Mathematics Year 1, Calculus and Applications I

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Problem Sheet 4

1. (a) Let $S_2 := \sum_{i=1}^{n} i^2$. Compute

$$\sum_{i=1}^{n}[(i+1)^3 - i^3] = \sum_{i=1}^{n}[i^3 + 3i^2 + 3i + 1 - i^3] = 3\sum_{1}^{n} i^2 + 3\sum_{1}^{n} i + \sum_{1}^{n} 1$$
$$= 3S_2 + \frac{3}{2}n(n+1) + n.$$

The first sum in the equation above is a telescoping series and is equal to $2^3 - 1^3 + 3^3 - 2^3 + \ldots + (n+1)^3 - n^3 = (n+1)^3 - 1$. Hence, after a little algebra,

$$S_2 = \frac{1}{6}n(n+1)(2n+1).$$

(b) Partition $[0,1]$ into $x_i = ih$ where $h = 1/n$ (note, $x_0 = 0$ and $x_n = 1$). The upper Riemann sum is

$$\sum_{1}^{n} h(ih)^2 = h^3 \sum_{1}^{n} i^3 = \frac{1}{n^3}.\frac{1}{6}n(n+1)(2n+1) \to \frac{1}{3} \quad \text{as} \quad n \to \infty.$$

2. Need to show $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} e^{i/n} = e - 1$. Let $r = e^{1/n}$. Then

$$\sum_{i=1}^{n} \frac{1}{n} e^{i/n} = \frac{1}{n}\sum_{i=1}^{n} r^i = \frac{1}{n} r \frac{1-r^n}{1-r} = \frac{(1/n)e^{1/n}}{(e^{1/n}-1)}(e-1) \to (e-1) \quad \text{as} \quad n \to \infty.$$

[E.g. use L'Hôpital's rule on $\lim_{x\to 0} \frac{x}{e^x - 1}$.]

3. Take any partition of $[0,1]$ and calculate the lower Riemann sum L, and upper Riemann sum U. Since any interval of real numbers, however small, contains an infinite number of rationals and irrationals, it follows that

$$L = 0, \qquad U = 1.$$

This is the case for *any* partition, hence $L \neq U$ for any partition and so the function is not Riemann integrable.

4. First part straightforward. Using this result we see that $\frac{d}{dx}\log(\sec x + \tan x) = \sec x$, hence we have the antiderivative of $\sec x$ as required.

The integral result can be applied on intervals that satisfy $\cos x \neq 0$ and $\sec x + \tan x > 0$, i.e. $\frac{1+\sin x}{\cos x} > 0$ which is only possible if $\cos x > 0$. Hence the appropriate interval is $(-\pi/2, \pi/2)$ since $\cos(\pm\pi/2) = 0$.

5. (i) Substitute $x = \tan\theta$, i.e. $dx = \sec^2\theta d\theta$ and $x^2 + 1 = tan^2\theta + 1 = \sec^2\theta$:

$$\int \frac{1}{(x^2+1)^3}dx = \int \frac{\sec^2\theta}{\sec^6\theta}d\theta = \int \cos^4\theta d\theta = \int \frac{(1+\cos 2\theta)^2}{4}d\theta$$
$$= \int \left(\frac{1+2\cos 2\theta}{4} + \frac{1+\cos 4\theta}{8}\right)d\theta = \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} + K.$$

(ii) Use partial fractions $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$. Solving we find $A = 1/3$, $B = -1/3$, $C = -2/3$, and the integral is

$$\frac{1}{3}\int \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right) dx = \frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{2}\frac{(2x+1)}{(x^2+x+1)} - \frac{(3/2)}{(x+1/2)^2+(3/4)}\right) dx,$$

where I have completed the square in the last term. Now we can integrate to find

$$\frac{1}{3}\log(x-1) - \frac{1}{2}\log(x^2+x+1) - (3/2)\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2} + K.$$

(iii) Write $\frac{x^3+1}{x^3-1} = 1 + \frac{2}{x^3-1}$, and the integral of the second piece has just been done.

(iv) Use integration by parts. Write

$$\begin{aligned}\int x^3\sqrt{x^2+1}dx = \int x^2 \frac{d}{dx}\left(\frac{1}{3}(x^2+1)^{3/2}\right) dx &= x^2\frac{1}{3}(x^2+1)^{3/2} - \int 2x\frac{1}{3}(x^2+1)^{3/2}dx \\ &= \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + K.\end{aligned}$$

(v) Use the trigonometric substitution $u = \sin x$ so that $du = \cos x dx$, and the integral becomes

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x + \sin^3 x}dx = \int \frac{du}{u(1+u^2)} = \int \left(\frac{1}{u} - \frac{u}{1+u^2}\right) du$$

$$= \log(u) - \frac{1}{2}\log(1+u^2) = \left[\log\frac{\sin x}{\sqrt{1+\sin^2 x}}\right]_{\pi/6}^{\pi/2} = \log(1/\sqrt{2}) - \log(1/\sqrt{5}) = \frac{1}{2}\log(5/2)$$

6. If $n = 1$, $I_1 = \tan^{-1} x$. Start with I_{n-1}

$$\begin{aligned}I_{n-1} = \int \frac{dx}{(x^2+1)^{n-1}} &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+1)^n}dx \\ &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{x^2+1-1}{(x^2+1)^n}dx \\ &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)I_n,\end{aligned}$$

therefore, as required,

$$2(n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}.$$

7. We will need the double angle formulas

$$\begin{aligned}\sin A\cos B &= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right], \\ \sin A\sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right], \\ \cos A\cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right].\end{aligned}$$

(a) For any two integers m, n

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2} \left[\sin(m+n)x + \sin(m-n)x \right] dx = 0.$$

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(m-n)x - \cos(m+n)x \right] = \left\{ \begin{array}{ll} 0 & \text{if } m \neq n \\ \pi & \text{if } n = m \end{array} \right. ,$$

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(m-n)x + \cos(m+n)x \right] = \left\{ \begin{array}{ll} 0 & \text{if } m \neq n \\ \pi & \text{if } n = m \end{array} \right. .$$

(b) We assume

$$f(x) = a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx, \qquad (*)$$

and need to calculate the a's and b's. The constant a_0 can be found immediately by integrating over $[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} f(x) dx = 2\pi a_0 + \sum_{k=1}^{N} \int_{-\pi}^{\pi} (a_b \cos kx + b_k \sin kx) dx = 2\pi a_0 \quad \Rightarrow$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

i.e. it is the average (or mean) value of the function over the domain.
Next, take any integer $m \geq 1$, multiply (*) by $\cos mx$ and integrate between $-\pi$ and π. Using the *orthogonality* results from part (a) we see that only the term containing a_m in the sum will survive to give

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_m \int_{-\pi}^{\pi} \cos^2 mx dx = \pi a_m \quad \Rightarrow \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx.$$

Now do the same calculation but multiply by $\sin mx$ and integrate to find

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx.$$

(c) For $f(x)$ defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } |x| \leq \pi/2 \\ 0 & \text{otherwise} \end{array} \right. .$$

we have by direct application of the formulas in part (b)

$$a_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos kx dx = \frac{1}{\pi} \left[\frac{\sin kx}{k} \right]_{-\pi/2}^{\pi/2} = \frac{2}{k\pi} \sin \frac{k\pi}{2},$$

$$b_k = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin kx dx = \frac{1}{\pi} \left[-\frac{\cos kx}{k} \right]_{-\pi/2}^{\pi/2} = 0.$$

The last result could have been anticipated because the given $f(x)$ is an even function of x on $[-\pi/2, \pi/2]$ and $\sin kx$ is odd there, hence the product $f(x) \sin kx$ is also odd, implying that the integral must be zero.

8. (i) Write $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$. The first integral is a finite number. For the second integral we have

$$\int_1^\infty e^{-x^2}dx < \int_1^\infty xe^{-x^2}dx = \frac{1}{2e},$$

hence the integral is bounded.

(ii) Again, given any $M > 0$ write $\int_0^\infty \frac{x^3}{(1+x^2)^2}dx = \int_0^M \frac{x^3}{(1+x^2)^2}dx + \int_M^\infty \frac{x^3}{(1+x^2)^2}dx$; the first integral is a finite number and for the second integral, taking $M > 1$ we have

$$\int_M^\infty \frac{x^3}{(1+x^2)^2}dx > \int_M^\infty \frac{x^3}{(x^2+x^2)^2}dx = \frac{1}{4}\int_M^\infty \frac{1}{x}dx.$$

The last integral diverges so by comparison our original integral also diverges.

(iii) For $M > 0$ write $\int_0^\infty \frac{1}{\sqrt{x+x^3}}dx = \int_0^M \frac{1}{\sqrt{x+x^3}}dx + \int_M^\infty \frac{1}{\sqrt{x+x^3}}dx$, and consider separately $\lim_{a\to 0+}\int_a^M \frac{1}{\sqrt{x+x^3}}dx$ and $\lim_{b\to\infty}\int_M^b \frac{1}{\sqrt{x+x^3}}dx$. We have

$$\lim_{a\to 0+}\int_a^M \frac{1}{\sqrt{x+x^3}}dx = \lim_{a\to 0+}\int_a^M \frac{1}{\sqrt{x}\sqrt{1+x^2}}dx \le \lim_{a\to 0+}\int_a^M \frac{1}{\sqrt{x}}dx$$

$$= \lim_{a\to 0+} 2(\sqrt{M}-\sqrt{a}) = 2\sqrt{M}.$$

Similarly,

$$\int_M^\infty \frac{dx}{\sqrt{x+x^3}} < \int_M^\infty \frac{dx}{x^{3/2}} = \lim_{b\to\infty}\int_M^b \frac{dx}{x^{3/2}} = \lim_{b\to\infty} 2(M^{-1/2}-b^{-1/2}) = 2/\sqrt{M},$$

i.e. also bounded.

(iv)

$$\int_0^1 \frac{\sin^2 x}{1+x^2}dx < \int_0^1 \frac{1}{1+x^2}dx = \pi/4.$$

(v) Useful to compare with a function we know how to integrate. Let $f(x) = x - \log(1+x)$. We have $f(0) = 0$ and $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \ge 0$ for $x \ge 0$. Hence $f(x)$ is increasing and as a result

$$x > \log(1+x) \qquad \text{for} \qquad x > 0.$$

Hence, given $0 < \varepsilon < 1$ we have

$$\int_\varepsilon^1 \frac{dx}{\log(1+x)} > \int_\varepsilon^1 \frac{dx}{x} = \log(1/\varepsilon),$$

and sending $\varepsilon \to 0+$ proves that the integral is divergent.

(vi) This is an improper integral whose integrand does not decay for large x. In fact infinitely more rapid oscillations take place and we can intuitively expect that there is cancellation to give a finite result. [In fact this and its cosine sister are known as Fresnel integrals and come up generically in wave propagation, optics etc.]

Again we split the integral $\int_0^\infty \sin(x^2)dx = (\int_0^1 + \int_1^\infty)\sin(x^2)dx$. The first integral is perfectly fine and is equal to a finite number (we cannot find this in closed form). For the second integral we integrate by parts

$$\int_1^b \sin(x^2)dx = \int_1^b \frac{x\sin(x^2)}{x}dx = \left[-\frac{1}{2x}\cos(x^2)\right]_1^b - \frac{1}{2}\int_1^b \frac{\cos(x^2)}{x^2}dx$$

$$\leq \frac{1}{2}\cos 1 - \frac{1}{2b}\cos b^2 + \frac{1}{2}\int_1^b \frac{dx}{x^2}dx = \frac{1}{2}\cos 1 - \frac{1}{2b}\cos b^2 + \frac{1}{2}(1-\frac{1}{b}),$$

and as b becomes large we see that the integral is bounded above by 1.

Combining this with the integral over $[0,1]$ proves that the Fresnel integral converges.

9. To prove that $\int_0^1 \frac{x^3}{2-\sin^4 x}dx \leq \frac{1}{4}\log 2$ we use the inequality $\sin x \leq x$ in the interval $[0,1]$ of interest. [In fact we have $|x| \leq |\sin x|$ for all $x \in \mathbb{R}$ - you have already proved this elsewhere by showing that the function $f(x) = x - \sin x$ is increasing.] This implies that $2 - \sin^4 x \geq 2 - x^4$, hence

$$\int_0^1 \frac{x^3}{2-\sin^4 x}dx \leq \int_0^1 \frac{x^3}{2-x^4}dx = \left[-\frac{1}{4}\log(2-x^4)\right]_0^1 = \frac{1}{4}\log(2).$$

For the second integral, since we are on the interval $[0,\pi/2]$ we have $\cos x > 0$, hence

$$\left|\int_0^{\pi/2} \frac{x-\pi/2}{2+\cos x}dx\right| \leq \int_0^{\pi/2} \frac{|x-\pi/2|}{2}dx = \frac{1}{2}\int_0^{\pi/2}(\frac{\pi}{2}-x)dx = \frac{\pi^2}{16}.$$

10. Proof of the *integral mean value theorem*: *Let f and g be continuous on $[a,b]$ with $g(x) \geq 0$ for $x \in [a,b]$. Then there exists a number c between a and b with*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Proof: Since f is continuous on $[a,b]$ it must have a maximum M and a minimum m on $[a,b]$, i.e. $m \leq f(x) \leq M$. Since $g(x) \geq 0$, we have for all $x \in [a,b]$

$$mg(x) \leq f(x)g(x) \leq Mg(x),$$

and by the properties of integrals we have in turn

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

If $\int_a^b g(x)dx = 0$ then we also have $\int_a^b f(x)g(x)dx = 0$ and the result follows. If $\int_a^b g(x)dx \neq 0$, we have

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M,$$

and by the Intermediate Value Theorem there must be a number c between a and b so that

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

and the result of the theorem follows.

An example that violates the conclusion of the theorem if we drop $g(x) \geq 0$ is, $f(x) = \sin x$, $g(x) = \sin x$, $a = -\pi/2$, $b = \pi/2$. Then $\int_{-\pi/2}^{\pi/2} fgdx \neq 0$ but $f(c)\int_{-\pi/2}^{\pi/2} \sin xdx = 0$ for all $c \in [-\pi/2, \pi/2]$.

11. These are fairly straightforward. I will do a couple of them.

    x^2 on $[0,1]$. Here we have $\mu = \int_0^1 x^2dx = 1/3$ and $\sigma^2 = \int_0^1 (x^2 - 1/3)^2 dx = \frac{4}{45}$.

    $f(x) = \begin{cases} 1 & \text{on } [0,1] \\ 2 & \text{on } (1,2] \end{cases}$. Here $\mu = \frac{1}{2}\left(\int_0^1 1dx + \int_1^2 2dx\right) = \frac{3}{2}$. Next, $(f(x)-\mu)^2$ is equal to $(1-3/2)^2 = 1/4$ on $[0,1]$ and $(2-3/2)^2 = 1/4$ on $(1,2]$. Hence $\sigma^2 = \frac{1}{2}\int_0^2 (1/4)dx = 1/4$.

12. (a)

    $$\mu = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\sum_{i=1}^n \int_{x_{i-1}}^{x_i} k_i dx = \frac{1}{b-a}\sum_{i=1}^n k_i(x_i - x_{i-1})$$

    Hence

    $$\sigma^2 = \frac{1}{b-a}\int_a^b (f(x)-\mu)^2 dx = \frac{1}{b-a}\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x)-\mu)^2 dx = \frac{1}{b-a}\sum_{i=1}^n (k_i-\mu)^2(x_i-x_{i-1}).$$

    (b) If the partition consists of equally spaced points, we have $x_i - x_{i-1} = \frac{b-a}{n}$, hence the formulas are

    $$\mu = \frac{1}{n}\sum_{i=1}^n k_i,$$

    $$\sigma^2 = \frac{1}{n}\sum_{i=1}^n (k_i - \mu)^2.$$

    (c) The standard deviation of a step function is a sum of non-negative terms (in both cases of uniform and non-uniform partitions). The only way this can be zero is if the function is a constant (and hence equal to its average).

    (d) For a list of numbers $a_1, a_2, \ldots, a_n$ the mean is $\mu = \frac{1}{n}\sum_{i=1}^n a_i$ and $\sigma^2 = \frac{1}{n}\sum_{i=1}^n (a_i - \mu)^2$, so completely analogous to the uniform partition result for functions.

    (e) If its standard deviation is zero, then the numbers are equal.

13. (a) Calculate using definition of $f(x)$:

    $$\int_a^b g(x)\delta_n(x)dx = \int_{-1/2n}^{1/2n} g(x)ndx = \int_{-1/2}^{1/2} g(y/n)dy \to g(0) \qquad \text{as} \quad n \to \infty,$$

    since g is a continuous function.

(b) Using the result above, we have

$$\lim_{n\to\infty}\int_a^t g(x)\delta_n(x)dx = \begin{cases} g(0) & t>0 \\ 0 & t<0 \end{cases}.$$

(c) Here take $g(x)=1$ and use the results above to find

$$\lim_{n\to\infty}\int_{t_1}^{t_2} \delta_n(x)dx = \begin{cases} 0 & t_1<t_2<0 \\ 0 & 0<t_1<t_2 \\ 1 & t_1<0<t_2 \end{cases}.$$

(d) We have shown in (c) that the anti-derivative of $\delta(x)$ is the function $H(x)$ (called the Heaviside function) defined by

$$H(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}.$$

14. Begin by constructing $f(x)$: For $0 \le x < 1$ we have $f(x)=1$, and for $1 \le x < 2$, $f(x)=2$. Hence, for $0 \le x \le 1$ we have $F(x)=\int_0^x dx = x$, and for $1 \le x \le 2$, $F(x)=\int_0^1 dx + \int_1^x 2dx = 2x-1$:

$$F(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2x-1 & \text{for } 1 \le x \le 2 \end{cases}.$$

The function $F(x)$ is continuous but $F'(1)$ does not exist. This does not contradict the fundamental theorem of calculus since $f(x)$ is not continuous. Recall the statement of the FTC:

> Fundamental Theorem of Calculus
>
> *If f is Riemann integrable on $[a,b]$ and $F(x)=\int_a^x f(t)dt$, then $F(x)$ is continuous on $[a,b]$. If in addition f is continuous on $[a,b]$, then F is differentiable on $[a,b]$ and $F'=f$.*

15. Can evaluate the integral using integration by parts:

$$\int_1^n \log x dx = [x\log x]_1^n - \int_1^n x.\frac{1}{x}dx = n\log n - (n-1).$$

Now partition $[1,n]$ in unit intervals as instructed to find the upper Riemann sum U and lower Riemann sum L:

$$U = \log 2 + \log 3 + \ldots + \log n = \log[n!],$$

$$L = \log(1) + \log(2) + \ldots + \log(n-1) = \log[(n-1)!],$$

therefore we have the inequality

$$\log[(n-1)!] \le n\log n - (n-1) \le \log[n!],$$

and using $n \log n - (n-1) = \log(n^n) + \log\left(e^{-(n-1)}\right) = \log\left(n^n e^{-n} e\right)$, along with the fact that log is monotonic increasing we obtain the desired result

$$(n-1)! \le n^n e^{-n} e \le n!$$

Using this we can bound $n!/n^n$ as follows

$$e^{-n} e \le \frac{n!}{n^n} \le n e^{-n} e \;\; \Rightarrow \;\; e^{-1} e^{1/n} \le \left(\frac{n!}{n^n}\right)^{1/n} \le n^{1/n} e^{-1} e^{1/n}.$$

As n gets large we know that $e^{-1/n} \to 1$ and $n^{1/n} \to 1$ (why?), hence by the squeezing theorem the result $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} = 1/e$ follows.

16. (a) This question involves integration by parts. Calculate

$$I_0 = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$$

$$J_0 = \int_0^\infty e^{-x} \cos x \, dx = \left[e^{-x} \sin x\right]_0^\infty + \int_0^\infty e^{-x} \sin x \, dx = \left[-e^{-x} \cos x\right]_0^\infty - J_0$$

Hence $J_0 = 1/2$.
[Alternatively use complex variables and write $e^{-x} \cos x = \mathcal{R}[e^{(i-1)x}]$ so that $J_0 = \mathcal{R}\left[\frac{e^{(i-1)x}}{(i-1)}\right]_0^\infty = \mathcal{R}\left(\frac{1+i}{2}\right) = 1/2$.]

$$I_n = \left[-e^{-x}(\sin x)^n\right]_0^\infty + \int_0^\infty n e^{-x} (\sin x)^{n-1} \cos x \, dx = n J_{n-1}.$$

For any non-negative integer n, let

$$I_n = \int_0^\infty e^{-x} (\sin x)^n dx, \qquad J_n = \int_0^\infty e^{-x} (\sin x)^n \cos x dx.$$

To get the last expression we write $e^{-x} = d(-e^{-x})$ and integrate that first by parts, i.e.

$$J_n = \left[-e^{-x}(\sin x)^n \cos x\right]_0^\infty - \int_0^\infty (-e^{-x}) \left[-(\sin x)^{n+1} + n(\sin x)^{n-1} \cos^2 x\right] \, dx$$

$$= \int_0^\infty e^{-x} \left[n(\sin x)^{n-1} - (1+n)(\sin x)^{n+1}\right] \, dx = n I_{n-1} - (n+1) I_{n+1} \;\; \Rightarrow$$

$$J_n = n I_{n-1} - (n+1) I_{n+1} \qquad (*_1)$$

as required.

Note: J_n can also be integrated by parts by noting that $(\sin x)^n \cos x = d\left(\frac{(\sin x)^{n+1}}{n+1}\right)$ and integrating this first, to immediately obtain the alternative expression

$$J_n = \left[\frac{e^{-x}(\sin x)^{n+1}}{n+1}\right]_0^\infty + \int_0^\infty e^{-x} \frac{(\sin x)^{n+1}}{n+1} dx \;\; \Rightarrow \;\; J_n = \frac{1}{n+1} I_{n+1}. \quad (*_2)$$

As we will see, this is of course consistent with the recursion formula for I_n we find next.

(b) Use $n = 1$ in the formula $I_n = nJ_{n-1}$ to find $I_1 = J_0 = 1/2$.

From (*) $J_1 = I_0 - 2I_2 = I_0 - 2(2J_1)$, hence $5J_1 = 1$ as needed.

We have $I_n = nJ_{n-1}$. Evaluate $(*_1)$ at $n-1$ to find $J_{n-1} = (n-1)I_{n-2} - nI_n$ and substitute into the expression for I_n to get

$$I_n = n(n-1)I_{n-2} - n^2 I_n \qquad \Rightarrow \qquad I_n = \frac{n(n-1)}{(1+n^2)} I_{n-2}. \qquad (*_3)$$

Note: This result should of course be consistent with equating $(*_1) = (*_2)$ which implies $I_{n+1} = \frac{n(n+1)}{1+(n+1)^2} I_{n-1}$ which is identical to $(*_3)$ once we shift the index $n+1 \to n$.

To find the recursion for J, we now eliminate I_{n-1} and I_{n+1} in $(*_1)$ to find

$$J_n = n(n-1)J_{n-2} - (n+1)^2 J_n \qquad \Rightarrow \qquad J_n = \frac{n(n-1)}{1+(n+1)^2} J_{n-2}. \qquad (*_4)$$

(c) We need to calculate explicit expressions from $(*_3)$ and $(*_4)$ for $n \geq 2$. In both cases we have starting values $I_0 = 1$, $I_1 = 1/2$ and $J_0 = 1/2$, $J_1 = 1/5$, and all other values will be in terms of these. Inspection of $(*_3)$ and $(*_4)$ shows that all even indices will involve I_0 or J_0 and odd indices will involve I_1 and J_1. Lets calculate a few terms and you will get the general formula.

$$\begin{array}{cc}
I_2 = \frac{2\cdot 1}{(1+2^2)} I_0 & I_3 = \frac{3\cdot 2}{(1+3^2)} I_1 \\
I_4 = \frac{4\cdot 3}{(1+4^2)} I_2 = \frac{4!}{(1+2^2)(1+4^2)} I_0, & I_5 = \frac{5\cdot 4}{(1+5^2)} I_3 = \frac{5!}{(1+3^2)(1+5^2)} I_1 \\
\cdots & \cdots \\
I_{2k} = (2k)! \prod_{p=1}^{k} \left(\frac{1}{1+(2p)^2} \right), & I_{2k+1} = \frac{1}{2}(2k+1)! \prod_{p=1}^{k} \left(\frac{1}{1+(2p+1)^2} \right)
\end{array}$$

Similarly for the Js we have

$$\begin{array}{cc}
J_2 = \frac{2\cdot 1}{(1+3^2)} J_0 & J_3 = \frac{3\cdot 2}{(1+4^2)} J_1 \\
J_4 = \frac{4!}{(1+3^2)(1+5^2)} J_0 & J_5 = \frac{5!}{(1+4^2)(1+6^2)} J_1 \\
\cdots & \cdots \\
J_{2k} = \frac{1}{2}(2k)! \prod_{p=1}^{k} \left(\frac{1}{(1+(2p+1)^2} \right) & J_{2k+1} = \frac{1}{5}(2k+1)! \prod_{p=1}^{k} \left(\frac{1}{(1+(2p+2)^2} \right)
\end{array}$$

We can compare directly I_{2k} with J_{2k} found above; the products in J_{2k} are smaller since $\frac{1}{(1+(2p+1)^2} < \frac{1}{1+(2p)^2}$, in addition to the factor of $1/2$. Hence $I_{2k} > J_{2k}$. Analogous reasoning shows also that $I_{2k+1} > J_{2k+1}$ and so $I_n > J_n$ for all $n \geq 0$. This is reasonable since comparison of the integrands of J_n and I_n, respectively shows that

$$\left| e^{-x} (\sin x)^n \cos x \right| \leq \left| e^{-x} (\sin x)^n \right|.$$