MATH96032 Geometry I: Algebraic Curves

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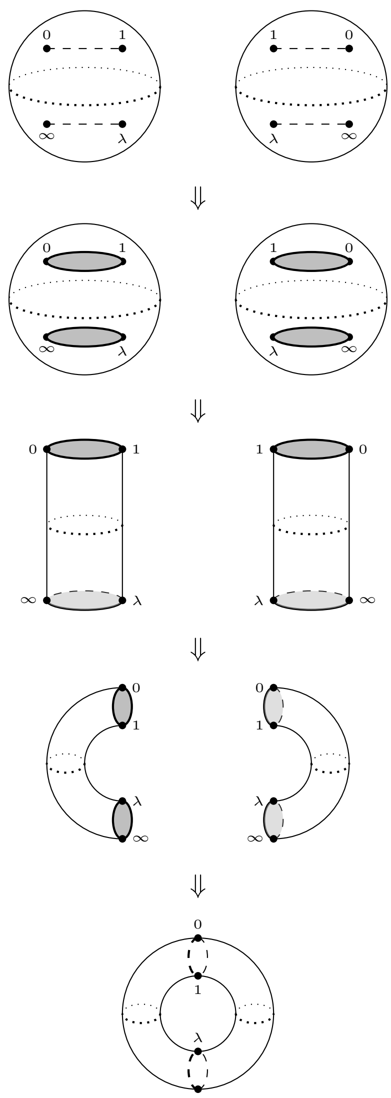

Syllabus

Affine plane algebraic curves. Projective space. Plane projective curves. Projectivisation. Points at infinity. Singularities. Smoothness. Intersections of plane curves. Resultants. Multiplicities. Bézout's theorem. Conics. Cubic curves. Riemann surfaces. Genus. Ramification. The Riemann-Hurwitz formula. The degree-genus formula.

[0]These lecture notes are partially based on the typed notes by David Kurniadi Angdinata for the 2018–2019 module. Many thanks to David for making his notes available!

Contents

1 Introduction

Content of the course

Geometry is the study of shapes. The kind of shapes we consider determines the tools that we can use to analyze them and thus the branch of geometry in which we are working. Algebraic geometry studies the geometric properties of *algebraic sets*, shapes that arise as solution spaces of systems of polynomial equations over a field (or, more generally, a commutative ring). It employs techniques from algebra, topology, complex analysis, differential geometry, number theory, ... Conversely, algebraic geometry often furnishes a geometric intuition for problems arising in algebra and number theory, and geometric tools to solve them.

The basic philosophy of algebraic geometry can be summarized as follows. Let K be a field, and consider a system of finitely many polynomial equations

$$\begin{cases} P_1(x_1,\dots,x_n) & = & 0 \\ & \vdots & \\ P_r(x_1,\dots,x_n) & = & 0 \end{cases}$$

with $P_j \in K[x_1,\dots,x_n]$ for $j = 1,\dots,r$. The set S of all the points in K^n satisfying these equations is called an *algebraic set*[1] in K^n. Algebraic geometry provides a dictionary between

- the geometric properties of the algebraic set S;
- the algebraic properties of the commutative ring $K[x_1,\dots,x_n]/(P_1,\dots,P_r)$, the quotient of the polynomial ring by the ideal generated by the polynomials that appear in our system of equations.

This leads to a beautiful and powerful interplay between geometry and algebra.

Example 1.1.

- Let K be a field and let $P(x)$ be a polynomial with coefficients in K. Then the set $\{a \in K \mid P(a) = 0\}$ is an algebraic subset of K. It is equal to K if $P = 0$ and a finite set of points otherwise; conversely, every finite set of points in K can be realized as the zero set of a polynomial. Thus the algebraic subsets of K are precisely K itself and the finite subsets of K.
- The unit circle $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}$ is an algebraic set in $\mathbb{R}^2$.
- Consider an integer $n \geq 3$. By Fermat's Last Theorem, which was proved by Andrew Wiles in 1994, the algebraic set $\{(x,y) \in \mathbb{Q}^2 \mid x^n + y^n = 1\}$ in $\mathbb{Q}^2$ has only finitely many elements, namely, $(1,0)$ and $(0,1)$ if n is odd, and $(\pm 1, 0)$ and $(0, \pm 1)$ if n is even. Wiles's proof makes heavy use of algebraic geometry (the theory of elliptic curves). If we consider the same equation over the field $\mathbb{C}$ of complex numbers, then the solution set becomes infinite: we can choose an arbitrary value of x and then solve for y. Thus the nature of algebraic sets and the flavour of algebraic geometry depends strongly on the choice of base field. A fundamental insight in algebraic geometry is that there exist deep and surprising interactions between the arithmetic properties of equations over fields like $\mathbb{Q}$ or finite fields, and the geometric shape of the solution sets over the complex numbers.

In this module we will mainly study plane curves over $\mathbb{C}$; these are algebraic sets that can be defined by a single polynomial equation in $\mathbb{C}^2$ (or in the projective plane $\mathbb{P}^2_{\mathbb{C}}$ which will be defined later in the course). Besides the finite sets, these are the most elementary algebraic sets one can study, but their geometry is already quite rich and intricate and their study illustrates many essential features of algebraic geometry. This is why we will focus on algebraic curves in this course. Important questions that will be addressed are:

- When do two polynomials in $\mathbb{C}[x,y]$ define the same curve in $\mathbb{C}^2$? The answer to this question is a special case of *Hilbert's Nullstellensatz*, one of the cornerstones of algebraic geometry and an essential part of the dictionary between algebra and geometry.

[1] Some authors call this an *algebraic variety*, but other authors, including Fulton, use a more restrictive definition of algebraic variety.

- What can be said about the intersection of two plane curves? What is the number of intersection points? This will be discussed in *Bézout's theorem*, one of the main results in this module.

Exercise 1.2. ☜

- Is $\mathbb{Z} \subseteq \mathbb{R}$ an algebraic set?
- Is the unit square
$$\{(x, y) \in [0,1] \times [0,1] \,|\, x \in \{0,1\} \text{ or } y \in \{0,1\}\}$$
an algebraic set in $\mathbb{R}^2$?

Recommended reading

This course is intended as a first encounter with algebraic geometry. It will focus on one-dimensional algebraic varieties over the complex numbers. Good references are:

- F. Kirwan, Complex algebraic curves, 1992. This is the main reference for this course; it contains additional background, historical references, and helpful pictures.
- W. Fulton, Algebraic curves, an introduction to algebraic geometry, 1969. The 2008 edition of this book is freely accessible on the author's website: `http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf`. This book uses more algebraic machinery then we will use in this course, and less complex analysis; it is a great introduction for the algebraically inclined students.

These references are included as background material and sources of additional exercises, but they are not mandatory reading: all the examinable material is included in the lecture notes, the problem sheets and the coursework assignments (as well as the mastery material for students in years 4 and 5).

Students who want to learn more about algebraic geometry can consult "An invitation to algebraic geometry" by K. Smith, L. Kahanpää, P. Kekäläinen and W. Traves. This book goes beyond the scope of the course and is included only for more advanced students.

The exercises

It is **extremely important** to do the exercises in the notes, they will help you digest the material. Some of the exercises are elementary and mainly test whether you have correctly understood the definitions and results; they form a good check to see if you are ready to move on to the next part of the course. More challenging exercises, comparable to the hardest 35% on the exam, are marked with $\star$. Optional exercises beyond exam level are marked with $\star\star$. Solutions to the non-optional exercises are provided in Appendix D, and hints to the starred exercises in Appendix C. The optional exercises can be discussed on the Blackboard forum.

The recommended exercises are marked with the symbol ☜ in the margin. You should do these exercises before moving on to the next lecture.

2 Affine plane curves

We start with a bit of notation. Let K be a field, such as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. A *monomial* in the variables $x_1, \ldots, x_n$ is an expression of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$. We often use multi-index notations: setting $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$, we denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *degree* of the monomial x^α is the natural number $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

A polynomial over K in the variables $x_1, \ldots, x_n$ is a sum

$$P(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$$

where the coefficients c_α lie in K and only finitely many of them are non-zero. The set of such polynomials is denoted by $K[x_1, \ldots, x_n]$. If $P = 0$ (that is, all the coefficients c_α are zero) then we say that the degree of P is $-\infty$; otherwise, the degree of P is defined to be the maximum of the set

$$\{|\alpha| \mid \alpha \in \mathbb{N}^n,\, c_\alpha \neq 0\}.$$

In other words, it is the maximum of the degrees of the monomials that appear in P with a non-zero coefficient. The degree of P is denoted by $\deg(P)$. A polynomial is called *constant* if it has degree 0 or $-\infty$; this is equivalent to saying that it lies in the subset K of $K[x_1, \ldots, x_n]$.

Example 2.1. The expression

$$P(x_1, x_2, x_3) = 3 + x_1^2 x_2 + x_2^4 x_3^{10}$$

defines a polynomial over $\mathbb{Q}$ of degree 14; the multi-indices α with non-zero coefficient are $(0, 0, 0)$, $(2, 1, 0)$ and $(0, 4, 10)$.

Exercise 2.2. Show that $K[x_1, \ldots, x_n]$ is a ring with respect to the usual addition and multiplication rules for polynomials. Prove that, if P and Q are polynomials in this ring and λ and μ are elements of K, then the degree of $\lambda P + \mu Q$ is at most $\max\{\deg(P), \deg(Q)\}$. Give an example of polynomials $P, Q \in K[x]$ such that

$$\deg(P + Q) < \max\{\deg P, \deg Q\}.$$

Show that $\deg PQ = \deg P + \deg Q$, where we apply the convention that $-\infty + a = -\infty$ for all a in $\mathbb{Z} \cup \{-\infty\}$. Deduce that the invertible elements in $K[x_1, \ldots, x_n]$ are precisely the non-zero constant polynomials.

Definition 2.3. A polynomial $P \in K[x_1, \ldots, x_n]$ is called *reducible* if there exist non-constant polynomials $Q, R \in K[x_1, \ldots, x_n]$ such that $P = Q \cdot R$. We say that P is *irreducible* if it is not constant and not reducible. Constant polynomials are neither reducible, nor irreducible.

Example 2.4. The polynomial $x^2 - y^2$ in $\mathbb{C}[x, y]$ is reducible, because we can write it as $(x - y)(x + y)$. The polynomial $x^2 - y^2 + 1$ is irreducible in $\mathbb{C}[x, y]$: you can prove by brute force that it cannot be written as a product of two non-constant polynomials. Note that reducibility of a polynomial also depends on the base field K: the polynomial $x^2 + y^2$ is irreducible in $\mathbb{R}[x, y]$ but reducible in $\mathbb{C}[x, y]$, where we can write it as $(x + iy)(x - iy)$.

Every non-constant polynomial $P \in K[x_1, \ldots, x_n]$ can be written as a product of irreducible factors $P = P_1 \cdots P_k$: simply keep splitting off non-constant factors as long as some factors are reducible. This process necessarily terminates in finite time because the degree of some factor drops in every step.

One can show (but this requires some work) that this factorization is unique up to the order of the factors and up to multiplication of the factors with non-zero constants in K. In the lingo of commutative algebra, we say that $K[x_1, \ldots, x_n]$ is a *unique factorization domain* (often abbreviated to *UFD*). This relies on the fact that irreducible polynomials have the prime property: if R is an irreducible polynomial that divides the product of two polynomials P and Q, then R divides P or Q.

Example 2.5. The product $x^2 + y^2 = (x + iy)(x - iy)$ is a factorization of $x^2 + y^2$ into irreducible polynomials. We can swap the factors in this product or multiply one factor with a constant $\lambda \in \mathbb{C}^*$ and the other with $1/\lambda$, but otherwise this factorisation is unique.

Definition 2.6. A non-constant polynomial $P \in K[x_1, \ldots, x_n]$ has a *no repeated factors* if it is not divisible by the square of a non-constant polynomial. This is equivalent to saying that

$$P = P_1 \cdot \cdots \cdot P_k,$$

where $P_1, \ldots, P_k$ are irreducible polynomials which are distinct in the following strong sense: there do not exist $\lambda \in K^*$ and distinct elements $\ell, m \in \{1, \ldots, k\}$ satisfying $P_m = \lambda P_\ell$.

Exercise 2.7. Prove the equivalence of the two definitions. ☜

Definition 2.8. An *affine plane curve* defined over K is a subset of K^2 of the form

$$C = \{(x, y) \in K^2 \mid P(x, y) = 0\}$$

where P is a non-constant polynomial in $K[x, y]$.

The zero set of a constant polynomial P in $K[x, y]$ is K^2 if $P = 0$ and empty otherwise; these cases do not correspond to our geometric picture of a curve, which is why we excluded them in the definition. Every affine plane curve can be defined by a polynomial with no repeated factors: by grouping multiple irreducible factors, every non-constant polynomial P in $K[x, y]$ can be written as

$$P = cP_1^{a_1} \ldots P_k^{a_k}, \qquad c \in K^*,\ a_i \in \mathbb{Z}_{>0}$$

where $P_1, \ldots, P_k$ are irreducible polynomials in $K[x, y]$ such that there do not exist $\lambda \in K^*$ and distinct elements $\ell, m \in \{1, \ldots, k\}$ satisfying $P_m = \lambda P_\ell$. Then the zero set of P in K^2 coincides with the zero set of the polynomial

$$Q = P_1 \cdot \cdots \cdot P_k,$$

which has no repeated factors. For instance, the equations $y^2(x + 1)^3 = 0$ and $y(x + 1) = 0$ define the same affine plane curve; the second polynomial has no repeated factors.

Example 2.9.

- Let $a, b, c \in K$ with $(a, b) \neq (0, 0)$. Then the affine plane curve

  $$\{(x, y) \in K^2 \mid ax + by + c = 0\}$$

  is a line in the vector space K^2.

- Conics in $\mathbb{R}^2$ are examples of affine plane curves. For instance, if $a, b \in \mathbb{R}^*$, then the affine plane curve

  $$\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\}$$

  is an ellipse, and

  $$\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\}$$

  is a hyperbola.

- Beware that, with our definition, affine plane curves may be unexpectedly small and not correspond to the geometric picture of a curve when the base field K is not algebraically closed. For instance, the zero set of the polynomial $x^2 + y^2 + 1$ is empty in $\mathbb{R}^2$, because the square of a real number is always non-negative. Another example is the zero set in $\mathbb{Q}^2$ of the Fermat polynomial $x^n + y^n - 1$, which is finite when $n \geq 3$. The moral is that the true geometry of a polynomial equation may only become visible if we pass to an algebraically closed field extension, like $\mathbb{C}$. For this reason, if one uses the classical language of algebraic geometry (as we will do in this course) it is usually assumed that the base field K is algebraically closed. To deal with more general base fields, the more modern language of *schemes*, developed by Grothendieck and his school in the 1950s and 1960s, is better suited. However, it is also considerably more abstract, which is why one continues to use the classical approach for a first encounter with algebraic geometry.

We will now investigate how affine plane curves can be decomposed into more elementary pieces.

Lemma 2.10. *The union of two affine plane curves is again an affine plane curve.*

Proof. Let $P_1, P_2 \in K[x,y]$ be non-constant polynomials, defining affine plane curves

$$C_1 = \{(x,y) \in K^2 \mid P_1(x,y) = 0\}, \qquad C_2 = \{(x,y) \in K^2 \mid P_2(x,y) = 0\}.$$

Then $Q = P_1P_2 \in K[x,y]$ is a non-constant polynomial such that

$$C_1 \cup C_2 = \{(x,y) \in K^2 \mid Q(x,y) = 0\}.$$

It follows that $C_1 \cup C_2$ is an affine plane curve. $\square$

Exercise 2.11. Write down an equation for the affine plane curve in $\mathbb{R}^2$ that is the union of the lines through pairs of distinct vertices of the unit square.

If P is a non-constant polynomial in $K[x_1, \dots, x_n]$ and $P = P_1 \cdots P_k$ is a factorisation into irreducible polynomials, then the zero set of P in K^n is the union of the zero sets of the factors $P_1, \dots, P_k$. In particular, for $n = 2$, every affine plane curve can be written as a union of finitely many affine plane curves defined by irreducible polynomials. We will soon show that this decomposition is unique.

Very different questions can be approached through algebraic curves. In applications to number theory, we often try to determine the set of *rational points* on an algebraic curve.

Definition 2.12. Let $C \subset K^2$ be an affine plane curve over K. If K_0 is a subfield of K, then a *K_0-rational point* on C is a point $(x_0, y_0) \in C$ such that $x_0, y_0 \in K_0$. The set of K_0-rational points on C is denoted by $C(K_0)$.

Thus, if P is a non-constant polynomial in $K[x,y]$ whose zero set is C, then a K_0-rational point on C is simply a solution of the equation $P(x,y) = 0$ with coordinates in K_0. Note that we do not require that the polynomial P has coefficients in K_0; if it does, then the set $C(K_0)$ is nothing but the affine plane curve over K_0 defined by $P = 0$.

Example 2.13. As we have already discussed, the Fermat equation $x^n + y^n = 1$ has only finitely many solutions in $\mathbb{Q}^2$ if $n \geq 3$, and each of these solutions satisfies $x = 0$ or $y = 0$. This determines the set $C(\mathbb{Q})$ of $\mathbb{Q}$-rational points on the curve C in $\mathbb{C}^2$ defined by the equation $x^n + y^n = 1$. The proof of this result uses highly sophisticated techniques from algebraic geometry and goes far beyond the scope of this course. However, we can use some elementary planar geometry to describe the solutions in the case $n = 2$.

For $n = 2$, the set of $\mathbb{R}$-rational points on C is the unit circle

$$C(\mathbb{R}) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We can write down an algebraic parametrisation of C by considering lines through the point $p = (-1, 0)$, using a stereographic projection. For every $t \in \mathbb{R}$ we denote by L_t the line through p with slope t; it is given explicitly by the equation

$$L_t = \{(x,y) \in \mathbb{R}^2 \mid y = t(x+1)\}.$$

This line meets C in two points, p and $p_t = (x(t), y(t))$. We can determine the coordinates of p_t by solving the system of equations for $L_t \cap C$:

$$\begin{cases} y = t(x+1), \\ x^2 + y^2 = 1. \end{cases}$$

Substituting the first equation into the second yields two solutions for $x(t)$. The first solution is $x = -1$ and corresponds to the point $p = (-1, 0)$. The second one is

$$x(t) = \frac{1-t^2}{1+t^2}, \qquad y(t) = \frac{2t}{1+t^2}. \tag{1}$$

We obtain a bijection

$$\mathbb{R} \to C \setminus \{p\},\ t \mapsto p_t = (x(t), y(t))$$

since every point of $C \setminus \{p\}$ lies on a unique line through p, and this line has finite slope. Note that $p_t \to p = (-1, 0)$ as $t \to \infty$, so that we can think of C as being parameterized by $\mathbb{R} \cup \{\infty\}$. The advantage of this parametrisation is that it is given by rational functions: the coordinates $x(t)$ and $y(t)$ are of the form $P(t)/Q(t)$, where P and Q are polynomials with coefficients in $\mathbb{Q}$. This guarantees that $x(t)$ and $y(t)$ are rational numbers whenever $t \in \mathbb{Q}$. Conversely, if $x(t)$ and $y(t)$ are rational then the line connecting p and $p_t = (x(t), y(t))$ has rational slope, so that $t \in \mathbb{Q}$. Thus the set of solutions in $\mathbb{Q}^2$ of the equation $x^2 + y^2 = 1$ is given by

$$(-1, 0) \cup \{(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) \mid t \in \mathbb{Q}\}.$$

We can push this method a little further to find the triples (a, b, c) where a, b and c are coprime integers satisfying

$$a^2 + b^2 = c^2. \tag{2}$$

Note that $c \neq 0$ since otherwise, a and b are also equal to 0 and the triple is not coprime. Then equation (2) is equivalent to the equation

$$(a/c)^2 + (b/c)^2 = 1$$

whose rational solutions we have just determined. Now it is an easy **exercise** to show that the triples we are looking for are given by

$$\pm(u^2 - v^2, 2uv, u^2 + v^2)$$

where u and v are coprime integers that are not both odd, and

$$\pm\left(\frac{u^2 - v^2}{2}, uv, \frac{u^2 + v^2}{2}\right)$$

where u and v are odd coprime integers (these solutions correspond to the rational solution given by $t = u/v$ if $v \neq 0$ and to the rational solution $(-1, 0)$ if $v = 0$).

3 Complex curves and the Nullstellensatz

Let $P \in \mathbb{R}[x, y]$ be a polynomial with coefficients in $\mathbb{R}$. Then we can consider the real affine plane curve

$$C_{\mathbb{R}} = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

However, P can also been seen as a polynomial with coefficients in $\mathbb{C}$, and it will often be easier to study the complex affine plane curve

$$C_{\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}.$$

Once we understand the geometry of $C_{\mathbb{C}}$, we can then try to analyze its set of $\mathbb{R}$-rational points, for instance by exploiting the fact that they are the fixed points of the involution on $C_{\mathbb{C}}$ that replaces coordinates of a point by their complex conjugates. This is by no means an easy step, and there are still many open problems about the possible shapes of $C_{\mathbb{R}}$!

We have already indicated the main problem with looking directly at $C_{\mathbb{R}}$: this subset of $\mathbb{R}^2$ may be unexpectedly small and not resemble a curve at all. Let us look at another example illustrating this issue.

Example 3.1. Let $t \in \mathbb{R}$ and consider

$$P_t(x, y) = x^2 + y^2 - t,$$

and the real plane curve

$$C_{t,\mathbb{R}} = \{(x, y) \in \mathbb{R}^2 \mid P_t(x, y) = 0\}.$$

- If $t > 0$, then $C_{t,\mathbb{R}}$ is a circle with radius $\sqrt{t}$.
- If $t = 0$, then $C_{0,\mathbb{R}} = \{(0, 0)\}$.
- If $t < 0$, then $C_{t,\mathbb{R}} = \emptyset$.

A related problem is that it is not so clear when two polynomials $P, Q \in \mathbb{R}[x, y]$ define the same real plane curve, that is, when they have the same zero set in $\mathbb{R}^2$.

Example 3.2. Consider the polynomials

$$P(x, y) = x^2y + y + x^3 + x, \qquad Q(x, y) = x^2 + 2xy + y^2.$$

Then, since $P(x, y) = (x + y)(x^2 + 1)$ and $Q(x, y) = (x + y)^2$, the polynomials P and Q define the same affine plane curve in $\mathbb{R}^2$, namely, the line in $\mathbb{R}^2$ defined by $y = -x$.

These difficulties disappear when working with curves in $\mathbb{C}^2$, essentially because $\mathbb{C}$ is algebraically closed by the fundamental theorem of algebra.

Theorem 3.3 (Fundamental theorem of algebra)**.** *Let $P \in \mathbb{C}[x]$ be a non-constant polynomial. Then P has at least one complex root: there exists $a \in \mathbb{C}$ such that $P(a) = 0$.*

As an immediate consequence, we see that complex affine plane curves are never finite.

Proposition 3.4. *Let $P \in \mathbb{C}[x, y]$ be a non-constant polynomial. Then the algebraic curve*

$$C = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}$$

is not a bounded subset of $\mathbb{C}^2$. In particular, it is not compact[2]*, and it contains infinitely many points.*

Proof. Because P is not constant, x or y will show up in a monomial in P. Assume that x occurs; otherwise, we can swap x and y. By grouping together monomials with the same degree in x, we can write $P(x, y)$ as

$$P(x, y) = P_0(y) + \cdots + x^d P_d(y), \qquad d \geq 1,$$

where $P_0(y), \ldots, P_d(y)$ are polynomials in y alone, with $P_d(y) \neq 0$. Let M be a real number. It suffices to find a point (x_0, y_0) in C with $|y_0| > M$.

Since $P_d(y) \neq 0$, this polynomial has only finitely many roots, so that there exists $y_0 \in \mathbb{C}$ satisfying $P_d(y_0) \neq 0$ and $|y_0| > M$. The polynomial $P(x, y_0)$ in the variable x is non-constant, and thus has at least one root x_0. In this way, we find a point (x_0, y_0) in C with $|y_0| > M$. □

[2]See appendix A for a review of some basic notions from topology.

Example 3.5. The circle with radius -1,

$$C_{-1,\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 + 1 = 0\},$$

from Example 3.1 has no $\mathbb{R}$-rational points but infinitely many complex points: for each value $y_0 \in \mathbb{C}$ we can solve the equation $x^2 + (y_0)^2 + 1 = 0$ to find either one or two values x_0 in $\mathbb{C}$ such that (x_0, y_0) lies in $C_{-1,\mathbb{C}}$. Similarly, the circle with radius 0,

$$C_{0,\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 0\},$$

has only one $\mathbb{R}$-rational point but infinitely many complex points: since

$$x^2 + y^2 = (x + iy)(x - iy)$$

it is the union of the complex lines defined by $x = -iy$ and $x = iy$, which intersect at the origin $(0, 0)$.

Remark 3.6. Visualizing complex affine plane curves is not easy, since they live inside the real vector space space $\mathbb{C}^2 \cong \mathbb{R}^4$ of dimension 4. When making sketches we will often either draw a cartoon of real dimension one inside a real plane, or project the complex curve onto a space with three real dimensions. These are only coarse approximations of the actual geometric picture; for instance, we cannot draw two real planes intersecting at a unique point as in Example 3.5.

We now turn to the question when two polynomials define the same complex affine plane curve. The following theorem is a special case of a more general result called *Hilbert's Nullstellensatz*, which will be covered in more advanced algebraic geometry modules.

Theorem 3.7 (Consequence of Hilbert's Nullstellensatz)**.** *Let $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ be two polynomials. Then*

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n \mid P(x_1, \dots, x_n) = 0\} = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid Q(x_1, \dots, x_n) = 0\}$$

if and only if there exist

$$P_1, \dots, P_k \in \mathbb{C}[x_1, \dots, x_n], \qquad a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Z}_{>0}, \qquad \lambda, \mu \in \mathbb{C}^*,$$

such that

$$\begin{cases} P(x_1, \dots, x_n) = \lambda P_1^{a_1} \dots P_k^{a_k}, \\ Q(x_1, \dots, x_n) = \mu P_1^{b_1} \dots P_k^{b_k}. \end{cases}$$

Proof. The "if" part is easy: when P and Q can be written in this way, then the zero sets of P and Q are both equal to the union of the zero sets of the polynomials $P_1, \dots, P_k$. The "only if" part is much deeper and relies in an essential way on the fact that $\mathbb{C}$ is algebraically closed. The proof requires tools from commutative algebra and is omitted; it will be covered in more advanced algebraic geometry modules. See also the optional Exercise 3.8 below. □

Exercise 3.8 ($\star\star$)**.** Reduce the "only if" implication of Theorem 3.7 to the special case where P and Q have no repeated factors. Now prove this special case for $n = 2$. You may freely use the following fact: if $P \in \mathbb{C}[x, y]$ is a non-constant polynomial with no repeated factors, then for all but finitely many values x_0 in $\mathbb{C}$, the polynomial $P(x_0, y)$ in $\mathbb{C}[y]$ has no multiple roots.

Theorem 3.7 implies in particular that two non-constant polynomials in $\mathbb{C}[x, y]$ define the same affine plane curve if and only if they have the same irreducible factors (ignoring multiplicities). Thus, the relation between a polynomial P in $\mathbb{C}[x, y]$ and the geometric shape of its zero set in $\mathbb{C}^2$ is much more transparent than in $\mathbb{R}^2$. We have already seen several examples that show that Theorem 3.7 is false for affine plane curves in $\mathbb{R}^2$: certain polynomials over $\mathbb{R}$ have so few real zeros (or even none at all) that the zero set does not capture enough information about the polynomial. The Nullstellensatz guarantees that this cannot happen over the field $\mathbb{C}$. The theorem becomes particularly elegant if we assume that P and Q have no repeated factors.

Exercise 3.9. Let $P, Q \in \mathbb{C}[x, y]$ be polynomials with no repeated factors. Deduce from Theorem 3.7 that P and Q define the same complex affine plane curve if and only if there is a non-zero constant $\lambda \in \mathbb{C}^*$ such that $P = \lambda Q$.

Thus the affine plane curve defined by a polynomial P with no repeated factors remembers (almost) the whole polynomial P! More precisely, we obtain a bijective correspondence between the set of affine plane curves in $\mathbb{C}^2$ and the set of non-constant polynomials in $\mathbb{C}[x, y]$ with no repeated factors up to multiplicative constants in $\mathbb{C}^*$. This correspondence is the bridge between algebra and geometry. We can use it to set up a dictionary that translates algebraic properties into geometry and vice versa. For instance, we will now establish geometric interpretations of division relations between polynomials, and of unique factorisation into irreducible polynomials.

Proposition 3.10. *Let $P, Q \in \mathbb{C}[x, y]$ be non-constant polynomials with no repeated factors, and let C and D be their respective zero sets. Then C is contained in D if and only if the polynomial P divides Q in the ring $\mathbb{C}[x, y]$.*

Proof. The "if" implication is trivial: if P divides Q then every zero of P is also a zero of Q. Conversely, assume that C is contained in D. Let $P = P_1 \cdots P_k$ and $Q = Q_1 \cdots Q_\ell$ be factorizations into irreducible polynomials. Let J be the set of indices j in $\{1, \dots, \ell\}$ such that Q_j does not divide P, and consider the polynomial

$$R = P \cdot \prod_{j \in J} Q_j.$$

By our choice of the set J, this polynomial has no repeated factors. The zero set of R coincides with the union of C and D, because every factor that we omitted from Q already divides P. This union is equal to D since C is contained in D. Thus the Nullstellensatz implies that $Q = \lambda R$ for some λ in $\mathbb{C}^*$. Since P divides R, it also divides Q. □

Just like non-constant polynomials can be factorized into irreducible polynomials, the elementary building blocks of affine plane curves are the *irreducible* affine plane curves.

Definition 3.11. A complex affine plane curve C is called *irreducible* if, whenever D is an affine plane curve contained in C, we have $C = D$.

Exercise 3.12. Let $P \in \mathbb{C}[x, y]$ be non-constant polynomials with no repeated factors. Show that the zero set of P is irreducible if and only if P is irreducible. Deduce that every affine plane curve C in $\mathbb{C}^2$ can be written as a finite union of distinct irreducible affine plane curves in a unique way. These curves are called the *irreducible components* of C.

It follows that there exists a bijective correspondence between irreducible affine plane curves and irreducible polynomials in $\mathbb{C}[x, y]$ up to multiplication with constants in $\mathbb{C}^*$.

Definition 3.13. The *degree* of an affine plane curve C in $\mathbb{C}^2$ is the degree of any polynomial P in $\mathbb{C}[x, y]$ with no repeated factors such that C is the zero set of P. We denote this invariant by $\deg C$.

This notion is well-defined because P is unique up to a factor in $\mathbb{C}^*$, so that the degree of P does not depend on the choice of the defining polynomial P. The degree of a curve is a measure for the complexity of the curve: the higher the degree, the more complicated the polynomial needed to define it.

Example 3.14.

- A line in $\mathbb{C}^2$ is the same thing as an affine plane curve of degree one.
- A *conic* C in $\mathbb{C}^2$ is the zero set of a polynomial $P(x, y)$ of degree two. Thus a conic has degree two, unless P has a repeated factor; in that case,

  $$P(x, y) = L(x, y)^2$$

  for some polynomial $L(x, y)$ of degree one, and C is a line.

- The affine plane curve
$$\{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + 1\}$$
has degree three, because its equation has no repeated factors (in fact, it is even irreducible). As we will see later, this is an example of an *elliptic curve*.

Exercise 3.15. Let P and Q be non-constant polynomials in $\mathbb{C}[x,y]$, and let C and D be their respective zero sets in $\mathbb{C}^2$.

- Show that C and D have a common irreducible component if and only if P and Q are both divisible by some non-constant polynomial R in $\mathbb{C}[x,y]$. Otherwise, we say that C and D have *no common component*.

- Show that
$$\deg C \cup D \leq \deg C + \deg D$$
and that equality holds if and only if C and D have no common component.

4 Projective space

The main reason to introduce projective geometry is to obtain a more satisfactory intersection theory, ultimately culminating in *Bézout's theorem* (Theorem 11.16). The basic motivating example is that of two distinct lines in $\mathbb{C}^2$: they will usually intersect in a unique point, but if they happen to be parallel, the intersection is empty. Completing the affine plane $\mathbb{C}^2$ into the projective plane $\mathbb{P}^2_{\mathbb{C}}$ fixes this problem by adding points at infinity, corresponding to all possible slopes of lines in $\mathbb{C}^2$. Two parallel lines in $\mathbb{C}^2$ will then intersect in $\mathbb{P}^2_{\mathbb{C}}$ at the point at infinity corresponding to their common slope.

On the algebraic side, this idea is implemented by introducing a triple of so-called *homogeneous coordinates* u, v and w and substituting the affine coordinates x and y by $x = u/w$ and $y = v/w$. We can then send the coordinates x and y to infinity by setting $w = 0$.

Example 4.1. Consider two distinct lines

$$L_1 = \{(x, y) \in \mathbb{C}^2 \mid a_1x + b_1y + c_1 = 0\}, \qquad L_2 = \{(x, y) \in \mathbb{C}^2 \mid a_2x + b_2y + c_2 = 0\}$$

where the coefficients of the equations lie in $\mathbb{C}$ and $(a_1, b_1) \neq (0, 0) \neq (a_2, b_2)$. Then L_1 and L_2 are parallel if and only if (a_1, b_1) and (a_2, b_2) are proportional, which is equivalent to saying that

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0.$$

Substituting $x = u/w$ and $y = v/w$ and mutiplying the resulting equations by w, we find homogeneous equations

$$\begin{cases} a_1u + b_1v + c_1w = 0, \\ a_2u + b_2v + c_2w = 0. \end{cases}$$

The assumption that L_1 and L_2 are distinct guarantees that this homogeneous system has rank 2, so that its solution set is a one-dimensional complex vector space

$$\{\lambda(u_0, v_0, w_0) \mid \lambda \in \mathbb{C}\}$$

for some non-zero vector (u_0, v_0, w_0) in $\mathbb{C}^3$. This vector space corresponds to a point of $\mathbb{P}^2_{\mathbb{C}}$, namely, the point p with *homogeneous coordinates* (u_0, v_0, w_0) (this will be made precise below). This is the unique intersection point of our two lines in $\mathbb{P}^2_{\mathbb{C}}$. If L_1 and L_2 are not parallel then $w_0 \neq 0$ and we can identify our intersection point p with the point $(u_0/w_0, v_0/w_0)$ in $\mathbb{C}^2$. If L_1 and L_2 are parallel then $w_0 = 0$ and p is the point at infinity corresponding to the common slope of L_1 and L_2.

In order to make this construction rigorous, we first need to formally define the projective plane $\mathbb{P}^2_{\mathbb{C}}$, and then to devise a way to turn algebraic curves in $\mathbb{C}^2$ into algebraic curves in $\mathbb{P}^2_{\mathbb{C}}$ by adding the appropriate points at infinity.

More generally, we will define the projective n-space $\mathbb{P}^n_{\mathbb{C}}$ for every integer $n \geq 0$. Let $\underline{0} = (0, \dots, 0) \in \mathbb{C}^{n+1}$ be the origin of the $(n+1)$-dimensional complex affine space. We define a relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$: for all $x, y \in \mathbb{C}^{n+1} \setminus \{\underline{0}\}$, we declare that $x \sim y$ if and only if the vectors x and y are proportional, that is, there exists $\lambda \in \mathbb{C}^*$ such that $y = \lambda x$. It is easy to see that $\sim$ is an equivalence relation.

Definition 4.2. The *n-dimensional complex projective space* $\mathbb{P}^n_{\mathbb{C}}$ (often also denoted by $\mathbb{P}^n(\mathbb{C})$, or simply $\mathbb{P}^n$), is the quotient of $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$ by the equivalence relation $\sim$.

Note that $\mathbb{P}^0_{\mathbb{C}}$ consists of a single point. We call $\mathbb{P}^1_{\mathbb{C}}$ the *projective line* over $\mathbb{C}$, and $\mathbb{P}^2_{\mathbb{C}}$ the *projective plane* over $\mathbb{C}$. The equivalence classes of $\sim$ are precisely the one-dimensional subspaces of the complex vector space $\mathbb{C}^{n+1}$, with the origin $\underline{0}$ omitted. Thus we can think of the points in $\mathbb{P}^n_{\mathbb{C}}$ as the one-dimensional subspaces of $\mathbb{C}^{n+1}$ (equivalently, the lines through the origin of $\mathbb{C}^{n+1}$).

When $x = (x_0, \dots, x_n)$ is a point of $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$, its equivalence class in $\mathbb{P}^n_{\mathbb{C}}$ will be denoted by $[x] = [x_0, \dots, x_n]$, and the $(n+1)$-tuple $(x_0, \dots, x_n)$ is called a tuple of *homogeneous coordinates* of the point $[x] \in \mathbb{P}^n_{\mathbb{C}}$. Thus if y is another element of $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$, then $[x] = [y]$ in $\mathbb{P}^n_{\mathbb{C}}$ if and only if there exists an element λ in $\mathbb{C}^*$ such that $y = \lambda x$.

A convenient way to understand the geometry of $\mathbb{P}^n_{\mathbb{C}}$ is to cover it by copies of the affine n-space $\mathbb{C}^n$. For any i in $\{0,\dots,n\}$, the i-th *affine chart* of $\mathbb{P}^n_{\mathbb{C}}$ is the subset

$$U_i = \{[x] = [x_0,\dots,x_n] \in \mathbb{P}^n_{\mathbb{C}} \mid x_i \neq 0\}.$$

Since the homogeneous coordinates of a point of $\mathbb{P}^n_{\mathbb{C}}$ are not all zero, $\mathbb{P}^n_{\mathbb{C}}$ is the union of the affine charts $U_0,\dots,U_n$. We define a map

$$\begin{array}{rccc} \phi_i \quad : & \mathbb{C}^n & \longrightarrow & U_i \\ & y = (y_1,\dots,y_n) & \longmapsto & [y_1,\dots,y_i,1,y_{i+1},\dots,y_n] \end{array}$$

that inserts a homogeneous coordinate 1 on the i-th position. Note that $\phi_i(y)$ is indeed a point in U_i.

Lemma 4.3. *The map ϕ_i is a bijection, and its inverse is given by*

$$\begin{array}{rccc} \psi_i \quad : & U_i & \longrightarrow & \mathbb{C}^n \\ & [x] = [x_0,\dots,x_n] & \longmapsto & \left(\dfrac{x_0}{x_i},\dots,\dfrac{x_{i-1}}{x_i},\dfrac{x_{i+1}}{x_i},\dots,\dfrac{x_n}{x_i}\right) \end{array}.$$

Proof. We first show that the map ψ_i is well-defined. The coordinate x_i is different from 0 since $[x]$ lies in U_i, so that we can divide by x_i. Moreover, the image $\psi_i([x])$ is independent of the choice of homogeneous coordinates: if $[x_0,\dots,x_n] = [x'_0,\dots,x'_n]$ then these tuples are proportional so that $x_j/x_i = x'_j/x'_i$ for every j in $\{0,\dots,n\}$.

Thus, it is enough to show that both $\phi_i\circ\psi_i$ and $\psi_i\circ\phi_i$ are the identity. This follows from a straightforward calculation: we have

$$\psi_i(\phi_i(y_1,\dots,y_n)) = \psi_i[y_1,\dots,y_i,1,y_{i+1},\dots,y_n] = \left(\frac{y_1}{1},\dots,\frac{y_i}{1},\frac{y_{i+1}}{1},\dots,\frac{y_n}{1}\right) = (y_1,\dots,y_n).$$

Similarly,

$$\phi_i(\psi_i[x_0,\dots,x_n]) = \phi_i\left(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right) = \left[\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},1,\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right] = [x_0,\dots,x_n].$$

□

We can use the maps ϕ_i and ψ_i to identify each affine chart U_i with a copy of $\mathbb{C}^n$. In particular, this allows us to define a topology on $\mathbb{P}^n_{\mathbb{C}}$ (see Appendix A for a review of some basic notions from topology). We say that a subset U of $\mathbb{P}^n_{\mathbb{C}}$ is open if $\psi_i(U\cap U_i)$ is open in $\mathbb{C}^n$ with respect to the usual metric topology on $\mathbb{C}^n$, for every i in $\{0,\dots,n\}$. We endow the affine charts U_i with the topology induced from $\mathbb{P}^n_{\mathbb{C}}$.

Exercise 4.4. Show that U_i is open in $\mathbb{P}^n_{\mathbb{C}}$ for every i in $\{0,\dots,n\}$, and that ϕ_i and ψ_i are homeomorphisms. ☜

Exercise 4.5. There is another natural way to define a topology on $\mathbb{P}^n_{\mathbb{C}}$, using the projection map

$$\pi\colon \mathbb{C}^{n+1}\setminus\{\underline{0}\} \to \mathbb{P}^n_{\mathbb{C}},\ x\mapsto [x].$$

If we consider the topology on $\mathbb{C}^{n+1}\setminus\{\underline{0}\}$ induced from its embedding into $\mathbb{C}^{n+1}$, then we can endow $\mathbb{P}^n_{\mathbb{C}}$ with the *quotient topology* with respect to π. In down-to-earth terms, a subset U in $\mathbb{P}^n_{\mathbb{C}}$ is open with respect to this quotient topology if and only if its preimage $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1}$. Prove that this topology coincides with the one defined above. In particular, the projection map π is continuous.

Exercise 4.6 ($\star$)**.** Prove that $\mathbb{P}^n_{\mathbb{C}}$ is compact and Hausdorff. ☜

Example 4.7. We can think of the projective line $\mathbb{P}^1_C$ as being obtained by gluing two copies of $\mathbb{C}$. Indeed, $\mathbb{P}^1_C$ is the union of the affine charts U_0 and U_1, which are both homeomorphic to $\mathbb{C}$ *via* the maps

$$\psi_0\colon U_0\to\mathbb{C},\ [x_0,x_1]\mapsto x_1/x_0, \quad \psi_1\colon U_1\to\mathbb{C},\ [x_0,x_1]\mapsto x_0/x_1.$$

These homeomorphisms both identify the intersection $U_0 \cap U_1$ with $\mathbb{C}^*$; thus our two copies of $\mathbb{C}$ are glued together by identifying the open subsets $\mathbb{C}^*$ *via* the map

$$\psi_1 \circ \phi_0 \colon \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto 1/z.$$

We can also use this description to prove that $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to the 2-sphere S^2. We embed S^2 as the unit sphere in $\mathbb{R}^3$ and we identify $\mathbb{C}$ with the horizontal plane in $\mathbb{R}^3$ by viewing the real and complex parts of $z \in \mathbb{C}$ as real coordinates. Let

$$p \colon S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$$

be the homeomorphism given by stereographic projection from the north pole $(0,0,1)$. It identifies the south pole $(0,0,-1)$ with $0 \in \mathbb{C}$. Now consider the map

$$q \colon \mathbb{C}^* \to S^2 \setminus \{(0,0,1),(0,0,-1)\},\ z \mapsto p^{-1}(1/z).$$

This is also a homeomorphism because taking inverses defines a homeomorphism of $\mathbb{C}^*$ onto itself. When z tends to zero, then $1/z$ tends to infinity and $p^{-1}(1/z)$ tends to the north pole. It follows that q extends to a continuous bijection

$$\overline{q} \colon \mathbb{C} \to S^2 \setminus \{(0,0,-1)\},\ z \mapsto p^{-1}(1/z).$$

We have set up the construction is such a way that

$$p^{-1} \circ \psi_0 \colon U_0 \to S^2 \setminus \{(0,0,1)\} \text{ and } \overline{q} \circ \psi_1 \colon U_1 \to S^2 \setminus \{(0,0,-1)\}$$

coincide on $U_0 \cap U_1$. Therefore, they glue to a continuous map $\mathbb{P}^1_{\mathbb{C}} \to S^2$, which is bijective, and thus a homeomorphism because the source is compact and the target is Hausdorff.

Likewise, the projective n-space $\mathbb{P}^n_{\mathbb{C}}$ is obtained by gluing $n+1$ copies of $\mathbb{C}^n$; the gluing maps can be computed from the formulas for ϕ_i and ψ_i. The choice of an index i in $\{0,\dots,n\}$ determines an open embedding $\mathbb{C}^n \to \mathbb{P}^n_{\mathbb{C}}$, namely, the composition of ϕ_i with the inclusion map of U_i into $\mathbb{P}^n_{\mathbb{C}}$. We will now describe the complement of this embedding. We set

$$\mathcal{P}_i = \mathbb{P}^n_{\mathbb{C}} \setminus U_i = \{[x_0,\dots,x_n] \in \mathbb{P}^n \mid x_i = 0\}.$$

Exercise 4.8. Assume that $n \geq 1$. Show that the function

$$\begin{array}{cccc} \theta_i & : & \mathbb{P}^{n-1}_{\mathbb{C}} & \longrightarrow & \mathcal{P}_i \\ & & [x_0,\dots,x_{n-1}] & \longmapsto & [x_0,\dots,x_{i-1},0,x_i,\dots,x_{n-1}] \end{array},$$

which inserts a homogeneous coordinate 0 at the i-th position, is a homeomorphism if we endow $\mathcal{P}_i$ with the topology induced from $\mathbb{P}^n_{\mathbb{C}}$.

Thus we can also think of $\mathbb{P}^n_{\mathbb{C}}$ as being obtained from $\mathbb{C}^n$ by attaching a copy of $\mathbb{P}^{n-1}_{\mathbb{C}}$; then $\mathcal{P}_i$ is called the *hyperplane at infinity*. Beware that this depends on the choice of an index i, since we have $n+1$ different ways to embed $\mathbb{C}^n$ into $\mathbb{P}^n_{\mathbb{C}}$, given by the $n+1$ affine charts. By induction on n, we can then further decompose the set $\mathbb{P}^n_{\mathbb{C}}$ into a disjoint union

$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \dots \sqcup \mathbb{C}^0$$

where we have implicitly chosen an affine chart in $\mathbb{P}^m_{\mathbb{C}}$ for each m in $\{0,\dots,n-1\}$, and identified it with $\mathbb{C}^m$.

Example 4.9.

- We have $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^1 \sqcup \mathbb{C}^0$. In other words, $\mathbb{P}^1_{\mathbb{C}}$ is obtained by adding a *point at infinity* to the complex line $\mathbb{C}$.
- We have $\mathbb{P}^2_{\mathbb{C}} = \mathbb{C}^2 \sqcup \mathbb{P}^1_{\mathbb{C}}$. Thus $\mathbb{P}^2_{\mathbb{C}}$ is obtained by adding a projective *line at infinity* to the complex plane $\mathbb{C}^2$.

5 Projective plane curves

In Section 2, we have defined an affine plane curve in $\mathbb{C}^2$ as the zero set of a non-constant polynomial in $\mathbb{C}[x, y]$. We will now formulate a similar definition for algebraic curves in $\mathbb{P}^2_{\mathbb{C}}$, which are called *projective* plane curves. We will also set up a construction to turn affine plane curves into projective plane curves by adding points at infinity.

A natural first attempt is to define projective plane curves as zero sets of non-constant complex polynomials P in the homogeneous coordinates x_0, x_1, x_2. However, this does not quite work, because P might vanish on one tuple of homogeneous coordinates of a point in $\mathbb{P}^2_{\mathbb{C}}$ and not on another, so that the zero set of P is not always well-defined.

Example 5.1. If

$$P(x_0, x_1, x_2) = x_0^2 + x_1 + x_2,$$

then $P(1, -1, 0) = 0$, but $P(2, -2, 0) \neq 0$, while $(1, -1, 0)$ and $(2, -2, 0)$ define the same point $[1, -1, 0] = [2, -2, 0]$ of $\mathbb{P}^2_{\mathbb{C}}$.

The solution is to consider only a special class of polynomials P, namely, the *homogeneous* polynomials.

Definition 5.2. A polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$ is called *homogeneous* of degree $d \in \mathbb{N}$ if all its monomials have degree d; that is,

$$P(x_0, \dots, x_n) = \sum_{\alpha \in \mathbb{N}^{n+1},\ |\alpha| = d} c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{C}.$$

Example 5.3.

- The polynomial $Q(x_0, z_1, z_2) = x_0^4 - 2x_0^2 x_1 x_2 + 5x_1 x_2^3$ is homogeneous of degree four.
- The polynomial $R(x_0, x_1, x_2) = x_0^2 + x_1 + x_2$ is not homogeneous because it contains a monomial of degree two and also a monomial of degree one.
- The zero polynomial 0 is homogeneous of any degree $d \in \mathbb{N}$. This causes a small conflict of terminology: when we view 0 as a general polynomial, the convention is that it has degree $-\infty$ (in order to be consistent with the formula for the degree of a product of polynomials). When we view 0 as a *homogeneous* polynomial, we say that it has any degree $d \in \mathbb{N}$; this is motivated by the theory of graded rings in commutative algebra.

The key property of homogeneous polynomials is that their zero set is closed under the operation of rescaling coordinates, as a consequence of the following result.

Proposition 5.4. *Let $P \in \mathbb{C}[x_0, \dots, x_n]$ be a polynomial. Then P is homogeneous of degree $d \in \mathbb{N}$ if and only if*

$$P(\lambda a_0, \dots, \lambda a_n) = \lambda^d P(a_0, \dots, a_n)$$

for all $\lambda \in \mathbb{C}$ and all $a = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$.

Proof. The "only if" implication is obvious: if $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ is a monomial of degree $|\alpha| = d$, then

$$(\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n} = \lambda^{|\alpha|} a^\alpha.$$

The analogous property then holds for $\mathbb{C}$-linear combinations of monomials of degree d, which are precisely the homogeneous polynomials of degree d.

Conversely, assume that

$$P(\lambda a_0, \dots, \lambda a_n) = \lambda^d P(a_0, \dots, a_n)$$

for all $\lambda \in \mathbb{C}$ and all $a = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$. For every polynomial Q in $\mathbb{C}[x_0, \dots, x_n]$, we can view $Q(yx_0, \dots, yx_n) - y^d Q(x_0, \dots, x_n)$ as a complex polynomial in the variables $x_0, \dots, x_n, y$. Our assumption means that, for $Q = P$, this polynomial vanishes on the whole of $\mathbb{C}^{n+2}$. Since $\mathbb{C}$ is infinite, it follows that $P(yx_0, \dots, yx_n) = y^d P(x_0, \dots, x_n)$ (see Problem Sheet 1). Writing P as a $\mathbb{C}$-linear combination of monomials, one sees that this can only happen when each monomial has degree d. $\square$

Definition 5.5. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$, and let $[a]$ be a point in $\mathbb{P}^n_{\mathbb{C}}$, with homogeneous coordinates $a = (a_0, \dots, a_n)$ in $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$. Then we say that $P([a]) = 0$ if $P(a_0, \dots, a_n) = 0$.

Proposition 5.4 guarantees that this notion is well-defined, since it does not depend on the choice of the tuple a of homogeneous coordinates. Beware that, when $P(a) \neq 0$, the value of P at a is not invariant under rescaling, so that it does not make sense to speak about the value of P at the point $[a]$.

Example 5.6. The polynomial

$$P(x_0, x_1, x_2) = x_0^2 + x_1^2 - x_2^2$$

is homogeneous of degree two. The triples $(\lambda, 0, \lambda)$ with $\lambda \in \mathbb{C}^*$ all define the same point p in $\mathbb{P}^2_{\mathbb{C}}$, and P vanishes on all these triples. Thus p lies in the zero set of P.

Definition 5.7. A *projective plane curve* over $\mathbb{C}$ is a subset of $\mathbb{P}^2_{\mathbb{C}}$ of the form

$$C = \{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid P(x_0, x_1, x_2) = 0\}$$

where P is a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$.

Example 5.8. A *line* in $\mathbb{P}^2_{\mathbb{C}}$ is a projective plane curve defined by a homogeneous polynomial $P = c_0x_0 + c_1x_1 + c_2x_2$ of degree one, with $c_0, c_1, c_2 \in \mathbb{C}$, not all zero.

Many constructions and results for affine plane curves carry over to the projective setting.

Proposition 5.9. *If P is a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$, then the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$ is infinite.*

Proof. We may assume that the variable x_0 appears in P; otherwise, we can permute the variables. Writing P as a $\mathbb{C}$-linear combination of degree d monomials, one sees that $P(x_0, x_1, 1)$ is a non-constant polynomial in $\mathbb{C}[x_0, x_1]$. This polynomial has infinitely many zeroes, by Proposition 3.4; each of these zeros (a_0, a_1) gives rise to a separate zero $[a_0, a_1, 1]$ of P in $\mathbb{P}^2_{\mathbb{C}}$. □

Theorem 5.10 (Projective Nullstellensatz)**.** *Let P and Q be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors. Then P and Q have the same zero set in $\mathbb{P}^2_{\mathbb{C}}$ if and only if $Q = \lambda P$ for some $\lambda \in \mathbb{C}^*$.*

Proof. The "if" implication is trivial, so let us prove the "only if" implication. Assume that P and Q have the same zero set in $\mathbb{P}^2_{\mathbb{C}}$. If this set is empty then P and Q lie in $\mathbb{C}^*$ by Proposition 5.9, and the result holds. Thus we may assume that this set is non-empty; then P and Q are either zero or have positive degree.

If a is an element of $\mathbb{C}^3 \setminus \{\underline{0}\}$ then $P(a) = 0$ if and only if $P([a]) = 0$, and the analogous statement applies to Q. Thus P and Q have the same zero set in $\mathbb{C}^3 \setminus \{\underline{0}\}$. Moreover, P and Q both vanish at $\underline{0}$, since they are zero or have positive degree. Therefore, P and Q have the same zero set in $\mathbb{C}^3$. Now the result follows from the affine Nullstellensatz (Theorem 3.7). □

Lemma 5.11. *Let $P \in \mathbb{C}[x_0, \dots, x_n]$ be a non-zero homogeneous polynomial of degree $d \geq 0$. Assume that we can write P as a product $Q \cdot R$ with $Q, R \in \mathbb{C}[x_0, \dots, x_n]$. Then the polynomials Q and R are homogeneous.*

Proof. Since P is homogeneous of degree d, we have

$$y^d P(x_0, \dots, x_n) = P(yx_0, \dots, yx_n) = Q(yx_0, \dots, yx_n)R(yx_0, \dots, yx_n)$$

in $\mathbb{C}[x_0, \dots, x_n, y]$. By the additivity of degrees with respect to multiplication, we have $d = \deg(Q) + \deg(R)$. On the other hand, $Q(yx_0, \dots, yx_n)$ has degree $\deg(Q)$ in the variable y, and the analogous statement holds for R. By unique factorization, it follows that $y^{\deg(Q)}$ divides $Q(yx_0, \dots, yx_n)$ and $y^{\deg(R)}$ divides $R(yx_0, \dots, yx_n)$. This is only possible when every monomial in Q has degree $\deg(Q)$ and every monomial in R has degree $\deg(R)$. □

Lemma 5.11 implies that the factors in any factorization of P into irreducible polynomials are themselves homogeneous. This allows us to copy the remaining results from Section 3 to the projective setting. First of all, every projective plane curve can be realized as the zero set of a non-constant homogeneous polynomial with no repeated factors: simply pick any homogeneous polynomial defining the curve, write a factorization into irreducible polynomials, and delete recurring factors. The resulting polynomial is still homogeneous, by Lemma 5.11. We can then define the *degree* of a projective plane curve just like in the affine case.

Definition 5.12. Let C be a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$. The *degree* of C is the degree of any homogeneous polynomial with no repeated factors whose zero set equals C.

This definition does not depend on the choice of P, by Theorem 5.10. Just like in the affine case, a projective plane curve of degree 1 is called a *line*, and the zero set of a homogeneous degree 2 polynomial P in $\mathbb{P}^2_{\mathbb{C}}$ is called a *conic*; this is either a line (if P is the square of a linear polynomial) or a projective plane curve of degree 2.

Exercise 5.13. Show that three points $[a_0, a_1, a_2]$, $[b_0, b_1, b_2]$ and $[c_0, c_1, c_2]$ in $\mathbb{P}^2_{\mathbb{C}}$ are collinear if and only if the matrix

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

has determinant 0. Deduce that there exists a unique line through any two distinct points in $\mathbb{P}^2_{\mathbb{C}}$.

We can also generalize the decomposition into irreducible components to the projective case.

Definition 5.14. A projective plane curve C is called *irreducible* if, for every projective plane curve D contained in C, we have $D = C$.

Exercise 5.15. Let P and Q be homogeneous polynomials of positive degrees in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors, and let C and D be their respective zero sets in $\mathbb{P}^2_{\mathbb{C}}$. Show that C is contained in D if and only if P divides Q. Deduce that C is irreducible if and only if P is irreducible, and that we can write every projective plane curve C as a finite union of distinct irreducible projective plane curves in a unique way. These irreducible curves are called the *irreducible components* of C. We say that two projective plane curves have *no common component* if they do not share any irreducible component.

A key difference between affine and projective plane curves lies in their topological properties. Affine plane curves are subsets of $\mathbb{C}^2$ and thus inherit an induced topology from $\mathbb{C}^2$. Likewise, projective plane curves inherit a topology from $\mathbb{P}^2_{\mathbb{C}}$. Whereas affine plane curves are *never* compact (they are unbounded in $\mathbb{C}^2$, by Proposition 3.4), projective plane curves are *always* compact.

Proposition 5.16. *Projective plane curves are compact and Hausdorff.*

Proof. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with zero set C in $\mathbb{P}^2_{\mathbb{C}}$. Since $\mathbb{P}^2_{\mathbb{C}}$ is compact and Hausdorff by Exercise 4.6, we only need to prove that C is closed in $\mathbb{P}^2_{\mathbb{C}}$. Since the affine charts U_0, U_1 and U_2 form an open cover of $\mathbb{P}^2_{\mathbb{C}}$, it suffices to show that $C \cap U_i$ is closed in U_i for every i in $\{0, 1, 2\}$. By symmetry, it is enough to show that $C \cap U_2$ is open in U_2. But if we use the homeomorphism ψ_2 to identify U_2 with $\mathbb{C}^2$, then $C \cap U_2$ is the zero set of the polynomial $P(x_0, x_1, 1)$ in $\mathbb{C}^2$. This zero set is closed in U_2 because the polynomial $P(x_0, x_1, 1)$ defines a continuous map $\mathbb{C}^2 \to \mathbb{C}$. $\square$

6 Affine vs projective plane curves

Given a projective plane curve C in $\mathbb{P}^2_{\mathbb{C}}$, we can view the intersection of C with the affine chart

$$U_2 = \{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_2 \neq 0\}$$

as an affine plane curve, provided it is non-empty. Indeed, if C is the zero set of a homogeneous polynomial $P(x_0, x_1, x_2)$ in $\mathbb{C}[x_0, x_1, x_2]$, then the image of $C \cap U_2$ under the homeomorphism

$$\psi_2 \colon U_2 \to \mathbb{C}^2,\ [x, y, 1] \mapsto (x, y)$$

is the zero set of the polynomial $P(x, y, 1) \in \mathbb{C}[x, y]$. There is nothing particular about the affine chart U_2, the analogous property holds for U_0 and U_1.

Conversely, every affine plane curve in $\mathbb{C}^2$ can be extended in a natural way to a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$, up to the choice of an affine chart. Here we will work with the chart U_2, but one can pass to any other chart by swapping x_2 with another variable. Let P be a polynomial in $\mathbb{C}[x, y]$ of positive degree d. The *homogenization* of the polynomial P is given by

$$\overline{P}(x_0, x_1, x_2) = x_2^d P\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) \in \mathbb{C}[x_0, x_1, x_2].$$

Note that this is indeed a polynomial because the factor x_2^d gets rid of all the powers of x_2 that appear in the denominators. Moreover, it is a *homogeneous* polynomial of degree d, because $\overline{P}(\lambda x_0, \lambda x_1, \lambda x_2) = \lambda^d \overline{P}(x_0, x_1, x_2)$ for all $\lambda \in \mathbb{C}$. Thus, $\overline{P}$ defines a projective plane curve $\overline{C}$ in $\mathbb{P}^2_{\mathbb{C}}$. It is an extension of the curve C in the sense that $\psi_2(\overline{C} \cap U_2)$ is the zero locus of $\overline{P}(x, y, 1) = P(x, y)$, and therefore coincides with C.

Definition 6.1. The curve $\overline{C}$ is called the *projectivization* of C.

It follows from the Nullstellensatz that $\overline{C}$ only depends on C, and not on the polynomial P.

Example 6.2. Let C be the affine plane curve in $\mathbb{C}^2$ defined by the polynomial $P(x, y) = x^2 - y^3 - 1 = 0$ of degree $d = 3$. Then its projectivization $\overline{C}$ is the zero set in $\mathbb{P}^2_{\mathbb{C}}$ of the homogeneous polynomial

$$\overline{P}(x_0, x_1, x_2) = x_2^3 \left(\left(\frac{x_0}{x_2}\right)^2 - \left(\frac{x_1}{x_2}\right)^3 - 1\right) = x_0^2 x_2 - x_1^3 - x_2^3.$$

A quick way to write down $\overline{P}$ is to substitute x and y by x_0 and x_1 and then add a power of x_2 to each monomial in such a way that the polynomial becomes homogeneous of degree d. An even quicker way is to work in homogeneous coordinates (x, y, z) and to add a suitable power of z to each monomial, but we will be careful to distinguish the affine coordinates x and y from the homogeneous coordinates x_0 and x_1: the correct relation is $x = x_0/x_2$ and $y = x_1/x_2$.

Exercise 6.3. Let C be an affine plane curve in $\mathbb{C}^2$. Show that $\overline{C}$ does not contain the line at infinity L_∞ in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_2 = 0$. Show that, when D is a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ such that $\psi_2(D \cap U_2) = C$, we have $D = \overline{C}$ or $D = \overline{C} \cup L_\infty$. Thus $\overline{C}$ is the smallest extension of C to a projective plane curve. ☜

Exercise 6.4 ($\star\star$)**.** Show that $\overline{C}$ is the closure of $\psi_2^{-1}(C)$ in $\mathbb{P}^2_{\mathbb{C}}$. This is surprisingly hard! The standard proof uses the *Weierstrass preparation theorem*, which you can Google if you don't know what it is.

As Exercise 6.3 indicates, if we start from a projective plane curve D in $\mathbb{P}^2_{\mathbb{C}}$, it is not quite true that we can always recover D as the projectivization of $\psi_2(D \cap U_2)$. For one thing, if $D = L_\infty$ then $D \cap U_2$ is empty and not an affine plane curve. But even if $D \cap U_2$ is non-empty, if may happen that D contains L_∞ as an irreducible component, in which case the projectivization of $\psi_2(D \cap U_2)$ is strictly smaller than D.

Example 6.5. The projective plane curve C defined by the homogeneous polynomial $P(x_0, x_1, x_2) = x_2(x_0 x_1 - x_2^2)$ consists of two irreducible components: the line at infinity L_∞, and the conic C' defined by $x_0 x_1 - x_2^2 = 0$. The homogenization of $P(x, y, 1) = xy - 1$ only recovers the factor $x_0 x_1 - x_2^2$, so that the projectivization of $\psi_2(C \cap U_2)$ is equal to C'.

However, if we discard the projective plane curves that contain L_∞, then we do get a bijective correspondence between affine and projective plane curves.

Proposition 6.6. *The map*

$$\{\textit{affine plane curves in } \mathbb{C}^2\} \to \{\textit{projective plane curves in } \mathbb{P}^2_{\mathbb{C}} \textit{ that do not contain } L_\infty\}$$

that sends an affine plane curve C to its projectivization $\overline{C}$ is a bijection. Its inverse is the map $D \mapsto \psi_2(D \cap U_2)$ that restricts a projective plane curve to the affine chart U_2.

Proof. We have already argued that the restriction to U_2 of the projectivization of an affine plane curve C gives back C. Thus, it suffices to show that when D is a projective plane curve that does not contain L_∞, then $\psi_2(D \cap U_2)$ is an affine plane curve whose projectivization equals D. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ of positive degree d with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is D. The assumption that D does not contain L_∞ is equivalent to the property that x_2 does not divide P. This implies that $P(x, y, 1)$ is not a constant polynomial, so that its zero set $C = \psi_2(D \cap U_2)$ is an affine plane curve. By Exercise 6.3, the projectivization $\overline{C}$ of C is the unique projective plane curve that extends C and does not contain L_∞. Since D also has this property, we have $\overline{C} = D$. $\square$

From now on, we will leave the identification of the affine chart U_2 with $\mathbb{C}^2$ *via* the homeomorphism ψ_2 implicit: we will think of U_2 as a copy of $\mathbb{C}^2$ sitting inside $\mathbb{P}^2_{\mathbb{C}}$, with affine coordinates $x = x_0/x_2$ and $y = x_1/x_2$ (and, similarly, we will think of U_0 and U_1 as two other copies of $\mathbb{C}^2$ sitting inside $\mathbb{P}^2_{\mathbb{C}}$). This makes it possible to speak directly of affine plane curves C in U_2 and their projectivizations $\overline{C}$ in $\mathbb{P}^2_{\mathbb{C}}$.

Definition 6.7. If C is an affine plane curve in $\mathbb{C}^2$, then the points of $\overline{C} \setminus C$ are called the *points at infinity* of C.

The points at infinity capture the asymptotic directions of the curve C, as illustrated by the following examples.

Example 6.8. We return to our motivating example for projective geometry, Example 4.1. Let L_1 and L_2 be two parallel lines in U_2, with respective equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$. For each j in $\{1, 2\}$, the projectivization $\overline{L}_j$ is then defined by the homogeneous linear equation $ax_0 + bx_1 + c_j x_2 = 0$. We find the points at infinity by intersecting with the line at infinity defined by $x_2 = 0$. Thus, we see that L_1 and L_2 have the same point at infinity, namely, the point $[b, -a, 0]$.

Example 6.9. Let C be the hyperbola in U_2 defined by $xy - x - 1 = 0$. Then $\overline{C}$ is defined by $x_0x_1 - x_0x_2 - x_2^2 = 0$, and C has two points at infinity, namely, $[1, 0, 0]$ and $[0, 1, 0]$. These are also the points at infinity of the asymptotes of the curve C, the lines defined by $y = 1$ and $x = 0$.

How many points at infinity can an affine plane curve C have? This number turns out to be intimately related to the degree of C; the higher the degree, the more asymptotic directions the curve can have. We will deduce a precise statement from the following elementary lemma.

Lemma 6.10. *Let $P \in \mathbb{C}[x_0, x_1]$ be a non-zero homogeneous polynomial of degree $d \geq 1$. Then there exist $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ in $\mathbb{C}$, with $(\alpha_j, \beta_j) \neq (0, 0)$ for $j \in \{1, \ldots, d\}$, such that*

$$P(x_0, x_1) = \prod_{j=1}^{d} (\alpha_j x_0 + \beta_j x_1).$$

The zero set of P in $\mathbb{P}^1_{\mathbb{C}}$ then consists of the points $[\beta_j, -\alpha_j]$ for $j = 1, \ldots, d$.

Proof. We may assume that P is not divisible by x_1; otherwise, we can simply put the largest possible power of x_1 in front. Since P is homogeneous of degree d, we can write $P(x_0, x_1) = (x_1)^d P\left(\frac{x_0}{x_1}, 1\right)$. The assumption that P is not divisible by x_1 guarantees that $P(x_0/x_1, 1)$ is a polynomial of degree d in the variable x_0/x_1. By the fundamental theorem of algebra, we can write it as a product of linear polynomials:

$$P\left(\frac{x_0}{x_1}, 1\right) = \prod_{j=1}^{d} (\alpha_j \frac{x_0}{x_1} + \beta_j).$$

Multiplying with x_1^d yields the desired result. $\square$

Proposition 6.11. *Let C be an affine plane curve in U_2, defined by a polynomial $P(x,y)$ of degree $d \geq 1$ in $\mathbb{C}[x,y]$. We write $P = P_d + P_{d-1} + \ldots + P_0$ where P_i is a homogeneous polynomial of degree i in $\mathbb{C}[x,y]$ for each i in $\{0,\ldots,d\}$. We identify the line L_∞ at infinity with $\mathbb{P}^1_{\mathbb{C}}$ by means of the homeomorphism*

$$L_\infty \to \mathbb{P}^1_{\mathbb{C}},\ [x,y,0] \mapsto [x,y].$$

Then the points at infinity of C are precisely the points in the zero set of $P_d(x_0,x_1)$ in $\mathbb{P}^1_{\mathbb{C}}$. This set is non-empty and has at most d elements.

Proof. By definition, the points at infinity of C are the zeros of the polynomial

$$\overline{P}(x_0,x_1,0) = \left(x_2^d P\left(\frac{x_0}{x_2},\frac{x_1}{x_2}\right)\right)|_{x_2=0} = \left(P_d(x_0,x_1) + x_2P_{d-1}(x_0,x_1) + \ldots + x_2^d P_0(x_0,x_1)\right)|_{x_2=0}$$

which is precisely $P_d(x_0,x_1)$. By Lemma 6.10, this set is non-empty and has at most d distinct elements. □

A slightly different perspective is offered by taking the zero set of the homogeneous degree d polynomial $P_d(x_0,x_1)$ inside the affine plane $\mathbb{C}^2$: factoring P_d as in Lemma 6.10, we see that this zero set is the union of the lines through $(0,0)$ whose points at infinity are also points at infinity of C. These lines indicate the asymptotic directions of C. Note that they are not necessarily asymptotes in the usual sense, because those need not pass through the origin (for instance, in Example 6.9, one of the asymptotes is the line defined by $y = 1$, rather than the line $y = 0$).

Proposition 6.11 gives a quick way to compute the points at infinity in concrete examples.

Example 6.12. Let a and b be elements in $\mathbb{R}_{>0}$. The ellipse defined by $ax^2 + by^2 = 1$ in U_2 has two imaginary points at infinity, which form the zero set of the homogeneous polynomial $ax_0^2 + bx_1^2$ in the line at infinity: $[\sqrt{b}, \pm\sqrt{a}i, 0]$. The parabola given by $y = x^2$ has a unique point at infinity: $[0,1,0]$, corresponding to the direction of the y-axis. The hyperbola defined by $ax^2 - by^2 = 1$ has two real points at infinity: $[\sqrt{b}, \pm\sqrt{a}, 0]$. These are also the points at infinity of its two real asymptotes.

Exercise 6.13. Let C be an affine plane curve in $\mathbb{C}^2$, with projectivization $\overline{C}$. Show that the irreducible components of $\overline{C}$ are the projectivizations of the irreducible components of C. In particular, C is irreducible if and only if $\overline{C}$ is irreducible.

7 Smoothness and singularities

Singularity theory is the study of the local shape of algebraic sets. The essential distinction is that between *smooth* and *singular* points, where the smooth case is the generic one and singularities are points where something "special" occurs. Let us illustrate this rather vague statement by means of a simple example.

(a) The parabola $y = x^2$ (b) The cusp $y^3 = x^2$

Figure 1: Smooth versus singular

We have drawn the set of $\mathbb{R}$-rational points on two affine plane curves, the *parabola* defined by $y = x^2$ and the *cusp* defined by $y^3 = x^2$. The only reason to restrict to $\mathbb{R}$-rational points is that the sets of all $\mathbb{C}$-rational points are objects of real dimension 2 in $\mathbb{C}^2 \cong \mathbb{R}^4$, and therefore much harder to visualize. The parabola looks roughly the same around each of its points. On the cusp, however, we notice a difference between the origin and the other points: the cusp looks "pinched" at the origin, reflecting the fact that this is a singular point. On the algebraic side, the presence of singularities is detected by the vanishing of the partial derivatives of a polynomial equation. Given a polynomial P in $\mathbb{C}[x, y]$, we will denote by $\partial_x P$ and $\partial_y P$ the partial derivatives of P with respect to the variables x and y.

Definition 7.1. Let P be a non-constant polynomial in $\mathbb{C}[x, y]$ with no repeated factors, and denote by C its zero set in $\mathbb{C}^2$. Let $p = (a, b)$ be a point in C. We say that the affine plane curve C is *singular* at p, or that p is a singular point of C, if $\partial_x P(a, b) = \partial_y P(a, b) = 0$. Otherwise, we say that C is *smooth* at p, or that p is a smooth point of C. The curve C is called *singular* if it has at least one singular point, and *smooth* if it is smooth at each of its points.

If C is smooth at p, then the *tangent line* of C at p is the line in $\mathbb{C}^2$ defined by

$$\partial_x P(a, b)(x - a) + \partial_y P(a, b)(y - b) = 0.$$

These definitions only depend on the curve C and the point p, and not on the choice of P, since P is determined by C up to a factor in $\mathbb{C}^*$ by the Nullstellensatz. The tangent line of C at a smooth point p is the best approximation of C by a line in $\mathbb{C}^2$ around the point p.

Example 7.2.

- Every line in $\mathbb{C}^2$ is smooth.
- The affine plane curve in $\mathbb{C}^2$ defined by $x^2 + y^2 - 1 = 0$ is smooth, because it does not contain any points where $2x = 2y = 0$.
- The affine plane curve in $\mathbb{C}^2$ defined by $y^2 - x^2(x + 1) = 0$ is singular at the point $(0, 0)$ and smooth at all other points (see Figure 2 for a picture of the $\mathbb{R}$-rational points).
- The affine plane curve in $\mathbb{C}^2$ defined by $x^2 - y^3 = 0$ is singular at the point $(0, 0)$ and smooth at all other points (see Figure 1).

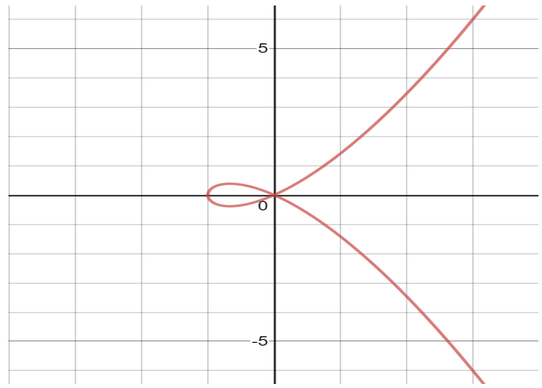

Figure 2: The node $y^2 = x^2(x+1)$

Exercise 7.3. For which values of $\lambda \in \mathbb{C}$ is the affine plane curve in $\mathbb{C}^2$ defined by

$$y^2 - x(x-1)(x-\lambda) = 0$$

singular? What are the singular points?

We can formulate analogous definitions for projective plane curves.

Definition 7.4. Let P be a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors, and denote by C its zero set in $\mathbb{P}^2_{\mathbb{C}}$. Let $p = [a, b, c]$ be a point in C. We say that the projective plane curve C is *singular* at p, or that p is a singular point of C, if $\partial_{x_i} P(a, b, c) = 0$ for each i in $\{0, 1, 2\}$. Otherwise, we say that C is *smooth* at p, or that p is a smooth point of C.

The curve C is called *singular* if it has at least one singular point, and *smooth* if it is smooth at each of its points.

Note that this definition does not depend on the choice of homogeneous coordinates (a, b, c), because the partial derivatives of a homogeneous polynomial of degree $d > 0$ are homogeneous polynomials of degree $d - 1$. It also only depends on C, and not on the polynomial P, since P is determined by C up to a factor in $\mathbb{C}^*$, by the projective Nullstellensatz.

Exercise 7.5. Determine the singular points of the projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ defined by the following equations:

1. $x_0^2 + x_1^2 + x_2^2 = 0$,
2. $x_0 x_1^2 - x_2^3 = 0$

We could also have defined singular and smooth points by restricting the curve C to any affine chart that contains the point p, and applying the definition in the affine case. We will now show that this is equivalent to our definition, using the following elementary result.

Proposition 7.6 (Euler's relation)**.** *Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of positive degree d. Then*

$$x_0 \cdot \partial_{x_0} P + x_1 \cdot \partial_{x_1} P + x_2 \cdot \partial_{x_2} P = d \cdot P.$$

Proof. It suffices to prove the equality in the case where P is a monomial; then it follows from a straightforward calculation. There is also an alternative proof that uses the expression

$$P(\lambda x_0, \lambda x_1, \lambda x_2) = \lambda^d \cdot P(x_0, x_1, x_2)$$

from Proposition 5.4. Taking partial derivatives with respect to λ, applying the chain rule and finally setting $\lambda = 1$ yields the desired equality. $\square$

Proposition 7.7. *Let C be a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ and let p be a point of C. Let U be an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ that contains p. Let $D = C \cap U$, viewed as an affine plane curve in $\mathbb{C}^2$. Then C is singular at p if and only if D is singular at p.*

Proof. Permuting the variables (x_0, x_1, x_2), we may assume that U is the affine chart U_2. Then $p = [a, b, 1]$ for some complex numbers a and b, and we need to prove that C is singular at p if and only if D is singular at (a, b).

Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. Let $Q = P(x, y, 1)$; this is a polynomial in $\mathbb{C}[x, y]$ with no repeated factors whose zero set equals D. If $\partial_{x_i} P(a, b, 1) = 0$ for every i in $\{0, 1, 2\}$, then $\partial_x Q(a, b) = \partial_y Q(a, b) = 0$.

Conversely, if $\partial_x Q(a, b) = \partial_y Q(a, b) = 0$ then $\partial_{x_i} P(a, b, 1) = 0$ for i in $\{0, 1\}$. Since p lies on C, we have $P(a, b, 1) = 0$, so that Euler's relation implies that also $\partial_{x_2} P(a, b, 1) = 0$. $\square$

Exercise 7.8 ($\star$)**.** Let C be an affine or projective plane curve over $\mathbb{C}$, and let p be a smooth point of C. Show that p has an open neighbourhood in C that is homeomorphic to $\mathbb{C}$. *Hint: look up the statement of the implicit function theorem.* Give an example of an affine plane curve D over $\mathbb{C}$ and a singular point q of D that does not have any open neighbourhood homeomorphic to $\mathbb{C}$.

Definition 7.9. Let P be a non-zero homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ of positive degree with no repeated factors. Let C be the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$ and let $p = [a, b, c]$ be a smooth point of C. The *projective tangent line* of C at p is the projective line defined by

$$\sum_{i=0}^{2} \partial_{x_i} P(a, b, c) \cdot x_i = 0.$$

This definition does not depend on the choice of homogeneous coordinates (a, b, c), since rescaling these coordinates amounts to multiplying the equation for the projective tangent line by a non-zero constant. The projective Nullstellensatz guarantees that it only depends on the curve C and the point p, and not on the polynomial P. Our definition is once again compatible with the one in the affine case after restricting to an affine chart.

Exercise 7.10. Let C be a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ and let p be a smooth point of C. Let U be an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ that contains the point p, and let $D = C \cap U$ be the restriction of C to U, viewed as an affine plane curve in $\mathbb{C}^2$. Show that the projective tangent line of C at p is the projectivization of the tangent line of D at p (with respect to the chosen affine chart U).

Proposition 7.11. *Let C be an affine or projective plane curve over $\mathbb{C}$, and let p be a point of C that lies on at least two distinct irreducible components of C. Then C is singular at p.*

Proof. We will prove the affine case; the projective case is entirely similar (and can also be reduced to the affine case by working on an affine chart). We can write C as the zero set of a polynomial $P = QR$ in $\mathbb{C}[x, y]$ where P has no repeated factors and Q and R are non-constant polynomials such that $Q(p) = R(p) = 0$. The Leibniz rule for partial derivatives now implies that p is a singular point of C: we have

$$\partial_x P(p) = \partial_x Q(p) R(p) + Q(p) \partial_x R(p) = 0$$

and the analogous equalities hold for the variable y. $\square$

We will soon prove that two *projective* plane curves always intersect (Theorem 10.10). Proposition 7.11 then implies that reducible projective plane curves are always singular. Thus smooth projective plane curves are always irreducible, and checking that a projective plane curve is smooth is often easier than proving directly that a defining polynomial is irreducible (of course, this method is not always conclusive, since there are examples of singular irreducible projective plane curves).

In the *affine* case, there are plenty of examples of reducible smooth plane curves; for instance, a union of two parallel lines.

8 Projective transformations

In linear algebra, you have seen that the choice of a basis allows one to identify an n-dimensional real vector space V with $\mathbb{R}^n$, and you have worked out how a different choice of basis affects the coordinate representations of points in V. The chosen basis had no influence on the linear algebra in V (e.g., the study of its sub-vector spaces, the dimensions of their intersections, and so on). It simply offered an explicit way to write down elements in this vector space. Likewise, in your algebra modules, you have learnt to think about isomorphic groups as different manifestations of the "same" group: the elements and group operation may be represented in a different way (for instance, the symmetric group $\mathcal{S}_3$ versus the dihedral group of order 6) but the abstract algebraic structure is the same.

A similar philosophy applies to projective geometry: projective spaces have interesting groups of symmetries, the *projective transformations* that will be defined below. Projective geometry then studies those properties of objects in projective spaces that are invariant under projective transformations. For instance, we do not consider the property that $[0,0,1]$ lies on a projective plane curve C as a truly geometric statement, because it is not invariant under projective transformations: it is an accidental property that depends on a particular choice of coordinates. On the other hand, being an irreducible projective plane curve, or being a singular point on a projective plane curve, are examples of geometric properties: as we will see, they are invariant under projective transformations (Exercise 8.4).

Definition 8.1. A *projective transformation* of $\mathbb{P}^n_{\mathbb{C}}$ is a map of the form

$$\Phi_A\colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}},\ [x_0,\dots,x_n] \mapsto \left(A\begin{bmatrix} x_0 \\ \vdots \\ x_n\end{bmatrix}\right)^t = \left[\ \sum_{j=0}^n a_{0j}x_j, \quad \cdots \quad , \sum_{j=0}^n a_{nj}x_j\ \right]$$

where $A=(a_{ij})_{i,j=0}^n$ is an invertible $(n+1)\times(n+1)$-matrix with coefficients in $\mathbb{C}$.

Note that this map is well-defined: if $(x_0,\dots,x_n)$ is a tuple of homogeneous coordinates, then

$$A(x_0,\dots,x_n)^t \neq \underline{0}$$

since A is invertible, so that this is again a tuple of homogeneous coordinates defining a point in $\mathbb{P}^n_{\mathbb{C}}$; moreover, this point does not depend on the choice of the homogeneous coordinates $(x_0,\dots,x_n)$, because rescaling $(x_0,\dots,x_n)$ amounts to rescaling $A(x_0,\dots,x_n)^t$. The following properties follow immediately from the definition:

- the map $\Phi_{\mathrm{Id}_{n+1}}$ associated with the identity matrix is the identity;
- when B is another invertible $(n+1)\times(n+1)$-matrix with coefficients in $\mathbb{C}$, then $\Phi_{AB}=\Phi_A\circ\Phi_B$;
- the map Φ_A is a bijection with inverse $\Phi_{A^{-1}}$.

In particular, the projective transformations of $\mathbb{P}^n_{\mathbb{C}}$ form a group with respect to composition.

Example 8.2. The projective transformations of $\mathbb{P}^1_{\mathbb{C}}$ are the maps of the form

$$\Phi\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}},\ [x_0,x_1]\mapsto[ax_0+bx_1, cx_0+dx_1]$$

where a,b,c,d are elements of $\mathbb{C}$ such that $ad-bc\neq 0$. These maps are also called *Möbius transformations.* On the affine chart $U_1\cong\mathbb{C}$ they induce a map

$$U_1\setminus P\to U_1,\ x\mapsto \frac{ax+b}{cx+d}$$

where $x=x_0/x_1$ is the affine coordinate on U_1, and $P=\{-d/c\}$ when $c\neq 0$ and $P=\emptyset$ otherwise. This example already shows that projective transformations need not respect the points at infinity: we have $\Phi([1,0])=[1,0]$ if and only if $c=0$.

Exercise 8.3. When do two invertible invertible matrices A and B of rank $n+1$ give rise to the same projective transformation $\Phi_A = \Phi_B \colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$?

Exercise 8.4. Let A be an invertible matrix of rank 3 with coefficients in $\mathbb{C}$, with associated projective transformation $\Phi_A \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$. Let C be a projective plane curve over $\mathbb{C}$ of degree d.

1. Show that $D = \Phi_A(C)$ is again a projective plane curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. In particular, C is a line if and only if D is a line.

2. Show that C is irreducible if and only if D is irreducible. More generally, show that Φ_A induces a bijection between the sets of irreducible components of C and D.

3. Show that a point p of C is smooth if and only if D is smooth at $q = \Phi_A(p)$; in that case, show that the projective tangent line of D at q is the image under Φ_A of the projective tangent line of C at p.

We say that two projective plane curves C and D over $\mathbb{C}$ are *projectively equivalent* if there exists a projective transformation $\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(C) = D$. Exercise 8.4 gives a few examples of properties of projective plane curves that are invariant under projective equivalence. The upshot is that, to study these and similar properties, we can apply a projective coordinate transformation to simplify the equations.

Exercise 8.5. Let p, q and r be three distinct points on $\mathbb{P}^1_{\mathbb{C}}$. Show that there exists a unique projective transformation Φ of $\mathbb{P}^1_{\mathbb{C}}$ that maps p, q and r to $[0,1]$, $[1,0]$ and $[1,1]$, respectively (with respect to the affine chart U_1, these are the points 0, ∞ and 1). If s is a point on $\mathbb{P}^1_{\mathbb{C}}$ different from p, q and r, what is the value of the affine coordinate x_0/x_1 on U_1 at the point $\Phi(s)$?

This result can be extended to projective transformations of $\mathbb{P}^2_{\mathbb{C}}$. We have already seen that projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ map lines to lines. It is easy to check that no three of the points

$$[1,0,0],\ [0,1,0],\ [0,0,1],\ [1,1,1]$$

lie on a line. Thus a projective transformation that maps a given quadruple of points in $\mathbb{P}^2_{\mathbb{C}}$ to these points can only exist if no three of the given points are collinear. It turns out that this condition is also sufficient; this is the analog in projective geometry of the fact that any linearly independent set of d elements in a vector space of dimension d form a basis. A quadruple (p_1, p_2, p_3, p_4) of points in $\mathbb{P}^2_{\mathbb{C}}$ such that no three of them are collinear is called a *projective basis* of $\mathbb{P}^2_{\mathbb{C}}$.

Proposition 8.6. *Let p_1, p_2, p_3, p_4 be points in $\mathbb{P}^2_{\mathbb{C}}$ such that no three of them are collinear. Then there exists a unique projective transformation $\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ that maps p_1, p_2, p_3 and p_4 to $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$, respectively.*

Proof. Since projective transformations are invertible, it suffices to prove that there exists a unique projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ that maps $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$ to p_1, p_2, p_3 and p_4, respectively.

For each i in $\{1,2,3,4\}$ we pick a tuple (p_{0i}, p_{1i}, p_{2i}) of homogeneous coordinates of the point p_i. Since no three of these points are collinear, any three columns of the matrix

$$P = \begin{pmatrix} p_{01} & p_{02} & p_{03} & p_{04} \\ p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{pmatrix}$$

are linearly independent, by Exercise 5.13.

Let e_1, e_2 and e_3 be the standard basis vectors of $\mathbb{C}^3$, viewed as column vectors. We want to find an invertible rank 3 matrix $A = (a_{ij})_{i,j=0}^2$ with complex coefficients a_{ij} such that Ae_i is proportional to $(p_{0i}, p_{1i}, p_{2i})^t$ for $i = 1,2,3$, and such that $A(1,1,1)^t$ is proportional to $(p_{04}, p_{14}, p_{24})^t$. Since Φ_A is invariant under multiplication of A with a scalar in $\mathbb{C}^*$, we may assume that $A(1,1,1)^t$ is *equal* to $(p_{04}, p_{14}, p_{24})^t$; we then need to show that there exists a unique such matrix A.

Let P' be the invertible 3×3 matrix consisting of the first three columns of P. The first condition means that

$$A = P' \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

for some λ, μ and ν in $\mathbb{C}^*$. The second condition then says that

$$P'(\lambda, \mu, \nu)^t = (p_{04}, p_{14}, p_{24})^t.$$

Since P' is invertible, this equation has a unique solution (λ, μ, ν) in $\mathbb{C}^3$; and since any three columns of P are linearly independent, this solution lies in $(\mathbb{C}^*)^3$. $\square$

Exercise 8.7. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ and let p be a point on L. Show that there exists a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(L)$ is the zero set of x_0 and $\Phi(p) = [0, 0, 1]$.

9 Conics

A *real affine conic* is the zero set in $\mathbb{R}^2$ of a polynomial $P(x,y)$ of degree 2 in $\mathbb{R}[x,y]$. The conic is called *degenerate* if P is reducible in $\mathbb{C}[x,y]$, and *non-degenerate* otherwise. Non-degenerate conics in $\mathbb{R}^2$ are either empty (like the zero set of x^2+y^2+1) or belong to one of three types, depending on the number of $\mathbb{R}$-rational points at infinity:

1. ellipse, with no real points at infinity;
2. parabola, with a unique real point at infinity;
3. hyperbola, with two real points at infinity.

This distinction disappears if we work in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. A *complex projective conic* is the zero set in $\mathbb{P}^2_{\mathbb{C}}$ of a non-zero homogeneous polynomial $Q(x_0,x_1,x_2)$ of degree 2 in $\mathbb{C}[x_0,x_1,x_2]$. The conic is called *degenerate* if Q is reducible in $\mathbb{C}[x_0,x_1,x_2]$, and *non-degenerate* otherwise. A degenerate conic is either a line[3] or a union of two distinct lines. We will now show that all non-degenerate complex projective conics are projectively equivalent: we can always find a projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ mapping one such conic to any other.

Theorem 9.1. *Every non-degenerate conic C in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to the smooth conic*

$$C_0 = \{[x_0,x_1,x_2] \in \mathbb{P}^2_{\mathbb{C}} \,|\, x_1^2 + x_0x_2 = 0\}.$$

Proof. By Proposition 8.6, we may assume that $[0,0,1]$ lies on C. Then C is the zero set of a homogeneous quadratic polynomial of the form

$$Q(x_0,x_1,x_2) = ax_0^2 + bx_1^2 + cx_0x_1 + dx_0x_2 + ex_1x_2$$

with $a,b,c,d,e \in \mathbb{C}$. If $d=e=0$ then Q is a product of two linear factors by Lemma 6.10, contradicting the assumption that C is non-degenerate. Thus d or e is non-zero, which means that $[0,0,1]$ is a smooth point of C. By Exercises 8.4 and 8.7, we may also assume that the projective tangent line of C at $[0,0,1]$ is defined by $x_0=0$, which means that $e=0$ and $d\neq 0$. Then $b\neq 0$ because otherwise, Q would be divisible by x_0 and C would be degenerate. Multiplying Q with $1/b$, we can reduce to the case where $b=1$.

This leaves us with

$$Q(x_0,x_1,x_2) = x_1^2 + x_0(ax_0 + cx_1 + dx_2).$$

Now we apply the projective transformation Φ_A where A is the invertible matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & c & d \end{pmatrix}$$

This operation transforms C into $\Phi_A(C) = C_0$. □

Corollary 9.2. *A conic in $\mathbb{P}^2_{\mathbb{C}}$ is smooth if and only if it is irreducible.*

Proof. If C is reducible then it is a union of two distinct lines, which intersect in a point. This intersection point is singular, by Proposition 7.11. Now assume that C is irreducible. If C is non-degenerate then it is smooth, by Exercise 8.4 and Theorem 9.1. If C is degenerate then it is a line, and thus also smooth. □

Remark 9.3. Curves of degree $d \geq 3$ may be singular without being reducible. For instance, let C be the zero set in $\mathbb{P}^2_{\mathbb{C}}$ of the homogeneous degree polynomial $P(x_0,x_1,x_2) = x_2^3 - x_0x_1^2$ of degree 3. Then C is singular at $[1,0,0]$, but it is irreducible.

Exercise 9.4. Show that every conic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to exactly one of the following projective plane curves: ☜

[3] We should really think of such a degenerate conic as a double line, appearing as a limit of a union of two distinct lines when the lines move towards each other. The classical language of algebraic geometry is ill-equipped to make sense of such double lines; this structure is preserved in the more flexible theory of *schemes*, which is widely used today.

1. the line $\{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_0 = 0\}$,
2. the union of two lines $\{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_0^2 + x_1^2 = 0\}$,
3. the smooth conic $\{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_0^2 + x_1^2 + x_2^2 = 0\}$.

Exercise 9.5. Let C be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Show that there exists a symmetric 3×3 matrix $A = (a_{ij})_{i,j=0}^2$, which is unique up to a factor in $\mathbb{C}^*$, such that C is the zero set of the homogeneous quadratic polynomial

$$P(x_0, x_1, x_2) = \sum_{i,j=0}^{2} a_{ij} x_i x_j = (x_0, x_1, x_2) A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

Show that C is non-degenerate if A has rank 3, a union of two lines if A has rank 2, and a line if A has rank 1.

Example 9.6. A famous and important construction involving conics is the stereographic projection. We have already seen an instance of this construction in Example 2.13.

Let C be a non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$. After a projective transformation, we may assume that it contains the point $[0, 0, 1]$. Every line L through $[0, 0, 1]$ in $\mathbb{P}^2_{\mathbb{C}}$ intersects the line at infinity defined by $x_2 = 0$ in a unique point p_L, and this yields a bijective correspondence between the set of lines through $[0, 0, 1]$ and $\mathbb{P}^1_{\mathbb{C}}$. Explicitly, the line L defined by $ax_0 + bx_1 = 0$ corresponds to the point $p_L = [b, -a]$.

If L is not tangent to C at $[0, 0, 1]$, then it intersects C in a unique point q_L different from $[0, 0, 1]$. This is a particular case of Bézout's theorem that will be addressed later. We can give an ad hoc proof by applying a projective transformation to reduce to the case where C is defined by $x_1^2 + x_0 x_2 = 0$. Then the projective tangent line to C at $[0, 0, 1]$ is defined by $x_0 = 0$. Any other line L through $[0, 0, 1]$ has an equation of the form $ax_0 + x_1 = 0$, and intersects C in $[0, 0, 1]$ and $q_L = [1, -a, -a^2]$. Conversely, for every point q on C different from $[0, 0, 1]$, there exists a unique line through $[0, 0, 1]$ and q. If L is the projective tangent line to C at $[0, 0, 1]$, we set $q_L = [0, 0, 1]$.

In this way, we obtain a bijective correspondence $L \mapsto q_L$ between the set of lines through $[0, 0, 1]$ and the set C. This in turn gives rise to a bijection $q_L \mapsto p_L$ between C and $\mathbb{P}^1_{\mathbb{C}}$; geometrically, it is obtained by projecting C onto the line at infinity from the point $[0, 0, 1]$. This bijection is an example of an *isomorphism* of algebraic curves as defined in algebraic geometry, and we find that every non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. This notion of isomorphism will be formally introduced in Section 15. Beware that a non-degenerate conic is *not* projectively equivalent to a line in $\mathbb{P}^2_{\mathbb{C}}$, since these projective plane curves have different degrees.

10 Resultants and weak Bézout

One of the most fundamental theorems about projective plane curves is *Bézout's theorem.* It states that two projective plane curves C and D in $\mathbb{P}^2_{\mathbb{C}}$ with no common component, of respective degrees d and e, intersect in $d \cdot e$ points. A few immediate comments are in order to interpret this statement correctly.

- If C and D have a common component then they have infinitely many points in common, by Proposition 5.9.
- It is essential to work with *projective* plane curves, the statement is false for affine plane curves because we miss the intersection points at infinity. For instance, two distinct parallel lines in $\mathbb{C}^2$ have no intersection point, while they both have degree one.
- Bézout's theorem is only correct if we count intersection points with suitable *multiplicities.* Intersection multiplicities appear when C and D intersect in a non-generic way.

Example 10.1. Let C be the conic defined by $x_0^2 + x_1x_2 = 0$ and let D be the line defined by $x_1 = 0$. Then $C \cap D$ consists of the unique point $p = [0, 0, 1]$, while Bézout's theorem predicts two intersection points. The reason for this discrepancy is that D is tangent to C at the point p. Perturbing the equation for D, replacing it by $ax_0 + (1 + b)x_1 + cx_2 = 0$ with a, b and c complex numbers close to 0, we would get two distinct intersection points instead of one. So we will need to define intersection multiplicities in such a way that the point p is counted twice.

Another source of intersection multiplicities are singularities of the curves C and D. Let C be the degenerate conic defined by $x_0x_1 = 0$ and let D be the line defined by $x_0 + x_1 = 0$. Then C and D again intersect at the unique point $p = [0, 0, 1]$, while Bézout's theorem predicts two intersection points. This time, the reason is that p is a singular point of C. Perturbing the equation of C, for instance replacing it by $x_0x_1 + ax_2^2 = 0$ with a a complex number close to 0, the curve C becomes smooth and intersects D in two distinct points. Our definition of intersection multiplicities should also take such examples into account.

Introducing the proper definitions and proving Bézout's theorem will require quite a bit of work, and we will proceed in several steps. In this section, we will prove a weak form of Bézout's theorem that does not require any multiplicities: we will show that $C \cap D$ is non-empty and contains at most $d \cdot e$ distinct points.

The algebraic foundation underpinning our proof is the *resultant*, which controls when two polynomials in $\mathbb{C}[x]$ have a common root. Let $P(x) = a_0 + \cdots + a_dx^d$ and $Q(x) = b_0 + \cdots + b_ex^e$, where the coefficients a_i and b_j are complex numbers, and the leading coefficients a_d and b_e are different from zero. Our aim is to write down an explicit equation in the coefficients a_i and b_j that describes when P and Q have a common root.

Lemma 10.2. *Two non-zero polynomials $P(x)$ and $Q(x)$ in $\mathbb{C}[x]$ have a common root in $\mathbb{C}$ if and only if there exist non-zero polynomials $R(x)$ and $S(x)$ in $\mathbb{C}[x]$ with $\deg(R) \leq \deg(P) - 1$ and $\deg(S) \leq \deg(Q) - 1$ such that*

$$P(x) \cdot S(x) - Q(x) \cdot R(x) = 0. \tag{3}$$

Proof. If $\lambda \in \mathbb{C}$ is a common root of $P(x)$ and $Q(x)$, then we can write $P(x) = (x - \lambda)R(x)$ and $Q(x) = (x - \lambda)S(x)$ in $\mathbb{C}[x]$, and equation (3) is satisfied.

Conversely, assume that there exist non-zero polynomials $R(x)$ and $S(x)$ in $\mathbb{C}[x]$ satisfying the conditions in the statement. This implies in particular that $P(x)$ and $Q(x)$ have positive degrees. We write

$$P(x) = c(x - \lambda_1) \cdots (x - \lambda_d)$$

where $\lambda_1, \ldots, \lambda_d$ lie in $\mathbb{C}$ and the factor c lies in $\mathbb{C}^*$. The polynomial $P(x)$ divides $Q(x)R(x)$ but it does not divide $R(x)$ because the degree of $R(x)$ is at most $d - 1$. Thus at least one of the factors $x - \lambda_i$ must divide $Q(x)$, because these factors are prime. This implies that at least one of the numbers λ_i is a common root of $P(x)$ and $Q(x)$. $\square$

The advantage of the condition (3) is that we can express it as a system of linear equations. We want to determine whether there exist non-zero polynomials

$$R(x) = -r_0 - \ldots - r_{d-1}x^{d-1}, \qquad S(x) = s_0 + \ldots + s_{e-1}x^{e-1}$$

in $\mathbb{C}[x]$ satisfying (3):

$$(a_0 s_0 + b_0 r_0) + (a_1 s_0 + a_0 s_1 + b_1 r_0 + b_0 r_1)x + \ldots + (a_d s_{e-1} + b_e r_{d-1})x^{d+e-1} = 0.$$

Treating the coefficients r_i and s_j as variables, this amounts to checking whether the homogeneous system of linear equations

$$A \cdot (s, r)^t = 0$$

has a non-trivial solution, where A is the $(d+e) \times (d+e)$ matrix

$$A = \begin{pmatrix} a_0 & 0 & \ldots & 0 & b_0 & 0 & \ldots & \ldots & \ldots & 0 \\ a_1 & a_0 & \ldots & 0 & b_1 & b_0 & \ldots & \ldots & \ldots & 0 \\ a_2 & a_1 & \ddots & \vdots & b_2 & b_1 & \ddots & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & \ddots & \ldots & 0 \\ \vdots & \vdots & \ldots & a_1 & b_e & b_{e-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ldots & a_2 & 0 & b_e & \ddots & \ddots & \ddots & b_0 \\ a_d & a_{d-1} & \ldots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & b_1 \\ 0 & a_d & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_d & 0 & 0 & \ldots & \ldots & \ldots & b_e \end{pmatrix}$$

consisting of e columns filled with the coefficients a_i, and d columns filled with the coefficients b_j, all shifted down by one step in each new column. From linear algebra, we know that this system has a non-trivial solution if and only if $\det A = 0$. This motivates the following definition.

Definition 10.3. Let $P, Q \in \mathbb{C}[x]$ be as above. Then the *resultant* $\mathbf{R}_{P,Q}$ of P and Q is the determinant of the $(d+e) \times (d+e)$ matrix A:

$$\mathbf{R}_{P,Q} = \det A.$$

It follows directly from the definition that $\mathbf{R}_{Q,P} = (-1)^{de}\mathbf{R}_{P,Q}$. Lemma 10.2 can now be reformulated in the following way.

Theorem 10.4. *Two non-zero polynomials P and Q in $\mathbb{C}[x]$ have a common root if and only if the resultant $\mathbf{R}_{P,Q}$ is equal to zero.*

Proof. This follows immediately from Lemma 10.2 and the subsequent discussion. □

The interest of Theorem 10.4 is that it provides a necessary and sufficient condition for the existence of a common root in the form of a polynomial equation in the coefficients of P and Q. This equation can easily be checked on a computer.

Example 10.5.

1. The resultant of $P(x) = x^2 - 1$ and $Q(x) = x^2 + 3x + 2$ is given by

$$\mathbf{R}_{P,Q} = \det \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 0$$

reflecting the fact that P and Q have the common root -1.

2. Let $P(x) = a_0 + a_1 x$ and $Q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ with $a_1 \neq 0$ and $b_3 \neq 0$. Then the resultant of P and Q is given by
$$\mathbf{R}_{P,Q} = \det \begin{pmatrix} a_0 & 0 & 0 & b_0 \\ a_1 & a_0 & 0 & b_1 \\ 0 & a_1 & a_0 & b_2 \\ 0 & 0 & a_1 & b_3 \end{pmatrix}$$
and a direct calculation shows that this determinant is equal to $-a_1^3 Q(-a_0/a_1)$. This again illustrates that $\mathbf{R}_{P,Q} = 0$ if and only if P and Q have a common root.

3. Recall that a polynomial $P(x) \in \mathbb{C}[x]$ has a double root if and only if P and its derivative $\partial_x P$ have a common root, that is, if and only if $\mathbf{R}_{P,\partial_x P} = 0$. If $P(x) = ax^2 + bx + c$ is a polynomial of degree 2, then
$$\mathbf{R}_{P,\partial_x P} = \det \begin{pmatrix} c & b & 0 \\ b & 2a & b \\ a & 0 & 2a \end{pmatrix} = -a\Delta$$
where $\Delta = b^2 - 4ac$ is the discriminant of P. Thus we recover the well-known fact that $P(x)$ has a double root if and only if its discriminant is zero.

In order to prove the weak form of Bézout's theorem, we need a similar construction in the case where P and Q are homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ of respective degrees d and e. The trick is to view these as polynomials in the variable x_2 with coefficients in $\mathbb{C}[x_0, x_1]$: we can write
$$P(x_0, x_1, x_2) = \sum_{i=0}^{d} a_i(x_0, x_1) x_2^i, \qquad Q(x_0, x_1, x_2) = \sum_{j=0}^{e} b_j(x_0, x_1) x_2^j,$$
where $a_i \in \mathbb{C}[x_0, x_1]$ is a homogeneous polynomial of degree $d - i$, and $b_j \in \mathbb{C}[x_0, x_1]$ is homogeneous of degree $e - j$. In particular, $a_d = P(0, 0, 1)$ and $b_e = Q(0, 0, 1)$ are elements of $\mathbb{C}$.

Definition 10.6. Assume that $P(0, 0, 1) \neq 0$ and $Q(0, 0, 1) \neq 0$. The *resultant* of P and Q is the determinant of the $(d + e) \times (d + e)$ matrix
$$A = \begin{pmatrix}
a_0 & 0 & \dots & 0 & b_0 & 0 & \dots & \dots & \dots & 0 \\
a_1 & a_0 & \dots & 0 & b_1 & b_0 & \dots & \dots & \dots & 0 \\
a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & \dots & \dots & 0 \\
\vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & \ddots & \dots & 0 \\
\vdots & \vdots & \dots & a_1 & b_e & b_{e-1} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \dots & a_2 & 0 & b_e & \ddots & \ddots & \ddots & b_0 \\
a_d & a_{d-1} & \dots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & b_1 \\
0 & a_d & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & a_d & 0 & 0 & \dots & \dots & \dots & b_e
\end{pmatrix}.$$

The difference with Definition 10.3 for univariate polynomials is that now the entries of this matrix are polynomials in $\mathbb{C}[x_0, x_1]$, so that also $\mathbf{R}_{P,Q}$ is a polynomial in $\mathbb{C}[x_0, x_1]$. It is compatible with the previous definition in the sense that for all u and v in $\mathbb{C}$, the complex number $\mathbf{R}_{P,Q}(u, v)$ is the resultant of the polynomials $P(u, v, x)$ and $Q(u, v, x)$ in $\mathbb{C}[x]$, because these polynomials still have degrees d and e (by our assumption that $P(0, 0, 1)$ and $Q(0, 0, 1)$ are non-zero), and addition and multiplication of polynomials are compatible with pointwise evaluation.

Theorem 10.7. *Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials of respective degrees d and e such that $P(0, 0, 1) \neq 0$ and $Q(0, 0, 1) \neq 0$. Then $\mathbf{R}_{P,Q}$ is a homogeneous polynomial of degree de in $\mathbb{C}[x_0, x_1]$. The polynomials P and Q have a non-constant common factor if and only if $\mathbf{R}_{P,Q} = 0$ in $\mathbb{C}[x_0, x_1]$.*

Proof. We denote by $(c_{ij})_{i,j=1}^{d+e}$ the entries of the matrix A in the definition of the resultant. Then to prove the first part of the statement, it suffices to show that for every permutation σ of $\{1,\dots,d+e\}$, the product $\prod_{i=1}^{d+e} c_{i\sigma(i)}$ is homogeneous of degree de. The factor $c_{i\sigma(i)}$ is either 0 or homogeneous of degree $d-i+\sigma(i)$ if $\sigma(i)\leq e$, and either 0 or homogeneous of degree $\sigma(i)-i$ if $\sigma(i)>e$. Thus $\prod_{i=1}^{d+e} c_{i\sigma(i)}$ is homogeneous of degree

$$\sum_{j=1}^{e}(d-\sigma^{-1}(j)+j)+\sum_{j=e+1}^{d+e}(j-\sigma^{-1}(j)) = de+\sum_{j=1}^{d+e} j-\sum_{j=1}^{d+e}\sigma^{-1}(j) = de.$$

This includes the possibility that this product is 0, since 0 is homogeneous of every degree in $\mathbb{N}$.

If P and Q have a non-constant common factor, then this factor must have positive degree in the variable x_2: otherwise, P and Q would be divisible by a homogeneous polynomial of positive degree in $\mathbb{C}[x_0,x_1]$, contradicting the assumption that $P(0,0,1)\neq 0$ and $Q(0,0,1)\neq 0$. We can view P and Q as polynomials in the variable x_2 over the fraction field $F=\mathbb{C}(x_0,x_1)$ of the polynomial ring $\mathbb{C}[x_0,x_1]$. Repeating the argument for univariate polynomials with $\mathbb{C}[x]$ replaced by $F[x]$, we find that the homogeneous system of linear equations $A(s,r)^t=0$ has a non-zero solution, so that $\det(A)=0$ in F. It follows that $\mathbf{R}_{P,Q}=0$ in $\mathbb{C}[x_0,x_1]$. For the proof of the converse implication, we refer to Exercise 10.8 below. $\square$

Exercise 10.8 ($\star$)**.** Finish the proof of Theorem 10.7 by going through the proof of the univariate case. Assume that $\mathbf{R}_{P,Q}=0$. Show that P and Q are divisible by a polynomial $T(x_2)$ in $F(x_2)$ of positive degree in x_2. Now prove that we can take $T(x_2)$ to lie in $\mathbb{C}[x_0,x_1,x_2]$.

Exercise 10.9. Let P be a non-constant homogeneous polynomial in $\mathbb{C}[x_0,x_1,x_2]$ of degree d with no repeated factors, and let C be the projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by $P=0$.

1. Assume that $P(0,0,1)\neq 0$. Show that $\partial_{x_2}P(0,0,1)\neq 0$ and that the resultant $\mathbf{R}_{P,\partial_{x_2}P}$ is not identically zero.

2. ($\star$) Let p be a point in $\mathbb{P}^2_{\mathbb{C}}$ that does not lie on C. Show that all but finitely many lines L through p in $\mathbb{P}^2_{\mathbb{C}}$ intersect the curve C in precisely d points.

We are now ready to prove the weak version of Bézout's theorem.

Theorem 10.10 (Weak Bézout theorem)**.** *Let C and D be projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ of degrees $d=\deg(C)$ and $e=\deg(D)$. Then*

1. C and D intersect in at least one point;

2. C and D have at most de distinct points of intersection, unless they have a common component.

Proof. Let $P,Q\in\mathbb{C}[x_0,x_1,x_2]$ be homogeneous polynomials with no repeated factors whose zero sets in $\mathbb{P}^2_{\mathbb{C}}$ are C and D, respectively. Then P and Q have degrees d and e.

(1) Given finitely many projective curves in $\mathbb{P}^2_{\mathbb{C}}$, one can always find a point $p\in\mathbb{P}^2_{\mathbb{C}}$ that does not lie on the union of these curves; see Problem sheet 1. Let $p\in\mathbb{P}^2_{\mathbb{C}}$ be a point that lies neither on C nor on D. Applying a projective transformation, we may assume that $p=[0,0,1]$, so that $P(0,0,1)$ and $Q(0,0,1)$ are both non-zero. Then the resultant $\mathbf{R}_{P,Q}$ is a homogeneous polynomial of degree de in $\mathbb{C}[x_0,x_1]$, by Theorem 10.7.

By Lemma 6.10, we can find a couple (u,v) in $\mathbb{C}^2\setminus\{(0,0)\}$ such that $\mathbf{R}_{P,Q}(u,v)=0$. But $\mathbf{R}_{P,Q}(u,v)$ is nothing but the resultant of the polynomials $P(u,v,x)$ and $Q(u,v,x)$ in $\mathbb{C}[x]$, so that these polynomials have a common root w in $\mathbb{C}$. Then the point $[u,v,w]$ lies in $C\cap D$.

(2) Assume that $C\cap D$ contains a set $\mathcal{S}=\{p_0,\dots,p_{de}\}$ of $de+1$ distinct points. We will prove that C and D have a common component. We choose a triple of homogeneous coordinates (p_{0i},p_{1i},p_{2i}) for each point p_i in $\mathcal{S}$. By the same argument as in (1), we may assume that $[0,0,1]$ does not lie on $C\cup D$, and also not on any of the lines through two distinct points of $\mathcal{S}$. This means that $P(0,0,1)$ and $Q(0,0,1)$ are non-zero, and that

$$\det\begin{pmatrix}0 & p_{0i} & p_{0j}\\ 0 & p_{1i} & p_{1j}\\ 1 & p_{2i} & p_{2j}\end{pmatrix}\neq 0$$

for all distinct elements i and j in $\{0, \ldots, de\}$. Thus, each couple (p_{0i}, p_{1i}) defines a distinct point $[p_{0i}, p_{1i}]$ in $\mathbb{P}^1_{\mathbb{C}}$.

For every i in $\{0, \ldots, de\}$, the polynomials $P(p_{0i}, p_{1i}, x)$ and $Q(p_{0i}, p_{1i}, x)$ have the common root $x = p_{2i}$, so that their resultant $\mathbf{R}_{P,Q}(p_{0i}, p_{1i})$ vanishes. It follows that the homogeneous polynomial $\mathbf{R}_{P,Q}$ of degree de in $\mathbb{C}[x_0, x_1]$ has at least $de+1$ distinct zeros in $\mathbb{P}^1_{\mathbb{C}}$. By Lemma 6.10, this can only happen when $\mathbf{R}_{P,Q} = 0$, so that P and Q have a common non-constant factor, and C and D have a common component. □

Let us discuss a few interesting applications of the weak Bézout theorem.

Corollary 10.11. *An irreducible projective plane curve C in $\mathbb{P}^2_{\mathbb{C}}$ of degree d has at most $d(d-1)$ singular points.*

Proof. Let $P(x_0, x_1, x_2)$ be a homogeneous polynomial with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. Swapping the variables if necessary, we may assume that the variable x_0 occurs in P; then $Q = \partial_{x_0} P$ is a non-zero homogeneous polynomial of degree $d-1$ in $\mathbb{C}[x_0, x_1, x_2]$, and its zero set in $\mathbb{P}^2_{\mathbb{C}}$ is either empty (if $d = 1$) or a projective plane curve D of degree at most $d-1$. Since C is irreducible, and $C \neq D$ because these curves have different degrees if $D \neq \emptyset$, we know that C and D have no common component. Each singular point of C lies in $C \cap D$, and the cardinality of $C \cap D$ is at most $d(d-1)$, by the weak Bézout theorem. □

Exercise 10.12. Use Exercise 10.9 to show that Corollary 10.11 remains valid for reducible projective plane curves C. If $C_1, \ldots, C_r$ are the irreducible components of C, how are the singularities of C related to those of the curves C_i?

The bound in Corollary 10.11 is far from optimal; we will deduce a sharper bound from the strong form of Bézout's theorem in Exercise 11.19. The optimal bound for irreducible projective plane curves of degree d is $(d-1)(d-2)/2$, but proving this requires some further results from singularity theory.

Our next application was already announced at the end of Section 7.

Corollary 10.13. *Every smooth projective plane curve C in $\mathbb{P}^2_{\mathbb{C}}$ is irreducible.*

Proof. It follows from the weak Bézout's theorem that two irreducible components of C must intersect; each intersection point of distinct irreducible components is singular, by Proposition 7.11. □

As a final application, we prove a famous theorem by Pascal.

Proposition 10.14 (Pascal's mystic hexagon)**.** *Let C be a non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$, and let $p_1, \ldots, p_6$ be six distinct points in C. Denote by L_1, L_2 and L_3 the unique lines containing $\{p_1, p_2\}$, $\{p_3, p_4\}$, and $\{p_5, p_6\}$, respectively. Denote by M_1, M_2 and M_3 the unique lines containing $\{p_4, p_5\}$, $\{p_6, p_1\}$, and $\{p_2, p_3\}$. For every i in $\{1, 2, 3\}$, let q_i be the unique intersection point of L_i and M_i. Then q_1, q_2 and q_3 are collinear.*

In the special case Pascal was considering, C is a non-degenerate real conic in $\mathbb{R}^2$. Then $p_1, \ldots, p_6$ form a hexagon, and the theorem implies that in the projective plane, the intersection points of the three pairs of lines passing through opposite sides of the hexagon are collinear, as illustrated by Figure 3.

Proof. We may assume that q_1, q_2 and q_3 are distinct, since otherwise, the result is trivial. These points do not belong to the set $\mathcal{S} = \{p_1, \ldots, p_6\}$: otherwise, one of the lines L_i or M_i would contain three distinct points in $\mathcal{S}$ and therefore intersect C in at least 3 points. The weak Bézout theorem then implies that this line is an irreducible component of C, contradicting the assumption that C is non-degenerate.

Consider the projective plane curves

$$L = L_1 \cup L_2 \cup L_3, \qquad M = M_1 \cup M_2 \cup M_3$$

in $\mathbb{P}^2_{\mathbb{C}}$. These are two curves of degree 3 containing $\mathcal{S}$ as well as the points q_1, q_2 and q_3.

Since C is non-degenerate, it does not contain any line; in particular, it has no common component with L or M. Thus the weak Bézout theorem implies that $C \cap L$ and $C \cap M$ contain at most $2 \cdot 3 = 6$ points. Both of these intersections contain the set $\mathcal{S}$ of cardinality 6, and are therefore are equal to $\mathcal{S}$. In particular, the points q_1, q_2 and q_3 do not lie on C.

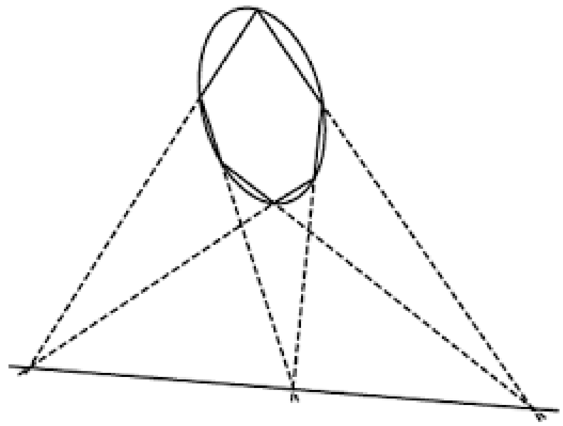

Figure 3: Pascal's mystic hexagon. *Source: Wolfram MathWorld.*

Now let p_0 be a point on C that does not belong to $\mathcal{S}$; then it does not lie in L or M. Let P and Q be homogeneous polynomials with no repeated factors whose zero sets are L and M, respectively. Since $P(p_0) \neq 0$ and $Q(p_0) \neq 0$, we can find λ and μ in $\mathbb{C}^*$ such that the homogeneous polynomial $\lambda P + \mu Q$ vanishes at p_0: simply pick a triple of homogeneous coordinates (p_{00}, p_{10}, p_{20}) for p_0 and set $\lambda = Q(p_{00}, p_{10}, p_{20})$ and $\mu = -P(p_{00}, p_{10}, p_{20})$. Let D be the zero locus of $\lambda P + \mu Q$ in $\mathbb{P}^2_{\mathbb{C}}$. Since P and Q have degree 3, the projective plane curve D has degree at most 3.

The set $\mathcal{S}$ lies in $L \cap M$ and therefore also in D, because P and Q vanish at all the points in $\mathcal{S}$. The point p_0 also lies in D. Thus C and D intersect at least in the seven points $p_0, \ldots, p_6$. Now it follows from the weak Bézout theorem that C and D have a common component; otherwise, they would intersect in at most $2 \cdot 3 = 6$ points. But C is irreducible, so that this common component must be C itself, which means that C is contained in D. The curve D is not equal to C, because it contains q_1, q_2 and q_3, which lie in the zero sets of P and Q. It follows that D must have another irreducible component E, of degree $3 - 2 = 1$. This is a line in $\mathbb{P}^2_{\mathbb{C}}$ that contains $D \setminus C$, and, in particular, the points q_1, q_2 and q_3. $\square$

The introduction of the parameters λ and μ in the proof is a special case of a *linear system*; these will be discussed in detail in Section 14. The idea is to put the curves L and M (which correspond to $(\lambda, \mu) = (1, 0)$ and $(0, 1)$, respectively) in a continuous family parameterized by $[\lambda, \mu] \in \mathbb{P}^1_{\mathbb{C}}$. By picking a member of this family containing p_0, we deform the degree 3 curves L and M into a union of the conic C and a line.

11 Multiplicities and strong Bézout

In this section we will prove the strong form of Bézout's theorem. As we have explained at the beginning of Section 10, this requires a rigorous definition of intersection multiplicities. The first step towards this definition is the notion of the multiplicity of a zero of a polynomial.

If $P(x) \in \mathbb{C}[x]$ is a non-zero polynomial in one variable, then the *multiplicity* of P at a point $a \in \mathbb{C}$ is the order of vanishing of P at a. This is the unique non-negative integer m such that $P(x) = (x-a)^m Q(x)$ with Q a polynomial in $\mathbb{C}[x]$ satisfying $Q(a) \neq 0$.

In order to get a characterization that generalizes to polynomials in several variables, we observe that the multiplicity of P at a is also the smallest non-negative integer α such that $\partial^\alpha P(a) \neq 0$. Here $\partial^\alpha P$ denotes the derivative of P of order α, with the convention that $\partial^0 P = P$.

Now let P be a polynomial in $\mathbb{C}[x_0, \dots, x_n]$. We denote the partial derivatives of P using multi-index notation: if α is a tuple in $\mathbb{N}^{n+1}$, then $\partial^\alpha P$ is the polynomial $\partial_{x_0}^{\alpha_0} \cdots \partial_{x_n}^{\alpha_n} P$ obtained by differentiating α_i times with respect to each variable x_i. Since P is a polynomial, the order of differentiation does not matter. We adopt the convention that $\partial^{(0,\dots,0)} P = P$. Recall that $|\alpha| = \alpha_0 + \ldots + \alpha_n$.

Example 11.1. If $n = 1$ then $\partial^{(2,1)} P = \partial_{x_0} \partial_{x_0} \partial_{x_1} P$.

Definition 11.2. Let P be a non-zero polynomial in $\mathbb{C}[x_0, \dots, x_n]$ and let $a = (a_0, \dots, a_n)$ be a point in $\mathbb{C}^{n+1}$. The *multiplicity* $\operatorname{mult}_a P$ of P at a is the smallest non-negative integer m such that there exists $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = m$ and $\partial^\alpha P(a) \neq 0$.

Such a non-negative integer m exists, because, up to some non-zero factors, the partial derivatives $\partial^\alpha P(a)$ are the coefficients in the Taylor expansion of P around the point a. Thus if P is non-zero, then at least one of these partial derivatives is different from zero. This argument also shows that $\operatorname{mult}_a P \leq \deg(P)$, and that $\operatorname{mult}_a(PQ) = \operatorname{mult}_a P + \operatorname{mult}_a Q$ for every non-zero polynomial Q in $\mathbb{C}[x_0, \dots, x_n]$, since the Taylor expansion of PQ around a is the product of the Taylor expansions of P and Q.

If P is homogeneous, then the same holds for all of its partial derivatives, so that $\partial^\alpha P(a) = 0$ if and only if $\partial^\alpha P(\lambda a)$ for all λ in $\mathbb{C}^*$. Thus $\operatorname{mult}_a P = \operatorname{mult}_{\lambda a} P$, and we can define the multiplicity of P at a point p of $\mathbb{P}^n_{\mathbb{C}}$ as the multiplicity of P at any tuple of homogeneous coordinates of p. We can now use this definition to define the multiplicity of a projective plane curve at a point.

Definition 11.3. If C is a projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ and P is a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set is C, then the multiplicity $\operatorname{mult}_p C$ of C at a point p of $\mathbb{P}^2_{\mathbb{C}}$ is the multiplicity of P at p.

The multiplicity $\operatorname{mult}_p C$ is well-defined, because the polynomial P is determined by C up to a factor in $\mathbb{C}^*$, by the projective Nullstellensatz. It always satisfies $\operatorname{mult}_p C \leq \deg(P) = \deg(C)$.

Example 11.4. We have $\operatorname{mult}_p C = 0$ if and only if $P(a) \neq 0$, that is, p does not lie on C. Moreover, $\operatorname{mult}_p C = 1$ if and only if $P(a) = 0$ but one of the first-order partial derivatives of P does not vanish at a; this means precisely that p is a smooth point of C. It follows that $\operatorname{mult}_p C \geq 2$ if and only if p is a singular point of C. The multiplicity $\operatorname{mult}_p C$ can be viewed as a measure for the singularity of C at p: the higher the multiplicity, the "worse" the singularity is.

Example 11.5.

1. The polynomial $P = x_0^2 x_1 + x_0 x_1 x_2$ has multiplicity three at $a = (0,0,0)$: we have $P(a) = 0$ and all the partial derivatives of order 1 or 2 vanish at a, but the partial derivative $\partial^{(1,1,1)} P(a)$ is non-zero. Note that we can immediately read off the multiplicity from the Taylor expansion of P around a (which in this case is simply the given expression for P): the lowest order terms have degree 3. This argument also shows that, more generally, the multiplicity of a non-zero homogeneous polynomial of degree d in $\mathbb{C}[x_0, \dots, x_n]$ at the origin of $\mathbb{C}^{n+1}$ is equal to d.

2. Let $P = x_0^2 x_2 - x_1^2(x_1 + x_2)$ and let C be the projective plane curve defined by $P = 0$. Then C has multiplicity two at $p = [0,0,1]$. Indeed, $P(0,0,1) = 0$ and all first-order partial derivatives of P vanish at $(0,0,1)$, but the second-order derivative $\partial^{(0,2,0)} P = -6x_1 - 2x_2$ does not vanish at $(0,0,1)$. We can again directly read off this result from the Taylor expansion $P = x_0^2 - x_1^2 + x_0^2(x_2 - 1) - x_1^3 - x_1^2(x_2 - 1)$ around $(0,0,1)$: the lowest order term $x_0^2 - x_1^2$ has degree 2.

Exercise 11.6. Show that the multiplicity of a projective plane curve C at a point p in $\mathbb{P}^2_{\mathbb{C}}$ is invariant under projective transformations $\Phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$, in the sense that $\operatorname{mult}_p C = \operatorname{mult}_{\Phi(p)} \Phi(C)$.

Exercise 11.7. Let P be a non-zero homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$ and let $a = (a_0, a_1, \dots, a_n)$ be a point in $\mathbb{C}^{n+1}$. Let i be an element of $\{0, \dots, n\}$ such that $a_i \neq 0$. Show that the multiplicity of P at a is equal to the multiplicity of $P(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ at the point $(a_0/a_i, \dots, a_{i-1}/a_i, a_{i+1}/a_i, \dots, a_n/a_i)$.

Deduce that the multiplicity of a projective plane curve C at a point p in $\mathbb{P}^2_{\mathbb{C}}$ is compatible with restriction to affine charts: if $U \cong \mathbb{C}^2$ is an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ containing p, and $Q \in \mathbb{C}[x, y]$ is a non-constant polynomial with no repeated factors whose zero set in $\mathbb{C}^2$ is $C \cap U$, then $\operatorname{mult}_p C = \operatorname{mult}_p Q$.

The following result will be a key ingredient in the proof of the strong Bézout theorem. It is essentially a reformulation in the projective setting of the fundamental theorem of algebra: a degree d polynomial in $\mathbb{C}[x]$ has precisely d roots when counted with multiplicities.

Lemma 11.8. *Let $P \in \mathbb{C}[x_0, x_1]$ be a non-zero homogeneous polynomial of degree $d > 0$, and let $p_1, \dots, p_r$ be the zeros of P in $\mathbb{P}^1_{\mathbb{C}}$. Then*

$$\sum_{i=1}^{r} \operatorname{mult}_{p_i} P = d.$$

Proof. By Lemma 6.10, we can write

$$P(x_0, x_1) = \prod_{j=1}^{d} (\alpha_j x_0 + \beta_j x_1)$$

where the couples (α_j, β_j) lie in $\mathbb{C}^2 \setminus \{(0, 0)\}$. Then the multiplicity of P at a point p of $\mathbb{P}^1_{\mathbb{C}}$ is the number of indices j in $\{1, \dots, d\}$ such that $p = [\beta_j, -\alpha_j]$. It follows that the sum of the multiplicities of all zeros of P in $\mathbb{P}^1_{\mathbb{C}}$ is equal to d. $\square$

As we have seen in Example 10.1, singularities are not the only source of intersection multiplicities: we also need to encode tangencies between curves. In order to come to a definition of intersection multiplicities that takes both singularities and tangencies into account, we will consider the multiplicity of the resultant of two polynomials that define our given projective plane curves.

Definition 11.9. Let C and D be projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ with no common component that do not contain $[0, 0, 1]$. Assume that $[0, 0, 1]$ does not lie on a line through two distinct points of $C \cap D$. Let P and Q be non-constant homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero sets are C and D. Then the *intersection multiplicity* $\mathbf{I}(p, C, D)$ of C and D at a point $p = [p_0, p_1, p_2]$ in $C \cap D$ is defined by

$$\mathbf{I}(p, C, D) = \operatorname{mult}_{(p_0, p_1)} \mathbf{R}_{P,Q}.$$

Example 11.10. Let us check that this definition gives the expected answer in the case where D is a line. Since D does not contain $[0, 0, 1]$, we can choose a defining polynomial of the form $Q(x_0, x_1, x_2) = d_0 x_0 + d_1 x_1 + x_2$ with d_0 and d_1 in $\mathbb{C}$. Let $p = [1, p_1, p_2]$ be a point on D that lies in the affine chart U_0 (the case where p lies in U_1 is similar). The equation for D on this chart is $d_0 + d_1 x + y = 0$, and the equation for C on U_0 is $P(1, x, y) = 0$. A natural definition of the intersection multiplicity of C and D at p would be to make the substitution $y = -d_0 - d_1 x$ in $P(1, x, y)$ and to compute the order of vanishing of $P(1, x, -d_0 - d_1 x)$ at $x = p_1$. We will verify that this answer is compatible with Definition 11.9.

A direct generalization of Example 10.5(2) shows that $\mathbf{R}_{P,Q} = P(x_0, x_1, -d_0 x_0 - d_1 x_1)$ (check this!). Thus the intersection multiplicity of C and D at p is equal to the multiplicity of the polynomial $P(x_0, x_1, -d_0 x_0 - d_1 x_1)$ at $(1, p_1)$, and this is the same thing as the order of vanishing of $P(1, x, -d_0 - d_1 x)$ at $x = p_1$, by Exercise 11.7.

If the degrees of P and Q are higher than one, then we can no longer use these equations to write x_2 in terms of the variables x_0 and x_1, but the resultant takes care of this issue for us.

The peculiar condition in Definition 11.9 that $[0, 0, 1]$ does not lie on any line through two distinct points in $C \cap D$ is due to the choice involved in the definition of the resultant: we singled out the variable x_2

and decided to view P and Q as polynomials in x_2 with coefficients in $\mathbb{C}[x_0, x_1]$. The zero set of $\mathbf{R}_{P,Q}$ in $\mathbb{C}^2 \setminus \{(0,0)\}$ is the set of couples (a,b) such that $P(a,b,x)$ and $Q(a,b,x)$ have a common root; equivalently, such that $C \cap D$ contains a point of the form $[a,b,c]$ for some c in $\mathbb{C}$. Our non-collinearity condition now guarantees that this point $[a,b,c]$ is *unique*, since any two points of this form are collinear with $[0,0,1]$. Thus we do not lose information by projecting $C \cap D$ to $\mathbb{P}^1_{\mathbb{C}}$ *via* the map $[a,b,c] \mapsto [a,b]$.

Example 11.11. Let $P = (x_0 - x_2/2)^2 + x_1^2 - x_2^2$ and $Q = (x_0 + x_2/2)^2 + x_1^2 - x_2^2$. These polynomials define two conics in $\mathbb{P}^2_{\mathbb{C}}$, which we denote by C and D. The intersection $C \cap D$ consists of four points: $[0, \sqrt{3}, \pm 2]$ and $[1, \pm i, 0]$. At each of these points, the curves C and D have intersection multiplicity 1 (this will be further justified in Corollary 12.10), so that these multiplicities add up to $2 \cdot 2 = 4$ in accordance with Bézout's theorem. However, the resultant gives the wrong result: the multiplicity of

$$\mathbf{R}_{P,Q} = \det \begin{pmatrix} x_0^2 + x_1^2 & 0 & x_0^2 + x_1^2 & 0 \\ -x_0 & x_0^2 + x_1^2 & x_0 & x_0^2 + x_1^2 \\ -3/4 & -x_0 & -3/4 & x_0 \\ 0 & -3/4 & 0 & -3/4 \end{pmatrix} = -3x_0^2(x_0^2 + x_1^2)$$

at $(0, \sqrt{3})$ is equal to 2, rather than 1, because the resultant adds up the multiplicities at the two points $[0, \sqrt{3}, \pm 2]$.

To circumvent the condition in Definition 11.9, we can apply a suitable projective transformation. By Problem sheet 1, we can always find a point q that does not lie on C, D or any line through two distinct points of $C \cap D$. If Φ is any projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ that maps q to $[0,0,1]$, then $\Phi(C)$ and $\Phi(D)$ satisfy the conditions of Definition 11.9. To use such a transformation to define the intersection multiplicities, we must first check that Definition 11.9 is invariant under projective transformations.

Proposition 11.12. *Let C and D be projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ with no common component, let p be a point of $C \cap D$, and let Φ be a projective transformation of $\mathbb{P}^2_{\mathbb{C}}$. Assume that C and D satisfy the conditions in Definition 11.9, and that these conditions are also satisfied by $\Phi(C)$ and $\Phi(D)$. Then $\mathbf{I}(p, C, D) = \mathbf{I}(\Phi(p), \Phi(C), \Phi(D))$.*

Proof. We omit the proof; an argument can be found in Chapter 8, §7 of the book *Ideals, varieties and algorithms* by D. Cox, J. Little and D. O'Shea (not examinable). The idea is to argue that $\mathbf{I}(p, C, D) \geq \mathbf{I}(\Phi(p), \Phi(C), \Phi(D))$. Then the proof of the strong Bézout theorem below shows that both the sum of the numbers $\mathbf{I}(p, C, D)$, and the sum of the numbers $\mathbf{I}(\Phi(p), \Phi(C), \Phi(D))$, over all p in $C \cap D$ are equal to $\deg(C) \cdot \deg(D)$. This turns our inequalities into equalities $\mathbf{I}(p, C, D) = \mathbf{I}(\Phi(p), \Phi(C), \Phi(D))$. □

We can now formulate the general definition of intersection multiplicities.

Definition 11.13. Let C and D projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$, and let p be a point of $\mathbb{P}^2_{\mathbb{C}}$.

1. If p does not lie in $C \cap D$, then $\mathbf{I}(p, C, D) = 0$.

2. If p lies in $C \cap D$ and C and D have no common component, then the intersection multiplicity of C and D at p is defined by $\mathbf{I}(p, C, D) = \mathbf{I}(\Phi(p), \Phi(C), \Phi(D))$, where Φ is any projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(C)$ and $\Phi(D)$ satisfy the conditions of Definition 11.9. This definition does not depend on the choice of Φ, by Proposition 11.12.

3. If p lies in $C \cap D$ but not on a common component of C and D, then we denote by C' and D' the unions of the irreducible components of C, resp. D, which are not a common component of C and D, and we set $\mathbf{I}(p, C, D) = \mathbf{I}(p, C', D')$.

4. If p lies on a common component of C and D, then $\mathbf{I}(p, C, D) = \infty$.

This definition is well-suited to prove the strong Bézout theorem, but it does not provide much intuition about the geometric meaning of the intersection multiplicities. The axiomatic characterization in Section 12 is more conceptual; there we will deduce a list of basic properties confirming that the intersection multiplicity behaves as one would expect. Before moving on to the strong Bézout theorem, we will only discuss two essential results. The first one, formulated as an exercise, confirms that the intersection multiplicity recognizes intersection points.

Exercise 11.14. Let C and D be projective plane curves over $\mathbb{C}$ and let p be a point in $\mathbb{P}^2_{\mathbb{C}}$. Show that $\mathbf{I}(p, C, D) > 0$ if and only if p lies in $C \cap D$.

The second result states that $\mathbf{I}(p, C, D)$ picks up at least the multiplicities of the singularities of C and D at p.

Proposition 11.15. *Let C and D be projective plane curves over $\mathbb{C}$, and let p be a point in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathbf{I}(p, C, D) \geq \operatorname{mult}_p C \cdot \operatorname{mult}_p D$.*

Proof. The inequality is trivially satisfied when p lies on a common component of C and D, so we may assume that this is not the case. Then we can immediately reduce to the case where C and D have no common component. After applying a projective transformation, we may assume that $p = [1, 0, 0]$ and that C and D satisfy the conditions of Definition 11.9. Let P and Q be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero sets in $\mathbb{P}^2_{\mathbb{C}}$ are C and D, respectively. We denote by d and e the degrees of P and Q.

By Exercise 11.7, we have $\operatorname{mult}_p C = \operatorname{mult}_{(0,0)} P(1, x, y)$, $\operatorname{mult}_p D = \operatorname{mult}_{(0,0)} Q(1, x, y)$, and $\operatorname{mult}_{(1,0)} \mathbf{R}_{P,Q} = \operatorname{mult}_0 \mathbf{R}_{P,Q}(1, x)$. For notational convenience, we denote $\operatorname{mult}_{(0,0)} P(1, x, y)$ and $\operatorname{mult}_{(0,0)} Q(1, x, y)$ by μ and ν. We write

$$P(1, x, y) = \sum_{i,j \geq 0} \gamma_{ij} x^i y^j, \qquad Q(1, x, y) = \sum_{i,j \geq 0} \delta_{ij} x^i y^j$$

where the coefficients γ_{ij} and δ_{ij} are complex numbers. Then $\mathbf{R}_{P,Q}(1, x)$ is the determinant of the matrix

$$\begin{pmatrix}
\sum_{i\geq 0}\gamma_{i0}x^i & 0 & \dots & 0 & \sum_{i\geq 0}\delta_{i0}x^i & 0 & \dots & \dots & \dots & 0 \\
\sum_{i\geq 0}\gamma_{i1}x^i & \sum_{i\geq 0}\gamma_{i0}x^i & \dots & 0 & \sum_{i\geq 0}\delta_{i1}x^i & \sum_{i\geq 0}\delta_{i0}x^i & \dots & \dots & \dots & 0 \\
\sum_{i\geq 0}\gamma_{i2}x^i & \sum_{i\geq 1}\gamma_{i1}x^i & \ddots & \vdots & \sum_{i\geq 0}\delta_{i2}x^i & \sum_{i\geq 0}\delta_{i1}x^i & \ddots & \dots & \dots & 0 \\
\vdots & \vdots & \ddots & \sum_{i\geq 0}\gamma_{i0}x^i & \vdots & \vdots & \ddots & \ddots & \dots & 0 \\
\vdots & \vdots & \dots & \sum_{i\geq 0}\gamma_{i1}x^i & \sum_{i\geq 0}\delta_{ie}x^i & \sum_{i\geq 0}\delta_{i,e-1}x^i & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \dots & \sum_{i\geq 0}\gamma_{i2}x^i & 0 & \sum_{i\geq 0}\delta_{ie}x^i & \ddots & \ddots & \ddots & \sum_{i\geq 0}\delta_{i0}x^i \\
\sum_{i\geq 0}\gamma_{id}x^i & \sum_{i\geq 0}\gamma_{i,d-1}x^i & \dots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \sum_{i\geq 0}\delta_{i1}x^i \\
0 & \sum_{i\geq 0}\gamma_{id}x^i & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \sum_{i\geq 0}\gamma_{i,d-1}x^i & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \sum_{i\geq 0}\gamma_{id}x^i & 0 & 0 & \dots & \dots & \dots & \sum_{i\geq 0}\delta_{ie}x^i
\end{pmatrix}.$$

By the definition of the multiplicity, we know that $\gamma_{ij} = 0$ whenever $i+j < \mu$, and similarly $\delta_{ij} = 0$ whenever $i + j < \nu$. Therefore, if we divide the r-th row of this matrix by $x^{\mu+\nu+1-r}$ for all r in $\{1, \dots, \mu + \nu\}$, the s-th column by $x^{s-1-\nu}$ for all s in $\{1, \dots, \nu\}$, and the $(e + s)$-th column by $x^{s-1-\mu}$ for all s in $\{1, \dots, \mu\}$, then the resulting matrix still has entries in $\mathbb{C}[x]$. It follows that $\mathbf{R}_{P,Q}(1, x)$ is divisible by x to the power

$$\sum_{r=1}^{\mu+\nu}(\mu + \nu + 1 - r) + \sum_{s=1}^{\nu}(s - 1 - \nu) + \sum_{s=1}^{\mu}(s - 1 - \mu) = \mu\nu$$

so that

$$\mathbf{I}(p, C, D) = \operatorname{mult}_0 \mathbf{R}(1, x) \geq \mu\nu.$$

□

With our definition of intersection multiplicities at hand, we can finally state and prove the strong form of Bézout's theorem.

Theorem 11.16 (Strong Bézout theorem)**.** *Let C and D be projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ with no common component, and let $d = \deg(C)$ and $e = \deg(D)$. Then*

$$\sum_{p \in C \cap D} \mathbf{I}(p, C, D) = de.$$

Proof. After a projective transformation, we may assume that $[0, 0, 1]$ is not contained in $C \cup D$ or in any of the lines through two distinct points in $C \cap D$. Let $p_1, \ldots, p_r$ be the points in $C \cap D$ (we know that this intersection is finite by the weak Bézout theorem). For each of the points p_i, we choose a triple of homogeneous coordinates (p_{0i}, p_{1i}, p_{2i}). Since all these points are different from $[0, 0, 1]$ and no two of them are collinear with $[0, 0, 1]$, we know that the couples (p_{0i}, p_{1i}) define r distinct points in $\mathbb{P}^1_{\mathbb{C}}$.

Let P and Q be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero sets in $\mathbb{P}^2_{\mathbb{C}}$ are C and D. By the solution to Exercise 11.14, the points $[p_{0i}, p_{1i}]$ are precisely the zeros of the resultant $\mathbf{R}_{P,Q}$ in $\mathbb{P}^1_{\mathbb{C}}$. By definition, we have

$$\mathbf{I}(p_i, C, D) = \operatorname{mult}_{(p_{0i}, p_{1i})} \mathbf{R}_{P,Q}$$

for every i in $\{1, \ldots, r\}$. By Theorem 10.7, $\mathbf{R}_{P,Q}$ is a homogeneous polynomial of degree de, so that the result follows from Lemma 11.8. $\square$

Corollary 11.17. *Let C and D be projective plane curves in $\mathbb{P}^2_{\mathbb{C}}$ with no common component, and let $d = \deg(C)$ and $e = \deg(D)$. Then*

$$\sum_{p \in C \cap D} \operatorname{mult}_p C \cdot \operatorname{mult}_p D \leq de.$$

Proof. This follows immediately from the strong Bézout theorem and Proposition 11.15. $\square$

Example 11.18. Consider the projective conics C and D defined by $x_0x_2 - x_1^2 = 0$ and $x_0x_2 + x_1^2 = 0$, respectively. Then C and D have two intersection points: $p_1 = [0, 0, 1]$ and $p_2 = [1, 0, 0]$. Let us compute the intersection multiplicities of C and D at these points.

Since $[0, 0, 1]$ lies on $C \cup D$, we first need to perform a change of coordinates. Let Φ be the projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ that maps $[x_0, x_1, x_2]$ to $[x_0, x_2, x_1]$. Then $\Phi(C)$ is the projective conic defined by $x_0x_1 - x_2^2 = 0$, and $\Phi(D)$ is the projective conic defined by $x_0x_1 + x_2^2 = 0$. These conics are smooth, and therefore irreducible; since they are distinct, they have no common component. The resultant of $P = x_0x_1 - x_2^2$ and $Q = x_0x_1 + x_2^2$ is given by

$$\mathbf{R}_{P,Q} = \det \begin{pmatrix} x_0x_1 & 0 & x_0x_1 & 0 \\ 0 & x_0x_1 & 0 & x_0x_1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 4x_0^2x_1^2.$$

It has multiplicity 2 at $(0, 1)$ and $(1, 0)$. It follows that $\Phi(C)$ and $\Phi(D)$ have intersection multiplicity 2 at $\Phi(p_1) = [0, 1, 0]$ and $\Phi(p_2) = [1, 0, 0]$, so that C and D have intersection multiplicity 2 at p_1 and p_2.

Note that this result also follows from Bézout's theorem and the invariance of intersection numbers under projective transformations: the transformation that swaps x_0 and x_2 preserves C and D but exchanges p_1 and p_2. Thus the intersection multiplicities at p_1 and p_2 are the same; since they need to add up to $2 \cdot 2 = 4$ by Bézout's theorem, we conclude that they are both equal to 2.

Exercise 11.19. Show that an irreducible projective plane curve over $\mathbb{C}$ of degree d has at most $d(d-1)/2$ singular points. ☜

Exercise 11.20. Show that every projective plane curve C over $\mathbb{C}$ of degree 3 with at least 2 singular points is reducible.

Exercise 11.21 ($\star$)**.** Let C be an irreducible projective plane curve of degree d over $\mathbb{C}$, and let p be a point of C. Set $m = \operatorname{mult}_p C$. Show that for all but finitely many lines L through p in $\mathbb{P}^2_{\mathbb{C}}$, the line L meets C in exactly $d - m + 1$ points. Deduce that there exists a line in $\mathbb{P}^2_{\mathbb{C}}$ that meets C in precisely d points.

Remark 11.22. Our definition of the intersection multiplicity $\mathbf{I}(p, C, D)$ is somewhat artificial: first, there is the matter of its apparent coordinate dependence, and some non-trivial work is required to prove Proposition 11.12. Another quirk is that $\mathbf{I}(p, C, D)$ should be a *local* invariant of C and D around the point p, while the resultant uses global information (equations for C and D in the projective plane). A more natural definition of intersection multiplicities is given in Fulton's *Algebraic curves*, but it requires more tools from commutative algebra. This definition is coordinate independent; a comparison with Definition 11.9 then gives an alternative proof for Proposition 11.12.

12 Axiomatic characterization of intersection multiplicities

Our definition of intersection multiplicities was convenient to deduce Bézout's theorem, and has the advantage of being explicit. However, it does not provide much geometric insight, and some basic properties (like the invariance under projective transformations) are difficult to prove directly. In this section, we will establish an axiomatic characterization of intersection multiplicities, which is closer to our geometric intuition.

Theorem 12.1 (Axiomatic characterization of intersection multiplicities)**.** *Let p be a point of $\mathbb{P}^2_{\mathbb{C}}$. There is a unique way to associate with each couple (P,Q) of non-zero homogeneous polynomials in $\mathbb{C}[x_0,x_1,x_2]$ an invariant $\mathbf{I}(p,P,Q)$ in $\mathbb{N}\cup\{\infty\}$ that satisfies the following six axioms.*

1. Symmetry: $\mathbf{I}(p,P,Q)=\mathbf{I}(p,Q,P)$.
2. Detects intersection points: $\mathbf{I}(p,P,Q)\neq 0$ *if and only if* $P(p)=Q(p)=0$.
3. Detects common components: $\mathbf{I}(p,P,Q)=\infty$ *if and only if P and Q have a common irreducible factor that vanishes at p.*
4. Transversality: *suppose that P and Q both have degree one and that Q is not of the form λP for some $\lambda\in\mathbb{C}^*$. If p is the unique point in $\mathbb{P}^2_{\mathbb{C}}$ such that $P(p)=Q(p)=0$, then $\mathbf{I}(p,P,Q)=1$.*
5. Additivity: *if R is another non-zero homogeneous polynomial in $\mathbb{C}[x_0,x_1,x_2]$, then* $\mathbf{I}(p,P,QR)=\mathbf{I}(p,P,Q)+\mathbf{I}(p,P,R)$.
6. Deformation: *assume that* $\deg(Q)\geq\deg(P)$, *and let R be a homogeneous polynomial in $\mathbb{C}[x_0,x_1,x_2]$ of degree* $\deg(Q)-\deg(P)$ *such that $Q+PR\neq 0$. Then* $\mathbf{I}(p,P,Q)=\mathbf{I}(p,P,Q+PR)$.

Moreover, if P and Q are non-constant and have no repeated factors, and we denote by C and D their zero sets in $\mathbb{P}^2_{\mathbb{C}}$, then $\mathbf{I}(p,P,Q)=\mathbf{I}(p,C,D)$.

The deformation axiom is the least geometric: the idea is that, when we restrict Q to the zero set of P, its order of vanishing at a point does not change if we deform Q by adding a multiple of P. Beware that, when P and Q are non-constant homogeneous polynomials in $\mathbb{C}[x_0,x_1,x_2]$ with zero sets C and D in $\mathbb{P}^2_{\mathbb{C}}$, then the equality $\mathbf{I}(p,C,D)=\mathbf{I}(p,P,Q)$ does not necessarily hold if P and Q have repeated factors. For instance, Q and Q^2 have the same zero set, but the additivity axiom implies that $\mathbf{I}(p,P,Q^2)=2\mathbf{I}(p,P,Q)$.

It follows from Theorem 12.1 that the invariant $\mathbf{I}(p,C,D)$ satisfies the geometric translations of all the axioms in the statement.

Exercise 12.2.

1. Formulate the geometric versions of the first five axioms in Theorem 12.1 (the geometric translation of the deformation axiom is not so obvious and requires the language of linear systems).
2. Let C, D and E be complex projective plane curves, and let p be a point in C that does not lie in a common component of C and D or a common component of C and E. Deduce from the axioms that $\mathbf{I}(p,C,D\cup E)\leq\mathbf{I}(p,C,D)+\mathbf{I}(p,C,E)$, and that equality holds if and only if p does not lie in a common component of D and E.

Before moving on to the proof of Theorem 12.1, let us see in a concrete example how the axioms allow us to compute the intersection multiplicity $\mathbf{I}(p,P,Q)$.

Example 12.3. We will compute $\mathbf{I}(p,P,Q)$ for $p=[0,0,1]$, $P=x_0x_2-x_1^2$ and $Q=x_0x_2+x_1^2$. We have

$$\begin{aligned}
\mathbf{I}(p,P,Q) &= \mathbf{I}(p,x_0x_2-x_1^2,2x_1^2) && \text{by deformation,}\\
&= \mathbf{I}(p,x_0x_2-x_1^2,2)+2\mathbf{I}(p,x_0x_2-x_1^2,x_1) && \text{by additivity,}\\
&= 2\mathbf{I}(p,x_1,x_0x_2-x_1^2) && \text{by detection of intersection points and symmetry,}\\
&= 2\mathbf{I}(p,x_1,x_0x_2) && \text{by deformation,}\\
&= 2\mathbf{I}(p,x_1,x_0)+2\mathbf{I}(p,x_1,x_2) && \text{by additivity,}\\
&= 2 && \text{by detection of intersection points and transversality.}
\end{aligned}$$

Since P and Q have no repeated components, this is also the intersection multiplicity at p of the curves defined by $P = 0$ and $Q = 0$.

The proof of Theorem 12.1 will occupy most of the remainder of this section. It is split up in two steps. We will first prove that the axioms characterize $\mathbf{I}(p, C, D)$ uniquely. Afterwards, we will prove in Proposition 12.6 that such an invariant exists, and we will establish its relation with intersection multiplicities of curves.

Proposition 12.4. *There is at most one invariant $\mathbf{I}(p, P, Q)$ that satisfies all the axioms in Theorem 12.1.*

Proof. Since the axioms are coordinate-independent, we may assume that $p = [1, 0, 0]$. Indeed, if we could find two distinct invariants $\mathbf{I}(q, P, Q)$ and $\mathbf{I}'(q, P, Q)$ at another point q in $\mathbb{P}^2_{\mathbb{C}}$ satisfying all the axioms, then we could pick an invertible rank 3 matrix A over $\mathbb{C}$ such that the projective coordinate transformation Φ_A maps q to $p = [0, 0, 1]$, and produce two distinct invariants $\mathbf{I}(p, P, Q) = \mathbf{I}(q, P((x_0, x_1, x_2)A^t), Q((x_0, x_1, x_2)A^t))$ and $\mathbf{I}'(p, P, Q) = \mathbf{I}'(q, P((x_0, x_1, x_2)A^t), Q((x_0, x_1, x_2)A^t))$ satisfying all the axioms with respect to the point p.

So, suppose that we have two invariants $\mathbf{I}(p, \cdot, \cdot)$ and $\mathbf{I}'(p, \cdot, \cdot)$ that satisfy all the axioms. We will show that $\mathbf{I}(p, P, Q) = \mathbf{I}'(p, P, Q)$ for all non-zero homogeneous polynomials P and Q in $\mathbb{C}[x_0, x_1, x_2]$. Denote by r and s the degrees of the polynomials $P(1, 0, x)$ and $Q(1, 0, x)$ in $\mathbb{C}[x]$. By symmetry, we may assume that $r \leq s$. We will prove the result by induction on r.

First, suppose that $r \leq 0$. By the detection of intersection points, the equality $\mathbf{I}(p, P, Q) = \mathbf{I}'(p, P, Q)$ is satisfied when P or Q does not vanish at p. Thus, by additivity and symmetry, we can reduce to the case where $P(p) = Q(p) = 0$ and P is irreducible. By the detection of common components, we can exclude the case where Q is divisible by P.

Now $P(1, 0, x)$ is a constant polynomial that vanishes at $x = 0$, so that $P(1, 0, x) = 0$. This implies that $P(x_0, x_1, x_2)$ is divisible by x_1. Since we are assuming that P is irreducible, it follows that $P = ax_1$ for some a in $\mathbb{C}^*$. Now we can use the deformation axiom to remove all the terms in Q that are divisible by x_1; this brings us to the case where Q is a non-zero homogeneous polynomial in $\mathbb{C}[x_0, x_2]$. Factoring Q into linear homogeneous polynomials, we deduce from the additivity and transversality axioms that $\mathbf{I}(p, P, Q) = \mathbf{I}'(p, P, Q) = \operatorname{mult}_0 Q(1, 0, x)$: this is the number of linear factors in Q that vanish at the point p.

Therefore, we may assume that $s \geq r \geq 1$, and that $\mathbf{I}(p, P', Q') = \mathbf{I}'(p, P', Q')$ for all non-zero homogeneous polynomials P' and Q' in $\mathbb{C}[x_0, x_1, x_2]$ such that the degree of $P'(1, 0, x)$ or $Q'(1, 0, x)$ is strictly less than r. We may again assume that $P(p) = Q(p) = 0$ and that P is irreducible and does not divide Q. Dividing P and Q by suitable elements in $\mathbb{C}^*$, we can reduce to the case where $P(1, 0, x)$ and $Q(1, 0, x)$ are monic. We set

$$R(x_0, x_1, x_2) = x_0^{\deg(P)+s-r} Q(x_0, x_1, x_2) - x_0^{\deg(Q)} x_2^{s-r} P(x_0, x_1, x_2).$$

This is a homogeneous polynomial such that the degree of $R(1, 0, x) = Q(1, 0, x) - x^{s-r} P(1, 0, x)$ is at most $s - 1$. Moreover, R is non-zero because P is irreducible and does not divide x_0 or Q.

By detection of intersection points, additivity and deformation, we have

$$\mathbf{I}(p, P, Q) = \mathbf{I}(p, P, x_0^{\deg(P)+s-r} Q(x_0, x_1, x_2)) = \mathbf{I}(p, P, R)$$

and the same equalities hold for $\mathbf{I}'$. Replacing Q by R and repeating the argument, we arrive at a situation where the degree of Q becomes strictly less than r, where we can invoke the induction hypothesis. □

Now we will prove the existence of the invariant $\mathbf{I}(p, P, Q)$ and its relation with intersection multiplicities of curves. We start with an auxiliary lemma that gives an alternative description of the resultant for univariate polynomials.

Lemma 12.5. *Let P and Q be non-constant polynomials in $\mathbb{C}[x]$. We write them as*

$$P(x) = a(x - \lambda_1) \cdots (x - \lambda_d), \qquad Q(x) = b(x - \mu_1) \cdots (x - \mu_e),$$

with a and b in $\mathbb{C}^*$, and $\lambda_1, \ldots, \lambda_d$ and $\mu_1, \ldots, \mu_e$ in $\mathbb{C}$. Then

$$\mathbf{R}_{P,Q} = a^e b^d \prod_{\substack{i=1,\ldots,d \\ j=1,\ldots,e}} (\mu_j - \lambda_i).$$

Proof. Dividing P by a amounts to dividing e columns by a in the matrix whose determinant computes the resultant, so that the resultant gets divided by a^e. The analogous property holds for Q and b. Thus we may assume that $a = b = 1$.

If we treat the λ_i and μ_j as complex parameters, then the proof of Theorem 10.7 shows that $\mathbf{R}_{P,Q}$ is a homogeneous polynomial of degree de in the variables $\lambda_1, \ldots, \mu_d$ and $\mu_1, \ldots, \mu_e$. Moreover, for every i and every j, it vanishes if we specialize λ_i and μ_j to the same complex number, because then the polynomials P and Q then have a common root. It follows that $\mathbf{R}_{P,Q}$ is divisible by $\lambda_i - \mu_j$ for all i and j, and thus also by the polynomial

$$\prod_{\substack{i=1,\ldots,d \\ j=1,\ldots,e}} (\mu_j - \lambda_i).$$

Since $\mathbf{R}_{P,Q}$ and this product both have degree de, they differ by a factor in $\mathbb{C}^*$. So, to prove that they are equal, we may first specialize all of the variables λ_i to 0 and all of the variables μ_j to 1. This brings us to the case $P = x^d$ and $Q = (x-1)^e$, where it is straightforward to check that $\mathbf{R}_{P,Q} = 1$. □

Proposition 12.6. *There exists an invariant $\mathbf{I}(p, P, Q)$ that satisfies all the axioms in Theorem 12.1. If P and Q are non-constant and have no repeated factors, and we denote by C and D their zero sets in $\mathbb{P}^2_{\mathbb{C}}$, then $\mathbf{I}(p, P, Q) = \mathbf{I}_p(C, D)$.*

Proof. We can construct $\mathbf{I}(p, P, Q)$ by mimicking the definition of intersection multiplicities for curves. If P and Q have a common irreducible factor that vanishes at p, then $\mathbf{I}(p, P, Q) = \infty$. Otherwise, we remove all common irreducible factors of P and Q (the result is well-defined only up to a factor in $\mathbb{C}^*$, but this will have no impact on the construction). If $P(p) \neq 0$ or $Q(p) \neq 0$, then we set $\mathbf{I}(p, P, Q) = 0$.

In the remaining case, P and Q have no common irreducible factor and $P(p) = Q(p) = 0$. We choose a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $[0, 0, 1]$ does not lie in the zero sets of $P \circ \Phi^{-1}$ and $Q \circ \Phi^{-1}$ or on a line through two distinct common zeroes of $P \circ \Phi^{-1}$ and $Q \circ \Phi^{-1}$ (this is a slight abuse of notation, because P and Q are not functions on $\mathbb{P}^2_{\mathbb{C}}$; we write $P \circ \Phi^{-1}$ and $Q \circ \Phi^{-1}$ for the homogeneous polynomials that we obtain by choosing an invertible rank 3 matrix A such that $\Phi = \Phi_A$ and evaluating P and Q in the triple $(x_0, x_1, x_2)(A^{-1})^t$). Then we pick homogeneous coordinates (q_0, q_1, q_2) for $q = \Phi(p)$, and we set

$$\mathbf{I}(p, P, Q) = \operatorname{mult}_{(q_0,q_1)} \mathbf{R}_{P\circ\Phi^{-1}, Q\circ\Phi^{-1}}.$$

This again requires checking that this definition does not depend on Φ, which can be done like in Proposition 11.12 (not examinable). This definition immediately implies the relation with intersection multiplicities of curves.

Let us check that our definition satisfies all the axioms in Theorem 12.1. Detection of common components is baked into the definition; thus, we may assume that P and Q have no common irreducible factor (and neither do P and R in the additivity axiom). We can easily reduce to the case where $P(p) = Q(p) = 0$ (in the additivity axiom, swap the roles of Q and R if necessary) . Replacing p, P and Q by $\Phi(p)$, $P \circ \Phi^{-1}$ and $Q \circ \Phi^{-1}$, we may assume that Φ is the identity. Then $\mathbf{I}(p, P, Q) = \operatorname{mult}_{(p_0,p_1)} \mathbf{R}_{P,Q}$ where (p_0, p_1, p_2) is a triple of homogeneous coordinates for p.

Symmetry follows easily from the definition: swapping P and Q amounts to swapping columns in the matrix whose determinant computes the resultant $\mathbf{R}_{P,Q}$; this can only affect the sign of the resultant, and has no impact on its multiplicity. You have already shown in Exercise 11.14 that the intersection multiplicity detects intersection points (the argument remains valid if P and Q have repeated factors). Transversality follows from the fact that the resultant of two non-proportional linear homogeneous polynomials is itself a linear homogeneous polynomial, so that its multiplicity at any point is at most 1.

To check the additivity axiom, we can apply a projective transformation to further reduce to the case where $R(0, 0, 1) \neq 0$ and $[0, 0, 1]$ is not collinear with two distinct common zeros of P and QR (then the same holds for P and R, because every common zero of P and R is also a common zero of P and QR). Since $P(p) = Q(p) = 0$, this implies that the line through $[0, 0, 1]$ and p contains no common zeros of P and R, except possibly p itself. Thus, if $R(p) \neq 0$, then $\mathbf{R}_{P,R}$ does not vanish at (p_0, p_1) so that $\operatorname{mult}_{(p_0,p_1)} \mathbf{R}_{P,R} = 0$. This means that $\operatorname{mult}_{(p_0,p_1)} \mathbf{R}_{P,R}$ is equal to $\mathbf{I}(p, P, R)$ even when $R(p) \neq 0$, so that it suffices to show that for all non-constant homogeneous polynomials P, Q and R in $\mathbb{C}[x_0, x_1, x_2]$ that do not vanish at $[0, 0, 1]$, we have $\mathbf{R}_{P,QR} = \mathbf{R}_{P,Q} \cdot \mathbf{R}_{P,R}$. The result then follows from the fact that the multiplicity of a polynomial at a point is additive under multiplication of polynomials. Furthermore, since two complex polynomials are equal if they take the same value at each point, it is enough to prove that $\mathbf{R}_{P,QR}(a,b) = \mathbf{R}_{P,Q}(a,b) \cdot \mathbf{R}_{P,R}(a,b)$ for all complex numbers a and b. Because taking resultants commutes with specialization, we can substitute x_0 and x_1 by a and b in P, Q and R, and reduce the problem to the case where P, Q and R are polynomials in one variable. Now the result follows from Lemma 12.5.

Finally, we prove the deformation axiom. We may assume that P, Q and $Q + PR$ do not vanish at $[0,0,1]$; then it is enough to show that $\mathbf{R}_{P,Q} = \mathbf{R}_{P,Q+PR}$. Set $d = \deg(P)$ and $e = \deg(Q)$. Then we can write $R = r_0(x_0,x_1)+\ldots+r_{e-d}(x_0,x_1)x_2^{e-d}$ where each coefficient $r_i(x_0,x_1)$ is a homogeneous polynomial of degree $e-d-i$ in $\mathbb{C}[x_0,x_1]$. Now we obtain the matrix whose determinant computes $\mathbf{R}_{P,Q+PR}$ from that of $\mathbf{R}_{P,Q}$ by adding $r_i(x_0,x_1)$ times the $(i+j)$-th column to the $(e+j)$-th column, for every i in $\{0,\ldots,e-d\}$ and every j in $\{1,\ldots,d\}$. This has no influence on the determinant of the matrix, so that $\mathbf{R}_{P,Q} = \mathbf{R}_{P,Q+PR}$. ☐

Remark 12.7. The existence part of the proof of Theorem 3.18 in Kirwan's *Complex algebraic curves* is incomplete, because Kirwan does not prove coordinate independence of the intersection multiplicities. She constructs the invariant $\mathbf{I}(p,P,Q)$ by picking any coordinate transformation such that $[0,0,1]$ does not lie in the zero sets of P and Q, and also not on any line through two distinct common zeroes of P and Q (she uses the reference point $[1,0,0]$ instead, but that does not change the argument). Without invoking a form of coordinate independence, it is not possible to check the deformation axiom in this way, because we have no systematic way of picking coordinates that work for (P,Q) and $(P,Q+PR)$ simultaneously (for instance, in making a choice for (P,Q), we would need to know R in advance to make sure that $Q + PR$ does not vanish at $[0,0,1]$).

Definition 12.8. Let C and D be projective plane curves over $\mathbb{C}$. Let p be a point in $C \cap D$ such that C and D are smooth at p. We say that C and D intersect *transversally* at p if $\mathbf{I}(p,C,D) = 1$. Otherwise (that is, if $\mathbf{I}(p,C,D) \geq 2$), we say that C and D are *tangent* at p.

The transversality axiom expresses that two distinct lines intersect transversally at their unique point of intersection. The following result shows that our definition is compatible with the previously defined notion of projective tangent line.

Proposition 12.9. *Let C and D be projective plane curves over $\mathbb{C}$. Let p be a point in $C \cap D$ such that C and D are smooth at p. Then C and D are tangent at p if and only if they have the same projective tangent line at p. In particular, a line L in $\mathbb{P}^2_{\mathbb{C}}$ is tangent to C at a smooth point p if and only if L is the projective tangent line to C at p.*

Proof. We first show that the projective tangent line of C at a smooth point p only depends on the irreducible component of C that contains p. Note that this component is unique, by Proposition 7.11. Let P be a homogeneous polynomial in $\mathbb{C}[x_0,x_1,x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C, and write $P = P_1 \cdot P_2$ where P_1 is an irreducible factor of P that vanishes at p; then $P_2(p) \neq 0$, and the zero set of P_1 is the unique irreducible component C' of C through p. Let (p_0,p_1,p_2) be a triple of homogeneous coordinates for p. Then, for every i in $\{0,1,2\}$, we have

$$\partial_{x_i} P(p_0,p_1,p_2) = P_2(p_0,p_1,p_2) \cdot \partial_{x_i} P_1(p_0,p_1,p_2)$$

by the Leibniz rule, so that the vector of partial derivatives of P at (p_0,p_1,p_2) is a non-zero scalar multiple of the vector of partial derivatives of P_1 at (p_0,p_1,p_2). This implies that the projective tangent line to C at p coincides with the projective tangent line to C' at p. Likewise, the projective tangent line to D at p coincides with the projective tangent line to the unique irreducible component D' of D that contains p. Moreover, $\mathbf{I}(p,C,D) = \mathbf{I}(p,C',D')$ by symmetry, additivity and the detection of intersection points, so that C and D are tangent at p if and only if C' and D' are tangent at p. Therefore, we may replace C and D by C' and D', and assume that C and D are irreducible.

If $C = D$ then $\mathbf{I}(p,C,D) = \infty$ and the result is trivial. Otherwise, C and D have no common component, and to prove this case we take another look at the proof of Proposition 11.15. We follow the assumptions and notations in that proof. By picking a suitable coordinate transformation, we may also assume that $[0,0,1]$ does not lie on the tangent lines to C and D at the point p. Since in our current setting, C and D are smooth at p, we have $\mu = \nu = 1$. We need to show that $x^{-1}\mathbf{R}_{P,Q}(1,x)$ vanishes at $x = 0$ if and only if C and D have the same projective tangent line at p.

The first row of the matrix in the proof of Proposition 11.15 is divisible by x, because $\gamma_{00} = \delta_{00} = 0$. Dividing the first row by x and setting $x = 0$, we get a matrix whose first two rows have $(\gamma_{10}, \gamma_{01})^t$ in the first column, $(\delta_{10}, \delta_{01})^t$ in the $(e+1)$-th column, and zeros everywhere else. Computing the determinant of this matrix by developing the first 2 rows, we see that the value of $x^{-1}\mathbf{R}_{P,Q}(1, x)$ at $x = 0$ equals

$$(-1)^{e+1} \det \begin{pmatrix} \gamma_{10} & \delta_{10} \\ \gamma_{01} & \delta_{01} \end{pmatrix}$$

times the determinant of the matrix

$$\begin{pmatrix}
\gamma_{01} & 0 & \dots & 0 & \delta_{01} & 0 & \dots & \dots & \dots & 0 \\
\gamma_{02} & \gamma_{01} & \dots & 0 & \delta_{02} & \delta_{01} & \dots & \dots & \dots & 0 \\
\gamma_{03} & \gamma_{02} & \ddots & \vdots & \delta_{03} & \delta_{02} & \ddots & \dots & \dots & 0 \\
\vdots & \vdots & \ddots & \gamma_{01} & \vdots & \vdots & \ddots & \ddots & \dots & 0 \\
\vdots & \vdots & \dots & \gamma_{02} & \delta_{0e} & \delta_{0,e-1} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \dots & \gamma_{03} & 0 & \delta_{0e} & \ddots & \ddots & \ddots & \delta_{01} \\
\gamma_{0d} & \gamma_{0,d-1} & \dots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \delta_{02} \\
0 & \gamma_{0d} & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \gamma_{0,d-1} & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \gamma_{0d} & 0 & 0 & \dots & \dots & \dots & \delta_{0e}
\end{pmatrix}.$$

The first factor is zero if and only if C and D have the same projective tangent line at p, because these tangent lines are defined by $\gamma_{10}x_1 + \gamma_{01}x_2 = 0$ and $\delta_{10}x_1 + \delta_{01}x_2 = 0$. Therefore, it suffices to show that the second factor is never zero. This second factor is the resultant of the polynomials $y^{-1}P(1, 0, y)$ and $y^{-1}Q(1, 0, y)$. The only common root of the polynomials $P(1, 0, y)$ and $Q(1, 0, y)$ is $y = 0$, by our assumption that $[0, 0, 1]$ does not lie on a line through two distinct intersection points of C and D. Moreover, $y = 0$ is a simple root of $P(1, 0, y)$ and $Q(1, 0, y)$, by the assumption that $[0, 0, 1]$ does not lie on the tangent lines to C and D at the point p. It follows that $y^{-1}P(1, 0, y)$ and $y^{-1}Q(1, 0, y)$ have no common root, so that their resultant is non-zero.

As a special case, we find that a line L in $\mathbb{P}^2_{\mathbb{C}}$ is tangent to C at p if and only if L is the projective tangent line to C at p, since the projective tangent line to L at any of its points is L itself. □

Corollary 12.10. *Let C and D be projective plane curves and let p be a point of $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathbf{I}(p, C, D) = 1$ if and only if $p \in C \cap D$, the curves C and D are smooth at p, and the projective tangent lines of C and D at p are distinct.*

Proof. The "if" implication follows from Proposition 12.9. Conversely, if $\mathbf{I}(p, C, D) = 1$ then C and D are smooth at p by Proposition 11.15, and Proposition 12.9 implies that the projective tangent lines to C and D at p are distinct. □

Exercise 12.11. Let C be the conic in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_0^2 + x_1x_2 = 0$. Construct a smooth conic in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to C in one point and that intersects C transversally in two other points.

13 Cubic curves

A *cubic curve* is a projective plane curve defined by a homogeneous equation $P(x_0, x_1, x_2) = 0$ of degree 3. We say that the curve is *non-degenerate* if P is irreducible. In this section, we will investigate the geometry of cubic curves.

Non-degenerate projective plane curves of degrees 1 and 2 can be put into a standard form by means of a projective transformation: every line is projectively equivalent to the line defined by $x_0 = 0$, and every non-degenerate conic is projectively equivalent to the conic defined by $x_0x_2 + x_1^2 = 0$ (Theorem 9.1). Cubic curves display a richer geometry: there is a one-parameter family of projective equivalence classes of smooth non-degenerate cubic curves.

Theorem 13.1. *For every $\lambda \in \mathbb{C} \setminus \{0, 1\}$, the projective plane curve C_λ over $\mathbb{C}$ defined by*

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

is a smooth non-degenerate cubic. Every smooth non-degenerate cubic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to C_λ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

Proof. Set $P = x_1^2x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2)$. If $\partial_{x_1} P$ vanishes at a point $p = [p_0, p_1, p_2]$ of C_λ then we have $p_1p_2 = 0$ and $p_0(p_0 - p_2)(p_0 - \lambda p_2) = 0$. Moreover, $\partial_{x_0} P$ vanishes at p if and only if p_0 is a multiple root of the polynomial $x(x - p_2)(x - \lambda p_2)$. Since $\lambda \notin \{0, 1\}$, this implies that $p_2 = 0$. Then we also have $p_0 = 0$, so that $p = [0, 1, 0]$. The remaining partial derivative $\partial_{x_2} P$ does not vanish at $[0, 1, 0]$, so that P and its first order partial derivatives have no common zeros. This implies that P does not have repeated factors, because all the zeros of a repeated factor are common zeros of P and its first order partial derivatives by the Leibniz rule. Moreover, P is irreducible, by Corollary 10.13. It follows that C_λ is a smooth non-degenerate cubic.

The second part of the statement is a consequence of a stronger statement that we will formulate in Theorem 13.11 below. □

Dehomogenizing the equation for C_λ with respect to x_2, we find the so-called *Legendre equation* $y^2 = x(x-1)(x-\lambda)$ for the affine plane curve $C_\lambda \cap U_2$. The parameter λ in Theorem 13.1 is not uniquely determined by the curve C: one can show that C_λ and $C_{\lambda'}$ are projectively equivalent if and only if $j(\lambda) = j(\lambda')$ where $j(\lambda)$ is the *j-invariant*

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

The factor 2^8 is included in the definition for arithmetic purposes that play no role in our discussion.

To finish the proof of Theorem 13.1, we need to make some further preparations. The key to understanding the geometry of a smooth non-degenerate cubic C is an analysis of its *inflection points*.

Definition 13.2. Let C be a projective plane curve over $\mathbb{C}$. Let p be a smooth point of C and let L be the projective tangent line to C at p. We say that p is an *inflection point*, or *flex point*, of C if $\mathbf{I}(p, C, L) \geq 3$.

We always have $\mathbf{I}(p, C, L) \geq 2$, by Proposition 12.9. An inflection point is characterized by the property that the intersection multiplicity of C with the tangent line is higher than expected.

Example 13.3. Every point of a line is an inflection point; a non-degenerate conic C has no inflection points, because $\mathbf{I}(p, C, L) \leq \deg(C)\deg(L) = 2$ for every line L.

We can recognize inflection points of a curve by means of the *Hessian matrix* of an equation for the curve.

Definition 13.4. Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial. The *Hessian matrix* of P is the symmetric matrix $\mathbf{H}_P$ whose entries are the second order derivatives of P:

$$\mathbf{H}_P = \begin{pmatrix} \partial_{x_0}^2 P & \partial_{x_0}\partial_{x_1} P & \partial_{x_0}\partial_{x_2} P \\ \partial_{x_0}\partial_{x_1} P & \partial_{x_1}^2 P & \partial_{x_1}\partial_{x_2} P \\ \partial_{x_0}\partial_{x_2} P & \partial_{x_1}\partial_{x_2} P & \partial_{x_2}^2 P \end{pmatrix}.$$

The *Hessian* of P is defined by $\mathcal{H}_P(x_0, x_1, x_2) = \det \mathbf{H}_P$.

If P has degree 1, then $\mathcal{H}_P = 0$. If P has degree $d \geq 2$, then each entry in the Hessian matrix is homogeneous of degree $d-2$, so that the Hessian $\mathcal{H}_P$ is a homogeneous polynomial of degree $3(d-2)$ in $\mathbb{C}[x_0, x_1, x_2]$.

Exercise 13.5. Let A be an invertible matrix over $\mathbb{C}$ of rank 3. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ and let $Q(x_0, x_1, x_2) = P((x_0, x_1, x_2)A)$. Write the Hessian of Q in terms of the Hessian of P. Deduce that the Hessian of P vanishes at (p_0, p_1, p_2) if and only if the Hessian of Q vanishes at $(p_0, p_1, p_2)A^{-1}$.

In calculations it is often convenient to express the Hessian in the following way.

Lemma 13.6. *Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree $d > 1$. Then*

$$x_2^2 \cdot \mathcal{H}_P = (d-1)^2 \cdot \det \begin{pmatrix} \partial_{x_0}^2 P & \partial_{x_0}\partial_{x_1} P & \partial_{x_0} P \\ \partial_{x_0}\partial_{x_1} P & \partial_{x_1}^2 P & \partial_{x_1} P \\ \partial_{x_0} P & \partial_{x_1} P & \frac{d}{d-1} P \end{pmatrix}.$$

There is nothing special about the variable x_2: swapping variables we obtain analogous results for x_0 and x_1.

Proof. Multiplying the last row and column of the Hessian matrix by x_2, we find that

$$x_2^2 \cdot \mathcal{H}_P = \det \begin{pmatrix} \partial_{x_0}^2 P & \partial_{x_0}\partial_{x_1} P & x_2\partial_{x_0}\partial_{x_2} P \\ \partial_{x_0}\partial_{x_1} P & \partial_{x_1}^2 P & x_2\partial_{x_1}\partial_{x_2} P \\ x_2\partial_{x_0}\partial_{x_2} P & x_2\partial_{x_1}\partial_{x_2} P & x_2^2\partial_{x_2}^2 P \end{pmatrix}.$$

For every i in $\{0, 1, 2\}$, the partial derivative $\partial_{x_i} P$ is homogeneous of degree $d-1$, so that Euler's relation (Proposition 7.6) tells us that

$$(d-1)\partial_{x_i} P = x_0\partial_{x_0}\partial_{x_i} P + x_1\partial_{x_1}\partial_{x_i} P + x_2\partial_{x_2}\partial_{x_i} P.$$

When applied to P itself, Euler's relation yields

$$dP = x_0\partial_{x_0} P + x_1\partial_{x_1} P + x_2\partial_{x_2} P.$$

Therefore, adding x_0 times the first column and x_1 times the second column to the final column of the matrix, we may replace the final column by $(d-1) \cdot (\partial_{x_0} P, \partial_{x_1} P, x_2\partial_{x_2} P)^t$. Now we add x_0 times the first row and x_1 times the second row to the final row; this replaces the final row by

$$(d-1)\left(\partial_{x_0} P, \partial_{x_1} P, dP\right),$$

whence the desired formula. $\square$

Proposition 13.7. *Let C be a projective plane curve over $\mathbb{C}$ and let p be a smooth point of C. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors such that C is the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$. Then p is an inflection point of C if and only if the Hessian $\mathcal{H}_P$ vanishes at p.*

Proof. By Exercise 13.5, we may assume that $p = [0, 0, 1]$ and that the tangent line to C at p is defined by $x_0 = 0$. Multiplying P with a non-zero constant, we may also assume that $\partial_{x_0} P(0, 0, 1) = 1$. We set $Q(x, y) = P(x, y, 1)$. Lemma 13.6 implies that

$$\mathcal{H}_P(0,0,1) = (d-1)^2 \cdot \det \begin{pmatrix} \partial_x^2 Q(0,0) & \partial_x\partial_y Q(0,0) & 1 \\ \partial_x\partial_y Q(0,0) & \partial_y^2 Q(0,0) & 0 \\ 1 & 0 & 0 \end{pmatrix} = -(d-1)^2\partial_y^2 Q(0,0).$$

By Example 11.10, p is an inflection point of C if and only if the polynomial $Q(0, y)$ has a zero of multiplicity at least 3 at $y = 0$. Taking the Taylor expansion of Q around $(0, 0)$, we see that this is equivalent to saying that $\partial_y^2 Q(0, 0) = 0$. $\square$

Example 13.8. Suppose that C is the projectivization of the graph of a polynomial function $x \mapsto Q(x)$, where Q is a polynomial of degree d in $\mathbb{C}[x]$. Then C is defined by the equation $x_1 x_2^{d-1} = \overline{Q}(x_0, x_2)$ where $\overline{Q}$ is the homogenization of Q. By Lemma 13.6 and Proposition 13.7, a point $p = [p_0, p_1, 1]$ in $C \cap U_2$ is an inflection point of C if and only if

$$\det \begin{pmatrix} \partial_x^2 Q(p_0) & 0 & \partial_x Q(p_0) \\ 0 & 0 & -1 \\ \partial_x Q(p_0) & -1 & 0 \end{pmatrix} = 0,$$

that is, if and only if $\partial_x^2 Q(p_0) = 0$. So we see that our definition of an inflection point is closely related to the definition for differentiable functions in real analysis, except that there one also requires that the derivative changes sign; this extra condition has no meaning when we work over the complex numbers (or any other field without an ordering).

Exercise 13.9 ($\star$)**.** Let C be an irreducible projective plane curve over $\mathbb{C}$ with infinitely many inflection points. Show that C is a line. *Hint: let L be the tangent line at p; compare the intersection multiplicities at p of L and C, resp. the zero set of $\mathcal{H}_P$.*

Let P be an irreducible homogeneous polynomial of degree $d \geq 3$, and denote by C the zero set of P in $\mathbb{P}^2_{\mathbb{C}}$. We have already observed that $\mathcal{H}_P$ is is homogeneous of degree $3(d-2)$; moreover, it is non-zero, because otherwise every smooth point of C would be an inflection point, contradicting Exercise 13.9. Therefore, the zero set of $\mathcal{H}_P$ in $\mathbb{P}^2_{\mathbb{C}}$ is a projective plane curve of degree at most $3(d-2)$. It is called the *Hessian curve* of C and denoted by $\mathcal{H}_C$. By the projective Nullstellensatz, $\mathcal{H}_C$ only depends on C, and not on P. By Exercise 13.5, the Hessian curve is preserved by projective transformations.

Corollary 13.10. *Every smooth projective curve C over $\mathbb{C}$ of degree $d \geq 3$ has at least one inflection point, and the number of its inflection points is at most $3d(d-2)$.*

Proof. Since C is smooth, it is irreducible by Corollary 10.13. The set of inflection points of C is precisely the intersection of C with its Hessian curve $\mathcal{H}_C$, by Proposition 13.7. It follows from Exercise 13.9 that C and $\mathcal{H}_C$ have no common components. Now Bézout's theorem implies that $C \cap \mathcal{H}_C$ is non-empty and consists of at most $3d(d-2)$ points. □

We are finally ready to prove Theorem 13.1. Since every smooth cubic curve has an inflection point by Corollary 13.10, Theorem 13.1 is a consequence of the following stronger statement.

Theorem 13.11. *Let C be a smooth non-degenerate cubic in $\mathbb{P}^2_{\mathbb{C}}$ and let p be an inflection point of C. Then there exists a projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\Phi(p) = [0, 1, 0]$ and such that $\Phi(C)$ has an equation of the form*

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

with $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

Proof. Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. We can write

$$P(x_0, x_1, x_2) = a x_1^3 + b x_0 x_1^2 + c x_1^2 x_2 + d x_0^2 x_1 + e x_0 x_1 x_2 + f x_1 x_2^2 + Q(x_0, x_2)$$

where a, b, c, d, e, f are complex numbers and Q is a homogeneous polynomial of degree 3 in the variables x_0 and x_2.

After a projective transformation, we may assume that $p = [0, 1, 0]$ and that the projective tangent line to C at p is defined by $x_2 = 0$. In terms of our equation, this implies that $P(0, 1, 0) = a = 0$, $\partial_{x_0} P(0, 1, 0) = b = 0$, and $\partial_{x_2} P(0, 1, 0) = c \neq 0$. Moreover, $\mathcal{H}_P(0, 1, 0) = 0$; by Lemma 13.6 (with x_1 and x_2 swapped), this means that

$$\det \begin{pmatrix} \partial_{x_0}^2 P(0,1,0) & \partial_{x_0}\partial_{x_2} P(0,1,0) & 0 \\ \partial_{x_0}\partial_{x_2} P(0,1,0) & \partial_{x_2}^2 P(0,1,0) & c \\ 0 & c & 0 \end{pmatrix} = 0.$$

Since $c \neq 0$, we deduce that $\partial^2_{x_0} P(0,1,0) = 2d = 0$.

We have reduced our expression for P to

$$P(x_0,x_1,x_2) = cx_1^2x_2 + ex_0x_1x_2 + fx_1x_2^2 + Q(x_0,x_2) = c\left(x_1 + \frac{ex_0+fx_2}{2c}\right)^2 x_2 + R(x_0,x_2),$$

where $R \in \mathbb{C}[x_0,x_2]$ is again a homogeneous polynomial of degree three. Now consider the projective transformation

$$\Phi_1 \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},\ [x_0,x_1,x_2] \mapsto [x_0, x_1 + \tfrac{ex_0+fx_2}{2c}, x_2]$$

which is well-defined because $c \neq 0$. It preserves the point p, and $\Phi_1(C)$ is the zero set of a homogeneous polynomial of the form $x_1^2x_2 - S(x_0,x_2)$ where $S \in \mathbb{C}[x_0,x_2]$ is still a homogeneous polynomial of degree three. Thus, we may assume that C itself is defined by $x_1^2x_2 - S(x_0,x_2) = 0$.

By Lemma 6.10, we can write S as a product of linear factors. Since C is irreducible, the variable x_2 does not divide S. It follows that

$$S(x_0,x_2) = \delta(x_0 - \alpha x_2)(x_0 - \beta x_2)(x_0 - \gamma x_2)$$

where $\alpha,\beta,\gamma,\delta$ are complex numbers such that $\delta \neq 0$. Applying a further projective transformation to rescale the variable x_0, we can reduce to the case where $\delta = 1$.

Since C is smooth, the complex numbers α,β,γ are distinct: if the polynomial $(x-\alpha)(x-\beta)(x-\gamma)$ had a double root ε, then $[\varepsilon,0,1]$ would be a singular point of C. We can thus apply the projective transformation

$$\Phi_2 \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},\ [x_0,x_1,x_2] \mapsto \left[\frac{x_0 - \alpha x_2}{\beta - \alpha}, \eta x_1, x_2\right],$$

where η is a square root of $(\beta-\alpha)^{-3}$, to finally reduce to the case where $\alpha = 0$, $\beta = 1$ and $\gamma \in \mathbb{C} \setminus \{0,1\}$. Setting $\lambda = \gamma$, we obtain the desired expression. □

Corollary 13.12. *Every smooth non-degenerate cubic C in $\mathbb{P}^2_{\mathbb{C}}$ has precisely nine inflection points.*

Proof. Since C and its Hessian curve $\mathcal{H}_C$ have no common components by Exercise 13.9, it is enough to show that $\mathcal{H}_C$ has degree 3 and intersects C transversally at each intersection point. The result then follows from Bézout's theorem.

Let P be a homogeneous polynomial in $\mathbb{C}[x_0,x_1,x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. Then the Hessian curve $\mathcal{H}_C$ is the zero set of the homogeneous polynomial $\mathcal{H}_P$ of degree 3. Thus, to prove that $\mathcal{H}_C$ has degree 3, we need to show that $\mathcal{H}_P$ has no repeated factors. Assume the contrary: then $\mathcal{H}_P$ is divisible by the square of a linear homogeneous polynomial L in $\mathbb{C}[x_0,x_1,x_2]$. By Bézout's theorem, P and L have a common zero p in $\mathbb{P}^2_{\mathbb{C}}$, which is then an inflection point of C. Applying a projective transformation, we may assume that $p = [0,1,0]$ and that

$$P = x_1^2x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

for some $\lambda \in \mathbb{C} \setminus \{0,1\}$. A direct calculation shows that $\partial_{x_0}\mathcal{H}_P(0,1,0) = 24$. On the other hand, since $\mathcal{H}_P$ is divisible by L^2 and $L(0,1,0) = 0$, the Leibniz rule implies that $\partial_{x_0}\mathcal{H}_P(0,1,0) = 0$. This contradiction shows that $\mathcal{H}_P$ has no repeated factors.

Now let p be any inflection point of C. To show that C and $\mathcal{H}_C$ intersect transversally at p, it suffices to show that $\mathcal{H}_C$ is smooth at p, and that C and $\mathcal{H}_C$ have distinct projective tangent lines at p, by Corollary 12.10. We may again assume that $p = [0,1,0]$ and

$$P = x_1^2x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

for some $\lambda \in \mathbb{C} \setminus \{0,1\}$. Then the projective tangent line to C at p is defined by $x_2 = 0$. Our calculation that $\partial_{x_0}\mathcal{H}_P(0,1,0) = 24 \neq 0$ implies that $\mathcal{H}_C$ is smooth at p and that its tangent line at p is not defined by $x_2 = 0$. □

Remark 13.13. An *elliptic curve* is a pair (C, o) where C is a smooth non-degenerate cubic curve and o is a point of C. Elliptic curves are extremely important in geometry, analysis and number theory; for instance, they played a pivotal role in the proof of Fermat's last theorem. A key property of elliptic curves is that they carry a natural commutative group structure, with o serving as the identity element. One can always re-embed[4] C in $\mathbb{P}^2_{\mathbb{C}}$ in such a way that o becomes an inflection point; then the group structure on C is defined by saying that three points of C add up to o if and only if they are the three intersection points of C with a line in $\mathbb{P}^2_{\mathbb{C}}$ (taking multiplicities into account). See Theorem 3.38 in Kirwan's book.

[4] Making sense of this requires one to define *morphisms* between projective plane curves, which we will do in the chapter on Riemann surfaces.

14 Linear systems

It often occurs in algebraic geometry that a family of geometric objects can be parameterized in a natural way by an algebraic set in affine or projective space. A basic example is the set of lines through the origin of $\mathbb{C}^{n+1}$, which can be identified with the projective n-space $\mathbb{P}^n_{\mathbb{C}}$. In this section, we will study a generalization of this idea: *linear systems* of projective plane curves. We will see how certain families of projective plane curves can be parameterized by a projective space $\mathbb{P}^N_{\mathbb{C}}$, and how properties of this family can be deduced from geometry in the parameter space $\mathbb{P}^N_{\mathbb{C}}$ [5].

Here is a typical question that we will discuss: let $p_1, \ldots, p_k$ be distinct points in $\mathbb{P}^2_{\mathbb{C}}$. When can we find a projective plane curve C of given degree d that passes through $p_1, \ldots, p_k$? How many are there? Can we also prescribe the multiplicities $\operatorname{mult}_{p_i} C$? We will solve such questions by studying the space of all projective plane curves of degree d and analyzing which kind of subset is cut out by conditions of the form $p_i \in C$ or $\operatorname{mult}_{p_i} C = \mu_i$.

Example 14.1. There exists a unique projective plane curve C of degree $d = 1$ through $k = 2$ distinct points p_1 and p_2 in $\mathbb{P}^2_{\mathbb{C}}$, namely, the unique projective line through these points. We always have $\operatorname{mult}_{p_1} C = \operatorname{mult}_{p_2} C = 1$.

For $d = 1$ and $k = 3$, there usually does not exist a projective plane curve of degree d through k distinct points, except when the points happen to be collinear.

To properly develop the theory of linear systems, we need a slight generalization of the notion of projective plane curve. A *divisor* in $\mathbb{P}^2_{\mathbb{C}}$ is a formal expression of the form

$$D = \sum_{i=1}^{r} m_i C_i$$

where $C_1, \ldots, C_r$ are distinct irreducible projective plane curves and $m_1, \ldots, m_r$ are integers. The coefficient m_i is then called the *multiplicity* of D along the curve C_i. We say that the divisor D is *effective* if $m_i \geq 0$ for all i, and that D is *reduced* if $m_i = 1$ for all i. The *support* of the divisor D is the union of the curves C_i such that $m_i \neq 0$. The *trivial* divisor is the divisor $D = 0$; it has empty support.

A reduced non-trivial divisor is nothing but a projective plane curve: we can identify every projective plane curve C with the divisor $C_1 + \ldots + C_r$ where $C_1, \ldots, C_r$ are the irreducible components of C. Thus we can think of divisors as generalizations of curves, obtained by assigning integer multiplicities to the irreducible components. The key role of divisors is that they provide a geometric interpretation for homogeneous polynomials with repeated factors. To every non-zero homogeneous polynomial P in $\mathbb{C}[x_0, x_1, x_2]$ we can attach an effective divisor $\operatorname{div}(P)$ in the following way. If P is constant then we set $\operatorname{div}(P) = 0$. Otherwise, we pick a factorization $P = P_1^{m_1} \cdots P_r^{m_r}$ into irreducible polynomials where $m_1, \ldots, m_r$ are positive integers and such that P_i does not divide P_j when $i \neq j$ in $\{1, \ldots, r\}$. The zero set of each factor P_i in $\mathbb{P}^2_{\mathbb{C}}$ is an irreducible projective plane curve C_i, and we set

$$\operatorname{div}(P) = m_1 C_1 + \ldots + m_r C_r.$$

This definition does not depend on the choice of a factorization of P.

Exercise 14.2. Show that the map $P \mapsto \operatorname{div}(P)$ defines a bijective correspondence between the set of non-zero homogeneous polynomials P in $\mathbb{C}[x_0, x_1, x_2]$ up to a factor in $\mathbb{C}^*$, and the set of effective divisors in $\mathbb{P}^2_{\mathbb{C}}$. ☜

The *degree* of a divisor $D = m_1 C_1 + \ldots + m_r C_r$ is defined by

$$\deg D = \sum_{i=1}^{r} m_i \deg C_i.$$

When D is non-trivial and reduced, this definition agrees with the definition for projective plane curves, because the degree of a projective plane curve is the sum of the degrees of its irreducible components. If P is a non-zero homogeneous polynomial, then the degree of $\operatorname{div}(P)$ is equal to the degree of P.

[5] There are also many interesting examples of families parameterized by more complicated algebraic sets; the general study of such parameter spaces is the theory of *moduli spaces* in algebraic geometry.

Definition 14.3. For every integer $d \geq 0$, the *complete linear system of degree* d in $\mathbb{P}^2_{\mathbb{C}}$ is the set of all effective divisors of degree d on $\mathbb{P}^2_{\mathbb{C}}$. This set is denoted by $\mathcal{L}_d$.

By Exercise 14.2, we can identify $\mathcal{L}_d$ with the set of non-zero homogeneous degree d polynomials in $\mathbb{C}[x_0, x_1, x_2]$ up to a non-zero factor in $\mathbb{C}^*$. The set of all homogeneous degree d polynomials in $\mathbb{C}[x_0, x_1, x_2]$ is a complex vector space with a basis given by the degree d monomials in the variables x_0, x_1 and x_2. Choosing an ordering on these monomials (for instance, the lexicographical ordering on the exponents), we obtain an isomorphism of this vector space with $\mathbb{C}^{N_d+1}$ where $N_d + 1$ is the number of elements $\alpha \in \mathbb{N}^3$ with $|\alpha| = d$, that is,

$$N_d = \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}.$$

Thus, we can identify $\mathcal{L}_d$ with the set of elements in $\mathbb{C}^{N_d+1} \setminus \{\underline{0}\}$ up to a factor in $\mathbb{C}^*$ – this is precisely the projective space $\mathbb{P}^{N_d}_{\mathbb{C}}$. This means that we can think of effective divisors of degree d as points in $\mathbb{P}^{N_d}_{\mathbb{C}}$.

Definition 14.4. A *linear system of degree* d in $\mathbb{P}^2_{\mathbb{C}}$ is the set of effective divisors of degree d corresponding to a linear subspace of $\mathcal{L}_d \cong \mathbb{P}^{N_d}_{\mathbb{C}}$.

By a *linear subspace*, we mean the zero set of a finite system of homogeneous linear equations in the homogeneous coordinates on $\mathbb{P}^{N_d}_{\mathbb{C}}$. Thus, linear systems of degree d in $\mathbb{P}^2_{\mathbb{C}}$ are obtained by imposing homogeneous linear relations on the coefficients of homogeneous polynomials of degree d.

Equivalently, a linear subspace of $\mathbb{P}^{N_d}_{\mathbb{C}}$ is a subset of the form $(V \setminus \{\underline{0}\})/\sim$ where V is a subspace of $\mathbb{C}^{N_d+1}$ and $\sim$ is the equivalence relation defining projective space. The dimension of the linear subspace is defined to be $\dim(V) - 1$. For instance, the linear subspaces of $\mathbb{P}^2_{\mathbb{C}}$ are the empty set (dimension -1), the points (dimension 0), the projective lines (dimension 1), and $\mathbb{P}^2_{\mathbb{C}}$ itself (dimension 2).

Example 14.5. A basic but important example of linear systems appears in the theory of *projective duality.* A non-trivial effective divisor of degree 1 in $\mathbb{P}^2_{\mathbb{C}}$ is the same thing as a projective line. The set of all lines in $\mathbb{P}^2_{\mathbb{C}}$ is the complete linear system $\mathcal{L}_1$ of degree 1. Giving a projective line in $\mathbb{P}^2_{\mathbb{C}}$ amounts to giving an equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$ up to rescaling by a factor in $\mathbb{C}^*$, where $(a_0, a_1, a_2) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$. Thus the lines in $\mathbb{P}^2_{\mathbb{C}}$ are parameterized by $[a_0, a_1, a_2] \in \mathbb{P}^{N_1}_{\mathbb{C}}$, with $N_1 = 2$. This parameter space is called the *dual projective plane*, to distinguish it from the original copy of $\mathbb{P}^2_{\mathbb{C}}$.

We can define linear subspaces of $\mathcal{L}_1 \cong \mathbb{P}^2_{\mathbb{C}}$ by imposing incidence conditions. Consider a point $p = [p_0, p_1, p_2]$ in $\mathbb{P}^2_{\mathbb{C}}$. Then the set of lines passing through p corresponds to the set of points $[a_0, a_1, a_2]$ in the dual $\mathbb{P}^2_{\mathbb{C}}$ such that $a_0p_0 + a_1p_1 + a_2p_2 = 0$, and this is precisely the line in the dual $\mathbb{P}^2_{\mathbb{C}}$ defined by $p_0x_0 + p_1x_1 + p_2x_2 = 0$. Thus, points in $\mathbb{P}^2_{\mathbb{C}}$ correspond to lines in the dual projective plane.

Projective duality allows one to get free theorems by dualizing existing theorems. For instance, the dual of the statement that two distinct lines intersect at a unique point is the statement that two distinct points p and q lie on a unique line (namely, the line corresponding to the intersection point of the two lines dual to p and q). A more subtle example is *Brianchon's theorem*, which is obtained by dualizing Pascal's Mystic Hexagon theorem (Proposition 10.14). We will not discuss projective duality any further in this course, but interested students may consult `https://en.wikipedia.org/wiki/Duality_(projective_geometry)` (non-examinable).

Example 14.6. Every conic C in $\mathbb{P}^2_{\mathbb{C}}$ supports a unique effective divisor of degree 2: if C is a line then this is the divisor $2C$, and otherwise it is the sum of the irreducible components of C. Thus we can think of $\mathcal{L}_2$ as the space of conics in $\mathbb{P}^2_{\mathbb{C}}$. If our conic is defined by

$$a_0x_2^2 + a_1x_1x_2 + a_2x_1^2 + a_3x_0x_2 + a_4x_0x_1 + a_5x_0^2 = 0$$

then it corresponds to the point $[a_0, \ldots, a_5]$ in $\mathcal{L}_2 \cong \mathbb{P}^{N_2}_{\mathbb{C}}$, with $N_2 = 5$ (here we used the lexicographical order on the exponents).

Example 14.7. An effective divisor of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ is *almost* the same thing as a cubic, but not quite. Let P_1 and P_2 be non-zero homogeneous linear polynomials in $\mathbb{C}[x_0, x_1, x_2]$, defining lines L_1 and L_2. If we set $P = P_1^2 \cdot P_2$, then $\operatorname{div}(P) = 2L_1 + L_2$. But the cubic defined by $P = 0$ is simply the union of L_1 and L_2, and we can no longer tell whether P_1 or P_2 appeared with exponent 2 in the factorization of P. In other words, different effective divisors of degree 3 can be supported on the same cubic. In such cases, it is often more natural to work with divisors instead of curves, since we retain more information about the defining polynomial P.

Exercise 14.8. A *pencil* of projective plane curves of degree d is a linear system of degree d and dimension 1. Let P and Q be non-zero homogeneous polynomials such that $\operatorname{div}(P)$ and $\operatorname{div}(Q)$ are distinct members of the pencil. Show that the pencil consists precisely of the divisors $\operatorname{div}(\lambda P + \mu Q)$ with $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Deduce that every member of the pencil contains in its support the intersection of the supports of $\operatorname{div}(P)$ and $\operatorname{div}(Q)$.

Since the divisors $\operatorname{div}(\lambda P + \mu Q)$ only depend on (λ, μ) up to scaling by a factor in $\mathbb{C}^*$, we can use λ and μ as homogeneous coordinates to identify the pencil with $\mathbb{P}^1_{\mathbb{C}}$.

For every integer $d \geq 0$, each projective transformation Φ of $\mathbb{P}^2_{\mathbb{C}}$ acts on $\mathcal{L}_d$ by sending an effective degree d divisor $m_1C_1 + \ldots + m_rC_r$ to $m_1\Phi(C_1) + \ldots + m_r\Phi(C_r)$. The map $\mathcal{L}_d \to \mathcal{L}_d$ induced by Φ is a projective transformation of $\mathcal{L}_d \cong \mathbb{P}^{N_d}_{\mathbb{C}}$, because Φ acts linearly on the vector space of homogeneous polynomials of degree d. In particular, Φ preserves linear subspaces of $\mathbb{P}^{N_d}_{\mathbb{C}}$, and their dimensions.

This defines an action of the group of projective transformations on the space $\mathcal{L}_d$. The orbits of this action are the projective equivalence classes of divisors. For instance, the action on $\mathcal{L}_1 \cong \mathbb{P}^2_{\mathbb{C}}$ has a unique orbit because all lines are projectively equivalent; the action on $\mathcal{L}_2 \cong \mathbb{P}^5_{\mathbb{C}}$ has exactly three orbits corresponding to the three types of conics (line, two lines, non-degenerate) by Exercise 9.4; and the action on $\mathcal{L}_3 \cong \mathbb{P}^9_{\mathbb{C}}$ has infinitely many orbits, because there exist infinitely many projective equivalence classes of smooth non-degenerate cubics.

Now let us return to the motivating question from the beginning of this section: how can we count the number of curves of fixed degree with prescribed multiplicities at finitely many given points? It will be more convenient to count divisors, rather than curves. If $D = m_1C_1 + \ldots + m_rC_r$ is an effective divisor in $\mathbb{P}^2_{\mathbb{C}}$ and p is a point of $\mathbb{P}^2_{\mathbb{C}}$, then we define

$$\operatorname{mult}_p D = \sum_{i=1}^{r} m_i \operatorname{mult}_p C_i.$$

Note that $\operatorname{mult}_p D \leq m_1 \deg(C_1) + \ldots + m_r \deg(C_r) = \deg(D)$.

Exercise 14.9. Let P be a non-zero homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$. Show that, for every point p in $\mathbb{P}^2_{\mathbb{C}}$, we have $\operatorname{mult}_p \operatorname{div}(P) = \operatorname{mult}_p P$.

Proposition 14.10. *Let p be a point in $\mathbb{P}^2_{\mathbb{C}}$, and let d and μ be non-negative integers such that $\mu \leq d$. Then the set S of effective degree d divisors D such that $\operatorname{mult}_p D \geq \mu$ is a linear system of dimension*

$$N_{d,\mu} = \frac{d(d+3)}{2} - \frac{\mu(\mu+1)}{2}.$$

Proof. After projective transformation, we may assume that $p = [0, 0, 1]$. Then by Exercise 11.7, the elements of S correspond to the non-zero homogeneous polynomials P in $\mathbb{C}[x_0, x_1, x_2]$ of the form

$$P = \sum_{\substack{\alpha=(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{N}^3 \\ |\alpha|=d}} a_\alpha x^\alpha$$

such that $a_\alpha = 0$ when $\alpha_0 + \alpha_1 \leq \mu - 1$. Adding the zero polynomial, we obtain a complex vector space of dimension

$$|A| = N_d + 1 - \sum_{\alpha_0=0}^{\mu-1} (\mu - \alpha_0) = \frac{d(d+3)}{2} + 1 - \frac{\mu(\mu+1)}{2}.$$

The quotient of this vector space by the equivalence relation $\sim$ used to define projective space is a linear subspace of $\mathcal{L}_d = \mathbb{P}^{N_d}_{\mathbb{C}}$ of dimension $N_{d,\mu}$. □

Corollary 14.11. *Let $p_1, \dots, p_r$ be distinct points in $\mathbb{P}^2_{\mathbb{C}}$, and let $d, \mu_1, \dots, \mu_r$ be non-negative integers such that $\mu_i \leq d$ for every i. Then the set S of effective divisors D of degree d in $\mathbb{P}^2_{\mathbb{C}}$ such that $\operatorname{mult}_{p_i} D \geq \mu_i$ for every i is a linear system of dimension at least*

$$\frac{d(d+3)}{2} - \sum_{i=1}^{r} \frac{\mu_i(\mu_i+1)}{2}.$$

In particular, if this lower bound is non-negative, then S is non-empty.

Proof. The set S is the intersection of r linear systems as in Proposition 14.10. The intersection of r linear subspaces of $\mathbb{P}^{N_d}_{\mathbb{C}}$ of codimensions $c_1, \dots, c_r$ is a linear subspace of codimension at most $c_1 + \ldots + c_r$, because the corresponding statement holds for subspaces of $\mathbb{C}^{N_d+1}$ by linear algebra. □

The lower bound in Corollary 14.11 may be strict, because the conditions imposed on the coefficients are not necessarily independent. For instance, if $d = 1$ and $r = 3$, and p_1, p_2, p_3 are distinct collinear points with prescribed multiplicities $\mu_1 = \mu_2 = \mu_3 = 1$, then the set S in the corollary consists of a unique element, namely, the unique line through p_1, p_2, p_3. Thus the dimension of the linear system is 0, whereas the bound in the corollary is negative.

To conclude this section, we will discuss a few further examples and applications.

Example 14.12. Let p be a point in $\mathbb{P}^2_{\mathbb{C}}$. Let S be the set of effective degree 2 divisors D in $\mathbb{P}^2_{\mathbb{C}}$ such that $\operatorname{mult}_p D \geq 2$. Then S is a linear system of dimension 2, by Proposition 14.10. Non-degenerate conics are smooth and therefore have multiplicity 1 at each of their points. Therefore, the elements of S are the double lines through p and the unions of two lines intersecting at p. In other words, giving an element of S is equivalent to choosing a pair of lines through p. This allows us to understand in a geometric way why the dimension of S is 2: we have one degree of freedom in choosing a line through p, because the lines through p form a pencil of degree 1 (a line in the dual projective plane $\mathcal{L}_1$).

Exercise 14.13 ($\star$). Let $p_1, \dots, p_5$ be five distinct points in $\mathbb{P}^2_{\mathbb{C}}$ such that no four of them are collinear.

1. Prove that the set of conics in $\mathbb{P}^2_{\mathbb{C}}$ containing $p_1, \dots, p_4$ is a pencil. How many of these conics are degenerate, resp. reducible, if three of the points $p_1, \dots, p_4$ are collinear? And if no three of these points are collinear?

2. Prove that there exists a unique conic through $p_1, \dots, p_5$, and that this conic is non-degenerate if and only if no three of the points are collinear.

Proposition 14.14. *Let $p_1, \dots, p_8$ be eight distinct points in $\mathbb{P}^2_{\mathbb{C}}$ and suppose that no four of the points lie on a line and no seven on a conic. Then the set S of effective degree 3 divisors in $\mathbb{P}^2_{\mathbb{C}}$ containing the points $p_1, \dots, p_8$ is a pencil.*

Proof. Since the complete linear system $\mathcal{L}_3$ has dimension 9, the expected dimension of the linear system S is $9 - 8 = 1$, but we need to show that the point conditions are all independent. It suffices to show that, for every non-negative integer $r \leq 7$, there exists an effective divisor D of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ that contains $p_1, \dots, p_r$ but not p_{r+1}. Adding the points $p_{r+2}, \dots, p_8$ to $p_1, \dots, p_r$ only makes the statement stronger, so we may assume right away that $r = 7$.

First, suppose that three of the points $p_1, \dots, p_7$ are collinear; without loss of generality, we may then assume that p_1, p_2, p_3 are collinear. Let L_1 be the unique line through these points. This line does not contain p_8, because no four points in the given set are collinear. Now we can find two other lines L_2 and L_3 that do not contain p_8 and such that $L_2 \cup L_3$ contain $p_4, \dots, p_7$ (if two of these points are collinear with p_8, we just need to make sure not to put both of them on the same line). Then $D = L_1 + L_2 + L_3$ is an effective divisor of degree 3 that contains $p_1, \dots, p_7$ but not p_8.

It remains to settle the case where no three of the points $p_1, \dots, p_7$ are collinear. We relabel the points such that p_1 and p_7 are not collinear with p_8, and neither are p_6 and p_7. By Exercise 14.13 there is a unique conic C through $p_1, \dots, p_5$. This conic is non-degenerate since otherwise, three of the points would be collinear. If C does not contain p_8 then we can take $D = C + L$ where L is the line through p_6 and p_7. If C contains p_8 then it does not contain p_6, because no seven of the given points lie on a conic. The unique conic C' through $p_2, \dots, p_6$ intersects C in at most 4 points by Bézout's theorem, so that $C \cap C' = \{p_2, \dots, p_5\}$. In particular, C' does not contain p_8. Swapping p_1 and p_6 brings us back to the previous case. □

Corollary 14.15 (Cayley–Bacharach theorem)**.** *Let C_1 and C_2 be two cubics in $\mathbb{P}^2_{\mathbb{C}}$ intersecting in exactly nine distinct points $p_1, \ldots, p_9$. Then every cubic in $\mathbb{P}^2_{\mathbb{C}}$ that contains $p_1, \ldots, p_8$ also passes through p_9.*

Proof. No four points among $p_1, \ldots, p_8$ are collinear: any line L through four of these points would intersect C_1 and C_2 in at least 4 points and thus be contained in C_1 and C_2 by Bézout's theorem, contradicting the assumption that $C_1 \cap C_2$ contains only nine points. Similarly, no seven points among $p_1, \ldots, p_8$ lie on a conic: such a conic C_0 would be non-degenerate by the assumption that no four points are collinear, and Bézout's theorem would then again force C_1 and C_2 to contain C_0.

It follows that $p_1, \ldots, p_8$ satisfy the conditions of Proposition 14.14, so that the set of effective divisors of degree 3 through $p_1, \ldots, p_8$ is a pencil containing C_1 and C_2. By Exercise 14.8, every member of this pencil contains $C_1 \cap C_2$; in particular, it also contains p_9. $\square$

15 Riemann surfaces

Smooth projective plane curves can also be studied through the lens of complex analysis: they are examples of *Riemann surfaces*, which are the smooth curves that can be defined in the setting of complex analytic geometry (the name *surface* refers to the fact that these objects have two real dimensions). This opens a new perspective on projective plane curves that is particularly useful if we want to understand their topology. We will explore this perspective in the remainder of the course. One of the main results will be that smooth projective plane curves are topologically classified by a single invariant, called the *genus*. It is a remarkable fact that the genus can be computed topologically, through complex geometry and through algebraic geometry, giving rise to a beautiful interplay between these mathematical subfields.

A *topological surface* is a Hausdorff topological space that can be covered by countably many opens homeomorphic to opens in $\mathbb{R}^2$ or, equivalently, in $\mathbb{C}$. A *complex atlas* on a topological surface X consists of an open cover $\{U_i \mid i \in I\}$ of X and, for each $i \in I$, a homeomorphism $\psi_i \colon U_i \to V_i$ where V_i is an open subset of $\mathbb{C}$. These homeomorphisms are required to be compatible in the sense that, for all i and j in I, the map

$$\psi_j \circ \psi_i^{-1} \colon \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$$

is biholomorphic. The pairs (U_i, ψ_i) are called the *charts* of the atlas, and the map $\psi_j \circ \psi_i^{-1}$ is called the *transition function* between the charts (U_i, ψ_i) and (U_j, ψ_j). Two atlases $\{(U_i, \psi_i) \mid i \in I\}$ and $\{(U'_j, \psi'_j) \mid j \in J\}$ are called *equivalent* if their union is still an atlas; this means that the charts ψ_i are all compatible with the charts ψ'_j. Each equivalence class of atlases contains a unique maximal element, namely, the union of all the atlases in the equivalence class. An equivalence class of atlases on X is called a *complex structure* on X.

Definition 15.1. A *Riemann surface* is a topological surface X equipped with a complex structure, that is, an equivalence class of atlases[6].

By an atlas for a Riemann surface X, we then mean any atlas in this equivalence class. A *chart* for a Riemann surface X is a pair (U, ψ) that appears as a chart in some atlas for X.

Exercise 15.2. Show that every collection of charts for X that covers X is an atlas for X. ☜

It follows directly from the definition that every open subset U of X has a natural structure of a Riemann surface, obtained by restricting the charts on X to U. We will always assume that open subsets of Riemann surfaces are endowed with this induced complex structure. The key purpose of the complex structure on a Riemann surface X is that it allows us to speak about holomorphic functions $f \colon X \to \mathbb{C}$.

Definition 15.3. Let X be a Riemann surface. A function $f \colon X \to \mathbb{C}$ is called *holomorphic* if, for every chart (U, ψ) for X, the function $f \circ \psi^{-1} \colon \psi(U) \to \mathbb{C}$ is holomorphic.

Note that $\psi(U)$ is an open subset of $\mathbb{C}$, so we know what it means for a function on this domain to be holomorphic. The compatibility of the charts for X implies that every chart is itself holomorphic. The following exercise states that holomorphy of a function can be tested on the charts of any atlas.

Exercise 15.4. Let $\{(U_i, \psi_i) \mid i \in I\}$ be an atlas for X. Show that a function $f \colon X \to \mathbb{C}$ is holomorphic if and only if $f \circ \psi_i^{-1} \colon \psi_i(U_i) \to \mathbb{C}$ is holomorphic for every i in I. ☜

We can think of a chart $\psi \colon U \to \mathbb{C}$ for X as a holomorphic coordinate function on U: the holomorphic functions on U are those that can be expressed as holomorphic functions in the coordinate ψ. In more informal terms, if $\{(U_i, \psi_i) \mid i \in I\}$ is an atlas for X then we can think of X as being glued together from the open subsets $V_i = \psi_i(U_i)$ of $\mathbb{C}$; then a function on X is holomorphic if and only if its restriction to each V_i is holomorphic.

Exercise 15.5. Prove that being holomorphic is a local property: for every open cover of X, a function $f \colon X \to \mathbb{C}$ is holomorphic if and only if its restriction to each member of the open cover is holomorphic.

[6] It is often included in the definition that X is connected, but we will not make this assumption.

Local properties of holomorphic functions on opens in $\mathbb{C}$ generalize immediately to Riemann surfaces, since one can reduce to the case of opens in $\mathbb{C}$ by looking at charts. For instance, a holomorphic function $f\colon X \to \mathbb{C}$ is always continuous: this can be checked after restricting f to U for every chart (U, ψ) in an atlas for X, and there it follows from the fact that $f \circ \psi^{-1}$ is continuous and ψ is a homeomorphism. One can similarly show that the zeros of a holomorphic function are always isolated, because of the identity theorem. The same type of argument also shows that, for every Riemann surface X, the following functions $X \to \mathbb{C}$ are holomorphic:

- sums and products of holomorphic functions on X;
- the inverse of a non-vanishing holomorphic function on X;
- the composite of a holomorphic function $f\colon X \to \mathbb{C}$ with a holomorphic function $g\colon V \to \mathbb{C}$ for some open V in $\mathbb{C}$ containing $f(X)$.

Example 15.6. Every open U in $\mathbb{C}$ is a Riemann surface with respect to the atlas $\{(U, \mathrm{Id}_{\mathrm{U}})\}$, where Id_U is the identity function on U. The holomorphic functions on U are the ones you know from complex analysis.

Example 15.7 (The Riemann sphere)**.** We can turn $\mathbb{P}^1_{\mathbb{C}}$ into a Riemann surface by means of the affine charts defined in Section 4: the set $\{(U_0, \psi_0), (U_1, \psi_1)\}$ is an atlas, because the function

$$\psi_1 \circ \psi_0^{-1}\colon \psi_0(U_0 \cap U_1) \to \psi_1(U_0 \cap U_1)$$

is biholomorphic (it is equal to $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto 1/z$). This Riemann surface is called the *Riemann sphere*, because it is homeomorphic to S^2, as we have seen in Example 4.7[7].

Example 15.8 (Complex tori)**.** A *lattice* in $\mathbb{C}$ is a subgroup Λ of $(\mathbb{C}, +)$ of the form $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where ω_1 and ω_2 are elements of $\mathbb{C}$ that are linearly independent over $\mathbb{R}$ (that is, non-zero elements such that ω_1/ω_2 does not lie in $\mathbb{R}$). The quotient group $\mathbb{C}/\Lambda$ of residue classes modulo Λ carries a natural topology: if we denote by $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ the projection map, then a subset U of $\mathbb{C}/\Lambda$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. We can visualize the topological space $\mathbb{C}/\Lambda$ by starting from the parallellogram in $\mathbb{C}$ with vertices 0, ω_1, ω_2 and $\omega_1 + \omega_2$; then $\mathbb{C}/\Lambda$ is obtained by identifying opposite sides (explain!). The shape we obtain is a two-dimensional *real torus* (a donut).

We can construct an atlas on $\mathbb{C}/\Lambda$ by considering all the connected opens V in $\mathbb{C}$ that contain at most one element in each residue class modulo Λ. Then $U = \pi(V)$ is open in $\mathbb{C}/\Lambda$ and π restricts to a homeomorphism $\pi|_V\colon V \to U$, because $\pi^{-1}(U)$ is the disjoint union of the translates of V by elements in Λ. The opens U equipped with the maps $(\pi|_V)^{-1}\colon U \to V$ form an atlas on $\mathbb{C}/\Lambda$, because the transition functions between these charts are translations by elements in Λ. The resulting Riemann surface is called the *complex torus* associated with the lattice Λ. For every open subset U of $\mathbb{C}/\Lambda$, a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi\colon \pi^{-1}(U) \to \mathbb{C}$ is holomorphic.

Even though the topological space $\mathbb{C}/\Lambda$ does not depend on Λ (all of these spaces are homeomorphic to the torus), the complex structure very much depends on Λ. The reason is that a real linear transformation of $\mathbb{C}$ that transforms one lattice into another is usually *not* holomorphic: it is only holomorphic when it is given by multiplication with a complex number.

Exercise 15.9. Let X be a Riemann surface. Let U be an open subset of X, let V be an open subset of $\mathbb{C}$, and let $\psi\colon U \to V$ be a bijective map that is holomorphic with respect to the complex structure on X. Show that (U, ψ) is a chart for X.

Exercise 15.10 ($\star$)**.** Give two non-equivalent atlases on $\mathbb{C}$.

We will now show that the set of smooth points on a projective plane curve over $\mathbb{C}$ can be made into a Riemann surface in a natural way (the same then holds for affine plane curves, because these are open subsets of their projectivizations).

[7]More generally, the charts (U_i, ψ_i) on $\mathbb{P}^n_{\mathbb{C}}$ form an atlas that defines an n-dimensional complex structure on $\mathbb{P}^n_{\mathbb{C}}$, turning $\mathbb{P}^n_{\mathbb{C}}$ into a *complex manifold*. These objects will be studied in detail in the Manifolds course.

Theorem 15.11. *Let C be a projective plane curve over $\mathbb{C}$, and let S be its set of singular points. Set $C^o = C \setminus S$. Then C^o is a topological surface, and there exists a unique complex structure on C^o such that, for each affine chart U_i on $\mathbb{P}^2_{\mathbb{C}}$ and every polynomial P in $\mathbb{C}[x, y]$, the function $U_i \cap C^o \to \mathbb{C}$ defined by P is holomorphic.*

Proof. The key ingredient of the proof is the complex version of the implicit function theorem. The statement we will use is the following: let Q be a non-constant polynomial in $\mathbb{C}[x, y]$, and let D be the affine plane curve defined by $Q = 0$. Let (a, b) be a point in D such that $\partial_y Q(a, b) \neq 0$. Then there exists an open neighbourhood U of (a, b) in D such that the projection

$$\psi \colon U \to \mathbb{C},\ (x, y) \mapsto x$$

is a homeomorphism onto an open subset V of $\mathbb{C}$, and such that the composition of $\psi^{-1} \colon V \to U$ with the projection $\mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto y$ is holomorphic. In other words, we can identify U with the graph of a holomorphic function $V \to \mathbb{C}$. Swapping the roles of x and y, we get an analogous statement when $\partial_x Q(a, b) \neq 0$.

We first prove the uniqueness part of the theorem. Suppose that C^o has been equipped with the structure of a Riemann surface satisfying the properties in the statement. By the implicit function theorem, we can cover C^o with opens U contained in an affine chart U_i such that one of the affine coordinate functions on U_i defines a homeomorphism $\psi \colon U \to V$ with V open in $\mathbb{C}$. Since this affine coordinate function is polynomial on U_i, it is holomorphic. By Exercise 15.9, the collection of all such pairs (U, ψ) is an atlas for C^o, which completely determines its complex structure.

Next, we prove the existence of a complex structure as in the statement. The space C^o is Hausdorff because it is a subspace of the Hausdorff space $\mathbb{P}^2_{\mathbb{C}}$. It is not hard to show that every open cover of an open in $\mathbb{P}^2_{\mathbb{C}}$ has a countable subcover: since $\mathbb{P}^2_{\mathbb{C}}$ is covered by three open affine charts homeomorphic to $\mathbb{C}^2$, it suffices to prove this property for $\mathbb{C}^2$; every open in $\mathbb{C}^2$ is a union of open balls with rational radius and such that the coordinates of the centers have rational real and imaginary parts. It follows that every open cover of the subspace C^o of $\mathbb{P}^2_{\mathbb{C}}$ also has a countable subcover.

So, it suffices to construct an atlas on C^o with the required properties. Consider the collection $\mathcal{A}$ of all the pairs (U, ψ) such that:

- the set U is an open in $C^o \cap U_i$ for some affine chart U_i on $\mathbb{P}^2_{\mathbb{C}}$;
- the map ψ is the restriction to U of one of the affine coordinate functions on U_i;
- the map ψ is a homeomorphism from U onto an open subset V of $\mathbb{C}$;
- the composition of $\psi^{-1} \colon V \to U$ with the other affine coordinate function on U_i is holomorphic.

By the implicit function theorem, these opens U cover C^o. To show that the pairs (U, ψ) in $\mathcal{A}$ form an atlas, we need to check that any two of them are compatible. Let (U, ψ) and (U', ψ') be two pairs in $\mathcal{A}$, where U lives in U_i and U' lives in U_j for some (not necessarily distinct) affine charts U_i and U_j on $\mathbb{P}^2_{\mathbb{C}}$. If we denote by x and y the affine coordinate functions on U_i, then the restrictions to $U_i \cap U_j$ of the affine coordinate functions on U_j are among the functions x, $1/x$, y, $1/y$, x/y, y/x. In any case, the function $\psi' \circ \psi^{-1}$ on $\psi(U \cap U')$ is a quotient of holomorphic functions with non-vanishing denominator, so that it is again holomorphic.

It remains to show that, for each affine chart U_i on $\mathbb{P}^2_{\mathbb{C}}$ and every polynomial P in $\mathbb{C}[x, y]$, the function $U_i \cap C^o \to \mathbb{C}$ defined by P is holomorphic. Since sums and products of holomorphic functions are again holomorphic, it is enough to prove this for $P = x$ and $P = y$. Being holomorphic is a local property, so that we can check the result after restricting P to opens U that appear in a chart (U, ψ) of the atlas $\mathcal{A}$. Then either x or y is equal to the function ψ, which is holomorphic, and the remaining coordinate function is also holomorphic because the composition of ψ^{-1} with this coordinate function is holomorphic by the definition of our atlas. □

Let X and Y be Riemann surfaces. A *morphism* $h \colon Y \to X$ of Riemann surfaces is a continuous map such that, for every open U in X and every holomorphic function $f \colon U \to \mathbb{C}$, the composite

$$f \circ h \colon h^{-1}(U) \to \mathbb{C}$$

is holomorphic with respect to the complex structure on Y. It follows immediately from the definition that a composition of morphisms of Riemann surfaces is again a morphism of Riemann surfaces. If $\{V_i \mid i \in I\}$ is an open cover of Y, then a map $h\colon Y \to X$ is a morphism of Riemann surfaces if and only if the restriction of h to each open V_i is a morphism of Riemann surfaces, because continuity and holomorphy are local properties.

An isomorphism of Riemann surfaces is a bijective morphism whose inverse is still a morphism of Riemann surfaces. We define morphisms and isomorphisms between smooth projective plane curves over $\mathbb{C}$ by viewing them as Riemann surfaces by means of Theorem 15.11.

Exercise 15.12. Let X and Y be Riemann surfaces, and let $\mathcal{A}$ be an atlas for X. Show that a map $h\colon Y \to X$ is a morphism of Riemann surfaces if and only if, for every chart (U, ψ) in $\mathcal{A}$, the set $h^{-1}(U)$ is open in Y and the function $\psi \circ h\colon h^{-1}(U) \to \mathbb{C}$ is holomorphic with respect to the complex structure on Y. ☜

Exercise 15.13. Let $h\colon Y \to X$ be a morphism of Riemann surfaces. Assume that Y is connected and h is not constant. Show that the image of h is open in X. Deduce that h is surjective if we moreover assume that Y is non-empty and compact, and X is connected. This implies that a morphism from a connected compact Riemann surface to a connected non-compact Riemann surface is always constant. In particular, every holomorphic function on a connected compact Riemann surface is constant. ☜

Exercise 15.14. Show that every bijective morphism of Riemann surfaces is an isomorphism.

Example 15.15. If X is a Riemann surface then a morphism from X to the Riemann surface $\mathbb{C}$ with atlas $\{(\mathbb{C}, \mathrm{Id}_{\mathbb{C}})\}$ is the same thing as a holomorphic function on X.

Example 15.16. If U is an open subset of a Riemann surface X, then the inclusion map $U \to X$ is a morphism of Riemann surfaces.

Example 15.17. Every projective transformation

$$\Phi\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}},\ [x_0, x_1] \mapsto [ax_0 + bx_1, cx_0 + dx_1]$$

is an isomorphism of Riemann surfaces. Since Φ is bijective and its inverse is again a projective transformation, we only need to prove that Φ is a morphism of Riemann surfaces. Using Exercise 15.12, we can check this on the atlas $\{(U_0, \psi_0), (U_1, \psi_1)\}$ from Example 15.7. By symmetry, it suffices to show that the set $V = \Phi^{-1}(U_0)$ is open in $\mathbb{P}^1_{\mathbb{C}}$, and that $\psi_0 \circ \Phi$ is holomorphic on V.

Openness of V is obvious, because its complement consists of a single point. Checking holomorphy amounts to showing that $\psi_0 \circ \Phi \circ \psi_0^{-1}$ is holomorphic on $\psi_0(V \cap U_0)$ and $\psi_0 \circ \Phi \circ \psi_1^{-1}$ is holomorphic on $\psi_1(V \cap U_1)$. These functions are given explicitly by

$$\mathbb{C} \setminus S,\ x \mapsto \frac{c + dx}{a + bx}, \qquad \mathbb{C} \setminus T,\ x \mapsto \frac{cx + d}{ax + b}$$

where $S = \{-a/b\}$ if $b \neq 0$ and $S = \emptyset$ otherwise, and $T = \{-b/a\}$ if $a \neq 0$ and $T = \emptyset$ otherwise. These are indeed holomorphic functions.

Example 15.18. One can show that, up to isomorphism, the complex tori from Example 15.8 are precisely the smooth non-degenerate cubics in $\mathbb{P}^2_{\mathbb{C}}$. For every $\lambda \in \mathbb{C} \setminus \{0, 1\}$, the curve defined by

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ that can be determined explicitly by means of the theory of elliptic integrals (not examinable). Moreover, it can be proved that two smooth non-degenerate cubics in $\mathbb{P}^2_{\mathbb{C}}$ are isomorphic if and only if they are projectively equivalent.

Exercise 15.19.

1. Let C be a smooth projective plane curve and let $\Phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a projective transformation. Show that the map $\Phi\colon C \to \Phi(C)$ is an isomorphism of Riemann surfaces.

2. Show that every line in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.

3. Beware that isomorphic projective plane curves are not always projectively equivalent, because the isomorphism does not keep track of the embedding into $\mathbb{P}^2_{\mathbb{C}}$. Show that every non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and therefore to a line in $\mathbb{P}^2_{\mathbb{C}}$ (*hint: stereographic projection*). These curves are not projectively equivalent, because they have different degrees.

16 Morphisms of Riemann surfaces and ramification

Our principal aim in the remainder of the course is to describe the topology of smooth projective plane curves over $\mathbb{C}$. Recall that projective plane curves are compact by Proposition 5.16; one can also show that they are always connected, but this is surprisingly hard to show and we will simply accept this fact. Therefore, we will more generally analyse the topology of a connected compact Riemann surface X. Our main strategy will be to consider a morphism h from X to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, and to study its *ramification* to deduce information about the topology of X. This notion will be rigorously introduced in the present chapter: we will prove that, locally on the source and target, every non-constant morphism of Riemann surfaces looks like the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^m$ around 0, and the ramification is then measured by the exponent m. The ramification of a morphism $h\colon X \to \mathbb{P}^1_{\mathbb{C}}$ will tell us how the fibers $h^{-1}(y)$ vary as y runs through $\mathbb{P}^1_{\mathbb{C}}$, giving valuable clues on the shape of X.

Exercise 15.13 tells us that holomorphic functions defined on a connected compact Riemann surface X are always constant, and therefore do not carry any interesting information. For this reason, one instead studies *meromorphic* functions to probe the geometry of X. Let V be an open subset of X and let x_0 be a point of V. Let $f\colon V \setminus \{x_0\} \to \mathbb{C}$ be a holomorphic function and let (U, ψ) be a chart on V such that U contains x_0. The order of f at x_0 is then defined to be the order of $f \circ \psi^{-1}$ at $\psi(x_0)$ (see Appendix B), and denoted by $\operatorname{ord}_{x_0} f$.

Exercise 16.1. Prove that $\operatorname{ord}_{x_0} f$ does not depend on the choice of the chart (U, ψ).

We say that f has a removable singularity at x_0 if $\operatorname{ord}_{x_0} f \geq 0$; then f extends to a holomorphic function on V. We say that f has a pole of order m at x_0 if $m = -\operatorname{ord}_{x_0} f$ is a positive integer. Finally, we say that f has an essential singularity at x_0 if $\operatorname{ord}_{x_0} f = -\infty$. The function f is called *meromorphic* at x_0 if it has a removable singularity or a pole at x_0.

A meromorphic function on X is a holomorphic function $f\colon X \setminus S \to \mathbb{C}$ for some finite[8] subset S of X such that f is meromorphic at every point of S. We identify two meromorphic functions on X if they agree on $X \setminus T$ for some finite subset T of X. In other words, we treat two meromorphic functions as the same object if they become equal after removing finitely many points from their domains. Every meromorphic function on X has a maximal domain of definition, whose complement is the finite set of poles of the function.

Exercise 16.2. Prove the following variant of the identity theorem for Riemann surfaces: if X is a connected Riemann surface, S is a finite subset of X and f is a non-constant holomorphic function on $X \setminus S$ with a pole at every point of S, then $f^{-1}(0)$ is a discrete closed subset of X. Here *discrete* means that every point of $f^{-1}(0)$ has an open neighbourhood in X that does not contain any other points of $f^{-1}(0)$.

The meromorphic functions on a connected compact Riemann surface X form a field with respect to pointwise addition and multiplication: meromorphic functions that are not identically zero are invertible, because they have only finitely many zeros by the identity theorem and compactness of X.

Lemma 16.3. *Let W be an open subset of $\mathbb{C}$, let w_0 be a point of W, and let f be a non-vanishing holomorphic function on $W \setminus \{w_0\}$ with a pole at w_0. Define the function $g\colon W \to \mathbb{C}$ by $g(w) = 1/f(w)$ for $w \neq w_0$ and $f(w_0) = 0$. Then g is holomorphic.*

Proof. We write $f(w) = (w - w_0)^{-m} \widetilde{f}(w)$, where m is the order of the pole of f at w_0 and $\widetilde{f}$ is a non-vanishing holomorphic function on W. Then $g(w) = (w - w_0)^m / \widetilde{f}(w)$ so that g is holomorphic. □

Proposition 16.4. *Let X be a connected compact Riemann surface. Every meromorphic function on X extends uniquely to a morphism $X \to \mathbb{P}^1_{\mathbb{C}}$. This defines a bijection between the set of meromorphic functions on X and the set of morphisms $X \to \mathbb{P}^1_{\mathbb{C}}$ which are not constant with value ∞.*

Proof. Let S be a finite subset of X and let $f\colon X \setminus S \to \mathbb{C}$ be a holomorphic function that is meromorphic on X. We may assume that every point of S is a pole of f. We extend f to a map $\overline{f}\colon X \to \mathbb{P}^1_{\mathbb{C}}$ by sending each point of S to ∞.

[8] To compare this with the definition of a meromorphic function on an open subset of $\mathbb{C}$ in Appendix B, note that, since X is compact, the discrete closed subsets of X are precisely the finite subsets.

Let us prove that $\overline{f}$ is a morphism of Riemann surfaces. Let T be the set of zeros of f on $X \setminus S$. If f is not identically zero on X, then its zeros are isolated in X, so that T is finite because X is compact. The charts $(U_0, x_1/x_0)$ and $(U_1, x_0/x_1)$ form an atlas of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$. By Exercise 15.12, it suffices to show that $V_0 = \overline{f}^{-1}(U_0)$ and $V_1 = \overline{f}^{-1}(U_1)$ are open in X, and that $(x_1/x_0) \circ \overline{f}|_{V_0}$ and $(x_0/x_1) \circ \overline{f}|_{V_1}$ are holomorphic.

The set V_1 is equal to the open subset $X \setminus S$ of X, and the composition of x_0/x_1 with $\overline{f}|_{V_1}$ is equal to the holomorphic function f on $X \setminus S$. The set V_0 is equal to the open subset $X \setminus T$ of X, and $(x_1/x_0) \circ \overline{f}|_{V_0}$ is equal to $1/f$ on $X \setminus (S \cup T)$, and equal to 0 at every point of S. Choosing a chart on X around each point of S, we deduce from Lemma 16.3 that $(x_1/x_0) \circ \overline{f}|_{V_0}$ is holomorphic.

The extension of f to a morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ is unique, by continuity. It remains to show that, conversely, every morphism $h\colon X \to \mathbb{P}^1_{\mathbb{C}}$ that is not constant with value ∞ is the extension of a meromorphic function on X. Set $S = h^{-1}(\infty)$ and $T = h^{-1}(0)$. The function

$$f = (x_0/x_1) \circ h\colon X \setminus S \to \mathbb{C}$$

is holomorphic by the definition of a morphism of Riemann surfaces. The set T consists precisely of the zeros of this function. Similarly,

$$(x_1/x_0) \circ h\colon X \setminus T \to \mathbb{C}$$

is holomorphic, and its set of zeros is S. The identity theorem implies that the closed subset S of X is discrete, and therefore finite because X is compact. Moreover, $1/f$ has a removable singularity at every point of S because $(x_1/x_0) \circ h$ is holomorphic at these points. It follows that f is meromorphic. By construction, h coincides with the extension $\overline{f}$ of f. □

Remark 16.5. It is a deep fact that there exists a non-constant meromorphic function on every connected compact Riemann surface X. More precisely, the field of meromorphic functions on X is a finitely generated extension of $\mathbb{C}$ of transcendence degree one. From this result, one can deduce that every compact Riemann surface is isomorphic to a smooth projective curve over $\mathbb{C}$ (but not necessarily to a *plane* curve; one can define more general projective curves that live in $\mathbb{P}^3_{\mathbb{C}}$ but do not admit an embedding in $\mathbb{P}^2_{\mathbb{C}}$). This is a very special property in dimension one; in higher dimensions there are many examples of compact complex manifolds that are not algebraic. See the Manifolds module for more on this topic.

Example 16.6. A rational function on $\mathbb{C}$ is a meromorphic function of the form $P(x)/Q(x)$ where P and Q are polynomials in $\mathbb{C}[x]$ and Q is not identically zero. Identifying $\mathbb{C}$ with the chart U_1 on $\mathbb{P}^1_{\mathbb{C}}$, we can view $P(x)/Q(x)$ as a meromorphic function on $\mathbb{P}^1_{\mathbb{C}}$. Indeed, it is also meromorphic at ∞ because on the chart U_0 we can write it as a quotient of $P(x)/x^m$ and $Q(x)/x^m$, which are polynomials in $1/x$ when m is sufficiently large. Thus, every rational function defines a morphism $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

Exercise 16.7 ($\star$)**.** Show that the meromorphic functions on $\mathbb{P}^1_{\mathbb{C}}$ are precisely the rational functions. Beware that there are many meromorphic functions on $\mathbb{C}$ that are not rational; these functions are not meromorphic at ∞.

Proposition 16.8. *Let $h\colon Y \to X$ be a morphism of Riemann surfaces. Assume that Y is connected and h is not constant. Then for every $x \in X$, the set $h^{-1}(x)$ is a discrete closed subset of Y. In particular, if Y is also compact, then $h^{-1}(x)$ is finite.*

Proof. The set $h^{-1}(x)$ is closed in Y because h is continuous. Let us show that it is also discrete. Let (U, ψ) be a chart for X such that x lies in U. Let V be the set of all the points y in Y such that h is constant with value x on some open neighbourhood of y. This set is open in Y by definition. If y_0 is an accumulation point of $h^{-1}(x)$ in Y, then it follows from the identity theorem that $\psi \circ h$, and therefore h, are constant on an open neighbourhood of y_0 in Y. Thus V is also closed. It is not equal to Y because h is not constant, so that V must be empty because Y is connected. It follows that $h^{-1}(x)$ has no accumulation points in Y. □

Morphism of Riemann surfaces admit a simple description locally on the source.

Proposition 16.9. *Let $h\colon Y \to X$ be a morphism of Riemann surfaces. Assume that Y is connected and h is not constant. Let y_0 be a point of Y, and set $x_0 = h(y_0)$. Let (U, ψ) be a chart for X such that $x_0 \in U$ and $\psi(x_0) = 0$. Then there exists a chart (V, φ) for Y with the following properties:*

- *the point y_0 lies in V, and $\varphi(y_0) = 0$;*
- *the set $h(V)$ is contained in U;*
- *the image $\varphi(V)$ is an open disk with center 0 in $\mathbb{C}$, and $\psi \circ h \circ \varphi^{-1}$ is the function*

$$\varphi(V) \to \mathbb{C},\ z \mapsto z^m$$

for some positive integer m.

This integer m does not depend on the choice of the charts.

Proof. Let (V, φ) be a chart for Y such that $y_0 \in V \subset h^{-1}(U)$. Adding a constant to the function φ, we may assume that $\varphi(y_0) = 0$. Then $f = \psi \circ h \circ \varphi^{-1}$ is a holomorphic function on $\varphi(V) \subset \mathbb{C}$ that maps 0 to 0. The function f is not identically zero on any open neighbourhood of 0, by our assumption that h is not constant. Therefore, we can write it as $f(z) = z^m g(z)$ where m is a positive integer and g is a holomorphic function on $\varphi(V)$ that does not vanish at 0. Shrinking V around y_0, we may assume that $\varphi(V)$ is an open disk around 0 in $\mathbb{C}$, and g is the m-th power of a holomorphic function $\widetilde{g}$.

The function

$$\xi \colon \varphi(V) \to \mathbb{C},\ z \mapsto z\widetilde{g}(z)$$

is a holomorphic function mapping 0 to 0. Applying the Leibniz rule, we see that its derivative at 0 does not vanish, because $\widetilde{g}(0) \neq 0$. Now the inverse function theorem applies that ξ defines a biholomorhic map from an open neighbourhood of 0 in $\varphi(V)$ to some open neighbourhood of 0 in $\mathbb{C}$. Shrinking V around y_0 and composing the chart φ with ξ to create a new chart for Y, we arrive in the situation where $f(z) = z^m$: the effect of replacing φ by $\xi \circ \varphi$ is that f gets replaced by $f \circ \xi^{-1} = \xi^m \circ \xi^{-1}$.

The integer m does not depend on the chosen charts: it is the smallest positive integer such that y_0 has an open neighbourhood W in Y with the property that, for all y in $W \setminus \{y_0\}$, the number of points in $W \cap h^{-1}(h(y))$ is equal to m. Indeed, for every open disk D around 0 in $\varphi(V)$, the open neighbourhood $W = \varphi^{-1}(D)$ has this property, and these neighbourhoods form a basis of open neighbourhoods of y_0 in Y. ☐

Definition 16.10. The integer m in Proposition 16.9 is called the *ramification degree* of h at y_0, and denoted by $v_h(y_0)$. We say that h is *ramified* at y_0 if $v_h(y_0) \geq 2$; then we also call y_0 a *ramification point* or *branch point* of h. Otherwise, we say that h is *unramified* at y_0.

When Y is no longer connected but h is not constant on any connected component of Y, we can still define the ramification degree of h at a point y_0 of Y by restricting h to the connected component of Y containing y_0.

Exercise 16.11. Let $h \colon Y \to X$ be a morphism of Riemann surfaces. Assume that Y is connected and h is not constant. Let y_0 be a point of Y and set $x_0 = h(y_0)$. Let (U, ψ) be a chart for X such that $x_0 \in U$, and let (V, φ) be a chart for Y such that $y_0 \in V \subset h^{-1}(U)$. Let f be the holomorphic function $\psi \circ h \circ \varphi^{-1}$ on $\varphi(V)$.

Prove that $v_h(y_0) = \mathrm{ord}_{\varphi(y_0)}(f - \psi(x_0))$. Deduce that h is ramified at y_0 if and only if the derivative f' vanishes at $\varphi(y_0)$, and that the set of branch points of h is a discrete closed subset of Y. In particular, if Y is compact, then this set is finite.

Example 16.12. Let $P(x, y)$ be a non-constant polynomial in $\mathbb{C}[x, y]$ with no repeated factors, and assume that the affine plane curve C defined by $P(x, y) = 0$ is smooth. Then C is an open subset of the set of smooth points of the projectivization $\overline{C}$, so that we can view C as a Riemann surface by means of Theorem 15.11. The projection onto the second coordinate defines a morphism of Riemann surfaces $h \colon C \to \mathbb{C}$. We assume that P is not divisible by $y - b$ for any $b \in \mathbb{C}$, so that h is not constant on any connected component of C.

Let (a, b) be a point of C. If $\partial_x P(a, b) \neq 0$ then the restriction of h to some open neighbourhood of (a, b) in C is a chart for C. With respect to this chart, h is the identity, so that h is unramified at (a, b). If $\partial_x P(a, b) = 0$ then $\partial_y P(a, b) \neq 0$ because C is smooth, and the projection onto the first coordinates defines a chart (V, φ) for C on some open neighbourhood V of (a, b). Then V is the graph of the holomorphic function $f = h \circ \varphi^{-1}$ on $\varphi(V)$.

Differentiating the expression $P(x, f(x)) = 0$ with respect to x and using that $\partial_x P(a, b) = 0$, we find that for every positive integer m, we have $\partial_x^n P(a, b) = 0$ for all $1 \leq n < m$ if and only if $f^{(n)}(a) = 0$ for all $1 \leq n < m$. By Exercise 16.11, the smallest positive integer m with this property is the ramification degree of h at (a, b). It follows that h is ramified at a point (a, b) of C if and only $\partial_x P(a, b) = 0$, and the ramification degree is equal to the multiplicity of $P(x, b)$ at $x = a$.

If we slightly perturb the value a, then Proposition 16.9 implies that the solution $x = a$ for the equation $P(x, b) = 0$ "branches" into m different solutions close to a, where m is the ramification degree of h at (a, b). This explains the name "branch point".

Proposition 16.13. *Let $h\colon Y \to X$ be a non-constant morphism of connected compact Riemann surfaces. Let x be a point of X. Then the number*

$$d(x) = \sum_{y \in h^{-1}(x)} v_h(y)$$

depends only on h, and not on x.

Proof. The sum in the statement is finite because $h^{-1}(x)$ is a finite set, by Proposition 16.8. We want to prove that the function

$$d\colon X \to \mathbb{N},\ x \mapsto d(x)$$

is constant on X. Since X is connected, it suffices to show that d is locally constant.

Let x_0 be a point of X. Let $y_0, \ldots, y_n$ be the points in $h^{-1}(x_0)$. For each i in $\{0, \ldots, n\}$ we pick an open V_i around y_i in Y like in Proposition 16.9. These have the property that for every y in $V_i \setminus \{y_i\}$, the morphism h is unramified at y by Exercise 16.11, and the cardinality of $h^{-1}(h(y)) \cap V_i$ is equal to the ramification degree of h at y_i. Shrinking the sets V_i around the points y_i, we may assume that they are disjoint.

The set $Y \setminus (\cup_{i=0}^n V_i)$ is closed in Y, and thus compact. This implies that its image in X is compact, and therefore closed because X is Hausdorff. The complement of this image is an open subset U of X that contains x_0, because $h^{-1}(x_0)$ is contained in the union of the sets V_i. Moreover, $h^{-1}(U)$ is also contained in the union of the sets V_i. Since $h(V_i)$ is open in X for every i, the set $W = \cap_{i=0}^n h(V_i)$ is an open in X that contains x_0. For every x in $(U \cap W) \setminus \{x_0\}$, we have that $h^{-1}(x)$ is contained in $\cup_{i=0}^n V_i$ and intersects each of the sets V_i, so that

$$d(x) = |h^{-1}(x)| = \sum_{i=0}^n |h^{-1}(x) \cap V_i| = \sum_{i=0}^n v_h(y_i) = d(x_0)$$

and d is constant on $U \cap W$. ☐

Definition 16.14. The number $d(x)$ from Proposition 16.13 is called the *degree* of the morphism h, and denoted by $\deg(h)$.

Since h has only finitely many ramification points by Exercise 16.11, the degree of h is also equal to the maximal number of elements of a fiber of h.

Example 16.15. We will construct a projective version of Example 16.12. Let P be a non-constant homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors such that $P(0, 0, 1) \neq 0$. Let C be the projective plane curve defined by $P = 0$, and assume that C is smooth. Consider the map

$$h\colon C \to \mathbb{P}^1_{\mathbb{C}},\ [x_0, x_1, x_2] \mapsto [x_0, x_1].$$

It follows easily from Theorem 15.11 that this is a morphism of Riemann surfaces.

Let $p = [p_0, p_1, p_2]$ be a point on C and let L be the line through p and $[0, 0, 1]$. We will show that the ramification degree of h at p is equal to the intersection multiplicity $\mathbf{I}(p, C, L)$. In particular, h is ramified at p if and only if L is tangent to C at p. To prove this, we may assume that $p_0 = 1$. The affine plane curve

$C_0 = C \cap U_0$ is defined by $P(1, x, y) = 0$, and the line $L_0 = L \cap U_0$ is defined by $x = p_1$. With respect to the chart U_0 on the target $\mathbb{P}^1_{\mathbb{C}}$, the restriction of h to C_0 is given by $(x, y) \mapsto x$. It follows from Example 16.12 (with the roles of x and y exchanged) that the ramification degree of h at p is equal to the order of vanishing of $P(1, p_1, y)$ at $y = p_2$, which is precisely the intersection multiplicity of C and L at p, by Example 11.10.

The points in $h^{-1}(h(p))$ are exactly the intersection points of C and L. Thus,

$$\deg(h) = \sum_{q \in C \cap L} \mathbf{I}(q, C, L)$$

which is equal to $\deg(P)$ by Bézout's theorem.

17 The degree-genus formula

In this chapter, we will build upon the results in the preceding chapters to determine the homeomorphism type of a smooth projective plane curve. We need to borrow some general facts from the theory of topological surfaces, which will be presented without proofs.

Recall that a *topological surface* is a Hausdorff topological space that can be covered by countably many opens homeomorphic to opens in $\mathbb{R}^2$. Connected compact topological surfaces S can be classified in the following way. One can show that S always admits a finite *triangulation* $\mathcal{T}$: a subdivision of S into finitely many triangles such that the intersection of two triangles is either empty or a union of common vertices and common edges. The fact that S is a topological surface then implies that each edge is contained in precisely two triangles.

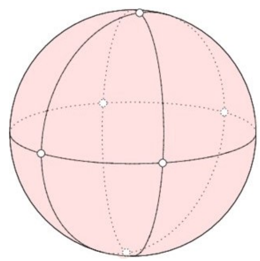

Figure 4: A triangulation of the sphere. *Source: H. Edelbsrunner and J. Harer,* Computational topology: an introduction. *American Mathematical Society, 2010.*

It turns out that, up to homeomorphism, the surface S is completely determined by the following two invariants:

1. *orientability*: we say that S is *orientable* if we can pick an orientation on the boundary of each triangle in $\mathcal{T}$ in such a way that each edge receives two *opposite* orientations from the triangles that contain the edge (see Figure 5). Otherwise, we say that S is non-orientable.

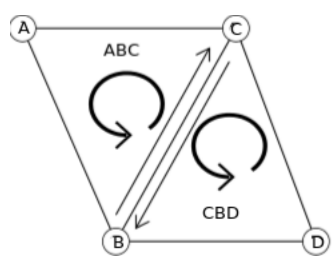

Figure 5: Two triangles with compatible orientations. *Source: https://triangleinequality.wordpress.com.*

   One can show that orientability of S does not depend on the choice of the triangulation $\mathcal{T}$, and that S is non-orientable if and only if we can embed the Möbius strip into S. Beware that the Möbius strip itself is not a topological surface, because it has a boundary: points on the boundary do not have an open neighbourhood homeomorphic to an open in $\mathbb{R}^2$. Orientability is preserved under homeomorphisms, because the image of a triangulation under a homeomorphism is again a triangulation, which can be oriented in the same manner.

   For instance, the sphere is orientable, so that everyone on earth agrees on what "clockwise" means, and the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ is non-orientable because it contains a copy of the Möbius strip (see the non-examinable document on the real projective plane on Blackboard). After travelling once around the middle circle of the Möbius strip, the hands of our clock will move in the opposite direction.

2. *the Euler characteristic*: denote by v, e and f the numbers of vertices, edges and triangles (also called *faces*) in the triangulation $\mathcal{T}$. Then the integer $\chi(S) = v - e + f$ only depends on S, and not on

the choice of the triangulation[9]. It is called the *Euler characteristic* of S. The Euler characteristic is invariant under homeomorphisms because homeomorphisms map triangulations to triangulations with the same numbers of vertices, edges and faces. On Figure 4 you can verify that the Euler characteristic of the sphere is equal to $6 - 12 + 8 = 2$; this fact is known as *Euler's theorem.* If S is orientable, then the Euler characteristic is always even, and one defines the *genus* $g(S)$ by $g(S) = (2 - \chi(S))/2$. For instance, the genus of the sphere is equal to 0.

Connected compact Riemann surfaces (and, in particular, smooth projective plane curves over $\mathbb{C}$) are always orientable. The reason is essentially that biholomorphic functions between opens in $\mathbb{C}$ preserve the orientation on $\mathbb{C}$ because of the Cauchy-Riemann relations. The classification theorem states that every orientable connected compact topological surface S is homeomorphic to precisely one surface in Figure 6.

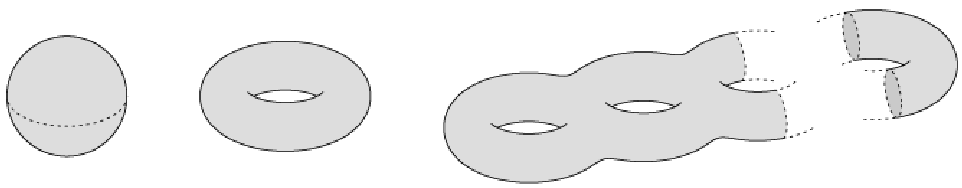

Figure 6: The sphere (genus 0), the torus (genus 1), and the connected sum of $g \geq 2$ tori (genus g). *Source: the Manifold Atlas Project.*

The genus $g(S)$ is always non-negative and can take any value in $\mathbb{Z}_{\geq 0}$. It is equal to the number of "holes" in the surface. An orientable connected compact topological surface S is completely determined by its genus $g(S)$: if $g(S) = 0$ then S is homeomorphic to the sphere; if $g(S) > 0$ then S can be constructed by gluing $g(S)$ tori together as in the picture. This gluing construction is called the *connected sum* and will be further discussed in the Algebraic Topology module.

For the sake of completeness, we will also briefly discuss the classification of non-orientable surfaces, but this paragraph relies on the definition of the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ and is non-examinable. Just like all orientable connected compact topological surfaces of positive genus can be constructed by gluing tori, all non-orientable connected compact topological surfaces can be constructed by gluing real projective planes. If you have studied the definition of the real projective plane, it is a good exercise to check that it has Euler characteristic 1. The classification theorem states that for each non-orientable connected compact topological surface S, the Euler characteristic is at most 1, and S can be constructed by gluing $2 - \chi(S)$ copies of the real projective plane $\mathbb{P}^2_{\mathbb{R}}$. In particular, S is completely determined by its Euler characteristic. Gluing two copies of $\mathbb{P}^2_{\mathbb{R}}$ we get a topological surface that is called the *Klein bottle* (see Figure 7). Non-orientable compact topological surfaces cannot be embedded in $\mathbb{R}^3$ without self-intersections, so we cannot make a realistic picture.

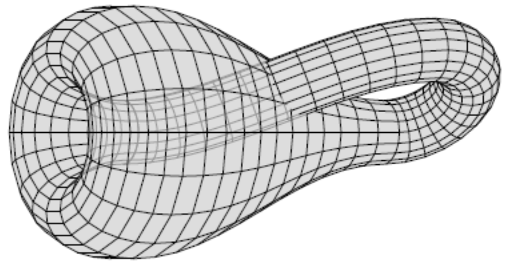

Figure 7: The Klein bottle. *Source: the Manifold Atlas Project.*

Since a smooth projective plane curve over $\mathbb{C}$ is always orientable, its topology is completely determined by its genus. The genus can be computed algebraically by means of the following formula.

[9]This is a special case of a deeper theorem about simplicial homology that will be discussed in the Algebraic Topology module.

Theorem 17.1 (Degree-genus formula)**.** *If C is a smooth projective plane curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d, then the genus of C is given by*

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

The remarkable thing about this result is that it relates purely algebraic information (the degree) to purely topological information (the genus).

Example 17.2. Lines and non-degenerate conics in $\mathbb{P}^2_{\mathbb{C}}$ have genus 0 and are homeomorphic to the sphere. Smooth non-degenerate cubics in $\mathbb{P}^2_{\mathbb{C}}$ have genus 1 and are homeomorphic to the torus.

There are no smooth projective plane curves of genus 2 in $\mathbb{P}^2_{\mathbb{C}}$, because 2 cannot be written as $(d-1)(d-2)/2$ with $d \in \mathbb{Z}_{>0}$: the smallest genus beyond 1 that is realized by a smooth projective plane curve is $(4-1)(4-2)/2 = 3$. One can show that there exist connected compact Riemann surfaces of any genus $g \in \mathbb{N}$, but some of them will not be isomorphic to smooth projective plane curves. However, they are always isomorphic to smooth algebraic curves in $\mathbb{P}^3_{\mathbb{C}}$, which have not been discussed in this course.

Exercise 17.3. As we have explained in Example 15.18, the smooth non-degenerate cubics in $\mathbb{P}^2_{\mathbb{C}}$ are precisely the complex tori $\mathbb{C}/\Lambda$, up to isomorphism. Prove directly that $\mathbb{C}/\Lambda$ has genus 1 by constructing a triangulation and computing the Euler characteristic.

We will deduce Theorem 17.1 from the following result.

Theorem 17.4 (Riemann-Hurwitz formula)**.** *Let $h \cdot Y \to X$ be a non-constant morphism between connected compact Riemann surfaces. Then the Euler characteristics of X and Y are related by*

$$\chi(Y) = \deg(h) \cdot \chi(X) - \sum_{y \in Y} (v_h(y) - 1).$$

Equivalently, their genera are related by

$$2 - 2g(Y) = \deg(h)(2 - 2g(X)) - \sum_{y \in Y} (v_h(y) - 1).$$

Note that the sum $\sum_{y \in Y}(v_h(y) - 1)$ is finite because h has only finitely many ramification points, so that $v_h(y) - 1 = 0$ for all but finitely many y in Y.

Proof. We will only explain the main idea of the proof, omitting the technical details. Set $d = \deg(h)$ and let $R_h \subset Y$ be the set of ramification points of h. The key point is that for every point x in X, the fiber $h^{-1}(x)$ consists of precisely d points when x does not lie in $h(R_h)$; if x lies in $h(R_h)$, then the cardinality $|h^{-1}(x)|$ of $h^{-1}(x)$ is strictly smaller than d, and it is related to the ramification degrees of h at the points in $h^{-1}(x)$ by means of the formula

$$d - |h^{-1}(x)| = \sum_{y \in h^{-1}(x)} (v_h(y) - 1) \tag{4}$$

deduced from Proposition 16.13.

Let x_0 be a point of $X \setminus h(R_h)$. Then h is unramified at every point of $h^{-1}(x_0)$, so that $h^{-1}(x_0)$ consists of precisely d points $y_1, \ldots, y_d$. Around each point y_i, we can pick an open V_i like in Proposition 16.9, with $m = 1$. Then h is a homeomorphism from V_i onto its image, because this holds for the map $\psi \circ h \circ \varphi^{-1}$ in Proposition 16.9 whenever $m = 1$. Shrinking each open V_i, we may assume that they are disjoint. Defining U as in the proof of Proposition 16.13 and replacing each open V_i by $V_i \cap h^{-1}(U)$, we find that $h^{-1}(U)$ is a disjoint union of d open sets in Y, each of which is mapped homeomorphically onto U by the morphism h. One says that $h \colon Y \setminus R_h \to X \setminus h(R_h)$ is a *topological covering* of degree d; these will be further studied in the Algebraic Topology module.

We can use this result to construct triangulations $\mathcal{T}_X$ and $\mathcal{T}_Y$ of X and Y with the following poperties:

1. The vertices, edges and triangles in $\mathcal{T}_X$ are precisely the images under h of the vertices, edges, resp. triangles in $\mathcal{T}_Y$.

2. Each ramification point of h is a vertex of $\mathcal{T}_Y$.
3. For every vertex V in $\mathcal{T}_X$, the fiber $h^{-1}(V)$ is a set of vertices of $\mathcal{T}_Y$. For every edge E in $\mathcal{T}_X$, the inverse image $h^{-1}(E)$ is a union of d distinct edges of $\mathcal{T}_Y$, and the analogous property holds for triangles.

Using these triangulations, we can compare the Euler characteristics of X and Y. Let v_X, e_X and f_X be the numbers of vertices, edges and triangles in $\mathcal{T}_X$, and define v_Y, e_Y and f_Y similarly. Then $e_Y = d \cdot e_X$ and $f_Y = d \cdot f_X$. Moreover, since the vertices of $\mathcal{T}_Y$ are precisely the points of Y that are mapped to vertices in $\mathcal{T}_X$, we deduce from equation (4) that

$$v_Y = d \cdot v_X - \sum_{y \in Y}(v_h(y) - 1).$$

The definition of the Euler characteristic now implies that

$$\chi(Y) = v_Y - e_Y + f_Y = d(v_X - e_X + f_X) - \sum_{y \in Y}(v_h(y) - 1) = d \cdot \chi(X) - \sum_{y \in Y}(v_h(y) - 1).$$

□

Corollary 17.5. *If X and Y are connected compact Riemann surfaces and $g(X) > g(Y)$, then every morphism $h\colon Y \to X$ is constant.*

Proof. This follows from the Riemann-Hurwitz formula, because the genus is non-negative and $v_h(y) - 1 \geq 0$ for every y in Y. □

Example 17.6. Let C be the smooth projective plane curve defined by the Legendre equation

$$x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Set $p = [0, 1, 0]$ and consider the function

$$f\colon C \setminus \{p\} \to \mathbb{C},\ [x_0, x_1, x_2] \mapsto x_0/x_2.$$

This function is well-defined because p is the only point of C where $x_2 = 0$, and the quotient x_0/x_2 does not depend on the choice of homogeneous coordinates (x_0, x_1, x_2). The function f is holomorphic on $C \setminus \{p\}$ and meromorphic at p, because we can write it as a quotient of holomorphic functions on each affine chart.

The meromorphic function f induces a morphism

$$h = \overline{f}\colon C \to \mathbb{P}^1_{\mathbb{C}}$$

such that $h([x_0, x_1, x_2]) = [x_0, x_2]$ if $x_2 \neq 0$, and $h([0, 1, 0]) = [1, 0]$. The restriction of this morphism to the affine curve $C \cap U_2 = C \setminus \{p\}$ with affine equation $y^2 = x(x-1)(x-\lambda)$ is given by

$$C \cap U_2 \to \mathbb{C},\ (x, y) \mapsto x,$$

the projection onto the x-axis. We see that h has degree 2, because $h^{-1}([x, 1])$ consists of two points for all but finitely many x in $\mathbb{C}$. The ramification points of h are $p_1 = [0, 0, 1]$, $p_2 = [1, 0, 1]$, $p_3 = [\lambda, 0, 1]$ and $p_4 = q = [0, 1, 0]$, which are mapped to $q_1 = [0, 1]$, $q_2 = [1, 1]$, $q_3 = [\lambda, 1]$ and $q_4 = [1, 0]$, respectively. Indeed, for each i in $\{1, 2, 3, 4\}$, the fiber $h^{-1}(q_i)$ only contains the point p_i. The ramification degree of h at p_i equals 2, by Proposition 16.13. Applying the Riemann-Hurwitz formula to the morphism h, we find that $\chi(C) = 2\chi(\mathbb{P}^1_{\mathbb{C}}) - 4 = 0$ and $g(C) = 1$. Since every smooth non-degenerate cubic in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic (even projectively equivalent) to a curve in the Legendre family, this tells us that all smooth non-degenerate cubics have genus 1.

A pictorial representation of the morphism h is given on the cover page of the lecture notes (courtesy of David Angdinata). It shows how, starting from two copies of the Riemann sphere, we can reconstruct the curve C by making a cut between the points q_1 and q_2 and between the points q_3 and q_4, and gluing the two copies into a torus. The projection back to the sphere $\mathbb{P}^1_{\mathbb{C}}$ is two-to-one except above the points q_1, q_2, q_3 and q_4. This illustrates how we can reconstruct the topology of C from a morphism to $\mathbb{P}^1_{\mathbb{C}}$ together with information about its ramification points.

Finally, we deduce the degree-genus formula from the Riemann-Hurwitz formula.

Proof. Let C be a smooth projective curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d. If $d = 1$ then C is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and $g(C) = 0$, so that the degree-genus formula is satisfied. Thus, we may assume that $d \geq 2$.

We know by Exercise 13.9 that C has only finitely many inflection points. Therefore, after a projective transformation, we may assume that $[0, 0, 1]$ does not lie on C or on any tangent line at an inflection point of C. Now consider the morphism

$$h \colon C \to \mathbb{P}^1_{\mathbb{C}},\ [x_0, x_1, x_2] \mapsto [x_0, x_1]$$

from Example 16.15. As we have seen in that example, the morphism h has degree $d = \deg(C)$, and h is unramified at a point p in C unless the tangent line L to C at p passes through $[0, 0, 1]$. In that case, the ramification degree of h at p equals $\mathbf{I}(p, C, L) = 2$ because p is not an inflection point, by our choice of projective coordinates. Now the Riemann-Hurwitz formula implies that

$$2 - 2g(C) = 2d - N$$

where N is the number of ramification points of h. Thus, we must show that $N = d(d - 1)$.

Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. If p is a point of C, then the tangent line to C at p passes through $[0, 0, 1]$ if and only if $\partial_{x_2} P(p) = 0$. Therefore, the ramification points of h are the intersection points of C with the zero set D of $\partial_{x_2} P$. Since $P(0, 0, 1) \neq 0$, the variable x_2 appears in P, so that $\partial_{x_2} P$ is a non-zero polynomial of degree $d - 1$. We will prove that $\partial_{x_2} P$ has no repeated factors and that $\mathbf{I}(p, C, D) = 1$ for every point p in $C \cap D$. The curves C and D have no common component because C is irreducible and of strictly larger degree than D, so that Bézout's theorem implies that

$$N = |C \cap D| = \sum_{p \in C \cap D} \mathbf{I}(p, C, D) = \deg(C) \cdot \deg(D) = d(d - 1).$$

First, we observe that if p is a point of $C \cap D$, then p is not an inflection point of C by our choice of projective coordinates. This implies that the Hessian $\mathcal{H}_P$ does not vanish at p, so that $\partial_{x_0}\partial_{x_2} P(p)$, $\partial_{x_1}\partial_{x_2} P(p)$ and $\partial^2_{x_2} P(p)$ are not all zero, and p is a smooth point of D.

Assume that $\partial_{x_2} P$ is divisible by Q^2 for some non-constant homogeneous polynomial Q in $\mathbb{C}[x_0, x_1, x_2]$. Then P and Q have a common zero p in $\mathbb{P}^2_{\mathbb{C}}$ by weak Bézout. This point is then also a zero of $\partial_{x_2} P$, and therefore lies on $C \cap D$. But the Leibniz rule now implies that all the first order partial derivatives of $\partial_{x_2} P$ vanish at p, because $Q(p) = 0$. This contradicts the smoothness of D at p, and it follows that $\partial_{x_2} P$ has no repeated factors.

Since the curve C is smooth by assumption, it only remains to show that C and D have distinct projective tangent lines at each point p of $C \cap D$; it then follows from Corollary 12.10 that $\mathbf{I}(p, C, D) = 1$. The projective tangent line to C at p passes through $[0, 0, 1]$ because p lies in D. Thus, it is enough to show that the projective tangent line to D at p does not pass through $[0, 0, 1]$, which is equivalent to showing that $\partial^2_{x_2} P(p) \neq 0$. Swapping the coordinates x_0 and x_1 if necessary, we may assume that $p = [1, a, b]$ for some a and b in $\mathbb{C}$. Applying the formula for the Hessian in Lemma 13.6 with the roles of x_0 and x_2 interchanged, we get

$$\begin{aligned}
\mathcal{H}_P(1, a, b) &= (d-1)^2 \cdot \det \begin{pmatrix} \partial^2_{x_2} P(1, a, b) & \partial_{x_1}\partial_{x_2} P(1, a, b) & \partial_{x_2} P(1, a, b) \\ \partial_{x_1}\partial_{x_2} P(1, a, b) & \partial^2_{x_1} P(1, a, b) & \partial_{x_1} P(1, a, b) \\ \partial_{x_2} P(1, a, b) & \partial_{x_1} P(1, a, b) & \frac{d}{d-1} P(1, a, b) \end{pmatrix} \\
&= (d-1)^2 \cdot \det \begin{pmatrix} \partial^2_{x_2} P(1, a, b) & \partial_{x_1}\partial_{x_2} P(1, a, b) & 0 \\ \partial_{x_1}\partial_{x_2} P(1, a, b) & \partial^2_{x_1} P(1, a, b) & \partial_{x_1} P(1, a, b) \\ 0 & \partial_{x_1} P(1, a, b) & 0 \end{pmatrix}.
\end{aligned}$$

Since p is not an inflection point of C, we know that $\mathcal{H}_P(1, a, b) \neq 0$, so that $\partial^2_{x_2} P(1, a, b) \neq 0$. $\square$

Exercise 17.7. Let C be a smooth projective plane curve of degree $d \geq 2$ in $\mathbb{P}^2_{\mathbb{C}}$.

1. Let p be a point of $\mathbb{P}^2_{\mathbb{C}}$. Show that at most $d(d-1)$ distinct lines through p are tangent to C.
2. Show that there exists a point p in $\mathbb{P}^2_{\mathbb{C}}$ such that $d(d-1)$ distinct lines through p are tangent to C. You may freely use that a smooth projective plane curve has only finitely many *bitangents* (lines that are tangent to the curve in at least 2 distinct points).
3. ($\star$) Deduce that the dual curve C^* from the 2020–2021 coursework 2 has degree $d(d-1)$.

18 What's next?

Now that you have all fallen in love with algebraic geometry and want to learn more, what are the next steps? Algebraic geometry is a vast field of research, and the literature may seem daunting at first. There is no canonical Royal Road into the subject, but there are excellent textbooks at every level of study, depending on your taste and interests.

- To advance any deeper into algebraic geometry territory, a backpack of commutative algebra is indispensable. Standard references are *Introduction to commutative algebra* by Atiyah and MacDonald, and *Undergraduate commutative algebra* by Reid. The latter reference is set up specifically as a preparation for algebraic geometry.

- To continue your study of curves, you can consult the more algebraic treatment in Fulton's *Algebraic curves.* The perspective of Riemann surfaces is further developed in Miranda's *Algebraic curves and Riemann surfaces.* If you are interested in the connections with number theory, good references are *Rational points on elliptic curves* by Silverman and Tate, and *An invitation to arithmetic geometry* by Lorenzini.

- An introduction to higher-dimensional algebraic varieties is provided in *An invitation to algebraic geometry* by Smith, Kahanpää, Kekäläinen and Traves, and in *Algebraic geometry: a first course* by Harris. An overview of "classical" algebraic geometry is given in *Lectures on curves, surfaces and projective varieties* by Beltrametti, Carletti, Gallarati and Monti Bragadin. It emphasizes geometry over algebra and explains the motivation behind some more abstract constructions in modern algebraic geometry.

- The modern language of algebraic geometry is the theory of schemes. This is a beautiful and powerful framework, but the path towards it is rather steep because it uses technology from commutative algebra, category theory, algebraic topology and differential geometry. An excellent reference with a lot of examples and motivation is Vakil's *The rising sea*, currently available at the author's website `http://math.stanford.edu/~vakil/216blog/`. I also recommend *Algebraic Geometry I* by Görtz and Wedhorn, although it is a bit drier than Vakil's text. A great introduction geared towards arithmetic geometry is Liu's *Algebraic geometry and arithmetic curves.* Another classic is Hartshorne's *Algebraic geometry*, which was one of the first textbooks on the topic; however, I would not recommend it for a first encounter with schemes, because the style is too terse.

A First aid topology

Here's a quick review of some basic definitions that were covered in the module on metric spaces and topology.

- A *topological space* is a set X, equipped with a collection $\mathcal{T}$ of subsets of X, which are called the *open sets* of the topology. These need to satisfy the following axioms:
  - $\emptyset$ and X are open;
  - any (possibly infinite) union of open sets is open;
  - any *finite* intersection of open sets is open.

  A subset of the topological space X is called *closed* if its complement is open.

- A metric space, such as $(\mathbb{C}^n, \|\cdot\|)$, carries a natural topology, called the *metric topology*, whose open sets are the unions of open balls.

- A subset $Y \subseteq X$ of a topological space X inherits a topology from X, called the *induced topology*. Its open sets are the sets of the form $Y \cap U$, where U is an open subset of X.

- A topological space X is called *connected* if it cannot be written as a disjoint union of two non-empty open subsets. This is equivalent to saying that $\emptyset$ and X are the only subsets of X that are both open and closed. The topological spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are connected for every $n \geq 0$. If X is a topological space, and U and V are connected subspaces with non-empty intersection, then the union $U \cup V$ is again connected. If X is non-empty, a *connected component* of X is a maximal connected subspace (the usual convention is that the empty space is connected but has no connected components). Every topological space is the disjoint union of its connected components.

- A topological space X is called *compact* if every cover of X by open subsets has a finite subcover. That is, if $U_i, i \in I$ are open subsets of X whose union equals X, then there exists a finite subset J of I such that X is already the union of the sets U_j with $j \in J$. A closed subset of a compact space is compact. The *Heine-Borel theorem* states that a subset of $\mathbb{R}^n$ or of $\mathbb{C}^m$ is compact with respect to the metric topology if and only if it is closed and bounded.

- If x is a point of a topological space X, then an *open neighbourhood* of x in X is an open subset of X that contains x. We say that X is *Hausdorff* if we can find disjoint open neighbourhoods around every pair of distinct points in X. A metric topology is always Hausdorff, since in a metric space we can find disjoint open balls around each pair of distinct points, by the triangle inequality.

- There are some inconsistencies in the literature regarding the notions of *limit point* and *accumulation point*. The only notion that will be important for us is what I have called an *accumulation point*; I have opted for this terminology because it is not used for anything else in the literature I am aware of. If X is a topological space and Y is a subset of X, then a point $x \in X$ is called an *accumulation point* of Y if every open neighourhood of x in X intersects Y in at least one point different from x. According to this convention, a *limit* point of Y is then either an accumulation point of Y or a point in Y; however, in many sources, "limit point" is also used to denote an accumulation point. The set X is closed if and only if it contains all of its limit points, which is also equivalent to saying that it contains all of its accumulation points, because the other limit points are already in Y. The term *limit point* only appears in two places in the solutions to the exercises, and you can replace it by *accumulation point* without any loss of information.

- A map $f\colon Y \to X$ between topological spaces is *continuous* if and only if $f^{-1}(U)$ is open in Y for every open subet U of X. This is equivalent to saying that $f^{-1}(F)$ is closed in Y for every closed subset F of X. We say that f is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous. In that case, f identifies the open sets in X and Y, so that we can think of X and Y as two copies of the same topological space. A continuous bijection $f\colon Y \to X$ is not always a homeomorphism because it may happen that f^{-1} is not continuous; however, it is always a homeomorphism if Y is compact and X is Hausdorff (because a closed subset of a compact space is compact, the image of a compact space under a continuous map is compact, and a compact subspace of a Hausdorff space is closed).

Whenever we speak about the topology on $\mathbb{C}^n$ for some $n \geq 0$, this always refers to the metric topology induced by the usual norm

$$\|(a_1, \ldots, a_n)\| = |a_1|^2 + \ldots + |a_n|^2$$

on $\mathbb{C}^n$.

B Reminders on complex analysis

Let U be an open subset of $\mathbb{C}$. A function $f\colon U \to \mathbb{C}$ is called *holomorphic* if it satisfies one of the following equivalent conditions.

- The function f is complex differentiable: for every point $z_0 \in U$, the function
$$U \setminus \{z_0\} \to \mathbb{C},\ z \mapsto \frac{f(z)-f(z_0)}{z-z_0}$$
has a limit at z_0.
- The function f admits a power series expansion around every point $z_0 \in U$: for some open disk D in U with center z_0, there exists a sequence of complex numbers $(c_n)_{n\geq 0}$ such that
$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$
for all $z \in D$. This then holds on *every* open disk around z_0 contained in U.
- Cauchy's integral formula holds for f: for every closed disk $\overline{D}$ in U and every point z_0 in its interior, we have
$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz,$$
where γ is the boundary of the disk $\overline{D}$, oriented counterclockwise.

Sums and products of holomorphic functions are holomorphic, and the inverse of a non-vanishing holomorphic function is holomorphic. The composite of two holomorphic functions is again holomorphic. The *maximum modulus principle* implies that if U is a connected open subset of $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ is a holomorphic function whose absolute value $|f|$ achieves a maximum on U, then f is constant. The *open mapping theorem* states that, if U is a connected open subset of $\mathbb{C}$, then the image of every non-constant holomorphic function $f\colon U \to \mathbb{C}$ is open. One of the most striking properties of holomorphic functions is the *identity theorem*: if U is a connected open subset of $\mathbb{C}$ and f is a holomorphic function on U such that $f^{-1}(0)$ has an accumulation point[10] in U, then f is identically zero on U. This implies that, if g and h are holomorphic functions on U such that the set $\{z \in U \mid g(z) = h(z)\}$ has an accumulation point in U, then $g = h$ on the whole of U (apply the identity theorem to $f = g - h$).

If U is an open subset of $\mathbb{C}$, then a holomorphic function $f\colon U \to \mathbb{C}$ is called *biholomorphic* if it is a bijection onto an open subset $V = f(U)$ of $\mathbb{C}$, and $f^{-1}\colon V \to U$ is holomorphic. Contrary to what happens for real differentiable functions, an injective holomorphic function $f\colon U \to \mathbb{C}$ is automatically biholomorphic: the image $V = f(U)$ is always open by the open mapping theorem, and f' does not vanish on U so that we can apply the inverse function theorem to deduce that f^{-1} is holomorphic.

If D is an open disk in $\mathbb{C}$ with center z_0, and g is a holomorphic function on the punctured disk $D^* = D \setminus \{z_0\}$, then g admits on D^* a unique *Laurent series expansion*
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$
whose complex coefficients c_n can be computed by means of the residue formula
$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$
where γ is a circle around z_0 in D, oriented counterclockwise. The *order* of f at z_0 is the infimum of the set of integers n such that $c_n \neq 0$. It is denoted by $\mathrm{ord}_{z_0} f$. We say that f has an *essential singularity* at z_0 if $\mathrm{ord}_{z_0} f = -\infty$; otherwise, f is called *meromorphic* at z_0. Assume that we are in the latter case and set

[10] An accumulation point of a subset S of U is a point $z \in U$ such that z is a limit point of $S \setminus \{z\}$.

$m = \mathrm{ord}_{z_0} f$. If f is identically zero on D^*, then $m = +\infty$. Otherwise, m is an integer, and we can write f as $f = (z - z_0)^m g$ where g is a holomorphic function on D such that $g(z_0) \neq 0$. We say that f has a *removable singularity* if $m \geq 0$; this is equivalent to saying that f extends to a holomorphic function on D, and this function then has a zero of order m at z_0. If $m < 0$, we say that f has a *pole* of order $-m$ at z_0.

If U is an open subset of $\mathbb{C}$, then a *meromorphic function* on U is a holomorphic function f on $U \setminus S$ for some discrete closed subset S of U such that f is meromorphic at every point of S. This is equivalent to saying that f can be written as a quotient g/h of holomorphic functions g and h on U such that h is not identically zero on any connected component of U. We identify two meromorphic functions on U if they are equal on $U \setminus T$ for some discrete closed subset T of U (in other words, removing a discrete closed subset from the domain does not alter the meromorphic function). If U is connected, then the meromorphic functions on U form a field with respect to pointwise addition and multiplication.

C Hints to starred exercises

Exercise 4.6

A union of finitely many compact subspaces is still compact. Try to find a compact subset C_i in each affine chart U_i such that $\mathbb{P}^n_{\mathbb{C}}$ is the union of $C_0, \ldots, C_n$.

Exercise 7.8

For the first part of the exercise, you can either use the implicit function theorem for complex analytic functions, or write everything in real coordinates and use the implicit function theorem for real continuously differentiable functions. For the second part of the exercise, let D be the affine plane curve defined by $xy = 0$ and take $q = (0, 0)$.

Exercise 10.8

Show that there exists an irreducible polynomial $T(x_2)$ in $F[x_2]$ that divides P and Q in the ring $F[x_2]$. Multiplying T with a suitable element of F, we may then assume that T lies in $\mathbb{C}[x_0, x_1, x_2]$ and that, if we view T as a polynomial in x_2 with coefficients in $\mathbb{C}[x_0, x_1]$, its coefficients have no common irreducible factor in $\mathbb{C}[x_0, x_1]$. Now prove that P/T lies in $\mathbb{C}[x_0, x_1, x_2]$.

Exercise 11.21

Reduce to the case $p = [1, 0, 0]$. Now you can show that for all but finitely many λ in $\mathbb{C}$, the line defined by $x_2 = \lambda x_1$ intersects C in precisely $d - m + 1$ points. This amounts to proving that the polynomial $P(x, 1, \lambda)$ in $\mathbb{C}[x]$ has $d - m$ distinct roots (why?). Which coefficients of the polynomial P are certainly equal to zero if we know that P has multiplicity m at p? What is the degree of $P(x, 1, \lambda)$? And when does this polynomial have multiple roots?

Exercise 13.9

Show that, when P is an irreducible homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ of degree $d \geq 2$, then P does not divide its Hessian $\mathcal{H}_P$. Weak Bézout then implies that the curve C defined by $P = 0$ has only finitely many inflection points. Reduce to the case where $p = [0, 0, 1]$ is an inflection point of C, and the tangent line to C at p is given by $x_0 = 0$.

Exercise 14.13

(1) To prove that you get a pencil, you need to show that all the point conditions are independent. Argue that it is enough to find a conic that contains p_1, p_2 and p_3, but not p_4. Construct such a conic.
(2) Take two distinct non-degenerate conics from the pencil in (1). Show that they cannot both contain p_5.

Exercise 15.10

What do you know about bounded holomorphic functions on $\mathbb{C}$? Does this property still hold on the open unit disk in $\mathbb{C}$?

Exercise 16.7

Reduce to the case where the function is holomorphic and non-vanishing on $\mathbb{C}$. Show that the induced morphism $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is not surjective, and deduce that it must be constant.

Exercise 17.7

How can we express the degree of a projective plane curve in terms of intersections with lines? What does this mean for the dual curve C^* in the dual projective plane?

D Solutions to the exercises

Exercise 1.2

The set $\mathbb{Z}$ is not an algebraic set in $\mathbb{R}$: we have seen that, for every field K, the algebraic subsets of K are the finite sets and K itself. The unit square is also not an algebraic set in $\mathbb{R}^2$: if $P(x,y) \in \mathbb{R}[x,y]$ is a polynomial vanishing on the segment $[0,1] \times \{0\}$, then $P(x,0)$ has infinitely many roots and therefore must be the zero polynomial. This implies that $P(x,y)$ vanishes on the entire x-axis. Thus every algebraic subset of $\mathbb{R}^2$ that contains $[0,1] \times \{0\}$ must also contain the x-axis.

Exercise 2.2

It is straightforward to check that $K[x_1,\ldots,x_n]$ is a ring (check the axioms by brute force) and it follows immediately from the definition that $\deg(P+Q) \leq \max\{\deg(P),\deg(Q)\}$. The inequality can be strict because of cancellations in the sum: for instance, if $P = x$ and $Q = 1-x$ then P and Q have degree 1 but the degree of $P+Q$ is zero.

To prove that the degree is additive under multiplication of polynomials, we may assume that P and Q are non-zero, since otherwise their product is zero and the result follows from the conventions on addition with $-\infty$. It is obvious from the definition that $\deg(PQ) \leq \deg(P) + \deg(Q)$, so we will only prove the converse inequality. Write $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$ and $Q = \sum_{\beta\in\mathbb{N}^n} b_\beta x^\beta$, where the coefficients a_α and b_β lie in K and only finitely many of them are non-zero. Consider the set of α in $\mathbb{N}^n$ such that $a_\alpha \neq 0$ and $|\alpha| = \deg(P)$. Let α_0 be the largest element in this set with respect to the lexicographical ordering (= the "alphabetical" ordering of tuples of numbers; google it if you haven't seen this before). We define β_0 accordingly. Then $a_{\alpha_0} b_{\beta_0} x^{\alpha_0+\beta_0}$ is a non-zero term in the polynomial expansion of the product PQ (there are no other terms with monomial $x^{\alpha_0+\beta_0}$ because of our choice of α_0 and β_0). Thus $\deg(PQ) \geq |\alpha_0 + \beta_0| = \deg(P) + \deg(Q)$.

An alternative, slick way to prove this is to introduce a new variable y and to observe that the degree of P is equal to the degree of $P(yx_1,\ldots,yx_n)$ viewed as a polynomial in the variable y with coefficients in the integral domain $K[x_1,\ldots,x_n]$. Now you just need to show that the degree is additive under multiplication of polynomials in one variable with coefficients in an integral domain; this follows from the fact that the product of the highest degree coefficients is non-zero.

It is clear that every non-zero constant polynomial in $K[x_1,\ldots,x_n]$ is invertible. Conversely, the additivity of the degree under multiplication implies that every invertible polynomial P in $K[x_1,\ldots,x_n]$ has degree 0, since $\deg(P) + \deg(P^{-1}) = \deg(1) = 0$.

Exercise 2.7

Let $P = P_1 \cdots P_k$ be a factorization of P into irreducible polynomials. If there exist λ, ℓ and m as in the statement of the exercise, then P is divisible by the square of the non-constant polynomial P_ℓ; thus P has a repeated factor. Conversely, assume that P is divisible by the square of a non-constant polynomial Q in $\mathbb{C}[x,y]$. Replacing Q by an irreducible factor, we can reduce to the case where Q is irreducible. By unique factorization into irreducible polynomials (up to multiplicative constants), there must exist distinct elements ℓ and m in $\{1,\ldots,k\}$ and constants λ,μ in $\mathbb{C}^*$ such that $Q = \lambda P_\ell = \mu P_m$. Then $P_m = (\lambda/\mu)P_\ell$.

Exercise 2.11

The lines through pairs of vertices of the unit square in $\mathbb{R}^2$ have defining polynomials x, $x-1$, y, $y-1$, $x-y$ and $x+y-1$. Thus their union is defined by the equation $x(x-1)y(y-1)(x-y)(x+y-1) = 0$.

Exercise 3.9

By Exercise 2.7, all the exponents a_j and b_j in the statement of Theorem 3.7 must be equal to 1 because P and Q have no repeated factors.

Exercise 3.12

Assume that C is irreducible, and let Q be a non-constant polynomial that divides P. Let D be the zero set of Q in $\mathbb{C}^2$. Then D is contained in C, so that $C = D$. The Nullstellensatz now implies that P and Q are equal up to a factor in $\mathbb{C}^*$. This means that P is irreducible.

Conversely, assume that P is irreducible. Let D be an affine plane curve contained in C and let Q be a defining polynomial for D with no repeated factors. Then Q divides P by Proposition 3.10, which implies that P and Q are equal up to a factor in $\mathbb{C}^*$ because P is irreducible. It follows that $C = D$, so that C is irreducible.

In general, we can factorize P into irreducible polynomials: $P = P_1 \cdots P_k$. Let $C_1, \ldots, C_k$ be the respective zero sets of these polynomials. These are distinct irreducible affine plane curves because P has no repeated factors, and C is their union. Now suppose that we can also write C as the union of distinct irreducible affine plane curves $D_1, \ldots, D_\ell$, with irreducible defining polynomials $Q_1, \ldots, Q_\ell$. Then P and $Q = Q_1 \cdots Q_\ell$ are polynomials with no repeated factors that define the same curve C, so that P and Q must be equal up to a factor in $\mathbb{C}^*$. The uniqueness of the irreducible factorization of polynomials now implies that $k = \ell$ and that the polynomials $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ are equal up to permutation of the indices and up to factors in $\mathbb{C}^*$. Thus $C_1, \ldots, C_k$ and $D_1, \ldots, D_k$ are equal up to permutation of the indices.

Exercise 3.15

The first part of the exercise is an immediate consequence of Proposition 3.10 and Exercise 3.12. If C and D have a common irreducible component and R is an irreducible defining polynomial of this component, then R divides P and Q. Conversely, if R is a non-constant polynomial that divides P and Q, then each irreducible component of the zero set of R is also an irreducible component of C and D.

We can write down a defining polynomial R with no repeated factors for the curve $C \cup D$ like in the proof of Proposition 3.10: we pick a factorization of Q into irreducible polynomials and we multiply P with all the irreducible factors of Q that do not divide P. The polynomial R divides PQ, and it is equal to PQ if and only if P and Q are not divisible by the same irreducible polynomial; equivalently, if and only if C and D have no common component. Since the degree is additive under the multiplication of polynomials, it follows that the degree of $C \cup D$ is at most $\deg(C) + \deg(D)$, and equality holds if and only if C and D have no common component.

Exercise 4.4

By symmetry, it suffices to consider the case $i = 0$. In order to prove that U_0 is open in $\mathbb{P}^n_{\mathbb{C}}$, we need to show that $\psi_j(U_0 \cap U_j)$ is open in $\mathbb{C}^n$ for every j in $\{0, \ldots, n\}$. This is obvious for $j = 0$, because $\psi_0(U_0) = \mathbb{C}^n$. If $j > 0$ then $\psi_j(U_0 \cap U_j)$ is the complement in $\mathbb{C}^n$ of the j-th coordinate hyperplane, which is an open subset of $\mathbb{C}^n$.

Now we prove that ψ_0 and ϕ_0 are homeomorphisms. Since these maps are mutually inverse bijections, it is enough to show that they are continuous. The continuity of ϕ_0 follows directly from the definition of the topology on $\mathbb{P}^n_{\mathbb{C}}$: if V is an open subset of U_0 then it is also open in $\mathbb{P}^n_{\mathbb{C}}$ since U_0 is open; thus $\phi_0^{-1}(V) = \psi_0(V)$ is open in $\mathbb{C}^n$.

Let us prove that ψ_0 is also continuous. Let V be an open subset of $\mathbb{C}^n$ and set $U = \psi_0^{-1}(V)$. We will show that U is open in $\mathbb{P}^n_{\mathbb{C}}$. This requires us to check that $V_j = \psi_j(U \cap U_j)$ is open in $\mathbb{C}^n$ for every j in $\{0, \ldots, n\}$. For $j = 0$ we simply find $V_0 = V$, so we may assume that $j > 0$. Then $V_j = \phi_j^{-1}(\psi_0^{-1}(V))$ so that V_j is the inverse image of V under the map

$$\psi_0 \circ \phi_j \colon \{y = (y_1, \ldots, y_n) \in \mathbb{C}^n, | \, y_1 \neq 0\} \to \mathbb{C}^n$$

that sends y to the tuple

$$\left(\frac{y_2}{y_1}, \ldots, \frac{y_j}{y_1}, \frac{1}{y_1}, \frac{y_{j+1}}{y_1}, \ldots, \frac{y_n}{y_1}\right)$$

in $\mathbb{C}^n$. This map is continuous, so that V_j is open.

Exercise 4.5

Let U be a subset of $\mathbb{P}^n_{\mathbb{C}}$. We need to prove that U is open in $\mathbb{P}^n_{\mathbb{C}}$ if and only if $V = \pi^{-1}(U)$ is open in $\mathbb{C}^{n+1}$.

For every i in $\{0, \dots, n\}$, let $W_i = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_i \neq 0\}$, and consider the map

$$\pi_i \colon W_i \to \mathbb{C}^n,\ (x_0, \dots, x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

This map fits into a commutative diagram

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "\pi_i"'] & \mathbb{C}^{n+1} \setminus \{\underline{0}\} \arrow[d, "\pi"] \\ \mathbb{C}^n \arrow[r, "\phi_i"'] & \mathbb{P}^n_{\mathbb{C}} \end{tikzcd}$$

where the top horizontal arrow is the inclusion map of W_i into $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$. The commutativity of this diagram implies that $V \cap W_i$ is the inverse image of $\psi_i(U \cap U_i) = \phi_i^{-1}(U)$ under π_i.

Since the sets W_i form an open cover of $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$, we know that V is open if and only if $V \cap W_i$ is open for every i. Thus it suffices to show that $V \cap W_i$ is open in W_i if and only if $\psi_i(U \cap U_i)$ is open in $\mathbb{C}^n$, for every i in $\{0, \dots, n\}$. By symmetry, it suffices to prove this for $i = 0$.

The "if" implication follows directly from the continuity of π_0. For the "only if" implication we observe that π_0 has a continuous section

$$\sigma \colon \mathbb{C}^n \to W_0,\ (y_1, \dots, y_n) \mapsto (1, y_1, \dots, y_n),$$

where "section" means that $\pi_0 \circ \sigma$ is the identity on $\mathbb{C}^n$. If $V \cap W_0$ is open in W_0, then

$$\psi_0(U \cap U_0) = (\pi_0 \circ \sigma)^{-1}(\phi_0^{-1}(U)) = \sigma^{-1}((\phi_0 \circ \pi_0)^{-1}(U)) = \sigma^{-1}(V \cap W_0)$$

is open in $\mathbb{C}^n$.

Exercise 4.6

Let D be the closed unit polydisk in $\mathbb{C}^n$:

$$D = \{(y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_i| \leq 1 \text{ for all } i \in \{1, \dots, n\}\}.$$

This is a compact subset of $\mathbb{C}^n$. For each i in $\{0, \dots, n\}$, we set $D_i = \phi_i(D)$. We will show that $\mathbb{P}^n_{\mathbb{C}}$ is the union of the sets $D_0, \dots, D_n$; since each of these sets is compact, it then follows that $\mathbb{P}^n_{\mathbb{C}}$ is compact. Let $[x] = [x_0, \dots, x_n]$ be a point of $\mathbb{P}^n_{\mathbb{C}}$. Permuting the variables if necessary, we may assume that $|x_0| \geq |x_i|$ for all i in $\{0, \dots, n\}$. Then $x_0 \neq 0$ and $|x_i/x_0| \leq 1$ for all i, so that $[x]$ lies in D_0.

Now we prove that $\mathbb{P}^n_{\mathbb{C}}$ is Hausdorff. Let $[a]$ and $[b]$ be two distinct points in $\mathbb{P}^n_{\mathbb{C}}$. We need to find disjoint open neighbourhoods of $[a]$ and $[b]$, respectively, in $\mathbb{P}^n_{\mathbb{C}}$. If $[a]$ and $[b]$ lie in a common affine chart U_i, then we can pick disjoint open balls around $\psi_i([a])$ and $\psi_i([b])$ in $\mathbb{C}^n$ and map them back to U_i by means of the homeomorphism ϕ_i. Thus, we may assume that $[a] = [a_0, \dots, a_n]$ lies in U_0 but not in U_1, and that $[b] = [b_0, \dots, b_n]$ lies in U_1 but not in U_0. Then $a_1 = b_0 = 0$.

Let $B_{[a]}$ and $B_{[b]}$ be the open unit balls of radius 1 in $\mathbb{C}^n$ around the points $\psi_0([a])$ and $\psi_1([b])$. Then $\phi_0(B_{[a]})$ and $\phi_1(B_{[b]})$ are open neighbourhoods around $[a]$ and $[b]$, and they are disjoint: every point $[c]$ in $\phi_0(B_{[a]})$ satisfies $c_0 \neq 0$ and $|c_1/c_0| < 1$ while every point $[c]$ in $\phi_1(B_{[b]})$ satisfies $c_1 \neq 0$ and $|c_0/c_1| < 1$.

Exercise 4.8

By symmetry, it suffices to consider the case $i = n$. The map θ_n is obviously a bijection. Thus, it suffices to show that a subset V of $\mathcal{P}_n$ is open if and only if $U = \theta_n^{-1}(V)$ is open in $\mathbb{P}^{n-1}_{\mathbb{C}}$.

Let $\iota\colon \mathbb{P}^{n-1}_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ be the composite of θ_n with the inclusion map $\mathcal{P}_n \to \mathbb{P}^n_{\mathbb{C}}$. This map fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n \setminus \{\underline{0}\} & \xrightarrow{\widetilde{\iota}} & \mathbb{C}^{n+1} \setminus \{\underline{0}\} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^{n-1}_{\mathbb{C}} & \xrightarrow{\iota} & \mathbb{P}^n_{\mathbb{C}} \end{array}$$

where $\widetilde{\iota}$ maps $(x_0, \dots, x_{n-1})$ to $(x_0, \dots, x_{n-1}, 0)$ and we have made a slight abuse of notation by writing π for each of the two projection maps from Exercise 4.5.

If V is open then we can write it as $W \cap \mathcal{P}_n$ with W an open subset in $\mathbb{P}^n_{\mathbb{C}}$. Then

$$\pi^{-1}(U) = \pi^{-1}(\iota^{-1}(W)) = (\widetilde{\iota})^{-1}(\pi^{-1}(W))$$

is open in $\mathbb{C}^n \setminus \{\underline{0}\}$ by continuity of $\widetilde{\iota}$ and the projection map π. Thus, U is open in $\mathbb{P}^{n-1}_{\mathbb{C}}$ by Exercise 4.5.

Assume, conversely, that U is open in $\mathbb{P}^{n-1}_{\mathbb{C}}$. Then $\pi^{-1}(U)$ is open in $\mathbb{C}^n \setminus \{\underline{0}\}$ by continuity, and this implies that $\widetilde{W} = \pi^{-1}(U) \times \mathbb{C}$ is open in $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$. Setting $W = \pi(\widetilde{W})$ we have $\widetilde{W} = \pi^{-1}(W)$, because $\widetilde{W}$ is a union of fibers of π: a point x of $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$ lies in $\widetilde{W}$ if an only if λx lies in $\widetilde{W}$ for all $\lambda \in \mathbb{C}^*$. The fact that $\widetilde{W}$ is open in $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$ now implies that W is open in $\mathbb{P}^n_{\mathbb{C}}$, by Exercise 4.5. We have constructed W in such a way that $V = W \cap \mathcal{P}_n$, since

$$\theta_n^{-1}(V) = U = \pi(\pi^{-1}(U)) = \pi(\widetilde{\iota}^{-1}(\widetilde{W})) = \iota^{-1}(W) = \theta_n^{-1}(W \cap \mathcal{P}_n).$$

Exercise 5.13

These three points are collinear if and only if they all satisfy the same non-trivial homogeneous linear equation, which is equivalent to saying that the rows of the matrix are linearly dependent. It follows that, when $[b_0, b_1, b_2]$ and $[c_0, c_1, c_2]$ are distinct, there is a unique line through these points, defined by

$$\det \begin{pmatrix} x_0 & b_0 & c_0 \\ x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \end{pmatrix} = 0.$$

Exercise 5.15

Thanks to Theorem 5.10 and Lemma 5.11, we can simply copy the arguments from the affine case (Proposition 3.10 and Exercise 3.12).

Exercise 6.3

Let $P \in \mathbb{C}[x, y]$ be a polynomial of degree $d > 0$ with no repeated factors whose zero set is C. Showing that $\overline{C}$ does not contain L_∞ is equivalent to showing that the homogenization $\overline{P}$ is not divisible by x_2, by Exercise 5.15. We write $P(x, y) = P_d(x, y) + P_{d-1}(x, y) + \ldots + P_0(x, y)$ where each term P_i is a homogeneous polynomial of degree i in $\mathbb{C}[x, y]$. Then

$$\overline{P}(x_0, x_1, x_2) = x_2^d P\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) = P_d(x_0, x_1) + x_2 P_{d-1}(x_0, x_1) + \ldots + x_2^d P_0(x_0, x_1)$$

is not divisible by x_2 because $P_d \neq 0$.

Now let D be a projective plane curve that extends C. Then D is defined by an equation of the form $Q(x_0, x_1, x_2) = 0$ or $x_2 Q(x_0, x_1, x_2) = 0$ where $Q(x_0, x_1, x_2)$ is a homogeneous polynomial of degree $e > 0$ with no repeated factors that is not divisible by x_2. We will show that $Q = \lambda \overline{P}$ for some $\lambda \in \mathbb{C}^*$; then D is either $\overline{C}$ or the union of $\overline{C}$ and L_∞.

Since D extends C, we know that C is the zero locus of $Q(x, y, 1)$. If $R(x, y)$ is a polynomial of degree $f > 0$ in $\mathbb{C}[x, y]$ such that R^2 divides $Q(x, y, 1)$, then the square of

$$x_2^f R\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$$

also divides

$$x_2^e Q\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1\right) = Q(x_0, x_1, x_2)$$

which contradicts the fact that Q has no repeated factors. Thus $Q(x, y, 1)$ has no repeated factors. Now the affine Nullstellensatz implies that $Q(x, y, 1) = \lambda P(x, y)$ for some $\lambda \in \mathbb{C}^*$. It follows that $d = e$ and

$$Q(x_0, x_1, x_2) = x_2^d Q\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1\right) = \lambda x_2^d P\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) = \lambda \overline{P}(x_0, x_1, x_2).$$

Exercise 6.13

Let $C_1, \ldots, C_r$ be the irreducible components of C. Their projectivizations $\overline{C}_i$ are all distinct, because $\overline{C}_i \cap U_2 = C_i$. The union of the curves $\overline{C}_i$ is a projective plane curve extending C that does not contain the line at infinity, since each component C_i has only finitely many points at infinity. Thus $\overline{C}$ is the union of the projectivizations $\overline{C}_i$.

Therefore, it suffices to show that $\overline{C}$ is irreducible if C is irreducible. Let D be a projective plane curve contained in $\overline{C}$. Then D does not contain the line at infinity, so that $D \cap U_2$ is an affine plane curve contained in C. Since C is irreducible, it follows that $C = D \cap U_2$, so that $\overline{C} = \overline{D \cap U_2} = D$.

Exercise 7.3

For each $\lambda \in \mathbb{C}$, the singular points of this curve are the solutions of the system of equations

$$y^2 - x(x-1)(x-\lambda) = x(x-1) + x(x-\lambda) + (x-1)(x-\lambda) = 2y = 0.$$

This system has a solution if and only if $\lambda = 0$ or $\lambda = 1$; in those cases, the curve has a unique singular point, given by $(\lambda, 0)$.

Exercise 7.5

The first curve is smooth: there is no point $[x_0, x_1, x_2]$ in $\mathbb{P}^2_{\mathbb{C}}$ where $2x_0 = 2x_1 = 2x_2 = 0$, since at least one of the homogeneous coordinates must be non-zero. The second curve is singular at $[1, 0, 0]$ and smooth at all other points, because $[1, 0, 0]$ is the unique point where

$$x_0 x_1^2 - x_2^3 = x_1^2 = 2x_0 x_1 = -3x_2^2 = 0.$$

Exercise 7.8

If C is projective, we can intersect it with an affine chart containing p to reduce to the affine case, thanks to Proposition 7.7. The affine case follows from the implicit function theorem: let $P(x, y)$ be a non-constant polynomial in $\mathbb{C}[x, y]$ with no repeated factors whose zero set is $\mathbb{C}$. Swapping x and y if necessary, we may assume that $\partial_y P(p) \neq 0$. Then projecting onto the x-axis yields a homeomorphism between an open neighbourhood V of p in C and an open disk in $\mathbb{C}$; such a disk is homeomorphic to $\mathbb{C}$.

If D is the affine plane curve defined by $xy = 0$ and q is the origin of $\mathbb{C}^2$, then for every open neighbourhood W of q in D, the set $W \setminus \{q\}$ is disconnected: we can write it as the disjoint union of the non-empty open sets $\{(x, y) \in W \mid x \neq 0\}$ and $\{(x, y) \in W \mid y \neq 0\}$. But the complement of a point in $\mathbb{C}$ is always connected (even pathwise connected).

Exercise 7.10

The solution is quite similar to the proof of Proposition 7.7. We may again assume that $U = U_2$. In the notation of the proof of Proposition 7.7, the tangent line to p at D is defined by

$$\partial_x Q(a, b)(x - a) + \partial_y Q(a, b)(y - b) = 0$$

while the projective tangent line of C at p is defined by

$$\partial_{x_0} P(a,b,1)x_0 + \partial_{x_1} P(a,b,1)x_1 + \partial_{x_2} P(a,b,1)x_2 = 0.$$

We will prove that the second equation is the homogenization of the first. Since $\partial_x Q(a,b) = \partial_{x_0} P(a,b,1)$ and $\partial_y Q(a,b) = \partial_{x_1} P(a,b,1)$, this amounts to showing that

$$\partial_{x_2} P(a,b,1) = -a\partial_{x_0} P(a,b,1) - b\partial_{x_1} P(a,b,1).$$

This equality is obtained by evaluating Euler's relation at the point $(a,b,1)$.

Exercise 8.3

It is clear that $\Phi_A = \Phi_{\lambda A}$ for every invertible matrix A of rank $n+1$ and every $\lambda \in \mathbb{C}^*$. Conversely, assume that B is another invertible matrix of rank $n+1$ such that $\Phi_A = \Phi_B$. This means that Av is proportional to Bv for every column vector v in $\mathbb{C}^{n+1}$. We will prove that $B = \lambda A$ for some λ in $\mathbb{C}^*$. For every column vector v in $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$, we have $Bv = \lambda_v Av$ for some unique $\lambda_v \in \mathbb{C}^*$. Clearly, $\lambda_{\mu v} = \lambda_v$ for every μ in $\mathbb{C}^*$. Moreover, for each linearly independent pair of column vectors v and w in $\mathbb{C}^{n+1}$, we can write

$$\lambda_v Av + \lambda_w Aw = B(v+w) = \lambda_{v+w} Av + \lambda_{v+w} Aw.$$

Since A is invertible, the vectors Av and Aw are also linearly independent, so that $\lambda_v = \lambda_{v+w} = \lambda_w$. It follows that λ_v is independent of v, so that $B = \lambda A$ for some $\lambda \in \mathbb{C}^*$.

Exercise 8.4

Let $B = (b_{ij})_{i,j=0}^2$ be the inverse of the matrix A. Let P be a homogeneous polynomial of degree $d > 0$ in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors such that C is the zero set of P. Then $D = \Phi_A(C)$ is the zero set of the homogeneous polynomial

$$Q(x_0,x_1,x_2) = P(b_{00}x_0 + b_{01}x_1 + b_{02}x_2, b_{10}x_0 + b_{11}x_1 + b_{12}x_2, b_{20}x_0 + b_{21}x_1 + b_{22}x_2).$$

It is obvious that Q has degree at most d; it must then be equal to d, because we can recover P from Q by applying the inverse transformation. Moreover, Q has no repeated factors: if Q were divisible by the square of a homogeneous polynomial R of positive degree e, then P would be divisible by the square of the homogeneous degree e polynomial

$$R(a_{00}x_0 + a_{01}x_1 + a_{02}x_2, a_{10}x_0 + a_{11}x_1 + a_{12}x_2, a_{20}x_0 + a_{21}x_1 + a_{22}x_2).$$

Thus, D is a projective plane curve of degree d.

If D' is a projective plane curve strictly contained in D, then $\Phi_B(D')$ is a projective plane curve strictly contained in C. It follows that C is reducible if D is reducible. Reversing the roles of C and D, we then conclude that C is reducible if and only if D is reducible. More generally, if $C_1, \ldots, C_r$ are the irreducible components of C, then $\Phi_A(C_1), \ldots, \Phi_A(C_r)$ are distinct irreducible projective plane curves whose union equals D; thus they must be the irreducible components of D.

Let p be a point of C and set $q = \Phi_A(p)$. We pick a triple (p_0,p_1,p_2) of homogeneous coordinates for p; then $(q_0,q_1,q_2) = (p_0,p_1,p_2)A^t$ is a triple of homogeneous coordinates for q. It follows from the chain rule that

$$\begin{pmatrix} \partial_{x_0} Q(q_0,q_1,q_2) \\ \partial_{x_1} Q(q_0,q_1,q_2) \\ \partial_{x_2} Q(q_0,q_1,q_2) \end{pmatrix} = B^t \begin{pmatrix} \partial_{x_0} P(p_0,p_1,p_2) \\ \partial_{x_1} P(p_0,p_1,p_2) \\ \partial_{x_2} P(p_0,p_1,p_2) \end{pmatrix}$$

If C is smooth at p, then at least one of the partial derivatives $\partial_{x_i} P(p_0,p_1,p_2)$ is non-zero; since B is invertible, it then follows that at least one of the partial derivatives $\partial_{x_i} Q(q_0,q_1,q_2)$ is non-zero, so that D is smooth at q. The converse implication again follows by reversing the role of P and Q. Finally, the chain rule also implies that the projective tangent line of D at q is the image under Φ_A of the projective tangent line of C at p.

Exercise 8.5

The problem can be translated into a linear algebra problem in a straightforward way, as in the proof of Proposition 8.6. We will give a slightly different argument: we first show the existence of Φ, and will then prove its uniqueness.

We pick homogeneous coordinates (p_0, p_1), (q_0, q_1), (r_0, r_1) and (s_0, s_1) for the points p, q, r, s. Since a composition of projective transformations is again a projective transformation, we can reduce to the case $p = [0, 1]$ and $q = [1, 0]$ by first applying Φ_A with

$$A = \begin{pmatrix} p_1 & -p_0 \\ q_1 & -q_0 \end{pmatrix}.$$

Now $\Phi_A(r) = [r'_0, r'_1]$ with $r'_0 = p_1 r_0 - p_0 r_1$ and $r'_1 = q_1 r_0 - q_0 r_1$. This is a point different from $[0, 1]$ and $[1, 0]$; the projective transformation Φ_B associated with

$$B = \begin{pmatrix} r'_1 & 0 \\ 0 & r'_0 \end{pmatrix}$$

fixes $[0, 1]$ and $[1, 0]$, and maps r to $[1, 1]$. This proves the existence of Φ.

To prove the uniqueness of Φ, it suffices to show that any projective transformation of $\mathbb{P}^1_{\mathbb{C}}$ that fixes $[0, 1]$, $[1, 0]$ and $[1, 1]$ is equal to the identity. Let C be an invertible rank 2 matrix over $\mathbb{C}$. Then Φ_C fixes $[0, 1]$ and $[1, 0]$ if and only if C is a diagonal matrix, and Φ_C moreover fixes $[1, 1]$ if and only if C is a scalar multiple of the identity matrix; then Φ_C is the identity.

This observation has an interesting consequence: if p, q and r are distinct points in $\mathbb{P}^1_{\mathbb{C}}$ we can attach an invariant to any point $s = [s_0, s_1]$ in $\mathbb{P}^1_{\mathbb{C}}$ by considering the unique projective transformation Φ mapping p, q, r to $[0, 1]$, $[1, 0]$ and $[1, 1]$, and taking the affine coordinate of the point $\Phi(s)$ with respect to the affine chart U_1, with the convention that this coordinate is ∞ when $\Phi(s) = [1, 0]$. The *inverse* of this coordinate is called the *cross-ratio* of the points p, q, r, s and denoted by $c(p, q, r, s)$ (here we adopt the rule that $1/0 = \infty$ and $1/\infty = 0$). This is an element of $\mathbb{C} \cup \{\infty\}$; it is an intrinsic geometric invariant of the quadruple (p, q, r, s), in the sense that it does not change when we apply a projective transformation to $\mathbb{P}^1_{\mathbb{C}}$ (any such transformation would simply be reabsorbed by Φ). In other words, once we have chosen points to play the roles of $[0, 1]$, $[1, 0]$ and $[1, 1]$, the cross-ratio (or its inverse) can be used as a *canonical* coordinate function on $\mathbb{P}^1_{\mathbb{C}}$.

We can write down an explicit formula for the cross-ratio by using our construction of the projective transformation $\Phi = \Phi_{BA}$ above. We have

$$BA = \begin{pmatrix} p_1 r'_1 & -p_0 r'_1 \\ q_1 r'_0 & -q_0 r'_0 \end{pmatrix}$$

so that $\Phi(s) = [r'_1(p_1 s_0 - p_0 s_1), r'_0(q_1 s_0 - q_0 s_1)]$ and

$$c(p, q, r, s) = \frac{r'_0(q_1 s_0 - q_0 s_1)}{r'_1(p_1 s_0 - p_0 s_1)} = \frac{(p_1 r_0 - p_0 r_1)(q_1 s_0 - q_0 s_1)}{(q_1 r_0 - q_0 r_1)(p_1 s_0 - p_0 s_1)}.$$

This formula can be further simplified. If we suppose that none of the points p, q, r, s is equal to $[1, 0]$, then we can choose homogeneous coordinates in such a way that $p_1 = q_1 = r_1 = s_1 = 1$. Our expression then becomes

$$c(p, q, r, s) = \frac{(r_0 - p_0)(s_0 - q_0)}{(r_0 - q_0)(s_0 - p_0)}.$$

If one of the points p, q, r, s is equal to $[1, 0]$, we get a similar expression where the factors involving this point are omitted.

Exercise 8.7

Take a second point q on L, distinct from p. By Proposition 8.6, there exists a projective transformation of $\mathbb{P}^2_{\mathbb{C}}$ that maps p to $[0, 0, 1]$ and q to $[0, 1, 0]$. Then $\Phi(L)$ is a line passing through $[0, 1, 0]$ and $[0, 0, 1]$; the unique such line is the one defined by $x_0 = 0$.

Exercise 9.4

Every line in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to the line defined by $x_0 = 0$, by Exercise 8.7. The conic

$$D_0 = \{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_0^2 + x_1^2 + x_2^2 = 0\}$$

is projectively equivalent to the conic C_0 from Theorem 9.1 *via* the projective transformation that maps $[x_0, x_1, x_2]$ to $[x_0 + ix_2, x_1, x_0 - ix_2]$. Since every non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to C_0, it is also projectively equivalent to D_0.

If C is a union of two distinct lines then, by Proposition 8.6, we can apply a projective transformation to reduce to the case where these lines are defined by $x_0 = 0$ and $x_1 = 0$. The inverse of the projective transformation

$$\Phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}, [x_0, x_1, x_2] \mapsto [x_0 + ix_1, x_0 - ix_1, x_2]$$

maps C to the conic

$$\{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_0^2 + x_1^2 = 0\}.$$

It remains to show that these three cases correspond to different projective equivalence classes. A line has degree 1 and is therefore not projectively equivalent to a union of two distinct lines or a non-degenerate conic, which have degree 2. A non-degenerate conic is irreducible, and therefore not projectively equivalent to a union of two distinct lines.

Exercise 9.5

Let

$$Q(x_0, x_1, x_2) = c_{00}x_0^2 + c_{11}x_1^2 + c_{22}x_2^2 + c_{01}x_0x_1 + c_{02}x_0x_2 + c_{12}x_1x_2$$

be a homogeneous quadratic polynomial in $\mathbb{C}[x_0, x_1, x_2]$ whose zero set is C. This polynomial is unique up to a factor λ in $\mathbb{C}^*$. Then a direct calculation reveals that there is a unique symmetric 3×3-matrix $A = (a_{ij})_{i,j=0}^2$ over $\mathbb{C}$ such that

$$Q(x_0, x_1, x_2) = (x_0, x_1, x_2)A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

This matrix is given explicitly by $a_{ii} = c_{ii}$ and $a_{ij} = a_{ji} = c_{ij}/2$ whenever $i < j$. Since the coefficients c_{ij} are unique up to a factor λ in $\mathbb{C}^*$, the same holds for A.

If B is an invertible rank 3 matrix over $\mathbb{C}$, then $\Phi_{B^{-1}}(C)$ is the zero set of the quadratic polynomial

$$(x_0, x_1, x_2)B^tAB \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

The matrix B^tAB is still symmetric, and it has the same rank as A. By Exercise 9.4, we can find B such that B^tAB is a diagonal matrix with diagonal $(1, 0, 0)$, $(1, 1, 0)$ or $(1, 1, 1)$. In the first case, C is a line and A has rank 1; in the second case, C is a union of two distinct lines and A has rank 2; in the final case, C is non-degenerate and A has rank 3.

This exercise connects the classification of projective conics to the theory of symmetric matrices and the classification of so-called *quadratic forms*, an important topic in linear algebra, geometry and number theory.

Exercise 10.8

Our assumption that $P(0, 0, 1)$ and $Q(0, 0, 1)$ are non-zero implies that P and Q have degrees d and e in the variable x_2, since they must contain a term without the variables x_0 and x_1. The proof of the univariate case shows that there exist a polynomial $R(x_2)$ of degree at most $d-1$ and a polynomial $S(x_2)$ of degree at most $e-1$ in $F[x_2]$ such that $P(x_2)S(x_2) = Q(x_2)R(x_2)$. We can write $P(x_2)$ as a product $P_1(x_2) \cdots P_r(x_2)$ of irreducible polynomials $P_i(x_2)$ in $F[x_2]$. Then this product divides $Q(x_2)R(x_2)$ but not $R(x_2)$ because the degree of $R(x_2)$ is at most $d-1$. By the UFD property of $F[x_2]$, at least one of the polynomials $P_i(x_2)$

divides $Q(x_2)$ in $F[x_2]$; we pick such a factor and call it $T(x_2)$. Multiplying $T(x_2)$ with a non-zero constant in F, we can reduce to the case where $T(x_2)$ has coefficients in $\mathbb{C}[x_0,x_1]$, and where these coefficients have no common irreducible factor in $\mathbb{C}[x_0,x_1]$.

Now we have $P(x_2) = T(x_2)P'(x_2)$ and $Q(x_2) = T(x_2)Q'(x_2)$ for some polynomials $P'(x_2)$ and $Q'(x_2)$ in $F[x_2]$. We will prove that $P'(x_2)$ lies in $\mathbb{C}[x_0,x_1,x_2]$. The same argument then also applies to $Q'(x_2)$ and shows that P and Q have a common factor $T(x_2)$ in $\mathbb{C}[x_0,x_1,x_2]$.

Clearing denominators, we can write $P'(x_2)$ as $U(x_2)/V$ where V is a non-zero element in $\mathbb{C}[x_0,x_1]$ and $U(x_2)$ has coefficients in $\mathbb{C}[x_0,x_1]$ with no common irreducible factor. Then V divides $T(x_2)U(x_2)$ in $\mathbb{C}[x_0,x_1,x_2]$. Since the coefficients of $T(x_2)$ have no common irreducible factor in $\mathbb{C}[x_0,x_1]$ and the same holds for $U(x_2)$, the UFD property of $\mathbb{C}[x_0,x_1,x_2]$ implies that V lies in $\mathbb{C}^*$, so that $P'(x_2)$ lies in $\mathbb{C}[x_0,x_1,x_2]$.

Exercise 10.9

(1) Evaluating Euler's identity for P at the point $(0,0,1)$, we find

$$\partial_{x_2}P(0,0,1) = d\cdot P(0,0,1) \neq 0.$$

Now we argue by contradiction. If $\mathbf{R}_{P,\partial_{x_2}P}$ is identically zero, then it follows from Theorem 10.7 that there exists an irreducible polynomial Q in $\mathbb{C}[x_0,x_1,x_2]$ that divides P and $\partial_{x_2}P$. Writing $P = Q\cdot R$ and applying the Leibniz rule, we find that

$$\partial_{x_2}P = \partial_{x_2}Q\cdot R + Q\cdot\partial_{x_2}R$$

so that Q must also divide $\partial_{x_2}Q\cdot R$. The polynomial Q is homogeneous of positive degree and $Q(0,0,1)\neq 0$, so that applying Euler's identity again we find that $\partial_{x_2}Q(0,0,1)\neq 0$. In particular, $\partial_{x_2}Q\neq 0$. Therefore, the irreducible polynomial Q cannot divide $\partial_{x_2}Q$ for degree reasons, so that it must divide R; but then Q^2 divides P, contradicting the assumption that P has no repeated factors.

(2) After a projective transformation, we may assume that $p = [0,0,1]$, so that $P(0,0,1)\neq 0$. We will prove that for all but finitely many $\lambda\in\mathbb{C}$, the line L_λ through p defined by $x_1 = \lambda x_0$ intersects C in precisely d points.

All the intersection points of C and L_λ lie in the affine chart U_0, because $[0,0,1]$ is the only point of L_λ outside U_0 and this is not a point of C. Therefore, $C\cap L_\lambda$ consists of the points of the form $[1,\lambda,t]$ such that $P(1,\lambda,t) = 0$. Since $P(0,0,1)\neq 0$, the polynomial P contains a term of the form cx_2^d with $c\in\mathbb{C}^*$, so that $P(1,\lambda,t)$ has degree d in t. We must show that, for all but finitely many λ in $\mathbb{C}$, the polynomial $P(1,\lambda,t)$ has no multiple roots. This is equivalent to saying that $\mathbf{R}_{P,\partial_{x_2}P}(1,\lambda)\neq 0$. The resultant $\mathbf{R}_{P,\partial_{x_2}P}(1,\lambda)$ is a polynomial in λ, and it is not identically zero because $\mathbf{R}_{P,\partial_{x_2}P}$ is a non-zero homogeneous polynomial in $\mathbb{C}[x_0,x_1]$ by question 1. Therefore, $\mathbf{R}_{P,\partial_{x_2}P}(1,\lambda)$ has only finitely many roots.

Exercise ??

Let $P(x_0,x_1,x_2)$ be a homogeneous polynomial with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. After a projective transformation, we may assume that $P(0,0,1)\neq 0$; then it follows from Exercise 10.9 that $\partial_{x_2}P(0,0,1)\neq 0$. Therefore, $\partial_{x_2}P$ is a non-zero homogeneous polynomial of degree $d-1$ in $\mathbb{C}[x_0,x_1,x_2]$, and Exercise 10.9 also implies that P and $\partial_{x_2}P$ have no common components. Now it follows from weak Bézout that the number of common zeroes of P and $\partial_{x_2}P$ is at most $d(d-1)$.

The singularities of C are precisely the singularities of the irreducible components of C and the intersection points of distinct irreducible components of C. If C is affine, then the singularities of C are the singularities of the projectivization $\overline{C}$ contained in U_2, and the irreducible components of $\overline{C}$ are the projectivizations of the irreducible components of C (Exercise 6.13). Thus it suffices to solve the projective case.

Let P be a homogeneous polynomial with no repeated factors in $\mathbb{C}[x_0,x_1,x_2]$ with zero set C, and let $P = P_1\cdots P_r$ be a factorization into irreducible polynomials. Then the irreducible components of C are the zero sets $C_1,\ldots,C_r$ of the homogeneous polynomials $P_1,\ldots,P_r$ in $\mathbb{P}^2_{\mathbb{C}}$. Let $p = [p_0,p_1,p_2]$ be a point in C_1. If p also lies in different irreducible components of C, then we know by Proposition 7.11 that p is a singular point of C. Thus we may assume that $P_j(p_0,p_1,p_2)\neq 0$ for all j in $\{2,\ldots,r\}$.

By the Leibniz rule, we have

$$\partial_{x_i} P(p_0, p_1, p_2) = \sum_{j=1}^{r} \left(\partial_{x_i} P_j(p_0, p_1, p_2) \prod_{\ell \neq j} P_\ell(p_0, p_1, p_2) \right) = \partial_{x_i} P_1(p_0, p_1, p_2) \prod_{\ell=2}^{r} P_\ell(p_0, p_1, p_2)$$

for every i in $\{0, 1, 2\}$. Thus $\partial_{x_i} P(p_0, p_1, p_2) = 0$ for every i in $\{0, 1, 2\}$ if and only if $\partial_{x_i} P_1(p_0, p_1, p_2) = 0$ for every i in $\{0, 1, 2\}$. In other words, p is a singular point of C if and only if it is a singular point of the irreducible component C_1.

Exercise 11.6

This follows from a more general result: if P is a non-zero polynomial in $\mathbb{C}[x_1, \ldots, x_n]$, A is an invertible rank n matrix over $\mathbb{C}$, and b is a vector in $\mathbb{C}^n$, then the multiplicity of P at any point $a \in \mathbb{C}^n$ is equal to the multiplicity of the polynomial

$$Q(x_1, \ldots, x_n) = P((x_1, \ldots, x_n)A + b)$$

at the point $(a - b)A^{-1}$ (where we view a and b as row vectors).

The case where A is the identity matrix is trivial, since we obtain the Taylor expansion of Q around $a - b$ by taking the Taylor expansion of P around a and substituting $(x_1, \ldots, x_n)$ by $(x_1 + b_1, \ldots, x_n + b_n)$. Thus, we may assume that $a = b = \underline{0}$. Looking at the degrees of the terms in $P((x_1, \ldots, x_n)A)$ it is obvious that the multiplicity of Q at $\underline{0}$ is at least the multiplicity of P at $\underline{0}$. Repeating the argument with P and Q swapped, and replacing A by A^{-1}, we also obtain the converse inequality. Thus $\operatorname{mult}_{\underline{0}} P = \operatorname{mult}_{\underline{0}} Q$.

Exercise 11.7

Swapping the variables if necessary, we may assume that $i = n$. Since P is homogeneous, we can rescale the coordinates of $a = (a_0, \ldots, a_n)$ and assume that $a_n = 1$. Let ℓ be a non-negative integer. Then we need to show that all partial derivatives of P of orders up to ℓ vanish at a as soon as all partial derivatives of P of orders up to ℓ with respect to the variables $x_0, \ldots, x_{n-1}$ vanish at a.

We prove this by induction on ℓ. For $\ell = 0$ the statement is trivial (recall that the partial derivative of order 0 of P at a with respect to any variable x_i is equal to $P(a)$). So assume that $\ell > 0$ and that, for all homogeneous polynomials R in $\mathbb{C}[x_0, \ldots, x_n]$, all partial derivatives of R of orders up to $\ell - 1$ vanish at a as soon as all partial derivatives of R of order up to $\ell - 1$ with respect to the variables $x_0, \ldots, x_{n-1}$ vanish at a.

Suppose that all partial derivatives of P of orders up to ℓ with respect to the variables $x_0, \ldots, x_{n-1}$ vanish at a. We want to show that $\partial^\alpha P(a) = 0$ for all $\alpha = (\alpha_0, \ldots, \alpha_n)$ in $\mathbb{N}^{n+1}$ that satisfy $|\alpha| \leq \ell$. This is trivial when $\alpha_n = 0$, so we may assume that $\alpha_n > 0$. We denote by $e_0, \ldots, e_n$ the standard basis vectors in $\mathbb{N}^{n+1}$. If $\partial^{\alpha - e_n} P$ is constant, then $\partial^\alpha P = 0$. Otherwise, we can apply Euler's identity to the homogeneous polynomial $\partial^{\alpha - e_n} P$, which yields

$$\partial^\alpha P(a) = (\deg(P) - |\alpha| + 1)\partial^{\alpha - e_n} P(a) - \sum_{i=0}^{n-1} a_i \partial^{\alpha - e_n + e_i} P(a).$$

By induction on α_n, we may suppose that each term in the right hand side of this expression vanishes, so that $\partial^\alpha P(a) = 0$.

The last part of the exercise now follows easily: let P be a homogeneous polynomial with no repeated factors in $\mathbb{C}[x_0, x_1, x_2]$ whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. By symmetry, we may assume that p lies in the affine chart U_2; then $p = [a_0, a_1, 1]$ for some complex numbers a_0 and a_1. Hilbert's Nullstellensatz implies that $Q(x, y) = \lambda P(x, y, 1)$ for some $\lambda \in \mathbb{C}^*$, so that

$$\operatorname{mult}_p C = \operatorname{mult}_{(a_0, a_1, 1)} P(x_0, x_1, x_2) = \operatorname{mult}_{(a_0, a_1)} P(x_0, x_1, 1) = \operatorname{mult}_{(a_0, a_1)} Q.$$

Exercise 11.14

After a projective transformation, we may assume that $p \neq [0, 0, 1]$ and that $[0, 0, 1]$ does not lie in $C \cup D$ or on a line through two distinct points of $C \cap D$. Let P and Q be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$

with no repeated factors whose zero sets in $\mathbb{P}^2_{\mathbb{C}}$ are C and D, respectively. We choose a triple of homogeneous coordinates $[a_0, a_1, a_2]$ for the point p.

It is part of the definition of the intersection multiplicity that $\mathbf{I}(p, C, D) = 0$ when p is not an intersection point of C and D. Thus we may assume that p lies in $C \cap D$; then we need to show that $\mathbf{I}(p, C, D) > 0$. If p lies on a common component of C and D then $\mathbf{I}(p, C, D) = \infty > 0$ by definition, so that we may assume that this is not the case. Replacing C and D by the union of their non-common irreducible components, we reduce to the case where C and D have no common component. In this case, we need to show that $\mathbf{R}_{P,Q}(a_0, a_1) = 0$. This follows from the fact that $P(a_0, a_1, x)$ and $Q(a_0, a_1, x)$ have the common root a_2.

Exercise 11.19

We can simply copy the proof of Corollary 10.11, invoking the strong Bézout theorem and using the fact that $\mathbf{I}(p, C, D) \geq 2$ at every point p of $C \cap D$ that is a singular point of C, by Proposition 11.15.

Exercise 11.20

Let L be a line through two distinct singular points of C. At each of these two points, the intersection multiplicity of L and C is at least 2, by Proposition 11.15. Thus the sum of the intersection multiplicities exceeds $\deg(C) \cdot \deg(L) = 3$. By Bézout's theorem, this can only happen when C and L have a common component. This means that L is an irreducible component of C, so that C is reducible.

Exercise 11.21

After a projective transformation, we may assume that $p = [1, 0, 0]$. Let λ be a complex number and let L be the line through p defined by $x_2 = \lambda x_1$. We will show that, for all but finitely many values of λ, the intersection $C \cap L$ consists of precisely $d - m + 1$ distinct points. Let $P(x_0, x_1, x_2)$ be a homogeneous polynomial $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C. Since P has degree d and multiplicity m at $(1, 0, 0)$, it follows from Exercise 11.7 that it is of the form

$$P(x_0, x_1, x_2) = \sum_{\substack{i,j \geq 0 \\ m \leq i+j \leq d}} c_{ij} x_0^{d-i-j} x_1^i x_2^j$$

where the coefficients c_{ij} are complex numbers and $c_{ij} \neq 0$ for at least one couple $(i, j) \in \mathbb{N}^2$ such that $i + j = m$.

The intersection points of C and L different from p are all of the form $[x, 1, \lambda]$ with $x \in \mathbb{C}$. Thus we need to show that the polynomial $P(x, 1, \lambda)$ in $\mathbb{C}[x]$ has $d - m$ distinct roots for all but finitely many λ in $\mathbb{C}$. Setting $\ell = i + j$ we can write

$$P(x, 1, \lambda) = \sum_{m \leq \ell \leq d} (\sum_{j=1}^{\ell} c_{\ell-j,j} \lambda^j) x^{d-\ell}.$$

The leading term in this expression corresponds to $\ell = m$, and its coefficient is a non-zero polynomial in $\mathbb{C}[\lambda]$. Therefore, for all but finitely many λ, this coefficient is non-zero and the polynomial $P(x, 1, \lambda)$ has degree $d - m$. We may assume that $m < d$: otherwise, $P(x, 1, \lambda)$ is constant and the result is trivial.

Since C is irreducible, it has no common component with the zero set of the non-zero homogeneous polynomial $\partial_{x_0} P$ of degree $d - 1$. So there are only finitely many points in $\mathbb{P}^2_{\mathbb{C}}$ where P and $\partial_{x_0} P$ vanish simultaneously. This implies that for all but finitely many λ, the polynomial $P(x, 1, \lambda)$ and its derivative have no common root, and $P(x, 1, \lambda)$ has precisely $d - m$ distinct roots.

In particular, if we pick a smooth point p of C, then $m = 1$ and there exists a line through this point that intersects C in precisely d distinct points.

Note that the intersection multiplicity of C and L at p is at least m, by Proposition 11.15. By Bézout's theorem, the intersection multiplicities at the points of $C \cap L$ must add up to d. It follows that the intersection multiplicity at p is exactly m, and at the other points of $C \cap L$ it is equal to 1. The geometric meaning of this result is that all the points in $C \cap L$ besides p are smooth points of C, and L is not tangent to C at any of these points (this interpretation will be further justified in Section 12).

Exercise 12.2

Let C and D be projective plane curves over $\mathbb{C}$, and let p be a point of $\mathbb{P}^2_{\mathbb{C}}$. The first five axioms translate into the following properties.

1. Symmetry: $\mathbf{I}(p, C, D) = \mathbf{I}(p, D, C)$.
2. Detects intersection points: $\mathbf{I}(p, C, D) \neq 0$ if and only if p lies in $C \cap D$.
3. Detects common components: $\mathbf{I}(p, C, D) = \infty$ if and only if p lies on a common component of C and D.
4. Transversality: two distinct lines in $\mathbb{P}^2_{\mathbb{C}}$ have intersection multiplicity one at their unique point of intersection.
5. Additivity: if E is another projective plane curve over $\mathbb{C}$ that does not have a common component with D, then $\mathbf{I}(p, C, D \cup E) = \mathbf{I}(p, C, D) + \mathbf{I}(p, C, E)$.

We can use the additivity axiom to solve the second part of the exercise. Let E_1 and E_2 be the unions of the irreducible components of E that are, resp. are not, contained in D. Then D and E_2 have no common components, and the same holds for E_1 and E_2. Thus, the additivity axiom implies that $\mathbf{I}(p, C, D \cup E) = \mathbf{I}(p, C, D) + \mathbf{I}(p, C, E_2)$ and $\mathbf{I}(p, C, E) = \mathbf{I}(p, C, E_1) + \mathbf{I}(p, C, E_2)$, and by the detection of common components, all the terms in these equalities are finite. It follows that $\mathbf{I}(p, C, D \cup E) = \mathbf{I}(p, C, D) + \mathbf{I}(p, C, E) - \mathbf{I}(p, C, E_1)$. The term $\mathbf{I}(p, C, E_1)$ is always non-negative, and it is zero if and only if p does not lie in E_1, by the detection of intersection points.

Exercise 12.11

There are many different solutions to this exercise; one such solution is the smooth conic D defined by $x_0^2 - 2x_2^2 - x_1x_2 = 0$. The intersection $C \cap D$ consists of 3 points, namely, $p = [0, 1, 0]$, $q = [1, -1, 1]$ and $r = [1, 1, -1]$. By Bézout's theorem, the intersection multiplicities at these points add up to 4, so that C and D have intersection multiplicity 2 at one of these points, and intersection multiplicity 1 at the others. We can check this by explicit calculation: at p the projective tangent lines to C and D are both defined by $x_2 = 0$, so that C and D are tangent at p. At the points q and r, the tangent lines are distinct: at q they are given by $2x_0 + x_1 - x_2 = 0$ and $2x_0 - x_1 - 3x_2 = 0$, and at r they are given by $2x_0 - x_1 + x_2 = 0$ and $2x_0 + x_1 + 3x_2 = 0$. This confirms that C and D intersect transversally at q and r.

Exercise 13.5

A direct application of the chain rule shows that the Hessian matrix of Q is equal to $A \cdot \mathbf{H}_P((x_0, x_1, x_2)A) \cdot A^t$, so that the Hessian of Q equals $\det(A)^2 \mathcal{H}_P((x_0, x_1, x_2)A)$.

An alternative way to arrive at this formula is to use Taylor expansions. We can write the Taylor expansion of P around a point (p_0, p_1, p_2) as

$$P(x_0, x_1, x_2) = P(p) + DP(p) \begin{pmatrix} x_0 - p_0 \\ x_1 - p_1 \\ x_2 - p_2 \end{pmatrix} + \frac{1}{2}(x_0 - p_0, x_1 - p_1, x_2 - p_2)\mathbf{H}_P(p) \begin{pmatrix} x_0 - p_0 \\ x_1 - p_1 \\ x_2 - p_2 \end{pmatrix} + \text{higher order terms.}$$

The Hessian matrix $\mathbf{H}_P(p)$ is the unique symmetric matrix with this property, because any such matrix is completely determined by the second order partial derivatives of P at (p_0, p_1, p_2). Now set $(q_0, q_1, q_2) = (p_0, p_1, p_2)A^{-1}$. Then $Q(x_0, x_1, x_2) = P((x_0, x_1, x_2)A)$ is equal to

$$P(p) + DP(p)A^t \begin{pmatrix} x_0 - q_0 \\ x_1 - q_1 \\ x_2 - q_2 \end{pmatrix} + \frac{1}{2}(x_0 - q_0, x_1 - q_1, x_2 - q_2)A\mathbf{H}_P(p)A^t \begin{pmatrix} x_0 - p_0 \\ x_1 - p_1 \\ x_2 - p_2 \end{pmatrix} + \text{higher order terms.}$$

This is the Taylor expansion of Q around (q_0, q_1, q_2), so that we can read off that the Hessian matrix $\mathbf{H}_Q(q)$ is equal to the symmetric matrix $A \cdot \mathbf{H}_P(p) \cdot A^t$.

Exercise 13.9

We argue by contradiction. If C is not a line then it is the zero set of an irreducible homogeneous polynomial P of degree $d \geq 2$. We will prove that P does not divide its Hessian $\mathcal{H}_P$. Then it follows from Bézout's theorem that P and $\mathcal{H}_P$ have only finitely many common zeros in $\mathbb{P}^2_{\mathbb{C}}$, contradicting the assumption that C has infinitely many inflection points.

After a projective transformation, we may assume that $p = [0, 0, 1]$ is an inflection point of C, and that the tangent line L to C at p is defined by $x_0 = 0$. Set $Q(x, y) = P(x, y, 1)$. Isolating all the terms in Q that do not contain x, we can write

$$Q(x, y) = xR(x, y) + y^\mu S(y)$$

with $R(x, y) \in \mathbb{C}[x, y]$ and $S(y) \in \mathbb{C}[y]$ such that $S(0) \neq 0$. Since p is an inflection point and $L \cap U_2$ is defined by $x = 0$, we have $\mu = \mathbf{I}(p, C, L) \geq 3$. The fact that C is smooth at p then implies that $R(0, 0) \neq 0$. Now we use Lemma 13.6 to compute that

$$\mathcal{H}_P(0, y, 1) = (d-1)^2 \det \begin{pmatrix} 2\partial_x R(0, y) & \partial_y R(0, y) & R(0, y) \\ \partial_y R(0, y) & y^{\mu-2}T(y) & y^{\mu-1}U(y) \\ R(0, y) & y^{\mu-1}U(y) & \frac{d}{d-1} y^\mu S(y) \end{pmatrix}$$

where T and U are polynomials in $\mathbb{C}[y]$ such that $T(0) \neq 0$ and $U(0) \neq 0$. Developing the determinant from the last column, we see that it can be written as $-R(0, y)^2 y^{\mu-2}T(y)$ plus a term divisible by $y^{\mu-1}$. Since $R(0, 0)$ an $T(0)$ are both non-zero, it follows that the maximal power of y that divides $\mathcal{H}_P(0, y, 1)$ is $y^{\mu-2}$. Since $P(0, y, 1) = Q(0, y)$ is divisible by y^μ, we conclude that P does not divide $\mathcal{H}_P$.

Exercise 14.2

We construct an inverse for the map $P \mapsto \operatorname{div}(P)$. Let $D = m_1 C_1 + \ldots + m_r C_r$ be an effective divisor in $\mathbb{P}^2_{\mathbb{C}}$. For each curve C_i, we pick a homogeneous polynomial P_i in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C_i. This polynomial is unique up to a factor in $\mathbb{C}^*$ by the projective Nullstellensatz. Now we set $P_D = P_1^{m_1} \cdots P_r^{m_r}$. This is a non-zero homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ and it is again uniquely determined up to a factor in $\mathbb{C}^*$. It follows immediately from the definitions that the map $D \mapsto D_P$ is inverse to $P \mapsto \operatorname{div}(P)$.

Exercise 14.8

A pencil of degree d curves in $\mathbb{P}^2_{\mathbb{C}}$ corresponds to a two-dimensional subspace V of the vector space of homogeneous polynomials of degree d in $\mathbb{C}[x_0, x_1, x_2]$. By our assumptions, the polynomials P and Q belong to V, and they are linearly independent because they define different divisors in $\mathbb{P}^2_{\mathbb{C}}$. It follows that P and Q form a basis of V, and that our pencil consists of the divisors $\operatorname{div}(\lambda P + \mu Q)$ with $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Every common zero of P and Q in $\mathbb{P}^2_{\mathbb{C}}$ is also a zero of $\lambda P + \mu Q$ for every $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

Exercise 14.9

If P is constant then both sides of the desired equality are zero. Otherwise, we can write

$$P = P_1^{m_1} \cdots P_r^{m_r}$$

where $m_1, \ldots, m_r$ are positive integers and $P_1, \ldots, P_r$ are irreducible homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ such that P_i does not divide P_j when $i \neq j$. If we denote by C_i the zero set of P_i in $\mathbb{P}^2_{\mathbb{C}}$, then $\operatorname{div}(P) = m_1 C_1 + \ldots + m_r C_r$ and $\operatorname{mult}_p C_i = \operatorname{mult}_p P_i$ for every i. It follows that

$$\operatorname{mult}_p D = \sum_{i=1}^r m_i \operatorname{mult}_p P_i = \operatorname{mult}_p(\prod_{i=1}^r P_i^{m_i}) = \operatorname{mult}_p P.$$

Exercise 14.13

(1) The expected dimension of the linear system of conics through $p_1, \ldots, p_4$ is $5 - 4 = 1$, but we need to show that all these point conditions are independent. It suffices to show that for every non-negative integer $r \geq 3$, there exists a conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains $p_1, \ldots, p_r$ but not p_{r+1}. Adding $p_{r+2}, \ldots, p_4$ to $p_1, \ldots, p_r$ only makes the statement stronger, so it suffices to consider the case $r = 3$. Since the four points are not collinear, we can always find two lines in $\mathbb{P}^2_{\mathbb{C}}$ whose union is a conic that contains $p_1, \ldots, p_3$ but not p_4.

Assume that p_1 and p_2 and p_3 lie on a line L. Then every conic C in $\mathbb{P}^2_{\mathbb{C}}$ that contains $p_1, \ldots, p_4$ intersects L in at least 3 points, so that L must be contained in C by Bézout's theorem. Since C also contains p_4, and this point does not lie on L, every conic in our pencil is a union of two distinct lines, that is, reducible.

If no three points among $p_1, \ldots, p_4$ lie on a line, then there are precisely 3 degenerate conics through $p_1, \ldots, p_4$: three pairs of lines.

(2) If three of the points $p_1, \ldots, p_5$ lie on a line L, then the unique conic through these points is the union of L and the unique line through the remaining two points. So, assume that no three of the points are collinear. Then every conic C through $p_1, \ldots, p_5$ is non-degenerate, since otherwise, three points would lie on the same irreducible component of C and be collinear. We need to show that the condition that a conic contains p_5 is independent of the condition that it contains $p_1, \ldots, p_4$; then the linear system of conics through $p_1, \ldots, p_5$ has dimension 0, which means that it contains a unique element. It suffices to show that there exists a conic through $p_1, \ldots, p_4$ that does not contain p_5. Any degenerate conic through $p_1, \ldots, p_4$ has this property.

Exercise 15.2

Let $\mathcal{A}$ be the maximal atlas for X, the union of all atlases for X. Then every chart for X is contained in $\mathcal{A}$. It follows immediately from the definitions that every subset of $\mathcal{A}$ that covers X is an atlas for X, because all the charts in this collection are compatible.

Exercise 15.4

The "only if" implication is trivial, so it suffices to prove the "if" implication. Assume that $f \circ \psi_i^{-1}$ is holomorphic on $\psi_i(U_i)$ for every i in I, and let $\psi \colon U \to \mathbb{C}$ be a chart for X. We must prove that $f \circ \psi^{-1}$ is holomorphic on $\psi(U)$. For every $i \in I$, the restriction of $f \circ \psi^{-1}$ to $\psi(U \cap U_i)$ can be written as $(f \circ \psi_i^{-1}) \circ (\psi_i \circ \psi^{-1})$ which is a composition of holomorphic functions on opens in $\mathbb{C}$, and therefore itself holomorphic. Since the opens $\psi(U \cap U_i)$ cover $\psi(U)$, it follows that $f \circ \psi^{-1}$ is holomorphic.

Exercise 15.5

Let X be a Riemann surface and consider a function $f \colon X \to \mathbb{C}$. Let $\{U_i \mid i \in I\}$ be an open cover of X. For every chart (U, ψ) for X, the function

$$f \circ \psi^{-1} \colon \psi(U) \to \mathbb{C}$$

is holomorphic if and only if its restriction to $\psi(U \cap U_i)$ is holomorphic for every $i \in I$, because the sets $\psi(U \cap U_i)$ form an open cover of $\psi(U)$ and being holomorphic is a local property for functions on opens in $\mathbb{C}$. It follows that f is holomorphic if and only if, for every i in I, its restriction to U_i is holomorphic.

Exercise 15.9

The key point is that a holomorphic bijection between two opens in $\mathbb{C}$ is always biholomorphic. Let $\mathcal{A} = \{(U_i, \psi_i) \mid i \in I\}$ be an atlas for X. We need to show that

$$\psi \circ \psi_i^{-1} \colon \psi_i(U \cap U_i) \to \psi(U \cap U_i)$$

is biholomorphic for all i in I. Since $\psi \circ \psi_i^{-1}$ is bijective, it suffices to show that it is holomorphic. This follows immediately from the assumption that ψ is holomorphic.

Exercise 15.10

Let D be the open unit disk around 0 in $\mathbb{C}$ and pick your favourite homeomorphism $\psi\colon \mathbb{C} \to D$. Then $\{(\mathbb{C}, \mathrm{Id}_{\mathbb{C}})\}$ and $\{(\mathbb{C}, \psi)\}$ are two non-equivalent atlases for $\mathbb{C}$: with respect to the first atlas, all bounded holomorphic functions are constant, but with respect to the second atlas, there are plenty of non-constant bounded holomorphic functions (for instance, the function $z \circ \psi$ where z is the coordinate function on D).

Exercise 15.12

The "only if" implication follows directly from the definition of a morphism of Riemann surfaces, so that it suffices to prove the "if" implication. Assume that $h^{-1}(U)$ is open in Y and $\psi \circ h$ is holomorphic on $h^{-1}(U)$ for every chart (U, ψ) in the atlas for X. Since the sets $h^{-1}(U)$ form an open cover of Y, we can reduce to the case where $X = U$ and $Y = h^{-1}(U)$. Then $h = \psi^{-1} \circ (\psi \circ h)$ is a composite of continuous functions, and therefore continuous. If V is an open subset of U and $f\colon V \to \mathbb{C}$ is a holomorphic function, then we can similarly write $f \circ h\colon h^{-1}(V) \to \mathbb{C}$ as the composite of the holomorphic functions $\psi \circ h\colon h^{-1}(V) \to \psi(V)$ and $f \circ \psi^{-1}\colon \psi(V) \to \mathbb{C}$. Thus, $f \circ h$ is also holomorphic.

Exercise 15.13

It is enough to show that $h(V)$ is open in U for every chart (U, ψ) for X and every connected chart (V, φ) for Y such that $h(V) \subset U$, because these charts V form an open cover of Y. The map $\psi \circ h \circ \varphi^{-1}\colon \varphi(V) \to \mathbb{C}$ is a non-constant holomorphic map on a connected open subset of $\mathbb{C}$ (otherwise, h would be constant on the whole of Y by the identity theorem – see the proof of Proposition 16.8). By the open mapping theorem, the image of this map is open in $\mathbb{C}$. Since ψ is a homeomorphism onto $\psi(U)$, this implies that $h(V)$ is open in U.

If Y is compact, then $h(Y)$ is also compact, and therefore closed in X because X is Hausdorff. Thus, if we moreover assume that Y is non-empty and X is connected, then $h(Y) = X$ so that h is surjective. If X is not compact, then there are no surjective continuous maps from a compact space to X, so that any morphism from a connected compact Riemann surface to X must be constant.

Exercise 15.14

Let $h\colon Y \to X$ be a bijective morphism of Riemann surfaces. We need to show that $h^{-1}\colon X \to Y$ is again a morphism of Riemann surfaces. Let (V, φ) be a chart for Y. Then $h(V)$ is open in X by Exercise 15.13. By Exercise 15.12, it suffices to show that $\varphi \circ h^{-1}\colon h(V) \to \mathbb{C}$ is holomorphic.

Let (U, ψ) be a chart for $h(V)$. Then the map

$$\xi = \psi \circ h \circ \varphi^{-1}\colon (\varphi \circ h^{-1})(U) \to \psi(U)$$

is a holomorphic bijection, and therefore biholomorphic. We can write $\varphi \circ h^{-1}\colon U \to \mathbb{C}$ as the composite of the maps

$$\psi\colon U \to \psi(U), \qquad \xi^{-1}\colon \psi(U) \to (\varphi \circ h^{-1})(U).$$

Each of these maps is holomorphic, so that $\varphi \circ h^{-1}$ is holomorphic on each chart (U, ψ) for $h(V)$, and therefore on the whole of $h(V)$.

Exercise 15.19

(1) Since the inverse of Φ is again a projective transformation, we only need to show that Φ defines a morphism from C to $\Phi(C)$. It follows from Exercise 4.5 that Φ is continuous, because it is induced by a linear map $\mathbb{C}^3 \to \mathbb{C}^3$. For every affine chart U_i of $\mathbb{P}^2_{\mathbb{C}}$, we can cover $\Phi(C) \cap U_i$ by charts of the form $(U, x_j/x_i)$ where j is an element in $\{0, 1, 2\}$ distinct from i. Thus, by Exercise 15.12, it suffices to show that $x_j/x_i \circ \Phi$ is holomorphic on $\Phi^{-1}(U)$. This is a local property, that can be checked after restricting $x_j/x_i \circ \Phi$ to each of the affine charts on $\mathbb{P}^2_{\mathbb{C}}$. On each of these charts, we can write $x_j/x_i \circ \Phi$ as a quotient of two polynomial functions of degree one in the affine coordinates, with non-vanishing denominator. Such a quotient is holomorphic by the definition of the complex structure on C.

(2) Every line is projectively equivalent, and therefore isomorphic, to the line L in $\mathbb{P}^2_{\mathbb{C}}$ defined by $x_2 = 0$. By Exercise 15.14, it suffices to show that the bijective map

$$\theta\colon \mathbb{P}^1_{\mathbb{C}} \to L,\ [x_0, x_1] \mapsto [x_0, x_1, 0]$$

is a morphism of Riemann surfaces. By Theorem 15.11, the function x_1/x_0 is holomorphic on $L \cap U_0$, and x_0/x_1 is holomorphic on $L \cap U_1$. Since these functions are also bijective onto $\mathbb{C}$, Exercise 15.9 implies that $\{(U_0 \cap L, x_1/x_0), (U_1 \cap L, x_0/x_1)\}$ is an atlas for L. Thus θ is a morphism by Exercise 15.12.

(3) Every non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to the conic

$$C = \{[x_0, x_1, x_2] \in \mathbb{P}^2_{\mathbb{C}} \mid x_1^2 + x_0x_2 = 0\}.$$

We will construct an isomorphism $C \to \mathbb{P}^1_{\mathbb{C}}$ by projecting from $p = [0, 0, 1]$ onto the line L defined by $x_2 = 0$, which we can identify with $\mathbb{P}^1_{\mathbb{C}}$ by means of step (2). Consider the map

$$h\colon C \to L,\ [x_0, x_1, x_2] \mapsto \begin{cases} [x_0, x_1, 0] & \text{if } (x_0, x_1) \neq (0, 0); \\ [0, 1, 0] & \text{else.} \end{cases}$$

The map h is bijective: this follows from a direct computation and can also be deduced from Bézout's theorem, because for every point q of C, the image $h(q)$ is the unique intersection point of L with the line through p and q (resp. the tangent line to C at p if $p = q$). Therefore, we only need to show that h is a morphism of Riemann surfaces. By Exercise 15.12, it is enough to show that $V_0 = h^{-1}(U_0 \cap L)$ and $V_1 = h^{-1}(U_1 \cap L)$ are open in $\mathbb{C}$, and that $x_1/x_0 \circ h$ is holomorphic on V_0 and $x_0/x_1 \circ h$ is holomorphic on V_1. Openness of V_0 and V_1 is obvious, because the complement of each of these sets in C consists of a unique point. The function $x_1/x_0 \circ h$ on V_0 is again equal to x_1/x_0, which is holomorphic by the definition of the complex structure on C. It remains to prove that $x_0/x_1 \circ h$ is holomorphic on $V_1 = C \setminus \{[1, 0, 0]\}$. Using the equation $x_1^2 + x_0x_2 = 0$ for the conic C, we see that this function can be written as $-x_1/x_2$ on V_1, which is holomorphic.

Exercise 16.1

Changing the chart amounts to pre-composing $f \circ \psi^{-1}$ with a biholomorphic map. Therefore, it suffices to prove the following property: let $g\colon V \to U$ be a biholomorphic map between two connected opens in $\mathbb{C}$. Let v_0 be a point in V and set $u_0 = g(v_0)$. Then for every holomorphic function f on $U \setminus \{u_0\}$, we have $\mathrm{ord}_{u_0} f = \mathrm{ord}_{v_0}(f \circ g)$.

So let us prove this equality. We have that $\mathrm{ord}_{u_0}(f) = -\infty$ if and only if f cannot be written as a quotient of holomorphic functions on $U \setminus \{u_0\}$, and the analogous property holds for $f \circ g$ at v_0. But f is a quotient of holomorphic functions if and only if $f \circ g$ is a quotient of holomorphic functions. Thus, $\mathrm{ord}_{u_0}(f) = -\infty$ if and only if $\mathrm{ord}_{v_0}(f \circ g) = -\infty$.

Therefore, we may exclude this case an assume that f is meromorphic at u_0, so that $f \circ g$ is meromorphic at v_0. Since g is biholomorphic, the coefficient of $v - v_0$ in the Taylor expansion of $g(v)$ around v_0 is different from zero. Taking the Laurent expansion for $f(u)$ around u_0 and substituting u by the Taylor expansion of $g(v)$, we obtain a Laurent expansion for $f \circ g$ around v_0, and read off that $\mathrm{ord}_{u_0} f = \mathrm{ord}_{v_0}(f \circ g)$.

Exercise 16.2

It follows directly from the continuity of f that $f^{-1}(0)$ is closed in $X \setminus S$. To see that it is also closed in X, we remark that the limit of $|f|$ at each point of S is $+\infty$, because f has poles at all these points (reduce to the case where X is an open in $\mathbb{C}$ by taking charts). Therefore, S does not contain any limit points of $f^{-1}(0)$.

It remains to show that $f^{-1}(0)$ is discrete. Let V be the set of points x in $X \setminus S$ such that $f = 0$ on some open neighbourhood of x. This set is open by definition. If x is an accumulation point of $f^{-1}(0)$ and (U, ψ) is a chart of $X \setminus S$ with $x \in U$, then it follows from the identity theorem that $f \circ \psi^{-1} = 0$ on some open neighbourhood of $\psi(x)$ in $\psi(U) \subset \mathbb{C}$, so that $f = 0$ on some open neighbourhood of x in $X \setminus S$. Thus, every accumulation point of $f^{-1}(0)$ lies in V. In particular, every accumulation point of V is contained in V, so that V is both open and closed in X. Since X is connected and $V \neq X$ by our assumption that f is not identically zero, it follows that $V = \emptyset$, so that $f^{-1}(0)$ has no accumulation points.

Exercise 16.7

Let f be a meromorphic function on $\mathbb{P}^1_{\mathbb{C}}$. If f is constant then it is rational, so that we can exclude this case. Let S be the finite set consisting of zeros and poles of f in $\mathbb{C}$. Multiplying $f(z)$ with the rational function

$$\prod_{a \in S} (z-a)^{-\mathrm{ord}_a f}$$

we can reduce to the case where f is holomorphic and non-vanishing on $\mathbb{C}$. Let $\overline{f}\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be the morphism of Riemann surfaces induced by f. This morphism cannot be surjective, because it misses the values 0 and ∞ on $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. Thus, by Exercise 15.13, the morphism $\overline{f}$ is constant, so that f is rational.

Exercise 16.11

We have $f(\varphi(y_0)) - \psi(x_0) = 0$, and $f'(\varphi(y_0)) = 0$ if and only if $\mathrm{ord}_{\varphi(y_0)}(f - \psi(x_0)) \geq 2$, so that it suffices to prove the equality $v_h(y_0) = \mathrm{ord}_{\varphi(y_0)}(f - \psi(x_0))$.

Subtracting the constant $\psi(x_0)$ from the chart ψ, we may assume that $\psi(x_0) = 0$. By Exercise 16.1, the value of $\mathrm{ord}_{\varphi(y_0)} f$ does not depend on the chart (V, φ), so that we may also assume that $\varphi(y_0) = 0$ and that f is the function

$$\varphi(V) \to \mathbb{C},\ z \mapsto z^m$$

with $m = v_h(y_0)$. The order of this function at $\varphi(y_0) = 0$ is equal to m.

The set of zeros of f' on V is closed and discrete, by the identity theorem. Since this holds for all charts (U, ψ) and (V, φ) as in the statement of the exercise, and these charts (V, φ) cover Y, it follows that the set of ramification points of h is closed and discrete in Y (if S is a subset of Y then a point y of Y is a limit point, resp. accumulation point of S if and only if it is a limit point, resp. accumulation point of $S \cap V$ in any open V of Y that contains y).

Exercise 17.3

As explained in Example 15.8, we can construct $\mathbb{C}/\Lambda$ topologically by identifying opposite sides in a parallelogram to obtain a torus. A triangulation can be constructed by subdividing the parallelogram into two triangles by means of one of the diagonals. Taking the identifications of opposite sides into account, this triangulation has 2 triangles, 3 edges and 1 vertex. Thus, the Euler characteristic of of $\mathbb{C}/\Lambda$ equals $1 - 3 + 2 = 0$, and its genus equals 1.

Exercise 17.7

Let P be a homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ with no repeated factors whose zero set in $\mathbb{P}^2_{\mathbb{C}}$ is C.

(1) After a projective transformation, we may assume that $p = [0, 0, 1]$. For every point q of C, the tangent line to C at q passes through p if and only if $\partial_{x_2} P(q) = 0$. Since C is smooth and of degree $d \geq 2$, the variable x_2 occurs in P (otherwise we could factor P into a product of d linear homogeneous polynomials in $\mathbb{C}[x_0, x_1]$, and C would be a union of d distinct lines through p, and therefore singular at p). It follows that $\partial_{x_2} P$ is a non-zero homogeneous polynomial of degree $d-1$. By weak Bézout, there are at most $d(d-1)$ points q such that $P(q) = \partial_{x_2} P(q) = 0$. Note that P and $\partial_{x_2} P$ have no common irreducible factor because P is irreducible and $\partial_{x_2} P$ is a non-zero polynomial of strictly smaller degree.

(2) After a projective transformation, we may assume that $[0, 0, 1]$ does not lie on C, on the tangent line at an inflection point of C, or on a bitangent of C. Then it follows from the proof of the degree-genus formula that there are precisely $d(d-1)$ points on C whose tangent lines pass through $[0, 0, 1]$. These tangent lines are all distinct because no bitangents of C pass through $[0, 0, 1]$.

(3) By Bézout's theorem and Exercise 10.9, the degree of a projective plane curve D is equal to the maximal number of intersections points of D and a line in $\mathbb{P}^2_{\mathbb{C}}$ not contained in D. A line L in the dual projective plane $\mathcal{L}_1$ corresponds to the pencil $\mathcal{P}$ of lines through a point in $\mathbb{P}^2_{\mathbb{C}}$, and an intersection point of L with C^* corresponds to a line in $\mathcal{P}$ that is tangent to C. Now it follows from (1) and (2) that the degree of C^* is equal to $d(d-1)$.