Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Mathematics Pre-arrival course

Problem Sheet 3 – Linear Algebra, Sequences and Series

1. The 2×2 matrices A and B are defined by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ 3 & 2 \end{bmatrix}.$$

   Calculate

   (a) AB
   (b) BA
   (c) A^T
   (d) $B^T A^T$
   (e) det(A)
   (f) A^{-1}

2. Which of the following statements are true for all square matrices A, B and where I denotes the identity matrix:

$$I^2 = I, \quad A(B + I) = (B + I)A, \quad A(A + I) = (A + I)A.$$

3. Describe the transformation that the following matrices represent:

   (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4. Find all of the invariant points under the following transformations

   (a) $\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$

5. Given that A and B are invertible square matrices, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

6. Find all the values of k for which the matrix

$$\begin{bmatrix} k-1 & k-1 \\ k^2 & k+6 \end{bmatrix}$$

is singular.

7. The 3×3 matrix M is defined by

$$M = \begin{bmatrix} 3 & -2 & 2 \\ 2 & 3 & -1 \\ 1 & 3 & 1 \end{bmatrix}.$$

Calculate M^{-1} and solve the system of equations:

$$\begin{aligned} 3x - 2y + 2z &= -9 \\ 2x + 3y - z &= 12 \\ x + 3y + z &= 3 \end{aligned}$$

8. Find an expression in terms of n only for the following series:

(a) $\sum_{r=1}^{n}(3r-1)$
(b) $\sum_{r=1}^{n}(2r^2-r)$
(c) $\sum_{r=1}^{n} r(r^2-2)$
(d) $\sum_{r=1}^{n} r^2(r+1)$
(e) $\sum_{r=1}^{n}(2r^3-3r^2+5r-3)$
(f) $\sum_{r=1}^{n}(r+2)(r-5) + \sum_{r=1}^{n} r(2r+1)(3r-2)$

9. Use the method of differences to find a formula for the sums of the following series:

(a) $\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)}$
(b) $\sum_{r=1}^{n} \frac{1}{r(r+2)}$
(c) $\sum_{r=1}^{n} \frac{r}{(r+2)(r+3)(r+4)}$

10. Consider the following iterative geometric construction. We start from an equilateral triangle with side length 1. At each subsequent steps, we first divide each side of the polygon in three equal parts, we construct an equilateral triangle on the middle part and we delete the middle part itself. The Koch snowflake is the piecewise continuous curve that results from repeating this process infinitely. For a given step n in the process, we denote S_n the number of sides of the polygon, L_n the length of a given side and P_n the total perimeter of the snowflake.

(a) Sketch the first few steps of the construction process.
(b) Devise formulas for S_n, L_n and P_n.
(c) What can you say about the perimeter of the snowflake curve?
(d) What is the area enclosed by Koch's snowflake?

This curve is called the Koch snowflake; it is one of the earliest examples of fractals which appeared in 1904 in a paper entitled "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire" (in english, "On a continuous curve without tangents, constructible from elementary geometry") by Swedish mathematician Helge von Koch.