1. (i) The Lagrangian for a spherical pendulum is

$$L = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + g\cos\theta,$$

where g is a positive constant. Here θ and ϕ are spherical polar angles where the angle θ is measured with respect to the negative z-axis.

(a) Show that the Hamiltonian corresponding to the Lagrangian is

$$H = \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2\sin^2\theta} - g\cos\theta.$$

(b) Obtain Hamilton's equations for the pendulum (there are four equations).

(c) Consider solutions where $p_\theta = 0$. Describe these solutions physically and determine p_ϕ and $\dot{\phi}$ (write them as functions of θ).

(ii) Consider the time-dependent canonical transformation

$$Q = q\cos t + p\sin t, \quad P = -q\sin t + p\cos t.$$

(a) Verify that $\{Q, P\} = 1$.

(b) Find a generating function, $F(q, Q, t)$, for the transformation.

Hint: express p and P in terms of q, Q and t. Use $p = \dfrac{\partial F}{\partial q}$, $P = -\dfrac{\partial F}{\partial Q}$ to find F.

(c) Is $F(q, Q, t)$ defined for all t?