IMPERIAL

MATH70063

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebra 4

Date: Friday, May 2, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Let R be a ring. Provide a counterexample to each of the following false assertions:

(i) Let M be a flat R-module and $N \subset M$ a submodule. Then N is flat.

(5 marks)

(ii) Let M be a projective R-module and $N \subset M$ a submodule. Then N is projective.

(5 marks)

(iii) Let M be an injective R-module and $N \subset M$ a submodule. Then N is injective.

(5 marks)

(b) Show that every R-module admits a projective resolution. (5 marks)

(Total: 20 marks)

2. Let A and B be abelian groups. Let A_{tors} and B_{tors} denote their respective torsion subgroups.

(a) If A is torsion and B is divisible, show that

$$A \otimes_{\mathbb{Z}} B = 0\,.$$

(5 marks)

(b) Show that

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \cong \mathrm{Tor}_1^{\mathbb{Z}}(A_{\mathrm{tors}}, B_{\mathrm{tors}})\,.$$

(5 marks)

(c) Using parts (a) and (b) or otherwise, show that

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A_{\mathrm{tors}}\,.$$

(5 marks)

(d) Is it true that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A_{\mathrm{tors}}, B) \cong \mathrm{Ext}^1(A, B)\,?$$

Justify your answer. (5 marks)

(Total: 20 marks)

3. (a) Let R be a principal ideal domain and $I \subset R$ a non-zero ideal. Prove that R/I is an injective R/I-module.

(5 marks)

(b) Let k be a field. Consider k as a $k[x]/(x^2)$-module via the map $k[x]/(x^2) \to k$ sending $x \mapsto 0$.

(i) Write down a projective resolution of k by $k[x]/(x^2)$-modules. (5 marks)

(ii) Using part (a) or otherwise, write down an injective resolution of k by $k[x]/(x^2)$-modules. (5 marks)

(iii) Using part (i) or part (ii) or otherwise, compute

$$\mathrm{Ext}^i_{k[x]/(x^2)}(k, k)$$

for all $i \geq 0$. (5 marks)

(Total: 20 marks)

4. (a) Let $C_2 = \{1, \sigma\}$ be the group with two elements. Show that

$$\cdots \to \mathbb{Z}[C_2] \xrightarrow{\sigma-1} \mathbb{Z}[C_2] \xrightarrow{\sigma+1} \mathbb{Z}[C_2] \xrightarrow{\sigma-1} \mathbb{Z}[C_2]$$

together with the surjection $\mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}$ sending $n + m\sigma \mapsto n + m$ is a projective resolution of the trivial module $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module, and hence compute $H^i(C_2, \mathbb{Z})$ for all $i \geq 0$, where the C_2-action on $\mathbb{Z}$ is given by $\sigma = -1$. (7 marks)

(b) Let G be a finite cyclic group and let A be a finite $\mathbb{Z}[G]$-module. Show that

$$|H^n(G, A)| = |H^{n+1}(G, A)|$$

for all $n \geq 1$. (5 marks)

(c) Let S_3 be the symmetric group on three elements, and let A be the $\mathbb{Z}[S_3]$-module given by $\mathbb{Z}^3$ with S_3 acting by permuting the coordinates. Compute $H^1(S_3, A)$.

(8 marks)

(Total: 20 marks)

5. (a) Let I be the category whose objects are positive integers, and the set $\mathrm{Hom}_I(m,n)$ is empty unless $m \geq n$, in which case there is precisely one morphism $m \to n$. In this question we consider the left exact functor

$$\varprojlim : \mathrm{AbGrps}^I := \mathrm{Fun}(I, \mathrm{AbGrps}) \to \mathrm{AbGrps}$$

and its right derived functors $R^i \varprojlim$.

(i) Let p be a prime number. Consider the object $A_\bullet = (A_i, \varphi_{ji} : A_j \to A_i)$ in AbGrps^I given by

$$\cdots \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}\,.$$

Show that $\varprojlim A_\bullet = 0$. (7 marks)

(ii) Give an example of an object $A_\bullet = (A_i, \varphi_{ji} : A_j \to A_i)$ in AbGrps^I such that

$$R^1 \varprojlim A_\bullet \neq 0\,.$$

[You may use without proof the fact that $R^1 \varprojlim B_\bullet = 0$ if $B_\bullet$ satisfies the Mittag-Leffler condition]. (7 marks)

(b) Let R be a ring. Prove that the category R-Mod of R-modules has all finite colimits (that is, every finite diagram in R-Mod has a colimit in R-Mod). You may use without proof that R-Mod has direct sums. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70063

Algebra 4 (Solutions)

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1. (a) I heard each of these false statements claimed to be true in student conversations!

part seen ⇓

5, A

(i) Let $R = \mathbb{Z}/4\mathbb{Z}$. Then $R = M$ is a flat R-module but the submodule $N = 2\mathbb{Z}/4\mathbb{Z}$ is not flat. Indeed, tensor the injection $N \hookrightarrow R$ to get an induced map

$$N \otimes_R N \to N\,.$$

If N were flat then this would be an injection. It is clearly the trivial map (it sends $a \otimes b$ to ab), so N would have to be $\{0\}$. But

$$N \otimes_R N = 2\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/4\mathbb{Z}} 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \neq \{0\}\,.$$

part seen ⇓

5, A

(ii) Indeed, consider again $R = \mathbb{Z}/4\mathbb{Z}$, $M = R$ and $N = 2\mathbb{Z}/4\mathbb{Z}$. Then the map $\mathbb{Z}/4\mathbb{Z} \to 2\mathbb{Z}/4\mathbb{Z}$ does not split, so N is not projective.

seen ⇓

5, A

(iii) We showed that *every* R-module N is isomorphic to a submodule of an injective module. So just choose any module N which is not injective. For example $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module (because it is not divisible, say).

seen ⇓

5, A

(b) First recall that any module M is a quotient of the free module F freely generated as an R-module by the elements of M. In other words, F consists of $(r_m)_{m\in M}$, where each $r_m \in R$ and only finitely many r_m can be non-zero. There is a natural map of R-modules $F \to M$ sending $(r_m)_{m\in M} \in F$ to $\sum_{m\in M} r_m m \in M$. So apply this to our given projective module M to see there is a surjective map $\epsilon : P_0 \to M$, where P_0 is free, hence projective. Define $M_0 = \ker(\epsilon)$. Take a surjective map $P_1 \to M_0$ with P_1 projective. Now let $d : P_1 \to P_0$ be the composition $P_1 \to M_0 \to P_0$. It is easy to see that $\mathrm{im}[P_1 \to P_0] = M_0 = \ker[P_0 \to M]$. This gives the first bit of our resolution and proves the exactness at P_0. Now define $M_1 = \ker[P_1 \to P_0]$ and repeat the procedure to construct P_2, and so on.

2. (a) Suppose that A is torsion and B is divisible. Let $a \otimes b \in A \otimes_{\mathbb{Z}} B$. Choose an integer n such that $na = 0 \in A$. Since B is divisible, there exists some $c \in B$ such that $b = nc$. Hence

meth seen ⇓

5, A

$$a \otimes b = a \otimes nc = na \otimes c = 0 \otimes c = 0 \in A \otimes_{\mathbb{Z}} B\,.$$

seen ⇓

5, A

(b) Consider the short exact sequence

$$0 \to A_{\text{tors}} \to A \to A/A_{\text{tors}} \to 0\,.$$

Doing $- \otimes_{\mathbb{Z}} B$ gives a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, B) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \to \mathrm{Tor}_1^{\mathbb{Z}}(A/A_{\text{tors}}, B) \to \cdots\,.$$

Since A/A_{tors} is torsion-free it is flat (because $\mathbb{Z}$ is a PID), so $\mathrm{Tor}_1^{\mathbb{Z}}(A/A_{\text{tors}}, B) = 0$. Therefore

$$\mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, B) \cong \mathrm{Tor}_1^{\mathbb{Z}}(A, B)\,.$$

The same argument gives

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B_{\text{tors}}) \cong \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

and hence

$$\mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, B_{\text{tors}}) \cong \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

as desired.

meth seen ⇓

5, B

(c) Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0\,.$$

Doing $A_{\text{tors}} \otimes_{\mathbb{Z}} -$ gives a long exact sequence

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, \mathbb{Q}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, \mathbb{Q}/\mathbb{Z}) \to A_{\text{tors}} \to A_{\text{tors}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \cdots\,.$$

Since $\mathbb{Q}$ is torsion-free it is flat (because $\mathbb{Z}$ is a PID), so $\mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, \mathbb{Q}) = 0$. $A_{\text{tors}} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ by part (a) because A_{tors} is torsion and $\mathbb{Q}$ is divisible. So the sequence gives

$$\mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, \mathbb{Q}/\mathbb{Z}) \cong A_{\text{tors}}\,.$$

But we saw in part (b) that

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Tor}_1^{\mathbb{Z}}(A_{\text{tors}}, \mathbb{Q}/\mathbb{Z})\,.$$

unseen $\Downarrow$

5, D

(d) This is not true. An slick solution is to note that $\mathbb{Q}$ is not projective, so there must exist some abelian group A such that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, A) \neq 0$.

An explicit counterexample can be given as follows: $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_{\mathrm{tors}}, \mathbb{Z}) = 0$ because $\mathbb{Q}$ is torsion-free, but I claim that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq 0$. There are a few ways to see this. In general it is a bit tricky, so I expect only the most able students will be able to get these marks (they should be able to do it if they have read either one of the two references for the course, and/or studied the tutorial on endomorphisms of $\mathbb{Q}/\mathbb{Z}$). Some of the steps have been seen or similar things have been seen.

Anyway, here is one way to see $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq 0$: Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \,.$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ gives a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to &\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \\ &\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \cdots \,. \end{aligned}$$

It is easy to see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. Moreover $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$ (because $\mathbb{Z}$ is a projective $\mathbb{Z}$-module!). So we find that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})/\mathbb{Z} \,.$$

Now, taking $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, -)$ on the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \,.$$

gives a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) &\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \\ &\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) \to \cdots \,. \end{aligned}$$

We have $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$ because $\mathbb{Q}/\mathbb{Z}$ is torsion and $\mathbb{Q}$ is torsion-free. We also have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$ because $\mathbb{Q}$ is injective (it is divisible and $\mathbb{Z}$ is a PID, so it is injective). Hence

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

so

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})/\mathbb{Z} \,.$$

We saw in class that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \hat{\mathbb{Z}}$. Hence

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \hat{\mathbb{Z}}/\mathbb{Z} \neq 0 \,.$$

3. (a) Use Baer's criterion. Let I be a non-zero ideal of R. Then $I = (r)$ for some non-zero $r \in R$. A slick way to proceed using what they have seen in a coursework question is to let $K = \operatorname{Frac} R$ and note that K/R is an injective R-module (K is divisible since it is a field, and quotients of divisible modules are divisible). Hence $(K/R)[r] = \frac{1}{r}R/R \cong R/(r)$ is an injective $R/(r)$-module. Probably the way that students will do it is to apply Baer's criterion directly: let $J \subset R/(r)$ be an ideal and suppose we have an $R/(r)$-module map $f : J \to R/(r)$. We want to show that extends to an $R/(r)$-module map $g : R/(r) \to R/(r)$. Well, $J = (s) = sR/(r)$ for some $s|r$, and the map f is completely determined by the value $f(s) \in R/(r)$. Since f is an $R/(r)$-module map, we must have $(r/s) \cdot f(s) = 0$, i.e. $s|f(s)$. But then the map $g : 1 \mapsto f(s)/s$ extends f to all of $R/(r)$.

meth seen $\Downarrow$

$\boxed{5, \mathrm{C}}$

(b) (i) We have seen (as a special case of something in lectures) that

$$\xrightarrow{\times x} k[x]/(x^2) \xrightarrow{\times x} k[x]/(x^2) \to 0$$

together with the map $k[x]/(x^2) \twoheadrightarrow k$ is a projective resolution of k as a $k[x]/(x^2)$-module.

seen $\Downarrow$

$\boxed{5, \mathrm{A}}$

(ii) Using part (a), $k[x]/(x^2)$ is an injective module over itself. Then the complex

$$\xrightarrow{\times x} k[x]/(x^2) \xrightarrow{\times x} k[x]/(x^2) \to \cdots$$

together with the injection $k \hookrightarrow k[x]/(x^2)$ is an injective resolution of k as a $k[x]/(x^2)$-module.

meth seen $\Downarrow$

$\boxed{5, \mathrm{B}}$

(iii) Using part (i) (or part (ii)), the groups $\operatorname{Ext}^i_{k[x]/(x^2)}(k,k)$ are the cohomology groups of the complex

$$0 \to \operatorname{Hom}_{k[x]/(x^2)}(k[x]/(x^2), k) \to \operatorname{Hom}_{k[x]/(x^2)}(k[x]/(x^2), k) \to \cdots$$

But each of these maps is the zero map, so

$$\operatorname{Ext}^i_{k[x]/(x^2)}(k,k) \cong \operatorname{Hom}_{k[x]/(x^2)}(k[x]/(x^2), k)$$

for all $i \geq 0$. One can also notice $\operatorname{Hom}_{k[x]/(x^2)}(k[x]/(x^2), k) \cong k$ to get

$$\operatorname{Ext}^i_{k[x]/(x^2)}(k,k) \cong k$$

for all $i \geq 0$ for a cleaner answer.

meth seen $\Downarrow$

$\boxed{5, \mathrm{B}}$

4. (a) The sequence is a complex because $(\sigma-1)(\sigma+1)=\sigma^2-1=1-1=0$. $\mathbb{Z}[C_2]$ is obviously a projective $\mathbb{Z}[C_2]$-module. The only thing to check is that the cohomology is trivial everywhere apart from degree zero where it is $\ker(\mathbb{Z}[C_2]\twoheadrightarrow\mathbb{Z})$. Let $n+m\sigma\in\mathbb{Z}[C_2]$. Then $(\sigma\pm 1)(n+m\sigma)=(n\pm m)(\sigma\pm 1)$. Hence meth seen ⇓ 7, B

$$\ker(\sigma\pm 1)=\{n+m\sigma \mid n=\mp m\}=\mathbb{Z}(\sigma\mp 1)$$

and $\mathrm{im}(\sigma\pm 1)=\mathbb{Z}(\sigma\pm 1)$. Hence the cohomology vanishes everywhere in degrees ≥ 1 and is $\ker(\mathbb{Z}[C_2]\twoheadrightarrow\mathbb{Z})$ in degree 0.
Suppose that σ acts as -1. Then $H^i(C_2,\mathbb{Z})$ is the cohomology of

$$\mathbb{Z}\xrightarrow{\times -2}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{\times -2}\cdots$$

so we get

$$H^i(C_2,\mathbb{Z})=\begin{cases}0 & \text{if } i \text{ is even}\\ \mathbb{Z}/2\mathbb{Z} & \text{if } i \text{ is odd.}\end{cases}$$

unseen ⇓

(b) Let $\sigma\in G$ be a generator. Then the exact sequence of finite groups 5, C

$$0\to A^G\to A\xrightarrow{1-\sigma}A\to A_G\to 0$$

gives $|A^G|=|A_G|$. Now consider the exact sequence of finite groups

$$0\to\ker N/(1-\sigma)A\to A_G\xrightarrow{N}A^G\to A^G/NA\to 0\,.$$

Since $|A^G|=|A_G|$ we get that

$$|\ker N/(1-\sigma)A|=|A^G/NA|,.$$

But we saw in lectures that

$$H^n(G,A)\cong\ker N/(1-\sigma)A$$

if $n\geq 1$ is odd, and

$$H^n(G,A)\cong A^G/NA$$

if $n\geq 2$ is even.

sim. seen ⇓

(c) S_3 has $\mathbb{Z}/3\mathbb{Z}$ as a normal subgroup and the quotient is $\mathbb{Z}/2\mathbb{Z}$. So we have an inf-res sequence is 8, D

$$0\to H^1(\mathbb{Z}/2\mathbb{Z},A^{\mathbb{Z}/3\mathbb{Z}})\to H^1(S_3,A)\to H^1(\mathbb{Z}/3\mathbb{Z},A)\,.$$

We have seen that

$$H^1(\mathbb{Z}/3\mathbb{Z},A)\cong\ker N/(1-\sigma)A$$

where $\sigma\in\mathbb{Z}/3\mathbb{Z}$ is a generator. Now,

$$\ker N=\{(x,y,z)\in\mathbb{Z}^3\mid x+y+z=0\}$$

Rearranging a little gives $\ker N=(1-\sigma)A$, so $H^1(\mathbb{Z}/3\mathbb{Z},A)=0$. Hence

$$H^1(S_3,A)\cong H^1(\mathbb{Z}/2\mathbb{Z},A^{\mathbb{Z}/3\mathbb{Z}})\,.$$

But $A^{\mathbb{Z}/3\mathbb{Z}}=\{(x,y,z)\in\mathbb{Z}^3\mid x=y=z\}\cong\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ acting trivially. Hence

$$H^1(S_3,A)\cong H^1(\mathbb{Z}/2\mathbb{Z},A^{\mathbb{Z}/3\mathbb{Z}})\cong H^1(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})\cong\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})=0\,.$$

part seen $\Downarrow$

5. (a) (i) Part of the mastery reading material (§3.5 in Weibel) gives that

7, M

$$\varprojlim A_\bullet \cong \ker\left(\Delta : \prod_{i=1}^{\infty} A_i \to \prod_{i=1}^{\infty} A_i\right)$$

where $\Delta(\dots, a_i, \dots, a_1) = (\dots, a_i - \varphi_{i+1,i}(a_{i+1}), \dots, a_1 - \varphi_{2,1}(a_2))$.
In our case this gives that

$$\varprojlim A_\bullet = \{(a_1, a_2, \dots) \in \prod_{i=1}^{\infty} A_i \,|\, a_i = pa_{i+1} \text{for all } i\}.$$

So every coordinate a_i of every element in $\varprojlim A_\bullet$ must be divisible by p^n for every $n \geq 1$. The only such number is 0 (!!!). Hence

$$\varprojlim A_\bullet = 0.$$

part seen $\Downarrow$

(ii) Consider the object $A_\bullet$ from part (a) given by

7, M

$$\cdots \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}.$$

This sits in a short exact sequence

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

in AbGrps^I, where $B_\bullet$ is

$$\cdots \xrightarrow{=} \mathbb{Z} \xrightarrow{=} \mathbb{Z} \xrightarrow{=} \mathbb{Z}$$

and $C_\bullet$ is

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}.$$

Taking $\varprojlim -$ of this short exact sequence gives a long exact sequence

$$0 \to \varprojlim A_\bullet \to \varprojlim B_\bullet \to \varprojlim C_\bullet \to R^1 \varprojlim A_\bullet \to R^1 \varprojlim B_\bullet \to \cdots$$

This sequence is therefore

$$0 \to \mathbb{Z} \to \mathbb{Z}_p \to R^1 \varprojlim A_\bullet \to 0.$$

(the 0 at the end is by the Mittag-Leffler condition, the 0 at the beginning is by part (a), and the other two limits are by book work/lectures). Hence $R^1 \varprojlim A_\bullet \cong \mathbb{Z}_p/\mathbb{Z} \neq 0$.

part seen $\Downarrow$

(b) Let $A : I \to R\text{-Mod}$ be a functor where I has only finitely many objects. Write $A_i := A(i)$ for $i \in \mathrm{ob}\, I$. We saw in lectures that $\oplus_{i \in I} A_i$ exists in R-Mod. It is book work that $\varinjlim_{i \in I} A_i$ is the cokernel of

6, M

$$\bigoplus_{\varphi : i \to j} A_i \to \bigoplus_{i \in I} A_i \,;\, a_i[\varphi] \mapsto \varphi(a_i) - a_i.$$

Review of mark distribution:

Total A marks: 35 of 32 marks

Total B marks: 22 of 20 marks

Total C marks: 10 of 12 marks

Total D marks: 13 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70063 Algebra 4 Markers Comments

Question 1	A common mistake was to assert that Z/2Z is a submodule of Z. The construction of a projective resolution was done perfectly by the vast majority.
Question 2	Most students answered the (easier) first three parts very well. The hard question at the end involving Ext was found to be quite difficult.
Question 3	Baer's criterion was applied very well by the vast majority. The resolutions were proved well too (a special case of something in lectures.) The vast majority knew how to compute the Ext-groups using these resolutions, but a few sign errors crept in here and there.
Question 4	The vast majority answered (a) very well. This was similar to Q3. The harder question here was (b), since we hadn't seen it in lectures. Many students struggled, but some answered very well. We had seen the S3 calculation in class (where we called S3 the dihedral group D3). Most students knew to use inf-res, and most were able to do so well.
Question 5	The mastery question was very polarising: it was answered perfectly by some students and very poorly by others. A lot of students tried to write down what they knew about limits/colimits, which is a good idea when you are stuck, and earned a few marks doing so.