Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$\langle x, y \rangle = \langle y, x \rangle, \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \langle ax, y \rangle = a \langle x, y \rangle.$$

Solution: These are computations using the definition of the inner product, vector addition and scalar multiplication, and linearity properties of sums.

$$\langle x, y \rangle = \sum_{i=1}^{n} x^i y^i = \sum_{i=1}^{n} y^i x^i = \langle y, x \rangle.$$

$$\begin{aligned} \langle x + y, z \rangle &= \sum_{i=1}^{n} (x + y)^i z^i = \sum_{i=1}^{n} (x^i + y^i) z^i = \sum_{i=1}^{n} (x^i z^i + y^i z^i) \\ &= \sum_{i=1}^{n} x^i z^i + \sum_{i=1}^{n} y^i z^i = \langle x, z \rangle + \langle y, z \rangle. \end{aligned}$$

$$\langle ax, y \rangle = \sum_{i=1}^{n} a x^i y^i = a \sum_{i=1}^{n} x^i y^i = a \langle x, y \rangle.$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0 \tag{1}$$

Solution: We use the properties of the inner product established above to find:

$$\begin{aligned} \|x + ty\|^2 = \langle x + ty, x + ty \rangle &= \langle x, x + ty \rangle + \langle ty, x + ty \rangle \\ &= \langle x, x \rangle + \langle x, ty \rangle + \langle ty, x \rangle + \langle ty, ty \rangle \\ &= \langle x, x \rangle + 2t \langle x, y \rangle + t^2 \langle y, y \rangle \\ &= \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2. \end{aligned}$$

Since $\|x + ty\|^2 \geq 0$, we certainly have:

$$\|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0.$$

(c) By thinking of (1) as a quadratic in t, and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|\langle x, y \rangle| \leq \|x\| \|y\|. \tag{2}$$

When does equality hold?

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

Solution: (1) is a non-negative quadratic in t, so it can have at most one root. Thus the discriminant ($b^2 - 4ac$ with the usual conventions) must be non-positive, i.e.

$$4\langle x, y\rangle^2 - 4\|x\|^2\|y\|^2 \leq 0,$$

which gives the result on re-arrangement. Equality holds iff there exists $t \in \mathbb{R}$ such that $\|x + ty\| = 0$, which is the condition that x, y are parallel.

(d) Deduce the triangle inequality for the norm on $\mathbb{R}^n$.

Solution: Returning to (1) and setting $t = 1$, we have:

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2\end{aligned}$$

Since both sides are positive, we deduce that:

$$\|x+y\| \leq \|x\| + \|y\|.$$

(e) Show the reverse triangle inequality:

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\|$$

Solution: To see the above inequality, it is enough to show that

$$\|x\| - \|y\| \leq \|x-y\| \qquad \text{and} \qquad \|x\| - \|y\| \geq -\|x-y\|.$$

For the first one, we note that

$$\|x\| = \|(x-y) + y\| \leq \|x-y\| + \|y\|$$

which gives is the first inequality. For the second one, we note that

$$\|y\| = \|(y-x) + x\| \leq \|x-y\| + \|x\|$$

which gives the second inequality by rearranging the terms.

Exercise 1.2. Suppose $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

(i) Show that:

$$\max_{k=1,\ldots,n} \left|x^k\right| \leq \|x\|.$$

Solution: Fix an arbitrary k in $\{1, 2, \ldots, n\}$. Since $y \mapsto \sqrt{y}$ is an increasing map from $[0, +\infty)$ to $[0, +\infty)$, we have

$$\left|x^k\right| = \sqrt{(x^k)^2} \leq \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2} = \|x\|.$$

This implies that the maximum of all these numbers is bounded by $\|x\|$.

(ii) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|.$$

[Hint: write out $\|x\|^2$ in coordinates and estimate]

Solution: Writing out $\|x\|^2$, we have:

$$\|x\|^2 = \sum_{i=1}^{n} (x^i)^2 \leq \sum_{i=1}^{n} \max_{k=1,\ldots,n} (x^k)^2 = \sum_{i=1}^{n} \left(\max_{k=1,\ldots,n} \left|x^k\right| \right)^2 = n \left(\max_{k=1,\ldots,n} \left|x^k\right| \right)^2.$$

Taking square roots, we have:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|,$$

since both sides are positive.

Exercise 1.3. Suppose that $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i \to \infty} x_i = x, \qquad \lim_{i \to \infty} y_i = y.$$

(a) Show that

$$\lim_{i \to \infty} (x_i + y_i) = x + y.$$

Solution: Fix $\epsilon > 0$. By the convergence of (x_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{2}, \qquad \|y_j - y\| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. Then if $i \geq N$ we have:

$$\|x_i + y_i - (x + y)\| \leq \|x_i - x\| + \|y_i - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Show that

$$\lim_{i \to \infty} \langle x_i, y_i \rangle = \langle x, y \rangle,$$

and deduce that

$$\lim_{i \to \infty} \|x_i\| = \|x\|.$$

[Hint: Write $\langle x_i, y_i \rangle - \langle x, y \rangle = \langle x_i - x, y_i - y \rangle + \langle x_i - x, y \rangle + \langle x, y_i - y \rangle$ and use the Cauchy-Schwartz inequality (2)]

Solution: Fix $\epsilon > 0$, and without loss of generality assume $\epsilon < 1$. By the convergence of (x_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{3(1 + \|y\|)}, \qquad \|y_j - y\| < \frac{\epsilon}{3(1 + \|x\|)}.$$

(The reason for the above choices will be clear in a moment.)

Set $N = \max\{N_1, N_2\}$. Then, for all $i \geq N$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\langle x_i, y_i\rangle - \langle x, y\rangle| &= |\langle x_i, y_i\rangle - \langle x_i, y\rangle + \langle x_i, y\rangle - \langle x, y\rangle| \\
&= |\langle x_i, y_i - y\rangle + \langle x_i - x, y\rangle| \\
&\leq |\langle x_i, y_i - y\rangle| + |\langle x_i - x, y\rangle| \\
&\leq \|x_i\|\, \|y_i - y\| + \|x_i - x\|\, \|y\| \\
&= \|x_i - x + x\|\, \|y_i - y\| + \|x_i - x\|\, \|y\| \\
&\leq (\|x_i - x\| + \|x\|)\, \|y_i - y\| + \|x_i - x\|\, \|y\| \\
&\leq \|x_i - x\|\, \|y_i - y\| + \|x\|\, \|y_i - y\| + \|x_i - x\|\, \|y\| \\
&< \frac{\epsilon^2}{9(1 + \|y\|)(1 + \|x\|)} + \|x\| \frac{\epsilon}{3(1 + \|x\|)} + \|y\| \frac{\epsilon}{3(1 + \|y\|)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence of real numbers with $a_i \to a$ as $i \to \infty$. Show that

$$\lim_{i \to \infty} (a_i x_i) = ax.$$

[Hint: Write $a_i x_i - ax = (a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)$ and use the properties of the norm.]

Solution: Fix $\epsilon > 0$, and without loss of generality assume $\epsilon < 1$. By the convergence of (a_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{3(1 + |a|)}, \qquad |a_j - a| < \frac{\epsilon}{3(1 + \|x\|)}.$$

Set $N = \max\{N_1, N_2\}$. Then if $i \geq N$ we have:

$$\begin{aligned}
\|a_i x_i - ax\| &= \|(a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)\| \\
&\leq \|(a_i - a)(x_i - x)\| + \|(a_i - a)x\| + \|a(x_i - x)\| \\
&= |a_i - a|\, \|x_i - x\| + |a_i - a|\, \|x\| + |a|\, \|x_i - x\| \\
&< \frac{\epsilon^2}{9(1 + |a|)(1 + \|x\|)} + \epsilon \frac{\|x\|}{3(1 + \|x\|)} + \epsilon \frac{|a|}{3(1 + |a|\,)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$