Linear Algebra **Unseen Problem 4**

This week's problem is about *conjugacy classes* in the general linear group $GL(n, \mathbb{C})$. In case you haven't seen the definition, for a group G, we can define an equivalence relation $\sim$ on G by

$$x \sim y \Leftrightarrow \exists g \in G \text{ such that } y = g^{-1}xg.$$

The equivalence classes are called the conjugacy classes of G. So the conjugacy class containing $x \in G$ is $\{g^{-1}xg : g \in G\}$.

In the group $GL(n, \mathbb{C})$, the conjugacy class of an element A is $\{P^{-1}AP : P \in GL(n, \mathbb{C})\}$, which is just the set of all matrices similar to A.

(i) Suppose $A \in GL(n, \mathbb{C})$ has finite order k (recall this means that k is the least positive integer such that $A^k = I$). Show that every element in the conjugacy class of A also has order k.

(ii) Suppose $A \in GL(n, \mathbb{C})$ has finite order k. Show that A is diagonalisable. (Hint: there is a relevant question on Problem Sheet 4.)

(iii) Calculate the number of conjugacy classes of elements of order 2 in $GL(n, \mathbb{C})$.

(iv) Calculate the number of conjugacy classes of elements of order 3 in $GL(n, \mathbb{C})$.

(v) Can you generalize to some higher orders? Start with $GL(2, \mathbb{C})$.