MATH50001 Problems Sheet 8
Solutions

1a)

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$
$$= \frac{\partial^2}{\partial x^2} + \frac{1}{i}\frac{\partial}{\partial y}\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} = \Delta.$$

1b) Clearly

$$\Delta|f(z)|^2 = \Delta(u^2+v^2)$$
$$= 2u(u''_{xx}+u''_{yy}) + 2v(v''_{xx}+v''_{yy}) + 2((u'_x)^2 + (v'_x)^2 + (u'_y)^2 + (v'_y)^2).$$

Since f is holomorphic we have

$$\Delta u = u''_{xx} + u''_{yy} = 0 \quad \text{and} \quad \Delta v = v''_{xx} + v''_{yy} = 0.$$

Besides using the Cauchy-Riemann equations $u'_x = v'_y$ and $u'_y = -v'_x$ we find

$$\Delta|f(z)|^2 = 2\Big((u'_x)^2 + (-u'_y)^2 + (u'_y)^2 + (u'_x)^2\Big) = 4\Big((u'_x)^2 + (u'_y)^2\Big)$$
$$= 4\left|2\,\frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right)\right|^2 = 4\,|2\partial u/\partial z|^2 = 4\,|f'_z(z)|^2.$$

1c) It follows from the proof of **1.b** and the Cauchy-Riemann equations that

$$|f'(z)|^2 = (u'_x)^2 + (u'_y)^2 = u'_x v'_y - u'_y v'_x = \det\begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix}.$$

2. Harmonic conjugates.

a) For $u = x^3 - 3xy^2 - 2y$ we have $u'_x = 3x^2 - 3y^2$, $u''_{xx} = 6x$ and $u'_y = -6xy - 2$, $u''_{yy} = -6x$. Thus we have

$$u''_{xx} + u''_{yy} = 6x - 6x = 0$$

and it shows that u is harmonic.

Cauchy-Riemann equations imply

$$v'_y = u'_x = 3x^2 - 3y^2.$$

Integrating the latter w.r.t. y we find

$$v = 3x^2y - y^3 + F(x),$$

and differentiating it w.r.t. x we have

$$v_x = 6xy + F'(x) = -u'_y = 6xy + 2.$$

So $F'(x) = 2$ and $F(x) = 2x + c$, $c \in \mathbb{R}$. This implies

$$\begin{aligned} v &= 3x^2y - y^3 + 2x + c, \\ f &= u + iv = x^3 - 3xy^2 - 2y + 3ix^2y - iy^3 + 2ix + ic \\ &= (x+iy)^3 + 2i(x+iy) + ic \end{aligned}$$

or $f(z) = z^3 + 2iz + ic$.

b) If $u = x - xy$, then $u''_{xx} = 0$, $u''_{yy} = 0$, and thus u is harmonic.
Using the Cauchy-Riemann equations we find $v'_y = u'_x = 1 - y$ and integrating this w.r.t. y we obtain

$$v = y - y^2/2 + F(x).$$

Differentiating the latter w.r.t. x we arrive at

$$v'_x = F'(x) = -u'_y = x$$

and therefore $F(x) = x^2/2 + c$, $v = y - y^2/2 + x^2/2 + c$;
$f = u + iv = x - xy + iy + i\dfrac{x^2}{2} - i\dfrac{y^2}{2} + ic = (x+iy) + i\dfrac{(x+iy)^2}{2} + ic$
or $f = z + iz^2/2 + ic$, $c \in \mathbb{R}$.

c) For any $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} \Delta u = u''_{xx} + u''_{yy} &= (e^x \cos y\,(x+1))'_x - (ye^x \sin y)'_x \\ &+ (xe^x\,(-\sin y))'_y - (e^x(\sin y + y\cos y))'_y \\ &= e^x \cos y\,(x+1) + e^x \cos y - ye^x \sin y \\ &- xe^x \cos y - e^x(\cos y + \cos y - y \sin y) = 0. \end{aligned}$$

Using the C-R equation $u'_x = v'_y$ and integrating by parts we derive

$$\begin{aligned} v = \int u'_x\,dy &= \int \left(e^x \cos y\,(x+1) - y\,e^x \sin y\right)\,dy \\ &= e^x \sin y\,(x+1) + ye^x \cos y - \int e^x \cos y\,dy \\ &= e^x \sin y\,(x+1) + ye^x \cos y - e^x \sin y + C(x). \end{aligned}$$

The second C-R equation $v_x = -u'_y$ gives

$$\begin{aligned} &e^x \sin y\,(x+1) + e^x \sin y + ye^x \cos y - e^x \sin y + C'(x) \\ &= xe^x\,\sin y + e^x(\sin y + y\cos y). \end{aligned}$$

This implies $C'(x) = 0$ and thus $C(x) = c = \mathrm{const} \in \mathbb{R}$.

Finally we obtain

$$v(x,y) = xe^x \sin y + ye^x \cos y + c.$$

Moreover,

$$\begin{aligned} f(z) &= u + iv = xe^x \cos y - ye^x \sin y + i\,(xe^x \sin y + ye^x \cos y + c) \\ &= (x+iy)\,e^x(\cos y + i\sin y) + ic = (x+iy)\,e^{x+iy} + ic = z\,e^z + ic, \end{aligned}$$

where $c \in \mathbb{R}$. Then the equation

$$f(i\pi) = i\pi e^{i\pi} + ic = -i\pi + ic = 0 \quad \Longrightarrow \quad c = \pi.$$

Answer: $f(z) = ze^x + i\pi$.

3. We have

$$\begin{aligned} 0 = \Delta g(x,y) = \Delta|f(z)|^2 = 4|f_z'(z)|^2 \quad &\Rightarrow \quad f_z'(z) = 0 \\ &\Rightarrow \quad u_x' = v_x' = u_y' = v_y' \equiv 0. \end{aligned}$$

This implies $f(z) \equiv$ constant.

4. Since u is harmonic we have $\Delta u = 0$. Therefore

$$\Delta u^2 = 2(\Delta u)\,u + 2\,\nabla u \cdot \nabla u = 2|\nabla u|^2 = 2\left((u_x')^2 + (u_y')^2\right) \geq 0.$$

Moreover, since both u_x' and u_y' are harmonic we also have

$$\Delta^2(u^2) = 2\Delta|\nabla u|^2 = 2\left(\Delta(u_x')^2 + \Delta(u_y')^2\right) \geq 0.$$

5. We first check that $u_x' = v_y'$. Indeed, since φ and ψ are harmonic we obtain

$$\begin{aligned} u_x' &= \varphi_{xx}''\,\varphi_y' + \varphi_x'\,\varphi_{yx}'' + \psi_{xx}''\,\psi_y' + \psi_x'\,\psi_{yx}'' \\ &= -\varphi_{yy}''\,\varphi_y' + \varphi_x'\,\varphi_{yx}'' - \psi_{yy}''\,\psi_y' + \psi_x'\,\psi_{yx}'' \\ &= -\frac{1}{2}\,\left((\varphi_y')^2\right)_y' + \frac{1}{2}\,\left((\varphi_x')^2\right)_y' - \frac{1}{2}\,\left((\psi_y')^2\right)_y' + \frac{1}{2}\,\left((\psi_x')^2\right)_y' = v_y'. \end{aligned}$$

The second C-R equation says $u_y' = -v_x'$ and we have

$$\begin{aligned} u_y' &= \varphi_{xy}''\,\varphi_y' + \varphi_x'\,\varphi_{yy}'' + \psi_{xy}''\,\psi_y' + \psi_x'\,\psi_{yy}'' \\ &= \varphi_{xy}''\,\varphi_y' - \varphi_x'\,\varphi_{xx}'' + \psi_{xy}''\,\psi_y' - \psi_x'\,\psi_{xx}'' \\ &= \frac{1}{2}\,\left((\varphi_y')^2\right)_x' - \frac{1}{2}\,\left((\varphi_x')^2\right)_x' + \frac{1}{2}\,\left((\psi_y')^2\right)_x' - \frac{1}{2}\,\left((\psi_x')^2\right)_x' = -v_y'. \end{aligned}$$

6)

The the mapping $w = f(z)$ must satisfy the Cross-Ratios Möbius Transformation. We have

$$\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right) = \left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right)$$

where $z_1 = 2$, $z_2 = i$ and $z_3 = -1$ and $w_1 = 2i$, $w_2 = -$, and $w_3 = -2i$, respectively. This implies

$$\left(\frac{z-2}{z+1}\right)\left(\frac{i+1}{i-2}\right) = \left(\frac{w-2i}{w+2i}\right)\left(\frac{-2+2i}{-2-2i}\right),$$

$$\left(\frac{z-2}{z+1}\right)\left(-\frac{1+3i}{5}\right) = \left(\frac{w-2i}{w+2i}\right)(-i) \quad \Rightarrow$$

$$w = \frac{(16-2i)\,z + (-2+4i)}{(1-2i)\,z - (2+11i)}.$$

6′)

The the mapping $w = f(z)$ must satisfy the Cross-Ratios Möbius Transformation. We have

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where $z_1 = 2$, $z_2 = 1+i$ and $z_3 = 0$ and $w_1 = 1$, $w_2 = i$, and $w_3 = -i$, respectively. This implies

$$\left(\frac{z-2}{z}\right)\left(\frac{1+i}{1+i-2}\right) = \left(\frac{w-1}{w+i}\right)\left(\frac{i-(-i)}{i-1}\right),$$

$$\frac{z-2}{z}(-i) = \frac{w-1}{w+i}\,\frac{2i}{i-1}, \quad \Rightarrow \quad \frac{z-2}{z} = \frac{w-1}{w+i}(1+i) \quad \Rightarrow$$

$$(z-2)(w+i) = z(1+i)\,(w-1) \quad \Rightarrow \quad w(z-2-z-iz) = -z(1+i)-i(z-2)$$

Finally we have

$$w = \frac{(1+2i)z-2i}{iz+2}.$$

7. We first find where f maps the boundary of the set $\operatorname{Im} z > 0$. It is enough to check it with three points, for example, $z_1 = -1$, $z_2 = 0$, $z_3 = 1$. Such points map to

$$w_1 = \frac{-1-i}{-1+i} = i, \qquad w_2 = -1, \qquad w_1 = \frac{1-i}{1+i} = -i.$$

This implies that the real line $\operatorname{Im} z = 0$ maps onto the unit circle $|w| = 1$. Now we only need to find out if the image of $\operatorname{Im} z > 0$ is $w : |w| < 1$ or $w : |w| > 1$. Clearly if we take $w = i$ we obtain

$$f(i) = 0.$$

Therefore, $\Omega = \{w \in : |w| < 1\}$.

8. Let

$$w = f(z) = \frac{az + b}{cz + d}.$$

Since the image of $z_1 = -2i$ equals $w_1 = 0$ we can choose $a = 1$ and $b = 2i$ (not that all the coefficients a, b, c and d could be chosen up to a multiplication by the same non-zero complex number). The from the $f(0) = 1$ we obtain

$$\frac{2i}{d} = 1 \quad \Longrightarrow \quad d = 2i.$$

Finally, the condition $f(-2) = i$ implies

$$\frac{-2 + 2i}{-2c + 2i} = i$$

which defines $c = -1$.

Answer: $f(z) = \frac{z+2i}{-z+2i}$. The points $z_1 = -2i$, $z_2 = -2$ and $z_3 = 0$ belong to the circle

$$C_1 = \{z : |z + 1 + i| = \sqrt{2}\}$$

oriented anticlockwise. The same is true for the points $w_1 = 0$, $w_2 = i$ and $w_3 = 1$ that are lying on the circle

$$C_2 = \left\{ z : \left| z - \frac{1}{2} - \frac{i}{2} \right| = \frac{1}{\sqrt{2}} \right\}.$$

which is also oriented anticlockwise. This implies that the D_1 maps onto D_2.

Alternatively, in order to show that the D_1 maps inside D_2 we can, for example, take $z = -1 - i$ whose image is $\frac{1}{5} + i\frac{2}{5} \in D_2$.

9. Let $z_1 = -2$, $z_2 = -1 - i$ and $z_3 = 0$ onto the points $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$.

If $w = f(z)$ is a Möbius transformation that maps the distinct points (z_1, z_2, z_3) into the distinct points (w_1, w_2, w_3) respectively, then

$$\left(\frac{z - z_1}{z - z_3} \right) \left(\frac{z_2 - z_3}{z_2 - z_1} \right) = \left(\frac{w - w_1}{w - w_3} \right) \left(\frac{w_2 - w_3}{w_2 - w_1} \right),$$

for all z. Therefore, since $z_1 = -2$, $z_2 = -1 - i$ and $z_3 = 0$ onto the points $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$

$$\frac{z-(-2)}{z-0} \cdot \frac{-1-i-0}{-1-(-2)} = \frac{w-(-1)}{w-1} \cdot \frac{0-1}{0-(-1)},$$

$$\frac{z+2}{z} \cdot \frac{-1-i}{1-i} = \frac{w+1}{1-w}.$$

Since

$$\frac{-1-i}{1-i} = \frac{1}{i}$$

we have

$$\frac{z+2}{iz} = \frac{w+1}{1-w}.$$

$$(z+2)(1-w) = iz(w+1); \quad \Longrightarrow \quad z + 2 - zw - 2w = izw + iz.$$

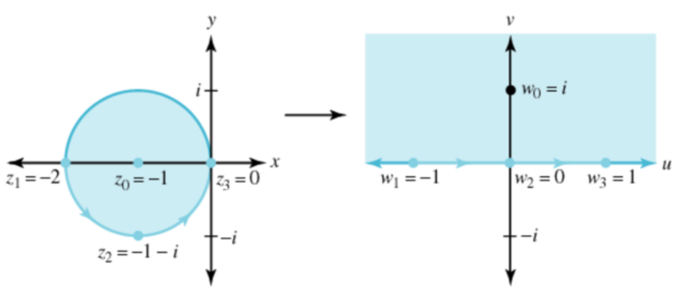
Finally

$$\Longrightarrow \quad w(iz + z + 2) = z + 2 - iz; \quad \Longrightarrow \quad w = \frac{z(1-i)+2}{z(1+i)+2}.$$

There are two possibilities to check that this transformation maps the disk $|z+1| < 1$ onto the upper half plane.

1. The points $z_1 = -2$, $z_2 = -1 - i$ and $z_3 = 0$ that belong to the circle $|z+1| = 1$, have their images on the real $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$, respectively. Because both ordered triple of of z-points and w-points have anticlockwise orientation we obtain that the disk $|z+1| < 1$ onto the upper half plane.

2. The transformation f maps the point $z_0 = -1$ (that is inside the disc $|z+1| = 1$) to $T(-1) = w_0 = i \in \{z : \operatorname{Im} z > 0\}$.



10. Let $f(z) = z^{\pi/\alpha}$. Then

$$\left\{f(re^{i\theta}) : r > 0,\ 0 < \theta < \alpha\right\} = \left\{r^{\pi/\alpha}\, e^{i\theta\pi/\alpha} : r > 0,\ 0 < \theta < \alpha\right\}$$
$$= \left\{\rho\, e^{i\varphi} : \rho > 0,\ 0 < \varphi < \pi\right\}.$$

11. First transform the sector onto the upper half-plane $\{z : \operatorname{Im} z > 0\}$ using $z \to z^4$. Then find a Möbius transformation mapping the half-plane to the disc. This is not unique, but one way is to map 0 (on the half-plane) to -1 (on the circle), and to map the inverse points i and $-i$ relative to the half-plane to the inverse points 1 and ∞ relative to the circle. We obtain the Möbius transformation $z \to (3z - i)/(z + i)$. The required conformal mapping is

$$w = f(z) = \frac{3z^4 - i}{z^4 + i}.$$