games are zero-sum.

For $n=3$:

We denote a_i = player A chooses i. Similarly for player B and b_j.

A \ B	b_1	b_2	b_3
a_1	0	1	−1
a_2	1	0	1
a_3	−1	1	0

unseen game.

seen similar method.

2 sensible payoffs + strategic form.

No pure strategy equilibria. We look for a pair of equaliser strategies. Owing to the symmetric structure of the game we seek a pair of ES of the form $\alpha^* = \beta^* = (p, 1-2p, p)$. If these are ES then we must have:

$$g(\alpha^*, b_1) = g(\alpha^*, b_2), \text{ or:}$$

$$1-3p = 2p$$

$$\Rightarrow p = \frac{1}{5}.$$

Thus: (α^*, β^*), where $\alpha^* = \beta^* = \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right)$, give a pair of max-min, min-max strategies; a solution of the game.

$\boxed{\text{Value} = \frac{2}{5}}$

Seen similar method.

3 an equilibrium of game

Seen similar

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Parts

(b). $\underline{n=4:}$

	b_1	b_2	b_3	b_4
a_1	0	1	−1	−1
a_2	1	0	1	−1
a_3	−1	1	0	1
a_4	−1	−1	1	0

unseen

We are given that $\alpha^* = (0, \frac{1}{2}, \frac{1}{2}, 0)$ is max-min for player A. We find that:

$$g(\alpha^*, b_1) = g(\alpha^*, b_4) = 0$$
$$g(\alpha^*, b_2) = g(\alpha^*, b_3) = \frac{1}{2}$$

Therefore if β^* is min-max for player B there cannot be any positive probability assigned to b_2 or b_3 (since playing b_1 or b_4 does better). Thus we seek a min-max strategy β^* for B of form:

$\beta^* = (q, 0, 0, 1-q)$.

2 reasonable argument as to form of β^*. unseen

Insisting that $g(a_2, \beta^*) = g(a_3, \beta^*)$ so player A is indifferent over a_2 and a_3 and can hence mix between them gives: $q - (1-q) = -q + 1 - q$

$$\Rightarrow \underline{\underline{q = \frac{1}{2}}}.$$

2 a min-max for B unseen

Hence $\underline{\beta^* = (\frac{1}{2}, 0, 0, \frac{1}{2})}$ is min-max for B. $\boxed{\text{Value} = 0}$

1 value.

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Question 5

Topic

Marks& seen/unseen

Parts

(c). $n = 5$:

	b_1	b_2	b_3	b_4	b_5
a_1	0	1	-1	-1	-1
a_2	1	0	1	-1	-1
a_3	-1	1	0	1	-1
a_4	-1	-1	1	0	1
a_5	-1	-1	-1	1	0

unseen

We are given $\alpha^* = (0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, 0)$ is max-min for player A. We find that:

$$g(\alpha^*, b_1) = g(\alpha^*, b_2) = g(\alpha^*, b_4) = g(\alpha^*, b_5) = \frac{-1}{5}.$$

$$g(\alpha^*, b_3) = \frac{4}{5}.$$

Thus if β^* is min-max for B then b_3 is played with 0 probability. We seek β^* of form:

$\beta^* = (p, q, 0, r, 1-p-q-r)$. If β^* is min-max for B it must be the case that:

2

Sensible argument on form of β^*

$$g(a_2, \beta^*) = g(a_3, \beta^*) = g(a_4, \beta^*)$$

unseen

$$\Rightarrow p - r - (1-p-q-r) = -p + q + r - (1-p-q-r)$$

$$= -p - q + 1 - p - q - r$$

$$\Rightarrow 2p + q - 1 = 2q + 2r - 1 = -2p - 2q - r + 1$$

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Question 5	Topic	Marks& seen/unseen

Parts

(c). (continued.)

The first equality gives: $2p = q + 2r$ ①

Substituting this into the second equality gives:

$$2p + q - 1 = -2p - 2q - (2p - q)\frac{1}{2} + 1$$

$$\Rightarrow 2p + q - 1 = -3p - \frac{3}{2}q + 1$$

$$\Rightarrow 5p = 2 - \frac{5}{2}q$$

$$\Rightarrow q = \frac{4}{5} - 2p.$$

And substituting back into ①: $r = p - \frac{1}{2}q$

$$\Rightarrow r = 2p - \frac{2}{5}.$$

Then:

$$1 - p - q - r = 1 - p - \left(\frac{4}{5} - 2p\right) - \left(2p - \frac{2}{5}\right)$$

$= \frac{3}{5} - p$, so we find if β^* is min-max for B it must be of form:

$$\beta^* = \left(p, \frac{4}{5} - 2p, 0, 2p - \frac{2}{5}, \frac{3}{5} - p\right),$$

3 form of β^* determined. unseen

but what range of values for p are viable?

Well if α^* is max-min for A, clearly A cannot find a_1 or a_5 desirable against β^* (since they have zero probability in α^*). Thus;

$$g(a_1, \beta^*) \leqslant -\frac{1}{5} \quad \text{and} \quad g(a_5, \beta^*) \leqslant -\frac{1}{5}.$$

unseen

Setter's initials SJB

Checker's initials

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Question 5 | Topic | Marks& seen/unseen

Parts

(c). (continued.)

These result in:

$$\frac{4}{5}-2p-\left(2p-\frac{2}{5}\right)-\left(\frac{3}{5}-p\right)\leqslant-\frac{1}{5}$$

$$\Rightarrow 3p\geqslant\frac{4}{5},\ \text{or}\ \ \underline{\underline{p\geqslant\frac{4}{15}}},\ \text{and:}$$

$$-p-\left(\frac{4}{5}-2p\right)+\left(2p-\frac{2}{5}\right)\leqslant-\frac{1}{5}$$

$$\Rightarrow 3p\leqslant 1,\ \text{or}\ \ \underline{\underline{p\leqslant\frac{1}{3}=\frac{5}{15}}}.$$

(which were the required bounds on p).

$$\boxed{\text{Value}=-\frac{1}{5}.}$$

unseen

2 range of p

unseen

1 value.

(d). A quick check shows:

we found:

$\underline{n=1:}$ $\underline{V=0}$

$\underline{n=2:}$ $\underline{V=\frac{1}{2}}$

$\underline{n=3:}$ $\underline{V=\frac{2}{5}}$

$\underline{n=4:}$ $\underline{V=0}$

$\underline{n=5:}$ $\underline{V=-\frac{1}{5}}$ or <0

So we conclude $\underline{n=1}$ and $\underline{n=4}$ are the only values for n for which this is a fair game. We 'loosely' justify why for $n\geqslant 6$ we expect value $\neq 0$: As the game grows only additional '-1' terms really extend the payoff matrix (it remains 0 on diagonal with $+1$ on the next diagonals), so growing this structure only stands to benefit B.

Intuitively: if n is huge, B could choose any value randomly and it is highly likely A chooses a value 'far' away from B's: so $V\to-1$ as $n\to\infty$

unseen

1 $n=1,4$.

unseen

1 sensible argument as to why no more.

Setter's initials SJB

Checker's initials