Imperial College
London

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Analysis 1A

Lecture 5 - Proof of completeness axiom and more about supremums and infimums

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WLOG we may assume $S \neq \emptyset$ has a positive element $0 \leq s \in S$ - for instance by replacing S by $S + a := \{s + a \colon s \in S\}$ by a big enough $a \in \cancel{N}$, this only translates the supremum by a.

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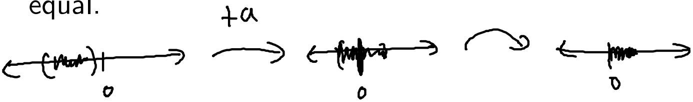
Assuming now that S has a positive element, we also know S is bounded above by some $N \in \mathbb{N}_{>0}$. We can replace finding the supremum of S by finding the supremum of $S \cap [0, N]$: one has a sup if and only if the other one does, and the two suprema are equal.



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We now don't have to worry about any negative elements of S!

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Leading integer. Write each $s \in S$ as a decimal $s_0.\, s_1 s_2 s_3 \ldots$ not ending in $\overline{9}$. Since $s \in [0, N]$ we see that $s_0 \in \{0, 1, \ldots, N\}$, a finite set. So the set of leading integers s_0 is finite, so has a maximum $a_0 \geq 0$.

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First decimal place. So $S \cap [a_0, a_0 + 1)$ is nonempty and we may replace S by it (same easy exercise). All its elements are of the form $a_0.\, s_1 s_2 \ldots$ with $s_1 \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_1 value; call it a_1.

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Second decimal place. So can replace S by $S \cap \big[a_0.\, a_1,\, a_0.\, (a_1 + 1)\big)$. (If $a_1 = 9$ we mean $S \cap [a_0.9,\, a_0 + 1)$.) Every $s \in S$ has decimal expansion $a_0.\, a_1 s_2 s_3 \ldots$ with $s_2 \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_2 value; call it a_2.

Step 1 - continued inductively

n-th decimal place. Assume I've defined $a_0, \ldots, a_{n-1}$ and shown that

$$S \cap \left[a_0.\, a_1 \ldots a_{n-1},\ a_0.\, a_1 \ldots (a_{n-1}+1)\right)$$

is nonempty and has the same upper bounds as the original S. Any element is $s = a_0.\, a_1 \ldots a_{n-1} s_n s_{n+1} \ldots$ with $s_n \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_n value; call it a_n.

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Our inductive procedure has produced for us a decimal expansion

$$a_0.a_1 a_2 \ldots a_n \ldots \text{ with } a_0 \in \mathbb{N} \text{ and } a_j \in \{0, 1, \ldots, 9\}\ .$$

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If this decimal has repeating 9's, then we assume we have rounded up so that we have

$$x = a_0.a_1 a_2 \ldots a_n \ldots \in \mathbb{R}\ .$$

Step 2 - Check x is an upper bound

Let $s \in S$, then $s = s_0 . s_1 s_2 \cdots\cdots$

Want to show $x \geq s$

By construction, we either have

$s_0 < a_0 \leftarrow$ We are done

$s_0 = a_0$

In the second case, we have

$s_1 < a_1 \leftarrow$ We are done

$s_1 = a_1$

$\vdots$ (Exercise: Finish w/ induction)

Step 3 - Check x is the least upper bound

Suppose $b<x$ and b is an upper bound for S

Suppose that n is the first digit where b differs from x

$$b = a_0.a_1a_2\cdots a_{n-1}b_n\cdots$$

with $\underline{\underline{b_n < a_n}}$

But by construction, $\exists s \in S$ of the form

$$s = a_0.a_1\cdots a_{n-1}a_n\cdots$$

And so $s > b$ ∎

Exercise 2.36

A student is trying to prove there exists $0 < x \in \mathbb{R}$ such that $x^2 = 2$. Since

$$S \ := \ \{0 < a \in \mathbb{R} \ : \ a^2 < 2\},$$

is nonempty ($1 \in S$) and bounded above by 2 (if $a \geq 2$ then $a^2 \geq 4$ so $a \notin S$) they set $x := \sup S > 0$.

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Next they give this proof that $x^2 \not< 2$. Is any step wrong?

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4. So $x + \epsilon \in S$ but $x + \epsilon > x = \sup S$ **Contradiction!**

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3. So if we take $0 < \epsilon < \frac{2-x^2}{2x}$ then $(x+\epsilon)^2 < 2$. ✓ wrong
4. So $x + \epsilon \in S$ but $x + \epsilon > x = \sup S$ **Contradiction!**
5. Nothing wrong, full marks for the student.

(2) $2 > (x+\varepsilon)^2 \Rightarrow \varepsilon < \dfrac{2-x^2}{2x}$ OK

(3) $\varepsilon < \dfrac{2-x^2}{2x} \Rightarrow (x+\varepsilon)^2 < 2$ Not true

$(x+\varepsilon)^2 = x^2 + 2\underset{\swarrow 4\varepsilon}{\varepsilon x} + \underset{\swarrow \varepsilon}{\varepsilon^2} \leq x^2 + 5\varepsilon$ if $\varepsilon \leq 1$

So $(x+\varepsilon)^2 < 2$ if $\varepsilon = \min(1, (2-x^2)/10)$

Exercise 2.37

Let $S = \{x \in \mathbb{Z} : x^2 < 3\}$. Then S is nonempty and bounded above. What is $\sup S$?

1. 0
2. 1 ✓
3. 2
4. 3
5. $\sqrt{3}$
6. Something else

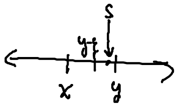
$S = \{-1, 0, 1\}$ $\quad$ $\sup(S) = 1$

Proposition 2.38

Suppose $\emptyset \neq S \subset \mathbb{R}$ and y is an upper bound for S.
Then $y = \sup S \iff \forall \epsilon > 0\ \exists s \in S : s > y - \epsilon$.

"$\Longrightarrow$" Suppose $y = \sup(S)$.
Let $\varepsilon > 0$, $y - \varepsilon < y$, so $y - \varepsilon$ is not an upper bound for S
Therefore $\exists s \in S$, $s > y - \varepsilon$.

"$\Longleftarrow$" We know y is an upper-bound and
$\longrightarrow$ $\forall \varepsilon > 0$, $\exists s \in S$: $s > y - \varepsilon$.



WTS $y = \sup(S)$
Suppose $x < y$ and x is an upper-bound for S
Set $\varepsilon = \frac{y-x}{2}$, then $\exists s \in S$, $s > y - \left(\frac{y-x}{2}\right)$ Rearrange $\Longrightarrow s > x$
Could also choose $\varepsilon = y - x$

Completeness Axiom $\Rightarrow$ Archimedean Axiom

For any $x \in \mathbb{R}$, $\exists\ N \in \mathbb{N}$ such that $N \geq x$.