IMPERIAL

**MATH70056**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

# Algebraic Geometry

**Date:** Wednesday, April 30, 2025

**Time:** Start time 14:00 – End time 16:30 (BST)

**Time Allowed**: 2.5 hours

**This paper has 5 Questions.**

***Please Answer All Questions in 1 Answer Booklet***

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO**

As in the lectures and in the lecture notes, we always assume that $k$ is an algebraically closed field: in particular, one can use the Nullstellensatz. Notice that $k$ can be of any characteristic. As in the lectures and in the lecture notes, for $I \subseteq k[X_1, ..., X_n]$, $V(I) \subseteq \mathbb{A}^n$ is the zero set of $I$, and, for $V \subseteq \mathbb{A}^n$, $I(V) \subseteq k[X_1, ..., X_n]$ is the ideal of polynomials vanishing on $V$, and $k[V]$ denotes the ring of regular functions of an affine variety. You can assume any result proven in the lecture notes and the problem sheets, unless if you are explicitly asked to reprove it.

1. Zariski topology:

(a) (i) Give, with a proof, all the closed subsets of $\mathbb{A}^1$. (2 marks)

(ii) Describe, with a proof, all the closed subsets of $V(XY) \subset \mathbb{A}^2$. (3 marks)

(b) (i) Define what it means for a ring to be Noetherian. (2 marks)

(ii) We recall that the ring of functions $k[V]$ of an affine variety $V$ is Noetherian. What is the consequence of the Noetherianity of $k[V]$ in terms of the closed subsets of $V$? (2 marks)

(iii) Prove that any affine variety is quasi-compact. Specifically, if a collection $(U_i)_{i \in I}$ of open subsets satisfies $V = \bigcup_{i \in I} U_i$, then there exists is a finite subset $i_0, ..., i_k \in I$ such that $V = U_{i_0} \cup ... \cup U_{i_k}$. (4 marks)

(c) (i) Prove that the variety $V(Y^2 - X^3) \subset \mathbb{A}^2$ is irreducible. (3 marks)

(ii) Give, with detailed arguments, the decomposition into irreducible components of the variety $V(XY, Z^3 - X - Y^2) \subset \mathbb{A}^3$. (4 marks)

(Total: 20 marks)

2. Regular and rational maps:

(a) Note that although regular maps are always continuous with respect to the Zariski topology, a continuous function in the Zariski topology is not necessarily regular.

Consider $V = V(Y^2 - X^3) \subset \mathbb{A}^2$, which is irreducible from 1)c)i), and:

$$\begin{aligned} \phi :& \mathbb{A}^1 \to V \\ & t \mapsto (t^2, t^3) \end{aligned}$$

(i) Prove that $\phi$ is birational. (5 marks)

(ii) Prove that $\phi$ is bijective, and that its inverse map $\psi : V \to \mathbb{A}^1$ is continuous for the Zariski topology. (4 marks)

(iii) Prove that the inverse map of $\phi$ is not regular, and hence $\phi$ is not an isomorphism. (4 marks)

(b) A group variety is a quasi-projective variety $V$ with a group structure given by a multiplication map $\mu : V \times V \to V$ and and inverse map $i : V \to V$ which are both regular.

(i) Prove that the variety $M_{n,m}(k)$ of $n \times m$ matrices with addition as the group operation is a group variety. (3 marks)

(ii) Prove that the variety $GL_n(k)$ of invertible $n \times n$ matrices with multiplication as the group operation is a group variety. (4 marks)

(Total: 20 marks)

3. Projective varieties:

(a) Consider the quasiprojective variety:

$$X := \{((x_1, ..., x_n), [y_1 : ... : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} | x_i y_j - x_j y_i = 0 \, \forall i, j\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \quad (1)$$

and consider the first projection $f : X \to \mathbb{A}^n$ (it is called the blow up of $\mathbb{A}^n$ at the point $0$).

(i) Show that the restriction $f' : f^{-1}(\mathbb{A}^n - \{0\}) \to \mathbb{A}^n - \{0\}$ of $f$ is an isomorphism, and that $f^{-1}(0) \simeq \mathbb{P}^{n-1}$. (4 marks)

(ii) Show that $X$ is the Zariski closure of $f^{-1}(\mathbb{A}^n - \{0\})$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Deduce that $X$ is irreducible, and $f$ is a birational equivalence. (4 marks)

(b) Consider $Gr(2, 4)$, the set of lines in $\mathbb{P}^3$ (or equivalently, the set of affine planes in $k^4$). We will introduce in this question the Plücker coordinates, giving a bijection between $Gr(2, 4)$ and a quadric in $\mathbb{P}^5$. We denote by $[z_{01} : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}]$ the homogeneous coordinates of a point in $\mathbb{P}^5$.

(i) Consider a line $L \subset \mathbb{P}^3$, and two distinct points $[x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3] \in L$. Prove that the vector:

$$\begin{aligned} &(x_i y_j - x_j y_i)_{0 \le i < j \le 3} \\ =&(x_0 y_1 - x_1 y_0, x_0 y_2 - x_2 y_0, x_0 y_3 - x_3 y_0, x_1 y_2 - x_2 y_1, x_1 y_3 - x_3 y_1, x_2 y_3 - x_3 y_2) \end{aligned} \quad (2)$$

is nonzero, and that the corresponding point of $\mathbb{P}^5$ only depends on $L$, and not on the choice of $x, y \in L$. (4 marks)

(ii) We have obtained from the previous question a map $\phi : Gr(2, 4) \to \mathbb{P}^5$. Show that $\phi^{-1}([1 : 0 : 0 : 0 : 0 : 0])$ contains only the line:

$$L := \{[s : t : 0 : 0] \subset \mathbb{P}^3 | [s : t] \in \mathbb{P}^1\} \subset \mathbb{P}^3 \quad (3)$$

and deduce that $\phi$ is injective. (4 marks)

(iii) We assume that the image of $\phi$ lies in the hypersurface $V \subset \mathbb{P}^5$ defined by the quadratic polynomial $z_{01} z_{23} - z_{02} z_{13} + z_{03} z_{12}$.
Show that $\phi : Gr(2, 4) \to V$ is surjective (hence bijective from $ii$)).
Hint: Without loss of generality, you can restrict to the open subset of $V$ were $z_{01} \neq 0$, and try to define a section of $\phi$:

$$\{[1 : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}] | z_{23} - z_{02} z_{13} + z_{03} z_{12} = 0\} \to Gr(2, 4) \quad (4)$$

(4 marks)

(Total: 20 marks)

4. Dimension theory: For $V \subset \mathbb{P}^n$ an irreducible projective variety, consider the secant variety of $V$, denoted by $\bar{S}(V)$, which is defined to be the Zariski closure of:

$$S(V) := \bigcup_{(p,q)\in V\times V|p\neq q} L_{pq}$$

where $L_{pq} \subset \mathbb{P}^n$ denote the line passing through $p$ and $q$.

(a) Consider the set:

$$\Sigma := \{(p,q,r) \in V \times V \times \mathbb{P}^n | p \neq q, r \in L_{pq}\}.$$

Show that $\Sigma$ is irreducible, and deduce that $S(V)$ (and then also $\bar{S}(V)$) is irreducible. You may use the fact that the product of irreducible varieties is irreducible without proof.

(4 marks)

(b) By applying the fiber dimension theorem to the projections $\pi_{1,2} : \Sigma \to V \times V$, prove that $\dim(\Sigma) = 2\dim(V) + 1$, and deduce that $\dim(\bar{S}(V)) = \dim(S(V)) \leq 2\dim(V) + 1$.

(4 marks)

(c) For $n, m \in \mathbb{Z}^{>0}$, define the Segre embedding $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, and give an interpretation of its image $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ in terms of matrices. (4 marks)

(d) Considering the irreducible projective variety $V = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$, describe $S(V)$ and $\bar{S}(V)$ in term of matrices (you can assume $n, m \geq 1$). (4 marks)

(e) Consider the case $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, and its image $V = \sigma_{2,2}(V)$. Compute the dimension of $\bar{S}(V)$. Compare this computation with the estimation of $b$), and give an interpretation of the difference in terms of the third projection $\pi_3 : \Sigma \to \mathbb{P}^n$. (4 marks)

(Total: 20 marks)

5. Mastery material: tangent space and singular points:

(a) Prove that the cuspidal cubic curve $V = V(X^3 - Y^2) \subset \mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$, using smoothness argument. (You may use the fact that $V$ is birational to $\mathbb{A}^1$ as shown in part $2)a)i)$).

(5 marks)

(b) If $V = V(f_1, ..., f_r) \subset \mathbb{A}^n$, is it true that $T_xV = \cap_i \ker(d(f_i)_x)$? Give a proof or a counterexample. (4 marks)

(c) Consider the affine subvarieties $V_1 = V((Y - X^2), Z)$ and $V_2 = V((Z - X^2), Y)$ of $\mathbb{A}^3$.

(i) Determine the tangent space at each point of $V_1$. (Note that the result is the same for $V_2$, up to exchanging $Y$ and $Z$). (4 marks)

(ii) Consider $V = V_1 \cup V_2$. We know that $I(V) = I(V_1) \cdot I(V_2)$ (This does not need to be proven). Compute the tangent space at $(0, 0, 0)$ of $V$ (try to avoid lengthy computations). (4 marks)

(iii) Prove that $V$ is not isomorphic to an affine subvariety of $\mathbb{A}^2$. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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# MATH70056

## Algebraic Geometry (Solutions)

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As in the lectures and in the lecture notes, we always assume that $k$ is an algebraically closed field: in particular, one can use the Nullstellensatz. Notice that it can be of any characteristic. As in the lectures and in the lecture notes, for $I \subseteq k[X_1, ..., X_n]$, $V(I) \subseteq \mathbb{A}^n$ is the zero set of $I$, and, for $V \subseteq \mathbb{A}^n$, $I(V) \subseteq k[X_1, ..., X_n]$ is the ideal of polynomials vanishing on $V$, and $k[V]$ denotes the ring of regular functions of an affine variety. You can assume any result proven in the lecture notes and the problem sheets, unless if you are explicitly asked to reprove it.

1. *Zariski topology:* seen ⇓

(a) (i) *Give, with a proof, all the closed subsets of $\mathbb{A}^1$.* 2, A

We prove that the closed subset of $\mathbb{A}^1$ are $\mathbb{A}^1$ itself, and finite set of points.

$\mathbb{A}^1 = V(0)$ is closed in $\mathbb{A}^1$, and, given $x_1, ..., x_n \in \mathbb{A}^1$, $\{x_1, ..., x_n\} = V((X - x_1)...(X - x_n))$ is closed in $\mathbb{A}^1$.

Suppose that $V = V(I) \subset \mathbb{A}^1$ contains infinitely many point. Then any $f \in I$ vanishes at infinitely many point, hence $f = 0$, as $k$ is algebraically closed. Then $I = 0$ and $V = \mathbb{A}^1$. seen ⇓

(ii) *Describe, with a proof, all the closed subsets of $V(XY) \subset \mathbb{A}^2$.* 3, A

We have the decomposition into irreducible components $V = V(X) \cup V(Y)$, where both components are isomorphic to $\mathbb{A}^1$, and intersects at $(0,0)$. Given $F \subset V(XY)$, we have that $F$ is closed if and only if $F \cap V(X)$ and $F \cap V(Y)$ are closed. The closed subsets of $V(XY)$ are then:

- $V(XY)$ itself.
- The union of $V(Y)$ and a finite set of points of $V(X)$.
- The union of $V(X)$ and a finite set of points of $V(Y)$.
- A finite set of points of $V(XY)$. seen ⇓

(b) (i) *Define what it means for a ring to be Noetherian.* 2, A

A ring $R$ is Noetherian if and only if each ascending chain of ideals is stationary.

(ii) *We recall that the ring of functions $k[V]$ of an affine variety $V$ is Noetherian. What is the consequence of the Noetherianity of $k[V]$ in terms of the closed subsets of $V$?* seen ⇓ 2, A

Every descending chain of closed subsets of $V$ is stationnary. seen ⇓

(iii) *Prove that any affine variety is quasi-compact. Specifically, if a collection $(U_i)_{i \in I}$ of open subsets satisfies $V = \bigcup_{i \in I} U_i$, then there exists is a finite subset $i_0, ..., i_k \in I$ such that $V = U_{i_0} \cup ... \cup U_{i_k}$.* 4, B

Taking the complements of the last property, every ascending chain of open subsets of an affine variety is stationnary. We take some nonemempty open set $U_{i_0}$, an we form sequence $(i_0, ..., i_k, ...)$, such that, at each step $U_{i_{k+1}}$ is not included in $\bigcup_{l=0}^{k} U_{i_l}$. This procedure gives an ascending chain of open subsets $\bigcup_{l=0}^{k} U_{i_l}$, hence must terminates for some value $k$, giving that $V = U_{i_0} \cup ... \cup U_{i_k}$. meth seen ⇓

(c) (i) *Prove that the variety $V(Y^2 - X^3) \subset \mathbb{A}^2$ is irreducible.* 3, A

It suffice to prove that the polynomial $Y^2 - X^3 \in k[X, Y]$ is irreducible. As a polynomial in $Y$ with coefficient in $k[X]$, it is monic of degree 2, hence a nontrivial factorization would be of the form $Y^2 - X^3 = (Y - P(X))(Y - Q(X))$. Identification of the coefficients give $Q(X) = -P(X)$ and $P(X)^2 = X^3 + X$, which is impossible for degree reason. Then $Y^2 - X^3 - X$ is irreducible $\boxed{\text{meth seen} \Downarrow}$

(ii) *Give, with detailed arguments, the decomposition into irreducible components of the variety $V(XY, Z^3 - X - Y^2) \subset \mathbb{A}^3$.* $\boxed{4, \text{B}}$

We have:

$$\begin{aligned} V(XY, Z^3 - X - Y^2) &= V(X, Z^3 - X - Y^2) \cup V(Y, Z^3 - X - Y^2) \\ &= V(X, Z^3 - Y^2) \cup V(Y, Z^3 - X) \end{aligned} \tag{1}$$

The first component is isomorphic to $V(Y^2 - X^3) \subset \mathbb{A}^2$ by:

$$\begin{aligned} V(Y^2 - X^3) &\to V(X, Z^3 - Y^2) \\ (x, y) &\mapsto (0, y, x) \end{aligned} \tag{2}$$

hence it is irreducible, and the second component is isomorphic to $\mathbb{A}^1$ by:

$$\begin{aligned} \mathbb{A}^1 &\to V(Y, Z^3 - X) \\ t &\mapsto (t^3, 0, t) \end{aligned} \tag{3}$$

hence it is irreducible. The two components are of dimension $1$ and meet only at $(0, 0, 0)$, hence this is the decomposition into irreducible components.

2. *Regular and rational maps:*

(a) *Note that although regular maps are always continuous with respect to the Zariski topology, a continuous function in the Zariski topology is not necessarily regular. Consider $V = V(Y^2 - X^3) \subset \mathbb{A}^2$, which is irreducible from 1)c)i), and:*

$$\begin{aligned} \phi :& \mathbb{A}^1 \to V \\ & t \mapsto (t^2, t^3) \end{aligned} \tag{4}$$

$\boxed{\text{sim. seen} \Downarrow}$

(i) *Prove that $\phi$ is birational.* $\boxed{5, \text{A}}$

Consider the rational map:

$$\begin{aligned} \rho :& V \dashrightarrow \mathbb{A}^1 \\ & (x, y) \mapsto y/x \end{aligned} \tag{5}$$

its image is $\mathbb{A}^1 - \{0\}$, which is dense in $\mathbb{A}^1$, i.e. it is dominant, and one have $\phi \circ \rho = Id$. One find that $\phi$ is surjective, hence dominant, and $\rho \circ \phi = Id$, , i.e. $\rho$ is the birational inverse of $\phi$.

sim. seen ⇓

(ii) *Prove that $\phi$ is bijective, and that its inverse map $\psi : V \to \mathbb{A}^1$ is continuous for the Zariski topology.* 4, B

The map:

$$\psi : V \to \mathbb{A}^1$$
$$(x,y) \mapsto \begin{cases} \frac{y}{x} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{6}$$

is giving an inverse to $\phi$, i.e. $\phi$ is bijective. The closed subset of $\mathbb{A}^1$ are $\mathbb{A}^1$ itself and finite set of points. $\psi^{-1}(\mathbb{A}^1) = V$, and the preimage of a finite subset of $\mathbb{A}^1$ is a finite subset of $V$, which is closed. Therefore the preimage of any closed subset by $\psi$ is a closed subset of $V$, hence $\psi$ is continuous for the Zariski topology.

sim. seen ⇓

(iii) *Prove that the inverse map of $\phi$ is not regular, and hence $\phi$ is not an isomorphism.* 4, A

Because $\mathbb{A}^1$ is affine, if $\psi$ were regular, it would be defined globally by a polynomial $g(X,Y)$ such that $g(T^2,T^3) = T$, which is impossible for degree reasons, hence $\psi$ is not regular.

(b) *A group variety is a quasi-projective variety $V$ with a group structure given by a multiplication map $\mu : V \times V \to V$ and and inverse map $i : V \to V$ which are both regular.*

unseen ⇓

(i) *Prove that the variety $M_{n,m}(k)$ of $n \times m$ matrices with addition as the group operation is a group variety.* 3, A

We can identify $M_{n,m}(k)$ with $\mathbb{A}^{nm}$ using the coordinates, hence it is a quasi-projective variety. The addition, and the opposite of a matrix is polynomial in the coordinates, hence they are given by regular maps.

sim. seen ⇓

(ii) *Prove that the variety $GL_n(k)$ of invertible $n \times n$ matrices with multiplication as the group operation is a group variety.* 4, B

We identify as before the set of $n \times n$ matrices with $\mathbb{A}^{n^2}$ using coefficients. The map $det : \mathbb{A}^{n^2} \to k$ is polynomial in the coefficients, hence its zero set is closed. $Gl_n(k)$ is the complementary of this closed subset, hence it is open in $\mathbb{A}^{n^2}$, then it is a quasi-projective variety. The multiplication of matrices is polynomial in the coefficients, hence it give a regular map $\mu : Gl_n(k) \times Gl_n(k) \to Gl_n(k)$. The inverse of a matrice can be expressed as a rational fraction:

$$M^{-1} = \frac{\tilde{M}}{det(M)} \tag{7}$$

where $\tilde{M}$ is the matrix obtained by taking the determinant of the $(n-1) \times (n-1)$ submatrices, hence is polynomial in the coefficients, and $\det(M) \neq 0$ on the open subset $Gl_n(k)$. It defines then a regular map $Gl_n(k) \to G_n(k)$.

3. *Projective varieties:*

(a) *Consider the quasiprojective variety:*

$$X := \{((x_1, ..., x_n), [y_1 : ... : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} | x_i y_j - x_j y_i = 0 \ \forall i, j\} \\ \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \tag{8}$$

*and consider the first projection $f : X \to \mathbb{A}^n$ (it is called the blow up of $\mathbb{A}^n$ at the point $0$).*

unseen ⇓

(i) *Show that the restriction $f' : f^{-1}(\mathbb{A}^n - \{0\}) \to \mathbb{A}^n - \{0\}$ of $f$ is an isomorphism, and that $f^{-1}(0) \simeq \mathbb{P}^{n-1}$*

4, A

Consider the regular map:

$$g : (\mathbb{A}^n - \{0\}) \to f^{-1}(\mathbb{A}^n - \{0\}) \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \\ (x_1, ..., x_n) \mapsto ((x_1, ..., x_n), [x_1 : ... : x_n]) \tag{9}$$

obviously, $f' \circ g = Id$, and:

$$g \circ f'((x_1, ..., x_n), [y_1 : ... : y_n]) = ((x_1, ..., x_n), [x_1 : ... : x_n]) \\ = ((x_1, ..., x_n), [y_1 : ... : y_n]) \tag{10}$$

the last equality holding because the equations of $X$ gives that $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are nonvanishing colinear vectors. Then $g$ is the inverse isomorphism of $f'$.

It follows directly from the equations defining $X$ that:

$$f^{-1}(0) = \{((0, ..., 0), [y_1 : ... : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \simeq \mathbb{P}^n \tag{11}$$

unseen ⇓

(ii) *Show that $X$ is the Zariski closure of $f^{-1}(\mathbb{A}^n - \{0\})$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Deduce that $X$ is irreducible, and $f$ is a birational equivalence.*

4, C

$X$ is defined by polynomial equations in $x, y$ which are homogeneous of degree $1$ in the $y$ variables, hence it is closed in $\mathbb{A}^n \times \mathbb{P}^{n-1}$, and then contains the closure of $f^{-1}(\mathbb{A}^n - \{0\})$. Conversely, we have to show that, for any $[y_1 : ... : y_n] \in \mathbb{P}^{n-1}$, $((0, ..., 0), [y_1 : ... : y_n])$ lies in the closure of $f^{-1}(V - \{0\})$. Consider the line:

$$\{((ty_1, ..., ty_n), [y_1 : ... : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} | t \in \mathbb{A}^1)\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \tag{12}$$

for $t \in \mathbb{A}^1 - \{0\}$, $(ty_1, ..., ty_n), [y_1 : ... : y_n]) \in f^{-1}(\mathbb{A}^n - \{0\})$. As $\mathbb{A}^1$ is the Zariski closure of $\mathbb{A}^1 - \{0\}$, we have that $((0, ..., 0), [y_1 : ... : y_n])$ is in the Zariski closure of $f^{-1}(\mathbb{A}^n - \{0\})$.

By $a$), $f^{-1}(\mathbb{A}^n - \{0\}) \simeq \mathbb{A}^n - \{0\}$ is irreducible, then its Zariski closure $X$ is irreducible. By $a$), $f$ restricts to an isomorphism from the (necessarily dense) open subset $f^{-1}(\mathbb{A}^n - \{0\})$ to the dense open subset $\mathbb{A}^n - \{0\}$, hence it is a birational equivalence.

(b) *Consider $Gr(2, 4)$, the set of lines in $\mathbb{P}^3$ (or equivalently, the set of affine planes in $k^4$). We will introduce in this question the Plücker coordinates, giving a bijection between $Gr(2, 4)$ and a quadric in $\mathbb{P}^5$. We denote by $[z_{01} : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}]$ the homogeneous coordinates of a point in $\mathbb{P}^5$.*

unseen $\Downarrow$

(i) *Consider a line $L \subset \mathbb{P}^3$, and two distinct points $[x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3] \in L$. Prove that the vector:*

$$\begin{aligned} &(x_iy_j - x_jy_i)_{0\le i<j\le 3} \\ =&(x_0y_1 - x_1y_0, x_0y_2 - x_2y_0, x_0y_3 - x_3y_0, x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2) \end{aligned} \tag{13}$$

*is nonzero, and that the corresponding point of $\mathbb{P}^5$ only depends on $L$, and not on the choice of $x, y \in L$.* 4, C

If the above vector were zero, it would gives that $(x_0, x_1, x_2, x_3)$ and $(y_0, y_1, y_2, y_3)$ would be colinear, i.e. $x = y$, a contradiction. Consider an other choice of two distinct points $[x'_0 : x'_1 : x'_2 : x'_3], [y'_0 : y'_1 : y'_2 : y'_3] \in L$. Then, the vectors $(x_i)$ and $(y_i)$ spans the same plane that $(x'_i)$ and $(y'_i)$, which means that there is an invertible $2 \times 2$ matrix $M$ such that:

$$\begin{pmatrix} x'_0 & y'_0 \\ x'_1 & y'_1 \\ x'_2 & y'_2 \\ x'_3 & y'_3 \end{pmatrix} = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \cdot \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \tag{14}$$

which gives:

$$\det \begin{pmatrix} x'_i & y'_i \\ x'_j & y'_j \end{pmatrix} = \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} \cdot \det \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \tag{15}$$

with $\det \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \neq 0$ which is independents of $i, j$. Hence the two nonzero vectors $(x_iy_j - x_jy_i)_{0\le i<j\le 3}$ and $(x'_iy'_j - x'_jy'_i)_{0\le i<j\le 3}$ are colinear, i.e. define the same point of $\mathbb{P}^5$.

unseen $\Downarrow$

(ii) *We have obtained from the previous question a map $\phi : Gr(2, 4) \to \mathbb{P}^5$. Show that $\phi^{-1}([1 : 0 : 0 : 0 : 0 : 0])$ contains only the line:*

$$L := \{[s : t : 0 : 0] \subset \mathbb{P}^3 | [s : t] \in \mathbb{P}^1\} \subset \mathbb{P}^3 \tag{16}$$

*and deduce that $\phi$ is injective.* 4, D

Considering the two distinct points $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0] \in L$, we obtain by direct computation that $\phi(L) = [1 : 0 : 0 : 0 : 0 : 0]$. Consider $L'$ such that $\phi(L') = [1 : 0 : 0 : 0 : 0 : 0]$, and two distinct points $[x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3] \in L'$. As $x_0y_1 - x_1y_0 = 1 \neq 0$, we can find a linear combination $(x'_i), (y'_i)$ of the vectors $(x_i), (y_i)$ such that $x'_0 = 1, x'_1 = 0, y'_0 = 0, y'_1 = 1$. We have then two distinct points $[1 : 0 : x'_2 : x'_3], [0 : 1 : y'_2 : y'_3] \in L'$. Now, the equations gives $1 \cdot y'_2 = x'_3 \cdot 0$, i.e. $y'_2 = 0$, and similarly $y'_3 = x'_2 = x'_3 = 0$. Then $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0] \in L'$, i.e. $L' = L$.

We have proven that $\phi$ is injective at $L$. Given any line $L' \subset \mathbb{P}^3$, by a linear change of coordinate, we can ensure $L' = L$, i.e. $\phi$ is injective.

unseen $\Downarrow$

(iii) *We assume that the image of $\phi$ lies in the hypersurface $V \subset \mathbb{P}^5$ defined by the quadratic polynomial $z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12}$.*

*Show that $\phi : Gr(2,4) \to V$ is surjective (hence bijective from $ii$)).*
*Hint: Without loss of generality, you can restrict to the open subset of $V$ were $z_{01} \neq 0$, and try to define a section of $\phi$:*

$$\{[1 : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}] | z_{23} - z_{02}z_{13} + z_{03}z_{12} = 0\} \to Gr(2,4) \quad (17)$$

4, D

Consider $[1 : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}] \in V$. Consider the line $L'$ passing through the points $[1 : 0 : -z_{12} : -z_{13}], [0 : 1 : z_{02} : z_{03}] \in \mathbb{P}^3$. The equation of $V$ gives:

$$(-z_{12}) \cdot z_{03} - (-z_{13}) \cdot z_{02} = z_{23} \quad (18)$$

which give $\phi(L) = [1 : z_{02} : z_{03} : z_{12} : z_{13} : z_{23}]$. We obtain that $\phi$ is surjective over the open subset of $V$ on which $z_{01} \neq 0$. A similar reasoning, permuting the arrows, would give that $\phi$ is surjective over the open subset of $V$ on which $z_{ij} \neq 0$ for $0 \leq i < j \leq 3$, i.e. $\phi$ is surjective.

4. *Dimension theory:*

*For $V \subset \mathbb{P}^n$ an irreducible projective variety, consider the secant variety of $V$, denoted by $\bar{S}(V)$, which is defined to be the Zariski closure of:*

$$S(V) := \bigcup_{(p,q) \in V \times V | p \neq q} L_{pq} \quad (19)$$

*where $L_{pq} \subset \mathbb{P}^n$ denote the line passing through $p$ and $q$.*

unseen ⇓

(a) *Consider the set:*

$$\Sigma := \{(p,q,r) \in V \times V \times \mathbb{P}^n | p \neq q, r \in L_{pq}\}.$$

*Show that $\Sigma$ is irreducible, and deduce that $S(V)$ (and then also $\bar{S}(V)$) is irreducible. You may use the fact that the product of irreducible varieties is irreducible without proof.*

4, B

Consider the open subset $U \subset V \times V$ defined by the condition $p \neq q$: as $V$ is irreducible, $V \times V$ is irreducible, and then $U$ is irreducible. Consider the regular map:

$$\begin{aligned} \phi : & U \times \mathbb{P}^1 \to V \times V \times \mathbb{P}^n \\ & ([p_0 : ... : p_n], [q_0 : ... : q_n], [s : t]) \mapsto ([p_0 : ... : p_n], [q_0 : ... : q_n], [s : t]) \end{aligned} \quad (20)$$

As $U$ is irreducible, then the image of $\phi$, which is $\Sigma$, is irreducible. As $S(V)$ is the image of the regular projection $\pi_3 : \Sigma \to \mathbb{P}^n$, and $\Sigma$ is irreducible, $S(V)$ is irreducible, and its Zariski closure $\bar{S}(V)$ is irreducible too.

seen ⇓

(b) *By applying the fiber dimension theorem to the projections $\pi_{1,2} : \Sigma \to V \times V$, prove that $\dim(\Sigma) = 2\dim(V) + 1$, and deduce that $\dim(\bar{S}(V)) = \dim(S(V)) \leq 2\dim(V) + 1$.*

4, C

By the above, $\Sigma$ and $V \times V$ are irreducible. For any $(p, q) \in V \times V$ such that $p \neq q$, $(p, q, p) \in \Sigma$, i.e. $(p, q) \in Im(\pi_{1,2})$, which means that $\pi_{1,2}$ is dominant. The fiber of $\pi_{1,2}$ over the dense open subset $p \neq q$ is $L_{pq} \simeq \mathbb{P}^1$, which is irreducible of dimension 1. By the second part of the fiber dimension theorem, over a dense open subset of $V \times V$, each irreducible component of the fibers must have dimension $\dim(\Sigma) - \dim(V \times V)$, hence necessarily $\dim(\Sigma) - \dim(V \times V) = 1$. Then:

$$\dim(\Sigma) = \dim(V \times V) + 1 = 2\dim(V) + 1 \tag{21}$$

But $S$ is the image of $\Sigma$ under the regular map $\pi_3$, then $\dim(S) \leq 2\dim(V) + 1$, and the same results holds for $\bar{S}$, as $\dim(\bar{S}(V)) = \dim(S(V))$.

seen ⇓

(c) *For $n, m \in \mathbb{Z}^{>0}$, define the Segre embedding $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, and give an interpretation of its image $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ in terms of matrices.*

4, A

For $N = (n+1)(m+1) - 1$, consider the homogeneous coordinates $(z_{ij})_{0\leq i\leq n, 0\leq j\neq m}$. The Segre embedding is defined by:

$$\begin{aligned} \sigma_{n,m} :& \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N \\ & ([x_0 : ... : x_n], [y_0 : ... : y_n]) \mapsto [(x_i y_j)_{0\leq i\leq n, 0\leq j\neq m}] \end{aligned} \tag{22}$$

interpreting the $(z_{ij})_{0\leq i\leq n, 0\leq j\neq m}$ in terms of $(n+1)(m+1)$ matrices, the image $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ consist of matrices which are the product of a nonzero line and a nonzero column, i.e. matrices of rank 1.

unseen ⇓

(d) *Considering the irreducible projective variety $V = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$, describe $S(V)$ and $\bar{S}(V)$ in term of matrices (you can assume $n, m \geq 1$).*

4, D

Given $p = [(z_{ij})_{0\leq i\leq n, 0\leq j\neq m}]$, $q = [(z'_{ij})_{0\leq i\leq n, 0\leq j\neq m}]$, the elements of $L_{pq}$ are the $[(sz_{ij} + tz'_{ij})_{0\leq i\leq n, 0\leq j\neq m}]$ for $(s, t) \neq (0, 0)$. As the columns $(z_{ij})_{0\leq i\leq n}$ are colinear, and the columns $(z'_{ij})_{0\leq i\leq n}$ are colinear, the columns $(sz_{ij} + tz'_{ij})_{0\leq i\leq n}$ forms a vector space of dimension $\leq 2$. Then $S(V)$ is contained in the closed subset of $\mathbb{P}^N$ of matrices of rank $\leq 2$.

Conversely, given any nonzero matrix $(z_{ij})_{0\leq i\leq n, 0\leq j\neq m}$ of rank 2, consider two columns $(z_{ij_1})_{0\leq i\leq n}$, $(z_{ij_2})_{0\leq i\leq n}$ which generates the image, and write any other column as $(\lambda_j z_{ij_1} + \mu_j)_{0\leq i\leq n}$, with $\lambda_{j_1} = 1, \mu_{j_1} = 0$ and $\lambda_{j_1} = 2, \mu_{j_2} = 1$. Then, defining $p := [(\lambda_j z_{ij_1})_{0\leq i\leq n, 0\leq j\neq m}$, $q := [(\mu_j z_{ij_2})_{0\leq i\leq n, 0\leq j\neq m}$, we have $p, q \in V$, $p \neq q$, and $[(z_{ij})_{0\leq i\leq n, 0\leq j\neq m}] \in L_{pq}$. Then the closed subset of $\mathbb{P}^N$ of matrices of rank $\leq 2$ is contained in $S(V)$.

Then $S(V)$ is the closed subset of $\mathbb{P}^N$ of matrices of rank $\leq 2$, and $\bar{S}(V) = S(V)$.

unseen ⇓

(e) *Consider the case $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, and its image $V = \sigma_{2,2}(V)$. Compute the dimension of $\bar{S}(V)$. Compare this computation with the estimation of $b$), and give an interpretation of the difference in terms of the third projection $\pi_3 : \Sigma \to \mathbb{P}^n$.*

4, D

Bu $iii$), $\bar{S}(V)$ is the set of $3 \times 3$ matrices of rank $\leq 2$. It is the vanishing locus of the determinant, a nonvanishing homogeneous polynomial of degree 3. Then $\bar{S}(V)$

is a hypersurface in $\mathbb{P}^8$, which has dimension 7. The estimation of $ii)$ gives:

$$\begin{aligned}\dim(\bar{S}(V)) &\leq 2\dim(\sigma_{2,2}(V)) + 1 \\ &= 2\dim(\mathbb{P}^2 \times \mathbb{P}^2) + 1 = 2 \times 2\dim(\mathbb{P}^2) + 1 = 9\end{aligned} \tag{23}$$

then this estimation is not sharp (one says that the secant variety is defective). This is due to the fact that the map $\pi_3 : \Sigma \to \mathbb{P}^n$ is not injective: indeed, in the above section, we had to do arbitrary choices to define $p, q$.

5. *Mastery material: tangent space and singular points*

seen $\Downarrow$

(a) *Prove that the cuspidal cubic curve $V = V(X^3 - Y^2) \subset \mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$, using smoothness argument.(You may use the fact that $V$ is birational to $\mathbb{A}^1$ as shown in part $2)a)i)$).* 5, M

As $V$ is birational to $\mathbb{A}^1$, it has dimension 1. If it were isomorphic to $\mathbb{A}^1$, each point would be regular, i.e would have a tangent space of dimension 1. But:

$$\frac{\partial(X^3 - Y^2)}{\partial X} = 3X^3 \quad \frac{\partial(X^3 - Y^2)}{\partial Y} = 2X^3 \tag{24}$$

hence:

$$\frac{\partial(X^3 - Y^2)}{\partial X}|_{(0,0)} = 0 \quad \frac{\partial(X^3 - Y^2)}{\partial Y}|_{(0,0)} = 0 \tag{25}$$

which gives $T_{(0,0)}V = k^2$ has dimension 2, a contradiction.

seen $\Downarrow$

(b) *If $V = V(f_1, ..., f_r) \subset \mathbb{A}^n$, is it true that $T_xV = \cap_i \ker(d(f_i)_x)$? Give a proof or a counterexample.* 4, M

This is false. Consider $V = V(X^2) \subset \mathbb{A}^1$. Then $V = \{0\}$, $I(V) = (X)$, then $dX_0 \neq 0$, which gives $T_0V = \ker(dX_0) = \{0\} \subset k$. But $d(X^2)_0 = 0$, which gives $\ker(d(X^2)_0) = k$.

(c) *Consider the affine subvarieties $V_1 = V((Y - X^2), Z)$ and $V_2 = V((Z - X^2), Y)$ of $\mathbb{A}^3$.*

meth seen ⇓

(i) *Determine the tangent space at each point of $V_1$. (Note that the result is the same for $V_2$, up to exchanging $Y$ and $Z$).* 4, M

The points of $V_1$ are of the form $(x, x^2, 0)$. We have:

$$\begin{aligned} d(Y-X^2)_{(x,x^2,0)}(a,b,c) &= \frac{\partial(Y-X^2)}{\partial X}|_{(x,x^2,0)}a + \frac{\partial(Y-X^2)}{\partial Y}|_{(x,x^2,0)}b + \frac{\partial(Y-X^2)}{\partial Z}|_{(x,x^2,0)}c \\ &= -2xa + b \\ d(Z)_{(x,x^2,0)}(a,b,c) &= \frac{\partial Z}{\partial X}|_{(x,x^2,0)}a + \frac{\partial Z}{\partial Y}|_{(x,x^2,0)}b + \frac{\partial Z}{\partial Z}|_{(x,x^2,0)}c \\ &= c \end{aligned} \tag{26}$$

Hence:

$$\begin{aligned} T_{(x,x^2,0)}V_1 = (\ker d(Y-X^2)_{(x,x^2,0)}) \cap (d(Z)_{(x,x^2,0)}) \\ = k(1, 2x, 0) \end{aligned} \tag{27}$$

In particular, $T_{(0,0,0)}V_1 = k(1,0,0)$ is the $X$ axis.

meth seen ⇓

(ii) *Consider $V = V_1 \cup V_2$. We know that $I(V) = I(V_1) \cdot I(V_2)$ (This does not need to be proven). Compute the tangent space at $(0,0,0)$ of $V$ (try to avoid lengthy computations).* 4, M

We have from the assumption:

$$I(V) = ((Y-X^2)(Z-Y^2), (Y-X^2)Y, Z(Z-X^2), ZY) \tag{28}$$

We have that $T_{(0,0,0)}V \subseteq k^3$ is the intersection of the kernel of the differentials of each of these four polynomials. But each of these polynomials have no term of total degree 1, hence each of their partial derivatives, as a polynomial in $k[X,Y,Z]$ has no constant term, and then vanishes when evaluated at $(0,0,0)$, which means that each of their differential vanishes at $(0,0,0)$. Hence $T_{(0,0,0)}V = k^3$.

unseen ⇓

(iii) *Prove that $V$ is not isomorphic to an affine subvariety of $\mathbb{A}^2$.* 3, M

For $x \in V$, $T_xV$ as a vector space is independent of the embedding of $V$ in an affine space. For $x = (0,0,0)$, $T_xV$ has dimension 3. But if $V$ were isomorphic to an affine subvariety of $\mathbb{A}^2$, $T_xV$ would be included in $k^2$, hence have dimension $\leq 2$, giving a contradiction.

**Review of mark distribution:**

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

**MATH70056 Algebraic Geometry Markers Comments**

| | |
|---|---|
| Question 1 | The question 1, really close to the lectures and the problem sheets, was well performed by the students in general. The question b)iii) about the proof of quasicompactness was not well done in general, even if it was in the lectures: indeed, taking an arbitrary sequence of open subsets is not sufficient. In the question c)i), a few misreading's and computational errors appear. |
| Question 2 | The answer to question 2 was pretty unequal. Concerning question 2)a), it was well done in general, but a few students continue to make mistakes, such as considering that a regular bijection must be an isomorphism, or that a continuous map is closed. In question 2)b)i), a number of the students have neglected the need to invert the determinant, but a lot of other students were remembering pretty well Cramer's formula. |
| Question 3 | Question 3a)i) was pretty well done, even if a lot of students made pretty intricate arguments to show that f was an isomorphism, albeit an explicit inverse was easy to write down. The proof of irreducibility in question 3)a)ii) was pretty subtle, and was well done by a few students, but a lot of other students were less cautious (eg, assuming that a continuous function is always closed). Question 3)b) was harder, and some students had some really good ideas about this: every try, even partial, was rewarded. |
| Question 4 | The question was pretty advanced, with the use of the fiber dimension theorem, and some linear algebra, and was pretty well done by a lot of students. Unfortunately, very few students have seen in question 4)d) that one obtains matrices of rank less than or equal to 2, and then few were able to do the numerical application of 4)e). |
| Question 5 | This question was a lot of routine computations about tangent spaces and was pretty well done by most of the students. Unfortunately, in question 5)b)ii), a lot of students were misled, using a wrong formula to compute the tangent space of a variety in terms of those of its irreducible components, and have not seen that all the differentials were vanishing at (0,0,0), so they were unable to do 5)b)iii). |