Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 0

1. In this first problem, we want to obtain the general solution of

$$\frac{d^2y}{dx^2} = 5e^{2x}$$

This is a first-order ODE of the form $dy/dx = f(x)$, we can obtain the general solution of this type of equations by direct integration. Integrating once, we get

$$\frac{dy}{dx} = \int 5e^{2x} dx = \frac{5e^{2x}}{2} + A$$

where A is an arbitrary constant of integration. We must integrate a second time to obtain

$$y = \int \left(\frac{5e^{2x}}{2} + A \right) dx = \frac{5e^{2x}}{4} + Ax + B$$

where B is a second constant of integration. Integrating this second-order ODE led to two integration constants which would be determined if we were provided with initial conditions for y and dy/dx.

2. In this problem, we are trying to find the general solutions of these separable first-order ODEs of the form $dy/dx = f(x)g(y)$ which is obtained from

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

(a) This equation is separable with $f(x) = e^x$ and $g(y) = 3y$. We thus write the solution as

$$\int \frac{1}{3y} dy = \int e^x dx \Rightarrow \frac{1}{3} \ln y = e^x + C \Rightarrow y = De^{3e^x}$$

where C (and D) is an arbitrary constant.

(b) This equation is separable with $f(x) = e^{-x}$ and $g(y) = 1/y$. We thus write the solution as

$$\int y dy = \int e^{-x} dx \Rightarrow \frac{y^2}{2} = C - e^{-x} \Rightarrow y = \pm\sqrt{D - 2e^{-x}}$$

where C (and D) is an arbitrary constant.

(c) This equation is separable with $f(x) = 1/x$ and $g(y) = \tan y$. We thus write the solution as

$$\int \frac{1}{\tan y} dy = \int \frac{1}{x} dx \Rightarrow \ln(\sin y) = \ln x + C \Rightarrow y = \sin^{-1}(Dx)$$

where C (and D) is an arbitrary constant.

(d) This equation is separable with $f(x) = 4x$ and $g(y) = y$. We thus write the solution as

$$\int \frac{1}{y} dy = \int 4x dx \Rightarrow \ln y = 2x^2 + C \Rightarrow y = De^{2x^2}$$

where C (and D) is an arbitrary constant.

3. To find the general solution of the following equation

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1,$$

we use the substitution $z = y/x$, i.e. $y = zx$. The product rule tells us that

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

Using this substitution, we write the above equation as

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1 \Rightarrow z + x\frac{dz}{dx} = z^2 + z + 1 \Rightarrow \frac{dz}{dx} = \frac{z^2+1}{x}$$

which brings us back to a separable first-order ODE, which we can solve by writing

$$\int \frac{1}{z^2+1}dz = \int \frac{1}{x}dx \Rightarrow \tan^{-1} z = \ln x + C \Rightarrow z = \tan(\ln Dx)$$

and finally, we conclude that

$$y = x\tan(\ln Dx)$$

with D an arbitrary constant.

4. (a) We can write this equation in standard form as $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = 2$ and $Q(x) = e^{-2x}$ for which the integrating factor is given by $\mathcal{I} = e^{\int P(x)dx} = e^{2x}$. Provided an integrating factor, we know that the general solution is written

$$\mathcal{I}y = \int \mathcal{I}Q(x)dx$$

so that we write

$$e^{2x}y = \int e^{2x}e^{-2x}dx = \int 1dx = x + C$$

with C an arbitrary constant. Therefore, we find that

$$y = (x + C)e^{-2x}$$

(b) This equation is already in standard form with $P(x) = -\tan x$ and $Q(x) = 1$. The integrating factor is thus given by $\mathcal{I} = e^{\int P(x)dx} = \cos x$. Provided an integrating factor, we know that the general solution is written

$$\mathcal{I}y = \int \mathcal{I}Q(x)dx$$

so that we write

$$\cos xy = \int \cos xdx = \sin x + C$$

with C an arbitrary constant. Therefore, we find that

$$y = \tan x + C\sec x$$

5. For homogeneous second-order linear equations with constant coefficients, we know that the general form of the solution depends on the roots of the associated auxiliary equation. If the equation is written

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

then the auxiliary equation reads

$$ak^2 + bk + c = 0$$

(a) The auxiliary equation reads $k^2 + 3k - 10 = 0$; this can be factorized as follows

$$(k-2)(k+5) = 0 \Rightarrow k = 2 \quad \text{and} k = -5$$

Thus, there exists two real roots to the auxiliary equation and the general solution is thus written as a sum of exponentials

$$y = Ae^{2x} + Be^{-5x}$$

(b) The auxiliary equation reads $k^2 + 2k + 4 = 0$; this equation has two complex roots which we write

$$k = -1 \pm i\sqrt{3}$$

Thus, there exists two distinct complex roots to the auxiliary equation and the general solution is thus written as

$$y = e^{-x}\left[A\cos\sqrt{3}x + B\sin\sqrt{3}x\right]$$

(c) The auxiliary equation reads $k^2 + 8k + 16 = 0$; this equation can be factorized as follows

$$(k+4)(k+4) = 0$$

Thus, we obtain twice the same root $k = -4$; in this case, the solution is written

$$y = (A + Bx)e^{-4x}$$

6. (a) We start by looking for the solution to the homogeneous equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0$$

whose solution we found earlier to be

$$y_1(x) = Ae^{2x} + Be^{-5x}$$

We now note that the RHS of the inhomogeneous equation is a polynomial of degree 2, the particular integral can thus be found by trying a solution of the form

$$y_2 = ax^2 + bx + c$$

Differentiating this twice, we write

$$\frac{dy_2}{dx} = 2ax + b \quad \text{and} \quad \frac{d^2y_2}{dx^2} = 2a$$

We then substitute this in the inhomogeneous equation and by equating coefficient of x^2, x and constants, we obtain a set of equations satisfied by the constants a, b and c:

$$2a + 3(2ax + b) - 10(ax^2 + bx + c) = 3x^2$$

which leads to

$$\begin{cases} -10a & = 3 \\ 6a - 10b & = 0 \\ 2a + 3b - 10c & = 0 \end{cases}$$

Solving these equations, we find the following constants

$$a = -\frac{3}{10}, \quad b = -\frac{9}{50} \quad and \quad -\frac{57}{500}$$

leading to the following general solution

$$y(x) = y_1(x) + y_2(x) = -\frac{3}{10}x^2 - \frac{9}{50}x - \frac{57}{500} + Ae^{2x} + Be^{-5x}$$

(b) We start by looking for the solution to the homogeneous equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

whose auxiliary equation is $k^2 + k - 12 = 0$. Factorizing this quadratic, we find that this equation has two distinct real roots $k = 3$ and $k = -4$. So the general solution to the homogeneous equation is

$$y_1(x) = Ae^{3x} + Be^{-4x}$$

The RHS of the inhomogeneous problem is of the form $4e^{2x}$ and so a particular integral can be found by trying a solution of the form $y_2 = \alpha e^{2x}$. Substituting this ansatz back in the inhomogeneous equation, we find

$$4\alpha e^{2x} + 2\alpha e^{2x} - 12\alpha e^{2x} = 4e^{2x}$$

so that

$$4\alpha + 2\alpha - 12\alpha = 4 \Rightarrow \alpha = -2/3$$

The general solution is then written

$$y(x) = y_1(x) + y_2(x) = Ae^{3x} + Be^{-4x} - \frac{2}{3}e^{2x}$$

The particular solution is then found by applying the initial conditions $y(0) = 7$ and $y'(0) = 0$ and we find that A and B are solutions to the following equations

$$\begin{cases} A + B - 2/3 & = 7 \\ 3A - 4B - 4/3 & = 0 \end{cases}$$

which we solve to obtain

$$A = 32/7 \quad \text{and} \quad B = 65/21$$

Finally, the particular solution is

$$y(x) = \frac{32}{7}e^{3x} + \frac{65}{21}e^{-4x} - \frac{2}{3}e^{2x}$$

7. Here, we want to find the Fourier series of period 2π which represent the following functions on the interval $-\pi < x < \pi$ ...

(a) ... $f(x) = x$. This function is odd about $x = 0$, therefore $a_n = 0 \Rightarrow x = \sum_{n=1}^{\infty} b_n \sin_n x$, $|x| < \pi$. Further, we know that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = \frac{1}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\underbrace{\int_{-\pi}^{\pi}\cos nx dx}_{0}\right\} = -\frac{2\cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$$

which leads to

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx, \quad |x| < \pi$$

with $f(\pi) = f(-\pi) = 0$.

(b) ... $f(x) = x^2$. This function is even about $x = 0$, therefore $b_n = 0 \Rightarrow x^2 = a_0/2 + \sum_{n=1}^{\infty} a_n \cos_n x$, $|x| < \pi$. Further, we know that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = \frac{4(-1)^n}{n^2}$$

(by integration by parts twice). This leads to

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \quad |x| < \pi$$

with $f(\pi) = f(-\pi) = \pi^2$.

(c) ... $f(x) = \sinh x$. This function is odd about $x = 0$, therefore $a_n = 0 \Rightarrow \sinh x = \sum_{n=1}^{\infty} b_n \sin_n x$, $|x| < \pi$. Further, we know that

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh x \sin nx dx \\ &= \frac{1}{\pi}\left\{\underbrace{[\cosh x \sin nx]_{-\pi}^{\pi}}_{0} - n\int_{-\pi}^{\pi}\cosh x\cos nx dx\right\} \\ &= -\frac{n}{\pi}\left\{[\sinh x \cos nx]_{-\pi}^{\pi} + \underbrace{\int_{-\pi}^{\pi} n\sinh x \sin nx dx}_{nb_n\pi}\right\} \end{aligned}$$

This allows us to write

$$(1+n^2)b_n = -\frac{2n}{\pi}(-1)^n\sinh\pi \Rightarrow b_n = (-1)^{n+1}\frac{2n}{\pi(1+n^2)}\sinh\pi$$

which leads to

$$\sinh x = \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{1+n^2}\sin nx, \quad |x| < \pi$$

with $f(\pi) = f(-\pi) = 0$.

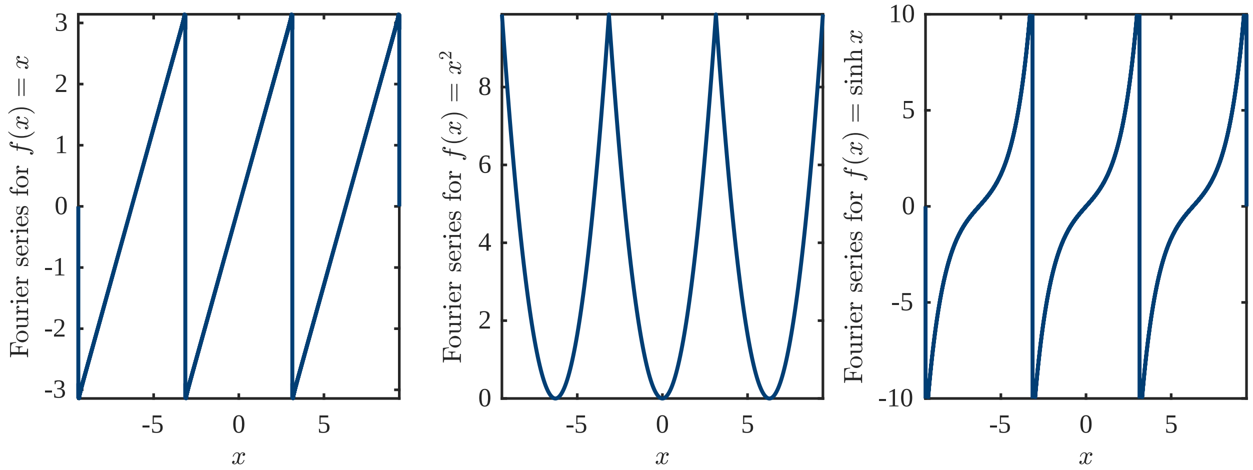

Figure 1: Representation of the Fourier series over the interval $(-\pi, \pi)$.

8. In this question, we want to obtain the Fourier expansion of $f(x) = \cos\alpha x$, where α is not an integer. This function is even about $x = 0$, therefore we know that $b_n = 0$ and $\cos\alpha x = a_0/2 + \sum_{n=1}^{\infty} a_n \cos_n x$, $|x| < \pi$. Further, we know that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\alpha x dx = \frac{2\sin\alpha\pi}{\alpha\pi}$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\alpha x\cos nx dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(n+\alpha)x + \cos(n-\alpha)x]dx \\
&= \frac{1}{2\pi}\left[\frac{\sin(n+\alpha)x}{n+\alpha} + \frac{\sin(n-\alpha)x}{n-\alpha}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\sin(\alpha\pi+n\pi)}{n+\alpha} - \frac{\sin(\alpha\pi-n\pi)}{n-\alpha}\right) \\
&= \frac{\cos n\pi}{\pi}\left(\frac{\sin\alpha\pi}{n+\alpha} - \frac{\sin\alpha\pi}{n-\alpha}\right) = 2\alpha(-1)^n\frac{\sin\alpha\pi}{\pi(\alpha^2-n^2)}
\end{aligned}$$

Therefore, the Fourier expansion reads

$$\cos\alpha x = \frac{\sin\alpha\pi}{\alpha\pi} + \sum_{n=1}^{\infty}(-1)^n\frac{2\alpha\sin\alpha\pi}{\pi(\alpha^2-n^2)}\cos nx, \quad |x| \le \pi.$$

When $\alpha \to m$, all terms in the series converge to 0 except for a_m, indeed we have

$$a_m = \lim_{\alpha\to m}(-1)^n\frac{2\alpha\sin\alpha\pi}{\pi(\alpha^2-n^2)} = \frac{2(-1)^m}{\pi}\lim_{\alpha\to m}\frac{\alpha}{\alpha+m}\frac{\sin\alpha\pi}{\alpha-m} = \frac{2(-1)^m}{\pi}\frac{m}{2m}\lim_{\alpha\to m}\frac{\sin\alpha\pi}{\alpha-m}$$

We can use L'Hopital's rule to evaluate the above limit and write

$$\lim_{\alpha\to m}\frac{\sin\alpha\pi}{\alpha-m} = \lim_{\alpha\to m}\pi\cos\alpha\pi = (-1)^m\pi$$

Thus, $a_m = \frac{2(-1)^m}{\pi}\frac{m}{2m}(-1)^m\pi = 1$ and the Fourier series converges to $\cos mx$ as $\alpha \to m$.

9. The function

$$f(x) = \begin{cases} 1 + (x/\pi), & -\pi < x < 0 \\ 1 - (x/\pi), & 0 \le x \le \pi \end{cases}$$

is even about $x = 0$ which leads to a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_n x, \quad |x| < \pi$$

with

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{x}{\pi}\right)dx + \int_{0}^{\pi}\left(1-\frac{x}{\pi}\right)dx\right] = 1,$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{x}{\pi}\right)\cos nx dx + \int_{0}^{\pi}\left(1-\frac{x}{\pi}\right)\cos nx dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{x}{\pi}\right)\cos nx dx \\
&= -\frac{2}{n^2\pi^2}\left((-1)^n - 1\right)
\end{aligned}$$

where we have used integration by parts. We thus conclude that

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}, \quad |x| \le \pi.$$

Substituting $x = 0$ in the previous result, we obtain $f(0) = 1 = 1/2 + (4/\pi^2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$ which leads to

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

10. For the function $f(x) = x(\pi - x)$, $0 \le x \le \pi$, we want to derive the Fourier half-range sine and cosine expansions.

Sine series In this case, we write

$$x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx = \frac{4}{\pi n^3}(1 - (-1)^n) = \begin{cases} 0, & n \text{ even} \\ 8/(\pi n^3), & n \text{ odd} \end{cases}$$

where we have used integration by parts. So we conclude

$$x(\pi - x) = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{(2k-1)^3}, \quad 0 \le x \le \pi.$$

Cosine series Similarly here, we write

$$x(\pi - x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) dx = \frac{\pi^2}{3},$$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx dx = -\frac{2}{n}(1 + (-1)^n) = \begin{cases} -4/n^2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

where we have used integration by parts. So we conclude

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{\cos 2kx}{k^2}, \quad 0 \le x \le \pi.$$

Using these expansions, we find that

(a) substituting $x = 0$ in the Fourier cosine series, we find $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$;

(b) substituting $x = \pi/2$ in the Fourier cosine series, we find $\frac{\pi^2}{4} = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$;

(c) substituting $x = \pi/2$ in the Fourier sine series, we find $\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi/2}{(2k-1)^3} \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$.

11. Let $f(x) = 1 + (x/L),\ 0 < x < L$. Its half-range Fourier sine series is defined as

$$1 + \frac{x}{L} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L (1 + x/L)\sin(n\pi x/L)dx$. Integrating by parts, we find

$$b_n = \frac{2}{n\pi}\left(1 - 2(-1)^n\right)$$

and so we conclude that

$$1 + \frac{x}{L} = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - 2(-1)^n)\sin\left(\frac{n\pi x}{L}\right)$$

12. Let $a > 0$, find the Fourier transforms of the following functions ...

(a) ... $f(x) = \exp(-a|x|)$. Here, we write

$$\begin{aligned}\mathcal{F}\{\exp(-a|x|)\} &= \int_{-\infty}^{\infty} \exp(-a|x|)e^{-i\omega x}dx \\ &= \int_{-\infty}^{0} e^{(a-i\omega)x}dx + \int_{0}^{\infty} e^{-(a+i\omega)x}dx \\ &= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}\end{aligned}$$

(b) ... $f(x) = \operatorname{sgn}(x)\exp(-a|x|)$, where $\operatorname{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$. Here, we write

$$\begin{aligned}\mathcal{F}\{\operatorname{sgn}(x)\exp(-a|x|)\} &= \int_{-\infty}^{\infty} \operatorname{sgn}(x)\exp(-a|x|)e^{-i\omega x}dx \\ &= \int_{-\infty}^{0} (-1)e^{(a-i\omega)x}dx + \int_{0}^{\infty} e^{-(a+i\omega)x}dx \\ &= -\frac{1}{a-i\omega} + \frac{1}{a+i\omega} = -\frac{2i\omega}{a^2+\omega^2}\end{aligned}$$

(c) ... $f(x) = 2a/(a^2+x^2)$. We know from (a) that if $f(x) = \exp(-a|x|)$, then $\widehat{f}(\omega) = 2a/(a^2+\omega^2)$. So we can write $\widehat{f}(x) = 2a/(a^2+x^2)$ and using the symmetry formula, we obtain

$$\mathcal{F}\{f(x)\} = \mathcal{F}\left\{\widehat{f}(x)\right\} = 2\pi f(-\omega) = 2\pi\exp(-a|\omega|)$$

(d) ... $f(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise. Here, we write

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \int_{-1}^{1} (1-x^2)e^{-i\omega x}dx \\ &= \int_{-1}^{1} (1-x^2)\cos(\omega x)dx - i\int_{-1}^{1} (1-x^2)\sin(\omega x)dx\end{aligned}$$

The second integral is clearly zero as we are integrating an odd function, while the first integral has an even integrand and so can be rewritten

$$\mathcal{F}\{f(x)\} = 2\int_{0}^{1} (1-x^2)\cos(\omega x)dx = -\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega$$

where we have integrated by parts twice to obtain the final result.

(e) ... $f(x) = \sin(ax)/(\pi x)$. Here, we will use a classical result: the Fourier transform of a rectangular pulse of width a. This function is defined as

$$h(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{otherwise} \end{cases}$$

Its Fourier transform is thus given by

$$\begin{aligned}\mathcal{F}\{h(x)\} &= \int_{-\infty}^{\infty} h(x)e^{-i\omega x}dx = \int_{-a}^{a} e^{-i\omega x}dx \\ &= -\frac{1}{i\omega}\left[e^{-i\omega x}\right]_{-a}^{a} = -\frac{1}{i\omega}\left[e^{-i\omega a} - e^{i\omega a}\right] \\ &= \frac{2}{\omega}\sin\omega a\end{aligned}$$

Then, by the symmetry formula, we obtain that $\mathcal{F}\left\{\frac{2}{x}\sin ax\right\} = 2\pi h(-\omega) = 2\pi h(\omega)$, as h is even. This allows us to conclude that

$$\mathcal{F}\{\sin(ax)/(\pi x)\} = h(\omega)$$

By applying the inversion formula to the transforms obtained for (a) and (d), we want to establish the following results:

$$\int_0^\infty \frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-a}}{2a}, \text{ if } a>0 \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{\sin x - x\cos x}{x^3}dx = \frac{\pi}{2}$$

☐ Firstly, from (a), $\mathcal{F}\{\exp(-a|x|)\} = 2a/(a^2+\omega^2)$. Using the inversion formula, we have

$$\exp(-a|x|) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2a}{a^2+\omega^2}e^{i\omega x}d\omega = \frac{a}{\pi}\left[\int_{-\infty}^\infty \frac{2a}{a^2+\omega^2}\cos\omega x d\omega + i\int_{-\infty}^\infty \frac{2a}{a^2+\omega^2}\sin\omega x d\omega\right]$$

The second integral is zero since the integrand is odd in ω, while the first integral has an even integrand and so we can write

$$\exp(-a|x|) = \frac{2a}{\pi}\int_0^\infty \frac{\cos x\omega}{(a^2+\omega^2)}d\omega$$

Setting $x=1$ in the previous result, we obtain

$$\frac{\pi e^{-a}}{2a} = \int_0^\infty \frac{\cos\omega}{a^2+\omega^2}d\omega$$

☐ Secondly, from (d), if we define $g(x) = 1-x^2$ for $|x|\le 1$ and zero otherwise, then by inversion, we get

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(-\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega\right)e^{i\omega x}d\omega$$

If we set $x=0$ and rearrange the terms, we obtain the desired result.

13. Let f be the function defined as

$$f(x) = \begin{cases} 2d-|x|, & \text{for } |x|\le 2d, \\ 0, & \text{otherwise.}\end{cases}$$

Its Fourier transform is defined as

$$\widehat{f}(\omega) = \int_{-2d}^{2d}(2d-|x|)e^{-i\omega x}dx = \int_{-2d}^{2d}(2d-|x|)\cos(\omega x)dx - i\int_{-2d}^{2d}(2d-|x|)\sin(\omega x)dx$$

The second integral is zero since the integrand is odd in x, while the first integral has an even integrand and so, we can write

$$\widehat{f}(\omega) = 2\int_0^{2d}(2d-|x|)\cos(\omega x)dx = \frac{2}{\omega^2}(1-\cos(2\omega d)) = \frac{4}{\omega^2}\sin^2(\omega d)$$

where we have integrated by parts at the last step. So we conclude that

$$\widehat{f}(\omega) = (2/\omega)^2\sin^2(\omega d).$$

Therefore, we have that

$$\left|\widehat{f}(\omega)\right|^2 = \frac{16}{\omega^4}\sin^4(\omega d).$$

But we have that

$$\int_{-\infty}^\infty (f(u))^2du = \int_{-2d}^{2d}(2d-|u|)^2du = 2\int_{-2d}^{2d}(2d-u)^2du = \cdots = \frac{16}{3}d^3$$

Using the energy theorem, we have

$$\frac{32\pi d^3}{3} = 16\int_{-\infty}^\infty \frac{\sin^4(\omega d)}{\omega^4}d\omega$$

and setting $d=1$, we get

$$\int_{-\infty}^\infty \frac{\sin^4 x}{x^4}dx = \frac{2\pi}{3}$$

as required.

14. Let $f(x) = \exp(-cx)H(x)$, where H is the Heaviside function and c is a positive constant, then its Fourier transform is defined as follows

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-cx} H(x) e^{-i\omega x} dx = \int_{0}^{\infty} e^{-(c+i\omega)x} dx = \frac{1}{c + i\omega}$$

The convolution theorem gives us that

$$\mathcal{F}^{-1}\left\{\widehat{g}(\omega)\widehat{h}(\omega)\right\} = g(x) * h(x)$$

Now, let

$$\begin{aligned}\widehat{g}(\omega) &= \frac{1}{a + i\omega} \Rightarrow g(x) = e^{-ax}H(x)\\ \widehat{h}(\omega) &= \frac{1}{b + i\omega} \Rightarrow g(x) = e^{-bx}H(x)\end{aligned}$$

Therefore, we have

$$\mathcal{F}^{-1}\left\{\frac{1}{a + i\omega}\frac{1}{b + i\omega}\right\} = \left(e^{-ax}H(x)\right) * \left(e^{-bx}H(x)\right)$$

The RHS of the equation above reads

$$\begin{aligned}\mathrm{RHS} &= \left(e^{-ax}H(x)\right) * \left(e^{-bx}H(x)\right)\\ &= \int_{-\infty}^{\infty} \exp(-a(x-u))H(x-u)\exp(-bu)H(u)du\\ &= \int_{0}^{\infty} \exp(-a(x-u))H(x-u)\exp(-bu)du\end{aligned}$$

but the function $H(x-u)$ is nonzero (and equal to 1) only if $0 < u < x$, therefore we obtain

$$\mathrm{RHS} = \int_{0}^{x} \exp(-ax)\exp((a-b)u)du = \frac{\exp(-bx) - \exp(-ax)}{a - b}$$

for $x > 0$ and $\mathrm{RHS} = 0$ if $x < 0$.

15. In this problem, we are asked to write down:

(a) the most general linear first-order PDE in two variables:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

(b) the most general linear second-order PDE in two variables:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} + c(x,y)\frac{\partial^2 u}{\partial x \partial y} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y)$$

(c) the most general semilinear first-order PDE in two variables:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

16. The following PDEs are:

(a) $x\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ ... is a second-order linear homogeneous PDE

(b) $\dfrac{\partial^2 u}{\partial x^2} + u\dfrac{\partial u}{\partial y} = xyu$ ...is a second-order semilinear PDE

(c) $\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial u}{\partial t} = \sin(u)$ ... is a second-order quasilinear PDE

(d) $\left(\dfrac{\partial u}{\partial t}\right)^2 + \dfrac{\partial^3 u}{\partial x^3} = 0$ ... is a third-order quasilinear PDE

(e) $\left(\dfrac{\partial u}{\partial t}\right)^2 + \left(\dfrac{\partial u}{\partial x}\right)^3 = 0$ ... is a first-order fully nonlinear PDE

(f) $x^3\dfrac{\partial u}{\partial x} - u^3\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial t^2} = x^5 + t^4$ ... is a second-order semilinear PDE

17. We define the operator $\mathcal{L}$ by

$$\mathcal{L}u(x,y) = a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} + c(x,y)\frac{\partial^2 u}{\partial x \partial y}$$

First, we can state that $\mathcal{L}$ is an operator which involves the second-order partial derivatives of the function u, it is a differential operator. Let $(\alpha, \beta) \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \mapsto \mathbb{R}$ twice differentiable, we have the following:

$$\begin{aligned}
\mathcal{L}(\alpha u + \beta v) &= a(x,y)\frac{\partial^2}{\partial x^2}(\alpha u + \beta v) + b(x,y)\frac{\partial^2}{\partial y^2}(\alpha u + \beta v) + c(x,y)\frac{\partial^2}{\partial x \partial y}(\alpha u + \beta v) \\
&= \alpha a(x,y)\frac{\partial^2 u}{\partial x^2} + \beta a(x,y)\frac{\partial^2 v}{\partial x^2} + \alpha b(x,y)\frac{\partial^2 u}{\partial y^2} + \beta b(x,y)\frac{\partial^2 v}{\partial y^2} + \alpha c(x,y)\frac{\partial^2 u}{\partial x \partial y} + \beta c(x,y)\frac{\partial^2 v}{\partial x \partial y} \\
&= \alpha\left[a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} + c(x,y)\frac{\partial^2 u}{\partial x \partial y}\right] + \beta\left[a(x,y)\frac{\partial^2 v}{\partial x^2} + b(x,y)\frac{\partial^2 v}{\partial y^2} + c(x,y)\frac{\partial^2 v}{\partial x \partial y}\right] \\
&= \alpha \mathcal{L}u + \beta \mathcal{L}v
\end{aligned}$$

where we have used the linearity of partial differentiation. We conclude that $\mathcal{L}$ is a linear differential operator.