Imperial College London

MATH70063

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebra 4

Date: 5 June 2023
Time: 10:00 – 12:30 (BST)
Time Allowed: 2.5hrs
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their answers to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

(1) In this paper, R is an associative ring with 1. All R-modules are assumed to be left R-modules, unless explicitly stated otherwise.

(2) When answering a question, or part of a question, you are permitted to quote statements from other questions or parts even if you have not answered them.

(3) You are permitted to use without proof any statement that is proved in the lecture notes, provided that you make it clear which statement you are using. Unless instructed otherwise, you must justify all other statements that you make.

1. (a) (i) Give the definition of a *projective* R-module. (1 mark)

(ii) Prove that for an R-module A the following three statements are equivalent:

(P1) A is projective,

(P2) The functor $\mathrm{Hom}_R(A, -)$ is an exact functor from the category of R-modules to the category of abelian groups.

(P3) For all R-modules B, we have $\mathrm{Ext}^1_R(A, B) = 0$ (6 marks)

(b) Prove that every R-module M admits a projective resolution. (3 marks)

(c) Let R be a principal ideal domain (PID). Provide either a complete proof or a counterexample for the following statements.

(i) Every submodule of a projective module is projective. (2 marks)

(ii) Every quotient of a free module is free. (2 marks)

(iii) Every flat module is torsion-free. (2 marks)

(d) Let $R = \mathbb{Z}/18$ and M be the ideal generated by 3. Is M a projective R-module? (4 marks)

(Total: 20 marks)

2. (a) Let R be a principal ideal domain (PID).

(i) Give the definition of a *divisible* R-module. (1 mark)

(ii) Apply Baer's criterion to show that all divisible modules are injective. (2 marks)

(iii) Is the quotient of an injective R-module again injective? (Justify your answer.) (3 marks)

(b) Consider the abelian groups $G_1 = \mathbb{R}/\mathbb{Z}$ and $G_2 = \mathbb{Z} \oplus \mathbb{Z}$. Find injective abelian groups I_i and injective homomorphisms $\varphi_i \colon G_i \to I_i$ for $i = 1, 2$. (4 marks)

(c) Let $R = \mathbb{Z}/4$.

(i) Prove that $\mathbb{Z}/4$ is an injective R-module. (2 marks)

(ii) For all $n \geq 0$, compute $\mathrm{Ext}^n_R(\mathbb{Z}/2, \mathbb{Z}/2)$ using an injective resolution of $\mathbb{Z}/2$. (4 marks)

(d) Show that if A is an abelian group, then $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is a flat $\mathbb{Z}$-module. (4 marks)

(Total: 20 marks)

3. (a) Consider two R-modules A and C.

   (i) Give the definition of an *extension of C by A.* (1 mark)

   (ii) Define what it means for two extensions of C by A to be *equivalent*. (2 marks)

   (iii) Give the definition of *Baer sum* of two extensions of C by A. (3 marks)

(b) Consider the short exact sequence of abelian groups

$$(\xi): \qquad 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

   (i) Compute explicitly the Baer sum of the extension ξ with itself. (5 marks)

   (ii) What is the order of the class of extension ξ in the group $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$? (Justify your answer.)

(3 marks)

(c) In the category of abelian groups, find projective resolutions $P_\bullet \to \mathbb{Z}$, $Q_\bullet \to \mathbb{Z}$ and $R_\bullet \to \mathbb{Z}/2$ and a short exact sequence

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$$

of complexes of abelian groups lifting the short exact sequence ξ above in part (b).

(6 marks)

(Total: 20 marks)

4. (a) Let G be a group.

   (i) Give the definition of the *group ring* $\mathbb{Z}[G]$. (2 marks)

   (ii) Let M be a G-module. Give the definition of the *group M^G of G-invariants of M.*

(2 marks)

(b) Let G be a finite group. Prove that the group $\mathbb{Z}[G]^G$ of G-invariants is isomorphic to $\mathbb{Z}N$ where $N = \sum_{g \in G} g \in \mathbb{Z}[G]$ is the norm element. (4 marks)

(c) (i) Let M be a G-module for a group G. Prove that the zeroth cohomology group $H^0(G, M)$ is isomorphic to the group M^G of G-invariants of M. (2 marks)

   (ii) For all $n \geq 0$, find the n-th cohomology group $H^n(\mathbb{Z}, M)$. (4 marks)

(d) Let A, B and C be abelian groups. Consider the natural homomorphism of abelian groups

$$\Psi \colon \mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, C) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}}(B, C))$$

which sends a homomorphism $f \colon A \otimes_{\mathbb{Z}} B \to C$ to the homomorphism $\Psi(f) \colon A \to \mathrm{Hom}_{\mathbb{Z}}(B, C)$ such that the homomorphism $\Psi(f)|_a \colon B \to C$ sends b to $f(a \otimes b)$. Prove that Ψ is an isomorphism.

(6 marks)

(Total: 20 marks)

5. (a) Let $\mathcal{I}$ be a category with no non-trivial morphism such that $|\mathcal{I}|$ is a set, and let $\mathcal{A}$ be an abelian category. Prove that the colimit of a functor $F\colon \mathcal{I} \to \mathcal{A}$ is isomorphic to the coproduct $\coprod_{i\in|\mathcal{I}|} \mathcal{F}(i)$. (3 marks)

(b) Prove that any abelian group G is isomorphic to the direct limit of its finitely generated subgroups. (4 marks)

(c) Let A and B be abelian groups.

(i) Prove that for all $n \in \mathbb{Z}_{\geq 1}$, we have $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n, B) = B/nB$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, B) = B[n] = \{b \in B\colon nb = 0\}$. (3 marks)

(ii) Show that $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.
(*Hint: apply part (b).*) (6 marks)

(iii) Prove that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion subgroup of B.
(*Hint: you may use without proof the fact that $\mathbb{Q}/\mathbb{Z}$ is the direct limit of its finite subgroups.*) (4 marks)

(Total: 20 marks)

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Algebra IV

Algebra IV (Solutions)

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1. (a) (i) An R-module M is projective if for every surjective morphism $\beta\colon B \to C$ of R-modules and for every morphism of R-modules $f\colon M \to C$, there is a morphism of R-modules $g\colon M \to B$ such that $f = \beta \circ g$. $\boxed{\text{seen} \Downarrow}$ $\boxed{1, \text{A}}$

(ii) We first show that (P1) and (P2) are equivalent. We know from the lecture note that the functor $\text{Hom}_R(A, -)$ is left-exact for every R-module A. So it's enough to show that A is projective if and only if $\text{Hom}_R(A, -)$ sends surjective maps of R-modules to the surjective maps of abelian groups. By definition, A is projective if and only if for every surjective map $B \to C$, the map $\text{Hom}_R(A, B) \to \text{Hom}_R(A, C)$ is surjective, which is just another way to of saying that $\text{Hom}_R(A, -)$ sends surjective maps to surjective maps. $\boxed{2, \text{A}}$

The next step is to show that if (P1) holds, then (P3) holds. To compute $\text{Ext}^1_R(A, B)$, we may replace A by its projective resolution, which is itself, so the claim follows. $\boxed{2, \text{A}}$

Finally, we show that if (P3) holds, then (P2) holds. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of R-modules, then taking the functor $\text{Hom}_R(A, -)$ gives the long exact sequence

$$0 \to \text{Hom}_R(A, A_1) \to \text{Hom}_R(A, A_2) \to \text{Hom}_R(A, A_3) \to \text{Ext}^1_R(A, A_1) \to \dots$$

But $\text{Ext}^1_R(A, A_1) = 0$ by our assumption, so we are done. $\boxed{2, \text{A}}$

$\boxed{\text{seen} \Downarrow}$

(b) Let F_0 be the free module $\oplus_{m \in M} R.m$ generated by all elements $m \in M$. We have a surjective map $\varphi_0\colon F \to M$, which sends every generator m in F to $m \in M$. Then we consider the kernel $\ker \varphi_0$, and via the same procedure construct a surjective map $\varphi_1\colon F_1 \to \ker \varphi_0$, so continuing this process gives an exact chain complex

$$\dots \to F_2 \to F_1 \to F_0 \to M \to 0$$

so that each of F_i's are free and so projective. $\boxed{3, \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(c(i)) True. As R is a PID, all projective modules are free, so we have a submodule of a free module which is free as R is PID, and so it is projective. $\boxed{2, \text{B}}$

$\boxed{\text{sim. seen} \Downarrow}$

(c(ii)) False. For instance we have the surjective map $\mathbb{Z} \to \mathbb{Z}/2$ in the category of $\mathbb{Z}$-modules, but we know $\mathbb{Z}/2$ is not free. $\boxed{2, \text{A}}$

meth seen $\Downarrow$

(c(iii)) True. Take $a \in R$, and consider the map of R-modules $\Psi_a \colon R \to R$ sending 1 to a which is injective as R has no zero divisor. If M is a flat R-module, then the induced map $\Psi \otimes M \colon R \otimes_R M \to R \otimes_R M$ is also injective. Identification of M with $R \otimes_R M$ via $m \mapsto 1 \otimes m$ allows us to rewrite this map as the map $M \to M$ sending m to am, thus $am \neq 0$. 2, B

unseen $\Downarrow$

(d) No. We know the ideal M is $3.\mathbb{Z}/18$ which is isomorphic to $\mathbb{Z}/6$. If it's projective, then the short exact sequence

$$0 \to \mathbb{Z}/3 \xrightarrow{.6} \mathbb{Z}/18 \to \mathbb{Z}/6 \to 0$$

of R-modules split. But the cyclic group of order 18 is not isomorphic to the direct sum of cyclic groups of order 3 and 6. 4, B

2.(a(i)) An R-module M is divisible if for every $m \in M$ and for every $0 \neq r \in R$, there is an $m' \in M$ such that $m = r.m'$. seen $\Downarrow$ 1, A

seen $\Downarrow$

(a(ii)) Let $f\colon I \to M$ be a map of R-modules where $I \subset R$ is an ideal. Since R is PID, we have $I = aR$ for some $a \in R$. As M is divisible, we can write $f(a) = am$ for some $m \in M$. Then the morphism $f\colon I \to M$ extends to a map of R-modules $R \to M$ which sends 1 to m, thus the claim follows from Baer's criterion. 2, A

unseen $\Downarrow$

(a(iii)) Yes, since R is a PID, injective modules are the same as divisible modules, so we only need to show the quotient of a divisible module is again divisible. Let A be divisible and let $q\colon A \to B$ be a surjective map. Given $b \in B$ and $r \in R$, pick a preimage $a \in A$ and an element a' such that $r.a' = a$, then $r.f(a') = f(a) = b$, so B is divisible. 3, C

meth seen $\Downarrow$

(b) We know $G_1 = \mathbb{R}/\mathbb{Z}$ is divisible and $R = \mathbb{Z}$ is a PID, so G_1 is injective itself. Thus we can take $I_1 = G_1$. 1, A

For G_2, we know $\mathbb{Q}$ is a divisible abelian group, so it is an injective abelian group. We know a direct sum of injective modules is injective, so $I_2 := \mathbb{Q}\oplus\mathbb{Q}$ is an injective abelian group. We also have a natural injective map $\varphi_2\colon G_2 = \mathbb{Z}\oplus\mathbb{Z} \to I_2 = \mathbb{Q}\oplus\mathbb{Q}$ sending $(1,1)$ to $(1,1)$. 3, B

meth seen $\Downarrow$

(c(i)) We apply Baer's criterion to show that $R = \mathbb{Z}/4$ is an injective R-module. The only non-trivial ideal $I \subset R$ is $2R$ which is isomorphic to $\mathbb{Z}/2$. There is exactly one non-zero morphism $\mathbb{Z}/2 \to \mathbb{Z}/4$ which is multiplication by 2, so the claim follows from Baer's criterion. 2, B

unseen $\Downarrow$

(c(ii)) An injective resolution of $\mathbb{Z}/2$ as $\mathbb{Z}/4$-module is given by 1, D

$$0 \to \mathbb{Z}/2 \xrightarrow{.2} \mathbb{Z}/4 \xrightarrow{.2} \mathbb{Z}/4 \xrightarrow{.2} \mathbb{Z}/4 \xrightarrow{.2} \dots$$

Applying the functor $\mathrm{Hom}_R(\mathbb{Z}/2, -)$ to the injective resolution, we get the complex

$$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \dots$$

Therefore $\mathrm{Ext}^n_R(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for all $n \geq 0$. 3, C

unseen $\Downarrow$

(d) Since $\mathbb{Z}$ is a PID, we only need to show $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is torsion-free. Take a non-zero arbitrary element $m = \sum_{i=1}^{k} q_i \otimes a_i \in \mathbb{Q} \otimes_{\mathbb{Z}} A$. We can pick $d \in \mathbb{Z}$ so that each q_i

can be written as a fraction with denominator d. So $m = \frac{1}{d} \otimes a$ for some $a \in A$. Moreover $a \in A$ is not a torsion element as $m \neq 0$. Let $A' \subset A$ be the infinite cyclic group generated by a. Then for every non-zero $n \in \mathbb{Z}$, the multiplication map $A' \xrightarrow{.n} A'$ is injective. Since $\mathbb{Q}$ is a flat abelian group, we get the injective morphism of abelian groups

$$\mathbb{Q} \otimes_{\mathbb{Z}} A' \xrightarrow{\mathsf{id} \otimes .n} \mathbb{Q} \otimes_{\mathbb{Z}} A'$$

Thus $n.\left(\frac{1}{d} \otimes a\right) \neq 0$ and m is not a torsion element. So $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is torsion-free. $\boxed{\text{4, D}}$

3. (a) (i) A short exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called an extension of C by A. $\boxed{\text{seen} \Downarrow}$ $\boxed{1, \text{A}}$

(ii) We call two extensions of R-modules $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C \to 0$ equivalent if there is a commutative diagram $\boxed{2, \text{A}}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

(iii) Given two extensions of R-modules $\xi : \ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and $\xi' : \ 0 \to A \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C \to 0$. We define

$$X = B \times_C B' / \{(\alpha(a), -\alpha'(a)) | \ a \in A\}$$

where $B \times_C B' = \{(b, b') \in B \oplus B' | \ \beta(b) = \beta'(b)\} \subset B \oplus B'$. Clearly, X has the structure of an R-module. Then we define the Baer sum of two extensions ξ and ξ' to be

$$0 \to A \xrightarrow{\tilde{\alpha}} X \xrightarrow{\tilde{\beta}} C \to 0$$

where $\tilde{\alpha}(a) = (\alpha(a), 0)$ and $\tilde{\beta}(b, b') = \beta(b)$. $\boxed{3, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(b(i)) Using the same notations as in part (a(iii)), we have

$$\mathbb{Z} \times_{\mathbb{Z}/2} \mathbb{Z} = \{(b, b') \in \mathbb{Z} \oplus \mathbb{Z} | \ \beta(b) = \beta(b)\}$$

where $\beta \colon \mathbb{Z} \to \mathbb{Z}/2$ is the quotient map in ξ which sends $n \in \mathbb{Z}$ to $n \mod 2$. Thus we have

$$X = \{(b, b + 2k) | \ b, k \in \mathbb{Z}\} / \{(2a, -2a) | \ a \in \mathbb{Z}\}$$

Consider the subgroup $X_1 \subset X$ generated by $(2, 0)$ which is isomorphic to $\mathbb{Z}$ and the subgroup $X_2 \subset X$ generated by $(1, -1)$ which is isomorphic to $\mathbb{Z}/2$.
The map $\tilde{\alpha} \colon \mathbb{Z} \to X$ in the Baer sum sends 1 to the coset of $(2, 0)$, so $\tilde{\alpha}(\mathbb{Z}) = X_1$. The map $\tilde{\beta} \colon X \to \mathbb{Z}/2$ in the Baer sum sends $(b, b + 2k)$ to $b \mod 2$, so $(1, -1)$ goes to 1. Thus the map $\tilde{\beta}|_{X_1} = 0$, and $\tilde{\beta}$ has a section $\mathbb{Z}/2 \to \mathbb{Z}$ sending 1 to $(1, -1)$. This implies that $X = X_1 \oplus X_2$ and the Baer sum of ξ with itself is a split extension. $\boxed{5, \text{D}}$

$\boxed{\text{unseen} \Downarrow}$

(b(ii)) Since $\mathbb{Z}$ is torsion-free, the original extension ξ is not split. Hence its class in $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$ is non-zero. By part (i), its Baer sum with itself is split. Hence the order of its class in $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$ is 2. $\boxed{3, \text{B}}$

(c) This question is an application of Horseshoe Lemma: we first start with trivial free resolution $P_\bullet = 0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ and a two-term free resolution $R_\bullet = 0 \to \mathbb{Z} \xrightarrow{.2} \mathbb{Z} \to 0$ of $\mathbb{Z}/2$, then Horseshoe Lemma says that there is a projective resolution $Q_\bullet$ of the middle term $\mathbb{Z}$ in ξ such that $Q_n = P_n \oplus R_n$ for every $n \in \mathbb{Z}$ and $P_\bullet \to \mathbb{Z}$, $Q_\bullet \to \mathbb{Z}$ and $R_\bullet \to \mathbb{Z}/2$ form a short exact sequence of exact complexes. Thus to complete the answer we only need to describe morphisms in the complex $Q_\bullet$. We have the following commutative diagram such that all rows and columns are exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, ".1"] \arrow[d, "\psi_1"] & \mathbb{Z} \arrow[r] \arrow[d, ".2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\pi_1"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\pi_2"] \arrow[d, "\psi_2"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, ".2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We know each row is an split short exact sequence, so $\pi_1 \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the natural injection sending m to $(m, 0)$. Also $\pi_2 \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is the natural surjection which sends (m, n) to n. $\boxed{3, \text{D}}$

First consider the square consisting of π_1 and ψ_2. We get $\psi_2(m, 0) = 2m$. Then the square consisting of π_2 and ψ_2 implies that $\psi_2(m, n) \equiv n \mod 2$. Thus one possible choice for ψ_2 is to send $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ to $2m + n$.

Now consider the square consisting of ψ_1 and π_2, then we get $\pi_2(\psi_1(c)) = 2c$ for $c \in \mathbb{Z}$. On the other hand, since the complex $Q_\bullet \to \mathbb{Z}$ is exact, we must have $\mathrm{im}(\psi_1) = \ker(\psi_2)$. Thus ψ_1 must send $c \in \mathbb{Z}$ to $(-c, 2c) \in \mathbb{Z} \oplus \mathbb{Z}$. $\boxed{3, \text{D}}$

4. (a) (i) The group ring of G is a free $\mathbb{Z}$-module with generators g for $g \in G$. The addition is defined naturally via the free $\mathbb{Z}$-module structure. The multiplication is defined by the condition that the product of generators attached to $g, h \in G$ is the generator attached to $g.h$ where $(.)$ is the group operation in G. seen $\Downarrow$

2, A

(ii) The subgroup $M^G \subset M$ consists of all elements $m \in M$ so that $gm = m$ for all $g \in G$. 2, A

seen $\Downarrow$

(b) Take $x \in \mathbb{Z}[G]^G$ so by definition $g.x = x$ for every $g \in G$. Suppose G has n elements $g_1 = 1, g_2, \ldots, g_n$, then we can write $x = \sum_{i=1}^n x_i g_i$. For every $i_0 \in [2, n]$, we know

$$\sum_{i=1}^{n} x_i g_{i_0}^{-1} g_i = g_{i_0}^{-1} x = x = \sum_{i=1}^{n} x_i g_i$$

So the components corresponding to identity $g_1 = 1$ must be the same in the first and last expressions, i.e. $x_{i_0} = x_1$. This implies that $x = x_1.N$ where N is the norm element. Since we clearly have $N \in \mathbb{Z}[G]^G$, the claim follows. 4, A

seen $\Downarrow$

(c(i)) By definition, we have $H^0(G, M) = \mathsf{Hom}_G(\mathbb{Z}, M)$. We claim it is isomorphic to the group M^G via the map

$$\begin{aligned} \Psi \colon \mathsf{Hom}_G(\mathbb{Z}, M) &\to M^G \\ f &\mapsto f(1). \end{aligned}$$

Note that since Z is a trivial G-module, we have $g.f(1) = f(g.1) = f(1)$, i.e. $f(1) \in M^G$. Clearly, the map Ψ is a morphism of abelian groups which is injective as the function f is completely determined with its image at 1. Moreover, Ψ is surjective: take $m \in M^G$, then one can define the map $f \colon \mathbb{Z} \to M$ which sends $k \in \mathbb{Z}$ to km. Since m is G-invariant, f is a morphism of G-modules. Thus Ψ is an isomorphism. 2, A

meth seen $\Downarrow$

(c(ii)) Let $G = \mathbb{Z} = \{t^i \colon i \in \mathbb{Z}\}$ with t as a generator. As have seen in the lectures, the exact sequence

$$0 \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\sigma} \mathbb{Z} \to 0$$

gives a projective resolution of the trivial $\mathbb{Z}$-module $\mathbb{Z}$. Here σ sends $p(t) \in \mathbb{Z}[\mathbb{Z}]$ to $p(t = 1)$. Then taking $\mathsf{Hom}_G(-, M)$ from the above free resolution gives the complex

$$0 \to \mathsf{Hom}_G(\mathbb{Z}[\mathbb{Z}], M) \to \mathsf{Hom}_G(\mathbb{Z}[\mathbb{Z}], M) \to 0.$$

Since $\mathsf{Hom}_R(R, M) \cong M$ for every ring R, this complex is $0 \to M \xrightarrow{1-t} M \to 0$. Thus

$$H^0(\mathbb{Z}, M) = M^{\mathbb{Z}} \qquad \text{and} \qquad H^1(\mathbb{Z}, M) = M_{\mathbb{Z}}.$$

Moreover, $H^n(\mathbb{Z}, M) = 0$ for $n > 1$. 4, B unseen $\Downarrow$

(d) If $f \neq 0$, then there exists $a \in A$ and $b \in B$ such that $f(a \otimes b) \neq 0$. This implies that $\Psi(f)|_a(b) = f(a \otimes b) \neq 0$, so $\Psi(f) \neq 0$ and so Ψ is injective.

Take a homomorphism $g \colon A \to \mathrm{Hom}_{\mathbb{Z}}(B, C)$. Define the map $G \colon A \times B \to C$ sending $a \otimes b$ to the value $g(a)|_b$. Since $g(a)$ is a homomorphism, G is additive in the second argument, and since g is a homomorphism, F is additive in the first argument. Then by the universal property of tensor product, G factors through a homomorphism $f \colon A \otimes_{\mathbb{Z}} B \to C$. Thus $\Psi(f) = g$ which proves Ψ is surjective. $\boxed{6, \mathrm{C}}$

5. (a) Since there is no non-trivial morphism in the category $\mathcal{I}$, a colimit of the functor $F\colon \mathcal{I} \to \mathcal{A}$ is an object $L \in \mathcal{A}$ with morphisms $\alpha_i\colon F(i) \to L$ for every i in the set $|\mathcal{I}|$, so that if there is another object $L' \in \mathcal{A}$ with morphisms $\alpha'_i\colon F(i) \to L'$, then there is a unique morphism $h\colon L \to L'$ so that $h \circ \alpha_i = \alpha'_i$. Since colimit is unique up to isomorphism, the definition of coproduct immediately gives $L \cong \coprod_{i\in\mathcal{I}} F(i)$.

$\boxed{3, \text{M}}$

(b) Define the category $\mathcal{I}$ as follows: $|\mathcal{I}|$ is the collection of all finitely generated subgroups of G, i.e. for every $i \in |\mathcal{I}|$, there is a corresponding finitely generated subgroup $H_i \subset G$. For every $i, j \in |\mathcal{I}|$, there is a unique morphism $e_{i,j}$ from i to j if and only if $H_i \subset H_j$. Then clearly $\mathcal{I}$ is a filtered category.

Consider the functor F from $\mathcal{I}$ to the category of abelian groups, so that $F(i) = H_i$. Then we prove

$$\text{colim}_{i\in I}\mathcal{F}(i) = \lim_{\rightarrow} \mathcal{F}(i) = G$$

We clearly have natural inclusions $\alpha_i\colon F(i) = H_i \to G$ for all $i \in |\mathcal{I}|$ which are compatible with the morphisms $e_{i,j}$ in the category $\mathcal{I}$. If G' is another abelian group with morphisms $\alpha'_i\colon H_i \to G'$ which are compatible with the inclusions, then we define the morphism $h\colon G \to G'$ as follows. For every $0 \neq g \in G$, we consider the subgroup H_{i_0} generated by g, then define $h(g) = \alpha'_{i0}(g)$. This defines a group homorphism because α'_i's are group homomorphisms and they are compatible with the inclusions of f.g. subgroups. Moreover, we have $h \circ \alpha_i = \alpha'_i$ for every $i \in |\mathcal{I}|$. This completes the proof.

$\boxed{4, \text{M}}$

(c) (i) We have a natural 2-term projective resolution $0 \to \mathbb{Z} \xrightarrow{.n} \mathbb{Z}$ of abelian group $\mathbb{Z}/n$. Then tensoring by B gives the complex

$$0 \to B \xrightarrow{.n} B \to 0.$$

Thus we have $\text{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n, B) = B/nB$ and $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, B) = \ker([n]) = B[n]$.

$\boxed{3, \text{M}}$

(ii) Since Tor functor commutes with direct limits, part (b) implies that $\text{Tor}^1_{\mathbb{Z}}(A, B)$ is the direct limit of $\text{Tor}^1_{\mathbb{Z}}(A_\alpha, B)$ where A_α's are finitely generated abalian subgroups of A. We know a direct limit of torsion groups is a torsion group, thus we may assume that A is finitely generated, that is $A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_r$ for integers $m, p_1, \ldots, p_r$. As $\mathbb{Z}^m$ is projective, $\text{Tor}^1_{\mathbb{Z}}(\mathbb{Z}^m, B)$ vanishes, so we have

$$\text{Tor}_1^{\mathbb{Z}}(A, B) \cong \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_1, B) \oplus \cdots \oplus \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_r, B).$$

By part (i), we have $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_i, B) \cong \{b \in B | p_i.b = 0\}$ which proves $\text{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.

$\boxed{6, \text{M}}$

(iii) We know $\mathbb{Q}/\mathbb{Z}$ is the direct limit of its finite subgroups, each of which is isomorphic to $\mathbb{Z}/p$ for some integer p. Since Tor functor commutes with direct limits, we get

$$\mathsf{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = \lim_{\rightarrow} \mathsf{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) = \lim_{\rightarrow} B[p] = \cup_p \{b \in B : pb = 0\}$$

which is the torsion subgroup of B. $\boxed{4, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70063	1	In question (1), the students demonstrated excellent understanding overall. The majority of this question focused on definitions and Lemmas covered in the lecture notes.In part (a), a few students missed the proof of P3 implying P1. In part (b), some students failed to provide an explanation for the existence of a surjective map from a free module to an arbitrary module. Regarding part (c)(ii), the counter-examples provided by certain students did not meet the requirement of being a PID ring. Lastly, in part (d), some students mistakenly considered the ring Z/3 instead of the ideal generated by 3, which is isomorphic to Z/6.
MATH70063	2	Students did very well in question (2) as well. In part (a)(iii), some students asserted, without proof, that the quotient of a divisible module is divisible. In part (c), the majority of students successfully constructed an injective resolution for Z/2. However, they encountered difficulties in applying the Hom functor correctly to obtain an accurate complex for computing cohomologies and Ext groups. Part (d) proved to be the most challenging part of the question, with only a few students delivering comprehensive proof for it.
MATH70063	3	In part (c(iii)), a notable observation was that some students provided only the middle term of the short exact sequence without mentioning the morphisms involved in the sequence.Most students did a great job for part (b). For students, part (c) posed the greatest challenge in this question. Only a few students managed to produce a correct commutative diagram.
MATH70063	4	In part (a(i)), some students provided the elements of the group ring without explaining what the addition and multiplication are.Most of the students demonstrated great work in part (b). In part (c(i)), there were instances where some students unnecessarily complicated the task by constructing a projective resolution, and some of them were assuming that G is a finite group. However, they overlooked the key point that the zeroth cohomology can be computed simply as Hom_G(Z, M), without the need for a resolution.Part (d) presented the greatest challenge. Some students attempted to construct an inverse, but they did not provide a proof that their function is a homomorphism of abelian groups. Additionally, there were instances where students overlooked crucial details in verifying that their function is a two-sided inverse.
MATH70063	5	In the mastery question (5), the majority of students demonstrated excellent proficiency, particularly in parts (a) and (c).In part (b), some students made a mistake by attempting to use part (a). They failed to recognize that the filtered category mentioned in part (a) has no non-trivial morphism.