Imperial College
London

MATH97033 MATH97141

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Theory

Date: Tuesday, 25 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let G be a finite group. Which of the following subgroups are characteristic in G (provide a proof or give a counter example). (4 marks)

   (i) the centre $Z(G)$ of G;

   (ii) a proper subgroup of the centre of G;

   (iii) the derived subgroup of G;

   (iv) the largest normal p-subgroup in G.

(b) Let N and K be normal subgroups in a group G. Is the commutator subgroup $[N, K]$ equal to $N \cap K$? Provide a proof or give a counterexample.

(3 marks)

(c) Suppose G and H are groups, with centers of order 2. The central product $G * H$ is defined as the factor group $(G \times H)/ < (z_1, z_2) >$, where z_1 and z_2 are non-trivial central elements of G and H respectively. Calculate with justification the number of elements of every possible order in the central product $Q_8 * Q_8$.

(4 marks)

(d) Show that the central products $Q_8 * Q_8$ and $D_8 * Q_8$ are not isomorphic.

(5 marks)

(e) Show that the automorphism group of Q_8 is isomorphic to S_4.

(4 marks)

(Total: 20 marks)

2. Let a finite group G act on a finite set Ω of size n and $a \in \Omega$. Define the stabilizer $G(a)$ of a in G.

(a) Define the following properties of such an action:

(3 marks)

(i) transitive and imprimitive with blocks of size k;

(ii) primitive;

(iii) 3-transitive;

(b) What is the kernel of the action of G on Ω? Justify your answer.

(2 marks)

(c) Let G act transitively on Ω and let N be an intransitive non-trivial normal subgroup in G. Show that the orbits of N on Ω have the same length and form an system of blocks of G on Ω.

(3 marks)

(d) Let G act transitively on Ω and let H be a subgroup of G with $G(a) < H < G$. Show that $\{h(a) : h \in H\}$ is a block of G.

(3 marks)

(e) Prove that the action of G on Ω is 3-transitive if and only if it is transitive and the action of $G(a)$ on $\Omega \setminus \{a\}$ is 2-transitive.

(5 marks)

(f) Let $a = (123)$ and $b = (123)(456)$. Calculate $C_{S_6}(a)$ and $C_{S_6}(b)$. Are they isomorphic?

(4 marks)

(Total: 20 marks)

3. Let G be a finite group.

(a) State the Sylow theorems (no proof required)

(2 marks)

(b) Describe the structure of a group with exactly two distinct composition series.

(4 marks)

(c) Is it possible to decide from the composition series if a group is solvable? If so, explain how. If not, give a counterexample to explain why it is not possible.

(2 marks)

(d) Is it possible to decide from the composition series if a group is nilpotent? If so, explain how. If not, give a counterexample to explain why it is not possible.

(3 marks)

(e) Give an application of the Frattini argument.

(3 marks)

(f) Deduce the simplicity of A_7 from that of A_6.

(6 marks)

(Total: 20 marks)

4. Let q be a power of a prime number p, $n \geq 2$ be an integer, and $V_n(q)$ be an n-dimensional vector space over $GF(q)$.

(a) What is the order of the general linear group $GL(V_n(q)) \cong GL_n(q)$ of $V_n(q)$? Justify your answer. (2 marks)

(b) What is the order of the stabilizer in $GL_n(q)$ of an unordered basis set in $V_n(q)$? (2 marks)

(c) Determine the kernel of the action of $GL_n(q)$ on the set of all subspaces of $V_n(q)$. (5 marks)

(d) By considering the action of $SL_2(4)$ on the set of one dimensional subspaces in $V_2(4)$, prove that $PSL_2(4)$ is isomorphic to A_5. (5 marks)

(e) By considering the action of $GL_2(5)$ on the set of 1-dimensional subspaces of $V_2(5)$, show that the symmetric group $Sym(\Omega)$ of a set Ω of size 6 contains a subgroup isomorphic to S_5 which acts transitively of Ω. (3 marks)

(f) Let Ω be the set of non-zero vectors of $V_n(2)$. Show that the action of $GL_n(2)$ on Ω is 2-transitive, but not 3-transitive for $n \geq 3$. (3 marks)

(Total: 20 marks)

5. (a) Let Ω be a set of size $2n$ and G be a permutation group on Ω, which has a system of blocks of size 2. Prove that $|G| \leq 2^n n!$.

(4 marks)

(b) Show that the upper bound in (a) is attained for every n.

(4 marks)

(c) Let $G = S_2 \wr S_n$ be the semidirect product of $N = {S_2}^n$ and S_n permuting the direct factors. Find the subgroups of N which are normal subgroups of G.

(4 marks)

(d) Classify the index 2 subgroups of $G = S_2 \wr S_n$.

(4 marks)

(e) Provide a set of generators of a Sylow 3-subgroup of S_9.

(4 marks)

(Total: 20 marks)

MATH96024/MATH97033/MATH97141 Solutions (2020/2021)

S1.
(a) In this part $\alpha \in \mathrm{Aut}\,(G)$ (i) $Z(G)\,\mathrm{char}\,G$: if $z \in Z(G)$ then $e = \alpha([z,g]) = [\alpha(z), \alpha(g)]$ for any $g \in G$ and hence $\alpha(z) \in Z(G)$. (ii) No: $G = C_2 \times C_2$, $H \cong C_2$, α is of order 3; (iii) $G'\mathrm{char}\,G$: $\alpha([x,y]) = [\alpha(x), \alpha(y)] \in G'$; (iv) if P is maximal normal p-subgroup in G the $Pn\,\mathrm{char}\,G$: suppose that $\alpha(P) \neq P$ and put $Q := P\alpha(P)$, which is a normal p-subgroup in G and if $\alpha(P) \neq P$ then $Q > P$, a contradiction.
[3pt. seen]

(b) Let $n \in N$ and $k \in K$. Then, $x := [n,k]$ is contained in N since $x = n^{-1}(k^{-1}nk)$ and in K since $x = (n^{-1}k^{-1}n)k$. Although the commutator subgoup $[N,K]$ might be smaller than $N \cap K$ for instance if the whole G is abelian then $[N,K] = 1$, while $N \cap K$ could be larger.
[4pt. seen]

(c) Denote by $Q_8^{(i)}$ the central factors of the central product. Let $x \in Q_8^{(1)}$ and $y \in Q_8^{(2)}$. Then, the image of (x,y) in the central product has order 1 if $x \in Z(D_8)$, $y \in Z(Q_8)$ and $o(x) = o(y)$, which gives one element of order 1 and one element of order 2 if x, y are as above with $o(x) \neq o(y)$. Now assume that $x \notin Z(Q_8^{(1)})$ and $y \in Z(Q_8^{(2)})$ (or $x \in Z(Q_8^{(1)})$ and $y \notin Z(Q_8^{(2)})$), then the order of the image is equal to $o(x)$ (or $o(y)$). Finally, if $x \notin Z(Q_8^{(1)})$ and $y \notin Z(Q_8^{(2)})$, then the order of image of (x,y) is 2 if $o(x) = o(y)$ and 4, if $o(x) \neq o(y)$. This gives 1 element of order 1, 19 elements of order 2 and 12 elements of order 4.
[4pt. seen]

(d) We calculate the numbers of elements of every order in $D_8 * Q_8$. Arguing as in (c) we have for $x \in D_8$ and $y \in Q_8$. Then, the image of (x,y) in the central product has order 1 if $x \in Z(D_8)$, $y \in Z(Q_8)$ and $o(x) = o(y)$, which gives one element of order 1 and one element of order 2 if x, y are as above with $o(x) \neq o(y)$. Now assume that $x \notin Z(D_8)$ and $y \in Z(Q_8)$ (or $x \in Z(D_8)$ and $y \notin Z(Q_8)$), then the order of the image is equal to $o(x)$ (or $o(y)$). Finally, if $x \notin Z(D_8)$ and $y \notin Z(Q_8)$, then the order of image of (x,y) is 2 if $o(x) = o(y)$ and 4, if $o(x) \neq o(y)$. This gives 1 element of order 1, 11 elements of order 2 and 20 elements of order 4. Sonce the numbers are different from that in (c) the groups are nonisomorphic
[5pt. unseen]

(e) Let $Q_8 =< a, b \mid a^4 = b^4 = 1, z := b^2 = a^2, b^{-1}ab = a^{-1} >$. The set of elements of order 4 are $F = \{a, a^3, b, b^3, ab, ba\}$. Define a graph Γ on F where two element-vertices are adjacent iff they generate the same subgroup, so that the edges are

$\{a, a^3\}$, $\{b, b^3\}$, $\{ab, ba\}$. Then every automorphism of Q_8 induces an automorphism of Γ (since the vertices generate the whole group). The aitomorphism group of the graph is $S_3 \wr S_2 \cong S_4 \times S_2$. The inner automorphisms induce an elementary abelian subgroup of order 2^2 which swaps the ends in an even number of edges. The mapings $s : a \mapsto b \mapsto ba^3 \mapsto a$; $t : a \mapsto b \mapsto a, ab \mapsto ba$ induce automorphisms of order 3 and 2, which together with the inner automorphisms generate an S_4-subgroup. Thus it only remains to shoe that swapping all the edges of Γ is not an automorphis. But such an automoprhism would invert all the alements of the group and this is possible if the group is abelian while Q_8 is not.
[4pt. unseen]

S2. The stabilizser is the subgroup $G(a) = \{g \mid g \in G, g(a) = a\}$. **[1pt. seen]** (a) (i) G is transitive if for any $a, b \in \Omega$ there is $g \in G$ such that $g(a) = b$ and there is a subset B of size k in Ω with $1 < k < n$ such that for every $g \in G$ either $g(B) = B$ or $g(B) \cap B = \emptyset$; (ii) G is primitive if whenever $B \subseteq \Omega$ so that $g(B) = B$ or $g(B) \cap B = \emptyset$ for any $g \in G$, then either $|B| = 1$ or $B = \Omega$; (iii) G is 3-transitive if for any two triple $\{a, b, c\}$ and $\{x, y, z\}$ of pairwise distinct elements of Ω there is $g \in G$ such that $g(a) = x$, $g(b) = y$, $g(c) = z$.
[2pt. seen]

(b) The kernel is the largest subgroup in the element stabiliser $G(a)$ which is normal in G. If N is normal in G and $N \leq G(a)$ then $N \leq G(b)$ since $G(b) = g^{-1}G(a)g$ for $g : a \mapsto b$ for $b \in G$ (a result from the lectures) hence N is contained in $G(b)$ and N is in the kernel of the action. On the other hand, if N is the kernel then N is normal in G and is contained in $G(a)$. This gives the result.
[2pt. seen]

(c) Let O be an orbit of N and $g \in G$. Then, for all $n \in N$, $n \cdot g(O) = g \cdot n_1(O) = g(O)$, so $g(O)$ is also an orbit of N, so either $g(O) = O$ or $g(O) \cap O = \emptyset$. Hence, $\{g(O) | g \in G\}$ is an imprimitivity system of G on Ω.
[3pt. seen]

(d) Let $g \in G$, $g(a^H) \neq a^H$ and $g(a^H) \cap a^H \neq \emptyset$. Take $b \in g(a^H) \cap a^H$. Then $b = gh'(a)$ for some $h' \in H$, so that $h^{-1}gh'(a) = a$, so $h^{-1}g \in G(a) \subset H$. This implies that $g \in H$ contrary to our assumption.
[3pt. seen]

(e) If G is 3-transitive that clearly $G(a)$ acts transitively on ordered triple of distinct elements starting with a and thus $G(a)$ acts 2-transitively on $\Omega \setminus \{a\}$. On the other hand, suppose that G is transitive on Ω and $G(a)$ is transitive in the set of ordered pairs of $\Omega \setminus \{a\}$. Let $\alpha = (a = a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3))$ and $\gamma = (c_1, c_2, c_3)$ be triples of distinct elements of Ω. We claim that there is an element on G which sends β onto γ. First map β of to α. By transitivity there is $g \in G$ with $g(b_1) = a$. By 2-transitivity there is $h \in G(a)$ with $gh(b_2) = a_2$ and $gh(b_3) = a_3$ Therefore $gh(\beta) = \alpha$. Similarly there is $f \in G$ with $f(\gamma) = \alpha$ and finally $ghf^{-1}(\beta) = \gamma$. This gives the result, since β and γ are chosen arbitrary.
[5pt.unseen]

(f) Since the centraliser preserves both the set of fixed poinr and the support, $C_{S_6}(123) = C_3 \times S_3$. Among the elements commuting with $a = (1\,2\,3)(4\,5,6)$ are a, $b = (1\,2\,3)(4\,5\,6)$ and $c = (1\,4)(2\,5)(3\,6)$ which generate $C_3 \times S_3$, counting the number of elements of the cyclic type of a in S_6 we decide that this is the full centraliser and we get the required isomorphism.
[4pt.unseen]

S3. (a) Theere are four Sylow's theorem which we combine in one statement. Let G be a finite group and let p be a prime number and let p^a be the highest power of p which divises the order of G then a subgroup P is said to be a Sylow p-subgroup in G if it has order p^a. Then (a) a Sylow p-subgroup exusts; (b) all Sylow p-subgroups are conjugate in G, (c) the number of Sylow p subgroups in G is 1 modulo p; (d) every p-subgroup is contained in a Sylow p-subgroup.
[**2pt seen**]

(b) Let $G > G_1 > \{1\}$ and $G > H_1 > \{1\}$ be the composition series, where $G_1, H_1, G/G_1, G/H_1$ are simple groups. Note that $G_1 \cap H_1$ is a normal subgroup of G, which is properly contained in G_1 and H_1. Hence, $G_1 \cap H_1 = \{1\}$. Furthermore, $G_1 H_1$ is a normal subgroup containing G_1 and H_1 properly. Hence, $G_1 H_1 = G$, so $G \cong G_1 \times H_1$. If $\alpha : G_1 \to H_1$ is an isomorphism then $\{g, \alpha(g)\} \mid g \in G_1\}$ is a third normal subgroup which leads to the third composition series. Hence G_1 and H_1 are non-isomorphic.
[**4pt seen**]

(c) G is solvable if and only if all the composition factors are abelian.
[**2pt unseen**]

(d) A nilpotent group is solvable, although S_4 and ${C_2}^2 \times C_3 \times C_2$ have the same composition factors, however S_4 is not nilpotent and the direct product of cyclic groups is nilpotent.
[**3pt unseen**]

(e) **Frattini argument:** Let G be a finite group and N be a normal subgroup of G and P is a Sylow p-subgroup of G with $P \subseteq N$. Then, $G = N N_G(P)$. It is used to show the normaliser of the normaliser of a Sylow subgroup is just the normaliser.
[**3pt seen**]

(f) Let N be a nontrivial normal subgroup of A_7. Then, $N \cap A_6$ is a normal subgroup of A_6, so either $N \cap A_6 = A_6$ or $N \cap A_6 = 1$. Since the action of A_7 on the 7 set is primitive, the action of N is transitive. If $N \cap A_6 = A_6$, then $N = A_7$ by the order reason. If $N \cap A_6 = 1$, then $|N| = 7$, $N \cong C_7$ which is impossible since by Frattini argumnent $N_{A_7}(C_7)$ has order 21 which is less than the order of A_7. [**6pt. unseen**]

S4.

(a) The order of $GL_n(q)$ is the number of bases, which can be calculated recurrently: b_i should be picked in a way that it is not in the span of $b_1, ..., b_{i-1}$. There are q^{i-1} vectors in this span, so the number of bases is $\prod_{i=1}^{n}(q^n - q^i)$. **[2pt seen]**

(b) Let $(b_1, ..., b_n)$ and $(c_1, ..., c_n)$ be two (ordered) bases. Then, the transformation which sends $\sum_{i=1}^{n} \alpha_i b_i$ to $\sum_{i=1}^{n} \alpha_i c_i$ is the unique linear transformation which sends B to C. Thus for an unordered basis we have the order of the stabilizer equal to $n!$. **[2pt seen]**

(c) First, let us determine the kernel on 1-subspaces: $f(v) = \alpha v$, $f(u) = \beta u$, $f(v+u) = \alpha v + \beta u$ must be proportional to $v + u$, so $\alpha = \beta$. So, the kernel is $K_1 = \{\lambda I_n | \lambda \in GF(q) \setminus \{0\}\}$. It is clear that K_1 preserves all the subspaces in $V_n(q)$, hence it is the required kernel.
[5pt unseen]

(d) By considering the action of $SL_2(4)$ of order $15 \cdot 12 = 60 \cdot 3$ on the set of 1-subspaces on $V_2(4)$ of size $5 = 4 + 1$ we obtain a subgroup of order 60 of S_5. The only such subgroup is A_5.
[5pt unseen]

(e) Consider the action of $GL_2(5)$ on the set Ω of 6 1-dimensional subspaces of $V_2(5)$. Then this action is transitive and has order $(25 - 1)(25 - 5)/4 = 120$. In order to finilize the proof let us prove that a subgroup G of S_6 of order 120 is isomorphic to S_5. Consider the action of S_6 on the set Δ of left cosets of G. The action is faithful since the only proper normal subgroup A_6 of S_6 is not contained in G by the order reason. Thus, the action is a subgroup of order 360 in the symmetric group of Δ. Hence, it is the whole symmetric group of Δ and G is the stabiliser of an element which must be isomorphic to S_5 as claimed. **[3pt unseen]**

(f) In $V_n(2)$ every 1-subspace contains a unique non-zero vector. Hence, the 2-transitivity follows from (a). On the other hand, a linearly independent triple (x, y, z) and $(x, y, x + y)$ are contained in different orbits, so the action is not 3-transitive.
[3pt unseen]

S5. (a) Let $\Omega = B_1 \cup B_2 \cup ... \cup B_n$ be a system of blocks, where $B_i = \{x_{2i-1}, x_{2i}\}$. Let N be the subgroup of the G, which stabilize the blocks B_i as a set. Moreover, N induces on each block B_i an action of order 2. Moreover, if N_i is the subgroup of N, which stabilizes the blocks $B_1, ... B_{i-1}$, then $[N_i : N_{i-1}] \leq 2$. Hence, $|N| \leq 2^n$. Moreover, G/N is a permutation group of the set of the blocks $B_1, ..., B_n$ so $|G/N| \leq n!$. So, $|G| = |N||G/N| \leq 2^n n!$.
[4pt unseen]

(b) Note that the wreath product $S_2 \wr S_n$ has order $2^n n!$. Moreover, its permutation action of degree $2n$ has blocks of size 2. Thus, the upper bound in (a) is attained. **[4pt unseen]**

(c) Let $t_i = (x_{2i-1}, x_{2i})$ be a generating transposition set for N, so that each $n \in N$ can be written in form $n = t_1^{\alpha_1} ... t_n^{\alpha_n}$, where $\alpha_i \in \{0, 1\}$. Note that $N_1 =< t_1 ... t_n >$ is normal in G, because $t_1 ... t_n$ is invariant under G. Moreover, $N_{n-1} =< t_1^{\alpha_1} ... t_n^{\alpha_n} \mid 2 | \sum_{i=1}^{n} \alpha_i >$ because it has index 2. Let us prove that the are no more normal subgroups. Indeed, suppose $M \leq N$ is normal in G and $M \neq N$, $M \neq \{1\}$. Then, without loss of generality we may assume that there exists $m \in M$, $m = t_1 t_2^0 t_3^{\alpha_3} ... t_n^{\alpha_n}$. Denote $s = (x_0, x_3)(x_1, x_4)$. Then, $s^{-1} m s = t_1^0 t_2 t_3^{\alpha_3} ... t_n^{\alpha_n}$. So, $m(s^{-1} m s) = t_1 t_2 \in N$. By applying conjugation we can obtain that $t_i t_j \in N$, for all pairs i, j. The following set generates N_{n-1}, so we are done.
[4pt unseen]

(d) If K is a subgroup of order 2 in G, then $[N : N \cap K] \leq 2$. So, according to part (c), either $N \cap K = N$ or $N \cap K = N_{n-1}$. Similarly, $K/N \cong A_n$ or S_n. Therefore, we obtain that $K = N_{n-1} \rtimes S_n$, $K = N \rtimes A_n$ or $\{ns : n \in N, s \in S_n \mid sgn(s) = \sum_{i=1}^{n} \alpha_i (\bmod 2)\}$.
[4pt unseen]

(e) Note that $C_3 \wr C_3$ is a Sylow 3-subgroup of S_9. Hence, the set of permutations $(1, 2, 3)$, $(4, 5, 6)$, $(7, 8, 9)$, $(1, 4, 7)\,(2, 5, 8)\,(3, 6, 9)$ is such a generating set. **[4pt unseen]**

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96024 MATH97033 MATH97141	1	This question was done well by most, difficulties with Q1 (iv), it was unseen.
MATH96024 MATH97033 MATH97141	2	Done well, Q2 (b) was understood by most as request of the definition of the kernel, I gave full mark for such answer. Q2 (f) was unseen but done well by most, I am pleased.
MATH96024 MATH97033 MATH97141	3	Quite a few students included the full proof of the simplicity for An in Q3 (f). I gave full mark for this but the expected solution was much easier.
MATH96024 MATH97033 MATH97141	4	Overall good response, some details were missing by some leading to partial credit.
MATH96024 MATH97033 MATH97141	5	Out of a few candidates only one attemted (c) and noone (d). You could do better.