MATH50010: Probability for Statistics
Problem Sheet 8

1. A flea jumps randomly on vertices $\{1, 2, 3\}$ according to the transition probabilities shown in Figure 1. Let X_t be the position of the flea at time t $(t = 0, 1, ...)$.

   (a) Write down the transition matrix $\boldsymbol{P}$.

   (b) Find $P(X_2 = 3|X_0 = 1)$.

   (c) Suppose that the flea is equally likely to start at any vertex at time 0. Find the probability distribution of X_1.

   (d) Suppose that the flea begins at vertex 1 at time 0. Find the probability distribution of X_2.

   (e) Suppose that the flea is equally likely to start on any vertex at time 0. Find the probability of obtaining the trajectory $(3, 2, 1, 1, 3)$.

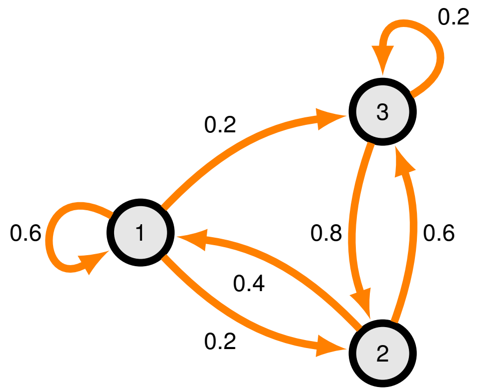

Figure 1: Transition diagram For Question 1

(a)

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

(b) By the Chapman Kolmogorov equations,

$$\begin{aligned} \Pr(X_2 = 3|X_0 = 1) &= \sum_{i\in\{1,2,3\}} \Pr(X_2 = 3 \mid X_1 = i) \Pr(X_1 = i \mid X_0 = 1) \\ &= P_{11}P_{13} + P_{12}P_{23} + P_{13}P_{33} \\ &= 0.6 \times 0.2 + 0.2 \times 0.6 + 0.2 \times 0.2 \\ &= 0.28. \end{aligned}$$

Alternatively, one could compute the matrix P^2 and extract the entry $(1, 3)$ to get the same answer.

(c) *As the flea is equally likely to start in any state, the initial distribution is* $\pi = (1/3, 1/3, 1/3)^T$. *The distribution of* X_1 *is*

$$\pi^T P = (1/3, 1/3, 1/3)\begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix} = (1/3, 1/3, 1/3)^T.$$

So the flea is equally likely to be in states 1, 2, or 3 after the first step.

(d) *The initial distribution of* X_0 *is* $\pi^T = (1, 0, 0)$. *We need to compute the distribution of* X_2 *which is given by* $\pi^T P^2$.

$$\pi^T P^2 = (0.6, 0.2, 0.2)P = (0.44, 0.28, 0.28)^T.$$

(e) *Let* $\pi = (1/3, 1/3, 1/3)^T$ *be the initial distribution. Using the Markov property, we can write*

$$\begin{aligned} \Pr(X_0 = 3, X_1 = 2, X_2 = 1, X_3 = 1, X_4 = 3) &= \pi_3 p_{32} p_{21} p_{11} p_{13} \\ &= 1/3 \times 0.8 \times 0.4 \times 0.6 \times 0.2 \\ &= 0.0128. \end{aligned}$$

2. Suppose a gambler has \$1 initially. At each round, he either wins \$1 with probability p or loses \$1 with probability $q = 1 - p$. The game ends when the gambler obtains $\$N$. Find the probability that the gambler goes broke, i.e., that his capital reaches \$0. What is the fate of a gambler who faces an opponent who is infinitely rich? (A reasonable model for an individual playing against a casino, who will always take the gambler's bet.)

Let B_i *be the event that the gambler becomes broke if he starts with* $\$i$ *and let* W *be the event that the gambler wins the first game. Define* $h_i = \mathbb{P}(B_i)$. *Then, using the law of total probability*

$$\begin{aligned} h_i &= \mathbb{P}(B_i) \\ &= \mathbb{P}(B_i \mid W)\mathbb{P}(W) + \mathbb{P}(B_i \mid W^C)\mathbb{P}(W^C) \\ &= \mathbb{P}(B_i \mid W)p + \mathbb{P}(B_i \mid W^C)(1-p). \end{aligned}$$

Consider the term $\mathbb{P}(B_i \mid W)$, *the probability that the gambler becomes broke when he starts with* $\$i$ *even though he wins the first round. By the Markov property, this is equivalent to the probability that the gambler becomes broke given that he starts with* $\$i + 1$. *So,* $\mathbb{P}(B_i \mid W) = \mathbb{P}(B_{i+1}) = h_{i+1}$. *Similarly,* $\mathbb{P}(B_i \mid W^C) = h_{i-1}$ *so*

$$h_i = p h_{i+1} + (1-p) h_{i-1}.$$

In particular,

$$\begin{aligned} (1-p)h_i + p h_i &= p h_{i+1} + (1-p) h_{i-1} \\ \implies \qquad h_{i+1} - h_i &= \left(\frac{1-p}{p}\right)(h_i - h_{i-1}) \end{aligned}$$

which can be iterated to obtain

$$h_{i+1} - h_i = \left(\frac{1-p}{p}\right)^i (h_1 - h_0) = \left(\frac{1-p}{p}\right)^i (h_1 - 1),$$

as $h_0 = 1$. Then,

$$\begin{aligned} h_i - h_0 &= \sum_{k=0}^{i-1} (h_{k+1} - h_k) \\ &= (h_1 - 1) \sum_{k=0}^{i-1} \left(\frac{1-p}{p} \right)^k . \end{aligned}$$

Summing over the geometric series and using $h_0 = 1$ (since starting at 0 means that we are already broke) gives

$$h_i = 1 + (h_1 - 1) \begin{cases} \frac{1-[(1-p)/p]^i}{(2p-1)/p} & p \neq 1/2, \\ i & p = 1/2. \end{cases}$$

As $h_N = 0$ (since we win when we have $\$N$, we can solve for h_1 to obtain

$$h_1 = 1 + \begin{cases} \frac{-(2p-1)/p}{1-[(1-p)/p]^N} & p \neq 1/2, \\ -1/N & p = 1/2. \end{cases}$$

Substituting this into the previous result

$$h_i = 1 - \begin{cases} \frac{1-[(1-p)/p]^i}{1-[(1-p)/p]^N} & p \neq 1/2, \\ i/N & p = 1/2. \end{cases}$$

This gives the probability of going broke when starting with $\$i$. When playing against an infinitely rich casino, we take $N \to \infty$ to obtain

$$h_i = 1 - \begin{cases} 1 - [(1-p)/p]^i & p > 1/2, \\ 0 & p \leq 1/2. \end{cases}$$

So, when $p \leq 1/2$ the gambler will go broke with probability one when he plays against an infinitely rich opponent (irrespective of his starting point).

3. Consider the two Markov chains below and decide which are irreducible and which are periodic:

   (a) A random walk on a cycle with state space $\mathcal{E} = \{0, 1, \cdots, M-1\}$. At each step the walk increases by 1 (mod M) with probability p and decreases by 1 (mod M) with probability $1-p$. That is:

   $$p_{ij} = \begin{cases} p & \text{if } j \equiv i + 1 \text{mod } M \\ 1-p & \text{if } j \equiv i - 1 \text{mod } M \\ 0 & \text{otherwise} \end{cases}$$

   (b) Simple symmetric random walk on $\mathbb{Z}^{\mathrm{d}}$. At each step the walk moves from its current site to one of its $2d$ neighbours chosen uniformly at random. That is:

   $$p_{ij} = \begin{cases} 1/2d & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

   where $|i-j| = |i_1 - j_1| + \cdots + |i_d - j_d|$ for states $i = (i_1, ..., i_d)$, $j = (j_1, \cdots, j_d)$.

(a) *The random walk on the cycle is irreducible since every site is accessible from every other. It has period 2 if M is even, and is aperiodic if M is odd, since we can always get back in an even number of steps or M steps, and if M is odd gcd$\{M, 2\} = 1$.*

(b) *The random walk on $\mathbb{Z}^{\mathrm{d}}$ is irreducible and has period 2 for any d.*

4. Consider the random walk on $\{0, 1, 2, \ldots\}$, where $p_{01} = 1$ and for $i > 0$,

$$p_{ij} = \begin{cases} q & j = i - 1 \\ p & j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $p + q = 1$.

Let h_i be the probability of hitting 0 when the chain starts from $X_0 = i$.

(a) Explain why h_i satisfies

$$h_0 = 1 \qquad h_i = ph_{i+1} + qh_{i-1}, \quad i \geq 1.$$

(b) Show that if $u_i = h_{i-1} - h_i$, then $u_i = \left(\frac{q}{p}\right)^{i-1} u_1$.

(c) Hence determine h_i, distinguishing between the cases $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$.

(a) *By definition,*

$$h_i = \Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_0 = i\right).$$

Applying the law of total probability,

$$\begin{aligned} h_i = &\Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_1 = i + 1, X_0 = i\right) \Pr(X_1 = i + 1 | X_0 = i) \\ &+ \Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_1 = i - 1, X_0 = i\right) \Pr(X_1 = i - 1 | X_0 = i). \end{aligned}$$

Applying the Markov property, for $i \geq 1$ this is

$$h_i = ph_{i+1} + qh_{i-1}, \quad i \geq 1.$$

The boundary case $i = 0$ is trivial.

(b) *For $i \geq 1$, we see that*

$$(p + q)h_i = h_i = ph_{i+1} + qh_{i-1}.$$

Rearranging gives

$$p(h_i - h_{i+1}) = q(h_{i-1} - h_i),$$

so that if $u_i = h_{i-1} - h_i$, we have

$$u_i = \frac{q}{p} u_{i-1}.$$

So recursing gives the result.

(c) Substituting back for h_i, we get

$$h_i = h_{i-1} - u_i = h_{i-2} - u_{i-1} - u_i = \ldots = h_0 - \sum_{k=1}^{i} u_k = 1 - u_1 \sum_{k=1}^{i} \left(\frac{q}{p}\right)^{k-1}.$$

Summing the geometric series gives ($p \neq q$)

$$h_i = 1 - \frac{u_1 \left(1 - \left(\frac{q}{p}\right)^i\right)}{1 - \frac{q}{p}}.$$

As indicated in lecture notes, we find the hitting probabilities, and consequently u_1, by requiring the **minimal non-negative solution** .

When $p > q$, h_i is a decreasing function of u_1 and an increasing function of i, so for the minimal non-negative solution set $h_i \to 0$ as $i \to \infty$. This then gives

$$u_1 = 1 - \frac{q}{p},$$

so that $h_i = \left(\frac{q}{p}\right)^i$. A smaller choice of u_1 would lead to a solution that is not minimal non-negative; a larger choice of u_1 would lead to a solution with negative values.

For $p < q$, since $\frac{q}{p} > 1$, h_i is unbounded unless $u_1 = 0$, so that $h_i = 1$ for all $i \geq 0$.

For $p = q$, we get that $h_i = 1 - u_1 i$, so that minimal non-negativity requires $u_1 = 0$ and $h_i = 1$ for all $i \geq 0$.

5. Extend the idea of the previous question to the more general birth-death chain on $\{0, 1, 2, \ldots\}$ for which $p_{i\,i+1} = p_i$ and $p_{i\,i-1} = q_i = 1 - p_i$, with zero probability for all other transitions, and $p_i, q_i > 0$ for all $i \geq 1$.

   (a) Show that $h_i = p_i h_{i+1} + q_i h_{i-1}$ and deduce that $u_i = \frac{q_i}{p_i} u_{i-1}$, for $u_i = h_{i-1} - h_i$.

   (b) Write u_i in terms of $\gamma_i = \prod_{k=1}^{i} \frac{q_k}{p_k}$ and u_1.

   (c) Determine u_1 and show that the chain is transient if and only if $\sum_{i=1}^{\infty} \gamma_i < \infty$.

(a) By definition,

$$h_i = \Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_0 = i\right).$$

Applying the law of total probability as before,

$$\begin{aligned} h_i =& \Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_1 = i+1, X_0 = i\right) \Pr(X_1 = i+1 | X_0 = i) \\ &+ \Pr\left(\bigcup_{n=0}^{\infty} \{X_n = 0\} | X_1 = i-1, X_0 = i\right) \Pr(X_1 = i-1 | X_0 = i). \end{aligned}$$

Applying the Markov property, for $i \geq 1$ this is

$$h_i = p_i h_{i+1} + q_i h_{i-1}, \quad i \geq 1.$$

The boundary case $i = 0$ is trivial.

(b) *For $i \geq 1$, we see that*

$$(p_i + q_i)h_i = h_i = p_i h_{i+1} + q_i h_{i-1}.$$

Rearranging gives

$$p_i(h_i - h_{i+1}) = q_i(h_{i-1} - h_i),$$

so that if $u_i = h_{i-1} - h_i$, we have

$$u_{i+1} = \frac{q_i}{p_i} u_i = \prod_{k=1}^{i} \frac{q_k}{p_k} u_1 = \gamma_i u_1.$$

(c) *Substituting back for h_i, we get*

$$h_i = h_{i-1} - u_i = h_{i-2} - u_{i-1} - u_i = \ldots = h_0 - \sum_{k=1}^{i} u_k = 1 - u_1 \sum_{k=0}^{i} \gamma_k.$$

For the minimal non-negative solution we set $u_1 = \frac{1}{\sum_{k=0}^{\infty} \gamma_k}$ if the sum in the denominator is finite, and $u_1 = 0$ otherwise. The condition for transience follows from considering the definition of h_i.

6. Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Find π, the stationary distribution of P.

The stationary distribution must satisfy $\pi P = \pi$. Extracting individual entries, we obtain the following equations

$$\begin{aligned} \pi_3/2 &= \pi_1 \\ \pi_1 + \pi_2/2 &= \pi_2 \\ \pi_2/2 + \pi_3/2 &= \pi_3. \end{aligned}$$

Solving this system of equations we have

$$\pi_3 = \pi_2 = 2\pi_1,$$

so $\pi = c(1, 2, 2)$ for any $c \in \mathbb{R}$. As π must be a distribution we must have $c(1 + 2 + 2) = 1$ so

$$\pi = (1/5, 2/5, 2/5).$$

Another approach is to find the left-eigenvectors of P.

For discussion

7. Consider the random walk on $\mathbb{Z}$ with

$$p_{i,j} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \end{cases}$$

Show that all states are transient if $p \neq 1/2$ and recurrent if $p = 1/2$.

[Hint: You may use without proof Stirlings formula: $n! \sim \sqrt{2\pi n}(n/e)^n$ as $n \to \infty$ where $a_n \sim b_n$ here means $a_n/b_n \to 1$]

Suppose we start at state $X_0 = 0$. From lectures, we know that we cannot return to 0 after an odd number of steps. Hence, $p_{0,0}(2n+1) = 0$ for any $n \geq 0$. We therefore consider $m = 2n$ for some $n \geq 1$. To get from 0 to 0 in $2n$ steps means that we must take n steps to the left and n steps to the right in any order, the number of such sequences is equal to the number of ways of choosing n items from $2n$. Hence,

$$p_{0,0}(2n) = \binom{2n}{n} p^n(1-p)^n = \frac{(2n)!}{(n!)^2} p^n(1-p)^n.$$

We can now apply Stirlings formula to approximate the ratio of factorials by

$$\frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{4\pi n}(2n/e)^{2n}}{2\pi n(n/e)^{2n}} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

Hence,

$$p_{0,0}(2n) \sim \frac{2^{2n}}{\sqrt{\pi n}} p^n(1-p)^n = \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

If $p = 1/2$, $4p(1-p) = 1$ and so there exists an N such that for all $n \geq n$, $p_{0,0}(2n) \geq \frac{1}{\sqrt{\pi n}}$. Hence,

$$\sum_{n=1}^{\infty} p_{0,0}(2n) \geq \sum_{n=N}^{\infty} p_{0,0}(2n) \geq \sum_{n=N}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

and so the random walk is recurrent.

On the other hand, if $p \neq 1/2$, $4p(1-p) = r < 1$, so for some N and all $n \geq N$, $p_{0,0}(2n) \leq r^n$ so,

$$\sum_{n=1}^{\infty} p_{0,0}(2n) = N + \sum_{n=N}^{\infty} p_{0,0}(2n) \leq N + \sum_{n=N}^{\infty} r^n < \infty$$

and so the random walk is transient.