

Section A

A.1. Consider the following two real matrices:

$$A := \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$$

Calculate $A^2 - B^2$ and $(A - B)(A + B)$.

The relevant parts of the notes for this question are Definitions 2.2.1, 2.2.2, and 2.2.3.

A.2. Solve the following system of simultaneous linear equations by finding the augmented matrix and applying row operations.

$$\begin{array}{ll} \text{(a)} & \begin{array}{rcl} x_1 - 2x_2 + x_3 - x_4 & = & 8 \\ 3x_1 - 6x_2 + 2x_3 & = & 18 \\ x_3 - 2x_4 & = & 5 \\ 2x_1 - 2x_2 + 3x_4 & = & 4 \end{array} & \text{(b)} & \begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 2 \\ 3x_1 - 8x_2 + 2x_3 & = & 5 \\ 2x_1 - 5x_2 + x_3 & = & 1 \end{array} \\ \text{(c)} & \begin{array}{rcl} x_1 - 2x_3 + x_4 & = & 0 \\ 2x_1 - x_2 + x_3 - 3x_4 & = & 0 \\ 4x_1 - 3x_2 - x_3 - 7x_4 & = & 4 \end{array} & \text{(d)} & \begin{array}{rcl} -x_2 + x_3 - 3x_4 & = & 0 \\ x_1 + 3x_2 + x_3 - x_4 & = & 0 \\ 2x_1 + 5x_2 + 3x_3 - 5x_4 & = & 0 \end{array} \end{array}$$

If you're having trouble, you may wish to review Example 2.3.4 from the notes.

A.3. Which of these matrices A is invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

If you're having trouble, you may wish to review Example 2.3.4 from the notes.

A.4. Prove Theorem 2.4.4. The key definition is 2.4.1.

A.5. Exercise 2.5.9. Theorem 2.5.8 will be useful here.

Seen B

B.1. Use Gaussian elimination to find every solution to the following equation:

$$\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

B.2. Let A be an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Suppose $Ax = b$ has at least one solution $x_0 \in \mathbb{R}^m$. Show all solutions are of the form $x = x_0 + h$, where h solves $Ah = 0$.

B.3. Let A and B be square $n \times n$ matrices with real entries. For each of the following statements, either give a **proof**, or find a **counterexample with** $n = 2$.

- (i) If $AB = 0$ then A and B cannot both be invertible.
- (ii) If A and B are invertible then $A + B$ is invertible.
- (iii) If A and B are invertible then AB is invertible.
- (iv) If A and B are invertible and $(AB)^2 = A^2B^2$, then $AB = BA$.
- (v) If $ABA = 0$ and B is invertible then $A^2 = 0$.
- (vi) If $ABA = I$ then A is invertible and $B = (A^{-1})^2$.
- (vii) If A has a left inverse B and a right inverse C then $B = C$.

B.4. Let $n \geq 2$ and let $A_n = (a_{ij})$ be the $n \times n$ matrix such that

$$\begin{aligned} a_{i-1,i} &= 1 \text{ for } i = 2, \dots, n, \\ a_{i+1,i} &= 1 \text{ for } i = 1, \dots, n-1, \end{aligned}$$

and $a_{ij} = 0$ for all other i, j . Write down A_2, A_3 and A_4 . Prove that A_n is invertible for all even values of n , and is not invertible for all odd values of n . Find A_2^{-1} and A_4^{-1} .