Network Science
Spring 2024
Problem sheet 7

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1. Consider $f(x,t)$ and $f_i(t)$ where the former is a solution to the 1-D diffusion equation, and the latter is a solution to the graph diffusion equation.

   (a) How does the total amount of "stuff", $F_g(t) = \sum_{i=1}^{N} f_i(t)$, vary with time for graph diffusion?

   (b) How does the total amount of "stuff", $F(t) = \int_{-\infty}^{\infty} f(x,t)dx$ vary with time? Assume that f and $\partial f/\partial x \to 0$ as $|x| \to \infty$.

2. Last year, you were introduced to the *incidence matrix* for an N-node directed graph with no self-loops or multiedges. The links are numbered from 1 to L and then the incidence matrix is defined as

$$E_{ij} = \begin{cases} 1 & \text{if link } i \text{ points to node } j \\ -1 & \text{if link } i \text{ points from node } j \text{ to another node} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

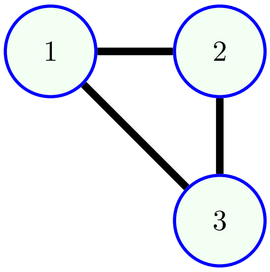
   (a) Consider the "equivalent" simple undirected graph where two nodes are linked if there is a directed link from one node to the other in the original directed graph. Show that the Laplacian matrix for this undirected graph is related to the incidence matrix of the directed graph by, $\mathbf{L} = \mathbf{E}^T\mathbf{E}$

   (b) Consider an arbitrary real matrix $\mathbf{F}$. Show that the eigenvalues of $\mathbf{M} = \mathbf{F}^T\mathbf{F}$ are non-negative (note that $\mathbf{M}$ is always symmetric)

3. In lecture 10, the graph diffusion equation was introduced,

$$d\mathbf{n}/dt = -\alpha\mathbf{L}\mathbf{n},$$

   where $\mathbf{L}$ is the (symmetric) Laplacian matrix for an N-node undirected graph, $\mathbf{n} \in \mathbb{R}^N$, and $\alpha > 0$. Show that this system of N coupled ODEs can be written as a system of N uncoupled ODEs. This will require a change of variable, $\mathbf{g} = \mathbf{W}\mathbf{n}$, and you will need to determine $\mathbf{W}$. Will these N uncoupled ODEs all be distinct from each other? Hint: consider the orthogonal diagonalization of $\mathbf{L}$



4. Consider the "triangle graph" shown above. Solve the graph diffusion equation on this graph with $\alpha = 1$, $n_1(t=0) = 1$ and $n_2(t=0) = n_3(t=0) = 0$

5. Consider the following coupled system of ODEs on a complete N-node undirected graph with Laplacian matrix, $\mathbf{L}$:

$$\frac{dx_i}{dt} = x_i - \alpha \sum_{j=1}^{N} L_{ij} x_j, i = 1, \ldots, N. \tag{2}$$

The initial condition for each node is $x_i(t=0) = y_i$.
The eigenvalues of $\mathbf{L}$ are $\lambda_1 = 0$ and $\lambda_i = N$ for $i = 2, 3, ..., N$.

(a) What is the solution of this system when $\alpha = 0$?

(b) Show that with an appropriate choice of $\mathbf{B}$, the transformation, $\mathbf{x} = \mathbf{B}\mathbf{w}$, allows (2) to be written as a decoupled system of equations,

$$\frac{d\mathbf{w}}{dt} = (\mathbf{I} - \alpha\mathbf{\Lambda})\mathbf{w} \tag{3}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{L}$

(c) Show that (2) synchronizes when $\alpha > 1/N$: $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$.