IMPERIAL

MATH50003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Linear Algebra and Numerical Analysis

Date: Tuesday, May 7, 2024

Time: 10:00 – 13:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Throughout the paper, you may use any results from the course that you require provided you state them clearly. You may also use previous parts of a question to solve later parts.

1. (a) State the *Cayley-Hamilton theorem.* (2 marks)

(b) Let V be a finite-dimensional vector space, and $T : V \mapsto V$ a linear map.

(i) Define what is meant by a T-*invariant subspace* of V.

(ii) Define the *minimal polynomial* of T, and prove that it is unique.

(iii) Let W be a nonzero T-invariant subspace of V, and let T_W be the restriction of T to W. Prove that the minimal polynomial of T_W divides the minimal polynomial of T.

(6 marks)

(c) Let F be a field, and let $V = F^4$, a 4-dimensional vector space over F. Let $T : V \mapsto V$ be the linear map defined by $T(v) = Av$ for all $v \in V$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Find the characteristic and minimal polynomials of T.

(ii) Suppose $F = \mathbb{F}_2$, the field of 2 elements. Determine whether V has a T-invariant subspace that is not equal to 0 or V.

(iii) Suppose $F = \mathbb{F}_3$, the field of 3 elements. Determine whether V has a T-invariant subspace that is not equal to 0 or V. (12 marks)

(Total: 20 marks)

2. (a) Let n be a positive integer, and let $\lambda \in \mathbb{C}$. Define the $n \times n$ *Jordan block* $J_n(\lambda)$. (1 mark)

(b) State the Jordan Canonical Form theorem for matrices over $\mathbb{C}$. (2 marks)

(c) Let $V = \mathbb{C}^4$, and let $T : V \mapsto V$ be the linear map defined by $T(v) = Av$ for all $v \in V$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Find the Jordan Canonical Form of T.

(ii) Find a Jordan basis of V for the linear map T. (7 marks)

(d) For any positive integer r, find the Jordan Canonical Form of the r^{th} power $J_n(1)^r$. Justify your answer.

(6 marks)

(e) Let A be an $n \times n$ matrix over $\mathbb{C}$, and suppose that the only eigenvalue of A is 1. Prove that A is similar to A^r for any positive integer r. (4 marks)

(Total: 20 marks)

3. (a) Let V be a finite-dimensional vector space over a field F, and assume that the characteristic of F is not 2.

   (i) Define what is meant by a *symmetric bilinear form* on V.

   (ii) Define also what is meant by a *quadratic form* $Q : V \mapsto F$.

   (iii) What is meant by the statement that the quadratic form Q is *non-degenerate*?

   (iv) Let $Q : V \mapsto F$ be a non-degenerate quadratic form. Prove that there exists $v \in V$ such that $Q(v) \neq 0$.

(7 marks)

(b) Let $V = F^n$, and let $Q : V \mapsto F$ and $Q' : V \mapsto F$ be quadratic forms. What is meant by the statement that Q and Q' are *equivalent*? (1 mark)

(c) Now let $V = \mathbb{Q}^2$, a 2-dimensional vector space over the rationals $\mathbb{Q}$, and let $Q : V \mapsto \mathbb{Q}$ be the quadratic form

$$Q(x) = x_1^2 + 2x_1x_2 - x_2^2 \quad (x = (x_1, x_2) \in \mathbb{Q}^2).$$

   (i) Find a quadratic form $Q'(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$ that is equivalent to Q, where $\lambda_1, \lambda_2 \in \mathbb{Q}$. (3 marks)

   (ii) Which of the following equations have a solution $x \in \mathbb{Q}^2$:

   (1) $Q(x) = 0$ with $x \neq (0, 0)$? (2 marks)

   (2) $Q(x) = 2$? (2 marks)

   (3) $Q(x) = 3$? (5 marks)

(Total: 20 marks)

4. (a) What are the following three numbers when rounded to the nearest 16-bit floating point number ($F_{16} = F_{15,5,10}$)? (4 marks)

$$1, 1 + 2^{-11}, 1 + 2^{-10} + 2^{-11}$$

(b) Assuming f is thrice-differentiable in $[x-h, x+h]$, show the central-differences approximation satisfies

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \delta$$

where

$$|\delta| \leq \frac{h^2}{6} \sup_{x-h \leq t \leq x+h} |f'''(t)|.$$

(5 marks)

(c) Consider a double-dual number of the form

$$a + b\epsilon + c\delta + d\epsilon\delta$$

such that $\epsilon^2 = \delta^2 = 0$ and $a, b, c, d \in \mathbb{R}$. You may assume ϵ and δ commute: $\epsilon\delta = \delta\epsilon$.

(i) What is the formula for representing the product of two double-dual numbers

$$(a + b\epsilon + c\delta + d\epsilon\delta)(e + f\epsilon + g\delta + h\epsilon\delta)$$

as a double-dual number, where $a, b, c, d, e, f, g, h \in \mathbb{R}$? (3 marks)

(ii) Prove for $k = 2, 3, \ldots$ that

$$(a + b\epsilon + c\delta + d\epsilon\delta)^k = a^k + kba^{k-1}\epsilon + kca^{k-1}\delta + (kda^{k-1} + k(k-1)bca^{k-2})\epsilon\delta.$$

(4 marks)

(iii) Explain how double-dual numbers can be used to compute the second derivative of a polynomial p at the point x by choosing a, b, c, d appropriately when evaluating $p(a + b\epsilon + c\delta + d\epsilon\delta)$. (4 marks)

(Total: 20 marks)

5. (a) Consider a bidiagonal matrix

$$U = \begin{bmatrix} 1 & u_1 & & \\ & \ddots & \ddots & \\ & & 1 & u_{n-1} \\ & & & 1 \end{bmatrix} \in F_{\sigma,Q,S}^{n\times n}$$

and a vector $\mathbf{x} \in F_{\sigma,Q,S}^n$, where $F_{\sigma,Q,S}$ is a set of floating-point numbers.

(i) Denoting matrix-vector multiplication implemented using floating point arithmetic as $\mathbf{b} := \texttt{bidiagmul}(U, \mathbf{x})$, express the entries $b_k := \mathbf{e}_k^\top \mathbf{b}$ in terms of u_k and $x_k := \mathbf{e}_k^\top \mathbf{x}$, using rounded floating-point operations $\oplus$ and $\otimes$. Only use $O(1)$ floating-point operations per entry. (3 marks)

(ii) Assuming all operations involve normal floating numbers, show that your approximation has the form

$$U\mathbf{x} = \texttt{bidiagmul}(U, \mathbf{x}) + \boldsymbol{\delta}$$

where, for ϵ_{m} denoting machine epsilon,

$$\|\boldsymbol{\delta}\|_\infty \leq \frac{3}{2}\epsilon_{\mathrm{m}}\|U\|_\infty\|\mathbf{x}\|_\infty.$$

Here we use the matrix norm $\|A\|_\infty := \max_k \sum_{j=1}^n |a_{kj}|$ where $a_{kj} = \mathbf{e}_k^\top A\mathbf{e}_j$ and the vector norm $\|\mathbf{x}\|_\infty := \max_k |x_k|$.

(4 marks)

(b) Compute the Cholesky factorisation of the matrix

$$\begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

(6 marks)

(c) (i) A real reflection is a matrix of the form $I - 2\mathbf{v}\mathbf{v}^\top$ where $\|\mathbf{v}\| = 1$. Show that any diagonal matrix D_k such that

$$\mathbf{e}_j^\top D_k \mathbf{e}_j = \begin{cases} -1 & k = j \\ 1 & \text{otherwise} \end{cases}.$$

can be written as a reflection. (3 marks)

(ii) Recall every matrix has a QR factorisation where Q is a product of Householder reflections. Use this to show that all orthogonal matrices can be expressed as a product of reflections. (4 marks)

(Total: 20 marks)

6. (a) Write the polynomial that interpolates $\sin z$ at the points $[1, \mathrm{i}, -1, -\mathrm{i}]$ as

$$p(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3$$

where $c_k \in \mathbb{R}$. (6 marks)

(b) Consider the Chebyshev polynomials of the 2nd kind: $U_n(x) = 2^n x^n + O(x^{n-1})$ orthogonal with respect to $\sqrt{1-x^2}$ on $[-1, 1]$.

(i) Show that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

and that

$$xU_0(x) = \frac{U_1(x)}{2}, \qquad xU_n(x) = \frac{U_{n-1}(x) + U_{n+1}(x)}{2}.$$

You may use the trigonometric identity

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta).$$

(5 marks)

(ii) Consider the eigen-decomposition $J_n = Q\mathrm{diag}[x_1, \ldots, x_n]Q^\top$ of the matrix

$$J_n := \begin{bmatrix} 0 & 1/2 & & \\ 1/2 & 0 & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Relate this to the Jacobi matrix associated with $U_n(x)$ to give an explicit formula for x_j in terms of $\theta_j = \pi j/(n+1)$ and hence show that the entries of the eigenvector matrix are

$$\mathbf{e}_k^\top Q \mathbf{e}_j = \sqrt{\frac{2}{n+1}} \sin k\theta_j.$$

You may use without proof the diagonalisation formula for truncated Jacobi matrices and the sums-of-squares formula

$$2\sum_{k=0}^{n-1} \sin^2(k+1)\theta_j = n+1.$$

(5 marks)

(iii) Derive an explicit formula for the Gaussian quadrature rule for $w(x) = \sqrt{1-x^2}$ on $[-1, 1]$, valid for all n. (4 marks)

(Total: 20 marks)

Linear Algebra and Numerical Analysis MATH 50003/50012/50016 Solutions 2024

1. (a) Cayley-Hamilton Thm: If V is a finite-dimensional vector space, and $T : V \mapsto V$ a linear map with characteristic polynomial $c(x)$, then $c(T) = 0$. (**2 marks, category A**)

(b) (i) A T-invariant subspace is a subspace W such that $T(W) \subseteq W$. (**1 mark, A**)

(ii) The minimal polynomial of T is a monic polynomial $m(x)$ of minimal degree such that $m(T) = 0$. (**1 mark, A**)

To show it is unique, suppose $m_1(x)$ is another such poly. Then $r(x) = m(x) - m_1(x)$ has smaller degree and satisfies $r(T) = 0$, and hence by definition of $m(x)$, we have $r(x) = 0$. So $m_1(x) = m(x)$. (**2 marks, seen, A**)

(iii) Let $m(x)$ be the minimal poly of T. Then $m(T)(v) = 0$ for all $v \in W$, so $m(T_W) = 0$. By a standard result from lectures, this implies that the minimal poly of T_W divides $m(x)$. (**2 marks, seen, B**)

(c) (i) Characteristic poly $c_A(x)$ is $x^4 - x^2 - 1$ (**2 marks, A**) .

This is also the minimal poly: the simplest way to show this is to note that A is the companion matrix of $x^4 - x^2 - 1$, and by a standard result has min pol equal to its char pol.

A much less simple way is to observe that the standard basis of V can be written as $e_1, Ae_1, A^2e_1, A^3e_1$. If there was a monic poly $p(x)$ of degree less than 4 such that $p(A) = 0$, this would imply that these 4 vectors are linearly dependent, Hence there is no such poly, and so $m_A(x)$ has degree 4 and is therefore equal to $c_A(x)$. (**2 marks, B**)

(ii) Let $F = \mathbb{F}_2$. Then $c_A(x) = m_A(x) = x^4 + x^2 + 1$ factorizes as $(x^2 + x + 1)^2$. From this factorization it follows that $W = \mathrm{Ker}(T^2 + T + I)$ must be nonzero and also is not equal to V, and it is T-invariant. Explicitly, $W = \mathrm{Sp}(e_1+e_4,\ e_1+e_2+e_3)$. (**4 marks, C**)

(iii) Let $F = \mathbb{F}_3$. We claim that $m_A(x) = x^4 - x^2 - 1$ is irreducible in $\mathbb{F}_3[x]$: it has no roots in $\mathbb{F}_3$, so has no linear factors, so the only possible factorization is $m_A = f_1 f_2$ with f_1, f_2 monic irreducible quadratics. The monic irred quadratics are $x^2 + 1$, $x^2 + x - 1$, $x^2 - x - 1$ and we check that none of these divides $m_A(x)$. Hence $m_A(x)$ is irreducible, as claimed.

If W is a nonzero T-invariant subspace, then by (biii) the min poly of T_W divides $m_A(x)$. Hence as m_A is irreducible, the min pol of T_W is equal to m_A, and so $\dim W = 4$ and $W = V$. So there are no T-invariant subslaces apart form 0 and V. (**4 marks, D**)

2. (a) The Jordan block $J_n(\lambda)$ is the $n \times n$ matrix

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \vdots & \\ & & & \cdots & \lambda \end{pmatrix}.$$

(**1 mark, A**)

The Jordan Canonical Form theorem states that any $n \times n$ matrix over $\mathbb{C}$ is similar to a block diagonal matrix of the form $J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, and this is unique apart from the order in which the blocks are written. (**2 marks, A**)

(b) (i) Compute that the char poly of A is $(x-1)^4$; also $(A-I)^2 = 0$, so the min poly is $(x-1)^2$. Also rank$(A-I) = 2$, so the geometric mult of the eigenvalue 1 is 2, so there are two Jordan blocks in the JCF. Hence the JCF is $J_2(1) \oplus J_2(1)$. (**3 marks, A**)

(ii) Apply an algorithm given in the lecture notes. Let $N_1 = \text{Ker}(A-I)$. Compute that $N_1 = \text{Sp}(e_1 - e_3, e_1 + e_4)$. Now write down a basis of V mod N_1: for example e_1, e_2. These are then cyclic vectors for the two Jordan blocks, and a Jordan basis is $(A-I)e_1, e_1,\ (A-I)e_2, e_2$, which is

$$e_3 + e_4,\ e_1,\ -e_1 + e_3,\ e_2.$$

(**4 marks, B**)

(c) Write $J = J_n(1)$. This is equal to $I + J_n(0)$. Hence $J^r = (I + J_n(0))^r = I + rJ_n(0) + \cdots$ has 1's on the diagonal and r's above the diagonal. Hence J^r has only one eigenvalue 1, and rank$(J^r - I) = n - 1$ so the geometric mult of this eigenvalue is 1. So the JCF of J^r has one block, hence is $J_n(1)$. In other words, $J_n(1)^r \sim J_n(1)$ (where $\sim$ denotes similarity). (**6 marks, D – similar problem set on a problem sheet**)

(d) We are given the only evalue of A is 1. Hence the JCF of A is $J_{n_1}(1) \oplus \cdots \oplus J_{n_k}(1)$. Hence by (c),

$$A^r \sim J_{n_1}(1)^r \oplus \cdots \oplus J_{n_k}(1)^r \sim J_{n_1}(1) \oplus \cdots \oplus J_{n_k}(1) \sim A.$$

(**4 marks, C**)

3. (a) (i) A symmetric bilinear form is a map $(\,,\,)$ from $V \times V \mapsto F$ which is both left- and right-linear, and satisfies $(u, v) = (v, u)$ for all $u, v \in V$. (**2 marks, A**)

(ii) A quadratic form $Q : V \mapsto F$ is a map of the form $Q(v) = (v, v)$ for all $v \in V$, where $(\,,\,)$ is a symmetric bilinear form. (**1 mark, A**)

(iii) Q is non-degenerate if the corresponding symmetric bilinear form is, ie. it satisfies the condition

$$(u, v) = 0 \ \forall v \in V \Rightarrow u = 0.$$

(**1 mark, A**)

(iv) We prove this by contradiction. Suppose $Q(v) = 0$ for all $v \in V$. Then for $x, y \in V$,

$$0 = Q(x + y) = (x + y, x + y) = Q(x) + Q(y) + (x, y) + (y, x) = 0 + 0 + 2(x, y).$$

Since $2 \neq 0$ in F, this implies that $(x, y) = 0$ for all $x, y \in V$, a contradiction as Q is non-degenerate. (**3 marks, seen as part of a bigger proof, B**)

(b) (iv) We know from lectures that there are matrices A, B such that $Q(v) = v^T Av$ and $Q'(v) = v^T Bv$ for all $v \in V$. We say Q, Q' are similar if $\exists$ an invertible matric P such that $B = P^T AP$. (**1 mark, A**)

(c) (i) Note that $Q(x) = (x_1 + x_2)^2 - 2x_2^2$. Hence Q is equivalent to Q' where

$$Q'(x) = x_1^2 - 2x_2^2.$$

(**3 marks, B**)

(ii) Since $Q \sim Q'$, the equation $Q(x) = k$ has a solution $x \in \mathbb{Q}^2$ iff $Q'(x) = k$ has a solution.

(1) $Q'(x) = 0$ has no nonzero solution in $\mathbb{Q}^2$, since it would give $\left(\frac{x_1}{x_2}\right)^2 = 2$, whereas $\sqrt{2}$ is irrational. (**2 marks, B**)

(2) $Q'(x) = 2$ has a solution $x = (2, 1)$. (**2 marks, B**)

(3) Now we show $Q'(x) = 3$ has no solution in $\mathbb{Q}^2$. Suppose $x = (x_1, x_2)$ is a solution. Then clearing denominators, there are integers a, b, c (not all 0) such that

$$a^2 - 2b^2 = 3c^2. \quad (*)$$

We may also assume that a, b, c have no common factor greater than 1 (otherwise we can divide the equation by such a factor).

Then $a^2 \equiv 2b^2 \bmod 3$. If $a, b \not\equiv 0 \bmod 3$ then $a^2, b^2 \equiv 1 \bmod 3$, and so we get $1 \equiv 2 \bmod 3$, a contradiction. Hence a or b is $\equiv 0 \bmod 3$, and so $a \equiv b \equiv 0 \bmod 3$. But then equation (*) shows that c is also divisible by 3, so a, b, c have a common factor 3, a contradiction. (**5 marks, D**)

For (c), similar problems set on exercise sheet

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH50003

Numerical Analysis (Solutions)

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1. (a) We have

meth seen ⇓

$$fl(1) = 1.$$

Since its at the half-way point we round down to have a 0 in the last bit:

$$fl(1 + 2^{-11}) = fl((1.00000000001)_2) = 1$$

On the other hand, for the next example we round up to have a 0:

$$fl(1 + 2^{-10} + 2^{-11}) = fl((1.00000000011)_2) = (1.000000001)_2 = 1 + 2^{-9}$$

4, A

(b) We have

seen ⇓

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(t_1)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(t_2)$$

where $t_1, t_2 \in [x-h, x+h]$. Thus we get

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \underbrace{\frac{h^2}{12}(f'''(t_1) + f'''(t_2))}_{-\delta}$$

where

$$|\delta| \leq \frac{h^2}{12}(|f'''(t_1)| + |f'''(t_2)|) \leq \frac{h^2}{6} \sup_{x-h \leq t \leq x+h} |f'''(t)|$$

5, A

(c) (i)

meth seen ⇓

$$(a+b\epsilon+c\delta+d\epsilon\delta)(e+f\epsilon+g\delta+h\epsilon\delta) = ae+(be+af)\epsilon+(ag+ce)\delta+(ah+de+bg+cf)\epsilon\delta.$$

3, B

(ii) Either one can treat it it as a dual within a dual:

meth seen ⇓

$$\begin{aligned}(a + b\epsilon + \delta(c + d\epsilon))^k &= (a + b\epsilon)^k + k(c + d\epsilon)(a + b\epsilon)^{k-1}\delta \\ &= a^k + kba^{k-1}\epsilon + k(c + d\epsilon)(a^{k-1} + (k-1)ba^{k-2}\epsilon)\delta \\ &= a^k + kba^{k-1}\epsilon + k(ca^{k-1} + (da^{k-1} + (k-1)bca^{k-2})\epsilon)\delta \\ &= a^k + kba^{k-1}\epsilon + kca^{k-1}\delta + (kda^{k-1} + k(k-1)bca^{k-2})\epsilon\delta.\end{aligned}$$

Alternatively, we can do a proof by induction using the formula in part (i). Assuming it's valid up to k with $k = 1$ being trivial, we have

$$\begin{aligned}(a + b\epsilon + c\delta + d\epsilon\delta)^{k+1} &= (a + b\epsilon + c\delta + d\epsilon\delta)(a + b\epsilon + c\delta + d\epsilon\delta)^k \\ &= (a + b\epsilon + c\delta + d\delta\epsilon) \\ &\quad \times (a^k + kba^{k-1}\epsilon + kca^{k-1}\delta + (kda^{k-1} + k(k-1)bca^{k-2})\epsilon\delta) \\ &= a^{k+1} + (ka^kb + ba^k)\epsilon + (ka^kc + ca^k)\delta \\ &\quad + (kda^k + k(k-1)bca^{k-1} + bkca^{k-1} + ckba^{k-1} + da^k)\epsilon\delta \\ &= a^{k+1} + (k+1)ba^k\epsilon + (k+1)ca^k\delta \\ &\quad + ((k+1)da^k + k(k+1)bca^{k-1})\epsilon\delta.\end{aligned}$$

4, C

(iii) Setting $a = x$, $b = c = 1$ and $d = 0$ we see that

$\boxed{\text{unseen} \Downarrow}$

$$(x + \epsilon + \delta)^k = x^k + kx^{k-1}\epsilon + kx^{k-1}\delta + k(k-1)x^{k-2}\epsilon\delta$$

Therefore if $p(x) = \sum_{k=0}^n c_k x^k$ we have

$$\begin{aligned} p(x + \epsilon + \delta) &= \sum_{k=0}^n c_k (x + \epsilon + \delta)^k = \sum_{k=0}^n c_k (x^k + kx^{k-1}\epsilon + kx^{k-1}\delta + k(k-1)x^{k-2}\epsilon\delta) \\ &= p(x) + p'(x)\epsilon + p'(x)\delta + p''(x)\epsilon\delta \end{aligned}$$

and the second derivative can be recovered from the $\epsilon\delta$ term. $\boxed{4, \text{D}}$

2. (a) (i) The entries are for $k = 1, \dots, n-1$ $\boxed{\text{meth seen} \Downarrow}$

$$b_k = x_k \oplus (u_k \otimes x_{k+1})$$

and $b_n = x_n$. $\boxed{3, \text{A}}$

(ii) Note that

$$\|U\|_\infty = 1 + \max_k |u_k|$$ $\boxed{\text{unseen} \Downarrow}$

so that $1 + |u_k| \le \|U\|_\infty$.
We deduce that

$$b_k = (x_k + u_k x_{k+1}(1+\delta_1))(1+\delta_2) = x_k + u_k x_{k+1} + \underbrace{u_k x_{k+1}\delta_1 + (x_k + u_k x_{k+1})\delta_2 + u_k x_{k+1}\delta_1\delta_2}_{\epsilon_k}$$

where $|\delta_1|, |\delta_2| \le \epsilon_{\mathrm{m}}/2$. Using the rather naive bound $|\delta_1||\delta_2| \le \epsilon_{\mathrm{m}}^2/4 \le \epsilon_{\mathrm{m}}/2$
We bound

$$\begin{aligned}
|\epsilon_k| &\le |u_k||x_{k+1}||\delta_1| + (|x_k| + |u_k||x_{k+1})||\delta_2| + |u_k||x_{k+1}||\delta_1||\delta_2| \\
&\le \|\mathbf{x}\|_\infty \epsilon_{\mathrm{m}}/2(|u_k| + 1 + |u_k| + |u_k|) \\
&\le \frac{3}{2}\|\mathbf{x}\|_\infty \epsilon_{\mathrm{m}} \|U\|_\infty.
\end{aligned}$$

$\boxed{4, \text{D}}$

(b) Write $\boxed{\text{sim. seen} \Downarrow}$

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{5} & & \\ 2/\sqrt{5} & 1 & \\ 0 & & 1 \end{bmatrix}}_{L_1} \begin{bmatrix} 1 & \\ & A_2 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{5} & 2/\sqrt{5} & 0 \\ & 1 & \\ & & 1 \end{bmatrix}}_{L_1^\top}$$

where

$$\begin{aligned}
A_2 &= \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6/5 & 2 \\ 2 & 5 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \sqrt{6/5} & \\ 2\sqrt{5/6} & 1 \end{bmatrix}}_{L_2} \begin{bmatrix} 1 & \\ & A_3 \end{bmatrix} L_2^\top.
\end{aligned}$$

where

$$A_3 = 5 - 20/6 = 5/3 = \sqrt{5/3}\sqrt{5/3}.$$

Thus

$$L = \begin{bmatrix} \sqrt{5} & & \\ 2/\sqrt{5} & \sqrt{6/5} & \\ 0 & 2\sqrt{5/6} & \sqrt{5/3} \end{bmatrix}.$$

$\boxed{6, \text{A}}$

(c) (i) Consider $\mathbf{v} = \mathbf{e}_k$. Then $\boxed{\text{unseen} \Downarrow}$

$$\mathbf{e}_j^\top (I - 2\mathbf{e}_k\mathbf{e}_k^\top)\mathbf{e}_j = 1 - 2\delta_{kj} = \begin{cases} -1 & k = j \\ 1 & \text{otherwise} \end{cases}.$$

$\boxed{3, \text{B}}$

(ii) Let Q be an orthogonal matrix. We know from the QR factorisation with Householder reflections there exists a sequence of reflections so that $\boxed{\text{unseen} \Downarrow}$

$$Q = Q_k \cdots Q_1 R$$

where R is upper triangular. By choosing the right signs in the Householder reflections we can ensure that R has positive diagonal. One can use uniqueness of QR factorisations with positive diagonals to confirm that $Q = Q_k \cdots Q_1$. Alternatively, we can deduce

$$I = Q^\top Q = R^\top Q_1^\top \cdots Q_k^\top Q_k \cdots Q_1 R = R^\top R.$$

We know the diagonal of R cannot be zero since Q is invertible. For $j > 1$ we have

$$0 = \mathbf{e}_1 I \mathbf{e}_j = \mathbf{e}_1 R^\top R \mathbf{e}_j = R[1,1]R[1,j]$$

hence the first row of R is zero apart from the first entry. We can continue by induction: assume $R[\ell, j] = 0$ for all $\ell < k$ and $j > \ell$. Then for $j > k$ we have

$$0 = \mathbf{e}_k^\top I \mathbf{e}_j = \mathbf{e}_k^\top R^\top R \mathbf{e}_j = \sum_{\ell=1}^k R[\ell,k]R[\ell,j] = R[k,k]R[k,j]$$

hence $R[k,j] = 0$ as well. Thus R is diagonal and from $R^\top R = I$ we know it is orthogonal. Thus the diagonal entries are ± 1. From the previous part these can also be written as products of reflections. $\boxed{4, \text{D}}$

3. (a) We can use the DFT: sim. seen ⇓

$$\begin{bmatrix} c_0 \\ \vdots \\ c_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} \sin 1 \\ \sin -i \\ \sin(-1) \\ \sin i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ \sin 1 + \sinh 1 \\ 0 \\ \sin 1 - \sinh 1 \end{bmatrix}$$

Thus the polynomial is

$$p(z) = \frac{\sin 1 + \sinh 1}{2} z + \frac{\sin 1 - \sinh 1}{2} z^3.$$

This can also be derived using Lagrange polynomials. 6, A

(b) (i) Let seen ⇓

$$p_n(x) := \frac{\sin(n+1)\mathrm{acos}\, x}{\sin \mathrm{acos}\, x}.$$

We verify that $p_n(x) = U_n(x)$ by showing that it satisfies (1) its a graded basis (2) orthogonality and (3) right normalisation constant. First (2): with the change of variables $x = \cos\theta$, $dx = -\sin\theta d\theta$,

$$\begin{aligned} \int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx &= \int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx \\ &= \int_0^{\pi} \frac{\sin((n+1)\theta)\sin((m+1)\theta)\sqrt{1-\cos^2\theta}}{\sin^2\theta} \sin\theta d\theta \\ &= \int_0^{\pi} \sin((n+1)\theta)\sin((m+1)\theta)d\theta = 0 \end{aligned}$$

for $n \neq m$ (which can be shown by expanding in complex exponentials). (1/3) follow from deriving the 3-term recurrence. We have $p_0(x) = 1$ and

$$xp_0(x) = \cos\theta \frac{\sin\theta}{\sin\theta} = \frac{\sin 2\theta}{2\sin\theta} = p_1(x)/2$$

hence $p_1(x) = 2x + O(1)$. Now assume $p_n(x) = 2^n x^n + O(x^n - 1)$. We find

$$xp_n(x) = \cos\theta \frac{\sin(n+1)\theta}{\sin\theta} = \frac{\sin(n+2)\theta + \sin n\theta}{2\sin\theta} = \frac{p_{n+1}(x) + p_{n-1}(x)}{2}$$

hence

$$p_{n+1}(x) = 2xp_n(x) + O(x^{n-1}) = 2^{n+1}x^{n+1} + O(x^n)$$

Thus $p_n(x) = U_n(x)$ and we have also verified the 3-term recurrence. 5, B

(ii) The Jacobi matrix for $U_n(x)$ is sim. seen ⇓

$$J = \begin{bmatrix} 0 & 1/2 & & \\ 1/2 & 0 & 1/2 & \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

and we see that J_n is a truncation of J. We thus know that x_n are the roots of $U_n(x)$. We find for $\theta_j = \pi j/(n+1)$ and $x_j = \cos\theta_j$ that

$$U_n(x_j) = \frac{\sin(n+1)\theta_j}{\sin\theta_j} = 0.$$

Note that because the Jacobi matrix is symmetric $U_n(x)$ are up to a constant equal to the orthonormal polynomials. Since

$$\int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \int_0^\pi \sin^2\theta d\theta = \frac{\pi}{2}$$

we know that $q_n(x) = \sqrt{2/\pi}U_n(x)$. The eigenvector matrix is then

$$\mathbf{e}_k^\top Q\mathbf{e}_j = q_{k-1}(x_j)/\alpha_j = \sqrt{2/\pi}\frac{\sin k\theta_j}{\alpha_j \sin\theta_j}$$

where, using the sums-of-squares formula, we have

$$\begin{aligned}\alpha_j^2 &= \sum_{k=0}^{n-1} q_k(x_j)^2 = \frac{2}{\pi\sin^2\theta_j}\sum_{k=0}^{n-1}\sin^2(k+1)\theta_j \\ &= \frac{n+1}{\pi\sin^2\theta_j}.\end{aligned}$$

Thus

$$\mathbf{e}_k^\top Q\mathbf{e}_j = \sqrt{2}\frac{\sin k\theta_j}{\sqrt{n+1}}.$$

$\boxed{5, \mathrm{C}}$

(iii) Using the previous part, we know the Gauss quadrature rule has roots $x_j = \cos\theta_j$ and weights

$\boxed{\text{meth seen} \Downarrow}$

$$w_j = \frac{1}{\alpha_j^2} = \frac{\pi\sin^2\theta_j}{n+1}.$$

Thus the explicit formula is

$$\frac{\pi}{n+1}\sum_{j=1}^{n}\sin^2\theta_j f(x_j).$$

This problem can also be solved without solving the previous part by recalling

$$w_j = \int_{-1}^{1} w(x)\mathrm{d}x(\mathbf{e}_1^\top Q\mathbf{e}_j)^2 = \frac{\pi}{2}\frac{2}{n+1}\sin^2\theta_j = \frac{\pi\sin^2\theta_j}{n+1}.$$

$\boxed{4, \mathrm{B}}$

Review of mark distribution:
Total A marks: 24 of 24 marks
Total B marks: 15 of 15 marks
Total C marks: 9 of 9 marks
Total D marks: 12 of 12 marks
Total marks: 60 of 60 marks

MATH50003 Linear Algebra and Numerical Analysis

Question	Marker's comment
1	Marks were disappointingly low on average for this question. Parts (a) and (b), which were short bookwork questions, were quite well done. But part (c), asking for computations with a nice 4x4 matrix, was not at all well done -- not many candidates noticed that the matrix was a companion matrix (for which we know from lectures how to instantly compute the characteristic and minimal polys), and then not very many were able to factorise the characteristic poly over F_2, or give a proper proof that it is irreducible over F_3.
3	The question is on the easier side but the performance of the students appeared quite poor. Even the answers to the parts where the student is just required to state a definition were poor. It is important to make sure that you understand the definitions in order to achieve good mark in an exam.
4	Students overall did very well on this question, including unseen material.
5	Marks were generally on the low side for Q5. Particularly in part c. with candidates unsure of what was expected, in both parts, particularly c(ii) was a very wordy solution, and it was obvious that candidates were unsure of what was expected (householder method).nbsp; Part c(i) was also confusing to candidates, though trivial.Part a(ii) marks were lost as the bounds were not clearlynbsp; defined by candidates when answering. Since, the final answer was given in the exam paper, nearly all candidates simply worked (using vague language, incorrect maths) to fit their result with that stated, Many were confused with how to deal with (eps/2 * eps/2) term when defining bounds, or reducing this to a coarse bound as stated in the solution.Part b, Cholesky factorisation was done well by most, with a few making some trivial errors for which they were not heavily penalised. The majority gained full 6 marks.