Introduction to Quantum Mechanics – Problem sheet 1 Solutions

1. Constants of motion

(a) To show $E = H(p, q)$ is conserved in time we take the time derivative

$$\frac{dE}{dt} = \frac{\partial H}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial t}$$

Inserting Hamilton's canonical equations $\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}, \frac{\partial q}{\partial t} = -\frac{\partial H}{\partial p}$ we find

$$\frac{dE}{dt} = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0.$$

(b) C_1, C_2 are constants of motion and $\frac{\partial C_{1,2}}{\partial t} = 0$. Thus we have

$$\{H, C_1\} = 0 \quad \text{and} \quad \{H, C_2\} = 0.$$

The time evolution of C_3 is given by

$$\begin{aligned}
\frac{dC_3}{dt} &= \{H, C_3\} \\
&= \{H, \{C_1, C_2\}\}
\end{aligned}$$

We use the Jacobi identity $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$ to obtain

$$\begin{aligned}
\frac{dC_3}{dt} &= = -[\{C_1, \{C_2, H\}\} + \{C_2, \{H, C_1\}\}] \\
&= 0
\end{aligned}$$

2. Two-dimensional harmonic oscillator

(a) The Hamiltonian function of the system is given by

$$H = \sum_j \frac{1}{2m}p_j^2 + \frac{1}{2}m\omega_j^2 q_j^2.$$

We have

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} - m\omega_j^2 q_j,$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \frac{1}{m}p_j.$$

Combining these gives

$$\ddot{q}_j = \frac{1}{m}\dot{p}_j = -\omega_j^2 q_j.$$

This is solved by

$$q_j = A_j \sin\omega_j t + B_j \cos\omega_j t.$$

Taking derivatives gives us

$$p_j = m\omega_j(A_j \cos\omega_j t - B_j \sin\omega_j t).$$

Solving for initial conditions at t=0 gives

$$q_j(0) = B_j \text{ and } p_j(0) = m\omega_j A_j.$$

Therefore

$$q_j = \frac{p_j(0)}{m\omega_j}\sin\omega_j t + q_j(0)\cos\omega_j t.$$

$$p_j = p_j(0)\cos\omega_j t - m\omega_j(0)q_j(0)\sin\omega_j t.$$

(b) Plots of the trajectories in position space $\mathbb{R}^2$ (spanned by $q_{1,2}$) for the initial conditions given in the question.
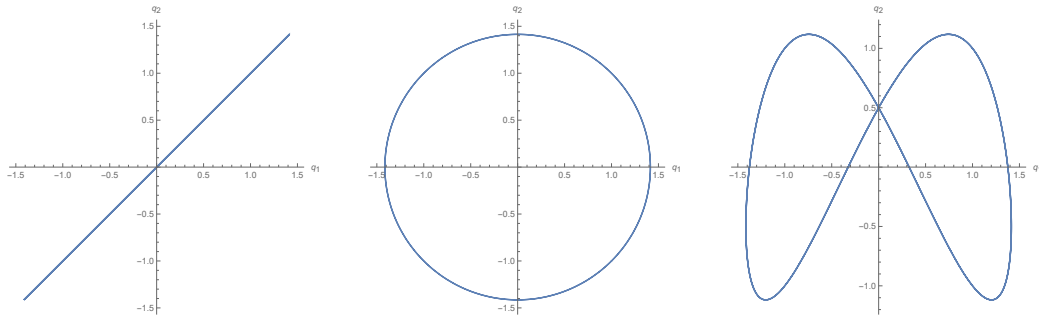

Figure 1: from left to right (i),(ii),(iii)

(c) We have

$$H_j = \frac{1}{2m}p_j^2 + \frac{1}{2}m\omega_j^2 q_j^2.$$

To show the H_j are conserved we take the time derivative

$$\frac{dH_j}{dt} = \{H, H_j\} = \sum_i \frac{\partial H}{\partial p_i} \frac{\partial H_j}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial H_j}{\partial p_i}.$$

A quick way to solve this problem would be to notice

$$\{H_1, H_2\} = 0,$$

and thus we immediately have the $\{H, H_j\} = 0$. One could also evaluate this Poisson bracket by brute force to find

$$\{H, H_1\} = \sum_i \frac{\partial}{\partial p_i} \left(\sum_k \frac{1}{2m} p_k^2 + \frac{1}{2} m\omega_k^2 q_k^2 \right) \frac{\partial}{\partial q_i} \left(\frac{1}{2m} p_1^2 + \frac{1}{2} m\omega_1^2 q_1^2 \right)$$

$$- \frac{\partial}{\partial q_i} \left(\sum_k \frac{1}{2m} p_k^2 + \frac{1}{2} m\omega_k^2 q_k^2 \right) \frac{\partial}{\partial p_i} \left(\frac{1}{2m} p_1^2 + \frac{1}{2} m\omega_1^2 q_1^2 \right)$$

$$= \omega_1^2 q_1 p_1 - \omega_1^2 q_1 p_1$$

$$= 0.$$

Similar for H_2.

(d)   (i) We have

$$\frac{dL}{dt} = \{H, L\} = \{H_1, L\} + \{H_2, L\}.$$

Using the properties of the Poisson bracket we find

$$\{H_1, L\} = -\frac{1}{2m} p_1 p_2 - \frac{1}{2} m\omega^2 q_1 q_2,$$

and

$$\{H_2, L\} = \frac{1}{2m} p_2 p_1 + \frac{1}{2} m\omega^2 q_1 q_2,$$

and thus indeed

$$\frac{dL}{dt} = \{H, L\} = \{H_1, L\} + \{H_2, L\} = 0.$$

(ii) Using our result from 1b) we conclude that

$$K = \{H_2, L\} = -\{H_1, L\} = \frac{1}{2m} p_1 p_2 + \frac{1}{2} m\omega^2 q_1 q_2$$

is another conserved quantity

3. Pendulum

(a)

$$H = \frac{p^2}{2mL^2} - mgl\cos(q)$$

The time evolution is given by Hamilton's canonical equations

$$\dot{p} = -\frac{\partial H}{\partial q} = -mgL\sin q$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{1}{mL^2} p.$$

(b)   Differentiating the second equation we obtain

$$\ddot{q} = \frac{1}{mL^2}\dot{p} = -\frac{g}{L}\sin q.$$

(c)   For small displacements we only consider the lowest terms order terms in the Taylor expansion, thus $\sin q \approx q$

$$\ddot{q} \approx -\frac{g}{L}q.$$

For small displacements the system behaves like the simple harmonic oscillator with $\omega = \sqrt{\frac{g}{L}}$.

(d)   The system evolves along contours of constant energy of the Hamiltonian (plotted below). For small energies we have oscillatory motion around the stable fixed point at $q = 0$ (when the pendulum is at rest), and for larger energies we observe rotor orbits where the pendulum traverses a series of full 360 degree rotations. These two types of motion are separated by the separatrix, that passes through the unstable fixed point at $q = \pi$, corresponding to the pendulum balancing up-side down.
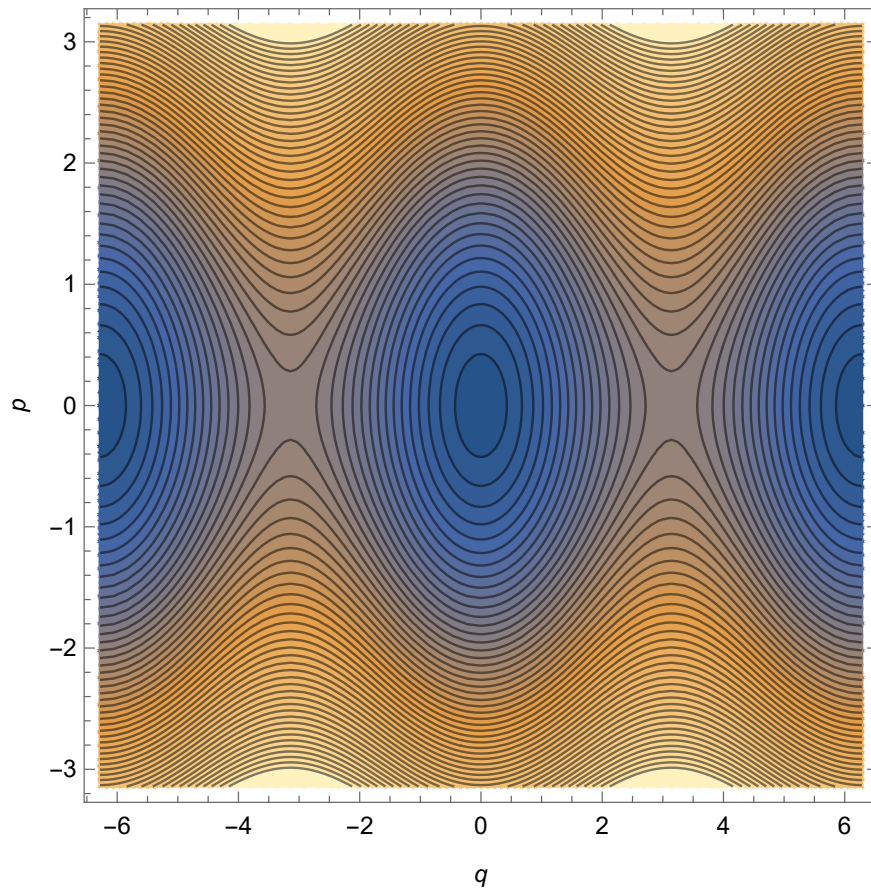

Figure 2: A contour plot of the Hamiltonian $H(q,p)$ with $m = g = L = 1$,