MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Extra Material 1: Separation of variables

In this document, we provide a rigorous justification of the separation of variables technique that was introduced in Chapter 1.

Proposition 1 (Separation of variables)**.** *Consider the differential equation*

$$\dot{x} = g(t)h(x)\,,$$

where $I, J \subset \mathbb{R}$ are two intervals and $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are two continuous functions. Then the following statements hold.

(i) Every zero x_0 of the function h leads to a constant solution $\lambda(t) \equiv x_0$ on I.

(ii) Let $h(x) \neq 0$ for all $x \in J_0$, where $J_0 \subset J$ is a subinterval of J. Let $H : J_0 \to \mathbb{R}$ be an antiderivative of the function $x \mapsto \frac{1}{h(x)}$, and $H^{-1} : H(J_0) \to J_0$ be the inverse of H. Furthermore, let $G : I \to \mathbb{R}$ be an antiderivative of g. Then for all $\alpha \in \mathbb{R}$, the function $\lambda_\alpha(t) := H^{-1}(G(t)+\alpha)$ is a solution to the above differential equation, on every interval $I_0 \subset I$, on which it is defined.

Proof. (i) Suppose that $h(x_0) = 0$ and define $\lambda(t) = x_0$ for all $t \in I$. Then $\dot{\lambda}(t) = 0$ and $g(t)h(\lambda(t)) = g(t)h(x_0) = 0$ for all $t \in I$, so λ is a solution to this differential equation.

(ii) Suppose that for some $\alpha \in \mathbb{R}$ and $I_0 \subset I$, we have $G(t) + \alpha \in H(J_0)$ for all $t \in I_0$, so that the function $\lambda_\alpha(t) := H^{-1}(G(t) + \alpha)$ is well defined. We show now that λ_α is a solution. Firstly, we note that the inverse function theorem implies that H^{-1} is differentiable, and due to $H(H^{-1}(x)) = x$ for all $x \in H(J_0)$, we get with the chain rule that $H'(H^{-1}(x))(H^{-1})'(x) = 1$, which implies that

$$(H^{-1})'(x) = \frac{1}{H'(H^{-1}(x))} \quad \text{for all } \ x \in H(J_0)\,,$$

This implies using the chain rule that for all $t \in I_0$, we have

$$\dot{\lambda}_\alpha(t) = (H^{-1})'(G(t)+\alpha)\dot{G}(t) = \frac{g(t)}{H'(H^{-1}(G(t)+\alpha))} = g(t)h(H^{-1}(G(t)+\alpha) = g(t)(h(\lambda_\alpha(t)))\,.$$

This shows that λ_α solves the differential equation $\dot{x} = g(t)h(x)$. □

Remark. Note that the differentiability of H^{-1} follows in this situation from the inverse function theorem.