Imperial College London

Lecture 02: Point Estimation

Statistical Modelling I

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Outline

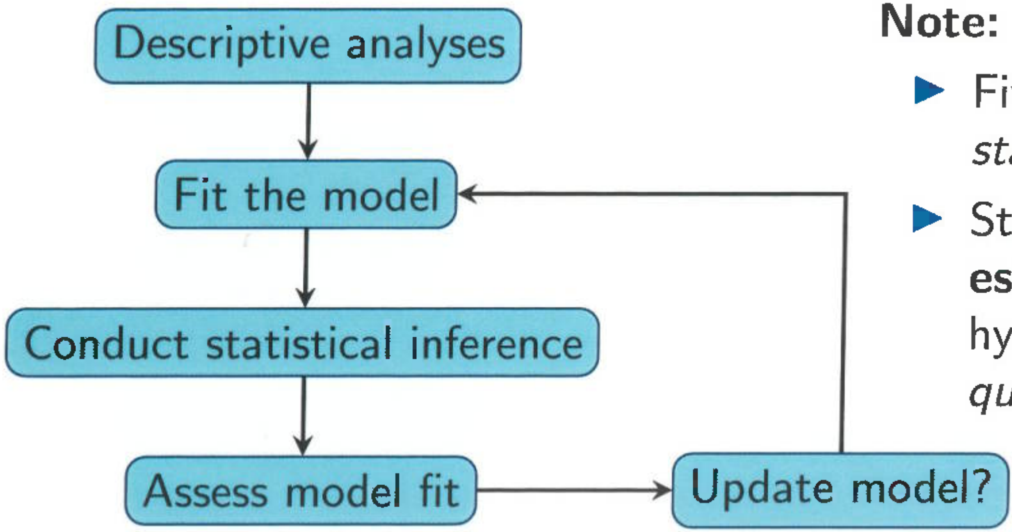
1. Point Estimation

2. Properties of Estimators

3. Worked Examples

Point Estimation

Statistical Analysis



Note:

- Fit the model $\Leftrightarrow$ **estimate** θ in the *statistical model*
- Statistical inference $\Leftrightarrow$ **point estimate**, interval estimate, hypothesis test to address *scientific question*

WE NEED TO FIND THE DISTRIBUTION P_θ WHICH "BEST DESCRIBES THE DATA"

Statistics, Estimates and Estimators

Data $y_1, \dots, y_n$ is one realisation of $Y_1, \dots, Y_n$.

DEFINITION

- **Statistic**: a function t of observable random variables
- **Estimate** (of θ): $t(y_1, \dots, y_n)$
- **Estimator** (of θ): $T = t(Y_1, \dots, Y_n)$

ESTIMATE $t(y_1, \dots, y_n) = t(Y_1(\omega), \dots, Y_n(\omega))$, $\omega \in \Omega$

Example: $Y_1, \ldots, Y_n$ iid $N(\theta, \sigma^2) \Rightarrow$ how to estimate θ?

Candidate Estimators

- Sample mean:

$$\frac{1}{n}\sum_{i=1}^{n} Y_i$$

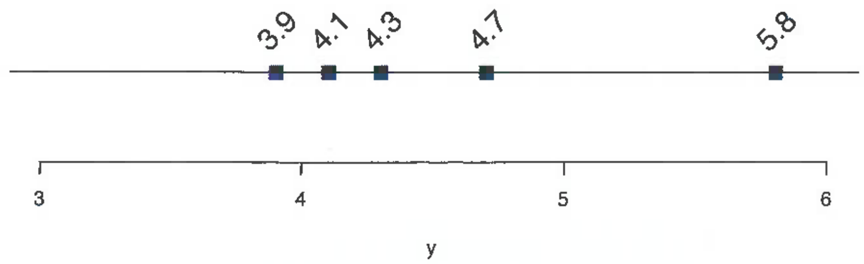
- Median (n odd):

$$Y_{(1)} < Y_{(2)} < \cdots < Y_{(n+1)/2} < \cdots < Y_{(n)}$$

- k-Trimmed mean:

$$\frac{1}{n-2k}\sum_{i=k+1}^{n-k} Y_{(i)}$$

Data

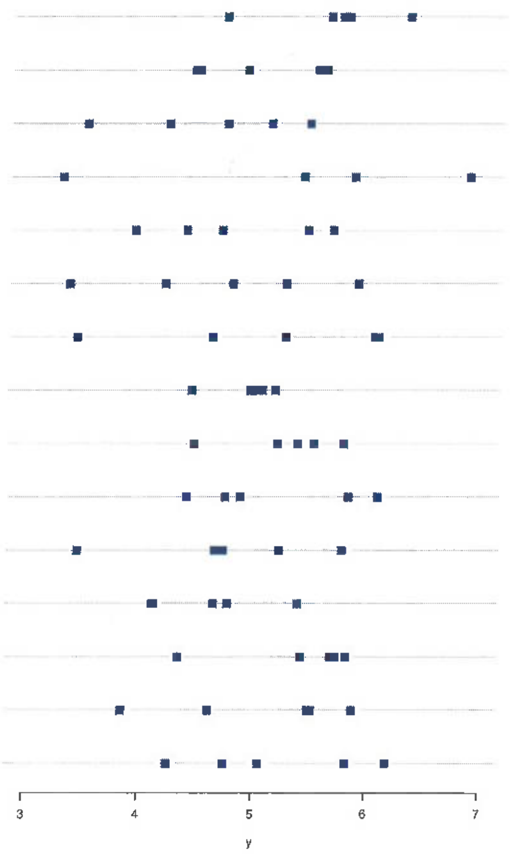


Candidate Estimates

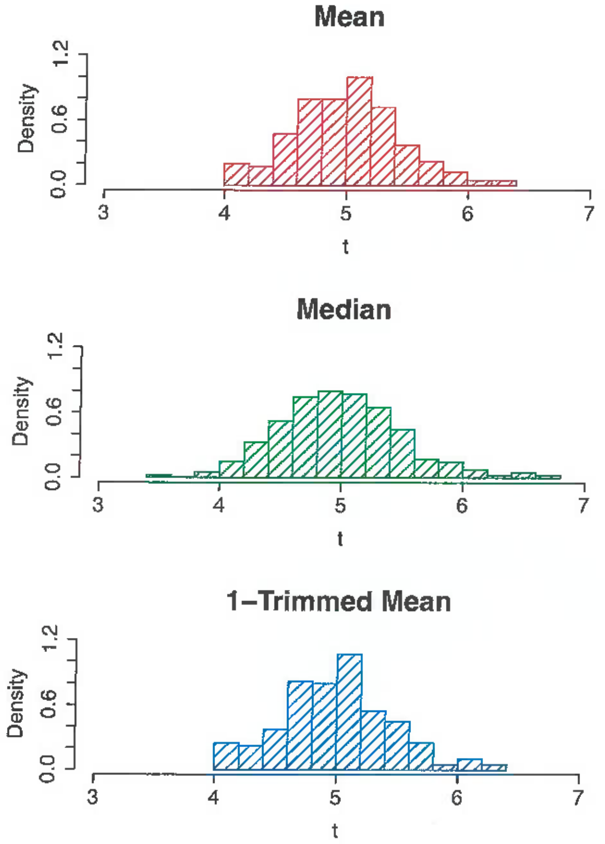
- Sample mean: $\frac{3.9+4.1+4.3+4.7+5.8}{5}$
- Median: 4.3
- 1-Trimmed mean: $\frac{4.1+4.3+4.7}{3}$

Example: $Y_1, \ldots, Y_n$ iid $N(\theta, \sigma^2) \Rightarrow$ repeat the experiment

New data sets ($n = 5$)



Sampling distributions of estimates



Properties of Estimators

Properties of estimators

Key idea: $T = t(Y_1, \ldots, Y_n)$ is a random variable and summaries of its sampling distribution

$$P_\theta(T \in \mathcal{A}) \qquad E_\theta(T) \qquad Var_\theta(T) \qquad etc \ldots$$

can be computed. Comparing different estimators means comparing the properties of their summaries.

Common properties of estimators:

- Bias
- Standard error
- Mean square error

Definition: Bias (general)

If $\Theta \subset \mathbb{R}^k$, $g(\theta)$ for $g : \Theta \to \mathbb{R}$ and T is an estimator of $g(\theta)$, then $bias_\theta(T) = E_\theta(T) - g(\theta)$

Example: $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$ iid, $\theta = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times (0, \infty)$ unknown

$g(\theta) = g((\mu, \sigma^2)) = \mu$

A CANDIDATE ESTIMATOR FOR $g(\theta)$ IS THE SAMPLE MEAN

Definition: Unbiased Estimator

If $bias_\theta(T) = 0$ for all $\theta \in \Theta$, then T is unbiased for $g(\theta)$

Example: $X \sim \text{Binomial}(n, p)$, $p \in [0, 1]$ unknown

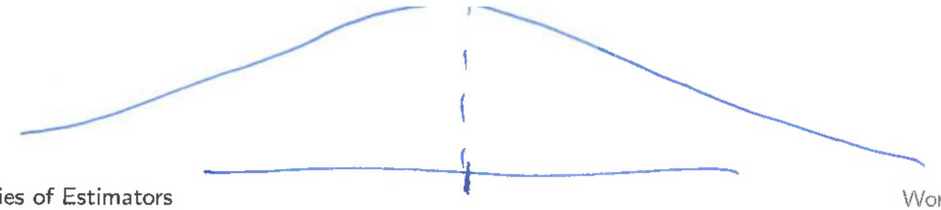
$$S = X/n \qquad T = \frac{X+1}{n+2}$$

$$E_p[S] = \frac{E_p[X]}{n} = \frac{np}{n} = p$$

$$E_p[T] = \frac{E_p[X]+1}{n+2} = \frac{np+1}{n+2}$$

$$bias_p(T) = E_p[T] - p = \frac{np+1}{n+2} - p = \frac{1-2p}{n+2}$$

$bias(T) = 0$ IFF $p = \frac{1}{2}$



Definition: Standard Error and MSE

Let T be an estimator for $\theta \in \Theta \subset \mathbb{R}$.

The standard error (SE) is the standard deviation of the sampling distribution of T:
$SE_\theta(T) = \sqrt{Var_\theta(T)} = \sqrt{E[(T-E[T])^2]}$

The mean square error (MSE) of T is defined by $MSE_\theta(T) = E_\theta[(T-\theta)^2]$.

WHEN
$MSE(T) = VAR(T)$? → WHEN THE ESTIMATOR IS UNBIASED

$MSE(T) = VAR(T) + BIAS(T)^2$

Worked Examples

Example: $Y_1, \ldots, Y_n$ iid with mean μ and variance σ^2 $\Rightarrow$ estimating $\theta = (\mu, \sigma^2)$ $\Theta = (\mathbb{R}, [0, \infty))$

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$E[\bar{Y}] = \mu, \quad \forall \mu \in \mathbb{R}$

Q: IS $\bar{Y}$ AN ESTIMATOR FOR μ^2?

$VAR(\bar{Y}) = VAR\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n}$

$E[\bar{Y}^2] = VAR(\bar{Y}) + E[\bar{Y}]^2 = \frac{\sigma^2}{n} + \mu^2 \neq \mu^2$

$\cancel{E[Y_i^2] = VAR(Y_i) + E[Y_i^2]}$

$E[Y_i^2] = VAR(Y_i) + E[Y_i]^2 = \sigma^2 + \mu^2$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$E[Y_i^2] - 2E[Y_i\bar{Y}] + E[\bar{Y}^2]$

$E[s^2] = \frac{1}{n-1}\sum_{i=1}^{n} E[(Y_i - \bar{Y})^2] =$

$= \frac{1}{n-1}\sum_{i=1}^{n} E[(Y_i)^2] - n E[\bar{Y}^2]$

$= \frac{1}{n-1}\sum_{i=1}^{n} \sigma^2 + \mu^2 - n\left(\frac{\sigma^2}{n} + \mu^2\right)$

$= \frac{1}{n-1} n\left(\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2\right)$

$= \frac{1}{n-1} n\left(\frac{\sigma^2(n-1)}{n}\right) = \sigma^2$

Example: $X \sim \text{Binomial}(n, p) \Rightarrow$ estimating p

$$S = X/n \qquad\qquad T = \frac{X+1}{n+2}$$

$$MSE(S) = VAR(S) = \frac{VAR(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

$$MSE(T) = VAR(T) + BIAS(T)^2 = \frac{np(1-p)}{(n+2)^2} + \frac{(1-2p)^2}{(n+2)^2}$$

IF $p=0$ OR $p=1$

$$MSE(S) = 0 < MSE(T)$$

IF $p = \frac{1}{2}$ THEN

$$MSE(S) = \frac{1}{4n} > MSE(T) = \frac{n}{4(n+2)^2}$$