Imperial College London

MATH97014 MATH70015

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 1

Date: 20 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. **The principles of quantum mechanics**

Consider a system on the Hilbert space $\mathbb{C}^3$ and a Hamiltonian $\hat{H}$ represented by the matrix

$$\hat{H} = \begin{pmatrix} -E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{pmatrix},$$

with $E \in \mathbb{R}$. Let another observable $\hat{A}$ be described by the matrix

$$\hat{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{pmatrix},$$

with $a \in \mathbb{R}$.

(a) Calculate the eigenvalues and a set of normalised eigenvectors of $\hat{A}$. (4 marks)

(b) (i) Assume that at a given time the system is in the state which is an eigenstate of the Hamiltonian with eigenenergy E, when a measurement of the observable A is performed. What are the possible outcomes of this measurement? Calculate the probabilities of the different measurement outcomes, and the expectation value of A.

(6 marks)

(ii) How can one prepare a system in a given eigenstate of the energy using measurements? (2 marks)

(c) In another experiment, with an unknown initial state, at time $t = 0$ the measurement of the observable A yields the result a. What is the probability that a subsequent measurement of A at time $t > 0$ yields the same result a? (8 marks)

(Total: 20 marks)

2. **Quantum harmonic oscillator and momentum representation**

Consider the harmonic oscillator Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{I}\right),$$

with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ with $[\hat{a},\hat{a}^\dagger] = \hat{I}$.

(a) We introduce the operators $\hat{Q} = \frac{\sqrt{\hbar}}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right)$ and $\hat{P} = \frac{\sqrt{\hbar}}{\sqrt{2}\mathrm{i}}\left(\hat{a}-\hat{a}^\dagger\right)$.

(i) Calculate the commutator $[\hat{Q},\hat{P}]$. (3 marks)

(ii) Verify that the harmonic oscillator Hamiltonian expressed in terms of $\hat{Q}$ and $\hat{P}$ is given by

$$\hat{H} = \frac{\omega}{2}\left(\hat{P}^2 + \hat{Q}^2\right).$$

(4 marks)

(iii) What are the eigenvalues E_n of $\hat{H}$? (2 marks)

(b) The eigenstates $|q\rangle$ of the scaled position operator $\hat{Q}$ belong to the eigenvalues $q \in \mathbb{R}$, so that $\hat{Q}|q\rangle = q|q\rangle$. They fulfil the generalised orthonormality condition $\langle q'|q\rangle = \delta(q-q')$, and form a resolution of the identity,

$$\int_{-\infty}^{\infty} |q\rangle\langle q|\mathrm{d}q = \hat{I}.$$

The eigenstates $|p\rangle$ of the scaled momentum operator $\hat{P}$ belonging to the eigenvalues $p \in \mathbb{R}$ (i.e. $\hat{P}|p\rangle = p|p\rangle$) fulfil the generalised orthonormality condition $\langle p'|p\rangle = \delta(p-p')$, form a resolution of the identity

$$\int_{-\infty}^{\infty} |p\rangle\langle p|\mathrm{d}p = \hat{I},$$

and we have

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}.$$

(i) Show that

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\langle q|\psi\rangle\mathrm{d}q.$$

(2 marks)

(ii) Show that

$$\langle p|\hat{Q}|\psi\rangle = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\tilde{\psi}(p) = \langle p|\psi\rangle.$$

(5 marks)

(iii) Express $\langle p|\hat{H}|\psi\rangle$ in terms of $\langle p|\psi\rangle$.

(4 marks)

(Total: 20 marks)

3. **Particle hopping on a lattice**

Consider a particle on a one-dimensional discrete lattice, modelled on a Hilbert space spanned by an orthonormal basis $\{|n\rangle\}$, with integer n, where the particle is located on the n-th lattice site with certainty if the system is in the state $|n\rangle$. We define the "shift" operators

$$\hat{K} = \sum_n |n-1\rangle\langle n|, \quad \text{and} \quad \hat{K}^\dagger = \sum_n |n+1\rangle\langle n|,$$

and the discrete position operator

$$\hat{N} = \sum_n n|n\rangle\langle n|.$$

A Hamiltonian of the form

$$\hat{H} = -\frac{E_0}{2}\left(\hat{K} + \hat{K}^\dagger\right) + F\hat{N},$$

where E_0 and F are real positive constants, describes the system of a particle that can hop between neighbouring sites, subject to a static force F.

(a) Verify the commutation relations

$$[\hat{N}, \hat{K}] = -\hat{K}, \quad [\hat{N}, \hat{K}^\dagger] = \hat{K}^\dagger, \quad [\hat{K}, \hat{K}^\dagger] = 0.$$

(4 marks)

(b) Consider the special case $F = 0$.

(i) Insert the ansatz $|\phi\rangle = \sum_n c_n|n\rangle$ into the eigenvalue equation for $\hat{H}$ and verify that $c_n = \mathrm{e}^{\mathrm{i}n\kappa}$ with $\kappa \in [0, 2\pi]$ yields a set of eigenvectors. What are the corresponding eigenvalues of $\hat{H}$?

(5 marks)

(ii) Starting from the Schrödinger equation $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|\psi\rangle$, derive the Heisenberg equation of motion for the expectation value of an observable $\hat{A}$ with $\frac{\partial}{\partial t}\hat{A} = 0$.

(3 marks)

(c) Consider the general case $F \neq 0$.

(i) State the Heisenberg equations for the expectation values of $\hat{K}$, $\hat{K}^\dagger$ and $\hat{N}$, simplify using the commutators from part (a).

(4 marks)

(ii) Solve the equations of motion in (i) with initial conditions $\langle\hat{K}(t=0)\rangle = \mathrm{e}^{\mathrm{i}\kappa_0}$, $\langle\hat{K}^\dagger(t=0)\rangle = \mathrm{e}^{-\mathrm{i}\kappa_0}$, and $\langle\hat{N}(t=0)\rangle = N_0$.

(4 marks)

(Total: 20 marks)

4. **A quantum wave function**

Consider a quantum particle, the state of which is described by the wave function

$$\psi(x) = A\mathrm{e}^{-\frac{1}{2}|x-4|},$$

with a real and positive normalisation constant A.

(a) Find the A such that the wave function $\psi(x)$ is normalised to one. (4 marks)

(b) What is the expectation value $\langle x \rangle$ and the uncertainty Δx of the position. (9 marks)
Hint: You may find the integral $\int_0^\infty x^2 e^{-x} \mathrm{d}x = 2$ useful.

(c) Calculate the probability of finding the particle at a position with $x \leq 0$ and the probability of finding it at a position $x \geq 0$. (7 marks)

(Total: 20 marks)

5. **Mastery Question: Husimi functions**

(a) Consider the set of coherent states $|z\rangle$. These can be defined as eigenstates $|z\rangle$ of the annihilation operator $\hat{a}$, such that $\hat{a}|z\rangle = z|z\rangle$, where $z \in \mathbb{C}$. In the basis of harmonic oscillator states they have the form

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle.$$

Show that these states form a resolution of the identity with

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \hat{I},$$

where $\mathrm{d}^2 z = \mathrm{dRe}(z)\mathrm{dIm}(z)$. (5 marks)

Hint: Introduce polar coordinates $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi]$, so that $z = r\mathrm{e}^{\mathrm{i}\theta}$, and $\mathrm{d}^2 z = r\mathrm{d}r\mathrm{d}\theta$. The integral $\int_0^\infty \xi^n \mathrm{e}^{-\xi} \mathrm{d}\xi = n!$ for integer n should be useful.

(b) We define the Husimi representation Q of a quantum state $|\psi\rangle$ as

$$Q(z, z^*) = \frac{1}{\pi} |\langle z|\psi\rangle|^2.$$

(Note that this differs by an overall normalisation factor from the Husimi distribution defined in the lecture notes.)

Assume that the state $|\psi\rangle$ is normalised to one, and show that

(i) $\int Q(z, z^*)\mathrm{d}^2 z = 1.$ (2 marks)

(ii) $\langle \hat{a} \rangle = \int z Q(z, z^*)\mathrm{d}^2 z,$ (2 marks)

(iii) $\langle \hat{a}^\dagger \hat{a} \rangle = \int |z|^2 Q(z, z^*)\mathrm{d}^2 z - 1.$ (3 marks)

(c) (i) Show that

$$\frac{\partial}{\partial z} Q(z, z^*) = -z^* Q(z, z^*) + \frac{1}{\pi} \langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle.$$

(4 marks)

(ii) Consider a harmonic oscillator Hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{I})$. Starting from the Schrödinger equation for a state $|\psi\rangle$, derive an equation of motion for the time-dependent Husimi function $Q(z, z^*, t)$. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH60015/70015/97014

Solutions

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1. **The principles of quantum mechanics**

(a) From the characteristic polynomial of $\hat{A}$ we find for the eigenvalues λ of $\hat{A}$ $\boxed{\text{sim. seen} \Downarrow}$

$$-\lambda^3 + a^2\lambda = 0,$$

and thus $\lambda_0 = 0$ and $\lambda_\pm = \pm a$. For the components of the eigenvector ϕ_0 we find from $\hat{A}|\phi_0\rangle = 0$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is,

$$y_0 = 0, \text{ and } x_0 = -z_0.$$

Together with the normalisation condition that yields

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

up to an arbitrary phase factor $e^{i\varphi}$. Similarly we find the remaining eigenvectors

$$\phi_+ = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \text{and} \quad \phi_- = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

$\boxed{4, \text{A}}$

(b) (i) The system is in the state $\psi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. The probability to measure an eigenvalue of an observable is given by the squared modulus of the projection of the state onto the corresponding eigenspace of the observable. $\boxed{\text{sim. seen} \Downarrow}$
We can directly read off the probabilities for the different eigenvalues from the eigenvectors of $\hat{A}$ as

$$P(a) = \frac{1}{4}, \quad P(0) = \frac{1}{2}, \quad \text{and} \quad P(-a) = \frac{1}{4}.$$

$\boxed{3, \text{B}}$

The expectation value can be either deduced from $\langle\psi|\hat{A}|\psi\rangle$ via vector and matrix multiplications, or we calculate

$$\langle\hat{A}\rangle = \sum_j P(\lambda_j)\lambda_j = aP(a) - aP(-a) = a\left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

$\boxed{3, \text{B}}$

(ii) A measurement of the energy with outcome E collapses the quantum state to the corresponding eigenstate of $\hat{H}$. Thus if we measure the energy and the outcome is E the state is prepared. If there is another outcome the system needs to be perturbed in some way and the measurement needs to be repeated until the desired outcome is observed. $\boxed{2, \text{C}}$

(c) A measurement of the value a at time $t = 0$ projects the system onto the corresponding eigenstate of $\hat{A}$, sim. seen $\Downarrow$

$$\psi(t=0) = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

which is not an eigenstate of $\hat{H}$. The time-dependent wave function is then given by 2, C

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}|\psi(0)\rangle.$$

1, C

We can use the method of stationary states to find $\psi(t)$. For this purpose we need to represent $\psi(0)$ as a linear superposition of the eigenstates of $\hat{H}$, the canonical basis χ_j, as

$$\psi(t=0) = \frac{1}{2}\chi_- + \frac{1}{\sqrt{2}}\chi_0 + \frac{1}{2}\chi_+,$$

Therefore the state at time t is given by

$$\psi(t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}Et/\hbar}\chi_- + \frac{1}{\sqrt{2}}\chi_0 + \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et/\hbar}\chi_+,$$

3, B

The probability to measure the result a again is given by

$$\begin{aligned} P(1) = |\langle\phi_a|\psi(t)\rangle|^2 &= \left|\frac{1}{2}\mathrm{e}^{\mathrm{i}Et/\hbar}\cdot\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et/\hbar}\cdot\frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right|^2 \\ &= \left|\frac{1}{2}\left(1+\cos(Et/\hbar)\right)\right|^2 \\ &= \frac{1}{4}\left(1+\cos(Et/\hbar)\right)^2, \end{aligned}$$

which oscillates between zero and one. 2, B

2. **Quantum harmonic oscillator**

(a) (i) By definition we have

seen/sim.seen ⇓

$$[\hat{Q},\hat{P}] = \frac{\hbar}{2\mathrm{i}}[\hat{a}+\hat{a}^\dagger, \hat{a}-\hat{a}^\dagger].$$

That is

$$[\hat{Q},\hat{P}] = -\mathrm{i}\frac{\hbar}{2}([\hat{a}^\dagger,\hat{a}] - [\hat{a},\hat{a}^\dagger] = \mathrm{i}\hbar[\hat{a},\hat{a}^\dagger] = \mathrm{i}\hbar\hat{I}.$$

3, A

(ii) From the definition of $\hat{P}$ and $\hat{Q}$ we find

seen/sim.seen ⇓

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{Q}+\mathrm{i}\hat{P}), \quad \text{and} \quad \hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{Q}-\mathrm{i}\hat{P}).$$

Thus

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{2\hbar}(\hat{Q}-\mathrm{i}\hat{P})(\hat{Q}+\mathrm{i}\hat{P}) \\
&= \frac{1}{2\hbar}(\hat{Q}^2+\hat{P}^2+\mathrm{i}\hat{Q}\hat{P}-\mathrm{i}\hat{P}\hat{Q}) \\
&= \frac{1}{2\hbar}(\hat{Q}^2+\hat{P}^2+\mathrm{i}[\hat{Q},\hat{P}]) \\
&= \frac{1}{2\hbar}(\hat{Q}^2+\hat{P}^2-\hbar\hat{I})
\end{aligned}$$

and thus we have

$$\hat{H} = \frac{\omega}{2}(\hat{P}^2+\hat{Q}^2).$$

4, A

(iii) With $\hat{H}|n\rangle = E_n|n\rangle$ the eigenvalues of $\hat{H}$ are given by

seen ⇓

$$E_n = \hbar\omega(n+\frac{1}{2}),$$

where n is a non-negative integer.

2, A

(b) (i) We insert an identity to find

sim. seen ⇓

$$\langle p|\psi\rangle = \int_{-\infty}^{+\infty}\langle p|q\rangle\langle q|\psi\rangle\mathrm{d}q.$$

Inserting $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}$ as given in the question we verify that

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\langle q|\psi\rangle\mathrm{d}q.$$

2, B

unseen ⇓

(ii) To calculate $\langle p|\hat{Q}|\psi\rangle$ we write

$$
\begin{aligned}
\langle p|\hat{Q}|\psi\rangle &= \int_{-\infty}^{+\infty} \langle p|q\rangle\langle q|\hat{Q}|\psi\rangle \mathrm{d}q \\
&= \int_{-\infty}^{+\infty} q\langle p|q\rangle\langle q|\psi\rangle \mathrm{d}q \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q)\mathrm{d}q \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{i}\hbar\frac{\partial}{\partial p}(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q))\mathrm{d}q \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q)\mathrm{d}q\right) \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\left(\int_{-\infty}^{+\infty} \langle p|q\rangle\langle q|\psi\rangle \mathrm{d}q\right) \\
&= \mathrm{i}\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial p}\tilde{\psi}(p).
\end{aligned}
$$

$\boxed{5,\ \mathrm{D}}$

(iii) We have by definition of the states $|p\rangle$ that

$\boxed{\text{unseen} \Downarrow}$

$$\langle p|\hat{P}^2|\psi\rangle = p^2\langle p|\psi\rangle = p^2\tilde{\psi}(p).$$

Following the same argument as in part (b)(ii) we have

$$\langle p|\hat{Q}^2|\psi\rangle = -\hbar^2\frac{\partial^2}{\partial p^2}\tilde{\psi}(p).$$

and thus in summary we have

$$\langle p|\hat{H}|\psi\rangle = \frac{\omega}{2}\left(-\hbar^2\frac{\partial^2}{\partial p^2} + p^2\right)\tilde{\psi}(p).$$

$\boxed{4,\ \mathrm{D}}$

3. **Particle hopping on a lattice**

(a) We calculate $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}[\hat{K},\hat{N}] &= \sum_{n,n'} n' \left(|n-1\rangle\langle n|n'\rangle\langle n'| - |n'\rangle\langle n'|n-1\rangle\langle n|\right)\\ &= \sum_{n,n'} n' \left(|n-1\rangle\langle n'|\delta_{n,n'} - |n'\rangle\langle n|\delta_{n-1,n'}\right)\\ &= \sum_{n} (n|n-1\rangle\langle n| - (n-1)|n-1\rangle\langle n|)\\ &= \sum_{n} |n-1\rangle\langle n| = \hat{K}.\end{aligned}$$

And similarly

$$\begin{aligned}[\hat{K}^\dagger,\hat{N}] &= \sum_{n,n'} n' \left(|n+1\rangle\langle n|n'\rangle\langle n'| - |n'\rangle\langle n'|n+1\rangle\langle n|\right)\\ &= \sum_{n,n'} n' \left(|n+1\rangle\langle n'|\delta_{n,n'} - |n'\rangle\langle n|\delta_{n+1,n'}\right)\\ &= \sum_{n} (n|n+1\rangle\langle n| - (n+1)|n+1\rangle\langle n|)\\ &= -\sum_{n} |n+1\rangle\langle n| = -\hat{K}^\dagger.\end{aligned}$$

And finally,

$$\begin{aligned}[\hat{K},\hat{K}^\dagger] &= \sum_{n,n'} \left(|n-1\rangle\langle n|n'+1\rangle\langle n'| - |n'+1\rangle\langle n'|n-1\rangle\langle n|\right)\\ &= \sum_{n,n'} \left(|n-1\rangle\langle n'|\delta_{n,n'+1} - |n'+1\rangle\langle n|\delta_{n-1,n'}\right)\\ &= \sum_{n} (|n-1\rangle\langle n-1| - |n\rangle\langle n|)\\ &= \sum_{n} |n\rangle\langle n| - \sum_{n} |n\rangle\langle n| = 0\end{aligned}$$

$\boxed{4, \text{A}}$

(b) (i) We have $\boxed{\text{unseen} \Downarrow}$

$$\hat{K}|n\rangle = \sum_{n'} |n'-1\rangle\langle n'|n\rangle = \sum_{n'} |n'-1\rangle\delta_{n,n'} = |n-1\rangle$$

and similarly

$$\hat{K}^\dagger|n\rangle = |n+1\rangle.$$

That is,

$$\hat{H}|n\rangle = -\frac{E_0}{2}\left(|n+1\rangle + |n-1\rangle\right).$$

Inserting the ansatz into $\hat{H}|\phi\rangle = E|\phi\rangle$ then yields

$$-\frac{E_0}{2}(c_{n+1}+c_{n-1}) = Ec_n.$$

Inserting the ansats $c_n = \mathrm{e}^{\mathrm{i}n\kappa}$ yields $\boxed{2, \text{D}}$

$$-\frac{E_0}{2}\mathrm{e}^{\mathrm{i}n\kappa}(\mathrm{e}^{\mathrm{i}\kappa}+\mathrm{e}^{\mathrm{i}\kappa}) = E\mathrm{e}^{\mathrm{i}n\kappa},$$

or

$$-\frac{E_0}{2}(\mathrm{e}^{\mathrm{i}\kappa}+\mathrm{e}^{\mathrm{i}\kappa}) = E.$$

That is, the eigenvector equation is fulfilled by this ansatz, for the corresponding eigenvalues

$$E = -E_0\cos(\kappa).$$

$\boxed{3,\ \mathrm{C}}$

(ii) We have for a normalised state $|\psi\rangle$ $\boxed{\text{seen} \Downarrow}$

$$\langle\hat{A}\rangle := \langle\psi|\hat{A}|\psi\rangle.$$

That is with $\frac{\partial\hat{A}}{\partial t} = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \langle\dot{\psi}|\hat{A}|\psi\rangle + \langle\psi|\hat{A}|\dot{\psi}\rangle,$$

where the dot denotes the derivative with respect to time. From the Schrödinger equation we know

$$|\dot{\psi}\rangle = -\frac{\mathrm{i}}{\hbar}\hat{H}|\psi\rangle$$

and the adjoint equation gives

$$\langle\dot{\psi}| = \frac{\mathrm{i}}{\hbar}\langle\psi|\hat{H},$$

as $\hat{H}^\dagger = \hat{H}$. And therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \frac{\mathrm{i}}{\hbar}\langle\psi|\hat{H}\hat{A} - \hat{A}\hat{H}|\psi\rangle = \frac{\mathrm{i}}{\hbar}\langle[\hat{H},\hat{A}]\rangle.$$

$\boxed{3,\ \mathrm{A}}$

(c) (i) The Heisenberg equation for the expectation value of $\hat{K}$ is given by $\boxed{\text{unseen} \Downarrow}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{K}\rangle = \frac{\mathrm{i}}{\hbar}\langle[\hat{H},\hat{K}]\rangle = -\frac{\mathrm{i}}{\hbar}F\langle\hat{K}\rangle,$$

and the one for the adjoint operator is consequently given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{K}^\dagger\rangle = \frac{\mathrm{i}}{\hbar}F\langle\hat{K}^\dagger\rangle,$$

For the expectation value of $\hat{N}$ we find the dynamical equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{N}\rangle = -\frac{\mathrm{i}}{\hbar}\frac{E_0}{2}\left(\langle\hat{K}\rangle - \langle\hat{K}^\dagger\rangle\right).$$

$\boxed{4,\ \mathrm{C}}$

(ii) The equations for $\hat{K}$ and $\hat{K}^\dagger$ are straightforwardly solved by

$$\langle \hat{K}(t) \rangle = \mathrm{e}^{\mathrm{i}(\kappa_0 - \frac{F}{\hbar}t)}$$

and

$$\langle \hat{K}^\dagger(t) \rangle = \mathrm{e}^{-\mathrm{i}(\kappa_0 - \frac{F}{\hbar}t)}.$$

Inserting the time dependent expectation values of $\hat{K}$ and $\hat{K}^\dagger$ into the equation for $\hat{N}$ we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N} \rangle &= -\frac{\mathrm{i}}{\hbar}\frac{E_0}{2}\left(\mathrm{e}^{\mathrm{i}(\kappa_0 - \frac{F}{\hbar}t)} - \mathrm{e}^{-\mathrm{i}(\kappa_0 - \frac{F}{\hbar}t)}\right) \\ &= \frac{E_0}{\hbar}\sin(\kappa_0 - \frac{F}{\hbar}t) \end{aligned}$$

Integrating this yields

$$\langle N(t) \rangle = N_0 + \frac{E_0}{F}\cos(\kappa_0 - \frac{F}{\hbar}t) - \frac{E_0}{F}\cos(\kappa_0).$$

$\boxed{4, \mathrm{D}}$

4. **A quantum wave function**

(a) We have

$\boxed{\text{sim. seen} \Downarrow}$

$$
\begin{aligned}
||\psi||^2 &= A^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-|x-4|} \mathrm{d}x \\
&= A^2 \left(\int_{-\infty}^{0} \mathrm{e}^{y} \mathrm{d}y + \int_{0}^{+\infty} \mathrm{e}^{-y} \mathrm{d}y \right) \\
&= 2A^2 \int_{0}^{+\infty} \mathrm{e}^{-y} \mathrm{d}y.
\end{aligned}
$$

where we have made the substitution $y = x - 4$.
We calculate

$$
\int_{0}^{+\infty} \mathrm{e}^{-y} \mathrm{d}y = -\left[\mathrm{e}^{-y}\right]_0^{\infty} = 1.
$$

Thus we find

$$
||\psi||^2 = 2A^2.
$$

That is, $\psi(x)$ is normalised to one for $A = \frac{1}{\sqrt{2}}$. $\boxed{4, \text{A}}$

(b) The wave function is symmetric around a single maximum, thus this is the expectation value $\boxed{\text{sim. seen} \Downarrow}$

$$
\langle \hat{x} \rangle = 4.
$$

To calculate the uncertainty we need to calculate the expectation value of $\hat{x}^2$ first. $\boxed{2, \text{A}}$
We calculate

$$
\begin{aligned}
\langle \hat{x}^2 \rangle &= \int_{-\infty}^{+\infty} x^2 |\psi(x)|^2 \mathrm{d}x \\
&= \frac{1}{2} \left(\int_{-\infty}^{0} (y+4)^2 e^{y} \mathrm{d}y + \int_{0}^{+\infty} (y+4)^2 e^{-y} \mathrm{d}y \right) \\
&= \frac{1}{2} \left(\int_{-\infty}^{0} (y^2+8y+16) e^{y} \mathrm{d}y + \int_{0}^{+\infty} (y^2+8y+16) e^{-y} \mathrm{d}y \right)
\end{aligned}
$$

In part (a) we have calculated

$$
\int_{0}^{+\infty} e^{-y} \mathrm{d}y = 1 = \int_{-\infty}^{0} e^{y} \mathrm{d}y
$$

Further, we notice that we have

$$
\int_{-\infty}^{0} y e^{y} \mathrm{d}y = -\int_{0}^{+\infty} y e^{-y} \mathrm{d}y.
$$

Thus, it remains to calculate $\boxed{2, \text{B}}$

$$
\int_{-\infty}^{0} y^2 e^{y} \mathrm{d}y = \int_{0}^{+\infty} y^2 e^{-y} \mathrm{d}y.
$$

This can be done using integration by parts, but the hint already provides the solution

$$
\int_{0}^{\infty} y^2 e^{y} \mathrm{d}y = 2.
$$

In summary we have

$$\begin{aligned}\langle \hat{x}^2 \rangle &= \frac{1}{2}\left(\int_{-\infty}^{0}(y^2+8y+16)e^{y}\mathrm{d}y + \int_{0}^{+\infty}(y^2+8y+16)e^{-y}\mathrm{d}y\right)\\ &= \frac{1}{2}\left(2+16+2+16\right) = 18.\end{aligned}$$

And for the uncertainty we finally find

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \sqrt{18-16} = \sqrt{2}.$$

$\boxed{\text{5, A}}$

(c) The probability to find the particle at a position $x \geq 0$ is given by $\boxed{\text{sim. seen} \Downarrow}$

$$P(x \geq 0) = \int_0^\infty |\psi(x)|^2 \mathrm{d}x = \frac{1}{2}\int_0^\infty \mathrm{e}^{-|x-4|}\mathrm{d}x$$

We calculate $\boxed{\text{2, A}}$

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-|x-4|}\mathrm{d}x &= \int_0^4 \mathrm{e}^{x-4}\mathrm{d}x + \int_4^\infty \mathrm{e}^{4-x}\mathrm{d}x\\ &= \mathrm{e}^{-4}\int_0^4 \mathrm{e}^{x}\mathrm{d}x + \mathrm{e}^{4}\int_4^\infty \mathrm{e}^{-x}\mathrm{d}x\\ &= \mathrm{e}^{-4}(\mathrm{e}^4-1) + \mathrm{e}^4\mathrm{e}^{-4}\\ &= 2-\mathrm{e}^{-4}\end{aligned}$$

And thus

$$P(x \geq 0) = 1 - \frac{\mathrm{e}^{-4}}{2} \approx 0.98,$$

the probability to find the particle at negative positions is simply given by

$$P(x \leq 0) = 1 - P(x \geq 0) = \frac{\mathrm{e}^{-4}}{2} \approx 0.02.$$

$\boxed{\text{5, B}}$

5. **Mastery Question: Husimi functions**

(a) We have $\boxed{\text{unseen} \Downarrow}$

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \sum_{n,m} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int \mathrm{d}^2 z \mathrm{e}^{-|z|^2} z^{*m} z^n.$$

Introducing polar coordinates $z = r\mathrm{e}^{\mathrm{i}\theta}$ this becomes

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \sum_{n,m} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int_0^\infty r\mathrm{d}r \mathrm{e}^{-r^2} r^{n+m} \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\mathrm{i}(n-m)\theta}.$$

The last integral reduces to $2\pi\delta_{nm}$ and thus the expression simplifies to

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \sum_{n} \frac{2|n\rangle\langle n|}{n!} \int_0^\infty r r^{2n} \mathrm{e}^{-r^2} \mathrm{d}r.$$

Substituting $\zeta = r^2$ and thus $\mathrm{d}\zeta = 2r\mathrm{d}r$ we have

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \sum_{n} \frac{|n\rangle\langle n|}{n!} \int_0^\infty \zeta^n \mathrm{e}^{-\zeta} \mathrm{d}\zeta.$$

Using that $\int_0^\infty \xi^n \mathrm{e}^{-\xi} \mathrm{d}\xi = n!$ we finally find

$$\int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} = \sum_{n} |n\rangle\langle n| = \hat{I}.$$

$\boxed{5, \text{M}}$

(b) (i) Using the resolution of the identity from part (a),we have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\langle\psi|\psi\rangle &= \int \frac{\mathrm{d}^2 z}{\pi} \langle\psi|z\rangle\langle z|\psi\rangle \\ &= \int \mathrm{d}^2 z Q(p,q)\end{aligned}$$

Since $\langle\psi|\psi\rangle = 1$ this verifies the statement. $\boxed{2, \text{M}}$

(ii) We insert an identity into $\langle\psi|\hat{a}|\psi\rangle$ and use that $\hat{a}|z\rangle = z|z\rangle$ to find

$$\begin{aligned}\langle\psi|\hat{a}|\psi\rangle &= \int \frac{\mathrm{d}^2 z}{\pi} \langle\psi|\hat{a}|z\rangle\langle z|\psi\rangle \\ &= \int \frac{\mathrm{d}^2 z}{\pi} z\langle\psi|z\rangle\langle z|\psi\rangle \\ &= \int zQ(z,z^*)\mathrm{d}^2 z.\end{aligned}$$

$\boxed{2, \text{M}}$

(iii) We first write $\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - \hat{I}$. Thus

$$\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{a}\hat{a}^\dagger\rangle - 1.$$

Now we insert an identity and use that $\hat{a}|z\rangle = z|z\rangle$ and hence $\langle z|\hat{a}^\dagger = \langle z|z^*$ to find

$$\langle\psi|\hat{a}\hat{a}^\dagger|\psi\rangle = \int |z|^2 Q(z,z^*)\mathrm{d}^2 z.$$

Thus, in summary we confirm

$$\langle\hat{a}^\dagger\hat{a}\rangle = \int |z|^2 Q(z,z^*)\mathrm{d}^2 z - 1.$$

$\boxed{3, \text{M}}$

(c) (i) We have

unseen ⇓

$$Q = \frac{1}{\pi}\langle z|\psi\rangle\langle\psi|z\rangle = \frac{\mathrm{e}^{-|z|^2}}{\pi}\sum_{n,m}\frac{z^{*n}z^m}{\sqrt{n!m!}}\langle n|\psi\rangle\langle\psi|m\rangle.$$

Taking the derivative with respect to z we find

$$\frac{\partial Q}{\partial z} = -z^*Q + \frac{\mathrm{e}^{-|z|^2}}{\pi}\sum_{n,m}\frac{mz^{*n}z^{m-1}}{\sqrt{n!m!}}\langle n|\psi\rangle\langle\psi|m\rangle.$$

On the other hand,

$$\begin{aligned}\langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle &= \mathrm{e}^{-|z|^2}\sum_{n,m}\frac{z^{*n}z^m}{\sqrt{n!m!}}\sqrt{m+1}\langle n|\psi\rangle\langle\psi|m+1\rangle \\ &= \mathrm{e}^{-|z|^2}\sum_{n,m}\frac{z^{*n}z^m}{\sqrt{n!(m+1)!}}(m+1)\langle n|\psi\rangle\langle\psi|m+1\rangle\end{aligned}$$

Relabelling we can rewrite this as

$$\langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle = \mathrm{e}^{-|z|^2}\sum_{n,m}\frac{mz^{*n}z^{m-1}}{\sqrt{n!m!}}\langle n|\psi\rangle\langle\psi|m+1\rangle$$

That is,

$$\frac{\partial Q}{\partial z} = -z^*Q + \frac{1}{\pi}\langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle.$$

4, M

(ii) We have by definition

$$\frac{\partial Q}{\partial t} = \frac{1}{\pi}\left(\langle z|\dot{\psi}\rangle\langle\psi|z\rangle + \langle z|\psi\rangle\langle\dot{\psi}|z\rangle\right).$$

Inserting the Schrödinger equation and its adjoint this yields

$$\frac{\partial Q}{\partial t} = \frac{\mathrm{i}}{\pi\hbar}\left(\langle z|\hat{H}|\psi\rangle\langle\psi|z\rangle - \langle z|\psi\rangle\langle\psi|\hat{H}|z\rangle\right).$$

Now inserting the harmonic oscillator Hamiltonian we find

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \frac{\mathrm{i}\omega}{\pi}\left(\langle z|\hat{a}^\dagger\hat{a}|\psi\rangle\langle\psi|z\rangle - \langle z|\psi\rangle\langle\psi|\hat{a}^\dagger\hat{a}|z\rangle\right) \\ &= \frac{\mathrm{i}\omega}{\pi}\left(z^*\langle z|\hat{a}|\psi\rangle\langle\psi|z\rangle - z\langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle\right).\end{aligned}$$

From part (c)(i) we know that

$$\langle z|\psi\rangle\langle\psi|\hat{a}^\dagger|z\rangle = \pi\frac{\partial Q}{\partial z} + \pi z^*Q,$$

and

$$\langle z|\hat{a}|\psi\rangle\langle\psi|z\rangle = \pi\frac{\partial Q}{\partial z^*} + \pi zQ.$$

Thus we have

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \frac{\mathrm{i}\omega}{\pi}\left(\pi z^*\frac{\partial Q}{\partial z^*} + \pi|z|^2Q - \pi z\frac{\partial Q}{\partial z} - \pi|z|^2Q\right) \\ &= \mathrm{i}\omega\left(z^*\frac{\partial Q}{\partial z^*} - z\frac{\partial Q}{\partial z}\right).\end{aligned}$$

4, M

We could, if we felt like it, rewrite this in terms of p and q using

$$\frac{\partial}{\partial z} = \frac{\sqrt{\hbar}}{\sqrt{2}}\left(\frac{\partial}{\partial q} - \mathrm{i}\frac{\partial}{\partial p}\right)$$

and

$$\frac{\partial}{\partial z^*} = \frac{\sqrt{\hbar}}{\sqrt{2}}\left(\frac{\partial}{\partial q} + \mathrm{i}\frac{\partial}{\partial p}\right)$$

to find

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \mathrm{i}\omega\left(\frac{1}{2}(q-\mathrm{i}p)\left(\frac{\partial Q}{\partial q} + \mathrm{i}\frac{\partial Q}{\partial p}\right) - \frac{1}{2}(q+\mathrm{i}p)\left(\frac{\partial Q}{\partial q} - \mathrm{i}\frac{\partial Q}{\partial p}\right)\right)\\ &= \mathrm{i}\omega\left(\mathrm{iq}\frac{\partial \mathrm{Q}}{\partial \mathrm{p}} - \mathrm{ip}\frac{\partial \mathrm{Q}}{\partial \mathrm{q}}\right)\\ &= \omega\left(\frac{\partial Q}{\partial q}p - \frac{\partial Q}{\partial p}q\right).\end{aligned}$$

Which happens to agree with the classical Liouville equation.

Review of mark distribution:
Total A marks: 33 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Quantum Mechanics 1_MATH60015 MATH97014 MATH70015	1	On the whole this question was on the easier side. But marks were lost for a lot of minor errors in calculations, particularly in part a, and in the final section of part c when calculating the probability. Many failed to even attempt the "prose" question part b (ii) and there were many partial answers. Often the working wasn't shown. Marks were awarded for results and calcuations. for part a. Overall there were many high marks in this question and only a few candidates clearly didn't know what to do or didn't attempt the question.
Quantum Mechanics 1_MATH60015 MATH97014 MATH70015	2	Question 2 worked fairly well for most. Unfortunately, there was a typo in part b that hadn't been spotted during the exam, which confused many candidates. I was very generous with the marking of this part as a result.
Quantum Mechanics 1_MATH60015 MATH97014 MATH70015	3	In part a marks were lost if no dummy indices were used, leading to nonsensical expressions. In part b(i) some solutions were to sketchy to receive full marks, and often the last steps were missing. Part c did not pose conceptual challenges for most, but some marks were lost in mistakes in the (solutions to) the differential equations, in particular the one for N.
Quantum Mechanics 1_MATH60015 MATH97014 MATH70015	4	Part a was mostly correct, but often the working wasn't shown, and some people used the symmetry wrong and went along wrong directions. Also in part b, the most common issue was not to show the calculations, and confusions about the absolute value and the symmetry. A few candidates did not know the formual for the uncertainty. Part c had the most problems, probably because of time issues.
Quantum Mechanics 1_MATH60015 MATH97014 MATH70015	5	I admit that the mastery question this year was quite hard. I hadn't intended it to be, but it turned out this way. A large number of students did not attempt part c at all, or got almost none of it right. Part a on the other hand was better, and part b went well for many.