ME4 Q4

$$\min \quad x_1 - 4x_2 + x_3^4$$

$$\text{s.t.} \quad x_1 + x_2 + x_3^2 \leq 2$$

$$x_1 \geq 0 \checkmark \quad x_2 \geq 0 \checkmark$$

i) Cost is convex $\nabla^2 f = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 12x_3^2 \end{bmatrix} \succeq 0$

$$\boxed{x_3^* = 0 \quad x_2^* = 2 \\ x_1^* = 0}$$

Constraints are convex $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 2 \end{bmatrix} \succeq 0$

(Do KKT).

ii) <u>Duality</u>: $\mathcal{X} = \mathbb{R}^3$

$$L(\underline{x}, \lambda) = x_1 - 4x_2 + x_3^4 + \lambda_1(x_1 + x_2 + x_3^2 - 2) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

$$\min_{x \in \mathbb{R}^3} L(\underline{x}, \lambda) = x_1(1 + \lambda_1 - \lambda_2) + x_2(-4 + \lambda_1 - \lambda_3) + x_3^4 + \lambda_1 x_3^2 - 2\lambda_1$$

$$\Rightarrow \min L = \begin{cases} \min\limits_{\underline{x} \in \mathbb{R}^3} x_3^4 + \lambda_1 x_3^2 - 2\lambda_1 & 1 + \lambda_1 - \lambda_2 = 0 \text{ and } -4 + \lambda_1 - \lambda_3 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_3} \; x_3^4 + \lambda_1 x_3^2 - 2\lambda_1$$

$f' = 0 \Rightarrow 4x_3^3 + 2\lambda_1 x_3 = 0$

$$2x_3\underbrace{(2x_3^2 + 2\lambda_1)}_{\overset{?}{=}0} = 0$$

$\overset{?}{=} 0$, not possible because $\lambda_1 \geq 0$

$\boxed{x_3 = 0}$

$$\min_{x \in \mathbb{R}^2} L = \begin{cases} -2\lambda_1 & 1 + \lambda_1 - \lambda_2 = 0 \\ & -4 + \lambda_1 - \lambda_3 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual:

$$\max \; -2\lambda_1$$

s.t. $\underline{\lambda} \geq 0$

$1 + \lambda_1 - \lambda_2 = 0$

$-4 + \lambda_1 - \lambda_3 = 0$

$\rightarrow \lambda_1 = 4$

$\lambda_2 = 5$

$\lambda_3 = 0$

$\wedge \; \boxed{x_3 = 0}$

$$\lambda_1 (x_1 + x_2 + \underbrace{x_3^2}_{=0} - 2) = 0$$

$\lambda_2 (-x_1) = 0$

$\lambda_3 (-x_2) = 0$

$\lambda_1 = 4 \Rightarrow x_1 + x_2 - 2 = 0$

$\lambda_2 = 5 \Rightarrow \boxed{x_1 = 0}$

$\Rightarrow \boxed{x_2 = 2}$

EXTRA: Using KKT for the primal.

$$\mathcal{L}(\underline{x},\underline{\lambda}) = x_1 - 4x_2 + x_3^4 + \lambda_1(x_1+x_2+x_3^2-2) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

$$\nabla_x L = 0 \iff \begin{aligned} &1+\lambda_1-\lambda_2=0 && (1)\\ &-4+\lambda_1-\lambda_3=0 && (2)\\ &4x_3^3+2\lambda_1 x_3=0 && (3)\end{aligned}$$

$$\lambda_1(x_1+x_2+x_3^2-2)=0 \quad (4)$$

$$\lambda_2(-x_1)=0 \quad (5)$$

$$\lambda_3(-x_2)=0 \quad (6)$$

$$\lambda_1,\lambda_2,\lambda_3 \geq 0$$

Case $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$

$\lambda_1 \neq 0 \Rightarrow x_1 + x_2 + x_3^2 - 2 = 0$

$\lambda_2 \neq 0 \Rightarrow x_1 = 0$

$\Rightarrow x_2 + x_3^2 - 2 = 0$

Also, (3) $\iff 2x_3(2x_3^2+\lambda_1)=0$

$\hookrightarrow \lambda_1 \neq 0 \Rightarrow x_3 = 0$

$x_2 = 2$

Therefore $\underline{x}^* = (0,2,0)$ solves KKT + convexity + Slater ($\hat{x}=(0,0,0)$; $0+0+0^2<2$)

$\Rightarrow$ KKT are nec. and sufficient $\Rightarrow \underline{x}^*$ is optimal.