Question: Which of the following functions defines a metric on $\mathbb{R}$?

A. $d(x,y) = |x^{2023} - y^{2023}|$,

B. $d(x,y) = |x^{2024} - y^{2024}|$,

C. $d(x,y) = |x-y|^{2023}$,

D. $d(x,y) = |x-y|^{2024}$.

Solution: (The answer is A)

A. $d(x,y) = |x^{2023} - y^{2023}|$ is a metric because the map $x \mapsto x^{2023}$ is a monotone map. See Problem Sheet 6 for more details.

B. $d(x,y) = |x^{2024} - y^{2024}|$ is not a metric, since $d(1,-1) = 0$ but $1 \neq -1$.

C. $d(x,y) = |x-y|^{2023}$ is not a metric since the triangle inequality does not hold:

$$2^{2023} = d(1,3) \nleq d(1,2) + d(2,3) = 1^{2023} + 1^{2023}.$$

D. $d(x,y) = |x-y|^{2024}$ is not a metric as well, since the triangle inequality does not hold:

$$2^{2024} = d(1,3) \nleq d(1,2) + d(2,3) = 1^{2024} + 1^{2024}.$$

Question: Consider the discrete metric d_{disc} on the set $\mathbb{R}^2$ and let

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}.$$

What are the interior A°, the closure $\overline{A}$, and the boundary ∂A of the set A in the metric space $(\mathbb{R}^2, d_{disc})$?

A. $A^\circ = A$, $\overline{A} = A$, and $\partial A = \emptyset$.

B. $A^\circ = A$, $\overline{A} = A$, and $\partial A = \mathbb{R}^2$.

C. $A^\circ = A$, $\overline{A} = A$, and $\partial A = A$.

D. $A^\circ = \emptyset$, $\overline{A} = \mathbb{R}^2$, and $\partial A = \emptyset$.

Solution: (The answer is A) Let (X, d) be a discrete metric space in general, and $A \subseteq X$. We have $B_{1/2}(x) = \{x\}$. For any $x \in A$,we have $B_{1/2}(x) \subset A$. Thus, any $x \in A$ in an interior point of A. The interior of A, A° is the collection of all the interior points of A, therefore $A^\circ = A$. For any $x \in X$, $B_{1/2}(x) \cap A = \{x\}$ does not contain any other point than x, so x is not a limit point of A. The closure of A is the union of A with its limit points, so $\overline{A} = A$. No point of X is a boundary point of A, because $B_{1/2}(x)$ does not contain any other points than x. Therefore, $\partial A = \emptyset$.

Question: Let (X, d) be an arbitrary metric space, and let V and W be arbitrary subsets of X. Which of the following properties is always true?

A. $\overline{(V^\circ)} = \overline{V}$.

B. $\partial V = \overline{V} \setminus V^\circ$.

C. $\overline{V \setminus W} = \overline{V} \setminus \overline{W}$.

D. $\partial(V \cup W) = \partial V \cup \partial W$.

Solution: (The answer is B)

A. This is not true, for example in $(\mathbb{R}, d_1)$, we have $\overline{(\mathbb{Q}^\circ)} = \overline{\emptyset} = \emptyset$, but $\overline{\mathbb{Q}} = \mathbb{R}$.

B. This is true. If $z \in \partial V$, then every neighbourhood of z contains a point from V. Thus, $z \in \overline{V}$. On the other hand, since every neighbourhood of z contains a point from the complement of V, z cannot be in V°. Therefore, $z \in \overline{V} \setminus V^\circ$. By a similar argument, if $z \in \overline{V} \setminus V^\circ$, then $z \in \partial V$.

C. This is not true, for example, let $V = \mathbb{R}$, $W = \mathbb{Q}$ in the Euclidean metric space $(\mathbb{R}, d_1)$. Then, $\overline{V \setminus W} = \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$ but $\overline{V} \setminus \overline{W} = \mathbb{R} \setminus \mathbb{R} = \emptyset$. In fact, this relation can be proved using the relation in the previous item and some set theoretic manipulations (taking complement and intersections).

D. This is not correct; in $(\mathbb{R}, d_1)$ for $V = \mathbb{Q}$ and $W = \mathbb{R} \setminus \mathbb{Q}$ we have $\partial(V \cup W) = \partial \mathbb{R} = \emptyset$ but $\partial V \cup \partial W = \mathbb{R} \cup \mathbb{R} = \mathbb{R}$.

Question: Let $C([0,1])$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the supremum metric

$$d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Let

$$\Omega = \{f \in C([0,1]) \mid f(1/2) > 5\}.$$

Is the set Ω open in $(C([0,1]), d_\infty)$?

Solution: (The answer is A) The set Ω is open in the metric d_∞. To see this, let f be an arbitrary element in Ω. We have $f(1/2) > 5$. Let $\delta = f(1/2) - 5$. It is easy to see that the ball $B_\delta(f)$ is contained in Ω in the metric d_∞.

One can also show that the map $\Phi(f) = f(1/2)$ is continuous on $(C([a,b]), d_\infty)$, and the set $(5, +\infty)$ is open in $\mathbb{R}$, so $\Omega = \phi^{-1}((5, +\infty))$ must be open.

Question: Consider the Euclidean metric space $(\mathbb{R}, d_1)$ where $d_1(x, y) = |x - y|$, and let

$$X = \{1/n \mid n = 1, 2, 3, \dots\}$$

and

$$Y = \{1/n \mid n = 1, 2, 3, \dots\} \cup \{0\}.$$

Which of the following statements in true?

A. Both X and Y are not compact in $(\mathbb{R}, d_1)$,

B. Both X and Y are compact in $(\mathbb{R}, d_1)$,

C. X is compact but Y is not compact in $(\mathbb{R}, d_1)$,

D. X is not compact but Y is compact in $(\mathbb{R}, d_1)$.

Solution: (The answer is D) There are three ways to prove this.

Y is closed and bounded, so by Heine-Borel theorem, it must be compact.

Alternatively, let $\mathcal{R}$ be a arbitrary open cover for Y. There must be an element $U \in \mathcal{R}$ which contains 0. But, since $1/n$ converges to 0 with respect to d_1 and U is open, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $1/n \in U$. We may choose a finite number of elements from $\mathcal{R}$ which cover the elements $1/1, 1/2, 1/3, \dots, 1/(N-1)$, and combine them with U to get a finite subcover of $\mathcal{R}$ for Y.

On the other, hand X is not compact. That is because any compact set must be closed, but X is not closed. The sequence $1/n$ belongs to X but its limit does not belong to X. Alternatively, one can see that $\mathbb{R} = \{(1/n - 1/n^3, 1/n + 1/n^3) \mid n = 1, 2, 3, \dots\}$ is an open cover for X which has no finite subcover.