

MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to the mid-term exam

Question 1

(i) Any solution $\lambda : I \rightarrow \mathbb{R}$ to this differential equation is continuously differentiable. This follows from the solution identity $\dot{\lambda}(t) = |\lambda(t)|$ for all $t \in I$, since the right hand side of this equation is a continuous function. [3 points]

(ii) Let $f(x) = |x|$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned}\lambda_0(t) &= 1, \\ \lambda_1(t) &= 1 + \int_0^t f(\lambda_0(s)) \, ds = 1 + \int_0^t ds = 1 + t, \\ \lambda_2(t) &= 1 + \int_0^t f(\lambda_1(s)) \, ds = 1 + \int_0^t |1 + s| \, ds = 1 + \int_1^{t+1} |s| \, ds = 1 + \left[\frac{s^2 \operatorname{sgn}(s)}{2} \right]_{s=1}^{s=t+1} \\ &= \begin{cases} 1 + \frac{1}{2}(t+1)^2 - \frac{1}{2} = \frac{1}{2}t^2 + t + 1 & : t > -1, \\ 1 - \frac{1}{2}(t+1)^2 - \frac{1}{2} = -\frac{1}{2}t^2 - t & : t \leq -1. \end{cases}\end{aligned}$$

[7 points = 1 point for λ_0 ; 2 points for λ_1 ; 4 points for λ_2]

Question 2

(i) Suppose that f is continuous and satisfies

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\| \quad \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^d,$$

where $K > 0$ is a constant. Then every initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$, admits a unique solution $\lambda : [t_0 - h, t_0 + h] \rightarrow \mathbb{R}^d$, where $h := \frac{1}{2K}$. [2 points; no need to specify h precisely]

(ii) Let $u_1, u_2 \in X := C^0([t_0 - h, t_0 + h], \mathbb{R}^d)$. Then for all $t \in [t_0 - h, t_0 + h]$,

$$\begin{aligned}\|P(u_1)(t) - P(u_2)(t)\| &= \left\| \int_{t_0}^t (f(s, u_1(s)) - f(s, u_2(s))) \, ds \right\| \\ &\leq \left| \int_{t_0}^t \|f(s, u_1(s)) - f(s, u_2(s))\| \, ds \right| \\ &\leq K \left| \int_{t_0}^t \|u_1(s) - u_2(s)\| \, ds \right| \leq K \left| \int_{t_0}^t \|u_1 - u_2\|_\infty \, ds \right| \\ &\leq Kh\|u_1 - u_2\|_\infty.\end{aligned}$$

This implies by taking the supremum over all $t \in [t_0 - h, t_0 + h]$ that

$$\|P(u_1) - P(u_2)\|_\infty = \sup_{t \in [t_0 - h, t_0 + h]} \|P(u_1)(t) - P(u_2)(t)\| \leq Kh\|u_1 - u_2\|_\infty.$$

Thus, P is a contraction for $h < \frac{1}{K}$. A natural choice as in (i) would be $h = \frac{1}{2K}$. [8 points]

(iii) $\dot{x} = tx$. Due to t being unbounded, the right hand side is not globally Lipschitz continuous, but it follows from Exercise 1 that solutions exist globally. [4 points = 3 points for example + 1 point for justification]

Question 3

The local version of the Picard–Lindelöf theorem implies that there exist a $h > 0$ and a solution $\mu : [-h, h] \rightarrow \mathbb{R}^d$ satisfying $\mu(0) = y_0$. Define $T := \frac{h}{2}$. Translation invariance implies that the function $\lambda : [-T, 3T] \rightarrow \mathbb{R}^d$, $\lambda(t) := \mu(t - T)$, is a solution to the differential equation, and we have $\lambda(T) = \mu(0) = y_0$. The function λ satisfies $\lambda(0) = \mu(-\frac{h}{2})$, so the statement is correct with $x_0 = \mu(-\frac{h}{2})$.