

MATH50001 Problems Sheet 3 Solutions

1). Because $1/(z^2 - 4)$ is holomorphic at all points

$$\{z : |z| \leq 4\} \setminus \left\{ \{z : |z+2| \leq 1\} \cup \{z : |z-2| \leq 1\} \right\}.$$

we obtain

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = \oint_{\gamma_1} \frac{1}{(z+2)(z-2)} dz + \oint_{\gamma_2} \frac{1}{(z+2)(z-2)} dz,$$

where $\gamma_1 : |z+2| = 1$ and $\gamma_2 : |z-2| = 1$.

Note that

$$\frac{1}{(z+2)(z-2)} = -\frac{1}{4} \frac{1}{z+2} + \frac{1}{4} \frac{1}{z-2}.$$

It follows that

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = \frac{1}{4} \oint_{\gamma_1} \left(\frac{1}{z-2} - \frac{1}{z+2} \right) dz + \frac{1}{4} \oint_{\gamma_2} \left(\frac{1}{z-2} - \frac{1}{z+2} \right) dz.$$

Because $1/(z-2)$ is holomorphic inside and on γ_1 its integral around γ_1 is zero (by the Cauchy-Goursat theorem). Similarly, the contour integral of $1/(z+2)$ around γ_2 vanishes. Besides,

$$\oint_{\gamma_1} \frac{1}{z+2} dz = 2\pi i \quad \text{and} \quad \oint_{\gamma_2} \frac{1}{z-2} dz = 2\pi i.$$

Consequently

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = -\frac{1}{4} 2\pi i + \frac{1}{4} 2\pi i = 0.$$

2).

$$\oint_{\gamma} \frac{1}{z^3 - 1} dz = \oint_{\gamma} \frac{1}{(z-1)(z-e^{i2\pi/3})(z-e^{i4\pi/3})} dz.$$

Note that

$$\begin{aligned} \frac{1}{(z-1)(z-e^{i2\pi/3})(z-e^{i4\pi/3})} \\ = \frac{1}{3} \frac{1}{z-1} - \frac{e^{-i\pi/3}}{3} \frac{1}{z-e^{i2\pi/3}} + \frac{e^{i\pi/3}}{3} \frac{1}{z-e^{i4\pi/3}} \end{aligned}$$

Both functions $1/(z-1)$ and $1/(z-e^{i4\pi/3})$ are holomorphic inside and on $\gamma = \{z : |z-i| = 1\}$. Therefore

$$\oint_{\gamma} \frac{1}{z-1} dz = \oint_{\gamma} \frac{1}{z-e^{i4\pi/3}} dz = 0.$$

Since $e^{i2\pi/3}$ is inside the domain bounder by γ we have

$$\oint_{\gamma} \frac{1}{z - e^{i2\pi/3}} dz = 2\pi i.$$

Thus

$$\oint_{\gamma} \frac{1}{z^3 - 1} dz = -\frac{e^{-i\pi/3}}{3} 2\pi i = -\frac{1 - i\sqrt{3}}{6} 2\pi i = -\frac{\pi(\sqrt{3} + i)}{3}.$$

3). a) No, b) Yes, c) No.

4).

$$\oint_{\gamma} \frac{e^z \sin z}{z - 5} dz = 2\pi i e^z \sin z \Big|_{z=5} = 2\pi i e^5 \sin 5.$$

5). Let $\operatorname{Re} z > 0$, choose any $\beta > |z|$. We consider the interval $\gamma_1 = [i\beta, -i\beta]$, half circle

$$\gamma_2 = \{z : z = \beta e^{i\theta}, -\pi/2 < \theta < \pi/2\}$$

and denote $\gamma = \gamma_1 \cup \gamma_2$. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta - z} d\eta = \frac{1}{2\pi i} \int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z} d\eta + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\eta)}{\eta - z} d\eta.$$

Let $r = \min_{\eta \in \gamma_2} |\eta - z|$. Using our assumptions we find that if $\eta \in \gamma_2$

$$\left| \frac{f(\eta)}{\eta - z} \right| = \left| \frac{f(\eta)\eta^k}{\eta^k(\eta - z)} \right| \leq \frac{M}{|\eta|^k} \frac{1}{|\eta - z|} \leq \frac{M}{\beta^k r}.$$

According to the ML-inequality

$$\left| \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\eta)}{\eta - z} d\eta \right| \leq \frac{M}{2\pi \beta^k r} \cdot \pi \beta = \frac{M}{2\beta^{k-1} r} \rightarrow 0,$$

as $\beta \rightarrow \infty$. Thus

$$f(z) = \frac{1}{2\pi i} \lim_{\beta \rightarrow \infty} \int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z} d\eta = -\frac{1}{2\pi i} \lim_{\beta \rightarrow \infty} \int_{-i\beta}^{i\beta} \frac{f(\eta)}{\eta - z} d\eta.$$

6). We use Cauchy's theorem

$$\begin{aligned}
f(z_0) &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - z_0} dz \\
&= \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - z_0} dz - \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - \bar{z}_0^{-1}} dz \\
&= \frac{1}{2i\pi} \oint_{|z|=1} \frac{-\bar{z}_0^{-1} + z_0}{(z - z_0)(z - \bar{z}_0^{-1})} f(z) dz \\
&= \frac{1}{2i\pi} \oint_{|z|=1} \frac{-\bar{z}_0^{-1}(1 - |z_0|^2)}{(z - z_0)(z - \bar{z}_0^{-1})} f(z) dz \\
&= \frac{1}{2i\pi} \oint_{|z|=1} \frac{(1 - |z_0|^2)}{(z - z_0)(1 - z\bar{z}_0)} f(z) dz.
\end{aligned}$$

7)

Let

$$\begin{aligned}
\gamma_1 &= \{z : z = 2e^{i\varphi}, \varphi \in [0, \pi]\}, & \gamma_2 &= \{z : z = x, -2 < x < -1\}, \\
\gamma_3 &= \{z : z = e^{i\varphi}, \varphi \in [\pi, 0]\}, & \gamma_4 &= \{z : z = x, 1 < x < 2\}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\gamma_1} \frac{z}{\bar{z}} dz &= \int_0^\pi \frac{2e^{i\varphi}}{2e^{-i\varphi}} 2i e^{i\varphi} d\varphi = 2i \int_0^\pi e^{3i\varphi} d\varphi = \frac{2}{3} (e^{3i\pi} - 1) = -\frac{4}{3}, \\
\int_{\gamma_2} \frac{z}{\bar{z}} dz &= \int_{-2}^{-1} dx = x \Big|_{-2}^{-1} = -1 - (-2) = 1, \\
\int_{\gamma_3} \frac{z}{\bar{z}} dz &= \int_\pi^0 \frac{e^{i\varphi}}{e^{-i\varphi}} i e^{i\varphi} d\varphi = i \int_\pi^0 e^{3i\varphi} d\varphi = \frac{1}{3} (1 - e^{3i\pi}) = \frac{2}{3}, \\
\int_{\gamma_4} \frac{z}{\bar{z}} dz &= \int_1^2 dx = x \Big|_1^2 = 1.
\end{aligned}$$

Finally we have

$$\int_{\gamma} \frac{z}{\bar{z}} dz = \sum_{i=1}^4 \int_{\gamma_i} \frac{z}{\bar{z}} dz = -\frac{4}{3} + 1 + \frac{2}{3} + 1 = \frac{4}{3}.$$

8a)

Let us introduce curves $\gamma_n = \{z : |z - n| = 1/2\}$, $n = 0, 1, \dots, k$. Then by using the Deformation Theorem we have

$$I_k = \oint_{\gamma} \frac{dz}{z(z-1)\dots(z-k)} dz = \sum_{n=0}^k \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz.$$

Note that for any $n : 0 < n < k$ the Cauchy's integral formula implies

$$\oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz = 2\pi i \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{n-k} = 2\pi i \frac{(-1)^{k-n}}{n!(k-n)!},$$

where the cases $n = 0$ and $n = k$ are treated similarly. Thus

$$\sum_{n=0}^k \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz = 2\pi i \sum_{n=0}^k \frac{(-1)^{k-n}}{n!(k-n)!}.$$

8b)

Using Cauchy's integral formula we obtain (for $k \geq 1$)

$$J_k = \oint_{\gamma} \frac{(z-1)(z-2)\dots(z-k)}{z} dz = 2\pi i (-1)^k k!.$$

9. Step 1. We first prove that if $\gamma \subset \mathbb{C}$ is a simple closed piecewise-smooth curve bounding Ω and f and f'_z are continuous inside and on γ then

$$\oint_{\gamma} f(z) dz = 2i \iint_{\Omega} \frac{\partial f(z)}{\partial \bar{z}} dx dy.$$

Indeed, this is a direct corollary of Green's theorem

$$\begin{aligned} \oint_{\gamma} f(z) dz &= \oint_{\gamma} (u + iv) d(x + iy) = \oint_{\gamma} u dx - v dy + i \oint_{\gamma} v dx + u dy \\ &= \iint_{\Omega} (-v'_x - u'_y + i u'_x - i v'_y) dx dy \\ &= 2i \iint_{\Omega} \frac{1}{2} \left(u'_x - \frac{1}{i} u'_y \right) + \frac{i}{2} \left(v'_x - \frac{1}{i} v'_y \right) dx dy \\ &= 2i \iint_{\Omega} \frac{d}{d\bar{z}} (u + iv) dx dy = 2i \iint_{\Omega} \frac{d}{d\bar{z}} f(z) dx dy. \end{aligned}$$

Step 2. Let $r > 0$ is small enough, so that $B_r(z_0) = \{z : |z - z_0| \leq r\} \subset D$. Using the same argument as in the proof of the Deformation theorem we

find

$$\begin{aligned} & \oint_{|z-z_0|=1} \frac{f(z)}{z-z_0} dz - \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz \\ &= 2i \iint_{D \setminus B_r(z_0)} \frac{d}{d\bar{z}} \left(\frac{f(z)}{z-z_0} \right) dx dy = 2i \iint_{D \setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0} dx dy, \end{aligned}$$

where we have used the fact that $1/(z-z_0)$ is holomorphic in $D \setminus B_r(z_0)$ and therefore $d(z-z_0)^{-1}/d\bar{z} = 0$. It remains to note that

$$\lim_{r \rightarrow 0} \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz = \lim_{r \rightarrow 0} \oint_{|z-z_0|=r} \frac{f(z_0) + f(z) - f(z_0)}{z-z_0} dz = 2\pi i f(z_0)$$

and that

$$\lim_{r \rightarrow 0} \iint_{D \setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0} dx dy = \iint_D \frac{df(z)/d\bar{z}}{z-z_0} dx dy.$$