IMPERIAL

MATH70140

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometric Complex Analysis

Date: Tuesday, May 6, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

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- Throughout this paper, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, and $B(z, r)$ denotes the Euclidean ball of radius r about z.

- You may use earlier part(s) of a problem in order to answer later part(s), even if you have not answered the earlier part(s).

- You may use any results from the lectures to answer the problems below, provided you state the result clearly.

1. (a) State the *Schwarz Lemma* (without proof). (5 marks)

   (b) Prove that for every $a \in \mathbb{D}$, $\phi_a(z) = (a - z)/(1 - \overline{a}z)$ is a biholomorphic map from $\mathbb{D}$ to $\mathbb{D}$. (5 marks)

   (c) Let Ω be a simply connected open set strictly contained in $\mathbb{C}$, and $h : \Omega \to \Omega$ be a holomorphic map. Prove that for every $z \in \Omega$ satisfying $h(z) = z$, we have $|h'(z)| \leq 1$. (5 marks)

   (d) Assume that Ω_1 and Ω_2 are simply connected open sets, both strictly contained in $\mathbb{C}$, $f : \Omega_1 \to \Omega_2$ is a holomorphic map, and $g : \Omega_1 \to \Omega_2$ is biholomorphic. Prove that if $f(z) = g(z)$ for some $z \in \Omega_1$, then $|f'(z)| \leq |g'(z)|$. (5 marks)

   (Total: 20 marks)

2. (a) What is the Poincaré conformal metric ρ on the unit disk $\mathbb{D}$? (4 marks)

   (b) Let $\Omega \subseteq \mathbb{C}$ be an open set, and $\eta : \Omega \to [0, \infty)$ be a conformal metric. Explain how η defines a metric d_η on Ω (no need to prove that it is a metric). (5 marks)

   In Parts (c) and (d), let d_ρ denote the metric induced on $\mathbb{D}$ from the Poincaré metric ρ.

   (c) Let $C \subset \mathbb{D}$ be an arbitrary compact set, and z_0 be an arbitrary point in $\mathbb{D}$. Prove that there is $r > 0$ such that C is contained in the ball of radius r about z_0 with respect to d_ρ. (5 marks)

   (d) Let $f_n : \mathbb{D} \to \mathbb{D}$, for $n \geq 1$, be a sequence of holomorphic maps, and assume that there are $z_0 \in \mathbb{D}$ and $w \in \partial D$ such that $\lim_{n\to\infty} f_n(z_0) = w$. Prove that the sequence f_n converges uniformly on compact subsets of $\mathbb{D}$ to the constant function $f \equiv w$. (6 marks)

   (Total: 20 marks)

3. (a) State the *Koebe Distortion theorem* (without proof). (6 marks)

   (b) Prove that for every univalent map $f : \mathbb{D} \to \mathbb{C}$, we have

$$\frac{1}{81} \leq \frac{\operatorname{area}(f(B(0.1, 0.1)))}{\operatorname{area}(f(B(-0.1, 0.1)))} \leq 81.$$

   (7 marks)

Question continues overleaf

(c) Let $f : \mathbb{D} \to \mathbb{C}$ be an arbitrary univalent map satisfying $f(0) = 0$ and $f'(0) = 1$. Prove that for every $r \in [0, 1)$, the Euclidean ball of radius $\frac{(1-r)^2}{4(1+r)^3}$ about $f(r)$ is contained in $f(\mathbb{D})$.

(7 marks)

(Total: 20 marks)

4. Let $\Omega \subseteq \mathbb{C}$ be an open set, and $\mathcal{F}$ be a family of holomorphic maps from Ω to $\mathbb{C}$.

(a) Define what it means for the family $\mathcal{F}$ to be *normal*. (6 marks)

(b) Assume that a sequence of maps f_n in $\mathcal{F}$ converges uniformly on compact subsets of Ω to some holomorphic map $f : \Omega \to \mathbb{C}$. Prove that for every $w \in \Omega$, we have $f_n''(w) \to f''(w)$, as $n \to \infty$. (6 marks)

(c) Let $P_0(z) = z + z^2$, and define the sequence of maps $P_n : \mathbb{C} \to \mathbb{C}$ according to

$$P_{n+1} = P_n \circ P_0, \qquad n \geq 0.$$

Is the family of maps $\{P_n : \mathbb{C} \to \mathbb{C}\}_{n \geq 0}$ normal? Justify your answer. (8 marks)

(Total: 20 marks)

5. (a) Let $\Omega \subseteq \mathbb{C}$ be an open set. What is a *quasi-conformal* map from Ω to $\mathbb{C}$? (4 marks)

(b) State the continuous version of the *Measurable Riemann Mapping Theorem* (without proof). (4 marks)

For parts (c), (d), and (e) below, consider the function $\mu : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined as

$$\mu(z) = \frac{1}{2} e^{i(2\arg(z)+\pi)}.$$

(c) Show that the Beltrami equation

$$\frac{\partial f}{\partial \overline{z}}(z) = \mu(z) \frac{\partial f}{\partial z}(z), \quad \forall z \in \mathbb{C} \setminus \{0\},$$

has a solution $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. (4 marks)

(d) If $g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a solution of the Beltrami equation with coefficient μ, what is the Dilatation quotient of g at $1+i$? Justify your answer. (4 marks)

(e) Assume that $g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a solution of the Beltrami equation with coefficient μ. Prove that for every $\theta_0 \in \mathbb{R}$, $k(z) = g(e^{i\theta_0} z)$ is also a solution of the Beltrami equation with coefficient μ. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

Geometric Complex Analysis

MATH60140/70140 (Solutions)

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1. (a) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then, $\boxed{\text{seen} \Downarrow}$

   (i) for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|$;

   (ii) $|f'(0)| \leq 1$;

   (iii) if either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then f is a rotation about 0. $\boxed{5, \text{A}}$

(b) The map ϕ_a is defined and holomorphic at every $z \in \mathbb{C}$, except at $z = 1/\overline{a}$ where the denominator becomes 0. However, since $|a| < 1$, we have $|1/\overline{a}| = 1/|a| > 1$, and therefore, $1/\overline{a} \notin \mathbb{D}$. Hence, ϕ_a is holomorphic on $\mathbb{D}$. $\boxed{\text{seen} \Downarrow}$

To see that ϕ_a maps $\mathbb{D}$ into $\mathbb{D}$, fix an arbitrary $z \in \mathbb{C}$ with $|z| = 1$. Observe that

$$|\phi_a(z)| = \left|\frac{a-z}{1-\overline{a}z}\right| = \left|\frac{a-z}{1-\overline{a}z}\right| \cdot \frac{1}{|\overline{z}|} = \left|\frac{a-z}{\overline{z}-\overline{a}}\right| = 1,$$

since $z\overline{z} = |z|^2 = 1$. By the maximum principle, $|\phi_a(z)| \leq 1$ on $\mathbb{D}$.

We observe that

$$\phi_a(a) = 0, \text{ and } \phi_a(0) = a.$$

Then,

$$\phi_a \circ \phi_a(0) = 0, \text{ and } \phi_a \circ \phi_a(a) = a.$$

By the Schwarz lemma, this implies that $\phi_a \circ \phi_a$ must be the identity map of $\mathbb{D}$. It follows that ϕ_a is both one-to-one and onto from $\mathbb{D}$ to $\mathbb{D}$. $\boxed{5, \text{A}}$

(c) By the Riemann mapping theorem, there is a biholomorphic map $\phi : \mathbb{D} \to \Omega$ satisfying $\phi(0) = z$. Consider the map $G : \mathbb{D} \to \mathbb{D}$ defined as $G = \phi^{-1} \circ h \circ \phi$. We have $\boxed{\text{sim. seen} \Downarrow}$

$$G(0) = \phi^{-1} \circ h \circ \phi(0) = \phi^{-1} \circ h(z) = \phi^{-1}(z) = 0.$$

Thus by the Schwarz lemma, we must have $|G'(0)| \leq 1$. This implies that

$$\left|\frac{1}{\phi'(0)} \cdot h'(z) \cdot \phi'(0)\right| \leq 1,$$

which implies that $|h'(z)| \leq 1$. $\boxed{5, \text{B}}$

(d) We may apply the property in part (b) to the map $g^{-1} \circ f : \Omega_1 \to \Omega_1$ at z, and conclude that $\boxed{\text{unseen} \Downarrow}$

$$|(g^{-1})'(f(z)) \cdot f'(z)| = |(g^{-1})'(g(z)) \cdot f'(z)| = |\frac{1}{g'(z)} \cdot f'(z)| \leq 1,$$

which implies that $|f'(z)| \leq |g'(z)|$. $\boxed{5, \text{C}}$

2. (a) The Poincare metric on $\mathbb{D}$ is defined as the function $\rho(z) = 1/(1-|z|^2)$. seen $\Downarrow$ 4, A

(b) Given z and w in Ω, let $\Gamma_{z,w}$ denote the set of all piece-wise C^1 curves $\gamma : [0,1] \mapsto \Omega$ satisfying $\gamma(0) = z$ and $\gamma(1) = w$. We define the length of the curve γ with respect to η as meth seen $\Downarrow$

$$\ell_\eta(\gamma) = \int_a^b \Big\| \frac{\partial \gamma(t)}{\partial t} \Big\|_{\eta, \gamma(t)} \, dt = \int_a^b \eta(\gamma(t)) \cdot \Big| \frac{\partial \gamma(t)}{\partial t} \Big| \, dt.$$

Then, the metric d_η is defined as

$$d_\eta(z,w) = \inf_{\gamma \in \Gamma_{z,w}} \ell_\eta(\gamma).$$

5, A

(c) Because C is compact in $\mathbb{D}$, there is $\delta \in (0,1)$ such that C is contained in the Euclidean ball $B(0,\delta)$. By making $\delta \in (0,1)$ larger if necessary, we may assume that z_0 is also contained in $B(0,\delta)$.

Evidently, $\rho \leq 1/(1-\delta^2)$ on $B(0,\delta)$, and for every $w \in B(0,\delta)$ there is a straight line of length at most 2δ from z_0 to w. The length of this line segment, with respect to η is at most $2\delta/(1-\delta^2)$. Thus, $d_\eta(w, z_0) \leq 2\delta/(1-\delta^2)$. So we may choose $r = 2\delta/(1-\delta^2)$. 5, B

(d) Let C be an arbitrary compact set in $\mathbb{D}$, and $\epsilon > 0$. By Part (c), there is $r > 0$ such that C is contained in the ball $B_\rho(z_0, r)$ with respect to the metric d_ρ. By a theorem in the lectures, any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is distance non-increasing with respect to the Poincare metric. This implies that for every $n \geq 1$, unseen $\Downarrow$

$$f_n(C) \subseteq f_n(B_\rho(z_0, r)) \subset B_\rho(f_n(z_0), r).$$

We claim that the right hand side of the above equation is contained in $B(z_n, \epsilon/2)$, of large enough n. To see that, let $z_n = f_n(z_0)$. By a theorem in the lectures $\phi_{z_n} = (z_n - z)/(1 - \overline{z_n} z)$ is an isometry of the d_ρ metric. thus,

$$B_\rho(z_n, r) = \phi_{z_n}(B_\rho(0, r)).$$

By a result in the lectures $B_\rho(0,r) = B(0,r')$ for some $r' \in (0,1)$. Combining the above, we note that

$$f_n(C) \subset \phi_{z_n}(B(0,r')).$$

On the other hand, for $z \in B(0,r')$, we have

$$|\phi_{z_n}(z) - z_n| = \frac{z(1-|z_n|^2)}{1 - \overline{z_n} z} \to 0$$

as $n \to \infty$. This convergence is uniform, since $|z| \leq r'$ Thus, by making n large enough, we may assume that $\phi_{z_n}(B(0,r'))$ is contained in $B(z_n, \epsilon/2)$. We can also assume that $|z_n - w| < \epsilon/2$. Therefore, for all $z \in C$, $|f_n(z) - w| \leq \epsilon$, for large enough n depending only on C. 6, D

3. (a) For every univalent map $f : \mathbb{D} \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = 1$, and every $z \in \mathbb{D}$, we have

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}. \tag{1}$$

Moreover, an equality holds at some $z \neq 0$, if and only if f is a suitable rotation of the Koebe function.

seen ⇓

6, A

(b) First we note that by adding a constant and multiplying by a non-zero constant to f, the ratio of the areas in the question does not change. So we may assume that $f(0) = 0$ and $f'(0) = 1$.

meth seen ⇓

By the Koebe Distortion theorem, for all $z \in B(0, 0.2)$ we have

$$\frac{1}{3} \leq \frac{25}{54} = \frac{1-0.2}{(1+0.2)^3} \leq |f'(z)| \leq \frac{1+0.2}{(1-0.2)^3} = \frac{75}{32} \leq 3.$$

Therefore, for $X = B(0.1, 0.1)$ and $X = B(-0.1, 0.1)$, we have

$$\text{area}(f(X)) \leq 3^2 \,\text{area}(X) = 9 \cdot \pi \cdot 0.1^2,$$

and

$$\text{area}(f(X) \geq (1/3)^2 \,\text{area}(X) = 9^{-1}\pi \cdot 0.1^2.$$

Combining the above inequalities we get

$$\frac{\text{area}(f(B(0.1, 0.1))}{\text{area}(f(B(-0.1, 0.1))} \leq \frac{9 \cdot \pi \cdot 0.1^2}{9^{-1}\pi \cdot 0.1^2} \leq 81,$$

and

$$\frac{\text{area}(f(B(0.1, 0.1))}{\text{area}(f(B(-0.1, 0.1))} \geq \frac{9^{-1} \cdot \pi \cdot 0.1^2}{9\pi \cdot 0.1^2} \leq 1/81.$$

7, B

(c) Consider the map $\phi : \mathbb{D} \to \mathbb{C}$, defined as

$$\phi(z) = \frac{f((1-r)z + r)}{(1-r)f'(r)}.$$

For $z \in \mathbb{D}$, $|(1-r)z + r| \leq |(1-r)z| + r < 1 - r + r = 1$, and hence ϕ is defined on $\mathbb{D}$. We have $\phi'(0) = (1-r)f'(r)/((1-r)f'(r)) = 1$, and since f is univalent, ϕ must be univalent.

By the $1/4$-theorem applied to ϕ,

$$B(0, 1/4) \subset \phi(\mathbb{D}) = f(B(r, 1-r))/((1-r)f'(r)),$$

and hence

$$B(0, (1-r)f'(r)/4) \subset f(B(r, 1-r)) \subset f(\mathbb{D}).$$

Combining with the lower bound in part (a), we conclude the result in part c.

7, D

4. (a) Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ be a family (set) of holomorphic maps from Ω to $\mathbb{C}$. We say that the family $\mathcal{F}$ is *normal*, if for every sequence of maps $(f_n)_{n\geq 1}$ in $\mathcal{F}$ there is a subsequence $(f_{n_k})_{k\geq 1}$ and a holomorphic map $f : \Omega \to \mathbb{C}$ such that $(f_{n_k})_{k\geq 1}$ converges to f uniformly on compact subsets of Ω. seen ⇓ 6, A

(b) By a theorem in the lectures, if a sequence of holomorphic maps $f_n : \Omega \to \mathbb{C}$ converges uniformly on compact sets to some holomorphic map $f : \Omega \to \mathbb{C}$, then the sequence of maps $f_n' : \Omega \to \mathbb{C}$ converges uniformly on compact sets to the map $f' : \Omega \to \mathbb{C}$. We may repeat applying this theorem to the sequence of maps f_n to get the desires result. meth seen ⇓

Alternatively, one is likely to use the Cauchy integral formula for the second derivative, and then use the uniform convergence of the sequence on a curve or a circle around w, and conclude the result. 6, B

(c) No. First note that for all $n \geq 1$, meth seen ⇓

$$P_{n+1}'(z) = P_n'(P_0(z)) \cdot P_0'(z),$$

which using $P_n(0) = 0$ implies that for all $n \geq 1$, we have $P_n'(0) = 1$. Differentiating the above equation one more time, we get,

$$P_{n+1}''(z) = P_n''(P_0(z))P_0'(z)P_0'(z) + P_n'(P_0(z))P_0''(z).$$

At $z = 0$, this gives us

$$P_{n+1}''(0) = P_n''(0) + 2, = \cdots = 2(n+1).$$

By induction, the above relation implies that $P_{n+1}''(0) = 2(n+1)$. Therefore, $P_n''(0) \to \infty$, as $n \to \infty$. By the property in Part (b), the sequence P_n cannot have a convergent subsequence. 8, C

5. (a) Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an orientation preserving homeomorphism. We say that $f : \Omega \to \mathbb{C}$ is K-quasi-conformal if we have $\boxed{\text{seen} \Downarrow}$

(i) f is absolutely continuous on lines,

(ii) for almost every $z \in \Omega$ we have $K_f(z) \leq K$.

An orientation preserving homeomorphism $f : \Omega \to \mathbb{C}$ is called quasi-conformal, if it is K-quasi-conformal for some $K \geq 1$. $\boxed{4, \text{M}}$

(b) Let Ω be an open set in $\mathbb{C}$, and let $\mu : \Omega \to \mathbb{D}$ be a continuous map with $\sup_{z\in\Omega} |\mu(z)| < 1$. Then, there is a quasi-conformal map $f : \Omega \to \mathbb{C}$ such that the Beltrami equation $\boxed{\text{unseen} \Downarrow}$

$$\partial f/\partial \overline{z} = \mu \partial f/\partial z$$

holds on Ω.

Moreover, the solution f is unique if we assume that $f(z_1) = z_1$ and $f(z_2) = z_2$, for some distinct points z_1 and z_2 in Ω. $\boxed{4, \text{M}}$

(c) The function μ is continuous on $\mathbb{C} \setminus \{0\}$, because

$$e^{i(2\arg(z)+\pi)} = -(e^{i\arg(z)})^2 = -(\cos(\arg(z) + i\sin(\arg(z)))^2.$$

On the other hand, $|\mu(z)| \leq 1/2 < 1$. Then, the Riemann mapping Theorem applies, which gives us a quasi-conformal map $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ which satisfies the Beltrami equation.

Because g is a homeomorphic, the image of g is not all of $\mathbb{C}$. The post-composition of g by any holomorphic map also satisfies the same Beltrami equation. Thus, we may choose a translation by a constant, so that the image of g omits 0. $\boxed{4, \text{M}}$

(d) By definition, the dilatation quotient is given by the formula $\boxed{\text{sim. seen} \Downarrow}$

$$K_g(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|},$$

and hence $K_g(1 + i) = \frac{1+1/2}{1-1/2} = 4/3$. Note that K_g does not depend on the solution, but the coefficient μ. $\boxed{4, \text{M}}$

(e) Let $R(z) = e^{i\theta_0}z$, and $k(z) = g(e^{i\theta_0}z)$. With the notations $w = R(z)$, and the complex chain rule,

$$\frac{\partial k}{\partial z}(z_0) = \frac{\partial (g \circ R)}{\partial z}(z_0) = (\frac{\partial g}{\partial w}(R(z_0)))\cdot\frac{\partial R}{\partial z}(z_0) + (\frac{\partial g}{\partial \overline{w}}(R(z_0)))\cdot\frac{\partial \overline{R}}{\partial z}(z_0) = \frac{\partial g}{\partial w}(e^{i\theta_0}z)\cdot e^{i\theta_0}.$$

and

$$\frac{\partial k}{\partial \overline{z}}(z_0) = \frac{\partial (g \circ R)}{\partial \overline{z}}(z_0) = (\frac{\partial g}{\partial w}(R(z_0)))\cdot\frac{\partial R}{\partial \overline{z}}(z_0) + (\frac{\partial g}{\partial \overline{w}}(R(z_0)))\cdot\frac{\partial \overline{R}}{\partial \overline{z}}(z_0) = \frac{\partial g}{\partial \overline{w}}(e^{i\theta_0}z)\cdot e^{-i\theta_0}.$$

Combining the above equations, and using the notation $w_0 = R(z_0)$, we see that $k(z)$ satisfies the Beltrami equation with coefficient μ, that is,

$$\begin{aligned}
\frac{\partial k}{\partial \overline{z}}(z_0) &= \frac{\partial g}{\partial \overline{w}}(w_0) \cdot e^{-i\theta_0} \\
&= \mu(w_0)\frac{\partial g}{\partial w}(w_0) \cdot e^{-i\theta_0} \\
&= \mu(z_0)e^{2i\theta_0}\frac{\partial g}{\partial w}(w_0) \cdot e^{-i\theta_0} \\
&= \mu(z_0)k(z_0).
\end{aligned}$$

$\boxed{4, \text{M}}$

Review of mark distribution:
Total A marks: 31 of 32 marks
Total B marks: 23 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 13 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70140 Geometric Complex Analysis Markers Comments

Question 1	Most students did well in this problem. there is a good understanding of the schwarz lemma and how to apply it. Part b was part of the proof of the classification of the automorphisms of the disk, and some students used the full classification to answer the special case.
Question 2	Parts a and b are routine, and most students did well. a good number of students worked hard to get part c. Part d was challenging, but still some students showed deep understanding.
Question 3	Part b was very similar to a problem in the assessed course work, where most students did well. but somehow very few managed to do it in the exam.
Question 4	Part c of the problem was difficult, but students show creativity in dealing with the problem. Indeed, not a single student use the solution provided in the solution key, but a good number of students either completely or partially did the problem.
Question 5	Overall students show very good understanding of the material, but there is clear weaknesses in the understanding of the prerequisite for the module. Most students state that the arg functions continuous on C.