IMPERIAL

MATH70033

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebraic Curves

Date: Wednesday, May 15, 2024
Time: 14:00 – 16:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

If you are working on one part of a question, you may assume the results of any previous parts, even if you have not solved them.

1. Let $C \subset \mathbb{P}^2(\mathbb{C})$ be the projective plane curve defined by $x^3 + y^3 + z^3 + 6xyz = 0$.

   (a) Show that $p = [1, 0, -1]$ is a smooth point of C, and find the tangent line T_pC. (5 marks)

   (b) Prove that $p = [1, 0, -1]$ is an inflection point of C. (4 marks)

   (c) Prove that C is a smooth curve. (5 marks)

   (d) Find the genus of C as a Riemann surface. (3 marks)

   (e) Let $C' \subset \mathbb{P}^2(\mathbb{C})$ be the curve defined by $x^2 + xy + 2z^2 = 0$. Write down a nonzero homogeneous polynomial $h(y, z)$ with the property that any point $[a, b, c] \in C \cap C'$ satisfies $h(b, c) = 0$. (You do not have to explicitly evaluate any determinants that may arise along the way.) (3 marks)

(Total: 20 marks)

2. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

   (a) Prove that any nine points of $\mathbb{P}^2(\mathbb{K})$ lie on some plane cubic. (4 marks)

   (b) State the strong version of Bézout's theorem over $\mathbb{K}$. (3 marks)

   (c) Let $C \subset \mathbb{P}^2(\mathbb{K})$ be an irreducible curve of degree 5.

      (i) Prove that no three singular points of C are collinear. (4 marks)

      (ii) Prove that C has at most six singular points. (5 marks)

   (d) Prove that the degree-6 curve $C = \{x^4z^2 - (y^2 - z^2)^3 = 0\} \subset \mathbb{P}^2(\mathbb{K})$ has two singular points of multiplicity 3, at $[0, \pm 1, 1]$. (4 marks)

(Total: 20 marks)

3. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Given an irreducible plane curve $C \subset \mathbb{P}^2(\mathbb{K})$ of degree at least 2, we define the *dual curve* $\check{C} \subset \mathbb{P}^2(\mathbb{K})$ to be the Zariski closure of the set

$$\left\{ [a,b,c] \in \mathbb{P}^2(\mathbb{K}) \;\middle|\; \begin{array}{c} T_pC = \{ax+by+cz=0\} \text{ for} \\ \text{some smooth point } p \in C \end{array} \right\}.$$

(a) Prove that $\check{C}$ contains infinitely many points. (3 marks)

(b) Write $C = \{F(x,y,z) = 0\}$, where F has degree $d \geq 2$, and suppose that $[1,0,0] \notin C$.

(i) Show that if $a \neq 0$ and the line $ax+by+cz=0$ is tangent to C at some smooth point of C, then the resultant

$$G(y,z) = \mathcal{R}_{F,ax+by+cz}(y,z)$$

has a double root. (5 marks)

(ii) With G as given in part (b)(i), prove that there are $a \neq 0$ and $b \in \mathbb{K}$ such that $G(1,0) \neq 0$. (3 marks)

(iii) Prove that $\check{C}$ satisfies an equation of the form $H(a,b,c) = 0$, where $H \in \mathbb{K}[a,b,c]$ is a nonzero polynomial. (5 marks)

(c) Find an equation defining $\check{C}$ in the case where $C = \{xy^3 - z^4 = 0\}$. (4 marks)

(Total: 20 marks)

4. Let $f : S \to S'$ be a non-constant holomorphic map between compact, connected Riemann surfaces.

(a) Give a definition of the *ramification index* $v_f(p)$ at a point $p \in S$. (3 marks)

(b) Prove that f is surjective. (4 marks)

(c) Prove that if S has genus g and S' has genus h, then $g \geq h$. (4 marks)

(d) Let $S = \{x^7 + y^7 + z^7 = 0\} \subset \mathbb{P}^2(\mathbb{C})$ and $S' = \{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2(\mathbb{C})$.
Define a map $\phi : S \to S'$ by $\phi([x,y,z]) = [x^3z, y^3x, z^3y]$.
In this part you may assume without proof that S and S' are smooth and ϕ is holomorphic.

(i) Prove that in fact $\phi(S) \subset S'$. (2 marks)

(ii) Determine the set $\phi^{-1}([1,0,0])$. (2 marks)

(iii) Prove that ϕ has no ramification points, and determine (with proof) its degree. (5 marks)

(Total: 20 marks)

5. For this question you may assume all Riemann surfaces are compact and connected, and that the Riemann–Roch theorem is true for any such Riemann surface.

(a) State the Riemann–Roch theorem. (3 marks)

(b) Let S be a Riemann surface of genus g, and let D be a divisor on S.

(i) Prove that if $\deg(D) = 0$ and $\ell(D) > 0$, then D is a principal divisor and we have $\ell(D) = 1$. (4 marks)

(ii) Prove that if $\deg(D) = 2g-2$, then $\ell(D) \leq g$ with equality if and only if D is a canonical divisor. (4 marks)

(c) Let S be a Riemann surface of genus g, and fix a point $p \in S$ and a positive integer $n \geq 2g$.

(i) Prove that $\ell(np) > \ell((n-1)p)$. (4 marks)

(ii) Prove that there is a non-constant meromorphic function $f : S \to \mathbb{C} \cup \{\infty\}$ such that f is holomorphic on $S \setminus \{p\}$ and has a pole of order exactly n at p. (3 marks)

(iii) Prove that there is a holomorphic map $S \to \mathbb{P}^1(\mathbb{C})$ of degree n. (2 marks)

(Total: 20 marks)

Module:	MATH60033/MATH70033
Setter:	Sivek
Checker:	Corti
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Version:	Final version

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH60033/MATH70033 Algebraic Curves

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 5 marks; 1(b) 4 marks; 1(d) 3 marks; 2(b) 3 marks; 2(c)(i) 4 marks; 2(d) 4 marks; 3(b)(i) 5 marks; 4(a) 3 marks; 4(d)(ii) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(e) 3 marks; 2(a) 4 marks; 3(a) 3 marks; 3(c) 4 marks; 4(b) 4 marks; 4(d)(i) 2 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
1(c) 5 marks; 3(b)(ii) 3 marks; 4(c) 4 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
2(c)(ii) 5 marks; 3(b)(iii) 5 marks; 4(d)(iii) 5 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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Temporary front page

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May – June 2024

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Algebraic Curves

Date: Wednesday, 15th May 2024
Time: 14:00-16:30
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

If you are working on one part of a question, you may assume the results of any previous parts, even if you have not solved them.

1. Let $C \subset \mathbb{P}^2(\mathbb{C})$ be the projective plane curve defined by $x^3 + y^3 + z^3 + 6xyz = 0$.

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(iii) Prove that there is a holomorphic map $S \to \mathbb{P}^1(\mathbb{C})$ of degree n. (2 marks)

(Total: 20 marks)

MATH60033 Algebraic Curves

Question	Marker's comment
1	People did well on this question. nbsp;Most points were lost for omitting some cases in part (c), or for not giving the resultant in part (e).
2	In (a), several students miscounted the number of monomials in a basis, or otherwise didn't show that there is a *nonzero* solution to the corresponding system of equations. nbsp;In c(ii), applying Bézout's theorem to the curves {F=0} and {F_x=0} doesn't give strong enough constraints; instead one should use (a) to find a cubic passing through seven singular points of C. nbsp;Part (d) mostly went well, but it would have saved everyone a lot of work to pass to an affine chart such as {z=1}.
3	This question was too hard, and I think most people postponed it to the end of the exam and ran out of time. nbsp;For b(i), the key observation needed was the fact that C has intersection multiplicity at least 2 with any tangent line T_pC, and by definition this means that the corresponding resultant has at least a double root.
4	In d(ii), many people said that if phi(x,y,z) = [1,0,0] then x^3 y = 1. nbsp;This is not necessarily true -- all you really know is that x^3 y is nonzero -- and led some students to conclude wrongly that there are only three or four solutions.

MATH70033 Algebraic Curves

Question	Marker's comment
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4	In d(ii), many people said that if phi(x,y,z) = [1,0,0] then x^3 y = 1. nbsp;This is not necessarily true -- all you really know is that x^3 y is nonzero -- and led some students to conclude wrongly that there are only three or four solutions.
5	Students mainly encountered difficulty applying lemmas about the various spaces in Riemann-Roch: for example, if deg(D) lt; 0 then l(D) = 0 in c(i). nbsp;Part b(i) was harder than expected; the goal was to say that if deg(D)=0 and l(D) gt; 0, then for some nonzero f in L(D) the divisor (f)+D is both effective and of degree zero, hence equal to zero.