Imperial College London

MATH97223 MATH97238

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Classical Dynamics

Date: Wednesday, 19 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Show that if the integral

$$S = \int_a^b L(y(x), y'(x))dx$$

is stationary for fixed $y(a)$ and $y(b)$ then

$$H = y' \frac{\partial L}{\partial y'} - L,$$

is a constant. (6 marks)

(b) The motion of a particle of unit mass on a sphere of radius R is governed by the Lagrangian

$$L = \frac{R^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \alpha\dot{\phi}\cos\theta,$$

where θ and ϕ are spherical polar coordinates and α is a non-zero constant. Obtain the equations of motion and determine the period of circular orbits of the form $\theta =$constant.

(8 marks)

(c) Obtain the Lagrangian $L(q, \dot{q})$ corresponding to the Hamiltonian

$$H(q, p) = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right).$$

(6 marks)

(Total: 20 marks)

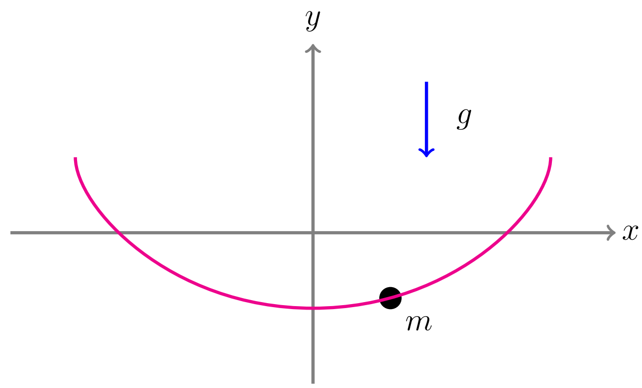
2. (a) A particle of mass m moving in two dimensions obeys the equations of motion

$$m\ddot{x} = ay + bx^2, \qquad m\ddot{y} = ax + c,$$

where a, b and c are constants. Obtain a Lagrangian for the motion of the particle.

(6 marks)

(b) A bead of mass m moves without friction on a cycloid-shaped wire as in the diagram below.



The cycloid is defined through

$$x = R(s + \sin s), \quad y = -R\cos s,$$

where s is a parameter ($-\pi \leq s \leq \pi$) and R is a positive constant. The (downwards) acceleration due to gravity has magnitude g.

(i) Use a suitable Lagrangian to obtain the equation of motion for the bead.
Hint: use the parameter s as the generalised coordinate so that the Lagrangian is a function of s and $\dot{s}$. (10 marks)

(ii) Show that for small s the equation of motion is approximately

$$4R\ddot{s} + gs = 0.$$

(4 marks)

(Total: 20 marks)

3. (a) Consider the time-dependent canonical transformation

$$Q = q\cosh t + p\sinh t, \quad P = q\sinh t + p\cosh t.$$

(i) Verify that $\{Q, P\} = 1$. (3 marks)

(ii) Find a type 2 generating function, $F_2(q, P)$, for the canonical transformation.

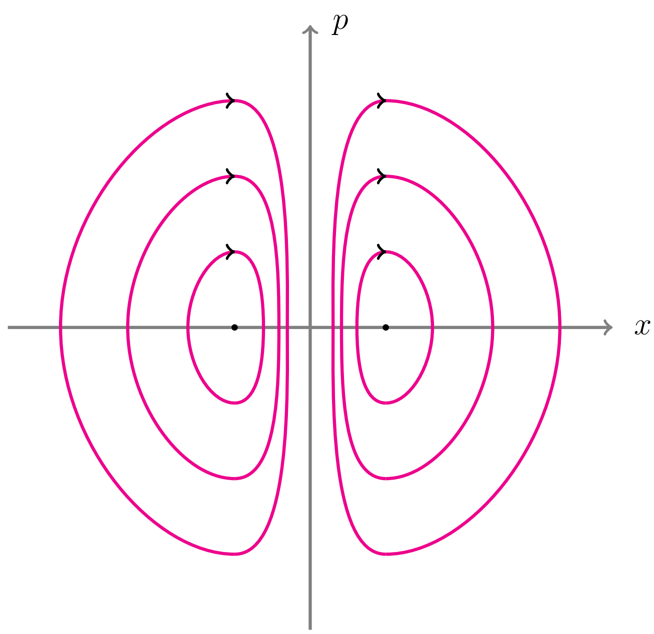
$$\text{Hint: Use} \quad p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}.$$

(7 marks)

(b) The dynamics generated by the Hamiltonian

$$H = p^2 + x^2 + \frac{1}{x^2},$$

exhibits libration as shown in the phase-space diagram below



Write H as a function of the action variable

$$J = \oint p \, dx,$$

and comment on the result. (10 marks)

Hint: evaluate J using the substitution $u = x^2$ and the integral formula

$$\int_a^b \frac{\sqrt{(u-a)(b-u)}}{u} \, du = \frac{\pi}{2}\left(a + b - 2\sqrt{ab}\right) \qquad (b > a > 0).$$

(Total: 20 marks)

4. The Lagrangian for a symmetric top fixed at one point is $L = T - V$ where

$$T = \frac{I_1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2,$$

and

$$V = Mgl\cos\theta.$$

ϕ, θ and ψ are Euler angles, M is the total mass and l is the distance between the fixed point and the centre of mass. $I_1 = I_2$ and I_3 are the principal moments of inertia.

(a) Use

$$p_\phi\dot{\phi} + p_\theta\dot{\theta} + p_\psi\dot{\psi} = 2T,$$

to show that the Hamiltonian is

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta.$$

(8 marks)

(b) Use Hamilton's equations to obtain the equation of motion

$$I_1\ddot{\theta} = -\frac{(p_\phi - p_\psi\cos\theta)(p_\psi - p_\phi\cos\theta)}{I_1\sin^3\theta} + Mgl\sin\theta.$$

(6 marks)

(c) Fix $p_\psi = p_\phi$ and find an approximate form for equation of motion quoted in part (b) assuming θ is small. How large must p_ψ be for the solutions of the approximate ODE to be periodic?
(6 marks)

(Total: 20 marks)

5. (a) Consider the Hamiltonian

$$H = \frac{1}{2}\left({p_x}^2 + \lambda {p_y}^2\right) + \frac{1}{2}\left(x^2 + \lambda y^2\right) + x^2 y - \frac{1}{3}y^3,$$

where λ is a constant. For $\lambda = 1$, H is the Hénon-Heiles Hamiltonian. For what λ is

$$\alpha = -p_x p_y + xy + xy^2 - \frac{1}{3}x^3,$$

a constant of the motion? Comment on the result. (10 marks)

(b) The Hénon-Heiles Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3.$$

Consider solutions to the Hénon-Heiles system where $x = 0$. Show that there are periodic solutions for $0 < E < \frac{1}{6}$. Are there non-periodic solutions for $0 < E < \frac{1}{6}$?
Hint: consider the potential energy on the line $x = 0$. (10 marks)

(Total: 20 marks)

Answers to May 2021 Timed Remote Assessment

1. (a) If S is stationary then $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}.$$

Therefore

$$\frac{dH}{dx} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'} - L\right) = y''\frac{\partial L}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{dL}{dx} = y''\frac{\partial L}{\partial y'} + y'\frac{\partial L}{\partial y} - \frac{dL}{dx}.$$

We require the derivative of $L(y(x), y'(x))$ with respect to x; using the chain rule

$$\frac{dL}{dx} = \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Hence H is constant.

(6 marks, seen, A)

(b) The E-L equations are

$$R^2(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) + \alpha\sin\theta\dot{\phi} = 0, \qquad \frac{d}{dt}[R^2\sin^2\theta\dot{\phi} - \alpha\cos\theta] = 0.$$

$\theta =$constant yields $\sin\theta\dot{\phi}(R^2\cos\theta\dot{\phi} - \alpha) = 0$ and $\ddot{\phi} = 0$. Accordingly,

$$\dot{\phi} = \frac{\alpha}{R^2\cos\theta}.$$

The period is

$$T = \frac{2\pi}{|\dot{\phi}|} = \frac{2\pi R^2|\cos\theta|}{|\alpha|}.$$

(8 marks, seen similar, B)

(c)

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2q^4\right).$$

$\dot{q} = \partial H/\partial p = pq^4$ so that $p = \dot{q}/q^4$.

$$L = p\dot{q} - H = \frac{\dot{q}^2}{q^4} - \frac{1}{2q^2} - \frac{\dot{q}^2}{2q^4} = \frac{\dot{q}^2}{2q^4} - \frac{1}{2q^2}.$$

(6 marks, seen similar, C)

(Total: 20 marks)

2. (a) $m\ddot{x} = -\partial V/\partial x = ay + bx^2$, $m\ddot{y} = -\partial V/\partial y = ax + c$, which integrate to $V = -axy - \frac{1}{3}by^3 - cy$ (dropping a constant of integration). Accordingly,

$$L = T - V = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + axy + \frac{1}{3}by^3 + cy.$$

(6 marks, seen similar, A)

(b) (i) $\dot{x} = \dot{s}dx/ds = R(1 + \cos s)\dot{s}$, $\dot{y} = R\dot{s}\sin s$ $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mR^2[(1 + \cos s)^2 + \sin^2 s]\dot{s}^2 = mR^2(1 + \cos s)\dot{s}^2$.

$V = mgy = -mgR\cos s$. A suitable Lagrangian is

$$L = T - V = mR^2(1 + \cos s)\dot{s}^2 + mgR\cos s.$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial s}\right) - \frac{\partial L}{\partial s} = \frac{d}{dt}\left[2mR^2(1 + \cos s)\dot{s}\right] + mgR\sin s = 0.$$

(10 marks, seen similar, B)

(ii) For small s, $\cos s \approx 1 - \frac{1}{2}s^2 + ..$, $\sin s \approx s + ...$ The equation of motion is approximately

$$4mR^2\ddot{s} + mgRs = 0,$$

hence the result.

(4 marks, seen similar, C)

(Total: 20 marks)

3. (a) (i)

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \cosh t \cdot \cosh t - \sinh t \cdot \sinh t = 1.$$

(3 marks, seen similar, A)

(ii) Express p and Q as functions of q and P. From the second equation

$$p = -q \tanh t + \frac{P}{\cosh t} = \frac{\partial F_2}{\partial q}$$

Substituting this into the first equation yields

$$Q = q\left(\cosh t - \frac{\sinh^2 t}{\cosh t}\right) + \tanh t P = \frac{q}{\cosh t} + \tanh t P = \frac{\partial F_2}{\partial P}.$$

These integrate to give

$$F_2(q, P) = -\frac{q^2 \tanh t}{2} + \frac{qP}{\cosh t} + \frac{P^2 \tanh t}{2}.$$

(7 marks, seen similar, A)

(b) Taking x positive

$$J = \oint p \; dx = 2\int_{x_1}^{x_2} \sqrt{\alpha - x^2 - \frac{1}{x^2}} \; dx,$$

where $\alpha = H$ and x_1 and x_2 are the roots of $\alpha - x^2 - 1/x^2$ $(x_2 > x_1)$. The factor of 2 takes into account the lower part of the trajectory in the phase space diagram. Using the substitution $u = x^2$

$$J = 2\int_{u_1}^{u_2} \sqrt{\alpha - u - \frac{1}{u}} \; \frac{du}{2\sqrt{u}} = \int_{u_1}^{u_2} \frac{\sqrt{\alpha u - u^2 - 1}}{u} \; du.$$

This can be evaluated via the given integral formula with $\alpha u - u^2 - 1 = (b-u)(u-a) = (a+b)u - u^2 - ab$:

$$J = \frac{\pi}{2}\left(a + b - 2\sqrt{ab}\right) = \frac{\pi}{2}\left(\alpha - 2\sqrt{1}\right),$$

so that $\alpha = 2 + 2J/\pi$

The result, $H = 2 + 2J/\pi$, implies that the frequency of libration is independent of the energy $(\nu = \partial H/\partial J = 2/\pi)$.

(10 marks, unseen, D)

(Total: 20 marks)

4. (a) The momenta conjugate to the Euler angles are

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = I_1\dot\theta, \qquad p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta),$$

and

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1\sin^2\theta\dot\phi + I_3\cos\theta(\dot\psi + \dot\phi\cos\theta),$$

The formula for p_ϕ can be written in the form $I_1\sin^2\theta\dot\phi = p_\phi - \cos\theta p_\psi$, and so

$$T = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - \cos\theta p_\psi)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3}.$$

The Hamiltonian is

$$\begin{aligned} H &= p_\phi\dot\phi + p_\theta\dot\theta + p_\psi\dot\psi - L = 2T - (T - V) = T + V \\ &= \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta. \end{aligned}$$

(8 marks, seen similar , A)

(b) $\dot\theta = \partial H/\partial\dot\theta = p_\theta/I_1$ giving

$$I_1\ddot\theta = \dot p_\theta = -\frac{\partial H}{\partial\theta}$$

$$\begin{aligned} &= -\frac{(p_\phi - p_\psi\cos\theta)p_\psi}{I_1\sin\theta} + \frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^3\theta}\cos\theta + Mgl\sin\theta. \\ &= -\frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^3\theta}\left[p_\psi\sin^2\theta - (p_\phi - p_\psi\cos\theta)\cos\theta\right] + Mgl\sin\theta \\ &\quad -\frac{(p_\phi - p_\psi\cos\theta)(p_\psi - p_\phi\cos\theta)}{I_1\sin^3\theta} + Mgl\sin\theta. \end{aligned}$$

(6 marks, seen similar, C)

(c) Fixing $p_\psi = p_\phi$ the equation reduces to

$$I_1\ddot\theta = -\frac{p_\psi^2(1-\cos\theta)^2}{I_1\sin^3\theta} + Mgl\sin\theta.$$

For small θ, $1 - \cos\theta \approx \frac{1}{2}\theta^2$ and $\sin\theta \approx \theta$ so that

$$\ddot\theta = -\omega^2\theta, \qquad \omega^2 = \frac{p_\psi^2}{4I_1^2} - \frac{Mgl}{I_1}.$$

The solutions are periodic if $\omega^2 > 0$ (simple harmonic motion), that is if

$$p_\psi^2 > 4I_1 Mgl.$$

(6 marks, unseen, D)

(Total: 20 marks)

5. (a)

$$\begin{aligned}
\dot{\alpha} = \frac{1}{2}\{\alpha, H\} &= \{xy + xy^2 - \tfrac{1}{3}x^3, p_x^2 + \lambda p_y^2\} \\
&+\{-p_x p_y, \tfrac{1}{2}(x^2 + \lambda y^2) + x^2 - \tfrac{1}{3}\} \\
&= p_x(y + y^2 - x^2) + \lambda p_y(x + 2xy) \\
&+p_y(x + 2xy) + p_x(\lambda y + x^2 - y^2) \\
&= p_x y + \lambda p_y x + p_y x + \lambda p_x y.
\end{aligned}$$

This is zero if $\lambda = -1$.

For $\lambda = -1$ the system is Liouville integrable. It cannot be integrable at $\lambda = 1$ as the Hénon-Heiles system is not integrable. (10 marks, unseen)

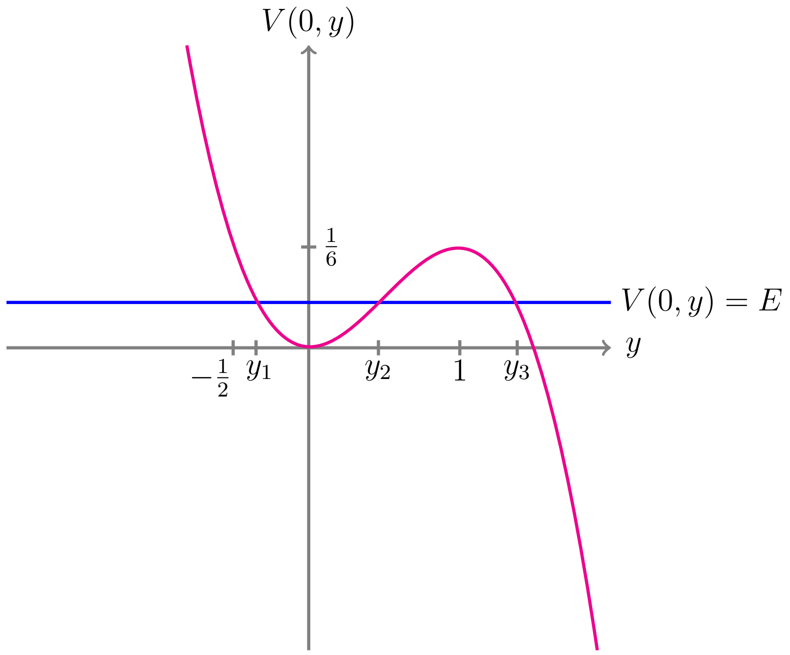
(b) Hamilton's equations give $\dot{p}_x = \dot{x}$, $\dot{p}_y = \dot{y}$ and

$$\ddot{x} = \dot{p}_x = -\frac{\partial H}{\partial x} = -x - 2xy, \quad \ddot{y} = \dot{p}_y = -\frac{\partial H}{\partial y} = -y - x^2 + y^2.$$

$x = 0$ solves the first equation and reduces the second equation to

$$\ddot{y} = -y + y^2.$$

The potential energy on the line $x = 0$ is $V(x = 0, y) = \frac{1}{2}y^2 - \frac{1}{3}y^3$. This has a local maximum at $y = 1$. As the potential energy is $1/6$ at this point there are periodic solutions where $0 < E < \frac{1}{6}$. These solutions oscillate between $y = y_1$ and $y = y_2$ where $-\frac{1}{2} < y_1 < y_2 < 1$. Here y_1 and y_2 are solutions of $V(0, y) = E$. If $0 < E < \frac{1}{6}$ there is a third root $y_3 > 1$ (as shown in the diagram). For $y > y_3$ the motion is not periodic; $y \to \infty$ as $t \to \infty$.



(10 marks, unseen)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96065 MATH97223 MATH97238	1	This question comprising 3 independent subparts was well done.
MATH96065 MATH97223 MATH97238	2	This question was well answered. The last part (b) (ii) was rather easy. The marker intended that it would provide a check on part (b) (i). However, most students were able to answer (b) (i) without this hidden hint.
MATH96065 MATH97223 MATH97238	3	This question was very well answered. Most students did not note that in part (b) all orbits have the same frequency.
MATH96065 MATH97223 MATH97238	4	Question 4 was well answered. Students were not told that part (c) relates to a sleeping top. This was to encourage students to solve the problem rather than look up the result.
MATH96065 MATH97223 MATH97238	5	This question proved challenging. Students did not realise that the answer to part (a) could not be 1 (contradicting the non-integraility of Henon-Heiles). Part (b) was also problematic - all was required was to discuss periodic and non-periodic orbits in the simple potential V(y)=y^2/2-y^3/3. Low marks on this question were compensated by generally high marks in the first four questions.