The total marks for this test is 40, with 10 marks for each problem.

Problem 1. Prove that the set

$$U = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1\}$$

is open in $\mathbb{R}^2$.

Problem 2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = (e^y \sin(x), e^x \cos(y)).$$

- Is the map f differentiable at every point in $\mathbb{R}^2$? Justify your answer.
- Is the map f continuously differentiable at every point in $\mathbb{R}^2$? Justify your answer.

Problem 3. Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y, z) = \begin{cases} \frac{xyz+x^2y}{\|(x,y,z)\|^2} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

is continuous at every point in $\mathbb{R}^2$

Problem 4. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$$f(x, y) = \begin{cases} (x, y^2, 0) & \text{if } y \geq 0, \\ (x, 0, -y^2) & \text{if } y < 0. \end{cases}$$

(a) Find the directional derivatives of f at $(0, 0)$ in the directions $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(b) Is the map f differentiable at $(0, 0)$? Justify your answer.

Solution to Problem 1: Let $p = (x_0, y_0) \in U$ be an arbitrary point. By the definition of U, we have $x_0 \in (0, 1)$. Define

$$\delta = \min\{x_0, 1 - x_0\} > 0.$$

We claim that $B_\delta(p) \subset U$. To see this, let $q = (x, y) \in B_\delta(p)$ be an arbitrary point. We have

$$|x - x_0| \leq \|q - p\| < \delta.$$

This implies that

$$-\delta < x - x_0 < \delta.$$

By the definition of δ, the above inequality gives us four inequalities,

$$x - x_0 < x_0, \quad x - x_0 < 1 - x_0, \qquad -x_0 < x - x_0, \quad x_0 - 1 < x - x_0.$$

The second inequality gives us $x < 1$, and the third inequality gives us $x > 0$. This implies that $q = (x, y) \in U$. As $q \in B_\delta(p)$ was arbitrary, we conclude that $B_\delta(p) \subset U$.

[3pts for the correct understanding of the notion of open sets, 3pts for correct value of δ, 4pts for correct complete details.]

Solution to Problem 2: (a) From Analysis I, the components of f are differentiable functions in x and y. The partial derivative of f are

$$D_1 f(x, y) = (e^y \cos(x), e^x \cos(y)), \qquad D_2 f(x, y) = (e^y \sin(x), -e^x \sin(y)).$$

Both of these maps are continuous on $\mathbb{R}^2$. By a theorem in the lectures, if the partial derivatives of f are continuous on an open set (here $\mathbb{R}^2$), f is differentiable at all points in $\mathbb{R}^2$.
[5pts = 2pt for each partial derivative+3pt for the latter argument.]

(b) By a theorem in the lectures, the derivative of f at (x, y) in the standard basis of $\mathbb{R}^2$ is the matrix

$$Df(x, y) = \begin{pmatrix} e^y \cos(x) & e^y \sin(x) \\ e^x \cos(y) & -e^x \sin(y) \end{pmatrix}$$

All entries in the above matrix are continuous on $\mathbb{R}^2$, therefore, $Df(x, y)$ depends continuously on $(x, y) \in \mathbb{R}^2$.
[5pts =2pt for the matrix+3pts for the latter argument.]

Solution to Problem 3: We have seen in the lectures that for every $(x, y, z) \in \mathbb{R}^3$,

$$|x| \leq \|(x, y, z)\|, \quad |y| \leq \|(x, y, z)\|, \quad |z| \leq \|(x, y, z)\|.$$

These imply that for all $(x, y, z) \neq (0, 0, 0)$, we have

$$|xyz + x^2 y| \leq |xyz| + |x^2 y| = |x||y||z| + |x|^2 |y| \leq \|(x, y, z)\|^3 + \|(x, y, z)\|^3$$

and hence

$$\frac{|xyz + x^2 y|}{\|(x, y, z)\|^2} \leq \frac{\|(x, y, z)\|^3 + \|(x, y, z)\|^3}{\|(x, y, z)\|^2} \leq 2\|(x, y, z)\|^1.$$

Therefore,

$$0 \leq \lim_{(x,y,z)\to(0,0,0)} \frac{|xyz + x^2y|}{\|(x,y,z)\|^2} = 0.$$

At any point $(x,y,z) \neq (0,0,0)$, the numerator is a continuous function, and the denominator is a non-zero continuous function. Thus f is continuous at every $(x,y,z) \neq (0,0,0)$.

[5pts for stating the inequality (1), 5pts for completing the argument.]

Solution to Problem 4: Part (a): The directional derivatives are

$$\begin{aligned} D_{e_1} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{(t,0,0)}{t} = (1,0,0). \end{aligned}$$

$$D_{e_2} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t}.$$

There are two possibilities for $f(0,t)$, depending on the sign of t.

$$\lim_{t\to 0^+} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0^+} \frac{(0,t^2,0)}{t} = (0,0,0)$$

$$\lim_{t\to 0^-} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0^+} \frac{(0,0,-t^2)}{t} = (0,0,0).$$

These imply that $D_{e_2}(0,0) = (0,0,0)$.
[2pts for the derivative in direction of e_1 and 3pts for the derivative in direction e_2.]

Part (b): We claim that f is differentiable at $(0,0)$, and its derivative is the linear map Λ, which maps e_1 to $(1,0,0)$ and e_2 to $(0,0,0)$. That is, $\Lambda(x,y) = (x,0,0)$. Let $(x,y) \in \mathbb{R}^2$ be an arbitrary point. We have

$$\begin{aligned} f((0,0)+(x,y)) - f(0,0) - \Lambda[(x,y)] &= f(x,y) - (x,0,0) \\ &= \begin{cases} (0,y^2,0) & \text{if } y \geq 0 \\ (0,0,-y^2) & \text{if } y < 0 \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\|f((0,0)+(x,y)) - f(0,0) - \Lambda[(x,y)]\|}{\|(x,y)\|} &\leq \lim_{(x,y)\to(0,0)} \frac{y^2}{\|(x,y)\|} \\ &\leq \lim_{(x,y)\to(0,0)} \frac{\|(x,y)\|^2}{\|(x,y)\|} = 0. \end{aligned}$$

This shows that f is differentiable at $(0,0)$.
[2pts for stating the correct linear map, 3pts for showing the limit is 0.]