Problem Sheet 2

1). Show that the principle value integral

$$I(x) = ⨍_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x} dt = \frac{\pi}{2} - \pi x^2, \quad -1 < x < 1,$$

(as seen in section 2.8 page 63), using

(a). The Plemelj formulae,

(b). Contour integration of a suitable function around an appropriate contour.

2). Show that the principle value integral

$$I(x) = ⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-2)(t-x)} dt = \frac{-\pi\sqrt{3}}{x-2} - \pi, \quad -1 < x < 1,$$

(as seen in section 2.9 page 65).

3). Show that if

$$\frac{1}{\pi}\int_{-1}^{1} \left(\frac{\sqrt{3}}{(t-2)\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}} \right) dt = 0,$$

where A is a constant, then we must have $A = 1$ (as seen in section 2.9 page 65).

4). Solve

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x} dt = \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1,$$

subject to the condition that $f(0) = 1$.

5). Solve

$$\frac{1}{\pi}\int_{-1}^{1} f(t) \log|t-x| dt = 1 + \arcsin(x), \quad -1 < x < 1.$$

6). Show that with $\lambda =$ constant, the equation

$$\frac{1}{\pi}\int_{-1}^{1} f(t) \log|t-x| dt + \lambda \int_{-1}^{1} f(t) dt = 1, \quad -1 < x < 1,$$

has a solution unless λ takes a particular value λ_0. Calculate λ_0 and solve for $f(x)$ when $\lambda \neq \lambda_0$.