MATH50010 – Autumn 2023 – Midterm

You should state carefully any results from lectures that are used.

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ unless otherwise stated.

(a) (1 mark) Define mathematically what it means for $\mathcal{F}$ to be a sigma algebra.

Let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is said to be a sigma algebra if

(a) $\emptyset \in \mathcal{F}$.

(b) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.

(c) If $A_1, A_2, \dots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

(b) (3 marks) Let $\Omega = \{1, 2, 3, 4\}$. Give an example of a sigma algebra $\mathcal{F}$ on Ω and two functions $X_1, X_2 : \Omega \to \mathbf{R}$ such that X_1 is a random variable with respect to $\mathcal{F}$ but X_2 is not a random variable with respect to $\mathcal{F}$. Briefly justify your answers.

[There are many possible solutions. 1 mark for defining $\mathcal{F}$ as a valid sigma algebra. 1 mark for defining X_1 such that it is a random variable with respect to $\mathcal{F}$, and 1 mark for defining X_2 that is not a random variable with respect to $\mathcal{F}$.]

One solution is $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\}$ with $X_1(\omega) = \begin{cases} 1 & \omega \in \{1,2\} \\ 0 & \omega \in \{3,4\} \end{cases}$ and $X_2(\omega) = \begin{cases} 1 & \omega \in \{1,3\} \\ 0 & \omega \in \{2,4\} \end{cases}$. Then, for any Borel set B, $X_1^{-1}(B) \in \mathcal{F}$, however for, e.g. $B = \{0\}$, $X_2^{-1}(B) = \{2, 4\} \notin \mathcal{F}$ so X_2 is not a random variable with respect to $\mathcal{F}$.

(c) (4 marks) Prove that for any random variable, X, and $x \in \mathbf{R}$, $Pr(X < x) = \lim_{x_n \to x} Pr(X \leq x_n)$ for any strictly increasing sequence $x_n \uparrow x$.

Define the increasing sequence of events

$$A_n = \{\omega \in \Omega : -\infty < X(\omega) \leq x_n\}.$$

Then for any $y < x$, since $x_n \uparrow x$ there exists $n \in \mathbf{N}$ such that $y \leq x_n$. Therefore, for any $\omega \in \Omega$ such that $X(\omega) < x$, it follows that there exists $n \in \mathbf{N}$ such that $X(\omega) \leq x_n$, so

$$\{\omega \in \Omega : -\infty < X(\omega) < x\} \subseteq \bigcup_{n=1}^{\infty} A_n.$$

Additionally, since $x_n \leq x$ for all $n \in \mathbf{N}$, $\bigcup_{n=1}^{\infty} A_n \subseteq \{\omega \in \Omega : -\infty < X(\omega) < x\}$ so the result follows by the continuity property since

$$Pr(X < x) = Pr(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} Pr(A_n) = \lim_{n \to \infty} Pr(X \leq x_n).$$

In the remainder of the question, let X and Y be absolutely continuous random variables with joint probability density function given by

$$f_{XY}(x, y) = \lambda^2 \exp\{-\lambda(x + y)\} \quad \text{for} \quad x > 0, y > 0 \tag{1}$$

and zero otherwise, where $\lambda > 0$ is a constant.

(d) (3 marks) Derive the marginal density functions f_X and f_Y and cumulative distribution functions F_X and F_Y of the random variables X and Y.

We see that the density factorizes, $f_{XY}(x,y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})$ for all $x > 0, y > 0$ (and since it is 0 elsewhere it still factorizes). So we can extract $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$. Note that these are the densities of exponential random variables so we know they integrate to 1.

[Directly integrating out Y is also valid]

For $x \in (0,1)$,

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = \left[-\frac{\lambda}{\lambda} e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x},$$

so that

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & 0 < x \end{cases}$$

By symmetry,

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ 1 - e^{-\lambda y} & 0 < y \end{cases}$$

[2 mark for pdfs, 1 mark for cdfs, ranges must be correct]

(e) (1 mark) Calculate $Cov(X, Y)$, the covariance between X and Y.

Since the joint density of X, Y factorizes for all $(x, y) \in \mathbf{R}^2$, X and Y are independent. Therefore from lecture notes, $Cov(X, Y) = 0$.

(f) (3 marks) Determine the joint probability density function of the random variables $S = X + Y$ and $D = X - Y$.

We can write $X = \frac{S+D}{2}, Y = \frac{S-D}{2}$. Then the Jacobian determinant is $\left|\det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\right| = 1/2$. Hence, we can write the joint pdf of S and D as

$$f_{S,D}(s,d) = 1/2 f_X(\frac{S+D}{2}) f_Y(\frac{S-D}{2}) = \frac{\lambda^2}{2} \exp^{-\lambda(\frac{s+d}{2} + \frac{s-d}{2})} = \frac{\lambda^2}{2} \exp^{-\lambda s}$$

for $s + d > 0, s - d > 0$, or equivalently $s > |d|$.

[Other methods are acceptable. 2 marks for derivation of pdf of S, D, 1 mark for range.]

(g) (2 marks) Calculate $Cov(S, D)$, the covariance between S and D from part (f). Are S and D independent?

By Part (d), X and Y have the same distribution so $E[X - Y] = 0$ and $E[(X - Y)(X + Y)] = E[X^2 - Y^2] = 0$ so $Cov(S, D) = 0$. However, S and D are not independent since their joint density does not factorize for all $s, d \in \mathbf{R}^2$.

(h) (3 marks) Let $U_1 \sim$ Uniform$[0, 1]$ and $U_2 \sim$ Uniform$[0, 1]$ be independent uniform random variables. Explain how to use U_1 and U_2 to obtain a sample of $X + Y$.

We can obtain a sample of X by the probability integral transform $X = F_X^{-1}(U_1) = -\frac{1}{\lambda}\log(1 - U_1)$. Similarly, we can apply the same transform to U_2 to obtain $Y = -\frac{1}{\lambda}\log(1 - U_2)$ since by (d) X and Y have the same distribution. Since U_1 and U_2 are independent, so are X and Y. Therefore, a sample of $X + Y$ can be obtained from $X + Y = -\frac{1}{\lambda}\log(1 - U_1) - \frac{1}{\lambda}\log(1 - U_2)$.