

1. Show that $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$.¹

2. Let $PL(a_n)$ be the set of all limits of convergent subsequences of (a_n) , i.e.,

$$PL(a_n) = \{ L \in \mathbb{R} \mid \text{there is some subsequence } (a_{n_k}) \text{ such that } a_{n_k} \rightarrow L \text{ as } k \rightarrow \infty \}.$$

Elements of $PL(a_n)$ are also called partial limits of (a_n) .

(a) For each one of the following items, give an example, without proof, of a sequence (a_n) such that $PL(a_n) = S$.

i. $S = \{1, \dots, m\}$.

ii. $S = \mathbb{N}$.

(b) Is there a sequence (a_n) such that $PL(a_n) = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$? You are not required to justify your answer, just come up with an answer – yes or no. You will prove the correct answer in a further question.

3. Let (a_n) be a sequence, $L \in \mathbb{R}$. Prove that $L \in PL(a_n)$ if and only if for every $\epsilon > 0$, the set $\{ n \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon \}$ is infinite.

4. Prove that if (a_n) is a sequence and there is a sequence L_n of partial limits of $PL(a_n)$ such that $L_n \rightarrow L$, then L is also a partial limit of (a_n) .

5. In this question we give yet another definition of \limsup :

Let (a_n) be a sequence. Show that

$$\lim_{m \rightarrow \infty} \left(\sup_{n \geq m} a_n \right) = \sup(PL(a_n))$$

in the sense that if one exists, so does the other and they are equal.

¹Hint: Problem Sheet 5, Question 1