1. Consider the equation

$$\begin{cases} \partial_t u + (u^3 + u)\partial_x u = 0, \\ u\Big|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where $u_0(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0. \end{cases}$

(a) Plot schematically the characteristics and show that the solution $u(t,x)$ has the form $u(t,x) = u_0(x - c_u t)$, $t > 0$. Determine the shock speed c_u using the Rankine-Hugoniot condition. (8 marks)

(b) Let $F(z) := z^3 + z$ and let $u(t,x)$ be a C^1-smooth solution of (1) (with the appropriate initial data). Prove that $v(t,x) := F(u(t,x))$ solves the Burgers equation

$$\partial_t v + v\partial_x v = 0. \tag{2}$$

(5 marks)

(c) Solve the Burgers equation (2) with the initial data $v\Big|_{t=0} = F(u_0(x))$ and show that its shock solution $v(t,x) \neq F(u(t,x))$, where u is a shock solution of (1) found in Part (a).

(7 marks)

(Total: 20 marks)

2. Let us consider the wave equation

$$\begin{cases} \partial_{tt} u = \partial_{xx} u, & x \in \mathbb{R}, t > 0, \\ u\Big|_{t=0} = \phi_0(x), \quad \partial_t u\Big|_{t=0} = \phi_1(x). \end{cases}$$

(a) Using the d'Alembert formula write down the solution with $\phi_1(x) \equiv 0$ and $\phi_0 = \phi(x)$ some given function.

(4 marks)

(b) Assume that $\phi(x) \equiv 0$ for $|x| \geq 1$, but non-zero otherwise. Plot the region in the (x, t) plane where the solution $u(t, x) = 0$ (for all $\phi(x)$ satisfying the above property). (3 marks)

(c) Assuming that $\phi(x) \equiv 0$ for $x \geq 0$ express the value of $u(t, 0) \equiv \psi(t)$ in terms of the function ϕ. Using this formula, solve the initial-boundary value problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u, & x > 0, t > 0, \\ u\Big|_{t=0} = 0, \quad \partial_t u\Big|_{t=0} = 0, \\ u\Big|_{x=0} = \psi(t). \end{cases}$$

(5 marks)

(d) Using the formula for the partial solution found in Part (c), find a general solution of the initial-boundary value problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u, & x > 0, \ t > 0, \\ u\Big|_{t=0} = \phi_0(x), \quad \partial_t u\Big|_{t=0} = \phi_1(x), \\ u\Big|_{x=0} = \psi(t). \end{cases}$$

Hint: reduce the problem to the particular case $\psi(t) \equiv 0$, do the reflection about the line $x = 0$ and use the d'Alembert formula for the whole line $x \in \mathbb{R}$. (8 marks)

(Total: 20 marks)

3. Let us consider the fully non-linear parabolic equation

$$\begin{cases} (\partial_t u)^5 = \Delta u, & x \in \mathbb{R}^3 \\ u\Big|_{t=0} = u_0. \end{cases} \tag{3}$$

(a) Assuming that $u(t,x) = T(t)X(r)$, $r = |x|$ write down the ODEs for the functions T and X.

Hint: $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ for radially symmetric functions. (5 marks)

(b) Find the solution of the X-equation in the form $X(r) = Cr^\alpha$ (compute C and α).

(5 marks)

(c) Solve the $T(t)$ equation and find the solution of (3) with the initial data $u\Big|_{t=0} = u_0(x) = r^\alpha$ (α is the same as in Part (b)).

(6 marks)

(d) Prove that the solution $u(t,x)$ from Part (c) vanishes identically for $t \geq t_0 > 0$. Find the value t_0.

(4 marks)

(Total: 20 marks)

4. (a) Using the formula for the harmonic function in polar coordinates in $\mathbb{R}^2$

$$u(r,\theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta))(C_n r^n + D_n r^{-n}),$$

find the solution of

(i) Interior boundary value problem

$$\begin{cases} \Delta u_{int} = 0, x \in \Omega = B_1(0)(\text{ unit ball in } \mathbb{R}^2) \\ u\Big|_{\partial\Omega} = x^2 + y^2 + xy. \end{cases}$$

(6 marks)

(ii) Exterior boundary value problem

$$\begin{cases} \Delta u_{ext} = 0, \quad x \in \mathbb{R} \setminus B_1(0) \\ u\Big|_{\partial\Omega} = x^2 + y^2 + xy, \\ u(x,y) \to \text{const}, \text{ when } r \to \infty. \end{cases}$$

(6 marks)

(b) Compute

$$\partial_n u_{int}\Big|_{\partial\Omega} - \partial_n u_{ext}\Big|_{\partial\Omega},$$

as well as

$$\partial_n u_{int}\Big|_{\partial\Omega} + \partial_n u_{ext}\Big|_{\partial\Omega},$$

where n is an exterior normal to Ω. (4 marks)

(c) Explain why the function

$$\tilde{u}(x,y) = \begin{cases} u_{int}(x,y), \text{ if } (x,y) \in \overline{B_1(0)} \\ u_{ext}(x,y), \text{ if } (x,y) \in \mathbb{R}^2 \setminus B_1(0). \end{cases}$$

is not harmonic in the whole space $\mathbb{R}^2$.

Hint: You may use the Weyl theorem which claims that any harmonic function is C^∞-smooth without proving it. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH50008

Partial Differential Equations in Action (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. (a) The equations for characteristics are sim. seen ⇓

$$\frac{du}{dt} = 0 \text{ on } \frac{dx}{dt} = u^3 + u,$$

which leads to

$$x(t) = (u_0^3(\xi) + u_0(\xi))t + \xi = \begin{cases} 2t + \xi, & \text{if } \xi \leq 0 \\ \xi, & \text{if } \xi > 0. \end{cases}$$

4, B

So we see that, for any $t > 0$, however small, characteristics from $\xi > 0$ cross with the characteristic emanating from $\xi < 0$. Thus, the shock is formed. To determine the shock speed we use the Rankine-Hugoniot condition. In our case, the flux function $q(u) = \frac{u^4}{4} + \frac{u^2}{2}$ and sim. seen ⇓

$$c_u = s'(t) = \frac{\frac{u_-^4}{4} + \frac{u_-^2}{2} - \frac{u_+^4}{4} - \frac{u_+^2}{2}}{u_- - u_+} = \frac{\frac{1}{4} + \frac{1}{2}}{1} = \frac{3}{4}.$$

Thus, we obtain that

$$u(t, x) = \begin{cases} 1, & \text{if } x \leq \frac{3}{4}t \\ 0, & \text{if } x > \frac{3}{4}t, \end{cases}$$

or, in other words, $u(t, x) = u_0(x - \frac{3}{4}t)$, as required. 4, B

(b) Since function $u(t, x)$ is C^1-smooth, we can differentiate $v(t, x)$ and obtain unseen ⇓

$$\partial_t v = (3u^2 + 1)\partial_t u \text{ and } \partial_x v = (3u^2 + 1)\partial_x u.$$

Substituting this derivatives to the Burgers we get the final result

$$\partial_t v + v\partial_x v = (3u^2 + 1)\partial_t u + (u^3 + u)(3u^2 + 1)\partial_x u = 0.$$

5, A

sim. seen ⇓

(c) First we find that

$$v_0(x) = F(u_0(x)) = \begin{cases} 2, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0, \end{cases}$$

and the equations for characteristics are

$$v(t, x) = v_0(\xi) \text{ on } x(t) = v_0(\xi)t + \xi = \begin{cases} 2t + \xi, & \text{if } \xi \leq 0 \\ \xi, & \text{if } \xi > 0. \end{cases}$$

Let us compute the shock speed, by the Rankine-Hugoniot condition 3, A

$$s'(t) = \frac{1}{2}u_- = 1,$$

which leads to the solution

$$v(t, x) = \begin{cases} 2, & \text{if } x \leq t \\ 0, & \text{if } x > t, \end{cases}$$

whereas 3, A

$$F(u(t, x)) = \begin{cases} 2, & \text{if } x \leq \frac{3}{4}t \\ 0, & \text{if } x > \frac{3}{4}t, \end{cases}$$

1, A

meth seen $\Downarrow$

2. (a) The d'Alembert formula reads $u(t,x) = \varphi(x+t) + \psi(x-t)$, where φ and ψ are arbitrary functions to be found from the initial conditions at $t = 0$:

$$\phi(x) = \varphi(x) + \psi(x), \quad 0 = \varphi'(x) - \psi'(x)$$

integrating the second equation, we get $\varphi(x) = \psi(x) + C$ and from the first equation, we conclude that $\psi(x) = \frac{1}{2}\phi(x) - \frac{C}{2}$, $\varphi(x) = \frac{1}{2}\phi(x) + \frac{1}{2}C$. The constant C disappears from the formula for $u(t,x)$ and we arrive at

$$u(t,x) = \frac{1}{2}(\phi(x-t) + \phi(x+t)).$$

4, C

meth seen $\Downarrow$

(b) Let now the support of ϕ lies in the segment $[-1,1]$ (i.e. $\phi(x) = 0$ for $|x| > 1$). Then the first wave ($\phi(x-t)$) is zero outside of the strip $\Pi_+ := (x,t) \in \mathbb{R}^2 : -1 < x - t < 1$. Analogously, the second wave ($\phi(t+x)$) vanishes outside of the strip $\Pi_- := (x,t) \in \mathbb{R}^2 : -1 < x + t < 1$. Then, the function $u(t,x)$ is zero outside of the union of these two strips. This area consists of 3 triangles: $\{t \geq |x| + 1\}$, $\{t \geq 0,\ t \leq x - 1\}$ and $\{t \geq 0, t \leq -x - 1\}$, see the picture below

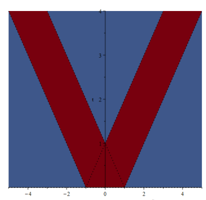

Figure 1: 0-region is filled by blue

3, A

sim. seen $\Downarrow$

(c) We use the formula from Part (a). Since $\phi(x) = 0$ for $x \geq 0$, the function $\phi(x+t)$ vanishes at $x = 0$, $t \geq 0$, so we have $u(t,0) = \frac{1}{2}\phi(-t)$. Thus, if we want to have $u(t,0) = \psi(t)$, we need to take $\phi(x) = 2\psi(-x)$ and the solution of the initial-boundary value problem is $u(t,x) = \tilde{\psi}(t+x)$, where $\tilde{\psi}(x) = 0$ for $x \leq 0$ and $\tilde{\psi}(x) = \psi(x)$ for $x \geq 0$.

5, D

sim. seen $\Downarrow$

(d) We use the partial solution $u = u_p(t,x)$ found in Part (c). Then the difference $v - u_p$ solves the equation with the same initial data, but with zero boundary condition at $x = 0$. To solve the equation for v we use the reflection (odd extension) $v(t,-x) = -v(t,x)$, $x \leq 0$ which solves the string equation on the whole line with the initial data $\tilde{\varphi}_0(x) := \varphi_{0,odd}(x)$, $\tilde{\varphi}_1(x) = \varphi_{1,odd}(x)$. Using the d'Alembert formula, we finally get the desired solution

3, C

3, C

$$u(t,x) = \tilde{\psi}(t+x) + \frac{1}{2}\tilde{\varphi}_0(x-t) + \frac{1}{2}\tilde{\varphi}_0(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{\varphi}_1(s)\, ds.$$

2, C

3. (a) Using the ansatz $u(t,x) = T(t)X(r)$, we get meth seen ⇓

$$T'(t)^5 X(r)^5 = T(t)(X''(r) + \frac{2}{r}X'(r))$$

and separating variables, we get $T'(t)^5 = cT(t)$, $X''(r) + \frac{2}{r}X'(t) = cX(r)^5$. 5, A

(b) Inserting $X(r) = Cr^{\alpha}$, we get $C\alpha(\alpha-1)r^{\alpha-2} + 2C\alpha r^{\alpha-2} = cC^5 r^{5\alpha}$. From this equation, we get $\alpha - 2 = 5\alpha$, i.e. $\alpha = -\frac{1}{2}$ and $-\frac{1}{4} = cC^4$. From here we conclude that $c < 0$ and $C = \sqrt[4]{\frac{1}{-4c}}$. sim. seen ⇓ 5, D

(c) The equation for $T(t)$ reads $T'(t) = c^{1/5}T^{1/5}(t)$. Separating variables, we get unseen ⇓

$$\frac{5}{4}(T(t)^{4/5} - T(0)^{4/5}) = c^{1/5}t, \quad T(t) = \left(T(0)^{\frac{4}{5}} + \frac{4}{5}c^{1/5}t\right)_+^{\frac{5}{4}},$$

where $z_+ = \max\{z, 0\}$. The corresponding solution of the PDE is

$$u(t,x) = \sqrt[4]{\frac{1}{-4c}}\left(T(0)^{\frac{4}{5}} + \frac{4}{5}c^{1/5}t\right)_+^{\frac{5}{4}} \frac{1}{\sqrt{r}}$$

6, D

(d) To satisfy the initial condition, we need to take $T(0) = \sqrt[4]{-4c}$ and we see that the constant c actually cancels out, so we may take without loss of generality $c = -1$ and $T(0) = \sqrt{2}$. Thus, unseen ⇓

$$u(t,x) = \frac{1}{\sqrt{2}}\left(\sqrt[5]{4} - \frac{4}{5}t\right)_+^{\frac{5}{4}} \frac{1}{\sqrt{|x|}}.$$

Thus, the solution remains strictly positive for $t < t_0 := 5\,4^{-4/5}$ and becomes identically zero for $t \geq t_0$. 4, A

4. meth seen ⇓

(a) (i) First, we need to write the boundary condition in polar coordinates

$$\phi(x,y) = x^2 + y^2 + xy = r^2 + r\cos\theta\sin\theta = 1 + \frac{1}{2}\sin(2\theta).$$

For the interior problem all of the coefficients C_0, D_n must be zero, so we get $D_0 = 1$ and $B_2 = 1/2$, $C_2 = 1$. Thus,

$$u_{int}(x,y) = 1 + \frac{1}{2}r^2\sin(2\theta) = 1 + xy.$$

6, B

(ii) For the exterior problem, $C_n = 0$ for $n = 0, 1, 2, \cdots$, so $D_0 = 1$, $D_2 = 1$. $B_2 = 1/2$ and

meth seen ⇓

$$u_{ext}(x,y) = 1 + \frac{1}{2}r^{-2}\sin(2\theta) = 1 + \frac{xy}{(x^2+y^2)^2}.$$

6, B

(b) Since $\partial_n = \partial_r$, we have

sim. seen ⇓

$$\partial_n u_{int}\big|_{\partial\Omega} = r\sin(2\theta) = \sin(2\theta).$$

. Analogously,

$$\partial_n u_{ext}\big|_{\partial\Omega} = -r^{-3}\sin(2\theta) = -\sin(2\theta).$$

Thus, $\partial_n u_{int}\big|_{\partial\Omega} + \partial_n u_{int}\big|_{\partial\Omega} = 0$, $\partial_n u_{int}\big|_{\partial\Omega} - \partial_n u_{int}\big|_{\partial\Omega} = 2\sin(2\theta)$

4, A

(c) The function $\tilde{u}(x,y)$ is not C^2-smooth since it has the jump of normal derivative on the unit sphere of size $2\sin(2\theta)$, so by the Weyl theorem, it is not a harmonic function on the whole plane.

unseen ⇓

4, A

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks