Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2023-2024

Partial Differential Equations in Action **MATH50008**

Coursework 2

Instructions: The deadline to submit this coursework is on **Friday 22 March at 1pm (UK time)**. The **neatness, completeness and clarity of the answers** will contribute to the final mark. You can turn in handwritten or typed solutions (for instance, using LaTeX). You should upload your answers to this coursework as a single PDF via the Turnitin Assignment called **Coursework 2** which you will find in the *Assessments* folder of our Blackboard site. On the front page of your submission, you must **not** indicate your first name or last name (as papers will be marked anonymously), but make sure that you indicate your CID. Your submission filename **must** have the following format: `MATH50008_CW2_[CID].pdf`, where [CID] is your college ID.

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1. **Total: 8 Marks** Consider a thermally conducting rectangular plate such that $0 \leq x \leq L_1$ and $0 \leq y \leq L_2$. The steady-state temperature profile $u(x, y)$ in the plate is governed by Laplace's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
Solve this equation subject to the following boundary conditions:
$$u(0, y) = 0, \quad u(L_1, y) = T_1, \quad u(x, 0) = 0, \quad u(x, L_2) = T_2$$
[Hint: use the superposition principle to decompose this problem into two simpler problems to solve.]

2. **Total: 12 Marks** Here, we consider the following problem
$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = 1, \quad 0 < x < \pi \\ &u(x, 0) = f(x), \quad 0 < x < \pi \\ &\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0 \end{aligned}$$
Note that this is an inhomogeneous problem with homogeneous Neumann boundary conditions.

   (a) 4 Marks Using the method of separation of variables, show that the most general solution to the associated homogeneous problem is given by
$$u(x, t) = T_0(t) + \sum_{n=1}^{\infty} T_n(t) \cos(nx)$$
where $T_n(t),\ n \geq 0$ are functions you do not need to determine at this point.

   (b) 4 Marks Find the general solution of the inhomogeneous problem. To do this, assume that the solution of the inhomogeneous problem is of the form obtained in (a) and determine the functions $T_n(t)$.

   (c) 4 Marks Finally, we assume that $f(x) = \cos x + \cos 2x$. Show that the particular solution is given by
$$u(x, t) = 1 - e^{-t} + e^{-2t} \cos(x) + e^{-5t} \cos(2x)$$