IMPERIAL

MATH70051

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Vortex Dynamics

Date: Thursday, May 1, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. The streamfunction $\psi(x,y)$ of a steady, incompressible flow of an ideal fluid in an (x,y) plane is given to be

$$(x,y) = -\frac{1}{4\pi}\log(x^2+y^2) + \frac{\gamma y^2}{2}.$$

(a) Find an expression for each of the two velocity components $\mathbf{u} = (u(x,y), v(x,y))$.

(2 marks)

(b) Determine all stagnation points of the flow. (4 marks)

(c) Find the circulation

$$\oint_{C_1} \mathbf{u}.\mathbf{dx},$$

where C_1 is the circle described by $x^2+y^2=1$. (4 marks)

(d) Find the circulation

$$\oint_{C_2} \mathbf{u}.\mathbf{dx},$$

where C_2 is the circle described by $(x-2)^2+y^2=1$. (3 marks)

(e) The streamfunction is now modified to

$$\tilde{\psi}(x,y) = -\frac{1}{4\pi}\log(x^2+y^2) + \frac{\gamma y}{2}.$$

Find a complex potential $\tilde{w}(z)$ associated with this modified flow where $z = x + \mathrm{i}y$.

(3 marks)

(f) Let $\tilde{\mathbf{u}}$ be the flow associated with the streamfunction given in part (e). Find the circulation

$$\oint_{C_1} \tilde{\mathbf{u}}.\mathbf{dx},$$

where C_1 is the circle described by $x^2+y^2=1$. (2 marks)

(g) Find the circulation

$$\oint_{C_2} \tilde{\mathbf{u}}.\mathbf{dx},$$

where C_2 is the circle described by $(x-2)^2+y^2=1$. (2 marks)

(Total: 20 marks)

2. Consider an incompressible ideal fluid flowing in an unbounded domain in the presence of an impenetrable flat plate occupying the interval $-1 \leq x \leq 1$ in an (x, y) plane.

(a) Let $z = x + \mathrm{i}y$ and consider the fluid motion taking place in this complex z plane. Write down a conformal mapping

$$z = f(\zeta)$$

from the interior of the unit disc $|\zeta| < 1$ in a parametric complex ζ plane to the fluid region exterior to the flat plate. (2 marks)

(b) Determine the inverse conformal mapping function

$$\zeta = f^{-1}(z).$$

(4 marks)

(c) There is a point vortex of circulation Γ in the fluid at a complex location $z_\alpha = x_\alpha + \mathrm{i}y_\alpha$ near the plate. Except for this point vortex, the flow exterior to the plate is irrotational. Far from the flat plate, the flow is stagnant. Show that a complex potential for the flow is given by

$$w_1(z) = -\frac{\mathrm{i}\Gamma}{2\pi} \log\left(\frac{z - \sqrt{z^2-1} - (z_\alpha - \sqrt{z_\alpha^2 - 1})}{z - \sqrt{z^2-1} - (\overline{z_\alpha} + \sqrt{\overline{z_\alpha}^2 - 1})}\right).$$

(6 marks)

(d) Suppose now that, in addition to the flow generated by the point vortex, there is an imposed far-field flow that is uniform with speed U parallel to the x-axis. Find the new complex potential $w_2(z)$. (2 marks)

(e) Suppose that the imposed far-field flow with speed U in part (d) is now directed at a non-zero angle χ to the real axis with $0 < \chi < \pi$. Find the associated complex potential $w_3(z)$.

(6 marks)

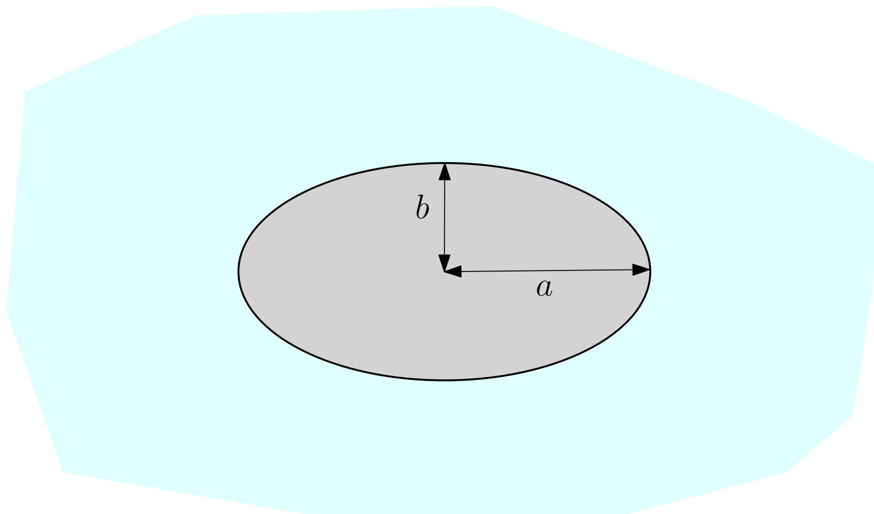
(Total: 20 marks)

3. The vorticity field $\omega(x, y)$ of a two-dimensional incompressible fluid in an unbounded $\mathbf{x} = (x, y)$ plane exterior to a solid impenetrable obstacle with boundary that is an ellipse described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$\omega(x, y) = x^2 + y^2.$$



There is no circulation around the obstacle and, as $|\mathbf{x}| \to \infty$, the streamfunction ψ has the far-field behaviour

$$\to -\frac{(x^2 + y^2)^2}{16} + \text{constant}, \qquad |\mathbf{x}| \to \infty.$$

(a) Introduce the change of variable $(x, y) \mapsto (z, \overline{z})$ where $z = x + \mathrm{i}y$ and, in terms of these variables, write down the boundary value problem for the streamfunction associated with the flow exterior to the obstacle. (4 marks)

(b) Suppose that $a = b = 1$ so that the impenetrable obstacle is a circular disc of unit radius. Find the streamfunction associated with the flow exterior to it. (6 marks)

(c) Suppose now that $a > b$ so that the boundary of the impenetrable obstacle is an ellipse with semi-major axis aligned with the x axis.

(i) Find a conformal mapping from the unit disc in a complex ζ plane to the fluid region exterior to the obstacle. (4 marks)

(ii) Use the conformal mapping found in part (i) to find the streamfunction of the flow around this obstacle as a function of ζ. (6 marks)

(Total: 20 marks)

4. (a) Using $z = x + \mathrm{i}y$ and its complex conjugate $\overline{z} = x - \mathrm{i}y$, find the streamfunction $\quad_{SBR}(z, \overline{z})$ associated with solid body rotation of an incompressible fluid in an (x, y) plane with angular velocity velocity Ω. (2 marks)

(b) A point vortex of circulation Γ and another of circulation 2Γ are rotating with constant angular velocity Ω about the origin in an (x, y)-plane. The vortex of circulation Γ is at a distance a from the origin $z = 0$, and the vortex of circulation 2Γ is at a distance b from $z = 0$.

(i) Determine a/b. (4 marks)

(ii) Find an expression for Ω as a function of a. (4 marks)

(c) Consider incompressible flow of an ideal fluid on the surface of a sphere of unit radius.

(i) Write down an expression, in terms of spherical polar angles (θ, ϕ) where $\theta = 0$ corresponds to the North pole of the sphere, for the stereographic projection from the surface of the sphere to a complex ζ plane through its equator. (1 mark)

(ii) Hence, show that

$$\sin\theta = \frac{2\sqrt{\zeta\overline{\zeta}}}{1 + \zeta\overline{\zeta}}.$$

(1 mark)

(d) Find an expression $\psi_{SBR}(\zeta, \overline{\zeta})$ for the streamfunction associated with solid body rotation of the fluid on the spherical surface with angular velocity Ω about the axis through North and South poles. (2 marks)

[*Hint:* the zonal velocity u_ϕ and meridional velocity u_θ of an incompressible flow with streamfunction ψ are given by

$$u_\phi - \mathrm{i}u_\theta = \frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}.$$

]

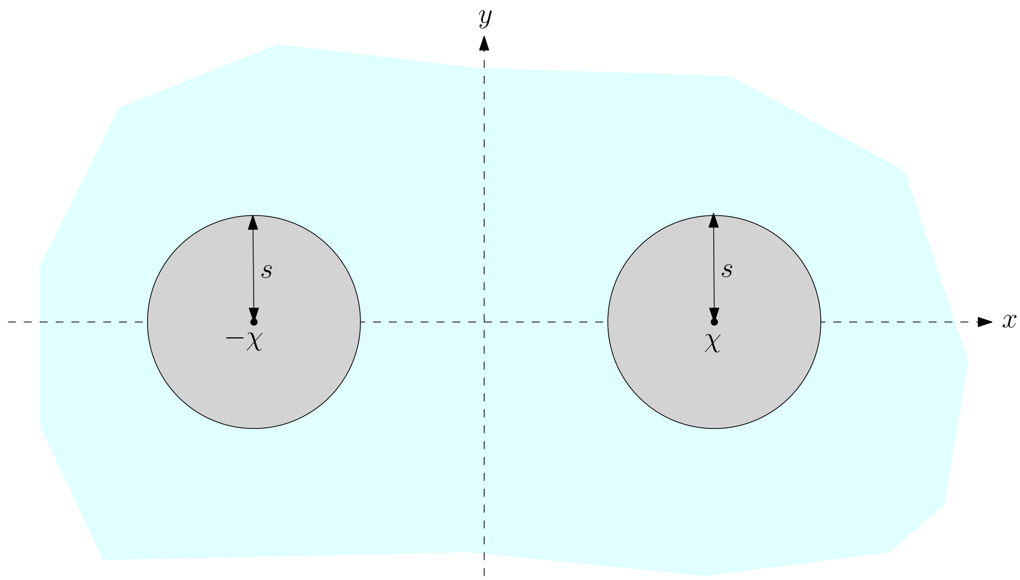
(e) A point vortex of circulation Γ and another of circulation 2Γ are rotating with constant angular velocity Ω about the axis through the North and South poles of the sphere. The vortex of circulation Γ has polar angle θ_a and the vortex of circulation 2Γ has polar angle θ_b. Show that

$$\frac{\sin(\theta_a)}{\sin(\theta_b)} = 2.$$

(6 marks)

(Total: 20 marks)

5. Consider unbounded incompressible, irrotational flow of an ideal fluid around two circular obstacles, both of radius s, centred at $(\pm\chi, 0)$ on the x-axis in an (x, y) plane.



Let $z = x + \mathrm{i}y$ and consider the fluid motion taking place in this complex z plane.

(a) Let $0 < \rho < 1$. Show that the Möbius mapping

$$z = f(\zeta) = R\,\frac{\sqrt{\rho} - \zeta}{\sqrt{\rho} + \zeta}$$

transplants the boundaries of the annulus $\rho < |\zeta| < 1$ in a parametric complex ζ plane to the two boundaries of the circular obstacles in the z plane provided the parameters R and ρ are chosen so that

$$R = \sqrt{\chi^2 - s^2}, \qquad \rho = \frac{\chi - R}{\chi + R} = \frac{\chi - \sqrt{\chi^2 - s^2}}{\chi + \sqrt{\chi^2 - s^2}}.$$

(4 marks)

(b) Suppose that, far from the circular obstacles, there is a uniform flow of speed U parallel to the x axis. Suppose also that there is no circulation around either of the circular obstacles. Write down all the conditions that must be satisfied by a complex potential $w(z)$ of such a flow. (3 marks)

(c) Define the functions

$$P(\zeta, \rho) \equiv (1 - \zeta)\hat{P}(\zeta, \rho) \qquad \text{with} \qquad \hat{P}(\zeta, \rho) \equiv \prod_{k=1}^{\infty} (1 - \rho^{2k}\zeta)(1 - \rho^{2k}/\zeta).$$

You may assume that this infinite product is convergent for $0 \leq \rho < 1$. Show that $P(\zeta, \rho)$ satisfies the two functional relations

$$P(1/\zeta, \rho) = -(1/\zeta)P(\zeta, \rho), \qquad P(\rho^2\zeta, \rho) = -(1/\zeta)P(\zeta, \rho).$$

(2 marks)

...question continues on the next page

(d) Define the function

$$K(\zeta,\rho) \equiv \frac{\zeta}{P(\zeta,\rho)}\frac{\partial P}{\partial \zeta}(\zeta,\rho).$$

(i) Use part (c) to show that $K(\zeta,\rho)$ satisfies the two functional relations

$$K(1/\zeta,\rho) = 1 - K(\zeta,\rho), \qquad K(\rho^2\zeta,\rho) = K(\zeta,\rho) - 1.$$

(2 marks)

(ii) Show also that $K(\zeta,\rho)$ has simple poles at $\zeta = \rho^{2n}$ for any integer n with the local behaviour

$$K(\zeta,\rho) \sim \frac{1}{\zeta - 1} + \text{a locally analytic function}$$

as $\zeta \to 1$. (1 mark)

(e) Define the composed function

$$W(\zeta) = w(f(\zeta)),$$

where $w(z)$ is the complex potential discussed in part (b) and $f(\zeta)$ is the conformal mapping in part (a). Based on your answer to part (b), write down all the conditions that must be satisfied by $W(\zeta)$. (3 marks)

(f) Show that

$$W(\zeta) = -2RU(K(-\zeta/\sqrt{\rho},\rho) - K(-\zeta\sqrt{\rho},\rho))$$

satisfies all the conditions required of $W(\zeta)$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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M70051

Vortex dynamics (Solutions)

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1. (a) The velocity components are $\boxed{\text{sim. seen} \Downarrow}$

$$u = \frac{\partial \psi}{\partial y} = \gamma y - \frac{y}{2\pi(x^2+y^2)}, \qquad v = -\frac{\partial \psi}{\partial x} = \frac{x}{2\pi(x^2+y^2)}.$$

$\boxed{2, \text{A}}$

(b) The stagnation points where $u = v = 0$ must be located on $x = 0$ due to the form of v. Then, on setting $u = 0$, i.e., $\boxed{\text{sim. seen} \Downarrow}$

$$u = y\left(\gamma - \frac{1}{2\pi y^2}\right) = 0$$

it is necessary that

$$y = \pm\frac{1}{\sqrt{2\pi\gamma}}.$$

Hence there are two stagnation points at

$$\left(0, \pm\frac{1}{\sqrt{2\pi\gamma}}\right).$$

$\boxed{4, \text{A}}$

(c) By inspection, the logartihmic singularity is indicative of a point vortex at $z = 0$, indeed $\boxed{\text{sim. seen} \Downarrow}$

$$-\frac{1}{4\pi}\log(x^2+y^2) = \text{Im}\left[-\frac{\text{i}}{2\pi}\log z\right]$$

so there is a point vortex of unit circulation at the origin. The vorticity associated with the other term in the streamfunction is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\gamma$$

which is everywhere uniform. Therefore, the circulation around the unit circle is

$$1 - \pi\gamma$$

where 1 comes from the point vortex, and $-\pi\gamma$ is the area of the circle multiplied by $-\gamma$ (by Stokes theorem the circulation is the area integral of the vorticity inside the curve). $\boxed{4, \text{A}}$

(d) By simillar arguments, the point vortex is no longer inside the circle described by $(x-2)^2 + y^2 = 1$ although it has the same area as the unit circle in part (c). Therefore, the circulation in this case is $\boxed{\text{sim. seen} \Downarrow}$

$$-\pi\gamma.$$

$\boxed{3, \text{A}}$

(e) It is clear from the answer to part (c) that $\boxed{\text{sim. seen} \Downarrow}$

$$\tilde{w}(z) = -\frac{\text{i}}{2\pi}\log z + \frac{\gamma z}{2}$$

has an imaginary part that equals $\tilde{\psi}$. $\boxed{3, \text{A}}$

(f) The flow is now irrotational except for the same unit-circulation point vortex at the origin. Therefore the circulation around the unit circle is $\boxed{\text{sim. seen} \Downarrow}$

$$\oint_{C_1} \tilde{\mathbf{u}}.\mathbf{dx} = 1.$$

$\boxed{2, \text{A}}$

(g) Since the point vortex is no longer inside C_2, then $\boxed{\text{sim. seen} \Downarrow}$

$$\oint_{C_2} \tilde{\mathbf{u}}.\mathbf{dx} = 0.$$

$\boxed{2, \text{A}}$

2. (a) The required conformal mapping is the Joukowski mapping sim. seen ⇓

$$z = f(\zeta) = \frac{1}{2}\left(\frac{1}{\zeta} + \zeta\right).$$

The point $\zeta = 0$ is transplanted to $z = \infty$. On parametrizing $|\zeta| = 1$ by $e^{\mathrm{i}\theta}$ for $0 \le \theta \le 2\pi$ it is clear that

$$z = f(\zeta) = \frac{1}{2}\left(\frac{1}{\zeta} + \zeta\right) = \frac{1}{2}\left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right) = \cos\theta$$

which covers the interval $x \in [-1, 1]$. 2, A

(b) To find the inverse, note that sim. seen ⇓

$$\zeta^2 - 2z\zeta + 1 = 0$$

which is a quadratic equation for ζ with solution

$$\zeta = f^{-1}(z) = z - \sqrt{z^2 - 1},$$

where the minus sign in front of the square root has been selected in order that $\zeta \to 0$ as $z \to \infty$.

4, C

(c) The complex potential for a point vortex of circulation Γ at $\zeta = \alpha$ in the unit disc, where sim. seen ⇓

$$z_\alpha = f(\alpha),$$

is well known to be

$$\mathcal{G}_0(\zeta, \alpha) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right).$$

By the conformal invariance of the relevant boundary value problem, it follows that, to within an inconsequential constant, the required complex potential is

$$w_1(z) = \mathcal{G}_0(\zeta, \alpha) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right).$$

Now if

$$\alpha = f^{-1}(z_\alpha) = z_\alpha - \sqrt{z_\alpha^2 - 1},$$

then

$$\frac{1}{\overline{\alpha}} = \frac{1}{\overline{z_\alpha} - \sqrt{\overline{z_\alpha}^2 - 1}} = \overline{z_\alpha} + \sqrt{\overline{z_\alpha}^2 - 1}.$$

Therefore,

$$w_1(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{z - \sqrt{z^2-1} - (z_\alpha - \sqrt{z_\alpha^2 - 1})}{z - \sqrt{z^2-1} - (\overline{z_\alpha} + \sqrt{\overline{z_\alpha}^2 - 1})}\right) + \text{constant}.$$

6, B

(d) Adding in a uniform flow parallel to the x-axis is straightforward: it is just necessary to add Uz to $w_1(z)$ since the former already satisfies the streamline condition on the flat plate. Therefore, sim. seen ⇓

$$w_2(z) = w_1(z) + Uz.$$

2, A

(e) Adding in a flow of speed U making angle χ to the real axis requires

sim. seen $\Downarrow$

$$w_3(z) \sim U e^{-\mathrm{i}\chi} z, \qquad \text{as } |z| \to \infty$$

On considering the composed analytic function

$$W_3(\zeta) = w_3(f(\zeta))$$

then it is necessary that

$$W_3(\zeta) \sim \frac{U e^{-\mathrm{i}\chi}}{2\zeta}$$

as $\zeta \to 0$ in view of the fact that the conformal mapping is such that

$$z \sim \frac{1}{2\zeta}$$

as $\zeta \to 0$. In order to ensure that the streamline condition is satisfied it is natural to consider

$$W_3(\zeta) = \frac{U e^{-\mathrm{i}\chi}}{2\zeta} + \frac{U e^{\mathrm{i}\chi}\zeta}{2}$$

which has the correct far-field behaviour and is clearly real on $|\zeta| = 1$ ensuring that the associated streamfunction (its imaginary part) is constant. Therefore,

$$\begin{aligned} w_3(z) &= w_1(z) + \frac{U e^{-\mathrm{i}\chi}}{2} \frac{1}{z - \sqrt{z^2-1}} + \frac{U e^{\mathrm{i}\chi}}{2}\left(z - \sqrt{z^2-1}\right) \\ &= w_1(z) + \frac{U e^{-\mathrm{i}\chi}}{2}\left(z + \sqrt{z^2-1}\right) + \frac{U e^{\mathrm{i}\chi}}{2}\left(z - \sqrt{z^2-1}\right). \end{aligned}$$

6, B

3. (a) The vorticity-streamfunction relation in (x, y) variables is sim. seen $\Downarrow$

$$\nabla^2\psi = -\omega = -(x^2 + y^2).$$

In terms of $(z, \overline{z})$, this is

$$4\frac{\partial^2\psi}{\partial z\partial\overline{z}} = -z\overline{z} \tag{1}$$

The boundary condition on the boundary of the solid object, given that it is impenetrable, is

$$\psi = \text{constant} = 0 \qquad \text{without loss of generality}$$

which guarantees that it is a streamline. The far-field boundary condition is given in the question and is restated as

$$\psi \to -\frac{|z|^4}{16}, \qquad |z| \to \infty.$$

4, A

(b) We can integrate (1) directly with respect to $\overline{z}$: sim. seen $\Downarrow$

$$\frac{\partial\psi}{\partial z} = -\frac{z\overline{z}^2}{8} + f'(z) \tag{2}$$

where $f'(z)$ is an arbitrary analytic function in the fluid region $|z| > 1$ and decaying as $|z| \to \infty$ in order to be consistent with the specified far-field boundary condition. Integrating again with respect to z gives

$$\psi = -\frac{z^2\overline{z}^2}{16} + f(z) + \overline{f(z)}, \tag{3}$$

where we have added the function of $\overline{z}$ that renders this result real, as required. It is easy to check that $f(z) = 0$ provides a solution since, on the boundary of the object,

$$\psi = -\frac{z^2\overline{z}^2}{16} = -\frac{1}{16} \tag{4}$$

so the boundary is a streamline. On the other hand, insisting that $\psi = 0$ on the boundary $|z| = 1$, as specified in part (a), means that the constant should be chosen so that

$$\psi = \frac{1 - z^2\overline{z}^2}{16} \tag{5}$$

which satisfies both the far-field condition and the boundary condition on $|z| = 1$.

6, B

(c) (i) The required conformal mapping is well-known from lectures to be sim. seen $\Downarrow$

$$z = Z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta, \qquad \alpha, \beta \in \mathbb{R}, \tag{6}$$

where

$$\alpha + \beta = a, \qquad \alpha - \beta = b.$$

4, A

(ii) The general solution for ψ was found earlier to be sim. seen $\Downarrow$

$$\psi = -\frac{z^2\overline{z}^2}{16} + f(z) + \overline{f(z)}, \tag{7}$$

where $f(z)$ is analytic in the fluid region and tending to a constant as $|z| \to \infty$. To find $f(z)$, introduce the composed function

$$F(\zeta) \equiv f(Z(\zeta)). \tag{8}$$

As a function of ζ, this function must have a convergent Taylor series inside the disc $|\zeta| < 1$ which corresponds, under the mapping, to the fluid region with $\zeta \to 0$ corresponding to $|z| \to \infty$. On the boundary of the object, corresponding to $|\zeta| = 1$, we require

$$0 = \psi = -\frac{z^2\overline{z}^2}{16} + f(z) + \overline{f(z)} = -\frac{Z(\zeta)^2\overline{Z}(1/\zeta)^2}{16} + F(\zeta) + \overline{F}(1/\zeta), \tag{9}$$

where we have used the fact that $\overline{\zeta} = 1/\zeta$ on this curve. Hence

$$F(\zeta) + \overline{F}(1/\zeta) = \frac{Z(\zeta)^2\overline{Z}(1/\zeta)^2}{16}. \tag{10}$$

On substituting the mapping,

$$\begin{aligned} F(\zeta) + \overline{F}(1/\zeta) &= \frac{1}{16}\left[\left(\frac{\alpha}{\zeta} + \beta\zeta\right)^2 \left(\alpha\zeta + \frac{\beta}{\zeta}\right)^2\right] \\ &= \frac{1}{16}\left[\left(\frac{\alpha^2}{\zeta^2} + 2\alpha\beta + \beta^2\zeta^2\right)\left(\alpha^2\zeta^2 + 2\alpha\beta + \frac{\beta^2}{\zeta^2}\right)\right] \\ &= \frac{1}{16}\left[\frac{\alpha^2\beta^2}{\zeta^4} + \frac{2\alpha\beta^3}{\zeta^2} + \beta^4 + \frac{2\alpha^3\beta}{\zeta^2}\right. \\ &\qquad \left. + 4\alpha^2\beta^2 + 2\alpha\beta^3\zeta^2 + \alpha^4 + 2\alpha^3\beta\zeta^2 + \alpha^2\beta^2\zeta^4\right]. \end{aligned} \tag{11}$$

We can now read off $F(\zeta)$ since it must have the aforementioned Taylor series:

$$F(\zeta) = \frac{1}{32}(\alpha^4 + 4\alpha^2\beta^2 + \beta^4) + \frac{1}{16}\left(\alpha^2\beta^2\zeta^4 + 2\alpha\beta(\beta^2 + \alpha^2)\zeta^2\right). \tag{12}$$

Therefore, as a function of ζ,

$$\psi = -\frac{|Z(\zeta)|^4}{16} + 2\mathrm{Re}[F(\zeta)], \tag{13}$$

where $Z(\zeta)$ is given in (6) and $F(\zeta)$ is given in (12). $\boxed{6, \mathrm{C}}$

4. (a) Solid body rotation with angular velocity Ω corresponds to a flow with uniform vorticity $\omega(x, y) = 2\Omega$ so, using the well-known streamfunction-vorticity relation for a 2D flow, sim. seen $\Downarrow$

$$\omega(x, y) = 2\Omega = -\nabla^2 \psi = -4\frac{\partial^2 \psi}{\partial z \partial \overline{z}},$$

where the change of variables $(x, y) \mapsto (z, \overline{z})$ has been employed to rewrite the Laplacian operator. On integration with respect to each variable,

$$\psi = \psi_{SBR}(z, \overline{z}) = -\frac{\Omega z \overline{z}}{2},$$

where no additional terms due to irrotational strain have been added because the motion is pure solid body rotation. 2, B

(b) (i) We expect the two vortices to be at $z = a$ and $z = -b$ in a corotating frame in which the vortex pair is steady. The complex velocity field in this frame is sim. seen $\Downarrow$

$$u - \mathrm{i}v = -\frac{\mathrm{i}\Gamma}{2\pi}\frac{1}{z-a} - \frac{\mathrm{i}\Gamma}{\pi}\frac{1}{z+b} + \mathrm{i}\Omega\overline{z}, \tag{14}$$

where the relation

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial \psi}{\partial z} \tag{15}$$

has been used together with the result of part (a) and the expressions for the complex potential associated with the two point vortices, namely,

$$-\frac{\mathrm{i}\Gamma}{2\pi}\log(z-a) - \frac{\mathrm{i}\Gamma}{\pi}\log(z+b).$$

The streamfunction associated with the two point vortices is the imaginary part of this complex potential. The condition that the vortex at $z = a$ is steady is

$$-\frac{\mathrm{i}\Gamma}{\pi}\frac{1}{a+b} + \mathrm{i}\Omega a = 0$$

or

$$\Omega a = \frac{\Gamma}{\pi}\frac{1}{a+b} \tag{16}$$

The condition that the vortex at $z = -b$ is steady is

$$\frac{\mathrm{i}\Gamma}{2\pi}\frac{1}{a+b} - \mathrm{i}\Omega b = 0$$

or

$$\Omega b = \frac{\Gamma}{2\pi}\frac{1}{a+b} \tag{17}$$

A ratio of (16) and (17) gives

$$\frac{a}{b} = 2, \qquad b = \frac{a}{2}. \tag{18}$$

4, D

(ii) It follows from the calculations of part (i) that sim. seen $\Downarrow$

$$\Omega = \frac{2\Gamma}{3\pi a^2}. \tag{19}$$

4, D

(c) This result is familiar from lectures: $\boxed{\text{sim. seen} \Downarrow}$

$$\zeta = \cot(\theta/2)e^{\mathrm{i}\phi}.$$

It follows, using familiar double angle formulas, that

$$|\zeta|^2 = \cot^2(\theta/2) = \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} = \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} = \frac{\cos\theta + 1}{1 - \cos\theta} \tag{20}$$

so that

$$|\zeta|^2(1 - \cos\theta) = 1 + \cos\theta, \qquad \cos\theta = \frac{|\zeta|^2 - 1}{1 + |\zeta|^2}. \tag{21}$$

Hence

$$\sin^2\theta = 1 - \cos^2\theta = \frac{(1 + |\zeta|^2)^2 - (|\zeta|^2 - 1)^2}{(1 + |\zeta|^2)^2} = \frac{4|\zeta|^2}{(1 + |\zeta|^2)^2} \tag{22}$$

and therefore

$$\sin\theta = \frac{2\sqrt{\zeta\overline{\zeta}}}{1 + \zeta\overline{\zeta}}.$$

$\boxed{2, \text{C}}$

(d) From the hint, $\boxed{\text{sim. seen} \Downarrow}$

$$u_\phi - \mathrm{i}u_\theta = \frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}$$

so, for solid body rotation with angular velocity Ω, at any location with longitudinal angle θ.

$$u_\phi - \mathrm{i}u_\theta = \Omega\sin\theta = \frac{2\zeta}{\sin\theta}\frac{\partial\psi_{SBR}}{\partial\zeta}$$

or

$$\frac{\Omega\sin^2\theta}{2\zeta} = \frac{\partial\psi_{SBR}}{\partial\zeta}.$$

From part (c),

$$\frac{\partial\psi_{SBR}}{\partial\zeta} = \frac{2\Omega\overline{\zeta}}{(1 + \zeta\overline{\zeta})^2}.$$

Integration with respect to ζ yields

$$\psi_{SBR}(\zeta, \overline{\zeta}) = -\frac{2\Omega}{1 + \zeta\overline{\zeta}}$$

where no other function of integration has been added because the flow is in pure solid body rotation. $\boxed{2, \text{D}}$

(e) From the hint, the (complex) velocity in a corotating frame with angular velocity Ω about the $N - S$ axis is related to $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial\psi}{\partial\zeta} = -\frac{\Gamma}{4\pi}\left[\frac{1}{\zeta - a} - \frac{\overline{\zeta}}{1 + \zeta\overline{\zeta}}\right] - \frac{\Gamma}{2\pi}\left[\frac{1}{\zeta + b} - \frac{\overline{\zeta}}{1 + \zeta\overline{\zeta}}\right] - \frac{2\Omega\overline{\zeta}}{(1 + \zeta\overline{\zeta})^2}.$$

The condition for the vortex at a to be steady is

$$0 = -\frac{\Gamma}{2\pi}\left[\frac{1}{a + b} - \frac{a}{1 + a^2}\right] - \frac{2\Omega a}{(1 + a^2)^2}$$

or

$$\frac{2\Omega a}{(1 + a^2)^2} = -\frac{\Gamma}{2\pi}\left[\frac{1}{a + b} - \frac{a}{1 + a^2}\right] = -\frac{\Gamma}{2\pi}\left[\frac{1 - ab}{(a + b)(1 + a^2)}\right]$$

or, after simplification,

$$\frac{2\Omega a}{(1+a^2)} = -\frac{\Gamma}{2\pi}\left[\frac{1-ab}{(a+b)}\right]. \tag{23}$$

The condition for the vortex at $-b$ to be steady is

$$0 = -\frac{\Gamma}{4\pi}\left[\frac{1}{-b-a} + \frac{b}{1+b^2}\right] + \frac{2\Omega b}{(1+b^2)^2}$$

or

$$\frac{2\Omega b}{(1+b^2)^2} = \frac{\Gamma}{4\pi}\left[\frac{1}{-b-a} + \frac{b}{1+b^2}\right] = -\frac{\Gamma}{4\pi}\left[\frac{1-ab}{(a+b)(1+b^2)}\right].$$

or, after simplification,

$$\frac{2\Omega b}{(1+b^2)} = -\frac{\Gamma}{4\pi}\left[\frac{1-ab}{(a+b)}\right]. \tag{24}$$

A ratio of (23) and (24) leads to

$$\frac{a}{b}\left(\frac{1+b^2}{1+a^2}\right) = 2.$$

Hence

$$\frac{b+1/b}{a+1/a} = 2.$$

If $a = \cot(\theta_a/2)$ and $b = \cot(\theta_b/2)$ then this implies

$$\frac{\cot(\theta_b/2) + \tan(\theta_b/2)}{\cot(\theta_a/2) + \tan(\theta_a/2)} = 2. \tag{25}$$

Since

$$\cot(\theta_a/2) + \tan(\theta_a/2) = \frac{\cos(\theta_a/2)}{\sin(\theta_a/2)} + \frac{\sin(\theta_a/2)}{\cos(\theta_a/2)} = \frac{1}{\sin(\theta_a/2)\cos(\theta_a/2)} = \frac{2}{\sin(\theta_a)} \tag{26}$$

then (25) says that

$$\frac{\sin(\theta_a)}{\sin(\theta_b)} = 2. \tag{27}$$

[Note: a check, not required from the students, is that for small angles θ_a and θ_b the planar result in part (b) should be retrieved, which is the case since the distance a from the south pole S in a tangent plane though S is, to leading order, equal to θ_a given that the sphere has unit radius.] $\boxed{6, \mathrm{D}}$

5. (a) Given

sim. seen ⇓

$$z = R\left(\frac{\sqrt{\rho}-\zeta}{\sqrt{\rho}+\zeta}\right)$$

then

$$\frac{z}{R} = \left(\frac{1-\zeta/\sqrt{\rho}}{1+\zeta/\sqrt{\rho}}\right).$$

The right hand is recognized as the (self-inverse) Cayley mapping therefore

$$\frac{\zeta}{\sqrt{\rho}} = \frac{1-z/R}{1+z/R} = \frac{R-z}{R+z}, \qquad \text{or} \qquad \zeta = \sqrt{\rho}\left(\frac{R-z}{R+z}\right).$$

Therefore the circle

$$|\zeta|^2 = c^2$$

corresponds to

$$\rho\left(\frac{R-z}{R+z}\right)\left(\frac{R-\overline{z}}{R+\overline{z}}\right) = c^2$$

which can be rearranged to

$$|z|^2 - R\left(\frac{1+c^2/\rho}{1-c^2/\rho}\right)(z+\overline{z}) + R^2 = 0.$$

A circle centred at $\chi \in \mathbb{R}$ and of radius s in the z plane has equation

$$|z-\chi|^2 = s^2, \qquad \text{or} \qquad |z|^2 - \chi(z+\overline{z}) + \chi^2 - s^2 = 0.$$

Therefore, it is necessary that

$$R^2 = \chi^2 - s^2$$

and

$$\chi = R\left(\frac{1+c^2/\rho}{1-c^2/\rho}\right)$$

which rearranges to

$$\frac{c^2}{\rho} = \frac{\chi-R}{\chi+R}.$$

Therefore, setting $c = 1$ or $c = \rho$ gives

$$\rho = \frac{\chi-R}{\chi+R} = \frac{\chi-\sqrt{\chi^2-s^2}}{\chi+\sqrt{\chi^2-s^2}}.$$

4, M

(b) The complex potential $w(z)$ must be locally analytic everywhere outside the two circular obstacles except at infinity As $|z| \to \infty$, it is necessary that

sim. seen ⇓

$$w(z) \sim Uz + \text{constant} + \mathcal{O}(1/z) \tag{28}$$

in order to provide the uniform flow. Since there is no circulation around either obstacle then $w(z)$ must be single-valued around both obstacles. In order that the two obstacle boundaries are streamlines it is also necessary that

$$\text{Im}[w(z)] = \text{constant}, \qquad \text{on } |z \pm \chi| = s.$$

3, M

(c) Clearly,

sim. seen ⇓

$$P(\zeta^{-1},\rho)=(1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta)=\frac{(\zeta-1)}{\zeta}\frac{P(\zeta,\rho)}{1-\zeta}=-\zeta^{-1}P(\zeta,\rho).$$

Also

$$\begin{aligned}P(\rho^2\zeta,\rho)&=(1-\rho^2\zeta)\prod_{k=1}^{\infty}(1-\rho^{2(k+1)}\zeta)(1-\rho^{2(k-1)}\zeta^{-1})\\&=(1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta)\\&=-\frac{(1-\zeta)}{\zeta}\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta)\\&=-\zeta^{-1}P(\zeta,\rho).\end{aligned}$$

2, M

(d) (i) On taking a derivative of the first identity with respect to ζ we find

sim. seen ⇓

$$-\frac{1}{\zeta^2}P'(1/\zeta,\rho)=\frac{1}{\zeta^2}P(\zeta,\rho)-\frac{1}{\zeta}P'(\zeta,\rho).$$

On division by the original identity, and multiplication by ζ, we find

$$-\frac{(1/\zeta)P'(1/\zeta,\rho)}{P(1/\zeta,\rho)}=-1+\frac{\zeta P'(\zeta,\rho)}{P(\zeta,\rho)}$$

which implies

$$K(1/\zeta,\rho)=1-K(\zeta,\rho).$$

The two identities in part (a) imply

$$P(\rho^2\zeta,\rho)=P(1/\zeta,\rho).$$

On taking a derivative of this, we find

$$\rho^2P'(\rho^2\zeta,\rho)=-\frac{1}{\zeta^2}P'(1/\zeta,\rho).$$

On division by the original identity, and multiplication by ζ, we find

$$K(\rho^2\zeta,\rho)=-K(1/\zeta,\rho)$$

which implies, using the identity just derived, that

$$K(\rho^2\zeta,\rho)=K(\zeta,\rho)-1.$$

2, M

(ii) It follows on taking a logarithmic derivative of the infinite product representation of $P(\zeta)$ that

$$K(\zeta,\rho)=-\frac{\zeta}{1-\zeta}-\sum_{n=1}^{\infty}\left(\frac{\rho^{2n}\zeta}{1-\rho^{2n}\zeta}-\frac{\rho^{2n}/\zeta}{1-\rho^{2n}/\zeta}\right).$$

which shows that, as $\zeta\to1$,

$$K(\zeta,\rho)\sim\frac{1}{\zeta-1}+\text{a locally analytic function.}$$

$\boxed{1, \mathrm{M}}$

$\boxed{\text{sim. seen} \Downarrow}$

(e) The complex potential $W(\zeta)$ is analytic in the annulus except for a simple pole at $\zeta = -\sqrt{\rho}$, the preimage of $z = \infty$. This is because

$$z \sim \frac{2R\sqrt{\rho}}{\zeta + \sqrt{\rho}}$$

as $\zeta \to -\sqrt{\rho}$ which means $z \to \infty$ where it is required that

$$w(z) \sim Uz + \ldots.$$

Therefore,

$$W(\zeta) \sim \frac{2RU\sqrt{\rho}}{\zeta + \sqrt{\rho}}$$

as $\zeta \to -\sqrt{\rho}$. Since there is no circulation around either obstacle then $W(\zeta)$ must be single-valued in the annulus. In order that the two obstacle boundaries are streamlines it is also necessary that

$$\mathrm{Im}[W(\zeta)] = \text{constant}, \qquad \text{on } |\zeta| = \rho, 1.$$

$\boxed{3, \mathrm{M}}$

$\boxed{\text{unseen} \Downarrow}$

(f) It is clear that the given $W(\zeta)$ is single-valued in the annulus since the function $K(\zeta,\rho)$ is clearly so. The given $W(\zeta)$ also has a simple pole at $\zeta = -\sqrt{\rho}$, indeed, as $\zeta \to -\sqrt{\rho}$, and using the property of $K(\zeta,\rho)$ near $\zeta = 1$ established in part (d),

$$\begin{aligned} W(\zeta) = -2RU(K(-\zeta/\sqrt{\rho},\rho) - K(-\zeta\sqrt{\rho},\rho)) &\sim -2RU\left(\frac{1}{-\zeta/\sqrt{\rho} - 1}\right) + \ldots \\ &\sim \frac{2RU\sqrt{\rho}}{\zeta + \sqrt{\rho}} + \ldots \end{aligned}$$

as required. There are no other singularities in the annulus. It only remains to check that the imaginary part of $W(\zeta)$ is constant on $|\zeta| = \rho, 1$. On $|\zeta| = 1$, and using the functional properties of $K(\zeta,\rho)$ derived in part (d),

$$\begin{aligned} \overline{W(\zeta)} = W(1/\zeta) &= -2RU(K(-1/(\zeta\sqrt{\rho}),\rho) - K(-\sqrt{\rho}/\zeta,\rho)) \\ &= -2RU(1 - K(-\zeta\sqrt{\rho},\rho) - (1 - K(-\zeta/\sqrt{\rho},\rho)) = W(\zeta) \end{aligned}$$

confirming that its imaginary part is zero. Similarly, on $|\zeta| = \rho$,

$$\begin{aligned} \overline{W(\zeta)} = W(\rho^2/\zeta) &= -2RU(K(-\rho^2/(\zeta\sqrt{\rho}),\rho) - K(-\rho^2\sqrt{\rho}/\zeta,\rho)) \\ &= -2RU(K(-1/(\zeta\sqrt{\rho}),\rho) - 1 - (K(-\sqrt{\rho}/\zeta,\rho) - 1)) \\ &= -2RU(1 - K(-\zeta\sqrt{\rho},\rho) - (1 - K(-\zeta/\sqrt{\rho},\rho)) = W(\zeta) \end{aligned}$$

confirming that its imaginary part is zero.

$\boxed{5, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

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MATH70051 Vortex Dynamics Markers Comments

Question 1	This question was intended as a relatively easy "warm-up" question on basic principles. It was mostly answered well.
Question 2	This question was answered well by the students, with part (e) proving the most difficult.
Question 3	A well answered question on basic boundary value problems for two-dimensional stream functions with vorticity.
Question 4	One of the more difficult questions on the paper, but still answered quite well. There were quite a few algebraic slips, which were penalized only gently.
Question 5	This mastery question had elements in common with this year's coursework exercise, so it was answered better than might have been expected if the students had come to it blind, with no previous experience.