M40007: Introduction to Applied Mathematics

Imperial College London

1 Resistive circuit theory and rank-nullity

Suppose that a given connected graph with n nodes and m edges is a resistive circuit. The net current out of the nodes in a graph representing a resistive electric circuit is represented by the vector

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w}, \tag{1}$$

where $\mathbf{w}$ is the vector of currents in the conductors that make up of the edges of the graph.

Suppose Kirchhoff's current law, KCL, holds at all the nodes in the graph. Then

$$\mathbf{f} = 0 \tag{2}$$

which implies, from (1), that

$$\mathbf{A}^T\mathbf{w} = 0. \tag{3}$$

On taking a transpose of this equation we find that

$$\mathbf{w}^T\mathbf{A} = 0. \tag{4}$$

This means that the vector of currents in the circuit must lie in the left null space of $\mathbf{A}$. But, by the rank-nullity theorem, the left null space of $\mathbf{A}$ is orthogonal to the column space of $\mathbf{A}$. Thus $\mathbf{w}$ is orthogonal to the column space of $\mathbf{A}$. Recall that the column space of $\mathbf{A}$ is the space of vectors that can be written as

$$\mathbf{A}\mathbf{x} \tag{5}$$

for some vector of coefficients $\mathbf{x}$. The rank-nullity theorem says that $\mathbf{w}$ is orthogonal to the space of such vectors. Expressed differently, unless $\mathbf{w}$ is the zero vector, there is no vector $\mathbf{x}$ for which we can write

$$\mathbf{w} = \mathbf{A}\mathbf{x}. \tag{6}$$

Physically, since by Ohm's law currents are set up in the conductors of a circuit by voltage drops between the nodes, this means that there is no set of voltages $\mathbf{x}$ at the nodes which will produce the set of currents represented by $\mathbf{w}$.

Since, under the assumption that KCL holds at *all* the nodes in the circuit, there is no admissible set of voltages at the nodes that can produce a non-zero current in the circuit, then in order to get a non-zero current the rank-nullity theorem tells us we are forced to *relax the assumption* that KCL holds at all the nodes.

Mathematically this means that we must have *at least one* of the elements of $\mathbf{f}$ to be non-zero. Recall that KCL holding at a node means the element of $\mathbf{f}$ corresponding to that node is zero.

Actually, using the rank-nullity theorem again, we can be more precise about this. Equation (1) implies that

$$-\mathbf{f} = \mathbf{A}^T \mathbf{w}. \tag{7}$$

This means that $-\mathbf{f}$ lies in the column space of $\mathbf{A}^T$: it can be written as a linear combination, with coefficients being the elements of the vector $\mathbf{w}$, of the columns of $\mathbf{A}^T$. But the column space of $\mathbf{A}^T$ is, by definition, the row space of $\mathbf{A}$. The rank-nullity theorem tells us that the right null space of $\mathbf{A}$ is perpendicular to its row space. Since we know that $\mathbf{x}_0$ – the n-dimensional vector with all entries equal to 1 – lies in the right null space of $\mathbf{A}$, then necessarily

$$\mathbf{x}_0^T \mathbf{f} = 0. \tag{8}$$

Denoting the entries of $\mathbf{f}$ by f_i for $i = 1, \ldots, n$ (8) is equivalent to

$$\sum_{i=1}^{n} f_i = 0. \tag{9}$$

In summary, to have a non-zero current in a circuit represented by a given connected graph, at least some of the nodes of the graph must be connected to an external current source in such a way that the net current sources $\{f_i | i = 1, \ldots, n\}$ at each of the n nodes satisfies (9). The mathematics tells us this, and no electrical engineer will be surprised by it.

2 Two-point source/sink circuits

Given these constraints, the simplest first case to consider is $\mathbf{f}$ given by

$$\mathbf{f} = (+1, -1, \underbrace{0, 0, \cdots, 0, 0}_{n-2 \text{ copies}})^T. \tag{10}$$

We have chosen just *two* of the nodes for special treatment: in this case, the nodes corresponding to first two elements of $\mathbf{f}$, although any choice will do. Let us call these node 1 and node 2. These two special nodes will be the chosen *boundary nodes*. At all other nodes, KCL is assumed to hold, as encoded in (10). The choice (10) clearly satisfies the condition

$$\mathbf{x}_0^T \mathbf{f} = \sum_{i=1}^{n} f_i = 0. \tag{11}$$

The choice (10) corresponds to a net current $+1$ entering the circuit at ("diverging from") node 1 and a net current -1 entering the circuit at ("diverging from") node 2. It is natural to think of a positive net current into the circuit as a *current source* and a negative net current into the circuit as a *current sink*. Physically, this is easy to set up in practice by connecting the $+$ and $-$ electrodes of a battery to nodes 1 and 2. A battery is an external source of current.

2.1 The Neumann problem

One of the simplest two-point source/sink problems to consider is to specify the vector $\mathbf{f}$ of net currents is given by (10) and to ask: what are the voltages at all the nodes? Let $\mathbf{x}$ be the vector containing the required voltages.

It is easiest to answer this problem in the context of a specific circuit. Figure 1 shows our familiar graph with these physical conditions at the boundary nodes 1 and 2 imposed. The voltage at node 2 has been set equal to zero for reasons to be explained next.

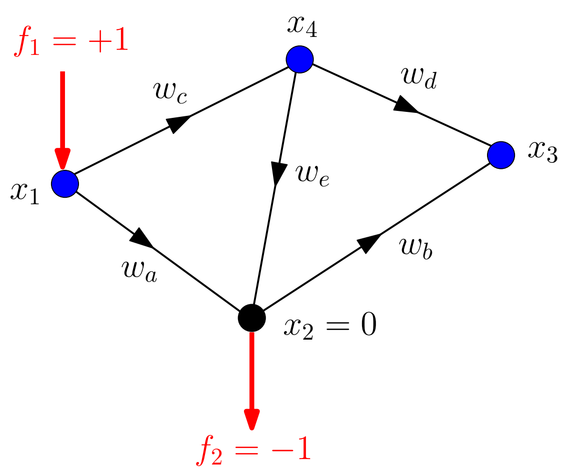

Figure 1: The Neumann problem with node 2 grounded: $x_2 = 0$. A unit current enters the circuit at node 1 and leaves it at node 2. KCL holds at the other nodes.

In this example,

$$\mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \tag{12}$$

where the zeros are entered for all nodes for which KCL holds. The basic relation between $\mathbf{f}$ and $\mathbf{x}$ is given by

$$\mathbf{K}\mathbf{x} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \mathbf{x} = \mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

This looks like a familiar linear algebra problem and it is therefore tempting to suggest that the solution is simply

$$\mathbf{x} = \mathbf{K}^{-1}\mathbf{f}. \tag{14}$$

But this is *not* the case because $\mathbf{K}$ is a singular matrix and its inverse $\mathbf{K}^{-1}$ does not exist. We know $\mathbf{K}$ is singular because

$$\mathbf{K}\mathbf{x}_0 = \mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \mathbf{A}^T(\mathbf{A}\mathbf{x}_0) = 0, \tag{15}$$

where the final equality follows because the non-trivial vector $\mathbf{x}_0$ is in the right null space of $\mathbf{A}$. But (15) implies that $\mathbf{x}_0$ is also in the right null space of the n-by-n matrix $\mathbf{K}$. Hence, by the rank-nullity theorem, the rank of $\mathbf{K}$ is at most $n-1$. In fact, we can argue that the rank of $\mathbf{K}$ is exactly $n-1$. For suppose the some other non-zero vector $\hat{\mathbf{x}}_0$ that is linearly independent of $\mathbf{x}_0$ also lies in the right null space of $\mathbf{K}$. Then

$$\mathbf{K}\hat{\mathbf{x}}_0 = 0. \tag{16}$$

This implies that

$$\hat{\mathbf{x}}_0^T\mathbf{K}\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0^T\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}}_0 = ||\mathbf{A}\hat{\mathbf{x}}_0||^2 = 0, \tag{17}$$

where $||\mathbf{a}||^2$ denotes the Euclidean norm of the vector. The only vector with a zero Euclidean norm is the zero vector implying that we must have

$$\mathbf{A}\hat{\mathbf{x}}_0 = 0. \tag{18}$$

Therefore $\hat{\mathbf{x}}_0$ lies in the right null space of $\mathbf{A}$. However the right null space of $\mathbf{A}$ is the 1-dimensional space spanned by $\mathbf{x}_0$. We have therefore reached a contradiction with our initial assumption that $\hat{\mathbf{x}}_0$ that is linearly independent of $\mathbf{x}_0$. We conclude that $\mathbf{x}_0$ is the *only* null vector of $\mathbf{K}$ and that the rank of $\mathbf{K}$ is $n-1$.

Grounding a node: Since $\mathbf{x}_0$ is a right null vector of $\mathbf{K}$ – indeed its only right null vector to within rescaling – then if $\mathbf{x}$ is a solution to (13) then so too is $\mathbf{x} + c\mathbf{x}_0$ where c is any constant since

$$\mathbf{K}(\mathbf{x} + c\mathbf{x}_0) = \mathbf{K}\mathbf{x} + c\mathbf{K}\mathbf{x}_0 = 0. \tag{19}$$

Physically, adding $c\mathbf{x}_0$ to any solution $\mathbf{x}$ corresponds to adding the same additional voltage c to all the nodes. Adding the same voltage to all nodes does not change the currents flowing in the circuit since those depend, according to Ohm's law, only on the voltage drops, which are unchanged.

It is clear that eliminating the singular nature of $\mathbf{K}$ is equivalent to removing this freedom to add an arbitrary voltage c to all the nodes. One way to do this is to *ground* one of the nodes; that is, to insist that some chosen node has *zero* voltage. If this is to be enforced then one loses the freedom to add an arbitrary voltage c to all the nodes. The choice of which node to ground is ours. Returning to the circuit in

Figure 1 let us ground node 2. This is equivalent to setting

$$x_2 = 0. \tag{20}$$

We can therefore redefine the vector of unknowns, and a vector $\hat{\mathbf{f}}$, to be

$$\hat{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \hat{\mathbf{f}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{21}$$

where we have removed the second element of the vectors $\mathbf{x}$ and $\mathbf{f}$. The system (13) with the 2nd column and 2nd row removed gives the *grounded system*:

$$\hat{\mathbf{K}}\hat{\mathbf{x}} = \hat{\mathbf{f}}, \qquad \hat{\mathbf{K}} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}. \tag{22}$$

Removing the second column is equivalent to setting $x_2 = 0$ which we decided was a good idea. Removing the second row, however, corresponds to eliminating an equation. This is more concerning, and we will return to it later.

First, since the *grounded Laplacian matrix* $\hat{\mathbf{K}}$ is now non-singular, the required solution is

$$\hat{\mathbf{x}} = \hat{\mathbf{K}}^{-1}\hat{\mathbf{f}} = \begin{pmatrix} 5/8 \\ 1/8 \\ 1/4 \end{pmatrix}. \tag{23}$$

The equation we deleted in removing row 2 of the system (13) was

$$-x_1 + 3x_2 - x_3 - x_4 = -1. \tag{24}$$

However it is easily checked that our solution (23), with $x_2 = 0$, satisfies this equation automatically since

$$-\frac{5}{8} + 3 \times 0 - \frac{1}{8} - \frac{1}{4} = -1. \tag{25}$$

This is no accident, of course, since we originally constrained the vector $\mathbf{f}$ appearing on the right hand side of the system (13) to satisfy (11). Had we not done so, the deleted equation (24) would *not* have been satisfied by our solution (23).

3 The Dirichlet problem

A variant of the simple two-point Neumann problem just considered is shown in Figure 2. Now we specify that the voltage at node 1 is +1 and node 2 is grounded:

$$x_1 = 1, \qquad x_2 = 0. \tag{26}$$

At nodes 3 and 4 it is assumed that the KCL holds. The challenge is to find the voltages x_3 and x_4 at nodes 3 and 4. We expect the associated distribution of net currents at the nodes to have the form

$$\mathbf{f} = \begin{pmatrix} f \\ -f \\ 0 \\ 0 \end{pmatrix}, \tag{27}$$

where f is also to be determined. It is the net current entering the circuit at node 1 and exiting at node 2. Figure 2 shows a schematic of this *Dirichlet problem*. In this problem it is the values of the voltage potential that are given at the boundary nodes 1 and 2.

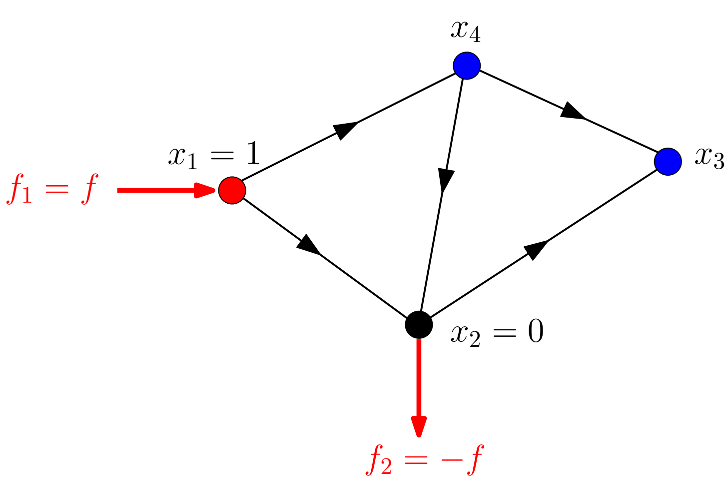

Figure 2: The Dirichlet problem with boundary nodes 1 and 2 set to voltages $x_1 = 1$ and $x_2 = 0$ respectively. The challenge is to find x_3 and x_4, and the net current f entering the circuit at node 1 and leaving it at node 2.

It is easy to quickly determine the solution of this Dirichlet problem having already found the solution to the Neumann problem just considered. This is a consequence of the linearity of the system. The Neumann problem was solved by the vectors

$$\mathbf{x} = \begin{pmatrix} 5/8 \\ 0 \\ 1/8 \\ 1/4 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

A net current +1 enters the circuit at node 1 and -1 leaves it at node 2. The voltage at node 1 is $x_1 = 5/8$ and the voltage at node 2 is zero, $x_2 = 0$, because it is grounded. Suppose we multiply the solution (28) by 8/5 then, by the linearity of the system, it remains a solution of the system but now with

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1/5 \\ 2/5 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 8/5 \\ -8/5 \\ 0 \\ 0 \end{pmatrix}. \tag{29}$$

This satisfies the conditions of the Dirichlet problem depicted in Figure 2 with $f = 8/5$. It is the required solution to the Dirichlet problem.

Suppose, however, that we had not noticed this. How else could we solve the problem?

We still have the basic relation between the voltages $\mathbf{x}$ and the net currents at the nodes $\mathbf{f}$ given by

$$\mathbf{Kx} = \mathbf{f}, \qquad \mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}. \tag{30}$$

It is useful to write this out explicitly,

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f \\ -f \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

since it reveals this to be a slightly curious linear system. The unknown quantities are x_3, x_4 and f. The first two are voltages and sit on the left-hand side of (31); on the other hand, f is the net current into the circuit at node 1 and it sits on the right-hand side of (31).

The Schur complement: To solve such non-standard linear systems, it is natural to introduce the notion of the *Schur complement* of a sub-block matrix in $\mathbf{K}$. Let us introduce the sub-block decomposition

$$\mathbf{K} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{pmatrix}, \tag{32}$$

where

$$\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \tag{33}$$

and let

$$\hat{\mathbf{x}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \qquad \hat{\mathbf{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{f}} = \begin{pmatrix} f \\ -f \end{pmatrix}. \tag{34}$$

We can now write the linear system (31) as

$$\mathbf{K}\mathbf{x} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}} \\ \hat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}} \\ \mathbf{0} \end{pmatrix} = \mathbf{f}, \tag{35}$$

where $\mathbf{0}$ denotes the vector $(0,0)^T$. By carrying out the block matrix multiplication this system can be seen to be equivalent to the two systems

$$\begin{aligned} \mathbf{P}\hat{\mathbf{e}} + \mathbf{Q}^T\hat{\mathbf{x}} &= \hat{\mathbf{f}}, \\ \mathbf{Q}\hat{\mathbf{e}} + \mathbf{R}\hat{\mathbf{x}} &= \mathbf{0}. \end{aligned} \tag{36}$$

It can be verified that $\mathbf{R}$ is invertible so the second of these can be solved for $\hat{\mathbf{x}}$:

$$\hat{\mathbf{x}} = -\mathbf{R}^{-1}\mathbf{Q}\hat{\mathbf{e}}. \tag{37}$$

On substitution of this result into the first of equations (36), we find

$$\hat{\mathbf{f}} = \mathbf{P}\hat{\mathbf{e}}_1 - \mathbf{Q}^T\mathbf{R}^{-1}\mathbf{Q}\hat{\mathbf{e}} = [\mathbf{P} - \mathbf{Q}^T\mathbf{R}^{-1}\mathbf{Q}]\hat{\mathbf{e}}. \tag{38}$$

The matrix

$$\mathbf{P} - \mathbf{Q}^T\mathbf{R}^{-1}\mathbf{Q} \tag{39}$$

is known as the *Schur complement* of the matrix $\mathbf{R}$ in $\mathbf{K}$.

We leave it as an exercise for the reader to confirm that the results (36) and (38) retrieve the result (29) derived earlier.

Figure 3 gives a schematic representation of all the voltages, edge currents, and the net currents at the boundary nodes for this Dirichlet problem. The edge currents can be now calculated using Ohm's law if required.

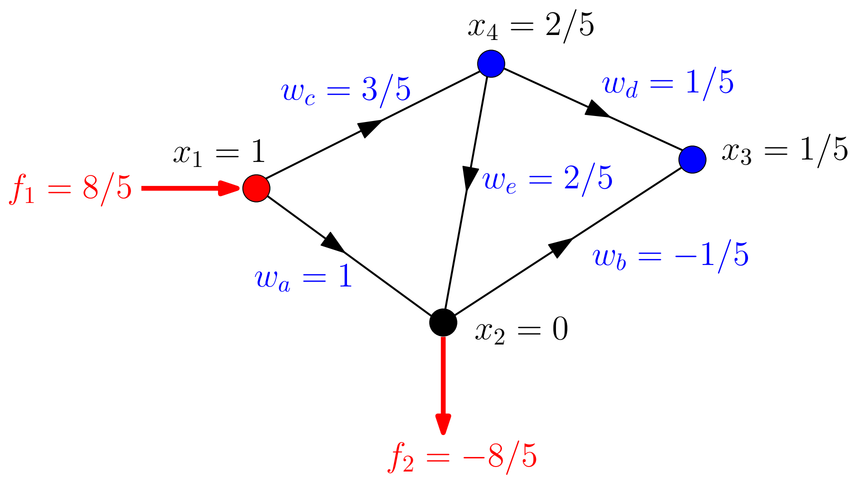

Figure 3: Summary of the solution of the Dirichlet problem with node 1 set to unit voltage and node 2 grounded.

4 Effective conductance

The *effective conductance* C_{eff} of a given circuit with unit voltage specified at one node, call it the + node, and a second grounded node, call it the - node, is defined to be the net current into the circuit at the + node. This same current leaves the circuit at the - node.

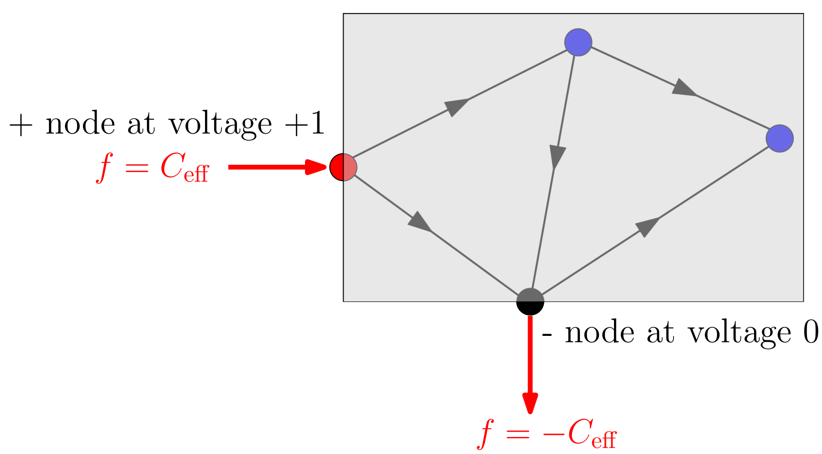

Figure 4: The idea of effective conductance C_{eff}: it is the net current flowing in a circuit between two selected nodes, a + node and a - node, with unit voltage drop between them. The circuit can then be viewed as a "black box" conductor, with an input and an output, with this effective conductance.

Figure 4 shows the idea behind the notion of effective conductance. If one thinks of the given circuit as replaced by a "black box" conductor with a single input (the

+ node) at voltage x_+ and a single output (the - node) at voltage x_- then if one measures the current f then Ohm's law for the black box conductor will take the form

$$f = -C_{\text{eff}}(x_- - x_+), \tag{40}$$

where C_{eff} is the effective conductance. On evaluating (40) for $x_+ = 1, x_- = 0$, so that there is unit voltage drop between the input and output nodes, we see that the value of C_{eff} is the value of the current flowing into the circuit at the + node.

5 The conductance matrix C

In the circuit example considered so far, all the conductors were assumed to have unit conductance. But this is a special case. It is important to generalize the analysis to conductors with any specified conductance value.

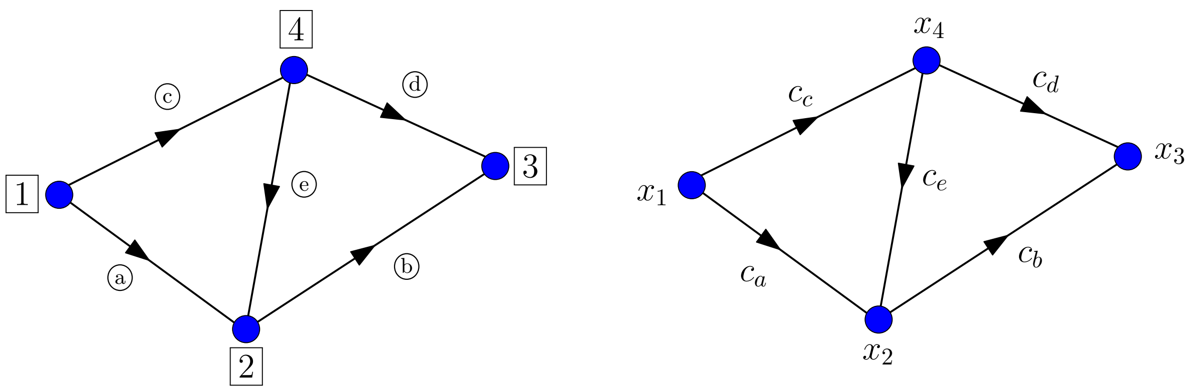

Figure 5: A circuit with different conductances.

As before, the 5-dimensional vector of voltage potential differences $\mathbf{e}$ is

$$\mathbf{e} \equiv \mathbf{A}\mathbf{x} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \\ x_2 - x_4 \end{pmatrix}. \tag{41}$$

To express Ohm's law for a set of conductors of arbitrary conductances it is necessary to introduce the diagonal m-by-m matrix

$$\mathbf{C} = \begin{pmatrix} c_a & 0 & 0 & 0 & 0 \\ 0 & c_b & 0 & 0 & 0 \\ 0 & 0 & c_c & 0 & 0 \\ 0 & 0 & 0 & c_d & 0 \\ 0 & 0 & 0 & 0 & c_e \end{pmatrix} \tag{42}$$

where (positive) conductances appear as the diagonal elements. Ohm's Law in each of the 5 conductors can now be expressed as

$$\mathbf{w} = -\mathbf{C}\mathbf{e} = -\mathbf{C}\mathbf{A}\mathbf{x}. \tag{43}$$

When all the conductances are equal to 1, $\mathbf{C}$ is just the identity matrix and we retrieve the case considered earlier.

The net currents out of each node is still given by

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w}. \tag{44}$$

Combining this with (43) leads to a modified relation between $\mathbf{f}$ and $\mathbf{x}$:

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w} = -\mathbf{A}^T(-\mathbf{C}\mathbf{A}\mathbf{x}) = \mathbf{A}^T\mathbf{C}\mathbf{A}\mathbf{x}. \tag{45}$$

The special matrix

$$\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A} \tag{46}$$

is the *weighted Laplacian matrix*. When $\mathbf{C} = \mathbf{I}$ it reduces to the regular Laplacian matrix already introduced.

In summary, for a general connected graph with edges that are conductors with arbitrary conductance values, the vector $\mathbf{f}$ of net current sources at the node is related to the vector $\mathbf{x}$ of node voltages via

$$\mathbf{f} = \mathbf{A}^T\mathbf{C}\mathbf{A}\ \mathbf{x}. \tag{47}$$

Quick construction: Just as there was a quick way to work out how to compute the regular Laplacian matrix – without first computing the incidence matrix – there is also a quick way to generate a weighted Laplacian matrix. This is left as an exercise for the reader.

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6 Exercises

Question 1: Prove that, when all the conductances of the conductors are strictly positive, $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ is a positive semi-definite matrix. That is, show that for any $\mathbf{x} \neq 0$,

$$\mathbf{x}^T\mathbf{K}\mathbf{x} \geq 0. \tag{48}$$

Question 2: Write down for which vectors $\mathbf{x}$ it is true that

$$\mathbf{x}^T\mathbf{K}\mathbf{x} = 0. \tag{49}$$

Question 3: Introduce the subblock decomposition

$$\mathbf{K} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{pmatrix}, \tag{50}$$

where the square matrix $\mathbf{P}$ is p-by-p for $1 \leq p \leq n-1$. Show that $\mathbf{R}$ is a positive definite matrix. That is, show that for any suitable $\mathbf{x} \neq 0$,

$$\mathbf{x}^T \mathbf{R} \mathbf{x} > 0. \tag{51}$$

7 Effective conductance of two conductors in parallel

Suppose two conductors are "in parallel" between two nodes as shown in Figure 6. This means there are two edges between the same two nodes. One of the conductors has conductance c_a and the other has conductance c_b. What is the effective conductance of this circuit between nodes 1 and 2?

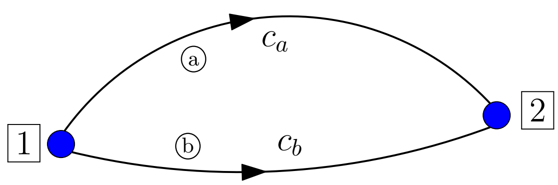

Figure 6: Two conductors in parallel.

This graph has 2 nodes and 2 edges. The incidence matrix is

$$\mathbf{A} = \begin{array}{c} \begin{array}{cc} \boxed{1} & \boxed{2} \end{array} \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{edge (a)} \\ \text{edge (b)} \end{array} \tag{52}$$

The conductance matrix is

$$\mathbf{C} = \begin{array}{c} \begin{array}{cc} \text{(a)} & \text{(b)} \end{array} \\ \begin{pmatrix} c_1 & 0 \\ 0 & c_b \end{pmatrix} \end{array} \begin{array}{l} \\ \text{edge (a)} \\ \text{edge (b)} \end{array} \tag{53}$$

Therefore

$$\mathbf{f} = \mathbf{A}^T\mathbf{C}\mathbf{A}\,\mathbf{x} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_a + c_b \\ -(c_a + c_b) \end{pmatrix} = \begin{pmatrix} C_{\text{eff}} \\ -C_{\text{eff}} \end{pmatrix}. \tag{54}$$

The effective conductance can now be read off as

$$C_{\text{eff}} = c_a + c_b. \tag{55}$$

We see that the effective conductance of two conductors in parallel is the *sum* of the individual conductances.

8 Effective conductance of two conductors in series

Suppose now that two conductors are "in series" between three nodes as shown in Figure 7. One of the conductors has conductance c_a and the other has conductance c_b. What is the effective conductance of this circuit between nodes 1 and 3?

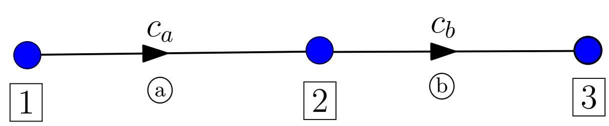

Figure 7: Two conductors in parallel.

The incidence matrix is

$$\begin{array}{cc} & \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \\ \mathbf{A} = & \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{array}{l} \text{edge (a)} \\ \text{edge (b)} \end{array} \end{array} \tag{56}$$

The conductance matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{(a)} & \text{(b)} \end{array} \\ \mathbf{C} = & \begin{pmatrix} c_1 & 0 \\ 0 & c_b \end{pmatrix} \begin{array}{l} \text{edge (a)} \\ \text{edge (b)} \end{array} \end{array} \tag{57}$$

We know the voltages at nodes 1 and 3, but not at node 2. We therefore set

$$\mathbf{x} = \begin{pmatrix} 1 \\ \phi_2 \\ 0 \end{pmatrix}, \tag{58}$$

where ϕ_2 is to be found. Therefore

$$\begin{aligned} \mathbf{f} = \mathbf{A}^T\mathbf{C}\mathbf{A}\ \mathbf{x} &= \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \phi_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_a & -c_a & 0 \\ -c_a & c_a + c_b & -c_b \\ 0 & -c_b & c_b \end{pmatrix} \begin{pmatrix} 1 \\ \phi_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_a(1-\phi_2) \\ (c_a + c_b)\phi_2 - c_a \\ -c_b\phi_2 \end{pmatrix}. \end{aligned} \tag{59}$$

But this is supposed to equal

$$\mathbf{f} = \begin{pmatrix} C_{\text{eff}} \\ 0 \\ -C_{\text{eff}} \end{pmatrix} \tag{60}$$

because KCL holds at node $\boxed{2}$ while the input current at node 1, or C_{eff}, is the quantity we seek. On setting (59) equal to (60) the middle equation tells us that

$$\phi_2 = \frac{c_a}{c_a + c_b}. \tag{61}$$

When this result is substituted into either the first or third equation the effective conductance for two conductors in series follows as

$$C_{\text{eff}} = \frac{c_a c_b}{c_a + c_b}. \tag{62}$$

Thus the effective conductance is the product of the two individual conductances divided by their sum.

9 Effective resistance

Some authors prefer to work with resistances rather than conductances; resistance is simply the inverse of the conductance:

$$R_a = \frac{1}{c_a}. \tag{63}$$

Accordingly, a "conductor" is instead referred to as a "resistor". In the same way, the effective resistance R_{eff} is related to effective conductance via

$$R_{\text{eff}} = \frac{1}{C_{\text{eff}}}. \tag{64}$$

It follows that the effective resistance of two resistors in parallel is

$$R_{\text{eff}} = \frac{1}{C_{\text{eff}}} = \frac{1}{c_a + c_b} = \frac{1}{1/R_a + 1/R_b} = \frac{R_a R_b}{R_a + R_b}, \tag{65}$$

where

$$R_a = \frac{1}{c_a}, \qquad R_b = \frac{1}{c_b}. \tag{66}$$

Simillarly, the effective resistance of two resistors in series is

$$R_{\text{eff}} = \frac{1}{C_{\text{eff}}} = \frac{c_a + c_b}{c_a c_b} = \frac{1}{c_a} + \frac{1}{c_b} = R_a + R_b, \tag{67}$$

i.e., the sum of the two resistances.

10 Calculating effective conductance

The rules for two conductors in parallel and in series can be used as a shortcut for calculating the effective conductance of more complicated circuits: it can be a valuable tool. By noticing pairs of conductors in series and in parallel, and then replacing them with other conductors with the relevant *effective* conductance, a successive reduction of a circuit to a set of "equivalent circuits" – in the sense of having the same effective conductance – is sometimes possible.

Figure 8 gives an example. It shows how the rule for two conductors in parallel and in series can be used (twice each) to retrieve the known value

$$C_{\text{eff}} = \frac{8}{5} \tag{68}$$

for this circuit. In this case, by a reduction to a sequence of equivalent circuits, the circuit can be reduced to a single equivalent graph with two nodes connected by a single conductor.

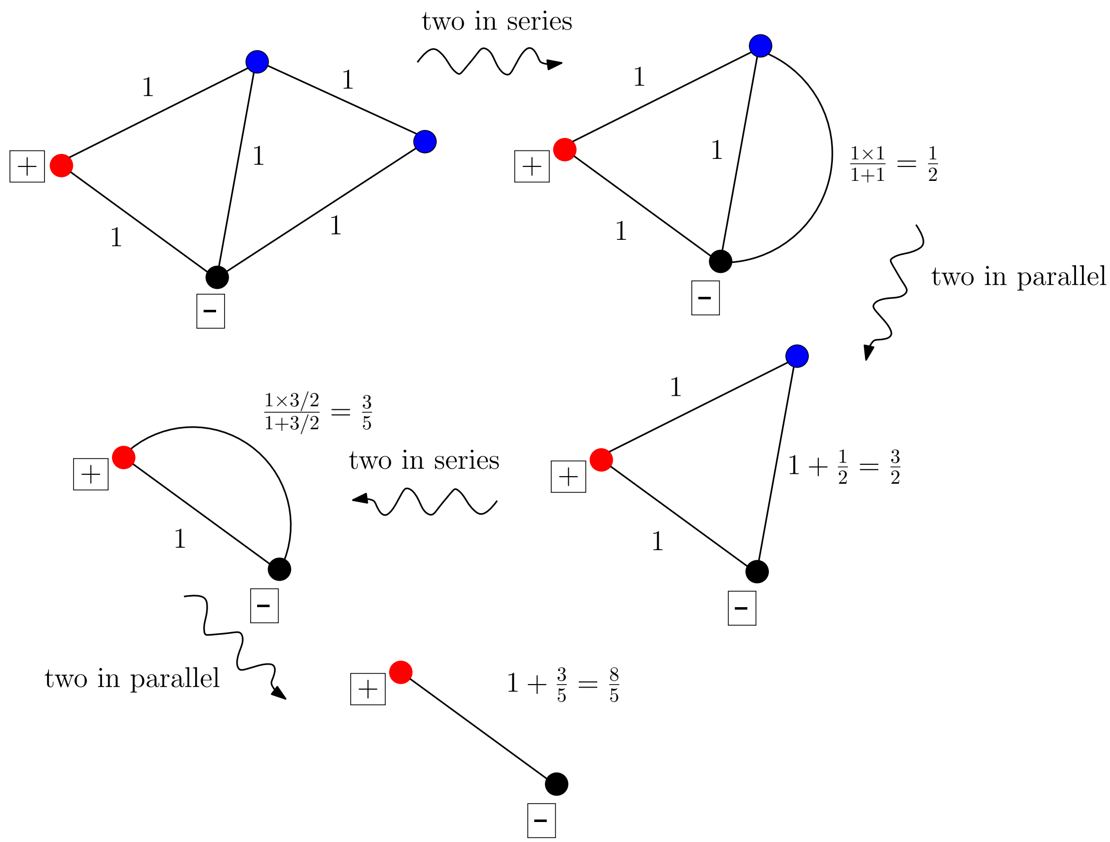

Figure 8: Calculating effective conductance by reduction to equivalent circuits. In this figure the quantity next to each edge is its conductance.

A complete reduction of this kind to a single conductor of known effective conductance is not always possible. Yet the same ideas can be used to at least simplify the calculation of the effective conductance.

Figure 9 shows another example. This graph has $n = 6$ nodes and $m = 7$ edges. All conductors are assumed to have unit conductance. Node 1 is set to unit voltage and node 6 is grounded.

Suppose we proceed in the usual way without making any reductions to equivalent circuits. Then the Laplacian matrix is

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix} \tag{69}$$

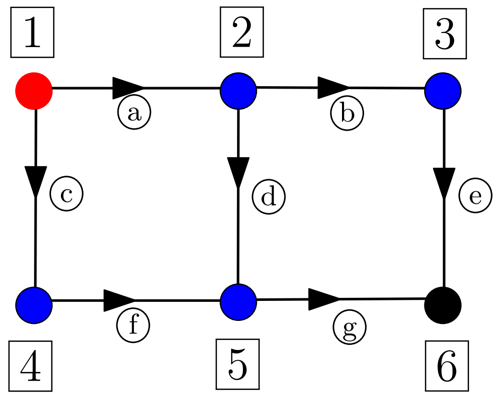

Figure 9: Calculating effective conductance by reduction to equivalent circuits. In this figure the quantity next to each edge is its conductance.

The *grounded Laplacian*, with node 6 grounded, gives the linear system

$$\hat{\mathbf{K}}\hat{\mathbf{x}} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \hat{\hat{\mathbf{x}}} \end{pmatrix} = \begin{pmatrix} C_{\text{eff}} \\ \mathbf{0} \end{pmatrix}, \tag{70}$$

where $\mathbf{0}$ denotes a 4-dimensional zero vector and

$$\hat{\hat{\mathbf{x}}} = (x_2, x_3, x_4, x_5)^T. \tag{71}$$

This can be solved using a method based on Schur complements. On writing

$$\hat{\mathbf{K}}\hat{\mathbf{x}} = \begin{pmatrix} 2 & \mathbf{q}^T \\ \mathbf{q} & \mathbf{R} \end{pmatrix} \hat{\mathbf{x}} = \begin{pmatrix} C_{\text{eff}} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{q} = (-1, 0, -1, 0)^T \tag{72}$$

with

$$\mathbf{R} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} \tag{73}$$

then we find

$$\mathbf{q} + \mathbf{R}\hat{\hat{\mathbf{x}}} = \mathbf{0}, \qquad \hat{\hat{\mathbf{x}}} = -\mathbf{R}^{-1}\mathbf{q}. \tag{74}$$

It follows that

$$C_{\text{eff}} = 2 + \mathbf{q}^T\hat{\hat{\mathbf{x}}} = 2 - \mathbf{q}^T\mathbf{R}^{-1}\mathbf{q}. \tag{75}$$

This is an expression for the effective conductance, but it still requires finding the inverse of a 4-by-4 matrix $\mathbf{R}$. This is cumbersome by hand, but it is easily done using a numerical linear algebra package.

Alternative approach: This calculation of the inverse of this 4-by-4 matrix can, however, be avoided by using the rules for "two conductors in series" to reduce the circuit to an equivalent one whose effective conductance is easier to calculate. Conductors c and f are in series and can be replaced by a single conductor with effective conductance

$$\frac{1\times 1}{1+1}=\frac{1}{2} \tag{76}$$

and similarly for conductors b and e. Figure 10 shows this reduction to an equivalent circuit.

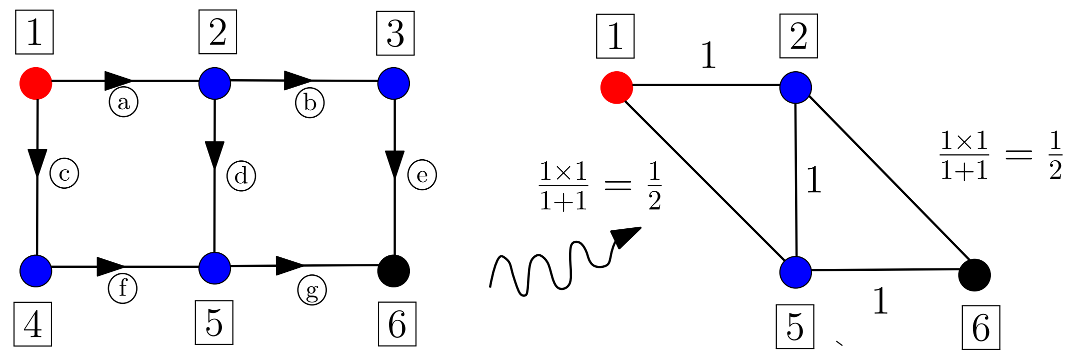

Figure 10: Removal of two conductors to produce an equivalent circuit with the same effective conductance. In the reduced circuit the quantity next to each edge is its conductance.

The Laplacian matrix of the equivalent graph on the right in Figure 10 is

$$\mathbf{K}=\begin{array}{cc} \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} \end{array} & \\ \begin{pmatrix} 3/2 & -1 & -1/2 & 0 \\ -1 & 5/2 & -1 & -1/2 \\ -1/2 & -1 & 5/2 & -1 \\ 0 & -1/2 & -1 & 3/2 \end{pmatrix} & \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{5} \\ \boxed{6} \end{array} \end{array} \tag{77}$$

With node 6 grounded, the linear system to solve is now

$$\hat{\mathbf{K}}\hat{\mathbf{x}}=\begin{pmatrix} 3/2 & -1 & -1/2 \\ -1 & 5/2 & -1 \\ -1/2 & -1 & 5/2 \end{pmatrix}\begin{pmatrix} 1 \\ x_2 \\ x_5 \end{pmatrix}=\begin{pmatrix} C_{\text{eff}} \\ 0 \\ 0 \end{pmatrix}. \tag{78}$$

This linear system is clearly smaller. With a view to using Schur complements again, it can be written as

$$\hat{\mathbf{K}}\hat{\mathbf{x}} = \begin{pmatrix} 3/2 & \mathbf{e}_1^T \\ \mathbf{e}_1 & \mathbf{R} \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} C_{\text{eff}} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{e}_1 = (-1, -1/2)^T \tag{79}$$

with

$$\mathbf{R} = \begin{pmatrix} 5/2 & -1 \\ -1 & 5/2 \end{pmatrix}. \tag{80}$$

By inspection the system to determine x_2 and x_5 is

$$\mathbf{R} \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5/2 & -1 \\ -1 & 5/2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \tag{81}$$

which is easily solved by hand to find

$$x_2 = \frac{4}{7}, \qquad x_5 = \frac{3}{7}. \tag{82}$$

With these determined the effective conductance follows as

$$C_{\text{eff}} = \frac{3}{2} + \mathbf{e}_1^T \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \frac{3}{2} - x_2 - \frac{x_5}{2} = \frac{5}{7}. \tag{83}$$

The reader is encouraged to confirm that (75) gives the same answer.

Later on we will give more evidence to argue why this procedure of "reduction to equivalent circuits" works.