3.8 Rank of a Matrix

Definition 3.8.1. Let A be an $m \times n$ matrix with entries from a field F. Define:

- The *Row Space of A (RSp(A))* as the span of the rows of A. This is a subspace of F^n.

- The *Row Rank of A* is $dim(RSp(A))$.

- The *Column Space of A (CSp(A))* as the span of the columns of A. This is a subspace of F^m.

- The *Column Rank of A* is $dim(CSp(A))$.

Example 3.8.2. Let $F = \mathbb{R}$ and $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$. Then,

$$RSp(A) = Span\{(3 \quad 1 \quad 2), (0 \quad -1 \quad 1)\},$$
$$CSp(A) = Span\left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

Now the row vectors $(3 \quad 1 \quad 2)$ and $(0 \quad -1 \quad 1)$ are linearly independent so $dim(RSp(A)) = 2$, so the column rank is 2. The set

$$\left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

is linearly dependent as

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So

$$CSp(A) = Span\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

which is linearly independent, so $dimCSp(A) = 2$.

Procedure 3.8.3.

Calculating the row rank of a matrix A.

- <u>Step 1</u>: Reduce A to row echelon form using row operations:

$$A_{ech} = \begin{pmatrix} 1 & * & * & * & * & \ldots \\ 0 & 0 & 1 & * & * & \ldots \\ 0 & 0 & 0 & 1 & * & *\ldots \\ \vdots & & & & & \\ 0 & \ldots & & & & \end{pmatrix}$$

(Actually it doesn't matter whether the leading entries in each row are 1s or not.)

- <u>Step 2</u>: The row rank of A is the number of non-zero rows in A_{ech}. In fact it the non-zero

rows of A_{ech} form a basis for $RSp(A)$.

It will be enough to show:

Example 3.8.4. Find the row rank of $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ -1 & 4 & 15 \end{pmatrix}$

Example 3.8.5. Find the dimension of

$$W = Span\{(\ -1 \quad 1 \quad 0 \quad 1\ ), (\ 2 \quad 3 \quad 1 \quad 0\ ), (\ 0 \quad 1 \quad 2 \quad 3\ )\} \subseteq \mathbb{R}^4.$$

We can find the column rank of a matrix in a very similar way to finding the row rank of a matrix.

Procedure 3.8.6. The columns of A are the rows of A^T so we can apply Procedure 3.8.3 to A^T.

Alternatively: use column operations to resuce A to "column echelon form and then count the non-zero columns.

Example 3.8.7. Let $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ -1 & 4 & 15 \end{pmatrix}$. Find the column rank of A. This equals the row rank

of A^T.

Theorem 3.8.8. For any matrix A the row rank of A is equal to the column rank of A.

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Example 3.8.9. Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 \end{pmatrix}$

Note that $r_3 = r_1 + r_2$, so a basis for $RSp(A)$ is

$$\{\underbrace{(1,2,-1,0)}_{v_1} \quad , \quad \underbrace{(-1,1,0,1)}_{v_2}\}$$

Write the rows as linear combinations of v_1 and v_2:

$$r_1 =$$
$$r_2 =$$
$$r_3 =$$

These co-efficients are the λ_{ij}'s from the proof:

$$\lambda_{11} = \qquad \lambda_{12} =$$
$$\lambda_{21} = \qquad \lambda_{22} =$$
$$\lambda_{31} = \qquad \lambda_{32} =$$

According to the proof, a spanning set for $CSp(A)$ is:

$$\begin{pmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{pmatrix} = \qquad , \begin{pmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{pmatrix} =$$

Check this is really a spanning set for $CSP(A)$: Let $w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Now we have:

$$c_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$c_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} =$$

$$c_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} =$$

$$c_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} =$$

So it is indeed the case that $\{w_1, w_2\}$ spans $CSp(A)$.

Definition 3.8.10. Let A be a matrix. The *rank of A* written $rank(A)$ or $rk(A)$, is the row rank of A (or the column rank since they are the same).

Proposition 3.8.11. Let A be an $n \times n$ matrix with entried in F, then the following statements are equivalent:

1. $rank(A) = n$ ("A has full rank").

2. The rows of A form a basis for F^n.

3. The columns of A form a basis for F^n.

4. A is invertible (so $det(A) \neq 0$, etc.).

4 Linear Transformations

4.1 Introduction

Definition 4.1.1. Suppose V, W are vector spaces over a field F. Let $T : V \longrightarrow W$ be a function from V to W. We say:

- T *preserves addition* if for all $v_1, v_2 \in V$ we have $T(v_1 + v_2) = T(v_1) + T(v_2)$. (i.e. if $T(v_1) = w_1, T(v_2) = w_2$ for $w_1, w_2 \in W$ we have $T(v_1 + v_2) = w_1 + w_2$.

- T *preserves scalar multiplication* if for all $v \in V$, $\lambda \in F$, $T(\lambda v) = \lambda T(v)$.

- T is a *linear transformation* (or *linear map*) if it:
    1. preserves addition.
    2. preserves scalar multiplication

Example 4.1.2.

(a) The identity map $T : V \longrightarrow V$ is obviously a linear transformation.

(b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T(x, y) = x + y$ is a linear transformation.

(c) Let V be the space of all polynomials in x over $\mathbb{R}$ (i.e. $V = \mathbb{R}[x]$). Defin $T : V \longrightarrow V$ by $T(f(x)) = \frac{d}{dx} f(x)$. Then T is a linear map.

(d) Let $V = \mathbb{C}$ (as a 1-dimensional vector space over $\mathbb{C}$). The map $T(z) = \bar{z}$ is **not** a linear map:

(e) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by $T(x, y, z) = (xyz)^{\frac{1}{3}}$ then:

(f) Lots of functions preserve neither addition nor scalar multiplication, e.g., for $\mathbb{R} \to \mathbb{R}$ the functions taking $x \mapsto x + 1$, $x \mapsto x^2$, and $x \mapsto e^x$.

Proposition 4.1.3. Let A be an $m \times n$ matrix over F. Define $T : F^n \longrightarrow F^m$ (spaces of column vectors), by $T(v) = Av$ (for $v \in F^n$. Then T is a linear transformation.

Proposition 4.1.4. *Basic Properties of linear transformations*

Let $T : V \longrightarrow W$ be a linear map. Write 0_V, 0_W for the zero vectors in V and W respectively. We have:

1. $T(0_v) = 0_W$

2. Suppose $v = \lambda_1 v_1 + ... + \lambda_k v_k$ for $\lambda_i \in F$, $v_i \in V$. Then $T(v) = \lambda_1 T(v_1) + ... + \lambda_k T(v_k)$.