15 Hyperbolic PDEs

15.1 First-order PDE

Consider a quasi-linear equation of the type

$$a\,\frac{\partial z}{\partial x} + b\,\frac{\partial z}{\partial y} = c, \tag{15.1}$$

where $a,\ b,\ c$ are functions of $x,\ y$. This can be simplified by changing the variables $x,\ y$ to $s,\ t$ so that Eqn. 15.1 becomes

$$\left(\frac{\partial z}{\partial t}\right)_s = c. \tag{15.2}$$

Suitable variables s and t are identified by comparing Eqn. 15.1 with the identity

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}, \tag{15.3}$$

provided s and t are chosen such that

$$\frac{\partial x}{\partial t} = a, \quad \text{and} \quad \frac{\partial y}{\partial t} = b, \tag{15.4}$$

in which case Eqn. 15.1 reduces to Eqn. 15.2.

The variable s is defined by noting that, if s is constant, Eqns. 15.4 gives a coordinate direction along which the PDE reduces to an *ordinary differential equation* (ODE)

$$\frac{dy}{dx} = \frac{\partial y/\partial t}{\partial x/\partial t} = \frac{b}{a}, \tag{15.5}$$

which can be integrated (in principle – at least). The "*constant*" of integration is an arbitrary function of s (since s has been held constant) and any convenient function can be chosen to define s. Hence

$$s = f(x, y), \tag{15.6}$$

where f is a known function which defines s explicitly. The variable t can be defined by integrating **either** of the expressions 15.4, *e.g.* the first of these gives

$$t = \int \frac{dx}{a} - g(s), \tag{15.7}$$

the integration carried out with s held constant; Eqn. 15.6 is re-arranged in the form

$$y = y(x, s)$$

and used to express $a(x, y)$ as $a(x, s)$. The arbitrary function of s in Eqn. 15.7 may be chosen for convenience; it is usually taken to be zero.

Equation 15.2 is then integrated, first of all expressing c as a function of s and t by means of Eqns. 15.6 and 15.7,

$$z = \int c\ dt + h(s). \tag{15.8}$$

The integration is carried out keeping s constant. The arbitrary function $h(s)$ is determined by inserting the given boundary condition into Eqn. 15.8.

Observe, generally we have the **characteristic** relations

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}. \tag{15.9}$$

We may choose what approach and variable to use, based on ease of integration of these **characteristic** variables. As an example, or summary, note

$$\frac{dz}{dy} = \frac{c}{b}, \quad \frac{dz}{dx} = \frac{c}{a} \text{ along } \frac{dy}{dx} = \frac{b}{a}.$$

15.1.1 Example solution to first-order PDE

Consider the equation

$$y\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 2,$$

where U is known along an initial segment Γ along $y = 0$ and $0 \leq x \leq 1$ as indicated in Fig. 15.1. The PDE reduces to an ODE on the characteristic curve

$$\frac{dy}{dx} = \frac{1}{y}, \tag{15.10}$$

along which the solution is given by

$$\frac{dU}{dy} = 2. \tag{15.11}$$

Integration of Eqn. 15.10 gives $x = y^2/2 + A$ where A is a constant for each characteristic. So for a characteristic through $R(x_r, 0), A = x_r$. Thus the equation defining this path is

$$y^2 = 2(x - x_r)$$

and the solution on this characteristic will be given by $U = 2y + B$, with B a constant determined from the initial condition along $y = 0$ – if $U = U_r$ at $R(x_r, 0)$ then $B = U_r$, *i.e.*

$$U = 2y + U_r. \tag{15.12}$$

Since initial values of U are known along Γ or the segment P–Q, where $0 \leq x_r \leq 1$, it follows that the solution is only known in the region bounded by and including the terminating characteristics $y^2 = 2x$ and $y^2 = 2(x-1)$; outside this region (non-hatched area in Fig. 15.1) the solution will be undefined.

Along the $y^2 = 2x$ characteristic the solution U will be uniquely determined by the value of U_o at $P(0, 0)$, *i.e.* $U = 2y + U_o$. In other words the initial values for U on the initial curve $y^2 = 2x$ can not be arbitrarily prescribed, but is dependent upon the Γ initial data path defined*.

Importantly, observe that the initial data U_r prescribed along Γ need **not be smooth**, it may well be **discontinuous**, but the solution along the characteristic path $y^2 = 2(x - x_r)$ will still be uniquely defined to be $U = 2y + U_r$.

*see G.D. Smith for further details

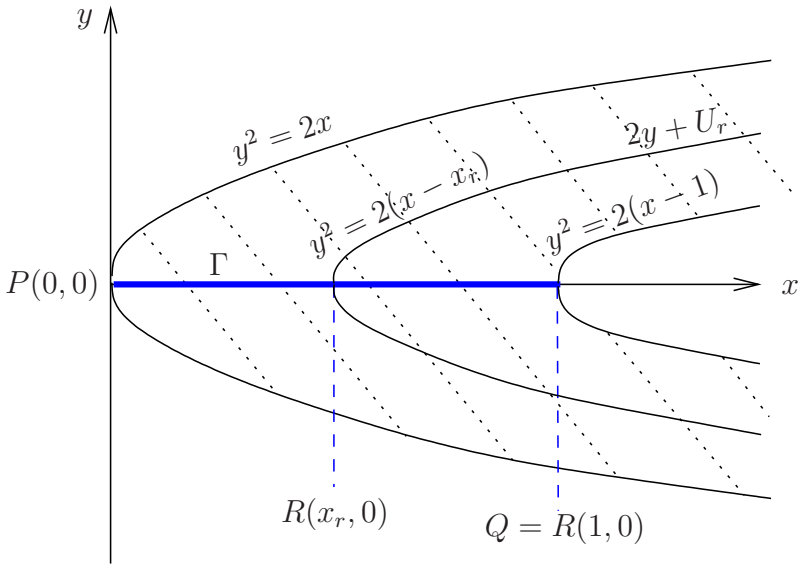

Figure 15.1: Solution domain, $\Gamma \in 0 \leq x \leq 1,\ y = 0$ (blue solid line) is the path along which the initial conditions are prescribed.

15.2 The one-dimensional linear advection equation and Upwinding

Consider the problem for $u(x,\, t)$

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \ \text{ for } \ t > 0, \ \ \text{initial condition } \ u(x,\, 0) = f(x)\ , \tag{15.13}$$

where c is a **positive** constant. The characteristics of this equation are

$$dx/dt = c \quad \text{or} \quad x - ct = \quad \text{constant.}$$

Furthermore, u is constant along each characteristic and so the solution is $u = f(x - ct)$. This represents a wave travelling in the positive x-direction with speed c, **without change of magnitude or shape** as shown in Figure 15.2 – we emphasis that the exact solution of Eqn. 15.13 says that the initial function $f(x)$ propagates unaltered in form in the positive x-direction as time progresses. The PDE simply translates as t progress, the profile $f(x)$ with velocity c to the right if $c > 0$ and to the left if $c < 0$.

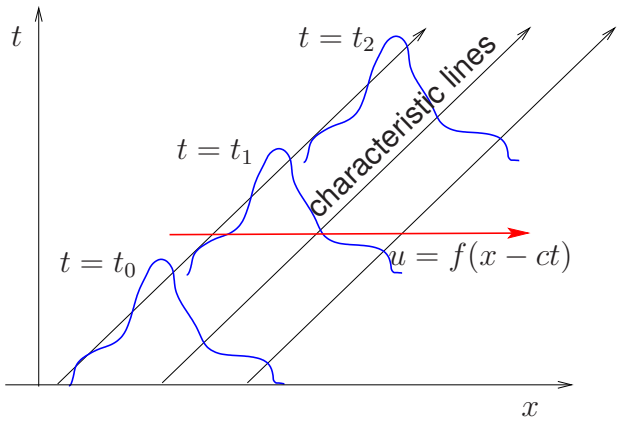

Figure 15.2: Propagation of a function $u = f(x)$ with time t.

Can we develop finite-difference methods (FDMs) with these properties? In the present context by this we mean, that our FDM reproduces precisely, or close to as possible, the expected behaviour of $f(x)$ propagating unaltered in form as time progresses.

For future reference, we observe that for a Fourier mode with $f(x) = \exp(ix/h) \equiv e^{in\xi}$. Here x/h is non-dimensional and may be viewed as a *wavenumber*; similarly (ck) has dimensions of a length, since c is a speed and k a time-scale, therefore $q = ck/h$ is non-dimensional and may too be associated as a wavenumber, in the following

$$u_n^{j+1} \equiv u(nh,\,(j+1)k) = u_n^j \exp(-i\xi q) \;\text{ where }\; q = \frac{ck}{h} > 0. \tag{15.14}$$

The value $q = ck/h$ is a very important dimensionless number called the **Courant number** (also referred to as the **Courant-Friedrichs-Lewy** (**CFL**)-condition). We shall see this plays an important role when considering the effectiveness of FD discretisations in hyperbolic problems. It may also be viewed as

$$\frac{ck}{h} \equiv \;\text{ ratio of }\; \frac{\textit{physical distance moved}}{\textit{grid-spacing}}.$$

The real solution therefore has a growth factor $\lambda \exp(-i\xi q)$. We note that for the model equation (15.13) $|\lambda| = 1$ so that there is no growth or decay. We can model our simple equation in many different ways. We shall use a forward difference for u_t and consider 6 schemes:

$$\frac{\partial u}{\partial t} \approx \frac{U_n^{j+1} - U_n^j}{-q} = \begin{cases} \textbf{A}: & \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \textbf{B}: & U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \textbf{C}: & U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \textbf{D}: & \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \textbf{E}: & \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \textbf{F}: & \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

(We shall also later consider Keller's 'Box scheme'). We use the Fourier method to investigate the stability of all of these schemes, looking for a solution $U_n^j = \lambda^j \exp(in\xi)$. We find:

$$\lambda = \begin{cases} \textbf{A}: & 1 - iq\sin\xi \\ \textbf{B}: & 1 - q(e^{i\xi} - 1) \\ \textbf{C}: & 1 - q(1 - e^{-i\xi}) \\ \textbf{D}: & \frac{1}{4}\left[iq\sin\xi \pm \sqrt{4 - q^2\sin^2\xi}\right]^2 \quad \text{(two steps)} \\ \textbf{E}: & \dfrac{2 - iq\sin\xi}{2 + iq\sin\xi} \\ \textbf{F}: & 1 - iq\sin\xi - 2q^2\sin^2\tfrac{1}{2}\xi \end{cases}$$

We recall that if

1. If $|\lambda| > 1 + O(k)$, the method is unstable.
2. If $|\lambda| < 1$, it is dissipative.

3. While if $|\lambda| = 1$, it is conservative. We mean (*loosely*) that the solution propagates in time without change, and is also exact as the $t-$variable progresses.

We see therefore that

$$|\lambda|^2 = \begin{cases} \textbf{A}: & 1+q^2\sin^2\xi & & \text{stable only if } q^2 = O(k)\ i.e.\ k \sim h^2 \\ \textbf{B}: & 1+2q(1+q)(1-\cos\xi) & \geq 1 \quad \forall\, q,\xi & \text{hopelessly unstable} \\ \textbf{C}: & 1-2q(1-q)(1-\cos\xi) & \leq 1 \quad \forall\, q \leq 1 & \text{stable, dissipative} \\ \textbf{D}: & 1 & & \text{provided } q^2\sin^2\xi \leq 4 \quad \text{or} \quad \tfrac{1}{2}q \leq 1 \\ \textbf{E}: & 1 & \forall\, q & \text{stable and conservative} \\ \textbf{F}: & 1-4q^2(1-q^2)\sin^4\tfrac{1}{2}\xi & \leq 1 \text{ if } q \leq 1 & \text{dissipative, but less than } \textbf{C} \end{cases}$$

We may also be interested in $\arg\lambda$, which determines the **phase** of each mode. In case **E**, for example,

$$\arg\lambda = -2\tan^{-1}(\tfrac{1}{2}q\xi) \approx -q\xi \quad \text{when} \quad \xi \quad \text{is small.}$$

All the above schemes agree well with the exact solution, for which $\arg\lambda = -q\xi$, when ξ is small. The waves with higher values of ξ are **dispersive**; their phase velocity varies with frequency. A computer demonstration will illustrate the effects of this. Although the amplitude of each wave component may be conserved, phase differences develop which alter the shape of the whole.

We can see from the above the importance of the CFL condition, $q \leq 1$. The Courant or CFL number is a non-dimensional number that plays a central role in the numerical solution of hyperbolic equations. If c can be thought of a speed, $q = ck/h \equiv c\Delta t/\Delta x$, as alluded to earlier, can be thought of a distance measured in grid points that a particle or information reaches to, in an increment of time $k \equiv \Delta t$.

Another very important conclusion we can draw is that backwards, or **Upwind** differences are a good idea. Case **C** is stable provided the CFL condition holds, whereas **B** is unconditionally unstable. If $c < 0$, the stable scheme is **B**. In general, the **Upwind** scheme can be written

$$U_n^{j+1} = U_n^j - sc\Delta U_n^j + s|c|\delta^2 U_n^j \qquad \text{where} \quad s = \tfrac{k}{2h}. \tag{15.15}$$

The need for upwind differences can be interpreted in terms of Eqn. 15.13's characteristics, $x - ct =$ *constant*, which must pass through the **Numerical Domain of Dependence**.

Above we have a very simple equation with constant coefficients (*i.e. the* c). In this very special case, setting $q = 1$ for some of the discretisation schemes above, the numerical scheme reproduces the exact solution with no error. However in more complex hyperbolic systems with varying coefficients maintaining this *exactness* is not that straightforward and should not be expected, unless one goes to considerable effort towards honouring the true physical propagation paths or characteristics of the equations set.

15.3 Upwinding for simultaneous equations

Let $\mathbf{u}(x,\, t)$ be a p-dimensional vector satisfying the equation

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{d}$$

where A is a $p \times p$ matrix, which for simplicity we shall assume to be constant. We shall investigate how the upwinding method generalises to this problem, by **diagonalising** the matrix A, which we assume to have p distinct eigenvalues $\lambda_1, \ldots \lambda_p$ and corresponding eigenvectors. We make the linear transformation $\mathbf{u} = S\mathbf{v}$ where S is the matrix whose columns are the eigenvectors of A, so that

$$S^{-1}AS = D \equiv \mathrm{diag}(\lambda_1,\, \lambda_2, \ldots, \lambda_p) \qquad \text{and so} \quad A = SDS^{-1}\ .$$

Then v_i, the i-th component of $\mathbf{v}$, satisfies the equation

$$(v_i)_t + \lambda_i (v_i)_x = (S^{-1}\mathbf{d})_i\ .$$

We have thus separated the problem into p separate ones, each with its own characteristic family, $dx/dt = \lambda_i$, and we can use **upwinding** on each one in turn. Note that if any of the λ_i are complex, then some of the p equations are elliptic, and we cannot use a time-stepping approach for them. We assume here that all the λ_i are real. From (10.9), the upwind scheme for $\mathbf{v}$ is

$$\mathbf{V}_n^{j+1} = \mathbf{V}_n^j - sD\Delta\mathbf{V}_n^j + sD^+\delta^2\mathbf{V}_n^j + kS^{-1}\mathbf{d}_n^j\ ,$$

where $D^+ \equiv \mathrm{diag}(|\lambda_1|,\, |\lambda_2|, \ldots, |\lambda_p|)$, and $\mathbf{V}$ is the finite-difference approximation to $\mathbf{v}$. Then transforming back, defining $\mathbf{U} = S\mathbf{V}$ so that $\mathbf{V} = S^{-1}U$, we find

$$\mathbf{U}_n^{j+1} = \mathbf{U}_n^j - sA\Delta\mathbf{U}_n^j + sA^+\delta^2\mathbf{U}_n^j + k\mathbf{d}_n^j \ \text{ where } \ A^+ = SD^+S^{-1}\ . \tag{15.16}$$

This is the required generalisation of upwinding for p simultaneous equations. If all the eigenvalues of A are positive, then $D^+ = D$ and $A^+ = A$, while $A^+ = -A$ if they are all negative. Otherwise, A^+ bears no simple relation to A, and we must be very careful how we define "**up**" when upwinding. The stability condition for Eqn. 15.16 is

$$\frac{k}{h}\max_{i=1\ldots p}|\lambda_i| \le 1\ . \tag{15.17}$$

16 Dissipation and Dispersion

16.1 Summary of findings through numerical experiments

We found on solving the very simple linear advection equations the following:

1. CFL > 1 : All methods examined tend to blow up schemes (A,B,C & F), *i.e.* were found to be unstable numerically.
2. CFL $= 1$: Schemes A, B were unstable, C & F give exact result.
3. CFL < 1 : Schemes C & F display dispersive, dissipative behaviour and out of phase with exact solution result. Results exasperated at regions where sudden changes, or large gradients in the u-field arise.
4. CFL < 1 : With greater resolution in grid, a reduction in the dispersive, dissipative behaviour of schemes C & F arose. Dissipation effects of scheme F were less than C. Dispersive behaviour in C was not evident; scheme F displays relatively weaker dissipation but significant dispersive behaviour.

Thus the numerical discretisations gave rise to *advection, dissipation, dispersion* and *phase differences*. These mechanisms are central to the behaviour of PDEs and their discrete models, and together account for most linear phenomena. We expect that the advection equation will advect (of course!), but how and why do the other effects arise? The phenomena that emerge from this simple study turn out to be fundamental, in understanding and being able to control modelling accuracy, when we attempt to discretise more complicated problems.

Looked at, from the viewpoint of advection of **energy**, consider the equation

$$u_t + cu_x = 0,$$

which with sufficiently smooth initial data $u(x,0) = u_o(x)$, has the solution

$$u(x,t) = u_o(x - ct).$$

With this PDE, energy propagates at a finite speed and is **conserved** : by which we mean nothing is *lost, gained (or dissipated)*. However, not always does all the energy propagate at exactly the same finite speed. Consider the dissipation (or diffusion) given by the parabolic one-dimensional (heat) equation

$$u_t = u_{xx},$$

with sufficiently well-behaved initial data $u(x,0) = u_o(x)$, the solution using Fourier transforms, can be written as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x - i\alpha^2 t}\hat{u}_o(\alpha)\, d\alpha, \tag{16.1a}$$

$$= \frac{1}{4\pi t}\int_{-\infty}^{\infty} e^{-(x-s)^2/4t} u_o(s)\, ds. \tag{16.1b}$$

Here $\hat{u}_o(\alpha)$ is the Fourier transform of the initial data with α the spatial wavenumber. Physically, the above solution asserts that the oscillatory components in the initial data of

wavenumber α decay at a rate of $e^{-\alpha^2 t}$. The solution will then be composed of increasingly smooth wave components.

The simplest example of dispersion is modelled by the equation

$$u_t = i\, u_{xx},$$

the one-dimensional **Schrodinger equation**. Similar to the the heat equation, the solution to this can be expressed as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x - i\alpha^2 t}\hat{u}_o(\alpha)\, d\alpha, \tag{16.2a}$$

$$= \frac{1}{4i\pi t}\int_{-\infty}^{\infty} e^{i(x-s)^2/4t} u_o(s)\, ds. \tag{16.2b}$$

giving quite different behaviour. Dispersion is observed in the form of solutions that (rather than decaying as $t \to \infty$, as for the heat equation) break into oscillatory **wave packets**. Figure 16.1 illustrates these features.

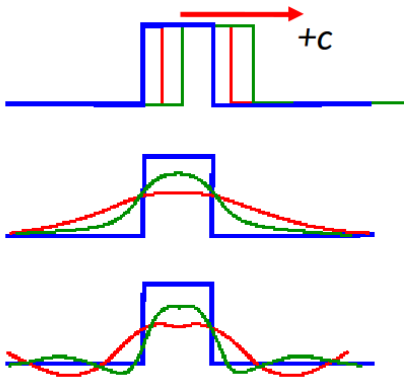

Figure 16.1: Propagation of a top hat initial condition in time under an advection (top), dissipation (middle) and dispersive (bottom) equation. Initial condition (blue) and later time instances shown.

Diffusion refers to a phenomenon where some quantity spreads out in space as time goes on. Dissipation is used to refer to loss of energy. The effects are related. If you consider the heat equation

$$T_t = \mu \nabla^2 T, \tag{16.3}$$

and start with an initial condition with high temperature in some small region, then the temperature distribution spreads out. This would be called diffusion. If you calculate

$$\int_\Omega T^2(x,t)\, dx, \tag{16.4}$$

this quantity will decrease with time. So there is a loss of energy. (This is not really energy in physical sense if T is temperature, you can substitute velocity here). Both are caused by the Laplacian term. If something spreads out, its square integral will probably decrease.

Dispersion is related to wave phenomena. For a linear wave equation

$$u_t + c u_x = 0, \quad \text{or} \quad u_{tt} = c^2 u_{xx} \tag{16.5}$$

you can resolve the solution into Fourier modes. Then each mode will travel at the same speed. We say such equations are **non-dispersive**. These two equations are also **non-dissipative**. For smooth solutions with periodic boundary conditions, they conserve all integrals of the form

$$\int_{\Omega} |u(x,t)|^P \, dx. \tag{16.6}$$

An equation like

$$u_t + cu_x = \nu u_{xxx}, \tag{16.7}$$

would have solutions whose Fourier modes travel at different speed depending on ν and wave number. We say these equations exhibit **dispersive** behaviour. We observe these phenomena at the level of PDE and we can map them to the behaviour of numerical schemes.

The effect of dispersion, therefore, is that often spurious oscillations or wiggles occur in solutions that include sharp gradients, discontinuities, or shocks. This is usually manifested by high-frequency oscillations trailing the particular effect.

If we solve a non-dissipative and non-dispersive PDE with a numerical scheme, we want the numerical solutions to be non-dissipative and non-dispersive too. As we have shown, sadly this is not possible since numerical solutions exhibit these behaviours even if the exact solution does not. Then we say that the numerical scheme is dissipative and/or dispersive and the challenge is to devise discretisation schemes which at least minimise these effects which are not present in the exact solution.

16.2 Modified Equations

Solutions of finite-difference schemes applied to partial differential equations are typically thought of as approximations to the exact solutions of the continuous problem. It is interesting and instructive to start with the discrete finite-difference equation and ask the question *"which partial differential equation has this discrete equation as an exact solution"*. We consider the one-dimensional diffusion equation, discretised in space by a centered second-order finite-difference scheme and in time by a forward Euler method, namely

$$\frac{U_n^{j+1} - U_n^j}{k} = \mathcal{D}\left(\frac{U_{n-1}^j - 2U_n^j + U_{n+1}^j}{h^2}\right),$$

where $\mathcal{D}$ is the diffusion, k the time step and h the spatial step interval. Employing a Taylor series about U_n^j, we obtain for the time and space derivative, respectively,

$$\frac{U_n^{j+1} - U_n^j}{k} = \frac{\partial u}{\partial t} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \dots$$

and

$$\frac{U_{n-1}^j - 2U_n^j + U_{n+1}^j}{h^2} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4},$$

which results in

$$\frac{U_n^{j+1} - U_n^j}{k} - \mathcal{D}\frac{U_{n-1}^j - 2U_n^j + U_{n+1}^j}{h^2} = \frac{\partial u}{\partial t} - \mathcal{D}\frac{\partial^2 u}{\partial x^2} - \mathcal{D}\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \dots$$

As both the time and space step tend to zero, the discrete equation approaches the continuous equation, which renders the chosen scheme **consistent** in space and time. The term on the right-hand side confirms a second-order accuracy in space and a first-order accuracy in time. Moreover, the largest error term is dissipative in nature (due to the second derivative in time and the fourth derivative in space) – using the fact that

$$\frac{\partial^2 u}{\partial t^2} = \mathcal{D}\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2}) = \mathcal{D}^2\frac{\partial^4 u}{\partial x^4}.$$

Hence by a judicious choice of $\mathcal{D}, k$ and h, one may eliminate the largest error terms in the above equation for the most accurate representation of the FD to the exact PDE, but the space and time step required to meet this, is very restrictive, and so seldom used in practice, *i.e.*

$$\frac{\mathcal{D}k}{h^2} = \frac{1}{6}.$$

Linear Advection equation, up-winded scheme :
Consider the discretisation of $u_t + cu_x = 0$,

$$U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j), \quad \text{with} \quad q = ck/h.$$

A Taylor series expansion around U_n^j, gives

$$u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + \cdots - u + q\left(u - \left[u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + \ldots\right]\right) = 0.$$

On simplification, and noting $q = ck/h$, we obtain

$$u_t + cu_x = \frac{1}{2}(chu_{xx} - ku_{tt}) - \frac{1}{6}(k^2u_{ttt} + ch^2u_{xxx}).$$

Next we use, assuming c to be a constant $u_{tt} = -cu_{xt} = -cu_{tx} = c^2u_{xx}$, while ignoring the $u_{ttt},\ u_{xxx}$ terms as being even smaller order, followed by simplification, we get

$$u_t + cu_x = \frac{ch}{2}\left(1 - \frac{ck}{h}\right)u_{xx},$$

i.e.

$$u_t + cu_x = \frac{ch}{2}(1 - q)u_{xx}. \tag{16.8}$$

A similar analysis for the Lax-Wendroff discretisation

$$U_n^{j+1} = (1 - q^2)U_n^j + \frac{q(q-1)}{2}U_{n+1}^j + \frac{q(q+1)}{2}U_{n-1}^j$$

gives

$$u_t + cu_x = -\frac{ch^2}{6}(1 - q^2)u_{xxx}. \tag{16.9}$$

Comparing Eqns.16.8 and 16.9, and recalling results from the numerical experimentation, we observe the following

1. Setting $q = 1$, namely the CFL condition, both expressions recover the exact PDE.
2. In Eqn. 16.8, the up-winded scheme gives a u_{xx} term factored by $ch(1-q)/2$ *diffusion-like* term. Thus provided $1 - q < 1$, we may expect our numerical solutions to exhibit diffusion or dissipation effects. While for $q > 1$ a negative diffusion term appears, in which case we should expect our numerical method to be unstable (ill-posed).
3. In Eqn. 16.9 the u_{xxx} term leads to dispersive behaviour, rather than dissipation.
4. We also observe, that the magnitude of the error is smaller for the Lax-Wendroff method, compared to up-winded scheme C. Since scheme F is a higher-order method (hence the $O(h^2)$ dispersive term, compared to the $O(h)$ dissipation term in scheme C.
5. Dispersive term gives rise to oscillating solution and also a shift in main peak location, *i.e.* phase error as observed in the numerical experiments.

16.3 Dispersion-dissipation analysis

When solving PDES analytically a common approach is to assume that the solution $u(x,t)$ has a wave form

$$u(x,t) = \hat{u}e^{i(\omega t+\alpha x)} \tag{16.10}$$

where ω is a frequency and α the wavenumber; the wavelength is given by

$$\lambda = \frac{2\pi}{\alpha}.$$

Consider the equation

$$u_t = \nu u_{xx}, \tag{16.11}$$

using Eqn. 16.10 gives

$$i\omega + \nu\alpha^2 = 0, \tag{16.12}$$

from which it is clear that Eqn. 16.10 can only be satisfied if

$$\omega = i\nu\alpha^2.$$

We call Eqn. 16.12 the **dispersion relationship**. The solution to Eqn. 16.11 thus becomes

$$u(x,t) = \hat{u}e^{-\nu\alpha^2 t}\, e^{i\alpha x}. \tag{16.13}$$

This indicates that the wave does not move and decays with time. Likewise for the linear advection equation

$$u_2 + cu_x = 0, \quad \text{with } c \text{ a constant value.} \tag{16.14}$$

We get the dispersion relationship

$$\omega = -c\alpha,$$

and hence

$$u(x,t) = \hat{u}e^{i\alpha(x-ct)}. \tag{16.15}$$

In this case, assuming ω was real, the solution propagates with speed c and with no decay in amplitude. Also note that the speed of propagation is independent of the frequency ω.

The decay (or growth) and propagation of Fourier modes is an important aspect in the behaviour of solutions to PDEs. With finite difference (F-D) discretisation of the PDE an aspect to consider is upon obtaining the discretised solution, is how well do they decay and the F-D derived solution propagation characteristics match those of the original PDE. The numerical scheme will be unstable if some of the Fourier modes grow without bound. In what follows, **dissipation** in solutions of PDEs is when the Fourier modes do not grow with time and at least one mode decays. A PDE is **non-dissipative** if they neither grow nor decay, while we define **dispersion** when solutions of PDEs exhibit differing Fourier mode wavelengths propagating at different speeds. In this respect, observe that Eqn. 16.13 is dissipative for $\nu > 0$ for all wave-numbers $\alpha \neq 0$, while Eqn. 16.15 is neither dissipative nor dispersive.

Next consider

$$u_t + cu_{xxx} = 0, \tag{16.16}$$

for which the dispersion relationship is

$$\omega = \alpha^3 c, \tag{16.17}$$

and hence the Fourier mode representing a solution to Eqn. 16.16 is

$$u(x,t) = \hat{u}\, e^{i\alpha(x+\alpha^2 ct)}. \tag{16.18}$$

There are two aspects here:

1. Unlike Eqn. 16.15 the Fourier mode propagates in the opposite direction with speed $\alpha^2 c$;
2. Fourier modes with different wave-numbers α propagate with different speeds $\alpha^2 c$.

So Eqn. 16.16 is dispersive but non-dissipative: *i.e.* the amplitude neither decays nor grows (with either space or time). With some thought, one should be able to see that PDEs containing even ordered x-derivatives will be dissipative. PDEs containing only odd derivatives in x will be non-dissipative and involve propagating waves, and be dispersive when the order is greater than one.

Finally consider the equation

$$u_t + au_x - \nu u_{xx} + cu_{xxx} = 0, \tag{16.19}$$

for which the dispersion relationship is

$$\omega = -a\alpha + i\nu\alpha^2 + c\alpha^3, \tag{16.20}$$

and hence the solution for the α-mode will be

$$u(x,t) = \hat{u}e^{-\nu\alpha^2 t}\, e^{i\alpha\left[x-(a-c\alpha^2)t\right]}. \tag{16.21}$$

The dissipation term is $e^{-\nu\alpha^2 t}$, while the propagating term is

$$e^{i\alpha\left[x-(a-c\alpha^2)t\right]},$$

thus the PDE is dissipative and dispersive due to the dependence of the term $(a - c\alpha^2)$ term on α^2, and becomes larger the more larger α is.

16.4 Dispersion and dissipation in discretised equations

The discrete analogue of the Fourier mode Eqn. 16.10 is

$$u_k^n = \hat{u} e^{i(n\omega\ \Delta t + \alpha k\ \Delta x)}, \tag{16.22}$$

and thus for information about dispersion and dissipation for all wavelengths we consider $\alpha\ \Delta x$ in the range $0 \leq \alpha\ \Delta x \leq \pi$. Furthermore for what follows, the dispersion relation $\omega = \omega(\alpha)$ will in general be complex, hence we set

$$\omega = a + i\ b,$$

with (a, b) considered real only. Substituting $\omega = a + ib$ into Eqn. 16.22 and a rewrite gives

$$u_k^n = \hat{u} e^{-bn\ \Delta t}\ e^{i(na\ \Delta t + k\alpha\ \Delta x)} = \hat{u}(e^{-b\ \Delta t})^n\ e^{i\alpha(k\ \Delta x - (-a/\alpha)n\ \Delta t)}. \tag{16.23}$$

From this we see that

- If $b > 0$ for some α, the difference equation is dissipative.
- If $b < 0$ for some α, the solution grows without bound, and thus the scheme will be unstable.
- If $b = 0$ for all α, the scheme will be non-dissipative.

Furthermore

- If $a = 0$ for all α, there is no wave propagation.
- If $a \neq 0$ for some α, wave propagation with velocity $-a/\alpha$ occurs.
- If a/α is a non-trivial function of α, dispersive effects will arise.

Let us consider the linear advection Eqn. 16.5 (with c positive) discretised with an up-winded scheme, namely :

$$U_j^{n+1} = U_j^n - q(U_j^n - U_{j-1}^n), \tag{16.24}$$

where $q = ck/h$. Substitution of (16.22) into the above and simplification gives

$$e^{i\omega k} = 1 - q + q\left[\cos(\alpha h) - i\sin(\alpha h)\right]. \tag{16.25}$$

We next write $\omega = a + ib$ and hence

$$e^{iak} e^{-bk} = 1 - q + q\left[\cos(\alpha h) - i\sin(\alpha h)\right]. \tag{16.26}$$

Which thus gives the relative dissipation error $\mathcal{E}_D$

$$\mathcal{E}_D = e^{-bk} = \sqrt{(1-q)^2 + q^2 + 2q(1-q)\cos(\alpha h)}. \tag{16.27}$$

For stability we require that $b > 0$ for some αh. Setting $\alpha h = 0$ we get that

$$e^{-bk} = 1,$$

that the modes neither decay nor grow, while $\alpha \neq 0$, all modes decay for $|q|< 1$. Setting $q = 1$ thus gives $b = 0$, which implies that the scheme will be non-dissipative for all α.

Next setting $q = 1 - \epsilon$, with $\epsilon \ll 1$, it can be shown that

$$e^{-bk} = 1 - 2\epsilon \sin^2(\alpha h/2) + \frac{\epsilon^2}{2} \sin^2(\alpha h) + \dots \,, \tag{16.28}$$

which suggests the scheme will be dissipative for $q < 1$ values.

Whether the scheme is dispersive or not we consider the imaginary part of Eqn. 16.26, namely

$$e^{iak} = \cos(ak) + i \sin(ak) = \frac{1 - q + q\,[\cos(ah) - i \sin(ah)]}{|(1 - q + q\,[\cos(ah) - i \sin(ah)])|}, \tag{16.29}$$

hence

$$\tan(ak) = \frac{-q \sin(\alpha h)}{1 - q + q \cos(\alpha h)},$$

i.e. hence

$$a = -\frac{1}{k} \tan^{-1} \left(\frac{q \sin(\alpha h)}{1 - q + q \cos(\alpha h)} \right) = -\frac{1}{k} \tan^{-1} \left(\frac{q \sin(\alpha h)}{1 - 2q \sin^2(\alpha h/2)} \right). \tag{16.30}$$

This is in general dispersive since a is a nonlinear function of α. This expression then requires a careful examination to ascertain behaviour of the low frequency (αh small) and high frequency (αh near π) waves. For $\alpha h \ll 1$ (small), we obtain

$$ak = -q\alpha h + O(\alpha h)^3. \tag{16.31}$$

Thus dispersive effects should be expected for the low frequency modes. The high frequency modes require a similar but separate analysis, namely for $\alpha h \leq \pi$ limit.

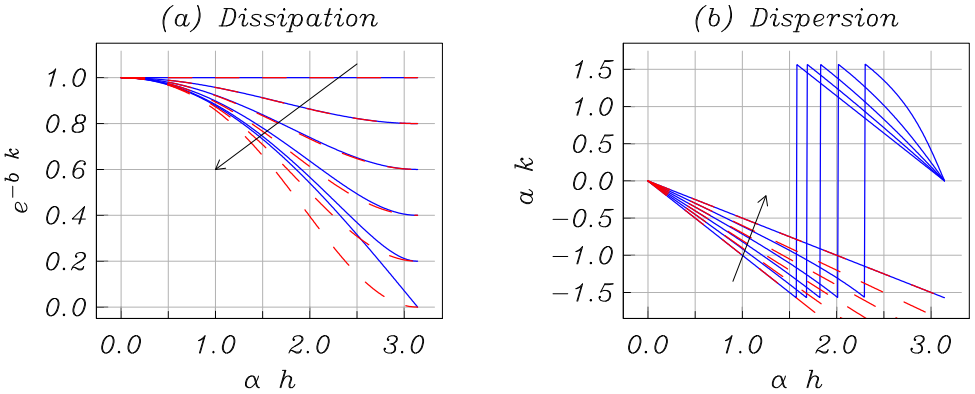

Figure 16.2: Dissipation and dispersive effects arising in Eqn. (16.24) as q and αh vary; $q = (1,\ 0.9,\ 0.8,\ 0.7,\ 0.6,\ 0.5)$ (arrow in direction of decreasing q) (a) Dissipation given by Eqn. 16.27; (b) Dispersive behaviour given by Eqn. 16.30. The approximate results Eqn. (16.28) and Eqn. (16.31) are shown by the red dashed lines.

The relative dispersion error $\mathcal{E}_p$ is thus

$$\mathcal{E}_p = \left[\frac{\tan^{-1} \left(\frac{-q \sin(\alpha h)}{1 - 2q \sin^2(\alpha h/2)} \right)}{-q\alpha h} \right],$$

Further details see, chapter 7 of Thomas, J. W. "Numerical Partial Differential Equations – Finite Difference Methods"