LA rev 1

Basis change

$T : V \longrightarrow V$ /F field

$\dim V = n$ $n \in \mathbb{N}$

B, C bases of V

$[T]_C = {}_C[id]_B \, [T]_B \, {}_B[id]_C$

Example:

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(x_1, x_2) = \begin{pmatrix} 3x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

$E = \{ e_1 , e_2 \}$

$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$[T]_E = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

$[T]_B$

↳ Apply def

OR

↳ Use basis change

$[T]_B = {}_B[id]_E \, [T]_E \, {}_E[id]_B$

$${}_B[id]_E = \left({}_E[id]_B \right)^{-1}$$

$${}_E[id]_B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

↳ Apply def

OR

↳ invert ${}_E[id]_B$

$${}_B[id]_E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Apply b.c. formula

$$[T]_B = \begin{pmatrix} 5 & 1 \\ -13 & -4 \end{pmatrix}$$

LA rev 2

With def

$$T(1,2) = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -3 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 13 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Matrices:

$M \in M_n(F)$ F field, $n \in \mathbb{N}$

$$M = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

M is upper-triang.

Eq.

$M e_j \in \operatorname{Span}(e_1, \dots, e_j)$

$\forall j = 1, \dots, n$

(j-th col

$$\begin{matrix} * \\ \vdots \\ * & \leftarrow j \\ 0 \\ \vdots \\ 0 \end{matrix}$$

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in the sol:

$u^T A v = u^T A^T v \quad \forall u, v \in F^n$

$\Rightarrow A = A^T$

Id. principle

$A, B \in M_n(F)$ $n \in \mathbb{N}$, F field

$\{u_1, \dots, u_n\}$ basis of F^n

$A = B \iff A u_i = B u_i \quad \forall i = 1, \dots, n$

$A \in M_n(F)$ $A = (a_{ij})$

$e_i^T A e_j = a_{ij}$

if $B \in M_n(F)$ and

$e_i^T B e_j = e_i^T A e_j \Rightarrow A = B$.

LA rev 3

Groups

Euler's totient

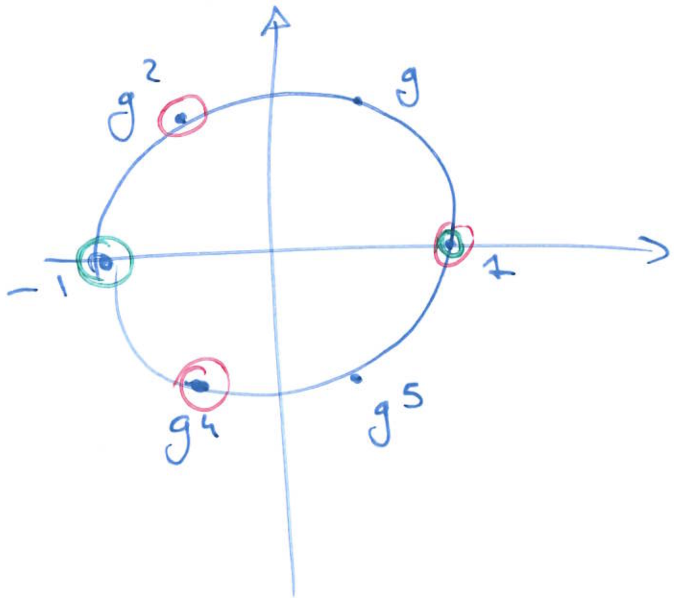
$n \in \mathbb{N}$

$\phi(n) = \#\{m \in \mathbb{N} \ m \leq n \mid \gcd(m,n) = 1\}$

$\phi(6) = 2 = (2-1)\cdot(3-1)$

Example:

$g := e^{2\pi i/6}$ $\quad G := \langle g \rangle \leq \mathbb{C}^{\times}$



How many g^i are there s.t. $G = \langle g^i \rangle$?

Thm 1.22 (notes)

$\langle g^5 \rangle = G$ because $\gcd(5, 6) = 1$

$\phi(6) = 2$

For a cyclic group $G = \langle g \rangle$ $\quad |G| = n$

there are $\phi(n)$ integers m $1 \leq m \leq n$ s.t. $\langle g^m \rangle = G$.

LA rev 4

Coset:

$H \leq G \qquad g \in G$

Def

$gH := \{gh \mid h \in H\}$

left coset

Def: G finite

$|G : H| = \#$ distinct left cosets of H in G

Th: (Lagrange) G finite

$|G| = |G : H| \, |H|$

Example:

$G = S_3 := \mathrm{Sym}([3])$

$\alpha = (1\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$H = \langle \alpha \rangle = \{\alpha^0, \alpha^1, \dots\}$

$|H| = 2$ because

$\alpha^2 = id$

List the cosets of H in G.

- id

  $id\,H = H$

- Choose $\beta \in G \setminus H$

  e.g. $\beta = (1\ 3)$

  (this is not in H because $H = \{id, \alpha\}$)

$\beta H = \{\beta, \beta\alpha\}$

$= \{(1\ 3), \underbrace{(1\ 3)(1\ 2)}_{=(1\ 2\ 3)}\}$

- Choose $\gamma \notin H$

  $\gamma \notin \beta H$

e.g.

$$\gamma = (2\,3)$$

$$\gamma H = \{\gamma, \gamma\alpha\}$$
$$= \{(2,3), \underbrace{(2,3)(12)}_{(1\,3\,2)}\}$$

Stop because H has $3 = 6/2$ cosets by Lagrange.

<u>Dihedral group</u>

$n \in \mathbb{N}$ $\qquad D_{2n}$

- subgroup of S_n containing the symmetries of the regular n-gon.

- $D_{2n} = \langle a, b \rangle$

$$a = (1)\,(2, n)\,(3, n-1) \cdots$$
$$b = (1, n)\,(2, n-1) \cdots$$

- $ab = (1, 2, 3, \ldots, n)$.

LA

- Compute determinants
  - Laplace
  - Row reduction
  - Combination of the two
- Cramer's rule

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \det(A) = -2$$

Cofactors

$$C = \begin{pmatrix} 4 & -3 \\ -2 & +1 \end{pmatrix} \qquad C^T = \begin{pmatrix} 4 & -2 \\ -3 & +1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

- Eigenvalues etc.
  - Find the eigenspaces of $T: V \to V$ linear
  - Solving linear recurrences (e.g. Fibonacci)
- Orthogonal vectors in $\mathbb{R}^n$
  - Extract orthogonal bases with Gram-Schmidt
  - Spectral theorem
    Every <u>real</u> symmetric matrix is diagonalisable by a orthogonal transform

Group Theory

- Definition of a group
  Set with a binary operation
  - associativity
  - identity
  - inverses..

- Examples of groups
  - Finite cyclic groups
    $(\mathbb{Z}_n, +)$ $n \in \mathbb{N} \setminus \{0\}$
  - Infinite cyclic group(s)
    $(\mathbb{Z}, +)$
  - Symmetric group $Sym(X)$
    S_n
    bijections $\{1, \dots, n\}$

  - Invertible matrices
    $GL_n(F) = \{A \in M_n(F) \mid \det(A) \neq 0\}$
    ↳ Multiplicative group of a field F
    $F^\times$
    (if F finite $\Rightarrow$ $F^\times$ is cyclic ...)

- Subgroups
  Symmetric S_n
  - A_n subgroup of even perm.
  - V Klein four-group in S_4
  - D_{2n} dihedral group. (generators:
    $a = (1)(2, n)(3, n-1) \dots$
    $b = (1, n)(2, n-1) \dots$

  $GL_n(F)$
  - $SL_n(F)$
  - $F = \mathbb{R}$ $O_n(\mathbb{R})$

- Cosets
  - Lagrange's theorem
- Homomorphisms of groups
  - Kernel (inj. <=> kernel trivial)
  - Image (is a sbg of the target)
- ~~Symme~~ S_n
  - dec. in disjoint cycles.