

Unseen 6

MATH40003 Linear Algebra and Groups

Term 2, 2022/23

1. Suppose $q, n \in \mathbb{N}$ and \mathbb{F}_q is a field with q elements.
 - (a) Find a formula (depending on n and q) for the number of bases v_1, \dots, v_n of \mathbb{F}_q^n (consider choosing v_1, v_2, \dots in turn).
 - (b) What is the order of $GL_n(\mathbb{F}_q)$, the group of all invertible $n \times n$ matrices over \mathbb{F}_q .
 - (c) Suppose $0 \neq \alpha \in \mathbb{F}_q$. Show that the number of matrices in $GL_n(\mathbb{F}_q)$ with determinant α does not depend on the choice of α . When do these matrices form a subgroup of $GL_n(\mathbb{F}_q)$?
 - (d) Find a formula (depending on n and q) for the order of $SL_n(\mathbb{F}_q)$, the group of all $n \times n$ matrices over \mathbb{F}_q with determinant 1.
2. Let (G, \cdot) be a finite group and let $A, B \subseteq G$ be subsets. Prove that if $|A| + |B| > |G|$ then $G = AB$ where $AB = \{a.b \mid a \in A, b \in B\}$
3. Let F be a finite field. Prove that every element of F is a sum of two squares, i.e., for every $a \in F$, there are $b_1, b_2 \in F$ such that $a = b_1^2 + b_2^2$. Is it true that every $n \in \mathbb{N}$ is a sum of two squares of \mathbb{N} ?
4. Let X be a set and let $G \leq \text{Sym}(X)$ be a subgroup. G *acts freely* if $\forall x \in X, g, h \in G: g(x) = h(x) \implies g = h$. G *is transitive* if $\forall x, y \in X, \exists g \in G: g(x) = y$. Prove that if G is transitive and acts freely, then $|G| = |X|$.