Practice Exam Solutions

Pure

ALGEBRAIC CURVES SOLUTION

We may assume that $p := [0, 0, 1]$ is a singular point of C after taking a projective transformation. Then C must be the zero set of an irreducible polynomial of the form:

$$P(x_0, x_1, x_2) = \alpha x_0^3 + \beta x_0^2 x_1 + \gamma x_0 x_1^2 + \delta x_1^3 + Q(x_0, x_1)x_2,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and Q is a homogeneous polynomial of degree 2 in $\mathbb{C}[x_0, x_1]$. Note that Q cannot be the zero polynomial, or else P would be reducible. Then we can write Q as a product of two homogeneous linear polynomials in x_0 and x_1. Note that the singular point p must be in the zero set of Q. Then we have two options:

First, suppose the zero set of Q is a line through p. This means $Q = L^2$ for some linear polynomial L. After a projective transformation that fixes p, we can assume this line is given by $x_1 = 0$ such that $Q = \lambda x_1^2$ for some $\lambda \in \mathbb{C}^*$. We can assume, by scaling, that $\lambda = -1$. Then going back to the equation for P, this means that α is non-zero since then P would be divisible by x_1 and by scaling x_0, we can further assume that $\alpha = 1$. This gives the first desired equation.

The other option is that the zero set of Q consists of two distinct lines through p. Again, we can find a projective transformation that fixes p such that $Q = \lambda x_0 x_1$. Then in P, both α and γ must be non-zero since otherwise P would be divisible by either x_0 or x_1. Then rescaling x_0 and x_1, we can assume that $\alpha = \gamma = 1$. Then by rescaling x_2, we can assume that $\lambda = 1$ which gives the second option.

The first part of the question is out of 7 points. Of course, this is just one way to do it but the key observation is the form of P by assuming a singular point. Writing this and noting that Q must be non-zero should warrant 3 points immediately. Then splitting it up to the two separate cases should be 2 points each.

The statement about p being the unique singular point can be checked by explicitly computing partial derivatives of the two equations and stating that the number of singular points is preserved under projective transformation.

For each of the possible equations, reward 1 point for a correct partial derivative computation and solving the resulting system of equations. Also give one point for stating that projective transformations preserve singularities. This last part should then be out of a possible 3 points.

(1) (a) An R-module M is *finitely generated* if there exists a finite set of elements $m_1, \ldots, m_d$ of M such that every element m of M can be written as $r_1 m_1 + \cdots + r_d m_d$ for some $r_1, \ldots, r_d \in R$. **[Cat A, 1]**

An R-module M is Noetherian if every increasing chain of submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$$

eventually stabilizes; that is, there exists an integer n such that $M_i = M_n$ for all $i \geq n$. **[Cat A, 1]**

A ring R is a Noetherian ring if R is Noetherian as an R-module. **[Cat A, 1]**

(b) Suppose M is Noetherian and let N be a submodule of N. Suppose N is not finitely generated. Then inductively define submodules as follows: set $N_0 = \{0\}$, and for each $i > 0$, choose an element n_i in N but not N_{i-1}, and let N_i be the submodule generated by $\{n_1, n_2, \ldots, n_i\}$. (Such an n_i always exists since N is not finitely generated, so $N_i \subsetneq N$.) Then we have an infinite ascending chain:

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \ldots$$

of submodules of M so M is not Noetherian. **[Cat A, 2]**

Conversely, suppose that every submodule of M is finitely generated, and let

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$$

be an ascending chain of submodules of M. Let N be the union of the N_i; then N is a submodule of M and thus generated by a finite set $n_1, \ldots, n_d$. Since these are in N, each lives in some N_i, so there exists n such that n_i is an element of N_n for all i. Then N_n contains N, so $N_n = N$. For all $i > n$, $N_n \subseteq N_i \subseteq N$, so $N_i = N$ and the tower stabilizes. **[Cat A, 3]**

(c) Let M be Noetherian and suppose that no quotient M/N of M is simple. We construct a strictly increasing chain of submodules of M inductively as follows: fix a submodule N_0 of M, not equal to M. For each i, M/N_i the quotient is nonzero and not simple, so there exists a submodule J_i of M/N_i that is nonzero and not all of N_i. Let N_{i+1} be the preimage of J_i under the map $M \to M/N_i$. Then N_{i+1} strictly contains N_i and is strictly contained in M. We thus obtain an infinite chain:

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots$$

contradicting the fact that M is Noetherian. **Cat A, 8**

(d) There are many possible examples. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$. Then any nonzero submodule of M is of the form $n\mathbb{Z}$ for some nonzero integer n, and this is not simple since it properly contains $2n\mathbb{Z}$. **Cat B, 4**

Mock Exam Question for Commutative Algebra

For every ring R let $R[[X]]$ denote the formal power series ring in the variable X over R, and let $\mathcal{N}(R)$ denote the nilradical of R.

a) **(10 points)** Assume that R is Noetherian. Show that

$$\mathcal{N}(R[[X]]) = \{\sum_{n=0}^{\infty} a_n X^n \in R[[X]] : a_n \in \mathcal{N}(R) \text{ for every } n\}.$$

b) **(5 points)** Assume that $R = k$ is a field. Show that every non-zero element of $k[[X]]$ is of the form $X^n u(X)$ for some $n \in \mathbb{Z}_{\geq 0}$ and an invertible element $u(X) \in k[[X]]^*$. Deduce that the only prime ideals of $k[[X]]$ are (0) and (X).

c) **(10 points)** Assume that R is an Artinian ring. Show that the Krull dimension of $R[[X]]$ is one. (Hint: use the previous part.)

Solutions:

a) **(5 points)** Let $f(X) = \sum_{n=0}^{\infty} a_n X^n \in R[[X]]$. Consider the ideal of R generated by a_i, by the Noetherian hypothesis, this can be generated by only finitely many of the a_i, say, $a_n \in (a_1, \ldots, a_s)$ for all n. Then, we can write $a_n = \sum_{i=1}^{s} c_{ni} a_i$ for all n, hence

$$f(X) = \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} (\sum_{i=0}^{s} c_{ni} a_i) X^n = \sum_{i=0}^{s} a_i f_i(X)$$

where $f_i(X) = \sum_{n=0}^{\infty} c_{ni} X^n$. Thus, if a_i are nilpotent in R, then $a_i f_i(X)$ are nilpotent in $R[[X]]$, and so $f(X)$ which is a finite sum of nilpotent elements is nilpotent (by binomial expansion).

(5 points) Conversely, suppose $f(X)$ is nilpotent, so $f(X)^N = 0$ for some N, then $0 = f(X)^N = a_0^N + X(\ldots)$, hence $a_0^N = 0$. Suppose by induction that $a_0, \ldots, a_k$ are nilpotent, then $p_k(X) = \sum_{n=0}^{k} a_n X^n$ is nilpotent (by binomial expansion) and so $f(X) - p_k(X) = a_{k+1} X^{k+1} + \ldots$ is nilpotent, which implies that a_{k+1} is nilpotent.

b) **(5 points)** We claim that $f(X) = a_0 + a_1 X + a_2 X^2 + \ldots$ is invertible if and only if a_0 is nonzero. If it exists, we can write $f^{-1}(X) = b_0 + b_1 X + b_2 X^2 + \ldots$, then $a_0 b_0 = 1$. We have

$$\begin{aligned} 1 &= (a_0 + a_1 X + a_2 X^2 + \ldots)(b_0 + b_1 X + b_2 X^2 + \ldots) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1) X + \ldots \end{aligned}$$

This can be solved for $b_0, b_1, \ldots$ successively from $a_0 b_0 = 1$, $a_1 b_0 + a_0 b_1 = 0$,...etc. which establishes our claim.

Now, given an arbitrary $f(X) = a_0 + a_1 X + a_2 X^2 + \ldots$ let n be the smallest non-zero a_i, then we can write $f(X) = X^n(a_n + a_{n+1}X + \ldots)$ which is $X^n u(X)$ for some invertible $u(X)$ from what we showed above.

Given a prime ideal $\mathfrak{p} \subset k[[X]]$, if it is not (0) then it has a non-zero element $X^n u(X)$ for some unit $u(X)$ then multiplying with the inverse of $u(X)$ we see that X^n is in the ideal, and now since $\mathfrak{p}$ is prime, this implies that X is in $\mathfrak{p}$. Now the ideal $(X) \subset \mathfrak{p}$ and (X) is maximal since its complement consists of elements of the form $X^n u(X)$ for $n = 0$, hence are invertible. Thus, $(X) = \mathfrak{p}$ as required.

c) **(5 points)** Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be the maximal ideals of R. Since these are all prime ideals of R as R is Artinian, we know that $\mathcal{N}(R) = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$. On the other hand, the ideal $\mathfrak{m}_i[[X]]$ are prime in $R[[X]]$. To see this, note that the quotient $R[[X]]/\mathfrak{m}_i[[X]] \simeq (R/\mathfrak{m}_i)[[X]]$ is an integral domain. Moreover, by part a), we have that $\mathcal{N}(R[[X]]) = \mathfrak{m}_1[[X]] \cap \ldots \cap \mathfrak{m}_r[[X]]$.

(5 points) Since nilradical is intersection of all primes, and the prime ideals $\mathfrak{m}_i[[X]]$ do not contain each other, it follows that $\mathfrak{m}_i[[X]]$ are minimal prime ideals in $R[[X]]$. Finally, note that the prime ideals in $R[[X]]$ that contain $\mathfrak{m}_i[[X]]$ are in one-to-one correspondence with the prime ideals of $(R/\mathfrak{m}_i)[[X]]$ which we know from part b) are either (0) or (X). It follows that the chains of prime ideals $\mathfrak{m}_i[[X]] \subset \mathfrak{m}_i[[X]] + (X)$ are maximal length chains and they are all maximal length chains of prime ideals, so Krull dimension of $R[[X]]$ is 1.

Justin Trias

Solutions: Mock exam question on elliptic curves

1. **Let E be the elliptic curve $y^2 = (x-20)(x-26)(x+46)$.**

This is the elliptic curve with Cremona label 6336ck2. Denote by $T = E(\mathbb{Q})^{\mathrm{tor}}$.

(a) **Show that the torsion subgroup of $E(\mathbb{Q})$ has order 4.** (10 marks)

The points of order 2 are given by $(20,0)$,$(26,0)$,$(-46,0)$. We wish to show that these points, together with $\mathcal{O}$, are all the torsion points. The discriminant of E is equal to

$$\Delta = -(20-26)^2(20+46)^2(26+46)^2 = -2^{10}\cdot 3^8 \cdot 11^2.$$

Since $5 \nmid 2\Delta$, the torsion subgroup of $E(\mathbb{Q})$ maps injectively to $\overline{E}(\mathbb{F}_5)$. The equation of E reduces to $y^2 = x(x-1)(x+1)$. Over $\mathbb{F}_5$, we find the solutions $(0,0)$, $(1,0)$,$(2,1)$,$(4,0)$,$(2,4)$,$(3,2)$,$(3,3)$. Thus, the group $\overline{E}(\mathbb{F}_5)$ has order 8.

Similarly, we can consider the points of E over $\mathbb{F}_7$; reducing the equation modulo 7, we obtain $y^2 = (x+1)(x+2)(x+4)$, for which we get the solutions $(0,1)$,$(0,6)$,$(1,3)$,$(1,4)$,$(2,3)$,$(2,4)$,$(3,0)$,$(4,3)$,$(4,4)$,$(5,0)$,$(6,0)$. So the group $\overline{E}(\mathbb{F}_7)$ has order 12. Since the torsion subgroup T of $E(\mathbb{Q})$ has order at least 4 and injects in a group of order 8 and a group of order 12 it must have order 4.

(b) **Show that the points $(2,144)$ and $(18,32)$ generate a subgroup of $E(\mathbb{Q})$ which is isomorphic to $\mathbb{Z}^2$.** (10 marks)

As we showed in the lectures, there is an injective group homomorphism

$$\delta\colon E(\mathbb{Q})/2E(\mathbb{Q}) \hookrightarrow (\mathbb{Q}^\times/(\mathbb{Q}^\times)^2)^3,$$

whose image is contained in the subgroup of triples (b_1,b_2,b_3) such that $b_1b_2b_3 = 1$. The map δ is defined as

$$\delta((x,y)) = (x-20, x-26, x+46)$$

for all points (x,y) that are not 2-torsion, and is suitably extended to 2-torsion points. Denote $P = (2,14)$ and $Q = (18,32)$. Let H be the subgroup of $E(\mathbb{Q})$ generated by P and Q. In order to show that H is isomorphic to $\mathbb{Z}^2$, it suffices to prove the following:

1. $\delta(H)$ is an $\mathbb{F}_2$-vector space of dimension 2;
2. The intersection of $\delta(H)\cap\delta(T)$ is trivial.

For (1), we compute

$$\delta(P) = (-2,-6,3), \qquad \delta(Q) = (-2,-2,1),$$

which shows that $\delta(H)$ has dimension 2 as a $\mathbb{F}_2$-vector space and $\delta(H) = \{(1,1,1),(-2,-6,3),(-2,-2,1),(1,3,3)\}$.

For (2) we compute

$$\delta((20,0)) = (-11,-6,66) \qquad \delta((26,0)) = (6,3,2) \qquad \delta((-46,0)) = (-66,-2,33),$$

so (2) is satisfied as well.

(Total: 20 marks)

Group Theory Solution

The Sylow theorems:

Sylow I: Let $|G| = p^a m$, where p is prime and p does not divide m. Then G has a subgroup of order p^a.

Sylow II: If $n_p(G)$ denotes the number of Sylow p-subgroups of G, then $n_p(G) \equiv 1 \bmod p$.

Sylow III: Let Q be a p-subgroup of G. Then there exists $P \in Syl_p(G)$ such that $Q \leq P$.

Sylow IV: $Syl_p(G)$ is a single conjugacy class of subgroups of G; that is, for any $P, Q \in Syl_p(G)$, there exists $g \in G$ such that $Q = {}^gP$.

Burnside's transfer theorem: Let p be prime, $P \in Syl_p(G)$, and suppose that $P \leq Z(N_G(P))$. Then G has a normal p-complement (ie. a normal subgroup N such that $G = PN$ and $P \cap N = 1$).

5 marks, 1 for each theorem

(a) Let $|G| = 108 = 2^2 \cdot 3^3$. Suppose G is simple. Then $n_3(G) \neq 1$, so by Sylow II and the standard result that $n_p(G)$ divides $|G|$, we have $n_3(G) = 4$. But then by Sylow IV, the action of G by conjugation on $Syl_3(G)$ gives a homomorphism $\pi : G \mapsto S_4$ with nontrivial image. As G is simple, $\text{Ker}\pi = 1$, so G is isomorphic to a subgroup of S_4. This is a contradiction, as $|G| = 108$ does not divide $|S_4| = 24$. **5 marks**

(b) Let $|G| = pqr$ and take $p > q > r$. Suppose G is simple. Then $n_p(G) > 1$. As $n_p(G) \equiv 1 \bmod p$ and divides qr, it follows that $n_p(G) = qr$. Hence if $P \in Syl_p(G)$, then $|G : N_G(P)| = qr = |G : P|$, and so $P = N_G(P)$. Also $|P| = p$ implies that P is abelian. Hence $P = C_G(P) = N_G(P)$ and so Burnside's theorem shows that G has a normal p-complement. So G is not simple. **5 marks**

(c) Let $|G| = p^2(p+1)$. Suppose G simple. Then $n_p(G) = p + 1$. So if $P \in Syl_p(G)$, then $|G : N_G(P)| = p + 1 = |G : P|$ and $P = N_G(P)$. Also $|P| = p^2$ so P is abelian (standard result). So as in (b), Burnside's theorem implies G ius not simple. **5 marks**

2. (a) (i) **2 marks, seen** Define $\mathfrak{g}^{(1)} = [\mathfrak{g}\mathfrak{g}]$ and $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}\mathfrak{g}^{(n)}]$ for $n \geq 1$. The Lie algebra $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

(ii) **2 marks, seen** Define $\mathfrak{g}^1 = [\mathfrak{g}\mathfrak{g}]$ and $\mathfrak{g}^{n+1} = [\mathfrak{g}^n\mathfrak{g}]$ for $n \geq 1$. The Lie algebra $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^n = 0$ for some n.

(b) **2 marks, seen** The centre of a nilpotent Lie algebra is non-zero. If $\mathfrak{g}$ is nilpotent and $\mathfrak{g}^m \neq 0$, but $\mathfrak{g}^{m+1} = 0$, then $\mathfrak{g}^m$ is contained in the centre of $\mathfrak{g}$.

(b) **8 marks, seen** The Lie algebra $\mathfrak{g}'$ is nilpotent. By a corollary of Lie's theorem, if $\mathfrak{g}$ is solvable, then the vector space $\mathfrak{g}$ has a basis in which $\mathrm{ad}(x)$ is upper triangular for every $x \in \mathfrak{g}$. Thus for any $y \in \mathfrak{g}'$ the matrix of $\mathrm{ad}(y)$ is strictly upper triangular, hence nilpotent. Since $\mathrm{ad}(y)(\mathfrak{g}') \subset \mathfrak{g}'$, we see that every linear transformation in the image of the adjoint representation of $\mathfrak{g}'$ is nilpotent. By a corollary of Engel's theorem, $\mathfrak{g}'$ is nilpotent.

(d) **6 marks, unseen** We have $\mathfrak{g}^{(1)} = \mathbb{C}f_1 \oplus \mathbb{C}f_2$ and $\mathfrak{g}^{(2)} = 0$, so $\mathfrak{g}$ is solvable.

We have $\mathfrak{g}^n = \mathbb{C}f_1 \oplus \mathbb{C}f_2$ for every $n \geq 1$, so $\mathfrak{g}$ is not nilpotent.

If $ae+bf_1+cf_2$ is in the centre of $\mathfrak{g}$, then $[e, ae+bf_1+cf_2] = bf_1+\sqrt{-1}cf_2$ hence $b = c = 0$. Thus the centre is contained in the span of e, but $[e, f_1] = f_1$, so the centre of $\mathfrak{g}$ is zero.

MANIFOLDS - MATH70058 - SOLUTIONS

(1) (**4 points**) We may write

$$\omega = d\left(\frac{1}{2}x^2dy - \frac{1}{2}z^2dx\right).$$

Thus ω is exact and since S^2 is a compact manifold without border, Stokes' Theorem implies that $\int_{S^2}\omega = 0$

(2) (**3 points**) Consider the the inclusion $i\colon S^1 \to \mathbb{R}^2$ as the circle of radius one. Let

$$\omega = i^*(x_2dx_1 - x_1dx_2).$$

We want to show that ω is a volume form. Consider the vector field $V = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$. Then for all $x = (x_1, x_2) \in S^1$, we have $\omega(V)(x) = x_2^2 + x_1^2 = 1$. Thus ω is no-where zero and, therefore, it is a volume form. It follows that S^1 is orientable.

(3) (**6 points**)Consider the inclusion $i\colon S^2 \to \mathbb{R}^3$ as the sphere of radius one. Let

$$\omega = i^*(x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2).$$

We want to show that ω is a volume form on S^2. Consider the vector fields:

$$V_1 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} \qquad V_2 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3}.$$

Then for any $x = (x_1, x_2, x_3) \in S^2$, we have

$$\omega(V_1, V_2)(x) = x_3^3 + x_2^2x_3 + x_3x_1^2 = x_3.$$

Thus, ω is non-zero if $x_3 \neq 0$. By symmetry, it follows easily that ω is no-where zero and therefore it is a volume form.

(4) (**5 points**) In general, there is no such η. For example, let $X = \mathbb{R}$ and let f be a smooth function such that $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x > 1$. Let $\omega = df$. Then ω is an exact 1-form which has support contained in the interval $[0, 1]$. In particular, ω has compact support. Assume that there exists η with compact support and such that $\omega = d\eta$. Let $g = f - \eta$. Then $dg = 0$ and, in particular, g is a constant function, such that $g(x) = 0$ for x sufficiently negative and $g(x) = 1$ for x sufficiently large, a contradiction.

(5) (**2 points**) Let C be a circle inside X with centre at the origin and with radius sufficiently large so that ω is zero along C. Consider the manifold $Y \subset X$ whose

boundary is $C \cup S^1$. Then, by Stokes' Theorem, we have
$$\int_{S^1} \omega = \int_{S^1} \omega - \int_C \omega = -\int_{\partial Y} \omega = -\int_Y d\omega = 0,$$
where the sign is due to the choice of the orientation on ∂Y. Thus, the result follows.

MARKOV PROCESSES 2022-23: SOLUTIONS TO MOCK EXAMINATION

(*Bernoulli-Laplace urn*).
(i) $(1+x)^{2d} \equiv (1+x)^d.(1+x)^d$. Equate coefficients of x^d left and right:

$$\binom{2d}{d} = \sum_{i=0}^{d} \binom{d}{i}.\binom{d}{d-i} = \sum_i \binom{d}{i}^2,$$

so the π_i in $HG(d)$ sum to 1, so $HG(d)$ is a probability distribution. **[5]**
(ii) For $i \mapsto i+1$, one more black ball must go into I. So a white ($d-i$ of these) goes from I to II, and a black (again, $d-i$ of these) from II to I. Similarly, for $i+1 \mapsto i$, a black ($i+1$ of these) goes from I to II, and a white ($i+1$ of these) from II to I. For $i \mapsto i$, a black from I is interchanged with a black from II, or a white with a white (pr. $i(d-i)/d^2$ each). So

$$p_{i,i+1} = \Big(\frac{d-i}{d}\Big)^2, \quad p_{i,i-1} = \Big(\frac{i}{d}\Big)^2, \quad p_{i,i} = 2i(d-i)/d^2,$$
$$p_{ij} = 0 \ (j \neq i, i \pm 1).$$ **[6]**

(iii) With $P = (p_{ij})$ as above and $\pi = HG(d)$,

$$\pi_i p_{i,i+1} = \frac{1}{\binom{2d}{d}} \binom{d}{i}^2 . \Big(\frac{d-i}{d}\Big)^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2,$$

and similarly

$$\pi_{i+1} p_{i+1,i} = \frac{1}{\binom{2d}{d}} \binom{d}{i+1}^2 . \Big(\frac{i+1}{d}\Big)^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2 = \pi_i p_{i,i+1},$$

proving detailed balance, and so reversibility, with $\pi = HG(d)$ as limit distribution. **[6]**
(iv) $\pi_i = 1/\mu_i$ by the Erdös-Feller-Pollard theorem, so $\mu_0 = 1/\pi_0 = \binom{2d}{d}$. **[2]**
(v) By Stirling's formula,

$$\mu_0 \sim \frac{\sqrt{2\pi} e^{-2d} (2d)^{2d+\frac{1}{2}}}{(\sqrt{2\pi} e^{-d} d^{d+\frac{1}{2}})^2} = \frac{4^d}{\sqrt{\pi d}}.$$

Now as d is already very large (of the order of Avogadro's number 6×10^{23}), 4^d is astronomically vast – effectively infinite. This reconciles *microscopic reversibility* with *macroscopic irreversibility* in Statistical Mechanics. **[6]**
[Seen – Problems]

Imperial College London
Department of Mathematics
Lecturer: P.-F. Rodriguez

MATH 60029
Fall 2022

Sample question

1. Let $X = L^p(0,2)$ for some $1 \le p < \infty$, endowed with the usual $\|\cdot\|_{L^p(0,2)}$-norm, and assume the functions in X to be real-valued. For $f \in X$, define

$$(Tf)(t) = t^2 f(t),\ t \in (0,2).$$

a) Show that $T \in \mathcal{L}(X)$.

b) Determine $\|T\|_{\mathcal{L}(X)}$.

Solution.

a) We first argue that T is linear. Plainly, for $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$, one has $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$. To see that T is bounded, observe that for $f \in X$,

$$\|Tf\|^p_{L^p(0,2)} = \int_0^2 t^{2p}|f(t)|^p dt \le 2^{2p}\|f\|^p_{L^p(0,2)}.$$

Taking the p-th root on both sides, it follows that T is bounded.

b) The previous part gives that

$$\|T\|_{\mathcal{L}(X)} = \sup_{\|f\|_{L^p(0,2)} \le 1} \|Tf\|_{L^p(0,2)} \le 4.$$

for any value of p. We now argue that this bound is sharp. To this effect, consider for integer $n \ge 1$ the function

$$f_n(t) = \begin{cases} 0, & \text{if } 0 < t < 2 - \frac{1}{n}, \\ n^{\frac{1}{p}}, & \text{if } 2 - \frac{1}{n} < t < 2. \end{cases}$$

Clearly, $f_n \in X$ and $\|f_n\|_{L^p(0,2)} = 1$ for all n. Hence,

$$\|T\|_{\mathcal{L}(X)} \ge \|Tf_n\|_{L^p(0,2)} \ge \Big(2 - \frac{1}{n}\Big)^2 \|f_n\|_{L^p(0,2)} = \Big(2 - \frac{1}{n}\Big)^2.$$

Letting $n \to \infty$ yields that $\|T\|_{\mathcal{L}(X)} \ge 4$. It follows that

$$\|T\|_{\mathcal{L}(X)} = 4.$$

MATH60132/MATH70132 Mathematical Logic.

Solution:

(a) **((i) Standard definition; (ii) is in the notes, but unseen in this form: I would not expect so much detail.)**

(i) This means that there is an injective function from A to B.

1 mark (A)

(ii) The first statement does not require AC: it is the Cantor - Schröder - Bernstein Theorem and the proof given in the notes does not use AC.

The second statement does require AC (for general A, B). Indeed, it implies that every set A can be well-ordered. To see this, take an ordinal β where there is no injective function $\beta \to A$ (Hartogs' Lemma, which holds in ZF). By the statement there is therefore an injective function $A \to \beta$ and A can be well ordered using this. **4 marks (2A, 2B)**

(b) **Similar to examples in the notes and on previous exams in 2018 and 2019**

Note that by Cantor's Theorem, $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. We claim that all of the other sets have cardinality equal to $\kappa = |\mathbb{R}|$.

Using characteristic functions, $|\mathcal{P}(\mathbb{N})| = 2^\omega$ (where $2 = \{0, 1\}$). We have an injective function $2^\omega \to \mathbb{R}$ by viewing an element of 2^ω as a sequence of 0's and 1's after the decimal point of a real number.

We can consider a real number as a sequence of rational numbers (by truncating its decimal expansion) and so we have an injective function $\mathbb{R} \to \mathbb{Q}^\mathbb{N}$.

We have $|\mathbb{Q}| = \omega$ and so $\mathbb{Q}^\mathbb{N}$ has cardinality ω^ω. By a general result, this is equal to 2^ω.

Thus we have

$$2^\omega = \mathbb{Q}^\mathbb{N} \geq |\mathbb{R}| \geq 2^\omega = |\mathcal{P}(\mathbb{N})|$$

and so all of these cardinalities are equal.

We have $\omega < \kappa$ (by the above and Cantor's Theorem again). By definition, $\kappa = |X| + |\mathbb{Q}|$ and (by a consequence of the Fundamental Theorem of Cardinal Arithmetic) this is $max(|X|, \omega)$. So $|X| = \kappa$.

8 marks (2A, 3B, 3C)

(c) **((ii) is unseen, therefore harder.)**

(i) This means that every non-empty subset of A has a least element. **1 marks (A)**

(ii) Suppose not. So $B = \{a \in A : f(a) \neq a\}$ is non-empty and therefore has a least element, b. For every $a < b$ we therefore have $a \notin B$ and so $f(a) = a$. We also have $f(b) < b$. Thus $f(f(b)) = f(b)$. So $f(b)$ and b have the same image under f, contradicting injectivity of f and $f(b) < b$. **4 marks (D)**

(iii) Without the assumption that the ordering is a well ordering, the result is false in general. For example take $(A; \leq)$ to be $(\mathbb{Z}; \leq)$ (with the usual ordering) and $f(a) = a - 1$. **2 marks (A)**

Solutions

(a) (2 marks for right formula for B, 1 mark for choosing right domain for B) The bilinear form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ associated with L is

$$B[u,v] = \int_\Omega (\nabla u) \cdot (\nabla v) + (b \cdot \nabla u)v + cuv \ .$$

(b) (1 mark for imposing boundary condition with $H_0^1(\Omega)$, 2 marks for rest of formulation)

We say that u is such a weak solution if $u \in H_0^1(\Omega)$ and, for all $v \in H_0^1(\Omega)$,

$$B[u,v] = \langle f, v \rangle_{L^2(\Omega)} \ .$$

(c) (1 mark for invoking Poincare, 3 marks for boundedness, 3 marks for coercivity)

By the Poincare inequality, there exists some $\tilde{C} > 0$ such that, for any $g \in H_0^1(\Omega)$,

$$\int_\Omega |g|^2 \ \le \tilde{C} \int_\Omega |\nabla g|^2$$

In particular, this tells us that $\|g\|_{H_0^1} := \|\nabla g\|_{L^2}$ is a equivalent to the H^1 norm on H_0^1.

In order to apply the Lax-Milgram theorem we need to verify boundness and coercivity of the bilinear form B on H_0^1.

For boundedness we note that

$$\big|B[u,v]\big| \le \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + |b| \times \|\nabla u\|_{L^2}\|v\|_{L^2} \le (1 + \tilde{C}|b|)\|u\|_{H_0^1}\|v\|_{H_0^1} \ .$$

For coercivity we note that

$$B[u,u] \ge \|\nabla u\|_{L^2}^2 - |b| \times \|\nabla u\|_{L^2}\|u\|_{L^2} \ge \big(1 - \tilde{C}|b|\big)\|u\|_{H_0^1}^2 \ .$$

Therefore, if we set $C_1 = 1/\tilde{C}$ then we can apply the Lax-Milgram theorem when $|b| \le C_1$.

(d) (2 marks for using the right type of cut-off, 3 points for estimating the three terms of B correctly, 2 points for putting things together at the end)
Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be smooth, radial (centered around x), and satisfy $0 \le \xi \le 1$, $\xi = 1$ on $B(x, R_1)$ and $\xi = 0$ on $\mathbb{R}^n \setminus B(x, R_2)$.

Note that we can choose some constant $D > 0$, independent of x, R_1, R_2, and u such that

$$\sup_{y \in \mathbb{R}^n} \|\nabla \xi(y)| \le \frac{D}{R_2 - R_1} \ .$$

This is because ξ changes from 1 to 0 over an interval of length $R_2 - R_1$.

We look at the three terms of $B[u,\xi^2 u]$. For the first term we have

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla(\xi^2 u) &= \int_{B(x,R_2)} \xi^2|\nabla u|^2 + 2\xi u(\nabla\xi\cdot\nabla u)\\ &\geq \int_{B(x,R_2)} \xi^2|\nabla u|^2 - \frac{1}{2}\int_{B(x,R_2)} \xi^2|\nabla u|^2 - 2\int_{B(x,R_2)} |\nabla\xi|^2\,|u|^2\\ &\geq \frac{1}{2}\int_{B(x,R_2)} \xi^2|\nabla u|^2 - \frac{2D^2}{(R_2-R_1)^2}\int_{B(x,R_2)} |u|^2\end{aligned}$$

For the other two terms we have

$$\Big|\int_\Omega (b\cdot\nabla u)\xi^2 u\Big| \leq \frac{1}{4}\int_{B(x,R_2)} \xi^2|\nabla u|^2 + D'\int_{B(x,R_2)} |u|^2\ ,$$
$$\Big|\int_\Omega c\xi^2 u^2\Big| \leq D''\int_{B(x,R_2)} |u|^2\ .$$

Now recalling that $B[u,\xi^2 u]=0$ and putting together these three estimates gives

$$\Big(\frac{1}{2}-\frac{1}{4}\Big)\int_{B(x,R_2)} \xi^2|\nabla u|^2 \leq \Big(D'+D''+\frac{2D^2}{(R_2-R_1)^2}\Big)\int_{B(x,R_2)} |u|^2\ .$$

We are done upon observing that $R_2-R_1\in(0,1)$ and

$$\int_{B(x,R_1)} |\nabla u|^2 \leq \int_{B(x,R_2)} \xi^2|\nabla u|^2\ .$$

MOCK EXAM FOR NUMBER THEORY - PROBLEM AND SOLUTION

AMBRUS PÁL

Question 1.

(a) Find the continued fraction expansion of $\sqrt{7}$. (6 marks)
(b) Find the two smallest solutions to $x^2 - 7y^2 = 1$, where x and y are both strictly positive and solutions are ordered by the value of y. (7 marks)
(c) Find the two smallest solutions to $x^2 - 7y^2 = 2$, where x and y are both strictly positive and solutions are ordered by the value of y. (7 marks)

Solution. (a) The continued fraction of $\sqrt{7}$ is given by $[2; 1, 1, 4, 1, 1, 4, \ldots]$. (3 marks)

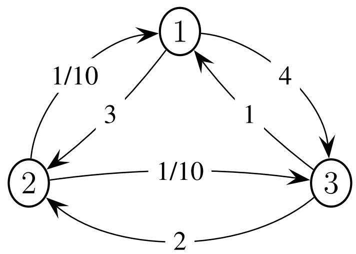
(b) The first convergent of the continued fraction that gives a solution is $\frac{8}{3}$, so the first solution is $(8, 3)$. We have $(8 + 3\sqrt{7})^2 = 127 + 48\sqrt{7}$, so the second solution is $(127, 48)$. (7 marks)

(c) The smallest solution is clearly $(3, 1)$, by inspection. We have $(3 + \sqrt{7})(8 + 3\sqrt{7}) = 45 + 17\sqrt{7}$, so the second solution is $(45, 17)$. (10 marks)

Statistics

Applied Probability – Solutions meth seen $\Downarrow$

1. (a) (i) The transition diagram is given by



(ii) According to lectures, the transition probabilities of the corresponding embedded jump chain are given by $p_{ij} = -g_{ij}/g_{ii}$ for all $i, j \in E$ provided that $g_{ii} \neq 0$. Hence, the transition matrix of the embedded jump chain is given by

2, A

meth seen $\Downarrow$

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{3}{7} & \frac{4}{7} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

(iii) We observe that the jump chain has one communicating class, i.e. it is an irreducible Markov chain. Since the class is finite and closed, we conclude that the class and hence all states are positive recurrent.

3, A

meth seen $\Downarrow$

We know that if a state is recurrent (transient) for the jump chain, then it is recurrent (transient) for the continuous-time Markov chain. So we conclude that all states are recurrent for the continuous-time Markov chain.

2, A

(b) (i) Let $t \geq 0$. We recall that $\mathrm{E}(N_t) = \mathrm{Var}(N_t) = \lambda t$ since $N_t \sim \mathrm{Poi}(\lambda t)$. Using the linearity of the expectation and the fact that deterministic constants do not impact the variance we get $\mathrm{E}(Y_t) = t + \lambda t$ and $\mathrm{Var}(N_t) = \lambda t$.

meth seen $\Downarrow$

3, A

(ii) Let $t \geq 0$. We use the law of total expectation to deduce that

unseen $\Downarrow$

$$\begin{aligned}
\mathrm{E}(X_t) &= \mathrm{E}(B_{t+N_t}) = \mathrm{E}(\mathrm{E}(B_{t+N_t}|N_t)) \\
&= \sum_{n=0}^{\infty} \mathrm{E}(B_{t+N_t}|N_t = n)\mathrm{P}(N_t = n) = \sum_{n=0}^{\infty} \mathrm{E}(B_{t+n}|N_t = n)\mathrm{P}(N_t = n) \\
&\overset{\text{indep. of } B \text{ and } N}{=} \sum_{n=0}^{\infty} \mathrm{E}(B_{t+n})\mathrm{P}(N_t = n) = \sum_{n=0}^{\infty} 0\mathrm{P}(N_t = n) = 0,
\end{aligned}$$

since the Brownian motion has zero mean.

3, C

[**1 mark for computation, 2 marks for justifications**]

Similarly, we have

$$\begin{aligned}
\mathrm{E}(X_t^2) &= \mathrm{E}(B_{t+N_t}^2) = \mathrm{E}(\mathrm{E}(B_{t+N_t}^2|N_t)) \\
&= \sum_{n=0}^{\infty} \mathrm{E}(B_{t+N_t}^2|N_t = n)\mathrm{P}(N_t = n) = \sum_{n=0}^{\infty} \mathrm{E}(B_{t+n}^2|N_t = n)\mathrm{P}(N_t = n) \\
&\overset{\text{indep. of } B \text{ and } N}{=} \sum_{n=0}^{\infty} \mathrm{E}(B_{t+n}^2)\mathrm{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} (t+n)\mathrm{P}(N_t = n) = t + \mathrm{E}(N_t) = t + \lambda t,
\end{aligned}$$

since $E(B_x^2) = x$ for all $x \geq 0$. Hence, $\mathrm{Var}(X_t) = t + \lambda t$.

2, C

[**1 mark for computation, 1 mark for justifications**]

(iii) Without loss of generality, we assume that $0 \leq s < t$. Recall that $\mathrm{Cov}(X_s, X_t) = \mathrm{E}(X_sX_t) - \mathrm{E}(X_s)\mathrm{E}(X_t) = \mathrm{E}(X_sX_t)$ since $\mathrm{E}(X_s) = \mathrm{E}(X_t) = 0$. Then

unseen ⇓

$$\begin{aligned}
\mathrm{E}(X_sX_t) &= \mathrm{E}(\mathrm{E}(X_sX_t|N_s, N_t)) \\
&= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \mathrm{E}(B_{s+N_s}B_{t+N_t}|N_s = n, N_t = m)\mathrm{P}(N_s = n, N_t = m) \\
&= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \mathrm{E}(B_{s+n}B_{t+m}|N_s = n, N_t = m)\mathrm{P}(N_s = n, N_t = m) \\
&\overset{\text{indep. of } B \text{ and } N}{=} \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \mathrm{E}(B_{s+n}B_{t+m})\mathrm{P}(N_s = n, N_t = m) \\
&= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} (s+n)\mathrm{P}(N_s = n, N_t = m),
\end{aligned}$$

where we used the result from lectures that $\mathrm{E}(B_xB_y) = \min\{x, y\}$. Here we have that $s < t$, which also implies that $N_s < N_t$, hence, $s + n < s + m$. Applying the law of total probability, leads to

$$\begin{aligned}
\mathrm{E}(X_sX_t) &= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} (s+n)\mathrm{P}(N_s = n, N_t = m) \\
&= \sum_{n=0}^{\infty}(s+n)\mathrm{P}(N_s = n) = s + \lambda s.
\end{aligned}$$

Hence, for $0 \leq s < t$, $\mathrm{Cov}(X_s, X_t) = s + \lambda s$. We can conclude that, for $s, t \geq 0$, $\mathrm{Cov}(X_s, X_t) = (1 + \lambda)\min\{s, t\}$.

5, D

[3 marks for the calculations, 2 marks for the justifications]
[Students may apply the law of total covariation for an alternative (quicker) solution, but need to give proper justifications as well.]

Time Series Analysis – Solutions sim. seen ⇓

2. (a) Here $\{X_t\}$ is a stationary process with mean value $\mu = E\{X_t\}$, and variance s_0. By definition,

$$\begin{aligned}
\hat{s}_0 &= \frac{1}{N}\sum_{t=1}^{N}(X_t-\overline{X})^2 = \frac{1}{N}\sum_{t=1}^{N}([X_t-\mu]-[\overline{X}-\mu])^2\\
&= \frac{1}{N}\sum_{t=1}^{N}\left([X_t-\mu]^2 - 2[X_t-\mu][\overline{X}-\mu]+[\overline{X}-\mu]^2\right)\\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t-\mu]^2 - 2[\overline{X}-\mu][\overline{X}-\mu]+[\overline{X}-\mu]^2\\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t-\mu]^2 - [\overline{X}-\mu]^2.
\end{aligned}$$

Taking the expectation of both sides and noting that $E\{\overline{X}\}=\mu$ yields

$$\begin{aligned}
E\{\hat{s}_0\} &= \frac{1}{N}\sum_{t=1}^{N}E\{[X_t-\mu]^2\} - E\{[\overline{X}-\mu]^2\}\\
&= \mathrm{var}\,\{X_t\} - \mathrm{var}\,\{\overline{X}\} = s_0 - \mathrm{var}\,\{\overline{X}\},
\end{aligned}$$

the desired result. 5

(b) Let sim. seen ⇓

$$J(f) \equiv (1/\sqrt{N})\sum_{t=1}^{N}(X_t-\mu)\mathrm{e}^{-\mathrm{i}2\pi ft}.$$

By the spectral representation theorem $X_t-\mu = \int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi f't}\,\mathrm{d}Z(f')$, where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{dZ(f)\}=0$. Thus

$$\begin{aligned}
J(f) &= (1/\sqrt{N})\sum_{t=1}^{N}\left(\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi f't}\,\mathrm{d}Z(f')\right)\mathrm{e}^{-\mathrm{i}2\pi ft}\\
&= (1/\sqrt{N})\int_{-1/2}^{1/2}\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi(f-f')t}\,\mathrm{d}Z(f')\\
&= \int_{-1/2}^{1/2}F(f-f')\,\mathrm{d}Z(f'),
\end{aligned}$$

where $F(f) = (1/\sqrt{N})\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi ft}$.
Now it is given that,

$$\widehat{S}(f) \equiv |J(f)|^2 = (1/N)\left|\sum_{t=1}^{N}(X_t-\mu)\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2.$$

Because $\{Z(\cdot)\}$ has orthogonal increments, we therefore have

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2}\mathcal{F}(f-f')S(f')\,\mathrm{d}f',$$

where

$$\mathcal{F}(f) \equiv |F(f)|^2 = (1/N)\left|\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2.$$

5

(c) Now

unseen $\Downarrow$

$$\begin{aligned}\operatorname{var}\{\overline{X}\} &= E\{(\overline{X}-\mu)^2\} = (1/N^2)E\left\{\left(\sum_{t=1}^{N}(X_t-\mu)\right)^2\right\}\\ &= (1/N)E\{\widehat{S}(0)\},\end{aligned}$$

and from (ii), $E\{\widehat{S}(0)\} = \int_{-1/2}^{1/2} \mathcal{F}(f)S(f)\,\mathrm{d}f$ (by symmetry of spectral density function), and of course $s_0 = \int_{-1/2}^{1/2} S(f)\,\mathrm{d}f$, so that the result follows from part (i), i.e.,

$$E\{\hat{s}_0\} = s_0 - \operatorname{var}\{\overline{X}\} = \int_{-1/2}^{1/2}\left(1-\frac{1}{N}\mathcal{F}(f)\right)S(f)\,\mathrm{d}f.$$

$\boxed{5}$

(d) The function $(1/N)\mathcal{F}(f)$ is non-negative and symmetric about $f=0$ where it takes the max value 1. It is multi-lobed. The main lobe decreases from 1 to zero at $\pm 1/N$. Other much smaller side-lobes are to be found between the zeros at $k/N, k \in \mathbb{Z}\setminus\{0\}$ and the sidelobes decrease with increasing $|f|$.

So $\left(1-\frac{1}{N}\mathcal{F}(f)\right)$ is close to 1 for $(1/N) \le |f| \le 1/2$, but decreases to zero as $|f|$ decreases from $1/N$ to zero. Since $s_0 = \int_{-1/2}^{1/2} S(f)\mathrm{d}f$, we can expect a large discrepancy between s_0 and $E\{\hat{s}_0\}$ if the SDF for frequencies $0 \le |f| \le (1/N)$ largely determines the value of s_0, i.e., if most power in $\{X_t\}$ is concentrated in frequencies $0 \le |f| \le (1/N)$.

$\boxed{5}$

Consumer Credit Risk Modelling – Solutions

seen ⇓

3. (a)

$$s(\mathbf{x}) = \log\left(\frac{P(Y=0|\mathbf{X}=\mathbf{x})}{P(Y=1|\mathbf{X}=\mathbf{x})}\right)$$

1

(b) Therefore

$$\begin{aligned}\exp(s(\mathbf{x})) &= \frac{P(Y=0|\mathbf{X}=\mathbf{x})}{1-P(Y=0|\mathbf{X}=\mathbf{x})}\\ \Rightarrow (1-P(Y=0|\mathbf{X}=\mathbf{x}))\exp(s(\mathbf{x})) &= P(Y=0|\mathbf{X}=\mathbf{x})\\ \Rightarrow \exp(s(\mathbf{x})) &= P(Y=0|\mathbf{X}=\mathbf{x})\,(1+\exp(s(\mathbf{x})))\\ \Rightarrow P(Y=0|\mathbf{X}=\mathbf{x}) &= \frac{\exp(s(\mathbf{x}))}{1+\exp(s(\mathbf{x}))} = \frac{1}{1+\exp(-s(\mathbf{x}))}\end{aligned}$$

2

(c) For each observation i, probability of outcome $Y=0$ is given by $P(Y=0|\mathbf{X}=\mathbf{x}_i)$ if $y_i=0$ and $P(Y=1|\mathbf{X}=\mathbf{x}_i)$ if $y_i=1$. This can be expressed as

$$P(Y=y_i|\mathbf{X}=\mathbf{x}_i) = P(Y=0|\mathbf{X}=\mathbf{x}_i)^{1-y_i}(1-P(Y=0|\mathbf{X}=\mathbf{x}_i))^{y_i}$$

Assuming independence between observations, the likelihood is

$$\prod_{i=1}^{n} P(Y=y_i|\mathbf{X}=\mathbf{x}_i)$$

and so log-likelihood is

$$\begin{aligned}\ell &= \sum_{i=1}^{n} \log P(Y=y_i|\mathbf{X}=\mathbf{x}_i)\\ &= \sum_{i=1}^{n} (1-y_i)\log P(Y=0|\mathbf{X}=\mathbf{x}_i) + y_i(1-P(Y=0|\mathbf{X}=\mathbf{x}_i))\\ &= \sum_{i=1}^{n} (1-y_i)\log\left(\frac{1}{1+e^{-(\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) + y_i\log\left(\frac{1}{1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}}\right)\end{aligned}$$

by substituting the logistic function from part (b) and the linear form of s. *Note: students lose one mark if they do not mention that data is assumed to be independent.*

3

(d) Therefore

$$\begin{aligned}\ell &= \sum_{i=1}^{n} (1-y_i)\log\left(\frac{e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}}{1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}}\right) - y_i\log\left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right)\\ &= \sum_{i=1}^{n} (1-y_i)(\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i) - (1-y_i)\log\left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right) - y_i\log\left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right)\\ &= \sum_{i=1}^{n} (1-y_i)(\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i) - \log\left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right)\end{aligned}$$

2

(e) Take derivative ℓ with respect to β_0 to find stationary point,

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_0} &= \sum_{i=1}^{n} (1-y_i) - \left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right)^{-1} e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i} \\ &= \sum_{i=1}^{n} \left(1 - y_i - \frac{1}{1+e^{-(\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) = 0\end{aligned}$$

and with respect to each element β_j in $\boldsymbol{\beta}$ to find stationary point,

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_j} &= \sum_{i=1}^{n} (1-y_i)x_{ij} - \left(1+e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i}\right)^{-1} x_{ij} e^{\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i} \\ &= \sum_{i=1}^{n} x_{ij} \left(1 - y_i - \frac{1}{1+e^{-(\beta_0+\boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) = 0\end{aligned}$$

for each coefficient $j \in \{1, \cdots, m\}$.

$\boxed{4}$

(f) $\boxed{\text{seen} \Downarrow}$

- ∗ FICO score is negatively associated with default and is statistically significant.
- ∗ Property type CO is positively associated with default, relative to LH, and is not statistically significant.
- ∗ Property type PD is negatively associated with default, relative to LH, but is statistically significant.
- ∗ Property type O is negatively associated with default, relative to LH, but is not statistically significant.
- ∗ Loan-to-value is positively associated with default and is statistically significant.
- ∗ Having more than one borrowers is negatively associated with default and is statistically significant.

Note: 1/2 mark for direction and 1/2 mark for statistical significance for each point. Lose 1 mark if association of categorical levels is not made relative to the excluded category.

$\boxed{5}$

(f) The Naive Bayes method follows from the log-odds formulation of a scorecard if we assume independence between the covariates in the model, and using Bayes Rule. Therefore, before building such model, we should conduct some association and correlation analyses between the explanatory variables. If we find significant association or correlation between them, then the Naive Bayes model would be misspecified and the inference would not be reliable. $\boxed{\text{unseen} \Downarrow}$

$\boxed{3}$

Stochastic Simulation – Solutions seen ⇓

4. (a) (i) The idea of inversion is to construct a CDF of a given density and then invert this and evaluate the inverse at $U \sim \text{Unif}(0,1)$. Any similar answer should be given the mark. 2

(ii) By definition of CDF

$$F_X(x) = \int_0^x p(x')\mathrm{d}x'$$
$$= \lambda \int_0^x e^{-\lambda x'}\mathrm{d}x',$$
$$= -e^{-\lambda x'}\Big|_0^x,$$
$$= 1 - e^{-\lambda x}.$$

1 mark can be given for if the CDF expression written correctly (integral starts from zero as the density defined on $x \geq 0$). 2

(iii) We set $y = F_X(x)$ and leave x alone for this. Therefore

$$y = 1 - e^{-\lambda x}, \implies \qquad e^{-\lambda x} = 1 - y$$
$$\implies -\lambda x = \log(1-y)$$
$$\implies x = -\lambda^{-1}\log(1-y).$$

Therefore,

$$F_X^{-1}(x) = -\lambda^{-1}\log(1-x).$$

[1 mark can be given for students who partially derive this.] 2

(iv) This is also true since the distribution of U and $1-U$ is the same. This is a trivial transformation of random variables. unseen ⇓ 1

(b) (i) The CDF is described by sim. seen ⇓

$$F_X(x) = \int_{-\infty}^x p(x')\mathrm{d}x'.$$

Therefore, we can write

$$F_X(c) = \int_{-\infty}^c p(x)\mathrm{d}x,$$
$$= \int_{-\infty}^{\infty} \mathbf{1}_{\{x\leq c\}}(x)p(x)\mathrm{d}x.$$

Correct description of the integral gets 1 mark. The test function by inspecting above is given by

$$\varphi(x) = \mathbf{1}_{\{x\leq c\}}(x).$$

2

(ii) Assume we have samples $X_i \sim p(x)$ for $i = 1, \ldots, N$

$$F_X(c) = \frac{1}{N}\sum_{i=1}^N \varphi(X_i),$$
$$= \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{X_i\leq c}(x).$$

The first line only above gets 1 mark. 2

(iii) The practical implementation of this estimator is simple. We sample from $X_i \sim N(0,1)$ and keep the samples if $X_i \leq c$, if they are smaller than c and count the number of these samples (say c_N). Then the estimator is given by c_N/N. $\boxed{1}$

(c) The rejection sampler **cannot be used** to sample from p using q described in the question. The exam taker must provide the reasoning by mentioning the condition $p(x) \leq Mq(x)$ for q to be used to sample from p (1 mark). Then deriving the ratio (1 mark)

$$\frac{p(x)}{q(x)} = \frac{\sqrt{2\pi}}{\pi(1+x^2)} e^{x^2/2},$$

and arguing that this function is not upper-bounded (it grows to infinity as $x \to \infty$ (2 marks). Therefore, M s.t. $p(x) \leq Mq(x)$ **cannot be found**. $\boxed{2}$

(d) (i) We have

$$p(x) = \frac{1}{2\sqrt{2}} \left[1 + \frac{x^2}{2}\right]^{-3/2}$$
$$q(x) = \frac{1}{\pi(1+x^2)}.$$

We want to find

$$M = \sup_{x \in \mathbb{R}} \frac{p(x)}{q(x)}.$$

For this we typically compute $\log p(x)/q(x)$ and find maximisers. If we compute (up to constant that do not depend on x)

$$\log \frac{p(x)}{q(x)} =^c -\frac{3}{2} \log\left(1 + \frac{x^2}{2}\right) + \log(1+x^2).$$

If we compute the derivative and set it to zero

$$\frac{\mathrm{d} \log \frac{p(x)}{q(x)}}{\mathrm{d}x} = -\frac{3}{2} \frac{x}{1 + \frac{x^2}{2}} + \frac{2x}{1+x^2} = 0,$$

one can see that one root is $x = 0$ (**1 mark**). To find others, we move the first term to the r.h.s. and obtain

$$\frac{3x}{2+x^2} = \frac{2x}{1+x^2},$$

and obtain

$$3 + 3x^2 = 4 + 2x^2,$$

means that we obtain $x \pm 1$ (**1 mark**). We need to now determine which roots are minima. We can compute the second derivative

$$\frac{\mathrm{d}^2 \log \frac{p(x)}{q(x)}}{\mathrm{d}x^2} = -\frac{3}{2}\left(\left(1 + \frac{x^2}{2}\right)^{-1} - x\left(1 + \frac{x^2}{2}\right)^{-2} x\right) + 2\left((1+x^2)^{-1} - x(1+x^2)^{-2} 2x\right).$$

By inspection, it can be seen that

$$\frac{\mathrm{d}^2 \log \frac{p(x)}{q(x)}}{\mathrm{d}x^2}(0) > 0 \qquad \text{hence it is a minimum,}$$

and $x \pm 1$ are the maxima (**1 mark**) since the second derivative is positive evaluated at $x \pm 1$. The value of the ratio at these points are the same (due to symmetry) and is given by (**2 marks**)

$$M = \frac{p(\pm 1)}{q(\pm 1)} = \frac{2\pi}{3\sqrt{3}}.$$

$\boxed{5}$

(ii) The procedure is given as follows

- Sample $X' \sim p(x)$,
- Sample $U \sim \mathrm{Unif}(0, 1)$,
- Accept the sample X' if

$$u \leq \frac{p(x)}{Mq(x)}.$$

$\boxed{1}$

Applied

sim. seen ⇓

1. (a) Since the plates are infinitely large, the flow is two dimensional with velocity field $(u, v) = \big(u(y,t), v(y,t)\big)$.

The continuity equation reduces to $\dfrac{\partial v}{\partial y} = 0$, which implies that v is independent of y. Hence $v \equiv 0$ since $v = 0$ at $y = 0$ and h.

The $y-$momentum equation implies $\dfrac{\partial p}{\partial y} = 0$, which is automatically satisfied.

The $x-$momentum equation yields:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

The no-slip boundary conditions imply that

$$u(h,t) = U_1 \cos(\Omega_1 t), \quad u(0,t) = U_2 \cos(\Omega_2 t).$$

4, A

(b) (i) Substitution of the assumed form for $u(y,t)$ into the equation (1) gives

meth seen ⇓

$$\mathrm{i}\,\Omega_1 f = \nu f'', \quad \text{i.e.} \quad f'' - \frac{\mathrm{i}\,\Omega_1}{\nu} f = 0,$$

which is a second-order linear equation with constant coefficients. The general solution for f is found as

$$f = A \exp\{\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma y\} + B \exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma y\}, \tag{2}$$

where $\sigma = \sqrt{\Omega_1/\nu}$.

The boundary conditions can be written as

$$u(0,t) = 0, \quad u(h,t) = U_1 \cos(\Omega_1 t) = \frac{1}{2} U_1\, \mathrm{e}^{\mathrm{i}\,\Omega_1 t} + c.c.,$$

application of which gives

$$f(0) = 0, \quad f(h) = U_1/2. \tag{3}$$

Substituting the general solution (2) into (3), we obtain

$$A + B = 0, \quad A \exp\{\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} + B \exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} = U_1/2.$$

It follows that

$$A = -B = \frac{U_1/2}{\exp\{\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} - \exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\}},$$

use of which in (2) yields

$$f(y) = \frac{U_1}{2} \frac{\exp\{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} y\} - \exp\{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} y\}}{\exp\{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} h\} - \exp\{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} h\}}.$$

Hence,

$$u(y,t) = \frac{U_1}{2} \left[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} y} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} y}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}} h}} \right] \mathrm{e}^{\mathrm{i}\,\Omega_1 t} + c.c. \tag{4}$$

8, B

(ii) (a) When $(2\nu/\Omega_1)^{\frac{1}{2}} \gg h$, $(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h \ll 1$ and $(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}y \ll 1$. Performing Taylor expansions in $f(y)$,

unseen ⇓

$$\mathrm{e}^{\pm(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h} \approx 1 \pm (1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h, \quad \mathrm{e}^{\pm(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}y} \approx 1 \pm (1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}y,$$

we obtain

$$f(y) \approx \frac{U_1}{2}(y/h),$$

and hence

$$u \approx U_1 \cos(\Omega_1 t)(y/h).$$

The flow appears to be quasi-steady (i.e. a quasi-steady plane Couette flow) in this limit.

(b) When $(2\nu/\Omega_1)^{\frac{1}{2}} \ll h$, it follows that $(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h \gg 1$, and

$$f(y) = \frac{U_1}{2}\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(y-h)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(y+h)}}{1-\mathrm{e}^{-2(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h}} \approx \mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(y-h)},$$

when the exponentially terms in the numerator and denominator are neglected. Thus

$$u \approx U_1\,\mathrm{e}^{(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(y-h)}\cos\Big((\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(h-y) - \Omega_1 t\Big).$$

The flow motion is confined in a layer near the upper wall, with an $O(\sqrt{2\nu/\Omega})$ width, i.e. $y - h = O(\sqrt{2\nu/\Omega}) \ll h$. Outside this layer, the velocity is exponentially. This is practically an infinite Stokes layer with the remote fixed plate playing a negligible role.

4, D

unseen ⇓

(c) By the substitution $\tilde{y} = h - y$, the problem is converted to one for which the oscillator plate is at $\tilde{y} = h$ and the fixed plate is at $\tilde{y} = 0$, whereas the equation for u remains unchanged. The solution is thus equivalent to (4) with y being replaced by $h-y$, and Ω_1 and U_1 by Ω_2 and U_2 respectively, which gives

$$u(y,t) = \frac{U_2}{2}\Big[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h}}\Big]\mathrm{e}^{\mathrm{i}\,\Omega_2 t} + c.c. \qquad (5)$$

Of course, the solution may alternatively be found by imposing the appropriate conditions for this case, which become

$$A\exp\{\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} + B\exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} = 0, \quad A + B = U_2/2$$

from which one obtains

$$A = -B\exp\{-2\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} = -\frac{(U_2/2)\exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\}}{\exp\{\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\} - \exp\{-\,\mathrm{e}^{\mathrm{i}\,\pi/4}\,\sigma h\}},$$

where $\sigma = \sqrt{\Omega_2/\nu}$. It follows that

$$f(y) = \frac{U_2}{2}\Big[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h}}\Big],$$

which leads to the same solution as (5).

Since the governing equation and boundary conditions for $u(y,t)$ are linear, the principle of superposition may be applied to give the solution for the general case where both plates oscillate,

$$u(y,t) = \frac{U_1}{2}\left[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(h-y)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}(h-y)}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_1}{2\nu})^{\frac{1}{2}}h}}\right]\mathrm{e}^{\mathrm{i}\,\Omega_1 t}$$

$$+\frac{U_2}{2}\left[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}(h-y)}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega_2}{2\nu})^{\frac{1}{2}}h}}\right]\mathrm{e}^{\mathrm{i}\,\Omega_2 t} + c.c.$$

$\boxed{4,\ \mathrm{C}}$

1. **MATH60004/70004 Asymptotic Methods (Solutions)**

$$\varepsilon y'' + (x+\beta)y' - xy = 0, \qquad y(0) = 0, \quad y(1) = 1.$$

sim. seen ⇓

(a–i) The outer expansion $y = y_0 + \ldots$ yields, at leading order,

$$(x+\beta)y_0' - xy_0 = 0,$$

which can be solved using separation of variables

$$\frac{y_0'}{y_0} = \frac{x}{x+\beta} \quad \Rightarrow \quad \ln y_0 = \int 1 - \frac{\beta}{x+\beta}\,\mathrm{d}x = x - \beta\ln(x+\beta) + \text{const}$$
$$\Rightarrow \quad y_0 = a_0(x+\beta)^{-\beta}\mathrm{e}^{x},$$

or an integrating factor

$$y_0' - \frac{x}{x+\beta}y_0 = 0 \quad \Rightarrow \quad \left[y_0 \exp\left(-\int \frac{x}{x+\beta}\,\mathrm{d}x\right)\right]' = 0$$
$$\Rightarrow \quad y_0 = a_0 \exp\left(\int \frac{x}{x+\beta}\,\mathrm{d}x\right) = a_0 \exp\left(x - \beta\ln(x+\beta)\right) = a_0(x+\beta)^{-\beta}\mathrm{e}^{-x}.$$

We assume there is no boundary layer at $x = 1$, so imposing $y_0(1) = 1$ yields

$$1 = a_0(1+\beta)^{-\beta}\mathrm{e} \quad \Rightarrow \quad a_0 = (1+\beta)^{\beta}\mathrm{e}^{-1}, \quad y_0 = \left(\frac{1+\beta}{x+\beta}\right)^{\beta}\mathrm{e}^{x-1}.$$

3, A

(a–ii) Rescaling $x = \varepsilon^{\alpha}X$ with $y(x) = Y(X)$ yields

$$\varepsilon^{1-2\alpha}Y'' + \varepsilon^{-\alpha}(\beta + \varepsilon^{\alpha}X)Y' - \varepsilon^{\alpha}XY = 0.$$

The first two terms must balance, giving $\alpha = 1$. Thus,

$$x = \varepsilon X, \quad Y'' + (\beta + \varepsilon X)Y' - \varepsilon^2 XY = 0.$$

2, A

The leading-order equation for $Y \sim Y_0$, with the boundary condition $Y(0) = 0$, is

$$Y_0'' + \beta Y_0' = 0, \quad Y_0(0) = 0 \quad \Rightarrow \quad Y_0 = A_0(1 - \mathrm{e}^{-\beta X}).$$

The constant A_0 is determined by matching:

As $x \to 0,\ y_0 \to (1+1/\beta)^{\beta}\mathrm{e}^{-1}$; as $X \to \infty,\ Y_0 \to A_0$.

Hence $A_0 = (1+1/\beta)^{\beta}\mathrm{e}^{-1}$, and so

$$Y_0 = (1+1/\beta)^{\beta}\mathrm{e}^{-1}(1 - \mathrm{e}^{-\beta X}).$$

3, A

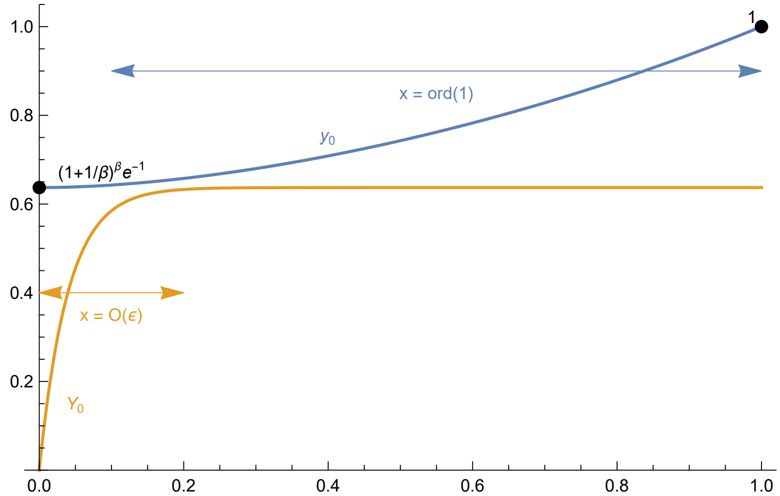
(a-iii) Similarly attempting to find a boundary layer at $x = 1$ yields, with $x = 1 + \varepsilon X$,

$$Y_0'' + (1+\beta)Y_0' = 0 \quad \Rightarrow \quad Y_0 = A_0 + B_0\mathrm{e}^{-(1+\beta)X}.$$

The exponential term diverges as $X \to -\infty$, which cannot match the outer solution, so the boundary-layer solution would just be a trivial constant.

2, B

(a-iv)



2, B

(a-v) The common overlap behaviour between the inner and outer is

$$y \approx A_0 = (1+1/\beta)^\beta \mathrm{e}^{-1},$$

so adding the two and subtracting this yields the additive composite

$$\begin{aligned} y &\approx \left(\frac{1+\beta}{x+\beta}\right)^\beta \mathrm{e}^{x-1} - (1+1/\beta)^\beta \mathrm{e}^{-1}\mathrm{e}^{-\beta x/\varepsilon} \\ &= (1+1/\beta)^\beta \mathrm{e}^{-1}\left[(1+x/\beta)^{-\beta}\mathrm{e}^{x} - \mathrm{e}^{-\beta x/\varepsilon}\right]. \end{aligned}$$

2, B

unseen ⇓

(b) For $\beta = 0$, we have

$$\varepsilon y'' + xy' - xy = 0.$$

The boundary layer should still be at $x = 0$, for the same reason as before, so the outer solution is obtained by setting $\beta = 0$ in the previous result, yielding

$$y_0 = \mathrm{e}^{x-1}.$$

1, C

Rescaling $x = \varepsilon^\alpha X$ yields

$$\varepsilon^{1-2\alpha} Y'' + XY' - \varepsilon^\alpha XY = 0,$$

and we see that the third term is negligible so the first two must balance, leading to $\alpha = 1/2$. Thus,

$$x = \varepsilon^{1/2}X, \quad Y'' + XY' - \varepsilon^{1/2}XY = 0, \quad Y(0) = 0.$$

2, C

The leading-order equation for $Y \sim Y_0$ is then

$$Y_0'' + XY_0' = 0, \quad Y_0(0) = 0 \quad \Rightarrow \quad Y_0' = A_0\mathrm{e}^{-X^2/2} \quad \Rightarrow \quad Y_0 = A_0\int_0^X \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi,$$

where the boundary condition was applied when choosing the lower limit of the integral.

As $X \to \infty$ we have $Y_0 \to \sqrt{\pi/2}\,A_0$, so matching to the outer value $y_0(0) = \mathrm{e}^{-1}$ yields

$$\sqrt{\pi/2}\,A_0 = \mathrm{e}^{-1} \quad \Rightarrow \quad A_0 = \mathrm{e}^{-1}\sqrt{2/\pi}, \quad Y_0 = \mathrm{e}^{-1}\sqrt{2/\pi}\int_0^X \mathrm{e}^{-\xi^2/2}\,\mathrm{d}\xi.$$

3, C

MATH60005/70005: Optimisation (Autumn 22-23)

Consider the problem

$$\begin{aligned} \min\ & x_1^2 + 2x_2^2 + x_1 \\ \text{subject to}\ & x_1 + x_2 \leq \gamma\,, \end{aligned}$$

with $\gamma \in \mathbb{R}$ a parameter.

i) (6 marks) Prove that for any $\gamma \in \mathbb{R}$, this problem has a unique solution (do not solve it).

Answer. The cost is equivalent to minimizing a quadratic function $\mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = [1\ 0]^\top .$$

It is clear that the eigenvalues of $\mathbf{A}$ are 1 and 2, and therefore this is a strictly convex function, with an affine constraint for any γ. Under these conditions, there exists a unique optimal solution.

ii) (8 marks) Solve the problem, expressing the general solution as a function of γ.

Answer. We proceed via KKT conditions, which in this case are sufficient. The Lagrangian is given by

$$L(\mathbf{x}, \lambda) = x_1^2 + 2x_2^2 + x_1 + \lambda(x_1 + x_2 - \gamma)\,,$$

leading to ($\nabla_{\mathbf{x}} L = 0$)

$$\begin{aligned} 2x_1 + 1 + \lambda &= 0\,, \\ 4x_2 + \lambda &= 0\,, \\ \lambda(x_1 + x_2 - \gamma) &= 0\,, \\ \lambda &\geq 0\,. \end{aligned}$$

We analyse this system by cases. If $\lambda = 0$, then it follows that $x_1 = -1/2$ and $x_2 = 0$. This is a feasible solution if and only if $\gamma \geq -1/2$. In the second case, we assume $\lambda > 0$, which leads to the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \gamma \\ -1 \\ 0 \end{bmatrix}.$$

This solution of this system is given by

$$
\begin{aligned}
x_1 &= \frac{1}{6}(4\gamma - 1)\,,\\
x_2 &= \frac{1}{6}(2\gamma + 1)\,,\\
\lambda &= \frac{1}{6}(-8\gamma - 4)\,.
\end{aligned}
$$

For this solution to be feasible, we require λ to be positive, which is guaranteed if $\gamma < -1/2$. Expressing the solution as a function of γ yields

$$
\mathbf{x}^* = \begin{cases} \frac{1}{6}(4\gamma - 1, 2\gamma + 1) & \text{if} \quad \gamma < -\frac{1}{2}\,,\\ (-\frac{1}{2}, 0) & \text{if} \quad \gamma \geq -\frac{1}{2}\,. \end{cases}
$$

iii) (6 marks) Let $g(\gamma)$ be the optimal value of the problem for a given value of γ. Write an explicit expression for $g(\gamma)$ and determine whether this is a convex function or not.

Answer. Evaluating the solution from (c) into the cost leads to

$$
g(\gamma) = \begin{cases} \frac{1}{12}(8\gamma^2 + 8\gamma - 1) & \text{if} \quad \gamma < -\frac{1}{2}\,,\\ -\frac{1}{4} & \text{if} \quad \gamma \geq -\frac{1}{2}\,. \end{cases}
$$

This is a continuously differentiable function defined by parts. Note that $g'(-\frac{1}{2}) = 0$. Each part is convex in its domain (quadratic function and constant). However, this function is not twice continuously differentiable, it jumps at $\gamma = -\frac{1}{2}$. We need to show convexity using the first-order characterization. Since the convexity is clear at the interior of each piece, we need to verify that

$$
(g'(\gamma_1) - g'(\gamma_2))(\gamma_1 - \gamma_2) \geq 0\,,
$$

for all $\gamma_1 < -\frac{1}{2}$, $\gamma_2 \geq -\frac{1}{2}$, for which we have

$$
\frac{8}{12}(2\gamma_1 + 1)(\gamma_1 - \gamma_2) \leq 0\,,
$$

as both terms are always negative.

1 MATH 60006/MATH 70006: Applied Complex Analysis
January Mock

(a). Let $k(x) = 12e^{-3|x|}$. Then, introduce:

$$f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$g_-(x) = \begin{cases} 0, & x \geq 0 \\ 12\int_0^\infty e^{-3|x-y|} f(y)\,dy, & x < 0. \end{cases}$$

Then, we can rewrite the equation as:

$$12\int_0^\infty f(y) e^{-3|x-y|}\,dy = f_+'(x) + 8f_+(x) + g_-(x), \quad (1)$$

1 re-write equation $\forall x$

valid for $-\infty < x < \infty$. Recalling the fact that the Fourier transform of $f_+'(x)$ is given by:

$$-f_+(0) - isF_+(s),$$

i.e. since $f_+(0) = 1$ here: $-1 - isF_+(s)$, means that taking Fourier transforms of both sides of (1) gives:

$$\hat{K}(s)F_+(s) = -1 - isF_+(s) + 8F_+(s) + G_-(s),$$

or $$K(s)F_+(s) + G_-(s) = 1, \quad (2)$$

2 Take Fourier transforms.

where $K(s) = 8 - is - \hat{K}(s)$.

Here $F_+(s)$ denotes the right-sided Fourier transform of $f_+(x)$, $G_-(s)$ denotes the left-sided Fourier transform of $g_-(x)$ and $\hat{K}(s)$ denotes the ordinary Fourier transform of $k(x)$.

Let's calculate :

$$\hat{K}(s) = 12\int_{-\infty}^{\infty} e^{-3|x|} e^{isx}\,dx$$

$$= 12\int_{-\infty}^{0} e^{(3+is)x}\,dx + 12\int_{0}^{\infty} e^{(is-3)x}\,dx$$

$$= 12\left[\frac{e^{(3+is)x}}{3+is}\right]_{-\infty}^{0} + 12\left[\frac{e^{(is-3)x}}{is-3}\right]_{0}^{\infty}$$

$$= 12\left(\frac{1}{3+is} - \frac{1}{is-3}\right),$$ provided $-3<\mathrm{Im}\{s\}<3$ so the integrals converge.

$$= \underline{\underline{\frac{72}{s^2+9}}}.$$

1 Compute/give $\hat{K}(s)$

It follows that: $$K(s) = 8 - is - \frac{72}{s^2+9}$$

$$= \frac{(8-is)(s^2+9)-72}{s^2+9}$$

$$= \frac{-is(s-i)(s+9i)}{(s+3i)(s-3i)}$$

1 Determine $K(s)$

For convergence and analyticity of $F_+(s)$ we require that $|f_+(x)| < Ae^{(3-\delta)x}$, as $x\to\infty$, for some $\delta>0$.
($A>0$ constant)

$\Rightarrow F_+(s)$ is analytic in $\{s : \mathrm{Im}\{s\} > 3-\delta\}$.

Similarly, for $G_-(s)$, we can show $g_-(x) = Be^{3x}$
(B constant) $\Rightarrow G_-(s)$ is analytic in $\{s : \mathrm{Im}\{s\} < 3\}$.

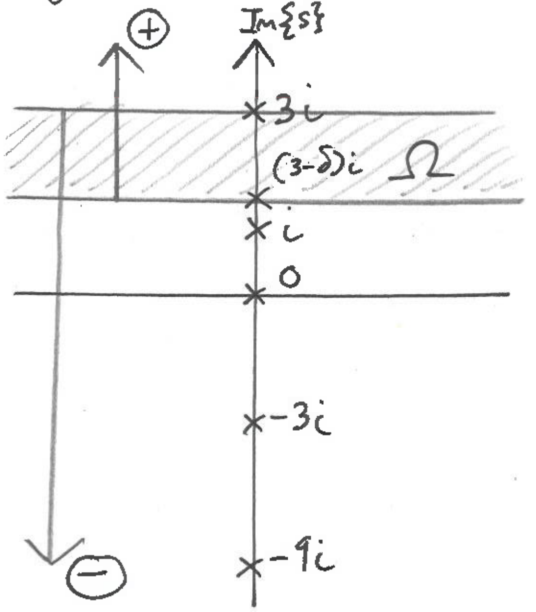
So, we take the $\oplus$ and $\ominus$ regions to be:

$\oplus = \{s : \mathrm{Im}\{s\} > 3-\delta\}$, where $0<\delta<2$

$\ominus = \{s : \mathrm{Im}\{s\} < 3\}$.

Hence the strip of analyticity, Ω, where $\oplus$ and $\ominus$ overlap is given by: $\Omega = \{s : \underbrace{3-\delta}_{\alpha} < \mathrm{Im}\{s\} < \underbrace{3}_{\beta}\}$, where $0<\delta<2$.

3 Determine Ω + check conditions. Give α, β.



} $0<\delta<2$ keeps this above $s=i$

$K(s)$: analytic provided $-3<\mathrm{Im}\{s\}<3$ ✓

non-zero provided $s \neq 0, i, -9i$ ✓

We decompose $K(s) = K_+(s)K_-(s)$, where:

$$K_+(s) = \frac{-is(s-i)(s+9i)}{s+3i}, \quad K_-(s) = \frac{1}{s-3i},$$

2 product decomposition

then ② gives: $K_+(s)F_+(s) + \dfrac{G_-(s)}{K_-(s)} = \dfrac{1}{K_-(s)} = R(s) = s-3i$.

Now we can write: $R(s) = R_+(s) + R_-(s)$, where

$$R_+(s) = s-3i, \quad R_-(s) = 0,$$

1 Sum decomposition (Note: Any decomp. works here).

leading to: $\underbrace{K_+(s)F_+(s) - R_+(s)}_{\text{analytic in } \oplus} = \underbrace{-\dfrac{G_-(s)}{K_-(s)}}_{\text{analytic in } \ominus}, \quad s \in \Omega$

SJB

Now since LHS is analytic in $\oplus$ and RHS is analytic in $\ominus$, and $\oplus/\ominus$ overlap in Ω, then the function:

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in \oplus \\ -\dfrac{G_-(s)}{K_-(s)}, & s \in \ominus, \end{cases}$$

is <u>entire</u> by analytic continuation.

1

E(s) definition + entire

Let's now consider $s \to \infty$ in $\oplus$. We know from lectures, as $s \to \infty$;

$$F_+(s) = \frac{i f_+(0)}{s} - \frac{f_+'(0)}{s^2} + O\left(\frac{1}{s^3}\right)$$

$$= \frac{i}{s} + \frac{5}{s^2} + O\left(\frac{1}{s^3}\right),$$ using the given conditions.

Now expanding $K_+(s)$:

$$K_+(s) = \frac{-i\cancel{8}(s-i)(s+9i)}{\cancel{8}\left(1+\frac{3i}{s}\right)}$$ $$\left(1+\frac{3i}{s}\right)^{-1} = 1 - \frac{3i}{s} + \dots$$

$$= -i(s-i)(s+9i)\left[1 - \frac{3i}{s} + O\left(\frac{1}{s^2}\right)\right]$$

$$= -is^2 + 5s + O(1),$$

So expanding $E(s)$ in $\oplus$ as $s \to \infty$ leads to:

$$K_+(s)F_+(s) - R_+(s) \sim \left(-is^2 + 5s + O(1)\right)\left(\frac{i}{s} + \frac{5}{s^2} + O\left(\frac{1}{s^3}\right)\right) - \left[s - 3i\right]$$

$$\sim (s - 5i + 5i - s + 3i) + O\left(\frac{1}{s}\right)$$

$$\sim 3i + O\left(\frac{1}{s}\right) \to \underline{\underline{3i}} \text{ as } s \to \infty.$$

Hence, by Liouville's theorem: $\underline{\underline{E(s) \equiv 3i}}$, for all s.

3

$s \to \infty$ expansion + determine E(s) via Liouville

SSR

Therefore: $F_+(s)K_+(s) - R_+(s) = 3i$

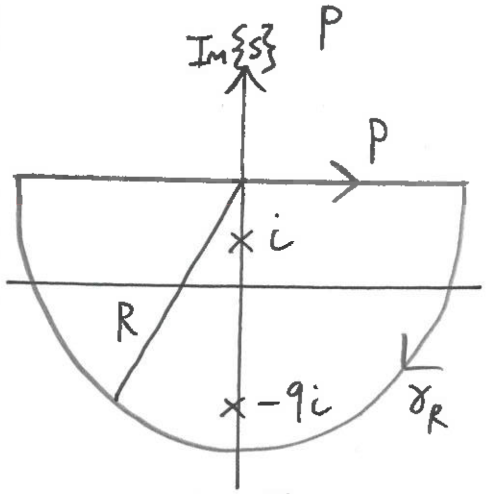
$$\Rightarrow F_+(s) = \frac{s(s+3i)}{-is(s-i)(s+9i)} = \underline{\underline{\frac{i(s+3i)}{(s-i)(s+9i)}}}, \text{ as required.}$$

1 Final manipulation.

(b). To retrieve $f_+(x)$, apply the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}\,ds,$$ where P is a horizontal line in the ⊕ region.

1 Apply inversion



- For $x<0$, $f_+(x) = 0$ from lectures.
- For $x>0$, close P in the LHP as shown and take $R\to\infty$. Let $\gamma = P + \gamma_R$.
- In $\lim_{R\to\infty}$, $\int_{\gamma_R} \to 0$ by exponential decay.

1 contour + poles inside.

By the residue theorem:

clockwise integration.

$$\oint_\gamma F_+(s)e^{-isx}\,ds = -2\pi i\left(\text{Res}\{F_+(s)e^{-isx}, s=i\} + \text{Res}\{F_+(s)e^{-isx}, s=-9i\}\right)$$

$$= -2\pi i\left[\frac{-4}{10i}e^{x} + \frac{-6}{10i}e^{-9x}\right]$$

$$= \frac{4\pi}{5}e^{x} + \frac{6\pi}{5}e^{-9x}.$$

both simple poles so residues straightforward.

1 Apply residue theorem + calculate residues.

$$\Rightarrow f_+(x) = \frac{1}{2\pi}\left(\frac{4\pi}{5}e^{x} + \frac{6\pi}{5}e^{-9x}\right), \quad x \geq 0,$$

or:

$$\boxed{f(x) = \frac{2}{5}e^{x} + \frac{3}{5}e^{-9x}}, \quad x \geq 0.$$

1 Final answer.

20

SSB

1. [The entire question concerns basic theory; all results were mentioned in class but (b) and (c)(ii) not spelled out in detail.]

(a) (i) - f is *topologically mixing* if for any two nonempty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that

$$f^n(U) \cap V \neq \emptyset, \forall n \geq N\,.$$

(2 marks)

- f has *sensitive dependence (on initial conditions)* if there exists a $\delta > 0$ such that for all $x \in X$ and all $\varepsilon > 0$, there exists a $y \in B_\varepsilon(x)$ and an $n \in \mathbb{N}$ with

$$d(f^n(x), f^n(y)) \geq \delta\,.$$

(2 marks)

(ii) Take $\delta > 0$ such that there are points x_1, x_2 with $d(x_1, x_2) > 4\delta$. Let $V_i = B_\delta(x_i)$. Suppose $x \in X$ and U is a neighbourhood of x. Then by topological mixing there are N_1, N_2 such that $f^n(U) \cap V_1 \neq \emptyset\ \forall n > N_1$ and $f^n(U) \cap V_2 \neq \emptyset\ \forall n > N_2$. For $n > \max(N_1, N_2)$ there are points $y_1, y_2 \in U$ such that $f^n(y_1) \in V_1$ and $f^n(y_2) \in V_2$; hence $d(f^n(y_1), f^n(y_2)) \geq 2\delta$. By the triangle inequality this implies that $d(f^n(y_1), f^n(x)) \geq \delta$ or $d(f^n(y_2), f^n(x)) \geq \delta$.

(4 marks)

(b) Let $\omega = \omega_0\omega_1 \ldots \omega_{n-1} \ldots$ and $\tilde{\omega} \in C_{\omega_0\omega_1\ldots\omega_{n-1}}$ then

$$D(\omega, \tilde{\omega}) = \sum_{i=0}^{\infty} \frac{\delta_{\omega_i,\tilde{\omega}_i}}{3^i} = \sum_{i=n}^{\infty} \frac{\delta_{\omega_i,\tilde{\omega}_i}}{3^i} = \frac{1}{3^n}\sum_{i=0}^{\infty} \frac{\delta_{\omega_{i+n},\tilde{\omega}_{i+n}}}{3^i} \leq \frac{1}{3^n}\frac{3}{3-1} = \frac{1}{2 \cdot 3^{n-1}} < 3^{1-n}.$$

In addition, if $\tilde{\omega} \notin C_{\omega_0\omega_1\ldots\omega_{n-1}}$ then $D(\omega, \tilde{\omega}) \geq 3^{1-n}$ since $\omega_i = \tilde{\omega}_i$ for some $i \in \{0, \ldots, n-1\}$.

(4 marks)

(c) (i) μ is an invariant measure of f: all $A \in \mathcal{B}(X)$, $f_*\mu(A) := \mu(f^{-1}(A)) = \mu(A)$. For all $B \in \mathcal{B}(X)$, $h_*\mu = \mu(h^{-1}(B))$. Then $h_*\mu(g^{-1}(B)) = \mu(h^{-1} \circ g^{-1}(B)) = \mu((g \circ h)^{-1}(B)) = \mu((h \circ f)^{-1}(B)) = \mu(f^{-1} \circ h^{-1}(B)) = \mu(h^{-1}(B)) = h_*\mu(B)$. (It may be noted that $h : X \to Y$ need not be invertible for $h^{-1} : \mathcal{B}(Y) \to \mathcal{B}(X)$ to be well defined as $h^{-1}(B) = \{A \in \mathcal{B}(X) \mid h(A) = B\}$.) (4 marks)

(ii) μ is ergodic iff $\forall A \in \mathcal{B}(X)$ satisfying $f^{-1}(A) = A$, we have $\mu(A) \in \{0, 1\}$. If $g^{-1}(B) = B$ then $f^{-1}(h^{-1}(B)) = (h \circ f)^{-1}(B) = (g \circ h)^{-1}(B) = h^{-1} \circ g^{-1}(B) = h^{-1}(B)$. The latter implies that $h_*\mu(B) := \mu(h^{-1}(B)) \in \{0, 1\}$ and thus that $h_*\mu$ is ergodic. (4 marks)

(Total: 20 marks)

Answers to January 2023 Mock Question

1. (a) $\mathbf{F} = yz\cos(xy)\mathbf{i} + xz\cos(xy)\mathbf{j} + \sin(xy)\mathbf{k} = -\nabla V$, with $V = -z\sin(xy)$, so that a Lagrangian is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + z\sin(xy).$$

(4 marks, seen similar)

(b) $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$.

Standard result $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$.

$(y\dot{x} - x\dot{y}) = r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) - r\cos\theta(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) = -r^2\dot{\theta}$.

Accordingly,

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \alpha r^2\dot{\theta}.$$

(4 marks, seen similar)

θ is cyclic so that $p_\theta = \partial L/\partial\dot{\theta} = r^2\dot{\theta} - \alpha r^2$ is a constant of the motion

A second constant of the motion is (the energy)

$$p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

(4 marks, unseen)

(c) (i) The Hamilton-Jacobi equation is

$$(1 + q^2)\frac{\partial S}{\partial q} + \frac{\partial S}{\partial t} = 0.$$

Writing $S = W(q) - \alpha t$ yields

$$(1 + q^2)\frac{dW}{dq} = \alpha,$$

so that

$$W = \alpha\int\frac{dq}{1 + q^2} = \alpha\tan^{-1}q,$$

ignoring an additive constant. A complete solution is

$$S = \alpha\tan^{-1}q - \alpha t.$$

(4 marks, seen similar)

(ii) The new coordinate is

$$\beta = \frac{\partial S}{\partial\alpha} = \tan^{-1}q - t,$$

so that $q(t) = \tan(t + \beta)$.

$$p(t) = \frac{\alpha}{1 + q^2} = \alpha\cos^2(t + \beta).$$

Alternatively, directly solve Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} = 1 + q^2, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -2qp,$$

to get the same results.

(4 marks, seen similar)

(Total: 20 marks)

1. (a) The constants k_1 and k_2 have dimensions $[\mathrm{Time}]^{-1}$, while k_4 is a concentration. The constant k_3 has dimensions $[\mathrm{Concentration}]\,[\mathrm{Time}]^{-1}$. Let us therefore divide the equation by k_4 to form a dimensionless measure of concentration and by k_1 to obtain a dimensionless measure of time. seen simil ⇓ 2

$$\frac{d\left(\frac{g}{k_4}\right)}{d\left(k_1 t\right)} = \frac{s_0}{k_4} - \frac{k_2}{k_1}\frac{g}{k_4} - \frac{k_3}{k_1 k_4}\frac{\left(g/k_4\right)^2}{1+\left(g/k_4\right)^2}.$$

Now multiply across by $k_1 k_4/k_3$ and identify a *timescale* $t_0 = k_4/k_3$ (Note this does indeed have the correct units). Then the equation is

$$\frac{d\left(\frac{g}{k_4}\right)}{d\left(\frac{t}{t_0}\right)} = \frac{s_0 k_1}{k_3} - \frac{k_2 k_4}{k_3}\frac{g}{k_4} + \frac{\left(g/k_4\right)^2}{1+\left(g/k_4\right)^2}.$$

Now we identify the dimensionless concentration $x = g/k_4$, the dimensionless time $\tau = t/t_0$ and the dimensionless groups $s = s_0 k_1/k_3$ and $r = k_2 k_4/k_3$. (Note that the dimensionless groups are indeed dimensionless. For example, 2

$$[s] = \frac{[\mathrm{Concentration}]\,[\mathrm{Time}]^{-1}}{[\mathrm{Concentration}]\,[\mathrm{Time}]^{-1}} = 1.)$$

Following standard practice, we replace τ by t and remind ourselves that this is now dimensionless. The equation is thus

$$\frac{dx}{dt} = s - rx + \frac{x^2}{1+x^2}.$$

1

(b) For $s=0$ the equation is $\dot{x} = f\left(x;r\right)$, where $f = -rx + \frac{x^2}{1+x^2}$. The fixed points are $x=0$ and the two solutions to $x^2 - \frac{1}{r}x + 1 = 0$, which gives

$$x_* = \frac{\frac{1}{r} \pm \sqrt{\frac{1}{r^2} - 4}}{2}.$$

These fixed points are real and positive for $r < r_c$, where $r_c = \frac{1}{2}$. 2

Now, to investigate the growth rates associated with these fixed points. That is, let us form linear perturbations about the fixed points x_*:

$$\delta\dot{x} = \alpha\delta x, \qquad \alpha = f'\left(x_*, r\right).$$

Now $f'\left(x,r\right) = -r + \frac{2x}{\left(1+x^2\right)^2}$ and $x_*^2 + 1 = \frac{1}{r}x_*$, so that

$$\alpha = -r + \frac{2x_*}{\left(x_*/r\right)^2} = -r + \frac{2r^2}{x_*} = \frac{2r^2 - rx_*}{x_*}.$$

Hence,

$$\alpha_\pm = \frac{4r^2 - 1 \mp \sqrt{1 - \frac{r}{r_c}}}{2x_*}.$$

Fig. 1 shows the bifurcation curve and the growth rates for this system. 3

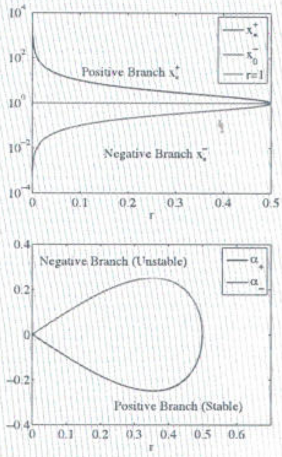


Figure 1: Bifurcation diagram for $s = 0$. Here $r_c = \frac{1}{2}$; Growth rates corresponding to steady states. The positive branch (larger steady state) is stable while the negative branch (smaller steady state) is unstable.

(c) We are interested in the equation $\dot{x} = f(x; r, s)$, where $f(x; r, s) = s - rx + \frac{x^2}{1+x^2}$. For r and s sufficiently small, there are three fixed points x_1, x_2 and x_3 corresponding to the three real zeros of f. The smallest and largest of the fixed points are stable while the intermediate fixed point is unstable. This is shown in Fig. 2(a). If we increase s away from zero, the graph of f changes, and this is shown in Fig. 2(b). Indeed, for s sufficiently large, the equilibria x_1 and x_2 collide and vanish in a saddle-node bifurcation. For s very large, only the x_3 fixed point survives, as shown in Fig. 2(c). $\boxed{\text{seen simil} \Downarrow}$ $\boxed{2}$

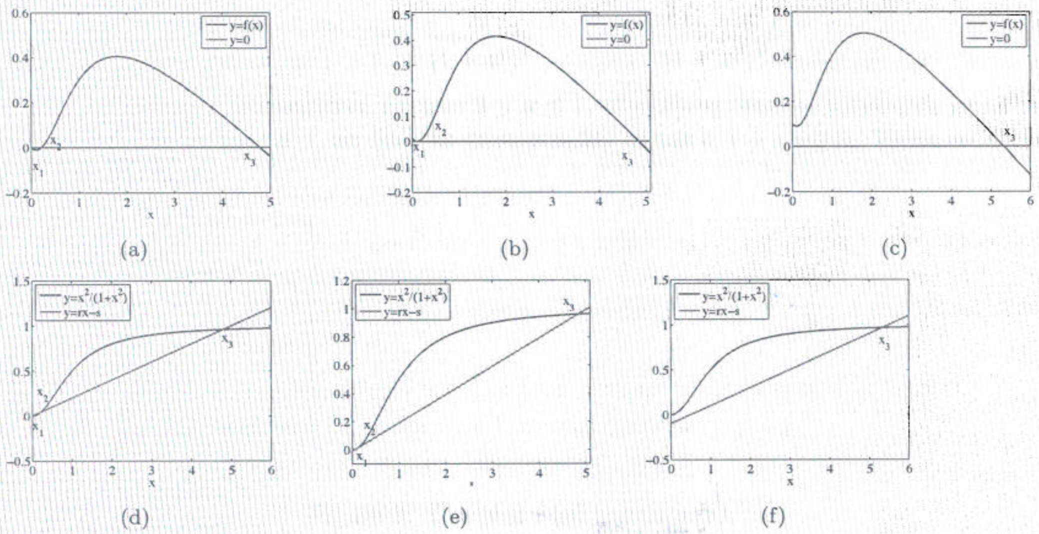


Figure 2: (a)–(c) The vector field $f(x; r, s)$ for various values of r and s. For sufficiently large s, the fixed points x_1 and x_2 always disappear in a saddle-node bifurcation; (d)–(f) An equivalent graphical method of finding the roots x_1, x_2 and x_3 is to plot $y = \frac{x^2}{1+x^2}$ and $y = rx - s$ and to find the intersection of these curves. The plots in (d)–(f) correspond to those in (a)–(c).

In this way, by increasing s to a sufficiently large value, the only steady state the system can select is the x_3 equilibrium, and this is stable. Increasing s from 0 to a $\boxed{2}$ large fixed value $s_{\max}$ will cause the concentration to go from zero to x_3 $(s = s_{\max})$. By decreasing s back to zero, the system will not move away from this stable concentration level. Thus, a biological switch has been achieved. $\boxed{1}$

(d) For a steady state to exist, we need the curves $rx - s$ and $\frac{x^2}{1+x^2}$ to intersect, since this condition is equivalent to the requirement that $f(x; r, s) = 0$. The intersection may occur in at most three places, as we have seen in Fig. 2. $\boxed{3}$

At a bifurcation point, these steady states collide, which will occur if the curves are tangent to one another. That is, their derivatives must be equal. (Recall that at a saddle node bifurcation, the zeroth and first derivatives of the nullclines must be equal.) This gives

$$r = \frac{2x}{(1+x^2)^2}, \qquad s = \frac{x^2(1-x^2)}{(1+x^2)^2}.$$

To keep s non-negative, we take $0 \le x \le 1$. Identifying these as parametric equations in the parameter x, we find the corresponding curves in $r - s$ space. These are shown in Fig. 3. $\boxed{2}$

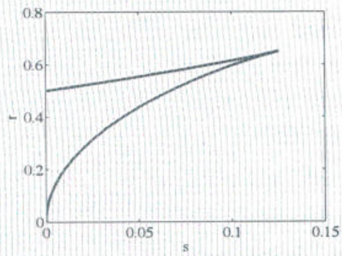


Figure 3: The plot $r(x)$ versus $s(x)$.

Q. **Quantum harmonic oscillator and momentum representation**

(a) Applying $\hat{H}$ in position representation to $\phi(x)$ yields

$$\langle x|\hat{H}|\phi\rangle = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x) + \frac{1}{2}x^2\phi(x).$$

(1 mark)

We have

$$\frac{\partial}{\partial x}\phi_0(x) = -x\phi_0(x),$$

and

$$\frac{\partial^2}{\partial x^2}\phi_0(x) = (x^2-1)\phi_0(x),$$

and thus

$$\langle x|\hat{H}|\phi_0\rangle = \frac{1}{2}\phi_0(x),$$

that is, $\phi_0(x)$ is an eigenfunction of $\hat{H}$ with eigenvalue $\frac{1}{2}$.

(3 marks)

For $\phi_1(x)$ we find

$$\frac{\partial^2}{\partial x^2}\phi_1(x) = (-3+x^2)\phi_1(x),$$

that is

$$\langle x|\hat{H}|\phi_1\rangle = \frac{3}{2}\phi_0(x),$$

which means that $\phi_1(x)$ is an eigenfunction of $\hat{H}$ with eigenvalue $\frac{3}{2}$.

(4 marks)

(b) (i) We insert an identity $\int_{-\infty}^{\infty}|q\rangle\langle q|\mathrm{d}q = \hat{I}$, to find

$$\langle p|\psi\rangle = \int_{-\infty}^{+\infty}\langle p|q\rangle\langle q|\psi\rangle\mathrm{d}q.$$

Inserting $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}$ as given in the question we verify that

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\langle q|\psi\rangle\mathrm{d}q.$$

(3 marks)

(ii) To calculate $\langle p|\hat{q}|\psi\rangle$ we write

$$\begin{aligned}
\langle p|\hat{q}|\psi\rangle &= \int_{-\infty}^{+\infty} \langle p|q\rangle\langle q|\hat{q}|\psi\rangle \mathrm{d}q \\
&= \int_{-\infty}^{+\infty} q\langle p|q\rangle\langle q|\psi\rangle \mathrm{d}q \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q)\mathrm{d}q \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{i}\hbar\frac{\partial}{\partial p}(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q))\mathrm{d}q \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q)\mathrm{d}q\right) \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\left(\int_{-\infty}^{+\infty} \langle p|q\rangle\langle q|\psi\rangle\mathrm{d}q\right) \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial p}\tilde{\psi}(p).
\end{aligned}$$

(5 marks)

(iii) We have by definition of the states $|p\rangle$ that

$$\langle p|\hat{p}^2|\psi\rangle = p^2\langle p|\psi\rangle = p^2\tilde{\psi}(p).$$

Following the same argument as in part (b)(ii) we have

$$\langle p|\hat{q}^2|\psi\rangle = -\hbar^2\frac{\partial^2}{\partial p^2}\tilde{\psi}(p).$$

and thus in summary we have

$$\langle p|\hat{H}|\psi\rangle = \frac{\omega}{2}\left(-\hbar^2\frac{\partial^2}{\partial p^2} + p^2\right)\tilde{\psi}(p).$$

(4 marks)

Question

Consider the Cauchy problem for the Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & (t,x) \in (0,+\infty) \times \mathbb{R} \\ \rho(0,x) = g(x), & x \in \mathbb{R} \end{cases} \qquad g(x) = \begin{cases} 0 & x \leq 0, \\ 1 & 0 < x \leq 1, \\ 0 & x > 1. \end{cases} \tag{1}$$

(a) (i) Solve the characteristic system associated to the problem (1) and draw the characteristic lines. (4 marks)

(ii) Assume that $t \leq 2$. Compute the shock curve, find the unique entropy solution and draw the characteristic lines. (12 marks)

(b) For $t > 2$, compute the shock curve, find the unique entropy solution and draw the characteristic lines. (4 marks)

Solution

seen ⇓

4, A

(a) (i) The ODEs to solve for the characteristic system is

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = z, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = 0, \tag{2}$$

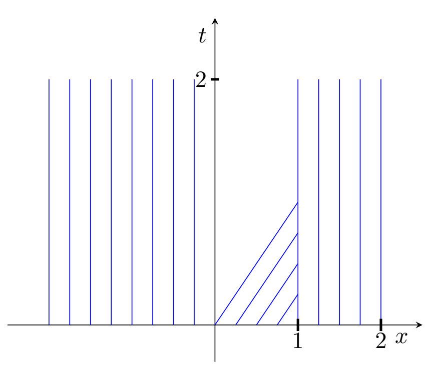
with initial conditions

$$t(0) = 0, \quad x(0) = \tau, \quad z(0) = g(\tau). \tag{3}$$

Solving the system, we obtain the formula

$$x = \tau + g(\tau)t = \begin{cases} \tau, & \tau \leq 0, \\ \tau + t, & 0 < \tau \leq 1, \\ \tau, & \tau > 1. \end{cases} \tag{4}$$

The characteristic lines are shown in the figure below



seen ⇓

12, B

(ii) The region

$$S := \{(t,x) : 0 < x \leq t\}$$

is not covered by the characteristics. We then connect the states 0 and 1 through a rarefaction wave, that is a solution of the form $\rho(t,x) = x/t$ in the region S. (We could have chosen to form a shock but since we are passing from 0 to 1 this shock wouldn't be entropic).

On the other hand, from (4), we see that the characteristic lines intersect at time $t = 0$ and $x = 1$. At $x = 1$ the initial datum g has a decreasing discontinuity, we can therefore expect the formation of a shock discontinuity. We compute the shock curve $(t, \sigma(t))$ emanating from $(0, 1)$ appealing to the Rankine-Hugoniot condition. In particular, for the Burgers equation we know that σ satisfies

$$\sigma'(t) = \frac{1}{2}\frac{\rho_+(t,\sigma(t))^2 - \rho_-(t,\sigma(t))^2}{\rho_+(t,\sigma(t)) - \rho_-(t,\sigma(t))} = \frac{1}{2}(\rho_+(t,\sigma(t)) + \rho_-(t,\sigma(t))) \tag{5}$$

$$\sigma(0) = 1. \tag{6}$$

In addition, the shock is entropic if

$$\rho_+(t,\sigma(t)) < \sigma'(t) < \rho_-(t,\sigma(t)). \tag{7}$$

The value at the right of the shock will always be $\rho_+ \equiv 0$, meaning that an entropic shock is such that $\sigma'(t) > 0$ for all $t \geq 0$. To determine ρ_-, we observe that on the left of the shock we have the region where $\rho \equiv 1$ and $\rho(t,x) = x/t$ on S. But since $\sigma'(t) > 0$, the shock curve does not enter in the region S at least for some time $t^* > 0$. Consequently, we have that

$$\rho_-(t,\sigma(t)) \equiv 1, \qquad \text{for } t \leq t^*. \tag{8}$$

With this, we find $\sigma'(t) = 1/2$ for $t \leq t^*$. Therefore

$$\sigma(t) = t/2 + 1, \qquad t \leq t^*,$$

so that the shock curve is the straight line $(t, t/2+1)$. The shock enters in the region S if

$$t = \frac{t}{2} + 1 \quad \Longrightarrow \quad t = 2. \tag{9}$$

In particular, we can take $t^* \leq 2$.

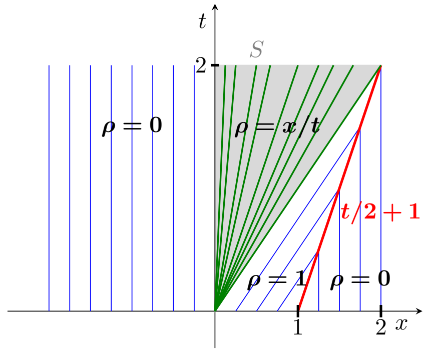
For $t \leq 2$, we can define the solution as

$$\rho(t,x) = \begin{cases} 0, & x \leq 0, \\ x/t, & 0 < x \leq t, \\ 1, & t < x \leq t/2+1, \\ 0, & x > t/2+1, \end{cases} \qquad \text{for all } 0 \leq t \leq 2. \tag{10}$$

Notice that this is an entropic solution since

$$0 = \rho_+(t,\sigma(t)) < \sigma'(t) = 1/2 < \rho_-(t,\sigma(t)) = 1, \qquad \text{for all } 0 \leq t \leq 2.$$

Thanks to Theorem 3.7 in Chapter 2 of the lecture notes, we know that (10) is the unique entropy solution since also g is bounded. In the picture below you find the characteristics with the rarefaction wave and the shock line for the solution (10).



unseen ⇓

4, D

(b) When $t = 2$, the straight line $(t, t/2+1)$ enters in the region $S = \{(t,x) : 0 < x < t\}$. This implies that we need to modify the shock curve since we will not see anymore the value $\rho \equiv 1$ as in the previous case. In particular, on the left of the shock we now have

$$\rho_-(t, \sigma(t)) = \frac{\sigma(t)}{t}, \qquad t > 2. \tag{11}$$

Since $\sigma(2) = 2$, the ODE we need to solve is given by

$$\sigma'(t) = \frac{\sigma(t)}{2t}, \qquad \text{for } t > 2 \tag{12}$$

$$\sigma(2) = 2. \tag{13}$$

The solution is

$$\sigma(t) = \sqrt{2t}.$$

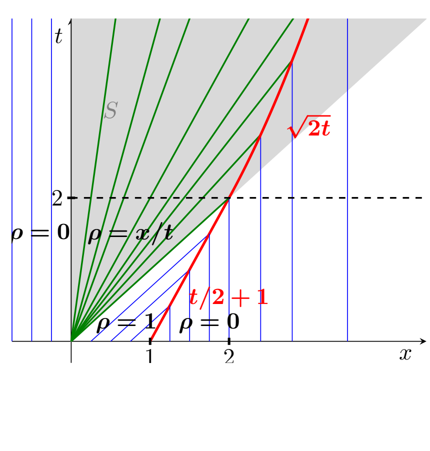
We only have to check that this shock is entropic, namely

$$0 = \rho_+(t, \sigma(t)) < \sigma'(t) = \frac{1}{\sqrt{2t}} < \rho_+(t, \sigma(t)) = \frac{\sqrt{2}}{\sqrt{t}}. \tag{14}$$

Therefore, the unique entropy solution for $t > 2$ is

$$\rho(t,x) = \begin{cases} 0, & x \le 0, \\ x/t, & 0 < x \le \sqrt{2t}, \\ 0, & x > \sqrt{2t}, \end{cases} \qquad \text{for all } t > 2, \tag{15}$$

where we also use that $\rho(2, x)$ is bounded to apply Theorem 3.7 in Chapter 2 of the lecture notes. In the picture below you find the characteristics with the rarefaction wave and the shock line for the solution $\rho(t,x)$ for all $t \ge 0$.



1. (a) The norm on the set of continuous functions is the uniform norm, which we denote $\|\cdot\|_\infty$. Write T for the application

$$T(f) = f * \phi.$$

First, we need to check that T maps the set of continuous functions to itself. Observe that, by linearity of the integral and the triangle inequality,

$$\begin{aligned}|T(f)(x) - T(f)(y)| &= \left|\int f(z)\phi(x-z)\,dz - \int f(z)\phi(y-z)\,dz\right| \\ &\lesssim \int |f(z)|\,|\phi(x-z) - \phi(y-z)|\,dz\end{aligned}$$

We now fix $\epsilon > 0$. On the one hand, it is possible to bound $|f(z)|$ by $\|f\|_\infty$. On the other hand, the function ϕ, being continuous and compactly supported, is uniformly continuous. Therefore, there exists δ such that $|\phi(a) - \phi(b)| < \epsilon$ as soon as $a, b \in \mathbb{R}$ and $|a - b| < \delta$. If $|x - y| < \delta$, this lets us bound the above by

$$|T(f)(x) - T(f)(y)| \leq \int_S \|f\|_\infty \epsilon\,dx,$$

where $S \subset \mathbb{R}$ is the support of $z \mapsto \phi(x-z) - \phi(y-z)$. Since ϕ is compactly supported, the size of this support can be uniformly bounded (in x and y) by a constant M. Thus, we can bound

$$|T(f)(x) - T(f)(y)| \leq M\|f\|_\infty \epsilon.$$

Summarizing, for any $\epsilon > 0$, we can find $\delta > 0$ such that $|T(f)(x) - T(f)(y)| < M\|f\|_\infty \epsilon$ if $|x - y| < \delta$. This shows that T is continuous. $\boxed{6, \mathrm{B}}$

(b) Next, we want to bound the norm of Tf. This can be achieved as follows: if $x \in \mathbb{R}$

$$\begin{aligned}|Tf(x)| &\leq \left|\int f(z)\phi(x-z)\,dz\right| \leq \int |f(z)|\,|\phi(x-z)|\,dz \\ &\leq \|f\|_\infty \int |\phi(x-z)|\,dz \leq \|f\|_\infty \|\phi\|_{L^1}.\end{aligned}$$

Therefore,

$$\|Tf\|_\infty \lesssim \|f\|_\infty \|\phi\|_{L^1},$$

which proves the boundedness of T. $\boxed{6, \mathrm{B}}$

(c) We just saw that the operator norm of T can be bounded by

$$\|T\|_{\mathcal{C}\to\mathcal{C}} \leq \|\phi\|_{L^1}.$$

We now claim that

$$\|T\|_{\mathcal{C}\to\mathcal{C}} = \|\phi\|_{L^1}.$$

In order to prove this equality, we need to find $f \in \mathcal{C}$ such that

$$\|Tf\|_\infty = \|\phi\|_{L^1}\|f\|_\infty.$$

Define the function F by:

* $F(x) = 0$ if $\phi(-x) = 0$
* $F(x) = 1$ if $\phi(-x) > 0$
* $F(x) = -1$ if $\phi(-x) < 0$.

Then
$$TF(0) = \int F(-x)\phi(x)\,dx = \int |\phi(x)|\,dx = \|\phi\|_{L^1}$$
by definition of F. However, F is in L^∞, but not in $\mathcal{C}$ if ϕ changes sign. This can be remedied by an approximation argument, finding a sequence of functions (f_n) in $\mathcal{C}$ such that
$$\|f_n\|_{L^\infty} = 1 \qquad \text{and} \qquad f_n \to F \quad \text{in } L^1.$$
Then
$$\lim_{n\to\infty} \frac{\|Tf_n\|_{L^\infty}}{\|f_n\|_{L^\infty}} = \frac{\|TF\|_{L^\infty}}{1} = \|\phi\|_{L^1},$$
which proves the desired identity. $\boxed{7, \text{C}}$

Marking scheme.

The marking for subquestions with high marks refers to a correct and detailed solution.

- A correct derivation with omissions of details in the derivation may lower the mark of a relative value of $20 - 40\%$, leaving $60 - 80\%$ of the full mark.
- Light mistakes that do not compromise the spirit of the derivation and that are due to lack of attention lower the relative value of $30 - 50\%$, leaving $50 - 70\%$ of the full mark.
- Mistakes on the substance of the result and/or important mistakes/omissions in the derivation lower the mark to zero or to a residual $20 - 40\%$ of the full mark if the solution maintains elements of interest.

Solution.

a.1) Consider

$$dX_t = \frac{1}{3}(X_t)^{1/3}dt + (X_t)^{2/3}dW_t,\ X_0 = x_0.$$

$\sigma(x) = (x)^{2/3}$. The equivalent Stratonovich SDE is obtained by changing the drift by

$$\frac{1}{3}x^{1/3} \to \frac{1}{3}x^{1/3} - \frac{1}{2}\sigma(x)\frac{d}{dx}\sigma(x) =$$

$$= \frac{1}{3}x^{1/3} - \frac{1}{2}(x)^{2/3}\frac{2}{3}x^{-1/3} = \frac{1}{3}(x)^{1/3} - \frac{1}{3}(x)^{1/3} = 0.$$

So the equivlent Stratonovich SDE has zero drift,

$$dX_t = X_t^{2/3} \circ dW_t.$$

a.2) Now we know that the Stratonovich SDE obeys the formal rules of calculus. Let's try to solve the SDE by separating variables:

$$\frac{dX_t}{X_t^{2/3}} = 1 \circ dW_t$$

Integrate both sides

$$\int_{X_0}^{X_t} \frac{dX}{X^{2/3}} = \int_0^t 1 \circ dW_t$$

leading to

$$3(X_t^{1/3} - X_0^{1/3}) = W_t$$

and therefore

$$X_t^{1/3} = \frac{W_t}{3} + X_0^{1/3}$$

Taking the cube on both sides, our solution is

$$X_t = \left({X_0}^{1/3} + \frac{W_t}{3}\right)^3.$$

b) Let's check this is correct. Write

$$X_t = \left({X_0}^{1/3} + \frac{W_t}{3}\right)^3 = Z_t^3, \quad Z_t = {X_0}^{1/3} + \frac{W_t}{3}.$$

Let's differentiate X_t as a function of Z_t using Ito's formula.

$$dX_t = 3Z^2dZ + \frac{1}{2}6Z\ dZdZ = 3Z^2dZ + 3Z\ dZdZ$$

i.e.

$$dX_t = 3Z^2dZ + \frac{1}{3}Z_tdt$$

As $X = Z^3$, we have $Z = X^{1/3}$ and

$$dX_t = 3X_t^{2/3} d \underbrace{\left({X_0}^{1/3} + \frac{W_t}{3} \right)}_{Z_t} + \frac{1}{3} X_t^{1/3} dt$$

or

$$dX_t = \frac{1}{3} X_t^{1/3} dt + X_t^{2/3} dW_t$$

which is our initial Ito SDE.

c) Consider

$$dX_t = \frac{1}{3}(X_t)^{1/3} dt + (X_t)^{2/3} dW_t, \ X_0 = 0.$$

Let's try a constant solution $X_t = k$, for a constant k. Given that $X_0 = 0$ and that the solution is constant in time, we need to have $k = 0$. Then $dX_t = dk = d0 = 0$ and the SDE reads

$$0 = \frac{1}{3}(0)^{1/3} dt + (0)^{2/3} dW_t$$

leading to the identity

$$0 = 0$$

so that the equation is satisfied and indeed $X_t = 0$ is a solution. Hence in the case $X_0 = 0$ we have at least two solutions: the previous solution we found

$$X_t = \left({X_0}^{1/3} + \frac{W_t}{3} \right)^3 = \left(\frac{W_t}{3} \right)^3.$$

The new solution we found is

$$X_t = 0.$$

The two solutions are clearly different.

It was reasonable not to expect uniqueness as the drift and diffusion coefficients $x^{1/3}$ and $x^{2/3}$ do not satisfy the Lipschitz plus linear growth condition. In particular, note that the coefficients do not admit first derivative in zero, as the derivative grows larger and larger as we approach zero from either direction. This means the growth near zero for cubic root functions is much stronger than linear. We therefore don't expect to have both existence and uniqueness to be guaranteed. Indeed, while we found existence, there is no uniqueness.

1.a *(5 marks, difficult, unseen, D)*

Writing
$$\frac{\partial}{\partial ct'} = \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial ct'}\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial ct} + \beta\gamma\frac{\partial}{\partial x}$$
and similarly $\partial_{x'} = \beta\gamma\partial_{ct} + \gamma\partial_x$ gives
$$\frac{\partial^2}{\partial ct'^2} = \gamma^2\left(\frac{\partial^2}{\partial ct^2} + 2\beta\frac{\partial^2}{\partial ct\partial x} + \beta^2\frac{\partial^2}{\partial x^2}\right)$$
and similarly for $\partial^2_{x'}$, so that
$$\boxed{\frac{\partial^2}{\partial ct'^2} - \frac{\partial^2}{\partial x'^2} = \gamma^2(1-\beta^2)\frac{\partial^2}{\partial ct^2} + \gamma^2(\beta^2-1)\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial ct^2} - \frac{\partial^2}{\partial x^2}}$$
As the other coordinates do not transform, $\partial_{y'} = \partial_y$ and $\partial_{z'} = \partial_z$, the desired statement results.

1.b *(5 marks, easy, A)*

This is trivial, $\boxed{\frac{1}{c}\dot{\phi} + \nabla\cdot\mathbf{A} = 0}$ and as a Lorentz scalar $\boxed{\text{it does not transform}}$.

1.c *(5 marks, medium, C)*

From $\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \nabla\phi$ follows $\nabla\cdot\mathbf{E} = -\frac{1}{c}\nabla\cdot\dot{\mathbf{A}} - \nabla^2\phi$. Using the Lorenz gauge, $\nabla\cdot\dot{\mathbf{A}} = -\frac{1}{c}\ddot{\phi}$ and therefore $\boxed{\nabla\cdot\mathbf{E} = \frac{1}{c^2}\ddot{\phi} - \nabla^2\phi}$.

1.d *(5 marks, medium, B)*

In the vacuum $\nabla\cdot\mathbf{E} = 4\pi\rho = 0$ and so we are looking for solutions of $\ddot{\phi}/c^2 = \nabla^2\phi$, which is solved by travelling waves of the form $\boxed{\phi(t,\mathbf{x}) = f(\omega t + \mathbf{k}\cdot\mathbf{x})}$ with dispersion relation $\omega = kc$. Any wave-like solution is accepted here.