Imperial College London

MATH97039 MATH97148

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fourier Analysis and Theory of Distributions

Date: Thursday, 6 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let the function on $[-\pi, \pi]$ be given by the formula $f(x) = 1$ for $x \in [-\pi, 0)$ and $f(x) = 5$ for $x \in [0, \pi]$.

   (i) Compute its Fourier series.

   (ii) Obtain the value of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. Justify your answer.

   (b) Let f and g be continuous real-valued periodic functions with the same period on the real line whose Fourier series coincide. Does it follow that $f = g$? Give a short justification for your answer.

2. (a) Let f be twice differentiable on the real line and $f, \frac{d}{dx}f, \frac{d^2}{dx^2}f \in L_1(-\infty, \infty)$. Prove that the Fourier transform of f belongs to $L_1(-\infty, \infty)$.

   (b) Find the Fourier-Stieltjes transform of the function f given by $f(x) = 1$ for $x < 1$, and $f(x) = 3$ for $x \geq 1$.

   (c) Let h be a function of bounded variation on the real line, and H be its Fourier-Stieltjes transform. Let f be a differentiable function on the real line such that both f and its Fourier transform G are in $L_1(-\infty, \infty)$. Prove that

$$\int_{-\infty}^{\infty} f(x)dh(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda)H(-\lambda)d\lambda.$$

3. Let E be a normed linear space, and E^* its adjoint (the space of linear continuous functionals on E).

   (a) Let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^*. Show that

      (i) If $x_n \overset{w}{\to} x$ and $x_n \overset{w}{\to} y$ for some $x, y \in E$, then $x = y$ ($\overset{w}{\to}$ denotes weak convergence)

      (ii) If $f_n \overset{w*}{\to} f$ and $f_n \overset{w*}{\to} g$ for some $f, g \in E^*$, then $f = g$ ($\overset{w*}{\to}$ denotes weak* convergence)

   (b) Show that the convergence of a sequence in E^* in the weak* topology of E^* is equivalent to its convergence on all points of E.

4. Here $\mathcal{D}$ and $\mathcal{D}'$ denote the space of infinitely differentiable functions on the real line with compact support and the space of distributions on it, respectively.

   (a) Let f be a linear functional on $\mathcal{D}$. Show that $f \in \mathcal{D}'$ if there exist $C, k > 0$ such that

$$|f(\phi)| \leq C \sup_x \left| \frac{d^k}{dx^k} \phi(x) \right|, \qquad \phi \in \mathcal{D}.$$

   (b) Find the Fourier transform in the class of tempered distributions of the function $x \sin x$.

5. For any $s = 0, 1, 2, \dots$ define the Sobolev space $H_s(\mathbb{R})$ of order s as the space of tempered distributions u on the real line such that the distributional derivatives $u^{(k)}$, $k = 0, \dots, s$ ($u^{(0)} = u$) are regular (i.e. identified with functions) and in $L_2(-\infty, \infty)$. The norm on $H_s(\mathbb{R})$ is

$$||u||_s = \left(\sum_{k=0}^{s} ||u^{(k)}||^2_{L^2(-\infty,\infty)} \right)^{1/2}.$$

(a) Let $u \in H_s$. Show that the norm $||u||_s$ is equivalent to the norm given by

$$N_s(u) = \left(\int_{-\infty}^{\infty} (1 + x^2)^s |F[u](x)|^2 dx \right)^{1/2},$$

where F denotes the Fourier transform.

(b) Let the function f be defined as $f(x) = e^{-x}$ for $x > 0$, $f(x) = 0$ otherwise. What is the largest s such that $f \in H_s(\mathbb{R})$?

	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen

Parts

ai (unseen)

Note that $g(x) = f(x) - 3$ is odd ($g(-x) = -g(x)$, $x \neq 0$) so all coefficients of $\cos kx$ vanish and the Fourier series of $g(x)$ is $\sum_{k=1}^{\infty} b_k \sin kx$,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x)-3)\sin kx\,dx =$$

$$= \frac{1}{\pi}\left(-2\int_{-\pi}^{0} \sin kx\,dx + 2\int_{0}^{\pi} \sin kx\,dx\right)$$

$$= \frac{2}{\pi k}\left(1-(-1)^k - ((-1)^k - 1)\right) =$$

$$= \frac{4}{\pi k}\left(1-(-1)^k\right) = \begin{cases} 0, & k \text{ even} \\ 8/\pi k, & k \text{ odd} \end{cases}$$

Thus the Fourier series of $f(x)$ is $3 + \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin(2k+1)x$ — 7

aii

At $x = \pi/2$, $f(x)$ is differentiable (in particular satisfies Dini condition) and so $f(\pi/2) = 3 + 8/\pi \sum_{k=0}^{\infty} \frac{1}{2k+1}(-1)^k$ — 6

so $2 = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \Rightarrow \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \pi/4$

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			1

	EXAMINATION SOLUTIONS 2021-22	Course
Question 1		Marks & seen/unseen
Parts b	The function $f-g$ has all its Fourier coefficients equal 0, i.e. $f-g$ is orthogonal to all elements of a complete orthogonal system in $L_2[-\pi,\pi]$. Therefore $f-g=0$ in L_2, so $f(x)=g(x)$ a.e. Since f, g are continuous, it follows that $f(x)=g(x)$ everywhere.	seen similar 7
	Setter's initials Checker's initials	Page number 2

Question		Marks & seen/unseen
2		
Parts		
a	Under these conditions we can write $F[f^{(2)}] = (i\lambda)^2 F[f]$, so $$\lvert F[f]\rvert \leqslant \frac{\lvert F[f^{(2)}](\lambda)\rvert}{\lambda^2} \leqslant \frac{C}{\lambda^2},$$ $\lambda \to \infty$. Since $F[f^{(2)}(\lambda)]$ is bounded on $\mathbb{R}$ as any Fourier transform. Since $F[f](\lambda)$ is also continuous, $F[f]$ is integrable on the real line and $\int_{-\infty}^{\infty} \lvert F[f](\lambda)\rvert d\lambda < \infty$.	Seen similar 7
b	$$\int_{-\infty}^{\infty} e^{i\lambda x} df = e^{i\lambda}(3-1) = 2e^{i\lambda}$$	unseen 3

Course

Question 2

Marks & seen/unseen

Parts c

unseen

10

Since $f \in L_1$ and is differentiable (so satisfies Dini condition) we can apply the inversion formula for the Fourier transform, and since $G \in L_1$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) e^{i\lambda x} d\lambda,$$

$$\int_{-\infty}^{\infty} f\, dh = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) e^{i\lambda x} d\lambda \, dh(x).$$

Since $G \in L_1$ we can apply Fubini thm to change the order of integration. This gives

$$\int_{-\infty}^{\infty} f\, dh = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, G(\lambda) \int_{-\infty}^{\infty} e^{i\lambda x} dh(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) H(-\lambda)\, d\lambda.$$

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EXAMINATION SOLUTIONS 2021-22

Course

Question 3

Marks & seen/unseen

Parts

unseen

a i

Let $x_n \xrightarrow{w} x$, $x_n \xrightarrow{w} y$, i.e.

for any $f \in E^*$

$$f(x_n) \to f(x), \quad f(x_n) \to f(y)$$

$$\Rightarrow f(x-y) = 0$$

But by a corollary to Hahn-Banach thm if $x - y = z \neq 0$ there exists an ~~xxxxxxxx~~ $f_0 \in E^*$ such that $f_0(z) \neq 0$.

Therefore $x = y$.

5

a ii

Let $f_n \xrightarrow{w^*} f$, $f_n \xrightarrow{w^*} g$, i.e.

for any $x \in E$,

$$f_n(x) \to f(x), \quad f_n(x) \to g(x)$$

$$\Rightarrow g(x) - f(x) = 0, \text{ i.e.}$$

$g - f = 0$ since $(g-f)(x) = 0$ $\forall x \in E$.

Thus $f = g$.

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Setter's initials

Checker's initials

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts b	By linearity, sufficient to verify the statement of convergence to 0 in E^*. 1) Let $f_n(x) \to 0 \quad \forall x \in E$ Take any neighbourhood of 0 in weak-$*$ topology $U = \{ f : \lvert f(x_j) \rvert < \varepsilon,\ j = 1, 2 \ldots, k \}$ Since k is finite and $f_n(x_j) \to 0$, $n \to \infty$ for $j = 1, 2 \ldots, k$, there is N s.t. $\lvert f_n(x_j) \rvert < \varepsilon \ \forall n > N$, and for all $j = 1, 2 \ldots, k$, i.e. $f_n \in U$, $\forall n > N$, i.e. $f_n \to 0$ in the weak-$*$ topology. 2) Let $\forall U$-neighb. of zero $\exists N$ s.t. $f_n \in U \ \forall n > N$. Take $x \in E$, $U_{x,\varepsilon} = \{ f : \lvert f(x) \rvert < \varepsilon \}$. Thus $f_n(x) \to 0$, $n \to \infty$.	seen similar 10
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		Page number 6

	EXAMINATION SOLUTIONS 2020-21	Course
Question 4		Marks & seen/unseen
Parts		
a	We need to show that f is continuous. It suffices to verify that $f(\varphi_n) \to 0$ whenever $\varphi_n \to 0$ in D. The latter means that there is an interval $[a,b]$ such that $\varphi_n(x)=0$ outside of it and any derivative $\varphi_n^{(k)} \to 0$ uniformly on $[a,b]$. Thus, $$|f(\varphi_n)| \leq C \sup_{x\in[a,b]} |\varphi_n^{(k)}(x)| \to 0$$ as $n\to\infty$	unseen 10
b	$(F[xe^{iax}], \varphi) = (xe^{iax}, \psi)$, where $\psi = F[\varphi]$. $$(xe^{iax}, \psi) = \int_{-\infty}^{\infty} x e^{iax} \psi(x)\,dx =$$	
	Setter's initials	Checker's initials	Page number 7

EXAMINATION SOLUTIONS 2021-22

Course

Question 4

Marks & seen/unseen

Parts b

$$= \frac{2\pi}{i} \varphi'(a) = 2\pi i \left(\delta'(x-a), \varphi \right)$$

So since $x \sin x = \frac{x}{2i}\left(e^{ix} - e^{-ix}\right)$,

$$F[x \sin x] =$$

$$= \pi\left(\delta'(x-1) - \delta'(x+1) \right).$$

10

Setter's initials

Checker's initials

Page number 8

Question 5

Marks & seen/unseen

Parts

a

unseen

12

By Plancherel equality and the identity $F[u'] = i\lambda F[u]$ we obtain

$$\|u^{(k)}\|^2_{L_2} = \frac{1}{2\pi}\|F[u^{(k)}]\|^2_{L_2} =$$

$$= \frac{1}{2\pi}\|(i\lambda)^k F[u](\lambda)\|^2_{L_2} =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^{2k} |F[u](\lambda)|^2 d\lambda .$$

Therefore

$$\sum_{k=0}^{s} \|u^{(k)}\|^2_{L_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\lambda^2+\dots+\lambda^{2s})\cdot$$

$$\cdot|F[u](\lambda)|^2 d\lambda \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\lambda^2)^s |F[u]|^2 d\lambda$$

Thus $\|u\|^2_s \leq C\, N_s(u)$, $C = \frac{1}{2\pi}$

Conversely,

$$N_s(u) = \int_{-\infty}^{\infty}(1+\lambda^2)^s |F[u](\lambda)|^2 d\lambda \leq$$

$$\leq C_1 \int_{-\infty}^{\infty}(1+\lambda^2+\dots+\lambda^{2s})|F[u](\lambda)|^2 d\lambda =$$

Setter's initials

Checker's initials

$= 2\pi C_1 \, \|u\|_s^2$ for some $C_1 > 0$.

b $\quad (f', \varphi) = -(f, \varphi') =$ unseen

$= -\int_0^\infty e^{-x} \varphi' \, dx = \varphi(0) - \int_0^\infty \varphi e^{-x} dx,$ 8

so $f' = \delta - f$ which is not regular. On the other hand $\|f\|_{L_2} < \infty$. Thus $f \in H_0(\mathbb{R})$, but $f \notin H_1(\mathbb{R})$, i.e. $s = 0$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96030 MATH97039 MATH97148	1	There were sometimes computational mistakes in computing Fourier coefficients leading to wrong result
MATH96030 MATH97039 MATH97148	2	Mostly well done
MATH96030 MATH97039 MATH97148	3	In 3 a i, one can use Hahn-Banach corollary we mentioned in the course
MATH96030 MATH97039 MATH97148	4	in 4 a use of characterisation of convergent sequence in D is helpfull; in 4 b there were sometimes mistakes in comutation
MATH96030 MATH97039 MATH97148	5	in 5a one can use a connection between derivative and Fourier transform; in 5b note that the distributional derivative of f is not regular.