IMPERIAL

MATH70063

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebra 4

Date: Wednesday, May 22, 2024
Time: 14:00 – 16:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Compute the following. You should justify your answers.

   (i) $\mathrm{Hom}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/9\mathbb{Z})$

   (ii) $\mathrm{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$

   (iii) $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$

   (iv) $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

(8 marks)

(b) Let R be an associative ring with unit, and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of left R-modules. Let M be a left R-module, and consider the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0 \qquad (\star)$$

For each of the three non-zero terms in this sequence, either prove that $(\star)$ is exact at this term for all M or give a counterexample to this statement.

(9 marks)

(c) Define what it means for the R-module M to be *projective*. How does your answer to part (b) change if M is assumed to be projective? (3 marks)

(Total: 20 marks)

2. (a) Let R be an associative ring with unit, let M and N be left R-modules, and let $n \geq 0$. Give a brief explanation of how the groups $\mathrm{Ext}^n_R(M, N)$ are defined and computed. (You should state the main points but omit all proofs.) (5 marks)

(b) Let A and B be abelian groups.

(i) Define what it means for an abelian group E to be an *extension* of A by B. Define what it means for two such extensions to be *equivalent*. (3 marks)

(ii) Suppose that E_1 and E_2 are extensions of A by B. Define the *Baer sum* of E_1 and E_2. (3 marks)

(iii) State a theorem that relates extensions of A by B to the functor Ext^1. (3 marks)

(c) Use parts (a) and (b) to show that every extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$ in the category of abelian groups is split. (6 marks)

(Total: 20 marks)

3. (a) Let a and b be integers with $a, b \geq 2$. Let A and B be the $\mathbb{Z}$-modules given by $A = \mathbb{Z}/a\mathbb{Z}$ and $B = \mathbb{Z}/b\mathbb{Z}$. Compute the following. You should justify your answers.

   (i) $A \otimes_{\mathbb{Z}} B$

   (ii) $A \otimes_{\mathbb{Z}} \mathbb{Q}$

(5 marks)

(b) Let n be an integer with $n > 2$ and let R be the ring $\mathbb{Z}/n\mathbb{Z}$. Show that R is injective as a left R-module. (You may use results from lectures without proof provided that you state them clearly.) (7 marks)

(c) Let S be an associative ring with unit. Suppose that

$$0 \longrightarrow A \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_n \longrightarrow B \longrightarrow 0$$

is an exact sequence of right S-modules, and that each P_i is projective. Let M be any left S-module. Show that

$$\mathrm{Tor}_m^S(A, M) \cong \mathrm{Tor}_{n+m}^S(B, M)$$

for all $m > 0$. (8 marks)

(Total: 20 marks)

4. (a) Let G be a group and M be a G-module. Define the *group cohomology* $H^k(G, M)$.

(2 marks)

(b) Let G be a finite group and M be a finite G-module. Show that $H^k(G, M)$ is finite for all $k \geq 0$.

(2 marks)

(c) Let n be an integer with $n \geq 2$ and let G be the cyclic group of order n. Let M be a trivial G-module – that is, an abelian group equipped with a trivial action of G. Show that

$$H^k(G, M) = \begin{cases} M & k = 0 \\ M[n] & k \text{ odd, } k \geq 1 \\ M/nM & k \text{ even, } k \geq 2 \end{cases}$$

where $M[n]$ denotes the n-torsion subgroup of M – that is, the subgroup of M consisting of the elements x such that $nx = 0$. (You may use results from lectures without proof provided that you state them clearly.)

(8 marks)

(d) Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups. Use part (c) to show that there is a long exact sequence

$$0 \longrightarrow A[n] \longrightarrow B[n] \longrightarrow C[n] \longrightarrow A/nA \longrightarrow B/nB \longrightarrow C/nC \longrightarrow 0$$

(8 marks)

(Total: 20 marks)

5. (a) Let $\mathcal{A}$ be an abelian category, let I be a set, and suppose that A_i, $i \in I$, are objects of $\mathcal{A}$. Define what is meant by the *coproduct* $\coprod_{i\in I} A_i$. (2 marks)

(b) Recall that a category $\mathcal{I}$ is small if and only if $|\mathcal{I}|$ is a set, and that an abelian category $\mathcal{A}$ is cocomplete if and only if $\operatorname{colim}_{i\in\mathcal{I}} F(i)$ exists for every small category $\mathcal{I}$ and every functor $F\colon \mathcal{I} \to \mathcal{A}$. Let $\mathcal{A}$ be an abelian category. Show that the following are equivalent:

1. The coproduct $\coprod_{i\in I} A_i$ exists for every set of objects $\{A_i \in |\mathcal{A}| : i \in I\}$.
2. $\mathcal{A}$ is cocomplete.

(10 marks)

(c) Show that any abelian group G is isomorphic to the direct limit of its finitely generated subgroups. (8 marks)

(Total: 20 marks)

MATH70063: FINAL EXAM WITH SOLUTIONS

Difficulty summary: Category A 32/80, Category B 19/80, Category C 13/80, Category D 16/80. This excludes the mastery question.

(1) (a) Compute the following. You should justify your answers.

(i) $\mathrm{Hom}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/9\mathbb{Z})$
(ii) $\mathrm{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$
(iii) $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$
(iv) $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

(8 marks, seen similar, A)

Solution:

(i) The map is determined by the image of $1 \in \mathbb{Z}/6\mathbb{Z}$ in $\mathbb{Z}/9\mathbb{Z}$, which must be a 6-torsion element of $\mathbb{Z}/9\mathbb{Z}$. Addition of maps corresponds to addition of these image elements. The 6-torsion elements are 0, 3, and 6, so $\mathrm{Hom}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$. (2 marks)

(ii) Same argument but for 9-torsion elements of $\mathbb{Z}/6\mathbb{Z}$. These are 2, 4, and 6, so $\mathrm{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$. (2 marks)

(iii) Let x be any element of $\mathbb{Q}/\mathbb{Z}$, and let f be any element of $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$. Choose y such that $3y = x$; this exists because $\mathbb{Q}$ is divisible. Then $f(x) = f(3y) = 3f(y) = 0$, so f is the zero map. Hence $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = \{0\}$. (2 marks)

(iv) The only 3-torsion elements of $\mathbb{Q}/\mathbb{Z}$ are the equivalence classes of 0, 1/3, and 2/3. Hence, by the argument of (i) again, $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$. (2 marks)

(b) Let R be an associative ring with unit, and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of left R-modules. Let M be a left R-module, and consider the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0 \qquad (\star)$$

For each of the three non-zero terms in this sequence, either prove that $(\star)$ is exact at this term for all M or give a counterexample to this statement.

(9 marks, seen similar, B)

Solution:

(i) The sequence is exact at $\mathrm{Hom}_R(M, A)$. For if $f \in \mathrm{Hom}_R(M, A)$ is such that it maps to zero in $\mathrm{Hom}_R(M, B)$ then $\alpha \circ f(x)$ is zero for all $x \in M$, and therefore $f(x) \in \ker \alpha$ for all $x \in M$. But α is injective by assumption, so $f(x) = 0$ for all $x \in M$. Thus f is the zero map, as required. (2 marks)

(ii) The sequence is exact at $\mathrm{Hom}_R(M, B)$. The sequence is evidently a complex here, because $\beta \circ \alpha = 0$. Furthermore, if $f \in \mathrm{Hom}_R(M, B)$ is such that it maps to zero in $\mathrm{Hom}_R(M, C)$ then $\beta \circ f(x) = 0$ for all $x \in M$. Thus $f(x) \in \ker \beta$, and by exactness of the original sequence there exists a unique element $a(x) \in A$ such that $\alpha(a(x)) = f(x)$. Define a map $g\colon M \to A$ by $x \mapsto a(x)$. This is well-defined as a map of sets, and the uniqueness of $a(x)$ guarantees that g is also a homomorphism of R-modules. By construction $g \in \mathrm{Hom}_R(M, A)$ maps to f, which proves exactness at $\mathrm{Hom}_R(M, B)$. (4 marks)

(iii) The sequence is not in general exact at $\mathrm{Hom}_R(M, B)$ – a counterexample is any M such that $\mathrm{Ext}^1(M, A)$ is non-zero. (3 marks)

(c) Define what it means for M to be a projective R-module. How does your answer to part (b) change if M is assumed to be projective?

(3 marks, seen, A)

Solution: If M is projective then the sequence $(\star)$ is exact. We only need to check surjectivity of $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$, which is immediate from the definition of projectivity of M. (3 marks)

(2) (a) Let R be an associative ring with unit, let M and N be left R-modules, and let $n \geq 0$. Sketch how the groups $\mathrm{Ext}^n_R(M, N)$ are defined and computed.

(5 marks, seen, A)

Solution: Main points:

- Ext^n_R are the right derived functors of Hom_R.
- They are computed by first taking an injective resolution of B:

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

  (such resolutions exist)
- ... and then applying the left-exact functor $\mathrm{Hom}_R(A, -)$ to get a complex:

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A, I^0) \longrightarrow \mathrm{Hom}_R(A, I^1) \longrightarrow \cdots$$

- Then $\mathrm{Ext}^n_R(A, B)$ is the nth homology group of this complex.
- It is independent of the choice of injective resolution of B.

(5 marks)

(b) Let A and B be abelian groups.

(i) Define what it means for an abelian group E to be an extension of A by B. Define what it means for two such extensions to be equivalent.

(3 marks, seen, A)

(ii) Suppose that E_1 and E_2 are extensions of A by B. Define the Baer sum of E_1 and E_2. (3 marks, seen, A)

(iii) State a theorem that relates extensions of A by B to the functor Ext^1.

(3 marks, seen, A)

Solution: This is all bookwork.

(i) E is an extension of A by B if and only if there exists a short exact sequence of abelian groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

Two such extensions are equivalent if there is a commutative diagram of group homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where the left-hand vertical arrow is the identity map on B and the right-hand vertical arrow is the identity map on A. (3 marks)

(ii) The Baer sum of extensions

$$0 \longrightarrow B \xrightarrow{\alpha_1} E_1 \xrightarrow{\beta_1} A \longrightarrow 0$$

and

$$0 \longrightarrow B \xrightarrow{\alpha_2} E_2 \xrightarrow{\beta_2} A \longrightarrow 0$$

is

$$0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0$$

where E is the homology of the complex

$$B \xrightarrow{(\alpha_1,-\alpha_2)} E_1 \oplus E_2 \xrightarrow{\beta_1-\beta_2} A$$

The map α is induced by $(\alpha_1, 0)$ and the map β is induced by β_1. (3 marks)

(iii) There is a one-to-one correspondence between equivalence classes of extensions of A by B and $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$. This correspondence identifies the Baer sum of equivalence classes with addition in the $\mathbb{Z}$-module $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$, and identifies the equivalence class of split extensions with the zero element in $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$. (3 marks)

(c) Use parts (a) and (b) to show that every extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$ in the category of abelian groups is split. (6 marks, seen similar, C)

Solution: It suffices to show that $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$ vanishes, where $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/3\mathbb{Z}$. Consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow A \longrightarrow 0$$

where the second arrow is multiplication by 2. This is a free, and hence projective, resolution of A in the category of abelian groups. We now compute $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$ using the balancing theorem, by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, B)$ to the above resolution and then taking cohomology. We obtain the complex:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \xrightarrow{2} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

and since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) = B$ this is

$$\mathbb{Z}/3\mathbb{Z} \xrightarrow{2} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Multiplication by 2 gives an isomorphism $\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$, and therefore all cohomology groups of this complex vanish. Thus $\mathrm{Ext}^1(A, B) = 0$, and every extension of A by B is split, as required. (6 marks)

(3) (a) Let a and b be integers with a, $b \geq 2$. Let A and B be the $\mathbb{Z}$-modules given by $A = \mathbb{Z}/a\mathbb{Z}$ and $B = \mathbb{Z}/b\mathbb{Z}$. Compute the following. You should justify your answers.

(i) $A \otimes_{\mathbb{Z}} B$

(ii) $A \otimes_{\mathbb{Z}} \mathbb{Q}$

(5 marks, seen, A)

Solution:

(i) I claim that $A \otimes_{\mathbb{Z}} B$ is isomorphic to A/bA. The bilinear map $A \times B \to A/bA$ given by $(x, y) \mapsto [xy]$ induces, by the universal property of tensor product, a ring homomorphism $\phi : A \otimes_{\mathbb{Z}} B \to A/bA$ such that $\phi(x \otimes y) = [xy]$. The map $\psi : A/bA \to A \otimes_{\mathbb{Z}} B$ that sends $[x]$ to $x \otimes 1$ is evidently an inverse to ϕ, so $A \otimes_{\mathbb{Z}} B \cong A/bA$. And now $A/bA \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$ by the Chinese Remainder Theorem. (3 marks)

(ii) Let $x \in A$ and $q \in \mathbb{Q}$. Then $x \otimes q = (ax) \otimes (q/a) = 0 \otimes (q/a) = 0$. Thus $A \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$. (2 marks)

(b) Let n be an integer with $n > 2$ and let R be the ring $\mathbb{Z}/n\mathbb{Z}$. Show that R is injective as a left R-module. (7 marks, unseen, C)

Solution: We apply Baer's criterion for injectivity (Theorem 1.8 in the lecture notes). Any ideal $I \subset R$ is principal, so we can write $I = aR$ for some $a > 0$ with $a|n$. A map of R-modules $f : I \to R$ is determined by the image $f(a)$ of a, and $f(a)$ is annihilated by $b = n/a$. But the set of elements annihilated by b is I, so $f(a) \in I$ and therefore $f(a) = as$ for some $s \in R$. We can therefore extend f to a map $R \to R$ by sending $1 \in R$ to s. Baer's criterion now implies that R is injective. (7 marks)

(c) Let S be an associative ring with unit. Suppose that

$$0 \longrightarrow A \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_n \longrightarrow B \longrightarrow 0$$

is an exact sequence of right S-modules, and that each P_i is projective. Let M be any left S-module. Show that

$$\mathrm{Tor}^S_m(A, M) \cong \mathrm{Tor}^S_{n+m}(B, M)$$

for all $m > 0$. (8 marks, unseen, B)

Solution: First observe that if

$$0 \longrightarrow K \longrightarrow P \longrightarrow L \longrightarrow 0$$

is a short exact sequence of right S-modules and P is projective then $\mathrm{Tor}^S_m(K, M) = \mathrm{Tor}^S_{m+1}(L, M)$. This follows from the long exact sequence of Tor groups

$$\cdots \longrightarrow \mathrm{Tor}_{m+1}(P, M) \longrightarrow \mathrm{Tor}_{m+1}(L, M) \longrightarrow$$

$$\mathrm{Tor}_m(K, M) \longrightarrow \mathrm{Tor}_m(P, M) \longrightarrow \mathrm{Tor}_m(L, M) \longrightarrow \cdots$$

and the fact that P is projective, and hence flat.

Now break the given long exact sequence into short exact sequences

$$0 \longrightarrow A \longrightarrow P_1 \longrightarrow K_1 \longrightarrow 0$$

$$0 \longrightarrow K_1 \longrightarrow P_2 \longrightarrow K_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow K_n \longrightarrow P_n \longrightarrow B \longrightarrow 0$$

where K_i is the kernel of $P_i \to P_{i+1}$, and apply the above result n times.

(8 marks)

(4) (a) Let G be a group and M be a G-module. Define the group cohomology $H^k(G, M)$.

(2 marks, seen, A)

Solution: $H^k(G, M) = \mathrm{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ where $\mathbb{Z}$ is a trivial G-module. (2 marks)

(b) Let G be a finite group and M be a finite G-module. Show that $H^k(M, G)$ is finite for all $k \geq 0$. (2 marks, seen, B)

Solution: In lectures we showed that $H^k(M, G)$ is the kth cohomology of a complex

$$0 \longrightarrow M \xrightarrow{d} \mathrm{Fun}(G, M) \xrightarrow{d} \mathrm{Fun}(G \times G, M) \xrightarrow{d} \cdots$$

All terms in this complex are finite, and so its cohomology groups are finite too. (2 marks)

(c) Let n be an integer with $n \geq 2$ and let G be the cyclic group of order n. Let M be a trivial G-module – that is, an abelian group equipped with a trivial action of G. Show that

$$H^k(G, M) = \begin{cases} M & k = 0 \\ M[n] & k \text{ odd}, k \geq 1 \\ M/nM & k \text{ even}, k \geq 2 \end{cases}$$

where $M[n]$ denotes the n-torsion subgroup of M – that is, the subgroup of M consisting of the elements x such that $nx = 0$. (8 marks, seen similar, D)

Solution: Let G be the cyclic group of order n with generator s, let $N \in \mathbb{Z}[G]$ be the norm element $1+s+\cdots+s^{n-1}$, let $N|_M$ denote the map $M \to M$ defined by N, and let M be any G-module. In lectures we proved (as Theorem 5.6) that:

$$H^k(G, M) = \begin{cases} M^G & k = 0 \\ \ker(N|_M)/(1-s)M & k \text{ odd}, k \geq 1 \\ M^G/NM & k \text{ even}, k \geq 2 \end{cases}$$

In the case at hand, M is a trivial G-module and so $M^G = G$. Furthermore $N|_M$ is multiplication by n. This proves the statement for k even. For k odd, observe that $(1-s)M$ is zero, and that $\ker N|_M$ is the kernel of multiplication by n on M, that is, the n-torsion subgroup $M[n]$. (8 marks)

(d) Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups. Use part (c) to show that there is a long exact sequence

$$0 \longrightarrow A[n] \longrightarrow B[n] \longrightarrow C[n] \longrightarrow A/nA \longrightarrow B/nB \longrightarrow C/nC \longrightarrow 0$$

(8 marks, unseen, D)

Solution:

Equip A, B, and C with trivial G-module structures, where G is the cyclic group of order n. The long exact sequence for group cohomology:

$$\begin{tikzcd} 0 \arrow[r] & H^0(G,A) \arrow[r] & H^0(G,B) \arrow[r] & H^0(G,C) \arrow[r] & {} \\ {} \arrow[r] & H^1(G,A) \arrow[r] & H^1(G,B) \arrow[r] & H^1(G,C) \arrow[r] & \cdots \end{tikzcd}$$

plus part (c) gives a long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & {} \\ {} \arrow[r] & A[n] \arrow[r] & B[n] \arrow[r] & C[n] \arrow[r] & {} \\ {} \arrow[r] & A/nA \arrow[r] & B/nB \arrow[r] & C/nC \arrow[r] & {} \\ {} \arrow[r] & A[n] \arrow[r] & B[n] \arrow[r] & C[n] \arrow[r] & {} \\ {} \arrow[r] & A/nA \arrow[r] & B/nB \arrow[r] & C/nC \arrow[r] & \cdots \end{tikzcd}$$

The first three terms here are just the original sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

hence the right-hand map on the top row of the long exact sequence is surjective. By exactness this means that the first map on the second row of the long exact sequence is injective, and by periodicity of the sequence the same is true for the first map on the fourth, sixth, eighth, ... rows. Thus the last map on the third, fifth, seventh, ... rows is surjective, by exactness, and we obtain a six term exact sequence

$$0 \longrightarrow A[n] \longrightarrow B[n] \longrightarrow C[n] \longrightarrow A/nA \longrightarrow B/nB \longrightarrow C/nC \longrightarrow 0$$

as required. (8 marks)

(5) (a) Let $\mathcal{A}$ be an abelian category, let I be a set, and suppose that A_i, $i \in I$, are objects of $\mathcal{A}$. Define what is meant by the coproduct $\coprod_{i \in I} A_i$.

(2 marks, seen)

Solution: A *coproduct* of $\{A_i : i \in I\}$ is an object $X \in |\mathcal{A}|$ together with morphisms $f_i \colon A_i \to X$, $i \in I$, such that for every object $Y \in |\mathcal{A}|$ and every set of morphisms $g_i \colon A_i \to Y$ there exists a unique morphism $g \colon X \to Y$ such that the following diagram commutes for all $i \in I$:

$$\begin{tikzcd} A_i \arrow[d, "f_i"'] \arrow[dr, "g_i"] & \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

(2 marks)

(b) Recall that a category $\mathcal{I}$ is small if and only if $|\mathcal{I}|$ is a set, and that an abelian category $\mathcal{A}$ is cocomplete if and only if $\operatorname{colim}_{i\in\mathcal{I}} F(i)$ exists for every small category $\mathcal{I}$ and every functor $F\colon \mathcal{I} \to \mathcal{A}$. Let $\mathcal{A}$ be an abelian category. Show that the following are equivalent:

(i) The coproduct $\coprod_{i\in I} A_i$ exists for every set of objects $\{A_i \in |\mathcal{A}| : i \in I\}$.
(ii) $\mathcal{A}$ is cocomplete.

(10 marks, seen)

Solution: A set I determines a small category $\mathcal{I}$ with $|\mathcal{I}| = I$ and only identity morphisms. To give a functor $\mathcal{I} \to \mathcal{A}$ is exactly the same as to give a set of objects $A_i \in |\mathcal{A}|$, $i \in I$, and the colimit of this functor is the coproduct $\coprod_{i\in I} A_i$. So (ii) implies (i). It remains to show that (i) implies (ii). For this, let $\mathcal{I}$ be a small category, let $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ be a functor, write A_i for $\mathcal{F}(i)$ where $i \in |\mathcal{I}|$, and consider the map

$$\coprod_{\substack{i,j\in|\mathcal{I}| \\ f\in \operatorname{Hom}_{\mathcal{I}}(i,j)}} A_i \longrightarrow \coprod_{i\in|\mathcal{I}|} A_i \qquad (\star\star)$$

which is given, on the factor A_i indexed by $f \in \operatorname{Hom}_{\mathcal{I}}(i,j)$, by $f - \mathrm{id}_{A_i}$. The universal property satisfied by the cokernel of this map is exactly the universal property that defines $\operatorname{colim}_{i\in\mathcal{I}} \mathcal{F}(i)$. Thus the colimit exists, and so (ii) implies (i). (10 marks)

(c) Show that any abelian group G is isomorphic to the direct limit of its finitely generated subgroups. (8 marks, seen as exercise without solution provided)

Solution: Consider the partially ordered set I where elements $i \in I$ are in one-to-one correspondence with finitely generated subgroups $G_i \subseteq G$, and $i \leq j$ if and only if $G_i \subseteq G_j$. This is a filtered poset, because any two elements i, $j \in I$ are bounded above by the element of I that corresponds to the subgroup generated by G_i and G_j. The filtered poset I determines a small filtered category $\mathcal{I}$ with $|\mathcal{I}| = I$, and consider the functor $\mathcal{F}$ from $\mathcal{I}$ to the category of abelian groups that sends $i \in |\mathcal{I}|$ to the subgroup G_i. Since the category of abelian groups admits coproducts, part (b) implies that the colimit $\operatorname{colim}_{i\in\mathcal{I}} cF(i)$ exists. Let us call this H.
To give a map out of the colimit, $f : H \to X$, is to give a map out of each subgroup $f_i : G_i \to X$ such that these maps are compatible under inclusions $G_i \subseteq G_j$. Thus the inclusion maps $G_i \to G$ define a map $H \to G$. It remains to show that this map is an isomorphism. The concrete construction of the coproduct in (b) shows that this map is surjective: any element $g \in G$ lies in at least one finitely generated subgroup (if $g \neq e$ then we can take this to be the cyclic subgroup generated by g, for example) and so g occurs in at least one of the factors on the right-hand side of $(\star\star)$. Furthermore the map is injective: if $[g]$ is an element of the cokernel of $(\star\star)$ that maps to the identity in G then we see, by representing the equivalence class $[g]$ by an element of the cyclic subgroup C_g generated by g, that g has to be the identity element of G. Otherwise the inclusion $C_g \subseteq G$ would not be injective, which is a contradiction. Thus $G \cong H$, as claimed. (8 marks)