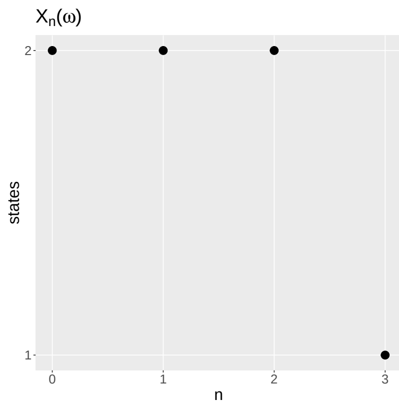
1. Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n\in\{0,1,2,3\}}$ on the state space $E = \{1,2\}$. We denote the (one-step) transition matrix by $\mathbf{P} = (p_{ij})_{i,j\in E}$ and the corresponding marginal distribution of X at time n by $\boldsymbol{\nu}^{(n)}$ for $n \in \mathbb{N}_0$.

   Let $\mathbf{P} = \begin{pmatrix} 0.25 & 0.75 \\ 0.2 & 0.8 \end{pmatrix}$, $\boldsymbol{\nu}^{(0)} = (0.5, 0.5)$.

   The following picture depicts a sample path of this Markov chain. What is the probability of such a realisation?



   Please record your answer in decimals (e.g. if your answer is $\frac{1}{2}$, write 0.5 and not 1/2).

Solution: We compute

$$\mathrm{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 1) = \nu_2^{(0)} p_{22} p_{22} p_{21} = 0.5 \times 0.8 \times 0.8 \times 0.2 = \frac{64}{1000} = 0.064.$$

2. Multiple answer: Which of the following matrices are stochastic?

$$a)\begin{pmatrix}\alpha & 1-\alpha\\ \beta & 1-\beta\end{pmatrix}\qquad \text{for } \alpha,\beta\in(0,1),$$
$$b)\begin{pmatrix}\alpha & 1-\alpha\\ \beta & 1-\beta\end{pmatrix}\qquad \text{for } \alpha,\beta\in\mathbb{R},$$
$$c)\begin{pmatrix}3 & -2\\ -2 & 3\end{pmatrix}$$
$$d)\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$
$$e)\begin{pmatrix}0.3 & 0.7\\ 0 & 0.9\end{pmatrix}$$
$$f)\begin{pmatrix}1-\sqrt{2} & \sqrt{2}\\ 0.5 & 0.5\end{pmatrix}$$

Solution: a)

3. Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Find $p_{23}(2)$, i.e. the 2-step transition probability of going from state 2 to state 3.

Please record your answer in decimals (e.g. if your answer is $\frac{1}{2}$, write 0.5 and not 1/2).

Solution: Using Chapman-Kolmogorov, we find that

$$\begin{aligned} p_{23}(2) &= \sum_{i=1}^{3} p_{2i}p_{i3} = p_{21}p_{13} + p_{22}p_{23} + p_{23}p_{33} \\ &= 0.1 \times 0.4 + 0.2 \times 0.7 + 0.7 \times 0.6 = 0.6. \end{aligned}$$

4. Multiple answer: Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.25 & 0 & 0.75 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

Which of the following statements are correct?

a) State 1 communicates with itself.
b) State 2 communicates with itself.
c) State 3 communicates with itself.
d) States 2 and 3 communicate with each other.
e) States 1 and 3 communicate with each other.
f) The transition matrix is not stochastic.
g) The Markov chain is irreducible.

Solution: a), b), c), d), e), g)

5. Multiple choice: Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on a countable state space E with n-step transition matrix $\mathbf{P}_n = (p_{ij}(n))_{i,j\in E}$ for $n \in \mathbb{N}$. Suppose $i \in E$ is a null recurrent state of the Markov chain. Which statement is correct?

   a) $\lim_{n\to\infty} p_{ki}(n) = 1$ for all $k \in E$.
   b) $\lim_{n\to\infty} p_{ik}(n) = 0$ for all $k \in E$.
   c) $f_{ii} = \infty$.
   d) None of the above

Solution: b) There are two scenarios: If k is either transient or null-recurrent, then the statement holds by a theorem from lectures for those k. If k is positive recurrent, then it needs to be in a different closed communicating class than i, hence $p_{ik}(n) = 0$ for all n and also $\lim_{n\to\infty} p_{ik}(n) = 0$.

6. Multiple choice: Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1,2,3,4,5\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.8 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0.8 & 0 \\ 0.25 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The communicating classes of this Markov chain are given by

a) $\{1,2\},\{3\},\{4\},\{5\}$
b) $\{1,2,3\},\{4\},\{5\}$
c) $\{1,2,3,4\},\emptyset$
d) $\{1,2,3,4\},\{5\}$
e) $\{1,2,3,4,5\}$
f) None of the above.

Solution: d)

7. Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1,2,3,4,5\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.8 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0.8 & 0 \\ 0.25 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

What is the period of state 3?

Solution: 1

8. Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.8 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.6 & 0.3 \end{pmatrix}$$

Compute the stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$. What is π_1?

Please record your answer in decimals (e.g. if your answer is $\frac{1}{2}$, write 0.5 and not 1/2).

Solution: Solving $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ leads to $\boldsymbol{\pi} = (1/10, 62/90, 19/90)$. Hence $\pi_1 = 0.1$.

Note that $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi} \Leftrightarrow \boldsymbol{\pi}(\mathbf{P}-I) = \mathbf{0} \Leftrightarrow (\boldsymbol{\pi}(\mathbf{P}-I))^\top = \mathbf{0}^\top$. We also note that $\pi_1+\pi_2+\pi_3 = 1$. This leads to the following system of equations:

$$A = \begin{pmatrix} -0.9 & 0.1 & 0.1 \\ 0.8 & -0.3 & 0.6 \\ 0.1 & 0.2 & -0.7 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = (0, 0, 0, 1)^\top,$$

where we need to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then we conclude that $\boldsymbol{\pi} = \mathbf{x}^\top = (1/10, 31/45, 19/90)$.

9. Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3, 4\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0.1 & 0 & 0.4 & 0.5 \end{pmatrix}$$

Find the return probability f_{44}.

Please record your answer in decimals (e.g. if your answer is $\frac{1}{2}$, write 0.5 and not 1/2).

Solution: We note that $f_{44}(1) = 0.5, f_{44}(2) = 0.4 \times 0.7, f_{44}(3) = 0.4 \times 0.3 \times 0.7, \ldots$ and $f_{44}(n) = 0.4 \times 0.3^{n-2} \times 0.7$ for $n \geq 2$. Then

$$f_{44} = \sum_{n=1}^{\infty} f_{44}(n) = 0.5 + 0.4 \times 0.7 \times \sum_{n=2}^{\infty} 0.3^{n-2} = 0.5 + 0.4 \times 0.7 \times \frac{1}{1-0.3} = 0.9.$$

10. Consider a time-homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3, 4\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0.1 & 0 & 0.4 & 0.5 \end{pmatrix}$$

How many null recurrent states does this Markov chain have?

Please record your answer as a number, i.e. if your answer is one state, write 1.

Solution: 0