IMPERIAL

MATH70001

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics I

Date: Tuesday, May 6, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Explain the concepts of *fluid particle* and *continuum hypothesis*, stating the relevant length scales and limits involved. For a gas consisting of two types of molecules with masses m_1 and m_2, describe how the density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{V}(\mathbf{x}, t)$ of a fluid particle at an arbitrary point $\mathbf{x}$ and time t may be defined in terms of the masses and velocities of the molecules.

(6 marks)

(b) An interface separating two fluids in motion is composed of fluid particles that remain on it all the time, and mathematically is described by $F(\mathbf{x}, t) = 0$. Derive the kinematic condition

$$\frac{\partial F}{\partial t} + (\mathbf{V} \cdot \nabla)F = 0, \qquad (*)$$

where $\mathbf{V}$ is the velocity of the flow.

Show that the condition $(*)$ remains valid for a moving *impermeable* surface, $F(\mathbf{x}, t) = 0$, that serves as a boundary of a flow field with velocity $\mathbf{V}$, where each point of the surface moves with a specified velocity $\mathbf{V}_s$, i.e. $\dfrac{d\mathbf{x}}{dt} = \mathbf{V}_s$, and 'impermeable surface' refers to one that does not allow fluid particles to migrate through.

Write down the condition if the moving surface is *porous* such that fluid particles go through the surface in its normal direction $\mathbf{n}$ with a velocity, $\mathcal{V}\,\mathbf{n}$, relative to that of the surface.

(6 marks)

(c) When Newton's second law is applied to the motion of a volume of fluid, which occupies a region $\mathcal{D}(t)$ at time t, we have

$$\frac{D}{Dt}\iiint_{\mathcal{D}} \rho \mathbf{V} d\tau = \iint_{S} \mathbf{p}_n \, ds + \iiint_{\mathcal{D}} \rho \, \mathbf{f} \, d\tau,$$

where $\mathbf{f} = (f_1, f_2, f_3)$ is the body force per unit mass exerted at each point in $\mathcal{D}$, and $\mathbf{p}_n$ is the stress (i.e. surface force per unit area) on S, the surface enclosing $\mathcal{D}$.

(i) Given that the stress $\mathbf{p}_n = (p_1, p_2, p_3)$ can be expressed in terms of the stress tensor, $\mathcal{P} = (p_{ij})_{3\times 3}$, and the unit normal vector of S, $\mathbf{n}$, as $p_i = p_{ji} n_j$. Show that

$$\rho \frac{DV_i}{Dt} = \frac{\partial p_{ji}}{\partial x_j} + \rho \, f_i; \qquad (**)$$

here and in the rest of this question Einstein's convention is assumed.
Hint: You may use without proof the results:

$$\frac{D}{Dt}\iiint_{\mathcal{D}} \rho \, G \, d\tau = \iiint_{\mathcal{D}} \rho \frac{DG}{Dt} d\tau, \qquad \iint_{S} \mathbf{G} \cdot \mathbf{n} \, ds = \iiint_{\mathcal{D}} \nabla \cdot \mathbf{G} \, d\tau.$$

(4 marks)

(ii) Applying $(**)$ to a so-called polar fluid, which has the constitutive relation,

$$p_{ij} = -p\delta_{ij} + \mu\Big(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\Big) + \eta \frac{\partial^2}{\partial x_k \partial x_k}\Big(\frac{\partial V_i}{\partial x_j} - \frac{\partial V_j}{\partial x_i}\Big),$$

where μ and η are constants, show that

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} - \eta \nabla^2(\nabla^2 \mathbf{V}) + \rho \, \mathbf{f}.$$

(4 marks)

(Total: 20 marks)

2. Consider the incompressible flow between two infinitely large parallel rigid plates at $y = 0$ and $y = h$. The upper plate is kept stationary, while the lower plate oscillates with velocity $U\sin(\Omega t)$ in its own plane, where t is the time variable. Let x be the coordinate parallel to the plates, and u and v denote the velocity components in the x and y directions respectively. There is no pressure gradient in the x direction so that the flow is driven by the oscillatory plate only. The fluid has kinematic viscosity ν.

(a) Show that the Navier-Stokes equations admit an exact solution of the form

$$(u, v) = \Big(u(y,t),\, 0\Big).$$

Derive the equation satisfied by $u(y,t)$ and specify the boundary conditions on $u(y,t)$.

(4 marks)

(b) After an initial transient period, a time-periodic state of the flow is established.

(i) The solution $u(y,t)$ for this periodic state may be sought in the form

$$u(y,t) = Uf(y)\mathrm{e}^{\mathrm{i}\Omega t} + c.c.,$$

where $c.c.$ stands for complex conjugate. Deduce the equation and boundary conditions that $f(y)$ satisfies. Determine $f(y)$ and hence $u(y,t)$.

(8 marks)

(ii) Simplify the solution in each of the following limiting cases:

(a) $(2\nu/\Omega)^{\frac{1}{2}} \gg h$; (b) $(2\nu/\Omega)^{\frac{1}{2}} \ll h$.

Comment briefly on the nature of the flow in each case.

(4 marks)

(c) Suppose that the initial state of the flow between the plates is at rest, i.e. $u(y,0) = 0$ for $y \in [0,h]$. The full solution for $u(y,t)$ satisfying the initial condition may be sought by writing

$$u(y,t) = u_p(y,t) + \tilde{u}(y,t),$$

where $u_p(y,t)$ is the solution found in Part b(i) for the established periodic state. Write down the equation as well as the boundary and initial conditions that $\tilde{u}(y,t)$ satisfies. Find the solution for $\tilde{u}$.

Hint: The equation for $\tilde{u}$ may be solved using the method of variable separation.

You may use the identity

$$\int_0^h e^{a(y-h)}\sin(n\pi y/h)dy = \frac{n\pi/h}{a^2 + (n\pi/h)^2}\Big[e^{-ah} - (-1)^n\Big],$$

which is valid for an arbitrary constant a and integer n.

(4 marks)

(Total: 20 marks)

3. Consider the flow due to discharge of water from a pipe with cross-section area A placed vertically into the Earth's gravity field g, and connected to a horizontal pipe whose cross-section area is $B = kA$ as is shown in Figure 1. The vertical pipe is filled with water to a height h_0 with its top end open to the atmosphere where the pressure is p_a, while the horizontal pipe is open, in both the left and right directions, to an environment with pressure $p_a + \Delta p$; shutters FF' and GG' are used to block the water. When the water is completely settled, the shutters are opened simultaneously at time $t = 0$. The ensuing water motion in both pipes is assumed to be incompressible, *inviscid* and *one-dimensional*, and the small distances between the shutters and the centre of vertical pipe are negligible. The instantaneous height and length of the water columns in the vertical and horizontal pipes at time t are denoted by $h(t)$ and $l(t)$, respectively.

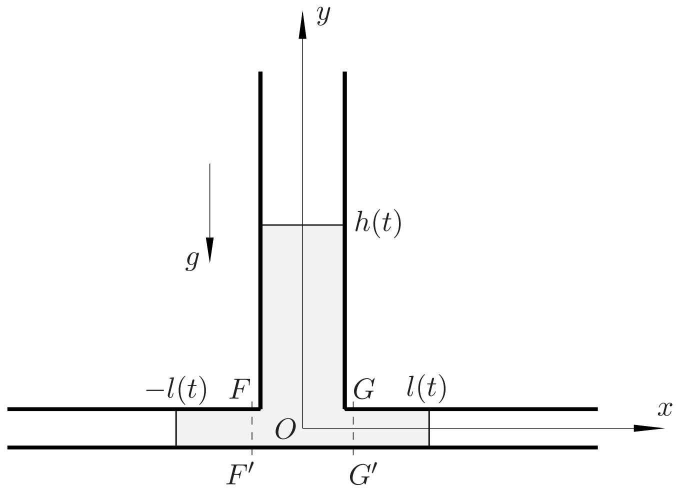


Figure 1: A diagram of the layout and the flow.

(i) Use the Euler equations governing inviscid flows, or otherwise, to determine the pressure distribution $p(y)$ in the vertical pipe between $y = 0$ and $y = h(t)$.

Repeat the analysis for the horizontal pipe to find the pressure distribution between $x = 0$ and $x = l(t)$.

Deduce the relation between $h(t)$ and $l(t)$.

(10 marks)

(ii) On the assumption that at point O the pressure is the same for the horizontal and vertical pipes, show that $h(t)$ satisfies the differential equation,

$$\Big(1 - \frac{1}{4k^2}\Big)hh'' + \frac{h_0}{4k^2}h'' + gh = \Delta p/\rho.$$

Solve this equation for the case of $k = B/A = 1/2$ to find $h(t)$ that satisfies the initial conditions that $h(0) = h_0$, $h'(0) = 0$ and $l(0) = 0$.

(6 marks)

(iii) Using the solution found in Part (ii) for $k = 1/2$, show that the water in the vertical pipe can be discharged completely if $\Delta p < \rho g h_0/2$, and determine the time that the discharge takes. Discuss what happens when $\Delta p > \rho g h_0/2$.

(4 marks)

(Total: 20 marks)

4. (a) Let C be a streamline connecting two points, (x_1, y_1) and (x_2, y_2), in a two-dimensional inviscid flow. Let vector (F_x, F_y) denote the force that the fluid exerts on C, whose unit normal vector is $\mathbf{n} = (dy/ds, -dx/ds)$, where $ds = \sqrt{(dx)^2 + (dy)^2}$ is the arc length increment along C.

(i) Show that the components F_x and F_y of the force in the x and y directions respectively are given by the contour integrals of the pressure p,

$$F_x = -\int_C p\, dy, \quad F_y = \int_C p\, dx,$$

and deduce that

$$F_x - iF_y = -i\int_C p\, d\bar{z},$$

where $\bar{z}$ is the complex conjugate of $z = x + iy$.

(3 marks)

(ii) Suppose that the flow is irrotational, and represented by a complex potential $w(z)$. In the far field, the pressure and the modulus of the velocity, p_∞ and V_∞, are both constant. Show that

$$F_x - iF_y = -i\Big(p_\infty + \frac{1}{2}\rho V_\infty^2\Big)(\bar{z}_2 - \bar{z}_1) + \frac{1}{2}i\rho\int_C \Big(\frac{dw}{dz}\Big)^2 dz,$$

where ρ is the density of the fluid.

(5 marks)

(b) Consider a two-dimensional inviscid irrotational flow represented by the complex potential

$$w(z) = V_\infty z + \frac{q}{2\pi}\ln z, \qquad (*)$$

which is a superposition of a uniform flow with velocity $V_\infty > 0$ and a two-dimensional source ($q > 0$) located at $z = 0$.

(i) Introduce polar coordinates (r, ϑ) by writing $z = r\, e^{i\vartheta}$, and find the velocity potential $\varphi(r, \vartheta)$ and stream function $\psi(r, \vartheta)$.

(2 marks)

(ii) Find the stagnation point of the flow and the contour of the stream function passing through the stagnation point, and show that as $x \to \infty$ along the contour, $y \to d$, where d is a constant.

Explain why the complex potential ($*$) can represent a flow past a rigid body/surface of a certain shape, which you are expected to determine; illustrate it with a sketch.

Hint: You may use without proof the formulae that the radial and azimuthal velocities, V_r and V_ϑ, are given by

$$V_r = \frac{\partial\varphi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}, \quad V_\vartheta = \frac{1}{r}\frac{\partial\varphi}{\partial\vartheta} = -\frac{\partial\psi}{\partial r}.$$

(6 marks)

(iii) Using the result in Part (a), or otherwise, calculate F_x and F_y of the force that the fluid exerts on the body contour found in b (ii) above.

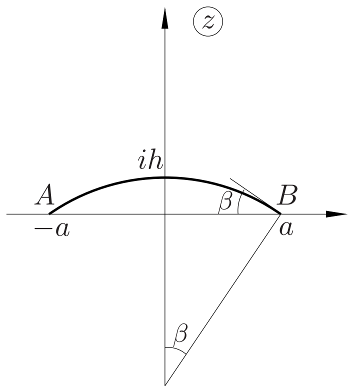
Hint: You may evaluate the contour integral using the Residue Theorem:

$$\oint_{\tilde{C}} \Big(\frac{dw}{dz}\Big)^2 dz = 2\pi i \times \text{(sum of the residues at simple poles of } \Big(\frac{dw}{dz}\Big)^2 \text{ within } \tilde{C}).$$

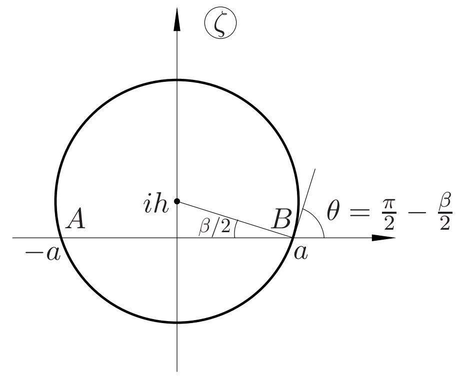
(4 marks)

(Total: 20 marks)

5. Consider an incompressible inviscid irrotational flow past a (slitted) circular arc, which has a depth h and a chord length $2a$. A Cartesian coordinate system is used such that the chord is aligned with the x-axis and bisected by the y-axis as is illustrated in Figure 2(a). The modulus of the velocity of the oncoming flow far from the arc is V_∞, and the angle of attack is α.



(a) z-plane.



(b) ζ-plane.

Figure 2: Joukowskii transformation.

It is known that the Joukowskii transformation,

$$z = \frac{1}{2}\Big(\zeta + \frac{a^2}{\zeta}\Big),$$

with its inversion being $\zeta = z + \sqrt{z^2 - a^2}$, maps the exterior of the circular arc onto the exterior of the circle centred at $\zeta = ih$ on the ζ-plane, as is shown in Figure 2(b).

(i) Find the radius and centre of the circle on which the arc sits, expressing your results in terms of a and h. Verify that the centre in the z-plane is mapped to $\zeta = ih$ in the ζ-plane.

(3 marks)

(ii) The complex potential $W(\zeta)$ in the auxiliary ζ-plane may assume the form

$$W(\zeta) = \tilde{V}_\infty\Big[(\zeta - ih)e^{-i\alpha} + \frac{a^2 + h^2}{(\zeta - ih)e^{-i\alpha}}\Big] + \frac{\Gamma}{2\pi i}\ln(\zeta - ih).$$

Determine $\tilde{V}_\infty$ in terms of V_∞ such that the far-field condition in the z-plane is satisfied.

Show that if the Joukowskii-Kutta condition is to be satisfied, Γ must take the value,

$$\Gamma = -2\pi V_\infty \sqrt{a^2 + h^2}\,\sin(\alpha + \beta/2).$$

(7 marks)

(iii) The results in (ii) may help understand how a parachute works if it is modelled by such an arc-shaped surface with an infinite span in the third direction. A parachute descends through the air at a constant vertical velocity $V_\perp$ while drifting horizontally at a speed $-V_\parallel$ (in the direction opposite to the x-axis). Assuming that the air is at rest at infinity, what is the force in the vertical direction that the air exerts on the parachute per unit spanwise length?

Hint: The lift produced by the flow considered in (ii) is $L = -\rho\Gamma V_\infty$.

(4 marks)

Question continues on the next page.

(iv) For the special case of $\alpha = \pi/2$, one may take $\Gamma = 0$ in order to respect the symmetry. Let z_+ be an arbitrary point on the upper side of the arc, while z_- be the corresponding point on the lower side. Find the expressions for the velocities at z_+ and z_-. On which side, the *upper* or *lower*, of the arc is the velocity greater in magnitude?

What is the total force in the vertical direction that the fluid exerts on the arc? Explain or reconcile your observations by referring to the relevant feature of the solution.

Hint: You may use the result that z_+ and z_- are mapped to

$$\zeta_\pm = z \pm i\sqrt{a^2 - z^2},$$

where $z = z_+$, so that $\zeta_- = a^2/\zeta_+$.

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

Math60001, Math70001

Fluid Dynamics 1 (Solutions)

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1. (a) A fluid particle is a collection of fluid molecules in a small volume $V \sim l^3$ with length scale l in the range $\lambda \ll l \ll L$, where λ denotes the mean free path (i.e. the mean distance molecules travel between two consecutive collisions) and L is the size of the fluid field. On the scale L, the volume is so small that this collection of molecules behaves like a 'particle', which occupies the centre of the volume, say $\mathbf{x}$. With the condition $l \gg \lambda$, the volume is sufficiently large that many collisions take place to ensure that statistically averaged (i.e. macroscopic) properties of all the molecules in the volume can be defined and attached to the fluid particle at $\mathbf{x}$. As this can be done for each $\mathbf{x}$, fluid particles can be considered as being continuously distributed in space, and the fluid as a continuum. seen $\Downarrow$

unseen $\Downarrow$

For a gas consisting of two types of molecules, the density of a fluid particle at a point $\mathbf{x}$ may be defined as

$$\rho(\mathbf{x}, t) = (N_1 m_1 + N_2 m_2)/V,$$

and the velocity as

$$\mathbf{V}(\mathbf{x}, t) = \frac{1}{N_1 m_1 + N_2 m_2} \sum (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2),$$

where N_1 and N_2 denote the numbers of molecules of type 1 and 2 respectively in a small volume (e.g. a sphere) centred at $\mathbf{x}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the velocities of the molecules. Here the key hypothesis made is that the averaged quantities become independent of the shape and size of the volume when $l \gg \lambda$. 6, A

(b) Let $\mathbf{x}(t)$ be the instantaneous position of a particle on the interface. Since it remains on the interface, there follows seen $\Downarrow$

$$F\big(\mathbf{x}(t), t\big) = 0.$$

Differentiation of the equation with respect to t using the Chain Rule yields

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_j}\frac{dx_j}{dt} = 0.$$

Noting that $\dfrac{d\mathbf{x}}{dt} = \mathbf{V}$, we arrive at the required condition

$$\frac{\partial F}{\partial t} + (\mathbf{V} \cdot \nabla)F = 0.$$

2, A

For a moving boundary, a material point $\mathbf{x}(t)$ on the surface obeys the constraints: $F\big(\mathbf{x}(t), t\big) = 0$ and $\dfrac{d\mathbf{x}}{dt} = \mathbf{V}_s$, from which follows unseen $\Downarrow$

$$\frac{\partial F}{\partial t} + (\mathbf{V}_s \cdot \nabla)F = 0, \quad \text{i.e.} \quad \mathbf{V}_s \cdot \nabla F = -\frac{\partial F}{\partial t}.$$

If the flow is assumed viscous, then the no-slip condition $\mathbf{V} = \mathbf{V}_s$ on the boundary is imposed, leading immediately to the required equation. However, for inviscid flows only the impermeability condition holds. Noting that the unit normal vector is given by $\mathbf{n} = \nabla F/|\nabla F|$, by projection the velocity of the moving boundary normal to the surface is $(\mathbf{V}_s \cdot \nabla F)/|\nabla F|$, which must, according to the impermeability condition, be equal to the velocity of the fluid in the direction $\mathbf{n}$:

$$(\mathbf{V} \cdot \nabla F)/|\nabla F| = (\mathbf{V}_s \cdot \nabla F)/|\nabla F| = -\frac{\partial F}{\partial t}/|\nabla F|,$$

from which the required condition also follows.

If the fluid penetrates through the boundary in the normal direction with a relative speed $\mathcal{V}$, we have

$$(\mathbf{V}\cdot\nabla F)/|\nabla F| - (\mathbf{V}_s\cdot\nabla F)/|\nabla F| = \mathcal{V},$$

which is, on using the relation $(\mathbf{V}_s\cdot\nabla F) = -\dfrac{\partial F}{\partial t}$, recast as

$$\frac{\partial F}{\partial t} + (\mathbf{V}\cdot\nabla F) = \mathcal{V}|\nabla F|.$$

$\boxed{4,\ \mathrm{C}}$

(c) Consider the i-th component of the momentum equation, which may, by applying the first of the given identity, be rewritten as $\boxed{\text{seen} \Downarrow}$

$$\iiint\limits_{\mathcal{D}} \rho\frac{DV_i}{Dt}d\tau = \iint\limits_{S} p_i\,ds + \iiint\limits_{\mathcal{D}} \rho\, f_i\, d\tau.$$

Noting that $p_i = p_{ji}n_j = (p_{1i}, p_{2i}, p_{3i})\cdot\mathbf{n}$, we have

$$\iint\limits_{S} p_i\,ds = \iint\limits_{S}(p_{1i}, p_{2i}, p_{3i})\cdot\mathbf{n}\,ds = \iiint\limits_{\mathcal{D}} \nabla\cdot(p_{1i}, p_{2i}, p_{3i})d\tau = \iiint\limits_{\mathcal{D}} \frac{\partial p_{ji}}{\partial x_j}\,d\tau,$$

after applying the Divergence Theorem. Hence

$$\iiint\limits_{\mathcal{D}} \rho\frac{DV_i}{Dt}d\tau = \iiint\limits_{\mathcal{D}} \Big[\frac{\partial p_{ji}}{\partial x_j} + \rho f_i\Big]\,d\tau.$$

As $\mathcal{D}$ is arbitrary, there follows the required equation

$$\rho\frac{DV_i}{Dt} = \frac{\partial p_{ji}}{\partial x_j} + \rho f_i.$$

$\boxed{4,\ \mathrm{B}}$

(d) With the given constitutive relation for 'polar fluids', which also reads, $\boxed{\text{unseen} \Downarrow}$

$$p_{ji} = -p\delta_{ji} + \mu\Big(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\Big) + \eta\frac{\partial^2}{\partial x_k\partial x_k}\Big(\frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}\Big),$$

we have

$$\frac{\partial p_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ji} + \mu\Big(\frac{\partial}{\partial x_i}\big(\frac{\partial V_j}{\partial x_j}\big) + \frac{\partial^2 V_i}{\partial x_j\partial x_j}\Big) + \eta\frac{\partial^2}{\partial x_k\partial x_k}\Big(\frac{\partial}{\partial x_i}\big(\frac{\partial V_j}{\partial x_j}\big) - \frac{\partial^2 V_i}{\partial x_j\partial x_j}\Big).$$

Use of the continuity equation, $\nabla\cdot\mathbf{V} = \dfrac{\partial V_j}{\partial x_j} = 0$, leads to

$$\frac{\partial p_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 V_i}{\partial x_j\partial x_j} - \eta\frac{\partial^2}{\partial x_k\partial x_k}\big(\frac{\partial^2 V_i}{\partial x_j\partial x_j}\big).$$

The three terms on the right-hand side are the i-th components of $-\nabla p$, $\mu\nabla^2\mathbf{V}$ and $-\eta\nabla^2(\nabla^2\mathbf{V})$, respectively. There follows the required momentum equation in the vector form. $\boxed{4,\ \mathrm{D}}$

2. (a) Since the plates are infinitely large, the flow is two dimensional with velocity field $(u, v) = \big(u(y,t), v(y,t)\big)$. $\boxed{\text{seen} \Downarrow}$

The continuity equation reduces to $\dfrac{\partial v}{\partial y} = 0$, implying that v is independent of y. Hence $v \equiv 0$ since $v = 0$ at $y = 0$ and h.

The $y-$momentum equation simplifies to $\dfrac{\partial p}{\partial y} = 0$, which is automatically satisfied.

With $\partial p/\partial x = 0$, the $x-$momentum equation yields:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

The no-slip boundary conditions imply that

$$u(0,t) = U \sin(\Omega t), \quad u(h,t) = 0.$$

$\boxed{4, \text{A}}$

(b) (i) Substitution of the assumed form for $u(y,t)$ into equation (1) gives $\boxed{\text{meth seen} \Downarrow}$

$$\mathrm{i}\,\Omega f = \nu f'', \quad \text{i.e.} \quad f'' - \frac{\mathrm{i}\,\Omega}{\nu} f = 0,$$

which is a second-order linear equation with constant coefficients. The general solution for f is found as

$$f = A \exp\{\sigma y\} + B \exp\{-\sigma y\}, \tag{2}$$

where $\sigma = \sqrt{\mathrm{i}\,\Omega/\nu} = (1+\mathrm{i})(\Omega/2\nu)^{1/2}$. The boundary conditions can be written as

$$u(0,t) = U \sin(\Omega t) = -\frac{1}{2}\mathrm{i}\, U \,\mathrm{e}^{\mathrm{i}\,\Omega t} + c.c., \quad u(h,t) = 0,$$

application of which gives

$$f(0) = -\mathrm{i}/2, \quad f(h) = 0. \tag{3}$$

Substituting the general solution (2) into (3), we obtain

$$A + B = -\mathrm{i}/2, \quad A \exp\{\sigma h\} + B \exp\{-\sigma h\} = 0,$$

from which follows

$$A = \frac{(\mathrm{i}/2)\exp\{-\sigma h\}}{\exp\{\sigma h\} - \exp\{-\sigma h\}}, \qquad B = -\frac{(\mathrm{i}/2)\exp\{\sigma h\}}{\exp\{\sigma h\} - \exp\{-\sigma h\}}.$$

Inserting A and B back into (2), we have $f(y)$, use of which in the assumed form for u gives

$$u(y,t) = \mathrm{i}\,\frac{U}{2}\left[\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}(y-h)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}(y-h)}}{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}h} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}h}}\right]\mathrm{e}^{\mathrm{i}\,\Omega t} + c.c. \tag{4}$$

$\boxed{8, \text{B}}$

(ii) (a) When $(2\nu/\Omega)^{\frac{1}{2}} \gg h$, $(\frac{\Omega}{2\nu})^{\frac{1}{2}}h \ll 1$ and $(\frac{\Omega}{2\nu})^{\frac{1}{2}}y \ll 1$. Performing Taylor expansions in $f(y)$, $\boxed{\text{sim. seen} \Downarrow}$

$$\mathrm{e}^{\pm(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}h} \approx 1 \pm (1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}h, \quad \mathrm{e}^{\pm(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}y} \approx 1 \pm (1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}y,$$

we obtain $f(y) \approx \frac{1}{2}\,\mathrm{i}(y-h)/h$, and hence

$$u(y,t) \approx U\sin(\Omega t)(h-y)/h.$$

The flow appears to be quasi-steady (i.e. a quasi-steady plane Couette flow).

(b) When $(2\nu/\Omega)^{\frac{1}{2}} \ll h$, it follows that $(\frac{\Omega}{2\nu})^{\frac{1}{2}}h \gg 1$, and

$$f(y) = \frac{1}{2}\,\mathrm{i}\,\frac{\mathrm{e}^{(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}(y-2h)} - \mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}y}}{1-\mathrm{e}^{-2(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}h}} \approx -\frac{1}{2}\,\mathrm{i}\,\mathrm{e}^{-(1+\mathrm{i})(\frac{\Omega}{2\nu})^{\frac{1}{2}}y},$$

when the exponentially small terms are neglected. Thus

$$u \approx U\,\mathrm{e}^{-(\frac{\Omega}{2\nu})^{\frac{1}{2}}y}\sin\Big(\Omega t - (\frac{\Omega}{2\nu})^{\frac{1}{2}}y)\Big).$$

The flow is confined in a layer near the lower wall, with an $O(\sqrt{2\nu/\Omega}\,)$ width, i.e. $y = O(\sqrt{2\nu/\Omega}\,) \ll h$. Outside this layer, the velocity is exponentially small. This is practically the Stokes layer for infinite domain with the remote fixed plate playing a negligible role. $\boxed{\text{4, C}}$

(c) With decomposition $u = u_p + \tilde{u}$, $\tilde{u}$ satisfies the equation $\boxed{\text{unseen} \Downarrow}$

$$\frac{\partial\tilde{u}}{\partial t} = \nu\frac{\partial^2\tilde{u}}{\partial y^2},$$

the boundary conditions: $\tilde{u}(0,t)=0$ and $\tilde{u}(h,t)=0$, and the initial condition

$$\tilde{u}(y,0) = -u_p(y,0) = -\,\mathrm{i}\,\frac{U}{2}\Big[\frac{\mathrm{e}^{a(y-h)}-\mathrm{e}^{-a(y-h)}}{\mathrm{e}^{ah}-\mathrm{e}^{-ah}}\Big] + c.c.,$$

where we have put $a = (1+\mathrm{i})(\Omega/2\nu)^{\frac{1}{2}}$.

Following the suggestion, we seek solution for $\tilde{u}$ in the variable separation form: $\tilde{u} = T(t)Y(y)$. Substitution into the equation of $\tilde{u}$ gives

$$Y''(y)/Y = T'(t)/(\nu T) = -\lambda^2.$$

Hence $Y = A\sin(\lambda y) + B\cos(\lambda y)$. Applying the boundary condition at $y=0$, we infer that $B=0$, while the boundary condition at $y=h$ requires $\lambda = n\pi/h$. It follows that $Y = A\sin(n\pi y/h)$, $T = \mathrm{e}^{-(n\pi/h)^2\nu t}$ and thus

$$\tilde{u} = \sum_{n=1}^{\infty} A_n\sin(n\pi y/h)\,\mathrm{e}^{-(n\pi/h)^2\nu t}\,.$$

Imposing the initial condition, we have

$$\sum_{n=1}^{\infty} A_n\sin(n\pi y/h) = -\,\mathrm{i}\,\frac{U}{2}\Big[\frac{\mathrm{e}^{a(y-h)}-\mathrm{e}^{-a(y-h)}}{\mathrm{e}^{ah}-\mathrm{e}^{-ah}}\Big] + c.c.$$

To determine the coefficients A_n, we multiply both sides by $\sin(n\pi y/h)$ and integrate across the channel:

$$\frac{h}{2}A_n = -\,\mathrm{i}\,\frac{U}{2}\int_0^h\Big[\frac{\mathrm{e}^{a(y-h)}-\mathrm{e}^{-a(y-h)}}{\mathrm{e}^{ah}-\mathrm{e}^{-ah}}\Big]\sin(n\pi y/h)\mathrm{d}y + c.c.$$

$$= \mathrm{i}\,\frac{U}{2}\frac{n\pi/h}{a^2+(n\pi/h)^2} + c.c. = \frac{(n\pi/h)\Omega/\nu}{(n\pi/h)^4+(\Omega/\nu)^2}U,$$

where the given identity is used for $a = \pm(1+\mathrm{i})(\Omega/2\nu)^{1/2}$ ($a^2 = \mathrm{i}\,\Omega/\nu$). Hence

$$A_n = \frac{(2n\pi)(\Omega/\nu)/h^2}{(n\pi/h)^4+(\Omega/\nu)^2}U = \frac{(2n\pi)(\Omega/\nu)h^2}{(n\pi)^4+(\Omega/\nu)^2h^4}U.$$

$\boxed{\text{4, D}}$

3. (i) Apply the Euler equations to the motion in the vertical pipe. Since the flow in this pipe is assumed to be one-dimensional, $u = 0$. It follows from the continuity equation that $\dfrac{\partial v}{\partial y} = 0$, which means that at any time, the fluid velocity at any point in the vertical pipe is the same, equal to that the moving top end of the fluid column. With $h(t)$ being the position of the top end, we have

sim. seen ⇓

$$v = h'(t).$$

Using this in the y-momentum equation and noting that $f_y = -g$, we obtain

$$h''(t) = -g - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

which is integrated to yield

$$p = -\rho(h'' + g)y + C_1(t). \tag{5}$$

The function $C_1(t)$ is determined by noting that the pressure on the top end is equal to the atmospheric pressure, i.e. $p = p_a$ at $y = h(t)$, which gives

$$C_1(t) = p_a + \rho(h'' + g)h.$$

Substitution of this into (5) shows that at any point $y < h(t)$ in the vertical pipe,

$$p = p_a + \rho(h'' + g)(h - y). \tag{6}$$

In particular, by setting $y = 0$ in (6) the pressure at point O is found to be

$$p_0 = p_a + \rho(h'' + g)h. \tag{7}$$

Since the horizontal pipe is symmetric with respect to point O, it suffices to consider the right-hand side branch of the pipe. As the flow in the horizontal pipe is assumed to be one-dimensional, we have $v = 0$, with which the continuity equation reduces to $\dfrac{\partial u}{\partial x} = 0$, indicating that u remains the same along the horizontal pipe, equal to that at the right end,

$$u = l'(t).$$

Using this in the x-momentum equation, and noting that $f_x = 0$, we obtain

$$l'' = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

integration of which with respect to x yields

$$p = -\rho l'' x + C_2(t). \tag{8}$$

Imposing the pressure condition at the front of the horizontal column, $p = p_a + \Delta p$ at $x = l(t)$, from which follows

$$C_2(t) = p_a + \Delta p + \rho l'' l.$$

Now substitution of this back into (8) shows that everywhere in the horizontal pipe the pressure is given by

$$p = p_a + \Delta p + \rho l''(l - x). \tag{9}$$

Setting $x = 0$ in (9) gives the pressure at point O,

$$p_0 = p_a + \Delta p + \rho l'' l. \tag{10}$$

[Alternatively, results (7) and (10) can be obtained by using the Cauchy-Lagrange integral (unsteady Bernoulli equation).

For the vertical pipe, noting that the gravity force has the potential $U = gz + C$, we have

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + gy = C_1(t). \tag{11}$$

Since $v = h'$, the velocity potential φ can be found from $\dfrac{\partial \varphi}{\partial y} = v = h'$, which gives

$$\varphi = h'y + \varphi_0(t).$$

The above is inserted into (11):

$$h''y + \frac{1}{2}h'^2 + \frac{p}{\rho} + gy = \hat{C}_1(t),$$

where φ_0 is absorbed into $\hat{C}_1$. Applying the above result at $y = h$ and at $y = 0$, we have

$$h''h + \frac{1}{2}h'^2 + \frac{p_a}{\rho} + gh = \frac{1}{2}h'^2 + \frac{p_0}{\rho},$$

from which follows $p_0 = p_a + \rho(h'' + g)h$, precisely equation (7).

For the horizontal pipe, the Cauchy-Lagrange integral reads

$$\frac{\partial \varphi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} = C_2(t). \tag{12}$$

Since $u = l'$, the velocity potential φ is found from $\dfrac{\partial \varphi}{\partial x} = u = l'$ as

$$\varphi = l'x + \varphi_0(t),$$

substitution of which into (12) leads to

$$l''x + \frac{1}{2}l'^2 + \frac{p}{\rho} = \hat{C}_2.$$

Applying this to $x = 0$ and $x = l$, we have

$$\frac{1}{2}l'^2 + \frac{p_0}{\rho} = l''l + \frac{1}{2}l'^2 + \frac{p_a + \Delta p}{\rho},$$

i.e. $p_0 = p_a + \Delta p + \rho l'' l$, which is precisely equation (10).]

Mass conservation requires that the volume of the fluid in the vertical and horizontal columns remains a constant, taking its initial value $h_0 A$, that is, $hA + 2Bl = h_0 A$. Noting that $B = kA$, we have

$$h(t) - h_0 = -2kl(t). \tag{13}$$

$\boxed{10,\ \mathrm{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(ii) Since equations (7) and (10) represent the same quantity, we have

$$p_a + \rho(h'' + g)h = p_a + \Delta p + \rho l'' l,$$

which leads to the equation

$$(h'' + g)h = \Delta p/\rho + l'' l. \tag{14}$$

Combing (13) and (14), we arrive at the required equation for $h(t)$,

$$\Big(1 - \frac{1}{4k^2}\Big)h''h + \frac{h_0}{4k^2}h'' + gh = \Delta p/\rho, \tag{15}$$

which is nonlinear except when $k = 1/2$. $\boxed{3,\ \mathrm{A}}$

For $k = 1/2$, equation (15) simplifies to a linear equation

$$h'' + \frac{g}{h_0}h = \Delta p/(\rho h_0). \tag{16}$$

The general solution of (16) is found as

$$h = B_1 \sin\Big(\sqrt{\frac{g}{h_0}}\,t\Big) + B_2 \cos\Big(\sqrt{\frac{g}{h_0}}\,t\Big) + \frac{\Delta p}{\rho g}. \tag{17}$$

The constants B_1 and B_2 are determined by using the initial conditions:

$$h = h_0 \text{ at } t = 0; \qquad h' = 0 \text{ at } t = 0,$$

imposing which on (17) gives

$$B_2 + \Delta p/(\rho g) = h_0, \quad B_1 = 0.$$

It follows that $B_2 = h_0 - \Delta p/(\rho g)$, and

$$h(t) = h_0\Big[1 - \frac{\Delta p}{\rho g h_0}\Big]\cos\Big(\sqrt{\frac{g}{h_0}}\,t\Big) + \frac{\Delta p}{\rho g}. \tag{18}$$

$\boxed{3,\ \mathrm{B}}$

(iii) The solution (18) indicates that when $\Delta p < \rho g h_0$, h is a monotonic decreasing function of $t \in [0, \pi\sqrt{h_0/g}\,)$, with the minimum being $-h_0 + \dfrac{2\Delta p}{\rho g}$, attained at $t = \pi\sqrt{h_0/g}$. The water in the vertical pipe can be completely discharged if the minimum is negative, i.e. if $\Delta p < \rho g h_0/2$. The time T taken is determined by $h(T) = 0$, from which we obtain

$$T = \sqrt{\frac{h_0}{g}}\cos^{-1}\Big(\frac{-\Delta p}{\rho g h_0 - \Delta p}\Big). \tag{19}$$

When $\Delta p > \rho g h_0/2$, the vertical water column would oscillate periodically with frequency $\sqrt{g/h_0}$ about the mean height $\Delta p/(\rho g)$. $\boxed{4,\ \mathrm{D}}$

4. (a) (i) The surface force exerted by the fluid on a surface element formed by ds and unit length in the spanwise direction is $-p\,\mathbf{n}\,ds$, which can, by using the (given) expression for $\mathbf{n}$, be expressed as $-p(dy, -dx) = (-pdy, pdx)$. Integrating along C, we obtain the required expressions for F_x and F_y. **seen ⇓**

The expressions obtained for F_x and F_y are combined to give

$$F_x - \mathrm{i}\,F_y = -\int_C p\,(dy + idx) = -\,\mathrm{i}\int_C p\,(dx - \mathrm{i}\,dy) = -i\int_C p\,d\bar{z}. \qquad (20)$$

3, A

(ii) In terms of $w(z)$, the complex conjugate velocity $\bar{V} = u - iv = \frac{dw}{dz}$. Let V be the modulus of $\bar{V}$. Then Bernoulli's equation can be written as **meth seen ⇓**

$$p = p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2 \equiv p_0 - \frac{1}{2}\rho V^2.$$

Substitution into (20) shows that

$$F_x - \mathrm{i}\,F_y = -\,\mathrm{i}\,p_0(\bar{z}_2 - \bar{z}_1) + \mathrm{i}\,\frac{1}{2}\rho\int_C V^2 d\bar{z}. \qquad (21)$$

Let ϑ denote the angle made by dz with the x-axis (see Figure 1). Then

$$dz = |dz|e^{\mathrm{i}\,\vartheta} \quad \text{and} \quad d\bar{z} = |dz|e^{-\,\mathrm{i}\,\vartheta}. \qquad (22)$$

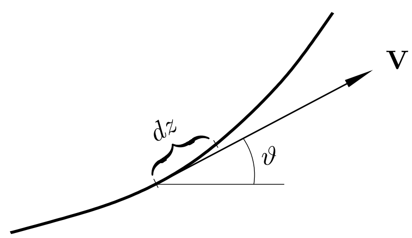

Figure 1: The tangent direction and the velocity on a streamline.

Since the velocity vector is tangent to the streamline, we can write $u + \mathrm{i}\,v = Ve^{\mathrm{i}\,\vartheta}$, and its complex conjugate $\overline{V}(z) \equiv u - iv = Ve^{-\,\mathrm{i}\,\vartheta}$, and hence

$$V = \overline{V}e^{\mathrm{i}\,\vartheta}.$$

Substitution of this and (22) into (21) yields

$$F_x - \mathrm{i}\,F_y = -\,\mathrm{i}\,p_0(\bar{z}_2 - \bar{z}_1) + \mathrm{i}\,\frac{\rho}{2}\int_C \overline{V}^2 e^{\mathrm{i}\,2\vartheta}|dz|e^{-\,\mathrm{i}\,\vartheta} = -\,\mathrm{i}\,p_0(\bar{z}_2 - \bar{z}_1) + \mathrm{i}\,\frac{\rho}{2}\int_C \overline{V}^2 dz,$$

that is,

$$F_x - \mathrm{i}\,F_y = -\,\mathrm{i}\Big(p_\infty + \frac{1}{2}\rho V_\infty^2\Big)(\bar{z}_2 - \bar{z}_1) + \mathrm{i}\,\frac{\rho}{2}\int_C \Big(\frac{dw}{dz}\Big)^2 dz. \qquad (23)$$

(This is a modified (generalised) version of the Blasius-Chaplygin formula to an open streamline.)

5, B

sim. seen ⇓

(b) (i) Substituting $z = r\,e^{\mathrm{i}\,\vartheta}$ into $w(z)$, we have

$$w = V_\infty r(\cos\vartheta + \mathrm{i}\sin\vartheta) + \frac{q}{2\pi}(\ln r + \mathrm{i}\,\vartheta).$$

The real and imaginary parts of $w(z)$ are the velocity potential and stream function, respectively:

$$\varphi(r,\vartheta) = V_\infty r\cos\vartheta + \frac{q}{2\pi}\ln r, \quad \psi(r,\vartheta) = V_\infty r\sin\vartheta + \frac{q}{2\pi}\vartheta.$$

2, A

(ii) Using the relations between the velocity (V_r, V_ϑ) and φ or ψ, we obtain

$$V_r = \frac{\partial\varphi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\vartheta} = V_\infty \cos\vartheta + \frac{q}{2\pi}r^{-1}, \quad V_\vartheta = \frac{1}{r}\frac{\partial\varphi}{\partial\vartheta} = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\vartheta.$$

A stagnation point corresponds to

$$V_r = V_\infty \cos\vartheta + \frac{q}{2\pi}r^{-1} = 0, \quad V_\vartheta = -V_\infty \sin\vartheta = 0. \tag{24}$$

From the second equation, we find that $\vartheta = 0$ or $\vartheta = \pi$, and correspondingly from the first equation of (24) follows that

$$\vartheta_s = 0, \quad r_s = -q/(2\pi V_\infty),$$

or

$$\vartheta_s = \pi, \quad r_s = q/(2\pi V_\infty). \tag{25}$$

2, A

Since r must be positive and $q > 0$, the stagnation point is given by (25), and the stream function contour going through it is determined by $\psi(r,\vartheta) = \psi(r_s,\pi) = q/2$ as

$$V_\infty r\sin\vartheta + \frac{q}{2\pi}\vartheta = \frac{q}{2}, \quad \text{i.e.} \quad y = \frac{q}{2V_\infty}(1 - \frac{\vartheta}{\pi}).$$

Clearly, y monotonically increases from $y = 0$ to $d \equiv q/(2V_\infty)$ as ϑ decreases from π to 0. Since $r\sin\vartheta \to d$ (finite), r must tend to ∞ and so must $x = r\cos\vartheta$ as $\vartheta \to 0$. The contour is symmetric about the x-axis, and is shown in figure 2. Since contours of stream function are streamlines, the

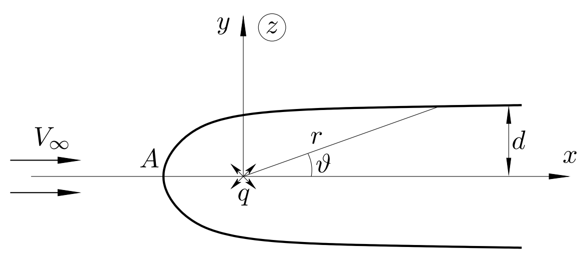

Figure 2: The contour of the body.

velocity is in the tangent direction of the contour so that the impermeability condition is satisfied. Furthermore, the domain exterior to the contour contains no singularity, and thus the complex potential represents the flow past a body with the shape given by the contour.

4, C

(iii) The body contour C connects $z_1 = \infty + \mathrm{i}\,d$ and $z_2 = \infty - \mathrm{i}\,d$ $(\bar{z}_2 - \bar{z}_1 = 2\,\mathrm{i}\,d)$. Applying the result in Part a(ii) for C and the $w(z)$, we have

unseen ⇓

$$F_x - \mathrm{i}\,F_y = \left(p_\infty + \frac{1}{2}\rho V_\infty^2\right)(2d) + \mathrm{i}\,\frac{\rho}{2}\int_C \left(V_\infty^2 + \frac{qV_\infty}{\pi z} + \frac{q^2}{4\pi^2 z^2}\right)dz. \tag{26}$$

We form a closed loop $\tilde{C}$ by adding a bypass path, a vertical line γ connecting $a - \mathrm{i}\,d$ and $a + \mathrm{i}\,d$ with $a \to \infty$. Let the integrals along C, $\tilde{C}$ and γ be denoted by I_C, $I_{\tilde{C}}$ and I_γ, respectively. I_C is decomposed as

$$I_C = I_{\tilde{C}} - I_\gamma.$$

The closed contour $\tilde{C}$ encloses a simple pole, $z = 0$, of the integrand, and use of the Residual Theorem gives

$$I_{\tilde{C}} = 2\pi \, \mathrm{i}(qV_\infty)/\pi = 2\, \mathrm{i}\, qV_\infty,$$

Along γ, $z = a + \mathrm{i}\, y$ and $dz = \mathrm{i}\, dy$ with $y \in [-d, d]$ and $a \to \infty$. Thus

$$I_\gamma = 2\, \mathrm{i}\, d\, V_\infty^2 = \mathrm{i}\, qV_\infty,$$

and $I_C = \mathrm{i}\, qV_\infty$, use of which in (26) yields

$$F_x - \mathrm{i}\, F_y = 2d\big(p_\infty + \frac{1}{2}\rho V_\infty^2\big) - \frac{1}{2}\rho\, q\, V_\infty = \frac{q}{V_\infty} p_\infty.$$

Note that $F_y = 0$ as expected on the basis of the symmetry of the flow. $\boxed{4,\ \mathrm{D}}$

5. (i) Let the distance of the centre on the z-plane to the chord be denoted as d. The radius of the circle is $d+h$. The geometry indicates that $\boxed{\text{seen} \Downarrow}$

$$d^2 + a^2 = (d+h)^2,$$

which gives

$$d = (a^2 - h^2)/(2h).$$

The radius is $d + h = (a^2 + h^2)/(2h)$.
The centre is at $z = -\mathrm{i}\,d = -\mathrm{i}(a^2 - h^2)/(2h)$. It is mapped to

$$\begin{aligned}\zeta = z + \sqrt{z^2 - a^2} &= -\mathrm{i}(a^2 - h^2)/(2h) + \sqrt{-(a^2 - h^2)^2/(2h)^2 - a^2} \\ &= -\mathrm{i}(a^2 - h^2)/(2h) + \mathrm{i}(a^2 + h^2)/(2h) = \mathrm{i}\,h.\end{aligned}$$

$\boxed{3,\ \text{M}}$

(ii) For the given complex potential $\boxed{\text{sim. seen} \Downarrow}$

$$W(\zeta) = \tilde{V}_\infty\Big[(\zeta - \mathrm{i}\,h)e^{-\mathrm{i}\,\alpha} + \frac{a^2 + h^2}{(\zeta - \mathrm{i}\,h)e^{-\mathrm{i}\,\alpha}}\Big] + \frac{\Gamma}{2\pi\,\mathrm{i}}\ln(\zeta - \mathrm{i}\,h), \qquad (27)$$

the complex conjugate velocity in the z-plane is calculated as

$$\overline{V} = \frac{dW}{d\zeta}\frac{d\zeta}{dz} = \frac{dW}{d\zeta}\Big/\frac{dz}{d\zeta} = \Big\{\tilde{V}_\infty\Big[e^{-\mathrm{i}\,\alpha} - \frac{a^2 + h^2}{(\zeta - \mathrm{i}\,h)^2 e^{-\mathrm{i}\,\alpha}}\Big] + \frac{\Gamma}{2\pi\,\mathrm{i}(\zeta - \mathrm{i}\,h)}\Big\}\frac{2}{1 - a^2/\zeta^2}.$$

Taking the limit $\zeta \to \infty$ (which corresponds to $z \to \infty$), we have $\overline{V} \to 2\tilde{V}_\infty e^{-\mathrm{i}\,\alpha}$.
In order to represent the physical flow, we take

$$\tilde{V}_\infty = V_\infty/2.$$

The trailing edge of the arc is $z = a$, which is mapped to $\zeta = a$. The figure shows that for $\zeta = a$,

$$\zeta - \mathrm{i}\,h = \sqrt{a^2 + h^2}e^{-\mathrm{i}\,\beta/2}. \qquad (28)$$

In order to satisfy the Joukowskii-Kutta condition, i.e. to keep the velocity at the trailing edge finite, we set

$$\frac{1}{2}V_\infty\Big[e^{-\mathrm{i}\,\alpha} - \frac{a^2 + h^2}{(\zeta - \mathrm{i}\,h)^2 e^{-\mathrm{i}\,\alpha}}\Big] + \frac{\Gamma}{2\pi\,\mathrm{i}(\zeta - \mathrm{i}\,h)} = 0,$$

which may, after using (28), be written as

$$\frac{1}{2}V_\infty\Big[e^{-\mathrm{i}\,\alpha} - e^{\mathrm{i}(\alpha+\beta)}\Big] + \frac{\Gamma}{2\pi\,\mathrm{i}}\frac{e^{\mathrm{i}\,\beta/2}}{\sqrt{a^2 + h^2}} = 0.$$

We find that

$$\Gamma = -2\pi V_\infty\sqrt{a^2 + h^2}\sin(\alpha + \beta/2), \qquad (29)$$

$\boxed{7,\ \text{M}}$

unseen $\Downarrow$

(iii) The flow around a descending and drifting parachute is equivalent to a uniform flow past the arc with a far-field velocity, whose modulus $V_\infty = \sqrt{V_\parallel^2 + V_\perp^2}$, and an angle of attack α satisfying

$$\tan\alpha = V_\perp / V_\parallel. \tag{30}$$

According to Joukowskii-Kutta formula, the lift produced, which is perpendicular to the direction of the flight (the far-field velocity), is $L = -\rho\Gamma V_\infty$. The vertical component is then

$$\begin{aligned} F_y &\equiv -\rho\Gamma V_\infty \cos\alpha = 2\pi\rho V_\infty^2 \sqrt{a^2+h^2}\,\sin(\alpha+\beta/2)\cos\alpha \\ &= 2\pi\rho V_\infty^2 \sqrt{a^2+h^2}\,\Big[\cos\alpha\sin(\beta/2)+\sin\alpha\cos(\beta/2)\Big]\cos\alpha \\ &= 2\pi\rho V_\infty^2 (h + a\tan\alpha)/(1+\tan^2\alpha), \end{aligned}$$

where use has been made of the geometry shown in figure 2(b) in the exam paper. Noting the relation (30), we have

$$F_y = 2\pi\rho V_\infty^2 \big[h + a(V_\perp/V_\parallel)\big] V_\parallel^2/(V_\perp^2 + V_\parallel^2) > 0.$$

4, M

unseen $\Downarrow$

(iv) With $\Gamma = 0$, $\alpha = \pi/2$ and $\zeta - \mathrm{i}\,h = \sqrt{a^2+h^2}\,\mathrm{e}^{i\vartheta}$, we have

$$\frac{dW}{d\zeta} = \frac{1}{2}V_\infty\Big[e^{-\mathrm{i}\,\alpha} - \frac{a^2+h^2}{(\zeta-\mathrm{i}\,h)^2 e^{-\mathrm{i}\,\alpha}}\Big] = -\mathrm{i}\,V_\infty e^{-\mathrm{i}\,\vartheta}\cos\vartheta, \tag{31}$$

and

$$\overline{V} = \frac{dW}{d\zeta}\frac{2\zeta^2}{\zeta^2 - a^2}. \tag{32}$$

The complex conjugate velocities at z_+ and z_- on the arc are given by

$$\overline{V}(z_+) = \frac{dW}{d\zeta}\frac{2\zeta_+^2}{\zeta_+^2 - a^2},$$

$$\overline{V}(z_-) = \frac{dW}{d\zeta}\frac{2\zeta_-^2}{\zeta_-^2 - a^2} = \frac{dW}{d\zeta}\frac{2}{1 - a^2/\zeta_-^2} = -\frac{dW}{d\zeta}\frac{2a^2}{\zeta_+^2 - a^2},$$

where we have used the relation $\zeta_- = a^2/\zeta_+$. It follows that the ratio

$$|\overline{V}(z_+)|/|\overline{V}(z_-)| = |\zeta_+|^2/a^2 \ge 1,$$

where the estimate $|\zeta_+| \ge a$ may be inferred from the geometry in figure 2(b) in the exam paper. Alternatively, we note that

$$\zeta_+ = \sqrt{a^2+h^2}\cos\vartheta + \mathrm{i}\big(h + \sqrt{a^2+h^2}\sin\vartheta\big) \quad (-\beta/2 \le \vartheta \le \pi + \beta/2),$$

and hence

$$|\zeta_+|^2 = a^2 + 2h^2 + 2h\sqrt{a^2+h^2}\sin\vartheta \ge a^2 + 2h^2 - 2h^2 \ge a^2.$$

The fluid speed over the upper side of the arc is greater and hence the pressure must be lower by Bernoulli's equation. One might expect that a net upward force in the vertical direction is produced. However, this is not the case as the drag (the force in the far-field velocity direction) is known to be identically zero.

This apparently 'contradiction' can be reconciled by considering the local behaviour of the flow near the trailing and leading edges, $z \to \pm a$. In these limits,

$$\zeta^2 - a^2 = 2z^2 - 2a^2 + 2z\sqrt{z^2 - a^2} \to \pm 2a\sqrt{z^2 - a^2},$$

and $\vartheta \to -\beta/2$, $\pi + \beta/2$, use of which in (31) and (32) shows that

$$\overline{V} \to \mp \mathrm{i}\, aV_\infty e^{\pm \mathrm{i}\,\beta/2} \cos(\beta/2)/\sqrt{z^2 - a^2}.$$

Due to the characteristic square-root singularity, a 'tip suction force', in the direction tangent to the body contour, is produced at each edge. The horizontal components, being in opposite directions, cancel out, while the vertical components both point downwards, and their sum balances the vertical force arising from the pressure difference over the upper and lower surfaces of the arc. $\boxed{6,\ \mathrm{M}}$

5 **Review of mark distribution:**

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70001 Fluid Dynamics I Markers Comments

Question 1	The overall performance was fine, but the marks on Parts (a) and (b) turned out to be somewhat disappointing since most (but not all) of the material is essentially book work. This aspect appeared to be overlooked during revision. Advice: revision should pay sufficient attention to theoretical foundation.
Question 2	This is actually a very straightforward question except Part (c) made it fairly long. It is understandable that some struggled with time. This was taken into account in marking. Quite a few fell on solving the standard ODE. Advice: competence with ODEs is always important.
Question 3	All except very few did well on this question.
Question 4	Overall performance was OK. The performance on Part (a), which generalises the Blasius-Chaplygin formula, was somewhat disappointing with the common oversight on the relation between the complex velocity and the tangent of the streamline. Only a small number of students managed to do Part (c), which is difficult part of the question; the majority did not realize the contour is not closed.
Question 5	The marks are low on this question. Part (iii) was actually straightforward, but unfortunately most did not recognise this. Part (iv) is hard, and also makes the question (and indeed this entire paper) long. This will be taken into account in processing the marks.