Mathematical Logic (MATH60132 and MATH70132)
2024-25, Coursework 1

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 10 February 2025. The coursework is marked out of 20 and the marks per question are indicated below.
The work which you submit should be your own, unaided work. Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet, generative AI agent or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1]

(a) **(1 mark)** Use results from the notes to show that, if $\Gamma \cup \{\phi\}$ is a set of L-formulas, then $\Gamma \vdash_L \phi$ if and only if for every valuation v with $v(\Gamma) = T$ we have $v(\phi) = T$. In the case where Γ is a finite set, how does this result allow you to check using truth tables (involving only the propositional variables in Γ) whether or not $\Gamma \vdash_L \phi$?

(b) **(2 marks)** Is $((p_2 \to (p_1 \to p_3))$ a consequence of $\{(p_1 \to ((\neg p_2) \to p_3)), ((\neg p_3) \to (p_1 \to p_2))\}$? Give reasons for your answers.

(c) **(4 marks)** Suppose Γ_n is a set consisting of n L-formulas (where $n \in \mathbb{N}$).

(i) In the case $n = 4$, show that there exists a set Γ_n with the property that Γ_n is inconsistent and every subset of Γ_n of size $n-1$ is consistent. You do not necessarily need to say explicitly what the formulas in Γ_n are.
[Hint: we may take the formulas in Γ_4 to involve only variables p_1, p_2.]

(ii) Do (i) for general $n \in \mathbb{N}$. [The hint in (i) does not apply in general.]

[2] We say that L-formulas ϕ, ψ are *L-equivalent* if $\vdash_L (\phi \to \psi)$ and $\vdash_L (\psi \to \phi)$. In this question you should give syntactic arguments, not involving truth tables, valuations, or use of the Completeness Theorem for L. You may use results from the notes and problem sheets about theorems of L.

(a) **(1 mark)** Show that if β is an L-formula, then $(\neg(\neg\beta))$ is L-equivalent to β.

(b) **(2 marks)** Prove that if χ, η are L-formulas, then $((\chi \to \eta) \to ((\neg\eta) \to (\neg\chi)))$ is a theorem of L.

(c) **(4 marks)** If α is a subformula of the L-formula ϕ, then there are L-formulas $\phi_1, \ldots, \phi_k$ such that: ϕ_1 is α; ϕ_k is ϕ; and (for $i < k$) we have that ϕ_{i+1} is one of $(\neg\phi_i)$, $(\phi_i \to \chi_i)$ or $(\chi_i \to \phi_i)$, for some formula χ_i.

Give a **syntactic** proof of the following result:

Suppose α is a subformula of ϕ. Let $\hat{\alpha}$ be an L-formula which is L-equivalent to α and let $\hat{\phi}$ be the L-formula obtained by replacing α by $\hat{\alpha}$ in ϕ. Then ϕ and $\hat{\phi}$ are L-equivalent.
[Hint: Prove this by induction on k, treating the cases $k = 1, 2$ as the base case.]

[3] In the following, you should give syntactic arguments, not involving truth tables, valuations, or use of the Completeness Theorem for L. You may use results from the notes and problem sheets about theorems of L. Suppose ϕ, ψ are L-formulas. In L we define $(\phi \wedge \psi)$ to be $(\neg(\phi \to (\neg\psi)))$. Prove the following (you may use 2(b) if you wish):

(a) **(3 marks)** $\vdash_L ((\phi \wedge \psi) \to \phi)$;

(b) **(3 marks)** $\{\psi, \phi\} \vdash_L (\phi \wedge \psi)$.