Analysis II, Complex Analysis

CW2 Solutions

Q1. (5p)

a) 3p. Assume that u and v are harmonic in an open set Ω and that v is a harmonic conjugate to u in Ω. Define functions

$$U(x,y) = e^{u^2(x,y)-v^2(x,y)}\cos(2u(x,y)v(x,y))$$
$$V(x,y) = e^{u^2(x,y)-v^2(x,y)}\sin(2u(x,y)v(x,y)).$$

Show that U and V are harmonic in Ω and that V is a harmonic conjugate to U.

Solution: It is enough to check that the function $F(z) = U(x,y) + iV(x,y)$ ($z = x + iy$) is holomorphic in Ω.

1 mark

By hypothesis, the function $f(z) = u(x,y) + iv(x,y)$ is holomorphic in Ω. We have

$$\begin{aligned} F(z) = U(x,y) + iV(x,y) &= e^{u^2(x,y)-v^2(x,y)}\,(\cos(2u(x,y)v(x,y)) \\ &\qquad + i\sin(2u(x,y)v(x,y))) \\ &= e^{u^2(x,y)-v^2(x,y)}\,e^{2iu(x,y)v(x,y)} \\ &= e^{(u(x,y)+iv(x,y))^2} \\ &= e^{f^2(z)}. \end{aligned}$$

1 mark

Therefore $F(z)$ is holomorphic in Ω. Thus U and V are harmonic in Ω and V is a harmonic conjugate to U.

1 mark

b) 2p. Let $u = u(x,y)$ be harmonic for all $(x,y) \in \mathbb{R}^2$. Show that if there is $M \in \mathbb{R}$ such that $u \le M$, then $u = const$.

Solution: If u is harmonic in $\mathbb{R}^2$, then it has a harmonic conjugate v such that $f = u + iv$ is entire. 1 mark

Thus

$$F = e^{f-M} = e^{u+iv-M}$$

is also entire. Besides

$$|f(z)| = \left|e^{u+iv-M}\right| = \left|e^{u-M}\right| < 1.$$

By Liouville's theorem, f is constant and thus u is also constant.

1 mark

Q 2. (5p)

a) 2p. Find the poles and their orders of the function

$$\frac{1}{e^z - 1} - \frac{1}{z}.$$

Answer: $z = 2\pi i k$, $k = \pm 1, \pm 2, \dots$ are poles of order one. (Note $k \neq 0$.)

2 marks

b) 3p. Find the Laurent expansion for the function

$$f(z) = \frac{1}{(z+2)^3} \quad \text{for} \quad |z| > 2.$$

Solution: We have

$$\begin{aligned}
\frac{1}{(z+2)^3} &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{z+2}\right) = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{z(1+2/z)}\right) \\
&= \frac{1}{2}\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}\frac{(-2)^n}{z^{n+1}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{d^2}{dz^2}\left(\frac{(-2)^n}{z^{n+1}}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-2)^n(n+1)(n+2)}{z^{n+3}}.
\end{aligned}$$

3 marks

Q3. (5p) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{16+x^2}\,dx.$$

Solution: Note first that

$$2\sin^2 x = 1-\cos 2x.$$

Since $\sin 2x$ is odd we obtain

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{16+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{16+z^2}\,dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-e^{2ix}}{16+x^2}\,dx.$$

Denote

$$f(z) = \frac{1-e^{2iz}}{16+z^2}.$$

Let $R > 4$ and introduce

$$\gamma = \gamma_1 \cup \gamma_2,$$

where

$$\gamma_1 = \{z = z+iy,\ x\in(-R,R), y=0\}, \quad \gamma_2 = \{z:\ z = R\,e^{\imath\theta}, \theta\in(0,\pi)\}.$$

1 mark

Within γ the function $f(z)$ has one simple pole at $z_0 = 4i$.

$$\oint_{\gamma} \frac{1-e^{2iz}}{16+z^2}\,dz = 2\pi i\,\mathrm{Res}\left[\frac{1-e^{2iz}}{16+z^2}, 4i\right]$$
$$= 2\pi i\,\frac{1-e^{-8}}{8i} = \frac{\pi}{4}(1-e^{-8}).$$

2 marks

Using the ML inequality we find

$$\left|\int_{\gamma_2} \frac{1-e^{2iz}}{16+z^2}\,dz\right| = \left|\int_0^{\pi} \frac{1-e^{2iRe^{i\theta}}}{16+R^2e^{2i\theta}}\,R\,i\,e^{i\theta}\,d\theta\right|$$
$$\le \max_{\theta\in(0,\pi)}\left|\frac{1-e^{2iR(\cos\theta+i\sin\theta)}}{16+R^2e^{2i\theta}}\,R\right|\pi \le \max_{\theta\in(0,\pi)}\left|\frac{1-e^{-R\sin\theta}}{R^2-16}\,R\right| \to 0,$$
$$\text{as}\quad R\to\infty.$$

1 mark

Besides,

$$\frac{1}{2}\int_{\gamma_1} \frac{1-e^{2iz}}{16+z^2}\,dz = \frac{1}{2}\int_{-R}^{R} \frac{1-e^{2ix}}{16+x^2}\,dx = \frac{1}{2}\int_{-R}^{R} \frac{1-\cos 2x}{16+x^2}\,dx$$
$$\to \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{16+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\sin^2 x}{16+x^2}\,dx,$$

as $R \to \infty$. 1 mark

Answer:

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{16+x^2}\,dx = \frac{\pi}{8}\,(1-e^{-8}).$$

Q4. (5p)

a) 2p. Let $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ and let $f(z)$ be a continuous function on γ. Prove that

$$\overline{\oint_\gamma f(z)\,dz} = -\oint_\gamma \overline{f(z)}\,\frac{dz}{z^2}.$$

Solution: Indeed,

$$\begin{aligned}\overline{\oint_\gamma f(z)\,dz} &= \overline{\int_0^{2\pi} f(e^{i\theta})ie^{i\theta}\,d\theta} = -\int_0^{2\pi} \overline{f(e^{i\theta})}\;ie^{-i\theta}\,d\theta \\ &= -\int_0^{2\pi} \overline{f(e^{i\theta})}\;i\frac{e^{i\theta}}{e^{2i\theta}}\,d\theta = -\oint_\gamma \overline{f(z)}\,\frac{dz}{z^2}.\end{aligned}$$

2 marks

b) 3p. Prove that for a real $\lambda > 1$ there is a unique solution to the equation

$$ze^{\lambda - z} = 1$$

in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Solution:

The equation $ze^{\lambda-z} = 1$ is equivalent to $ze^\lambda = e^z$ and also to $ze^\lambda - e^z = 0$. Denote by $f(z) = ze^\lambda$, $g(z) = -e^z$.

1 mark

Let $z : |z| = 1$. Then, since $\lambda > 1$, we have

$$|g(z)| = |-e^z| = \left|\sum_{n=0}^{\infty} \frac{z^n}{n!}\right| \le \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} = e < e^\lambda = |ze^\lambda| = |f(z)|.$$

The equation $ze^\lambda = f(z) = 0$ has only one solution. Therefore by using Rouche's theorem we obtain that the equation $ze^\lambda - e^z = 0$ and thus $ze^{\lambda-z} = 1$ has exactly one solution in the unit disc.

2 marks