Imperial College London
MATH 50004/50015 Multivariable Calculus
Mid-Term Examination Date: 16th November 2023
Duration: 40 minutes
Total Marks: 25

(a) In this part of the question you may assume the relation

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl},$$

with summation over i implied.

(i) [**3 marks**] Show, using subscript notation, that

$$(\text{curl}\mathbf{A}) \cdot (\text{curl}\mathbf{A}) = \left(\frac{\partial A_k}{\partial x_j}\right)^2 - \frac{\partial A_k}{\partial x_j}\frac{\partial A_j}{\partial x_k},$$

with summation over j and k implied.

(ii) [**3 marks**] Show that the j th component of $\mathbf{A} \times \text{curl}\mathbf{A}$ is given by

$$A_k \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right),$$

with summation over k implied.

(iii) [**4 marks**] You are given that $\mathbf{A}$ is a solenoidal vector field. Use this information, together with the results to parts (i) and (ii), to simplify the expression

$$\text{div}(\mathbf{A} \times \text{curl}\mathbf{A}) - (\text{curl}\mathbf{A}) \cdot (\text{curl}\mathbf{A}).$$

(b) In this part of the question you may assume Green's theorem in the form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R (\text{curl}\mathbf{F}) \cdot \mathbf{k}\; dx\, dy, \tag{1}$$

in the usual notation.

Consider the closed curve C given by a triangle with vertices $(0,0), (1,0)$ and $(1/2, 1/2)$, and the vector field

$$\mathbf{F} = e^{x+y}\,\mathbf{i} + xy\,\mathbf{j},$$

(i) [**1 mark**] In which direction should the curve C be traversed to be consistent with Green's theorem?

(ii) [**3 marks**] Express the equations for the sides of the triangle in parametric form.

(iii) [**6 marks**] Evaluate the left-hand side of (1) for $\mathbf{F}$ and C given above.

(iv) [**5 marks**] Evaluate the corresponding right-hand side of (1) using horizontal strips to evaluate the double integral. Why is it more complicated to use vertical strips in this case?