

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**  
**May-June 2022**

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Group Representation Theory**

Date: 26 May 2022

Time: 12:00 – 14:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD  
WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS  
ANSWERED AND PAGE NUMBERS PER QUESTION.**

All answers must be accompanied by adequate justification. All vector spaces are over the complex numbers. You may use results from lectures, lecture notes, problem sheets, and coursework, without further justification. You may use preceding parts in your solution to later parts (even if you did not solve the former).

1. (a) Let  $G$  be a finite group and  $(V, \rho)$  a finite-dimensional representation. Let  $Z(G)$  denote the centre of  $G$ :

$$Z(G) = \{z \in G \mid zh = hz, \forall h \in G\}.$$

Consider the statements for  $g \in G$ :

(A)  $\rho(g) = \lambda I$  for some  $\lambda \in \mathbb{C}$ ;

(B)  $g \in Z(G)$ .

(i) Show that, if  $\rho$  is injective, then (A) implies (B). (3 marks)

(ii) Show that, if  $(V, \rho)$  is irreducible, then (B) implies (A). (3 marks)

- (b) Let  $G = C_n$  be the cyclic group of finite order  $n \geq 1$ . Find the number of non-isomorphic three-dimensional representations of  $G$ . (6 marks)

- (c) Let  $U, W$  be irreducible representations of a finite group  $G$  and let  $V$  be any finite-dimensional representation of  $G$ . Show that the multiplicity of  $W$  in  $U \otimes V$  equals the multiplicity of  $U$  in  $V^* \otimes W$ .

*Hint: Give an isomorphism  $\text{Hom}_{\mathbb{C}}(U \otimes V, W) \cong \text{Hom}_{\mathbb{C}}(U, \text{Hom}_{\mathbb{C}}(V, W))$ .*

(8 marks)

(Total: 20 marks)

2. Fix  $n \geq 1$  and let  $G = \{g^i h^j z^k \mid 0 \leq i, j, k < n\}$  be the group of order  $n^3$  with multiplication defined by  $gz = zg, hz = zh, gh = zhg$  and  $g^n = h^n = z^n = 1$ .

- (a) Prove that for every irreducible representation  $(V, \rho)$  of  $G$ , we have  $\rho(z) = \lambda_\rho I$  for some  $\lambda_\rho \in \mathbb{C}$  with  $\lambda_\rho^n = 1$ . Moreover, show that this  $\lambda_\rho \in \mathbb{C}$  is uniquely determined by the isomorphism class of  $\rho$ . (4 marks)

- (b) Prove that  $z$  is in the kernel of every one-dimensional representation of  $G$ . (3 marks)

- (c) Prove that  $G$  has  $n^2$  non-isomorphic one-dimensional representations. (5 marks)

- (d) Prove that  $G$  has an irreducible representation of dimension  $n$ .

*Hint: Let  $g$  act by a diagonal matrix with eigenvalues  $1, \zeta, \dots, \zeta^{n-1}$  for  $\zeta$  a primitive  $n$ -th root of unity, and let  $h$  act by a cyclic permutation matrix.* (5 marks)

- (e) When  $n = p$  is a prime number, classify all of the irreducible representations of  $G$ .

*Hint: Find irreducible representations for each possible value of  $\lambda_\rho$ .* (3 marks)

(Total: 20 marks)

3. Consider the following incomplete character table:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
size	1	5	5	5	16	16	16	16
$\chi_U$	1	1	1	1	$\zeta$	$\zeta^2$	$\zeta^3$	$\zeta^4$
$\chi_V$	5	-3	1	1	0	0	0	0

Here  $\zeta$  is a primitive fifth root of unity.

- (a) Find four other one-dimensional irreducible representations, using the tensor product. (3 marks)
- (b) You are also given the following information: there is an automorphism of the group which induces a cyclic permutation of the three conjugacy classes of size 5. Using this, find two more characters of five-dimensional irreducible representations. (4 marks)
- (c) Find the number of irreducible representations of the group  $G$ . Using this, explain how to complete the character table using the answers to (a) and (b). (4 marks)
- (d) Compute the abstract symmetry group of the character table (a symmetry acts by simultaneously permuting some rows and permuting some columns). (Here “abstract” means that the elements don’t have to be explicitly written as permutations of the table.) (5 marks)
- (e) The following fact is true: for any finite group  $G$  and any irreducible representation  $V$ ,  $\dim V$  is a factor of  $|G|$ . Using this fact, show that if  $G$  is a finite nonabelian group with a prime number  $p$  of one-dimensional representations up to isomorphism, and with all other irreducible representations of dimension  $n$ , then  $n = p$ . (4 marks)

(Total: 20 marks)

4. Let  $A$  be a finite-dimensional algebra with exactly two simple modules up to isomorphism, of dimensions 1 and 4. Call them  $V_1$  and  $V_4$ , respectively.

- (a) Show that  $\dim A \geq 17$ , and that equality holds if and only if  $A$  is semisimple. (6 marks)
- (b) Let  $W$  be an  $A$ -module of dimension 3. Show that  $\chi_W = 3\chi_{V_1}$ .  
*Hint: Recall from lectures the formula  $\chi_U = \chi_{U_1} + \chi_{U/U_1}$ , for  $U_1 \subseteq U$  a subrepresentation.* (5 marks)
- (c) Suppose that  $\dim A = 18$ . Show that for every finite-dimensional  $A$ -module  $W$ , there is a submodule  $U \subseteq W$  satisfying  $U \cong V_1^m = V_1 \oplus \cdots \oplus V_1$  ( $m$  times), for some  $m$ , so that  $W/U$  is semisimple. (6 marks)
- (d) Again assume  $\dim A = 18$ . If  $W$  is an  $A$ -module such that  $\chi_W = n\chi_{V_4}$ , show that  $W \cong V_4^n = V_4 \oplus \cdots \oplus V_4$  ( $n$  times).  
*Hint: Use part (c).* (3 marks)

(Total: 20 marks)

5. (a) Let  $D_n$  be the dihedral group acting on a regular  $n$ -gon (of size  $2n$ ) and  $C_n < D_n$  the subgroup of rotational symmetries. Let  $V$  be the vector space of labellings of the vertices of the  $n$ -gon by complex numbers, viewed as a  $D_n$ -representation by applying the symmetries.
- (i) For  $\zeta \in \mathbf{C}$  an  $n$ -th root of unity, let  $V_\zeta \subseteq V$  be the one-dimensional subspace of labellings of the form  $(a, \zeta a, \zeta^2 a, \dots, \zeta^{n-1} a)$ , going in cyclic counterclockwise order. Show that  $V = \bigoplus_\zeta V_\zeta$  is a decomposition of  $V$  into irreducible representations of  $C_n$  (here we sum  $\zeta$  over all  $n$ -th roots of unity). (3 marks)
  - (ii) Give an isomorphism  $\text{Res}_{C_n}^{D_n} \text{coInd}_{C_n}^{D_n} V_\zeta \cong V_\zeta \oplus V_{\zeta^{-1}}$ . (3 marks)
  - (iii) Show that  $\text{coInd}_{C_n}^{D_n} V_\zeta$  is an irreducible representation of  $D_n$  if and only if  $\zeta \neq \zeta^{-1}$ . (3 marks)
  - (iv) Using this, give an explicit decomposition of  $V$  into irreducible representations of  $D_n$ , and relate these to the representations found in part (iii). (3 marks)
- (b) Let  $N \leq G$  be a normal subgroup, for  $G$  any group (not necessarily finite). Suppose that  $[G : N] = k < \infty$  and let  $g_1, \dots, g_k \in G$  be coset representatives. Let  $\text{Ad}_g : N \rightarrow N$  be the map  $\text{Ad}_g(n) = gng^{-1}$ .
- (i) Let  $(W, \rho_W)$  be an irreducible finite-dimensional representation of  $N$ . Prove that  $\chi_{\text{coInd}_N^G W}$  is only nonzero on  $N$ , and  $\chi_{\text{coInd}_N^G W}|_N = \sum_i \chi_W \circ \text{Ad}_{g_i}$ . (3 marks)
  - (ii) Deduce that, if  $V \subseteq \text{Res}_N^G \text{coInd}_N^G W$  is a subrepresentation, then  $\dim V$  is a multiple of  $\dim W$ .  
*Hint: see the hint in Problem 4b.* (2 marks)
  - (iii) Now suppose that, for every  $g \notin N$ , we have  $\chi_W \circ \text{Ad}_g \neq \chi_W$ . Show that  $\text{coInd}_N^G W$  is irreducible. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60039/70039/97035

Group Representation Theory (Solutions)

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1. (a) Let  $G$  be a finite group and  $(V, \rho)$  a finite-dimensional representation. Let  $Z(G)$  denote the centre of  $G$ :

$$Z(G) = \{z \in G \mid zh = hz, \forall h \in G\}.$$

Consider the statements for  $g \in G$ :

(A)  $\rho(g) = \lambda I$  for some  $\lambda \in \mathbb{C}$ ;

(B)  $g \in Z(G)$ .

sim. seen ↓

- (i) Show that, if  $\rho$  is injective, then (A) implies (B).

Here  $\rho(g)\rho(h) = \lambda\rho(h) = \rho(h)\rho(g)$ . So  $\rho(gh) = \rho(hg)$ . By injectivity,  $gh = hg$ . As  $h$  was arbitrary,  $g$  is central.

3, A

sim. bookwork ↓

- (ii) Show that, if  $(V, \rho)$  is irreducible, then (B) implies (A).

This is a consequence of Schur's Lemma, as explained in lectures. In more detail, if  $g \in Z(g)$ , then  $\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g)$  for all  $h \in G$ , so  $\rho(g)$  is  $G$ -linear. As  $V$  is a finite-dimensional complex representation, Schur's Lemma states that (A) holds.

3, A

sim. seen ↓

- (b) Let  $G = C_n$  be the cyclic group of order  $n < \infty$ . Find the number of non-isomorphic three-dimensional representations of  $G$ .

Every finite-dimensional representation is a direct sum of irreducible representations: since  $G$  is finite, this is a consequence of Maschke's theorem discussed in lectures. In more detail, Maschke's Theorem states that all subrepresentations of a finite-dimensional complex representation of a finite group have a complementary subrepresentation; by induction on the dimension we obtain that there is a decomposition as a direct sum of irreducible subrepresentations. All irreducible representations are one-dimensional (a consequence of Schur's Lemma—every central element acts by a scalar as in part (a), and a scalar representation is only irreducible if the dimension is one; alternatively, this is part of the classification of irreducible representations of cyclic groups). There are  $n$  of these (either by explicit classification or because  $G$  has  $n$  conjugacy classes). There is a unique decomposition into irreducible representations up to isomorphism (by Schur's Lemma, as explained in lectures).

Put together we see that the number of  $m$ -dimensional representations of  $C_n$  up to isomorphism is the number of ways of partitioning  $m$  into  $n$  cells, labelled by the irreducible representations. This is given by  $\binom{m+n-1}{m}$ . (For example, when  $m = 2$  we have  $\binom{n+1}{2} = n + \binom{n}{2}$ , where  $n$  represents a sum of two isomorphic one-dimensional representations and  $\binom{n}{2}$  represents a sum of two non-isomorphic ones). In particular in the case at hand we get  $\binom{n+2}{3}$ .

6, B

sim. seen ↓

- (c) Let  $U, W$  be irreducible representations of a finite group  $G$  and let  $V$  be any finite-dimensional representation of  $G$ . Show that the multiplicity of  $W$  in  $U \otimes V$  equals the multiplicity of  $U$  in  $V^* \otimes W$ .

*Hint: Give an isomorphism  $\text{Hom}_{\mathbb{C}}(U \otimes V, W) \cong \text{Hom}_{\mathbb{C}}(U, \text{Hom}_{\mathbb{C}}(V, W))$ .*

The isomorphism desired in the hint is  $T \mapsto T'$  where  $T'(u)(v) = T(u \otimes v)$ . This is an isomorphism: the inverse sends  $T'$  to the linear map given by the universal property for tensor product from the bilinear map  $U \times V \rightarrow W, (u, v) \mapsto T'(u)(v)$ . The isomorphism is compatible with any structure of representation of  $G$  on  $U, V$ , and  $W$ , by bilinearity of the tensor product.

Moreover since  $G$  is finite,  $U$  and  $W$  are finite-dimensional. For finite-dimensional vector spaces we have  $\text{Hom}(X, Y) \cong \text{Hom}(Y, X)^*$ . In particular if  $X$  is irreducible then we get that the multiplicity of  $X$  in  $Y$ ,  $\dim \text{Hom}_G(X, Y)$ , also equals  $\dim \text{Hom}_G(Y, X)$ , a fact also mentioned in lectures.

Now by the first paragraph we have an isomorphism  $\text{Hom}_G(U \otimes V, W) \cong \text{Hom}_G(U, \text{Hom}_{\mathbb{C}}(V, W))$ . Since  $V$  is finite dimensional, the second space is isomorphic to  $\text{Hom}_G(U, V^* \otimes W)$ . Taking dimensions and applying the preceding paragraph, the multiplicity of  $W$  in  $U \otimes V$  equals the multiplicity of  $U$  in  $V^* \otimes W$ .

8, D

2. Fix  $n \geq 1$  and let  $G = \{g^i h^j z^k \mid 0 \leq i, j, k < n\}$  be the group of order  $n^3$  with multiplication defined by  $gz = zg, hz = zh, gh = zhg$  and  $g^n = h^n = z^n = 1$ .

sim. seen ↓

- (a) Prove that for every irreducible representation  $(V, \rho)$  of  $G$ , we have  $\rho(z) = \lambda_{\rho}I$  for some  $\lambda_{\rho} \in \mathbb{C}$  with  $\lambda_{\rho}^n = 1$ . Moreover, show that this  $\lambda_{\rho} \in \mathbb{C}$  is uniquely determined by  $\rho$  up to isomorphism.

The first statement is a consequence of Schur's Lemma, because  $z$  is central. The uniqueness of  $\lambda_{\rho}$  follows because  $\lambda_{\rho}I$  is invariant under conjugation by an invertible transformation (or,  $\lambda_{\rho}$  is the unique eigenvalue of  $\rho(z)$ ).

4, A

- (b) Prove that  $z$  is in the kernel of every one-dimensional representation of  $G$ .

sim. seen ↓

This is because  $z = ghg^{-1}h^{-1}$ , and if  $V$  is one-dimensional, then  $\text{GL}(V) = \mathbb{C}^{\times} I_V$  is abelian.

3, A

- (c) Prove that  $G$  has  $n^2$  non-isomorphic one-dimensional representations.

sim. seen ↓

Each representation is determined by the image of  $g$  and of  $h$ , each of which are multiples of the identity by  $n$ -th roots of unity. There are  $n^2$  such images. Conversely, each pair  $\lambda_g, \lambda_h$  of  $n$ -th roots of unity defines a representation  $\rho(g^i h^j z^k) = \lambda_g^i \lambda_h^j I$ . These are non-isomorphic and every one-dimensional representation is isomorphic to one of these, by any linear isomorphism of one-dimensional vector spaces. (Alternatively, one could have reduced the question to one-dimensional matrix representations from the beginning, simplifying this discussion slightly.)

5, B

- (d) Prove that  $G$  has an irreducible representation of dimension  $n$ .

unseen ↓

*Hint: Let  $g$  act by a diagonal matrix with eigenvalues  $1, \zeta, \dots, \zeta^{n-1}$  for  $\zeta$  a primitive  $n$ -th root of unity, and let  $h$  act by a cyclic permutation matrix.*

Following the hint, consider the  $n$ -dimensional matrix representation  $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C})$  where  $\rho(g)$  is the diagonal matrix with entries given (in order) and  $\rho(h)$  is the cyclic permutation matrix  $P_{(123\dots n)}$ . Then  $\rho(h)\rho(g)\rho(h)^{-1}$  is the diagonal matrix with entries  $(\zeta^{n-1}, 1, \zeta, \dots, \zeta^{n-2}) = \zeta^{-1}\rho(g)$ . So  $\rho(g)\rho(h) = \zeta\rho(h)\rho(g)$ . We may therefore define  $\rho(z) = \zeta$  and we obtain a representation of  $G$ . To see that it is irreducible, note that any subrepresentation, to be fixed by  $\rho(g)$ , must be spanned by eigenvectors of  $\rho(g)$  (either by linear algebra since  $\rho(g)$  is diagonalisable, or by Maschke's theorem applied to the cyclic group  $\langle g \rangle \cong C_n$ ). This means that every subrepresentation is spanned by some of the standard basis vectors  $e_1, \dots, e_n$ . But to be fixed by the cyclic permutation matrix this must be either all of these or none. Thus  $\rho$  is irreducible.

5, C

- (e) When  $n = p$  is a prime number, classify all of the irreducible representations of  $G$ .

*Hint: Find irreducible representations for each possible value of  $\lambda_\rho$ .*

unseen ↓

By part (d) we can construct an irreducible representation with  $\lambda_\rho$  equal to every primitive  $n$ -th root of unity. For  $n = p$  prime this means every  $n$ -th root of unity except the identity. For  $\lambda_\rho = 1$  we can construct  $n^2$  one-dimensional representations by part (c). Put together the sum of squares of dimensions of these irreducible representations is  $(p-1)p^2 + p^2 = p^3 = |G|$ . By a theorem from lectures this implies we have found all of the irreducible representations of  $G$ : they are all isomorphic to one of the ones constructed in parts (c) and (d).

3, D

3. Consider the following incomplete character table:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
size	1	5	5	5	16	16	16	16
$\chi_U$	1	1	1	1	$\zeta$	$\zeta^2$	$\zeta^3$	$\zeta^4$
$\chi_V$	5	-3	1	1	0	0	0	0

Here  $\zeta$  is a primitive fifth root of unity.

sim. seen ↓

- (a) Find four other one-dimensional irreducible representations, using the tensor product.

We can tensor  $\chi_U$  with itself arbitrarily many times, and we get the same character as  $\chi_U$  except with  $\zeta$  replaced by every fifth root of unity. That is a total of five one-dimensional representations, four new ones.

3, A

- (b) You are also given the following information: there is an automorphism of the group which induces a cyclic permutation of the three conjugacy classes of size 5. Using this, find two more characters of five-dimensional irreducible representations. By cyclically permuting these conjugacy classes we get the characters  $(5, 1, -3, 1, 0, 0, 0, 0)$  and  $(5, 1, 1, -3, 0, 0, 0, 0)$ . These are the characters of the irreducible representations given by composing  $\rho_V$  with the automorphisms. This yields two new five-dimensional irreducible characters.

sim. seen ↓

4, B

- (c) Find the number of irreducible representations of the group  $G$ . Using this, explain how to complete the character table using the answers to (a) and (b). There are eight conjugacy classes, hence by results from lectures there are eight irreducible representations. Adding up the previously obtained representations, we get eight, so that the table is complete:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
size	1	5	5	5	16	16	16	16
$\chi_{\mathbb{C}}$	1	1	1	1	1	1	1	1
$\chi_U$	1	1	1	1	$\zeta$	$\zeta^2$	$\zeta^3$	$\zeta^4$
$\chi_{U^{\otimes 2}}$	1	1	1	1	$\zeta^2$	$\zeta^4$	$\zeta$	$\zeta^3$
$\chi_{U^{\otimes 3}}$	1	1	1	1	$\zeta^3$	$\zeta$	$\zeta^4$	$\zeta^2$
$\chi_{U^{\otimes 4}}$	1	1	1	1	$\zeta^4$	$\zeta^3$	$\zeta^2$	$\zeta$
$\chi_V$	5	-3	1	1	0	0	0	0
$\chi_{V'}$	5	1	-3	1	0	0	0	0
$\chi_{V''}$	5	1	1	-3	0	0	0	0

4, A

unseen ↓

- (d) Compute the abstract symmetry group of the character table (a symmetry acts by simultaneously permuting some rows and permuting some columns). (Here “abstract” means that the elements don’t have to be explicitly written as permutations of the table.)

Visibly we can permute the three conjugacy classes of size five arbitrarily, which gives a symmetry group  $S_3$  (each permutation induces a corresponding permutation of the five-dimensional irreducible representations). We can also permute the size 16 conjugacy classes by all powers of the cyclic permutation  $(1342)$  which sends  $\chi_U$  to  $\chi_{U^{\otimes 2}}$ . These symmetries are independent and we get the overall symmetry group  $S_3 \times C_4$ .

(Remark: the outer automorphism group of the one whose table is above is actually an index-two subgroup of this symmetry group, the dicyclic group of order twelve.)

5, C

sim. seen ↓

- (e) The following fact is true: for any finite group  $G$  and any irreducible representation  $V$ ,  $\dim V$  is a factor of  $|G|$ . Using this fact, show that if  $G$  is a finite nonabelian group with a prime number  $p$  of one-dimensional representations up to isomorphism, and with all other irreducible representations of dimension  $n$ , then  $n = p$ .

If there are  $k$  representations of dimension  $n$ , then by the sum of squares formula,  $|G| = p + kn^2$ . By the fact that  $G$  is finite and nonabelian, we must have  $k \geq 1$  (e.g., by the sum of squares formula, for otherwise  $|G| = p$  is prime and hence  $G$  is cyclic and abelian, a contradiction). We have  $n | (p + kn^2)$ . This is equivalent to

$n \mid p$ . Note that since  $G$  is nonabelian, not every irreducible representation Since we are assuming  $n \neq 1$ , we get  $n = p$ .

4, A

4. Let  $A$  be a finite-dimensional algebra with exactly two simple modules up to isomorphism, of dimensions 1 and 4. Call them  $V_1$  and  $V_4$ , respectively.

- (a) Show that  $\dim A \geq 17$ , and that equality holds if and only if  $A$  is semisimple.

sim. seen ↓

We saw in lectures that the homomorphism  $A \rightarrow \text{End}(V_1) \oplus \text{End}(V_4)$  is surjective. Taking dimensions, and applying the fact that  $\text{End}(V_1) \oplus \text{End}(V_4)$  is semisimple (part of Artin–Wedderburn's theorem, or true because it is a semisimple module over itself, isomorphic to  $V_1 \oplus V_4^4$ ), we conclude the result.

6, A

- (b) Let  $W$  be an  $A$ -module of dimension 3. Show that  $\chi_W = 3\chi_{V_1}$ .

Hint: Recall from lectures the formula  $\chi_U = \chi_{U_1} + \chi_{U/U_1}$ , for  $U_1 \subseteq U$  a subrepresentation.

sim. seen ↓

We can find some simple submodule  $W_1 \subseteq W$ , and by dimension reasons it must be of dimension one. Iterating this for  $W/W_1$ , we find  $W_2 \subseteq W$ , with  $W_2 \supseteq W_1$  and  $W_2/W_1$  of dimension one. So we have a chain  $0 \subsetneq W_1 \subsetneq W_2 \subsetneq W$ . By assumption  $V_1 \cong W_1 \cong W_2/W_1 \cong W/W_2$ , since there is only one one-dimensional  $A$ -module up to isomorphism. Now we apply the fact mentioned in the hint (twice), and we obtain that  $\chi_W = 3\chi_{V_1}$ .

5, B

- (c) Suppose that  $\dim A = 18$ . Show that for every finite-dimensional  $A$ -module  $W$ , there is a submodule  $U \subseteq W$  satisfying  $U \cong V_1^m = V_1 \oplus \cdots \oplus V_1$  ( $m$  times), for some  $m$ , so that  $W/U$  is semisimple.

unseen ↓

Let  $U_1 \subseteq A$  be the kernel of the surjection  $A \rightarrow \text{End}(V_1) \oplus \text{End}(V_4)$ . By the rank-nullity theorem,  $\dim U_1 = 1$ . So  $U_1 \cong V_1$  by hypothesis. As explained in lectures, there exists a surjection  $\varphi : A^m \rightarrow W$  for some  $m$  (e.g.,  $m = \dim W$ ). The image of  $U_1^m \cong V_1^m$  is a quotient of a semisimple module and we saw in lectures it must also be semisimple and isomorphic to a direct sum of copies of  $V_1$ . Call this image  $U := \varphi(U_1^m)$ . Now the quotient module  $W/U$  has trivial action of  $U$ , so the homomorphism  $A \rightarrow \text{End}(W/U)$  factors through  $\text{End}(V_1) \oplus \text{End}(V_4)$ . As it is therefore a module over this semisimple quotient,  $W/U$  must be semisimple.

6, D

- (d) Again assume  $\dim A = 18$ . If  $W$  is an  $A$ -module such that  $\chi_W = n\chi_{V_4}$ , show that  $W \cong V_4^n = V_4 \oplus \cdots \oplus V_4$  ( $n$  times).

Hint: use part (c).

unseen ↓

Applying the formula from before,  $\chi_W = \chi_U + \chi_{W/U}$ . We can write  $\chi_U = k\chi_{V_1}$  for some  $k \geq 0$  and  $\chi_{W/U} = a\chi_{V_1} + b\chi_{V_4}$  for  $a, b \geq 0$ . Then  $n\chi_{V_4} = (k+a)\chi_{V_1} + b\chi_{V_4}$ . By linear independence of characters,  $k+a=0$  and  $b=n$ . So  $U=0$ . Thus  $W$  is semisimple. Therefore by linear independence of characters again, we have  $W \cong V_4^n$ .

3, C

5. (a) Let  $D_n$  be the dihedral group acting on a regular  $n$ -gon (of size  $2n$ ) and  $C_n < D_n$  the subgroup of rotational symmetries. Let  $V$  be the vector space of labellings of the vertices of the  $n$ -gon by complex numbers, viewed as a  $D_n$ -representation by applying the symmetries.

- (i) For  $\zeta \in \mathbb{C}$  an  $n$ -th root of unity, let  $V_\zeta \subseteq V$  be the one-dimensional subspace of labellings of the form  $(a, \zeta a, \zeta^2 a, \dots, \zeta^{n-1} a)$ , going in cyclic counterclockwise order. Show that  $V = \bigoplus_\zeta V_\zeta$  is a decomposition of  $V$  into irreducible representations of  $C_n$ .

It is clear that  $V_\zeta \subseteq V$  is a subrepresentation of  $C_n$ , with action of the generator  $g$  (the counterclockwise rotation by  $2\pi/n$ ) given by multiplication by  $\zeta^{-1}$ . We get a map  $\bigoplus_\zeta V_\zeta \rightarrow V$ . If the kernel is nonzero, it must contain an irreducible subrepresentation which must be of the form  $V_\zeta$  for some  $\zeta$  by results from lectures, but that is impossible since  $V_\zeta$  is a subspace of  $V$ . So the map is injective. By the rank-nullity theorem this injective map is also surjective, so  $V = \bigoplus_\zeta V_\zeta$  as desired. (Alternatively, we know that  $V$  is a direct sum of its eigenspaces under the action of the generator of  $C_n$ , and each eigenspace is nonzero and contains  $V_\zeta$ , hence must actually equal  $V_\zeta$  for dimension reasons.)

unseen ↓

3, M

sim. seen ↓

- (ii) Give an isomorphism  $\text{Res}_{C_n}^{D_n} \text{coInd}_{C_n}^{D_n} V_\zeta \cong V_\zeta \oplus V_{\zeta^{-1}}$ .

We see that  $\text{Res}_{C_n}^{D_n} \text{coInd}_{C_n}^{D_n} V_\zeta$  has dimension  $[D_n : C_n] \dim V_\zeta = 2$  and is spanned by  $V_\zeta$  and  $\rho(\sigma)V_\zeta$  for  $\sigma$  any reflection in  $D_n$ . For  $v \in V_\zeta$ , we have  $\rho(g)\rho(\sigma)v = \rho(\sigma)\rho(g^{-1})v = \zeta\rho(\sigma)v$  (here  $g$  is the generator of  $C_n$ ), so that  $\rho(\sigma)V_\zeta \cong V_{\zeta^{-1}}$ . This proves the isomorphism.

- (iii) Show that  $\text{coInd}_{C_n}^{D_n} V_\zeta$  is an irreducible representation of  $D_n$  if and only if  $\zeta \neq \zeta^{-1}$ .

If  $\zeta \neq \zeta^{-1}$  then we see that the only two  $C_n$ -subrepresentations of  $\text{coInd}_{C_n}^{D_n} V_\zeta$  are  $V_\zeta$  or  $V_{\zeta^{-1}}$ . But neither of these is closed under the action of  $\sigma$ . So  $\text{coInd}_{C_n}^{D_n} V_\zeta$  is irreducible. On the other hand, if  $\zeta = \zeta^{-1}$ , then  $v \pm \rho(\sigma)v$  both span one-dimensional subrepresentations of  $\text{coInd}_{C_n}^{D_n} V_\zeta$ : it is clearly closed under the action of  $g$  and  $\sigma$  and these generate  $D_n$ .

3, M

sim. seen ↓

- (iv) Using this, give an explicit decomposition of  $V$  into irreducible representations of  $D_n$ , and relate these to the representations found in part (iii).

We get that  $V$  is a direct sum of the two-dimensional irreducible representations  $V_\zeta \oplus V_{\zeta^{-1}}$  for  $\zeta \neq \zeta^{-1}$ , and the one-dimensional spaces  $V_\zeta$  for  $\zeta = \pm 1$  ( $-1$  only in the case that  $n$  is even). The latter are closed under the action of  $D_n$  by inspection. The former representations are isomorphic to  $\text{coInd}_{C_n}^{D_n} V_\zeta$ , whereas the latter are isomorphic to a summand of  $\text{coInd}_{C_n}^{D_n} V_\zeta$  (if we pick  $\sigma$  to be a reflection which fixes at least one vertex, then the summand is the one spanned by  $v + \rho(\sigma)v$  for any nonzero  $v \in V_\zeta$ ).

3, M

unseen ↓

- (b) Let  $N \leq G$  be a normal subgroup, for  $G$  any group (not necessarily finite). Suppose that  $[G : N] = k < \infty$  and let  $g_1, \dots, g_k \in G$  be coset representatives. Let  $\text{Ad}_g : N \rightarrow N$  be the map  $\text{Ad}_g(n) = gng^{-1}$ .

- (i) Let  $(W, \rho_W)$  be an irreducible finite-dimensional representation of  $N$ . Prove that  $\chi_{\text{coInd}_N^G W}$  is only nonzero on  $N$ , and  $\chi_{\text{coInd}_N^G W}|_N = \sum_i \chi_W \circ \text{Ad}_{g_i}$ . Mackey's formula for  $\chi_{\text{coInd}_N^G W}$  gives both statements, as explained in the

seen ↓

coinduction notes. More precisely, there it is written:

$$\chi_{\text{coInd}_H^G V}(h) = \begin{cases} \sum_{Hg \in H \setminus G} \chi_V(ghg^{-1}), & h \in H, \\ 0 & h \notin H. \end{cases} \quad (1)$$

Note that  $\chi_V(ghg^{-1}) = \chi_V \circ \text{Ad}_G$ , and now we substitute  $N$  for  $H$  and  $W$  for  $V$ .

2, M

- (ii) *Deduce that, if  $V \subseteq \text{Res}_N^G \text{coInd}_N^G W$  is a subrepresentation, then  $\dim V$  is a multiple of  $\dim W$ .*

*Hint: see the hint in Question 4b.*

sim. seen ↓

By that hint, we have  $\chi_{\text{Res}_N^G \text{coInd}_N^G} = \chi_V + \chi_{\text{Res}_N^G \text{coInd}_N^G / V}$ . Note that  $\rho_W \circ \text{Ad}_{g_i}$  is irreducible since  $W$  is and that its character is  $\chi_W \circ \text{Ad}_{g_i}$ . By the preceding part and linear independence of characters we must have that  $\chi_V$  is a sum of characters of the form  $\chi_W \circ \text{Ad}_{g_i}$ . But then, evaluating at the identity element, its dimension is a multiple of  $\dim W$ .

3, M

- (iii) *Now suppose that, for every  $g \notin N$ , there exists a conjugacy class  $\mathcal{C} \subseteq N$  with  $g\mathcal{C}g^{-1} \neq \mathcal{C}$ . Show that  $\text{coInd}_N^G W$  is irreducible.*

unseen ↓

Under this hypothesis,  $\chi_{\text{coInd}_N^G W}$  is a sum of characters of irreducible representations of  $N$  which are all distinct (since any two distinct elements of  $G/N$  must induce a different permutation of the conjugacy classes of  $N$  by conjugation). Any proper nonzero subrepresentation, by the previous part, would have character which is a sum of only some but not all of these characters. But then this character would not be fixed under  $\text{Ad}_{g_i}$  for all  $g_i$ . That's impossible since characters are invariant under conjugation. Thus  $\text{coInd}_N^G W$  is irreducible.

3, M

Total A marks: 30 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 13 of 12 marks

Total D marks: 17 of 16 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60039/70039/97035	1	This was mostly well received by the candidates. Problem (2) involved a little combinatorics. Some students noticed that (3) has an easier solution using character theory.
	2	In contrast to recent years, this problem was fairly accessible to students. Many had difficulty with (d), explicitly showing that the given formula in the hint defines an irreducible representation (I intended this to be done directly, but some students tried to find an approach via character theory or otherwise). Part (e) was easy for the students who noticed that the construction in (d) actually gives several representations, not just one.
	3	This was quite well received. There were some difficulties in finding the symmetry of the table in (b) --- not something we did much of in the course, although it was mentioned. Many students gave too much symmetry. Particularly hard was noticing the symmetry C_4 of the 4x4 square in the table coming from representations of dimension one.
	4	This was the hardest question for the students, partly because this topic is at the end of the course and a bit different in spirit from the preceding material (although definitely thematically linked). Particularly challenging is part (c), to which no student found a solution.
	5	The mastery question was more accessible than in previous years, with several students managing to solve it. The last parts were the most difficult, perhaps partly due to time constraints.