1 (a) The p_i be t the probability that a random walker, starting at interior node i, reaches nodes n before node 0. It is clear that

$$p_0 = 0, \qquad p_n = 1. \tag{1}$$

Using a basic law of probability, at any interior node $i = 1, \cdots, n-1$,

$$p_i = \frac{1}{2}p_{i-1} + \frac{1}{2}p_{i+1}. \tag{2}$$

That is, $\mathbf{p}$ is a harmonic potential on this graph. The easiest way to solve this problem is to solve it explicitly for small values of n and to spot a pattern. For $n = 2$, it is easy to find

$$p_1 = \frac{1}{2}. \tag{3}$$

For $n = 3$, we find

$$p_1 = \frac{1}{3}, \qquad p_2 = \frac{2}{3}. \tag{4}$$

For $n = 4$, we find

$$p_1 = \frac{1}{4}, \qquad p_2 = \frac{2}{4}, \qquad p_3 = \frac{3}{4}. \tag{5}$$

The pattern is easy to see. For any n,

$$p_i = \frac{i}{n}, \qquad i = 1, \cdots, n-1. \tag{6}$$

Having spotted this solution, it is easy to check directly it satisfies the set of equations (2) for any n. But, from the uniqueness principle for harmonic potentials, this must be the required solution.

1 (b) Let h_i be the probability that a random walker, starting at interior node i, **never** reaches two specified ("boundary") nodes 0 and n. It is clear that

$$h_0 = h_n = 0. \tag{7}$$

Using a basic law of probability, at any interior node i,

$$h_i = \frac{1}{2}h_{i-1} + \frac{1}{2}h_{i+1} \tag{8}$$

which means that $\mathbf{h}$ is a harmonic potential on this graph. It must therefore satisfy the Maximum Principle which says that both the maximum and minimum of this potential are attained at the boundary. But the boundary values are zero. Hence we conclude

$$h_i = 0, \qquad \text{for all } i = 0, 1, \cdots, n. \tag{9}$$

By the definition of h_i this means that the walker is certain to reach the boundary nodes.

2 (a) If UCL is node 0 and Imperial is node n then, by inspection of the figure, we find $n = 18$. Tower hill station corresponds to node $i = 7$. Hence, by the results of problem 1(a), the required probability starting at Tower Hill is

$$p_{\text{Tower Hill}} = \frac{7}{18}. \tag{10}$$

If UCL is node 0 and Imperial is node n then, from the figure, we find $n = 9$. Paddington station corresponds to node $i = 4$. Hence, by the results of problem 1(a), the required probability starting at Paddington is

$$p_{\text{Paddington}} = \frac{4}{9} = \frac{8}{18}. \tag{11}$$

The tourist is therefore more likely to reach Imperial before UCL if he starts at Paddington.

2 (b) This question is asking us for the escape probability p_{esc} to Euston Square from South Kensington station. By the analogy with electric circuits we know this is

$$p_{\text{esc}} = \frac{C_{\text{eff}}}{2}, \tag{12}$$

where the denominator 2 is the total number of edges out of South Kensington. Going clockwise from South Kensington to Euston Square there are $n = 9$ conductors (with unit conductance) in series; the effective conductance of this route $C^{(1)}_{\text{eff}}$ satisfies

$$\frac{1}{C^{(1)}_{\text{eff}}} = n = 9, \tag{13}$$

where we have used the fact that *resistances* of conductors in series can be added up to give the effective resistance. Similarly, going anticlockwise from South Kensington to Euston Square there are $n = 18$ conductors (with unit conductance) in series; the effective conductance of this route $C^{(2)}_{\text{eff}}$ satisfies

$$\frac{1}{C^{(2)}_{\text{eff}}} = n = 18. \tag{14}$$

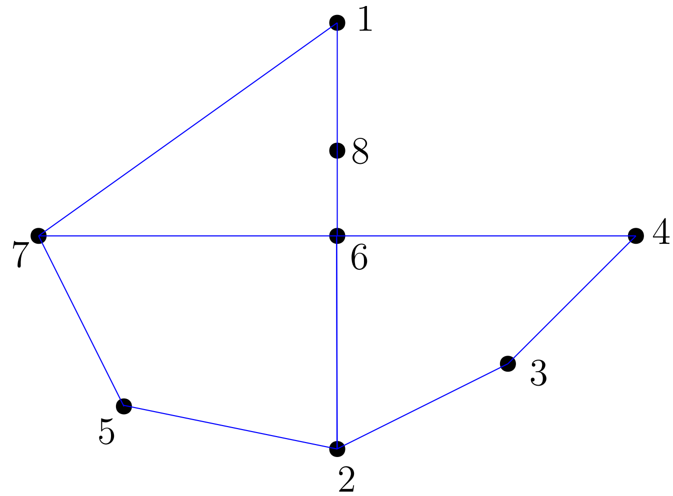
The Circle Line is therefore equivalent to two conductors in parallel with conductances 1/9 and 1/18. The effective conductance C_{eff} is therefore

$$C_{\text{eff}} = C^{(1)}_{\text{eff}} + C^{(2)}_{\text{eff}} = \frac{1}{9} + \frac{1}{18} = \frac{1}{6}. \tag{15}$$

Therefore the probability we seek is

$$p_{\text{esc}} = \frac{C_{\text{eff}}}{2} = \frac{1}{12}. \tag{16}$$

3 (a) Let p_i be the probability of reaching Warren Street before a Piccadilly Line station starting the walk at node i where the number the nodes as follows:



Clearly,

$$p_1 = 1, \qquad p_2 = p_3 = p_4 = p_5 = 0.$$

Using the basic laws of probability we know that

$$\begin{aligned} p_6 &= \frac{1}{4}\left[p_2 + p_4 + p_7 + p_8\right], \\ p_7 &= \frac{1}{3}\left[p_1 + p_5 + p_6\right], \\ p_8 &= \frac{1}{2}\left[p_1 + p_6\right]. \end{aligned}$$

Equivalently, the probability potential is harmonic at these interior nodes. This system becomes

$$\begin{aligned} 4p_6 &= p_7 + p_8, \\ 3p_7 &= 1 + p_6, \\ 2p_8 &= 1 + p_6. \end{aligned}$$

and is readily solved to find

$$p_6 = \frac{5}{19}, \qquad p_7 = \frac{8}{19}, \qquad p_8 = \frac{12}{19}.$$

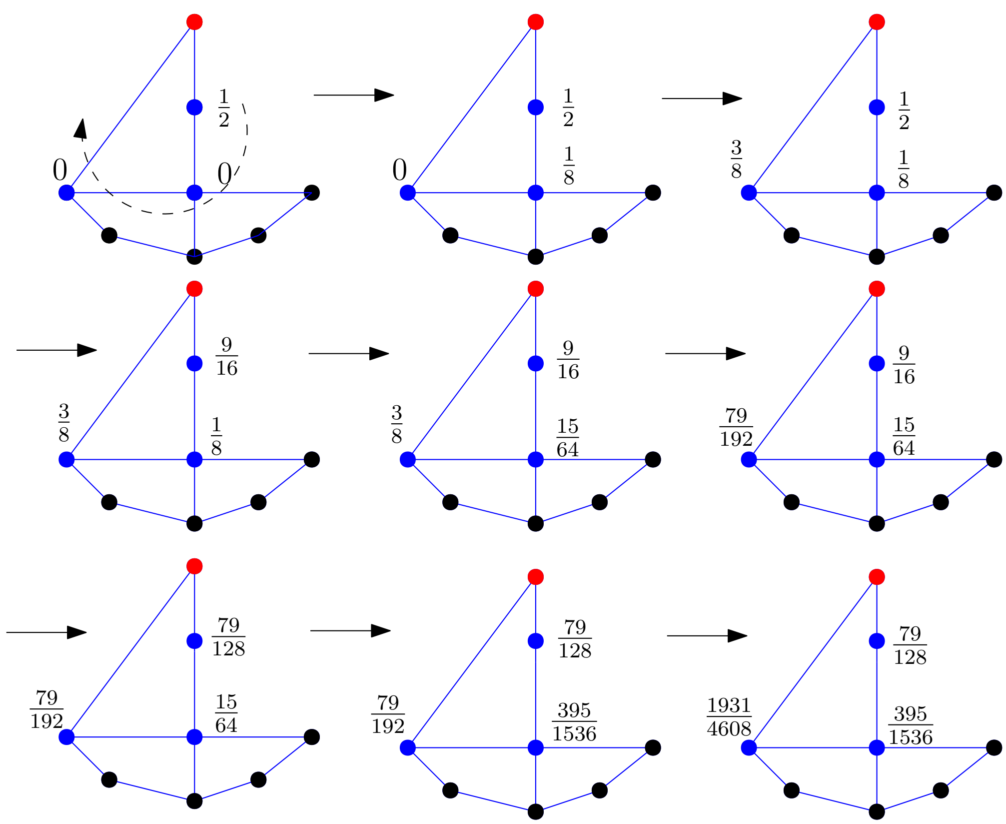
Hence the probability required for part (a) is

$$p_7 = \frac{8}{19}.$$

3 (b) The probability required for part (b) is

$$p_6 = \frac{5}{19}.$$

3 (c) Starting with all blue nodes in the figure at zero voltage (the red node is at voltage 1 and the black nodes are grounded) we carry out 3 loops in the direction shown (via nodes 8, 6 and 7, in that order, and then repeat):



After 3 iterations around the loop, we find

$$p_6 \approx \frac{395}{1536} = 0.2573 \text{ correct to 4 decimal places} \tag{17}$$

and

$$p_7 \approx \frac{1931}{4608} = 0.4191 \text{ correct to 4 decimal places} \tag{18}$$

and

$$p_6 \approx \frac{79}{128} = 0.6172 \text{ correct to 4 decimal places} \tag{19}$$

The correct values are

$$p_6 \approx \frac{5}{19} = 0.2632 \text{ correct to 4 decimal places} \tag{20}$$

and

$$p_7 \approx \frac{8}{19} = 0.4211 \text{ correct to 4 decimal places} \tag{21}$$

and

$$p_6 \approx \frac{12}{19} = 0.6316 \text{ correct to 4 decimal places} \tag{22}$$

The maximum relative error is just over 2%.

3 (d) The Laplacian can be constructed by the usual rules to be

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 3 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 4 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{bmatrix}, \tag{23}$$

where, to compute the required probability we have split $\mathbf{K}$ up into useful sub-blocks:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}. \tag{24}$$

The system to solve is

$$\begin{bmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{0} \end{bmatrix}, \tag{25}$$

where

$$\mathbf{e}_1 = [0\ 1]^T, \qquad \mathbf{f}_1 = [-f\ f]^T. \tag{26}$$

It is clear that

$$\mathbf{x} = -\mathbf{R}^{-1}\mathbf{Q}\mathbf{e}_1. \tag{27}$$

Hence, on introducing the vector

$$\mathbf{e}_2 = [0\ 0\ 0\ 1\ 0\ 0]^T \tag{28}$$

a formula for the probability we require is

$$-\mathbf{e}_2^T \mathbf{R}^{-1} \mathbf{Q} \mathbf{e}_1. \tag{29}$$

Optional part: Using `MATLAB` this quantity is found to be 0.6301 (any other means of calculating this can be used).

3 (e) This question is asking for the escape probability p_{esc} to Warren Street from Leicester Square. It is related to the quantity f introduced in the solution of part (d) via the formula

$$p_{\text{esc}} = \frac{f}{3} \tag{30}$$

where the denominator 3 is the number of edges out of Leicester Square (node 2). From part (d) we find

$$\mathbf{P}\mathbf{e}_1 + \mathbf{Q}^T \mathbf{x} = \mathbf{P}\mathbf{e}_1 - \mathbf{Q}^T \mathbf{R}^{-1} \mathbf{Q} \mathbf{e}_1 = [\mathbf{P} - \mathbf{Q}^T \mathbf{R}^{-1} \mathbf{Q}]\mathbf{e}_1 = \mathbf{f}_1. \tag{31}$$

Hence a formula for the escape probability is

$$\frac{1}{3}\mathbf{e}_1^T [\mathbf{P} - \mathbf{Q}^T \mathbf{R}^{-1} \mathbf{Q}]\mathbf{e}_1.$$

Optional part: Using `MATLAB` this quantity is found to be 0.2557 (any other means of calculating can be used).