Imperial College London

Department of Mathematics

M3B Mathematics of Business

Question	Examiner's Comments
Q 1	– Overall result: mean 11.3, standard deviation 4.3. – In question c)(ii) it should rather be "intercept" than "intersection". This was announced orally during the exam. – When you make a case distinction, please beware that the converse of $x_1 < x_2$ is $x_1 \geq x_2$, rather than $x_1>2$.
Q 2	– Overall result: mean 7.7, standard deviation 4.9. – For c)(iii) Beware that utilities are usually not injective. That is, $u(x) = u(y)$ implies that $x \sim y$, but not necessarily $x=y$. – Moreover, that a utility is increasing means that $x \leq y$ implies $u(x) \leq u(y)$. However, $u(x)\leq u(y)$ does not necessarily imply that $x\leq y$ (you are in a higher dimensional space!) – In (a) the point was to prove that $u(x) \leq u(y)$ iff x is less preferred than y. – When you want to prove an equivalence, please make sure that you prove both implications. – When you differentiate a function like in (b)(ii) please don't supress the arguments of the function. The argument is quite important here! – A crucial point for the proof of the inequality at (2) is that you show that the two integrands coincide at the left endpoint of the integration region. Then you can derive by the Slutsky Equation that the integrand on the right hand side grows faster than the one on the left hand side. This means the the integrand on the left hand side will be smaller than the one on the right hand side, such that one obtains the respective result on the level of the integrals.
Q 3	– Overall result: mean 13.5, standard deviation 4.7. – In part (h), neither the supply curve nor the demand curve change. That means the market price and quantity will not be a point were the to curves intersect. That is, because we are not in an equilibrium any more. Indeed, this means that there is a whole range of possible market prices. One can only determine the lower and upper bound of the price and the actual outcome will depend on the negotiation power of either side.
Q 4	– Overall result: mean 9.5, standard deviation 3.9. – For (a)(iii) some of you mentioned a factory that produces pipelines. However, some answers implied that the radius of the pipelines is an input factor. But that's a very abstract input factor. It should rather be the material that is used. – In c) it is not advantagous if you give more than one result and the results are mutually exclusive (something like "GDP increases because of A and GDP decreases because of B") – In part c)(iv) one short-coming only. You couldn't earn more marks with mentioning more short-comings. However, you were supposed to explain that short-coming, not only claim it. Finally, you were asked how to solve the short-coming of GDP as a measure of the county's welfare. You were not asked how to solve a country's welfare problems, that's something else (so answers that suggested to incentivise enviromental behaviour missed the point). Moreover, some of you mentioned that healthcare and education were ignored by GDP. That's not true, only some aspects are ignored that are done without a payment.

Imperial College
London

M3B

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematics of Business & Economics

Date: Tuesday, 29 May 2018

Time: 2:00 PM - 4:00 PM

Time Allowed: 2 hours

This paper has 4 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. Consider the firm Leo having two non-negative input goods and one output good. Suppose that the technology of the firm is a Leontieff technology. That is, the production function is of the form

$$f\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}, \qquad f(x_1, x_2) = \min\{x_1, x_2\}.$$

(a) Sketch the input requirement set of Leo. That is, sketch the set $f^{-1}([y, \infty))$ for some $y \geq 0$. Also indicate the isoquant $f^{-1}(\{y\})$ in the graph.

(b) (i) Suppose that Leo produces with some arbitrary input bundle $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$. Now suppose that the input of good 1 changes from x_1 to some $x_1' = x_1 + \epsilon$ where $\epsilon > 0$. How does Leo need to adjust the input of good 2 in order to keep the output constant? Consider explicitly the cases $x_1 < x_2$ and $x_1 \geq x_2$.

(ii) What do we mean by the marginal rate of technical substitution (MRTS)?

Use part (b)(i) to calculate the MRTS for Leo at some arbitrary point $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$, explicitly considering the cases $x_1 < x_2$ and $x_1 \geq x_2$. Describe possible problems. Relate your solution to the isoquant in (a).

Suppose the two input goods have prices $w_1 > 0$ and $w_2 > 0$. A necessary first-order condition for the cost minimisation problem for some fixed output $y > 0$ is given by

$$\mathrm{MRTS}(x_1^*, x_2^*) = -\frac{w_1}{w_2} \tag{1}$$

for the cost minimising input bundle $\underline{x}^* = (x_1^*, x_2^*)$ if the production function is differentiable. Graphically, the general situation can be illustrated as in Figure 1 for a firm with increasing, continuous and quasi-concave production function f.

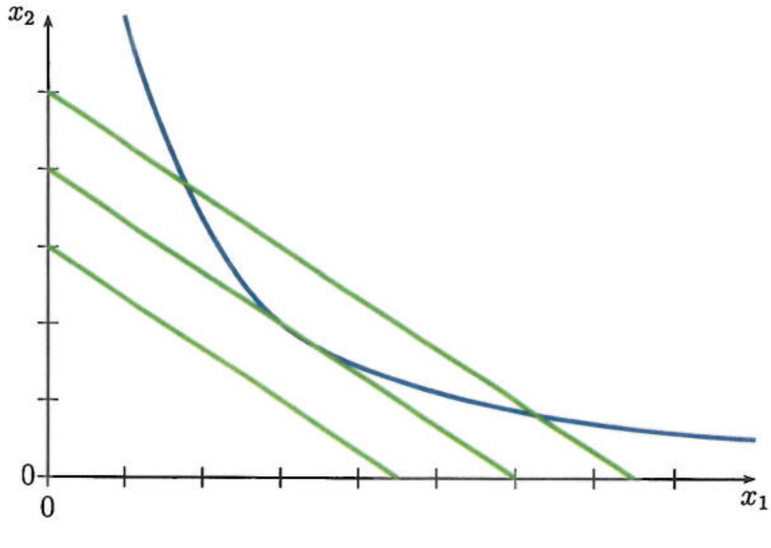

Figure 1: Several lines with slope $-w_1/w_2$ and different intersections. The curve is the isoquant $f^{-1}(\{y\})$.

[CONTINUED]

(c) Use Figure 1 to answer the following questions:

(i) Why is the cost minimising consumption bundle $\underline{x}^*$ necessarily on the isoquant $f^{-1}(\{y\})$?

(ii) What does a line with slope $-w_1/w_2$ and intersection $K \geq 0$ represent in this context economically?

(iii) What does the condition given by Equation (1) mean graphically?

(iv) Depict the cost minimising utility bundle $\underline{x}^* = (x_1^*, x_2^*)$ in Figure 1.

(v) Suppose that the price for good 1 increases to $w_1' > w_1$ whereas the price for good 2 remains constant. Determine the new cost minimising input bundle graphically using Figure 1. (The graph needs only to be indicative, rather than exact.)

(d) Now consider again Leo's production function. Determine the cost minimising input bundle for Leo graphically, similar to the graphical approach shown in Figure 1. To this end, consider the following situations.

(i) $y = 2$ and $w_1 = w_2 = 1$.

(ii) $y = 2$, $w_1 = 2$ and $w_2 = 1$.

Use the same graph for the two situations. Make sure that you label the axis correctly to allow for an exact/quantitative solution, rather than only an indicative one.

2. (a) Consider a finite consumption set $X = \{x_1, \dots, x_n\}$ with a complete and transitive preference relation $\preceq$. Define a utility function

$$u\colon X \to \mathbb{R}, \qquad u(x) = \#\{y \in X \colon y \preceq x\}.$$

That is, $u(x)$ gives the number of elements in X that are less preferred than x.
Show that u represents the preference relation $\preceq$.

(b) Let $X = \mathbb{R}^2_{\geq 0}$ and let $u\colon X \to \mathbb{R}$ be a utility function. Let $\preceq$ be a preference relation on X induced by u. That is

$$x \preceq x' \iff u(x) \leq u(x')$$

for any $x, x' \in X$. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $\phi' > 0$.

(i) Show that u and $\phi \circ u\colon X \to \mathbb{R}$ induce the same preference relation.

(ii) Assume that u is differentiable. Show that u and $\phi \circ u$ have the same marginal rate of substitution.

(c) Assume that Joseph has utility $u\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ which is strictly increasing, continuously differentiable, and strictly quasi-concave. That is, for any $\underline{x}, \underline{y} \in \mathbb{R}^2_{\geq 0}$ and for any $t \in (0,1)$

$$u((1-t)\underline{x} + t\underline{y}) > \min\{u(\underline{x}), u(\underline{y})\}.$$

Joseph has a budget of $m > 0$, and the two goods have strictly positive prices $p_1, p_2 > 0$.

(i) Show that there exists a consumption bundle $\underline{x}^*(\underline{p}, m) \in \mathbb{R}^2_{\geq 0}$, depending on the prices $\underline{p} = (p_1, p_2)$ and the budget m, which Joseph can afford and which maximises his utility.

(ii) Show that any utility maximising and affordable bundle $\underline{x}^*(\underline{p}, m) = \big(x_1^*(\underline{p}, m), x_2^*(\underline{p}, m)\big)$ satisfies

$$p_1 x_1^*(\underline{p}, m) + p_2 x_2^*(\underline{p}, m) = m.$$

Give an interpretation for this fact.

(iii) Show that the utility maximising affordable consumption bundle $\underline{x}^*(\underline{p}, m)$ is unique.

[CONTINUED]

In parts c(i) to c(iii) you have shown the existence and uniqueness of Joseph's Marshallian demand function $\underline{x}^* = (x_1^*, x_2^*)$. Furthermore, let $\underline{x}_H^* = (x_{H,1}^*, x_{H,2}^*)$ be his Hicksian demand function, v his indirect utility function, and e his expenditure function.

(iv) Consider an increase in price for good 1. That is, consider the new price vector $\underline{q} = (q_1, q_2) \in \mathbb{R}^2_{>0}$ where $q_1 > p_1$ and $q_2 = p_2$. Of course, Joseph is not happy about this fact.

- Explain why
$$e\big(\underline{q}, v(\underline{p}, m)\big) - m$$
is the extra that Joseph has to pay to arrive at the initial level of utility.
- Show that
$$\begin{aligned} e\big(\underline{q}, v(\underline{p}, m)\big) - m &= e\big(\underline{q}, v(\underline{p}, m)\big) - e\big(\underline{p}, v(\underline{p}, m)\big) \\ &= \int_{p_1}^{q_1} x_{H,1}^*\big((z, p_2), v(\underline{p}, m)\big)\,\mathrm{d}z. \end{aligned}$$
- Assume that good 1 is a normal good. Using Slutsky's equation, show that
$$\int_{p_1}^{q_1} x_1^*\big((z, p_2), m\big)\,\mathrm{d}z \le \int_{p_1}^{q_1} x_{H,1}^*\big((z, p_2), v(\underline{p}, m)\big)\,\mathrm{d}z. \qquad (2)$$
- Why do many economists still prefer to work with the left-hand side of Equation (2)?

3. Consider a consumer with a utility function given by $u\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$

$$u(x_1, x_2) = x_1^a x_2^b$$

for constants $a, b > 0$.

(a) Determine the Marshallian demand for goods 1 and 2 with prices $p_1, p_2 > 0$ and budget $m \geq 0$.

(b) For $a = b = 1$ the result in (a) for good 1 should be $x_1^*(\underline{p}, m) = x_1^*(p_1, m_1) = \frac{m}{2p_1}$. Suppose there is a market for good 1 consisting of I individuals whose preferences can be represented with the same utility function $u(x_1, x_2) = x_1 x_2$. Let $m_i \geq 0$, $i \in \{1, \ldots, I\}$, be the budget of each individual.
Determine the market demand for good 1.

(c) Consider J firms producing good 1 in a perfectly competitive market (meaning that they are price takers). Since they all use the same production function, their individual cost functions coincide. Assume the cost is given by

$$c(y) = y^2,$$

where y is the output.

(i) Compute each firm's profit maximising output y^* as a function in the price p.

(ii) Compute the industry supply for good 1.

(d) Compute the equilibrium price p^* and the equilibrium quantity Q^*.

(e) Sketch a graph of the market demand and industry supply (only qualitatively). Use the conventions that the horizontal axis indicates the quantity and the vertical axis indicates the price. Depict the equilibrium price p^* and the equilibrium quantity Q^*.

(f) Illustrate the producers' and the consumers' surplus in the same graph.

(g) Give the mathematical definition of the producers' and the consumers' surplus and compute them, if possible.

(h) A new study shows that the consumption of good 1 can do harm to the health of the consumers. Therefore, the government decides to introduce a maximal quantity Q_0. Suppose that $Q_0 = Q^*/2$.

* Illustrate the new situation (sketch a new graph) and depict the deadweight loss.
* Discuss where the market price will be in this scenario? (You can also use the graph for illustrative purposes.)

4. Throughout this exercise you may assume that all underlying production functions and utility functions are strictly increasing, quasi-concave and as smooth as you need them. Moreover, all prices are strictly positive.

(a) Give real-world examples for the following situations.

(i) A firm with a Leontieff production function (two inputs is sufficient).

(ii) A firm with a linear production function (two inputs is sufficient).

(iii) A firm with a production function that exhibits increasing returns to scale.

(b) For each statement (i) to (iii), say whether it is true or false? Justify your answer by providing thorough economical explanation or mathematical proof or counter-example (as indicated). You can assume that firms are in a competitive market, that is, they are price takers for output and supply. Moreover, assume that firms produce one product only.

(i) The expenditure function, for fixed level of utility, is positively homogeneous of degree 1 in the prices. (Proof or counter-example, and explanation)

(ii) Any profit maximising input bundle is also a cost minimising input bundle. (Proof or counter-example)

(iii) Any cost minimising input bundle is also a profit maximising input bundle. (Proof or counter-example)

(c) (i) State the definition of the 'Gross Domestic Product (GDP)'.

(ii) What is the effect of a deflation (negative inflation) on GDP? You can focus on the primary / short-run effect.

(iii) What is the effect on GDP if all alien residents obtain domestic citizenship? You can focus on the primary / short-run effect.

(iv) Mention and explain one of the short-comings of GDP as a measure of a country's overall welfare. Explain how one could solve this short-coming.

TEMPORARY FRONT PAGE

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematics for Business and Economics – **Solution (v2)**

Date: Tuesday, 29th May 2018
Time: 14:00 – 16:00
Time Allowed: 2 Hours
This paper has *4 Questions.*
Candidates should start their solutions to each question in a new main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. Consider the firm Leo having two non-negative input goods and one output good. Suppose that the technology of the firm is a Leontieff technology. That is, the production function is of the form

$$f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}, \qquad f(x_1, x_2) = \min\{x_1, x_2\}.$$

(a) Sketch the input requirement set of Leo. That is, sketch the set $f^{-1}([y, \infty))$ for some $y \geq 0$. Also indicate the isoquant $f^{-1}(\{y\})$ in the graph.

2 marks seen routine

Solution: See Figure 1.

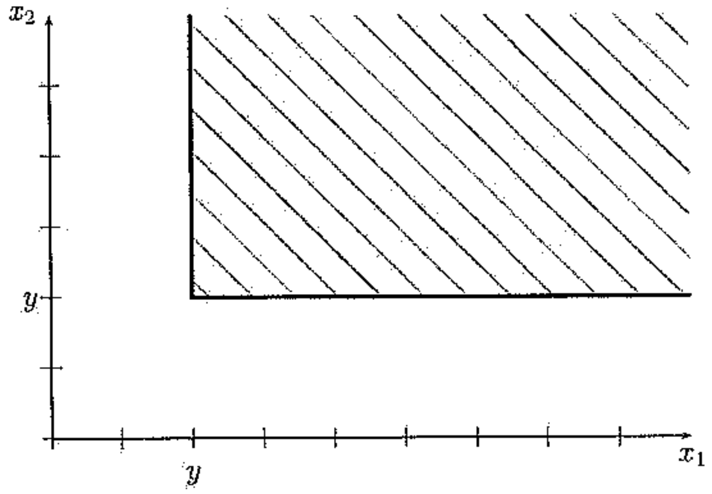

Figure 1: The solid blue line is the isoquant $f^{-1}(\{y\})$. The dashed area in light blue corresponds to the input requirement set $f^{-1}([y, \infty))$.

(b) (i) Suppose that Leo produces with some arbitrary input bundle $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$. Now suppose that the input of good 1 changes from x_1 to some $x_1' = x_1 + \epsilon$ where $\epsilon > 0$. How does Leo need to adjust the input of good 2 in order to keep the output constant? Consider explicitly the cases $x_1 < x_2$ and $x_1 \geq x_2$.

2 marks not seen medium

Solution: If $x_2 \leq x_1$, we have that $f(x_1, x_2) = f(x_1', x_2)$. That means the output does not change.

If $x_1 < x_2$, we have that $f(x_1, x_2) = x_1$ and the output increases to $\min\{x_1', x_2\}$. That means we need to reduce x_2 to the level of x_1 to ensure that output remains the same. So $x_2' = x_1$. Note that this change in the input of good 2 is independent of the size of ϵ, but only depends on the difference between x_1 and x_2.

(ii) What do we mean by the marginal rate of technical substitution (MRTS)?

Use part (b)(i) to calculate the MRTS for Leo at some arbitrary point $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$, explicitly considering the cases $x_1 < x_2$ and $x_1 \geq x_2$. Describe possible problems. Relate your solution to the isoquant in (a).

4 marks not seen mixed (2 routine, 2 upper medium)

Solution: The MRTS at point $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$ is the relative marginal change of input of good 2 in reaction to a marginal change in good 1 in order to keep the initial output constant.

[1 mark, routine]

If they give the formula for the MRTS involving partial derivatives, they also get 1 mark. However, they will struggle solving the concrete question for the Leontieff technology.

We confine ourselves to the situation in (b)(i), that is, to an increase in x_1. So we divide the reaction in x_2 by ϵ and consider the ratio for $\epsilon \to 0$. We obtain

$$\mathrm{MRTS}(x_1, x_2) = \begin{cases} 0, & \text{for } x_2 \leq x_1 \\ -\infty, & \text{for } x_1 < x_2. \end{cases}$$

Here, the minus sign is due to the fact that we have to *reduce* the input of good 2 in order to keep the output constant. Since the absolute reaction of x_2 is independent of ϵ, we end up with $-\infty$ in the limit of the ratio when ϵ tends to 0.

[2 marks, hardest]

Indeed, we can see that the MRTS describes the 'slope' of the isoquant in (a)(i) where we consider the isoquant as a function in x_1.

[1 mark, routine]

Suppose the two input goods have prices $w_1 > 0$ and $w_2 > 0$. A necessary first-order condition for the cost minimisation problem for some fixed output $y > 0$ is given by

$$\mathrm{MRTS}(x_1^*, x_2^*) = -\frac{w_1}{w_2} \tag{1}$$

for the cost minimising input bundle $\underline{x}^* = (x_1^*, x_2^*)$ if the production function is differentiable. Graphically, the general situation can be illustrated as in Figure 2 for a firm with increasing, continuous and quasi-concave production function f.

(c) Use Figure 2 to answer the following questions:

(i) Why is the cost minimising consumption bundle $\underline{x}^*$ necessarily on the isoquant $f^{-1}(\{y\})$?

2 marks similar seen upper medium/hardest

Solution: The firm needs to produce at least y. Thus, necessarily $\underline{x}^* \in f^{-1}([y, \infty))$. Assume that $\underline{x}^* \in f^{-1}((y, \infty))$. Since (y, ∞) is open and f is continuous, also $f^{-1}((y, \infty))$ is open. But that means that if $\underline{x}^* \in f^{-1}((y, \infty))$ then there is another $\underline{x}' \in f^{-1}((y, \infty))$ with $\underline{x}' < \underline{x}$ (componentwise). Since the prices are strictly positive, this implies that $\underline{x}'$ has lower costs than $\underline{x}^*$, which leads to a contradiction. Hence $\underline{x}^* \in f^{-1}(\{y\})$.

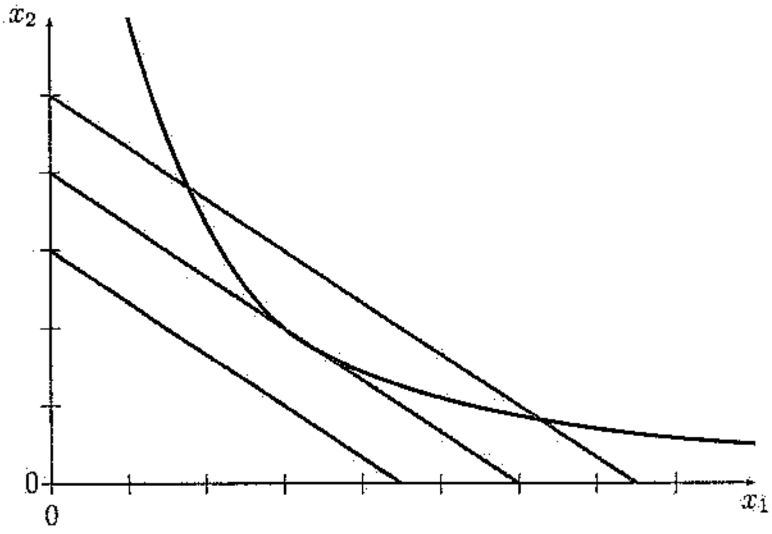

Figure 2: Several lines with slope $-w_1/w_2$ and different intersections. The curve is the isoquant $f^{-1}(\{y\})$.

(ii) What does a line with slope $-w_1/w_2$ and intersection $K \geq 0$ represent in this context economically?

2 marks similar seen upper medium

Solution: For positive prices, a line with slope $-w_1/w_2$ and intersection $K \geq 0$ in the $x_1 - x_2$–plane can be represented as the set of input bundles (x_1, x_2) satisfying

$$x_2 = K - \frac{w_1}{w_2} x_1.$$

This is equivalent to

$$w_1 x_1 + w_2 x_2 = c,$$

where $c = Kw_2$. That means a line with slope $-w_1/w_2$ and intersection $K \geq 0$ represents possible input bundles with the same total cost $c = Kw_2$ at prices w_1 and w_2. (If you wish you could call them 'isocost lines'. – *There is no mark for mentioning this expression.*)

(iii) What does the condition given by Equation (1) mean graphically?

2 marks similar seen medium

Solution: It means we have costs such that the associated isocost line is tangential to the isoquant. Indeed, with lower costs (lower x_2-axis intersection) the isocost line does not contain a bundle that can produce y.

(iv) Depict the cost minimising utility bundle $\underline{x}^* = (x_1^*, x_2^*)$ in Figure 2.

1 marks unseen routine

Solution: See Figure 3

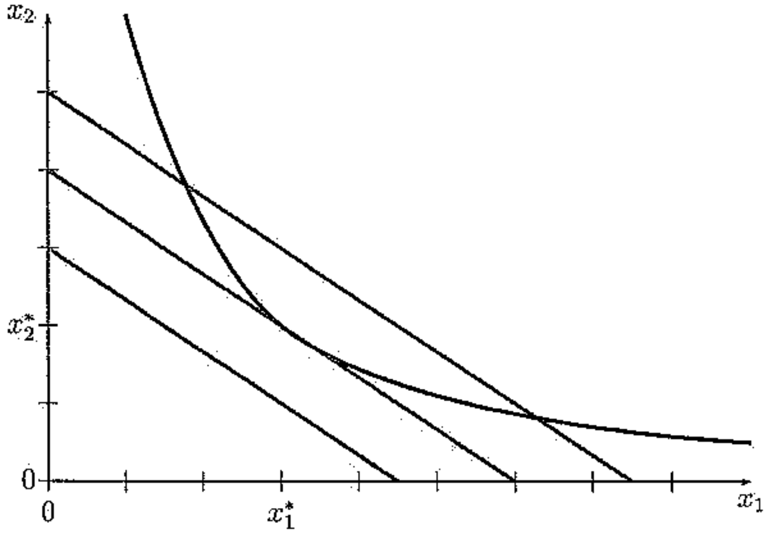

Figure 3: Solution to problem 1(c)(iv).

(v) Suppose that the price for good 1 increases to $w_1' > w_1$ whereas the price for good 2 remains constant. Determine the new cost minimising input bundle graphically using Figure 2. (The graph needs only to be indicative, rather than exact.)

2 marks unseen medium

Solution: See Figure 4. The new isocost line is 'steeper' and the new cost minimising bundle $(x_1'^*, x_2'^*)$ is such that $x_1'^* < x_1^*$ and $x_2'^* > x_2^*$.

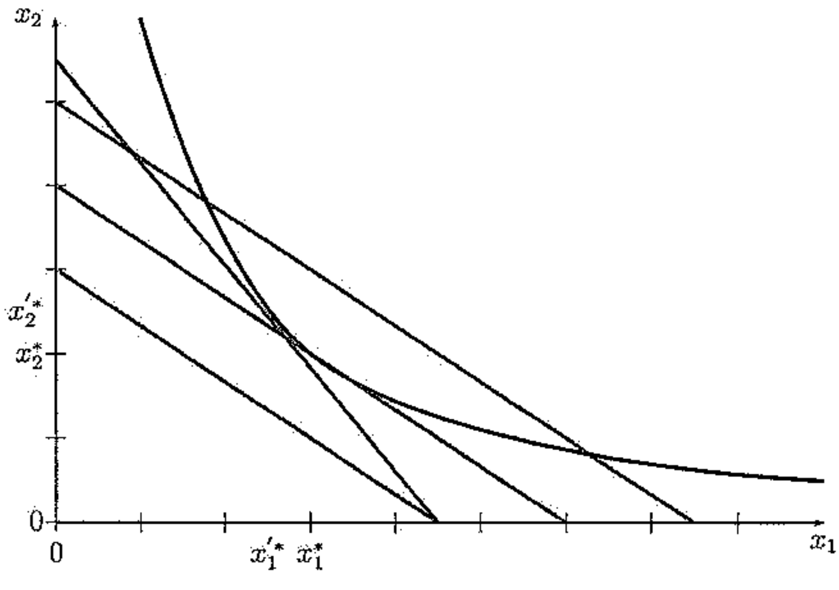

Figure 4: The red line is the new isocost line [*only indicative picture*].

(d) Now consider again Leo's production function. Determine the cost minimising input bundle for Leo graphically, similar to the graphical approach shown in Figure 2. To this end, consider the following situations.

(i) $y = 2$ and $w_1 = w_2 = 1$.

(ii) $y = 2$, $w_1 = 2$ and $w_2 = 1$.

Use the same graph for the two situations. Make sure that you label the axis correctly to allow for an exact / quantitative solution, rather than only an indicative one.

3 marks unseen 1 medium, 2 upper medium

Solution:

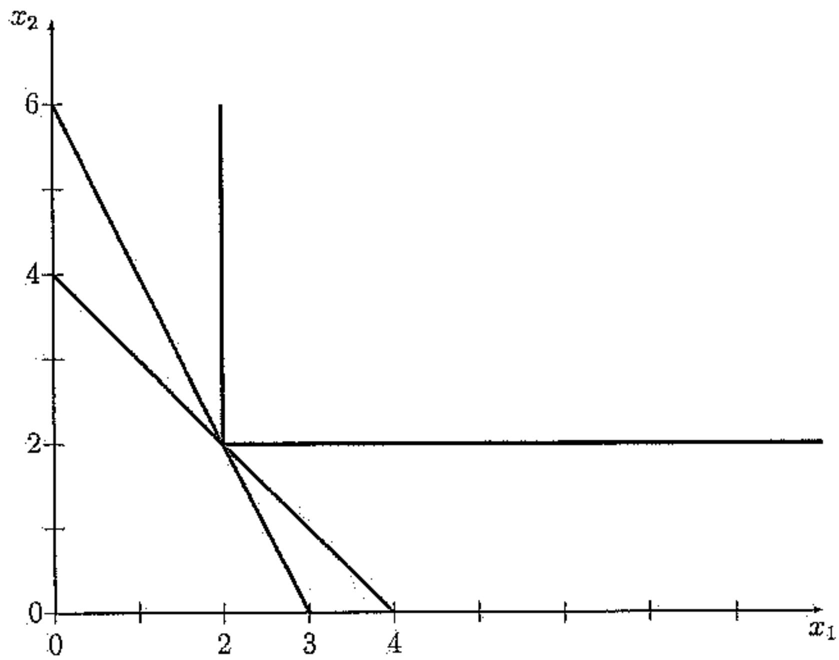

Figure 5: Solution to 1(d).

See Figure 5. The blue curve is the isoquant $f^{-1}(\{2\})$. The green line represents the situation for prices $\underline{w} = (1, 1)$. The red line shows the situation for prices $\underline{w} = (2, 1)$. In both situations, the cost minimising input bundle is $\underline{x}^* = (y, y) = (2, 2)$.

[1 pt for qualitatively correct graph, 1 point for correct labels, 1 point for correct solution for $\underline{x}^*$.]

2. (a) Consider a finite consumption set $X = \{x_1, \dots, x_n\}$ with a complete and transitive preference relation $\preceq$. Define a utility function

$$u: X \to \mathbb{R}, \qquad u(x) = \#\{y \in X : y \preceq x\}. \tag{2}$$

That is, $u(x)$ gives the number of elements in X that are less preferred than x.
Show that u represents the preference relation $\preceq$.

3 marks similar seen routine

Solution: We have to show that for any $x, x' \in X$

$$u(x) \leq u(x') \iff x \preceq x'.$$

"$\Longleftarrow$": Assume that $x \preceq x'$ for some $x, x' \in X$. Invoking completeness and transitivity of $\preceq$ we obtain that $\{y \in X : y \preceq x\} \subseteq \{y \in X : y \preceq x'\}$. So $u(x) \leq u(x')$.

"$\Longrightarrow$": Assume that $x' \prec x$ for some $x, x' \in X$. Again, with completeness and transitivity we obtain that $\{y \in X : y \preceq x'\} \subseteq \{y \in X : y \preceq x\}$. However, since x is strictly preferred over x', $x \notin \{y \in X : y \preceq x'\}$. That means

$$u(x') \leq u(x) - 1.$$

(b) Let $X = \mathbb{R}^2_{\geq 0}$ and let $u: X \to \mathbb{R}$ be a utility function. Let $\preceq$ be a preference relation on X induced by u. That is

$$x \preceq x' \iff u(x) \leq u(x')$$

for any $x, x' \in X$. Let $\phi: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $\phi' > 0$.

(i) Show that u and $\phi \circ u: X \to \mathbb{R}$ induce the same preference relation.

1 mark seen routine

Solution: The fact that $\phi' > 0$ implies that ϕ is strictly increasing. Then we obtain for all $x, x' \in X$.

$$x \preceq x' \iff u(x) \leq u(x') \iff \phi(u(x)) \leq \phi(u(x')).$$

(ii) Assume that u is differentiable. Show that u and $\phi \circ u$ have the same marginal rate of substitution.

2 marks unseen routine

Solution: This follows with the chain rule. The marginal rate of substitution of $\phi \circ u$ at point x is

$$-\frac{\frac{\partial \phi(u(x))}{\partial x_1}}{\frac{\partial \phi(u(x))}{\partial x_2}} = -\frac{\phi'(u(x))\frac{\partial u(x)}{\partial x_1}}{\phi'(u(x))\frac{\partial u(x)}{\partial x_2}} = -\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}}$$

which coincides with the marginal rate of substitution of u at x.

(c) Assume that Joseph has utility $u\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ which is strictly increasing, continuously differentiable, and strictly quasi-concave. That is, for any $\underline{x}, \underline{y} \in \mathbb{R}^2_{\geq 0}$ and for any $t \in (0,1)$

$$u((1-t)\underline{x} + t\underline{y}) > \min\{u(\underline{x}), u(\underline{y})\}.$$

Joseph has a budget of $m > 0$, and the two goods have strictly positive prices $p_1, p_2 > 0$.

(i) Show that there exists a consumption bundle $\underline{x}^*(\underline{p}, m) \in \mathbb{R}^2_{\geq 0}$, depending on the prices $\underline{p} = (p_1, p_2)$ and the budget m, which Joseph can afford and which maximises his utility.

2 marks seen in the lecture upper medium

Solution: Joseph's budget set is $B_{\underline{p},m} = \{x \in \mathbb{R}^2_{\geq 0} \colon p_1x_1 + p_2x_2 \leq m\}$. So $B_{\underline{p},m}$ contains the consumption bundles Joseph can afford. Since $B_{\underline{p},m}$ is an intersection of half-spaces, it is closed. Since prices are strictly positive, it is bounded. Hence, $B_{\underline{p},m}$ is compact. u is continuous and we know that any continuous function attains a maximum over a compact set.

(ii) Show that any utility maximising and affordable bundle $\underline{x}^*(\underline{p}, m) = \big(x_1^*(\underline{p}, m), x_2^*(\underline{p}, m)\big)$ satisfies

$$p_1x_1^*(\underline{p}, m) + p_2x_2^*(\underline{p}, m) = m. \tag{3}$$

Give an interpretation for this fact.

2 marks seen in the lecture medium

Solution: Since the utility maximising bundle needs to be affordable, we definitely have $p_1x_1^*(\underline{p}, m) + p_2x_2^*(\underline{p}, m) \leq m$. Assume that this inequality is strict. Then there is another bundle $\underline{x}'$ that is also in the budget set and that is componentwise strictly larger than $\underline{x}^*(\underline{p}, m)$. Since u is strictly increasing, $u(\underline{x}') > u(\underline{x}^*(\underline{p}, m))$. So $\underline{x}^*(\underline{p}, m)$ cannot be utility maximising unless (3) holds.

Interpretation: In order to maximise utility, one needs to spend the entire budget. (This is known as 'Walras' Law'. – *They do not need to mention the name 'Walras' Law' to get the mark. It is really about the interpretation.*)

(iii) Show that the utility maximising affordable consumption bundle $\underline{x}^*(\underline{p}, m)$ is unique.

2 markst unseen hardest

Solution: From (ii) we know that any affordable utility maximising consumption bundle lies on the budget line. Assume there are two utility maximising consumption bundles $\underline{x}^*(\underline{p}, m)$ and $\underline{x}'^*(\underline{p}, m)$, both lying on the budget line. So $u(\underline{x}^*(\underline{p}, m)) = u(\underline{x}'^*(\underline{p}, m))$. Since the budget line is convex, we can use the strict quasi-concavity of u and conclude that for any bundle $\underline{x}''$ between $\underline{x}^*(\underline{p}, m)$ and $\underline{x}'^*(\underline{p}, m)$ we obtain that

$$u(\underline{x}'') > \min\{u(\underline{x}^*(\underline{p}, m)), u(\underline{x}'^*(\underline{p}, m))\} = u(\underline{x}^*(\underline{p}, m)).$$

This is a contradiction to the assumption that $\underline{x}^*(\underline{p}, m)$ is utility maximising.

In parts c(i) to c(iii) you have shown the existence and uniqueness of Joseph's Marshallian demand function $\underline{x}^* = (x_1^*, x_2^*)$. Furthermore, let $\underline{x}_H^* = (x_{H,1}^*, x_{H,2}^*)$ be his Hicksian demand function, v his indirect utility function, and e his expenditure function.

(iv) Consider an increase in price for good 1. That is, consider the new price vector $\underline{q} = (q_1, q_2) \in \mathbb{R}^2_{>0}$ where $q_1 > p_1$ and $q_2 = p_2$. Of course, Joseph is not happy about this fact.

- Explain why
$$e\big(\underline{q}, v(\underline{p}, m)\big) - m$$
is the extra that Joseph has to pay to arrive at the initial level of utility.
- Show that
$$\begin{aligned} e\big(\underline{q}, v(\underline{p}, m)\big) - m &= e\big(\underline{q}, v(\underline{p}, m)\big) - e\big(\underline{p}, v(\underline{p}, m)\big) \\ &= \int_{p_1}^{q_1} x_{H,1}^*\big((z, p_2), v(\underline{p}, m)\big)\,\mathrm{d}z. \end{aligned}$$
- Assume that good 1 is a normal good. Using Slutsky's equation, show that
$$\int_{p_1}^{q_1} x_1^*\big((z, p_2), m\big)\,\mathrm{d}z \le \int_{p_1}^{q_1} x_{H,1}^*\big((z, p_2), v(\underline{p}, m)\big)\,\mathrm{d}z. \qquad (4)$$
- Why do many economists still prefer to work with the left-hand side of Equation (4)?

1+2+4+1 marks seen in the lecture 1 routine, 2 medium, 1 upper medium, 4 hardest

Solution:

- With prices $\underline{p}$ and budget m, Joseph can attain a maximal utility of $v(\underline{p}, m)$. Due to (ii) he then spends his whole budget m. If prices increase to $\underline{q}$ and he wants to have the same level of utility $v(\underline{p}, m)$ he needs to spend at least $e(\underline{q}, v(\underline{p}, m))$. So $e(\underline{q}, v(\underline{p}, m)) - m$ is the extra that Joseph has to pay to arrive at the initial level of utility.. *[1 mark, routine]*
- From the lecture, we have the identity
$$e\big(\underline{p}, v(\underline{p}, m)\big) = m,$$
which proves the first equality. One can re-write this difference as an integral and use the fundamental theorem of calculus:
$$\begin{aligned} e\big(\underline{q}, v(\underline{p}, m)\big) - e\big(\underline{p}, v(\underline{p}, m)\big) &= \int_{p_1}^{q_1} \frac{\mathrm{d}\, e\big((z, p_2), v(\underline{p}, m)\big)}{\mathrm{d}z}\,\mathrm{d}z \\ &= \int_{p_1}^{q_1} x_{H,1}^*\big((z, p_2), v(\underline{p}, m)\big)\,\mathrm{d}z. \end{aligned}$$
The last step is due to Shephard's lemma. *[2 marks – identical to lecture, medium]*

- We know that
$$x_1^*(\underline{p},m) = x_{H,1}^*\big(\underline{p}, v(\underline{p},m)\big). \tag{5}$$
That means both integrands in (4) coincide at their left endpoint. Slutsky's equation helps to compare how fast the integrands grow. We obtain that for $p_1 \leq z \leq q_1$
$$\frac{\mathrm{d}\, x_{H,1}^*\big((z,p_2), v(\underline{p},m)\big)}{\mathrm{d}z} - \frac{\mathrm{d}\, x_1^*\big((z,p_2),m\big)}{\mathrm{d}z} = \frac{\mathrm{d}\, x_1^*\big((z,p_2),m\big)}{\mathrm{d}m} x_1^*\big((z,p_2),m\big) \geq 0,$$
where we used the fact that the good is a normal good. So Hicksian demand grows faster than Marshallian demand. With (5) we obtain that
$$x_1^*\big((z,p_2),m\big) \leq x_{H,1}^*\big((z,p_2), v(\underline{p},m)\big), \qquad z \geq p_1.$$
With the monotonicity of the integral, the claim finally follows.
[4 marks – hardest – They saw the assertion in the lecture, but without a proof.]
- The advantage of the left-hand side in (4) is that one can empirically observe Marshallian demand. On the other hand, Hicksian demand is rather a theoretical / conceptual quantity that cannot be observed empirically.
There are also other points one could mention and that are worth marks: The left hand side of (4) is antisymmetric in the price change. This is not true for the right hand side.
[1 marks – upper medium – We had several discussions in that direction during the lecture.]

3. Consider a consumer with a utility function given by $u\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$

$$u(x_1, x_2) = x_1^a x_2^b$$

for constants $a, b > 0$.

(a) Determine the Marshallian demand for goods 1 and 2 with prices $p_1, p_2 > 0$ and budget $m \geq 0$.

3 marks seen 2 routine, 1 upper medium

Solution: The Marshallian demand is

$$\underset{p_1x_1+p_2x_2=m}{\arg\max}\ u(x_1,x_2) = \underset{x_2=\frac{m}{p_2}-\frac{p_1}{p_2}x_1}{\arg\max}\ \log(u(x_1,x_2)) = \underset{x_1\geq 0}{\arg\max}\, a\log(x_1) + b\log\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$$

[One does not need the log-transformation of the utility. But it can make life easier. Also students do not have to mention that, formally, this transformation could make problems since it maps 0 to $-\infty$. However, this is consistent – indeed, the strictly smallest utility one can have with u is 0 and the smallest utility with $\log(u(\cdot))$ is $-\infty$.]

Since the Marshallian demand exists and is unique, we only have to check the first order condition *[1 mark, upper medium]*. That is,

$$\frac{a}{x_1} - \frac{p_1}{p_2}\frac{b}{\frac{m}{p_2} - \frac{p_1}{p_2}x_1} = 0.$$

This yields

$$x_1^* = \frac{am}{p_1(a+b)}.$$

Using the constraint $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$ (or with a symmetry argument) we obtain

$$x_2^* = \frac{bm}{p_2(a+b)}.$$

[Alternatively, one can also use a Lagrangian approach...]

(b) For $a = b = 1$ the result in (a) for good 1 should be $x_1^*(\underline{p}, m) = x_1^*(p_1, m_1) = \frac{m}{2p_1}$. Suppose there is a market for good 1 consisting of I individuals whose preferences can be represented with the same utility function $u(x_1, x_2) = x_1x_2$. Let $m_i \geq 0$, $i \in \{1, \ldots, I\}$, be the budget of each individual.
Determine the market demand for good 1.

2 marks
seen
routine

Solution: The market demand is the sum of the individual Marshallian demand functions. Therefore, it is a function in the individual budgets $m_1, \dots, m_I$ and in the price of good 1. (Since the Marshallian demand for good 1 does not depend on the price for good 2, we shall ignore the price for good 2 and denote the price for good 1 with $p > 0$.)

$$X(p, m_1, \dots, m_I) = \sum_{i=1}^{I} x_1^*(p, m_i) = \frac{M}{2p},$$

where $M := \sum_{i=1}^{I} m_i$.

(c) Consider J firms producing good 1 in a perfectly competitive market (meaning that they are price takers). Since they all use the same production function, their individual cost functions coincide. Assume the cost is given by

$$c(y) = y^2,$$

where y is the output.

(i) Compute each firm's profit maximising output y^* as a function in the price p.

2 marks
seen
routine

Solution: The profit function of each firm is

$$\pi(y, p) = py - c(y) = py - y^2.$$

It is a strictly concave function such that there is a unique maximum. This maximum can be found with the first order condition

$$\frac{\partial}{\partial y}\pi(y, p) = p - 2y = 0,$$

which yields the optimal output

$$y^*(p) = \frac{p}{2}.$$

(ii) Compute the industry supply for good 1.

1 marks
seen
routine

Solution: This is just the sum of the individual output functions. Since they coincide, industry supply is simply given by

$$Y(p) = J\frac{p}{2}.$$

(d) Compute the equilibrium price p^* and the equilibrium quantity Q^*.

2 marks
seen
1
rou-
tine,
1
medium

Solution: In the equilibrium, supply meets demand. That is

$$X(p^*, m_1, \dots, m_I) = Y(p^*) \quad \Longleftrightarrow \quad \frac{M}{2p^*} = \frac{J}{2}p^*.$$

This yields the only possible (since non-negative) solution $p^* = \sqrt{\frac{M}{J}}$.
The associated equilibrium quantity is

$$Q^* = Y(p^*) = X(p^*, m_1, \dots, m_I) = \frac{1}{2}\sqrt{JM}.$$

(e) Sketch a graph of the market demand and industry supply (only qualitatively). Use the conventions that the horizontal axis indicates the quantity and the vertical axis indicates the price. Depict the equilibrium price p^* and the equilibrium quantity Q^*.

2 marks seen routine

Solution: See Figure 6.

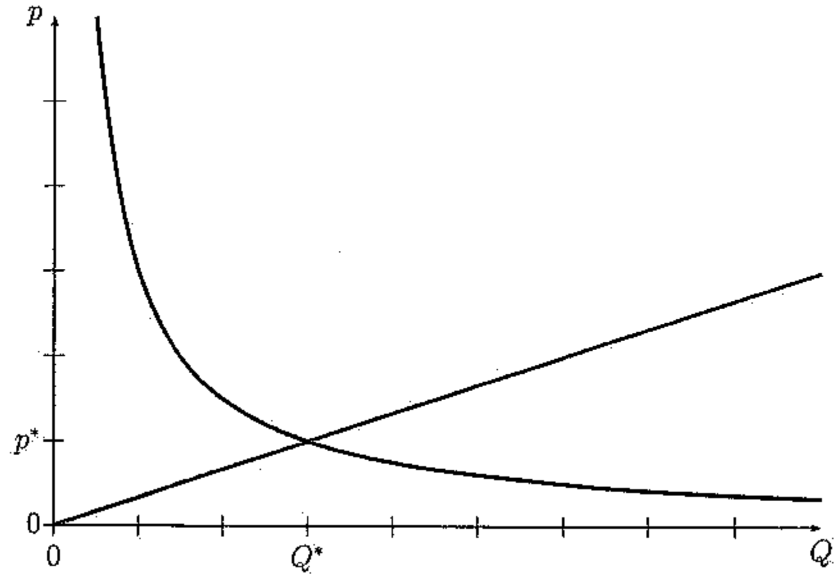

Figure 6: Straight line (green) is the industry supply. Blue curve is the market demand.

(f) Illustrate the producers' and the consumers' surplus in the same graph.

2 marks seen routine

Solution: See Figure 7.

(g) Give the mathematical definition of the producers' and the consumers' surplus and compute them, if possible.

3 marks seen medium

Solution: Producers' surplus (PS):

$$PS = \int_0^{p^*} Y(p)\,\mathrm{d}p = \frac{1}{2}p^*Q^* = \frac{1}{4}M.$$

Consumers' surplus (CS):

$$CS = \int_{p^*}^{\infty} X(p, m_1, \ldots, m_I)\,\mathrm{d}p = \int_{p^*}^{\infty} \frac{M}{2p}\,\mathrm{d}p = \infty,$$

since the integral diverges.

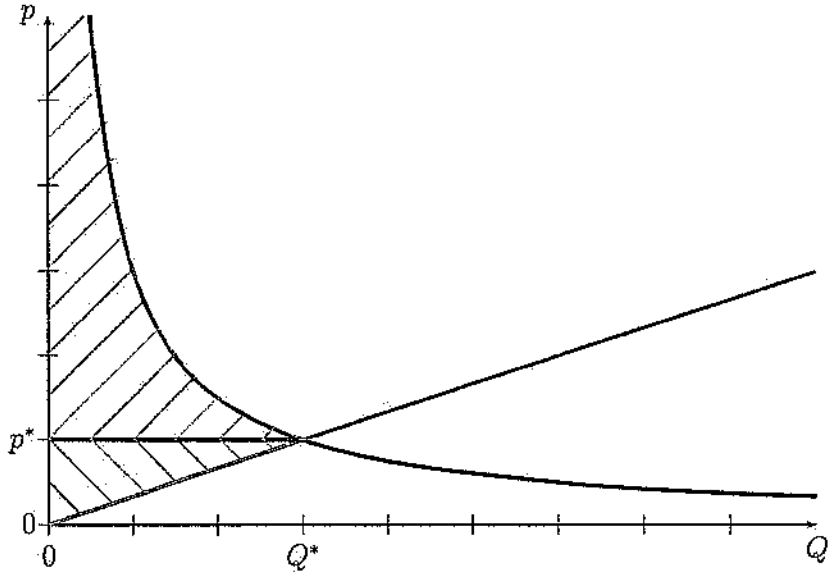

Figure 7: The area in light green illustrates the producers' surplus. The area in light blue is the consumers' surplus.

(h) A new study shows that the consumption of good 1 can do harm to the health of the consumers. Therefore, the government decides to introduce a maximal quantity Q_0. Suppose that $Q_0 = Q^*/2$.

* Illustrate the new situation (sketch a new graph) and depict the deadweight loss.
* Discuss where the market price will be in this scenario? (You can also use the graph for illustrative purposes.)

3 marks
unseen
hardest

Solution: See Figure 8. It is not clear where the new price will be. What we can say is that the price will be at least $p_{\min}$ which is the quantity such that supply is Q_0, that is

$$X(p_{\min}) = Q_0.$$

At Q_0, the consumers' aggregate reservation price is $p_{\max}$, which is the quantity such that

$$Y(p_{\max}) = Q_0$$

and the price will not exceed $p_{\max}$. In summary, the price p_0 will be

$$p_0 \in [p_{\min}, p_{\max}].$$

(See figure 8.)

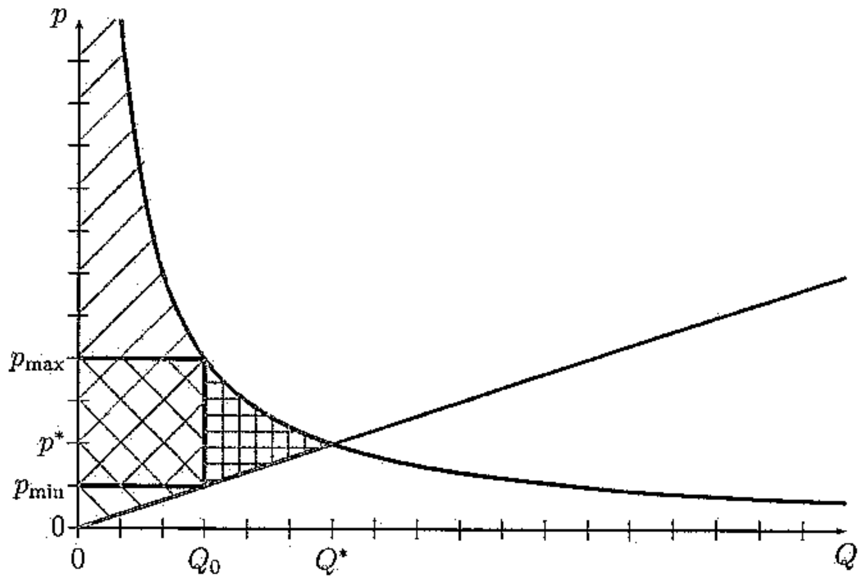

Figure 8: The gray area is the deadweight loss.

4. Throughout this exercise you may assume that all underlying production functions and utility functions are strictly increasing, quasi-concave and as smooth as you need them. Moreover, all prices are strictly positive.

   (a) Give real-world examples for the following situations.

      (i) A firm with a Leontieff production function (two inputs is sufficient).

      1 mark seen routine

      Solution: One can use any example where two goods are perfect complements. In the lecture, the standard example was shovels / spades and workers. But also other solutions are possible.

      (ii) A firm with a linear production function (two inputs is sufficient).

      1 mark seen routine

      Solution: One can use any example where two goods are perfect substitutes. E.g. ordinary coffee and organic coffee.

      (iii) A firm with a production function that exhibits increasing returns to scale.

      1 mark seen routine

      Solution: A firm building pipelines. Assume that output is measured in the capacity of the pipeline and input is raw material and land. Then the capacity scales with the square of the diameter of the pipeline. However, it is a good assumption that raw material and land is proportional to the diameter of the pipeline.

(b) For each statement (i) to (iii), say whether it is true or false? Justify your answer by providing thorough economical explanation or mathematical proof or counter-example (as indicated). You can assume that firms are in a competitive market, that is, they are price takers for output and supply. Moreover, assume that firms produce one product only.

(i) The expenditure function, for fixed level of utility, is positively homogeneous of degree 1 in the prices. (Proof or counter-example, and explanation)

2 marks seen 1 routine, 1 medium

Solution: True.

Proof: The expenditure function at prices $\underline{p} \in \mathbb{R}^n_{\geq 0}$ and utility $k \in \mathbb{R}$ is defined as

$$e(\underline{p}, k) = \min_{\underline{x} \in u^{-1}([k,\infty))} \underline{p}^\top \underline{x}.$$

That means for any $t > 0$

$$e(t\underline{p}, k) = \min_{\underline{x} \in u^{-1}([k,\infty))} t\underline{p}^\top \underline{x} = t \min_{\underline{x} \in u^{-1}([k,\infty))} \underline{p}^\top \underline{x} = te(\underline{p}, k).$$

The explanation is that prices and expenditure need to have the same unit (currency). That means if one changes the unit (currency) in which prices are reported, also the expenditure needs to change to that unit.

(ii) Any profit maximising input bundle is also a cost minimising input bundle. (Proof or counter-example)

2 marks unseen, but discussed hardest

Solution: True.

Proof: Consider a firm with production function $f\colon X \to [0,\infty)$, $X \subseteq \mathbb{R}^n_{\geq 0}$, with input prices $\underline{w} \in \mathbb{R}^n_{\geq 0}$ and output price $p \geq 0$. Then the firm's profit with input bundle $\underline{x} \in X$ is given by

$$\pi(\underline{x}, p, \underline{w}) = pf(\underline{x}) - \underline{w}^\top \underline{x},$$

where $\underline{w}^\top \underline{x}$ are the costs. Suppose that the profit maximising input bundle for prices p and $\underline{w}$ is $\underline{x}^*$. Let $y^* = f(\underline{x}^*)$ be the profit maximising output. Then $\underline{x}^*$ needs also minimise costs, given the output y^*. Indeed, if there were another input bundle $\underline{z}$ with $f(\underline{z}) = f(\underline{x}^*)$ and $\underline{w}^\top \underline{z} < \underline{w}^\top \underline{x}^*$, then

$$\pi(\underline{z}, p, \underline{w}) = pf(\underline{z}) - \underline{w}^\top \underline{z} = pf(\underline{x}^*) - \underline{w}^\top \underline{z} > pf(\underline{x}^*) - \underline{w}^\top \underline{x}^* = \pi(\underline{x}^*, p, \underline{w}),$$

which is a contradiction to the assumption that $\underline{x}^*$ is profit maximising.

(iii) Any cost minimising input bundle is also a profit maximising input bundle. (Proof or counter-example)

2 marks unseen hardest

Solution: False.

Proof (by counter-example): One can choose a cost minimising input bundle for a certain output, but still one needs to determine the profit maximising output. Example: $f\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, $f(x_1, x_2) = \min\{x_1, x_2\}$. If input prices are strictly positive, any cost minimising input bundle for output y is $\underline{x}^* = (y, y)$. Suppose that $p = w_1 = w_2 = 1$ then the profit maximising output is $y^* = 0$. But we could still produce with a cost minimising bundle, say $(2, 2)$ and have a smaller profit.

(c) (i) State the definition of the 'Gross Domestic Product (GDP)'.

3 marks seen routine

Solution: The Gross Domestic Product (GDP) measures the nominal gross value of all goods and services produced in a certain country in a certain period of interest.
What is important are the following points:

- It measures the gross value.
- It is a nominal value, not a real value.
- It has a spatio-temporal restriction.

(ii) What is the effect of a deflation (negative inflation) on GDP? You can focus on the primary / short-run effect.

2 marks seen medium

Solution: Assuming that the real output of an economy does not change, GDP decreases since it is a nominal measure (that means output quantities are multiplied with their (market) prices).

(iii) What is the effect on GDP if all alien residents obtain domestic citizenship? You can focus on the primary / short-run effect.

2 marks seen medium

Solution: There is no effect to GDP since GDP is measured as the gross output produced by all residents independent of their respective nationality (in contrast to the Gross National Product).

(iv) Mention and explain one of the short-comings of GDP as a measure of a country's overall welfare. Explain how one could solve this short-coming.

4 marks seen and unseen routine, upper medium, and hardest (1+1+2)

Solution:
In the lecture, we have mentioned the following short-comings:
[1 mark, routine]

1. Inflation increases the GDP.
2. Tendency to commercialise those services such as child-rearing (kindergartens), care for elderly people (retirement home), or voluntary work.
3. Increase of industry production despite adverse effects to the environment (for example in the former German Democratic Republic). /
   GDP ignores adverse externalities of production.
4. There might be incentives to cause some depreciation in order to re-build infrastructure (destroy streets on order to rebuild them). Also in business, there might be an incentive to build products with a high deterioration rate.
5. People might be pushed into (dependent) work despite their preferences.

The following explanations are not in the written notes. We have just discussed them orally. Here are possible explanations: [1 mark, upper medium]

1. GDP is a nominal value so real output is multiplied with market prices.
2. Only paid services and products are measured in the GDP.
3. If nobody pays for those externalities, GDP fails to measure them.
4. GDP is a *gross* quantity. It only considers values that are *added* to an economy's 'stock'.
5. There is no payment associated to the consumption of leisure. Therefore, GDP ignores it (and the associated utility people get from it).

Here are possible solutions to the problem. We have not discussed about them in the lecture (though, I have encouraged them to think about them). [2 marks, hardest]

1. One needs to 'deflate' GDP. That means, at least to measure GDP growth in a sensible way, one should multiply the respective outputs of different time periods with the same price level.
2. One could estimate the value of this voluntary work and try to include it in GDP.
3. One would need to try to measure those externalities and incorporate those costs into a modified version of GDP.
4. One needs to consider the 'Net Domestic Product' (NDP). This is just GDP minus all depreciations.
5. Maybe, the whole approach of measuring welfare in monetary terms is doubtful. There are alternative attempts to measure the aggregate overall happiness of a society (e.g. Nepal pursues this approach). But also those alternative approaches are delicate both from a theoretical perspective and in terms of implementation.

[Of course, one needs to be a bit flexible and open when one marks this exercise.]