IMPERIAL

**MATH70059**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Differential Topology**

Date: Thursday, May 30, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer All Questions in 1 Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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1. (a) Compute the de Rham cohomology groups of $T^2 = S^1 \times S^1$. Give a basis for each of these. (8 marks)

(b) Let $\omega$ be a compactly supported $n$-form on $\mathbb{R}^n$, where $n \geq 1$. Is it true that there exists a compactly supported $(n-1)$-form on $\mathbb{R}^n$ whose exterior derivative is $\omega$? Justify your answer. (4 marks)

(c) Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Suppose that the curl of $V$ is zero, i.e.,

$$\operatorname{curl}(V) = \left( \frac{\partial V^3}{\partial x^2} - \frac{\partial V^2}{\partial x^3}, \frac{\partial V^1}{\partial x^3} - \frac{\partial V^3}{\partial x^1}, \frac{\partial V^2}{\partial x^1} - \frac{\partial V^1}{\partial x^2} \right) = 0.$$

Show that there is a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $V = \nabla f$. (8 marks)

(Total: 20 marks)

2. (a) Let $M$ and $N$ be connected, compact, oriented manifolds of dimension $n \geq 1$. Let $F : M \to N$ be smooth.

(i) Define the degree of $F$. (2 marks)

(ii) Suppose $M$ is the boundary of a connected, compact, oriented $(n+1)$-manifold $X$, and that $F$ extends smoothly to a map $\tilde{F} : X \to N$. What is the degree of $F$? (5 marks)

(b) Consider the 2-torus $T^2 = S^1 \times S^1$. Let $S$ denote the submanifold

$$S := \{(x, y) \in T^2 : x = y\}.$$

Determine the mod 2 intersection number of $S$ with itself. (5 marks)

(c) Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = e^{|z|^2} z + \cos(|z|^2)(z^9 + 25).$$

Show that there is at least one point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$. (8 marks)

(Total: 20 marks)

3. (a) Let $M$ be an oriented smooth manifold of dimension $n$. What is the relationship between the groups $H^p_{dR}(M)$ and $H^{n-p}_c(M)^*$? (2 marks)

(b) (i) Define the Betti numbers $b_p(M)$ and the Euler characteristic $\chi(M)$ of a compact smooth manifold.

(2 marks)

(ii) Compute the Euler characteristic of a compact, oriented, smooth manifold of dimension seven. (2 marks)

(c) Let $M$ and $N$ be smooth manifolds of dimension $k$ and $l$ respectively. Suppose $k$ and $l$ are both at least one, and let $n = k + l$. Show that the product $M \times N$ is not diffeomorphic to the $n$-dimensional sphere. (6 marks)

(d) Consider the open halfplane

$$H = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

The subset $M \subset H$ given by

$$M = H \setminus \{(\tfrac{1}{n}, 0) \in \mathbb{R}^2 : n = 1, 2, 3, \dots\}$$

is open, and hence is a submanifold. Show that $H^1_{dR}(M)$ is not finite-dimensional.

(8 marks)

(Total: 20 marks)

4. (a) Let $M \subset \mathbb{R}^{12}$ be a smooth submanifold of dimension 4. Show that there is a constant vector field on $\mathbb{R}^{12}$ whose projection onto the normal space $N_pM$ is nonzero for every $p \in M$.

(6 marks)

(b) Let $M$ and $N$ be smooth manifolds, both of dimension at least one, and consider a smooth map $F : M \to N$.

(i) Suppose $F$ is an immersion. Is it then true that $F$ has at least one regular value in $N$?

(3 marks)

(ii) Let $U$ be an open subset of $N$ and assume $U \subset F(M)$. Is it possible that every point in $U$ is a critical value of $F$? (4 marks)

(c) Let $\Sigma_1$ and $\Sigma_2$ be codimension-$n$ submanifolds of $\mathbb{R}^{2n}$, where $n \geq 1$. Suppose $\Sigma_1$ is compact. A submanifold $\Sigma \subset \mathbb{R}^{2n}$ is said to be a translate of $\Sigma_1$ if there is a $v \in \mathbb{R}^{2n}$ such that

$$\Sigma = \{x + v : x \in \Sigma_1\}.$$

Show that every tubular neighbourhood of $\Sigma_1$ contains a translate of $\Sigma_1$ which intersects $\Sigma_2$ in at most finitely many points. (7 marks)

(Total: 20 marks)

5. (a) Consider the sphere $S^{2n}$, where $n \geq 1$. Give an example of a tangent vector field on $S^{2n}$ whose index sum is 2. Conclude that every tangent vector field on $S^{2n}$ must vanish somewhere.

(5 marks)

(b) Let $M$ be a compact manifold, and consider a smooth section of the tangent bundle $f : M \to TM$. Let $M_0$ denote the submanifold of $TM$ traced out by its zero section. Show that the mod 2 intersection number of $f$ with $M_0$ is equal to $\chi(M)$ mod 2.

(8 marks)

(c) Let $M$ be a compact $n$-dimensional submanifold of $\mathbb{R}^{n+1}$, where $n$ is assumed to be even. Let $G : M \to S^n$ denote the (outward pointing) Gauss map of $M$. Show that $\deg(G) = \frac{1}{2}\chi(M)$. (**Hint:** Consider an appropriate tubular neighbourhood of $M$. You will also need the fact that the antipodal map of $S^n$ is orientation reversing when $n$ is even.)

(7 marks)

(Total: 20 marks)

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# MATH70059

# Differential Topology (Solutions)

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1. (a) Since $T^2$ is connected we have $H^0(T^2) \cong \mathbb{R}$. A generator is given by any nonzero constant function. Since $T^2$ is also compact and orientable we know from the lectures that $H^2(T^2) \cong \mathbb{R}$, and a generator is given by any orientation 2-form. Finally, we use the Künneth formula to obtain

meth seen ⇓

$$H^1(T^2) \cong H^0(S^1) \otimes H^1(S^1) \oplus H^1(S^1) \otimes H^0(S^1) \cong \mathbb{R}^2.$$

If $\theta$ is an orientation form on $S^1$ then the forms $(\theta, 0)$ and $(0, \theta)$ induce a basis for $H^1(T^2)$ (indeed, both forms are clearly closed, and they fail to integrate to zero on $S^1 \times \{p\}$ and $\{p\} \times S^1$, respectively, so they are not exact). 8, A

(b) Since $\omega$ is an $n$-form it is automatically closed, and hence induces a cohomology class $[\omega]_c \in H^n_c(\mathbb{R}^n)$. We showed in lectures that the integration map $[\omega]_c \mapsto \int_{\mathbb{R}^n} \omega$ is an isomorphism from $H^n_c(\mathbb{R}^n)$ to $\mathbb{R}$. So $[\omega]_c = 0$, i.e. $\omega = d\gamma$ for some compactly supported $\gamma$, if and only if $\int_{\mathbb{R}^n} \omega = 0$. 4, A

seen ⇓

(c) We define a 1-form $\omega$ by

unseen ⇓

$$\omega = V^1 dx^1 + V^2 dx^2 + V^3 dx^3.$$

Direct computation gives

$$d\omega = \left( \frac{\partial V^2}{\partial x^1} - \frac{\partial V^1}{\partial x^2} \right) dx^1 \wedge dx^2 + \left( \frac{\partial V^3}{\partial x^1} - \frac{\partial V^1}{\partial x^3} \right) dx^1 \wedge dx^3 + \left( \frac{\partial V^3}{\partial x^2} - \frac{\partial V^2}{\partial x^3} \right) dx^2 \wedge dx^3.$$

Since the curl of $V$ is zero we conclude that $\omega$ is closed. By the Poincaré lemma, $H^1(\mathbb{R}^3) = 0$, so $\omega$ is also exact. That is, there is a smooth function $f$ such that $df = \omega$. In other words, $\nabla f = V$. 8, B

2. (a) (i) The degree of $F$, denoted $\deg(F)$, is the unique integer such that $\int_M F^*\omega = \deg(F)\int_N \omega$ for every $\omega \in \Omega^n(N)$. seen ⇓ 2, A

(ii) We may write $F = \tilde{F} \circ \iota$, where $\iota$ is the inclusion of $M$ in $X$. Using Stokes' theorem we compute

$$\int_M F^*\omega = \int_M \iota^*\tilde{F}^*\omega = \int_X d\tilde{F}^*\omega = \int_X \tilde{F}^* d\omega = 0$$

for every $\omega \in \Omega^n(N)$. By the definition of the degree, it follows that $\deg(F) = 0$. 5, A

(b) We are interested in the mod 2 intersection number of $S$ with itself. It suffices to compute the mod 2 intersection number of $\tilde{S}$ with $S$, where $\tilde{S}$ is homotopic to $S$ and intersects $S$ transversally. But it is easy to construct $\tilde{S}$ so that it does not intersect $S$ at all. This shows that the mod 2 intersection number is zero. One way to construct $\tilde{S}$ is as follows. Let $\tilde{S} := \{(R_\theta x, x) \in T^2 : x \in S^1\}$, where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a clockwise rotation by $\theta \in (0, 2\pi)$ and we are identifying $S^1$ with the unit circle in $\mathbb{R}^2$. unseen ⇓ 5, C

(c) We view $z \in \mathbb{C}$ as the point $(\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^2$ when it is convenient. Note that $f$ defines a vector field on $\mathbb{R}^2$, namely $v(z) := (\operatorname{Re} f(z), \operatorname{Im} f(z))$. Suppose $v$ has no zeroes in the ball $B_R(0)$. Then $\hat{v}(z) := v(z)/|v(z)|$ defines a smooth map from $B_R(0)$ into $S^1$. The restriction of this map to $\partial B_R(0)$ has degree zero (by, e.g., (a)(ii) immediately above). On the other hand, if $\hat{v}$ points outwards at every point of $\partial B_R(0)$, then its restriction to $\partial B_R(0)$ must have degree one, since it is homotopic to the identity on $\partial B_R(0)$. The outward unit normal to $\partial B_R(0)$ is $z/R$. So we will have arrived at a contradiction if we can show that $v(z) \cdot z$ is positive for every $z \in \partial B_R(0)$, for some sufficiently large $R > 0$. Observe that unseen ⇓

$$v(z) \cdot z = \operatorname{Re}(f(z)\bar{z}).$$

We have

$$f(z)\bar{z} = e^{|z|^2}|z|^2 + \cos(|z|^2)(z^9 + 25)\bar{z}.$$

Since

$$|\cos(|z|^2)(z^9 + 25)\bar{z}| \le |z|^{10} + 25|z| < e^{|z|^2}|z|^2$$

whenever $|z| = R$ is sufficiently large, we see that

$$\operatorname{Re}(f(z)\bar{z}) > 0$$

whenever $|z| = R$ is sufficiently large. This gives the desired contradiction, so $v$ (and hence $f$) must vanish somewhere. 8, D

3. (a) Poincaré duality asserts that

$$H^p_{dR}(M) \cong H^{n-p}_c(M)^*$$

for every oriented smooth manifold $M$.

seen ⇓

2, A

(b) (i) For a compact smooth manifold $M$ each of the groups $H^p_{dR}(M)$ is finite-dimensional. The $p$-th Betti number of $M$ is

seen ⇓

$$b_p = \dim(H^p_{dR}(M)).$$

The Euler characteristic of $M$ is

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p.$$

2, A

(ii) Let $M$ be a compact oriented 7-manifold. By Poincaré duality we have $b_p = b_{7-p}$ for each $p$, and hence

meth seen ⇓

$$\chi(M) = (b_0 - b_7) - (b_1 - b_6) + (b_2 - b_5) - (b_3 - b_4) = 0.$$

2, A

(c) We proceed by contradiction. Suppose $M \times N$ is diffeomorphic to $S^n$. Then any orientation form on $S^n$ can be pulled back by a diffeomorphism and restricted to each factor to yield orientation forms on $M$ and $N$. So we see that $M$ and $N$ must be orientable. We also know that $M \times N$ is connected and compact, since $S^n$ is connected and compact. It follows that $M$ and $N$ are both connected and compact as well. We conclude that $H^0_{dR}(M) \cong \mathbb{R}$ and $H^l_{dR}(N) \cong \mathbb{R}$. The Künneth formula tells us that

seen ⇓

$$H^l(M \times N) \cong H^0_{dR}(M) \otimes H^l_{dR}(N) \oplus \ldots \cong \mathbb{R} \oplus \ldots,$$

so the dimension of $H^l(M \times N)$ is at least one. But $1 \leq l < n$, and $n \geq 2$, so by assumption

$$0 = H^l(S^n) \cong H^l(M \times N).$$

This is a contradiction.

6, B

(e) We claim that $H^1_{dR}(M)$ is not finite-dimensional. To see this we apply the Mayer–Vietoris theorem. Let

unseen ⇓

$$U = M \cap \{(x, y) : x > \tfrac{1}{2} + 10^{-1}\}, \qquad V = M \cap \{(x, y) : x < 1 - 10^{-1}\}.$$

Then $M = U \cup V$, $U \cap V$ is contractible and $U$ is homotopy equivalent to $S^1$. Moreover, it is not difficult to see that $V$ is homotopy equivalent to $M$. By the Mayer–Vietoris theorem,we have an exact sequence

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M)$$
$$\to H^1(U) \oplus H^1(V) \to H^1(U \cap V).$$

We use: the fact that $U$, $V$ and $U \cap V$ are all connected; $H^1(U) \cong \mathbb{R}$; $H^p(V) \cong H^p(M)$; and $H^1(U \cap V) = 0$, to see that we have an exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(M) \to \mathbb{R} \oplus H^1(M) \to 0.$$

We read off that the map $H^1(M) \to \mathbb{R} \oplus H^1(M)$ is both injective and surjective. But no such map exists if $H^1(M)$ is finite-dimensional, so this proves the claim.

8, D

4. (a) Consider $Z := TM \setminus \{(x,v) \in TM : x \in M,\ v = 0\}$. Let $F : Z \to \mathbb{RP}^{11}$ be the map sending $(x,v)$ to $[v]$, where $[v]$ is the equivalence class of nonzero vectors parallel to $v$. Then $F$ is smooth, and since the dimension of $Z$ is strictly less than that of $\mathbb{RP}^{11}$, $F(Z)$ has measure 0 in $\mathbb{RP}^{11}$. In particular, the complement of $F(Z)$ is dense in $\mathbb{RP}^{11}$. Let $[v_0]$ be any point in $\mathbb{RP}^{11} \setminus F(Z)$. By construction, the vector field $x \mapsto v_0(x)$ is never tangent to $M$, so its normal component is always nonzero. meth seen ⇓

6, B

(b) (i) No. If $F$ is an immersion then the rank of $dF$ equals $\dim M$ everywhere. Suppose $\dim M < \dim N$. Then $F$ has no regular values. seen ⇓

3, A

(ii) Sard's theorem asserts that the set of critical values of $F$ has measure 0 in $N$. No open subset of $N$ has measure 0, so it is not possible that every point in $U$ is a critical value.

4, A

(c) Let $\iota : \Sigma_1 \to \mathbb{R}^{2n}$ denote the inclusion map. Define $F : \Sigma_1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $F(x,v) = \iota(x) + v$. Clearly $F$ is a submersion, since $dF(x,v)(0,w) = w$ for every $w \in \mathbb{R}^{2n}$. In particular, $F$ is transverse to $\Sigma_2$. By parametric transversality, $F(\cdot, v)$ is transverse to $\Sigma_2$ for almost every $v \in \mathbb{R}^{2n}$. Choose a sequence $v_k \to 0$ such that $F(\cdot, v_k)$ is transverse to $\Sigma_2$ for every $k$. Then the image of $F(\cdot, v_k)$ is a translate of $\Sigma_1$ which intersects $\Sigma_2$ in at most finitely many points (since the intersection is a codimension-two submanifold of $\Sigma_1$). Moreover, given any tubular neighbourhood of $\Sigma_1$, $F(\cdot, v_k)$ will map into that neighbourhood when $k$ is large. unseen ⇓

7, C

5. (a) We may view $S^{2n}$ as the unit sphere in $\mathbb{R}^{2n+1}$. Let $v$ be a constant unit vector on $\mathbb{R}^{2n+1}$, and define a tangent vector field $u$ on $S^{2n}$ by projecting onto the tangent space at each point. Explicitly, $u(x) = v - (x \cdot v)x$. Clearly $u$ has isolated zeroes at the antipodes $v$ and $-v$. Choosing a chart near $v$ so that we may view $u$ as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, we see that $du$ is -1 times the identity at $v$, so its determinant is $(-1)^{2n} = 1$, and hence the index of $u$ at this zero is 1. At $-v$ we have that $du$ equals the identity, so the index of $u$ at $-v$ is also 1. Thus the index sum of $u$ is 2. The index sum is the same for every smooth tangent vector field with isolated zeroes, and a smooth tangent vector field with no zeroes would have index sum equal to zero. So we conclude that every smooth tangent vector field vanishes somewhere. $\boxed{5, \mathrm{M}}$

(b) Any two sections of $TM$ are homotopic (a homotopy can be constructed by a fiberwise linear interpolation), and the mod 2 intersection number is invariant under homotopy. Moreover, there exists a section of $TM$ which intersects $M_0$ transversally. Therefore, we may assume without loss of generality that $f$ intersects $M_0$ transversally. Since $M$ is compact, and $M_0$ is diffeomorphic to $M$, we then have that $f^{-1}(0)$ is some finite set of points. In addition, the zeroes of $f$ are all nondegenerate, and so their indices are all $\pm 1$. The Poincaré–Hopf theorem asserts that the index sum of $f$ is equal to $\chi(M)$. So the index sum mod 2 is equal to $\chi(M)$ mod 2. But since the indices are all $\pm 1$, the index sum mod 2 is just the number of zeroes mod 2, i.e., the mod 2 intersection number of $f$ with $M_0$. $\boxed{8, \mathrm{M}}$

(c) Let $N$ denote the set of points in $\mathbb{R}^{n+1}$ whose distance to $M$ is at most $\varepsilon$. If $\varepsilon$ is sufficiently small then $N$ is (the closure of) a tubular neighbourhood of $M$. Let $G_N : \partial N \to S^n$ be the (outward pointing) Gauss map of $\partial N$. From the proof of the Poincaré–Hopf theorem, we know that $\deg(G_N) = \chi(M)$. So it suffices to show that $\deg(G_N) = 2\deg(G)$, where $G : M \to S^n$ is the Gauss map of $M$. To see this we first note that $\partial N$ consists of two connected components, namely

$$\partial N^+ = \{x + \varepsilon G(x) : x \in M\}, \qquad \partial N^- = \{x - \varepsilon G(x) : x \in M\}.$$

Let $G_N^\pm$ denote the restriction of $G_N$ to $\partial N^\pm$. The map $F^\pm(x) := x \pm \varepsilon G(x)$ is a diffeomorphism from $M$ to $\partial N^\pm$, and $G_N^\pm \circ F^\pm = \pm G$. The degree of a composition is the product of the degrees, so we have

$$\deg(G_N^\pm)\deg(F^\pm) = \deg(G_N^\pm \circ F^\pm) = \deg(\pm G).$$

The map $F^+$ is an orientation preserving diffeomorphism, so $\deg(F^+) = 1$, and hence

$$\deg(G_N^+) = \deg(G).$$

The map $F^-$ is an orientation reversing diffeomorphism (since $\partial N^-$ carries the opposite orientation to $M$), so $\deg(F^-) = -1$, and the antipodal map on $S^n$ is orientation reversing for $n$ even. Consequently,

$$\deg(G_N^-) = -\deg(-G) = \deg(G).$$

Putting this all together we obtain

$$\deg(G_N) = \deg(G_N^+) + \deg(G_N^-) = 2\deg(G),$$

as required. $\boxed{7, \mathrm{M}}$

**Review of mark distribution:**

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks