ZF6 Axiom scheme of Specification (or Comprehension)

Suppose $P(x, y_1, \dots, y_r)$ is a formula in our 1st order language. Then we have an axiom

$$(\forall A)(\forall y_1)\dots(\forall y_r)(\exists B)(\forall x)$$
$$(x \in B) \leftrightarrow ((x \in A) \wedge P(x, y_1, \dots, y_r))$$

ie. "given a set A and sets $y_1, \dots, y_r$ we can form the set

refer to these as parameters

$$B = \{ x \in A : P(x, \underbrace{y_1, \dots, y_r}) \text{ holds} \}$$
$$\subseteq A .$$

Eg 1) Let C be a non-empty set & $A \in C$.

Then
$$\bigcap C = \{ x \in A : \underbrace{(\forall z)((z \in C) \to (x \in z))}_{P(x, C)} \}$$

Ex. This doesn't depend on the A here.

2) $A \times B = \{ x \in \mathcal{P}(\mathcal{P}(A \cup B)) : (\exists a)(\exists b)((a \in A) \wedge (b \in B) \wedge (x = \{\{a\}, \{a, b\}\})) \}$

Ex: Can form $B^A \subseteq \mathcal{P}(A \times B)$.

(1)

ZF7 Axiom of infinity

(3.2.1) Def. ① For a set a the successor of a is

$$a^+ = a \cup \{a\}$$

Eg

$$\emptyset^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\} = 1$$

$$1^+ = \{\emptyset\} \cup \{1\} = \{\emptyset, 1\} = 2$$

$$2^+ = \{\emptyset, 1\} \cup \{2\} = \{\emptyset, 1, 2\} = 3.$$

② A set A is inductive if

$$(\emptyset \in A) \wedge ((\forall x)((x \in A) \to (x^+ \in A)))$$

The axiom of infinity ZF7 is

$$(\exists A)((\emptyset \in A) \wedge ((\forall x)((x \in A) \to (x^+ \in A))))$$

(3.2.2) Def. Let A be any inductive set. Using Specification can form

$$\mathbb{N} = \{x \in A : \text{if } B \text{ is an inductive set then } x \in B\}$$

Note: this doesn't depend on choice of A.

Notation: Also denote $\mathbb{N}$ by ω (or ω_0).

(3.2.3) Theorem ① $\mathbb{N}$ is an inductive set. If B is any inductive set then $\mathbb{N} \subseteq B$

② "Proof by induction works for $\mathbb{N}$"

~~Some~~ Suppose $P(x)$ is a 1^{st} order fmla such that

(i) $P(\emptyset)$ holds &

(ii) For every $k \in \mathbb{N}$ if $P(k)$ holds, then $P(k^+)$ holds.

Then $P(n)$ holds for all $n \in \mathbb{N}$

Pf: ① Ex.

② Consider $B \subseteq \mathbb{N}$

$B = \{ k \in \mathbb{N} : P(k) \text{ holds} \}$

By (i), (ii) B is an inductive set. So by (i)

$B = \mathbb{N}$. #.

Could develop arithmetic in $\mathbb{N}$ (using n^+ as $n+1$)

"Ex" For $m, n \in \mathbb{N}$, if we write $m \leq n$ to mean

$$(m = n) \vee (m \in n)$$

then this is a linear order on $\mathbb{N}$ & is a well ordering.

3.3 Linear orderings

(3.3.1) Def. A linear ordering $(A; \leq)$ is a well ordering (or well-ordered set) if every non-empty subset of A has a least element.

Examples. (Informal)

$(\mathbb{Z}; \leq)$ is not a w.o. set

$(\mathbb{N}; \leq)$ is a w.o. set

Any subset of a w.o. is a w.o. set.

(3.3.2) Def. Suppose $\mathcal{A}_1 = (A_1 ; \leq_1)$ and $\mathcal{A}_2 = (A_2 ; \leq_2)$ are linear orderings. Say these are isomorphic or similar if there is a bijection $\alpha : A_1 \to A_2$ st. $\forall a, b \in A_1$

$$a \leq_1 b \iff \alpha(a) \leq_2 \alpha(b)$$

α is called a similarity between $\mathcal{A}_1, \mathcal{A}_2$.

Write $\mathcal{A}_1 \simeq \mathcal{A}_2$.

If $\beta : A_1 \to A_2$ is injective and $\forall a, b \in A_1$

$$a \leq_1 b \Rightarrow \beta(a) \leq_2 \beta(b)$$

say that β is order preserving.

(3.3.3) Def. ($\mathcal{A}_1, \mathcal{A}_2$ as in 3.3.2)

(1) the reverse lexicographic product

$$\mathcal{A}_1 \times \mathcal{A}_2 = (A_1 \times A_2 ; \leq)$$

is defined by

$$(a_1, a_2) \leq (a_1', a_2')$$

$\iff$ either $a_2 <_2 a_2'$

or $a_2 = a_2'$ and $a_1 \leq_1 a_1'$.

" In $\mathcal{A}_2$, replace every elt of $\mathcal{A}_2$ by a copy of $\mathcal{A}_1$ ".

Example: 1 $\{0,1\} \times \mathbb{N}$

$$\{0,1\} \times \mathbb{N} \cong \mathbb{N}.$$

2 $\mathbb{N} \times \{0,1\}$

$$\mathbb{N} \times \{0,1\} \not\cong \mathbb{N}.$$

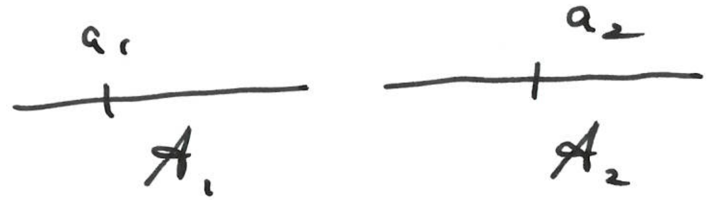
(3.3.3) (2) Sum

Given $\mathcal{A}_i = (A_i ; \leq_i)$ l.o.

Regard A_1, A_2 as disjoint.

(or replace A_1, A_2 by disjoint sets $A_1 \times \{0\}$ & $A_2 \times \{1\}$)

Define $\mathcal{A}_1 + \mathcal{A}_2 = (A_1 \cup A_2 ; \leq)$



where $a_1 \leq a_2$ for all $a_1 \in A_1$ & $a_2 \in A_2$ & all other orderings as in $\mathcal{A}_1, \mathcal{A}_2$.

Eg $(\mathbb{N}; \leq) + (\mathbb{N}; \leq)$

$$\cong \mathbb{N} \times \{0,1\}.$$

(3.3.4) Lemma. With this notation

1. $\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_1 \times \mathcal{A}_2$ are linear orderings.

2. If $\mathcal{A}_1, \mathcal{A}_2$ are well ordered sets, then so are $\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}_1 \times \mathcal{A}_2$.

[Eg $\mathbb{N} \times \mathbb{N}$ is well-ordered..]

Pf: 1. Ex.

2. Eg $\mathcal{A}_1 \times \mathcal{A}_2$

Let $\emptyset \neq X \subseteq A_1 \times A_2$.

Consider $Y = \{ b \in A_2 : \exists a \in A_1$ with $(a, b) \in X \}$.

$\subseteq A_2$.

Let d be the least elt. of Y.

Consider

$Z = \{ a \in A_1 : (a, d) \in X \}$.

this has a least elt. $c \in A_1$.

$(c, d) \in X$ and is the least elt. of X. #.