Generalizing the method we used above gives the following proposition and theorem.

Proposition 4.6. *Suppose* $a_n \geq 0\ \forall n$ *(*$\iff s_n = \sum_{i=1}^n a_i$ *is monotonically increasing), Then the following two facts are true:*

1. $\sum_{n=1}^\infty a_n$ *converges if and only if* (s_n) *is bounded above.*
2. *Similarly* $\sum_{n=1}^\infty a_n$ *diverges to* $+\infty$ *(i.e.* $\forall M > 0\ \exists N \in \mathbb{N}$ *such that* $s_n > M\ \ \forall n \geq N$*) if and only if* (s_n) *is unbounded.*

Proof. Since (s_n) is monotonic increasing, we have by Proposition 3.16 and Theorem 3.21 that

$$s_n \text{ is bounded} \iff s_n \text{ is convergent.}$$

For the second statement, s_n is unbounded $\iff \forall M > 0\ \exists N \in \mathbb{N}_{>0}$ such that $s_N > M$. But s_N is monotonic, so this is $\iff \forall M > 0\ \exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ s_n > M$. And this is the definition of $s_n \to +\infty$. □

We now give a very useful convergence test for positive series.

Theorem 4.7: Comparison test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
Moreover, $0 \leq \sum_{n=1}^\infty a_n \leq \sum_{n=1}^\infty b_n$.

Proof. Call the partial sums A_n, B_n respectively. Then

$$0 \ \leq\ A_n \ \leq\ B_n \ \leq\ \lim_{n\to\infty} B_n \ =\ \sum_{i=1}^\infty b_i.$$

So A_n is bounded and monotonically increasing $\implies$ convergent.

We are done since in previous exercise we have shown that $A_n \leq B_n$ and $A_n \to A$, $B_n \to B$ implies that $A \leq B$. □

> **Exercise 4.8** (Converse of Comparison Test.)**.** If $0 \leq a_n \leq b_n$ then $\sum a_n$ diverges to $+\infty \implies \sum b_n$ diverges to $+\infty$.

Remark 4.9. So from $\sum \frac{1}{n^2}$ convergent (Example 4.5) we can now deduce $\sum \frac{1}{n^\alpha}$ convergent for $\alpha \geq 2$ by the Comparison Test. In fact we can improve on this.

Example 4.10. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent for $\alpha > 1$.

Proof. (Cf. proof of divergence of $\sum \frac{1}{n}$ in Example 4.4.) Arrange the partial sums as follows:

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots \;= 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \dots + \frac{1}{7^\alpha}\right) + \left(\frac{1}{8^\alpha} + \dots + \frac{1}{15^\alpha}\right) + \left(\frac{1}{16^\alpha} + \dots + \frac{1}{31^\alpha}\right) + \dots$$

Bound the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha} + \dots + \frac{1}{(2^{k+1}-1)^\alpha}\right) \;\le\; \frac{1}{2^{k\alpha}} + \dots + \frac{1}{2^{k\alpha}} \;=\; \frac{2^k}{2^{k\alpha}} \;=\; \frac{1}{2^{k(\alpha-1)}}\,.$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for $n \le 2^{k+1} - 1$ we have

$$s_n \;<\; \sum_{i=0}^{k} \frac{1}{2^{i(\alpha-1)}} \;=\; \frac{1 - \frac{1}{2^{(k+1)(\alpha-1)}}}{1 - \frac{1}{2^{(\alpha-1)}}} \;\le\; \frac{1}{1 - \frac{1}{2^{\alpha-1}}}\,.$$

(It is here we used $\alpha > 1$, so $\left|\frac{1}{2^{\alpha-1}}\right| < 1$, so top and bottom of the central fraction are > 0.)

So partial sums are monotonic and bounded above $\implies$ convergent. $\square$

Theorem 4.11: Algebra of limits for series

If $\sum a_n$, $\sum b_n$ are convergent then so is $\sum(\lambda a_n + \mu b_n)$, to

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) \;=\; \lambda \sum_{n=1}^{\infty} a_n \;+\; \mu \sum_{n=1}^{\infty} b_n.$$

Proof. Partial sum (to n terms) of LHS is

$$\sum_{i=1}^{n}(\lambda a_i + \mu b_i) \;=\; \lambda \sum_{i=1}^{n} a_i + \mu \sum_{i=1}^{n} b_i \;\longrightarrow\; \lambda \sum_{i=1}^{\infty} a_i + \mu \sum_{i=1}^{\infty} b_i$$

as $n \to \infty$ by the algebra of limits for sequences. So the partial sums converge. $\square$

We have proved enough results about sequences to make these proofs very quick.

4.2 Absolute convergence

Definition. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Remark 4.12. It is possible for a series to be convergent (that is, its sequence of partial sums converges), but not absolutely convergent!

Example 4.13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent (by Example 4.4), but it is convergent.

Rough Working:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots \;=\; (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \dots$$

with kth bracket $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$. This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$. We saw this is convergent in Example 4.5. So cancellation between consecutive terms is enough to make series converge by comparison with $\sum \frac{1}{k(k-1)}$.

Proof. Fix $\epsilon > 0$. Then use 2 things

$$\sum \frac{1}{2k(2k-1)} \quad \text{is convergent to } L \text{ say} \tag{1}$$

$$\frac{(-1)^{n+1}}{n} \longrightarrow 0 \tag{2}$$

By (1) $\exists\, N_1$ such that $\forall n \geq N_1$, $\left|\sum_{k=1}^{n} \frac{1}{2k(2k-1)} - L\right| < \epsilon$.

By (2) $\exists\, N_2$ such that $\forall n \geq N_2$, $\left|\frac{(-1)^{n+1}}{n}\right| < \epsilon$.

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$, setting $j := \lfloor \frac{n}{2} \rfloor$ we have:

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{2j-1} - \frac{1}{2j}\right) + \delta \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} + \delta, \end{aligned}$$

where

$$\delta = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \qquad \text{satisfies } |\delta| \le \epsilon \text{ for } n \ge N_2 \text{ by (2).}$$

So

$$|s_n - L| \;\le\; \left| \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} - L \right| \;+\; |\delta| \;<\; \epsilon + \epsilon$$

for all $n \ge 2N+1$ (so that $\lfloor \frac{n}{2} \rfloor \ge N \ge N_1$ and $n \ge N \ge N_2$) by (1) and (2). □

Definition. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if and only if the series $\sum_{n=1}^{\infty} a_n$ is convergent but it is **not** absolutely convergent (that is, $\sum_{n=1}^{\infty} |a_n|$ diverges to infinity).

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ above is an example of a conditionally convergent series.

While it is possible for a series to be convergent without being absolutely convergent, the next theorem shows that if a series is absolutely convergent it **must** be convergent.

Theorem 4.14

Let $(a_n)_{n\ge 0}$ be a real or complex sequence.
If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Let $s_n = \sum_{i=1}^{n} |a_i|$ and $\sigma_n = \sum_{i=1}^{n} a_i$ be the partial sums.

Fix $\epsilon > 0$. We're assuming that s_n converges, so it is Cauchy:

$$\exists N_\epsilon \text{ such that } n > m \ge N_\epsilon \implies |s_n - s_m| < \epsilon \iff |a_{m+1}| + \cdots + |a_n| < \epsilon,$$

i.e. the terms in the tail of the series contribute little to the sum. So by the triangle inequality,

$$|a_{m+1} + \cdots + a_n| < \epsilon \implies |\sigma_n - \sigma_m| < \epsilon$$

and (σ_n) is Cauchy, and so convergent. □

Tail of $\sum a_n$ even smaller than tail of $\sum |a_n|$

Example 4.15. For $z \in \mathbb{C}$ the power series $\sum_{n=1}^{\infty} z^n$ is absolutely convergent for $|z| < 1$ and divergent for $|z| \geq 1$.

Proof. $\sum_{n=1}^{\infty} z^n$ is absolutely convergent because in Example 4.1 we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$.
For $|z| \geq 1$, the individual terms z^n have $|z^n| \geq 1$, so $z^n \not\to 0$, so $\sum z^n$ is divergent by Theorem 4.2. □

4.3 Tests for convergence

We already met the first test:

Theorem 4.7: Comparison I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Recall proof: $s_n = \displaystyle\sum_{i=1}^{n} a_i$ is monotonic increasing and bounded above by $\displaystyle\sum_{i=1}^{\infty} b_i \in \mathbb{R}$.

Theorem 4.16: Comparison II: Sandwich Test

Suppose $c_n \leq a_n \leq b_n$ $\forall n$ and $\sum c_n$, $\sum b_n$ are both convergent.
Then $\sum a_n$ is convergent.

Proof. We use the Cauchy criterion. $\forall \epsilon > 0$ $\exists N \in \mathbb{N}_{>0}$ such that $\forall n > m > N$,

$$\left| \sum_{i=m+1}^{n} b_i \right| < \epsilon, \quad \left| \sum_{i=m+1}^{n} c_i \right| < \epsilon$$

since the partial sums of b_i, c_i are Cauchy. Therefore

$$-\epsilon < \sum_{i=m+1}^{n} c_i \leq \sum_{i=m+1}^{n} a_i \leq \sum_{i=m+1}^{n} b_i < \epsilon$$

which implies

$$\left| \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i \right| < \epsilon,$$

i.e. the partial sums $\sum_{i=1}^{n} a_i$ form a Cauchy sequence. $\square$

Theorem 4.17: Comparison III

If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

Remark 4.18. While writing $\frac{a_n}{b_n} \to L$ makes sense, writing $a_n \to Lb_n$ does not make sense (why)!

It doesn't mean anything; b_n is not a single real number

Proof. Set $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$,

$$\left|\frac{a_n}{b_n} - L\right| < 1 \implies \left|\frac{a_n}{b_n}\right| < |L| + 1 \implies |a_n| < (|L|+1)|b_n|.$$

So now by the comparison test $\sum_{n\geq N} |b_n|$ convergent $\implies \sum_{n\geq N} |a_n|$ convergent. By the next exercise this gives the result. $\square$

Exercise 4.19. Fix $N \in \mathbb{N}_{>0}$. Then $\sum_{n\geq N} c_n$ is convergent if and only if $\sum_{n\geq 1} c_n$ is convergent.

Only the tail matters! The rest is finite

We call a sequence a_n *alternating* if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ $\forall n$ (or the opposite).

Theorem 4.20: Alternating Series Test

Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

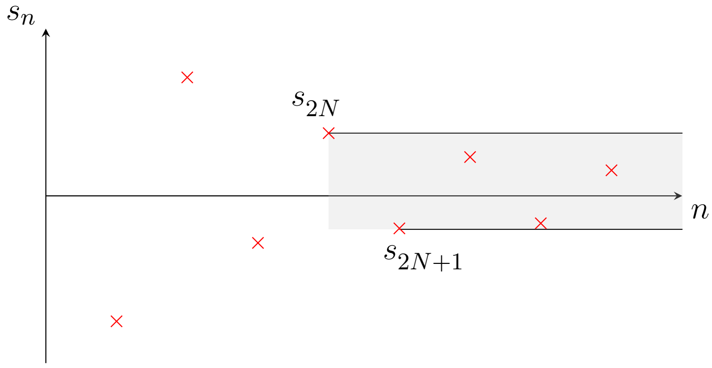
Monotonically decreasing to 0

Proof. Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. Consider the partial sums $s_n = \sum_{i=1}^{n} (-1)^i b_i$.

We claim

(1) $s_i \leq s_{2n}$ $\forall i \geq 2n$,

(2) $s_i \geq s_{2n+1}$ $\forall i \geq 2n+1$.



Indeed if $i = 2j \geq 2n$ is even then

$$s_{2j} \;=\; s_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} \;\leq\; s_{2n}$$

by monotonicity, while if $i = 2j+1 > 2n$ is odd then $s_{2j+1} = s_{2j} - b_{2j+1} \leq s_{2j} \leq s_{2n}$.

Similarly if $i = 2j + 1 \geq 2n + 1$ is odd then

$$s_{2j+1} \;=\; s_{2n+1} + \underbrace{(b_{2n+2} - b_{2n+3})}_{\geq 0} + \cdots + \underbrace{(b_{2j} - b_{2j+1})}_{\geq 0} \;\geq\; s_{2n+1},$$

while if $i = 2j + 2 > 2n + 1$ is even then $s_{2j+2} = s_{2j+1} + b_{2j+2} \geq s_{2j+1} \geq s_{2n+1}$.

The upshot is that $\forall n, m \geq 2N + 1$,

$$s_{2N+1} \;\leq\; s_n,\, s_m \;\leq\; s_{2N},$$

and so

$$|s_n - s_m| \;\leq\; s_{2N} - s_{2N+1} \;=\; b_{2N+1}.$$

But $b_n \downarrow 0$ so $\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ b_n < \epsilon$. Thus (s_n) is Cauchy. $\square$

Exercise 4.21. What do you think about the infinite sum

$+--+--$
$+--+--$
$\dots$

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \dots?$$

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$ ✓
5. Other

If we bracket the finite partial sums as

$$\left(1-\frac{1}{2}\right)-\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{5}\right)-\frac{1}{6}+\left(\frac{1}{7}-\frac{1}{8}\right)-\frac{1}{9}+\left(\frac{1}{10}-\frac{1}{11}\right)-\dots$$

you can show the sum of the bracketed terms converges by the alternating series test where as the remaining terms add to something unboundedly negative. So you can show (exercise!) that the partial sum $\to -\infty$.

Alternatively bracket the partial sums differently as

$$1-\frac{1}{2}+\left(-\frac{1}{3}+\frac{1}{4}\right)-\frac{1}{5}+\left(-\frac{1}{6}+\frac{1}{7}\right)-\frac{1}{8}+\left(-\frac{1}{9}+\frac{1}{10}\right)-\frac{1}{11}+\dots$$

$$< \ 1-\frac{1}{2}-\frac{1}{5}-\frac{1}{8}-\frac{1}{11}-\dots \ < \ 1-\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\right) \ \longrightarrow \ -\infty$$

and turn that into a proof (ex!).

Exercise 4.22. The alternating sequence $a_n = \begin{cases} \frac{1}{n^2}+\frac{1}{n} & n \text{ even,} \\ -\frac{1}{n^2} & n \text{ odd,} \end{cases}$ has sum $\sum a_n$ which is

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$ ✓
4. Divergent to $-\infty$
5. Other

It is alternating but $|a_n|$ is **not** monotonically decreasing, so the alternating series test does **not** apply.

It *does* apply to $\sum \frac{(-1)^n}{n^2}$ of course, so this sum converges. Use this, and the fact that the other bit $\sum \frac{1}{n}$ diverges to $+\infty$, to show that $\sum a_n$ diverges to $+\infty$.

Theorem 4.23: Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Idea: Expect, eventually, $a_{N+k} \approx a_N r^k$ so that $\sum_{k\geq 0}|a_{N+k}| \approx |a_N|\sum_{k\geq 0} r^k = \frac{|a_N|}{1-r}$. More realistically, bound $|a_{N+k}|$ by $|a_N|(r+\epsilon)^k$, choosing ϵ so that $r+\epsilon<1$.

Proof. Let $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon \implies |a_{n+1}| < (r+\epsilon)|a_n| = \widetilde{r}|a_n|,$$

where we set $\widetilde{r} := r + \epsilon = \frac{1+r}{2} < 1$.

Inductively

$$|a_{N+k}| < \widetilde{r}|a_{N+k-1}| < \ldots < \widetilde{r}^k|a_N|.$$

So, setting $C := \widetilde{r}^{-N}|a_N|$,

$$|a_k| < \widetilde{r}^{k-N}|a_N| = C\widetilde{r}^k \quad \forall k \geq N.$$

Therefore, for $n \geq N$,

$$\sum_{k=N}^{n}|a_k| \leq C\sum_{k=N}^{n}\widetilde{r}^k = \frac{C(\widetilde{r}^N - \widetilde{r}^{n+1})}{1-\widetilde{r}} \leq \frac{C\widetilde{r}^N}{1-\widetilde{r}}$$

since $\widetilde{r} < 1$. So partial sums $\sum_{i=1}^{n}|a_n|$ are monotonically increasing, and bounded above once $n \geq N$ (and therefore for all n). Thus they converge. □

Remark 4.24. The ratio test only applies when a_n decays exponentially in n. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Example 4.25. Consider the complex sequence

$$a_n = \frac{100^n(\cos n\theta + i\sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100^{n+1}/(n+1)!}{100^n/n!} = \frac{100}{n+1} \longrightarrow 0.$$

So by the ratio test, $\sum a_n$ is absolutely convergent (and so convergent).

Theorem 4.26: Root Test

If $|a_n|^{1/n} \to r < 1$, then $\sum a_n$ is absolutely convergent.

Remark 4.27. Again, writing $|a_n| \to r^n$ does not make sense.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$,

$$\left| |a_n|^{1/n} - r \right| < \epsilon \implies |a_n|^{1/n} < r + \epsilon$$

Set $\widetilde{r} := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \widetilde{r}^n$, and we can conclude just as in the proof of the Ratio Test. $\square$