	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 1	Topic BASIC MATERIAL	Marks& seen/unseen

Parts

(a). A pair of strategies, α^* for player A and β^* for player B, are said to be in <u>equilibrium</u> if:

$$g_A(\alpha^*,\beta^*) \geq g_A(\alpha,\beta^*), \quad \forall \alpha \in \mathbb{A}_S$$
$$g_B(\alpha^*,\beta^*) \geq g_B(\alpha^*,\beta), \quad \forall \beta \in \mathbb{B}_S.$$

3 A seen definition

(b). A strategy $\alpha \in \mathbb{A}_S$ is <u>strictly dominated</u> by another strategy $\alpha' \in \mathbb{A}_S$ if:

$$g_A(\alpha,\beta) < g_A(\alpha',\beta), \quad \forall \beta \in \mathbb{B}_S.$$

We say $\alpha \in \mathbb{A}_S$ is <u>weakly dominated</u> by $\alpha' \in \mathbb{A}_S$ if the above inequality becomes $\leqslant$, where the inequality remains strict for at least one possible β. ~~soto~~

2 A seen definition

(c). Suppose α' strictly dominates α for A.

This means that α is never part of an equilibrium since player A could always deviate to α' to do better.

Denoting the original game by G and the game with α' removed by G', we can say that any equilibrium in G doesn't contain α, so therefore

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 1

Topic

Parts

Marks& seen/unseen

(c). (contined.) is a possible set of strategies in G' where it is also an equilibrium.

2 A seen proof

On the other hand, any equilibrium in G' is a possible set of strategies in G where it is either an equilibrium, or, if not, then player A must be able to benefit by deviating to the strategy α.

But if this were the case then they could deviate to α' to do better which is a possible strategy in G', violating the fact we started with an equilibrium of G'.

Thus G and G' have the same equilibria. □

3 B seen proof

(d). G is termed degenerate if at least one of the players has a mixed strategy that assigns positive probability to exactly k pure strategies such that the other player has more than k best responses to that mixed strategy.

1 A seen definition

Setter's initials SJB

Checker's initials

Page number 2

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 1

Topic

Marks& seen/unseen

Parts

(e).

(i). We produce a strategic form of the game: (x_i, y_j) means X produces i, Y produces j.

	y_0	y_2	y_4	y_6	y_8
x_0	0,0	0,32	0,48	0,60	0,64
x_2	32,0	24,24	20,40	16,48	12,48
x_4	48,0	40,20	32,32	24,36	16,32
x_6	60,0	48,16	36,24	24,24	12,16
x_8	64,0	48,12	32,16	16,12	0,0

Seen similar game

$\boxed{A}$ 1 Seen similar

Now x_0 is strictly dominated by x_2, x_4 or x_6, so we can delete x_0 from the game. Similarly, by the symmetry of the payoffs, we can delete y_0 from the game.

x_2 is strictly dominated by x_4, so we can delete x_2 from the game. Similarly y_2 can be removed.

$\boxed{B}$ 1 Seen similar

Upon the deletion of these pure strategies, x_8 is now strictly dominated by x_6, so we can delete x_8. Similarly we can delete y_6.

$\boxed{B}$ 1 Seen similar

We are left with the game:

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 1

Topic

Marks& seen/unseen

Parts

(e). (i). (continued.)

X \ Y	y_4	y_6
x_4	32,32	24, $\boxed{36}$
x_6	36, $\boxed{24}$	24, $\boxed{24}$

This game has <u>three</u> pure strategy equilibria, at (x_6, y_4), (x_6, y_6) and (x_4, y_6). Note that the game is also <u>degenerate</u>, owing to each players remaining weakly dominated strategy. One can check the best response condition, but it is straightforward to see that we also find the infinite set of equilibria $(x_6, (q, 1-q))$, for any $0 \leq q \leq 1$ and $((p, 1-p), y_6)$, for any $0 \leq p \leq 1$.

This gives <u>all</u> equilibria in this 2×2 game, which should be then extended to give all equilibria of the original game.

B

1 identify pure equilibria

seen similar

D

2

unseen

← This should be done for final B mark

(ii). We have: $g_X(x,y) = 16x - x^2 - xy$. For fixed y, X's best response is to play x such that $\frac{\partial g_X}{\partial x} = 16 - 2x - y = 0$.

i.e. play x such that: $x = 8 - \frac{y}{2}$

C

1

seen similar

Setter's initials SJB

Checker's initials

Page number 4

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 1 | Topic

Marks& seen/unseen

Parts (e). (ii). (continued.)

Due to the symmetry, Y's best response against fixed x is to play y such that: $y = 8 - \frac{x}{2}$.

1 C seen similar

Solving this pair of equations simultaneously leads to $(x,y) = (\frac{16}{3}, \frac{16}{3})$. The unique equilibrium of the game.

1 C seen similar

Q1 Total: 20

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 2	Topic Zero-Sum Games	Marks& seen/unseen

Parts

(a). A strategy $\alpha^* \in A_S$ is an <u>equaliser strategy</u> for A if and only if:

$$g(\alpha^*, b) = \text{constant}, \quad \forall b \in B_S.$$

2 [A] Seen definition (in bimatrix games)

(b). Assume α^*, β^* are ES for A and B respectively. Then:

$$g(\alpha^*, b) = c_1, \quad \forall b \in B_S.$$
$$g(a, \beta^*) = c_2, \quad \forall a \in A_S.$$

$$\Rightarrow g(\alpha^*, \beta^*) = \sum_i p_i g(a_i, \beta^*), \text{ by writing } \alpha^* = \sum_i p_i a_i$$
$$= c_2 \sum_i p_i$$
$$= c_2 = g(a, \beta^*), \quad \forall a \in A_S.$$

Similarly $g(\alpha^*, \beta^*) = c_1 = g(\alpha^*, b), \ \forall b \in B_S$.

Thus, α^* and β^* are mutual best responses and form an equilibrium of the game.

Seen proof (in bimatrix games) [A] 3

Setter's initials	Checker's initials	Page number
SJB		6

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 2

Topic

Marks& seen/unseen

Parts

(c). (i).

		b_1	b_2	b_3	b_4
A	a_1	$\boxed{0}$	2	3	4
	a_2	2	$\boxed{0}$	3	4
	a_3	3	3	$\boxed{0}$	4
	a_4	4	4	4	$\boxed{0}$

(Columns: B)

Let a_i / b_i represent the choice that A/B chooses $i \in \{1,2,3,4\}$.

Zero-sum game.

No pure strategy equilibria or obvious dominated strategies.

unseen game. A 2 seen similar

We look for a pair of equaliser strategies, α^*, β^* for A and B. Let $\alpha^* = (p, q, r, 1-p-q-r)$. Then, if α^* is an ES for A, we have:

C 1 unseen ES with four pure strats.

$$g(\alpha^*, b_1) = g(\alpha^*, b_2) = g(\alpha^*, b_3) = g(\alpha^*, b_4)$$

$$\Rightarrow 2q + 3r + 4(1-p-q-r) = 2p + 3r + 4(1-p-q-r)$$

$$= 3p + 3q + 4(1-p-q-r) = 4(p+q+r).$$

The first equality leads to $\underline{p = q}$. In the second equality this gives: $\underline{r = \frac{4}{3}p}$. In the third equality this gives: $4 + 6p = 8\left(\frac{10}{3}p\right) \Rightarrow \underline{p = \frac{6}{31}}$

Thus:

$$\underline{\alpha^* = \left(\frac{6}{31}, \frac{6}{31}, \frac{8}{31}, \frac{11}{31}\right)} \text{ is an ES for A.}$$

B 2 seen similar

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 2

Topic

Marks& seen/unseen

Parts

(c). (i). (continued.)

From the symmetrical payoff structure, we note that:
$\beta^* = \alpha^* = \left(\frac{6}{31}, \frac{6}{31}, \frac{8}{31}, \frac{11}{31}\right)$ is also an ES for B.
Thus (α^*, β^*) together form an equilibrium and hence a solution to the game, which has value:

$$V_1 = g(\alpha^*, b_4) = 4\left(\frac{6}{31} + \frac{6}{31} + \frac{8}{31}\right) = \underline{\underline{\frac{80}{31}}}.$$

$\boxed{B}$ 2 Seen similar.

← For part (iii)!

(ii).

		B: b_1	b_2	b_3	b_4
A	a_1	0	1	1	1
	a_2	1	0	2	2
	a_3	1	2	0	3
	a_4	1	2	3	0

We consider the sub-game in which A never plays a_1 (this can be shown to be strictly dominated in fact, e.g. by $\frac{1}{2}a_3 + \frac{1}{2}a_4$) and B never plays b_4:

unseen game.

$\boxed{A}$ 1 Seen similar

		B: b_1	b_2	b_3
A	a_2	1	0	2
	a_3	1	2	0
	a_4	1	2	3

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 2	Topic	Marks& seen/unseen
Parts		
(c). (ii). (continued).	Clearly b_1 is an ES for B in this sub-game. It is also straightforward to see $\alpha = \frac{1}{2}a_2 + \frac{1}{2}a_3$ is an ES for A here. (or can calculate)	2 seen similar C; 1 seen similar B
	This suggests that (α^*, β^*), where: $\alpha^* = (0, \frac{1}{2}, \frac{1}{2}, 0)$, $\beta^* = (1, 0, 0, 0)$ is a solution to the original game. We check this: $g(\alpha^*, b_4) = \frac{5}{2} > 1 = g(\alpha^*, b_j)$, $j = 1, 2, 3$. So player B has no incentive to switch to b_4. Also: $g(a_1, \beta^*) = 0 < 1 = g(a_i, \beta^*)$, $i = 2, 3, 4$, so A has no incentive to switch to a_1 (in fact if you spotted a_1 was strictly dominated earlier this check is unnecessary!).	D; 3 Seen Similar (unseen in zero-sum game)
	Thus (α^*, β^*) is a <u>solution</u> to this game. This game has value, $V_2 = 1$.	← part (iii).
(iii).	We found $V_1 = \frac{80}{31} > 1 = V_2$, so the first game is more profitable for player A.	D; 1 values.
		Q2 Total: 20
	Setter's initials SJB Checker's initials	Page number 9

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic: Cooperation + Congestion Games

Parts

Marks& seen/unseen

(a).

(i).

		B	
		b_l	b_h
A	a_l	$\frac{l-c}{2}, \frac{l-c}{2}$	$l-c, 0$
	a_h	$0, l-c$	$\frac{h-c}{2}, \frac{h-c}{2}$

unseen game

1 A

seen similar

(ii). We look at builder A's payoffs:

		B	
		b_l	b_h
A	a_l	$\frac{l-c}{2}$	$l-c$
	a_h	0	$\frac{h-c}{2}$

we check:

$$0 < c < 2l-h$$

$$\Rightarrow h < h+c < 2l \quad (+h)$$

$$\Rightarrow h-2c < h-c < 2l-2c \quad (-2c)$$

$$\Rightarrow \frac{h-2c}{2} < \underline{\frac{h-c}{2} < l-c} \quad (\div 2)$$

(for 2nd column)

unseen inequality manipulations

1 A

$$0 < \underbrace{c < 2l-h}$$

$$\Rightarrow h < 2l-c$$

$$\Rightarrow h-l < l-c$$

but $h > l \Rightarrow 0 < h-l < l-c$

$$\Rightarrow \underline{0 < l-c} \Rightarrow \underline{0 < \frac{l-c}{2}}$$

(for 1st column)

1 A

$\Rightarrow a_l$ is a max-min strategy for A giving threat level: $\boxed{t_A = \frac{l-c}{2}}$ By the symmetry: $\boxed{t_B = \frac{l-c}{2}}$

1 A

seen similar

Setter's initials: SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

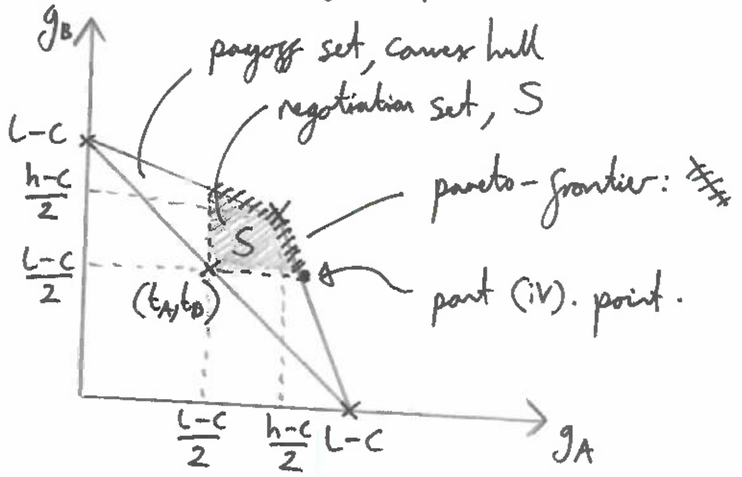
Question 3

Topic

Marks& seen/unseen

Parts

(a). (iii). $0 < \frac{l-c}{2} < \frac{h-c}{2} < l-c$ from part (ii).



Seen similar

$\boxed{A}$

3 (1: payoff set. 1: S 1: pareto-frontier).

(iv). Neither builder can expect the other to accept anything less than their threat level, this means the most either builder could expect to get is the extremum of points within the negotiation set where g_A (for A) or g_B (for B) is highest.

The point where the profit is thus maximised for A is labelled on the diagram in part (iii).

unseen

1 $\boxed{D}$

This point lies on the line passing through $(l-c, 0)$ and $\left(\frac{h-c}{2}, \frac{h-c}{2}\right)$, with equation:

unseen

1 $\boxed{D}$

$$\frac{y-0}{\left(\frac{h-c}{2}\right)-0} = \frac{x-(l-c)}{\left(\frac{h-c}{2}\right)-(l-c)}$$

$$\Rightarrow y = (h-c)\cdot\left(\frac{x-l+c}{h+c-2l}\right).$$

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic

Parts

Marks& seen/unseen

(a). (iv). (continued.) The value $x = m$ is thus obtained on this line when $y = \frac{l-c}{2}$, so plugging these values in gives:

$$\frac{(l-c)}{2} \cdot \left(\frac{1}{(h-c)}\right) \cdot (h+c-2l) - (c-l) = m$$

$$\Rightarrow m = \frac{(l-c)}{2(h-c)}\left(h+c-2l+2(h-c)\right)$$

$$\Rightarrow m = \frac{(l-c)(3h-2l-c)}{2(h-c)}, \text{ as required.}$$

3 $\boxed{D}$ unseen

(v). By the symmetry property of the Nash bargaining solution it must lie in the pareto-optimal frontier and on the line $y = x$, hence the solution is $\left(\frac{h-c}{2}, \frac{h-c}{2}\right)$, the builders should both make the higher bid £h.

2 $\boxed{B}$ seen similar

(b). (i). The price of anarchy (POA) in a congestion game is defined to be:

$$\text{POA} = \frac{\text{worst average cost per user in any equilibrium}}{\text{Average cost per user in social optimum}}.$$

1 $\boxed{A}$ seen definition

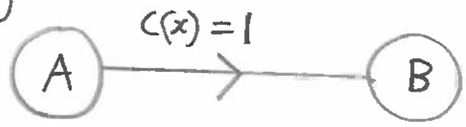
Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 3	Topic	Marks& seen/unseen

Parts

(b). (ii). e.g.

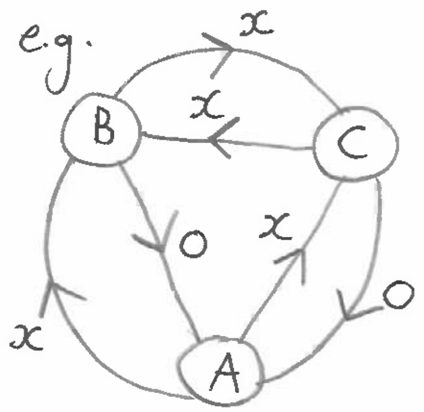


1 user going from $A \to B$.

$$POA = \frac{1}{1} = 1.$$

B 2 unseen

(iii). e.g.



4 users, destinations + origins in table:

User, i	O_i	D_i
1	A	B
2	A	C
3	B	C
4	C	B

unseen

C

Social optimum: all pay $\underline{\underline{1}}$ on the routes with x.

Worst equilibrium: all take indirect routes via 2 edges.

user:	route:	cost:
1	$A \to C \to B$	3
2	$A \to B \to C$	3
3	$B \to A \to C$	2
4	$C \to A \to B$	2

Av. cost per user $= \frac{5}{2}$.

+ is equilibrium

$\Rightarrow$ $\underline{\underline{POA = \frac{5}{2}}}$.

3 (this example seen)

[Many Answers Possible (b)(ii), (iii).]

Q3: Total: 20

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 4

Topic Impartial Games

Marks& seen/unseen

Parts

(a). If $G \equiv *m$ for an impartial game G, then we call m the <u>Nim value</u> of G (here $*m$ represents a single Nim pile of size m).

2 $\boxed{A}$ seen definition

(b). The copycat principle for an impartial game G states that:

$$G + G \equiv 0.$$

1 $\boxed{A}$ seen theorem

<u>Proof:</u> $G + G \equiv 0 \iff G+G$ is a losing game. To show this we will use top down induction by showing that every option of $G+G$ is winning. Indeed, any option of $G+G$ is of the form $G'+G$ for an option G' of G. But then the next player has the winning move to the game $G'+G'$ which is a losing game by the inductive hypothesis (since $G'+G'$ is <u>simpler</u> than $G+G$). It remains to take the losing game $0+0$ as a base case and we are done. $\square$

3 $\boxed{A}$ seen proof

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 4 | Topic

Marks& seen/unseen

Parts

Rest: Unseen
Game:

(c).

(i). Blue (player B) wins in this position. Whichever row red makes a move in, player B makes the counter move (closing the gap between the red and blue counter to zero again) in the same row. In this way blue always has a move left over red so wins.

A 1 seen similar principle

B 1 seen similar principle

(ii). A single row corresponds to a Nim pile which can be reduced, by closing the gap between the counters, or can be increased (if there is additional space left in the grid) by widening the gap between the counters. This corresponds precisely to the game of <u>poker Nim</u> where we know that a single poker-Nim pile of size m has Nim value $*m$. The gap between the counters in Northcott's game corresponds to this size of the poker Nim pile, and thus has Nim value $*m$.

B 2 unseen

C 1 unseen but poker Nim seen.

(iii). We draw out the grid and label the equivalent poker-Nim piles on the right side.

We also make the comment that a move for a player takes place in a single row, leaving the other rows unchanged. Therefore Northcott's game with multiple rows is a <u>game sum</u> of single row Northcott's games.

Setter's initials SJB

Checker's initials

Page number 15

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 4

Topic

Marks& seen/unseen

Parts (c). (iii). (continued).

	a	b	c	d	e	f	
1			r		b		$\equiv *1$
2	r			b			$\equiv *2$
3		r				b	$\equiv *3$
4			r			b	$\equiv *2$
5		r	b				$\equiv 0$

B 2 unseen

The Nim value of this position is then:

$*1 + *2 + *3 + *2 + 0 \equiv \underline{\underline{*2}}$ (a game sum of single row Northcott's games)

$\Rightarrow$ The position is <u>winning</u> for red. (Nim value $\neq 0$).

C 1 seen similar

Calling the game G, we have that $G + *2 \equiv 0$ by the copycat principle. A winning move from G therefore corresponds to adding the Nim-sum $*2$ to G, which we can do by adding $*2$ to any one of the five Poker-Nim piles. We check each case to see if the resulting new pile size is a legal move in the Northcott's game:

unseen C 1 Some form of reasoning

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 4

Topic

Marks& seen/unseen

Parts

(c). (iii). (contined.)

row 1 : $*1 + *2 \equiv *3$ ✓ → increasing the gap size from 1 to 3 here is a possible winning move.

row 2 : $*2 + *2 \equiv 0$ ✓ → this corresponds to reducing the gap size in this row to 0; another possible winning move.

row 3 : $*3 + *2 \equiv *1$ → this corresponds to reducing the gap from 3 to 1 ✓

row 4 : $*2 + *2 \equiv 0$ ✓ → reduce the gap to 0 in this row.

row 5 : $*0 + *2 \equiv *2$ X → this corresponds to increasing the gap size to 2 in this row. This is <u>not</u> a possible legal move for red in this row as there is insufficient remaining space to do this for them in this row.

D 2 unseen

(iv). The position is still winning for blue, with Nim value $*2$, as calculated before.

D 1 unseen

The ~~gap~~ <u>reducing</u> winning moves for red are also possible for blue (the gap size is the same for both players); so the winning moves in rows 2, 3 and 4

D 1 unseen

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Question	Topic	Course: Introduction to Game Theory
4		Marks& seen/unseen

Parts

(c). (iv). (contined.)

also hold for blue. However the winning move necessary in row 1 (increasing the gap size to 3) is not available to blue but the winning move in row 5 (increasing the gap to size 2) is available for blue.

1 unseen D

[One may question why this game can be seen as impartial → indeed the players use different pieces and thus, as seen, sometimes have different moves available to them. The key point is that the winning moves that reduce a gap between counters (reduce the respective Nim pile) are available to either player: (in essence it wouldn't matter whether they moved in the blue or red counter).

The extra poker-Nim like moves (increasing the gap sizes) are not needed to win → they never are: this is the whole consequence of the mex rule.]

Q4 Total: 20

Setter's initials: SJB

Checker's initials:

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 5	Topic Mastery	Marks& seen/unseen
Parts		
(a).	Let $\alpha^* = k\alpha + (1-k)\hat{\alpha}$, now writing $\alpha^* = (p_1^*, \dots, p_n^*)$, $\alpha = (p_1, \dots, p_n)$, $\hat{\alpha} = (\hat{p}_1, \dots, \hat{p}_n)$, then we have that $p_i^* \geq 0$ for all i and: $$\sum_{i=1}^{n} p_i^* = \sum_{i=1}^{n} (kp_i + (1-k)\hat{p}_i) = k\sum_i p_i + (1-k)\sum_i \hat{p}_i = k + 1 - k = 1.$$	1 seen proof
	So $\alpha^* \in A_S$ as needed. Now for any mixed strategy $\tilde{\alpha} \in A_S$ we have: $g_A(\alpha, \beta) \geq g_A(\tilde{\alpha}, \beta)$ and $g_A(\hat{\alpha}, \beta) \geq g_A(\tilde{\alpha}, \beta)$, since (α, β) and $(\hat{\alpha}, \beta)$ are equilibria. Consequently: $$g_A(\alpha^*, \beta) = k g_A(\alpha, \beta) + (1-k) g_A(\hat{\alpha}, \beta) \geq k g_A(\tilde{\alpha}, \beta) + (1-k) g_A(\tilde{\alpha}, \beta) = g_A(\tilde{\alpha}, \beta),$$	seen proof 3
	showing α^* is a best response to β. Since β is a BR to $\hat{\alpha}$ and α it is also a best response to α^*, hence (α^*, β) is also an equilibrium for any $k \in [0,1]$. $\square$	seen proof 1
	Setter's initials SJB	Checker's initials
		Page number 19

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic

Marks& seen/unseen

Parts

(b). Let's start with the subgame with a_1 removed:

		B	
A	b_1	b_2	b_3
a_2	$0, \boxed{4}$	$3, 2$	$2, 1$
a_3	$1, 0$	$2, 2$	$0, \boxed{3}$

• No pure strategy equilibria.

• Let $\alpha = (p, 1-p)$

$\beta = (q, r, 1-q-r)$

If α is an ES for A then:

$$g_B(\alpha, b_1) = 4p = g_B(\alpha, b_2) = 2 = g_B(\alpha, b_3) = 3 - 2p$$

This leads to $p = \frac{1}{2}$ which satisfies both equalities, so the subgame is degenerate. Thus in any equilibrium in this subgame A plays $\alpha_1 = \frac{1}{2}a_2 + \frac{1}{2}a_3$. For this to be the case, A needs to be made indifferent, so we insist:

$$g_A(a_2, \beta) = 3r + 2(1-q-r) = 2 - 2q + r$$

and

$$g_A(a_3, \beta) = q + 2r,$$ are equal.

This leads to $r = 2 - 3q$, giving:

$$\beta_1 = (k_1, 2-3k_1, 2k_1 - 1), \text{ where } k_1 \in \left[\tfrac{1}{2}, \tfrac{2}{3}\right]$$

where the bounds on k_1 ensure that β_1 remains a valid mixed strategy for B. (α_1, β_1) form all equilibria in this subgame.

Seen method.

unseen degeneracy in a 3x3 game.

3

1st Subgame equilibria.

Setter's initials SJB

Checker's initials

Page number

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic

Marks& seen/unseen

Parts

(b). (continued).

We now check whether these remain equilibria in the original game.

In the subgame: $g_A(a_2,\beta_1) = g_A(a_3,\beta_1) = 4-5k_1$.

But:

$$g_A(a_1,\beta_1) = 2k_1+2k_1-1 = 4k_1-1,$$ so the equilibria from the subgame remain equilibria of the original game provided: $4k_1-1 \leq 4-5k_1$

$$\Leftrightarrow \underline{\underline{k_1 \leq \tfrac{5}{9}}}$$

This means: $\underline{((0,\tfrac{1}{2},\tfrac{1}{2}),(k_1,2-3k_1,2k_1-1))}$, are equilibria provided $\underline{\underline{k_1 \in [\tfrac{1}{2},\tfrac{5}{9}]}}$.

Seen method. unseen degeneracy in 3×3 game.

2

check if sols. of full game.

Now remove a_2:

		B		
		b_1	b_2	b_3
A	a_1	2, $\boxed{1}$	0, $\boxed{1}$	1, $\boxed{1}$
	a_3	1, 0	2, 2	0, $\boxed{3}$

- If A puts any +ve probability on a_3 here, then B plays b_3 as b_3 weakly dominates b_2 and b_1.

Thus in any equilibrium in this subgame, A plays a_1.

Denoting $\beta_2 = (q, r, 1-q-r)$ (abuse of notation re-using q and r here), for a_1 to remain a BR for A against β_2 we must have that:

Setter's initials SJB

Checker's initials

Page number 21

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic

Marks& seen/unseen

Parts (b). (continued).

$g_A(a_1, \beta_2) \geqslant g_A(a_3, \beta_2)$, i.e.

$1 + q - r \geqslant q + 2r \iff r \leqslant \frac{1}{3}$

This results in all equilibria in this subgame being:

$$(a_1, \underbrace{(k_2, k_3, 1-k_2-k_3)}_{\beta_2}), \text{ where } k_3 \in [0, \tfrac{1}{3}], \quad k_2 \in [0, 1-k_3]$$

Seen methods

unseen degeneracy in 3x3 game.

3

2nd subgame equilibria.

To check if these remain equilibria in the original game we check against a_2. $g_A(a_1, \beta_2) = 1 + k_2 - k_3$, but $g_A(a_2, \beta_2) = 2 - 2k_2 + k_3$, so we need to ensure that:

$$2 - 2k_2 + k_3 \leqslant 1 + k_2 - k_3$$

$$\iff k_2 \geqslant \frac{1+2k_3}{3}.$$

This means:

$(a_1, (k_2, k_3, 1-k_2-k_3))$, are all equilibria provided $k_3 \in [0, \frac{1}{3}]$, $k_2 \in [\frac{1+2k_3}{3}, 1-k_3]$.

(we may check here that $\frac{1+2k_3}{3} < 1 - k_3$ for all $k_3 \in [0, \frac{1}{3}]$ which gives $k_3 < \frac{2}{5}$ ✓ so we are fine).

2

check if sols. of full game.

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 5	Topic	Marks& seen/unseen
Parts (b). (continued).	Finally, the subgame with a_3 removed results in the same conclusion: that A will play a_1 in equilibrium, so the calculation will find the equilibria we already discovered in the second subgame.	− 3rd subgame. repeats equilibria.
	Any remaining equilibria we have not found must be where player A mixes over all three of their pure strategies. For this to be possible we need: (denoting $\beta_3 = (q, r, 1-q-r)$, again abusing q, r notation) $$g_A(a_1,\beta_3) = g_A(a_2,\beta_3) = g_A(a_3,\beta_3),$$ giving: $$1+q-qr = 2-2q+r = q+2r$$ $$\Leftrightarrow \quad q=\frac{5}{9},\ r=\frac{1}{3},\ \text{so: } \beta_3 = \left(\frac{5}{9},\frac{3}{9},\frac{1}{9}\right).$$	unseen 3×3 degeneracy. seen methods. 3
	To find the strategies A can play in this instance we observe that: (a_1,β_3) and $((0,\frac{1}{2},\frac{1}{2}),\beta_3)$ (both found earlier) are equilibria. Hence, employing part (a); $$(\alpha^*,\beta_3), \text{ where } \alpha^* = k_4 a_1 + (1-k_4)\alpha_1,$$ $$k_4 \in [0,1],\ \alpha_1 = \tfrac{1}{2}a_2 + \tfrac{1}{2}a_3$$ are the remaining set of equilibria where player A mixes over all three pure strategies.	unseen application of theorem. 2 Q5: Total 20
	Setter's initials SJB	Checker's initials	Page number 23