Imperial College London

MATH97225 MATH70038

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Graph Theory

Date: 25 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

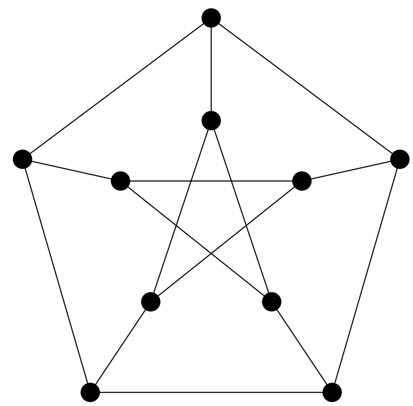
SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let G be a graph. A set of vertices D is said to be *dominating* if for every $v \in V_G$ either $v \in D$ or there is a $d \in D$ such that dEv. A set D is said to be *independent* if for all $d_1, d_2 \in D$ there is no edge e such that $\epsilon(e) = \{d_1, d_2\}$.

(a) Find an independent dominating set for the following graphs.

(i) K_n. You must justify your answer. (2 marks)

(ii) The Petersen Graph.



(2 marks)

(b) Prove that every finite graph has an independent dominating set. (6 marks)

(c) Let G, H be graphs, let $D \subseteq V_G$ and let $\phi : G \to H$ be a homomorphism.

(i) Prove that if ϕ is an isomorphism and D is independent and dominating then $\phi(D)$ is also independent and dominating.

(4 marks)

(ii) Find an example of G, H, D, and ϕ such that D is dominating but $\phi(D)$ is not dominating. (2 marks)

(iii) Find an example of G, H, D, and ϕ such that D is independent but $\phi(D)$ is not independent. (4 marks)

(Total: 20 marks)

2. A *simple directed graph* is a simple graph where for every edge there is a *start vertex* and *end vertex*, rather than just two endpoints. Standard notation for simple directed graphs is to write $v \to w$ if there is an edge that starts at v and ends at w. Paths in a simple directed graph must travel along edges the correct way, i.e. if $x \to y$ then x, y is a path, but y, x is not.

(a) How many simple directed graphs are there on $\{v_1, \ldots, v_n\}$, where the order of the vertices matters? You must justify your answer. (4 marks)

(b) A *tournament* is a simple directed graph where there is an edge between every pair of vertices.

(i) A tournament is *transitive* if for all vertices u, v, w:

$$(u \to v \text{ and } v \to w) \Rightarrow u \to w.$$

Prove that for all n there is a transitive tournament with n vertices. (2 marks)

(ii) Prove that every tournament has a path that visits every vertex. (6 marks)

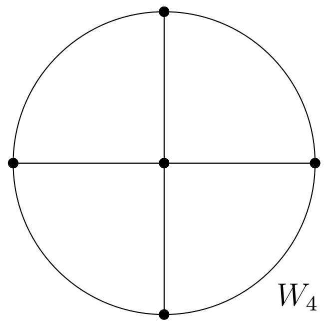
(c) The degree of a vertex in a simple directed graph is the number of edges with that vertex as either a start vertex or an end vertex. Find a simple directed graph where every vertex has even degree, but there is not an Eulerian tour. (2 marks)

(d) If G is a tournament, and H is a tournament such that the vertices of H are a subset of the vertices of G, we say that $H \leq G$ if and only if the direction of the edges in H is the same as the direction of the edges in G.

Find a tournament G such that for all n there is a tournament H such that $|H| > n$ and $G \not\leq H$. (6 marks)

(Total: 20 marks)

3. The *wheel* graph W_n is a graph with vertex set $\{w_0, \ldots, w_n\}$. There are edges that make $w_1, \ldots, w_n$ into an n-cycle, and edges from w_0 to any other vertex, but there are no other edges.



(a) (i) Prove that W_n is 4-colourable for all n. You may apply any results from the module you choose. (2 marks)

(ii) Prove that there is an n such that W_n is not 3-colourable. (2 marks)

(iii) Find a maximal subset $X \subseteq \{W_n : n \in \mathbb{N}\}$ such that if $W \in X$ then W is 3-colourable. You must prove that your subset is both maximal and only contains 3-colourable graphs. (6 marks)

(b) Prove that W_n does not have an Eulerian tour for any $n > 2$. (2 marks)

(c) Recall that $\mathbb{P}(\mathcal{G}(n+1, p) \cong W_n)$ is the probability that a random graph on $n+1$ elements (where the edges have probability p of being included) is isomorphic to W_n. Prove that $\mathbb{P}(\mathcal{G}(n+1, p) \cong W_n)$ converges to 0 as $n \to \infty$. (8 marks)

(Total: 20 marks)

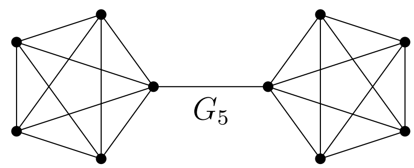
4. (a) Let G and H be graphs. Recall that $\mathrm{ex}(n, G)$ is the smallest number such that for all graphs K, if $|K| = n$ and $\| K \| \geq \mathrm{ex}(n, G)$ then $G \subseteq K$.

Let $\mathrm{ex}_2(n, G, H)$ be the smallest number such that for all graphs K, if $|K| = n$ and $\| K \| \geq \mathrm{ex}_2(n, G, H)$ then $G, H \subseteq K$. Prove that if $|G|, |H| < n$ then

$$\mathrm{ex}_2(n, G, H) = \max(\mathrm{ex}(n, G), \mathrm{ex}(n, H)).$$

(3 marks)

(b) Let G_m be the graph consisting of two disjoint copies of K_m with a single extra edge.



(i) Suppose $n > 2m$. Prove $\mathrm{ex}(n, G_m) \leq \| T(n, 2m) \|$, where $T(n, 2m)$ is the Turán graph with n many vertices and $2m$ many parts. (2 marks)

(ii) Suppose that $n = (2m - 1)x + c$, where $c < m$ and $x \geq 1$. Prove that

$$\mathrm{ex}(n, G_m) > \frac{x}{2}(2m - 2)(2m - 3) + \max(\frac{1}{2}(c - 1)(c - 2), 0) + xc.$$

(4 marks)

(c) Let N be a network. A flow f contains a *circulation* if there is a cycle $v_1, \ldots, v_n$, using edges $e_1, \ldots, e_n$ such that:

* $\sigma(e_i) = v_i$ for all i, and
* if $i < n$ then $\tau(e_i) = v_{i+1}$, and
* $\tau(e_n) = v_1$, and
* $f(e_i) > 0$ for all i.

(i) Prove that for every f with a circulation there is a flow g without a circulation such that $v(f) = v(g)$. (8 marks)

(ii) Give an example of a network where a maximal flow cannot have a circulation.

(1 mark)

(iii) Give an example of a network where a maximal flow can have a circulation. (2 marks)

(Total: 20 marks)

5. Let X and Y be disjoint sets of vertices in a graph G, and let $d \in (0,1]$. You may find the defintions of the density $d(X,Y)$ and what it means for (X,Y) to be an ϵ-regular pair useful for this question.

Any results you use must be from the lecture notes, and must be stated explicitly.

(a) Let $\epsilon \in (0, \frac{d}{2})$. Suppose that (X,Y) is an ϵ-regular pair with density greater than or equal to d. Prove that if $|X| = |Y| = n$ then there are at most ϵn vertices in X with less than $\frac{dn}{2}$ neighbours in Y. (6 marks)

(b) Let $\epsilon \in (0, \frac{d}{2})$. Suppose that $\{V_0, \dots, V_k\}$ is an ϵ-partition of G, such that $\epsilon < \frac{1}{2k}$ and the density of the ϵ-regular pairs of the partition is at least d. Suppose that $|V_1|, \dots, |V_k| = n$. Prove that there is a path of length k in G. (14 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60038/70038/97225

Graph Theory (Solutions)

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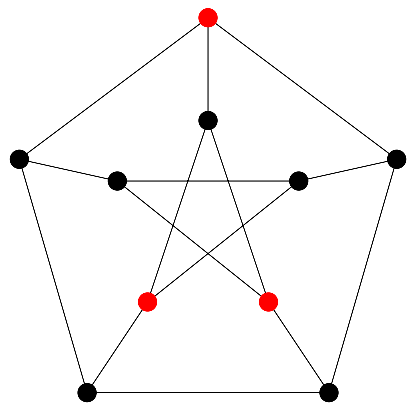
1. (a) (i) Every vertex of K_n has an edge to every other edge, so every non-empty set of vertices is dominating, and any set of vertices that contains more than 1 vertex is not independent. Therefore every set that contains exactly one vertex of K_n is an independent dominating set. unseen $\Downarrow$ 2, A

(ii) There are many examples, including:



2, A

(b) Let G be a graph. We construct an set $\{v_1, \dots, v_k\}$ inductively, which will be independent dominating. We will keep track of which vertices are adjacent to $\{v_1, \dots, v_i\}$ at each stage via sets which we will call X_i. This will let us ensure that the result is both independent and dominating. Let $v_1 \in V_G$, and let $X_1 = \{w \in V_G : v_1 E w\} \cup \{v_1\}$, i.e. the set of all vertices that are adjacent to v_1, together with v_1 itself. If $X_1 = V_G$ then $\{v_1\}$ is an independent dominating set, otherwise we continue the construction. unseen $\Downarrow$

Suppose we have defined $v_1, \dots, v_n$ and $X_1, \dots, X_n$, and that $X_n \neq V_G$. Let $v_{n+1} \in V_G \setminus X_n$. Let $X_{n+1} = \{w \in V_G : v_{n+1} E w\} \cup X_n \cup \{v_{n+1}\}$. If $X_{n+1} = V_G$ then $\{v_1, \dots, v_{n+1}\}$ is an independent dominating set, otherwise we continue the construction.

Since G is finite, this process will eventually stop. Therefore G has an independent dominating set. 6, C

(c) (i) Let $\phi : G \to H$ be an isomorphism, and let D be an independent dominating set of G.

Let $w \in V_G$. Since D is a dominating set of G, there is a $d \in D$ such that $dE\phi^{-1}(w)$. Since ϕ is an isomorphism, we also have that $\phi(d)Ew$. Therefore $\phi(D)$ is a dominating set of H. 2, A

Suppose that there are $d_1, d_2 \in D$ such that $\phi(d_1)E\phi(d_2)$. Then $\phi^{-1}(\phi(d_1))E\phi^{-1}(\phi(d_2))$, and therefore $d_1 E d_2$. This contradicts the assumption that D is an independent set, so $\phi(D)$ is also an independent set. 2, A

(ii) Let G be the graph with a single vertex, and let $H = G \oplus G$, and let $D = V_G$, which is clearly dominating. Then either of the two homomorphisms from G to H maps D to a set that is not dominating. 2, A

(iii) Let G be the graph with vertices a_1, a_2, b_1, b_2 be such that $a_1 E a_2$ and $b_1 E b_2$, and has no other edges.

Let H be the graph with vertices c_1, c_2 be such that $c_1 E c_2$.

Then let $\phi(a_1) = \phi(b_1) = c_1$, and let $\phi(a_2) = \phi(b_2) = c_2$. Then the set $\{a_1, b_2\}$ is independent, but $\phi(\{a_1, b_2\}) = \{c_1, c_2\}$ is not independent. 4, B meth seen $\Downarrow$

2. (a) There are $\binom{n}{2}$ many possible edges in $\{v_1, \ldots, v_n\}$. Between vertices v_i and v_j there are three possibilities, an edge starting at v_i, an edge starting at v_j, and no edge. Therefore there are

$$3^{\binom{n}{2}}$$

many directed graphs with the given labelling. $\boxed{4, \mathrm{A}}$

(b) (i) Let G be the tournament on vertices $\{v_1, \ldots, v_n\}$ such that there is an edge that starts at v_i and ends at v_j if and only if $i < j$.

unseen $\Downarrow$

$\boxed{2, \mathrm{B}}$

(ii) We prove this by inducting on the size of the tournament.
If G has a single vertex, then the path that starts and ends at that single vertex is a path that visits every vertex.
Suppose that if G is a tournament such that $|G| < n$ then G has a path that visits every vertex. Let T be a tournament on the set $\{v_1, \ldots, v_n\}$. Consider the following tournaments:

$$A := \{v \in G \,:\, v \to v_1\}\, B := \{v \in G \,:\, v_1 \to v\}$$

where A and B inherit the direction of edges from T. If A is empty then B is non-empty. Since $|B| = |T| - 1 < n$, there is a path which visits every vertex of B. We call this path $b_1, \ldots, b_j$. Then, since $v \to b_1$, we know that $v, b_1, \ldots, b_j$ is a path, and so we have a path that visits every vertex.
Similarly, if B is empty, then $|A| = |T| - 1 < n$, so there is a path $a_1, \ldots, a_i$ which visits every vertex in A. Then $a_i \to v$, so $a_1, \ldots, a_i, v$ is a path that visits every vertex.
Suppose both A and B are non-empty. Then $|A|, |B| < |T| = n$, so by the inductive hypothesis, there is a path that visits every vertex of A, which we call $a_1, \ldots, a_i$, and a path that visits every vertex of B, which we call $b_1, \ldots, b_j$. By assumption, $a_i \to v_1 \to b_1$, so

$$a_1, \ldots, a_i, v_1, b_1, \ldots, b_j$$

is a path, which visits every vertex of T. $\boxed{6, \mathrm{D}}$

(c) In the graphs defined in the solutions to (b)(i), every tour can use at most one edge that ends at v_n. If we take $n = 5$ then every vertex has even degree. $\boxed{2, \mathrm{B}}$

(d) Let G be the tournament on vertices a,b,and c, such that $a \to b$, $b \to c$, and $c \to a$. If H is one of the graphs defined in (b)(i), then $G \not\leq H$. $\boxed{6, \mathrm{B}}$

3. (a) (i) **Either**: State that W_n is planar, and apply Wagner's Theorem. meth seen $\Downarrow$

Or: Remove the central vertex. In the remaining graph, every vertex has degree 2, and we can prove by induction that this means that it is 3 colourable. The central vertex can be given a fourth colour, so we have a 4 colouring of W_n. 2, A

(ii) W_3 is actually K_4, and hence is 4-colourable. 2, A

(iii) If n is even, then w_0 can be coloured red, w_{2i} for $i > 0$ can be coloured blue, and w_{2i-1} for $i > 0$ can be coloured green. Therefore W_n for even n is 3-colourable. 2, C

Suppose n is odd. Then w_0 must have a different colour to all the other vertices. Therefore W_n is 3-colourable if and only if C_n is 2-colourable. If C_n is 2-colourable, then w_2 must have a different colour to w_1, and w_3 must have a different colour to w_2, and hence the same colour as w_1, etc. Since n is odd, this means that w_n has the same colour as w_1, but this is a contradiction, as $w_1 E w_n$. Therefore C_n is not 2-colourable, and hence W_n is not 3-colourable. Therefore $\{W_n : n \text{ is even}\}$ is a maximal set of 3-colourable wheels. 4, C

(b) For all n, the vertices w_i where $i > n$ have degree 3. Therefore for all n, the wheel W_n has vertices with odd degree, and therefore does not have an Eulerian tour. 2, A

(c) Let's first calculate the probability that $\mathcal{G}(n+1, p)$ is equal to W_n with a given labelling. There are n many edges in the outer ring. and n-many "spokes", so there are $2n$ edges in W_n. Therefore the probability that $\mathcal{G}(n+1, p)$ is equal to W_n with a given labelling is unseen $\Downarrow$

$$p^{2n}(1-p)^{\binom{n+1}{2}-2n} = p^{2n}(1-p)^{\frac{1}{2}n(n-5)}$$

2, D

There are $(n+1)!$ many ways of labelling W_n, and therefore the probability that $\mathcal{G}(n+1,p) \cong W_n$ is

$$a_n = (n+1)! p^{2n}(1-p)^{\frac{1}{2}n(n-5)}$$

Let's investigate $\frac{a_{n+1}}{a_n}$. 4, D

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+2)! p^{2n+2}(1-p)^{\frac{1}{2}(n+1)(n-4)}}{(n+1)! p^{2n}(1-p)^{\frac{1}{2}n(n-5)}} \\ &= (n+2)p^2(1-p)^{3n-4} \\ &\underset{n\to\infty}{\to} 0 \end{aligned}$$

Therefore $\sum a_n$ converges, so $a_n \to 0$ as $n \to \infty$. 2, A

Accept counting the number of labelled graphs isomorphic to W_n, and showing that the proportion of all labelled graphs isomorphic to W_n goes to zero.

Common other methods included taking an upper bound by considering the probability that G has the right density.

4. Note for anyone reading whose not familiar with the module: I've used in the lecture notes the standard notation that $G \subseteq H$ means that G is a not-necessarily-induced subgraph of H, while $G \leq H$ means that G is an induced subgraph of H. unseen ⇓

(a) If $|K| = n$ and $\| K \| \geq \max(\mathrm{ex}(n, G), \mathrm{ex}(n, H))$ then K has both G and H as subgraphs, and therefore

$$\mathrm{ex}_2(n, G, H) \leq \max(\mathrm{ex}(n, G), \mathrm{ex}(n, H)).$$

There is a graph K such that $|K| = n$ and $\| K \| < \max(\mathrm{ex}(n, G), \mathrm{ex}(n, H))$ such that K either does not have G as a subgraph, or does not have H as a subgraph. Then $\| K \| < \mathrm{ex}_2(n, G, H)$. Therefore

$$\mathrm{ex}_2(n, G, H) \geq \max(\mathrm{ex}(n, G), \mathrm{ex}(n, H)).$$

3, A

(b) (i) If $|H| = n$ and $\| H \| \geq \| T(n, 2m) \|$ then H contains K_{2m} as a subgraph. meth seen ⇓ G_m has $2m$ many vertices, and therefore $G_m \subseteq T(n, 2m) \subseteq H$. Therefore $\mathrm{ex}(n, G_m) \leq \| T(n, 2m) \|$. 2, A

(ii) Let H be the graph with vertices $h_1, \ldots, h_n$ and the following edges: unseen ⇓

- $\{h_1, \ldots, h_{x(2m-1)}\} \cong \oplus_{i=1}^{x} K_{2m-1}$. There are more than $\frac{x}{2}(2m-2)(2m-3)$ many such edges.
- $\{h_{x(2m-1)+1}, \ldots, h_{x(2m-1)+c}\} \cong K_c$. If $c < 2$ there are no such edges, but if $c > 2$ then there are more than $\frac{1}{2}(c-1)(c-2) > 0$ many edges.
- If $c > 0$ then from $h_{x(2m-1)+i}$ there is an edge to $h_{j(2m-1)+i}$ for all i and j. There are xc many such edges.

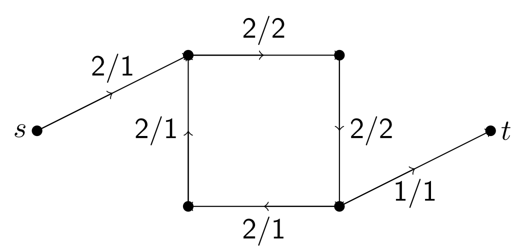
If K is a subgraph on H on $2m$ many vertices, then K cannot be contained entirely within one clique of H. If K were to be isomorphic to G_m, then $K \cap \{h_{x(2m-1)+1}, \ldots, h_{x(2m-1)+c}\}$ would have to be isomorphic to K_m, but $c < m$, so this is impossible. Therefore H does not have G_m as a subgraph. 4, D

(c) (i) A circulation cannot contain the sink. seen/sim.seen ⇓

The sink determines the value, so if we remove the circulation, we don't alter the value. We do this by defining $g(e_i)$ to be $f(e_i) - \min\{f(e_j) : e_j \text{ is in the circulation.}\}$ on the circulation, and $g(e_i)$ otherwise. If there are any other circulations, repeat. There are finitely many circulations, so we can repeat this process until no circulations remain. The result is a g which has value same as f, but contains no circulations. 2, A 6, B

(ii) The network with only a source and a sink, where the only edge has value 1. 1, A

(iii)



The numbers before the slash indicate the capacity of the edge, the number after the slash indicates the value of a flow f.

2, A

5. Note for anyone reading whose not familiar with the module: I've used in the lecture notes the standard terminology that paths cannot have repeated vertices in them. Walks are sequences of vertices without this restriction. seen ⇓

(a) Let $\epsilon \in (0, \frac{d}{2})$ and let (X, Y) be an ϵ-regular pair such that $d(X, Y) \geq d$.

Suppose $|X| = |Y| = n$, and let $A = \{x \in X : ||\{x\}, Y|| < \frac{dn}{2}\}$.

$||\{x\}, Y|| < \frac{dn}{2}$ for all $x \in A$, so $||A, Y|| < \frac{dn}{2}|A|$. Dividing both sides of this inequality by $|A||Y|$ gives $d(A, Y) < \frac{d}{2}$. We've assumed that $d(X, Y) > d$, so $d(A, Y) < d(X, Y)$. 3, M

We can get a lower bound for $|d(X, Y) - d(A, Y)|$ by using a lower bound for $d(X, Y)$ and an upper bound for $d(A, Y)$.

$$|d(X,Y) - d(A,Y)| \geq \left| d - \frac{d}{2} \right| = \frac{d}{2} > \epsilon$$

If there are more than $n\epsilon$ many vertices in X with less than $\frac{dn}{2}$ neighbours in Y then $|A| > \epsilon|X|$, so the ϵ-regularity of (X, Y) would imply that $|d(X, Y) - d(A, Y)| < \epsilon$. This gives a contradiction, so there must be less than $n\epsilon$ many such vertices. 2, M 1, M

(b) Since $\{V_0, \ldots, V_k\}$ is an ϵ-partition, there are at most $\epsilon k^2 < \frac{k}{2}$ many pairs (V_i, V_j) which are not ϵ-regular. I claim that there is a sequence of the form $(V_{i_0}, V_{i_1}), \ldots, (V_{i_{k-1}}, V_k)$ where each V_j occurs exactly once. Each sequence is uniquely described by a permutation σ which maps j to i_j, and hence by a permutation matrix. The sequence $(V_{i_0}, V_{i_1}), \ldots, (V_{i_{k-1}}, V_k)$ does not contain a pair that is not ϵ-regular if and only if the corresponding permutation matrix has a 0 in $\frac{k}{2}$ many specified entries.

We try and build a permutation matrix with 0 in $\frac{k}{2}$ many specified entries. We swap rows so that the first row has the most specified entries, the second row has the second most, and so on. Let a_i be the number of specified entries in the i'th row. Then there are $(k - a_i)$ many choices for the 1 entry in the first row. Therefore the number of permutation matrices with 0's in $\frac{k}{2}$ many specified entries is

$$\prod_{i=1}^{k}((k - i + 1) - a_i)$$

If $i \leq \frac{k}{2}$ then $k - i + 1 > \frac{k}{2}$, so $(k - i + 1) - a_i > 0$. If $i > \frac{k}{2}$ then $a_i = 0$, so $(k - i + 1) - a_i > 0$. Therefore every entry in this product is a positive integer, and so the result is a positive integer. Therefore there are permutation matrices with 0 in $\frac{k}{2}$ many specified entries.

Accept any argument that such a sequence exists. 7, M

There are at most ϵn many vertices in V_1 with less than $\frac{dn}{2}$ many neighbours in V_2. There are at most ϵn many vertices in V_2 with less than $\frac{dn}{2}$ many neighbours in V_3. Since $\epsilon < \frac{d}{2}$ this means that there is a $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1 E v_2$ and v_2 has at least $\frac{dn}{2}$ many neighbours in V_3.

As before, there are at most ϵn many vertices in V_3 with less than $\frac{dn}{2}$ many neighbours in V_4. Since $\epsilon < \frac{d}{2}$, this means that there is a v_3 such that $v_2 E v_3$ and v_3 has at least $\frac{dn}{2}$ many neighbours in V_4.

We repeat this process until we have found $v_1, \ldots, v_{k-1}$ such that $v_i E v_{i+1}$, and v_{k-1} has at least $\frac{dn}{2}$ many neighbours in V_k. We pick any of them to be v_k. Then $v_1, \ldots, v_k$ is a path of length k in G. $\boxed{7, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
	1	This question was very well done on average.
	2	Parts of this question acted as hints for each other. The transitive tournaments provide a great family of graphs to consider in parts (c) and (d). A common mistake in (a) was to not properly apply simplicity.
	3	This question was very well done on average.
	4	There were a number of details that needed to be considered in part (c) that were mostly missed. While most people hit upon the idea of removing the circulation by reducing the flow on the edges used in the circulation, very few considered the situation where a flow has more than 1 circulation.
	5	(a) was very standard, being quite similar to things you had seen before. (b) was quite tricky, it was common to just handwave the fact that you could find a path through the partition elements where each step in the path was via a regular pair, but this needed to be done carefully.