Exercise 8.1. Consider a discrete metric space $(X, \mathrm{d}_{\text{disc}})$, that is d_{disc} is a discrete metric on X. Show that d_{disc} induces the discrete topology on X.

Hint: Identify the open sets in the discrete metric.

Solution: Let $A \subset X$. Take any point $x \in A$. Recall that in the discrete metric we have $B_{1/2}(x) = \{x\}$, and so $B_{1/2}(x) \subset A$. Thus A is an open set in $(X, \mathrm{d}_{\text{disc}})$. Since A is arbitrary, we conclude that any set in $(X, \mathrm{d}_{\text{disc}})$ is open. By definition, these sets form the discrete topology on X.

Exercise 8.2. Let (X, τ_X) be a topological space, $Y \subset X$, and

$$\tau_Y = \{U \cap Y \mid U \in \tau_X\}.$$

Show that τ_Y is a topology on Y.

Hint: you need to verify the three properties for the topology, and use basic relations for unions and intersections of sets.

Solution: We must check that the collection τ_Y satisfies 3 properties of a topology on Y.

(T1) Since $U = \emptyset \in \tau_X$ and $\emptyset \cap Y = \emptyset$, we have that $\emptyset \in \tau_Y$. Also, since $X \in \tau_X$ and $X \cap Y = Y$, we have that $Y \in \tau_Y$.

(T2) Let V_α be arbitrary elements of τ_Y, for α in some set I. We need to show that $\cup_{\alpha\in I} V_\alpha$ belongs to τ_Y. To show that, first we note that by the definition of τ_Y, since for every $\alpha \in I$, $V_\alpha \in \tau_Y$, there is $U_\alpha \in \tau_X$ such that

$$V_\alpha = U_\alpha \cap Y.$$

By the distributive property of the union and intersection, we have

$$\cup_{\alpha\in I} V_\alpha = \cup_{\alpha\in I}(U_\alpha \cap Y) = (\cup_{\alpha\in I} U_\alpha) \cap Y.$$

Now, since $U_\alpha \in \tau_X$, for every $\alpha \in I$, and τ_X is a topology on X, we conclude that $\cup_{\alpha\in I} U_\alpha \in \tau_X$. Then, $(\cup_{\alpha\in I} U_\alpha) \cap Y$ belongs to τ_Y. By the above equation, we conclude that $\cup_{\alpha\in I} V_\alpha$ belongs to τ_Y.

(T3) The argument is similar to the one for (T2).

Let V_α be arbitrary elements of τ_Y, for α in a finite set I. We need to show that $\cap_{\alpha\in I} V_\alpha$ belongs to τ_Y. To show that, first we note that by the definition of τ_Y, since for every $\alpha \in I$, $V_\alpha \in \tau_Y$, there is $U_\alpha \in \tau_X$ such that

$$V_\alpha = U_\alpha \cap Y.$$

We have

$$\cap_{\alpha\in I} V_\alpha = \cap_{\alpha\in I}(U_\alpha \cap Y) = (\cap_{\alpha\in I} U_\alpha) \cap Y.$$

Now, since $U_\alpha \in \tau_X$, for every $\alpha \in I$, I is a finite set, and τ_X is a topology on X, we conclude that $\cap_{\alpha\in I} U_\alpha \in \tau_X$. Then, $(\cap_{\alpha\in I} U_\alpha) \cap Y$ belongs to τ_Y. By the above equation, we conclude that $\cap_{\alpha\in I} V_\alpha$ belongs to τ_Y.

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Questions marked with $*$ are optional

Exercise 8.3. Let τ_{Eucl} be the Euclidean topology on $\mathbb{R}$, that is τ_{Eucl} is the collection of all open sets in $(\mathbb{R}, \mathrm{d}_1)$. Show that the collection

$$\{U \times V \mid U \in \tau_{\text{Eucl}}, V \in \tau_{\text{Eucl}}\}.$$

is not a topology on $\mathbb{R} \times \mathbb{R}$. Is condition T2 satisfied? How about condition T3?

Hint: Consider the union of two boxes.

Solution: In order to show that the collection in the exercise is not a topology, it is enough to show that one of the three properties for the topology is not satisfied. The empty set can be written as $\emptyset \times \emptyset$, so it belongs to the above set. Also, the whole set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, so the whole set belongs to the collection. These show that T1 holds.

We claim that property T2 does not hold. To see that, consider the sets

$$(0,2) \times (0,2), \quad \text{and} \quad (1,3) \times (1,3).$$

Both of the above sets belong to the collection, since they are of the form $U \times V$ for some open sets in $\mathbb{R}$. However, their union does not belong to that collection. That is because, there are not open sets U and V in $\mathbb{R}$ such that

$$U \times V = ((0,2) \times (0,2)) \cup ((1,3) \times (1,3)).$$

That is because if the above relation holds, we must have

$$(0,3) \subset U, \quad \text{and} \quad (0,3) \subset V,$$

and hence

$$(0,3) \times (0,3) \subset U \times V \subset ((0,2) \times (0,2)) \cup ((1,3) \times (1,3)),$$

which is not true.

Property T3 is true. Indeed let U_1, V_1, U_2, V_2 be open sets in $\mathbb{R}$. That is because a point

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$$

if and only if both $x \in U_1 \cap U_2$ and $y \in V_1 \cap V_2$. Thus,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Since U_1 and U_2 are open in $\mathbb{R}$, $U_1 \cap U_2$ is open in $\mathbb{R}$. Similarly, Since V_1 and V_2 are open in $\mathbb{R}$, $V_1 \cap V_2$ is open in $\mathbb{R}$. The above equation shows that $(U_1 \times V_1) \cap (U_2 \times V_2)$ belongs to the collection in the exercise. This shows that T3 holds for 2 sets. Then, one may use induction to show that T3 holds for any finite collection.

Exercise 8.4. Let (A, τ) be a topological space, and let S and T be subsets of A. The following properties hold:

(i) if $S \subset T$ then $S^\circ \subset T^\circ$,

(ii) S is open in A if and only if $S = S^\circ$,

(iii)* S° is the largest open set contained in S.

Hint: Compare this to the corresponding exercise for the metric spaces, and see if those proofs can be adapted here.

Solution: (i) If $x \in S^\circ$, then there is an open set U in A such that $x \in U$ and $U \subset S$. As $S \subset T$, we must have $U \subset T$. Thus, $x \in U$, U is open, and $U \subset T$. This shows that $x \in T^\circ$.

(ii) First assume that S is open in A. By the definition of the interior of a set, we always that $S^\circ \subset S$. We need to show that $S \subset S^\circ$. Let $x \in S$ be an arbitrary point. Since S is an open set, there is $U \in \tau$ such that $x \in U$ and $U \subset S$. This immediately shows that $x \in S^\circ$.

Now assume that $S = S^\circ$. Let x be an arbitrary point in S. Since $S = S^\circ$, $x \in S^\circ$. By the definition of the interior of a set, there is $U_x \in \tau$ such that $x \in U_x$ and $U_x \subseteq S$. As $x \in S$ was arbitrary, we conclude that

$$S = \cup_{x \in S} U_x$$

Now, since every $U_x \in \tau$, by property T2 of topology, their union also belongs to τ. Thus, $S \in \tau$, in other words, S is open in A.

(iii) Now we show that S° is the largest open set contained in S. To see that, let Ω be an arbitrary open set contained in S. We need to show that $\Omega \subset S^\circ$. Fix an arbitrary $z \in \Omega$. Since $z \in \Omega$, $\Omega \in \tau$, and $\Omega \subset S$, we conclude that $x \in S^\circ$. Since $z \in \Omega$ was arbitrary, we conclude that $\Omega \subseteq S^\circ$.

Exercise 8.5. Let (X, d) be a metric space, and let τ be the topology on X induced from the metric d. Show that (X, τ) is a Hausdorff topological space.

Hint: For a pair of distinct points, consider the distance between those points, and use that to define balls around each of the two points, so that they do not intersect.

Solution: Let $x, y \in X$, $x \neq y$. Then $\mathrm{d}(x, y) = \epsilon > 0$. We claim that

$$B_{\epsilon/3}(x) \cap B_{\epsilon/3}(y) = \emptyset.$$

Assume in the contrary that there exists z in the left hand side of the above equation. Then, $\mathrm{d}(x, z) < \epsilon/3$ and $\mathrm{d}(y, z) < \epsilon/3$. By the triangle inequality,

$$\epsilon = \mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y) < 2\epsilon/3,$$

which is a contradiction.

Thus, we have disjoint open sets $B_{1/3}(x)$ and $B_{1/3}(y)$, with $x \in B_{1/3}(x)$ and $y \in B_{1/3}(y)$. This shows that X with the induced topology is a Hausdorff space.

Exercise 8.6. Assume that the topological spaces (X, τ_X) and (Y, τ_Y) are topologically equivalent. Then, (X, τ_X) is Hausdorff if and only if (Y, τ_Y) is Hausdorff.

Hint: By the hypothesis, there is a homeomorphism from (X, τ_X) to (Y, τ_Y). Use this map to send pairs of distinct open sets to pairs of distinct open sets.

Solution: Let $f : X \to Y$ be a homeomorphism. First assume that (Y, τ_Y) is Hausdorff. Let $x, y \in X$ with $x \neq y$. Then, since $f : X \to Y$ is injective, $f(x) \neq f(y)$. Since Y is Hausdorff, there are open sets U and V in Y such that

$$f(x) \in U, \quad f(y) \in V, \quad U \cap V = \emptyset.$$

Since f is continuous, the pre-images $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Clearly $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. We also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. If the intersection is not empty, there is $z \in f^{-1}(U) \cap f^{-1}(V)$, and hence $f(z) \in U \cap V$, which is not possible. As x and y were arbitrary distinct elements in X, this shows that (X, τ_X) is Hausdorff.

For the other direction, one can repeat the above argument, using the inverse homeomorphism $f^{-1} : Y \to X$.

Unseen Exercise. (unseen) Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $f : X \to Y$ be a continuous and injective map. Then, if Y is Hausdorff, then X is Hausdorff.

Solution: Let $x, y \in X$, $x \neq y$. Then, since $f : X \to Y$ is injective, $f(x) \neq f(y)$. Since Y is Hausdorff, there are open sets U and V in Y such that

$$f(x) \in U, \quad f(y) \in V, \quad U \cap V = \emptyset.$$

Since f is continuous, the pre-images $f^{-1}(U)$, $f^{-1}(V)$ are open in X. Clearly $x \in f^{-1}(U)$, $y \in f^{-1}(V)$. We also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, since otherwise, any $z \in f^{-1}(U) \cap f^{-1}(V)$ implies that $f(z) \in U \cap V$, which is not possible.