Imperial College London
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Solutions to Question Sheet 0

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

The first problem class will be on Monday of week 2. The following questions are about Lecture 1 and the last few topics from Term 1. The questions are to revise Term 1 material (including the basis change formula). There are few more question on Sheet 1 that can be solved after Lecture 1. Those will be solved in week 3 together with the rest of Sheet 1.

Question 1 Compute the determinant of the identity matrix I_n ($n \in \mathbb{N}$).

Solution: The question is posed in a way that does not reveal the answer. One could look at the notes but let us, just for fun, have a look at the first few cases and make an educated guess. For $n = 1$,

$$\det(1) = 1.$$

For $n = 2$ a Laplace expansion along the first row gives

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot \det(1) + 0 \cdot \det(0) = 1.$$

It is clear now what the guess should be and how to prove it. We claim $\det(I_n) = 1$ for all $n \in \mathbb{N}$. The base case is $n = 1$ and we have already proved it. The inductive hypothesis is that $\det(I_{n-1}) = 1$. By a Laplace expansion on the first row, the determinant of I_n is

$$1 \cdot \det(I_{n-1}) = 1$$

by the inductive hypothesis.

Question 2 Prove that the determinant is linear on the rows. This is Theorem 5.1.5 from the notes.

Solution: We prove the following statement: let $A, B, C \in M_n(F)$ and let $1 \leq \ell \leq n$. Suppose A, B, C are the same except in the ℓ^{th} row, where we have that the ℓ^{th} row of C is the sum of the ℓ^{th} rows of A and B. Then

$$\det(C) = \det(A) + \det(B).$$

Set

$$A = (a_{ij})_{ij}, \quad B = (b_{ij})_{ij}, \quad C = (c_{ij})_{ij}.$$

(This is my notation for "let a_{ij} be the (i, j)-entry of A, etc.). We proceed by induction on the size of the matrix. There is nothing to prove for $n = 1$. Assume the statement true for $(n-1) \times (n-1)$ matrices. We have two cases:

(i) If $\ell = 1$ it suffices to perform a Laplace expansion along the first row, we obtain

$$\det(A) = \sum_{j=1}^{n} (-1)^{\ell+j} a_{\ell j} \det(A_{\ell j})$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{\ell+j} b_{\ell j} \det(A_{\ell j})$$

$$\det(C) = \sum_{j=1}^{n} (-1)^{\ell+j} (a_{\ell j} + b_{\ell j}) \det(A_{\ell j}).$$

(ii) If $\ell > 1$, a Laplace expansion along the first row gives

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(B) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(B_{1j})$$

$$\det(C) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(C_{1j})$$

For $j = 1, \ldots, n$, row $(\ell - 1)$ of C_{1j} is the sum of rows $(\ell - 1)$ of A_{1j} and B_{1j}. Outside of row $(\ell - 1)$, the three matrices have the same entries. The statement is, therefore, true by the inductive hypothesis.

Question 3 If $A \in M_n(F)$ and $1 \leq i, j \leq n$, write down a formula for the (ℓ, m)-entry of A_{ij} (for $1 \leq \ell, m \leq n - 1$).

Solution: The submatrix A_{ij} is the one obtained from A after removing row i and column j. The (ℓ, m)-entry of A_{ij} is given by the following case by case definition, that corresponds to dividing the matrix A in the four quadrants delimited by row i and column j:

$$\begin{cases} a_{\ell m} & \ell < i \text{ and } m < j \\ a_{\ell,(m+1)} & \ell < i \text{ and } m \geq j \\ a_{(\ell+1),m} & \ell \geq i \text{ and } m < j \\ a_{(\ell+1),(m+1)} & \ell \geq i \text{ and } m \geq j. \end{cases}$$

Note that if, for instance, $i = n$ then the case $\ell \geq i$ never occurs so there is no issue with $\ell + 1$ being bigger than n.

Question 4 Prove that a lower triangular matrix has determinant equal to the product of the elements on the diagonal. (A matrix is said to be lower-triangular if all its entries above the diagonal are 0.)

Solution: This is an easy induction on the size n of A. The case $n = 1$ is obvious. We assume that the result is true for matrices of size $n - 1$. Let A be a lower-triangular matrix of size n. Let a_{ij} be the (i,j)-entry of A. Then $a_{ij} = 0$ for $j > i$. The first row has only one non-zero entry, namely a_{11}, thus a Laplace expansion along the first row gives

$$\det(A) = a_{11}\det(A_{11}).$$

Now, A_{11} is a lower-triangular matrix thus the result is true by the inductive hypothesis.
Prove an analogue for upper-triangular matrices.

Question 5 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 \end{pmatrix}.$$

(i) Prove T is a linear transformation.

Let

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}.$$

Let also E be the standard basis of $\mathbb{R}^3$.

(ii) Calculate ${}_E[T]_B$ and ${}_E[T]_C$.

(iii) Does it make sense to write ${}_B[T]_E$?

(iv) If you have not used it to solve (ii), write the change of basis matrix from C to B.

Solution:

(i) This should be a routine check at this point. Check that $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

(ii) For $i = 1, 2$, we call b_i (resp. c_i) the vectors in B (resp. C) (in the order they are written in the definition of B and C). Similarly, for $i = 1, 2, 3$, let e_i be the vectors of E. The matrix ${}_E[T]_B$ is given by the coordinates of $T(b_i)$ $(i = 1, 2)$ with respect to the basis E. We compute

$$T(b_1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2e_1 + e_3, \quad T(b_2) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 2e_2 + e_3.$$

Thus

$${}_E[T]_B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Similarly

$${}_E[T]_C = \begin{pmatrix} 1 & -1 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

(iii) It is often hard to remember which way we write the vectors. In any case, since we apply maps on the left of vectors, we usually write the basis of the source on the right of the map. My mnemonic trick is to remember that the vector has to be written in the basis that is closer to it when we apply the linear transformation. For instance ${}_E[T]_C$ takes vectors that are written with respect to the basis C. We shall see this notation makes sense because of the way we write the composition of two functions. In any case, here E is a basis of $\mathbb{R}^3$, so ${}_B[T]_E$ is not a meaningful notation.

(iv) We need to write ${}_B[id]_C$. In other words we need to express the vectors of C in the basis B. A quick computation gives

$$c_1 = \frac{1}{2}(b_1 - b_2), \qquad c_2 = -\frac{1}{2}(b_1 + b_2).$$

The change of basis matrix is

$${}_B[id]_C = 1/2 \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Check that ${}_E[T]_B\, {}_B[id]_C = {}_E[T]_C$.