MATH50001 Problems Sheet 6
Solutions

1) Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta$, $d\theta = \frac{dz}{iz}$,

$$\sin^2\theta = -\frac{1}{4}\left(z - \frac{1}{z}\right)^2 = -\frac{1}{4}\frac{(z^2-1)^2}{z^2} \quad \text{and} \quad \cos\theta = \frac{1}{2}\frac{z^2+1}{z}.$$

Therefore we find

$$\int_0^{2\pi} \frac{\sin^2\theta}{2+\cos\theta}\, d\theta = \frac{i}{4}\oint_{|z|=1|} \frac{(z^2-1)^2}{z^2\left(2 + \frac{1}{2}\frac{z^2+1}{z}\right)}\frac{dz}{z}$$

$$= \frac{i}{2}\oint_{|z|=1} \frac{(z^2-1)^2}{z^2(4z+z^2+1)}\, dz.$$

Within the disc $\{z : |z| < 1\}$ the integrant has one pole of order two at $z = 0$ and one more pole at $z = -2+\sqrt{3}$ of order one. Therefore we obtain

$$\frac{i}{2}\oint_{|z|=1} \frac{(z^2-1)^2}{z^2(4z+z^2+1)}\, dz$$

$$= \frac{i}{2}2\pi i\left(\text{Res}\left[\frac{(z^2-1)^2}{z^2(4z+z^2+1)}, 0\right] + \text{Res}\left[\frac{(z^2-1)^2}{z^2(4z+z^2+1)}, -2+\sqrt{3}\right]\right)$$

$$= -\pi \lim_{z\to 0}\frac{d}{dz}\left(\frac{(z^2-1)^2}{4z+z^2+1}\right) - 2\pi\sqrt{3} = 2\pi(2-\sqrt{3}).$$

Answer:

$$\int_0^{2\pi} \frac{\sin^2\theta}{2+\cos\theta}\, d\theta = 2\pi(2-\sqrt{3}).$$

2)

Let Log (z) be the principle value of $\log(z)$ and let

$$f(z) = -\text{Log}\,(z) + \int_1^z \frac{e^\eta}{\eta}\, d\eta.$$

a) Differentiating $f(z)$ with $z \in \mathbb{C} \setminus (-\infty, 0]$ we have

$$-\frac{1}{z} + \frac{e^z}{z} = \frac{e^z - 1}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}.$$

The latter series converges for all $z \in \mathbb{C}$ and thus defines an entire function $f'(z)$.

b) A primitive to f' can now be found by integrating

$$f(z) = \int \left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \right) \, dz = \sum_{n=1}^{\infty} \frac{z^n}{nn!} + C.$$

In order to find C we note that $f(1) = 0$ and thus

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{nn!}.$$

3) Let us introduce the parametrisation $z = e^{it}$, $t \in [0, 2\pi]$. Then

$$\overline{\oint_{\gamma} f(z) \, dz} = \int_0^{2\pi} \overline{f(e^{it})} e^{-it} \, (-i) \, dt$$
$$= -\int_0^{2\pi} \overline{f(e^{it})} e^{-2it} \, e^{it} \, i \, dt = -\oint_{\gamma} \frac{\overline{f(z)}}{z^2} \, dz.$$

4) There are two cases:

If $|w| < 1$, then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z(z-w)} = \frac{1}{2\pi i} \frac{1}{w} \oint_{\gamma} \left(\frac{1}{z-w} - \frac{1}{z} \right) \, dz$$
$$= \frac{1}{2\pi i} \frac{1}{w} \, (2\pi i - 2\pi i) = 0.$$

If $|w| > 1$, then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z(z-w)} = \frac{1}{2\pi i} \frac{1}{w} \oint_{\gamma} \left(\frac{1}{z-w} - \frac{1}{z} \right) \, dz$$
$$= -\frac{1}{2\pi i} \frac{1}{w} \oint_{\gamma} \frac{1}{z} \, dz = -\frac{1}{w}.$$

5) We argue by contradiction. Assume that for any $\varepsilon_n = 1/n$ there is a polynomial p_n, such that

$$\max_{z \in A} |p_n(z) - z^{-1}| < \frac{1}{n}.$$

This implies that p_n convergens uniformly on A to $1/z$. Let

$$\gamma = \left\{ z : \, |z| = \frac{r+R}{2} \right\}.$$

Since p_n is holomorphic

$$\oint_\gamma p_n(z)\,dz = 0.$$

Using that $p_n \to 1/z$ uniformly on A, we have

$$0 = \oint_\gamma p_n(z)\,dz \to \oint_\gamma \frac{1}{z}\,dz = 2\pi i.$$

6) We first find the number of roots of the equation $w(z) = z^3 + 5z + 1 = 0$ for $|z| < 1$. Denoting $f(z) = 5z$ and $g(z) = z^3 + 1$ we obtain that for $z : |z| = 1$

$$|g(z)| = |z^3 + 1| \leq 2 < 5 = |5z| = |f(z)|.$$

By using the Rouche's theorem we obtain that since $f(z) = 5z = 0$ has only one solution, the number of roots of $w(z)$ inside the unit disc equals one. Since the degree of $w(z)$ equals three and $w(z) \neq 0$ for $z : |z| = 1$, we conclude that the equation

$$z^3 + 5z + 1 = 0$$

has 2 zeros for $|z| > 1$.

7) On the circle $|z| = 3/2$, $|z^5| = 243/32$ and $|15z + 1| \geq 15|z| - 1 = 21.5$. Thus $|15z + 1| > |z|^5$. Hence there is no zero of the polynomial on the circle. If we now denote by $f(z) = 15z + 1$ and by $g(z) = z^5$, then by Rouche's Theorem we have $N(f + g) = N(f)$ inside $|z| = 3/2$. Since the equation $f(z) = 15z + 1 = 0$ has one solution $z_0 = -1/15$, we conclude that $z^5 + 15z + 1$ has one zero inside the circle $|z| < 3/2$.

On the circle $|z| = 2$, $|z^5| = 32$ and $|15z + 1| \leq 15|z| + 1 = 31$. Hence there is no zero of the polynomial on the circle and by Rouche's Theorem $N(z^5 + 15z + 1) = N(z^5) = 5$ inside $|z| = 2$. Thus we deduce that in the annulus $\{z : 3/2 < lzl < 2\}$ there are four zeros.

8) Let us split the function $w(z) = f(z)+g(z) = z^{100}+8z^{10}-3z^3+z^2+z+1$ such that

$$f(z) = 8z^{10} \qquad \text{and} \qquad g(z) = z^{100} - 3z^3 + z^2 + z + 1.$$

Then for $|z| = 1$ we have

$$|f(z)| = 8 > 7 = |z^{100}| + |3z^3| + |z^2| + |z| + 1 \geq |z^{100} - 3z^3 + z^2 + z + 1|.$$

Therefore the number of solutions of the equation $w(z) = 0$ inside the unit disc coincides with the number of solutions of $z^{10} = 0$, namely 10.

9)
a) Let us consider the case $z : |z| = 1$ and split the function $w(z) = 3z^9 + 8z^6 + z^5 + 2z^3 + 1$ as $f(z) = 8z^6$ and $g(z) = 3z^9 + z^5 + 2z^3 + 1$. Then

$$|f(z)| = 8 > 7 = |3z^9| + |z^5| + |2z^3| + 1 \geq |3z^9 + z^5 + 2z^3 + 1| = |g(z)|.$$

Therefore inside the unit disk there are 6 zeros of w.
b) Let us consider first the case $z : |z| = 2$. Denote $f(z) = 3z^9$ and $g(z) = 8z^6 + z^5 + 2z^3 + 1$. Then

$$\begin{aligned} |f(z)| = 32^9 = 1536 > 512 + 32 + 16 + 1 &= 8|z^6| + |z^5| + 2|z^3| + 1 \\ &\geq |8z^6 + z^5 + 2z^3 + 1| = |g(z)|. \end{aligned}$$

Therefore there are 9 roots of the equation $w(z) = 0$ inside the disc $|z| = 2$. Note that there are no roots of the equation $w(z) = 0$ on the circle $|z| = 1$. Therefore we conclude that there are 3 roots of the equation $w(z) = 0$ in annulus $\{z : 1 < |z| < 2\}$.

10) On the circle $|z| = 1$ we have $|az^n| = |a|$ and $|e^z| = e^{\cos\theta} \leq e$. Thus $|az^n| > |e^z|$, $|z| = 1$. The function $az^n - e^z$ has no roots on $|z| = 1$ and no poles. By Rouche's Theorem, $N(az^n - e^z) = N(az^n) = n$.