

LS

(1.3.1) Theorem. (Soundness of L)

Suppose ϕ is a theorem of L .

Then ϕ is a tautology.

(1.3.2) Notation

A (propositional) valuation v is an assignment of truth values to the prop. variables p_1, p_2, p_3, \dots

So $v(p_i) \in \{T, F\}$ (for $i \in \mathbb{N}$)

Using the truth table rules, this assigns a truth value $v(\phi) \in \{T, F\}$

to every L -formula ϕ , satisfying

$$v(\neg \phi) \neq v(\phi)$$

$$\text{and } v((\phi \rightarrow \psi)) = F$$

$$\Leftrightarrow v(\phi) = T \text{ \& } v(\psi) = F$$

(for all formulas ϕ, ψ)

Proof of 1.3.1: By induction on the length of a proof of ϕ it is enough to show that

(a) every axiom of L is a tautology

(b) MP preserves tautologies.

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(a) Use truth tables or argue as follows. Do A2.

Suppose for a contradiction that v is a valuation with (2)

$$v((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))) = F$$

then $v((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) = F$... (1)

$\& \quad v(\phi \rightarrow (\psi \rightarrow \chi)) = T$... (2)

By (1) $v(\phi \rightarrow \chi) = F$... (3)

$\& \quad v(\phi \rightarrow \psi) = T$... (4)

By (2) $v(\phi) = T \& \quad v(\chi) = F$

By this and (4) $v(\psi) = T$

this contradicts (2). #A1.
For A2.

Ex: Do A1 $\&$ A3. //

(b) If ϕ and $(\phi \rightarrow \psi)$ are tautologies then so is ψ .

If $v(\phi) = T \& \quad v(\phi \rightarrow \psi) = T \quad v(\psi) = T$. #W.
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(1.3.3) Thm (Generalisation of soundness)

Suppose Γ is a set of L-fmlas and ϕ is an L-fmla. Suppose

that $\Gamma \vdash_L \phi$.

then for every valuation v with $v(\Gamma) = T$

(i.e. $v(\psi) = T$ for all $\psi \in \Gamma$)

we have $v(\phi) = T$.

Pf: Almost same as 1.3.1. #

(1.3.4) Thm.

(Completeness/Adequacy theorem for L).

Suppose ϕ is a tautology. Then $\vdash_L \phi$.

Generalisation

(3)

Suppose that for every valuation v with $v(\Gamma) = T$ we have

$v(\phi) = T$. THEN $\Gamma \vdash_L \phi$.

Equivalently if $\Gamma \not\vdash_L \phi$ there is a valuation v with $v(\Gamma) = T$ and $v(\phi) = F$.

(1.3.6) Def ① A set Γ of L-fmlas is consistent if there is no L-fmla ψ such that $\Gamma \vdash_L \psi$ and $\Gamma \vdash_L (\neg \psi)$.

② Say Γ is complete if for every ψ $\Gamma \vdash_L \psi$ or $\Gamma \vdash_L (\neg \psi)$.

Remarks ① By 1.3.1 there
is not L -formula ϕ with
 $\vdash_L \phi$ and $\vdash_L (\neg \phi)$

(Say L is consistent) .
More generally if there is a val.
 v with $v(\Gamma) = T$ then
 Γ is consistent, by 1.3.3 .

② If v is a valuation and
 $\Gamma = \{ \phi : v(\phi) = T \}$
then Γ is consistent &
complete .

④

(1.3.7) Proposition Suppose Γ is a consistent set of L-formulas and $\Gamma \not\vdash_L \phi$. Then $\Gamma \cup \{(\neg\phi)\}$ is consistent.

Pf. Suppose not. So there is a formula ψ such that

$$\Gamma \cup \{(\neg\phi)\} \vdash \psi \dots \textcircled{1}$$

$$\& \Gamma \cup \{(\neg\phi)\} \vdash (\neg\psi) \dots \textcircled{2}$$

Apply DT to $\textcircled{2}$

$$\Gamma \vdash ((\neg\phi) \rightarrow (\neg\psi))$$

so by A3 + MP

$$\Gamma \vdash (\psi \rightarrow \phi) \dots \textcircled{3}$$

By $\textcircled{3}$, $\textcircled{1}$ & MP

obtain

$$\Gamma \cup \{(\neg\phi)\} \vdash \phi$$

By DT

$$\Gamma \vdash ((\neg\phi) \rightarrow \phi) \dots \textcircled{5}$$

$$1.2.7(c) \vdash_L (((\neg\phi) \rightarrow \phi) \rightarrow \phi)$$

This and $\textcircled{5}$ give

$$\Gamma \vdash \phi \text{ . Contradiction.}$$

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(1.3.8) Proposition (Lindenbaum Lemma)

⑥

Suppose Γ is a consistent set of L -formulas. Then there is a consistent set of L -formulas $\Gamma^* \supseteq \Gamma$ which is complete.

Pf. The set of L -formulas is countable, so we can list the L -formulas as

$\phi_0, \phi_1, \phi_2, \dots$

[Why countable? Alphabet

$\neg \rightarrow () P_1 P_2 \dots$

is countable. ~~Each~~ Formulas

are finite sequences of these.

the set of these is countable.]