Introduction to Quantum Mechanics – Solutions to problem sheet 9

1. Properties of the Pauli matrices

(a) The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Squaring each Pauli matrix yields

$$\begin{aligned} \sigma_x^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_y^2 &= \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_z^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

It follows that $\sigma_i^2 + \sigma_i^2 = 2\hat{I}$.

For $j \neq k$ we have $\sigma_j \sigma_k = i\sigma_l$, with j, k, l cyclic, and thus and thus $\sigma_j \sigma_k = -\sigma_k \sigma_j$ for $j \neq k$.

(b)

$$\begin{aligned} (\mathbf{A} \cdot \sigma)(\mathbf{B} \cdot \sigma) &= \sum_i (A_i \sigma_i) \sum_j (B_j \sigma_j) \\ &= \sum_i \sum_j A_i B_j \sigma_i \sigma_j \\ &= \sum_{i=j} A_i B_i \hat{I} + \sum_{i \neq j} A_i B_j \sigma_i \sigma_j \\ &= \sum_{i=j} A_i B_i \hat{I} + i \sum_i \epsilon_{ijk} A_i B_j \sigma_k \\ &= (\mathbf{A} \cdot \mathbf{B})\hat{I} + \mathrm{i}(\mathbf{A} \times \mathbf{B}) \cdot \sigma, \end{aligned}$$

as required.

2. Angular momentum matrices for $j = 1$

(a) The eigenvector of $\hat{L}_z$ belonging to the eigenvalue $m = 0$ is clearly given by

$$\phi_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} .$$

For the eigenvector $\phi_x = \begin{pmatrix} \phi_{x,1} \\ \phi_{x,2} \\ \phi_{x,3} \end{pmatrix}$ of $\hat{L}_x$ belongin to the eigenvalue zero we find from

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \phi_x = 0,$$

that $\phi_{x,2} = 0$, and $\phi_{x,3} = -\phi_{x,1}$. Together with the normalisation condition this yields

$$\phi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} ,$$

up to an arbitrary phase factor.
Similarly we find the eigenvector of $\hat{L}_x y$ belongin to the eigenvalue zero as

$$\phi_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ,$$

up to an arbitrary phase factor.
These vectors indeed form an orthogonal set:

$$\begin{aligned} \phi_x \cdot \phi_y &= \frac{1}{2}(1 + 0 - 1) = 0 \\ \phi_y \cdot \phi_z &= \frac{1}{\sqrt{2}}(0 + 0 + 0) = 0 \\ \phi_x \cdot \phi_z &= \frac{1}{\sqrt{2}}(0 + 0 + 0) = 0. \end{aligned} \tag{1}$$

(b) (i) For the expectation value of $\hat{L}_z$ we find

$$\langle \hat{L}_z \rangle = \frac{\hbar}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\frac{4}{7}\hbar.$$

The uncertainty ΔL_z is given by $\Delta L_z = \sqrt{\langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2}$. With

$$\langle \hat{L}_z^2 \rangle = \frac{\hbar^2}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{5}{7}\hbar^2,$$

that yields

$$\Delta L_z = \sqrt{\frac{35 - 16}{7^2}}\hbar = \frac{\sqrt{19}}{7}\hbar.$$

(ii) The probability that a measurement of $\hat{L}_j$ yields the outcome zero is givem by the projection of the wave function on the respective eigenvector, $P(L_j = 0) = |\langle\phi|\phi_j\rangle|^2$. Thus we find

$$
\begin{aligned}
P(L_x = 0) &= \left| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|^2 = \tfrac{1}{7} \\
P(L_y = 0) &= \left| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \tfrac{4}{7} \\
P(L_z = 0) &= \left| \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \tfrac{2}{7}.
\end{aligned}
$$

3. Dynamics of a spin $\frac{1}{2}$ system

The dynamics of the $s_j := \langle \hat{s}_j \rangle$ is governed by the Heisenberg equations

$$\dot{s}_j = \frac{i}{\hbar}\langle[\hat{H}, \hat{s}_j]\rangle.$$

Here we have

$$\hat{H} = \vec{B} \cdot \hat{\vec{\sigma}} = \frac{2}{\hbar}(B_x \hat{s}_x + B_y \hat{s}_y + B_z \hat{s}_z),$$

and thus

$$\dot{s}_j = \frac{2i}{\hbar^2}\langle[B_x \hat{s}_x + B_y \hat{s}_y + B_z \hat{s}_z, \hat{s}_j]\rangle.$$

With $[\hat{s}_j, \hat{s}_k] = i\hbar\epsilon_{jkl}\hat{s}_l$, this yields

$$\begin{aligned}
\dot{s}_x &= \tfrac{2i}{\hbar^2}\left(B_y\langle[\hat{s}_y, \hat{s}_x] + B_z[\hat{s}_z, \hat{s}_x]\right) = \tfrac{2}{\hbar}(B_y s_z - B_z s_y) \\
\dot{s}_y &= \tfrac{2i}{\hbar^2}\left(B_x\langle[\hat{s}_x, \hat{s}_y] + B_z[\hat{s}_z, \hat{s}_y]\right) = \tfrac{2}{\hbar}(B_z s_x - B_x s_z) \\
\dot{s}_z &= \tfrac{2i}{\hbar^2}\left(B_x\langle[\hat{s}_x, \hat{s}_z] + B_y[\hat{s}_y, \hat{s}_z]\right) = \tfrac{2}{\hbar}(B_x s_y - B_y s_x),
\end{aligned}$$

that is,

$$\dot{\vec{s}} = \frac{2}{\hbar}\vec{B} \times \vec{s}.$$

This dynamics conserves the quantity $\sum_j s_j^2$:

$$\begin{aligned}
\frac{d}{dt}\sum_j s_j^2 &= 2\sum_j \dot{s}_j s_j \\
&= \frac{4}{\hbar}\left(B_y s_x s_z - B_z s_x s_y + B_z s_x s_y - B_x s_y s_z + B_x s_z s_y - B_y s_x s_z\right) \\
&= 0.
\end{aligned}$$

Thus, the dynamics is confined to a sphere.

Note that the expectation value of $\hat{s}^2$ is a separate conserved quantity here.

The vector $\vec{s}$ moves with constant speed along circles perpendicular to the direction of $\vec{B}$, the resulting trajectories are sketched for an example where $\vec{B}||z$ in figure 1.

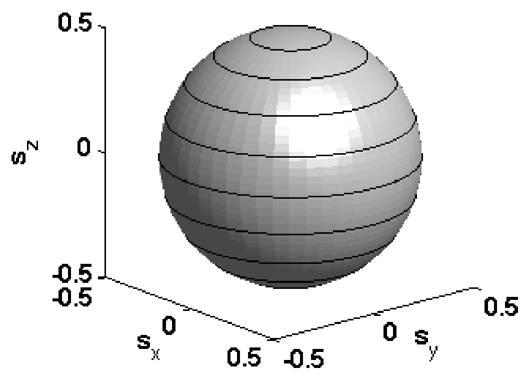

Figure 1: Dynamical orbits of $\vec{s}$ for $\vec{B}||z$.