Problem sheet 2 – question 8

Denote

$$I = \int_0^\infty f(z) e^{i\omega z^p} \, dz$$

Note that for z real, $e^{i\omega z^p}$ oscillates as $z \to \infty$. Consider the contour

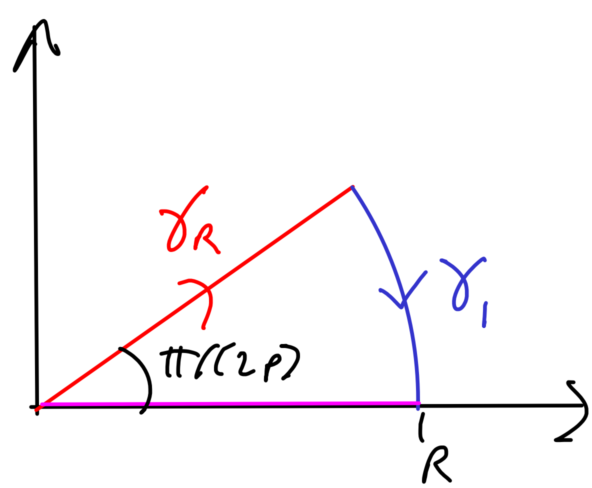
$$\gamma = \gamma_R \cup \gamma_1$$

where we define

$$\gamma_R = \{ z = t e^{i\pi/(2p)} \mid t \in [0, R] \}$$

$$\gamma_1 = \{ z = Re^{i\theta} \mid \theta \in [0, \pi/2p] \}$$

and we orient these as sketched here:



Assume $\rho \geq 2$. Then $\pi/(2\rho) \leq \pi/4$, and so $f(z)$ and hence $f(z)e^{iwz^\rho}$ are analytic in the region bounded by closed contour that is bounded by γ and the interval $[0,R]$ along the real axis. It follows from Cauchy's Theorem that the integral of $f(z)e^{iwz^\rho}$ around this contour is zero, i.e.,

$$\int_{\gamma \cup \overleftarrow{[0,R]}} f(z)\, e^{iwz^\rho}\, dz = 0$$

where we integrate along $[0,R]$ from R to 0, and along γ in the direction sketched above. This is true for all $R > 0$. Hence, taking $R \to \infty$, one may deduce that

$$I = \lim_{R \to \infty} \int_\gamma f(z) e^{iwz^\rho} dz$$

$$= \int_0^\infty f\left(t e^{i\pi/(2\rho)}\right) \cdot e^{i\omega t^\rho e^{i\pi/2}} e^{i\pi/(2\rho)}\, dt$$

$$+ \lim_{R\to\infty} \int_{\gamma_1} f(z) e^{i\omega z^\rho}\, dz$$

$$= e^{i\pi/(2\rho)} \int_0^\infty f\left(t e^{i\pi/(2\rho)}\right) \cdot e^{-\omega t^\rho}\, dt$$

$$+ \lim_{R\to\infty} \int_{\gamma_1} f(z) e^{i\omega z^\rho}\, dz$$

But

$$\int_{\gamma_1} f(z) e^{i\omega z^\rho}\, dz = \int_{\theta=\frac{\pi}{2\rho}}^{0} f(Re^{i\theta}) e^{i\omega (Re^{i\theta})^\rho} \cdot iRe^{i\theta}\, d\theta$$

$$\Rightarrow \left| \int_{\gamma_1} f(z) e^{iwz^p} dz \right|$$

$$\leq \int_{\theta=0}^{\pi/(2p)} |f(Re^{i\theta})| . e^{-wR^p \sin(p\theta)} . R \, d\theta$$

$\downarrow$ $f(z) = O(z^q)$ as $z \to \infty$

$\Rightarrow |f(Re^{i\theta})| \leq cR^q$ as $R \to \infty$

for some (bounded) constant c

$$\leq cR^{q+1} \int_0^{\pi/(2p)} e^{-wR^p \sin(p\theta)} d\theta$$

$\downarrow$ for $0 \leq \theta \leq \pi/(2p)$, we have

$0 \leq p\theta \leq \pi/2$ and so $\sin(p\theta) \geq \frac{2p\theta}{\pi}$

$$\leq cR^{q+1} \int_0^{\pi/(2p)} e^{-wR^p . 2p\theta/\pi} d\theta$$

$$= c R^{q+1} \left[\frac{e^{-2p w R^p \theta/\pi}}{-2p w R^p/\pi} \right]_{\theta=0}^{\frac{\pi}{2p}}$$

$$= \frac{\pi c R^{q+1-p}}{2pw} \left(1 - e^{-wR^p} \right)$$

$\to 0$ as $R \to \infty$ if and only if

$p - q - 1 > 0$

(note, $e^{-wR^p} \to 0$ as $R \to \infty$

as $w, R > 0$ and $p \geq 2$).

So

$$I = e^{i\pi/(2p)} \int_0^\infty f(t e^{i\pi/(2p)}) . e^{-wt^p} \, dt$$

if $p \geq 2$ and $p - 1 > q$.

b) Now suppose

$$I = \int_0^\infty f(z)\, e^{i\omega z^p}\, dz$$

where now $\omega \in \mathbb{C}$, $\mathrm{Re}\{\omega\} > 0$.

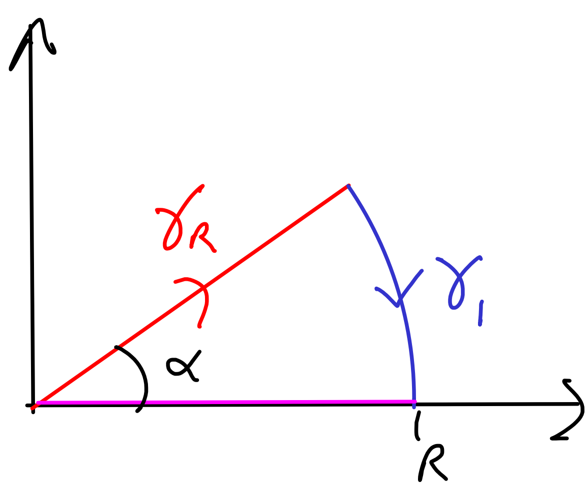
As before, we seek a contour γ of the form

$$\gamma = \gamma_R \cup \gamma_1$$

where now

$$\gamma_R = \{ z = t e^{i\alpha} \mid t \in [0, R] \}$$

$$\gamma_1 = \{ z = R e^{i\theta} \mid \theta \in [0, \alpha] \}$$



such that

$$I = \lim_{R\to\infty} \int_{\gamma_R} f(z) e^{i\omega z^\rho} dz$$

and α is such that the integrand $f(z)e^{i\omega z^\rho}$ tends to 0 exponentially as $|z|\to\infty$ along γ_R as $R\to\infty$, rather than oscillating. For this to be the case, $i\omega z^\rho$ should be real and negative along γ_R, i.e., we should have

$$\arg\{i\omega z^\rho\} = \pi \mod 2\pi \quad , \text{for } z\in\gamma_R \quad (1)$$

But along γ_R, we have $z = te^{i\alpha}$, $t\in[0,R]$. And $\omega = \mu e^{i\phi}$ for some μ and $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})$ (recall, $\mathrm{Re}\{\omega\}>0$). So (1) will hold provided

$$\frac{\pi}{2} + \phi + \rho\alpha = \pi \mod 2\pi$$

i.e., if

$$\alpha = \frac{1}{\rho}\left(\frac{\pi}{2} - \phi\right) \quad (2)$$

Notice that, assuming $\rho > 0$, α as given by (2) is contained in $(0, \pi/\rho)$, since $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\alpha = \frac{\pi}{2\rho}$ if $\phi = 0$ (as is the case for part (a)).

Furthermore, $f(z)$ must be analytic in the region that is bounded by the real z-axis and the ray along which $\arg z = \alpha$.

Then we can use Cauchy's Theorem to deform the contour of integration for I (as in part (a)).