

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
Summer 2025

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Stochastic Calculus with Applications to non-Linear Filtering**

**Date:** Friday, May 23, 2025

**Time:** Start time 10:00 – End time 12:30 (BST)

**Time Allowed:** 2.5 hours

**This paper has 5 Questions.**

***Please Answer All Questions in 1 Answer Booklet***

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO**

For the following questions, assume the set-up: Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $(\mathcal{F}_t)_{t \geq 0}$  be a filtration in  $\mathcal{F}$  and  $V$  be a standard one-dimensional  $\mathcal{F}_t$ -adapted Brownian motion under  $\mathbb{P}$ . Let  $f$  and  $\sigma$  be bounded Lipschitz real valued functions and let  $X$  be the  $\mathcal{F}_t$ -adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s) ds + \int_0^t \sigma(X_s) dV_s. \quad (1)$$

Assume that  $X_0$  has distribution  $\pi_0$  at time 0, is independent of  $V$  and  $\mathbb{E}[(X_0)^2] < \infty$ . Let  $W$  be a standard  $\mathcal{F}_t$ -adapted one-dimensional Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$  independent of  $X$ , and  $Y$  be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \quad (2)$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded measurable function. The process  $Y = \{Y_t, t \geq 0\}$  is called the observation process. Let  $\{\mathcal{Y}_t, t \geq 0\}$  be the filtration associated with the process  $Y$ , that is  $\mathcal{Y}_t = \sigma(Y_s, s \in [0, t])$ . The filtering problem consists in determining the conditional distribution  $\pi_t$  of the signal  $X_t$  given  $\mathcal{Y}_t$ . That is,  $\pi_t(A) = \mathbb{E}[I_A(X_t) | \mathcal{Y}_t]$  for any Borel set  $A \in \mathcal{B}(\mathbb{R})$  ( $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -field on  $\mathbb{R}$  and  $I_A$  is the indicator function of the set  $A$ ) and  $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t]$  for any bounded Borel measurable function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ .

1. (a) Let  $k > 0$  be a positive integer. Define  $B^k = \{B_t^k, t \geq 0\}$  to be the process defined as

$$B_t^k = \begin{cases} t^k V_{\frac{1}{t^k}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}. \quad (3)$$

- (i) Prove that the stochastic process  $B^k$  has continuous paths on  $[0, \infty)$ . [You can use without proof the fact that  $\lim_{s \rightarrow \infty} \frac{V_s}{s} = 0$ .] (3 marks)
- (ii) Prove that the increments  $B_t^k - B_s^k$ ,  $0 \leq s < t$  of the process  $B^k$  are normally distributed and compute their means and variances. (3 marks)
- (iii) Choose three arbitrary time instances  $0 \leq r \leq s \leq t$ . Is the increment  $B_t^k - B_s^k$  independent of  $B_r^k$ ? (3 marks)
- (b) (i) Give the definition of a martingale. (3 marks)
- (ii) Is the stochastic  $B^k$  a martingale? (3 marks)
- (c) (i) Give the definition of a standard one-dimensional Brownian motion. (3 marks)
- (ii) Find all the values of the positive integer  $k$  for which the corresponding process  $B^k$  is a Brownian motion? [No proof required for this part]. (2 marks)

(Total: 20 marks)

2. (a) Give the definition of a d-dimensional continuous semimartingale. (3 marks)
- (b) State Itô's formula as applied to semimartingales [no proof required]. (4 marks)
- (c) Suppose that  $S_t$  is the following process
- $$S_t = \sinh(t + W_t), t \geq 0. \quad (4)$$
- (i) Using Itô's formula, find two functions  $c_1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $c_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that
- $$dS_t = c_1(S_t)dt + c_2(S_t)dW_t, \quad t \geq 0. \quad (5)$$
- (5 marks)
- (ii) Prove that equation (5) has a unique solution, in other words that the process  $S$  is the unique solution of (5). [You can use without proof any of the results in the lectures.] (4 marks)
- (iii) Find the limit of  $S_t$  as  $t$  tends to  $\infty$ . [You can use without proof the fact that  $\lim_{t \rightarrow \infty} \frac{W_t}{t} = 0$ ] (4 marks)

(Total: 20 marks)

3. Let  $Z = \{Z_t, t > 0\}$  be the process defined by

$$Z_t = \exp \left( - \int_0^t h(X_s) dW_s - \frac{1}{2} \int_0^t h(X_s)^2 ds \right), \quad t \geq 0,$$

where  $h$  is the function appearing in the equation (2) satisfied by the observation process  $Y$ .

- (a) State the Novikov condition. (4 marks)
- (b) Prove that  $Z$  is an  $\mathcal{F}_t$ -adapted martingale. (5 marks)
- (c) Deduce the evolution equation satisfied by the process  $Z$ . (5 marks)
- (d) Let  $p$  be a positive constant,  $p \geq 2$ . Prove that

$$\sup_{t \in [0, T]} \mathbb{E}[Z_t^p] < \infty$$

for any  $T > 0$ . (6 marks)

(Total: 20 marks)

4. Let  $A$  be the second order differential operator

$$A\varphi = f\varphi' + \frac{1}{2}\sigma^2\varphi'', \quad \varphi \in C_b^2(\mathbb{R}),$$

where  $C_b^2(\mathbb{R})$  is the set of all bounded functions  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  twice differentiable with bounded and continuous first and second derivatives. Next let  $\mu = \{\mu_t, t \geq 0\}$  be a measure valued process satisfying the equation

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(A\varphi) ds + 2 \int_0^t (\mu_s(h\varphi) - \mu_s(h)\mu_s(\varphi)) ds, \quad \text{for any } \varphi \in C_b^2(\mathbb{R}), \quad (6)$$

where  $\mu_0$  is a probability measure and  $h$  is the function appearing in the equation (2) satisfied by the observation process  $Y$ . Let  $\mu(\mathbf{1}) = \{\mu_t(\mathbf{1}), t \geq 0\}$  be the associated total mass process, that is  $\mu_t(\mathbf{1}) := \mu_t(\mathbb{R})$  for any  $t \geq 0$  and let  $\nu = \{\nu_t, t \geq 0\}$  be the measure valued process defined as

$$\nu_t(\varphi) = \mu_t(\varphi) \exp\left(2 \int_0^t \mu_s(h) ds\right) \quad \text{for any } \varphi \in C_b^2(\mathbb{R}) \quad (7)$$

and for any  $t \geq 0$ .

- (a) Deduce the evolution equation satisfied by the mass process  $\mu(\mathbf{1}) = \{\mu_t(\mathbf{1}), t \geq 0\}$   
(3 marks)
- (b) Prove  $\mu_t$  is a probability measure for any  $t \geq 0$ .  
(5 marks)
- (c) Deduce the evolution equation satisfied by the process  $\nu(\varphi) = \{\nu_t(\varphi), t \geq 0\}$ .  
(6 marks)
- (d) Assume that the measure  $\nu_t$  is absolutely continuous with respect to the Lebesgue measure for any  $t \geq 0$ . Denote by  $\tilde{\nu}_t$  the density of the measure  $\nu_t$  with respect to the Lebesgue measure and assume that  $\tilde{\nu}_t \in C_b^2(\mathbb{R})$  for any  $t \geq 0$ . Deduce the partial differential equation satisfied by  $\tilde{\nu} = \{\tilde{\nu}_t, t \geq 0\}$ .  
(6 marks)

You may use any results given in the course without proof, provided that you make it clear which ones you are using.

(Total: 20 marks)

### Mastery Question

5. Assume that the pair  $(a, b)$  satisfies the following linear system of stochastic differential equations

$$\begin{aligned} da_t &= F_t a_t dt + \sigma_t dV_t, \quad a_0 = 1, \\ db_t &= H_t a_t dt + \mu_t dW_t, \quad b_0 = 1, \end{aligned}$$

where the real valued functions  $s \mapsto F_s$ ,  $s \mapsto h_s$ ,  $s \mapsto \sigma_s$ ,  $s \mapsto \mu_s$  defined on  $[0, \infty)$  are all continuous strictly positive functions.

- (a) Define  $\lambda_t = \exp\left(-\int_0^t F_s ds\right)$ . Prove that

$$a_t \lambda_t = 1 + \int_0^t \sigma_s \lambda_s dV_s.$$

(5 marks)

- (b) (i) Find the distribution of  $a_t$  for  $t > 0$ .

(4 marks)

- (ii) Find the distribution of  $(a_t, a_{2t})$ . for  $t > 0$ .

(4 marks)

- (c) Find the distribution of  $b_t$  for  $t > 0$ .

(4 marks)

- (d) Find the distribution of  $(a_t, b_{2t})$  for  $t > 0$ .

(3 marks)

You may use any results given in the lectures without proof, provided that you make it clear which ones you are using.

(Total: 20 marks)

## Marking Scheme

### Question 1. [20 marks]

(a)

(i) [3 marks, not seen] On  $(0, \infty)$ , the paths  $t \mapsto V_{\frac{1}{t^k}}$  are continuous as they are the composition of two continuous functions  $t \mapsto V_t$  and  $t \mapsto \frac{1}{t^k}$ . Therefore on  $(0, \infty)$  the paths  $t \mapsto B_t^k$  are continuous as they are the product of two continuous functions: the function  $t \mapsto t^k$  and the continuos paths  $t \mapsto V_{\frac{1}{t^k}}$ . Since  $\lim_{t \rightarrow 0} B_t^k = \lim_{t \rightarrow 0} t^k V_{\frac{1}{t^k}} = \lim_{s \rightarrow \infty} \frac{V_s}{s} = 0$ , the paths of the stochastic process  $B^k$  are also continuous at 0, hence  $B^k$  is continuous on  $[0, \infty)$ .

(ii) [3 marks, not seen] Observe that, for  $0 \leq s \leq t$  we have

$$B_t^k - B_s^k = t^k V_{\frac{1}{t^k}} - s^k V_{\frac{1}{s^k}} = (t^k - s^k) V_{\frac{1}{t^k}} + s^k \left( V_{\frac{1}{t^k}} - V_{\frac{1}{s^k}} \right)$$

and  $V_{\frac{1}{t^k}}, \left( V_{\frac{1}{t^k}} - V_{\frac{1}{s^k}} \right)$  are independent normally distributed random variables (following from the properties of the Brownian motion  $V$ ), it follows that  $B_t^k - B_s^k$  is a linear combination of two independent normally distributed random variables, hence is itself a normally distributed random variable with mean equal to the same linear combination of the component means and variance equal to the sum of the component variances multiplied by the square of the corresponding coefficients. Since  $V_{\frac{1}{t^k}} \sim N(0, \frac{1}{t^k})$  and  $\left( V_{\frac{1}{t^k}} - V_{\frac{1}{s^k}} \right) \sim N(0, \frac{1}{s^k} - \frac{1}{t^k})$ , it follows that  $B_t^k - B_s^k$  has mean  $(t^k - s^k) \times 0 + s^k \times 0 = 0$  and variance

$$(t^k - s^k)^2 \times \frac{1}{t^k} + s^{2k} \times \left( \frac{1}{s^k} - \frac{1}{t^k} \right) = t^k - s^k$$

Hence

$$B_t^k - B_s^k \sim N(0, t^k - s^k).$$

(iii) [3 marks, not seen] Observe that

$$\begin{aligned} B_t^k - B_s^k &= (t^k - s^k) V_{\frac{1}{t^k}} + s^k \left( V_{\frac{1}{t^k}} - V_{\frac{1}{s^k}} \right) \\ B_r^k &= r^k (V_{\frac{1}{r^k}} - V_{\frac{1}{s^k}}) + r^k (V_{\frac{1}{s^k}} - V_{\frac{1}{t^k}}) + r^k V_{\frac{1}{t^k}} \end{aligned}$$

So the pair  $(B_t^k - B_s^k, B_r^k)$  is formed of linear combinations of normally distributed zero mean independent random variables. It follows that  $B_t^k - B_s^k$  and  $B_r^k$  are jointly normally distributed random variable. Hence they are independent if the expected value of their product is 0. Indeed, we have

$$\begin{aligned} E[(B_t^k - B_s^k) B_r^k] &= (t^k - s^k) r^k \frac{1}{t^k} - s^k r^k \left( \frac{1}{s^k} - \frac{1}{t^k} \right) \\ &= r^k \left( (t^k - s^k) \frac{1}{t^k} - s^k \left( \frac{1}{s^k} - \frac{1}{t^k} \right) \right) = 0. \end{aligned}$$

(b)

(i) [3 marks, seen] Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  be a filtered probability space and let  $M = \{M_t, t \geq 0\}$  be a stochastic process defined on it. The process  $M$  is a martingale if

1.  $M_t$  is adapted to  $\mathcal{F}_t$  for any  $t \geq 0$
2.  $M_t$  is integrable,  $\mathbb{E}[|M_t|] < \infty, \forall t \geq 0$ .
3. For all  $s, t$  with  $s < t$  we have  $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$

If no explicit reference is made to the filtration used, a process is checked to be a martingale always with respect to its natural filtration.

(ii) [3 marks, seen similar] As no explicit reference is made to the filtration, we check the martingale property against the natural filtration of  $B^k$ :

$$\mathcal{B}_t^k = \sigma(B_s^k, s \in [0, t]) = \sigma(V_s, s \in [\frac{1}{t^k}, \infty)).$$

Of course,  $B_t^k$  is obviously  $\mathcal{B}_t^k$ -adapted and is also integrable as it is normally distributed, following from part (a). So we only need to prove that  $\mathbb{E}[B_t^k | \mathcal{B}_s^k] = B_s^k$ . Since  $B_t^k - B_s^k$  is independent of  $B_r^k$  for any  $r \leq s$  it follows that  $B_t^k - B_s^k$  is independent of  $\mathcal{B}_s^k$ . Hence

$$\mathbb{E}[B_t^k | \mathcal{B}_s^k] = \mathbb{E}[B_t^k - B_s^k | \mathcal{B}_s^k] + \mathbb{E}[B_s^k | \mathcal{B}_s^k] = \mathbb{E}[B_t^k - B_s^k] + \mathbb{E}[B_s^k | \mathcal{B}_s^k] = 0 + B_s^k = B_s^k$$

(c)

(i) [3 marks, seen] A real-valued stochastic process  $A = \{A_t, t \geq 0\}$  is a standard Brownian motion if the following properties are satisfied

1.  $A$  is continuous a.s. and  $A_0 = 0$ .
2.  $A$  has independent increments. That is, if  $0 \leq t_1 < t_2 < \dots < t_n$ , then the  $n - 1$  random variables

$$A_{t_2} - A_{t_1}, A_{t_3} - A_{t_2}, \dots, A_{t_n} - A_{t_{n-1}}$$

are independent random variables.

3. For any  $s < t$  the random variable  $A_t - A_s$  is normally distributed with mean 0 and variance  $t - s$ .

(ii) [2 marks, not seen]  $B^k$  is a Brownian motion if and only if  $k = 1$ .

## Question 2. [20 marks]

(a) [3 marks, seen] Let  $X_t$  be an  $\{\mathcal{F}_t\}$ -adapted  $d$ -dimensional process with continuous paths. If  $X_t$  can be decomposed as  $X_t = M_t + V_t$ ,  $t \geq 0$ , where  $M_t$  is an  $\{\mathcal{F}_t\}$ -adapted  $d$ -dimensional martingale with continuous paths and the paths of the  $d$ -dimensional process  $V_t$  are of finite variation, then we call  $X_t$  a  $d$ -dimensional continuous semimartingale.

(b) [4 marks, seen]

Let  $X_t^1, \dots, X_t^d$  be semi-martingales and  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  a function which is one time continuously differentiable with respect to  $t$  and twice with respect to  $x_i$ ,  $i = 1, 2, \dots, d$ . Then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d[X^i, X^j]_s. \end{aligned}$$

(c)

(i) [5 marks, seen similar] Let  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  be the function  $f(x) = \sinh(t+x)$ . We apply Itô's formula for this function to the Brownian motion  $W$  which is, in particular a semimartingale. Then, by Itô's formula, we get that (we use the fact that  $\langle W \rangle_t = t$ ):

$$\begin{aligned} \sinh(t + W_t) &= \sinh(0 + W_0) + \int_0^t \cosh(s + W_s) ds + \int_0^t \cosh(s + W_s) dW_s \\ &\quad + \frac{1}{2} \int_0^t \sinh(s + W_s) d\langle W \rangle_s \\ &= \int_0^t \left( \sqrt{1 + (\sinh(s + W_s))^2} + \frac{1}{2} \sinh(s + W_s) \right) ds \\ &\quad + \int_0^t \sqrt{1 + (\sinh(s + W_s))^2} dW_s \end{aligned}$$

It follows that the process  $S_t = \sinh(t + W_t)$  satisfies equation

$$dS_t = c_1(S_t) dt + c_2(S_t) dW_t$$

where

$$c_1(x) = \sqrt{1 + x^2} + \frac{x}{2}, \quad c_2(x) = \sqrt{1 + x^2}$$

(ii) [4 marks, seen similar] The drift and the diffusion coefficients in the equations are Lipschitz functions as their derivatives are bounded

$$c'_1(x) = \frac{x}{\sqrt{1 + x^2}} + \frac{1}{2}, \quad c_2(x) = \frac{x}{\sqrt{1 + x^2}}$$

and

$$\sup_{x \in R} |c'_1(x)| \leq \frac{3}{2}, \quad \sup_{x \in R} |c'_2(x)| \leq \frac{1}{2}$$

It follows that the equation satisfied by the process  $S$  has a unique solution in accordance with one of the theorems in the lectures.

**(ii) [4 marks, unseen]** We have that

$$\begin{aligned}\lim_{t \rightarrow \infty} (t + W_t) &= \lim_{t \rightarrow \infty} t \left(1 + \frac{W_t}{t}\right) \\ &= \lim_{t \rightarrow \infty} t \lim_{t \rightarrow \infty} \left(1 + \frac{W_t}{t}\right) \\ &= \infty (1 + 0) = \infty.\end{aligned}$$

Then since the function  $x \rightarrow \sinh(x)$  tends to  $\infty$  as  $x$  tends to  $\infty$ , we deduce that

$$\lim_{t \rightarrow \infty} S_t = \lim_{t \rightarrow \infty} \sinh(t + W_t) = \infty.$$

**Question 3. [20 marks]**

(a) [4 marks, seen] Novikov's condition states that if  $u = \{u_t, t > 0\}$  is a process defined as  $u_t = \exp\left(M_t - \frac{1}{2}\langle M \rangle_t\right)$  for  $M$  a continuous local martingale, then a sufficient condition for  $u$  to be a martingale is that

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle M \rangle_t \right) \right] < \infty, \quad 0 \leq t < \infty.$$

(b) [5 marks, seen] In this case the process  $t \rightarrow - \int_0^t h(X_s) dW_s$  is a local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by  $t \rightarrow \int_0^t h(X_s)^2 ds$ ). Moreover, since  $h$  is bounded, it follows that  $|h(X_s)| \leq \|h\|_\infty$  and hence

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle M \rangle_t \right) \right] = \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^t h(X_s)^2 ds \right) \right] \leq \exp \left( \frac{t \|h\|_\infty^2}{2} \right) < \infty, \quad 0 \leq t < \infty.$$

Hence, by Novikov's condition, the process  $Z = \{Z_t, t > 0\}$  is a martingale.

(c) [5 marks, seen] Let  $\xi = \{\xi_t, t > 0\}$  be the semimartingale defined by

$$\xi_t = - \int_0^t h(X_s) dW_s - \frac{1}{2} \int_0^t h(X_s)^2 ds, \quad t \geq 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned} Z_t &= \exp(\xi_t) \\ &= \exp(\xi_0) + \int_0^t \exp(\xi_s) d\xi_s + \frac{1}{2} \int_0^t \exp(\xi_s) d\langle \xi \rangle_s \\ &= 1 + \int_0^t Z_s \left( -h(X_s) dW_s - \frac{1}{2} h(X_s)^2 ds \right) + \frac{1}{2} \int_0^t Z_s h(X_s)^2 ds \\ &= 1 - \int_0^t Z_s h(X_s) dW_s. \end{aligned}$$

(d) [6 marks, not seen] Observe that

$$Z_t^p = \exp \left( \frac{1}{2} \int_0^t (p^2 - p) h(X_s)^2 ds \right) \bar{z}_t \leq \exp \left( \frac{(p^2 - p)t}{2} \|h\|_\infty^2 \right) \bar{z}_t,$$

where  $\bar{z} = \{\bar{z}_t, t > 0\}$  is the process defined by

$$\bar{z}_t = \exp \left( - \int_0^t ph(X_s) dW_s - \frac{p^2}{2} \int_0^t h(X_s)^2 ds \right), \quad t \geq 0.$$

Again, by Novikov's condition, the process  $\bar{z} = \{\bar{z}_t, t > 0\}$  is a martingale. Hence  $E[\bar{z}_t] = E[\bar{z}_0] = 1$  and

$$\sup_{t \in [0, T]} E[Z_t^p] \leq \sup_{t \in [0, T]} \exp \left( \frac{(p^2 - p)t}{2} \|h\|_\infty^2 \right) E[\bar{z}_t] = \exp \left( \frac{(p^2 - p)T}{2} \|h\|_\infty^2 \right) < \infty$$

for any  $T > 0$ .

#### Question 4. (20 marks)

(a). [3 marks, seen similar] Since  $A\mathbf{1} = 0$ , it follows from the equation that the mass process satisfies the following

$$\begin{aligned}\mu_t(\mathbf{1}) &= \mu_0(\mathbf{1}) + \int_0^t \mu_s(A\mathbf{1}) ds + 2 \int_0^t (\mu_s(h) - \mu_s(h)\mu_s(\mathbf{1})) ds \\ &= 1 + 2 \int_0^t \mu_s(h)(1 - \mu_s(\mathbf{1})) ds.\end{aligned}$$

(b). [5 marks, not seen] Let  $e = \{e_t, t \geq 0\}$  be the process defined as  $e_t = \mu_t(\mathbf{1}) - 1$ . It follows from the above equation that  $e$  satisfies the linear ordinary differential equation

$$\frac{de_t}{dt} = -2\mu_s(h)e_t, \quad t \geq 0, \quad e_0 = 0,$$

which has as the unique solution the trivial solution  $e_t = 0$ . Hence  $\mu_t(\mathbf{1}) = 1$  and therefore  $\mu_t$  is indeed a probability for all  $t \geq 0$ .

(c). [6 marks, not seen] Denote  $m = \{m_t, t \geq 0\}$  the process defined as

$$\begin{aligned}m_t &= \exp\left(2 \int_0^t \mu_s(h) ds\right), \quad t \geq 0, \\ m_t &= m_0 + 2 \int_0^t \mu_s(h)m_s ds, \quad t \geq 0.\end{aligned}$$

By the product rule

$$\begin{aligned}\nu_t(\varphi) &= \mu_t(\varphi)m_t \\ &= \mu_0(\varphi)m_0 + \int_0^t \mu_s(\varphi)dm_s + \int_0^t m_s d\mu_s(\varphi) \\ &= \mu_0(\varphi)m_0 + 2 \int_0^t \mu_s(\varphi)\mu_s(h)m_s ds \\ &\quad + \int_0^t m_s \mu_s(A\varphi) ds + 2 \int_0^t m_s(\mu_s(h\varphi) - \mu_s(h)\mu_s(\varphi)) ds \\ &= \mu_0(\varphi) + \int_0^t \nu_s(A\varphi) ds + 2 \int_0^t \nu_s(h\varphi) ds\end{aligned}$$

(d). [6 marks, seen similar]. We have that

$$\begin{aligned}\nu_t(\varphi) &= \int \varphi(x)\tilde{\nu}_t(x)dx \\ &= \mu_0(\varphi) + \int_0^t \nu_s(A\varphi) ds + 2 \int_0^t \nu_s(h\varphi) ds \\ &= \int_{\mathbb{R}} \varphi(x)\tilde{\nu}_0(x)dx + \int_0^t \int_{\mathbb{R}} (A\varphi(x) + 2h(x)\varphi(x))\tilde{\nu}_s(x)dx ds\end{aligned}$$

Let  $A^* : \mathcal{C}_b^2(\mathbb{R}) \rightarrow \mathcal{C}_b(\mathbb{R})$  be the operator

$$A^*\varphi = -(f\varphi)' + \frac{1}{2}(\sigma^2\varphi)''.$$

Then, assuming that  $\varphi \in \mathcal{C}_c^2(\mathbb{R})$ , where the following holds

$$\int A\varphi \tilde{\nu}_s dx = \int \varphi A^* \tilde{\nu}_s dx \quad (1)$$

We can deduce that

$$\begin{aligned} \int \varphi(x) \tilde{\nu}_t(x) dx &= \int \varphi(x) \tilde{\nu}_0(x) dx + \int_0^t \int \varphi(x) A^* \tilde{\nu}_s(x) dx ds \\ &= \int \varphi(x) \left[ \tilde{\nu}_0(x) + \int_0^t (A^* + 2h(x)) \tilde{\nu}_s(x) ds \right] dx \end{aligned}$$

Denote by  $g_t(x) = \tilde{\nu}_t(x) - \tilde{\nu}_0(x) - \int_0^t (A^* + 2h(x)) \tilde{\nu}_s(x) ds$ , so that

$$\int \varphi(x) g_t(x) dx = 0.$$

This means that  $g_t$  is orthogonal to any  $\varphi \in \mathcal{C}_c^2(\mathbb{R})$ . Since  $\tilde{\nu}_t \in C_b^2(\mathbb{R})$  it follows that  $g_t \in C_b(\mathbb{R})$ . It follows that  $g_t 1_{[-M,M]} \in L^2(\mathbb{R})$  and, since  $\mathcal{C}_c^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ , it follows that  $g_t 1_{[-M,M]} \perp g_t 1_{[-M,M]}$  for any  $M > 0$ . This implies that  $g_t = 0$ . We have deduced that

$$\frac{d\tilde{\nu}_t(x)}{dt} = (A^* + 2h(x)) \tilde{\nu}_t.$$

### Question 5. (20 marks)

5. Assume that the pair  $(a, b)$  satisfies the following linear system of stochastic differential equations

$$\begin{aligned} da_t &= F_t a_t dt + \sigma_t dV_t, \quad a_0 = 1, \\ db_t &= H_t a_t dt + \mu_t dW_t, \quad b_0 = 1, \end{aligned}$$

where the real valued functions  $s \mapsto F_s$ ,  $s \mapsto h_s$ ,  $s \mapsto \sigma_s$ ,  $s \mapsto \mu_s$  defined on  $[0, \infty)$  are all continuous.

a. Define  $\lambda_t = \exp\left(-\int_0^t F_s ds\right)$ . Prove that

$$a_t \lambda_t = 1 + \int_0^t \sigma_s \lambda_s dV_s.$$

b.

i.

Find the distribution of  $a_t$  for  $t > 0$ .

ii.

Find the distribution of  $(a_t, a_{2t})$  for  $t > 0$ .

c. Find the distribution of  $b_t$  for  $t > 0$ .

d. Find the distribution of  $(a_t, b_{2t})$  for  $t > 0$ .

(a). [5 marks, seen] By integration by parts

$$a_t \lambda_t = a_0 \lambda_0 + \int_0^t a_s d\lambda_s + \int_0^t \lambda_s da_s + \langle a, \lambda \rangle_t.$$

Since  $\lambda$  has no martingale part  $\langle a, \lambda \rangle_t = 0$ , and  $a_0 = \lambda_0 = 1$  we get

$$\begin{aligned} a_t \lambda_t &= 1 - \int_0^t a_s F_s \lambda_s ds + \int_0^t \lambda_s F_s a_s ds + \int_0^t \lambda_s \sigma_s dV_s \\ &= 1 + \int_0^t \sigma_s \lambda_s dV_s \end{aligned}$$

which gives us the identity.

(b) i. [4 marks, seen similar]. By using one of the results in the lectures,  $c_t := \int_0^t \sigma_s \lambda_s dV_s$  is a Gaussian random variable with zero mean and variance  $\int_0^t (\sigma_s \lambda_s)^2 ds$ . It follows that  $a_t$  is the sum between the Gaussian random variable  $\frac{1}{\lambda_t} a_t$  and the constant  $\frac{1}{\lambda_t}$ , hence it is itself Gaussian with mean  $m_t^a$  and variance  $p_t^a$  obtained by summing up the component variances. Hence

$$m_t^a = \frac{1}{\lambda_t}, \quad p_t^a = \frac{1}{\lambda_t^2} \int_0^t (\sigma_s \lambda_s)^2 ds.$$

(b) ii. [4 marks, not seen]. The pair of random variables  $(a_t, a_{2t})$  are individually Gaussian random variables. Moreover, they are obtained via linear transformations from the Brownian motion  $V$  and therefore  $(a_t, a_{2t})$  is (jointly) a Gaussian random vector. To identify the distribution of  $(a_t, a_{2t})$  we need their mean and the covariance matrix, that is

$$\begin{pmatrix} a_t \\ a_{2t} \end{pmatrix} = N \left( \begin{pmatrix} m_t^a \\ m_{2t}^a \end{pmatrix}, \begin{pmatrix} p_{11}^a & p_{12}^a \\ p_{21}^a & p_{22}^a \end{pmatrix} \right),$$

where

$$m_{it}^a = E[m_{it}^a], \quad p_{ij}^a = E[(a_{it} - m_{it}^a)(a_{jt} - m_{jt}^a)], \quad i, j = 1, 2.$$

From part i. we have that

$$m_{it}^a = \frac{1}{\lambda_{it}}, \quad p_{ii}^a = p_{it}^a = \frac{1}{\lambda_{it}^2} \int_0^{it} (\sigma_s \lambda_s)^2 ds, \quad i = 1, 2.$$

It only remains to compute

$$\begin{aligned} p_{12}^a &= p_{21}^a = E[(a_t - m_t^a)(a_{2t} - m_{2t}^a)] \\ &= \frac{1}{\lambda_t \lambda_{2t}} E \left[ \int_0^t \lambda_s \sigma_s dV_s \int_0^{2t} \lambda_s \sigma_s dV_s \right] \\ &= \frac{1}{\lambda_t \lambda_{2t}} E \left[ \left( \int_0^t \lambda_s \sigma_s dV_s \right)^2 \right] \\ &= \frac{1}{\lambda_t \lambda_{2t}} E \left[ \int_0^t (\sigma_s \lambda_s)^2 ds \right] \end{aligned}$$

In the above, we use the fact that  $\int_0^t \lambda_s \sigma_s dV_s$  and  $\int_t^{2t} \lambda_s \sigma_s dV_s$  are zero mean independent random variables (due to the independent of the increments of the Brownian motion).

(c) [4 marks, seen similar]. We have

$$\begin{aligned} b_t &= 1 + \int_0^t H_s a_s ds + \int_0^t \mu_s dW_s \\ &= 1 + \int_0^t \frac{H_s}{\lambda_s} \left( 1 + \int_0^s \sigma_r \lambda_r dV_r \right) ds + \int_0^t \mu_s dW_s \\ &= c_t^1 + \int_0^t c_s^2 dV_s + \int_0^t \mu_s dW_s \end{aligned}$$

where

$$\begin{aligned} c_t^1 &= 1 + \int_0^t \frac{H_s}{\lambda_s} ds \\ c_s^2 &= \sigma_s \lambda_s \left( \int_s^t \frac{H_r}{\lambda_r} dr \right) \end{aligned}$$

To obtain the second term in the above decomposition, observe that, by integration by parts, we have

$$\begin{aligned}\int_0^t \frac{H_s}{\lambda_s} \left( \int_0^s \sigma_r \lambda_r dV_r \right) ds &= \left( \int_0^t \frac{H_s}{\lambda_s} ds \right) \left( \int_0^t \sigma_r \lambda_r dV_s \right) - \int_0^t \sigma_r \lambda_r \left( \int_0^s \frac{H_s}{\lambda_s} dr \right) dV_s \\ &= \int_0^t c_s^2 dV_s,\end{aligned}$$

Since  $\int_0^t c_s^2 dV_s$  and  $\int_0^t \mu_s dW_s$  are independent Gaussian random variables ( $V$  and  $W$  are independent), it follows that  $b_t$  is the sum of the two independent Gaussian random variable plus the constant  $c_t^1$ , hence it is itself Gaussian with mean  $m_t^b$  and variance  $p_t^b$  obtained by summing up the component variances. Hence

$$m_t^b = c_t^1, \quad p_t^b = \int_0^t \left( (c_s^2)^2 + (\mu_s^2)^2 \right) ds.$$

**iv. [3 marks, not seen]** The pair of random variables  $(a_t, b_{2t})$  are individually Gaussian random variables. Moreover, they are obtained via linear/affine transformations from the mutually independent Brownian motions  $(V, W)$ . It follows that  $(a_t, b_{2t})$  is (jointly) a Gaussian random vector and

$$\begin{pmatrix} a_t \\ b_{2t} \end{pmatrix} = N \left( \begin{pmatrix} m_t^a \\ m_{2t}^b \end{pmatrix}, \begin{pmatrix} p_t^a & p_t^{ab} \\ p_t^{ab} & p_{2t}^b \end{pmatrix} \right),$$

where

$$p_t^{ab} = E [(a_t - m_t^a)(b_{2t} - m_{2t}^b)].$$

The only term that needs to be computed is  $p_t^{ab}$ . We have

$$\begin{aligned}p_t^{ab} &= E [(a_t - m_t^a)(b_{2t} - m_{2t}^b)] \\ &= \frac{1}{\lambda_t} E \left[ \int_0^t \lambda_s \sigma_s dV_s \left( \int_0^{2t} c_s^2 dV_s + \int_0^{2t} \mu_s dW_s \right) \right] \\ &= \frac{1}{\lambda_t} E \left[ \int_0^t \lambda_s \sigma_s dV_s \int_0^t c_s^2 dV_s \right] \\ &= \frac{1}{\lambda_t} \int_0^t \lambda_s \sigma_s c_s^2 ds.\end{aligned}$$

**MATH70055 Stochastic Calculus with Applications to non-Linear Filtering Markers****Comments**

- Question 1 Many good answers. In some case the students where caught unaware of the choice of the filtration
- Question 2 A lot of complete answers. Students understood well this part of the course.
- Question 3 Students found this question quite easy. Cleary a nice topic.
- Question 4 A mixed bag. Some excellent answers some not.
- Question 5 A challenging question. Despite that most students engaged well with it.