MATH50001/50017/50018 - Analysis II

Complex Analysis

Lecture 21

Swapping two limits

Theorem.
Uniform limit of a sequence of continuous functions is continuous. Namely, let f_n be a sequence of continuous functions on $[a, b]$ and let $f_n \to f$ uniformly as $n \to \infty$. Then f is continuous.

Remark. In this case if $x_0 \in [a, b]$. Then

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = f(x_0).$$

Proof. We want to show that for any $\varepsilon >$ there is $\delta > 0$ such that for $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

Indeed, let $\varepsilon > 0$. Since $f_n \to f$ uniformly there N_0 such that for any $n > N_0$

$$|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x \in [a, b].$$

Fixing $n > N_0$ and using continuity of f_n we find $\delta > 0$ such that if $|x - x_0| < \delta$

$$|f_n(x) - f_n(x_0)| < \varepsilon/3.$$

Finally we obtain

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon.$$

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\oint_T f_n(z)\,dz = 0, \qquad \text{for all}\, n.$$

By assumption$f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\oint_T f_n(z)\,dz = \oint_T f(z)\,dz.$$

Therefore

$$\oint_T f(z)\,dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω.

Univalent/conformal functions.

Consider a class S of univalent functions on a unit disc $\mathbb{D} = \{z : |z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$. For each $f \in S$ we have a Taylor series

$$f(z) = z + a_2 z^2 + a_3 z^3 \ldots, \qquad |z| < 1.$$

The leading example of a function from class S is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z\left(1 + z + z^2 + z^3 + \ldots\right)^2 = z + 2z^2 + 3z^3 \ldots.$$

The Koebe function maps the disc $\mathbb{D}$ on the

$$\Omega = \mathbb{C} \setminus (-\infty, -1/4)$$

Indeed, this could be seen by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

and observing that the function

$$w = \frac{1+z}{1-z}$$

maps conformally $\mathbb{D}$ onto $\operatorname{Re} w > 0$.

Closely relater to S is the class Σ of functions

$$g(z) = z + b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \dots .$$

which are holomorphic and univalent in $\{z : |z| > 1\}$.

Theorem. (The Area Theorem)

Let $g \in \Sigma$. Then

$$\sum_{n=1} n\, |b_n|^2 \leq 1.$$

Proof. We use the Green formula

$$\oint_\gamma P\,du + Q\,dv = \iint_\Omega \Big(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\Big)\,dudv.$$

Therefore applying the Green formula we find (with $Q = u, P = 0$, or $P = -v, Q = 0$)

$$\mathrm{Area}\,\Omega := |\Omega| = \int_\Omega du\,dv = \oint_\gamma u\,dv == -\oint_\gamma v\,du.$$

This also could be written as

$$\begin{aligned} |\Omega| = \frac{1}{2i}\,i\,\oint_\gamma (u\,dv - v\,du) = \frac{1}{2i}\,\oint_\gamma (u\,du + v\,dv) + \frac{1}{2i}\,i\,\oint_\gamma (u\,dv - v\,du) \\ = \frac{1}{2i}\,\oint_\gamma (u - iv)\,d(u + iv) = \frac{1}{2i}\,\oint_\gamma \overline{w}\,dw. \end{aligned}$$

Let $r > 1$ and let γ_r be the image under g of the circle $C_r = \{z : |z| = r\}$ and Ω_r be the set bounded by this image. Then

$$
\begin{aligned}
0 < |\Omega_r| = \frac{1}{2i} \oint_\gamma \overline{w}\, dw = \frac{1}{2i} \oint_{\{z\colon |z|=1\}} \overline{g(z)}\, g'(z)\, dz \\
= \frac{1}{2} \int_0^{2\pi} \left(r\, e^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n}\, r^{-n}\, e^{in\,\theta} \right) \\
\times \left(1 - \sum_{m=1}^{\infty} m\, b_m\, r^{-m-1}\, e^{-i(m+1)\theta} \right) r\, e^{i\theta}\, d\theta \\
= \pi \left(r^2 - \sum_{n=1}^{\infty} n\, |b_n|^2\, r^{-2n} \right), \qquad r > 1.
\end{aligned}
$$

Letting $r \to 1$ we complete the proof.

Corollary.

$$|b_n| \leq n^{-1/2}, \qquad n = 1, 2, 3, \ldots .$$

In particular, $|b_1| < 1$ with the equality iff g has the form

$$g(z) = z + b_0 + \frac{b_1}{z}, \quad \text{with} \quad |b_1| = 1.$$

Theorem. (mini-Bieberbach's Theorem)

If $f \in S$, then $|a_2| \leq 2$ with quality iff f is a rotation of the Koebe function.

Proof. It is easy to check that

$$g(z) = \left(f(1/z^2)\right)^{-1/2} = z - \frac{a_2}{2} z^{-1} + \cdots \in \Sigma.$$

The Area Theorem immediately implies $|a_2| \leq 2$ with the equality iff

$$g(z) = z - e^{i\theta}/z.$$

Computing f we find

$$f(z) = \frac{z}{(1 - e^{i\theta} z)^2} = e^{-i\theta} k(e^{i\theta} z).$$

Indeed,

$$\left(f(1/z^2\right)^{-1} = \frac{z^2 - e^{i\theta}}{z}.$$

Thank you

Good luck with the exam