MATH60142/70142 Mathematics of Business & Economics
Class test – Solutions

29 February 2024

Question (20 marks)

Suppose a firm's production is based on three input goods x_1, x_2, and x_3, with respective prices $w_1 > 0, w_2 > 0$, and $w_3 > 0$. The production function $f : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the firm is given via

$$f(x_1, x_2, x_3) = x_1^\alpha x_2^{\alpha/2} + 2x_3, \qquad \text{with } \alpha > 0. \tag{1}$$

For parts (a) to (f), assume that the quantity of good x_3 is fixed at some non-negative value, and the firm can only vary the goods x_1 and x_2 flexibly for production.

(a) **(2 marks)** Describe the economic notions of long-run and short-run. Which scenario is described above?

Solution: In the economic literature, short-run and long-run are inexact periods of time, defined implicitly by the number of production inputs $(x_1, \ldots, x_n)$ that may vary within such a timeframe: in the long-run, all inputs may vary, whereas in the short run, at least one input will be held constant. Since x_3 is fixed in the above description, it is considered a short-run scenario.

[1 mark for the correct description, and 1 mark for identifying the short-run scenario. Note: Definition is provided on page 11 in the lecture notes (LN).]

(b) **(3 marks)** Calculate the Marginal Rate of Technical Substitution $\text{MRTS}(x_1, x_2)$ of f, and briefly explain the economical meaning of the MRTS.

Solution: The MRTS is computed via

$$\text{MRTS}(x_1, x_2) = -\frac{\text{MP}_1(x_1, x_2)}{\text{MP}_2(x_1, x_2)} = -\frac{\partial_1 f(x_1, x_2)}{\partial_2 f(x_1, x_2)} = -\frac{\alpha x_1^{\alpha-1} x_2^{\alpha/2}}{\frac{\alpha}{2} x_1^\alpha x_2^{\alpha/2-1}} = -2\frac{x_2}{x_1} \tag{2}$$

The MRTS describes the rate of change of good x_2 w.r.t. good x_1, in order to keep the level of output constant (i.e., decreasing x_1 and increasing x_2 such that the output is fixed).

[1 mark for correct definition of the MRTS, 1 mark correct calculations of MRTS, 1 mark for correct economical understanding. Note: Definition of MRTS is provided in LN, p. 12, with an example on p. 13.]

(c) **(1 mark)** The firm wants to find the input bundle that minimises their cost while achieving some given level of output $\tilde{y}$. Write down the minimisation problem of the firm (you can assume that $\tilde{y} \geq 2x_3$).

Solution: With x_3 being fixed, for $\tilde{y} \geq 2x_3$, we seek to find

$$\underset{x_1, x_2 \in \mathbb{R}_{\geq 0}}{\text{argmin}}\; w_1 x_1 + w_2 x_2 + w_3 x_3 = \underset{x_1, x_2 \in \mathbb{R}_{\geq 0}}{\text{argmin}}\; w_1 x_1 + w_2 x_2, \quad \text{s.t., } f(\underline{x}) = x_1^\alpha x_2^{\alpha/2} + 2x_3 = \tilde{y}. \tag{3}$$

[1 mark for correct specification of the minimisation problem. Note: A cost minimisation example (without fixed costs) is provided in LN, p. 22. A (slightly different) cost minimisation example with fixed costs is provided on problem sheet (PS) 3, Q3.c), based on LN, p. 26]

(d) **(3 marks)** Write down the Lagrangian for the minimisation problem given in part (c), and derive the first-order conditions for minimisation of the Lagrangian.

Solution: The Lagrangian of the cost minimisation problem in part (c) is given via

$$\mathcal{L}(x_1, x_2, \lambda) = w_1x_1 + w_2x_2 - \lambda(x_1^{\alpha}x_2^{\alpha/2} + 2x_3 - \tilde{y}) \tag{4}$$

The first-order conditions are given by

$$\frac{\partial}{\partial\lambda}\mathcal{L}(x_1, x_2, \lambda) = 0 \Rightarrow x_1^{\alpha}x_2^{\alpha/2} + 2x_3 = \tilde{y} \tag{5}$$

$$\frac{\partial}{\partial x_1}\mathcal{L}(x_1, x_2, \lambda) = 0 \Rightarrow \lambda\alpha x_1^{\alpha-1}x_2^{\alpha/2} = w_1 \tag{6}$$

$$\frac{\partial}{\partial x_2}\mathcal{L}(x_1, x_2, \lambda) = 0 \Rightarrow \lambda\frac{\alpha}{2}x_1^{\alpha}x_2^{\alpha/2-1} = w_2 \tag{7}$$

[1 mark for correct specification of the Lagrangian. 2 marks for correct first-order conditions. Note: Cost minimisation via Lagrangian (without fixed costs) is covered in LN, p. 22.]

(e) **(3 marks)** Compute the cost minimising input bundle as a function of $\tilde{y}$, w_1, w_2, and x_3. [*You may assume that the second order condition is satisfied.*]

Solution: To obtain the cost minimising input bundle as a function of $\tilde{y}$, w_1, w_2, and x_3 (also denoted as the short-run conditional factor demand function), we simply solve the first-order conditions provided in part (d). By dividing (6) by (7), we obtain

$$2\frac{x_2}{x_1} = \frac{w_1}{w_2} \tag{8}$$

$$\Rightarrow x_1 = 2\frac{w_2}{w_1}x_2 \tag{9}$$

Substituting (9) in (5), we obtain

$$\left(2\frac{w_2}{w_1}\right)^{\alpha} x_2^{3\alpha/2} = y - 2x_3 \tag{10}$$

From that, we obtain the solution for the short-run conditional factor demand for good 2 via

$$x_2^*(w_1, w_2, x_3, \tilde{y}) = (y - 2x_3)^{2/(3\alpha)}\left(\frac{w_1}{2w_2}\right)^{2/3} \tag{11}$$

and thus, by substituting (11) back into (9), we obtain

$$x_1^*(w_1, w_2, x_3, \tilde{y}) = (y - 2x_3)^{2/(3\alpha)}\left(2\frac{w_2}{w_1}\right)^{1/3}. \tag{12}$$

(As a note, via (8), we can double check with part (b) that $\text{MRTS}(x_1^*, x_2^*) = -\frac{w_1}{w_2}$, as derived in the lecture, LN p. 22)

[1 mark for approach of solving the first-order conditions from part (d), then 2 marks for correctly deriving x_1^* and x_2^*.]

(f) **(1 mark)** Compute the firm's cost function $c_S^*(w_1, w_2, w_3, x_3, y)$.

Solution: This is straightforward, we simply compute the conditional short-run costs using the short-run conditional factor demand $x_1^*(w_1, w_2, x_3, \tilde{y})$ and $x_2^*(w_1, w_2, x_3, \tilde{y})$ obtained in part (e). This is leads to

$$c_S^*(w_1, w_2, w_3, x_3, y) = w_1 x_1^*(w_1, w_2, x_3, \tilde{y}) + w_2 x_2^*(w_1, w_2, x_3, \tilde{y}) + w_3 x_3 \tag{13}$$

$$= w_1 (y - 2x_3)^{2/(3\alpha)} \left(2\frac{w_2}{w_1}\right)^{1/3} + w_2 (y - 2x_3)^{2/(3\alpha)} \left(\frac{w_1}{2w_2}\right)^{2/3} + w_3 x_3 \tag{14}$$

[1 mark for correct short-run cost function.]

For the remaining parts (g) to (j), you can now assume that the firm can vary all three input goods x_1, x_2, x_3 flexibly.

(g) **(2 mark)** Explain briefly in words (no more than three sentences needed) how the additional flexibility of input good x_3 might change the cost function from part (f). *(No calculations needed.)*

Solution: Since all three input goods are now variable, we consider a long-run scenario. We have seen in the lecture that the conditional long-run costs $c^*(\underline{w}, y)$ are smaller, or maximal equal to the conditional short-run costs $c_S^*(\underline{w}, x_F, y)$, here $x_F = x_3$, since now the optimal amount for all input goods can be achieved.

[1 mark for correctly identifying that $c^*(\underline{w}, y) \leq c_S^*(\underline{w}, x_F, y)$, and 1 mark for providing the explanation of additional flexibility in the optimisation given in the long-run scenario. Note: This behaviour has been discussed for the general case in LN, p. 26.]

(h) **(2 mark)** Derive a value for α such that the production function is positively homogeneous of degree $k = 1$. Justify your reasoning.

Solution: The production function f is positively homogeneous of some degree k if

$$f(t\underline{x}) = t^k f(\underline{x}), \quad \forall t > 0, \forall \underline{x} \in \mathbb{R}^3_{\geq 0}. \tag{15}$$

We obtain

$$f(tx_1, tx_2, tx_3) = t^\alpha x_1^\alpha t^{\alpha/2} x_2^{\alpha/2} + 2tx_3 \tag{16}$$

$$= t \cdot t^{3\alpha/2 - 1} f(x_1, x_2, x_3), \tag{17}$$

which leads to positive homogeneity of degree $k = 1$ iff.

$$t^{3\alpha/2-1} = 1 \Leftrightarrow \frac{3}{2}\alpha - 1 = 0 \Leftrightarrow \alpha = \frac{2}{3}. \tag{18}$$

[1 mark for correct definition of positive homogeneity of degree k, and 1 mark for correctly deriving α. Note: Definition of pos. homogeneity is provided in LN, p.15, and it has been used several times in the lecture and on problem sheets.]

(i) **(2 mark)** For which values of α does f exhibit (I) constant, and (II) increasing returns to scale? Justify your reasoning.

Solution: Case (I) can directly be answered from part (h). From the lecture, we know that a positively homogeneous function of degree $k = 1$ has constant returns to scale, and thus:

$$(I): f \text{ exhibits CRTS} \Leftrightarrow \alpha = \frac{2}{3}. \tag{19}$$

Case (II): From the lecture, we know that f exhibits increasing returns to scale if it is positively homogeneous of some degree $k > 1$. From (17), we can see that this is the case, if $\frac{3}{2}\alpha > 1$. We can thus follow, that

$$(I) : f \text{ exhibits IRTS} \Leftrightarrow \alpha > \frac{2}{3}. \tag{20}$$

(As an alternative approach, the values of α for cases (I) and (II) can be re-derived via the definitions of CRTS and IRTS.)

[1 mark for correctly identifying the values for α for cases (I) and (II), respectively. Note: The definition of returns to scale are provided in LN, p.13-14. The definition of pos. homogeneity and its relation to returns to scale is provided in LN, p.15.]

(j) **(1 mark)** Provide some simplified real-world example for increasing returns to scale.

Solution: Various examples could be used here. During the lecture, we had discussed the occurrence of IRTS as an effect of specialization, or merging of firms. As a very specific example, lets consider a restaurant that employs 5 workers and has 3 grills, producing 50 burgers and 50 steaks. The restaurant now scales the number of workers to 15 and the number of grills to 9, producing an output of 200 burgers and 200 steaks. While the input has been tripled, the output has been more than tripled. This could be due to specialisation of the workers, now being responsible for a specific part of the production only.

[1 mark for an example of IRTS. Referring to merging, and/or specialisation effects are enough to obtain the mark.]