Imperial College London

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Analysis 1A

Lecture 9 - Uniqueness of limits, convergence implies bounded

Ajay Chandra

Example 3.13

Set $\delta = 10^{-1000000}$, $a_n = (-1)^n\delta$. Prove that a_n diverges, that is it does not converge (to any $a \in \mathbb{R}$).

Proof 1

Suppose, by contradiction, $a_n \to a \in \mathbb{R}$

Let $\varepsilon = \delta$. Then $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \delta$

Let $n_1 \geq N$ be odd

$|a_{n_1} - a| = |-\delta - a| = \underbrace{|a+\delta| < \varepsilon}_{|a-(-\delta)|<\varepsilon}$

$a \in \underbrace{(-\delta-\varepsilon, -\delta+\varepsilon)}_{(-2\delta,0)}$ $\quad \varepsilon = \delta$

$\Rightarrow a < 0$

$|a_{n_1+1} - a| = |\delta - a| < \varepsilon$ so $a \in \underbrace{(\delta-\varepsilon, \delta+\varepsilon)}_{(0,2\delta)} \Rightarrow a > 0$

$\#$

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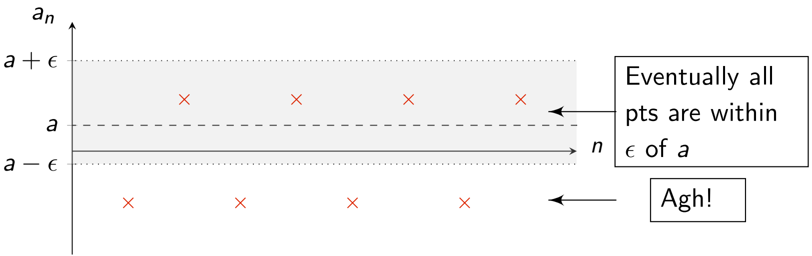

Example 3.13

Set $\delta = 10^{-1000000}$, $a_n = (-1)^n\delta$. Prove that a_n diverges, that is it does not converge (to any $a \in \mathbb{R}$).

<u>Proof 2</u>

Suppose, by contradiction, $a_n \to a \in \mathbb{R}$

Let $\varepsilon = \delta$, then $\exists N$ s.t $\forall n \geq N$, $|a_n - a| < \varepsilon = \delta$

By choosing n_1 even, n_2 odd $n_1, n_2 \geq N$

$$\underset{\substack{\parallel \\ 2\delta}}{|a_{n_1} - a_{n_2}|} = |a_{n_1} - a + a - a_{n_2}| \leq |a_{n_1} - a| + |a_{n_2} - a| < \underset{= 2\delta}{\delta + \delta}$$

$$\Rightarrow 2\delta < 2\delta$$

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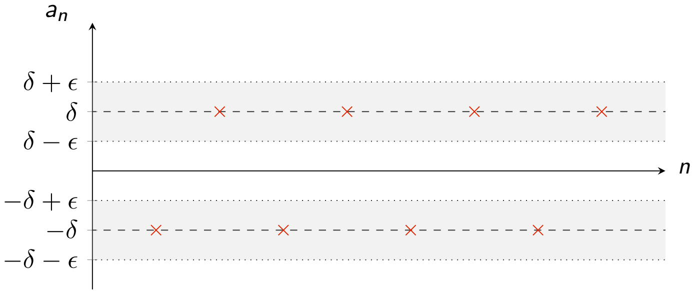
Proof 2: Assume for contradiction that $a_n \to a$



Theorem 3.14 - Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$.

<u>Proof</u>

Suppose, by contradiction, $a \neq b$.

Let $\varepsilon = \frac{|a-b|}{2} > 0$. Then since $a_n \to a$, $\exists N_1$ s.t

$\forall n \geq N_1$, $\underline{|a_n - a| < \varepsilon}$.

Since $a_n \to b$, $\exists N_2$, $\forall n \geq N_2$, $\underline{|a_n - b| < \varepsilon}$.

Now choose $n \geq \max(N_1, N_2)$

Then

$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |b - a_n| < \varepsilon + \varepsilon = 2\varepsilon = |a-b|$$

$$\Rightarrow |a-b| < |a-b| \quad \times$$

Theorem 3.14 - Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$.

Example 3.15

Let a_n be defined by $a_1 = a_2 = 0$ and $a_n = \frac{1}{n-2}$ for $n > 2$. Show $a_n \to 0$.

Which step is incorrect in this student's strategy?

Fix $\epsilon > 0$. We assume $n > 2$. Then

1. We want $|\frac{1}{n-2} - 0| = \frac{1}{n-2} < \epsilon$
2. $\implies n - 2 > 1/\epsilon$
3. $\implies n > 2 + 1/\epsilon$
4. $\implies n > 1/\epsilon$ $(*)$
5. So take $N > \max(1/\epsilon, 2)$, then
6. $\forall n \geq N$, $n > 1/\epsilon$ which is $(*)$
7. So $\frac{1}{n-2} \to 0$
8. More than one mistake
9. All correct

$\exists \varepsilon > 0$, $n = \max(1/\varepsilon, 2)$

s.t.

$\frac{1}{n-2} \not< \varepsilon$

Proposition 3.16

If (a_n) is convergent, then it is bounded. That is,

$$a_n \to a \Rightarrow \exists A \in \mathbb{R} \text{ such that } |a_n| \leq A \ \forall n$$

Note: If X is finite

$\max_{x \in X} |x|$ is a bound

$\{a_n : n \in \mathbb{N}_{>0}\}$ is bounded

$\big| |a_n| - |a| \big| \leq |a_n - a|$

Proof Let $\varepsilon = 1$, then $\exists N$ s.t $\forall n \geq N$, $|a_n - a| < 1$

$\Longrightarrow |a_n| < |a| + 1$

exercise

Let $A = \max\{|a_1|, |a_2|, \ldots\ldots, |a_N|, |a|+1\}$

Then, for any $n \in \mathbb{N}_{>0}$

$|a_n| \leq A$

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Notice $a_n = \frac{1}{n-7}$ is not a counterexample! It is not a well defined sequence of real numbers because a_7 is either not defined or not real.

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Instead we could take

$$a_n = \begin{cases} \frac{1}{n-7} & n \neq 7, \\ 0 & n = 7. \end{cases}$$

This is then indeed bounded as $\forall n \in \mathbb{N}$ we have

$$-1 = a_6 \leq a_n \leq a_8 = 1.$$