Mathematical Logic (MATH6/70132;P65)
Problem Sheet 4

[1] In each of the following cases a first-order language $\mathcal{L}_i$ and two $\mathcal{L}_i$-structures $\mathcal{A}_i$, $\mathcal{B}_i$ are given. In each case, write down a sentence of $\mathcal{L}_i$ which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$. Explain your answers briefly (your argument need not involve valuations).
(a) $\mathcal{L}_1$ has a single binary relation symbol R. The domain of $\mathcal{A}_1$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$. The domain of $\mathcal{B}_1$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$.
(b) $\mathcal{L}_2$ has a single binary relation symbol R. The domain of $\mathcal{A}_2$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$. The domain of $\mathcal{B}_2$ is $\mathbb{Q}$ (the set of rational numbers) and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$.
(c) $\mathcal{L}_3$ has a single unary function symbol f and a single binary relation symbol E. The domain of $\mathcal{A}_3$ is $\mathbb{N}$ and f is interpreted as the function $x_1 \mapsto x_1 + 1$. The domain of $\mathcal{B}_3$ is $\mathbb{Z}$ and f is interpreted as the function $x_1 \mapsto x_1 + 2$. In both structures E is interpreted as equality.
(d) $\mathcal{L}_4$ has a single binary relation symbol R. The domain of $\mathcal{A}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 3'. The domain of $\mathcal{B}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 5'.

[2] (a) Show (by giving an argument involving valuations) that for any formula ϕ the following formula is logically valid:

$$((\exists x_1)(\forall x_2)\phi \to (\forall x_2)(\exists x_1)\phi).$$

(b) Give an example of a formula ϕ and an interpretation where the following is false:

$$((\forall x_1)(\exists x_2)\phi \to (\exists x_2)(\forall x_1)\phi).$$

[3] Let $\mathcal{L}$ be a first-order language with a binary relation symbol R. A *strict partial order* is an $\mathcal{L}$-structure which is a model of the closed formula ϕ:

$$(\forall x_1)(\forall x_2)(\forall x_3)((\neg R(x_1, x_1)) \wedge ((R(x_1, x_2) \wedge R(x_2, x_3)) \to R(x_1, x_3))).$$

(So in a model of this formula, the interpretation of R behaves like $<$.)

(a) Show that in any model of ϕ the formula ψ given by:

$$(\forall x_1)(\forall x_2)(R(x_1, x_2) \to (\neg R(x_2, x_1)))$$

is true.

(b) Write down an $\mathcal{L}$-formula χ which has a model and is such that any $\mathcal{L}$-structure which is a model of χ is infinite.

[4] Suppose $\mathcal{L}$ is a first-order language and $\phi(x_1)$ is an $\mathcal{L}$-formula with a free variable x_1 and possibly other free variables. Under what circumstances is the formula

$$((\forall x_1)\phi(x_1) \to (\forall x_2)\phi(x_2))$$

logically valid? Justify your answer.

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