Network Science
Spring 2024
Review exercises (answers)

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1. (a) $\binom{N_b-1}{k}p^k(1-p)^{N_b-1-k}$. This is just the degree distribution for the standard G_{Np} model.
   (b) $q(N_b) + p(N_r - 1)$
   (c) qN_rN_b
   (d) $p > log(N_r)/N_r$
   (e) Let l_{br} be the number of links connecting blue and red nodes. Then $P(l_{br} = 0) = (1-q)^{(N^2)}$ where $N = N_r = N_b$. Then, if $q > 0$, we have $lim_{N\to\infty}P(l_{br} = 0) = lim_{N\to\infty}r^{(N^2)} = 0$ where $r = 1-q$, and the limit goes to zero since $0 \leq r < 1$. So there will be at least one link between the blue and red subnetworks, while the condition on p ensures the subnetworks themselves are connected.

2. (a) $\sum_{j=1}^{N}\left[a + k_j(t)\right]/c = 1$, so $c = aN + K(t)$, and since $K(t) = 2t$, we have $c = aN + 2t$
   (b) Let $N' = \binom{N}{2}$. There are $\binom{N'}{2}$ graphs with two distinct links and $\binom{N}{2}$ graphs with a single multiedge.
   (c) The graph will have zero or one multiedge after two iterations. The probability of having one multiedge is the probability of 1) distinct nodes being chosen during the first iteration and 2) the same two nodes being chosen during the second iteration as were chosen during the first. The probability of not generating a self-loop during the first iteration is $\frac{N-1}{N}$. The probability of choosing the same two nodes again is, $\frac{2(a+1)}{Nc(t=1)}$, so the probability of generating one multiedge which is also the expected number of multiedges is, $\frac{N-1}{N}\frac{2(a+1)}{N(aN+2)}$.
   (d) $\langle N_k(t+1)\rangle = \langle N_k(t)\rangle + \langle N_{k-1}(t)\rangle\left[\frac{1}{N} + \frac{(a+k-1)}{c(t)}\frac{(N-2)}{N}\right] - \langle N_k(t)\rangle\left[\frac{1}{N} + \frac{(a+k)}{c(t)}\frac{(N-1)}{N}\right] + \langle N_{k-2}(t)\rangle\left[\frac{(a+k-2)}{c(t)N}\right]$

3. (a) $D = 2$
   (b) $\overline{C} = 31/35$
   (c) $C_g = 3/7$
   (d) Let $N' = \binom{7}{2}$. Then, $P_G = p^9(1-p)^{N'-9}$
   (e) i. The expected number of links is $\frac{k_1k_2}{K-1} = \frac{2*6}{17} = 12/17$

       ii. Let $\gamma_{2j} = 1$ indicate the existence of a multiedge between nodes 2 and j. From PS4, $P(\gamma_{2,j} = 1) = k_j(k_j-1)/[(K-1)(K-3]$ with $K = 18$. The probability of node 2 not having a multiedge is, $1 - \sum_{j\neq 2}P(\gamma_{2,j} = 1) = 43/51$.

(f) A pair of nodes including node 1 has at most 1 common neighbor while node 1 has a larger degree than nodes 6 and 7 which also have 1 common neighbor. Since any node paired with node 1 will have the same degree as nodes 6 and 7, nodes 6 and 7 will have a larger cosine similarity.

(g) $x_1 = 2, x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 1$.

(h) i. $\overline{C} \to 1$

ii. $2\pi_l = 1 - \pi_{l-1}$. $\pi_\infty = 1/3 = k_1/K$.

4. (a) Let $z = \langle x_2 - x_1 \rangle$. Then, $z = z(t=0)e^{-\beta t}$.

(b) $\langle x_i(t) \rangle = \frac{x_0 e^{\beta(N-1)t}}{1 - x_0 + x_0 e^{\beta(N-1)t}}$ where $x_0 = \langle x_i(t=0) \rangle$.

(c) Let $\langle x_i \rangle = 1 - \epsilon z_i$. Then, $z_i(t) = z_i(0)e^{-\beta k_i t}$.

(d) i. The graph can be represented as a square with its corners corresponding to the four nodes.

ii. $\mathbf{A}\mathbf{v}_\omega = [\omega^{N-1} + \omega, 1 + \omega^2, \omega + \omega^3, ..., \omega^{N-3} + \omega^{N-1}, 1 + \omega^{N-2}]^T = (\omega + \omega^{-1})\mathrm{v}_\omega$ since $\omega^{-1} = \omega^{N-1}$ and $\omega^N = 1$.

iii. The eigenvalues can be written as $\lambda_j = 2cos(2\pi j/N), j = 1, ..., N$, and the general solution is $\mathbf{z}(t) = c_1\mathbf{v}_1 exp(\beta\lambda_1 t) + c_2\mathbf{v}_2 exp(\beta\lambda_2 t) + ... + c_N\mathbf{v}_N exp(\beta\lambda_N t)$ where $\mathbf{v}_i$ is the eigenvector corresponding to eigenvalue λ_i

iv. Set the initial condition to $\mathbf{z}(t=0) = [1 \quad -1 \quad 1 \quad -1 \quad ... \quad 1 \quad -1]^T$ This is an eigenvector with $\omega = -1$ and eigenvalue $\lambda = -2$. Then, it follows from the general solution that the solution t will be of the required form with $a = \exp(-2)$.

(e) $term\ 2 = A_{im} \langle (1 - x_i)x_j x_l x_m \rangle$, and $term\ 3 = A_{jm} \langle x_i(1 - x_j)x_l x_m \rangle$ These terms are related to disease transmission from node m to the node with the $(1 - x)$ term.

5. (a) i. $M = \frac{n}{K}\left[\eta(\eta - 1) - 2/n - \frac{1}{n}\eta(\eta - 1)\right]$ with $K = n\left[\eta(\eta - 1) + 2\right]$

ii. $n < \eta(\eta - 1) + 2$

(b) The matrix is symmetric, so its eigenvalues are real. It always has a zero eigenvalue, so the largest eigenvalue will be non-negative.

(c) $\mathbf{v}_4 = [1, 1, 1, 1]^T$

(d) The eigenvector corresponding to the smallest positive eigenvalue ($\lambda_3 = 1$) is used to construct the partition. Since the first two elements of the corresponding eigenvector, $\mathbf{v}_3$ are the same, nodes 1 and 2 will be placed in the same group.