Imperial College London

Department of Mathematics
Imperial College London
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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 5

1. In this problem, we consider the 1D wave equation with unit wave speed

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < \pi$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0, \quad \text{for all } t > 0$$

and initial conditions

$$u(x,0) = \sin x + 2\sin 7x \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for all } 0 \le x \le \pi$$

We seek separated solutions of the form $u(x,t) = X(x)T(t)$. Substituting in the wave equation, we obtain

$$XT'' = X''T \Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda^2$$

as we want a solution periodic in x (see boundary conditions). We can integrate these ODEs to conclude that

$$\begin{aligned} X(x) &= A_1 \sin \lambda x + A_2 \cos \lambda x \\ T(x) &= B_1 \sin \lambda t + B_2 \cos \lambda t \end{aligned}$$

Using the boundary conditions will give us conditions on the integration constants and the separation constant λ. Here, we find that

$$\begin{aligned} &u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A_2 = 0 \\ &u(L,t) = 0 \Rightarrow X(L) = 0 \Rightarrow \sin \lambda\pi = 0 \Rightarrow \lambda = n, \quad n \in \mathbf{N}^* \end{aligned}$$

The initial condition further provides us that

$$\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow T'(0) = 0 \Rightarrow B_1 = 0$$

Therefore, the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin nx \cos nt$$

Now, we are left to use the initial conditions to find the coefficients α_n. The initial conditions give us that

$$u(x,0) = \sin x + 2\sin 7x = \sum_{n=1}^{\infty} \alpha_n \sin nx$$

One would normally then calculate the half-range Fourier transform of the initial condition. However, here we can see that the initial condition takes the form of a finite combination of sine functions, so we can equate the coefficients of $\sin nx$ (by orthogonality of the set $\{\sin nx, n \in \mathbb{N}^*\}$ — if not convinced, show it!).

We find that

$$\begin{cases} a_1 = 1 \\ a_7 = 2 \\ a_n = 0, \quad \forall n \in \mathbb{N}^* \setminus \{2, 7\} \end{cases}$$

So the solution to this problem is given by

$$u(x,t) = \sin x \cos t + 2 \sin 7x \cos 7t$$

and is shown in Fig. 1.

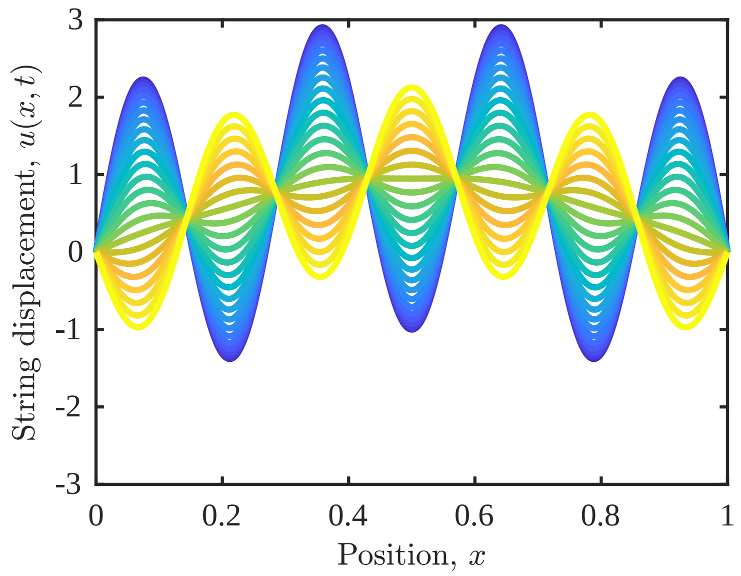

Figure 1: Solution to Q1 shown for $t \in [0, 0.1]$ (from blue to yellow).

2. In this problem, we assume that the function $y(x,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad t > 0, 0 < x < \infty$$

and is subject to the following initial and boundary conditions

$$\frac{\partial y}{\partial x}(0,t) = 0, \quad \text{for } t \geq 0,$$
$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad \text{for } 0 < x < \infty$$

As the problem is posed over $[0, \infty)$, we know that we need to take a half-range Fourier transform. We are faced with the choice of taking either a Fourier sine transform or a Fourier cosine transform. Now recall that

$$\mathcal{F}_c\{f''(x)\} = -f'(0) - \omega^2 \widehat{f}_c(\omega)$$
$$\mathcal{F}_s\{f''(x)\} = \omega f(0) - \omega^2 \widehat{f}_s(\omega)$$

So the sine transform of the second derivative requires the knowledge of $y(0,t)$ (which is here unknown) while the cosine transform involves $\partial y / \partial x$ at $x = 0$, which is given as zero. We therefore take a Fourier cosine transform to get

$$\frac{\partial^2 \widehat{y}_c}{\partial t^2} = -c^2 \omega^2 \widehat{y}_c$$

Solving the differential equation we obtain

$$\widehat{y}_c = A(\omega)\cos(\omega ct) + B(\omega)\sin(\omega ct)$$

The initial conditions impose

$$y(x,0) = 0 \Rightarrow \widehat{y}_c(\omega,0) = 0 \Rightarrow A(\omega) = 0$$

as well as

$$\frac{\partial y}{\partial t}(x,0) = g(x) \Rightarrow \frac{\partial \widehat{y}_c}{\partial t}(\omega,0) = \widehat{g}_c(\omega)$$

Hence, we obtain

$$B(\omega) = \frac{\widehat{g}_c(\omega)}{\omega c}$$

The solution in Fourier space is therefore:

$$\widehat{y}_c(\omega,t) = \frac{\widehat{g}_c(\omega)}{\omega c}\sin(\omega ct)$$

Using the inversion formula for the cosine transform, we find

$$y(x,t) = \frac{2}{\pi c}\int_0^\infty \frac{\widehat{g}_c(\omega)}{\omega}\cos(\omega x)\sin(\omega ct)d\omega$$

3. In this question, we explore the vertical vibrations of a uniform beam (instead of a string). It can be shown that the small vertical vibrations of a uniform beam are governed by the following fourth-order PDE

$$\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^4 u}{\partial x^4} = 0$$

We seek separated solutions of the form $u(x,t) = X(t)T(t)$. Substituting this in the 1D beam equation, we obtain

$$XT'' = -c^2X^{(4)}T \Rightarrow \frac{X^{(4)}}{X} = -\frac{T''}{c^2T} = \beta^4$$

as we are looking for oscillatory behavior. We can integrate these ODEs and conclude that

$$\begin{aligned}
&X^{(4)} = \beta^4 X \Rightarrow X(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x\\
&T'' = -c^2\beta^4 T \Rightarrow T(x) = a\cos\beta^2 ct + b\sin\beta^2 ct
\end{aligned}$$

We assume that our PDE is subject to the following boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(0,t) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(L,t) = 0, \quad \text{for all } t > 0$$

and initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for } 0 < x < L$$

The boundary conditions impose

$$X(0) = X''(0) = 0 \Rightarrow A + C = 0 \text{ and } -A + C = 0 \Rightarrow A = C = 0$$

and

$$\begin{aligned}
X(L) = X''(L) = 0 &\Rightarrow B\sin\beta L + D\sinh\beta L = 0 \text{ and } -B\sin\beta L + D\sin\beta L = 0\\
&\Rightarrow B\sin\beta L = 0 \text{ and } D\sin\beta L = 0\\
&\Rightarrow \beta L = n\pi,\ n\in\mathbb{N}^* \text{ and } D = 0
\end{aligned}$$

Finally, we conclude that

$$X(x) = B \sin\left(\frac{n\pi}{L} x\right)$$

Further, we also impose that $\partial u / \partial t = 0$ at $t = 0$, which implies

$$T'(t) = 0 \Rightarrow b = 0$$

and we conclude that

$$T(t) = a \cos\left(\frac{n^2\pi^2}{L^2} ct\right)$$

So the general solution of this problem reads

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n^2\pi^2}{L^2} ct\right) \sin\left(\frac{n\pi}{L} x\right)$$

Finally the initial conditions require that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L} x\right)$$

and so the coefficients α_n are given by

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

We note that the natural angular frequencies in this problem are given by

$$\omega_n = \frac{n^2\pi^2}{L^2} c \propto n^2$$

while in the vibrating string case (see Q1), we found that $\omega_n \propto n$.

4. In this problem, we want to show the uniqueness of the solution to the 1D wave equation Dirichlet problem. We have used a similar procedure in the lecture notes for the diffusion equation.

   Assume that we have found u_1 and u_2 two possible solutions of the following problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0 \\
&u(0,t) = u(L,t) = 0 \quad t > 0 \\
&u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 < x < L
\end{aligned}$$

   Consider the difference $U = u_1 - u_2$, then U satisfies

$$\begin{aligned}
&\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, t > 0 \\
&U(0,t) = U(L,t) = 0 \quad t > 0 \\
&U(x,0) = 0, \quad \frac{\partial U}{\partial t}(x,0) = 0, \quad 0 < x < L
\end{aligned}$$

   Note that these initial and boundary conditions imply that

$$\frac{\partial U}{\partial x}(x,0) = 0 \quad \text{and} \quad \frac{\partial U}{\partial t}(0,t) = \frac{\partial U}{\partial t}(L,t) = 0$$

We have seen in the lecture notes that the total energy of the vibrating string is given by

$$\varepsilon(t) = \frac{1}{2}\int_0^L \left[\rho_0 \left(\frac{\partial U}{\partial t}\right)^2 + \tau_0 \left(\frac{\partial U}{\partial x}\right)^2\right] dx$$

with $c^2 = \tau_0/\rho_0$, where τ_0 is the tension in the string and ρ_0 is the linear density.
Defining $E(t) = \varepsilon(t)/\rho_0$, we write

$$E(t) = \frac{1}{2}\int_0^L \left[\left(\frac{\partial U}{\partial t}\right)^2 + c^2 \left(\frac{\partial U}{\partial x}\right)^2\right] dx$$

We note in particular that $E(0) = 0$. Now taking a time derivative of $E(t)$, we find

$$\frac{dE}{dt} = \int_0^L \left[\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial t^2} + c^2 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t}\right] dx = \left[c^2 \frac{\partial U}{\partial x}\frac{\partial U}{\partial t}\right]_0^L + \int_0^L \left[\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial t^2} + c^2 \frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial x^2}\right] dx$$

after integrating by parts. The integral on the RHS is zero as U is solution of the wave equation. The integrated term is found to be zero on application of the boundary conditions. Therefore, we have

$$\frac{dE}{dt} = 0$$

So we have $E(t) = E(0) = 0$. Hence, we must have that $\partial U/\partial t$ and $\partial U/\partial x$ identically zero. We conclude that U is at most a constant! However, the initial condition tells us that $U \equiv 0$ and hence the solution is unique!

5. In this problem, we consider a uniform elastic membrane stretched over a rectangular frame of length L and width H. Here, $u(x, y, t)$ denotes the vertical displacement of the membrane at point $(x, y) \in [0, L] \times [0, H]$ from its equilibrium position (assumed to be $u(x, y) = 0$). As the membrane is clamped on the frame, we know that the boundary conditions are given by

$$\begin{cases} u(x, 0, t) = 0 \\ u(x, H, t) = 0 \\ u(0, y, t) = 0 \\ u(L, y, t) = 0 \end{cases}$$

We want to find an expression for the normal modes of vibration of the membrane. As the boundary conditions are homogeneous, the superposition principle applies and so we can try the method of separation of variables. As we are here dealing with a 2D problem, we will need to proceed in two steps.

- ☐ First, we proceed to a separation of variables between time and spatial variables and seek separated solutions of the form:

$$u(x, y, t) = S(x, y)T(t)$$

Substituting this in the 2D wave equation, we obtain

$$c^2(S_{xx}T + S_{yy}T) = ST''$$

where we have the subscript notation for the partial derivatives for the sake of simplicity. We can separate the variables a first time and write

$$\frac{T''}{c^2 T} = \frac{S_{xx} + S_{yy}}{S}$$

The LHS of this equation only depends on t, while the RHS only depends on (x, y). So we know that there exists a question such that

$$\frac{T''}{c^2T} = \frac{S_{xx} + S_{yy}}{S} = -\lambda$$

where we will assume $\lambda > 0$ as we want to get oscillatory behavior. One could check like in the lecture notes all the cases with (i) $\lambda < 0$, (ii) $\lambda = 0$, (iii) $\lambda > 0$. So we get the following ODE for T

$$T'' + c^2\lambda T = 0 \quad (\star)$$

☐ On the other hand, we also get the following PDE

$$S_{xx} + S_{yy} = -\lambda S$$

with boundary conditions

$$S(0, y) = S(L, y) = S(x, 0) = S(x, H) = 0$$

As these boundary conditions are homogeneous, this calls again for a separation of variables. For this problem, we seek solutions of the form

$$S(x, y) = X(x)Y(y)$$

and substituting this in the PDE for S, we obtain

$$X''Y + XY'' = -\lambda XY$$

which we can rewrite as

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

The first term on the LHS only depends on x while the second term only depends on y. We can separate the variables x and y by writing

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu, \quad \mu > 0$$

where the sign of μ is imposed by the BCs on X. This gives me the following set of ODEs to solve:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = X(L) = 0 \quad (\star\star) \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = Y(L) = 0 \quad (\star\star\star) \end{aligned}$$

We can see that unsurprisingly, we have had to introduce a second separation constant μ. This may seem confusing at first but let's just approach this problem systematically.

☐ We know from the 1D problems we have solved that the BCs are setting constraints on the values that (λ, μ) can take. Let us start with the ODE for X as it only involves one separation variable. We know that equation $(\star\star)$ has for general solution

$$X(x) = A\sin(\sqrt{\mu}x) + B\cos(\sqrt{\mu}x)$$

The boundary conditions on X imply that

$$\begin{cases} X(0) = 0 & \Rightarrow B = 0 \\ X(L) = 0 & \Rightarrow \sqrt{\mu}L = n\pi, n = 1, 2, \cdots \end{cases}$$

This means that the family of solutions of $(\star\star)$ is written

$$X_n(x) = A_n \sin(\mu_n x) \quad \text{and} \quad \mu_n = \left(\frac{n\pi}{L}\right)^2$$

☐ Now for each value of μ_n, $(\star\star\star)$ reads

$$Y'' + (\lambda - \mu_n)Y = 0$$

For the sake of simplicity, I can define $\nu_n = \lambda - \mu_n$. We know that in this case, the general solution to $(\star\star\star)$ is written

$$Y(y) = A\sin(\sqrt{\nu_n}y) + B\cos(\sqrt{\nu_n}y)$$

The boundary conditions on Y require that

$$\begin{cases} Y(0) = 0 & \Rightarrow B = 0 \\ Y(H) = 0 & \Rightarrow \sqrt{\nu_n}H = m\pi, \quad m = 1, 2, \cdots \end{cases}$$

where we have introduced a second index $m \in \mathbb{N}^*$. Finally, we go back to λ to find that

$$\lambda - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

so λ can take the values

$$\lambda_{n,m} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

parametrized by the indices $(n, m) \in \mathbb{N}^*$. Finally, we obtain in all generality the family of solutions

$$S_{n,m}(x, y) = X_n(x)Y_{n,m}(y) = \alpha_{n,m}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}x\right)$$

☐ Now going back to Equation $(\star)$, we had

$$T'' + c^2\lambda T = 0$$

so the function $T_{n,m}$ associated with indices (n, m) is given by

$$T_{n,m}(t) = a_{n,m}\sin\left(\sqrt{\lambda_{n,m}}ct\right) + b_{n,m}\cos\left(\sqrt{\lambda_{n,m}}ct\right)$$

So we conclude that the normal modes are given by

$$u_{n,m}(x, y, t) = \left[A_{n,m}\sin\left(\sqrt{\lambda_{n,m}}ct\right) + B_{n,m}\cos\left(\sqrt{\lambda_{n,m}}ct\right)\right]\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}x\right)$$

(Note that the spatial dependent only is called the modal shape). These normal modes of vibrations of the membrane have angular frequencies

$$\omega_n^2 = c^2\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right] = \frac{\pi^2\tau}{\rho}\left[\frac{n^2}{L^2} + \frac{m^2}{H^2}\right]$$

The 16 first normal modes are shown on Fig. 2.

6. In this problem, we consider the Schrödinger's equation for a non-relativistic free particle, which reads

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = i\hbar\frac{\partial\psi}{\partial t}$$

where $\psi(\mathbf{r}, t)$ is the quantum mechanical wavefunction of a particle of mass m. While ψ is a complicated physical concept to grasp, we can see that the Schrödinger's equation is is a partial differential equation, so we can take this problem purely mathematically.

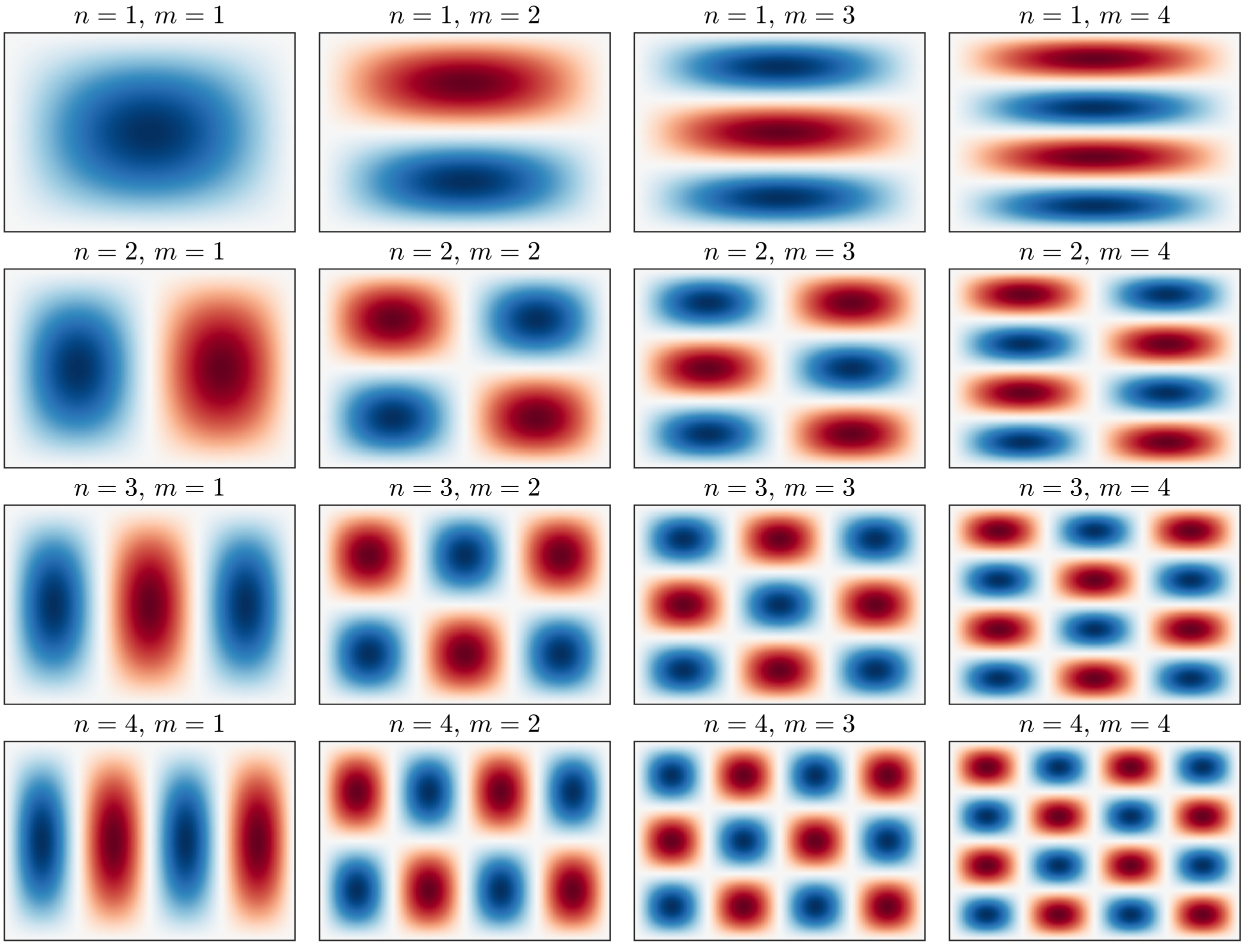

Figure 2: Top view of the 16 first normal modes for a vibrating membrane clamped on a rectangular frame, blue (resp., red) are positive vertical (resp. negative) displacements. The nodal lines correspond to the white region.

(a) Here, we are looking for a solution which is both separable in the four independent variables and that can be written in the form

$$\psi(x, y, z, t) = A \exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right]$$

Let us write $\psi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ and substitute it in Schrödinger's equation

$$-\frac{\hbar^2}{2m}\left(X''YZT + XY''ZT + XYZ''T\right) = i\hbar XYZT' \Rightarrow -\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T}$$

If the solution is to be separable **and** written in the form of a plane wave solution

$$\psi(x, y, z, t) = A \exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right] = Ae^{ik_x x}e^{ik_y y}e^{ik_z z}e^{-i\omega t}$$

then the constants k_i and ω must satisfy

$$-\frac{\hbar^2}{2m}(-k_x^2 - k_y^2 - k_z^2) = i\hbar(-i\omega)$$

The de Broglie relation states that the momentum of the particle $\mathbf{p}$ is related to the wave number $\mathbf{k}$ via $\mathbf{p} = \hbar\mathbf{k}$. The Einstein relation states that the energy of the particle $E = \hbar\omega$, where ω is the frequency of oscillation. Using both of these relationships, we find

$$p_x^2 + p_y^2 + p_z^2 = 2mE$$

(b) Solutions of the Schrödinger equation which vanish on any walls of a box fixed in between the planes, $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$ must be a product of sine waves of the form

$$u(x, y, z, t) = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) e^{-i\omega t}$$

where n_i's are integers. For the following relation

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = i\hbar\frac{T'}{T}$$

to be satisfied, we then require

$$-\frac{\hbar^2}{2m}\left[-\left(\frac{n_x\pi}{a}\right)^2 - \left(\frac{n_y\pi}{a}\right)^2 - \left(\frac{n_z\pi}{a}\right)^2\right] = i\hbar(-i\omega) \Rightarrow \frac{\hbar^2\pi^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2) = \hbar\omega = E$$

which shows that the value of the energy E is quantized and can only take the values given by the discrete integer values of n_x, n_y and n_z.

In this module, we cannot go more deeply in these concepts but note that for those of you who are interested in understanding this more deeply, there are two Quantum Mechanics electives offered by the department in Years 3/4.

7. Let's talk about the design of guitars!

(a) On a guitar, a pitch (or musical note) is produced by plucking an elastic string fixed at both ends. When a guitarist plays the open string, the string is fixed on one end at the bridge and on the other end at the nut. But a guitarist can play another pitch by pushing the string against the guitar neck to shorten it at locations specified by metal transverse bars called frets. The question is: where should the frets be placed in order for the guitar to be tuned on a twelve-tone equal-tempered scale.

This problem is a vibrating string problem with both ends fixed. The displacement of the string is governed by a 1D wave equation with homogeneous Dirichlet boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < L$$
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \quad \text{for all } t > 0$$

We have seen in the lecture notes that, in this case, the vibrating string admits normal modes whose angular frequencies are function of the wave speed

$$\omega_n = \frac{n\pi c}{L}$$

The lowest of these frequencies (i.e. the fundamental frequency) sets the pitch. We can write all these harmonics as

$$f_n = nf_0, \quad \text{with} \quad f_0 = \frac{c}{2L}$$

where the wavespeed $c = \sqrt{\tau/\rho}$ with τ the tension in the string and ρ the linear density. For a given string, by turning the tuning pegs on the guitar, one varies the tension in the string and thus sets the fundamental frequency f_0 for the open string. When fretted, the string will have a higher fundamental resonant frequency and so a higher pitch will be produced. The twelve-tone equally tempered scale is designed such that an octave is divided in twelve frequency intervals equally-spaced on a logarithmic scale. So starting from the fundamental frequency f_0 of the open string (with length L_0), the absolute frequencies on a twelve-tone equally tempered scale are given by

$$f_n = f_0 2^{n/12}$$

where n is the number of semitones between f_0 and the desired pitch. Said differently, the twelve-tone equally-tempered scale is such that the ratio of frequencies of two consecutive semitones is given by

$$\frac{f_{n+1}}{f_n} = 2^{1/12}$$

so if L_0 is the length of the open string (from nut to bridge), the frets should be placed such that the length of the fretted strings are given by

$$L_n = L_0 2^{-n/12}$$

If the open string has a length L_0, then we have the following:

Semitone	Frequency f_n/f_0	Fretted length L_n/L_0	Distance from nut ℓ_n/L_0
Unison	$2^{0/12} = 1$	1	0
Minor second	$2^{1/12} = 1.059$	0.943	0.056
Major second	$2^{2/12} = 1.122$	0.890	0.109
Minor Third	$2^{3/12} = 1.189$	0.840	0.159
Major Third	$2^{4/12} = 1.259$	0.793	0.206
Perfect fourth	$2^{5/12} = 1.334$	0.749	0.251
Augmented fourth	$2^{6/12} = 1.414$	0.707	0.292
Perfect fifth	$2^{7/12} = 1.498$	0.667	0.333
Minor sixth	$2^{8/12} = 1.587$	0.629	0.370
Major sixth	$2^{9/12} = 1.681$	0.594	0.405
Minor seventh	$2^{10/12} = 1.781$	0.561	0.439
Major seventh	$2^{11/12} = 1.887$	0.529	0.470
Octave	$2^{12/12} = 2$	0.5	0.5

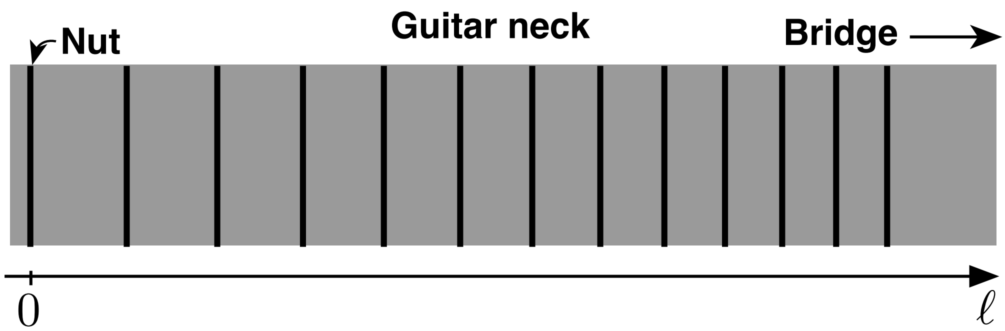

Figure 3: Schematic of the fretboard of a guitar.

(b) A guitar is a plucked string instrument. When plucking the string, one sets the initial displacement of the string and releases the string with zero initial velocity. One can say that the motion of the string then obeys the following IVBP

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < L$$
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \quad \text{for all } t > 0$$
$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for all } \ 0 < x < L$$

While the exact initial displacement depends on how your are plucking the string (with your finger or with a guitar pick), as a mathematician you can approximate the initial displacement to be a triangular shape (see Fig. 4) whose exact shape depends on where you are plucking the string $x = a$ $(0 < a < L)$.

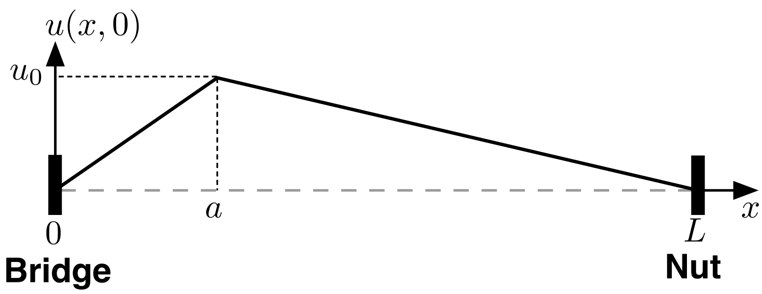

Figure 4: Initial displacement of a plucked string.

We have seen in lecture (please redo it!) that the solution to this problem is written as the following superposition of normal modes

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/L)\cos(n\pi ct/L) \quad \text{with} \quad \alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (\star)$$

In the case of this plucked string, we can write that

$$f(x) = \begin{cases} u_0 x/a, & 0 < x < a \\ u_0(L-x)/(L-a), & a < x < L \end{cases}$$

So the amplitudes of each normal are given by

$$\alpha_n = \frac{2u_0L^2}{n^2\pi^2 a(L-a)}\sin\left(\frac{n\pi}{L}a\right) \qquad (\star\star)$$

(To practice, do the integral!) For a given a, we see that the fundamental mode $n = 1$ will have the highest amplitude; the amplitude of the harmonics then decreases quickly as n increases. For a given mode n, we can see that its contribution to the solution (its amplitude) varies with the position a at which we pluck the string (see Fig. 5).

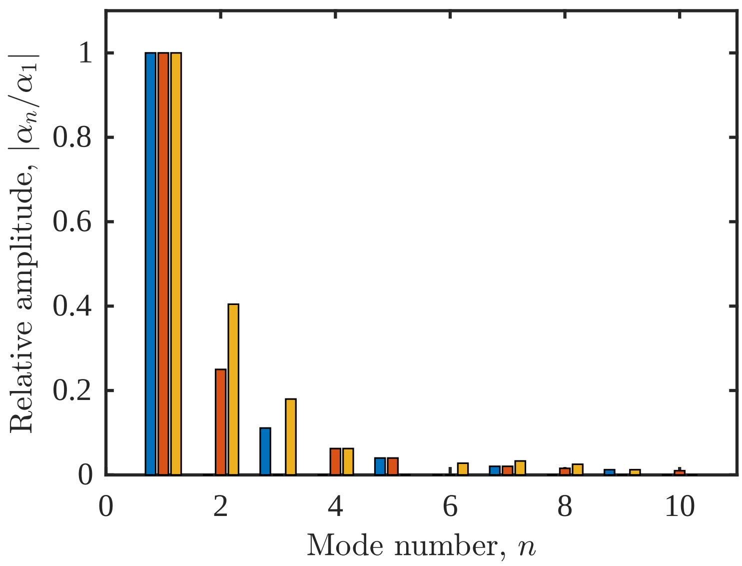

Figure 5: Relative amplitudes of the normal modes for $a = L/2$ (blue), $a = L/3$ (orange) and $a = L/5$ (yellow).

For a given frequency, the spectral content of the sound produced (i.e. the contribution of the harmonics) will change depending on where you pluck the string. Note for instance

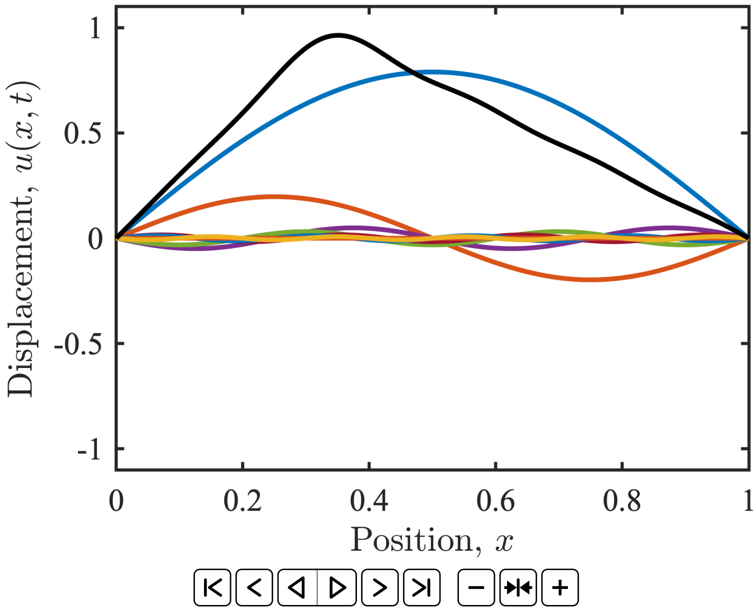

Figure 6: In this animation, we show the time evolution of the solution to the plucked string (with $a = 1/3$). The black curve represents the actual string (actually, the sum of the first 10 normal modes only in $(\star)$) and the colored lines represent the individual modes each oscillating at their own natural frequencies and with their own maximal amplitudes (given by $(\star\star)$).

that when plucking the string at $a = L/2$, we find that $A_{2n} = 0$, $n = 1, 2, \dots$. Similarly, when plucking the string at $a = L/3$, we find that $A_{3n} = 0$, $n = 1, 2, \dots$.

This is actually used by professional musician to modulate the sound produced. Musicians may say that the sound will be brighter when you pluck close to the bridge and mellower when you pluck closer to the neck. Technically, the answer is thus ”Yes! It matters where you pluck a guitar”, whether you can hear the difference is a different question and may depend on whether you were trained to hear these differences. Using a similar idea, by placing their finger at a node point, players of string instruments such as guitars, violins, etc., can produce so-called flageolet tones.

In Fig. 6, we show an animation of the solution to the plucked string problem (with $a = 1/3$). **Note that you may need to open the PDF in Adobe Acrobat Reader for the animation to work.**