Mathematical Logic (MATH60132 and MATH70132)
2024-25, Coursework 2

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 10 March 2025. The coursework is marked out of 20 and the marks per question are indicated below.
The work which you submit should be your own, unaided work. Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet, generative AI agent or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1] (5 marks) Suppose $\mathcal{L}$ is a 1st-order language, Δ is a set of $\mathcal{L}$-formulas and ϕ an $\mathcal{L}$-formula. We write $\Delta \models \phi$ to mean that for every valuation v (in an $\mathcal{L}$-structure $\mathcal{A}$), if $v[\Delta] = T$, then $v[\phi] = T$.

(i) Suppose that the variable x_i is not free in any formula in Δ. Show that if $\Delta \models \phi$, then $\Delta \models (\forall x_i)\phi$.

(ii) Give (with justification) an example where $\Delta \models \phi$ and $\Delta \not\models (\forall x_i)\phi$.

[Note that (i) is used in the proof of the Generalised Soundness Theorem (2.4.7 in the notes), so
Marks your solution to (i) should not quote that result.]

3 *Solution:* (i) Let v be any valuation (in an $\mathcal{L}$-structure $\mathcal{A}$) with $v[\Delta] = T$. We want to show that $v[(\forall x_i)\phi] = T$. So let w be a valuation (in $\mathcal{A}$) which is x_i-equivalent to v. We need to show that $w[\phi] = T$. As x_i is not free in any formula in Δ, v and w agree on the free variables in every formula in Δ. So by 2.3.3 we then have $w[\Delta] = v[\Delta] = T$. By the assumption $\Delta \vdash \phi$ we therefore have $w[\phi] = T$, as required.

2 (ii) Take a language with a single 1-ary relation symbol P and consider the structure $\mathcal{A}$ with domain $\{1, 2\}$ and P interpreted as the subset $\{1\}$. Let $\phi(x_1)$ be the formula $P(x_1)$ and $\Delta = \{\phi(x_1)\}$. So clearly $\Delta \models \phi(x_1)$. Let v be a valuation in $\mathcal{A}$ with $v(x_1) = 1$. Then $v[\phi(x_1)] = T$, but $v[(\forall x_1)\phi(x_1)] = F$.

[2] (11 marks) Let $\mathcal{L}^=$ be the 1st-order language (with equality) for groups, having a 2-ary function symbol $\cdot$ for the group operation and a constant symbol e for the identity element. For each natural number $n \geq 1$, let $\mathcal{C}_n$ denote the cyclic group of order 2^n, considered as a normal $\mathcal{L}^=$-structure. Let Φ be the set consisting of all closed $\mathcal{L}^=$-formulas ϕ having the property that:

there are only finitely many $n \geq 1$ with $\mathcal{C}_n \models (\neg\phi)$.

(i) Show that Φ is consistent.

(ii) Show that if χ is a closed $\mathcal{L}^=$-formula and $\Phi \vdash \chi$, then $\chi \in \Phi$.

(iii) Suppose $\mathcal{A}$ is a normal model of Φ (and denote its domain by A). Show that:

(a) $\mathcal{A}$ is an infinite abelian group.

(b) If $g \in A$ has finite order k, then k is a power of 2.

(c) For every odd number $m \in \mathbb{N}$ we have $\{a^m : a \in A\} = A$.

(d) $\mathcal{A}$ has exactly one subgroup H_n of order 2^n, for each $n \in \mathbb{N}$.

(e) If H_n is as in (d), then $H_n \subseteq H_{n+1}$.

[Hint: You may use that: if $G = \langle g \rangle$ is a cyclic group of finite order k and $m \in \mathbb{N}$, then g^m has order $k/\gcd(m,k)$; and if d divides k, then there is a unique subgroup of G of order d.]

(iv) Show that there is a normal model of Φ with an element of infinite order.

[Hint: Consider a language $\mathcal{L}_c^=$ consisting of $\mathcal{L}^=$ together with an extra constant symbol c and use the Compactness Theorem.]

Solution: (i) It is enough to show that each finite subset of Φ is consistent, and therefore it will
1 suffice that show that if $\phi_1, \ldots, \phi_n \in \Phi$, then there is some k with $\mathcal{C}_k$ a model of each ϕ_i (for $i \leq n$). Note that the condition on Φ means that for each $\phi \in \Phi$ there is some $N(\phi) \in \mathbb{N}$ with $\mathcal{C}_n \models \phi$ for all $n \geq N(\phi)$. So we just take $k \geq N(\phi_1), \ldots, N(\phi_n)$.

1 (ii) As deductions are finite, χ is a consequence of a finite subset $\{\phi_1, \ldots, \phi_n\}$ of Φ. Using the notation of (i), Generalised Soundness then implies that $\mathcal{C}_k \models \chi$ for $n \geq N(\phi_1), \ldots, N(\phi_n)$.

1 (iii) (a) Each $\mathcal{C}_n$ is an abelian group, so the group axioms and the formula $(\forall x)(\forall y)(x \cdot y = y \cdot x)$ are in Φ.

(b) For each k, there is a formula $\alpha_k(x_1)$ expressing that 'x_1 has order k':

$$(x_1^k = e) \wedge \bigwedge_{1 \leq i < k} (x_1^i \neq e).$$

1 (Using the usual shorthand notation for powers in a group written multiplicatively.) If k is not a power of 2, then $\mathcal{C}_n$ has no element of order k (using the given fact). So the formula $(\forall x_1)(\neg \alpha_k(x_1))$ is in Φ, and the result follows.

2 (c) For each odd m, we claim that the formula $(\forall x)(\exists y)(y^m = x)$ is true in all $\mathcal{C}_n$, and this suffices. Let g_n be a generator of $\mathcal{C}_n$. Then by the given fact, g_n^m has order 2^n and thus is a generator of $\mathcal{C}_n$. It follows that every element of $\mathcal{C}_n$ is an m-th power.

2 (d) Note that for $m \geq n$, $\mathcal{C}_m$ has a unique subgroup of order 2^n. It follows that the number of elements of order 2^n in $\mathcal{C}_m$ is equal to $K(n)$, the number of elements of order 2^n in $\mathcal{C}_n$ (easily seen to be $2^n - 2^{n-1}$, but we do not need this). Using the formula $\alpha_{2^n}(x_1)$ from part (b), we can write down a closed formula γ_n saying that 'there are exactly $K(n)$ elements of order 2^n' and any group in which this is true, there is a unique subgroup of order 2^n. By what we just noted, $\mathcal{C}_m \models \gamma_n$ for $n \geq m$, so $\gamma_n \in \Phi$. As $\mathcal{A} \models \gamma_n$, we obtain (d).

1 (e) This follows from (d): as H_{n+1} is cyclic of order 2^{n+1} it has a subgroup of order 2^n which must be must be H_n (by (d)).

2 (iv) Let $\mathcal{L}_c^=$ be as in the hint. Let β_k be the formula $(\neg \alpha_k(c))$ and consider the set of formulas $\Sigma = \Phi \cup \{\beta_k : k \in \mathbb{N}\}$. Suppose Σ_0 is a finite subset of Σ. Then for sufficiently large n we can interpret c as a generator of $\mathcal{C}_n$ and then we have $\mathcal{C}_n \models \Sigma_0$. Indeed, Σ_0 contains finitely many formulas in Φ, so there is some m such that $\mathcal{C}_n$ is a model for all of these for $n \geq m$. We then just need to choose n so that 2^n is greater than k for all of the β_k in Σ_0. So every finite subset of Σ has a normal model and therefore by the Compactness Theorem (for normal models), there is a normal model $\mathcal{B}$ of Σ. The interpretation of c in $\mathcal{B}$ is an element of infinite order in $\mathcal{B}$. As $\mathcal{B} \models \Phi$, we are done.

[3] (4 marks) Let $\mathcal{L}^E$ be a 1st-order language with equality and $\mathcal{A}$ a normal $\mathcal{L}^E$-structure. Prove that there is an $\mathcal{L}^E$-structure $\mathcal{B}$ which satisfies the axioms for equality Σ_E, is NOT a normal $\mathcal{L}^E$-structure and has $\hat{\mathcal{B}}$ isomorphic to $\mathcal{A}$ (in the notation of Section 2.6 of the notes).

4 *Solution:* Choose some $a \in A$ and let $B = A \cup \{a'\}$ where $a' \notin A$. We make B into an $\mathcal{L}^E$ structure $\mathcal{B}$ as follows: In $\mathcal{B}$, E is interpreted as an equivalence relation where all the classes have size 1 (as in $\mathcal{A}$) except that $\{a, a'\}$ is an equivalence class (i.e. $\bar{E}(a, a')$ holds).

The remaining relations on $\mathcal{B}$ are now determined by the requirements of the axioms of equality. So if R is an n-ary relation symbol interpreted by R_A and R_B in $\mathcal{A}$ and $\mathcal{B}$, and $b = (b_1, \ldots, b_n) \in B^n$, let $c \in A^n$ be obtained by replacing occurrences of a' in b by a. We then say that $R_B(b)$ holds in $\mathcal{B}$ iff $R_A(c)$ holds in $\mathcal{A}$.

Similarly if f is an n-ary function symbol, interpreted by f_A in $\mathcal{A}$ and f_B in $\mathcal{B}$, we define $f_B(b) = f_A(c)$ (where b, c are as above).

A constant symbol is interpreted in $\mathcal{B}$ as in its interpretation in $\mathcal{A}$.

By definition of the structure on $\mathcal{B}$, $\mathcal{B}$ satisfies the axioms for equality Σ_E and the map $\alpha : \mathcal{A} \to \hat{\mathcal{B}}$ given by $\alpha(e) = [e]$ for $e \in A$ is an isomorphism.