IMPERIAL

MATH70012

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: Tuesday, May 13, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Question 1

(Total: 20 marks)

On the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ endowed with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$, define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	4	10
$Y_1(\omega)$	4	2	0

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 1$, a stock whose initial price is $S_0 = 2$, and whose final price is S_1. Answer the following questions and justify carefully with proofs or counter-examples.

(a) Is this model free of arbitrage? (2 marks)

(b) Consider the derivative with payoff Y_1 at time 1. Is Y_1 replicable? (4 marks)

(c) Is the model (B, S) complete? (3 marks)

(d) Compute the set of all arbitrage-free prices for Y_1. (4 marks)

(e) Compute the smallest price p^* at which an infinitely risk-averse agent would be willing to sell Y_1. Is such p^* an arbitrage-free price? (3 marks)

(f) Let p be an arbitrage-free price of Y in the model (B, S). Suppose Y was sold at time 0 at price p. Is the model (B, S, Y) complete for some of the possible values of p? Is it complete for all the possible values of p? (4 marks)

Question 2

(Total: 20 marks)

Consider a multi-period binomial market model $(B, S) = (B_i, S_i)_{i=0}^N$, where B is a bond and S a stock. Let $E_i = E_i(t_1, k_1, t_2, k_2, a)$ be the value at time $i \leq N$ of the *extendible* call option, which is a call option with maturity t_1 and strike k_1 which also allows its buyer to choose at time t_1 to (not get paid at time t_1 and instead) extend the maturity of the option to time t_2 and change its strike price from k_1 to k_2, by paying an additional premium a to the seller of the option. Let $C_i(t, k)$ be the value at time $i \leq t$ of the call option with maturity $t \leq N$ and strike k. Use the notation $C_i^j := C_i(t_j, k_j)$ for $j = 1, 2$, $i \leq N$ to denote the two corresponding call options. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Write a formula to express E_{t_1} in terms of C^2, a, etc. (4 marks)

Assume from now on that $N = 2$, $B_0 = B_1 = B_2 = 1$, S *has parameters* $S_0 = 4, u = 2, d = \frac{1}{2}$ *(so e.g.* $S_1(H) = 8, S_1(T) = 2$*), and the exchange and call options have parameters* $t_1 = 1$, $k_1 = 4$, $t_2 = 2$, $k_2 = 10$.

(b) Is the model (B, S) arbitrage-free? (1 marks)

(c) Compute the values taken by the random variables C_1^1, C_1^2 at both $\omega \in \{H, T\}$. (6 marks)

(d) Compute the values taken by the random variable $E_1 = E_1(a)$ at both $\omega \in \{H, T\}$ for any $a \in \mathbb{R}$. (2 marks)

(e) Compute $E_0 = E_0(a)$ for any $a \in \mathbb{R}$, and draw the graph of $\mathbb{R} \ni a \mapsto E_0(a)$. (5 marks)

(f) Determine the values of $a \in \mathbb{R}$ for which the buyer of the extendible option would choose to exercise it at time 1 if $S_1(\omega) = 8$, i.e. if $\omega = H$. (2 marks)

Question 3 (Total: 20 marks)

In the framework of the N-period binomial model with constant parameters $S_0 > 0, u > 1 + r > d > 0$, let $S = (S_n)_{n=0}^N$ be the price process of the underlying,

$$A_n := \frac{1}{n+1} \sum_{i=0}^{n} S_i$$

its historical average up to time n, and fix $F \in \mathbb{R}$. Given predictable processes $L \leq U$, define T_n as the time spent by S in the corridor $[L, U]$ between times 1 and n; in other words, given $\mathcal{F}_{n-1}$-measurable L_n, U_n such that $L_n \leq U_n$ a.s. for all $n \geq 1$, define

$$T_n := T_n(L, U) := \sum_{i=1}^{n} 1_{\{L_i \leq S_i \leq U_i\}}.$$

Let V_n denote the arbitrage-free price at time n of the option with payoff $V_N := \frac{F}{N} T_N$ at maturity N. As usual $\mathbb{Q}$ denotes the risk-neutral measure, we take as filtration $\mathcal{F}$ the natural filtration of S, and when we say that a process is *Markov* we mean with respect to $(\mathbb{Q}, \mathcal{F})$. Prove all your assertions carefully or provide counter-examples.

(a) Is (S, A) a Markov process? (4 marks)

(b) If L, U are constants (meaning $L_n(\omega), U_n(\omega)$ do not depend on n nor on ω), is T a Markov process? (3 marks)

Assume from now on that L, U are given by $L_n := \frac{1}{2} A_{n-1}, U_n := 2A_{n-1}$ for all $n \geq 1$.

(c) Is (S, A, T) a Markov process? (4 marks)

(d) Which of the processes (S, A) and (S, A, T) are such that V_n can be written as a function v_n of the value of the process at time n, for every $n = 0, \ldots, N$? (3 marks)

(e) For each process as in the above item, write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \ldots, N - 1$. (6 marks)

Question 4 (Total: 20 marks)

Given a set $A := \{u, m, d\}$ made of the three possible values taken by a dice, and $N \in \mathbb{N} \setminus \{0\}$, consider the probability space $\Omega = A^N$, endowed with the σ-algebra $\mathcal{A}$ of all its subsets, and a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$. Let X_j represents the result of the j^{th} dice toss, i.e. $X_j(\omega) = \omega_j$ for all $\omega \in \Omega, j = 1, \ldots N$. On the filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the the natural filtration of X, build the N-period trinomial model with constant coefficients of a market (B, S^1, S^2), where the bond B has constant value $B_i = 1$ for all $i \leq N$, and the two stocks S^1, S^2 have initial values $S_0^1 = 6 = S_0^2$ and their values S_j^i are given by the following expressions, for $j = 1, \ldots, N$, $i = 1, 2$:

$$S_j^i := S_0^i a^i(X_1) a^i(X_2) \ldots a^i(X_j), \text{ where } \begin{array}{c||c|c|c} \omega_j & u & m & d \\ \hline a^1(\omega_j) & \frac{6}{3} & \frac{2}{3} & \frac{1}{3} \end{array}, \quad \begin{array}{c||c|c|c} \omega_j & u & m & d \\ \hline a^2(\omega_j) & \frac{3}{2} & \frac{2}{2} & \frac{1}{2} \end{array}.$$

Consider the *exchange option* which gives its buyer the right at time N to pay S_N^1 and receive in exchange S_N^2. Denote with E_j (resp. C_j) the arbitrage-free price of the exchange option (resp. of the call option on S^1 with strike price k and maturity N) at time $j = 0, \ldots, N$. Prove all your assertions carefully or provide counter-examples.

(a) Show that the market (B, S^1, S^2) admits a unique EMM (Equivalent Martingale Measure) $\mathbb{Q}$, compute (5 marks)

$$q_u := \mathbb{Q}(\{X_{j+1} = u\} | \mathcal{F}_j), \quad q_m := \mathbb{Q}(\{X_{j+1} = m\} | \mathcal{F}_j) \quad q_d := \mathbb{Q}(\{X_{j+1} = d\} | \mathcal{F}_j),$$

and notice that they do not depend on $j = 1, \ldots, N$ nor on $\omega \in \Omega$.

(b) Are the $(X_j)_{j=1}^N$ independent under $\mathbb{Q}$? (2 marks)

(c) Are the $(X_j)_{j=1}^N$ identically distributed under $\mathbb{Q}$? (1 marks)

(d) Is the process S^1 Markov under $\mathbb{Q}$? (2 marks)

(e) Is the process $S = (S^1, S^2)$ Markov under $\mathbb{Q}$? (2 marks)

(f) Show that, for every $j = 0, \ldots, N$, E_j admits the representation $E_j = e_j(S_j)$, where $e_j : \mathbb{R}^2 \to \mathbb{R}$ is a (deterministic) function. Write explicitly v_N and an explicit formula to express e_j in terms of e_{j+1} for $j = 0, \ldots, N-1$. (4 marks)

(g) Show that, for every $j = 0, \ldots, N$, C_j admits the representation $C_j = c_j(S_j)$, where $c_j : \mathbb{R}^2 \to \mathbb{R}$ is a (deterministic) function. Write explicitly c_N and an explicit formula to express c_j in terms of c_{j+1} for $j = 0, \ldots, N-1$. Is it possible to find such $(c_j(s^1, s^2))_j$ which do not depend on s^2? (4 marks)

Question 5

(Total: 20 marks)

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded) with prices S_t^1, S_t^2 at times $t = 0, 1$. As usual a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in the i^{th} asset. Denote with $\mu(\pi)$ the average, and with $\sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π).

Assume that the average μ of the returns of S, and their covariance matrix Σ, are given by

$$\mu = (2, 3)\,, \quad \Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

(a) Find a portfolio with average return of 50%. Does it involve any short-selling? (2 marks)

(b) Express with a formula the function $\mu \mapsto \sigma(\mu)$ whose graph G is the set of values (μ, σ) taken by all portfolios. (4 marks)

(c) Describe the asymptotic behaviour of $\mu \mapsto \sigma(\mu)$ for $\mu \to \infty$, and for $\mu \to -\infty$: what are its asymptotic tangents? (2 marks)

(d) Determine the intervals in which the function $\mu \mapsto \sigma(\mu)$ is increasing, and those in which it is decreasing. Compute its minimiser(s) and minimum. *Hint: consider* $\mu \mapsto \sigma^2(\mu)$. (4 marks)

(e) Carefully draw G, taking into account yours answers to items (c) and (d). (4 marks)

(f) Explicitly find two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (4 marks)

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2025

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Mathematical Finance: An Intro to Option Pricing

WITH SOLUTIONS INCLUDED

Date:
Time:
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

Question 1

(Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

On the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ endowed with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$, define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	4	10
$Y_1(\omega)$	4	2	0

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 1$, a stock whose initial price is $S_0 = 2$, and whose final price is S_1. Answer the following questions and justify carefully with proofs or counter-examples.

(a) Is this model free of arbitrage? (2 marks)

(b) Consider the derivative with payoff Y_1 at time 1. Is Y_1 replicable? (4 marks)

(c) Is the model (B, S) complete? (3 marks)

(d) Compute the set of all arbitrage-free prices for Y_1. (4 marks)

(e) Compute the smallest price p^* at which an infinitely risk-averse agent would be willing to sell Y_1. Is such p^* an arbitrage-free price? (3 marks)

(f) Let p be an arbitrage-free price of Y in the model (B, S). Suppose Y was sold at time 0 at price p. Is the model (B, S, Y) complete for some of the possible values of p? Is it complete for all the possible values of p? (4 marks)

Solution:

(a) The model is free of arbitrage since the down, middle and up factors d, m, u are respectively $1, 2, 5$ and so $1 + r = 2$ satisfies $d < 1 + r < u$.

Alternatively one can compute the set $\mathcal{M}$ of equivalent martingale measures and show that it is not empty. Recall that $\mathbb{Q} \in \mathcal{M}$ if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 2 = q_1 + 2q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

Subtracting twice the second line from the first line we get $0 = -q_1 + 3q_3$ and so $q_1 = 3q_3$ and the second line now gives $q_2 = 1 - q_1 - q_3 = 1 - 4q_3$. Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q_t := \begin{pmatrix} 3t \\ 1 - 4t \\ t \end{pmatrix} : t \in \left(0, \frac{1}{4}\right) \right\}, \tag{EMM}$$

which is non-empty.

(b) One possible approach is to compute explicitly the solution to the replication equation, which we will presently do. If X is a process, we denote with $\bar{X}$ the discounted process $(X_0, \frac{X_1}{1+r})$. The portfolio with initial wealth x and trading strategy h has discounted payoff $\bar{V}_1 = x + h(\bar{S}_1 - \bar{S}_0)$ equal to

$$x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \cdot \begin{pmatrix} 1-2 \\ 2-2 \\ 5-2 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+3h \end{pmatrix}.$$

Solving for $\bar{Y}_1 = \bar{V}_1$ gives

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+3h \end{pmatrix}$$

which has no solution, so Y_1 is not replicable.

Another possible solution is to show that $\mathbb{E}^{\mathbb{Q}}[\bar{Y}_1]$ is not constant across all $\mathbb{Q} \in \mathcal{M}$. Using (EMM) this means that $2(3t) + 1(1-4t) + 0(t) = 1 + 2t$ is constant over $t \in \left(0, \frac{1}{4}\right)$, which is clearly false.

(c) Clearly the market is not complete, since Y_1 is not replicable. Alternatively, it is incomplete since the vector space generated by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

has dimension 2 while the vector space of all possible values of derivatives is in this example $\mathbb{R}^3$, which is strictly bigger; in other words, the equation $\bar{X}_1 = \bar{V}_1$ does not have solution for arbitrary X_1, since it corresponds to a system of 3 linearly independent equations in 2 unknowns (which does not always have a solution).

Alternatively (EMM) shows that $\mathcal{M}$ is not a singleton, which implies that the market is not complete.

(d) The set of arbitrage free prices of Y_1 are given by the RNFP as

$$\{\mathbb{E}^{\mathbb{Q}}[\bar{Y}_1] : \quad \mathbb{Q} \in \mathcal{M}\} = \left\{1 + 2t : t \in \left(0, \frac{1}{4}\right)\right\} = \left(1, \frac{3}{2}\right).$$

(e) As always p^* is the supremum of the set of arbitrage-free prices, so $p^* = \frac{3}{2}$, which is not an arbitrage-free price (this happens iff the set of arbitrage-free prices is an *open* interval instead of a singleton, i.e. iff the derivative is not replicable).

(f) 1^{st} **solution:** For any choice of AFP p, the market (B, S, Y) is complete. Indeed, since Y was not replicable, the vector $\overline{Y}_1$ is independent of $\overline{B}_1, \overline{S}_1$. Since the 3 independent vectors $\overline{B}_1, \overline{S}_1, \overline{Y}_1$ have 3 components, they form a basis of $\mathbb{R}^3$, i.e. the market (B, S, Y) is complete.

2^{nd} **solution:** For an alternative solution, observe that, for any choice of AFP p, the market (B, S, Y) has only one EMM (thus it is complete), which is the unique EMM $\mathbb{Q}$ for the (B, S) market s.t.

$$p = \mathbb{E}^{\mathbb{Q}}[Y_1/(1+r)]. \tag{1}$$

Question 2

(Total: 20 marks)

(PARTIALLY) UNSEEN (we never considered the extendible option)

Consider a multi-period binomial market model $(B, S) = (B_i, S_i)_{i=0}^N$, where B is a bond and S a stock. Let $E_i = E_i(t_1, k_1, t_2, k_2, a)$ be the value at time $i \leq N$ of the *extendible* call option, which is a call option with maturity t_1 and strike k_1 which also allows its buyer to choose at time t_1 to (not get paid at time t_1 and instead) extend the maturity of the option to time t_2 and change its strike price from k_1 to k_2, by paying an additional premium a to the seller of the option. Let $C_i(t, k)$ be the value at time $i \leq t$ of the call option with maturity $t \leq N$ and strike k. Use the notation $C_i^j := C_i(t_j, k_j)$ for $j = 1, 2$, $i \leq N$ to denote the two corresponding call options. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Write a formula to express E_{t_1} in terms of C^2, a etc.) (4 marks)

Assume from now on that $N = 2$, $B_0 = B_1 = B_2 = 1$, S *has parameters* $S_0 = 4, u = 2, d = \frac{1}{2}$ *(so e.g.* $S_1(H) = 8, S_1(T) = 2$*), and the exchange and call options have parameters* $t_1 = 1$, $k_1 = 4$, $t_2 = 2$, $k_2 = 10$.

(b) Is the model (B, S) arbitrage-free? (1 marks)

(c) Compute the values taken by the random variables C_1^1, C_1^2 at both $\omega \in \{H, T\}$. (6 marks)

(d) Compute the values taken by the random variable $E_1 = E_1(a)$ at both $\omega \in \{H, T\}$ for any $a \in \mathbb{R}$. (2 marks)

(e) Compute $E_0 = E_0(a)$ for any $a \in \mathbb{R}$, and draw the graph of $\mathbb{R} \ni a \mapsto E_0(a)$. (5 marks)

(f) Determine the values of $a \in \mathbb{R}$ for which the buyer of the extendible option would choose to exercise it at time 1 if $S_1(\omega) = 8$, i.e. if $\omega = H$. (2 marks)

Solution:

(a) At time t_1 the options holder can choose to get either $C_{t_1}^1 = (S_{t_1} - k_1)^+$, or get a call option C^2 by paying a, and the value at time t_1 of this second choice is $C_{t_1}^2 - a$. So

$$E_{t_1} = \max(C_{t_1}^1, C_{t_1}^2 - a) = \max((S_{t_1} - k_1)^+, C_{t_1}^2 - a). \tag{2}$$

(b) Yes, since $\frac{1}{2} = d < 1 + r = 1 < u = 2$.

(c) By definition $C_1^1 = (S_1 - k_1)^+ = (S_1 - 4)^+$. Notice that $r = 0$ and

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}. \tag{3}$$

Since $C_2^2 = (S_2 - k_2)^+$, by taking conditional expectation with respect to the risk-neutral measure $\mathbb{Q}$, using the independence lemma we compute

$$C_1^2 = \mathbb{E}_1^{\mathbb{Q}}(S_2 - 10)^+ = \mathbb{E}_1^{\mathbb{Q}} \left(S_1 \frac{S_2}{S_1} - 10 \right)^+ = c(S_1),$$

where

$$c(s) := \mathbb{E}^{\mathbb{Q}} \left(s \frac{S_2}{S_1} - 10 \right)^+ = \frac{1}{3} (2s - 10)^+ + \frac{2}{3} \left(\frac{1}{2} s - 10 \right)^+ .$$

Since $S_1(H) = 8, S_1(T) = 2$ we get

$$C_1^1(H) = (8-4)^+ = 4, \quad C_1^2(H) = c(8) = \frac{1}{3}(16-10)^+ + \frac{2}{3}(4-10)^+ = 2 \tag{4}$$

$$C_1^1(T) = (2-4)^+ = 0, \quad C_1^2(T) = c(2) = \frac{1}{3}(4-10)^+ + \frac{2}{3}(1-10)^+ = 0 \tag{5}$$

(d) Using eqs. (2), (4) and (5) we get that

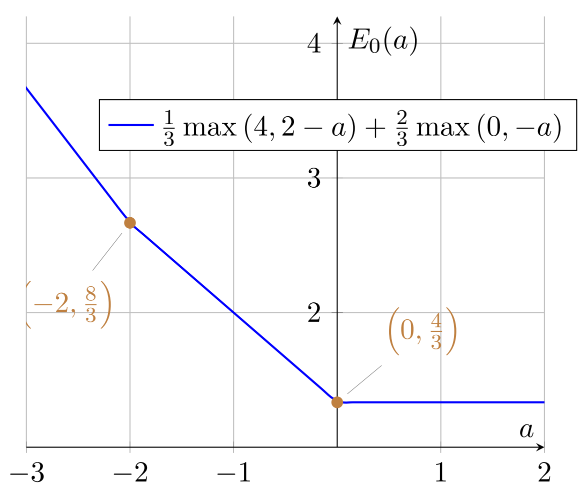
$$E_1(H) = \max(4, 2-a) = \begin{cases} 4 & \text{if } a \in [-2, \infty), \\ 2-a & \text{if } a \in (-\infty, -2], \end{cases} \tag{6}$$

$$E_1(T) = \max(0, -a) = a^- = \begin{cases} 0 & \text{if } a \in [0, \infty), \\ -a & \text{if } a \in (-\infty, 0], \end{cases} . \tag{7}$$

(e) Using eq. (3) and the RNPF we can compute

$$E_0(a) = \mathbb{E}^{\mathbb{Q}} E_1(a) = \frac{1}{3}\left(\max(4, 2-a) + 2a^-\right) = \begin{cases} \frac{4}{3} & \text{if } a \in [0, \infty), \\ \frac{4-2a}{3} & \text{if } a \in [-2, 0], \\ \frac{2}{3} - a & \text{if } a \in (-\infty, -2], \end{cases} ,$$

whose graph is as follows



(f) 1^{st} **solution:** The buyer would choose to exercise the extendible option at time t_1 iff $C_{t_1}^1 \geq C_{t_1}^2 - a$, which when $S_1(\omega) = 8$ happens iff $a \in [-2, \infty)$ by eq. (4).

2^{nd} **solution:** The buyer would choose to exercise the extendible option at time t_1 iff $E_{t_1} = C_{t_1}^1$, which when $S_1(\omega) = 8$ happens iff $a \in [-2, \infty)$ by eqs. (4) and (6).

Question 3

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

In the framework of the N-period binomial model with constant parameters $S_0 > 0, u > 1 + r > d > 0$, let $S = (S_n)_{n=0}^N$ be the price process of the underlying,

$$A_n := \frac{1}{n+1} \sum_{i=0}^{n} S_i$$

its historical average up to time n, and fix $F \in \mathbb{R}$. Given predictable processes $L \le U$, define T_n as the time spent by S in the corridor $[L, U]$ between times 1 and n; in other words, given $\mathcal{F}_{n-1}$-measurable L_n, U_n such that $L_n \le U_n$ a.s. for all $n \ge 1$, define

$$T_n := T_n(L, U) := \sum_{i=1}^{n} 1_{\{L_i \le S_i \le U_i\}}.$$

Let V_n denote the arbitrage-free price at time n of the option with payoff $V_N := \frac{F}{N} T_N$ at maturity N. As usual $\mathbb{Q}$ denotes the risk-neutral measure, we take as filtration $\mathcal{F}$ the natural filtration of S, and when we say that a process is *Markov* we mean with respect to $(\mathbb{Q}, \mathcal{F})$. Prove all your assertions carefully or provide counter-examples.

(a) Is (S, A) a Markov process? (4 marks)

(b) If L, U are constants (meaning $L_n(\omega), U_n(\omega)$ do not depend on n nor on ω), is T a Markov process? (3 marks)

Assume from now on that L, U are given by $L_n := \frac{1}{2} A_{n-1}, U_n := 2A_{n-1}$ for all $n \ge 1$.

(c) Is (S, A, T) a Markov process? (4 marks)

(d) Which of the processes (S, A) and (S, A, T) are such that V_n can be written as a function v_n of the value of the process at time n, for every $n = 0, \dots, N$? (3 marks)

(e) For each process as in the above item, write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \dots, N-1$. (6 marks)

Solution:

(a) $Y := (S, A)$ is a Markov process. To see this, let $Z_{n+1} := \frac{S_{n+1}}{S_n}$ and write

$$S_{n+1} = S_n Z_{n+1}, \quad A_{n+1} = \frac{n+1}{n+2} \left(\frac{S_{n+1}}{n+1} + A_n \right)$$

to conclude that we can write $Y_{n+1} = h_n (Y_n, Z_{n+1})$ for

$$h_n(s, a, z) := \left(sz, \frac{(n+1)}{n+2} \left(\frac{sz}{n+1} + a \right) \right).$$

That Y is Markov follows from the independence lemma: since Z_{n+1} is independent of $\mathcal{F}_n$ and Y_n is $\mathcal{F}_n$-measurable, it gives that

$$\mathbb{E}^{\mathbb{Q}}[f(Y_{n+1})|\mathcal{F}_n] = g(Y_n)$$

where g is the function

$$g(y) := \mathbb{E}^{\mathbb{Q}}[f(h_n(y, Z_{n+1}))] = \tilde{p} f(h_n(y, u)) + (1 - \tilde{p}) f(h_n(y, d)),$$

where as usual $\tilde{p} := \mathbb{Q}(X_{n+1} = H)$.

(b) T is not Markov, as you can see drawing its tree for a simple choice of the values of the parameters. For example, choosing $S_0 = 8, u = 2, d = 1/2, r = 0$ and constant barriers $L = 6, U = 99$, and drawing the tree of S and of T we immediately see the problem: $T_2(HH) = 2 = T_2(HT)$, yet $T_3(HTT) = 2$ is different from

$$T_3(HHH) = T_3(HHT) = T_3(HTH) = 3,$$

and so

$$\mathbb{E}_2^{\mathbb{Q}} f(T_3)(HH) = \tilde{p}f(3) + (1-\tilde{p})f(3) \neq \tilde{p}f(3) + (1-\tilde{p})f(2) = \mathbb{E}_2^{\mathbb{Q}} f(T_3)(HT)$$

whenever $f(3) \neq f(2)$, where $\tilde{p} = \mathbb{Q}(H) = 1/3$.

(c) Yes, $M := (S, A, T)$ is a Markov process. To see this, write

$$T_{n+1} = T_n + 1_{\{S_n Z_{n+1} \in [\frac{1}{2}A_n, 2A_n]\}}$$

to conclude that we can write $M_{n+1} = h_n(M_n, Z_{n+1})$ for

$$h_n(s, a, t, z) := \left(sz, \frac{(n+1)}{n+2}\left(\frac{sz}{n+1} + a\right), t + 1_{\{sz \in [\frac{1}{2}a, 2a]\}} \right). \tag{8}$$

That M is Markov then follows as in item (a).

(d) Since $V_N = v_N(M_n)$ for $v_N(s, a, t) := \frac{F}{N}t$, we can choose $W := M$. We cannot choose $W = (S, A)$ since we cannot express V_N as a function of (S_N, A_N).

(e) We only need to consider $W = M := (S, A, T)$. Then $v_N(s, a, t) := \frac{F}{N}t$, and the Markov property of W then implies that for any v_{n+1} there exists a v_n s.t.

$$v_n(M_n) = V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}] = \mathbb{E}_n^{\mathbb{Q}}[\frac{v_{n+1}(M_{n+1})}{1+r}].$$

To compute v_n explicitly for $n < N$ we use that $M_{n+1} = h_n(M_n, Z_{n+1})$ for h_n given by (8), and applying the independence lemma we get that

$$v_n(m) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[v_{n+1}(h_n(m, Z_{n+1}))] = \frac{1}{1+r}\left(\tilde{p}v_{n+1}(h_n(m, u)) + (1-\tilde{p})v_{n+1}(h_n(m, d))\right).$$

Question 4

(Total: 20 marks)

(PARTIALLY) UNSEEN

Given a set $A := \{u, m, d\}$ made of the three possible values taken by a dice, and $N \in \mathbb{N} \setminus \{0\}$, consider the probability space $\Omega = A^N$, endowed with the σ-algebra $\mathcal{A}$ of all its subsets, and a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$. Let X_j represents the result of the j^{th} dice toss, i.e. $X_j(\omega) = \omega_j$ for all $\omega \in \Omega, j = 1, \ldots N$. On the filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the the natural filtration of X, build the N-period trinomial model with constant coefficient of a market (B, S^1, S^2), where the bond B has constant value $B_i = 1$ for all $i \leq N$, and the two stocks S^1, S^2 have initial values $S_0^1 = 6 = S_0^2$ and their values S_j^i are given by the following expressions, for $j = 1, \ldots, N$, $i = 1, 2$:

$$S_j^i := S_0^i a^i(X_1) a^i(X_2) \ldots a^i(X_j), \quad \text{where} \quad \begin{array}{c||c|c|c} \omega_j & u & m & d \\ \hline a^1(\omega_j) & \frac{6}{3} & \frac{2}{3} & \frac{1}{3} \end{array}, \quad \begin{array}{c||c|c|c} \omega_j & u & m & d \\ \hline a^2(\omega_j) & \frac{3}{2} & \frac{2}{2} & \frac{1}{2} \end{array}.$$

Consider the *exchange option* which gives its buyer the right at time N to pay S_N^1 and receive in exchange S_N^2. Denote with E_j (resp. C_j) the arbitrage-free price of the exchange option (resp. of the call option on S^1 with strike price k and maturity N) at time $j = 0, \ldots, N$. Prove all your assertions carefully or provide counter-examples.

(a) Show that the market (B, S^1, S^2) admits a unique EMM (Equivalent Martingale Measure) $\mathbb{Q}$, compute (5 marks)

$$q_u := \mathbb{Q}(\{X_{j+1} = u\} | \mathcal{F}_j), \quad q_m := \mathbb{Q}(\{X_{j+1} = m\} | \mathcal{F}_j) \quad q_d := \mathbb{Q}(\{X_{j+1} = d\} | \mathcal{F}_j),$$

and notice that they do not depend on $j = 1, \ldots, N$ nor on $\omega \in \Omega$.

(b) Are the $(X_j)_{j=1}^N$ independent under $\mathbb{Q}$? (2 marks)

(c) Are the $(X_j)_{j=1}^N$ identically distributed under $\mathbb{Q}$? (1 marks)

(d) Is the process S^1 Markov under $\mathbb{Q}$? (2 marks)

(e) Is the process $S = (S^1, S^2)$ Markov under $\mathbb{Q}$? (2 marks)

(f) Show that, for every $j = 0, \ldots, N$, E_j admits the representation $E_j = e_j(S_j)$, where $e_j : \mathbb{R}^2 \to \mathbb{R}$ is a (deterministic) function. Write explicitly v_N and an explicit formula to express e_j in terms of e_{j+1} for $j = 0, \ldots, N-1$. (4 marks)

(g) Show that, for every $j = 0, \ldots, N$, C_j admits the representation $C_j = c_j(S_j)$, where $c_j : \mathbb{R}^2 \to \mathbb{R}$ is a (deterministic) function. Write explicitly c_N and an explicit formula to express c_j in terms of c_{j+1} for $j = 0, \ldots, N-1$. Is it possible to find such $(c_j(s^1, s^2))_j$ which do not depend on s^2? (4 marks)

Solution:

(a) By definition of EMM, q_u, q_m, q_d are determined by asking that $\mathbb{Q}(\Omega) = 1$ and $\mathbb{E}^{\mathbb{Q}}[S_{j+1}^i | \mathcal{F}_j] = S_j^i$ for $i = 1, 2$. Dividing the latter by $S_j^i > 0$, we see that it is equivalent to $\mathbb{E}^{\mathbb{Q}}[a^i(X_{j+1}) | \mathcal{F}_j] = 1$ for $i = 1, 2$, and so $\mathbb{Q}$ is an EMM iff

$$\begin{cases} \frac{6}{3} q_u + \frac{2}{3} q_m + \frac{1}{3} q_d = 1 \\ \frac{3}{2} q_u + \frac{2}{2} q_m + \frac{1}{2} q_d = 1 \\ q_u + q_m + q_d = 1 \end{cases}. \tag{9}$$

Calling R_j the j^{th} row of the above system, and replacing R_j by R_j' defined as follows

$$R_1' := 3R_1 - 2R_2, \quad R_2' := 2R_2 - R_3, \quad R_3' := R_3,$$

leads to the equivalent system

$$\begin{cases} 3q_u & = 1 \\ 2q_u + q_m & = 1 \\ \ q_u + q_m + q_d & = 1 \end{cases} \tag{10}$$

which obviously has the unique solution $q_u = q_m = q_d = \frac{1}{3}$, which does not depend on j, ω.

(b) Yes: since q_u, q_m, q_d are deterministic (i.e. they do not depend on ω), also $\mathbb{Q}(X_{j+1} \in B|\mathcal{F}_j)$ is deterministic for any $B \subseteq A$, since by definition

$$\mathbb{Q}(X_{j+1} \in B|\mathcal{F}_j) = \sum_{\omega_j \in B} q_{\omega_j}.$$

By the theorem seen in class, this is equivalent to saying that X_{j+1} is independent of $\mathcal{F}_j := \sigma(X_1, \ldots, X_j)$. Since this holds for any j, the conclusion follows.

(c) Yes: since q_u, q_m, q_d do not depend on j, so also doesn't $\mathbb{Q}(X_{j+1} \in B)$ for any $B \subseteq A$.

(d) Yes, since $S^1_{j+1} = S^1_j \cdot a^1(X_{j+1})$, and $a^1(X_{j+1})$ is $\sigma(X_{j+1})$-measurable and thus independent of $\mathcal{F}_j$, this follows from the corollary seen in class. In fact, using the independence lemma we can also explicitly compute

$$\mathbb{E}^{\mathbb{Q}}[f(S^1_{j+1})|\mathcal{F}_j] = g(S^1_j) \quad \text{for} \quad g(s) := \mathbb{E}^{\mathbb{Q}} f(s \cdot a^1(X_{j+1})) = \frac{1}{3}\left(f\left(\frac{6}{3}s\right) + f\left(\frac{2}{3}s\right) + f\left(\frac{1}{3}s\right)\right) \tag{11}$$

(e) Yes, since $S^i_{j+1} = S^i_j \cdot a^i(X_{j+1})$, and $a^i(X_{j+1})$ is $\sigma(X_{j+1})$-measurable and thus independent of $\mathcal{F}_j$ for each $i = 1, 2$, it follows from the corollary seen in class that both S^1 and S^2, and thus a fortiori also (S^1, S^2), are Markov. In fact, using the independence lemma we can also explicitly compute

$$\mathbb{E}^{\mathbb{Q}}[f(S^1_{j+1}, S^2_{j+1})|\mathcal{F}_j] = g(S^1_j, S^2_j) \tag{12}$$

$$\text{for} \quad g(s^1, s^2) := \mathbb{E}^{\mathbb{Q}} f(s^1 \cdot a^1(X_{j+1}), s^2 \cdot a^2(X_{j+1})) = \frac{1}{3}\left(f\left(\frac{6}{3}s^1, \frac{3}{2}s^2\right) + f\left(\frac{2}{3}s^1, \frac{2}{2}s^2\right) + f\left(\frac{1}{3}s^1, \frac{1}{2}s^2\right)\right). \tag{13}$$

(f) Such e_j's exist since S is Markov and $E_N = (S^2_N - S^1_N)^+ = e_N(S^1_N, S^2_N)$ for $e_N(s^1, s^2) := (s^2 - s^1)^+$. Since the interest rate r equals 0, by the Risk Neutral Pricing Formula the recursive expression for e_j is obtained by taking $f = e_{j+1}, g = e_j$ in eq. (13), i.e.

$$e_N(s^1, s^2) := (s^2 - s^1)^+, \quad e_j(s^1, s^2) = \frac{1}{3}\left(e_{j+1}\left(\frac{6}{3}s^1, \frac{3}{2}s^2\right) + e_{j+1}\left(\frac{2}{3}s^1, \frac{2}{2}s^2\right) + e_{j+1}\left(\frac{1}{3}s^1, \frac{1}{2}s^2\right)\right), \quad \text{for } j =$$

(g) There exists such c_j, which do not depend on s^2, since S^1 is Markov and $C_N = (S^1_N - K)^+ = c_N(S^1_N)$ for $c_N(s^1) := (s^1 - k)^+$. Since the interest rate r equals 0, by the Risk Neutral Pricing Formula the recursive expression for c_j is obtained by taking $f = c_{j+1}, g = c_j$ in eq. (11), i.e.

$$c_N(s^1) := (s^1 - k)^+, \quad c_j(s^1) = \frac{1}{3}\left(c_{j+1}\left(\frac{6}{3}s^1\right) + c_{j+1}\left(\frac{2}{3}s^1\right) + c_{j+1}\left(\frac{1}{3}s^1\right)\right), \quad \text{for } j = 0, \ldots, N-1.$$

Question 5 (Total: 20 marks)

SIMILARLY SEEN IN PREVIOUS FINAL EXAM

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded) with prices S_t^1, S_t^2 at times $t = 0, 1$. As usual a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in the i^{th} asset. Denote with $\mu(\pi)$ the average, and with $\sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π).

Assume that the average μ of the returns of S, and their covariance matrix Σ, are given by

$$\mu = (2, 3), \quad \Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

(a) Find a portfolio with average return of 50%. Does it involve any short-selling? (2 marks)

(b) Express with a formula the function $\mu \mapsto \sigma(\mu)$ whose graph G is the set of values (μ, σ) taken by all portfolios. (4 marks)

(c) Describe the asymptotic behaviour of $\mu \mapsto \sigma(\mu)$ for $\mu \to \infty$, and for $\mu \to -\infty$: what are its asymptotic tangents? (2 marks)

(d) Determine the intervals in which the function $\mu \mapsto \sigma(\mu)$ is increasing, and those in which it is decreasing. Compute its minimiser(s) and minimum. *Hint: consider* $\mu \mapsto \sigma^2(\mu)$. (4 marks)

(e) Carefully draw G, taking into account yours answers to items (c) and (d). (4 marks)

(f) Explicitly find two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (4 marks)

Solution:

(a) Since the return $R(\pi)$ of a portfolio (x_0, π) depends only on $\pi = (\pi_1, \pi_2)$ and is given by $R_1(\pi) = \pi_1 R_1^1 + \pi_2 R_1^2$, using the notation $u := \pi_1$, so that $\pi_2 = 1 - u$, and using eq. (14) we compute

$$\mu(u) := \mu(\pi) = \mathbb{E}[R_1(\pi)] = u\mu_1 + (1-u)\mu_2 = 2u + 3(1-u) = 3 - u, \tag{15}$$

and so $\mathbb{E}[R_1(\pi)] = \frac{50}{100}$ becomes $3 - u = \frac{1}{2}$, and solving for u gives $u = \frac{5}{2}$ and $\pi = (\frac{5}{2}, -\frac{3}{2})$. Since $\pi_2 < 0$, this portfolio involves short-selling the 2nd asset.

(b) To determine $\mu \mapsto \sigma(\mu)$, we use eq. (15) and analogously compute the variance of the portfolio $\pi = (u, 1-u)$ as

$$\sigma(u)^2 := \sigma(\pi)^2 = \pi^T \Sigma \pi = u^2 \sigma_1^2 + (1-u)^2 \sigma_2^2 + 2\rho u(1-u)\sigma_1\sigma_2$$

and so we get

$$\sigma(u)^2 = 4u^2 + (1-u)^2 \tag{16}$$

Inverting eq. (15) we get

$$u = 3 - \mu \tag{17}$$

which plugged into eq. (16) gives

$$\sigma^2 = 4(3-\mu)^2 + (\mu-2)^2 = 4(\mu^2 - 6\mu + 9) + (\mu^2 - 4\mu + 4) = 5\mu^2 - 28\mu + 40.$$

For later use, we record the useful alternative expression for σ^2:

$$\sigma(\mu)^2 = 5\mu^2 - 28\mu + 40 = 5\left(\mu^2 - 2\cdot\frac{14}{5}\mu\right) + 40 = 5\left(\mu - \frac{14}{5}\right)^2 + 40 - 5\left(\frac{14}{5}\right)^2. \tag{18}$$

In summary

$$\sigma(\mu) := \sqrt{5\mu^2 - 28\mu + 40} \in \mathbb{R}_+, \qquad \text{for } \mu \in \mathbb{R}. \tag{19}$$

(c) From eq. (18) we get that

$$|\sigma(\mu)| \approx \sqrt{5}\left|\mu - \frac{14}{5}\right| \quad \text{for 'big' values of } \left|\mu - \frac{14}{5}\right|, \tag{20}$$

i.e. the asymptotic tangents of $\mu \mapsto \sigma(\mu)$ as $\mu \to \pm\infty$ are $\pm\sqrt{5}\left(\mu - \frac{14}{5}\right)$.

(d) Since $\mathbb{R}_+ \ni x \mapsto x^2$ is a strictly increasing function, the intervals in which the functions $\mu \mapsto \sigma(\mu)$ and $\mu \mapsto \sigma^2(\mu)$ are increasing/decreasing coincide, and in particular these functions have the same minimiser(s). We will thus study $\mu \mapsto \sigma^2(\mu)$, which requires less involved calculations. The expression eq. (18) shows that the convex quadratic function $\mu \mapsto \sigma(\mu)^2$ obviously attains its unique minimum at

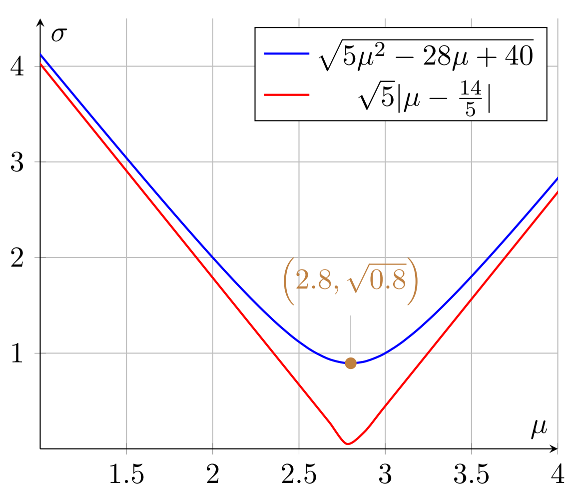
$$\mu = \frac{14}{5} = 3 - \frac{1}{5} = 2.8, \tag{21}$$

which can also be found by computing the derivative $\frac{d\sigma(\mu)^2}{d\mu} = 10\mu - 28$ and setting it to 0. Thus clearly on the interval $(-\infty, \frac{14}{5}]$ the quadratic convex function $\mu \mapsto \sigma^2(\mu)$ is strictly decreasing, and on $[\frac{14}{5}, \infty)$ it is strictly increasing.

(e) To draw G, notice that the minimum of the quadratic function $\sigma(\mu)^2$, which by eq. (21) is attained at $\mu = \frac{14}{5}$, equals

$$\sigma\left(\frac{14}{5}\right)^2 = 40 - 5\left(\frac{14}{5}\right)^2 = \frac{2^2}{5}(50 - 7^2) = \frac{4}{5} = 0.8.$$

Taking also into account eq. (20) and the fact that $\mu \mapsto \sigma(\mu)$ is a positive smooth (infinitely differentiable) function, we can draw G as follows



(f) For example one can take π^a and π^b which correspond to $\mu^a = 2$ and $\mu^b = \frac{14}{5} = 2.8$, which are found using eq. (17): $\pi^a = (1, 0), \pi^b = (\frac{1}{5}, \frac{4}{5})$. Indeed $\mu^a = 2 < \frac{14}{5} = \mu^b$, and since $\frac{14}{5}$ is the minimum of $\sigma(\mu)^2 = 5\mu^2 - 28\mu + 40$ we have $\sigma(\mu^a)^2 > \sigma(\mu^b)^2$.

The portfolio π^a is preferable, since the standard deviation $\sigma(\pi)$ of the returns of a portfolio π is normally used as the measure its risk, and so π^a has both a higher average return, and a lower risk, than π^b, and so any investor (which uses $\sigma(\pi)$ as the measure its risk) would prefer π^a to π^b, irrespectively of his/her attitude towards risk.

MATH70012 Mathematical Finance: An Introduction to Option Pricing Markers Comments

Question 1

Almost all students got very high marks in this easy question. The only somewhat common mistakes where: In part (e) taking p* as the infimum 1 instead of the supremum 1.5 of the interval (1,1.5) of Arbitrage-Free Prices. In (f) stating that the market was either never complete, or complete only for some values of p.

Question 2

Performance was somewhat bimodal in this question: while the majority of students got a high mark in the 15-20 range, there was quite a chunk of students who got low marks in the 0-8 range. I believe this was because this question, though not hard, had more original, unseen components than most of the exam; also unfortunately getting part (a) very wrong precluded getting some of the marks in parts (d,e,f), though a surprising number of students got (a) wrong and (d,e,f) completely right by being not self-consistent. An uncomfortably high % of students got part (c) wrong, though it was very standard.

Question 3

Here are some common mistakes in Question 3:
1. Claim A or T is Markov. It is the pair (S,A) or (S,A,T) which is Markov, not each component.
2. Misunderstand how T is defined. It does not include time zero.
3. Write S_{n+1} in the final expression. You must write it as $S_n \frac{S_{n+1}}{S_n}$ to apply the independence lemma.
4. Wrong calculation of A_{n+1} in terms of A_n, S_n and $\frac{S_{n+1}}{S_n}$.
5. You should write clearly what is $V_N(s, a, t)$.
6. You should write $\frac{1}{1+r}$ in part (e).
7. Very serious mistake: $V_n = \frac{F}{n} T_n$ only when n=N. You are asked to price this derivative, with the given terminal value V_N. Many students think the formula of V_N works for $n \in [0, N-1]$.
8. You should provide a counterexample for part (b).

Question 4

<Overall there was a wide range of performances on this question - marks range from very low to full marks.
(4A) This was answered correctly by most students, although there are varying degrees of the quality of presentation in the answers. The most common mistakes were that the wrong linear equations were calculated or that there was not enough, if any, explanation as to how the linear equations were obtained (e.g. the first equation is due to the definition of a probability measure, the second from S1 etc).
(4B, 4C) These were answered correctly by most. Although many explanations were sufficient and well-written, there were some which were clearer and more detailed than others. A few students wrote insufficient explanations or omitted explanations.
(4D, 4E) Most students came to the correct conclusion (that both processes are Markov) and explained the key reason why S^{1} is Markov, as well as why $S=(S^{1},S^{2})$ is Markov. The reason most students who did not obtain full marks on either of these questions failed to obtain full marks is that they explained only partially or insufficiently WHY the processes are Markov. There were noticeably fewer students who answered parts (F) and (G) correctly or even fully. Some students gave very good answers to at least one of the questions (if not both). (4F) Quite a few students gave incomplete answers or answers which were not completely correct. In particular, many did not identify correctly e_{N} - they have said that $e_{N}(s1,s2)=(s1-s2)$ as opposed to $e_{N}(s1,s2)=(s1-s2)^{+}$. A few did not have enough detail in their answers, or gave a partial answer - either proving the inductive step or only e_{N}.
(4G) Similar to (4F). Here, several people did identify that it is possible to find an expression that does not depend on s^{2}, but they have not given this expression (that only depends on s1) explicitly. Still, there were at least a quarter of students who answered parts (4F) and (4G) very well, too. As notified, and as some students noted in their scripts, there was a minor typo in (1F) - 'v_{N}' should be 'e_{N}'.

Question 5 The vast majority of students got marks in the 15-20 range in this question, which was very similar to past exam questions. Almost none managed to do part (c). Some students made algebraic mistakes in part (b) and lost some points in part (e).