MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 1

Section: Syllabus & Historical Remarks

- Holomorphic Functions: Definition using derivative, Cauchy-Riemann equations, Polynomials, Power series.

- Cauchy's Integral Formula: Complex integration along curves, Goursat's theorem, Local existence of primitives and Cauchy's theorem in a disc, Evaluation of some integrals, Homotopies and simply connected domains, Cauchy's integral formulas.

- Applications of Cauchy's integral formula: Morera's theorem, Sequences of holomorphic functions, Holomorphic functions defined in terms of integrals, Schwarz reflection principle.

- Meromorphic Functions: Zeros and poles. Laurent series. The residue formula, Singularities and meromorphic functions, The argument principle and applications, The complex logarithm.

- Harmonic functions: Definition, and basic properties, Maximum modulus principle. Conformal Mappings: Definitions, Preservation of Angles, Statement of the Riemann mapping theorem, Rational functions, Möbius transformations.

see Blackboard for Lectures and Problem Sheets

Section: Complex numbers

The complex number $i = \sqrt{-1}$ ie associated with solutions of the equation

$$x^2 + 1 = 0$$

that does not have real solutions. However, historically complex numbers came through the cubic equation

$$x^3 - ax - b = 0.$$

In 1515 Scipione del Ferro (1465-1526, Italian) found but not published the solution

$$x = \sqrt{3}\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}} + \sqrt{3}\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}$$

It was interesting that even if $\frac{b^2}{4} - \frac{a^3}{27} < 0$ the equation has real solutions for a, b real. This formula was published by Girolamo Cardano (1501-15-76, Italian) in 1545.

Scipione del Ferro

Girolamo Cardano

In 1572, Rafael Bombelli (1526-1572, Italian) published a book which spelled out rules of arithmetic for complex numbers and used them in Cardano's formula for finding real solutions of cubics.

Key later work is by John Wallis (1616 - 1703, English) and Leonhard Euler (1707-1783, Swiss). In particular, Euler clarified complex roots of unity and found the multiple roots. He used complex numbers extensively. He introduced i as the symbol for $\sqrt{-1}$ and linked the exponential and trigonometric functions in the famous formula

$$e^{it} = \cos t + i \sin t.$$

Carl Friedrich Gauss (1777-1855, German), who gave a proof of the Fundamental Theorem of Algebra in 1799.

It took almost another century before mathematicians as a community fully accepted complex numbers.

Rafael Bombelli

John Wallis

Leonhard Euler

Carl Friedrich Gauss

The founding fathers of complex analysis are:
Augustin-Louis Cauchy, Karl Weierstrass and Bernhard Riemann.

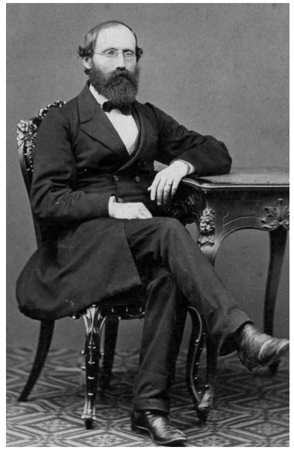
To A.-L. Cauchy - the central aspect is the differential and integral calculus of complex-valued functions of a complex variable. Here the fundamentals are the Cauchy integral theorem and Cauchy integral formula

Augustin-Louis Cauchy (1789 -1857) - French

To K. Weierstrass - sums and products and especially power series are the central object.

Karl Weierstrass (1815-1897) - German

To B. Riemann - conformal maps and associated geometry.



Bernhard Riemann (1826-1866) - German

- Riemann Hypothesis is still open (1859)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p\,\mathrm{prime}} \frac{1}{1-p^{-z}}.$$

This series converges if $\operatorname{Re} z > 1$.

If z is a complex number then in the above sum there some cancellation. In particular the Riemann Hypothesis states

$$\zeta(z) = 0 \quad \Longrightarrow \quad \operatorname{Re} z = 1/2.$$

Modern state of art

- The Mandelbrot set, Complex dynamics:

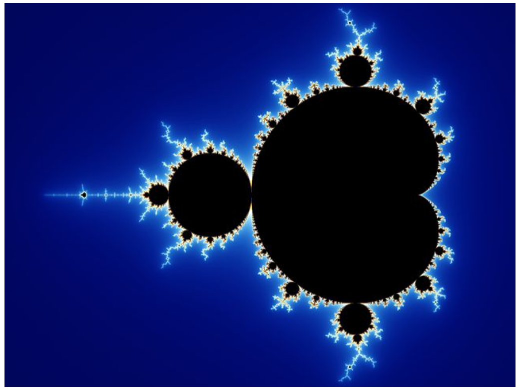
The Mandelbrot set is the set of complex numbers η for which the function

$$f_\eta(z) = z^2 + \eta$$

does not diverge when iterating from $z = 0$ so that the sequence

$$f_\eta(0), f_\eta(f_\eta(0)), f_\eta(f_\eta(f_\eta(0))), \ldots$$

remains bounded



Benoit Mandelbrot (1924 - 2010)

Section: Basic properties

A complex number takes the form $z = x + iy$, where x and y are real, $x, y \in \mathbb{R}$, and i is an imaginary number that satisfies $i^2 = -1$. We call x and y the real part and the imaginary part of z, respectively, and we write

$$x = \mathrm{Re}\,(z) \quad \text{and} \quad y = \mathrm{Im}\,(z).$$

The real numbers are complex numbers with zero imaginary parts. A complex number with zero real part is said to be purely imaginary.

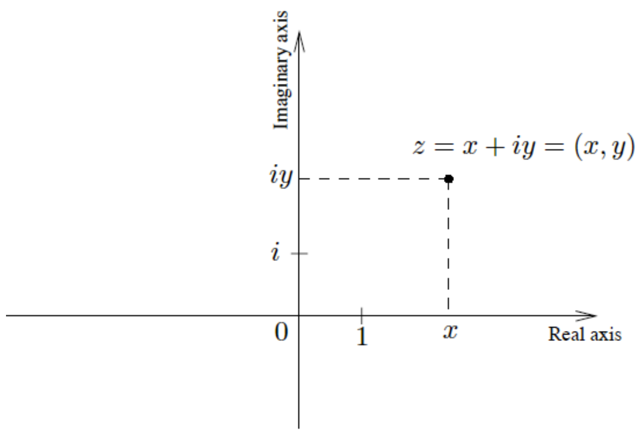
The complex conjugate of $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

The complex numbers can be visualized as the usual Euclidean plane:

$z = x + iy \in \mathbb{C}$ is identified with the point $(x, y) \in \mathbb{R}^2$.

- in this case 0 corresponds to the origin,
- i corresponds to $(0, 1)$.
- the x and y axis of $\mathbb{R}^2$ are called the real axis and imaginary axis respectively.



- **Polar coordinates.**

$$z = x + iy, \quad r = |z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}},$$

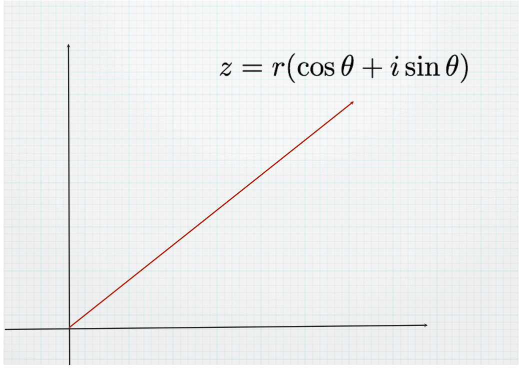
$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}.$$

and thus

$$z = r(\cos\theta + i\sin\theta).$$



Example. Let $z = 1 - i$. Then $r = \sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. Then

$$\theta = -\frac{\pi}{4} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

So $\arg z = -\pi/4 + 2\pi k$.

Definition. $\operatorname{Arg} z = \theta$ such that $-\pi < \theta \leq \pi$ is called the Principal value of the argument of z.

Example.

$$\operatorname{Arg}(1 - i) = -\frac{\pi}{4}.$$

Theorem. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Proof. Use elementary trigonometric formulae.

Corollary. (De Moivres formula)

$$z^n = r^n(\cos n\,\theta + i \sin n\,\theta), \quad n = 1, 2, 3, \ldots.$$

Abraham De Moivres (French, 1667-1754)

Remark. Theorem implies

$$\arg z_1 + \arg z_2 = \arg(z_1 \cdot z_2),$$

however,

$$\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg}(z_1 \cdot z_2).$$

WHY ???

Section: Sets in the complex plane

Definition. Let $z_0 \in \mathbb{C}$ and $r > 0$. Define the open disc $D_r(z_0)$

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The boundary of the open or closed disc is the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The unit disc is the disc centred at the origin and of radius one

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an interior point of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$. The interior of Ω consists of all its interior points.

Definition. A set Ω is open if every point in that set is an interior point of Ω. This definition coincides precisely with the definition of an open set in $\mathbb{R}^2$.

Definition. A set Ω is *closed* if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

A set is closed if and only if it contains all its limit points. The closure of any set Ω is the union of Ω and its limit points, and is often denoted by $\overline{\Omega}$.

Definition. The *boundary* of a set Ω is equal to its closure minus its interior, and is often denoted by $\partial\Omega$.

Definition. A set Ω is *bounded* if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$.

Definition. If Ω is bounded, we define its *diameter* by

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|.$$

Definition. A set Ω is said to be *compact* if it is closed and bounded. Arguing as in the case of real variables, one can prove the following.

Theorem. The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

An open covering of Ω is a family of open sets $\{U_\alpha\}$ (not necessarily countable) such that

$$\Omega \subset \cup_\alpha U_\alpha.$$

In analogy with the situation in $\mathbb{R}^2$, we have the following equivalent formulation of compactness.

Theorem. A set Ω is compact if and only if every open covering of Ω has a finite subcovering.

Another property of compactness is that of "nested sets".

Theorem. If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \ldots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Proof. Choose a point z_n in each Ω_n. The condition $\operatorname{diam}(\Omega_n) \to 0$ says that $\{z_n\}$ is a Cauchy sequence, therefore this sequence converges to a limit that we call w. Since each set Ω_n is compact we must have $w \in \Omega_n$ for all n. Finally, w is the unique point satisfying this property, for otherwise, if w' satisfied the same property with $w' \neq w$ we would have $|w' - w| > 0$ and the condition $\operatorname{diam}(\Omega_n) \to 0$ would be violated.

Definition. An open set Ω is *connected* if and only if any two points in Ω can be joined by a curve γ entirely contained in Ω.

Quizzes

Question 1: Let $z_1 = 1 + i$ and $z_2 = 2 + i$. Is it true that $z_2 > z_1$?

Answers:

A. Yes

B. No

Question 2: Let $z_1 = 1 + i$ and $z_2 = 2 + i$. Is it true that $|z_2| > |z_1|$?

Answers:

A. Yes

B. No

Question 3: Let $z = 1 + i$. What is $|z|$?

Answers:

A. $|z| = 1$

B. $|z| = 2$

C. $|z| = \sqrt{2}$

Question 4: Let $z = e^{in}$. What is $|z|$?

Answers:

A. $|z| < 1$

B. $|z| = 1$

C. $|z| > 1$

Thank you