MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Problem Sheet 5

Exercise 21 (Properties of the matrix exponential function).
Consider matrices $B, C, T \in \mathbb{R}^{d\times d}$ such that T is invertible. Show that

(i) if $C = T^{-1}BT$, then $e^C = T^{-1}e^B T$,

(ii) $e^{-B} = \left(e^B\right)^{-1}$,

(iii) if $BC = CB$, then $e^{B+C} = e^B e^C$,

(iv) if B is a block diagonal matrix $B = \operatorname{diag}(B_1, \dots, B_p)$ with matrices $B_1, \dots, B_p$, then $e^B = \operatorname{diag}(e^{B_1}, \dots, e^{B_p})$.

Exercise 22 (Transformation of phase portraits).
Consider an autonomous linear differential equation

$$\dot{x} = Ax\,,$$

where $A \in \mathbb{R}^{d\times d}$, and assume that you know the phase portrait for the corresponding system in Jordan normal form

$$\dot{x} = Jx\,, \quad \text{where } J = T^{-1}AT \text{ with the invertible transformation } T \in \mathbb{R}^{d\times d}\,.$$

Show that T maps the phase portrait of $\dot{x} = Jx$ onto the phase portrait of $\dot{x} = Ax$.
Hint. This has been deliberately formulated vaguely, and please try to first think about a more precise formulation of this question.

Exercise 23 (Computation of matrix exponential functions).

(i) Compute the matrix exponential function e^{At} for the two matrices

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\,,$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$.

Hint. Write $A = D + P$ with a diagonal matrix D, and use Exercise 21.

(ii) Compute for $\alpha = 0$ and $\alpha = -2$ the matrix exponential function e^{At} for

$$A = \begin{pmatrix} 1 & \alpha \\ 1 & -1 \end{pmatrix}\,,$$

and draw in each case a phase portrait of the linear system $\dot{x} = Ax$. Compute the Lyapunov exponent for each solution of this differential equation.

Exercise 24 (Bounded solutions of linear systems).
Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\,.$$

Determine for which values of the real parameters a, b, c, each of the following statements hold.

(i) All solutions converge to 0 for $t \to \infty$.

(ii) All solutions are bounded on $[0, \infty)$.

(iii) Only the trivial solution (i.e. the zero solution) is bounded on $[0, \infty)$.

(iv) All solutions are bounded on $\mathbb{R}$.

(v) Only the trivial solution is bounded on $\mathbb{R}$.

Exercise 25 (Optional challenging question).

Consider two tanks K_1 and K_2 that are both filled with 1,000 litres of saline solution (salt and pure water), with an initial concentration of 5% salt in tank K_1 and 2% salt in tank K_2. The tanks are connected via a pipe with a pump. The following process will be started at time $t_0 = 0$. Per minute, 60 litres of pure water will be pumped from outside into the tank K_1, and per minute, 80 litres of the content of tank K_1 are pumped into K_2, and 20 litres of the content of K_2 will be pumped to K_1 per minute. Furthermore, per minute 60 litres of the content of K_2 are released to the outside. Assume that the salt is distributed in the tanks uniformly at all times.

(i) What is the concentration of salt $s_1(t)$ in tank K_1 and $s_2(t)$ in tank K_2 for all $t \geq 0$? What happens in the limit $t \to \infty$? What happens to the proportion $\frac{s_1(t)}{s_2(t)}$ in this limit?

(ii) What is the asymptotic (i.e. long-term) behaviour for the concentration of salt $(s_1(t), s_2(t))$ if instead of pure water, every minute 60 litres of saline solution with 10% salt concentration flow into K_1?

Hint. Model this by a discretisation in small time steps and consider the continuous time limit in order to obtain a differential equation describing the salt concentrations.

Comments on importance and difficulty of the exercises. Exercise 21 is identical to Proposition 3.4, and the statement is crucial for obtaining the explicit representation of the matrix exponential function in Section 4 of Chapter 2. Exercise 22 is very important because it shows that the knowledge that we achieve from understanding phase portraits of systems in Jordan normal form can be transferred in a straightforward way to the original system. The challenge here is to understand what precisely needs to be proved, which requires a thorough understanding. The first part of Exercise 23 concerns the explicit representation of two-dimensional matrix exponential functions in Jordan form and was used in Section 2 of Chapter 2. The technique employed here is used again to solve the higher-dimensional case in Section 4 of Chapter 2. The second part of Exercise 23 relies on Exercise 22 and requires lots of rather elementary computations. Exercise 24 trains you in understanding the matrix exponential function and making conclusions about growth behaviour of solutions of this particular linear system. This question is very elementary. The optional challenging exercise is very interesting. Despite the relatively simple formulation of the problem, it may be quite tricky to find the correct translation to our material on linear systems. This question trains you to apply the course material to a practical problem.