M40007: Introduction to Applied Mathematics

Imperial College London

1 Continuum equations

In the problem of $n+1$ masses pulled away from a fixed wall the limit $n \to \infty$ was taken, but in such a way that the (eventually) infinite number of nodes remained confined to a finite interval. In this way, in the limit $n \to \infty$, that interval became "filled out" to form a continuous mass distribution. There we took the $n \to \infty$ limit of a *solution* of the discrete system we had derived.

We can proceed differently, however. Now we examine the possibility of taking the $n \to \infty$ limit at the level of the governing *equations* – rather than at the level of the solutions – and aim being to produce a set of governing equations to be solved. The linear matrix equations we have been solving up to now will turn into *differential equations* for continuous functions.

The mathematical framework we have built up applies to several different physical problems, as we have seen, but to be concrete, we will develop the continuum limit in the context of the electric circuit problem. With different interpretations of the context, the same mathematics carries over to other applications.

In the electric circuit problem we assigned a *conductance* to any conductor making up the edge of a graph but without giving any details about why the conductance takes that value. In reality, a conductor is a piece of physical material that conducts electricity: Figure 1 shows a piece of a conductor having length L, width H and and depth D. It is assumed that this conductor is made up of a single "homogeneous" material with uniform properties.

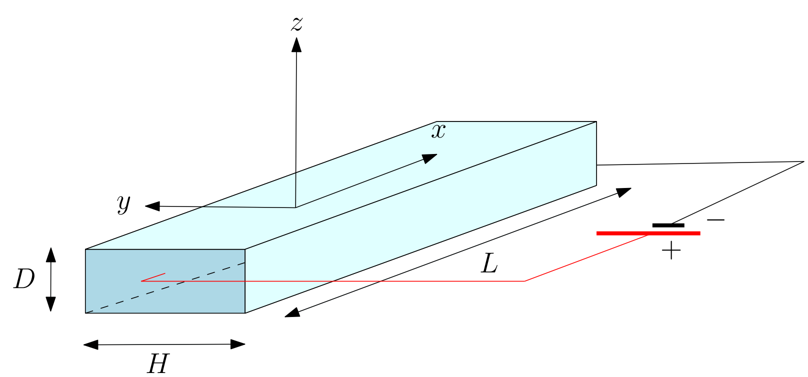

Figure 1: A physical conductor in three dimensions with length L, width H and height D. If a voltage drop is applied across the two ends, for example using a battery, the conductance is expected to depend on the dimensions according to (1).

Consider the two ends of this conductor, with cross-sectional area $D \times H$, at the two x locations distance L apart. Suppose a voltage drop is applied across these two ends as indicated in Figure 1. It is reasonable to suppose that its conductance c can be expressed as

$$c = \tilde{c} \times \frac{H \times D}{L} \tag{1}$$

for some constant $\tilde{c}$ which will depend on the material properties; this constant will be different if the block is made of copper or tungsten, for example. Applied scientists call the constant $\tilde{c}$ the *conductivity* of the material.

Relation (1) is natural because if we suppose H and D, and hence the cross-sectional area, are fixed then as L increases, so that the conductor gets longer, we expect the conductance to decrease since there is more material for the current to pass through. This is reflected in formula (1) which shows that c will decrease if L increases. On the other hand, if L remains fixed but either D or H, or both, increases then the cross-sectional area along which the current can flow is increased, meaning that the current will flow more easily and the conductance will increase, again as reflected in formula (1).

To make our models more realistic, and in particular to enable us to move to a continuum description, we now embed our graphs in $\mathbb{R}$, and also in $\mathbb{R}^2$. This means that the nodes will now be located at designated *positions* in space with edges that will have definite *lengths*. It then becomes natural to use the notion of *conductivity* rather than *conductance* since, as shown in (1), the conductivity allows us to "factor out" the expected dependence of conductance on the geometry of the conductor.

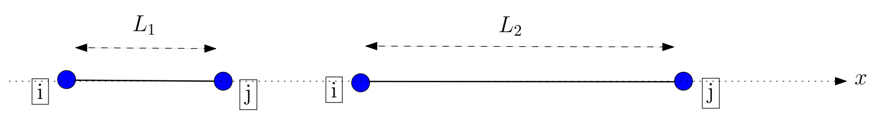

Figure 2: Two conductors embedded on the line $-\infty < x < \infty$, or $\mathbb{R}$, between nodes $\boxed{i}$ and $\boxed{j}$ of different lengths L_1 and L_2.

Conductivity in $\mathbb{R}$: Suppose first that our graph is embedded in the line $-\infty < x < \infty$. In such a case, in accordance with (1), we will write the conductance as

$$c = \frac{\hat{c}}{L}, \tag{2}$$

where we will refer to $\hat{c}$ as the *conductivity* of the wire. According to the definition of the conductivity given in (1), this is a slight abuse of terminology because, strictly speaking,

$$\hat{c} = \tilde{c} \times (H \times D), \tag{3}$$

where $\tilde{c}$ is actually the conductivity. Since $\hat{c} = \tilde{c}$ when $H \times D = 1$ then $\hat{c}$ is really the conductivity of a conductor with unit cross-sectional area. In any 1D model of a physical conductor it is implicitly assumed that the dimensions H and D of a conductor in the two transverse directions are constant as the longitudinal coordinate x varies – otherwise use of a 1D model would not be appropriate in the first place – and it is convenient mathematically to lump these constants into $\hat{c}$.

As shown in Figure 2 the conductance of the conductor with length L_2 is expected to have lower conductance than the conductor with length L_1 where, as indicated, $L_1 < L_2$. This qualitative result is consistent with formula (2) provided $\hat{c}$ is the same for both (e.g., both conductors are made of copper).

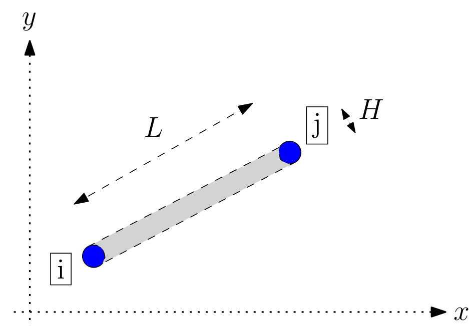

Figure 3: A conductor embedded in the (x, y) plane, or $\mathbb{R}^2$, with length L and width H between nodes $\boxed{i}$ and $\boxed{j}$.

Conductivity in $\mathbb{R}^2$: Consider now a two-dimensional situation where we embed the graph representing the circuit in the (x, y) plane, or $\mathbb{R}^2$. Now any wire between two points in space can have both a length L and a thickness H as shown in Figure 3. In such a case, in accordance with (1), we will now write the conductance as

$$c = \frac{H\hat{c}}{L}, \tag{4}$$

where we will refer to $\hat{c}$ as the conductivity of the wire. Strictly speaking, in view of (1),

$$\hat{c} = \tilde{c} \times D, \tag{5}$$

where $\tilde{c}$ is the *actual* conductivity; since $\hat{c} = \tilde{c}$ when $D = 1$ it is more accurate to call $\hat{c}$ the conductivity of unit length of conductor in the transverse z direction.

2 One-dimensional continuum limit

Consider a graph with $N + 1$ nodes in a line as shown in Figure 4 where we assume that $N \geq 2$. The incidence matrix for this graph, with the natural assignment of columns to nodes, and with all edge directions chosen to be from left to right, is the N-by-$(N + 1)$ dimensional matrix

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccccccc} \boxed{0} & \boxed{1} & \boxed{2} & . & . & \boxed{\text{N-1}} & \boxed{\text{N}} \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array} \tag{6}$$

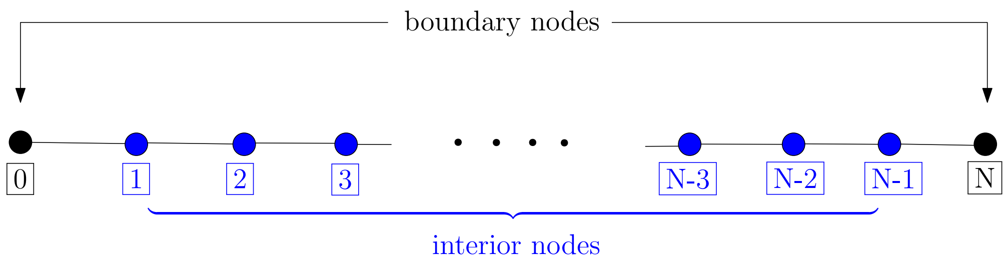

Figure 4: A graph with $N+1$ nodes in a line: nodes $\boxed{0}$ and $\boxed{\text{N}}$ are boundary nodes.

Let the vector of node variables, or potentials, be denoted by

$$\mathbf{x} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ . \\ . \\ \phi_{N-1} \\ \phi_N \end{pmatrix}. \tag{7}$$

As usual, the product

$$\mathbf{Ax} = \begin{pmatrix} \phi_1 - \phi_0 \\ \phi_2 - \phi_1 \\ \phi_3 - \phi_2 \\ . \\ . \\ \phi_{N-1} - \phi_{N-2} \\ \phi_N - \phi_{N-1} \end{pmatrix} = \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ . \\ . \\ \Delta\phi_{N-1} \\ \Delta\phi_N \end{pmatrix} \tag{8}$$

is the vector of potential differences across the edges where we have introduced a convenient notation

$$\Delta\phi_i \equiv \phi_i - \phi_{i-1}, \qquad i = 1, \dots, N \tag{9}$$

to denote the potential differences across the N edges.

Suppose now that this graph is embedded in the real interval $0 \leq x \leq 1$ on an infinite line $-\infty < x < \infty$. It is then natural to use the *location* of the node in this interval as the node label. In other words, rather than using $\boxed{\text{i}}$ for $i = 0, 1, \cdots, N$ we will use

$$x_i = \frac{i}{N}, \qquad i = 0, 1, \cdots, N \tag{10}$$

to label the nodes. With $\Delta x = 1/N$, these node labels can be written as

$$x_i = i\Delta x, \qquad i = 0, 1, \cdots, N. \tag{11}$$

The boundary node potentials are

$$x_0 = 0, \qquad x_N = 1. \tag{12}$$

All the interior nodes, corresponding to $i = 1, \cdots, N-1$, will lie in the interval $(0, 1)$. With this labelling, it is also natural to think of the values of the potential as the values of a *function*, $\Phi(x)$ say, evaluated at those points in space, i.e., we let

$$\phi_i = \Phi(x_i). \tag{13}$$

As for the edge variables let

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_{N-1} \\ w_N \end{pmatrix}. \tag{14}$$

Now that the graph is embedded on the line $-\infty < x < \infty$ a natural choice of edge labels is

$$x_i - \frac{\Delta x}{2}, \qquad i = 1, \dots, N \tag{15}$$

which is the location in the interval $(0, 1)$ of the *midpoint* of each edge. And just as was done for the node variables, these values of the edge variables can be thought of as values of a *function*, $W(x)$ say, evaluated at those points in space, i.e.,

$$w_i = W\left(x_i - \frac{\Delta x}{2}\right), \qquad i = 1, \dots, N. \tag{16}$$

Turning now to the conductance matrix $\mathbf{C}$, in view of the earlier discussion, and since the length of each edge is Δx, it is natural to define

$$c_i = \frac{\hat{c}_i}{\Delta x}, \qquad i = 1, \cdots, N \tag{17}$$

where $\hat{c}_i$ is the conductivity of the material in edge i. Once again, we let

$$\hat{c}_i = \hat{C}\left(x_i - \frac{\Delta x}{2}\right), \qquad i = 1, \cdots, N, \tag{18}$$

and view the conductivity of each edge to be the values of a conductivity distribution function, called $\hat{C}(x)$, evaluated at the midpoints of each edge. It follows that

$$\mathbf{C} = \begin{pmatrix} \hat{c}_1/\Delta x & 0 & 0 & 0 & 0 \\ 0 & \hat{c}_2/\Delta x & 0 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \hat{c}_{N-1}/\Delta x & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_N/\Delta x \end{pmatrix}. \tag{19}$$

The currents in the edges will now be given by

$$\begin{aligned} \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ w_{N-1} \\ w_N \end{pmatrix} &= -\mathbf{CAx} \\ &= -\begin{pmatrix} \hat{c}_1/\Delta x & 0 & 0 & 0 & 0 \\ 0 & \hat{c}_2/\Delta x & 0 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \hat{c}_{N-1}/\Delta x & 0 \\ 0 & 0 & 0 & 0 & \hat{c}_N/\Delta x \end{pmatrix} \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ . \\ . \\ \Delta\phi_{N-1} \\ \Delta\phi_N \end{pmatrix} \\ &= -\begin{pmatrix} \hat{c}_1\Delta\phi_1/\Delta x \\ \hat{c}_2\Delta\phi_2/\Delta x \\ . \\ . \\ \hat{c}_{N-1}\Delta\phi_{N-1}/\Delta x \\ \hat{c}_N\Delta\phi_N/\Delta x \end{pmatrix}. \end{aligned} \tag{20}$$

It is also useful to reformulate the divergence of the currents at each node, given by the vector $\mathbf{f}$, as a *current source density* vector $\hat{\mathbf{f}}$ where the elements of these two vectors are related by the scaling

$$f_i = \hat{f}_i \Delta x, \qquad i = 0, 1, \cdots, N. \tag{21}$$

And just as we have done for all other variables, we let

$$\hat{f}_i = F(x_i), \qquad i = 0, 1, \ldots, N \tag{22}$$

so the current source density $\hat{f}_i$ at each node is the value of a current source density function, $F(x)$ say, at the node location x_i. From the general theory the divergence of the currents is

$$-\mathbf{A}^T \mathbf{w} = \mathbf{f}, \tag{23}$$

where, in this case,

$$-\mathbf{A}^T \mathbf{w} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ w_{N-1} \\ w_N \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 - w_1 \\ . \\ . \\ w_{N-1} - w_{N-2} \\ w_N - w_{N-1} \\ -w_N \end{pmatrix}. \tag{24}$$

On introducing the notation

$$\Delta w_i = w_{i+1} - w_i, \qquad i = 1, \ldots, N, \tag{25}$$

and using (21), we can write (23) as

$$-\mathbf{A}^T \mathbf{w} = \begin{pmatrix} w_1 \\ \Delta w_1 \\ \Delta w_2 \\ . \\ . \\ \Delta w_{N-1} \\ -w_N \end{pmatrix} = \begin{pmatrix} \Delta x \hat{f}_0 \\ \Delta x \hat{f}_1 \\ \Delta x \hat{f}_2 \\ . \\ . \\ \Delta x \hat{f}_{N-1} \\ \Delta x \hat{f}_N \end{pmatrix}. \tag{26}$$

At the two boundary nodes, this implies the equations,

$$w_1 = \Delta x \hat{f}_0 = f_0, \qquad -w_N = \Delta x \hat{f}_N = f_N. \tag{27}$$

At the interior nodes, (26) implies

$$\Delta w_i = \Delta x \hat{f}_i, \qquad i = 1, \ldots, N-1, \tag{28}$$

or

$$\frac{\Delta w_i}{\Delta x} = \hat{f}_i, \qquad i = 1, \ldots, N-1. \tag{29}$$

2.1 The limit $N \to \infty$

In the discrete case the expression of Ohm's law,

$$\mathbf{w} = -\mathbf{CAx} \tag{30}$$

was combined with the equation for the divergence of the currents at the nodes, namely,

$$-\mathbf{A}^T\mathbf{w} = \mathbf{f} \tag{31}$$

to deduce

$$-\mathbf{A}^T(-\mathbf{CAx}) = \mathbf{A}^T\mathbf{CAx} = \mathbf{f} \tag{32}$$

and to arrive at the fundamental equation

$$\mathbf{Kx} = \mathbf{f}, \qquad \mathbf{K} = \mathbf{A}^T\mathbf{CA}. \tag{33}$$

We will now do the same thing, but also consider the limit $N \to \infty$ or, equivalently from (10) , $\Delta x \to 0$. In this way we hope to produce the continuum limit of the governing equations.

First note that as $N \to \infty$,

$$\frac{\Delta\phi_i}{\Delta x} = \frac{\phi_i - \phi_{i-1}}{\Delta x} = \frac{\Phi(x_i) - \Phi(x_i - \Delta x)}{\Delta x} \to \frac{d\Phi(x_i)}{dx_i}, \tag{34}$$

where we have used the definition of the derivative of the function $\Phi(x)$. Hence, from (18), as $N \to \infty$ a typical term on the right hand side of (20) has the limit

$$-\hat{c}_i\frac{\Delta\phi_i}{\Delta x} \to -C(x_i)\frac{d\Phi(x_i)}{dx_i}. \tag{35}$$

From (16), in the same limit $N \to \infty$, the left hand side of (20) has the limit

$$w_i \to W(x_i). \tag{36}$$

Hence as $N \to \infty$ the limit of (20) is, using (35) and (36),

$$W(x_i) = -C(x_i)\frac{d\Phi(x_i)}{dx_i}, \qquad 0 < x_i < 1. \tag{37}$$

Notice that in the limit $N \to \infty$, the independent variable x_i is any rational number in the interval $(0, 1)$ so, since we originally supposed the graph to be embedded on an x axis, it is convenient to simply rename the independent variable x_i to be x, namely,

$$x_i \mapsto x, \tag{38}$$

and to think of x now as a continuous variable in the interval $0 < x < 1$. Then (37) becomes

$$W(x) = -C(x)\frac{d\Phi(x)}{dx}, \qquad 0 < x < 1. \tag{39}$$

Similarly, as $N \to \infty$, using (16) and (22) in (29) gives the result that, at the interior nodes,

$$\frac{dW}{dx_i} = F(x_i), \qquad 0 < x_i < 1 \tag{40}$$

or, with the renaming of variables in (38),

$$\frac{dW}{dx} = F(x), \qquad 0 < x < 1. \tag{41}$$

Just as we did in equation (32) we can combine (37) and (41) to form

$$-\frac{d}{dx}\left(C(x)\frac{d\Phi(x)}{dx}\right) = F(x). \tag{42}$$

This is an *ordinary differential equation* satisfied by the voltage distribution function $\Phi(x)$. As expected, it depends on the conductivity distribution function $C(x)$ and the current source density function $F(x)$.

Note: In the continuous limit, the operation of multiplication of a vector by a matrix becomes the action of a differential operator on a function. The conductivity and current source density become continuous functions of position, namely $C(x)$ and $F(x)$.

Note: Equations (37) and (27) imply that, at the two boundary nodes $x = 0$ and $x = 1$,

$$-C(0)\frac{d\Phi(0)}{dx} = f_0, \qquad C(1)\frac{d\Phi(1)}{dx} = f_\infty. \tag{43}$$

Each is just a statement of how the divergence of the currents at each endpoint of the interval is related to the derivatives of the potential function $\Phi(x)$ there.

Note: In the discrete case, for a connected graph we know that an $N+1$ dimensional vector

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \\ 1 \end{pmatrix} \tag{44}$$

is in the right nullspace of the incidence matrix $\mathbf{A}$. Consequently,

$$\mathbf{x}_0^T(-\mathbf{A}^T\mathbf{w}) = 0. \tag{45}$$

From (24) equation (45) says that

$$w_1 + \left(\sum_{i=1}^{N-1} \Delta w_i\right) - w_N = 0. \tag{46}$$

This can be rewritten as

$$\sum_{i=1}^{N-1} \Delta w_i \frac{\Delta x}{\Delta x} = \sum_{i=1}^{N-1} \frac{\Delta w_i}{\Delta x}\Delta x = w_N - w_1. \tag{47}$$

On taking the limit $N \to \infty$, so that $\Delta x \to 0$, on use of (16), and on renaming the independent variable x_i as x as in (38), this becomes

$$\int_0^1 \frac{dW}{dx}dx = W(1) - W(0). \tag{48}$$

This is the *fundamental theorem of calculus*. It is interesting to see it emerge from (45) and consideration of the right null vector of an incidence matrix.

3 Alternative argument: 1D continuum limit

There is another way to derive the same result which relies on local arguments rather than on use of the incidence matrix. For convenience, it will be assumed that all edges in the graph have equal conductance although this assumption can be easily relaxed.

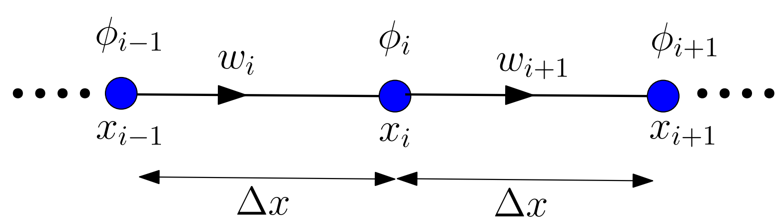

Figure 5: Typical interior nodes on a linear graph.

Consider a typical interior node labelled x_i and its two immediate neighbours as shown in Figure 5. Again we are thinking of the graph embedded on the real line $\mathbb{R}$ so the location x_i on the line $-\infty < x < \infty$ is now used as the node label. Let the currents in the two edges, each of length Δx, connected to node at x_i be w_i and

w_{i+1}, as indicated in Figure 5, and let the conductances of each be

$$c = \frac{\hat{c}}{\Delta x} \tag{49}$$

where $\hat{c}$ is the conductivity constant; this is just equation (2) with $L = \Delta x$. According to Ohm's law the currents in the two edges are given by

$$\begin{aligned} w_{i+1} &= -c(\phi_{i+1} - \phi_i) = -\frac{\hat{c}}{\Delta x}(\phi_{i+1} - \phi_i), \\ w_i &= -c(\phi_i - \phi_{i-1}) = -\frac{\hat{c}}{\Delta x}(\phi_i - \phi_{i-1}). \end{aligned} \tag{50}$$

The conservation of current at at the node at x_i says that

$$w_{i+1} - w_i = f_i = \hat{f}_i \Delta x, \tag{51}$$

where, as in (21), we have introduced the *current source density* $\hat{f}_i$. If Kirchhoff's current law holds at node at x_i then $f_i = \hat{f}_i = 0$ but we are allowing here for a more general formulation. On substituting (50) into (51) it is found that

$$-\frac{\hat{c}}{\Delta x}\left[(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})\right] = \hat{f}_i \Delta x. \tag{52}$$

This can be rewritten as

$$-\frac{\hat{c}}{(\Delta x)^2}\left[(\phi_{i+1} - 2\phi_i + \phi_{i-1})\right] = \hat{f}_i. \tag{53}$$

As in the previous section we now assume that ϕ_i is the value of a continuous function of position on the line, $\Phi(x)$ say, evaluated at $x = x_i$:

$$\phi_i = \Phi(x_i), \qquad \phi_{i+1} = \Phi(x_i + \Delta x), \qquad \phi_{i-1} = \Phi(x_i - \Delta x). \tag{54}$$

The same is done with the current source density $\hat{f}_i$:

$$\hat{f}_i = F(x_i), \tag{55}$$

where $F(x)$ is the *current source density function*. Equation (53) then becomes

$$-\frac{\hat{c}}{(\Delta x)^2}\left[\Phi(x_i + \Delta x) - 2\Phi(x_i) + \Phi(x_i - \Delta x)\right] = F(x_i). \tag{56}$$

Renaming the independent variable

$$x_i \mapsto x \tag{57}$$

as before, and assuming that $\Phi(x)$ is a differentiable function that admits a Taylor expansion, then as $\Delta \to 0$,

$$\begin{aligned} \Phi(x + \Delta x) &= \Phi(x) + \Delta x \frac{d\Phi}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2\Phi}{dx^2} + \cdots \\ \Phi(x - \Delta x) &= \Phi(x) - \Delta x \frac{d\Phi}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2\Phi}{dx^2} + \cdots \end{aligned} \tag{58}$$

It follows that (56) becomes

$$-\frac{\hat{c}}{(\Delta x)^2}\left[(\Delta x)^2 \frac{d^2\Phi}{dx^2} + \cdots \right] = F(x). \tag{59}$$

and as $\Delta x \to 0$ has the limit

$$-\hat{c}\frac{d^2\Phi(x)}{dx^2} = F(x). \tag{60}$$

This is precisely the earlier result (42) when the conductivity is uniform: $C(x) = \hat{c}$.

4 2D continuum limit

The second approach to the 1D continuum limit just described in §3 turns out to be the easiest way to generalize the analysis to the 2D continuum limit. To do so, consider the graph given by the two-dimensional *grid* embedded in a Cartesian (x, y) plane, a portion of which is shown in Figure 6. As just done in the 1D case, the conductivity is assumed to be uniform.

Consider a typical interior node labelled (x_i, y_j) and its four immediate neighbours as shown in Figure 6. Again we are thinking of the graph embedded in $\mathbb{R}^2$ so the location (x_i, y_j) in $\mathbb{R}^2$ is now used as the node label. The potential at the node (x_i, y_j) is denoted by

$$\phi_{i,j}. \tag{61}$$

Let the currents in the two edges parallel to the x-axis, each of length Δx, connected to node at (x_i, y_j) be $u_{i,j}$ and $u_{i+1,j}$, as indicated in Figure 6 and, in accordance with (4), let the conductances of each be

$$c = \frac{\hat{c}\Delta y}{\Delta x}, \tag{62}$$

where $\hat{c}$ is the conductivity. As indicated in Figure 6 we think of each edge of the graph as a conductor that is not just a line but has been "smeared out", that is, given a width $H = \Delta y$ in the direction transverse to its length $L = \Delta x$. Hence (62) is just a restatement of equation (4) for these values of H and L.

Similarly, let the currents in the two edges parallel to the y-axis, each of length Δy, connected to node at (x_i, y_j) be $v_{i,j}$ and $v_{i,j+1}$, as indicated in Figure 6 and let

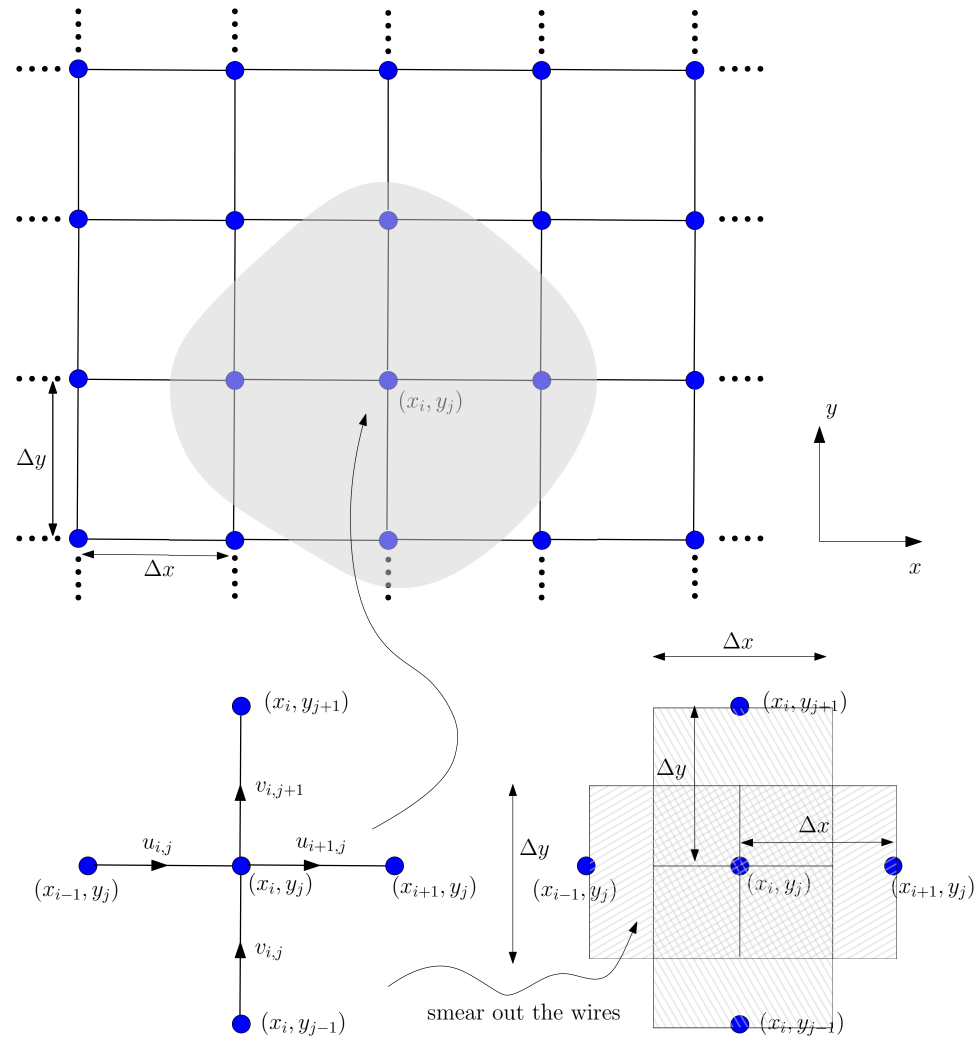

Figure 6: Typical interior nodes on a linear graph.

the conductances of each be

$$c = \frac{\hat{c}\Delta x}{\Delta y}, \tag{63}$$

where $\hat{c}$ is the conductivity and, again, we take each conductor to be smeared out, that is, given a width $H = \Delta x$ in the direction transverse to its length $L = \Delta y$. Again, (63) is another restatement of equation (4).

According to Ohm's law the currents in the four edges are given by

$$\begin{aligned}
u_{i+1,j} &= -c(\phi_{i+1,j} - \phi_{i,j}) = -\frac{\hat{c}\Delta y}{\Delta x}(\phi_{i+1,j} - \phi_{i,j}), \\
u_{i,j} &= -c(\phi_{i,j} - \phi_{i-1,j}) = -\frac{\hat{c}\Delta y}{\Delta x}(\phi_{i,j} - \phi_{i-1,j}), \\
v_{i,j+1} &= -c(\phi_{i,j+1} - \phi_{i,j}) = -\frac{\hat{c}\Delta x}{\Delta y}(\phi_{i,j+1} - \phi_{i,j}), \\
v_{i,j} &= -c(\phi_{i,j} - \phi_{i,j-1}) = -\frac{\hat{c}\Delta x}{\Delta y}(\phi_{i,j} - \phi_{i,j-1}).
\end{aligned} \tag{64}$$

The conservation of current at at the node at (x_i, y_j) says that

$$u_{i+1,j} - u_{i,j} + v_{i,j+1} - v_{i,j} = f_{i,j} = \hat{f}_{i,j}\Delta x \Delta y, \tag{65}$$

where we have introduced the *current source density* $\hat{f}_{i,j}$ related to the divergence of the current $f_{i,j}$ at the node at (x_i, y_j) as shown. If Kirchhoff's current law holds at node at x_i then $f_{i,j} = \hat{f}_{i,j} = 0$ but we are allowing here for a general formulation.

We can now substitute (64) into (65) to find

$$\begin{aligned}
-\frac{\hat{c}\Delta y}{\Delta x}\left[(\phi_{i+1,j} - \phi_{i,j}) - (\phi_{i,j} - \phi_{i-1,j})\right] - \frac{\hat{c}\Delta x}{\Delta y}\left[(\phi_{i,j+1} - \phi_{i,j}) - (\phi_{i,j} - \phi_{i,j-1})\right] \\
= \hat{f}_{i,j}\Delta x \Delta y.
\end{aligned} \tag{66}$$

This can be rewritten as

$$\begin{aligned}
-\frac{\hat{c}}{(\Delta x)^2}\left[(\phi_{i+1,j} - \phi_{i,j}) - (\phi_{i,j} - \phi_{i-1,j})\right] - \frac{\hat{c}}{(\Delta y)^2}\left[(\phi_{i,j+1} - \phi_{i,j}) - (\phi_{i,j} - \phi_{i,j-1})\right] \\
= \hat{f}_{i,j}.
\end{aligned} \tag{67}$$

If we now do the same thing done in the previous section and assume that

$$\phi_{i,j} = \Phi(x_i, y_j), \tag{68}$$

where we have introduced a continuous *potential function* $\Phi(x, y)$ whose value at (x_i, y_j) gives the potential $\phi_{i,j}$ at that node. This means

$$\phi_{i,j} = \Phi(x_i, y_j), \qquad \phi_{i+1,j} = \Phi(x_i + \Delta x, y_j), \qquad \phi_{i-1,j} = \Phi(x_i - \Delta x, y_j) \tag{69}$$

and

$$\phi_{i,j+1} = \Phi(x_i, y_j + \Delta y), \qquad \phi_{i,j-1} = \Phi(x_i, y_j - \Delta y). \tag{70}$$

We do the same thing with the current source density $\hat{f}_i$:

$$\hat{f}_{i,j} = F(x_i, y_j), \tag{71}$$

where we call $F(x,y)$ the current source density function. Equation (67) becomes

$$\begin{aligned} &-\frac{\hat{c}}{(\Delta x)^2}\left[\Phi(x_i+\Delta x, y_j) - 2\Phi(x_i, y_j) + \Phi(x_i - \Delta x, y_j)\right] \\ &-\frac{\hat{c}}{(\Delta y)^2}\left[\Phi(x_i, y_j+\Delta y) - 2\Phi(x_i, y_j) + \Phi(x_i, y_j - \Delta y)\right] = F(x_i, y_j). \end{aligned} \tag{72}$$

We can now rename independent variables

$$x_i \mapsto x, \qquad y_j \mapsto y \tag{73}$$

and, assuming that $\Phi(x,y)$ is suitably differentiable, then as $\Delta x, \Delta y \to 0$ we can use the Taylor expansions

$$\begin{aligned} \Phi(x+\Delta x, y) &= \Phi(x,y) + \Delta x \frac{\partial \Phi}{\partial x}(x,y) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \Phi}{\partial x^2}(x,y) + \ldots, \\ \Phi(x-\Delta x, y) &= \Phi(x,y) - \Delta x \frac{\partial \Phi}{\partial x}(x,y) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \Phi}{\partial x^2}(x,y) + \ldots, \\ \Phi(x, y+\Delta y) &= \Phi(x,y) + \Delta y \frac{\partial \Phi}{\partial y}(x,y) + \frac{(\Delta y)^2}{2!}\frac{\partial^2 \Phi}{\partial y^2}(x,y) + \ldots, \\ \Phi(x, y-\Delta y) &= \Phi(x,y) - \Delta y \frac{\partial \Phi}{\partial y}(x,y) + \frac{(\Delta y)^2}{2!}\frac{\partial^2 \Phi}{\partial y^2}(x,y) + \ldots \end{aligned} \tag{74}$$

to see that (72) becomes

$$-\frac{\hat{c}}{(\Delta x)^2}\left[(\Delta x)^2 \frac{\partial^2 \Phi}{\partial x^2} + \ldots\right] - \frac{\hat{c}}{(\Delta y)^2}\left[(\Delta y)^2 \frac{\partial^2 \Phi}{\partial y^2} + \ldots\right] = F(x,y). \tag{75}$$

As $\Delta x, \Delta y \to 0$ this becomes the *partial differential equation*

$$-\hat{c}\left(\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2}\right) = F(x,y). \tag{76}$$

The differential operator appearing in (76) is important so we give it a special symbol:

$$\nabla^2 \Phi \equiv \frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2}. \tag{77}$$

This is the two-dimensional *Laplacian operator*. It is the continuous analogue, in this two-dimensional embedding of the grid graph in $\mathbb{R}^2$, of the Laplacian matrix $\mathbf{K}$

appearing in the discrete case. We can now write (76) as

$$-\hat{c}\ \nabla^2\Phi(x,y) = F(x,y). \tag{78}$$

Current densities: From the statement (64) of Ohm's law we have

$$u_{i,j} = -\hat{c}\Delta y\left(\frac{\phi_{i,j}-\phi_{i-1,j}}{\Delta x}\right) = -\hat{c}\Delta y\left(\frac{\Phi(x_i,y_j)-\Phi(x_i-\Delta x,y_j)}{\Delta x}\right). \tag{79}$$

Recall that this is just one component in the vector relation

$$\mathbf{w} = -\mathbf{CAx}. \tag{80}$$

The quantity $u_{i,j}$ is the total current along the edge shown in Figure 6 but, as also seen there, the edge gets smeared out to have width Δy in the transverse direction so it is natural to set

$$u_{i,j} = \Delta y \times j^{(x)}_{i,j}, \tag{81}$$

where $j^{(x)}$ is a *current density function in the x direction*. Relation (81) tells us that on multiplying the current density function by the width Δy one finds the current. Similarly,

$$v_{i,j} = -\hat{c}\Delta x\left(\frac{\phi_{i,j}-\phi_{i,,j-1}}{\Delta y}\right) = -\hat{c}\Delta x\left(\frac{\Phi(x_i,y_j)-\Phi(x_i,y_j-\Delta y)}{\Delta y}\right). \tag{82}$$

This is the total current along the edge shown in Figure 6 but, as also seen there, the edge gets smeared out to have width Δx in the transverse direction so it is natural to set

$$v_{i,j} = \Delta x \times j^{(y)}_{i,j}, \tag{83}$$

where $j^{(y)}$ is a *current density in the y direction*. It follows on comparing (79) with (81), and (82) with (83), that

$$\begin{aligned} j^{(x)}_{i,j} &= -\hat{c}\left(\frac{\Phi(x_i,y_j)-\Phi(x_i-\Delta x,y_j)}{\Delta x}\right), \\ j^{(y)}_{i,j} &= -\hat{c}\left(\frac{\Phi(x_i,y_j)-\Phi(x_i,y_j-\Delta y)}{\Delta y}\right). \end{aligned} \tag{84}$$

In the limit $\Delta x, \Delta y \to 0$, and on use of (74), the limiting forms are

$$j^{(x)} = -\hat{c}\frac{\partial\Phi}{\partial x}, \qquad j^{(y)} = -\hat{c}\frac{\partial\Phi}{\partial y}, \tag{85}$$

where we have renamed independent variables according to (73). The two components of the current density function in the x and y directions can be collected in a two-dimensional vector

$$\mathbf{j} = \begin{pmatrix} J^{(x)}(x,y) \\ J^{(y)}(x,y) \end{pmatrix}. \tag{86}$$

From (85),

$$\mathbf{j} = -\hat{c} \begin{pmatrix} \partial\Phi/\partial x \\ \partial\Phi/\partial y \end{pmatrix}. \tag{87}$$

In view of this, it is convenient to introduce another differential operator called the two-dimensional *gradient operator*

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \tag{88}$$

in terms of which (87) can be written as

$$\mathbf{j} = -\hat{c}\nabla\Phi. \tag{89}$$

Equation (89) is just the *continuous analogue* of (80) so it is natural to identify

$$\mathbf{A} \longleftrightarrow \nabla. \tag{90}$$

The gradient operator acting on a function is the continuous analogue, in this two-dimensional setting, of multiplication of a vector of potentials by the incidence matrix $\mathbf{A}$.

We can also revisit the statement (65) of the conservation of current at each node, i.e.,

$$u_{i+1,j} - u_{i,j} + v_{i,j+1} - v_{i,j} = f_{i,j} = \hat{f}_{i,j}\Delta x\Delta y, \tag{91}$$

and write it in terms of the current densities using (81) and (83):

$$\Delta y(j^{(x)}_{i+1,j} - j^{(x)}_{i,j}) + \Delta x(j^{(y)}_{i,j+1} - j^{(y)}_{i,j}) = \hat{f}_{i,j}\Delta x\Delta y. \tag{92}$$

Recall that (92) is just one component of the matrix relation

$$-\mathbf{A}^T\mathbf{w} = \mathbf{f}. \tag{93}$$

On division by $\Delta x\Delta y$ equation (92) can be rewritten

$$\frac{j^{(x)}_{i+1,j} - j^{(x)}_{i,j}}{\Delta x} + \frac{j^{(y)}_{i,j+1} - j^{(y)}_{i,j}}{\Delta y} = \hat{f}_{i,j}. \tag{94}$$

As usual, we let the current densities $j^{(x)}_{i,j}$ and $j^{(y)}_{i,j}$ be the values of the two continuous functions $J^{(x)}(x,y)$ and $J^{(y)}(x,y)$ at discrete points, i.e.,

$$j_{i,j}^{(x)} = J^{(x)}(x_i, y_j), \qquad j_{i,j}^{(y)} = J^{(y)}(x_i, y_j). \tag{95}$$

Then in the limit $\Delta x, \Delta y \to 0$, (94) becomes

$$\frac{\partial J^{(x)}}{\partial x} + \frac{\partial J^{(y)}}{\partial y} = F(x,y). \tag{96}$$

In view of this it is convenient to introduce the two-dimensional *divergence operator* which acts on a two-dimensional vector according to

$$\nabla.\mathbf{j} \equiv \frac{\partial J^{(x)}}{\partial x} + \frac{\partial J^{(y)}}{\partial y} \tag{97}$$

in terms of which (96) becomes

$$\nabla.\mathbf{j} = F(x,y). \tag{98}$$

This is the continuous analogue of (93) so it is natural to identify

$$-\mathbf{A}^T \longleftrightarrow \nabla. \tag{99}$$

The divergence operator acting on a two-dimensional vector is the continuous analogue, in this two-dimensional setting, of multiplication of a vector of edge variables by the matrix $-\mathbf{A}^T$.

Finally, note that combining the continous version of Ohm's law (89) with the continuous version of the statement on the divergence of currents (98) we find

$$\nabla.\mathbf{j} = \nabla.(-\hat{c}\nabla\Phi) = -\hat{c}\nabla^2\Phi = F(x,y) \tag{100}$$

which, of course, is (78).