MATH50001 - Problems Sheet 7

1.* Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Prove that either $p(z) = z^n$, or there is a point z_0, $|z_0| = 1$, such that $|p(z)| > 1$.
[Hint: Use the maximum modulus principle and the fact that $q(z) = z^n p(1/z)$ is also a polynomial of degree n].

2.* Is there a holomorphic function f in the open unit disc and such that $|f(z)| = e^{|z|}$?

3.* Prove Schwarz's Lemma: If f is holomorphic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| \le 1$, then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.

4. Prove
$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx = \pi\, e^{-|\xi|}, \qquad \xi \in \mathbb{R}.$$
Show also that the 'inverse Fourier transform" of $\pi e^{-|\xi|}$ equals
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\, e^{-|\xi|}\, e^{ix\xi}\, d\xi = \frac{1}{1+x^2}.$$

5.* Find that for any $n = 2, 3, 4, \ldots$ we have
$$\int_0^{\infty} \frac{1}{1+x^n}\, dx = \frac{\pi/n}{\sin \pi/n}.$$

6. Show that if $0 < a < 1$, then
$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin \pi a}.$$

7. Show that
$$\int_{-\infty}^{\infty} \frac{x-1}{x^5-1}\, dx = \frac{4\pi}{5}\, \sin \frac{2\pi}{5}.$$

8.* Evaluate
$$\int_0^{\infty} \cos(x^2)\, dx.$$