MATH50004 Differential Equations
Spring Term 2020/21
Solutions to the mid-term exam

Question 1

(i) $\lambda_n = \lambda_{n+1}$ implies that $\lambda_n(t) = x_0 + \int_{t_0}^t f(s, \lambda_n(s))\,\mathrm{d}s$. We differentiate this identity and apply the fundamental theorem of calculus to get $\dot\lambda_n(t) = f(t, \lambda_n(t))$.

[3 points; note this was also proved in Proposition 2.1, and an answer involving citing this result is good enough]

(ii) Consider $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t,x) = 0$ and $(t_0, x_0) = (0,0)$. It follows that $\lambda_n(t) \equiv 0$ for all $n \in \mathbb{N}_0$.

[3 points; justification is not necessary, since it is obvious for all correct examples]

(ii) With $f(x) = 1 - x$, we get for all $t \in J$ that

$$\begin{aligned}
\lambda_0(t) &= 0\,,\\
\lambda_1(t) &= 0 + \int_0^t f(\lambda_0(s))\,\mathrm{d}s = t\,,\\
\lambda_2(t) &= 0 + \int_0^t f(\lambda_1(s))\,\mathrm{d}s = \int_0^t (1-s)\,\mathrm{d}s = \left[s - \frac{1}{2}s^2\right]_{s=0}^{s=t} = t - \frac{1}{2}t^2\,.
\end{aligned}$$

[6 points = 1 point for λ_0; 2 points for λ_1; 3 points for λ_2]

Question 2

(i) The fastest way to see this is using Proposition 2.14, which implies that the conditions are satisfied if f is continuously differentiable, and this is clear since $f'(t,x) = \left(\frac{e^{t^2}2t}{x^2}, -\frac{2e^{t^2}}{x^3}\right)$ is obviously continuous.

[4 points; note that it also suffices to say that f is continuously differentiable without computing the derivative, since this is quite clear and it was argued similarly in solutions of exercises. An alternative weaker condition than continuous differentiability was discussed in Quiz 3, Question 5, and it suffices to show that $(t,x) \mapsto \frac{\partial f}{\partial x}(t,x)$ is continuous. An alternative direct proof is more involved and requires to somewhat mimic the proof of Proposition 2.14 by involving the mean value inequality/theorem involving partial derivatives with respect to x; note that in this case, also continuity of f needs to be checked/mentioned.]

(ii) We have $\dot\lambda_{max}(t) = \frac{e^{t^2}}{(\lambda_{max}(t))^2} > 0$, so λ_{max} is monotonically increasing. Since $\lambda_{max}(t_0) = x_0$, this implies that $\lambda_{max}(t) \le x_0$ for all $t \in (I^-(t_0,x_0), t_0]$, and since the differential equation is only defined on the domain D, where $x > 0$, the statement follows.

[4 points]

(iii) Since $I_-(t_0,x_0)$ is finite, Theorem 2.17 on the boundary behaviour implies that either the maximal solution is unbounded for $t \le t_0$, or we have that $\lim_{t \to I_-(t_0,x_0)} \operatorname{dist}((t, \lambda_{max}(t)), \partial D) = 0$. The first possibility is excluded due to (ii). Then the second possibility implies that, since $\partial D = \mathbb{R} \times \{0\}$, we have $\lim_{t \to I_-(t_0,x_0)} \lambda_{max}(t) = 0$.

[4 points]

Question 3

Assume for contradiction that $I_+(t_0, x_0)$ is finite. Then consider the interval $J = [t_0, I_+(t_0, x_0)]$, and note that the right hand side f is bounded on $J \times \mathbb{R}$, i.e. $M := \sup_{(t,x)\in J\times\mathbb{R}} |f(t,x)| < \infty$. Proposition 2.1 implies that

$$\lambda_{max}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_{max}(s))\,\mathrm{d}s\,,$$

which yields for $t \in [t_0, I_+(t_0, x_0))$ the estimate

$$|\lambda_{max}(t)| \leq |x_0| + \int_{t_0}^{t} \underbrace{|f(s, \lambda_{max}(s))|}_{\leq M}\,\mathrm{d}s \leq |x_0| + M(I_+(t_0, x_0) - t_0) < \infty\,,$$

and this contradicts the theorem on the boundary behaviour (Theorem 2.17).