15 Hyperbolic PDEs

15.1 First-order PDE

Consider a quasi-linear equation of the type

$$a\,\frac{\partial z}{\partial x} + b\,\frac{\partial z}{\partial y} = c, \tag{15.1}$$

where $a,\ b,\ c$ are functions of $x,\ y$. This can be simplified by changing the variables $x,\ y$ to $s,\ t$ so that Eqn. 15.1 becomes

$$\left(\frac{\partial z}{\partial t}\right)_s = c. \tag{15.2}$$

Suitable variables s and t are identified by comparing Eqn. 15.1 with the identity

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}, \tag{15.3}$$

provided s and t are chosen such that

$$\frac{\partial x}{\partial t} = a, \quad \text{and} \quad \frac{\partial y}{\partial t} = b, \tag{15.4}$$

in which case Eqn. 15.1 reduces to Eqn. 15.2.

The variable s is defined by noting that, if s is constant, Eqns. 15.4 gives a coordinate direction along which the PDE reduces to an *ordinary differential equation* (ODE)

$$\frac{dy}{dx} = \frac{\partial y/\partial t}{\partial x/\partial t} = \frac{b}{a}, \tag{15.5}$$

which can be integrated (in principle – at least). The "*constant*" of integration is an arbitrary function of s (since s has been held constant) and any convenient function can be chosen to define s. Hence

$$s = f(x, y), \tag{15.6}$$

where f is a known function which defines s explicitly. The variable t can be defined by integrating **either** of the expressions 15.4, *e.g.* the first of these gives

$$t = \int \frac{dx}{a} - g(s), \tag{15.7}$$

the integration carried out with s held constant; Eqn. 15.6 is re-arranged in the form

$$y = y(x, s)$$

and used to express $a(x, y)$ as $a(x, s)$. The arbitrary function of s in Eqn. 15.7 may be chosen for convenience; it is usually taken to be zero.

Equation 15.2 is then integrated, first of all expressing c as a function of s and t by means of Eqns. 15.6 and 15.7,

$$z = \int c\,dt + h(s). \tag{15.8}$$

The integration is carried out keeping s constant. The arbitrary function $h(s)$ is determined by inserting the given boundary condition into Eqn. 15.8.

Observe, generally we have the **characteristic** relations

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}. \tag{15.9}$$

We may choose what approach and variable to use, based on ease of integration of these **characteristic** variables. As an example, or summary, note

$$\frac{dz}{dy} = \frac{c}{b}, \quad \frac{dz}{dx} = \frac{c}{a} \text{ along } \frac{dy}{dx} = \frac{b}{a}.$$

15.1.1 Example solution to first-order PDE

Consider the equation

$$y\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 2,$$

where U is known along an initial segment Γ along $y = 0$ and $0 \le x \le 1$ as indicated in Fig. 15.1. The PDE reduces to an ODE on the characteristic curve

$$\frac{dy}{dx} = \frac{1}{y}, \tag{15.10}$$

along which the solution is given by

$$\frac{dU}{dy} = 2. \tag{15.11}$$

Integration of Eqn. 15.10 gives $x = y^2/2 + A$ where A is a constant for each characteristic. So for a characteristic through $R(x_r, 0)$, $A = x_r$. Thus the equation defining this path is

$$y^2 = 2(x - x_r)$$

and the solution on this characteristic will be given by $U = 2y + B$, with B a constant determined from the initial condition along $y = 0$ – if $U = U_r$ at $R(x_r, 0)$ then $B = U_r$, *i.e.*

$$U = 2y + U_r. \tag{15.12}$$

Since initial values of U are known along Γ or the segment P–Q, where $0 \le x_r \le 1$, it follows that the solution is only known in the region bounded by and including the terminating characteristics $y^2 = 2x$ and $y^2 = 2(x-1)$; outside this region (non-hatched area in Fig. 15.1) the solution will be undefined.

Along the $y^2 = 2x$ characteristic the solution U will be uniquely determined by the value of U_o at $P(0, 0)$, *i.e.* $U = 2y + U_o$. In other words the initial values for U on the initial curve $y^2 = 2x$ can not be arbitrarily prescribed, but is dependent upon the Γ initial data path defined*.

Importantly, observe that the initial data U_r prescribed along Γ need **not be smooth**, it may well be **discontinuous**, but the solution along the characteristic path $y^2 = 2(x - x_r)$ will still be uniquely defined to be $U = 2y + U_r$.

*see G.D. Smith for further details

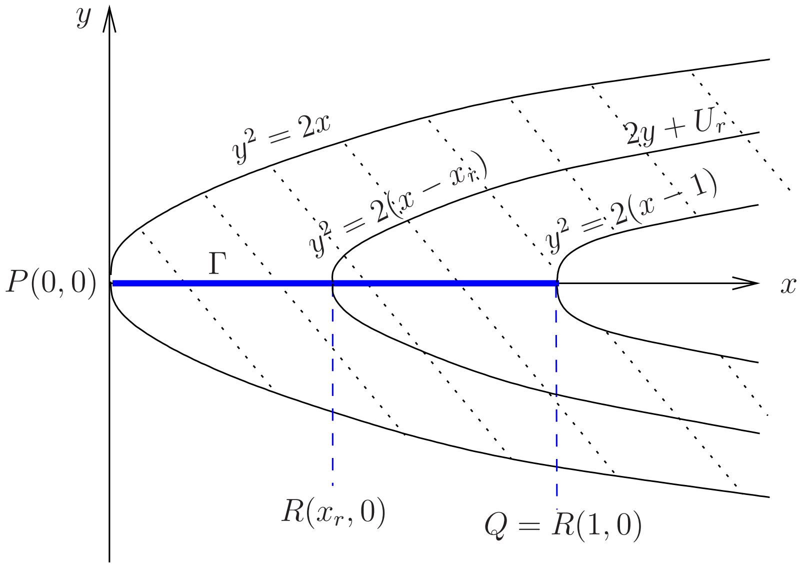

Figure 15.1: Solution domain, $\Gamma \in 0 \le x \le 1,\ y = 0$ (blue solid line) is the path along which the initial conditions are prescribed.

15.2 The one-dimensional linear advection equation and Upwinding

Consider the problem for $u(x,\, t)$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \ \text{ for } \ t > 0, \ \text{ initial condition } \ u(x,\, 0) = f(x)\ , \tag{15.13}$$

where c is a **positive** constant. The characteristics of this equation are

$$dx/dt = c \quad \text{or} \quad x - ct = \quad \text{constant.}$$

Furthermore, u is constant along each characteristic and so the solution is $u = f(x - ct)$. This represents a wave travelling in the positive x-direction with speed c, **without change of magnitude or shape** as shown in Figure 15.2 – we emphasis that the exact solution of Eqn. 15.13 says that the initial function $f(x)$ propagates unaltered in form in the positive x-direction as time progresses. The PDE simply translates as t progress, the profile $f(x)$ with velocity c to the right if $c > 0$ and to the left if $c < 0$.

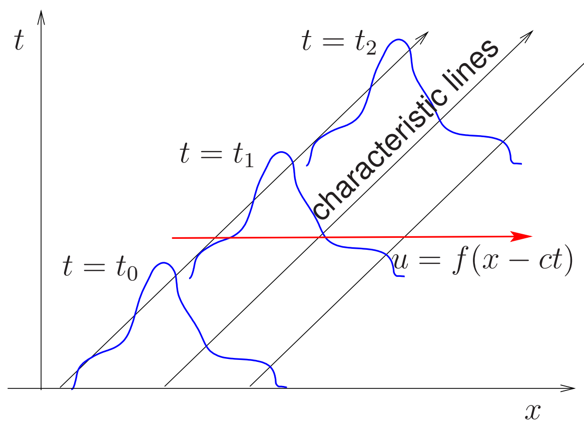

Figure 15.2: Propagation of a function $u = f(x)$ with time t.

Can we develop finite-difference methods (FDMs) with these properties? In the present context by this we mean, that our FDM reproduces precisely, or close to as possible, the expected behaviour of $f(x)$ propagating unaltered in form as time progresses.

For future reference, we observe that for a Fourier mode with $f(x) = \exp(ix/h) \equiv e^{in\xi}$. Here x/h is non-dimensional and may be viewed as a *wavenumber*; similarly (ck) has dimensions of a length, since c is a speed and k a time-scale, therefore $q = ck/h$ is non-dimensional and may too be associated as a wavenumber, in the following

$$u_n^{j+1} \equiv u(nh,\,(j+1)k) = u_n^j \exp(-i\xi q) \text{ where } q = \frac{ck}{h} > 0. \tag{15.14}$$

The value $q = ck/h$ is a very important dimensionless number called the **Courant number** (also referred to as the **Courant-Friedrichs-Lewy** (**CFL**)-condition). We shall see this plays an important role when considering the effectiveness of FD discretisations in hyperbolic problems. It may also be viewed as

$$\frac{ck}{h} \equiv \text{ ratio of } \frac{\textit{physical distance moved}}{\textit{grid-spacing}}.$$

The real solution therefore has a growth factor $\lambda \exp(-i\xi q)$. We note that for the model equation (15.13) $|\lambda| = 1$ so that there is no growth or decay. We can model our simple equation in many different ways. We shall use a forward difference for u_t and consider 6 schemes:

$$\frac{\partial u}{\partial t} \approx \frac{U_n^{j+1} - U_n^j}{-q} = \begin{cases} \textbf{A}: & \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \textbf{B}: & U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \textbf{C}: & U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \textbf{D}: & \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \textbf{E}: & \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \textbf{F}: & \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

(We shall also later consider Keller's 'Box scheme'). We use the Fourier method to investigate the stability of all of these schemes, looking for a solution $U_n^j = \lambda^j \exp(in\xi)$. We find:

$$\lambda = \begin{cases} \textbf{A}: & 1 - iq\sin\xi \\ \textbf{B}: & 1 - q(e^{i\xi} - 1) \\ \textbf{C}: & 1 - q(1 - e^{-i\xi}) \\ \textbf{D}: & \frac{1}{4}\left[iq\sin\xi \pm \sqrt{4 - q^2\sin^2\xi}\right]^2 & \text{(two steps)} \\ \textbf{E}: & \dfrac{2 - iq\sin\xi}{2 + iq\sin\xi} \\ \textbf{F}: & 1 - iq\sin\xi - 2q^2\sin^2\frac{1}{2}\xi \end{cases}$$

We recall that if

1. If $|\lambda| > 1 + O(k)$, the method is unstable.
2. If $|\lambda| < 1$, it is dissipative.

3. While if $|\lambda| = 1$, it is conservative. We mean (*loosely*) that the solution propagates in time without change, and is also exact as the $t-$variable progresses.

We see therefore that

$$|\lambda|^2 = \begin{cases} \textbf{A}: & 1 + q^2\sin^2\xi & & \text{stable only if } q^2 = O(k)\ i.e.\ k \sim h^2 \\ \textbf{B}: & 1 + 2q(1+q)(1-\cos\xi) & \geq 1 \quad \forall\, q,\xi & \text{hopelessly unstable} \\ \textbf{C}: & 1 - 2q(1-q)(1-\cos\xi) & \leq 1 \quad \forall\, q \leq 1 & \text{stable, dissipative} \\ \textbf{D}: & 1 & & \text{provided } q^2\sin^2\xi \leq 4 \quad \text{or} \quad \tfrac{1}{2}q \leq 1 \\ \textbf{E}: & 1 & \forall\, q & \text{stable and conservative} \\ \textbf{F}: & 1 - 4q^2(1-q^2)\sin^4\tfrac{1}{2}\xi & \leq 1 \text{ if } q \leq 1 & \text{dissipative, but less than } \textbf{C} \end{cases}$$

We may also be interested in $\arg\lambda$, which determines the **phase** of each mode. In case **E**, for example,

$$\arg\lambda = -2\tan^{-1}(\tfrac{1}{2}q\xi) \approx -q\xi \quad \text{when} \quad \xi \quad \text{is small.}$$

All the above schemes agree well with the exact solution, for which $\arg\lambda = -q\xi$, when ξ is small. The waves with higher values of ξ are **dispersive**; their phase velocity varies with frequency. A computer demonstration will illustrate the effects of this. Although the amplitude of each wave component may be conserved, phase differences develop which alter the shape of the whole.

We can see from the above the importance of the CFL condition, $q \leq 1$. The Courant or CFL number is a non-dimensional number that plays a central role in the numerical solution of hyperbolic equations. If c can be thought of a speed, $q = ck/h \equiv c\Delta t/\Delta x$, as alluded to earlier, can be thought of a distance measured in grid points that a particle or information reaches to, in an increment of time $k \equiv \Delta t$.

Another very important conclusion we can draw is that backwards, or **Upwind** differences are a good idea. Case **C** is stable provided the CFL condition holds, whereas **B** is unconditionally unstable. If $c < 0$, the stable scheme is **B**. In general, the **Upwind** scheme can be written

$$U_n^{j+1} = U_n^j - sc\Delta U_n^j + s|c|\delta^2 U_n^j \qquad \text{where} \quad s = \frac{k}{2h}. \tag{15.15}$$

The need for upwind differences can be interpreted in terms of Eqn. 15.13's characteristics, $x - ct =$ *constant*, which must pass through the **Numerical Domain of Dependence**.

Above we have a very simple equation with constant coefficients (*i.e. the* c). In this very special case, setting $q = 1$ for some of the discretisation schemes above, the numerical scheme reproduces the exact solution with no error. However in more complex hyperbolic systems with varying coefficients maintaining this *exactness* is not that straightforward and should not be expected, unless one goes to considerable effort towards honouring the true physical propagation paths or characteristics of the equations set.

15.3 Upwinding for simultaneous equations

Let $\mathbf{u}(x,\, t)$ be a p-dimensional vector satisfying the equation

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{d}$$

where A is a $p \times p$ matrix, which for simplicity we shall assume to be constant. We shall investigate how the upwinding method generalises to this problem, by **diagonalising** the matrix A, which we assume to have p distinct eigenvalues $\lambda_1, \ldots \lambda_p$ and corresponding eigenvectors. We make the linear transformation $\mathbf{u} = S\mathbf{v}$ where S is the matrix whose columns are the eigenvectors of A, so that

$$S^{-1}AS = D \equiv \mathrm{diag}(\lambda_1,\, \lambda_2, \ldots, \lambda_p) \qquad \text{and so} \quad A = SDS^{-1}\, .$$

Then v_i, the i-th component of $\mathbf{v}$, satisfies the equation

$$(v_i)_t + \lambda_i (v_i)_x = (S^{-1}\mathbf{d})_i\, .$$

We have thus separated the problem into p separate ones, each with its own characteristic family, $dx/dt = \lambda_i$, and we can use **upwinding** on each one in turn. Note that if any of the λ_i are complex, then some of the p equations are elliptic, and we cannot use a time-stepping approach for them. We assume here that all the λ_i are real. From (10.9), the upwind scheme for $\mathbf{v}$ is

$$\mathbf{V}_n^{j+1} = \mathbf{V}_n^j - sD\Delta\mathbf{V}_n^j + sD^+\delta^2\mathbf{V}_n^j + kS^{-1}\mathbf{d}_n^j\, ,$$

where $D^+ \equiv \mathrm{diag}(|\lambda_1|,\, |\lambda_2|, \ldots, |\lambda_p|)$, and $\mathbf{V}$ is the finite-difference approximation to $\mathbf{v}$. Then transforming back, defining $\mathbf{U} = S\mathbf{V}$ so that $\mathbf{V} = S^{-1}U$, we find

$$\mathbf{U}_n^{j+1} = \mathbf{U}_n^j - sA\Delta\mathbf{U}_n^j + sA^+\delta^2\mathbf{U}_n^j + k\mathbf{d}_n^j \ \text{ where } \ A^+ = SD^+S^{-1}\, . \tag{15.16}$$

This is the required generalisation of upwinding for p simultaneous equations. If all the eigenvalues of A are positive, then $D^+ = D$ and $A^+ = A$, while $A^+ = -A$ if they are all negative. Otherwise, A^+ bears no simple relation to A, and we must be very careful how we define “**up**" when upwinding. The stability condition for Eqn. 15.16 is

$$\frac{k}{h} \max_{i=1\ldots p} |\lambda_i| \leq 1\, . \tag{15.17}$$