MATH50001 Problems Sheet 7
Solutions

1) Let us first prove that if $|p(e^{i\theta})| \leq 1$, then $p(z) = z^n$. Indeed, consider

$$q(z) = z^n p(1/z) = 1 + a_{n-1}z + \cdots + a_0 z^n.$$

By using the maximum modulus principle we obtain

$$\max_{|z|\leq 1} |q(z)| = \max_{|z|=1} |q(z)| = \max_{|z|=1} |e^{in\theta} p(e^{-i\theta}| \leq 1,$$

where we also have used the assumption $|p(e^{-i\theta}| \leq 1$. This implies

$$a_{n-1} = \cdots = a_0 = 0$$

and thus $p(z) = z^n$.

2) Assume that such a function exists. Since it does not vanish we have $|(f(z))^{-1}| = e^{-|z|} \leq 1$. However $|f(0)| = 1$ and therefore by the maximum modulus principle we have that f is constant. The constant function cannot satisfy $|f(z)| = e^{|z|}$.

3) Consider the function $g(z) = f(z)/z$. Since f is holomorphic in $\mathbb{D}$ and $f(0) = 0$, we conclude that $g(z)$ is holomorphic in $\mathbb{D}$.
Consider g in $D_\rho = \{z : |z| < \rho\}$, where $\rho < 1$. By the maximum modulus principle $|g|$ has its maximum on the boundary $\gamma_\rho = \{z : |z| = \rho\}$. Since $|f(z)| \leq 1$, $z \in \mathbb{D}$, we have

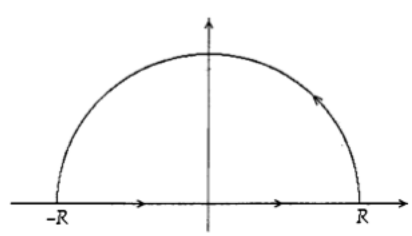
$$|g(z)| = \frac{|f(z)|}{\rho} \leq \frac{1}{\rho}, \qquad \forall z \in \gamma_\rho.$$

Fixing $z \in D_\rho \subset \mathbb{D}$ and letting $\rho \to 1$ we obtain $|g(z)| \leq 1$ and thus $|f(z)| \leq |z|$ for any $z \in \mathbb{D}$.

4)
a. Let first $\xi < 0$ and consider

$$\oint_\gamma \frac{e^{-i\xi z}}{1 + z^2}\, dz,$$

where $\gamma = \gamma_1 \cup \gamma_2$



$$\gamma_1 = \{z : \ z = x + i0, \ -R < x < R\},$$
$$\text{and} \quad \gamma_2 = \{z : \ z = R\,e^{i\theta}, \ 0 \le \theta \le \pi\}, \quad R > 1.$$

Then

$$\oint_\gamma \frac{e^{-i\xi z}}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left[\frac{e^{-i\xi z}}{1+z^2}, i\right] = 2\pi i\,\frac{e^{\xi}}{2i} = \pi\,e^{-|\xi|}.$$

Note that since $0 \le \theta \le \pi$ we have $\sin\theta > 0$. Therefore by using the ML-inequality we find

$$\Big|\int_{\gamma_2} \frac{e^{-i\xi z}}{1+z^2}\,dz\Big| \le \pi R \max\Big|\frac{e^{-i\xi\,R\,(\cos\theta + i\sin\theta)}}{1+R^2\,e^{2i\theta}}\Big| \le \frac{\pi R}{R^2-1} \to 0,$$

as $R \to \infty$.
Finally

$$\pi\,e^{-|\xi|} = \pi\,e^{\xi} = \oint_\gamma \frac{e^{-i\xi z}}{1+z^2}\,dz = \lim_{R\to\infty}\left(\int_{\gamma_1}\frac{e^{-i\xi z}}{1+z^2}\,dz + \int_{\gamma_2}\frac{e^{-i\xi z}}{1+z^2}\,dz\right)$$
$$= \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{-i\xi x}}{1+x^2}\,dx + \int_{\gamma_2}\frac{e^{-i\xi z}}{1+z^2}\,dz\right) = \int_{-\infty}^{\infty}\frac{e^{-i\xi x}}{1+x^2}\,dx.$$

b. If $\xi > 0$ then by substituting $x = -y$ we have

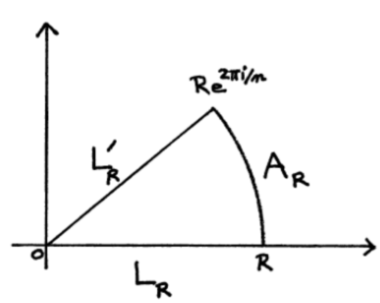
$$\int_{-\infty}^{\infty}\frac{e^{-i\xi x}}{1+x^2}\,dx = \int_{-\infty}^{\infty}\frac{e^{-i\,(-\xi)y}}{1+y^2}\,dy.$$

and thus reduce the the problem to the case **1.a**.

5) Consider

$$\oint_\gamma \frac{1}{1+z^n}\,dz,$$

where $\gamma = \gamma_1 \cup \gamma_2$ is defined by



$$\gamma_1 = \{z : z = x + i0,\ 0 < x < R\}, \quad R > 1,$$
$$\gamma_2 = \{z : z = R\, e^{i\theta},\ 0 \le \theta \le 2\pi/n\},$$
$$\gamma_3 = \{z : z = r\, e^{i2\pi/n},\ r \in [R, 0]\}.$$

The only singularity of the function $1/(1 + z^n)$ internal for γ is the point $e^{i\pi/n}$. Therefore

$$\oint_\gamma \frac{1}{1+z^n}\, dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^n}, e^{i\pi/n}\right] = 2\pi i \frac{1}{n e^{i\pi(n-1)/n}} = -\frac{2\pi i}{n}\, e^{\pi i/n}.$$

Moreover

$$\int_{\gamma_1} \frac{1}{1+z^n}\, dz \to \int_0^\infty \frac{1}{1+x^n}\, dx, \qquad R \to \infty,$$

$$\int_{\gamma_2} \frac{1}{1+z^n}\, dz \to 0, \qquad R \to \infty$$

and

$$\int_{\gamma_3} \frac{1}{1+z^n}\, dz \to -e^{2\pi i/n} \int_0^\infty \frac{1}{1+x^n}\, dx, \qquad R \to \infty.$$

Finally we obtain

$$(1 - e^{2\pi i/n}) \int_0^\infty \frac{1}{1+x^n}\, dx = -\frac{2\pi i}{n}\, e^{\pi i/n},$$

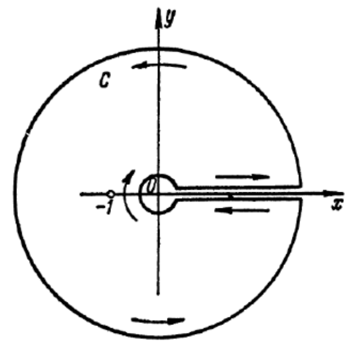
which is equivalent to

$$\frac{e^{\pi i/n} - e^{-\pi i/n}}{2i} \int_0^\infty \frac{1}{1+x^n}\, dx = \frac{\pi}{n}.$$

6) Consider

$$f(z) = \frac{z^{a-1}}{1+z}, \quad z = re^{i\theta}, \quad 0 \le \theta < 2\pi$$

and consider the contour



$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4,$$

where

$$\gamma_1 = \{z: z = re^{i0},\ r \in [\varepsilon, R]\}, \quad R > 1,$$
$$\gamma_2 = \{z: z = R\,e^{i\theta},\ 0 \le \theta < 2\pi\},$$
$$\gamma_3 = \{z: z = r\,e^{i2\pi},\ r \in [R, \varepsilon)]\},$$
$$\gamma_4 = \{z: z = \varepsilon\,e^{i\theta},\ \theta \in (2\pi, 0]\}.$$

Then

$$\oint_\gamma \frac{z^{a-1}}{1+z}\,dz = 2\pi\,i\,\mathrm{Res}\left[\frac{z^{a-1}}{1+z}, e^{i\pi}\right] = 2\pi\,ie^{i\pi(a-1)} = -2\pi\,i\,e^{i\pi a}.$$

Moreover, as $\varepsilon \to 0$ and $R \to \infty$ we obtain

$$\int_{\gamma_1} \frac{z^{a-1}}{1+z}\,dz = \int_\varepsilon^R \frac{r^{a-1}}{1+r}\,dr \to \int_0^\infty \frac{r^{a-1}}{1+r}\,dr,$$
$$\Big|\int_{\gamma_2} \frac{z^{a-1}}{1+z}\,dz\Big| \le 2\pi\,R\,\frac{R^{a-1}}{R-1} \to 0,$$
$$\int_{\gamma_3} \frac{z^{a-1}}{1+z}\,dz = \int_R^\varepsilon \frac{r^{a-1}\,e^{i2\pi(a-1)}}{1+r\,e^{i2\pi}}\,e^{i2\pi}\,dr \to -e^{i2\pi a}\int_0^\infty \frac{r^{a-1}}{1+r}\,dr,$$
$$\Big|\int_{\gamma_4} \frac{z^{a-1}}{1+z}\,dz\Big| \le 2\pi\,\varepsilon\,\frac{\varepsilon^{a-1}}{1-\varepsilon} \to 0.$$

Therefore we have

$$\left(1 - e^{i2\pi a}\right)\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = -2\pi\,i\,e^{i\pi a}.$$

and finally

$$\sin \pi a \int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \pi.$$

7) Let

$$\gamma_1 = \{z = x + iy : -R \le x \le R,\ y = 0\},$$
$$\gamma_2 = \{z : z = R\,e^{i\theta},\ 0 < \theta < \pi\}, \qquad R > 1,$$

and $\gamma = \gamma_1 \cup \gamma_2$. Let

$$f(z) = \frac{z-1}{z^5-1}.$$

The simple poles of f in the upper half-plane are at the points $z_1 = e^{2i\pi/5}$ and $z_2 = e^{4i\pi/5}$. The point $z = 1$ is a removable singularity of f. Therefore

$$\oint_\gamma \frac{z-1}{z^5-1}\,dz = 2i\pi\Big(\mathrm{Res}\,[f, z_1] + \mathrm{Res}\,[f, z_2]\Big)$$
$$= 2i\pi\Big(\lim_{z\to z_1} f(z)(z-z_1) + \lim_{z\to z_2} f(z)(z-z_2)\Big)$$

By using l'Hopital's rule we find

$$\lim_{z\to z_1} f(z)(z-z_1) = \frac{[(z-1)(z-z_1)]'}{[(z^5-1)]'}\Big|_{z=z_1} = \frac{e^{2i\pi/5}-1}{5\,(e^{2i\pi/5})^4}$$
$$= \frac{e^{2i\pi/5}(e^{2i\pi/5}-1)}{5\,e^{2i\pi}} = \frac{e^{4i\pi/5}-e^{2i\pi/5}}{5}$$

and also

$$\lim_{z\to z_2} f(z)(z-z_2) = \frac{e^{4i\pi/5}-1}{5(e^{4i\pi/5})^4} = \frac{e^{8i\pi/5}-e^{4i\pi/5}}{5}.$$

Therefore

$$\oint_\gamma \frac{z-1}{z^5-1}\,dz = \frac{2i\pi}{5}\Big(e^{4i\pi/5} - e^{2i\pi/5} + e^{8i\pi/5} - e^{4i\pi/5}\Big)$$
$$= \frac{2i\pi}{5}\Big(-e^{2i\pi/5} + e^{-2i\pi/5}\Big)$$
$$= -\frac{2i\pi}{5}\,2i\,\sin(2\pi/5) = \frac{4\pi}{5}\,\sin(2\pi/5).$$

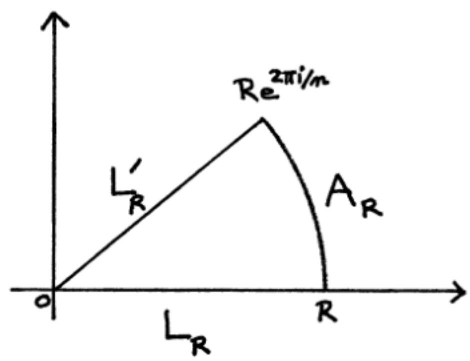
Moreover, if $z \in \gamma_2$, then by using the ML inequality we obtain

$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \le \int_{\gamma_2} |f(z)|\,dz = \int_{\gamma_2}\Big|\frac{z-1}{z^5-1}\Big| \le \pi R \max_{z\in\gamma_2}\Big|\frac{z-1}{z^5-1}\Big|$$
$$\le \pi R\,\frac{R+1}{R^5-1} \to 0, \quad \text{as} \quad R\to\infty.$$

Thus

$$\int_{-\infty}^{\infty} \frac{x-1}{x^5-1}\,dx = \lim_{R\to\infty}\Big(\oint_\gamma \frac{z-1}{z^5-1}\,dz - \int_{\gamma_2} f(z)\,dz\Big) = \frac{4\pi}{5}\,\sin(2\pi/5).$$

8) Consider
$$\oint_\gamma e^{iz^2}\,dz,$$
where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$,



$$\gamma_1 = \{z : z = x + i0,\ 0 < x < R\},$$
$$\gamma_2 = \{z : z = Re^{i\theta},\ 0 \le \theta \le \pi/4\},$$
$$\gamma_3 = \{z : z = te^{i\pi/4},\ t \in (R, 0]\}.$$

Since e^{iz^2} is holomorphic we obtain
$$\oint_\gamma e^{iz^2}\,dz = \int_{\gamma_1} e^{iz^2}\,dz + \int_{\gamma_2} e^{iz^2}\,dz + \int_{\gamma_3} e^{iz^2}\,dz =: I_1 + I_2 - I_3 = 0.$$
Note that
$$I_1 = \int_0^R e^{ix^2}\,dx \quad \text{and} \quad I_3 = \int_0^R e^{i(e^{i\pi/4}t)^2}\,e^{i\pi/4}\,dt = \frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}\,dt.$$
Now,
$$|I_2| = \left|\int_{\gamma_2} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2e^{i2\theta}}\,Ri\,e^{i\theta}d\theta\right|$$
$$\le R\int_0^{\pi/4}\left|e^{iR^2(\cos 2\theta + i\sin 2\theta)}\right|d\theta = R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta.$$
It is known that $\sin 2\theta \ge 4\theta/\pi$ **(show this)** and therefore
$$|I_2| \le R\int_0^{\pi/4} e^{-R^2 4\theta/\pi}\,d\theta = \frac{\pi}{4R}\,(1 - e^{-R^2}) \to 0, \quad R \to \infty.$$

Therefore

$$\int_0^\infty e^{ix^2}\,dx = \lim_{R\to\infty}\int_0^R e^{ix^2}\,dx = \lim_{R\to\infty} I_3 = \lim_{R\to\infty}\frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}\,dt = \frac{(1+i)\sqrt{\pi}}{2\sqrt{2}},$$

where we used $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$.

Finally comparing real parts we obtain

$$\int_0^\infty \cos(x^2)\,dx = \sqrt{\pi/8}.$$