Imperial College London

Introduction to University Mathematics   **MATH40001/MATH40009**

Part I – Problem Sheet 3: Binary relations

This is the last Part I sheet. If you want to try problems related to Part I material (or any other material from the undergraduate course here) in Lean, I'll be happy to talk about it in the MLC (4th floor computer room) on Thursdays from 5pm every week during Imperial's term dates all year.
After this you'll start Part II, where Dr Lawn will explain how to define the natural numbers axiomatically. The natural number game goes through this material in Lean and a lot of people have found this game helpful in the past.

1. ✦ Are the following definitions of R binary relations on the real numbers?

   (a) $R(x, y) \iff x - y$;
   (b) $R(x, y) \iff x^2 < y + 1$;
   (c) $R(x, y, z) \iff x + y = z$;
   (d) $R(x, y) \iff x/y > 0$;
   (e) $R(x, y) \iff (x < y \implies y < x)$.

2. ✦ For each of the sets X and binary relations R below, figure out whether R is (a) reflexive, (b) symmetric, (c) antisymmetric, (d) transitive.

   (a) Let X be the set $\{1, 2\}$ and define R like this: $R(1, 1)$ is true, $R(1, 2)$ is true, $R(2, 1)$ is true and $R(2, 2)$ is false.
   (b) Let $X = \mathbb{R}$ and define $R(a, b)$ to be the proposition $a = -b$.
   (c) Let $X = \mathbb{R}$ and define $R(a, b)$ to be false for all real numbers a and b.
   (d) Let X be the empty set and define R to be the empty binary relation (we don't have to say what its value is on any pair (a, b) because no such pairs exist).

3. ✦ Let X be the set of subsets of the integers. Examples of elements of X would be the set of even numbers, or the set of prime numbers.

   (a) Prove that the binary relation $\subseteq$ on X is a partial order (just to be clear, I mean the binary relation $R(A, B)$ defined by $R(A, B) \iff A \subseteq B$).
   (b) Is $\subseteq$ a total order? Give a proof or a counterexample.

4. ✦ Say $\sim$ is an equivalence relation on $\mathbb{R}$. Say we know that $1 \sim \pi$, that $2 \sim \pi$ and that $3 \sim 2$ (i.e., we know they're all true). Prove using only the axioms of an equivalence relation that $3 \sim 1$.

5. ✦✦✦ In this question you can assume all standard facts about the complex numbers. Prove that there is no total order $\leq$ on the complex numbers satisfying the properties that if $0 \leq a$ and $0 \leq b$ then $0 \leq ab$, and that if $a \leq b$ then for all t we have $t + a \leq t + b$. Note that $\leq$ doesn't have to restrict to the usual $\leq$ on real numbers, e.g. we allow crazy new definitions of $\leq$ on the complexes such that $3 \leq 2$ or whatever.

6. ✦✦ If R is a binary relation on a set X, let's define the *opposite* binary relation R^{op} by $R^{op}(x, y) = R(y, x)$. For example the opposite binary relation to $<$ on the real numbers is $>$.

   Prove that if R is transitive then R^{op} is transitive. Is it true that if R is antisymmetric then R^{op} is antisymmetric?

7. ✦✦ Let X be a set, and say R is a binary relation on X which is symmetric and transitive. We will show that R is reflexive. So let $x \in X$ be arbitrary, and choose $y \in X$ such that $R(x, y)$ is true. Then $R(y, x)$ is true by symmetry, and now because $R(x, y) \wedge R(y, x)$ is true, by transitivity we deduce that $R(x, x)$ is true. Because x was arbitrary, this means that R is reflexive, QED.

   Find another question earlier on this problem sheet which this question contradicts. Where's the mistake?

8. ✦✦ ("pulling back binary relations".) Say $f : X \to Y$ is a function, and S is a binary relation on Y. Define a binary relation R on X in the following way: if $a, b \in X$ then we say $R(a, b) = S(f(a), f(b))$. In other words, a is related to b via the relation R if and only if $f(a)$ is related to $f(b)$ via the relation S.

   Give proofs or counterexamples to the following statements.

   (a) If S is reflexive then R is reflexive.
   (b) If S is symmetric then R is symmetric.
   (c) If S is antisymmetric then R is antisymmetric.
   (d) If S is transitive then R is transitive.
   (e) If S is a partial order than R is a partial order.
   (f) If S is a total order than R is a total order.
   (g) If S is an equivalence relation then R is an equivalence relation.

9. ✦✦ ("pushing forward binary relations".) Say $f : X \to Y$ is a function, and R is a binary relation on X. Define a binary relation A on Y by saying that for $y_1, y_2 \in Y$, $A(y_1, y_2)$ is true if and only if for all $x_1, x_2 \in X$ such that $f(x_i) = y_i$ for $1 \le i \le 2$, $R(x_1, x_2)$ is true. Now define another binary relation E on Y by saying that for $y_1, y_2 \in Y$, $E(y_1, y_2)$ is true if and only if there exists $x_1, x_2 \in X$ such that $f(x_i) = y_i$ for $1 \le i \le 2$ and $R(x_1, x_2)$ is true.

   In the below questions, don't let X and Y be sets like the natural numbers or real numbers, experiment with examples where X and Y are sets with just one or two elements. Try to *prove* the things which I'm claiming are false, see where you get stuck, and then write down a simple counterexample.

   (a) Give an example to show that the relations A and E might not be equal.
   (b) Give an example to show that if R is an equivalence relation then A might not be.
   (c) Give an example to show that if R is an equivalence relation then E might not be.

10. ✦✦✦ Let X be a fixed set. If $\phi : X \to A$ is a function then let's define the equivalence relation R_ϕ on X associated to ϕ by $R_\phi(s, t) \iff \phi(s) = \phi(t)$ (you can check it's an equivalence relation if you like). Of course as ϕ and A vary we can get lots of different equivalence relations on X in this way.

   We say two surjections $f : X \to Y$ and $g : X \to Z$ are *pals* if there exists a bijection $h : Y \to Z$ such that $g = h \circ f$. Prove that f and g are pals if and only if the equivalence relations R_f and R_g on X associated to f and g are equal (by which I mean that for all $s, t \in X$, we have $R_f(s, t) \iff R_g(s, t)$).

   NB if X is is a collection of plastic red, green, yellow and blue squares and triangles, Y is the abstract set {red,green,yellow,blue} and Z is the concrete set whose four elements are the pile of red shapes in X, the pile of yellow shapes in X, the pile of green shapes in X and the pile of blue shapes in X, then the natural maps $X \to Y$ (sending a shape to its colour) and $X \to Z$ (sending a shape to the pile it's in) are pals.