Imperial College London

MATH97061 MATH70055

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Stochastic Calculus with Applications to Non-linear Filtering

Date: 16 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

For the following questions, assume the set-up: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$ and V be a standard one-dimensional $\mathcal{F}_t$-adapted Brownian motion under $\mathbb{P}$. Let f and σ be bounded Lipschitz continuous real valued functions and let X be the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s)\, ds + \int_0^t \sigma(X_s)\, dV_s. \tag{1}$$

Assume that X_0 has distribution π_0 at time 0, is independent of V and $\mathbb{E}\left[(X_0)^2\right] < \infty$. Let W be a standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X, and Y be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. The process $Y = \{Y_t,\ t \geq 0\}$ is called the observation process. Let $\{\mathcal{Y}_t,\ t \geq 0\}$ be the filtration associated with the process Y, that is $\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t])$. The filtering problem consists in determining the conditional distribution π_t of the signal X_t given $\mathcal{Y}_t$. That is, $\pi_t(A) = \mathbb{E}[I_A(X_t) \mid \mathcal{Y}_t]$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$ and I_A is the indicator function of the set A and $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]$ for any bounded Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$.

1. Let $k > 0$ be a positive integer and let $B^k = \{B_t^k,\ t \geq 0\}$ to be the process defined as

$$B_t^k = \begin{cases} t^k W_{\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}. \tag{3}$$

   (a) Prove that the stochastic process B^k has continuous paths on $[0, \infty)$. [You can use without proof the fact that $\lim_{t\mapsto\infty} \frac{W_t}{t} = 0$].

   (6 marks)

   (b) Prove that the increments $B_t^k - B_s^k$, $0 \leq s < t$ of the process B^k are normally distributed.

   (4 marks)

   (c) Compute the means and variances of the increments $B_t^k - B_s^k$, $0 \leq s < t$.

   (4 marks)

   (d) Prove that the stochastic process B^k is a standard Brownian motion if and only if $k = 1$. [You can use without proof any of the results in the lectures.]

   (6 marks)

(Total: 20 marks)

2. Suppose that S is a geometric Brownian motion satisfying the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = 1. \tag{4}$$

(a) Prove that equation (4) has a unique solution. [You can use without proof any of the results in the lectures.] (3 marks)

(b) Prove that

$$S_t = \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

for any $t > 0$.

(5 marks)

(c) Prove that S is a martingale if and only if $\mu = 0$.

(8 marks)

(d) Prove that $\lim_{t\to 0} S_t = 0$ if $2\mu < \sigma^2$. [You can use without proof the fact that $\lim_{t\to\infty} \frac{W_t}{t} = 0$.]

(4 marks)

(Total: 20 marks)

3. Recall that X is the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation (1) and that f and σ are bounded Lipschitz continuous real valued functions and assume that X_0 is bounded. Let $\rho_t : \mathcal{B}(\mathbb{R}) \mapsto \mathbb{R}$ be the set function defined as

$$\rho_t(A) := \mathbb{E}\left[I_A(X_t)\left(\int_0^t (\sin(X_s))^2 ds\right)\right]$$

for any $A \in \mathcal{B}(\mathbb{R})$. Let $C_b^2(\mathbb{R})$ be the set of all bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$ twice differentiable with bounded first and second derivatives.

(a) Prove that ρ_t is a positive finite measure for any $t > 0$.

(6 marks)

(b) For any $\varphi \in C_b^2(\mathbb{R})$, let $q(\varphi) = \{q_t(\varphi), t \geq 0\}$ be the process defined as

$$q_t(\varphi) = \varphi(X_t)\left(\int_0^t (\sin(X_s))^2 ds\right).$$

Find the evolution equation satisfied by q_t.

(6 marks)

(c) Find the evolution equation satisfied by the process $\rho(\varphi) = \{\rho_t(\varphi),\ t \geq 0\}$ for arbitrary $\varphi \in C_b^2(\mathbb{R})$.

(8 marks)

(Total: 20 marks)

4. Let ρ_t be the measure defined in Question 3 and let $\rho_t^n : \mathcal{B}(\mathbb{R}) \mapsto \mathbb{R}$ be the measure defined as

$$\rho_t^n(A) = \frac{t}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[I_A(X_t)(\sin(X_{\frac{it}{n}}))^2\right]$$

for arbitrary $A \in \mathcal{B}(\mathbb{R})$.

(a) Prove that that there exist a positive constant m such that

$$\mathbb{E}[|X_u - X_v|] \leq m\sqrt{|u - v|}$$

for any $u, v \in [0, t]$, where m depends on t, but it is independent of u and v.

(6 marks)

(b) Prove that there exists a constant c independent of n such that

$$|\rho_t(\varphi) - \rho_t^n(\varphi)| \leq \frac{c}{\sqrt{n}}$$

for arbitrary $\varphi \in C_b^2(\mathbb{R})$.

(7 marks)

(c) For arbitrary $t > 0$, let $\bar{\rho}_t$ and $\bar{\rho}_t^n$ be the normalized versions of ρ_t and ρ_t^n, respectively, defined as

$$\bar{\rho}_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(\mathbb{R})}, \qquad \bar{\rho}_t^n(\varphi) = \frac{\rho_t^n(\varphi)}{\rho_t^n(\mathbb{R})}$$

for arbitrary $\varphi \in C_b^2(\mathbb{R})$. Prove that there exists a constant $\bar{c}$ such that

$$|\bar{\rho}_t(\varphi) - \bar{\rho}_t^n(\varphi)| \leq \frac{\bar{c}}{\sqrt{n}}$$

for arbitrary $\varphi \in C_b^2(\mathbb{R})$. [You can assume without proof that the normalization constants are strictly positive, i.e., $\rho_t(\mathbb{R}) > 0$ and $\rho_t^n(\mathbb{R}) > 0$.]

(7 marks)

(Total: 20 marks)

Mastery Question

5. Define $Z = \{Z_t, t > 0\}$ to be the process

$$Z_t = \exp\left(-\int_0^t h(X_s)dW_s - \frac{1}{2}\int_0^t (h(X_s))^2\, ds\right),\ t \geq 0.$$

and let $\tilde{\mathbb{P}}$ be a probability measure which is absolutely continuous with respect to $\mathbb{P}$, defined such that

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t, \qquad \forall t \geq 0.$$

Moreover let $z = \{z_t, t > 0\}$ be the process defined by

$$z_t = \exp\left(\int_0^t \pi_s(h)dY_s - \frac{1}{2}\int_0^t \pi_s(h)^2\, ds\right),\ t \geq 0.$$

(a) Using Novikov's condition prove that the process $z = \{z_t, t > 0\}$ is a $\mathcal{Y}_t$-adapted martingale under $\tilde{\mathbb{P}}$. [You can use, without proof, the fact that, under $\tilde{\mathbb{P}}$, the process Y is a Brownian motion.]

(5 marks)

(c) Deduce the evolution equation satisfied by the martingale z.

(3 marks)

(d) Prove that

$$\sup_{t\in[0,1]} \tilde{\mathbb{E}}[z_t^m] < \infty$$

for any $m \in \mathbb{R}$. (7 marks)

(e) Prove that

$$\sup_{t\in[0,1]} \mathbb{E}[z_t^m] < \infty$$

for any $m \in \mathbb{R}$.

(5 marks)

(Total: 20 marks)

Marking Scheme

Question 1. [20 marks]

(a) [6 marks, not seen] On the open interval $(0, \infty)$, the paths $t \mapsto W_{\frac{1}{t}}$ are continuous as they are the composition of two continuous functions: $t \mapsto W_t$ and $t \mapsto \frac{1}{t}$. Therefore on $(0, \infty)$ the paths $t \mapsto B_t^k$ are continuous as they are the product of two continuous functions: the function $t \mapsto t^k$ and the continuos paths $t \mapsto W_{\frac{1}{t}}$. Since

$$\lim_{t \mapsto 0} B_t^k = \lim_{t \mapsto 0} t^k W_{\frac{1}{t}} = \lim_{t \mapsto \infty} \frac{1}{t^{k-1}} \frac{W_t}{t} = 0,$$

the paths of the stochastic process B^k are also continuous at 0, hence continuous on $[0, \infty)$.

(b) [4 marks, seen similar] For $0 \leq s \leq t$, observe that

$$B_t^k - B_s^k = t^k W_{\frac{1}{t}} - s^k W_{\frac{1}{s}} = \left(t^k - s^k\right) W_{\frac{1}{t}} - s^k \left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right)$$

and $W_{\frac{1}{t}}$, $\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right)$ are independent normally distributed random variables (using from the properties of the Brownian motion W). It follows that $B_t^k - B_s^k$ is a linear combination of two independent normally distributed random variables, hence it is itself a normally distributed random variable.

(c) [4 marks, seen similar] The mean of $B_t^k - B_s^k$ is equal to the mean of the same linear combination of the component means and the variance of $B_t^k - B_s^k$ is equal with the sum of the component variances multiplied by the square of the corresponding coefficients. Since $W_{\frac{1}{t}} \sim N\left(0, \frac{1}{t}\right)$ and

$$\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right) \sim N\left(0, \frac{1}{s} - \frac{1}{t}\right),$$

it follows that $B_t^k - B_s^k$ has mean

$$\mathbb{E}\left[B_t^k - B_s^k\right] = \left(t^k - s^k\right) \mathbb{E}\left[W_{\frac{1}{t}}\right] - s^k \mathbb{E}\left[W_{\frac{1}{s}} - W_{\frac{1}{t}}\right] = 0.$$

Since the mean of $B_t^k - B_s^k$ is 0 the variance of $B_t^k - B_s^k$ is equal to the second moment. So using the independence of $W_{\frac{1}{t}}$and $\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right)$ we deduce that

$$\begin{aligned} \mathbb{E}\left[\left(B_t^k - B_s^k\right)^2\right] &= \left(t^k - s^k\right)^2 \mathbb{E}\left[\left(W_{\frac{1}{t}}\right)^2\right] + s^{2k} \mathbb{E}\left[\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right)^2\right] \\ &= \frac{\left(t^k - s^k\right)^2}{t} + s^{2k} \times \left(\frac{1}{s} - \frac{1}{t}\right) \\ &= \frac{1}{t}\left(\left(t^k - s^k\right)^2 + s^{2k-1}(t - s)\right) \end{aligned}$$

(d) [2+4 marks, not seen]
$\Longrightarrow$ If B^k is a standard Brownian motion, using one of the results in the lectures, if follows that $\mathbb{E}\left[B_t^k B_s^k\right] = s$ for any $0 \leq s \leq t$. Since

$$\mathbb{E}\left[B_t^k B_s^k\right] = t^k s^k \mathbb{E}\left[W_{\frac{1}{t}} W_{\frac{1}{s}}\right] = t^k s^k \frac{1}{t} = t^{k-1} s^k$$

this can only be true if $k = 1$.
$\Longleftarrow$ For $k = 1$, the process B^k is continuous, has normally distributed increments with mean 0 and variance

$$\mathbb{E}\left[\left(B_t^k - B_s^k\right)^2\right] = \frac{1}{t}\left((t-s)^2 + s\,(t-s)\right) = t - s.$$

The only other property that needs to be checked is the independent increments property. For this it is enough to show that, for arbitrary $0 \leq r \leq s \leq t$, the random variable $B_t^k - B_s^k$ is indepenedent of B_r^k. Since the pair $\left(B_t^k - B_s^k, B_r^k\right)$ is formed of linear combinations of normally distributed zero mean independent random variables, it follows that $B_t^k - B_s^k$ and B_r^k are jointly normally distributed random variable. Hence they are independent if the expected value of their product is 0. Indeed, we have

$$\mathbb{E}\left[\left(B_t^k - B_s^k\right) B_r^k\right] = \mathbb{E}\left[B_t^k B_r^k\right] - \mathbb{E}\left[B_s^k B_r^k\right] = r - r = 0.$$

This implies that the process B^k is indeed a Brownian motion if $k = 1$.

Question 2. [20 marks]

(a) [3 marks, seen similar] Both the drift and the difussion coefficients in equation (4) are Lipschitz continuous functions (they are linear) therefore the equation has a unique soultion in accordance with one of the theorems in the lectures.
(b) [5 marks, seen similar] Let $\nu = \{\nu_t, t > 0\}$ be the semi-martingale defined by

$$\nu_t = \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t, \qquad t \geq 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned} S_t &= \exp(\nu_t) \\ &= 1 + \int_0^t \exp(\nu_s)\, d\nu_s + \frac{1}{2}\int_0^t \exp(\nu_s)\, d\langle \nu \rangle_s \\ &= 1 + \int_0^t S_s \left(\sigma dW_s + \left(\mu - \frac{\sigma^2}{2}\right) ds\right) + \frac{\sigma^2}{2}\int_0^t S_s ds \\ &= 1 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s \end{aligned}$$

(c) [3+5 marks, not seen]
$\Longrightarrow$ S is a semi-martingale with a finite variation part given

$$\mu \int_0^t S_s ds.$$

If S is a martingale, it follows that the finite variation part of S must be 0 (by the uniqueness of the Doob-Meyer decomposition). Since S is non-zero, the only way to ensure that the finite variation part of S is 0 is to have $\mu = 0$.
$\Longleftarrow$ If $\mu = 0$, then S is given by the sum

$$S_t = 1 + \sigma \int_0^t S_s dW_s.$$

To show that S is a martingale is suffices to prove that the stochastic integral is a genuine martingale, which is ensured by showing that

$$\mathbb{E}\left[\int_0^t S_s^2 ds\right] = \int_0^t \mathbb{E}\left[S_s^2\right] ds < \infty. \tag{1}$$

From (b) we deduce that

$$\mathbb{E}\left[S_s^2\right] = \mathbb{E}\left[\exp\left(2\sigma W_s + 2\left(\mu - \frac{\sigma^2}{2}\right)s\right)\right] = e^{\left(2\mu - \sigma^2\right)s}\mathbb{E}\left[\exp\left(2\sigma W_s\right)\right] = e^{\left(2\mu + \sigma^2\right)s}$$

which indeed implies (1) (when $\mu = 0$).
(d) [4 marks, not seen] Observe that

$$\lim_{t\to\infty} \frac{\nu_t}{t} = \sigma \lim_{t\to\infty} \frac{W_t}{t} + \left(\mu - \frac{\sigma^2}{2}\right) = \left(\mu - \frac{\sigma^2}{2}\right) < 0.$$

It follows that

$$\lim_{t\to\infty} \nu_t = \lim_{t\to\infty} \frac{\nu_t}{t} \times \lim_{t\to\infty} t = -\infty$$

and therefore

$$\lim_{t\to\infty} S_t = \lim_{t\to\infty} \exp\left(\nu_t\right) = 0.$$

Question 3. (20 marks)

(a) [6 marks (1+2+3), seen similar] We need to prove that ρ and ρ^n are non-negative, finite and countably additive set functions. We show this for ρ only as the arguments for ρ^n are identical.

i. For any $A \in B(\mathbb{R})$, $I_A(X_t)\left(\int_0^t (\sin(X_s))^2\, ds\right)$ is a non-negative random variable and therefore also its expectation

$$\rho(A) = \mathbb{E}\left[I_A(X_t)\left(\int_0^t (\sin(X_s))^2\, ds\right)\right]$$

will be non-negative.

ii. Since

$$\rho(\mathbb{R}) = \mathbb{E}\left[I_{\mathbb{R}}(X_t)\left(\int_0^t (\sin(X_s))^2\, ds\right)\right] \leq \mathbb{E}\left[\left(\int_0^t (\sin(X_s))^2\, ds\right)\right] \leq t,$$

it follows that the measure ρ has finite mass.

iii. Let $\xi = \int_0^t (\sin(X_s))^2\, ds$. For mutually disjoint sets $A_i \in B(\mathbb{R})$, $i = 1, 2, ...,$ we have by the monotone convergence theorem that

$$\begin{aligned}
\rho\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathbb{E}\left[I_{\cup_{i=1}^{\infty} A_i}(X_t)\xi\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\infty} I_{A_i}(X_t)\xi\right] \\
&= \mathbb{E}\left[\lim_{N\mapsto\infty}\sum_{i=1}^{N} I_{A_i}(X_t)\xi\right] \\
&= \lim_{N\mapsto\infty}\mathbb{E}\left[\sum_{i=1}^{N} I_{A_i}(X_t)\xi\right] = \sum_{i=1}^{\infty}\rho(A_i).
\end{aligned}$$

Note that for $t = 0$ the measure is trivially null for any arbitrary set in $B(\mathbb{R})$.

b. (6 marks, seen similar) By Itô's formula we have that

$$\begin{aligned}
\varphi(X_t) &= \varphi(X_0) + \int_0^t A\varphi(X_s)ds + \int_0^t \sigma\varphi'(X_s)dV_s \\
\tilde{z}_t &= 0 + \int_0^t (\sin(X_s))^2\, ds,
\end{aligned}$$

where $\tilde{z}_t := \int_0^t (\sin(X_s))^2\, ds$ and A is the infinitesimal generator of the process X. Note that both $\varphi(X_.)$ and $\tilde{z}$ are semi-martingales and $[\varphi(X_.), \tilde{z}]_t = 0$ since the martingale part of $\tilde{z}$ is null. Then, by integration by parts, we get that

$$\begin{aligned} q_t(\varphi) &= q_0(\varphi) + \int_0^t \varphi(X_s) d\tilde{z}_s + \int_0^t \tilde{z}_s d\varphi(X_s) \\ &= \int_0^t \varphi(X_s) (\sin(X_s))^2 \, ds + \int_0^t \tilde{z}_s A\varphi(X_s) ds + \int_0^t \tilde{z}_s \sigma \varphi'(X_s) dV_s. \end{aligned} \tag{2}$$

In the above we used that fact that $\tilde{z}_0 = 0$ and hence $q_0(\varphi) = 0$.
b. (8 marks, not seen) We obtain the evolution equation for $\rho(\varphi)$ by taking expectation of identity 2. We have, for any $\varphi \in C_b^2(\mathbb{R})$, that

$$\rho_t(\varphi) = \mathbb{E}\left[\int_0^t \varphi(X_s) (\sin(X_s))^2 \, ds\right] + \mathbb{E}\left[\int_0^t \tilde{z}_s A\varphi(X_s) ds\right] + \mathbb{E}\left[\int_0^t \tilde{z}_s \sigma \varphi'(X_s) dV_s\right] \tag{3}$$

$$\begin{aligned} &= E[q_0(\varphi)] + E\left[\int_0^t \varphi(X_s)\tilde{z}_s h(X_s) ds\right] + E\left[\int_0^t \tilde{z}_s A\varphi(X_s) ds\right] \\ &\quad + E\left[\int_0^t \tilde{z}_s \sigma \varphi'(X_s) dV_s\right] \\ &= \pi_0(\varphi) + \int_0^t E[A\varphi(X_s)\tilde{z}_s] ds + \int_0^t E[\varphi(X_s)h(X_s)\tilde{z}_s] ds \\ &= \pi_0(\varphi) + \int_0^t m_s(A\varphi) ds + \int_0^t m_s(h\varphi) ds. \end{aligned}$$

First we show that each of the integrants on the right hand side of 3 are uniformly bounded processes on any compact interval, as,

$$\begin{aligned} \sup_{s\in[0,t]} \left|\varphi(X_s) (\sin(X_s))^2\right| &\leq \|\varphi\|_\infty \\ \sup_{s\in[0,t]} |\tilde{z}_s A\varphi(X_s)| &\leq t\|A\varphi\|_\infty \\ \sup_{s\in[0,t]} |\tilde{z}_s \sigma \varphi'(X_s)| &\leq t\|\varphi'\|_\infty \|\sigma\|_\infty \end{aligned}$$

It follows that we can apply Fubini in the first two terms of of 3. Also the last term is a square integrable martingale and therefore its expectation is equal to 0. We deduce that

$$\begin{aligned} \rho_t(\varphi) &= \int_0^t \mathbb{E}\left[\varphi(X_s) (\sin(X_s))^2\right] ds + \int_0^t \mathbb{E}\left[\tilde{z}_s A\varphi(X_s)\right] ds \\ &= \int_0^t p_s\left(\tilde{\varphi}\right) ds + \int_0^t \rho_s(A\varphi) ds, \end{aligned}$$

where p is the prior distribution of the process X and $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$, $\tilde{\varphi}(x) := \varphi(x) (\sin(x))^2$ for any $x \in \mathbb{R}$.

Question 4. (20 marks)

(a). [6 marks seen similar] Using the fact that f and σ are bounded, we have by Itô's isometry theorem, that

$$\begin{aligned}
\mathbb{E}[|X_u - X_v|] &\leq \mathbb{E}[|X_u - X_v|^2]^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\left|\int_u^v f\left(X_s\right) ds\right|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\left|\int_u^v \sigma\left(X_s\right) dV_s\right|^2\right]^{\frac{1}{2}} \\
&\leq ||f||_\infty |u - v| + \mathbb{E}\left[\int_u^v \sigma\left(X_s\right)^2 ds\right]^{\frac{1}{2}} \\
&\leq ||f||_\infty |u - v| + ||\sigma||_\infty \sqrt{|u - v|}. \\
&\leq (||f||_\infty \sqrt{|u + v|} + ||\sigma||_\infty)\sqrt{|u - v|}.
\end{aligned}$$

for $0 \leq u \leq v \leq t$ Hence the required inequality holds with $m = \left(\sqrt{2t}\, ||f||_\infty + ||\sigma||_\infty\right)^2$.

b. (7 marks, not seen) Let $\xi_n = \frac{t}{n}\sum_{i=0}^{n-1}\left(\sin(X_{\frac{it}{n}})\right)^2$. Since a is Lipschitz we have that

$$|\xi - \xi_n| = \left|\sum_{i=0}^{n-1}\int_{\frac{it}{n}}^{\frac{(i+1)t}{n}} \left((\sin(X_s))^2 - \left(\sin(X_{\frac{it}{n}})\right)^2\right) ds\right|$$

Observe that, by the mean value theorem, there exists $\theta\left(X_s, X_{\frac{it}{n}}\right)$ in between X_s and $X_{\frac{it}{n}}$ such that

$$\begin{aligned}
\left|(\sin(X_s))^2 - \left(\sin(X_{\frac{it}{n}})\right)^2\right| &= 2\left|\sin\left(\theta\left(X_s, X_{\frac{it}{n}}\right)\right)\cos\left(\theta\left(X_s, X_{\frac{it}{n}}\right)\right)\right|\left|X_s - X_{\frac{it}{n}}\right| \\
&\leq 2\left|X_s - X_{\frac{it}{n}}\right|.
\end{aligned}$$

Then

$$\mathbb{E}\left[|\xi - \xi_n|\right] \leq 2\sum_{i=0}^{n-1}\int_{\frac{it}{n}}^{\frac{(i+1)t}{n}} \mathbb{E}\left[\left|X_s - X_{\frac{it}{n}}\right|\right] ds.$$

Using the fact that

$$\mathbb{E}[|X_u - X_v|] \leq m\sqrt{|u - v|} \quad \forall u, v \in [0, t],$$

we deduce that

$$\begin{aligned}
\mathbb{E}\left[|X_s - X_{t_i}|\right] &\leq m\sqrt{s - t_i} \\
&\leq \frac{m}{\sqrt{n}}.
\end{aligned}$$

Hence

$$\mathbb{E}\left[|\xi - \xi_n|\right] \le \frac{2mt}{\sqrt{n}}.$$

Finaly, we get that

$$\begin{aligned} |\rho(\varphi) - \rho^n(\varphi)| &\le \mathbb{E}\left[\varphi(X_t)\,|\xi - \xi_n|\right] \\ &\le ||\varphi||\,\mathbb{E}\left[|\xi - \xi_n|\right] \\ &\le \frac{d}{\sqrt{n}} \end{aligned}$$

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with $d = 2mt\,||\varphi||$.

(c). [6 marks, see similar] Observe first that if $\mathbf{1}$ is the function identicallly equal to 1, then $\rho_t^n(\mathbf{1}) = \rho_t^n(\mathbb{R})$ and $\rho_t(\mathbf{1}) = \rho_t(\mathbb{R})$. From **(b)** we deduce that

$$|\rho_t^n(\mathbb{R}) - \rho_t(\mathbb{R})| \le \frac{2mt}{\sqrt{n}}.$$

Next we have the following

$$\begin{aligned} \bar{\rho}_t(\varphi) - \bar{\rho}_t^n(\varphi) &= \frac{\rho_t(\varphi)}{\rho_t(\mathbb{R})} - \frac{\rho_t^n(\varphi)}{\rho_t^n(\mathbb{R})} \\ &= \frac{\rho_t(\varphi)}{\rho_t(\mathbb{R})} - \frac{\rho_t^n(\varphi)}{\rho_t(\mathbb{R})} + \frac{\rho_t^n(\varphi)}{\rho_t(\mathbb{R})} - \frac{\rho_t^n(\varphi)}{\rho_t^n(\mathbb{R})} \\ &= \frac{1}{\rho_t(\mathbb{R})}\left(\rho_t(\varphi) - \rho_t^n(\varphi)\right) + \frac{\rho_t^n(\varphi)}{\rho_t^n(\mathbb{R})\rho_t(\mathbb{R})}\left(\rho_t^n(\mathbb{R}) - \rho_t(\mathbb{R})\right). \end{aligned}$$

Since $\rho_t^n(\varphi) \le ||\varphi||_\infty\,\rho_t^n(\mathbb{R})$ we deduce from the above that

$$\begin{aligned} |\bar{\rho}(\varphi) - \bar{\rho}^n(\varphi)| &\le \frac{1}{\rho(\mathbb{R})}\,|\rho(\varphi) - \rho^n(\varphi)| + \frac{||\varphi||_\infty}{\rho(\mathbb{R})}\,|\rho^n(\mathbb{R}) - \rho(\mathbb{R})| \\ &\le \frac{\bar{c}}{\sqrt{n}}, \end{aligned}$$

where $\bar{c} = \dfrac{2d}{\rho(\mathbb{R})} > 0$.

Question 5. (20 marks)

(a) [5 marks, seen similar] Novikov's condition states that if $u = \{u_t, t > 0\}$ is a process defined as $u_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ for M a continuous local martingale, then a sufficient condition for u to be a martingale is that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] < \infty, \qquad 0 \le t < \infty.$$

In this case the process $t \to \int_0^t \pi_s(h)dY_s$ is a local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by $t \to \int_0^t \pi_s(h)^2 ds$. Moreover, since h is bounded and π_s is a probability measure, it follows that $|\pi_s(h)| \le \|h\|_\infty$ and hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \pi_s(h)^2\, ds\right)\right] \le \exp\left(\frac{t\,\|h\|_\infty^2}{2}\right) < \infty, \qquad 0 \le t < \infty.$$

Hence, by Novikov's condition, the process $z = \{z_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Moreover since the process π is adapted to the filtration $\mathcal{Y}_t$ the property remains true for z as well.

(c) [3 marks, seen similar] Let $\xi = \{\xi_t, t > 0\}$ be the semimartingale defined by

$$\xi_t = \int_0^t \pi_s(h)dY_s - \frac{1}{2}\int_0^t \pi_s(h)^2 ds, \qquad t \ge 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned}
z_t &= \exp\left(\xi_t\right) \\
&= \exp\left(\xi_0\right) + \int_0^t \exp\left(\xi_s\right) d\xi_s + \frac{1}{2}\int_0^t \exp\left(\xi_s\right) d\left\langle \xi\right\rangle_s \\
&= 1 + \int_0^t z_s\left(\pi_s(h)dY_s - \frac{1}{2}\pi_s(h)^2 ds\right) + \frac{1}{2}\int_0^t z_s\pi_s(h)^2 ds \\
&= 1 + \int_0^t z_s\pi_s(h)dY_s.
\end{aligned}$$

(d) [7 marks, seen similar] Observe that

$$\begin{aligned}
z_t^m &= \exp\left(\frac{m^2 - m}{2}\int_0^t \pi_s(h)^2 ds\right)\bar{z}_t \\
&\le \exp\left(\frac{t}{2}\left|m^2 - m\right| \|h\|_\infty^2\right)\bar{z}_t,
\end{aligned}$$

where $\bar{z} = \{\bar{z}_t, t > 0\}$ is the process defined by

$$\bar{z}_t = \exp\left(\int_0^t m\pi_s(h)dY_s - \frac{1}{2}\int_0^t (m\pi_s(h))^2\ ds\right),\ t \geq 0.$$

Again, by Novikov's condition, the process $\bar{z} = \{\bar{z}_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Hence

$$\tilde{\mathbb{E}}[\bar{z}_t] = \tilde{\mathbb{E}}[\bar{z}_0] = 1$$

and

$$\begin{aligned}\sup_{t\in[0,1]} \tilde{\mathbb{E}}[z_t^m] &\leq \sup_{t\in[0,T]} \exp\left(\frac{t}{2}\left|m^2 - m\right| \|h\|_\infty^2\right)\tilde{\mathbb{E}}[\bar{z}_t] \\ &= \exp\left(\frac{1}{2}\left|m^2 - m\right| \|h\|_\infty^2\right) < \infty\end{aligned}$$

for any $T > 0$.

(d) [5 marks, not seen] Since $\left.\dfrac{dP}{d\tilde{P}}\right|_{\mathcal{F}_t} = \dfrac{1}{Z_t}$, we deduce that

$$\mathbb{E}[z_t^m] = \tilde{\mathbb{E}}[z_t^m (Z_t)^{-1}] \leq \sqrt{\tilde{\mathbb{E}}[z_t^{2m}]\tilde{\mathbb{E}}[(Z_t)^{-2}]} = \sqrt{\tilde{\mathbb{E}}[z_t^{2m}]\mathbb{E}[(Z_t)^{-1}]}.$$

Since

$$\begin{aligned}(Z_t)^{-1} &= \exp\left(\int_0^t h(X_s)\,dW_s + \frac{1}{2}\int_0^t (h(X_s))^2\ ds\right) \\ &= e^{\int_0^t (h(X_s))^2\,ds}\exp\left(\int_0^t h(X_s)\,dW_s - \frac{1}{2}\int_0^t (h(X_s))^2\ ds\right) \\ &\leq e^{t||h||_\infty^2}\exp\left(\int_0^t h(X_s)\,dW_s - \frac{1}{2}\int_0^t (h(X_s))^2\ ds\right)\end{aligned}$$

we deduce, with a similar argument as that used in part (c), that

$$\sup_{t\in[0,1]} \mathbb{E}[(Z_t)^{-1}] = e^{t||h||_\infty^2} < \infty.$$

The result follows from part (c) after replacing m by $2m$ and using the fact that

$$\sup_{t\in[0,1]} \tilde{\mathbb{E}}[(Z_t)^{-2}] \leq \sqrt{\sup_{t\in[0,1]} \tilde{\mathbb{E}}[z_t^{2m}]\ \sup_{t\in[0,1]} \tilde{\mathbb{E}}[(Z_t)^{-2}]}.$$

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ExamModuleCode	QuestionNumber	Comments for Students
Stochastic Calculus with Applications to Non-linear Filtering_MATH97061 MATH70055	1	A mixed bag here. Some students gave perfect answers, some not at all.
Stochastic Calculus with Applications to Non-linear Filtering_MATH97061 MATH70055	2	This question was answered uniformly well by all students
Stochastic Calculus with Applications to Non-linear Filtering_MATH97061 MATH70055	3	The question was answered reasonably well by the students
Stochastic Calculus with Applications to Non-linear Filtering_MATH97061 MATH70055	4	The question was answered reasonably well by the students
Stochastic Calculus with Applications to Non-linear Filtering_MATH97061 MATH70055	5	The first three parts of the question were answered well. The last part less so.