

**Partial Differential Equations in Action**

**MATH50008**

**Solutions to Coursework 2**

1. **Total: 8 Marks** Here, we consider a thermally conducting rectangular plate such that  $0 \leq x \leq L_1$  and  $0 \leq y \leq L_2$ . The steady-state temperature profile in the plate is governed by Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We want to solve this equation subject to the following boundary conditions:

$$u(0, y) = 0, \quad u(L_1, y) = T_1, \quad u(x, 0) = 0, \quad u(x, L_2) = T_2$$

Here the problem is that the way the problem is formulated, we do not have a problem with homogeneous boundary conditions and so we do not know how to apply the method of separation of variables. Now, using the superposition principle, we write that the solution to this problem can be decomposed as  $u(x, y) = u_1(x, y) + u_2(x, y)$ , where  $u_1$  and  $u_2$  are solutions to the following problems with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_x^2 u_1 + \partial_y^2 u_1 = 0 \\ u_1(0, y) = 0, \quad u_1(L_1, y) = T_1, \quad u_1(x, 0) = 0, \quad u_1(x, L_2) = 0 \end{cases}$$

$$\begin{cases} \partial_x^2 u_2 + \partial_y^2 u_2 = 0 \\ u_2(0, y) = 0, \quad u_2(L_1, y) = 0, \quad u_2(x, 0) = 0, \quad u_2(x, L_2) = T_2 \end{cases}$$

First, we look for separated solutions of the form  $u_2(x, t) = X(x)Y(y)$  which leads to

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

where the BVP problem for  $X$  has homogeneous boundary conditions. As we have homogeneous Dirichlet BCs in  $x$ , we easily obtain

$$X_n(x) = a_n \sin\left(\frac{n\pi x}{L_1}\right), \quad n \geq 1$$

following the same arguments as usual.

The ODE for  $Y$  is now given by

$$Y'' - \left(\frac{n\pi}{L_1}\right)^2 Y = 0$$

which has for solutions

$$Y(y) = A_n \cosh\left(\frac{n\pi y}{L_1}\right) + B_n \sinh\left(\frac{n\pi y}{L_1}\right)$$

So we get the following product solutions

$$u_{2,n}(x, y) = \left[ A_n \cosh\left(\frac{n\pi y}{L_1}\right) + B_n \sinh\left(\frac{n\pi y}{L_1}\right) \right] \sin\left(\frac{n\pi x}{L_1}\right), \quad n \geq 1$$

As  $u_2(x, 0) = 0$ , then we have  $A_n = 0$  for all  $n$ . Leading to the overall solution

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi y}{L_1}\right) \sin\left(\frac{n\pi x}{L_1}\right)$$

Further the second BC in  $y$  reads

$$T_2 = u_2(x, L_2) = \sum_{n=1}^{\infty} B_n \sinh\left(n\pi \frac{L_2}{L_1}\right) \sin\left(\frac{n\pi x}{L_1}\right)$$

The Fourier sine series of  $T_2$  is given by

$$T_2 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L_1}\right)$$

with

$$b_n = \frac{2}{L_1} \int_0^{L_1} T_2 \sin\left(\frac{n\pi x}{L_1}\right) dx$$

Recall that

$$\int_0^{L_1} \sin\left(\frac{n\pi x}{L_1}\right) dx = \frac{L_1}{n\pi} [1 - (-1)^n]$$

This leads to

$$B_n \sinh\left(n\pi \frac{L_2}{L_1}\right) = \frac{2T_2}{\pi n} [1 - (-1)^n]$$

and we finally conclude that

$$u_2(x, y) = \frac{2T_2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n} \frac{\sinh(n\pi y/L_1) \sin(n\pi x/L_1)}{\sinh(n\pi L_2/L_1)}$$

We can directly write by symmetry that

$$u_1(x, y) = \frac{2T_1}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n} \frac{\sinh(n\pi x/L_2) \sin(n\pi y/L_2)}{\sinh(n\pi L_1/L_2)}$$

and our final solution reads

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

2. **Total: 12 Marks** Here, we consider the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u &= 1, \quad 0 < x < \pi \\ u(x, 0) &= f(x), \quad 0 < x < \pi \\ \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0 \end{aligned}$$

(a) Let's first proceed to the method of separation of variables for the homogeneous problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u &= 0, \quad 0 < x < \pi \\ u(x, 0) &= f(x), \quad 0 < x < \pi \\ \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0 \end{aligned}$$

We look for solutions of the form  $u(x, t) = X(x)T(t)$  leading to

$$XT' - X''T + XT = 0 \Rightarrow \frac{T' + T}{T} = \frac{X''}{X} = -\lambda^2$$

where the sign of the separation constant is guided by the requirement that we are looking for non-trivial solutions.

In particular, this leads to the following boundary value problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$$

whose general solution is given by  $X(x) = A \cos \lambda x + B \sin \lambda x$ . The boundary conditions impose the following

$$X'(0) = 0 \Rightarrow B = 0$$

$$X'(\pi) = 0 \Rightarrow \sin \lambda \pi = 0 \Rightarrow \lambda \pi = n\pi, n \geq 0 \Rightarrow \lambda = n, n \geq 0$$

Finally, the solution thus reads:

$$u(x, t) = \sum_{n=0}^{\infty} T_n(t) \cos(nx) = T_0(t) + \sum_{n=1}^{\infty} T_n(t) \cos(nx) \quad (\star)$$

**4 Marks**

(b) Subbing  $(\star)$  in the original inhomogeneous PDE, we obtain

$$T'_0(t) + T_0(t) + \sum_{n=1}^{\infty} [T'_n(t) + n^2 T_n(t) + T_n(t)] \cos(nx) = 1$$

So using the Fourier cosine series of 1, we can identify term-by-term and we obtain

$$T'_0(t) + T_0(t) = 1$$

$$T'_n(t) + (n^2 + 1)T_n(t) = 0, n \geq 1$$

We can integrate both equation and write

$$T_0(t) = 1 + A_0 e^{-t}$$

$$T_n(t) = A_n e^{-(n^2+1)t}, n \geq 1$$

We thus conclude that the final solution to the inhomogeneous problem reads

$$u(x, t) = 1 + A_0 e^{-t} + \sum_{n=1}^{\infty} A_n e^{-(n^2+1)t} \cos(nx)$$

**4 Marks**

(c) Finally, if  $f(x) = \cos(x) + \cos(2x)$ , then by the orthogonality of the family of functions  $\{\cos(nx)\}_{n \geq 0}$ , we have

$$u(x, 0) = \cos(x) + \cos(2x) = 1 + A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

which leads to

$$\begin{cases} 1 + A_0 = 0 \Rightarrow A_0 = -1 \\ A_1 = 1 \\ A_2 = 1 \\ A_n = 0, \text{ otherwise} \end{cases}$$

The particular solution thus reads

$$u(x, t) = 1 - e^{-t} + e^{-2t} \cos(x) + e^{-5t} \cos(2x)$$

**4 Marks**