Mathematical Logic (MATH6/70132;P65)
Problem Class, week 8

[1] Let $\mathcal{L}^=$ be the usual language for rings, with binary function symbols $+, \cdot, -$ and constant symbols $0, 1$. Let Φ consist of the usual axioms for fields. So a field is a normal model of Φ.

Recall that for each prime number p there is a field $\mathbb{F}_p$ with p elements (take the integers modulo p).

Using the compactness theorem for normal models, prove the following:

Suppose ϕ is a closed $\mathcal{L}^=$-formula with the property that for infinitely many primes p, we have $\mathbb{F}_p \models \phi$. Then there is an infinite field F with $F \models \phi$.

If you know what the characteristic of a field is, show that we can also take F to be of characteristic 0.

[2] Suppose $\mathcal{L}^=$ is a first order language with equality ($=$) and a single binary relation symbol R.

(i) Write down a set Σ of closed $\mathcal{L}^=$-formulas such that the normal models of Σ are the normal $\mathcal{L}^=$-structures in which R is interpreted as an equivalence relation in which there are infinitely many equivalence classes and all equivalence classes are infinite.
(ii) Explain why any two countable normal models of Σ are isomorphic.

(iii) Find two non-isomorphic normal models of Σ with the same domain.

(iv) Prove that if $\mathcal{A}_1, \mathcal{A}_2$ are two normal models of Σ and ϕ is a closed $\mathcal{L}^=$-formula, then $\mathcal{A}_1 \models \phi \Leftrightarrow \mathcal{A}_2 \models \phi$.

[3] Suppose $\mathcal{L}^=$ is a language with equality and a single 2-ary relation symbol R. A graph $\mathcal{A} = \langle A; \bar{R} \rangle$ is a normal model of

$$(\forall x_1)(\forall x_2)(\neg R(x_1, x_1) \wedge (R(x_1, x_2) \to R(x_2, x_1))).$$

So $\bar{R}$ is symmetric and irreflexive. The elements of A are usually called *vertices*.

A clique in a graph is a set C of vertices such that any two distinct vertices in C are related by $\bar{R}$; a co-clique is a set K of vertices such that no pair of vertices in K is related by $\bar{R}$.

(i) For $n \in \mathbb{N}$, express the properties 'there is a clique of size n' and 'there is a co-clique of size n' by closed formulas μ_n and λ_n.

(ii) The infinite version of Ramsey's Theorem says that an infinite graph has an infinite clique or an infinite co-clique. Using this and the Compactness Theorem deduce the finite version of the theorem:

For every $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if $\mathcal{A}$ is a graph with at least N vertices, then $\mathcal{A}$ has a clique of size n or a co-clique of size n.