Imperial College London

MATH70057

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Riemannian Geometry

Date: 12 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Let (M, g) be a Riemannian manifold.

   (a) Define the length $\ell(\gamma)$ of a curve $\gamma : [a, b] \to M$.

   (3 marks)

   (b) Define the distance $\mathrm{dist}(p, q)$ between two points $p, q \in M$.

   (3 marks)

   (c) Show that $\ell(\gamma) = \mathrm{dist}(\gamma(a), \gamma(b))$ implies γ must be a geodesic, i.e. $\nabla_{\gamma'}\gamma' = 0$.

   (6 marks)

   (d) A geodesic $\gamma : [0, +\infty) \to M$ is called a *ray* if $\ell(\gamma|_{[a,b]}) = \mathrm{dist}(\gamma(a), \gamma(b))$, for all $0 \le a \le b$. Show that if (M, g) is complete and non-compact then it contains a ray.

   (8 marks)

(Total: 20 marks)

2. Let $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a smooth map and denote by $E : [0, a] \to \mathbb{R}$ the (energy) function

$$E(s) = \frac{1}{2}\int \langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\rangle dt.$$

   (a) Show that $\frac{d^2}{ds^2}E(s) =$

$$\int_0^a \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s}\right\rangle + \left\langle R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \frac{d}{dt}\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle - \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt.$$

   (10 marks)

   (b) Use the formula above to prove that a closed surface with positive curvature does not admit a closed geodesic which is both minimising and embedded. Hint: use a variation for which $\frac{\partial f}{\partial s}(0, t) = V(t)$ is a parallel vector field.

   (10 marks)

(Total: 20 marks)

3. (a) Define what it means for a map $f : (M, g) \to (M, g)$ to be an isometry of M.

(3 marks)

(b) What does it mean to say that a connection is symmetric and compatible with the metric?

(3 marks)

(c) Explain how to deduce Koszul formula (you can save time by just explaining the main relevant symmetry):

$$\begin{aligned} 2g(\nabla_X Y, Z) = X\left(g(Y,Z)\right) + Y\left(g(X,Z)\right) - Z\left(g(X,Y)\right) \\ - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X). \end{aligned}$$

(6 marks)

(d) Prove that the Levi-Civita connection is an isometric invariant.

(4 marks)

(e) Show that if $\dim M$ is odd and f is an orientation preserving isometry with a fixed point, then M admits a geodesic of fixed points, i.e. $\gamma : [0, a] \to M$ such that $f(\gamma(t)) = \gamma(t)$, for all $t \in [0, a]$.

(4 marks)

(Total: 20 marks)

4. Let (M, g) be a complete Riemannian manifold.

(a) Let $U \subset M$ be a proper open subset, i.e. $M \setminus U \neq \emptyset$. Show that U is not geodesically complete with respect to the inherited metric.

(4 marks)

(b) Let $f : M \to \mathbb{R}$ and $t \in \mathbb{R}$ a regular value of f. Show that $N = f^{-1}(t)$ is geodesically complete with respect to the inherited metric.

(6 marks)

(c) State the definition of a Jacobi vector field.

(3 marks)

(d) State the definition of conjugate points.

(3 marks)

(e) Assume that $R(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X$. Prove that M does not have conjugate points.

(4 marks)

(Total: 20 marks)

5. Let $p \in M$. Denote by $f : M \to \mathbb{R}$ the function $f(q) = (\mathrm{dist}(p,q))^2$.

(a) Prove that f is smooth on a neighborhood of p.

(5 marks)

(b) Let $q = \exp_p(x)$. Show that $\nabla f(q) = 2 \cdot d(\exp_p)_x(x)$ and, for $q \neq p$, $X(q) = \frac{\nabla f(q)}{|\nabla f(q)|}$ defines a unitary vector field which is normal to small geodesic spheres around p.

(5 marks)

(c) Argue that $\operatorname{Hess} f$ is positive definite in a small neighborhood around p. (Hint: you can use what you know about the Taylor expansion of the metric in normal coordinates)

(5 marks)

(d) Conclude that there exists a neighborhood of p which is strongly convex. (Hint: compute $\frac{d^2}{dt^2} f(\gamma(t))$ along a geodesic γ and show that $f|_\gamma$ is a convex function)

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

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MATH70057

Riemannian Geometry (Solutions)

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1. (a) $\ell(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt$. $\boxed{\text{seen} \Downarrow}$

(b) $\text{dist}(p, q) = \inf\{\ell(\gamma) : \gamma : [0, 1] \to M, \gamma(0) = p, \gamma(1) = q\}$. $\boxed{3, \text{A}}$

(c) Let $f : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a smooth map such that $f(0, t) = \gamma(t)$, $\gamma(s, a) = \gamma(a)$, $\gamma(s, b) = \gamma(b)$, for all $(s, t) \in (-\varepsilon, \varepsilon) \times [a, b]$. Witouth loss of generality we can assume that $|\frac{\partial f}{\partial t}(0, t)| = |\gamma'(t)| = 1$, for all t. Let $\ell(s)$ be the length of the curve $t \mapsto f(s, t)$. Since ℓ attains a minimum at $s = 0$, we must have $\boxed{\text{seen} \Downarrow}$ $\boxed{3, \text{A}}$ $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} \ell(s) \\
&= \frac{d}{ds}\bigg|_{s=0} \int_a^b \left\langle \frac{\partial f}{\partial t}(s, t), \frac{\partial f}{\partial t}(s, t) \right\rangle^{1/2} dt \\
&= \int_a^b \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}(0, t), \frac{\partial f}{\partial t}(0, t) \right\rangle dt \\
&= \int_a^b \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}(0, t), \frac{\partial f}{\partial t}(0, t) \right\rangle dt \\
&= \int_a^b \frac{d}{dt}\left\langle \frac{\partial f}{\partial s}(0, t), \frac{\partial f}{\partial t}(0, t) \right\rangle - \left\langle \frac{\partial f}{\partial s}(0, t), \frac{D}{dt}\frac{\partial f}{\partial t}(0, t) \right\rangle dt \\
&= \langle V(t), \gamma(t) \rangle \bigg|_{t=0}^{a} - \int_0^a \langle V(t), \nabla_{\gamma'(t)}\gamma'(t) \rangle dt,
\end{aligned}
$$

where $V(t) = \frac{\partial f}{\partial s}(0, t)$. This formula holds for all vector fields V along γ such that $V(0) = 0$ and $V(a) = 0$. Therefore, $\nabla_{\gamma'(t)}\gamma'(t) \equiv 0$ $\boxed{6, \text{C}}$

(d) Let $p \in M$. Since M is complete and non-compact there exists a sequence $x_n \in M$ such that $\text{dist}(p, x_n) = T_n \to +\infty$. Moreover, such distance is realised by a geodesic $\gamma_n : [0, T_n] \to M$, which without loss of generality we assume to be parametrised by unit speed. Since $\gamma_n(0) = 0$ and $v_n = \gamma_n'(0) \in T_pM$ is a unitary vector, we can assume, after passing to a subsequence if necessary, that $v_n \to v \in T_pM$ where $|v| = 1$. Let γ be the geodesic with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$. By the smooth dependence on the initial conditions, it follows that $\gamma_n \to \gamma$ on any compact interval $[0, T]$ and the convergence is smooth. Let $q_n = \gamma_n(T)$ and $q = \gamma(T)$. Then, by the triangle inequality $\ell(\gamma_n|_{[0,T]}) = \text{dist}(p, q_n) \leq \text{dist}(p, q) + \text{dist}(q, q_n)$. From the smooth dependence we obtain $\ell(\gamma_n|_{[0,T]}) \to \ell(\gamma|_{[0,T]})$ and $\text{dist}(q, q_n) \to 0$. We conclude $\boxed{\text{meth seen} \Downarrow}$

$$\ell(\gamma|_{[0,T]}) \leq \text{dist}(p, q),$$

which must be an equality by the definition of distance. $\boxed{8, \text{D}}$

2. (a) First, we use the compatibility of the connection with the metric: $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
\frac{d^2}{ds^2}E(s) &= \int_0^a \frac{d}{ds}\left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt \\
&= \int_0^a \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{D}{ds}\frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{D}{ds}\frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt \\
&= \int_0^a \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle + \left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt \\
&= \int_0^a \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle + \left\langle R\Big(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt.
\end{aligned}
$$

The formula follows by substituting the last term using:

$$
\frac{d}{dt}\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \right\rangle.
$$

$\boxed{10, \text{B}}$

(b) If $\gamma : [0,a] \to M$ is a closed embedded geodesic, then $\gamma(0) = \gamma(a) = p$ and $\gamma'(0) = \gamma'(a) = v$. In this case we can consider a variation such that $f(s,0) = f(s,a)$ and $\frac{\partial f}{\partial s}(0,t) = V(t)$ is a parallel vector field with $|V(t)| \equiv 1$ and $V(t) \perp \gamma'(t)$. If the geodesic is minising, then using the formula from the previous item we have $\boxed{\text{meth seen} \Downarrow}$

$$
0 \le \frac{d^2}{ds^2}E(0) = \int_0^a |V'(t)|^2 + \langle R(V,\gamma')V,\gamma'\rangle + \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\gamma' \right\rangle dt + \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle \Bigg|_{(s,t)=(0,0)}^{(0,a)}.
$$

The first and third terms are zero because $V' \equiv 0$, $\frac{D}{dt}\gamma' \equiv 0$. The third term is zero because the values of the maps coincide when $t = 0$ and $t = a$. Finally,$\langle R(V,\gamma')V,\gamma'\rangle = -K(\sigma) < 0$ is (minus) the sectional curvature of the plane $\sigma = \mathrm{span}\{V,\gamma'\}$. It follows, $0 \le \frac{d^2}{ds^2}E(0) < 0$, which is a contradiction. $\boxed{10, \text{A}}$

3. (a) We say that f is an isometry if f is a diffeomorphism such that $\langle v, w\rangle = \langle df_p(v), df_p(w)\rangle$, for all $p \in M$ and all $v, w \in T_pM$. seen ⇓ 3, A

(b) Symmetric means $\nabla_X Y - \nabla_Y X = [X, Y]$, for all $X, Y \in \chi(M)$ and compatible with the metric means $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$. seen ⇓ 3, A

(c) The relevant symmetry comes from combining both symmetry and compatibility with the metric, i.e. $\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle$. This induces a cyclic permutation of the roles of (X, Y, Z) while at the same time it changes the sign. So after three iterations we return to the same element with the right sign. seen ⇓ 6, B

(d) The Levi-Civita connection is symmetryc and compatible with the metric. Therefore, it satisfies Koszul's formula. Note that the elements of the right hand side of the formula only contain isometric invariants. In fact, the bracket $[X, Y]$ is a diffeomorphic invariant, meaning $[f^*X, f^*Y] = f^*[X, Y]$. Since g is non-degenerate, the formula characterises also the values of $\nabla_X Y$ and this can be used as the definition of the Levi-Civita connection. seen ⇓ 4, A

(e) Let $p \in M$ be a fixed point of f, i.e. $f(p) = p$. Then $df_p : T_pM \to T_pM$ is an orientation preserving orthonormal transformation of T_pM onto itself. Since $\dim T_pM$ is odd, it follows that there exists $v \in T_pM \setminus \{0\}$ such that $df_p(v) = v$. Let $\gamma(t) = \exp_p(tv)$ be the geodesic starting at p with velocity v. Since f is an isometry, then $\tilde{\gamma}(t) = f(\gamma(t))$ is also a geodesic of M. Since, $\tilde{\gamma}(0) = f(\gamma(0)) = f(p) = p = \gamma(0)$ and $\tilde{\gamma}'(0) = df_p(\gamma'(0)) = df_p(v) = v = \gamma'(0)$, it follows that $f(\gamma(t)) = \tilde{\gamma}(t) = \gamma(t)$. meth seen ⇓ 4, D

4. (a) Let $x \in \partial U$. Then, for all small $r > 0$, the geodesic ball $B_r(x)$ (with respect to the metric g on M) contains $p \in U$ and $q \in M \setminus U$. Let γ be a geodesic of M joining p and q, i.e. $\gamma(0) = p$ and $\gamma(1) = q$, which exists if $r > 0$ is small enough. Then, for a short time γ is a geodesic of (U, g). If this geodesic could be continued for all time in (U, g) then it would also be a geodesic of M, contradicting the fact that such geodesic leave U infinite time. meth seen ⇓ 4, B

(b) By Hopf-Rinow, it is enough to check that N is complete in the metric space sence. First, N is a closed set, by continuity of f, and by the Implicit Function Theorem, we know that N is in fact an embedded submanifold of M. Let $p_n \in N$ be a Cauchy sequence with respect to the inherited metric $(N, g|_N)$. Since $\mathrm{dist}_M(p_n, p_m) \leq \mathrm{dist}_N(p_n, p_m)$ it follows that p_n is also a Cauchy sequence in (M, g), which is complete. Therefore, there exists $p \in M$ such that $p_n \to p$. Since N is closed, it follows that $p \in N$. unseen ⇓ 6, C

(c) A vector field $t \mapsto J(t)$ along a geodesic γ, is a Jacobi vector field if it satisfies $\frac{D}{dt}\frac{D}{dt}J + R(J, \gamma')\gamma' = 0$. seen ⇓ 3, A

(d) Let $\gamma : [0, a] \to M$ be a geodesic. We say that $p = \gamma(0)$ and $q = \gamma(a)$ are conjugate, if there exists a non-zero Jacobi vector field along γ such that $J(0) = 0$ and $J(a) = 0$. seen ⇓ 3, A

(e) Let J be a Jacobi vector field along a geodesic $\gamma : [0, a] \to M$ with $|\gamma'| \equiv 1$. It is a general property of Jacobi vector fields that $\langle J(t), \gamma'(t)\rangle$ is an affine function. If $J(0) = 0$ and $J(a) = 0$, it follows that $J \perp \gamma'$, for all times. Therefore, the Jacobi equation in this case is $J'' - J = 0$. Then $\frac{1}{2}\frac{d^2}{dt^2}\langle J, J\rangle = \langle J'', J\rangle + \langle J', J'\rangle = |J|^2 + |J'|^2 \geq 0$. So $|J|^2$ is a positive convex function and must vanish between 0 and a. Therefore $J \equiv 0$ on this interval. meth seen ⇓ 4, D

5. (a) It is enough to see that the function is smooth on a normal coordinate chart. Let $U \subset T_pM$ be a neighborhood of 0 where normal coordinates are defined. We saw during the lectures that $|x| = \mathrm{dist}(p, \exp_p(x))$, this function is not smooth at 0. However, $f(\exp_p(x)) = |x|^2$ which is a smooth function on U.

seen ⇓

5, M

(b) The geodesic sphere of radius $r > 0$ and center p, is the set $S_r(p) = \{\exp_p(x), x \in T_pM, |x| = r\}$. By the previous item $|x| = \mathrm{dist}(p, \exp_p(x))$. It follows they are all level sets of f, in particular $\nabla f(\exp_p(x)) \perp T_{\exp_p(x)} S_{|x|}(p)$. Differentiating $f(\exp_p(x + tv)) = |x + tv|_p^2$, we obtain, by Gauss' lemma and the definition of gradient, that:

meth seen ⇓

$$\begin{aligned}
\langle \nabla f(q), d(\exp_p)_x(v) \rangle_q &= df_q(d(\exp_p)_x(v)) \\
&= 2\langle x, v \rangle_p \\
&= 2\langle d(\exp_p)_x(x), d(\exp_p)_x(v) \rangle_q.
\end{aligned}$$

Since $d(\exp_p)_x$ is injective, this implies $\nabla f(q) = 2 \cdot d(\exp_p)_x(x)$. Moreover, choosing $x = v$ we see that $\nabla f(q) \neq 0$ for $q \neq p$ since $|x|^2 \neq 0$ in that case.

5, M

(c) In normal coordinates, $f(x) = |x|^2$. Therefore,

meth seen ⇓

$$\begin{aligned}
\mathrm{Hess}\, f(\partial_i, \partial_j) &= \partial_i \partial_j f - \nabla_{\partial_i} \partial_j (f) \\
&= \partial_i \partial_j |x|^2 - \sum_k \Gamma_{ij}^k \partial + k(|x|^2) \\
&= 2\delta_{ij} - \sum_k 2\Gamma_{ij}^k(x) x_k.
\end{aligned}$$

Now, in normal coordinates $\Gamma_{ij}^k = O(|x|)$. It follows that $\mathrm{Hess}\, f = 2\,\mathrm{Id} + O(|x|^2)$, which is positive definite if $|x|$ is small enough.

5, M

(d) Note that

meth seen ⇓

$$\frac{d^2}{dt^2} f(\gamma(t)) = \frac{d}{dt} \langle \nabla f, \gamma'(t) \rangle = \langle \nabla_{\gamma'(t)} \nabla f, \gamma'(t) \rangle = \mathrm{Hess}\, f(\gamma', \gamma') > 0,$$

where we used that $\nabla_{\gamma'} \gamma' = 0$ and that $\mathrm{Hess}\, f$ is positive definite. It follows that $f = \mathrm{dist}(\cdot, p)$ is a convex function along any geodesic. In particular, the maximum of f along any geodesic is attained at the extremes of the geodesic. We conclude that a geodesic that joins two point inside of a geodesic ball must be entirely contained inside of the geodesic ball if the radius of the ball is sufficiently small.

5, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70057	1	Some people were confused on how to obtain a ray and used a limit of points rather than a limit of directions associated to a sequence of points. The hypothesis of completeness was relevant because it allows us to say a minimising geodesic exists.
MATH70057	2	Most people did well on this one.
MATH70057	3	Most people did well on this one.
MATH70057	4	For the most part people did well on this one. Some of you got confused on how to use Hopf-Rinow in part (b). An advantage of the theorem is that we can check topological conditions in order to conclude geometric properties. In this case, it was enough to check metric completeness to conclude geodesic completeness.
MATH70057	5	Most people did very well on this one. You asked many questions about this topic over office hours and it really shows. Well done!