MATH50001/50017/50018 - Analysis II
Complex Analysis

Lecture 9

Section: Taylor and Maclaurin series.

Theorem. (Taylor Expansion theorem)
Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 \ldots,$$

valid in all circles $\{z : |z - z_0| < r\} \subset \Omega$.

Brook Taylor

1685 – 1731,
English

Colin Maclaurin

1698 – 1746
Scottish

Proof. Let $\gamma = \{\eta : |\eta - z_0| = r\} \subset \Omega$ and let $z : |z - z_0| < r$.

$$\begin{aligned} f(z) = \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z}\, d\eta &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{(\eta - z_0) - (z - z_0)}\, d\eta \\ &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\eta - z_0}}\, d\eta \\ &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot \Bigg\{ 1 + \frac{z - z_0}{\eta - z_0} + \left(\frac{z - z_0}{\eta - z_0}\right)^2 + \dots \\ &\qquad + \left(\frac{z - z_0}{\eta - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\eta - z_0}\right)^n}{1 - \frac{z - z_0}{\eta - z_0}} \Bigg\}\, d\eta \end{aligned}$$

Using Cauchy's generalised integral formula applied to the first n terms we obtain

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n,$$

where

$$R_n = \frac{(z - z_0)^n}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)(\eta - z_0)^n} \, d\eta.$$

Let $M = \max_{\eta\in\gamma} |f(\eta)|$ and let $|z - z_0| = \rho$. Then by using the ML-inequality we obtain

$$|R_n| \leq \frac{\rho^n}{2\pi} \frac{M}{(r - \rho)\, r^n} (2\pi r) = \frac{rM}{r - \rho} \left(\frac{\rho}{r}\right)^n.$$

Since $\rho < r$ we conclude that $R_n \to 0$ as $n \to \infty$.

Definition. The expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 \ldots,$$

is called the Taylor series of f about z_0. The special case in which $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

is called the Maclaurin series for f.

Example.

$f(z) = e^z$, $f^{(n)}\Big|_{z=0} = 1$. Therefore

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \qquad R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Example.

$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$ $(R = 1)$.

Example.

$\operatorname{Log}(1-z)$. Note that

$$(\operatorname{Log}(1-z))' = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Integrating both sides we arrive at

$$\operatorname{Log}(1-z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} + C = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C,$$

where $C = \operatorname{Log}(1-0) = 0$.

Example.

$f(z) = \frac{1}{1+z}$ about $z_0 = i$.

$$\frac{1}{1+z} = \frac{1}{1+i+z-i} = \frac{1}{1+i} \cdot \frac{1}{1-\left(-\frac{z-i}{1+i}\right)}$$

$$= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1+i)^{n+1}} (z-i)^n.$$

where R is defined by the inequality

$$\frac{|z-i|}{|1+i|} < 1 \quad \text{or} \quad |z-i| < \sqrt{2}.$$

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Cauchy-Goursat's theorem implies

$$\oint_T f_n(z)\, dz = 0, \qquad \text{for all } n.$$

By assumption $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\oint_T f_n(z)\, dz = \oint_T f(z)\, dz.$$

Therefore

$$\oint_T f(z)\, dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω.

Remark. This is not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. WHY??

Remark. Consider

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

where f_n are holomorphic in $\Omega \subset \mathbb{C}$. Assume that the series converges uniformly in compact subsets of Ω, then the theorem guarantees that F is also holomorphic in Ω.

Theorem. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω. Then the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.

Proof. For any $\widetilde{\Omega} \subset \Omega$ such that $\overline{\widetilde{\Omega}} \subset \Omega$ and given $\delta > 0$ we define $\widetilde{\Omega}_\delta \subset \widetilde{\Omega}$ by

$$\widetilde{\Omega}_\delta = \{z \in \widetilde{\Omega} : \overline{D_\delta}(z) \subset \widetilde{\Omega}\}.$$

By the previous theorem it is enough to show that $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on $\widetilde{\Omega}_\delta$. For any holomorphic function F in Ω_δ we have

$$\begin{aligned} |F'(z)| &= \left| \frac{1}{2\pi i} \oint_{|\eta - z| = \delta} \frac{F(z)}{(\eta - z)^2} \, d\eta \right| \\ &\leq \frac{1}{2\pi} \max_{\eta \in \overline{\widetilde{\Omega}}} |F(\eta)| \, \frac{1}{\delta^2} \, 2\pi\delta \leq \frac{1}{\delta} \max_{\eta \in \overline{\widetilde{\Omega}}} |F(\eta)|. \end{aligned}$$

Applying this inequality to $F(z) = f_n - f$ we conclude the proof.

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω. Then F is holomorphic in Ω.

Thank you

Quizzes