Imperial College
London

MATH70011

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Classical Dynamics

Date: 30 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) The force on a particle of mass m is

$$\mathbf{F} = ax^2y\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k},$$

where a is a constant. For what a is the force conservative? For this value of a give a Lagrangian for the motion of the particle.

(6 marks)

(b) Consider the canonical transformation

$$Q = \sqrt{2p}\cos q, \quad P = -\sqrt{2p}\sin q.$$

Verify that $\{Q, P\} = 1$ and obtain a (type-one) generating function for the transformation.

(8 marks)

(c) Find a complete solution of the Hamilton-Jacobi equation for the Hamiltonian

$$H = (1 + q^2)p^2.$$

The following integral may be useful

$$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + \text{constant}.$$

(6 marks)

(Total: 20 marks)

2. A particle of unit mass moves without friction on a collapsing sphere of radius e^{-t} (the centre of the sphere is fixed).

(a) Assume there is no gravitational force acting on the particle.

(i) Using spherical polar angles θ and ϕ as generalised coordinates, show that one of the equations of motion is

$$\ddot{\theta} - 2\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0,$$

and obtain the other equation of motion.
Hint: in spherical polar coordinates r, θ, ϕ the kinetic energy of a particle of mass m has the form

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right).$$

(8 marks)

(ii) Find solutions of the equations of motion where $\theta = \pi/2$. For these solutions determine the kinetic energy as a function of time. (7 marks)

(b) What are the equations of motion if gravity is included? The acceleration due to gravity is the constant g.

(5 marks)

(Total: 20 marks)

3. A bead of unit mass moves on a heart-shaped wire $r = \lambda(2 - \sin\theta)$ without friction or gravity where r and θ are polar coordinates and λ is a positive constant.

(a) Obtain a Lagrangian for the motion of the bead.

Hint: in polar coordinates the kinetic energy of a particle of mass m is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

(5 marks)

(b) Using your result from part (a), or otherwise, show that a suitable Hamiltonian is

$$H = \frac{p_\theta^2}{2\lambda^2(5 - 4\sin\theta)}.$$

(5 marks)

(c) Express the energy as a function of the action variable

$$J = \oint p_\theta \, d\theta.$$

Hint: Your result should include a numerical constant defined as an integral. An explicit calculation of the integral is not required.

(5 marks)

(d) Suppose that λ is slowly doubled (adiabatically). What happens to the frequency?

(5 marks)

(Total: 20 marks)

4. The Hamiltonian for a symmetric top is

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta.$$

Here ϕ, θ, ψ are Euler angles, I_1 and I_3 are principal moments of inertia, M is the total mass, ℓ is the separation of the centre of mass and the fixed point and g is the acceleration due to gravity.

(a) Show that p_ϕ and p_ψ are constants of the motion.

(2 marks)

(b) Show that θ satisfies the equation of motion

$$I_1^2\ddot{\theta} = f(\cos\theta, p_\phi, p_\psi)\sin\theta,$$

where

$$f(u, p_\phi, p_\psi) = -\frac{p_\psi(p_\phi - p_\psi u)}{(1-u^2)} + \frac{(p_\phi - p_\psi u)^2 u}{(1-u^2)^2} + Mg\ell I_1.$$

(6 marks)

(c) Show that if

$$p_\phi p_\psi = Mg\ell I_1, \tag{1}$$

then $\theta = \pi/2$ (a precessing top) is a solution of Hamilton's equation. (6 marks)

(d) Determine the frequency of small oscillations (nutation) about $\theta = \pi/2$ assuming (1) from part (c) holds.

(6 marks)

(Total: 20 marks)

5. The time evolution of a physical system is governed by the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + q^2 + \frac{1}{q^2}\right).$$

Use the canonical transformation

$$Q = \log q, \qquad P = qp,$$

to solve Hamilton's equations for the system.

Hint: determine the new Hamiltonian $K(Q,P)$. Consider $\dot{P} - 2K$ and show that $\ddot{P}$ is proportional to P.

(20 marks)

(Total: 20 marks)

Classical Dynamics Solutions (2022-2023)

1. (a) The force on a particle of mass m is

$$\mathbf{F} = ax^2y\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}.$$

The force is conservative if $a = 3$ as $V = -x^3y - \frac{1}{4}z^4$ is a potential energy function. A suitable Lagrangian is

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + x^3y + \frac{1}{4}z^4.$$

(6 marks, seen similar A)

(b)

$$Q = -\sqrt{2p}\cos q, \quad P = \sqrt{2p}\sin q.$$

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \sqrt{2p}\sin q \cdot \frac{1}{\sqrt{2p}}\sin p - -\frac{1}{\sqrt{2p}}\cos q \cdot \sqrt{2p}\cos q$$
$$= \sin^2 q + \cos^2 q = 1.$$

$$p = \frac{Q^2}{2\cos^2 q} = \frac{\partial F}{\partial q}, \quad P = \sqrt{2p}\sin q = -\frac{Q}{\cos q}\cdot\sin q = -Q\tan q = -\frac{\partial F}{\partial Q},$$

with solution $F(q, Q) = \frac{1}{2}Q^2\tan q$. (8 marks, seen similar B)

(c) Hamilton-Jacobi

$$(1 + q^2)\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

Writing $S = W(q, \alpha) - \alpha t$ yields

$$(1 + q^2)\left(\frac{\partial W}{\partial q}\right)^2 = \alpha,$$

or

$$W = \sqrt{\alpha}\int \frac{dq}{\sqrt{1 + q^2}} = \sqrt{\alpha}\sinh^{-1} q,$$

dropping an additive constant.

(6 marks, seen similar B)

(Total: 20 marks)

2. (a) (i) The system is a free particle moving in three dimensions subject to the holonomic constraint $r = e^{-t}$. The unconstrained Lagrangian is the kinetic energy

$$L = T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$

The constraint is solved by inserting $r = e^{-t}$ giving

$$L = \frac{1}{2}\left[e^{-2t} + e^{-2t}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right].$$

The first term, $\frac{1}{2}e^{-2t}$, is independent of the coordinates θ and ϕ and does not affect the equations of motion. Dropping this gives

$$L = \frac{1}{2}e^{-2t}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right).$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(e^{-2t}\dot{\theta}\right) - e^{-2t}\sin\theta\cos\theta\dot{\phi}^2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \frac{d}{dt}\left(e^{-2t}\sin^2\theta\dot{\phi}\right) = 0.$$

These simplify to

$$\ddot{\theta} - 2\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0,$$

as required, and

$$\ddot{\phi} - 2\dot{\phi} + 2\cot\theta\ \dot{\theta}\dot{\phi} = 0.$$

(8 marks, seen similar A)

(ii) $\theta = \pi/2$ satisfies the first (given) equation and the second equation reduces to $\ddot{\phi} = 2\dot{\phi}$ with solution $\dot{\phi} = Ae^{2t}$. Integrating gives $\phi = \frac{1}{2}Ae^{2t} + B$ where A and B are constants. For these solutions the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) = \frac{1}{2}\left(e^{-2t} + A^2e^{2t}\right).$$

(7 marks, seen similar C)

(b) The (unconstrained) potential energy is $V = mgz = gz = gr\cos\theta$. Imposing the constraint, $V = e^{-t}g\cos\theta$, giving the constrained Lagrangian

$$L = \frac{1}{2}e^{-2t}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - e^{-t}g\cos\theta.$$

The equation of motion for ϕ is the same as in part (a). The Euler-Lagrange equation for θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}e^{-2t}\dot{\theta} - e^{-t}\sin\theta\cos\theta\dot{\phi}^2 - ge^{-t}\sin\theta,$$

which simplifies to

$$\ddot{\theta} - 2\dot{\theta} - \sin\theta\cos\theta\;\dot{\phi}^2 - ge^{t}\sin\theta = 0.$$

(5 marks, unseen D)

(Total: 20 marks)

3. (a) Here $\dot{r} = -\lambda \cos\theta\ \dot{\theta}$. Accordingly,

$$L = T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}\lambda^2\left[\cos^2\theta + (2-\sin\theta)^2\right]\dot{\theta}^2$$

$$= \frac{1}{2}\lambda^2\left(5 - 4\sin\theta\right)\dot{\theta}^2.$$

(5 marks, seen similar A)

(b) $p_\theta = \partial L/\partial\dot{\theta} = \lambda^2(5-4\sin\theta)\dot{\theta}$ and

$$H = p_\theta\dot{\theta} - L = \frac{p^2}{\lambda^2(5-4\sin\theta)} - \frac{p^2}{2\lambda^2(5-4\sin\theta)} = \frac{p^2}{2\lambda^2(5-4\sin\theta)}.$$

(5 marks, seen similar A)

(c) $p = \pm\sqrt{2E}\lambda\sqrt{5-4\cos\theta}$ so that

$$J = \oint p_\theta\ d\theta = \pm\sqrt{2E}\lambda C,$$

where

$$C = \int_0^{2\pi}\sqrt{5-4\sin\theta}\ d\theta,$$

is a constant. Accordingly,

$$E = \frac{J^2}{2C^2\lambda^2}.$$

(5 marks, seen similar C)

(d) The frequency is

$$\nu = \frac{dE}{dJ} = \frac{J}{C^2\lambda^2}.$$

Under an adiabatic doubling of λ, J does not change so the frequency is a quarter of its initial value. (5 marks, seen similar D)

(Total: 20 marks)

4.
$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta.$$

(a) Two of Hamilton's equations are

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = 0, \qquad \dot p_\psi = -\frac{\partial H}{\partial \psi} = 0,$$

so that p_ϕ and p_ψ are constants of the motion. (2 marks, seen similar A)

(b) Hamilton's equations give

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I_1},$$

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = -\frac{p_\psi\sin\theta(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta} + \frac{\cos\theta(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^3\theta} + Mg\ell\sin\theta.$$

Accordingly,

$$\begin{aligned} I_1^2\ddot\theta = I_1\dot p_\theta &= -\frac{p_\psi\sin\theta(p_\phi - p_\psi\cos\theta)}{\sin^2\theta} + \frac{\cos\theta(p_\phi - p_\psi\cos\theta)^2}{\sin^3\theta} + MgI_1\ell\sin\theta \\ &= \left[-\frac{p_\psi(p_\phi - p_\psi\cos\theta)}{1-\cos^2\theta} + \frac{\cos\theta(p_\phi - p_\psi\cos\theta)^2}{(1-\cos^2\theta)^2} + MgI_1\ell\right]\sin\theta \\ &= f(\cos\theta, p_\phi, p_\psi)\sin\theta, \end{aligned}$$

as required. (6 marks, seen similar A)

(c) If $\theta = \pi/2$, $\dot\theta = 0$ and so (from $\dot\theta = \partial H/\partial p_\theta = p_\theta/I_1$) $p_\theta = 0$. Inserting this together with $\cos\theta = 0$ and $\sin\theta = 1$ into the equation for $I_1^2\ddot\theta$ above, yields

$$0 = -\frac{p_\psi p_\phi}{I_1} + 0 + Mg\ell,$$

which holds if $p_\phi p_\psi = Mg\ell I_1$ (allowed as p_ϕ and p_ψ are constants of the motion).

(6 marks, seen similar B)

(d) For small u (neglecting terms of order u^2)

$$f(u, p_\phi, \psi) = -p_\psi(p_\phi - up_\psi) + up_\phi^2 + MgI_1\ell = u(p_\psi^2 + p_\phi^2)$$

Now $u = \cos\theta = -\sin(\theta - \pi/2) \approx -(\theta - \pi/2)$; for $\theta - \pi/2$ small

$$I_1^2 \frac{d^2}{dt^2}(\theta - \pi/2) = -(p_\psi^2 + p_\phi^2)(\theta - \pi/2),$$

with frequency $\sqrt{p_\psi^2 + p_\phi^2}/I_1$.

(6 marks, seen similar D)

(Total: 20 marks)

5. As the canonical transformation is independent of time the new Hamiltonian is

$$K = H = \frac{1}{2}\left(P^2 e^{-2Q} + e^{2Q} + e^{-2Q}\right)$$

Now

$$\dot{P} = -\frac{\partial K}{\partial Q} = P^2 e^{-2Q} - e^{2Q} + e^{-2Q} = 2K - 2e^{2Q}$$

As K is a constant of the motion

$$\ddot{P} = 0 - 4e^{2Q}\dot{Q} = -4e^{2Q}\frac{\partial K}{\partial P} = -4P,$$

with general solution

$$P = A\cos(2t+\beta). \tag{1}$$

$$q^2 = e^{2Q} = E - \frac{1}{2}\dot{P} = E + A\sin(2t+\beta),$$

so that

$$q(t) = \sqrt{E + A\sin(2t+\beta)}, \qquad p(t) = \frac{qp}{q} = \frac{A\cos(2t+\beta)}{\sqrt{E + A\sin(2t+\beta)}}. \tag{2}$$

The constant A can be fixed using

$$E = \frac{1}{2}\left(p^2 + q^2 + \frac{1}{q^2}\right) = \frac{1}{2q^2}\left((qp)^2 + q^4 + 1\right)$$

$$= \frac{1}{2q^2}\left(A^2\cos^2(2t+\beta) + E^2 + 2EA\sin(2t+\beta) + A^2\sin^2(2t+\beta) + 1\right)$$

$$= \frac{1}{2q^2}\left(A^2 + E^2 + 1 + 2EA\sin(2t+\beta)\right),$$

which requires $A^2 + 1 = E^2$ or $A = \pm\sqrt{E^2-1}$. The negative sign may be discarded as flipping the sign is equivalent to shifting β by π. Accordingly,

$$q(t) = \sqrt{E + \sqrt{E^2-1}\sin(2t+\beta)}, \quad p(t) = \frac{\sqrt{E^2-1}\cos(2t+\beta)}{\sqrt{E + \sqrt{E^2-1}\sin(2t+\beta)}}.$$

In marking the question it is expected that 12 marks would be awarded for a correct solution up to (1) and 16 marks for a correct solution up to (2).

(Total: 20 marks)

Category A marks: 1(a), 2(a)(i), 3(a)(b), 4(a)(b) 6+8+10+8=32

Category B marks: 1(b)(c), 4(c) 8+6+6=20

Category C marks: 1(c), 2(a)(ii), 3(c) 7+5=12

Category D marks: 2(b), 3(d), 4(d) 5+5+6=16

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60011/70011	1	This question was well answered by most students.
MATH60011/70011	2	Some candidates attempted to implement the constraint after determining the Euler Lagrange equations which is generally wrong (although in this case it is possible to recover the correct answers).In part (b) lack of knowledge of spherical polar coordinates hindered some candidates. This did not affect part (a) as the kinetic energy formula was given.
MATH60011/70011	3	Parts (a) and (b) were well answered. In part (c) some candidates got the range of integration wrong in the numerical constant.
MATH60011/70011	4	Parts (a) (b) and (c) were generally well-answered despite the slightly messy analysis.As expected part (d) proved much more difficult.
MATH70011	5	The Mastery question was a short unseen dynamics problem. The hint provided seemed to provide enough guidance so that the question was neither too easy nor too difficult. The marking scheme (which was devised in advance of the exam) worked quite well.Candidates who obtained q(t) and p(t) presented solutions with too many arbitrary constants (3 instead of 2).