Imperial College London

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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 2

1. Newton's second law tells us that the displacement $x(t)$ of the mass in a mass-spring-dashpot system satisfies the following ODE

$$m\frac{d^2x}{dt^2} = F_s + F_d$$

where m is the mass, F_s is the restoring force from the spring and F_d is the damping force from the dashpot. We can complete this equation with the following initial conditions

$$x(0) = 0, \quad \frac{dx}{dt}(0) = v_0$$

(a) We consider the case where there is no damping, $F_d = 0$, and where the spring is linear, $F_s = -kx$. We know that the dimensions of F_s are those of a force, i.e. $[F_s] = MLT^{-2}$, so we find that $[k] = MT^{-2}$. We will now nondimensionalize this initial value problem. We write

$$t = t_c\tau \quad \text{and} \quad x = x_c\xi$$

Using the chain rule, we notice that

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{t_c}\frac{d}{d\tau} \Rightarrow \frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{t_c^2}\frac{d^2}{d\tau^2}$$

With this change of variables, the equation for the mass-spring system becomes

$$\frac{m}{t_c^2}\frac{d^2}{d\tau^2}(x_c\xi) = -kx_c\xi$$

Given, the dimensions of k, we can write this equation in nondimensional form as

$$\Pi_1\frac{d^2\xi}{d\tau^2} = -\xi$$

where we have introduced the dimensionless group $\Pi_1 = \frac{m}{kt_c^2}$. The initial conditions are given by

$$\xi(0) = 0 \quad \text{and} \quad \frac{d\xi}{d\tau}(0) = \Pi_2$$

where Π_2 is the following dimensionless group

$$\Pi_2 = \frac{v_0t_c}{x_c}$$

We have two characteristic quantities to determine and two dimensionless groups so we set:

$$\Pi_1 = 1 \Rightarrow t_c = \sqrt{\frac{m}{k}}$$

$$\Pi_2 = 1 \Rightarrow x_c = v_0t_c \Rightarrow x_c = v_0\sqrt{\frac{m}{k}}$$

Following this choice of characteristic quantities, the equation reads in dimensionless form

$$\frac{d^2\xi}{d\tau^2} = -\xi \quad \text{with} \quad \xi(0) = 0 \quad \text{and} \quad \frac{d\xi}{d\tau}(0) = 1$$

(b) Now, in addition to a linear spring, we suppose that we also have linear damping, i.e.

$$F_d = -c\frac{dx}{dt}$$

We know that the dimensions of F_d are those of a force, i.e. $[F_d] = MLT^{-2}$, so we find that $[c] = MT^{-1}$. The differential equation now reads

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

Using the same scaling (i.e. keeping the definitions of x_c and t_c), we can write

$$\frac{m}{t_c^2}\frac{d^2}{d\tau^2}(x_c\xi) = -kx_c\xi - \frac{c}{t_c}\frac{d}{d\tau}(x_c\xi) \Rightarrow \frac{m}{kt_c^2}\frac{d^2\xi}{d\tau} = -\xi - \frac{c}{kt_c}\frac{d\xi}{d\tau}$$

By definition of t_c, we thus find that

$$\frac{d^2\xi}{d\tau} = -\xi - \varepsilon\frac{d\xi}{d\tau}$$

where we have defined ε as the following nondimensional parameter:

$$\varepsilon = \frac{c}{\sqrt{km}}$$

We can confirm that the weak damping limit corresponds to small $c \ll 1$ which implies small parameter value $\varepsilon \ll 1$.

2. In this problem, we consider the problem of capillary rise of water in a water strip.

(a) We know that J is a density flux. Knowing that the dimension of ρ are given by $[\rho] = ML^{-3}$, the following conservation equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

can only be dimensionally homogeneous if $[J] = ML^{-2}T^{-1}$.

(b) The flux J depends on the gravitational constant g, the strip width d, the density gradient $\frac{\partial\rho}{\partial x}$, and the surface tension σ of the water. Mathematically, we write that

$$J = f\left(g, d, \sigma, \frac{\partial\rho}{\partial x}\right)$$

We want to find the numbers (a, b, c, d) such that

$$[J] = \left[g^a d^b \sigma^c \left(\frac{\partial\rho}{\partial x}\right)^d\right]$$

These relevant quantities have the following dimensions

$$[J] = ML^{-2}T^{-1}, \quad [g] = LT^{-2}, \quad [d] = L, \quad \left[\frac{\partial\rho}{\partial x}\right] = ML^{-4}, [\sigma] = MT^{-2} \tag{1}$$

We can then write

$$ML^{-2}T^{-1} = L^aT^{-2a}L^bM^cT^{-2c}M^dL^{-4d}$$

or equivalently, the following system

$$\begin{cases} L: & a + b - 4d = -2 \\ T: & -2a - 2c = -1 \\ M: & c + d = 1 \end{cases}$$

Keeping d the undetermined variable, a solution to this system is given by

$$a = d - 1/2, \quad b = -3/2 + 3d, \quad c = 1 - d$$

We finally find that a dimensionally reduced from for J is given by

$$J = \frac{\sigma}{\sqrt{gd^3}} F(\Pi)$$

with

$$\Pi = \frac{gd^3}{\sigma} \frac{\partial \rho}{\partial x}$$

You should check that Π is dimensionless.

(c) Now if we want the flux function to depend linearly on the density gradient in such a way that $J = 0$ when $\partial\rho/\partial x = 0$, then we require that $F(\Pi) = \alpha\Pi$ where α is an arbitrary number (dimensionless). In this case, the PDE reduces to

$$\frac{\partial \rho}{\partial t} + \alpha\sqrt{gd^3} \frac{\partial^2 \rho}{\partial x^2} = 0$$

(d) Finally, we try to non-dimensionalize our problem. Let's define the following scaling variables:

$$\rho = \rho_c r, \quad x = x_c \xi, \quad t = t_c \tau$$

We know that then

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{1}{t_c} \frac{\partial}{\partial \tau} \\ \frac{\partial^2}{\partial x^2} &= \frac{1}{x_c^2} \frac{\partial^2}{\partial \xi^2} \end{aligned}$$

Let's then nondimensionalize the equation

$$\frac{\partial \rho}{\partial t} + \alpha\sqrt{gd^3} \frac{\partial^2 \rho}{\partial x^2} = 0 \Rightarrow \frac{1}{t_c} \frac{\partial r}{\partial \tau} + \alpha \frac{\sqrt{gd^3}}{x_c^2} \frac{\partial^2 r}{\partial \xi^2} = 0 \Rightarrow \frac{\partial r}{\partial \tau} + \alpha \frac{\sqrt{gd^3} t_c}{x_c^2} \frac{\partial^2 r}{\partial \xi^2} = 0$$

One can easily check that the quantity in front of the second order space derivative is dimensionless. Proceeding similarly for the initial conditions and boundary conditions, we write that the problem in dimensionless form reads

$$\begin{aligned} &\frac{\partial r}{\partial \tau} + \Pi_1 \frac{\partial^2 r}{\partial \xi^2} = 0 \\ &r = \Pi_2 \quad \text{for} \quad \xi = 0 \\ &\frac{\partial r}{\partial \xi} = 0 \quad \text{for} \quad \xi = \Pi_3 \end{aligned}$$

We have three characteristic values to determine and three dimensionless groups, so we set

$$\begin{aligned} &\Pi_3 = 1 \Rightarrow x_c = h \\ &\Pi_2 = 1 \Rightarrow \rho_c = \rho_0 \\ &\Pi_1 = 1 \Rightarrow \alpha\sqrt{gd^3} \frac{t_c}{x_c^2} = 1 \Rightarrow t_c = \frac{h^2}{\alpha\sqrt{gd^3}} \end{aligned}$$

Using this scaling, we obtain that the dimensionless problem reads:

$$\frac{\partial r}{\partial \tau} + \frac{\partial^2 r}{\partial \xi^2} = 0$$
$$r(0,\tau) = 1$$
$$\frac{\partial r}{\partial \xi}(1,\tau) = 0$$
$$r(\xi,0) = 0$$

3. Let's solve the following linear transport equation with variable coefficients

$$\frac{\partial u}{\partial t} + e^{x+t}\frac{\partial u}{\partial x} = 0$$

The characteristic equation for this PDE is given by

$$\frac{dx}{dt} = e^{x+t}$$

This ODE separates into

$$e^{-x}dx = e^{t}dt$$

Its solutions are thus given by

$$e^{-x} = -e^{t} + \xi \Rightarrow x = -\log(\xi - e^{t})$$

where ξ is an arbitrary constant to be determined. The general solution of this PDE is thus

$$u(x,t) = f(\xi) = f(e^{-x} + e^{t})$$

where f is an arbitrary differentiable function of one variable. If we consider the case where we set the initial conditions to be given by

$$u(x,0) = \phi(x)$$

Then, using the preceding result, we must have that $\phi(x) = u(x,0) = f(e^{x}+1)$. Let us find f in terms of ϕ. For that, we substitute $s = e^{x}+1$ or equivalently $x = -\log(s-1)$ and obtain $f(s) = \phi(-\log(s-1))$. So we conclude that the solution is given by

$$u(x,t) = \phi\left(-\log(e^{-x} + e^{t} - 1)\right)$$

In particular, if $\phi(x) = x^3$, we find that

$$u(x,t) = -\left[\log(e^{-x} + e^{t} - 1)\right]^3$$

4. Using the method of characteristics, we solve the following partial differential equations

   (a) We consider the following problem

$$x\frac{\partial u}{\partial t} + t^2\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = x^2, \text{ when } t = 0$$

   We first rewrite the PDE

$$\frac{\partial u}{\partial t} + \frac{t^2}{x}\frac{\partial u}{\partial x} = 0$$

   We know that u is constant along the curve satisfying the following characteristic equation

$$\frac{dx}{dt} = \frac{t^2}{x}$$

which we can solve easily as it can separated as

$$xdx = t^2 dt \Rightarrow x^2/2 - t^3/3 = C$$

where C is a constant. So the general solution of this PDE is given by

$$u(x, t) = f(x^2/2 - t^3/3)$$

where f is an arbitrary differentiable function of one variable. But we know that $u = x^2$ at $t = 0$, so we conclude that $x^2 = f(x^2/2)$. Now let $s = x^2/2$, we have $2s = f(s)$. We can thus conclude that

$$u(x, t) = 2\left(\frac{x^2}{2} - \frac{t^3}{3}\right) = x^2 - \frac{2}{3}t^3$$

(b) We consider the following problem

$$(1 + t)\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = x^5, \text{ when } t = 0$$

We first rewrite the PDE

$$\frac{\partial u}{\partial t} + \frac{x}{t+1}\frac{\partial u}{\partial x} = 0$$

which is allowed since $t > 0$. We know that u is constant along the curve satisfying the following characteristic equation

$$\frac{dx}{dt} = \frac{x}{t+1} \quad \text{with} \quad x(0) = \xi$$

which we can solve easily as it can be separated as

$$\frac{dx}{x} = \frac{dt}{t+1} \Rightarrow \log x = \log(t + 1) + C$$

where C is an integration constant to be determined. We use the fact that $x(0) = \xi$ to finally write $x = \xi(t + 1)$. We know that along these characteristics, the solution $u(x, t)$ is constant and so we find that

$$u(x, t) = \left[\frac{x}{t+1}\right]^5$$

(c) We consider the following problem

$$\cos x\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = t^4, \text{ when } x = 0$$

We first rewrite the PDE

$$\frac{\partial u}{\partial t} + \frac{t}{\cos(x)}\frac{\partial u}{\partial x} = 0$$

We know that u is constant along the curve satisfying the following characteristic equation

$$\frac{dx}{dt} = \frac{t}{\cos(x)}$$

which we can solve easily as it can be separated as

$$\cos(x)dx = tdt \Rightarrow \sin x - \frac{t^2}{2} = C$$

where C is a constant. So the general solution of this PDE is given by

$$u(x, t) = f(\sin(x) - t^2/2)$$

where f is an arbitrary differentiable function of one variable. But we know that $u = t^4$ at $x = 0$, so we conclude that $t^4 = f(-t^2/2)$. Now let $s = -t^2/2$, we have $4s^2 = f(s)$. We can thus conclude that

$$u(x, t) = 4\left(\sin x - \frac{t^2}{2}\right)^2$$

5. We consider the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - x^2 \frac{\partial u}{\partial x} = 0, \quad x > 0,\ t > 0, \\
&u(x,0) = e^{-x}, \quad x > 0 \\
&u(0,t) = 1, \quad t > 0
\end{aligned}$$

The characteristic equations are given by

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -x^2$$

Integrating these equations and using the initial conditions, we find

$$u = \exp(-\xi) \quad \text{on} \quad \frac{1}{x} = t + \frac{1}{\xi} \Rightarrow \xi(x,t) = \frac{x}{1-xt}$$

The solution is then given by

$$u(x,t) = \exp\left(-\frac{x}{1-xt}\right)$$

If you compare this example with the one we have seen in lecture, you will see that condition at $x = 0$ does not propagate into the $x > 0$ region, as the line $x = 0$ is the characteristic corresponding to $\xi = 0$.

6. We consider the PDE

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x^2 + y^2$$

and the following boundary conditions

$$u(x,y) = \begin{cases} 1 + x^2 & \text{on } y = 0 \\ 1 + y^2 & \text{on } x = 0 \end{cases}$$

We have here an equation of the kind

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)$$

with

$$a(x,y) = y, \quad b(x,y) = x, \quad c(x,y) = x^2 + y^2$$

In particular, we can write the following characteristic equations:

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow x^2 - y^2 = c_1$$

where c_1 is a constant. We also have

$$\begin{aligned}
\frac{du}{dx} = \frac{x^2+y^2}{y} &\Rightarrow du = \left(\frac{x^2}{y} + y\right) dx \\
&\Rightarrow du = ydx + x\left(\frac{x}{y}dx\right) \\
&\Rightarrow du = ydx + xdy \text{ (on the characteristics)} \\
&\Rightarrow du = d(xy) \\
&\Rightarrow u(x,t) = xy + c_2
\end{aligned}$$

where c_2 is a constant to be determined. We can conclude from this that the general solution of this PDE is given by

$$u(x, y) = xy + f(x^2 - y^2)$$

where f is an arbitrary differential function to be determined. To do so, let's have a look at the boundary conditions:

$$\begin{cases} 1 + x^2 = f(x^2) \Rightarrow f(t) = 1 + t & \text{if } t \geq 0 \\ 1 + y^2 = f(-y^2) \Rightarrow f(t) = 1 - t & \text{if } t \leq 0 \end{cases}$$

We can finally write that

$$u(x, y) = xy + 1 + \left|x^2 - y^2\right|$$

7. In this problem, we consider one more time the example of a pollutant transported along a thin and long water channel. The water stream moves with speed v. We consider that diffusion is negligible in this problem. If you suppose that due to biological decomposition, the pollutant is decays at a rate proportional to the pollutant density.

   (a) We are told that the pollutant decays at a rate proportional to the pollutant density. Let us call γ the proportionality constant, we know that the rate of decay reads

   $$r(x, t) = -\gamma u(x, t)$$

   Such that, the density of pollutant in the channel satisfies the following equation

   $$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -\gamma u(x, t)$$

   (b) Assume that the initial conditions are given by $u(x, 0) = f(x)$, let us try to find a solution to this problem! If we set

   $$c(x, t) = u(x, t)e^{\frac{\gamma}{v}x},$$

   we have the following partial derivative

   $$\frac{\partial c}{\partial x} = \left(\frac{\partial u}{\partial x} + \frac{\gamma}{v}u\right)e^{\frac{\gamma}{v}x}$$

   and

   $$\frac{\partial c}{\partial t} = \frac{\partial u}{\partial t}e^{\frac{\gamma}{v}x}$$

   Combining these results and using the original PDE, we find that

   $$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

   and the initial condition transforms to $c(x, t) = f(x)e^{\frac{\gamma}{v}x}$. From the lecture notes, we know that the solution to this easier problem is given by

   $$c(x, t) = f(x - vt)e^{\frac{\gamma}{v}(x - vt)}$$

   so by definition of $c(x, t)$, we finally find

   $$u(x, t) = f(x - vt)e^{-\gamma t}$$

   The solution is a travelling wave (i.e. the term $f(x - vt)$) but its amplitude is decaying with time due to the $e^{-\gamma t}$ term.

8. In this problem, we will derive partial differential equations modelling the flow of a fluid in 1D. Consider a one-dimensional flow in a pipe of cross-sectional area A (constant), and a fixed slice of fluid in between $x = a$ and $x = b$ (where $b > a$). We denote the fluid velocity $u(x, t)$ and its density $\rho(x, t)$. The quantity $\rho(x, t)u(x, t)$ is called the momentum density.

(a) The dimensions of the momentum density are given by

$$[\rho u] = ML^{-2}T^{-1}$$

(b) The momentum flux into the fluid slice on its left end is given by $\rho(a,t)u(a,t)Au(a,t)$; similarly, we can write that the momentum flux into the fluid slide on its right end is given by $-\rho(b,t)u(b,t)Au(b,t)$. The net gain in momentum of the slice is thus given by

$$\left[\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right] A$$

(c) What are the forces at play in the fluid slice? We assume that the effect of viscosity are negligible (otherwise, one would have to include viscous forces here!). We assume that the only external forces acting on the slice are pressure forces at the ends. The total amount of momentum in the slice is given by

$$\int_a^b \rho(x,t)u(x,t)Adx$$

with dimensions

$$\left[\int_a^b \rho(x,t)u(x,t)Adx\right] = MLT^{-1}$$

The rate of change of the momentum has the dimensions of a force (you can check this by hand or see Newton's second law for instance). Now the pressure force on the left end of the slice is given by $-p(a,t)A\mathbf{n}$, where $\mathbf{n}$ is the outward normal vector to the cross-section of the pipe, i.e. $\mathbf{n} = -\hat{\boldsymbol{x}}$ and the pressure is given by $p(a,t)A$. Similarly, one can write that the pressure force applied on the right end of the slice is given by $-p(b,t)A$.

The conservation principle for conservation of momentum thus states

$$\frac{d}{dt}\int_a^b \rho(x,t)u(x,t)Adx = \left[\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right] A + (p(a,t) - p(b,t))A$$

(d) As A is constant, we can simplify this equation and use the fundamental theorem of calculus to write

$$\int_a^b \left(\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x}\right) = 0$$

As the interval is arbitrary, the integrand itself must be zero and we obtain

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x} = 0$$

giving us the expected result.

(e) Proceeding as in the lectures, we can write that the conservation of mass takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

This is also called the continuity equation. By expanding the derivatives in the conservation of momentum equation, we write

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x} = 0 \Rightarrow \frac{\partial \rho}{\partial t}u + \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho u)}{\partial x} = -\frac{\partial p}{\partial x}$$

So using the continuity equation, we can simplify the expression above to obtain the Euler equations for 1D fluid flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$
$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

which finally gives us

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}$$

This is the celebrated Euler equation for a 1D flow. The Euler equation can also be obtained for more general 3D flows and then reads:

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p$$

As we saw above, we neglected the effects of viscosity in the fluid. In our derivation, we also implictly assumed that the fluid was incompressible. Thus, Euler's equation is a good model for what we call incompressible perfect fluids (i.e. fluids with no viscosity); a good example is superfluid helium, indeed Helium 4 has the property to see its viscosity vanish when its temperature is brought down below $2.17°\mathrm{K}$.

(f) Now suppose that mass is being created within the fluid such that, in the absence of fluid motion, the change in mass of a slice of fluid δx is

$$\delta m=r(x,t)A\delta x\delta t$$

in a time interval δt. Can we explain how the conservation laws for mass and momentum used above need to be modified to account for this mass creation.

The rate of change of mass in an arbitrary slice of fluid $[a,b]$ is equal to: the mass flux entering in $x=a$, minus the mass flux leaving in $x=b$, plus the rate of change of any extra mass generated inside the slice! So far, we have assumed that there was no source of mass in the lectures. Mathematically, we can write this as

$$\frac{d}{dt}\int_a^b\rho(x,t)Adx=-[\rho uA]_a^b+\int_a^b r(x,t)Adx$$

where $r(x,t)$ is the rate of creation of mass at position x and time t per unit volume. By the usual arguments, we can conclude this conservation equation in differential form takes the following form

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=r$$

In this new continuity equation, the right hand side is non-zero due to the mass creation. Following what we have derived above, we shall write that

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial p}{\partial x}=0$$

which we can simplify using our new continuity equation to obtain

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}-ru$$

which is the modified form of the Euler momentum equation.