Applied Complex Analysis - Lecture Two

Andrew Gibbs

January 2025

Some questions:

- How will the mastery level be evaluated?
- How can I catch up on complex analysis?
- How much coding will there be in the coursework?

Summary

- In the previous lecture, we reviewed complex *numbers*
- We saw that many properties of standard arithmetic generalise in a natural way.
- Issues arise with complex argument, which carry over to radicals, multiplying complex numbers, etc
- This time, we will review complex *functions*
- ... and contour integrals.

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The triangle inequality

For $z = a + bi$:

$$||a| - |b|| \leq |z| \leq ||a| + |b||$$

For $z_1, z_2 \in \mathbb{C}$:

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq ||z_1| + |z_2||$$

Proof: exercise.

Functions of complex variables

Functions of complex variables

Any function of a complex variable can be written as:

$$f(z) = u(x,y) + iv(x,y)$$

Elementary functions generalise to complex variables.

- Polynomials
- Rational functions
- The exponential function...

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The exponential function

Familiar Taylor expansion holds for complex variables (more on this later):

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Conjugation (of Euler's identity) provides trig identities:

$$\cos \theta = \frac{e^{i\theta} + e^{i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{i\theta}}{2i}$$

Periodicity (not enjoyed by real-valued exp)

$$e^{i\theta + 2\pi k i} = e^{i\theta}, \quad k \in \mathbb{Z}$$

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The complex logarithm

Due to periodicity of $\exp$, if defined as the inverse of $\exp$, $\log$ is multi-valued:

$$\log z = \log \left(|z|e^{i\arg z}\right) = \log |z| + i \arg z + 2\pi i j, \quad \text{for } j \in \mathbb{Z}.$$

- The inverse is an example of a *multi-valued* function.
- Mathematica demo
- This is distinct from $\exp(x)$ for $x \in \mathbb{R}$, which has a unique single-valued inverse $\log(x)$ for $x > 0$.
- Actually, $\exp z$ has a single-valued inverse for z in the complex strip $\operatorname{Im} z \in [-\pi, \pi)$.
- Due to $\arg$, not all real-valued properties of $\log$ generalise
- Can define $z^\alpha := \exp(\alpha \log z)$, also multi-valued for $\alpha \notin \mathbb{Z}$.

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Analytic/Holomorphic functions

This is where the magic happens. All of our complex analysis applications require this property. There are multiple equivalent definitions.

- Holomorphic: Complex differentiable in a neighbourhood of a point
- Satisfying the Cauchy Riemann equations
- Analytic: Has a locally convergent Taylor expansion
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The complex derivative

A function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at z if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is independent of the path taken for the limit, noting $h \in \mathbb{C}$.

- We say f is holomorphic (equiv. analytic) in $\Omega \subset \mathbb{C}$ if f is complex differentiable for all $z \in \Omega$.

- The complex derivative satisfies all of the same rules as the real derivative, nice!

- Before exploring theoretical consequences, let's consider some examples...

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Are these functions analytic?

- $f(z) = \text{Re}\{z\}$

- $f(z) = z^2$

- $f(z) = \overline{z}$

- $f(z) = \exp(az)$

- $f(z) = \sin(z)$

- Rational functions $P(z)/Q(z)$ for polynomials P, Q

- Compositions and products of analytic functions

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Domain of analyticity

- We have seen friendly looking functions which are not analytic anywhere

- If a $f(z)$ is analytic for all $z \in \mathbb{C}$, we say f is 'entire'

- Most functions we are interested are analytic in some non-trivial open $\Omega \subset \mathbb{C}$, called the 'domain of analyticity'.

- Conventionally, mathematicians say 'f is analytic' to mean 'f is analytic in some domain Ω'. This can be ambiguous, so it is often best to clarify Ω, if any confusion might arise.

- Confusingly - we will see later that for many f, we may have multiple choices for Ω, but we cannot take the union of these choices!

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Cauchy-Riemann equations

For $f(z) = u(x, y) + iv(x, y)$, the complex derivative $f'(z)$ exists if and only if the following hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Harmonic functions

For $\phi(x, y)$ real-valued, we say ϕ is *Harmonic* if:

$$\Delta\phi := \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

- There is a beautiful connection between *harmonic* functions and *analytic* functions: If $f(x, y) = u(x, y) + iv(x, y)$ is *analytic*, then u and v are *harmonic*.

- This means that complex analysis techniques can be used to solve Laplace problems, involving $\Delta\phi = 0$, such as ideal fluid flow and electrostatic potential. More on this later!

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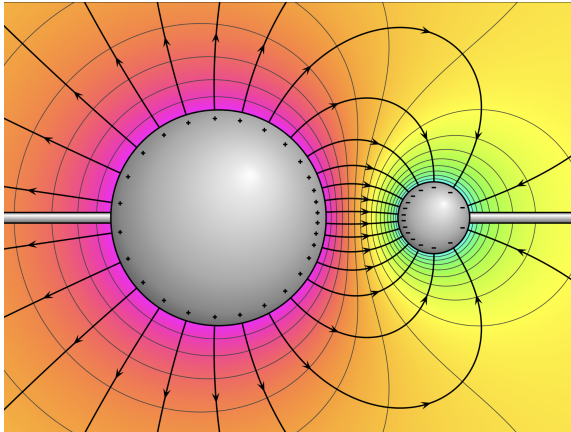For $\phi(x, y)$ real-valued, we say ϕ is *Harmonic* if:

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Ideal fluid flow
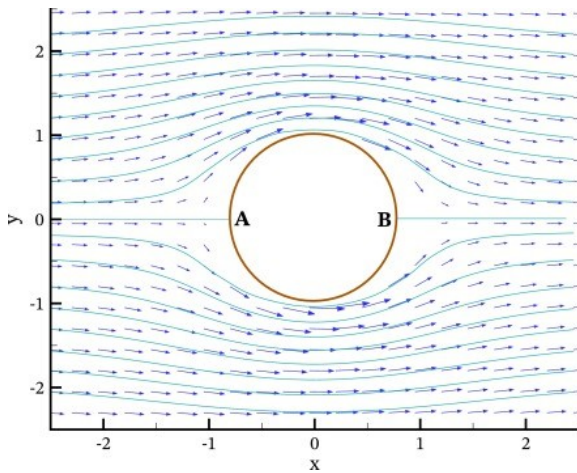
Electrostatic potential

Spoiler alert

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- Contour integrals of...
- analytic functions.

So let's talk about contour integrals!

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Contour Integrals

Contour integrals

Given a curve $\gamma = \{z(t) : t \in [a, b]\}$, we can define the contour integral

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

- Direction matters! If $\tilde{\gamma}$ is parametrised in the opposite direction, then

$$\int_{\tilde{\gamma}} f(z)dz = -\int_\gamma f(z)dz$$

- If γ is the disjoint union of γ_1 and γ_2, then

$$\int_{\gamma_1 + \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

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The ML principle

For a piecewise differentiable curve γ in the complex plane and a complex function $f(z)$, we have

$$\left| \int_\gamma f(z)dz \right| \leq \max_{z \in \gamma} \{|f(z)|\} \times \text{length}(\gamma).$$

- Proof uses definition of length of a contour.
- Example: $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz$ for $|f(z)| = O(z^{-3/2})$ as $|z| \to \infty$.
- Example: $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz$ for $f(z) = g(z)e^{iz}$, $|g(z)| = O(z^{-1/2})$ as $|z| \to \infty$.

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The ML principle

For a piecewise differentiable curve γ in the complex plane and a complex function $f(z)$, we have

$$\left| \int_\gamma f(z)dz \right| \leq \max_{z \in \gamma}\{|f(z)|\} \times \text{length}(\gamma).$$

- Proof uses definition of length of a contour.
- Example: $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz$ for $|f(z)| = O(z^{-3/2})$ as $|z| \to \infty$.
- Example: $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz$ for $f(z) = g(z)e^{iz}$, $|g(z)| = O(z^{-1/2})$ as $|z| \to \infty$.

Jordan's Lemma

For a function of the form $f(z) = e^{iaz}g(z)$, defined on a contour $\gamma_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, such that $g(z) \leq M_R$ for $z \in \gamma_R$, $a > 0$, we have

$$\left| \int_{\gamma_R} f(z)dz \right| \leq \frac{\pi}{a} M_R.$$

- Proof is a little more involved than ML.

- When applicable, this is a stronger result than the ML principle.

- Revisit previous example: $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz$ for $f(z) = g(z)e^{iz}$, $|g(z)| = O(z^{-1/2})$ as $|z| \to \infty$.

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Cauchy's Integral Theorem

Theorem:

Suppose f is analytic/holomorphic in some simply connected (no holes) open $D \subset \mathbb{C}$. For a closed contour $\gamma \subset D$, we have

$$\oint_\gamma f(z)dz = 0$$

(Proof omitted.)

Corollary:

If γ_1 and γ_2 have the same endpoints, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

- Proof
- Contour integrals of analytic functions depend only on the endpoints!
- For integrals defined over some contour γ_1, we can move the contour to another contour γ_2 which is more convenient!

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