Applied Complex Analysis - Lecture Thirteen

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The argument principle

For f meromorphic (all non-analytic points are are poles) and g analytic in Ω,

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f'(z)}{f(z)}g(z)dz = \sum_{a\in\{\text{zeros of } f\}} g(a)m_a - \sum_{b\in\{\text{poles of } f\}} g(b)m_b.$$

where m_a and m_b represent the order of the zeros and poles respectively, γ is a closed contour in Ω with no loops, such that $f(z) \neq 0$ for $z \in \gamma$.

- Proof
- Consequences and applications
- Approximation trick when $g = 1$
- Example: Root-finding

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Trapezium rule(s) for unbounded contours

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- Consider
$$I = \int_{-\infty}^{\infty} f(x)dx,$$
for some f analytic on $\mathbb{R}$, with appropriate decay of f such that $I < \infty$.
- We've seen techniques for evaluating these by hand - not always possible.
- For $h > 0$ we define the *unbounded* Trapezium rule $I_h \approx I$ as
$$I_h := h \sum_{j=-\infty}^{\infty} f(x_j),$$
where $x_j = jh$.
- We define the *truncated* Trapezium rule $I_h^{(N)} \approx I$ as
$$I_h^{(N)} := h \sum_{j=-N}^{N} f(x_j), \quad \text{for } N \in \mathbb{N}_0.$$

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Convergence theorem

Suppose $f(z)$ is analytic in the complex strip $|\mathrm{Im}(z)| < a$ for some $a > 0$. Suppose further that $f(z) \to 0$ uniformly as $|z| \to 0$ in the strip and

$$\int_{-\infty}^{\infty} |f(t + ia')| dt \leq M,$$

for all $a' \in (-a, a)$. Then I_h satisfies

$$|I - I_h| \leq \frac{2M}{e^{2\pi a/h} - 1}.$$

- This result is about the *unbounded* trapezium rule.
- This is called the *discretisation* error.
- Proof

The truncation error

Defined as, for $x_n = hn$

$$|I_h - I_h^{(N)}| = \left| h \sum_{n=-\infty}^{\infty} f(x_n) - h \sum_{n=-N}^{N} f(x_n) \right|$$

$$= \left| h \sum_{n=-\infty}^{-(N+1)} f(x_n) - h \sum_{n=N+1}^{\infty} f(x_n) \right|$$

- Practically, we care about

$$|I - I_h^{(N)}| \leq |I - I_h| + |I_h - I_h^{(N)}|$$

- Often, it is enough to bound by a constant multiplied by the first term.

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Bound on (half of the) truncation error

- Suppose that, for some $\alpha > 0$ independent of $y_0 > 0$, the function g satisfies the mild growth condition

$$g(y + \delta) - g(y) \geq \alpha\delta, \tag{1}$$

  for all $\delta > 0$ and $y > y_0$.
- Furthermore, suppose either that (i) the meshwidth h is independent of N, or (ii) the meshwidth $h \to 0$ as $N \to \infty$, but with a rate $1/N \ll h$.
- Then the positive terms in the truncation error satisfies:

$$h \sum_{n=N+1}^{\infty} e^{-g(hn)} = O(e^{-g(h(N+1))}), \quad N \to \infty.$$

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- Proof

Examples

- $$I = \int_{-\infty}^{\infty} e^{-x^2} \sqrt{(1+x^2)}\, dx,$$

- $$\operatorname{erfc}(z) = \frac{2e^{-z^2}}{\pi} \int_0^{\infty} \frac{e^{-z^2 t^2}}{t^2+1}\, dt, \quad z \in \mathbb{R}.$$

- $$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$