Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 2 - Decimal Expansions

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Finite decimals

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For $a_0 \in \mathbb{Z}$ and and $a_i \in \{0, 1, \ldots, 9\}$ we **define** the finite decimal $a_0.\, a_1 \ldots a_i$ as follows:
If $a_0 \geq 0$ then $a_0.\, a_1 \ldots a_i$ is set to be

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_i}{10^i} \ \in \ \mathbb{Q}\,.$$

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For $a_0 < 0$ we set $a_0.\, a_1 \ldots a_i$ to be given by $-\big(|a_0|.\, a_1 \ldots a_i\big)$.

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Eventually periodic decimals

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At school you became happy with the idea that

$$\begin{aligned}
0.\overline{3} &= 0.3333\ldots \\
&= 0.3 + 0.03 + 0.003 + 0.0003 + \ldots \\
&= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots
\end{aligned}$$

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&= \frac{3}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \ldots\right)
\end{aligned}
$$

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&= \frac{3}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \ldots\right) \\
&\stackrel{?}{=} \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{3}{10}\frac{10}{9} = \frac{1}{3}.
\end{aligned}
$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

for $|r| < 1$

Eventually periodic decimals

Now I'll grant you we can certainly justify

$$1 + x + x^2 + \cdots + x^n \;=\; \frac{1 - x^{n+1}}{1 - x}\,, \qquad x \neq 1,$$

by multiplying both sides by $1 - x$.

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$$1 + x + x^2 + \cdots + x^n \;=\; \frac{1 - x^{n+1}}{1 - x}\,, \qquad x \neq 1,$$

by multiplying both sides by $1 - x$.
But it will be many lectures before we know how to conclude that

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n + \ldots \quad &= \quad \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} \\ &= \quad \frac{1}{1 - x}\,, \quad -1 < x < 1, \end{aligned}$$

to justify the $\stackrel{?}{=}$ in the last slide.

$1.43\overline{57}$

$1.4357575757\ldots = 1 + \frac{4}{10} + \frac{3}{100} + \frac{57}{10^4}\left(\frac{1}{1-10^{-2}}\right)$

Eventually periodic decimals

So for now we simply take it as a **definition**:

For $a_0 \in \mathbb{Z},\ a_{i>0} \in \{0, 1, \ldots, 9\}$ we define

$$a_0.\, a_1 \ldots a_i \overline{a_{i+1} a_{i+2} \ldots a_j}$$

to be the *rational number*

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_i}{10^i} + \left(\frac{a_{i+1} a_{i+2} \ldots a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right) \quad (2.6)$$

motivated by the "fact" (which we've yet to prove) that the last fraction equals $1 + 10^{-(j-i)} + 10^{-2(j-i)} + \ldots$.

Eventually periodic decimals

There's a bit of checking that (2.6) is well-defined: for instance, you should check that our definition makes different expansions like $0.\overline{3}$ and $0.3\overline{3}$ and $0.\overline{33}$ all the same number....).

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Another good exercise:

Exercise 2.7

Consider two eventually periodic decimals differing in only one place:

$$a = a_0.a_1a_2\dots a_{n-1}a_na_{n+1}\dots, \qquad b = a_0.a_1a_2\dots a_{n-1}b_na_{n+1}\dots$$

Show using our definition (2.6) that $a < b$ if and only if $a_n < b_n$.

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To see the idea of the proof it is good to do an example and try to write $\frac{25}{11}$ as a decimal.

Eventually periodic decimals

$$\frac{25}{11} = a_0.\, a_1 a_2 a_3 \ldots$$

$\lfloor \cdot \rfloor$ floor
"nearest integer below"
"round down"

Eventually periodic decimals

$$\frac{25}{11} = a_0 .\, a_1 a_2 a_3 \ldots$$

To find $a_0 = \left\lfloor \frac{25}{11} \right\rfloor$ we divide 11 into 25:

$$25 = \mathbf{2} \times 11 + 3 \quad \Rightarrow \quad a_0 = 2. \qquad (*)$$

Eventually periodic decimals

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$$\frac{30}{11} = a_1 . a_2 a_3 \dots$$

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To find a_1 we take $\frac{25}{11} - 2 = 3/11 = 0.\ a_1 a_2 a_3 \dots$ Then multiply by 10 and take integer part, i.e. $a_1 = \left\lfloor \frac{30}{11} \right\rfloor$

$$30 = \mathbf{2} \times 11 + 8 \quad \Rightarrow \quad a_1 = 2.$$

$\frac{8}{11} = .a_2 \ldots$

$\frac{80}{11} = a_2 . a_3 \ldots$

Eventually periodic decimals

$$\frac{25}{11} = a_0 .\, a_1 a_2 a_3 \ldots$$

2.27

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Multiplying remainder 8 by 10 and dividing 11 in gives $a_2 = \left\lfloor \frac{80}{11} \right\rfloor$:

$$80 = \mathbf{7} \times 11 + 3 \quad \Rightarrow \quad a_2 = 7.$$

Eventually periodic decimals

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We have remainder 3 again, just as in $(*)$. Now everything repeats - the remainder determines the next step.

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Can check with definition (2.6) that $\dfrac{25}{11} = 2.\overline{27}$

All of $\mathbb{Q}$ can be written as eventually periodical decimals

Theorem 2.8

Any $x \in \mathbb{Q}$ is equal to an eventually periodic decimal expansion:
$x = a_0.\, a_1 \ldots a_i \overline{a_{i+1} a_{i+2} \ldots a_j}$
($a_0 \in \mathbb{Z},\ a_\ell \in \{0, 1, \ldots, 9\}$ for $\ell \geq 1$).

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Proof: Without loss of generality we take $x > 0$.
In our proof we write $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ for the non-integer part of x.

$\left\{\frac{5}{3}\right\} = \frac{2}{3}$ $\quad \lfloor 5/3 \rfloor = 1$

To write x as a decimal we let $a_0 := \lfloor x \rfloor$ and $e_0 := \{x\}$, so

$$x = a_0 + e_0, \qquad a_0 \in \mathbb{N},\ e_0 \in [0,1) \tag{2.9}$$

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Now repeat for $10e_0 \in [0,10)$, setting $a_1 := \lfloor 10e_0 \rfloor$ and error $e_1 := \{10e_0\}$,

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Inductively, given $a_i \in \{0,1,\ldots,9\}$ and $e_i \in [0,1)$ for $i < k$ we set the k-th digit $a_k := \lfloor 10e_{k-1} \rfloor$ and k-th error $e_k := \{10e_{k-1}\}$, so

$$10e_{k-1} \;=\; a_k + e_k, \qquad a_k \in \{0,1,\ldots,9\},\; e_k \in [0,1). \tag{2.10}$$

To write x as a decimal we let $a_0 := \lfloor x \rfloor$ and $e_0 := \{x\}$, so

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Plugging each equation into the former gives, for any k,

$$x \ = \ a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k} \ + \frac{e_k}{10^k}, \qquad e_k \in [0,1). \tag{2.11}$$

$$\frac{a_{k+1}}{10^{k+1}} + \frac{e_{k+1}}{10^{k+1}}$$

Now remember $x = p/q$ $(p, q \in \mathbb{N})$ is rational! So $q \times (2.9)$ tells us

$$p \;=\; qa_0 + r_0, \qquad qe_0$$

where $r_0 := qe_0 \in \{0, 1, \ldots, q-1\}$ is the remainder.

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Inductively we also have, for all $k \geq 1$,

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The set on the righthand side is finite so there must exist $1 \leq i < j$ with $r_i = r_j$ (and so $e_i = e_j$). Will show that x is given by

$$\begin{aligned} &a_0.\, a_1 a_2 \ldots a_i \overline{a_{i+1} a_{i+2} \ldots a_j} \\ &= a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_i}{10^i} \;+\; \frac{a_{i+1} a_{i+2} \ldots a_j}{10^j} \frac{1}{1 - 10^{i-j}}\ . \end{aligned}$$

Now remember $x = p/q$ $(p, q \in \mathbb{N})$ is rational! So $q \times (2.9)$ tells us

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$// \; \frac{e_i}{10^i}$

$$\begin{aligned} & a_0.\, a_1 a_2 \ldots a_i \overline{a_{i+1} a_{i+2} \ldots a_j} \\ &= a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_i}{10^i} \;+ \frac{a_{i+1} a_{i+2} \ldots a_j}{10^j} \frac{1}{1 - 10^{i-j}}\ . \end{aligned}$$

All we need to show is

$$\frac{e_i}{10^i} = \frac{a_{i+1} a_{i+2} \ldots a_j}{10^j} \frac{1}{1 - 10^{i-j}}\ .$$

Equivalently, we need to show

$$(10^{-i} - 10^{-j})e_i = \frac{a_{i+1}a_{i+2}\ldots a_j}{10^j} \ . \tag{2.13}$$

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On the other hand, we have that $10^{-i}e_i - 10^{-j}e_j$ can be written as a telescoping sum

$$\begin{aligned}(10^{-i}e_i - 10^{-i-1}e_{i+1}) + (10^{-i-1}e_{i+1} - 10^{-i-2}e_{i+2})\\ + \dots + (10^{-j+1}e_{j-1} - 10^{-j}e_j) \ .\end{aligned}$$

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On the other hand, we have that $10^{-i}e_i - 10^{-j}e_j$ can be written as a telescoping sum

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Since $10^{-k}e_k - 10^{-k-1}e_{k+1} = a_{k+1}10^{-k-1}$ (by 2.11) we obtain

$$10^{-i}e_i - 10^{-j}e_j = \frac{a_{i+1}}{10^{i+1}} + \cdots + \frac{a_j}{10^j}$$

which gives us (2.13). □

Uniqueness of decimal expansions

Not all eventually periodic decimals give *different* rational numbers: by (2.6),

$$0.\overline{9} \;=\; \left(\frac{9}{10}\right)\left(\frac{1}{1-10^{-1}}\right) \;=\; 1.$$

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This is actually the only sort of case we have to worry about!

Proposition 2.14

If $x \in \mathbb{Q}$ has two different decimal expansions, they're of the form

$$\begin{aligned} x &= a_0.a_1a_2\dots a_n\overline{9} \\ &= a_0.a_1a_2\dots(a_n+1) \quad \text{with } a_n \in \{0,1,\dots,8\}. \end{aligned}$$

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Sketch of proof: Suppose, without loss of generality, that the two expansions are:

$$\begin{aligned} x &= a_0.a_1a_2\ldots a_{n-1}a_na_{n+1}\ldots \\ &= a_0.a_1a_2\ldots a_{n-1}b_nb_{n+1}\ldots \end{aligned}$$

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with $a_n < b_n$. Then, using Exercise 2.7 along with some easier claims one can show

$$\begin{aligned} x &= a_0.a_1a_2\ldots a_na_{n+1}\ldots \\ &\leq a_0.a_1a_2\ldots a_n999\ldots \\ &= a_0.a_1a_2\ldots(a_n+1)000\ldots \\ &\leq a_0.a_1a_2\ldots b_nb_{n+1}\cdots = x. \end{aligned}$$

Therefore, the inequalities are both equalities and we're done. $\square$

We can now define the real numbers as

Definition: The real numbers

$$\mathbb{R} := \left\{ a_0.a_1a_2\ldots \ : \quad \begin{array}{c} a_0 \in \mathbb{Z},\ a_{i\geq 1} \in \{0,1,\ldots,9\}, \\ \nexists N \text{ such that } a_i = 9\ \forall i \geq N \end{array} \right\}$$

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Exercise 2.15:

Extend the definition of $<$ from $\mathbb{Q}$ to $\mathbb{R}$ and show that, $\forall x, y \in \mathbb{R}$ with $x < y$,

1. $\exists z \in \mathbb{Q} : x < z < y$, and
2. $\exists z \notin \mathbb{Q} : x < z < y$.

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Theorem 2.8 gives us a way to produce *many* explicit irrational numbers like

$$x = 0.1010010001\ldots \notin \mathbb{Q}.$$

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Theorem 2.8 gives us a way to produce *many* explicit irrational numbers like

$$x \;=\; 0.1010010001\ldots \;\notin\; \mathbb{Q}.$$

In the next lecture we will make precise the fact that $\mathbb{R}$ is much bigger than $\mathbb{Q}$.