IMPERIAL

MATH70032

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Tuesday, May 21, 2024
Time: 10:00 – 12:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be a regular curve parameterised by arc length.

   (i) Define the curvature of γ. (4 marks)

   (ii) Suppose now that $\gamma''(t) \neq 0$ for all $t \in \mathbb{R}$. Define the Frenet frame of γ at time t. (4 marks)

   (b) Consider the curve $\gamma\colon (0, \infty) \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\cos(t^2), \sin(t^2), \sqrt{3}t^2).$$

   (i) Compute the curvature of γ. (4 marks)

   (ii) Compute the Frenet frame of γ. (4 marks)

   (c) Consider some $r > 0$ and the curve $\gamma\colon (0, 2\pi) \to \mathbb{R}^2$ defined by

$$\gamma(t) = (r(t - \sin t), r(1 - \cos t)).$$

   Compute the length of γ. You may use, without proof, the double angle formula $2\sin^2 x = 1 - \cos(2x)$ for all x. (4 marks)

(Total: 20 marks)

2. (a) Let $S \subset \mathbb{R}^3$ be a regular connected surface, and let $f\colon S \to \mathbb{R}$ be a smooth function such that the differential of f satisfies $df_p = 0$ for all $p \in S$. Show that f is constant. (6 marks)

   (b) Consider the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + 3y^2 = 4e^z\} \subset \mathbb{R}^3.$$

   (i) Show that S is a regular surface. (4 marks)

   (ii) Compute the tangent plane to S at the point $(1, 1, 0)$. (4 marks)

   (iii) Compute the differential of the map $f\colon S \to \mathbb{R}$ defined by

$$f(x, y, z) = xz,$$

   at the point $(1, 1, 0) \in S$. Hint: you may wish to use a chart to compute this differential, but the final answer should not make reference to any chart (though the final answer can be expressed in terms of an appropriate basis for the tangent plane). (6 marks)

(Total: 20 marks)

3. Let $S \subset \mathbb{R}^3$ be a regular oriented surface with Gauss map $N\colon S \to \mathbb{S}^2$.

(a) Define the second fundamental form $A_p\colon T_pS \times T_pS \to \mathbb{R}$ of S at $p \in S$. (4 marks)

(b) Define the mean curvature of S. (4 marks)

(c) Let $\gamma\colon (a,b) \to S$ be smooth. Show that

$$A_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \langle N(\gamma(t)), \gamma''(t) \rangle,$$

for all $t \in (a,b)$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot product. (6 marks)

(d) Suppose there is a chart $\phi\colon U \to S$ such that

$$S = \{\phi(u,v) \mid (u,v) \in U\},$$

and define

$$\phi_t(u,v) = \phi(u,v) - tN(\phi(u,v)).$$

Suppose that $S_t = \{\phi_t(u,v) \mid (u,v) \in U\}$ is a regular surface for all t suitably small. Show that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(S_t) = 2\int_S H\,dA.$$

You may use, without proof, the fact that the mean curvature of S takes the form

$$H = \frac{eG + gG - 2fF}{EG - F^2},$$

where

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

are the matrices of the first and second fundamental forms of S respectively, with respect to the chart ϕ. (6 marks)

(Total: 20 marks)

4. (a) Let $\gamma\colon (a,b) \to \mathbb{R}^3$ be a regular curve defined by $\gamma(s) = (x(s), 0, z(s))$, where $(x'(s))^2 + (z'(s))^2 = 1$ and $x(s) > 0$ for all $s \in (a,b)$. Let S denote the surface of revolution obtained by rotating γ around the z axis,

$$S = \{(x(s)\cos\theta, x(s)\sin\theta, z(s)) \mid s \in (a,b), \theta \in (0,2\pi)\}.$$

(i) Show that the Christoffel symbols Γ^k_{ij} of the chart $\phi\colon (a,b) \times (0,2\pi) \to S$, defined by

$$\phi(s,\theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s)),$$

take the form

$$\Gamma^s_{ss} = \Gamma^\theta_{ss} = \Gamma^s_{s\theta} = \Gamma^s_{\theta s} = \Gamma^\theta_{\theta\theta} = 0,$$
$$\Gamma^s_{\theta\theta} = -x(s)x'(s), \qquad \Gamma^\theta_{s\theta} = \Gamma^\theta_{\theta s} = \frac{x'(s)}{x(s)}.$$

(6 marks)

(ii) Write down the geodesic equations of the chart ϕ. (2 marks)

(iii) Let $c(t) := \phi(s(t), \theta(t))$ be a unit speed geodesic in M (so that $|c'(t)|_g = 1$ for all t), and let

$$\varphi(t) = \cos^{-1}\left(\frac{\langle c'(t), \partial_\theta\phi\rangle}{|c'(t)||\partial_\theta\phi|}\right),$$

be the angle between $c'(t)$ and $(\partial_\theta\phi)(s(t),\theta(t))$. Show that,

$$x(s(t))\cos\varphi(t),$$

is independent of t. (6 marks)

(b) Let $S \subset \mathbb{R}^3$ be a regular oriented surface with non-positive Gauss curvature $K \leq 0$. Show that any two geodesics γ_1, γ_2 starting at $p \in S$ cannot meet again at another point $q \in S$ in such a way as to bound a region of S homeomorphic to a disk.

(6 marks)

(Total: 20 marks)

5. (a) Let $S \subset \mathbb{R}^3$ be a regular oriented surface.

(i) Show that the principal curvatures $\lambda_1(p)$, $\lambda_2(p)$ of S at a point $p \in S$ both solve the quadratic equation

$$\lambda^2 - 2H(p)\lambda + K(p) = 0$$

where H is the mean curvature and K is the Gauss curvature of S. You may use any results from the lectures, provided you state them clearly.

(4 marks)

(ii) Show that, at a point $p \in S$, the principal curvatures are equal if and only if

$$H(p)^2 = K(p).$$

(2 marks)

(b) Consider the torus of revolution

$$S = \{((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u) \mid u, v \in (0, 2\pi)\}.$$

Show that S has no umbilical points. You may use, without proof, the fact that Gauss and mean curvatures of S, with respect to a given chart, take the form

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{eG + gG - 2fF}{EG - F^2},$$

where

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

are the matrices of the first and second fundamental forms of S respectively.

(14 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60032/70032

Geometry of curves and surfaces (Solutions)

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1. (a) (i) Since γ is parameterised by arc length, the curvature is defined by seen $\Downarrow$

$$k(t) = |\gamma''(t)|.$$

4, A

(ii) Again, since γ is parameterised by arc length, seen $\Downarrow$

$$T(t) = \gamma'(t), \qquad N(t) = \frac{\gamma''(t)}{|\gamma''(t)|}, \qquad B(t) = T(t) \times N(t),$$

where the assumption $\gamma''(t) \neq 0$ is used. 4, A

(b) (i) First note that $|\gamma'(t)| = 4t$, so γ is not parameterised by arc length. Let $\sigma\colon (0,\infty) \to \mathbb{R}^3$, defined by sim. seen $\Downarrow$

$$\sigma(t) = \gamma\Big(\sqrt{\frac{t}{2}}\Big) = (\cos\frac{t}{2}, \sin\frac{t}{2}, \frac{\sqrt{3}t}{2}),$$

be the arc length reparameterisation. The curvature is given by

$$k(t) = |\sigma''(t)| = |-\frac{1}{4}(\cos\frac{t}{2}, \sin\frac{t}{2}, 0)| = \frac{1}{4}.$$

4, A

(ii) The Frenet frame is given by sim. seen $\Downarrow$

$$T(t) = \sigma'(t) = \frac{1}{2}(-\sin\frac{t}{2}, \cos\frac{t}{2}, \sqrt{3}), \qquad N(t) = \frac{T'(t)}{|T'(t)|} = -(\cos\frac{t}{2}, \sin\frac{t}{2}, 0),$$

$$B(t) = T(t) \times N(t) = \frac{1}{2}(\sqrt{3}\sin\frac{t}{2}, -\sqrt{3}\cos\frac{t}{2}, 1).$$

4, A

(c) One computes sim. seen $\Downarrow$

$$|\gamma'(t)| = |r(1-\cos t, \sin t)| = r\sqrt{2 - 2\cos t} = 2r\sqrt{\sin^2\frac{t}{2}} = 2r\sin\frac{t}{2},$$

where the final equality follows from the fact that $\sin\frac{t}{2} > 0$ for $t \in (0, 2\pi)$. Thus

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)| dt = \int_0^{2\pi} 2r\sin\frac{t}{2} dt = 8r.$$

4, B

2. (a) Consider a chart $\phi\colon U \to S$ and any $(u,v) \in U$. Since $df_{(\phi(u,v))} = 0$ for all u, v, seen ⇓

$$\frac{\partial f \circ \phi(u,v)}{\partial u} = df_{\phi(u,v)}(\partial_u \phi) = 0,$$

and

$$\frac{\partial f \circ \phi(u,v)}{\partial v} = df_{\phi(u,v)}(\partial_v \phi) = 0.$$

Thus there is a constant c such that $f \circ \phi\colon U \to \mathbb{R}$ satisfies $f \circ \phi \equiv c$. Since S is connected and can be covered by such charts, it follows that $f \equiv c$. 6, A

(b) (i) Note that $S = F^{-1}(0,0,0)$, where $F\colon \mathbb{R}^3 \to \mathbb{R}$ is defined by sim. seen ⇓

$$F(x,y,z) = x^4 + 3y^2 - 4e^z.$$

We showed in the lectures that such level sets are a regular surface if $\nabla F(x,y,z) \neq 0$ for all $(x,y,z) \in \mathbb{R}^3$, so it suffices to check this condition. We have

$$\nabla F(x,y,z) = (4x^3, 6y, -4e^z),$$

which is indeed non-zero for all (x,y,z) due to the fact that the final component can never vanish. 4, B

(ii) We also showed that, for such level sets, the tangent plane is given by the orthogonal complement of the gradient of F at the point in question. Thus sim. seen ⇓

$$T_{(1,1,0)}S = (\nabla F(1,1,0))^\perp = \{(x,y,z) \in \mathbb{R}^3 \mid 4x + 6y = 4z\}.$$

4, B

(iii) Consider the chart $\phi\colon U \to S$ defined by $\phi(u,v) = (u, v, \log \frac{u^4+3v^2}{4})$, where $U \subset \mathbb{R}^2$ is a small open neighbourhood of $(1,1)$. Note that $(1,1,0) = \phi(1,1)$ and sim. seen ⇓

$$f \circ \phi(u,v) = u \log \frac{u^4 + 3v^2}{4},$$

and so

$$\partial_u f \circ \phi(1,1) = \log \frac{u^4+3v^2}{4} + \frac{4u^4}{u^4+3v^2}\Big|_{(1,1)} = 1,$$

$$\partial_v f \circ \phi(1,1) = \frac{6uv}{u^4+3v^2}\Big|_{(1,1)} = \frac{3}{2}.$$

Thus

$$df_{(1,1,0)}(V^1(1,0,1) + V^2(0,1,3/2)) = V^1 + \frac{3V^2}{2},$$

since

$$\partial_u \phi(1,1) = (1,0,1), \qquad \partial_v \phi(1,1) = (0,1,3/2),$$

is a basis for $T_{(1,1,0)}S$. 6, B

3. (a) The second fundamental form is the negative of the quadratic form associated to the differential of the Gauss map: seen ⇓

$$A_p(X,Y) = -\langle X, dN_p(Y)\rangle,$$

for all $X, Y \in T_pS$, $p \in S$. 4, A

(b) The differential of the Gauss map can be viewed as a map seen ⇓

$$dN_p \colon T_pS \to T_pS,$$

in view of the fact that $T_{N(p)}\mathbb{S}^2 = T_pS$. The mean curvature is $-1/2$ times the trace of the differential of the Gauss map:

$$H(p) = -\frac{1}{2}\mathrm{tr}(dN_p)$$

Equivalently,

$$H(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2},$$

where $\lambda_1(p)$, $\lambda_2(p)$ are the (negatives of) the eigenvalues of the differential of the Gauss map dN_p (the principal curvatures). 4, A

(c) By definition of second fundamental form and differential, unseen ⇓

$$A(\gamma'(t), \gamma'(t)) = -\langle dN_{\gamma(t)}(\gamma'(t)), \gamma'(t)\rangle = -\langle \frac{dN(\gamma(t))}{dt}, \gamma'(t)\rangle$$
$$= \langle N(\gamma(t)), \gamma''(t)\rangle - \frac{d}{dt}\langle N(\gamma'(t)), \gamma'(t)\rangle.$$

The proof follows from noting that the second term on the right hand side vanishes in view of the fact that $\gamma'(t) \in T_{\gamma(t)}S$ and thus has vanishing dot product with the normal. 6, C

unseen ⇓

(d) For t suitably small ϕ_t is a chart for S_t. Thus

$$\mathrm{Area}(S_t) = \int_U \sqrt{(\partial_u\phi_t)^2(\partial_v\phi_t)^2 - \langle \partial_u\phi_t, \partial_v\phi_t\rangle^2}dudv.$$

Now

$$\partial_u\phi_t = \partial_u\phi - tdN(\partial_u\phi), \qquad \partial_v\phi_t = \partial_v\phi - tdN(\partial_v\phi),$$

and so

$$\frac{d}{dt}|_{t=0}\mathrm{Area}(S_t) = \int_U \frac{\partial_u\phi\cdot dN(\partial_u\phi)(\partial_v\phi)^2 + \partial_v\phi\cdot dN(\partial_v\phi)(\partial_u\phi)^2}{\sqrt{(\partial_u\phi)^2(\partial_v\phi)^2 - \langle\partial_u\phi, \partial_v\phi\rangle^2}}dudv$$
$$- \int_U \frac{\langle\partial_u\phi, \partial_v\phi\rangle(\partial_u\phi\cdot dN(\partial_v\phi) + \partial_v\phi\cdot dN(\partial_u\phi))}{\sqrt{(\partial_u\phi)^2(\partial_v\phi)^2 - \langle\partial_u\phi, \partial_v\phi\rangle^2}}dudv$$
$$= \int_U \frac{eG + gE - 2Ff}{\sqrt{EG - F^2}}dudv,$$

where the last equality uses the notation from the lectures. We showed in the lectures

$$H = \frac{1}{2}\frac{eG + gE - 2Ff}{EG - F^2},$$

and the proof follows. 6, D

4. (a) (i) One computes

$$\partial_s \phi = (x'(s)\cos\theta, x'(s)\sin\theta, z'(s)), \qquad \partial_\theta \phi(-x(s)\sin\theta, x(s)\cos\theta, 0),$$

and so the matrix of the first fundamental form takes the form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (x(s))^2 \end{pmatrix}.$$

The Christoffel symbols are then computed using the formula

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}\big(\partial_{x^i} g_{jl} + \partial_{x^j} g_{il} - \partial_{x^l} g_{ij}\big),$$

with $x^1 = s$, $x^2 = \theta$.

meth seen ⇓ 6, C

(ii) The geodesic equations take the form

$$s''(t) - x(s(t))x'(s(t))(\theta'(t))^2 = 0, \qquad \theta''(t) + 2\frac{x'(s(t))}{x(s(t))}\theta'(t)s'(t) = 0.$$

seen ⇓ 2, A

(iii) Since $|c'| \equiv 1$ and

$$c'(t) = \partial_s\phi s'(t) + \partial_\theta\phi\theta'(t),$$

it follows from the above computations that that

$$\begin{aligned}\frac{d}{dt}\big(x(s(t))\cos\varphi(t)\big) &= \frac{d}{dt}\big(x(s(t))^2\theta'(t)\big) \\ &= \Big(\theta''(t) + 2\theta'(t)s'(t)\frac{x'(s(t))}{x(s(t))}\Big)x(s(t))^2 = 0,\end{aligned}$$

where the final equality follows from the above geodesic equations.

unseen ⇓ 6, D

(b) Suppose, on the contrary that γ_1 and γ_2 meet at another point $q \in S$ in such a way that the region U which they bound is homoemorphic to a disk. Say $\gamma_1(a_1) = \gamma_2(a_2) = p$, $\gamma_1(b_1) = \gamma_2(b_2) = q$. Since a disk has Euler characteristic $\chi(U) = 1$, denoting $A = \gamma_1([a_1, b_1])$ and $A = \gamma_2([a_2, b_2])$, the Gauss–Bonnet theorem implies that

$$\int_{A\cup B} k_g ds + \int_U K dA + \theta_1 + \theta_2 = 2\pi.$$

where $\theta_1, \theta_2 \in [-\pi, \pi]$ are the exterior angles at p and q respectively. Since γ_1 and γ_2 are geodesics, the first term on the left vanishes. By assumption the second term on the left has a sign. Thus

$$\theta_1 + \theta_2 \geq 2\pi.$$

The exterior angles satisfy, however, $\theta_1, \theta_2 < \pi$ (as the geodesics would coincide if the were both equal to π), which is a contradiction.

meth seen ⇓ 6, D

5. (a) (i) We showed in the lectures that

unseen ⇓

$$H = \frac{\lambda_1 + \lambda_2}{2}, \qquad K = \lambda_1 \lambda_2.$$

Thus one checks

$$\lambda^2 - 2H\lambda + K = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0,$$

where the final equality holds when $\lambda = \lambda_1$ and $\lambda = \lambda_2$. 4, M

(ii) Using the first part of the question, the principal curvatures are equal if and only if the quadratic has a repeated root, which happens if and only if

$$H^2 = K.$$

2, M

(b) Consider the chart $\phi\colon (0, 2\pi)^2 \to S$ defined by

unseen ⇓

$$\phi(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u).$$

One computes

$$\partial_u \phi = (-\sin u \cos v, -\sin u \sin v, \cos u), \qquad \partial_v \phi = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0),$$

$$\partial_u^2 \phi = (-\cos u \cos v, -\cos u \sin v, -\sin u), \qquad \partial_u \partial_v \phi = (\sin u \sin v, -\sin u \cos v, 0),$$

$$\partial_v^2 \phi = (-(2 + \cos u)\cos v, -(2 + \cos u)\sin v, 0),$$

and

$$N = \frac{\partial_u \phi \times \partial_v \phi}{|\partial_u \phi \times \partial_v \phi|} = (-\cos u \cos v, -\cos u \sin v, -\sin u).$$

Thus

$$E = |\partial_u \phi|^2 = 1, \quad F = \langle \partial_u \phi, \partial_v \phi \rangle = 0, \quad G = |\partial_v \phi|^2 = (2 + \cos u)^2,$$

$$e = \langle N, \partial_u^2 \phi \rangle = 1, \quad f = \langle N, \partial_u \partial_v \phi \rangle = 0, \quad g = \langle N, \partial_v^2 \phi \rangle = \cos u(2 + \cos u).$$

It follows that

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\cos u}{2 + \cos u},$$

and

$$H = \frac{eG + gG - 2fF}{EG - F^2} = \frac{2 + 2\cos u}{2 + \cos u}.$$

Using the first part of the question, there is an umbilical point if and only if the quadratic has a repeated root, which happens if and only if

$$H^2 = K.$$

However,

$$H^2 - K = \frac{(2 + 2\cos u)^2}{(2 + \cos u)^2} - \frac{2 + 2\cos u}{2 + \cos u} = \frac{1 + 3(1 + \cos u)^2}{(2 + \cos u)^2} > 0.$$

14, M

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 18 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 18 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH60032 Geometry of Curves & Surfaces

Question	Marker's comment
1	Most of this question was well answered. nbsp;The most common mistake was to not re parameterise \gamma by arc length in part (b). nbsp;A few students mistakenly thought that the function \sin(t/2) takes different signs on the interval (0,2\pi).
2	The most common mistake in this question was in part (a). nbsp;Many students mistakenly tried to treat f as a function on \mathbb{R}^3 and, for example, compute \partial_x, \partial_y, and \partial_z of f.
3	This question was generally well answered. nbsp;Many students struggled to begin on part (d).
4	Question (a) (i) was well answered, but many students struggled with (ii) and (iii). nbsp;In part (b), many students incorrectly used a version of Gauss--Bonnet for regions with smooth boundary, despite the fact that the region that the two geodesics would bound will in general have two exterior angles.

MATH70032 Geometry of Curves & Surfaces

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2	The most common mistake in this question was in part (a). nbsp;Many students mistakenly tried to treat f as a function on \mathbb{R}^3 and, for example, compute \partial_x, \partial_y, and \partial_z of f.
3	This question was generally well answered. nbsp;Many students struggled to begin on part (d).
4	Question (a) (i) was well answered, but many students struggled with (ii) and (iii). nbsp;In part (b), many students incorrectly used a version of Gauss--Bonnet for regions with smooth boundary, despite the fact that the region that the two geodesics would bound will in general have two exterior angles.
5	This question was generally well answered, though some students seem to have run out of time when answering this question.