IMPERIAL

MATH50008

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Partial Differential Equations in Action

Date: Tuesday, May 14, 2024
Time: 10:00 – 12:00 (BST)
Time Allowed: 2 hours
This paper has 4 Questions.
Please Answer Each Question in a Separate Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) The Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) equation is a famous model of population dynamics. In 1D, it reads

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{M}\right)$$

where $u(x,t)$ is the number density of an animal specie.

(i) What are the dimensions of D, r and M? (3 marks)

(ii) Interpret physically the two terms appearing on the right-hand side of the FKPP equation. (3 marks)

(iii) First, consider that we are studying this equation on an infinite domain, with the following initial conditions:

$$u(x,0) = \begin{cases} M, & x < 0 \\ 0, & \text{otherwise} \end{cases}$$

Non-dimensionalize this PDE problem. (3 marks)

(iv) When working on a finite domain of size L, it is natural to set the characteristic lengthscale to L. Show that in this case the non-dimensionalization leads to the following reduced equation

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \beta\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{u}\left(1 - \tilde{u}\right)$$

where $\tilde{u}$, $\tilde{x}$ and $\tilde{t}$ are dimensionless variables and β is a dimensionless parameter whose expression you will determine. Interpret physically the parameter β. (4 marks)

(b) We consider the following quasilinear first-order PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u, \quad x, y \in \mathbb{R}$$

(i) Find the equation of the characteristics for this PDE. Draw the characteristics in the (x,y)-plane. (5 marks)

(ii) Find an explicit solution that satisfies the following condition $u(x,y) = 1$ on $x^2 + y^2 = 1$. (2 marks)

(Total: 20 marks)

2. We consider the one-dimensional inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

Here, we assume that the initial conditions are given by

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 2 - x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

(a) Draw the initial conditions. (2 marks)

(b) Find the equation of the characteristics for this problem. Draw them in the (x,t)-plane and point out any fan regions. Show that a shock forms at $t = 1$. (5 marks)

(c) Find the explicit solution $u(x,t)$ for $0 < t < 1$. (4 marks)

(d) Find the explicit solution after the shock has formed. To do so, you will need to determine the explicit solution for the shock path. Draw an amended diagram of characteristics including the shock path. Until when is the solution you just obtained valid? Justify your reasoning. (9 marks)

(Total: 20 marks)

3. In this question, we consider the damped wave equation given by

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

where $c = \sqrt{\tau_0/\rho_0}$ with τ_0 the tension in the string and ρ_0 the density of the string. We will impose homogeneous Dirichlet boundary conditions such that $u(0,t) = u(L,t) = 0$. You can assume that $\kappa < \pi c/L$.

(a) What are the dimensions of κ and c? (2 marks)

(b) Show that the general solution of this damped wave problem is given by

$$u(x,t) = e^{-\kappa t} \sum_{n=1}^{\infty} \left[\alpha_n \cos(2\pi f_n t) + \beta_n \sin(2\pi f_n t)\right] \sin\left(\frac{n\pi x}{L}\right)$$

where α_n and β_n are real constants and f_n are the frequencies of the normal modes whose expression you will determine. Make sure to justify your steps. (6 marks)

(c) What is the effect of the damping on the frequency of the normal modes? (2 marks)

(d) We denote $E(t)$ the total energy of the string. Show that

$$\frac{dE}{dt} \leq 0$$

You do not need to re-derive the energy of the string from first principles, you can simply quote its definition from the lecture notes. (6 marks)

(e) Find the particular solution in the case where the initial conditions are given by

$$u(x,0) = x(L-x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

You can use the fact that the Fourier series expansion of $f(x) = x(L-x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3}\left[1 - (-1)^n\right] \sin\left(\frac{n\pi x}{L}\right).$$

(4 marks)

(Total: 20 marks)

4. In this question, when taking Fourier transforms, we will follow the conventions introduced in the course, namely:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega$$

where $\widehat{f}(\omega)$ is the Fourier transform of the real function $f(x)$.

(a) First, we consider the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = f_1(x)$$
$$\frac{\partial u}{\partial t}(x,0) = f_2(x)$$

with f_1 and f_2 two known real-valued functions.

(i) Find the Fourier representation of the solution to this problem. (4 marks)

(ii) Using this result, show that an explicit real-space representation of the solution is given by d'Alembert's solution

$$u(x,t) = \frac{1}{2}\left[f_1(x-ct) + f_1(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

(4 marks)

(b) We now consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0$$

and assume that: (1) $u(x,y) \to 0$ as $y \to \infty$ and (2) $u(x,0) = f(x)$ where f is a piece-wise smooth real function with $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$; so that u is a bounded solution.

(i) Using a Fourier method, show that an explicit representation of the solution to this problem is given by

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2 + (x-\xi)^2} f(\xi)d\xi$$

(8 marks)

(ii) Use the previous result to find the explicit solution in the case where

$$f(x) = \begin{cases} 2, & |x| < 4 \\ 0, & \text{otherwise} \end{cases}$$

(4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH50008

PDEs in Action (Solutions)

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1. (a) Here, we consider the FKPP equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{M}\right)$$

where $u(x,t)$ is a number density.

unseen ⇓

(i) The dimensions of D, r and M are given by the principle of dimensional homogeneity and the fact that $[u] = L^{-3}$ as

$$[D] = L^2T^{-1}$$
$$[r] = T^{-1}$$
$$[M] = L^{-3}$$

3, A

(ii) The first term on the RHS of the FKPP equation is a diffusion term governing the spatial motion of the population. The second term on the RHS of the FKPP equation is a logistic term composed of a linear term $\propto u$ leading to an exponential growth of the population at short times with rate r and a quadratic term in $\propto -u^2$ which act as saturation and prevents the population from growing out of bounds (in that context, M is the maximal population density at any point in space x).

sim. seen ⇓

3, C

(iii) To non-dimensionalize this equation, we write

meth seen ⇓

$$u = u_c\tilde{u}, \quad x = x_c\tilde{x}, \quad t = t_c\tilde{t},$$

where u_c, x_c, t_c are characteristic scales to be determined. Subbing in the FKPP equation, we find

$$\frac{u_c}{t_c}\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{Du_c}{x_c^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + ru_c\tilde{u}\left(1 - \frac{u_c}{M}\tilde{u}\right) \Rightarrow \frac{\partial \tilde{u}}{\partial \tilde{t}} = \Pi_1\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \Pi_2\tilde{u}\left(1 - \Pi_3\tilde{u}\right)$$

where Π_1, Π_2 and Π_3 are three dimensionless groups. Similarly, the initial conditions are written

$$u_c\tilde{u}(x_c\tilde{x}, 0) = \begin{cases} M, & x_c\tilde{x} < 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow \tilde{u}(\tilde{x}, 0) = \begin{cases} \Pi_3, & \tilde{x} < 0 \\ 0, & \text{otherwise} \end{cases}$$

To determine the characteristic values, we set the three dimensionless groups to 1 and we obtain:

$$u_c = M, \quad t_c = 1/r, \quad x_c = \sqrt{D/R}$$

leading to the following nondimensional problem

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{u}\left(1 - \tilde{u}\right) \quad \text{with} \quad \tilde{u}(\tilde{x}, 0) = \begin{cases} 1, & \tilde{x} < 0 \\ 0, & \text{otherwise} \end{cases}$$

3, A

(iv) On a finite domain of size L, we fix $x_c = L$. Going back to our earlier non-dimensionalization, we see that this choice affects the value of Π_1 only (we are not setting Π_1 to 1 anymore!). We thus obtain the following nondimensional equation

unseen $\Downarrow$

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \beta \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{u}\,(1 - \tilde{u})$$

with $\beta = D/(rL^2)$.

2, A

We can also write $\beta = (1/r)/(L^2/D)$ and interpret it as a ratio of the exponential growth timescale (from the logistic term) to the diffusion timescale (the time needed to diffuse across the domain).

2, D

sim. seen $\Downarrow$

(b) Here, we consider the following quasilinear first-order PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u, \quad x, y \in \mathbb{R}$$

(i) The characteristics for this equation are governed by the following set of ODEs

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = y, \quad \frac{dz}{ds} = 4z \quad \Rightarrow \quad x = x_0 e^s, \quad y = y_0 e^s, \quad z = z_0 e^{4s}$$

where (x_0, y_0, z_0) is a point on the initial curve.

3, A

We conclude that the characteristics are the curves with equation $x/y = k$, with k a real constant. These are straight lines plotted in Fig. 1.

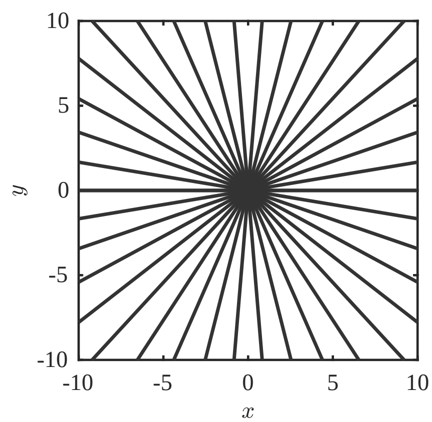

Figure 1: Characteristics associated with $xu_x + yu_y = 4u$.

2, A

(ii) The parametrization of the initial curve (a circle of radius 1) leads to $x_0 = \cos(t)$, $y_0 = \sin(t)$ and $z_0 = 1$. We thus conclude that

$$x(s,t) = \cos(t)e^s, \quad y(s,t) = \sin(t)e^s, \quad z(s,t) = e^{4s}$$

Here, we do not need to invert these relations, we can simply realize that $x^2 + y^2 = e^{2s}$ which finally leads to the following explicit solution

$$u(x,y) = (x^2 + y^2)^2$$

2, B

2. Here, we consider Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

where we have assumed that the initial conditions were given by

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 2 - x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$\boxed{\text{sim. seen} \Downarrow}$

(a) The initial conditions are given on Fig.2.

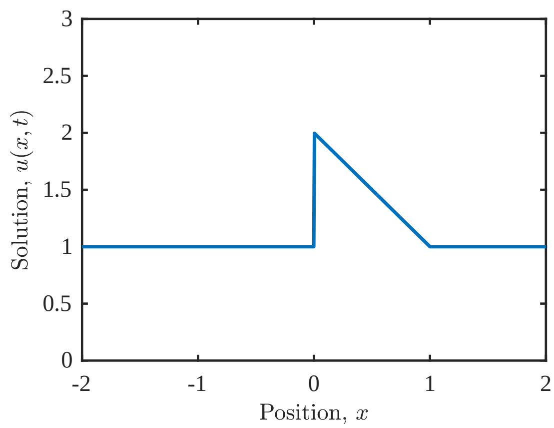

Figure 2: Sketch of the initial conditions for Q2.

$\boxed{2, \text{A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) The method of characteristics gives us here that

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u, \quad x(0) = \xi$$

which means that

$$u = u(\xi, 0) \quad \text{on} \quad x = u(\xi, 0)t + \xi$$

So based on the initial conditions, we obtain the following equation for the characteristics

$$\begin{cases} \text{I} — \xi < 0: & x = t + \xi \\ \text{II} — 0 < \xi < 1: & x = (2 - \xi)t + \xi \\ \text{III} — \xi > 1: & x = t + \xi \end{cases}$$

$\boxed{2, \text{B}}$

The diagram of characteristics is given on Fig. 3. $\boxed{2, \text{A}}$

To show that a shock forms, simply notice that all the characteristics from region II ($0 < \xi < 1$) are crossing in $t = 1$. Indeed, their equation is $x = (2 - \xi)t + \xi$, in $t = 1$, we thus have $x = 2$ for all ξ. $\boxed{1, \text{A}}$

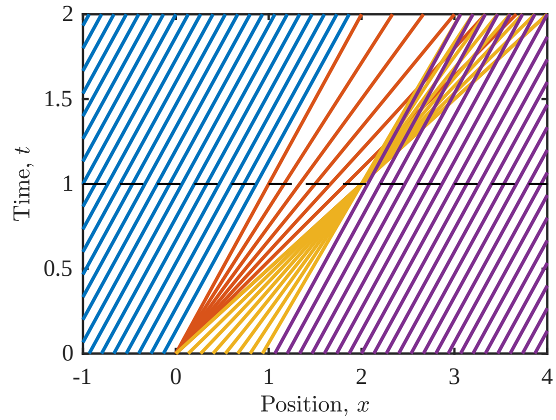

Figure 3: Diagram of characteristics with characteristics from region I (blue), region II (yellow), region III (purple). This diagram shows a rarefaction fan (shown in orange).

(c) We know that the solution is constant along the characteristics, so we can write

- ☐ For $\xi < 0$, we had $u = 1$ on $x(t) = t + \xi$ which we can invert and find $u(x,t) = 1$ for $x < t$;
- ☐ For $\xi > 1$, we had $u = 1$ on $x(t) = t + \xi$ which we can invert and find $u(x,t) = 1$ for $x > t+1$;
- ☐ For $0 < \xi < 1$, we had $u = 2 - \xi$ on $x(t) = (2-\xi)t + \xi$, i.e. on

$$\xi = \frac{x - 2t}{1 - t}$$

which leads to

$$u = 2 - \frac{x-2t}{1-t} = \frac{2-x}{1-t}$$

for $0 < (x-2t)/(1-t) < 1$, i.e. $2t < x < t+1$.

- ☐ In the fan region, $t < x < 2t$, we impose that the solution goes linearly from $u = 1$ on the left to $u = 2$ on the right, leading to $u(x,t) = x/t$.

The explicit solution is finally given by

$$u(x,t) = \begin{cases} 1, & \text{for } x < t \\ x/t, & \text{for } t < x < 2t \\ (2-x)/(1-t), & \text{for } 2t < x < t+1 \\ 1, & \text{for } x > t+1 \end{cases}$$

4, B

meth seen ⇓

(d) To find an explicit solution $u(x,t)$ for $t \geq 1$, we first need to deal with the shock! To do so, we want to apply the RH jump condition which requires the knowledge of the solution on either side of the shock. We easily have the solution in region III as along the yellow characteristics, $u(x,t) = 1$. The solution on the left of the shock will be coming from the fan region. Forgetting that a shock has formed, we can extend our explicit solution.

If we denote $s(t)$ the position of the shock, we thus have

After the shock: $u_+ = 1$ (which comes from region III)

Before the shock: $u_- = \dfrac{s(t)}{t}$ (which comes from the fan region)

The Rankine-Hugoniot jump condition reads

$$\frac{ds}{dt} = \frac{[u^2/2]}{[u]} = \frac{1}{2}(u_+ + u_-) = \frac{1}{2} + \frac{s}{2t}$$

subject to the initial condition $s(1) = 2$. $\boxed{2, \mathrm{B}}$

This ODE can be solved with an integrating factor and we get

$$s(t) = e^{\int dt/(2t)}\left(\int \frac{1}{2}e^{-\int dt/(2t)}dt\right) + Ce^{\int dt/(2t)} \Rightarrow s(t) = e^{\ln(t)/2}\left(\int \frac{dt}{2}e^{-\ln(t)/2}\right) + Ce^{\ln(t/2)}$$

where C is an integration constant. We finally obtain that

$$s(t) = \sqrt{t}\left(\int dt \frac{1}{2\sqrt{t}}\right) + C\sqrt{t} = t + C\sqrt{t}$$

Imposing that $s(1) = 2 = 1 + C\sqrt{1} = 1 + C \Rightarrow C = 1$, leading to

$$s(t) = t + \sqrt{t}$$

$\boxed{2, \mathrm{C}}$

The amended diagram of characteristics is given in Fig. 4. $\boxed{2, \mathrm{A}}$

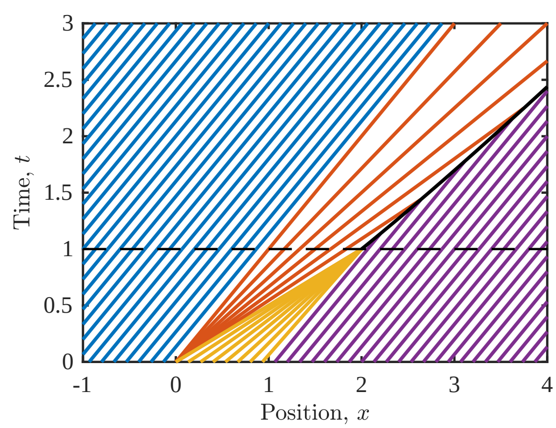

Figure 4: Amended diagram of characteristics for Q2 showing the shock path (black line).

The explicit solution can be read directly from this diagram and we obtain

$$u(x,t) = \begin{cases} 1, & \text{for } x < t \\ x/t, & \text{for } t < x < t + \sqrt{t} \\ 1, & \text{for } x > t + \sqrt{t} \end{cases}$$

$\boxed{2, \mathrm{B}}$

As $s(t) > t$, we know that this shock will follow this path forever as the regions used to apply the RH condition will not change (hence, the jump condition will remain the same). $\boxed{1, \mathrm{C}}$

3. We consider the damped wave equation given by

$$\frac{\partial^2 u}{\partial t^2} + 2\kappa \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

We will impose homogeneous Dirichlet boundary conditions ($u(0,t) = u(L,t) = 0$). You can assume that $\kappa < \pi c/L$.

sim. seen ⇓

(a) On dimensional homogeneity grounds, we have that

$$[\kappa] = T^{-1}$$
$$[c] = LT^{-1}$$

2, A

meth seen ⇓

(b) We follow the method of separation of variables and look for solutions of the form $u(x,t) = X(x)T(t)$. Subbing in the damped wave equation, we obtain

$$XT'' + 2\kappa XT' = c^2 X'' T \Rightarrow \frac{T'' + 2\kappa T'}{c^2 T} = \frac{X''}{X}$$

so we know that there exists a separation constant K such that

$$X'' + KX = 0$$

The homogeneous Dirichlet boundary conditions impose that $X(0) = X(L) = 0$ and so we require $K = \lambda^2 > 0$ to have non-trivial solutions. The general solution to the above second-order ODE is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

The BCs impose

$$X(0) = 0 \Rightarrow A = 0$$
$$X(L) = 0 \Rightarrow \lambda_n = \frac{n\pi}{L}, n \in \mathbb{Z}$$

leading to the family of solutions $X_n(x) = B_n \sin(n\pi x/L)$ with $n \in \mathbb{Z}$. 2, A

Back to the time-dependent ODE, we have

$$T'' + 2\kappa T' + c^2\lambda_n^2 T = 0$$

whose characteristic equation has for roots

$$r_\pm = -\kappa \pm \sqrt{\kappa^2 - c^2\lambda_n^2}$$

Further, $\kappa < \pi c/L$ imposes that for all $n \in \mathbb{Z}$, $\kappa^2 < c^2\lambda_n^2$ and so we have two complex roots leading to oscillatory solutions of the form

$$\begin{aligned} T_n(t) &= e^{-\kappa t}\left[C_n e^{i\sqrt{c^2\lambda_n^2 - \kappa^2}t} + D_n e^{-i\sqrt{c^2\lambda_n^2 - \kappa^2}t}\right] \\ &= e^{-\kappa t}\left[C_n' \cos\left(\sqrt{c^2\lambda_n^2 - \kappa^2}t\right) + D_n' \sin\left(\sqrt{c^2\lambda_n^2 - \kappa^2}t\right)\right] \end{aligned}$$

2, B

Defining $\alpha_n = C_n' B_n - C_{-n}' B_{-n}$ and $\beta_n = D_n' B_n - D_{-n}' B_{-n}$, we finally find that the general solution reads

$$u(x,t) = e^{-\kappa t} \sum_{n=1}^{\infty} \left[\alpha_n \cos(2\pi f_n t) + \beta_n \sin(2\pi f_n t)\right] \sin\left(\frac{n\pi x}{L}\right), \text{ with } f_n = \frac{1}{2\pi}\sqrt{c^2\lambda_n^2 - \kappa^2}$$

2, B

(c) The limit $\kappa = 0$ gives us the non-damped solution seen in lecture. As the frequency of the normal modes is given by $f_n = \frac{1}{2\pi}\sqrt{c^2\lambda_n^2 - \kappa^2}$, it is clear that damping has for effect to lower the frequencies of the normal modes.

unseen ⇓

2, C

unseen ⇓

(d) From lectures, we know that the total energy of the string is written

$$E(t) = \frac{1}{2}\int_0^L \left(\tau_0\left(\frac{\partial u}{\partial x}\right)^2 + \rho_0\left(\frac{\partial u}{\partial t}\right)^2\right) dx = \frac{\rho_0}{2}\int_0^L \left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

Taking a time-derivative, we get

$$\frac{dE}{dt} = \rho_0 \int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx$$

but as $u_{tt} = c^2 u_{xx} - 2\kappa u_t$, we have

$$\begin{aligned}
\frac{dE}{dt} &= \rho_0 \int_0^L \left(c^2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} - 2\kappa\left(\frac{\partial u}{\partial t}\right)^2 + c^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx \\
&= \rho_0 \int_0^L \left(c^2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) - 2\kappa\left(\frac{\partial u}{\partial t}\right)^2\right) dx \\
&= \rho c^2 \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^L - 2\kappa\rho_0 \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx
\end{aligned}$$

As for all $t > 0$, we have $u(0,t) = u(L,t) = 0$, then we conclude that $u_t(0,t) = u_t(L,t) = 0$ and thus we have

$$\frac{dE}{dt} = -2\kappa\rho_0 \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \le 0$$

6, D

meth seen ⇓

(e) We assume that the initial conditions are given by

$$u(x,0) = f(x) = x(L-x)$$
$$u_t(x,0) = 0$$

Setting $t = 0$ in the general solution, we obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)$$
$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} (-\kappa\alpha_n + 2\pi f_n \beta_n)\sin\left(\frac{n\pi x}{L}\right)$$

Using the Fourier series that was given for $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3} \left[1 - (-1)^n\right] \sin\left(\frac{n\pi x}{L}\right)$$

we conclude that

$$\begin{aligned} \alpha_n &= \frac{8L^2}{(2n+1)^3\pi^3} \\ \beta_n &= \frac{\kappa\alpha_n}{2\pi f_n} = \frac{8\kappa L^3}{(2n+1)^3\pi^3\sqrt{c^2n^2\pi^2 - \kappa^2L^2}} \end{aligned}$$

Leading to $\boxed{2, \mathrm{D}}$

$$u(x,t) = \frac{8L^2e^{-\kappa t}}{\pi^3} \sum_{n=0}^{\infty} \left[\frac{\cos\left(\sqrt{\frac{c^2n^2\pi^2}{L^2} - \kappa^2}t\right)}{(2n+1)^3} + \frac{\kappa L \sin\left(\sqrt{\frac{c^2n^2\pi^2}{L^2} - \kappa^2}t\right)}{(2n+1)^3\sqrt{c^2n^2\pi^2 - \kappa^2L^2}} \right] \sin\left(\frac{n\pi x}{L}\right)$$

$\boxed{2, \mathrm{D}}$

4. (a) First, we consider the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = f_1(x)$$
$$\frac{\partial u}{\partial t}(x,0) = f_2(x)$$

with f_1 and f_2 two known real-valued functions.

seen ⇓

(i) Take a Fourier transform of the wave equation to obtain the following ODE

$$\frac{\partial^2 \widehat{u}}{\partial t^2} + c^2\omega^2 \widehat{u} = 0$$

whose general solution is

$$\widehat{u}(\omega, t) = A(\omega)\cos(\omega ct) + B(\omega)\sin(\omega ct)$$

Setting $t = 0$ in $u(x,t)$ and $u_t(x,t)$, we obtain

$$\begin{cases} \widehat{f_1}(\omega) = A(\omega) \\ \widehat{f_2}(\omega) = \omega c B(\omega) \end{cases}$$

where $\widehat{f_1}$ and $\widehat{f_2}$ are the Fourier transforms of the real-valued functions f_1 and f_2. We finally conclude that the Fourier representation of the solution to our initial problem is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\widehat{f_1}(\omega)\cos(\omega ct) + \widehat{f_2}(\omega)\frac{\sin(\omega ct)}{\omega c}\right] e^{i\omega x} d\omega$$

4, A

(ii) Recall Euler's formulae

seen ⇓

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Thus, we can write

$$\begin{aligned}
\int_{-\infty}^{+\infty} \widehat{f_1}(\omega)\cos(\omega ct)e^{i\omega x} d\omega &= \frac{1}{2}\int_{-\infty}^{+\infty} \widehat{f_1}(\omega)\left(e^{i\omega ct} + e^{-i\omega ct}\right)e^{i\omega x} d\omega \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \widehat{f_1}(\omega)\left(e^{i\omega(x+ct)} + e^{i\omega(x-ct)}\right) d\omega \\
&= \frac{2\pi}{2}\left[f_1(x-ct) + f_1(x+ct)\right]
\end{aligned}$$

Similarly, we write

$$\begin{aligned}
\int_{-\infty}^{+\infty} \widehat{f_2}(\omega)\frac{\sin(\omega ct)}{\omega c}e^{i\omega x} d\omega &= \frac{1}{2}\int_{-\infty}^{+\infty} \widehat{f_2}(\omega)\frac{e^{i\omega ct} - e^{-i\omega ct}}{i\omega c}e^{i\omega x} d\omega \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \widehat{f_2}(\omega)\frac{e^{i\omega(x+ct)} - e^{i\omega(x-ct)}}{i\omega c} d\omega \\
&= \frac{1}{2c}\int_{-\infty}^{+\infty} \widehat{f_2}(\omega)\left(\int_{x-ct}^{x+ct} e^{i\omega\xi} d\xi\right) d\omega \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct}\left(\int_{-\infty}^{+\infty} e^{i\omega\xi}\widehat{f_2}(\omega) d\xi\right) d\omega = \frac{2\pi}{2c}\int_{x-ct}^{x+ct} f_2(\xi) d\xi
\end{aligned}$$

Combining these terms gives us what is known as d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[f_1(x-ct) + f_1(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

$\boxed{4, \text{A}}$

(b) We now consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and assume that: (1) $u(x,y) \to 0$ as $y \to \infty$ and (2) $u(x,0) = f(x)$ where f is a piece-wise smooth real function with $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$; so that u is a bounded solution.

$\boxed{\text{seen} \Downarrow}$

(i) Here, we use a Fourier method and take the Fourier transform of Laplace's equation with respect to the variable x as it is the only variable to vary over $\mathbb{R}$. If $u(x,y)$ is a solution of Laplace's equation then $\widehat{u}(\omega,y)$ is a solution to the following ODE

$$-\omega^2\widehat{u} + \frac{d^2\widehat{u}}{dy^2} = 0$$

with initial condition $\widehat{u}(\omega,0) = \widehat{f}(\omega)$. This general solution to this ODE is of the form $Ae^{\omega y} + Be^{-\omega y}$. If we impose the initial condition and the fact that $u(x,y) \to 0$ as $y \to \infty$, we can write

$$\widehat{u}(\omega,y) = \begin{cases} \widehat{f}(\omega)e^{-\omega y}, & \text{if} \quad \omega \geq 0 \\ \widehat{f}(\omega)e^{\omega y}, & \text{if} \quad \omega < 0 \end{cases}$$

The Fourier representation of the solution to our problem is thus written compactly

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-|\omega|y}e^{i\omega x}d\omega, \quad \text{with } \widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$\boxed{4, \text{B}}$

From here, we can write the following by formally changing the order of integration

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}d\xi\right)e^{-|\omega|y}e^{i\omega x}d\omega, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-|\omega|y}e^{i\omega(x-\xi)}d\omega\right)f(\xi)d\xi, \end{aligned}$$

The inner integral can be performed as follows

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\omega|y}e^{i\omega(x-\xi)}d\omega &= 2\Re\left[\int_0^{\infty} e^{-\omega y}e^{i\omega(x-\xi)}d\omega\right], \\ &= 2\Re\left[\frac{1}{y - i(x-\xi)}\right], \\ &= \frac{2y}{y^2 + (x-\xi)^2} \end{aligned}$$

And we conclude that the explicit representation of the solution is given by

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - \xi)^2} f(\xi) d\xi,$$

$\boxed{4, \text{C}}$

(ii) Let's find the explicit solution in the case where $\boxed{\text{unseen} \Downarrow}$

$$f(x) = \begin{cases} 2, & |x| < 4 \\ 0, & \text{otherwise} \end{cases}$$

From the previous question, we have

$$u(x, y) = \frac{1}{\pi} \int_{-4}^{4} \frac{2y}{y^2 + (x - \xi)^2} d\xi,$$

We proceed to the substitution $v = (\xi - x)/y$ such that we have $d\xi = ydv$; this leads to the following

$$u(x, y) = \frac{2}{\pi} \int_{(-4-x)/y}^{(4-x)/y} \frac{y^2}{y^2(1 + v^2)} dv = \frac{2}{\pi} \int_{(-4-x)/y}^{(4-x)/y} \frac{1}{1 + v^2} dv$$

We recognize in this last integral the arctangent. We thus write

$$u(x, y) = \frac{2}{\pi} \left[\arctan\left(\frac{4 - x}{y}\right) - \arctan\left(\frac{-4 - x}{y}\right) \right] = \frac{2}{\pi} \left[\arctan\left(\frac{4 - x}{y}\right) + \arctan\left(\frac{4 + x}{y}\right) \right]$$

where we have used the fact that $\arctan$ is an odd function. $\boxed{4, \text{D}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks

MATH50008 Partial Differential Equations in Action

Question	Marker's comment
1	This question contained two parts: (1) the non-dimensionalisation of a PDE and (2) a direct application of the method of characteristics for a quasi-linear first order PDE. Students seem to have found this question harder than was intended.Often students provided wrong dimensions for the number density (which is not to be confused with a linear density). The physical interpretation of the two terms appearing in the FKPP equation was often at best approximative; concise but clear answers were expected, often students wrote a lot but were too vague/imprecise.The method of characteristics in (b) caused more issues than I would have expected as problem was similar to problems seen in problem sessions.
2	This question dealt with the application of the method of characteristics for the inviscid Burgers' equation. It was a slight variation around a problem studied in the revision problem sheet. As expected a majority of students did well. The method was mostly understood; lost marks often are the result of algebra mistakes.Part (d) often caused issues, especially the integration of the shock path, students received partial marks for a correct method but leading to a wrong result. That being said, a number of students achieved full marks or close to full marks on this question.
3	The students were relatively at ease with Fourier series, but only few could use the energy of the equation.
4	This question was surprisingly very polarizing with a large fraction of students either obtaining very low marks or very high marks including some students obtaining full marks.I was surprised at the answers provided considering that a large fraction of the marks were given for subquestions which had been seen either in lecture or in problem sheets and the rest of the unseen marks were given for relatively simple algebra. From the results, it is clear that Fourier transform methods are not well understood by a large fraction of the students and basic results about Fourier transforms are not well known. Some students wrongly attempted to write a method of separation of variables on infinite domains.It is fair to assume that time was running out for a non-negligible fraction of students.