Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2023-2024

Partial Differential Equations in Action **MATH50008**

Midterm Exam

Instructions: This test is composed of **two questions**; you must attempt both questions. The **neatness, completeness and clarity of the answers** will contribute to the final mark.

1. **Total: 10 Marks** Consider the following first-order PDE

$$2\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = e^{x+5y} - u \quad \text{with} \quad x, y \in \mathbb{R}$$

   (a) Use the method of characteristics to show that the general solution to this PDE is given by

$$u(x, y) = f(x + y)e^{y/2} - \frac{1}{7}e^{x+5y}$$

   with f an arbitrary function to be determined.

   6 Marks

   (b) Find the particular solution to this PDE if you are given that $u = 0$ on the x-axis.

   4 Marks

2. **Total: 20 Marks** We consider the one-dimensional flow of a perfect fluid. In this case, the velocity $u(x, t)$ of the fluid is governed by the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

   Here, we assume that the initial velocity field is given by

$$u(x, 0) = \begin{cases} 3, & x < 0 \\ 3(1 - x/6), & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

   (a) Draw the initial conditions.

   2 Marks

   (b) Find the equation of the characteristics for this problem. Draw them in the (x, t)-plane and point out any fan regions. Show that a shock forms at $t = 0$.

   5 Marks

   (c) Find the explicit solution $u(x, t)$. To do so, you will need to first determine the explicit solution for the shock path. Until what time t_s is the solution you just obtained valid? Justify your reasoning.

   8 Marks

   (d) Finally, find the explicit solution valid for $t > t_s$ and draw an amended diagram of characteristics including the shock path.

   5 Marks