Generalizing the method we used above gives the following proposition and theorem.

Proposition 4.6.

Proof. Since (s_n) is monotonic increasing, we have by Proposition 3.16 and Theorem 3.21 that

$$s_n \text{ is bounded} \iff s_n \text{ is convergent.}$$

For the second statement, s_n is unbounded $\iff \forall M > 0\ \exists N \in \mathbb{N}_{>0}$ such that $s_N > M$. But s_N is monotonic, so this is $\iff \forall M > 0\ \exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ s_n > M$. And this is the definition of $s_n \to +\infty$. □

We now give a very useful convergence test for positive series.

Theorem 4.7: Comparison test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Proof. Call the partial sums A_n, B_n respectively. Then

$$0 \ \leq\ A_n \ \leq\ B_n \ \leq\ \lim_{n\to\infty} B_n \ =\ \sum_{i=1}^{\infty} b_i.$$

So A_n is bounded and monotonically increasing $\implies$ convergent.

We are done since in previous exercise we have shown that $A_n \leq B_n$ and $A_n \to A$, $B_n \to B$ implies that $A \leq B$. □

Exercise 4.8 (Converse of Comparison Test.)**.** If $0 \leq a_n \leq b_n$ then $\sum a_n$ diverges to $+\infty \implies \sum b_n$ diverges to $+\infty$.

Remark 4.9. So from $\sum \frac{1}{n^2}$ convergent (Example 4.5) we can now deduce $\sum \frac{1}{n^\alpha}$ convergent for $\alpha \geq 2$ by the Comparison Test. In fact we can improve on this.

Example 4.10. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent for $\alpha > 1$.

Proof. (Cf. proof of divergence of $\sum \frac{1}{n}$ in Example 4.4.) Arrange the partial sums as follows:

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots = 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \dots + \frac{1}{7^\alpha}\right) + \left(\frac{1}{8^\alpha} + \dots + \frac{1}{15^\alpha}\right) + \left(\frac{1}{16^\alpha} + \dots + \frac{1}{31^\alpha}\right) + \dots$$

Bound the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha} + \dots + \frac{1}{(2^{k+1}-1)^\alpha}\right) \le \frac{1}{2^{k\alpha}} + \dots + \frac{1}{2^{k\alpha}} = \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}.$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for $n \le 2^{k+1} - 1$ we have

$$s_n < \sum_{i=0}^{k} \frac{1}{2^{i(\alpha-1)}} = \frac{1 - \frac{1}{2^{(k+1)(\alpha-1)}}}{1 - \frac{1}{2^{(\alpha-1)}}} \le \frac{1}{1 - \frac{1}{2^{\alpha-1}}}.$$

(It is here we used $\alpha > 1$, so $\left|\frac{1}{2^{\alpha-1}}\right| < 1$, so top and bottom of the central fraction are > 0.)

So partial sums are monotonic and bounded above $\implies$ convergent. ☐

Theorem 4.11: Algebra of limits for series

If $\sum a_n$, $\sum b_n$ are convergent then so is $\sum(\lambda a_n + \mu b_n)$, to

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

Proof.

☐

4.2 Absolute convergence

Definition. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if

Remark 4.12. It is possible for a series to be convergent (that is, its sequence of partial sums converges), but not absolutely convergent!

Example 4.13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent (by Example 4.4), but it is convergent.

Rough Working:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots \;=\; (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \dots$$

with kth bracket $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$. This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$. We saw this is convergent in Example 4.5. So cancellation between consecutive terms is enough to make series converge by comparison with $\sum \frac{1}{k(k-1)}$.

Proof. Fix $\epsilon > 0$. Then use 2 things

$$\sum \frac{1}{2k(2k-1)} \quad \text{is convergent to } L \text{ say} \tag{1}$$

$$\frac{(-1)^{n+1}}{n} \quad \longrightarrow \quad 0 \tag{2}$$

By (1) $\exists\, N_1$ such that $\forall n \geq N_1$, $\left| \sum_{k=1}^{n} \frac{1}{2k(2k-1)} - L \right| < \epsilon$.

By (2) $\exists\, N_2$ such that $\forall n \geq N_2$, $\left| \frac{(-1)^{n+1}}{n} \right| < \epsilon$.

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$, setting $j := \lfloor \frac{n}{2} \rfloor$ we have:

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{2j-1} - \frac{1}{2j}\right) + \delta \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} + \delta, \end{aligned}$$

where

$$\delta = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \qquad \text{satisfies } |\delta| \le \epsilon \text{ for } n \ge N_2 \text{ by (2).}$$

So

$$|s_n - L| \le \left| \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} - L \right| + |\delta| < \epsilon + \epsilon$$

for all $n \ge 2N+1$ (so that $\lfloor \frac{n}{2} \rfloor \ge N \ge N_1$ and $n \ge N \ge N_2$) by (1) and (2). $\square$

Definition. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if and only if

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ above is an example of a conditionally convergent series.

While it is possible for a series to be convergent without being absolutely convergent, the next theorem shows that if a series is absolutely convergent it **must** be convergent.

Theorem 4.14

Let $(a_n)_{n \ge 0}$ be a real or complex sequence.

Proof. Let $s_n = \sum_{i=1}^{n} |a_i|$ and $\sigma_n = \sum_{i=1}^{n} a_i$ be the partial sums.

Fix $\epsilon > 0$. We're assuming that s_n converges, so it is Cauchy:

i.e. the terms in the tail of the series contribute little to the sum. So by the triangle inequality,

and (σ_n) is Cauchy, and so convergent. $\square$

Example 4.15. For $z \in \mathbb{C}$ the power series $\sum_{n=1}^{\infty} z^n$ is absolutely convergent for $|z| < 1$ and divergent for $|z| \geq 1$.

Proof. $\sum_{n=1}^{\infty} z^n$ is absolutely convergent because in Example 4.1 we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$.
For $|z| \geq 1$, the individual terms z^n have $|z^n| \geq 1$, so $z^n \not\to 0$, so $\sum z^n$ is divergent by Theorem 4.2. □

4.3 Tests for convergence

We already met the first test:

Theorem 4.7: Comparison I

Recall proof: $s_n = \sum_{i=1}^{n} a_i$ is monotonic increasing and bounded above by $\sum_{i=1}^{\infty} b_i \in \mathbb{R}$.

Theorem 4.16: Comparison II: Sandwich Test

Proof. We use the Cauchy criterion. $\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0}$ such that $\forall n > m > N$,

since the partial sums of b_i, c_i are Cauchy. Therefore

which implies

i.e. the partial sums $\sum_{i=1}^{n} a_i$ form a Cauchy sequence. $\square$

Theorem 4.17: Comparison III

Remark 4.18. While writing $\frac{a_n}{b_n} \to L$ makes sense, writing $a_n \to Lb_n$ does not make sense (why)!

Proof. Set $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$,

$$\left|\frac{a_n}{b_n} - L\right| < 1 \implies \left|\frac{a_n}{b_n}\right| < |L| + 1 \implies |a_n| < (|L|+1)|b_n|.$$

So now by the comparison test $\sum_{n\geq N} |b_n|$ convergent $\implies \sum_{n\geq N} |a_n|$ convergent. By the next exercise this gives the result. $\square$

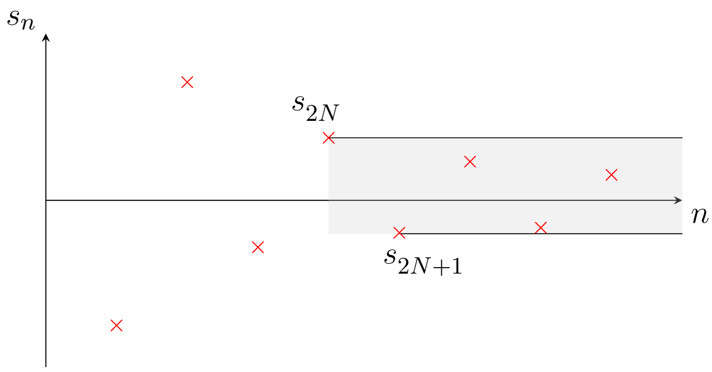
Exercise 4.19. Fix $N \in \mathbb{N}_{>0}$. Then $\sum_{n\geq N} c_n$ is convergent if and only if $\sum_{n\geq 1} c_n$ is convergent.

We call a sequence a_n *alternating* if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ $\forall n$ (or the opposite).

Theorem 4.20: Alternating Series Test

Proof. Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. Consider the partial sums $s_n = \sum_{i=1}^{n} (-1)^i b_i$.

We claim



Indeed if $i = 2j \geq 2n$ is even then

by monotonicity, while if $i = 2j+1 > 2n$ is odd then $s_{2j+1} = s_{2j} - b_{2j+1} \leq s_{2j} \leq s_{2n}$. Similarly if $i = 2j+1 \geq 2n+1$ is odd then

while if $i = 2j+2 > 2n+1$ is even then $s_{2j+2} = s_{2j+1} + b_{2j+2} \geq s_{2j+1} \geq s_{2n+1}$. The upshot is that $\forall n, m \geq 2N+1$,

and so

But $b_n \downarrow 0$ so $\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ b_n < \epsilon$. Thus (s_n) is Cauchy. $\square$

Exercise 4.21. What do you think about the infinite sum

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \ldots ?$$

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$
5. Other

Exercise 4.22. The alternating sequence $a_n = \begin{cases} \frac{1}{n^2} + \frac{1}{n} & n \text{ even,} \\ -\frac{1}{n^2} & n \text{ odd,} \end{cases}$ has sum $\sum a_n$ which is

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$
5. Other

Theorem 4.23: Ratio Test

Idea: Expect, eventually, $a_{N+k} \approx a_N r^k$ so that $\sum_{k\geq 0} |a_{N+k}| \approx |a_N| \sum_{k\geq 0} r^k = \frac{|a_N|}{1-r}$. More realistically, bound $|a_{N+k}|$ by $|a_N|(r+\epsilon)^k$, choosing ϵ so that $r+\epsilon < 1$.

Proof.

□

Remark 4.24. The ratio test only applies when a_n decays exponentially in n. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Example 4.25. Consider the complex sequence

$$a_n = \frac{100^n(\cos n\theta + i \sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}.$$

Then

So by the ratio test, $\sum a_n$ is absolutely convergent (and so convergent).

Theorem 4.26: Root Test

Remark 4.27. Again, writing $|a_n| \to r^n$ does not make sense.

Proof.

$\square$