Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 16

Series continued:

More on Absolute convergence, Series Tests

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Where we were last time:

Definition - Absolute Convergence

For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

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It is possible for a series to be convergent, but not absolutely convergent!

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It is possible for a series to be convergent, but not absolutely convergent!

Example 4.13

We note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent (remember the harmonic series), but show that it is convergent.

Definition

For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if and only if the series $\sum_{n=1}^{\infty} a_n$ is convergent but it is **not** absolutely convergent.

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While it is possible for a series to be convergent without being absolutely convergent, the next theorem shows that if a series is absolutely convergent it **must** be convergent.

Theorem 4.14

Let $(a_n)_{n\geq 0}$ be a real or complex sequence.
If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof $\sum |a_n|$ is convergent. $s_n = \sum_{j=1}^{n} |a_j|$, $s_n \to L$

Let $\sigma_n = \sum_{j=1}^{n} a_j$. We claim σ_n is Cauchy (therefore convergent)

Let $\varepsilon > 0$. Since s_n is Cauchy, $\exists N$ s.t. $\forall\ n \geq m \geq N$

$|s_n - s_m| = s_n - s_m = \sum_{j=m+1}^{n} |a_j| < \varepsilon$

Then $|\sigma_n - \sigma_m| = \left| \sum_{j=m+1}^{n} a_j \right| \leq \sum_{j=m+1}^{n} |a_j| < \varepsilon$ ∎

Example 4.15

Show that for $z \in \mathbb{C}$ the power series $\sum_{n=1}^{\infty} z^n$ is convergent for $|z| < 1$ and divergent for $|z| \geq 1$.

If $z^n \to 0$, $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N$ $|z^n - 0| = |z^n| < \varepsilon$.

Proof

Suppose $|z| \geq 1$, Then $\forall n$, $|z^n| \geq 1$ so $z^n \not\to 0$

so $\sum z^n$ cannot converge.

Suppose $|z| < 1$

We know that $\sum |z^n| = \sum |z|^n$ converges to $\frac{1}{1-|z|}$.

So $\sum z^n$ absolutely convergent, therefore convergent.

$\blacksquare$

Now we turn to discussing useful tests for investigating the convergence of a series. Our first batch, comparison tests!

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Theorem 4.7 - Comparison Test I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Theorem 4.16 - Comparison II: Sandwich Test

Suppose $c_n \leq a_n \leq b_n \ \forall n$ and $\sum c_n$, $\sum b_n$ are both convergent. Then $\sum a_n$ is convergent.

Theorem 4.17 - Comparison III

If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

Theorem 4.16 - Comparison II: Sandwich Test

Suppose $c_n \leq a_n \leq b_n$ $\forall n$ and $\sum c_n$, $\sum b_n$ are both convergent. Then $\sum a_n$ is convergent.

Proof

Let $\varepsilon > 0$, then $\exists N \in \mathbb{N}_{>0}$ s.t. $\forall n \geq m \geq N$

$$\left|\sum_{j=m+1}^{n} b_j\right| < \varepsilon \quad , \left|\sum_{j=m+1}^{n} c_j\right| < \varepsilon \quad \left(\text{since } \sum b_j \text{ and } \sum c_j \text{ are convergent}\right)$$

Then $-\varepsilon < \sum_{j=m+1}^{n} c_j \leq \sum_{j=m+1}^{n} a_j \leq \sum_{j=m+1}^{n} b_j < \varepsilon$

$\Rightarrow \left|\sum_{j=m+1}^{n} a_j\right| < \varepsilon$. So $s_n = \sum_{j=1}^{n} a_j$ is Cauchy,

therefore convergent. $\blacksquare$

Before proving Comparison III we state a useful result:

Exercise 4.19

Fix $N \in \mathbb{N}_{>0}$. Then $\sum_{n \geq N} c_n$ is convergent if and only if $\sum_{n \geq 1} c_n$ is convergent.

$$\sum_{n=10^6} c_n \text{ convergent} \iff \sum_{n=1}^{\infty} c_n \text{ convergent}$$

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Theorem 4.17 - Comparison III

If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

Don't know this

Idea: If $\left|\frac{a_n}{b_n}\right| \leq L \Rightarrow |a_n| \leq L|b_n|$

then if $\sum b_n$ is absolutely convergent, $|a_n| \leq L|b_n|$ so $\sum |a_n|$ is convergent

(Comparison test 1, $\sum L|b_n|$ convergent)

Pf Since $\frac{a_n}{b_n} \to L$, set $\varepsilon = 1$, $\exists N$ st $\forall n \geq N$

$\left|\frac{a_n}{b_n} - L\right| < 1 \to \left|\frac{a_n}{b_n}\right| < (|L|+1)$ so by comparison 1

$\sum_{n \geq} |a_n|$ converges since $\sum_{n \geq N} (|L|+1)|b_n|$ does.

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Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

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Theorem 4.20 - Alternating Series Test

Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. Consider the partial sums $s_n = \sum_{i=1}^{n} (-1)^i b_i$.

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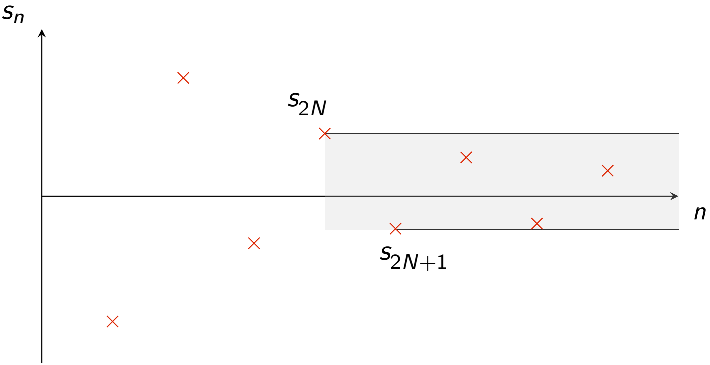

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Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. We claim

(1) $s_i \leq s_{2n} \ \forall i \geq 2n$,

(2) $s_i \geq s_{2n+1} \ \forall i \geq 2n+1$.

Exercise 4.21

What do you think about the infinite sum

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \ldots ?$$

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$
5. Other

Exercise 4.22

The alternating sequence $a_n = \begin{cases} \frac{1}{n^2} + \frac{1}{n} & n \text{ even}, \\ -\frac{1}{n^2} & n \text{ odd}, \end{cases}$

has sum $\sum a_n$ which is

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$
5. Other

Theorem 4.23 - Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

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Example 4.25

Let
$$a_n = \frac{100^n(\cos n\theta + i \sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}$$
Does the series $\sum_{n=1}^{\infty} a_n$ converge?

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Remark 4.24

The ratio test only applies when a_n decays at least exponentially in n. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Theorem 4.26: Root Test

If $|a_n|^{1/n} \to r < 1$, then $\sum a_n$ is absolutely convergent.