1. The exponential function can be characterised in many equivalent different ways. For example, if we assume there exists a function $f : \mathbb{R} \to \mathbb{R}$, satisfying:

   - $f(x + y) = f(x) \cdot f(y), \quad \forall x, y, \in \mathbb{R}$, and
   - f is differentiable at 0 with $f'(0) = 1$,

   then we can prove such a function must be unique by following these steps:

   (a) Show that $f(0) = 1$ and $f(x) \neq 0$, for all $x \in \mathbb{R}$.

   (b) Show that f is differentiable everywhere and $f'(x) = f(x)$, for all $x \in \mathbb{R}$.

   (c) Prove that f must be unique. *Hint: assume there is another function g satisfying the two items at the beginning. Is $\frac{f(x)}{g(x)}$ well-defined? Is it differentiable? If so, what is its derivative?*

   In addition, we can derive several useful properties of f:

   (d) Argue that $f(x) > 0$, for all $x \in \mathbb{R}$, and conclude that f is strictly increasing. *Hint: for the first part use item (a) and Bolzano's theorem.*

   (e) Show that for $q \in \mathbb{Q}$, $f(q) = f(1)^q$ and conclude that the image of f is the interval $(0, \infty)$.

   Thanks to the uniqueness we can now define the constant e as $e := f(1)$. Similarly, now the notation $f(x) = e^x$, for all $x \in \mathbb{R}$, is meaningful. However, note that we do not know yet that a function like f exists. Its existence will be proven once we define the concept of integral. At this point, we are only showing that only one such function can exist.

   Solution.

   (a) Note that $f(0) = f(0 + 0) = f(0) \cdot f(0) = f(0)^2 \geq 0$. If $f(0) = 0$, this would imply that $f(x) = f(x + 0) = f(x) \cdot f(0) = 0$, for all $x \in \mathbb{R}$, which would make f a constant function, contradicting $f'(0) = 1$. Therefore, it must be that $f(0) > 0$ and from the equality $f(0) = f(0)^2$ we conclude $f(0) = 1$. From $1 = f(0) = f(x + (-x)) = f(x) \cdot f(-x)$ we conclude that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

   (b) Using what we know so far, we can write:
   $$\frac{f(x+h) - f(x)}{h} = \frac{f(x)f(h) - f(x)}{h} = f(x)\frac{f(h) - 1}{h} = f(x)\frac{f(h) - f(0)}{h}.$$
   Taking the limit when $h \to 0$, we conclude that $f'(x) = f(x)$.

   (c) If another function g satisfies the items at the beginning then for all $x \in \mathbb{R}$, $g'(x) = g(x) \neq 0$. Therefore $\frac{f(x)}{g(x)}$ is well-defined and differentiable. Moreover,
   $$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)}{g(x)} - \frac{f(x)}{g(x)^2}g'(x) = \frac{f(x)}{g(x)} - \frac{f(x)}{g(x)} = 0.$$
   We conclude that $\frac{f(x)}{g(x)}$ is constant, but $f(0) = f'(0) = 1$ and $g(0) = g'(0) = 1$, therefore $f(x) = g(x)$.

(d) Since f is differentiable everywhere, it is also a continuous everywhere. Given that $f(x)$ is never 0 and $f(0) = 1$ applying Bolzano's theorem we see that $f(x) > 0$ for all $x \in \mathbb{R}$. Therefore, $f'(x) = f(x) > 0$, which implies that f is strictly monotone increasing.

(e) Given $n, m \in \mathbb{N}$ we have

$$f(1)^n = f(1 + \cdots + 1) = f(n) = f\left(m \cdot \frac{n}{m}\right) = f\left(\frac{n}{m} + \cdots + \frac{n}{m}\right) = f\left(\frac{n}{m}\right)^m.$$

We conclude that $f(\frac{n}{m}) = f(1)^{\frac{n}{m}}$. Combining this with the previous item, we can say that $f(q) = f(1)^q$, for all $q \in \mathbb{Q}$. Finally, since f is monotone increasing, $f(1) > f(0) = 1$. Therefore, $f(n) = f(1)^n \to +\infty$ as $n \to \infty$ and $f(-n) = f(1)^{-n} \to 0$ as $n \to \infty$. The second part of the claim then follows from the intermediate value theorem.

2. This problem is borrowed from the last problem sheet: let $f : (a, b) \to \mathbb{R}$ be a monotone increasing function. Show that the following one-sided limits exist and satisfy the inequality

$$\lim_{h \to 0^-} f(x + h) \leq f(x) \leq \lim_{h \to 0^+} f(x + h)$$

for all $x \in (a, b)$.

Solution. The situation is analogous to when you proved last semester that bounded monotone sequences have a limit. Note that it makes sense to think that

$$\lim_{h \to 0^-} f(x + h) = \ell = \sup\{f(y) : y < x\}$$

should hold. To make our intuition formal we must first do a few checks. First, in order to talk about its supremum, the set $\{f(y) : y < x\}$ must be non-empty and bounded from above. Note that $f(\frac{a+x}{2})$ is in the set and that $y < x$ implies $f(y) \leq f(x)$, i.e., the set is not empty and $f(x)$ is an upper bound. Then, the set has a supremum, which we call ℓ. We must have $\ell \leq f(x)$. We can finally try to prove $\lim_{h \to 0^-} f(x + h) = \ell$. Note that, since ℓ is the supremum, for all $\varepsilon > 0$ there exists $x_0 < x$, such that $f(x_0) \in (\ell - \varepsilon, \ell]$. Let $\delta = x - x_0$. Then $h \in (-\delta, 0)$ implies $x_0 = x - \delta < x + h < x$ and by monotonicity and our previous assumptions $\ell - \varepsilon < f(x_0) < f(x + h) < \ell$. In particular, $h \in (-\delta, 0)$ implies $|f(x + h) - \ell| < \varepsilon$, which is what we wanted to show. The computation for the limit from the right follows a similar argument.

3. Show that among all rectangles with a fixed perimeter, the square has the largest area.

Solution. Let the perimeter of the rectangle be $2p$. Then, the sides of the rectangle are x and $p-x$, where x ranges in the interval $(0,p)$. The area of the rectangle, as a function of x is $A(x)=x(p-x)=px-x^2$. This function is differentiable on the whole of $\mathbb{R}$. Since $A'(x)=p-2x$, the only critical point of A is $x=p/2$, which luckily lies in $(0,p)$ and corresponds to a square since $x=p/2$ implies $p-x=p/2$. To finish, we must only check that such point is a maximum, but this follows from the fact that $\lim_{|x|\to+\infty} A(x)=-\infty$.

4. Find $f'(0)$ if

$$f(x)=\begin{cases} g(x)\sin(\frac{1}{x}) & x\neq 0\\ 0 & x=0\end{cases}$$

and

$$g(0)=g'(0)=0.$$

Solution. Note that $g(x)=xh(x)$ where $h:\mathbb{R}\to\mathbb{R}$ is a continuous function. In fact, we can define $h(x):=g(x)/x$, when $x\neq 0$, and $h(0):=g'(0)=0$. It follows from this that h is continuous at 0. Moreover, for $x\neq 0$ we have

$$\frac{f(x)-f(0)}{x-0}=\frac{g(x)\sin(1/x)}{x}=\frac{xh(x)\sin(1/x)}{x}=h(x)\sin(1/x).$$

Taking limits we obtain $f'(0)=0$.

5. Show that $e^x\geq \frac{x^n}{n!}$, for all $x\geq 0$ and $n\in\mathbb{N}$, and conclude that the exponential function grows faster than any polynomial, i.e., $\lim_{x\to\infty}\frac{x^n}{e^x}=0$, for all $n\in\mathbb{N}$.

Solution. We can prove this by induction. First the case $n=0$ is the inequality $e^x\geq e^0=1$, which holds by (d) and (a) of Problem 1. Now, assume $e^x\geq\frac{x^n}{n!}$ holds for all $x\geq 0$. Let $f(x)=e^x-\frac{x^{n+1}}{(n+1)!}$. Note that $f(0)=1\geq 0$ and $f'(x)=e^x-\frac{x^n}{n!}\geq 0$ where the inequality holds for all $x\geq 0$ by the inductive hypothesis. In particular $f(x)$ is monotone increasing and $f(x)\geq f(0)>0$, for all $x\geq 0$. Finally, we conclude by taking the limit of $0\leq\frac{x^n}{e^x}\leq\frac{(n+1)!x^n}{x^{n+1}}=\frac{(n+1)!}{x}$ as $x\to\infty$.

6. Let

$$f(x)=\begin{cases} e^{-1/x} & x>0\\ 0 & x\leq 0\end{cases}.$$

Show that f is infinitely differentiable and that $f^{(n)}(0)=0$, for all $n\in\mathbb{N}$. (Note that this is a very special function, it interpolates smoothly between being constantly zero to being almost constantly one).

Solution. We first try compute $f^{(n)}(x)$, for $x > 0$. Using the chain rule $f^{(1)}(x) = f'(x) = e^{-1/x}\frac{d}{dx}(-\frac{1}{x}) = f(x) \cdot \frac{1}{x^2}$. Similarly, $f^{(2)}(x) = f(x)\frac{1}{x^4} + f(x)\frac{1}{x^2}(\frac{-2}{x^3})$ which is of the form $\frac{f(x)}{p(x)}$ for some polynomial $p(x)$. Prove by induction that this is true for all n, i.e. $f^{(n)}(x) = \frac{e^{-1/x}}{p(x)}$ and use the previous item to show that $\lim_{x\to 0} f^{(n)}(x) = 0$.

7. Let $A = (0, h)$ and $B = (p, q)$ points of $\mathbb{R}^2$, with p, q and h fixed positive numbers. Draw the two line segments joining A with $O = (x, 0)$ and then O with A. Show that the sum of both segments is minimised when the angles formed between the horizontal axis and the segments $\overline{AO}$ and $\overline{OB}$, are both equal.

   Solution. The lengths of AO and OB are, as functions of x, $\sqrt{x^2+h^2}$ and $\sqrt{q^2+(p-x)^2}$, respectively. The total distance we would like to minimise is $f(x) = \sqrt{x^2+h^2} + \sqrt{q^2+(p-x)^2}$. Since $q, h > 0$, $f(x)$ is differentiable for all $x \in \mathbb{R}$. In addition, $f(x) \geq 0$, for all $x \in \mathbb{R}$ and $\lim_{x\to+\infty} f(x) = +\infty$. This implies that f attains a minimum. The critical point equation for f is

$$0 = f'(x) = \frac{x}{\sqrt{x^2+h^2}} - \frac{p-x}{\sqrt{q^2+(p-x)^2}},$$

   which directly implies $x \in (0, p)$. Moreover, each quotient is already the cosine of the angle formed by the segments and the horizontal. Summarising, the minimum distance is attained for some $x \in (0, p)$ and the angles the segments form with respect to the horizontal line coincide in such configuration.

   To finish, we just need to justify there is only one such critical point in $(0, p)$. By squaring and inverting the expression above, we obtain

$$\frac{x^2+h^2}{x^2} = \frac{q^2+(p-x)^2}{(p-x)^2}.$$

   This is a degree two polynomial on x and solving it we get the two solutions $x = p(\frac{h}{h-q})$ and $x = p(\frac{h}{h+q})$, of which only the second one is in $(0, p)$.

8. You drive down a road whose speed limit is 60 miles per hour. An observer sees you at 12pm, and a second observer 35 miles away sees you at 12:30pm. Assuming they've watched their analysis lectures, how can they prove you were speeding?

   Solution. The mean value theorem guarantees that there is some time strictly between 12:00 and 12:30 when your velocity was

$$\frac{35 \text{ miles}}{\frac{1}{2} \text{ hour}} = 70 \text{ miles per hour.}$$

9. Let H_n denote the harmonic sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

   (a) Show using the mean value theorem that $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

(b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n \geq 2$, where $H_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$, and deduce that $\log(n+1) < H_n < \log(n) + 1$.

(c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty} (H_n - \log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma \approx 0.577\ldots$.)

Solution. (a) Since $\log(x)$ has derivative $\frac{1}{x}$, there is some $z \in (n, n+1)$ such that

$$\frac{\log(n+1) - \log(n)}{(n+1) - n} = \frac{1}{z} \in \left(\frac{1}{n+1}, \frac{1}{n}\right),$$

or equivalently $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$.

(b) We sum each side of the inequality $\frac{1}{k+1} < \log(k+1) - \log(k) < \frac{1}{k}$ from $k = 1$ to $n - 1$ to get

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \log(n) - \log(1) < \sum_{k=1}^{n-1} \frac{1}{k},$$

noticing that lots of cancellation occurs in the middle. Since $\log(1) = 0$, this is equivalent to $H_n - 1 < \log(n) < H_{n-1}$. The left half of this gives us $H_n < \log(n) + 1$, and when we replace n with $n + 1$ the right half gives us $\log(n+1) < H_n$, so we combine these to get $\log(n+1) < H_n < \log(n) + 1$.

(c) From part (b) we have

$$0 < \log(n+1) - \log(n) < H_n - \log(n) < 1,$$

so the sequence $a_n = H_n - \log(n)$ is bounded, and thus if it is monotone decreasing then it converges. We compute

$$a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log(n),$$

so if we let $f(x) = \frac{1}{x+1} - \log(x+1) + \log(x)$, then we want to show that $f(n) < 0$ for all integers $n \geq 1$. We have

$$\begin{aligned} f'(x) &= -\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x} \\ &= -\frac{1}{(x+1)^2} + \frac{1}{x(x+1)} = \frac{1}{x(x+1)^2} \end{aligned}$$

and so $f'(x) > 0$ for all $x > 0$, meaning that $f(x)$ is strictly increasing. But

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{1}{x+1} - \log\left(1 + \frac{1}{x}\right)\right) = 0.$$

Since $f(x)$ is strictly increasing, this implies that $f(x) < 0$ for all $x > 0$. (Proof: assuming otherwise, let $\epsilon = f(y)$ be a positive value of f. Then there is no $N > 0$ such that if $x \geq N$ then $|f(x) - 0| < \epsilon$, because as soon as $x > \max(y, N)$ we have $f(x) > f(y) = \epsilon$. This contradicts $f(x) \to 0$.) So in particular, for all integers $n \geq 1$ we have $a_{n+1} - a_n = f(n) < 0$, hence $a_{n+1} < a_n$.

10. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose there is a constant $C < 1$ such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. We will prove that f has exactly one fixed point, meaning there is a unique $y \in \mathbb{R}$ such that $f(y) = y$. Pick some $x_0 \in \mathbb{R}$ and let

$$x_{n+1} = f(x_n) \text{ for all } n \geq 0.$$

(a) Prove that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all n.

(b) Prove that the sequence (x_n) converges, and that if its limit is y then $f(y) = y$.

(c) Prove that f cannot have two different fixed points.

Solution. (a) If $x_{n+1} = x_n$ then $x_{n+2} = f(x_{n+1}) = f(x_n) = x_{n+1}$, and so both sides of the desired inequality are zero. Otherwise, the mean value theorem tells us that there is some t between x_n and x_{n+1} such that

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(t) \;\Rightarrow\; \left|\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right| = |f'(t)| \leq C.$$

Thus $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ as desired.

(b) Write $d = |x_1 - x_0|$. Then $|x_{n+1} - x_n| \leq C^n d$ by induction and part (a). The triangle inequality says that for any integers $m \geq n$,

$$\begin{aligned}|x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n| \\ &\leq C^{m-1}d + \cdots + C^{n+1}d + C^n d \\ &< \sum_{i=n}^{\infty} C^i d = \frac{C^n d}{1 - C}.\end{aligned}$$

For any $\epsilon > 0$ we can find $N \geq 0$ such that $\frac{C^N d}{1-C} < \epsilon$, since the left side approaches 0 as $N \to \infty$. Then given $m, n \geq N$ we have shown that

$$|x_m - x_n| < \frac{C^N d}{1 - C} < \epsilon,$$

which proves that the sequence (x_n) is Cauchy and hence convergent, say with limit y. Since $x_n \to y$ and f is continuous, we have $f(x_n) \to f(y)$. But $f(x_n) = x_{n+1} \to y$, so it must be the case that $f(y) = y$.

(c) Suppose that y and z are distinct fixed points of f. By the mean value theorem, there is some t between y and z such that

$$\frac{f(y) - f(z)}{y - z} = f'(t) \;\Rightarrow\; \frac{y - z}{y - z} = f'(t) \;\Rightarrow\; f'(t) = 1.$$

But this contradicts the assumption that $|f'(x)| \leq C < 1$ for all $x \in \mathbb{R}$.

11. Prove using l'Hôpital's rule that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

Solution. We write the limiting term as

$$\left(1+\frac{r}{x}\right)^x = e^{x\log\left(1+\frac{r}{x}\right)},$$

so by the continuity of $f(x)=e^x$ it will suffice to compute

$$\lim_{x\to\infty}\frac{\log\left(1+\frac{r}{x}\right)}{1/x} = \lim_{y\downarrow 0}\frac{\log(1+ry)}{y}.$$

The derivative of $\log(1+ry)$ is $\frac{r}{1+ry}$, so we apply l'Hôpital's rule to get

$$\lim_{y\downarrow 0}\frac{\log(1+ry)}{y} = \lim_{y\downarrow 0}\frac{r/(1+ry)}{1} = r,$$

and so $\lim\limits_{x\to\infty} x\log\left(1+\frac{r}{x}\right) = r$ and it follows that $\lim\limits_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r$.

12. The aim of this exercise is to prove $\lim\limits_{x\to\infty} xs^{x-1}=0$ for all $s\in(0,1)$.
    (a) Prove that for all $c>0$, there exists $N>0$ such that $\log(x)<cx$ for all $x\geq N$.
    (b) Prove for $s\in(0,1)$ that $\lim\limits_{x\to\infty} xs^x=0$, and that this implies the above claim.

Solution. (a) It's enough to prove that $\lim\limits_{x\to\infty}\frac{\log(x)}{x}=0$, since then there's an $N>0$ such that $0<\frac{\log(x)}{x}<c$ for all $x\geq N$. This limit exists by l'Hôpital's rule, which says that it is equal to $\lim\limits_{x\to\infty}\frac{1/x}{1}=0$.

(b) For any $c>0$, part (a) says that $0<xs^x<e^{cx}s^x=(e^cs)^x$ for all large enough x. Since $0<s<1$, we can choose a positive $c<\log(1/s)$ so that $0<e^cs<1$, and then

$$\lim_{x\to\infty}(e^cs)^x=0.$$

Thus the squeeze theorem says that $\lim\limits_{x\to\infty} xs^x=0$ as well. We conclude that

$$\lim_{x\to\infty} xs^{x-1} = \frac{1}{s}\left(\lim_{x\to\infty} xs^x\right) = 0.$$

13. (a) Prove that $f(x)=e^x$ is convex on all of $\mathbb{R}$.
    (b) Let $a,b>0$. Prove the *arithmetic mean–geometric mean inequality*

$$\frac{a+b}{2}\geq\sqrt{ab}$$

by using the convexity of e^x. (Hint: think about $\alpha=\log(a)$ and $\beta=\log(b)$.)

    (c) Prove for any $a,b>0$ and $s\in[0,1]$ that $sa+(1-s)b\geq a^sb^{1-s}$.

(d) Prove *Young's inequality*: for any $x, y \geq 0$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

Solution. (a) It suffices to check that that $f''(x) \geq 0$ for all x, and this is certainly true since $f''(x) = e^x$.

(b) Assuming $\alpha < \beta$ without loss of generality, the convexity of e^x implies for $\alpha < \frac{\alpha+\beta}{2} < \beta$ that

$$\frac{e^\alpha + e^\beta}{2} \geq e^{(\alpha+\beta)/2} = \sqrt{e^\alpha \cdot e^\beta}$$

which is equivalent to $\frac{a+b}{2} \geq \sqrt{ab}$.

(c) Since e^x is convex, we know that

$$se^\alpha + (1-s)e^\beta \geq e^{s\alpha+(1-s)\beta},$$

and the left side is $sa + (1-s)b$ while the right side is $(e^\alpha)^s(e^\beta)^{1-s} = a^s b^{1-s}$.

(d) We may assume that $x, y > 0$, since otherwise the inequality reduces to $\frac{x^p}{p} + \frac{y^q}{q} = 0$, which is true. We now use part (c), setting $s = \frac{1}{p}$ (so $1 - s = \frac{1}{q}$) and $(a, b) = (x^p, y^q)$, to get

$$\frac{x^p}{p} + \frac{y^q}{q} \geq (x^p)^{1/p}(y^q)^{1/q} = xy.$$