Analysis II, Term I

Dr. Davoud Cheraghi

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Introduction to the module

This is a continuation of Analysis I module you had in year-one. In that module, you have learned about the real numbers, completeness, convergence of sequences and series, continuity and differentiability of functions on an interval or $\mathbb{R}$, integral of a function on an interval. Analysis II is a single module in year-two, delivered during term I and term II.

The content of Analysis II in term I has two parts. In the first part we complete the study of analysis on Euclidean spaces, by introducing the concepts of converges of sequences in higher dimensional Euclidean spaces $\mathbb{R}^n$, and the continuity and differentiability of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. In the second part of the module, we generalise these notions of analysis on Euclidean spaces into a broader setting, called metric spaces and topological spaces. That is a setting where one can define the notions of converge of sequences, completeness of spaces, continuity of maps, etc. Many theorems you have learned in the previous analysis module extends into this setting, and indeed, one can give unified proofs to all those statements at once. Many theorems find a natural form in the setting of metric spaces, and you will see that the proof you already know for a statement can be adapted to the more general setting.

Any section/subsection marked with * is not examinable, but will be valuable in future courses, especially if you take pure analysis courses in your third year and beyond. You should certainly at least read through the notes on these sections, even if you choose not to attempt the questions. I will try to indicate in lectures when I'm covering those material.

Throughout this lecture notes, the definitions are numbered successively within each chapter, that is, in Chapter 1, you will see Definition 1.1, Definition 1.2, Definition 1.3, and so on. The same numbering mechanism applies to Examples, Exercises, and Remarks in each chapter. On the other hand, the results such as lemmas, propositions, corollaries, and theorems are collectively numbered in a successive fashion. That is, in Chapter 1, you will see Proposition 1.1, Theorem 1.2, Theorem 1.3, etc.

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Chapter 1

Differentiation in higher dimensions

1.1 Euclidean spaces

1.1.1 Preliminaries from analysis I

In this chapter we are going to extend some of the ideas that you saw last year (such as limits and continuity) to higher dimensions. The definitions are almost identical, so this should mostly feel like a review chapter to begin with, although some of the ideas we are going to approach from a different point of view.

Throughout these notes we frequently use the standard notations for the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the set of integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

the set of rational numbers

$$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\}\},$$

and the set of real numbers $\mathbb{R}$. The set of real numbers is obtained as the *completion* of $\mathbb{Q}$. We may add, multiply and subtract elements of $\mathbb{R}$, and we can divide by elements of $\mathbb{R} \setminus \{0\}$. Note that some authors use the notation $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$, but we will omit 0 from this set.

On $\mathbb{R}$ we have a notion of ordering $\leq$, so that we may say whether a real number is greater than, less than or equal to another. Moreover, $\mathbb{R}$ satisfies the **completeness axiom**, that is, if $A \subset \mathbb{R}$ is non-empty and bounded above, then A has a least upper bound. The standard notation for the least upper bound of A is $\sup(A)$.

An important function defined on all real numbers is the **modulus function**, defined as

$$|x| := \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

This function has the following properties:

(i) for all $x \in \mathbb{R}$, we have $|x| \geq 0$, with $|x| = 0$ if and only if $x = 0$,

(ii) for all x and y in $\mathbb{R}$, $|xy| = |x|\,|y|$,

(iii) for all x and y in $\mathbb{R}$,
$$|x+y| \leq |x| + |y| .$$

The third property in the above list is called the **triangle inequality** for the modulus function.

1.1.2 Euclidean space of dimension n

For $n \geq 1$, the n-**dimensional Euclidean space**, denoted by $\mathbb{R}^n$, is defined as the set of ordered n-tuples $(x^1, x^2, \dots, x^n)$, where each $x^i \in \mathbb{R}$, for $i = 1, 2, \dots, n$. Each such n-tuple is denoted by a single letter $x = (x^1, x^2, \dots, x^n)$ and will be referred to as a point in $\mathbb{R}^n$. The entries x^i are called the **coordinates** of x.

One may see each element of $\mathbb{R}^n$ as a row vector with n real components, or as a column vector with n real components. We do not make this distinction (unless when a matrix is acting on the point x. When a matrix M acts on a vector with the same components as x we use Mx^t to make it clear that x is viewed as a column vector. Here ^{t} denotes the transpose operation.)

We shall try to stick to the convention of using superscripts to label components of vectors, and subscripts to label different vectors, so that $x_1, x_2 \in \mathbb{R}^n$ are two different vectors, while $x^1, x^2 \in \mathbb{R}$ are the components of one vector.

If x and y are elements of $\mathbb{R}^n$ with

$$x = \left(x^1, \dots, x^n\right), \qquad y = \left(y^1, \dots, y^n\right),$$

we can add these two elements according to

$$x + y = \left(x^1 + y^1, \dots, x^n + y^n\right).$$

Moreover, for every $\lambda \in \mathbb{R}$, we define

$$\lambda x = \left(\lambda x^1, \dots, \lambda x^n\right).$$

With these definitions, $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$.

The **inner product**,

$$\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

is defined as

$$\left\langle (x^1, \ldots, x^n), (y^1, \ldots, y^n) \right\rangle = \sum_{i=1}^{n} x^i y^i .$$

Using the inner product, we may define the **length**, or **norm**, function

$$\|\cdot\| : \mathbb{R}^n \to [0, \infty)$$

as

$$\|x\| = \sqrt{\langle x, x \rangle} = \langle x, x \rangle^{1/2} .$$

Note that the inner product of two vectors is a real number, not a vector.

The norm function on $\mathbb{R}^n$ has the following properties:

(i) for all $x \in \mathbb{R}^n$, we have $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$,

(ii) for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \, \|x\|$,

(iii) for all x and y in $\mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\| . \tag{1.1}$$

The third property in the above list is called the **triangle inequality** for the norm on $\mathbb{R}^n$.

Remark 1.1. As we shall see later, these properties can be used in an abstract fashion to define more general "normed vector spaces". The norm gives us a useful notion of "distance" between two points, that is, the distance from x to y is given by $\|x - y\|$. Notice that if $n = 1$ we have $|\cdot| = \|\cdot\|$, and we will use either interchangeably in this case.

Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all x, y, and z in $\mathbb{R}^n$ and all $a \in \mathbb{R}$,

$$\langle x, y \rangle = \langle y, x \rangle , \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle , \qquad \langle ax, y \rangle = a \langle x, y \rangle .$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0. \tag{1.2}$$

(c) By thinking of (1.2) as a quadratic in t, and considering its possible roots, deduce the **Cauchy-Schwartz** inequality:

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| . \tag{1.3}$$

When does equality hold?

(d) Deduce the triangle inequality (1.1).

(e) Show the reverse triangle inequality:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|$$

Exercise 1.2. Suppose $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

(a) Show that:

$$\max_{k=1,\ldots,n} \left|x^k\right| \leq \|x\| \,. \tag{1.4}$$

(b) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|. \tag{1.5}$$

1.1.3 Convergence of sequences in Euclidean spaces

Now that we have a few definitions relating to $\mathbb{R}^n$, we're ready to revisit some concepts from first year analysis and see how they can be extended to higher dimensions.

A sequence in $\mathbb{R}^n$ is an ordered list

$$x_0, x_1, x_2, \ldots,$$

with each $x_i \in \mathbb{R}^n$, for $i = 0, 1, 2, \ldots$. This is often written $(x_i)_{i=0}^{\infty}$, or $(x_i)_{i\in\mathbb{N}}$. A very important concept relating to sequences is convergence.

Definition 1.1. A sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ **converges** to (the vector) $x \in \mathbb{R}^n$ if the following holds: For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

We then write:

$$x_i \to x, \quad \text{as } i \to \infty,$$

or

$$\lim_{i\to\infty} x_i = x.$$

One may compare the above definition to the one for convergence of a sequence of real numbers. Indeed, this notion is intimately related to convergence of real numbers, as stated in the next lemma.

Proposition 1.1. *The sequence of vectors $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ converges to the vector $x \in \mathbb{R}^n$ if and only if each component of x_i converges to the corresponding component of x. That is, if we write:*

$$x_i = (x_i^1, \ldots, x_i^n), \quad \textit{and} \quad x = (x^1, \ldots x^n),$$

then, $x_i \to x$ as $i \to \infty$ if and only if for all $k = 1, \ldots n$, $x_i^k \to x^k$ as $i \to \infty$.

Proof. Let us first assume that for all $k = 1, 2, \ldots, n$,

$$x_i^k \to x^k, \quad \text{as } i \to \infty.$$

Fix an arbitrary $\epsilon > 0$. Then, for each $k = 1, \ldots, n$, we apply the definition of convergence of $x_i^k \to x^k$ to $\epsilon/\sqrt{n}$ to obtain $N_k \in \mathbb{N}$ such that for all $i \geq N_k$ we have

$$\left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Let $N = \max\{N_1 \ldots, N_n\}$. Then, for every $i \geq N$, we have

$$\max_{k=1,\ldots,n} \left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Now, recall from the inequality in (1.4) that for every $y = (y^1, y^2, \ldots, y^n) \in \mathbb{R}^n$,

$$\|y\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|y^k\right|,$$

so we deduce

$$\|x_i - x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x_i^k - x^k\right| < \epsilon.$$

This establishes the result in one direction.

Now assume that

$$\lim_{i\to\infty} x_i = x.$$

Fix an arbitrary integer k with $1 \leq k \leq n$, and an arbitrary $\epsilon > 0$. We aim to show that $x_i^k \to x^k$, as $i \to \infty$. The definition of convergence of $x_i \to x$, as $i \to \infty$, with ϵ, gives us $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

Recall from Exercise 1.1, Equation (1.5) that for every $y = (y^1, y^2, \ldots, y^n) \in \mathbb{R}^n$,

$$\max_{k=1,\ldots,n} \left|y^k\right| \leq \|y\|.$$

In particular, for all $i \geq N$, we have

$$\left|x_i^k - x^k\right| \leq \max_{k=1,\ldots,n} \left|x_i^k - x^k\right| \leq \|x_i - x\| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this shows that x_i^k converges to x^k, as $i \to \infty$. $\square$

Exercise 1.3. Suppose that $(x_i)_{i=0}^\infty$ and $(y_i)_{i=0}^\infty$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i\to\infty} x_i = x, \quad \lim_{i\to\infty} y_i = y.$$

(a) Show that

$$x_i + y_i \to x + y \quad \text{as } i \to \infty.$$

(b) Show that

$$\langle x_i, y_i \rangle \to \langle x, y \rangle \qquad \text{as } i \to \infty,$$

deduce that

$$\|x_i\| \to \|x\| \qquad \text{as } i \to \infty.$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence in $\mathbb{R}$ with $a_i \to a$ as $i \to \infty$. Show that:

$$a_i x_i \to ax, \qquad \text{as } i \to \infty.$$