Mathematics Year 1, Calculus and Applications I

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Problem Sheet 5

1. Consider the function $f(x) = 1/x$ for $x \in [1, \infty)$. Calculate and compare the area under the curve, the surface area of the solid formed by revolving $f(x)$ about the $x-$axis, and the volume of the revolved solid. What do you conclude? [The revolved object is called *Gabriel's horn.*]

2. Find (i) $\int_0^1 \frac{dx}{8x^3+1}$ and (ii) $\int \frac{(1+x)^{3/2}}{x} dx$.

3. After a glitch, a manufacturer only produced chains of variable density that start off with unit value but then become a linear function of distance from one end of the chain to the other. An order was delivered but the customer emailed back angrily saying that instead of the chains hanging evenly over their one unit tables, they rested in such a way that one of the hanging pieces was twice as long as the other hanging piece. What is the density of the chain produced by the malfunctioning machine?

4. Write an integral representing the area of the surface obtained by revolving the graph of $1/(1 + x^2)$ about the $x-$axis. Do not compute the integral but show that it is less than $2\sqrt{5}\pi^2$ no matter how long an interval is taken. Show also that an improved bound is $\sqrt{91}\pi^2/4$.

5. (a) Find the volume of the solid obtained by revolving the region under the graph of the function $y = \frac{1}{(1-x)(1-2x)}$ on the interval $[5, 6]$ about the $y-$axis.
   (b) Find the centre of mass of the region under $1/(x^2 + 4)$ on $[1, 3]$.

6. As a circle of radius a and centre O rolls along a plane, the position of a point A on the circle's circumference is given parametrically by $x = a\theta - a \sin\theta$, $y = a - a\cos\theta$, where θ is the angle that AO makes with the vertical.
   (a) Find the distance travelled by A for $0 \le \theta \le 2\pi$. Is it bigger or smaller than the circle's circumference. Explain your finding.
   (b) Draw a diagram for one arch of the curve traced out by A and superimpose on it the circle when its centre is at $(\pi a, a)$, together with the line segment $0 \le x \le 2\pi a$ on the $x-$axis. Show that the three enclosed areas are equal.

7. A parametric curve in the plane is given by $x = f(t)$ and $y = g(t)$ with $(f(0), g(0)) = (0, 0)$ and $(f(1), g(1)) = (0, a)$. Show that the length of the curve for $0 \le t \le 1$ is at least equal to a. What can you say when the length is exactly equal to a?

8. Consider a sphere of radius r. Suppose the sphere is sliced into three pieces by two parallel planes that are a distance d apart, where $0 < d < r$. Show that the surface area of the middle piece is the same irrespective of where the cuts are made on the sphere.

9. (a) Consider a function $y = f(x)$ with $f(0) = 0$ and assume that its inverse $x = f^{-1}(y)$ exists. The function is rotated about the $y-$axis to produce a solid in

the region $0 \le y \le y_0$. Use infinitesimals to show that the desired volume of revolution is

$$V = \pi \int_0^{y_0} [f^{-1}(y)]^2 dy.$$

(b) A bowl is created as described above by rotating $y = f(x)$ about the $y-$axis, and is filled with water to a height h_0. At its bottom ($x = y = 0$) a little hole is bored of radius r, that when open allows for the fluid to drain from the bowl. The speed of the exiting fluid is given by *Torricelli's* [1] *law* that states that at any given instant the speed equals $\sqrt{2gh}$ where h is the instantaneous height of the liquid remaining in the bowl. Formulate a conservation law that describes the physics of the problem, namely, *the rate of change of the volume at any given instant decreases by the rate at which fluid is exiting the small hole of radius r* to derive the equation

$$\frac{dV}{dt} = -\pi r^2 \sqrt{2gh},$$

where V is the volume of fluid remaining in the bowl.

(c) Three different bowls are now created to be used as hourglasses. The functions describing the bowls are (using the notation of part (a)) (i) $y = \frac{1}{k}x^2$, where $k > 0$ has dimensions of length, (ii) $y = \alpha x$, and (iii) a hemispherical bowl of radius a centred at $(0, a)$. In all three cases the bowls are filled with liquid to an initial height h_0 (note that $0 < h_0 \le a$), and have identical small holes of radius r at the bottoms. At $t = 0$ the hole is opened and the bowls are allowed to drain empty. Find α and a so that all three bowls empty at the same time.

10. Professor X had an arduous day chasing an elusive minus sign, and so decided to settle down (at precisely 18:00 hours) and mix himself a nice refreshing Negroni[2] to tame his depleted spirits. He knew that there were two large ice cubes to use (the Negroni is served with a single large ice cube), but on opening the freezer he was faced with a precipitous dilemma: One ice-cube was indeed a cube, but the other one was a perfect sphere. In addition, the cube had a larger volume, in fact exactly 1.5 times that of the spherical one (he weighed them quickly and also ascertained that the cube had sides 2cm long). His stress and anxiety was indescribable. Which one should he use? At a blink of an eye the elusive minus sign was totally forgotten, and his train of thought was devoted to the mathematical problem at hand. Volume must be lost in proportion to the surface area, he reasoned, and a quick search convinced him that the constant of proportionality is approximately 10^{-2}mm/s. He assumed that the volumes retain their topology as they melt (i.e. cubes remain cubes and spheres remain spheres) and did a few calculations to predict that after T minutes the two ice cubes would have the same volume. Find T and also the time it takes for the cubic and spherical cubes to melt, hence recommending which one to use given that Professor X is partial to the lasting of his cold drinks!

11. Two particles of equal mass m_0 are at rest at $t = 0$ on a rod that is balanced at a fulcrum positioned at the origin. Mass A is at $x = L$ and mass B is at $x = -L$. For $t > 0$ mass A moves to the right with constant speed v_0 and while doing so its mass

[1] Evangelista Torricelli (1608-1647) was an Italian physicist and mathematician who is best known for the invention of the barometer

[2] An Italian cocktail, see https://en.wikipedia.org/wiki/Negroni

increases at a constant rate proportional to v_0 - let the constant of proportionality be k_A. Mass B moves to the left and while doing so its mass also increases at a rate proportional to its speed with constant of proportionality k_B.

(i) Find what the speed of mass B should be in order for the rod to remain balanced about its fulcrum in a horizontal position. What does your formula give in the special case $k_A = k_B$? Explain using your physical intuition.

(ii) Now suppose that particle B picks up mass at a rate proportional to the square of its speed - take the constant to be k_B again. Derive an equation that in principle gives the speed $v(t)$ in order to keep the rod balanced. [Note that you will not be able to solve this equation analytically since it involves integrals of v^2 and v - it is an *integral equation*.]