4.4 Rearrangement of Series

Warning. Do not rearrange series and sum them in a different order unless you are a professional who knows what you are doing and can *prove* the result is the same.

> Without a license you can rearrange partial sums only; they are finite so $a+b = b+a$ makes them behave. Infinite sums are more difficult beasts.

Example 4.28. $\sum(-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$

either this " = " $(1-1) + (1-1) + \cdots = 0$,

or this " = " $1 - (1-1) + (1-1) + \cdots = 1$.

A better (convergent) example:

Example 4.29. Let $a_n = \frac{(-1)^{n+1}}{n}$ so that $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent by the Alternating Series Test.

Exercise 4.30. $\sum_{n=1}^{\infty} a_n > \frac{1}{2}$.

(In fact $\sum a_n = \log 2$ can be seen by substituting $x = 1$ into the Taylor series $\log(1+x) = x - \frac{x^2}{2} + \dots$ even though $x = 1$ is on its radius of convergence.)

If you've not seen this integrate $\frac{1}{1+x} = 1 - x + x^2 - \dots$

Reorder $\sum a_n$ as follows:

$$\begin{array}{rcccccccc}
 & 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\
 & & -\frac{1}{2} & & -\frac{1}{4} & & -\frac{1}{6} & & \dots \\
= & 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\
-\frac{1}{2}\Big[& & 1 & & +\frac{1}{2} & & +\frac{1}{3} & & \dots\Big]
\end{array}$$

Terms with even denominator appear only in bottom row $(\times\frac{-1}{2})$.

Terms with odd denominator appear in the top row $(\times 1)$ and bottom row $(\times\frac{-1}{2})$, so $(\times\frac{1}{2})$ in total.

So we get $\frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots] = \frac{1}{2}\sum a_n$.

Thus reordering the sum can lead to a different result.

This happened because when I reordered I went along the bottom row twice as fast as I went along the top row (check you see this!). Since the top and bottom rows are both series which diverge to ∞, I'm computing $\infty - \infty$, and this can give me anything depending on how quickly I add up the first ∞ and how quickly I take away the second.

In fact we can rearrange $\sum \frac{(-1)^{n+1}}{n}$ to converge to anything we like.

Example 4.31. Rearrange $\sum \dfrac{(-1)^{n+1}}{n}$ to make it converge to your favourite number.

Pick your favourite number; call it π say. Then reorder the sum as follows.

1. Take only odd terms $a_{2n+1} > 0$ until their sum is $> \pi$. We can do this as $1 + \frac{1}{3} + \dots$ diverges to ∞!
2. Now take only even terms $a_{2n} < 0$ until sum gets $< \pi$.
3. Repeat 1 and 2 to fade.

We can do each step because $\sum a_{2n+1} \to +\infty$ and $\sum a_{2n} \to -\infty$. If we did not eventually use all the terms a_n then we must eventually only take terms of one type (without loss of generality the even terms < 0), but the even terms sum to $-\infty$ so our sum eventually drops below π and we start taking odd terms > 0 again.

Finally we sketch the proof that this reordered sum converges to π. Since $a_n \to 0$,

Draw picture with π and its ϵ corridor!

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n| < \epsilon. \tag{$*$}$$

Go down the reordered sequence to a point where we have used all $a_1, a_2, \dots, a_N$, and then further to the point where the partial sum crosses π. At this point, $(*)$ holds, so the sum is within ϵ of π. From this point on the sum is always within ϵ of π by design and by $(*)$. But this is the dictionary definition of the sum converging to π.

Definition (Rearrangement of a sequence) Given a bijection $n\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$, define $b_i := a_{n(i)}$. Then $(b_i)_{i\geq 1}$ is a *rearrangement* or *reordering* of $(a_n)_{n\geq 1}$.

Then the method of Example 4.31 shows that if (a_n) is any sequence such that

- $a_n \to 0$,
- $\displaystyle\sum_{n\colon a_n \geq 0} a_n \to +\infty$,
- $\displaystyle\sum_{n\colon a_n < 0} a_n \to -\infty$,

then we can rearrange the series $\sum a_n$ to make it converge to *any* real number we like by the algorithm above.

And we can make it diverge to $+\infty$ or to $-\infty$.
For instance, here's an algorithm to make it diverge to $+\infty$:

1. Pick only $a_n \geq 0$ terms until the partial sum is > 1,
2. Now pick only $a_n < 0$ terms until the partial sum is < 0,
3. Pick only $a_n \geq 0$ terms until the partial sum is > 2,
4. Now pick only $a_n < 0$ terms until the partial sum is < 1,

$$\vdots$$

$2k-1$. Pick only $a_n \geq 0$ terms until the partial sum is $> k$,

$2k$. Now pick only $a_n < 0$ terms until the partial sum is $< k-1$,

$$\vdots$$

> **Exercise 4.32.** Show this is a reordering and the sum diverges to $+\infty$.

> **Exercise 4.33.** If (a_n) is a sequence such that
>
> - $a_n \to 0$,
> - $\displaystyle\sum_{n\colon a_n \geq 0} a_n \to +\infty$,
> - $\displaystyle\sum_{n\colon a_n < 0} a_n$ converges,
>
> then any reordering of $\sum a_n$ will diverge to $+\infty$.

The "good case" is when

- $\displaystyle\sum_{n\colon a_n \geq 0} a_n$ converges,
- $\displaystyle\sum_{n\colon a_n < 0} a_n$ converges,

which imply $a_n \to 0$ of course. Together these are equivalent to $\sum_n a_n$ being absolutely convergent, and in this case any reordering will give the same sum.

Theorem 4.34

$\sum a_n$ is absolutely convergent $\iff$ (1) + (2) $\implies$ (3) + (4), where

(1) $\sum_{a_n \geq 0} a_n$ is convergent (to A say),

(2) $\sum_{a_n < 0} a_n$ is convergent (to B say),

(3) $\sum a_n = A + B$,

(4) $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n).

Idea: $\sum |a_n|$ is convergent so has a significant finite part and then a small "insignificant" tail. Any reordering covers all the finite part after finitely many terms, and then all that remains is insignificant: just a reordering of part of the tail.

Proof. Let $p_1, p_2, p_3, \ldots$ be the nonnegative $a_n \geq 0$ (so p_i is the ith nonnegative element of the sequence (a_n)).

Similarly let $n_1, n_2, n_3, \ldots$ be the negative $a_n < 0$.

Suppose $\sum a_n$ is absolutely convergent, and set $R := \sum_n |a_n|$. For any $n \in \mathbb{N}_{>0}$ the partial sum of the p_i satisfies

$$\sum_{i=1}^{n} p_i \;\leq\; \sum_{i=1}^{N} |a_i| \;\leq\; R,$$

for any N sufficiently large that $\{p_1, \ldots, p_n\} \subseteq \{a_1, \ldots, a_N\}$. Therefore the partial sums of the p_i are monotonically increasing, bounded above and so convergent (to A say), proving (1).

Similarly the partial sums of the n_i are monotonically decreasing, bounded below and so convergent (to B say), proving (2).

So if we fix any $\epsilon > 0$, then

$$\exists\, N_1 \text{ such that } n \geq N_1 \implies A - \epsilon < \sum_{i=1}^{n} p_i \leq A, \tag{A}$$

$$\exists\, N_2 \text{ such that } n \geq N_2 \implies B < \sum_{i=1}^{n} n_i < B + \epsilon. \tag{B}$$

In particular, by monotonicity,

$$0 \leq \sum_{i \in I} p_i < \epsilon \quad \text{for any } I \subset \{N_1 + 1,\, N_1 + 2,\, \ldots\}, \tag{C}$$

$$-\epsilon < \sum_{j\in J} n_j < 0 \quad \text{for any } J \subset \{N_2+1,\, N_2+2,\, \dots\}. \tag{D}$$

Using (A-D) we next show that any rearrangement (b_m) of (a_n) sums to $A+B$. This will prove (3) and (4).

Pick N is sufficiently large that both $\{p_1,\dots,p_{N_1}\}$ and $\{n_1,\dots,n_{N_2}\}$ are subsets of $\{b_1,\dots,b_N\}$. (I.e. go far enough down the sequence (b_m) that we've included all the "significant" p_i and n_j.) Then write the complement as $\{p_i\}_{i\in I}\cup\{n_j\}_{j\in J}$, where I is a set of indices $> N_1$ and J is a set of indices $> N_2$.

Hence $\forall n \geq N$,

$$\begin{aligned}
\left|\sum_{i=1}^{n} b_i - (A+B)\right| &= \left|\sum_{i=1}^{N_1} p_i - A + \sum_{j=1}^{N_2} n_j - B + \sum_{i\in I} p_i + \sum_{j\in J} n_j\right| \\
&\leq \left|\sum_{i=1}^{N_1} p_i - A\right| + \left|\sum_{j=1}^{N_2} n_j - B\right| + \sum_{i\in I} p_i + \sum_{j\in J} |n_j| \\
&< \epsilon+\epsilon+\epsilon+\epsilon
\end{aligned}$$

by (A), (B), (C) and (D) respectively.

Finally we prove that (1)+(2) $\implies \sum |a_n|$ is convergent. We fix $\epsilon > 0$ and use the same N_1, N_2, N as above so that $\forall n \geq N$, $\{a_1,\dots,a_n\}$ contains both $\{p_1,\dots,p_{N_1}\}$ and $\{n_1,\dots,n_{N_2}\}$. Therefore

$$\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{N'} p_i - \sum_{i=1}^{N''} n_i,$$

where $N' \geq N_1$ and $N'' \geq N_2$. Applying (A) and (B) to the RHS then gives

$$(A-\epsilon) - (B+\epsilon) < \sum_{i=1}^{n} |a_i| \leq A - B,$$

so $\sum |a_i|$ converges to $A-B$. □

4.5 Power Series

Let $[0,\infty]$ denote the set $[0,\infty)\cup\{+\infty\}$.

Theorem 4.35: Radius of Convergence

Fix a real or complex series (a_n) and consider the series $\sum a_n z^n$ for $z \in \mathbb{C}$.

Then $\exists R \in [0, \infty]$ such that

- $|z| < R \implies \sum a_n z^n$ is absolutely convergent, and
- $|z| > R \implies \sum a_n z^n$ is divergent.

Proof. Let $S = \{|z| : a_n z^n \to 0\}$, nonempty since $0 \in S$. Then define

$$R = \begin{cases} \sup S & \text{if } S \text{ bounded,} \\ \infty & \text{if } S \text{ unbounded.} \end{cases}$$

Now suppose $|z| < R$. Since $|z|$ not an upper bound for S there exists w such that $|w| > |z|$ and $a_n w^n \to 0$. In particular $|a_n w^n|$ is bounded by some A for all n. Thus

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n.$$

Therefore by comparison with the convergent series $\sum \left|\frac{z}{w}\right|^n$ (recall $\left|\frac{z}{w}\right| < 1$) we find $\sum |a_n z^n|$ is convergent.

On the other hand, if $|z| > R$ then $a_n z^n \not\to 0$ as $n \to \infty \implies \sum a_n z^n$ diverges. □

Notice how simple this was. If $|a_n w^n|$ is just bounded (so nowhere near convergent!) then $\sum |a_n z^n|$ is convergent for $|z| < |w|$ because $\left(\frac{|z|}{|w|}\right)^n$ decays exponentially as $n \to \infty$.

Stepping inside the radius of convergence makes $|a_n z^n|$ much smaller.

Remark 4.36. R is called the *radius of convergence* of $\sum a_n z^n$. Note we are not saying anything about its behaviour on the circle $|z| = R$.

Exercise 4.37. Consider the sequences

(a) $a_n = \frac{1}{n^2}$,

(b) $a_n = \frac{1}{n}$,

(c) $a_n = 1$.

Show their power series $\sum a_n z^n$ all have radius of convergence $R = 1$, and on $|z| = 1$ their behaviour is as follows,

(a) convergent everywhere on $|z| = 1$,
(Absolutely convergent because $\sum \frac{1}{n^2} < \infty$.)

(b) convergent somewhere,
(Convergent at $z = -1$ by alternating series test, not convergent at $z = 1$.)

(c) convergent nowhere on $|z| = 1$.
($a_n z^n \not\to 0$ as $n \to \infty$.)

In fact convergent $\forall z \neq 1$ with $|z| = 1$

The exponential-in-n behaviour of z^n makes the **ratio test** particularly useful for testing convergence of power series, for instance readily giving the following.

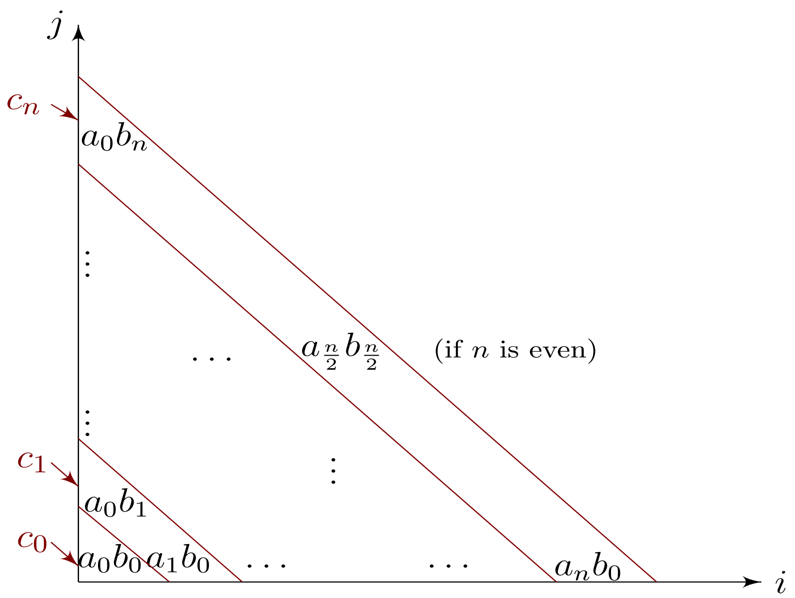
Exercise 4.38. Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to a \in [0, \infty]$ as $n \to \infty$.
Then $R = \frac{1}{a}$ is the radius of convergence of $\sum a_n z^n$.

4.5.1 Products of Series

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ \text{“}=\text{”} &\ a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \\ &= \sum_{n=0}^{\infty} c_n z^n,\end{aligned}$$

where $c_0 = a_0 b_0,\ c_1 = a_0 b_1 + a_1 b + 0, \dots,\ c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.



So we set $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ and ask when is the product $\sum a_n z^n \sum b_n z^n$ equal to $\sum c_n z^n$? We can also do this without the z^ns:

Definition. Given series $\sum a_n$, $\sum b_n$ their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^n a_i b_{n-i}$.

Notice we used power series to motivate this definition; it is not the only way we could collect all the terms $a_i b_j$ to turn $\sum a_i \sum b_j$ into a single sum. This is why we give it the specific name "Cauchy product".

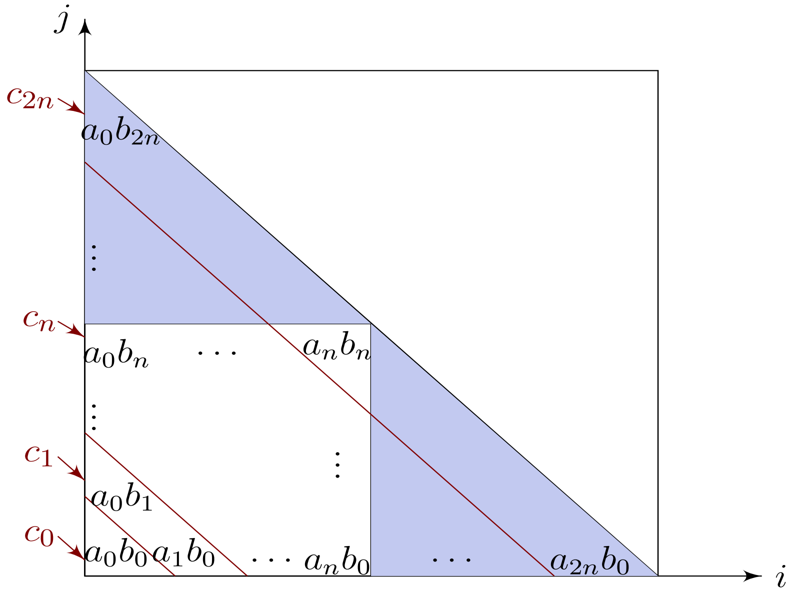
Theorem 4.39: Cauchy Product

If $\sum a_n, \sum b_n$ are absolutely convergent, then their Cauchy product $\sum c_n$ is absolutely convergent to $\left(\sum a_n\right) \cdot \left(\sum b_n\right)$.

Proof. (Non-examinable.) We try to control

$$\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j.$$

The first term is the sum of $a_i b_j$ over all (i, j) below the diagonal in the diagram below. The second term is the sum over the small square. Therefore the difference is the sum of $a_i b_j$ over (i, j) in the two shaded triangles.



By the triangle inequality

$$\left| \sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j \right| \le \sum |a_i b_j|,$$

where the right hand sum is over i, j in (2 shaded triangles) $\subset$ (big square minus small square). Thus it is less than the sum over (big square minus small square),

$$\left|\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j\right| \leq \sum_{i=0}^{2n}\sum_{j=0}^{2n} |a_i b_j| - \sum_{i=0}^{n}\sum_{j=0}^{n} |a_i b_j|. \tag{1}$$

Now we're in good shape because we're summing over the complement of the small square, i.e. we're in the tail of at least one of $\sum a_n$ or $\sum b_n$, and these are (absolutely) small. Since the partial sums $\sum_{i=0}^n |a_i|$ and $\sum_{j=0}^n |b_j|$ converge, their product $\sum_{i,j=0}^n |a_i b_j|$ also converges by the Algebra of limits for sequences (Theorem 3.19). In particular it defines a Cauchy sequence; fixing $\epsilon > 0$, there exists N_1 such that

$$m \geq n \geq N_i \implies \sum_{i,j=0}^{m} |a_i b_j| - \sum_{i,j=0}^{n} |a_i b_j| < \epsilon.$$

Taking $m = 2n$ and substituting into (1) gives us

$$n \geq N_1 \implies \left|\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_j b_j\right| \leq \epsilon. \tag{2}$$

Now we know that the partial sums $\sum_{i=0}^n a_i \to A$ and $\sum_{j=0}^n b_j \to B$, so by the Algebra of limits again,

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{n} b_j \longrightarrow AB.$$

This means that $\exists N_2$ such that

$$n \geq N_2 \implies \left|\sum_{i,j=0}^{n} a_i b_j - AB\right| < \epsilon.$$

Combined with (2) and the triangle inequality this gives

$$\left|\sum_{i=0}^{2n} c_i - AB\right| < 2\epsilon$$

for all $n \geq \max(N_1, N_2)$.

We can deal with $\sum_{i=0}^{2n+1} c_i$ in the same way by sandwiching it between the squares $0 \leq i, j \leq n$ and $0 \leq i, j \leq 2n+1$. The upshot is that $\exists N$ such that for all $k \geq N$,

$$\left|\sum_{i=0}^{k} c_i - AB\right| < 2\epsilon.$$

Thus $\sum_{i=0}^{k} c_i \to AB$. Finally to prove that $\sum c_n$ is absolutely convergent, just replace a_n, b_n by $|a_n|, |b_n|$ in the above proof. $\square$

Corollary 4.40. *If $\sum a_n z^n$ and $\sum b_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.*

Proof. By Theorem 4.39, for any $|z| < \min\{R_A, R_B\}$ we have $\sum a_n z^n$ and $\sum b_n z^n$ absolutely convergent $\implies \sum c_n z^n$ absolutely convergent $\implies |z| \leq R_C$. $\square$

So now apply the next Ex to $\min\{R_A, R_B\}$ and R_C

Exercise 4.41. Fix α, $\beta \in \mathbb{R}$. Prove that if $\left[x < \alpha \implies x \leq \beta\right]$ then $\alpha \leq \beta$.

Proof. If $\alpha > \beta$ then let $x := \frac{1}{2}(\alpha + \beta)$ so that $\beta < x < \alpha$ ※ $\square$

Example 4.42. $\sum z^n$ has $R_A = 1$.
$1 - z$ has $R_B = \infty$.
So their Cauchy product $\sum c_n z^n$ has $R_C \geq 1$.

Exercise: Check $c_0 = 1, c_n = 0\ \forall n \geq 1$, so the Cauchy product is 1 and in fact $R_C = \infty$.

Nonetheless, we can only say that $\sum c_n z^n = 1 = \left(\sum z^n\right)(1 - z)$ when $|z| < 1 = \min(R_A, R_B)$.