MATH40005: Probability and Statistics

Bridging lecture: Multivariate calculus

Partial derivatives

Suppose you have a bivariate function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = xy^2 + x^3y.$$

We would like to find the *partial derivative* with respect to x. Then we treat the variable y as a constant and differentiate the function $g(x) := f(x, y)$ in the usual way with respect to x. This leads to

$$\frac{\partial f(x, y)}{\partial x} = \frac{d}{dx}g(x) = y^2 + 3x^2y$$

Similarly, if we would like to find the *partial derivative* with respect to y, then we treat the variable x as a constant and differentiate the function $h(y) := f(x, y)$ in the usual way with respect to y. This leads to

$$\frac{\partial f(x, y)}{\partial y} = \frac{d}{dy}h(y) = 2xy + x^3$$

Partial derivatives

Recall $\frac{\partial f(x,y)}{\partial x} = y^2 + 3x^2 y$, $\frac{\partial f(x,y)}{\partial y} = 2xy + x^3$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = 2y + 3x^2$$

We note that it does not matter in which order we differentiate and we get the same result when we compute

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = 2y + 3x^2$$

Under mild technical assumptions, we have for a general function f that

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

Jacobian

Consider a transformation which maps $\mathbf{x} = (x_1, x_2)$ to $\mathbf{y} = (y_1, y_2)$. Then the *Jacobian* of the transformation is defined as the 2x2 matrix of all possible first order partial derivatives:

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

Bivariate integrals

$$\iint_A f(x,y)\,dy\,dx$$

Let $A \subseteq \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$. Then (under mild conditions), we have that the order of integration does not matter and that

$$\int_A f(x,y)dxdy = \int_A f(x,y)dydx,$$

and the joint integral can be computed by iteratively computing the univariate integrals. I.e., similar to the concept of partial derivatives, we can compute multiple integrals by treating the variable which is not the integration variable constant and integrate out one variable at a time.

Let $A = \{(x,y) : 0 \le x \le y \le 1\} \subseteq \mathbb{R}^2$ and $f(x,y) = xy^2 + x^3y$.

$$\int_A f(x,y)\,dxdy = \int_0^1 \int_0^y (xy^2 + x^3y)\,dxdy = \int_0^1 \left(\frac{1}{2}x^2y^2 + \frac{1}{4}x^4y \Big|_0^y \right) dy$$

$$= \int_0^1 \left(\frac{1}{2}y^4 + \frac{1}{4}y^5 \right) dy$$

$$= \frac{1}{2}\frac{1}{5}y^5 + \frac{1}{4}\frac{1}{6}y^6 \Big|_0^1 = \frac{1}{10} + \frac{1}{24} = \frac{17}{120}.$$

Fubini's theorem

$$\int_A f(x,y)\,dydx = \int_0^1 \int_x^1 (xy^2 + x^3y)\,dydx = \int_0^1 \left(\frac{1}{3}xy^3 + \frac{1}{2}x^3y^2 \Big|_x^1 \right) dx$$

$$= \int_0^1 \left(\frac{1}{3}x + \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{2}x^5 \right) dx$$

$$= \frac{1}{2}\frac{1}{3}x^2 + \frac{1}{2}\frac{1}{4}x^4 - \frac{1}{3}\frac{1}{5}x^5 - \frac{1}{2}\frac{1}{6}x^6 \Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{8} - \frac{1}{15} - \frac{1}{12} = \frac{17}{120}.$$

Change of variables formula

Let $f : \mathbb{R}^2 \to \mathbb{R}$. We define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x, y) = (u(x, y), v(x, y)),$$

and assume that T is a bijection from the domain $D \subseteq \mathbb{R}^2$ to some range $S \subseteq \mathbb{R}^2$. Then we can write $T^{-1} : S \to D$ for the inverse mapping of T, i.e. $(x, y) = T^{-1}(u, v)$. For the first component we write $x = x(u, v)$ and for the second $y = y(u, v)$. The *Jacobian determinant* of T^{-1} is defined as the determinant

$$J(u, v) = \det\left(\frac{\partial(x, y)}{\partial(u, v)}\right) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The change of variable formula states that (under mild conditions[3])

$$\int_D f(x, y)dxdy = \int_S f(x(u, v), y(u, v))|J(u, v)|dudv. \tag{34.1}$$

Change of variables formula: Polar coordinates

Suppose we want to compute the integral

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x^2+y^2)\right) dxdy.$$

Now we consider the (invertible) transformation to polar coordinates:

$$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

for $r > 0$ and $\theta \in [0, 2\pi]$. Then, we compute the Jacobian determinant $J(r,\theta)$ of the transformation as follows:

$$J(r,\theta) = \det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = \det\begin{pmatrix} \cos\theta & -r\sin(\theta) \\ \sin\theta & r\cdot\cos(\theta)\end{pmatrix} = r\cdot\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Change of variables formula: Polar coordinates

$$I = \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{1}{2}(r^2\cos^2(\theta) + r^2\sin^2(\theta))\right)|J(r,\theta)|drd\theta$$
$$= \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{1}{2}r^2\right) rdrd\theta.$$

You can now do another variable transformation (or integrate directly):
We set $u = r^2/2$, the $du = rdr$ and

$$I = \int_0^{2\pi} \left(\int_0^{\infty} e^{-u}du\right) d\theta = \int_0^{2\pi} \left(-e^{-u}|_{u=0}^{\infty}\right) d\theta = \int_0^{2\pi} 1 d\theta = 2\pi.$$

Hence, we have that $\sqrt{I} = \sqrt{2\pi}$.

Prove that the standard normal density integrates to 1!

$$\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}_{\phi(x) \geq 0} dx = 1. \tag{34.2}$$

$$\Leftrightarrow \quad \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \sqrt{2\pi}.$$

$$\left(\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\right)^2 = 2\pi.$$

$$\left(\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\right)^2 = \left(\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\right) \cdot \left(\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x^2 + y^2)\right) dxdy = I.$$

Since we have already shown that $I = 2\pi$, we can conclude that $\int_{-\infty}^{\infty} \phi(x) dx = 1$.