Example. $\mathcal{L}^=$ a language with $=$ $n \in \mathbb{N}$. Let

$$\sigma_n : (\exists x_1)\cdots(\exists x_n)\bigwedge_{1\le i<j\le n}(x_i \neq x_j)$$

If $\mathcal{A}$ is a normal $\mathcal{L}^=$-str. with domain A then

$$\mathcal{A} \models \sigma_n \iff |A| \geq n.$$

Ex: Suppose θ is a closed $\mathcal{L}^=$-fmla such that for every $n \in \mathbb{N}$ there is a normal model of θ with $\geq n$ elements. Then there is an infinite normal model of θ.

Pf Apply Compactness Thm (2.6.5) to $\Delta = \{\theta, \sigma_1, \sigma_2, \cdots\}$

Beginning Model theory.

(2.6.6) Thm. (Countable Downward Löwenheim-Skolem Theorem)

Suppose $\mathcal{L}^=$ is a countable 1st order language with equality and $\mathcal{B}$ is a normal $\mathcal{L}^=$-str. Then there is a countable normal $\mathcal{L}^=$-str. $\mathcal{A}$ s.t. for all ~~all~~ closed $\mathcal{L}^=$-fmlas ϕ

$$\mathcal{B} \models \phi \iff \mathcal{A} \models \phi.$$

Notation:

$Th(\mathcal{B}) = \{\text{closed } \phi : \mathcal{B} \models \phi\}$

the $\mathcal{L}^=$-theory of $\mathcal{B}$.

Pf: $Th(\mathcal{B}) \supseteq \Sigma_{=}$ (axioms for $=$).
as $\mathcal{B}$ is a normal $\mathcal{L}^{=}$-str.
and $Th(\mathcal{B})$ is consistent.
So by 2.6.4, $Th(\mathcal{B})$ has a countable normal model $\mathcal{A}$.

So $Th(\mathcal{B}) \subseteq Th(\mathcal{A})$.

If ϕ is closed & $\mathcal{B} \not\models \phi$ then $\mathcal{B} \models (\neg\phi)$, so $(\neg\phi) \in Th(\mathcal{A})$ ie $\mathcal{A} \models (\neg\phi)$

Thus $\phi \notin Th(\mathcal{A})$.
So $Th(\mathcal{B}) = Th(\mathcal{A})$.

#.

(2.7) Example / Application.

Linear orders.

$\mathcal{L}^{=}$: 1^{st} order language with $=$ and a 2-ary relation symbol $\leq$ (Nothing else.).

(2.7.1) Def. A linear order $\mathcal{A} = \langle A ; \leq_A \rangle$ is a normal model of:

ϕ_1: $(\forall x_1)(\forall x_2)(((x_1 \leq x_2) \wedge (x_2 \leq x_1)) \to (x_1 = x_2))$

ϕ_2: $(\forall x_1)(\forall x_2)(\forall x_3)(((x_1 \leq x_2) \wedge (x_2 \leq x_3)) \to (x_1 \leq x_3))$

ϕ_3: $(\forall x_1)(\forall x_2)((x_1 \leq x_2) \vee (x_2 \leq x_1))$

Say $\mathcal{A}$ is <u>dense</u> if also

ϕ_4: $(\forall x_1)(\forall x_2)(\exists x_3)$
$\big((x_1 < x_2) \to (x_1 < x_3) \wedge (x_3 < x_2)\big)$

where "$x_1 < x_2$" is
shorthand for $(x_1 \leq x_2) \wedge (x_1 \neq x_2)$

It is <u>without endpoints</u> if

ϕ_5 : $(\forall x_1)(\exists x_2)\,(x_2 < x_1)$

ϕ_6 : $(\forall x_1)(\exists x_2)(x_1 < x_2)$.

Let $\Delta = \{\phi_1, \dots, \phi_6\}$

(<u>Ex</u>: $\Delta \vdash \Sigma_=$)

Let $\mathcal{Q} = \langle \mathbb{Q} ; \leq \rangle$ ③

& $\mathcal{R} = \langle \mathbb{R} ; \leq \rangle$

(usual ordering)

These are normal models of Δ.

(2.7.2) <u>thm</u>. For every closed $\mathcal{L}^=$-fmla

ϕ we have

$\mathcal{Q} \models \phi \iff \mathcal{R} \models \phi$

$\iff \Delta \vdash \phi$.

(Say that Δ <u>axiomatizes</u>
$Th(\mathcal{Q})$ and $Th(\mathcal{R})$.)

(2.7.3) <u>Def / Result</u>.

(2.3.9 + 10, L12).

① Linear orders

$\mathcal{A} = \langle A; \leq_A \rangle$ &

$\mathcal{B} = \langle B; \leq_B \rangle$ are

isomorphic if there is a bijection $\alpha : A \to B$ s.t.

for all $a, a' \in A$

$a \leq_A a' \iff \alpha(a) \leq_B \alpha(a')$

② If $\mathcal{A}, \mathcal{B}$ are isomorphic then

$Th(\mathcal{A}) = Th(\mathcal{B})$.

(2.7.4) <u>Thm</u> (Fact; G. Cantor)

If $\mathcal{A}, \mathcal{B}$ are <u>countable</u> dense linear orderings without endpoints then $\mathcal{A}, \mathcal{B}$ are isomorphic.

(2.7.5) <u>Lemma</u>.

(Special case of Łos - Vaught test)

Δ as in 2.7.1.

Let $\Sigma_0 = \Delta \cup \Sigma_=$

then for every closed $\mathcal{L}^=$-fmla ϕ we have either

$\Sigma_0 \vdash \phi$ or $\Sigma_0 \vdash (\neg\phi)$.

<u>Pf</u>: Suppose not for some ϕ.

As Σ_0 is consistent we have $\Sigma_1 = \Sigma_0 \cup \{\neg\phi\}$

and $\Sigma_2 = \Sigma_0 \cup \{\neg\neg\phi\}$

are consistent (by 2.5.2).

By 2.6.4 Σ_1, Σ_2 have

countable normal models
$\mathcal{A}_1, \mathcal{A}_2$.
By 2.7.4 we have
that $\mathcal{A}_1$ & $\mathcal{A}_2$ are
isomorphic. But
$\mathcal{A}_1 \models \neg\phi$ &
$\mathcal{A}_2 \models \neg\neg\phi$.
that contradicts 2.7.3 (2).
#.

General Δ: Assume all
normal models of Δ are
infinite, Δ has a normal model
& any two countable normal
models of Δ are isomorphic. (5)

Conclusion: $\Sigma = \Delta \cup \Sigma_=$
is complete.