Imperial College London

MATH70017

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Tensor Calculus and General Relativity

Date: 16 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer Questions 1-3, and Questions 4-5 in Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Formula Sheet

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b.$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Equation of Parallel Transport:

$$\frac{dv^a}{d\lambda} + \Gamma^a_{bc} v^b \frac{dx^c}{d\lambda} = 0.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad \mathcal{R} = g^{bd} R_{bd}.$$

Einstein tensor:

$$G^{ab} = R^{ab} - \frac{1}{2} \mathcal{R} g^{ab}.$$

Schwarzschild metric:

$$ds^2 = c^2 d\tau^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \; d\phi^2),$$

where R is the Schwarzschild radius. In the equatorial plane ($\theta = \pi/2$)

$$ds^2 = c^2 d\tau^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 d\phi^2.$$

1. In this question K is an inertial frame with coordinates $x^\mu, \quad \mu = 0, 1, 2, 3.$

(a) State briefly what is meant by four-velocity and four-acceleration. (4 marks)

(b) A spacecraft is at rest in K. The spacecraft then accelerates in the x^1 direction until it reaches 80% of the speed of light. Give the components of the four-velocity before and after the acceleration phase. (4 marks)

(c) A photon rocket is a hypothetical propulsion system that provides thrust by converting mass into photons which are directed into an exhaust beam. The four-acceleration of a photon rocket moving in the x^1 direction of K has components

$$a^0 = -\frac{u^1}{m}\frac{dm}{d\tau}, \quad a^1 = -\frac{u^0}{m}\frac{dm}{d\tau}, \quad a^2 = a^3 = 0.$$

Here τ and m is the proper time and mass of the rocket, respectively.

Determine u^0 and u^1 as a function of the mass m assuming that $u^2 = u^3 = 0$.

Hint: use $u \cdot u = \eta_{\mu\nu} u^\mu u^\nu = c^2$ and the integrals

$$\int \frac{dp}{\sqrt{p^2 - b^2}} = \cosh^{-1}\frac{p}{b} + \text{constant}, \quad \int \frac{dp}{\sqrt{b^2 + p^2}} = \sinh^{-1}\frac{p}{b} + \text{constant}, \quad (b > 0).$$

(8 marks)

(d) Suppose the spacecraft in part (b) is a photon rocket. Determine the ratio of its final and initial mass. (4 marks)

(Total: 20 marks)

2. Cylindrical polar coordinates (ρ, θ, z) are related to cartesian coordinates (x, y, z) through

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta,$$

with z unchanged. The equation $\rho = \cosh z$ defines a *catenoid* which is a surface of revolution. The non-zero Christoffel symbols are

$$\Gamma^{\theta}_{z\theta} = \Gamma^{\theta}_{\theta z} = \tanh z, \quad \Gamma^{z}_{zz} = \tanh z, \quad \Gamma^{z}_{\theta\theta} = -\tanh z.$$

(a) (i) Starting from the standard metric

$$ds^2 = dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2,$$

show that the metric on the catenoid is

$$ds^2 = \cosh^2 z \left(d\theta^2 + dz^2\right).$$

(ii) Verify that $\Gamma^{z}_{\theta\theta} = -\tanh z$.

(7 marks)

(b) (i) Write down a system of ODEs describing the parallel transport of a contravariant vector around the circle $z =$constant. Hint: use θ as the parameter.

(ii) Solve the system of equations. Does $v^a(\theta + 2\pi) = v^a(\theta)$ hold for any z?

(7 marks)

(c) (i) Show that $h = \cosh^2 z \; d\theta/ds$ is constant along geodesics.

(ii) For what values of h do the geodesics cross the circle defined by $z = 0$?

(6 marks)

(Total: 20 marks)

3. (a) The covariant derivative of a tensor of type $(1,1)$ is given by

$$\nabla_c T^a_b = \partial_c T^a_b + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d.$$

(i) Show that $\nabla_c \delta^a_b = 0$.

(ii) Show that the inverse metric satisfies

$$\nabla_c g^{ab} = 0,$$

assuming that $\nabla_c g_{ab} = 0$. Hint: use the Leibniz property of covariant differentiation.

(5 marks)

(b) Show that in two dimensions only one component of the Riemann tensor is independent. Hint: Which of the 16 components of R_{abcd} are not automatically zero?.

(5 marks)

(c) Consider a N dimensional space where $N \neq 2$. If the Ricci tensor is zero the Einstein tensor is zero. Is the converse true? In other words, is $R^{ab} = 0$ if $G^{ab} = 0$? Justify your answer.

(5 marks)

(d) Show that

$$[\nabla_c, \nabla_d]\phi = 0,$$

where ϕ is a scalar field and the connection is torsion-free. Does the equation still hold in the presence of torsion?

(5 marks)

(Total: 20 marks)

4. *In this question the Schwarzschild radius is unity ($R = 1$).*

(a) (i) Show that in the Schwarzschild space-time

$$k = \left(1 - \frac{1}{r}\right)\frac{dt}{ds},$$

is constant along geodesics.

(ii) A space-probe is in free-fall directly towards the centre of a black hole. Use

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1,$$

to show that if

$$ck = 1,$$

the probe's trajectory is described by the equation

$$ct = f(r), \tag{1}$$

where

$$\frac{df(r)}{dr} = -r^{1/2}\left(1 - \frac{1}{r}\right)^{-1}. \tag{2}$$

(9 marks)

(b) Mission Control, which is located far away from the black hole, sends instructions to the probe using radio waves. The trajectory of photons in such a signal has the form

$$ct = g(r), \tag{3}$$

where

$$g(r) = -r - \log(r - 1) + C.$$

Here the constant C is different for each photon in the signal.

(i) Suppose a photon described by equation (3) is detected at some time, t_*, by the probe at $r = r_*$, what is the constant C? Hint: your answer may include $f(r_*)$; an explicit integration of (2) is not required. Note that C does not depend on t_*.

(ii) Determine the derivative

$$\frac{dC(r_*)}{dr_*},$$

where C is the function of r_* considered in part (i). Simplify your answer if possible.

(6 marks)

(c) The probe receives a message when $r = r_*$. The duration of the message as measured by the probe is $\delta\tau$. What was the duration of the signal sent by Mission Control? What happens if $r_* = 1$? Hint: consider dC/ds for $C = C(r_*(s))$. (5 marks)

(Total: 20 marks)

5. (a) Null geodesics in the equatorial plane of the Schwarzschild space-time satisfy

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - Ru^3 = \text{constant}, \tag{1}$$

and

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}Ru^2 = 0, \tag{2}$$

where $u = 1/r$. Briefly outline the derivation of these results.

(10 marks)

(b) Show that if a photon crosses the photon sphere (at $r = \frac{3}{2}R$) the trajectory starts or ends at $r = 0$ (as in Figure 1 below).

Hint: Use (2) to show that any minimum of r is greater than $\frac{3}{2}R$ (as in Figure 2 below). A minimum of r is equivalent to a maximum of u.

(5 marks)

(c) For what values of the impact parameter, a, does the photon cross the photon sphere?

(5 marks)

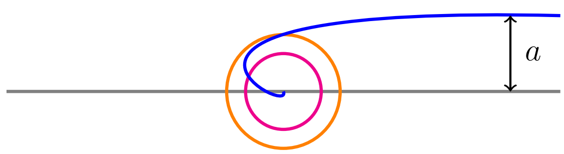

Figure 1: If a light ray crosses into the photon sphere (outer circle) it also crosses the event horizon (inner circle) and terminates at $r = 0$.

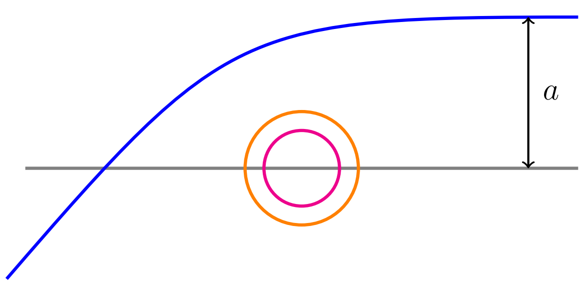

Figure 2: With a larger impact parameter, a, the light ray is deflected by the black hole. The curve does not cross the photon sphere.

(Total: 20 marks)

Tensor Calculus and General Relativity **Summer Term 2023**

Answers to 2022-2023 examination

1. (a) The four-velocity is the contravariant four vector, u, with components

$$u^\mu = \frac{dx^\mu}{d\tau},$$

where $x^\mu(\tau)$ gives the trajectory or worldline of a particle.

The four-acceleration is the contravariant four-vector, a, with components

$$a^\mu = \frac{du^\mu}{d\tau}.$$

(4 marks, bookwork, A)

(b) Using $u = \gamma(c, \mathbf{v})$ where $\gamma = (1 - v^2/c^2)^{-1/2}$:

before: $\gamma = 1$, $u = (c, 0, 0, 0)$

after: $v = \frac{4}{5}c$, $\gamma = (1 - \frac{16}{25})^{-1/2} = \frac{5}{3}$, $u = \frac{5}{3}(c, \frac{4}{5}c, 0, 0) = (\frac{5}{3}c, \frac{4}{3}, 0, 0)$.

(4 marks, seen similar, A)

(c) The given acceleration leads to the system of coupled ODEs

$$\frac{du^0}{d\tau} = -\frac{u^1}{m}\frac{dm}{d\tau}, \quad \frac{du^1}{d\tau} = -\frac{u^0}{m}\frac{dm}{d\tau}.$$

Using

$$u \cdot u = (u^0)^2 - (u^1)^2 = c^2,$$

$u^0 = \sqrt{c^2 - (u^1)^2}$ and $u^1 = \sqrt{(u^0)^2 - c^2}$ gives the decoupled system

$$du^0 = -\frac{\sqrt{(u^0)^2 - c^2}}{m}dm, \quad du^1 = -\frac{\sqrt{c^2 - (u^1)^2}}{m}dm.$$

Separating the variables and integrating using the given integrals gives

$$\cosh^{-1}\frac{u^0}{c} = -\log m + a = -\log Am, \quad \sinh^{-1}\frac{u^0}{c} = -\log m + b - \log Bm,$$

where A and B are integration constants. Accordingly,

$$u^0 = c\cosh(-\log(Am)) = \frac{c}{2}\left(Am + \frac{1}{Am}\right).$$

Similarly

$$u^0 = c \sinh(-\log(Bm)) = \frac{c}{2}\left(\frac{1}{Bm} - Bm\right).$$

The condition $u \cdot u = 1$ fixes $B = A$.

(8 marks, unseen, C)

(d) Before the acceleration fixes $Am_i = 1$ where m_i is the initial mass so that

$$u^0 = \frac{c}{2}\left(\frac{m}{m_i} + \frac{m_i}{m}\right).$$

Matching to the final value of u^0 which is $\frac{5}{3}c$ yields

$$\frac{5}{3} = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

where $x = m_f/m_i$. This has solutions $x = 3$ and $x = \frac{1}{3}$. Matching u^1 eliminates $x = 3$ so the ratio of the final and initial mass is $\frac{1}{3}$.

(4 marks, unseen, D)

(Total: 20 marks)

2. (a) (i) $\rho = \cosh z$, $d\rho = \sinh z dz$ so that

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 = \sinh^2 z dz^2 + \cosh^2 z d\theta^2 + dz^2 = \cosh^2 z(d\theta^2 + dz^2).$$

(ii) Here $g_{zz} = g_{\theta\theta} = \cosh^2 z$, $g^{\theta\theta} = g^{zz} = 1/\cosh^2 z$ with the off-diagonal elements zero.

$$\begin{aligned}\Gamma^z_{bc} &= \frac{1}{2} g^{zd} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \\ &= \frac{1}{2} g^{zz} \left(\partial_b g_{zc} + \partial_c g_{bz} - \partial_z g_{bc}\right).\end{aligned}$$

Accordingly,

$$\Gamma^z_{\theta\theta} = \frac{1}{2} g^{zz}(0 + 0 - \partial_z g_{\theta\theta}) = -\tanh z.$$

(7 marks, seen, A)

(b) (i) Equation of parallel transport

$$\frac{dv^a}{d\lambda} + \Gamma^a_{bc} v^b \frac{dx^c}{d\lambda} = 0.$$

For the circle z =constant use $\lambda = \theta$ as a parameter: The equation of parallel transport is

$$\begin{aligned}\frac{dv^\theta}{d\theta} + \Gamma^\theta_{b\theta} v^b \cdot 1 &= \frac{dv^\theta}{d\theta} + \Gamma^\theta_{z\theta} v^z = \frac{dv^\theta}{d\theta} + \tanh z v^z = 0, \\ \frac{dv^z}{d\theta} + \Gamma^z_{b\theta} v^b \cdot 1 &= \frac{dv^z}{d\theta} + \Gamma^z_{\theta\theta} v^\theta = \frac{dv^z}{d\theta} - \tanh z v^\theta = 0.\end{aligned}$$

(ii) Differentiating the first equation with respect to θ and using the second equation

$$\frac{d^2 v^\theta}{d\theta^2} + \tanh^2 z v^\theta = 0,$$

with general solution

$$v^\theta = A\cos(\theta \tanh z + \beta),$$

and

$$v^z = -\frac{1}{\tanh z} \frac{dv^\theta}{d\theta} = A \sin(\theta \tanh z + \beta).$$

$v^a(\theta + 2\pi) = v^a(\theta)$ only holds for $z = 0$.

(7 marks, seen similar, B)

(c) (i) As the metric does not depend on θ, $g_{\theta b}dx^b/ds = g_{\theta\theta}d\theta/ds = \cosh^2 z d\theta/ds$ is constant along geodesics.

(ii)

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = \cosh^2 z\left[\left(\frac{d\theta}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = \frac{h^2}{\cosh^2 z} + \cosh^2 z\left(\frac{dz}{ds}\right)^2 = 1,$$

At $z = 0$

$$\left(\frac{dz}{ds}\right)^2 = 1 - h^2,$$

For a geodesic to cross $z = 0$ requires $h^2 < 1$.

(6 marks, seen similar, C)

(Total: 20 marks)

3. (a) (i)

$$\nabla_c T^a_b = \partial_c T^a_b + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d.$$

$$\nabla_c \delta^a_b = \partial_c \delta^a_b + \Gamma^a_{dc}\delta^d_b - \Gamma^d_{bc}\delta^a_d = 0 + \Gamma^a_{bc} - \Gamma^a_{bc} = 0.$$

(ii)

$$\nabla_c \delta^a_b = \nabla_c g_{ad} g^{db} = (\nabla_c g_{ad}) g^{db} + g_{ad}\nabla_c g^{db} = 0 + g_{ad}\nabla_c g^{db} = 0.$$

(5 marks, seen similar, A)

(b) Due to the asymmetry in the first two and last two indices 12 components are automatically zero. The remaining four are

$$R_{1212}, \quad R_{1221}, \quad R_{2112}, \quad R_{2121}.$$

The last three are not independent as $R_{1221} = -R_{1212}$, $R_{2112} = -R_{1212}$, $R_{2121} = R_{1212}$.

(5 marks, seen similar, A)

(c) Yes, $R^{ab} = 0$ If $G^{ab} = 0$. Suppose $G^{ab} = R^{ab} - \frac{1}{2}\mathcal{R} g^{ab} = 0$. Contracting with g_{ab} gives $\mathcal{R} - \frac{1}{2}\mathcal{R}N = (1 - \frac{1}{2}N)\mathcal{R} = 0$ or $\mathcal{R} = 0$ provided $N \neq 2$. Hence $R^{ab} = G^{ab} = 0$.

(5 marks, unseen, C)

(d)

$$[\nabla_c, \nabla_d]\phi = \nabla_c \partial_d \phi - \nabla_d \partial_c \phi = (\partial_c \partial_d \phi - \Gamma^a_{dc}\partial_a \phi) - (\partial_d \partial_c \phi - \Gamma^a_{cd}\partial_a \phi)$$

$$= \Gamma^a_{cd}\partial_a \phi - \Gamma^a_{dc}\partial_a \phi = T^a_{cd}\partial_a \phi,$$

which vanishes if $T^a_{cd} = 0$. In the presence of torsion $[\nabla_c, \nabla_d]\phi$ is non-zero.

(5 marks, unseen, C)

(Total: 20 marks)

4. (a) (i) As the metric is independent of t, $g_{t\nu}dx^\nu/ds = c^2(1-R/r)dt/ds$ is constant along geodesics.

(i) Here

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = c^2\left(1-\frac{1}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{1}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2$$
$$- r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right] = 1.$$

As the probe is directly infalling θ and ϕ are constant so that the third and fourth terms above are zero. Using the result from part (i) we have

$$c^2k^2\left(1-\frac{1}{r}\right)^{-1} - \left(1-\frac{1}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 = 1.$$

Multiplying by $1-1/r$ gives

$$ck^2 - \left(\frac{dr}{ds}\right)^2 = 1-\frac{1}{r},$$

which simplifies to

$$\frac{dr}{ds} = -\frac{1}{r^{1/2}},$$

as $ck=1$ and the probe is infalling. Now $dt/dr = (dt/ds)/(dr/ds) = -r^{1/2}dt/ds = -kr^{1/2}/(1-1/r) = c^{-1}r^{1/2}/(1-1/r)$.

(9 marks, seen similar, A)

(b) (i) If the photon described by (3) is detected by the probe described by (1) at some time t_*, then $f(r_*) = g(r_*)$ so that

$$C = f(r_*) + r_* + \log(r_*-1).$$

(ii)

$$\frac{dC}{dr_*} = f'(r_*)+1+\frac{1}{r_*-1} = -r_*^{1/2}(1-1/r_*)^{-1}+\frac{r_*}{r_*-1} = \frac{-r_*^{3/2}+1}{r_*-1} = -\frac{r_*}{r_*^{1/2}+1}.$$

(6 marks, unseen, B)

(c)

$$\frac{dC}{ds} = \frac{dC}{dr_*}\frac{dr_*}{ds} = -r_*^{-1/2}\frac{dC}{dr_*} = \frac{r_*^{1/2}}{r_*^{1/2}+1}.$$

Interpreting $\delta C = c\delta t$ where δt is the duration of the signal as measured by the sender, and $\delta s = c\delta\tau$ where $\delta\tau$ is the duration measured by the probe

$$\delta t = \frac{{r_*}^{1/2}}{{r_*}^{1/2} + 1}\delta\tau.$$

As $r_* \to 1$, $\delta t = \frac{1}{2}\delta\tau$.

(5 marks, unseen, D)

(Total: 20 marks)

5. (a) *This is the argument given in the lectures. Other approaches are possible (e.g. as a limit of the massive orbit equations)*

Null geodesics satisfy the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0,$$

with

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0.$$

Here λ is a parameter. In the equatorial plane

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

As the metric is independent of t and ϕ

$$\tilde{k} = \left(1-\frac{R}{r}\right)\frac{dt}{d\lambda}, \qquad \tilde{h} = r^2\frac{d\phi}{d\lambda},$$

are constant along geodesics. Accordingly,

$$c^2\tilde{k}^2\left(1-\frac{R}{r}\right)^{-1} - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{\tilde{h}^2}{r^2} = 0.$$

Multiplying by $(1-R/r)$ and using

$$\frac{dr}{d\lambda} = -\frac{1}{u^2}\frac{du}{d\lambda} = -\frac{1}{u^2}\frac{d\phi}{d\lambda}\frac{du}{d\phi} = -\tilde{h}\frac{du}{d\phi},$$

yields

$$\tilde{h}^2\left[\left(\frac{du}{d\phi}\right)^2 + u^2 - Ru^3\right] = c^2\tilde{k}^2,$$

which is equivalent to (1). (2) Follows by differentiating (1) with respect to ϕ.

(10 marks, bookwork)

(b) If r has a minimum, $u = 1/r$ has a maximum. At any local maximum

$$\frac{d^2u}{d\phi^2} \leq 0.$$

Using (2) this requires that at a maximum

$$\frac{3R}{2}u^3 - u^2 \leq 0,$$

or $\frac{3}{2}Ru \leq 1$ or $\frac{3}{2}R \leq r$.

(5 marks, unseen)

(c) Far away from the black hole (where u is small) the trajectory is the line $u = \sin\phi/a$ which gives the boundary conditions $u(0) = 0$, $u'(0) = 1/a$ so the constant in (1) is $1/a^2$.

If the photon crosses the photon sphere $(du/d\phi)^2 > 0$ for $u = \frac{2}{3}R^{-1}$. Accordingly,

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{a^2} - u^2 + Ru^3 = \frac{1}{a^2} + \frac{1}{R^2}\left(-\frac{4}{9} + \frac{8}{27}\right) = \frac{1}{a^2} - \frac{4}{27R^2} > 0,$$

giving $a^2 < \frac{27}{4}R^2$ or $a < \frac{3}{2}\sqrt{3}R$.

(5 marks, unseen)

(Total: 20 marks)

Category A marks: 1 (a)(b) 8 marks, 2 (a) 7 marks , 3(a)(b) 10 marks, 4 (a) 9 marks

total 34/80

Category B marks: 2 (b) 7 marks, 3 (c) 5 marks , 4 (b) 6 marks

total 18/80

Category C marks 1 (c) 8 marks, 2 (c) 6 marks, 3 (d) 5 marks

total 19/80

Category D marks 1 (d) 4 marks, 4 (c) 5 marks

total 9/80

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60017/70017	1	This question was conveniently answered by most students; although, a lot of students lost marks on part (c). The definitions of four-velocity and four-acceleration are known. Subquestion (b) was often answered correctly, although some students either do not simplify their answers or entirely forgot to calculate the value of the boost. Subquestion (c) caused more issues; too many students did not see how to use the extensive hints. Quite a few students lead on long algebraic manipulations to put the equations in a form that is solvable (for instance squaring them or multiplying them etc.) which eventually lead to sign mistakes. Finally, success at part (d) was mostly contingent on successfully obtaining answers at (b) and (c) and so fewer students found the correct answer (for instance sign mistakes in (c) carry over to (d)). It is to be noted that very few students simplify their answers to (d) writing it as an exponential of hyperbolic function.
MATH60017/70017	2	This question was overall well done by students. Part (a)(i) posed no issue. Part (a)(ii) was often not justified enough, in particular, very little discussion of which metric components or Christoffel symbols are non-zero to justify the simplifications. When following the hint, part (b)(i) was answered correctly with the adequate level of justification. Solving the system of ODE posed surprisingly some problems to students who had forgotten the solution to the harmonic oscillator. The discussion of the periodicity of the velocity components was rarely done correctly with students often stating at most that z needs to be an integer (while technically zero is the only possible integer value z can take). Marks were awarded for "z needs to be an integer"-type answers. Part (c)(i) was done correctly and as the results was often quoted directly from the notes. Virtually no student answered correctly Part (c)(ii); students often equated z=0 to dz/ds=0 which is not correct (indeed, the geodesic was asked to cross the circle).

MATH60017/70017	3	This question had mostly independent subquestions. As such, while it was found much harder than the first two questions, students still could collect some marks. Overall, index notation is not mastered by many students. While Part (a)(i) was often answered correctly, it was too often not used in (a)(ii), while the hint clearly pointing at it. If not using the result of Part (a)(i), students often got lost in the index notation and could not satisfactorily answer (a)(ii). Most students understood (b) but the level of answers varied quite a lot with some answers not providing enough justification. Part (c) was not answered correctly in most cases; too few students thought about contracting by the metric to show the result (which comes about in three lines if using this trick) and instead fudged indices, or tried to raise and lower indices etc. A common mistake was to write the definition of the curvature using indices overlapping with the indices used in writing G^{ab}, which then led to wrong raising or lowering of indices and simplifications. Finally, only few students made mistakes in writing the definition of the commutator in (d). When this question was not answered correctly it is because students either did not write covariant derivatives at all (forgetting that the gradient of the scalar field is not a scalar field anymore) or wrote two covariant derivatives.
MATH60017/70017	4	Parts (a) and (b) were mostly well answered. In part (a) some candidates did not set phi to be constant straight away (direct infall). Although (b) is unseen sufficient hints were provided.As expected (c) was more challenging even though the heavy lifting is done in parts (a) and (b). Many students argued that the probe cannot receive signals as it passes through the event horizon. While it certainly cannot send signals back to the outside universe it can receive signals up until it hits the singularity at r=0.
MATH70017	5	Part (a) is a bookwork question worth 10 marks. Parts (b) and (c) are short unseen questions. All parts relate to light rays in the Schwarzschild space-time.Overall the answering of part (a) was weaker than expected. Many candidates developed the orbit equation for massive rather than massless particles (this is actually a viable approach if an appropriate limit is taken at the end).The answering of part (b) was as expected. Although the question is unseen students were able to get the required result.Few students made much progress with part (c). This part would have been improved by a hint.