Q1.

$$f(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{\infty} g(t)\,H(x-t)\,dt$$

$$= g(x) * H(x)$$

Convolution theorem: $\mathcal{F}\{f(x)\} = \mathcal{F}\{g(x) * H(x)\} =$

$$\hat{g}(w)\,\hat{H}(w) \Rightarrow$$

$$\mathcal{F}\{f(x)\} = \hat{f}(w) = \frac{\hat{g}(w)}{iw} + \pi\,\hat{g}(w)\,\delta(w) =$$

$$\frac{\hat{g}(w)}{iw} + \pi\,\hat{g}(0)\,\delta(w)$$

$$\hat{g}(0) = \int_{-\infty}^{\infty} g(x)\,dx \Rightarrow \hat{g}(0) = 0 \Rightarrow$$

$$\boxed{\hat{f}(w) = \frac{\hat{g}(w)}{iw}}$$

Q 2 $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1}{2}e^y$

$$u = \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{d}{dy}\left(\frac{1}{2}u^2\right)$$

$$\frac{d}{dy}\left(\frac{1}{2}u^2\right) = \frac{1}{2}e^y \Rightarrow \frac{u^2}{2} = \frac{1}{2}e^y + C_1$$

$$\left.\begin{array}{l} y(0) = 0 \\ y'(0) = u(0) = 1 \end{array}\right\} \Rightarrow C_1 = 0 \Rightarrow u^2 = e^y$$

$$u = \frac{dy}{dx} = e^{y/2} \Rightarrow \int e^{-y/2}\,dy = \int dx$$

$$(-2)e^{-y/2} = x + C_2 \Rightarrow$$

using $y(0) = 0 \Rightarrow C_2 = -2 \Rightarrow$

$$e^{-y/2} = 1 - \frac{x}{2} \Rightarrow$$

$$\frac{y}{2} = \ln\frac{2}{2-x} \Rightarrow y = \ln\left[\left(\frac{2}{2-x}\right)^2\right]$$

Q3 $\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{x}{4} = te^{-\frac{t}{2}} + 1$

a) ~~1st step~~ $L[x_{CF}] = 0$

$CF = e^{\lambda t} \Rightarrow 4\lambda^2 + 4\lambda + \lambda = 0 \Rightarrow$

$\lambda_1 = \lambda_2 = -\frac{1}{2} \Rightarrow x_{CF} = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t}$

2nd step $L[x_{PI}] = te^{-\frac{t}{2}} + 1$

Ansatz $x_{PI} = At^3 e^{-\frac{t}{2}} + B$

$\Rightarrow A = \frac{1}{6}, \ B = 4 \Rightarrow$

$$x_{GS} = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t} + \frac{1}{6} t^3 e^{-\frac{t}{2}} + 4$$

b) $\frac{dx}{dt} = u \Rightarrow$

$$\begin{cases} \frac{du}{dt} = -u - \frac{x}{4} + te^{-t/2} + 1 \\ \frac{dx}{dt} = u \end{cases}$$

$$u_{GS} = \frac{dx_{GS}}{dt} = -\frac{1}{2} C_1 e^{-\frac{1}{2}t} - \frac{1}{2} C_2 t e^{-\frac{1}{2}t} + C_2 e^{-\frac{1}{2}t} + \frac{1}{2} t^2 e^{-\frac{t}{2}} - \frac{1}{12} t e^{3-\frac{t}{2}}$$

$$\vec{x}_{GS} = \begin{pmatrix} x_{GS} \\ u_{GS} \end{pmatrix} = \underbrace{C_1 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-\frac{1}{2}t} + C_2 \begin{pmatrix} t \\ -\frac{1}{2}t + 1 \end{pmatrix} e^{-\frac{1}{2}t}}_{\vec{x}_{CF}} + \underbrace{\begin{pmatrix} \frac{1}{6} t^3 e^{-\frac{t}{2}} + 4 \\ \frac{1}{2} t^2 (1 - \frac{1}{6} t) e^{-\frac{t}{2}} \end{pmatrix}}_{\vec{x}_{PI}}$$