Introduction to Quantum Mechanics – Solutions to Problem sheet 8

1. Dynamics of a coherent state in the harmonic oscillator

An initial coherent state is given by

$$|\psi(t=0)\rangle = |z\rangle = e^{-|z|^2/2}\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle.$$

This state evolves according to

$$|\psi(t)\rangle = e^{-\hat{H}t/\hbar}|z\rangle.$$

Inserting in the Hamiltonian, and extracting terms independent of n from the sum, we find

$$\begin{aligned}|\psi(t)\rangle &= e^{-|z|^2/2}\sum_n \frac{e^{-i\omega(n+1/2)t}z^n}{\sqrt{n!}}|n\rangle \\ &= e^{-i\omega t/2}e^{-|z|^2/2}\sum_n \frac{(ze^{-i\omega t})^n}{\sqrt{n!}}|n\rangle.\end{aligned}$$

Defining $z(t) = ze^{-i\omega t}$, we have

$$|\psi(t)\rangle = e^{-i\omega t/2}e^{-|z|^2/2}\sum_n \frac{z(t)^n}{n!}|n\rangle.$$

Thus we find

$$\langle \hat{a}(t)\rangle = e^{-i\omega t}\langle a(0)\rangle.$$

Recalling that

$$\langle q\rangle = \sqrt{\frac{2\hbar}{m\omega}}\mathsf{Re}(\hat{a}(t)) \quad \text{and} \quad \langle p\rangle = \sqrt{2m\omega\hbar}\,\mathsf{Im}(\hat{a}(t)),$$

we confirm that

$$\begin{aligned}\langle\hat{q}\rangle(t) &= \cos(\omega t)\langle\hat{q}\rangle(0) + \frac{1}{m\omega}\sin(\omega t)\langle\hat{p}\rangle(0) \\ \langle\hat{p}\rangle(t) &= \cos(\omega t)\langle\hat{p}\rangle(0) - m\omega\sin(\omega t)\langle\hat{q}\rangle(0),\end{aligned}$$

as was required.

2. Dynamics for an anharmonic oscillator

(a) Since harmonic oscillator eigenstates are eigenstates of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, harmonic oscillator eigenstates are also eigenvalues of $\hat{H} = \mu\hat{N}^2$ with $\hat{H}\,|n\rangle = \mu n^2\,|n\rangle$.

(b) Decomposing in terms of eigenstates of the Hamiltonian (stationary states) we have

$$|\psi(t)\rangle = \sum_n \mathrm{e}^{-\mathrm{i}E_n t/\hbar}\psi_n |n\rangle \quad \text{with} \quad E_n = \mu n^2.$$

Our initial state is given by

$$|\psi(0)\rangle = |z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle ,$$

Combining these gives

$$\begin{aligned}|\psi(t)\rangle &= \mathrm{e}^{-\frac{|z|^2}{2}} \sum_n \mathrm{e}^{-\mathrm{i}\mu n^2 t/\hbar} \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= \mathrm{e}^{-\frac{|z|^2}{2}} \sum_n \frac{(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{\sqrt{n!}} |n\rangle .\end{aligned}$$

(c) We have

$$\begin{aligned}\langle\psi(t)|\hat{a}|\psi(t)\rangle &= \mathrm{e}^{-|z|^2} \sum_{m,n} \frac{(z^*\mathrm{e}^{\mathrm{i}\mu m t/\hbar})^m}{\sqrt{m!}} \frac{(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{\sqrt{n!}} \langle m|\hat{a}|n\rangle \\ &= \mathrm{e}^{-|z|^2} \sum_{m,n} \frac{(z^*\mathrm{e}^{\mathrm{i}\mu m t/\hbar})^m}{\sqrt{m!}} \frac{(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{\sqrt{n!}} \sqrt{n}\delta_{m,n-1} \\ &= \mathrm{e}^{-|z|^2} \sum_{n} \frac{(z^*\mathrm{e}^{\mathrm{i}\mu (n-1) t/\hbar})^{n-1}}{\sqrt{(n-1)!}} \frac{(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{\sqrt{n!}} \sqrt{n} \\ &= \mathrm{e}^{-|z|^2} \sum_{n} \frac{(z^*\mathrm{e}^{\mathrm{i}\mu (n-1) t/\hbar})^{n-1}(z\mathrm{e}^{-\mathrm{i}\mu n t/\hbar})^n}{(n-1)!} \\ &= z\mathrm{e}^{-|z|^2} \sum_{n} \frac{\left(|z|^2\right)^{n-1}}{(n-1)!} \mathrm{e}^{\mathrm{i}\mu(n-1)^2 t/\hbar}\mathrm{e}^{-\mathrm{i}\mu n^2 t/\hbar} \\ &= z\mathrm{e}^{-|z|^2} \sum_{n} \frac{\left(|z|^2\right)^{n-1}}{(n-1)!} \mathrm{e}^{-\mathrm{i}\mu(2n-1) t/\hbar} \\ &= z\mathrm{e}^{-|z|^2}\mathrm{e}^{-\mathrm{i}\mu t/\hbar} \sum_{n} \frac{\left(|z|^2\right)^{n-1}}{(n-1)!} \mathrm{e}^{-2\mathrm{i}\mu(n-1) t/\hbar},\end{aligned}$$

Where we have used $2n-1 = 2(n-1)+1$. Thus we find

$$\begin{aligned}\langle\psi(t)|\hat{a}|\psi(t)\rangle &= z\mathrm{e}^{-|z|^2}\mathrm{e}^{-\mathrm{i}\mu t/\hbar} \sum_{n} \frac{(|z|^2\mathrm{e}^{-2\mathrm{i}\mu t/\hbar})^{n-1}}{(n-1)!} \\ &= z\mathrm{e}^{-|z|^2}\mathrm{e}^{-\mathrm{i}\mu t/\hbar}\mathrm{e}^{|z|^2\mathrm{e}^{-2\mathrm{i}\mu t/\hbar}} \\ &= z\mathrm{e}^{-\mathrm{i}\mu t/\hbar}\mathrm{e}^{|z|^2(1-\mathrm{e}^{-2\mathrm{i}\mu t/\hbar})}\end{aligned}$$

3. Wei-Norman form of the time evolution operator for an $SU(2)$ Hamiltonian

(a)

$$\begin{aligned}[\hat{J}_0, \hat{J}_\pm] &= [\hat{J}_0, \hat{J}_1 \pm \mathrm{i}\hat{J}_2] \\ &= \mathrm{i}\hbar\hat{J}_2 \pm \mathrm{i}(-\mathrm{i}\hbar\hat{J}_1) \\ &= \pm\hbar(\hat{J}_1 \pm \mathrm{i}\hat{J}_2) \\ &= \pm\hbar\hat{J}_\pm\end{aligned}$$

$$\begin{aligned}[\hat{J}_+, \hat{J}_-] &= [\hat{J}_1 + \mathrm{i}\hat{J}_2, \hat{J}_1 - \mathrm{i}\hat{J}_2] \\ &= [\hat{J}_1, -\mathrm{i}\hat{J}_2] + [\mathrm{i}\hat{J}_2, \hat{J}_1] \\ &= -\mathrm{i}(\mathrm{i}\hbar\hat{J}_0) + \mathrm{i}(-\mathrm{i}\hbar\hat{J}_0) \\ &= 2\hbar\hat{J}_0\end{aligned}$$

(b) Hadamards lemma states

$$\mathrm{e}^{c\hat{X}}\hat{Y}\mathrm{e}^{-c\hat{X}} = \hat{Y} + c[\hat{X}, \hat{Y}] + \frac{c^2}{2}[\hat{X}, [\hat{X}, \hat{Y}]] + ... + \frac{c^n}{n!}[[\hat{X}, \hat{Y}]]_n + ...$$

if we set $c = -\frac{\mathrm{i}}{\hbar}s$ we obtain

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}s\hat{X}}\hat{Y}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}s\hat{X}} = \hat{Y} - \frac{\mathrm{i}}{\hbar}s[\hat{X}, \hat{Y}] - \frac{s^2}{2\hbar^2}[\hat{X}, [\hat{X}, \hat{Y}]] + ... + \frac{(-\mathrm{i}s)^n}{\hbar^n n!}[[\hat{X}, \hat{Y}]]_n + ...$$

Which is more similar to the form in the question, we wish to calculate

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}s\hat{J}_\pm/\hbar}\hat{J}_0\mathrm{e}^{\mathrm{i}s\hat{J}_\pm/\hbar} &= \hat{J}_0 - \frac{\mathrm{i}}{\hbar}s\underbrace{[\hat{J}_\pm, \hat{J}_0]}_{\mp\hbar J_\pm} - \frac{s^2}{2\hbar^2}\underbrace{[\hat{J}_\pm, \mp\hbar\hat{J}_\pm]}_{=0} \\ &= \hat{J}_0 \pm \mathrm{i}s\hat{J}_\pm \\ \mathrm{e}^{-\mathrm{i}s\hat{J}_0/\hbar}\hat{J}_\pm\mathrm{e}^{\mathrm{i}s\hat{J}_0/\hbar} &= \hat{J}_\pm - \frac{\mathrm{i}}{\hbar}s\underbrace{[\hat{J}_0, \hat{J}_\pm]}_{\pm\hbar J_\pm} - \frac{s^2}{2\hbar^2}\underbrace{[\hat{J}_0, \pm\hbar\hat{J}_\pm]}_{\hbar^2\hat{J}_\pm} + ... + \frac{(-\mathrm{i}s)^n}{\hbar^n n!}\underbrace{[[\hat{J}_0, \hat{J}_\pm]]_n}_{(\pm\hbar)^n\hat{J}_\pm} ... \\ &= \hat{J}_\pm \mathrm{e}^{\mp\mathrm{i}s} \\ \mathrm{e}^{-\mathrm{i}s\hat{J}_\pm/\hbar}\hat{J}_\mp\mathrm{e}^{\mathrm{i}s\hat{J}_\pm/\hbar} &= \hat{J}_\mp - \frac{\mathrm{i}}{\hbar}s\underbrace{[\hat{J}_\pm, \hat{J}_\mp]}_{\pm 2\hbar J_0} - \frac{s^2}{2\hbar^2}\underbrace{[\hat{J}_\pm, \pm 2\hbar J_0]}_{-2\hbar^2\hat{J}_\pm} + ... \frac{(-\mathrm{i}s)^3}{\hbar^3 3!}\underbrace{[\hat{J}_\pm, -2\hbar^2\hat{J}_\pm]}_{=0} \\ &= \hat{J}_\mp \mp 2\mathrm{i}s\hat{J}_0 + s^2\hat{J}_\pm\end{aligned}$$

(c)

$$\hat{U} = \mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\mathrm{e}^{-\mathrm{i}c_-(t)\hat{J}_-/\hbar}.$$

Taking the time derivative

$$\begin{aligned}\mathrm{i}\hbar\dot{\hat{U}} &= \dot{c}_0(t)\hat{J}_0\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\mathrm{e}^{-\mathrm{i}c_-(t)\hat{J}_-/\hbar} \\ &\quad + \dot{c}_+(t)\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\hat{J}_+\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\mathrm{e}^{-\mathrm{i}c_-(t)\hat{J}_-/\hbar} \\ &\quad + \dot{c}_-(t)\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\hat{J}_-\mathrm{e}^{-\mathrm{i}c_-(t)\hat{J}_-/\hbar} \\ &= \dot{c}_0(t)\hat{J}_0\hat{U} \\ &\quad + \dot{c}_+(t)\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\hat{J}_+\mathrm{e}^{\mathrm{i}c_0(t)\hat{J}_0/\hbar}\hat{U} \\ &\quad + \dot{c}_-(t)\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\mathrm{e}^{-\mathrm{i}c_+(t)\hat{J}_+/\hbar}\hat{J}_-\mathrm{e}^{\mathrm{i}c_+(t)\hat{J}_+/\hbar}\mathrm{e}^{\mathrm{i}c_0(t)\hat{J}_0/\hbar}\hat{U}.\end{aligned}$$

Using results from part (b) we have

$$\begin{aligned}
\mathrm{i}\hbar\dot{\hat{U}} &= \dot{c}_0(t)\hat{J}_0\hat{U} \\
&\quad + \dot{c}_+(t)\hat{J}_+\mathrm{e}^{-\mathrm{i}c_0(t)}\hat{U} \\
&\quad + \dot{c}_-(t)\mathrm{e}^{-\mathrm{i}c_0(t)\hat{J}_0/\hbar}\left(\hat{J}_- - 2\mathrm{i}c_+(t)\hat{J}_0 + c_+^2(t)\hat{J}_+\right)\mathrm{e}^{\mathrm{i}c_0(t)\hat{J}_0/\hbar}\hat{U} \\
&= \left(\dot{c}_0(t)\hat{J}_0 + \dot{c}_+(t)\hat{J}_+\mathrm{e}^{-\mathrm{i}c_0(t)} + \hat{J}_-\mathrm{e}^{\mathrm{i}c_0(t)} - 2\mathrm{i}c_+(t)\hat{J}_0 + c_+(t)\hat{J}_+\mathrm{e}^{-\mathrm{i}c_0(t)}\right)\hat{U}.
\end{aligned}$$

inserting the ansatz gives

$$\dot{c}_0(t)\hat{J}_0 + \dot{c}_+(t)\hat{J}_+\mathrm{e}^{-\mathrm{i}c_0(t)} + \hat{J}_-\mathrm{e}^{\mathrm{i}c_0(t)} - 2\mathrm{i}c_+(t)\hat{J}_0 + c_+(t)\hat{J}_+\mathrm{e}^{-\mathrm{i}c_0(t)} = a\hat{J}_0 + \frac{b}{2}(\hat{J}_+ + \hat{J}_-)$$

leading to the relations

$$\begin{aligned}
\dot{c}_0 - 2\mathrm{i}c_+\dot{c}_- &= a \\
\dot{c}_+\mathrm{e}^{-\mathrm{i}c_0} + \dot{c}_-c_+^2\mathrm{e}^{-\mathrm{i}c_0} &= \frac{b}{2} \\
\dot{c}_-\mathrm{e}^{-\mathrm{i}c_0} &= \frac{b}{2}.
\end{aligned}$$

re-arranging these expressions gives

$$\dot{c}_- = \frac{b}{2}\mathrm{e}^{-\mathrm{i}c_0}, \quad \dot{c}_0 = a + \mathrm{i}bc_+\mathrm{e}^{-\mathrm{i}c_0}, \quad \dot{c}_+ = \frac{b}{2}\left(\mathrm{e}^{\mathrm{i}c_0} - c_+^2\mathrm{e}^{-\mathrm{i}c_0}\right).$$