Imperial College London

M4P59/M5P59

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Advanced Analysis

Date: **Wednesday 25th May 2016**

Time: **14.00 – 16.30**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

Notation

$\mathbb{N}$ set of natural numbers;
$\mathbb{Z}^+ \equiv \mathbb{N} \cup \{0\}$
$\mathbb{R}$ set of real numbers;
$d\lambda \equiv \lambda(dx)$ the Lebesgue measure in $\mathbb{R}^n$;
$D = (D_j)_{j=1,\dots,n}$ - gradient in $\mathbb{R}^n$;
$\mathfrak{F}$ Fourier transform;
$\mathcal{S}$ Schwartz space.

Q1.

(1.i) Give the definition of a metric linear space, a normed space and a Banach space.

(1.ii) Explain giving reasons, which of the following linear spaces satisfy conditions of :

- a metric linear space;
- a normed space .

In both cases below, for a measurable function $f:\mathbb{R}\to\mathbb{R}$ we set

$$[f]\equiv\{g:\mathbb{R}\to\mathbb{R}|g=f \quad a.e.\}.$$

(1.ii.a) $\{[f]: \quad \exists\varepsilon\in(0,\infty) \qquad \Phi(\varepsilon f)<\infty\}$,
where

$$\Phi(\varepsilon f)\equiv\int(e^{\varepsilon^2 f^2}-1)d\nu$$

and

$$d\nu=e^{-\frac{x^4}{2}}d\lambda/\int e^{-\frac{x^4}{2}}d\lambda,$$

with $\|[f]\|\equiv\inf\{\xi>0:\Phi(f/\xi)\leq 1\}$ and $\rho([f],[g])\equiv\|[f]-[g]\|$;

(1.ii.b) Let $\{[f]: \quad \int|f|^{1/4}d\lambda<\infty\}$

with

$\|[f]\|\equiv\int|f|^{1/4}d\lambda$ and $\rho([f],[g])\equiv\|[f]-[g]\|$.

(1.iii) Define the space of Schwartz functions and the space of tempered distributions on $\mathbb{R}^m$.
Which of the following sequences is convergent in the space of

- Schwartz functions;
- tempered distributions ?

(1.iii.a) $f_n\equiv\sum_{k=1}^{7}(in)^k\sin(nx)$, $n\in\mathbb{N}$

(1.iii.b) $g_n\equiv(\sqrt{n}+(x/n)^2)\exp(-nx^2)$

Q2.

(2.i) Let $H_n(x)$, $n \in \mathbb{Z}^+$, denote normalised Hermite polynomials and let $\phi_n \equiv H_n(x)e^{-\frac{x^2}{2}}$. You may assume that the set $\{\phi_n\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis for $\mathbb{L}_2(\mathbb{R},\lambda)$ and that each ϕ_n is an eigenvector of the Fourier transform $\mathfrak{F}$ on $\mathbb{L}_2(\mathbb{R},\lambda)$. You may also assume that the Fourier transform is continuous in $\mathbb{L}_2(\mathbb{R},\lambda)$.
Show that

$$\|f\|_{\mathbb{L}_2(\mathbb{R},\lambda)} = \|\mathfrak{F}f\|_{\mathbb{L}_2(\mathbb{R},\lambda)}$$

where $\mathfrak{F}f$ denotes the Fourier transform of $f \in \mathbb{L}_2(\mathbb{R},\lambda)$;

(2.ii) Prove that for all $f \in \mathbb{L}_2(\mathbb{R},\lambda)$,

$$\mathfrak{F}^{\circ 2}f(x) = f(-x) \; .$$

(2.iii) Find a representation of functions in $\mathbb{L}_2(\mathbb{R},\lambda)$ satisfying

$$\mathfrak{F}f = f$$

in terms of the orthonormal basis $H_n(x)e^{-\frac{x^2}{2}}$, $n \in \mathbb{Z}^+$.

Q3.

(3.i) Prove, for any smooth compactly supported function f on $\mathbb{R}^3$ the following Sobolev inequality in $\mathbb{R}^3$:

$$\|f\|_{\frac{3}{2}} \leq C \sum_{j=1}^{3} \int |D_j f| d\lambda$$

with some constant $C \in (0,\infty)$ independent of f.

(3.ii) Prove for any smooth compactly supported function f on $\mathbb{R}^2$ the following inequality

$$\|f\|_{4/3} \leq C' (\int |Df| d\lambda + \|f\|_1) \ ,$$

with some constant $C' \in (0,\infty)$ independent of f.

Hint : You may assume Sobolev inequality for $n=4$.

Q4.

Let $\mathcal{S}(\mathbb{R})$ denote the space of Schwartz functions on the real line.
For $t \in (0,\infty)$ and $\alpha \in \left(\frac{1}{2},\infty\right)$, let P_t^α be a map defined for $f \in \mathcal{S}(\mathbb{R})$ as follows :

$$P_t^\alpha f := \mathfrak{F}^{-1}(e^{-t|k|^{2\alpha}}\mathfrak{F}f).$$

(4.i) Prove that the Fourier transform and its inverse are continuous on the space of Schwartz functions.
Using this or otherwise, prove that for $\alpha \in \mathbb{N}$

$$\forall t > 0, \qquad P_t^\alpha : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \quad \text{is continuous.;}$$

(4.ii) Prove or disprove each of the following two bounds for any Schwartz function f

$$\|P_t^\alpha f\|_2 \leq \|f\|_2$$

and

$$\|P_t^\alpha f\|_\infty \leq C\|f\|_1$$

with some constant $C \in (0,\infty)$ independent of f.

Q5.

(5.i) Let $p \in [1, \infty)$ and $k \in \mathbb{Z}^+$. Give the definition of of the space $W_0^{k,p}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain with a smooth boundary.

(5.ii)
Prove the following bound for $1 < p < n$

$$\| f \|_{\frac{np}{n-p}} \leq C' \quad \sum_{j=1}^{n} \| D_j f \|_p$$

with some constant $C' \in (0, \infty)$ independent of any smooth compactly supported function f on $\mathbb{R}^n$.
Hence conclude that for $f \in W_0^{k,p}(\Omega)$, $p \in [1, n)$, we have $f \in W_0^{k-1,p_1}(\Omega)$ with $p_1 \equiv \frac{np}{n-p}$.

(5.iii)
Prove that $W_0^{k,p}(\Omega)$ can be continuously embedded into $\mathbb{L}_{\frac{pn}{n-pk}}(\Omega, \lambda)$ for $1 \leq p < n/k$.