Mathematics Year 1, Calculus and Applications I

List of Quizzes

The number corresponds to the Recording number

19. (i) Consider the particular case of the centre of mass of a plate of area A that has unit density per unit area. Write down formulas for the centre of mass in terms of double integrals.

(ii) Now take A to be the region $0 \le x \le 1$, $0 \le y \le 1$ (unit density also). Compute the double integrals to show that the centre of mass is $(1/2, 1/2)$ as expected by intuition.

(iii) Now take A as in part (ii) but assume that the density in half the square between $0 \le x \le 1/2$ is 1, and the density in the remaining half of the square is ρ_0. Compute the centre of mass in this case.

(iv) Check that your formula in (iii) agrees with your answer in (ii) when $\rho_0 = 1$. What happens as ρ_0 tends to 0 and ∞? Explain using physical intuition.

Solution:

(i) Consider a region A in the plane. Given any point $(x, y) \in A$, consider a small rectangle of size $dx \times dy$ whose centre is (x, y). Now take a moment about the x axis, and since the density is unity, this moment is $y\,dx\,dy$, and summing over all rectangles we find $\int\int y\,dx\,dy$. Similarly, a moment about the $y-$axis gives $\int\int x\,dx\,dy$. (It is understood that the double integrals are over the region A.) Now if $(\overline{x}, \overline{y})$ is the centre of mass of the region, by balancing moments we find (using the fact that the total mass of the region is its area $A = \int\int dx\,dy$)

$$\overline{x} = \frac{\int\int x\,dx\,dy}{\int\int dx\,dy}, \qquad \overline{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}.$$

(ii) For the square region of uniform density we have

$$\overline{x} = \frac{\int_0^1\int_0^1 x\,dx\,dy}{\int_0^1\int_0^1 dx\,dy} = \frac{\int_0^1 (1/2)dy}{1} = \frac{1}{2}$$

and

$$\overline{y} = \frac{\int_0^1\int_0^1 y\,dx\,dy}{\int_0^1\int_0^1 dx\,dy} = \frac{\int_0^1 \left(\int_0^1 y\,dy\right)\,dx}{1} = \frac{1}{2}.$$

(iii) Due to symmetry we still have $\overline{y} = 1/2$. Now for $\overline{x}$ we have

$$\overline{x} = \frac{\int_0^1 \left(\int_0^{1/2} x dx + \int_{1/2}^1 \rho_0 x dx\right) dy}{\frac{1}{2}(1+\rho_0)} = \frac{\frac{1}{8} + \rho_0(\frac{1}{2} - \frac{1}{8})}{\frac{1}{2}(1+\rho_0)}$$

(iii) Setting $\rho_0 = 1$ we recover the result in item (i). As $\rho_0 \to 0$ we find $\overline{x} \to 1/4$ which is expected since the region with any mass is in $0 \le x \le 1/2$, $0 \le y \le 1$ and has uniform density there. Now sending $\rho_0 \to \infty$ we find $\overline{x} \to 3/4$ which again makes physical sense since the mass is now non-zero and of uniform density in the region $1/2 \le x \le 1$, $0 \le y \le 1$.

20. (a) Consider the functions $f(x) = x^m$ and $g(x) = x^n$ where $0 < m < n$. Find a condition satisfied by m and n that guarantees that the centre of mass of the region between the graphs $y = f(x)$ and $y = g(x)$ is *outside* the region.

(b) We know from Example 2 from Recording 20 that $m = 1$, $n = 2$ has the centre of mass within the region. What are the smallest integer pair m, n for which the centre of mass is outside the region?

Solution:

(a) The region between the graphs is in $0 \leq x \leq 1$. Since $m < n$ then $x^m \geq x^n$ in this interval, and hence the centre of mass is given by

$$\overline{x} = \frac{\int_0^1 x(x^m - x^n)dx}{\int_0^1 (x^m - x^n)dx}, \qquad \overline{y} = \frac{\int_0^1 \frac{1}{2}(x^m + x^n)(x^m - x^n)dx}{\int_0^1 (x^m - x^n)dx} = \frac{\int_0^1 \frac{1}{2}(x^{2m} - x^{2n})dx}{\int_0^1 (x^m - x^n)dx}$$

which give

$$\overline{x} = \frac{\frac{1}{m+2} - \frac{1}{n+2}}{\frac{1}{m+1} - \frac{1}{n+1}} = \frac{(m+1)(n+1)}{(m+2)(n+2)}, \qquad \overline{y} = \frac{(m+1)(n+1)}{(2m+1)(2n+1)}.$$

Now $(\overline{x}, \overline{y})$ must be inside the region, hence the condition is

$$\frac{(m+1)^n(n+1)^n}{(m+2)^n(n+2)^n} \leq \frac{(m+1)(n+1)}{(2m+1)(2n+1)} \leq \frac{(m+1)^m(n+1)^m}{(m+2)^m(n+2)^m}.$$

21. Use the results developed in Recording 21 to show that a circle of radius a has perimeter $2\pi a$ and area πa^2.

Solution: Here we use polar coordinates so that the perimeter $P = \int_0^{2\pi} r d\theta = \int_0^{2\pi} a\, d\theta = 2\pi a$, and the area $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2} a^2\, d\theta = \pi a^2$.

22. (a) For what real values of α (if any at all) does the series $\sum_{n=1}^{\infty} \frac{e^{\alpha n}}{n}$ converge?

(b) Calculate $\sum_{n=100}^{\infty} \frac{1}{n^{1/100}}$.

(c) Let $P = \sum_{n=1}^{K} 3^n$ and $Q = \sum_{n=1}^{K} (1/3)^n$. Find P, Q and PQ. Which ones converge as $K \to \infty$? Could you have anticipated this without calculations?

Solution:

(a) Necessary condition is $\alpha < 0$, because if $\alpha > 0$ the general term does not tend to 0 as $n \to \infty$, and if $\alpha = 0$ we have the harmonic series which is divergent. With $\alpha < 0$ the ratio test gives

$$\frac{a_{n+1}}{a_n} = e^{\alpha} \frac{n}{n+1} \to e^{\alpha} < 1.$$

[Note: Could do the whole thing with the ratio test from the outset.]

(b) Comparison test with $\int_{100}^{\infty} \frac{dx}{x^{1/100}}$ which diverges.

(c) These are geometric series and so we have

$$P = \frac{3 - 3^{K+1}}{(1-3)} = \frac{3^{K+1} - 3}{2}, \qquad Q = \frac{\frac{1}{3} - \frac{1}{3^{K+1}}}{1 - \frac{1}{3}} = \frac{1}{2} \frac{3^{K+1} - 3}{3^{K+1}}$$

Clearly P diverges but Q converges (to 1/2). PQ also diverges. Could have anticipated this: P cannot converge since its nth term does not tend to zero, and Q converges - it is a geometric series. Hence the product cannot converge.

23. (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n^2}$ converge?

(b) Does the series $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converge?

(c) Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n}$ convergent? Is it absolutely convergent?

Solution:

(a) Since $0 < \frac{\log n}{n^2} \to 0$ as $n \to \infty$, the series has alternating decreasing terms and hence converges by the Alternating Series Test.

(b) Again use the integral test - it is enough to show the existence of $\int_1^{\infty} \frac{\log x}{x^2} dx$. Integrate by parts

$$\lim_{M\to\infty} \int_1^M \frac{\log x}{x^2} dx = \lim_{M\to\infty} \left(\log x \left(-\frac{1}{x}\right)\Big|_1^M - \int_1^M (-1/x)(1/x)dx \right) = 1,$$

hence the integral converges.

(c) Alternating series with $\lim_{n\to\infty} \frac{\log n}{n} = 0$, hence converges. Not absolutely convergent since $\sum_1^{\infty} \frac{\log n}{n} > \sum_1^{\infty} \frac{1}{n} = \infty$. [Or use the integral test directly.]

24. (a) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by using (i) the comparison test, (ii) the integral test.

(b) Does the series $\sum_{n=1}^{\infty} \frac{3n+\sqrt{n}}{2n^{3/2}+2}$ converge or diverge?

(c) For what values of $p > 0$ does the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge?

Solution:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, and $\int_1^{\infty} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, hence convergence by the integral test.

(b) Diverges because (for example) $\frac{3n+\sqrt{n}}{2n^{3/2}+2} > \frac{3n}{3n^{3/2}} = \frac{1}{n^{1/2}}$ and we know $\sum_1^{\infty} \frac{1}{n^{1/2}}$ diverges by the integral test, for example.

(c) Use the integral test, i.e. we need to consider

$$\lim_{M\to\infty} \int_2^M \frac{dx}{x(\log x)^p} = \lim_{M\to\infty} \left[\frac{(\log x)^{-p+1}}{(-p+1)} \right]_2^M,$$

hence we require $p > 1$ for convergence. If $p = 1$ we need to consider $\int_2^{\infty} \frac{dx}{x \log x} = [\log(\log x)]_2^{\infty} = \infty$.