Imperial College London

MATH60010

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometric Mechanics

Date: 10 May 2023

Time: 14:00 – 16:00 (BST)

Time Allowed: 2 hrs

This paper has 4 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. **Euler-Lagrange equations for geodesics on $SO(3)$**

The Lie group transfomations of three dimensional rotations may be represented by 3×3 special orthogonal matrices, denoted $O \in SO(3)$ and satisfying, $O^T O = \mathbb{I}$, where $\mathbb{I}$ is the 3×3 identity matrix. By their defining relation, such matrices satisfy $O^{-1} = O^T$ and $\det O = 1$.

Geodesic motion on the manifold of $SO(3)$ rotations in three dimensions may be represented as a curve $O(t) \in SO(3)$ depending on time t. Its angular velocity is defined as the 3×3 matrix $\widehat{\Omega}$,

$$\widehat{\Omega}(t) := O^{-1}(t)\dot{O}(t) \in \mathfrak{so}(3)\,.$$

(a) Show that $\widehat{\Omega}(t) \in \mathfrak{so}(3)$ is skew symmetric, $\widehat{\Omega}^T = -\,\widehat{\Omega}$.

(2 marks)

(b) Show that the variational derivative $\delta\widehat{\Omega} = \frac{d}{ds}\Big|_{s=0}\widehat{\Omega}(t,s)$ of angular velocity $\widehat{\Omega} = O^{-1}\frac{d}{dt}O \in \mathfrak{so}(3)$ satisfies

$$\delta\widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,,$$

in which $\widehat{\Xi}(t) = O^{-1}(t)\delta O(t) = O^{-1}(t)\frac{d}{ds}\Big|_{s=0}O(t,s) \in \mathfrak{so}(3)$.

(2 marks)

(c) Compute the Euler-Lagrange equations for Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int_0^T L(\widehat{\Omega})\,dt\,,$$

for the quadratic Lagrangian $L : TSO(3) \to \mathbb{R}$, written using the Frobenius matrix pairing as,

$$L(\widehat{\Omega}) = -\frac{1}{2}\mathrm{tr}(\widehat{\Omega}\mathbb{A}\widehat{\Omega})\,,$$

in which $\mathbb{A}$ is a real symmetric positive-definite 3×3 matrix, $\mathbb{A} = \mathbb{A}^T$.

(12 marks)

(d) What is the symmetry group G of this Lagrangian? What is the implication of this symmetry group, G, according to Noether's theorem?

(4 marks)

(Total: 20 marks)

2. **Intersecting level sets**

The dynamical system for the divergence-free motion $\mathbf{x}(t) = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ along the intersection of two orthogonal circular cylinders is given by

$$\dot{x}_1(t) = x_2 x_3\,, \quad \dot{x}_2(t) = -\,x_1 x_3\,, \quad \dot{x}_3(t) = x_1 x_2\,.$$

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are two conserved functions, whose level sets are circular cylinders oriented, respectively, along the x_3-direction (H_1) and x_1-direction (H_2).

(2 marks)

(b) Show that the velocity $\dot{\mathbf{x}} \in T\mathbb{R}^3$ is divergence-free.

(2 marks)

(c) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of H_1. Show that the Poisson bracket on the circular cylinder $H_1 = const$ is symplectic.

(4 marks)

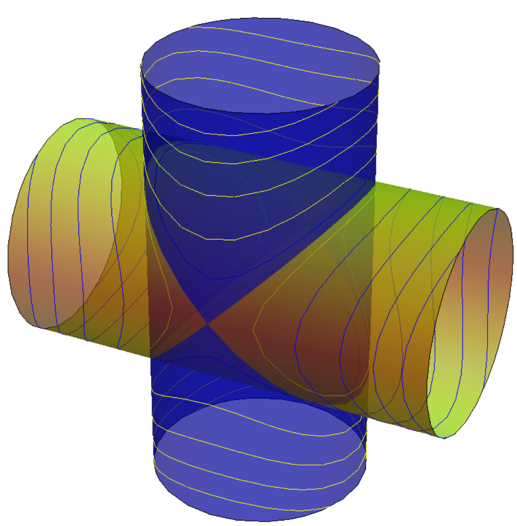
(d) Derive the equations of motion in cylindrical polar coordinates on a level set of H_1.

(6 marks)

(e) Express the equations of motion in the form of Newton's Law and show that these Newton's Law equations reduce to those for pendulum motion. The figure below may be helpful in the interpretation of this result.

(6 marks)

(Total: 20 marks)



3. **Hamilton-Pontryagin principle for heavy top dynamics**

Consider the following ***Hamilton-Pontryagin principle***

$$0 = \delta S = \delta \int_a^b \ell(\Omega, g^{-1}(t)e_3) + \Big\langle \Pi\,,\; g^{-1}\dot{g}(t) - \Omega \Big\rangle \, dt\,.$$

Physically, for variables $\Omega, \Pi, \Gamma \in \mathbb{R}^3$, constant $e_3 \in \mathbb{R}^3$, and Lie group $g = SO(3)$, the resulting dynamics would describe ***heavy top dynamics***.

(a) Calculate the equations of motion for this Hamilton-Pontryagin principle. That is, show that

$$\frac{d\Pi}{dt} = \mathrm{ad}^*_\Omega \Pi + \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma \quad \text{and} \quad \frac{\mathrm{d}\Gamma}{\mathrm{dt}} = \Omega\,.\,\Gamma\,,$$

where $\Omega := g^{-1}\dot{g}(t)\,,\quad \Gamma := g^{-1}(t)e_3,\quad \Omega\,.\,\Gamma$ denotes matrix Lie algebra action of $\Omega \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$. (The matrix Lie algebra action $\Omega\,.\,\Gamma$ in the heavy top case is $\Omega\,.\,\Gamma = -\Omega \times \Gamma$ for $\Omega \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$.)

$$\Pi := \frac{\partial \ell}{\partial \Omega} \quad \text{and} \quad \Big\langle \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma\,,\, \Xi \Big\rangle = \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, -\Xi\,.\,\Gamma \Big\rangle, \quad \Xi = g^{-1}\delta g\,, \quad (\Omega, \Xi) \in \mathfrak{g} \times V\,,$$

and $\Xi\,.\,\Gamma$ represents the Lie algebra action of $\Xi \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$.

(10 marks)

(b) Use the reduced Legendre transform to determine the corresponding Hamiltonian and express its partial derivatives with respect to Π and Γ.

(2 marks)

(c) Calculate the Hamiltonian equations in (Π, Γ) and write them in Lie-Poisson matrix form.

(4 marks)

(d) Calculate the corresponding Lie-Poisson bracket, $\{f, h\}(\Pi, \Gamma)$.

(4 marks)

(Total: 20 marks)

4. **Palais theorems**

(a) Show, for vector fields $u, v \in \mathfrak{X}(\mathbb{R}^3)$ and a k-form $\alpha \in \Lambda^k(\mathbb{R}^3)$ that from the product rule,

$$\pounds_u(v \,\lrcorner\, \alpha) - \pounds_v(u \,\lrcorner\, \alpha) = [u, v] \,\lrcorner\, \alpha + \Big[v \,\lrcorner\, (\pounds_u \alpha) - \pounds_v(u \,\lrcorner\, \alpha) \Big] \,,$$

for an arbitrary k-form α.

(6 marks)

(b) Use the formula in the previous part to show that the following theorem due to Palais holds for a 1-form $\alpha \in \Lambda^1(\mathbb{R}^3)$,

$$\pounds_u(v \,\lrcorner\, \alpha) - \pounds_v(u \,\lrcorner\, \alpha) = [u, v] \,\lrcorner\, \alpha + v \,\lrcorner\, (u \,\lrcorner\, d\alpha) \,.$$

(8 marks)

(c) Derive the vector calculus formula representing the Palais theorem if div$\boldsymbol{u} = 0$ and div$\boldsymbol{v} = 0$.

(6 marks)

(Total: 20 marks)

1. **Euler-Lagrange equations for geodesics on $SO(3)$**

seen ⇓

(a)

$$\widehat{\Omega}^T = (O^T\dot{O})^T = \Big(\frac{d}{dt}O^{-1}\Big)O = -\,O^{-1}\dot{O}O^{-1}O = -\,O^{-1}\dot{O} = -\,\widehat{\Omega}$$

xx

2, A

(b) The required variational formula

seen ⇓

$$\delta\widehat{\Omega} = \widehat{\Xi}^{\,\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,,$$

in which $\widehat{\Xi} = O^{-1}\delta O$ follows by subtracting the time derivative $\widehat{\Xi}^{\,\cdot} = (O^{-1}\delta O)^{\cdot}$ from the variational derivative $\delta\widehat{\Omega} = \delta(O^{-1}\dot{O})$ in the relations

$$\begin{aligned} \delta\widehat{\Omega} &= \delta(O^{-1}\dot{O}) = -\,(O^{-1}\delta O)(O^{-1}\dot{O}) + \delta\dot{O} = -\,\widehat{\Xi}\widehat{\Omega} + \delta\dot{O}\,, \\ \widehat{\Xi}^{\,\cdot} &= (O^{-1}\delta O)^{\cdot} = -\,(O^{-1}\dot{O})(O^{-1}\delta O) + (\delta O)^{\cdot} = -\,\widehat{\Omega}\widehat{\Xi} + (\delta O)^{\cdot}\,, \end{aligned}$$

and using equality of cross derivatives $\delta\dot{O} = (\delta O)^{\cdot}$.

xx

2, A

(c) (i) Taking matrix variations in this Hamilton's principle yields

meth seen ⇓

$$\begin{aligned} \delta S =: \int_a^b \Big\langle \frac{\delta L}{\delta\widehat{\Omega}}\,,\,\delta\widehat{\Omega}\Big\rangle\,dt &= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\frac{\delta L}{\delta\widehat{\Omega}}\right)dt \\ &= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\mathbb{A}\widehat{\Omega} + \delta\widehat{\Omega}\widehat{\Omega}\mathbb{A}\right)dt = -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\left(\mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A}\right)\right)dt \\ &= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\widehat{\Pi}\right)dt = \int_a^b \Big\langle\,\widehat{\Pi}\,,\,\delta\widehat{\Omega}\,\Big\rangle\,dt\,. \end{aligned}$$

This first step uses skew symmetry $\delta\widehat{\Omega}^T = -\delta\widehat{\Omega}$ and expresses the pairing in the variational derivative of S for matrices as the **Frobenius trace pairing**, e.g.,

$$\Big\langle\,M\,,\,N\,\Big\rangle =: \frac{1}{2}\mathrm{tr}\Big(M^T N\Big) = \frac{1}{2}\mathrm{tr}\Big(N^T M\Big)\,.$$

xx

3, B

(ii) The second step applies the variational derivative. After cyclically permuting the order of matrix multiplication under the trace in the 2nd line, the 3rd line substitutes

$$\widehat{\Pi} = \mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A} = \frac{\delta L}{\delta\widehat{\Omega}}\,.$$

xx

3, C

(iii) The third step substitutes the variational formula for $\delta\widehat{\Omega}$ into the stationary variation of the action

$$\delta S = 0 \quad \text{with} \quad S = \int L(\widehat{\Omega})\, dt \,,$$

leads to

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\widehat{\Pi}\right) dt = -\frac{1}{2}\int_a^b \mathrm{tr}\left((\widehat{\Xi}^{\,\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega})\,\widehat{\Pi}\right) dt \,.$$

Permuting cyclically under the trace again yields

$$\mathrm{tr}(\widehat{\Omega}\,\widehat{\Xi}\,\widehat{\Pi}) = \mathrm{tr}(\widehat{\Xi}\,\widehat{\Pi}\,\widehat{\Omega}) \,.$$

Integrating by parts (dropping endpoint terms) then yields the equation

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{tr}\left(\widehat{\Xi}\,(-\widehat{\Pi}^{\,\cdot} + \widehat{\Pi}\widehat{\Omega} - \widehat{\Omega}\widehat{\Pi}\,)\right) dt \,.$$

xx $\boxed{3,\ \mathrm{C}}$

(iv) Finally, invoking stationarity $\delta S = 0$ for an arbitrary variation $\widehat{\Xi} = O^{-1}\delta O$ yields geodesic dynamics on $SO(3)$ with respect to the metric $\mathbb{A}$ in the matrix commutator form,

$$\frac{d\widehat{\Pi}}{dt} = -\,[\,\widehat{\Omega}\,,\,\widehat{\Pi}\,] \qquad \text{with} \qquad \widehat{\Pi} = \mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A} = \frac{\delta L}{\delta\widehat{\Omega}} = -\,\widehat{\Pi}^T \,.$$

xx $\boxed{3,\ \mathrm{A}}$

(d) The symmetry group of the Lagrangian $\boxed{\text{unseen} \Downarrow}$

$$L(\widehat{\Omega}) = \frac{1}{2}\mathrm{tr}(\widehat{\Omega}^T\mathbb{A}\widehat{\Omega})$$

is the Adjoint action $\mathrm{Ad}_{SO(3)} : SO(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$. This Adjoint action preserves the eigenvalues of the 3×3 symmetric matrix $\mathbb{A}$. The corresponding conserved quantities are the three components of the spatial angular momentum.

$$\frac{d}{dt}\Big(\mathrm{Ad}^*_{O^{-1}(t)}\widehat{\Pi}(t)\Big) = \mathrm{Ad}^*_{O^{-1}(t)}\Big(\frac{d}{dt} + \mathrm{ad}^*_{\widehat{\Omega}}\widehat{\Pi}(t)\Big) = 0 \,.$$

xx $\boxed{4,\ \mathrm{B}}$

2. **Intersecting level sets** $\boxed{\text{meth seen} \Downarrow}$

(a)

$$H_1 = \frac{1}{2}(x_1^2 + x_2^2)\,, \quad H_2 = \frac{1}{2}(x_2^2 + x_3^2)$$

xx $\boxed{2, \text{A}}$

(b) We have $\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2 = \mathrm{curl}(H_1 \nabla H_2)$, and $div(curl\,\mathbf{v}(\mathbf{x})) = 0$ vanishes for any differentiable vector field $\mathbf{v}(\mathbf{x})$. $\boxed{\text{meth seen} \Downarrow}$

xx $\boxed{2, \text{A}}$

(c) $\boxed{\text{unseen} \Downarrow}$

$$\frac{dF}{dt}\,\mathrm{d}^3\mathrm{x} = \nabla\mathrm{F}\cdot\nabla\mathrm{H}_1 \times \nabla\mathrm{H}_2\,\mathrm{d}^3\mathrm{x} = -\,\mathrm{dH}_1 \wedge (\mathrm{dF}\wedge\mathrm{dH}_2)$$

and on a level set of H_1 the term $(\mathrm{dF}\wedge\mathrm{dH}_2)$ defines a canonical Poisson bracket.

xx $\boxed{4, \text{D}}$

(d) On a level set of $H_1 = \frac{1}{2}(x_1^2 + x_2^2)$ one defines cylindrical coordinates so that

$$x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,, \quad x_3 = p\,.$$

The dynamics for $H_2 = \frac{1}{2}(x_2^2 + x_3^2)$ in these coordinates is symplectic with Hamiltonian

$$H_2 = \frac{1}{2}(r^2\sin^2\theta + p^2)\,, \quad \dot\theta = \frac{\partial H_2}{\partial p} = p\,, \quad \dot p = -\,\frac{\partial H_2}{\partial\theta} = -\,r^2\sin\theta\cos\theta = -\,\frac{r^2}{2}\sin 2\theta$$

xx $\boxed{6, \text{B}}$

(e) In terms of $\phi = 2\theta$ one finds Newton's law,

$$\ddot\phi = -\,r^2\sin\phi$$

which is the equation for the planar motion of a simple pendulum.

Thus, planar pendulum motion is isomorphic to the divergence-free motion in $\mathbb{R}^3$ along the intersection of two orthogonal circular cylinders.

xx $\boxed{6, \text{B}}$

3. **Hamilton-Pontryagin principle** $\boxed{\text{sim. seen} \Downarrow}$

(a) First calculate

$$\delta\Omega = \delta(g^{-1}\dot{g}) = \frac{d\Xi}{dt} + \big[\Omega, \Xi\big] \quad \text{and} \quad \delta g^{-1}(t)e_3 = -\,\Xi\,.\,\Gamma \quad \text{with} \quad \Gamma := g^{-1}(t)e_3$$

Then insert the results into the variational principle to find

$$\begin{aligned}
0 = \delta S = \delta \int_a^b \ell(\Omega, g^{-1}(t)e_3) + \big\langle \Pi\,,\, g^{-1}\dot{g}(t) - \Omega \big\rangle \, dt \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle \Pi\,,\, \frac{d\Xi}{dt} + \big[\Omega, \Xi\big] \Big\rangle + \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, -\,\Xi\,.\,\Gamma \Big\rangle \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle \Pi\,,\, \frac{d\Xi}{dt} + \mathrm{ad}_\Omega \Xi \Big\rangle + \Big\langle \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma\,,\, \Xi \Big\rangle \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle -\,\frac{d\Pi}{dt} + \mathrm{ad}^*_\Omega \Pi + \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma\,,\, \Xi \Big\rangle\,.
\end{aligned}$$

xx $\boxed{8, \text{A}}$

(b) The Hamiltonian is found from the reduced Legendre transform, $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned}
h(\Pi, \Gamma) &= \big\langle \Pi\,,\, \Omega \big\rangle - \ell(\Omega, \Gamma) \\
dh &= \big\langle d\Pi\,,\, \Omega \big\rangle + \Big\langle \Pi - \frac{\partial \ell}{\partial \Omega}\,,\, \delta\Omega \Big\rangle - \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, \delta\Gamma \Big\rangle \\
\frac{\delta h}{\delta \Pi} &= \Omega \quad \text{and} \quad \frac{\partial h}{\partial \Gamma} = -\,\frac{\partial \ell}{\partial \Gamma}\,.
\end{aligned}$$

xx $\boxed{2, \text{A}}$

(c) Hamiltonian equations in Lie-Poisson matrix form are found as $\boxed{\text{unseen} \Downarrow}$

$$\frac{d}{dt}\begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} \mathrm{ad}^*_\square \Pi & \square \diamond \Gamma \\ \square\,.\,\Gamma & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial \Pi = \Omega \\ \partial h/\partial \Gamma = -\,\partial \ell/\partial \Gamma \end{bmatrix}$$

xx $\boxed{6, \text{D}}$

(d) The corresponding Lie-Poisson bracket is given by $\boxed{\text{unseen} \Downarrow}$

$$\frac{df(\Pi, \Gamma)}{dt} = \left\langle \begin{bmatrix} \partial f/\partial \Pi \\ \partial f/\partial \Gamma \end{bmatrix}, \begin{bmatrix} \mathrm{ad}^*_\square \Pi & \square \diamond \Gamma \\ \square\,.\,\Gamma & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial \Pi = \Omega \\ \partial h/\partial \Gamma = -\,\partial \ell/\partial \Gamma \end{bmatrix} \right\rangle =: \{f, h\}$$

xx $\boxed{4, \text{A}}$

4. **Palais theorems** seen ⇓

(a) From the product rule we have

$$\pounds_u(v \lrcorner \alpha) = (\pounds_u v) \lrcorner \alpha + v \lrcorner (\pounds_u \alpha) = [u,v] \lrcorner \alpha + v \lrcorner (\pounds_u \alpha)\,.$$

Subtracting $\pounds_v(u \lrcorner \alpha)$ from both sides yields the required identity.

xx 6, A

(b) Here is a straightforward proof. We have from the product rule in part (a) that unseen ⇓ meth seen ⇓

$$\pounds_u(v \lrcorner \alpha) - \pounds_v(u \lrcorner \alpha) = [u,v] \lrcorner \alpha + \Big[v \lrcorner (\pounds_u \alpha) - \pounds_v(u \lrcorner \alpha)\Big]\,.$$

But, upon inserting Cartan's magic formula into the bracketed terms, we find

$$\begin{aligned}\Big[v \lrcorner (\pounds_u \alpha) - \pounds_v(u \lrcorner \alpha)\Big] &= v \lrcorner\, d(u \lrcorner \alpha) + v \lrcorner (u \lrcorner\, d\alpha) \\ &\quad - d(v \lrcorner\, (u \lrcorner \alpha)) - v \lrcorner\, d(u \lrcorner \alpha)\end{aligned}$$

The first and last terms cancel on the right-hand side, and for $\alpha \in \Lambda^1$ we have $v \lrcorner\, (u \lrcorner \alpha) = 0$, so that

$$\pounds_u(v \lrcorner \alpha) - \pounds_v(u \lrcorner \alpha) = [u,v] \lrcorner \alpha + \Big[v \lrcorner (u \lrcorner\, d\alpha)\Big]\,.$$

If $\alpha \in \Lambda^3$ is a top form, then $d\alpha$ vanishes and we get a different interesting combination.

xx 8, D

(c) The famous relation in part (b) follows from evaluating its left hand side for a 1-form $\alpha = \boldsymbol{\alpha}\cdot d\boldsymbol{x} = \alpha_j dx^j$ unseen ⇓ meth seen ⇓

$$\pounds_u(v \lrcorner \alpha) - \pounds_v(u \lrcorner \alpha) = (\boldsymbol{u}\cdot\boldsymbol{\nabla})(\boldsymbol{v}\cdot\boldsymbol{\alpha}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{\alpha})\,,$$

with notation $\boldsymbol{u}\cdot\boldsymbol{\alpha} = \alpha_j u^j$, and equating this formula to the evaluation of the right hand side, as

$$\begin{aligned}[u,v] \lrcorner \alpha + v \lrcorner (u \lrcorner\, d\alpha) &= \big((\boldsymbol{u}\cdot\boldsymbol{\nabla})v^j - (\boldsymbol{v}\cdot\boldsymbol{\nabla})u^j\big)\alpha_j \\ &\quad + v^j(\boldsymbol{u}\cdot\boldsymbol{\nabla})\alpha_j - u^j(\boldsymbol{v}\cdot\boldsymbol{\nabla})\alpha_j\,.\end{aligned}$$

Here, side calculations for $v \lrcorner (u \lrcorner\, d\alpha)$ and $[u,v] \lrcorner \alpha$ are facilitated by using their vector forms

$$\begin{aligned}v \lrcorner (u \lrcorner\, d\alpha) &= \mathrm{curl}\boldsymbol{\alpha}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = v^j(\boldsymbol{u}\cdot\boldsymbol{\nabla})\alpha_j - u^j(\boldsymbol{v}\cdot\boldsymbol{\nabla})\alpha_j\,,\\ [u,v] \lrcorner \alpha &= -\boldsymbol{\alpha}\cdot\mathrm{curl}(\boldsymbol{u}\times\boldsymbol{v}) = \alpha_j(\boldsymbol{u}\cdot\boldsymbol{\nabla})v^j - \alpha_j(\boldsymbol{v}\cdot\boldsymbol{\nabla})u^j\,.\end{aligned}$$

Consequently,

$$\begin{aligned}[u,v] \lrcorner \alpha + v \lrcorner (u \lrcorner\, d\alpha) &= -\boldsymbol{\alpha}\cdot\mathrm{curl}(\boldsymbol{u}\times\boldsymbol{v}) + \mathrm{curl}\boldsymbol{\alpha}\cdot\boldsymbol{u}\times\boldsymbol{v} \\ &= \mathrm{div}(\boldsymbol{\alpha}\times(\boldsymbol{u}\times\boldsymbol{v})) = \mathrm{div}(\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{\alpha}) - \boldsymbol{v}(\boldsymbol{u}\cdot\boldsymbol{\alpha})) \\ &= (\boldsymbol{u}\cdot\boldsymbol{\nabla})(\boldsymbol{v}\cdot\boldsymbol{\alpha}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{\alpha}) \\ &= \pounds_u(v \lrcorner \alpha) - \pounds_v(u \lrcorner \alpha)\end{aligned}$$

xx 6, C

5. **Euler-Poincaré compressible adiabatic fluid dynamics**

$\boxed{\text{seen} \Downarrow}$

(a) The symmetry reduced Hamilton's principle with a Lagrangian functional for ideal fluid dynamics in our standard form with Euler-Poincaré variations is given by

$$
\begin{aligned}
0 = \delta S_{red} = \delta \int_0^T \ell(u,a)\,dt &= \int_0^T \Big\langle \frac{\delta \ell}{\delta u}\,,\,\delta u \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a}\,,\,\delta a \Big\rangle\,dt \\
&= \int_0^T \Big\langle \frac{\delta \ell}{\delta u}\,,\,\partial_t \xi - \mathrm{ad}_{\mathrm{u}}\xi \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a}\,,\,-\pounds_\xi a \Big\rangle\,dt \\
&= \int_0^T \Big\langle -(\partial_t + \mathrm{ad}^*_{\mathrm{u}})\frac{\delta \ell}{\delta \mathrm{u}} + \frac{\delta \ell}{\delta \mathrm{a}} \diamond \mathrm{a}\,,\,\xi \Big\rangle\,dt\,.
\end{aligned}
$$

The resulting Euler-Poincaré equations are

$$
(\partial_t + \mathrm{ad}^*_{\mathrm{u}})\frac{\delta \ell}{\delta \mathrm{u}} = \frac{\delta \ell}{\delta \mathrm{a}} \diamond \mathrm{a} \quad \text{with advection relation} \quad (\partial_{\mathrm{t}} + \pounds_{\mathrm{u}})\mathrm{a} = 0\,.
$$

xx $\boxed{6,\ \mathrm{M}}$

(b) The reduced Legendre transformation is $\boxed{\text{meth seen} \Downarrow}$

$$
h(\mu,a) := \langle \mu\,,\,u \rangle - \ell(u,a)\,,
$$

xx $\boxed{2,\ \mathrm{M}}$

(c) The functional derivatives $\boxed{\text{meth seen} \Downarrow}$

$$
\begin{aligned}
\delta h(\mu,a) &= \langle \delta\mu\,,\,u \rangle + \Big\langle \mu - \frac{\delta \ell}{\delta u}\,,\,\delta u \Big\rangle - \Big\langle \frac{\delta \ell}{\delta a}\,,\,\delta a \Big\rangle \\
&= \Big\langle \frac{\delta h}{\delta \mu}\,,\,\delta \mu \Big\rangle + \Big\langle \frac{\delta h}{\delta a}\,,\,\delta a \Big\rangle\,,
\end{aligned}
$$

yield the required variational relations

$$
\delta\mu : \frac{\delta h}{\delta \mu} = u, \qquad \delta u : \mu = \frac{\delta \ell}{\delta u}, \qquad \delta a : \frac{\delta h}{\delta a} = -\frac{\delta \ell}{\delta a}\,.
$$

xx $\boxed{2,\ \mathrm{M}}$

(d) The resulting Lie-Poisson equations may then be written as $\boxed{\text{sim. seen} \Downarrow}$

$$
(\partial_t + \mathrm{ad}^*_{\delta \mathrm{h}/\delta\mu})\mu = -\frac{\delta \mathrm{h}}{\delta \mathrm{a}} \diamond \mathrm{a} \quad \text{with advection relation} \quad (\partial_{\mathrm{t}} + \pounds_{\delta \mathrm{h}/\delta\mu})\mathrm{a} = 0\,.
$$

These equations are expressed in Lie-Poisson matrix form, as follows

$$
\partial_t \begin{bmatrix} \mu \\ a \end{bmatrix} = -\begin{bmatrix} \mathrm{ad}^*_\square \mu & \square \diamond a \\ \pounds_\square a & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta\mu \\ \delta h/\delta a \end{bmatrix}
$$

xx $\boxed{4,\ \mathrm{M}}$

(e) The Lie-Poisson bracket is seen to be dual to a semidirect-product Lie algebra by isolating multiples of the fluid variables, as follows. unseen ⇓

$$
\begin{aligned}
\frac{d}{dt}f(\mu,a) &= -\Big\langle \Big(\frac{\delta f}{\delta\mu}\,,\,\frac{\delta f}{\delta a}\Big)\,,\,\Big(\mathrm{ad}^*_{\delta\mathrm{h}/\delta\mu}\mu + \frac{\delta\mathrm{h}}{\delta\mathrm{a}}\diamond\mathrm{a}\,,\,\pounds_{\delta\mathrm{h}/\delta\mu}\mathrm{a}\Big)\Big\rangle \\
&=: -\Big\langle \frac{\delta f}{\delta\mu}\,,\,\mathrm{ad}^*_{\delta\mathrm{h}/\delta\mu}\mu + \frac{\delta\mathrm{h}}{\delta\mathrm{a}}\diamond\mathrm{a}\Big\rangle - \Big\langle \frac{\delta f}{\delta a}\,,\,\pounds_{\delta h/\delta\mu}a\Big\rangle \\
&= -\Big\langle \frac{\delta f}{\delta\mu}\,,\,\mathrm{ad}^*_{\delta\mathrm{h}/\delta\mu}\mu\Big\rangle - \Big\langle \frac{\delta f}{\delta a}\,,\,\pounds_{\delta h/\delta\mu}a\Big\rangle - \Big\langle \frac{\delta f}{\delta\mu}\,,\,\frac{\delta h}{\delta a}\diamond a\Big\rangle \\
&= -\Big\langle \mathrm{ad}_{\delta\mathrm{h}/\delta\mu}\frac{\delta\mathrm{f}}{\delta\mu}\,,\,\mu\Big\rangle - \Big\langle \pounds^T_{\delta h/\delta\mu}\frac{\delta f}{\delta a} - \pounds^T_{\delta f/\delta\mu}\frac{\delta h}{\delta a}\,,\,a\Big\rangle \\
&= -\Big\langle (\mu\,,\,a)\,,\,\Big(\pounds_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\,,\,\pounds^T_{\delta h/\delta\mu}\frac{\delta f}{\delta a} - \pounds^T_{\delta f/\delta\mu}\frac{\delta h}{\delta a}\Big)\Big\rangle =: \big\{f,h\big\}(\mu,a)\,.
\end{aligned}
$$

The last line pairs $(\mu,a)\in\mathfrak{X}^*\times V$ with the semidirect-product Lie algebra action of vector fields $(X,\bar{X})\in\mathfrak{X}$ acting on a vector space V^* dual to V with elements $(a^*,\bar{a}^*)\in V^*$. Formally, the action is

$$
\big[(X,a^*),(\bar{X},\bar{a}^*)\big] = \big([X,\bar{X}]\,,\,X\bar{a}^* - \bar{X}a^*\big)\,.
$$

Thus, the matrix form of the Hamiltonian equations defines the Lie-Poisson bracket in as the pairing of the semidirect-product Lie algebra $\mathfrak{X}Ⓢ V^*$ with its coordinates $\mu\in\mathfrak{X}^*$ dual to $\delta h/\delta\mu\in\mathfrak{X}$ and $a\in V$ dual to $\delta h/\delta a\in V^*$.

xx 6, M

Review of mark distribution:

Total A marks: 31 of 32 marks

Total B marks: 19 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 18 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60010	1	No Comments Received
MATH60010	2	No Comments Received
MATH60010	3	No Comments Received
MATH60010	4	No Comments Received