

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
May 2024**

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Fluid Dynamics 1**

Date: Wednesday, May 8, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer All Questions in 1 Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. Consider a steady two-dimensional flow with the velocity field

$$\mathbf{V} \equiv (u, v) = (x - \sqrt{2}y, \sqrt{2}x - y),$$

given in a Cartesian coordinate system  $(x, y)$ .

- (i) Verify that the flow is incompressible, and calculate the vorticity of this flow.

Show that the velocity field satisfies the Navier-Stokes equations for a suitable pressure field  $p(x, y)$ , which you are expected to determine. It is assumed that no body force is present.

(6 marks)

- (ii) Solve the initial-value problem,

$$\frac{dx}{dt} = x - \sqrt{2}y, \quad \frac{dy}{dt} = \sqrt{2}x - y; \quad x(0) = x_0, \quad y(0) = y_0.$$

Show that each particle would return repeatedly to its original position  $(x_0, y_0)$  after a certain period of time, which you are required to determine.

In experimental investigations of fluid motions, hydrogen bubbles are often used to visualise the flow under the assumption that each bubble follows a fixed fluid particle during its motion. Suppose that hydrogen bubbles are released *instantly* at  $t = 0$  from a unit circular wire,  $x_0^2 + y_0^2 = 1$ , so that each fluid particle is marked, and can be traced by the flow visualisation technique. Find the equation describing the loop of the bubbles at a subsequent time  $t > 0$ .

(8 marks)

- (iii) Calculate, by performing contour integration, the circulation around a unit circle  $x^2 + y^2 = 1$ , which is the initial shape of the loop of the bubbles. Verify your result using the relation between the circulation and the vorticity flux.

[Hint: A contour integral needs to be evaluated by parameterising the contour. The unit circle can be parameterised as  $x = \cos \theta$ ,  $y = \sin \theta$  with  $0 \leq \theta \leq 2\pi$ .]

What is the circulation around the loop of the bubbles at  $t > 0$ ?

(6 marks)

(Total: 20 marks)

2. It is known that the Earth is not a perfect sphere. It is also known that the pressure in the ocean increases with depth much faster than it decreases in the atmosphere. In view of this, one may determine the shape of the Earth by modelling it as a rotating volume of fluid surrounded by a vacuum, with the fluid being kept together through the action of the gravitational force. This force is assumed to have only a radial component, which is proportional to the distance from the Earth centre, namely,

$$f_r = -\alpha r.$$

The fluid motion is assumed to be symmetric with respect to the axis of the Earth. It is convenient to use the spherical polar coordinate system  $(r, \vartheta, \phi)$  as is shown in figure 1 on the next page, where the  $x$ -axis is oriented along the axis of the Earth's rotation. The velocity field of the flow is denoted by  $(V_r, V_\vartheta, V_\phi)$ , and the flow is governed by the Navier-Stokes equations in spherical polar coordinates given in (2) (see the next page), in which  $f_r = -\alpha r$ ,  $f_\vartheta = 0$  and  $f_\phi = 0$ .

Assume that the fluid rotates as a solid body so that its velocity field is given by

$$V_\phi = \Omega r \sin \vartheta, \quad V_r = 0, \quad V_\vartheta = 0,$$

where  $\Omega$  is the angular velocity of the Earth's rotation.

- (i) Deduce the pressure distribution,  $p(r, \vartheta)$ , within the fluid volume.

Given that the pressure in the surrounding vacuum is zero and the Earth's radius at the North pole is  $R_0$ , show that at any other meridional angle  $\vartheta$  (measured from the North pole) the distance,  $R$ , from the Earth surface to the centre is given by

$$R = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha} \sin^2 \vartheta}}. \quad (1)$$

(10 marks)

- (ii) Choose the plane S in figure 1 to coincide with the  $(x, y)$  plane, and show that equation (1) may be written as

$$x^2/a^2 + y^2/b^2 = 1.$$

Hence, conclude that the Earth is an ellipsoid. What are its principal axes  $a$  and  $b$ ?

(3 marks)

- (iii) Let  $M$  denote the total mass of the Earth. Express  $R_0$  in terms of  $M$ . You may use without proof that the volume of an ellipsoid with principle axes  $a$ ,  $b$  and  $c$  is given by  $\frac{4}{3}\pi abc$ .

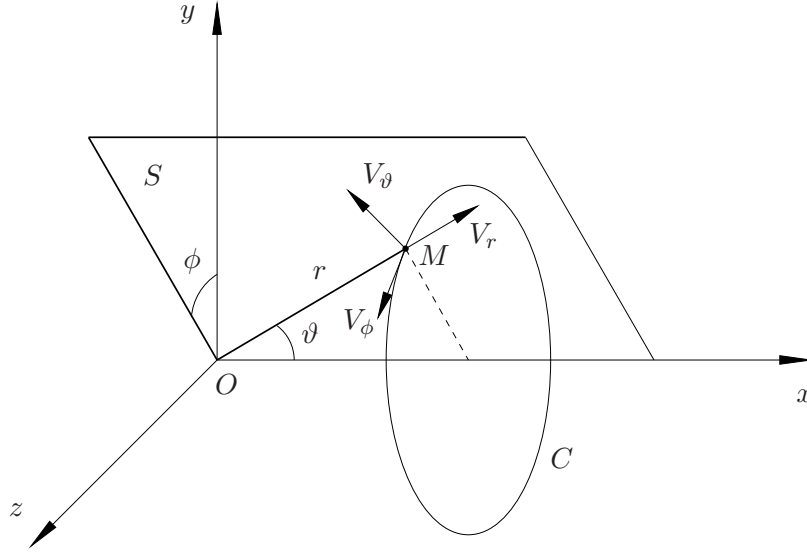
(4 marks)

- (iv) For the Earth,  $\Omega^2/\alpha$  is small. If there is a planet for which  $\Omega^2/\alpha$  is not small, then which of the assumptions made above would fail?

(3 marks)

(Total: 20 marks)

You may use without proof the Navier-Stokes equations in spherical polar coordinates (figure 1) given below.



**Figure 1:** Spherical polar coordinates.

$$\begin{aligned} \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\vartheta}{r} \frac{\partial V_r}{\partial \vartheta} + \frac{V_\phi}{r \sin \vartheta} \frac{\partial V_r}{\partial \phi} - \frac{V_\vartheta^2 + V_\phi^2}{r} = f_r - \frac{1}{\rho} \frac{\partial p}{\partial r} + \\ + \nu \left( \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \vartheta^2} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 V_r}{\partial \phi^2} + \frac{2}{r} \frac{\partial V_r}{\partial r} + \right. \\ \left. + \frac{1}{r^2 \tan \vartheta} \frac{\partial V_r}{\partial \vartheta} - \frac{2}{r^2} \frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2}{r^2 \sin \vartheta} \frac{\partial V_\phi}{\partial \phi} - \frac{2V_r}{r^2} - \frac{2V_\vartheta}{r^2 \tan \vartheta} \right), \end{aligned} \quad (2a)$$

$$\begin{aligned} \frac{\partial V_\vartheta}{\partial t} + V_r \frac{\partial V_\vartheta}{\partial r} + \frac{V_\vartheta}{r} \frac{\partial V_\vartheta}{\partial \vartheta} + \frac{V_\phi}{r \sin \vartheta} \frac{\partial V_\vartheta}{\partial \phi} + \frac{V_r V_\vartheta}{r} - \frac{V_\phi^2}{r \tan \vartheta} = f_\vartheta - \frac{1}{\rho r} \frac{\partial p}{\partial \vartheta} + \\ + \nu \left( \frac{\partial^2 V_\vartheta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V_\vartheta}{\partial \vartheta^2} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 V_\vartheta}{\partial \phi^2} + \frac{2}{r} \frac{\partial V_\vartheta}{\partial r} + \right. \\ \left. + \frac{1}{r^2 \tan \vartheta} \frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial V_\phi}{\partial \phi} + \frac{2}{r^2} \frac{\partial V_r}{\partial \vartheta} - \frac{V_\vartheta}{r^2 \sin^2 \vartheta} \right), \end{aligned} \quad (2b)$$

$$\begin{aligned} \frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + \frac{V_\vartheta}{r} \frac{\partial V_\phi}{\partial \vartheta} + \frac{V_\phi}{r \sin \vartheta} \frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} + \frac{V_\vartheta V_\phi}{r \tan \vartheta} = f_\phi - \frac{1}{\rho r \sin \vartheta} \frac{\partial p}{\partial \phi} + \\ + \nu \left( \frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V_\phi}{\partial \vartheta^2} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 V_\phi}{\partial \phi^2} + \frac{2}{r} \frac{\partial V_\phi}{\partial r} + \right. \\ \left. + \frac{1}{r^2 \tan \vartheta} \frac{\partial V_\phi}{\partial \vartheta} + \frac{2}{r^2 \sin \vartheta} \frac{\partial V_r}{\partial \phi} + \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial V_\vartheta}{\partial \phi} - \frac{V_\phi}{r^2 \sin^2 \vartheta} \right), \end{aligned} \quad (2c)$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r} \frac{\partial V_\vartheta}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial V_\phi}{\partial \phi} + \frac{2V_r}{r} + \frac{V_\vartheta}{r \tan \vartheta} = 0. \quad (2d)$$

3. (a) A general viscous incompressible flow is described in a Cartesian coordinate system  $(x_1, x_2, x_3)$ , in which the velocity field is denoted by  $\mathbf{V} = (V_1, V_2, V_3)$ , where each component  $V_i$  depends on  $x_j$  ( $j = 1, 2, 3$ ) and the time variable  $t$ . Application of the momentum conservation law leads to the equations

$$\rho \left[ \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right] = \frac{\partial p_{ij}}{\partial x_j},$$

where  $\rho$  is the fluid density, and the Einstein summation convention is assumed. For a Newtonian fluid, the stress tensor  $p_{ij}$  is expressed, through the constitutive relation, as

$$p_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right),$$

where  $p$  is the pressure, and  $\delta_{ij}$  the Kronecker delta. The dynamic viscosity  $\mu$  is a function of the temperature  $T$ , which depends on  $(x_1, x_2, x_3)$ , i.e.  $T = T(x_1, x_2, x_3)$ . Assume that the continuity equation remains as

$$\nabla \cdot \mathbf{V} = 0.$$

Use the equations above to derive the momentum equations in the Navier-Stokes equations. (4 marks)

- (b) Let  $(x, y, z) = (x_1, x_2, x_3)$  and apply the momentum equations derived in Part (a) to a steady uni-directional flow, where the only non-zero velocity component  $u$  is in the  $x$ -direction. The temperature  $T$  is assumed to be independent of  $x$ . Show that  $u$  is independent of  $x$  and satisfies the equation,

$$\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) = \frac{\partial p}{\partial x},$$

and explain why the pressure gradient  $\frac{\partial p}{\partial x}$  must be a constant. (4 marks)

- (c) Apply the result in Part (b) to a steady uni-directional flow through a channel between two infinitely large parallel plates, which are located at  $y = 0$  and  $y = h$ . The plate at  $y = 0$  is fixed, while the plate at  $y = h$  moves in its own plane at a constant velocity  $U$ , driving the flow between the plates. It is assumed that the pressure gradient  $\frac{\partial p}{\partial x} = 0$  and that

$$\mu = \mu_0 + \mu_1 T, \quad T = y,$$

where  $\mu_0$  and  $\mu_1$  are both constants.

- (i) Deduce the equation for  $u$  and solve it to show that the velocity distribution across the channel is

$$u(y) = U \ln \left[ 1 + (\mu_1/\mu_0)y \right] / \ln(1 + h\mu_1/\mu_0).$$

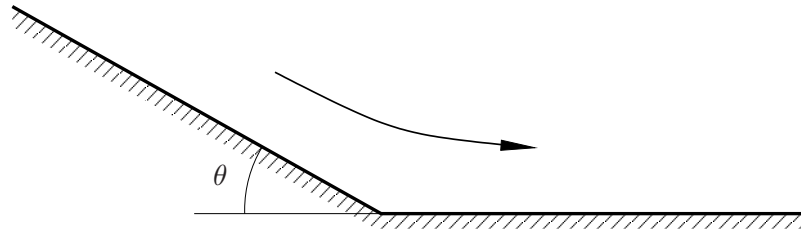
What does  $u(y)$  simplify to when  $\mu_1$ , or more precisely  $h\mu_1/\mu_0$ , is small? (6 marks)

- (ii) Consider the fluid in the region:  $0 \leq y \leq h$ ,  $0 \leq x \leq L$  and  $0 \leq z \leq 1$ . Calculate the viscous stresses that the plates exert on the fluid. Show that the viscous forces exerted by the lower and upper plates are in balance. Explain why this is the case.

(6 marks)

(Total: 20 marks)

4. Consider the two-dimensional irrotational inviscid flow of incompressible fluid past a corner formed between two semi-infinite flat plates as is illustrated in the diagram below.



The flow is described in the Cartesian coordinate system  $(x, y)$  with the origin at the apex of the corner and  $x$  being aligned with the horizontal plate. A complex position variable  $z = x + iy$  is introduced in order to use the conformal mapping technique.

- (i) Show that a conformal mapping of the power-function form,

$$\zeta = z^\alpha,$$

maps the physical region onto the upper half of the auxiliary  $\zeta$ -plane for a suitable value of  $\alpha$ , which you are required to determine.

Given that the complex potential  $W(\zeta) = V_\infty \zeta$  on the  $\zeta$ -plane (where  $V_\infty > 0$  is a real constant), find the complex potential  $w(z)$  on the physical plane. Determine the velocity  $V$  and the pressure  $p$  in terms of  $r = |z|$ , the distance to the apex.

(4 marks)

- (ii) Suppose now that a source with strength  $q > 0$  is added at the position  $z = de^{i(\pi-\theta)/2}$ , which is a point on the line bisecting the corner and at a distance  $d$  to the apex.

Determine the position of the source on the  $\zeta$ -plane, and hence find the appropriate complex potential  $W(\zeta)$  and the corresponding complex potential  $w(z)$  on the physical plane.

For the special case of  $\theta = \pi/2$ , explain how you can obtain the same complex potential  $w(z)$  by using the result in (i) and the idea of introducing appropriate image sources.

(8 marks)

- (iii) For the special case of  $\theta = 0$ , calculate the velocity field of the flow represented by the complex potential  $w(z)$ , and find all stagnation points.

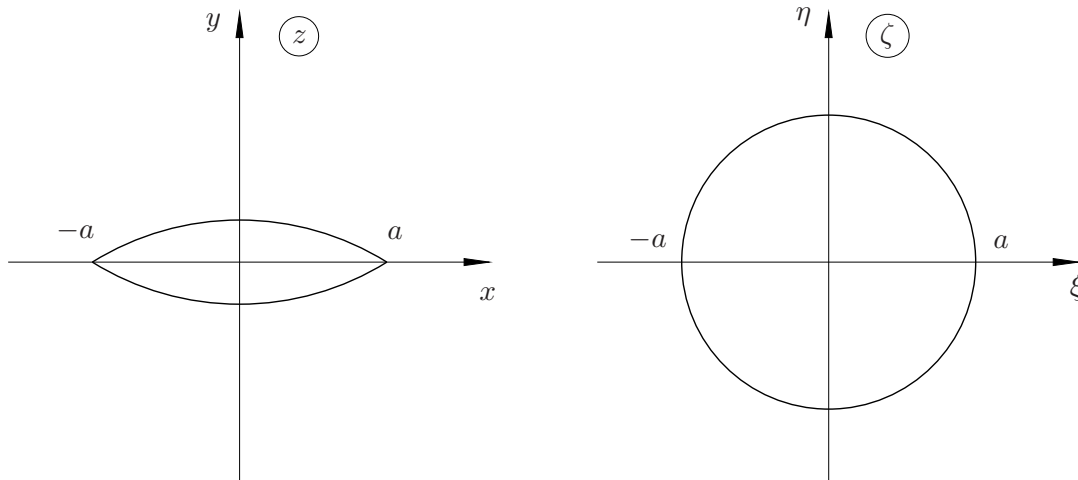
Discuss whether the complex potential  $w(z)$  may alternatively be viewed as representing an inviscid two-dimensional potential flow due to a uniform flow past a body (or bodies) with blunt leading edge(s); consider two cases: (a)  $d < q/(2\pi V_\infty)$  and (b)  $d > q/(2\pi V_\infty)$ .

(8 marks)

(Total: 20 marks)

5. Consider the two-dimensional inviscid irrotational flow past a symmetric aerofoil which consists of two circular arcs as is shown on the left-hand side of figure 2. The angle between the tangent of the arcs at the trailing edge is  $2\theta$  with  $0 \leq \theta < \pi/2$ . The modulus of the velocity in the free-stream far from the aerofoil is  $V_\infty$ , and the angle of attack is  $\alpha$ .

The solution for the flow is to be found by seeking a conformal mapping  $\zeta = \zeta(z)$ , which maps the aerofoil onto a circle on the auxiliary  $\zeta$ -plane; the circle is centred at  $\zeta = 0$  and has a radius  $a$  as is shown on the right-hand side of figure 2. The required mapping  $\zeta = \zeta(z)$  and the solution can be constructed by following the steps outlined below.



**Figure 2:** Geometry of the physical  $z$ -plane and the auxiliary  $\zeta$ -plane.

- (i) The aerofoil and the circle on the right-hand side of figure 2 are mapped, respectively, by the transformations

$$z_1 = \frac{z - a}{z + a}, \quad \zeta_1 = \frac{\zeta - a}{\zeta + a}.$$

Find their images on the  $z_1$ - and  $\zeta_1$ -planes, respectively.

Find the mapping between  $z_1$  and  $\zeta_1$  in the form of the power function

$$z_1 = \zeta_1^k,$$

where the value of  $k$  is to be determined in terms of  $\theta$ .

(7 marks)

- (ii) Write down the mapping between  $z$  and  $\zeta$ , and show that

$$\zeta = kz + \dots \quad \text{as } z \rightarrow \infty.$$

(3 marks)

*Question continues on the next page.*

- (iii) The complex potential in the auxiliary plane takes the form

$$W(\zeta) = \tilde{V}_\infty \left[ \zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}} \right] + \frac{\Gamma}{2\pi i} \ln \zeta.$$

Determine  $\tilde{V}_\infty$  in terms of  $V_\infty$  and  $k$ .

Show that for the Joukovskii-Kutta condition to be satisfied, we require that

$$\Gamma = -4\pi a(V_\infty/k) \sin \alpha.$$

(5 marks)

- (iv) Show that near the leading edge ( $z \rightarrow -a$ ), the complex conjugate velocity behaves as

$$u - iv = q(z + a)^\gamma,$$

where  $\gamma$  and  $q$  are constants that you are expected to find.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

MATH60001, MATH70001

Fluid Dynamics 1 (Solutions)

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1. (i) For the flow to be incompressible, the velocity field  $(u, v, w)$  must be divergence free,  $\text{div} \mathbf{V} = 0$  (the continuity equation), that is

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0,$$

and hence the flow is incompressible.

Since the flow is two-dimensional, the vorticity is in the  $z$ -direction, and that component is given by

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2},$$

and thus  $\boldsymbol{\omega} = 2\sqrt{2}\mathbf{k}$  with  $\mathbf{k}$  being the unit vector in the  $z$ -direction.

Substitution of  $(u, v, w)$  into the  $x$ - and  $y$ -momentum equations leads to

$$(x - \sqrt{2}y) + (\sqrt{2}x - y)(-\sqrt{2}) = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (x - \sqrt{2}y)\sqrt{2} + (\sqrt{2}x - y)(-1) = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \quad (1)$$

where we note that all viscous terms are identically zero (i.e. the flow is practically inviscid). The first equation is integrated with respect to  $x$  to give

$$p = \frac{1}{2}\rho x^2 + f(y),$$

where the function  $f(y)$  is determined by substituting the above into the second equation in (1), giving

$$\rho y = f'(y).$$

Thus we have  $f(y) = \frac{1}{2}\rho y^2 + p_0$ , and the pressure is given by

$$p = \frac{1}{2}\rho(x^2 + y^2) + p_0,$$

where  $p_0$  is a constant representing the pressure at  $(x, y, z) = (0, 0, 0)$ .

6, A

- (ii) The differential equations for  $x$  and  $y$  are *linear* and have *constant coefficients*, and can be cast into the standard form,

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}.$$

The solution is of exponential form, and can be found using the standard method, which sets

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{b} e^{\lambda t}.$$

Substitution into the system gives  $|\lambda I - A| = 0$ , that is,

$$(\lambda - 1)(\lambda + 1) + 2 = 0,$$

and so  $\lambda = \pm i$ . The solution for  $x$  can be written as

$$x = c_1 \cos t + c_2 \sin t.$$

It follows from the first equation in the system that

$$y = (x - \frac{dx}{dt})/\sqrt{2} = [(c_1 - c_2) \cos t + (c_2 + c_1) \sin t]/\sqrt{2}.$$

Impose the initial condition:

$$x_0 = c_1, \quad y_0 = (c_1 - c_2)/\sqrt{2},$$

from which we find  $c_1 = x_0$  and  $c_2 = (x_0 - \sqrt{2}y_0)$ . Thus we have

$$x = x_0 \cos t + (x_0 - \sqrt{2}y_0) \sin t = (\cos t + \sin t)x_0 - \sqrt{2} \sin t y_0, \quad (2)$$

$$y = y_0 \cos t + (\sqrt{2}x_0 - y_0) \sin t = \sqrt{2} \sin t x_0 + (\cos t - \sin t)y_0. \quad (3)$$

Clearly, both  $x$  and  $y$  are *periodic* functions of  $t$ , with the period  $t = 2\pi$ . Thus a particle would return to its original position with every time interval of  $2\pi$ .

The solution (2)-(3) represents the instantaneous position of a fluid particle which was at  $(x_0, y_0)$  when  $t = 0$ .

4, A

The line of bubble corresponds to the positions of all bubbles which were on  $x_0^2 + y_0^2 = 1$  at  $t = 0$ , imposing which on (2)-(3) allows us to determine the current shape of bubble line at  $t > 0$ . From (2)-(3), we can find  $(x_0, y_0)$ ,

$$x_0 = (\cos t - \sin t)x + \sqrt{2} \sin t y,$$

$$y_0 = -\sqrt{2} \sin t x + (\cos t + \sin t)y.$$

Use of the constraint  $x_0^2 + y_0^2 = 1$  leads to

$$[(\cos t - \sin t)^2 + 2 \sin^2 t]x^2 + [(\cos t + \sin t)^2 + 2 \sin^2 t]y^2 - 4\sqrt{2}(\sin^2 t)xy = 1.$$

This equation describes the shape of the bubble loop at time  $t$  (which can be shown to be an ellipse).

4, D

(iii) By its definition, the circulation corresponds to the line integral

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{r} = \oint_C [(x - \sqrt{2}y)dx + (\sqrt{2}x - y)dy].$$

In order to evaluate the integral, the contour, the unit circle here, is parameterised as  $x = \cos \theta$ ,  $y = \sin \theta$  ( $0 \leq \theta \leq 2\pi$ ), on which  $dx = -\sin \theta d\theta$  and  $dy = \cos \theta d\theta$ . Hence the line integral becomes a conventional definite integral,

$$\Gamma = \int_0^{2\pi} [(\cos \theta - \sqrt{2} \sin \theta)(-\sin \theta) + (\sqrt{2} \cos \theta - \sin \theta) \cos \theta] d\theta = 2\sqrt{2}\pi.$$

The circulation is related to the flux of the vorticity via the relation (which follows from Stoke's or Green's theorem)

$$\Gamma = \iint_S \omega \, dxdy = \iint_S (2\sqrt{2}) \, dxdy = 2\sqrt{2}\pi,$$

where we note that the area  $S$  enclosed by the unit circle is  $\pi$ .

4, B

The circulation at later times  $t > 0$  remains constant ( $\Gamma(t) = 2\sqrt{2}\pi$  conserved) according to **Kelvin's circulation theorem**, which is valid because (1) the line

consisting of the bubbles is a material line, and (2) the flow is actually inviscid (as was noted earlier).

Alternatively, the circulation can be calculated as follows. Let the bubble line at  $t > 0$  be denoted by  $\tilde{C}$ , and the enclosed region by  $\tilde{S}$ . Then

$$\Gamma(t) = \oint_{\tilde{C}} \mathbf{V} \cdot d\mathbf{r} = \iint_{\tilde{S}} \omega \, dxdy = (2\sqrt{2}) \iint_{\tilde{S}} dxdy.$$

Note that  $\tilde{S}$  is mapped from the initial region  $S$  through the trajectories  $(x_0, y_0) \rightarrow (x(t), y(t))$ , and so

$$\iint_{\tilde{S}} dxdy = \iint_S J \, dx_0 dy_0 = \iint_S dx_0 dy_0 = \pi,$$

where we use the fact that  $J = \left| \frac{\partial(x, y)}{\partial(x_0, y_0)} \right| = 1$  since the flow is assumed to be incompressible. The same result,  $\Gamma(t) = 2\sqrt{2}\pi$ , is obtained.

2, C

2. (i) Substitution of these into the given equations, (2a) and (2b), results in

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$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \Omega^2 r \sin^2 \vartheta - \alpha r, \quad (4a)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial \vartheta} = \Omega^2 r^2 \sin \vartheta \cos \vartheta. \quad (4b)$$

Equation (2c) is automatically satisfied as can easily be verified. Here we used the fact that the viscous terms vanish in all three momentum equations as is expected for motions in solid rotation.

Integrating (4a), we have

$$\frac{p}{\rho} = \frac{1}{2} \Omega^2 r^2 \sin^2 \vartheta - \frac{1}{2} \alpha r^2 + \Phi(\vartheta).$$

In order to find function  $\Phi(\vartheta)$ , we substitute the above into (4b) to obtain

$$\Phi'(\vartheta) = 0,$$

and hence  $\Phi(\vartheta) = C$  (constant). Thus the pressure everywhere inside the fluid (the Earth) is given by

$$p = \rho \left( \frac{1}{2} \Omega^2 r^2 \sin^2 \vartheta - \frac{1}{2} \alpha r^2 + C \right). \quad (5)$$

6, A

Since the fluid is in a state of solid rotation, there is no deformation (i.e. the rate-of-strain tensor is a null tensor), and hence on the surface of the fluid, the stress is simply the pressure, which should be zero, the pressure in the vacuum. This leads to the equation for the Earth surface as

$$\frac{1}{2} \Omega^2 R^2 \sin^2 \vartheta - \frac{1}{2} \alpha R^2 + C = 0. \quad (6)$$

Now the constant,  $C$ , can be found by using the condition that at the North pole ( $\vartheta = 0$ ) the radius  $R = R_0$ , which gives  $C = \frac{1}{2} \alpha R_0^2$ , with which the equation (6) for the Earth surface is

$$\frac{1}{2} \Omega^2 R^2 \sin^2 \vartheta - \frac{1}{2} \alpha R^2 + \frac{1}{2} \alpha R_0^2 = 0. \quad (7)$$

Solving this equation for  $R$ , we find that the Earth radius depends on the meridional angle  $\vartheta$  as

$$R = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha} \sin^2 \vartheta}}. \quad (8)$$

4, A

- (ii) In the  $(x, y)$  plane,  $z = 0$  and we have

$$R = \sqrt{x^2 + y^2}, \quad \sin \vartheta = y/R.$$

unseen ↓

Substituting the second equation into (8) leads

$$R^2 = R_0^2 / \left[ 1 - \frac{\Omega^2}{\alpha} \frac{y^2}{R^2} \right], \quad \text{i.e.} \quad R^2 - \frac{\Omega^2}{\alpha} y^2 = R_0^2,$$

which is arranged into

$$x^2/R_0^2 + \left( 1 - \frac{\Omega^2}{\alpha} \right) y^2/R_0^2 = 1.$$

This is the standard form of equation for ellipses, with the principal axes

$$a = R_0, \quad b = \frac{R_0}{\sqrt{1 - \Omega^2/\alpha}} > R_0.$$

Note that  $b/a$  increases with  $\Omega^2/\alpha$ , indicating that rotation flattens the Earth.

3, B

- (iii) Now determine  $R_0$  in terms of the mass of the Earth. The easiest is to use the formula for the volume of an ellipsoid,  $V = \frac{4}{3}\pi abc$ , where  $a$ ,  $b$  and  $c$  denote the three principal axes. For the present axisymmetric ellipsoid,

unseen ↓

$$a = R_0, \quad b = c = R_0 / \sqrt{1 - \frac{\Omega^2}{\alpha}}.$$

Thus the mass of the Earth (fluid in the ellipsoid),

$$M = \rho V = \frac{4}{3}\pi\rho R_0^3 \left[1 - \frac{\Omega^2}{\alpha}\right],$$

from which follows

$$R_0 = \left(\frac{3M}{4\pi\rho}\right)^{1/3} (1 - \Omega^2/\alpha)^{1/3}. \quad (9)$$

Alternatively, we calculate the mass (volume) by integrating over the  $(y, z)$  (equatorial) plane, where  $\vartheta = \pi/2$  and  $R = b$ . Let  $\tilde{r} = R \sin \vartheta (= y)$ . In terms of  $\tilde{r}$ , the mass of the Earth can be expressed as

$$M = 2\rho \int_0^b (\tilde{r} / \tan \vartheta) (2\pi \tilde{r}) d\tilde{r}, \quad (10)$$

while equation (7) can be written as

$$\frac{1}{2}\Omega^2 \tilde{r}^2 - \frac{1}{2}\alpha(\tilde{r} / \sin \vartheta)^2 + \frac{1}{2}\alpha R_0^2 = 0.$$

This is solved to obtain

$$\sin \vartheta = \frac{\sqrt{\alpha} \tilde{r}}{\sqrt{\Omega^2 \tilde{r}^2 + \alpha R_0^2}},$$

and it follows that

$$\cos \vartheta = \frac{\sqrt{(\Omega^2 - \alpha)\tilde{r}^2 + \alpha R_0^2}}{\sqrt{\Omega^2 \tilde{r}^2 + \alpha R_0^2}}, \quad 1/\tan \vartheta = \sqrt{(\Omega^2 - \alpha)\tilde{r}^2 + \alpha R_0^2} / (\sqrt{\alpha} \tilde{r}).$$

Use of the last expression in (10) gives

$$\begin{aligned} M &= \rho(4\pi/\sqrt{\alpha}) \int_0^b \tilde{r} \sqrt{(\Omega^2 - \alpha)\tilde{r}^2 + \alpha R_0^2} d\tilde{r} = \frac{4\pi\rho}{3\sqrt{\alpha}(\Omega^2 - \alpha)} \left[ (\Omega^2 - \alpha)\tilde{r}^2 + \alpha R_0^2 \right]^{3/2} \Big|_0^b \\ &= \frac{4\pi\rho}{3\sqrt{\alpha}(\Omega^2 - \alpha)} \left[ [(\Omega^2 - \alpha)b^2 + \alpha R_0^2]^{3/2} - \alpha^{3/2} R_0^3 \right] = \frac{4\pi\rho(\alpha^{3/2} R_0^3)}{3\sqrt{\alpha}(\alpha - \Omega^2)}, \end{aligned}$$

from which follows the same result (9).

4, D

- (iv) The expression for  $R$  indicates that the Earth is nearly spherical when  $\Omega^2/\alpha$  is small. As  $\Omega^2/\alpha$  increases, the deviation from sphere becomes significant. This would make the gravitational force deviate from the radial direction, an assumption made above. Indeed, the results obtained in (i)–(ii) suggests that as  $\Omega^2/\alpha \rightarrow 1$ , the planet would acquire a disc-like shape, in which case, the gravitational force would primarily be directed perpendicularly to and towards the axis of the rotation.

unseen ↓

3, D

3. (a) Substituting the constitutive relation into the momentum equations and differentiating using the Chain Rule, we obtain

sim. seen ↓

$$\rho \left[ \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right] = \frac{\partial p_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ij} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial V_i}{\partial x_j} \right) + \frac{d\mu}{dT} \frac{\partial T}{\partial x_j} \frac{\partial V_j}{\partial x_i} + \mu \frac{\partial^2 V_j}{\partial x_j \partial x_i}.$$

The last term vanishes since  $\frac{\partial^2 V_j}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_i} \left( \frac{\partial V_j}{\partial x_j} \right)$  and the continuity equation  $\frac{\partial V_j}{\partial x_j} = 0$  holds.

4, A

sim. seen ↓

- (b) For a steady flow,  $\frac{\partial V_i}{\partial t} = 0$ . Since  $u = u(x, y, z)$  is the only non-zero velocity component, it follows immediately from the continuity equation that

$$\frac{\partial u}{\partial x} = 0,$$

implying that  $u$  is independent of  $x$ . The  $y$ - and  $z$ -momentum equations reduce to

$$-\frac{\partial p}{\partial y} + \frac{d\mu}{dT} \frac{\partial T}{\partial x} \frac{\partial u}{\partial y} = 0, \quad -\frac{\partial p}{\partial z} + \frac{d\mu}{dT} \frac{\partial T}{\partial x} \frac{\partial u}{\partial z} = 0.$$

Since  $\partial T / \partial x = 0$ , we have  $\partial p / \partial y = 0$  and  $\partial p / \partial z = 0$ , implying that the pressure  $p$  could only be a function of  $x$ .

The momentum equation in the  $x$ -direction reduces to

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) + \frac{d\mu}{dT} \frac{\partial T}{\partial x} \frac{\partial u}{\partial x} = 0.$$

Since the last term on the left-hand side vanishes, the equation is the required one,

$$\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) = \frac{\partial p}{\partial x}.$$

The left-hand side depends only on  $y$  and  $z$ , whereas the right-hand side is a function of  $x$  only. It may therefore be inferred that both sides and hence the pressure gradient  $\partial p / \partial x$  must be a constant.

4, A

unseen ↓

- (c) (i) As the plates are infinitely large,  $u$  must be independent of  $z$ . Noting also that  $T$  is a function of  $y$  only, the equation for  $u$  reduces to

$$\frac{\partial}{\partial y} \left( (\mu_0 + \mu_1 y) \frac{\partial u}{\partial y} \right) = 0,$$

where we have set  $\partial p / \partial x = 0$ . It follows that

$$\frac{\partial u}{\partial y} = \frac{C_1}{\mu_0 + \mu_1 y}.$$

Integrating and using the boundary condition that  $u = 0$  at  $y = 0$ , we obtain

$$u = \frac{1}{\mu_1} \int_0^y \frac{C_1}{(y + \mu_0 / \mu_1)} dy = \frac{C_1}{\mu_1} \ln \left( \frac{y + \mu_0 / \mu_1}{\mu_0 / \mu_1} \right) = \frac{C_1}{\mu_1} \ln \left( 1 + (\mu_1 / \mu_0) y \right).$$

Now determine the integration constant  $C_1$  by imposing the boundary condition that  $u = U$  at  $y = h$ :

$$\frac{C_1}{\mu_1} \ln \left( 1 + (\mu_1 / \mu_0) h \right) = U,$$

which gives  $C_1/\mu_1$ , and using the latter the solution for  $u$  is written as

$$u = \frac{\ln(1 + y\mu_1/\mu_0)}{\ln(1 + h\mu_1/\mu_0)} U.$$

When  $h\mu_1/\mu_0 \ll 1$ , by Taylor expansion we have

$$\begin{aligned}\ln(1 + h\mu_1/\mu_0) &= h\mu_1/\mu_0 + O(h\mu_1/\mu_0), \\ \ln(1 + y\mu_1/\mu_0) &= y\mu_1/\mu_0 + O(y\mu_1/\mu_0),\end{aligned}$$

since  $0 \leq y \leq h$ . It follows that

$$u = Uy/h,$$

which represents the solution when the viscosity is constant.

6, B

(ii) The stress tensor at the lower plate has components:

unseen ↓

$$\begin{aligned}p_{11} &= p_{22} = p_{33} = -p, \quad p_{13} = p_{23} = 0; \\ p_{12}|_{y=0} &= \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \frac{\mu_0(\mu_1/\mu_0)}{\ln(1 + h\mu_1/\mu_0)} U = \frac{\mu_1}{\ln(1 + h\mu_1/\mu_0)} U.\end{aligned}$$

The lower surface at  $y = 0$  has unit normal direction  $\mathbf{n} = (0, 1, 0)$ . The shear stress of the fluid acting on the lower plate is given by  $p_{ij}n_j$ , which is

$$(p_{12}|_{y=0}, -p, 0). \quad (11)$$

The stress tensor at the upper plate at  $y = h$  has components:

$$\begin{aligned}p_{11} &= p_{22} = p_{33} = -p, \quad p_{13} = p_{23} = 0; \\ p_{12}|_{y=h} &= \mu \frac{\partial u}{\partial y} \Big|_{y=h} = (\mu_0 + \mu_1 h) \frac{\mu_1/\mu_0}{\ln(1 + h\mu_1/\mu_0)(1 + h\mu_1/\mu_0)} U \\ &= \frac{\mu_1}{\ln(1 + h\mu_1/\mu_0)} U.\end{aligned}$$

The upper surface at  $y = h$  has unit normal direction  $\mathbf{n} = (0, -1, 0)$ . The shear stress acting on the upper plate is given by  $p_{ij}n_j$ , which is

$$(-p_{12}|_{y=h}, p, 0). \quad (12)$$

Note that

$$p_{12}|_{y=0} = p_{12}|_{y=h}.$$

The stresses that the lower and upper plates exert on the fluid are of opposite sign to (11) and (12) respectively. The total viscous force in the  $x$ -direction is

$$(-p_{12}|_{y=0} + p_{12}|_{y=h})L = 0, \quad (13)$$

indicating that the the viscous forces themselves are in balance. This happens because the pressure gradient is zero so that the pressure forces acting on the cross sections at  $x = 0$  and at  $x = L$  are in balance. These forces are  $p|_{x=0}h$  and  $-p|_{x=L}h$  respectively, the sum of which is

$$hp|_{x=0} - hp|_{x=L} = 0.$$

It is worth noting that for  $0 \leq y \leq y$ ,

$$p_{12} = \mu \frac{\partial u}{\partial y} = (\mu_0 + \mu_1 y) \frac{\partial u}{\partial y} = C_1(\text{constant}).$$

6, C

4. (i) Under the mapping, the horizontal wall  $y = 0$  is mapped to  $\Im(\zeta) = 0$  (on the anticipation that  $\alpha > 0$ ). The other wall  $z = re^{i(\pi-\theta)}$  is mapped to  $\zeta = r^\alpha e^{i\alpha(\pi-\theta)}$ . Thus we need to choose  $\alpha$  such that  $\zeta$  is on the negative real axis on the complex  $\zeta$ -plane, that is,  $\alpha(\pi - \theta) = \pi$ , which gives

$$\alpha = \pi/(\pi - \theta).$$

The complex potential

$$w(z) = V_\infty \zeta = V_\infty z^{\frac{\pi}{\pi-\theta}}.$$

The complex conjugate velocity is thus found as

$$u - iv = \frac{dw}{dz} = \frac{\pi V_\infty}{\pi - \theta} z^{\frac{\theta}{\pi-\theta}}, \quad \text{and so} \quad V = |u - iv| = \frac{\pi V_\infty}{\pi - \theta} r^{\frac{\theta}{\pi-\theta}}.$$

From the Bernoulli equation follows the pressure

$$p = p_0 - \frac{1}{2} \rho \frac{(\pi V_\infty)^2}{(\pi - \theta)^2} r^{\frac{2\theta}{\pi-\theta}}.$$

- (ii) The position of the source,  $z = de^{i(\pi-\theta)/2}$ , is mapped to  $\zeta = d^{\frac{\pi}{\pi-\theta}} e^{i\pi/2} = d^\alpha i \equiv \tilde{d}$ , which is on the imaginary axis on the upper half of the  $\zeta$ -plane. In order to satisfy the impermeability condition at the wall on  $\Im(\zeta) = 0$ , an image source with the same strength  $q$  must be introduced at  $\tilde{d}^* = -d^\alpha i$ . The resulting complex potential on  $\zeta$ -plane is

$$W(\zeta) = V_\infty \zeta + \frac{q}{2\pi} \ln(\zeta - d^\alpha i) + \frac{q}{2\pi} \ln(\zeta + d^\alpha i) = V_\infty \zeta + \frac{q}{2\pi} \ln(\zeta^2 + d^{2\alpha}).$$

Substitution of  $\zeta = z^\alpha$  into the above gives the corresponding complex potential on physical plane,

$$w(z) = V_\infty z^\alpha + \frac{q}{2\pi} \ln(z^{2\alpha} + d^{2\alpha}). \quad (14)$$

For the special case  $\theta = \pi/2$ ,  $\alpha = 2$  and so the complex potential (14) becomes

$$w(z) = V_\infty z^2 + \frac{q}{2\pi} \ln(z^4 + d^4).$$

The physical region corresponds to a quarter plane. The physical source is at  $z = de^{i\pi/4} = (1+i)d/\sqrt{2}$ . Images at  $(1-i)d/\sqrt{2}$ ,  $(-1+i)d/\sqrt{2}$  and  $(-1-i)d/\sqrt{2}$  are introduced in order to satisfy the impermeability conditions on both horizontal and vertical walls. Thus the complex potential is

$$\begin{aligned} w(z) &= V_\infty z^2 + \frac{q}{2\pi} \left\{ \ln \left[ z - (1+i)d/\sqrt{2} \right] + \ln \left[ z - (1-i)d/\sqrt{2} \right] \right. \\ &\quad \left. + \ln \left[ z - (-1+i)d/\sqrt{2} \right] + \ln \left[ z + (1+i)d/\sqrt{2} \right] \right\} \\ &= V_\infty z^2 + \frac{q}{2\pi} \ln(z^4 + d^4), \end{aligned}$$

which is the same as that obtained by the method of conformal mapping.

- (iii) For  $\theta = 0$ ,  $\alpha = 1$  and so

$$w(z) = V_\infty z + \frac{q}{2\pi} \ln(z^2 + d^2),$$

and the complex conjugate velocity is given by

$$u - iv = V_\infty + \frac{q}{\pi} \frac{z}{(z^2 + d^2)}.$$

To find stagnation points, we set  $u - iv = 0$ :

$$V_\infty + \frac{q}{\pi} \frac{z}{(z^2 + d^2)} = 0 \quad \text{i.e.} \quad z^2 + \frac{q}{\pi V_\infty} z + d^2 = 0,$$

from which we obtain

$$z_s^\pm = \frac{1}{2} \left[ -\frac{q}{\pi V_\infty} \pm \sqrt{\left(\frac{q}{\pi V_\infty}\right)^2 - 4d^2} \right].$$

Interestingly, there exist two stagnation points even in the limit of  $d \rightarrow 0$ , but only one stagnation is present if we set  $d = 0$  in  $w(z)$ .

3, B

A possible body shape is represented by the streamline going through a stagnation point. Recalling that streamlines correspond to contours of the stream function, we set  $z = x + iy$  and write

$$w(z) = V_\infty(x + iy) + \frac{q}{2\pi} \ln(x^2 - y^2 + d^2 + 2ixy).$$

The imaginary part of  $w(z)$  gives the stream function

$$\psi(x, y) = V_\infty y + \frac{q}{2\pi} \theta_2,$$

where

$$\theta_2(x, y) = \arg(x^2 - y^2 + d^2 + 2ixy) = \tan^{-1} \left[ \frac{2xy}{x^2 - y^2 + d^2} \right].$$

When  $d < q/(2\pi V_\infty)$ , both stagnation points are on the real axis:  $z_s^\pm \equiv x_s^\pm$  with  $x_s^\pm < 0$ . As  $(x, y) \rightarrow (x_s^\pm, 0^+)$ ,  $\theta_2 \rightarrow 2\pi$ . The streamline going through the (front) stagnation point  $(x_s^-, 0)$  is

$$V_\infty y + \frac{q}{2\pi} \theta_2 = \psi(x_s^-, 0) = q,$$

which also goes through the (rear) stagnation point  $(x_s^+, 0)$ . The sources, at  $(0, \pm d)$ , are outside the region enclosed by this streamline, i.e. within the flow field. The  $w(z)$  does not represent a flow past the body with the shape given by the streamline.

When  $d > q/(2\pi V_\infty)$ ,  $z_s^\pm$  are complex conjugates. Due to symmetry, we consider  $z_s^+ = \left(-\frac{q}{2\pi V_\infty}, \sqrt{d^2 - \left(\frac{q}{2\pi V_\infty}\right)^2}\right) \equiv (x_s^+, y_s^+)$ , the streamline going through which is given by

$$V_\infty y + \frac{q}{2\pi} \theta_2 = \psi(x_s^+, y_s^+) = V_\infty \sqrt{d^2 - \left(\frac{q}{2\pi V_\infty}\right)^2} + \frac{q}{2\pi} \theta_{2,s}^+,$$

where

$$\theta_{2,s}^+ = \arg \left[ \left(\frac{q}{2\pi V_\infty}\right)^2 - \frac{iq}{2\pi V_\infty} \sqrt{d^2 - \left(\frac{q}{2\pi V_\infty}\right)^2} \right].$$

In the limit of  $d \gg 1$ ,  $z_s^+$  approaches  $(-q/(2\pi V_\infty), d)$ , the stagnation point for an isolated source, and the complex potential would represent the flow past a so-called Rankine body. For moderate  $d > q/(2\pi V_\infty)$ , we anticipate that  $w(z)$  would represent the flow past two bodies placed in the upper and lower planes, being represented by the streamlines going the stagnation points  $(x_s^+, y_s^+)$  and  $(x_s^-, y_s^-)$ , respectively.

5, D

5. (i) Consider the mapping of the physical  $z$ -plane onto  $z_1$ -plane,

sim. seen ↓

$$z_1 = \frac{z - a}{z + a}, \quad (15)$$

which is a linear-fractional transformation. An important fact about it is the 'circle property': it maps circles or circular arcs into circles in the extended complex plane. In particular, we note that (15) maps  $z = a$  to  $z_1 = 0$ , while  $z = -a$  is mapped to  $z_1 = \infty$ . This means that the upper and lower surfaces of the aerofoil are mapped onto two rays emanating from  $z_1 = 0$ .

To determine the orientation of the rays on the  $z_1$ -plane, recall that the angle  $\delta$  of rotation of any line when mapped from  $z$ -plane onto  $z_1$ -plane is given by

$$\delta = \arg \left( \frac{dz_1}{dz} \right).$$

Differentiating (15), we find

$$\frac{dz_1}{dz} = \frac{2a}{(z + a)^2}.$$

In particular, at the trailing edge of the aerofoil there is  $\frac{dz_1}{dz} \Big|_{z=a} = \frac{1}{2a}$ , which is a real positive quantity, i.e.  $\delta = 0$ . This means that the tangents to the upper and lower sides of the aerofoil do not experience any rotation. Their images on the  $z_1$ -plane remain being two rays, which are symmetric about the real axis (see figure 1). Note that the ray on the upper half plane is the image of the circular arc, and it makes an angle of  $\pi - \theta$  with the positive real axis. The ray on the lower half plane is the image of the lower surface of the aerofoil and it makes an angle of  $-(\pi - \theta)$  with the real axis. The interior of the airfoil is mapped onto the wedge on the left plane.

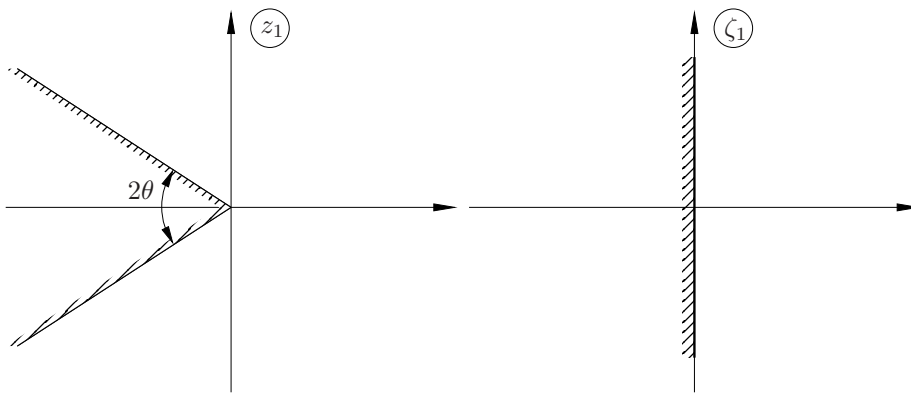


Figure 1: Images on  $z_1$ -plane (left) and  $\zeta_1$ -plane (right).

Similar to  $z_1$ , the mapping

$$\zeta_1 = \frac{\zeta - a}{\zeta + a}, \quad (16)$$

which is also linear-fractional transformation, maps the circle on the  $\zeta$ -plane onto a straight line, the angle between which and the positive real axis remains  $\pi/2$  as is shown in figure 1 (right plot).

It remains to establish the mapping

$$z_1 = \zeta_1^k. \quad (17)$$

The power  $k$  in (17) must be chosen to ensure that (17) provides proper correspondence of the boundaries of the regions in the  $z_1$ - and  $\zeta_1$ -planes. In particular, a point on the ray on the upper  $z_1$ -plane can be written as

$$z_1 = |z_1|e^{i(\pi-\theta)}, \quad (18)$$

while a point on the positive imaginary semi-axis on the  $\zeta_1$ -plane can be written as

$$\zeta_1 = |\zeta_1|e^{i\pi/2}. \quad (19)$$

Substitution of (18) and (19) into (17) results in  $|z_1|e^{i(\pi-\theta)} = |\zeta_1|^k e^{i\pi k/2}$ , which requires

$$(\pi - \theta) = \pi k/2 \quad \text{i.e.} \quad k = 2(1 - \theta/\pi).$$

3, M

(ii) Substituting (15) and (16) into (17) leads to the mapping between  $z$  and  $\zeta$

sim. seen ↓

$$\frac{z-a}{z+a} = \left( \frac{\zeta-a}{\zeta+a} \right)^k. \quad (20)$$

This may be viewed as a generalised Joukovskii transformation with the conventional one corresponding to  $k = 2$ . Write (20) as

$$\frac{1-a/z}{1+a/z} = \left( \frac{1-a/\zeta}{1+a/\zeta} \right)^k. \quad (21)$$

For  $z \gg 1$  and  $\zeta \gg 1$ , neglecting the  $O[(a/z)^2]$  and  $O[(a/\zeta)^2]$  terms, we have

$$\frac{1-a/z}{1+a/z} = (1-a/z)(1-a/z+\dots) = 1 - 2\frac{a}{z} + \dots,$$

$$\left( \frac{1-a/\zeta}{1+a/\zeta} \right)^k = \left( 1 - 2\frac{a}{\zeta} + \dots \right)^k = 1 - 2k\frac{a}{\zeta} + \dots.$$

Substituting these into (21), we find that

$$\zeta = kz + \dots \quad \text{as} \quad z \rightarrow \infty. \quad (22)$$

3, M

(iii) In order to determine  $\tilde{V}_\infty$ , we note that

sim. seen ↓

$$\frac{dW}{d\zeta} = \tilde{V}_\infty \left( e^{-i\alpha} - \frac{a^2}{\zeta^2} e^{i\alpha} \right) + \frac{\Gamma}{2\pi i \zeta} \rightarrow \tilde{V}_\infty e^{-i\alpha} \quad \text{as} \quad \zeta \rightarrow \infty.$$

At any point in the physical plane, the complex conjugate velocity

$$\overline{V}(z) = \frac{dw}{dz} = \frac{dW}{d\zeta} \frac{d\zeta}{dz} \rightarrow \tilde{V}_\infty e^{-i\alpha} k \quad \text{as} \quad \zeta \rightarrow \infty, \quad (23)$$

where use has been made of (22). Far from the aerofoil, the complex conjugate velocity in the physical plane is  $V_\infty e^{-i\alpha}$ . It follows that

$$\tilde{V}_\infty k = V_\infty \quad \text{i.e.} \quad \tilde{V}_\infty = V_\infty / k.$$

The complex potential in the auxiliary plane then reads

$$W(\zeta) = \frac{V_\infty}{k} \left( \zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}} \right) + \frac{\Gamma}{2\pi i} \ln \zeta. \quad (24)$$

The Joukovskii-Kutta condition implies that, in the physical  $z$ -plane, the flow should leave the aerofoil at its trailing edge with a finite velocity. The trailing edge  $z = a$  corresponds to  $\zeta = a$ . For the transformation (20),

$$\frac{dz}{d\zeta} \rightarrow k(2a)^{-k+1}(\zeta - a)^{k-1},$$

which indicates that  $\frac{dz}{d\zeta}$  vanishes at  $\zeta = a$  for  $k > 1$ , which is the case for  $\theta < \pi/2$ .

In order to render  $\bar{V}(z) = \frac{dW}{d\zeta} / \left( \frac{dz}{d\zeta} \right)$  finite, it is required that

$$\left. \frac{dW}{d\zeta} \right|_{\zeta=a} = \frac{V_\infty}{k} \left( e^{-i\alpha} - \frac{a^2}{a^2} e^{i\alpha} \right) + \frac{\Gamma}{2\pi i a} = 0.$$

which determines  $\Gamma$ :

$$\Gamma = -4\pi a(V_\infty/k) \sin \alpha.$$

5, M

(iv) Near the leading edge ( $z \rightarrow -a$  and  $\zeta \rightarrow -a$ ), the transformation (20) behaves as

unseen ↓

$$z + a \sim (-2a)^{-k+1}(\zeta + a)^k \quad \text{i.e.} \quad \zeta + a \sim (-2a)^{1-1/k}(z + a)^{1/k}.$$

It follows that

$$\begin{aligned} \frac{d\zeta}{dz} &\rightarrow k^{-1}(-2a)^{1-1/k}(z + a)^{-1+1/k}; \\ \bar{V}(z) \Big|_{z \rightarrow -a} &\rightarrow \left. \frac{dW}{d\zeta} \right|_{\zeta=-a} \frac{d\zeta}{dz} \rightarrow \left[ \frac{V_\infty}{k} \left( e^{-i\alpha} - \frac{a^2}{a^2} e^{i\alpha} \right) - \frac{\Gamma}{2\pi i a} \right] \\ &= \frac{i\Gamma}{\pi a} k^{-1} (-2a)^{1-1/k} (z + a)^{-1+1/k} \equiv q (z + a)^\gamma, \end{aligned}$$

where

$$\gamma = -1 + 1/k, \quad q = \frac{i\Gamma}{\pi a} k^{-1} (-2a)^{1-1/k} = -4i(V_\infty/k^2)(-2a)^{1-1/k} \sin \alpha.$$

Noting that  $\gamma = -1 + 1/k < 0$  for  $k > 1$ , we conclude that the velocity at the leading edge is singular.

5, M

**Review of mark distribution:**

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

Question   Marker's comment

- 1 Question 1 is rather long and algebraically involved. It also tests understanding of several concepts. There was impressive performance by a few students, but quite a few found parts of the questions challenging. This fact will be taken into account when setting Pass and Maximum marks. Given the length and difficulty level, the overall outcome was in line with expectation.
- 2 Question 2 was done well by the majority of students.
- 3 Very good performance by some, but quite a few either could not handle the case of temperature-dependent viscosity, or had difficulty in solving the differential equation correctly.
- 4 Question 4 is long, and Part (iii) is hard. The majority did well on parts (i) and (ii), but very few did substantial work on Part (iii). This fact will be taken into account when setting Pass and Maximum marks.

Question   Marker's comment

- 1 Question 1 is rather long and algebraically involved. It also tests understanding of several concepts. There was impressive performance by a few students, but quite a few found parts of the questions challenging. This fact will be taken into account when setting Pass and Maximum marks. Given the length and difficulty level, the overall outcome was in line with expectation.
- 2 Question 2 was done well by the majority of students.
- 3 Very good performance by some, but quite a few either could not handle the case of temperature-dependent viscosity, or had difficulty in solving the differential equation correctly.
- 4 Question 4 is long, and Part (iii) is hard. The majority did well on parts (i) and (ii), but very few did substantial work on Part (iii). This fact will be taken into account when setting Pass and Maximum marks.
- 5 Question 5 involved a unfamiliar transformation. Very good performance by a few who must have prepared thoroughly and understood the concepts in depth. Many could not tackled the hard parts. The question was testing but fair. The outcome was in line with expectation.