Imperial College
London

M3P5

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Thursday 25 May 2017

Time: 14:00 - 16:00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Draw two closed planar curves which are homotopic but do not have the same index.

(b) Let α be a planar curve parametrized by arclength and such that its curvature is $k_\alpha = \frac{1}{R}$, with R a constant. Show that α is a subset of a circle with radius R.
Hint: See the curve as a planar curve embedded in $\mathbb{R}^3$.

(c) Consider the function $f : [0,1] \to \mathbb{R}$, $f(t) = t(1-t)$. Does a closed curve $\alpha : [0,1] \to \mathbb{R}^2$ parametrized by arclength exist, such that $k_\alpha = f$?

2. (a) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Show that the surface S parametrized by $f : (x,y) \mapsto (x, y, g(x,y))$ is indeed a surface and compute the metric.

(b) Let S be a surface with second fundamental form A, metric g, mean curvature H and Gaussian curvature K. Show that the principal curvatures are solutions of the quadratic equation

$$\det(A - xg) = 0,$$

$x \in \mathbb{R}$, and write this equation in terms of H and K.

(c) Show using part b) that for a minimal surface the Gaussian curvature is always nonpositive.

3. (a) Consider the surface given by

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, c\theta), \theta \in \mathbb{R}, r \in (0,R),\ c \text{ a constant.}$$

Compute its Gaussian curvature and its mean curvature.

(b) Let S_1 and S_2 be two surfaces parametrized by

$$\begin{aligned} \phi_1(u,v) &= (\cosh v \cos u, \cosh v \sin u, v), \\ \phi_2(u,v) &= (\sinh v \sin u, -\sinh v \cos u, u) \end{aligned}$$

Show that the surfaces parametrized by $\phi_t(u,v) = \cos t\phi_1(u,v) + \sin t\phi_2(u,v)$ have the same Gaussian curvature K for all $t \in [0, \frac{\pi}{2}]$.

4. (a) Show that for the curvature k_α, the geodesic curvature k_g and the normal curvature k_n of a curve α at the point p of a surface the identity $k_\alpha^2 = k_g^2 + k_n^2$ holds.

(b) Let S be a surface parametrized by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3, \quad \phi(u,v) = (\cos u + v^2, \sin u, v),$$

and $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\quad \alpha(t) = (\cos(t), \sin(t), 0)$ a curve. Show that α is a geodesic on S.

Name

Solution Exam curves and surfaces

Problem 1

a) Draw two closed planar curves which are homotopic but do not have the same index.

(7pts, Seen) Solution: For example a heart shaped curve (index ± 1) and the same curve with a loop instead of the cusp (index 2).

b) Let α be a curve parametrized by arclength and such that its curvature is $k_\alpha = \frac{1}{R}$. Show that α is a subset of a circle with radius R.
Hint: See the curve as a planar curve embedded in $\mathbb{R}^3$.

(5pts, Seen similar) Solution: Up to rigid motion we can pick a parametrization by arclength of a planar circle with radius R in $\mathbb{R}^3$ to be given by $\alpha(t) = (\frac{1}{R}\cos(\frac{t}{R}), \frac{1}{R}\sin(\frac{t}{R}), 0)$.
Since it is planar, the torsion is 0 and the curvature is

$$k_\alpha = \|\alpha''\| = \frac{1}{R}.$$

By the fundamental theorem of curves in $\mathbb{R}^3$ two curves with same curvature and torsion are the same up to rigid motion. Hence α is congruent to a (subset of a) circle.

c) Consider the function $f : [0,1] \to \mathbb{R}$, $f(t) = t(1-t)$. Does a closed curve $\alpha : [0,1] \to \mathbb{R}^2$ parametrized by arclength exist, such that $k_\alpha = f$?

(8pts, not seen but easy) Solution The answer is no. Let namely $\alpha : [0,1] \to \mathbb{R}^2$ be a closed curve, $k_\alpha = f$ its curvature. Its index is an integer $n \in \mathbb{Z}$ given by

$$Ind(\alpha) = n = \frac{1}{2\pi} tot(k_\alpha) = \frac{1}{2\pi}\int_0^1 k_\alpha = \frac{1}{2\pi}\int_0^1 t(1-t)dt = \frac{1}{8\pi}.$$

This is a contradiction since $\frac{1}{8\pi}$ is not an integer.

Problem 2

a) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Show that the surface S parametrized by $f : (x,y) \mapsto (x, y, g(x.y))$ is indeed a surface and compute the metric.

(5pts, seen similar) Solution: S is actually the level set of the function $F(x,y,z) = z - g(x,y)$. And clearly $\nabla F(p) \neq 0$ for all p. Hence it is a surface.
The metric is given by $g = \begin{bmatrix} \partial_x F \cdot \partial_x F & \partial_x F \cdot \partial_y F \\ \partial_y F \cdot \partial_x F & \partial_y F \cdot \partial_y F \end{bmatrix} = \begin{bmatrix} 1 + (\partial_x g)^2 & \partial_x g \partial_y g \\ \partial_x g \partial_y g & 1 + (\partial_y g)^2 \end{bmatrix}$.

b) Let S be a surface with second fundamental form A, metric g, mean curvature H and Gaussian curvature K. Show that the principal curvatures are solutions of the quadratic equation

$$\det(A - xg) = 0,$$

$x \in \mathbb{R}$, and write this equation in terms of H and K.

(10pts, not seen) solution: Let $\phi : U \to S$ be a chart at $p \in S$. The principal curvatures of S at p are the eigenvalues of $-dN_p$, where N is the Gauss map of S at p. Let $\{B_{ij}\}_{i,j=1,2}$ be the matrix representing $-dN_p$. With respect to the basis $\{\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\}$, the second fundamental form is given by

$$A_{ij} = A_{ji} = dN_p(\frac{\partial \phi}{\partial x_j}) \cdot \frac{\partial \phi}{\partial x_i} = (\sum_{k=1}^{2} B_{kj} \frac{\partial \phi}{\partial x_k}) \frac{\partial \phi}{\partial x_i} = \sum_{k=1}^{2} B_{kj} g_{ki} = \sum_{k=1}^{2} g_{ik} B_{kj}.$$

Consequently $A = gB$ and $B = g^{-1}A$ since g is invertible by definition. Therefore the principal curvatures are eigenvalues of $g^{-1}A$ and consequently roots of the characteristic polynomial $P(x) := \det(g^{-1}A - \mathrm{Id}x) = \det(g^{-1})\det(A - gx)$, which proves the claim. But we know that $P(x) = x^2 - \mathrm{tr}(g^{-1}A)x + \det(g^{-1}A)$ and we have by definition $H = \frac{1}{2}\mathrm{tr}(-dN_p) = \frac{1}{2}\mathrm{tr}(g^{-1}A)$ and similarly $K = \det(g^{-1}A)$. Then we get for λ_1

$$\lambda_1^2 - 2\frac{1}{2}(\lambda_1 + \lambda_2)\lambda_1 + \lambda_1\lambda_2 = 0$$

and similarly for λ_2.

c) Show using part b) that for a minimal surface the Gaussian curvature is always non-positive.

(5pts, not seen) solution: Since the second fundamental form is symmetric, real solutions λ_1 and λ_2 of the quadratic equations in part b) always exist. Hence the discriminant $4H^2 - 4K \geq 0$ and consequently $H^2 \geq K$. Finally if the surface is minimal, $H = 0$, which proves the result.

Problem 3

a) Consider the surface given by

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, c\theta), \theta \in \mathbb{R}, r \in (0, R),\ c \text{ a constant.}$$

Compute its Gaussian curvature and its mean curvature.

(12 pts, seen similar) solution: First we compute all partial derivatives up to second order.

$$\partial_r\phi = (\cos\theta, \sin\theta, 0),\ \partial_\theta\phi = (-r\sin\theta, r\cos\theta, c)$$
$$\partial_r^2\phi = (0,0,0),\ \partial_\theta^2\phi = (-r\cos\theta, -r\sin\theta, 0),\ \partial_{r\theta}^2\phi = \partial_{\theta r}^2\phi = (-\sin\theta, \cos\theta, 0).$$

Now we compute the Gauss map

$$\frac{\partial_r\phi \times \partial_\theta\phi}{\|\partial_r\phi \times \partial_\theta\phi\|} = \frac{(c\sin\theta, -c\cos\theta, r)}{\sqrt{c^2+r^2}}.$$

And the first fundamental form $g = \begin{bmatrix} \partial_r\phi\cdot\partial_r\phi & \partial_r\phi\cdot\partial_\theta\phi \\ \partial_\theta\phi\cdot\partial_r\phi & \partial_\theta\phi\cdot\partial_\theta\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c^2+r^2 \end{bmatrix}$. Notice that $\det g = c^2 + r^2$ and $g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c^2+r^2} \end{bmatrix}$.

Finally we compute the second fundamental form

$$h = \begin{bmatrix} \partial_r^2\phi\cdot N & \partial_{r\theta}^2\phi\cdot N \\ \partial_{\theta r}^2\phi\cdot N & \partial_\theta^2\phi\cdot N \end{bmatrix} = \begin{bmatrix} 0 & \frac{-c}{\sqrt{c^2+r^2}} \\ \frac{-c}{\sqrt{c^2+r^2}} & 0 \end{bmatrix}$$

and we have $\det h = -\frac{-c^2}{c^2+r^2}$.

Hence $K = \frac{\det h}{\det g} = -\frac{-c^2}{(c^2+r^2)^2}$ and $H = \frac{1}{2}\mathrm{tr}(g^{-1}h) = 0$.

b) Let S_1 and S_2 be two surfaces parametrized by

$$\phi_1(u,v) = (\cosh v\cos u, \cosh v\sin u, v), \tag{1}$$
$$\phi_2(u,v) = (\sinh v\sin u, -\sinh v\cos u, u) \tag{2}$$

Show that the surfaces parametrized by $\phi_t(u,v) = \cos t\phi_1(u,v) + \sin t\phi_2(u,v)$ have the same Gaussian curvature K for all $t \in [0, \frac{\pi}{2}]$.

(8 pts, seen similar) solution: We first show that the two surfaces are locally isometric. We therefore compute their first fundamental form

$$g_1 = \begin{bmatrix} \partial_x\phi_1\cdot\partial_x\phi_1 & \partial_x\phi_1\cdot\partial_y\phi_1 \\ \partial_y\phi_1\cdot\partial_x\phi_1 & \partial_y\phi_1\cdot\partial_y\phi_1 \end{bmatrix} = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix} = g_2. \tag{3}$$

Hence

$$g_t = \begin{bmatrix} \partial_x\phi_t\cdot\partial_x\phi_t & \partial_x\phi_t\cdot\partial_y\phi_t \\ \partial_y\phi_t\cdot\partial_x\phi_t & \partial_y\phi_t\cdot\partial_y\phi_t \end{bmatrix} = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix} \tag{4}$$

since

$$\partial_x\phi_t\cdot\partial_x\phi_t = (\cos t\partial_x\phi_1 + \sin t\partial_x\phi_2)\cdot(\cos t\partial_x\phi_1 + \sin t\partial_x\phi_2) = \partial_x\phi_1\cdot\partial_x\phi_1 \tag{5}$$

as $\partial_x\phi_1\cdot\partial_x\phi_1 = \partial_x\phi_2\cdot\partial_x\phi_2$ and $\partial_x\phi_1\cdot\partial_x\phi_2 = 0$, and similarly for the other components of g_t. Hence all surfaces are locally isometric since they have the same first fundamental form. By Theorema Egregium all surfaces of the one-parameter family have the same Gaussian curvature.

Problem 4

a) Show that for the curvature k_α, the geodesic curvature k_g and the normal curvature k_n of a curve α at the point p of a surface the identity $k_\alpha^2 = k_g^2 + k_n^2$ holds.

(8 pts, seen) Solution: Let $\alpha(0) = p$. k_n is the projection of k_α in the direction of the Gauss map, and k_g the projection in the direction of the other normal to the curve. We have therefore

$$k_\alpha = k_n N(\alpha(0)) + k_g(\alpha'(0) \times N(\alpha(0))),$$

where $N(\alpha(0))$ and $\alpha'(0) \times N(\alpha(0))$ are orthogonal. Pythagoras theorem yields the result.

b) Let S be a surface parametrized by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3, \quad \phi(u,v) = (\cos u + v^2, \sin u, v),$$

and $\alpha : \mathbb{R} \to \mathbb{R}^3, \quad \alpha(t) = (\cos(t), \sin(t), 0)$ a curve. Show that α is a geodesic on S.

(12, seen similar) Solution: First notice that α is parametrized by arclength. Let $c : \mathbb{R} \to \mathbb{R}^2, c(t) = (t, 0)$. We have obviously $\alpha = \phi \circ c$ and α is a curve on the surface. It suffices to show that $\alpha''(t)$ is perpendicular to the tangent space $T_{\alpha(t)}S$ for all t. But $T_{\alpha(t)S}$ is spanned by

$$\frac{\partial \phi(c(t))}{\partial u} = (\sin(t), \cos(t), 0), \quad \frac{\partial \phi(c(t))}{\partial v} = (0,0,1).$$

The curvature vector $\alpha''(t) = (-\cos(t), -\sin(t), 0)$ is obviously perpendicular to these two vectors and hence normal to the surface, hence the geodesic curvature $k_g = 0$.

Problem 5

a) Let α be a curve on a surface S. Show that if it is a geodesic and α' is a principal direction at each point of S, then α is a planar curve.

(not seen, 10pts) Solution: We can assume that α is parametrized by arclength. Since it is a geodesic, the geodesic curvature satisfies $k_g = 0$ and the curvature vector is given by $\alpha'' = k_n N$, with k_n the normal curvature, and N the Gauss map. Hence the unit normal to the curve corresponds up to sign to the Gauss map.
But α' is a principal direction, hence an eigenvector of dN and we have

$$dN\alpha' = \frac{d(N(\alpha))}{dt} = \lambda\alpha' = \kappa T + \tau B,$$

by Frénet equations, with κ and τ the curvature and the torsion of α respectively. Since $\alpha' = T$ we get $\kappa = \lambda$ and $\tau = 0$, which yields the result.

b) Let S be a regular, compact connected orientable surface in $\mathbb{R}^3$ without boundary which is not homeomorphic to a sphere. Show that then there exist elliptic points, hyperbolic points and points of zero Gaussian curvature on S.

10pts, seen similar) Solution: Since S is not a sphere, the Euler characteristic is not 2. But we know that $\chi(S) = 2 - 2g$, with g the genus, for a compact surface. Hence $\chi(S) \leq 0$. Hence by Gauss-Bonnet theorem, since S has no boundary, $\int_S Kds \leq 0$, with K the Gaussian curvature of the surface.
But we know that a compact surface has at least one elliptic point, i.e a point where $K > 0$. Therefore we see immediately that K must attain a negative value at some point and, since K is continuous, we can use the intermediate value theorem to conclude that K is also zero at some point.

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________________ **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Part 1: Many people just drew the curves without any explanation.
Part 2: Most students thought of using the Fundamental Theorem, although it ended in many cases in circular reasoning
Part 3: Almost nobody used the index / total curvature and only a few students got it right.
Overall the most challenging question for students, unexpectedly.

Marker: Marie-Amélie LAWN

Signature: [signature] **Date:** 7/06/2017

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Part a) : Almost nobody thought of using the description of the surface as a level set but instead used the formal definition which mostley failed.
Part b) : Some students seemed to have difficulties with the linear algebra part. Otherwise OK.

Marker: Marie-Amelie LAW

Signature: [signature] **Date:** 07/06/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________________ **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Purely computational problem, most students performed well. Few tried to compute the Gaussian curvature directly for all t and did not think about theorema egregium.

Marker: Marie-Amélie Lawn

Signature: [signature] **Date:** 7/06/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Part a) : OK

Part b) : Method mostly applied correctly, although many computed the geodesic condition only in $t = 0$.

Marker: ______________________

Signature: ______________________ **Date:** ______________________

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ______________________ **Session: 2016-2107**

Question 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Part a) : Overall OK. Almost nobody thought of using the Frenet Frames.
Part b) : A bit disappointing as I thought it was an easy question. Many students did not think at all of using Gauss-Bonnet, or if they did, did not use $\chi(S) \leq 0$. Seemed overall a hard question.

Marker: Marie-Amélie LAWN

Signature: [signature] **Date:** 7/06/2017

Please return with exam marks (one report per marker)