Mathematical Logic (MATH6/70132;P65)
Problem Sheet 2

[1] Suppose ϕ is a formula of L.
(a) By giving a deduction in L, show that

$$(\neg(\neg\phi)) \vdash_L \phi.$$

(Hint: Let χ be an axiom. Start the deduction off with $((\neg(\neg\phi)) \to ((\neg(\neg\chi)) \to (\neg(\neg\phi))))$.)
(b) Show that $((\neg(\neg\phi)) \to \phi)$ is a theorem of L.
(c) Use (b) to show that $(\phi \to (\neg(\neg\phi)))$ is a theorem of L.

[2] Suppose Γ is a set of formulas of L and ϕ is a formula. Suppose that $\Gamma \vdash_L \phi$ and v is a valuation with $v(\psi) = T$ for all $\psi \in \Gamma$. Show that $v(\phi) = T$.

[3] (Do not spend a lot of time on this.) Think about how we might give careful proofs of the following facts, which we have been using a lot.

(a) (Unique reading lemma) Suppose ϕ is an L-formula. Then exactly one of the following occurs:

(i) ϕ is a propositional variable;

(ii) there exists a unique L-formula ψ such that ϕ is $(\neg\psi)$;

(iii) there exist unique L-formulas θ, χ such that ϕ is $(\theta \to \chi)$.

(b) Using (a), prove that if v is any function from the set of propositional variables of L to $\{T, F\}$, then there is a unique function w from the set of L-formulas to $\{T, F\}$ satisfying the following properties:

(i) $w(p_i) = v(p_i)$ for each propostional variable p_i;

(ii) for every L-formula ϕ we have $w(\phi) \neq w((\neg\phi))$;

(iii) for all L-formulas θ, χ we have $w((\theta \to \chi)) = F$ iff $w(\theta) = T$ and $w(\chi) = F$.

[4] A *ternary valuation* is a function f from the set of formulas of L to the set $\{0, 1, 2\}$ which satisfies the following 'truth table' rules:

$$f((\neg\phi)) = \begin{cases} 2 & \text{if } f(\phi) = 0, 1 \\ 0 & \text{if } f(\phi) = 2 \end{cases}$$

and

$$f((\phi \to \psi)) = \begin{cases} 0 & \text{if } f(\phi) \geq f(\psi) \\ f(\psi) & \text{otherwise} \end{cases}$$

A formula ϕ is called a *ternary tautology* if $f(\phi) = 0$ for all ternary valuations f.

(a) Let $\alpha(0) = \alpha(1) = T$ and $\alpha(2) = F$. Show that if f is a ternary valuation, then the composition $\alpha \circ f$ is an (ordinary) valuation.

(b) Show that the axioms of L of type A1 are ternary tautologies.

(c) Show that axioms of type A2 are ternary tautologies.

(d) Show that if $(\phi \to \psi)$ and ϕ are ternary tautologies then so is ψ.

(e) Show that the formula $(((\neg p) \to (\neg q)) \to (q \to p))$ is not a ternary tautology.

(f) Show that any formula of the form $((\psi \to \phi) \to ((\neg\phi) \to (\neg\psi)))$ is a ternary tautology.

(g) The formal system $\tilde{L}$ has the same formulas as L and deduction rule Modus Ponens, but has as axioms formulas of types A1 and A2 and all formulas as in (f). Explain why the formula in (e) is not a theorem of $\tilde{L}$.

David Evans, January 2025.