Additional Materials - Worksheet 5 Question 3

Solution

May 20, 2021

Why is $\text{var}(\bar{x}_1^*) = \frac{2n-1}{n}\sigma^2$?

Let $x_{i,j}$ be sample i from the j^{th} bootstrap and recall that

$$\bar{x}_i^* = \frac{1}{n}\sum_{j=1}^{n} x_{i,j}. \tag{1}$$

It is also necessary to note that $\mathbb{E}[x_{i,j}] = \mathbb{E}[\mathbb{E}[x_{i,j}|x_i]] = \mathbb{E}[\bar{x}] = \mu$. Also, for $i, j \neq l, k$,

$$\text{cov}(x_{i,j}, x_{l,k}) = \frac{\sigma^2}{n}. \tag{2}$$

It is necessary to consider the special case now where we wish to find the variance of $x_{i,j}$.

$$\begin{aligned} \text{var}(x_{i,j}) &= \mathbb{E}[\text{var}(x_{i,j}|x_i)] + \text{var}(\mathbb{E}[x_{i,j}|x_i]), && (3) \\ &= \frac{n-1}{n}\sigma^2 + \frac{\sigma^2}{n}, && (4) \\ &= \sigma^2. && (5) \end{aligned}$$

Therefore, now rearranging the necessary variance,

$$\begin{aligned} \text{var}(\bar{x}_1^*) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{1,i}, x_{1,j}), && (6) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\left[\left(\sum_{j\neq i}^{n}\text{cov}(x_{1,i}, x_{1,j})\right) + \text{cov}(x_{1,i}, x_{1,i})\right]. && (7) \end{aligned}$$

Now, arbitrarily $\text{cov}(x_{1,i}, x_{1,j}) = \sigma^2/n$ for all $i \neq j$ and the inner sum will contain $n-1$ elements. Furthermore, $\text{cov}(x_{1,i}, x_{1,i}) = \text{var}(x_{1,i}) = \sigma^2$. Thus,

$$\begin{aligned} \text{var}(\bar{x}_1^*) &= \frac{1}{n^2}\sum_{i=1}^{n}\left[(n-1)\sigma^2/n + \sigma^2\right]. && (8) \\ &= \frac{1}{n^2}\left(n(n-1)\sigma^2/n + n\sigma^2\right), && (9) \\ &= \frac{1}{n^2}\left((n-1)\sigma^2 + n\sigma^2\right), && (10) \\ &= \frac{(2n-1)\sigma^2}{n^2}. && (11) \end{aligned}$$