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Book: Landau + Lifshitz V2
Some chapters of V1

How light travels

1) Maybe like a ball?

Event: $\vec{r}, t$

position in space

"position" point in time

Event 1: Flash at one end of the room

$\vec{r}_0, t_0$

Event 2: Flash received at the other end of the room

$\vec{r}_1, t_1$

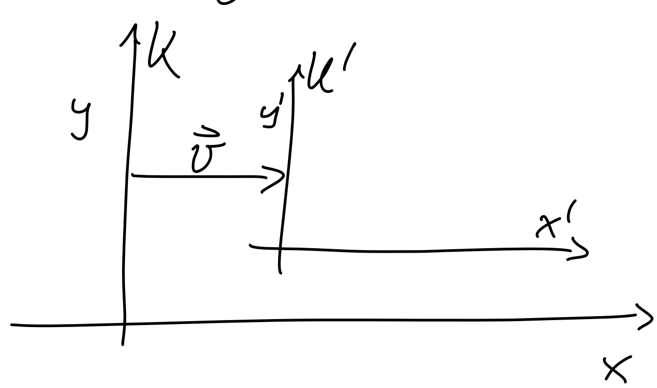
Perceived of light: $\frac{|\vec{r}_1 - \vec{r}_0|}{t_1 - t_0}$

Galilean transformation

Observer in frame K' moving relative to K with velocity $\vec{v}$.

Let's agree that the relative motion is along the x-axis.

K' moving relative to K with velocity $\vec{v}$.



Inertial frame of reference moves with constant velocity (i.e. is not accelerated).

Next time: Go through Galilean transformations.
Derive velocity of light in K, K'.
Compare with reality.

Lecture 2:

- complete classical picture
- move on to special relativity.

$$(\underset{\uparrow}{t}, \underbrace{x, y, z}_{\text{spatial coordinates}}) \qquad \text{event}$$

time of event

Event 1: Release of light

(t_1, x_1, y_1, z_1)

Event 2: Light detected

(t_2, x_2, y_2, z_2)

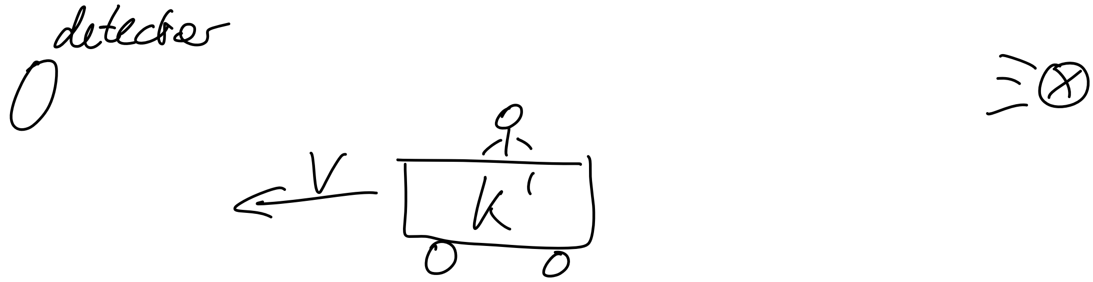
$y_1 = y_2 = 0$

$z_1 = z_2 = 0$

light has travelled distance $\quad x_2 - x_1$

time passed $\quad t_2 - t_1$

Observed speed of light $\longrightarrow c_k = \dfrac{x_2 - x_1}{t_2 - t_1}$



Simplification: $(0,0,0,0)$ is the same event in all frames

x, y, z axes of all frames are parallel

K' dashed coordinate system (moves with speed V away from K-origin)

K

$$\boxed{\begin{aligned} t' &= t \\ x' + Vt' &= x \\ y' &= y \\ z' &= z \end{aligned}}$$

$$(t_1, x_1, y_1, z_1) \rightarrow (\underset{\substack{\| \\ t_1}}{t_1'}, x_1', \underset{\substack{\| \\ y_1}}{y_1'}, \underset{\substack{\| \\ z_1}}{z_1'})$$

$$x_1' + Vt_1' = x_1 \quad \Rightarrow \quad x_1' = x_1 - Vt_1$$

In K' the coordinates of the release of light are

$$(t_1, x_1 - Vt_1, y_1, z_1)_{K'}$$

Detection

$$(t_2, x_2 - Vt_2, y_2, z_2)_{K'}$$

Perceived speed of light

$$c_{K'} = \frac{(x_2 - Vt_2) - (x_1 - Vt_1)}{t_2 - t_1} = \underbrace{\frac{x_2 - x_1}{t_2 - t_1}}_{c_K} - V$$

$$c_{K'} = c_K - V$$

This is not what is observed.

Instead every observer sees the same speed of light, c.

Introduce the notion of an interval

$$s^2 = -\underbrace{\left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]}_{\text{distance}^2 \text{ travelled}} + c^2 \underbrace{(t_2-t_1)^2}_{\text{time passed squared}}$$

If the two events are due to release of light and detection of light, then $s^2=0$

In the previous experiment $s=0$.

I hereby demand: If the interval of any two events is 0 in one inertial frame, then it's 0 in all inertial frames.

The transformation law that fixes everything:

$$\boxed{\begin{aligned} x &= \gamma(x' + ct'\beta) & \beta &= \frac{v}{c} \\ y &= y' \\ z &= z' & \gamma &= \frac{1}{\sqrt{1-v^2/c^2}} \\ ct &= \gamma(ct' + x'\beta) \end{aligned}}$$

Experiment in K: $\dfrac{x_2-x_1}{t_2-t_1} = c$

$$x' = \gamma(x - ct\beta)$$

$$y' = y$$

$$z' = z$$

$$ct' = \gamma (ct - x\beta)$$

$$x_1' = \gamma (x_1 - ct_1 \beta)$$
$$x_2' = \gamma (x_2 - ct_2 \beta)$$

$$ct_1' = \gamma (ct_1 - x_1 \beta)$$
$$ct_2' = \gamma (ct_2 - x_2 \beta)$$

$$\frac{x_2' - x_1'}{c(t_2' - t_1')} = \frac{(x_2 - x_1) - c\beta (t_2 - t_1)}{c(t_2 - t_1) - \beta (x_2 - x_1)}$$

$$c = \frac{x_2 - x_1}{t_2 - t_1}$$

$$= \frac{(x_2 - x_1) - \beta (x_2 - x_1)}{(x_2 - x_1) - \beta (x_2 - x_1)} = 1 \quad \text{Yes!}$$

$$\Rightarrow \frac{x_2' - x_1'}{t_2' - t_1} = c$$

It turns that the interval is invariant (i.e. it's the same in all frames of reference).

Lecture 3:

Lorentz transform

speed of K' relative to K

$$\beta = \frac{V}{c}$$

$$x = \gamma(x' + c\beta t')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(ct' + \beta x')$$

Lorentz-scalar are invariant under the Lorentz-transform (and normally in a non-trivial way).

The interval is a Lorentz-scalar.

$$s^2 = c^2(t_2 - t_1)^2 - \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$
$$\parallel$$
$$c^2(t_2' - t_1')^2 - \left[(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

Shortcut to transformations

$$x = \gamma(x' + c\beta t')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(ct' + \beta x')$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathcal{L}} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\beta = \frac{v}{c}$$

$$\gamma^2 (1 - \beta^2) = 1$$

Let's use this transformation to calculate $s^2(1,2,\text{undashed})$

$$ct_1 = \gamma (ct_1' + \beta x_1') \qquad ct_2 = \gamma (ct_2' + \beta x_2')$$
$$x_1 = \gamma (x_1' + c\beta t_1') \qquad x_2 = \gamma (x_2' + c\beta t_2')$$
$$y_1 = y_1' \qquad y_2 = y_2'$$
$$z_1 = z_1' \qquad z_2 = z_2'$$

$$s^2 = (ct_2 - ct_1)^2 - \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

$$= \gamma^2\left(c(t_2' - t_1') + \beta(x_2' - x_1')\right)^2$$

$$- \left[\gamma^2\left((x_2' - x_1') + c\beta(t_2' - t_1')\right)^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

cross terms cancel

$$= \gamma^2\left(c^2(t_2' - t_1')^2 + \beta^2(x_2' - x_1')^2\right)$$

$$- \left[\gamma^2\left((x_2' - x_1')^2 + c^2\beta^2(t_2' - t_1')^2\right) + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

$$= \underbrace{\gamma^2(1 - \beta^2)}_{1}\, c^2(t_2' - t_1')^2$$

$$- \left[\gamma^2 (x_2' - x_1')^2 (1 - \beta^2) + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

$$1$$

$$= c^2(t_2' - t_1')^2$$

$$- \left[(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

The interval in the undashed frame equals the interval in the dashed frame.

§3 in LL2 Landau Lifshitz

Proper time: Time that passes in a frame of reference where the clock is at rest.

Proper time is measured by the comoving clock.

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

infinitesimal interval (squared)

If the clock is at rest, then $dx = dy = dz = 0$

$$ds^2 = c^2 dt'^2$$

my frame of reference

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

$$dt'^2 = dt^2 - \underbrace{\frac{dx^2 + dy^2 + dz^2}{c^2}}_{\text{Can this be written more elegantly?}}$$

GP to check for past coursework

Lecture 4

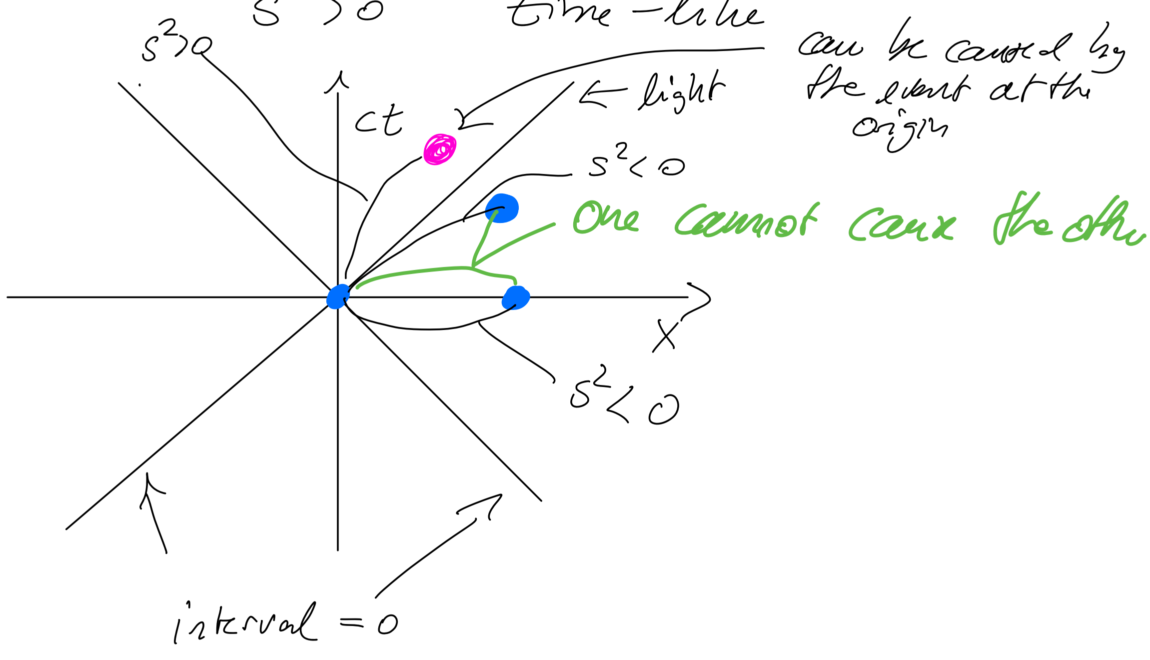
Interval squared

$s^2 = c^2t^2 - \ell^2$

Euclidean distance

$s^2 < 0$ space-like

$s^2 > 0$ time-like



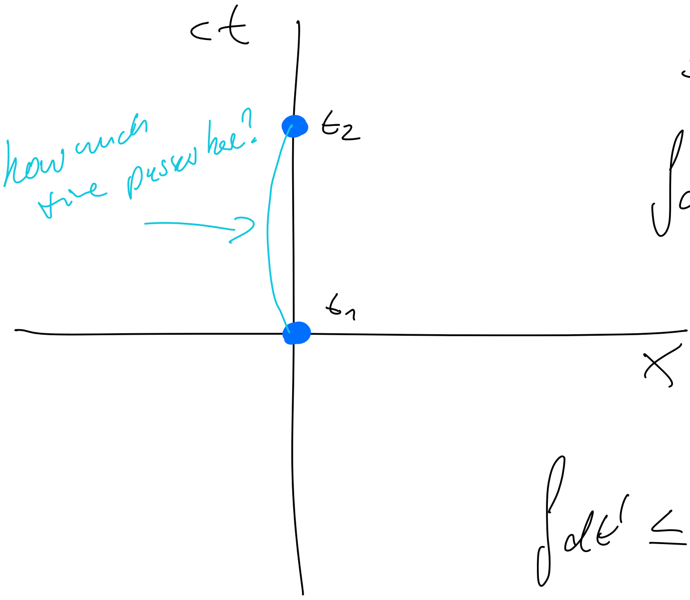
$$c^2dt'^2 = c^2dt^2 - (dx^2+dy^2+dz^2)$$

$$ds^2 = c^2dt^2\left(1 - \frac{dx^2+dy^2+dz^2}{c^2dt^2}\right) = c^2dt^2\left(1-\frac{V^2}{c^2}\right)$$

$$dt' = dt \underbrace{\sqrt{1-\frac{v^2}{c^2}}}_{\frac{1}{\gamma}} \Rightarrow \underbrace{\gamma\, dt' = dt}$$

Valid provided no displacement in the u' frame!

Similarly $ds = c\,dt'$ only if there is no displacement.



$$s = c(t_2 - t_1)$$

$$\int dt' = \int dt \underbrace{\sqrt{1-\frac{v^2}{c^2}}}_{\leq 1}$$

$$\int dt = t_2 - t_1$$

In this experiment:

$$\underset{\frac{s'}{c}}{\underset{\|}{\int dt'}} \leq \underset{\frac{s}{c}}{\underset{\|}{\int dt}}$$

Acceleration breaks the equivalence of the interval, the interval observed ~~at rest~~ in an inertial frame is always greater or equal to the interval observed in any other frame.

§4 Contravariant = physics

Covariant four-vectors

$$\boxed{(ct, x, y, z) = x^i}$$

$i = 0, 1, 2, 3$

Co is below

$x^1 = x$

$x_1 = -x$

$$x_i = (ct, -x, -y, -z)$$

F^{ik} $F_i{}^k$ F_{ik}

$x^i = (ct, x, y, z)$; $x_i = (ct, -x, -y, -z)$

$x^i x_i = (ct)^2 - x^2 - y^2 - z^2$ Einstein convention

every repeated index
to be summed over,
given one is contra
and the other is co

$(x^i x_i)\cdot(y^j y_j)$

$\cancel{x^i x^i}$ $\cancel{x_i x_i}$

$x^i x_i = \sum_i x^i x_i$

$x^i x_i = x_i x^i$; $x^i y_i = y^i x_i$?

Metric <u>tensor</u>

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$x_i = g_{ij} x^j$

$x^j = (ct, x, y, z) = (ct, \vec{r})$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{ij}$$

$$x_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}$$

$$x^i x_i = x^i g_{ij} x^j$$

$$\mathcal{L}^i{}_j = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz transform matrix

How do genrally transform a four-vector from K' to K

$$x^i = \mathcal{L}^i{}_j x'^j$$

$$x_i = g_{ik} x^k = g_{ik} \mathcal{L}^k{}_j x'^j$$

$$= g_{ik} \mathcal{L}^k{}_j g^{jl} x'_l$$

Matrices:

$$A = BC$$

$$(A)_{ij} = \sum_{k=0}^{3} (B)_{ik} (C)_{kj}$$

1) Show that g_{ij} is a tensor

2) Show ~~that~~ how g^{ij} can be obtained by raising indices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

g_{ik} $\quad$ $\mathcal{L}^k$ $\quad$ g^{jl}

$g^{\cdot\cdot}$ $\qquad$ $\mathcal{L}_j$ $\qquad$ g

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \hat{\mathcal{L}}^k{}_j$$

inverse transform (from k to k')

$$x'^i \xrightarrow{\mathcal{L}} x^i \qquad\qquad x^i = \mathcal{L}^i{}_j x'^j$$

$$x'_i \xrightarrow{\hat{\mathcal{L}}} x_i$$

$$x^i \xrightarrow{\hat{\mathcal{L}}} x'^i$$

$$x_i \xrightarrow{\mathcal{L}} x'_i$$

$$\boxed{g^{ij} = \mathcal{L}^i{}_k \mathcal{L}^j{}_l \, g'^{kl}}$$

$$\varepsilon^{ijkl} = \mathcal{L}^i{}_a \mathcal{L}^j{}_b \mathcal{L}^k{}_c \mathcal{L}^l{}_d \, \varepsilon'^{abcd}$$

$$\delta^i{}_j = \mathcal{L}^i{}_k \hat{\mathcal{L}}_j{}^l \, \delta'^k{}_l$$

$$\delta^{ij} \neq \mathcal{L}^i{}_k \mathcal{L}^j{}_l \, \delta'^{kl}$$

The transform is right, but on the left is generally not a Kronecker-δ, if there is a Kronecker-δ on the right.

Next time:

What event is $(ct, -x, -y, -z) = x_i$ ✓

Two outstanding items ✓

Complete Lorentz transform ✓

Addition of velocities → next time

Glimpse of <u>action</u>

$$\left.\begin{array}{l} x'^i = (ct, x, y, z) \\ x'_i = (ct, -x, -y, -z) \end{array}\right\} \text{ same event}$$

$$x'^i \xrightarrow{\mathcal{L}^i_j} x^i$$

$$x^i = \mathcal{L}^i{}_j\, x'^j$$
$$= x'^j \mathcal{L}^i{}_j$$
$$= x'^j \mathcal{L}^{T}{}_j{}^i$$

$$A'^{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$A = BC$ Linear algebra aside

$$\boxed{\begin{aligned} A_{ij} &= \sum_k B_{ik} C_{kj} \\ &= \sum_k C_{kj} B_{ik} \end{aligned}}$$

$$\mathcal{L} = \mathcal{L}^T$$

$$A^{ij} = \mathcal{L}^i{}_k \mathcal{L}^j{}_l A'^{kl}$$
$$= \mathcal{L}^i{}_k A'^{kl} \mathcal{L}^j{}_l$$
$$= \mathcal{L}^i{}_k A'^{kl} \mathcal{L}^{T}{}_l{}^j$$
$$= \mathcal{L}^i{}_k A'^{kl} \mathcal{L}_l{}^j$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & \beta\gamma^2 - \beta\gamma^2 & 0 & 0 \\ \beta\gamma^2 - \beta\gamma^2 & \beta^2\gamma^2 - \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^2(1-\beta^2)=1$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$B'^{i}{}_{j} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$B^{i}{}_{G} = B'^{i}{}_{j} = \mathcal{L}^{i}{}_{k} \, B'^{k}{}_{l} \, \hat{\mathcal{L}}_{j}{}^{l}$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B^{i}{}_{j} = \mathcal{L}^{i}{}_{k} \, \hat{\mathcal{L}}_{j}{}^{m} \, B'^{k}{}_{m}$$

$$G_{ij} = \hat{\mathcal{L}}_{i}{}^{k} \, \hat{\mathcal{L}}_{j}{}^{l} \, G_{kl}$$

$$B_{j}{}^{i} = \mathcal{L}^{i}{}_{k} \, \hat{\mathcal{L}}_{j}{}^{m} \, B'_{m}{}^{k}$$

Time dilation — time seems to pass more slowly in the moving frame

Lorentz contraction – objects seem shorter in the moving frame.

Measure the position of the two ends of a yard stick

$$(ct, x_2, 0, 0)$$
$$(ct, x_1, 0, 0)$$

] observations of the two end points of the yard stick

$$l = x_2 - x_1$$

What is l'? $\qquad l \leq l'$

$$(ct, \mathbf{r})$$

Next time: Transform $(ct, x_2, 0, 0)$
$(ct, x_1, 0, 0)$
to derive length in moving frame

Recap of L06:

$$x_i = \underbrace{\qquad\qquad}_{?} x'^i$$

$$x_i = g_{ij} x^j$$

$$x^j = \mathcal{L}^j{}_k x'^k$$

$$\Rightarrow x_i = g_{ij} \mathcal{L}^j{}_k x'^k$$

$$= \underbrace{g_{ij} \mathcal{L}^j{}_k g^{kl}}_{\hat{\mathcal{L}}} x'_l$$

Let's measure the length of a yard stick.

$$x^i = (ct, a_1, 0, 0) \longrightarrow x'^i$$
$$y^i = (ct, a_2, 0, 0) \longrightarrow y'^i$$

$$l = a_2 - a_1$$

$$x'^i = \hat{\mathcal{L}}^i{}_j x^j$$

$$x'^i = \big(\gamma(ct - \beta a_1),\ \gamma(a_1 - \beta ct),\ 0,\ 0\big)$$
$$y'^i = \big(\gamma(ct - \beta a_2),\ \gamma(a_2 - \beta ct),\ 0,\ 0\big)$$

Proper length $l_0 = \gamma(a_2 - \beta ct) - \gamma(a_1 - \beta ct)$ — Yard stick not moving in K'.

$$= \gamma (a_2 - a_1)$$

Lorentz contraction

$$l_0 = \gamma l \quad \Leftrightarrow \quad l = \sqrt{1 - v^2/c^2}\, l_0 \leq l_0$$

Conundrum: How can two sticks appear to be both shorter than the other, simultaneously. How can they overlap.

Time dilation

$$x'^i = (ct_1', 0, 0, 0) \qquad x'^i \Lambda_i^{\ j} = x^j$$

$$y'^i = (ct_2', 0, 0, 0)$$

$$x^i = (ct_1'\gamma, ct_1'\beta\gamma, 0, 0)$$

$$y^i = (ct_2'\gamma, ct_2'\beta\gamma, 0, 0)$$

$$\Delta t' = t_2' - t_1' = \Delta t_0$$

$$\Delta t = \gamma (t_2' - t_1')$$

Proper time $\Delta t_0 = \Delta t \sqrt{1 - v^2/c^2}$

Moving clocks appear to tick more slowly.

Today: Transformation of Velocities

On Tue: Differential ops.

§5 in LL2

$$dx = \gamma(dx' + \beta c dt')$$

$$dy = dy'$$

$$dz = dz'$$

$$c dt = \gamma(c dt' + \beta dx')$$

$$\vec{v} = (v_x, v_y, v_z) \qquad \vec{v}' = (v_x', v_y', v_z')$$

$$v_x = \frac{dx}{dt} = \frac{dx' + \beta c dt'}{\cancel{c} dt' + \frac{1}{c}\beta dx'} = \frac{v_x' + \beta c}{1 + \frac{\beta}{c} v_x'} = \frac{v_x' + V}{1 + \frac{V v_x'}{c^2}}$$

$$v_x' = \frac{dx'}{dt'}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{(dt' + \frac{1}{c}\beta dx')\gamma} = \frac{v_y'}{1 + \frac{V v_x'}{c^2}} \sqrt{1 - \frac{V^2}{c^2}}$$

$$v_z = \frac{v_z' \sqrt{1 - V^2/c^2}}{1 + \frac{V v_x'}{c^2}}$$

Simplifications

$$(1-\beta^2)\gamma^2 = 1$$

$$(1+\beta)\gamma = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1+\frac{v}{c}}{\sqrt{(1-\frac{v}{c})(1+\frac{v}{c})}}$$

$$= \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

$$F^{ij} \xrightarrow{\hat{\mathcal{L}}} F'^{ij}$$

$$F'^{ij} = \hat{\mathcal{L}}^i{}_k \hat{\mathcal{L}}^j{}_l F^{kl}$$

$$= \hat{\mathcal{L}}^i{}_k F^{kl} \hat{\mathcal{L}}^j{}_l$$

Differential operators

$$\frac{\partial}{\partial x^i} \cancel{\phi(x^i)}$$

$$\phi(ct, x, y, z)$$

$$\phi(ct, \vec{r})$$

$$\frac{\partial}{\partial x^i}\phi = ?\frac{\partial}{\partial x'^j}\phi$$

$$\frac{\partial}{\partial x^i}\phi = \underbrace{\frac{\partial x'^j}{\partial x^i}}_{\text{Jacobian}} \frac{\partial}{\partial x'^j}\phi$$

$$x'^j = \hat{\mathcal{L}}^j{}_i x^i$$

$$\frac{\partial x'^j}{\partial x^i} = \hat{\mathcal{L}}^j{}_i$$

$$\frac{\partial}{\partial x^i}\phi = \hat{\mathcal{L}}^j{}_i \frac{\partial}{\partial x'^j}\phi$$

$$\frac{\partial}{\partial x^i} = \hat{\mathcal{L}}^j{}_i \frac{\partial}{\partial x'^j}$$

$$\partial_i = \hat{\mathcal{L}}^j{}_i \, \partial'_j$$

$$x^i = \mathcal{L}^i{}_j \, x'^j$$

$$x_i = \hat{\mathcal{L}}_i{}^j \, x'_j$$

The gradient wrt contravariant coordinates transforms like a covariant four-vector.

$$\frac{\partial}{\partial x^i}\phi = \partial_i \phi = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \vec{\nabla}\phi\right)$$

$$\frac{\partial}{\partial x_i}\phi = \partial^i \phi = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, -\vec{\nabla}\phi\right)$$

Volume element

$$d\Omega = dx^0 dx^1 dx^2 dx^3$$

$$= c\,dt\, dx\, dy\, dz$$

$$J = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} = \left|\frac{\partial x'^i}{\partial x^j}\right|$$

$$x'^i = \hat{\mathcal{L}}^i{}_j \, x^j$$

$$\Rightarrow \quad J = \left|\hat{\mathcal{L}}^i{}_j\right|$$

$$\hat{\mathcal{L}} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma \begin{vmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \beta\gamma \ \ldots$$

$$= (\gamma^2 - \beta^2\gamma^2) \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\Rightarrow \quad d\Omega = d\Omega'$$

§7 Four-velocities (LL2)

x^i a four-vector
ds the infinitesimal interval

$$u^i := \frac{dx^i}{ds} \quad \text{is a four vector}$$

(by asking: What is $\frac{dx'^i}{ds'} = \frac{dx'^i}{ds} = \hat{f}^i{}_j \frac{dx^j}{ds}$)

Let's determine u^i for a moving particle

$$u^1 = \frac{dx^1}{ds}$$

$$ds = c\,dt \sqrt{1 - \frac{v^2}{c^2}}$$ ← based on the proper time of the moving particle

$$u^1 = \frac{dx^1}{c\,dt\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{c\sqrt{1-\frac{v^2}{c^2}}} \frac{dx^1}{dt} = \frac{v_x}{c\sqrt{1-\frac{v^2}{c^2}}}$$

$$u^2 = \frac{v_y}{c\sqrt{1-\frac{v^2}{c^2}}}$$

$$u^3 = \frac{v_z}{c\sqrt{1-\frac{v^2}{c^2}}}$$

dx^0

$$u^0 = \frac{c\,dt}{c\,dt\sqrt{1-\frac{v^2}{c^2}}}$$

$$u^i = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} , \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}} \right)$$

$$u_i = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} , \frac{-\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}} \right)$$

Contraction of u^i with itself.

$$u^i u_i = \frac{1}{1-\frac{v^2}{c^2}} - \frac{v^2}{c^2\left(1-\frac{v^2}{c^2}\right)} = 1$$

Every contraction of a four-vector with itself is Lorentz invariant.

$$u'^i u'_i = 1$$

Show this via

$$u'^i = \hat{L}^i{}_j u^j$$

$$u'_i = L_i{}^j u_j$$

$$u'^i u'_i = \underbrace{\hat{L}^i{}_j L_i{}^k}_{\text{Show that this is } \delta_j^k} u^j u_k = u^j u_j$$

LOG:

Lorentz transform question

$$F^{ij} = \hat{\mathcal{L}}^{i}{}_{k} \hat{\mathcal{L}}^{j}{}_{l} F'^{kl}$$

$$= F'^{kl} \hat{\mathcal{L}}^{i}{}_{k} \hat{\mathcal{L}}^{j}{}_{l}$$

$$\varepsilon^{ijkl} = \mathcal{L}^{i}{}_{a} \mathcal{L}^{j}{}_{b} \mathcal{L}^{k}{}_{c} \mathcal{L}^{l}{}_{d} \varepsilon'^{abcd}$$

$$= \mathcal{L}^{i}{}_{0} \mathcal{L}^{j}{}_{0} \mathcal{L}^{k}{}_{0} \mathcal{L}^{l}{}_{0} \varepsilon'^{0000}$$

$$+ \mathcal{L}^{i}{}_{0} \mathcal{L}^{j}{}_{0} \mathcal{L}^{k}{}_{0} \mathcal{L}^{l}{}_{1} \varepsilon'^{0001}$$

$$+ \dots$$

$$=$$

Analytical mechanics

§2 LL1 Principle of least action

$$\boxed{S = \int_{t_1}^{t_2} dt \, L(q(t), \dot{q}(t), t)}$$

action

Lagrangian

Principle of least action: The observed physics is the one for which S takes the minimal value.

$q(t)$, $\dot{q}(t)$

generalized coordinate (trajectory)

generalized velocity

Assume that we have $q(t)$ that minimises

$$S([q(t)], [\dot{q}(t)])$$

Now consider a slightly different path,

$$q(t) + \delta q(t)$$

$$\delta q(t_1) = 0 = \delta q(t_2)$$

Now consider

$$S' = S([q(t) + \delta q(t)], [\dot{q}(t) + \delta \dot{q}(t)])$$

To this end, Taylor expand ... L

$$S' = \int_{t_1}^{t_2} dt\, L\left(q(t)+\delta q(t),\ \dot q(t)+\delta \dot q(t),\ t\right)$$

unperturbed S

$$= \int dt\, L\left(q(t), \dot q(t), t\right)$$

$$+ \int dt\ \delta q(t)\ \frac{\partial L}{\partial q}\left(q(t), \dot q(t), t\right)$$

$$+ \int dt\ \delta \dot q(t)\ \frac{\partial L}{\partial \dot q}\left(q(t), \dot q(t), t\right)$$

$$+ \ \text{h.o.t.}$$

$$S' - S = \int dt \left(\delta q(t)\ \frac{\partial L}{\partial q} + \delta \dot q(t)\ \frac{\partial L}{\partial \dot q} \right) + \text{h.o.t.}$$

S minimal $\rightsquigarrow$

$$\int dt \left(\delta q(t)\ \frac{\partial L}{\partial q} + \underbrace{\delta \dot q(t)}_{\frac{d}{dt}\delta q(t)}\ \frac{\partial L}{\partial \dot q} \right) = 0$$

$$= \int_{t_1}^{t_2} dt\ \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q} \right) + \underbrace{\left[\delta q(t)\ \frac{\partial L}{\partial \dot q} \right]_{t_1}^{t_2}}_{=0 \text{ because } \delta q(t_1) = 0 = \delta q(t_2)}$$

integration by parts

$$\int_{t_1}^{t_2} dt\, \dot{g}(t)\, h(t) = -\int_{t_1}^{t_2} dt\, g(t)\, \dot{h}(t) + \left[g h \right]_{t_1}^{t_2}$$

When S is extremal, we have that

$$\int dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0$$

for any small $\delta q(t)$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}}$$

Euler-Lagrange equations (EL equations)

Let's consider an

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + g(q, \dot{q}, t)$$

What are the properties of g that leave the EL equations invariant?

Compare that to demanding that g leaves the action unchanged.

(*) $\int dt\, L - \int dt\, L' = 0$

Next: Derive the properties of g from ⊛

Show that this leaves EL eqns unchanged.

$$\int dt \; g(q(t), \dot{q}(t), t) \overset{!}{=} 0$$

Assume ...

$$S = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t), t)$$

Which class of functions $g(q, \dot{q}, t)$ can be added to L, s.t. S is unchanged.

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + g(q, \dot{q}, t)$$

$$S' = \int_{t_1}^{t_2} dt\, L'(q(t), \dot{q}(t), t)$$

$$= S + \int_{t_1}^{t_2} dt\, g(q(t), \dot{q}(t), t)$$

What's the effect of
$$g(q(t), \dot{q}(t), t) = \frac{d}{dt} f(q(t), \dot{q}(t), t)$$

$$\Rightarrow S' = S + \Big[f(q(t), \dot{q}(t), t) \Big]_{t_1}^{t_2}$$

$S' = S$ if

$$f(q(t_2), \dot{q}(t_2), t_2) - f(q(t_1), \dot{q}(t_1), t_1) = 0$$

Which g (and therefore which f) can I allow s.t. S' is identical S up to a quantity independent of the path

How about $f(q,\dot{q},t)$ being independent of $\dot{q}$.

Then

$$f(q(t_2),\dot{q}(t_2),t_2) - f(q(t_1),\dot{q}(t_1),t_1) = C'$$

independent of the path (given that $q(t_1)$ and $q(t_2)$ are fixed)

$$g(q(t),t) = \frac{d}{dt} f(q(t),t)$$

Plug this into S'' and determine the EL equations.

$$S' = \int_{t_1}^{t_2} dt\, L(q,\dot{q},t)$$

$$S'([q(t)]) = \int_{t_1}^{t_2} dt\, \overbrace{L(q,\dot{q},t) + \frac{d}{dt} f(q(t),t)}^{L'}$$

EL: $$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial L'}{\partial q} = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right)$$

See it

Say it

Sorted

$$\frac{d}{dt} f(q(t),t) = \underbrace{\dot{q}(t)\, f'(q(t),t)}_{\frac{\partial}{\partial p}\big|_{p=q(t)} f(p,t)} + \underbrace{\dot{f}(q(t),t)}_{\frac{\partial}{\partial t}\big|_{p=q(t)} f(p,t)}$$

$$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\Big(\dot{q}(t)\, f'(q(t),t) + \dot{f}(q(t),t)\Big)$$

$$= \frac{\partial L}{\partial q} + \dot{q}(t)\, f''(q(t),t) + \dot{f}'(q(t),t)$$

$$\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + f'(q(t),t)$$

Conclusion: We need $\dot{q}(t)\, f''(q(t),t) = -\dot{f}'(q(t),t)$

What I set out to see is that

if $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

then $\frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}}$

but that works out only if

$$\dot{q}(t)\, f''(q(t),t) + \dot{f}'(q(t),t) = \cancel{0}\ \frac{d}{dt} f'(q(t),t)$$

Solution: Next time! Ah! Just a missing term.

Last time

$$L'(q,\dot q,t) = L(q,\dot q,t) + \frac{d}{dt} f(q(t),t)$$

$$\frac{d}{dt} f(q(t),t) = \dot q(t)\, f'(q(t),t) + \dot f(q(t),t)$$

then

$$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \dot q(t) f''(q(t),t) + \dot f'(q(t),t)$$

$$\frac{\partial L'}{\partial \dot q} = \frac{\partial L}{\partial \dot q} + f'(q(t),t)$$

If want to find

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$$

$$\Longleftrightarrow$$

$$\frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L'}{\partial \dot q}$$

$$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \dot q f''(q,t) + \dot f'(q,t)$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} + f'(q(t),t)\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \dot q(t) f''(q(t),t) + \dot f'(q(t),t)$$

Yes indeed:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \iff \frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}}$$

We will use this to our advantage: Make the maths easier without spoiling the physics by using a "clever" $f(q,t)$.

Now on to conservation laws!

§6 Energy in LL1

Assume we have an L in place, plus EL

$$\boxed{\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}$$

$\nabla_q L = \frac{d}{dt}\nabla_{\dot{q}} L$

Assume $L(q(t),\dot{q}(t),t) = L(q(t),\dot{q}(t))$ not explicitly a function of time t.

$$\frac{d}{dt}L(q(t),\dot{q}(t)) = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \ddot{q}\frac{\partial L}{\partial \dot{q}}$$

$$= \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\Rightarrow \frac{d}{dt}\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\underbrace{L - \dot{q}\frac{\partial L}{\partial \dot{q}}}_{\text{"the energy"}} \text{ is conserved}$$

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

L not explicitly dependent of time

$\hat{=}$ time - translational invariance

$\Downarrow$ Emmy Noether

Energy conservation

§7 Momentum conservation

Demand that the Lagrangian is invariant under translation.

Consider multiple particles:

$$\delta L = L\Big(\underbrace{\{q+\vec{\varepsilon}\}}_{\substack{n \\ \text{coordinates}}}, \underbrace{\{\dot{q}\}}_{\substack{n \\ \text{velocities}}}\Big) - L(\{q\}, \{\dot{q}\}) \overset{!}{=} 0$$

$$\delta L = L(\{q\}, \{\dot{q}\}) + \sum_{a=1}^{n} (\nabla_{q_a} L \cdot \vec{\varepsilon})$$

$$- L(\{q\}, \{\dot{q}\}) + O(\varepsilon^2)$$

Consider small ε, so we need

$$\sum_{a=1}^{n} \left(\nabla_{q_a} L \cdot \vec{\varepsilon}\right) = 0 = \vec{\varepsilon} \cdot \sum_{a=1}^{n} \nabla_{q_a} L$$

for any small $\vec{\varepsilon}$

$$\Rightarrow \sum_{a=1}^{n} \nabla_{q_a} L = 0$$

$$\parallel$$

$$\sum \frac{d}{dt} \nabla_{\dot{q}_a} L = \frac{d}{dt} \underbrace{\sum \nabla_{\dot{q}_a} L}_{\text{momentum}}$$

Example (from the audience)

$$L = \frac{1}{2} m \dot{\vec{q}}^2 \rightsquigarrow \sum \nabla_{\dot{q}_a} L = m \dot{\vec{q}}$$

$$\vec{P} = \sum_{a=1}^{n} \vec{\nabla}_{\dot{q}_a} L \leftarrow \text{total momentum}$$

Only fools and horses

Cook up a simple L:

$L \neq \vec{q} \cdot \vec{g}$ not trans inv
not isotropic

$$L = \frac{\dot{\vec{q}}^4}{42} \qquad L = \frac{\dot{\vec{q}}^2}{42} \alpha$$

$\uparrow$ $1n.$

Exercise: Let's analyse coupling

$$L = \frac{1}{2} \alpha \ddot{q}^2$$

Relativistic again

§8 The principle of least action LL2

$S =$ integral.

§8 LL2 The principle least action

$$S = -\alpha \underbrace{\int_{\text{event }1}^{\text{event }2} ds}_{\substack{\text{(extremal for}\\ \text{any straight path)}}}$$

In mechanics:

$$S = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t), t)$$

$$ds = c\,dt \sqrt{1 - \frac{v^2}{c^2}}$$

from

$$ds^2 = c^2 dt'^2$$
$$= c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right)$$

$$\Rightarrow \quad S = -\alpha \underbrace{\int_{t_1}^{t_2} dt\, c \sqrt{1 - \frac{v^2}{c^2}}}_{\substack{\text{interval of a moving}\\ \text{particle}}}$$

By comparison $\boxed{L = -\alpha c \sqrt{1 - \frac{v^2}{c^2}}}$

In classical mechanics: $L = \frac{1}{2} m v^2$

is the most basic Lagrangian.

Let's Taylor expand:

$$L = -\alpha c \left(1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{8}\left(\frac{v^2}{c^2}\right)^2 + \dots\right.$$

$$(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^3 + \dots$$

$$L = -\alpha c + \frac{1}{2}\alpha \frac{v^2}{c} + \dots$$

Choose $\alpha = mc$, so that our relativistic Lagrangian recovers classical mechanics for $v \ll c$

$$① \quad \boxed{L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}}$$

What is $E = L - v\frac{\partial L}{\partial v}$?

$$\frac{\partial L}{\partial v} = \frac{v m c^2}{c^2}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\nabla_v v^2 = 2\vec{v}$$

$$E = L - v^2 m \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= -mc^2 \sqrt{1-\frac{v^2}{c^2}} - v^2 m \frac{1}{\sqrt{\quad}}$$

$$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \left(-1 + \frac{v^2}{c^2} - \frac{v^2}{c^2} \right) = - \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

After fixing the sign ($E = v \frac{\partial L}{\partial v} - L$), we have the energy

restmass (material property)

(2) $$\boxed{\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$= mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right)$$

$$= mc^2 + \frac{1}{2} m v^2 + \dots$$

$$m_{eff} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\vec{p} = \vec{\nabla}_v L$$

$$\vec{p} = \frac{\vec{v} m \cancel{c^2}}{\cancel{c^2}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{from above}$$

$$③ \quad \boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}} = m_{ef}\,\vec{v}$$

$$\vec{p} = \vec{v}\,\frac{\mathcal{E}}{c^2} \qquad \text{useful identity!}$$

Let's rewrite the formulae above in terms of four-vectors.

Lorentz scalar

$$S = -mc\int ds$$

$$ds = \sqrt{dx^i dx_i}$$

Consider $dx'^i = dx^i + \delta dx^i$ ← perturbed path (not a different frame)

perturbed path

$$\delta_1 S \stackrel{!}{=} 0$$

$$= \left(-mc\int ds\right) - \left(-mc\int ds'\right)$$

$$ds' = \sqrt{(dx^i + \delta dx^i)(dx_i + \delta dx_i)}$$

$$= \sqrt{dx^i dx_i + \underbrace{\delta dx^i dx_i + dx^i \delta dx_i}_{2\delta dx^i dx_i} + \delta dx^i \delta dx_i}$$

$$a^i b_i = a_k g^{ki} b_i = a_k b^k$$

Next time: Complete $\int ds - \int ds'$ and determine what that means for the path

Lecture 13 §9 Energy and momentum

$$S = -mc \int ds$$

What happens if I perturb the path?

$$S' = -m \int ds'$$

$$ds = \sqrt{dx^i dx_i}$$

$$ds' = \sqrt{(dx^i + \delta dx^i)(dx_i + \delta dx_i)}$$

$$= \sqrt{(dx^i + d\delta x^i)(dx_i + d\delta x_i)}$$

$$S' - S = \delta S \stackrel{!}{=} 0$$

$$ds' = \sqrt{dx^i dx_i} \sqrt{1 + \frac{2 d\delta x^i dx_i}{dx^i dx_i} + O(\delta^2)}$$

$$= ds \left(1 + \frac{d\delta x^i dx_i}{dx^i dx_i} + \dots\right)$$

$$S' - S = -mc\left(\int ds' - \int ds\right)$$

$$= -mc\left(\int ds \left(1 + \frac{d\delta x^i dx_i}{dx^i dx_i} + \dots\right) - \int ds\right)$$

$$= -mc \int ds \left(\frac{d\delta x^i dx_i}{dx^i dx_i} + \dots\right)$$

$$= -mc \int ds \left(\frac{d\delta x^i}{ds} \underbrace{\frac{dx_i}{ds}}_{\text{4-velocity}} + \dots\right)$$

$$\overset{\text{with... dropped}}{=} -mc\int_a^b ds\, \frac{d\delta x^i}{ds} u_i$$

$$= -mc\left[\delta x^i u_i\right]_a^b + mc\int_a^b ds\, \delta x^i \frac{du_i}{ds}$$

δx^i at endpoints vanish

$$S' - S \overset{!}{=} 0 \implies mc\int ds\, \delta x^i \frac{du_i}{ds} = 0 \quad \text{for all } \delta x^i$$

$$\implies \frac{du^i}{ds} = 0$$

Without external force the four-velocity remains constant.

(Boring physics: Nothing happens except rectilinear motion)

$u^i = \left(\gamma, \frac{\vec{v}}{c}\gamma\right)$ four-velocity

From before

$p^i = mc\, u^i$ four-momentum

$$p^i = (mc\gamma, m\vec{v}\gamma) = \Big(\underbrace{\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}}_{\mathcal{E}/c \text{ in L12}}, \underbrace{\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}}_{\vec{p}\text{ (relativistic momentum in L12)}}\Big)$$

$$p^i p_i = (mc\gamma, mc\vec{\beta}\gamma)(mc\gamma, -mc\vec{\beta}\gamma)$$

$$= m^2c^2\gamma^2 - m^2c^2\beta^2\gamma^2 = m^2c^2\gamma^2(1-\beta^2) = m^2c^2$$

$$p^i p_i = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)\left(\frac{\mathcal{E}}{c}, -\vec{p}\right) = \frac{\mathcal{E}^2}{c^2} - \vec{p}^{\,2}$$ (Lorentz invar)

$$\boxed{\Rightarrow \quad m^2c^4 = \mathcal{E}^2 - c^2\vec{p}^{\,2}}$$

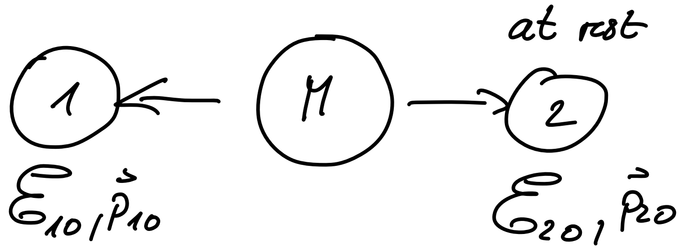
$g^i = \frac{dp^i}{ds}$ four-force

§11 Decay of particles LL2

Velocity, momentum, position, energy

$$\mathcal{E}^2 = m^2c^4 + c^2\vec{p}^{\,2}$$

particle with restmass M at rest



... decays into two particles of energy $\mathcal{E}_{10}$ $\mathcal{E}_{20}$

From earlier considerations

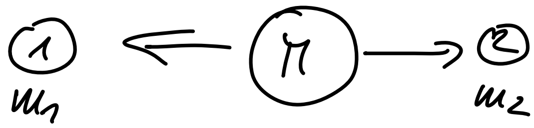
$$0 = \vec{p}_{10} + \vec{p}_{20} \quad \text{(momentum conservation)}$$

and energy conservation

$$Mc^2 = \mathcal{E}_{10} + \mathcal{E}_{20} \qquad\qquad \mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Next: Decay, collision, then charges

§11 Decay of particles



Energy conservation

$$\underline{\mathcal{E}_{00} = Mc^2 = \mathcal{E}_{10} + \mathcal{E}_{20}}$$

$$\left.\begin{aligned} \mathcal{E}_{10} &= \frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} = m_1 c^2 \gamma_1 \\ \mathcal{E}_{20} &= \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}} = m_2 c^2 \gamma_2 \end{aligned}\right\}$$

Momentum conservation

$$\underline{\vec{p}_0 = 0 = \vec{p}_1 + \vec{p}_2}$$

$$\vec{p}_1 = m_1 \vec{v}_1 \gamma_1$$

$$\vec{p}_2 = m_2 \vec{v}_2 \gamma_2$$

$\Rightarrow \quad \vec{v}_1 \parallel \vec{v}_2$

Summary:
$$m_1 \vec{v}_1 \gamma_1 + m_2 \vec{v}_2 \gamma_2 = 0$$
$$m_1 c^2 \gamma_1 + m_2 c^2 \gamma_2 = Mc^2$$
$$\Downarrow$$
$$m_1 \gamma_1 + m_2 \gamma_2 = M$$

Pick preferred direction (x)

$$m_1 v_1 \gamma_1 + m_2 v_2 \gamma_2 = 0$$

How are M, m_1 and m_2 related.

$$\gamma_{1,2} \geq 1$$

$$\Rightarrow \quad m_1 + m_2 \leq M$$

What are v_1, v_2 with M, m_1, m_2 given

Energy cons. $\quad m_1 \gamma_1 + m_2 \gamma_2 = M \quad \Rightarrow \quad \gamma_2 = \dfrac{M - m_1 \gamma_1}{m_2}$

Momentum cons. $\quad m_1 v_1 \gamma_1 = - m_2 v_2 \gamma_2$

$$= - \cancel{m_2} v_2 \frac{M - m_1 \gamma_1}{\cancel{m_2}}$$

$$m_1 v_1 \gamma_1 = - v_2 \left(M - m_1 \gamma_1 \right)$$

$$\boxed{m_1 \gamma_1 (v_1 - v_2) = - v_2 M}$$

Classical limit: Expect $v_1, v_2 \ll c$ $\quad (\Rightarrow \gamma_{1,2} \approx 1)$

$$v_1 = - v_2$$

$$m_1 = m_2 = \tfrac{1}{2} M$$

Check:

$$m_1 \cdot 1 \, (-2 v_2) = - v_2 M$$

$$m_1 = \tfrac{1}{2} M$$

$$\Rightarrow \quad m_1 \gamma_1 v_1 = - v_2 \left(M - m_1 \gamma_1 \right)$$

Is it possible that $\Pi < m_1 \gamma_1$ (not possible as $\Pi = \underbrace{m_1\gamma_1}_{>0} + \underbrace{m_2\gamma_2}_{>0}$

Both velocities +ve is not possible.
(not by momentum conservation anyway)

§13 Elastic collision of particles

(in 4-vector notation)

$$\underbrace{p_1^i + p_2^i}_{\text{before collision}} = \underbrace{p_1'^i + p_2'^i}_{\text{after collision}}$$

$$p^i = \left(\frac{\mathcal{E}}{c}, \vec{p}\right) \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_2'^i = p_1^i + p_2^i - p_1'^i$$

What is $p_1^i p_{1i} = \frac{\mathcal{E}^2}{c^2} - p_1^2 = m_1^2 c^2$

$\stackrel{?}{=} m^2(c^2 - v_1^2)$

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{\mathcal{E}^2}{c^2} - \vec{p}^{\,2} = \frac{m^2c^4}{c^2\left(1-\frac{v^2}{c^2}\right)} - \frac{m^2v^2}{1-\frac{v^2}{c^2}}$$

$$= \frac{m^2c^4 - m^2v^2c^2}{c^2 - v^2} = m^2c^2$$

Important: $\boxed{m^2c^4 = \mathcal{E}^2 - \vec{p}^{\,2}c^2}$

$$p_2'^{\,i} = p_1^i + p_2^i - p_1'^{\,i}$$

$$p_2'^{\,i} p'_{2i} = \left(p_1^i + p_2^i - p_1'^{\,i}\right)\left(p_{1i} + p_{2i} - p'_{1i}\right)$$

$$\Rightarrow \quad m_2'^{\,2}c^2 = m_1^2c^2 + m_2^2c^2 + m_1'^{\,2}c^2 + 2p_1^i p_{2i} - 2p_2^i p'_{1i} - 2p_1'^{\,i} p_{1i}$$

Next: Assume $m_1' = m_1$, $m_2' = m_2$ to make further progress.

Classical And?

$$\vec{p}_1{}' \cdot \vec{p}_2{}' = 0$$

$$\underbrace{p_1 + p_2}_{\substack{\shortparallel \\ 0}} = p_1{}' + p_2{}'$$

$$\underbrace{p_1{}^2 = \vec{p}_1{}'^2 + p_2{}'^2}_{\text{energies}} + 2\vec{p}_1{}'\vec{p}_2{}'$$

From yesterday:

$$m_2'^2 c^2 = m_1^2 c^2 + m_2^2 c^2 + m_1'^2 c^2$$
$$+ 2 p_1^i p_{2i} - 2 p_2^i p_{1i}' - 2 p_1'^i p_{1i}$$

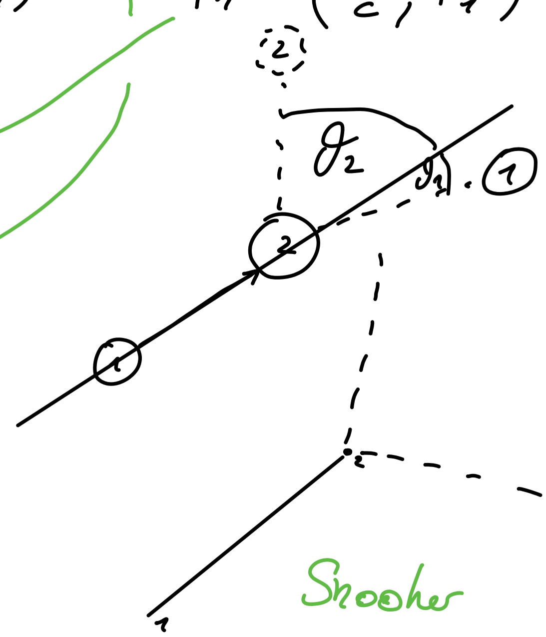
Now assume $m_1 = m_1' \wedge m_2 = m_2'$

$$0 = m_1^2 c^2 + p_1^i p_{2i} - p_2^i p_{1i}' - p_1'^i p_{1i}$$
$$0 = m_2^2 c^2 + p_2^i p_{1i} - p_1^i p_{2i}' - p_2'^i p_{2i}$$

Further assumption: $\vec{v}_2 = 0$ $m_2 c$ (at rest!)

$$p_2^i = \left(\frac{\mathcal{E}_2}{c}, 0\right) \qquad p_2'^i = \left(\frac{\mathcal{E}_2'}{c}, \vec{p}_2'\right)$$
$$p_1^i = \left(\frac{\mathcal{E}_1}{c}, \vec{p}_1\right) \qquad p_1'^i = \left(\frac{\mathcal{E}_1'}{c}, \vec{p}_1'\right)$$

$$p_1^i p_{2i} = \frac{\mathcal{E}_1 \mathcal{E}_2}{c^2}$$
$$p_2^i p_{1i}' = \frac{\mathcal{E}_1' \mathcal{E}_2}{c^2}$$
$$p_2'^i p_{2i} = \frac{\mathcal{E}_2' \mathcal{E}_2}{c^2}$$
$$p_1'^i p_{1i} = \frac{\mathcal{E}_1 \mathcal{E}_1'}{c^2} - \vec{p}_1 \vec{p}_1'$$
$$p_1^i p_{2i}' = \frac{\mathcal{E}_1 \mathcal{E}_2'}{c^2} - \vec{p}_1 \cdot \vec{p}_2'$$



Classically $m_1 = m_2 = m$

$\vec{p}_2 = \vec{0}$

$\vec{p}_1$

$\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$ momentum cons.

$$\frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} = \frac{\vec{p}_1^{\,\prime 2}}{2m} + \frac{\vec{p}_2^{\,\prime 2}}{2m}$$ energy conservation

$$\vec{p}_1^{\,2} + \vec{p}_2^{\,2} = \vec{p}_1^{\,\prime 2} + \vec{p}_2^{\,\prime 2}$$

$$\cancel{\vec{p}_1^{\,2}} + \cancel{\vec{p}_2^{\,2}} + \underbrace{2\vec{p}_1 \cdot \vec{p}_2}_{=0} = \cancel{\vec{p}_1^{\,\prime 2}} + \cancel{\vec{p}_2^{\,\prime 2}} + 2\vec{p}_1{}' \cdot \vec{p}_2{}'$$

$$2\vec{p}_1{}' \cdot \vec{p}_2{}' = 0$$

$$p_1^{\prime i} p_{1i} = \frac{\mathcal{E}_1 \mathcal{E}_1'}{c^2} - \vec{p}_1 \vec{p}_1{}' = \frac{\mathcal{E}_1 \mathcal{E}_1'}{c^2} - p_1 p_1' \cos\vartheta_1$$

$$p_1^{i} p_{2i}' = \frac{\mathcal{E}_1 \mathcal{E}_2'}{c^2} - \vec{p}_1 \cdot \vec{p}_2{}' = \frac{\mathcal{E}_1 \mathcal{E}_2'}{c^2} - p_1 p_2' \cos\vartheta_2$$

$$0 = m_1^2 c^2 + \frac{\mathcal{E}_1 \mathcal{E}_2}{c^2} - \frac{\mathcal{E}_1' \mathcal{E}_2}{c^2} - \left(\frac{\mathcal{E}_1 \mathcal{E}_1'}{c^2} - p_1 p_1' \cos\vartheta_1 \right)$$

$$0 = m_2^2 c^2 + \frac{\mathcal{E}_1 \mathcal{E}_2}{c^2} - \left(\frac{\mathcal{E}_1 \mathcal{E}_2'}{c^2} - p_1 p_2' \cos\vartheta_2 \right) - \frac{\mathcal{E}_2' \mathcal{E}_2}{c^2}$$

$$\frac{\frac{\mathcal{E}_1' \mathcal{E}_2}{c^2} + \frac{\mathcal{E}_1 \mathcal{E}_1'}{c^2} - m_1^2 c^2 - \frac{\mathcal{E}_1 \mathcal{E}_2}{c^2}}{p_1 p_1'} = \cos\vartheta_1$$

$$\frac{\frac{\mathcal{E}_2'\mathcal{E}_2}{c^2}+\frac{\mathcal{E}_1\mathcal{E}_1'}{c^2}-m_1^2c^2-\frac{\mathcal{E}_1\mathcal{E}_2}{c^2}}{p_1p_2'}=\cos\vartheta_2$$

$$\mathcal{E}^2=m^2c^4+p^2c^2$$

$$p_1=\sqrt{\frac{\mathcal{E}_1^2}{c^2}-m_1^2c^2}$$

$$p_1'=\sqrt{\frac{\mathcal{E}_1'^2}{c^2}-m_1^2c^2}$$

$$p_2'=\sqrt{\frac{\mathcal{E}_2'^2}{c^2}-m_2^2c^2}$$

Not too bad! All we need to know on top of m_1, m_2, $\mathcal{E}_1, \mathcal{E}_2$ (given by experiment) are $\mathcal{E}_1'$ for $\cos\vartheta_1$ and $\mathcal{E}_2'$ for $\cos\vartheta_2$.

$$\frac{\frac{\mathcal{E}_1'\mathcal{E}_2}{c^2}+\frac{\mathcal{E}_1\mathcal{E}_1'}{c^2}-m_1^2c^2-\frac{\mathcal{E}_1\mathcal{E}_2}{c^2}}{\sqrt{\frac{\mathcal{E}_1^2}{c^2}-m_1^2c^2}\sqrt{\frac{\mathcal{E}_1'^2}{c^2}-m_1^2c^2}}=\cos\vartheta_1$$

And similarly for $\cos\vartheta_2$

$$\Rightarrow\left(\frac{\mathcal{E}_1'\mathcal{E}_2}{c^2}+\frac{\mathcal{E}_1\mathcal{E}_1'}{c^2}-m_1^2c^2-\frac{\mathcal{E}_1\mathcal{E}_2}{c^2}\right)^2$$

$$=\cos^2\vartheta_1\left(\frac{\mathcal{E}_1^2}{c^2}-m_1^2c^2\right)\left(\frac{\mathcal{E}_1'^2}{c^2}-m_1^2c^2\right)$$

... and similar for $\cos^2\vartheta_2$

L16 Completing elastic collision of particles

$$\frac{\frac{\mathcal{E}_2'\mathcal{E}_2}{c^2}+\frac{\mathcal{E}_1\mathcal{E}_2'}{c^2}-m_2^2c^2-\frac{\mathcal{E}_1\mathcal{E}_2}{c^2}}{p_1p_2'}=\cos\vartheta_2$$

$$\frac{\mathcal{E}_2^2}{c^2}$$

$$\mathcal{E}^2=m^2c^4+p^2c^2$$

$$p_1=\sqrt{\frac{\mathcal{E}_1^2}{c^2}-m_1^2c^2}$$

$$p_1'=\sqrt{\frac{\mathcal{E}_1'^2}{c^2}-m_1^2c^2}$$

$$p_2'=\sqrt{\frac{\mathcal{E}_2'^2}{c^2}-m_2^2c^2}$$

$$\frac{\mathcal{E}_2'\mathcal{E}_2}{c^2}+\frac{\mathcal{E}_1\mathcal{E}_2'}{c^2}-m_2^2c^2-\frac{\mathcal{E}_1\mathcal{E}_2}{c^2}=\sqrt{\frac{\mathcal{E}_1^2}{c^2}-m_1^2c^2}\times\sqrt{\frac{\mathcal{E}_2'^2}{c^2}-m_2^2c^2}\cos\vartheta_2$$

$$\left[\mathcal{E}_2'(\mathcal{E}_2+\mathcal{E}_1)-(m_2^2c^4+\mathcal{E}_1\mathcal{E}_2)\right]^2=(\mathcal{E}_1^2-m_1^2c^4)(\mathcal{E}_2'^2-m_2^2c^4)\cos^2\vartheta_2$$

$$\mathcal{E}_2'^2\left[(\mathcal{E}_2+\mathcal{E}_1)^2-\cos^2\vartheta_2(\mathcal{E}_1^2-m_1^2c^4)\right]$$

$$-2\mathcal{E}_2'(\mathcal{E}_2+\mathcal{E}_1)(m_2^2c^4+\mathcal{E}_1\mathcal{E}_2)$$

$$= -\left(m_2^2c^4 + \mathcal{E}_1\mathcal{E}_2\right)^2 - m_2^2c^4\left(\mathcal{E}_1^2 - m_1^2c^4\right)\cos^2\vartheta_2$$

Stop here! What a mess. Let's simplify the numerator

$$\frac{\mathcal{E}_2'\mathcal{E}_2}{c^2} + \frac{\mathcal{E}_1\mathcal{E}_2'}{c^2} - m_2^2c^2 - \frac{\mathcal{E}_1\mathcal{E}_2}{c^2} \quad \frac{\mathcal{E}_2^2}{c^2}$$

$$= \frac{1}{c^2}\left(\mathcal{E}_2'\mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_2' - \mathcal{E}_2\mathcal{E}_2 - \mathcal{E}_1\mathcal{E}_2\right)$$

$$= \frac{1}{c^2}\ \mathcal{E}_2'(\mathcal{E}_2 + \mathcal{E}_1) - \mathcal{E}_2(\mathcal{E} + \mathcal{E}_1)$$

$$= \frac{1}{c^2}\left(\mathcal{E}_2' - \mathcal{E}_2\right)\left(\mathcal{E}_2 + \mathcal{E}_1\right)$$

$$\frac{\left(\mathcal{E}_2' - \mathcal{E}_2\right)\left(\mathcal{E}_2 + \mathcal{E}_1\right)}{c^2 p_1 p_2'} = \cos\vartheta_2 \qquad (\mathcal{E}_2' - \mathcal{E}_2)(\mathcal{E}_2' + \mathcal{E}_2)$$

$$\left(\mathcal{E}_2' - \mathcal{E}_2\right)^2\left(\mathcal{E}_2 + \mathcal{E}_1\right)^2 = \left(\mathcal{E}_1^2 - m_1^2c^4\right)\overbrace{\left(\mathcal{E}_2'^2 \ \mathcal{E}_2^2 \cancel{- m_2^2c^4}\right)}\cos^2\vartheta_2$$

$$\Rightarrow \left(\mathcal{E}_2' - \mathcal{E}_2\right)\left(\mathcal{E}_2 + \mathcal{E}_1\right)^2 = \left(\mathcal{E}_1^2 - m_1^2c^4\right)\left(\mathcal{E}_2' + \mathcal{E}_2\right)\cos^2\vartheta_2$$

$$\mathcal{E}_2'\left[\left(\mathcal{E}_2 + \mathcal{E}_1\right)^2 - \left(\mathcal{E}_1^2 - m_1^2c^4\right)\cos^2\vartheta_2\right]$$

$$= \mathcal{E}_2\left[\left(\mathcal{E}_2 + \mathcal{E}_1\right)^2 + \left(\mathcal{E}_1^2 - m_1^2c^4\right)\cos^2\vartheta_2\right]$$

$$\frac{(\mathcal{E}_2 + \mathcal{E}_1)^2 + (\mathcal{E}_1^2 - m_1^2c^4)\cos^2\vartheta_2}{}$$

$$\mathcal{E}_2' = \mathcal{E}_2 \, \frac{\quad}{(\mathcal{E}_2+\mathcal{E}_1)^2 - (\mathcal{E}_1^2 - m_1^2c^4)\cos^2\vartheta_2}$$

Let's move on!

Ch2 §15 Elementary particle in the theory of relativity

LL2

§16

(coupling charge)

$$S = -\int_a^b ds\, mc - \frac{e}{c}\int_a^b dx^i \, \overbrace{A_i(x^i)}^{\text{four potential}}$$

vector potential

$dx^i = (c\,dt, d\vec{r})$ $\qquad$ $\left[A^i = (\phi, \vec{A}) \right]$

scalar potential

$$= -\int_a^b ds\, mc - \frac{e}{c}\int_a^b dt\, c\,\phi + \frac{e}{c}\int_a^b d\vec{r}\cdot\vec{A}(\vec{r})$$

dt measured in the frame of reference of moving particle

$$= -\int_a^b ds\, mc - e\int_a^b dt\,\phi + \frac{e}{c}\int_a^b dt\, \vec{v}\cdot\vec{A}(\vec{r})$$

$ds = c\sqrt{1-\frac{v^2}{c^2}}\,dt$

$$= -\int_{t_a}^{t_b} dt \left[mc^2\sqrt{1-\frac{v^2}{c^2}} + e\phi - \frac{e}{c}\vec{A}\cdot\vec{v} \right]$$

$$L(\vec{r},\vec{v},t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi(\vec{r}) + \frac{e}{c}\vec{A}(\vec{r})\cdot\vec{v}$$

EL : $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

$\uparrow \qquad\qquad \uparrow$

$r \qquad\qquad v$

generalised Momentum : $\vec{P} = \nabla_{\vec{v}} L$

Energy : $\mathcal{H} = \vec{v} \cdot \nabla_{\vec{v}} L - L$

$$\boxed{\vec{\nabla}_{\vec{v}} L = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}(\vec{r})}$$

.... Carry on on Monday.

§16 Four-potential of a field

From Friday: $L(\vec{r}, \vec{v}, t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi(\vec{r}) + \frac{e}{c}\vec{A}(\vec{r})\cdot\vec{v}$

EL: $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

$$\underset{r}{\uparrow} \qquad\qquad \underset{v}{\uparrow}$$

generalised Momentum: $\vec{P} = \nabla_{\vec{v}} L$

energy: $\mathcal{H} = \vec{v}\cdot\nabla_{\vec{v}} L - L$

$$\boxed{\nabla_{\vec{v}} L = \vec{P} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}(\vec{r})}$$

momentum

$$\mathcal{H} = \vec{v}\cdot\nabla_{\vec{v}} L - L$$

$$= \vec{v}\cdot\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec{v}\cdot\vec{A} - \left(-mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi(\vec{r}) + \frac{e}{c}\vec{A}(\vec{r})\cdot\vec{v}\right)$$

$$= \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\left(v^2 + c^2\left(1-\frac{v^2}{c^2}\right)\right) + e\phi(\vec{r})$$

$$\boxed{\mathcal{H} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + e\phi(\vec{r})}$$

energy

scalar potential

$[m] = kg$

$[c] = \frac{m}{s}$

$[e] = As$

$[\phi] = V = \frac{kg\cdot\frac{m^2}{s^2}}{As}$

Two conserved quantities $\square$

§17 Equations of motion of a charge in a field

$$L(\vec r, \vec v, t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi(\vec r) + \frac{e}{c}\vec A(\vec r)\cdot\vec v$$

EL: $$\nabla_{\vec r} L = \frac{d}{dt}\nabla_{\vec v} L$$

$$\nabla_{\vec v} L = \frac{m\vec v}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec A(\vec r)$$

Ouch! $\vec A = \vec A(\vec r, t)$

$$\nabla_{\vec r} L = -e\nabla_{\vec r}\phi + \frac{e}{c}\nabla_{\vec r}\left(\vec A(\vec r)\cdot\vec v\right)$$

$$\nabla(\vec f\cdot\vec g) = (\vec f\cdot\vec\nabla)\vec g + (\vec g\cdot\vec\nabla)\vec f + \vec g\times(\vec\nabla\times\vec f) + \vec f\times(\vec\nabla\times\vec g)$$

$$\nabla_{\vec r}\left(\vec A(\vec r, t)\cdot\vec v\right) = (\vec v\cdot\vec\nabla)\vec A + \vec v\times(\vec\nabla\times\vec A)$$

$$\nabla_{\vec r} L = -e\nabla_{\vec r}\phi(\vec r, t) + \frac{e}{c}\left\{(\vec v\cdot\vec\nabla)\vec A + \vec v\times(\vec\nabla\times\vec A)\right\}$$

EL: $$\nabla_{\vec r} L = \frac{d}{dt}\nabla_{\vec v} L = \frac{d}{dt}\left\{\underbrace{\frac{m\vec v}{\sqrt{1-\frac{v^2}{c^2}}}}_{\vec p} + \frac{e}{c}\vec A(\vec r)_{,t}\right\}$$

$$\|$$

$$-e\nabla\phi + \frac{e}{c}\left((\vec v\cdot\vec\nabla)\vec A + \vec v\times(\vec\nabla\times\vec A)\right)$$

RHS: $\dot{\vec{p}} + \frac{e}{c}\partial_t \vec{A} + \frac{e}{c}\,(\vec{v}\cdot\vec{\nabla}_r)\vec{A}$

$$\vec{A} = \big(A_x(x,y,z),\ A_y(x,y,z),\ A_z(x,y,z)\big)$$

$$\frac{d}{dt}\vec{A} = \left(\frac{d}{dt}A_x,\ \frac{d}{dt}A_y,\ \frac{d}{dt}A_z\right)$$

$$\frac{d}{dt}A_x = \underbrace{\dot{x}\,\partial_x A_x + \dot{y}\,\partial_y A_x + \dot{z}\,\partial_z A_x}_{\vec{v}\cdot\vec{\nabla}_r}$$

$$\frac{d}{dt}A_y = \dot{x}\,\partial_x A_y + \dot{y}\,\partial_y A_y + \dot{z}\,\partial_z A_y$$

A_z . . . .

$$\frac{d}{dt}\vec{A} = (\dot{\vec{v}}\cdot\vec{\nabla}_r)\,\vec{A}$$

LHS: $-e\nabla\phi + \cancel{\frac{e}{c}\big((\vec{v}\cdot\vec{\nabla})\vec{A}} + \vec{v}\times(\vec{\nabla}\times\vec{A})\big)$

RHS: $= \dot{\vec{p}} + \frac{e}{c}\partial_t\vec{A} + \cancel{\frac{e}{c}\,(\vec{v}\cdot\vec{\nabla}_r)\vec{A}}$

$$\dot{\vec{p}} = -\frac{e}{c}\partial_t\vec{A}(\vec{r},t) - e\nabla_r\phi(\vec{r},t) + \frac{e}{c}\vec{v}\times(\vec{\nabla}_r\times\vec{A}(\vec{r},t))$$

Lorentz force

Electric field $\vec{E}(\vec{r},t) = -\frac{1}{c}\partial_t\vec{A} - \vec{\nabla}_r\phi$

Magnetic field $\vec{H}(\vec{r},t) = \vec{\nabla}_r\times\vec{A}$

$$\boxed{\dot{\vec{p}} = e\vec{E}(\vec{r},t) + \frac{e}{c}\vec{v}\times\vec{H}(\vec{r},t)}$$

How does the kinetic energy change in time?

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad ?$$

I want to show that $\frac{d}{dt}\mathcal{E} = \vec{v}\cdot\dot{\vec{p}}$

$$\frac{d}{dt}\mathcal{E} = m\frac{c^2}{c^2}\frac{\dot{\vec{v}}\cdot\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow \quad \dot{\vec{p}} = \frac{m\dot{\vec{v}}}{\sqrt{}} + m\vec{v}\frac{\dot{\vec{v}}\cdot\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$\vec{v}\cdot\dot{\vec{p}} = \frac{m\dot{\vec{v}}\cdot\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + m\frac{v^2}{c^2}\frac{\dot{\vec{v}}\cdot\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$= \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\dot{\vec{v}}\cdot\vec{v}\left(1+\frac{v^2/c^2}{1-\frac{v^2}{c^2}}\right)$$

$$= \frac{m\,\dot{\vec{v}}\cdot\vec{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{d}{dt}\mathcal{E}$$

indeed

Hurray!

$$\boxed{\frac{d}{dt}\mathcal{E} = \vec{v}\cdot\dot{\vec{p}}}$$

Next write $\vec{v}\cdot\dot{\vec{p}}$ in terms of the Lorentz force.

Canals!

Discussion of CW1

§18 Gauge invariance

Let's consider a four-potential shifted by $\partial_\mu f$

$$\partial_\mu f = \frac{\partial}{\partial x^\mu} f = \left(\frac{\partial f}{c\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$A'_\mu = A_\mu - \partial_\mu f$$

$$A^\mu = (\phi, \vec{A})$$

scalar potential

vector potential

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$\vec{H} = \vec{\nabla} \times \vec{A}$$

1) Determine shift of ϕ and $\vec{A}$ due to f

2) Plug into eqns for $\vec{E}$ and $\vec{H}$ and see what happens.

⊗ Post past problem sheets + course work

$$\int dx^i A_i = \int ds \frac{dx^i}{ds} A_i$$

four - potential

§18 continued

$$S = \int ds\,(-mc) - \frac{e}{c}\int dx^i A_i$$

A_i can be changed without changing the physics.

18.1 $\quad A'_\mu = A_\mu - \partial_\mu f$

Show that E and H that derive from A' are identical to E and H from A

$$A^\mu = (\phi, \vec{A}) \qquad A_\mu = (\phi, -\vec{A})$$

$$\boxed{\begin{aligned}\vec{E} &= -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi \\ \vec{H} &= \vec{\nabla}\times\vec{A}\end{aligned}}$$

Let's calculate $\vec{E}'$, $\vec{H}'$

$$\vec{E}' = -\frac{1}{c}\partial_t \vec{A}' - \vec{\nabla}\phi'$$

What are ϕ' and $\vec{A}'$?

$$A^\mu = (\phi, \vec{A})$$

$$A'^\mu = A^\mu - \partial^\mu f = A^\mu - \frac{\partial}{\partial x_\mu} f$$

$$\left[\begin{array}{l} \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \vec{\nabla}\right) \\ \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right) \end{array}\right.$$

$$A'^\mu = A^\mu - \left(\frac{\partial}{c\partial t} f, -\vec{\nabla} f\right)$$

$$\Rightarrow \quad \phi' = A'^0 = \phi - \frac{1}{c}\partial_t f$$

$$\vec{A}' = \vec{A} + \vec{\nabla} f$$

$$\Rightarrow \quad \vec{E}' = -\frac{1}{c}\partial_t \vec{A}' - \vec{\nabla}\phi'$$

$$= -\frac{1}{c}\partial_t \vec{A} \underbrace{- \frac{1}{c}\partial_t \vec{\nabla} f - \vec{\nabla}\phi + \vec{\nabla}\frac{1}{c}\partial_t f}_{\text{cancel}}$$

$$= \vec{E}$$

$$\vec{H}' = \vec{\nabla} \times \vec{A}'$$

$$= \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla} f)}_{\text{vanishes (curl of grad = 0)}} = \vec{H}$$

§19 Constant electromagnetic field

not time-dependent

(uniform will mean not space dependent)

Let's find a scalar and vector potential that accomodate given electric and magnetic fields.

$$\rightarrow \quad \vec{E} = -\vec{\nabla}\phi$$
$$\vec{H} = \vec{\nabla} \times \vec{A}$$

Let's consider the uniform case. $\vec{E}(t,x,y,z) = \vec{E}_0$

$\vec{H}(t,x,y,z) = \vec{H}_0$

$$\cancel{\phi = -\vec{E}_0 (x+y+z)}$$

$$\cancel{\phi \sim E_{0x} x = E_{0y} y = E_{0z} z}$$

$$\cancel{E_{0y} = \partial_y \phi}$$

$$\phi = -\left(E_{0x} x + E_{0y} y + E_{0z} z\right) = -\vec{E}_0 \cdot \vec{r}$$

$$\vec{\nabla} \times \vec{A} = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \vec{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

$$\vec{A} = (\vec{C}_1 \cdot \vec{r})\,\vec{e}_1 + (\vec{C}_2 \cdot \vec{r})\,\vec{e}_2 + (\vec{C}_3 \cdot \vec{r})\,\vec{e}_3$$
$$= C_{ij}\,\vec{r}$$

$\vec{\nabla} \times \vec{r} = 0$ Wild guess:

$$\vec{\nabla} \times (C_{ij}\,\vec{r}) = 0$$

$\vec{\nabla} \times (\vec{D} \times \vec{r})$?

In general

$$\vec{\nabla} \times (\vec{f} \times \vec{g}) = -(\vec{f} \cdot \vec{\nabla})\,\vec{g} + \vec{g}\,(\vec{\nabla} \cdot \vec{f})$$
$$-(\vec{g} \cdot \vec{\nabla})\,\vec{f} + \vec{f}\,(\vec{\nabla} \cdot \vec{g})$$

Signs?!

$$\vec{\nabla} \times (\vec{D} \times \vec{r}) = -\underbrace{(\vec{D} \cdot \vec{\nabla})}_{D_x\partial_x + D_y\partial_y + D_z\partial_z}\vec{r} + \vec{D}\underbrace{(\vec{\nabla} \cdot \vec{r})}_{3}$$
$$= -\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} + 3\,\vec{D} = 2\,\vec{D}$$

Choose $\vec{A} = \frac{1}{2}\,\vec{H}_0 \times \vec{r}$ then $\vec{\nabla} \times \vec{A} = \vec{H}_0$

Use this as a playground

Next up: §20 motion in constant + uniform electric field.

$$p'^i p_{1i} = p'^i p_{2i} \quad \not\Rightarrow \quad p_{1i} = p_{2i}$$

$$p_1^i p_{2i} = p_1'^i p_{2i}'$$

§20 Motion in a $\underbrace{\text{constant}}_{\text{constant in time}}$ $\underbrace{\text{uniform}}_{\text{constant in space}}$ electric field

$$\vec{E}(\vec{r},t) = \vec{E}$$
$$\vec{H}(\vec{r},t) = 0 \leftarrow \text{zero}$$

$$\dot{\vec{p}} = e\vec{E} + \cancel{\frac{e}{c}\vec{v}\times\vec{H}}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Choose $\quad \vec{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}$

Experiment in the x-y plane

$$\boxed{\begin{aligned} p_z &= 0 \\ p_y &= p_0 \\ p_x &= eEt \end{aligned}}$$

How can we relate $\vec{p}$ to $\vec{v}$ easily?

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \rightsquigarrow \vec{p} = \mathcal{E}\vec{v}/c^2$$

$\vec{p}(t) = \mathcal{E}(t)\,\vec{v}(t)/c^2$

What I want is $\vec{v}(t)$ and from there $x(t)$, $y(t)$, $z(t)$

$$\mathcal{E}^2 = \underset{\substack{\uparrow \\ \text{rest mass}}}{m^2}c^4 + c^2p^2 = m^2c^4 + c^2(p_z^2 + p_y^2 + p_x^2)$$

$$= \underbrace{m^2c^4 + c^2p_0^2}_{\mathcal{E}_0^2} + c^2(eEt)^2$$

$$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}$$

$$\frac{c^2\vec{p}(t)}{\sqrt{\mathcal{E}^2 + c^2(eEt)^2}} = \vec{v}(t) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\dot{z} = 0 \qquad z(t) = z_0$$

$$\dot{y} = \frac{c^2p_0}{\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}}$$

$$\dot{x} = \frac{c^2eEt}{\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}} = \frac{1}{eE}\frac{d}{dt}\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}$$

$$\frac{d}{ds}\sinh^{-1}(s) = \frac{1}{\sqrt{1+s^2}}$$

$$\cancel{\frac{d}{ds}\cosh^{-1}(s) \overset{?}{=} \frac{-1}{\sqrt{1+s^2}}}$$

$$\dot{y} = \frac{c^2p_0}{\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}} = \frac{c^2p_0}{\mathcal{E}_0}\,\frac{1}{\sqrt{1 + \frac{c^2(eE)^2}{\mathcal{E}_0^2}t^2}}$$

$$\int_0^t dt\,\dot{y} = \int dt\ \frac{c^2 p_0}{\mathcal{E}_0}\ \frac{1}{\sqrt{1+\frac{c^2(eE)^2}{\mathcal{E}_0^2}t^2}}$$

$$s = \frac{c(eE)}{\mathcal{E}_0}t$$

$$ds = \frac{c(eE)}{\mathcal{E}_0}dt$$

$$= \int_0^{\frac{c(eE)}{\mathcal{E}_0}t} ds\,\frac{\cancel{\mathcal{E}_0}}{\cancel{c}(eE)}\,\frac{\cancel{c^2}p_0}{\cancel{\mathcal{E}_0}}\ \frac{1}{\sqrt{1+s^2}} = \frac{p_0 c}{eE}\Big[\sinh^{-1}(s)\Big]_0^{\frac{c(eE)}{\mathcal{E}_0}t}$$

$$= \frac{p_0 c}{eE}\sinh^{-1}\left(\frac{c(eE)t}{\mathcal{E}_0}\right) = y(t)$$

$$\dot{x} = \frac{c^2 eEt}{\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}} = \frac{1}{eE}\frac{d}{dt}\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}$$

$$\leadsto \quad x(t) = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}$$

Trajectory in time:

$$x(t) = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + c^2(eEt)^2}$$

$$y(t) = \frac{p_0 c}{eE}\sinh^{-1}\left(\frac{c(eE)t}{\mathcal{E}_0}\right)$$

$$z(t) = z_0 \qquad \mathcal{E}_0 \sinh\left(\frac{eE}{p_0 c}y\right) = c(eE)t$$

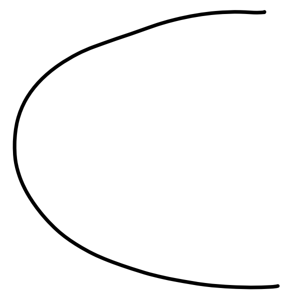
$$x(t) = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + \mathcal{E}_0^2\sinh^2\left(\frac{eE}{p_0 c}y\right)}$$

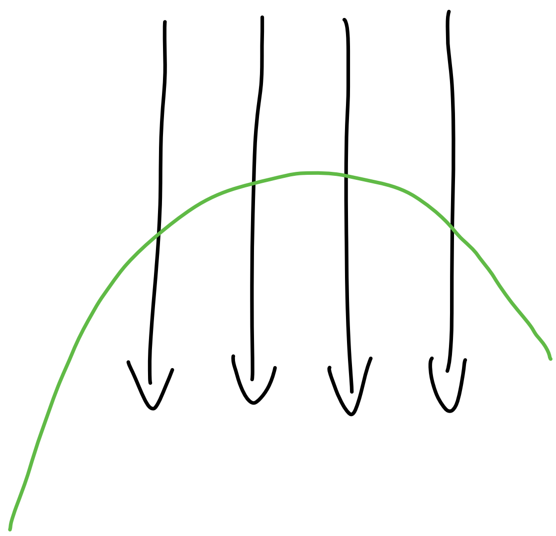
$$= \frac{\mathcal{E}_0}{eE} \sqrt{1 + \sinh^2 \ldots}$$

$$\cosh^2 - \sinh^2 = 1$$

$$= \frac{\mathcal{E}_0}{eE} \cosh\left(\frac{eE}{p_0 c} y\right)$$

$$= \frac{\mathcal{E}_0}{eE} \left(1 + \frac{1}{2}\left(\frac{eE}{p_0 c} y\right)^2 + \ldots\right)$$





Lecture 21:

§21 LL2 Motion in a constant uniform magnetic field

in time, in space

$\vec{H} \neq 0 \qquad \vec{E} = 0$

Lorentz force $\quad \dot{\vec{p}} = \frac{e}{c} \vec{v} \times \vec{H} \qquad$ (no $\vec{E}$ term)

We have seen that $\dot{\mathcal{E}} = e \vec{E} \cdot \vec{v} \implies \dot{\mathcal{E}} = 0$ here

$\mathcal{E}$ is const

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\mathcal{E}\vec{v}}{c^2} \qquad \mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

You can determine v^2 from here, but not $\vec{v}$.

$\mathcal{E}$ const $\implies \dot{\vec{p}} = \frac{\mathcal{E}}{c^2} \dot{\vec{v}}$

Let's turn the coordinate system, so that $\vec{H} \parallel \vec{e}_z$

$$\frac{\mathcal{E}}{c^2} \dot{\vec{v}} = \dot{\vec{p}} = \frac{e}{c} \vec{v} \times \vec{H} = \frac{e}{c} \begin{pmatrix} v_y H_z - v_z H_y \\ v_z H_x - v_x H_z \\ v_x H_y - v_y H_x \end{pmatrix}$$

$\left.\vec{H} = H \vec{e}_z\right]$

$$= \frac{e}{c} \begin{pmatrix} v_y H \\ -v_x H \\ 0 \end{pmatrix}$$

$$\dot{\vec{v}} = \frac{ec}{\mathcal{E}} \begin{pmatrix} v_y H \\ -v_x H \\ 0 \end{pmatrix}$$

$$\dot{v}_x = \frac{ec}{\mathcal{E}} v_y H$$

$$\dot{v}_y = -\frac{ec}{\mathcal{E}} v_x H$$

$$\dot{v}_z = 0$$

$w = v_x + i v_y$

How about
$$\dot{\vec{v}} = M \vec{v}$$

$$\rightsquigarrow \quad \vec{v}(t) = \exp(tM)\,\vec{v}_0$$

$$M = \frac{ecH}{\mathcal{E}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ evals: $\pm i$

$\lambda_+ + \lambda_- = 0$

$\lambda_+ \lambda_- = 1$

$$\vec{a}_+^{\,*} = \begin{pmatrix} -i \\ -1 \end{pmatrix} = -i\vec{a}_-$$

$$\vec{a}_-^{\,*} = \begin{pmatrix} 1 \\ i \end{pmatrix} = -i\vec{a}_+$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} i \\ -1 \end{pmatrix}}_{\vec{a}_+} = i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\vec{a}_-} = -i\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Write $\vec{v}(t) = \vec{a}_+ \alpha_+(t) + \vec{a}_- \alpha_-(t)$

$$\cancel{\vec{v}(t) =} \ \vec{a}_+ \alpha_+(t) + \vec{a}_- \alpha_-(t)$$
$$= e^{tM}\left(\vec{a}_+ \alpha_+(0) + \vec{a}_- \alpha_-(0)\right)$$
$$= e^{ti}\vec{a}_+ \alpha_+(0) + e^{-ti}\vec{a}_- \alpha_-(0)$$

actually
$$\vec{v}(t) = \exp\left(\frac{ecH}{\mathcal{E}} i t\right) \alpha_+(0)\vec{a}_+ + \exp\left(-\frac{ecH}{\mathcal{E}} i t\right) \alpha_-(0)\vec{a}_-$$

$\vec{v} \in \mathbb{R}^2$

$$\vec{v}^*(t) = \vec{v}(t) = \exp\left(-\frac{ecH}{\mathcal{E}} i t\right) \alpha_+^*(0)\, \vec{a}_+^*$$
$$+ \exp\left(\frac{ecH}{\mathcal{E}} i t\right) \alpha_-^*(0)\, \vec{a}_-^*$$

$$\alpha_+^*(0)\, \vec{a}_+^* = \alpha_-(0)\, \vec{a}_-$$
$$\parallel$$
$$-i\, \alpha_+^*(0)\, \vec{a}_- \qquad \Rightarrow \quad \alpha_-(0) = -i\, \alpha_+^*(0)$$

And I suppose
$$\alpha_+(0) = -i\, \alpha_-^*(0)$$

$$\Rightarrow \quad \vec{v}(t) = \exp\left(\frac{ecH}{\mathcal{E}} i t\right) \alpha_+(0)\, \vec{a}_+$$
$$-i \exp\left(-\frac{ecH}{\mathcal{E}} i t\right) \alpha_+^*(0)\, \vec{a}_-$$

$$\frac{ecH}{\mathcal{E}} = \omega$$

Use $-i\vec{a}_- = \vec{a}_+^*$ (Well seen!)

$$\vec{v}(t) = e^{i\omega t} \alpha_+(0)\, \vec{a}_+$$
$$+ e^{-i\omega t} \alpha_+^*(0)\, a_+^*$$
$$= 2\Re\left(e^{i\omega t} \alpha_+(0)\, \vec{a}_+\right)$$
$$= 2\Re\left(w\, e^{i(\omega t + \varphi)} \begin{pmatrix} i \\ -1 \end{pmatrix}\right)$$

real

$$\alpha_+(0) = w e^{i\varphi}$$

w

ω

$$= 2v \begin{pmatrix} -\sin(\omega t+\varphi) \\ -\cos(\omega t+\varphi) \end{pmatrix} = -2v \begin{pmatrix} \sin(\omega t+\varphi) \\ \cos(\omega t+\varphi) \end{pmatrix}$$

Velocity goes in a circle and from

$\dot{\vec{r}} = \vec{v}$ we have

$$\vec{r}(t) = \vec{r}(0) - 2\frac{v}{\omega} \begin{pmatrix} -\cos(\omega t+\varphi) \\ \sin(\omega t+\varphi) \end{pmatrix}$$

Particle goes in a circle in x-y plane and along a spiral path if $v_z \neq 0$

§22 Motion of a charge in constant and uniform electric and magnetic fields

To get rid of $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, we assume $v/c \ll 1$

$$\gamma \approx 1$$
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\dot{\vec{p}} = e\vec{E} + \frac{e}{c}\,\vec{v}\times\vec{H}$$

$$\dot{\vec{p}} \approx m\dot{\vec{v}} \qquad \text{(approximation)}$$

Choose coordinate system (rotate!) such that

$$\vec{H} = H\vec{e}_z$$
$$\vec{E} = \begin{pmatrix} 0 \\ E_y \\ E_z \end{pmatrix}$$

$$m\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = e\begin{pmatrix} 0 \\ E_y \\ E_z \end{pmatrix} + \frac{e}{c}\begin{pmatrix} v_y H \\ -v_x H \\ 0 \end{pmatrix}$$

$$m\,\dot{v}_z = eE_z \quad \Rightarrow \quad \boxed{z = \frac{eE_z}{2m}t^2 + v_{0z}t + z_0}$$

Choose origin of z, s.t. $z_0 = 0$

$$m\ddot{x} = \frac{e}{c}\dot{y}H$$
$$m\ddot{y} = eE_y - \frac{e}{c}\dot{x}H$$
$$mi\ddot{y} = eiE_y - \frac{ei}{c}\dot{x}H$$

$z = x + iy$

↑ This a complex number now. This is no longer the z-component

$$\frac{d}{dt}(\dot{x} + i\dot{y}) = \frac{e}{m}iE_y + \frac{eH}{cm}\underbrace{(\dot{y} - i\dot{x})}_{-i(i\dot{y}+\dot{x})}$$

ℑ ϑ θ

H Oliver Heaviside British Mathematician

$$\frac{d}{dt}\dot{z} = \frac{e}{m}iE_y - \frac{ieH}{cm}\dot{z}$$

$$e^{\frac{ieH}{cm}t}\frac{d}{dt}\dot{z} = e^{\frac{ieH}{cm}t}\frac{e}{m}iE_y - \frac{ieH}{cm}\dot{z}\,e^{\frac{ieH}{cm}t}$$

$$\frac{d}{dt}\left(e^{\frac{ieH}{cm}t}\dot{z}\right) = e^{\frac{ieH}{cm}t}\frac{ei}{m}E_y \qquad \frac{eH}{cm} = \omega$$

$$\frac{d}{dt}\left(e^{i\omega t}\dot{z}\right) = e^{i\omega t}c\omega i\frac{E_y}{H}$$

$$e^{i\omega t}\dot{z} = c\omega i\frac{E_y}{H}\int_{t_0}^{t} dt'\, e^{i\omega t'}$$

$$= \frac{cE_y}{H}\left[e^{i\omega t'}\right]_{t_0}^{t}$$

$$e^{i\omega t}\dot{z}(t) - e^{i\omega t_0}\dot{z}(t_0) = \frac{cE_y}{H}\left(e^{i\omega t} - e^{i\omega t_0}\right)$$

$$\underset{\overset{\parallel}{\dot{x}+i\dot{y}(t)}}{\dot{z}(t)} = \frac{cE_y}{H} - \underbrace{e^{i\omega(t_0-t)}}_{\substack{\cos(\omega(t_0-t))\\ +i\sin(\omega(t_0-t))}}\left(\frac{cE_y}{H} - \underbrace{\dot{z}(t_0)}_{\dot{x}(t_0)+i\dot{y}(t_0)}\right)$$

$$\dot{x}(t) = \frac{cE_y}{H} - \cos(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right) + \sin(\omega(t_0-t))\,\dot{y}(t_0)$$

$$\dot{y}(t) = -\sin(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right) - \cos(\omega(t_0-t))\left(-\dot{y}(t_0)\right)$$

Safe to assume that $\exists\, t_0$ s.t. $\dot{y}(t_0)=0$

Choose $\dot{y}(t_0)=0$

$$\dot{x}(t) = \frac{cE_y}{H} - \cos(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right)$$

$$\dot{y}(t) = -\sin(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right)$$

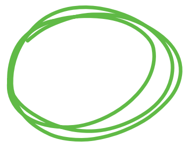
Really! Sign?

$$x(t) = \tilde{x}(t_0) + \frac{cE_y}{H}(t-t_0) - \frac{1}{\omega}\sin(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right)$$

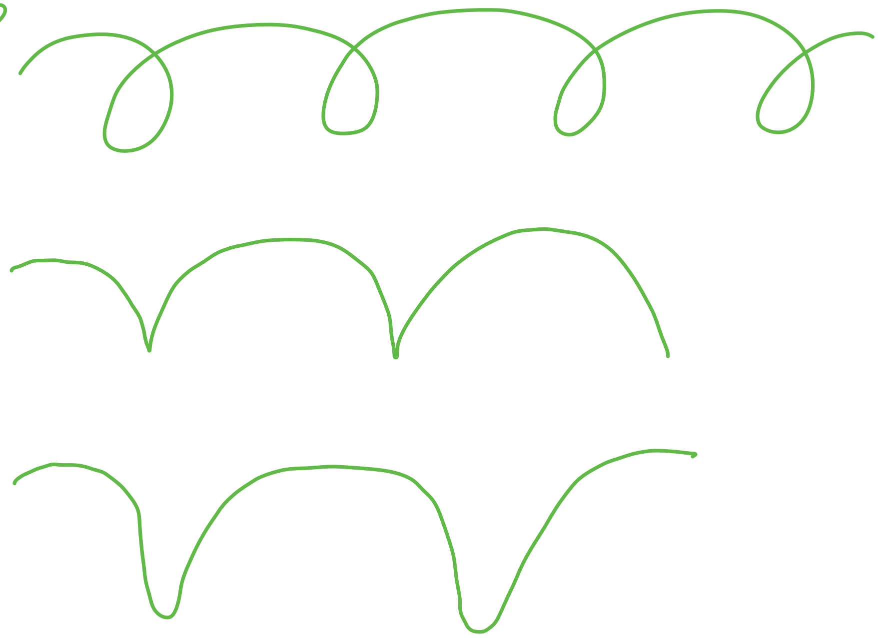
$$y(t) = \underbrace{\tilde{y}(t_0)}_{} + \frac{1}{\omega}\cos(\omega(t_0-t))\left(\frac{cE_y}{H} - \dot{x}(t_0)\right)$$

Includes not only $x(t_0)$, but also the lower limit from the integration on the right.

If $\frac{cE_y}{H} = 0$ then $x(t), y(t)$ is motion in a circle



For $\frac{cE_y}{H} > 0$



$$x(t) = \tilde{x}(t_0) + \frac{cE_y}{H}(t-t_0) - \frac{1}{\omega}\sin(\omega(t_0-t))\left(\frac{cE_y}{H} - \mathring{x}(t_0)\right)$$

$$y(t) = \tilde{y}(t_0) + \frac{1}{\omega}\cos(\omega(t_0-t))\left(\frac{cE_y}{H} - \mathring{x}(t_0)\right)$$

$$z = \frac{eE_z}{2m}t^2 + v_{0z}t + z_0 \qquad \mathring{z} = \frac{eE_z}{m}t + v_{0z}$$

and similarly

$$\frac{v}{c} \ll 1 \qquad \mathring{x} = \frac{cE_y}{H} + \cos(\omega(t_0-t))\left(\frac{cE_y}{H} - \mathring{x}(t_0)\right)$$

$\rightarrow$ To maintain $\frac{v}{c} \ll 1$

I need $\frac{eE_z}{m}t \ll c$

and $\frac{cE_y}{H} \ll c$

§23 The electromagnetic field tensor

We start with the variation of the action

$$\delta S = \delta \int \left(-mc\,ds - \frac{e}{c}A_i\,dx^i\right) \overset{!}{=} 0$$

$$ds = \sqrt{dx_i\,dx^i}$$

$$\delta ds = \frac{dx_i\,d\delta x^i}{\sqrt{dx_i\,dx^i}} = \frac{dx_i\,d\delta x^i}{ds}$$

1) $$\delta \int ds = \int \underbrace{\frac{dx_i}{ds}}_{\substack{u_i \\ \text{(four velocity)}}} d\delta x^i = \int u_i \frac{d\delta x^i}{ds} ds$$

$$= \underbrace{\left[\delta x^i u_i\right]}_{\substack{\text{vanishes because} \\ \delta x^i = 0 \text{ at the} \\ \text{end points.}}} - \int du_i \, \delta x^i$$

(as done before)

2) $$\delta \int A_i dx^i = \int A_i d\delta x^i = \int A_i \frac{d\delta x^i}{ds} ds$$

$$= \underbrace{\left[A_i \delta x^i\right]}_{=0} - \int ds \frac{dA_i}{ds} \delta x^i$$

3) $$\int \delta A_i dx^i = \int dx^i \frac{\partial A_i}{\partial x_k} \delta x_k$$

$$\delta A_i = \frac{\partial A_i}{\partial x_k} \delta x_k$$

$$\frac{dA_i}{ds} = \frac{\partial A_i}{\partial x_k} \frac{dx_k}{ds}$$

$$\delta S = \int du_i \, mc \, \delta x^i + \frac{e}{c} \int ds \frac{\partial A_i}{\partial x_k} \frac{dx_k}{ds} \delta x^i$$

$$- \frac{e}{c} \int dx^i \frac{\partial A_i}{\partial x_k} \delta x_k$$

$$= \int du_i\, mc\, \delta x^i + \frac{e}{c}\int dx_k \frac{\partial A_i}{\partial x_k}\delta x^i - \frac{e}{c}\int dx^i \frac{\partial A_i}{\partial x_k}\delta x_k$$

$$= \int du_i\, mc\, \delta x^i + \frac{e}{c}\int dx_k \frac{\partial A_i}{\partial x_k}\delta x^i - \frac{e}{c}\int dx^k \frac{\partial A_k}{\partial x_i}\delta x_i$$

$$= \int du_i\, mc\, \delta x^i + \frac{e}{c}\int dx^k \frac{\partial A_i}{\partial x^k}\delta x^i - \frac{e}{c}\int dx^k \frac{\partial A_k}{\partial x^i}\delta x^i$$

$$= \int du_i\, mc\, \delta x^i + \frac{e}{c}\int dx^k \underbrace{\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)}_{-F_{ik}}\delta x^i$$

electromagnetic field tensor

From this, derive four-Lorentz force next time.

$$\delta_i S = \int du_i \, mc \, \delta x^i + \frac{e}{c} \int dx^k \underbrace{\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right)}_{-F_{ik}} \delta x^i$$

$$= \int ds \left(\frac{du_i}{ds} mc + \frac{e}{c} \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right) \underbrace{\frac{dx^k}{ds}}_{u^k} \right) \delta x^i$$

$$\stackrel{!}{=} 0$$

EL-like conclusion:

$$\boxed{\frac{du_i}{ds} mc + \frac{e}{c} \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right) u^k = 0}$$

Lorentz-force in four-vector notation

$$\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} = -F_{ik}$$

$$-\frac{\partial A^i}{\partial x_k} + \frac{\partial A^k}{\partial x_i} = F^{ik}$$

$$A^k = (\phi, \vec{A})$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$\vec{H} = \vec{\nabla} \times \vec{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2}$$

$$= \frac{\partial A_y}{-\partial x} - \frac{\partial A_x}{-\partial y} = -H_z$$

$$F^{13} = \frac{\partial A^3}{\partial x_1} - \frac{\partial A^1}{\partial x^3} = -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} = H_y$$

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = \frac{\partial A_x}{c\partial t} + \frac{\partial \phi}{\partial x} = -E_x$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \qquad = \begin{pmatrix} -\frac{1}{c}\partial_t A_x - \partial_x \phi \\ -\frac{1}{c}\partial_t A_y - \partial_y \phi \\ -\frac{1}{c}\partial_t A_z - \partial_z \phi \end{pmatrix}$$

electromagnetic tensor

Transformation follows. (see CW11)

If $V \ll c$ then to leading in $\frac{V}{c}$

$$E_x = E_x'$$
$$E_y = E_y' + \frac{V}{c} H_z'$$
$$E_z = E_z' - \frac{V}{c} H_y'$$

$$H_x = H_x'$$
$$H_y = H_y' - \frac{V}{c} E_z'$$
$$H_z = H_z' + \frac{V}{c} E_y'$$

$$\left.\begin{aligned} \vec{E} &= \vec{E}' + \frac{1}{c}\vec{H}' \times \vec{V} \\ \vec{H} &= \vec{H}' - \frac{1}{c}\vec{E}' \times \vec{V} \end{aligned}\right\}$$ Any velocity in any direction, but $|\vec{V}| \ll c$

Assume that $\vec{H}'=0$ in K', how are $\vec{H}$ and $\vec{E}$ related

$$\vec{E} = \vec{E}'$$
$$\vec{H} = -\frac{1}{c}\vec{E}'\times\vec{V} = \frac{1}{c}\vec{V}\times\vec{E}$$

If $\vec{H}'$ vanishes in K', then there is a frame K, s.t. $\vec{H}$ and $\vec{E}$ are orthogonal in this frame.

Similarly if $\vec{E}'=0$, then $\vec{E} = -\frac{1}{c}\vec{V}\times\vec{H}$

§25 Invariants of the field

$$F^{ik}u_i u_k$$

$$u_i u^i = ? \qquad \text{Exercise!}$$

$$F^{ik}F_{ik} =$$

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \qquad F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}$$

$$= -E_x^2 - E_y^2 - E_z^2 - E_x^2 + H_z^2 + H_y^2$$
$$-E_y^2 + H_z^2 + H_x^2 - E_z^2 + H_y^2 + H_x^2 = 2\vec{H}^2 - 2\vec{E}^2$$

$$F_{ik}\,\varepsilon^{iklm}F_{lm} = -\underset{\sim}{8}\,\vec{H}\cdot\vec{E}$$

Two invariants: 1) $\vec{H}^2 - \vec{E}^2 = \vec{H}'^2 - \vec{E}'^2$

2) $\underbrace{\vec{H} \cdot \vec{E} = \vec{H}' \cdot \vec{E}'}$

If $\vec{H} \perp \vec{E}$ in one frame of reference, then $\vec{H} \perp \vec{E}$ in all frames of reference

Next time: Consider different cases of vanishing fields and study invariants.

Invariants $\vec{E}\cdot\vec{H} = \vec{E}'\cdot\vec{H}'$

$$\vec{E}^2 - \vec{H}^2 = \vec{E}'^2 - \vec{H}'^2$$

If $\vec{E}\cdot\vec{H}=0$ then we can always find a frame of reference such that either $\vec{E}=0$ or $\vec{H}=0$

(To clarify: There is only one such frame where either $\vec{E}$ or $\vec{H}$ vanish, we are not talking about two frames)

If $\vec{E}^2 - \vec{H}^2 > 0$ then $\vec{E}'^2 - \vec{H}'^2 > 0$, so we cannot have $\vec{E}'^2 = 0$.
$\Leftrightarrow \vec{E}=0$

Similarly, if $\vec{E}^2 - \vec{H}^2 < 0$ then we cannot find a frame where $\vec{H}'^2 = 0$.

Let's assume $\vec{E}\cdot\vec{H}=0$ and $\vec{E}\neq 0$. ~~...~~ Let's also assume $\vec{E}^2 - \vec{H}^2 > 0$, so there cannot be a frame with $\vec{E}'=0$, but there can be one with $\vec{H}'=0$.

$$H_x = H_x' \qquad H_y = \frac{H_y' - \frac{V}{c}E_z'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad H_z = \frac{H_z' + \frac{V}{c}E_y'}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$H_x' = H_x \qquad H_y' = \frac{H_y + \frac{V}{c}E_z}{\sqrt{1-\frac{V^2}{c^2}}} \qquad H_z' = \frac{H_z - \frac{V}{c}E_y}{\sqrt{1-\frac{V^2}{c^2}}}$$

Must choose x s.t. $H_x = 0$, otherwise $H_x' \neq 0$

$$\vec{H} = \begin{pmatrix} 0 \\ H_y \\ H_z \end{pmatrix} \qquad \vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

We know $\vec{H}\cdot\vec{E} = H_y E_y + H_z E_z = 0$

But the $H_y + \frac{V}{c}E_z = 0 \overset{?}{\Longleftrightarrow} H_z - \frac{V}{c}E_y = 0$

$$H_y + \frac{V}{c}E_z = 0$$

$$\underbrace{E_y H_y}_{-H_z E_z} + \frac{V}{c}E_z E_y = 0$$

$$\Rightarrow \quad -H_z + \frac{V}{c}E_y = 0$$

(Assumes $E_z \neq 0$)

$\Rightarrow$ The V I need is $H_y + \frac{V}{c}E_z = 0$

But what if $E_z \ll H_y$? $\underbrace{\text{We would need } |V| \gg c!}_{\frac{V}{c} = -\frac{H_y}{E_z}}$

$E^2 - H^2 > 0$ saves us.
(together with $\vec{H}\cdot\vec{E} = 0$)

$\vec{F} = \vec{E} + i\vec{H}$ has the nice property that $\vec{F}\cdot\vec{F}^* = \vec{E}^2 + \vec{H}^2$

and $\boxed{\vec{F}^2 = \vec{E}^2 - \vec{H}^2 + 2\varepsilon\, \vec{E}\cdot\vec{H}}$

Chapter 4 LL2

§26 The first pair of Maxwell's equations

$$\vec{H} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\, \vec{\nabla}\times\vec{A} - \underbrace{\vec{\nabla}\times\vec{\nabla}\phi}_{=0}$$

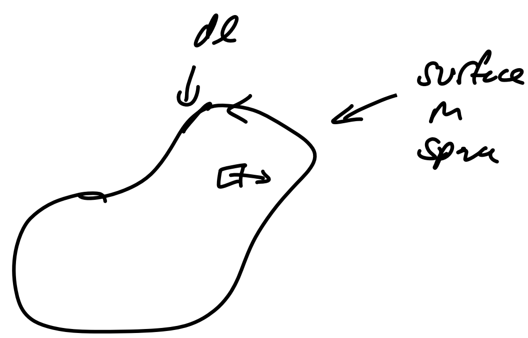
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t \vec{H}$$ Faradays law

$$\vec{\nabla}\cdot\vec{H} = \underbrace{\nabla\cdot\vec{\nabla}\times\vec{A}}_{=0} = 0$$ Gauss' law for magnetism

$$0 = \int d^3r\, \vec{\nabla}\cdot\vec{H} = \overset{\text{closed surface}}{\oint} d\vec{f}\cdot\vec{H}$$

(There are no sinks or sources of magnetic field lines)

$$-\frac{1}{c}\partial_t \overset{\text{open surface}}{\int} d\vec{f}\cdot\vec{H} = \int d\vec{f}\cdot\vec{\nabla}\times\vec{E} = \oint d\vec{l}\cdot\vec{E}$$



In four-notation the two Maxwell equations above can be written as

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

For all ikl

Next: Introduce a new term to the action

⋮

Derive two more Maxwell equations.

§27 The action of the electromagnetic field

$$S = \underset{\substack{\uparrow \\ \text{mass} \\ \text{only}}}{S_m} + \underset{\substack{\uparrow\uparrow \\ \text{interaction} \\ \text{of mass} \\ \text{and field} \\ \text{via charge}}}{S_{mf}}$$

$$S_m = -mc \int ds$$

for multiple particles:

$$S_m = -\sum_\alpha m_\alpha c \int ds_\alpha$$

particle index

$$S_{mf} = -\sum_\alpha \frac{e_\alpha}{c} \int dx^k_\alpha A_k$$

$$S_f = -\frac{1}{16\pi c} \int \underbrace{c\,dt\,dx\,dy\,dz}_{d\Omega} F^{ik} F_{ik}$$

$$S_f = \int dt\,dx\,dy\,dz \cdot c \quad \cancel{A^i A_i} \quad F^{ik}F_{ik}$$

$$c\,dt'\,dx'\,dy'\,dz' \leftrightarrow c\,dt\,dx\,dy\,dz$$

$$\begin{vmatrix} \frac{c\partial t'}{c\partial t} & \frac{\partial x'}{c\partial t} & \frac{\partial y'}{c\partial t} & \cdots \\ \frac{\partial x'}{c\partial t} & \frac{\partial x'}{\partial x} & \cdots & \\ & & & \end{vmatrix} = \begin{vmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{vmatrix} = 1$$

$$S = +\frac{1}{8\pi}\int dt \int dx\,dy\,dz \; E^2 - H^2$$

Lagrangian

$$L_f = \frac{1}{8\pi}\int dV \left(E^2 - H^2\right)$$

$dx\,dy\,dz$

§28 The four-dimensional current vector

~~Particles~~ Charges have position $\vec{r}_1(t), \vec{r}_2(t), \dots, \vec{r}_N(t)$

e_α

$$\rho(\vec{r},t) = \sum_\alpha^N e_\alpha \, \delta\left(\vec{r} - \vec{r}_\alpha(t)\right)$$

Charge density of N point charges

$$\int dV \, \rho(\vec{r},t) = \sum_\alpha^N e_\alpha$$

Remark: δ-function

$$\int_a^b dx \, \delta(x) f(x) = \begin{cases} f(0) & \text{if } 0 \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Current density $\vec{J}(\vec{r},t) = \rho(\vec{r},t)\,\vec{v}(\vec{r},t)$

velocity of the charge at this given point

How do ρ and $\vec{j}$ transform?

$$\underbrace{de}_{\text{infinitesimal charge}} \, dx^i$$

displacement

← transforms like any four-vector because de does not transform at all and dx^i is a four-vector

$$de = \rho \, dV$$

charge density

volume

$$de \, dx^i = \rho \, dV dx^i = \rho \, dV dt \frac{dx^i}{dt}$$

← transforms like a four-vector

$$= \rho \frac{1}{c} \underbrace{c\,dt\,dx\,dy\,dz}_{\text{Lorentz invariant}} \frac{dx^i}{dt}$$

$\rho \frac{1}{c} \frac{dx^i}{dt}$ transforms like a four-vector

$j^i = \rho \frac{dx^i}{dt}$ is a four vector

$$j^i = \left(\frac{\rho}{c}, \rho \vec{v}\right) = \left(\frac{\rho}{c}, \vec{j}\right)$$

Four-current (transforms like any other four-vector)

$$S_{mf} = -\sum_\alpha \frac{e_\alpha}{c} \int dx^\mu_\alpha A_\mu$$

$$= -\sum_\alpha \frac{e_\alpha}{c} \int dt \frac{dx^\mu_\alpha}{dt} A_\mu$$

This is a bit brutal

$$= -\frac{1}{c}\int dV \rho \int dt \frac{dx^k}{dt} A_k$$

Do this properly by expressing the integrand in terms of

$$\rho(\vec{r},t) = \sum_\alpha e_\alpha \delta(\vec{r}-\vec{r}_\alpha(t))$$

$$= -\frac{1}{c^2}\int dV dt c\, j^k A_k$$

Need

$$\sum e_\alpha \frac{x_\alpha^k(t)}{dt} = \int dV \rho \frac{x^k(t)}{dt}$$

$$\sum e_\alpha \left(c, \frac{d\vec{r}_\alpha(t)}{dt}\right)$$

velocity at $\vec{r}_\alpha$

$$= \int dV \sum e_\alpha \delta(\vec{r}-\vec{r}_\alpha(t)) \, (c, \vec{v}(\vec{r}_\alpha, t))$$

$$= \int dV \rho (c, \vec{v}) = \int dV \rho \frac{x^k(t)}{dt}$$

$$= -\frac{1}{c^2}\int d\Omega \, j^k A_k$$

$$S = -\sum_\alpha \int ds_\alpha m_\alpha c - \frac{1}{c^2}\int d\Omega A_k j^k - \frac{1}{16\pi c}\int d\Omega F^{ik} F_{ik}$$

Next: Learn more about currents

Maxwell's equations part II

... more on the current four-vector

$$de\, dx^i \; \ldots$$

$$j^i = (\rho c, \vec{j})$$

classical charge density

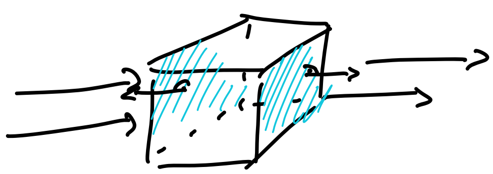
classical current

§29 Equation of continuity

$$\int_V dV \rho = q(V) \text{ the charge in volume } V$$

From physical considerations

$$\partial_t \int_V dV \rho = -\oint_{\partial V} d\vec{f} \cdot \vec{j} = -\int_V dV\, \vec{\nabla} \vec{j}$$



$$\vec{j} = \rho \underbrace{\vec{v}(x,t)}_{\text{the velocity of the charge density at } r,t}$$

charge per time going out

$$\hat{=}\; d\vec{f} \cdot \vec{j}$$

$$\int_V dV \partial_t \rho = \int_V dV (-\vec{\nabla}\vec{j})$$

$$\int dV \left(\partial_t \rho + \vec{\nabla}\vec{j}\right) = 0$$

For every volume $\Rightarrow$ $\partial_t \rho + \vec{\nabla}\vec{j} = 0$ continuity equation

$$j^i = (c\rho, \vec{j})$$

$$0 = \frac{\partial}{c\partial t} c\rho + \partial_x j_x + \partial_y j_y + \partial_z j_z = \partial_i j^i$$

$$\partial_i = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

covariant

$$\boxed{\partial_i j^i = 0}$$

continuity equ in 4-vector form.

4-dimensional ~~Gauss~~ divergence theorem

$$\boxed{\oint dS_i\, j^i = 0}$$

§ 30 The second pair of Maxwell equs.

$$S = -\sum_\alpha \int ds_\alpha m_\alpha c - \frac{1}{c^2}\int d\Omega\, A_k j^k - \frac{1}{16\pi c}\int d\Omega\, F^{ik} F_{ik}$$

Ask for $\delta_1 S'$ due to perturbations of the 4-potential

$$\delta_1 S' = -\frac{1}{c}\int d\Omega \left\{ \frac{1}{c} j^i \delta A_i + \frac{1}{8\pi} F^{ik} \delta F_{ik} \right\}$$

first order perturbation

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

$$\int dV\, \vec{\nabla}(\overset{=}{F} A) = \oint d\vec{f}\, (\overset{=}{F} A) \overset{!}{=} 0$$

$$\| \quad \int dV\, A \vec{\nabla}\overset{=}{F} + \int dV\, \overset{=}{F} \vec{\nabla} A$$

$$\delta F_{ik} = \frac{\partial}{\partial x^i}\delta A_k - \frac{\partial}{\partial x^k}\delta A_i$$

$$\delta_1 S' = -\frac{1}{c}\int d\Omega \left\{ \frac{1}{c} j^i \delta A_i + \frac{1}{8\pi} F^{ik} \frac{\partial}{\partial x^i}\delta A_k - \frac{1}{8\pi} F^{ik} \frac{\partial}{\partial x^k}\delta A_i \right\}$$

no surface term...

$$= -\frac{1}{c}\int d\Omega \left\{ \frac{1}{c} j^i \delta A_i - \frac{1}{8\pi}\left(\frac{\partial}{\partial x^i} F^{ik}\right)\delta A_k + \frac{1}{8\pi}\left(\frac{\partial}{\partial x^k} F^{ik}\right)\delta A_i \right\}$$

$$= -\frac{1}{c}\int d\Omega \left\{ \frac{1}{c} j^i \delta A_i - \frac{1}{8\pi}\left(\frac{\partial}{\partial x^k} F^{ki}\right)\delta A_i + \frac{1}{8\pi}\left(\frac{\partial}{\partial x^k} F^{ik}\right)\delta A_i \right\}$$

$$= -\frac{1}{c}\int d\Omega \left\{ \frac{1}{c} j^i \delta A_i - \frac{1}{8\pi}\left(\frac{\partial}{\partial x^k} F^{ki}\right)\delta A_i + \frac{1}{8\pi}\left(\frac{\partial}{\partial x^k} F^{ik}\right)\delta A_i \right\}$$

$$= -\frac{1}{c}\int d\Omega\, \delta A_i \left\{ \frac{1}{c} j^i - \frac{1}{8\pi}\frac{\partial F^{ki}}{\partial x^k} + \frac{1}{8\pi}\frac{\partial F^{ik}}{\partial x^k} \right\}$$

$$= -\frac{1}{c}\int d\Omega\, \delta A_i \left\{ \frac{1}{c} j^i + \frac{1}{8\pi}\frac{\partial F^{ik}}{\partial x^k} + \frac{1}{8\pi}\frac{\partial F^{ik}}{\partial x^k} \right\}$$

$$\delta S' = 0 \quad \Rightarrow$$

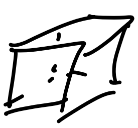
$$0 = \frac{1}{c} j^i + \frac{1}{8\pi}\frac{\partial F^{ik}}{\partial x^k} + \frac{1}{8\pi}\frac{\partial F^{ik}}{\partial x^k}$$

$$= \frac{1}{c} j^i + \frac{1}{4\pi}\frac{\partial F^{ik}}{\partial x^k}$$

F^{ik} is known in terms of $\vec{E}$ and $\vec{H}$ and so is j^i

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \qquad j^i = (c\rho, \vec{j})$$

$$\Rightarrow \quad \boxed{\begin{array}{ll} \vec{\nabla}\times\vec{H} = \frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{j} & \text{Ampère's law} \\ \vec{\nabla}\cdot\vec{E} = 4\pi\rho & \text{Gauss' law} \end{array}}$$



$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} d\vec{f}\cdot\vec{E}$$

displacement current

$$\int_S d\vec{f}\; \vec{\nabla}\times\vec{H} = \int_S d\vec{f}\left(\frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{J}\right)$$

$$\parallel$$

$$\oint d\vec{\ell}\cdot\vec{H}$$

Next: Summary of Maxwell in 4-Form.

Then: Light!

Summary of Maxwell's equns

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{H}$$ Faraday's law

$$\vec{\nabla} \cdot \vec{H} = 0$$ Gauss's law of magnetism

Or in 4-vector form:

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{J}$$ Ampere's law

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$ Gauss' law

Or in 4-vector form:

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i$$

Note:

$$\frac{\partial}{\partial x^i}\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c}\partial_i j^i$$

This is the correct first component. If it says $\frac{\rho}{c}$ somewhere, that's wrong.

RHS $$\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(c\rho, \vec{J}\right)$$

$$= \frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0$$ By continuity

$$\frac{\partial}{\partial x^i}\frac{\partial F^{ik}}{\partial x^k} = \frac{\partial}{\partial x^k}\frac{\partial F^{ik}}{\partial x^i} = -\frac{\partial}{\partial x^k}\frac{\partial F^{ki}}{\partial x^i}$$

$$= -\frac{\partial}{\partial x^i}\frac{\partial F^{ik}}{\partial x^k} \quad \Rightarrow \frac{\partial}{\partial x^i}\frac{\partial F^{ik}}{\partial x^k} = 0$$

Ch.5 $\longrightarrow$ Master material

Ch 6 Electromagnetic waves

§46 The wave equation

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{H}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\partial_t\vec{E} + \cancel{\frac{4\pi}{c}\vec{j}}$$

$$\vec{\nabla}\cdot\vec{E} = \cancel{4\pi\rho}$$

vacuum

$$\frac{1}{c}\partial_t^2\vec{E} = \vec{\nabla}\times\partial_t\vec{H} = -c\,\vec{\nabla}\times\vec{\nabla}\times\vec{E}$$

$$\nabla\times\nabla\times\vec{A} = -\underset{\uparrow\ \vec{\nabla}\cdot\vec{\nabla}}{\Delta}\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

$$\longrightarrow -\frac{1}{c^2}\partial_t^2\vec{E} = \Delta\vec{E} - \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{4\pi\rho=0}) = \Delta\vec{E}$$

$$-\frac{1}{c^2}\partial_t^2 E_x = (\partial_x^2+\partial_y^2+\partial_z^2)E_x$$

Wave eqn
(for every component)

$$\boxed{-\frac{1}{c^2}\partial_t^2 \vec{E} = \Delta \vec{E}}$$

Wave eqn
for $\vec{E}$

Correspondingly

$$\left.\begin{aligned} \vec{\nabla}\times\dot{\vec{E}} &= -\frac{1}{c}\partial_t^2 \vec{H} \\ \vec{\nabla}\times\vec{H} &= \frac{1}{c}\partial_t \vec{E} \end{aligned}\right\} \quad -\frac{1}{c}\partial_t^2 \vec{H} = c\vec{\nabla}\times\vec{\nabla}\times\vec{H}$$

$$\boxed{-\frac{1}{c^2}\partial_t^2 \vec{H} = \Delta \vec{H}}$$

Wave eqn
for $\vec{H}$

Derive a wave eqn for the potentials

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}$$

$$\vec{H} = \vec{\nabla}\times\vec{A}$$

From gauge invariance (the fact that $\vec{A}$ and ϕ are not uniquely determined) we can demand

$$\left.\begin{aligned} \phi &= 0 \\ \vec{\nabla}\vec{A} &= 0 \end{aligned}\right\} \text{ Different gauge}$$

$\Rightarrow \quad \partial_t \vec{H} = \vec{\nabla} \times \mathring{\vec{A}}$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{H}$$
$$\vec{\nabla} \cdot \vec{H} = 0$$
$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\partial_t \vec{E}$$
$$\vec{\nabla} \cdot \vec{E} = 0$$

$$-\frac{1}{c}\partial_t^2 \vec{A} = \partial_t \vec{E} = c\,\vec{\nabla} \times \vec{H} = c\,\vec{\nabla} \times \vec{\nabla} \times \vec{A}$$
$$= -c\,\Delta \vec{A}$$

$$\boxed{\frac{1}{c^2}\partial_t^2 \vec{A} = \Delta \vec{A}}$$

Wave eqn of the vector potential

Next: Wave eqn in 4-vector form
monochromatic waves
polarised light.

§46 Continued

The wave in 4-vector form

Maxwell's eqn in 4-vector form

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i$$

In vacuum $j^i = (\rho c, \vec{j}) = 0$

$$\left.\begin{aligned} &\frac{\partial F^{ik}}{\partial x^k} \overset{!}{=} 0 \\ &F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \end{aligned}\right\} \quad \frac{\partial^2 A^k}{\partial x_i \partial x^k} - \frac{\partial^2 A^i}{\partial x_k \partial x^k} = 0$$

Ludwig

By gauge invariance: $\frac{\partial A^k}{\partial x^k} = 0$ Lorenz-gauge

Add a four-gradient to the four-potential without changing $\vec{E}$ and $\vec{H}$ (and therefore F^{ik})

$$\Rightarrow \quad -\frac{\partial^2 A^i}{\partial x_k \partial x^k} = 0$$ wave eqn in 4-vector form

→ Spell out wave eqn and gauge to

find solutions.

§47 Plane waves

What are the solutions of the wave eqn.

$$-\frac{1}{c^2}\partial_t^2 \vec{E} = \Delta \vec{E}$$

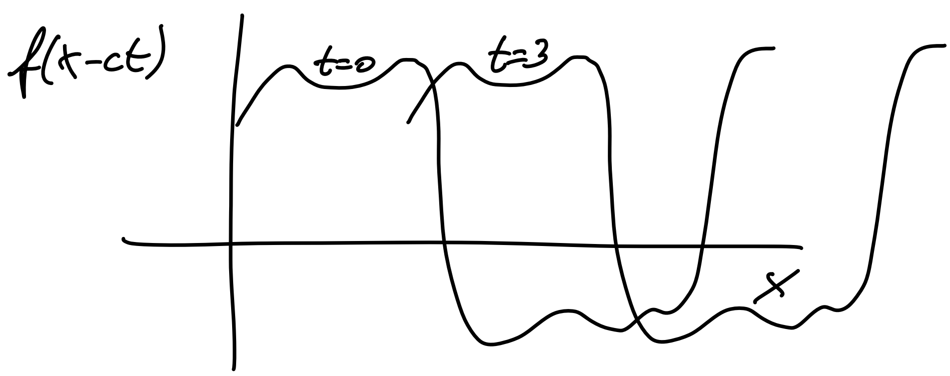
$$\vec{E}(x \pm ct)$$

$$\boxed{\frac{1}{c^2}\partial_t^2 \vec{A} = \Delta \vec{A}}$$

D'Alembert soln to the wave eqn.

Ans $\vec{A}(x \pm ct)$ Plane wave (plane in y,z)

Most important class of solutions, but surely not exhaustive.



Linear combinations of solutions are also solution.

Let's focus on $x-ct$ (or $t-\frac{x}{c}$)

Assume that $\vec{A} = \vec{A}(t - \frac{x}{c})$

What are $\vec{E}$ and $\vec{H}$?

$\phi = 0 \rightsquigarrow \vec{E} = -\frac{1}{c}\partial_t \vec{A}$

$$\Rightarrow \vec{E} = -\frac{1}{c}\vec{A}'$$

$$\vec{A}(t - \frac{x}{c}) = \vec{A}(t - \frac{\vec{r}\cdot\vec{n}}{c})$$

$$\vec{H} = \vec{\nabla} \times \vec{A}$$

$$\Rightarrow \vec{H} = \vec{\nabla} \times \vec{A}(t - \frac{x}{c}) = \left[\vec{\nabla}(t - \frac{x}{c})\right] \times \vec{A}'$$

$$= \begin{pmatrix} -\frac{1}{c} \\ 0 \\ 0 \end{pmatrix} \times \vec{A}' = -\frac{1}{c}\vec{n} \times \vec{A}'$$

director of the propagation $\vec{e}_x$.

$$\boxed{\vec{H} = \vec{n} \times \vec{E}}$$

The magnetic field is orthogonal to both $\vec{n}$ and $\vec{E}$

§47 Monochromatic plane waves

We consider solutions that are eigenfunctions of ∂_t^2

$$\partial_t^2 f = -\omega^2 f$$

and similar for ∂_x^2

$$-\frac{1}{c^2}\omega^2 f = \Delta\vec{A}$$

$$\Delta\vec{A} + \frac{\omega^2}{c^2}\vec{A} = 0$$

$$\vec{A} = \Re\left(\vec{A}_0 e^{-i\omega(t-\frac{x}{c})}\right)$$

$$x = \vec{n}\cdot\vec{r} \quad \text{(for more general directions)}$$

Dispersion relation

$$\vec{k} = \frac{\omega}{c}\vec{n} \quad \text{(definition)}$$

wave number

$$e^{-i\omega(t-\frac{x}{c})} = e^{-i\omega(t-\frac{\vec{n}\cdot\vec{r}}{c})}$$

$$= e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

(Fourier) frequency $\omega \leftrightarrow t$ direct time

reciprocal space / Fourier space $\vec{k} \leftrightarrow \vec{r}$ realspace

Consequences to $\vec{E}$ and $\vec{H}$ if $\vec{A} = \Re\left(\vec{A}_0 e^{-i(\omega t-\vec{k}\cdot\vec{r})}\right)$

$$\vec{E} = -\frac{1}{c}\partial_t\vec{A}$$

$$\vec{H} = \vec{\nabla}\times\vec{A}$$

For convenience, we drop $\Re$ $\quad \vec{A} = \vec{A}_0 e^{-i(\omega t-\vec{k}\cdot\vec{r})}$

$$\vec{E} = i\frac{\omega}{c}\vec{A} = i|\vec{k}|\vec{A}$$

$$\vec{H} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \vec{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} =$$

... complete tomorrow

introduce polarisation

redshift

§48 Monochromatic plane waves

For convenience, we drop $\Re$ $\quad \vec{A} = \vec{A}_0 e^{-i(\omega t - \vec{k}\cdot\vec{r})}$

$$\vec{E} = i\frac{\omega}{c}\vec{A} = i|\vec{k}|\vec{A}$$

$$\vec{H} = \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} \times \vec{A} = \begin{pmatrix}\partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x\end{pmatrix}$$

$\vec{k} \parallel \vec{e}_x$

$$= \begin{pmatrix} 0 \\ -A_z i k_x e^{-i(\omega t - \vec{k}\cdot\vec{r})} \\ A_y i k_x e^{-i(\omega t - \vec{k}\cdot\vec{r})} \end{pmatrix} = \vec{H}$$

$\vec{\nabla}\cdot\vec{E} = 0$ (from being in vacuum)

$\quad \shortparallel$

$i|\vec{k}| A_x i k_x e^{-i(\omega t - \vec{k}\cdot\vec{r})}$ $\qquad |\vec{k}| = k_x$

$$\vec{E} = \begin{pmatrix} 0 \\ A_y i|\vec{k}| e^{-i(\omega t - \vec{k}\cdot\vec{r})} \\ A_z i|\vec{k}| e^{-i(\omega t - \vec{k}\cdot\vec{r})} \end{pmatrix}$$

$A_z = A_y \neq 0$ $\qquad$ circularly polarised

$A_z \neq A_y$

$A_z, A_y \neq 0$ elliptically polarised

$A_z = 0 \quad A_y \neq 0$

linear polarised

$A_z \neq 0 \quad A_y = 0$

Going back to 4-vectors

$$\underbrace{\partial_i \partial^i}_{\Box} A^j = 0$$

d'Alembert operator

$$\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2 = \frac{1}{c^2}\partial_t^2 - \Delta$$

constant

Soln: $A^j = B^j \exp(-i k_i x^i)$

$$k^i = \left(\frac{\omega}{c}, \vec{k}\right)$$

$$\partial_\ell A^j = B^j(-i k_\ell) \exp(-i k_i x^i)$$

$$= -i k_\ell A^j$$

$$\partial_\ell \partial^\ell A^j = (-i k_\ell)(-i k^\ell) A^j = -k_\ell k^\ell A^j$$

$$k_\ell k^\ell = \frac{\omega^2}{c^2} - k^2 = 0$$

from the dispersion relation

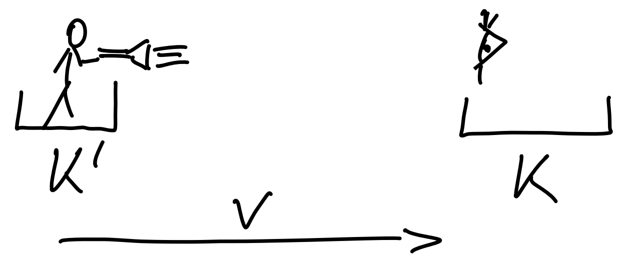
k^i is a 4-vector, as it can be extracted from the 4-potential A^j.

300 - 700 nm = λ wavelength of visible light

$\uparrow$

10^{-9}m

$$\frac{2\pi}{\lambda} = |k|$$



$$k'^{l} = \left(\frac{\omega'}{c}, \vec{k}'\right) \qquad k^{l} = \left(\frac{\omega}{c}, \vec{k}\right)$$

$$\frac{\omega}{c} = \frac{\frac{\omega'}{c} + \frac{v}{c} k_x'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\omega'}{c} \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\omega'}{c} \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

$$k_x' = \frac{\omega'}{c}, \quad k_y' = 0, \quad k_z' = 0$$

$v > 0 \Rightarrow \omega > \omega'$ blue shift

$v < 0 \Rightarrow \omega < \omega'$ red shift