$$\min \quad x_1^2 + x_2^2 + x_3^2$$
$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 \geq 4$$
$$x_3 \leq 1$$

i) KKT system.

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda_1 (4 - x_1 - 2x_2 - 3x_3) + \lambda_2 (x_3 - 1)$$

$$\nabla_{\underline{x}} L = 0 \iff \begin{cases} 2x_1 - \lambda_1 = 0 \\ 2x_2 - 2\lambda_1 = 0 \\ 2x_3 - 3\lambda_1 + \lambda_2 = 0 \end{cases} \quad + \quad \begin{cases} \lambda_1 (4 - x_1 - 2x_2 - 3x_3) = 0 \\ \lambda_2 (x_3 - 1) = 0 \\ \lambda_1 \geq 0, \ \lambda_2 \geq 0 \end{cases}$$

ii) Convex constraint + strictly convex cost

* Also need to mention coercivity for existence

$$\underline{x}^T \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \underline{x} \quad Q \succ 0$$

$\exists!$ optimal solution and KKT is sufficient.

iii) a) $\lambda_1 = \lambda_2 = 0 \implies x_1 = x_2 = x_3 = 0$, unfeasible

$$x_1 + 2x_2 + 3x_3 \cancel{\geq} 4$$

b) $\lambda_1 > 0$, $\lambda_2 = 0 \implies$

$$2x_1 = \lambda_1 \qquad 4 - x_1 - 2x_2 - 3x_3 = 0$$
$$x_2 = \lambda_1$$
$$2x_3 = 3\lambda_1$$

$$\implies \begin{array}{l} 2x_1 = x_2 \\ 2x_3 = 3x_2 \end{array} \implies 4 - \frac{x_2}{2} - 2x_2 - \frac{9}{2}x_2 = 0$$

$$x_2 = \frac{4}{7}, \ x_1 = \frac{2}{7}, \ x_3 = \frac{6}{7}$$

and $\lambda_1 = 4/7 > 0$ ✓.

The optimal solution is

$$\underline{x}^* = \frac{1}{7}(2, 4, 6)^T$$