Exercise 9.1. Consider the metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$. Show that all of the intervals $(a, b]$, $[a, b)$, $[a, +\infty)$, and $(-\infty, b]$ are not compact.

Exercise 9.2. Show that if A and B are compact subsets of a metric space (X, d), then $A \cup B$ is a compact set.

Exercise 9.3. Show that the ball

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in the metric space $(\mathbb{R}^2, \mathrm{d}_2)$ is not compact.

Exercise 9.4. Let (X, d) be a metric space, and A_1, A_2, $\dots$, A_n be a finite number of bounded sets in X. Then $\cup_{i=1}^n A_i$ is a bounded set in X.

Exercise 9.5. Let (X, d) be a non-empty metric space, and let $Z \subseteq X$. Show that Z is bounded if and only if there is $x \in X$ and $r \in \mathbb{R}$ such that $Z \subseteq B_r(x)$.

Exercise 9.6. Consider the set $\mathbb{R}$ with the discrete metric $\mathrm{d}_{\mathrm{disc}}$. The set $(0, 1)$ is closed and bounded in $(\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$, but it is not compact.

Exercise 9.7. Let (X, d) be a metric space, and assume that V_n, for $n \geq 1$, be a nest of non-empty closed sets in X.

(i) Show that if X is compact, then $\cap_{n\geq 1} V_n$ is not empty.

(ii) Give an example of a nest of non-empty closed sets V_n, for $n \geq 1$, in a metric space such that $\cap_{n\geq 1} V_n$ is empty.

Exercise 9.8. Show that if a metric space is sequentially compact, then it is bounded.

Exercise 9.9.* Let (X, d) be a sequentially compact metric space. Show that X is separable, that is, there is a countable dense set in X.

Exercise 9.10.* Let (X, d) be a sequentially compact metric space, and $\mathcal{R}$ be an open cover for X. Show that there is a countable sub-cover of $\mathcal{R}$ for X.

Exercise 9.11. Let (X, d) be a compact metric space, and assume that $f : X \to X$ is a continuous map such that for all $x \in X$, we have $f(x) \neq x$. Show that there is $\delta > 0$ such that for all $x \in X$, we have $\mathrm{d}(x, f(x)) \geq \delta$.

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional