MATH50001 Analysis II, Complex Analysis

Lecture 11

Last time:

Theorem. (Schwarz reflection principle)

Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in Ω such that $F|_{\Omega^+} = f$.

Proof. Let us define $F(z)$ for $z \in \Omega^-$ by

$$F(z) = \overline{f(\bar{z})}.$$

To prove that F is holomorphic in Ω^- we note that if $z, z_0 \in \Omega^-$ then $\bar{z}, \bar{z}_0 \in \Omega^+$ and since f is holomorphic in Ω^+ we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n.$$

Therefore

$$F(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n$$

and thus F is holomorphic in Ω^-.
Since f is real valued on I we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence F extends continuously up to I.

Section: The complex logarithm.

We have seen that to make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

Theorem. Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that:

(i) F is holomorphic in Ω,

(ii) $e^{F(z)} = z, \quad \forall z \in \Omega$,

(iii) $F(r) = \log r$ whenever r is a real number and near 1.

In other words, each branch $\log_{\Omega}(z)$ is an extension of the standard logarithm defined for positive numbers.

Proof.
We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

$$\log_{\Omega}(z) = F(z) = \int_{\gamma} f(z)\, dz,$$

where γ is any curve in Ω connecting 1 to z. Since Ω is simply connected, this definition does not depend on the path chosen. Then F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$. This proves (i).

To prove (ii), it suffices to show that $ze^{-F(z)} = 1$. Indeed,

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0.$$

Thus $ze^{-F(z)}$ is a constant. Using $F(1) = 0$ we find that this constant must be 1.

Section: Zeros of holomorphic functions.

Definition. We say that f has a zero of order m at $z_0 \in \mathbb{C}$ if

$$f^{(k)}(z_0) = 0, \qquad k = 0, 1, \ldots m-1,$$

and $f^{(m)}(z_0) \neq 0$.

Theorem. A holomorphic function f has a zero of order m at z_0 if and only if it can be written in the form

$$f(z) = (z - z_0)^m g(z),$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof.

$$f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \dots$$
$$= (z-z_0)^m \left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots \right).$$

Then $f(z) = (z-z_0)^m g(z)$ where g is defined by

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots.$$

The above series converges and thus g is holomorphic at z_0.

Conversely, if $f(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$, then $f^{(k)}(z_0) = 0$, $k = 0, 1 \dots, m-1$ and $f^{(m)}(z_0) = m!\, g(z_0) \neq 0$.

Corollary. The zeros of a non-constant holomorphic function are isolated; that is every zero has a neighbourhood inside of which it is the only zero.

Proof.

If z_0 is a zero of f of order m, then $f(z) = (z - z_0)^m g(z)$, where g is holomorphic at z_0 and $g(z_0) \neq 0$. This means that g is continuous and therefore there is a neighbourhood of z_0 in which $g(z) \neq 0$. Thus $f(z) \neq 0$ except for $z = z_0$.

Section: Laurent Series.

Definition. The series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1}$$
$$+ a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

is called Laurent series for f at z_0 where the series converges.

Example.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n, \qquad z \neq 0.$$

Theorem. (Laurent Expansion Theorem)
Let f be holomorphic in the annulus $D = \{z : r < |z - z_0| < R\}$.
Then $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}}\, d\eta,$$

and where γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

Pierre Alphonse Laurent
1813 – 1854 (French)

Thank you