Problem Sheet 3, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let $S \subset \mathbb{R}$ be a *helicoid*, that is, the surface parametrised by

$$\phi(u, v) = (u \sin(v), -u \cos(v), v).$$

Compute the Gaussian and mean curvatures of S at each point $\phi(u, v)$.

Problem 2. Let $S \subset \mathbb{R}^3$ be the graph of $z = f(x, y)$, where f is a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. Compute the Gaussian curvature of S at each point $(x, y, f(x, y))$.

Problem 3. Let $\gamma(t) = (x(t), z(t))$ be a plane curve (in the xz-plane) parametrised by arc length, with $x(t) > 0$ for all $t \in \mathbb{R}$. Let $S \subset \mathbb{R}^3$ denote the surface of revolution formed by rotating $\gamma(\mathbb{R})$ about the z-axis.

(a) Using the parametrisation

$$\phi(u, v) = (x(u) \cos(v), x(u) \sin(v), z(u))$$

of S, prove that the Gaussian curvature of S at a point $\phi(u, v)$ is given by

$$K(\phi(u, v)) = \frac{-x''(u)}{x(u)}.$$

(b) Characterise the planar points of S in terms of z and its derivatives.

Hint for part (a): at some point you may wish to differentiate the equation $(x')^2 + (z')^2 = 1$ to help simplify things, and prepare to do considerable calculations!

Problem 4. Let S be a regular surface and $C \subset S$ an *asymptotic line*, meaning that C is a regular curve whose normal curvature is zero.

(a) Prove that $K(p) \leq 0$ at all points $p \in C$,

(b) If the curvature of C is non-zero everywhere, then its torsion satisfies $|\tau(p)| = \sqrt{-K(p)}$.

Hint for part (a): parametrise the curve C by arc length, and use the definition of normal curvature in terms of the inner product with N, and show that $A(C', C') = 0$.

Hint for part (b): use the relation in part (a) to show that the normal to the curve n_C is orthogonal to the normal to the surface N. Then, think about the Frenet frames, and use the Frenet equations. This might be a difficult problem!

Problem 5. Let S be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, with the chart $\phi : \mathbb{R}^2 \to S$ given by

$$\phi(u, v) = (u, v, f(u, v)).$$

Compute the Christoffel symbols Γ_{ij}^k.