MATH50011 Statistical Modelling 1
Midterm Examination
From 9:00am to 9:50am

1. (a) Provide the definition of pivotal quantity. (2 marks)

   (b) Let T_n, $n \in \mathbb{N}$, be a sequence of estimators for a parameter $\theta \in \mathbb{R}$ such that $MSE_\theta(T_n) \to 0$ as $n \to \infty$. Prove that T_n is consistent (for this question only: if you use any results from the lectures you will need to prove them). (2 marks)

   For the remaining questions of this problem consider the following setting. Let $X_1, X_2, ...$ be a sequence of iid Exponential random variables with unknown parameter $\lambda \in (0, \infty)$. Recall that the pdf of an Exponential random variable is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

   (c) For fixed $n \in \mathbb{N}$, compute the maximum likelihood estimator (MLE) for $\theta := \frac{1}{\lambda}$ based on the sample $X_1, ..., X_n$. Denote this estimator by $\hat{\theta}_n$. (2 marks)

   (d) Show whether or not that $\hat{\theta}_n$ is an unbiased and consistent estimator for θ. (1 mark)

   (e) Let A_n, $n \in \mathbb{N}$, be a sequence of events such that $P(A_n) = \frac{1}{n}$, for every $n \in \mathbb{N}$. Let $Y_n = n\mathbf{1}_{A_n}$. Assume that $Y_1, ..., Y_n$ are independent from $X_1, ..., X_n$, for every $n \in \mathbb{N}$. Is $Y_n\hat{\theta}_n$ a consistent estimator for θ? Is it unbiased? Is it asymptotically Normal? Explain your answers in detail. (3 marks)

2. Let $X_1, X_2, ...$ be a sequence of iid Normal random variables with known mean μ and unknown variance $\sigma^2 > 0$.

   (a) Provide the definition of type 1 error and of type 2 error. (3 marks)

   (b) Show whether or not $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$ is an unbiased and consistent estimator for σ^2. (2 marks)

   (c) Let $\mu = 0$ and let $\sigma_0^2 > 0$. Build an exact test of level $\alpha = 0.05$ based on T_n for $H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 > \sigma_0^2$. (2 marks)

   (d) Construct the power function of the test built in point (c), making explicit the dependence on the parameter, and draw it. (2 marks)

   (e) Explain how your answers for points (c) and (d) would change if we had $H_0 : \sigma^2 \leq \sigma_0^2$ instead of $H_0 : \sigma^2 = \sigma_0^2$. (1 mark)

(Total 20 marks)