Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 1

*Questions marked by * are good candidates for discussion at the tutorials.*

1. Find the Fourier transforms of the following functions (with $a > 0$). Also, obtain the Fourier sine transform for the function in (ii) and Fourier cosine transform for the function in (iv).

   (i) $f(x) = \begin{cases} e^{-ax}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

   (ii) $f(x) = \operatorname{sgn}(x)\exp(-a\,|x|)$; [$\operatorname{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$].

   (iii) $f(x) = 2a/(a^2 + x^2)$;

   (iv) $f(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise;

   (v) $f(x) = \sin(ax)/(\pi x)$; (Hint: use the transform of a rectangular pulse from the lectures and the symmetry formula).

   From your result in part (v), deduce that
   $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

2. If a function has Fourier transform $\widehat{f}(\omega)$, find the Fourier transform of $f(x)\sin(ax)$ in terms of $\widehat{f}$.

3. By applying the inversion formula to the transforms obtained in 1(i) and 1(iv), establish the following results:
   $$\text{(i)} \int_0^\infty \frac{\cos x}{x^2 + a^2}\,dx = \frac{\pi e^{-a}}{2a} \text{ if } a > 0; \quad \text{(ii)} \int_{-\infty}^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{2}.$$

4.* Sketch the function given by
   $$f(x) = \begin{cases} 2d - |x| & \text{for } |x| \leq 2d, \\ 0 & \text{otherwise.} \end{cases},$$
   and show that $\widehat{f}(\omega) = (2/\omega)^2 \sin^2(\omega d)$.

   Use the energy theorem to demonstrate that
   $$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{2\pi}{3}.$$

5. Show that the Fourier transform of $\exp(-cx)H(x)$,where H is the Heaviside function and c is a positive constant, is given by $1/(c + i\omega)$. Hence use the convolution theorem to find the inverse Fourier transform of
   $$\frac{1}{(a + i\omega)(b + i\omega)},$$
   where $a > b > 0$.

6. Use the symmetry rule to show that

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}(\widehat{f}(\omega) * \widehat{g}(\omega)).$$

7. Suppose that $f(x)$ is a function such that $\widehat{f}(\omega) = 0$ for all ω with $|\omega| > M$, where M is a positive constant. Let $g(x) = \sin(ax)/(\pi x)$. Show that if the constant $a > M$:

$$f(x) * g(x) = f(x).$$

Hint: Use the transform of $g(x)$ from Q1(v).

8.* By considering suitable integration formulae, establish the following results involving the Dirac delta function:

(i) $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$; (ii) $x\delta'(x) = -\delta(x)$; (iii) $\delta(-x) = \delta(x)$.

Here $f(x)$ is continuous. [In each case multiply by an arbitrary continuous test function $\phi(x)$ and integrate from $-\infty$ to ∞].