1. (a) (i) The transition matrix is given by $\boxed{\text{meth seen} \Downarrow}$

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\boxed{2}$

(ii) We have a finite state space which can be divided into seven communicating classes: The classes $T_1 = \{1\}, T_2 = \{2\}, T_3 = \{3\}$ are not closed and hence transient. $\boxed{2}$

The classes $C_1 = \{4, 5\}, C_2 = \{6\}, C_3 = \{7\}$ and $C_4 = \{8\}$ are finite and closed and hence positive recurrent. $\boxed{2}$

(iii) Note that we do not have a unique stationary distribution since we have four closed (essential) communicating classes.

Let π denote the vector of all stationary distributions. According to lectures, we know that $\pi_i = 0$ for all transient states i. I.e. $\pi_1 = \pi_2 = \pi_3$. $\boxed{2}$

We determine the remaining components by solving four systems of equations: We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}, \qquad \mathbf{P}(C_i) := 1, \text{ for } i = 2, 3, 4.$$

We need to solve $(\pi_4, \pi_5)\mathbf{P}(C_1) = (\pi_4, \pi_5)$ which results in $\pi_4 = \pi_5$. $\boxed{2}$

Also, we need to solve $\pi_{i+4}\mathbf{P}(C_i) = \pi_{i+4}$ for $i = 2, 3, 4$, which is trivial. $\boxed{2}$

Then all possible stationary distributions are given by $\boldsymbol{\pi} := (0, 0, 0, \pi_4, \pi_4, \pi_6, \pi_7, \pi_8)$ for constants $\pi_4, \pi_6, \pi_7, \pi_8 \geq 0$ such that $2\pi_4 + \pi_6 + \pi_7 + \pi_8 = 1$ since $\pi_i \geq 0$ for $i = 1, \ldots, 8$ and $\sum_{i=1}^{8} \pi_i = 1$, and also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. $\boxed{2}$

(b) We show three properties. $\boxed{\text{seen} \Downarrow}$

1. Clearly for each $j \in E$ we have that $\pi_j \geq 0$ since it is a limit of a non-negative sequence. $\boxed{1}$
2. We get
$$\sum_{j \in E} \pi_j = \sum_{j \in E} \lim_{n \to \infty} p_{ij}(n) = \lim_{n \to \infty} \sum_{j \in E} p_{ij}(n) = 1,$$
since $\mathbf{P}_n$ is stochastic. $\boxed{2}$
3. Also, for all $j \in E$, we get
$$\begin{aligned} \pi_j &= \lim_{n \to \infty} p_{ij}(n) = \lim_{n \to \infty} \sum_{k \in E} p_{ik}(n-1)p_{kj} = \sum_{k \in E} \lim_{n \to \infty} p_{ik}(n-1)p_{kj} \\ &= \sum_{k \in E} \pi_k p_{kj}, \end{aligned}$$
where we used the Chapman-Kolmogorov equations. $\boxed{2}$

Note that we have used the finiteness of E to justify the interchange of summation and limit operations in (2.) and (3.). $\boxed{1}$

2. (a) For any $n \in \mathbb{N}$ and for any states $i_0, \ldots, i_{n+1} \in E$ we have

seen ⇓

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0) = \frac{\mathbb{P}(Y_k = i_k, 0 \leq k \leq n+1)}{\mathbb{P}(Y_k = i_k, 0 \leq k \leq n)}$$
$$= \frac{\mathbb{P}(X_{N-k} = i_k, 0 \leq k \leq n+1)}{\mathbb{P}(X_{N-k} = i_k, 0 \leq k \leq n)}.$$

Now we apply Bayes theorem and the Markov property of (X_n) to deduce that

$$\begin{aligned}
&\mathbb{P}(X_{N-k} = i_k, 0 \leq k \leq n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-k} = i_k, 1 \leq k \leq n+1)\mathbb{P}(X_{N-k} = i_k, 1 \leq k \leq n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-k} = i_k, 1 \leq k \leq n+1) \\
&= \mathbb{P}(X_N = i_0 | X_{N-1} = i_1)\mathbb{P}(X_{N-1} = i_1 | X_{N-2} = i_2) \cdots \mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1}) \\
&\quad \mathbb{P}(X_{N-n-1} = i_{n+1}) \\
&= \pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, Y_0 = i_0) &= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n} \cdots p_{i_1 i_0}}{\pi_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 i_0}} \\
&= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.
\end{aligned}$$

Similarly, we get that

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$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n) = \frac{\mathbb{P}(Y_{n+1} = i_{n+1}, Y_n = i_n)}{\mathbb{P}(Y_n = i_n)} = \frac{\mathbb{P}(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n)}{\mathbb{P}(X_{N-n} = i_n)}$$
$$= \frac{\mathbb{P}(X_{N-n} = i_n | X_{N-n-1} = i_{n+1})\mathbb{P}(X_{N-n-1} = i_{n+1})}{\mathbb{P}(X_{N-n} = i_n)} = \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}}.$$

So overall we have shown that for any $n \in \mathbb{N}$ and for any states $i_0, \ldots, i_{n+1} \in E$ we have that

$$\mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0) = \mathbb{P}(Y_{n+1} = i_{n+1} | Y_n = i_n)$$
$$= \frac{\pi_{i_{n+1}} p_{i_{n+1} i_n}}{\pi_{i_n}},$$

which completes the proof.

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(b) The Markov chain X is called **time–reversible** if the transition matrices of X and its time–reversal Y are the same.

seen ⇓

2

(c) Let Q be the transition matrix of $\{Y_n\}_{n \in \{0,1,\ldots,N\}}$. Then from (a) we have

seen ⇓

$$q_{ij} = p_{ji} \frac{\pi_j}{\pi_i},$$

thus for any $i, j \in E$ we have that

$$q_{ij} = p_{ij} \Leftrightarrow p_{ij} = p_{ji} \frac{\pi_j}{\pi_i} \Leftrightarrow p_{ij}\pi_i = p_{ji}\pi_j,$$

which are the detailed-balance equations.

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(d) $\boxed{\text{unseen} \Downarrow}$

(i) We already know that π is a distribution, so we only need to show that $\pi_j = \sum_{i\in E} \pi_i p_{ij}$, for all $j \in E$. We have

$$\begin{aligned}\sum_{i\in E} \pi_i p_{ij} &= \pi_{j-1}p_{(j-1)j} + \pi_j p_{jj} + \pi_{j+1}p_{(j+1)j} \\ &= \binom{2d}{d}^{-1} d^{-2} \underbrace{\left[\binom{d}{j-1}^2 (d-(j-1))^2 + \binom{d}{j}^2 2j(d-j) + \binom{d}{j+1}^2 (j+1)^2\right]}_{=:A}.\end{aligned}$$

Note that

$$\begin{aligned}\binom{d}{j-1}(d-j+1) &= \frac{d!(d-j+1)}{(d-j+1)!(j-1)!}\frac{j}{j} = j\binom{d}{j}, \\ \binom{d}{j+1}(j+1) &= \frac{d!(j+1)}{(d-j-1)!(j+1)!}\frac{(d-j)}{(d-j)} = (d-j)\binom{d}{j}.\end{aligned}$$

Hence

$$A = \binom{d}{j}^2 \left[j^2 + 2jd - 2j^2 + d^2 - 2dj + j^2\right] = \binom{d}{j}^2 d^2.$$

Overall, we get that for all $j \in E$ we have that

$$\sum_{i\in E} \pi_i p_{ij} = \binom{d}{j}^2 \binom{2d}{d}^{-1} = \pi_j.$$

$\boxed{5}$

(ii) We only need to show that the Markov chain satisfies the detailed-balance equations

$$p_{ij}\pi_i = p_{ji}\pi_j \text{ for all } i, j \in E.$$

We consider the four cases:

Case: $j = i-1$ We find that $p_{i(i-1)}\pi_i = p_{(i-1)i}\pi_{i-1}$ since

$$\begin{aligned}\binom{2d}{d} p_{i(i-1)}\pi_i &= \left(\frac{i}{d}\right)^2 \binom{d}{i}^2 = \frac{i^2(d!)^2}{d^2((d-i)!)^2(i!)^2} = \frac{((d-1)!)^2}{((d-i)!)^2((i-1)!)^2}, \\ \binom{2d}{d} p_{(i-1)i}\pi_{i-1} &= \frac{(d-i+1)^2}{d^2}\binom{d}{(i-1)}^2 = \frac{(d-i+1)^2(d!)^2}{d^2((d-i+1)!)^2((i-1)!)^2} \\ &= \frac{((d-1)!)^2}{((d-i)!)^2((i-1)!)^2}.\end{aligned}$$

Case: $j = i$ We find that $p_{ii}\pi_i = p_{ii}\pi_i$ is trivially true.

Case: $j = i+1$ We find that $p_{i(i+1)}\pi_i = p_{(i+1)i}\pi_{i+1}$ since

$$\begin{aligned}\binom{2d}{d} p_{i(i+1)}\pi_i &= \left(\frac{(d-i)}{d}\right)^2 \binom{d}{i}^2 = \frac{(d-i)^2(d!)^2}{d^2((d-i)!)^2(i!)^2} = \frac{((d-1)!)^2}{((d-i-1)!)^2(i!)^2}, \\ \binom{2d}{d} p_{(i+1)i}\pi_{i+1} &= \frac{(i+1)^2}{d^2}\binom{d}{(i+1)}^2 = \frac{(i+1)^2(d!)^2}{d^2((d-i-1)!)^2((i+1)!)^2} = \frac{((d-1)!)^2}{((d-i-1)!)^2(i!)^2}.\end{aligned}$$

All other cases: are trivially true. $\boxed{5}$

3. (a) For $t > 0$ we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in}[0,t]) = \mathbb{P}(N_t = 0) = e^{-\lambda t},$$

which is the survival function of the Exp(λ) distribution. $\boxed{2}$

(b) (i) Let $u > 0$. Then $\boxed{\text{sim. seen} \Downarrow}$

$$\mathbb{E}(e^{-uX_1}) = \int_0^\infty e^{-ux}\lambda e^{-\lambda x}dx = \int_0^\infty \lambda e^{-(\lambda+u)x}dx = \frac{\lambda}{\lambda+u}.$$

$\boxed{2}$

(ii) Let $u > 0$. Then

$$\begin{aligned}\mathbb{E}(e^{-uY}) &= \int_0^\infty e^{-uy}\frac{\lambda^n}{\Gamma(n)}y^{n-1}e^{-\lambda y}dy = \frac{\lambda^n}{\Gamma(n)}\int_0^\infty y^{n-1}e^{-(\lambda+u)y}dy\\ &= \frac{\lambda^n}{\Gamma(n)}\int_0^\infty z^{n-1}e^{-z}dz\frac{1}{(\lambda+u)^n} = \left(\frac{\lambda}{\lambda+u}\right)^n.\end{aligned}$$

$\boxed{2}$

(c) For $u > 0$, we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{E}[e^{-uT_n}] = \mathbb{E}[e^{-u(\sum_{i=1}^n X_i)}] = \prod_{i=1}^n \mathbb{E}[e^{-uX_i}] = (\mathbb{E}[e^{-uX_1}])^n = \left(\frac{\lambda}{\lambda+u}\right)^n$$

where we used the independence of the X_i in the second step and the identical distributions of the X_i in the third step. We notice (from (b)) that this is indeed the Laplace transform of a Gamma(n, λ) random variable, which concludes the proof. $\boxed{3}$

(d) **Case $t \le x \Leftrightarrow t - x \le 0$:** We have that $\boxed{\text{meth seen} \Downarrow}$

$$\mathbb{P}(Z_t > x) = \mathbb{P}(t - T_{N_t} > x) = \mathbb{P}(t - x > T_{N_t}) = 0,$$

since $T_n \ge 0$ for all $n \in \mathbb{N}_0$. $\boxed{1}$

Case $0 \le x < t$: Here we have that

$$\mathbb{P}(Z_t > x) = \mathbb{P}(t - x > T_{N_t}) = \sum_{n=0}^\infty \mathbb{P}(t - x > T_{N_t}, N_t = n),$$

where we used the law of total probability. The summands can be simplified as follows (when noting that $t - x > 0$): $\boxed{3}$

$$\begin{aligned}&\mathbb{P}(t - x > T_{N_t}, N_t = n) = \mathbb{P}(t - x > T_n, N_t = n)\\ &= \mathbb{P}(t - x > T_n, T_n \le t < T_{n+1}) = \mathbb{P}(t - x > T_n, T_n \le t < T_n + X_{n+1})\\ &= \mathbb{P}(t - x > T_n, t < T_n + X_{n+1}) = \int_0^\infty \mathbb{P}(t - x > T_n, t < T_n + X_{n+1}|T_n = y)f_{T_n}(y)dy\\ &= \int_0^\infty \mathbb{P}(t - x > y, t - y < X_{n+1})f_{T_n}(y)dy = \int_0^{t-x}\frac{\lambda^n}{\Gamma(n)}y^{n-1}e^{-\lambda y}e^{-\lambda(t-y)}dy\\ &= \frac{\lambda^n}{\Gamma(n)}e^{-\lambda t}\int_0^{t-x}y^{n-1}dy = \frac{\lambda^n}{n!}e^{-\lambda t}(t-x)^n,\end{aligned}$$

where we used the fact that X_{n+1} is independent of T_n and that $T_n \sim$ Gamma(n, λ) and $X_{n+1} \sim$ Exp(λ). So, overall we have $\boxed{5}$

$$\mathbb{P}(Z_t > x) = \sum_{n=0}^\infty \frac{\lambda^n}{n!}e^{-\lambda t}(t-x)^n = e^{-\lambda t}e^{\lambda(t-x)} = e^{-\lambda x}.$$

$\boxed{2}$

4. (a) The stochastic process $\{X_t\}_{t\geq 0}$ is called a birth-death process if it satisfies the following properties: $\boxed{\text{seen} \Downarrow}$

1. $\{X_t\}_{t\geq 0}$ is Markov chain on $E = \{0, 1, \dots\}$
2. The infinitesimal transition probabilities are (for $t \geq 0$, $\delta > 0$, $n, m \in \mathbb{N}_0$):

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \text{if } m = 0, \\ \lambda_n \delta + o(\delta) & \text{if } m = 1 \\ \mu_n \delta + o(\delta) & \text{if } m = -1 \\ o(\delta) & \text{if} |m| > 1 \end{cases}$$

3. The birth rates $\lambda_0, \lambda_1, \dots$ and the death rates $\mu_0, \mu_1, \dots$ satisfy

$$\lambda_i \geq 0 \quad \mu_i \geq 0 \quad \mu_0 = 0.$$

$\boxed{4}$

(b) (i) Let $t \geq 0$ and $\delta > 0$. Then $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} p_{i0}(t+\delta) &= \mathbb{P}(X_{t+\delta} = 0 | X_0 = i) = \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = 0 | X_t = k, X_0 = i)\mathbb{P}(X_t = k | X_0 = i) \\ &= \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = 0 | X_t = k)\mathbb{P}(X_t = k | X_0 = i) = \sum_{k\in\mathbb{N}_0} p_{k0}(\delta) p_{ik}(t) \\ &= p_{00}(\delta)p_{i0}(t) + p_{10}(\delta)p_{i1}(t) + o(\delta) = (1 - \lambda_0\delta)p_{i0}(t) + \mu_1\delta p_{i1}(t) + o(\delta), \end{aligned}$$

where we used the law of total probability, the Markov property and the infinitesimal transition probabilities of the birth-death process. Subtracting $p_{i0}(t)$ on both sides, dividing by δ and sending $\delta \to 0$ leads to

$$p'_{i0}(t) = -\lambda_0 p_{i0}(t) + \mu_1 p_{i1}(t) = -\alpha p_{i0}(t) + \mu p_{i1}(t), \quad t \geq 0.$$

Similarly, for $j \in \mathbb{N}$, we get $\boxed{3}$

$$\begin{aligned} p_{ij}(t+\delta) &= \mathbb{P}(X_{t+\delta} = j | X_0 = i) = \sum_{k\in\mathbb{N}_0} \mathbb{P}(X_{t+\delta} = j | X_t = k)\mathbb{P}(X_t = k | X_0 = i) \\ &= \sum_{k\in\mathbb{N}_0} p_{kj}(\delta) p_{ik}(t) \\ &= p_{(j-1)j}(\delta)p_{i(j-1)}(t) + p_{jj}(\delta)p_{ij}(t) + p_{(j+1)j}(\delta)p_{i(j+1)}(t) + o(\delta) \\ &= \lambda_{j-1}\delta p_{i(j-1)}(t) + (1 - (\lambda_j + \mu_j)\delta)p_{ij}(t) + \mu_{j+1}\delta p_{i(j+1)}(t) + o(\delta). \end{aligned}$$

Hence

$$\begin{aligned} p'_{ij}(t) &= \lambda_{j-1} p_{i(j-1)}(t) - (\lambda_j + \mu_j)p_{ij}(t) + \mu_{j+1} p_{i(j+1)}(t) \\ &= (\lambda(j-1) + \alpha)p_{i(j-1)}(t) - ((\lambda + \mu)j + \alpha)p_{ij}(t) + \mu(j+1)p_{i(j+1)}(t), \quad t \geq 0. \end{aligned}$$

$\boxed{4}$

(ii) We know that the marginal distribution of X_t is given by $\nu^{(t)} = \nu^{(0)}\mathbf{P}_t$, where the row vector $\nu^{(0)}$ satisfies $\nu_j^{(0)} = 1$ if $j = n_0$ and $\nu_j^{(0)} = 0$ otherwise. Hence $\boxed{\text{unseen} \Downarrow}$

$$M(t) = \mathbb{E}(X_t) = \sum_{j=0}^{\infty} j\mathbb{P}(X_t = j) = \sum_{j=1}^{\infty} j\nu_j^{(t)} = \sum_{j=1}^{\infty} j p_{n_0 j}(t).$$

$\boxed{3}$

(iii) Let $t \geq 0$ and $\delta > 0$. Using properties of conditional expectation, we get that $M(t+\delta) = \mathbb{E}(\mathbb{E}(X_{t+\delta}|X_t))$. Note that $\boxed{\text{unseen} \Downarrow}$ $\boxed{1}$

$$
\begin{aligned}
\mathbb{E}(X_{t+\delta}|X_t = i) &= \sum_{k\in\mathbb{N}_0} k\mathbb{P}(X_{t+\delta} = k|X_t = i) = \sum_{k\in\mathbb{N}_0} kp_{ik}(\delta) \\
&= (i-1)p_{i(i-1)}(\delta) + ip_{ii}(\delta) + (i+1)p_{i(i+1)}(\delta) + o(\delta) \\
&= (i-1)\mu_i\delta + i(1-(\lambda_i+\mu_i)\delta) + (i+1)\lambda_i\delta + o(\delta) \\
&= (i-1)i\mu\delta + i(1-((\lambda+\mu)i+\alpha)\delta) + (i+1)(i\lambda+\alpha)\delta + o(\delta) \\
&= (i^2-i)\mu\delta + i - ((\lambda+\mu)i^2+\alpha i)\delta + (i^2+i)\lambda\delta + \alpha(i+1)\delta + o(\delta).
\end{aligned}
$$

Hence $\boxed{3}$

$$
\begin{aligned}
M(t+\delta) &= (\mathbb{E}(X_t^2) - M(t))\mu\delta + M(t) - ((\lambda+\mu)\mathbb{E}(X_t^2) + \alpha M(t))\delta \\
&\quad + (\mathbb{E}(X_t^2) + M(t))\lambda\delta + \alpha(M(t)+1)\delta + o(\delta) \\
&= -M(t)\mu\delta + M(t) + M(t)\lambda\delta + \alpha\delta + o(\delta).
\end{aligned}
$$

Subtracting $M(t)$ on both sides, dividing by δ and sending $\delta \to 0$ leads to

$$M'(t) = \alpha + M(t)(\lambda - \mu), \quad t \geq 0.$$

$\boxed{2}$