Imperial College
London

MATH97008 MATH97088

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics 1

Date: 27th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. It is known that the Earth is not a perfect sphere. It is also known that the pressure in the ocean increases with depth much faster than it decreases in the atmosphere. Keeping this in mind, find the shape of the Earth by assuming that it may be thought of as a rotating volume of fluid surrounded by vacuum. The fluid is kept together through the action of the gravitational force. Assume that this force has only a radial component, which is proportional to the distance from the Earth's centre, namely

$$f_r = -\alpha r.$$

You need to perform the following tasks:

(a): Thanks to the fact that the fluid motion is symmetric with respect to the Earth's axis, it is convenient to use spherical polar coordinates (see Figure 1), where the Navier–Stokes equations are given by (2) on the next page. Place the x-axis along the axis of the Earth's rotation.

Assuming that the fluid rotates as a solid body, what are the velocity components V_r, V_ϑ and V_ϕ?

(3 marks)

(b): Given that the angular velocity of the Earth's rotation is Ω, and the Earth's radius at the North Pole is R_0, show that at any other meridional angle ϑ (measured from the North Pole), the distance R from the Earth's surface to the centre is given by

$$R = \frac{R_0}{\sqrt{1 - \dfrac{\Omega^2}{\alpha}\sin^2\vartheta}}. \tag{1}$$

(11 marks)

(c): Choose the plane surface S in Figure 1 to coincide with the (x, y) plane, and show that equation (1) may be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence, conclude that the Earth is an ellipsoid. What are its principal axes a and b?

(2 marks)

(d): For the Earth, Ω^2/α is small. If there is a planet for which Ω^2/α is not small, then which of the assumptions you have used will fail?

(4 marks)

(Total: 20 marks)

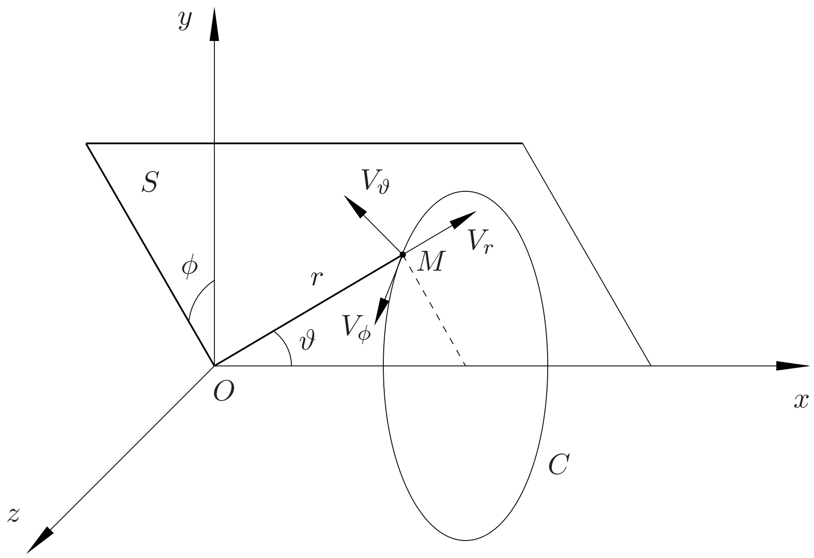

Figure 1: Spherical polar coordinates.

$$
\begin{aligned}
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_r}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_r}{\partial \phi} - \frac{V_\vartheta^2 + V_\phi^2}{r} &= f_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\
&+ \nu\biggl(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_r}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_r}{\partial r} \\
&+ \frac{1}{r^2\tan\vartheta}\frac{\partial V_r}{\partial \vartheta} - \frac{2}{r^2}\frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2}{r^2\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} - \frac{2V_r}{r^2} - \frac{2V_\vartheta}{r^2\tan\vartheta}\biggr), \qquad \text{(2a)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_\vartheta}{\partial t} + V_r \frac{\partial V_\vartheta}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_\vartheta}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_\vartheta}{\partial \phi} + \frac{V_r V_\vartheta}{r} - \frac{V_\phi^2}{r\tan\vartheta} &= f_\vartheta - \frac{1}{\rho r}\frac{\partial p}{\partial \vartheta} \\
&+ \nu\biggl(\frac{\partial^2 V_\vartheta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\vartheta}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_\vartheta}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_\vartheta}{\partial r} \\
&+ \frac{1}{r^2\tan\vartheta}\frac{\partial V_\vartheta}{\partial \vartheta} - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{2}{r^2}\frac{\partial V_r}{\partial \vartheta} - \frac{V_\vartheta}{r^2\sin^2\vartheta}\biggr), \qquad \text{(2b)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + \frac{V_\vartheta}{r}\frac{\partial V_\phi}{\partial \vartheta} + \frac{V_\phi}{r\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} + \frac{V_\vartheta V_\phi}{r\tan\vartheta} &= f_\phi - \frac{1}{\rho r\sin\vartheta}\frac{\partial p}{\partial \phi} \\
&+ \nu\biggl(\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\phi}{\partial \vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 V_\phi}{\partial \phi^2} + \frac{2}{r}\frac{\partial V_\phi}{\partial r} \\
&+ \frac{1}{r^2\tan\vartheta}\frac{\partial V_\phi}{\partial \vartheta} + \frac{2}{r^2\sin\vartheta}\frac{\partial V_r}{\partial \phi} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial V_\vartheta}{\partial \phi} - \frac{V_\phi}{r^2\sin^2\vartheta}\biggr), \qquad \text{(2c)}
\end{aligned}
$$

$$
\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\vartheta}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial V_\phi}{\partial \phi} + \frac{2V_r}{r} + \frac{V_\vartheta}{r\tan\vartheta} = 0. \qquad \text{(2d)}
$$

2. Analyse the following two viscous two-dimensional flows of incompressible fluid:

(a): The first one is the flow down an infinite flat slope under the action of the gravitational field $\mathbf{g}$. The angle between the slope and horizontal is α; see Figure 2. Assume that the fluid forms a layer of constant thickness h. Assume also that the flow is steady and none of the fluid-dynamic functions depends on the coordinate x measured down the slope. Your task is to find the velocity distribution across the layer.

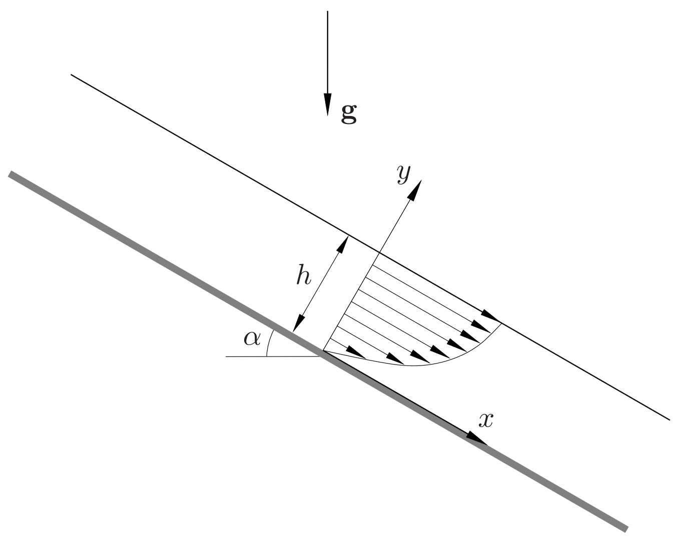

Figure 2: Fluid layer on the downslope.

Hint: Use the Navier–Stokes equations written in Cartesian coordinates with x directed down the slope (see Figure 2), and recall that the tangential stress

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

is zero at the upper edge of the fluid layer. (10 marks)

(b): Now consider an incompressible viscous fluid that occupies a semi-infinite region on one side of an infinite flat plate (as shown in Figure 3). The plate performs oscillatory motion in its

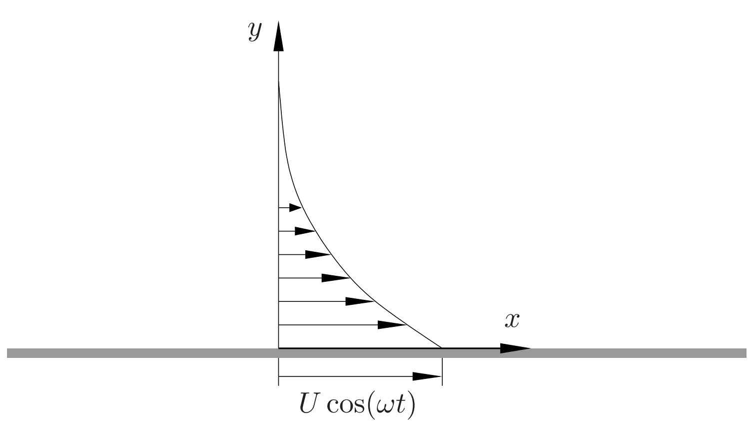

Figure 3: Flow above an oscillating plate.

plane with velocity given by

$$u = U\cos(\omega t). \tag{3}$$

Here U is the amplitude of the oscillations and ω is the frequency. Find the velocity distribution in the fluid above the surface.

Suggestion: Use Cartesian coordinates with the x-axis along the oscillating plate. Deduce that in the flow considered the Navier–Stokes equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Seek the solution to this equation in the form

$$u = f(y)e^{i\omega t} + \overline{f}(y)e^{-i\omega t},$$

where $f(y)$ is a complex-valued function, and $\overline{f}(y)$ is the complex conjugate of $f(y)$.

(10 marks)

(Total: 20 marks)

3. Consider an incompressible inviscid fluid flow past a sphere of radius a. The free-stream velocity is parallel to the x-axis, and its modulus is V_∞, as shown in Figure 4.

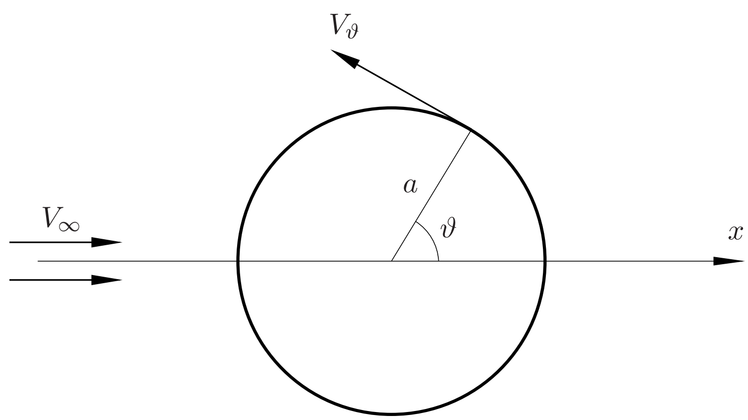

Figure 4: The flow past a sphere.

Assume that the flow is potential, and perform the following tasks:

(a) Formulate the boundary-value problem the velocity potential φ should satisfy when one deals with a steady inviscid flow past a motionless body.

(3 marks)

(b) Using (without proof) the fact that the uniform flow and the flow from a dipole are given by

$$\varphi_1 = V_\infty x, \qquad \varphi_2 = \frac{mx}{4\pi r^3},$$

show that the flow past a sphere is given by $\varphi = \varphi_1 + \varphi_2$ provided that the dipole moment m is appropriately chosen.

(5 marks)

(c) Show that the radial and meridional velocity components are given by

$$V_r = V_\infty \left(1 - \frac{a^3}{r^3}\right) \cos\vartheta, \qquad V_\vartheta = -V_\infty \left(1 + \frac{a^3}{2r^3}\right) \sin\vartheta. \tag{4}$$

Hint: You may use without proof the fact that the gradient of a scalar function $\Phi(r, \vartheta, \phi)$ is written in spherical polar coordinates as

$$\nabla\Phi = \frac{\partial \Phi}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial \Phi}{\partial \phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}\mathbf{e}_\vartheta,$$

with $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_\vartheta$ denoting the unit vectors in the radial, azimuthal, and meridional directions, respectively.

(4 marks)

(d) Now consider an arbitrary axisymmetric flow. Using the fact that for an axisymmetric flow the continuity equation for an incompressible fluid is written in spherical polar coordinates as

$$\frac{\partial}{\partial r}\big(r^2 \sin\vartheta\, V_r\big) + \frac{\partial}{\partial \vartheta}\big(r \sin\vartheta\, V_\vartheta\big) = 0,$$

introduce a scalar function $\psi(r, \vartheta)$, known as the *Stokes stream function*, such that

$$\frac{\partial \psi}{\partial r} = -r \sin\vartheta\, V_\vartheta, \qquad \frac{\partial \psi}{\partial \vartheta} = r^2 \sin\vartheta\, V_r, \tag{5}$$

and prove that the equation

$$\psi = \text{const}$$

defines the streamlines in the flow.

(4 marks)

(e) Combine (5) with (4), and show that for the flow past a sphere

$$\psi = \frac{V_\infty}{2}\left(r^2 - \frac{a^3}{r}\right)\sin^2\vartheta.$$

(4 marks)

(Total: 20 marks)

4. Consider the symmetrical flow past a parabola

$$y = \pm a\sqrt{x} \tag{6}$$

with the free-stream velocity V_∞ as shown in Figure 5.

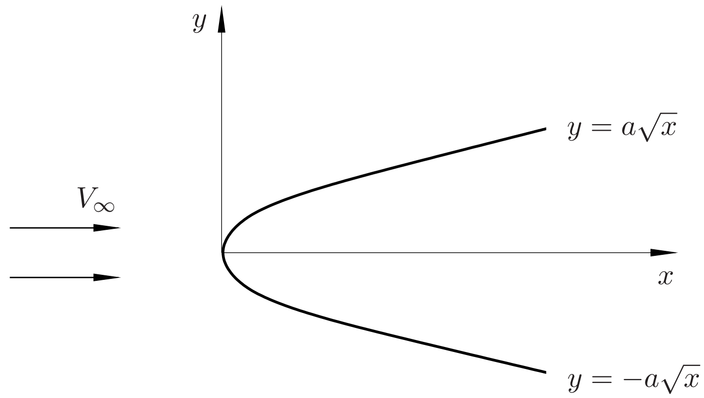

Figure 5: Symmetrical flow past a parabola.

You need to perform the following tasks:

(a) Prove that the conformal mapping of the region $\Im\{\zeta\} > h$ in the auxiliary ζ-plane onto the exterior of the parabola in the physical z-plane is given by

$$z = \zeta^2 + h^2.$$

What is the relationship between parameter h and constant a in (6)?

(5 marks)

(b) Prove that the complex potential of the flow in the auxiliary plane may be written as

$$W(\zeta) = b(\zeta - ih)^2, \tag{7}$$

where b is a real constant.

Suggestion: To perform this task, set $\zeta = \xi + ih$ in (7) and show that the imaginary part of W is constant, which proves that the body surface is a streamline.

(4 marks)

(c) Relate parameter b in (7) to the free-stream velocity V_∞ in the physical plane.

(4 marks)

(d) Find the modulus of the complex conjugate velocity $\overline{V}(z)$ on the parabola surface first in terms of ξ and then in term of y.

(4 marks)

(e) Calculate the parabola drag (per unit length in the spanwise direction)

$$D = 2\int_0^\infty (p - p_\infty)dy.$$

Here p denotes the local pressure and p_∞ the pressure at large distances from the parabola.

(3 marks)

(Total: 20 marks)

5. Consider inviscid irrotational flow past a symmetric aerofoil that is made of two circular arcs as shown on the left-hand side of Figure 6. The modulus of the velocity in the free-stream far from the aerofoil is V_∞, and the angle of attack is α.

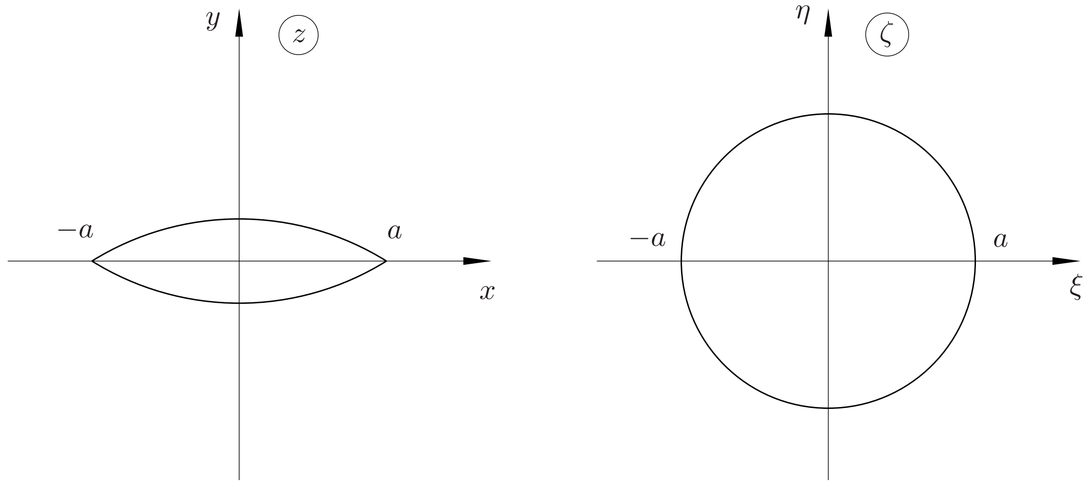

Figure 6: Physical z-plane and auxiliary ζ-plane for the generalised Joukovskii transformation (8).

(a): Demonstrate that the generalised Joukovskii transformation, written implicitly as

$$\frac{z-a}{z+a} = \left(\frac{\zeta - a}{\zeta + a}\right)^k, \tag{8}$$

maps the exterior of a circle of radius a in the auxiliary ζ-plane onto the exterior of the aerofoil. What should the parameter k in (8) be if the the angle between the upper and lower sides of the aerofoil at its leading (trailing) edge is 2θ?

Suggestion: In order to perform this task, introduce two additional planes

$$z_1 = \frac{z-a}{z+a} \quad \text{and} \quad \zeta_1 = \frac{\zeta - a}{\zeta + a},$$

and find the mapping between them in the form of the power function $z_1 = \zeta_1^k$. (10 marks)

(b): Show that at large z and ζ, equation (8) reduces to

$$\zeta = kz + \cdots \quad \text{as} \quad z \to \infty.$$

Hence, deduce that the free-stream velocity $\widetilde{V}_\infty$ in the auxiliary ζ-plane is

$$\widetilde{V}_\infty = V_\infty / k.$$

(5 marks)

(c): Write the complex potential in the auxiliary plane in the form

$$W(\zeta) = \frac{V_\infty}{k}\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}}\right) + \frac{\Gamma}{2\pi i}\ln\zeta,$$

where α is the angle between the free-stream velocity vector and the x-axis.

Argue that the Joukovskii-Kutta condition is satisfied if $dW/d\zeta$ is zero at the point $\zeta = a$ in the the auxiliary ζ-plane, and deduce that

$$\Gamma = -4\pi a \frac{V_\infty}{k} \sin\alpha.$$

(5 marks)

(Total: 20 marks)

Solutions

Problem 1. [unseen]

(a): Using Figure 1 the students should be able to see that the velocity components are given by

$$V_\phi = \Omega r \sin\vartheta, \qquad V_r = 0, \qquad V_\vartheta = 0, \tag{s1.1}$$

[3 marks – C]

(b): Substituting (s1.1) into equations (2a) and (2b) results in

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \Omega^2 r \sin^2\vartheta - \alpha r, \tag{s1.2a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} = \Omega^2 r^2 \sin\vartheta\cos\vartheta. \tag{s1.2b}$$

[3 marks – B]

Integrating the first of these we have

$$\frac{p}{\rho} = \tfrac{1}{2}\Omega^2 r^2 \sin^2\vartheta - \tfrac{1}{2}\alpha r^2 + \Phi(\vartheta). \tag{s1.3}$$

In order find function $\Phi(\vartheta)$, one needs to substitute (s1.3) into (s1.2b). This results in

$$\Phi'(\vartheta) = 0,$$

and we can conclude that $\Phi(\vartheta)$ is a constant, i.e.

$$\Phi(\vartheta) = C.$$

Thus the pressure everywhere inside the Earth is given by

$$p = \rho\Big(\tfrac{1}{2}\Omega^2 r^2 \sin^2\vartheta - \tfrac{1}{2}\alpha r^2 + C\Big). \tag{s1.4}$$

The equation for the Earth's surface is obtained by setting the pressure, p, to zero,

$$\tfrac{1}{2}\Omega^2 R^2 \sin^2\vartheta - \tfrac{1}{2}\alpha R^2 + C = 0. \tag{s1.5}$$

[4 marks – A]

Now the constant, C, can be found by taking into account that at the North pole, where $\vartheta = 0$, the radius $R = R_0$. We see that

$$C = \tfrac{1}{2}\alpha R_0^2,$$

which allows us to write the equation (s1.5) for the Earth surface as

$$\tfrac{1}{2}\Omega^2 R^2 \sin^2\vartheta - \tfrac{1}{2}\alpha R^2 + \tfrac{1}{2}\alpha R_0^2 = 0.$$ **[2 marks – A]**

It remains to solve this equation for R, and we will see that the Earth radius depends on the meridional angle ϑ as

$$R = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha}\sin^2\vartheta}}. \tag{s1.6}$$

[2 marks – B]

(c): Placing plane S into the (x, y)-plane, we can write

$$R = \sqrt{x^2 + y^2}, \qquad \sin\vartheta = \frac{y}{R}. \tag{s1.7}$$

Substituting the second equation on (s1.7) into (s1.6) and squaring both sides, we have

$$R^2 = \frac{R_0^2}{1 - \frac{\Omega^2}{\alpha}\frac{y^2}{R^2}},$$

or equivalently,

$$R^2 - \frac{\Omega^2}{\alpha} y^2 = R_0^2.$$

Now we use the first equation in (s1.7). We have

$$x^2 + \left(1 - \frac{\Omega^2}{\alpha}\right) y^2 = R_0^2.$$

This may be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with

$$a = R_0, \qquad b = \frac{R_0}{\sqrt{1 - \frac{\Omega^2}{\alpha}}}.$$

[2 marks – A]

(d): As the planet become more 'pancake-like', the gravitational force deviates from the radial direction.

[4 marks – D]

Problem 2. (a) [unseen], (b) [seen]

(a). The students are expected to remember that for a two-dimensional flow the Navier–Stokes equations are written in Cartesian coordinates as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{s2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{s2.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{s2.1c}$$

[2 marks – B]

With the x-axis directed downslope, the components of the body force are given by

$$f_x = g\sin\alpha, \qquad f_y = -g\cos\alpha.$$

Taking this into account, the x- and y-momentum equations (s2.1a), (s2.1b) are written as

$$0 = g\sin\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{s2.2}$$

$$0 = -g\cos\alpha - \frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{s2.3}$$

[2 marks – A]

Integrating (s2.3), we have

$$p = -\rho g\cos\alpha\, y + \Phi(x). \tag{s2.4}$$

To find $\Phi(x)$, we note that at the upper edge of the fluid layer

$$p\Big|_{y=h} = p_a.$$

We have

$$\Phi(x) = p_a + \rho g\cos\alpha\, h. \tag{s2.5}$$

Substituting (s2.5) back into (s2.4)

$$p = p_a + \rho g\cos\alpha(h-y),$$ **[3 marks – B]**

we see that the pressure does not depend on x, and therefore, equation (s2.2) reduces to

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g}{\nu}\sin\alpha.$$ **[1 mark – A]**

The general solution of this equation is

$$u = -\frac{g}{2\nu}\sin\alpha y^2 + C_1 y + C_2. \tag{s2.6}$$

Constants C_1 and C_2 may be found from the boundary conditions

$$u\Big|_{y=0} = 0, \qquad \frac{\partial u}{\partial y}\Big|_{y=h} = 0. \tag{s2.7}$$

The first of (s2.7) shows that

$$C_2 = 0. \tag{s2.8}$$

The second condition gives

$$C_1 = \frac{gh}{\nu}\sin\alpha. \tag{s2.9}$$

Substituting (s2.8) and (s2.9) back into (s2.6), we find

$$u = \frac{gh}{2\nu}\sin\alpha(2h-y)y.$$

[2 marks – A]

(b). Obviously, in this flow all the derivatives with respect to x are zeros. Therefore, it follows from the continuity equation (s2.1c) that

$$\frac{\partial v}{\partial y} = 0. \tag{s2.10}$$

Integrating (s2.10) with the impermeability condition on the plate surface

$$v\Big|_{y=0} = 0,$$

we have

$$v = 0$$

everywhere in the flow field.

[1 mark – A]

It is now easily seen that the x-momentum equation (s2.1a) reduces to

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{s2.11}$$

[2 marks – A]

It has to be solved with the boundary conditions

$$u\Big|_{y=0} = \tfrac{1}{2}Ue^{i\omega t} + \tfrac{1}{2}Ue^{-i\omega t}, \qquad u\Big|_{y=\infty} = 0. \tag{s2.12}$$

We represent the sought solution in the form

$$u = f(y)e^{i\omega t} + \overline{f}(y)e^{-i\omega t}. \tag{s2.13}$$

Substitution of (s2.13) into (s2.11) results in

$$i\omega f(y)e^{i\omega t} - i\omega\overline{f}(y)e^{-i\omega t} = \nu f''(y)e^{i\omega t} + \nu\overline{f}''(y)e^{-i\omega t}.$$

Separating the terms with $e^{i\omega t}$ from the terms with $e^{-i\omega t}$ we obtain the two equations

$$i\omega f = \nu f'', \qquad -i\omega\overline{f} = \nu\overline{f}''.$$

However, the second of these is simply the complex conjugate of the first equation. This means that in order to find function $f(y)$ we need to solve the equation

$$i\omega f = \nu f''. \tag{s2.14}$$

while the boundary conditions (s2.12) reduce to

$$f(0) = U, \qquad f(\infty) = 0. \tag{s2.15}$$

[3 marks – C]

The complementary solution to (s2.14) are sought in the form

$$f(y) = Ce^{\lambda y},$$

leading to the following characteristic equation

$$i\omega = \nu\lambda^2.$$

We see that

$$\lambda_{1,2} = \pm\sqrt{i\frac{\omega}{\nu}} = \pm e^{i\pi/4}\sqrt{\frac{\omega}{\nu}} = \pm(1+i)\sqrt{\frac{\omega}{2\nu}}.$$ **[2 marks – D]**

The general solution of equation (s2.14) is written as

$$f(y) = C_1e^{\lambda_1 y} + C_2e^{\lambda_2 y}. \tag{s2.16}$$

Since the real part of λ_1 is positive, $f(y)$ becomes infinitely large as $y \to \infty$ unless

$$C_1 = 0. \tag{s2.17}$$

In order to find C_2 we used the first of conditions (s2.15). We have

$$C_2 = U. \tag{s2.18}$$

Substituting (s2.17) and (s2.18) back into (s2.16), we have

$$f(y) = Ue^{-(1+i)\sqrt{\frac{\omega}{2\nu}}y}. \tag{s2.19}$$

It remains to substitute (s2.19) into (s2.13), and we arrive at the conclusion that

$$u = \Re\left\{Ue^{-\sqrt{\frac{\omega}{2\nu}}y}e^{i\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)}\right\} = Ue^{-\sqrt{\frac{\omega}{2\nu}}y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right).$$ **[2 marks – D]**

Problem 3. [unseen]

(a). If a rigid body is placed in a uniform flow with the free-stream $\mathbf{V}_\infty$, then the velocity potential φ should be found by solving the following problem

Problem *Find the velocity potential φ that satisfies Laplace's equation*

$$\nabla^2\varphi = 0 \tag{s3.1a}$$

everywhere inside the flow field. It should also satisfy the impermeability condition

$$\left.\frac{\partial\varphi}{\partial n}\right|_S = 0 \tag{s3.1b}$$

on the body's surface and the free-stream condition

$$\varphi = \mathbf{V}_\infty \cdot \mathbf{r} + \cdots \quad \text{as} \quad |\mathbf{r}| \to \infty \tag{s3.1c}$$

in the far field.

[3 marks – C]

(b). Since both φ_1 and φ_2 satisfy the Laplace equation, we only need to analyse the boundary conditions (s3.1a), its sum

$$\varphi = \varphi_1 + \varphi_2 = V_\infty x + \frac{mx}{4\pi r^3} \tag{s3.2}$$

is also a solution to the Laplace equation.

Let us now examine the boundary conditions (s3.1 b,c). As $r \to \infty$, the second term in (s3.2) tends to zero, reducing (s3.2) to

$$\varphi = V_\infty x,$$

which proves that the free-stream boundary condition is satisfied. Turning to the impermeability condition (s3.1b), we note that at any point situated outside the sphere, $x = r\cos\vartheta$, and equation (s3.2) may be written as

$$\varphi = V_\infty r\cos\vartheta + \frac{m\cos\vartheta}{4\pi r^2}. \tag{s3.3}$$

With a being the radius of the sphere, we have

$$\left.\frac{\partial\varphi}{\partial n}\right|_S = \left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = \left(V_\infty - \frac{m}{2\pi a^3}\right)\cos\vartheta.$$

Therefore, by choosing

$$m = 2\pi a^3 V_\infty, \tag{s3.4}$$

[1 mark – B]

we can satisfy the impermeability condition (s3.1b) for all values of ϑ, i.e. on the entire surface of the sphere. This proves that formula (s3.3) really represents the solution for the flow past the sphere. Substituting (s3.4) back into (s3.3) results in

$$\varphi = V_\infty\left(r + \frac{a^3}{2r^2}\right)\cos\vartheta. \tag{s3.5}$$

[4 marks – B]

(c). We know that the velocity vector is calculated as the gradient of φ. In the flow considered, the azimuthal velocity V_ϕ is, obviously, zero. The radial and meridional velocities are calculated as

$$V_r = \frac{\partial \varphi}{\partial r} = V_\infty \left(1 - \frac{a^3}{r^3}\right) \cos\vartheta, \qquad V_\vartheta = \frac{1}{r}\frac{\partial \varphi}{\partial \vartheta} = -V_\infty \left(1 + \frac{a^3}{2r^3}\right) \sin\vartheta. \tag{s3.6}$$

[4 marks – A]

(d). It follows from the continuity equation

$$\frac{\partial}{\partial r}\left(r^2 \sin\vartheta\, V_r\right) + \frac{\partial}{\partial \vartheta}\left(r \sin\vartheta\, V_\vartheta\right) = 0$$

that there exists a scalar function $\psi(r, \vartheta)$, such that

$$\frac{\partial \psi}{\partial r} = -r \sin\vartheta\, V_\vartheta, \qquad \frac{\partial \psi}{\partial \vartheta} = r^2 \sin\vartheta\, V_r, \tag{s3.7}$$

Solving (s3.7) for V_r and V_ϑ, we have

$$V_r = \frac{1}{r^2 \sin\vartheta}\frac{\partial \psi}{\partial \vartheta}, \qquad V_\vartheta = -\frac{1}{r \sin\vartheta}\frac{\partial \psi}{\partial r}. \tag{s3.8}$$

[2 marks – A]

In order to prove that the equation

$$\psi = \text{const}$$

defines the streamlines, we need to show that that

$$\nabla\psi \cdot \mathbf{V} = 0. \tag{s3.9}$$

In an axisymmetric flow

$$\nabla\psi = \frac{\partial \psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \vartheta}\mathbf{e}_\vartheta.$$

Consequently, equation (s3.9) is written as

$$V_r \frac{\partial \psi}{\partial r} + V_\vartheta \frac{1}{r}\frac{\partial \psi}{\partial \vartheta} = 0. \tag{s3.10}$$

Substituting (s3.8) into (s3.10) we see that equation (s3.9) really holds.

[2 marks – A]

(e). Substitution of (s3.6) into (s3.7) yields

$$\frac{\partial \psi}{\partial r} = V_\infty \left(r + \frac{a^3}{2r^2}\right) \sin^2\vartheta, \qquad \frac{\partial \psi}{\partial \vartheta} = V_\infty \left(r^2 - \frac{a^3}{r}\right) \sin\vartheta \cos\vartheta. \tag{s3.11}$$

If we integrate the first of equations (s3.11), then we will have

$$\psi = V_\infty \left(\frac{r^2}{2} - \frac{a^3}{2r}\right) \sin^2\vartheta + F(\vartheta). \tag{s3.12}$$

Substitution of (s3.12) into the second equation in (s3.12) shows that $F(\vartheta)$ is a constant, and may be disregarded since the stream function ψ is defined to within an arbitrary constant.

[4 marks – D]

Problem 4. [seen]

(a). In order to demonstrate that the function

$$z = \zeta^2 + h^2 \tag{s4.1}$$

maps the region in the auxiliary ζ-plane onto the exterior of the parabola in the physical z-plane, we note that for any point that lies on the boundary of the region in the ζ-plane (see Figure 1)

$$\zeta = \xi + ih. \tag{s4.2}$$

As ξ changes from $-\infty$ to ∞ the point moves along the boundary from left to right.

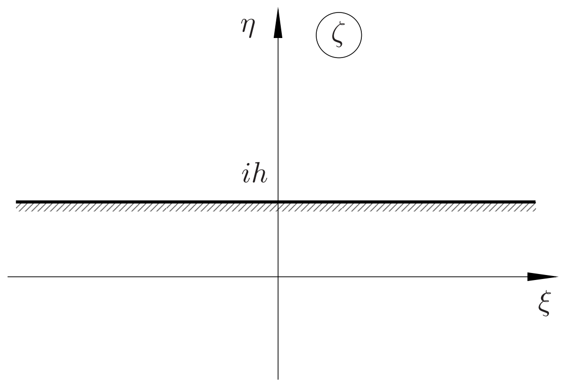

Figure 1: Auxiliary ζ-plane.

Substitution of (s4.2) into (s4.1) yields

$$z = x + iy = (\xi + ih)^2 + h^2 = \xi^2 + 2ih\xi.$$

Separating the real and imaginary parts in this equations we see that the boundary of the flow region in the physical plane is given by

$$x = \xi^2, \qquad y = 2h\xi. \tag{s4.3}$$

[3 marks – C]

Solving the second of equations (s4.3) for ξ we have

$$\xi = \frac{y}{2h}. \tag{s4.4}$$

Substitution of (s4.4) into the first equation in (s4.3) shows that the body contour in the z-plane is indeed parabolic:

$$x = \frac{y^2}{4h^2}. \tag{s4.5}$$

Notice that parameter h controls the "thickness" of the parabola. Comparing (s4.5) with (e6.1) we can conclude that

$$2h = a. \tag{s4.6}$$

[2 marks – A]

(b). The complex potential in the auxiliary plane

$$W(\zeta) = b(\zeta - ih)^2, \tag{s4.7}$$

represents the flow descending on the the flat surface as shown in Figure 2. Since the mapping (s4.1) doubles the angles, the streamline which lies along the positive imaginary axis and in the ζ-plane is mapped onto the streamline that coincides with the axis of symmetry in the z-plane.

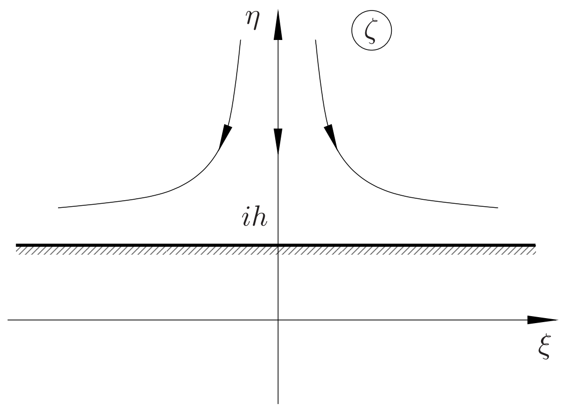

Figure 2: The flow in the auxiliary ζ-plane.

[2 marks – A]

To confirm that (s4.7) does indeed represent the required solution, we notice, first of all, that $W(\zeta)$ is analytic function. Second, we have to verify the impermeability condition on the body surface. For this purpose we substitute (s4.2) into (s4.7). We find that

$$W = b\xi^2.$$

Hence, the imaginary part of W representing the stream function is zero, which proves that the body surface is a streamline.

[2 marks – B]

(c). Thirdly, we need to make sure that the free-stream condition is satisfied. This is done as follows. We calculate the complex conjugate velocity as

$$\overline{V}(z) = \frac{dw}{dz} = \frac{dW}{d\zeta}\frac{1}{dz/d\zeta}. \tag{s4.8}$$

Using (s4.7) and (s4.1) we have

$$\overline{V}(z) = 2b\frac{\zeta - ih}{2\zeta}. \tag{s4.9}$$

[3 marks – B]

We see that

$$\overline{V} \to b \quad \text{as} \quad \zeta \to \infty.$$

Thus constant b should coincide with the free-stream velocity V_∞.

[1 mark – A]

(d). Using the fact that $b = V_\infty$, we can write (s4.9) as

$$\overline{V}(z) = V_\infty \frac{\zeta - ih}{\zeta}.$$

In particular on the body contour, where ζ is represented by (s4.2) we have

$$\overline{V}(z) = V_\infty \frac{\xi}{\xi + ih}.$$

The modulus of the velocity is

$$V = \left|\overline{V}(z)\right| = V_\infty \frac{|\xi|}{\sqrt{\xi^2 + h^2}}. \tag{s4.10}$$

It remains to substitute (s4.4) into (s4.10) and we will have

$$V = V_\infty \frac{|y|}{\sqrt{y^2 + 4h^4}}. \tag{s4.11}$$

[4 marks – D]

(e). Applying the Bernoulli theorem to the streamline that lies along the body contour we can write

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}. \tag{s4.12}$$

Substitution of (s4.12) into (s4.11) yields

$$p - p_\infty = \frac{\rho V_\infty^2}{2} \frac{4h^4}{y^2 + 4h^4}.$$

[2 marks – A]

Hence the x-component of the pressure force may be written as

$$D = 2 \int\limits_0^\infty (p - p_\infty)\, dy = 4\rho V_\infty^2 h^4 \int\limits_0^\infty \frac{dy}{y^2 + 4h^4}.$$

Introducing a new integration variable s such that

$$y = 2h^2 s,$$

we have

$$D = 4\rho V_\infty^2 h^4 \int\limits_0^\infty \frac{2h^2 ds}{4h^4(s^2+1)} =$$

$$= 2\rho V_\infty^2 h^2 \int\limits_0^\infty \frac{ds}{s^2+1} = 2\rho V_\infty^2 h^2 \arctan s \Big|_0^\infty = 2\rho V_\infty^2 h^2 \frac{\pi}{2}.$$

Recalling that $h = a/2$ we can finally conclude that

$$D = \frac{1}{4}\rho V_\infty^2 a^2 \pi.$$

[1 mark – A]

Problem 5. [unseen]

(a). We start with the mapping of the physical z-plane onto z_1-plane. As suggested, we use the transformation

$$z_1 = \frac{z-a}{z+a}. \tag{s5.1}$$

This is a linear-fractional transformation, and we know that it assumes the circle property, namely it maps circles or circular arcs into circles in the extended complex plane. In particular, we see that (s5.1) maps $z = a$ into $z_1 = 0$, while $z = -a$ is mapped into $z_1 = \infty$. This means that the upper and lower sides of the aerofoil are mapped onto two rays emerging from $z_1 = 0$, as shown in Figure 3.

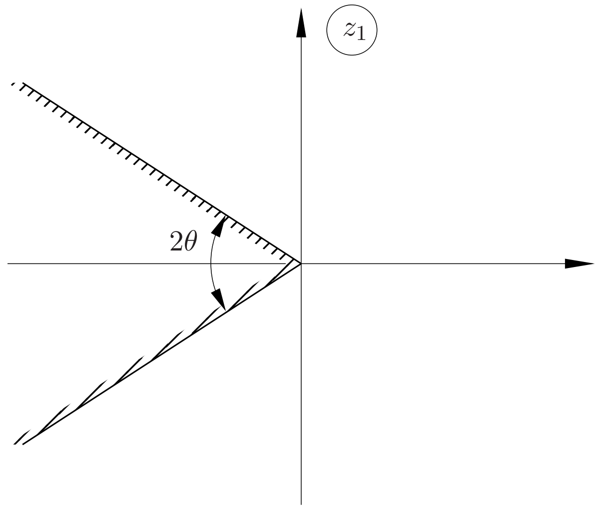

Figure 3: z_1-plane.

In order to determine the orientation of the rays in the z_1-plane, we recall that the angle δ of rotation of any line when mapped from z-plane into z_1-plane is given by

$$\delta = \arg\left(\frac{dz_1}{dz}\right).$$ **[2 Marks]**

Differentiating (s5.1), we find

$$\frac{dz_1}{dz} = \frac{2a}{(z+a)^2}.$$

In particular, at the trailing edge of the aerofoil

$$\left.\frac{dz_1}{dz}\right|_{z=a} = \frac{1}{2a},$$

which is a real positive quantity. This means that the tangents to the upper and lower sides of the aerofoil do not experience any rotation, $\delta = 0$. Hence, the two rays in the z_1-plane make the same angle, θ, with the negative real semi-axis as the upper and lower sides of the aerofoil at the trailing edge in the physical z-plane. **[2 Marks]**

The mapping

$$\zeta_1 = \frac{\zeta - a}{\zeta + a} \tag{s5.2}$$

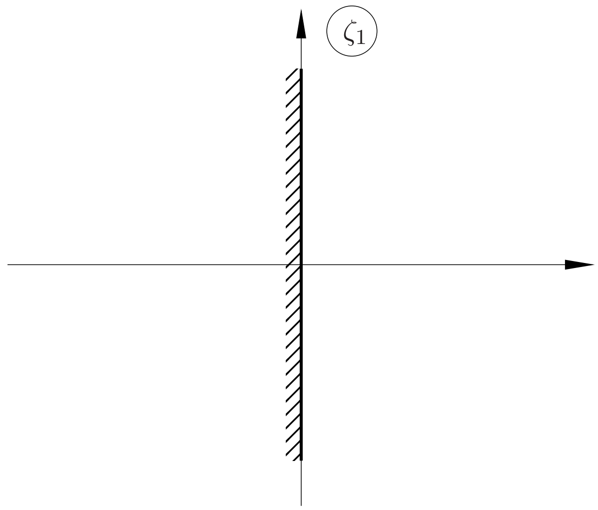

Figure 4: z_1-plane.

is analysed in the same way, and we can see that the the circle in the ζ-plane is mapped onto a straight line that lies along the imaginary axis in the ζ_1-plane, see Figure 4. Clearly, the exterior of the circle is mapped onto the right half-plane of the ζ_1-plane. [**2 Marks**]

It remains to study the mapping

$$z_1 = \zeta_1^k. \tag{s5.3}$$

In order to determine the power k in (s5.3) one needs to make sure that (s5.3) provides proper correspondence of the boundaries of the regions we are dealing with in the z_1- and ζ_1-planes. In particular, if we consider a point on the positive imaginary semi-axis in the ζ_1-plane, then we can write

$$\zeta_1 = |\zeta_1| e^{i\pi/2}. \tag{s5.4}$$

Substitution of (s5.4) into (s5.3) results in

$$z_1 = |\zeta_1|^k e^{ik\pi/2}. \tag{s5.5}$$

[**2 Marks**]

It is easily seen that point (s5.5) finds itself on the upper ray in Figure 3 provided that

$$k\frac{\pi}{2} = \pi - \theta.$$

Consequently,

$$k = 2\big(1 - \theta/\pi\big).$$ [**2 Marks**]

(b). Now we need to analyse the equation

$$\frac{1 - a/z}{1 + a/z} = \left(\frac{1 - a/\zeta}{1 + a/\zeta}\right)^k \tag{s5.6}$$

at large values of z and ζ. Disregarding the $O\big[(a/z)^2\big]$ terms, the left-hand side of this equation can be simplified as [**1 Mark**]

$$\frac{1 - a/z}{1 + a/z} = (1 - a/z)(1 - a/z + \cdots) = 1 - 2\frac{a}{z} + \cdots\,. \tag{s5.7}$$

Similarly, for the left-hand side of (s5.6) we have

$$\left(\frac{1 - a/\zeta}{1 + a/\zeta}\right)^k = \left(1 - 2\frac{a}{\zeta} + \cdots\right)^k = 1 - 2k\frac{a}{\zeta} + \cdots . \tag{s5.8}$$

[1 Mark]

Substituting (s5.7) and (s5.8) into (s5.6), we find that

$$\zeta = kz + \cdots \quad \text{as} \quad z \to \infty. \tag{s5.9}$$

In order to determine the modulus of the free-stream velocity, $\widetilde{V}_\infty$, in the auxiliary plane, we use the fact that, far from the aerofoil, the complex conjugate velocity in the physical plane is

$$\overline{V}(z) = V_\infty e^{-i\alpha}. \tag{s5.10}$$

We also know that at any point in the physical plane

$$\overline{V}(z) = \frac{dw}{dz} = \frac{dW}{d\zeta}\frac{d\zeta}{dz}. \tag{s5.11}$$

Using (s5.9) and (s5.10) in (s5.11), we find that

$$\frac{dW}{d\zeta} = \frac{V_\infty}{k}e^{-i\alpha} \quad \text{at} \quad \zeta = \infty,$$

[1 Mark]

with the modulus of the free-stream velocity being

$$\widetilde{V}_\infty = \frac{V_\infty}{k}.$$

[2 Marks]

(c). Consequently, the complex potential in the auxiliary plane should be written as

$$W(\zeta) = \frac{V_\infty}{k}\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}}\right) + \frac{\Gamma}{2\pi i}\ln\zeta, \tag{s5.12}$$

The Joukovskii-Kutta condition implies that, in the physical z-plane, the flow should leave the aerofoil at its trailing edge. **[2 Marks]**

This happens if the ζ-plane the stagnation point is situated at $\zeta = a$. Differentiation of (s5.12) gives

$$\frac{dW}{d\zeta} = \frac{V_\infty}{k}\left(e^{-i\alpha} - \frac{a^2}{\zeta^2}e^{i\alpha}\right) + \frac{\Gamma}{2\pi i\zeta}.$$

[2 Marks]

We need this derivative to become zero at $\zeta = a$, which gives the following equation for Γ:

$$\frac{V_\infty}{k}\left(e^{-i\alpha} - e^{i\alpha}\right) + \frac{\Gamma}{2\pi i a} = 0.$$

It is easily seen that

$$\Gamma = -4\pi a\frac{V_\infty}{k}\sin\alpha.$$

[1 Mark]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97008 MATH97088	1	This question does not require advanced mathematical skills, but relies on physical intuition. The majority of students did very well in this respect.	
MATH97008 MATH97088	2	In this question two typical problems of viscous fluid dynamics are considered. The students demonstrated a good knowledge of the standard methods used to solve such problems. Some students found it difficult to treat properly the complex representation of real solution.	
MATH97008 MATH97088	3	This problem concerned the potential flow theory. In my lectures I was concemtrating on two-dimensional flows. Here the students we supposed to extend the theory to tree-dimensional flows. I see from the answers that some students struggled with formulating the the boundary-value problem for the potential.	
MATH97008 MATH97088	4	This is atypical problem where the conformal mapping has to be used. The students did very well. However, some students found it difficult to justify the validity of the suggested form of the complex potemtial in the auxiliary plane.	
MATH97008 MATH97088	5	To solve this problem the students had to derive the generalised Joukovskii transformation. In the previous years, a significant proportion of students found problems of this sort difficult. I am happy that this year cohort did very well. However, some students did not demonstrate a solid knowledge of the properties of the linear-fractional transformation.	