Q7.1

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{-z^2t^2}}{t^2+1}\, dt$$

a) singularities at $t_0^2+1=0 \Rightarrow t_0 = \pm i$,
from notes we have

$$|I - I_n| = O(e^{-2\pi a/h}), \quad \forall a<1.$$

(Note we cannot choose $a=1$, as M is unbounded for this choice)

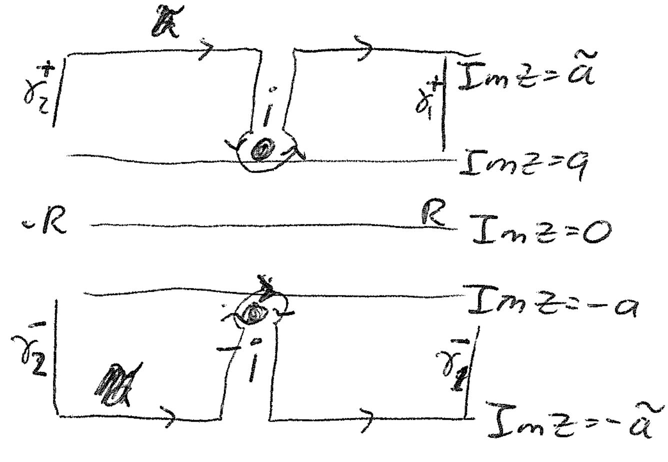
b) Truncation error: $O(e^{-z^2h^2N^2})$,

balancing
$$|I - I_n^{(N)}| \le |I - I_n| + |I_n - I_n^{(N)}|$$
$$= O(e^{-2\pi a/h}) + O(e^{-z^2h^2N^2})$$

$$\Rightarrow 2\pi a/h = z^2h^2N^2 \Rightarrow h^* = \left(\frac{2\pi a}{z^2N^2}\right)^{1/3},$$

~~so~~ since ~~choosing~~ $a \approx 1$, $h = \left(\frac{2\pi}{z^2N^2}\right)^{1/3}$ is appropriate.

c)



We can extend the integral representation of the error $I - I_n$ to a wider strip, (a to $\tilde{a}$), by collecting residues.

$$I_n - I = -\sum_{\pm} \int_{-\infty \pm ai}^{\infty \pm ai} \frac{f(\tilde{z})}{1-e^{\mp 2\pi i \tilde{z}/h}}\, d\tilde{z}$$

$$= \lim_{R\to\infty} -\sum_{\pm} \int_{-R\pm ai}^{R\pm ai} \frac{f(\tilde{z})}{1-e^{\mp 2\pi i \tilde{z}/h}}\, d\tilde{z}$$

$$= \lim_{R\to\infty}\Bigg[-\sum_{\pm} \pm 2\pi i \,\mathrm{Res}\left(\frac{f(\tilde z)}{1-e^{\mp 2\pi i \tilde z/h}}, \pm i\right) + \left\{\int_{\gamma_1^\pm} + \int_{\gamma_2^\pm} + \int_{-R\pm\tilde a i}^{R\pm \tilde a i}\right\} \frac{f(\tilde z)}{1-e^{\mp 2\pi i \tilde z/h}}\, d\tilde z\Bigg]$$

(where $f(\tilde z) = \exp(-z^2\tilde z^2)/(\tilde z^2+1)$)

in the limit the integrals over $\gamma_1^\pm$ & $\gamma_2^\pm$ tend to zero by identical arguments to the original proof, hence, for $\tilde a > 1$,

$$I_h - I = -\sum_{\pm} \pm 2\pi i\,\mathrm{Res}\left(\frac{f(\tilde z)}{1-e^{\mp 2\pi i\tilde z/h}}, \pm i\right) + \int_{-\infty\pm\tilde a i}^{\infty\pm\tilde a i} \frac{f(\tilde z)}{1-e^{\mp 2\pi i \tilde z/h}}\, d\tilde z, \quad (*)$$

noting the $\pm$ in front of the residue, because of the anti-clockwise & clockwise circular contours around $+i$ & $-i$ respectively.

Let's compute these,

$$\mathrm{Res}\left(\frac{f(\tilde z)}{1-e^{\mp 2\pi i \tilde z/h}}, \pm i\right) = \left.\frac{\exp(-z^2\tilde z^2)}{(1-e^{\mp 2\pi i\tilde z/h})\, 2\tilde z}\right|_{\tilde z = \pm i}$$

$$= \frac{\pm e^{-z^2}}{2i(1-e^{2\pi/h})},$$

thus rearranging $(*)$ we find

$$\underbrace{\left(I_h + \frac{2\pi e^{-z^2}}{1-e^{2\pi/h}}\right)}_{=:\tilde I_h} - I = \sum_{\pm}\int_{-\infty\pm\tilde a i}^{\infty\pm\tilde a i} \frac{f(\tilde z)\, d\tilde z}{1-e^{\mp 2\pi i\tilde z/h}} = O(\exp(-2\pi\tilde a/h))$$

as $h\to 0$, $\forall\, \tilde a > 1$, (bound holds for lower values of $\tilde a$ also)

d) Here

$$M = \int_{-\infty+\tilde{a}i}^{\infty+\tilde{a}i} |f(\tilde{z})|\, d\tilde{z}$$

$$= \int_{-\infty+\tilde{a}i}^{\infty+\tilde{a}i} \frac{e^{-z^2t^2}}{t^2+1}\, dt = O(e^{z^2\tilde{a}^2})$$

So accounting for this part of the error

$$|I_n - I| = O(\exp(\underbrace{z^2\tilde{a}^2 - 2\pi\tilde{a}/h}_{=\phi(\tilde{a})}))$$

$$\phi'(\tilde{a}) = 2z^2\tilde{a} - \frac{2\pi}{h}; \quad \phi'(\tilde{a}) = 0 \Rightarrow \tilde{a} = \frac{\pi}{z^2h}$$

$\phi''(\tilde{a}) = 2z^2 > 0 \Longrightarrow$ any critical point is a local minimum.

subbing the minimising value of $\tilde{a}$, $\phi(\tilde{a}) = \frac{z^2\pi^2}{z^4h^2} - \frac{2\pi\cdot\pi}{z^2h^2} = -\frac{\pi^2}{z^2h}$,

$$|I_n - I| = O(\exp(-\frac{\pi^2}{z^2h})).$$

Truncation error is

$$|I_n - I_n^{(N)}| = O(\exp(-z^2(hN)^2)).$$

Balancing: (triangle inequality again)

$$|I - I_n^{(N)}| = O(\exp(-\frac{\pi^2}{z^2h})) + O(\exp(-z^2(hN)^2))$$

set $\frac{-\pi^2}{z^2h} = -z^2h^2N^2 \Longrightarrow \boxed{h = \left(\frac{\pi^2}{z^4N^2}\right)^{\frac{1}{3}}}$

8b)

Given $\exp(i\omega t^p)$ for $\omega \in \mathbb{C}$ & $p \in \mathbb{N}$, want to choose a change of variable $t = se^{i\theta}$ for $s \in \mathbb{R}$, $\theta \in [0, 2\pi)$, such that

$$\exp(i\omega (se^{i\theta})^p) = \exp(-|\omega| s^p),$$

i.e. pure exponential decay.

Let $\omega = |\omega| \cdot e^{i\varphi}$, where $\varphi = \arg\omega$.

$$i\omega (se^{i\theta})^p = i|\omega| \cdot e^{i\varphi}(se^{i\theta})^p$$

$$= e^{\frac{i\pi}{2}} |\omega| e^{i\varphi} s^p e^{ip\theta}$$

Thus, the angle θ of the "steepest desent" deformation should satisfy

$$\frac{\pi}{2} + \varphi + p\theta = \pi \implies \theta = \left(\frac{\pi}{2} - \varphi\right)\frac{1}{p}$$

so that the complex arguments combine to give an angle along the negative real axis.

Choosing $\theta = \left(\frac{\pi}{2} - \arg\omega\right)\frac{1}{p}$ and the corresponding deformation along $\gamma_\theta = \{ se^{i\theta}, s \in \mathbb{R} \}$ means that f must be analytic in the region enclosed by $\mathbb{R}$ and γ_θ, so the deformation can be justified by Cauchy's theorem.