MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 1

Exercise 1.

Follow the hint and note that the function c solves a differential equation that does not depend on x, i.e. it is a differential equation of the form $\dot{x} = g(t)$; here g is just a function of time t. Such differential equations are easy to solve by integration, and the integration constant needs to be chosen so that the initial condition is fulfilled. By doing so, obtain the solution λ to the initial value problem. Now assume that there is another solution μ to this initial value problem. Demonstrate that $t \mapsto \lambda(t) - \mu(t)$ solves the initial value problem $\dot{x} = a(t)x$, $x(0) = 0$, and use an argument similar to Example 1.1 to complete the proof.

Exercise 2.

Argue by contradiction and assume that there exists a solution λ to the initial value problem. The initial condition implies that $\dot{\lambda}(0) = -1$, which determines the sign of λ for both negative values and positive values close to 0. Use the mean value theorem to create a contradiction.

Exercise 3.

The Picard iterates are obtained using elementary integrals using Definition 2.2 (see also Example 2.3), but note that in (ii), a vector of dimension two is integrated (with respect to a one-dimensional variable, given by time). This is defined componentwise as one-dimensional integrals, see text after Proposition 2.1 in the lecture notes.

Exercise 4.

Consider the difference $d(t) = \alpha(t) - \lambda(t)$ for $t \geq t_0$. Argue by contradiction and define $\tau := \inf\{t > t_0 : d(t) \leq 0\}$. Consider the two cases $\tau = t_0$ and $\tau > t_0$ and create a contradiction in each case. Use here an argument similar to Exercise 2, which is explained in the hint above (concerning the determination of the sign of λ).

Exercise 5.

The problem is readily solved when f has a zero. If f does not have a zero, consider the unique solution $\lambda : \mathbb{R} \to \mathbb{R}$ satisfying the initial condition $x(0) = 0$. Show then that $\lim_{t\to\infty} \lambda(t) = \infty$ or $\lim_{t\to\infty} \lambda(t) = -\infty$. Conclude that there exists an $a > 0$ with $\lambda(a) = 1$ or $\lambda(a) = -1$, and analyse this observation in the setting of the problem.