IMPERIAL

MATH70019

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Theory of Partial Differential Equations

Date: Friday, May 30, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Let $p > 0$ and consider the Cauchy problem given by

$$\begin{cases} \partial_t u + u^p \partial_x u = 0 \,, \\ u(0,x) = g(x) \,, \end{cases} \tag{1}$$

where the initial datum is

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x \leq 1 \\ 0 & x > 1 \,. \end{cases}$$

(a)

(i) Let $p = 1$, are there shocks? If so, compute the shock curve.

(5 marks)

(ii) For which values of $p > 0$ are there shocks for times $t < 1$?

(5 marks)

(b) Let $p = 1$ draw the characteristics and find the unique entropy solution.

(10 marks)

(Total: 20 marks)

2. Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded open set with an outer normal vector $\mathbf{n} \in \mathbb{S}^{n-1}$. Let $\mathbf{b} : [0,\infty) \times \Omega \to \mathbb{R}^n$ be a smooth divergence-free vector field, i.e. $\nabla \cdot \mathbf{b} = 0$, and with $\mathbf{b} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Let $\kappa \geq 0$ be the diffusivity parameter and consider the Cauchy problem

$$\begin{cases} \partial_t u_\kappa + \mathbf{b} \cdot \nabla u_\kappa = \kappa \Delta u_\kappa & \text{on } (0,\infty) \times \Omega\,, \\ \nabla u_\kappa \cdot \mathbf{n} = 0 & \text{on } [0,\infty) \times \partial\Omega\,, \\ u_\kappa(0,x) = g(x)\,. \end{cases} \tag{2}$$

You may assume existence of solutions in $C^2([0,\infty) \times \Omega)$.

(a) Let $f \in C^1([0,\infty) \times \Omega)$, then prove that for any $t \geq 0$

$$\int_\Omega \left(\mathbf{b}(t,x) \cdot \nabla f(t,x)\right) f(t,x) dx = 0\,.$$

(6 marks)

(b) Use part (a) to prove the following. Let u_κ be a solution of (2) with $\kappa \geq 0$ (including the case $\kappa = 0$!), then the following energy equality holds true for any $t \geq 0$

$$\int_\Omega |u_\kappa(t,x)|^2 dx + 2\kappa \int_0^t \int_\Omega |\nabla u_\kappa(s,x)|^2 dx = \int_\Omega |g(x)|^2 dx\,.$$

(4 marks)

(c) Use part (b) to prove uniqueness of solutions of the Cauchy problem (2) in $C^2([0,\infty) \times \Omega)$.

(6 marks)

(d) For any $\kappa > 0$ let u_κ be the unique solution to the Cauchy problem (2) with diffusivity parameter κ and let u_0 be the unique solution to the Cauchy problem (2) with $\kappa = 0$. Suppose that

$$\sup_{(t,x) \in [0,\infty) \times \Omega} |u_\kappa(t,x) - u_0(t,x)| \to 0\,, \quad \text{as } \kappa \to 0\,,$$

then prove that

$$\lim_{\kappa \to 0} \kappa \int_0^t \int_\Omega |\nabla u_\kappa(s,x)|^2 dx = 0\,.$$

(4 marks)

(Total: 20 marks)

3. Consider the following Cauchy problem with $u : [0, \infty) \times [-\pi, \pi] \to \mathbb{R}$

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = -2u \,, \text{on } (0, \infty) \times (-\pi, \pi) \,, \\ u(t, \pi) = u(t, -\pi) \,, \text{for any } t \,, \\ u(0, x) = 0 \,. \\ \partial_t u(0, x) = g(x) \,, \end{cases} \tag{3}$$

(a) Consider the N truncated Fourier series of $g(x)$, more precisely for any $N \in \mathbb{N}$

$$S_N(g)(x) := \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \,,$$

with $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx) dx$. Determine $A_n, B_n : [0, \infty) \to \mathbb{R}$ in terms of $\{a_n, b_n\}_n$ so that for any $N \in \mathbb{N}$

$$S_N(u)(t, x) = \frac{A_0(t)}{2} + \sum_{n=1}^{N} (A_n(t) \cos(nx) + B_n(t) \sin(nx)) \,.$$

is a solution to the Cauchy problem (3) with g replaced by $S_N(g)$. (6 marks)

(b) Suppose that $|a_n| \leq \frac{1}{n}$ and $|b_n| \leq \frac{1}{n}$ for any $n \in \mathbb{N}$. Prove that $\{S_N(u)\}_{N \in \mathbb{N}}$ converges in $C^0([0, \infty) \times [-\pi, \pi])$ to a function u. (4 marks)

(c) Suppose that $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) < \infty$. Prove rigorously that u is a classical solution to the Cauchy problem, where the initial data are attained in the uniform sense, i.e.

$$\lim_{\delta \to 0} \sup_{t \in (0, \delta), x \in [-\pi, \pi]} |u(t, x) - 0| + |\partial_t u(t, x) - g(x)| = 0 \,.$$

(10 marks)

(Total: 20 marks)

4. We say that $u \in C^\infty(\mathbb{R}^2)$ is a superharmonic function on $\mathbb{R}^2$ if it satisfies

$$\Delta u(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^2 . \tag{4}$$

(a) We denote by $B_r(x)$ the ball of radius r with center in x and by $\partial B_r(x)$ its boundary. Prove the following inequalities for any x and any $r > 0$

(i)

$$u(x) \geq \frac{1}{2\pi r} \int_{\partial B_r(x)} u(\sigma) d\sigma \,,$$

(ii)

$$u(x) \geq \frac{1}{\pi r^2} \int_{B_r(x)} u(y) dy \,.$$

(8 marks)

(b) We say that $u \in C^2(\mathbb{R}^2)$ is a subharmonic function on $\mathbb{R}^2$ if it satisfies $\Delta u(x) \geq 0$ for all $x \in \mathbb{R}^2$. Does a subharmonic function satisfy the same properties given in Part (a)? Prove it or give a counterexample. (2 marks)

(c) Let u be a subharmonic function. Is it true or false that the partial derivatives $\partial_{x_i} u$ for $i = 1, 2$ are subharmonic functions? Prove it or give a counterexample. (2 marks)

(d) Let u be a subharmonic function such that $\partial_{x_i} u$ is a subharmonic function. Suppose that $u(y) \leq 0$ for all $y \in \mathbb{R}^2$, then prove that u is constant. (8 marks)

(Total: 20 marks)

5. Given $f \in C^1[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$ and $\int_{-\pi}^{\pi} f = 0$, consider the following Cauchy problem with $u : [0,\infty) \times [-\pi,\pi] \to \mathbb{R}$

$$\begin{cases} \partial_t u - 2\partial_{xx} u = u - f \,, \text{on } (0,\infty) \times (-\pi,\pi)\,, \\ u(t,\pi) = u(t,-\pi)\,, \text{for any } t\,, \\ u(0,x) = 0\,. \end{cases}$$

(a) Write the Fourier series

$$f(x) = \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

and suppose that $\sum_n |a_n| + |b_n| < \infty$. Prove that there are functions $A_n, B_n : [0,\infty) \to \mathbb{R}$ expressed in terms of $\{a_n, b_n\}_n$ such that

$$S_N(u)(t,x) = \sum_{n=1}^{N} (A_n(t) \cos(nx) + B_n(t) \sin(nx))$$

converges in $C^0([0,\infty) \times (-\pi,\pi))$ as $N \to \infty$ to a solution u of the Cauchy problem.

(12 marks)

(b) Prove rigorously that there exists a function $v \in C^0(-\pi,\pi)$ such that

$$\lim_{t\to\infty} |u(t,x) - v(x)| = 0\,.$$

(8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60019/MATH70019

Theory of PDEs (Solutions)

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1. (a) (i) The characteristics satisfy the ODE

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4, A

1, B

$$\begin{cases} \frac{dx}{ds} = z, & x(0,a) = a, \\ \frac{dz}{ds} = 0, & z(0,a) = g(a). \end{cases}$$

The solution to this coupled ODE is given by $z(s,a) = g(a)$, and $x(s,a) = g(a)s + a$. The equation of the characteristics is given by

$$x = g(a)s + a. \tag{1}$$

Now we substitute the value of the initial datum $g(a)$, that is defined piecewise. Therefore, thanks to the definition of the initial datum, the first intersection point is at $(t,x) = (1,1)$. Then, yes, there are shocks. In order to determine the equation of the shock curve we use Rankine-Hugoniot condition, since we are looking for a weak solution. If the shock curve is parametrised by $x = \sigma(t)$, we find that it must satisfy the relation

$$\sigma'(t) = \frac{q(u_+(t,\sigma(t))) - q(u_-(t,\sigma(t)))}{u_+(t,\sigma(t)) - u_-(t,\sigma(t))},$$

where $u_+ = 0$ and $u_- = 1$ and $q(z) = \frac{z^2}{2}$. Hence, we obtain that the shock curve is a straight line arising from $(1,1)$ and given by

$$\sigma(t) = 1 + \frac{1}{2}(t-1).$$

2, C

3, D

(ii) For $p = 1$ it is clear from the previous point that the first intersection point is at time $t = 1$. So there are no shocks for times $t < 1$.
For $p \in (0,1)$, we consider the self intersection of the characteristics starting from $a = 1/2$ and $a = 1$, i.e. we impose

$$g(1/2)s + 1/2 = 1$$

substituting the value of $g(1/2)$ we have

$$\left(\frac{1}{2}\right)^p s = \frac{1}{2} \quad \Leftrightarrow \quad s = \left(\frac{1}{2}\right)^{1-p} < 1\,.$$

Therefore for any $p \in (0,1)$ there are shocks for times $t < 1$.
For $p \in (1,\infty)$, we consider the self intersection of the characteristics starting from $a = 1/2$ and $a = 0$, i.e. we impose

$$g(1/2)s + 1/2 = g(0)s$$

substituting the value of $g(1/2)$ and $g(0)$ we have

$$\left(\frac{1}{2}\right)^p s + \frac{1}{2} = s \quad \Leftrightarrow \quad s\left(1 - \left(\frac{1}{2}\right)^p\right) = \frac{1}{2} \quad \Rightarrow s < 1\,,$$

where in the last we used that $\left(\frac{1}{2}\right)^p < \frac{1}{2}$ for any $p > 1$. Therefore for any $p > 1$ there are shocks for times $t < 1$.

meth seen $\Downarrow$

4, A

6, B

(b) From the characteristics equation we have the following. For $x \le t < 1$, we get

$$a = x - t\,.$$

For $0 \le x \le 1$, $x \ge t$,

$$a = \frac{x-t}{1-t}\,.$$

For $x \ge 1$, $0 \le t < 1$,

$$a = x\,.$$

Thus, the solution we found for $t < 1$ is given by

$$u(t,x) = \begin{cases} 1, & \text{if } x \le t, 0 \le t < 1\,, \\ \frac{1-x}{1-t} & \text{if } t \le x < 1\,, 0 \le t < 1\,, \\ 0 & \text{if } x \ge 1\,, 0 \le t < 1\,. \end{cases}$$

For $t \ge 1$ we have

$$u(t,x) = \begin{cases} 1, & \text{if } x < \sigma(t)\,, \\ 0 & \text{if } x > \sigma(t)\,. \end{cases}$$

Finally, this is the unique entropic solution since it satisfies the entropy condition, that is given by

$$q'(u_+(t,\sigma(t))) < \sigma'(t) < q'(u_-(t,\sigma(t))),$$

where $u_+ = 0$ and $u_- = 1$.

2. (a) Using that $div(b) = 0$ $\boxed{\text{unseen} \Downarrow}$

$$\int_\Omega (b(t,x)\cdot\nabla u_\kappa(t,x))\,u_\kappa(t,x)dx = \int_\Omega div(b(t,x)u_\kappa(t,x))u_\kappa(t,x)dx$$
$$= \int_{\partial\Omega} |u_\kappa|^2 b\cdot\mathbf{n}$$
$$- \int_\Omega u_\kappa(t,x)\,(b(t,x)\cdot\nabla u_\kappa(t,x))\,dx\,,$$
$$= -\int_\Omega u_\kappa(t,x)\,(b(t,x)\cdot\nabla u_\kappa(t,x))\,dx$$

$\boxed{2,\ \mathrm{D}}$ $\boxed{2,\ \mathrm{C}}$ $\boxed{2,\ \mathrm{B}}$

where in the last we used the divergence theorem. Therefore, rearranging the terms we deduce that

$$2\int_\Omega (b(t,x)\cdot\nabla u_\kappa(t,x))\,u_\kappa(t,x)dx = 0$$

from which we conclude the exercise. $\boxed{\text{sim. seen} \Downarrow}$

(b) We multiply the equation by $2u_\kappa$ and integrate in space to get $\boxed{2,\ \mathrm{A}}$

$$\int_\Omega \frac{d}{dt}|u_\kappa(t,x)|^2dx + 2\int_\Omega (b(t,x)\cdot\nabla u_\kappa(t,x))\,u_\kappa(t,x)dx = 2\kappa\int_\Omega \Delta u_\kappa(t,x)u_\kappa(t,x)dx\,,$$ $\boxed{2,\ \mathrm{B}}$

where the right hand side, using the divergence theorem, is equal to

$$2\kappa\int_\Omega \Delta u_\kappa(t,x)u_\kappa(t,x)dx = \int_{\partial\Omega} u_\kappa\nabla u_\kappa\cdot\mathbf{n} - 2\kappa\int_\Omega |\nabla u_\kappa(t,x)|^2dx$$
$$= -2\kappa\int_\Omega |\nabla u_\kappa(t,x)|^2dx$$

therefore, using part (a) we deduce that

$$\frac{d}{dt}\int_\Omega |u_\kappa(t,x)|^2dx + 2\kappa\int_\Omega |\nabla u_\kappa(t,x)|^2dx = 0$$

integrating this identity in time we deduce the energy balance. $\boxed{\text{seen} \Downarrow}$

(c) Assume there are two solutions u_1, u_2 of the Cauchy problem. Then, there difference $w = u_1 - u_2$ solve the same PDE with initial datum $g \equiv 0$. Therefore, from part (b) we have the energy balance $\boxed{6,\ \mathrm{A}}$

$$\int_\Omega |w(t,x)|^2dx + 2\kappa\int_0^t\int_\Omega |\nabla w(s,x)|^2dxds = 0$$

for any $t \geq 0$. In particular we deduce $\int_\Omega |w(t,x)|^2dx = 0$ for any $t \geq 0$, then $w(t,x) = 0$ for any $t \geq 0$, for a.e. x. Since $w \in C^1$ we deduce $w \equiv 0$ and $u_1 \equiv u_2$ which implies uniqueness. $\boxed{\text{unseen} \Downarrow}$

(d) From part (b) we know that $\boxed{2,\ \mathrm{C}}$ $\boxed{2,\ \mathrm{D}}$

$$\int_\Omega |u_\kappa(t,x)|^2dx + 2\kappa\int_0^t\int_\Omega |\nabla u_\kappa(s,x)|^2dxds = \int_\Omega |g(x)|^2dx \tag{2}$$

and

$$\int_\Omega |u(t,x)|^2dx = \int_\Omega |g(x)|^2dx\,.$$

Using the assumption by dominated convergence theorem we have that

$$\lim_{\kappa\to 0}\int_\Omega |u_\kappa(t,x)|^2dx = \int_\Omega |u(t,x)|^2dx\,.$$

Therefore, taking the limit $\lim_{\kappa\to 0}$ in (2) we have

$$\lim_{\kappa\to 0}\kappa\int_0^t\int_\Omega |\nabla u_\kappa(s,x)|^2dxds = 0\,.$$

3. (a) Imposing that $S_N(u)(t,x) = \frac{A_0(t)}{2} + \sum_{n=1}^N A_n(t)\cos(nx) + B_n(t)\sin(nx)$ is a solution of the equation, we get the following ODEs system for the coefficient A_n

meth seen ⇓

4, A

2, B

$$\begin{cases} \partial_{tt} A_n = -(n^2+2)A_n \\ A_n(0) = 0\,,\ \partial_t A_n(0) = a_n\,, \end{cases}$$

and

$$\begin{cases} \partial_{tt} B_n = -(n^2+2)B_n \\ B_n(0) = 0\,,\ \partial_t B_n(0) = b_n\,, \end{cases}$$

from which we get

$$A_n(t) = \frac{a_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})\,,\ B_n(t) = \frac{b_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})$$

and we get the explicit formula

$$\begin{aligned} S_N(u)(t,x) = {} & \frac{a_0}{2\sqrt{2}}\sin(t\sqrt{2}) + \sum_{n=1}^N \frac{a_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})\cos(nx) \\ & + \sum_{n=1}^N \frac{b_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})\sin(nx)\,, \end{aligned}$$

(b) We define $u_0(t,x) = \frac{a_0}{2\sqrt{2}}\sin(t\sqrt{2})$ and $u_n(t,x) = \frac{a_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})\cos(nx) + \frac{b_n}{\sqrt{n^2+2}}\sin(t\sqrt{n^2+2})\sin(nx)$ so that

meth seen ⇓

2, A

2, C

$$S_N(u)(t,x) = \sum_{n=0}^N u_n(t,x)\,.$$

As seen in class to prove that $S_N(u)$ is a Cauchy sequence in C^0 it is sufficient to prove that

$$\sum_{n=0}^\infty \sup_{t,x} |u_n(t,x)| < \infty\,.$$

Observing that

$$\sup_{t,x} |u_n(t,x)| \le \frac{|a_n| + |b_n|}{n} \le \frac{2}{n^2}$$

for any $n \ge 1$. Since $\sum_{n\ge 0} \frac{2}{n^2} < \infty$ we get the thesis.

We define u the limit in the C^0 norm of the sequence $\{S_N(u)\}_{N\in\mathbb{N}}$.

meth seen ⇓

(c) We define

4, A

$$\omega_n = \sqrt{n^2+2}\,.$$

4, B

To prove that $u(t,x)$ is a solution of the PDE we need to prove that

2, D

- $\{\partial_{tt} S_N(u)\}_N$ is a Cauchy sequence in C^0.
- $\{\partial_{xx} S_N(u)\}_N$ is a Cauchy sequence in C^0.
- $\lim_{t\to 0} u(t,x) = 0$ and $\lim_{t\to 0}\partial_t u(t,x) = g(x)$.

To prove the first part we observe that for any $n \ge 1$

$$\sup_{t,x} |\partial_{tt} u_n(t,x)| \le \omega_n(|a_n| + |b_n|) \le 2n(|a_n| + |b_n|)\,.$$

Since $\sum_{n=0}^{\infty} n(|a_n|+|b_n|) < \infty$ we deduce that the sequence $\partial_{tt} S_N(u)(t,x) = \sum_{n=0}^{N} \partial_{tt} u(t,x)$ is a Cauchy sequence in C^0.

To prove the second part we observe that

$$\sup_{t,x} |\partial_{xx} u_n(t,x)| \leq \frac{n^2}{\omega_n}(|a_n|+|b_n|) \leq n(|a_n|+|b_n|)\,,$$

from which we conclude as in the first part.

To prove the third part, thanks to $\sum_n \sup_{t,x} |u_n(t,x)| < \infty$, we fix $\varepsilon > 0$ and we find N so that

$$\sum_{n=N+1}^{\infty} \sup_{t,x} |u_n(t,x)| < \frac{\varepsilon}{2}\,.$$

Now, by continuity of $\sin$ we fix $\delta > 0$ sufficiently small so that

$$|\sin(\omega_n t)| \leq \frac{\varepsilon}{2(N+1)\sup\{|a_0|,|b_0|,|a_1|,|b_1|,\dots,|a_n|,|b_n|\}}$$

for any $t \in (0,\delta)$ for any $n = 0,\dots,N$. Therefore, by definition of $u(t,x) = \sum_{n=0}^{\infty} u_n(t,x)$ we get

$$\sup_{t\in(0,\delta) x\in[-\pi,\pi]} |u(t,x)-0| \leq \sum_{n=0}^{N} \frac{\varepsilon}{2(N+1)} + \sum_{n=N+1}^{\infty} \sup_{t,x} |u_n(t,x)| < \varepsilon\,.$$

Proving that $\sup_{t\in(0,\delta) x\in[-\pi,\pi]} |\partial_t u(t,x) - g(x)| < \varepsilon$ is similar. We observe that

$$\partial_t u(t,x) - g(x) = \frac{a_0}{2}(\cos(t\sqrt{2})-1) + \sum_{n=1}^{\infty} a_n(\cos(t\sqrt{n^2+2})-1)\cos(nx)$$
$$+ \sum_{n=1}^{\infty} b_n(\cos(t\sqrt{n^2+2})-1)\sin(nx)\,,$$

and using that $\sum_n |a_n|+|b_n| < \infty$ and the continuity of $\cos(y) \to 1$ as $y \to 0$ we get that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{t\in(0,\delta)\,,x\in[-\pi,\pi]} |\partial_t u(t,x) - g(x)| \leq \sum_{n=0}^{N} \frac{\varepsilon}{2(N+1)} + \sum_{n=N+1}^{\infty} \sup_{t,x}(|\partial_t u_n(t,x)|+|g(x)|) < \varepsilon\,.$$

4. (a) Let us start from the second formula. For $r < R$ define $\boxed{\text{sim. seen} \Downarrow}$

$$\phi(r) = \frac{1}{2\pi r}\int_{\partial B_r(x)} u(y)dy \tag{3}$$

$\boxed{6, \text{A}}$ $\boxed{2, \text{C}}$

Perform the change of variables $y = x + ry'$. Then $y' \in \partial B_1(0)$, $dy = rdy'$ and

$$\phi(r) = \frac{1}{2\pi}\int_{\partial B_1(\mathbf{0})} u(x + ry')dy'. \tag{4}$$

Let $v(z) = u(x + rz)$ and observe that

$$\nabla v(z) = r\nabla u(x + rz), \qquad \Delta v(z) = r^2\Delta u(x + rz). \tag{5}$$

Then we have

$$\begin{aligned}\phi'(r) &= \frac{1}{2\pi}\int_{\partial B_1(0)} \partial_r u(x + ry')dy' = \frac{1}{2\pi}\int_{\partial B_1(0)} \nabla u(x + ry')\cdot y'dy' \\ &= \frac{1}{2\pi r}\int_{\partial B_1(0)} \nabla v(y')\cdot y'dy' = \frac{1}{2\pi r}\int_{B_1(0)} \Delta v(z)dz \\ &= \frac{r}{2\pi}\int_{B_1(0)} \Delta u(x + rz)dz \le 0. \end{aligned} \tag{6}$$

Thus, ϕ is non-increasing and since $\phi(r) \to u(x)$ as $r \to 0$, we get the first property. To obtain the second mean value property, multiply by r and integrate both sides between 0 and R the first property. We find

$$\frac{R^2}{2}u(x) \ge \frac{1}{2\pi}\int_0^R\left[\int_{\partial B_r(x)} u(y)dy\right]dr = \frac{1}{2\pi}\int_{B_R(x)} u(z)dz, \tag{7}$$

from which we get the second property. $\boxed{\text{meth seen} \Downarrow}$

(b) By multiplying by $-$ the previous inequalities it is clear that a subharmonic function satisfies the reverse inequalities $\boxed{2, \text{B}}$

$$u(x) \le \frac{1}{2\pi r}\int_{\partial B_r(x)} u(\sigma)d\sigma\,,$$

and

$$u(x) \le \frac{1}{\pi r^2}\int_{B_r(x)} u(y)dy\,.$$

To find a counterexample consider $u(x) = x^2$ which is subharmonic and for instance it does not satisfy the property for $x = 0$. $\boxed{\text{unseen} \Downarrow}$

(c) In general this property is false. Consider $u(x, y) = x^2 + \sin(x)$. It is clear that $\Delta u = 2 - \sin(x) \ge 0$. However, $\Delta\partial_x u = -\cos(x)$ that does not have a sign. $\boxed{2, \text{D}}$

(d) If u is subharmonic $\Delta u \ge 0$, then $-u$ is superharmonic $\Delta(-u) \le 0$ and also $\Delta(-\partial_{x_i}u) \le 0$. Then, by multiplying by $-$ the inequality in Part (a) applying to $\partial_{x_i}u$ and the divergence theorem we have $\boxed{\text{unseen} \Downarrow}$ $\boxed{5, \text{D}}$

$$\partial_{x_i}u(x_0)|B(x_0,1)|r^n \le \int_{B(x_0,r)} \partial_{x_i}u dx = \int_{\partial B(x_0,r)} un_i d\sigma \le \int_{\partial B(x_0,r)} |u|d\sigma\,,$$

$\boxed{2, \text{C}}$ $\boxed{1, \text{B}}$

where in the last we used $|n_i| \le 1$. Therefore, taking the absolute value in the previous inequality and using that $u \le 0$ we get

$$|\partial_{x_i}u(x_0)| \le -\frac{1}{r^n|B(x_0,1)|}\int_{\partial B(x_0,r)} ud\sigma \le -\frac{r^{n-1}|\partial B(x_0,1)|}{r^n|B(x_0,1)|}u(x_0)$$

where in the last we used again Part (a) applied to u. Using that in dimension 2 we have

$$|\partial_{x_i} u(x_0)| \leq \frac{2}{r} u(x_0)\,,$$

and sending $r \to \infty$ we get that

$$\nabla u \equiv 0$$

therefore, u is constant.

5. (a) The coefficient A_n, B_n solve the following ODEs

sim. seen $\Downarrow$

12, M

$$\begin{cases} \partial_t A_n = (1-2n^2)A_n - a_n\,, \\ A_n(0) = 0\,, \end{cases}$$

and

$$\begin{cases} \partial_t B_n = (1-2n^2)B_n - b_n\,, \\ B_n(0) = 0\,. \end{cases}$$

Therefore the coefficients are given by

$$A_n(t) = \frac{a_n}{1-2n^2}(1-e^{(1-2n^2)t})\,, \qquad B_n(t) = \frac{b_n}{1-2n^2}(1-e^{(1-2n^2)t})\,,$$

finally defining the functions u_n for any n so that

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t,x) := \sum_{n=1}^{\infty} A_n(t)\cos(nx) + B_n(t)\sin(nx)\,.$$

We need to prove that

* The sequence $\{S_N(u)\}_{N\in\mathbb{N}}$ defined as $S_N(u)(t,x) = \sum_{n=1}^{N} A_n(t)\cos(nx) + B_n(t)\sin(nx)$ is a Cauchy sequence in $C^0([0,\infty)\times\mathbb{R})$
* The sequences $\{\partial_t S_N(u)\}_{N\in\mathbb{N}}$ and $\{\partial_{xx} S_N(u)\}_{N\in\mathbb{N}}$ are Cauchy in C^0.
* $\lim_{t\to 0} u(t,x) = 0$.

For the first part we notice that there exists a constant $C>0$ such that

$$\sup_{t,x} |u_n(t,x)| \le C\frac{|a_n|}{n^2} \le C|a_n|\,,$$

which implies $\sum_{n\ge 1}\sup_{t,x}|u_n(t,x)| < \infty$, which implies that the sequence $\{S_N(u)\}_{N\in\mathbb{N}}$ is Cauchy in $C^0([0,\infty)\times\mathbb{R})$.

For the second part we observe that there exists a constant $C>0$ such that

$$\sup_{t,x} |\partial_t u_n(t,x)| + |\partial_{xx} u_n(t,x)| \le Cn^2\frac{|a_n|}{n^2} \le C|a_n|\,,$$

which implies that $\sum_{n\ge 1}\sup_{t,x}|\partial_t u_n(t,x)| + |\partial_{xx}u_n(t,x)| < \infty$, which implies that the sequences $\{\partial_t S_N(u)\}_{N\in\mathbb{N}}$ and $\{\partial_{xx} S_N(u)\}_{N\in\mathbb{N}}$ are Cauchy in $C^0([0,\infty)\times\mathbb{R})$.

For the third part we observe that for any $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$\sum_{n=N}^{\infty}\sup_{t,x}|u_n(t,x)| < \varepsilon$$

therefore

$$\begin{aligned} |u(t,x)| &\le \left|\sum_{n=1}^{N} A_n(t)\cos(nx) + B_n(t)\sin(nx)\right| + \sum_{n=N+1}^{\infty}\sup_{t,x}|u_n(t,x)| \\ &\le \left|\sum_{n=1}^{N} A_n(t)\cos(nx) + B_n(t)\sin(nx)\right| + \varepsilon\,, \end{aligned}$$

taking the limit as $t\to 0$ and using that $A_n(t)\to 0$ and $B_n(t)\to 0$ as $n\to\infty$ we conclude that

$$\lim_{t\to 0}|u(t,x)| < \varepsilon\,,$$

since ε is arbitrary we conclude the thesis.

(b) We define $v(x)$ as the limit of the Cauchy sequence $\{S_N(v)\}_{N\in\mathbb{N}}$ defined by

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$$S_N(v)(x) = \sum_{n=1}^{N} \frac{a_n}{1-2n^2}\cos(nx) + \frac{b_n}{1-2n^2}\sin(nx)\,,$$

that is a Cauchy sequence since $\sum_{n\geq 1} |a_n| + |b_n| < \infty$.
We now fix $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |a_n| + |b_n| < \varepsilon/2\,.$$

Then we have

$$|u(t,x) - v(x)| \leq \sum_{n=1}^{N} \frac{|a_n| + |b_n|}{1-2n^2} e^{(1-2n^2)t} + \sum_{n=N}^{\infty} \frac{|a_n| + |b_n|}{1-2n^2} e^{(1-2n^2)t}$$

using that $1-2n^2 \leq 0$ for any $n \geq 1$ we get

$$|u(t,x) - v(x)| \leq \sum_{n=1}^{N} \frac{|a_n| + |b_n|}{1-2n^2} e^{(1-2n^2)t} + \varepsilon/2$$

and thanks to the fact that $\lim_{t\to\infty} e^{(1-2n^2)t} = 0$ we conclude

$$\lim_{t\to\infty} |u(t,x) - v(x)| < \varepsilon\,,$$

since ε is arbitrary the thesis follows.

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70019 Theory of Partial Differential Equations Markers Comments

Question 1	The exercise was standard and had been covered in class, with the exception of Exercise 1 (ii). Students performed very well on the standard portion but encountered some difficulties with the non-standard part, Exercise 1 (ii).
Question 2	Exercises 2a), 2b), and 2c) were covered in class, and students performed well on these parts. Although Exercise 2d) was not covered in class, students still performed well. I'm pleased that they were able to come up with solutions they had not seen before.
Question 3	The exercise on the wave equation (without the right-hand side) was fully solved in class. However, students had difficulty proving that the sequence of approximate solutions is a Cauchy sequence and struggled with justifying the exchange of an infinite sum with a derivative. Although these aspects were discussed rigorously in class, very few students included all the necessary details in their solutions. Since this was a common issue, I did not deduct many points for the missing steps. Some students had problems solving an ordinary differential equation, which is a fundamental for this course.
Question 4	This was the more theoretical exercise. Part a was discussed in class, and the same proof applied well here, with the only change being the use of an inequality instead of an equality in the final step. Some students had difficulties with this part, which suggests they may not be studying the theory thoroughly and are instead focusing only on practicing exercises for the exam. Students performed well in part b and c. Part d was not covered in class, and I was pleased to see that some students attempted it with creative ideas, even if their solutions were not entirely correct.
Question 5	The exercise was a modification of the Fourier method used to solve PDEs, similar to those seen for the wave and heat equations. Students performed well on the first part, which was more standard, but encountered difficulties with the second part, which was more challenging. Additionally, some students struggled with solving an ordinary differential equation, a skill that should be well-established at the master's level in Mathematics.