IMPERIAL

MATH70029

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Functional Analysis

Date: Wednesday, May 29, 2024
Time: 10:00 – 12:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Note: unless specified otherwise, all linear spaces are over the field of real numbers $\mathbb{R}$ and L^p-spaces are endowed with their usual L^p-norm.

1. (a) (i) Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$ be a closed linear subspace. Show that $(A, \|\cdot\|)$ is complete. (2 marks)

(ii) Let

$$A = \Big\{(x_n) : x_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } x \overset{\text{def.}}{=} \lim_n x_n \text{ exists and } x \in \mathbb{R}\Big\}.$$

Show that $(A, \|\cdot\|)$ where $\|x\| = \sup_n |x_n|$ is complete. *Hint:* use (i). (8 marks)

(b) Consider $C = C[0,1]$ the space of continuous functions $f : [0,1] \to \mathbb{R}$. For $f \in C$ define

$$\|f\|^2 = \int_0^1 |f|^2 d\lambda$$

with λ denoting Lebesgue measure.

(i) For $n \in \mathbb{N}$ let

$$f_n(x) = \begin{cases} (2x)^n, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Is (f_n) Cauchy in $(C, \|\cdot\|)$? Justify your answer. (4 marks)

(ii) Show using (i) that $(C, \|\cdot\|)$ is not complete. (6 marks)

(Total: 20 marks)

2. Let $(X, \|\cdot\|)$ be a Banach space and $Y \subset X$ a closed subspace.

(a) Show that Y is also closed in the weak topology. (6 marks)

(b) Let $(y_n) \subset Y$ with y_n converging weakly to y as $n \to \infty$. Show that

$$\|x - y\| \leq \liminf_n \|x - y_n\|, \text{ for all } x \in X.$$

(6 marks)

(c) Let X be reflexive. Show that for any $x \in X$ there exists $y_0 \in Y$ such that

$$\|x - y_0\| = \inf_{y \in Y} \|x - y\|.$$

(8 marks)

(Total: 20 marks)

3. (a) True or false: the closed unit ball in a Banach space is weak-$*$ sequentially compact. Justify your answer. (4 marks)

(b) Let $1 < p < \infty$ and $f \in C_c^\infty(\mathbb{R})$, the space of smooth compactly supported functions. Define

$$f_n(x) = f(x - n), \quad x \in \mathbb{R},\, n \in \mathbb{N}.$$

(i) Show that f_n converges weakly in $L^p(\mathbb{R})$ and determine its limit. (12 marks)

(ii) Does f_n converge in $L^p(\mathbb{R})$? Justify your answer. (4 marks)

(Total: 20 marks)

4. Let $\ell^2 = \ell^2(\mathbb{C})$ and $A : \ell^2 \to \ell^2$ be given by

$$A(x_1, x_2, \dots) = (x_2, x_3, \dots), \quad \text{for } x = (x_1, x_2, \dots) \in \ell^2.$$

(a) Show that $A \in \mathcal{L}(\ell^2)$ and determine its operator norm $\|A\|$. (4 marks)

(b) Show that A is not compact. (6 marks).

(c) Is A self-adjoint? Justify your answer. (2 marks)

(d) Let $\sigma_p(A)$ and $\sigma(A)$ denote the point spectrum and the spectrum of A, respectively. Show that

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(A) \text{ and } \sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

(8 marks)

(Total: 20 marks)

5. Let $a < b$ be real numbers and $I = (a, b)$. Let $1 \leq p < \infty$.

(a) Recall the definition of the Sobolev space $W^{1,p}(I)$. Specify the meaning of the derivative in this context. (4 marks)

(b) Let $u \in L^p(I)$ and (u_k) be a bounded sequence in $W^{1,p}(I)$ with $\|u - u_k\|_{L^p(I)} \to 0$ as $k \to \infty$.

(i) If $p \neq 1$ prove that $u \in W^{1,p}(I)$. (12 marks)

(ii) Is the assumption $p \neq 1$ necessary for the conclusions of (i) to hold? (4 marks)

(Total: 20 marks)

Module: MATH60029/MATH70029
Setter: Rodriguez
Checker: Krasovsky
Editor: editor
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Version: Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH60029/MATH70029 Functional Analysis – **SOLUTIONS**

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a)(i) 2 marks; 1(b)(i) 4 marks; 1(b)(ii) 6 marks; 2(a) 6 marks; 3(a) 4 marks; 3(b)(ii) 4 marks; 4(a) 4 marks; 4(c) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(a)(ii) 8 marks; 2(b) 6 marks; 4(b) 6 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
3(b)(i) 12 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
2(c) 8 marks; 4(c) 8 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Functional Analysis – **SOLUTIONS**

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH60029 paper; 2.5 Hours for MATH70029 paper

This paper has *5 Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

Note: unless specified otherwise, all linear spaces are over the field of real numbers $\mathbb{R}$ and L^p-spaces are endowed with their usual L^p-norm.

1. (a) (i) (SEEN) Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$ be a closed linear subspace. Show that $(A, \|\cdot\|)$ is complete. (2 marks)

Solution: One needs to show that any Cauchy sequence $(x_n) \subset A$ converges in A. Since X is complete there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. But since A is closed the limit x belongs to A.

(ii) (SEEN SIMILAR) Let

$$A = \Big\{(x_n) : x_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } x \overset{\text{def.}}{=} \lim_n x_n \text{ exists and } x \in \mathbb{R}\Big\}.$$

Show that $(A, \|\cdot\|)$ where $\|x\| = \sup_n |x_n|$ is complete. *Hint:* use (i). (8 marks)

Solution: The space ℓ^∞ is complete so by (i) it is enough to show that $A \subset \ell^\infty$ is closed. Let $(x^k) \subset A$ be a sequence converging to $x \in \ell^\infty$, i.e.

$$\|x^k - x\| \to 0, \quad k \to \infty.$$

We need to show that $x \in A$, i.e. that x converges to a finite limit. By the triangle inequality, for all m, n, k

$$|x_m - x_n| \le |x_m - x_m^k| + |x_m^k - x_n^k| + |x_n^k - x_n| \le 2\|x - x^k\| + |x_m^k - x_n^k|.$$

Now let $\varepsilon > 0$ and first pick $k = k(\varepsilon)$ so that $\|x - x^k\| \le \frac{\varepsilon}{3}$. Then since $x^k \in A$ it converges hence it is Cauchy. Therefore $|x_m^k - x_n^k| \le \frac{\varepsilon}{3}$ whenever $n, m \ge N(\varepsilon)$. This shows that x is Cauchy, and therefore convergent.

(b) Consider $C = C[0,1]$ the space of continuous functions $f : [0,1] \to \mathbb{R}$. For $f \in C$ define

$$\|f\|^2 = \int_0^1 |f|^2 d\lambda$$

with λ denoting Lebesgue measure.

(i) (SEEN SIMILAR) For $n \in \mathbb{N}$ let

$$f_n(x) = \begin{cases} (2x)^n, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Is (f_n) Cauchy in $(C, \|\cdot\|)$? Justify your answer. (4 marks)

Solution: Using that $(a-b)^2 \le 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$ one finds that

$$\|f_n - f_m\|^2 \le 2\int_0^{\frac{1}{2}} (|f_n|^2 + |f_m|^2) d\lambda \le \frac{c}{n \wedge m}.$$

The claim follows.

(ii) (SEEN SIMILAR) Show using (i) that $(C, \|\cdot\|)$ is not complete. (6 marks)

Solution: If it were then the sequence (f_n) from (i) would have to converge to some $f \in C$. Suppose this were true. Then by a similar argument as in (i)

$$\int_0^{\frac{1}{2}} |f|^2 d\lambda \leq 2 \int_0^{\frac{1}{2}} (|f - f_n|^2 + |f_n|^2) d\lambda \leq 2\|f - f_n\|^2 + \frac{c}{n}$$

which implies by letting $n \to \infty$ that $f(x) = 0$ for $x \in [0, \frac{1}{2})$. Similarly

$$\int_{\frac{1}{2}}^1 |f - 1|^2 d\lambda = \int_{\frac{1}{2}}^1 |f - f_n|^2 d\lambda \leq \|f - f_n\|^2,$$

from which it follows that $f(x) = 1$ for $x \in (\frac{1}{2}, 1]$. Hence $f \notin C$.

(Total: 20 marks)

2. Let $(X, \|\cdot\|)$ be a Banach space and $Y \subset X$ a closed subspace.

(a) (SEEN SIMILAR) Show that Y is also closed in the weak topology. (6 marks)

Solution: Suppose that $(x_n) \subset Y$ converges weakly to $x \in X$. If $x \notin Y$, then by the property of separating points one can find $\ell \in X^*$ such that $\ell(y) = 0$ for all $y \in Y$ and $\ell(x) \neq 0$. But then $0 = \ell(x_n) \to \ell(x) \neq 0$ as $n \to \infty$ by weak convergence, a contradiction.

(b) (UNSEEN) Let $(y_n) \subset Y$ with y_n converging weakly to y as $n \to \infty$. Show that

$$\|x - y\| \leq \liminf_n \|x - y_n\|, \text{ for all } x \in X.$$

(6 marks)

Solution: By a corollary of Hahn-Banach (seen in class) one can find $\ell \in X^*$ with $\|\ell\|_* = 1$ and $\ell(x - y) = \|x - y\|$. But then

$$\|x - y\| = \ell(x - y) = \lim_n \ell(x - y_n) \leq \liminf_n \|x - y_n\|,$$

using in the last step that $\|\ell\|_* = 1$.

(c) (UNSEEN) Let X be reflexive. Show that for any $x \in X$ there exists $y_0 \in Y$ such that

$$\|x - y_0\| = \inf_{y \in Y} \|x - y\|.$$

(8 marks)

Solution: Pick a sequence (y_n) such that $\|x - y_n\|$ converges to the infimum on the right-hand side. Note that

$$\|y_n\| = \|x - y_n\| + \|x\|,$$

so (y_n) is bounded. Since X is reflexive, it has a weak limit $y_0 \in X$. By (a) one has $y_0 \in Y$. By (b) one further obtains that

$$\|x - y_0\| \leq \liminf_n \|x - y_n\| = \lim_n \|x - y_n\| = \inf_{y \in Y} \|x - y\|.$$

The reverse inequality is trivial and the claim follows.

(Total: 20 marks)

3. (a) (SEEN) True or false: the closed unit ball in a Banach space is weak-$*$ sequentially compact. Justify your answer. (4 marks)

Solution: False. The conclusions may not hold if the space is not separable.

(b) Let $1 < p < \infty$ and $f \in C_c^\infty(\mathbb{R})$, the space of smooth compactly supported functions. Define

$$f_n(x) = f(x-n), \quad x \in \mathbb{R},\, n \in \mathbb{N}.$$

(i) (UNSEEN) Show that f_n converges weakly in $L^p(\mathbb{R})$ and determine its limit. (12 marks)

Solution: We show that f_n converges weakly to 0. By the canonical isomorphism between $(L^p)^*$ and L^q where p and q are conjugate, it is enough to show that for all $g \in L^q(\mathbb{R})$,

$$\int f_n g \, d\lambda \to 0 \text{ as } n \to \infty$$

But noting that $\|f_n\|_p = \|f\|_p$ by translation invariance, using Hölder's inequality and the fact that $C_c^\infty(\mathbb{R})$ is dense in $L^q(\mathbb{R})$, one readily sees that the previous display follows at once if it holds for $g \in C_c^\infty(\mathbb{R})$. For such g, one simply notes that

$$\int f_n g \, d\lambda = 0$$

for large enough n since both f and g have compact support.

(ii) (SEEN SIMILAR) Does f_n converge in $L^p(\mathbb{R})$? Justify your answer. (4 marks)

Solution: No. If they did then the strong limit would have to be $g = 0$, for the weak limit coincides with it. But then one would have $\|f_n\|_p = \|f_n - g\|_p \to 0$ as $n \to \infty$ which cannot hold since $\|f_n\|_p = \|f\|_p$ for all n.

(Total: 20 marks)

4. Let $\ell^2 = \ell^2(\mathbb{C})$ and $A : \ell^2 \to \ell^2$ be given by

$$A(x_1, x_2, \dots) = (x_2, x_3, \dots), \quad \text{for } x = (x_1, x_2, \dots) \in \ell^2.$$

(a) (SEEN) Show that $A \in \mathcal{L}(\ell^2)$ and determine its operator norm $\|A\|$. (4 marks)

Solution: Linearity is straightforward and $\|Ax\|_2^2 \leq \|x\|_2^2$ so $A \in \mathcal{L}(\ell^2)$ and $\|A\| \leq 1$. Moreover $Ae_2 = e_1$ hence

$$1 = \|Ae_2\|_2 \leq \|A\| \cdot \|e_2\|_2 = \|A\|$$

so $\|A\| = 1$.

(b) (SEEN SIMILAR) Show that A is not compact. (6 marks).

Solution: The sequence e_2, $e_3, \dots$ is bounded but $(Ae_{n+1})_{n\geq 1} = (e_n)_{n\geq 1}$ does not have a convergent subsequence since $\|e_n - e_m\|_2 = c > 0$ for all $n \neq m$. Using a known characterization of compactness, it follows that A is not compact.

(c) (SEEN SIMILAR) Is A self-adjoint? Justify your answer. (2 marks)

Solution: No, since $\langle Ae_2, e_1 \rangle = 1 \neq \langle e_2, Ae_1 \rangle = 0$.

(d) (UNSEEN) Let $\sigma_p(A)$ and $\sigma(A)$ denote the point spectrum and the spectrum of A, respectively. Show that

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(A) \text{ and } \sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

(8 marks)

Solution: For the inclusion, one notes that $x = (x_1, x_2, \dots)$ with entries $x_n = \lambda^{n-1}$ satisfies $Ax = \lambda x$. Moreover one checks that $x \in \ell^2$ whenever $|\lambda| < 1$. As for the second conclusion, the inclusion $\sigma(A) \supset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ follows from the inclusion just shown since $\sigma_p(A) \subset \sigma(A)$ and the latter is a closed set. On the other hand, one has that $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$ so the reverse inclusion follows using (a).

(Total: 20 marks)

5. Let $a < b$ be real numbers and $I = (a, b)$. Let $1 \leq p < \infty$.

(a) Recall the definition of the Sobolev space $W^{1,p}(I)$. Specify the meaning of the derivative in this context. (4 marks)

Solution: This is the space of all u such that both u and its weak derivative u' belong to $L^p(I)$. The weak derivative is defined up to a.e. equivalence by the requirement that

$$\int_I u\varphi' d\lambda = -\int_I u'\varphi d\lambda,$$

for any test function $\varphi \in C_c^\infty(I)$.

(b) Let $u \in L^p(I)$ and (u_k) be a bounded sequence in $W^{1,p}(I)$ with $\|u - u_k\|_{L^p(I)} \to 0$ as $k \to \infty$.

(i) If $p \neq 1$ prove that $u \in W^{1,p}(I)$. (12 marks)

Solution: Let u'_k denote the weak derivative of u_k. By assumption, the sequence (u'_k) is bounded in $L^p(I)$. But the space $L^p(I)$ is reflexive hence (u'_k) has a subsequence which converges weakly in $L^p(I)$. Let $g \in L^p(I)$ denote the corresponding limit, which is attained along the subsequence $\Lambda \ni k$. For any $\varphi \in C_c^\infty(I)$, the maps

$$\ell : L^p(I) \to \mathbb{R}, \quad f \mapsto \ell(f) = \int_I f\varphi d\lambda$$
$$\ell' : L^p(I) \to \mathbb{R}, \quad f \mapsto \ell'(f) = -\int_I f\varphi' d\lambda$$

both belong to $(L^p(I))^*$. Since $\|u - u_k\|_{L^p(I)} \to 0$ implies weak convergence and by choice of Λ, we know that both

$$\ell'(u) = \lim_{k\in\Lambda, k\to\infty} \ell'(u_k) \text{ and } \ell(g) = \lim_{k\in\Lambda, k\to\infty} \ell(u'_k).$$

Now $\ell(u'_k) = \ell'(u_k)$ by definition of weak derivative and it follows that $\ell'(u) = \ell(g)$. Since φ was arbitrary this identifies g as the weak derivative of $u \in L^p(I)$ and $u \in W^{1,p}(I)$ follows.

(ii) Is the assumption $p \neq 1$ necessary for the conclusions of (i) to hold? (4 marks)

Solution: Yes. Consider $a = -1$, $b = 1$, $u = 1_{(0,1)}$ and

$$u_k(x) = \begin{cases} 0, & x < 0 \\ kx, & 0 \leq x < \frac{1}{k} \\ 1, & \frac{1}{k} < x \leq 1. \end{cases}$$

Then $u_k \in W^{1,1(I)}$ with $\|u_k\|_1 = 1 - \frac{1}{2k}$, $\|u'_k\|_1 = \frac{1}{k}k = 1$ and $\|u - u_k\|_1 = \frac{1}{2k} \to 0$ as $k \to \infty$. But $u \notin W^{1,1(I)}$, else it would have a continuous representative (obtained by suitable integration of its weak derivative).

(Total: 20 marks)

MATH60029 Functional Analysis

Question	Marker's comment
1	Solved mostly well. For 1(a)(ii) some students still struggle with epsilon/3 arguments. For 1(b)(ii) must argue convergence towards discontinuous limit is with respect to the given norm (L^2).
2	A fair number of students did not think about using the Hahn-Banach Theorem at all, in spite of similar questions discussed during class and on problem sheets.
3	Some students struggled to show 0 is the weak limit in b(i).
4	Solved mostly very well.

MATH70029 Functional Analysis

Question	Marker's comment
1	Solved mostly well. For 1(a)(ii) some students still struggle with epsilon/3 arguments.
2	A fair number of students did not think about using the Hahn-Banach Theorem at all, in spite of similar questions discussed during class and on problem sheets.
3	Some students struggled to show 0 is the weak limit in b(i).
4	Solved mostly very well.
5	none