Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 6 - Dedekind Cuts and Triangle inequalities

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Intuition

Suppose we have a construction of $\mathbb{R}$, e.g. by decimals. Then to every real number $r \in \mathbb{R}$ we can associate a semi-infinite subset S_r of $\mathbb{Q}$,

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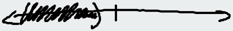
Definition

$\emptyset \neq S \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies (i) and (ii) below.

(i) If $s \in S$ and $s > t \in \mathbb{Q}$ then $t \in S$ (S is a semi-infinite interval to the left).



"S_r"

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Exercise 2.40

Convince yourself we can identify $\mathbb{Q} \subset \mathbb{R}$ via
$q \in \mathbb{Q} \leftrightarrow S_q := \{s \in \mathbb{Q} : s < q\}$.

We can again extend operations from $\mathbb{Q}$ to our newly constructed $\mathbb{R}$; eg if $S \subset \mathbb{Q}$ and $T \subset \mathbb{Q}$ are Dedekind cuts, we define

$$S + T := \{s + t \colon s \in S,\ t \in T\} \subset \mathbb{Q}.$$

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Exercise 2.41

Check this is a Dedekind cut (an element of $\mathbb{R}$!) and gives the usual $+$ on $\mathbb{Q}$: i.e. $S_{q_1} + S_{q_2} = S_{q_1+q_2}$.

$\{1,2\} + \{3,4\} = \{4,5,6\}$

different than $\cup$

$1+3=4$

$1+4=5$

$2+3=5$

$2+4=6$

Similarly we can define $<$ on $\mathbb{R}$ to be just $\subsetneq$ on Dedekind cuts, that is $S < T \iff S \subsetneq T$.

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$S \leq T \Leftrightarrow S \subset T$ "$\subseteq$" "$\subset$"

Exercise 2.42

Show a set of real numbers $A \subset \mathbb{R}$ is bounded above if and only if A is a collection of Dedekind cuts S all contained in some fixed interval $(-\infty, N)$ for some $N \in \mathbb{N}$.

$\subsetneq$ proper

$(-\infty, N) \cap \mathbb{Q}$

Exercise 2.43 - Completeness Axiom

If A is a nonempty collection of Dedekind cuts which is bounded above, define

$$\sup A \ := \ \bigcup_{S \in A} S \ \subset \ \mathbb{Q}.$$

Show this is also a Dedekind cut (i.e. a real number!) and check it is the least upper bound of A.

Triangle inequalities

Theorem 2.44 - main triangle inequality

For all $a, b \in \mathbb{R}$ we have

$$|a+b| \;\leq\; |a| + |b|.$$

Proof Suppose by contradiction that $|a+b| > |a|+|b|$

Then $\underline{|a+b| \cdot |a+b|} > (|a|+|b|)\,|a+b| > (|a|+|b|)^2 = \underline{|a|^2 + 2|a|\cdot|b| + |b|^2}$

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 \leq a^2 + 2|a|\cdot|b| + b^2 = |a|^2 + 2|a|\cdot|b| + |b|^2$$

Exercise 2.45

Why is this called the triangle inequality?
Give a direct proof *without squaring* by first proving $x \leq |x|$ by splitting into two cases.

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More triangle inequalities you will prove on Problem Sheet 2:

(a) $|x+y| \leq |x|+|y|$

(b) $|x+y| \geq |x|-|y|$

(c) $|x+y| \geq |y|-|x|$

(d) $|x-y| \geq \Big||x|-|y|\Big|$

(e) $|x| \leq |y|+|x-y|$

(f) $|x| \geq |y|-|x-y|$

(g) $|x-y| \leq |x-z|+|y-z|$

For instance, let's prove the inequality

(g) $|x - y| \ \leq \ |x - z| + |y - z|$

from

(a) $|x + y| \ \leq \ |x| + |y|$

$$|x-y| = |(x-z)+(z-y)| \leq |x-z| + |z-y| = |x-z| + |y-z|$$

Here are some different ways of saying a, b are close to each other; get used to them.

$$|a - b| < \epsilon \iff b - \epsilon < a < b + \epsilon$$

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$$\begin{aligned} |a-b| < \epsilon \quad &\iff \quad b-\epsilon < a < b+\epsilon \\ &\iff \quad a-\epsilon < b < a+\epsilon \\ &\iff \quad a \in (b-\epsilon, b+\epsilon) \end{aligned}$$

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Since, for all $\epsilon > 0$, $|a-b| < \epsilon$ implies $\big||a|-|b|\big| < \epsilon$, it follows that $\big||a|-|b|\big| \le |a-b|$ (show this carefully!).

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Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0,\ |x - a| < \epsilon$$

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mean for the number x? Which of the following is it equivalent to?

1. x is close to a
2. $x \in (a - \epsilon, a + \epsilon)$
3. $x = a$ ←
4. $x = a + \epsilon$
5. $x = a - \epsilon$
6. More than one of these
7. None of these

Suppose $x \neq a$ (by contradiction)

Then $|x-a| > 0$. Let $\varepsilon = \frac{1}{2}|x-a| > 0$

But then $|x-a| < \frac{1}{2}|x-a| \Rightarrow \frac{1}{2}|x-a| < 0 \Rightarrow |x-a| < 0$ ※