Imperial College London

M4/5P36

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Representations of Symmetric Groups

Date: Friday 02 June 2017

Time: 14:00 - 16:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Suppose G is a group and $H \leq G$ is a subgroup. We work over the field $F = \mathbf{C}$ of complex numbers. You may use, without proof, the Frobenius reciprocity theorem, $\mathrm{Hom}_G(\mathrm{Ind}_H^G V, W) \cong \mathrm{Hom}_H(V, \mathrm{Res}_H^G W)$ (as well as other results presented in lectures).

   (a) Let (V, ρ) be an irreducible representation of H of dimension ≥ 2. Using Frobenius reciprocity, prove that $\mathrm{Ind}_H^G V$ cannot have a subrepresentation of dimension 1.

   (b) Now let (V, ρ) and (V', ρ') be two nonisomorphic irreducible representations of H of dimension 1. Again using Frobenius reciprocity, show that no subrepresentation of $\mathrm{Ind}_H^G V$ of dimension 1 is isomorphic to a subrepresentation of $\mathrm{Ind}_H^G V'$.

   (c) Assume that the restriction of every irreducible representation of G to H is a direct sum of pairwise nonisomorphic irreducible representations of H. Assume also that the irreducible representations of H have dimensions 1, 1, and 2, that $[G : H] = 4$, and that G has an irreducible representation of dimension at least three. Prove that the dimensions of the irreducible representations of G with multiplicity are $1, 1, 2, 3, 3$. You may use, without proof, the statements to be proved in parts (a) and (b).

2. In this question we will compute characters of representations of the symmetric group.

   (a) Let $n \geq 2$ and work over $F = \mathbf{C}$. Prove directly from the basic combinatorial lemma (recalled below) and basic results from representation theory of finite groups that (i) $\mathrm{Hom}_{S_n}(S^\lambda, M^\mu) \neq 0$ implies $\lambda \trianglerighteq \mu$, and (ii) $\mathrm{End}_{S_n}(S^\lambda) = \mathbf{C}$.

   Basic combinatorial lemma (use without proof): If t is a λ-tableau and t' a μ-tableau and, for each row of μ, the corresponding entries of t' appear in distinct columns of t, then $\lambda \trianglerighteq \mu$.

   (b) Let F be an arbitrary field. Compute for S_3 the characters of the representations M^λ for all partitions λ of 3, justifying carefully your computation (you may use results from lectures, except not the answer itself).

   (c) Let $F = \mathbf{C}$. Demonstrate how to use parts (a) and (b) and basic character theory to compute the character table of S_3, without directly computing the character of any representation of S_3 (beyond that already computed in (b)).

   (d) Let $F = \mathbf{C}$. Use the Murnaghan-Nakayama rule to compute the character $\chi_{S^{(4,2,1)}}((1234)(56))$.

3. (a) Let p be prime and let $n \in \{p, p+1\}$. What are the irreducible representations of S_n over a field of characteristic p? How many of these are there up to isomorphism, in terms of the number of partitions of n? Justify your answer using results from lectures.

(b) Let F be an arbitrary field, and let $v_1, \ldots, v_n$ be the coordinate basis of std ($v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th coordinate). Consider the map $\Phi : \mathrm{std} \otimes \mathrm{sign} \to S^{(2,1^{n-2})}$ given by $\Phi(v_i \otimes 1) = (-1)^{i-1} e_{t_i}$ with t_i the following tableau:

$$t_1 = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and } t_i = \begin{array}{|c|c|} \hline 1 & i \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i-1 & \multicolumn{1}{c}{} \\ \cline{1-1} i+1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \text{ if } i > 1.$$

Show that Φ is a homomorphism and use this to prove that $S^{(2,1^{n-2})} \cong (\mathrm{std} / \mathrm{triv}) \otimes \mathrm{sign}$.

4. Let $\{1\} = G_1 \lneq G_2 \lneq \cdots \lneq G_n = G$ be a multiplicity-one chain of subgroups (i.e., the restriction of every irreducible representation of G_{i+1} to G_i is a direct sum of pairwise nonisomorphic irreducible subrepresentations for every $1 \leq i \leq n-1$).

(a) Let (V, ρ) be an irreducible representation of G, (W, θ) a one-dimensional representation of G, and $w \in W$ a nonzero vector. Show that if $\{v_i\}$ is a Gelfand-Tsetlin basis of V with respect to the G_i, then $\{v_i \otimes w\}$ is a Gelfand-Tsetlin basis of $V \otimes_{\mathbb{C}} W$.

(b) Now let $G_i = S_i$ for all i. Show that $\mathrm{Spec}(V \otimes_{\mathbb{C}} \mathrm{sign}) = -\mathrm{Spec}(V)$, where $-(c_1, \ldots, c_n) := (-c_1, \ldots, -c_n)$, and $-X := \{-x \mid x \in X\}$ for $X \subseteq \mathbb{C}^n$.

(c) Show that the content vector of a chain of partitions λ^i (with $[\lambda^1] \subsetneq \cdots \subsetneq [\lambda^n] = [\lambda]$) is negative the content vector of the transpose (or dual) chain, $(\lambda^i)^T$. Conclude that $\mathrm{Spec}(V^{\lambda^T}) = -\mathrm{Spec}(V^\lambda)$, with negation defined as in part (b), where V^μ denotes the irreducible representation of V corresponding to the partition μ via the Okounkov-Vershik approach.

(d) Conclude from parts (b) and (c) (which you should use without proof) and the Okounkov-Vershik approach that $V^{\lambda^T} \cong V \otimes \mathrm{sign}$.

5. Let λ be a partition of n and let V^λ be the irreducible complex representation of S_n associated to it by the Okounkov-Vershik approach. For a given partition λ, we denote by $S(\lambda)$ the sum of the contents of the boxes of λ.

(a) Prove that $\chi_{V^\lambda}((12)) = \frac{2\dim V^\lambda}{n(n-1)} S(\lambda)$.

(b) As in the lectures, let $H(2)_i := \langle X_i, X_{i+1}, (i, i+1)\rangle_{\text{alg}} \subseteq \mathbb{C}[S_n]$. Prove that $\mathbb{C}[S_{n-2}]$ and $H(2)_{n-1}$ commute with each other, and as a $\mathbb{C}[S_{n-2}] \otimes H(2)_{n-1}$-representation, V^λ decomposes as follows:

$$V^\lambda \cong \bigoplus_{[\mu]\subseteq[\lambda]||\mu|=n-2} V^\mu \otimes V^{\lambda\setminus\mu},$$

where $V^{\lambda\setminus\mu}$ is a certain irreducible representation of $H(2)_{n-1}$ for each μ appearing in the sum.

Moreover, prove that as a representation of $\mathrm{Perm}\{n-1, n\} \subseteq H(2)_{n-1}$, $V^{\lambda\setminus\mu}$ is the standard representation if $[\lambda] \setminus [\mu]$ is disconnected, the trivial representation if $[\lambda] \setminus [\mu]$ is a horizontal line of length two, and the sign representation if $[\lambda] \setminus [\mu]$ is a vertical line of length two.

(c) Using the statements of parts (a) and (b), prove that

$$\chi_{(12)(34)}(V^\lambda) = \sum_{[\mu]} \frac{2\dim V^\mu}{(n-2)(n-3)} S(\mu)(-1)^{\langle[\lambda]\setminus[\mu]\rangle},$$

where the sum is over μ such that $|\mu| = n-2$, $[\mu] \subseteq [\lambda]$, and $[\lambda] \setminus [\mu]$ is connected. In the formula, $\langle[\lambda] \setminus [\mu]\rangle$ is defined to be zero if $[\lambda] \setminus [\mu]$ is a horizontal line of two boxes, and to be one if $[\lambda] \setminus [\mu]$ is a vertical line of two boxes.

(d) Use the formula in (c) and the hook length formula to give a closed expression for $\chi_{S^{(k,k-1)}}((12)(34))$ for every k.

M4/5 P36 Solutions

Notation from the course.

We recall first some notation from the course used in the exam: The induced representation $\operatorname{Ind}_H^G V$ is defined as $F[G] \otimes_{F[H]} V$ where $H \le G$ is a subgroup and V a representation of H (viewed as a left $F[H]$-module), and F is the ground field. Given a partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ of n, $[\lambda]$ is the corresponding Young diagram, with rows of lengths λ_i; the size $|\lambda|$ is defined as $n = \sum_i \lambda_i$. Given a λ-tableau t (a filling of $[\lambda]$ by the numbers 1 through n), $\{t\}$ is the corresponding λ-tabloid (a λ-tableau modulo the equivalence relation of swapping entries of the same row). Then M^λ is the vector space with basis the λ-tabloids, and $e_t \in M^\lambda$ the corresponding polytabloid, defined by $e_t := \sum_{\sigma \in C_t} \operatorname{sign}(\sigma)\sigma\{t\}$, for $C_t \le S_n$ the column stabiliser subgroup, preserving the set of entries appearing in each column of t. Then S^λ is the span of the e_t for all λ-tableaux t. The representation M^λ is equipped with the nondegenerate symmetric bilinear form in which the tabloids form an orthonormal basis, and $D^\lambda := S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$. Recall that $\trianglerighteq$ is the dominance ordering, with $\lambda \trianglerighteq \mu$ meaning $\sum_{i=1}^k \lambda_i \ge \sum_{i=1}^k \mu_i$ for all $k \ge 1$ (where we identify $(\lambda_1, \dots, \lambda_m)$ with $(\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ for any number of zeros, and similarly for μ). The transpose (or dual) partition λ^T is the partition obtained by interchanging the rows and columns of $[\lambda]$, i.e., $[\lambda^T]$ is the mirror image of $[\lambda]$ by flipping across the diagonal line, which passes from the upper left corner of the diagram diagonally down and to the right. The standard, reflection, trivial, and sign representations of S_n are denoted std, refl, triv, and sign, respectively.

We recall some notation used in the Okounkov-Vershik approach to representations of the symmetric group. The Young-Jucys-Murphys elements are $X_i = (1i) + (2i) + \cdots + (i-1, i) \in \mathbb{C}[S_n]$, with $X_1 = 0$. Given a multiplicity-one chain of subgroups $\{1\} = G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n = G$ as in problem 4, a Gelfand-Tsetlin basis for an irreducible representation V of G satisfying $\operatorname{Res}_{G_{n-1}}^G V = V_1 \oplus \cdots \oplus V_m$ with V_i irreducible is inductively defined as the concatenation of Gelfand-Tsetlin bases for the V_i with respect to $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{n-1}$. In the case $G_i = S_i$, the symmetric groups, a Gelfand-Tsetlin basis is called a Young basis. The content of the box in the i-th row and j-th column of a Young diagram (written with English notation, with the first row and column being the box in the upper left corner) is defined as $j - i$. The content vector associated to a sequence of Young diagrams $[\lambda_1] \subset [\lambda_2] \subset \cdots \subset [\lambda_n]$ with $|\lambda_i| = i$ for all i is by definition the sequence of contents of the boxes $[\lambda_i] \setminus [\lambda_{i-1}]$ for $[\lambda_0] := \emptyset$. The spectrum of an irreducible representation V of S_n, denoted $\operatorname{Spec}(V)$, is defined as the the set of vectors $c \in \mathbb{C}^n$ where $c = (c_1, c_2, \dots, c_n)$ is the collection of eigenvectors under $X_1, \dots, X_n$ of a Young basis element of V. That is, for $v_1, \dots, v_m$ a Young basis, $\operatorname{Spec}(V) = \{(c_{i1}, \dots, c_{in}) \mid 1 \le i \le m\}$ where $X_j v_i = c_{ij} v_i$ for all i, j. The irreducible representation V^λ of S_n is defined as the one, up to isomorphism, whose spectrum consists of those content vectors with final Young diagram $[\lambda]$. In fact, $V^\lambda \cong S^\lambda$, but this fact will not be required.

Solutions to the exam problems.

1. (a) [Unseen, 5 points] Let W be a representation of G of dimension one. Then $\operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) \cong \operatorname{Hom}_H(V, \operatorname{Res}_H^G W)$ which by Schur's Lemma is zero. The result then follows from Maschke's theorem (since if a subrepresentation of $\operatorname{Ind}_H^G V$ were isomorphic to W, the subrepresentation would be a direct summand and hence the

isomorphism would extend to a nonzero homomorphism $\mathrm{Ind}_H^G V \to W$).

(b) [Unseen, 5 points] This is similar: let W be a representation of G of dimension one and V, V' two one-dimensional representations of H. Then $\mathrm{Hom}_G(\mathrm{Ind}_H^G V, W) \cong \mathrm{Hom}_H(V, \mathrm{Res}_H^G W)$, which by Schur's Lemma is nonzero if and only if $V \cong \mathrm{Res}_H^G W$ as W is one-dimensional. By assumption, is impossible that V and V' are both isomorphic to $\mathrm{Res}_H^G W$. Therefore as in (a), by Maschke's theorem, W cannot be isomorphic to a subrepresentation of both $\mathrm{Ind}_H^G V$ and $\mathrm{Ind}_H^G V'$.

(c) [Unseen, 10 points] Note first that $|H| = 1^2 + 1^2 + 2^2 = 6$ (in fact $H \cong S_3$) and hence $|G| = 24$. So the sums of the squares of the dimensions of the irreducible representations of G add to 24: this means we can have at most one irreducible of dimension four and none of dimension greater than four, for instance.

By Frobenius reciprocity and Maschke's theorem, every irreducible representation of G must be a summand of at least one representation of the form $\mathrm{Ind}_H^G V$ for V an irreducible representation of V; by the multiplicity-one assumption and Frobenius reciprocity each $\mathrm{Ind}_H^G V$ is a direct sum of nonisomorphic irreducible representations of G. Since $[G : H] = 4$ we have $\dim \mathrm{Ind}_H^G V = 4 \dim V$. Take V_2 to be the irreducible representation of H of dimension 2. Then $\mathrm{Ind}_H^G V_2$ has dimension eight and by (a) cannot decompose into any one-dimensional irreducible representations; it must decompose into representations of dimensions two through eight. By the first paragraph and the multiplicity-one assumption, the possible decompositions are: (A) $2, 2, 2, 2$, (B) $2, 3, 3$, or (C) $2, 2, 4$. We can eliminate (C) since $2^2 + 2^2 + 4^2 = 24$ already. We can eliminate (A) since then by sums of squares, we cannot also have a representation of dimension at least three. So (B) must be the case.

In case (B), call the irreducibles of G of dimension 3 W_3^1 and W_3^2. Then $\mathrm{Res}_H^G W_3^1 \cong V_2 \oplus V_1^1$ and $\mathrm{Res}_H^G W_3^2 \cong V_2 \oplus V_1^2$. It is not possible that $V_1^1 \cong V_1^2$ since then $\mathrm{Ind}_H^G V_1^1$ would contain both W_3^1 and W_3^2 by Frobenius reciprocity, but this has total dimension four only. So V_1^1 and V_1^2 are exactly the one-dimensional representations of H up to isomorphism. Therefore $\mathrm{Ind}_H^G V_1^1 \cong W_3^1 \oplus W_1^1$ and $\mathrm{Ind}_H^G V_1^2 \cong W_3^2 \oplus W_1^2$ for two one-dimensional representations W_1^1, W_1^2 of G, nonisomorphic by (b). Put together we get irreducibles of dimensions $1, 1, 2, 3, 3$, as in case (i).

2. (a) [Seen, 6 points] By Maschke's theorem, M^λ is a direct sum of S^λ and a complement (actually $(S^\lambda)^\perp$) so if $\mathrm{Hom}(S^\lambda, M^\mu) \ni \theta \neq 0$ then we can extend θ to $\widetilde{\theta} : M^\lambda \to M^\mu$. Let t be a λ-tableau. Then $\theta(e_t) \neq 0$ since S^λ is generated by e_t over S_n. We have $\theta(e_t) = \widetilde{\theta}(\kappa_t\{t\}) = \kappa_t \widetilde{\theta}(\{t\})$ for $\kappa_t = \sum_{\sigma \in C_t} \mathrm{sign}(\sigma)\sigma$ with C_t the column stabiliser of σ. Now let t' be a μ-tableau. For $\kappa_t\{t'\}$ to be nonzero then the entries of each row of t' appear in distinct columns of t, so by the basic combinatorial lemma, $\lambda \trianglerighteq \mu$.

In the case $\lambda = \mu$ then $\kappa_t\{t'\}$ must be a multiple of e_t, say ce_t, so that $\theta(e_t) = ce_t$, which implies that $\theta = c\,\mathrm{Id}$ as e_t generates S^λ over S_n. On the other hand $c\,\mathrm{Id} \in \mathrm{End}_{S_n}(S^\lambda)$ is nonzero for all $c \in \mathbf{C}$, so we get $\mathrm{End}_{S_n}(S^\lambda) = \mathbf{C}$.

(b) [Seen, 4 points] We have from a formula in lecture that $\chi_{M^\lambda}(\sigma) = \frac{|C_{S_n}(\sigma)|\cdot|C_\sigma \cap S_\lambda|}{|S_\lambda|}$ where $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_{\ell(\lambda)}}$ and C_σ is the conjugacy class of σ. Using this the table we get is:

λ	$[(1)]$	$[(12)]$	$[(123)]$
(3)	1	1	1
$(2,1)$	3	1	0
(1^3)	6	0	0

(c) [Seen, 6 points] Now we have $M^{(3)} = S^{(3)}$ so for the table of S^λ, the first line does not change. The second line changes by $\chi_{S^{(2,1)}} = \chi_{M^{(2,1)}} - \langle \chi_{M^{(2,1)}}, \chi_{S^{(3)}} \rangle \chi_{S^{(3)}} = \chi_{M^{(2,1)}} - \chi_{S^{(3)}} = (2, 0, -1)$. Finally the third line changes to $\chi_{S^{(1^3)}} = \chi_{M^{(1^3)}} - -\langle \chi_{M^{(1^3)}}, \chi_{S^{(3)}} \rangle \chi_{S^{(3)}} - \langle \chi_{M^{(1^3)}}, \chi_{S^{(2,1)}} \rangle \chi_{S^{(2,1)}} = \chi_{M^{(1^3)}} - \chi_{S^{(3)}} - 2\chi_{S^{(2,1)}} = (1, -1, 1)$. We get, for the S^λ as desired:

λ	$[(1)]$	$[(12)]$	$[(123)]$
(3)	1	1	1
$(2,1)$	2	0	-1
(1^3)	1	-1	1

(d) [Seen similar, 4 points] We need to sum over the signs of decompositions of $(4, 2, 1)$ into skew hooks of sizes 4, 2, and 1. Let us go in reverse order, deleting first a skew hook of size four. There is only one such skew hook, in the first and second rows, of height one. Deleting this we obtain the figure which is a vertical line of three boxes. Now there is again one way to delete a skew hook of size two, and it also has height one. Put together we get coefficient $(-1)^2 = 1$, which is the answer since it was the only decomposition.

3. (a) [Unseen but easy, 8 points] By results from lecture, the irreducible representations are of the form D^λ for λ a p-regular partition of p, which are all nonisomorphic, whereas the other D^λ are zero. To be a p-singular partition of p would require having p parts of the same size, hence one, so we get $D^{(1^n)} = 0$ and the other D^λ are the irreducible representations of S_p. So the irreducible representations of S_p are precisely, up to isomorphism, D^λ for $\lambda \vdash n$ with $\lambda \neq (1^n)$, and these are all nonisomorphic. The number of these is one less than the number of partitions of $n = p$.

Similarly, for $n = p+1$, the p-singular partitions are now (1^n) and $(2, 1^{n-1}) = (2, 1^p)$. So by the same reasoning, the irreducible representations are precisely, up to isomorphism, D^λ for $\lambda \vdash n = p+1$ and $\lambda \neq (1^n), (2, 1^n)$. The number of these is two less than the number of partitions of n.

(b) [Unseen, 12 points] We show that Φ is a homomorphism. We have $\sigma(v_i \otimes 1) = \operatorname{sign}(\sigma) v_{\sigma(i)} \otimes 1$. On the other hand, $\sigma\Phi(v_i \otimes 1) = (-1)^{i-1} e_{\sigma t_i}$ is the polytabloid associated to the tableau obtained from t_i by applying the permutation σ. We need to show these are equal for all $\sigma \in S_n$. Note that $e_{\sigma t_i} = \pm e_{t_{\sigma(i)}}$ so that we really only have to check the sign. If $\sigma(i) = i$ then $e_{t_{\sigma(i)}} = \operatorname{sign}(\sigma) e_{t_i}$, so the sign is correct in this case. It therefore suffices to check that for σ the permutation such that $\sigma(t_i) = t_j$, then $\operatorname{sign}(\sigma) = (-1)^{i-j}$. But $\sigma = (i, j, j-1, j-2, \ldots, i+1)$, so $\operatorname{sign}(\sigma) = (-1)^{i-j}$ as desired.

Next we prove that $\ker(\Phi) = \text{triv} \otimes \text{sign}$. Note that the $t_i, i > 1$ are exactly the standard $(2, 1^{n-2})$-tableau. Therefore they form a basis of $S^{(2,1^{n-2})}$ by a result from lecture. Since the v_i form a basis of std, the kernel is one-dimensional. It suffices therefore to show that $\Phi(\sum_i v_i \otimes 1) = 0$. This yields $\sum_i (-1)^{i-1} e_{t_i}$. We can explicitly see this is zero (it has a contribution of 1 and -1 as the coefficient of each tabloid); or alternatively since the kernel is a subrepresentation, it must be triv as we saw in the course that the only two subrepresentations of std are triv and refl (other than 0 and std itself), and for $n \geq 3$ the only one-dimensional one is triv. (For $n = 2$ it is obvious that $\text{triv} \otimes \text{sign}$ is the kernel since $S^{(2)} = M^{(2)} = \text{triv}$ and the map is $(a, b) \mapsto a - b$.)

By the first isomorphism theorem, we obtain $(\text{std} \otimes \text{sign})/(\text{triv} \otimes \text{sign}) \cong S^{(2,1^{n-2})}$. Now there is an isomorphism $(\text{std} / \text{triv}) \otimes \text{sign} \to (\text{std} \otimes \text{sign})/(\text{triv} \otimes \text{sign})$ sending $(a + \text{triv}) \otimes 1$ to $(a \otimes 1) + (\text{triv} \otimes \text{sign})$, which completes the proof.

4. (a) [Unseen, 8 points] Recall that the Gelfand-Tsetlin basis is indexed by chains of subrepresentations $V(1) \subseteq V(2) \subseteq \cdots \subseteq V(n) = V$ with $V(i)$ an irreducible representation of G_i, with corresponding basis element the unique vector v_j up to scaling such that $\mathbf{C}[G_i]v_j \cong V(i)$ for all i. Now in $V \otimes_{\mathbf{C}} W$, every subspace is of the form $U \otimes_{\mathbf{C}} W$, and it is clearly a G_i-subrepresentation if and only if $U \subseteq V$ is a G_i-subrepresentation. Moreover, if $\mathbf{C}[G_i]v_j \cong V(i)$ for all i, then $\mathbf{C}[G_i](v_j \otimes w) \cong V(i) \otimes_{\mathbf{C}} W \subseteq V \otimes_{\mathbf{C}} W$. Therefore if $\{v_i\}$ is a Gelfand-Tsetlin basis, so is $\{v_i \otimes w\}$.

   (b) [Seen, 4 points] Note that, if V is a representation of S_n and $v \in V$, then $X_i(v \otimes 1) = \sum_j (j, i) v \otimes \operatorname{sign}(ji) = (-X_i v) \otimes 1$. Thus, by part (a), the eigenvalues of X_i on Gelfand-Tsetlin basis elements of $V \otimes_{\mathbf{C}} \text{sign}$ are negative those for V.

   (c) [Seen, 4 points] Taking transpose merely interchanges rows and columns, which negates all the contents of boxes by definition. Therefore the chains of subdiagrams are the transposes of those we had before and the contents are all negated. Since, by definition, V^λ is the irreducible representation of S_n whose spectrum consists of content vectors of $[\lambda]$, it follows that $\operatorname{Spec}(V^{\lambda^T}) = -\operatorname{Spec}(V^\lambda)$.

   (d) [Seen, 4 points] By the Okounkov-Vershik approach, irreducible representations of S_n are completely classified by their spectrum. Hence by (b) and (c), we have $V^{\lambda^T} \cong V^\lambda \otimes \text{sign}$.

5. (a) [Unseen, 4 points] Note that $\sum_i X_i$ is just the sum of all transpositions. This is an element of the center of $\mathbf{C}[S_n]$, hence it acts by a scalar matrix on every irreducible representation by Schur's Lemma. This scalar can easily be computed by looking at what it is on any Young basis element: it is the sum of the contents of the diagram λ (which also gives another proof that $\sum_i X_i$ acts by a scalar in fact). On the other hand $\chi_{V^\lambda}(\sum_i X_i) = \binom{n}{2} \chi_{V^\lambda}((12))$ since every transposition is conjugate to (12). By the preceding, the LHS is $(\dim V^\lambda) \cdot S(\lambda)$. Putting this together gives the statement.

(b) [Seen similar, 8 points] We use the Young basis of V^λ. Since Young basis vectors of V^λ are in bijection with the spectrum $\mathrm{Spec}(V^\lambda)$, they are also in bijection with chains of partitions λ^i with $|\lambda^i| = i$ and $[\lambda^1] \subseteq \cdots \subseteq [\lambda^n] := [\lambda]$. Therefore they are also in bijection with the data of first a chain of partitions λ^i only up to $i = n-2$, together with, in the case that $[\lambda] \setminus [\mu]$ is disconnected, a choice of ordering of the remaining two boxes. On the span of the Young basis vectors with $\lambda^{n-2} = \mu$ (fixing the ordering if required of the remaining two boxes), the group S_{n-2} acts according to the representation V^μ. Similarly, on a Young basis vector X_{n-1} and X_n act by multiplication by the contents of the remaining two boxes (in the ordering given in the case $[\lambda] \setminus [\mu]$ is disconnected, or otherwise or taking first the box connected to $[\mu]$, then the remaining box). The element $(n-1, n)$, by results from lecture (since $H(2)_{n-1}$ is the image of $H(2)$ under the homomorphism ψ_{n-1} constructed in lecture) acts in the disconnected case by swapping the ordering and acts otherwise by 1 if the first content is one less than the second (i.e., $[\lambda] \setminus [\mu]$ is a horizontal line of length two) and by -1 if the first content is less than the second (i.e., $[\lambda] \setminus [\mu]$ is a vertical line of length two). Moreover as discussed in lecture these representations are all irreducible. This means that on the span of the Young basis vectors with $\lambda^{n-2} = \mu$, we obtain precisely the representation $V^\mu \otimes V^{\lambda\setminus\mu}$ with $V^{\lambda\setminus\mu}$ as explicitly described in the problem. It is evident that $(n-1, n)$ acts in the claimed way.

(c) [Unseen, 4 points] Since the trace on a tensor product is the product of the traces, we can compute the trace of $(12)(n-1, n)$ using the expression in (b). The trace of $(n-1, n)$ on $V^{\lambda\setminus\mu}$ is zero if $[\lambda]\setminus[\mu]$ is disconnected (since then $\mathrm{Perm}\{n-1, n\}$ acts via the standard representation), and the trace is 1 and -1 if $[\lambda]\setminus[\mu]$ is a horizontal or vertical line of length two, respectively (as these give the trivial and sign representations of $\mathrm{Perm}\{n-1. n\}$). Note also that the trace of $(12)(n-1, n)$ always equals the trace of $(12)(34)$ since these elements of S_n are conjugate. Put together, using (a), we get the claimed formula.

(d) [Unseen, 4 points] The only way to take a skew hook of length two (i.e., a horizontal or vertical line of length two of the form $[\lambda]\setminus[\mu]$) out of $\lambda = (k, k-1)$ is in the bottom row, using the last two boxes. By the formula from (c) we get:

$$\chi_{(12)(34)}(V^{(k,k-1)}) = \frac{2\dim V^{(k,k-3)}}{(2k-3)(2k-4)} S((k, k-3)).$$

Now $S((k, k-3) = 0+1+\cdots+(k-1)+(-1)+0+\cdots+(k-5) = \frac{k(k-1)}{2} + \frac{(k-4)(k-5)}{2} - 1 = \frac{k^2-k+k^2-9k+20-2}{2} = \frac{2k^2-10k+18}{2} = (k^2-5k+9)$. So we get

$$\chi_{(12)(34)}(V^{(k,k-1)}) = \frac{2\dim V^{(k,k-3)}}{(2k-3)(2k-4)}(k^2-5k+9).$$

Of course, by the hook length formula, $V^{(k,k-3)} = \frac{4(2k-3)!}{(k+1)!(k-3)!}$, so put together we get

$$\chi_{(12)(34)}(V^{(k,k-1)}) = \frac{8(2k-5)!(k^2-5k+9)}{(k+1)!(k-3)!}.$$

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M45 P36 Session: 2016-2107

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Overall, the exam was certainly not easy, but for an advanced course like this one, given also the exercises students were presented, it was fair. There were many opportunities to show understanding.

The first two parts of Q1 were reasonable. Most students forgot to invoke Maschke's theorem (so that subrepresentations are quotients as well).

The last part was one of the most difficult of the exam (despite requiring only preliminary material from the beginning of the course). Students had difficulty applying Frobenius Reciprocity in ~~a~~ somewhat deeper ways. In retrospect, a hint (decompose the induced representation of the two-dimensional irreducible representation of H) might have helped some students.

Marker: Travis Schedler

Signature: Travis Schedler Date: 5/6/17

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4S P36 **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question was the easiest of the exam. The first part was more difficult for many students even though it asked essentially to recall a proof from lecture. The other parts involved computations most students remembered how to perform.

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Department of Mathematics

Examiner's Comments

Exam: M4S P36 **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question was quite difficult for the students. The first part didn't require much more than recalling the classification of irreducible representations in characteristic p, but most students did not recall this correctly. The second part was more challenging, in principle a straightforward computation to verify ϕ is a homomorphism and compute its kernel, but it required greater confidence with tabloids.

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Department of Mathematics

Examiner's Comments

Exam: M4SP36 Session: 2016-2107

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question was among the easier ones on the exam. Mostly, it consisted of redoing carefully an essential result from lecture (that transpose ~~~~ diagrams correspond to ~~~~ irreducible representations of S_n differing by tensoring with the sign representation). Many students recalled how to do this. The hardest part was clearly explaining the proof in the first part, requiring some understanding of Gelfand-Tsetlin bases.

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Department of Mathematics

Examiner's Comments

Exam: M4SP36 **Session: 2016-2107**

Question 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This question was probably the hardest on the exam. It required students understand the structure of the Okounkov-Vershik approach and how to apply it to character formulae.

Part (a) was perhaps quite challenging because students did not all see the trick (to ~~add~~ sum all the Young-Jucys-Murphys elements). We had seen a similar but more difficult trick at the end of the course (involving the product).

Part (b) mostly only required understanding aspects of the final result from the course (partial proof of the Murnaghan-Nakayama rule) but was quite difficult.

Part (c) and, to a greater extent, part (d) were mostly straightforward computations but maybe students had little time left. (For (c) aspects of the preceding proof were again needed.)

Marker: Travis Schedler

Signature: Travis Schedler **Date:** 5/6/17

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