Seen B

B.1. Prove that the linear transformation you found in A.3(a) is unique.

The argument in A.3(a) shows that any linear transformation with the properties of A.3(a) must map the standard basis elements to $(1,1,0)$ and $(0,1,1)$. This totally determines where every vector must be mapped, and therefore the transformation is unique.

B.2. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Which of the following functions $T : V \longrightarrow V$ are linear transformations?

(a) $T(A) = A^2$ for all $A \in V$

(b) $T(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $A \in V$

(a) No, since $T(2I) = 4I \neq 2T(I)$.

(b) Yes; writing $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, we have

$$T(A_1 + A_2) = M(A_1 + A_2) = MA_1 + MA_2 = T(A_1) + T(A_2), \text{ and}$$
$$T(\lambda A) = M(\lambda A) = \lambda MA = \lambda T(A).$$

B.3. Let V be the vector space (over $\mathbb{R}$) of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are linear transformations (thinking of $\mathbb{R}$ as $\mathbb{R}^1$ in parts (a), (c) and (d))?

(a) $T_1 : V \to \mathbb{R}$ where $T_1(f) = f(1)$ (for $f \in V$).

(b) $T_2 : V \to V$ where $T_2(f) = f \circ f$ (for $f \in V$).

(c) $T_3 : \mathbb{R} \to V$ where $T_3(\mu)$ is the function $f_\mu \in V$ given by $f_\mu(x) = \mu x$ (for $\mu, x \in \mathbb{R}$).

(d) $T_4 : V \to \mathbb{R}$ where

$$T_4(f) = \int_{-\infty}^{\infty} f(x)\, dx$$

(a) If $f, g \in V$ and $\lambda \in$ $mathbbR$, then $T_1(f+g) = (f+g)(1) = f(1) + g(1) = T_1(f) + T_1(g)$ and $T_1(\lambda f) = \lambda f(1) = \lambda T_1(f)$. So T_1 is a linear transformation.

(b) Not a linear transformation. For example, consider $f \in V$ with $f(x) = x$. Then $T_2(2f) \neq 2T_2(f)$.

(c) This is a linear transformation. If $\mu_1, \mu_2, \lambda, x \in$ $mathbbR$, then $(T_3(\mu_1 + \mu_2))(x) = (\mu_1 + \mu_2)x = (T_3(\mu_1) + T_3(\mu_2))(x)$, so $T_3(\mu_1 + \mu_2) = T_3(\mu_1) + T_3(\mu_2)$. Similarly, $T_3(\lambda\mu_1)(x) = (\lambda\mu_1)x = \lambda(\mu_1 x) = \lambda T_3(\mu_1)(x)$. (The difficulty here is keeping track of the notation.)

(d) This preserves addition and scalar multiplication, but is not a linear transformation, as it's not a function! Let $f(x) = 1$ for all x.

$$T_4(f) = \infty$$

and is not an element of $\mathbb{R}$.

B.4. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly one vector $v \in \mathbb{R}^2$.

(b) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for no vector $v \in \mathbb{R}^2$.

(c) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for infinitely many vectors $v \in \mathbb{R}^2$.

(d) Show that there is no linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly two vectors $v \in \mathbb{R}^2$.

(a) $T(x_1, x_2) = (x_1, x_2, 0)$ is one example.

(b) $T(x_1, x_2) = (0,0,0)$.

(c) $T(x_1, x_2) = (x_1, 0)$.

(d) Suppose v_1 and v_2 are distinct vectors in $\mathbb{R}^2$ with $T(v_1) = T(v_2) = (1,0,0)$. Then

$$T(v_2 - v_1) = (1,0,0) - (1,0,0) = (0,0,0).$$

So for any $\lambda \in \mathbb{R}$ we have

$$T(v_1 + \lambda(v_2 - v_1)) = (1,0,0) + \lambda(0,0,0) = (1,0,0).$$

So we have infinitely many vectors $v = (1-\lambda)v_1 + \lambda v_2$ such that $T(v) = (1,0,0)$.

B.5. (a) Suppose V, W are vector spaces (over a field F) and $S, T : V \to W$ are linear transformations. Prove that $S + T : V \to W$ defined by $(S+T)(v) = S(v) + T(v)$ (for $v \in V$) is a linear transformation. If $\lambda \in F$, show that $\lambda S : V \to W$ defined by $(\lambda S)(v) = \lambda S(v)$ (for $v \in V$) is a linear transformation. Explain why the set U of all linear transformations from V to W is a vector space with these operations.

(b) In the case where $V = F^2$ and $W = F^3$, what is the dimension of the vector space U? What is the dimension of U for arbitrary finite dimensional vector spaces V and W?

(a) If $v_1, v_2 \in V$ then

$$(S+T)(v_1+v_2) = S(v_1+v_2)+T(v_1+v_2) = Sv_1+Sv_2+Tv_1+Tv_2 = (S+T)v_1+(S+T)v_2,$$

so $S+T$ preserves addition. And if $v \in V$ and $\lambda \in F$ then

$$(S+T)(\lambda v) = S(\lambda v) + T(\lambda v) = \lambda Sv + \lambda Tv = \lambda(Sv + Tv) = \lambda(S+T)v,$$

so λS preserves scalar multiplication.

If $v_1, v_2 \in V$ then

$$(\lambda S)(v_1 + v_2) = \lambda S(v_1 + v_2) = \lambda S v_1 + \lambda S v_2 = (\lambda S)v_1 + (\lambda S)v_2,$$

so $S + T$ preserves addition. And if $v \in V$ and $\mu \in F$ then

$$(\lambda S)(\mu v) = \lambda S(\mu v) = \lambda\mu S v = \mu\lambda S v = \mu(\lambda S)v,$$

so λS preserves scalar multiplication.

We have addition and scalar multiplication defined on U, so we just need to check that the vector space axioms are satisfied; this is routine. (The zero of U is the map which sends $v \mapsto 0_W$ for all $v \in V$. For $S \in U$, the negative $-S$ is the map $v \mapsto -(Sv)$.)

(b) The hard thing here is to prove that every linear transformation $S : F^2 \to F^3$ can be represented by a 3×2 matrix A with coefficients in F (we will cover this in detail in the course - but for now have a think about it). And it is clear that every 3×2 matrix A corresponds to an element S of U, given by $S(v) = Av$. So U is "essentially" just the space of 3×2 matrices, and this has dimension 6.

But let's turn that "essentially" into something rigorous. Let $\mathrm{Mat}_{3,2}(F)$ be the vector space of 3×2 matrices with entries from F. Then the map $S \mapsto A$ gives a bijection $\Phi : U \to \mathrm{Mat}_{2,3}(F)$. It is easy to check that this map is a linear transformation. Now since $\ker \Phi = \{0\}$ and $\mathrm{Im}\Phi = \mathrm{Mat}_{2,3}(F)$, Rank–Nullity tells us that $\dim U = \dim M_{2\times 3}(F)$.

In the general case, if $\dim V = m$ and $\dim W = n$, then $\dim U = mn$, by the same argument.

You could also prove this by finding an appropriate basis for U.