

Introduction to Quantum Mechanics – Problem sheet 6

1. **Harmonic oscillator eigenstates and Hermite polynomials** - *This question is just a little bit of practice easing back into the world of wave functions $\psi(x)$*

- (a) Verify directly that the functions

$$\phi_n(x) = H_n(x)e^{-\frac{x^2}{2}}$$

are indeed eigenfunctions of the operator

$$\hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2}x^2,$$

corresponding to the eigenvalues $n + \frac{1}{2}$.

- (b) Sketch the first five of the functions in (a).

2. **Exercise: Momentum representation** - *This is a very important exercise, and I will try to discuss it in the lecture too.*

Consider the momentum operator in position representation $\hat{p} : \psi(x) \rightarrow -i\hbar \frac{\partial}{\partial x} \psi(x)$.

- (a) Calculate the eigenvalues and eigenfunctions of \hat{p} . Does it have eigenfunctions in L^2 ?
(b) Which orthogonality condition do the eigenfunctions belonging to real eigenvalues fulfil?
(c) What is the connection between the coefficients of a wave function in coordinate and momentum basis respectively?

3. **Gaussian states and the lower limit of Heisenberg's uncertainty principle** - *This is some extra material relating to uncertainties in Chapter 4. This is “broadening your horizon” material, that isn't directly relevant to the core content of the Chapter, and the solution is a bit more fiddly.*

Prove that the lower bound of Heisenberg's uncertainty relation for position and momentum can only be reached for Gaussian states, which, in position representation are of the form

$$\psi(x) \propto \exp \left(-\alpha(x - q)^2 + \frac{i}{\hbar} p(x - q) + \frac{i\gamma}{\hbar} \right), \quad (1)$$

with $\alpha, q, p \in \mathbb{R}$ and $\gamma \in \mathbb{C}$. For this purpose, look back at the proof of the uncertainty relation and use the fact that the lower bound of the uncertainty relation is reached if and only if the state $|\chi\rangle = (\hat{a} - i\lambda\hat{b})|\psi\rangle$ is the zero vector. A vector $|\psi\rangle$ for which this would be the case is a state in which the uncertainty product of \hat{A} and \hat{B} is minimal. Write this condition for $\hat{A} = \hat{p}$ and $\hat{B} = \hat{q}$ in position representation to obtain a differential equation for the position representation $\psi(x) = \langle x|\psi\rangle$, and solve this to show that it is indeed of the form (1).