

Introduction to University Mathematics

MATH40001/MATH40009

Solutions to Group Coursework

1. **Total: 20 Marks**

- (a) An algorithm to solve all questions of this nature is “draw the truth table”. If we try this we see that if R is true and P is false (and Q is anything) then $(P \wedge Q) \vee R$ is true (because R is true) and $P \wedge (Q \vee R)$ is false (because P is false). So they’re not logically equivalent.
- (b) Let all the P_i be false. Then any claim of the form $P_i \implies$ anything is true, so the answer to the question is yes, and all P_i being false is an example.
- (c) $P \implies Q$ is always true unless P is true and Q is false, and $\neg Q \implies \neg P$ is true unless $\neg Q$ is true and $\neg P$ is false (i.e. unless Q is false and P is true), and so your pretty truth tables should have three Ts and an F and should match up exactly.
- (d)
 - i. If $Y = \{1, 2, 3\}$ then $P(Y)$ is false. Let’s show this by proving that the opposite statement, $\forall x \in \{1, 2, 3\}, \exists y \in Y, y \geq x$ is true. Let x be an arbitrary element of $\{1, 2, 3\}$. Let’s choose $y = 3$. Then $y \geq x$ must be true because $3 \geq 1, 3 \geq 2$ and $3 \geq 3$.
 - ii. For this case $P(Y)$ is true. The proof: let x be 3. Then for all $y \in Y$ we have $y < x$ because $x = 3$ and $y < 3$ because $y \in Y$.
 - iii. If Y is empty then $P(Y)$ is true. The proof: let x be 2 (or indeed any element of X). Now we have to prove $\forall y \in \emptyset, y < 2$, but this is true because its opposite, $\exists y \in \emptyset, y \geq 2$ is false. Indeed we can’t find y in the empty set which is at least 2, because we can’t find y in the empty set full stop.
- (e) Let A and B be all of \mathbb{R} . I claim that this works. To prove $Q(\mathbb{R}, \mathbb{R})$ is false it suffices to prove that the opposite statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y \geq x$ is true, and we can prove this by letting x be arbitrary and then setting $y = x$. To prove that $R(\mathbb{R}, \mathbb{R})$ is true, we let $y \in \mathbb{R}$ be arbitrary and then choose $x = y + 1$, and now we have to prove $y < y + 1$ which is obvious.
- (f) If $A = \{3\}$ and $B = \{4\}$ then both $Q(A, B)$ and $R(A, B)$ are false. Let’s prove their opposites, namely $\forall x \in \{3\}, \exists y \in \{4\}, y \geq x$ and $\exists y \in \{4\}, \forall x \in \{3\}, y \geq x$. For the first one we let x be an arbitrary element of $\{3\}$, so in fact $x = 3$. Now set $y = 4$ and we have to prove $4 \geq 3$, which is obvious. For the second, let’s use $y = 4$. Then choose an arbitrary element x of $\{3\}$, note that it must be 3, and again we have to prove $4 \geq 3$ which is obvious.
- (g) It is not possible. Indeed if A and B are sets and $Q(A, B)$ is true then we know $\exists x \in A, \forall y \in B, y < x$, so let’s define x_0 to be this element of A which makes the claim true. We then know that $\forall y \in B, y < x_0$. Now let’s prove $R(A, B)$. Let y be an arbitrary element of B . We need to find $x \in A$ with $y < x$. Let’s use $x = x_0$. We now have to prove $y < x_0$ but we know this already, from $Q(A, B)$.

2. **Total: 20 Marks**

- (a) Assume for a contradiction that $f \circ g$ is the identity function. We know that g is not injective, so let’s choose y_1 and y_2 in Y with $y_1 \neq y_2$ but $g(y_1) = g(y_2)$. Applying f to both sides we get $f(g(y_1)) = f(g(y_2))$. But $f(g(t)) = t$ by assumption, so $y_1 = y_2$, contradicting $y_1 \neq y_2$.
- (b) Say X has two elements (say 1 and 2 but it doesn’t matter), and say Y has one element (say 3). Let $f : X \rightarrow Y$ be the only possible map (sending 1 and 2 to 3), and let $g : Y \rightarrow X$ send 3 to 1. Then $f \circ g$ is a map from Y to Y so it must be the identity function (alternatively just note that $f(g(3)) = f(1) = 3$, but $g \circ f$ is not because $(g \circ f)(2) = g(f(2)) = g(3) = 1$).

- (c) First note that X must also be empty, because if there existed $x \in X$ then what could $f(x)$ be? It's an element of Y but Y has no elements. Hence $g \circ f$ must be injective because the opposite of this statement is that there exist two elements $a, b \in X$ with $g(f(a)) = g(f(b))$ and $a \neq b$, and such elements cannot exist because X is empty.
- (d) By lectures the composite of two injective functions is injective, and applying this twice we see that the composite of three injective functions is injective. Finally, a bijective function is injective by definition. Putting everything together we get the result.
- (e) i. This is true. Say $S \subseteq T$ and say $y \in f(S)$. We need to prove $y \in f(T)$. But $y \in f(S)$ means that there exists some $x \in S$ with $f(x) = y$. Let x_0 be such an x , so $f(x_0) = y$. Now because $x_0 \in S$ we know that $x_0 \in T$ because $S \subseteq T$. Hence there exists some $x \in T$ with $f(x) = y$ (namely $x = x_0$) and thus $y \in f(T)$, which was to be proved.
- ii. This is not true. For example if $X = \{1, 2\}$ and $Y = \{3\}$ and $f(1) = f(2) = 3$ and $S = \{1\}$ and $T = \{2\}$ then $f(S) = f(T) = \{3\}$ so $f(S) \subseteq f(T)$, but $S \not\subseteq T$ as $1 \in S$ and $1 \notin T$.
- iii. This is true. We have $y \in f(S \cup T)$ if and only if there exists $x \in S \cup T$ with $f(x) = y$, and $y \in f(S) \cup f(T)$ if and only if either there exists $x \in S$ with $f(x) = y$ or there exists $x \in T$ with $f(x) = y$. Any element in $S \cup T$ with this property is either in S or in T , and conversely any element in S or T with this property is in $S \cup T$, so the two sets are equal. Another way of thinking about it is that if $P = \{x \in X \mid f(x) = y\}$ then we need to check $(S \cup T) \cap P = (S \cap P) \cup (T \cap P)$ and this is true by drawing a Venn diagram or a truth table.
- iv. This is false, and we can use the same f , X and Y as in part (ii), but with a different S and T . Set $S = \{1, 2\}$ and $T = \{2\}$. Then $f(S) = f(T) = \{3\}$ and $f(S \setminus T) = f(\{1\}) = \{3\}$ but $f(S) \setminus f(T) = \{3\} \setminus \{3\}$ is empty.

3. **Total: 20 Marks**

- (a) One way to do this is the following. First, upon squaring

$$r^2 = \tan^2 \theta.$$

Now writing $\tan \theta$ in terms of $\sin \theta$ and $\cos \theta$ and rearranging gives

$$\begin{aligned} r^2 &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ \Rightarrow r^2 \cos^2 \theta &= \sin^2 \theta, \end{aligned}$$

which upon substitution of $x = r \cos \theta$ leads to

$$\begin{aligned} x^2 &= \sin^2 \theta = 1 - \cos^2 \theta \\ \Rightarrow 1 - x^2 &= \cos^2 \theta. \end{aligned}$$

Inverting this gives

$$\frac{1}{1 - x^2} = \sec^2 \theta = 1 + \tan^2 \theta = 1 + r^2,$$

and then using the fact that $r^2 = x^2 + y^2$, we have

$$\frac{1}{1 - x^2} = 1 + x^2 + y^2,$$

or

$$\begin{aligned} y^2 &= \frac{1}{1 - x^2} - x^2 - 1 \\ &= \frac{1 - (1 + x^2)(1 - x^2)}{1 - x^2} \\ &= \frac{x^4}{1 - x^2}. \end{aligned}$$

Finally square rooting gives

$$y = \frac{x^2}{\sqrt{1-x^2}},$$

where we have taken the positive square root since for $0 \leq \theta < \pi/2$, $\tan \theta$ is positive so the curve should lie in the first quadrant. The figure below shows a sketch of this curve (noting again that we only care about the portion of this curve within the first quadrant due to the range of θ values provided). There is a vertical asymptote at $x = 1$ which can be seen better from the cartesian representation of the curve.

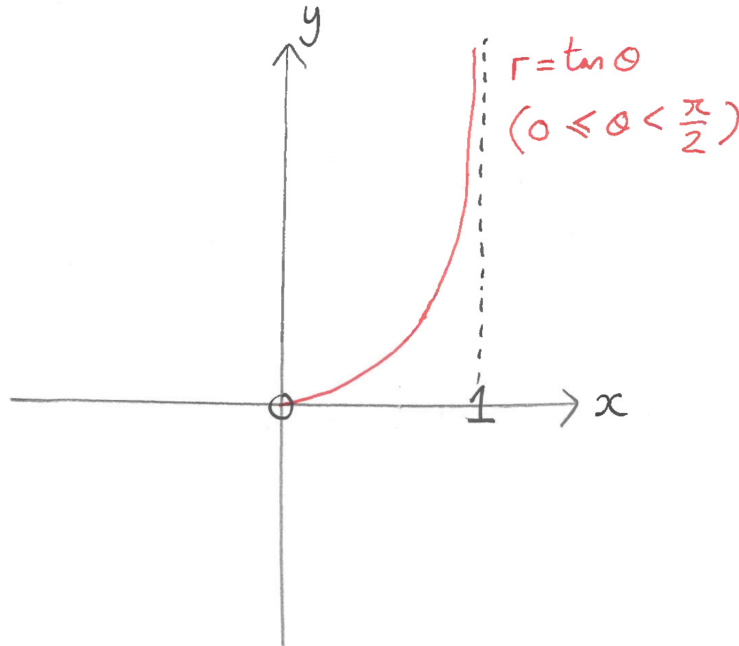


Figure 1: Sketch of $r = \tan \theta$ for $0 \leq \theta < \pi/2$.

- (b) i. The altitude through corner A contains the point A , so it is the set of points with position vectors \mathbf{x} satisfying

$$\mathbf{x} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{p},$$

where \mathbf{p} is a normal vector to the altitude. In particular, the altitude is defined as being perpendicular to the line segment BC so we can choose $\mathbf{p} = \mathbf{b} - \mathbf{c}$ for instance. This leads to the equation

$$\mathbf{x} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}),$$

as required.

- ii. The intersection point of the altitudes through A and B satisfies

$$\mathbf{x} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) \quad \text{and} \quad \mathbf{x} \cdot (\mathbf{a} - \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} - \mathbf{c}).$$

We can subtract these equations to obtain

$$\mathbf{x} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{c} \cdot (\mathbf{b} - \mathbf{a}),$$

which is precisely the equation that means the intersection point lies on the altitude through point C as well.

- iii. When $A(1, 2)$, $B(2, -1)$ and $C(0, 3)$, plugging these values into two of the altitude equations (say for the altitudes through A and B) gives

$$2x - 4y = -6 \quad \text{and} \quad x - y = 3.$$

Solving these equations simultaneously leads to the following solution $(x, y) = (9, 6)$, which are the coordinates of this intersection point (it is known as the **orthocentre** of the triangle).

- iv. Without loss of generality, let's assume the intersection point X lies along side AB of the triangle. We know that AX is orthogonal to BC and BX is orthogonal to AC by definition of the altitudes through A and B . But if X lies on side AB (and first let's assume X is not equal to A or B) then further we must have that AB is orthogonal to BC and also orthogonal to AC , which is nonsense since A and B are different points so this couldn't form a triangle. Thus the only way this can be true is if $X = A$ or $X = B$. In each case this corresponds to a right-angled triangle. So right-angled triangles are the only triangles for which the intersection point can lie on the boundary (and then it lies on the right-angled corner point).
- (c) i. First observe that for $z, w \neq -\frac{d}{c}, \infty$, then

$$\begin{aligned} f(z) = f(w) &\Rightarrow \frac{az + b}{cz + d} = \frac{aw + b}{cw + d} \\ &\Rightarrow (az + b)(cw + d) = (aw + b)(cz + d) \\ &\Rightarrow aczw + adz + bcw + bd = aczw + adw + bcz + bd \\ &\Rightarrow (ad - bc)(z - w) = 0 \\ &\Rightarrow z = w \quad \text{as } ad - bc \neq 0. \end{aligned}$$

It remains to check other cases, first assume $(c \neq 0)$, then observe

$$\begin{aligned} \frac{az + b}{cz + d} &= \frac{a}{c} \\ &\Rightarrow c(az + b) = a(cz + d) \\ &\Rightarrow ad - bc = 0, \end{aligned}$$

which cannot be, and so if $f(z) = f(w) = \frac{a}{c}$ then it follows that we must have $z = w = \infty$. Finally in the case where $f(z) = f(w) = \infty$, then by definition it follows that $z = w = -\frac{d}{c}$ if $c \neq 0$ and in the case where $c = 0$, then $f(z) = f(w) = \infty$. Hence f is injective.

- ii. For $z \neq -\frac{d}{c}, \infty$ define

$$w = \frac{-dz + b}{cz - a}.$$

When $z = -\frac{d}{c}$, let $w = \infty$ and when $z = \infty$, let $w = -\frac{d}{c}$ (except in the case where $c = 0$ where we should define $w = \infty$ when $z = \infty$). Now observe for $w \neq -\frac{d}{c}, \infty$ we have

$$\begin{aligned} f(w) &= \frac{a \left(\frac{-dz + b}{cz - a} \right) + b}{c \left(\frac{-dz + b}{cz - a} \right) + d} \\ &= \frac{-adz + ab + bcw - ab}{-cdz + bc + cdz - ad} \\ &= \frac{(ad - bc)z}{(ad - bc)} \\ &= z, \end{aligned}$$

since $ad - bc \neq 0$. With the definitions for the cases when $w = -\frac{d}{c}$ and $w = \infty$ earlier this proves f is a surjective function.

iii. Observe that

$$\begin{aligned} f(z) &= \frac{az + b}{cz + d} \\ &= \frac{\frac{a}{c}(cz + d) + b - \frac{ad}{c}}{(cz + d)} \\ &= \frac{a}{c} - \frac{1}{c}(ad - bc)\frac{1}{cz + d}, \end{aligned}$$

which collapses to a/c if $ad - bc = 0$. The function no longer becomes injective or surjective.