Network Science
Spring 2024
Problem sheet 5 Solutions

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1. Consider the set of graphs generated by the configuration model with degree sequence, d $= (3, 1, 1, 1)$).
   (a) How many distinct matchings are there? A "matching" is a graph corresponding to a particular set of pairings of the 6 stubs.
   Solution: There are $(6-1)*(6-3) = 15$ matchings.
   (b) How many distinct graphs can be generated? What is the probability of generating each of these distinct graphs? For this exercise, two graphs should be considered equivalent if 1) their node degrees match the given degree sequence and 2) the number of edges between each distinct pair of nodes is the same in both graphs.
   Solution: There are 4 distinct graphs. There is 1 graph with no self loops which corresponds to 6 matchings, so the probability of generating this graph is $6/15 = 2/5$. There are three distinct graphs with 1 self-loop. Each of these corresponds to 3 matchings (there are 3 pairings between stubs on node 1 that lead to a self loop on node 1), so the probability of generating each of these distinct graphs is $3/15 = 1/5$

2. Now consider the configuration model applied to a degree sequence with $k_1 = 3$, $k_2 = 2$, $k_3 = 1$ and with total degree $K >= 6$ (and even).
   (a) What is the expected number of self-loops on node 2?
   Solution: From lecture, we know that $\langle l_{ii} \rangle = k_i(k_i - 1)/[2(K-1)]$. In this case, $k_i = 2$, so $\langle l_{22} \rangle = 1/(K-1)$
   (b) What is the probability that nodes 3 and 1 are linked?
   Solution: The probability that the stub on node 3 links to any other stub is $1/(K-1)$, so, with 3 stubs on node 1, the answer is, $3/(K-1)$.
   (c) What is the probability that nodes 1 and 2 are linked?
   Solution: Consider $P(n_1 \sim n_2)$, the probability nodes 1 and 2 are linked, and label the 2 stubs on node 2 as s_1 and s_2. Also let $s_1 \sim n_1$ indicate that stub 1 links to node 1 and $s_1 \nsim n_1$ indicate that stub 1 does not link to node 1. For nodes 1 and 2 to be linked we need either (1) $s_1 \sim n_1$ *or* (2) $s_1 \nsim n_1, s_2 \sim n_1$. So,

$$P(n_1 \sim n_2) = P(s_1 \sim n_1) + P(s_1 \nsim n_1, s_2 \sim n_1).$$

   From the main property of the configuration model, $P(s_1 \sim n_1) = 3/(K-1)$. Next, rewrite the joint probability as a conditional probability:

$$P(s_1 \nsim n_1, s_2 \sim n_1) = P(s_1 \nsim n_1 | s_2 \sim n_1)P(s_2 \sim n_1),$$

   where as before $P(s_2 \sim n_1) = 3/(K-1)$. For the conditional probability, since $s_2 \sim n_1$, there are $K-3$ stubs available of which $K-5$ will satisfy $s_1 \nsim n_1$. Putting everything together,

$$P(n_1 \sim n_2) = \frac{3}{(K-1)} + \frac{3(K-5)}{(K-1)(K-3)} = \frac{6(K-4)}{(K-1)(K-3)}$$

3. Consider the set of graphs with N nodes generated by the configuration model with a specified degree sequence. If node i has degree 2, what is $\langle m_i \rangle$ the expected number of multiedges node i will form in a graph? A multiedge here is 2 links between node i and another node. Provide an approximate result in terms of N and the the moments of the degree sequence ($\overline{k}$, $\overline{k^2}$, ...) when the total degree $K \gg 1$.
Solution: Say that node j has degree k_j and $i \neq j$. Then the probability that both stubs on node i connect to node j is $k_j(k_j-1)/[(K-1)(K-3)]$ where K is the total degree for the graph. Note that this expression is correct even if $k_j = 1$. Let $\gamma_{ij} = 1$ if there is a multiedge between nodes i and j and zero otherwise. Then,

$$< m_i >= \sum_{j \neq i} P(\gamma_{ij} = 1) = \sum_{j \neq i} k_j(k_j-1)/[(K-1)(K-3)]$$

where the sum is over all nodes in the graph other than node i. This can be rewritten as a sum over all nodes with a "correction",

$$< m_i >= \sum_{j=1}^{N} k_j(k_j-1)/[(K-1)(K-3)] - 2/[(K-1)(K-3)].$$

When $K \gg 1$ this can be approximated as

$$< m_i > \approx \sum_{j=1}^{N} k_j(k_j-1)/K^2.$$

This can be rewritten as a sum over the degrees using N_k, the number of nodes with degree k and $p_k = N_k/N$,

$$< m_i > \approx \sum_{k=1}^{k_{max}} N_k k(k-1)/K^2 = \sum_{k=1}^{k_{max}} p_k k(k-1)/(N\overline{k}^2).$$

This last expression simplifies to, $\langle m_i \rangle \approx (\overline{k^2} - \overline{k})/(N\overline{k}^2)$.

4. Consider graphs generated by the configuration model with a specified degree sequence, d $= (k_1, k_2, ..., k_N)$. Choose a random stub on a node and follow it to the node it connects to. What is $\tilde{\pi}_k$, the probability that the connecting node has degree $k+1$?
Solution: From lecture, we know that the probability that the connecting node has degree k is, $\pi_k \approx k p_k/\overline{k}$. The numerator is the total number of stubs connected to nodes with degree k divided by N. This now changes to the total number of stubs connected to nodes with degree $k+1$ divided by N. It follows that $\tilde{\pi}_k \approx (k+1)p_{k+1}/\overline{k}$.