MATH50004 Differential Equations

Spring Term 2021/22

Solutions to the mid-term exam

Question 1

(i) With $f(x) = \cos(x)$, we get for all $t \in \mathbb{R}$ that

$$\begin{aligned}
\lambda_0(t) &= 0\,,\\
\lambda_1(t) &= 0 + \int_0^t f(\lambda_0(s))\,\mathrm{d}s = \int_0^t 1\,\mathrm{d}s = t\,,\\
\lambda_2(t) &= 0 + \int_0^t f(\lambda_1(s))\,\mathrm{d}s = \int_0^t \cos(s)\,\mathrm{d}s = [\sin(s)]_{s=0}^{s=t} = \sin(t)\,.
\end{aligned}$$

[6 points = 1 point for λ_0; 2 points for λ_1; 3 points for λ_2]

(ii) The right hand side is obviously continuous, and also is globally Lipschitz continuous in x with Lipschitz constant 1, since the derivative of the right hand side, $-\sin(x)$ is bounded by 1. These arguments were used in the course by applying the mean value theorem.

[2 points; this is an autonomous differential equation, so the continuity follows from Lipschitz continuity in x, and we should not be harsh if this is not mentioned. We also should not be too harsh if the mean value theorem is not been mentioned.]

(iii) We used a mapping $P : C^0\big([-h,h],\mathbb{R}^d\big) \to C^0\big([-h,h],\mathbb{R}^d\big)$ in the proof of global version of the Picard–Lindelöf theorem and showed that it is a contraction for suitable $h > 0$. The Picard iterates are iterates of this mapping, and Banach fixed point theorem then implies that they converge in the Banach space $C^0\big([-h,h],\mathbb{R}^d\big)$ to a fixed point, and this convergence corresponds to uniform convergence. It was shown in the lectures that a fixed point of P solves the differential equation.

[4 points; 1 point for mentioning the contraction P in the context of the Picard-Lindelöf theorem; 1 point for explaining the link between Picard iterates and P; 1 point for explaining uniform convergence; 1 point for pointing out that a fixed point of P solves the differential equation.]

Question 2

(i) We use the separation of variables to solve the initial value problem $x(0) = 0$ with $t_0 \in \mathbb{R}$. We get

$$\int_0^x \frac{1}{3y^{\frac{2}{3}}}\,\mathrm{d}y = \int_0^t \mathrm{d}s \quad\Longleftrightarrow\quad x^{\frac{1}{3}} = t \quad\Longleftrightarrow\quad x = t^3\,.$$

This implies that $\lambda(t) = t^3$ solves this initial value problem, and since 0 is a zero of the right hand side f, also $\mu(t) \equiv 0$ solves the initial value problem.

[6 points; 4 points for finding a solution with separation of variables (or guessing; note that a similar differential equation was on the problem sheets); 2 points for finding another solution]

(ii) The right hand side is clearly continuously differentiable. More precisely, note that the partial derivative w.r.t. x is given by $\frac{2}{\sqrt[3]{x}}$, which is continuous, and the partial derivative w.r.t. t is 0, so the derivative is continuous.

[2 points; it is okay to just write that the function is continuously differentiable without verifying this precisely]

(iii) For fixed $(t_0, x_0) \in D$, we consider $W^{1,x_0/2}(t_0, x_0)$. On this set, the right hand side is bounded by $M = 3\sqrt[3]{(3x_0/2)^2}$ and Lipschitz continuous with Lipschitz constant $K = 2/\sqrt[3]{x_0/2}$. Here, the mean value inequality was used and monotonicity of the right hand side and its derivative. Then $h = h(t_0, x_0) := \min\{1, \frac{1}{2K}, \frac{x_0}{2M}\}$.

[4 points; 1 point for choice of suitable neighbourhood; 1 point each for M and K; 1 point for finalising.]

Question 3

Consider $f(x) = 1$ and $\lambda(t) = t$ for $t \in \mathbb{R}$. It follows that any solution is of the form $\mu(t) = t + c$ for some $c \in \mathbb{R}$, and we have $\lambda(t + c) = t + c = \mu(t)$ for all $t \in \mathbb{R}$.

[6 points; this looks like in most cases, either one scores 0 or 6 points; points can be deduced if justification not correct. Correct example of f (without zeros!) but not of λ and without other insights should give 2 points.]