Answers to Problem Sheet 8

1. Take separation of particles to be a.

   Take masses to be m_1 and m_2. Coordinates: Use x, y, z as coordinates of centre of mass so that
$$T_{\text{trans}} = \frac{1}{2}(m_1+m_2)(\dot{x}^2+\dot{y}^2+\dot{z}^2).$$
   To describe the orientation of pair use spherical polar angles θ, ϕ.

   Distance of particle 1 from centre of mass $= m_2a/(m_1+m_2)$. For the second mass the distance is $m_1a/(m_1+m_2)$. Therefore the rotational kinetic energy is
$$T_{\text{rot}} = \frac{1}{2}m_1\left(\frac{m_2a}{m_1+m_2}\right)^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)+\frac{1}{2}m_2\left(\frac{m_1a}{m_1+m_2}\right)^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)$$
$$= \frac{1}{2}\frac{m_1m_2a^2}{m_1+m_2}\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right).$$
   Following the notes the potential energy can be approximated with $(m_1+m_2)\Phi(x,y,z)$. The complete Lagrangian is therefore
$$L = \frac{M}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{1}{2}\mu a^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-M\Phi(x,y,z),$$
   where $M = m_1+m_2$ is the total mass, and
$$\mu = \frac{m_1m_2}{m_1+m_2},$$
   is the *reduced mass*.

2.
$$L = \frac{1}{2}I\dot{\phi}^2+\frac{1}{2}ml^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)+mgl\cos\theta$$
   (i) $p_\theta = ml^2\dot{\theta}$, $p_\phi = (I+ml^2\sin^2\theta)\dot{\phi}$
$$H = p_\theta\dot{\theta}+p_\phi\dot{\phi}-L = \frac{p_\theta^2}{2ml^2}+\frac{p_\phi^2}{2(I+ml^2\sin^2\theta)}-mgl\cos\theta.$$
   (ii) If $I \gg ml^2$ then
$$(I+ml^2\sin^2\theta)^{-1} = I^{-1}(1+ml^2I^{-1}\sin^2\theta)^{-1} = \frac{1}{I}-\frac{ml^2}{I^2}\sin^2\theta+....,$$

so that

$$H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2I} - \frac{p_\phi^2}{2I^2} ml^2 \sin^2\theta - mgl\cos\theta + ...,$$

This is constant as the Hamiltonian has no explicit time dependence. Subtracting the constant $p_\phi^2/(2I)$ also gives a constant of the motion. Hence,

$$\frac{p_\theta^2}{2ml^2} - \frac{p_\phi^2}{2I^2} ml^2 \sin^2\theta - mgl\cos\theta,$$

is constant.

3.

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta,$$

ϕ and ψ are cyclic:

$p_\psi = I_3(\dot\psi + \dot\phi\cos\theta)$, p_ϕ constant.

Euler-Lagrange equation for θ

$$\frac{d}{dt}I_1\dot\theta = \frac{\partial L}{\partial\theta} = I_1\sin\theta\cos\theta\dot\phi^2 - I_3(\dot\psi + \dot\phi\cos\theta)\dot\phi\sin\theta + Mgl\sin\theta$$

$$= I_1\sin\theta\cos\theta\dot\phi^2 - p_\psi\dot\phi\sin\theta + Mgl\sin\theta = 0$$

if θ is constant

(i) If $p_\psi = 0$

$$0 = I_1\sin\theta\cos\theta\dot\phi^2 + Mgl\sin\theta$$

so that $\sin\theta = 0$ ($\theta = 0,\ \pi$ equilibria) or $I_1\cos\theta\dot\phi^2 = -Mgl$ so that $\cos\theta < 0$ or $\pi/2 < \theta < \pi$. This is essentially the same as for a spherical pendulum (after taking into account that in this case the θ angle is measured with respect to the positive z-axis).

(ii) For $p_\psi \neq 0$ and $\theta = \pi/2$, $p_\psi\dot\phi = Mgl$.

(iii) $\sin\theta = 0$ or

$$I_1\cos\theta\dot\phi^2 - p_\psi\dot\phi + Mgl = 0$$

giving

$$\dot\phi = \frac{p_\psi \pm \sqrt{p_\psi^2 - 4MI_1gl\cos\theta}}{2I_1\cos\theta}.$$

4. (i)

$$L = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

where

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \quad \omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \quad \omega_3 = \dot{\phi}\cos\theta + \dot{\psi},$$

are the components of angular velocity in terms of the Euler angles.

Here $p_\psi = I_3\omega_3$ and

$$\frac{\partial L}{\partial \psi} = I_1\omega_1(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi) + I_2\omega_2(-\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\cos\psi)$$

$$= I_1\omega_1\omega_2 - I_2\omega_2\omega_1 = \omega_1\omega_2(I_1 - I_2).$$

The Euler Lagrange equation for ψ gives the third Euler equation.

(ii)

$$p_\theta = I_1\omega_1\cos\psi - I_2\omega_2\sin\psi, \quad \frac{p_\phi - p_\psi\cos\theta}{\sin\theta} = I_1\omega_1\sin\psi + I_2\omega_2\cos\psi.$$

Some algebra gives

$$I_1\omega_1 = \left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)\sin\psi + p_\theta\cos\psi,$$

$$I_2\omega_2 = \left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)\cos\psi - p_\theta\sin\psi.$$

Accordingly

$$\{p_\psi, I_1\omega_1\} = -\frac{\partial}{\partial\psi}I_1\omega_1 = -I_2\omega_2, \quad \{p_\psi, I_2\omega_2\} = -\frac{\partial}{\partial\psi}I_2\omega_2 = I_1\omega_1.$$

or

$$\{I_3\omega_3, I_1\omega_1\} = -I_2\omega_2,$$

cyclic permutation gives

$$\{\omega_1, \omega_2\} = -\frac{I_3}{I_1 I_2}\omega_3.$$

5. $\dot{\omega}_3 = 0$ so thar ω_3 is constant. Inserting this into the first two Euler equations gives

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_3\omega_2, \quad I_1\dot{\omega}_2 = -(I_1 - I_3)\omega_3\omega_1.$$

Differentiating the first ODE and using the second equation to eliminate ω_2 gives

$$I_1\ddot{\omega}_1 = -\Omega^2\omega_1,$$

where

$$\Omega^2 = \frac{(I_1 - I_3)^2\omega_3^2}{I_1^2}.$$

Hence $\omega_1 = A\cos(\Omega t + \beta)$. Inserting this into the first ODE yields $\omega_2 = -A\sin(\Omega t + \beta)$ assuming $I_1 > I_3$.