

① 2.4 The formal system K_L

(2.4.1) Def. Suppose L is a 1st order language. The formal system K_L has as formulas the L -formulas and:

Axioms For L -formulas ϕ, ψ, χ :

$$\underline{A1} \quad (\phi \rightarrow (\psi \rightarrow \phi))$$

$$\underline{A2} \quad ((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$$

$$\underline{A3} \quad (((\neg \psi) \rightarrow (\neg \phi)) \rightarrow (\phi \rightarrow \psi))$$

$$\underline{K1} \quad ((\forall x_i) \phi(x_i) \rightarrow \phi(t))$$

whenever t is a term free for x_i in ϕ (ϕ can have other free vars.)
 [Eg. can take t to be x_i]

$$\underline{K2} \quad ((\forall x_i)(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\forall x_i) \psi))$$

if x_i is not free in ϕ .

Reduction rules MP From ϕ and $(\phi \rightarrow \psi)$ deduce ψ

Generalisation Gen From ϕ deduce $(\forall x_i) \phi$.

A proof in K_L is a finite sequence of L -formulas, each of which is an axiom or deduced from previous formulas using a deduction rule. A theorem of K_L is the last formula in some proof. Write $\vdash_{K_L} \phi$ (or $\vdash \phi$) if ϕ is a theorem of K_L . (2)

= 2.4.2 Def. Suppose Σ is a set of L -formulas and ψ is an L -formula. A deduction of ψ from Σ is a finite sequence of L -formulas ending with ψ , each of which is an axiom, a formula in Σ or obtained from formulas earlier in the deduction using MP or Gen with the restriction :
when Gen is used to deduce ~~All formulas~~ $(\forall x_i) \phi$ from ϕ x_i does not appear free in any formula of Σ used in the deduction of ϕ .

= Write $\Sigma \vdash_{K_L} \psi$ if there is a deduction of ψ from Σ .
(Say ψ is a consequence of Σ)

(2.4.3) Remarks (on the restriction) in Gen. (3)

(1) If the formulas in Σ are closed (no free variables), can ignore the restriction.

(2) Without the restriction: If x_i is free in $\phi(x_i)$ would have " $\{\phi\} \vdash (\forall x_i)\phi$ "

and this would contradict soundness of K_L .

(3) If $\Sigma' \subseteq \Sigma$ & $\Sigma' \vdash \psi$ then $\Sigma \vdash \psi$.

"(2.4.4) Theorem. Suppose ϕ is an L-formula which is a substitution instance of a propositional tautology. Then ϕ is a theorem of K_L .

Eg Prop. tautology $((\neg(\neg p_1)) \rightarrow p_1)$

Substitute ϕ_1 for p_1 : $((\neg(\neg \phi_1)) \rightarrow \phi_1)$

- a theorem of K_L .

(4)

Pf.: Let p_1, \dots, p_n be variables in a prop. tatalogy \mathcal{X}
 Suppose ϕ is obtained by substituting \mathcal{L} -formulas ψ_1, \dots, ψ_n in
 \mathcal{X} for p_1, \dots, p_n . By the Completeness Thm. for L (1.3.11)
 there is a proof in L of $\mathcal{X} : X_1, \dots, X_r$ where X_r is \mathcal{X} .
 If we substitute ψ_1, \dots, ψ_n in place of variables p_1, \dots, p_n in
 each X_i , we obtain a sequence of \mathcal{L} -formulas ϕ_1, \dots, ϕ_r
 where ϕ_r is ϕ and ϕ_1, \dots, ϕ_r is a proof in $K_{\mathcal{L}}$. #

(2.4.5) Theorem (Soundness of $K_{\mathcal{L}}$)

If $\vdash_{K_{\mathcal{L}}} \phi$ then $\vdash \phi$.

Pf.: Like the pf. for L (1.3.1)

- Show the axioms are logically valid
- Show the deduction rules preserve logical validity .

A_1, A_2, A_3 are substitution instances of propositional tautologies (5)

So by 2.2.14 they are logically valid.

K1 is logically valid by 2.3.7.

K2 $((\forall x_i)(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\forall x_i)\psi))$ [where x_i is not free in ϕ]

Suppose v is a valuation (in \mathcal{A}) with

$v[(\phi \rightarrow (\forall x_i)\psi)] = F$. Show $v[(\forall x_i)(\phi \rightarrow \psi)]$.

From this $v[\phi] = T$ & $v[(\forall x_i)\psi] = F$. So there is a val. v' which is x_i -equiv. to v and $v'[\psi] = F$.

v, v' agree on the free variables in ϕ . Then

$v'[\phi] = v[\phi] = T$. Thus $v'[(\phi \rightarrow \psi)] = F$.

As v, v' are x_i -equiv. $v[(\forall x_i)(\phi \rightarrow \psi)] = F$. //

MP If $\vdash \phi \wedge \vdash (\phi \rightarrow \psi)$ then $\vdash \psi$.

Gen If $\vdash \phi$ then $\vdash (\forall x_i)\phi$. //

(2.4.6) Cor. K_L is consistent : there is no formula ϕ ⑥
with $\vdash_{K_L} \phi$ and $\vdash_{K_L} (\neg \phi)$.
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