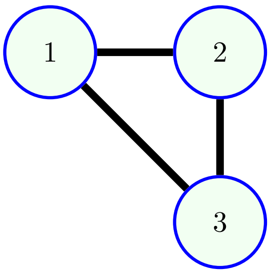
Network Science
Spring 2024
Problem sheet 8



1. Consider the "triangle graph" shown above. Solve the linearized naive network SI model for this graph with $\beta = 1$ and $\langle x_1(t=0)\rangle = 1$ and $\langle x_2(t=0)\rangle = \langle x_3(t=0)\rangle = 0$
Solution: Since the adjacency matrix is symmetric, the procedure is essentially the same as for question 4 on problem sheet 6. The eigenvalues of $\mathbf{A}$ are $\lambda_1, \lambda_2, \lambda_3 = 2, -1, -1$, with the same orthonormal eigenvectors that we have for the Laplacian, and then, $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$ with $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Let $\mathbf{g}_0 = \mathbf{V}^T \langle \mathbf{x}(t=0)\rangle$. Then $g_i = g_{0,i}e^{it}$ and $\langle \mathbf{x}\rangle = \mathbf{V}\mathbf{g}$. Symmetry dictates that nodes 2 and 3 will have the same solutions and at long times $\langle x_i\rangle$ will grow exponentially for all nodes.

2. Consider the naive network-SI model on an N-node complete graph. Derive an approximate nonlinear ODE for $\overline{x} = \frac{1}{N}\sum_{i=1}^{N}\langle x_i\rangle$ when N is large (the accuracy of the approximation should increase as N increases)
Solution: The naive network-SI model on a complete graph is,
$$d\langle x_i\rangle/dt = \beta\left(1 - \langle x_i\rangle\right)\left[\sum_{j=1}^{N}\langle x_j\rangle - \langle x_i\rangle\right].$$
Then take the sum of both sides over all nodes and divide by N:
$$\frac{d\overline{x}}{dt} = \beta\sum_{i=1}^{N}\left(1 - \langle x_i\rangle\right)\left[\overline{x} - \frac{\langle x_i\rangle}{N}\right]$$
For sufficiently large N, we can use the approximation, $\overline{x} - \frac{\langle x_i\rangle}{N} \approx \overline{x}$. The ODE then can be simplified to,
$$\frac{d\overline{x}}{dt} \approx \beta N\left(1 - \overline{x}\right)\overline{x}$$
which is usually simply referred to as *the* SI model.

3. Consider a "cloud" of particles undergoing random walks on a simple N-node graph. Let $n_i(t)$ and $n_j(t)$ be the number of particles on nodes i and j at time t. Given $n_i(t)$ and $n_j(t)$, what is the expected net flux of particles from node j to i between t and $t + \Delta t$. In other words, how many more particles are expected to move from node j to node i than from i to j during a time step? Give your answer in terms of $n_i(t)$, $n_j(t)$, k_i, and k_j. How does your result compare to the net flux of particles associated with diffusion on a graph?

(It may be helpful to work with an indicator random variable, X_{ija}, which is 1 if particle a on node j moves to i and is zero otherwise.)

Solution: Let X_{ija} be and indicator variable which is 1 if particle a on node j moves to i and is 0 otherwise. The net flux is then

$$j_{ij,net} = \sum_{a=1}^{n_j} X_{ija} - \sum_{a=1}^{n_i} X_{jia}.$$

Using linearity of expectation,

$$\langle j_{ij,net} \rangle = \sum_{a=1}^{n_j(t)} \langle X_{ija} \rangle - \sum_{a=1}^{n_i(t)} \langle X_{jia} \rangle ,$$

and $\langle X_{ija} \rangle = P(X_{ija} = 1) = 1/k_j$ with an analogous result for $\langle X_{jia} \rangle$. So the net flux is, $\langle j_{ij,net} \rangle = n_j(t)/k_j - n_i(t)/k_i$. The net flux in graph-diffusion does not depend on the node degrees.

4. Consider the following definition of a step of a random walk on a connected directed N-node graph with adjacency matrix $\mathbf{A}$. Each step begins with a Bernoulli trial with probability of success, π. If successful, the walker chooses an outward link uniformly at random and follows it to a neighboring node. If unsuccessful, the walker finds itself picked up and "dropped" back onto the graph which we model as selecting a node uniformly at random.

   (a) What is the master equation for this process? Let $p_i^{(l+1)}$ be the probability a walker is at node i after $l+1$ steps. You should derive an equation relating $p_i^{(l+1)}$ to the elements in the probability vector at iteration l, as well as N, $\mathbf{A}$, π and the node degrees (e.g. k_i^{out}).
   Solution: Consider the probability that walker is at node j at iteration $l+1$. It can reach this node via a step from a neighboring node or by being dropped onto the node. The probability that a walker is at node i at iteration l and then takes a step to j is $A_{ji}/k_i^{out} p_i^{(l)}$. The probability of j being the node a walker is dropped on is, $1/N$. Combining these probabilities, we find, $p_j^{(l+1)} = \sum_{i=1}^{N} \left[\pi p_i^{(l)} A_{ji}/k_i^{out}\right] + (1-\pi)/N$, and since the sum of the probability vector is 1, this can be rearranged as, $p_j^{(l+1)} = \sum_{i=1}^{N} \left[\pi A_{ji}/k_i^{out} + (1-\pi)/N\right] p_i^{(l)}$

   (b) Interpret the equation for the stationary state as a node centrality. What is the role of the probability, π?
   Solution: The equation for the stationary state is identical to the equation for PageRank centrality with π replacing $(1-m)$. This interpretation of PageRank as a random walk with "teleportation" was provided by Brin & Page in their initial paper on PageRank and Google.

5. Consider the following modification to the network-SI model: the probability that an infectious node becomes susceptible in time Δt is $\gamma \Delta t$. Taking this modification into account, and applying the limit $\Delta t \to 0$, derive the resulting system of ODEs known as the *network-SIS model*
   Solution: The master equation for the network-SI model is,

$$P(x_i(t+\Delta t) = 1) = P(x_i(t) = 1) + \beta \Delta t \sum_{j=1}^{N} A_{ij} P(x_i(t) = 0, x_j(t) = 1) + O(\Delta t^2).$$

The first term on the RHS is the probability that the node is infected at time t. We have to modify this to account for the possibility that the node recovers between t and $t+\Delta t$. Specifically, we replace $P(x_i(t)=1)$ with $P(x_i(t)=1, x_i \not\to 0)$ where $x_i \not\to 0$ indicates that an infectious node does not recover between t and $t+\Delta t$. Then,

$$P(x_i(t)=1, x_i \not\to 0) = P(x_i \not\to 0 | x_i(t)=1) P(x_i(t)=1) = (1-\gamma\Delta t) P(x_i(t)=1).$$

Replacing $P(x_i(t)=1)$ with this expression in the master equation, rewriting probabilities as expectations, and letting $\Delta t \to 0$ gives us the network-SIS model:

$$d\langle x_i \rangle /dt = \beta \sum_{l=1}^{N} A_{il} \langle s_i x_l \rangle - \gamma \langle x_i \rangle,$$

$$s_i = 1 - x_i.$$

Note that I have not considered the probability that a node both recovers and is then reinfected during time Δt. This effect will be $O(\Delta t^2)$ and will vanish when $\Delta t \to 0$.