1 Topics: Elementary set theory, the sample space, counting

1.1 Prerequisites: Lecture 1

Exercise 1- 1: (Suggested for personal/peer tutorial) Let A, B and C be three arbitrary events. Which of the following relationships are true? Justify your answers.

(a) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

(b) $(A \cup B) = (A \cap B^c) \cup B$.

(c) $(A^c \cap B) \cup (A \cap B^c) = (A \cup B) \cap (A \cap B)^c$.

(d) $(A \cup B)^c \cap C = A^c \cap B^c \cap C^c$.

(e) $(A \cap B) \cap (B^c \cap C) = \emptyset$.

Solution:

(a) TRUE: This is the distributive law.

(b) TRUE: We note that $\Omega = B \cup B^c$ and $A \cup B = (A \cup B) \cap \Omega$. Hence, by the distributive law, we have

$$(A \cap B^c) \cup B \overset{\text{distributivity}}{=} (A \cup B) \cap (B^c \cup B) = (A \cup B) \cap \Omega = A \cup B.$$

(c) TRUE: We apply the distributive law and De Morgan's law. Also, we note that, as above, $A \cup A^c = \Omega$ and $B \cup B^c = \Omega$. Hence

$$\begin{aligned}
&(A^c \cap B) \cup (A \cap B^c) \\
&\overset{\text{distributivity}}{=} [(A^c \cap B) \cup A] \cap [(A^c \cap B) \cup B^c] \\
&\overset{\text{distributivity}}{=} [(A^c \cup A) \cap (B \cup A)] \cap [(A^c \cup B^c) \cap (B \cup B^c)] \\
&\overset{A \cup A^c = \Omega, B \cup B^c = \Omega}{=} (B \cup A) \cap (A^c \cup B^c) \\
&\overset{\text{De Morgan}}{=} (A \cup B) \cap (A \cap B)^c.
\end{aligned}$$

Note that we sometimes use the *set difference* for the above expression and write

$$(A \cup B) \cap (A \cap B)^c = (A \cup B) \setminus (A \cap B).$$

(d) FALSE: $\omega \in (A \cup B)^c \cap C \implies \omega \in C$, but $\omega \in A^c \cap B^c \cap C^c \implies \omega \in C^c$ (CONTRADICTION).

(e) TRUE: $\omega \in (A \cap B) \cap (B^c \cap C) \implies \omega \in B$ and $\omega \in B^c \implies$ no such ω exists.

Exercise 1- 2: A coin is tossed three times. Let A be the event that there are exactly two heads, B the event that there are more heads than tails, and C the event that the last toss is a tail. Using the operations of union, intersection and complement, find in terms of A, B and C expressions for the events:

(a) There are more tails than heads.

(b) There are three heads.

(c) The first two tosses are heads.

Solution: We write e.g. HHT for the event head-head-tail etc. Then we have

$$A = \{HHT, HTH, THH\},$$
$$B = A \cup \{HHH\},$$
$$C = \{HHT, HTT, THT, TTT\}.$$

(a) B^c

(b) $A^c \cap B$

(c) $(A \cap C) \cup (A^c \cap B)$

Exercise 1- 3: A computer hardware company manufactures five apparently identical terminals, two of which are actually defective. An order for two terminals is received, and is filled by selecting two of the five.

(a) List the elements of the sample space corresponding to how the order is filled.

(b) Let A denote the event that the order is filled with two non-defective terminals. List the sample points in A.

Solution: Label the five terminals: $\underbrace{D_1, D_2,}_{\text{defective}} \underbrace{N_1, N_2, N_3.}_{\text{not defective}}$

(a)

$$\Omega = \{\{D_1, D_2\}, \{D_1, N_1\}, \{D_1, N_2\}, \{D_1, N_3\}, \{D_2, N_1\}, \{D_2, N_2\}, \{D_2, N_3\}, \{N_1, N_2\}, \{N_1, N_3\}, \{N_2, N_3\}\}$$

(b) $A = \{\{N_1, N_2\}, \{N_1, N_3\}, \{N_2, N_3\}\}$

1.2 Prerequisites: Lecture 2

Exercise 1- 4: Consider a horse race with 10 horses. We assume that all 10 horses will complete the race and that there are no ties. How many combinations are there for the first, second and third place winners?

Solution: For the first place, there are 10 possibilities, for the second place –after the first place has been decided – there are 9 possibilities left, and there are 8 possibilities for the third place. So by the multiplication principle there are $10 \cdot 9 \cdot 8 = 720$ combinations in total.

Exercise 1- 5: Consider buying one scoop of ice cream. You can choose either a cup or a cone and there are three possible flavours: chocolate (C), vanilla (V) and strawberry (S). Draw two possible trees to visualise the number of possibilities.

Solution: We have $2 \cdot 3 = 3 \cdot 2 = 6$ possibilities and the trees are given as follows:

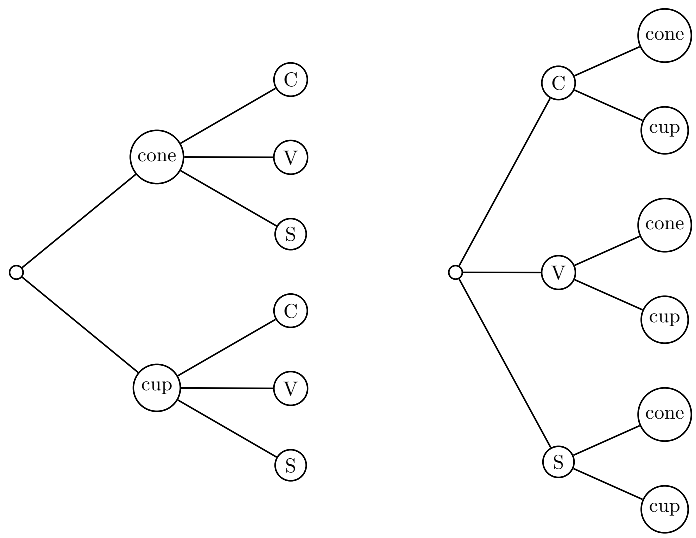

Figure 1: Illustration of the multiplication principle in the case of our ice cream example.

1.3 Prerequisites: Lecture 3

Exercise 1- 6: Revisit Exercise 1- 4 and describe how you can solve it using the idea of sampling without replacement.

Solution: Using the analogy of drawing balls from an urn, we can view the ten horses as ten labelled balls in the urn. We then draw subsequently. For the first draw, there are 10 balls, so 10 possible outcomes. For the second one, we have 9 balls left and for the third 8, which leads to $(10)_3 = 10 \cdot 9 \cdot 8 = 720$ possibilities.

Exercise 1- 7: Suppose we roll two fair dice and we would like to determine the probabilities of a sum of 11 and a sum of 12. Leibniz argued that both events are equally likely since each outcome can only be obtained in one way. Do you agree with Leibniz? Justify your answer!

Solution: It turns out that Leibniz's argument was flawed: While there is only one way of achieving a sum of 12 (when rolling $(6, 6)$), we can obtain a sum of 11 in two possible ways: $(5, 6)$ and $(6, 5)$. Hence the probability of getting a sum of 12 is given by $\frac{1}{6\cdot 6} = \frac{1}{36}$ and the probability of getting a sum of 11 is given by $\frac{2}{6\cdot 6} = \frac{1}{18}$, which are clearly not the same.