Imperial College London

MATH70031

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Markov Processes

Date: 2 June 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Define the *Ehrenfest urn* with d balls, and give its transition probabilities. (3 marks)

   (b) Define the *detailed balance* condition. Why is this important? (3 marks)

   (c) Assuming that the chain has invariant distribution the binomial distribution $\pi = (\pi_i) = (2^{-d}\binom{d}{i})$, show that the chain has detailed balance. (5 marks)

   (d) Conversely, assuming detailed balance, show that the invariant distribution π is as above. (5 marks)

   (e) What is the physical importance of this model? (4 marks)

(Total: 20 marks)

2. (a) Define the *gambler's ruin* problem, with total capital $a \in \mathbb{N}$ and success probability p for the gambler at each play.

   (4 marks)

   (b) For $p \neq \frac{1}{2}$, find the probability q_z that the gambler is ruined, starting with initial capital $z \in \{0, 1, \cdots, a\}$.

   (6 marks)

   (c) Repeat for the case $p = \frac{1}{2}$. (5 marks)

   (d) In a game of tennis, the server wins each point with probability p. Label the five states 'server loses, advantage out/(30-40), deuce/(40-40), advantage in/(40-30), server wins' $i = 0, 1, \cdots, 4$. Write π_i for the probability that the game reaches state i first, and use q_i as above. Find the probability that the server wins the game, in terms of the π_i and q_i. (5 marks)

(Total: 20 marks)

3. (a) For X, Y independent, Poisson distributed with parameters λ, μ, find the distribution of $X + Y$. (7 marks)

(b) For X_i independent, Bernoulli distributed with parameter p, $S_n := \sum_1^n X_i$, and N Poisson with parameter λ, find the probability generating functions of

(i) S_n (3 marks)

(ii) N (3 marks)

(iii) S_N, and the distribution of S_N. (7 marks)

(Total: 20 marks)

4. (a) In what respects is Brownian motion unrealistic as a model? (3 marks)

(b) For the Ornstein-Uhlenbeck velocity process $V = (V_t)$ for the momentum of a Brownian particle, give the defining stochastic differential equation and solve it. (6 marks)

(c) Find the covariance function of the process V. (5 marks)

(d) For $U = (U_t)$, $U_t := \int_0^t V_s ds$ $(V_0 = 0)$ the Ornstein-Uhlenbeck displacement process, which of the following is Markov, and why?

(i) V (2 marks)

(ii) U (2 marks)

(iii) (U, V) (2 marks)

(Total: 20 marks)

5. For $B = (B(t), t \geq 0)$ Brownian motion, T exponentially distributed with parameter λ and independent of B, the process B_λ is defined to be Brownian motion with drift λ beginning at the random time T:

$$B_\lambda(t) := B(t) + \lambda(t - T)_+.$$

(a) Why is B_λ not Markov, and how could one make it Markov? .

(2 marks)

(b) Describe the local behaviour of the paths of B_λ.

(2 marks)

(c) Find the mean $\mathbb{E}[B_\lambda(t)]$. (8 marks)

(d) Describe how you would find the covariance function of B_λ. Evaluate only those terms for which you can do this without calculation (in particular, there is no need to use calculus here).

(8 marks)

(Total: 20 marks)

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MATH 97216

Markov Processes (Solutions)

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1. (a) The Ehrenfest urn model with d balls has two urns, I and II say. At each stage, a ball is chosen at random (probability $1/d$ each), and changed to the other urn. The state is the number of balls in urn I (say). The transition probabilities are seen $\Downarrow$

$$p_{i,i+1} = (d-i)/d, \qquad p_{i,i-1} = i/d, \qquad p_{i,j} = 0 \quad \text{otherwise.}$$

3, A

(b) The *detailed balance* condition for a Markov chain with transition probability matrix $P = (p_{ij})$ with respect to a distribution $\pi = (\pi_k)$ is seen $\Downarrow$

$$\pi_j p_{jk} = \pi_k p_{kj} \qquad \forall j, k. \tag{DB}$$

Its importance is that *detailed balance is equivalent to reversibility* (in time), and then π is the invariant distribution of the chain. 3, A

(c) Assuming that the chain has invariant distribution the binomial $(2^{-d}\binom{d}{i})$, we check detailed balance: seen $\Downarrow$

$$j = i+1: \quad \pi_i \; p_{i,i+1} = 2^{-d}\binom{d}{i}.\frac{d-i}{d} = 2^{-d}.\frac{d!}{(d-i)!i!}.\frac{i+1}{d} = 2^{-d}\binom{d-1}{i};$$

$$\pi_{i+1} \; p_{i+1,i} = 2^{-d}\binom{d}{i+1}.\frac{i+1}{d} = 2^{-d}\frac{d!}{(d-i-1)!(i+1)!} = 2^{-d}\binom{d-1}{i}.$$

Comparing, this proves detailed balance, and so reversibility. 5, A

(d) Conversely, assuming reversibility/detailed balance, seen $\Downarrow$

$$i = 0: \quad \pi_1 = \pi_0.\frac{p_{0,1}}{p_{1,0}} = \pi_0.1/(1/d) = \pi_0.d;$$

$$i = 1: \quad \pi_2 = \pi_i.p_{1,2}/p_{2,1} = \pi_0.d.\frac{d-1}{d}/\frac{2}{d} = \pi_0.d(d-1)/2 = \pi_0\binom{d}{2}.$$

Continuing in this way gives

$$\pi_i = \pi_0.\frac{d(d-1)\cdots(d-i+1)}{1.2.\cdots i} = \pi_0.\binom{d}{i}.$$

Then $\sum_i \pi_i = 1$ gives $\pi_0.\sum\binom{d}{i} = 1$. But $\sum\binom{d}{i} = (1+1)^d = 2^d$, by the Binomial Theorem, so $\pi_0 = 2^{-d}$, and $\pi_i = 2^{-d}\binom{d}{i}$, as required. $\square$ 5, B

(e) The importance of this model (and its much earlier relative, the Bernoulli-Laplace urn) is that it reconciles reversibility in theory with its apparent opposite in practice. For, the mean recurrence time of e.g. the 'extreme' state 0 is $\mu_0 = 1/\pi_0$ (by the Erdős-Feller-Pollard theorem), $= 2^d$. For d the number of molecules in a quantity of gas (itself of the order of Avogadro's number, so $O(10^{23})$), this is so astronomically large as to be 'effectively infinite': if started in state 0, return there would never be seen in practice. seen $\Downarrow$ 4, B

2. (a) For $1 < z < a-1$, $q := 1-p$: the probability q_z of the server losing starting from z satisfies

$$q_z = pq_{z+1} + qq_{z-1}$$

(decomposing according to the result of the first play). The boundary conditions are $q_0 = 1$, $q_a = 0$, so this extends also to $1 \leq z \leq a-1$. This is a linear difference equation of the second order.

seen ⇓

4, A

(b) For $p \neq \frac{1}{2}$: by inspection, it has two particular solutions: $q_z = 1$; $q_z = (q/p)^z$. So the general solution is

seen ⇓

$$q_z = A + B(q/p)^z,$$

with A, B arbitrary constants. The boundary conditions give

$$A + B = 1, \qquad A + (q/p)^a B = 1.$$

Solving,

$$A = \frac{(q/p)^a}{(q/p)^a - 1}, \qquad B = \frac{-(q/p)^z}{(q/p)^a - 1}.$$

So the probability the gambler loses (is ruined) is

$$q_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1}.$$

6, A

(c) When $q = p = \frac{1}{2}$, by inspection, two particular solutions are $q_z = 1$, $q_z = z$. The general solution is $q_z = A + Bz$. The boundary conditions give

seen ⇓

$$A = 1, \qquad A + Ba = 0: \qquad B = -1/a: \qquad q_z = 1 - (z/a).$$

[Or, let $p \to \frac{1}{2}$ in (i) and use L'Hospital's rule.]

5, C

(d) Label the five states 'server loses, advantage out/(30-40), deuce/(40-40), advantage in/(40-30), server wins' $i = 0, 1, \cdots, 4$, and write π_i for the probability that the game reaches state i first. Then with $a = 4$ in the above, the probability that the server loses ('the gambler is ruined') is

seen ⇓

$$\sum_{i=0}^{4} \pi_i q_i.$$

5, D

3. (a) (i) For $X \sim P(\lambda)$, $Y \sim P(\mu)$, X, Y independent, meth seen $\Downarrow$

$$\begin{aligned}
\mathbb{P}(X+Y=n) &= \sum_0^n \mathbb{P}(X=k,\ Y=n-k) \\
&= \sum_0^n \mathbb{P}(X=k)\mathbb{P}(Y=n-k) \\
&= \sum_0^n e^{-\lambda}\lambda^k/k!\ \epsilon^{-\mu}\mu^{n-k}/(n-k)! \\
&= \sum_0^n e^{-(\lambda+\mu)}\binom{n}{k}\lambda^k\mu^{n-k} \\
&= \sum_0^n e^{-(\lambda+\mu)}(\lambda+\mu)^n :
\end{aligned}$$

$$X+Y \sim P(\lambda+\mu).$$

7, A

(b) (i) The X_i are independent Bernoulli, $X_i \sim B(p)$. Their probability generating function (PGF) is unseen $\Downarrow$

$$\mathbb{E}[t^X] = \mathbb{P}(X=0) + t\mathbb{P}(X=1) = q + pt.$$

So (PGFs multiply over independent sums) the PGF of S_n is

$$\mathbb{E}[t^{(S_n)}] = (q+pt)^n.$$

3, B

(ii) The PGF of $N \sim P(\lambda)$ is

$$\mathbb{E}[t^N] = \sum_0^\infty t^k \mathbb{P}(N=k) = \sum_0^\infty t^k.e^{-\lambda}\lambda^k/k! :$$

$$\mathbb{E}[t^N] = e^{-\lambda}.e^{\lambda t} = \exp\{-\lambda(1-t)\}.$$

3, C

(iii) So the PGF of S_N is

$$\begin{aligned}
\mathbb{E}[t^{(S_N)}] &= \sum_0^\infty \mathbb{E}[t^{(S_n)}]\ \mathbb{P}(N=n) \\
&= \sum_0^\infty e^{-\lambda}\lambda^n/n!\ (q+pt)^n \\
&= \sum_0^\infty e^{-\lambda}.\exp(\lambda(q+pt)) \\
&= \sum_0^\infty e^{-\lambda}.\exp(\lambda-\lambda p+\lambda pt) \\
&= \exp\{-\lambda p(1-t)\} :
\end{aligned}$$

$$S_N \sim P(\lambda p)$$

(this is *Poisson thinning*). 7, D

seen ⇓

4. (a) (i) Brownian motion (BM) is unrealistic, in at least two ways:
(a) As a process: because it is a fractal, so invariant under scaling, it fails to take account of the discreteness of matter at the atomic scale. (b) As a model of the diffusion of Brownian particles under molecular bombardment, it neglects the momentum of the particle. 3, A

(b) The *Ornstein-Uhlenbeck process* addresses this by using BM to model the *velocity process* $V = (V_t)$ of a Brownian particle by the following stochastic differential equation (SDE): for some $\beta > 0$ (the 'coefficient of frictional drag'), and $B = (B_t)$ BM, seen ⇓

$$dV_t = -\beta V_t + dB_t \qquad (V_0 = 0), \tag{OU}$$

The integrating factor $e^{\beta t}$ gives $d(e^{\beta t}V_t) = e^{\beta t}dB_t$. Integrating,

$$V_t e^{\beta t} = \int_0^t e^{\beta u} dB_u : \qquad V_t = \int_0^t e^{-\beta(t-u)} dB_u$$

(taking $V_0 = 0$ for convenience). (So V_t is Gaussian, has mean 0, and is continuous.) 6, B

(c) For the covariance: seen ⇓

$$cov(V_t, V_{t+s}) = \mathbb{E}\ [(\int_0^t e^{-\beta(t-u)} dB_u)(\int_0^{t+s} e^{-\beta(t+s-v)} dB_v] \qquad (s > 0).$$

Write $\int_0^{t+s} = \int_0^t + \int_t^{t+s}$. By independence of the Brownian increments in $[0,t]$ and $[t, t+s]$, the 2nd term then contributes 0, so

$$cov(V_t, V_{t+s}) = \mathbb{E}\ [(\int_0^t e^{-\beta(t-u)} dB_u)(\int_0^t e^{-\beta(t+s-v)} dB_v]$$

$$= e^{-2\beta t} e^{-\beta s} \int_0^t \int_0^t e^{\beta u} e^{\beta v} \mathbb{E}[dB_u dB_v].$$

By independence of Brownian increments, $\mathbb{E}[dB_u dB_v] = 0$ for $u \neq v$, but $\mathbb{E}[(dB_u)^2] = du$ (as may be quoted: Itô's isometry, the definition of a Brownian stochastic integral or Lévy's result on quadratic variation of BM). So

$$cov(V_t, V_{t+s}) = e^{-2\beta t} e^{-\beta s} \int_0^t e^{2\beta u} du = e^{-2\beta t} e^{-\beta s}[e^{2\beta t} - 1]/(2\beta)$$

$$= e^{-\beta s}[1 - e^{-2\beta t}]/(2\beta),$$

and similarly for $s < 0$:

$$cov(V_t, V_{t+s}) = e^{-\beta |s|}[1 - e^{-2\beta t}]/(2\beta).$$

5, C

(d) (a) The O-U *velocity* process V is Markov: the past before time t has played no role in the above. seen ⇓

(b) U is *not* Markov: knowing its position does not give full information about the future, as e.g. the direction of travel is needed to predict this.
(c) (U, V) is Markov: at time t, (U_t, V_t) give the initial conditions required to solve the SDE above. 6, D

5. (a) The process B_λ is not Markovian. For, at time t, its future depends on T, which is not given separately from B. unseen $\Downarrow$

To make the process Markovian, we augment the state-space from the univariate process B_λ to the bivariate process (B, T). 2, M

(b) The local behaviour of the paths is the same as that of B: continuous and nowhere differentiable a.s., as B is. unseen $\Downarrow$

2, M

(c) By the Conditional Mean Formula, unseen $\Downarrow$

$$\begin{aligned}
\mathbb{E}[B_\lambda(t)] &= \mathbb{E}[B(t)] + \mathbb{E}[(t-T)_+] = \mathbb{E}[(t-T)_+] \\
&= \mathbb{E}[\mathbb{E}[(t-u)|T=u]] \\
&= \int_0^\infty (t-u)_+ . \lambda e^{-\lambda u} du = e^{-\lambda t} \int_0^t v . \lambda e^{\lambda v} dv \\
&= e^{-\lambda t} . t e^{\lambda t} - e^{-\lambda t} \int_0^t e^{\lambda v} dv = t - e^{-\lambda t} . \frac{1}{\lambda} [e^{\lambda t} - 1] :
\end{aligned}$$

$$\mathbb{E}[B_\lambda(t)] = t - \frac{1}{\lambda}(1 - e^{-\lambda t}).$$

8, M

(d) To find $cov(B_\lambda(t), B_\lambda(t+s))$ $(t, s \geq 0)$, it suffices to find $\mathbb{E}[B_\lambda(t).B_\lambda(t+s)]$, by (c). To use independence of Brownian increments, write this as unseen $\Downarrow$

$$\mathbb{E}[(B(t) + (t-T)_+)(B(t) + [B(s+t) - B(t)] + (t+s-T)_+).$$

Multiplying out, we obtain six terms. Label them (i, j) ($i = 1, 2$ in the first factor, $j = 1, 2, 3$ in the second).

$(1,1)$-term: $\mathbb{E}[B(t)^2] = var\ B(t) = t$.

$(1,2)$-term: $\mathbb{E}[B(t)(B(t+s) - B(t))] = \mathbb{E}[B(t)]\mathbb{E}[B(t+s) - B(t)] = 0.0 = 0$, by independent increments.

$(1,3)$-term: $\mathbb{E}[B(t)(t+s-T)_+] = \mathbb{E}[\mathbb{E}[[B(t)(t+s-T)_+|T=u]], = 0$, as $\mathbb{E}[[B(t)(t+s-T)_+|T=u] = (t+s-u)_+\mathbb{E}[B(t)] = 0$.

$(2,1)$ and $(2,2)$ terms: also 0, as with $(1,3)$.

The $(2,3)$ term (not asked for) is just calculus, as in (c). 8, M

Review of mark distribution:
Total A marks: 31 of 32 marks
Total B marks: 18 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 18 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60031/70031	1	No Comments Received
MATH60031/70031	2	No Comments Received
MATH60031/70031	3	No Comments Received
MATH60031/70031	4	No Comments Received
MATH70031	5	No Comments Received