IMPERIAL

MATH70052

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Hydrodynamic Stability

Date: Tuesday, May 27, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Consider a system of two coupled equations,

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=\frac{1}{R}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)u+\frac{\partial v}{\partial y}\frac{\partial v}{\partial x},$$

$$\frac{\partial v}{\partial t}+a\frac{\partial v}{\partial x}=\frac{1}{R}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)v-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x},$$

which may be taken as a model for studying fluid motion and its stability, where $a>0$ and $R>0$ are real parameters. The boundary conditions for $u(x,y,t)$ and $v(x,y,t)$ are

$$u=v=0\ \text{ at }\ y=0;\qquad u=1,\quad v=\Gamma\ \text{ at }\quad y=1,$$

where Γ is a constant.

(i) Find the steady basic state, $u=U(y)$ and $v=V(y)$.

The basic state is perturbed by a small-amplitude disturbance, u' and v', such that

$$(u,v)=\Big(U(y),V(y)\Big)+\Big(u'(x,y,t),v'(x,y,t)\Big).$$

Write down the linearised equations and boundary conditions satisfied by u' and v'.

(5 marks)

(ii) Seek temporal normal-mode solutions of the form

$$\Big(u',v'\Big)=\Big(\bar{u}(y),\bar{v}(y)\Big)e^{\sigma t+i\alpha x}+c.c.,$$

where $c.c.$ stands for the complex conjugate. Show that the eigenvalue σ is

$$\sigma=-(a\alpha)i\pm\sqrt{\Gamma}\,\alpha-\Big(\alpha^2+(n\pi)^2\Big)/R\quad(n=1,\ 2,\ldots),$$

and find the corresponding eigenfunctions, $\bar{u}$ and $\bar{v}$.

Deduce that for the case of $\Gamma>0$, the basic state becomes unstable for

$$R>R_c=2\pi/\sqrt{\Gamma}.$$

Identify the upper and lower branches of the neutral curve in the limit $R\gg 1$.

(10 marks)

(iii) Consider now spatial normal modes, for which $\sigma=-i\omega$ with ω being real. Using the result in Part (ii) for $\Gamma>0$, find the eigenvalue α for spatially amplifying modes, and show that in the limit $R\gg 1$,

$$\alpha\to-\frac{\omega i}{\sqrt{\Gamma}-ia}+\Big[\frac{(n\pi)^2}{\sqrt{\Gamma}-ia}-\frac{\omega^2}{(\sqrt{\Gamma}-ia)^3}\Big]\frac{1}{R}.$$

(5 marks)

(Total: 20 marks)

2. Consider the flow between two concentric cylinders, which have radii R_1 and R_2, and rotate at angular velocities Ω_1 and Ω_2. In the cylindrical coordinates $(\hat{r}, \phi, \hat{z})$, the base flow has the velocity field $(\hat{V}_{\hat{r}}, \hat{V}_{\phi}, \hat{V}_{\hat{z}}) = (0, \hat{U}(\hat{r}), 0)$ and pressure $\hat{P}(\hat{r})$. The flow is perturbed by three-dimensional disturbances with velocities $(\hat{u}', \hat{v}', \hat{w}')$ and pressure $\hat{p}'$, which depend on all three coordinates $(\hat{r}, \phi, \hat{z})$ and time $\hat{t}$, and satisfy the equations

$$\frac{\partial \hat{u}'}{\partial \hat{t}} + \frac{\hat{U}}{\hat{r}}\frac{\partial \hat{u}'}{\partial \phi} + \frac{d\hat{U}}{d\hat{r}}\hat{v}' + \frac{\hat{U}}{\hat{r}}\hat{v}' = -\frac{1}{\rho \hat{r}}\frac{\partial \hat{p}'}{\partial \phi} + \nu\left(\frac{\partial^2 \hat{u}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{u}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{u}'}{\partial \hat{r}} - \frac{\hat{u}'}{\hat{r}^2} + \frac{1}{\hat{r}^2}\frac{\partial^2 \hat{u}'}{\partial \phi^2} + \frac{2}{\hat{r}^2}\frac{\partial \hat{v}'}{\partial \phi}\right),$$

$$\frac{\partial \hat{v}'}{\partial \hat{t}} + \frac{\hat{U}}{\hat{r}}\frac{\partial \hat{v}'}{\partial \phi} - 2\frac{\hat{U}}{\hat{r}}\hat{u}' = -\frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{r}} + \nu\left(\frac{\partial^2 \hat{v}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{v}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{v}'}{\partial \hat{r}} - \frac{\hat{v}'}{\hat{r}^2} + \frac{1}{\hat{r}^2}\frac{\partial^2 \hat{v}'}{\partial \phi^2} - \frac{2}{\hat{r}^2}\frac{\partial \hat{u}'}{\partial \phi}\right),$$

$$\frac{\partial \hat{w}'}{\partial \hat{t}} + \frac{\hat{U}}{\hat{r}}\frac{\partial \hat{w}'}{\partial \phi} = -\frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{z}} + \nu\left(\frac{\partial^2 \hat{w}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{w}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{w}'}{\partial \hat{r}} + \frac{1}{\hat{r}^2}\frac{\partial^2 \hat{w}'}{\partial \phi^2}\right),$$

$$\frac{\partial \hat{v}'}{\partial \hat{r}} + \frac{\hat{v}'}{\hat{r}} + \frac{\partial \hat{w}'}{\partial \hat{z}} + \frac{1}{\hat{r}}\frac{\partial \hat{u}'}{\partial \phi} = 0;$$

where ρ and ν, the density and kinematic viscosity, are constant.

Consider the case where $\hat{U} = \Omega \hat{r}$ $(\Omega_1 = \Omega_2 = \Omega)$ in the narrow-gap limit $h = R_2 - R_1 \ll R_1$.

(i) Introduce the non-dimensional independent and dependent variables as follows

$$\left.\begin{aligned} &\hat{t} = \Omega^{-1}\, t, \quad \hat{r} = R_1 + hy, \quad \hat{z} = hz; \\ &\hat{u}' = \Omega R_1\, u', \quad \hat{v}' = \Omega R_1\, v', \quad \hat{w}' = \Omega R_1\, w', \quad \hat{p}' = \rho \Omega^2 R_1 h\, p'. \end{aligned}\right\}$$

Deduce the equations governing (u', v', w') and p' in the limit $h/R_1 \ll 1$:

$$\frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial \phi} + 2v' = \frac{1}{Re}\Big[\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\Big], \qquad \frac{\partial v'}{\partial t} + \frac{\partial v'}{\partial \phi} - 2u' = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\Big[\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\Big],$$

$$\frac{\partial w'}{\partial t} + \frac{\partial w'}{\partial \phi} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\Big[\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\Big], \qquad \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

where Re is a parameter whose expression you are expected to determine.

(7 marks)

(ii) Seek solutions to the narrow-gap limit equations of the normal-mode form,

$$(u', v', w', p') = \Big(\bar{u}(y), \bar{v}(y), \bar{w}(y), \bar{p}(y)\Big)e^{\sigma t + i(\beta z + n\phi)} + c.c.,$$

where $c.c.$ denotes the complex conjugate.

Derive the equations satisfied by $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$, and show that these equations reduce to a system of two coupled equations,

$$(\sigma + in)\bar{u} - \frac{1}{Re}\Big[\frac{d^2}{dy^2} - \beta^2\Big]\bar{u} = -2\bar{v}, \tag{1}$$

$$(\sigma + in)\Big[\frac{d^2}{dy^2} - \beta^2\Big]\bar{v} - \frac{1}{Re}\Big[\frac{d^2}{dy^2} - \beta^2\Big]^2\bar{v} = -2\beta^2\bar{u}. \tag{2}$$

Specify the boundary conditions on $\bar{u}$ and $\bar{v}$.

(6 marks)

Question continues on the next page.

(iii) Show that with viscous effect included, the disturbances are damped with $\sigma_r < 0$. Deduce that modes are neutral in the inviscid limit.

Hint: Multiply equations (1) *and* (2) *by* $\bar{u}^*$ *and* $\bar{v}^*$ *(the complex conjugates of* $\bar{u}$ *and* $\bar{v}$*), respectively, integrate and perform integration by parts.*

(7 marks)

(Total: 20 marks)

3. Inviscid stability of an exactly parallel shear flow with velocity profile $U(y)$ is studied by introducing small-amplitude perturbations of the normal-mode form: $\Big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\Big)e^{i(\alpha x - \omega t)} + c.c.$. where $c.c.$ denotes the complex conjugate. You are given that $\Big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\Big)$ satisfies the equations,

$$i\alpha\bar{u} + \frac{d\bar{v}}{dy} = 0, \qquad i\alpha(U-c)\bar{u} + \frac{dU}{dy}\bar{v} = -i\alpha\bar{p}, \qquad i\alpha(U-c)\bar{v} = -\frac{d\bar{p}}{dy}, \tag{3}$$

where α is real, and $c = \omega/\alpha = c_r + ic_i$ is the (complex) phase speed. Assume that the shear flow occupies the region above a flat plate at $y = 0$.

(i) Show that the pressure $\bar{p}$ satisfies the equation

$$\frac{d}{dy}\Big[\frac{1}{(U-c)^2}\frac{d\bar{p}}{dy}\Big] - \frac{\alpha^2\bar{p}}{(U-c)^2} = 0. \tag{4}$$

Specify the boundary and far-field conditions in terms of $\bar{p}$ and its derivatives.

(5 marks)

(ii) Assuming that the profile $U(y)$ is sufficiently smooth, prove that if the flow is unstable, then c_r lies in the range

$$U_{\min} < c_r < U_{\max},$$

where $U_{\max}$ and $U_{\min}$ denote the maximum and minimum of $U(y)$, respectively.

Hint: Multiply equation (4) by $\bar{p}^*$, the complex conjugate of $\bar{p}$, followed by integration by parts.

(4 marks)

(iii) Suppose that $y = y_d$ is a location where $U(y)$ and/or $\dfrac{dU}{dy}$ is discontinuous. Show that

$$\Big[\bar{p}\Big]_{y_d^-}^{y_d^+} = 0, \qquad \Big[\frac{1}{(U-c)^2}\frac{d\bar{p}}{dy}\Big]_{y_d^-}^{y_d^+} = 0,$$

where $\Big[\ \cdot\ \Big]_{y_d^-}^{y_d^+}$ stands for the jump of the quantity across $y = y_d$.

Using the above results, or otherwise, explain why the result in Part (ii) remains valid even for a discontinuous profile.

(5 marks)

(iv) Verify the last conclusion in Part (iii) by solving the Rayleigh equation

$$(U-c)(\frac{d^2}{dy^2} - \alpha^2)\bar{v} - \frac{d^2U}{dy^2}\bar{v} = 0,$$

and finding $c = c_r + ic_i$ for a piecewise constant velocity profile,

$$U(y) = \begin{cases} U_2 & \text{for } y > h, \\ U_1 & \text{for } 0 < y < h, \end{cases}$$

where U_1 and U_2 are constants.

Verify that c_r and c_i satisfy (despite the profile being discontinuous) the inequality,

$$\Big[c_r - \frac{1}{2}\Big(U_{\max} + U_{\min}\Big)\Big]^2 + c_i^2 \le \Big[\frac{1}{2}\Big(U_{\max} - U_{\min}\Big)\Big]^2.$$

(6 marks)

(Total: 20 marks)

4. In a Cartesian coordinate system (x, y, z), a base flow has the velocity field $\Big(U(y), V, 0\Big)$ and the pressure $P(x)$. Note that unlike uni-directional flows, the base flow here has a non-zero velocity component in the y-direction, V, which is constant. In order to study its stability, a small three-dimensional disturbance is introduced, and the perturbed flow is written as

$$(u, v, w, p) = \Big(U(y), V, 0, P\Big) + \epsilon\Big(u'(x, y, z, t), v'(x, y, z, t), w'(x, y, z, t), p'(x, y, z, t)\Big),$$

where t is the time variable, and the amplitude of the perturbation $\epsilon \ll 1$. Suppose that the independent and dependent variables have been suitably non-dimensionalised such that $(\mathbf{u}, p) \equiv (u, v, w, p)$ satisfy the non-dimensional Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u},$$

where Re is the Reynolds number.

(a) Derive the linearised Navier-Stokes equations satisfied by the perturbation (u', v', w', p'). Show that v' satisfies the equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v' - \frac{d^2U}{dy^2}\frac{\partial v'}{\partial x} = Re^{-1}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 v'.$$

(9 marks)

(b) Suppose that solutions for the perturbation are sought in the normal-mode form,

$$(u', v', w', p') = \Big(\bar{u}(y), \bar{v}(y), \bar{w}(y), \bar{p}(y)\Big)\mathrm{e}^{\mathrm{i}(\alpha x + \beta z - \omega t)} + c.c.,$$

where α and β are real constants, while ω is complex valued in general, and $c.c.$ denotes the complex conjugate.

(i) Using the equation for v' derived in Part (a), or otherwise, find the equation satisfied by $\bar{v}$, and specify the boundary conditions for $\bar{v}$ at the walls located at y_1 and y_2.

Re-arrange the equation for $\bar{v}$ to a form that facilitates computation of $c \equiv \omega/\alpha$ as a generalised linear eigenvalue.

(6 marks)

(ii) Suppose that the base-flow velocity is $(U, V) = \left(1 - \mathrm{e}^{-y}, \dfrac{1}{Re}\right)$. Explain why Squire's theorem is valid for this flow.

(5 marks)

(Total: 20 marks)

5. When a three-dimensional boundary layer is perturbed by a disturbance, the perturbed flow field is written as

$$(u, v, w, p) = \Big(U(x, Y),\, Re^{-1/2}V(x, Y),\, W(x, Y),\, P\Big) + \epsilon(u',\, v',\, w',\, p')$$

in the Cartesian coordinate system (x, y, z), where x, y and z are non-dimensionalised by L, the distance to the leading edge, $Y = Re^{1/2}y$ and Re is the Reynolds number based on L. The parameter $\epsilon \ll 1$ measures the magnitude of the disturbance. The base flow is of wall-jet type, whose velocities have the near-wall and far-field behaviours:

$$(U, W) \to (\lambda_1 Y,\, \lambda_3 Y) \quad \text{as} \quad Y \to 0; \quad (U, W) \to (0, 0) \quad \text{as} \quad Y \to \infty,$$

where λ_1 and λ_3 are functions of x. The flow field $(\mathbf{u}, p) = (u, v, w, p)$ is governed by the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}.$$

(i) Derive the linearised equations governing the perturbation (u', v', w') and p', which are functions of x, Y and z but are assumed to be independent of t, e.g. $u' = u'(x, Y, z)$. Indicate the terms which represent the non-parallel-flow effect, and explain Prandtl's local parallel-flow approximation.

(5 marks)

(ii) Suppose that in the main layer (deck) where $Y = O(1)$, the solution expands as

$$(u',\, v',\, w',\, p') = \Big(\bar{u}(x, Y),\, Re^{-\frac{1}{14}}\bar{v}(x, Y),\, \bar{w}(x, Y),\, Re^{-\frac{1}{7}}\bar{p}(x, Y)\Big)E + c.c.,$$

where $c.c.$ stands for the complex conjugate, and $E = e^{\mathrm{i}Re^{3/7}(\alpha x + \beta z)}$. Derive the equations governing $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$, and verify that they have the solution

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{w} = A\frac{\partial W}{\partial Y}, \quad \bar{v} = -iA(\alpha U + \beta W), \quad \bar{p} = -A\int_{\infty}^{Y}(\alpha U + \beta W)^2 dY,$$

where A is a constant.

Explain why it is necessary to introduce a viscous sublayer (i.e lower deck). (5 marks)

(iii) Deduce that the lower deck has a width corresponding to $Y = O(Re^{-1/7})$ and that the solution expands as

$$(u',\, v',\, w',\, p') = \Big(\tilde{u}(x, \tilde{y}),\, Re^{-\frac{3}{14}}\tilde{v}(x, \tilde{y}),\, \tilde{w}(x, \tilde{y}),\, Re^{-\frac{1}{7}}\tilde{p}(x, \tilde{y})\Big)E + c.c.$$

Introduce $\tilde{y} = Re^{1/7}Y$, and write down the equations governing $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$.

Determine the pressure $\tilde{p}$, and specify the boundary conditions at $\tilde{y} = 0$ and the matching condition as $\tilde{y} \to \infty$.

(7 marks)

(iv) Based on the results in Part (iii), comment on the nature of instability, and determine the order of correction that non-parallelism would produce.

(3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70052

Hydrodynamic Stability (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. (i) The basic state satisfies the equation $\dfrac{d^2U}{dy^2}=0$, and $\dfrac{d^2V}{dy^2}=0$ so $U=c_2+c_1y$ and similarly for V. The boundary conditions at $y=0,1$ require that $c_2=0$ and $c_1=1$. Hence

sim. seen ⇓

$$U(y)=y, \qquad V(y)=\Gamma y.$$

The linearised equations for the disturbance:

$$\frac{\partial u'}{\partial t}+a\frac{\partial u'}{\partial x}=\frac{1}{R}\Big[\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\Big]+\Gamma\frac{\partial v'}{\partial x}, \tag{1}$$

$$\frac{\partial v'}{\partial t}+a\frac{\partial v'}{\partial x}=\frac{1}{R}\Big[\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\Big]-\frac{\partial u'}{\partial x}. \tag{2}$$

The boundary conditions are

$$u'=v'=0 \quad \text{at} \quad y=0; \qquad u'=v'=0 \quad \text{at} \quad y=1.$$

5, A

(ii) For disturbances of the assumed normal-mode form, $\dfrac{\partial}{\partial t}\to\sigma, \quad \dfrac{\partial}{\partial x}\to i\alpha.$ Substitution of the normal mode disturbances into (1)-(2) yields

sim. seen ⇓

$$\frac{d^2\bar u}{dy^2}+\big(-\alpha^2-R\sigma-iRa\,\alpha\big)\bar u=-\Gamma(i\alpha R)\bar v, \quad \frac{d^2\bar v}{dy^2}+\big(-\alpha^2-R\sigma-iRa\,\alpha\big)\bar v=(i\alpha R)\bar u,$$

which satisfy the boundary conditions

$$\bar u=\bar v=0 \quad \text{at} \quad y=0,1.$$

The two equations are combined to give

$$\Big[\frac{d^2}{dy^2}+\big(-\alpha^2-R\sigma-iRa\,\alpha\big)\Big]^2\bar u=\Gamma(\alpha R)^2\bar u, \tag{3}$$

while the boundary conditions imply that

$$\bar u=\frac{d^2\bar u}{dy^2}=0 \quad \text{at} \quad y=0,1.$$

By inspection, the function $\bar u=A\sin(n\pi y)$ satisfies the boundary conditions. (Such solutions for $\bar u$ can of course be obtained by solving (3) subject to the boundary conditions, but the procedure could be long). Inserting $\bar u$ into (3), we have

$$\big[-(n\pi)^2+\big(-\alpha^2-R\sigma-iRa\,\alpha\big)\big]^2=\Gamma(\alpha R)^2,$$

from which follows the required result (dispersion relation),

$$\sigma=-ia\,\alpha\pm\sqrt{\Gamma}\,\alpha-\big((n\pi)^2+\alpha^2\big)/R. \tag{4}$$

Substituting σ and $\bar u=A\sin(n\pi y)$ into the equation for $\bar v$, we have

$$\mp\sqrt{\Gamma}(\alpha R)\bar v=(i\alpha R)A\sin(n\pi y),$$

which gives

$$\bar v=\mp(i/\sqrt{\Gamma}\,)A\sin(n\pi y).$$

6, C

For $\Gamma > 0$, the negative sign gives a $\sigma_r < 0$, and hence we focus on the positive sign, for which the temporal growth rate is given by

$$\sigma_r = \sqrt{\Gamma}\,\alpha - \big((n\pi)^2 + \alpha^2\big)/R.$$

The neutral curve is given by

$$R = \big((n\pi)^2 + \alpha^2\big)/(\sqrt{\Gamma}\,\alpha).$$

The minimum of R is attained when $2\alpha^2 - (n\pi)^2 - \alpha^2 = 0$, i.e. when $\alpha = n\pi$, at which $R = 2n\pi/\sqrt{\Gamma}$. The critical Reynolds number is given by $n = 1$ as

$$R_c = 2\pi/\sqrt{\Gamma}.$$

The lower branch of the neutral curve is

$$\alpha \sim (\pi^2/\sqrt{\Gamma})R^{-1}.$$

The upper branch follows the scaling law

$$\alpha \sim \sqrt{\Gamma}R.$$

$\boxed{4, \text{A}}$

(iii) Putting $\sigma = -i\omega$ in the dispersion relation (4) gives $\boxed{\text{unseen} \Downarrow}$

$$\alpha^2 - (\sqrt{\Gamma} - ia)R\,\alpha + (n\pi)^2 - R\omega\, i = 0,$$

where we take the sign on the anticipation that temporally growing modes correspond to spatially amplifying ones.
The spatial eigenvalue α is found as

$$\alpha = \frac{1}{2}\Big[(\sqrt{\Gamma} - ia)R \pm \sqrt{(\sqrt{\Gamma} - ia)^2R^2 - 4\big[(n\pi)^2 - R\omega i\big]}\,\Big].$$

In the limit $R \gg 1$,

$$\alpha \to \frac{1}{2}(\sqrt{\Gamma} - ia)R\left\{1 \pm \left[1 - \frac{2\big((n\pi)^2 - R\omega i\big)}{(\sqrt{\Gamma} - ia)^2R^2} - \frac{2\big((n\pi)^2 - R\omega i\big)^2}{(\sqrt{\Gamma} - ia)^4R^4} + \dots\right]\right\}.$$

The positive sign gives a spurious solution and is discarded.For the negative sign,

$$\alpha \to -\frac{\omega i}{\sqrt{\Gamma} - ia} + \frac{1}{R}\Big[\frac{(n\pi)^2}{\sqrt{\Gamma} - ia} - \frac{\omega^2}{(\sqrt{\Gamma} - ia)^3}\Big].$$

$\boxed{5, \text{B}}$

2. (i) Substituting normalised variables in the question into the linearized Navier-Stokes equations, we have unseen $\Downarrow$

$$\frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial \phi} + 2v' = -\frac{h}{R_1}\frac{1}{r}\frac{\partial p'}{\partial \phi}$$
$$+ \frac{1}{Re}\left[\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} + \frac{h}{R_1}\frac{1}{r}\frac{\partial u'}{\partial y} + (\frac{h}{R_1})^2\frac{1}{r^2}\Big(-u' + \frac{\partial^2 u'}{\partial \phi^2} + 2\frac{\partial v'}{\partial \phi}\Big)\right],$$

$$\frac{\partial v'}{\partial t} + \frac{\partial v'}{\partial \phi} - 2u' = -\frac{\partial p'}{\partial y}$$
$$+ \frac{1}{Re}\left[\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} + \frac{h}{R_1}\frac{1}{r}\frac{\partial v'}{\partial y} + (\frac{h}{R_1})^2\frac{1}{r^2}\Big(-v' + \frac{\partial^2 v'}{\partial \phi^2} - 2\frac{\partial u'}{\partial \phi}\Big)\right],$$

$$\frac{\partial w'}{\partial t} + \frac{\partial w'}{\partial \phi} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\left[\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} + \frac{h}{R_1}\frac{1}{r}\frac{\partial w'}{\partial y} + (\frac{h}{R_1})^2\frac{1}{r^2}\frac{\partial^2 w'}{\partial \phi^2}\right],$$

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} + \frac{h}{R_1}\frac{v'}{r} + \frac{h}{R_1}\frac{1}{r}\frac{\partial u'}{\partial \phi} = 0.$$

Here we have put

$$r = 1 + (h/R_1)y,$$

and Re is the Reynolds number given by

$$Re = \Omega_1 h^2/\nu.$$

5, A

In the limit $h/R_1 \ll 1$, $r \approx 1$ so that the equations reduce to

$$\frac{\partial u'}{\partial t} + \frac{\partial u'}{\partial \phi} + 2v' = \frac{1}{Re}\Big[\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\Big],$$

$$\frac{\partial v'}{\partial t} + \frac{\partial v'}{\partial \phi} - 2u' = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\Big[\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\Big],$$

$$\frac{\partial w'}{\partial t} + \frac{\partial w'}{\partial \phi} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\Big[\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\Big],$$

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0.$$

2, A

(ii) Substitution of the assumed normal-mode form of solutions into the above equations leads to unseen $\Downarrow$

$$(\sigma + in)\bar{u} + 2\bar{v} = \frac{1}{Re}\Big[\frac{\partial^2}{\partial y^2} - \beta^2\Big]\bar{u}, \tag{5a}$$

$$(\sigma + in)\bar{v} - 2\bar{u} = -\frac{\partial \bar{p}}{\partial y} + \frac{1}{Re}\Big[\frac{\partial^2}{\partial y^2} - \beta^2\Big]\bar{v}, \tag{5b}$$

$$(\sigma + in)\bar{w} = -i\beta\bar{p} + \frac{1}{Re}\Big[\frac{\partial^2}{\partial y^2} - \beta^2\Big]\bar{w}, \tag{5c}$$

$$\frac{\partial \bar{v}}{\partial y} + i\beta\bar{w} = 0. \tag{5d}$$

3, A

Multiplying $(5c)$ by $i\beta$ and using $(5d)$, we obtain

$$(\sigma+in)(-\frac{\partial\bar{v}}{\partial y})=\beta^2\bar{p}+\frac{1}{Re}\Big[\frac{\partial^2}{\partial y^2}-\beta^2\Big](-\frac{\partial\bar{v}}{\partial y}).$$

Differentiating this equation with respect to y, multiplying β^2 to $(5b)$ and taking the sum of the resulting equations to eliminate the pressure $\bar{p}$, we have

$$(\sigma+in)\Big[\frac{\partial^2}{\partial y^2}-\beta^2\Big]\bar{v}+2\beta^2\bar{u}=\frac{1}{Re}\Big[\frac{\partial^2}{\partial y^2}-\beta^2\Big]^2\bar{v}. \tag{6}$$

Equations $(5a)$ and (6) form the coupled system as required.

On noting that $\hat{r}=R_1$ and R_2 correspond to $y=0$ and 1, respectively, the no-slip boundary conditions are translated to

$$\bar{u}=\bar{v}=\frac{dv}{dy}=0 \text{ at } y=0,1,$$

where the last follows from $\bar{w}=0$. $\boxed{3, \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(iii) Multiplying $(5a)$ by $\bar{u}^*$ (the complex conjugate of $\bar{u}$) and integrating by parts, we have

$$(\sigma+in)I_u+2\int_0^1\bar{u}^*\bar{v}dy=-\frac{1}{Re}J_u, \tag{7}$$

where

$$I_u=\int_0^1|\bar{u}|^2dy,\quad J_u=\int_0^1\Big[|\frac{d\bar{u}}{dy}|^2+\beta^2|\bar{u}|^2\Big]dy.$$

Similarly, multiplying (6) by $\bar{v}^*$ and integration parts give

$$-(\sigma+in)J_v+2\beta^2\int_0^1\bar{u}\bar{v}^*dy=\frac{1}{Re}K_v, \tag{8}$$

where J_v is the same as J_u provided that $\bar{u}$ is replaced by $\bar{v}$, and

$$K_v=\int_0^1\Big[|\frac{d^2\bar{v}}{dy^2}|^2+2\beta^2|\frac{d\bar{v}}{dy}|^2+\beta^4|v|^2\Big]dy.$$

Multiplying (7) by β^2, and substracting from it the complex conjugate of (8), we obtain,

$$(\sigma+in)\,\beta^2I_u+(\sigma^*-in)J_v=-\frac{1}{Re}(\beta^2J_v+K_v).$$

The real part of the equation is

$$\sigma_r(\beta^2I_u+J_v)=-\frac{1}{Re}(\beta^2J_v+K_v),$$

which shows that $\sigma_r<0$ (i.e. all modes are damped), since all the integrals are positive.

In the inviscid limit $Re\to\infty$, $\sigma_r\to 0$, indicating that modes are neutral. $\boxed{7, \text{D}}$

3. (i) Eliminating $\bar{u}$ from the first two equations, we obtain $\boxed{\text{sim. seen} \Downarrow}$

$$-(U-c)\frac{d\bar{v}}{dy}+\frac{dU}{dy}\bar{v}=-i\alpha\bar{p}. \tag{9}$$

The y-momentum equation gives

$$\bar{v}=-\frac{1}{i\alpha(U-c)}\frac{d\bar{p}}{dy}, \tag{10}$$

substitution of which into (9) leads to an equation for $\bar{p}$,

$$(U-c)\frac{d}{dy}\Big[\frac{\frac{d\bar{p}}{dy}}{U-c}\Big]-\frac{\frac{dU}{dy}\frac{d\bar{p}}{dy}}{U-c}=\alpha^2\bar{p}.$$

This can be rewritten into the required form

$$\frac{d}{dy}\Big[\frac{1}{(U-c)^2}\frac{d\bar{p}}{dy}\Big]=\frac{\alpha^2\bar{p}}{(U-c)^2}, \tag{11}$$

as can be verified by using the product rule.
At the plate at $y=0$, the impermeability condition, $\bar{v}=0$, is satisfied when

$$\frac{d\bar{p}}{dy}=0.$$

The far-field condition as $y\to\infty$ is

$$\bar{p}\to 0.$$

$\boxed{5, \text{B}}$

(ii) Multiplying equation (11) by $\bar{p}^*$, the complex conjugate of $\bar{p}$, and integrating from $y=0$ to ∞, we have $\boxed{\text{unseen} \Downarrow}$

$$\int_0^\infty \bar{p}^*\frac{d}{dy}\Big[\frac{1}{(U-c)^2}\frac{d\bar{p}}{dy}\Big]dy=\int_0^\infty\frac{\alpha^2|\bar{p}|^2}{(U-c)^2}dy.$$

Perform integration by parts with respect to y on the left-hand side:

$$\bar{p}^*\frac{\frac{d\bar{p}}{dy}}{(U-c)^2}\Big|_{y=0}^{\infty}-\int_0^\infty\frac{1}{(U-c)^2}\Big|\frac{d\bar{p}}{dy}\Big|^2dy=\int_0^\infty\frac{\alpha^2|\bar{p}|^2}{(U-c)^2}dy. \tag{12}$$

Noting that the contributions from the boundaries are zero due to the boundary and far-field conditions, we rewrite the key result as

$$\int_0^\infty\frac{Q(y)}{(U-c)^2}dy=0, \tag{13}$$

where

$$Q(y)=\Big|\frac{d\bar{p}}{dy}\Big|^2+\alpha^2|\bar{p}|^2\geq 0.$$

On noting that $(U-c)^{-2}=\big[(U-c_r)^2-c_i^2+2c_i(U-c_r)i\big]/\big[(U-c_r)^2+c_i^2\big]^2$, the real and imaginary parts of equation (13) can be written as

$$\int_0^\infty\tilde{Q}(y)\big[(U-c_r)^2-c_i^2\big]dy=0; \tag{14}$$

$$2c_i\int_0^\infty\tilde{Q}(y)(U-c_r)dy=0, \tag{15}$$

where we have put

$$\tilde{Q}(y) = \frac{Q(y)}{\left[(U-c_r)^2 + c_i^2\right]^2} \geq 0.$$

Since $c_i > 0$, relation (15) indicates that $(U - c_r)$ must change its sign, i.e.

$$U_{\min} < c_r < U_{\max}. \qquad (16)$$

4, D

(iii) Integrating the y-momentum equation with respect to y across the discontinuity, from $y_d - \epsilon$ to $y_d + \epsilon$, and taking the limit $\epsilon \to 0$, we have

unseen ⇓

$$\Big[\bar{p}\Big]_{y_d^-}^{y_d^+} = (-i\alpha)\int_{y_d-\epsilon}^{y_d+\epsilon} (U-c)\bar{v}dy \to 0.$$

Similarly, integrating (11) from $y_d - \epsilon$ to $y_d + \epsilon$ gives, in the limit $\epsilon \to 0$, the result

$$\Big[\frac{\frac{d\bar{p}}{dy}}{(U-c)^2}\Big]_{y_d^-}^{y_d^+} = \int_{y_d-\epsilon}^{y_d+\epsilon} \frac{\alpha^2 \bar{p}}{(U-c)^2} dy \to 0.$$

When a discontinuity is present at y_d, the proof in Part (ii) can be refined by noting that in the integration by parts, the left-hand side of (12) contains extra terms

$$-\bar{p}^* \frac{\frac{d\bar{p}}{dy}}{(U-c)^2}\Big|_{y_d+\epsilon} + \bar{p}^* \frac{\frac{d\bar{p}}{dy}}{(U-c)^2}\Big|_{y_d-\epsilon}.$$

These two terms cancel out due to the jump conditions, which imply that $\bar{p}$, $\dfrac{d\bar{p}}{dy}/(U-c)^2$ and hence $\bar{p}^*\dfrac{d\bar{p}}{dy}/(U-c)^2$ are actually continuous at y_d.

5, D

unseen ⇓

(iv) For the given profile, $U'' = 0$ and hence the Rayleigh equation reduces to $\dfrac{d^2\bar{v}}{dy^2} - \alpha^2\bar{v} = 0$. The solution can be written as

$$\bar{v} = \begin{cases} C^+ e^{-\alpha y} & y > h, \\ C_1 e^{-\alpha y} + C_2 e^{\alpha y} & 0 < y < h. \end{cases}$$

Now apply the jump conditions across the discontinuity $y = h$. The continuity of $(U-c)\bar{v}' - U'\bar{v}$ and $\bar{v}/(U-c)$ implies

$$-\alpha(U_2 - c)C^+ e^{-\alpha h} = -\alpha(U_1 - c)C_1 e^{-\alpha h} + \alpha(U_1 - c)C_2 e^{\alpha h}, \qquad (17)$$

$$C^+ e^{-\alpha h}/(U_2 - c) = \left[C_1 e^{-\alpha h} + C_2 e^{\alpha h}\right]/(U_1 - c). \qquad (18)$$

The boundary condition $\bar{v}(0) = 0$ gives $C_2 = -C_1$, which is used in (17) and (18) to eliminate C_2. Taking the ratio of the resulting equations, we obtain

$$-(U_2 - c)^2 = (U_1 - c)^2/\tanh(\alpha h), \quad \pm i(U_1 - c) = (U_2 - c)\gamma,$$

where put $\gamma = \sqrt{\tanh(\alpha h)}$, and hence

$$c = (\gamma U_2 \mp iU_1)/(\gamma \mp i) = \left[(\gamma^2 U_2 + U_1) \pm i\gamma(U_2 - U_1)\right]/(1+\gamma^2) \equiv c_r + ic_i,$$

where the positive sign gives growing modes. Clearly,

$$\min(U_1, U_2) \leq c_r = (\gamma^2 U_2 + U_1)/(1+\gamma^2) \leq \max(U_1, U_2).$$

Furthermore,

$$\Big[c_r - \frac{U_1+U_2}{2}\Big]^2 + c_i^2 = \frac{(1-\gamma^2)^2}{4(1+\gamma^2)^2}(U_2 - U_1)^2 + \frac{\gamma^2}{(1+\gamma^2)^2}(U_2 - U_1)^2 = \Big[\frac{U_2 - U_1}{2}\Big]^2.$$

6, C

4. (a) Substitution into the Navier-Stokes equations followed by linearisation gives the equations for the perturbation: sim. seen ⇓

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} = 0, \tag{19}$$

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+V\frac{\partial u'}{\partial y}+\frac{\partial U}{\partial y}v' = -\frac{\partial p'}{\partial x}+Re^{-1}\nabla^2 u', \tag{20}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+V\frac{\partial v'}{\partial y} = -\frac{\partial p'}{\partial y}+Re^{-1}\nabla^2 v', \tag{21}$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}+V\frac{\partial w'}{\partial y} = -\frac{\partial p'}{\partial z}+Re^{-1}\nabla^2 w', \tag{22}$$

where the Laplacian operator

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Eliminate the pressure p' as follows. First $\frac{\partial}{\partial y}(20)-\frac{\partial}{\partial x}$(21), we have

$$\Big[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\Big]\Big(\frac{\partial u'}{\partial y}-\frac{\partial v'}{\partial x}\Big)+\frac{\partial U}{\partial y}(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y})+\frac{d^2U}{dy^2}v'=Re^{-1}\nabla^2\Big(\frac{\partial u'}{\partial y}-\frac{\partial v'}{\partial x}\Big). \tag{23}$$

Then $\frac{\partial}{\partial y}(22)-\frac{\partial}{\partial z}$(21), we obtain

$$\Big[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\Big]\Big(\frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z}\Big)+\frac{\partial U}{\partial y}\frac{\partial w'}{\partial x}=Re^{-1}\nabla^2\Big(\frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z}\Big). \tag{24}$$

4, A

Now $\frac{\partial}{\partial x}$(23) $+\frac{\partial}{\partial z}$(24):

$$\begin{aligned}\Big[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\Big]\Big\{\frac{\partial}{\partial y}\Big(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\Big)-\frac{\partial^2 v'}{\partial x^2}-\frac{\partial^2 v'}{\partial z^2}\Big\}+\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}&\\ +\frac{\partial U}{\partial y}\frac{\partial}{\partial x}\Big[(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y})+\frac{\partial w'}{\partial z}\Big]&\\ =Re^{-1}\nabla^2\Big\{\frac{\partial}{\partial y}\Big(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\Big)-\frac{\partial^2 v'}{\partial x^2}-\frac{\partial^2 v'}{\partial z^2}\Big\}&.\end{aligned} \tag{25}$$

Use of the continuity equation (19) shows that

$$\frac{\partial}{\partial y}\Big(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\Big)-\frac{\partial^2 v'}{\partial x^2}-\frac{\partial^2 v'}{\partial z^2}=-\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)^2 v'.$$

Noting this and the continuity equation (19), we may write (25) as

$$-\Big[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\Big]\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)v'+\frac{d^2U}{dy^2}\frac{\partial v'}{\partial x}=-Re^{-1}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)^2 v',$$

which is the required equation for v'. 5, B

$\boxed{\text{sim. seen} \Downarrow}$

(b) (i) Substituting $v' = \bar{v}(y)\mathrm{e}^{i(\alpha x+\beta z-\omega t)} + c.c.$ into the above equation, noting that

$$\frac{\partial}{\partial x} \to i\alpha, \quad \frac{\partial^2}{\partial x^2} \to -\alpha^2, \quad \frac{\partial^2}{\partial z^2} \to -\beta^2,$$

we obtain

$$\mathcal{L}\bar{v} \equiv \Big(U - \frac{\omega}{\alpha} + \frac{iV}{\alpha}\frac{d}{dy}\Big)\Big(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\Big)\bar{v} - \frac{d^2U}{dy^2}\bar{v} - \frac{1}{i\alpha Re}\Big(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\Big)^2\bar{v} = 0,$$

where we define the operator $\mathcal{L}$.

[Alternatively, one can substitute the normal-mode form expressions for the perturbation into (19)-(22), and then eliminate the pressure $\bar{p}$, $\bar{u}$ and $\bar{w}$ to obtain the equation for $\bar{v}$.]

The boundary conditions for $\bar{v}$ are

$$\bar{v} = \frac{d\bar{v}}{dy} = 0 \quad \text{at} \quad y = y_1,\, y_2,$$

where $d\bar{v}/dy = 0$ follows from $\bar{u} = \bar{w} = 0$ and the continuity equation.

Let $c = \omega/\alpha$. The equation $\mathcal{L}\bar{v} = 0$ above are arranged to

$$\Big(U + \frac{iV}{\alpha}\frac{d}{dy}\Big)\Big(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\Big)\bar{v} - \frac{d^2U}{dy^2}\bar{v} - \frac{1}{i\alpha Re}\Big(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\Big)^2\bar{v} = c\Big(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\Big)\bar{v}.$$

Discretisation of the equation, using finite difference or spectral methods, leads naturally to a generalised linear eigenvalue for c. $\boxed{6,\ \mathrm{A}}$

(ii) With $V = 1/Re$, Squire's theorem remains valid because the transformation,

$$\alpha_* = (\alpha^2 + \beta^2)^{1/2}, \quad \omega_* = (\alpha^2 + \beta)^{1/2}\omega/\alpha, \quad Re_* = \alpha Re/\alpha_*,$$

converts the equation for a three-dimensional mode (α, β, ω) at Re into an equation for the corresponding two-dimensional mode $(\alpha_*, 0, \omega_*)$ at $Re_* = \alpha Re/\alpha_* < Re$:

$$\Big(U - \frac{\omega_*}{\alpha_*} + \frac{i}{\alpha^* Re_*}\frac{d}{dy}\Big)\Big(\frac{d^2}{dy^2} - \alpha_*^2\Big)\bar{v} - \frac{d^2U}{dy^2}\bar{v} - \frac{1}{i\alpha_* Re_*}\Big(\frac{d^2}{dy^2} - \alpha_*^2\Big)^2\bar{v} = 0,$$

while the boundary conditions remain the same. $\boxed{5,\ \mathrm{B}}$

5. (i) Substituting the expression for the perturbed flow into the N-S equations, and neglecting nonlinear terms, we obtain the linearised equations for the perturbation, sim. seen ⇓

$$\frac{\partial u'}{\partial x} + Re^{1/2}\frac{\partial v'}{\partial Y} + \frac{\partial w'}{\partial z} = 0, \tag{26}$$

$$U\frac{\partial u'}{\partial x} + \underline{\frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial U}{\partial Y}v' + W\frac{\partial u'}{\partial z} = -\frac{\partial p'}{\partial x} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})\Big]u', \tag{27}$$

$$U\frac{\partial v'}{\partial x} + \underline{Re^{-\frac{1}{2}}\frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial Y} + \frac{\partial V}{\partial Y}v'} + W\frac{\partial v'}{\partial z} = -Re^{\frac{1}{2}}\frac{\partial p'}{\partial Y} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})\Big]v', \tag{28}$$

$$U\frac{\partial w'}{\partial x} + \underline{\frac{\partial W}{\partial x}u' + V\frac{\partial w'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial W}{\partial Y}v' + W\frac{\partial w'}{\partial z} = -\frac{\partial p'}{\partial z} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})\Big]w'. \tag{29}$$

The underlined terms represent the non-parallel-flow effects, caused by the streamwise variation of U and W and by the presence of a normal velocity $Re^{-1/2}V$, which is associated with the streamwise variation of U.

Local parallel-flow approximation neglects the underlined terms in (27)-(29), and treats the variation of U and W with x as being parametric; the latter means that at each location, the profiles are 'frozen' so that normal-mode solutions may be sought. The viscous terms are retained (leading to the Orr-Sommerfeld type of equations). 5, M

(ii) Substituting the normal-mode form into (26)-(29), and noting that the operators unseen ⇓

$$\frac{\partial}{\partial x} \to iRe^{3/7}\alpha, \quad \frac{\partial}{\partial z} \to iRe^{3/7}\beta, \tag{30}$$

when acting on the perturbation of the assumed form, we obtain the equations

$$i\alpha\bar{u} + \frac{\partial \bar{v}}{\partial Y} + i\beta\bar{w} = 0, \quad i\alpha U\bar{u} + \frac{\partial U}{\partial Y}\bar{v} + i\beta W\bar{u} = 0, \quad i\alpha U\bar{w} + \frac{\partial W}{\partial Y}\bar{v} + i\beta W\bar{w} = 0, \tag{31}$$

$$(i\alpha U + i\beta W)\bar{v} = -\frac{\partial \bar{p}}{\partial Y}. \tag{32}$$

The expressions for $(\bar{u}, \bar{v}, \bar{w})$ and $\bar{p}$ given in the question satisfy (31) and (32) with $\bar{v}$ satisfying the required boundary condition $\bar{v} = 0$ at $Y = 0$. [Alternatively, multiplying $i\alpha$ and $i\beta$ to the second and third equations in (31), taking the sum and using the first equation in (31), one obtains

$$-i(\alpha U + \beta W)\frac{\partial \bar{v}}{\partial Y} + i\frac{\partial}{\partial Y}(\alpha U + \beta W)\bar{v} = 0.$$

This is a first-order ordinary differential equation for $\bar{v}$ and has the solution as given, substitution of which into the second and third equations in (31) gives the solution for $\bar{u}$ and $\bar{w}$, while use of $\bar{v}$ in (32) yields the solution for $\bar{p}$.]

Note that

$$\bar{u} \to \lambda_1 A, \quad \bar{v} \to -iA(\alpha\lambda_1 + \beta\lambda_3)Y, \quad \bar{w} \to \lambda_3 A \quad \text{as} \quad Y \to 0. \tag{33}$$

Because $\bar{u}$ and $\bar{w}$ do not satisfy the required no-slip condition, a viscous wall layer (lower deck) is required. 5, M

$\boxed{\text{sim. seen} \Downarrow}$

(iii) Let $Y = O(d)$ with $d \ll 1$ in the lower deck, where $U = \lambda_1 Y = O(d)$ and $W = \lambda_3 Y = O(d)$. It follows that the inertia terms $U\dfrac{\partial u'}{\partial x}$ and $W\dfrac{\partial u'}{\partial z} \sim O(dRe^{3/7}u')$, while the viscous diffusion $\dfrac{\partial^2 u'}{\partial Y^2} \sim O(u'/d^2)$. The balance between the two in the x-momentum equation,

$$dRe^{3/7}u' \sim u'/d^2,$$

suggests that $d = O(Re^{-1/7})$, which corresponds to $y = Re^{-1/2}Y = O(Re^{-9/14})$.

The asymptotic behaviour of the main-deck solution, (33), suggests that in the lower deck $u', w' = O(1)$ as in the main layer, but $v' = O(Re^{-1/14}d) = O(Re^{-3/14})$ as can be deduced by the matching principle. Similarly, $p' = O(Re^{-1/7})$. Therefore, in the lower deck, the solution should expand as

$$(u',\, v',\, w',\, p') = (\tilde{u},\, Re^{-\frac{3}{14}}\tilde{v},\, \tilde{w},\, Re^{-\frac{1}{7}}\tilde{p})E + c.c.$$

$\boxed{\text{unseen} \Downarrow}$

Substituting this into (26)-(28) and using the fact that $U = Re^{-1/7}(\lambda_1\tilde{y})$ and $W = Re^{-1/7}(\lambda_3\tilde{y})$ as well as the relations in (30), we obtain

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\tilde{y}} + i\beta\tilde{w} = 0, \tag{34}$$

$$i\alpha\lambda_1\tilde{y}\tilde{u} + \lambda_1\tilde{v} + i\beta\lambda_3\tilde{y}\tilde{u} = -i\alpha\tilde{p} + \frac{d^2\tilde{u}}{d\tilde{y}^2}, \tag{35}$$

$$i\alpha\lambda_1\tilde{y}\tilde{w} + \lambda_3\tilde{v} + i\beta\lambda_3\tilde{y}\tilde{w} = -i\beta\tilde{p} + \frac{d^2\tilde{w}}{d\tilde{y}^2}, \tag{36}$$

plus $d\tilde{p}/d\tilde{y} = 0$ so that $\tilde{p}$ is a constant.

To determine $\tilde{p}$, we note that the main-deck solution has the asymptote

$$\bar{p} \to -\alpha^2 A \int_\infty^0 \big[U + (\beta/\alpha)W\big]^2 dY \quad \text{as} \quad Y \to 0.$$

Matching $\tilde{p}$ with $\bar{p}$ gives

$$\tilde{p} = \alpha^2 I A \quad \text{where} \quad I = \int_0^\infty \big[U + (\beta/\alpha)W\big]^2 dY; \tag{37}$$

this is the pressure-displacement relation.

Matching $\tilde{u}$ and $\tilde{w}$ with their main-layer counterparts requires

$$(\tilde{u},\, \tilde{w}) \to (\lambda_1 A,\, \lambda_3 A) \quad \text{as} \quad \tilde{y} \to \infty.$$

$\boxed{7,\ \text{M}}$

$\boxed{\text{sim. seen} \Downarrow}$

(iv) The dominant terms in the equations indicate that the instability is a result of interplay among the shear, viscous effect and the (self-induced) pressure.

Note that the leading order terms, including the inertial and viscous terms, in the x- and z-momentum equations are of $O(Re^{2/7})$, while the terms representing non-parallelism, e.g. $\frac{\partial U}{\partial x}u'$, is $O(Re^{-1/7})$, and hence non-parallelism contributes an $O(Re^{-3/7})$ correction.

$\boxed{3,\ \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70052 Hydrodynamic Stability Markers Comments

Question 1	Most did quite well on this.
Question 2	Most did quite well on this.
Question 3	Most did quite well on this.
Question 4	Except a few, most struggled on this question, even though it wasn't a difficult question. Many didn't manage to reduce the system to a single equation (Part a), and then stumbled on arranging the equation into a form with phase speed c appearing a linear eigenvalue, and on Squire theorem.
Question 5	Except two students, all did rather poorly on this. That the paper is long was a (possible the main) reason, but better preparation on the topic would have helped. The maximum mark will be set taking into consideration the fact that this is a long paper.