5 MATH40003 Linear Algebra and Groups, Term 2

5.0 Preparatory material

I will assume that you have seen determinants, eigenvalues and eigenvectors of 2×2 and 3×3 matrices before. If you have not, or you need to revise this, then you can look at the following preparatory material and do some of the exercises on Question Sheet 1. The lectures start with Section 5.1.

5.0.1 2×2 Determinants

Definition: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with entries in a field F. Then the **determinant** of A is

$$\det(A) := ad - bc.$$

Instead of $\det(A)$ we may also write $|A|$.

For example, $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2$.

We list some simple properties of 2×2 determinants which will generalize to larger matrices. These relate to row operations, so we will often write matrices in terms of their rows:

$$A = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

We have

(i) $\det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \det \begin{pmatrix} R_1 \\ R_2' \end{pmatrix} = \det \begin{pmatrix} R_1 \\ R_2 + R_2' \end{pmatrix}$.

(ii) $\det \begin{pmatrix} \lambda R_1 \\ R_2 \end{pmatrix} = \lambda \det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$, for $\lambda \in F$.

(iii) $\det \begin{pmatrix} R_1 + \lambda R_2 \\ R_2 \end{pmatrix} = \det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$, for $\lambda \in F$.

(iv) $\det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\det \begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$.

(v) $\det \begin{pmatrix} R_1 \\ R_1 \end{pmatrix} = 0$ and $\det \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = 0$.

(vi) $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$.

(vii) $\det(A^T) = \det(A)$.

REMARK Note that (vii) implies that properties analogous to (i)-(v) hold for the columns of A also.

These properties are easily verified directly from the definition. For instance, (i) says that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a & b \\ e & f \end{pmatrix} = \det\begin{pmatrix} a & b \\ c+e & d+f \end{pmatrix}$. We can verify this by expanding out both sides

$$\det\begin{pmatrix} a & b \\ c+e & d+f \end{pmatrix} = a(d+f)-b(c+e) = (ad-bc)+(af-be) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a & b \\ e & f \end{pmatrix}.$$

□

In particular, properties (ii)-(iv) tell us what happens to determinants when we apply elementary row operations.

COROLLARY 5.0.1. *If A, B are 2×2 matrices and B is obtained from A by elementary row operations then* $\det(B) = \alpha \det(A)$, *for some non-zero* $\alpha \in F$. *In particular,* $\det(B) = 0$ *if and only if* $\det(A) = 0$.

We can now prove the following in a way which generalises:

PROPOSITION 5.0.2. *Let A be a 2×2 matrix. Then A is invertible if and only if* $\det(A) \neq 0$.

Proof: Recall what happens with Gaussian elimination: we either obtain the identity matrix, or a matrix with a row of zeros. If A is row-equivalent to the identity matrix, then A is invertible. Moreover, in this case, the above Corollary shows that $\det(A) \neq 0$. In the other case, A is row-equivalent to a matrix with a row of zeros and is not invertible. It is clear that the determinant of a matrix with a row of zeros is 0, so in this case $\det(A) = 0$, again by the Corollary. □

5.0.2 3×3 Determinants

DEFINITION: Let $A = (a_{ij})$ be a 3×3 matrix over a field F. Then the **determinant** of A is

$$\begin{aligned}\det(A) \ :=\ & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}.\end{aligned}$$

The correct way to remember this formula, which will generalise to larger matrices, is to write the determinant in terms of 2×2 determinants. For $1 \leq i, j \leq 3$, we define the ij^{th}

minor of A to be the 2×2 matrix obtained by deleting the i^{th} row and the j^{th} column from A. For example,

$$A_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}.$$

Then the following gives a formula for **expanding the determinant along the 1^{st} row**:

LEMMA 5.0.3. *If $A = (a_{ij})$ is a 3×3 matrix then*

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}).$$

Proof: Expand out the right hand side and compare with the definition of determinant. □

Indeed, more generally, we can expand along any row or column. For $1 \leq l \leq 3$ we have

$$\begin{aligned} \det(A) &= \sum_{j=1}^{3} (-1)^{l+j} a_{lj} \det(A_{lj}) && \text{expanding along the } l^{\text{th}} \text{ row}, \\ &= \sum_{i=1}^{3} (-1)^{i+l} a_{il} \det(A_{il}) && \text{expanding along the } l^{\text{th}} \text{ column}. \end{aligned}$$

For example, let us compute the determinant of the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$.

Expanding along the first row, we get

$$\det(A) = 1\det \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} - (-1)\det \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 1\det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 1 + 5 + 2 = 8.$$

Alternatively, expanding down the second column, we get

$$\det(A) = -(-1)\det \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 0 - 1\det \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = 5 + 0 + 3 = 8.$$

The following theorem (which we will prove later for $n \times n$ matrices) gives properties analogous to those we already listed for 2×2 determinants.

THEOREM 5.0.4. *Let I_3 denote the (3×3) identity matrix. Then*

(i) $\det(I_3) = 1$.

(ii) Suppose A, B are 3×3 matrices and B is obtained from A by performing one row operation. Then

Type I: interchange two rows	$\det(B) = -\det(A)$,
Type II: multiply a row by $0 \neq \alpha \in F$	$\det(B) = \alpha\det(A)$,
Type III: add a multiple of one row to another	$\det(B) = \det(A)$.

(iii) If A, B are 3×3 matrices and B is row equivalent to A then $\det(A) = 0$ if and only if $\det(B) = 0$.

In particular, this means we can use row operations to evaluate determinants:

EXAMPLES:

(1)
$$\det\begin{pmatrix}1 & -1 & 1\\ 2 & 0 & -1\\ 1 & 1 & 2\end{pmatrix} = \det\begin{pmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 0 & 2 & 1\end{pmatrix} = \det\begin{pmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 0 & 0 & 4\end{pmatrix}$$

$$= 2.4.\det\begin{pmatrix}1 & -1 & 1\\ 0 & 1 & -\frac{3}{2}\\ 0 & 0 & 1\end{pmatrix} = 8\det\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = 8\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = 8.$$

(2) Let x, y, z be real numbers. Then

$$\begin{aligned}\det\begin{pmatrix}x & x^2 & x^3\\ y & y^2 & y^3\\ z & z^2 & z^3\end{pmatrix} &= xyz\det\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix} = xyz\det\begin{pmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{pmatrix}\\ &= xyz(y-x)(z-x)\det\begin{pmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 1 & z+x\end{pmatrix}\\ &= xyz(y-x)(z-x)\det\begin{pmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 0 & z-y\end{pmatrix}\\ &= xyz(y-x)(z-x)(z-y).\end{aligned}$$

□

EXACTLY the same proof as for the 2×2 case gives:

PROPOSITION 5.0.5. *Let A be a 3×3 matrix over a field F. Then A is invertible if and only if $\det(A)\neq 0$.*

5.0.3 Eigenvalues and eigenvectors

Let A be an $n\times n$ matrix over a field F. We are interested in non-zero vectors $v\in F^n$ such that

$$Av = \lambda v \quad \text{for some} \quad \lambda\in F.$$

These play a crucial role in many part of mathematics and applications.

DEFINITION: Let A be an $n\times n$ matrix over a field F and let $\lambda\in F$. Then $v\in F^n$ with $v\neq 0$ is called an **eigenvector** of A with **eigenvalue** λ if

$$Av = \lambda v\,.$$

How could one find such eigenvectors and eigenvalues? The trick is to note that

$$Av = \lambda v \Leftrightarrow (\lambda I_n - A)v = 0$$

and there is a **non-zero** $v \in F^n$ satisfying this if and only if the matrix $(\lambda I_n - A)$ is not invertible. This gives:

PROPOSITION 5.0.6. *Suppose that A is an $n \times n$ matrix over a field F. Then $\lambda \in F$ is an eigenvalue of A if and only if* $\det(\lambda I_n - A) = 0$.

We will see that $\det(\lambda I_n - A)$ is polynomial in λ of degree n. It is called the *characteristic polynomial* of A. So finding the eigenvalues in F of A amounts to finding the roots in F of this polynomial. Note that there will be at most n eigenvalues of A.

EXAMPLE: Let $F = \mathbb{R}$. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det \begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2 - 4 = (\lambda - 3)(\lambda + 1).$$

So the eigenvalues of A (in $\mathbb{R}$) are 3 and -1. $\square$

Once we know the eigenvaules of the matrix A, we can find the corresponding eigenvectors by considering each eigenvalue λ in turn and solving the system of linear equations:

$$(A - \lambda I_n)v = 0.$$

The non-zero solutions give us the eigenvectors. This is a finite problem (as we have at most n eigenvalues), though it's usually a bit tedious unless you really like solving systems of linear equations.

EXAMPLE: In the above example, the eigenvectors are obtained by considering the systems

$$(A - 3I_2)x = 0 \quad \text{and} \quad (A + I_2)y = 0,$$

and hence we have the corresponding eigenvectors

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

together with all non-zero scalar multiples of x and y. $\square$

Suppose we have a basis $v_1, \ldots, v_n$ of F^n consisting of eigenvectors of A. Let $Av_i = \lambda_i v_i$. If Q is the matrix whose columns are $v_1, \ldots, v_n$, then using the change of basis formula we obtain that $Q^{-1}AQ$ is the *diagonal matrix* whose diagonal entries are $\lambda_1, \ldots, \lambda_n$ (so the ii-entry is λ_i and if $i \neq j$ the ij-entry is 0).

EXAMPLE: In the above example we have $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then you can check that

$$Q^{-1}AQ = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\square$