Imperial College London

MATH97018 MATH97096

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Biology

Date: Friday, 7 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. One approach to control a pest population is to introduce a population of sterile pests to compete with the fertile ones. The number of sterile pests is n and the number of fertile pests is N.

(a) A simple model describing the number of fertile pests is

$$\frac{dN}{dt} = \left(\frac{aN}{N+n} - b\right) N - kN(N+n), \tag{1}$$

where a, b, and k are positive constants and $a > b$.

(i) Take $n = 0$. Find the stable size of the pest population, N^*. (3 marks)

(ii) Now consider the case where $n > 0$ and maintained at a constant value. Find the fixed points of the system and assess their stability. (4 marks)

(iii) In terms of N^*, find n_c, the minimum value of n needed to guarantee that the fertile pests are eradicated. (3 marks)

(iv) Sketch the bifurcation diagram where n is the bifurcation parameter. Indicate the flow for different values of n. Use a dashed line to indicate unstable branches and a solid line to indicate stable branches. What kind of bifurcation occurs at n_c? Is there bistability and if so, over what range of n? (3 marks)

(b) Suppose that the fertile pests are motile and that the spatial dynamics of the pest population can be described by the reaction-diffusion equation

$$\frac{\partial N}{\partial t} = \left(\frac{aN}{N+n} - b\right) N - kN(N+n) + D\frac{\partial^2 N}{\partial x^2},$$

where D is the pest diffusion coefficient. Assume that the sterile population is $n = \gamma N$.

(i) The equation emits a travelling wave solution of the form $N(x,t) = u(x - ct)$ that propagates from left to right if the conditions at infinity are

$$N \to C \text{ as } x \to -\infty \text{ and } N \to 0 \text{ as } x \to \infty$$

where C is a positive constant. What is the value of C in terms of the parameters in the problem, and further, what is the permissible range of γ? (4 marks)

(ii) Find an expression for the wavespeed, c. You may use without proof results established in the notes for the Fisher-Kolmogorov (FK) equation. (3 marks)

(Total: 20 marks)

2. Consider the following reduced model for an enzyme-based catalytic reaction,

$$\frac{dx}{dt} = \alpha(yx^\gamma - x)$$
$$\frac{dy}{dt} = 1 - yx^\gamma$$

where $\gamma > 1$ and α is a positive constant.

(a) Find the equations of the nullclines for the system. (3 marks)

(b) Sketch the nullclines for the case of $\gamma = 2$ and indicate the direction of the flow in different regions of phase space. (4 marks)

(c) Take $\gamma > 1$. Find the fixed point and determine how its stability and type (saddle, node, spiral, etc) depend on α. (4 marks)

(d) At what value of α does the system undergo a Hopf bifurcation for $\gamma > 1$? Compute the period of the limit cycle just after the bifurcation occurs. (4 marks)

(e) Consider again the case where $\gamma = 2$. What do you expect to occur when $\alpha \gg 1$? Justify your answer by sketching trajectories in phase space and/or analysing the above equations in the correct limit. (5 marks)

(Total: 20 marks)

3. Consider the dimensionless activator-inhibitor reaction-diffusion system

$$\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + \nabla^2 u,$$
$$\frac{\partial v}{\partial t} = u^2 - v + d\nabla^2 v$$

for $\boldsymbol{x} \in \Omega$ and $t > 0$. The parameters b and d are positive constants.

(a) (i) Find the spatially homogenous steady state and the values of b for which it is stable. (4 marks)

(ii) For which values of d will a Turing instability develop? (5 marks)

(iii) Find the minimum, k_-, and maximum, k_+, unstable wavenumbers. (3 marks)

(b) Suppose that the domain Ω is the surface of a cylinder of length L and radius R. Accordingly, we can use cylindrical coordinates where we have $z \in (0, L)$ along the length of the cylinder and $\theta \in [0, 2\pi)$ around it. In these coordinates, the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial z^2} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \theta^2}.$$

(i) The function, $\phi(z, \theta) = \cos\left(\frac{2\pi n z}{L}\right) e^{im\theta}$ where m is an integer and n is a positive integer, satisfies

$$\nabla^2 \phi + k^2 \phi = 0,$$

with boundary conditions

$$\frac{\partial \phi}{\partial z}(0, \theta) = 0, \quad \frac{\partial \phi}{\partial z}(L, \theta) = 0.$$

Find the expression for the wavenumber squared, k^2. (4 marks)

(ii) Take the cylinder length L to be fixed. Suppose that the parameters b and d are such that the stripe pattern with $n = N$ is unstable, but a spotted pattern with $n = N$ is not. Based on this observation, find an expression for the maximum value of the cylinder aspect ratio, R/L. You may leave your answer in terms of k_+. (4 marks)

(Total: 20 marks)

4. Let $X(t)$ be the random variable for the total population size of a birth-death-immigration process and $p_i(t) = \text{Prob}\{X(t) = i\}$. The infinitesimal transition probabilities are,

$$p_{i+j,i}(\Delta t) = \text{Prob}\{X(t+\Delta t) = i+j | X(t) = i\}$$
$$= \begin{cases} \mu i \Delta t + o(\Delta t), & j = -1 \\ (\lambda i^2 + \nu)\Delta t + o(\Delta t), & j = 1 \\ 1 - (\lambda i^2 + \mu i + \nu)\Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1, 0, 1. \end{cases}$$

where λ, μ, and ν are positive constants. Take the initial population to be $X(0) = N$.

(a) Based on the infinitesimal transition probabilities, what is the corresponding deterministic model for the population size? (4 marks)

(b) From the transition probabilities, derive the forward Kolmogorov equation,

$$\frac{dp_i}{dt} = \left(\lambda(i-1)^2 + \nu\right) p_{i-1} + \mu(i+1)p_{i+1} - \left(\lambda i^2 + \mu i + \nu\right) p_i.$$

for $p_i(t)$. What should the initial condition be for each p_i? (5 marks)

(c) From the forward Kolmogorov equation (or otherwise), derive the differential equation for the mean, $m(t)$, of the process. If necessary, utilise the variance, $\sigma^2(t)$, to express terms in the righthand side of the equation. Compare the resulting equation with that of the deterministic model and discuss how you expect the mean to differ from the solution to the deterministic model. (7 marks)

(d) Take $\lambda = 0$ and compute the stationary probability distribution. (4 marks)

(Total: 20 marks)

5. (I) Recall the Fitzhugh-Nagumo model,

$$\frac{dv}{dt} = f(v) - w + I_a,$$
$$\frac{dw}{dt} = bv - \gamma w,$$

for neurons, where a, b, γ and I_a are positive constants and $f(v) = v(a-v)(v-1)$. Consider a phase oscillator,

$$\frac{d\theta}{dt} = \omega$$

where θ is the phase angle, related to the limit cycle just after the Hopf bifurcation occurs as I_a is increased. Find an expression for the frequency ω. (4 marks)

(II) (a) Suppose a phase oscillator, $\theta(t)$, is coupled to another phase oscillator,

$$\frac{d\Theta}{dt} = \Omega,$$

such that

$$\frac{d\theta}{dt} = \omega + \Gamma \cos\left(2(\Theta - \theta)\right)$$

where Ω and Γ are positive constants.

(i) For what range of parameters does a steady phase difference, $\phi = \Theta - \theta$, exist? (3 marks)

(ii) What is the range of **stable** steady phase differences for $\phi \in [-\pi, \pi]$? (4 marks)

(b) Suppose that three oscillators are coupled, such that

$$\frac{d\theta_1}{dt} = \omega_1 + \Gamma\cos(\theta_2 - \theta_1)$$
$$\frac{d\theta_2}{dt} = \omega_2 + \Gamma\cos(\theta_1 - \theta_2) + \Gamma\cos(\theta_3 - \theta_2)$$
$$\frac{d\theta_3}{dt} = \omega_3 + \Gamma\cos(\theta_2 - \theta_3).$$

(i) What conditions must be met for steady phase differences to exist? (4 marks)

(ii) Suppose that $\omega_1 = \omega_2 = \omega_3$. Find all the possible steady phase differences that lie in $[-\pi, \pi] \times [-\pi, \pi]$ and assess their stability. Sketch the phase portrait. (5 marks)

(Total: 20 marks)

	EXAMINATION SOLUTIONS 2020-21	Course MATH B.0
Question 1		Marks & seen/unseen
Parts		

(a) $$\frac{dN}{dt} = \left(\frac{aN}{N+n} - b\right)N - kN(N+n)$$

(i) FOR $n = 0$, WE HAVE — SEEN SIMILAR

$$\frac{dN}{dt} = (a-b)N - kN^2$$

$$= (a-b)N\left[1 - \frac{kN}{a-b}\right]$$

THIS IS THE LOGISTIC EQUATION

$$\frac{dN}{dt} = rN\left[1 - \frac{N}{K}\right]$$

WITH $r = a - b$ AND

$$K = \frac{a-b}{k}$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts		

THUS, THE STABLE SYSTEM SIZE IS

$$\boxed{N^* = K = \frac{a-b}{k}}$$

3A

(ii) FOR A CONSTANT n,

SEEN SIMILAR

WE CONSIDER

$$0 = N\left[\frac{aN}{n+N} - b - k(N+n)\right]$$

THUS, WE HAVE

$\boxed{N = 0}$ AND

$$\frac{aN}{n+N} - b - k(N+n) = 0$$

WHICH GIVES

$$k(N+n)^2 - (a-b)(N+n) + an = 0.$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen

Parts

SOLVING FOR N, WE FIND

$$N_{\pm} = -n + \frac{a-b}{2k} \pm \sqrt{\frac{(a-b)^2}{4k^2} - \frac{an}{k}}$$

WHICH IN TERMS OF N^* IS

$$\boxed{N_{\pm} = -n + \frac{N^*}{2} \pm \sqrt{\frac{N^{*2}}{4} - \frac{a}{a-b} N^* n}}$$

4A

SEEN SIMILAR

STABILITY

FIRST CONSIDER $N=0$.

$$f(N) = \frac{aN^2}{n+N} - bN - k(N^2 + nN)$$

$$f'(N) = \frac{2aN}{n+N} - \frac{aN^2}{(n+N)^2} - b - k(2N + n)$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts	$f'(0) = -b - kn < 0$	
	$\boxed{N=0 \text{ IS STABLE.}}$	
	SINCE THE STABILITY OF THE FIXED POINTS WILL ALTERNATE, WE HAVE	
	$\boxed{N = N_- \text{ IS UNSTABLE}}$	
	AND	
	$\boxed{N = N_+ \text{ IS STABLE.}}$	3B
(iii)	THE MINIMUM n IS GIVEN BY	SEEN SIMILAR
	$$\frac{N^{*2}}{4} - \frac{a}{a-b} N^* n_c = 0$$	
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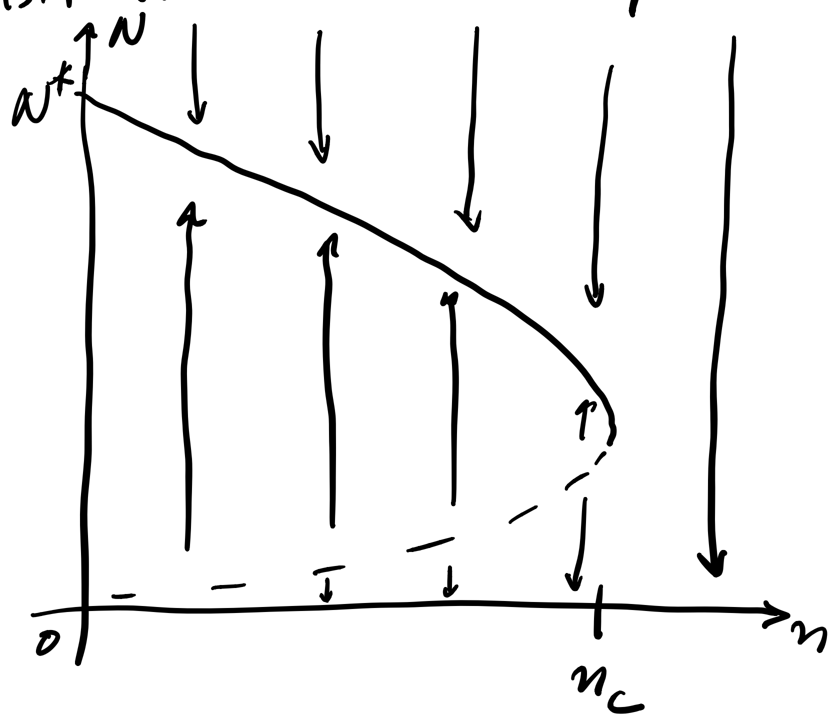
	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts	AND THEREFORE	

$$n_c = \frac{1}{4}\left(\frac{a-b}{a}\right) N^*$$

(iv) BIFURCATION DIAGRAM



SADDLE-NODE BIFURCATION AT $n = n_c$.

BISTABILITY FOR $0 < n < n_c$.

3B

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts		
(b) (i)	$$\frac{\partial N}{\partial t} = \left(\frac{aN}{N+n} - b\right)N - kN(N+n) + D\frac{\partial^2 N}{\partial x^2}$$ For $n = \gamma N$, $$\frac{\partial N}{\partial t} = \left(\frac{a}{1+\gamma} - b\right)N - k(1+\gamma)N^2 + D\frac{\partial^2 N}{\partial x^2}$$ $$= rN\left(1 - \frac{N}{K}\right) + D\frac{\partial^2 N}{\partial x^2} \quad (*)$$	UNSEEN ⇓
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen

Parts

WHERE

$$r = \left(\frac{a}{1+\gamma} - b \right)$$

AND

$$K = \frac{\frac{a}{1+\gamma} - b}{k(1+\gamma)}$$

SINCE $K > 0$

WE HAVE THAT

$$\frac{a}{1+\gamma} - b > 0$$

AND THEREFORE,

$$\gamma < \frac{a-b}{b} .$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts	SINCE THIS IS THE FISHER - KOLMOGOROV EQN, C MUST BE THE STABLE FIXED POINT OF THE LOGISTIC EQUATION, HENCE $$C = \frac{\frac{a}{1+\gamma} - b}{k(1+\gamma)}.$$	4C
(ii)	FOR THE DIMENSIONLESS F-K	UNSEEN
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts	EQUATION, $$\frac{du}{d\tau} = u(1-u) + \frac{\partial^2 u}{\partial X^2},$$ THE WAVE SPEED IS $c = 2$. OUR EQUATION CAN BE PUT INTO THIS FORM BY CONSIDERING $$N = Ku, \quad t = \tau/r$$ AND $x = (D/r)^{1/2} X$. BASED ON THESE SCALINGS,	
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2020-21	Course
Question 1		Marks & seen/unseen
Parts	THE WAVE SPEED IS $$c = 2\sqrt{Dr}$$ $$= 2\sqrt{D\left(\frac{a}{1+\gamma} - b\right)}$$	3D
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen

Parts

$$\frac{dx}{dt} = \alpha \left(y x^{\gamma} - x \right)$$

$$\frac{dy}{dt} = 1 - y x^{\gamma}$$

(a) NULLCLINES — SEEN SIMILAR ↓

$$0 = \alpha \left(y x^{\gamma} - x \right)$$

$$0 = 1 - y x^{\gamma}$$

THUS,

$$\boxed{y = 1/x^{\gamma}}$$

AND

$$\boxed{x = 0, \quad y = 1/x^{\gamma - 1}}$$

3A

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EXAMINATION SOLUTIONS 2020-21

Course

Question 2

Marks & seen/unseen

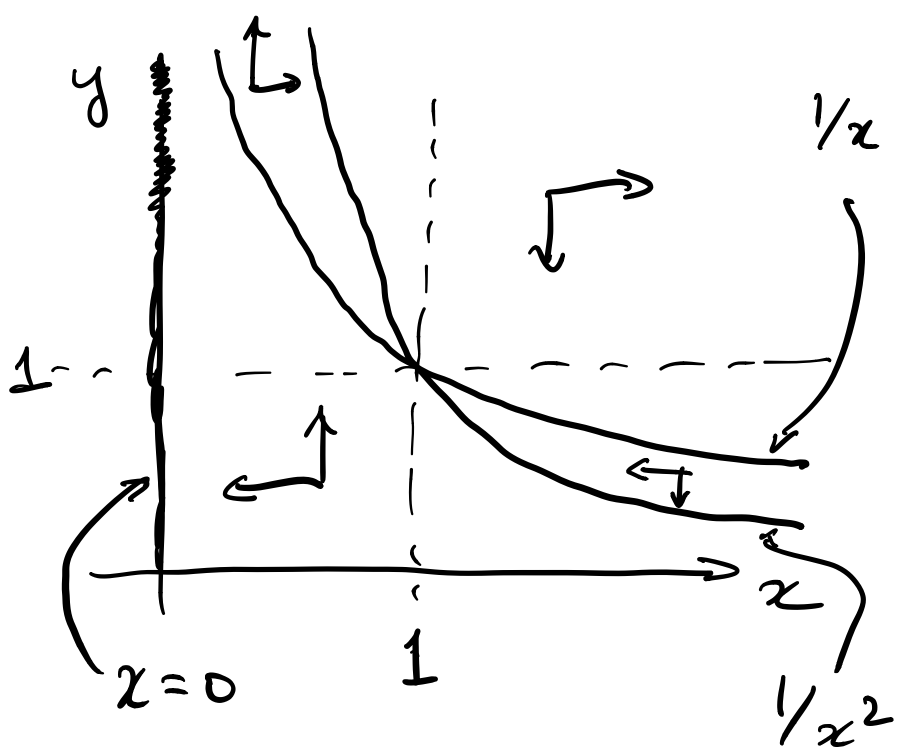
Parts

(b) FOR $\gamma = 2$, WE HAVE

SEEN SIMILAR

$$\frac{dx}{dt} = \alpha(x^2 y - x)$$

$$\frac{dy}{dt} = 1 - x^2 y$$



4A

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts (c)	FIXED POINT SUBSTITUTING $y = 1/x^{\gamma}$ INTO $0 = \alpha(yx^{\gamma} - x)$, WE SEE THAT $x = 1$ AND THEREFORE $y = 1$. THUS, THE FIXED POINT IS $(1, 1)$.	SEEN SIMILAR
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EXAMINATION SOLUTIONS 2020-21

Course

Question 2

Marks & seen/unseen

Parts

STABILITY

DEFINING,

$$f(x,y) = \alpha(yx^{\gamma} - x)$$

$$g(x,y) = 1 - yx^{\gamma}$$

WE HAVE

$$\frac{\partial f}{\partial x} = \alpha(y\gamma x^{\gamma-1} - 1)$$

$$\frac{\partial f}{\partial y} = \alpha x^{\gamma}$$

$$\frac{\partial g}{\partial x} = -\gamma y x^{\gamma-1}$$

$$\frac{\partial g}{\partial y} = -x^{\gamma}$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen

Parts

THUS,

$$\frac{\partial f}{\partial x}(1,1) = \alpha(\gamma - 1)$$

$$\frac{\partial f}{\partial y}(1,1) = \alpha$$

$$\frac{\partial g}{\partial x}(1,1) = -\gamma$$

$$\frac{\partial g}{\partial y}(1,1) = -1$$

THE JACOBIAN IS THEN

$$J = \begin{bmatrix} \alpha(\gamma-1) & \alpha \\ -\gamma & -1 \end{bmatrix}$$

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Question 2

Marks & seen/unseen

Parts

THE EIGENVALUES ARE

$$\lambda_{\pm} = \frac{1}{2}\left\{ \alpha(\gamma-1) - 1 \pm \sqrt{[\alpha(\gamma-1)-1]^2 - 4\alpha} \right\}$$

AS $\alpha > 0$, WE WILL EITHER HAVE NODES IF

$$[\alpha(\gamma-1) - 1]^2 - 4\alpha > 0$$

OR SPIRALS IF

$$[\alpha(\gamma-1) - 1]^2 - 4\alpha < 0.$$

	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen

Parts

THE NODES OR SPIRALS WILL BE STABLE IF

$$\alpha(\gamma-1)-1<0.$$

AND UNSTABLE IF

$$\alpha(\gamma-1)-1>0.$$

BASED ON THESE CONDITIONS, WE CAN SOLVE FOR α TO FIND

STABLE NODE: $0<\alpha\leq\alpha_1$

STABLE SPIRAL: $\alpha_1<\alpha<\alpha_0$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts	UNSTABLE SPIRAL: $\alpha_0 < \alpha < \alpha_2$ UNSTABLE NODE: $\alpha_2 \leq \alpha < \infty$. WHERE $\alpha_0 = \dfrac{1}{\gamma - 1}$ $\alpha_1 = \left(\dfrac{\sqrt{\gamma} - 1}{\gamma - 1}\right)^2$ $\alpha_2 = \left(\dfrac{\sqrt{\gamma} + 1}{\gamma - 1}\right)^2$	4B
(d)	THE SYSTEM UNDERGOES A HOPF BIFURCATION WHEN THE SPIRAL	SEEN SIMILAR
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EXAMINATION SOLUTIONS 2020-21

Course

Question

Marks & seen/unseen

Parts

BECOMES UNSTABLE AT $\alpha = \alpha_0$.

THE PERIOD OF THE LIMIT CYCLE IS GIVEN BY

$$T = \frac{2\pi}{\omega}$$

WHERE

$$\omega = \frac{1}{2}\sqrt{\left[\alpha_0(\gamma-1)-1\right]^2 - 4\alpha_0}$$

$$= \frac{1}{\sqrt{\gamma-1}}$$

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Checker's initials

Page number

	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts	AND THEREFORE	
	$$T = 2\pi\sqrt{r-1}$$	4 C
(e)	FOR $r=2$, WE HAVE	UNSEEN
	$$\frac{dx}{dt} = \alpha(yx^2 - x)$$	
	$$\frac{dy}{dt} = 1 - yx^2$$	
	FOR $\alpha >> 1$, x EVOLVES RAPIDLY	
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts	AND THE SYSTEM TO THE STABLE FIXED POINT OF THE 1D SYSTEM $$\frac{dx}{dt} = \alpha\left(y x^2 - x\right)$$ AT A CONSTANT VALUE OF y. THE FIXED POINTS ARE	
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts		

$x=0$ AND $x = 1/y$

DEFINING

$$f(x) = \alpha(y x^2 - x),$$

WE HAVE

$$f'(x) = \alpha(2yx - 1)$$

THUS

$f'(0) = -\alpha < 0 \Rightarrow$ STABLE

$f'(1/y) = \alpha > 0 \Rightarrow$ UNSTABLE

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts	THUS, WE WILL HAVE $x \to 0$ RAPIDLY AND THEREFORE $$\frac{dy}{dt} = 1$$ $$\Rightarrow y = t + \underset{\text{CONSTANT.}}{K}$$ AFTER x IS NEARLY ZERO.	
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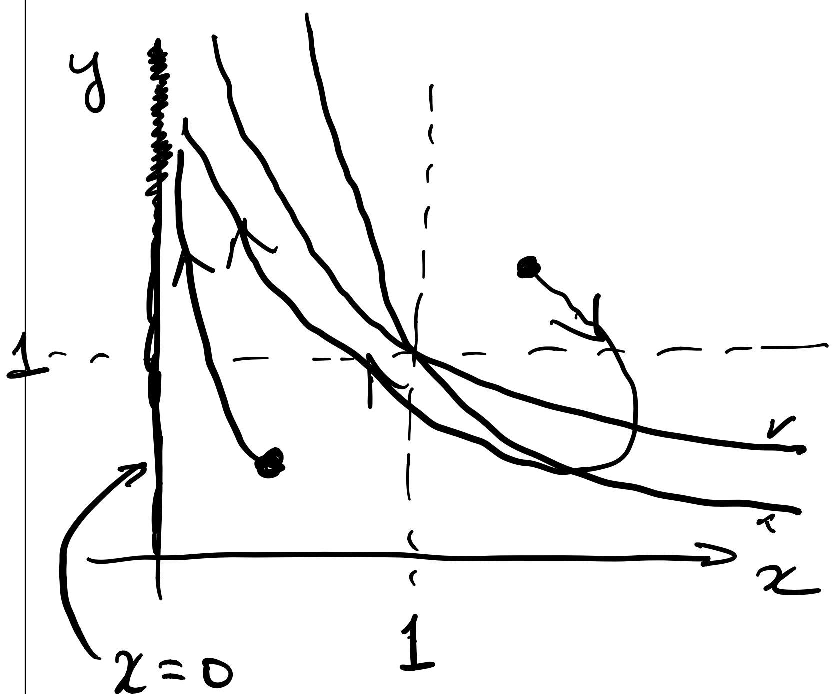
	EXAMINATION SOLUTIONS 2020-21	Course
Question 2		Marks & seen/unseen
Parts	POSSIBLE TRAJECTORIES 	SD
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		
(a) i	STEADY STATE IS GIVEN BY $$\frac{u^2}{v} - bu = 0$$ $$u^2 - v = 0$$ THUS, $v = u^2$ AND THEREFORE, $$u = 1/b$$ $$v = 1/b^2$$	SEEN SIMILAR
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen

Parts

STABILITY OF THE STEADY STATE.

DEFINE

$$f(u,v) = \frac{u^2}{v} - bu$$

$$g(u,v) = u^2 - v$$

$$\frac{\partial f}{\partial u} = \frac{2u}{v} - b$$

$$\frac{\partial f}{\partial v} = -\frac{u^2}{v^2}$$

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EXAMINATION SOLUTIONS 2020-21	Course
Question 3	Marks & seen/unseen

Parts

$$\frac{\partial g}{\partial u} = 2u$$

$$\frac{\partial g}{\partial v} = -1$$

$$J(u,v) = \begin{bmatrix} \frac{2u}{v} - b & -\frac{u^2}{v^2} \\ 2u & -1 \end{bmatrix}$$

$$J\left(\frac{1}{b}, \frac{1}{b^2}\right) = \begin{bmatrix} b & -b^2 \\ 2/b & -1 \end{bmatrix}$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts	FOR STABILITY, $\text{TRACE}(J) = b - 1 < 0$ AND $\det(J) = b > 0.$ THUS, $0 < b < 1.$	4A
(ii)	FOR PATTERN FORMATION TO OCCUR, WE	SEEN SIMILAR
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		

MUST ALSO HAVE

$$-1 + db > 0$$

AND

$$(db - 1)^2 - 4db > 0.$$

THUS, $d > 1/b$

AND

CONSIDERING

$$(db)^2 - 6db + 1 = 0$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts	$db = 3 \pm 2\sqrt{2},$ WE MUST HAVE $db > 3 + 2\sqrt{2}.$	SA
(iii)	THE MINIMUM AND MAXIMUM WAVENUMBERS ARE	
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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		

$$k_-^2 = \frac{(db-1) - \sqrt{(db-1)^2 - 4db}}{2d}$$

AND

$$k_+^2 = \frac{(db-1) + \sqrt{(db-1)^2 - 4db}}{2d}$$

3B

UNSEEN

(b) FOR

(i)

$$\varphi(z,\theta) = \cos\left(\frac{2\pi n z}{L}\right) e^{im\theta},$$

WE HAVE

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		

$$\frac{\partial \varphi}{\partial z} = -\frac{2\pi n}{L}\sin\left(\frac{2\pi n z}{L}\right) e^{jm\theta}$$

$$\frac{\partial^2 \varphi}{\partial z^2} = -\left(\frac{2\pi n}{L}\right)^2 \varphi(z,\theta)$$

AND

$$\frac{\partial \varphi}{\partial \theta} = jm \quad \varphi(z,\theta),$$

$$\frac{\partial^2 \varphi}{\partial \theta^2} = -m^2 \varphi(z,\theta)$$

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		

THUS,

$$\nabla^2 \varphi = -\left[\left(\frac{2\pi n}{L} \right)^2 + \frac{m^2}{R^2} \right] \varphi$$

AND THEREFORE

$$k^2 = \left(\frac{2\pi n}{L} \right)^2 + \frac{m^2}{R^2}$$

4D

(ii) IF ONLY $n = N$ IS UNSTABLE

UNSEEN

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	EXAMINATION SOLUTIONS 2020-21	Course
Question 3		Marks & seen/unseen
Parts		

THEN WE HAVE

$$\left(\frac{2\pi N}{L}\right)^2 + \left(\frac{1}{R}\right)^2 > k_+^2$$

SOLVING FOR R/L, WE SEE THAT

$$\left(\frac{R}{L}\right)^2 < \frac{1}{K_+^2 L^2 - (2\pi N)^2}$$

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THUS, THE MAXIMUM VALUE IS

$$\left(\frac{R}{L}\right)^2 = \frac{1}{k_+^2 L^2 - [2\pi N]^2 4D}$$

	EXAMINATION SOLUTIONS 2020-21	Course
Question 4		Marks & seen/unseen
Parts		
(a)	LET n BE THE POPULATION SIZE.	SEEN SIMILAR
	THE DETERMINISTIC MODEL IS	
	$$\frac{dn}{dt} = \nu + \lambda n^2 - \mu n$$	4A
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Parts		
(b)	THE GENERATOR MATRIX IS $q_{ii} = -\lambda i^2 - \mu i - \nu$ $q_{i+1,i} = \lambda i^2 + \nu$ $q_{i-1,i} = \mu i$ IN GENERAL, THE FKE'S $\frac{dp}{dt} = Qp$	SEEN SIMILAR
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Parts	AND THEREFORE,	

$$\frac{dp_i}{dt} = q_{i,i-1}\, p_{i-1} + q_{ii}\, p_i + q_{i,i+1}\, p_{i+1}$$

$$= \left[\lambda (i-1)^2 + \nu\right] p_{i-1} - \left[\lambda i^2 + \mu i + \nu\right] p_i + \mu (i+1)\, p_{i+1}$$

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Parts

THE INITIAL CONDITION IS

$$p_i(0) = \begin{cases} 1, & i=N \\ 0, & \text{OTHERWISE.} \end{cases}$$

5A

(C) MULTIPLYING THE FKE BY i AND SUMMING OVER i, WE HAVE THAT

SEEN SIMILAR

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$$\sum_i i \frac{dp_i}{dt} = \sum_i i\left(\lambda(i-1)^2 + \nu\right) p_{i-1}$$

$$+ \sum_i i \mu (i+1) p_{i+1}$$

$$- \sum_i i(\lambda i^2 + \mu i + \nu) p_i$$

WE CAN REWRITE THE FIRST 2 SUMS AS

$$\sum_i (i+1)\left[\lambda i^2 + \nu\right] p_i$$

AND

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Parts

$$\sum_i (i-1)\mu i\, p_i.$$

USING THESE EXPRESSIONS, WE HAVE,

$$\frac{dm}{dt} = \lambda \sum_i i^2 p_i - \mu m + \nu$$

IN TERMS OF THE VARIANCE

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Parts	WE HAVE $$\frac{dm}{dt} = \lambda m^2 - \mu m + \nu + \lambda \sigma^2.$$ WE SEE THAT THE EQUATION DIFFERS FROM THE DETERMINISTIC MODEL WITH THE ADDITIONAL	
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Parts

TERM $\lambda\sigma^2 > 0$.

THUS, WE EXPECT $m(t) \geq n(t)$. 7 B

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Question 4

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Parts

(d) THE STATIONARY DISTRIBUTION IS GIVEN BY

SEEN SIMILAR

$$\pi_i = \frac{\nu^i}{\mu^i\, i!}\pi_0, \quad i \geq 1$$

$$= \left(\frac{\nu}{\mu}\right)^i \frac{1}{i!}$$

AND

$$\pi_0^{-1} = 1 + \sum_{i=1}^{\infty}\left(\frac{\nu}{\mu}\right)^i \frac{1}{i!}$$

$$= e^{\nu/\mu}$$

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Parts	Thus $$\pi_i = \frac{1}{i!}\left(\frac{\nu}{\mu}\right)^i e^{-\nu/\mu}$$	4c
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Parts

SEEN SIMILAR

(I) FIXED POINT

AT

$$f(v) - w + I_a = 0$$

$$bv - \gamma w = 0$$

STABILITY,

$$J(v,w) = \begin{bmatrix} \frac{df}{dv}(v^*) & -1 \\ b & -\gamma \end{bmatrix}$$

HOPF BIFURCATION

WHEN

$$\text{trace}(J) = 0$$

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Parts		

AND

$$\det(J) > 0.$$

$$\text{trace}(J) = \frac{df}{dv}(v^*) - \gamma = 0,$$

$$\Rightarrow \frac{df}{dv}(v^*) = \gamma.$$

$$\det(J) = -\left[\frac{df}{dv}(v^*)\right]\gamma$$

$$+ b$$

$$= b - \gamma^2$$

THEREFORE THE

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Question 5

Marks & seen/unseen

Parts

EIGENVALUES ARE

$$\lambda = \pm\sqrt{\gamma^2 - b}$$

AND HENCE

$$\omega = \sqrt{b - \gamma^2}$$

4

(II) (a) UNSEEN

$$\frac{d\Theta}{dt} = \Omega$$

$$\frac{d\theta}{dt} = \omega + \Gamma \cos[2(\Theta - \theta)]$$

SET $\varphi = \Theta - \theta$

AND THEREFORE

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Question 5		Marks & seen/unseen
Parts	$$\frac{d\varphi}{dt} = \Omega - \omega - \Gamma\cos 2\varphi$$ STEADY PHASE DIFFERENCE $$0 = \Omega - \omega - \Gamma\cos 2\varphi$$ Thus, $$\cos 2\varphi = \frac{\Omega - \omega}{\Gamma}$$ REQUIRE THAT $$\left|\frac{\Omega - \omega}{\Gamma}\right| < 1.$$	3
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Question 5

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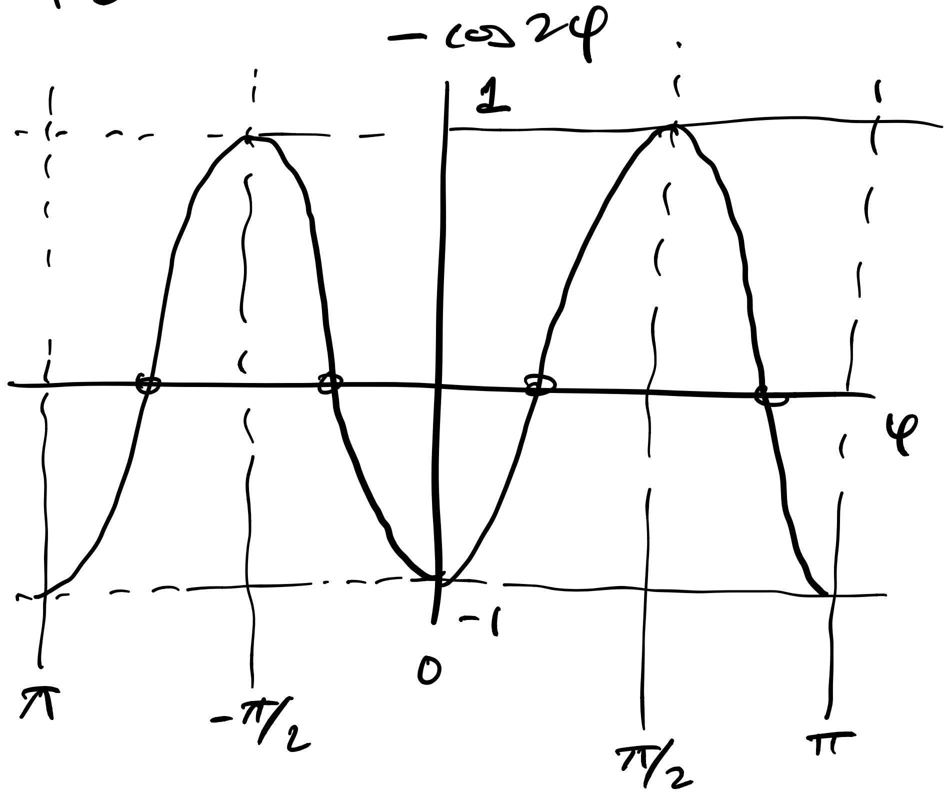
Parts

UNSEEN

TO DETERMINE THE RANGE OF STABLE FIXED POINTS, WE CONSIDER THE PLOT



WE SEE THAT $\varphi = -\pi/4, \frac{3\pi}{4}$ ARE STABLE

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Parts	ADJUSTING $\frac{\Omega-\omega}{\Gamma}$ WILL RAISE/LOWER THE CURVE ABOVE AND SHIFT THESE FIXED POINTS. FOR $\frac{\Omega-\omega}{\Gamma}=1$, THE STABLE FIXED POINTS MOVE TO $\varphi=0$ AND $\varphi=\pi$ WHILE IF $\frac{\Omega-\omega}{\Gamma}=-1$ WE HAVE $\varphi=-\pi/2$ AND $\varphi=\pi/2$	
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THUS THE STABLE FIXED POINTS ARE IN THE RANGE

$$\boxed{-\frac{\pi}{2} < \varphi < 0 \quad \text{AND} \quad \frac{\pi}{2} < \varphi < \pi.}$$

4

UNSEEN

(b)

WE CONSIDER

$$\varphi_1 = \theta_2 - \theta_1$$

AND

$$\varphi_2 = \theta_3 - \theta_2$$

WHICH OBEY

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Parts		

$$\frac{d\varphi_1}{dt} = \Omega_1 + \Gamma \cos\varphi_2$$

$$\frac{d\varphi_2}{dt} = \Omega_2 - \Gamma \cos\varphi_1$$

WHERE $\Omega_1 = \omega_L - \omega_1$, $\Omega_2 = \omega_3 - \omega_2$

$$\frac{d}{dt}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix} + \begin{bmatrix}0 & \Gamma\\ -\Gamma & 0\end{bmatrix}\begin{bmatrix}\cos\varphi_1\\ \cos\varphi_2\end{bmatrix}$$

STEADY PHASE DIFFERENCE

$$0 = \begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix} + \begin{bmatrix}0 & \Gamma\\ -\Gamma & 0\end{bmatrix}\begin{bmatrix}\cos\varphi_1\\ \cos\varphi_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos\varphi_1\\ \cos\varphi_2\end{bmatrix} = \frac{1}{\Gamma^2}\begin{bmatrix}0 & -\Gamma\\ \Gamma & 0\end{bmatrix}\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}$$

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Question 5		Marks & seen/unseen
Parts	THUS	
	$\cos\varphi_1 = -\frac{\Omega_1}{\Gamma}$	
	$\cos\varphi_2 = \frac{\Omega_2}{\Gamma}$	
	WE REQUIRE THEN	
	$\left\|\frac{\Omega_1}{\Gamma}\right\| \leq 1$ AND	
	$\left\|\frac{\Omega_2}{\Gamma}\right\| \leq 1.$	4
(ii)	IF $\omega_1 = \omega_2 = \omega_3$	UNSEEN
	THEN $\Omega_1 = \Omega_2 = 0.$	

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Parts

THEREFORE THE STEADY PHASE DIFFERENCES ARE GIVEN BY

$$\cos\varphi_1 = 0$$

$$\cos\varphi_2 = 0$$

$$\boxed{\text{THUS, } (\varphi_1, \varphi_2) = \left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \left(\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right), \left(-\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right)}$$

STABILITY

$$\frac{d\varphi_1}{dt} = \Gamma\cos\varphi_2$$

$$\frac{d\varphi_2}{dt} = -\Gamma\cos\varphi_1$$

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Parts		

$$J(\varphi_1, \varphi_2) = \begin{bmatrix} 0 & -\Gamma \sin\varphi_2 \\ \Gamma\sin\varphi_1 & 0 \end{bmatrix}$$

EIGENVALUES,

$$\lambda^2 + \Gamma^2 \sin\varphi_2 \sin\varphi_1$$

$$\lambda_\pm = \pm \Gamma \sqrt{-\sin\varphi_2 \sin\varphi_1}$$

THUS, FOR

$$\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ AND } \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$$

$$\lambda_\pm = \pm i\Gamma$$

$\boxed{\text{CENTRE.}}$ NEUTRALLY STABLE

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Parts		

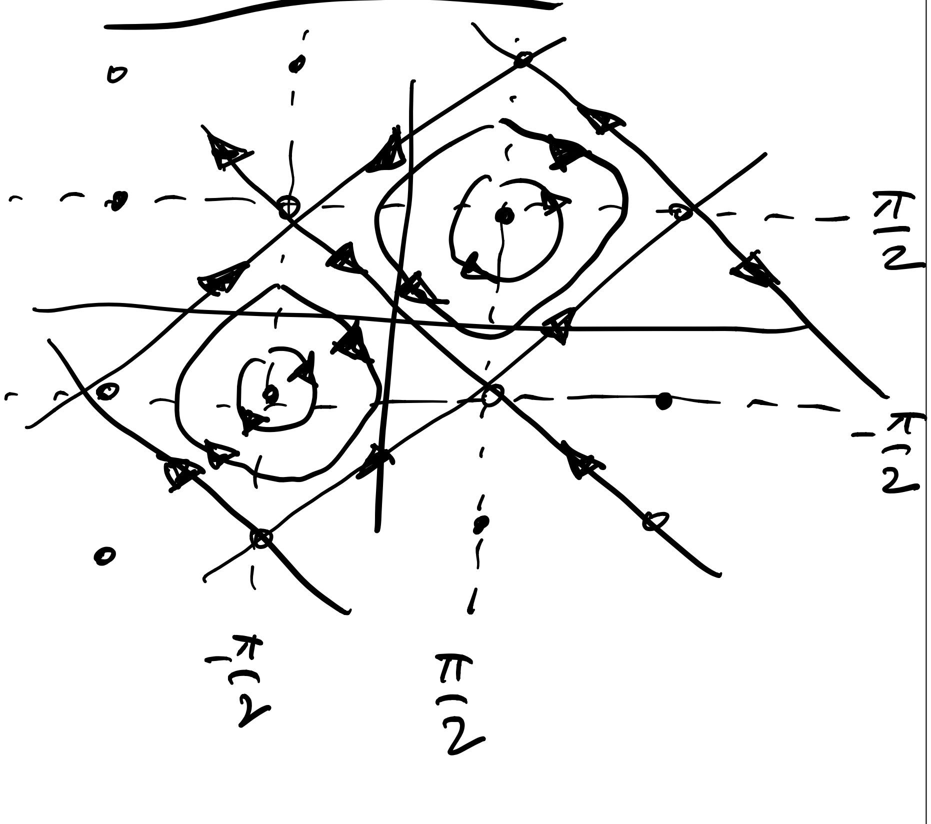
FOR

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ AND $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$

$\lambda_{\pm} = \pm \Gamma$

SADDLE , UNSTABLE

PHASE PORTRAIT



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If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96006 MATH97018 MATH97096	1	This question was found challenging by most students. Many were able to find the stable fixed point in the first part of the question, but solutions began to degrade after that. Very few students made it to the end and were able to provide a correct wave speed.
MATH96006 MATH97018 MATH97096	2	Responses improved for this question which was somewhat more standard. Most students correctly identified the fixed points and were able to provide some, if not all, of the stability analysis. The final part of the problem was found challenging and only very few students were able to provide a correct response.
MATH96006 MATH97018 MATH97096	3	Again, responses for most of this question were good. By and large the conditions for a Turing instability were identified by most students. Some students were scared off by the second part of the problem, but those that did attempt it did well on finding k^2. Correct responses for the final part, however, were limited.
MATH96006 MATH97018 MATH97096	4	For this problem, many students provided a correct response of how to derive the forward Kolmogorov equation, but some students missed steps going from the transfer probabilities to the forward Kolmogorov equation provided. Most students attempted deriving the mean, but some made errors withthe indices along the way, or missed that the equation is different from the deterministic one.

MATH96006 MATH97018 MATH97096	5	For the mastery, for the first part regarding the Fitzhugh-Nagumo equations, there were some correct answers, but many incorrect ones where b = \gamma was assumed. For part a, the condition on the parameters were widely obtained, but the range of stable \phi proved more problematic. This extended to b where many students correctly found the conditions, but then did not successfully analyse the fixed points.