MATH50010: Probability for Statistics
Problem Sheet 6

1. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with rate 1.

   (a) Show that $Y_n = \min\{X_1 \ldots X_n\}$ has an exponential distribution with rate n.

   (b) Write down the CDF of $Z_n = \max\{X_1 \ldots X_n\}$

   (c) Show that the sequence of random variables $Y_n = Z_n - \log n$ converges in distribution to a random variable Y, which has the Gumbel distribution

$$F_Y(y) = \exp(-\exp(-y)), \qquad y \in \mathbf{R}.$$

   (a)

$$\Pr(Y_n > y) = \prod_{i=1}^{n} \Pr(X_i > y) = e^{-ny},$$

   so $\Pr(Y_n > y) = 1 - e^{-ny}$*, which is the CDF of an exponential distribution with rate* n*.*

   (b) By lecture notes,

$$\Pr(Z_n \leq z) = \prod_{i=1}^{n} \Pr(X_i \leq z) = (1 - e^{-z})^n$$

   (c) First note that $Z_n < z$ *iff* $X_i < z$ *for each* $i = 1, 2, \ldots n$*. This gives*

$$\Pr(Z_n - \log n \leq x) = \Pr(Z_n \leq x + \log n) = \left(1 - e^{-x-\log n}\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n$$

   Taking the limit as $n \to \infty$ *then gives the result, on recalling the limit definition of the exponential function.*

2. Suppose that the random variable X has mgf, $M_X(t)$ given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}.$$

   Find the probability distribution, expectation, and variance of X.

   [Hint: Consider M_X *and its definition.]*

   By definition of mgfs for discrete variables, we can deduce immediately that since

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x),$$

   $\Pr(X = x)$ *is just the coefficient of* e^{tx} *in the expression for* M_X*. Hence* $\Pr(X = 1) = 1/8$, $\Pr(X = 2) = 1/4$ *and* $\Pr(X = 3) = 5/8$*. Now* $\mathrm{E}(X^r) = M_X^{(r)}(0)$*, so that*

$$\begin{aligned} \mathrm{E}(X) &= M_X^{(1)}(0) = \tfrac{1}{8} + 2\tfrac{1}{4} + 3\tfrac{5}{8} = \tfrac{5}{2}, \\ \mathrm{E}(X^2) &= M_X^{(2)}(0) = \tfrac{1}{8} + 4\tfrac{1}{4} + 9\tfrac{5}{8} = \tfrac{27}{4}, \end{aligned}$$

   so therefore

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = \frac{1}{2}.$$

3. Suppose that X is a continuous random variable with pdf

$$f_X(x) = \exp\{-(x+2)\}, \quad \text{for} \quad -2 < x < \infty.$$

Find the mgf of X, and *hence* find the expectation and variance of X.

For this pdf,

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,\mathrm{d}x = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\,\mathrm{d}x = e^{-2}\int_{-2}^{\infty} e^{-(1-t)x}\,\mathrm{d}x \\ &= \frac{e^{-2}}{1-t}\int_{-2(1-t)}^{\infty} e^{-y}\,\mathrm{d}y \;=\; \frac{e^{-2}}{1-t}\left[-e^{-y}\right]\Big|_{-2(1-t)}^{\infty} \;=\; \frac{e^{-2t}}{1-t}, \ \textit{for}\ t<1. \end{aligned}$$

Now

$$M_X^{(1)}(t) = \tfrac{e^{-2t}}{(1-t)^2}(2t-1), \qquad M_X^{(2)}(t) = \tfrac{e^{-2t}}{(1-t)^3}\left[1+(2t-1)^2\right],$$

so that $M_X^{(1)}(0) = -1 = \mathrm{E}(X)$ *and* $M_X^{(2)}(0) = 2 = \mathrm{E}(X^2) \Longrightarrow Var(X) = 1$.

4. Suppose $Z \sim \mathrm{N}(0,1)$.

   (a) Find the mgf of Z, and also the pdf and the mgf of the random variable X, where

   $$X = \mu + \frac{1}{\lambda}Z,$$

   for parameters μ and $\lambda > 0$.

   (b) Find the expectation of X, and the expectation of the function $g(X)$, where $g(x) = e^x$. Use both the definition of the expectation directly and the mgf of X and compare the complexity of your calculations.

   (c) Suppose now Y is the random variable defined in terms of X by $Y = e^X$. Find the pdf of Y, and show that the expectation of Y is

   $$\exp\left\{\mu + \frac{1}{2\lambda^2}\right\}.$$

   (d) Let random variable T be defined by $T = Z^2$. Find the pdf and mgf of T.

   (a) To calculate the mgf

   $$\begin{aligned} M_Z(t) &= \mathrm{E}(e^{tZ}) = \int_{-\infty}^{\infty} e^{zt}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}\,\mathrm{d}z = e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(z-t)^2}{2}\right\}\,\mathrm{d}z \\ &= e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{u^2}{2}\right\}\,\mathrm{d}u = e^{t^2/2}, \end{aligned}$$

   where we completed the square in z, and then set $u = z - t$, as the integrand is a pdf.
   Now, using the transformation theorem for univariate one-to-one transformations we have $X = \mu + \frac{1}{\lambda}Z$ *implies* $Z = \lambda(X-\mu)$, *so*

   $$f_X(x) = f_Z(\lambda(x-\mu))\,\lambda = \frac{\lambda}{\sqrt{2\pi}}\exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}, \qquad x \in \mathbb{R}.$$

   To calculate the mgf of X,

   $$M_X(t) = \mathrm{E}\left(e^{t(\mu+Z/\lambda)}\right) = e^{\mu t}M_Z(t/\lambda) = \exp\left\{\mu t + \frac{t^2}{2\lambda^2}\right\}.$$

(b) Using the definition of expectation,

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\mu + t\lambda^{-1}\right)\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,\lambda^{-1}\,\mathrm{d}t \ \textit{[with } t = \lambda(x-\mu)] \\
&= \mu \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,\mathrm{d}t + \lambda^{-1}\int_{-\infty}^{\infty} t\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,\mathrm{d}t \\
&= \mu,
\end{aligned}$$

as the first integral is 1, and the second integral is zero, as the integrand is an odd function about zero. Hence

$$\mathrm{E}(X) = \mu.$$

Alternately, we could use the mgf result

$$\mathrm{E}(X) = \frac{\mathrm{d}}{\mathrm{d}s}\left\{M_X(s)\right\}_{s=0} = M_X^{(1)}(0),$$

to compute

$$\mathrm{E}(X) = \frac{\mathrm{d}}{\mathrm{d}s}\left\{\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \left\{\left(\mu + \frac{s}{\lambda^2}\right)\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \mu.$$

The expectation of $g(X) = e^X$ is

$$\begin{aligned}
\mathrm{E}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1}\right\}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,\lambda^{-1}\,\mathrm{d}t, \qquad \textit{[setting } t = \lambda(x-\mu)] \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1} - \frac{t^2}{2}\right\}\,\mathrm{d}t \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t^2 - 2t\lambda^{-1} - 2\mu\right)\right\}\,\mathrm{d}t.
\end{aligned}$$

Completing the square in the exponent, we have

$$t^2 - 2t\lambda^{-1} - 2\mu = \left(t - \lambda^{-1}\right)^2 - \left(2\mu + \lambda^{-2}\right)$$

and hence

$$\begin{aligned}
\mathrm{E}[g(X)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t - \lambda^{-1}\right)^2 + \left(\mu + \frac{1}{2\lambda^2}\right)\right\}\,\mathrm{d}t \\
&= \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(t - \lambda^{-1}\right)^2\right\}\,\mathrm{d}t = \exp\left\{\mu + \frac{1}{2\lambda^2}\right\},
\end{aligned}$$

as the integral is equal to 1 since it is the integral of a pdf for all choices of λ.
Alternatively, simply note that $\mathrm{E}(e^X) \equiv M_X(1)$.
In this case, it appears that it is simpler to use MGFs to calculate the expectations.

(c) If $Y = e^X$, the support of Y is $\mathbb{Y} = R^+$. From first principles

$$F_Y(y) = \Pr(Y \le y) = \Pr\left(e^X \le y\right) = \Pr(X \le \log y) = F_X(\log y),$$

so by differentiation

$$f_Y(y) = f_X(\log y)\,\frac{1}{y}, \quad \textit{for} \;\; y > 0.$$

Note that the function $g(t) = e^t$ is a monotone increasing function, with $g^{-1}(t) = \log t$, so that we can use the transformation result directly. That is,

$$f_Y(y) = f_X(g^{-1}(y))\,|J(y)| \qquad \textit{where} \qquad |J(y)| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{g^{-1}(t)\right\}_{t=y}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{\log t\right\}_{t=y}\right| = \frac{1}{y}.$$

Hence

$$f_Y(y) = \frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda^2}{2}(\log y - \mu)^2\right\}, \;\; \textit{for} \;\; y > 0.$$

For the expectation, we have from first principles

$$\begin{aligned}
\mathrm{E}(Y) &= \int_0^\infty y f_Y(y)\,\mathrm{d}y = \int_{-\infty}^\infty y\,\frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda}{2}(\log y - \mu)^2\right\}\,\mathrm{d}y \\
&= \int_{-\infty}^\infty \left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda^2}{2}(t - \mu)^2\right\}\,e^t\,\mathrm{d}t \;\;=\;\; \exp\left\{\mu + \frac{1}{2\lambda^2}\right\},
\end{aligned}$$

where $t = \log y$, as the integral is precisely the one carried out above. Note that the expectation could be written down immediately as $M_X(1)$. This illustrates the transformation/expectation result that, if $Y = g(X)$, then

$$\mathrm{E}(Y) = \mathrm{E}\left[g(X)\right].$$

(d) If $T = Z^2$, then from first principles

$$\begin{aligned}
F_T(t) &= \Pr(T \le t) = \Pr(Z^2 \le t) = \Pr(-\sqrt{t} \le Z \le \sqrt{t}) \\
\Longrightarrow f_T(t) &= \tfrac{1}{2\sqrt{t}}\left[f_Z(\sqrt{t}) + f_Z(-\sqrt{t})\right] = \tfrac{1}{\sqrt{2\pi}}t^{-1/2}\exp\left\{-\tfrac{t}{2}\right\}, \quad t > 0,
\end{aligned}$$

and hence

$$\begin{aligned}
M_T(t) &= \mathrm{E}(e^{tT}) = \int_{-\infty}^\infty e^{tx} f_T(x)\,\mathrm{d}x = \int_0^\infty e^{tx}\frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{x}{2}\right\}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{(1-2t)x}{2}\right\}\,\mathrm{d}x \\
&= \left(\frac{1}{1-2t}\right)^{1/2}\int_0^\infty \frac{1}{\sqrt{2\pi y}}\exp\left\{-\frac{y}{2}\right\}\,\mathrm{d}y = \left(\frac{1}{1-2t}\right)^{1/2},
\end{aligned}$$

where $y = (1-2t)x$, as the integrand is a pdf.

5. Suppose that X is a random variable with pmf/pdf f_X and mgf M_X. The *cumulant generating function* of X, K_X, is defined by $K_X(t) = \log[M_X(t)]$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{K_X(t)\}_{t=0} = \mathrm{E}(X), \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\{K_X(t)\}_{t=0} = \mathrm{Var}(X).$$

We have $K_X(t) = \log M_X(t)$, *hence*

$$K_X^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}s}\{K_X(t)\}_{s=t} = \frac{\mathrm{d}}{\mathrm{d}s}\{\log M_X(t)\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathrm{E}(X),$$

as $M_X(0) = 1$. *Similarly*

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\{M_X(t)\}^2},$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\{M_X(0)\}^2} = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2,$$

so that $K_X^{(2)}(0) = \mathrm{Var}(X)$.

6. Using the central limit theorem, construct Normal approximations to random variables with each of the following distributions,

   (a) Binomial distribution, $X \sim \text{Binomial}(n,\theta)$;

   (b) Negative Binomial distribution, $X \sim \text{Negative Binomial}(n,\theta)$.

The key is to find iid random variables $X_1, \ldots, X_n$ *such that*

$$X = \sum_{i=1}^n X_i,$$

and then to use the Central Limit Theorem for large n*:*

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{\mathcal{D}} Z \sim \text{Normal}(0,1), \quad \textit{so that } X = \sum_{i=1}^n X_i \overset{\text{approx}}{\sim} \text{Normal}(n\mu, n\sigma^2),$$

where μ =E(X_i) *and* σ^2 =Var(X_i).

(a) $X \sim \text{Binomial}(n,\theta) \Longrightarrow X = \sum_{i=1}^n X_i$ *where* $X_i \sim \text{Bernoulli}(\theta)$, *so that* μ =*E*$(X_i) = \theta$ *and* σ^2 =*Var*$(X_i) = \theta(1-\theta)$, *and hence*

$$Z_n = \frac{\sum_{i=1}^n X_i - n\theta}{\sqrt{n\theta(1-\theta)}} \sim \text{Normal}(0,1) \Longrightarrow X \overset{\text{approx}}{\sim} \text{Normal}(n\theta, n\theta(1-\theta)).$$

(b) $X \sim \text{Negative Binomial}(n,\theta) \Longrightarrow X = \sum_{i=1}^n X_i$ *where* $X_i \sim \text{Geometric}(\theta)$, *so that* μ =*E*$(X_i) = 1/\theta$ *and* σ^2 =*Var*$(X_i) = (1-\theta)/\theta^2$, *and hence*

$$Z_n = \frac{\sum_{i=1}^n X_i - n\frac{1}{\theta}}{\sqrt{n\left((1-\theta)/\theta^2\right)}} \xrightarrow{\mathcal{D}} \text{Normal}(0,1) \Longrightarrow X \overset{\text{approx}}{\sim} \text{Normal}\left(\frac{n}{\theta}, \frac{n(1-\theta)}{\theta^2}\right).$$

For discussion

7. Suppose we observe a sequence of random variables from a uniform distribution, $X_i \stackrel{\text{iid}}{\sim} \text{UNIFORM}(0,1)$, for $i = 1, 2, \ldots$. We wish to investigate the asymptotic distribution of the sample median of the first n variables in this sequence. We assume n is odd for simplicity; then M_n is the middle value in the ordered list of the first n variables. Let

$$\begin{aligned} M_n &= \text{median}(X_1, \ldots, X_n), \text{ where } n \text{ is odd} \\ &= r^{\text{th}} \text{ order statistic with } r = (n+1)/2. \end{aligned}$$

(a) First, we will derive the CDF of M_n. Let J_n be the number of the $X_1, \ldots, X_n$ that are less than or equal to x. Explain why $M_n \le x$ if and only if *at least* r of the first n of the X_i are less than or equal to x. What is the distribution of J_n?

(b) Show that

$$F_{M_n}(x) = \Pr\left(L_n \ge \frac{n+1-2nx}{2\sqrt{nx(1-x)}}\right),$$

where L_n is a transformation of J_n that converges in distribution to $Z \sim N(0,1)$ as $n \to \infty$.

(c) Show that M_n has a degenerate limit

$$\lim_{n\to\infty} F_{M_n}(x) = \begin{cases} 0 & \text{if } x < 1/2, \\ \frac{1}{2} & \text{if } x = 1/2, \\ 1 & \text{if } x > 1/2. \end{cases}$$

(d) As in the central limit theorem, we seek a rescaling of M_n that has a non-degenerate distribution. Consider the variable $S_n = (M_n - \frac{1}{2})n^p$, for some power p. First, write down F_{S_n} in terms of F_{M_n}.

(e) Show that

$$\lim_{n\to\infty} F_{S_n}(s) = \Pr\left(Z \ge \frac{\frac{1}{2} - sn^{1-p}}{\sqrt{\frac{n}{4} - s^2 n^{1-2p}}}\right),$$

where $Z \sim N(0,1)$.

(f) Find the value of p that gives rise to a non-degenerate distribution.

(g) Deduce that M_n has an approximate normal distribution as n becomes large, and state (in terms of n) its mean and variance.

(a) Since $\Pr(X_i \le x) = x$ *for each* i*, and the variables are independent, we know that* $J_n \sim \text{BINOMIAL}(n, p = x)$*. Clearly,* $\Pr(M_n \le x) = \Pr(J_n \ge r)$ *since for the median to be* $\le x$*, we need at least* r *of the* x_i*'s to be* $\le x$*. Hence,*

$$F_{M_n}(x) = \Pr(J_n \ge r) = \sum_{j=r}^{n} \binom{n}{j} x^j (1-x)^{n-j}.$$

(b) *By the normal approximation to the binomial, we know that as* $n \to \infty$

$$\frac{J_n - nx}{\sqrt{nx(1-x)}} \xrightarrow{\mathcal{D}} \mathbb{N}(0,1).$$

Transforming $\Pr(J_n \geq r)$ *and using* $r = \frac{n+1}{2}$ *gives the result stated since*

$$\Pr(J_n \geq r) = \Pr\left(\frac{J_n - nx}{\sqrt{nx(1-x)}} \geq \frac{r - nx}{\sqrt{nx(1-x)}}\right) = \Pr\left(L_n \geq \frac{(n+1)/2 - nx}{\sqrt{nx(1-x)}}\right)$$

(c) *Considering the point of* L_n *to which* $M_n = x$ *corresponds,*

$$\lim_{n\to\infty} \frac{n+1-2nx}{2\sqrt{nx(1-x)}} = \lim_{n\to\infty} \frac{n(1-2x)+1}{2\sqrt{nx(1-x)}} = \begin{cases} -\infty & \text{if } x < 1/2, \\ 0 & \text{if } x = 1/2, \\ \infty & \text{if } x > 1/2. \end{cases}$$

Applying F_Z*, the cdf of a standard normal variable, then gives the result stated:*

$$\lim_{n\to\infty} F_{M_n}(x) = \begin{cases} F_Z(-\infty) = 0 & \text{if } x < 1/2, \\ F_Z(0) = \frac{1}{2} & \text{if } x = 1/2, \\ F_Z(\infty) = 1 & \text{if } x > 1/2. \end{cases}$$

(d)

$$F_{S_n}(s) = \Pr(S_n \leq s) = \Pr\left(\left(M_n - \frac{1}{2}\right)n^p \leq s\right) = \Pr\left(M_n \leq \frac{1}{2} + sn^{-p}\right) = F_{M_n}\left(\frac{1}{2} + sn^{-p}\right)$$

(e) *From the earlier representation in terms of* J_n *and* L_n*, and using* $x = 1/2 + sn^{-p}$ *in part (b),*

$$\lim_{n\to\infty} F_{S_n}(s) = \lim_{n\to\infty} \Pr\left(\frac{J_n - \frac{n}{2} - sn^{1-p}}{\sqrt{n(\frac{1}{2} + sn^{-p})(\frac{1}{2} - sn^{-p})}} \geq \frac{1 - 2sn^{1-p}}{2\sqrt{n(\frac{1}{2} + sn^{-p})(\frac{1}{2} - sn^{-p})}}\right) = \Pr(Z \geq c_n) \tag{1}$$

Where again $Z \sim \mathbb{N}(0,1)$ *and*

$$c_n = \frac{1 - 2sn^{1-p}}{2\sqrt{n(\frac{1}{2} + sn^{-p})(\frac{1}{2} - sn^{-p})}} = \frac{\frac{1}{2} - sn^{1-p}}{\sqrt{\frac{n}{4} - s^2 n^{1-2p}}}.$$

(f) *To avoid a degenerate asymptotic distribution, we want to pick* p *so that* $\lim_{n\to\infty} c_n = c$*, where* c *is finite. By inspection try* $p = \frac{1}{2}$ *:*

$$c_n = \frac{\frac{1}{2} - s\sqrt{n}}{\sqrt{\frac{n}{4} - s^2}} \longrightarrow -2s \text{ as } n \to \infty.$$

(g) For $p = 1/2$ as above,

$$\lim_{n\to\infty} F_{S_n}(s) = \Pr(Z \geq -2s) = \Pr\left(-\frac{Z}{2} \leq s\right) = \Pr\left(\frac{Z}{2} \leq s\right).$$

Thus, $S_n \xrightarrow{\mathcal{D}} N(0, \frac{1}{4})$ and and because $M_n = \frac{S_n}{\sqrt{n}} + \frac{1}{2}$ by definition of S_n, we have $M_n \overset{\text{approx}}{\sim} N(\frac{1}{2}, \frac{1}{4n})$.