IMPERIAL

MATH70010

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometric Mechanics

Date: Friday, May 9, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. For $(q,p) \in T^*M$ and $(q, \pounds_\xi q) \in TM$, two pairings denoted as

$$\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle : T^*M \times TM \mapsto \mathbb{R} \quad \text{and} \quad \langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$$

imply two types of operations dual to the Lie derivative $\pounds_\xi$ for elements ξ in Lie algebra $\mathfrak{g}$.

The transpose $(\pounds_\xi^T p)$ of Lie derivative $\pounds_\xi$ is defined as $\Big\langle\!\!\Big\langle\, \pounds_\xi^T p\,,\, q \,\Big\rangle\!\!\Big\rangle = \Big\langle\!\!\Big\langle\, p\,,\, \pounds_\xi q \,\Big\rangle\!\!\Big\rangle$.

The diamond operation $(\diamond)$ is defined as $\Big\langle\, p \diamond q\,,\, \xi \,\Big\rangle = \Big\langle\!\!\Big\langle\, p\,,\, -\pounds_\xi q \,\Big\rangle\!\!\Big\rangle$.

Prove the following Proposition:

The Euler–Poincaré equation

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{1}$$

defined on the dual Lie algebra $\mathfrak{g}^*$ is equivalent to the following implicit variational principle,

$$\delta S(\xi, q, \dot q, p) = \delta \int_a^b l(\xi, q, \dot q, p)\, dt = 0,$$

for an action integral S constrained by the reconstruction formula $\dot q + \pounds_\xi q = 0$ for Lie derivative defined by $\pounds_\xi q := \frac{d}{d\epsilon} q(t,\epsilon)|_{\epsilon=0}$ with $\xi = \dot g g^{-1} \in \mathfrak{g}$

$$S(\xi, q, \dot q, p) = \int_a^b l(\xi, q, \dot q, p)\, dt = \int_a^b \Big[l(\xi) + \Big\langle\!\!\Big\langle\, p\,,\, \dot q + \pounds_\xi q \,\Big\rangle\!\!\Big\rangle \Big]\, dt\,. \tag{2}$$

(a) Begin by showing that stationarity variations of the constrained action in (2) imply the following set of equations:

$$\frac{\delta l}{\delta \xi} = p \diamond q\,, \quad \dot q = -\,\pounds_\xi q\,, \quad \dot p = \pounds_\xi^T p\,.$$

(8 marks)

(b) Finish by expanding the time derivative

$$\frac{d}{dt}\Big\langle\, \frac{\delta l}{\delta \xi}\,,\, \eta \,\Big\rangle = \frac{d}{dt}\Big\langle\, p \diamond q\,,\, \eta \,\Big\rangle$$

for a fixed Lie algebra element $\eta \in \mathfrak{g}$, to obtain the Euler–Poincaré Equation (1).

(12 marks)

(Total: 20 marks)

2. Consider a free particle of mass $m = 1$ moving on the Lobachevsky half-plane $\mathbb{H}^2$. Its Lagrangian is the kinetic energy corresponding to the Lobachevsky metric. Namely,

$$L = \frac{1}{2}\left(\frac{\dot{x}^2 + \dot{y}^2}{y^2}\right). \tag{3}$$

This Lagrangian is invariant under the Lie group of translations $T_\tau : (x, y) \mapsto (x + \tau, y)$ along the x axis and scalings $S_\sigma : (x, y) \mapsto (e^\sigma x, e^\sigma y)$ centered at $(x, y) = (0, 0)$.

(a) Compute the Lie algebra of vector fields X_T and X_S whose flows generate the invariance Lie group T_τ, S_σ of this Lagrangian and compute their commutator $[X_T, X_S]$. (6 marks)

(b) Show that the quantities

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y} \tag{4}$$

are invariant under the Lie-symmetry transformations T_τ and S_σ.

(4 marks)

(c) Relate the canonical momenta (p_x, p_y) of the Lagrangian obtained from the Lobachevsky metric to the invariant quantities (u, v). Then use this relation to determine the Lie–Poisson bracket bracket $\{u, v\}$ from the canonical brackets. (4 marks)

(d) Transform the Lagrangian in (3) to its Lie-symmetry invariant quantities (u, v). Then Legendre transform the resulting reduced Lagrangian $\ell(u, v)$ to obtain the reduced Hamiltonian. Finally, write the reduced Hamiltonian dynamics in terms of the Lie–Poisson bracket bracket $\{u, v\}$ for invariant quantities (u, v).

(6 marks)

(Total: 20 marks)

3. The formula determining the momentum map for the cotangent-lifted action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$ as

$$\langle J, \xi \rangle = \langle p, \pounds_\xi q \rangle,$$

where $(q, p) \in T_q^* Q$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Define appropriate pairings and determine the momentum maps explicitly for the following actions:

(a) $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$. (4 marks)

(b) $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad–action $\mathrm{ad} \colon \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$ (4 marks)

(c) AqA^{-1} for $A \in GL(3, R)$ acting on $q \in GL(3, R)$ by matrix conjugation (4 marks)

(d) Aq for left action of $A \in SO(3)$ on $q \in SO(3)$ (4 marks)

(e) AqA^T for $A \in GL(3, R)$ acting on $q \in \mathrm{Sym}(3)$, that is $q = q^T$. (4 marks)

(Total: 20 marks)

4. The objective here is to use a constrained Hamilton's principle to derive the Euler–Poincaré equation with $\xi = \dot{g}g^{-1} \in \mathfrak{g}$

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}$$

for the following action integral S constrained by the reconstruction formula $\dot{q} + \pounds_\xi q = 0$ for Lie derivative defined by $\pounds_\xi q := \frac{d}{d\epsilon} q(t,\epsilon)|_{\epsilon=0}$:

$$\begin{aligned} S(\xi, q, \dot{q}, p) &= \int_a^b l(\xi, q, \dot{q}, p)\, dt \\ &= \int_a^b \Big[l(\xi) + \Big\langle\!\Big\langle\, p\,,\, \dot{q} + \pounds_\xi q \,\Big\rangle\!\Big\rangle \Big]\, dt\,. \end{aligned}$$

Here, the pairing $\langle\!\langle\, \cdot\,,\, \cdot\,\rangle\!\rangle : T^*M \times TM \mapsto \mathbb{R}$ for a configuration manifold, M.

(a) Compute the Hamilton's principle variations of the action integral, $\delta S = 0$ (6 marks)

(b) Identify the independent variational derivatives. (6 marks)

(c) Combine the independent variational derivatives to derive the corresponding the Euler–Poincaré equation.

(8 marks)

(Total: 20 marks)

5. (a) For Lie group elements $g \in G$ write $\xi = \dot{g}g^{-1}$ and $\eta = g'g^{-1}$ in natural notation and express the partial derivatives $\dot{g} = \partial g/\partial t$ and $g' = \partial g/\partial \epsilon$ using the right translations as $\dot{g} = \xi \circ g$ and $g' = \eta \circ g$. Assume $\dot{g}' = g'^{\cdot}$ and derive the relation

$$\xi' - \dot{\eta} = -[\xi, \eta] = -\operatorname{ad}_\xi \eta.$$

(2 marks)

(b) Use the relation in part (a) and the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$ to derive the Euler–Poincaré equation from the reduced variational principle, as

$$\delta \int_a^b l(\xi(t))dt = -\int_a^b \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi} + \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle dt\,.$$

(10 marks)

(c) Show that combining the Euler–Poincaré equation and the reduced Legendre transformation $h(\mu) := \langle \mu, \xi \rangle - l(\xi)$ implies the equivalence relation

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} \quad \Longleftrightarrow \quad \frac{d\mu}{dt} = -\operatorname{ad}^*_{\delta h/\delta \mu}\mu.$$

(2 marks)

(d) Show the equivalence relation in the previous part implies that the Lie–Poisson equations may be written in the Poisson bracket form

$$\dot{f} = \{f, h\}\,,$$

where $f\colon \mathfrak{g}^* \to \mathbb{R}$ is an arbitrary smooth function and the bracket is the (right) Lie–Poisson bracket given by

$$\{f, h\}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right] \right\rangle.$$

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

M70010

Geometric Mechanics (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. (a) The variations of the action integral in the problem statement are given by $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
\delta S &= \int_a^b \Big\langle \frac{\delta l}{\delta \xi}\,,\,\delta\xi \Big\rangle + \Big\langle\!\Big\langle \frac{\delta l}{\delta p}\,,\,\delta p \Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle \frac{\delta l}{\delta q}\,,\,\delta q \Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle p\,,\,\pounds_{\delta\xi}\, q \Big\rangle\!\Big\rangle\,dt \\
&= \int_a^b \Big\langle \frac{\delta l}{\delta \xi} - p\diamond q\,,\,\delta\xi \Big\rangle + \Big\langle\!\Big\langle \delta p\,,\,\frac{dq}{dt} + \pounds_\xi q \Big\rangle\!\Big\rangle - \Big\langle\!\Big\langle \frac{dp}{dt} - \pounds_\xi^T p\,,\,\delta q \Big\rangle\!\Big\rangle\,dt\,,
\end{aligned}
$$

These are the required variations. $\boxed{\text{4 marks for each of the } 1^{st} \text{ and } 3^{rd} \text{ tems}}$ $\boxed{8,\ \text{A}}$

(b) The Euler–Poincaré equation emerges from elimination of (q,p) using these formulas and the properties of the diamond operation that arise from its definition, as follows, for any *fixed* vector $\eta\in\mathfrak{g}$: $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
\Big\langle \frac{d}{dt}\frac{\delta l}{\delta\xi}\,,\,\eta \Big\rangle &= \frac{d}{dt}\Big\langle \frac{\delta l}{\delta\xi}\,,\,\eta \Big\rangle\,, \\
\text{[Definition of } \diamond\text{]} &= \frac{d}{dt}\Big\langle p\diamond q\,,\,\eta \Big\rangle = \frac{d}{dt}\Big\langle\!\Big\langle p\,,\,-\pounds_\eta q \Big\rangle\!\Big\rangle\,, \\
\text{[Definition of } \pounds_\xi^T\text{]} &= \Big\langle\!\Big\langle \pounds_\xi^T p\,,\,-\pounds_\eta q \Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle p\,,\,\pounds_\eta \pounds_\xi q \Big\rangle\!\Big\rangle\,, \\
\text{[Transpose, } \diamond \text{ and } \mathrm{ad}\text{]} &= \Big\langle\!\Big\langle p\,,\,-\pounds_{[\xi,\,\eta]} q \Big\rangle\!\Big\rangle = \Big\langle p\diamond q\,,\,\mathrm{ad}_\xi \eta \Big\rangle\,, \\
\text{[Definition of } \mathrm{ad}^*\text{]} &= \Big\langle\, \mathrm{ad}^*_\xi \frac{\delta l}{\delta\xi}\,,\,\eta \Big\rangle\,.
\end{aligned}
$$

This is the required Euler–Poincaré equation. $\boxed{\text{3 marks for each step}}$ $\boxed{12,\ \text{B}}$

2. (a) meth seen $\Downarrow$

The pull-back of $T_\tau : (x,y) \mapsto (x+\tau, y)$ reads $T_\tau^* f(x,y) = f(x+\tau, y)$ is the flow of $X_T = \partial_x$, since $\frac{d}{d\tau} T_\tau^* f(x,y)\big|_{\tau=0} = \partial_x f(x,y)$.
Similarly, $S_\sigma : (x,y) \mapsto (e^\sigma x, e^\sigma y)$ is the flow of $X_S = x\partial_x + y\partial_y$,
since $S_\sigma^* f(x,y) = f(e^\sigma x, e^\sigma y)$ & $\frac{d}{d\sigma} S_\sigma^* f(x,y)\big|_{\sigma=0} = (x\partial_x + y\partial_x) f(x,y)$,
The Lie algebra commutator of these vector fields is $[X_T, X_S] = X_T$

6, C

(b) By substitution of the actions of T_τ and S_σ on (x,y), the quantities meth seen $\Downarrow$

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y}$$

are invariant under these symmetry transformations. 4, A

(c) Hamilton's principle for the Lobachevsky metric Lagrangian yields meth seen $\Downarrow$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2} = p_x = \frac{u}{y} \quad \text{and} \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{y^2} = p_y = \frac{v}{y}\,.$$

Solving these formulas for u and v implies the Lie–Poisson bracket we seek, which is dual to the Lie symmetry algebra $[X_T, X_S] = X_T$. Namely,

$$\{u, v\} = \{y p_x, y p_y\} = y p_x = u\,.$$

4, C

(d) By substitution of the definitions of u and v in terms of (x, p_x) and (y, p_y), the Lobachevsky metric Lagrangian transforms into its Lie-symmetry invariants as meth seen $\Downarrow$

$$\ell(u,v) = \frac{1}{2}(u^2 + v^2)\,.$$

Via the reduced Legendre transform, the corresponding invariant Hamiltonian is obtained as

$$h(\mu,\nu) = \langle \mu, u \rangle + \langle \nu, v \rangle - \ell(u,v)\,.$$

Variations of $h(\mu,\nu)$ yield

$$\delta h(\mu,\nu) = \langle \delta\mu, u \rangle + \langle \delta\nu, v \rangle + \Big\langle \mu - \frac{\partial \ell}{\partial u}, \delta u \Big\rangle + \Big\langle \nu - \frac{\partial \ell}{\partial v}, \delta v \Big\rangle\,.$$

Consequently, one finds $\mu = u$, $\nu = v$ and one may write the invariant Hamiltonian as

$$h(u,v) = \frac{1}{2}(u^2 + v^2)\,.$$

Now using the Lie–Poisson bracket from the previous part, one finds the desired equations for the invariant quantities,

$$\dot{u} = \{u, h\} = uv \quad \text{and} \quad \dot{v} = \{v, h\} = -\,u^2\,.$$

6, B

3. (a) $p \cdot \xi \times q = q \times p \cdot \xi \Rightarrow J = q \times p$. $\boxed{\text{seen} \Downarrow}$

The pairing is scalar product of vectors. $\boxed{4, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(b) $\langle p, \mathrm{ad}_\xi q\rangle = -\langle \mathrm{ad}^*_q p, \xi\rangle \Rightarrow J = \mathrm{ad}^*_q p$

The pairing is $\langle \cdot, \cdot \rangle$: $\mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$ $\boxed{4, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(c) Compute $T_e(AqA^{-1}) = \xi q - q\xi = [\xi, q]$ for $\xi = A'(0) \in gl(3, R)$ acting on $q \in GL(3, R)$ by matrix Lie bracket $[\cdot, \cdot]$.

For the matrix pairing $\langle A, B\rangle = \mathrm{Tr}(A^T B)$, we have

$$\mathrm{Tr}\big(p^T[\xi, q]\big) = \mathrm{Tr}\big((pq^T - q^T p)^T \xi\big) \Rightarrow J = pq^T - q^T p.$$

$\boxed{4, \text{A}}$

(d) Compute $T_e(Aq) = \xi q$ for $\xi = A'(0) \in \mathfrak{so}(3)$ acting on $q \in SO(3)$ by left matrix multiplication. $\boxed{\text{meth seen} \Downarrow}$

For the matrix pairing $\langle A, B\rangle = \mathrm{Tr}(A^T B)$, we have

$$\mathrm{Tr}\big(p^T \xi q\big) = \mathrm{Tr}\big((pq^T)^T \xi\big) \Rightarrow J = \tfrac{1}{2}\big(pq^T - q^T p\big),$$

where we have used antisymmetry of the matrix $\xi \in \mathfrak{so}(3)$. $\boxed{4, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(e) Compute $T_e\big(AqA^T\big) = \xi q + q\xi^T$ for $\xi = A'(0) \in gl(3, R)$ acting on $q \in \mathrm{Sym}(3)$.

For the matrix pairing $\langle A, B\rangle = \mathrm{Tr}(A^T B)$, we have

$$\mathrm{Tr}\big(p^T\big(\xi q + q\xi^T\big)\big) = \mathrm{Tr}\big(q\big(p^T + p\big)\xi\big) = \mathrm{Tr}\big((2qp)^T \xi\big) \Rightarrow J = 2qp,$$

where we have used symmetry of the matrix $\xi q + q\xi^T$ to choose $p = p^T$.

(The momentum canonical to the symmetric matrix $q = q^T$ should be symmetric to have the correct number of components.) $\boxed{4, \text{A}}$

4. (a) Stationary variations of the action integral $\boxed{\text{seen} \Downarrow}$

$$S(\xi, q, \dot q, p) = \int_a^b \Big[l(\xi) + \Big\langle\!\Big\langle\, p\,,\, \dot q + \pounds_\xi q \,\Big\rangle\!\Big\rangle \Big]\, dt$$

are given by

$$\begin{aligned} 0 = \delta S &= \int_a^b \Big\langle\, \frac{\delta l}{\delta \xi}\,,\, \delta\xi \,\Big\rangle + \Big\langle\!\Big\langle\, \frac{\delta l}{\delta p}\,,\, \delta p \,\Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle\, \frac{\delta l}{\delta q}\,,\, \delta q \,\Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle\, p\,,\, \pounds_{\delta\xi} q \,\Big\rangle\!\Big\rangle\, dt \\ &= \int_a^b \Big\langle\, \frac{\delta l}{\delta \xi} - p \diamond q\,,\, \delta\xi \,\Big\rangle + \Big\langle\!\Big\langle\, \delta p\,,\, \dot q + \pounds_\xi q \,\Big\rangle\!\Big\rangle - \Big\langle\!\Big\langle\, \dot p - \pounds_\xi^T p\,,\, \delta q \,\Big\rangle\!\Big\rangle\, dt\,. \end{aligned}$$

$\boxed{6, \text{D}}$

$\boxed{\text{meth seen} \Downarrow}$

(b) Stationarity of the variational principle implies the following set of variational derivatives:

$$\frac{\delta l}{\delta \xi} = p \diamond q\,, \quad \dot q = -\,\pounds_\xi q\,, \quad \dot p = \pounds_\xi^T p\,.$$

$\boxed{6, \text{B}}$

$\boxed{\text{meth seen} \Downarrow}$

(c) The Euler–Poincaré equation emerges from elimination of (q, p) using these formulas and the properties of the diamond operation that arise from its definition, as follows, for any vector $\eta \in \mathfrak{g}$:

$$\begin{aligned} \Big\langle\, \frac{d}{dt}\frac{\delta l}{\delta \xi}\,,\, \eta \,\Big\rangle \quad &= \quad \frac{d}{dt}\Big\langle\, \frac{\delta l}{\delta \xi}\,,\, \eta \,\Big\rangle = \frac{d}{dt}\Big\langle\, p \diamond q\,,\, \eta \,\Big\rangle\,, \\ \text{[Definition of } \diamond] \quad &= \quad \frac{d}{dt}\Big\langle\, p \diamond q\,,\, \eta \,\Big\rangle = \frac{d}{dt}\Big\langle\!\Big\langle p\,,\, -\pounds_\eta q \Big\rangle\!\Big\rangle\,, \\ \text{[Variational derivatives]} \quad &= \quad \Big\langle\!\Big\langle \pounds_\xi^T p\,,\, -\pounds_\eta q \Big\rangle\!\Big\rangle + \Big\langle\!\Big\langle p\,,\, \pounds_\eta \pounds_\xi q \Big\rangle\!\Big\rangle\,, \\ \text{[Transpose, } \diamond \text{ and } \mathrm{ad}] \quad &= \quad \Big\langle\!\Big\langle p\,,\, -\pounds_{[\xi,\eta]} q \Big\rangle\!\Big\rangle = \Big\langle p \diamond q\,,\, \mathrm{ad}_\xi \eta \Big\rangle\,, \\ \text{[Definition of } \mathrm{ad}^*] \quad &= \quad \Big\langle \mathrm{ad}_\xi^*(p \diamond q)\,,\, \eta \Big\rangle = \Big\langle\, \mathrm{ad}_\xi^* \frac{\delta l}{\delta \xi}\,,\, \eta \,\Big\rangle\,. \end{aligned}$$

This is the Euler–Poincaré equation. $\boxed{\text{2 marks for each step}}$ $\boxed{8, \text{D}}$

5. (a) The difference of the mixed partial derivatives implies the desired formula, $\boxed{\text{seen} \Downarrow}$

$$\xi' - \dot{\eta} = \nabla\xi \cdot \eta - \nabla\eta \cdot \xi = -\left[\xi, \eta\right] = -\,\mathrm{ad}_\xi \eta.$$

(Note the minus sign in the last two terms.) $\boxed{\text{meth seen} \Downarrow}$

(b) The reduced variational principle produces the Euler–Poincaré equations, as follows

$$\begin{aligned}
\delta \int_a^b l(\xi(t))dt &= \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle dt = \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} - \mathrm{ad}_\xi \eta \right\rangle dt \\
&= \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} \right\rangle dt - \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \mathrm{ad}_\xi \eta \right\rangle dt \\
&= -\int_a^b \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle dt + \left\langle \frac{\delta l}{\delta \xi}\,, \,\eta \right\rangle \Bigg|_a^b \\
&= -\int_a^b \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle dt\,.
\end{aligned}$$

$\boxed{\text{seen} \Downarrow}$

(c) The reduced Legendre transform $h(\mu) := \langle \mu, \xi \rangle - l(\xi)$ yields

$$\delta h(\mu) = \langle \xi, \delta\mu \rangle + \left\langle \mu - \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle = \langle \xi, \delta\mu \rangle \quad \text{so} \quad \frac{\delta h}{\delta \mu} = \xi\,.$$

Hence, by substitution the Euler–Poincaré equations for $l(\xi)$ are equivalent to the Lie–Poisson equations for h:

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = -\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \quad \Longleftrightarrow \quad \frac{d\mu}{dt} = -\mathrm{ad}^*_{\delta h/\delta\mu}\mu\,.$$

$\boxed{\text{seen} \Downarrow}$

(d)

$$\frac{df}{dt} = -\left\langle \mathrm{ad}^*_{\delta h/\delta\mu}\,\mu, \frac{\delta f}{\delta \mu} \right\rangle = -\left\langle \mu, \mathrm{ad}_{\delta h/\delta\mu}\,\frac{\delta f}{\delta \mu} \right\rangle = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right] \right\rangle = \{f, h\}(\mu)$$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 24 of 20 marks

Total C marks: 10 of 12 marks

Total D marks: 14 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

MATH70010 Geometric Mechanics Markers Comments

Question 1	The goals and sequence of steps for answering were laid out very explicitly for the variational derivation of the Euler--Poincaré equations, as seen many times in the lectures and course notes for the MATH60010 and MATH70010 modules in Geometric Mechanics. This derivation is one of the fundamental elements of the module. The students all scored well in answering this question.
Question 2	Question 2 required an application of the goals of Question 1 for geodesic motion on the Lobachevsky half-plane. The students had seen the problem assigned early in the course notes. However, the complete solution to the problem only appeared considerably later in the notes. (The students had been told in the preface to the course notes that solutions to early exercises in the notes were likely to be given later in the notes; so, Question 2 was not given for marked course work.) The student scores were mixed and relatively high.
Question 3	Question 3 asked for applications of the given definition of the cotangent-lift momentum map for the action of a Lie group on a smooth manifold. Parts (a) and (b) were immediate applications of the given Lie algebra action. Parts (c), (d) and (e) required obtaining the Lie algebra action from the given Lie group action. The student scores were mixed and relatively high.
Question 4	Question 4 asked essentially the same question as Question 1, rephrased however conversely and with less explicit instructions on the progression of the solution. Scores were mixed and generally lower than the corresponding results for Question 1.
Question 5	No Comments