Imperial College London

MATH70021

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Advanced Topics in Partial Differential Equations

Date: 19 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Simplify the following expressions containing δ-functions:

$$1)\ \ x\delta(x), \quad 2)\ \ x\delta'(x), \quad 3)\ \ x^2\delta'(x), \quad 4)\ \ x^2\delta''(x).$$

Hint: Use the mathematically rigorous definition of $\delta(x) \in D'(\mathbb{R})$, not the "physical" one.

(4 marks)

(b) Prove the Leibnitz rule

$$(\psi(x)l(x))' = \psi(x)'l(x) + \psi(x)l(x)'$$

for $\psi(x) \in C^\infty(\mathbb{R})$ and $l \in D'(\mathbb{R})$. (4 marks)

(c) Prove that the general solution for the equation

$$xl(x) = 1, \quad l(x) \in D'(\mathbb{R})$$

is given by the expression

$$l = V.p.\frac{1}{x} + C\delta(x),$$

where $\delta(x)$ is the Dirac δ-function and C is an arbitrary constant. (6 marks)

(d) Find a general solution of the ODE

$$xl(x)' + l(x) = 0, \quad l \in D'(\mathbb{R}).$$

(6 marks)

(Total: 20 marks)

2. (a) Find the spectrum of $\mathcal{A}_1 := -\partial_x^2 + \partial_x$ in $L^2(\mathbb{R})$. (6 marks)

(b) Find the spectrum of $\mathcal{A}_n := -\Delta + \partial_{x_1}$ in $L^2(\mathbb{R}^n)$, $n \geq 2$. (8 marks)

(c) Prove that the point spectrum of $\mathcal{A}_1$ is empty. (6 marks)

(Total: 20 marks)

3. Consider the following problem

$$\begin{cases} -\Delta u + u - a(x)v = f_1, \\ -\Delta v + v + a(x)u = f_2, \\ u\Big|_{\partial\Omega} = v\Big|_{\partial\Omega} = 0, \end{cases}$$

where $a(x), f_1(x), f_2(x) \in C^\infty(\bar{\Omega})$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

(a) Using the Lax-Milgram theorem, prove the existence and uniqueness of a weak solution $u, v \in H_0^1(\Omega)$ for this problem. (8 marks)

(b) Assuming that u, v are smooth, prove that the function $w(x) = u^2(x) + v^2(x)$ satisfies

$$-\Delta w + w \leq f_1^2(x) + f_2^2(x).$$

(5 marks)

(c) Show that

$$u^2(x) + v^2(x) \leq \max_{x\in\bar{\Omega}}\{f_1^2(x) + f_2^2(x)\}$$

for any $x \in \bar{\Omega}$. (7 marks)

(Total: 20 marks)

4. Let $\Omega = (0, 1)$ and $u \in W^{1,1}(\Omega)$ with $u(0) = 0$.

(a) Prove that

$$\|u\|_{C(\bar{\Omega})} \leq \|u'\|_{L^1(\Omega)}. \tag{1}$$

(5 marks)

(b) Check that (1) is sharp and find the extremals for it. (5 marks)

(c) Let now $u \in W_0^{1,1}(\Omega)$. Prove that

$$\|u\|_{C(\bar{\Omega})} \leq \frac{1}{2}\|u'\|_{L^1(\Omega)}. \tag{2}$$

(5 marks)

(d) Prove that $\frac{1}{2}$ is the best constant in (2).

(5 marks)

(Total: 20 marks)

5. Let us consider the heat equation

$$\partial_t u = \Delta u, u\Big|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary; and $u(t, \cdot) \not\equiv 0$ for all $t \geq 0$. Let also

$$y(t) = \log \|u(t)\|_{L^2}^2.$$

(a) Find $y'(t)$. Express them in terms of u and its spatial derivatives only.

(4 marks)

(b) Find $y''(t)$. Express them in terms of u and its spatial derivatives only.

(7 marks)

(c) Prove that $y(t)$ is convex. (4 marks)

(d) Show that

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2}^{t/T} \|u(T)\|_{L^2}^{\frac{T-t}{T}}$$

holds for all $T > 0$ and $0 \leq t \leq T$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60021/70021

Advanced Topics in Partial Differential Equations (Solutions)

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1. (a) For any function $\phi(x) \in D(\mathbb{R})$, we have

sim. seen ⇓

1) $\langle x\delta(x), \phi(x)\rangle = \langle \delta(x), x\phi(x)\rangle = 0$, i.e., $x\delta(x) = 0$; 1, A

2) $\langle x\delta'(x), \phi(x)\rangle = \langle \delta'(x), x\phi(x)\rangle = -\langle \delta(x), \phi(x) + x\phi'(x)\rangle = -\phi(0)$, i.e., $x\delta'(x) = -\delta(x)$; 1, A

3) $\langle x^2\delta'(x), \phi(x)\rangle = \langle \delta'(x), x^2\phi(x)\rangle = -\langle \delta(x), 2x\phi(x) + x^2\phi'(x)\rangle = 0$, i.e., $x^2\delta'(x) = 0$; 1, A

4) $\langle x^2\delta''(x), \phi(x)\rangle = \langle \delta''(x), x^2\phi(x)\rangle = \langle \delta(x), 2\phi(x) + 4x\phi'(x) + x^2\phi''(x)\rangle = 2\phi(0)$, i.e., $x^2\delta''(x) = 2\delta(x)$. 1, A

(b) Let us take an arbitrary function $\phi(x) \in D(\mathbb{R})$. Then by the definition of the weak derivative the left-hand side is equal to

unseen ⇓

$$\langle (\psi(x)l(x))', \phi(x)\rangle = -\langle \psi(x)l(x), \phi'(x)\rangle = -\langle l(x), \psi(x)\phi'(x)\rangle,$$

whereas the right-hand side gives

$$\langle \psi'(x)l(x), \phi(x)\rangle + \langle \psi(x)l'(x), \phi(x)\rangle =$$
$$\langle l(x), \psi'(x)\phi(x)\rangle - \langle l(x), (\psi(x)\phi(x))'\rangle = -\langle l(x), \psi(x)\phi'(x)\rangle.$$

Thus, the RHS=LHS as required. 4, A

seen ⇓

(c) The naive solution $l(x) = 1/x$ is *wrong* since $1/x$ does not belong to $L^1_{loc}(x)$ and does not define a distribution. So, we need to proceed in a more accurate way. Let $l \in D'(\mathbb{R})$ be a solution of this equation. This means

$$\langle l, x\varphi\rangle = \int_{\mathbb{R}} \varphi(x)\, dx.$$

From here we see that if $\psi \in D(\mathbb{R})$ is such that $\psi(0) = 0$, then $\varphi(x) := \psi(x)/x \in D(\mathbb{R})$ and

$$\langle l, \psi\rangle = \int_{\mathbb{R}} \frac{\psi(x)}{x}\, dx, \quad \psi(0) = 0$$

For the case $\psi(0) \neq 0$, we fix some function $\tilde{\psi}(x) \in D(\mathbb{R})$ satisfying $\tilde{\psi}(0) = 1$. For instance, we may take an even function with such a property. Then, the function $\psi - \psi(0)\tilde{\psi}$ vanishes at zero and we know how y acts on it and, therefore, for arbitrary $\psi \in D(\Omega)$, we have

$$\langle l, \psi\rangle = \langle l, \psi - \psi(0)\tilde{\psi}\rangle + \psi(0)\langle l, \tilde{\psi}\rangle = \int_R \frac{\psi(x) - \psi(0)\tilde{\psi}(x)}{x}\, dx + C\psi(0)$$

and, therefore, $l = V.p.\,\frac{1}{x} + C\delta(x)$. 6, A

(d) First we notice that

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$$xl'(x) + l(x) = (xl(x))'$$

and consequently $xl(x) = \text{const}$. The part c) of this question gives us the general solution 3, B

$$l(x) = C_1 V.p.\,\frac{1}{x} + C_2\delta(x),$$

where C_1 and C_2 are constants. 3, A

2. (a) By definition, $\lambda \in \mathbb{C}$ belongs to the spectrum of $\mathcal{A}_1$ if the operator $(\mathcal{A}_1 - \lambda)^{-1}$ does not exist. Or, in other words, when the equation $\mathcal{A}_1 u - \lambda u = g$ is not uniquely solvable in $L^2(\mathbb{R})$ for all $g \in L^2(\mathbb{R})$. To solve this equation, we use the Fourier transform and write out it in the equivalent form $\boxed{\text{meth seen} \Downarrow}$

$$(|\xi|^2 + i\xi - \lambda)\hat{u}(\xi) = \hat{g}(\xi), \quad \text{or} \quad \hat{u}(\xi) = \frac{1}{(|\xi|^2 + i\xi - \lambda)}\hat{g}(\xi) := \frac{\hat{g}(\xi)}{K_\lambda(\xi)}$$

By Plancherel equality, $\|u\|_{L^2} \le C\frac{1}{\inf_{\xi\in\mathbb{R}}|K_\lambda(\xi)|}\|g\|_{L^2}$. So, the equation is uniquely solvable if this infimum is greater than zero. In this case $\lambda \notin \sigma(\mathcal{A}_1)$. Since $|K_\lambda(\xi)| \to \infty$ as $\xi \to \infty$ for every fixed λ, then the infimum is actually a minimum and is attained in some finite point ξ. It is non-zero if and only if $K_\lambda(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Thus, the spectrum lies on the curve $K_\lambda(\xi) = 0$, $\xi \in \mathbb{R}$. This curve is a parabola. Indeed

$$K_\lambda(\xi) = 0 \;\Leftrightarrow\; \operatorname{Re}\lambda = \xi^2,\ \operatorname{Im}\lambda = \xi \;\Leftrightarrow\; \operatorname{Re}\lambda = (\operatorname{Im}\lambda)^2.$$

Any point on this parabola belongs to the spectrum since $\frac{1}{K_\lambda(\xi)}$ has a pole at any point of this parabola, so $\hat{u} \in L^2$ not for all $g \in L^2$ for any such λ. Finally the spectrum is the parabola $x = y^2$ on a complex plane. $\boxed{6, \text{B}}$

(b) We again do the Fourier transform and get the equivalent equation $\boxed{\text{meth seen} \Downarrow}$

$$(|\xi|^2 + i\xi_1 - \lambda)\hat{u}(\xi) = \hat{g}(\xi).$$

By the same reasons as in Part (a),

$$\sigma(\mathcal{A}_n) = \{\lambda \in \mathbb{C},\ |\xi|^2 + i\xi_1 - \lambda) = 0 \ \text{ for some } \ \xi \in \mathbb{R}^n\}$$

Thus, $x = \operatorname{Re}\lambda = |\xi|^2 = \xi_1^2 + |\xi'|^2$, $y = \operatorname{Im}\lambda = \xi_1$ and

$$x = y^2 + |\xi'|^2,$$

where $|\xi'|^2 = \xi_2^2 + \dots + \xi_n^2$. The boundary of the spectrum is the same parabola as in Part 1, but the presence of an arbitrary nonnegative term $|\xi'|^2$ gives all the points satisfying $x \ge y^2$ and this is exactly *the interior* of the parabola. $\boxed{8, \text{D}}$

(c) For the point spectrum, we need to solve the eigenvalue problem $\mathcal{A}_1 u = \lambda u$ in $L^2(\mathbb{R})$ or which is the same as the equation $\boxed{\text{sim. seen} \Downarrow}$

$$(\xi^2 + i\xi - \lambda)\hat{u}(\xi) = 0$$

for $\hat{u} \in L^2(\mathbb{R})$. Thus, $\hat{u}(\xi) = 0$ for all ξ such that $K_\lambda(\xi) \neq 0$. But, for every fixed $\lambda \in \mathbb{C}$, we have at most one point ξ (if λ is on the parabola) where $K_\lambda(\xi) = 0$. Thus, $\hat{u}(\xi) = 0$ almost everywhere which gives zero function in $L^2(\mathbb{R})$ and the point spectrum is absent. $\boxed{6, \text{B}}$

3. (a) To apply the Lax-Milgram Theorem, we introduce the bilinear form in the space $H_0^1(\Omega)^2$, $\{u,v\} \in H_0^1(\Omega)^2$, via meth seen ⇓

$$[\{u,v\},\{\varphi,\psi\}] := \int_\Omega (\nabla u\cdot\nabla\varphi + \nabla v\cdot\nabla\psi + u\varphi + v\psi - av\phi + au\psi)\,dx$$

and the linear functional $l(\{\varphi,\psi\}) = \int_\Omega (f_1\varphi + f_2\psi)\,dx$ on this space. Then the weak formulation of our problem reads: to find $\{u,v\}\in H_0^1(\Omega)^2$ such that

$$[\{u,v\},\{\varphi,\psi\}] = l(\{\varphi,\psi\})$$

for all test functions $\{\varphi,\psi\}$. Let us check the conditions of LMT. Since the functions f_1, f_2 are smooth, in particular in $L^2(\Omega)$, the continuity of the linear functional l is obvious. Let us check the boundedness of the form $[\cdot,\cdot]$. This follows in a straightforward way from the Cauchy-Schwarz:

$$\begin{aligned}
|[\{u,v\},\{\varphi,\psi\}]| \le \|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2} + \|u\|_{L^2}\|\varphi\|_{L^2} + \|\nabla v\|_{L^2}\|\nabla\psi\|_{L^2} + \\
+ \|v\|_{L^2}\|\psi\|_{L^2} + \|a\|_{L^\infty}\|v\|_{L^2}\|\varphi\|_{L^2} + \|u\|_{L^2}\|\psi\|_2 \le \\
\le C(\|u\|_{H_0^1} + \|v\|_{H_0^1})(\|\varphi\|_{H_0^1} + \|\psi\|_{H_0^1})
\end{aligned}$$

and the boundedness is proved. Let us verify the coercivity. The terms containing a cancel out and we get

$$[\{u,v\},\{u,v\}] = \|u\|^2_{H_0^1} + \|v\|^2_{H_0^1} = \|\{u,v\}\|^2_{H_0^1}.$$

Thus, all conditions are verified and by LMT, we have the unique solvability of our problem. 8, D

(b) The solution of our problem is smooth due to the elliptic regularity theorem, but you need not to prove this. Just assuming that everything is smooth we calculate the Laplacian of w unseen ⇓

$$\begin{aligned}
-\Delta w = -2u\Delta u - 2\nabla u\cdot\nabla u - 2v\Delta v - 2\nabla v\cdot\nabla v \le \\
\le 2u(f_1 + av - u) + 2v(f_2 - au - v) = -2(u^2+v^2) + 2(uf_1 + vf_2).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality an using that $xy \le \frac12 x^2 + \frac12 y^2$, we get the desired point-wise estimate: $-\Delta w + w \le f_1^2(x) + f_2^2(x)$. 5, C

(c) Let $V_f := \max_{x\in\bar\Omega}\{f_1(x)^2 + f_2(x)^2\}$. Then this constant satisfies meth seen ⇓

$$-\Delta V_f + V_f = V_f \ge f_1(x)^2 + f_2(x)^2,$$

so the function $z := w - V_f$ is a subsolution: $-\Delta z + z \le 0$, $z\big|_{\partial\Omega} \le 0$ and by the maximum principle, we have $w(x) \le V_f$ for all $x\in\bar\Omega$ and returning to u and v:

$$u^2(x) + v^2(x) \le \max_{x\in\bar\Omega}\{f_1^2(x) + f_2^2(x)\}$$

for any $x\in\bar\Omega$. 7, C

4. (a) We start with the function $u \in C_0^\infty[0,1]$ and use the Newton-Leibnitz formula with the fact that $u(0) = 0$: $\boxed{\text{sim. seen} \Downarrow}$

$$u(x) = \int_0^x u'(s)\,ds, \quad |u(x)| \leq \int_0^x |u'(s)|\,ds \leq \|u\|_{L^1(0,1)}.$$

Approximating any function $u \in W^{1,1}(0,1)$ with $u(0) = 1$ by smooth functions we get the desired inequality. $\boxed{5, \text{A}}$

(b) From the previous estimate, we see that it becomes the equality if and only if $u'(x)$ does not change sign. Thus, the extremals are exactly all absolutely continuous monotone functions vanishing at 0. In particular, the inequality becomes equality if $u(x) = x$. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{5, \text{A}}$

(c) We now have zero boundary conditions at both endpoints, so we may write the following version of the Newton-Leibnitz formula: $\boxed{\text{sim. seen} \Downarrow}$

$$u(x) = \frac{1}{2}\left(\int_0^x u'(s)\,ds - \int_x^1 u'(s)\,ds\right).$$

Then

$$|u(x)| \leq \frac{1}{2}\int_0^x |u'(s)|\,ds + \frac{1}{2}\int_x^1 |u'(s)|\,ds = \frac{1}{2}\|u'(s)\|_{L^1}.$$

$\boxed{5, \text{A}}$

(d) Let the maximum of $u(x)$ be attained at some point $x_0 \in (0,1)$. Then, the last inequality becomes equality if and only if the function $u(x)$ is monotone increasing for $x \leq x_0$ and monotone decreasing for $x \geq x_0$ (or $-u(x)$ satisfies this property). Any such a function which corresponds to any $x_0 \in (0,1)$ will be an extremal. In particular $u(x) = \frac{1}{2} - |x - \frac{1}{2}|$ will be an extremal. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{5, \text{B}}$

5. (a) We use the chain rule, the equation and integration by parts to compute these derivatives: unseen ⇓

$$y'(t) = \frac{d_t\|u(t)\|_{L^2}^2}{\|u(t)\|^2} = 2\frac{(\partial_t u, u)}{\|u(t)\|^2} = 2\frac{(\Delta u, u)}{\|u(t)\|^2} = -2\frac{\|\nabla u(t)\|_{L^2}^2}{\|u(t)\|_{L^2}^2}.$$

and 4, M

(b) unseen ⇓

$$y''(t) = -4\frac{(\nabla\partial_t u, \nabla u)}{\|u(t)\|_{L^2}^2} + 4\frac{\|\nabla u\|_{L^2}^2(\partial_t u, u)}{\|u(t)\|_{L^2}^4} = 4\frac{\|\Delta u\|_{L^2}^2}{\|u(t)\|_{L^2}^2} - 4\frac{\|\nabla u\|_{L^2}^4}{\|u(t)\|_{L^2}^4}.$$

7, M

(c) Integrating by parts, we have sim. seen ⇓

$$\|\nabla u\|^2 = (\nabla u, \nabla u) = -(\Delta u, u) \leq \|\Delta u\|_{L^2}\|u\|_{L^2}$$

Taking a square of this inequality and applying it to the last term in the RHS of the expression for $y''(t)$, we end up with the desired inequality $y''(t) \geq 0$. 4, M

(d) Since $y(t)$ is convex, we have $y(t) \leq \frac{t}{T}y(0) + \frac{T-t}{T}y(t)$ and taking the exponent, we end up with the desired inequality. meth seen ⇓

5, M

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60021/70021	1	No Comments Received
MATH60021/70021	2	No Comments Received
MATH60021/70021	3	No Comments Received
MATH60021/70021	4	No Comments Received
MATH70021	5	No Comments Received