Imperial College London

MATH97051 MATH70058

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**Manifolds**

Date: 09 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. (a) Let $X$ be a manifold of dimension $n$ and let $x \in X$. Show that, for any non-negative integer $m < n$, there exists a compact submanifold $Y \subset X$ of dimension $m$ such that $x \in Y$.

(5 marks)

(b) Let $X = \mathbb{P}^n_{\mathbb{R}}$ be the real $n$-dimensional projective space. Let $m < n$ and let

$$Y = \{[x_0, \dots, x_n] \in X \mid x_0 = \dots = x_m = 0\}.$$

Show that $Y$ is a submanifold of $X$. (5 marks)

(c) Let $d > 0$ be a positive integer.

(i) Let $p \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ be a smooth function such that

$$p(\lambda x) = \lambda^d p(x)$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the function

$$P \colon \mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$$

defined by $P([x]) = [p(x)]$ is a well defined smooth function. (5 marks)

(ii) Let $F \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that

$$F(\lambda x) = \lambda^d F(x)$$

for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Let $c \in \mathbb{R} \setminus \{0\}$. Show that

$$Z = \{x \in \mathbb{R}^n \mid F(x) = c\}$$

is a submanifold of $\mathbb{R}^n$. (5 marks)

(Total: 20 marks)

2. (a) Let $X$ be a manifold, let $x \in X$ and let $v \in T_x X$. Show that there exists a vector field $\xi$ on $X$ such that $\xi(x) = v$.

(5 marks)

(b) Let $X$ be a manifold and let $Y \subset X$ be a submanifold. Let $\xi$ be a vector field on $Y$. Show that there exists a vector field $\tilde{\xi}$ on $X$ such that

$$\tilde{\xi}(y) = \xi(y)$$

for all $y \in Y$. (5 marks)

(c) Let $X$ be a manifold and let $D$ be a derivation on $X$. Let $f, g \in C^\infty(X)$ such that $g(x) \neq 0$ for all $x \in X$.

Show that

$$D\left(\frac{f}{g}\right) = \frac{gDf - fDg}{g^2}.$$

(5 marks)

(d) Let $X$ be a two-dimensional torus and let $x \in X$. Find a vector field on $X$ which vanishes exactly at $x$.

(5 marks)

(Total: 20 marks)

3. (a) Let $X, Y, Z$ be manifolds and let $F\colon X \to Y$ and $G\colon Y \to Z$ be smooth functions with constant rank. Determine if $G \circ F\colon X \to Z$ has constant rank (justify your answer).

(4 marks)

(b) Let $S^1$ be the unit circle.

(i) Determine if there exists a submersion $F\colon S^1 \to \mathbb{R}$ (justify your answer).

(4 marks)

(ii) Determine if there exists a submersion $G\colon S^1 \times \mathbb{R} \to \mathbb{R}^2$ (justify your answer).

(4 marks)

(c) Show that the tangent bundle of $S^1 \times \mathbb{R}$ is trivial.

(3 marks)

(d) Let $X$ be a manifold. Show that the tangent bundle of $X$ is trivial if and only if its cotangent bundle is trivial. (5 marks)

(Total: 20 marks)

4. (a) Let $f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3)$ be smooth functions such that, for all $x \in \mathbb{R}^3$, there exists $i \in \{1,2,3\}$ such that $f_i(x) \neq 0$ and let $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \in \Omega^1(\mathbb{R}^3)$.

Show that there exists a 2-form $\omega_2 \in \Omega^2(\mathbb{R}^3)$ such that $\omega \wedge \omega_2$ is a volume form on $\mathbb{R}^3$.

(3 marks)

(b) Let $X$ be a manifold and let $\omega_1, \omega_2 \in \Omega^1(X)$ be two 1-forms on $X$ such that $\omega_2$ is nowhere zero and

$$\omega_1 \wedge \omega_2 = 0.$$

Show that there exists a smooth function $f \in C^\infty(X)$ such that $\omega_1 = f\omega_2$. (7 marks)

(c) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ be a $(n-1)$-form on $\mathbb{R}^n$ such that

$$\int_S \omega = 0$$

for every sphere $S$ contained in $\mathbb{R}^n$. Show that $d\omega = 0$. (5 marks)

(d) Show that the $n$-dimensional torus $T^n$ is orientable. (5 marks)

(Total: 20 marks)

5. (a) Let $X$ be a manifold and let $D \subset T_X$ be a distribution of rank one. Show that $D$ is involutive. (4 marks)

(b) Let $X = S^3 \subset \mathbb{R}^4$ be the unit 3-dimensional sphere centred at the origin and consider the functions

$$V(x, y, z, w) = (y, -x, w, -z) \qquad \text{and} \qquad W(x, y, z, w) = (z, -w, -x, y).$$

(i) Show that $V$ and $W$ define vector fields on $X$. (2 marks)

(ii) Show that $V$ defines a distribution $D_1$ on $X$. (2 marks)

(iii) Show that $D_1$ is involutive. (3 marks)

(iv) Find an integral submanifold with respect to $D_1$ passing through the point $(x_0, y_0, z_0, w_0) \in X$. (3 marks)

(v) Show that $V$ and $W$ span a distribution $D_2$ on $X$. (3 marks)

(vi) Determine if $D_2$ is integrable (justify your answer). (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

# MATH97051

# Manifolds (Solutions)

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1. (a) Let $(U, f)$ be a chart on $X$ such that $x \in X$ and let $\tilde{U} = f(U) \subset \mathbb{R}^n$. Since $\tilde{U}$ is an open subset, there exists a $m$-dimensional sphere $S \subset \tilde{U}$ such that $f(x) \in S$. In particular, $S$ is a submanifold and, therefore, for any $z \in S$ there exists a chart $(V, g)$ of $\tilde{U}$, where $g\colon V \to \tilde{V} \subset \mathbb{R}^n$, $z \in V$ and $g(S \cap V) = \tilde{V} \cap A$ where $A \subset \mathbb{R}^n$ is an affine subspace. Let $Y = f^{-1}(S) \subset X$. Then $Y$ is compact and for any point $y \in Y$ there exists a chart $(f^{-1}(V), g \circ f)$ so that $y \in f^{-1}(V)$ and $g \circ f(Y \cap f^{-1}(V)) = \tilde{V} \cap A$, as requested. $\boxed{5, \mathrm{B}}$

(b) Let $y = [x_0, \ldots, x_n] \in Y$. Then there exists $k \in \{m+1, \ldots, n\}$ such that $x_k \neq 0$. Consider the open subset $U_k = \{[x_0, \ldots, x_n] \mid x_k \neq 0\}$ and the homeomorphism

$$f_k\colon U_k \to \mathbb{R}^n \qquad [x_0, \ldots, x_n] \mapsto \left(\frac{x_0}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_n}{x_k}\right).$$

Then $(U_k, f_k)$ is a chart of $X$ such that $f(U_k \cap Y)$ is the affine subspace

$$\{(t_0, \ldots, t_{n-1}) \in \mathbb{R}^n \mid t_0 = \cdots = t_m = 0\}.$$

Thus, $Y$ is a submanifold. $\boxed{5, \mathrm{A}}$

(c) By assumption, $[p(x)] \in \mathbb{P}^n_{\mathbb{R}}$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Assume that $[x] = [x']$. Then there exist $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x' = \lambda x$ and, in particular,

$$[p(x')] = [\lambda^d p(x)] = [p(x)].$$

Thus, $P$ is well defined.

Let $[x_0, \ldots, x_n] \in \mathbb{P}^n_{\mathbb{R}}$ and let $(y_0, \ldots, y_n) = p(x_0, \ldots, x_n)$. Then there exists $k \in \{0, \ldots, n\}$ such that $x_k \neq 0$. Moreover, there exists $m \in \{0, \ldots, n\}$ such that $y_m \neq 0$. Let $(U_k, f_k)$ and $(U_m, f_m)$ be defined as in the previous Exercise, so that $[x_0, \ldots, x_n] \in U_k$ and $[y_0, \ldots, y_n] \in U_m$. We have that

$$f_m \circ P \circ f_k^{-1}\colon f_k(P^{-1}(U_m)) \to \mathbb{R}^n$$

is given by

$$(t_0, \ldots, t_{n-1}) \mapsto \left(\frac{p_0(t_0, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1})}{p_m(t_0, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1})}, \ldots, \frac{p_n(t_0, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1})}{p_m(t_0, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1})}\right)$$

where $p_j(x)$ is the $j$-coordinate of $p(x)$ for $j = 0, \ldots, n$. Thus, $f_m \circ P \circ f_k^{-1}$ is a smooth function, which implies that $P$ is smooth as well. $\boxed{5, \mathrm{B}}$

(d) Note that $F(0) = 0$ and therefore $0 \notin Z$. For any $t \in \mathbb{R}$ and for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{n} x_i \cdot \frac{\partial}{\partial x_i} F(x) = \frac{d}{dt} F((1+t)x)|_{t=0} = \frac{d}{dt}\left((1+t)^d F(x)\right)|_{t=0} = dF(x).$$

Therefore, if $x \in Z$, then there exists $i \in \{1, \dots, n\}$ such that $\frac{\partial}{\partial x_i} F(x) \neq 0$, i.e. $x$ is a regular point. Thus, $c$ is a regular value and, in particular, $Z$ is a submanifold of $X$. $\boxed{5, \text{D}}$

2. (a) Let $n$ be the dimension of $X$. Let $(U, f)$ be a chart on $X$ such that $x \in U$. Let $\tilde{U} = f(U) \subset \mathbb{R}^n$ and let $\Delta_f \colon T_x X \to T_{f(x)} \tilde{U} \simeq \mathbb{R}^n$ be the induced isomorphism. Let $V$ be the constant vector field so that $V(z) = v$ for all $z \in \tilde{U}$ and let $\rho \colon \tilde{U} \to [0,1]$ be a bump function on $\tilde{U}$ such that $\rho(f(x)) = 1$. Then the induced vector field $\xi' = \Delta_f^{-1} \circ (\rho \cdot V) \circ f$ on $U$ is such that $\xi'(x) = v$ and it is zero, outside a neighbourhood of $x$. In particular, it can be extended to a smooth vector field $\xi$ on $X$ such that $\xi(x) = v$. $\boxed{5, \text{A}}$

(b) We first assume that $X$ is an open subset of $\mathbb{R}^n$ and that $Y \subset X$ is the intersection of $X$ with a $k$-dimensional affine subspace $A \subset \mathbb{R}^n$. After taking an affine map, we may assume that $A$ is the standard affine subspace, i.e.

$$Y = \{(x_1, \dots, x_n) \in X \mid x_{k+1} = \dots = x_n = 0\}.$$

The vector field $\xi$ is then induced by a smooth function $Y \to \mathbb{R}^k = A$. We may define

$$\tilde{\xi} \colon X \to A \subset \mathbb{R}^n \qquad (x_1, \dots, x_n) \mapsto \xi(x_1, \dots, x_k)$$

and it follows that $\tilde{\xi}$ defines a vector field on $X$ such that $\tilde{\xi}(y) = \xi(y)$ for all $y \in Y$, as requested.

We now consider the general case. Let $\{(U_i, f_i)\}_{i \in I}$ be an atlas of $X$ and let $\{f_i \colon X \to [0,1]\}_{i \in I}$ be a partition of the unity associated to the open covering $\{U_i\}_{i \in I}$. We may assume that, for all $i \in I$ such that $U_i \cap Y \neq \emptyset$, there exists an affine subspace $A_i \subset \mathbb{R}^n$ such that $f_i(U_i \cap Y) = f_i(U_i) \cap A_i$. Let $\xi_i$ be the vector field induced by $\xi$ on $f_i(U_i \cap Y)$. Then, as above, there exists a vector field $\tilde{\xi}_i$ on $f_i(U_i)$ such that $\tilde{\xi}_i(z) = \xi_i(z)$ for all $z \in f_i(U_i \cap Y)$. Let $\tilde{\xi}'_i$ be the induced vector field on $U_i$ and let $\tilde{\xi}'_i$ be the zero vector field on each $U_i$ such that $U_i \cap Y = \emptyset$. Then,

$$\tilde{\xi} = \sum_{i \in I} f_i \cdot \tilde{\xi}'_i$$

satisfies the requested property. $\boxed{5, \text{B}}$

(c) By the Leibniz rule, we have

$$D(f) = D\left(\frac{f}{g} \cdot g\right) = D\left(\frac{f}{g}\right) \cdot g + \frac{f}{g} \cdot D(g).$$

Thus, the claim follows immediately. $\boxed{5, \text{A}}$

(d) After possibly translating, we may write $X = \mathbb{R}^2/\mathbb{Z}^2$ so that $x = [0]$. Let $q\colon \mathbb{R}^2 \to X$ be the quotient map. Then, the tangent bundle of $X$ is trivial and it is spanned by the vector fields

$$v_i = q_* \frac{\partial}{\partial x_i} \qquad \text{for } i = 1, 2.$$

Thus, any section of $T_X$ can be written as

$$V = f_1 v_1 + f_2 v_2$$

where $f_1, f_2 \in C^\infty(X)$. Smooth functions on $X$ are induced by periodic smooth functions on $\mathbb{R}^2$. For example, we can take

$$f_1(x_1, x_2) = \sin^2(2\pi x_1) + \sin^2(2\pi x_2) \qquad \text{and} \qquad f_2 = 0.$$

$\boxed{5, \text{B}}$

3. (a) $G \circ F$ does not have constant rank in general. For example, consider

$$F\colon X = \mathbb{R} \to Y = \mathbb{R}^2 \qquad x \mapsto (x, x^2)$$

and

$$G\colon Y = \mathbb{R}^2 \to Z = \mathbb{R} \qquad (y, z) \mapsto z.$$

Then, both $F$ and $G$ have constant rank equal to one but

$$G \circ F\colon X = \mathbb{R} \to Z = \mathbb{R} \qquad x \mapsto x^2$$

has rank zero at $x = 0$ and rank one elsewhere. $\boxed{4, \text{C}}$

(b) (i) There is no submersion $F\colon S^1 \to \mathbb{R}$. Indeed, since $S^1$ is compact, if $F\colon S^1 \to \mathbb{R}$ is smooth, it would admit a maximum at $x \in S^1$. In particular, the rank of $F$ at $x$ is zero and $F$ is not a submersion. $\boxed{4, \text{A}}$

(ii) There exists a submersion $G\colon S^1 \times \mathbb{R} \to \mathbb{R}^2$. Indeed, by considering $S^1$ as the quotient $S^1 = \mathbb{R}/Z$, such a smooth function $G$ is induced by a smooth function $\tilde{G}\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is periodic with respect to the first entry. E.g. If we consider the function

$$\tilde{G}\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad (t, x) \mapsto (e^x \cos(2\pi t), e^x \sin(2\pi t))$$

it is easy to see that $\tilde{G}$ is a smooth function with constant rank equal to one. Thus, it induces a smooth function $G\colon S^1 \times \mathbb{R} \to \mathbb{R}^2$, as requested. $\boxed{4, \text{D}}$

(c) Both the tangent bundle of $S^1$ and the tangent bundle of $\mathbb{R}$ are trivial. Thus there exist nowhere vanishing vector fields $\xi_1$ and $\xi_2$ on $S^1$ and $\mathbb{R}$ respectively. Recall that, for any $(x,y) \in S^1 \times \mathbb{R}$, we have that $T_{(x,y)}(S^1 \times \mathbb{R}) = T_x S^1 \oplus T_y \mathbb{R}$. Thus,

$$\tilde{\xi}_1 = (\xi_1, 0) \qquad \tilde{\xi}_2 = (0, \xi_2)$$

are vector fields on $S^1 \times \mathbb{R}$ such that, for any $(x,y) \in S^1 \times \mathbb{R}$, the vectors $\tilde{\xi}_1(x,y), \tilde{\xi}_2(x,y)$ are linearly independent. It follows that the tangent bundle of $S^1 \times \mathbb{R}$ is trivial. 3, A

(d) Let $n$ be the dimension of $X$. Assume that the tangent bundle of $X$ is trivial. Then there exist vector fields $v_1, \dots, v_n$ such that, for all $x \in X$, the vectors $v_1(x), \dots, v_n(x)$ are linearly independent. Since, for each $x \in X$, the cotangent space $T_x^* X$ is dual to the tangent space $T_x X$, we can define $\omega_1(x), \dots, \omega_n(x)$ as the induced dual basis. It remains to show that $\omega_1, \dots, \omega_n$ are smooth. To this end, we may work locally around a point $x \in X$ and we may assume that $X$ is an open subset of $\mathbb{R}^n$. Consider the standard basis,

$$\xi_i(x) = \frac{\partial}{\partial x_i} \qquad \text{for } i = 1, \dots, n.$$

Then there exist smooth function $f_{i,j} \in C^\infty(X)$, with $i, j = 1, \dots, n$, such that

$$v_i = \sum_{j=1}^n f_{i,j} \xi_j \qquad i = 1, \dots, n$$

and the determinant of the matrix $A = (f_{i,j})$ is non-zero at every point $x \in X$. In particular, if $A^{-1} = (g_{i,j})$ then $g_{i,j}$ is smooth for all $i, j = 1, \dots, n$ and

$$\omega_i = \sum_{j=1}^n g_{i,j} dx_j \qquad i = 1, \dots, n.$$

Thus, our claim follows. 5, C

4. (a) Let

$$U_i = \{x \in \mathbb{R}^3 \mid f_i(x) \neq 0\} \qquad \text{for } i = 1, 2, 3.$$

Then $U_i$ is open in $\mathbb{R}^3$ and, by assumption, $\bigcup_{i=1}^3 U_i = \mathbb{R}^3$. On $U_1$, define the 2-form

$$\eta_1 = \frac{1}{f_1} dx_2 \wedge dx_3.$$

Similarly, define $\eta_2$ and $\eta_3$ on $U_2$ and $U_3$ respectively. Then $\omega \wedge \eta_i = dx_1 \wedge dx_2 \wedge dx_3$ is a volume form on $U_i$. Let $g_i \colon \mathbb{R}^3 \to [0,1]$, with $i = 1, 2, 3$, be a partition of the unity associated to the open cover $\{U_i\}_{i=1,2,3}$ and let $\tilde{\eta}_i$ be the extension of $g_i \eta_i$ which is zero outside $U_i$. Then

$$\omega_2 = \sum_{i=1}^3 \tilde{\eta}_i$$

is a 2-form on $\mathbb{R}^3$ such that $\omega \wedge \omega_2$ is a volume on $\mathbb{R}^3$. 3, C

(b) We first assume that $X$ is an open subset of $\mathbb{R}^n$. We may write

$$\omega_1 = \sum_{i=1}^n g_i dx_i \qquad \text{and} \qquad \omega_2 = \sum_{i=1}^n g_i' dx_i$$

where $g_1, \dots, g_n, g_1', \dots, g_n'$ are smooth functions on $X$. Since $\omega_2$ is nowhere zero and $\omega_1 \wedge \omega_2 = 0$ it follows that, at each point $x \in X$, the rank of the matrix

$$\begin{pmatrix} g_1(x) & \dots & g_n(x) \\ g_1'(x) & \dots & g_n'(x) \end{pmatrix}$$

is exactly one. It follows that there exists a smooth function $f$ on $X$ such that $g_i = f \cdot g_i'$ for all $i = 1, \dots, n$. Thus, $\omega_1 = f\omega_2$, as requested.

We now consider the general case. Let $\{(U_i, h_i)\}_{i \in I}$ be an atlas of $X$. By the previous case, it follows that for each $i \in I$, there exists a smooth function $f_i$ on $U_i$ such that

$$\omega_1 = f_i \omega_2 \qquad \text{for all } i \in I.$$

In particular, for any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, we have that

$$(f_i - f_j)\omega_2 = 0 \qquad \text{on } U_i \cap U_j.$$

Since $\omega_2$ is nowhere vanishing, it follows that $f_i = f_j$ on $U_i \cap U_j$. Therefore, the functions $f_i$ define a smooth function $f \in C^\infty(X)$ such that $\omega_1 = f\omega_2$. $\boxed{7, \text{D}}$

(c) Suppose by contradiction that there exists $x \in \mathbb{R}^n$ such that $d\omega$ is not zero in a neighbourhood $U$ of $x$. After possibly replacing $\omega$ by $-\omega$, we may assume that $d\omega$ is a volume form on $U$ and, in particular, if $B$ is a ball contained in $U$, then

$$\int_B d\omega > 0.$$

By Stokes theorem, if $S = \partial B$ then $\int_S \omega > 0$, a contradiction. $\boxed{5, \text{A}}$

(d) Let $X = T^n$. It is enough to show that $X$ admits a volume form. Recall that the cotangent bundle of $X$ is trivial and, therefore, there exist 1-forms $\omega_1 \dots, \omega_n$ which are linearly independent at each point of $X$. It follows that the $n$-form $\omega = \omega_1 \wedge \dots \wedge \omega_n$ is nowhere zero and, therefore, either $\omega$ or $-\omega$ is a volume form. $\boxed{5, \text{A}}$

4. (a) Being involutive is a local property. Thus, we can replace $X$ by a smaller open subset of $X$ and, in particular, we may assume that $D$ is a trivial vector bundle. Thus, $D$ admits a nowhere zero section $s$. Let $V$ and $W$ be vector fields on $X$ such that $V(x), W(x) \in D_x$ for all $x \in X$. Since $D$ has rank one, there exist smooth function $f$ and $g$ on $X$ such that

$$V = f \cdot s \qquad \text{and} \qquad W = g \cdot s.$$

It follows that

$$[V, W] = [f \cdot s, g \cdot s] = (f \cdot s(g) - g \cdot s(f)) \cdot s.$$

Thus $[V, W]_x \in D_x$ and the claim follows. $\boxed{4, \mathrm{M}}$

(b) (i) Since $X = \{F(x, y, z, w) = 0\}$ where $F(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 1$, it follows that, at the point $(x, y, z, w) \in X$, the tangent space of $X$ is given by

$$\{(a, b, c, d) \mid ax + by + cz + dw = 0\}.$$

Thus, at each point $(x, y, z, w) \in X$, the vectors $V(x, y, z, w)$ and $W(x, y, z, w)$ are tangent to $X$. $\boxed{2, \mathrm{M}}$

(ii) For each $(x, y, z, w) \in X$, the vector $V(x, y, z, w)$ is non-zero. Thus, $V$ defines a sub-line bundle of $T_X$ and, therefore, a distribution. $\boxed{2, \mathrm{M}}$

(iii) Since $D_1$ has rank one, Ex. (a) implies that $D$ is involutive. $\boxed{3, \mathrm{M}}$

(iv) Consider the subset

$$Y = \{(x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t, z_0 \cos t + w_0 \sin t, -z_0 \sin t + w_0 \cos t) \mid t \in [0, 2\pi]\} \subset \mathbb{R}^3.$$

Then, it is easy to check that $Y$ is a one dimensional submanifold of $X$ passing through the point $(x_0, y_0, z_0, w_0)$ and such that the tangent space of $X$ at each point $(x, y, z, w) \in Y$ coincides with $D_1|_{(x,y,z,w)}$. Thus, $Y$ is an integral submanifold of $X$. $\boxed{3, \mathrm{M}}$

(v) It is easy to check that, for each point $(x, y, z, w) \in X$, the vectors $V(x, y, z, w)$ and $W(x, y, z, w)$ are linearly independent. Thus, they span a rank two distribution on $X$. $\boxed{3, \mathrm{M}}$

(vi) An easy calculation shows that the vector field $[V, W]$ is given by

$$[V, W](x, y, z, w) = 2(w, z, -y, -x)$$

and it is easy to show that $[V, W]$ is not a section of $D_2$. Thus, $D_2$ is not involutive and, in particular, $D$ is not integrable. $\boxed{3, \mathrm{M}}$

**Review of mark distribution:**
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

**If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.**
**Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.**

| ExamModuleCode | QuestionNumber | Comments for Students |
|---|---|---|
| Manifolds_MATH97051 MATH70058 | 1 | In ex (a), it is required that the submaniofld is compact. Therefore, it is not enough to choose a chart and consider an affine subspace here. |
| Manifolds_MATH97051 MATH70058 | 2 | In Ex. (b), Y is taken to be closed. |
| Manifolds_MATH97051 MATH70058 | 3 | Overall good results. |
| Manifolds_MATH97051 MATH70058 | 4 | Regarding Ex. (b), note that it is not enough to solve it locally, as it is necessary to show that the functions obtained on the charts would glue together. |
| Manifolds_MATH97051 MATH70058 | 5 | Overall good results. In ex. a), it is not true that two vector fields in a rank one distribution are multiple to each other by a scalar, as it might not be constant. |