L3.

(1.1.12) Example.

(1) $\{\wedge, \vee\}$ is not adequate

[if ϕ just uses these

$F_\phi(T, \ldots, T) = T$.].

(2) $\{\neg, \leftrightarrow\}$ is not adequate.

//

(1.1.13) Example.

NOR connective $\downarrow$

('neither .. nor ... ')

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

$\{\downarrow\}$ is adequate

$(\neg p)$ is l.e. to $(p \downarrow p)$

$(p \wedge q)$ l.e. to

$((p \downarrow p) \downarrow (q \downarrow q))$.

So as $\{\neg, \wedge\}$ is adequate

then $\{\downarrow\}$ is adequate. //

Semantics of propositional logic.

(1.2) A formal system for propositional logic.

Idea. Try to generate all tautologies from certain 'basic assumptions' (axioms) using deduction rules.

[1.2.1 in typed notes: Def. of 'formal system']

(1.2.2) Def. The formal system L for propositional logic has the following ingredients

Alphabet of symbols:

variables: $p_1 \quad p_2 \quad p_3 \quad \ldots$

connectives: $\neg \quad \to$

punctuation: $) \quad ($

Formulas Certain finite sequences ('strings') of symbols from the alphabet constructed as follows: (as in 1.1.2)

i) any variable is a formula; (of L)

ii) if ϕ, ψ are formulas of L then so are $(\neg \phi)$ and $(\phi \to \psi)$

iii) Any formula of L arise in this way.

L-formulas

Axioms Suppose ϕ, ψ, χ are L-formulas. The following are axioms of L:

(A1) $(\phi \to (\psi \to \phi))$

(A2) $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

(A3) $(((\neg \psi) \to (\neg \phi)) \to (\phi \to \psi))$

Deduction rule 'Modus Ponens' (MP)

From ϕ $(\phi \to \psi)$ deduce ψ

A *proof* in L is a finite sequence of L-fmlas $\phi_1, \phi_2, \ldots, \phi_n$ such that each ϕ_i is

either an *axiom* or is obtained from earlier formulas in the proof using the deduction rule MP.

The final formula in a proof is a *theorem of* L.

(the n here is called the *length* of the proof).

$$\phi_1 \cdots \phi_k \cdots (\phi_k \to \phi_i) \cdots \phi_i \cdots$$

Write $\vdash_L \phi$

(\vdash)

to mean 'ϕ is a theorem of L'.

<u>Note</u>: (1) Any axiom is a theorem of L.

(2) Every formula in a proof is a theorem of L.

<u>Aim</u>: The theorems of L are precisely the tautologies (using $\neg$ $\rightarrow$).

(1.2.3) <u>Example</u>. Suppose ϕ is any L-fmla. Then $\vdash_L (\phi \rightarrow \phi)$.

Here is a proof in L:

1. $\underbrace{(\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi))}_{\text{call this } \chi}$ (Axiom A1)

2. $(\chi \rightarrow ((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)))$ (Axiom A2)

3. $((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi))$ (1, 2 + MP)

4. $(\phi \rightarrow (\phi \rightarrow \phi))$ (Axiom A1)

5. $(\phi \rightarrow \phi)$ (3, 4 + MP).

"Theorem 0"

(1.2.4) Def. Suppose Γ is a set of L-fmlas. A deduction from Γ is a finite sequence

$$\phi_1, \phi_2, \ldots, \phi_n$$

of L-fmlas. such that each ϕ_i is either

an axiom

a formula in Γ

or is obtained from previous formulas $\phi_1, \ldots, \phi_{i-1}$ by applying the deduction rule MP.

(n: length of the deduction)

Write $\Gamma \vdash_L \phi$ if there is a deduction from Γ which ends with ϕ.

Say 'ϕ is a consequence of Γ'.

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