S1 Solution

1. (a) The sequential probability test will require fewer observations on average to achieve the same size and power. $\boxed{\text{seen} \Downarrow}$

   Also acceptable: Approximate thresholds can be obtained for the sequential probability ratio test even when the distribution of the likelihood ratio test statistic is unknown. $\boxed{2}$

   (b) $A \le \frac{1-\alpha_2}{\alpha_1} \iff \alpha_2 \le 1 - A\alpha_1$ and $B \ge \frac{\alpha_2}{1-\alpha_1} \iff \alpha_2 \le B - B\alpha_1$. The shaded region is a quadrilateral delimited by the the axes and the lines $\alpha_2 = 1 - 4\alpha_1$ and $\alpha_2 = \frac{1}{4} - \frac{1}{4}\alpha_1$. $\boxed{\text{part seen} \Downarrow}$

   When the two lines intersect $1 - 4\alpha_1 = \frac{1}{4} - \frac{1}{4}\alpha_1 \implies \alpha_1 = \frac{1}{5}$ and $\alpha_2 = 1 - 4\frac{1}{5} = \frac{1}{5}$. Hence the four vertices of the shaded region are $(0,0)$, $(0,\frac{1}{4})$, $(\frac{1}{5},\frac{1}{5})$ and $(\frac{1}{4},0)$.
   [3 points for the graph and 3 points for the vertex coordinates] $\boxed{6}$

   (c) $\frac{f_1(0)}{f_0(0)} = \frac{0.2}{0.5} = 0.4$ and $\frac{f_1(1)}{f_0(1)} = \frac{0.8}{0.5} = 1.6$.
   Hence $\lambda_1 = 1.6$, $\lambda_2 = 1.6 \cdot 0.4 = 0.64$, and $\lambda_3 = 0.64 \cdot 0.4 = 0.256$.
   Since $\lambda_3 \le \frac{1}{3}$ the algorithm terminates after 3 observations and we accept H_0. $\boxed{4}$

   (d) (i)

$$\begin{aligned}
\frac{P_1(S=(x_1,\dots,x_n))}{P_0(S=(x_1,\dots,x_n))} &= \cancel{\frac{P_1(S=(x_1,\dots,x_n)|N=n)P(N=n)}{P_0(S=(x_1,\dots,x_n)|N=n)P(N=n)}} \\
&= \cancel{\frac{P_1(S=(x_1,\dots,x_n)|N=n)}{P_0(S=(x_1,\dots,x_n)|N=n)}} \\
&= \frac{\prod_{m=1}^{n} f_1(x_m)}{\prod_{m=1}^{n} f_0(x_m)} \\
&= \lambda_n
\end{aligned}$$

Correction made after exam.

$\boxed{2}$

   Because $S = (x_1,\dots,x_n)$ leads to accepting H_1, we must have $\lambda_n \ge A$. Hence $\frac{P_1(S=(x_1,\dots,x_n))}{P_0(S=(x_1,\dots,x_n))} \ge A$. $\boxed{2}$

   (ii) Let $Q_1 = \{s : S = s$ leads to the decision "Accept H_1"$\}$. Then the event "H_1 is accepted" is the same as the event $S \in Q_1$. By the law of total probability $P_1(S \in Q_1) = \sum_{s\in Q_1} P_1(S = s)$. By (d)(i) for each $s \in Q_1$ $P_1(S = s) \ge AP_0(S = s)$.
   Hence $P_1(S \in Q_1) \ge \sum_{s \in Q_1} AP_0(S = s) = A\sum_{s\in Q_1} P_0(S = s) = AP_0(S \in Q_1)$. $\boxed{4}$