Imperial College London

# Analysis 1

## Revision Lecture for Fall Term Analysis 1 2022

Ajay Chandra

## Basic notions: convergence of series

- A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent/divergent if and only if its sequence of partial sums $(s_n)_{n=1}^{\infty}$ is convergent/divergent, here

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- It is possible for $\sum_{n=1}^{\infty} a_n$ to be convergent but not absolutely convergent - this is often called **conditional convergence.**

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- Important example series to keep in mind:
  - $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is absolutely convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.
  - $\sum_{n=1}^{\infty} r^n$ is absolutely convergent for $|r| < 1$ and divergent for $|r| \geq 1$.

## Rearrangement of series

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Here is an example rearrangement question.

## Question

Given a sequence $(a_n)_{n=1}^{\infty}$, suppose that we define

$$S = \left\{ a \in \mathbb{R} : \begin{array}{c} \text{there exists a rearrangement} \\ \text{of } \sum_{n=1}^{\infty} a_n \text{ converging to } a \end{array} \right\}.$$

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- **True/False:** It is possible for $S$ to be empty.
- **True/False:** It is possible for $S$ to be countably infinite.
- **True/False:** It is possible for $S$ to be uncountably infinite.

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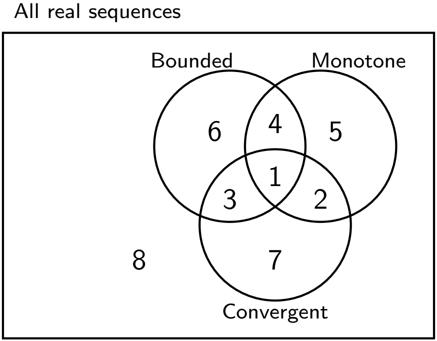
  - Some examples of sets we know are uncountably infinite sets : $\mathbb{R}$, power set of countably infinite set.

**A good problem on sequences:**

Below is a Venn Diagram of all real sequences with three regions corresponding to bounded, monotone, and convergent sequences. List all the numbered regions in this Venn Diagram that must be empty, that is every region where it would be impossible to find any sequences.



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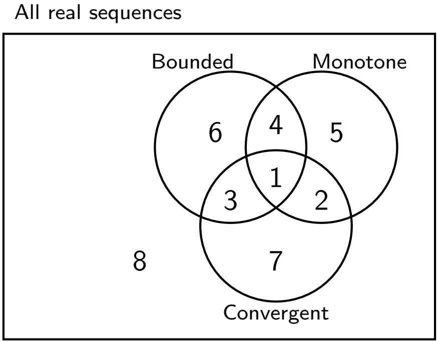


**2** - cannot be (convergent but not bounded),
**4** - cannot be (bounded and monotone but not convergent),
**7** - cannot be (convergent but not bounded)

For sequences, crucial tools in proofs:

- Every convergent sequence is bounded.
- Bounded (above/below) and monotone (increase/decrease) implies convergent.
- Cauchy implies convergent.
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Definitions to memorize (intuitively and in terms of quantifiers):

- $a_n$ is bounded.
- $a_n$ is unbounded.
- $a_n \to a$.
- $a_n$ is convergent.
- $a_n$ is divergent
- $a_n$ diverges to infinity.
- $a_n$ is Cauchy.