Imperial College London

MATH70034

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebraic Topology

Date: 10 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) (i) Give the definition of *homotopy equivalence* of topological spaces. (2 marks)

(ii) Prove that homotopy equivalence is an equivalence relation. (4 marks)

(iii) Give an example of two topological spaces that are homotopy equivalent but are not homeomorphic. (Justify your answer. You can use any results from the course if you state them clearly.) (4 marks)

(b) The 2-dimensional torus T is the quotient of the square $[0,1]\times[0,1]$ by the equivalence relation that identifies $(x,0)$ with $(x,1)$ for all $x \in [0,1]$, and $(0,y)$ with $(1,y)$ for all $y \in [0,1]$. Let $X \subset T$ be the image of $[0,1] \times \{0\}$. Let $Y \subset T$ be the image of $\{0\} \times [0,1]$.

(i) Does there exist a retraction $T \to X$? (3 marks)

(ii) Does there exist a retraction $T \to X \cup Y$? (7 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(Total: 20 marks)

2. (a) Let G_1, G_2, H be groups and let $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$ be homomorphisms. Give the definition of the *amalgamated product* $G_1 *_H G_2$. (2 marks)

(b) State the Seifert–van Kampen theorem. (No proof is required.) (3 marks)

(c) Consider the following rays in $\mathbb{R}^3$:

$$R_1 = \{(x,0,0)|x \geq 0\}, \quad R_2 = \{(0,y,0)|y \geq 0\}, \quad R_3 = \{(0,0,z)|z \geq 0\}.$$

Let $X_1 = \mathbb{R}^3 \setminus R_1$, $X_2 = \mathbb{R}^3 \setminus (R_1 \cup R_2)$, $X_3 = \mathbb{R}^3 \setminus (R_1 \cup R_2 \cup R_3)$. Determine the fundamental group of X_n for $n = 1,2,3$. (Justify your answer. You can use any results from the course if you state them clearly.) (8 marks)

(d) In the notation of part (c), do there exist covering spaces $f \colon X_2 \to X_1$ and $g \colon X_3 \to X_2$? (In each case, construct such a coverning space or prove that it does not exist.) (7 marks)

(Total: 20 marks)

3. (a) Let X be a topological space and let $A \subset X$ be a subspace. Give the definition of relative homology groups $H_n(X, A)$ for $n \geq 0$. (3 marks)

(b) Explain the construction of the long exact sequence of homology of a pair (X, A) and define the maps in this sequence. (8-9 sentences will suffice. No proofs are required.) (6 marks)

(c) The Möbius strip M is the quotient of $[0, 1] \times [0, 1]$ by the equivalence relation that identifies $(0, y)$ with $(1, 1-y)$ for $0 \leq y \leq 1$. Let $\partial M \subset M$ be the boundary of M, that is, the image of the union of $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$.

(i) Determine $H_n(\partial M)$ for $n \geq 0$. (2 marks)

(ii) Determine $H_n(M)$ for $n \geq 0$. (4 marks)

(iii) Determine $H_n(M, \partial M)$ for $n \geq 0$. (5 marks)

(Justify your answers. You can use all results from the course if you state them clearly.)

(Total: 20 marks)

4. Let n be a positive integer. The group $\mathbb{Z}/2$ acts on $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i^2 = 1\}$ so that the generator of $\mathbb{Z}/2$ sends a vector $(x_0, \ldots, x_n)$ to $-(x_0, \ldots, x_n)$. We define the *real projective space* $\mathbb{RP}^n$ as the quotient topological space $S^n/(\mathbb{Z}/2)$. (Two points of S^n are equivalent if and only if they are in the same orbit of $\mathbb{Z}/2$.)

(a) Prove that the quotient map $S^n \to \mathbb{RP}^n$ is a covering space and determine its degree. (6 marks)

(b) Determine the fundamental group of $\mathbb{RP}^n$. (4 marks)

(c) Determine the positive integers n such that $\mathbb{RP}^n$ is homeomorphic to S^n. (2 marks)

(d) Construct a Δ-complex structure on $\mathbb{RP}^2$. (Drawing a clear picture would be sufficient.) Hence, or otherwise, determine the homology groups of $\mathbb{RP}^2$. (8 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(Total: 20 marks)

5. (a) State the Mayer–Vietoris theorem describing the eponymous long exact sequence.

(2 marks)

(b) Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a short exact sequence of abelian groups. Prove that if there is a homomorphism $r\colon B \to A$ such that $r\alpha = \mathrm{id}_A$, then B is isomorphic to $A \oplus C$. (6 marks)

(c) Let X be a topological space. Using the Mayer–Vietoris long exact sequence, or otherwise, determine the homology groups of $X \times S^1$ in terms of the homology groups of X.

(12 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH60034/70034

Algebraic topology (Solutions)

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1. (a) (i) The topological spaces X and Y are homotopy equivalent if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y. seen $\Downarrow$ 2, A

(ii) It is clear that this relation is reflexive and symmetric. Let us prove that it is transitive. We have topological spaces X, Y, Z and continuous maps $f\colon X \to Y$, $g\colon Y \to X$, $h\colon Y \to Z$, $r\colon Z \to Y$ such that $gf \cong \mathrm{id}_X$, $fg \cong \mathrm{id}_Y$, $rh \cong \mathrm{id}_Y$, $hr \cong \mathrm{id}_Z$. Consider continuous maps $hf\colon X \to Z$ and $gr\colon Z \to X$. From $rh \cong \mathrm{id}_Y$ we obtain $grhf \cong gf \cong \mathrm{id}_X$. From $fg \cong \mathrm{id}_Y$ we obtain $hfgr \cong hr \cong \mathrm{id}_Z$. This proves that X is homotopy equivalent to Z. seen $\Downarrow$ 4, A

(iii) The line $\mathbb{R}$ is homotopy equivalent to the point $\{0\}$ because the unique map $r\colon \mathbb{R} \to \{0\}$ is a deformation retraction of the inclusion $i\colon \{0\} \to \mathbb{R}$. Indeed, $ri = \mathrm{id}_{\{0\}}$, and a homotopy from ir to $\mathrm{id}_{\mathbb{R}}$ is given by the continuous function $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by $F(x,t) = tx$. However, these topological spaces are not homeomorphic because one of them is finite and the other is infinite. unseen $\Downarrow$ 4, B

(b) (i) Yes. We have $T = X \times Y$, where X and Y are homeomorphic to S^1. The projection to the first factor $p\colon T \to X$ is a retraction because the composition of the inclusion map $X \hookrightarrow T$ with p is id_X. unseen $\Downarrow$ 3, B

(ii) No. The space $X \cup Y$ is the wedge product $S^1 \vee S^1$. Let $x_0 = X \cap Y$ be the image of the point $(0,0) \in [0,1] \times [0,1]$. By lectures, $\pi_1(S^1 \vee S^1, x_0) \cong F_2$ is the free group of rank 2, whereas $\pi_1(T, x_0) \cong \mathbb{Z}^2$ is the free abelian group of rank 2. If a retraction exists, the composition of induced maps $\pi_1(S^1 \vee S^1, x_0) \to \pi_1(T, x_0) \to \pi_1(S^1 \vee S^1, x_0)$ is the identity. This gives a surjective homomorphism of the abelian group $\mathbb{Z}^2$ onto F_2, which is non-abelian. A contradiction. unseen $\Downarrow$ 7, D

2. (a) Let N be the smallest normal subgroup of the free product $G_1 * G_2$ which contains $f_1(x)f_2^{-1}(x)$ for all $x \in H$. Then $G_1 *_H G_2$ is defined as the quotient group $(G_1 * G_2)/N$. seen ⇓ 2, A

(b) Let X be a topological space. Let $U_1, U_2 \subset X$ be path-connected open subsets such that $X = U_1 \cup U_2$. Assume that $U_1 \cap U_2$ is non-empty and path-connected. Let $x_0 \in U_1 \cap U_2$. Let $i_1 : U_1 \cap U_2 \hookrightarrow U_1$ and $i_2 : U_1 \cap U_2 \hookrightarrow U_2$ be the inclusion maps. Then $\pi_1(X, x_0)$ is isomorphic to the amalgamated product of $i_{1*} : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_1, x_0)$ and $i_{2*} : \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_2, x_0)$. seen ⇓ 3, A

(c) We have seen that the unit sphere S^2 is a deformation retract of $\mathbb{R}^3 \setminus \{0\}$. The same construction gives that S^2 minus n points is a deformation retract of X_n. The stereographic projection shows that S^2 without one point is homeomorphic to $\mathbb{R}^2$. Thus X_1 is simply connected. Next, $\mathbb{R}^2 \setminus \{0\}$ deformation retracts to S^1, thus, by a result from lectures, the fundamental group of X_2 is isomorphic to $\mathbb{Z}$. Finally, the plane $\mathbb{R}^2$ without two points deformation retracts to the wedge product $S^1 \vee S^1$. Thus X_3 is homotopy equivalent to $S^1 \vee S^1$. By lectures, $\pi_1(S^1 \vee S^1, x_0)$, where x_0 is the point where the two circles are joined, is the free group of rank 2. This is also the fundamental group of X_3. unseen ⇓ 8, C

(d) Such covering spaces do not exist. Otherwise, by lectures, there is an injective homomorphism $f_* : \pi_1(X_2, x_0) \to \pi_1(X_1, f(x_0))$. By part (c) and a result from lectures, this gives an embedding of $\mathbb{Z}$ into the trivial group, which is clearly impossible. By part (c), an injective homomorphism $g_* : \pi_1(X_3, x_0) \to \pi_1(X_2, f(x_0))$ gives an embedding of the non-abelian group F_2 into the abelian group $\mathbb{Z}$, which is impossible. unseen ⇓ 7, B

3. (a) Let $C_n(X)$, $n \geq 0$, be the free abelian group generated by the singular n-simplices $\Delta^n \to X$. The differentials $\partial\colon C_n(X) \to C_{n-1}(X)$ give rise to the chain complex $C_\bullet(X)$. Likewise, we have the chain complex $C_\bullet(A)$. The quotient groups $C_n(X)/C_n(A)$ inherit the differentials from $C_\bullet(X)$ so these groups form a chain complex $C_\bullet(X,A)$. The relative homology group $H_n(X,A)$ is the n-th homology group of $C_\bullet(X,A)$. seen ⇓ 3, A

(b) By the solution to part (a) we have a short exact sequence of chain complexes seen ⇓

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X,A) \to 0.$$

The associated long exact sequence of homology is

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \ldots$$

The map $H_n(A) \to H_n(X)$ is induced by the inclusion $A \hookrightarrow X$. The map $H_n(X) \to H_n(X,A)$ is induced by the quotient map $C_n(X) \to C_n(X)/C_n(A)$. Let us describe the map $H_n(X,A) \to H_{n-1}(A)$. An element $a \in H_n(X,A)$ comes from $b \in C_n(X)$, and then $\partial_n(b) \in C_{n-1}(X)$ comes from a unique $c \in C_{n-1}(A)$. We have $\partial_{n-1}(c) = 0$, so c defines a homology class $d \in H_{n-1}(A)$. But b is well-defined by a modulo the images of $C_{n+1}(X)$ and $C_n(A)$. Taking this into account implies that d depends only on a; this defines the required map $H_n(X,A) \to H_{n-1}(A)$. 6, A

(c) (i) It is clear that ∂M is homeomorphic to S^1. By lectures, we have $H_1(\partial M) \cong H_0(\partial M) \cong \mathbb{Z}$, $H_i(\partial M) = 0$ for $i \geq 2$. seen ⇓ 2, A

(ii) Let C be the image of $[0,1] \times \{1/2\}$ in M. It is clear that C is homeomorphic to S^1. Write $i\colon C \to M$ for the inclusion map. Sending (x,y) to $(x, \frac{1}{2})$ defines a retraction $r\colon M \to C$. Define $F_t(x,y) = (x, \frac{1}{2} + (y - \frac{1}{2})t)$. This is a continuous function $M \times [0,1] \to M$ such that $F_0(x,y) = (x, \frac{1}{2})$, $F_1(x,y) = (x,y)$, so F_t is a homotopy between ir and id_M. Thus C is a deformation retract of M, hence, by a result from lectures, i induces isomorphisms of homology groups. Thus $H_1(M) \cong H_0(M) \cong \mathbb{Z}$, $H_i(M) = 0$ for $i \geq 2$. meth seen ⇓ 4, A

(iii) Let $j\colon \partial M \to M$ be the inclusion map. By parts (i) and (ii) the long exact sequence of the pair $(M, \partial M)$ is unseen ⇓

$$0 \to \mathbb{Z} \xrightarrow{j_*} \mathbb{Z} \to H_1(M,\partial M) \to \mathbb{Z} \xrightarrow{j_*} \mathbb{Z} \to H_0(M,\partial M) \to 0.$$

In degree 0 the map j_* is an isomorphism, because $H_0(\partial M) \cong \mathbb{Z}$ and $H_0(M) \cong \mathbb{Z}$ are both generated by the class of point in ∂M. Next, the composition of $j\colon \partial M \to M$ and the retraction $r\colon M \to C$ from part (ii) is a covering map $S^1 \to S^1$ of degree 2. Thus it sends the generator of $H_1(\partial M)$ to twice the generator of $H_1(C)$. Since r induces an isomorphism $H_1(C) \to H_1(M)$, we see that in degree 1 the map $j_*\colon \mathbb{Z} \to \mathbb{Z}$ is the multiplication by 2. Now the exactness of our sequence implies that $H_1(M,\partial M) \cong \mathbb{Z}/2$ and $H_0(M,\partial M) = 0$. 5, C

4. (a) Let $g\colon (x_0,\dots,x_n)\mapsto -(x_0,\dots,x_n)$. This is clearly a homeomorphism $S^n\to S^n$. Let $p\colon S^n\to \mathbb{RP}^n$ be the quotient map. Every point $x\in S^n$ has an open neighbourhood U such that $gU\cap U=\emptyset$. Then $V=p(U)$ is an open neighbourhood of $p(x)$ in $\mathbb{RP}^n$. The restriction of p to U is a continuous bijection $U\to V$. From the definition of the quotient topology we see that it sends open sets to open sets, so the inverse map is continuous. Hence $p\colon U\to V$ is a homeomorphism. We have $p^{-1}(V)=U\cup gU$, so V is evenly covered for p. Thus $p\colon S^n\to\mathbb{RP}^n$ is a covering space. There are two sheets over V, so the degree of p is 2. meth seen ⇓ 6, A

(b) The group $\mathbb{Z}/2$ acts freely on S^n preserving the fibres of $p\colon S^n\to\mathbb{RP}^n$, so this is a Galois covering with $\mathrm{Aut}(S^n/\mathbb{RP}^n)\cong\mathbb{Z}/2$. The topological space S^n is path-connected and locally path-connected. If $n\geq 2$, then S^n is simply connected (as discussed in lectures). In this case, by a result from lectures, $\mathrm{Aut}(S^n/\mathbb{RP}^n)$ is the quotient of $\pi_1(\mathbb{RP}^n,x_0)$ by the image of the trivial group $\pi_1(S^n,\tilde{x}_0)$, thus $\pi_1(\mathbb{RP}^n,x_0)\cong\mathbb{Z}/2$. If $n=1$, then the orbits of $\mathbb{Z}/2$ are the fibres of the map $S^1\to S^1$ given by $z\mapsto z^2$, where S^1 is the unit circle in the complex plane $\mathbb{C}$. Thus $\mathbb{RP}^1$ is homeomorphic to S^1, and so $\pi_1(\mathbb{RP}^1,x_0)\cong\mathbb{Z}$. unseen ⇓ 4, B

(c) From part (b) we see that this holds for $n=1$. If $n\geq 2$, then $\mathbb{RP}^n$ is not homeomorphic to S^n because S^n is simply connected (by lectures) whereas $\pi_1(\mathbb{RP}^n,x_0)\cong\mathbb{Z}/2$, by part (b). unseen ⇓ 2, B

(d) Let $D_+\subset S^2$ be given by $x\geq 0$, $z\geq 0$, and let $D_-\subset S^2$ be given by $x\leq 0$, $z\geq 0$. It is clear that C and D are each homeomorphic to the 2-simplex. Let P be the point $(0,1,0)$ (identified with $(0,-1,0)$). Let $Q=(0,0,1)$. Let A be the semi-circle in $S^2\cap\{z=0\}$ from $(0,1,0)$ to $(0,-1,0)$ (identified with the semi-circle in $S^2\cap\{z=0\}$ from $(0,-1,0)$ to $(0,1,0)$). Let $B\subset S^2$ be the arc given by $x=0$, $-1\leq y\leq 0$, in the direction from P to Q, and C the arc $x=0$, $0\leq y\leq 1$ in the direction from Q to P. Then $\mathbb{RP}^2$ is the union of the interiors of simplices A,B,C,P,Q,D_+,D_-, so this gives a Δ-structure on $\mathbb{RP}^2$. unseen ⇓

The chain complex $\Delta_\bullet$ is $0\to\mathbb{Z}^2\to\mathbb{Z}^3\to\mathbb{Z}^2\to 0$. For an appropriate choice of orientation, we have $\partial_2(D_+)=A+B+C$, $\partial_2(D_-)=-A+B+C$, $\partial_1(A)=0$, $\partial_1(B)=Q-P$, $\partial_1(C)=P-Q$. Thus ∂_2 is injective and $\mathrm{Im}(\partial_2)\cong\mathbb{Z}^2$ with generators $2A$ and $A+B+C$, and $\mathrm{Ker}(\partial_1)\cong\mathbb{Z}^2$ with generators A and $A+B+C$. We deduce that $H_2(\mathbb{RP}^2)=0$, $H_1(\mathbb{RP}^2)\cong\mathbb{Z}/2$, $H_0(\mathbb{RP}^2)\cong\mathbb{Z}$. 8, D

5. (a) Let X be a topological space. Let $A, B \subset X$ be such that X is the union of the interiors of A and B, then there is a long exact sequence

seen $\Downarrow$

$$\ldots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(X) \to H_{i-1}(A \cap B) \to \ldots$$

Here the second arrow is induced by the inclusions of $A \cap B$ into A and B, and the third arrow is induced by the inclusions of A and B into X.

2, M

(b) Let $B \to A \oplus C$ be the homomorphism (r, β). Let us prove that it is injective. Let $x \in B$. If $r(x) = 0$ and $\beta(x) = 0$, then $x = \alpha(y)$ for some $y \in A$ by the exactness of the sequence. But then $r(x) = y = 0$, so $x = 0$. Let us prove that (r, β) is surjective. Take any $y \in A$ and $z \in C$. Since β is surjective, there is an $x \in B$ such that $\beta(x) = z$. Let $x_1 = x - \alpha(r(x)) + \alpha(y)$. Then $\beta(x_1) = z$ and $r(x_1) = y$. Thus (r, β) is surjective and hence an isomorphism.

meth seen $\Downarrow$

6, M

(c) The unit circle $S^1 \subset \mathbb{C}$ is the union of two closed intervals given by $e^{\pi i x}$, where $x \in [-\varepsilon, 1+\varepsilon]$ and $x \in [1-\varepsilon, 2+\varepsilon]$, for a small $\varepsilon > 0$. We denote these intervals I_1 and I_2. Let $A = X \times I_1$ and $B = X \times I_2$. It is clear that both A and B are homotopy equivalent to X, and $A \cap B$ is homotopy equivalent to the disjoint union of two copies of X. By a result from lectures, we have $H_i(A \cap B) \cong H_i(X) \oplus H_i(X)$ for $i \geq 0$. Since $X \times S^1$ is the union of the interiors of A and B, we have the Mayer–Vietoris exact sequence

unseen $\Downarrow$

$$\ldots \to H_i(X) \oplus H_i(X) \to H_i(X) \oplus H_i(X) \to H_i(X \times S^1) \to H_{i-1}(X) \oplus H_{i-1}(X) \to \ldots$$

From the description of maps in part (a) we see that the second map sends $(a, b) \in H_i(X) \oplus H_i(X)$ to $(a+b, a+b) \in H_i(X) \oplus H_i(X)$. The kernel and the cokernel of this map are both isomorphic to $H_i(X)$. This gives an exact sequence

$$0 \to H_i(X) \xrightarrow{\alpha} H_i(X \times S^1) \longrightarrow H_{i-1}(X) \to 0,$$

for all $i \geq 1$, and an isomorphism $H_0(X \times S^1) \cong H_0(X)$. Let $p \colon X \times S^1 \to X$ be the projection to the first factor. Then $r_* \alpha$ is the identity on $H_i(X)$. From part (b) we conclude that $H_i(X \times S^1) \cong H_i(X) \oplus H_{i-1}(X)$ for $i \geq 1$.

12, M

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60034/70034	1	This question was done well by most students.
MATH60034/70034	2	Many students struggled with part (c) and, consequently, part (d). Some confused rays with lines.
MATH60034/70034	3	Question (c) (iii) had a subtlety: for homology groups of degree 1 the map from Z to Z is not the identity but multiplication by 2.
MATH60034/70034	4	A harder question, with a lengthy calculation in part (d).
MATH70034	5	This question turned out to be harder than expected. Part (c) required some ingenuity.