Imperial College London

MATH70006

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Complex Analysis

Date: 5 June 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. The function $f(x)$, integrable over the interval $[-1, 1]$, satisfies the integral equation

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = 3\log 2 - 2x - \frac{2}{3}x^3, \qquad -1 < x < 1.$$

(a) Show that $f(x)$ can be expressed in the functional form

$$f(x) = \frac{2x^3 + x + A}{\sqrt{1-x^2}},$$

where A is a constant. (15 marks)

(b) Determine the value of the constant A. You may use the fact that

$$\int_{-1}^{1} \frac{\log|t|}{\sqrt{1-t^2}}dt = -\pi\log 2.$$

(3 marks)

(c) At the points where $x = \pm 1$, is $f(x)$ bounded/unbounded? (2 marks)

[The Hilbert inversion formula

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}} ⨍_{-1}^{1} \frac{g(t)\sqrt{1-t^2}}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

for the singular integral equation

$$\frac{1}{\pi} ⨍_{-1}^{1} \frac{f(t)}{t-x}dt = g(x), \quad -1 < x < 1,$$

may be quoted without proof. Here A is an arbitrary constant and $⨍$ represents the principal value integral.]

(Total: 20 marks)

~~2. Consider the Legendre equation~~

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0, \qquad -1<x<1,$$

~~where $n=0,1,2,\cdots$, which admits orthogonal polynomial solutions $P_n(x)$, known as the Legendre polynomials.~~

~~(a) Show that these polynomials are orthogonal with respect to the weight function~~

$$w(x)=1.$$

~~(3 marks)~~

~~(b) Write down the orthogonality relations and the Rodrigues formula for $P_n(x)$. Use the Rodrigues formula to find the first three Legendre polynomials $P_0(x)$, $P_1(x)$ and $P_2(x)$ (up to normalisation).~~

~~(4 marks)~~

~~(c) It can be shown that $P_n(x)$ can be represented by the following integral~~

$$P_n(x)=\frac{1}{2\pi i}\oint_C \frac{(z^2-1)^n}{2^n(z-x)^{n+1}}dz,$$

~~where C is a circle (of finite radius) with x in its interior, and we integrate around C in the positive orientation (anti-clockwise). Letting $G(x,y)$ denote the generating function for the Legendre polynomials, i.e for sufficiently small y;~~

$$G(x,y)=\sum_{n=0}^{\infty}P_n(x)y^n,$$

~~use the above integral representation for $P_n(x)$ to show that, for sufficiently small y, we may express $G(x,y)$ as~~

$$G(x,y)=\frac{1}{\sqrt{1-2xy+y^2}}.$$

~~In doing so you may swap the order of any integrations and summations that you encounter.~~

~~(10 marks)~~

~~(d) Verify that the generating function found in part (c) produces $P_0(x)$, $P_1(x)$ and $P_2(x)$ as found in part (b) (up to normalisation).~~

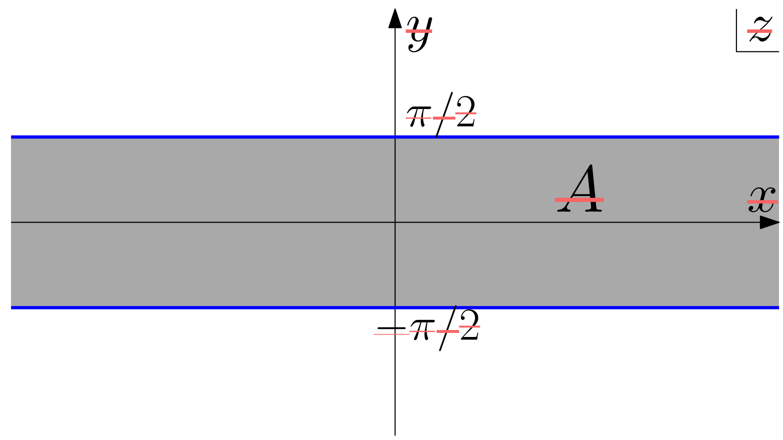
~~(3 marks)~~

~~(Total: 20 marks)~~

3. (a) ~~Define what it means for a mapping $\zeta = f(z)$ from the complex z-plane to the complex ζ-plane to be **conformal** at a point $z = z_0$.~~

~~(2 marks)~~

~~Figure 1 shows a 'strip'-region, bounded by the lines $y = -\pi/2$, $y = \pi/2$ and extending to infinity as $x \to \pm\infty$, shaded in grey and labelled region A in the z-plane.~~



~~Figure 1: An infinite strip in the z-plane.~~

(b) (i) ~~Determine the region, which we will label region B, of the complex ζ_1-plane that region A is mapped to via the transformation $\zeta_1 = e^z$.~~

~~(4 marks)~~

(ii) ~~Now show that the transformation $\zeta = \frac{1-\zeta_1}{1+\zeta_1}$ maps this new region B in the ζ_1-plane to region C; the interior of the unit disc in the ζ-plane.~~

~~(5 marks)~~

(c) ~~A point source of current of strength m is located at the origin inside a strip conductor occupying region A of the z-plane. Everywhere inside the conductor (except at $z = 0$) the voltage ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, and the two edges of the conductor are grounded, i.e~~

$$\phi = 0, \quad y = \pm\frac{\pi}{2}.$$

~~Determine the complex potential, $w(z)$, for this set-up, where $\phi = \mathrm{Re}\{w(z)\}$.~~ ~~(5 marks)~~

(d) ~~Show that, local to $z = 0$, the complex potential found in part (c) is such that~~

$$w(z) = \frac{m}{2\pi}\log z + O(1).$$

~~(4 marks)~~

~~(Total: 20 marks)~~

4. The function $f(x)$ satisfies the integral equation

$$f(x) = \lambda \int_0^\infty f(y) e^{-|x-y|} dy + e^{-x},$$

for $x \geq 0$, with $f(0) = 2$. The parameter $\lambda \in \mathbb{R}$ is restricted to $\lambda > \frac{1}{2}$. You may use the notation $p^2 = 2\lambda - 1$ to simplify any calculations.

(a) Using the Wiener-Hopf method, and taking the strip of analyticity to be $\{s : \alpha < \text{Im}\{s\} < \beta\}$, for values α, $\beta > 0$ which you should define carefully, show that for $\text{Im}\{s\} > \alpha$ the right-sided Fourier transform $F_+(s) \equiv \int_0^\infty f(x) e^{isx} dx$ is given by

$$F_+(s) = \frac{2is}{(s+p)(s-p)}.$$

(16 marks)

(b) Hence show that for $x \geq 0$

$$f(x) = 2\cos(px).$$

(4 marks)

(Total: 20 marks)

5. Consider two-dimensional Stokes flow in the complex z-plane with streamfunction ψ, which can be represented by

$$\psi = \text{Im}\{\bar{z}f(z) + g(z)\},$$

where $f(z)$ and $g(z)$ are analytic functions known as the Goursat functions and the bar represents the complex conjugate. The velocity field of the flow, (u, v), is known to satisfy the relationship

$$u - iv = 2i\frac{\partial \psi}{\partial z}.$$

(a) Show that

$$u - iv = -\overline{f(z)} + \bar{z}f'(z) + g'(z).$$

[The dash represents the derivative with respect to the function argument.] (2 marks)

The combination of Goursat functions

$$f(z) = 0, \quad g'(z) = \frac{\alpha i}{z - z_0},$$

corresponds to the fluid flow across the **entire plane** generated by a **rotlet** of strength $\alpha \in \mathbb{R}$ positioned at a point z_0.

(b) Find a simplified expression for the streamfunction for this flow. Give a sketch of the streamlines.

(4 marks)

Now consider a rotlet of strength $\alpha \in \mathbb{R}$ positioned at a point z_0 in Stokes flow contained in the infinite region above a flat wall running along the real axis in the z-plane.

(c) Assuming the no-slip condition along the wall ($u - iv = 0$ on the wall), determine the Goursat functions for this fluid flow.

(10 marks)

The torque acting on a body or region with boundary C about a point z_0 inside the body/region can be calculated via the formula

$$\text{Torque} = -2\text{Re}\left\{\overline{\oint_C g'(z)dz}\right\}.$$

[Here Re$\{\}$ denotes the real part of the expression within the parenthesis and the integration is taken in the positive (anti-clockwise) orientation.]

(d) Calculate the torque about the rotlet at z_0 acting on a small region of fluid containing z_0 in its interior in the case when the rotlet is in free-space (fluid across the entire plane). What happens to this torque when the rotlet is above the wall?

(4 marks)

(Total: 20 marks)

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts

(a). Differentiating both sides gives:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt = 2+2x^2, \quad -1<x<1.$$

Diff. 2 $\boxed{A}$ seen similar

Now applying the Hilbert inversion formula, we find:

$$f(x) = -\frac{2}{\pi\sqrt{1-x^2}}\underbrace{\int_{-1}^{1}\frac{(1+t^2)\sqrt{1-t^2}}{t-x}\,dt}_{=I(x)} + \frac{A}{\sqrt{1-x^2}}$$

Hilbert. 1 $\boxed{A}$ seen

Evaluating $I(x)$ using contour integration:

Consider the expression, as $z\to\infty$:

$$\frac{(1+z^2)\sqrt{z^2-1}}{z-x} = \frac{(1+z^2)\sqrt{1-\frac{1}{z^2}}}{1-\frac{x}{z}}$$

$$= (1+z^2)\left[1-\frac{1}{2z^2}+O\left(\frac{1}{z^4}\right)\right]\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\frac{x^3}{z^3}+O\left(\frac{1}{z^4}\right)\right)$$

$$=(1+z^2)\left(1+\frac{x}{z}+\frac{(x^2-\frac{1}{2})}{z^2}+\frac{(x^3-\frac{1}{2}x)}{z^3}+O\left(\frac{1}{z^4}\right)\right)$$

$$= z^2+xz+\left(x^2+\frac{1}{2}\right)+\frac{(x^3+\frac{1}{2}x)}{z}+O\left(\frac{1}{z^2}\right) \quad (*)$$

(Note: Could be done later on γ_R integral)

Expand. 2 $\boxed{B}$ seen similar

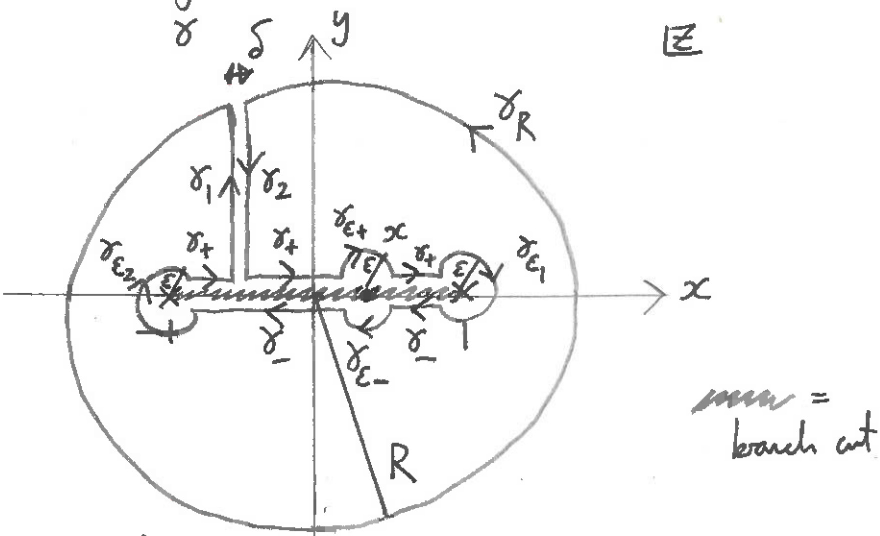
Setter's initials SJB	Checker's initials	Page number 1

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts (a). (continued).

Let $h(z) = \dfrac{(1+z^2)\sqrt{z^2-1}}{z-x} - \left(z^2+xz+\left(x^2+\tfrac{1}{2}\right)\right)$,

and consider: $\oint_\gamma h(z)\,dz$, where γ is as shown below:



contour. A 1 seen

We take the branch of $\sqrt{z^2-1}$ with a branch cut along $(-1,1)$ and which behaves like z as $|z|\to\infty$.

b.cut. A 1 seen

We consider the limit as $R\to\infty$, $\varepsilon\to 0$ and $\delta\to 0$.

$h(z)$ is analytic everywhere inside γ, hence

$$\oint_\gamma h(z)\,dz = 0, \text{ by Cauchy's theorem.}$$

Cauchy. A 1 seen similar

Now let's evaluate integrals along the separate components of γ:

- Those along γ_1 and γ_2 cancel one another out.
- On $\gamma_{\varepsilon_1}, \gamma_{\varepsilon_2}$, $|h(z)|\sim O(1)$, but $dz\sim O(\varepsilon)$, so these integrals are 0 as $\varepsilon\to 0$.

A 1 seen

Setter's initials SJB	Checker's initials	Page number 2

Parts (a). (continued).

• Next, for our choice of branch we have, for $z \in \gamma_\pm$, $\sqrt{z^2-1} = \pm i\sqrt{1-x^2}$, and hence: } 1 $\boxed{B}$ seen

$$\int_{\gamma_+} h(z)\,dz + \int_{\gamma_-} h(z)\,dz = 2i \underbrace{\int_{-1}^{1} \frac{(1+t^2)\sqrt{1-t^2}}{t-x}\,dt}_{=I(x)},$$

taking into account that we integrate along γ_- from right to left. } 1 $\boxed{A}$ seen

• Furthermore, on $\gamma_{\varepsilon+}$ we have $z = x+\varepsilon e^{i\theta}$ where θ varies from π to 0, while on $\gamma_{\varepsilon-}$, $z = x+\varepsilon e^{i\theta}$ where θ varies from 0 to $-\pi$. Keeping in mind our choice of branch, it follows that:

$$\int_{\gamma_{\varepsilon+}} h(z)\,dz \to \int_{\pi}^{0} \frac{i\sqrt{1-x^2}}{\varepsilon e^{i\theta}}\left(1+(x+\varepsilon e^{i\theta})^2\right) i\varepsilon e^{i\theta}\,d\theta$$

$$\to \pi(1+x^2)\sqrt{1-x^2} \quad \text{as } \varepsilon \to 0,$$

and

$$\int_{\gamma_{\varepsilon-}} h(z)\,dz \to \int_{0}^{-\pi} \frac{-i\sqrt{1-x^2}}{\varepsilon e^{i\theta}}\left(1+(x+\varepsilon e^{i\theta})^2\right) i\varepsilon e^{i\theta}\,d\theta$$

$$\to -\pi(1+x^2)\sqrt{1-x^2} \quad \text{as } \varepsilon \to 0,$$

where the terms $-(z^2+xz+(x^2+\frac{1}{2}))$ in $h(z)$ clearly cancel when we integrate and sum the integrals along $\gamma_{\varepsilon+}$ and $\gamma_{\varepsilon-}$. } 2 $\boxed{B}$ seen similar

Parts (a). (continued).

• Finally, for $z \in \gamma_R$, using our expansion $(*)$, we pick up the residue at infinity, giving

$$\int_{\gamma_R} h(z)\,dz = 2\pi i\left(x^3 + \tfrac{1}{2}x\right).$$

γ_R contribution. 1 $\boxed{C}$ seen similar

Hence, summing all our contributions up:

$$2iI(x) + 2\pi i\left(x^3 + \tfrac{1}{2}x\right) = 0$$

$$\Rightarrow \underline{I(x) = -\pi\left(x^3 + \tfrac{1}{2}x\right)}$$

1 $\boxed{A}$ seen similar

(Alternative Method to get $I(x)$: Plemelj + Cauchy transforms)

Alternative 12 marks:

Introduce the Cauchy transform of $(1+x^2)\sqrt{1-x^2}$ on $[-1,1]$:

$$C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{(1+t^2)\sqrt{1-t^2}}{t-z}\,dt.$$

2

The first Plemelj formula gives:

$$I(x) = \pi i\left(C_+(x) + C_-(x)\right), \quad -1<x<1. \qquad (**)$$

1

Now let's take $z = x \in \mathbb{R} > 1$. Let $t = \cos\theta$. Then:

1

$$C(x) = \frac{1}{2\pi i}\int_{\pi}^{0} \frac{\sin\theta\,(2-\sin^2\theta)}{\cos\theta - x}(-\sin\theta)\,d\theta$$

Parts (a). (continued).

$$= \frac{1}{2\pi i}\int_0^\pi \frac{(1-\cos^2\theta)(1+\cos^2\theta)}{\cos\theta - x}\,d\theta$$

$$= \frac{1}{2\pi i}\int_0^\pi \frac{(x+\cos\theta)^2(x-\cos\theta)^2 - 2x^2(x+\cos\theta)(x-\cos\theta) + (x^4-1)}{x-\cos\theta}\,d\theta$$

$$= \frac{-1}{2\pi i}\int_0^\pi \big(x^3 + \underbrace{x^2\cos\theta}_{} + x\cos^2\theta + \underbrace{\cos^3\theta}_{}\big)\,d\theta \qquad \int = 0 \text{ as odd functions about } \frac{\pi}{2}$$

$$+ \frac{(x^4-1)}{2\pi i}\underbrace{\int_0^\pi \frac{d\theta}{x-\cos\theta}}_{=\frac{\pi}{\sqrt{x^2-1}},\ \text{from lectures}}$$

$$= \frac{-1}{2\pi i}\left[\pi x^3 + \frac{\pi}{2}x\right] + \frac{(x^4-1)}{2i\sqrt{x^2-1}}$$

$$= \frac{1}{2i}\left(-x^3 - \frac{1}{2}x + (x^2+1)\sqrt{x^2-1}\right), \quad x>1.$$

5

Hence by analytic continuation;

$$C(z) = \frac{1}{2i}\left(-z^3 - \frac{1}{2}z + (z^2+1)\sqrt{z^2-1}\right).$$

1

Now $\sqrt{z^2-1}\Big|_{z=x\pm i\delta\ (\delta\ll 1)} = \pm i\sqrt{1-x^2}$ upon introducing the branch cut as before.

1

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts (a). (continued).	So, we have: $$C_{\pm}(x) = \frac{1}{2i}\left(-x^3-\frac{1}{2}x + (x^2+1)(\pm i\sqrt{1-x^2})\right).$$ Then using Plemelj (**) gives: $$\underline{I(x) = -\pi\left(x^3+\frac{x}{2}\right)}, \text{ as before.}$$	1
	Thus: $$f(x) = \frac{-2}{\pi\sqrt{1-x^2}}\left(-\pi\left(x^3+\frac{x}{2}\right)\right) + \frac{A}{\sqrt{1-x^2}}$$ $$\Rightarrow \boxed{f(x) = \frac{2x^3+x+A}{\sqrt{1-x^2}}}, \text{ as required.}$$	part of 1 mark for $I(x)$.
(b).	Substituting for $f(x)$ back into the original equation, and setting $x=0$, leads to: $$\underbrace{\frac{1}{\pi}\int_{-1}^{1}\left(\frac{2t^3+t}{\sqrt{1-t^2}}\log\lvert t\rvert\right)dt}_{=0,\text{ odd function}} + \underbrace{\frac{A}{\pi}\int_{-1}^{1}\frac{\log\lvert t\rvert}{\sqrt{1-t^2}}}_{=-\pi\log 2,\text{ as given.}} = 3\log 2$$ $$\Rightarrow \boxed{A=-3}$$	1 C seen; odd. 1 D seen similar; A value 1 C seen similar
(c).	Hence: $f(x) = \dfrac{2x^3+x-3}{\sqrt{1-x^2}}$. This is <u>regular</u> at $x=1$ as the numerator vanishes, but <u>singular</u> at $x=-1$.	2 D unseen
	Setter's initials SJB	Checker's initials	Total 20	Page number 6

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(a).

$$\frac{d}{dx}\left(\underbrace{(1-x^2)}\frac{dy}{dx}\right) + n(n+1)y = 0.$$

$\underline{p(x) = 1-x^2}$, $\underline{q(x) = 0}$.

Then:

$$w(x) = \exp\left\{\int^x \frac{q(u)}{p(u)}\,du\right\} = \exp\{0\} = \underline{\underline{1}}.$$

1 A seen

2 A seen

(b). Orthogonality relations:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \text{non-zero constant}, & m = n \end{cases}$$

1 A seen

Rodrigues formula:

$$\boxed{P_n(x) = \frac{d^n}{dx^n}\left((1-x^2)^n\right)}$$

1 A seen

$$\Rightarrow P_0(x) = \underline{\underline{1}}$$

$$P_1(x) = \frac{d}{dx}(1-x^2) = \underline{\underline{-2x}}$$

1 A seen

$$P_2(x) = \frac{d^2}{dx^2}\left((1-x^2)^2\right) = \underline{\underline{-4+12x^2}}$$

1 A seen

Setter's initials SJB	Checker's initials	Page number 7

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(c). Making use of the given integral representation for $P_n(x)$, we can write:

$$G(x,y) = \sum_{n=0}^{\infty} P_n(x) y^n$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \left(\frac{(z^2-1)^n}{2^n (z-x)^{n+1}} \right) dz \cdot y^n$$

1 B unseen

$$= \frac{1}{2\pi i} \oint_C \left(\sum_{n=0}^{\infty} \left(\frac{y(z^2-1)}{2(z-x)} \right)^n \right) \frac{dz}{z-x}$$

1 B unseen

$$= \frac{1}{2\pi i} \oint_C \left(\frac{2(z-x)}{2(z-x) - y(z^2-1)} \right) \frac{dz}{z-x},$$

where we have used the fact that

$$\sum_{n=0}^{\infty} \left(\frac{y(z^2-1)}{2(z-x)} \right)^n = \frac{1}{1 - \frac{y(z^2-1)}{2(z-x)}}$$

$$= \frac{2(z-x)}{2(z-x) - y(z^2-1)}.$$

sum. 2 C unseen

Setter's initials SJB	Checker's initials	Page number 8

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts (c). (continued).

Tidying up the resulting integral:

$$G(x,y) = \frac{1}{\pi i}\oint_C \frac{dz}{2(z-x)-y(z^2-1)}.$$

Now observe that:

$$2(z-x)-y(z^2-1) = -yz^2+2z+y-2x$$

$$= -y\left(z^2-\frac{2}{y}z+\frac{2x}{y}-1\right)$$

$$= -y(z-z_+)(z-z_-),$$

where $z_\pm = \frac{1}{y}\left(1 \pm \sqrt{1-2xy+y^2}\right).$

Thus we have:

$$G(x,y) = \frac{-1}{\pi i y}\oint_C \frac{dz}{(z-z_+)(z-z_-)}.$$

2 $\boxed{D}$ unseen

Now for y sufficiently small:

$$z_+ \sim \frac{1}{y}\left(1+(1+O(y))\right) \sim \frac{2}{y} \to \infty, \text{ but}$$

$$z_- \sim \frac{1}{y}\left(1-(1-xy+O(y^2))\right) \sim x.$$

Thus, the integrand is analytic in the interior of C, <u>except</u> for a simple pole at $z=z_-$ (since we are told x is in the <u>interior</u> of C).

$\boxed{D}$ 2 unseen

Setter's initials SJB	Checker's initials	Page number 9

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen
Parts (c). (continued).	Thus, by the residue theorem: $$G(x,y) = \frac{-1}{\pi i y}\left(2\pi i \cdot \frac{1}{z_- - z_+}\right)$$ $$= \frac{-2}{y} \cdot \frac{1}{\left(\frac{-2}{y}\right)\sqrt{1-2xy+y^2}}$$ $$= \frac{1}{\sqrt{1-2xy+y^2}}, \text{ as required.}$$	res. theorem. $\boxed{C}$ 1 unseen. result. $\boxed{D}$ 1 unseen
(d).	$$G(x,y) = \left(1+(-2xy+y^2)\right)^{-\frac{1}{2}}$$ $$= \left(1 - \frac{1}{2}(-2xy+y^2) - \frac{1}{2}\left(\frac{-3}{2}\right)\frac{1}{2!}(-2xy+y^2)^2 + \cdots\right)$$ $$= 1 + xy - \frac{1}{2}y^2 + \frac{3}{8}(4x^2y^2) + O(y^3)$$ $$= 1 + xy + \frac{1}{2}(3x^2-1)y^2 + O(y^3)$$ $\Rightarrow p_0(x) = 1$ ✓ agrees with (b). $p_1(x) = x$ ✓ agrees with (b). (multiply by -2) $p_2(x) = \frac{1}{2}(3x^2-1)$ ✓ agrees with (b). (multiply by 8)	expand. $\boxed{B}$ 2 seen similar. compare. $\boxed{B}$ 1 seen similar. Total: 20
	Setter's initials SJB / Checker's initials	Page number 10

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (a).	The mapping $\zeta = f(z)$ is called conformal at $z = z_0$ if: • $f(z)$ is analytic at z_0, • $f'(z_0) \neq 0$.	1 $\boxed{A}$ seen 1 $\boxed{A}$ seen
	Alternative: The mapping $\zeta = f(z)$ is called conformal at $z = z_0$ if it preserves the angle between any two different arcs passing through z_0.	Alternative: 2 marks 2
(b). (i).	On the upper side of the strip, $z = x + i\frac{\pi}{2}$, where $-\infty < x < \infty$. Thus: $\zeta_1 = e^{z} = e^{x} e^{i\frac{\pi}{2}} = ie^{x}$, hence ζ_1 ranges from 0 to "$i\infty$", i.e. along the positive imaginary axis in the ζ_1-plane.	2 $\boxed{A}$ seen similar
	Similarly, on the lower side, $z = x - i\frac{\pi}{2}$, where $-\infty < x < \infty$ and we find: $\zeta_1 = -ie^{x}$, giving us the negative imaginary axis.	1 $\boxed{A}$ seen similar
	To determine which side of this line we have let's check a point:	
	Setter's initials SJB	Checker's initials

EXAMINATION SOLUTION 22-23

Course: Applied Complex Analysis

Question 3

Topic CONFORMAL MAPPING

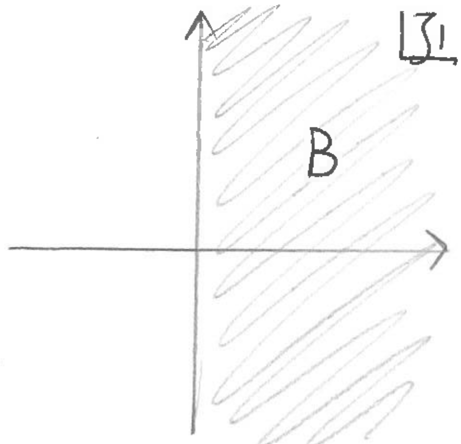
Marks& seen/unseen

Parts

(b). (i). (continued).

When $z=0$, $\zeta_1 = e^z = e^0 = 1$, so we have the right-half plane in the ζ_1-plane as region B.

1 A Seen similar



(ii). $\zeta = \dfrac{1-\zeta_1}{1+\zeta_1}$ is a Mobiüs map which we know maps circles to circles (straight lines being circles with infinite radius). Since ζ_1 doesn't take the value -1 on B or its boundary the image of B in the ζ-plane is a circle with finite radius.

Let $\zeta = \xi + i\eta$ and $\zeta_1 = \xi_1 + i\eta_1$. Then we find

$$\xi + i\eta = \frac{(1-\xi_1) - i\eta_1}{(1+\xi_1)+i\eta_1} \cdot \frac{(1+\xi_1)-i\eta_1}{(1+\xi_1)-i\eta_1}$$

$$= \frac{(1-\xi_1^2-\eta_1^2)}{(1+\xi_1)^2+\eta_1^2} - i\frac{2\eta_1}{(1+\xi_1)^2+\eta_1^2}.$$

algebra. 1 A Seen similar

Setter's initials SJB

Checker's initials

Page number

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

(b) (ii). (continued).

But the boundary in the ζ_1-plane is given by $\xi_1 = 0$. This leads to:

$$\xi = \frac{1-\eta_1^2}{1+\eta_1^2}, \quad \eta = \frac{-2\eta_1}{1+\eta_1^2},$$

$\xi_1 = 0$ 1 $\boxed{B}$ seen similar

and we see that:

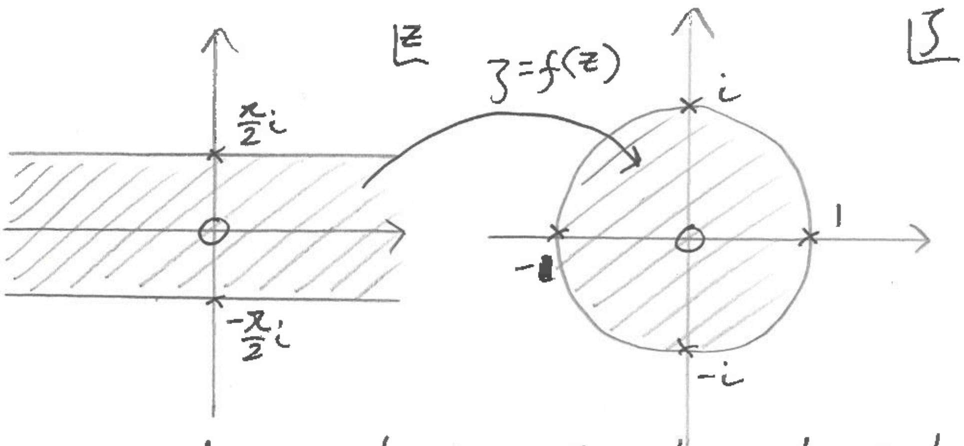
$$\xi^2 + \eta^2 = \frac{1}{(1+\eta_1^2)^2}\left(\underbrace{(1-\eta_1^2)^2 + (-2\eta_1)^2}_{\substack{= 1 - 2\eta_1^2 + \eta_1^4 + 4\eta_1^2 \\ = 1 + 2\eta_1^2 + \eta_1^4 = (1+\eta_1^2)^2}}\right) = 1,$$

$\xi^2+\eta^2 = \cdots$ $\boxed{B}$ 2 seen similar

i.e. the unit circle in the ζ-plane.
To check we have the interior, test a point: when $\zeta_1 = 1$, $\zeta = 0$, giving the interior.

1 $\boxed{A}$ seen similar

(c).



Consider the conformal mapping from the complex z-plane to the complex ζ-plane given by:

Setter's initials SJB	Checker's initials	Page number 13

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (c) (continued)	$$\zeta = \frac{1-e^{z}}{1+e^{z}}.$$	1 B seen similar
	This maps the interior of the strip to the interior of the unit-disc. Now in the ζ-plane the source at $z=0$ is transplanted to $\zeta=0$. In the ζ-plane we can make an <u>ansatz</u> for the complex-potential due to the symmetries. Try: $$w(\zeta) = \frac{m}{2\pi}\log\zeta.$$	$w(\zeta)$ 1 C seen similar
	This corresponds to a source of strength m at $\zeta=0$. To check the boundary conditions are satisfied: $$w(\zeta) = \underbrace{\frac{m}{2\pi}\log\lvert\zeta\rvert}_{=\phi} + \underbrace{\frac{m}{2\pi} i \arg\{\zeta\}}_{=\psi}$$ when $\lvert\zeta\rvert=1$, we want $\phi=0$, but we see this is already satisfied by our ansatz. Hence the solution in the ζ-plane is given by $w(\zeta)$, and thus the solution in the z-plane is given by:	BC's. 2 C seen similar
	$$\boxed{w(z) = \frac{m}{2\pi}\log\left(\frac{1-e^{z}}{1+e^{z}}\right)},$$ by use of the conformal mapping and the conformal invariance of Laplace's equation.	result 1 B seen similar
	Setter's initials SJB	Checker's initials	Page number 14

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

Parts

(d). We have:

$$w(z) = \frac{m}{2\pi}\log\left(\frac{1-e^z}{1+e^z}\right)$$

$$= \frac{m}{2\pi}\log\left(\frac{1-(1+z+\frac{1}{2}z^2+\cdots)}{1+(1+z+\frac{1}{2}z^2+\cdots)}\right),$$

upon expanding for z close to 0.

expand. 1 D unseen

$$= \frac{m}{2\pi}\log\left(\frac{-z(1+O(z))}{2(1+O(z))}\right)$$

$$= \frac{m}{2\pi}\log\left(-\frac{z}{2}(1+O(z))(1-O(z))\right),$$

where we use $(1+x)^{-1} = 1-x+\cdots$ to move the expansion on the denominator to the numerator.

re-arrange. 1 D unseen

$$= \frac{m}{2\pi}\log\left(-\frac{z}{2}(1+O(z))\right)$$

$$= \frac{m}{2\pi}\log z + \frac{m}{2\pi}\log\left(-\frac{1}{2}+O(z)\right)$$

$$= \frac{m}{2\pi}\log z + O(1)$$

result. 2 D unseen

Total: 20

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen

Parts

(a). Let $k(x) = \lambda e^{-|x|}$ and $p(x) = -e^{-x}$.

Introduce:

$$f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$g_-(x) = \begin{cases} 0, & x \geq 0 \\ \int_0^\infty f(y) k(x-y)\, dy, & x < 0. \end{cases}$$

Then we can rewrite the equation as:

$$\lambda \int_0^\infty f(y) e^{-|x-y|} dy = f_+(x) + g_-(x) + p_+(x), \quad -\infty < x < \infty.$$

1 $\boxed{A}$ seen

Taking the Fourier transform of both sides results in:

$$\hat{K}(s) F_+(s) = F_+(s) + G_-(s) + P_+(s),$$

1 $\boxed{A}$ seen

where $F_+(s)$ and $P_+(s)$ are the right-sided Fourier transforms of $f_+(x)$ and $p_+(x)$ respectively, $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$ and $\hat{K}(s)$ is the ordinary Fourier transform of $k(x)$.

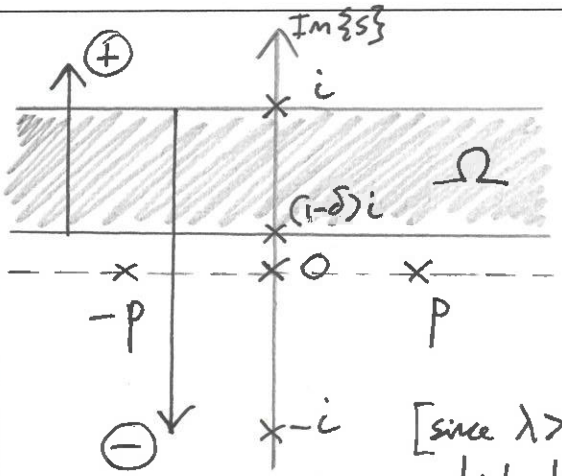
We can calculate: $\hat{K}(s) = \dfrac{2\lambda}{s^2+1}$,

$\hat{K}$ 1 $\boxed{A}$ seen

Setter's initials SJB	Checker's initials	Page number 16

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen
Parts (a). (continued).	and $$P_+(s) = -\int_0^\infty e^{-x} e^{isx}\,dx = -\int_0^\infty e^{(is-1)x}\,dx$$ $$= -\left[\frac{e^{(is-1)x}}{is-1}\right]_0^\infty = \frac{1}{is-1} = \frac{-i}{s+i},$$ provided $\mathrm{Im}\{s\} > -1$.	P_+ 1 $\boxed{A}$ seen similar
	Thus re-arranging we find: $$K(s)F_+(s) + G_-(s) = -P_+(s), \quad (*)$$ where $$K(s) = 1 - \hat{K}(s) = \frac{s^2+1-2\lambda}{s^2+1} = \frac{s^2-p^2}{s^2+1},$$ where $p^2 = 2\lambda - 1$ as given.	1 $\boxed{A}$ seen similar
	• From lectures we know if $\lvert f_+(x)\rvert < Ae^{(1-\delta)x}$ as $x\to\infty$, for some $\delta > 0$ (A constant), then $F_+(s)$ is analytic in $\{s : \mathrm{Im}\{s\} > 1-\delta\}$. Similarly we know that $G_-(s)$ is analytic in $\{s : \mathrm{Im}\{s\} < 1\}$. Thus we take the ⊕ and ⊖ regions to be: $$\oplus = \{s : \mathrm{Im}\{s\} > 1-\delta\}$$ $$\ominus = \{s : \mathrm{Im}\{s\} < 1\},$$ where $0<\delta<1$ and hence the strip of analyticity to be the region: $$\Omega = \{s : \underbrace{1-\delta}_{=\alpha} < \mathrm{Im}\{s\} < \underbrace{1}_{=\beta}\}.$$	α, β 2 $\boxed{B}$ seen similar
	Setter's initials SJB Checker's initials	Page number 17

Parts (a). (continued).



} $K(s)$ analytic + non-zero in Ω ✓ (avoids $p, -p, i, -i$)

$P_+(s)$ analytic in Ω ✓ (avoids $-i$)

[since $\lambda > \frac{1}{2}$, zeros of $K(s)$ are $\pm p$ which lie on real axis].

We now decompose $K(s)$ as: $K(s) = K_+(s)K_-(s)$, where:

$$K_+(s) = \frac{(s+p)(s-p)}{s+i}, \quad K_-(s) = \frac{1}{s-i},$$

K decomp. $\boxed{D}$ 2 Seen similar

then $(*)$ gives:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R(s) = \frac{-P_+(s)}{K_-(s)} = \frac{i(s-i)}{s+i}.$$

Now decompose $R(s)$ as: $R(s) = R_+(s) + R_-(s)$, where:

$$R_+(s) = \frac{i(s-i)}{s+i}, \quad R_-(s) = 0.$$

R decomp. $\boxed{D}$ 2 Seen similar

Then our equation becomes:

$$\underbrace{K_+(s)F_+(s) - R_+(s)}_{\text{analytic in } \oplus} = \underbrace{\frac{-G_-(s)}{K_-(s)}}_{\text{analytic in } \ominus}, \quad s \in \Omega$$

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen

Parts (a). (continued).

Now since ⊕, ⊖ overlap in Ω, then by analytic continuation:

$$E(s) = \begin{cases} F_+(s)K_+(s) - R_+(s), & s \in \oplus \\ -\dfrac{G_-(s)}{K_-(s)}, & s \in \ominus, \end{cases} \text{ is \underline{entire}.}$$

$E(s)$ A 1 seen

Consider now $s \to \infty$ in ⊕:

$$K_+(s)F_+(s) - R_+(s) \sim (s + O(1))\left(\frac{2i}{s} + O\left(\frac{1}{s^2}\right)\right) - \left(i + O\left(\frac{1}{s}\right)\right)$$

$= f_+(0)$

$$\sim 2i - i + O\left(\frac{1}{s}\right)$$

$$\sim i + O\left(\frac{1}{s}\right)$$

$$\to i \text{ as } s \to \infty.$$

expand. C 2 seen similar

Hence, by Liouville's theorem: $\underline{E(s) \equiv i \text{ for all } s}$.

Liouville. A 1 seen ~~similar~~

Therefore: $F_+(s)K_+(s) - R_+(s) = i$

$$\Rightarrow F_+(s) = \frac{i + R_+(s)}{K_+(s)}$$

$$\Rightarrow \boxed{F_+(s) = \frac{2is}{(s+p)(s-p)}}, \text{ as required.}$$

result. A 1 seen similar

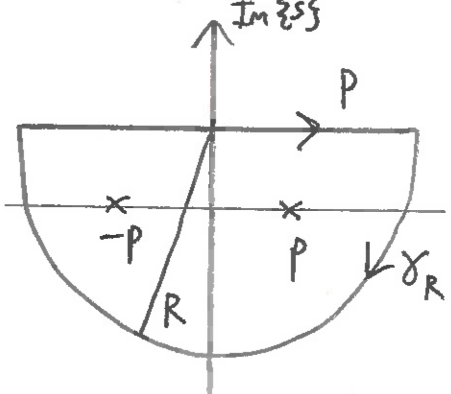
Setter's initials SJB	Checker's initials	Page number 19

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen

Parts (b).

To retrieve $f_+(x)$, apply the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}\,ds,$$ where P is a horizontal line in the $\oplus$ region. — 1 B seen



• For $x \geq 0$, we close P with a semi-circle γ_R of radius R below P and take $R \to \infty$. From lectures $\int_{\gamma_R} \to 0$ as $R \to \infty$. — 1 B seen similar

Thus, by the residue theorem:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}\,ds = \frac{1}{2\pi}\oint_\gamma \frac{2ise^{-isx}}{(s+p)(s-p)}\,ds$$

— residue theorem. 1 B seen similar

$$= \frac{1}{2\pi}\left(2i\left(\frac{pe^{-ipx}}{2p} + \frac{-pe^{ipx}}{-2p}\right)\right) \times 2\pi i \times (-1)$$

clockwise integration

$$= 2\left(\frac{e^{ipx} + e^{-ipx}}{2}\right)$$

$$\Rightarrow \boxed{f(x) = 2\cos(px)}, \quad x \geq 0, \text{ as required.}$$

— 1 C result. seen similar

Total: 20

	Setter's initials SJB	Checker's initials	Page number 20

EXAMINATION SOLUTION 22-23

Course: Applied Complex Analysis

Question 5

Topic: COMPLEX METHODS FOR BIHARMONIC EQUATION

Marks& seen/unseen

Parts

(a).

$$\psi = \mathrm{Im}\{\bar{z} f(z) + g(z)\}$$
$$= \frac{\bar{z} f(z) + g(z) - z\overline{f(z)} - \overline{g(z)}}{2i}$$

1 seen

Hence:

$$u - iv = 2i\frac{\partial \psi}{\partial z}$$
$$= \bar{z} f'(z) + g'(z) - \overline{f(z)} + 0,$$

giving the required result.

1 seen

(b).

$$\psi = \mathrm{Im}\{\bar{z}(0) + \alpha i \log(z - z_0) + c\},$$ where c is a constant, upon integrating $g'(z)$. The constant c can be safely set to 0 should we want as it just scales ψ by a constant. Thus:

1 seen similar

$$\psi = \mathrm{Im}\{\alpha i(\log|z - z_0| + i\arg\{z - z_0\})\}$$
$$= \underline{\underline{\alpha \log|z - z_0|}}, \text{ since } \alpha \in \mathbb{R}.$$

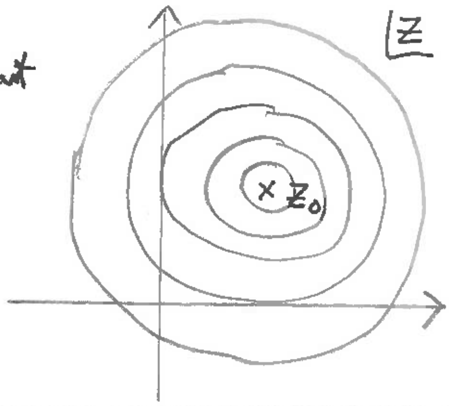
1 seen similar

Streamlines are when $\psi = \text{constant}$

$\Rightarrow \log|z - z_0| = \text{const.}$

$\Rightarrow |z - z_0| = \text{const.}$

i.e. circles about z_0.

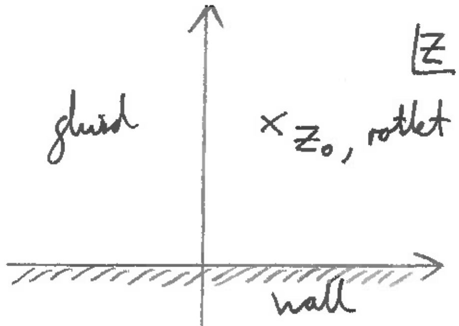


2 seen similar

Setter's initials: SJB

Checker's initials:

Parts (c).



We know that local to z_0 we have: (due to the rotlet)

$$f(z) = \text{analytic}$$
$$g'(z) = \frac{\alpha i}{z - z_0} + \text{analytic},$$

hence we introduce the unknown functions:

$$\left.\begin{aligned} f(z) &= f_R(z), \\ g'(z) &= \frac{\alpha i}{z-z_0} + g_R'(z), \end{aligned}\right\} (*)$$

which are <u>analytic</u> in the fluid region. — 1 seen

The no-slip condition on the wall gives:

$$u - iv = -\overline{f(z)} + \bar{z} f'(z) + g'(z) = 0, \text{ and}$$

upon plugging in $(*)$ we find: — 2 seen

$$-\overline{f_R(z)} + \bar{z} f_R'(z) + \frac{\alpha i}{z - z_0} + g_R'(z) = 0.$$

Now on the wall, $\bar{z} = z$, since it lies on the real axis, hence the above becomes: — 1 seen

EXAMINATION SOLUTION 22-23

Course: Applied Complex Analysis

Question 5

Topic COMPLEX METHODS FOR BIHARMONIC EQUATION

Marks& seen/unseen

Parts (c). (continued)

$$-\bar{f}_R(z) + z f_R'(z) + \frac{\alpha i}{z - z_0} + g_R'(z) = 0 .$$

This is now an equation involving <u>analytic</u> functions, and so holds everywhere in the fluid region off the wall by analytic continuation. 1 Seen

• Now we know $f_R(z)$ (and hence $f_R'(z)$) and $g_R'(z)$ are <u>analytic</u> in the fluid region (by definition).

Hence the pole at z_0 in the term $\frac{\alpha i}{z - z_0}$ must be cancelled out by the term $-\bar{f}_R(z)$.

Thus we make the ansatz:

$$\bar{f}_R(z) = \frac{\alpha i}{z - z_0},$$

2 Seen similar

then:

$$f_R(z) = \frac{-\alpha i}{z - \bar{z}_0},$$

and:

$$f_R'(z) = \frac{\alpha i}{(z - \bar{z}_0)^2}.$$

Plugging these back into the equation gives:

$$z\left(\frac{\alpha i}{(z - \bar{z}_0)^2}\right) + g_R'(z) = 0, \text{ or}$$

$$g_R'(z) = -(z - \bar{z}_0 + \bar{z}_0)\left(\frac{\alpha i}{(z - \bar{z}_0)^2}\right)$$

$$= \frac{-\alpha i}{z - \bar{z}_0} - \frac{\alpha i \bar{z}_0}{(z - \bar{z}_0)^2}.$$

2 Seen similar

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 5	Topic COMPLEX METHODS FOR BIHARMONIC EQUATION	Marks& seen/unseen

Parts

(c). (continued).

Finally putting everything back into (*) the Goursat functions for the flow are given by:

$$\boxed{\begin{aligned} f(z) &= \frac{-\alpha i}{z-\bar{z}_0}, \\ g'(z) &= \frac{\alpha i}{z-z_0} - \frac{\alpha i}{z-\bar{z}_0} - \frac{\alpha i \bar{z}_0}{(z-\bar{z}_0)^2}. \end{aligned}}$$

1 seen similar

We know this is everything possible except for possibly an added constant (Goursat functions are unique up to an additive constant).

(d). For the rotlet in free-space: $f(z) = 0$

$$g'(z) = \frac{\alpha i}{z-z_0}.$$

Thus:

$$\text{Torque} = -2\,\text{Re}\left\{ \oint_C \frac{\alpha i}{z-z_0}\,dz \right\},$$ where C is a small closed contour containing z_0 in its interior. Hence, by the residue theorem:

1 unseen

$$\begin{aligned} \text{Torque} &= -2\,\text{Re}\left\{ 2\pi i \,\text{Res}\left\{ \frac{\alpha i}{z-z_0}, z_0 \right\} \right\} \\ &= -2\,\text{Re}\left\{ 2\pi i (\alpha i) \right\} \\ &= -2\,\text{Re}\left\{ -2\pi\alpha \right\} = \underline{\underline{4\pi\alpha}}, \text{ since } \alpha \in \mathbb{R}. \end{aligned}$$

2 unseen

	Setter's initials SJB	Checker's initials	Page number 24

	EXAMINATION SOLUTION 22-23	Course: Applied Complex Analysis
Question 5	Topic COMPLEX METHODS FOR BIHARMONIC EQUATION	Marks& seen/unseen
Parts (d). (continued).	When the rotlet is above the wall we need to use the new Goursat function $g'(z)$ found in part (c). However the point $\bar{z}_0$ lies <u>outside</u> the fluid region and so when using the residue theorem to evaluate the torque integral, once again only the pole at z_0 gives any contribution and we find the torque remains the same: Torque $= 4\pi\alpha$.	1 unseen Total: 20

Setter's initials SJB

Checker's initials

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60006/70006	1	No Comments Received
MATH60006/70006	2	No Comments Received
MATH60006/70006	3	No Comments Received
MATH60006/70006	4	No Comments Received
MATH70006	5	No Comments Received