Introduction to Quantum Mechanics – Problem sheet 8

1. **Dynamics of a coherent state in the harmonic oscillator** - *This is an exercise that re-derives a result we can deduce from the Ehrenfest theorem using the method of stationary states and some basic facts of the harmonic oscillator. Fairly straight forward practice of various concept learned so far.*

   Consider the harmonic oscillator with the Hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. In the notes we have shown that the Ehrenfest theorem implies that the equations of motions that the time-dependent expectation values of $\hat{q}$ and $\hat{p}$ follow the classical dynamics, i.e.

$$\begin{aligned}\langle\hat{q}\rangle(t) &= \cos(\omega t)\langle\hat{q}\rangle(0) + \frac{1}{m\omega}\sin(\omega t)\langle\hat{p}\rangle(0)\\ \langle\hat{p}\rangle(t) &= \cos(\omega t)\langle\hat{p}\rangle(0) - m\omega\sin(\omega t)\langle\hat{q}\rangle(0),\end{aligned}$$

   independently of the initial state. Consider an initially coherent state $|z\rangle$ (as defined in Question 1 on Problem sheet 5) and use the method of stationary states to confirm this result for this special initial state.

2. **Dynamics for an anharmonic oscillator** - *You should be able to solve parts (a) and (b) quickly, part (c) is much more fiddly/advanced.*

   Consider the Hamiltonian $\hat{H} = \mu(\hat{a}^\dagger\hat{a})^2$, where μ has the dimension of an energy. (*Remark*: Hamiltonians of this type can be realised in the lab using ultracold atoms.)

   (a) What are the eigenvalues and eigenstates of $\hat{H}$?

   (b) Use the method of stationary states to show that the time evolution of an initial coherent state $|z\rangle$ is given by

$$|\psi(t)\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_n \frac{(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{\sqrt{n!}}|n\rangle,$$

   where $|n\rangle$ are the harmonic oscillator eigenstates.

   (c) Verify that

$$\langle\psi(t)|\hat{a}|\psi(t)\rangle = z\mathrm{e}^{-\mathrm{i}\mu t/\hbar}\mathrm{e}^{-|z|^2(1-\mathrm{e}^{-2\mathrm{i}\mu t/\hbar})}.$$

3. Wei-Norman form of the time evolution operator for an $SU(2)$ Hamiltonian - *Advanced question, though part (a) and (b) are easier - (a) is a simple commutator practice, and part (b) is an application of the Hadamard lemma.*

Let us consider a Hamiltonian of the form $\hat{H} = a(t)\hat{J}_3 + b(t)\hat{J}_1$, where $a, b \in \mathbb{R}$ are real parameters that may depend on time, and the operators $\hat{J}_{1,2,3}$ are a basis of the $su(2)$ algebra, with commutator relations

$$[\hat{J}_1, \hat{J}_2] = \mathrm{i}\hbar J_3, \quad [\hat{J}_2, \hat{J}_3] = \mathrm{i}\hbar J_1, \quad [\hat{J}_3, \hat{J}_1] = \mathrm{i}\hbar J_2.$$

We shall now derive the time evolution operator $\hat{U}(t)$ that yields the time-dependent state from the initial state as $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, and that fulfils the differential equation

$$\mathrm{i}\hbar \frac{d}{dt}\hat{U} = \hat{H}\hat{U}, \tag{1}$$

with $\hat{U}(0) = \hat{I}$, in a useful form called the *Wei-Norman* form.

(a) We can introduce the operators $\hat{J}_\pm = \hat{J}_1 \pm \mathrm{i}\hat{J}_2$, and $\hat{J}_0 = \hat{J}_3$. Check that these fulfil the commutation relations

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm, \quad \text{and} \quad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_0$$

(b) Use the Hadamard lemma to verify that

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}s\hat{J}_\pm/\hbar}\hat{J}_0\mathrm{e}^{\mathrm{i}s\hat{J}_\pm/\hbar} &= \hat{J}_0 \pm \mathrm{i}s\hat{J}_\pm \\
\mathrm{e}^{-\mathrm{i}s\hat{J}_0/\hbar}\hat{J}_\pm\mathrm{e}^{\mathrm{i}s\hat{J}_0/\hbar} &= \hat{J}_\pm\mathrm{e}^{\mp\mathrm{i}s} \\
\mathrm{e}^{-\mathrm{i}s\hat{J}_\pm/\hbar}\hat{J}_\mp\mathrm{e}^{\mathrm{i}s\hat{J}_\pm/\hbar} &= \hat{J}_\mp \mp 2\mathrm{i}s\hat{J}_0 + s^2\hat{J}_\pm
\end{aligned}$$

(c) Insert the ansatz

$$\hat{U} = \mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\mathrm{e}^{-\mathrm{i}c_-(t)\hat{J}_-/\hbar}$$

into the differential equation (1) and use the results from part (a) and (b) to verify that the $c_{0,\pm}(t)$ have to fulfil the dynamical equations

$$\dot{c}_- = \frac{b}{2}\mathrm{e}^{-\mathrm{i}c_0}, \quad \dot{c}_0 = a + \mathrm{i}bc_+\mathrm{e}^{-\mathrm{i}c_0}, \quad \dot{c}_+ = \frac{b}{2}\left(\mathrm{e}^{\mathrm{i}c_0} - c_+^2\mathrm{e}^{-\mathrm{i}c_0}\right).$$