① ② Substituting terms for variables

(2.3.5) Example

An example where

$$\mathcal{A} \models (\forall x_1)\phi(x_1)$$

but where we have a term t_1 and a valuation v in $\mathcal{A}$ with $v[\phi(t_1)] = F$.

= Take:

$\phi(x_1)$ free free

$$((\forall x_2)R(x_1, x_2) \rightarrow S(x_1))$$

t_1 is the term x_2

$\phi(t_1)$ is

$$((\forall x_2)R(x_2, x_2) \rightarrow S(x_2))$$

$\mathcal{A}$ domain $\mathbb{N} = \{0, 1, 2, \ldots\}$

$R(x_1, x_2)$ interpreted as "$x_1 \leq x_2$"

$S(x_1)$ interpreted as "$x_1 = 0$"

Then

$$\mathcal{A} \models (\forall x_1)((\forall x_2)R(x_1, x_2) \rightarrow S(x_1))$$

But if $v(x_2) = 1$

then $v[\phi(t_1)] = F$

"If for all x_2 $x_2 \leq x_2$, then $x_2 = 0$"

(2.3.6) **Def.**

Let ϕ be an $\mathcal{L}$-fmla

x_i a variable

t a term of $\mathcal{L}$.

Say that <u>t is free for x_i in ϕ</u>

if there is no variable x_j in t such that x_i occurs free within the scope of a quantifier $(\forall x_j)$ in ϕ.

So t is <u>not</u> free for x_i in ϕ

if

$$\phi : \;\ldots\ldots (\forall x_j) \overbrace{\ldots \underset{\uparrow \text{ free}}{x_i} \ldots}^{\text{scope}} \ldots\ldots$$

$$t \qquad \ldots x_j \ldots$$

In example 2.3.5

t_1 is <u>not</u> free for x_1

in ϕ $\qquad (i=1, \; j=2)$.

(2.3.7) Thm.

Suppose $\phi(x_1)$ is an $\mathcal{L}$-fmla (possibly with other free vars.).
Let t be a term free for x_1 in ϕ. Then

$$\models ((\forall x_1)\phi(x_1) \to \phi(t)).$$

In particular if $\mathcal{A}$ is an $\mathcal{L}$-str. with $\mathcal{A} \models (\forall x_1)\phi(x_1)$ then $\mathcal{A} \models \phi(t)$.

[Contrast with 2.3.5.].

Follows from

(2.3.8) Lemma. With this notation, suppose v is a valuation in $\mathcal{A}$.
Let v' be the val. in $\mathcal{A}$ x_1-equiv. to v with $v'(x_1) = v(t)$.
Then $v[\phi(t)] = T$
$\iff$ $v'[\phi(x_1)] = T$.

Pf: See notes. Not examinable.
#

Lemma $\Rightarrow$ Thm.

Suppose $v[\phi(t)] = F$
Take v' as in the Lemma.
So $v'[\phi(x_1)] = F$
and v' is x_1-equiv. to v.
So $v[(\forall x_1)\phi(x_1)] = F$.
$\therefore$ $v[((\forall x_1)\phi(x_1) \to \phi(t))] = T$
#

③ Comparing $\mathcal{L}$-structures.

(2.3.9) Def. Suppose $\mathcal{L}$ is a 1^{st}-order language with rel., fn. + constant symbols $(R_i : i \in I)$, $(f_j, j \in J)$, $(c_k : k \in K)$ and R_i of arity n_i, f_j of arity m_j. Consider $\mathcal{L}$-sts:

$$\mathcal{A} = \langle A ; (R_i^A : i \in I), (f_j^A : j \in J), (c_k^A : k \in K) \rangle$$

$$\mathcal{B} = \langle B ; (R_i^B : i \in I), (f_j^B : j \in J), (c_k^B : k \in K) \rangle.$$

A function $\alpha : A \to B$ is an isomorphism (from $\mathcal{A}$ to $\mathcal{B}$) if

α is a bijection and

a) $$R_i^A(a_1, \ldots, a_{n_i}) \iff R_i^B(\alpha(a_1), \ldots, \alpha(a_{n_i}))$$

for $i \in I$ and $a_1, \ldots, a_{n_i} \in A$.

b) $$\alpha(f_j^A(a_1, \ldots, a_{m_j})) = f_j^B(\alpha(a_1), \ldots, \alpha(a_{m_j}))$$

for $j \in J$ and $a_1, \ldots, a_{m_j} \in A$.

c) $$\alpha(c_k^A) = c_k^B \qquad (\text{for } k \in K).$$

Note: 1) This gives the usual notion for groups, rings, graphs, ...

2) If α is an isomorphism as above then α^{-1} is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$.

(2.3.10) Theorem. Suppose α is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Let v be a valuation in $\mathcal{A}$ and let w be the val. in $\mathcal{B}$ with $w(x_i) = \alpha(v(x_i))$. Then

(i) For every term t $\quad w(t) = \alpha(v(t))$.

(ii) If ϕ is an $\mathcal{L}$-fmla then v satisfies ϕ in $\mathcal{A}$ $\iff$ w satisfies ϕ in $\mathcal{B}$.

(iii) In particular if ϕ is a closed $\mathcal{L}$-fmla then

$$\mathcal{A} \models \phi \iff \mathcal{B} \models \phi .$$

Pf: (i) By induction on the length of t using 2.3.9 (b),(c).

(ii) By induction on the number connectives + quantifiers in ϕ. 6

Base case: ϕ atomic. By 2.3.9 (a) + (i).

Ind. step: ϕ is $\underbrace{(\neg\psi), (\psi \to \chi)}_{\text{Ex.}}$ or $(\forall x_i)\phi$

Suppose $v[(\forall x_i)\psi] = F$. There is v' (in $\mathcal{A}$) x_i-equiv. to v with $v'[\psi] = F$.

Let w' be the val. in $\mathcal{B}$ with $w'(x_j) = \alpha(v'(x_j))$

Then w' is x_i-equiv. to w. By ind. hypothesis $w'[\psi] = F$. (as $v'[\psi] = F$).

So $w[(\forall x_i)\psi] = F$. //.

(iii) By (ii). #.

(See typed notes for a bit more detail.)