3.3 Subsequences

> **Definition.** A *subsequence* of (a_n) is a new sequence $b_i = a_{n(i)}$ $\forall i \in \mathbb{N}_{>0}$, where $n(1) < n(2) < \cdots < n(i) < \dots$ $\forall i$.

Formally $n(\,\cdot\,)$ is a function $\mathbb{N}_{>0} \to \mathbb{N}_{>0}$ sending $i \mapsto n(i)$ which is strictly monotonically increasing. "Just go down the sequence faster, missing some terms out".

Exercise 3.32. Prove this implies $n(i) \geq i$ by induction.
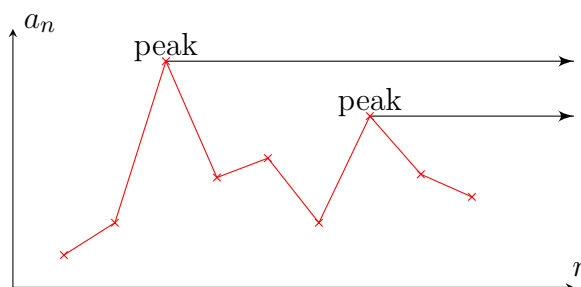
Example 3.33. $a_n = (-1)^n$ has subsequences:

- $b_n = a_{2n}$, so $b_n = 1$ $\forall n \implies b_n \to 1$.

- $c_n = a_{2n+1}$, so $c_n = -1$ $\forall n \implies c_n \to -1$.

- $d_n = a_{3n}$, so $d_n = (-1)^n$ $(= a_n)$ doesn't converge.

- $e_n = a_{n+17}$, so $e_n = (-1)^{n+1}$ $(= -a_n)$ doesn't converge.

Next we work up to the following technical-sounding but vitally important:

> **Theorem 3.34: Bolzano-Weierstrass**
>
>

Remark 3.35. Of course it will have *many* convergent subsequences, and they may converge to different limits; think of $a_n = (-1)^n$ for instance.

Cheap proof. Use "peak points" of (a_n):

We say that j is a *peak point* if and only if $a_k < a_j$ $\forall k > j$. Either

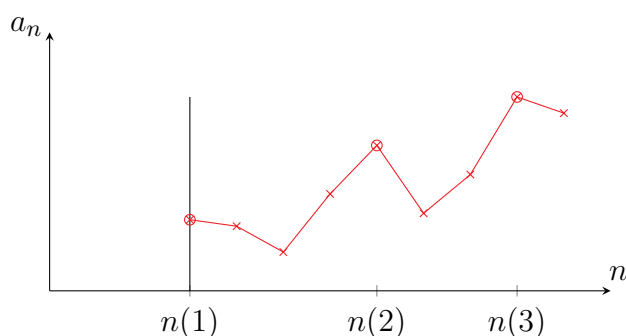1. (a_n) has a finite number of peak points, or

2. (a_n) has an infinite number of peak points.

Case 1: We go beyond the (finitely many) peak points: pick $n(1) \geq \max\{j_1, \ldots, j_k\}$ where $\{j_1, \ldots, j_k\}$ are the peak points.

$a_{n(1)}$ is not a peak point $\implies \exists\, n(2) > n(1)$ such that $a_{n(2)} \geq a_{n(1)}$.
$a_{n(2)}$ not a peak point $\implies \exists\, n(3) > n(2)$ such that $a_{n(3)} \geq a_{n(2)}$.
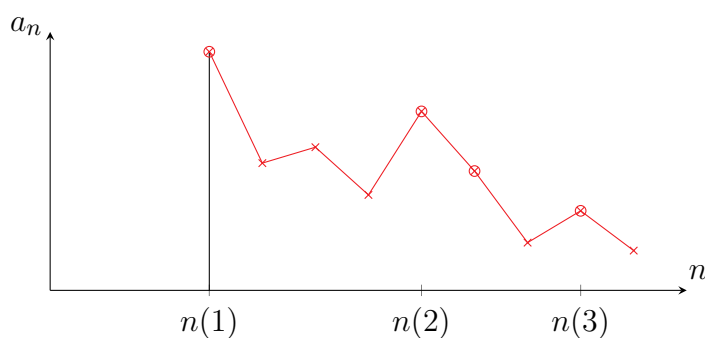Recursively no peak points beyond $n(1) \implies$ we get $n(i) > n(i-1) > \cdots > n(1)$ such that $a_{n(i)} \geq a_{n(i-1)} \;\forall i$.

So $a_{n(i)}$ is a monotonically increasing subsequence of a_n.
$(a_n)_{n \geq 1}$ bounded $\implies (a_{n(i)})_{i \geq 1}$ is bounded $\implies a_{n(i)} \uparrow \sup\{a_{n(i)} : i \in \mathbb{N}_{>0}\}$ by Theorem 3.21.

Case 2: There are infinitely many peak points, so we may call them $n(1), n(2), \ldots$ where $n(1) < n(2) < \ldots$. Then we choose our sequence to be $a_{n(i)}$.

Now $n(i+1) > n(i)$ and $a_{n(i)}$ is a peak point $\implies a_{n(i+1)} \leq a_{n(i)}$. Thus the subsequence $(a_{n(i)})_{i \geq 1}$ is monotonically decreasing and bounded $\implies$ convergent (to $\inf\{a_{n(i)} : i \in \mathbb{N}_{>0}\}$). $\qquad\square$

Exercise 3.36. Give an example of an unbounded sequence with a convergent subsequence.

Exercise 3.37. Given an example, with proof, of a sequence for which every subsequence is divergent.

Exercise 3.38. Give an example of an unbounded sequence that has at least three convergent subsequences that converge to three different limits.

Proposition 3.39. *If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.*

Proof.

$\square$
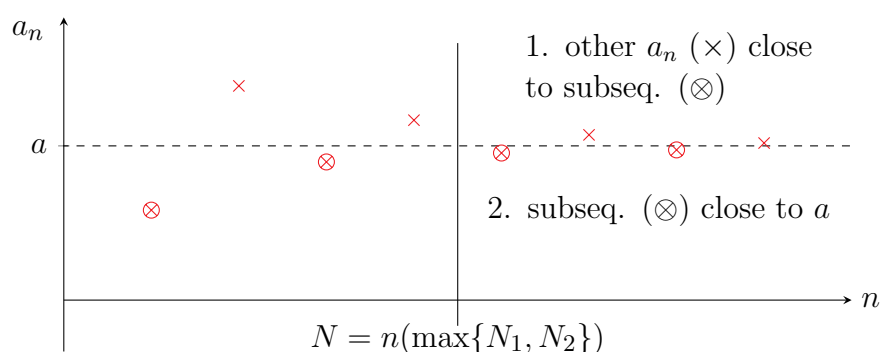
This gives us another proof that $(-1)^n$ is not convergent, because if $(-1)^n \to a$, then by Proposition 3.39, $(-1)^{2n} \to a$ and $(-1)^{2n+1} \to a \implies a = 1$ and $a = -1$ ※

Bolzano-Weierstrass $\implies$ the Cauchy theorem

We also get another proof of "Cauchy $\implies$ convergence" using Bolzano-Weierstrass.

Proof #2 of Cauchy $\implies$ Convergence. We know from Lemma 3.26 that a_n is bounded (by $\max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$ remember). So by Bolzano-Weierstrass, $\exists$ a convergent subsequence $(a_{n(i)})_{i \geq 1}$ such that $a_{n(i)} \to a$ as $i \to \infty$ for some $a \in \mathbb{R}$. So fix $\epsilon > 0$. We have:

(1) $\exists N_1$ such that $\forall n, m \geq N_1$, $|a_n - a_m| < \epsilon$ (Cauchy)

(2) $\exists N_2$ such that $\forall i \geq N_2$, $|a_{n(i)} - a| < \epsilon$ (convergent subsequence)

Set $N = n(\max\{N_1, N_2\}) \geq \max\{N_1, N_2\} \geq N_1$. Then $\forall n \geq N$ we have

$$
\begin{aligned}
|a_n - a| &= |(a_n - a_N) + (a_N - a)| \\
&\leq |a_n - a_N| + |a_N - a| \\
&< \epsilon + \epsilon = 2\epsilon,
\end{aligned}
$$

the first $< \epsilon$ being by the Cauchy property (1) and the second $< \epsilon$ being from the convergence of the subsequence property (2) (since a_N is in the subsequence). $\qquad \square$

Above, we used the following lemma.

Lemma 3.40. *Fix $c > 0$. Then $a_n \to a$ if and only if*

$$
\boxed{} \tag{$*$}
$$

Proof. $\Longrightarrow$. Fix $\epsilon > 0$ and let $\epsilon' := c\epsilon$. Then by the definition of convergence applied to $\epsilon' > 0$ we find

$$
\exists N \in \mathbb{N} : \; n \geq N \Longrightarrow |a_n - a| < \epsilon',
$$

which is $(*)$.

$\Longleftarrow$. Fix $\epsilon > 0$. Set $\epsilon' = \epsilon/c > 0$. Then $(*)$ applied to $\epsilon' > 0$ implies

$$
\exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N_\epsilon \Longrightarrow |a_n - a| < c\epsilon' = \epsilon. \qquad \square
$$

Warning. Do not let c depend on ϵ (nor N! or n).

E.g. if we let $c = \epsilon^{-1}$ then $(*)$ becomes $\forall \epsilon > 0$, $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$, $|a_n - a| < 1$. This is not a good definition of convergence; for instance it would say that the sequence $a_n = 1 \; \forall n$ converges to $\frac{3}{2}$!

Bolzano-Weierstrass $\Longleftarrow$ the Cauchy theorem

We can also go the other way round: the Cauchy theorem $\Longrightarrow$ Bolzano-Weierstrass.

Proof 2 of Bolzano-Weierstrass. Take a bounded sequence (a_n). We want to find a Cauchy subsequence, which will therefore be convergent.

Since $a_n \in [-R, R] \; \forall n$, repeatedly subdivide to make this interval smaller. Then either

1. $a_n \in [-R, 0]$ for infinitely many n or

2. $a_n \in [0, R]$ for infinitely many n, (or both).

Pick one of these intervals which contain a_n for infinitely many n, call it $[A_1, B_1]$ of length R.

Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ with contain a_n for infinitely many a_ns in it, with length $R/2$. Etc.

Recursively we get a sequence of intervals $[A_n, B_n]$ of length $2^{1-n}R$ which are nested – i.e. $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$ – with each containing a_n for infinitely many n.

Now we use a *diagonal argument.*
Choose $n(1)$ so that $a_{n(1)} \in [A_1, B_1]$.
Choose $n(2) > n(1)$ so that $a_{n(2)} \in [A_2, B_2]$. (Recall there are infinitely many a_n in each $[A_k, B_k]$, so we can do this.)
Recursively choose $n(k+1) > n(k)$ so that $a_{n(k+1)} \in [A_{k+1}, B_{k+1}]$.

Claim: the subsequence $a_{n(i)}$ is convergent.

Fix $\epsilon > 0$. Take $N > \frac{2R}{\epsilon}$, so that $2^{1-N}R < 2N^{-1}R < \epsilon$. Then $\forall i, j \geq N$ we have $n(i) \geq i \geq N$ and $n(j) \geq j \geq N$, so $a_{n(i)}, a_{n(j)} \in [A_N, B_N]$ and

$$|a_{n(i)} - a_{n(j)}| \ < \ 2^{1-N}R \ < \ \epsilon.$$

Therefore $(a_{n(i)})$ is Cauchy and so convergent. $\hspace{1cm}$ □

Definition. We say $a_n \to +\infty$ if and only if

Remark 3.41. Recall this is not the same as (but it does imply) a_n being divergent!

Exercise 3.42. Suppose $a_n > 0 \ \forall n$. Show $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$.

4 Series

*Maths is not a spectator sport. How well you do comes down solely to the time you spend **doing** maths.*

- Richard Thomas, annually

> **Definition.** An (infinite) series is an expression
>
> $$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots ,$$
>
> where $(a_i)_{i \geq 1}$ is a sequence.

For now, it is **not** a real number. It is just a formal expression. We could write $\sum_{n=1}^{\infty} n$, for instance, without worrying about convergence (just as we write $a_n = n$ without worrying about convergence).

Partial sums

Given a sequence a_n we get a series (formal expression!) $\sum_{n=1}^{\infty} a_n$ and another sequence of **partial sums**

$$(*)$$

Recall in Exercise 3.2 you proved that a_n and s_n determine each other – they are equivalent information. In other words, the sequence (a_n) determines the sequence (s_n) by $(*)$, and conversely we can recover (a_n) from the (s_n) by

4.1 Convergence of Series

> **Definition.** We say that the series $\sum a_n$ "converges to $A \in \mathbb{R}$" if and only if

We often write A as $\sum_{n=1}^{\infty} a_n$. In other words, if $\sum_{n=1}^{\infty} a_n$ converges (to A) then we use the same notation to denote the real number A.

We can obviously let the sum be from $n = 0$, or over n even, or ...

Example 4.1. Consider $a_n = x^n$, $n \geq 0$, so that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

The partial sums are

$$s_n = \sum_{i=0}^{n} x^i = 1 + x + \cdots + x^n.$$

Therefore

$$x s_n = x + \cdots + x^n + x^{n+1},$$

so

$$s_n - x s_n = 1 - x^{n+1},$$

giving

$$s_n = \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & x \neq 1, \\ n + 1 & x = 1. \end{cases}$$

So for $|x| < 1$, we see that

$$s_n = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x} \longrightarrow \frac{1}{1 - x} \quad \text{as } n \to \infty.$$

(Recall from the question sheet that $r^n \to 0$ if $|r| < 1$.)

So (s_n) is convergent and we can finally say $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$ if $|x| < 1$.

For $|x| \geq 1$, $a_n = x^n \nrightarrow 0$ as $n \to \infty$. So $\sum a_n = \sum x^n$ is *not* a real number (does not converge) by the next result.

Theorem 4.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$.

Remark 4.3. Converse is *not* true. E.g. $a_n = \frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ is *not* convergent.

Example 4.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Proof. Uses a slight trick. Arrange the partial sum as follows:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \cdots + \frac{1}{7}\right)$$
$$+ \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{31}\right) + \ldots$$

We can bound the kth bracketed term from below:

$$\left(\frac{1}{2^k} + \cdots + \frac{1}{(2^{k+1} - 1)}\right) > \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} = \frac{2^k}{2^{k+1}} = \frac{1}{2}.$$

In particular then

$$s_{2^{k+1}-1} > 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{k \text{ terms}} = 1 + \frac{k}{2}$$

is arbitrarily large. But if s_n converged, it would be bounded: $|s_n| \le C \ \forall n$. So we get the contradiction (to the Archimedean property) $1 + \frac{k}{2} \le C \ \forall k \in \mathbb{N}$. $\square$

Example 4.5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. (Using a trick; we will give another proof soon.) First show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

is convergent, using $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$
\begin{aligned}
s_n &= \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \longrightarrow 1 \ \text{ as } n \to \infty.
\end{aligned}
$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent to 1.

So now compare the partial sums σ_n of $\sum \frac{1}{n^2}$ to those of $\sum \frac{1}{n(n+1)} = 1$.

$$
\begin{aligned}
\sigma_n &= \sum_{i=1}^{n}\frac{1}{i^2} = 1 + \sum_{j=1}^{n-1}\frac{1}{(j+1)^2} \\
&\leq 1 + \sum_{j=1}^{n-1}\frac{1}{j(j+1)} \\
&= 1 + s_{n-1}.
\end{aligned}
$$

$s_{n-1} \uparrow 1$ because $\frac{1}{n(n+1)} > 0$. So $s_{n-1} < 1 \ \forall n \implies \sigma_n < 2 \implies$ bounded above monotonic increasing sequence $\implies \sigma_n$ is convergent $\implies \sum \frac{1}{n^2}$ is convergent. $\quad\square$