IMPERIAL

MATH70008

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamical Systems

Date: Tuesday, May 13, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3 - 5x^2 + \frac{13}{4}x, \quad \text{for all } x \in \mathbb{R}.$$

(a) Calculate the fixed points of f. (4 marks)

(b) Determine whether the fixed points of the map f are attractive or repulsive (justify your answer rigorously). (8 marks)

(c) Provide an upper bound for the number of eventually fixed points of f with pre-period two and justify your answer. (4 marks)

(d) Is there any dense orbit for the map f? Justify your answer. (4 marks)

(Total: 20 marks)

2. Let (X, d) be a metric space, suppose X has finite cardinality, and consider a continuous map $f : X \to X$.

(a) Give the definition of omega-limit set $\omega(x)$ for $x \in X$. (2 marks)

(b) Show that for any $x \in X$, the omega-limit set $\omega(x)$ is given by a periodic orbit. (8 marks)

(c) Assume that f is invertible. For a point $x \in X$ denote by $\alpha(x)$ the omega-limit set of x with respect to f^{-1}. Show that $\alpha(x) = \omega(x)$ for all $x \in X$. (6 marks)

(d) Define a Dirac measure δ_P concentrated on a periodic orbit P of f with prime period bigger than one (in the course, we have seen that such measure is invariant for f and you do not need to show this). Answer the following questions providing a rigorous justification:

(i) Is δ_P ergodic? (2 marks)

(ii) Is δ_P mixing? (2 marks)

(Total: 20 marks)

3. (a) Let (X, d) be a compact metric space and $f : X \to X$ be continuous.

(i) What properties of the map f are sufficient to show that f is chaotic? (clarify which property in the definition of chaotic mapping is redundant under the given assumptions) (2 marks)

(ii) Give the definition of topological entropy $h_{top}(f)$ for f. (2 marks)

(iii) Show that topological entropy is invariant under topological conjugation. (4 marks)

(b) Consider the Bernoulli shift σ on the the metric space Σ_k^+ of infinite sequences with k symbols endowed with the metric

$$d_\Sigma(a, b) = \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i}; \quad \text{for all } a, b \in \Sigma_k^+.$$

(i) Prove that the Bernoulli shift is chaotic. (4 marks)

(ii) Prove that $h_{top}(\sigma) = \ln(k)$. (8 marks)

(Total: 20 marks)

4. Consider the logistic map $g : [0, 1] \to [0, 1]$, defined by

$$f(x) = 4x(1 - x), \quad \text{for all } x \in [0, 1].$$

(a) Which theorem does guarantee that f has an invariant probability measure? Provide its statement. (2 marks)

(b) The map f is topologically conjugated to the tent map $T : [0, 1] \to [0, 1]$, defined by $T(x) = 1 - |2x - 1|$ for all $x \in [0, 1]$, via $h : [0, 1] \to [0, 1]$, $x \mapsto \sin^2(\pi x/2)$. Use this fact to provide a rigorous justification for the following statements:

(i) The measure $\mu : \mathcal{B} \to [0, 1]$ defined by

$$\mu(A) = \int_A \frac{1}{\pi\sqrt{x(1-x)}}\, dx$$

is invariant with respect to g. (5 marks)

(ii) The measure μ is mixing. (2 marks)

(c) Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 - f^i(x)},$$

exists for almost every $x \in [0, 1]$ and calculate the limit. (5 marks)

(d) Use the fact that μ is mixing to show that the map f is topologically transitive. (6 marks)

(Total: 20 marks)

5. Consider a probability space $(X, \mathcal{F}, \mu)$ and a measurable mapping $f : X \to X$ and assume that μ is invariant with respect to f. Let $Y := X \times X$ be equipped with the σ-algebra $\mathcal{F} \otimes \mathcal{F}$ (generated by the semi-ring of sets of the form $A \times B$, with $A, B \in \mathcal{F}$. You have seen in the course that the measure $\nu := \mu \otimes \mu$, defined by $\nu(A \times B) = \mu(A)\mu(B)$ for $A, B \in \mathcal{F}$, and extended by Carathéodory's Theorem, is invariant with respect to the mapping $f \times f : Y \to Y$, $(f \times f)(x_1, x_2) := (f(x_1), f(x_2))$ (you do not need to prove it here).

(a) Show that if $\nu = \mu \otimes \mu$ is ergodic with respect to $f \times f$, then μ is weakly mixing with respect to f. (6 marks)

(b) Assume that $X = [0, 1]$ and the Lebesgue measure μ is weakly mixing for f. Consider $x \in X$, $0 < \varepsilon < 1/2$.

(i) Can you prove that for Lebesgue-almost every $y_1, y_2 \in B_\varepsilon(x)$ there is $n \in \mathbb{N}$ with $n \geq 1$ such that $|f^n(y_1) - f^n(y_2)| < 2\varepsilon$?

(4 marks)

(ii) Show that for almost every $(y_1, y_2) \in [0, 1] \times [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |f^i(y_1) - f^i(y_2)| = \frac{1}{3}$$

(8 marks)

(iii) Are there any two invariant measures ν_1, ν_2 on the σ-algebra $\mathcal{F} \otimes \mathcal{F}$, with $\nu_1 \neq \nu_2$ such that $\nu = \gamma\nu_1 + (1 - \gamma)\nu_2$ for some $\gamma \in (0, 1)$? Justify your answer. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60008/70008

Dynamical Systems (Solutions)

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1. (a) The fixed points are calculated solving $f(x) = x$ for $x \in \mathbb{R}$, which yields meth seen $\Downarrow$

1, A

$$0 = x^3 - 5x^2 + \frac{9}{4}x = x(x^2 - 5x + \frac{9}{4})$$

which has roots in $0, 1/2, 9/2$. 3, A

(b) In order to determine wether the fixed points are attractive or repelling, calculate the derivative of f, meth seen $\Downarrow$

$$f'(x) = 3x^2 - 10x + \frac{13}{4}$$

and note that

$$f'(0) = 13/4 > 1, \quad f'(9/2) = 76/4 > 1, \quad f'(1/2) = -1$$

Due to Prop 2.12 the fixed points in 0 and $9/2$ are hyperbolic and repulsive. 2, A

On the other hand $x = 1/2$ is not hyperbolic. In order to apply Prop. 2.15, we consider the Schwarzian derivative of f 2, B

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Hence, noting that $f''(1/2) = -7$ and $f'''(1/2) = 6$ we obtain $Sf(1/2) = -135/2 < 0$. Hence, $x = 1/2$ attractive. 4, B

(c) The eventually fixed point of pre-period two can be found solving the equation $f^{1+2}(x) = f^2(x)$ which for the given map yields a polynomial equation of degree 27. Hence, the fundamental theorem of algebra guarantees the existence of at most twenty-seven real solutions, which correspond to at most twenty-seven eventually fixed points of pre-period two unseen $\Downarrow$

1, A

3, B

unseen $\Downarrow$

(d) Sketching the cobweb diagram of the dynamical systems induced by f it is possible to note that any initial condition smaller than zero decreases monotonically to $-\infty$, (alternatively, every initial condition greater than $9/2$ increases monotonically to $+\infty$). 2, B

Consequently, there cannot be any dense orbit for this map. 2, C

2. (a) Let $f : X \to X$ be a map on a metric space X, and let $x \in X$. A point $x^* \in X$ is called omega limit point of x if there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ with seen $\Downarrow$

$$\lim_{k\to\infty} n_k = \infty, \quad \text{and} \quad x^* = \lim_{k\to\infty} f^{n_k}(x).$$

The set of all omega limit points of x is denoted $\omega(x)$ and called the omega limit set of x. 2, A

unseen $\Downarrow$

(b) Firstly, show that every point $x \in X$ is eventually periodic. Since X is finite, the orbit of x

$$O_f^+(x) = \{f^i(x) \mid i \in \mathbb{N}_0\}$$

is finite. Since $\mathbb{N}_0$ is infinite, there exist $n, m \in \mathbb{N}_0$ with $f^n(x) = f^m(x)$. Without any loss of generality, let's assume that $n < m$. Then,

$$f^{m-n}(f^n(x)) = f^m(x) = f^n(x)$$

which implies that the point $f^n(x)$ is periodic of period $m - n$, and thus x is eventually periodic. Note that 4, D

$$O_f^+(x) = \{f^i(x) \mid i = 0, \dots, n\} \cup \{f^i(f^n(x)) \mid i = 0, \dots, m - n - 1\}$$

and

$$O_f^+(f^n(x)) = \{f^i(x) \mid i = 0, \dots, m - 1\} = O_f^+(f^k(x)) \quad \text{for all } k \geq n.$$

Proposition 3.14 implies that

$$\omega(x) = \bigcap_{k\in\mathbb{N}_0} \overline{O_f^+(f^k(x))} = \bigcap_{k\geq n} \overline{O_f^+(f^k(x))} = O_f^+(f^n(x))$$

which is a periodic orbit and finishes the proof. 4, D

(c) If f is invertible, then from (b) applied to $g = f^{-1}$ it must be that $\alpha(x)$ is a periodic orbit for every $x \in X$. Moreover, reasoning as for (b), there exists $n \in \mathbb{N}_0$ such that $g^n(x) \in \alpha(x)$ and it is a periodic point (both for f and g). Consequently, $\omega(g^n(x)) = \alpha(x)$. However, since $x = f^n(g^n(x))$, then it also holds that $\omega(x) = \omega(g^n(x)) = \alpha(x)$, which concludes the proof. unseen $\Downarrow$ 2, B 4, C

seen $\Downarrow$

(d) (i) Let $P = \{a_1, ..., a_n\}$be a periodic orbit of a measurable function $f : X \to X$. Let $\delta_P := \frac{1}{n}\sum_{i=1}^{n} \delta_{a_i}$ be the Dirac measure on P. We show that δ_P is an ergodic invariant measure. We know already that δ_P is invariant. Any measurable set A such that $f^{-1}(A) = A$ must contain either all points of P or none of them. Hence, A has either measure one or zero with respect to δ_P. 2, A

(ii) Dirac measures on periodic orbit of prime period greater than one are not weakly mixing and therefore also not mixing (Proposition 5.55). 2, A

3. (a) (i) A continuous mapping $f : X \to X$ on a metric space X that is not equal to a periodic orbit is chaotic if and only if f is topologically transitive and the periodic points are dense in X. Therefore, sensitive dependence on initial conditions is redundant under the question's assumptions. seen ⇓ 2, A

(ii) Let (X, d) be a compact metric space and $f : X \to X$ be continuous. Let $n \in \mathbb{N}$ and $\varepsilon > 0$, and consider the metric $d_n : X \times X \to \mathbb{R}_0^+$ defined by seen ⇓

$$d_n(x, y) = \max_{i \in \{0,\ldots,n-1\}} d\big(f^i(x), f^i(y)\big), \quad \text{for all } x, y \in X.$$

Then a set $A \subset X$ is called (n, ε)-spanning if for all $x \in X$, there exists a $y \in A$ with $d_n(x, y) < \varepsilon$. Denote by $Span(n, \varepsilon)$ the smallest cardinality of an (n, ε)-spanning set (which depends on f). Then the topological entropy of f is defined by 2, A

$$h_{top}(\sigma) = \lim_{\varepsilon \to 0^+} h(\sigma, \varepsilon) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \ln Span(n, \varepsilon)$$

seen ⇓

(iii) Let d_X and d_Y be the metrics on the metric spaces X and Y, respectively. Since X is compact, the conjugacy h is uniformly continuous. That means that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(h(x), h(y)) < \varepsilon.$$

It follows that

$$d_X(f(x), f(y)) < \delta \implies d_Y(g(h(x)), g(h(y))) = d_Y(h(f(x)), h(f(y))) < \varepsilon.$$

Consequently,

$$h(B_\delta^{n,X}(x))) \subset B_\varepsilon^{n,Y}(h(x)) \quad \text{for all} x \in X,$$

where $B_r^{n,W}(w)$ is the ball of radius r cantered in $w \in W$ with respect to the metric d_n^W induced by the metric on W. Since h is surjective, we can cocnlude that $Span_f(n, \delta) \geq Span_g(n, \varepsilon)$, which means that 2, A

$$\limsup_{n \to +\infty} \frac{1}{n} \ln Span_f(n, \delta) \geq \limsup_{n \to +\infty} \frac{1}{n} \ln Span_g(n, \varepsilon),$$

which gives us one direction of the desired invariance when $\varepsilon \to 0$ (which also implies $\delta \to 0$). Since h is topological a conjugacy the result follows applying the same reasoning to h^{-1}. 2, A

(b) (i) Consider the Bernoulli shift $\sigma : \Sigma_k^+ \times \Sigma_k^+$. The Bernoulli shift is topologically transitive since it has a dense orbit and Σ_k^+ is perfect (Proposition 3.27). Moreover, the periodic points of σ are dense in Σ_k^+ (Proposition 3.19). Finally, σ is continuous (Proposition 3.20). Therefore, since Σ_k^+ is not a periodic orbit of σ, from 3(a)(i) we have that σ is chaotic. (alternatively note that σ has sensitive dependence on initial conditions (Example 4.3). seen ⇓ 4, A meth seen $\Downarrow$

(ii) Fix $n \in \mathbb{N}$ and $\varepsilon > 0$, and choose $m \in \mathbb{N}$ such that $(k-1)/2^m < \varepsilon$. Now, let us consider the set $A_{n,\varepsilon}$ constructed as follows: for each finite sequence $x_0, \ldots, x_{n+m}$ with $x_j \in \{0, \ldots, k-1\}$ pick exactly one sequence $a \in I_{x_0,\ldots,x_{n+m}}$. Note that the cardinality of $A_{n,\varepsilon}$ is k^{n+m}. Then, for every $\xi \in \Sigma_k^+$, there is $a \in A_{n,\varepsilon}$ such that $\xi_i = a_i$ for all $i \in \{0, \ldots, n+m\}$. Therefore, for every $j \in \{0, \ldots, n-1\}$ we have

$$d_\Sigma(\sigma^j(\xi), \sigma^j(a)) \leq \frac{k-1}{2^m} < \varepsilon.$$

Hence, $A_{n,\varepsilon}$ is (n,ε)-spanning. Thus,

$$h(\sigma, \varepsilon) = \limsup_{n\to+\infty} \frac{1}{n} \ln Span(n,\varepsilon) \leq \ln(k),$$

which implies

$$h_{top}(\sigma) = \lim_{\varepsilon\to 0^+} h(\sigma,\varepsilon) \leq \ln(k).$$

We now look at (n,ε)-separated sets. Fix $n \in \mathbb{N}$ and $0 < \varepsilon < 1/2^n$. Let $m \in \mathbb{N}$, $m \geq 1$ such that 4, D

$$\frac{1}{2^{n+m-1}} \geq \varepsilon, \quad \text{and} \quad \frac{k-1}{2^{n+m}} < \varepsilon.$$

Now, let us consider the set $A_{n,\varepsilon}$ constructed as follows: for each finite sequence $x_0, \ldots, x_{n+m-2}$ with $x_j \in \{0, \ldots, k-1\}$ pick exactly one sequence $a \in I_{x_0,\ldots,x_{n+m-2}}$. Note that the cardinality of $A_{n,\varepsilon}$ is k^{n+m-1}. Note also, that for every $a, b \in A_{n,\varepsilon}$, there is $i \in \{0, \ldots, n+m-2\}$ such that $a_i \neq b_i$, and thus,

$$d_n(a,b) \geq \frac{1}{2^{n+m-1}} \geq \varepsilon$$

Hence, $A_{n,\varepsilon}$ is (n,ε)-separated. Thus,

$$h(\sigma, \varepsilon) = \limsup_{n\to+\infty} \frac{1}{n} \ln Sep(n,\varepsilon) \geq \ln(k),$$

which implies

$$h_{top}(\sigma) = \lim_{\varepsilon\to 0^+} h(\sigma,\varepsilon) \leq \ln(k).$$

Therefore, we can conclude that $h_{top}(\sigma) = \ln(k)$. 4, D

4. (a) (Theorem of Krylov–Bogolubov). Let X be a compact metric space and $f : X \to X$ be a continuous map. Then there exists a probability measure $\mu : \mathcal{B}(X) \to \mathbb{R}_0^+$ that is invariant with respect to f.

seen $\Downarrow$

2, A

sim. seen $\Downarrow$

(b) (i) We recall that the Lebesgue measure λ is invariant for the tent map (see Proposition 5.26)

1, A

Hence we can apply Proposition 5.43 to obtain an invariant measure for f, i.e. the push-forward $h_*\lambda$ of the Lebesgue measure via the measurable conjugacy h. For any measurable set $A \subset [0,1]$ with respect to the Borel σ-algebra, we have that

2, A

$$h_*\lambda(A) = \lambda(h^{-1}(A)) = \int_{h^{-1}(A)} d\lambda = \int_A |(h^{-1})'(y)|\, dy = \int_A \frac{1}{\pi\sqrt{y(1-y)}}\, dy.$$

Therefore, $\mu = h_*\lambda$ and it is invariant.

2, C

seen $\Downarrow$

(ii) The following result seen in the course applies. Let $(X, \mathcal{F}, \mu)$ be a probability space and $f : X \to X$ be measurable such that μ is mixing with respect to f. Consider a space Y with σ-algebra $\mathcal{G}$ and a measurable mapping $g : Y \to Y$ such that g is measurable conjugate to f via a conjugacy $h : X \to Y$. Then the push-forward $\nu := h_*\mu$ is mixing with respect to g.

2, A

seen/sim.seen $\Downarrow$

(c) Since μ is mixing for f, then it is in particular ergodic.

1, B

Hence we can apply Birkhoff's ergodic theorem to the observable ω, that is for μ-a.e. $x \in [0,1]$ one has

2, B

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega(g^i(x)) = \int_{[0,1]} \omega\, d\mu = \int_0^1 \frac{\sqrt{1-x}}{\pi\sqrt{x(1-x)}}\, dx = \int_0^1 \frac{\sqrt{1}}{\pi\sqrt{x}}\, dx = \frac{2}{\pi}$$

2, B

unseen $\Downarrow$

(d) Let $U, V \in [0,1]$ open sets. Hence, $\lambda(U), \lambda(V) > 0$, which implies that $\mu(U), \mu(V) > 0$. Moreover, since μ is mixing from 4(b)(ii), we have that

$$\lim_{n\to\infty} \mu(U \cap f^{-n}(V)) = \mu(U)\mu(V) > 0.$$

Therefore, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\mu(U \cap f^{-n}(V)) > \varepsilon$ for all $n \geq n_0$, which in turn implies that $U \cap f^{-n}(V) \neq \varnothing$ for all $n \geq n_0$.

4, C

Therefore f is topologically mixing, which implies topological transitivity.

2, B

5. (a) The following result seen in the course applies (See Theorem 5.60 (v)⇒(i)). Let $(X, \mathcal{B}, \mu)$ be a probability space and $f : X \to X$. If $\nu = \mu \otimes \mu$ is ergodic with respect to $f \times f$, then μ is weakly mixing with respect to f. seen ⇓

Proof. For any $A, B \in \mathcal{B}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\big(\mu(f^{-i}(A)\cap B) - \mu(A)\mu(B)\big)^2$$
$$= \frac{1}{n}\sum_{i=0}^{n-1}\big(\mu(f^{-i}(A)\cap B)\big)^2 - 2\mu(A)\mu(B)\frac{1}{n}\sum_{i=0}^{n-1}\big(\mu(f^{-i}(A)\cap B)\big) + \big(\mu(A)\mu(B)\big)^2$$
$$= \frac{1}{n}\sum_{i=0}^{n-1}(\mu\otimes\mu)\big((f\times f)^{-i}(A\times A)\cap B\times B\big)$$
$$- 2\mu(A)\mu(B)\frac{1}{n}\sum_{i=0}^{n-1}\big(\mu(f^{-i}(A)\cap B)\big) + \big(\mu(A)\mu(B)\big)^2$$

Since $\mu\otimes\mu$ is ergodic with respect to $f \times f$, and thus μ is ergodic with respect to f, this converges to

$$(\mu\otimes\mu)(A\times A)(\mu\otimes\mu)(B\times B) - 2\big(\mu(A)\mu(B)\big)^2 + \big(\mu(A)\mu(B)\big)^2 = 0.$$

Then, the following result applies and concludes the proof: let $a_{n n\in\mathbb{N}}$ be a bounded sequence of non-negative numbers. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} a_i = 0 \quad \text{if and only if} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} a_i^2 = 0.$$

6, M

unseen ⇓

(b) (i) Since $\nu = \mu \otimes \mu$ is invariant with respect to $f \times f$, then Poincaré's recurrence Theorem applies to the open set $B_\varepsilon(x) \times B_\varepsilon(x) \subset Y$ which clearly has positive measure for ν. Therefore, for ν-a.e. $(y_1, y_2) \in B_\varepsilon(x) \times B_\varepsilon(x)$ we have that there are infinitely many $n \in \mathbb{N}$ such that $(f^n(y_1), f^n(y_2)) \in B_\varepsilon(x) \times B_\varepsilon(x)$ and thus, $|f^n(y_1) - f^n(y_2)| < 2\varepsilon$.

4, M

unseen ⇓

(ii) Due to Theorem 5.60, we know that if μ is weakly mixing with respect to f, then $\nu = \mu \otimes \mu$ is ergodic with respect to $f \times f$. Hence, we can apply Birkhoff's ergodic theorem to the observable $m : [0,1]\times[0,1] \to [0,1]$ defined by $m(p,q) = |p-q|$ for all $(p,q) \in [0,1]\times[0,1]$, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|f^i(y_1) - f^i(y_2)| = \int_{[0,1]^2} m\,d\nu = \int_0^1\int_0^1 |x_1 - x_2|\,dx_2\,dx_1$$
$$= \int_0^1 \Big[\int_0^{x_1}(x_1 - x_2)\,dx_2 + \int_{x_1}^1 (x_2 - x_1)\,dx_2\Big]\,dx_1$$
$$= \int_0^1 \Big[\Big(x_1^2 - \frac{x_1^2}{2}\Big) + \Big(\frac{1}{2} - x_1 - \frac{x_1^2}{2} + x_1^2\Big)\Big]\,dx_1$$
$$= \int_0^1 \Big[x_1^2 - x_1 + \frac{1}{2}\Big]\,dx_1 = \frac{1}{3}$$

8, M

(iii) No, this is not possible because ν ergodic if and only if ν is an extremal point of the set of invariant measures for f. seen ⇓ 2, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70008 Dynamical Systems Markers Comments

Question 1 This question was correctly answered by the majority of students. However, some of you did not recognise that the Schwarzian derivative had to be studied to inspect the stability of the non' hyperbolic equilibrium at 1/2. The majority also showed a solid understanding of preimages of a point and provided a sound upper bound for the number of eventually fixed points with pre-period two. Others used the fundamental theorem of Algebra. Both solutions are correct. In order to show that a dense orbit does not exist, most of you correctly analysed the trajectories of points close to the unstable equilibria.

Question 2 Most students responded correctly to this question. The majority showed troubles organising a logical answer for b) and c), and while providing good intuitive answers, they lacked the rigorous analytical formalism to prove their claim.

Question 3 Some students had difficulties recalling the results we studied on chaotic maps losing easy points in a-i). Some of you had also troubles showing the invariance of topological entropy under topological conjugation despite having seen the case of semi-conjugations as an exercise in the lectures. Most of you only provided an upper bound for the topological entropy of the Bernoulli shift forgetting to use (n,ep)-separated sites for the lower bound.

Question 4 Most students have correctly cited Krylov-Bogolyubov's theorem. In regards to b), many students have shown a basic understanding of the concept of push-forward measure but have committed mistakes in applying to definition to obtain the measure mu. Moreover, most of you have correctly used Birkhoff Ergodic theorem to solve c). However, some of you have made mistakes in their calculations or (more worryingly) have forgotten the mu-a.a. scope of the theorem.

Question 5 The solution of a) required knowing the proof Theorem 5.60. The step (v)⇒(i) was the answer to the question and most of you respond correctly.
(i) aimed at checking that you are familiar with Poincare Recurrence Theorem and many of you were not able to respond losing important points in a relatively easy question.
For question b)(ii) you had to combine theorem 5.60, BET and the integration in several variables. Most of you have correctly done so but some have made mistakes along the way.
Most students have also correctly answered the last question which required knowing that ergodic measure are extremal.