MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 16

Theorem. (Rouche's Theorem)

Let f and g be holomorphic in an open set Ω and let $\gamma \subset \Omega$ be a simple, closed, piecewise-smooth curve that contains in its interior only points of Ω.
If $|\mathsf{g}(z)| < |\mathsf{f}(z)|$, $z \in \gamma$, then the sums of the orders of the zeros of $\mathsf{f} + \mathsf{g}$ and f inside γ are the same.

Eugène Rouché

1832 - 1910 (France)

Published in Journal of the École Polytechnique, 1862.

Example. Show that $N(z^5 + 3z^2 + 6z + 1) = 1$ inside the curve $|z| = 1$.

Proof. Let $f(z) = 6z + 1$ and $g(z) = z^5 + 3z^2$. If $|z| = 1$, then $|g(z)| < |f(z)|$. Indeed

$$|g(z)| = |z^5 + 3z^2| \leq |z^5| + 3|z^2| = 4.$$

$$|f(z)| = |6z + 1| \geq 6|z| - 1 = 5 > 4 \geq |g(z)|.$$

Since $6z + 1 = 0$ has only one zero $z = -1/6$, then $N(f) = N(f + g) = 1$.

Example. Show that all roots of $w(z) = z^7 - 2z^2 + 8 = 0$ are inside the annulus $1 < |z| < 2$.

Proof.
1. Consider first $\gamma = \{z : |z| = 2\}$. Let $f(z) = z^7$ and $g(z) = -2z^2 + 8$. If $|z| = 2$, then $|f(z)| = 2^7 = 128$ and

$$|g(z)| = |-2z^2 + 8| \leq 2|z^2| + 8 = 2 \cdot 2^2 + 8 = 16 < 128 = |f(z)|.$$

Since $|f(z)| > |g(z)|$, $|z| = 2$, then the number of roots of w inside the curve $|z| = 2$ coincides with the number of roots of $f(z) = z^7 = 0$ and equals 7.

2. Let now $\gamma = \{z : |z| = 1\}$ and let $f(z) = 8$ and $g(z) = z^7 - 2z^2$. Then

$$|z^7 - 2z^2| \leq |z^7| + 2|z|^2 \leq 3 < 8.$$

The equation $f(z) = 0$ has no solutions. This implies that all zeros of $f + g$ are outside $\gamma = \{z : |z| = 1\}$.

Section: Open mapping theorem and Maximum modulus principle.

Definition. A mapping is said to be *open* if it maps open sets to open sets.

Theorem. (Open mapping theorem) If f is holomorphic and non-constant in an open set $\Omega \subset \mathbb{C}$, then f is open.

Proof. Let w_0 belong to the image of f, $w_0 = f(z_0)$. We must prove that all points w near w_0 also belong to the image of f.
Define $g(z) = f(z) - w_1$. Then

$$g(z) = (f(z) - w_0) + (w_0 - w_1) = F(z) + G(z),$$

where we choose w_1 later. Let $\delta > 0$ such that the disc $\{z : |z - z_0| \leq \delta\}$ is contained in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$.
It is possible because zeros of holomorphic functions are isolated.

We then select $\varepsilon > 0$ so that we have $|f(z) - w_0| \geq \varepsilon$ on the circle $C_\delta = \{z : |z - z_0| = \delta\}$ (we can take $\varepsilon = \min_{z \in C_\delta} |f(z) - w_0|$).

Let us now consider w_1 as an arbitrary point in the open disk $|w - w_0| < \varepsilon$. Thus we have $|F(z)| > |G(z)|$ on the circle $C_\delta = \{z : |z - z_0| = \delta\}$. By Rouché's theorem we conclude that $g(z) = F(z) + G(z) = f(z) - w_1$ has a zero inside C_δ since F has one. Thus for an arbitrary $w_1 \in \{w : |w - w_0| < \varepsilon\}$ there is $z_1 \in \{z : |z - z_0| < \delta\}$ such that $f(z_1) = w_1$.
The proof is complete.

Theorem. (Maximum modulus principle)

If f is a non-constant holomorphic function is an open set $\Omega \subset \mathbb{C}$, then $|f|$ cannot attain a maximum in Ω.

Proof. Suppose that $|f|$ did attain a maximum at $z_0 \subset \Omega$. Since f is holomorphic it is an open mapping, and therefore, if $D \subset \Omega$ is a small open disc centred at z_0, its image $f(D)$ is open and contains $f(z_0)$. Then there is r_0 such that for any $r : 0 < r < r_0$ the circle $\gamma_r \subset D$. By the Cauchy's integral formula we have

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0} \, dz \right| = [z - z_0 = re^{i\theta}] \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta} + z_0)|}{r} \, r \, d\theta.$$

This implies that either $|f(re^{i\theta})| = |f(z_0)|$ or there is $\theta \in [0, 2\pi]$ such that $z = re^{i\theta} \in D$ and $|f(z)| > |f(z_0)|$. This is a contradiction.

Corollary.
Suppose that Ω is an open set and its closure $\overline{\Omega}$ is compact. If f is holomorphic on Ω and continuous on $\overline{\Omega}$ then

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega}|f(z)|.$$

Remark. The hypothesis that $\overline{\Omega}$ is compact (that is, bounded) is essential for the conclusion.

Indeed, consider $f(z) = e^{-iz^2}$ in $\Omega = \{z = x + iy : x > 0, y > 0\}$.

Section: Evaluation of Definite integrals.

Example. Evaluate

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\, d\theta.$$

Solution.
Let $z = e^{i\theta}$, where $0 \le \theta \le 2\pi$. Then $dz = ie^{i\theta} d\theta = iz\, d\theta$. Replacing

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$$

we obtain

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\, d\theta = \oint_{|z|=1} \frac{1}{2-\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = 2i \oint_{|z|=1} \frac{1}{z^2-4z+1}\, dz.$$

Note that

$$\frac{1}{z^2-4z+1} = \frac{1}{(z-2-\sqrt{3})(z-2+\sqrt{3})}.$$

Out of its two poles only the one $z = 2-\sqrt{3}$ is interior to $\gamma = \{z : |z| = 1\}$. Therefore

$$2i \oint_{|z|=1} \frac{1}{z^2-4z+1}\, dz = 2i \cdot 2\pi i \,\mathrm{Res}\left[\frac{1}{z^2-4z+1}, 2-\sqrt{3}\right]$$

$$= -4\pi \lim_{z\to 2-\sqrt{3}} \frac{z-2+\sqrt{3}}{(z-2-\sqrt{3})(z-2+\sqrt{3})} = -4\pi\left(-\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Quizzes

Thank you