Unseen problem sheet 1

Analysis 1 - Spring 2023

Choose TWO of the following problems for your CW1 on 2 Feb 2023.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

   - if x is irrational then $f(x) = 0$.
   - if $x = \frac{n}{m}$, where n and m are co-prime, then $f(x) = m$.

   Let I be an open interval. Prove that f is unbounded on I. Is f continuous anywhere?

2. Prove that if f and g are continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$, then $f = g$.

3. A function f is said to be *periodic* if and only if there is some $T > 0$ such that $f(x + T) = f(x)$ for all x, with T being referred to as the *period* of f. Prove that every non-constant, continuous, periodic function has a minimum period. Is this still true if f is not continuous?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   (a) Suppose that there is some $x_0 \in \mathbb{R}$ such that f is continuous on x_0. Prove that f is continuous on all $\mathbb{R}$.

   (b) Suppose that there is some $x_0 \in \mathbb{R}$ such that f is continuous on x_0. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

   (c) Suppose that there is an interval I such that f is monotone on I. Prove that there is some $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

   $f(x + y) = f(x) + f(y)$ is known as *Cauchy's functional equation.* The fact that there are non-linear functions that satisfy this equation is very interesting, but rather complicated, so if you've got some free time you may like to look it up.

5. Prove that the finite union of bounded sets is bounded.

6. (A limited version of the Heine-Borel Theorem)

   An *open cover* of $A \subset \mathbb{R}$ is a set of open bounded intervals $\{U_i : i \in I\}$ such that

   $$A \subseteq \bigcup_{i \in I} U_i$$

   Note that I can be finite, countable, or uncountable. Prove that the following are equivalent:

   (a) A is a closed and bounded interval.

(b) If $\{U_i : i \in I\}$ is an open cover of A then there is a finite $I_0 \subset I$ such that $\{U_i : i \in I_0\}$ is an open cover of A.

Hint for (a) $\Rightarrow$ (b): If $\{U_i : i \in I\}$ is an open cover of A with no finite I_0, then if we cut A into 2^n many pieces, then at least one of those pieces need infinitely many U_i to cover it.

7. Let $I = [a, b]$ for some a and b. Suppose that for all $c \in I$ there is a $\delta(c) > 0$ such that f is increasing on $I \cap [c - \delta(c), c + \delta(c)]$. Prove that f is increasing on I.

8. Let (a_n) be a sequence and $a \in \mathbb{R}$. Prove that $a_n \to a$ if and only if for every open set $U \subseteq \mathbb{R}$ such that $a \in U$, there is some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

9. Determine, with proof, whether each of the following is true or false.

   (a) If f is continuous and bounded on the interval (a, b) (meaning there exist $M, L \in \mathbb{R}$ such that $\forall x \in (a, b) : M \leq f(x) \leq L$), then f is uniformly continuous on (a, b).

   (b) If f is bounded on $\mathbb{R}$ and uniformly continuous on every interval $[a, b]$ where $a, b \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

   (c) If f is bounded, continuous and monotonic on $(0, 1)$, then f is uniformly continuous on $(0, 1)$.

   (d) If f is uniformly continuous on $(0, 1)$, then f is bounded on $(0, 1)$.

10. A function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ is *Lipschitz continuous* if there exists some $C \in \mathbb{R}$ such that $\forall x, y \in A : |f(x) - f(y)| \leq C|x - y|$. Prove the following or give a counterexample.

    (a) If f is Lipschitz continuous, then it is uniformly continuous.

    (b) If f is uniformly continuous, then it is Lipschitz continuous.

11. Find a function $f : [0, 1] \to \mathbb{R}$ such that the following hold:

    - Let $0 \leq a < b \leq 1$ and pick a value c between $f(a)$ and $f(b)$. Then there is some $x \in [a, b]$ such that $f(x) = c$.
    - f is *not* continuous.

12. Let $f : \mathbb{R} \to \mathbb{R}$ such that for every open interval $I \subseteq \mathbb{R}$: $f(I) := \{f(x) | x \in I\} = \mathbb{R}$.

    (a) Prove that for every $a, b \in \mathbb{R}$ and c between $f(a)$ and $f(b)$, there is some $x \in [a, b]$ such that $f(x) = c$.

    (b) Prove that f is *nowhere continuous*, i.e., f is not continuous for any $x \in \mathbb{R}$.

13. The previous problem shows that the converse of the Intermediate Value Theorem fails miserably for functions satisfying $f(I) = \mathbb{R}$ for every interval I. Here we will construct such a function.

    Define $a \sim b$ iff $a - b \in \mathbb{Q}$.

(a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$ and every equivalence class is countable.

(b) For every $x \in \mathbb{R}$, let $cl(x)$ be the equivalence class of x with respect to $\sim$. Show that for every $x \in \mathbb{R}$: $cl(x)$ is dense in $\mathbb{R}$, namely, for every interval I, there is some $y \in cl(x) \cap I$.

(c) Let $\mathcal{P} := \{cl(x) | x \in \mathbb{R}\}$ be the set of all equivalence classes of $\sim$.
Show that there is a bijection from $\mathcal{P}$ to $\mathbb{R}$.
You may need to use the following fact from set theory:

Let $\mathcal{C}$ be an infinite set of countable non-empty sets such that $A \cap B = \emptyset$ for every $A, B \in \mathcal{C}$. Then there is a bijection between $\mathcal{C}$ and $\bigcup_{A \in \mathcal{C}} A$.

(d) Let $g : \mathcal{P} \to \mathbb{R}$ be a bijection. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = g(cl(x))$. Prove that $f(I) = \mathbb{R}$ for every interval I. Deduce that f satisfies the intermediate value property, but is nowhere continuous.