Theory of the Consumer

We now focus on the theory of the consumer, where we will formalise the notion of consumer preferences and show how optimal behaviour of the consumer with respect to their preferences will lead to a specification of the demand function.
In the course of our analysis, we will see a lot of similarities and analogies to the Theory of the Firm.

Preferences & Utility

We start by considering the goods consumed by a consumer.

Define the ***consumption bundle*** for a particular consumer to be the quantities of a collection of goods that the consumer is willing to consume:

$$\underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_{\geq 0}.$$

The set of possible consumption bundles is referred to as the ***consumption set***; this is usually taken to be some *closed and convex set*

$$X \subseteq \mathbb{R}^n_{\geq 0}.$$

Consumers are assumed to have preferences between bundles $\underline{x}, \underline{x}' \in X$:

- $\underline{x} \preccurlyeq \underline{x}'$ means that the consumer has a preference for bundle $\underline{x}'$ over bundle $\underline{x}$.

i.e., the consumer wants $\underline{x}'$ at least as much as they want $\underline{x}$

- $\underline{x} \prec \underline{x}'$ means that the consumer has a *strict* preference for $\underline{x}'$ over $\underline{x}$.

i.e., the consumer wants $\underline{x}'$ more than they want $\underline{x}$

(so $\underline{x} \prec \underline{x}' \Leftrightarrow (\underline{x} \preccurlyeq \underline{x}' \wedge \underline{x}' \not\preccurlyeq \underline{x})$)

- $\underline{x} \sim \underline{x}'$ denotes indifference between $\underline{x}$ and $\underline{x}'$.

(so $\underline{x} \sim \underline{x}' \Leftrightarrow (\underline{x} \preccurlyeq \underline{x}' \wedge \underline{x}' \preccurlyeq \underline{x})$)

We are working under the condition that the preference relation satisfies the three axioms of a **complete weak order** on X. That is

- Completeness: $\forall \underline{x}, \underline{x}' \in X$, $\underline{x} \precsim \underline{x}'$ or $\underline{x}' \precsim \underline{x}$
  (i.e., any two bundles can be compared for preference.)
- Reflexivity: $\forall \underline{x} \in X$, $\underline{x} \precsim \underline{x}$
- Transitivity: $\forall \underline{x}, \underline{x}', \underline{x}'' \in X$, if $\underline{x} \precsim \underline{x}'$ and $\underline{x}' \precsim \underline{x}''$
  then $\underline{x} \precsim \underline{x}''$

Beware that reflexivity actually follows from completeness.

In addition, the following assumptions are *useful* but not necessary: (Axioms of consumer preferences)

Continuity

$\forall \underline{x} \in X$, the sets $\{\underline{x}' \in X : \underline{x} \precsim \underline{x}'\}$ and $\{\underline{x}' \in X : \underline{x}' \precsim \underline{x}\}$ are both closed. (One definition of a closed set is that any sequence of points in the set that converges, converges to a point in the set.) Roughly speaking, if bundles $\underline{x}'$ and $\underline{x}''$ are very similar, and $\underline{x}'$ is preferred to $\underline{x}$, then so should $\underline{x}'$ be. Or, if $\underline{x}$ is preferred to $\underline{x}'$, it should also be preferred to $\underline{x}''$.

Weak / Strong Monotonicity ('More is preferable to less')

$$\underline{x} \le \underline{x}' \quad \Longrightarrow \quad \underline{x} \preccurlyeq \underline{x}' \quad \text{(weak)}$$

$$\underline{x} \le \underline{x}' \text{ and } \underline{x} \ne \underline{x}' \quad \Longrightarrow \quad \underline{x} \prec \underline{x}' \quad \text{(strong)}$$

Local nonsatiation

$\forall \underline{x} \in X$ and $\forall \varepsilon > 0$, $\exists \underline{x}' \in X$ with $\|\underline{x} - \underline{x}'\| < \varepsilon$ and $\underline{x} \prec \underline{x}'$.

i.e., for any bundle $\underline{x}$, there is always another bundle $\underline{x}'$ arbitrarily close to $\underline{x}$ that is strictly preferred to it.

(Strict) Convexity

Convexity:

$\forall \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \precsim \underline{x}'$ and $\underline{x} \precsim \underline{x}''$

$$\underline{x} \precsim t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in [0,1].$$

Strict convexity:

$\forall \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \precsim \underline{x}'$ and $\underline{x} \precsim \underline{x}''$ and $\underline{x}' \neq \underline{x}''$

$$\underline{x} \prec t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in (0,1).$$

Note – we have not yet used the symbols $\succcurlyeq$ or $\succ$; we can use this as would be expected, i.e.

$$\underline{x} \precsim \underline{x}' \iff \underline{x}' \succsim \underline{x}$$

but it is no more than a notational convenience.

How does a consumer decide between bundles in some subset of X? How do we judge the suitability, or usefulness, of a consumption bundle $\underline{x}$? More to the point, how can we, as economists, model the unobserved preference allocation of consumers?

It is useful to model consumer preferences by a ***utility function***, which we define to be a real mapping $u: X \to \mathbb{R}$.
We say that u ***represents the preference relation*** $\preccurlyeq$ if

$$\forall \underline{x}, \underline{x}' \in X: \quad u(\underline{x}') \leq u(\underline{x}) \iff \underline{x}' \precsim \underline{x}$$

- If only the ordering imposed by a utility function is relevant, one speaks of an **ordinal utility**. If u is an ordinal utility, any strictly increasing transformation of u represents the same preferences.

That is, if one is only interested in whether a consumer prefers $\underline{x}$ to $\underline{x}'$ and not by how much the consumer prefers $\underline{x}$ to $\underline{x}'$, then one considers an ordinal utility.

Eg. The preferences $\underline{x} \prec \underline{x}' \prec \underline{x}''$ can be represented by the utility function

$$u(\underline{x}) = 1, \quad u(\underline{x}') = 3, \quad u(\underline{x}'') = 8$$

or by

$$v(\underline{x}) = 2, \quad v(\underline{x}') = 5, \quad v(\underline{x}'') = 10$$

The functions u and v are said to be ordinally equivalent. And if $g(u)$ is a strictly increasing transformation of u, then it, too, will be ordinally equivalent to u.

Note that we will only consider ordinal utilities.

- If one wants to compare different utility differences, say $|u(\underline{x}) - u(\underline{x}')|$, i.e., if one is interested in by how much a consumer prefers, say, $\underline{x}$ to $\underline{x}'$,

  one speaks of a **cardinal utility.** Cardinal utilities are in general only preserved by affine and increasing transformations. (eg, $u \mapsto 2u+1$, but not $u \mapsto -2u+1$).
  (so as to preserve order)

Existence of an (ordinal) utility function: (Debreu's Theorem, 1954)

Suppose a consumption set X is imbued with a preference relation that is complete, transitive, continuous and strongly monotonic. Then there exists a continuous utility function $u : X \to \mathbb{R}$ that represents this preference relation.

Note – the assumption of strong monotonicity can be dropped, though the proof is more complex.

Proof:

Outline:

- We will consider bundles of goods that contain the same amount of each good, i.e. 'homogeneous' bundles;
- We will show that if, for every $\underline{x} \in X$, there exists a homogeneous bundle to which the consumer is indifferent, then the level of the homogeneous bundle can be taken as an appropriate utility function, i.e. one that preserves the ordering of $\succcurlyeq$;
- We will then show that such a homogeneous bundle exists and is unique.

Details:

Let $\underline{e} = (1, \dots, 1)$ be a length n vector of 1's. ($\underline{e}$ is a homogeneous bundle of level 1.)

Suppose that for any consumption bundle $\underline{x} \in X$ there exists $u(\underline{x}) \in \mathbb{R}$ such that $u(\underline{x}) . \underline{e} \sim \underline{x}$ (†).

We will now show that $u(\underline{x})$ represents the preference relation $\succcurlyeq$.

Indeed, for any $\underline{x}, \underline{x}' \in X$ with $\underline{x} \neq \underline{x}'$

$$u(\underline{x}) > u(\underline{x}') \Rightarrow u(\underline{x}).\underline{e} > u(\underline{x}').\underline{e} \quad \text{(component-wise)}$$

$\downarrow$ strong monotonicity

$$\Rightarrow u(\underline{x}).\underline{e} \succ u(\underline{x}').\underline{e}$$

$\downarrow$ by transitivity and (†)

$$\Rightarrow \quad \underline{x} \succ \underline{x}' \quad (1)$$

Similarly, one can show that $u(\underline{x}) \leq u(\underline{x}') \Rightarrow \underline{x} \precsim \underline{x}'$ (2)

It follows from (1) and (2) that $u(\underline{x}') \leq u(\underline{x}) \Leftrightarrow \underline{x}' \precsim \underline{x}$.

So $u(\underline{x})$ represents the preference relation $\precsim$.

Now, to prove the existence of $u(\underline{x})$, let $\underline{x} \in X \subseteq \mathbb{R}^n_{\geq 0}$.

Define

$B = \{t \in \mathbb{R} \mid t\underline{e} \succsim \underline{x}\}$ and $W = \{t \in \mathbb{R} \mid t\underline{e} \precsim \underline{x}\}$.

Note that

$$\left(\max_i x_i\right) \cdot \underline{e} \geq \underline{x} \Rightarrow \left(\max_i x_i\right) \cdot \underline{e} \succsim \underline{x}$$

$$\Rightarrow \left(\max_i x_i\right) \in B$$

So B is non-empty.

And $0 \cdot \underline{e} \precsim \underline{x}$, so W is also non-empty.

Also, by the continuity of $\precsim$, B and W are both <u>closed</u>.

Then, since B is non-empty and closed it has upper and lower bounds which are also contained in B. Similarly for W. Set

$$t^* = \inf B \in B \quad (\text{a lower bound of } B)$$

and let

$$t_n = t^* - \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Then
$$t_n < t^* \Rightarrow t_n \notin B$$
$$\Rightarrow t_n \cdot \underline{e} \prec \underline{x}$$
$$\Rightarrow t_n \in W$$

Furthermore, $t_n \to t^*$ as $n \to \infty$, and W is closed, so $t^* \in W$. But then since $t^* \in B$ and $t^* \in W$ then $t^* \underline{e} \sim \underline{x}$, so $u(\underline{x}) = t^*$ exists.

Finally, we can prove the uniqueness of $u(\underline{x})$ as follows. Suppose $\underline{x} \sim u_1(\underline{x})\underline{e}$ and also that $\underline{x} \sim u_2(\underline{x})\underline{e}$.

Then,
$$u_1(\underline{x})\underline{e} \succsim \underline{x} \succsim u_2(\underline{x})\underline{e}$$
by transitivity
$$\Rightarrow u_1(\underline{x})\underline{e} \succsim u_2(\underline{x})\underline{e}$$

by monotonicity, for $a, b \in \mathbb{R}$,
$$a\underline{e} \geq b\underline{e} \Rightarrow a\underline{e} \succsim b\underline{e}$$
and $$b\underline{e} > a\underline{e} \Rightarrow b\underline{e} \succ a\underline{e}$$
So in fact
$$a\underline{e} \succsim b\underline{e} \Leftrightarrow a\underline{e} \geq b\underline{e}$$
$$\Leftrightarrow a \geq b$$

$$\Rightarrow \quad u_1(\underline{x}) \geq u_2(\underline{x})$$

But similarly, one can show that $u_2(\underline{x}) \geq u_1(\underline{x})$

Hence $u_1(\underline{x}) = u_2(\underline{x})$. //