L10

$\mathcal{L}$

$\mathcal{A}$ $\mathcal{L}$-str.

v a valuation in $\mathcal{A}$

ϕ $\mathcal{L}$-fmla

v satisfies ϕ (in $\mathcal{A}$)
$v[\phi] = T$ } 2.2.7

$\mathcal{A} \models \phi$ ϕ is true in $\mathcal{A}$

$\models \phi$ ϕ logically valid

Also: if Γ is a set of $\mathcal{L}$-fmlas + ϕ an $\mathcal{L}$-fmla

$\Gamma \models \phi$

means: for every $\mathcal{L}$-str. $\mathcal{A}$ and valuation v in $\mathcal{A}$ if $v[\Gamma] = T$ then $v[\phi] = T$.

(2.2.8) Def: (Shorthand)

1) If ϕ is an $\mathcal{L}$-fmla and x_i is a variable then $(\exists x_i)\phi$ is shorthand for $(\neg(\forall x_i)(\neg\phi))$

2) As before, $(\phi \vee \psi)$ is an abbreviation for $((\neg\phi) \to \psi)$ etc.

(2.2.9) Lemma. Suppose $\mathcal{A}$ is an ~~$\mathcal{L}$~~ ~~$\mathcal{L}$~~ $\mathcal{L}$-str. and v a valuation in $\mathcal{A}$.

then v satisfies $(\exists x_1)\phi$

if and only if there is a valuation w which is x_1-equivalent to v and where $w[\phi] = T$.

<u>Pf:</u> <u>$\Rightarrow$</u> : Suppose v satisfies $(\neg(\forall x_1)(\neg\phi))$ (in $\mathcal{A}$).

So (by 2.2.7)

$$v[(\forall x_1)(\neg\phi)] = F.$$

then (by 2.2.7 (ii) (c)).

there is a valuation w x_1-equivalent to v with $w[(\neg\phi)] = F$.

So for this w

$$w[\phi] = T.$$

\#.

<u>$\Leftarrow$</u> : Ex.

Examples. (2.2.10)

a) $(\forall x_1)(\exists x_2) R(x_1, x_2)$

is true in $\langle \mathbb{Z}; < \rangle$

is false in $\langle \mathbb{N}; > \rangle$

1) Suppose ϕ is an $\mathcal{L}$-fmla.

$$\Big((\exists x_1)(\forall x_2)\phi \to (\forall x_2)(\exists x_1)\phi \Big)$$

is logically valid. (Ex.)

2) $$\Big((\forall x_2)(\exists x_1)\phi \to (\exists x_1)(\forall x_2)\phi \Big).$$

is not logically valid.

(Give an example)

Some logically valid formulas.

Consider the propositional formula

χ $\quad (p_1 \to (p_2 \to p_1))$

Suppose $\mathcal{L}$ is a 1^{st} order language

& ϕ_1, ϕ_2 are $\mathcal{L}$-fmlas.

Substitute $\left.\begin{array}{l}\phi_1 \text{ in place of } p_1 \\ \phi_2 \text{ in place of } p_2\end{array}\right)$ in χ

Obtain

θ : $(\phi_1 \to (\phi_2 \to \phi_1))$

This is an $\mathcal{L}$-formula

and θ is logically valid:

Suppose v is a valuation

(in a $\mathcal{L}$-str. $\mathcal{A}$)

Suppose $v[\theta] = F$.

ie $v[(\phi_1 \to (\phi_2 \to \phi_1))] = F$

By 2.2.7

$v[\phi_1] = T$

& $v[(\phi_2 \to \phi_1)] = F$

so $v[\phi_2] = T$ &

$v[\phi_1] = F$. Contradiction

(2.2.11) Def. Suppose χ is a ~~prop~~ L-fmla involving prop. vars. $p_1, \dots, p_n$. Suppose $\mathcal{L}$ is a 1^{st}-order language and $\phi_1, \dots, \phi_n$ are $\mathcal{L}$-fmlas. A substitution instance of χ is obtained by replacing p_i by ϕ_i (for $i \leq n$) in χ. Call the result θ

(2.2.12) Thm.

(1) θ is a $\mathcal{L}$-fmla.

(2) If v is a valuation in an $\mathcal{L}$-str. $\mathcal{A}$, let w be the prop. val. with $w(p_i) = v[\phi_i]$ (for $i \leq n$). Then

$$\underbrace{v[\theta]}_{\mathcal{L}\text{-fmla}} = w(\underbrace{\chi}_{\text{prop. fmla}}).$$

(3) If χ is a tautology, then θ is logically valid.

Pf: 1) Omit.

3) : from (2).

2) By induction on the number of connectives in χ

Base case χ is p_i.

Just by def. of w.

Inductive step Two cases

a) χ is $(\neg \alpha)$

b) χ is $(\alpha_1 \to \alpha_2)$

for $\mathcal{L}$-fmlas $\alpha, \alpha_1, \alpha_2$.

a) Ex.

b) θ is $(\theta_1 \to \theta_2)$

where θ_1, θ_2 are obtained by making the substution in α_1, α_2.

$w(\chi) = F$

$\Leftrightarrow$ $w(\alpha_1) = T$ & $w(\alpha_2) = F$

$\Leftrightarrow$ $v[\theta_1] = T$ & $v[\theta_2] = F$

ind hyp.

$\Leftrightarrow$ $v[(\theta_1 \to \theta_2)] = F$

$\Leftrightarrow$ $v[\theta] = F$.

This does the inductive step.

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This does not give all logically valid $\mathcal{L}$-fmlas.