Imperial College London

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Analysis 1A

Lecture 7 - Defining the convergence of sequences

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A *sequence* is a function $a : \mathbb{N}_{>0} \to \mathbb{R}$.

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We will denote a by $a_1, a_2, a_3, \ldots$ or $(a_n)_{n\in\mathbb{N}_{>0}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark 3.1

a_i 's could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}_{>0}\} \subset \mathbb{R}$.

E.g. $(a_n) = 1, 0, 1, 0, \ldots$ is different from $(b_n) = 1, 0, 0, 1, 0, 0, 1, \ldots$

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- Via a **closed formula**, like $a_n = \frac{1-(-1)^n}{2}$ for the sequence above,
- By a **recursion**, e.g. the Fibonacci sequence $F_1 = 1 = F_2$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ (so (F_n) is $1, 1, 2, 3, 5, 8, 13, \dots$)

We can also define a sequence by summation, e.g.
$a_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
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Exercise 3.2

Show any sequence (a_n) can be written as a series $a_n = \sum_{i=1}^{n} b_i$ for an appropriate choice of sequence (b_n).

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)". We will spend a while exploring various formulations before we choose our definitive definition.

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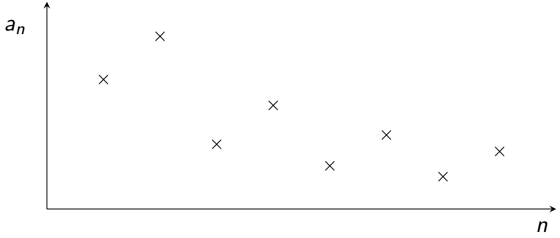
Idea 1: a_n should get closer and closer to a. Not necessarily monotonically, e.g. for:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0.$$



Idea 2: But notice that $\frac{1}{n}$ also gets closer and closer to -73.6! So we want to say instead that a_n gets "*as close as we like to a*" or "*arbitrarily close to a*".

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We will measure this with $\epsilon > 0$: we say a_n gets to within ϵ of a by

$$|a_n - a| \; < \; \epsilon \qquad \text{or} \qquad a_n \in (a - \epsilon, a + \epsilon).$$

We phrase "a_n gets *arbitrarily* close to a" by "*a_n gets to within ϵ of a for* **any** $\epsilon > 0$".

Exercise 3.3

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Leaving aside what they mean by "sufficiently large" for now, which of these sequences converges (to some $a \in \mathbb{R}$) according to their definition?

1. $0, 1, 0, 1, \ldots$
2. $1, 1, 1, 1, \ldots$ $\leftarrow$
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
4. $a_n = 2^{-n}$
5. More than one of these
6. None of these

Idea 3: Dedekind said that once n is large enough, $|a_n - a|$ is less than every $\epsilon > 0$, but that means it's zero, i.e. $a_n = a$. The problem they missed is that if we take smaller ϵ we will usually have to take bigger n to make $|a_n - a| < \epsilon$.

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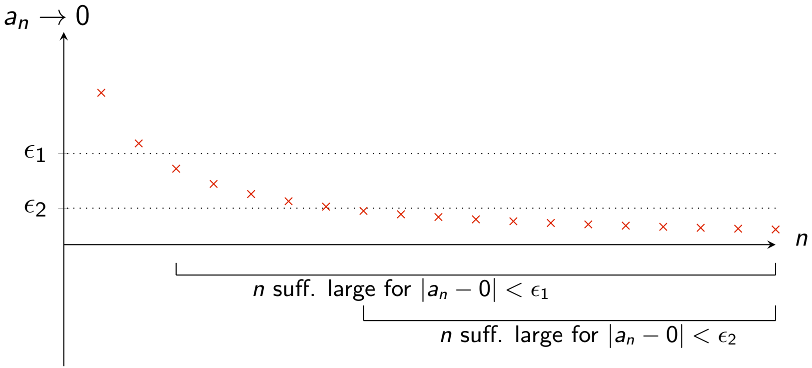
So we want to say that to get *arbitrarily close to the limit a* (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. If I change $\epsilon > 0$ to be smaller, I may have to go further down the sequence to get within ϵ of a.


$a_n \to 0$
ϵ_1
ϵ_2
n
n suff. large for $|a_n - 0| < \epsilon_1$
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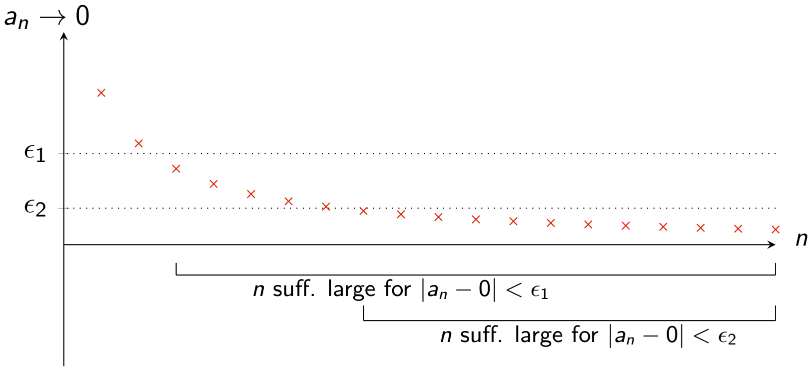

Don't fall for the same trap as Dedekind - there will not be a "n sufficiently large" that works for all ϵ at once! (Unless $a_n \equiv a$ eventually.)

That is, we want to *reverse* the order of specifying n and ϵ: only once we've seen how small ϵ is do we know how big to take n. If we chose a smaller ϵ we can then choose a larger n.

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For *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon > 0$" to "$\forall \epsilon > 0, \ \exists n$". *This allows n to depend on ϵ.*

Exercise 3.4

Dedekind takes your point, and modifies his definition of $a_n \to a$ to

$$\forall \epsilon > 0\ \exists n \in \mathbb{N}_{>0} \text{ such that } |a_n - a| < \epsilon.$$

Which of these sequences converges to $a = 0$ according to his new definition?

1. $0, 1, 0, 1, \ldots$ ✓
2. $1, 1, 1, 1, \ldots$ ✗
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ ✓
4. $a_n = 2^{-n}$ ✓
5. More than one of these ←
6. None of these

Definition (Convergence)

We say that $a_n \to a$ as $n \to \infty$ if and only if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N}_{>0} \text{ such that } \forall n \geq N, \ |a_n - a| < \epsilon.$$

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Convergence **in words**

However close ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}_{>0}$) beyond which ($n \geq N$) *all* a_n are indeed that close to the limit a ($|a_n - a| < \epsilon$).

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Remark 3.5

N depends on ϵ! For a while we will sometimes denote it N_ϵ, as a reminder. We often write ($a_n \to a$ as $n \to \infty$) as just ($a_n \to a$) or ($\lim_{n\to\infty} a_n = a$).

Example 3.6

Prove $\frac{1}{n} \to 0$ as $n \to \infty$.

<u>Pf</u> Let $\varepsilon > 0$. Let $N_\varepsilon = \lceil 1/\varepsilon \rceil + 1$

Then for $n \geq N_\varepsilon$

$$\left| 1/n - 0 \right| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon. \qquad \blacksquare$$

Since $N_\varepsilon > \frac{1}{\varepsilon}$

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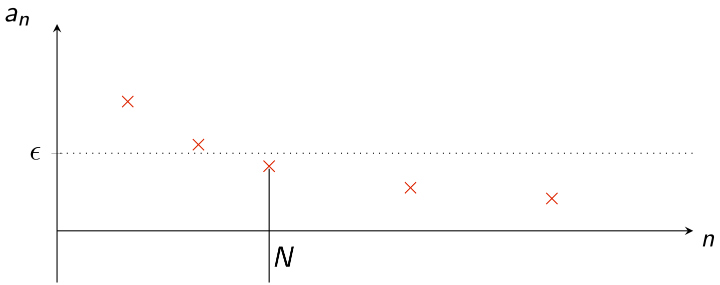

How to prove $a_n \to a$

$$\boxed{\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon \ \forall n \geq N_\epsilon}$$

(I) Fix $\epsilon > 0$.

(II) Calculate $|a_n - a|$.

(II′) Find a good estimate $|a_n - a| \leq b_n$.

(III) Try to solve $b_n < \epsilon$. $\quad (*)$

(IV) Find $N_\epsilon \in \mathbb{N}_{>0}$ such that $(*)$ holds whenever $n \geq N_\epsilon$.

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see the examples next lecture!).