Imperial College London

MATH97049 MATH70032

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: 12 May 2022
Time: 14:00 – 16:30 (BST)
Time Allowed: 2:30 hours
Upload Time Allowed: 30 minutes
This paper has 5 Questions.
Candidates should start their solutions to each question on a new sheet of paper.
Each sheet of paper should have your CID, Question Number and Page Number on the top.
Only use 1 side of the paper.
Allow margins for marking.
Any required additional material(s) will be provided.
Credit will be given for all questions attempted.
Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

You may use any results from the lectures, provided you state them clearly.

1. (a) Find a re-parametrisation of the curve

$$\phi(t) = (e^t \sin t, e^t, e^t \cos t), \quad t \in [0, +\infty)$$

by arc length. (5 marks)

(b) Show that the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x = 1 + y^2\}$$

is a regular curve. (5 marks)

(c) Calculate the curvature of the curve C at each point on C. (5 marks)

(d) Let $A \in \mathbb{R}^2$, $B \in \mathbb{R}^2$, and $K \in (0, +\infty)$ be arbitrary. Is there a regular curve $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = A$, $\gamma(1) = B$, and the curvature of γ at every point on γ is at least K? (Prove the non-existence or provide a sketch of an example.) (5 marks)

(Total: 20 marks)

2. Consider the set

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid 5x^2 + y^2 = (2 + \sin z)^2\}.$$

(a) Prove that C is a regular surface. (5 marks)

(b) Find the tangent plane of C at $p = (0, 5/2, \pi/6)$. (5 marks)

Consider the regular surface

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

(c) Prove that the map $F : C \to D$ defined as

$$F(x, y, z) = \left(\frac{\sqrt{5}x}{2 + \sin z}, \frac{y}{2 + \sin z}, z \right)$$

is a smooth map. (5 marks)

(d) Find the differential $DF_p : T_pC \to T_{F(p)}D$, where $p = (0, 5/2, \pi/6)$. (5 marks)

(Total: 20 marks)

3. Consider the regular surface

$$S = \{(u, v, w) \in \mathbb{R}^3 \mid w = v^3 - u^3\}.$$

(a) Calculate the first fundamental form of S at each point on S. (5 marks)

(b) Calculate the second fundamental form of S at each point on S. (5 marks)

(c) Calculate the Gaussian curvature of S at each point on S. (5 marks)

(d) Show that $\{(u, v, w) \in S \mid u = v\})$ is a geodesic on S. (5 marks)

(Total: 20 marks)

4. (a) Is the set

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{|y|}\}$$

a regular surface in $\mathbb{R}^3$? Justify your answer. (6 marks)

(b) Let S be a regular surface, $p \in S$, and $v \in T_pS$ be a unit vector. Let $\alpha : (-\epsilon, \epsilon) \to S$ be a smooth map satisfying $\alpha(0) = p$ and $\alpha'(0) = v$. Show that the normal curvature of α at p is independent of the choice of the curve α and only depends on the direction v.

(7 marks)

(c) Let S be a regular surface in $\mathbb{R}^3$ which is homeomorphic to the round cylinder

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, -1 < |z| < 1\}.$$

Assume that the Gaussian curvature of S at every point is negative. Prove that there is a unique simple closed geodesic on S. (7 marks)

(Total: 20 marks)

5. (a) Consider the regular surface with boundary

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, -1 \leq z \leq +1\},$$

and let

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 < 1/4\}.$$

Calculate the Euler characteristic of $E \setminus G$. (5 marks)

(b) Give an example of a regular surface S such that there is a single point on S where the surface is planar. You do not need to present a precise formula or make precise calculations, but you need to justify your answer. (5 marks)

(c) Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$. Assume that there is a smooth map $F : S_1 \to S_2$ such that for every $p \in S_1$, $DF_p : T_pS_1 \to T_{F(p)}S_2$ is injective. Show that if S_2 is orientable, S_1 must be orientable. (5 marks)

(d) Let S be a regular surface in $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$, and assume that the intersection of S with the xy-plane is a regular curve. Show that S is either parabolic or planar at every $p = (x, y, 0) \in S$.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH60032/70032/97049

Geometry of Curves and Surfaces (Solutions)

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1. (a) We have

sim. seen ⇓

$$\phi'(t) = \left(e^t \sin t + e^t \cos t, e^t, e^t \cos t - e^t \sin t\right),$$

and hence

$$\begin{aligned}|\phi'(t)| &= \big(e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\cos t \sin t + e^{2t} \\ &\qquad + e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t \sin t.\big)^{1/2} \\ &= \sqrt{3}e^t.\end{aligned}$$

(2 pt)

Consider the map

$$h(t) = \ell(\phi([0,t]) = \int_0^t |\phi'(u)|du = \sqrt{3}e^u \,|_{u=0}^{u=t} = \sqrt{3}(e^t - 1).$$

We have $h^{-1}(s) = \log(s/\sqrt{3} + 1)$, with $s \in [0, +\infty)$.

(2 pt)

By a theorem in the lectures, the map

$$\phi \circ h^{-1}(s) = \left(\frac{s}{\sqrt{3}} + 1\right)\sin(\log(\frac{s}{\sqrt{3}} + 1)), \frac{s}{\sqrt{3}} + 1, (\frac{s}{\sqrt{3}} + 1)\cos(\log(\frac{s}{\sqrt{3}} + 1)).$$

is parametrised by arc length.

(1 pt) 5, A

(b) Consider the map $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined as

unseen ⇓

$$\gamma(t) = (1 + t^2, t).$$

For $(x, y) \in C$, $\gamma(y) = (1 + y^2, y) = (x, y)$. On the other hand, for every $t \in \mathbb{R}$, $\gamma(t)$ belongs to C. Therefore, the image of γ is equal to C.

(2 pt)

We note that γ is infinitely differentiable with all derivatives continuous. Thus, it is a smooth curve. Moreover, $\gamma'(t) = (2t, 1)$, which is non-zero at every point, which shows that γ is a regular curve.

(3 pt) 5, A

(c) To calculate the curvature of C, first we calculate the signed curvature of C using a formula in the lecture notes, that is,

meth seen ⇓

$$\kappa(\gamma(t)) = \frac{\langle \gamma''(t), n(t)\rangle}{|\gamma'(t)|^2},$$

where $n(t)$ is a unit normal to the curve γ at $\gamma(t)$.
With $\gamma(t) = (1 + t^2, t)$, we have

$$\gamma'(t) = (2t, 1), \quad \gamma''(t) = (2, 0), \quad n(t) = \frac{(-1, 2t)}{|(-1, 2t)|} = \frac{(-1, 2t)}{(1 + 4t^2)^{1/2}}.$$

Thus,

$$\kappa(\gamma(t)) = \frac{-2}{(1 + 4t^2)^{3/2}}.$$

(3 pt)

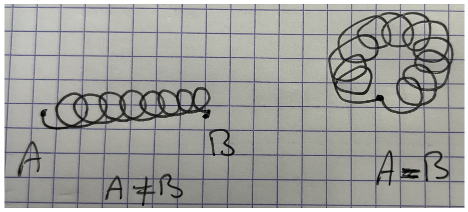
Since the curvature of a curve is the absolute value of the signed curvature, the curvature of C at $\gamma(t)$ is

$$k(\gamma(t)) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

(2 pt) 5, B

(d) Yes. Recall from the lectures that the curvature of a circle of radius r is $1/r$. Thus, by making r small, one can make the curvature arbitrarily large. If we perturb the circle slightly so that the curve, and its first and second derivatives only change slightly, the curvature of the curve only changes slightly. Repeatedly using this approach, we may build a curve with large curvature at every point, as in the figures below: $\boxed{\text{sim. seen} \Downarrow}$

(3 pt for drawing one of the curves, 2 pt for drawing the second one)



$\boxed{5, \text{B}}$

2. (a) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}$ defined as $\boxed{\text{meth seen} \Downarrow}$

$$F(x, y, z) = 5x^2 + y^2 - (2 + \sin z)^2.$$

This is a smooth function on $\mathbb{R}^3$, and we have $F^{-1}(0) = C$. We note that

$$\nabla F(x, y, z) = (10x, 2y, -2(2 + \sin z) \cos z) .$$

Then, $\nabla F(x, y, z) = 0$ implies that $x = y = 0$. But there is no point on C with $x = y = 0$ since $|\sin z| \le 1$. Therefore, ∇F is non-zero at every point on C. This shows that C is a regular level set. By a result in the lectures, every regular level set is a regular surface. $\boxed{5, \text{A}}$

(b) By a result in the lectures, the tangent plane to C at p, T_pC, is the set of all vectors which are orthogonal to $\nabla F(p)$. We have $\boxed{\text{meth seen} \Downarrow}$

$$\nabla F(0, 5/2, \pi/6) = (0, 5, -2(2 + 1/2) \cos \pi/6) = (0, 5, -5\sqrt{3}/2).$$

Equivalently, the tangent plane to C at p, is the set of all $(x, y, z) \in \mathbb{R}^3$ such that

$$y - \sqrt{3}z/2 = 0.$$

$\boxed{5, \text{A}}$

(c) In order to show that F is smooth, we need to show that for every chart $\phi : U \to C$, the map $F \circ \phi : U \to \mathbb{R}^3$ is a smooth map. The map $F : \mathbb{R}^3 \to \mathbb{R}^3$ has continuous partial derivatives of all orders, so it is smooth on $\mathbb{R}^3$. Let $\phi : U \to C$ be an arbitrary chart for S. By definition of charts, ϕ is smooth as a map from U to $\mathbb{R}^3$. Therefore, by the chain rule, the map $F \circ \phi : U \to \mathbb{R}^3$ is a smooth map. $\boxed{5, \text{B}}$

(d) We have

$$D_xF(x, y, z) = \left(\frac{\sqrt{5}}{2 + \sin z}, 0, 0\right), \quad D_yF(x, y, z) = \left(0, \frac{1}{2 + \sin z}, 0\right),$$

$$D_zF(x, y, z) = \left(\frac{-\sqrt{5}x \cos z}{(2 + \sin z)^2}, \frac{-y \cos z}{(2 + \sin z)^2}, 1\right).$$

Therefore,

$$D_xF(0, 5/2, \pi/6) = (2/\sqrt{5}, 0, 0), \quad D_yF(0, 5/2, \pi/6) = (0, 2/5, 0),$$
$$D_zF(0, 5/2, \pi/6) = (0, -\sqrt{3}/5, 1).$$

In order to identify $DF_p : T_pC \to T_{F(p)}D$, it is enough to find the value of this map on a basis for the tangent plane T_pC. By part (b), we may choose a basis for T_pC, say

$$\left\{v_1 = (0, \sqrt{3}, 2), v_2 = (1, \sqrt{3}, 2)\right\}.$$

Then,

$$DF_{(0,5/2,\pi/6)}(v_1) = \sqrt{3}D_yF(0, 5/2, \pi/6) + 2DF_z(0, 5/2, \pi/6) = (0, 0, 2),$$

$$\begin{aligned} DF_{(0,5/2,\pi/6)}(v_2) &= 1D_xF(0, 5/2, \pi/6) + \sqrt{3}D_yF(0, 5/2, \pi/6) + 2DF_z(0, 5/2, \pi/6) \\ &= (2/\sqrt{5}, 0, 2)). \end{aligned}$$

$\boxed{5, \text{C}}$

3. (a) Because S is the graph of a function, we have seen in the lectures that the map $\phi : \mathbb{R}^2 \to S$ defined as sim. seen $\Downarrow$

$$\phi(u,v) = (u, v, v^3 - u^3)$$

is a chart for S. We have

$$\phi_u(u,v) = (1, 0, -3u^2),$$

$$\phi_v(u,v) = (0, 1, 3v^2),$$

and

$$\langle \phi_u(u,v), \phi_u(u,v) \rangle = 1 + 9u^4,$$
$$\langle \phi_u(u,v), \phi_v(u,v) \rangle = -9u^2v^2,$$
$$\langle \phi_v(u,v), \phi_v(u,v) \rangle = 1 + 9v^4.$$

By a result in the lectures, *the matrix of the first fundamental form in the basis* $\{\phi_u, \phi_v\}$ is

$$g_{\phi(u,v)} = \begin{pmatrix} \langle \phi_u(u,v), \phi_u(u,v) \rangle & \langle \phi_v(u,v), \phi_u(u,v) \rangle \\ \langle \phi_u(u,v), \phi_v(u,v) \rangle & \langle \phi_v(u,v), \phi_v(u,v) \rangle \end{pmatrix} = \begin{pmatrix} 1 + 9u^4 & -9u^2v^2 \\ -9u^2v^2 & 1 + 9v^4 \end{pmatrix}.$$

5, A

(b) We have meth seen $\Downarrow$

$$\phi_{uu}(u,v) = (0, 0, -6u), \quad \phi_{uv}(u,v) = (0,0,0), \quad \phi_{vv}(u,v) = (0,0,6v).$$

The unit normal vector to the surface at $\phi(u,v)$ is

$$N(\phi(u,v)) = \frac{\phi_u(u,v) \times \phi_v(u,v)}{|\phi_u(u,v) \times \phi_v(u,v)|} = \frac{(3u^2, -3v^2, 1)}{(9u^4 + 9v^4 + 1)^{1/2}}.$$

By a result in the lectures, *the matrix of the second fundamental form in the basis* $\{\phi_u, \phi_v\}$ is

$$A_{\phi(u,v)} = \begin{pmatrix} \langle \phi_{uu}(u,v), N(\phi(u,v)) \rangle & \langle \phi_{vu}(u,v), N(\phi(u,v)) \rangle \\ \langle \phi_{uv}(u,v), N(\phi(u,v)) \rangle & \langle \phi_{vv}(u,v), N(\phi(u,v)) \rangle \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-6u}{(9u^4+9v^4+1)^{1/2}} & 0 \\ 0 & \frac{6v}{(9u^4+9v^4+1)^{1/2}} \end{pmatrix}$$

5, A

(c) Again, by a result in the lectures, the Gaussian curvature of S at $\phi(u,v)$ is given by

$$K(\phi(u,v)) = \frac{\det A_{\phi(u,v)}}{\det g_{\phi(u,v)}} = \frac{-36uv}{(9u^4 + 9v^4 + 1)^2}.$$

5, B

(d) The curve in part d may be parametrised as $\gamma(t) = (\sqrt{2}t/2, \sqrt{2}t/2, 0)$, for $t \in \mathbb{R}$. We have $\gamma'(t) = ((\sqrt{2}/2, \sqrt{2}/2, 0))$, which shows that γ is a regular curve parametrised by arc length. We have $\gamma''(t) = (0,0,0)$. Thus, the geodesic curvature at $\gamma(t)$ is unseen $\Downarrow$

$$k_g(\gamma(t)) = \langle \gamma'', N(\gamma(t) \times \gamma'(t)) \rangle \equiv 0.$$

This shows that γ is a geodesic on S. 5, C

4. (a) No, it is not. By a result in the lectures, any regular surface in $\mathbb{R}^3$ is locally the graph of a smooth function of either (x, y), (y, z), or (x, z). The set W near the point $(0, 0, 0)$ is not the graph of any such function. It cannot be the graph of a function of (x, z) because for any (x, z) near $(0, 0)$ there are two points (x, z^2, z) and $(x, -z^2, z)$ near $(0, 0, 0)$ in W. It cannot be the graph of a function of (y, z) since there are infinitely many x near 0 satisfying $(x, y, z) \in W$. The only possibility left is that it is the graph of a function of (x, y). meth seen ⇓

(3 pts for applying the result, and looking into the three cases)

There is a unique function, $g(x, y) = (x, y, \sqrt{|y|})$, which maps a neighbourhood of $(0, 0)$ in $\mathbb{R}^2$ into W. However, this function is not smooth at $(0, 0)$; it is not even differentiable at $(0, 0)$.

(2 pts for ruling out this case) 6, C

(b) Since the curvature of a parametrise curve only depends on its image, we may assume that α is parametrised by arc-length. meth seen ⇓

(1 pt)

By definition, the curvature of α at p is $\alpha''(0)$. Also, by definition, the normal curvature of α at p is

$$k_{n,\alpha}(p) = \langle \alpha''(0), N(p) \rangle$$

where N is the unit normal vector to the surface.

For every $t \in (-\epsilon, \epsilon)$, $\alpha'(t)$ belongs to $T_{\alpha(t)}S$ and $N(\alpha(t))$ is normal to $T_{\alpha(t)}S$, we have

$$\langle \alpha'(t), N(\alpha(t)) \rangle \equiv 0.$$

(2 pts)

Differentiating the above equation with respect to t, we obtain

$$\langle \alpha''(t), N(\alpha(t)) \rangle = -\langle \alpha'(t), (N(\alpha(t))' \rangle = -\langle \alpha'(t), dN_{\alpha(t)}(\alpha'(t)) \rangle.$$

(2 pts)

At $t = 0$, this gives us

$$k_{n,\alpha}(p) = -\langle v, dN_p(v) \rangle = A_p(v, v),$$

where A_p is the second fundamental form of S at p. The right hand side of the above equation does not depend on α, but only depends on $\alpha'(0)$. Therefore, the left hand side of the above equation only depends on $\alpha'(0)$.

(2 pts) 7, D

(c) Assume in the contrary that there are two simple closed geodesic on S, say γ_1 and γ_2. We consider two cases: $\gamma_1 \cap \gamma_2 = \emptyset$ and $\gamma_1 \cap \gamma_2 \neq \emptyset$. meth seen ⇓

First assume that $\gamma_1 \cap \gamma_2 = \emptyset$. Let U be the closure of the region on S bounded by γ_1 and γ_2. Then, U is a regular surface with boundary, which is homeomorphic to a closed round cylinder, and $\partial U = \gamma_1 \cup \gamma_2$. By a result in the lectures, the Euler characteristic of U is 0. Applying the Gauss-Bonnet thm to U, we obtain

$$0 = 2\pi\chi(U) = \int_U K\, dA + \int_{\partial U} k_g\, ds = \int_U K\, dA + \int_{\gamma_1 \cup \gamma_2} 0\, ds = \int_U K\, dA.$$

This contradicts $K < 0$ at every point on S.

(3 pts)

Now assume that $\gamma_1 \cap \gamma_2 \neq \emptyset$. The curves γ_1 and γ_2 cannot intersect tangentially. That is because if $\gamma_1(t_1) = \gamma_2(t_2)$ and $\gamma_1'(t_1) = \pm\gamma_2(t_2)$, by the uniqueness of the solution of the differential equations for geodesics, we must have $\gamma_1 = \gamma_2$. Also, it follows from the Jordan curve theorem that each of γ_1 and γ_2 divides S into two connected component. These imply that $\gamma_1 \cap \gamma_2$ contains at least two points.

The set $\{(s,t) \in \mathbb{R}^2 \mid \gamma_1(s) = \gamma_2(t)\}$ is a closed set. This is because, geodesics are continuous maps. Since γ_1 and γ_2 are distinct, we can find (s_1, t_1) and (s_2, t_2) such that $\gamma_1(s_1) = \gamma_2(t_1)$, $\gamma_1(s_2) = \gamma_2(t_2)$, and for all $s \in (s_1, s_2)$ and all $t \in (t_1, t_2)$, $\gamma_1(s) \neq \gamma_2(t)$. In other words, the two curves $\gamma_1([s_1, s_2]$ and $\gamma_2([t_1, t_2])$ only meet at their end points.

(2 pts)

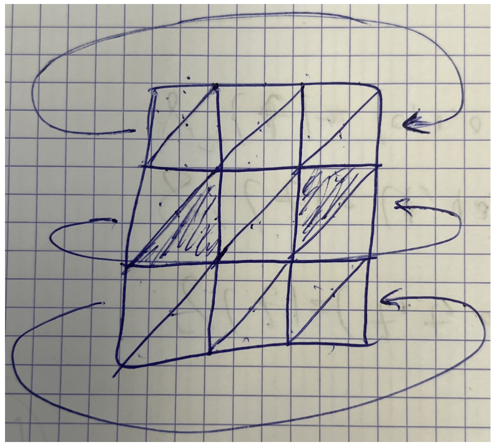
Let U be the closure of the region bounded by these two curves. Then, U is homeomorphic to a closed disk. By a result in the lectures, the Euler characteristic of U is 1. Thus, by the Gauss-Bonnet thm, we must have

$$2\pi = \int_U K\,dA + \int_{\gamma_1([s_1,s_2]) \cup \gamma_2([t_1,t_2])} 0\,ds = \int_U K\,dA.$$

This is also a contradiction since $K < 0$ on S.

(2 pts) $\boxed{7, \text{D}}$

5. (a) The set $E \setminus G$ is obtained from the round cylinder by removing two distinct regions each homeomorphic to an open disk. We may triangulate the set $E \setminus G$ as follows: meth seen $\Downarrow$



In this triangulation, the number of faces $F = 16$, the number of edges $E = 30$, and the number of vertexes $V = 12$. Then,

$$\chi(E \setminus G) = V - E + F = 12 - 30 + 16 = -2.$$

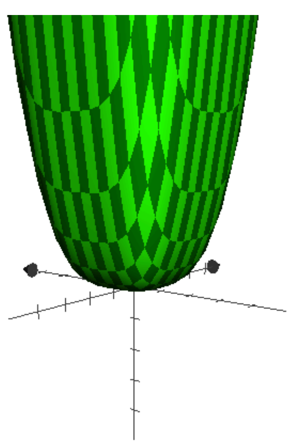
5, M

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with $f(0,0) = 0$, $f(x,y) > 0$ for all $(x,y) \neq (0,0)$, and f is tangent of order 2 to the xy plane at $(0,0)$. That is, $f_x(0,0) = f_y(0,0) = f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = 0$. We may choose f such that the second partial derivatives with respect to xx and yy are positive everywhere else. Let S be the graph of the function f in $\mathbb{R}^3$. Then S is planar at $(0,0)$, but not planar at any other point. meth seen $\Downarrow$

Indeed, by lectures, if there is a coordinate (x,y) near (x_0,y_0) in which the Taylor series expansion of f is of the form

$$f(x,y) = \lambda_1(x-x_0)^2 + \lambda_2(y-y_0)^2 + \text{ higher order terms}$$

then the Gaussian curvature of the graph of f at (x_0,y_0) is $\lambda_1\lambda_2$. In the above example, the local expansion of f near $(0,0)$ starts with degree three terms, the Gaussian curvature at $(0,0)$ must be zero, but since f_{xx} and f_{yy} are positive at other points, the Gaussian curvature is non-zero at all other points on the surface.



5, M

(c) Because S_2 is orientable, there is a continuous unit normal vector N_2 from S_2 to the unit sphere $\mathbb{S}^2$. meth seen ⇓

By the hypothesis, for every $p \in S_1$, $dF_p : T_pS_1 \to T_{F(p)}S_2$ is injective. By the Inverse Function Theorem for surfaces, there is a neighbourhood U_p of p in S_1 such that $F : U_p \to F(U_p)$ is a diffeomorphism. Shrinking U_j if necessary, we may assume that $F(U_j)$ is contained in the image of a single chart. We may write $S_1 = \cup_{j\in J} U_j$ such that F is a diffeomorphism on each U_j.

For each j, there are $E_{j,1} : F(U_j) \to \mathbb{R}^3$ and $E_{j,2} : F(U_j) \to \mathbb{R}^3$ such that at every $q \in F(U_j)$, $\{E_{j,1}(q), E_{j,2}(q)\}$ is a basis for T_qS_2, and the ordered list $(E_{j,1}(q), E_{j,2}(q), N_2(q))$ is a positively oriented bases for $\mathbb{R}^3$. For example, we may choose $E_{j,1}$ and $E_{j,2}$ as the first partial derivatives of the chart for $F(U_j)$.

Now, since $F : U_j \to F(U_j)$ is a diffeomorphism, there are $G_{j,1} : U_j \to \mathbb{R}^3$ and $G_{j,2} : U_j \to \mathbb{R}^3$ such that $dF(G_{j,1}) = E_{j,1}$ and $dF(G_{j,1}) = E_{j,1}$ at each point on U_j. It follows that $\{G_{j,1}(p), G_{j,2}(p)\}$ forms a basis for T_pS_1. Let use define $N_1 : U_j \to \mathbb{R}^3$ as the unit vector in the direction of $G_{j,1} \times G_{j,2}$. Because $G_{j,1}$ and $G_{j,2}$ are continuous, N_1 must be continuous on U_j.

If $U_m \cap U_n \neq \emptyset$, $N_1 : U_m \to \mathbb{S}^2$ agrees with $N_1 : U_n \to \mathbb{S}^2$ on $U_m \cap U_n$. To see this, let $p \in U_m \cap U_n$. By the above paragraph, $(E_{m,1}(F(p)), E_{m,2}(F(p)), N_2(F(p)))$ and $(E_{n,1}(F(p)), E_{n,2}(F(p)), N_2(F(p))))$ are positively oriented bases for $\mathbb{R}^3$. This means that there is a linear map $T_2 : \mathbb{R}^3$ to $\mathbb{R}^3$ which maps $(E_{m,1}(F(p)), E_{m,2}(F(p)), N_2(F(p)))$ to $(E_{n,1}(F(p)), E_{n,2}(F(p)), N_2(F(p))))$, and has a positive determinant. Since T_2 preserves the last vector $N_2(F(p))$, we conclude that the determinant of the restriction $T_2 : T_{F(p)}S_2 \to T_{F(p)}S_2$ is positive. The linear map $dF^{-1}_{F(p)} \circ T_2 \circ dF_p$ sends the pair of vectors $(G_{m,1}(p), G_{m,2}(p))$ to the pair $(G_{n,1}(p), G_{n,2}(p))$, and it has a positive determinant (the determinant of any linear map and its inverse have the same sign). This implies that there is a linear map $T_1 : \mathbb{R}^3$ to $\mathbb{R}^3$ which maps the triple $(G_{m,1}(p), G_{m,2}(p), N_1(p))$ to $(G_{n,1}(p), G_{n,2}(p), N_1(p))$, and has a positive determinant. This implies that $G_{m,1}(p) \times G_{m,2}(p)$ and $G_{n,1}(p) \times G_{n,2}(p)$ lie in the same direction. 5, M

(d) Let w be an arbitrary point in the xy-plane which also belongs to S. Let us choose a regular curve $\alpha : (-\epsilon, +\epsilon) \to S \cap xy$-plane such that $\alpha(0) = w$. By the hypothesis, the unit normal vector to the plane $N(\alpha(t))$ is constant (this is either $(0,0,1)$ or $(0,0,-1)$). Differentiating this equation with respect to t, we obtain

$$0 = (N(\alpha(t))'(0) = dN_{\alpha(0)}(\alpha'(0)).$$

Thus, the map $dN_w : T_wS \to T_wS$ is not invertible, it maps a non-zero vector to 0. This implies that $\det dN_p : T_wS \to T_wS$ is zero. By definition, this determinant is the Gaussian curvature of S at w.

If λ_1 and λ_2 are the principle curvatures of S at w, we have $\lambda_1\lambda_2 = 0$. There are two possibilities, either both λ_1 and λ_2 are zero and hence S is planar at w, or only one of them is 0 and hence S is parabolic at w. 5, M

Review of mark distribution:
Total A marks: 30 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 16 of 12 marks
Total D marks: 14 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Geometry of Curves and Surfaces_MATH60032 MATH97049 MATH70032	1	Parts a, b, and c are routine questions in this module, and the students have practiced these before. Most students were able to do these. Part b was new, and few students were able to do it.
Geometry of Curves and Surfaces_MATH60032 MATH97049 MATH70032	2	similar problems have been presented in the coursework,
Geometry of Curves and Surfaces_MATH60032 MATH97049 MATH70032	3	Parts a b and c are routine calculations, and are based on some theorems in the lectures. There has been enough practice in the problem sheets. Part d may have been new, but fairly doable.
Geometry of Curves and Surfaces_MATH60032 MATH97049 MATH70032	4	Part a can be answered using a proposition in the lecture notes. without that proportion it will be difficult to answer it. Many students identified the link, but a considerable number of students tried to do it with bare hands, and often not successful.
Geometry of Curves and Surfaces_MATH60032 MATH97049 MATH70032	5	This is a collection of few problems which require a little more thinking. To my surprise many students were able to deal with these problems.