IMPERIAL

MATH70089

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Stochastic Processes

Date: Friday, May 10, 2024

Time: 14:00 – 15:30 (BST)

Time Allowed: 1.5 hours

This paper has 6 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

The open-book material allowed during the examinations consists of any material provided by the lecturers and annotated by the students, i.e. annotated lecture notes, annotated slides, and annotated problem class sheets. Books and electronic devices are not allowed.

1. (General Concepts)

(i) Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$ be a filtered probability space. Assume that both S and T are stopping times. Prove that $S + T$ is a stopping time.

(ii) Please explain in plain English what it means for the Wiener process to be "self-similar."

[Total 8 marks]

2. Let $\{S_n\}$ be a simple random walk in one dimension, with $S_0 = 0$, and let

$$\tau = \tau_{[0,5]^c} = \inf\{n : S_n \notin [0,5]\}$$

be the first time the random walk exits the set $\{0, 1, 2, 3, 4, 5\}$.

(i) Calculate $\mathbf{E} S_{\tau-1}$.

[Total 15 marks]

3. Let W be the Wiener process.

(i) Show that $Z_t = \sqrt{\epsilon} W_{\frac{t}{\epsilon}}, \ \epsilon > 0$ is a Wiener process by the Gaussian characterization theorem.

(ii) Let $c > 0$. Show that $V(t) = \frac{1}{c} W(c^2 t)$ is a Wiener process by Lévy's martingale characterization theorem.

[Total 12 marks]

4. Let $W(t)$ denote a Brownian motion with continuous paths, starting at zero. Fix M, a given positive number, and let τ denote any stopping time for W for which $\mathbb{P}(\tau \leq M) = 1$.

(i) Find $\sup_\tau \mathbb{E}[W(\tau)]$.

(ii) Let $X(\omega) = \sup_{0 \leq t \leq M} W(t, \omega)$. Find $\mathbb{E}[X]$. Why is $X > 0$ w.p. 1? Why is this not contradictory to (a)?
HINT: Use a result from the reflection principle (Week 4, Lecture 1).

(iii) Find $\sup_\tau \mathbb{E}\left[W^2(\tau)\right]$.

[Total 17 marks]

5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$ be a filtered probability space and let S and T be two stopping times such that $\mathbb{P}(S \leq T) = 1$.

(i) Define the process $C_n := \mathbb{1}_{S<n\leq T}$. Prove that $(C_n)_{n\geq 1}$ is a predictable process. That is: for all $n \geq 1$, prove that C_n is $\mathcal{F}_{n-1}$ measurable.

(ii) Let the process $(X_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-supermartingale and define the process Y_n as follows:

$$Y_0 := 0, \quad Y_n := \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

Prove that $(Y_n)_{n\geq 0}$ is also a $(\mathcal{F}_n)$-supermartingale.

(iii) Recall that S and T are two stopping times such that $\mathbb{P}(S \leq T) = 1$. Prove that if $(X_n)_{n\geq 0}$ is a supermartingale then for all $n \geq 0$,

$$\mathbb{E}[X_{n\wedge T}] \leq \mathbb{E}[X_{n\wedge S}].$$

[Total 17 marks]

6. Let W be the standard Wiener process. Assume that $S_0 > 0$, $\sigma > 0$, and $\mu \in \mathbb{R}$ are constants. Recall that the stochastic process $S = (S_t)_{t \geq 0}$ given by

$$S_t := S_0 \exp\left(\sigma W_t + (\mu - \sigma^2/2)t\right),$$

is called *geometric Brownian motion*.

(i) Discuss the behaviour of S_t as $t \to \infty$ in the case that $\mu = \sigma^2/2$. HINT: Recall the law of the iterated logarithm.

[Total 6 marks]

 End of examination paper

Module:	MATH70089
Setter:	Ernst
Checker:	Ray
Editor:	Varty
External:	Woods
Date:	April 7, 2024

MSc EXAMINATIONS (STATISTICS)

MATH70089 Stochastic Processes

Time: 1 hour 30 minutes

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1. (General Concepts)

(i) Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$ be a filtered probability space. Assume that both S and T are stopping times. Prove that $S + T$ is a stopping time.

ANSWER: (SEEN) We can write the relevant event as

$$\{\omega \in \Omega : S(\omega) + T(\omega) \leq n\} = \bigcup_{l=0}^{n} \left(\{\omega \in \Omega : S(\omega) = l\} \cap \{\omega \in \Omega : T(\omega) \leq n - l\}\right).$$

Since all the events on the right-hand side of the above equation are $\mathcal{F}_n$ measurable, $S + T$ is a stopping time. [5 marks]

(ii) Please explain in plain English what it means for the Wiener process to be "self-similar."

ANSWER: (SEEN) The Wiener process is "self-similar" because it is invariant in law under scaling of time and space. [3 marks]

[Total 8 marks]

2. Let $\{S_n\}$ be a simple random walk in one dimension, with $S_0 = 0$, and let

$$\tau = \tau_{[0,5]^c} = \inf\{n : S_n \notin [0, 5]\}$$

be the first time the random walk exits the set $\{0, 1, 2, 3, 4, 5\}$.

(i) Calculate $\mathbf{E}S_{\tau-1}$.
ANSWER: (SEEN SIMILAR) Note that $E|X_i| = 0 < \infty$ and $E_\tau < \infty$. Then by Wald's equation, $ES_\tau = EX_1 E_\tau = 0$. Thus $(-1)P(S_\tau = -1)+6P(S_\tau = 6) = 0$. Since $P(S_\tau = 6) + P(S_\tau = -1) = 1$, we easily find that $P(S_\tau = 6) = \frac{1}{7}$ $P(S_\tau = -1) = \frac{6}{7}$. Now $S_{\tau-1}$ is either 0 or 5, being 0 precisely when $S_\tau = -1$ and being 5 when $S_\tau = 6$. Thus $ES_{\tau-1} = 0 \cdot P(S_{\tau-1} = 0) + 5 \cdot P(S_{\tau-1} = 5) = 0 \cdot \frac{6}{7} + 5 \cdot \frac{1}{7} = \frac{5}{7}$.
[15 marks]

[Total 15 marks]

3. Let W be the Wiener process.

(i) Show that $Z_t = \sqrt{\epsilon} W_{\frac{t}{\epsilon}}$, $\epsilon > 0$ is a Wiener process by the Gaussian characterization theorem.

ANSWER: (SEEN) (i) First, we note that for any $\epsilon > 0$ and $Z_0 = 0$, the paths of Z_t are almost surely continuous. (ii) Secondly, we show that Z_t is jointly Gaussian by writing the vector

$$(Z_{t_1}, ..., Z_{t_k}) = \sqrt{\epsilon}\left(W_{\frac{t_1}{\epsilon}}, ..., W_{\frac{t_k}{\epsilon}}\right)$$

and recalling that the Wiener process is jointly Gaussian. (iii) We note that $\mathbb{E}[Z_t] = 0$ and

$$\mathbb{E}[Z_t Z_s] = \epsilon \mathbb{E}\left[W_{\frac{t}{\epsilon}} W_{\frac{s}{\epsilon}}\right] = \epsilon \min(t/\epsilon, s/\epsilon) = \min(t, s).$$

[5 marks]

(ii) Let $c > 0$. Show that $V(t) = \frac{1}{c} W(c^2 t)$ is a Wiener process by Lévy's martingale characterization theorem.

ANSWER: (SEEN SIMILAR) We proceed to verify all four conditions of Lévy's martingale characterization theorem. (i)

$$V(0) = \frac{1}{c} W(0) = 0$$

a.s. (ii) We note that the paths of $t \mapsto V(t) = \frac{1}{c} W(c^2 t)$ are a.s. continuous. (iii)+(iv): We need to verify that $V(t)$ and $|V(t)|^2 - t$ are martingales with respect to the filtration

$$\begin{aligned}
\mathcal{G}_t &= \sigma\{V(s) : 0 \le s \le t\} \\
&= \sigma\{W(c^2 s) : 0 \le s \le t\} \\
&= \sigma\{W(s) : 0 \le s \le c^2 t\} \\
&= \mathcal{F}_{c^2 t}.
\end{aligned}$$

Note that if $s < t, c^2 s < c^2 t$. We now proceed to prove that $V(t)$ is a martingale with respect to $\mathcal{G}_t$:

$$\begin{aligned}
\mathbb{E}[V(t) \mid \mathcal{G}_s] &= \mathbb{E}\left[\frac{1}{c} W(c^2 t) \mid \mathcal{F}_{c^2 s}\right] \\
&= \frac{1}{c}\mathbb{E}\left[W(c^2 t) \mid \mathcal{F}_{c^2 s}\right] \\
&= \frac{1}{c} W(c^2 s) = V(s).
\end{aligned}$$

[This question continues on the next page . . .]

We complete the argument by showing that $|V(t)|^2 - t$ is a martingale with respect to $\mathcal{G}_t$:

$$\begin{aligned}\mathbb{E}\left[|V(t)|^2 - t \mid \mathcal{G}_s\right] &= \mathbb{E}\left[\frac{1}{c^2}\left|W(c^2t)\right|^2 - t \mid \mathcal{F}_{c^2s}\right] \\ &= \frac{1}{c^2}\mathbb{E}\left[\left|W(c^2t)\right|^2 - c^2t \mid \mathcal{F}_{c^2s}\right] \\ &= \frac{1}{c^2}\left(\left|W(c^2s)\right|^2 - c^s\right) \\ &= |V(s)|^2 - s.\end{aligned}$$

[7 marks]

[Total 12 marks]

4. Let $W(t)$ denote a Brownian motion with continuous paths, starting at zero. Fix M, a given positive number, and let τ denote any stopping time for W for which $\mathbb{P}(\tau \leq M) = 1$.

(i) Find $\sup_\tau \mathbb{E}[W(\tau)]$.
ANSWER: (SEEN) All uniformly bounded stopping times achieve the same value, zero. [3 marks]

(ii) Let $X(\omega) = \sup_{0 \leq t \leq M} W(t, \omega)$. Find $\mathbb{E}[X]$. Why is $X > 0$ w.p. 1? Why is this not contradictory to (a)?
HINT: Use a result from the reflection principle (Week 4, Lecture 1).
ANSWER: (SEEN SIMILAR) By the reflection principle,

$$\mathbb{P}(X > a) = 2\mathbb{P}(W(M) > a)\, da = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{M}}}^{\infty} e^{-\frac{t^2}{2}}\, dt.$$

Thus for any $a > 0$, the probability that the supremum of W is greater than a tends to one as $a \downarrow 0$. It is not contradictory to (a) because there is no stopping time which stops at a positive value w.p. 1, but it is surprising.

[8 marks]

(iii) Find $\sup_\tau \mathbb{E}\left[W^2(\tau)\right]$.
ANSWER: (SEEN SIMILAR) Since $W^2(t) - t, t \geq 0$ is a martingale, and τ is uniformly bounded, $\mathbb{E}\left[W^2(\tau)\right] = \mathbb{E}[\tau]$. Then set $\tau \equiv M$ which achieves the maximum value, M. [6 marks]

[Total 17 marks]

5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$ be a filtered probability space and let S and T be two stopping times such that $\mathbb{P}(S \leq T) = 1$.

(i) Define the process $C_n := \mathbb{1}_{S<n\leq T}$. Prove that $(C_n)_{n\geq 1}$ is a predictable process. That is: for all $n \geq 1$, prove that C_n is $\mathcal{F}_{n-1}$ measurable.
ANSWER: (SEEN) We proceed to write

$$\begin{aligned}\{\omega : C_n(\omega) = 1\} &= \{\omega : S(\omega) < n, T(\omega) \geq n\} \\ &= \{\omega : S(\omega) \leq n-1\} \cap \{\omega : T(\omega) \leq n-1\}^c.\end{aligned}$$

Now notice that all the events on the right-hand side are $\mathcal{F}_{n-1}$ measurable. Thus, C_n is a predictable process.

[5 marks]

(ii) Let the process $(X_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-supermartingale and define the process Y_n as follows:

$$Y_0 := 0, \quad Y_n := \sum_{k=1}^{n} C_k(X_k - X_{k-1}).$$

Prove that $(Y_n)_{n\geq 0}$ is also a $(\mathcal{F}_n)$-supermartingale.
ANSWER: (SEEN SIMILAR) We know that Y_n is $\mathcal{F}_n$ measurable. We now look at the supermartingale condition:

$$\begin{aligned}\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] &= Y_n + \mathbb{E}[C_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n] \\ &= Y_n + C_{n+1}\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \\ &\leq Y_n.\end{aligned}$$

where in the second step we have used the result from part (i). Note that by definition $C_{n+1} \geq 0$.

[7 marks]

(iii) Recall that S and T are two stopping times such that $\mathbb{P}(S \leq T) = 1$. Prove that if $(X_n)_{n\geq 0}$ is a supermartingale then for all $n \geq 0$,

$$\mathbb{E}[X_{n\wedge T}] \leq \mathbb{E}[X_{n\wedge S}].$$

ANSWER: (SEEN SIMILAR) We note that we can write Y_n as

$$Y_n = X_{n\wedge T} - X_{n\wedge S}.$$

We then apply the result from (ii), which finishes the argument. [5 marks]

[Total 17 marks]

6. Let W be the standard Wiener process. Assume that $S_0 > 0$, $\sigma > 0$, and $\mu \in \mathbb{R}$ are constants. Recall that the stochastic process $S = (S_t)_{t \geq 0}$ given by

$$S_t := S_0 \exp\left(\sigma W_t + (\mu - \sigma^2/2)t\right),$$

is called *geometric Brownian motion*.

(i) Discuss the behaviour of S_t as $t \to \infty$ in the case that $\mu = \sigma^2/2$. HINT: Recall the law of the iterated logarithm.

ANSWER: (SEEN SIMILAR) If $\mu = \frac{1}{2}\sigma^2$, then $S_t = S_0 e^{\sigma W_t}$. From the law of the iterated logarithm, we have that

$$\limsup_{t \to \infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1$$

and that

$$\liminf_{t \to \infty} \frac{W_t}{\sqrt{2t \log \log t}} = -1.$$

Thus, as $t \to \infty$, W oscillates between $+\infty$ and $-\infty$. Thus, as $t \to \infty$, S_t will oscillate between 0 and $+\infty$.

[6 marks]

[Total 6 marks]

 End of examination paper