Imperial College London
Department of Mathematics

Question Sheet 6 - Probl. Class week 9

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

Problem sheet released on Monday of week 8. All questions can be attempted before the problem class on Monday of week 9. Solutions will be released after the problem class.

Question 1 Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Prove that each of the following is an equivalence relation on G (where g, h are elements of G):

(i) $g \sim_1 h$ if and only if there is $k \in G$ with $h = kgk^{-1}$;

(ii) $g \sim_2 h$ if and only if $h^{-1}g \in H$.

In the case where $(G, .)$ is the group $(\mathbb{R}^2, +)$ and H is the subgroup $\{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, describe geometrically the $\sim_2$-equivalence classes. What are the $\sim_1$-equivalence classes?

Question 2 Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.

(i) Show that $H \cap K$ is a subgroup of G.

(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$.

Question 3 Which of the following groups are cyclic?

(a) S_2.

(b) $\mathrm{GL}(2, \mathbb{R})$.

(c) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \{1, -1\} \right\}$ under matrix multiplication.

(d) $(\mathbb{Q}, +)$.

Question 4 Let G and H be finite groups. Let $G \times H$ be the set $\{(g, h) \mid g \in G, h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

(a) Show that $(G \times H, *)$ is a group.

(b) Show that if $g \in G$ and $h \in H$ have orders a, b respectively, then the order of (g, h) in $G \times H$ is the lowest common multiple of a and b.

(c) Show that if G and H are both cyclic, and $\gcd(|G|, |H|) = 1$, then $G \times H$ is cyclic. Is the converse true?

Question 5 Find an example of each of the following:

(a) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{C})$.

(b) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{R})$.

(c) an element of infinite order in the group $\mathrm{GL}(2, \mathbb{R})$.

(d) an element of order 12 in the group S_7.

Question 6 Prove that if $\{x_1, \ldots, x_n\}$ is any finite subset of $(\mathbb{Q}, +)$, then the subgroup $\langle x_1, \ldots, x_n \rangle$ is cyclic.