Definition: Asymptotic $1-\alpha$ confidence interval

[redacted]

A sequence of random intervals I_n is called an asymptotic $1-\alpha$ CI for θ if

$$\lim_{n\to\infty} P_\theta(\theta \in I_n) \geq 1-\alpha \quad \forall \theta \in \Theta.$$

If $\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \xrightarrow{d} N(0,1)$, then (approximately)

$$\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \dot{\sim} N(0,1)$$

and we can use the LHS as a pivotal quantity.

Simplification

Suppose $\hat{\sigma}_n$ is consistent for $\sigma(\theta)$. Thus, $\hat{\sigma}_n \xrightarrow{P_\theta} \sigma(\theta)$ for all θ.

By Slutsky's lemma and the fact that $X \sim N(0, \sigma^2(\theta))$ implies $X/\sigma(\theta) \sim N(0,1)$,

$$\underbrace{\frac{\sigma(\theta)}{\hat{\sigma}_n}}_{\to 1} \cdot \underbrace{\frac{\sqrt{n}\,(T_n - \theta)}{\sigma(\theta)}}_{\to N(0,1)} = \frac{\sqrt{n}\,(T_n-\theta)}{\hat{\sigma}_n}$$

$$\sqrt{n}\frac{T_n - \theta}{\hat{\sigma}_n} \xrightarrow{d} N(0,1).$$

Using the LHS as the pivotal quantity leads to the approximate confidence limits

$$T_n \pm c_{\alpha/2}\hat{\sigma}_n/\sqrt{n}$$

where $\Phi(c_{\alpha/2}) = 1 - \alpha/2$.

Under mild regularity conditions, the quantity $\hat{\sigma}_n/\sqrt{n}$ estimates $\mathrm{SE}(T_n)$. Hence:

$$T_n \pm c_{\alpha/2}\,\mathrm{SE}(T_n).$$

Example: $Y \sim \text{Binomial}(n, \theta)$, $\theta \in (0,1)$ unknown

BY CLT

$\sqrt{n}\frac{Y/n-\theta}{\sqrt{\theta(1-\theta)}}$ is approx. $N(0,1)$, so

$\Phi(c_{\alpha/2}) = 1 - \frac{\alpha}{2}$

$$P(-c_{\alpha/2} \leq \underbrace{\frac{Y - n\theta}{\sqrt{n\theta(1-\theta)}}} \leq c_{\alpha/2}) \approx 1 - \alpha$$

$\frac{Y}{n} \xrightarrow{P} \theta$

BY CMT

$\sqrt{\frac{Y}{n}\left(1-\frac{Y}{n}\right)} \xrightarrow{P} \sqrt{\theta(1-\theta)}$

Using the (asymptotic) pivotal quantity

$$\sqrt{n}\frac{Y/n - \theta}{\sqrt{\frac{Y}{n}(1 - \frac{Y}{n})}}$$

leads to the confidence limits

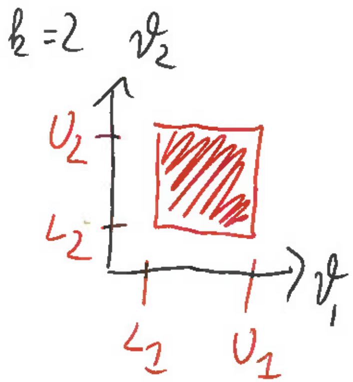
$$\underline{\frac{y}{n} \pm \frac{c_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{y}{n}(1 - \frac{y}{n})}}$$

Simultaneous confidence intervals

Extension of CIs to ≥ 1 parameter

Suppose $\theta = (\theta_1, \ldots, \theta_k)^t \in \Theta \subset \mathbb{R}^k$ and we have $(L_i(Y), U_i(Y))$ such that

$$\forall \theta : \quad P_\theta(L_i(Y) < \theta_i < U_i(Y) \text{ for } i = 1, \ldots, k) \geq 1 - \alpha$$



then

$$\{ \ (L_i(y), U_i(y)), \qquad i = 1, \ldots, k \ \}$$

is a $1 - \alpha$ **simultaneous confidence interval** for $\theta_1, \ldots, \theta_k$.

Can we construct simultaneous confidence intervals from one-dimensional confidence intervals?

The Bonferroni Correction

[redacted]

Suppose $[L_i, U_i]$ *is a* $1-\alpha/k$ *confidence interval for* θ_i, $i = 1, \ldots, k$.

Then $[(L_1, \ldots, L_k)^t, (U_1, \ldots, U_k)^t] = (L_1, U_1) \times \cdots \times (L_k, U_k)$ *is a* $1-\alpha$ *simultaneous confidence interval for* $(\theta_1, \ldots, \theta_k)^t$.

Proof

$P(A \cup B) \leq P(A) + P(B)$

$$P(\theta_i \in [L_i, U_i], i = 1, \ldots, k) = 1 - P\left(\bigcup_{i=1}^{k} \{\theta_i \notin [L_i, U_i]\}\right) \geq 1 - \sum_{i=1}^{k} \underbrace{P(\theta_i \notin [L_i, U_i])}_{\leq \alpha/k} \geq 1 - \alpha.$$

Example: different coverage probabilities

$\alpha_1 = 0.01$ $\alpha_2 = 0.03$

Suppose $[L_1, U_1]$ is a 99% confidence interval for θ_1 and $[L_2, U_2]$ is a 97% confidence interval for θ_2. Then $[L_1, U_1] \times [L_2, U_2]$ is a 96% simultaneous confidence interval for the parameter vector (θ_1, θ_2).

$$P(\theta_i \in [L_i, U_i],\ i=1,2) \geq 1 - \alpha_1 - \alpha_2 = 0.96$$

Example: Bonferroni corrections are conservative

Suppose $X_1, \ldots, X_n \sim N(\mu, 1)$, $Y_1, \ldots, Y_n \sim N(\theta, 1)$ independent with (μ, θ) being the unknown parameter.

One-dimensional CIs:

$$I = (\bar{X} - c_{\alpha/2}/\sqrt{n}, \bar{X} + c_{\alpha/2}/\sqrt{n}) \qquad J = (\bar{Y} - c_{\alpha/2}/\sqrt{n}, \bar{Y} + c_{\alpha/2}/\sqrt{n})$$

for $\Phi(c_{\alpha/2}) = 1 - \alpha/2$, are $1 - \alpha$ confidence intervals for μ and θ

Bonferroni correction: $I \times J$ is a $\underline{1 - 2\alpha}$ confidence region for (μ, θ). $\Rightarrow \quad = 0.80$

Actual coverage probability: I and J are independent, thus for $I \times J$

$$\underline{P_{(\mu,\theta)}((\mu, \theta) \in I \times J)} = \underline{P_{(\mu,\theta)}(\mu \in I)P_{(\mu,\theta)}(\theta \in J)} = \underline{(1 - \alpha)^2}. \quad = 0.81$$

BONF. TELLS YOU THAT $P_{(\mu,\theta)}((\mu,\theta) \in I \times J) \geq 1 - 2\alpha = 0.80$

For $\alpha = 0.1$, Bonferroni guarantees coverage probability of 80%, whereas the actual probability is $0.9^2 = 0.81$.

Looking ahead

We continue to work with and generalize CIs as we look toward **hypothesis testing** and then **linear models**.

Imperial College London

Lecture 09: Hypothesis Testing

Statistical Modelling I

Dr. Riccardo Passeggeri

Outline

Introduction

Motivation

- **(Point) estimator**: one number only (does not reflect uncertainty)
- **Confidence interval**: random interval that contains the true parameter with a certain probability
- **Hypothesis test**: decision rule to choose between one of two statements about the true parameter

Definition: Null Hypothesis and Alternative Hypothesis, Hypothesis Test and Rejection Region

- The two complementary hypotheses in a hypothesis testing problem are called the **null hypothesis** and the **alternative hypothesis**, denoted by H_0 and H_1, respectively.
- A **hypothesis test** is a rule that specifies for which values of the sample $X_1, \dots, X_n$ the decision is made to accept H_0 as true and for which values to reject H_0 and accept H_1 as true.
- The **rejection region** or **critical region** is a set of values for the test statistic for which H_0 is rejected. *i.e.* if the observed test statistic is in the critical region then we reject H_0 and accept H_1.

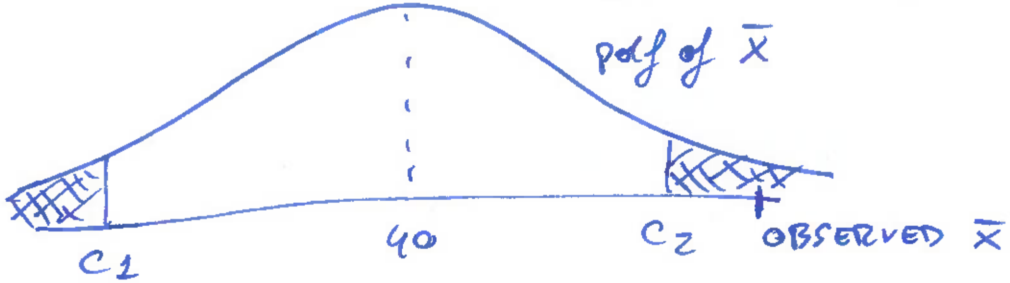
$X_1, \dots, X_{12} \overset{iid}{\sim} N(\mu, 1.21^2)$

$H_0: \mu = 40 \qquad H_1: \mu \neq 40$

$X_1, \dots, X_{12} \overset{iid}{\sim} N(40, 1.21^2)$

$\bar{X} \sim N\left(40, \frac{1.21^2}{12}\right)$

$\left[\frac{\sqrt{12}(\bar{X} - 40)}{1.21} \sim N(0,1) \right]$



$R = (-\infty, c_1] \cup [c_2, \infty)$

H_0 AND H_1 LEAD TO Θ_0 AND Θ_1 S.T.

$\Theta_0 \cup \Theta_1 = \Theta$ AND $\Theta_0 \cap \Theta_1 = \emptyset$

IN THIS EXAMPLE $\Theta_0 = \{40\}$

Two Types of Errors

	H_0 true	H_0 false
do not reject H_0	✓	Type II error
reject H_0	Type I error	✓

Level of a test: A test is of level α $(0 < \alpha < 1)$ if

$$P_\theta(\text{reject } H_0) \leq \alpha \quad \forall \theta \in \Theta_0. \quad \Longleftrightarrow P_\theta(Y \in R) \leq \alpha \quad \forall \theta \in \Theta_0$$

Usually α is small, e.g. 0.01 or 0.05.
Loosely speaking: the probability of a type I error is less than α.
There is no such bound for the probability of a type II error.

$\cancel{R = \bigcup_{\theta \in \Theta_0} R_\theta}$ $\qquad$ $R = \bigcap_{\theta \in \Theta_0} R_\theta$

Power of a Test

Definition: Power

Setup: Θ parameter space, $\Theta_0 \subset \Theta$, $\Theta_1 = \Theta \setminus \Theta_0$. Consider

$$H_0 : \theta \in \Theta_0 \text{ v.s. } H_1 : \theta \in \Theta_1$$

Suppose we have some test for this hypothesis.
The *power function* is defined as the mapping

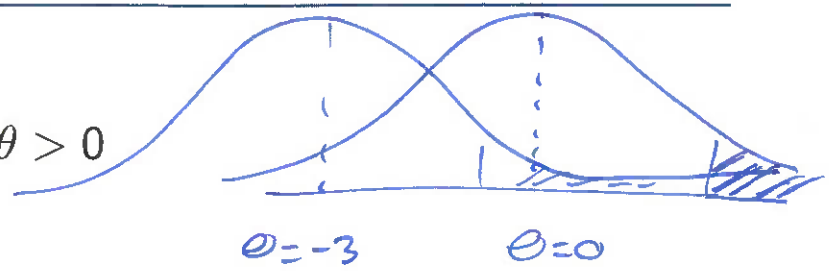
$$\beta : \Theta \to [0,1], \beta(\theta) = \underline{P_\theta(\text{reject } H_0)} = P_\theta(Y \in R)$$

If $\theta \in \Theta_0$ then we want $\beta(\theta)$ to be small.
If $\theta \in \Theta_1$ then we want $\beta(\theta)$ to be large.

$\theta \in \Theta$

Example: $X \sim N(\theta, 1)$, $\theta \in \mathbb{R}$ unknown

$$H_0 : \theta \leq 0 \quad \text{against} \quad H_1 : \theta > 0$$



Level α test

$$\Theta = \mathbb{R}, \Theta_0 = (-\infty, 0], \Theta_1 = (0, \infty)$$

Rejection region

$$\boxed{R = [c, \infty)}$$

Choose c s.t. $\Phi(c) = 1 - \alpha$. Then

$$P_\theta(\text{reject } H_0) = P_\theta(X \geq c)$$

$P_{\theta=0}(\text{reject } H_0) = \alpha$

Power of the test