Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
January 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Practice Exam

Date: January 2023

Time:

Time Allowed: 1.5 hours

It is recommended candidates answer 3 questions, but you can complete any number of questions you wish.

Candidates should start their solutions to each question in a new exam booklet.

Supplementary books may only be used after the relevant main book(s) are full. Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Pure

ALGEBRAIC CURVES

Question (10 points): Let C be an irreducible projective plane curve of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ and assume that C is singular. Show that C is projectively equivalent to the zero set of one of the following polynomials:

- $x_0^3 + ax_0^2x_1 + bx_0x_1^2 + cx_1^3 - x_1^2x_2$
- $x_0^3 + dx_0^2x_1 + ex_0x_1^2 + x_1^3 + x_0x_1x_2$

where $a, b, c, d, e \in \mathbb{C}$. Conclude that C has a unique singular point.

Algebra III: Rings and Modules
Mock Exam Question, 2022-23

John Nicholson

In this test, a ring will be as defined in the course, i.e. a not necessarily commutative ring with a multiplicative unit 1. You may use any results from lectures provided they are clearly stated (and provided they are not the statement you are being asked to prove).

1. (a) Give the definitions of a *finitely generated R-module*, a *Noetherian ring*, and a *Noetherian R-module.*

   (b) Show that if R is a ring and M is an R-module, then M is Noetherian if and only if every submodule of M is finitely generated.

   (c) Recall that a nonzero R-module M is *simple* if the only R-submodules of M are M itself and the zero module. Show that every nonzero Noetherian R-module has a simple quotient, that is, for any nonzero Noetherian R-module M, there exists an R-submodule N of M such that M/N is simple.

   (d) Give (with proof) an example of a ring R and an R-module M such that M has no simple submodules.

Mock Exam Question for Commutative Algebra

For every ring R let $R[[t]]$ denote the formal power series ring in the variable t over R, and let $\mathcal{N}(R)$ denote the nilradical of R.

a) Assume that R is Noetherian. Show that

$$\mathcal{N}(R[[t]]) = \{\sum_{n=0}^{\infty} a_n t^n \in R[[t]] : a_n \in \mathcal{N}(R) \text{ for every } n\}.$$

b) Assume that $R = k$ is a field. Show that every non-zero element of $k[[t]]$ is of the form $t^n u$ for some $n \in \mathbb{Z}_{\geq 0}$ and an invertible element $u \in k[[t]]^*$. Deduce that the only prime ideals of $k[[t]]$ are (0) and (t).

c) Assume that R is an Artinian ring. Show that the Krull dimension of $R[[t]]$ is one. (Hint: use the previous part.)

Justin Trias

Mock exam question on elliptic curves

1. Let E be the elliptic curve $y^2 = (x-20)(x-26)(x+46)$.

   (a) Show that the torsion subgroup of $E(\mathbb{Q})$ has order 4. (10 marks)

   (b) Show that the points $(2, 144)$ and $(18, 32)$ generate a subgroup of $E(\mathbb{Q})$ which is isomorphic to $\mathbb{Z}^2$. (10 marks)

(Total: 20 marks)

Group Theory Question

State the four Sylow theorems, and also Burnside's transfer theorem.

Using these results, together with any other standard results from the lectures, prove the following:

(a) If G is a group of order 108, then G is not simple.

(a) If G is a group of order pqr, where p, q, r are distinct primes, then G is not simple.

(b) If G is a group of order $p^2(p+1)$, where p is prime, then G is not simple.

In this paper $\mathfrak{g}$ is a finite-dimensional Lie algebra over the field of complex numbers $\mathbb{C}$. Write $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ for the derived algebra of $\mathfrak{g}$.

2. (a) (i) Give the definition of a *solvable* Lie algebra.

   (ii) Give the definition of a *nilpotent* Lie algebra.

   (b) What can you say about the centre of a nilpotent Lie algebra?

   (c) Let $\mathfrak{g}$ be a solvable Lie algebra. What can you say about $\mathfrak{g}'$?

   (d) Let $\mathfrak{g}$ be the Lie algebra which is a vector space with basis e, f_1, f_2 such that

$$[e, f_1] = f_1, \quad [e, f_2] = \sqrt{-1} f_2, \quad [f_1, f_2] = 0.$$

   (i) Is $\mathfrak{g}$ solvable?

   (ii) Is $\mathfrak{g}$ nilpotent?

   (iii) What is the centre of $\mathfrak{g}$?

   (Justify your answers.)

MANIFOLDS - MATH70058

(1) **(4 points)** Let $X = S^2 \subset \mathbb{R}^3$ be the unit sphere and let
$$\omega = xdx \wedge dy + zdx \wedge dz \in \Omega^2(\mathbb{R}^3).$$
Compute $\int_X \omega$.

(2) **(3 points)** Show that the circle S^1 is orientable.

(3) **(6 points)** Write down explicitly a volume form on S^2.

(4) **(5 points)** Let $p \geq 1$ be an integer and let X be a non-compact manifold. Let ω be an exact p-form on X with compact support. Determine if there exists a $(p-1)$-form η with compact support and such that $\omega = d\eta$ (justify your answer).

(5) **(2 points)** Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $i \colon S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion. Let ω be a closed 1-form on X with compact support. Show that
$$\int_{S^1} i^*\omega = 0.$$

MARKOV PROCESSES 2022-23: MOCK EXAMINATION

(i) Show that

$$\pi_i := \binom{d}{i}^2 \Big/ \binom{2d}{d} \qquad (i = 0, \cdots, d)$$

defines a probability distribution $\pi = (\pi_i)$ (the *hypergeometric distribution* $HG(d)$).
(ii) There are $2d$ balls, d black and d white; these are divided between two urns, I and II, d balls to each urn. At each stage, a ball is chosen at random from each urn and the two are interchanged; the state of the system is the number of black balls in the first urn. Write down the transition probability matrix $P = (p_{ij})$ of the Markov chain thus defined.
(iii) Show that the chain has detailed balance with respect to $HG(d)$.
(iv) Find the recurrence time of state 0.
(v) What is the physical importance of this model for large d?

N. H. Bingham

Imperial College London
Department of Mathematics
Lecturer: P.-F. Rodriguez

MATH 60029
Fall 2022

Sample question

1. Let $X = L^p(0,2)$ for some $1 \leq p < \infty$, endowed with the usual $\|\cdot\|_{L^p(0,2)}$-norm, and assume the functions in X to be real-valued. For $f \in X$, define

$$(Tf)(t) = t^2 f(t),\ t \in (0,2).$$

a) Show that $T \in \mathcal{L}(X)$.

b) Determine $\|T\|_{\mathcal{L}(X)}$.

MATH60132/MATH70132 Mathematical Logic.

1. (a) Suppose A and B are sets.

   (i) Define the notation $|A| \leq |B|$. (1 mark)

   (ii) Does either of the following statements require the Axiom of Choice for its proof (assuming the ZF axioms)? Explain your answer briefly.

   · If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

   · At least one of $|A| \leq |B|$ or $|B| \leq |A|$ holds.

   (4 marks)

   (b) Compare the cardinalities of the following sets:

   * the set $\mathcal{P}(\mathbb{N})$ of subsets of the set of natural numbers;
   * the set $\mathcal{P}(\mathbb{R})$ of subsets of the set of real numbers;
   * the set $\mathbb{Q}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to the set of rational numbers;
   * the set X of irrational real numbers.

   In your answer you may assume the Axiom of Choice. Any general results about cardinal arithmetic which you wish to use should be clearly indicated. (8 marks)

   (c) Suppose that $(A; \leq)$ is a linearly ordered set.

   (i) What does it mean to say that $(A; \leq)$ is a *well ordered* set? (1 mark)

   (ii) Suppose that $(A; \leq)$ is a well ordered set and $f : A \to A$ is an injective function with the property that $f(a) \leq a$ for all $a \in A$. Prove that $f(a) = a$ for all $a \in A$.

   (4 marks)

   (iii) Is the result in (c)(ii) necessarily true without the assumption that the linear ordered set $(A; \leq)$ is well ordered? Give a proof or a counterexample. (2 marks)

(Total: 20 marks)

Mock Exam Problem

MATH71035 Analytic Methods in PDE, Fall 2022

Instructions

Time limit: 30 minutes.

Question

Let $\Omega = B(0,1) \subset \mathbb{R}^n$ be the open ball of radius 1 around the origin, $b = (b_i)_{i=1}^n \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

Define the elliptic operator

$$L = -\Delta + \sum_{i=1}^{n} b_i \partial_i + c \, .$$

(a) (3 marks) Write down the bilinear form associated to L.

(b) (3 marks) Given $f \in L^2(\Omega)$, state what it means for u to be a weak solution to the problem

$$Lu = f \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \, .$$

(c) (7 marks) Show there exists a constant $C_1 > 0$ such that if $|b| < C_1$ and $c = 0$ then the bilinear form in part (a) satisfies the assumptions of the Lax-Milgram theorem.

(d) (7 marks) Let $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ again be arbitrary.

Show that there exists a constant $C_2 > 0$ such that following holds.

For any weak solution $u \in H^1(\Omega)$ to $Lu = 0$ on Ω, any $R_1, R_2 > 0$ and $x \in \Omega$ with $B(x, R_1) \subset B(x, R_2) \subset\subset \Omega$, one has

$$\int_{B(x,R_1)} u^2 \leq \frac{C_2}{(R_2 - R_1)^2} \int_{B(x,R_2)} u^2 \, .$$

Hint: Choose $v = \xi^2 u$ in the weak formulation, where ξ is an appropriate cutoff.

MOCK EXAM FOR NUMBER THEORY - PROBLEM

AMBRUS PÁL

Question 1.

(a) Find the continued fraction expansion of $\sqrt{7}$. (6 marks)

(b) Find the two smallest solutions to $x^2 - 7y^2 = 1$, where x and y are both strictly positive and solutions are ordered by the value of y. (7 marks)

(c) Find the two smallest solutions to $x^2 - 7y^2 = 2$, where x and y are both strictly positive and solutions are ordered by the value of y. (7 marks)

Statistics

Applied Probability

1. (a) Consider a continuous-time, time-homogeneous, minimal Markov chain $X = (X_t)_{t\geq 0}$ on the state space $E = \{1, 2, 3\}$ with generator given by

$$\mathrm{G} = \begin{pmatrix} -7 & 3 & 4 \\ 1/10 & -1/5 & 1/10 \\ 1 & 2 & -3 \end{pmatrix}.$$

   (i) Draw the transition diagram of this Markov chain. (2 marks)

   (ii) Find the (one-step) transition matrix of the embedded jump chain. (3 marks)

   (iii) For each state in the state space, justify whether it is recurrent or transient for X. (2 marks)

(b) Suppose that $B = (B_t)_{t\geq 0}$ denotes a standard Brownian motion. Let $N = (N_t)_{t\geq 0}$ denote a Poisson process with rate $\lambda > 0$. Suppose that B and N are independent of each other. Define the stochastic processes $Y = (Y_t)_{t\geq 0}$ with $Y_t = t + N_t$ and $X = (X_t)_{t\geq 0}$ with $X_t = B_{Y_t}$ for all $t \geq 0$.

   (i) Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$, for $t \geq 0$. (3 marks)

   (ii) Find $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$, for $t \geq 0$. (5 marks)

   (iii) Find $\mathrm{Cov}(X_s, X_t)$, for $s, t \geq 0$. (5 marks)

(Total: 20 marks)

Time Series Analysis

2. Let $X_1, \ldots, X_N$ be a sample of size N from a stationary process $\{X_t\}$ with a non-zero mean μ and spectral density function $S(f)$. At lag $\tau = 0$ both the unbiased and biased estimators of the autocovariance sequence reduce to

$$\widehat{s}_0 \equiv \frac{1}{N} \sum_{t=1}^{N} \left(X_t - \overline{X} \right)^2 .$$

(a) Show that $E\{\widehat{s}_0\} = s_0 - \text{var}\,\{\overline{X}\}$, where $s_0 = \text{var}\,\{X_t\}$. (5 marks)

(b) Define the spectral estimator where the exact mean is known and subtracted as

$$\widehat{S}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} (X_t - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 .$$

Use the spectral representation theorem to show that the mean of the spectral estimator $\widehat{S}(f)$ is given by

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \mathrm{d}f' ,$$

where $\mathcal{F}(f)$ denotes Fejer's kernel given by

$$\mathcal{F}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 .$$

(5 marks)

(c) Demonstrate that

$$\text{var}\,\{\overline{X}\} = (1/N) E\{\widehat{S}(0)\},$$

and hence that

$$E\{\widehat{s}_0\} = \int_{-1/2}^{1/2} \left(1 - \frac{1}{N} \mathcal{F}(f) \right) S(f)\, \mathrm{d}f.$$

(5 marks)

(d) Sketch the form of $(1/N)\mathcal{F}(f)$ and hence describe the kind of spectrum $S(f)$ that would give rise to a large discrepancy between $E\{\widehat{s}_0\}$ and s_0.

(5 marks)

(Total: 20 marks)

Consumer Credit Risk Modelling

3. Use the following notation in this question:

   - Let $Y \in \{0,1\}$ denote an outcome: $Y = 1$ for default, $Y = 0$ for non-default.
   - Let $\mathbf{X}$ denote a random vector of m predictor variables.
   - Let D be a training data set of n independent observations of $\mathbf{X}$ and Y, given as $\mathbf{x}_1, \cdots, \mathbf{x}_n$ and $y_1, \cdots, y_n$, respectively.

   Suppose we have a scorecard s defined such that

$$s(\mathbf{x}) = \beta_0 + \boldsymbol{\beta} \cdot \mathbf{x}$$

   where β_0 is an intercept and $\boldsymbol{\beta}$ is a vector of coefficients, and $s(\mathbf{x})$ is the log-odds score of $Y = 0$, conditional on $\mathbf{X} = \mathbf{x}$.

   (a) Give the formula that expresses $s(\mathbf{x})$ in terms of Y and $\mathbf{x}$. (1 mark)

   (b) Derive the logistic function that expresses the probability $P(Y = 0|\mathbf{X} = \mathbf{x})$ in terms of the scorecard $s(\mathbf{x})$. (2 marks)

   (c) Show that the log-likelihood function to model outcome $Y = 0$ based on scorecard s, treating β_0 and $\boldsymbol{\beta}$ as model parameters, and using data D, is the logistic regression log-likelihood,

$$\ell = \sum_{i=1}^{n} (1 - y_i) \log\left(\frac{1}{1 + e^{-(\beta_0 + \boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) + y_i \log\left(\frac{1}{1 + e^{\beta_0 + \boldsymbol{\beta}\cdot\mathbf{x}_i}}\right).$$

   (3 marks)

   (d) Show that ℓ can be rewritten as

$$\ell = \sum_{i=1}^{n} (1 - y_i)(\beta_0 + \boldsymbol{\beta} \cdot \mathbf{x}_i) - \log\left(1 + e^{\beta_0 + \boldsymbol{\beta}\cdot\mathbf{x}_i}\right).$$

   (2 marks)

   (e) The log-likelihood ℓ is maximized with respect to β_0 and $\boldsymbol{\beta}$ by finding the stationary point. Show that the stationary point of ℓ is found by solving the system of equations,

$$\sum_{i=1}^{n} \left(1 - y_i - \frac{1}{1 + e^{-(\beta_0 + \boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) = 0$$

   and

$$\sum_{i=1}^{n} x_{ij} \left(1 - y_i - \frac{1}{1 + e^{-(\beta_0 + \boldsymbol{\beta}\cdot\mathbf{x}_i)}}\right) = 0$$

   for $j \in \{1, \cdots, m\}$ where x_{ij} is the jth predictor variable of $\mathbf{x}_i$. (4 marks)

(f) Logistic regression is used to build a model of non-default for mortgage loans. The coefficient estimates on four predictor variables are shown in the table below.

Note that Property type is a categorical variable with four levels: CO=condominium, PD=planned development, O=others, LH=leasehold. The levels are introduced as indicator variables in the model with LH the excluded category. Number of borrowers is also included as an indicator variable.

Predictor variable	Coefficient estimate	Standard error	z	P-value
Intercept	-6.16	1.13	-5.46	<0.0001
FICO score	0.0161	0.0014	11.4	<0.0001
Property type CO	-0.580	0.284	-2.04	0.0415
Property type PD	0.506	0.151	3.35	0.0008
Property type O	1.270	1.160	1.09	0.274
Property type LH	0	-	-	-
Loan-to-value	-0.0658	0.0069	-9.50	<0.0001
Number of borrowers > 1	0.530	0.146	3.62	0.0003

Describe the direction and statistical significance of the associations of each of the predictor variables with default. Use a significance level $\alpha = 0.01$. (5 marks)

(g) Would you have considered building a Naive Bayes classifier to get more accurate probability estimates? Justify your answer. (3 marks)

(Total: 20 marks)

Stochastic Simulation

4. (a) Consider the probability density function (PDF)

$$p(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

(i) Describe the general idea of the inversion method to sample from $p(x)$. (2 marks)

(ii) Derive the cumulative distribution function (CDF) of $p(x)$, denoted $F_X(x)$ for $x \geq 0$. (2 marks)

(iii) Derive $F_X^{-1}(x)$. Based on this describe how you can sample from exponential distribution using a uniform sample $U \sim \text{Unif}(0,1)$. (2 marks)

(iv) Justify the following sampler for the exponential variable

$$U \sim \text{Unif}(0,1) \qquad X = -\lambda^{-1}\log(U).$$

Explain why this also gives $X \sim \text{Exp}(\lambda)$. (1 mark)

(b) Let $X \sim p(x) = \mathcal{N}(0,1)$. Let $F_X(x)$ be the CDF of X and assume that we would like to estimate $F_X(c)$ for a given value $c \in \mathbb{R}$. Describe the Monte Carlo (MC) estimator for this task via the following:

(i) Describe the integration problem and your test function $\varphi(x)$ clearly. (2 marks)

(ii) Describe the MC estimator in terms of the test function you derived. (2 marks)

(iii) Describe, in words, how this estimator can be implemented. (1 mark)

(c) Determine whether a rejection sampler can be implemented to sample from a Cauchy distribution using a standard Normal to construct the corresponding envelope. If yes, find M. If not, justify your answer mathematically. Our target (Cauchy) distribution is given as

$$p(x) = \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbb{R}$$

and the Gaussian density (proposal) is given as

$$q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \qquad x \in \mathbb{R}.$$

(2 marks)

(d) Consider the following density

$$p(x) = \frac{1}{2\sqrt{2}}\left[1 + \frac{x^2}{2}\right]^{-3/2}.$$

We choose a proposal $q(x)$ which is Cauchy density (as given in part (c)).

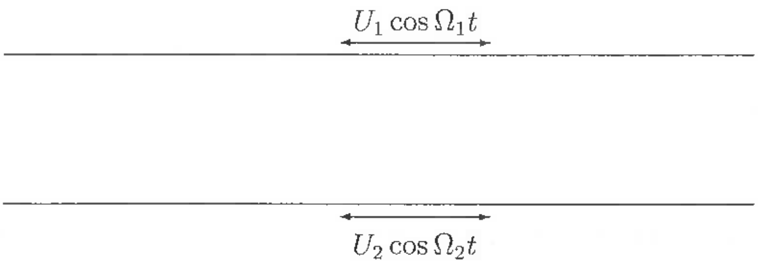
(i) Evaluate the constant $M = \sup_{x\in\mathbb{R}} \frac{p(x)}{q(x)}$ by working out all options (determine maxima and minima). (5 marks)

(ii) Describe the rejection sampling procedure using M. (1 mark)

(Total: 20 marks)

Applied

1. An incompressible fluid of kinematic viscosity ν is contained between two infinitely large parallel rigid plates at $y = 0$ and $y = h$. The upper and lower plates oscillate with velocities $U_1 \cos \Omega_1 t$ and $U_2 \cos \Omega_2 t$ in their own planes (see figure), where t is the time variable. Let x be the coordinate parallel to the plates, and u and v denote the velocity components in the x and y directions respectively. There is no pressure gradient in the x direction so that the flow is driven by the oscillatory plates only.



(a) Show that the Navier-Stokes equations admit an exact solution of the form

$$(u, v) = \big(u(y,t),\, 0\big).$$

Derive the equation satisfied by $u(y,t)$ and specify the boundary conditions on $u(y,t)$.

(4 marks)

(b) Consider the special case where $U_2 = 0$, i.e. the lower surface is fixed.

(i) Seek the solution for the velocity $u(y,t)$ of the form

$$u(y,t) = U_1\big(f(y)\,\mathrm{e}^{\mathrm{i}\,\Omega_1 t} + c.c.\big),$$

where $c.c.$ stands for complex conjugate. Deduce the equation and boundary conditions that $f(y)$ satisfies. Determine $f(y)$ and hence $u(y,t)$.

(8 marks)

(ii) Simplify the solution in each of the following limiting cases:

(a) $(2\nu/\Omega_1)^{\frac{1}{2}} \gg h$; (b) $(2\nu/\Omega_1)^{\frac{1}{2}} \ll h$.

Comment briefly on the nature of the flow in each case.

(4 marks)

(c) Suppose instead that $U_1 = 0$ but $U_2 \neq 0$, i.e. the upper plate is fixed while the lower plate oscillates. Find the solution for $u(y,t)$ in this case.

Use this result and that in (b)(i), deduce the solution for $u(y,t)$ in the general case of $U_1 \neq 0$ and $U_2 \neq 0$.

(4 marks)

(Total: 20 marks)

MATH60004/70004 Asymptotic Methods

1. Consider the singular perturbation problem

$$\varepsilon y'' + (x+\beta)y' - xy = 0, \qquad y(0) = 0, \quad y(1) = 1,$$

where β is a constant and $\varepsilon \searrow 0$.

(a) For the case $\beta > 0$,

(i) Determine the leading-order outer solution $y(x) = y_0(x) + \cdots$, assuming that the boundary layer is at $x = 0$.

(3 marks)

(ii) Deduce the width of the boundary layer at $x = 0$ and an appropriate inner variable X, and find the leading-order inner solution $y(x) = Y_0(X) + \cdots$.

(5 marks)

(iii) Explain briefly why the boundary layer cannot be at $x = 1$.

(2 marks)

(iv) Sketch the graphs of the inner and outer solutions, indicating the respective regions in which these solutions are valid.

(2 marks)

(v) Use the leading-order outer and inner solutions to construct an additive composite solution.

(2 marks)

(b) Find the leading-order outer and inner solutions for the case $\beta = 0$.

[You may use the result that $\displaystyle\int_0^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$.]

(6 marks)

(Total: 20 marks)

MATH60005/70005: Optimisation (Autumn 22-23)

Consider the problem

$$\begin{aligned} \min\ & x_1^2 + 2x_2^2 + x_1 \\ \text{subject to}\ & x_1 + x_2 \leq \gamma\,, \end{aligned}$$

with $\gamma \in \mathbb{R}$ a parameter.

i) (6 marks) Prove that for any $\gamma \in \mathbb{R}$, this problem has a unique solution (do not solve it).

ii) (8 marks) Solve the problem, expressing the general solution as a function of γ.

iii) (6 marks) Let $g(\gamma)$ be the optimal value of the problem for a given value of γ. Write an explicit expression for $g(\gamma)$ and determine whether this is a convex function or not.

1. The function $f(x)$ satisfies the integral-differential equation

$$\frac{df(x)}{dx} + 8f(x) = 12\int_0^\infty f(y)e^{-3|x-y|}dy,$$

for $x \geq 0$, with $f(0) = 1$ and $f'(0) = -5$.

(a) Using the Wiener-Hopf method, and taking the strip of analyticity to be $\{s : \alpha < \text{Im}\{s\} < \beta\}$, for suitable values α, β which you should define carefully to allow for the largest solution set possible, show that for $\text{Im}\{s\} > \alpha$ the right-sided Fourier transform $F_+(s) \equiv \int_0^\infty f(x)e^{isx}dx$ is given by

$$F_+(s) = \frac{i(s+3i)}{(s-i)(s+9i)}.$$

(16 marks)

(b) Hence find $f(x)$ for $x \geq 0$.

(4 marks)

(Total: 20 marks)

1.

(a) Let X be a metric space with metric $d : X \times X \to \mathbb{R}$ and $f : X \to X$ be continuous.

(i) Give definitions of the following properties: (4 marks)

- f is topologically mixing.
- f has sensitive dependence (on initial conditions).

(ii) Show that f has sensitive dependence if f is topologically mixing (unless X consists of a single point). (4 marks)

(b) Let Σ_3^+ denote the set of half-infinite sequences $\omega = \omega_0\omega_1 \ldots$ with $\omega_i \in \{0, 1, 2\}$ for all $i \in \mathbb{N}_0$, endowed with the metric

$$D(\omega, \tilde{\omega}) := \sum_{i=0}^{\infty} \frac{\delta_{\omega_i,\tilde{\omega}_i}}{3^i}, \text{ where } \delta_{\omega_i,\tilde{\omega}_i} := \begin{cases} 0 & \text{if } \omega_i = \tilde{\omega}_i. \\ 1 & \text{if } \omega_i \neq \tilde{\omega}_i. \end{cases} \tag{1}$$

Show that the cylinder set

$$C_{\omega_0\omega_1\ldots\omega_{n-1}} := \{\tilde{\omega} \in \Sigma_3^+ \mid \omega_i = \tilde{\omega}_i, \ i = 0, \ldots, n-1\}.$$

is an open ball in Σ_3^+ around any point $\omega \in C_{\omega_0\omega_1\ldots\omega_{n-1}}$ of radius 3^{1-n}. (4 marks)

(c) Let X, Y be metric spaces and $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be the associated Borel-measurable spaces. Let $f : X \to X$, $g : Y \to Y$ and $h : X \to Y$ be measurable such that $h \circ f = g \circ h$. Suppose that μ is an invariant measure of f.

(i) Show that $h_*\mu := \mu \circ h^{-1}$ is an invariant measure of g. (4 marks)

(ii) Show that $h_*\mu$ is ergodic if μ is ergodic. (4 marks)

(Total: 20 marks)

Classical Dynamics

1. (a) A particle of mass m is subject to a conservative force

$$\mathbf{F} = yz\cos(xy)\mathbf{i} + xz\cos(xy)\mathbf{j} + \sin(xy)\mathbf{k}.$$

Give a Lagrangian for this system. (4 marks)

(b) The Lagrangian for a particle of unit mass moving in the plane is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \alpha(y\dot{x} - x\dot{y}),$$

where α is a constant. Obtain the Lagrangian in polar coordinates r, θ. Identify any constants of the motion. (8 marks)

(c) The time-evolution of a physical system is governed by the Hamiltonian

$$H = (1 + q^2)p.$$

(i) Obtain a complete solution of the Hamilton-Jacobi equation for this system.

(ii) Using your solution from part (i), or otherwise, determine $q(t)$ and $p(t)$.

(8 marks)

(Total: 20 marks)

1. Consider a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be switched on to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

where the k's are positive constants. The production of g is stimulated by s_0 at a rate k_1 and by an autocatalytic or positive feedback process (the nonlinear term). There is also a linear degradation of g at rate k_2.

(a) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2}$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

(b) Show that if $s = 0$ there are 2 positive fixed points x^* if $r < r_c$, where r_c is to be determined. Determine the growth rate α by forming linear perturbations around the fixed point x_*.

(c) Assume that initially there is no gene product, *i.e.* $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on); what happens to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

(d) Find parametric equations for the bifurcation curves in (r, s) space, and classify the bifurcations that occur.

Q. Quantum harmonic oscillator and momentum representation

Consider a particle in a one-dimensional harmonic potential, described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2.$$

In what follows, we will use rescaled units with $m = 1 = \omega$, and $\hbar = 1$, to simplify calculations.

(a) Consider the normalised wave functions

$$\phi_0(x) = \left(\frac{1}{\pi}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}x^2}, \quad \text{and} \quad \phi_1(x) = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} x\mathrm{e}^{-\frac{1}{2}x^2}.$$

Verify that $\phi_0(x)$ and $\phi_1(x)$ are eigenfunctions of the Hamiltonian in position representation. What are the corresponding eigenvalues?

(8 marks)

(b) The eigenstates $|q\rangle$ of the position operator $\hat{q}$ belong to the eigenvalues $q \in \mathbb{R}$, so that $\hat{q}|q\rangle = q|q\rangle$. They fulfil the generalised orthonormality condition $\langle q'|q\rangle = \delta(q - q')$, and form a resolution of the identity,

$$\int_{-\infty}^{\infty} |q\rangle\langle q|\mathrm{d}q = \hat{I}.$$

The eigenstates $|p\rangle$ of the momentum operator $\hat{p}$ belonging to the eigenvalues $p \in \mathbb{R}$ (i.e. $\hat{p}|p\rangle = p|p\rangle$) fulfil the generalised orthonormality condition $\langle p'|p\rangle = \delta(p-p')$, form a resolution of the identity

$$\int_{-\infty}^{\infty} |p\rangle\langle p|\mathrm{d}p = \hat{I},$$

and we have

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}pq}.$$

(i) Show that

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}pq}\psi(q)\mathrm{d}q,$$

with $\psi(q) = \langle q|\psi\rangle$.

(3 marks)

(ii) Show that

$$\langle p|\hat{q}|\psi\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}\tilde{\psi}(p),$$

with $\tilde{\psi}(p) = \langle p|\psi\rangle$.

(5 marks)

(iii) Express $\langle p|\hat{H}|\psi\rangle$ in terms of $\tilde{\psi}(p)$.

(4 marks)

(Total: 20 marks)

Question

Consider the Cauchy problem for the Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & (t,x) \in (0,+\infty) \times \mathbb{R} \\ \rho(0,x) = g(x), & x \in \mathbb{R} \end{cases} \qquad g(x) = \begin{cases} 0 & x \leq 0, \\ 1 & 0 < x \leq 1, \\ 0 & x > 1. \end{cases} \tag{1}$$

(a) (i) Solve the characteristic system associated to the problem (1) and draw the characteristic lines. (4 marks)

(ii) Assume that $t \leq 2$. Compute the shock curve, find the unique entropy solution and draw the characteristic lines. (12 marks)

(b) For $t > 2$, compute the shock curve, find the unique entropy solution and draw the characteristic lines. (4 marks)

MOCK EXAM : FUNCTION SPACES AND APPLICATIONS

For ϕ a continuous, non-negative, compactly supported function on $\mathbb{R}$, consider the application

$$f \mapsto \phi * f.$$

Show that it is a bounded application on $\mathcal{C}(\mathbb{R})$. What is its norm?

1. **MATH70130 - Stochastic Differential Equations in Financial Modelling**

Mock exam

Consider the Ito SDE

$$dX_t = \frac{1}{3}(X_t)^{1/3}dt + (X_t)^{2/3}dW_t, \ X_0 = x_0$$

where the initial condition is a deterministic constant.

a) Do not try to prove existence and uniqueness of a solution a priori, invoking a theorem. Try to find one explicit solution using calculus and then check it is fine a posteriori. In particular:

a.1) Transform the SDE in Stratonovic form

[4 Points]

a.2) Use separation of variables and / or change of variables to solve the equation.

[6 Points]

b) Check a posteriori that the solution you found satisfies the given Ito SDE.

[5 Points]

c) Take the case $X_0 = 0$. Show that the solution is not unque by providing a second solution. [Hint: you can find easily a constant second solution in this case.] Why was it reasonable not to expect uniqueness in the first place?

[5 Points]

[End of Mock problem]

Special Relativity and Electromagnetism

1. (a) Show that the d'Alembert operator transforms as

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big) = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\Big)$$

directly using the Jacobian of the transform $ct = \gamma(ct' + x'\beta)$ and $x = \gamma(x' + ct'\beta)$. This follows immediately from $\Box = \partial_i \partial^i$ being a Lorentz scalar, but the question is asking for an explicit transform of coordinates.

(5 marks)

(b) Express the Lorenz gauge condition

$$\frac{\partial}{\partial x^i} A^i = 0$$

in terms of the scalar and vector potentials $A^i = (\phi, \mathbf{A})$ and state how it transforms.

(5 marks)

(c) Show that

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi = \nabla \cdot \mathbf{E}$$

with Lorenz gauge. (5 marks)

(d) State a class of non-trivial solutions of

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi = \nabla \cdot \mathbf{E}$$

in the vacuum, where $j^i = 0$. (5 marks)

(Total: 20 marks)