Graph Theory Notes
2023 preliminary version

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Preface to the 2023 preliminary version

These are the notes that Dr. Barham wrote last year as he taught the Graph theory course. I (Michele) inherited them and my contribution to this version is limited to correcting some typos and changed the layout, and made some minor changes to the first Section.

As this is the first time I am teaching the course I am planning to check the bits I teach every week for more typos, releasing an updated version every week, but for now this is the complete set of notes. If you do not like deforestation, I suggest you **do not print anything beyond Section 6**, as what remains after that is just a long list of photographs.

Preface to the 2022 edition

I'd like to thank the following mathematicians for the help that their work has been in creating the content for these notes:

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Most of all, I'd like to thank the following students for their assistance in eliminating mistakes from these notes:

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1 Introduction

Welcome to Graph Theory. Get ready to hear the name 'Erdős' a lot.

A graph is a collection of points (called vertices) with lines (called edges) drawn between them. Graphs are the usual way of formalising problems about networks into mathematics. Rumours, fluid flows, disease transmission, language syntax, and many more areas of study have been brought into the ever increasing domain of mathematics via graph theory.[1] This is not what this course is about.

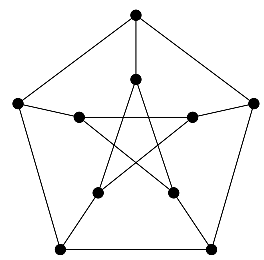


Figure 1: The Petersen Graph

The structure of graphs makes then spectacularly well suited for computer analysis, and thus graphs could almost be regarded as a playground for Computer Science. It's almost impossible to have a discussion about Complexity Theory, the study of how 'hard' a computational problem is, without drawing on examples from Graph Theory. This is not what this course is about.

This course is devoted to the many natural mathematical questions that arise in this situation. In my mind (although not in the table of contents) this course is split into 2 sections. The first is the elementary (in the mathematical, rather than judgemental, sense) graph theory studied before the 20$^{\text{th}}$ Century, most notably by Euler. We'll be preoccupied with constructions, invariants, and subgraphs in this section, and we'll be thinking about questions like "Can I draw this graph without any of the edges crossing." or "Can I cross each edge exactly once, while ending up where I started?"

The second is the extremely subtle and sophisticated graph theory that started in the 20$^{\text{th}}$ Century with Ramsey's Theorem. This area is concerned with the *eventual* theory of graphs. Do all graphs over a certain size have the same property? In other words, is the size of the counterexamples to a potential theorem bounded, and if so, bounded by what? If $\mathbb{G}_n$ is the set of all graphs with n vertices, does the proportion $\mathbb{G}_n$ with a certain property have a limit? Probability has a number of extremely beautiful and powerful applications here, but we will take a quite naive approach to Probability, and you will not require any specialist knowledge of the subject. I'll describe how everything we do can be put into Kolmogorov's axiomatisation of Probability, but not actually work in it, so don't feel the need to revise.

1.1 Graphs and Subgraphs

Let's define what we're going to be working with all term.

Definition 1.1 (Graph)**.** A *graph* is a triple $G = (V_G, E_G, \varepsilon_G)$ where

i) V_G is a set of *vertices*,

ii) E_G is a set of *edges*,

iii) A function $\varepsilon : E \to \{\{u, v\} \mid u, v \in V\}$ attaching each edge to a pair of vertices. This is called the *endpoint map* of G.

Definition 1.2 (Simple Graph)**.** A *simple graph* is a graph $G = (V, E, \varepsilon)$ such that both the following conditions are verified

i) ε is injective, i.e. there is at most one edge between any two vertices,

ii) For every $e \in E$ and every $v \in V$,

$$\varepsilon_G(e) \neq \{v, v\}$$

i.e. there is no edge joining a vertex to itself.

Definition 1.3 (Neighbours)**.** Suppose G is a graph and let $u, v \in V_G$. We write uE_Gv to mean that there is an $e \in E_G$ such that $\varepsilon_G(e) = \{u, v\}$. In this case we say that u is a *neighbour* of v.

[1] It's only a matter of time before we complete our coup d'etat of all human thought. We've driven the Geographers out of Meteorology, and look at our progress in Biology.

Definition 1.4 (Incident edges)**.** Suppose that G is a graph and $e, f \in E_G$ are edges of G. We say that e and f are *incident* when there are $u, u, w \in V_G$ such that $\varepsilon(e) = \{u, v\}$ and $\varepsilon(f) = \{v, w\}$. In other words, we say that two edges are incident when they share (at least) one endpoint.

Definition 1.5 ($|G|$ and $\|G\|$)**.** We define the *order* of G as $|G| = |V|$ and the *number of edges* of G as $\|G\| = |E|$.

Definition 1.6. Let G and H be graphs. We say that H is a *subgraph* of G (written as $H \subseteq G$) if

1. $V_H \subseteq V_G$,
2. $E_H \subseteq E_G$,
3. $\varepsilon_H(e) = \varepsilon_G(e)$ for all $e \in E_H$.

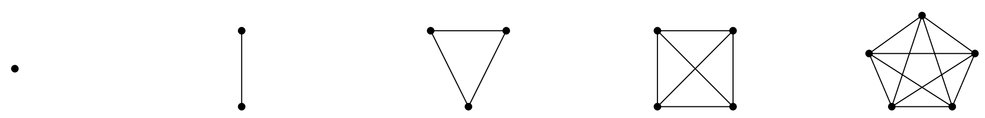
We say that H is an *induced subgraph* of G (written as $H \leq G$) if $H \subseteq G$ and

4. if $e \in E_G$ is such that $\varepsilon_G(e) \subseteq V_H$ then $e \in E_H$.

Question 1.7. Can you think of an example of a subgraph that is not induced? (Hint: What about the empty graph)

Example 1.8. Let $n \geq 1$. Then K_n, the complete graph with n-vertices, is the following graph:

- $V_{K_n} = \{v_1, v_2, \dots, v_n\}$
- $E_{K_n} = \{\{v_i, v_j\} : i \neq j\}$
- $\varepsilon_{K_n}(\{v_i, v_j\}) = \{v_i, v_j\}$ for all $i \neq j$.



K_1 K_2 K_3 K_4 K_5

Figure 2: Complete Graphs

Example 1.9. Let $n \geq 1$. Then P_n, the n-long path, is the following graph:

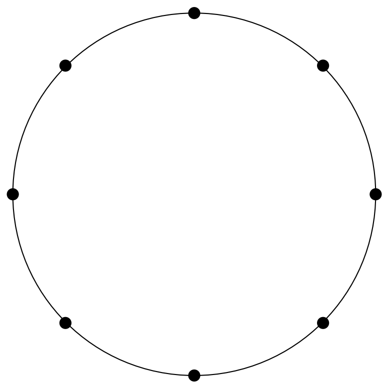
- $V_{P_n} = \{v_1, v_2, \dots, v_n\}$
- $E_{P_n} = \{\{v_i, v_{i+1}\} : 1 \leq i < n\}$
- $\varepsilon_{P_n}(\{v_i, v_{i+1}\}) = \{v_i, v_{i+1}\}$ for all $1 \leq i < n$.

P_n

Figure 3: Paths

Example 1.10. Let $n \geq 3$. Then C_n, the n-long cycle, is the following graph:

- $V_{C_n} = \{v_1, v_2, \dots, v_n\}$
- $E_{C_n} = \{\{v_i, v_{i+1}\} : 1 \leq i < n\} \cup \{\{v_1, v_n\}\}$
- $\varepsilon_{C_n}(\{v_i, v_j\}) = \{v_i, v_j\}$ for all i, j.



C_8

Figure 4: Cycles

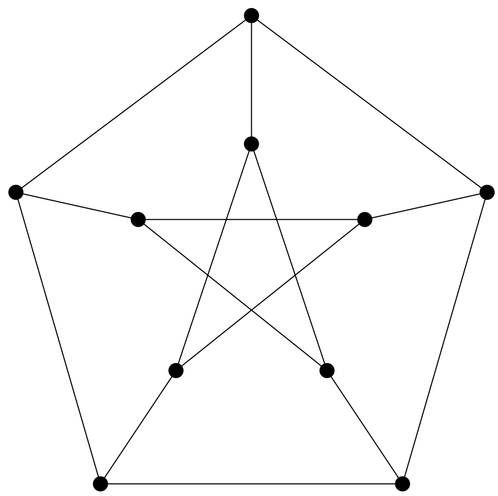


Figure 5: The Petersen Graph

Example 1.11. The Petersen Graph is pictured below.

Graphs are often discussed in terms of which subgraphs they don't have, not just the subgraphs that they do have.

Definition 1.12. Let G be a graph. We say that G is *n-partite* if there is a partition $V_1, \ldots, V_n$ of V_G such that uEv only if u and v belong to different parts. If G is 2-partite then we say that G is *bipartite* (and the usual Latin prefixes for 3-partite, 4-partite, etc.). If G is n-partite for some n, but we don't want to specify the n, then we say that G is *multipartite.*

Proposition 1.13. *If G is n partite then it does not contain K_{n+1} as a subgraph.*

Proof. Let G be an n-partite graph. If G does not contain K_n then it can't contain K_{n+1}, so let's assume that there are $v_1, \ldots, v_n \in V_G$ such that $v_i E v_j$ for all i and j. There are n of the v_i, all of which lie in different parts, therefore if we take any $u \in V_G$, then that u must lie in the same part as one of the v_i's, and hence $\neg u E v_i$. Therefore G cannot contain K_{n+1}. □

1.2 Homomorphisms

As with any branch of Pure Mathematics, once you have defined a type of object, you then define what the sub-objects are, and what a homomorphism is. Only then may you get on with doing the stuff you defined the objects to do.

All the propositions in this subsection have their proofs as exercises. I'm sorry. I *promise* you that this is not a sign of things to come, it's just that these propositions make excellent practice questions, and it's hard to get good questions this early in a course.

Definition 1.14. Let $G = (V_G, E_G, \varepsilon_G)$ and $H = (V_H, E_H, \varepsilon_H)$ be graphs. A *homomorphism* is a function $\phi : V_G \to V_H$ such that if there is an edge $e \in E_G$ such that $\varepsilon_G(e) = \{g, h\}$ then there is an edge $f \in E_H$ such that $\varepsilon_H(f) = \{\phi(g), \phi(h)\}$.

Proposition 1.15. *If $\phi : G \to H$ and $\psi : H \to K$ are homomorphisms, then $\psi \circ \phi$ is also a homomorphism.*

Proof. Exercise on Problem Sheet 1. □

Definition 1.16. If $\phi : G \to H$ is a bijective homomorphism whose inverse is a homomorphism then we say that ϕ is an *isomorphism*. We write $G \cong H$ if there is an isomorphism from G to H.

Proposition 1.17. *$\cong$ is an equivalence relation on the class of all graphs.*

Proof. Exercise on Problem Sheet 1. □

Definition 1.18. If $\phi : G \to G$ is an isomorphism, then we call ϕ an *automorphism*. The set of all automorphisms of G is written as $\mathrm{Aut}(G)$.

Proposition 1.19. *For all graphs G, the set $\mathrm{Aut}(G)$ with $\circ$ (as defined in Definition 1.14) and the identity map forms a group.*

Proof. Exercise on Problem Sheet 1. □

1.3 First Invariants

An invariant of a graph is some number (or other some other kind of object) that can be derived from that graph (the method needs to be the same for every graph). A good invariant will be the same for G and H if $G \cong H$, although we may allow there to be G and H with $G \ncong H$ but with the same invariant.

Definition 1.20. Let G be a graph and let $v \in V_G$. The *degree* of v is written as $\deg(v)$, and the *average degree* is written as $\mathrm{AvDeg}(G)$. They are defined as follows:

$$\begin{array}{rcl} \deg(v) & = & |\{u \in V_G \; uEv\}| \\ \mathrm{AvDeg}(G) & = & \frac{1}{|G|} \sum_{v \in V_G} \deg(v) \end{array}$$

Proposition 1.21. *If $G \cong H$ then $\mathrm{AvDeg}(G) = \mathrm{AvDeg}(H)$.*

Lemma 1.22. *Let G be a graph. If $\mathrm{AvDeg}(G) > k$ then $|G| > k$.*

Proof. If $|G| \le k$ then $\deg(v) < k$ for all $v \in V_G$. Then

$$\begin{array}{rcl} \mathrm{AvDeg}(G) & = & \frac{1}{|G|} \sum_{v \in V_G} \deg(v) \\ & < & \frac{1}{|G|} \sum_{v \in V_G} k \\ & = & k \end{array}$$

□

Definition 1.23. Let G be a graph. The density of G, written as $\mathrm{den}(G)$, is equal to $\frac{\|G\|}{|G|}$.

Proposition 1.24. *Let G be a graph. Then $\mathrm{den}(G) = \frac{1}{2}\mathrm{AvDeg}(G)$.*

1.4 Graph Constructions

Construction Technique 1.25. Let G be a graph. The inverse of G, written as $\overline{G}$, is the graph with the same vertex set, but there is an edge between x and y in $\overline{G}$ if and only if there is no edge between x and y in G.

Construction Technique 1.26. Let G be a graph. The dual of G is the graph H whose vertex set is E_G. There is an edge between e and f if and only if they share an endpoint.

Construction Technique 1.27. Let G be a graph. A vertex deletion of G is the induced subgraph on $V_G \setminus \{v\}$. An edge deletion of G is the graph with vertex set V_G and edge set $E_G \setminus \{e\}$. If it is clear whether x is a vertex or edge, we may write $G \setminus x$ for the graph with x deleted.

Construction Technique 1.28. Let G be a graph and let $e \in E_G$. *Bisecting* e involves replacing e by a new vertex of degree 2, joined to the end-points of e exactly.

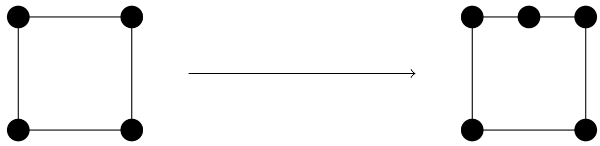


Figure 6: Bisecting Example

Construction Technique 1.29. Let G be a graph and let $e \in E_G$. Suppose that $\varepsilon_G(e) = \{a, b\}$. Contracting e involves deleting e, a and b, and replacing them by a single vertex v_e. We also require that $v_e E u$ if and only if uEa or uEb.

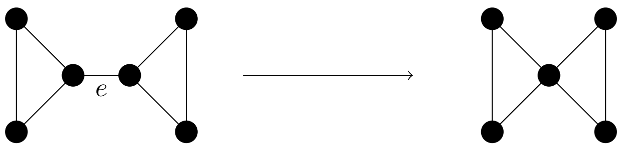


Figure 7: Contracting e

Definition 1.30. Let G and H be graphs. If we can construct H from G by applying finitely many deletions and contractions, then we say that H is a minor of G.

Example 1.31. K_5 is a minor of the Petersen graph.

2 Special Subgraphs

A vast amount of graph theory is concerned with which graphs do and do not contain certain special subgraphs (although I will confess to being quite... *liberal* to what properties I consider to be about special subgraphs), and so this is what we'll be doing for the rest of this section. To study this, we'll need to know what these special subgraphs are.

2.1 Connectedness

Definition 2.1. Let G be a group. A *walk* $(v_1, \dots, v_n)$ is a finite sequence of vertices such that $v_i E v_{i+1}$ for all i. We call v_1 the *start* and v_n the *end*

If a walk does not contain the same edge twice, the walk is called a *tour*. If a walk does not contain the same vertex twice (with an exception made for when $n > 2$ and $v_1 = v_n$), the walk is called a *path*. If a walk starts and ends at the same vertex, the walk is said to be *closed*. A closed path is called a *cycle*.

Lemma 2.2. *There is a walk from u to v if and only if there is a path from u to v.*

Proof Sketch. If we have a walk from u to v then we have a finite sequence with possibly repeated entries, say $v_i = v_j$. We delete from the sequence everything from v_i to v_{j-1}. Rinse, and repeat.

For the other direction, note that every path is a walk. □

Proposition 2.3. *Let $\sim$ be the relation on graph G given by $u \sim v$ if and only if there is a path from u to v. This $\sim$ is an equivalence relation.*

Proof Sketch. Reflexivity is due to the 1 element path.

Symmetry is due to the fact that the reverse of a path is still a path.

Transitivity is due to the fact that if we concatenate two paths, we get a walk. □

Definition 2.4. The equivalence classes of $\sim$ are called *connected components*.

We say that graph G is *connected* if and only if G has a single connected component.

Example 2.5. All the examples of graphs in Section 1.1 are connected. To build non-connected graphs, we can take the *disjoint union* of graphs. If G and H are graphs, then their disjoint union, written as $G \bigoplus H$, is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$ (we have to make sure G and H don't share any common labels). The endpoint map is equal to ε_G on E_G and ε_H on E_H.

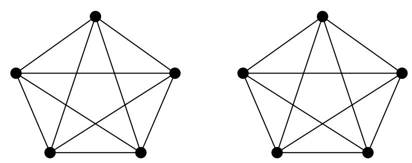


Figure 8: $K_5 \bigoplus K_5$

Definition 2.6. Let G be a graph. We say that G is k-connected if and only if $|G| > k$ and $G \setminus A$ is connected for every $A \subseteq V_G$ such that $|A| < k$.

$$\kappa(G) = \max\{k \in \mathbb{N} : G \text{ is } k\text{-connected}\}$$

Obviously if G is $k+1$-connected then G is k-connected.

Example 2.7. K_n is $(n-1)$-connected but not n-connected.

So far we've been doing quite simple things with our graphs. This will not last, and even these quite early definitions can result in some very complicated proofs. This next theorem is shows that if we have large average degree, then there has to be a very connected subgraph of similar density. It's hard, and not examinable, but I think going through the proof will be good for you.

Theorem 2.8 (Mader 1972)**.** *Let $k \in \mathbb{N}$ be non-zero. If G is a graph with* $\mathrm{AvDeg}(G) \geq 4k$ *then G has a $(k+1)$-connected subgraph H such that* $\mathrm{den}(H) > \mathrm{den}(G) - k$.

Proof. Note that $\mathrm{AvDeg}(G) \geq 4k$ implies that $\mathrm{den}(G) > 2k$ by Proposition 1.24.

We will be considering $\mathcal{G} = \{H \leq G : |H| \geq 2k \text{ and } \|H\| > \mathrm{den}(G)(|H|-k)\}$. Let's start by showing that $\mathcal{G}$ is non-empty.

Since $\mathrm{AvDeg}(G) \geq 4k$, Lemma 1.22 lets us assume that $|G| > 2k$. Also $|G|\mathrm{den}(G) = \|G\|$, so $\|G\| > \mathrm{den}(G)(|G| - k)$. Therefore $\mathcal{G}$ is non-empty, so we can choose $H_0 \in \mathcal{G}$ such that $|H_0|$ is minimal.

Since $|H_0|$ is minimal, if we delete any vertex from H_0 (say v) then $H_0 \setminus v \notin \mathcal{G}$. This means that either $|H_0| = 2k$ or $\|H_0 \setminus v\| \leq \mathrm{den}(G)(|H_0 \setminus v| - k)$. If $|H_0| = 2k$ then

$$\begin{array}{rcl} \|H_0\| & = & k\mathrm{den}(G) \\ & > & 2k^2 \\ & > & \binom{2k}{2} \end{array}$$

and so H_0 has far too many edges to be a simple graph. Therefore we may assume that $\|H_0 \setminus v\| \leq \mathrm{den}(G)(|H_0 \setminus v| - k)$ for all $v \in G$. Therefore every vertex of H_0 has degree greater than $\mathrm{den}(G)$, so $|H_0| > \mathrm{den}(G)$.

$$\begin{array}{rcl} \|H_0\| & > & \mathrm{den}(G)(|H_0| - k) \\ \mathrm{den}(H_0) & > & \mathrm{den}(G) - k\frac{\mathrm{den}(G)}{|H_0|} \\ & > & \mathrm{den}(G) - k \end{array}$$

If H_0 is $(k+1)$-connected then we're done. Suppose that there is an X such that $H_0 \setminus X$ is disconnected. Let K_1 be a connected component of $H_0 \setminus X$ and let $K_2 = H_0 \setminus (K_1 \cup X)$. If $v \in K_1$ then, from before, the degree of v is greater than $\mathrm{den}(G)$, so v is attached to at least $\mathrm{den}(G)$ many vertices in $K_1 \cup X$. Therefore $|K_1 \cup X| > 2k$. Similarly, $|K_2 \cup X| > 2k$.

Since H_0 is a minimal element of $\mathcal{G}$, we have that neither $K_1 \cup X$ nor $|K_2 \cup X|$ are in G. Therefore

$$\|K_1 \cup X\| \leq \mathrm{den}(G)(|K_1 \cup X| - k) \quad \text{and} \quad \|K_2 \cup X\| \leq \mathrm{den}(G)(|K_2 \cup X| - k)$$

However, every edge of H_0 that is not contained in X is contained in exactly one of $K_1 \cup X$ and $K_2 \cup X$.

$$\begin{array}{rcl} \|H_0\| & \leq & \|K_1 \cup X\| + \|K_2 \cup X\| \\ & \leq & \mathrm{den}(|K_1| + |K_2| + 2|X| - 2k) \\ & = & \mathrm{den}(|H_0| - k + (|X| - k)) \\ & \leq & \mathrm{den}(|H_0| - k) \end{array}$$

which contradicts our assumption that H_0 is in $\mathcal{G}$. We're done! □

2.2 Matchings

So far, our focus has been on the vertices of a graph, and the edges been regarded more as a property that pairs of vertices can have, rather than as objects in their own right. A matching is a set of edges, so we're going to have to shed that notion quite quickly.

Definition 2.9. Let G be a graph, and let $M \subseteq E_G$. We say that M is a *matching* if and only if $\varepsilon_G(e) \cap \varepsilon_G(f) = \emptyset$ for all $e, f \in M$. If $v \in V_G$ is such that there is an $e \in M$ with $v \in \varepsilon_G(e)$ then we say that v is *matched* by M. We say that v is *unmatched* otherwise.

A matching is *maximal* if and only if it is of maximal size, and is *perfect* if and only if it matches every vertex.

The most natural setting for considering matchings is bi-partite graphs.

Example 2.10. Recall that a bi-partite graph is one in which there is a partition of the vertices into two parts, such that every edge occurs between, rather than within, the parts.

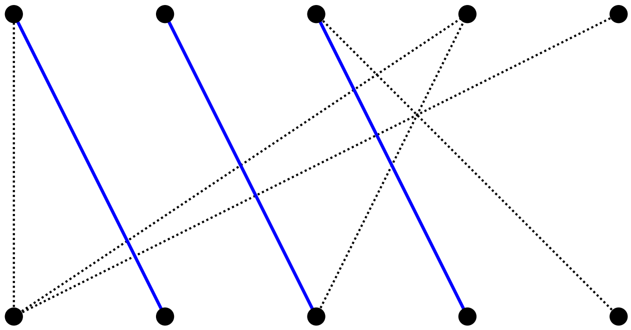


Figure 9: An example of a matching.

It's easy to see that for there to be a *perfect matching* of a bi-partite graph (a matching which matches every vertex) it is necessary for the cardinality of the parts to be equal, but the example itself shows that this is not a sufficient condition.

Matching is the dual concept to coverings, which I will define now.

Definition 2.11. Let G be a graph, and let $C \subseteq V_G$. We say that C is a *cover* if and only if for all $e \in E_G$ there is a $v \in C$ such that $v \in \varepsilon_G(e)$. A cover is *minimal* if and only if it is of minimal size.

The fact that minimal covers and maximal matchings exist is quite easy, so I'm going to put it in an exercise sheet.

The duality between matchings and coverings is a very strong one, as proved by the next theorem by Dénes Kőnig. There are many results known as Kőnig's (n.b. the accent is a double acute rather than an diaeresis), but the situation is even more complicated than the usual "Which Cauchy's Theorem do you mean?" problem, as we also have the problem of whether the theorem is named after Dénes or his father Gyula.

Lemma 2.12. *Let G be a graph with matching M and cover C. Then $|M| \leq |C|$.*

Proof. Every edge of G is covered by C, so there is a function $f : M \to C$ such that $f(m) \in \varepsilon_G(m)$ for all $m \in M$. However, M is a matching, so this f must be injective, therefore $|M| \leq |C|$. □

Lemma 2.13. *Let G be a graph with connected components $C_1, \ldots, C_n$ and matching M. Then M is a maximal matching if and only if $M|_{C_i}$ is a maximal matching of the induced subgraph C_i for all i.*

Proof. Exercise. □

Theorem 2.14 (Kőnig's Theorem)**.** *Let G be a bipartite graph, and let M and C be a maximal matching and minimal covering of G respectively. Then $|M| = |C|$.*

Proof. Let $A, B \subseteq V_G$ be the parts of G, and let M be a maximal matching. Let U be the set of all unmatched vertices of G. If $U \cap A$ is empty then M matches every vertex in A, and we can take A as a minimal cover of G, which completes the theorem.

Suppose $U \cap A$ is non-empty. We construct W inductively as follows:

Zero Step if $a \in U \cap A$ then $a \in W_0$.

Odd Step Suppose we have defined W_{2n}. If there is an $e \notin M$ such that $\varepsilon_G(e) = \{a, b\}$ and $a \in W_{2n}$ then $b \in W_{2n+1}$.

Even Step Suppose we have defined W_{2n-1}. If there is an $e \in M$ such that $\varepsilon_G(e) = \{a, b\}$ and $b \in W_{2n-1}$ then $a \in W_{2n}$.

$W := \bigcup_{n \in \mathbb{N}} W_n$. We call the sequences of edges that witness that a vertex v is in W an *alternating path*. Let $e \in E_G$. If e is used in the construction of W then e has an endpoint in $B \cap W$. If e is not used in the construction of W, but e has an endpoint in $A \cap W$ (called a) then $e \in M$, otherwise it would subsequently be used in the construction of W. However, to reach a in the first place required the use of an edge in M, and since M is a matching this is impossible. Therefore $C := (A \setminus W) \cup (B \cap W)$ is a cover.

If $c \in A \setminus W$ then $c \notin U$, so c is the endpoint of some edge in M. If $c \in B \cap W$ and c is not matched by M then since $c \in B$, we reach c with an alternating path of odd length (which we will call P), ending in edge not contained in M. But then $(M \setminus P) \cup (P \setminus M)$ is not only a matching, but also one of greater size, which contradicts the maximality of M.

Therefore $|C| = |M|$. □

2.3 Flows and Cuts

In the introduction, I mentioned that graphs were able to model flows. If we associate a number, representing the maximum flow, to each edge, we have an excellent model to study the flow of stuff through a restricted network. What I didn't mention is that this activity is also good for pure graph theory. Many interesting theorems about graphs are corollaries of a theorem called *The Max-flow Min-cut Theorem*, which is the goal of this section.

Definition 2.15. A *network*, often denoted by N consists of the following:

1. a vertex set V,
2. an edge set E,
3. a start vertex map $\sigma : E \to V$,
4. an end vertex map $\tau : E \to V$,
5. a source vertex $s \in V$, with $\forall e \in E\, \tau(e) \neq s$,
6. a sink vertex $t \in V$, with $\forall e \in E\, \sigma(e) \neq t$,
7. a capacity function $c : E \to \mathbb{R}^{\geq 0}$.

Typically you would also have a capacity function for the vertices, to model the maximum flow they can take, but we don't need that for our purposes.

Example 2.16. It is possible to turn any network into a graph by letting $\varepsilon(e) = \{\sigma(e), \tau(e)\}$, and forgetting all the other information. Most of the networks we'll be considering turn into a simple, connected graph, but this is not necessarily the case.

Definition 2.17. A *flow* in network N is a function $f : E \to \mathbb{R}^{\geq 0}$ such that:

F1 $f(e) \leq c(e)$ for all e.

F2 if u is neither s nor t then
$$\sum_{\tau(e)=u} f(e) = \sum_{\sigma(e)=u} f(e)$$

The *value* of flow f is $v(f) = \sum_{\tau(e)=t} f(e)$, and a flow is said to be maximal if it's value is maximal.

Remark 2.18. Let f be a flow on network N. The value of f is equal to $\sum_{\sigma(e)=s} f(e)$. This fact can be deduced from Lemma 2.24 quite easily, so I won't bother proving it now.

Example 2.19. Let N be any network. Then the zero flow, which assigns 0 to every edge is, indeed a flow. Some networks may only have this trivial flow.

Proposition 2.20. *Let N be a network. N admits a non-zero flow if and only if there is a sequence $v_1, \ldots, v_n$ of vertices such that:*

- *$v_1 = s$,*
- *for all i there is an $e_i \in E$ such that $\sigma(e_i) = v_i$ and $\tau(e_i) = v_{i+1}$,*
- *$v_n = t$.*

Definition 2.21. Let N be a network. A *cut*, denoted as P, is a subset of the vertices of N such that $s \in P$ but $t \notin P$. Let $E(P) = \{e \in E : \sigma(e) \in P \text{ and } \tau(e) \notin P\}$ and $E'(P) = \{e \in E : \tau(e) \in P \text{ and } \sigma(e) \notin P\}$.

Definition 2.22. Let N be a network and let P be a cut of N. The capacity of P is

$$c(P) = \sum_{e \in E(P)} c(e)$$

Proposition 2.23. *There is a cut M such that $c(M) = \inf\{c(P) : P \text{ is a cut of } N\}$.*

Proof. Even through capacities of cuts take values in $\mathbb{R}$, our networks are finite, so really we're asking for there to be a least value in a finite set of real numbers, not an arbitrary set. □

Lemma 2.24. *If P is a cut, and f is a flow, then $v(f) \leq c(P)$ and*

$$v(f) = \sum_{e \in E(P)} f(e) - \sum_{e \in E'(P)} f(e).$$

Proof Hint. A full proof is an exercise. Property F2 from Definition 2.17 will tell you how you can relate $v(f)$, which is the flow into t, to the flow into the vertices that are connected to t, and so on. □

Theorem 2.25 (The Max-flow Min-cut Theorem (Ford and Fulkerson 1956))**.** *Let N be a network. Then*

$$\max\{v(f) : f \text{ is a flow of } N\} = \min\{c(P) : P \text{ is a cut of } N\}$$

Proof. Let f be a maximal flow. We're going to recursively define a cut P such that $c(P) = v(f)$.

0. Let $s \in P$.
1. If $x \in P$ and there is an edge e such that:
   - $\tau(e) = x$, and
   - $0 < f(e)$,

   then $\sigma(e) \in P$.
2. If $x \in P$ and there is an edge e such that:
   - $\sigma(e) = x$, and
   - $f(e) < c(e)$,

   then $\tau(e) \in P$.

If $t \in P$ then there is a sequence of vertices $v_1, \ldots, v_n$ such that:

- $v_1 = s$,
- $v_n = t$,
- $v_i \in P$ for all i, and
- for all i there is an e_i such that $\{\sigma(e_i), \tau(e_i)\} = \{v_i, v_{i+1}\}$.

If $\sigma(e_i) = v_i$ then we say that e_i is a forward edge, a reverse edge otherwise. Note that e_1 and e_{n-1} have to be forward edges, as edges only start at s and edges can only end at t.

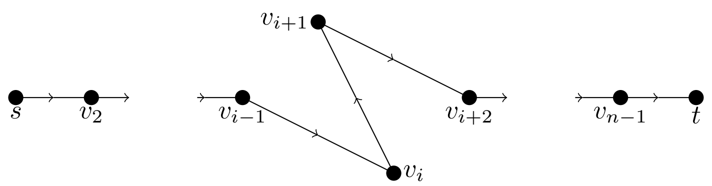


Figure 10: The Path from the $t \notin P$ Argument

If e_i is a forward edge then let $\alpha_i = c(e_i) - f(e_i)$. If e_i is a reverse edge then let $\alpha_i = f(e_i)$. For all i, we have that $\alpha_i > 0$.

Let $\alpha = \min(\{\alpha_i : 1 \leq i \leq n\})$. This α is non-zero. We may then define f' as follows:

$$f'(e) = \begin{cases} f(e) + \alpha & e \text{ is a forward edge} \\ f(e) - \alpha & e \text{ is a reverse edge} \end{cases}$$

Then $v(f') > v(f)$ contradicting our assumption that f is maximal.

Therefore $s \in P$ but $t \notin P$, so P is a cut. We will now investigate the capacity of P. By construction, if $e \in E(P)$ then $f(e) = c(e)$ and if $e \in E'(P)$ then $f(e) = 0$.

By Lemma 2.24.

$$\begin{aligned} v(f) &= \textstyle\sum_{e\in E(P)} f(e) - \sum_{e\in E'(P)} f(e) \\ &= \textstyle\sum_{e\in E(P)} c(e) \\ &= c(P) \end{aligned}$$

Therefore there is a cut whose capacity is equal to the value of a maximal flow, and by Lemma 2.24 is therefore minimal. □

Let's relate this back to simple graphs:

Definition 2.26. Let G be a graph an let $x, y \in V_G$ be such that $\neg xEy$.

- $\kappa(x, y)$ is the size of a minimal set of vertices X such that:
  1. $G \setminus X$ is disconnected, and
  2. x and y lie in different connected components.
- Two paths P and Q, which go from x to y, are distinct if $P \cap Q = \{x, y\}$. Then $\rho(x, y)$ is the number of pairwise distinct paths from x to y.

Corollary 2.27 (Menger's Theorem)**.** *Let G be a simple graph and let $x, y \in V_G$ be such that $\neg xEy$. Then $\kappa(x, y) = \rho(x, y)$.*

Proof Sketch. We will turn G into a network N and then apply the Max-flow Min-cut Theorem.

- The vertices of N are the vertices of G.
- For every edge $e \in E_G$ there are two edges e, e'.
- For the start map, if $\varepsilon(e) = \{u, v\}$ then $\sigma(e) = u$ and $\sigma(e') = v$.
- For the end map, pick the opposite to the start map, so $\tau(e) = v$ and $\tau(e') = u$.
- x is the source vertex, so delete the edges going in to x.
- y is the sink vertex, so delete the edges going from y.
- The capacity of each edge is 1.

$\kappa(x, y)$ is the capacity of a minimum cut of N, while $\rho(x, y)$ is the value of a maximal flow (exercise). □

3 Graph Colourings

3.1 Planar Graphs

Definition 3.1. A graph G is *planar* if and only if it is possible to draw the graph in a Euclidean plane without any of the edges crossing each other.

Remark 3.2. If you'd like a more technical definition of planar, here it is!
A graph G is planar if and only if

1. there is an injective map $\mathrm{Vert} : V_g \to \mathbb{R}^2$,
2. there is a map $\mathrm{Ed} : E_G \to \{f : [0,1] \to \mathbb{R}^2 \,:\, f \text{ is continuous}\}$,
3. $\{\mathrm{Ed}(e)(0), \mathrm{Ed}(e)(1)\} = \mathrm{Vert}(\varepsilon_G(e))$ for all $e \in E_G$,
4. $\mathrm{im}(\mathrm{Ed}(e_1)) \cap \mathrm{im}(\mathrm{Ed}(e_2)) = \emptyset$ for all $e_1, e_2 \in E_G$.

The vaguer definition is nicer, isn't it?

Example 3.3. The definition of planar only requires that there is a way of drawing the graph without crossing the edges, not that all pictures of the graph have no crossing edges. Most ways of drawing most planar graphs will have crossings.

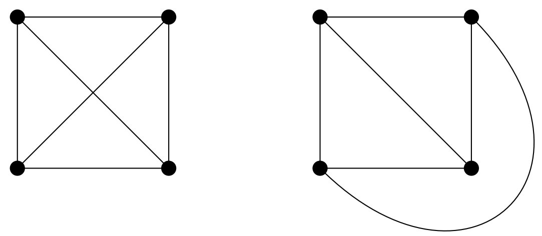


Figure 11: K_4 is planar

Proposition 3.4. *A graph is planar if and only if it can be drawn on the surface of a sphere without any edges crossing.*

Proof. Both directions of this proof rely on the following bijection between $\mathbb{R}^2$ and $S_2 \setminus \{N\}$, where N is the north pole of S_2. To start with, we embed both into $\mathbb{R}^3$.

$$\begin{aligned} \mathbb{R}_2 &= \{(x,y,0) \,:\, x,y \in \mathbb{R}\} \\ S_2 &= \{(x,y,z) \,:\, x^2+y^2+(z-1)^2 = 1\} \\ N &= (0,0,2) \end{aligned}$$

For any point $(x,y,0)$, the line joining $(x,y,0)$ and N intersects $S_2 \setminus \{N\}$ at exactly one point. f is the bijection that maps $(x,y,0)$ to that point.

Let G be a planar graph. Then applying f to a planar drawing of G gives a drawing of G on the surface of a sphere with no edges crossing. If we have a drawing of G on S_2, then we can rotate S_2 so none of the edges or vertices lie on N. Then applying f^{-1} to the drawing gives us a planar drawing of G. □

Lemma 3.5. *If H is a minor of planar graph G then H is also planar.*

Proof. If H is a minor of G then we can construct H by applying a finite sequence of contractions and deletions to G. It is clear that if G is planar then deleting an edge or a vertex will result in a planar graph.

Suppose that G is a planar graph with edge e. Let x and y be the endpoints of e. When e is contracted, e, x and y are replaced by a single vertex v, and there are edges from v to any vertex z such that zEx or zEy.

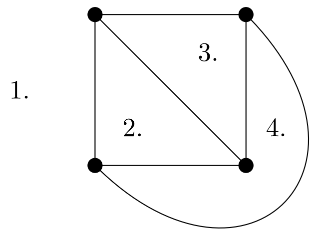
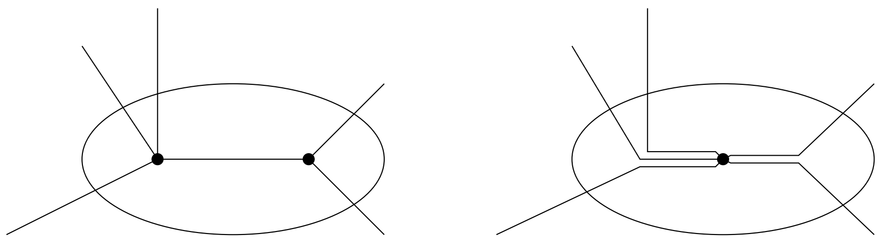


Figure 13: The faces of K_4

If we take any planar drawing of G we can draw a region that only contains x, y, e and any edges that have either x or y as an endpoint. If we redraw this region by placing v on the midpoint of e, and drawing the edges that connect to x or y by bringing them close to x or y resp. and then having them follow the line where e was, we can arrive at a planar drawing of the contracted graph.



(a) Before (b) After

Figure 12: Contractions preserve being planar

□

Lemma 3.6. *If G has less than 5 vertices, or if G is connected and* $\mathrm{den}(G) \leq 1$ *then G is planar.*

Definition 3.7. Let G be a planar graph. A *face* of graph G is one of the regions that a planar drawing of G divides the plane into. The set of faces is called F_G.

Example 3.8. K_4 has 4 faces.

Definition 3.9. Let G be a planar graph. The *Euler characteristic* of G, written as $\chi_E(G)$, is

$$|V_G| - |E_G| + |F_G|$$

Theorem 3.10 (Euler's Formula)**.** *If G is connected then $\chi_E(G) = 2$.*

Proof. Suppose G is connected, which implies that $|G| \leq \|G\| + 1$. Suppose $\|G\| = |G| - 1$. If G has an enclosed face, then the boundaries of the face form a cycle, and we can delete and edge while still keeping G connected. Therefore there is only one face, and $\chi_E(G) = |G| - (|G| - 1) + 1 = 2$.

Now suppose that there is a $k \in \mathbb{N}$ such that if H is a connected planar graph such that $\|H\| \leq |H| + k - 1$ then $\chi_E(H) = 2$. We also suppose that $\|G\| = |G| + k$. Since $\|G\| > |G| - 1$ there is at least one edge e such that $G \setminus \{e\}$ is still connected. Then $\chi_E(G \setminus \{e\}) = 2$. Reintroducing e will add one edge, but will also cut one face into two, so $\chi_E(G) = \chi_E(G \setminus \{e\}) - 1 + 1 = 2$

Therefore, by induction, $\chi_E(G) = 2$ for all connected planar graphs. □

Corollary 3.11. *Both K_5 and $K_{3,3}$ are not planar.*

Proof. Suppose that $G = K_5$ is planar. We'll find a contradiction by calculating F_G in two different ways. $|G|$ is connected, so $\chi(G) = 2$, and $|G| = 5$ and $\|G\| = 10$, so $|F_G| = 7$.

Each edge boarders exactly 2 faces, and each face has each at least 3 edges, so $\frac{2}{3}|E_V| \geq |F_G|$, and therefore $|F_G| < 7$. Our two calculations for the value of F_G have produced two different values, so we have a contradiction. Therefore K_5 is not planar.

$K_{3,3}$ is left as an exercise. □

Theorem 3.12 (Wagner's Theorem)**.** *Let G be a graph. Then G is not planar if and only if K_5 or $K_{3,3}$ is a minor of G.*

Proof. $\Leftarrow$ is exactly Lemma 3.5.

For $\Rightarrow$, we will consider a graph which has neither K_5 or $K_{3,3}$ as a minor, with a minimal number of vertices. We will show that if G does not have a high degree of connectivity then G is planar, but once we can assume that G has a large degree of connectivity, that will give us enough edges to find either K_5 or $K_{3,3}$

Let G be a graph, which does not have K_5 or $K_{3,3}$ and a minor, and such that if we delete any vertex from G then we get a planar graph and let $x \in V_G$. If $G \setminus \{x\}$ is disconnected, then let $C_1, \ldots, C_n$ be its connected components. $C_i \cup \{x\}$ is planar for each i, due to the minimality of G. Using Proposition 3.4, we can transform any drawing of $C_i \cup \{x\}$ to one where x borders the outside face. We can then place each of these drawings in the same plane, and draw an edge from each copy of x to a new vertex y.

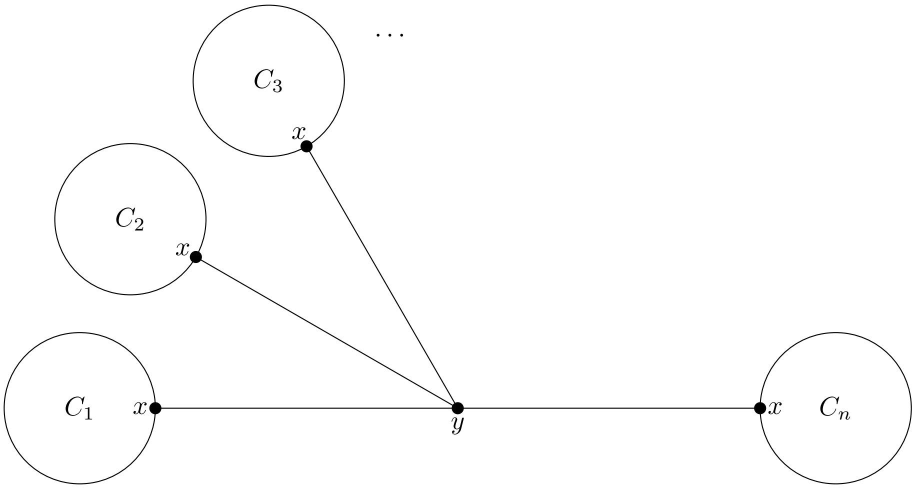


Figure 14: G is 1-connected

G is a minor of this new, planar graph, and is therefore planar. We now assume that G is at least 2-connected.

Suppose that G is at exactly 2-connected, i.e. there are vertices a and b such that $G \setminus a, b$ is disconnected. Let C be a connected component of $G \setminus \{a, b\}$. Then both $C \cup \{a, b\}$ and $G \setminus C$ are planar. We may assume that it is possible to draw both these graphs with a and b bordering the same face, otherwise adding an edge between a and b would result in a non-planar graph, which contradicts the minimality of G. Using Proposition 3.4, we can transform $C \cup \{a, b\}$ and $G \setminus C$ to have that face on the outside. We can glue these drawings of $C \cup \{a, b\}$ and $G \setminus C$ together to get a planar drawing of G.

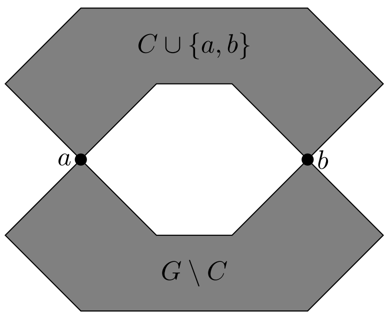


Figure 15: G is 2-connected

Therefore we may assume that G is least 3-connected. If we contract some edge e, we obtain a 2-connected planar graph, called G/e. Suppose that for all e, this G/e is not 3-connected, i.e. there are $a, b \in G/e$ such that if we delete a and b the result is disconnected. Either a or b has to be the vertex that replaced e, as otherwise removing a and b would disconnect G as well as G/e. Therefore for every edge $e \in E_G$ there is a vertex $c(e)$ such that $\{c(e)\} \cup \varepsilon_G(e)$ disconnects G.

For each $e \in E_G$, let $A^+(e)$ be the largest connected component of $G \setminus \{c(e)\} \cup \varepsilon_G(e)$, and let $A^-(e)$ be the union of the remaining connected components. Let $f \in E_G$ be such that $A^+(f)$ has the maximal number of vertices. There must be a vertex $a^+ \in A^+(f)$ and one in $a^- \in A^-$ such that $a^+Ec(f)$ and $a^-Ec(f)$, as otherwise $\varepsilon_G(f)$ would disconnect G on its own, and G is at least 3-connected. Let f' be the edge between $c(f)$ and a^-. If $c(f') \in A^-(f)$ then $A^+ \cup \varepsilon_G(f)$ is contained in a connected component of $G \setminus \{c(f')\} \cup \varepsilon_G(f')$, contradicting the maximality of $A^+(f)$.

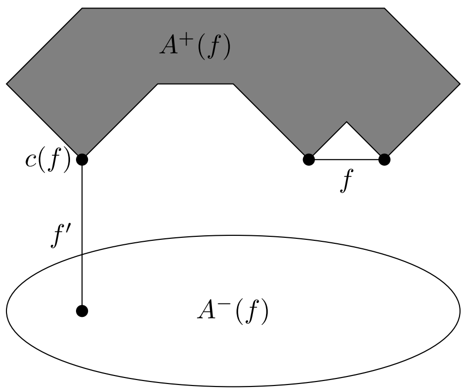


Figure 16: G/e is 3-connected

Therefore $c(f') \in A^+(f) \cup \varepsilon_G(f)$. Suppose $A^+(f) \cup \varepsilon_G(f)$ is disconnected after we delete $c(f')$. Then every path from one connected component of $(A^+(f) \cup \varepsilon_G(f)) \setminus c(f')$ to another has to pass through both $c(f)$ and $\varepsilon_G(f)$, so removing $\{c(f), c(f')\}$ would disconnect G, which is a contradiction. Now suppose that $A^+(f) \cup \varepsilon_G(f)$ is connected after we delete $c(f')$. Then G is still connected after we delete $\varepsilon_G(f') \cup \{c(f')\}$, another contradiction.

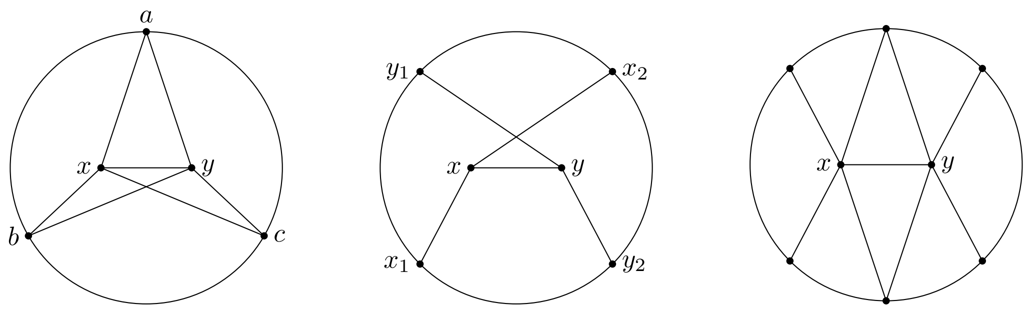
Therefore there is an e such that G/e is 3-connected. G/e is planar, so fix a drawing of G/e and delete the vertex that replaced e. The result is at least 2-connected, so every face is bounded by a cycle, including the face that contained e's replacement. We label the vertices in this cycle as $a_1, \ldots, a_n$. We write $C(a_i, a_j, a_k)$ if we pass through a_j when we go from a_i to a_k clockwise. Now draw two dots with an edge between them in this face, and label these points as x, y and e.

Let $X := \{a_i : xEa_i\}$ and $Y := \{a_i : yEa_i\}$. There are three cases to consider.

1. Suppose that there are distinct $a, b, c \in X \cap Y$. If we contract edges of the cycle so that only a, b and c remain distinct, we find that $\{x, y, a, b, c\}$ is isomorphic to K_5, and thus G has K_5 as a minor, which is a contradiction.

2. Suppose that there are distinct $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $C(x_1, y_1, x_2)$ and $C(y_1, x_2, y_2)$. We can delete edges to ensure that $x_1, x_2 \notin Y$ and $y_1, y_2 \notin X$. Then we can contract the edges of the cycle so that only x_1, x_2, y_1 and y_2 remain distinct. Then there is an edge from any element of $\{x, y_1, y_2\}$ to any element of $\{y, x_1, x_2\}$. Therefore $\{x, x_1, x_2, y, y_1, y_2\}$ is isomorphic to $K_{3,3}$, and thus G has $K_{3,3}$ as a minor, which is a contradiction.

3. If neither of the other two hold then if we draw x nearer to X and y nearer to Y then we have a planar drawing of G.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 17: The Three Sub-cases of the G/e is 3-connected Case

Therefore all non-planar graphs contain either K_5 or $K_{3,3}$ as a minor. □

Definition 3.13. A planar graph is said to be *maximal* if and only if adding an edge would result in a non-planar graph, or if the graph is complete.

Proposition 3.14. *If G is a maximal planar graph then in every drawing, every face is a triangle, and therefore $\|G\| = 3|G| - 6$.*

Proof. Suppose that G is a maximal planar graph, but that G has a drawing where one of the faces is not a triangle. Then there is a face with at least four edges, and therefore there are vertices u and v such that there is no edge between u and v. Drawing an edge between u and v can be done entirely within one face, and therefore G is not maximal.

Since every face is a triangle, every face has 3 edges bordering it. Each edge borders two faces, and therefore $3|F_G| = 2|E_G|$. We can substitute this into Euler's formula to get that

$$\begin{array}{rcl} 2 & = & \chi_E(G) \\ & = & |V_G| - |E_G| + |F_G| \\ & = & |V_G| - |E_G| + \frac{2}{3}|E_G| \\ 6 & = & 3|G| - \|G\| \end{array}$$

□

Lemma 3.15. *If G is planar then there is a $v \in V_G$ such that* $\deg(v) \leq 5$.

Proof. Suppose that every vertex in planar graph G has degree greater than 5. Therefore there are at least 6 edges joined to each vertex, and exactly two vertices attached to each edge, therefore $3|G| \leq \|G\|$. Proposition 3.14 shows that $\|G\| \leq 3|G| - 6$, so $\|G\| \leq \|G\| - 6$. A contradiction.

Therefore there is at least 1 vertex of order 5 or less. □

3.2 Tours and Cycles

Definition 3.16. Recall that a tour is a walk which does not use any edge twice. A tour is closed if it ends at the same place it started.

A tour is known as *Eulerian* if it is closed and each edge is used exactly once.

The concept of an Eulerian tour was created by Leonhard Euler (1707-1783) to examine the famous 'Seven Bridges of Königsburg' problem. The city of Königsburg had seven bridges, the problem asked for a walk through the city crossing each bridge exactly once, and ending on the landmass you started on.

It's not possible. Here's Euler's general result that proves it.

Lemma 3.17. *Let G be a graph, and let $v \in V_G$. If there's an Eulerian tour starting at v then there's an Eulerian tour starting at any vertex*

Exercise Hint. Write out the sequence of vertices visited in order. □

Theorem 3.18. *There is an Eulerian tour on connected graph G if and only if every vertex of G has even degree.*

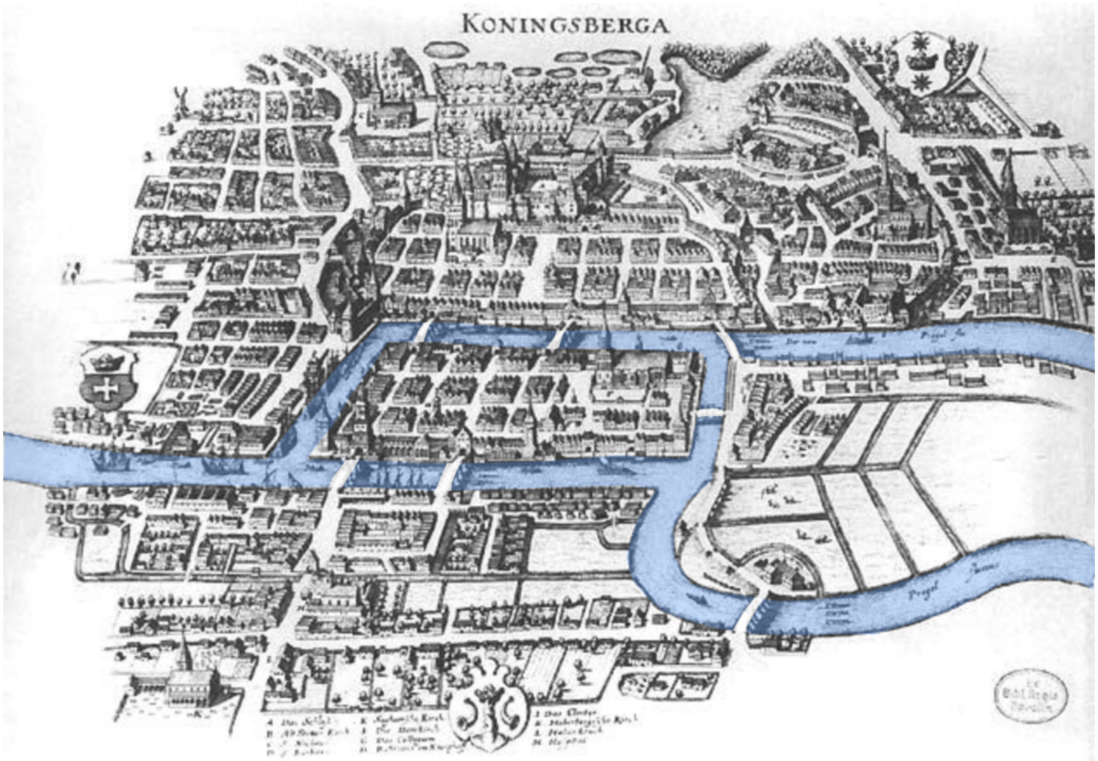


Figure 18: Königsburg in 1651

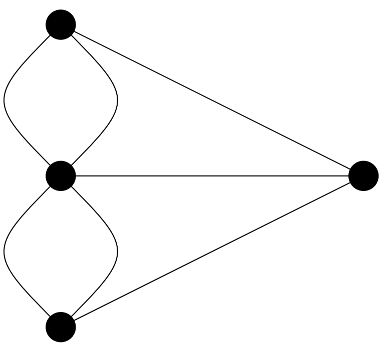


Figure 19: Formalising the Königsburg Bridge Problem

Figure 20: Kalingrad in 2020

$\Rightarrow$. Let G be a graph with vertex v such that $\deg(v)$ is odd, and suppose that there is an Eulerian tour on G. By Lemma 3.17, we can start that tour at G.

Let $v_1, \ldots, v_n$ be the vertices visited in the tour, starting with $v_1 = v$, and let i_j be all the times when the $v_{i_j} = v$. Since $i_1 = 1$, there is an odd number of edges attached to v that are yet to be traversed by the tour when we are at v_{i_1}.

When we travel from v_{i_j} to $v_{i_{j+1}}$ we use two edges joined to v, one to leave and one to return. Therefore at v_{i_j} , there are an odd number of edges attached to v that are yet to be traversed by the tour. Since $v_n = v$, this means that the tour never crosses every edge. $\square$

$\Leftarrow$. Let G be a connected graph such that every vertex of G has even degree. We will prove that G has an Eulerian tour by induction on $\|G\|$.

Suppose $\|G\| = 0$. Since G is connected, there must be only a single vertex in G. This graph has a trivial Eulerian tour.

Suppose that if H is a connected graph, such that every vertex of H has even degree, and $\|H\| < \|G\|$ then H has an Eulerian tour. Since every vertex of G has even degree, there can be no vertices with degree 1, and hence the set of all cycles in G is non-empty. Let T be the tour of maximal length, as a non-induced subgraph (i.e. T only contains the edges used in the maximal tour). If T is an Eulerian tour, then we're done, so assume that $G \neq T$.

T has an Eulerian tour, so every vertex of T has even degree. Therefore every vertex of $G \setminus E_T$ has even degree. Let $C_1, \ldots, C_n$ be the connected components of $G \setminus E_T$. If there is exactly one connected component, then let t be a vertex of T. The degree of t in $G \setminus E_T$ must be non-zero, otherwise t is a connected component all of its own. Let x and y be neighbours of t. Since $G \setminus E_T$ is connected there is a path from x to y, which we call P. We can extend T by replacing one of the instances of t by t, x, P, y, t. The result is a longer path, and therefore this contradicts our assumption that T was maximal.

Every vertex of C_i has even degree, and $\|C_i\| < \|G\|$, so C_i has an Eulerian tour, which we call S_i. Each of the C_i must contain one of the vertices of T, as otherwise G would already be disconnected. By Lemma 3.17, we can assume that each S_i starts at a vertex of T, say v_i.

Then the tour that follows T, but includes S_i when v_i is first reached, is an Eulerian tour of G. $\square$

The Potsdam Agreement transferred sovereignty of Königsburg to the USSR, and the city was renamed 'Kalingrad'. The extensive British bombing campaign and the long Soviet siege of the city had largely destroyed the city's infrastructure (including the bridges), which has been rebuilt quite differently.

Since the new graph has a vertex with odd degree, the Kalingrad bridge problem also has no solution. However, the Kalingrad graph is different from the Königsburg graph, as it only has one vertex of odd degree, rather than four.

Proposition 3.19. *Let G be a graph. There is a tour that uses every edge, which is not necessarily closed, if and only if there are at most two vertices of odd degree.*

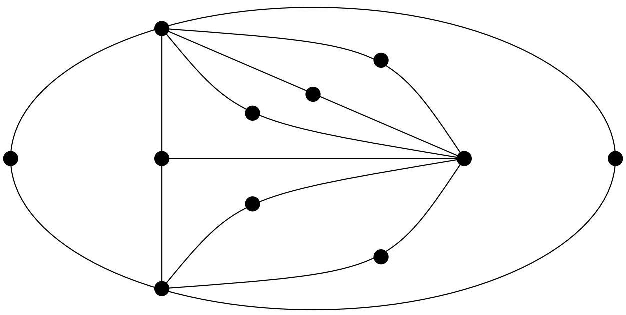


Figure 21: Formalising the Kalingrad Bridge Problem

Proof. Exercise. □

Remark 3.20. While a lot of our definitions and theorems lift naturally, or at least acceptably, to infinite graphs, Eulerian tours need more work. Obviously we need to use tours that are not necessarily closed, but we also makes more sense to consider a tour that not only never ends, but one that never starts as well. Consider the graph $\mathbb{Z}$ with nEm iff $n = m \pm 1$. This graph is as nice as they come for crossing all the edges exactly once, so it makes sense to adapt our definitions to permit an Eulerian tour on this graph.

Definition 3.21. A bi-infinite Eulerian tour is a sequence of vertices $\dots, v_{-1}, v_0, v_1, \dots$ such that:

1. $v_i E V_{i+1}$ for all i, and
2. for every edge e there is a unique i such that $\varepsilon(e) = \{v_i, v_{i+1}\}$.

Theorem 3.22. *Let G be a graph. G has a bi-infinite Eulerian tour if and only if all of the following:*

1. *G is connected,*
2. *G has countably many vertices,*
3. *every vertex of G has either even or infinite degree,*
4. *if A and B are a partition of the vertices then there are odd or infinitely many edges between A and B.*

$\Rightarrow$. The following graphs show that 1-4 are necessary conditions.

1. Let R be the Rado graph. Then $R \oplus R$ is disconnected and does not have a bi-infinite Eulerian tour.
2. Let K be a complete graph on uncountably many vertices. A bi-infinite Eulerian tour can only cover countably many edges, so K does not have an Eulerian tour.
3. Let N be the graph with vertex set $\mathbb{N}$, where nEm if and only if $n = m \pm 1$.
4. Let W be the graph with vertex set $\mathbb{Z} \times \{0, 1\}$, where $(n, i)E(m, j)$ if and only if:
   - $n = m \pm 1$ and $i = j$, or
   - $i \neq j$, and $m, n \in \{0, 1\}$.

   There are two edges between $A = \mathbb{Z} \times \{0\}$ and $B = \mathbb{Z} \times \{1\}$. Any potential bi-infinite Eulerian tour must contain the vertices $(-10, 1)$, $(10, 1)$, $(-10, 0)$, and $(10, 0)$. But there is no way to do that without crossing the same edge twice.

The rest is an exercise. □

$\Leftarrow$. Exercise. □

Definition 3.23. Recall that a cycle is a path that starts at the same vertex as it started. A cycle is said to be Hamiltonian if and only if it visits every vertex exactly once.

Theorem 3.24 (Dirac's Theorem (1952)). *Let G be a graph such that $|G| \geq 3$ and* $\min\{\deg(v) : v \in V_G\} \geq \frac{|G|}{2}$. *Then G has a Hamiltonian cycle.*

Proof. G can't have a Hamiltonian cycle if G is disconnected, so we need to rule that out. Suppose that G is disconnected. Then G has a connected component C such that $|C| \leq \frac{|G|}{2}$. Even if C is complete, if $v \in V$ then $\deg(v) \leq |C| - 1 < \frac{|G|}{2}$. Therefore G is connected.

Let P be a path in G with maximal length, and let $v_1, \ldots, v_k$ be the vertices of P. If $v_1 E x$ then $x = v_i$ for some i, otherwise $x, v_1, \ldots, v_k$ would be a longer path. Similarly, if $x E v_k$ then $x = v_i$ for some i. The degree of v_1 is greater than or equal to $\frac{|G|}{2}$ and $k \leq |G|$, therefore $v_1 E v_i$ for more than half of the possible values of i. This means that the set $\{v_i : v_1 E v_{i+1}\}$ contains more than half the possible vertices on path P, and therefore there is an i_0 such that $v_1 E v_{i_0+1}$ and $v_{i_0} E v_k$. We can therefore traverse P as $v_1, v_2, \ldots, v_{i_0}, v_k, v_{k-1}, \ldots, v_{i_0+1}, v_1$ to obtain a cycle.

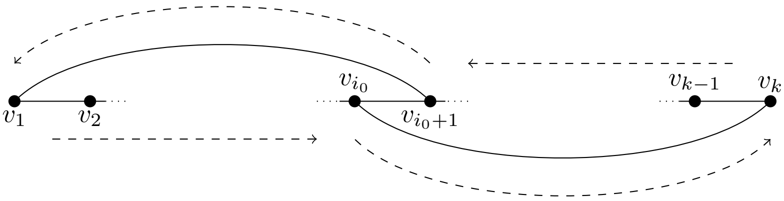
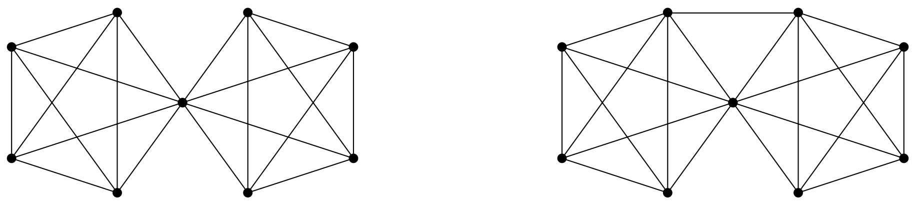


Figure 22: The Cycle from Dirac's Theorem

If P covers G then we're done, so suppose that there is an $x \in V_G \setminus P$. Since G is connected there is a $v \in P$ such that $x E v$. Since we're able to traverse P in a cycle, we can start that cycle at any point, including v. Suppose that if we start at v then the last vertex we reach before we return to v is u. Then $x, v, \ldots, u$ is a path longer than P, contradicting the maximality of P. Therefore the cycle we built in the last paragraph is a Hamiltonian cycle. □

Example 3.25. Dirac's Theorem cannot be improved, there are graphs where the minimum degree is equal to $\lfloor \frac{|G|}{2} \rfloor$ which do no contain a Hamiltonian cycle. For example, we can construct G by gluing two copies of K_n together at a vertex. This graph cannot have a Hamiltonian cycle, as every walk that visits every vertex will have to pass through the joining vertex at least twice. However, the minimal degree of this graph is n, which is equal to $\lfloor \frac{|G|}{2} \rfloor$.

However, Dirac's Theorem only gives us a sufficient condition for the existence of a Hamiltonian cycle, not a necessary one. By adding a new edge between vertices of minimal degree in the previous example, we find a graph which has a Hamiltonian cycle, but does not satisfy the conditions of Dirac's Theorem.



(a) Optimal

(b) Not Necessary

Figure 23: Dirac's Theorem is Optimal, but Not Necessary

3.3 Colourings

Definition 3.26. Let G be a graph, and let $n \in \mathbb{N}$. An *n-colouring* of G is a map $c : V_G \to \{1, \ldots, n\}$ such that if uEv then $c(u) \neq c(v)$.

The least n such that there is an n-colouring of G is written as $\chi_C(G)$.

Proposition 3.27. $\chi_C(K_n) = n$.

If T is a tree such that $|T| > 1$ then $\chi_C(T) = 2$.

Proof. Between any two vertices in K_n there is an edge, so no two vertices can have the same colour. If we colour every vertex of K_n with a different colour, we get a valid colouring. Therefore $\chi_C(K_n) = n$.

Let T be a tree. Pick any vertex $v \in V_T$, and colour it 1. Colour every vertex attached to v with 2, the vertices attached to those vertices with 1, and so on. This is a 2-colouring of T, so $\chi_C(T) = 2$. □

Example 3.28. For every $1 < i \leq n$ there is a connected graph G such that $|G| = n$ and $\chi_C(G) = i$.

Let G have vertices $v_1, \dots, v_n$. We specify that the induced subgraph on $\{v_1, \dots, v_{i-1}\}$ is isomorphic to K_{i-1}. For v_j where $i \geq j$, we require that $v_j E v_k$ if and only if $k < i$.

Proposition 3.29. *If G is a graph, and $H_1, H_2 \leq G$ are such that:*

1. *$H_1 \cup H_2 = G$,*
2. *if $h_1 \in H_1$ and $h_2 \in H_2$ and $h_1 E h_2$ then either h_1 or h_2 is contained in $H_1 \cap H_2$, and*
3. *$H_1 \cap H_2$ is a complete graph,*

then $\chi_C(G) = \max\{\chi_C(H_1), \chi_C(H_2)\}$

Proof. Exercise □

Definition 3.30. Let G be a graph. $P_G : \mathbb{N} \to \mathbb{N}$ is the function which is defined as follows:

$$n \mapsto |\{c : V_G \to \{1, \dots, n\} : c \text{ is any colouring function.}\}|$$

P_G is known as the *chromatic polynomial* of G (justification that this is a polynomial follows immediately.)

Proposition 3.31. *P_G is a polynomial for all graphs G.*

Proof. If G is the 1-vertex graph then $P_G(n) = n$ for all n. If $G = K_{m+1}$ then $P_G(n) = \prod_{i=0}^{m}(n - i)$.

Suppose that G is a graph such that if

- $|H| < |G|$, or
- $|H| = |G|$ and $\|H\| > \|G\|$,

then P_H is a polynomial.

There are $u, v \in V_G$ such that there is no edge between u and v. Every colouring of G either gives u the same colour as v, or a different one. Any colouring of G that gives u and v a different colour is also a colouring of A, the graph obtained by adding a new edge e to G which joins u and v. Any colouring of G that gives u and v the same colour is also a colouring of B, the graph obtained by contracting e in A.

$|B| < |G|$, and $|A| = |G|$ and $\|A\| \geq \|G\|$, so both P_A and P_B are polynomials. Since $P_G = P_A + P_B$, this means that P_G is also a polynomial. □

Proposition 3.32. *Let G be a graph. The following are equivalent:*

1. *G is n-partite but not $(n-1)$-partite.*
2. *$n = \min\{x \in \mathbb{N} : P_G(x) \neq 0\}$.*
3. *$\chi_C(G) = n$.*

Moreover $K_n \leq G$ then $\chi_C(G) \geq n$.

Proof. Exercise □

Theorem 3.33 (Erdős 1959)**.** *For every $k \geq 3$ there is a graph H with girth $g(H) > k$ and chromatic number $\chi_C(H) > k$.*

This theorem is hard to prove. The girth is the length of the shortest cycle, so a large girth means that the graph locally looks like a tree. If the girth of H is greater than k then the set of all vertices reachable from vertex x with a path of length at most k is a tree, and that means that the largest K_n we can embed into H is K_2. Therefore the 'moreover' of Proposition 3.32 can give a very poor lower bound for $\chi_C(H)$.

We will prove this, but it's the grand finale of this course, so you'll have to be patient. In literature, this technique is called 'foreshadowing'.

Theorem 3.34 (The 5-colour Theorem)**.** *Let G be any finite planar graph. Then $\chi_C(G) \leq 5$.*

Proof. Every graph with at most 5 vertices has a 5-colouring.

Let G be planar graph with $\chi_C G \geq 6$ such that every smaller graph is 5 colourable. By Lemma 3.15, there is a $v \in V_G$ such that $\deg(v) \leq 5$. Suppose $\deg(v) \leq 4$. Then $G \setminus v$ has a 5 colouring, giving the neighbours of v at most 4 different colours. If we colour v with the fifth colour, we have a 5 colouring of G.

Suppose that $\deg(v) = 5$. As before, $G \setminus v$ has a 5 colouring. If there is a colouring of $G \setminus v$ such that the neighbours of G have 4 or fewer colours, then we can colour v with one of the missing colours to obtain a 5 colouring of G.

Let x_1, x_2, x_3, x_4, and x_5 be the vertices which share an edge with v (labelled clockwise starting at a), and suppose they all have different colours. For convenience, assume that x_i has colour i. Now consider the induced subgraph on the set $X(1,3) := \{v \in V_G : \text{ the colour of } v \text{ is 1 or 3}\}$. Suppose that x_1 and x_3 are in different connected components of $X(1,3)$. If we swap the colours of the component contain x_1, then we still have a valid colouring of G, which gives x_1 and x_3 the same colour, so we can colour v with 1 to get a 5-colouring of G.

Therefore there is a path $P \subseteq X(1,3)$ from x_1 to x_3. Consider the region enclosed by the cycle that starts at v, goes to x_1, follows P, then returns to v. This region contains x_2, and does not include x_4. Therefore any path from x_2 to x_4 has to pass through a vertex with colour 1 or 3, and therefore $X(2,4)$ is disconnected.

Thus G also has a 5 colouring. □

Theorem 3.35 (The 4-colour Theorem)**.** *Let G be any finite planar graph. Then $\chi_C(G) \leq 4$.*

Proof. Proving this takes some serious work, with well over a thousand cases to consider. The proof of the 4-colour theorem was one of the first computer aided proofs, and to this day all proofs of this theorem are computer aided. □

Remark 3.36. The 4-colour theorem was first proved by Appel (born in the USA) and Haken (born in Germany) in 1976, who were both working at the University of Illinois at Urbana-Champaign. This suggests that the correct name is 'The 4-color Theorem'. However, the theorem was first conjectured by Guthrie (born in the UK) when he tried to colour a map of the counties of England, so actually 'The 4-colour Theorem' is correct. This is, of course, how spelling is decided. The language and dialect used by the writer is totally irrelevant. For example, the correct spelling in Greek is '$\tau o\ \theta\varepsilon\omega\rho\eta\mu\alpha\ \tau\omega\nu\ \tau\eta\sigma\sigma\alpha\rho\omega\nu\ \chi\rho\omega\mathbf{u}\mu\alpha\tau\omega\nu$', not '$\tau o\ \theta\varepsilon\omega\rho\eta\mu\alpha\ \tau\omega\nu\ \tau\eta\sigma\sigma\alpha\rho\omega\nu\ \chi\rho\omega\mu\alpha\tau\omega\nu$'.

Corollary 3.37. *Let G be any planar graph. Then there is a 4-colouring of G.*

Proof. This result is a corollary of the 4-colour Theorem, but is also an application of the Compactness Theorem of Propositional Logic. If you're familiar with that theorem, it's not a hard proof, but if you're not then it is. □

4 Limiting Behaviour

So far we have be studying the structure of graphs, usually finite but occasionally infinite. In this section we'll be looking at the structure of the set of finite graphs. Given how expressive and varied graphs can be, you might expect this to be not the most tractable of subjects. However, there is a remarkable amount that can be said about the set of all finite graphs.

4.1 Ramsey Theory

Ramsey Theory starts with Ramsey's Theorem, which says that, for any n and m, if you have a large enough graph then you have to have either K_n or $\overline{K_m}$ as an induced subgraph. Let's establish some notation for this.

Definition 4.1. Let $l, k \in \mathbb{N} \setminus \{0\}$. If it exists, we define $R(k, l) \in \mathbb{N}$ to be the smallest number such that if $|G| \geq R(k, l)$ then either $K_k \leq G$ or $\overline{K_l} \leq G$.

Lemma 4.2. *$R(1, n) = R(n, 1) = 1$ and $R(2, n) = R(n, 2) = n$ for all $n \in \mathbb{N}$. i.e.:*

Every non-empty graph contains both K_1 and $\overline{K_1}$ as an induced subgraph. If $|G| = n$ and $K_2 \not\leq G$ then $G \cong \overline{K_n}$, and if $|G| = n$ and $\overline{K_2} \not\leq G$ then $G \cong K_n$.

Proof. K_1 is also $\overline{K_1}$, is also every one element graph, and so every non-empty graph has K_1 and $\overline{K_1}$ as an induced subgraph, so $R(1, l) = R(k, 1) = 1$.

If a graph G does not have K_n as an induced subgraph, then either $|G| < n$ or $G = \overline{K_m}$, with $n \leq m$, showing that $R(2, n) = n$ for all n. A similar observation shows that $R(n, 2) = n$. □

Lemma 4.3. *If $R(k-1, l)$ and $R(k, l-1)$ exist, then so does $R(k, l)$.*

Proof. Let G be any graph such that $|G| \geq R(k-1, l) + R(k, l-1)$. Let $v \in V_G$, and let

$$A_v = \{w \in V_G : vE_G w\}$$

If $|A_v| \geq R(k-1, l)$ then there is a $B \leq G$ such that $B \subseteq A_v$ and $B \cong K_{k-1}$. Thus $B \cup \{v\} \cong K_k$, and $K_k \leq G$.

If $|A_v| < R(k-1, l)$ then $|V_G \setminus A_v| \geq R(k, l-1)$ so there is a $C \leq G \setminus E_G$ such that $C \cup \{v\} \cong \overline{K_l}$. □

Theorem 4.4 (Ramsey's Theorem)**.** *$R(k, l) \in \mathbb{N}$ exists for all $l, k \in \mathbb{N} \setminus \{0\}$.*

Proof. We prove by induction on $k + l$ that $R(k, l)$ always exists.

Suppose that $k + l = 2$ then $k = 1$ and $l = 1$, and Lemma 4.2 shows that $R(1, 1) = 1$.

Suppose that if $k' + l' < n$ then $R(k', l')$ exists, and let k, l be such that $k + l = n$. Then $(k-1) + l < n$ and $k + (l-1) < n$, so $R(k-1, l)$ and $R(k, l-1)$ both exist, and Lemma 4.3 shows that $R(k, l)$ exists. □

We can extract an upper bound for $R(k, l)$ from the proof of Lemma 4.3.

Proposition 4.5.

$$R(k, l) \leq \binom{k+l-2}{k-1}$$

Proof. $R(2, 2) = 2$, and $\binom{2}{1} = 2$ so this is true for $k = 2$ and $l = 2$.

Suppose that if $i + j < k + l$ then $R(i, j) \leq \binom{i+j-2}{i-1}$. Then

$$\begin{aligned} R(k, l) &\leq R(k-1, l) + R(k, l-1) \\ &\leq \binom{(k-1)+l-2}{(k-1)-1} + \binom{k+(l-1)-2}{k-1} \\ &\leq \binom{k+l-2}{k-1} \end{aligned}$$

□

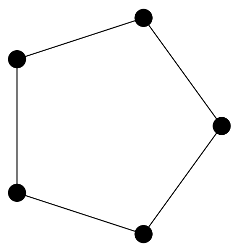
The exact value of $R(k, l)$ is for most values of k and l, an open problem, but there are some values which have been calculated.

Example 4.6. Lemma 4.2 calculates the exact values of $R(2, n)$ and $R(2, n)$. The graph C_5 proves that $R(3, 3) > 5$, so $R(3, 3) = 6$:

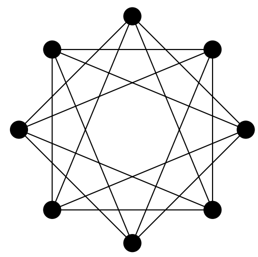
Please see https://mathworld.wolfram.com/RamseyNumber.html for a list of the currently known values and best estimates for the values of $R(k, l)$ for small k and l.

k\l	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003
8	1	8	36	120	330	792	1716	3432	6435
9	1	9	45	165	495	1287	3003	6435	12870

Figure 24: The Upper Bounds from Proposition 4.5



(a) $R(3,3) > 5$



(b) $R(4,3) > 8$

Ramsey style results exist in the infinite world as well as the finite one. One particularly lovely result is the Erdős-Dushnik-Miller Theorem, which shows that $R(|\mathbb{N}|, |X|) = |X| = R(|X|, |\mathbb{N}|)$ for all infinite X. We won't be able to prove this here, as it requires a lot of familiarity with Set Theory. The result I'm going to present here seems further from Graph Theory than it is. Part of the Exercise Sheets is to deduce the finite Ramsey Theorem from this statement.

Theorem 4.7 (Infinite Ramsey's Theorem)**.** *Let X be an infinite set. $\mathcal{P}_n(X) := \{A \subseteq X : |A| = n\}$ and let $f : \mathcal{P}_n(X) \to \{1, \ldots, c\}$ be a function. Then there is an infinite $M \subseteq X$ such that f is constant on $\mathcal{P}_n(M)$.*

Proof. This is proved via induction on n. If $n = 1$ then we can note that $X = \bigcup_{i=1}^{c} f^{-1}(i)$. Since X is infinite then at least one of the $f^{-1}(i)$ must be infinite.

Suppose that if $r \leq n$ then for every $f : \mathcal{P}_r(X) \to \{1, \ldots, c\}$ there is an infinite $M \subseteq X$ such that f is constant on $\mathcal{P}_n(M)$. Let $g : \mathcal{P}_{n+1}(X) \to \{1, \ldots, c\}$. We'll construct our M for g inductively.

Let $x_0 \in X$ and let $h_0 : \mathcal{P}_n(X \setminus \{x_0\}) \to \{1, \ldots, c\}$ be the function given by $h_0(A) = g(A \cup \{x_0\})$. By the induction hypothesis, there is an $X_0 \subset X \setminus \{x_0\}$ such that h_0 is constant on $\mathcal{P}_n(X_0)$.

Suppose we have defined x_i and X_i. Let $x_{i+1} \in X_i$, and let $h_i : \mathcal{P}_n(X_i \setminus \{x_{i+1}\}) \to \{1, \ldots, c\}$ be given by $h_i(A) = g(A \cup \{x_{i+1}\})$. Then, by the induction hypothesis, there is an $X_{i+1} \subseteq X_i \setminus \{x_{i+1}\}$ such that h_i is constant on $\mathcal{P}_n(X_{i+1})$.

Let $M' = \{x_i : i \in \mathbb{N}\}$. If $A, B \in \mathcal{P}_{n+1}(M')$ and the minimum i such that x_i is in A is equal to the minimum j such that x_j is in B then $g(A) = g(B)$. Let $f : M' \to \{1, \ldots, c\}$ be defined by $f(x_i) = g(\{x_i, x_{i+1}, \ldots, x_{i+n}\})$. By the base case, there is an infinite $M \subseteq M' \subseteq X$ such that f is constant on M. Then g is constant on $\mathcal{P}_{n+1}(M)$. $\square$

The results so far have been about when a complete graph, or the inverse of a complete graph, has to occur as a subgraph. It makes just as much sense to ask the same question where graphs are more complicated objects.

Definition 4.8. Let G and H be finite simple graphs. If it exists, $R(G, H)$ is the natural number such that if K is a graph and $|K| \geq R(G, H)$ then either $G \subseteq K$ or $H \subseteq \overline{K}$. Note that $R(K_n, K_m) = R(n, m)$.

Proposition 4.9. $R(G, H) \leq R(|G|, |H|)$.

Proof. $G \subseteq K_{|G|}$ and $H \subseteq K_{|H|}$. If K is such that $|K| \geq R(|G|, |H|)$ then either $G \subseteq K_{|G|} \subseteq K$ or $\overline{K_{|H|}} \subseteq K$, so $H \subseteq K_{|H|} \subseteq \overline{K}$. □

Proposition 4.10. *Let T be a tree (recall that a tree is a connected graph where every two vertices have a unique path between them). Then $R(T, K_n) = (|T| - 1)(n - 1) + 1$.*

Proof. Let G be the graph that has $n - 1$ many connect components, each of which is isomorphic to $K_{|T|-1}$. Then $T \not\subseteq K_{|T|-1}$, so $T \not\subseteq G$. But $\overline{G}$ is a complete $n - 1$-partite graph, so $K_n \not\subseteq \overline{G}$. Therefore $R(T, K_n) > |G| = (|T| - 1)(n - 1)$.

Suppose that $|G| > (|T| - 1)(n - 1)$. Let c be a colouring of G. If there is a set $X \subseteq G$ such that $|c(X)| = 1$ and $|X| \geq n$ then $\overline{K_n} \leq G$, so $K_n \subseteq \overline{G}$. Suppose that if $X \subseteq G$ is such that $|c(X)| = 1$ then $|X| < n$. Therefore we may assume that $\chi_C(G) \geq |T|$.

Let $H \subseteq G$ be a minimal subset of G such that $\chi_C(H) = \chi_C(G)$. This implies that H is connected. If $v \in H$ then $\chi(H \setminus \{v\}) < \chi_C(G)$. If there is a $v \in H$ such that $\deg(v) \leq \chi_C(G) - 2$ then we can extend a colouring of $H \setminus \{v\}$ that uses less than $\chi_C(G)$ colours to a colouring of H that uses less than $\chi_C(G)$ colours. Therefore if $v \in H$ then $\deg(v) \geq |T| - 1$.

It's an exercise to show that $T \subseteq H$. □

4.2 Extremal Graph Theory

Extremal Graph Theory is the subject dedicated to finding *extremal* graphs. Say we're interested in a certain property, e.g. whether a graph has K_n as a subgraph. A graph G would be extremal in this situation if $\|G\| = \max\{\|H\| : K_n \not\leq H\}$. The main theorem of this section, Turán's Theorem, can be viewed as a result from Extremal Graph Theory, but has many facets to it other than the one I'm presenting here.

Definition 4.11. If G and H be graphs. We say that G is H-free if there are no $K \subseteq G$ such that $K \cong H$.

Definition 4.12. Let H be a graph. If it exists, $\mathrm{ex}(n, H)$ (the ex stands for 'extremal'.) is the smallest number such that for all graphs G such that $|G| = n$, if $\|G\| \geq \mathrm{ex}(n, H)$ then $H \subseteq G$.

Since we're asking for $H \subseteq G$, rather than $H \leq G$, we have that $\mathrm{ex}(n, H)$ will eventually exist if $n \geq |H|$.

Proposition 4.13. *If $|H| \leq n$ then $\mathrm{ex}(n, H) \leq \binom{n}{2}$ for all n and H.*

Proof. If $|H| \leq n$ then $H \subseteq K_n$, and $\|K_n\| = \binom{n}{2}$. □

Proposition 4.14. *If $n > 1$ then $\mathrm{ex}(n, K_2) = 1$ and $\mathrm{ex}(n, \emptyset) = 0$.*

Proof. Easy exercise. □

Definition 4.15. Let $n, r \in \mathbb{N}$. Then $T(n, r)$, the (n, r)-Turán Graph, is the following graph:

- $V_{T(n,r)} = \{v_1, v_2, \ldots, v_n\}$
- $E_{T(n,r)} = \{\{v_i, v_j\} : 1 \leq i < n \text{ and } (\frac{i}{r} - \lfloor \frac{i}{r} \rfloor) \neq (\frac{j}{r} - \lfloor \frac{j}{r} \rfloor)\}$
- $\varepsilon_{T(n,r)}(\{v_i, v_j\}) = \{v_i, v_j\}$ for all i, j.

The Turán graphs are the complete multipartite graphs, where the vertices are spread evenly across the parts.

Theorem 4.16 (Turán's Theorem)**.** *Let $r > 1$. If G is a graph such that $|G| = n$ and*

$$\|G\| = \max\{\|G\| : |G| = n \text{ and } G \text{ is } K_{r+1} \text{ free}\}$$

Then $G \cong T(n, r)$.

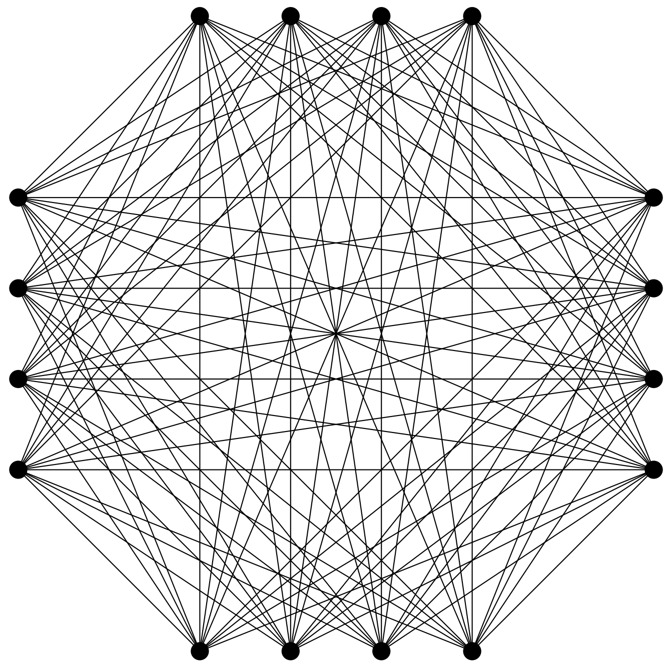


Figure 26: $T(16, 4)$, a Turán Graph

Proof. Let G be such a graph. We'll start by showing that G is complete multipartite. Let $v, w \in V_G$, we define the following relation on V_G:

$$v \sim w \Leftrightarrow \neg vEw$$

If this is an equivalence relation, then G is a complete multipartite graph. Reflexivity and symmetry are immediate, but transitivity is more challenging. Suppose for a contradiction that it isn't, i.e. that there are u, v, w such that $\neg uEv$ and $\neg vEw$, but uEw.

Suppose that $\deg(v) < \deg(u)$. Then consider G', the graph obtained by deleting v and adding v', which is joined to exactly the vertices that u is joined to. If there is a copy of K_{r+1} in G', called A', then $v' \in A'$, otherwise A' is also contained in G. However, $\neg v'Eu$, so $u \notin A'$. However, if $A = (A' \setminus \{v'\}) \cup \{u\}$ then A must also be a copy of K_{r+1}, as u joined to everything that v' is. Therefore there is no such A', and G' is K_{r+1}-free. However, the total number of edges in G' is greater than the total number of edges in G, as $\deg(u) > \deg(v)$, contradicting our assumption that G is extremal, showing that $\deg(v) \geq \deg(u)$. A similar argument shows that $\deg(v) \geq \deg(w)$.

Let G' be the graph obtained by deleting both u and w and replacing them with copies of v, which we will call v_u and v_w. If G' is not K_{r+1} free then there must be an $A' \leq G'$ such that $A' \cong K_{r+1}$. This A' must contain exactly one of v_u and v_w. If it does not contain either, then $A' \leq G$ as well. It cannot contain both, as $\neg v_u E v_w$. Once again, since $A \cong K_{r+1}$, we also have that $(A' \setminus \{v_u\}) \cup \{v\} \cong K_{r+1}$. Therefore G' is also K_{r+1}-free. However, when we constructed G', we replaced the edge uv by two new edges, so $\|G\| < \|G'\|$, contradicting our assumption that G is extremal.

Therefore G is complete i-partite for some i. We now wish to show that $i = r$. By selecting a vertex from each part, we obtain a copy of K_i, so $i \leq r$. If $i < r$ then we can pick any vertex $v \in V_G$. Then by drawing an edge between v and every vertex in the same part as v, we obtain a K_r-free graph with more edges than G, so $i = r$.

Suppose there are $u, v \in V_G$ such that $|[v]| < |[u]| - 1$, i.e. the part containing u of G has at least 2 more vertices than the part containing v. Then moving a vertex from $[u]$ to $[v]$ will increase the number of edges without introducing a copy of K_{r+1}. Therefore G is $T(n, r)$.

□

Corollary 4.17. $\mathrm{ex}(n, K_m) = \|T(n, m-1)\| + 1$

Corollary 4.18 (Mantel's Theorem (1907))**.** *If G is a graph such that $K_3 \not\subseteq G$ then $\|G\| \leq \frac{|G|^2}{4}$.*

Proof. Suppose that G is K_3-free. Then $\|G\| \leq \|T(|G|, 2)\|$. If $|G|$ is even then $T(|G|, 2)$ is the complete bipartite graph where both parts have $\frac{|G|}{2}$ many vertices. Therefore $\|T(|G|, 2)\| = \left(\frac{|G|}{2}\right)^2$. If $|G|$ is odd, say $2n+1$ then $T(|G|,2)$ is the complete bipartite graph where one part has n vertices, and the other has $(n+1)$. Therefore $\|T(|G|,2)\| = n.(n+1) \leq \left(\frac{|G|}{2}\right)^2$. □

Turán's Theorem is the best possible result for describing the K_n-free graphs. If G is K_n-free then G is a subgraph of $T(n|G|,n)$, and Mantel's Theorem, or other versions of it, give an optimal upper-bound of $\|G\|$. The situation for G-free graphs is more complicated, but significant progress is still possible.

The proof of the following theorem is not in the syllabus of this course, but that's due to reasons of volume rather than difficulty. Diestel's proof using the Szemerédi Regularity Lemma, the statement and proof of which is the basis for this course's mastery material, but more elementary proofs exist.

Theorem 4.19 (The Erdős-Stone Theorem (1946))**.** *Let $r \geq 2$ and $s \geq 1$ be natural numbers. Let $\varepsilon > 0$ be a real number. Then there is an $n(r,s,\varepsilon) \in \mathbb{Z}$ such that if G is a graph with $|G| \geq n(r,s,\varepsilon)$ and $\|G\| \geq \|T(|G|,r-1)\| + \varepsilon|G|^2$ then $T(rs,r) \subseteq G$.*

I feel guilty introducing this theorem without giving a proof, but I think you'll forgive me when you see what we do with it. This theorem has a lot of applications and importance, but the reason why I introduced this without proof is for the following corollary:

Corollary 4.20. *If H is any graph such that $\|H\| > 0$ then*

$$\lim_{n\to\infty} \mathrm{ex}(n,H)\binom{n}{2}^{-1} = \frac{\chi_C(H)-2}{\chi_C(H)-1}$$

Proof. Since this is a corollary of the Erdős-Stone Theorem, we will be talking about the Turán graphs. To that end, let's examine $\lim\limits_{n\to\infty} \|T(n,r-1)\|\binom{n}{2}^{-1}$ for a fixed r. Let a and b be the greatest and the least natural numbers such that $r-1|a$ and $r-1|b$ and:

$$a \leq n \leq b$$

So by examining $\|T(a,r-1)\|$ and $\|T(b,r-1)\|$ we obtain bounds for $\|T(n,r-1)\|$.

$$\frac{a^2}{2}\left(\frac{r-2}{r-1}\right) \leq \|T(n,r-1)\| \leq \frac{b^2}{2}\left(\frac{r-2}{r-1}\right)$$

These bounds show that if the subsequence of $\|T(n,r-1)\|\binom{n}{2}^{-1}$ obtained where n is divisible by $r-1$ has a limit, then the whole sequence has that limit.

$$\begin{aligned} \frac{a^2}{2}\left(\frac{r-2}{r-1}\right)\binom{a}{2}^{-1} &= \frac{a^2}{2}\left(\frac{r-2}{r-1}\right)\frac{2!(a-2)!}{a!} \\ &= \frac{r-2}{r-1}\frac{a}{a-1} \\ &\to \frac{r-2}{r-1} \text{ as } a \to \infty \end{aligned}$$

Therefore we have calculated the limit for the Turán graphs.

$$\lim_{n\to\infty} \|T(n,r-1)\|\binom{n}{2}^{-1} = \frac{r-2}{r-1}$$

Let's see how we can relate the limit for the Turán graphs to the general problem.

Let H be a graph. Suppose that $\chi_C(H) = r$. Since H does not have an $r-1$ colouring, $H \not\subseteq T(m,r-1)$ for all m, and therefore $\|T(m,r-1)\| \leq \mathrm{ex}(m,H)$ for all m. This means that there is an s such that $H \subseteq T(s,r)$, and therefore if $m > s$ then $\mathrm{ex}(m,H) \leq \mathrm{ex}(m,T(s,r))$.

Let $\varepsilon > 0$. By the Erdős-Stone theorem, there is an $n(r,\lceil\frac{s}{r}\rceil,\varepsilon)$ such that if G is any graph such that $|G| \geq n(r,\lceil\frac{s}{r}\rceil,\varepsilon)$ and $\|G\| \geq \|T(|G|,r-1)\| + \varepsilon|G|^2$ then $T(s,r) \subseteq T(r\lceil\frac{s}{r}\rceil,r) \subseteq G$, and therefore $\mathrm{ex}(m,T(s,r)) < \|T(|G|,r-1)\| + \varepsilon|G|^2$.

Finally, if $n \geq s$ and $n \geq n(r,\lceil\frac{s}{r}\rceil,\varepsilon)$ then

$$\begin{aligned} \|T(n,r-1)\|\binom{n}{2}^{-1} \leq \mathrm{ex}(n,H)\binom{n}{2}^{-1} &\leq \mathrm{ex}(n,T(s,r))\binom{n}{2}^{-1} \\ &< \|T(n,r-1)\|\binom{n}{2}^{-1} + \varepsilon n^2\binom{n}{2}^{-1} \\ &\leq \|T(n,r-1)\|\binom{n}{2}^{-1} + 4\varepsilon \end{aligned}$$

Applying the limits we calculated earlier, and noting that this is valid for all $\varepsilon > 0$, we find that:

$$\lim_{n\to\infty} \mathrm{ex}(n,H)\binom{n}{2}^{-1} = \frac{\chi_C(H)-2}{\chi_C(H)-1}$$

□

4.3 Random Graphs

How you you prove that a graph with a certain property exists? Of course, you construct it! Well, what if you need a graph with that property which has n-vertices? You could construct one inductively, take a smaller graph with the property, and add vertices and edges without damaging the property. Well, some properties don't like you doing that, and will hate you for trying. If I'm adding edges to a k-connected graph, I'm going to have to be ludicrously careful not to change k. Many graph properties are like this, the chromatic number, the diameter, the girth, and so many more.

Paul Erdős and Alfréd Rényi created the *probabilistic method* to circumvent this. They constructed a graph using a random method, and then showed that the probability of a constructing n-vertex graph with the desired property was never zero. Thus if you needed a graph of size greater than n, the probability that no such graph exists is 0. It is *almost certain* that the graph exists.

Since Graph Theory is part of mathematics, every good solution will turned into a better problem, leading to the branch of graph theory known as Random Graph Theory.

Definition 4.21. Let $V = \{v_1, \dots v_n\}$, and let $p \in [0,1]$. We construct a random graph G, by adding an edge between each (v_i, v_j) with probability p. Whether an edge between v_i and v_j is added or not is independent of all the other pairs. We write $G \in \mathcal{G}(n,p)$ to indicate that G is constructed this way.

Proposition 4.22. *Let $G \in \mathcal{G}(n,p)$. Given X, a specific labelled graph on n vertices with m vertices, $\mathbb{P}(G = X) = p^m(1-p)^{\binom{n}{2}-m}$.*

The approach to probability will be rather naive, but I think everyone who knows enough probability theory to worry about this will know enough to resolve those worries with just a few hints. Everything we do here can be translated into Kolmogorov's axiomatisation of probability theory. We would take all labelled n-vertex graphs as the sample space, the power space of the sample space as the event space, and use Proposition 4.22 to build the probability measure. We would call the resulting probability space $\mathcal{G}(n,p)$. Then statements such as "$G \in \mathcal{G}(n,p)$ has no isolated vertices." would then correspond to events in $\mathcal{G}(n,p)$, and so it makes sense to talk about their probability.

Theorem 4.23 (Erdős 1947)**.** *For every $k \geq 3$, the Ramsey number of k satisfies*

$$R(k,k) > 2^{\frac{k}{2}}$$

Proof. We'll prove that if $n \leq 2^{\frac{k}{2}}$ and $G \in \mathcal{G}(n, \frac{1}{2})$ then $\mathbb{P}(G \text{ contains a copy of } K_k \text{ or } \overline{K_k}) < 1$. This will show that there are graphs of size n with neither K_k nor $\overline{K_k}$ as induced subgraphs, proving that $n < R(k,k)$.

First, $\mathbb{P}(G \text{ contains a copy of } K_k) \leq \binom{n}{k}\frac{1}{2}^{\binom{k}{2}}$, as there are $\binom{n}{k}$ many k-element subsets, with $\binom{k}{2}$ possible edges between those vertices, the probability that one of those possible edges is actually an edge is $\frac{1}{2}$.

$$\begin{aligned}
\mathbb{P}(G \text{ contains a copy of } K_k) &\leq \binom{n}{k}\frac{1}{2^{\binom{k}{2}}} \\
&= \frac{n!}{k!(n-k)!}2^{-\frac{1}{2}k(k-1)} \\
&< \frac{n^k}{2^k}2^{-\frac{1}{2}k(k-1)} \\
&\leq (2^{\frac{k}{2}})^k 2^{-\frac{1}{2}k(k-1)-k} \\
&= 2^{-\frac{k}{2}} \\
&< \frac{1}{2}
\end{aligned}$$

Since the probability of drawing an edge in G is $\frac{1}{2}$,

$$\mathbb{P}(G \text{ contains a copy of } \overline{K_k}) = \mathbb{P}(G \text{ contains a copy of } K_k) < \frac{1}{2}$$

So $\mathbb{P}(G \text{ contains a copy of } K_k \text{ or } \overline{K_k}) < 1$, and therefore there are n-vertex graphs with no induced subgraphs isomorphic to K_k or $\overline{K_k}$. We have shown that $R(k,k) > 2^{\frac{k}{2}}$ □

We can model the various invariants that graphs can have as random variables, for which we can calculate the expectation and variance exactly as you would expect.

Lemma 4.24. *Let $G \in \mathcal{G}(n,p)$, and let $\mathrm{Cy}_k(G)$ be the set of distinct k-cycles in G.*

$$\mathbb{E}(|\mathrm{Cy}_k(G)|) = \frac{n!}{(2k)(n-k)!}p^k$$

Proof. A k-cycle is a sequence of k distinct vertices, of which there are $n(n-1)\ldots\ldots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}$ many. If the sequence $(v_1, \ldots, v_k)$ is a cycle, then the sequences $(v_i, v_{i+1}, \ldots\ldots, v_k, v_1, \ldots, v_{i-1})$ and $(v_k, \ldots, v_1)$ also represent that cycle, so there are $\frac{n!}{(2k)(n-k)!}$ many potential cycles in G. Let $\mathcal{C}_k$ be the set of potential k-cycles. Note that this C_k is shared by all the possible graphs on n variables.

For each $C \in \mathcal{C}_k$, let $X_C : \mathcal{G}(n,p)$ be the random variable defined by:

$$X_C(G) = \begin{cases} 1 & C \in \mathrm{Cy}_k(G) \\ 0 & C \notin \mathrm{Cy}_k(G) \end{cases}$$

There are k many edges in a k-cycle, so $\mathbb{P}(X_C(G) = 1) = p^k$.

$|\mathrm{Cy}_k(G)| = \sum_{C\in\mathcal{C}} X_C(G)$ so we're now able to calculate the expectation of $|\mathrm{Cy}_k(G)|$.

$$\begin{aligned} \mathbb{E}(|\mathrm{Cy}_k(G)|) &= \sum_{C\in\mathcal{C}} \mathbb{E}(X_C(G)) \\ &= \sum_{C\in\mathcal{C}} \mathbb{P}(X_C(G) = 1) \\ &= \sum_{C\in\mathcal{C}} p^k \\ &= \tfrac{n!}{(2k)(n-k)!} p^k \end{aligned}$$

□

The classic application of Random Graphs is to get graphs with both high chromatic number and high girth. Recall that the chromatic number is the minimum number of colours needed to give each vertex a colour without adjacent vertices sharing a colour. The girth of a graph is the length of its shortest cycle (so must be at least 3). Keeping the chromatic number high is easy, make sure you add as many edges as possible. Keeping the girth high is also each, make sure you add as few edges as possible. The fact that you can do both at the same time is, frankly, miraculous.

Theorem 4.25 (Erdős 1959)**.** *For every $k \geq 3$ there is a graph H with girth $g(H) > k$ and chromatic number $\chi_C(H) > k$.*

Now that you've been told that the proof of this uses random graphs, you may already have a proof in mind. By picking n large and keeping p above a certain value, we can get a high chromatic number. By picking p very small, we can make sure that the expected number of cycles is small, which makes it feasible that there is only a few long cycles, which would guarantee a large girth. We may need to make p depend on n to make sure that the largeness of n doesn't overpower the smallness of p.

This doesn't work. If we pick p small enough to make it very unlikely that there are any short cycles, then p is too small to guarantee a large chromatic number. But if we press on with a small p that still gives a large chromatic number, the result is a graph that, while probably having small girth, will be an easy starting point for a more traditional conclusion to the proof.

First let's investigate getting a large chromatic number. A chromatic number will be small if there's a large number of isolated vertices.

Lemma 4.26. *Let $k > 0$, let $G \in \mathcal{G}(n, p(n))$, and let $\bar{K}(G) = \max\{n \in \mathbb{N} : \overline{K_n} \leq G\}$. If $p(n) \geq 16\frac{k^2}{n}$ then*

$$\lim_{n\to\infty} \mathbb{P}(\bar{K}(G) \geq \frac{n}{2k}) = 0$$

Proof. There are $\binom{n}{r}$ many r-element subsets of G, and each one has probability $(1-p)^{\binom{r}{2}}$ of being isomorphic to $\overline{K_r}$. I'm going to put the whole chain of inequalities in one chunk, and then write the explanations afterwards.

$$\begin{aligned} \mathbb{P}(r \leq \bar{K}(G)) &= \mathbb{P}(G \text{ has an } r\text{-element subset isomorphic to } \overline{K_r}) && (1) \\ &\leq \binom{n}{r}(1-p)^{\binom{r}{2}} && (2) \\ &\leq 2^n e^{-p\binom{r}{2}} && (3) \\ &\leq 2^n e^{-p\frac{r^2}{4}} && (4) \\ &\leq 2^n e^{-p\frac{n^2}{16k^2}} && (5) \\ &\leq 2^n e^{-n} && (6) \\ &\to 0 && (7) \end{aligned}$$

(3) The 2^n in this line bounds $\binom{n}{r}$, while $(1-p) \leq e^{-p}$ gives the second half (let $y = e^x - x - 1$, show that if $x > 0$ then $y > 0$ using calculus.)

(4) $\binom{r}{2} = \frac{r!}{2!(r-2)!} = \frac{r(r-1)}{2} = \frac{r^2}{2} - \frac{r}{2} \geq \frac{r^2}{2} - \frac{r}{2}.\frac{r}{2} = \frac{r^2}{4}$. Since the exponent is negative, decreasing the absolute value of the exponent increases the final result.

(5) We're interested in $\lim_{n\to\infty} \mathbb{P}(\bar{K}(G) \geq \frac{n}{2k})$, so we may assume that $r = \lceil \frac{n}{2k} \rceil$.

(6) $p(n) \geq 16\frac{k^2}{n}$.

(7) $2^n e^{-n} = (\frac{2}{e})^n$ plus basic Analysis.

□

Proof of Theorem 4.25. Let n be a natural number, let $k \geq 3$, let $\varepsilon \in (0, \frac{1}{k})$ and let $p = n^{\varepsilon - 1}$. Let $G \in \mathcal{G}(n,p)$. We want to find a graph with girth greater than k, so we want to make sure that there are no cycles of length at most k. Lemma 4.24 will help us calculate the expected number of such cycles, and we'll be using the notation from that lemma and its proof.

$$\begin{aligned}
\mathbb{E}(|\bigcup_{i=3}^{k} \mathrm{Cy}_i(G)|) &= \sum_{i=3}^{k} \mathbb{E}(|\mathrm{Cy}_i(G)|) \\
&= \sum_{i=3}^{k} \frac{n!}{(2i)(n-i)!} p^i \\
&\leq \frac{1}{2} \sum_{i=3}^{k} (np)^i \\
&\leq \frac{k-2}{2} n^k p^k
\end{aligned}$$

We can now use Markov's Inequality[2] to bound the probability that there are lots of these cycles to the expectation.

$$\begin{aligned}
\mathbb{P}(|\bigcup_{i=3}^{k} \mathrm{Cy}_i(G)| > \tfrac{n}{2}) &\leq \tfrac{2}{n} \mathbb{E}(|\bigcup_{i=3}^{k} \mathrm{Cy}_i(G)|) \\
&\leq \tfrac{2}{n} \tfrac{k-2}{2} n^k p^k \\
&= (k-2) n^{k-1} (n^{\varepsilon - 1})^k \\
&= (k-2) n^{k\varepsilon - 1}
\end{aligned}$$

$\varepsilon \in (0, \frac{1}{k})$, so $\lim_{n\to\infty} \mathbb{P}(|\bigcup_{i=3}^{k} \mathrm{Cy}_i(G)| > \frac{n}{2}) = 0$.

No matter what ε and k we choose, if n is large enough then $16\frac{k^2}{n} < n^{\varepsilon-1}$, so we can pick a p that ensures that both this result and Lemma 4.26 apply. Therefore there is an n such that

$$\begin{aligned}
\mathbb{P}(|\bigcup_{i=3}^{k} \mathrm{Cy}_i(G)| > \tfrac{n}{2}) &< \tfrac{1}{2} \\
\mathbb{P}(\bar{K}(G) \geq \tfrac{n}{2k}) &< \tfrac{1}{2}
\end{aligned}$$

Therefore the probability that G is a graph such that:

- G has less than $\frac{n}{2}$ cycles of length at most k, and
- G has at most $\frac{n}{2k}$ isolated points.

is non-zero, and thus such a graph exists. Let H' be one such graph. We construct H by deleting a vertex from each cycle with length at most k. The girth of H is at least k, so we now turn our attention to the chromatic number of H.

$$k < \frac{n}{2} . \frac{1}{\bar{K}(H')} \leq \frac{|H|}{\bar{K}(H)} \leq \chi_C(H).$$

That last one is due to the fact that the largest equivalence class of vertices with all the same colour is at most the size of the largest independent set. So each equivalence class has at most $\bar{K}(H)$ elements, and there are $\chi_C(H)$ many of them. □

[2] Markov's Inequality says that if X is a random variable and $a > 0$ then $\mathbb{P}(X \geq a) \leq \mathbb{E}(X)/a$

5 Mastery Material

5.1 Szemerédi's Regularity Lemma

Chapter 7.4 of Diestel's Graph Theory, available online from Imperial College Library.

6 Appendix: Algorithms

Computational Graph Theory, the study of computer programs to decide questions about graphs, is a major field of study for both Mathematics and Computer Science. This course will take a very abstract approach to this topic. Any algorithms will be written in pseudocode, which can be read and understood as long as you speak English. You're welcome to try turning any of these algorithms into programs, but I won't be able to help you. My practical programming experience was entirely in BASIC, on my primary school's BBC Micro, during one rainy lunch-break in the 1990's.

In any situation where and algorithm is appropriate, we may assume that the graph or graphs are finite, and their vertices are labelled with natural numbers. An algorithm consists of numbered instructions. In each instruction, we are allowed to:

1. Indicate that this is where the algorithm starts.
2. Return a value and end the algorithm.
3. Assign a symbol as a variable, specifying where the variable is allowed to run and an initial value. E.g. Let n be a natural number, equal to 0. Let v be any vertex. Let A be a set of vertices, equal to $\emptyset$.
4. Alter a variable. e.g. Add 1 to n, and call the new value n.
5. Order the algorithm to move to a particular valued step, e.g. GOTO Step 4.
6. Combine the other operations in an if-then-otherwise step. E.g. If $n = 0$ then remove v from A, otherwise end.
7. Call upon an already established algorithm as a subroutine.

For ease of reading, its allowed to combine instructions when writing things out. It's also advisable to add comments to your pseudocode to make the reader better equipped to understand what you're doing. My comments will be after a semi-colon, and in `typewriter font`. Please devise your own convention that fits the format you're making notes in, e.g. instructions are left-aligned, while comments are right-aligned.

Algorithm 6.1. This algorithm, called DEG, is designed to calculate the degree of a specified vertex v in graph G.

I.1 START; `we wish to know the degree of` v.

I.2 Let n be a natural number, starting at 0. Let m be a natural number, starting at 1; m `will be used to count which vertex we're on, while` n `will keep track of how many are attached to` v.

I.3 If vEv_m then GOTO 4, otherwise GOTO 5.

I.4 Add 1 to n.

I.5 If $m = |G|$ then GOTO 7, otherwise add 1 to m.

I.6 GOTO 2

I.7 OUTPUT n; `the degree of` v `is equal to` n.

When running DEG on vertex v in graph G, we write DEG(G, v).

For an algorithm to be valid, it must eventually reach an OUTPUT instruction. We're also interested in how many steps each algorithm takes to actually complete.

Definition 6.2. The number of steps an algorithm A takes on graph G for vertex v, written as $\#(A, G, v,)$, is the total number of times each instruction is visited before END is reached. We omit the v if the algorithm does not need an initial vertex.

Example 6.3. In Algorithm 6.1 I.6 sends us back to I.2, so we may get sent to I.2 a lot. In this case, if $u = |G|$ then we won't reach I.6, so we'll pass through I.2-I.6 $|G|$ many times. This means that $2+4|G| \leq \#(\text{DEG}, G, v) \leq 2+5|G|$ steps to run the algorithm on graph G. This will attain its minimum if $\deg(v) = 0$ and maximum if $\deg(v) = |G|$.

How long an algorithms takes to run on a specific input doesn't tell us how fast or slow the algorithm is in general. We may have gotten extremely lucky with our choice of G.

Definition 6.4. Let A be an algorithm, and let $f : \mathbb{N} \to \mathbb{N}$ be a function. We say that A runs in $O(f)$ time if there is some $n_0 \in \mathbb{N}$ and constants $\alpha, \beta \in \mathbb{R}^{>0}$ such that if $n > n_0$ then

$$\alpha f(n) < \max\{\#(A, G, v) : |G| = n \text{ and } v \in G\} < \beta f(n)$$

If A runs in $O(p)$ for some polynomial p, then we say that A runs in *polynomial time*. If A runs in $O(n)$, then we say that A runs in *linear time*. If A runs in $O(1)$, then we say that A runs in *constant time*.

If A runs in $O(e^n)$, then we say that A runs in *exponential time*. If A runs in $O(e^{e^n})$ for some polynomial p, then we say that A runs in *double exponential time*.

If A runs in $O(\log(n))$, then we say that A runs in *logarithmic time*.

We've already shown that Algorithm 6.1 runs in linear time. For every algorithm in this course, either we will calculating its complexity, or you will calculate its complexity. If you're familiar with the practical implementation of algorithms, then you may be worried about how to encode a graph for a program to work on it, or how long it may take for the algorithm to order the vertices, or lots of other practical considerations. I am not worried.

Remark 6.5. If p and q are polynomials such that $\deg(p) = \deg(q)$ then A runs in $O(p)$ time if and only if A runs in $O(q)$ time.

When calculating the complexity of an algorithm, it is extremely important to (and very easy to forget to!) take the complexity of all the subroutines into account, as you will see in the next example.

Algorithm 6.6. This algorithm, called AvDEG, calculates the average degree of a graph.

I.1 START; `we wish to know the average degree of` G`.`

I.2 Let n be a natural numbers, starting at 0, and let m be a natural number, starting at 1; m `will be used to count which vertex we're on, while` n `will keep track of the total degree of the vertex.`

I.3 Add DEG(G, v_m) to n.

I.4 If $m = |G|$ then GOTO 5, otherwise add 1 to m and GOTO 3.

I.5 Divide n by $|G|$.

I.6 OUTPUT n; `the average degree of` G `is equal to` n`.`

This algorithm runs instructions I.3 and I.4 once for each vertex in G, so naively we may think that AvDEG runs in linear time. However, instruction I.3 calls DEG as a subroutine, which also runs in linear time. Therefore, it runs $|G|$ many steps once for each vertex. This means that the complexity of AvDEG is quadratic.

7 Appendix: Biographies

7.1 Kenneth Appel

Figure 27: Kenneth Appel (1932-2013)

7.2 Gabriel Dirac

I was unable to find a photo licensed for free educational uses. Gabriel Dirac (1925-1984). Son of Physicist Paul Dirac.

7.3 Paul Erdős

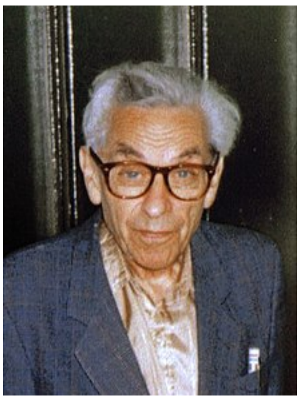


Figure 28: Paul Erdős (1913-1996). Photo from 1993.

7.4 Leonhard Euler

Figure 29: Leonhard Euler (1707-1783). Portrait by Jakob Emanuel Handmann, from 1753

7.5 Lester Ford Jr.

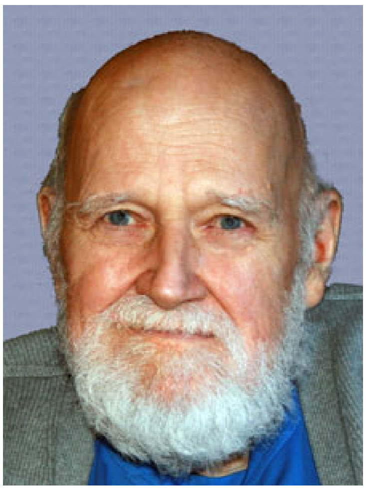


Figure 30: Lester Randolph Ford Jr. (1927-2017)

7.6 Delbert Fulkerson

Figure 31: Delbert Ray Fulkerson (1924-1976)

7.7 Wolfgang Haken

Figure 32: Wolfgang Haken (Born 1928)

7.8 William Hamilton

Figure 33: William Hamilton (1805-1865).

7.9 Francis Guthrie

Figure 34: Francis Guthrie (1831-1899)

7.10 Andrey Kolmogorov

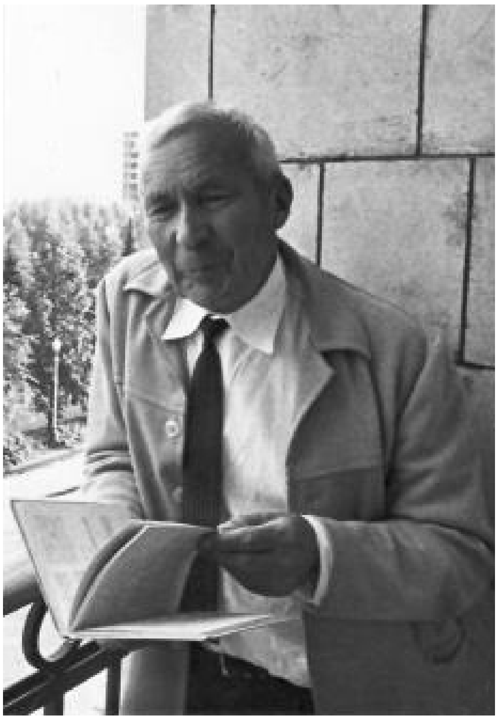


Figure 35: Andrey Kolmogorov

7.11 Denes König

Figure 36: Dénes Kőnig (1884-1944).

Wrote the first ever Graph Theory textbook.

7.12 W. Mader

I can't even find out what the W. stands for.

7.13 Andrey Markov

Figure 37: Andrey Markov (1856-1922). Photo from 1970's

7.14 Karl Menger

Figure 38: Karl Menger (1902-1985). Photo from 1970
Also known for the Menger Sponge.

7.15 Julius Petersen

Figure 39: Julius Petersen (1839-1910) (right)

7.16 Frank Ramsey

Figure 40: Frank Ramsey (1903-1930). Photo from c. 1921

7.17 Alfréd Rényi

Figure 41: Alfréd Renyi (1921-1970) with his wife, Kató Rényi (1924-1969). Photo from 1966 ©MFO

7.18 Pal Turán

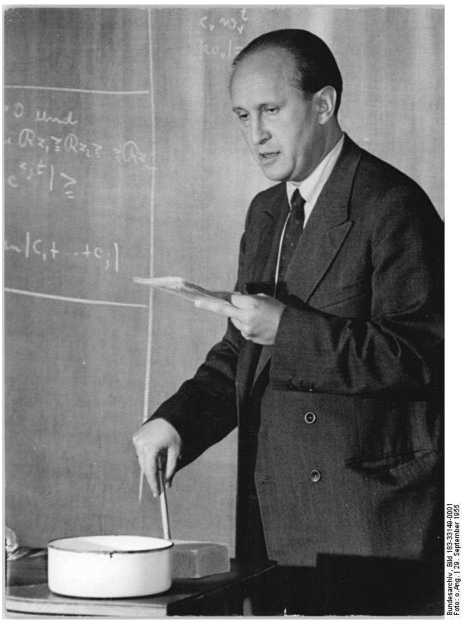


Figure 42: Pal Turán (1910-1976). Photo from 1955

7.19 Klaus Wagner

Figure 43: Klaus Wagner (1910-2000) (right). Photo from 1972.