IMPERIAL

**MATH70130**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Stochastic Differential Equations in Financial Modelling**

Date: Monday, April 29, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer Each Question in a Separate Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. OU SDE.

Consider the particular Ornstein Uhlenbeck (OU) SDE

$$dX_t = -a(t)X_t\ dt + \sigma(t)\ dW_t,$$

where $X_0 = x_0$ is the deterministic initial condition and where $a, \sigma$ are smooth deterministic functions of time, with $|a(t)|$ and $|\sigma(t)|$ all bounded above by $K$, namely

$$\max(|a(t)|, |\sigma(t)|) \leq K \ \text{ for all } \ t \geq 0,$$

with $K$ a positive real constant.

(a) Prove that this SDE admits a unique global solution. (6 marks)

(b) Calculate the solution without using Stratonovich calculus. [Hint: Set $Y_t = \exp(\int_0^t a(s)ds)X_t$ and work with $Y$, then go back to $X$.] (8 marks)

(c) Calculate the expected value of the solution at a time $T > 0$. You will use one property of expectations of Ito integrals. Give an informal intuitive reason for this property. (6 marks)

(Total: 20 marks)

2. Black-Scholes Option Pricing, Short Risk-Reversal.

Given a stock with price $S_t$ at time $t$, $t \geq 0$, consider a payoff $Y$ that is short a call option with strike $K_1$ and long a put option with strike $K_2$, with $K_2 < S_0 < K_1$, both options with maturity $T$. In formula, the payoff is $Y = -(S_T - K_1)^+ + (K_2 - S_T)^+$ and is called a bear (or short) risk reversal payoff.

(a) Draw a plot of this payoff as a function of $S_T$. Explain what type of investor would be interested in buying this payoff and what views on the stock marcket this investor would have. (5 marks)

(b) Price the short risk reversal in a Black-Scholes model, where the stock price dynamics under the the physical measure $P$ is

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$

with $S_0 = s_0$ the deterministic initial condition, and where interest rates $r$ are constant and deterministic. Here $W^P$ is a Brownian motion under the measure $P$. You can use the formula for a call option without deriving it. Derive the formula for the put option, either through put-call parity or through risk neutral valuation. (10 marks)

(c) Calculate the delta of the short risk reversal, namely the sensitivity of its price to the initial stock price $s_0$. How does the short risk reversal change with $s_0$? (5 marks)

(Total: 20 marks)

3. Option pricing Bachelier (long Straddle).

Consider a stock market where the stock price $S$ follows the Bachelier dynamics, assuming the risk free interest rate is zero, namely $r = 0$,

$$dS_t = \sigma \, dW_t, \quad S_0 = s_0,$$

where $W$ is a Brownian motion under the risk neutral measure $Q$, with deterministic initial stock price $s_0 > 0$. $\sigma$ is a positive real volatility. Consider a straddle payoff on $S$, with maturity $T$ and strike $K = s_0 e^{rT} = s_0$, namely

$$Y = (S_T - K)^+ + (K - S_T)^+.$$

(a) Compute the straddle price at time 0 in the Bachelier model, call it $V_0$. (10 marks)

(b) Compute the delta of the straddle in the Bachelier model, namely $\frac{\partial V_0}{\partial s_0}$ and determine conditions under which delta is positive. (6 marks)

(c) Compute the vega of the straddle in the Bachelier model, namely $\frac{\partial V_0}{\partial \sigma}$ and comment on how the straddle price varies with $\sigma$. (4 marks)

(Total: 20 marks)

4. Expected Shortfall of a stock position in the Bachelier model.

Consider the dynamics of an equity asset price $S$ in the Bachelier model, under both probability measures $P$ (the Physical or Historical measure) and $Q$ (the risk neutral measure), with stock dynamics $dS_t = \mu dt + \sigma dW_t$, with $\mu$ and $\sigma$ deterministic constants, $\sigma > 0$ and where $W$ is a Brownian motion under $P$. Assume the risk-free interest rate is equal to zero, $r = 0$.

(a) Define Expected Shortfall (ES) for a time horizon $T$ with confidence level $\alpha$ for a general portfolio.

(6 marks)

(b) Compute ES for horizon $T$ and confidence level $\alpha$ for a portfolio with $N$ units of equity, where the equity price follows the Bachelier process above.

(6 marks)

(c) Explain one drawback of ES as a risk measure

(4 marks)

(d) Is the equity dynamics you used for ES the same you would have used to price an equity call option in the Bachelier model? Discuss any possible differences. (4 marks)

(Total: 20 marks)

5. Mastery question.

In the course we developed the theory of option pricing, included also in the lecture notes, but we didn't discuss the theory in the problem classes. This is a question on the derivation of the Black Scholes partial differential equation and on the risk neutral measure whose solution can be based on what we learned in class and in the lecture notes. You don't need to add material from independent research.

(a) Define the Black–Scholes model, its economy and the main assumptions behind it. Write equations for the two assets in the model and comment on what the equations imply.

(5 marks)

(b) Sketch the derivation of the Black Scholes partial differential equation for the price of a call option on the stock with maturity $T$ and strike $K$ (without solving it) using the self financing condition and Ito's formula, assuming that the claim price at time $t > 0$ can be written in the form $V(t, S_t)$, where $V(t, S)$ is a function of $t$ and $S$ that is once continuously differentiable in $t$ and twice continuously differentiable in $S$. (5 marks)

(c) The price you derived in the previous part is a price by replication. Explain how one can arrive at a different notion of price that is more related to expectations of future cash flows, and what theorem one needs to invoke in order to obtain this interpretation starting from the partial differential equation. (5 marks)

(d) The answer to the previous question involves a new probability measure that is different from the initial physical measure $P$ postulated for the model. This measure is usually called $Q$, the risk neutral measure. Name which theorem allows you to find the relationship between $P$ and $Q$ and write this relationship for the Black Scholes model using a Radon-Nykodym derivative $dQ/dP$.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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# Math60130/70130

# Stochastic Differential Equations in Financial Modeling (Solutions)

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1. (a) A sufficient condition for existence and uniqueness of a global strong solution is that we have two conditions regarding Lipschitz continuity and linear growth. We know from the theory that for the SDE $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = Z$ with $Z$ independent of $\sigma(\{W_t,\ t \leq T)\}$ and $E[Z^2] < +\infty$, and with $\mu : [0,T] \times R \to R$ (the drift) and $\sigma : [0,T] \times R \to R$ (the diffusion coefficient) being measurable, if we have global Lipschitz continuity

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$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y| \text{ for all } t \in [0,T] \text{ and all } x \in R$$

and linear growth

$$|\mu(t,x)| + |\sigma(t,x)| \leq K'(1+|x|)| \quad \text{for all } t \in [0,T] \text{ and all } x \in R$$

for two constants $K, K'$, then our SDE has a unique global solution $X_t$.

Let's check our conditions. In our case $\mu(t,x) = -a(t)x,\ \sigma(t,x) = \sigma$ are both measurable functions (being constant and linear), $X_0 = Z = x_0$ is deterministic, and thus trivially independent of $W$ and with finite mean square $E(Z^2) = x_0^2 < \infty$, and we can see that

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| = |-a(t)x + a(t)y| + |\sigma - \sigma| = |a(t)||x-y| \text{ for all } t \text{ and } x, y$$

so that the Lipschitz condition is satisfied by taking $K$ as the upper bound of the function $|a(t)|$, which is bounded by assumption. As for linear growth,

$$|\mu(t,x)| + |\sigma(t,x)| = |a(t)||x| + \sigma \leq K|x| + \sigma \leq \max(K,\sigma)(1+|x|) \quad \text{for all } t \text{ and } x$$

shows that also the linear growth condition is satisfied with $K' = \max(K,\sigma)$. Hence our SDE admits a unique global solution.

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(b) To calculate the solution without using Stratonovich let's use the hint. Calculate

$$dY_t = d\left(\exp\left(\int_0^t a(s)ds\right)\ X_t\right) =$$

$$= X_t d\exp\left(\int_0^t a(s)ds\right) + \exp\left(\int_0^t a(s)ds\right) dX_t =$$

$$= X_t \exp\left(\int_0^t a(s)ds\right) d\left(\int_0^t a(s)ds\right)$$

$$+ \exp\left(\int_0^t a(s)ds\right)((-a(t)X_t)dt + \sigma(t)dW_t) =$$

$$= X_t \exp\left(\int_0^t a(s)ds\right) a(t)dt$$

$$+ \exp\left(\int_0^t a(s)ds\right)((-a(t)X_t)dt + \sigma(t)dW_t)$$

$$= \exp\left(\int_0^t a(s)ds\right)\sigma(t)dW_t.$$

Thus

$$dY_t = \exp\left(\int_0^t a(s)ds\right)\sigma(t)dW_t.$$

This is a very easy SDE to integrate, as $Y$ is not on the right hand side. We simply integrate both sides between $0$ and $T$:

$$Y_T - Y_0 = \int_0^T \exp\left(\int_0^t a(s)ds\right)\sigma(t)dW_t.$$

Recalling that $Y_t = \exp\left(\int_0^t a(s)ds\right) X_t$ for all $t > 0$ and substituting this in the last equation above for $Y_T$ we get:

$$\exp\left(\int_0^T a(s)ds\right) X_T - X_0 = \int_0^T \exp\left(\int_0^t a(s)ds\right)\sigma(t)dW_t.$$

Now multiply both sides for $\exp\left(-\int_0^T a(s)ds\right)$ to get

$$\begin{aligned} &X_T - e^{-\int_0^T a(s)ds} X_0 \\ &= e^{-\int_0^T a(s)ds}\left[\int_0^T e^{\int_0^t a(s)ds}\sigma(t)dW_t\right]. \end{aligned}$$

leading to

$$X_T = e^{-\int_0^T a(s)ds}\left[X_0 + \int_0^T e^{\int_0^t a(s)ds}\sigma(t)dW_t\right]$$

or

$$X_T = e^{-\int_0^T a(s)ds} x_0 + \int_0^T e^{-\int_t^T a(s)ds}\sigma(t)dW_t.$$

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(c) We compute

$$E[X_T] = E[e^{-\int_0^T a(s)ds} x_0] + E\left[\int_0^T e^{-\int_t^T a(s)ds}\sigma(t)dW_t\right].$$

Now, the first quantity inside expectation on the right hand side is deterministic, so we can remove expectation. Moreover, we recall that the expected value of an Ito integral is zero. We get

$$E[X_T] = e^{-\int_0^T a(s)ds} x_0.$$

The informal intuition behind the Ito integral having zero expectation is that the increments of Brownian motions $dW_t$ are independent and each has zero expectation, so adding them up even when multiplied by deterministic constants gives a final result of zero.

2. (a) To draw a plot of $Y$ it is best to re-write it in different areas of the $S$ domain. We note that the two options will have different values depending on $S_T > K_1$ or $S_T > K_2$ so we distinguish three cases: meth seen $\Downarrow$ 5, A

(i) $S_T < K_2$, (ii) $K_2 < S_T < K_1$, (iii) $K_1 < S_T$.

Let us look at the three cases:

(i) $S_T < K_2 \implies$ the put option is in the money, the call option is out of the money and it is worth zero. The payoff is $Y = K_2 - S_T$.

(ii) $K_2 < S_T < K_1 \implies$ the put option is out of the money, and the call too. So the payoff is zero, $Y = 0$.

(iii) $K_1 < S_T \Rightarrow$ the put option is out of the money and has zero payoff, while the call is in-the-money and has positive payoff $S_T - K_1$, so short the call is $-(S_T - K_1) = K_1 - S_T$.

We get

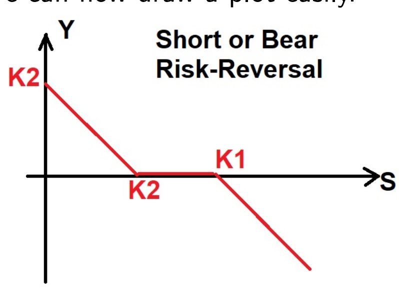
$$Y = -(S_T - K_1)^+ + (K_2 - S_T)^+ = \begin{cases} K_2 - S_T & \text{for } S_T \leq K_2 \\ 0 & \text{for } K_2 < S_T < K_1 \\ K_1 - S_T & \text{for } S_T \geq K_1 \end{cases}$$

If we include the initial price of $Y$ in the payoff itself, the initial price may be positive or negative depending on the strikes and other parameters. We would then have to shift the plot of the initial price to include the initial price of the trade in the overall payoff.

We can also write the payoff using indicator functions:

$$Y = (K_2 - S_T)1_{\{S_T \leq K_2\}} + (K_1 - S_T)1_{\{S_T > K_1\}}.$$

We can now draw a plot easily.




What kind of investor would buy this payoff? The payoff decreases with the stock, except in the interval $[K_2, K_1]$ where it stays constant to zero.

The payoff will make more money if the stock moves below $K_2$, and the more it moves below $K_2$ the more money it makes. The extreme case is the stock going to zero, which would give a value of $K_2$ to the payoff. This is the maximum value the payoff can take.

If the stock is between $K_2$ and $K_1$ the payoff is worth nothing, as both options expire out of the money.

Finally, if the stock is larger than $K_1$ then the put is worth nothing but the short call gives a negative payoff $K_1 - S_T$, and the payoff becomes negative, the more negative the more the stock becomes larger compared to $K_1$. Note that here the loss is potentially unlimited, as there is no bound for the stock to grow, as opposed for the put options where the stock could not go below zero.

It follows that an investor will buy this payoff only if she expects the stock price to move significantly below $K_2$ and will not be interested in buying this payoff (or might sell it) if she expects the stock to grow significantly above $K_1$.

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(b) To price the risk reversal we need the price of a put with strike $K_2$ and maturity $T$ minus the price of a call with strike $K_1$ and maturity $T$, both prices in the Black Scholes model. For the call we recall that

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$$V_{BS}^{CALL}(0, S_0, K_1, T, \sigma, r) = s_0 \Phi(d_1^{(1)}) - K_1 e^{-rT} \Phi(d_2^{(1)})$$

where $\Phi$ is the CDF of a standard normal and where

$$d_1^{(1)} = \frac{\ln(s_0/K_1) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \; d_2^{(1)} = d_1^{(1)} - \sigma\sqrt{T}.$$

For the put, we derive its price by put-call parity. Write the argument and the derivation here, as it has been done in the lecture, using the put call parity and the price of a forward contract. We get

$$V_{BS}^{PUT}(0, S_0, K_2, T, \sigma, r) = K_2 e^{-rT} \Phi(-d_2^{(2)}) - s_0 \Phi(-d_1^{(2)}).$$

$$d_1^{(2)} = \frac{\ln(s_0/K_2) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \; d_2^{(2)} = d_1^{(2)} - \sigma\sqrt{T}.$$

We can now calculate the Short Risk Reversal (SRR) price as

$$V_{BS}^{SRR}(0) = -V_{BS}^{CALL}(0, K_1) + V_{BS}^{PUT}(0, K_2) =$$

$$= -s_0 \Phi(d_1^{(1)}) + K_1 e^{-rT} \Phi(d_2^{(1)}) + K_2 e^{-rT} \Phi(-d_2^{(2)}) - s_0 \Phi(-d_1^{(2)}) =$$

$$= K_1 e^{-rT} \Phi(d_2^{(1)}) + K_2 e^{-rT} \Phi(-d_2^{(2)}) - s_0 \left( \Phi(d_1^{(1)}) + \Phi(-d_1^{(2)}) \right).$$

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(c) From the fact that the risk reversal is

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$$V_{BS}^{SRR}(0) = -V_{BS}^{CALL}(0, s_0, K_1) + V_{BS}^{PUT}(0, s_0, K_2)$$

we can calculate the delta quickly as

$$\frac{\partial V_{BS}^{SRR}(0)}{\partial s_0} = -\frac{\partial V_{BS}^{CALL}(0, s_0, K_1)}{\partial s_0} + \frac{\partial V_{BS}^{PUT}(0, s_0, K_2)}{\partial s_0}.$$

We know from memory (otherwise derive it, see lecture notes) that, in the basic theory of Black Scholes, the delta of a call option is

$$\frac{\partial V_{BS}^{CALL}(0, s_0, K_1)}{\partial S_0} = \Phi(d_1^{(1)}).$$

For the delta of a put, we use again put-call parity to derive the delta of a put from the delta of call and forward contract (see lecture notes). We obtain the formula

$$\frac{\partial V_{BS}^{PUT}(0, s_0, K_2)}{\partial S_0} = -\Phi(-d_1^{(2)}).$$

The total delta of the SRR is

$$\frac{\partial V_{BS}^{SRR}(0)}{\partial s_0} = -\frac{\partial V_{BS}^{CALL}(s_0, K_1)}{\partial s_0} + \frac{\partial V_{BS}^{PUT}(s_0, K_2)}{\partial s_0} = -\Phi(d_1^{(1)}) - \Phi(-d_1^{(2)}).$$

Note that the delta of the SRR is negative, meaning that the price of the SRR will go down when the stock $S_0$ increases. This is in agreement with our intuition of the payoff and the discussion in point a), as the payoff is decreasing in $S_T$.

3. (a) We know from the lectures that the formula for a call option in a Bachelier model is

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$$V^{Call}_{BaM}(0, s_0, K, T, \sigma) = (s_0 - K)\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right).$$

If you don't recall the formula, derive it, it's basic integration. Here $\Phi$ is the standard normal CDF and $p_N$ is the standard normal $PDF$.

For the put option, we can use put-call parity:

$$V_{Call} - V_{Put} = V_{Forward}.$$

Recall the price of a forward when $r = 0$ is $s_0 - K$, so that

$$V_{Call} - V_{Put} = s_0 - K \quad \Rightarrow V_{Put} = V_{Call} - (s_0 - K).$$

It follows that

$$V_{Straddle} = V_{Call} + V_{Put} = 2V_{Call} - (s_0 - K)$$

and recalling the call formula above,

$$V^{Str}_{BaM}(0, s_0, K, T, \sigma) = 2\left[(s_0 - K)\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\right] - (s_0 - K)$$

(b) From the previous point we obtained that

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$$V^{Str}_{BaM}(0, s_0, K, T, \sigma) = 2\left[(s_0 - K)\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\right] - (s_0 - K)$$

We can take a partial derivative with respect to $s_0$ to compute delta. We obtain

$$\frac{\partial V^{Str}_{BaM}}{\partial s_0} = 2\left[\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + (s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}}\right.$$
$$\left. + \sigma\sqrt{T}p'_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}}\right] - 1$$

where $p'_N$ is the first derivative of the standard normal PDF and we used the fact that $\Phi' = p_N$. We now need to compute

$$p'_N(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}(-x) = -xp_N(x)$$

so that, substituting,

$$\frac{\partial V^{Str}_{BaM}}{\partial s_0} = 2\left[\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + (s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}}\right.$$
$$\left. - (s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}}\right] - 1 = 2\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) - 1$$

This quantity sign depends on the argument of $\Phi$. Given that $\Phi(0) = 1/2$ and that $\Phi$ is increasing, we see that

$$\frac{\partial V^{Str}_{BaM}}{\partial s_0} = 2\left[\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) - \frac{1}{2}\right] = 2\left[\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) - \Phi(0)\right]$$

from which it will follow that delta is positive (the straddle price increases with $s_0$) when $s_0 > K$, and it is negative when $s_0 < K$.

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(c) For the vega, we start again from

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$$V^{Str}_{BaM}(0, s_0, K, T, \sigma) = 2\left[(s_0 - K)\Phi\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\right] - (s_0 - K)$$

and differentiate with respect to $\sigma$.

$$\frac{\partial V^{Str}_{BaM}}{\partial\sigma} = 2\left[(s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{s_0 - K}{\sqrt{T}}\frac{-1}{\sigma^2}\right.$$

$$\left. + \sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}p'_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{s_0 - K}{\sqrt{T}}\frac{-1}{\sigma^2}\right] = \dots$$

Using again $p'_N(x) = -xp_N(x)$ we get

$$= 2\left[(s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{s_0 - K}{\sqrt{T}}\frac{-1}{\sigma^2}\right.$$

$$\left. + \sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right) - (s_0 - K)p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)\frac{s_0 - K}{\sqrt{T}}\frac{-1}{\sigma^2}\right] = 2\sqrt{T}p_N\left(\frac{s_0 - K}{\sigma\sqrt{T}}\right)$$

Given that the normal PDF is always positive, being a probability density, vega is also always positive. This tells us that the straddle price will always increase with the volatility.

4. (a) To define ES we need first to define value at Risk (VaR). VaR is related to the potential loss on our portfolio over the time horizon $T$. Define this loss $L_T$ as the difference between the value of the portfolio today (time 0) and in the future $T$.

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$$L_T = \text{Portfolio}_0 - \text{Portfolio}_T.$$

VaR with horizon $T$ and confidence level $\alpha$ is defined as that number $q = q_{T,\alpha}$ such that

$$P[L_T < q] = \alpha$$

so that our loss at time $T$ is smaller than $q$ with $P$-probability $\alpha$. Recall that ES is then defined as the expectation of the loss conditional on the loss exceeding VaR:

$$\text{ES}_{T,\alpha} = E^{\mathbb{P}}[L_T | L_T > \text{VaR}_{T,\alpha}] = \frac{E^{\mathbb{P}}[L_T 1_{\{L_T > \text{VaR}_{T,\alpha}\}}]}{1-\alpha}$$

[see lecture notes for the steps to get to the last expression]

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(b) In the Bachelier model the equity process follows the dynamics

$$dS_t = \mu\, dt + \sigma\, dW_t, \quad s_0,$$

where $\mu, \sigma$ are positive constants and $W$ is a Brownian motion under the physical measure $P$.

We know that $S_T$ can be written as

$$S_T = S_0 + \mu T + \sigma W_T, \tag{1}$$

and recalling the distribution of $W_T \sim \sqrt{T}\mathcal{N}(0,1)$,

$$S_T = s_0 + \mu T + \sigma\sqrt{T}\mathcal{N}(0,1) \tag{2}$$

so that in our case $L_T = N(S_0 - S_T)$, namely

$$L_T = N\left(S_0 - (S_0 + \mu T + \sigma\sqrt{T}\mathcal{N}(0,1))\right)$$

$$= N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right)$$

Hence, if $q = VaR_{T,\alpha}$, we get

$$\alpha = P[L_T < q] = P\left[N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right) < q\right]$$

$$\alpha = P\left[-\mathcal{N}(0,1) < \frac{\frac{q}{N} + \mu T}{\sigma\sqrt{T}}\right] = \Phi\left(\frac{\frac{q}{N} + \mu T}{\sigma\sqrt{T}}\right)$$

where we used the fact that $-\mathcal{N}(0,1)$ is still distributed as the standard normal. Then, taking $\Phi^{-1}$ on both sides,

$$\Phi^{-1}(\alpha) = \frac{\frac{q}{N} + \mu T}{\sigma\sqrt{T}} \tag{3}$$

and therefore

$$q = N(-\mu T + \sigma\sqrt{T}\Phi^{-1}(\alpha)).$$

This is our $VaR_{T,\alpha}$ for the stock position. To compute ES we need to look at

$$\begin{aligned}
\mathsf{ES}_{T,\alpha} &= \frac{E^{\mathbb{P}}[L_T 1_{\{L_T > \mathsf{VaR}_{T,\alpha}\}}]}{1-\alpha} \\
&= \frac{E^{\mathbb{P}}[N(S_0 - S_T) 1_{\{N(S_0-S_T) > \mathsf{VaR}_{T,\alpha}\}}]}{1-\alpha} \\
&= \frac{E^{\mathbb{P}}[N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right) 1_{\{N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right) > q\}}]}{1-\alpha} = \dots
\end{aligned}$$

We can compute the expectation through an integral:

$$\begin{aligned}
&E^{\mathbb{P}}[N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right) 1_{\{N\left(-\mu T - \sigma\sqrt{T}\mathcal{N}(0,1)\right) > q\}}] = \\
&= \int_{-\infty}^{+\infty} \left[N\left(-\mu T - \sigma\sqrt{T}x\right) 1_{\{N\left(-\mu T - \sigma\sqrt{T}x\right) > q\}}\right] p_{\mathcal{N}(0,1)}(x) dx = \\
&= \int_{-\infty}^{+\infty} N\left(-\mu T - \sigma\sqrt{T}x\right) 1_{\{x < (-q - N\mu T)/(N\sigma\sqrt{T})\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{(-q-N\mu T)/(N\sigma\sqrt{T})} N\left(-\mu T - \sigma\sqrt{T}x\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= -N\mu T \int_{-\infty}^{\frac{-q-N\mu T}{N\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - N\sigma\sqrt{T} \int_{-\infty}^{\frac{-q-N\mu T}{N\sigma\sqrt{T}}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= -N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) - N\sigma\sqrt{T} \int_{-\infty}^{(-q-N\mu T)/(N\sigma\sqrt{T})} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= -N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) + N\sigma\sqrt{T} \int_{-\infty}^{(-q-N\mu T)/(N\sigma\sqrt{T})} \frac{1}{\sqrt{2\pi}} d\left(e^{-\frac{x^2}{2}}\right) \\
&= -N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) + N\sigma\sqrt{T} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) \Big|_{-\infty}^{\frac{-q-N\mu T}{N\sigma\sqrt{T}}} = \\
&= -N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) + N\sigma\sqrt{T} \left(\phi(x)\right) \Big|_{-\infty}^{\frac{-q-N\mu T}{N\sigma\sqrt{T}}} = \\
&-N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) + N\sigma\sqrt{T} \phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right)
\end{aligned}$$

where $\phi$ is the density of the standard normal, $\phi(x) = p_{\mathcal{N}(0,1)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and we used that $\lim_{x\to-\infty} \phi(x) = 0$. Substituting back we get

$$\dots = ES_{T,\alpha} = \frac{-N\mu T \Phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right) + N\sigma\sqrt{T} \phi\left(\frac{-q-N\mu T}{N\sigma\sqrt{T}}\right)}{1-\alpha}$$

*The above is acceptable as final solution with full marks*. However, using (3), this can be simplified further as

$$\begin{aligned}
\dots = ES_{T,\alpha} &= \frac{-N\mu T \Phi\left(-\Phi^{-1}(\alpha)\right) + N\sigma\sqrt{T} \phi\left(-\Phi^{-1}(\alpha)\right)}{1-\alpha} \\
&= \frac{-N\mu T\left(1 - \Phi\left(\Phi^{-1}(\alpha)\right)\right) + N\sigma\sqrt{T} \phi\left(-\Phi^{-1}(\alpha)\right)}{1-\alpha} \\
&= \frac{-N\mu T(1-\alpha) + N\sigma\sqrt{T} \phi\left(-\Phi^{-1}(\alpha)\right)}{1-\alpha} \\
&= -N\mu T + \frac{N\sigma\sqrt{T} \phi\left(-\Phi^{-1}(\alpha)\right)}{1-\alpha}
\end{aligned}$$

(c) ES does not completely look at the tail structure of the Loss, but does so only in expectation. So if $99\%$ VaR is 10 billions, we can have the remaining $1\%$ loss concentrated

seen ⇓

4, A

(i) either on 10.1 billions,

(ii) or on 10 trillions,

as two stylized cases, without VaR being able to tell us anything on whether we are in case (i) or (ii).

ES does a little better than VaR, in that it averages the tail. The average in case (ii) will be much larger than the average of case (i), thus alerting one to more risk in case (ii). Still, it won't tell us exactly how the tail risk looks like or where exactly the loss is concentrated on the tail.

Another problem of ES is that it is homogeneous with respect to the portfolio size. Namely, if $k$ is a positive constant, then

$VaR(k\ Portfolio) = k\ VaR(Portfolio)$

and

$ES(k\ Portfolio) = k\ ES(Portfolio)$.

This is unrealistic and completely neglects liquidity risk. Buying one million of shares is more than one million times risky than buying one share. Placing the order for one million shares will move the whole market and change the share price (theory of maket impact/market microstructure) with potential additional losses due to market impact, whereas placing the order for one share willl not move the market. Liquidity risk strongly disagrees with the homogeneous assumption.

meth seen ⇓

(d) No the dynamics is not the same, to price an option we need to use the risk neutral dynamics, where the drift parameter $\mu$ of $S$ is replaced by the risk free rate $r = 0$ of the bank account. So to price an option we need to use the dynamics under $Q$, $dS_t = \sigma dW^Q$. To compute value at risk or expected shortfall the dynamics that is relevant up to the risk horizon is the dynamics under $P$, i.e. the dynamics with drift $\mu$.

4, B

5. (a) Black Scholes economy consists of two assets: 5, M

$$
\begin{aligned}
dB_t &= B_t r dt, \quad B_0 = 1, \\
dS_t &= \mu\ S_t\ dt + \sigma\ S_t\ dW_t, \quad 0 \le t \le T,
\end{aligned}
$$

The second asset (a stock) is risky and its price is described by the process $S_t$. Furthermore, it is assumed that

* (i) there are no transaction costs in trading the stock;
* (ii) the stock pays no dividends or other distributions;
* (iii) shares are infinitely divisible;
* (iv) short selling is allowed without any restriction or penalty. Short selling: investor borrows a security and sells it on the market, planning to buy it back later for less money to give it back to the lender and make a profit. This assumes total absence of credit /default risk.

These assumptions are Black & Scholes' *ideal conditions*.

Note that the stock price is a positive process, as should be, being a geometric Brownian motion. Note that the cash account $B$ is $B_t = e^{rt}$ and grows exponentially at the risk free rate. This is a risk free asset.

5, M

(b) We now assume that the value of the simple claim at time $t$ is a function of the underlying stock $S$ at the same time, namely $V_t = V(t, S_t)$. This is the candidate claim (option) value at time $t$. Assume the function $V(t, S)$ of time $t$ and of the stock price $S$ to have regularity $V \in C^{1,2}([0,T] \times \mathbb{R}^+)$. In other terms, we assume $V$ is twice continously differentiable with respect to $S$ and once continuously differentiable with respect to $t$. Apply Ito's formula to $V$:

$$
dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)\ dt + \frac{\partial V}{\partial S}(t, S_t)\ dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\ dS_t\ dS_t. \tag{4}
$$

Substituting the equation for $dS_t = \mu S_t dt + \sigma S_t dW_t$ and recalling that $dt\ dt = 0,\ dt\ dW_t = 0,\ dW_t dW_t = dt$, we get

$$
\begin{aligned}
dV(t, S_t) &= \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2\right) dt \\
&+ \frac{\partial V}{\partial S}(t, S_t)\sigma S_t dW_t.
\end{aligned} \tag{5}
$$

Now, if we are looking to attain our claim with a self financing strategy $\phi^S, \phi^B$, so that $V(t, S_t) = \phi_t^S S_t + \phi_t^B B_t$, this will have to satisfy the self financing condition, namely

$$
dV(t, S_t) = \phi_t^B dB_t + \phi_t^S\ dS_t. \tag{6}
$$

Compare this last Eq to Eq (4) in the previous slide and match the $dS_t$ terms. We get, for each $0 \le t \le T$,

$$
\phi_t^S = \frac{\partial V}{\partial S}(t, S_t), \quad \phi_t^B = (V_t - \phi_t^S S_t)/B_t. \tag{7}
$$

where the first equation comes from the matching, and the second equation comes by construction, as the value of the strategy at time $t$ must be $V$ itself, and clearly

$V(t, S_t) = \phi_t^B \ B_t + \ \phi_t^S \ S_t$. In other terms, to get the second equation solve $V(t, S_t) = \phi_t^B \ B_t + \ \phi_t^S \ S_t$ in $\phi_t^B$.

Now we can explicit the self financing condition for $\phi$:

$$\begin{aligned} dV_t &= \phi_t^B dB_t + \phi_t^S dS_t \\ &= \left[ V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t) S_t \right] r dt + \frac{\partial V}{\partial S}(t, S_t) S_t (\mu dt + \sigma dW_t). \end{aligned} \tag{8}$$

Then by equating (5) and (8) (ITO + SELF FINANCING), we obtain that $V_t$ satisfies

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t) \sigma^2 S_t^2 = r V(t, S_t), \tag{9}$$

which is the celebrated Black and Scholes partial differential equation with terminal condition $V_T = (S_T - K)^+$.

5, M

(c) The Feynman-Kac Theorem allows to interpret the solution of a parabolic PDE such as the Black and Scholes PDE in terms of expected values of a diffusion process. In general, given suitable regularity and integrability conditions, the solution of the PDE

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) b(x) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, x) \sigma^2(x) = r V(t, x), \quad V(T, x) = f(x), \tag{10}$$

can be expressed as

$$V(t, x) = e^{-r(T-t)} E_{t,x}^Q \{ f(X_T) | \mathcal{F}_t \} \tag{11}$$

where the diffusion process $X$ has dynamics starting from $x$ at time $t$

$$dX_s = b(X_s) ds + \sigma(X_s) dW_s^Q \ , s \geq t, \ X_t = x \tag{12}$$

under the probability measure $\mathbb{Q}$ under which the expectation $E_{t,x}^Q \{\cdot\}$ is taken. The process $W^Q$ is a standard Brownian motion under $Q$.

By applying this theorem to the Black and Scholes setup, with $b(x) = rx$, $\sigma(x) = \sigma \ x$ (so that the general PDE of the theorem coincides with the BeS PDE) we obtain:

The unique no-arbitrage price of the integrable contingent claim $Y = (S_T - K)^+$ (European call option) at time $t$, $0 \leq t \leq T$, is given by

$$V_{BS}(t) = E^Q \left( e^{-r(T-t)} Y | \mathcal{F}_t \right). \tag{13}$$

The expectation is taken with respect to the so-called **martingale measure** $Q$, i.e. a probability measure $Q \sim P$ under which the risky–asset price $S_t / B_t = e^{-rt} S_t$ measured with respect to the risk-free asset price $B_t$ is a martingale, which is equivalent to $S$ having drift rate $r$ under $Q$:

$$dS_t = S_t [r dt + \sigma dW_t^Q], \quad 0 \leq t \leq T, \tag{14}$$

(d) We apply Girsanov's theorem to move from

$$d\ S_t = \mu S_t\ dt + \sigma S_t\ dW_t^P$$

to

$$d\ S_t = r S_t\ dt + \sigma S_t\ dW_t^Q$$

and obtain the Radon Nykodym derivative connecting $Q$ with $P$.

$$\frac{dQ}{dP} = \exp\left\{ -\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T \right\}. \tag{15}$$

Note that the Novikov condition needed for Girsanov to work is satisfied trivially:

$$E\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu S_t - r S_t}{\sigma S_t}\right)^2 dt\right)\right] = \exp\left(\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) < +\infty$$

The quantity

$$\frac{\mu - r}{\sigma}$$

is called "market price of risk", or sometimes, in finance circles, the Sharpe ratio. It tells us how much better the stock $S$ is doing with respect to the risk free rate, divided by the volatility. In the real world the stock local growth rate or "return" is $\mu$. So $\mu - r$ is the difference between $S$'s return and the risk free rate, telling us how much better $S$ is doing than a cash account $B$.

**Review of mark distribution:**

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

# MATH60130 Stochastic Differential Equations in Financial Modelling

| Question | Marker's comment |
| --- | --- |
| 1 | First marker comments.nbsp;This question was expected to be easy, having been covered in the lectures and being a special case of the mock exam problem to which students had been alerted, also written in the lecture notes. The performances were mixed, with marks ranging from 0 to 20 but quite a number ranging around high marks 17-20.Point a),nbsp; existence and uniqueness of solutions of the linear SDE, some students commented on measurability of a(t) and \sigma(t) in t but not of a(t) x in x. Also, somenbsp; students didn't comment about measurability at all, or the condition on x0 at all, focusing only on Lipschitz and linear growth, but I was lenient for this.On the core Lipschitz and linear growth conditions, a few students used \mu(t,x) = a(t) instead of a(t)x, and got a wrong condition. I was surprised by this common error, as it was very clear in the mock exam and in the lectures that $-a(t)x$ was the drift, and not $-a(t)$ alone. Other students made basic mistakes with absolute values, like \|-ax + ay \|= a\|x-y\| instead of the correct \|a\|\|x-y\| and other similar very basic errors.nbsp;nbsp;Point b), solving the SDE, was again relatively easy, integration of a linear SDE with a suggested change of variable that made the integration completely trivial. Here again I found occasional basic mistakes on integration and differentiation. The errors in point b) carried over to point c) (computing the expectation of the solution) for quite a few students, although I have marked keeping in mind also correctness in context. Some students couldn't explain properly the intuition behind the Wiener integral having zero mean, which was discussed at length several times in class and also in the mock exam in the lecture notes the students had been alerted to, during the lectures. A few students, rather than calculating the expectation from the SDE solution theynbsp; had derived, decided to derive an ODE for the mean and solve it. This was basically an overkill, as the other method was much faster, but I gave full marks for this too if correct.nbsp;nbsp;Overall this question went a little worse than I expected, being one of the simplest possible SDEs and having been done and emphasized in one of the mock exams in more general form, although quite a number of students got full marks 20/20 or close to full marks.nbsp; |

# MATH70130 Stochastic Differential Equations in Financial Modelling

| Question | Marker's comment |
|---|---|
| 1 | First marker comments.nbsp;This question was expected to be easy, having been covered in the lectures and being a special case of the mock exam problem to which students had been alerted, also written in the lecture notes. However, the performances were relatively poor, with marks ranging from 7 to 20 (excluding cases with zero input) and quite a number ranging around 10.Point a),nbsp; existence and uniqueness of solutions of the linear SDE, many students commented on measurability of a(t) and $\sigma(t)$ in t but not of a(t) x in x. Also, many students didn't comment about measurability or the condition on x0 at all, but I was lenient for this.On the core Lipschitz and linear growth conditions, quite a few students used $\mu(t,x) = a(t)$ instead of a(t)x, and got a wrong condition. I was surprised by this common error, as it was very clear in the mock exam and in the lectures that $-a(t)x$ was the drift, and not $-a(t)$ alone. Other students made basic mistakes with absolute values that I wasn't expecting in year 4/MSc, like $\|-ax + ay \|= a\|x-y\|$ instead of the correct $\|a\|\|x-y\|$ and other similar errors.nbsp;Point b), solving the SDE, was again relatively easy, integration of a linear SDE with a suggested change of variable that made the integration completely trivial. Here again I found basic mistakes on integration and differentiation that I wasn't expecting from students in year 4/MSc. A couple of students, despite me asking not to use Stratonovich calculus in the problem, did. The errors in point b) carried over to point c) (computing the expectation of the solution) for quite a few students, although I have marked keeping in mind also correctness in context. Some students couldn't explain properly the intuition behind the Wiener integral having zero mean, which was discussed at length several times in class and also in the mock exam the students had been alerted to during the lectures.nbsp;Overall this question went worse than I expected, being one of the simplest possible SDEs and having been done and emphasized in one of the mock exams in more general form.nbsp; |