IMPERIAL

MATH70012

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: Tuesday, May 14, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Question 1 (Total: 20 marks)

Consider the following one period trinomial model: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\omega_i) = 1/3$ for $i = 1, 2, 3$, a bank account B with interest rate $r = 0$, and one stock S with $S_0 = 10$ and

$$S_1(\omega) = \begin{cases} 20, & \text{if } \omega = \omega_1, \\ 15, & \text{if } \omega = \omega_2, \\ \frac{15}{2}, & \text{if } \omega = \omega_3. \end{cases}$$

Consider a forward contract on the stock, which has payoff $F_1(K) := S_1 - K$ at time 1 and cost $F_0(K) := 0$ at time 0; and a derivative with time 1 payoff $X_1(\omega_i) := x_i, i = 1, 2, 3$ and time 0 cost X_0. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) What is a replicating strategy for the forward contract? (3 marks)

(b) Fix $K = 8$. Explicitly build an arbitrage in the market $(B, S, F(K))$. (4 marks)

(c) For what value(s) of K is the market $(B, S, F(K))$ is arbitrage-free? (5 marks)

From now on, fix one value K^* of K. such that $(B, S, F(K))$ is arbitrage-free.

(d) Is the $(B, S, F(K^*))$ market complete? (3 marks)

(e) If $X_0 = 0$, for which values of $x = (x_i)_{i=1,2,3} \in \mathbb{R}^3$, is the derivative X replicable in the $(B, S, F(K^*))$ market? Does the answer change if $X_0 = 5$? (5 marks)

Question 2

(Total: 20 marks)

Let P be a probability on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ such that $\mathsf{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. Define the random variable

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	8	12	4

(1)

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price $B_0 = 1$ (all values/prices are given in £, unless otherwise specified), and interest rate $r = 1$, one stock S with initial price $S_0 = 4$ and final price S_1 as in eq. (1). Denote with P_t the value at time $t \in \{0, 1\}$ of a put option P (with underlying S) with strike price 10. Denote with V the value (in £) of your portfolio, and with W the value of your portfolio when using S as a numeraire (i.e. unit of measure of value), so that $W := V/S$. You can use (without having to prove it) that the set $\mathcal{M} = \mathcal{M}_B$ of EMM (Equivalent Martingale Measures) relative to the numeraire B (i.e. $\mathcal{M}_B$ is the set of probabilities $\mathsf{Q} \sim \mathsf{P}$ s.t. $\mathbb{E}^{\mathsf{Q}}[S_1/B_1] = S_0/B_0$)) is given by the formula

$$\mathcal{M}_B = \left\{ \begin{pmatrix} 1-2s \\ s \\ s \end{pmatrix} : s \in \left(0, \frac{1}{2}\right) \right\}. \tag{2}$$

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) free of arbitrage? (1 marks)

(b) Compute the set $\mathcal{P}_B$ of all arbitrage-free prices of P. (3 marks)
Hint: prices in £, and in units of B, coincide at time 0 since $B_0 = £1$.

(c) Is the put option P replicable? (2 marks)

(d) Let R be a probability on Ω. Prove that the RNP (Risk-Neutral Pricing) formula $\mathbb{E}^{\mathsf{R}}[W_1] = W_0$ relative to the numeraire S holds for all portfolios iff it holds for a portfolio made of only one bond, i.e. for $W = B/S$. (4 marks)

(e) Determine the set $\mathcal{M}_S$ of EMM relative to the numeraire S. (4 marks)
Hint: by item (d), a probability $\mathsf{R} \sim \mathsf{P}$ belongs to $\mathcal{M}_S$ iff $\mathbb{E}^{\mathsf{R}}[B_1/S_1] = B_0/S_0$.

(f) Find the set $\mathcal{P}_S$ of all arbitrage-free prices of P in units of S, doing your calculation using the RNP formula with numeraire S and EMMs $\mathsf{R} \in \mathcal{M}_S$. (4 marks)

(g) What it the relation between prices $\mathcal{P}_B$ (of P) in units of B and the prices $\mathcal{P}_S$ in units of S? Confirm that your answers in items (b) and (f) coincide, after converting the values expressed in units of B to the values expressed in units of S. (2 marks)

Question 3

(Total: 20 marks)

Consider the binomial model $(B_i, S_i)_{i=0}^N$ with expiration $N \in \mathbb{N}$, interest rate $r = 0$ and stock price

$$S_k := S_0 + \sum_{i=1}^{k} Y_i \quad \text{for all } k = 1, \dots, N,$$

where $Y_i, i = 1, \dots, N$ are IID random variables which take the values $d, u \in \mathbb{R}$ with probability $\frac{1}{2}$. Assume $u > 0 > d, S_0 + Nd > 0$, so in particular $S_i > 0$ a.s. for all $i \leq N$. Define

$$M_k := \min_{i=0,1,\dots,k} S_i, \quad L_k := S_k - M_k, \quad \text{for all } k = 0, \dots, N,$$

and consider the *floating lookback call option*, i.e. the option which at time N pays the amount $V_N := L_N$, and denote with V_k its arbitrage-free price at time $k = 0, \dots, N$. As usual let $\mathcal{F}$ be the natural filtration of S and $\mathbb{Q}$ the risk-neutral measure. When we ask if a process is Markov, we mean with respect to $(\mathcal{F}, \mathbb{Q})$. Prove all your assertions carefully or provide counter-examples.

(a) Prove that the $(Y_i)_i$ are IID under $\mathbb{Q}$, and compute $\tilde{p} := \mathbb{Q}(Y_i = u)$. (2 marks)

(b) Is M a Markov process? (3 marks)

(c) Is $C := (S, M)$ a Markov process? (3 marks)

(d) Write an explicit formula for a function h_n which satisfies $L_{n+1} = h_n(L_n, Y_{n+1})$. (3 marks)
Hint: use that $s - \min(a, b) = \max(s - a, s - b)$ *for all* $a, b, c \in \mathbb{R}$.

(e) Is L a Markov process? (2 marks)

(f) Which of the processes $S, M, (S, M)$, L, are such that, for every $n = 0, \dots, N$, V_n can be written as a function f_n of the value of the process at time n? (2 marks)

(g) For each process as in item (f), write explicitly f_N, and an explicit formula to express f_n in terms of f_{n+1}, for $n = 0, \dots, N - 1$. If there is more than one such process, explain which one is preferable to use for numerical implementations, and why. (5 marks)

Question 4

(Total: 20 marks)

Model a risky asset $S = (S_n)_{n=0}^{N}$ with a N-period binomial model with constant up factor u and down factor d and initial value $S_0 > 0$, and a bank account with constant interest rate r, and assume $d < 1 + r < u$. Let V_n be the arbitrage-free price at time n of the *American* call option on S with strike price $K > 0$ which has not yet been exercises; recall that, when exercising the option at time $n \leq N$, the buyer gets the payoff $I_n := S_n - K$. Let τ^* be the smallest optimal exercise time of this option, given by the formula

$$\tau^* = \inf\{n \in \{0, \ldots, N\} : V_n = I_n\},$$

where we use the usual convention that $\inf \emptyset := \infty$. As usual let $\mathcal{F}$ be the natural filtration of S, $\mathbb{Q}$ be the risk-neutral measure, and 1_A denote the indicator function of A (i.e. $1_A = 1$ on A, $1_A = 0$ on A^c). Answer the following questions and justify carefully with either proofs or counter-examples.

(a) Write the equation which allows to compute V_n by using V_{n+1} and I_n for $n = 0, \ldots, N-1$. (5 marks)

(b) Prove that S is Markov (with respect to $(\mathcal{F}, \mathbb{Q})$). (2 marks)

(c) Prove by backward induction that V_n admits the representation $V_n = f_n(S_n)$ for some $f_n : \mathbb{R} \to \mathbb{R}$, for all $n = 0, 1, ..., N$. (2 marks)

(d) Find a formula, as explicit as possible, to express f_n in terms of f_{n+1} (for any $n = 0, \ldots N-1$), and give a formula for f_N. (5 marks)

(e) Let τ be an arbitrary stopping time. Is it true that for every $0 \leq n \leq N$ there exists $a_n : \{0,1\} \times \mathbb{R} \to \{0,1\}$ such that $1_{\{\tau=n\}} = a_n(1_{\{\tau<n\}}, S_n)$? If such $a = (a_n)_n$ exists, give an explicit expression for it; otherwise, give an explicit counter-example. (3 marks)

(f) Does $a = (a_n)_n$ as in (e) exist when $\tau = \tau^*$? If such a exists, give an explicit expression for it; otherwise, give an explicit counter-example. (3 marks)

Question 5

(Total: 20 marks)

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded), whose prices S_t^1, S_t^2 at times $t = 0, 1$ have values

$$S_0^1 = 60, \qquad S_0^2 = 20, \qquad \begin{array}{r|c|r} \omega & H & T \\ \hline S_1^1(\omega) & 40 & 80 \\ \hline S_1^2(\omega) & 20 & 140 \end{array} \tag{3}$$

Assume that the event $\{H\}$ has probability $\frac{1}{3}$. We'll use the following standard notation: $R_1^i := \frac{S_1^i}{S_0^i} - 1$ denotes the return of asset i between time 0 and 1 and $\mu_i := \mathbb{E}[R_1^i]$ its average for $i = 1, 2$, while $\Sigma := \mathrm{Cov}(R_1)$ denotes the covariance matrix of the returns (i.e. $\Sigma_{i,j} := \mathbb{E}(R_1^i R_1^j) - \mathbb{E}(R_1^i)\mathbb{E}(R_1^j)$, $i = 1, 2$), a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in asset S^i for $i = 1, 2$. Denote with $\mu = \mu(\pi)$ the average, and with $\sigma = \sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π).

(a) Prove that the average returns μ and the covariance matrix Σ are given by the formulas (5 marks)

$$\mu = \left(\frac{1}{9}, 4\right), \quad \Sigma = \frac{8}{9^2}\begin{pmatrix} 1 & 9 \\ 9 & 9^2 \end{pmatrix}. \tag{4}$$

(b) Conclude that the correlation ρ between R_1^1 and R_1^2 equals 1. (1 marks)

(c) Find a portfolio with average return of $\frac{50}{3}$%. Does it involve any short-selling? (4 marks)

(d) Find the equation whose solution is the set S of values (σ, μ) taken by all portfolios, and draw S. (6 marks)

(e) Find explicitly two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (4 marks)

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH60012/MATH70012 Mathematical Finance: An Intro to Option Pricing

WITH SOLUTIONS INCLUDED

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes, for students who took the module in 2022/23 (also in 2021/22, if you exclude the mastery material)

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a-e) 3+4+5+3+5=20 marks; , 2(a-c) 1+3+2=6 marks; 3(e,f) 2+2=4 marks; 4(b) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2(d-f) 4+4+4=12 marks; 3 (a-c) 2+3+3=8 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(g) 2 marks; 3(d,g) 3+5=8 marks; 4(c) 2 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
4(a,d-f) 5+5+3+3=16 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Finance: An Intro to Option Pricing

WITH SOLUTIONS INCLUDED

Date: Tuesday, 14th May 2024
Time: 10:00 – 12:30
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

Question 1

(Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

Consider the following one period trinomial model: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\omega_i) = 1/3$ for $i = 1, 2, 3$, a bank account B with interest rate $r = 0$, and one stock S with $S_0 = 10$ and

$$S_1(\omega) = \begin{cases} 20, & \text{if } \omega = \omega_1, \\ 15, & \text{if } \omega = \omega_2, \\ \frac{15}{2}, & \text{if } \omega = \omega_3. \end{cases}$$

Consider a forward contract on the stock, which has payoff $F_1(K) := S_1 - K$ at time 1 and cost $F_0(K) := 0$ at time 0; and a derivative with time 1 payoff $X_1(\omega_i) := x_i, i = 1, 2, 3$ and time 0 cost X_0. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) What is a replicating strategy for the forward contract? (3 marks)

(b) Fix $K = 8$. Explicitly build an arbitrage in the market $(B, S, F(K))$. (4 marks)

(c) For what value(s) of K is the market $(B, S, F(K))$ is arbitrage-free? (5 marks)

From now on, fix one value K^* of K. such that $(B, S, F(K))$ is arbitrage-free.

(d) Is the $(B, S, F(K^*))$ market complete? (3 marks)

(e) If $X_0 = 0$, for which values of $x = (x_i)_{i=1,2,3} \in \mathbb{R}^3$, is the derivative X replicable in the $(B, S, F(K^*))$ market? Does the answer change if $X_0 = 5$? (5 marks)

Solution:

(a) You should start with initial capital $S_0 - \frac{K}{1+r}$, borrow $\frac{K}{1+r}$ from the bank and use all the resulting S_0 to buy one stock. This way at time 1 you will own one stock, worth S_1, and owe to the bank $(1+r)\frac{K}{1+r}$, so your wealth will be $S_1 - K$, so this is a replicating strategy.

(b) At time 0 buy one forward at cost $F_0(K) = 0$, sell one share of the stock at price $S_0 = 10$, and deposit the resulting 10 in the bank; this strategy has cost zero. At time one, your payoff is

$$F_1(K) - S_1 + S_0(1+r) = S_1 - 8 - S_1 + 10 = 2 > 0,$$

so this strategy is an arbitrage.

(c) As the forward contract is replicable, and its replicating strategy has initial cost $F_0 := S_0 - \frac{K}{1+r} = 10 - K$, whereas the forward contract has cost 0, the market has an arbitrage if $F_0 \neq 0$, i.e. if $K \neq 10$. If instead $K = 10$ then there is no arbitrage, since the (B, S) market (and thus the $(B, S, F(K))$ market) is free of arbitrage: indeed this is a trinomial model with **one** stock which satisfies the condition $d < 1 + r < u$ (since $d = 15/20 = 3/4$, $1 + r = 1$, $u = 20/10 = 2$).

(d) The market is not complete. Indeed the forward contract is replicable, so the set of attainable payoffs in the $(B, S, F(K^*))$ market is the same as in the (B, S) market; this is a trinomial model, so it is not complete.

(e) Recall that X_1 is replicable iff $\mathbb{E}^{\mathbb{Q}}[X_1]$ is constant across all EMM $\mathbb{Q}$. The set of EMM for the $(B, S, F(K^*))$ market is identified with the set of $q \in \mathbb{R}^3$ for which

$$q_1 + q_2 + q_3 = 1, \quad 20q_1 + 15q_2 + \frac{15}{2}q_3 = 10,$$

and $q_i > 0$ for all i, i.e. with $\{\frac{1}{3}(3t, 1-5t, 2+2t) : t \in (0, \frac{1}{5})\}$. Imposing that

$$\mathbb{E}^{\mathbb{Q}}[X_1] = \frac{1}{3}(3tx_1 + (1-5t)x_2 + (2+2t)x_3) = \frac{1}{3}(3x_1 - 5x_2 + 2x_3)t + \frac{1}{3}(x_2 + 2x_3) \quad (1)$$

does not depend on t amounts to asking that

$$3x_1 - 5x_2 + 2x_3 = 0. \quad (2)$$

This condition does not depend on the value of X_0, so nothing changes if $X_0 = 5$ instead of $X_0 = 0$.

Question 2 (Total: 20 marks)

(PARTIALLY) UNSEEN

Let P be a probability on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ such that $\mathsf{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. Define the random variable

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	8	12	4

(3)

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price $B_0 = 1$ (all values/prices are given in £, unless otherwise specified), and interest rate $r = 1$, one stock S with initial price $S_0 = 4$ and final price S_1 as in eq. (3). Denote with P_t the value at time $t \in \{0, 1\}$ of a put option P (with underlying S) with strike price 10. Denote with V the value (in £) of your portfolio, and with W the value of your portfolio when using S as a numeraire (i.e. unit of measure of value), so that $W := V/S$. You can use (without having to prove it) that the set $\mathcal{M} = \mathcal{M}_B$ of EMM (Equivalent Martingale Measures) relative to the numeraire B (i.e. $\mathcal{M}_B$ is the set of probabilities $\mathsf{Q} \sim \mathsf{P}$ s.t. $\mathbb{E}^{\mathsf{Q}}[S_1/B_1] = S_0/B_0$)) is given by the formula

$$\mathcal{M}_B = \left\{ \begin{pmatrix} 1-2s \\ s \\ s \end{pmatrix} : s \in \left(0, \frac{1}{2}\right) \right\}. \tag{4}$$

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) free of arbitrage? (1 marks)

(b) Compute the set $\mathcal{P}_B$ of all arbitrage-free prices of P. (3 marks)
Hint: prices in £, and in units of B, coincide at time 0 since $B_0 = £1$.

(c) Is the put option P replicable? (2 marks)

(d) Let R be a probability on Ω. Prove that the RNP (Risk-Neutral Pricing) formula $\mathbb{E}^{\mathsf{R}}[W_1] = W_0$ relative to the numeraire S holds for all portfolios iff it holds for a portfolio made of only one bond, i.e. for $W = B/S$. (4 marks)

(e) Determine the set $\mathcal{M}_S$ of EMM relative to the numeraire S. (4 marks)
Hint: by item (d), a probability $\mathsf{R} \sim \mathsf{P}$ belongs to $\mathcal{M}_S$ iff $\mathbb{E}^{\mathsf{R}}[B_1/S_1] = B_0/S_0$.

(f) Find the set $\mathcal{P}_S$ of all arbitrage-free prices of P in units of S, doing your calculation using the RNP formula with numeraire S and EMMs $\mathsf{R} \in \mathcal{M}_S$. (4 marks)

(g) What it the relation between prices $\mathcal{P}_B$ (of P) in units of B and the prices $\mathcal{P}_S$ in units of S? Confirm that your answers in items (b) and (f) coincide, after converting the values expressed in units of B to the values expressed in units of S. (2 marks)

Solution:

(a) 1^{st} **solution:** Yes, since the 3 values d, m, u of S_1/S_0 satisfy $d < 1 + r < u$, because

$$d = \frac{4}{4} = 1, \quad 1 + r = 2, \quad u = \frac{12}{4} = 3.$$

2^{nd} **solution:** As you were told in eq. (4), the set $\mathcal{M} = \mathcal{M}_B$ of EMMs with respect to the numeraire B is not empty; by the FTAP (Fundamental Theorem of Asset pricing) it follows that the model is arbitrage-free.

3^{rd} **solution:** As we show in item (e), the set $\mathcal{M} = \mathcal{M}_S$ of EMMs with respect to the numeraire S is not empty; by the FTAP (Fundamental Theorem of Asset pricing) it follows that the model is arbitrage-free (with the same proof as when using B as a numeraire).

(b) Since the put has payoff $P_1 = (10 - S_1)^+ = (2, 0, 6)$, using eq. (4) and $P_1/B_1 = (1, 0, 3)$ allows to use the RNP formula to compute

$$\mathcal{P}_B = \{\mathbb{E}^{\mathsf{Q}}[P_1/B_1] : \mathsf{Q} \in \mathcal{M}\} = \left\{1(1 - 2s) + 0s + 3s : s \in \left(0, \frac{1}{2}\right)\right\} = \left(1, \frac{3}{2}\right). \tag{5}$$

(c) Since $\mathcal{P}_B$ is not a singleton, P is not replicable.

(d) The RNP formula $\mathbb{E}^{\mathsf{R}}[W_1] = W_0$ automatically holds for portfolios made of only stocks, since in this case $V = S$ and so $W_t = S_t/S_t = 1$ for all t. Since any portfolio value is a linear combination of stocks and bonds, and the RNP formula is linear and it holds for portfolios made of only stocks, then the RNP formulas holds for any portfolio iff it holds for one made of only bonds.

In formulas: one implication is trivial, and the opposite one holds since multiplying $\mathbb{E}^{\mathsf{R}}[B_1/S_1] = B_0/S_0$ times k and the trivial identity $\mathbb{E}^{\mathsf{R}}[S_1/S_1] = S_0/S_0$ times h gives $\mathbb{E}^{\mathsf{R}}[V_1/S_1] = V_0/S_0$ for $V = kB + hS$ for every $k, h \in \mathbb{R}$, i.e. RNP formula holds for all portfolios.

(e) As usual we identify R with $r_i := \mathsf{R}(\{\omega_i\}), i = 1, 2, 3$. Then $\mathsf{R} \in \mathcal{M}_S$ iff the r_i's satisfy $r_i > 0$ for all i, $\sum_i r_i = 1$ and $\sum_i r_i \frac{B_1}{S_1}(\omega_i) = \frac{B_0}{S_0}$. Using eq. (7) we get $\mathsf{R} \in \mathcal{M}_S$ iff

$$\begin{cases} \frac{1}{4} = \frac{1}{4} r_1 + \frac{1}{6} r_2 + \frac{1}{2} r_3 \\ 1 = r_1 + r_2 + r_3 \\ r_i > 0 \text{ for } i = 1, 2, 3 \end{cases}.$$

Multiplying the first eq. times 12 and substituting r_3 from the second eq. gives

$$3 = 3r_1 + 2r_2 + 6(1 - r_1 - r_2), \quad \text{i.e. } 3r_1 + 4r_2 = 3,$$

and so

$$r_1 = 1 - \frac{4}{3} r_2, \quad r_3 = 1 - r_1 - r_2 = \frac{1}{3} r_2,$$

and imposing $r_i > 0$ we obtain

$$\mathcal{M}_S = \left\{ \begin{pmatrix} 1 - \frac{4}{3}s \\ s \\ \frac{1}{3}s \end{pmatrix} : s \in \left(0, \frac{3}{4}\right) \right\}. \tag{6}$$

(f) Since the put has payoff $P_1 = (10 - S_1)^+ = (2, 0, 6)$, using eq. (6) and $P_1/S_1 = (\frac{1}{4}, 0, \frac{3}{2})$ allows to use the RNP formula to compute

$$\mathcal{P}_S = \{\mathbb{E}^{\mathsf{R}}[P_1/S_1] : \mathsf{R} \in \mathcal{M}_S\} = \left\{ \frac{1}{4}(1 - \frac{4}{3}s) + 0s + \frac{3}{2}(\frac{1}{3}s) : s \in \left(0, \frac{3}{4}\right) \right\} = \left(\frac{1}{4}, \frac{3}{8}\right).$$

(g) Since, at time 0, one share of the stock is worth £S_0, prices in units of £ and of shares of S are linked via the relation $S_0\mathcal{P}_S = \mathcal{P}_B$, i.e. $\mathcal{P}_B = \{S_0 p : p \in \mathcal{P}_S\}$. This is indeed satisfied, since

$$\mathcal{P}_B = \left(1, \frac{3}{2}\right) = 4\left(\frac{1}{4}, \frac{3}{8}\right) = S_0\mathcal{P}_S.$$

Though you were not asked to do it, we now show that indeed $\mathcal{M}$ is given by eq. (4). The discounted stock value $\overline{S}_1 = \frac{S_1}{B_1} = \frac{1}{1+r}S_1$ is given by

$$\frac{S_1}{B_1} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}. \tag{7}$$

Recall that $\mathbb{Q}$ is an EMM if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 4 = 4q_1 + 6q_2 + 2q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system of equalities has solution $q_3 = q_2, q_1 = 1 - 2q_2$, and imposing $q_i > 0$ we obtain that the set of (q's corresponding to the set of) EMM is given by eq. (4).

Question 3 (Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Consider the binomial model $(B_i, S_i)_{i=0}^N$ with expiration $N \in \mathbb{N}$, interest rate $r = 0$ and stock price

$$S_k := S_0 + \sum_{i=1}^{k} Y_i \quad \text{for all } k = 1, \ldots, N,$$

where $Y_i, i = 1, \ldots, N$ are IID random variables which take the values $d, u \in \mathbb{R}$ with probability $\frac{1}{2}$. Assume $u > 0 > d, S_0 + Nd > 0$, so in particular $S_i > 0$ a.s. for all $i \leq N$. Define

$$M_k := \min_{i=0,1,\ldots,k} S_i, \quad L_k := S_k - M_k, \quad \text{for all } k = 0, \ldots, N,$$

and consider the *floating lookback call option*, i.e. the option which at time N pays the amount $V_N := L_N$, and denote with V_k its arbitrage-free price at time $k = 0, \ldots, N$. As usual let $\mathcal{F}$ be the natural filtration of S and $\mathbb{Q}$ the risk-neutral measure. When we ask if a process is Markov, we mean with respect to $(\mathcal{F}, \mathbb{Q})$. Prove all your assertions carefully or provide counter-examples.

(a) Prove that the $(Y_i)_i$ are IID under $\mathbb{Q}$, and compute $\tilde{p} := \mathbb{Q}(Y_i = u)$. (2 marks)

(b) Is M a Markov process? (3 marks)

(c) Is $C := (S, M)$ a Markov process? (3 marks)

(d) Write an explicit formula for a function h_n which satisfies $L_{n+1} = h_n(L_n, Y_{n+1})$. (3 marks)
Hint: use that $s - \min(a, b) = \max(s - a, s - b)$ *for all* $a, b, c \in \mathbb{R}$.

(e) Is L a Markov process? (2 marks)

(f) Which of the processes $S, M, (S, M)$, L, are such that, for every $n = 0, \ldots, N$, V_n can be written as a function f_n of the value of the process at time n? (2 marks)

(g) For each process as in item (f), write explicitly f_N, and an explicit formula to express f_n in terms of f_{n+1}, for $n = 0, \ldots, N-1$. If there is more than one such process, explain which one is preferable to use for numerical implementations, and why. (5 marks)

Solution: We will use the usual notation, i.e. assume that $Y_i = u$ corresponds to the i^{th} coin toss X_i resulting in H, and $Y_i = d$ corresponds to the i^{th} coin toss X_i resulting in T, and so e.g. $\{Y_1 = u, Y_2 = d\}$ is denoted with HT. As usual $a \wedge b := \min(a, b), a \vee b := \max(a, b)$.

(a) Since the up and down factors U, D are

$$U_n := \frac{S_n + u}{S_n}, \quad D_n := \frac{S_n + d}{S_n}, \tag{8}$$

the corresponding risk neutral probabilities are

$$\tilde{P}_n = \frac{(1 + r) - D_n}{U_n - D_n} = \frac{(1 + r) - \frac{S_n + d}{S_n}}{\frac{S_n + u}{S_n} - \frac{S_n + d}{S_n}} = \frac{S_n(1 + r) - (S_n + d)}{(S_n + u) - (S_n + d)} = \frac{rS_n - d}{u - d}.$$

Thus, since $r = 0$, we get

$$\tilde{P}_n = \frac{-d}{u - d}, \qquad \tilde{Q}_n = 1 - \tilde{P}_n = \frac{u}{u - d},$$

and so $\tilde{P}_n$ is constant $=: \tilde{p}$, which shows that the coin tosses are IID under $\mathbb{Q}$. Thus the $(Y_i)_i$ are IID under $\mathbb{Q}$, since $\{Y_i = u\} = \{X_i = H\}$, $\{Y_i = d\} = \{X_i = T\}$.

(b) No, as can easily been seen by evaluating $\mathbb{E}^{\mathbb{Q}}[f(M_3)|\mathcal{F}_2]$ at the points HH and HT and seeing that they differ, despite the fact that $M_2(HH) = S_0 = M_2(HT)$, and so $\mathbb{E}^{\mathbb{Q}}[f(M_3)|\mathcal{F}_2]$ is not $\sigma(M_2)$-measurable. Indeed, assume for example $u = 1, d = -1, S_0 = 10, r = 0$, then S/B is a $\mathbb{P}$-martingale and so $\mathbb{Q} = \mathbb{P}$ and

$$\mathbb{E}[f(M_3)|\mathcal{F}_2](HH) = \frac{1}{2}(f(10) + f(10)) \neq \frac{1}{2}(f(10) + f(9)) = \mathbb{E}[f(M_3)|\mathcal{F}_2](HT)$$

holds for some f (any f such that $f(9) \neq f(10)$).

(c) Yes, since

$$C_{n+1} = (S_n + Y_{n+1}, M_n \wedge (S_n + Y_{n+1})) = a_n(C_n, Y_{n+1}),$$

where $a_n((s,m),y) := (s + y, m \wedge (s + y))$, and Y_{n+1} is independent of $\mathcal{F}_n$, and so by the independence lemma

$$\mathbb{E}^{\mathbb{Q}}[f(C_{n+1})|\mathcal{F}_n] = g(C_n), \quad \text{for } g(c) := \mathbb{E}^{\mathbb{Q}}[f(a_n(c, Y_{n+1}))]. \tag{9}$$

(d) Using that $s - \min(a, b) = \max(s - a, s - b)$ we get

$$L_{n+1} = S_{n+1} - M_{n+1} = S_{n+1} - (M_n \wedge S_{n+1}) = (S_{n+1} - M_n) \vee 0 = (L_n + Y_{n+1}) \vee 0,$$

and so $h_n(l, y) := (l + y) \vee 0 = (l + y)^+$.

(e) Yes, since $L_{n+1} = h_n(L_n, Y_{n+1})$ and Y_{n+1} is independent of $\mathcal{F}_n$, and so by the independence lemma

$$\mathbb{E}^{\mathbb{Q}}[f(L_{n+1})|\mathcal{F}_n] = g(L_n) \quad \text{for } g(l) := \mathbb{E}^{\mathbb{Q}}[f(h_n(l, Y_{n+1}))]. \tag{10}$$

(f) Since $V_N = L_N = S_N - M_N$, we can express V_N as a function of L_N and of C_N, but not of M_N nor of S_N. Moreover, S, L and C are Markov, whereas M is not. So, there exist such functions $(f_n)_n$ for L, C, but not for M, S.

(g) We should choose L, since it takes fewer number of values than (S, M). Using (9) and $V_N = S_N - M_N$ we find that for C the formulas are

$$f_N(s,m) := s - m, \quad f_n(s,m) := \tilde{p} f_{n+1}(a_n(s,m,u)) + (1 - \tilde{p}) f_{n+1}(a_n(s,m,d)) \text{ for } n = 0, \dots, N-1.$$

Using (10) and $V_N = L_N$ we find that the formulas for L are

$$f_N(l) := l, \quad f_n(l) := \tilde{p} f_{n+1}(h_n(l,u) + (1 - \tilde{p}) f_{n+1}(h_n(l,d)) \text{ for } n = 0, \dots, N-1.$$

Question 4

(Total: 20 marks)

(PARTIALLY) UNSEEN

Model a risky asset $S = (S_n)_{n=0}^N$ with a N-period binomial model with constant up factor u and down factor d and initial value $S_0 > 0$, and a bank account with constant interest rate r, and assume $d < 1 + r < u$. Let V_n be the arbitrage-free price at time n of the *American* call option on S with strike price $K > 0$ which has not yet been exercises; recall that, when exercising the option at time $n \leq N$, the buyer gets the payoff $I_n := S_n - K$. Let τ^* be the smallest optimal exercise time of this option, given by the formula

$$\tau^* = \inf\{n \in \{0, \dots, N\} : V_n = I_n\},$$

where we use the usual convention that $\inf \emptyset := \infty$. As usual let $\mathcal{F}$ be the natural filtration of S, $\mathbb{Q}$ be the risk-neutral measure, and 1_A denote the indicator function of A (i.e. $1_A = 1$ on A, $1_A = 0$ on A^c). Answer the following questions and justify carefully with either proofs or counter-examples.

(a) Write the equation which allows to compute V_n by using V_{n+1} and I_n for $n = 0, \dots, N-1$. (5 marks)

(b) Prove that S is Markov (with respect to $(\mathcal{F}, \mathbb{Q})$). (2 marks)

(c) Prove by backward induction that V_n admits the representation $V_n = f_n(S_n)$ for some $f_n : \mathbb{R} \to \mathbb{R}$, for all $n = 0, 1, ..., N$. (2 marks)

(d) Find a formula, as explicit as possible, to express f_n in terms of f_{n+1} (for any $n = 0, \dots N-1$), and give a formula for f_N. (5 marks)

(e) Let τ be an arbitrary stopping time. Is it true that for every $0 \leq n \leq N$ there exists $a_n : \{0,1\} \times \mathbb{R} \to \{0,1\}$ such that $1_{\{\tau=n\}} = a_n(1_{\{\tau<n\}}, S_n)$? If such $a = (a_n)_n$ exists, give an explicit expression for it; otherwise, give an explicit counter-example. (3 marks)

(f) Does $a = (a_n)_n$ as in (e) exist when $\tau = \tau^*$? If such a exists, give an explicit expression for it; otherwise, give an explicit counter-example. (3 marks)

Solution:

(a)

$$V_n := \max\left(I_n, \mathbb{E}^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}|\mathcal{F}_n]\right), \quad n = 0, 1, \dots, N-1.$$

(b) Since S_{n+1}/S_n is independent of $\mathcal{F}_n$, the independence lemma gives that

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n \frac{S_{n+1}}{S_n})] = g(S_n) \quad \text{for} \quad g(s) := \mathbb{E}_n^{\mathbb{Q}}[f(s\frac{S_{n+1}}{S_n})], \quad n = 0, 1, \dots, N-1. \tag{11}$$

(c) By assumption, for all $n \leq N$, $I_n = h_n(S_n)$ with

$$h_n(s) := s - K,$$

and $V_N = I_N^+ = f_N(S_N)$ with $f_N(s) := (s-K)^+$. Reasoning by backward induction, assume that $V_{n+1} = f_{n+1}(S_{n+1})$ for some function f_{n+1} and $n+1 \le N$, then apply first the formula in part (a) and then eq. (11) to get

$$V_n = \max\left(h_n(S_n), \mathbb{E}_n^{\mathbb{Q}}\left[\frac{f_{n+1}(S_{n+1})}{1+r}\right]\right) = f_n(S_n)$$

for

$$f_n := \max(h_n, g_n), \qquad g_n(s) := \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[f_{n+1}\left(s\frac{S_{n+1}}{S_n}\right)\right].$$

(d) Since $\mathbb{E}^{\mathbb{Q}}[f_{n+1}(s\frac{S_{n+1}}{S_n})] = \tilde{p} f_{n+1}(su) + (1-\tilde{p}) f_{n+1}(sd)$ the previous item gives

$$f_N(s) := (s-K)^+, \quad f_n(s) := \max\left(s-K, \frac{1}{1+r}\left(\tilde{p} f_{n+1}(su) + (1-\tilde{p}) f_{n+1}(sd)\right)\right), n = 0, 1, \ldots, N-1$$

(e) Simple counter-examples shows that such a does not exist for all τ. To find one such counter-example, it helps to understand that the intuitive meaning of the existence of such a, which is that, if $\omega \notin \{\tau < n\}$, then we can determine whether $\tau(\omega) = n$ or not by knowing only the value of $S_n(\omega)$ (i.e. without the need to know also the value of $(S_i(\omega))_{i<n}$).

Let us now build a some possible counter-example explicitly. We fix $S_0 = 4, u = 2, d = \frac{1}{2}$.

1^{st} **example:** Define

$$\tau := \inf\{n \ge 0 : \Sigma_n \ge 17\}, \qquad \text{where } \Sigma_n := \sum_{i=0}^{n} S_i.$$

Then τ is a hitting time of the adapted process Σ, so it is a stopping time. Then

$$\{\tau < 3\} = \cup_{i=0}^{2}\{\Sigma_i \ge 17\} = \{S_0 = 4, S_1 = 8, S_2 = 16\} = \{HH\},$$

so HTT and TTH both belong to $\{\tau < 3\}^c \cap \{S_3 = 2\}$, yet $HTT \in \{\tau = 3\}$ whereas $TTH \notin \{\tau = 3\}$, so such a cannot exist (because, for any function a_3, $a_3(1_{\{\tau<3\}}, S_3)$ takes the same value at HTT and at TTH, whereas $1_{\{\tau=3\}}$ does not).

2^{nd} **example:** Define $S_{-1} := 0$ and

$$\sigma := \inf\{n \ge 0 : S_{n-1} = 8\}.$$

Then σ is a hitting time of the adapted (even predictable) process $(S_{n-1})_n$, so it is a stopping time. Then

$$S_2(HT) = 4 = S_2(TH), \quad \{\sigma < 2\} = \emptyset$$

and yet $\{\sigma = 2\} = \{H\}$ contains HT but does not contain TH. Thus $1_{\{\sigma=2\}}(HT) = 1 \neq 0 = 1_{\{\sigma=2\}}(TH)$ and yet, for any function a, $a(S_2, 1_{\{\sigma<2\}})$ takes the same value $a(4,0)$ at HT and at TH, and so $a(S_2, 1_{\{\sigma<2\}})$ cannot equal $1_{\{\sigma=2\}}$.

(f) Since $V_n = f_n(S_n), I_n = h_n(S_n)$, we get that

$$\tau^* = \inf\{n \in \{0, \dots, N\} : f_n(S_n) = h_n(S_n)\},$$

and so

$$\{\tau^* = n\} = \{\tau^* < n\}^c \cap \{f_n(S_n) = h_n(S_n)\}.$$

In particular, for $\tau = \tau^*$ there exists a such that $1_{\{\tau=n\}} = a_n(1_{\{\tau<n\}}, S_n)$, and a is given by

$$a_n(b, s) := \begin{cases} 0 & \text{if } b = 1; \\ 0 & \text{if } b = 0, s \in \{f_n \neq h_n\}; \\ 1 & \text{if } b = 0, s \in \{f_n = h_n\}. \end{cases}, \qquad n = 0, \dots, N$$

Question 5 (Total: 20 marks)

SIMILARLY SEEN IN PREVIOUS FINAL EXAM

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded), whose prices S_t^1, S_t^2 at times $t = 0, 1$ have values

$$S_0^1 = 60, \qquad S_0^2 = 20, \qquad \begin{array}{c|c|c} \omega & H & T \\ \hline S_1^1(\omega) & 40 & 80 \\ \hline S_1^2(\omega) & 20 & 140 \end{array} \tag{12}$$

Assume that the event $\{H\}$ has probability $\frac{1}{3}$. We'll use the following standard notation: $R_1^i := \frac{S_1^i}{S_0^i} - 1$ denotes the return of asset i between time 0 and 1 and $\mu_i := \mathbb{E}[R_1^i]$ its average for $i = 1, 2$, while $\Sigma := \mathrm{Cov}(R_1)$ denotes the covariance matrix of the returns (i.e. $\Sigma_{i,j} := \mathbb{E}(R_1^i R_1^j) - \mathbb{E}(R_1^i)\mathbb{E}(R_1^j)$, $i = 1, 2$), a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in asset S^i for $i = 1, 2$. Denote with $\mu = \mu(\pi)$ the average, and with $\sigma = \sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π).

(a) Prove that the average returns μ and the covariance matrix Σ are given by the formulas (5 marks)

$$\mu = \left(\frac{1}{9}, 4\right), \quad \Sigma = \frac{8}{9^2}\begin{pmatrix} 1 & 9 \\ 9 & 9^2 \end{pmatrix}. \tag{13}$$

(b) Conclude that the correlation ρ between R_1^1 and R_1^2 equals 1. (1 marks)

(c) Find a portfolio with average return of $\frac{50}{3}$%. Does it involve any short-selling? (4 marks)

(d) Find the equation whose solution is the set S of values (σ, μ) taken by all portfolios, and draw S. (6 marks)

(e) Find explicitly two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (4 marks)

Solution:

(a) First we compute the returns

$$\begin{array}{c|c|c} \omega & H & T \\ \hline R_1^1(\omega) & -\frac{1}{3} & \frac{1}{3} \\ \hline R_1^2(\omega) & 0 & 6 \end{array} \tag{14}$$

and then compute their average as

$$\mu_1 = \mathbb{E}[R_1^1] = \frac{1}{3}\left(-\frac{1}{3} + 2 \cdot \frac{1}{3}\right) = \frac{1}{9}, \qquad \mu_2 = \mathbb{E}[R_1^2] = \frac{1}{3}(0 + 2 \cdot 6) = 4, \tag{15}$$

and finally we can compute Σ as follows

$$\sigma_1^2 := \Sigma_{1,1} = \mathbb{E}[(R_1^1)^2] - (\mathbb{E}[R_1^1])^2 = \frac{1}{3}\left(\frac{1}{9} + 2\cdot\frac{1}{9}\right) - \left(\frac{1}{9}\right)^2 = \frac{1}{9} - \frac{1}{9^2} = \frac{8}{9^2},$$
$$\sigma_2^2 := \Sigma_{2,2} = \mathbb{E}[(R_1^2)^2] - (\mathbb{E}[R_1^2])^2 = \frac{1}{3}\left(0 + 2\cdot 6^2\right) - 4^2 = 24 - 16 = 8,$$
$$\Sigma_{1,2} = \Sigma_{2,1} = \mathbb{E}[R_1^1 R_1^2] - \mathbb{E}[R_1^1]\mathbb{E}[R_1^2] = \frac{1}{3}\left(0 + 2\cdot\frac{1}{3}\cdot 6\right) - \frac{1}{9}\cdot 4 = \frac{4}{3} - \frac{4}{9} = \frac{8}{9},$$

thus proving eq. (13).

(b) The correlation of the returns is

$$\rho := \frac{\Sigma_{1,2}}{\sigma_1\sigma_2} = \frac{\frac{8}{9}}{\sqrt{\frac{8}{9^2}}\sqrt{8}} = 1. \tag{16}$$

(c) Since the return $R(\pi)$ of a portfolio (x_0, π) depends only on $\pi = (\pi_1, \pi_2)$ and is given by $R_1(\pi) = \pi_1 R_1^1 + \pi_2 R_1^2$, using the notation $u := \pi_1$, so that $\pi_2 = 1 - u$, and using eq. (13) we compute

$$\mu(u) := \mu(\pi) = \mu^T\pi = u\mu_1 + (1-u)\mu_2 = \frac{u}{9} + 4(1-u) = \frac{36 - 3u}{9} = \frac{12 - u}{3}, \tag{17}$$

and so $\mu(\pi) = \frac{50}{3}\cdot\frac{1}{100} = \frac{1}{6}$ becomes $\frac{12-u}{3} = \frac{1}{6}$, and solving for u gives $12 - u = \frac{1}{2}$, and so $u = 12 - \frac{1}{2} = 11.5$ and $\pi = (11.5, -10.5)$. Since $\pi_2 < 0$, this portfolio involves short-selling the second asset.

(d) Since Σ is not invertible, S is given by the union of two half-lines. To describe S in formulas, we compute the standard deviation $\sigma(u)$ of the portfolio $\pi = (u, 1-u)$ as

$$\sigma(u)^2 = \pi^T\Sigma\pi = u^2\sigma_1^2 + (1-u)^2\sigma_2^2 + 2\rho u(1-u)\sigma_1\sigma_2$$

and we use eq. (13) to get

$$\sigma(u)^2 = \frac{8}{9^2}\left(u^2 + 9^2(1-u)^2 + 2\cdot 9u(1-u)\right) = \frac{8}{9^2}\left(u + 9(1-u)\right)^2 = \frac{8}{9^2}\left(9 - 8u\right)^2. \tag{18}$$

Inverting eq. (17) we get

$$u = 12 - 3\mu \tag{19}$$

which plugged into eq. (18) gives

$$\sigma = \frac{\sqrt{8}}{9}|9 - 8(12 - 3\mu)| = \frac{\sqrt{8}}{9}|24\mu - 87|,$$

i.e.

$$S := \{(\sigma,\mu) \in \mathbb{R}_+ \times \mathbb{R} : \sigma = \frac{\sqrt{8}}{9}|24\mu - 87|\}, \tag{20}$$

which is a union of the half-lines starting from the point $(\sigma, \mu) = (0, \frac{87}{24}) = (0, \frac{29}{8})$ and with slope $\pm\frac{9}{24\sqrt{8}} = \pm\frac{3}{16\sqrt{2}}$.

(e) For example one can take π^a and π^b which correspond to the points $(0, \frac{29}{8})$ and $(\frac{87\sqrt{8}}{9}, 0) = (\frac{58\sqrt{2}}{3}, 0)$ in S, since $\frac{58\sqrt{2}}{3} > 0$ and $0 < \frac{29}{8}$. To find $\pi^a = (u^a, 1-u^a), \pi^b = (u^b, 1-u^b)$ we apply eq. (19) and get

$$u^a = 12 - 3 \cdot \frac{29}{8} = 3(4 - \frac{29}{8}) = 3(\frac{32-29}{8}) = \frac{9}{8}, \qquad u^b = 12 - 3 \cdot 0 = 12.$$

The portfolio π^a is preferable, since the standard deviation $\sigma(\pi)$ of the returns of a portfolio π is normally used as the measure its risk, and so π^a has both a higher average return, and a lower risk, than π^b, and so any investor (which uses $\sigma(\pi)$ as the measure its risk) would prefer π^a to π^b, irrespectively of his/her attitude towards risk.

MATH60012 Mathematical Finance: An Introduction to Option Pricing

Question	Marker's comment
1	Q1: Considering how simple this question was (most of it could be solved after 2 weeks of class), I was surprised and disappointed by the students' middling performance in it.In items (a,b,c) only a minority of students gave fully correct solutions; also, of those who did, many obtained such solutions via time-consuming algebraic calculations instead of providing the short and intuitive answer, thus displaying little understanding.Most answered item (d) correctly, though a number of student wrote there that the forward contract is not replicable, which was logically inconsistent with item (a).Very few got full points in item (e). Many wrote as answer that x must belong to the span of B and S (which is true, though nearly tautological), without deriving the equation which x must satisfy for this to be true.Many students showed that they did not understand the crucial definition of replicating portfolio, either by getting wrong the required initial capital in item (a), or by writing in item (e) that whether X is replicable or not depends on the initial price of X; and quite a number used the wrong definition of arbitrage in item (b).
2	In Q2 nbsp;a good percentage of students either got 20/20, or lost points only in item (d) (mostly left empty) or sometimes in item (f) (mostly by forgetting to divide by S_1). The majority of the remaining students got at most 6/20, and solved only items (a,b,c). I believe this divide is explained by the fact that items (d-f) were easy but essentially unseen, in that the students only had seen similar exercises using the bond as numeraire. So those students who were only able to reproduce seen arguments solved only (a,b,c), whereas those which had actually understood the topic were able to solve almost everything. The weakest students also lost time by doing unnecessary calculations in item (a,b,c), instead of using the fact that I had provided them with the set of EMMs.nbsp;
3	Generally, most students do well, except for part (a), most students do not know the true meaning of u and d, so they got the wrong probability from the start. For part (b), most are correct, but some of them did not use a concrete counterexample to prove their arguments. For part (c)(d)(e), most know they should use the independence lemma, but only a small number of students really write it explicitly, and some of them forgot to put f() in their arguments. For part (f), most are correct, but some think S is also the answer. For part (f), most can write the function at the last time point, but for the backward induction, they seem to be confused about where to put f_{n+1} and some even did not write any f(), just used L and C.

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Question	Marker's comment
4	On the whole the marks for this question vary considerably, with some achieving full marks (or close to full marks) and others gaining very few marks.Many students attempted part (a). There was a range of success regarding attempts at part (a). Some students did achieve full marks on this question, but several did not. The most common (by far) mistakes were:- to not take account of both possibilities (not mentioning I_{n} and writing down only the conditional expectation, and not taking the maximum between them)-to not discount (divide by (1+r)) the conditional expectation or discounted the wrong term (whole of max(In, …)).nbsp;Most students attempted part (b). Many students achieved full marks on part (b), for including at least two of the three points:- the fact that S_{n-1}/S_{n} is independent of F_n—the independence lemma- the conclusion that the conditional expectation of f(S_{n+1}) given F_{n} is a function of S_{n} for any (appropriate) function f.nbsp;Some students who did not achieve full marks on this part did so because of a lack of explanation or gaps in explanation (mainly only mentioning one or none of the points above). Some students proved that S is a martingale rather than a Markov process.nbsp; Part (c) was done well by many students - many did address the base case of the induction as well as the inductive step. Some students who did not achieve full marks forgot to include the base case and a few did not attempt a proof by backward induction.nbsp; Fewer students were successful on part (d). As with part (a), a significant number forgot to include that f_n is the maximum of the function derived from the conditional expectation and the function derived from the intrinsic value of the option (I.E. max(s-K,…))For f_N, a noticeable number of students wrote that f_{N}(s)=s-K and not its positive part (the correct answer).Many students did not give any substantial answer for parts (e) and (f), let alone a correct one for either. Many students either did not attempt these two parts or attempted only one of them. Many attempts consisted only of ‘Yes’ or ‘No’ with no (or little/irrelevant) justification and were often incorrect anyway. Some attempts at part (e) made the mistake of thinking that the question asked for the stopping time in the preamble of the question, as opposed to any arbitrary stopping time. Aside from the fully correct attempts, more successful attempts captured the intuition behind why the answer was false but did not provide a counterexample. Some partially successful attempts at (f) contained the correct intuition of what the function in question ought to be, but made minor errors e.g. wrong multiplication, wrong case etc.Few students achieved full marks for one of part (e) or (f).nbsp;

MATH70012 Mathematical Finance: An Introduction to Option Pricing

Question	Marker's comment
1	Considering how simple this question was (most of it could be solved after 2 weeks of class), I was surprised and disappointed by the students' middling performance in it.In items (a,b,c) only a minority of students gave fully correct solutions; also, of those who did, many obtained such solutions via time-consuming algebraic calculations instead of providing the short and intuitive answer, thus displaying little understanding.Most answered item (d) correctly, though a number of student wrote there that the forward contract is not replicable, which was logically inconsistent with item (a).Very few got full points in item (e). Many wrote as answer that x must belong to the span of B and S (which is true, though nearly tautological), without deriving the equation which x must satisfy for this to be true.Many students showed that they did not understand the crucial definition of replicating portfolio, either by getting wrong the required initial capital in item (a), or by writing in item (e) that whether X is replicable or not depends on the initial price of X; and quite a number used the wrong definition of arbitrage in item (b).
2	In Q2 nbsp;a good percentage of students either got 20/20, or lost points only in item (d) (mostly left empty) or sometimes in item (f) (mostly by forgetting to divide by S_1). The majority of the remaining students got at most 6/20, and solved only items (a,b,c). I believe this divide is explained by the fact that items (d-f) were easy but essentially unseen, in that the students only had seen similar exercises using the bond as numeraire. So those students who were only able to reproduce seen arguments solved only (a,b,c), whereas those which had actually understood the topic were able to solve almost everything. The weakest students also lost time by doing unnecessary calculations in item (a,b,c), instead of using the fact that I had provided them with the set of EMMs.nbsp;
3	Generally, most students do well, except for part (a), most students do not know the true meaning of u and d, so they got the wrong probability from the start. For part (b), most are correct, but some of them did not use a concrete counterexample to prove their arguments. For part (c)(d)(e), most know they should use the independence lemma, but only a small number of students really write it explicitly, and some of them forgot to put f() in their arguments. For part (f), most are correct, but some think S is also the answer. For part (f), most can write the function at the last time point, but for the backward induction, they seem to be confused about where to put f_{n+1} and some even did not write any f(), just used L and C.

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4	On the whole the marks for this question vary considerably, with some achieving full marks (or close to full marks) and others gaining very few marks. Most students attempted part (a). There was a range of answers to part (a). Some students did achieve full marks on this question, but several did not. The most common (by far) mistakes were:- to not take account of both possibilities (not mentioning I_{n} and writing down only the conditional expectation, and not taking the maximum between them)-to not discount (divide by (1+r)) the conditional expectation or discounted the wrong term (whole of max(In, …)).nbsp; Most students attempted part (b). Many students achieved full marks on part (b), for including at least two of the three points:- the fact that S_{n-1}/S_{n} is independent of F_n—the independence lemma- the conclusion that the conditional expectation of $f(S_{n+1})$ given F_{n} is a function of S_{n} for any (appropriate) function f.nbsp;Some students who did not achieve full marks on this part did so because of a lack of explanation or gaps in explanation (mainly only mentioning one or none of the points above). Some students proved that S is a martingale rather than a Markov process.nbsp; Part (c) was done well by many students - many did address the base case of the induction as well as the inductive step. Some students who did not achieve full marks forgot to include the base case and a few did not attempt a proof by backward induction.nbsp; Fewer students were successful on part (d). As with part (a), a significant number forgot to include that f_n is the maximum of the function derived from the conditional expectation and the function derived from the intrinsic value of the option (I.E. max(s-K,…))For f_N, a noticeable number of students wrote that $f_{N}(s)=s-K$ and not its positive part (the correct answer). Many students did not give any substantial answer for parts (e) and (f), let alone a correct one for either. Many students either did not attempt these two parts or attempted only one of them. Many attempts consisted only of ‘Yes’ or ‘No’ with no (or little/irrelevant) justification and were often incorrect anyway. Some attempts at part (e) made the mistake of thinking that the question asked for the stopping time in the preamble of the question, as opposed to any arbitrary stopping time. Aside from the fully correct attempts, more successful attempts captured the intuition behind why the answer was false but did not provide a counterexample. Some partially successful attempts at (f) contained the correct intuition of what the function in question ought to be, but made minor errors e.g. wrong multiplication, wrong case etc.Few students achieved full marks for one of part (e) or (f).nbsp;
5	Most students did well in this question, stumbling at most in the algebra required to solve item (d)