Mathematical Logic (MATH6/70132;P65)
Problem Sheet 2: notes on solutions

[1] Suppose ϕ is a formula of L.
(a) By giving a deduction in L, show that

$$(\neg(\neg\phi)) \vdash_L \phi.$$

(Hint: Let χ be an axiom. Start the deduction off with $((\neg(\neg\phi)) \to ((\neg(\neg\chi)) \to (\neg(\neg\phi))))$.)
(b) Show that $((\neg(\neg\phi)) \to \phi)$ is a theorem of L.
(c) Use (b) to show that $(\phi \to (\neg(\neg\phi)))$ is a theorem of L.

Solution (a) Let ψ be an axiom. Omit some brackets.

1. $\neg\neg\phi \to (\neg\neg\psi \to \neg\neg\phi)$ (axiom A1)
2. $\neg\neg\phi$ (assumption)
3. $\neg\neg\psi \to \neg\neg\phi$ (1,2 and MP)
4. $(\neg\neg\psi \to \neg\neg\phi) \to (\neg\phi \to \neg\psi)$ (axiom A3)
5. $\neg\phi \to \neg\psi$ (3, 4 and MP)
6. $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$ (axiom A3)
7. $\psi \to \phi$ (5, 6 and MP)
8. ψ (axiom)
9. ϕ (7, 8 and MP)

(b) This follows immediately from (a) and the Deduction Theorem.

(c) By (b) (applied to the formula $(\neg\phi)$) we know that

$$((\neg\neg\neg\phi) \to \neg\phi)$$

is a theorem of L. As an Axiom of type A3 we have

$$(((\neg\neg\neg\phi) \to \neg\phi) \to (\phi \to (\neg\neg\phi))).$$

Using Modus Ponens it then follows that $(\phi \to (\neg\neg\phi))$ is a theorem of L.

[2] Suppose Γ is a set of formulas of L and ϕ is a formula. Suppose that $\Gamma \vdash_L \phi$ and v is a valuation with $v(\psi) = T$ for all $\psi \in \Gamma$. Show that $v(\phi) = T$.

Solution:

The proof is by induction on the length n of the deduction of ϕ from Γ. The base case $n = 1$ is where ϕ is an axiom of L or an element of Γ. In the first case $v(\phi) = T$ for any (propositional) valuation (as the axioms of L are tautologies). In the second case, there is nothing to prove.

So now suppose that the result holds for formulas which have a deduction of length $< n$ from Γ. If ϕ is an axiom or a formula in Γ, then we are in the base case again. Otherwise, ϕ is obtained by Modus Ponens, so there are formulas ψ and $(\psi \to \phi)$ appearing earlier in the deduction of ϕ from Γ. Let v be any valuation giving value T to all formulas in Γ. Then by inductive hypothesis $v(\psi \to \phi) = T$ and $v(\psi) = T$. By definition of a valuation, it follows that $v(\phi) = T$, as required.

[3] (a) Give a careful proof of the following facts, which we have been using a lot.

(a) (Unique reading lemma) Suppose ϕ is an L-formula. Then exactly one of the following occurs:

(i) ϕ is a propositional variable;

(ii) there exists a unique L-formula ψ such that ϕ is $(\neg\psi)$;

(iii) there exist unique L-formulas θ, χ such that ϕ is $(\theta \to \chi)$.

(b) Using (a), prove that if v is any function from the set of propositional variables of L to $\{T, F\}$, then there is a unique function w from the set of L-formulas to $\{T, F\}$ satisfying the following properties:

(i) $w(p_i) = v(p_i)$ for each propostional variable p_i;

(ii) for every L-formula ϕ we have $w(\phi) \neq w((\neg\phi))$;

(iii) for all L-formulas θ, χ we have $w((\theta \to \chi)) = F$ iff $w(\theta) = T$ and $w(\chi) = F$.

Solution: (a) A formula ϕ is a non-empty string $s_1 s_2 \ldots s_k$ of symbols from the alphabet. If symbol s_i is an opening bracket (then there is a unique $j > i$ such that s_j is the corresponding closing bracket). In this case, the substring $s_i \ldots s_j$ is a formula. All of this can be proved by induction on the number of opening brackets in the formula (which will be the same as the number of connectives in the formula and the number of closing brackets in the formula). The corresponding closing bracket can be located using a parity argument: assign $+1$ to (and -1 to) and 0 to all other symbols. Find the smallest j with $\sum_{i \leq k \leq j} s_k = 0$.

Now suppose ϕ is not just a single variable (ie. $k > 1$). So either

(A) ϕ is $(\neg\psi)$ for some formula ψ; or

(B) ϕ is $(\theta \to \chi)$ for some formulas θ, χ.

Note that (A) holds iff the second symbol in ϕ is $\neg$. Of course, if ψ, ψ' are formulas and the strings $(\neg\psi)$ and $(\neg\psi')$ are identical, then the strings ψ, ψ' are identical. So this gives the uniqueness in case (A).

It remains to show that if $\theta, \chi, \theta', \chi'$ are formulas and the strings $(\theta \to \chi)$ and $(\theta' \to \chi')$ are identical, then θ is equal to θ' and χ is equal to χ'. It will suffice to prove that θ and θ' are the same. If θ has length 1, then it is a variable, and this is the opening symbol of θ'. So θ' is also just the variable (as it's a formula with no opening bracket). If θ has length greater than 1, then its opening symbol is a (. Its partner closing bracket is uniquely determined in ϕ. But the same pair of brackets then determine θ' and so θ, θ' are equal.

(b) Note first that if w, w' are functions with these properties, then $w(\eta) = w'(\eta)$ for all formulas η: the proof is a straightforward induction on the number of connectives in η. So the issue here is the existence of w. We define $w(\eta)$ by induction on n, the number of connectives in η, assuming that w has been defined on formulas with fewer than n connectives (and that it has the properties (i-iii) for such formulas). By (a) we split into two (non-overlapping) cases. Either η is $(\neg\phi)$ for some formula ϕ or η is $(\theta \to \chi)$ for some formulas θ, χ. Moreover, in each of these cases there is no ambiguity about what ϕ, θ, χ are (by (a)). Thus, we can simply define $w(\eta)$ so that (ii) or (iii) holds, using the values of $w(\phi)$ (or $w(\theta), w(\chi)$) which we already have.

[4] A *ternary valuation* is a function f from the set of formulas of L to the set $\{0, 1, 2\}$ which satisfies the following 'truth table' rules:

$$f((\neg\phi)) = \begin{cases} 2 & \text{if } f(\phi) = 0, 1 \\ 0 & \text{if } f(\phi) = 2 \end{cases}$$

and

$$f((\phi \to \psi)) = \begin{cases} 0 & \text{if } f(\phi) \geq f(\psi) \\ f(\psi) & \text{otherwise} \end{cases}$$

A formula ϕ is called a *ternary tautology* if $f(\phi) = 0$ for all ternary valuations f.

(a) Let $\alpha(0) = \alpha(1) = T$ and $\alpha(2) = F$. Show that if f is a ternary valuation, then the composition $\alpha \circ f$ is an (ordinary) valuation.

(b) Show that the axioms of L of type A1 are ternary tautologies.

(c) Show that axioms of type A2 are ternary tautologies.

(d) Show that if $(\phi \to \psi)$ and ϕ are ternary tautologies then so is ψ.

(e) Show that the formula $(((\neg p) \to (\neg q)) \to (q \to p))$ is not a ternary tautology.

(f) Show that any formula of the form $((\psi \to \phi) \to ((\neg\phi) \to (\neg\psi)))$ is a ternary tautology.

(g) The formal system $\tilde{L}$ has the same formulas as L and deduction rule Modus Ponens, but has as axioms formulas of types A1 and A2 and all formulas as in (f). Explain why the formula in (e) is not a theorem of $\tilde{L}$.

Solution:

(a) Suppose f is a ternary valuation and let $v = \alpha \circ f$. Then for L-formulas ϕ, ψ:
$v(\neg\phi) = F \iff f(\neg\phi) = 2 \iff f(\phi) = 0, 1 \iff v(\phi) = T$. Also
$v(\phi \to \psi) = F \iff f(\phi) < f(\psi) = 2 \iff v(\phi) = T$ and $v(\psi) = F$.

(b) Suppose f is a ternary valuation and $f(\phi \to (\psi \to \phi)) \neq 0$. Then $f(\phi) < f(\psi \to \phi) \neq 0$. From the latter we get $f(\psi \to \phi) = f(\phi)$, a contradiction.

(c) Suppose f is a ternary valuation and let
$n = f((\psi \to (\phi \to \chi)) \to ((\psi \to \phi) \to (\psi \to \chi)))$. Suppose for a contradiction that $n \neq 0$. Then

$$f(\psi \to (\phi \to \chi)) < f((\psi \to \phi) \to (\psi \to \chi)) = n \tag{1}$$

and so (as $n > 0$) we must also have

$$f((\psi \to \phi)) < f((\psi \to \chi)) = n. \tag{2}$$

Similarly, it then follows that

$$f(\psi) < f(\chi) = n. \tag{3}$$

Note that: if $f(\psi \to \phi) = 0$, then $f(\phi) \leq f(\psi) < f(\chi)$ (by (3)); if $f(\psi \to \phi) \neq 0$, then $f(\phi) = f(\psi \to \phi) < n = f(\chi)$ (by (2) and (3)). Thus $f(\phi) < f(\chi)$. It follows that $f(\phi \to \chi) = f(\chi)$ and so (by (3)) $f(\psi \to (\phi \to \chi)) = f(\chi) = n$. This contradicts (1).

(d) If $f(\phi \to \psi) = 0$ and $f(\phi) = 0$ then we must have $0 \geq f(\psi)$, whence $f(\psi) = 0$.

(e) Consider a ternary valuation f with $f(p) = 1$ and $f(q) = 0$. The formula then has value 1 as given by f.

(f) Suppose f is a ternary valuation and f of the formula is non-zero. Then $f(\psi \to \phi) < f(\neg\phi \to \neg\psi) = f(\neg\psi) = 2$. So $f(\neg\phi) = 0$ and thus $f(\phi) = 2$ and $f(\psi) = 0, 1$, which gives $f(\psi \to \phi) = 2$, a contradiction.

(g) Any theorem of $\tilde{L}$ is a ternary tautology: the axioms of this system are ternary tautologies, and the deduction rule preserves ternary tautologies. Thus the formula in (e) is not a theorem of $\tilde{L}$.