Classical Dynamics **Autumn Term 2022**

Answers to Test 2

1. (a)
$$L = \frac{1}{2}e^{2q}\left(\dot{q}^2 - 1\right).$$

(i) $p = \partial L/\partial \dot{q} = e^{2q}\dot{q}$ so $\dot{q} = e^{-2q}p$

$$H = p\dot{q} - L = p \cdot e^{-2q}p - \frac{1}{2}e^{2q}(e^{-4q}p^2 - 1) = \frac{1}{2}e^{-2q}p^2 + \frac{1}{2}e^{2q}.$$

(ii) Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = e^{-2q}p, \qquad \dot{p} = -\frac{\partial H}{\partial q} = e^{-2q}p^2 - e^{2q}.$$

(iii) $Q = e^q$, $P = e^{-q}p$. As this is time-independent

$$K = H = \frac{1}{2}e^{-2q}p^2 + \frac{1}{2}e^{2q} = \frac{1}{2}(P^2 + Q^2).$$

Hamilton's equations are

$$\dot{Q} = \frac{\partial K}{\partial P} = P, \qquad \dot{P} = -\frac{\partial K}{\partial Q} = -Q.$$

Combining the two equations gives $\ddot{Q} = -Q$ with general solution $Q = A\cos(t+\alpha)$ and so $P = \dot{Q} = -A\sin(t+\alpha)$.

(iv) From the CT $q = \log Q$ and $p = e^q P = QP$. Using the result of part (iii)

$$q = \log\cos(t+\alpha) + \log A, \qquad p = -\frac{A^2}{2}\sin(2t+2\alpha).$$

The motion is not periodic as $q(t)$ is defined for a finite range of t values. Note that solutions for the new Hamiltonian are periodic! [14 marks]

(b)
$$H = xyp_z + yzp_x + xzp_y.$$

(i) Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} = yz, \quad \dot{y} = \frac{\partial H}{\partial p_y} = zx, \quad \dot{z} = \frac{\partial H}{\partial p_z} = xy,$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -yp_z - zp_y, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -zp_x - xp_z, \quad \dot{p}_z = -\frac{\partial H}{\partial z} = -xp_y - yp_x.$$

(ii)

$$\{x^2, H\} = 2x\{x, H\} = 2x\frac{\partial H}{\partial p_x} = 2xyz.$$

Similarly $\{y^2, H\} = 2xyz$ and $\{z^2, H\} = 2xyz$. Accordingly, $\{x^2 - y^2, H\} = 0$ and $\{x^2 - z^2, H\} = 0$.

(iii) Setting $x = y$ and $x^2 - z^2 = 1$ gives

$$\dot{z} = xy = x^2 = z^2 + 1,$$

or

$$\frac{dz}{1 + z^2} = dt,$$

which integrates to $\tan^{-1} z = t + c$ or $z = \tan(t + c)$ where c is an arbitrary constant. Inserting this into Hamilton's equation for $\dot{x}$ yields

$$\dot{x} = yz = xz = x \tan(t + c),$$

or

$$\frac{dx}{x} = \tan(t + c)dt,$$

which integrates to $\log x = -\log \cos(t + c)$+constant, or

$$x = \frac{A}{\cos(t + c)}.$$

The condition $x^2 - z^2 = 1$ fixes $A = \pm 1$. [11 marks]

[Total: 25 marks]