Classical Dynamics **Autumn Term 2024**

Problem Sheet 3

1. The time evolution of a system is described by the Lagrangian $L(q_i, \dot{q}_i)$ which has no explicit time-dependence. Here q_i $(i = 1, .., N)$ are generalised coordinates. Use the Euler-Lagrange equations to show that

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - L$$

is a constant of the motion.

2. A bead of mass m moves without friction or gravity on a heart-shaped wire described in polar coordinates by the equation $r = 1 + \cos\theta$. Show that a Lagrangian for this system is

$$L(\theta, \dot{\theta}) = m(1 + \cos\theta)\dot{\theta}^2.$$

Find the general solution of the equation of motion and explain why the solutions are only valid for a finite time interval (excluding the trivial solutions $\theta =$constant).

Hint: Obtain $\theta(t)$ by solving the first order ODE $H = p_\theta\, \dot{\theta} - L =$constant.

3. (i) Show that the Lagrangian

$$L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

leads to the Lorentz force law for a (non-relativistic) charged particle

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r})$$

in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

Hint: Check that the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

yields the x-component of the Lorentz force law. Here A_x, A_y and A_z are arbitrary functions of x, y and z.

(ii) Consider the constant magnetic field $\mathbf{B} = b\mathbf{k}$ (or $B_x = B_y = 0$, $B_z = b$). Find a vector potential, $\mathbf{A}$, for this magnetic field and hence obtain a Lagrangian for a charged particle in the magnetic field. Are any of the coordinates (x, y and z) cyclic?

4. (i) The Lagrangian

$$L = \frac{R^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right),$$

describes the motion of a spherical pendulum in the absence of gravity (a spherical pendulum on a space station). Here R is a constant. Identify any circular orbits of the form $\phi =$ constant or $\theta =$ constant.

(ii) A different spherical pendulum is found aboard an abandoned alien spacecraft. The Lagrangian for this pendulum is

$$L = \frac{R^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + \alpha(1 - \cos\theta)\dot{\phi},$$

where α is a non-zero constant. Identify any circular orbits of the form $\phi =$ constant or $\theta =$ constant. Comment on the result.