M40007: Introduction to Applied Mathematics

Imperial College London

1 The shift matrix S

For a *general* graph, and assuming one naturally assigns the k-th column of the Laplacian matrix $\mathbf{K}$ to node $\boxed{\text{k}}$, then any renumbering of the nodes must usually be expected to change the corresponding Laplacian matrix. But consider the two choices of node labels for the graph shown in Figure 1. Let $\mathbf{K}$ be the Laplacian matrix of the graph on the left, and let $\hat{\mathbf{K}}$ be the Laplacian of the graph on the right.

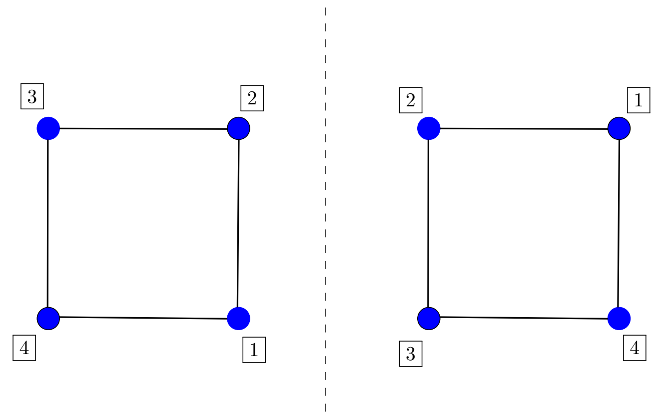

Figure 1: A graph with two different choices of node labels.

It is clear from inspection of these particular graphs that we should expect the Laplacian matrices of each to be identical: $\hat{\mathbf{K}} = \mathbf{K}$. This is because if the graph on the left with its assigned node labels is simply rotated anticlockwise by 90^{o} it is indistinguishable from the graph, and its assigned node labels, on the right. Indeed, it is straightforward to deduce that the Laplacian matrix for each graph is

$$\mathbf{K} = \begin{array}{c} \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array} = \hat{\mathbf{K}}. \tag{1}$$

Consider now the matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Suppose the graphs in Figure 1 are viewed as electric circuits with the vector of voltages in the graph on the left given by the 4-dimensional vector $\mathbf{x}$ and those in the circuit on the right by $\hat{\mathbf{x}}$. Similarly, let $\mathbf{f}$ and $\hat{\mathbf{f}}$ denote the vector of current

divergences in each circuit. As usual, the voltages and current divergences are related by the relations
$$\mathbf{K}\mathbf{x} = \mathbf{f}, \qquad \hat{\mathbf{K}}\hat{\mathbf{x}} = \hat{\mathbf{f}}. \tag{3}$$

Suppose also that the voltages happen to be related by

$$\hat{\mathbf{x}} = \mathbf{S}\mathbf{x} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix}. \tag{4}$$

Such an assignment of voltages in the circuit on the right now makes it indistinguishable from the circuit on the left as far as the voltages and the flow of currents is concerned. Because the currents in the two circuits will be identical, we therefore expect $\mathbf{f}$ and $\hat{\mathbf{f}}$ to satisfy
$$\hat{\mathbf{f}} = \mathbf{S}\mathbf{f} = \begin{pmatrix} f_2 \\ f_3 \\ f_4 \\ f_1 \end{pmatrix}. \tag{5}$$

If (4) holds then, on use of (1),

$$\hat{\mathbf{K}}\hat{\mathbf{x}} = \mathbf{K}\hat{\mathbf{x}} = \mathbf{K}\mathbf{S}\mathbf{x} = \hat{\mathbf{f}}. \tag{6}$$

But since we also expect (5) to hold then, together with (3),

$$\hat{\mathbf{f}} = \mathbf{S}\mathbf{f} = \mathbf{S}\mathbf{K}\mathbf{x}. \tag{7}$$

In combination (6) and (7) imply that

$$\hat{\mathbf{f}} = \mathbf{K}\mathbf{S}\mathbf{x} = \mathbf{S}\mathbf{K}\mathbf{x}, \qquad \text{or} \qquad (\mathbf{K}\mathbf{S} - \mathbf{S}\mathbf{K})\mathbf{x} = 0. \tag{8}$$

But such a relation must hold for any set of voltages $\mathbf{x}$ set up in the circuit on the left. The conclusion is that
$$\mathbf{K}\mathbf{S} = \mathbf{S}\mathbf{K}, \tag{9}$$
which says that $\mathbf{K}$ and $\mathbf{S}$ are *commuting matrices*. Recall that two arbitrarily chosen square matrices will not, in general, commute.

It is an easy matter to confirm by direct multiplication, using (1) and (2), that (9) holds.

The commutativity of the two matrices $\mathbf{K}$ and $\mathbf{S}$ encodes a symmetry of the circuit. The matrix $\mathbf{S}$ given in (2) is interesting. Rewriting (4) as

$$\mathbf{S}\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix} \tag{10}$$

shows that, on multiplying $\mathbf{x}$ by $\mathbf{S}$, a cyclic shift has taken place: each element is shifted *up* a position in the vector, with the first element rotating round to take the final position in the vector. For this reason, we refer to $\mathbf{S}$ as a *shift matrix*. Repeated use of this property on the individual columns of $\mathbf{S}$ leads easily to

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{11}$$

2 Circulant matrices

Further inspection reveals that the Laplacian matrix $\mathbf{K}$ in (1) can be written in terms of these products of the shift matrix $\mathbf{S}$ as

$$\mathbf{K} = 2\mathbf{I} - \mathbf{S} - \mathbf{S}^3, \tag{12}$$

where $\mathbf{I}$ is the 4-by-4 identity matrix. Any matrix that can be written as

$$c_0\mathbf{I} + c_1\mathbf{S} + c_2\mathbf{S}^2 + c_3\mathbf{S}^3 \tag{13}$$

for some set of real-valued coefficients $\{c_j | j = 0, \ldots, 3\}$ is called a real *circulant matrix*. The Laplacian matrix $\mathbf{K}$ is an example of such a 4-by-4 circulant matrix, as is $\mathbf{S}$ itself.

It should be clear that to define an N-by-N circulant matrix simply requires generalization of the 4-by-4 matrix $\mathbf{S}$ just studied to its N-by-N analogue

$$\mathbf{S} = N \text{ rows} \left\{ \overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & 0 & 0 \\ & \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix}}^{N \text{ columns}} \right. .$$

A real-valued N-by-N circulant matrix is defined to be one having the form

$$c_0\mathbf{I} + c_1\mathbf{S} + c_2\mathbf{S}^2 + \cdots + c_{N-1}\mathbf{S}^{N-1} \tag{14}$$

for some set of real coefficients $\{c_j | j = 0, \ldots, N-1\}$ and where $\mathbf{I}$ now denotes the N-by-N identity matrix.

Circulant matrices have interesting properties. For now, these are explored in the $N = 4$ case but it should be clear from the following analysis how to generalize it to other values of N.

3 On eigenvectors of commuting matrices

The 4-by-4 matrix $\mathbf{S}$ in (2) is a square matrix and we can ask about its eigenvectors. Suppose

$$\mathbf{Sx} = \lambda\mathbf{x}, \qquad \mathbf{x} \neq 0, \tag{15}$$

so that $\mathbf{x}$ is an eigenvector of $\mathbf{S}$ with eigenvalue λ. Suppose too that the dimension of the eigenspace associated with this eigenvalue is *unity* so that the eigenspace is the one-dimensional space spanned by $\mathbf{x}$. Operating on (15) by $\mathbf{K}$ produces

$$\mathbf{KSx} = \lambda\mathbf{Kx}. \tag{16}$$

But $\mathbf{K}$ and $\mathbf{S}$ commute so

$$\mathbf{SKx} = \lambda\mathbf{Kx}. \tag{17}$$

This means that $\mathbf{Kx}$ is *also* an eigenvector of $\mathbf{S}$ with the same eigenvalue λ. Since, by assumption, this eigenspace is spanned by the vector $\mathbf{x}$ we conclude that

$$\mathbf{Kx} = \hat{\lambda}\mathbf{x} \tag{18}$$

for some constant $\hat{\lambda}$. Equation (18) says that $\mathbf{x}$ is *also* an eigenvector of $\mathbf{K}$ but with eigenvalue $\hat{\lambda}$ that is generally not the same as λ.

It is important to realize that these observations on the eigenvectors of commuting matrices pertain to *any* commuting matrices. In particular, the commuting matrices need not be circulant.

4 Eigenvectors of the shift matrix S

Suppose a non-zero vector $\mathbf{x}$ is an eigenvector of the 4-by-4 matrix $\mathbf{S}$ in (2) with eigenvalue λ then

$$\mathbf{Sx} = \lambda\mathbf{x}. \tag{19}$$

Operating on this equation with $\mathbf{S}$ gives

$$\mathbf{S}^2\mathbf{x} = \lambda\mathbf{S}\mathbf{x} = \lambda^2\mathbf{x}. \tag{20}$$

Similarly,

$$\mathbf{S}^3\mathbf{x} = \lambda^2\mathbf{S}\mathbf{x} = \lambda^3\mathbf{x} \tag{21}$$

and

$$\mathbf{S}^4\mathbf{x} = \lambda^3\mathbf{S}\mathbf{x}. = \lambda^4\mathbf{x}. \tag{22}$$

We could carry on indefinitely, but we will stop because it is evident from the observation in (10) that if we carry out the cyclic shift operation 4 times – equivalently, if we operate on a vector with the matrix $\mathbf{S}$ four times in succession – then all elements of the vector are back where they started and the vector is unchanged. In other words,

$$\mathbf{S}^4\mathbf{x} = \mathbf{x}. \tag{23}$$

On combining (22) and (23), we infer that

$$(\lambda^4 - 1)\mathbf{x} = 0. \tag{24}$$

Since $\mathbf{x} \neq 0$, the eigenvalues of $\mathbf{S}$ are therefore among the 4th roots of unity, i.e., the solutions of

$$\lambda^4 = 1 = e^{2\pi \mathrm{i} n}, \qquad n \in \mathbb{Z}, \tag{25}$$

implying the four possible choices

$$\lambda = \lambda_n \equiv e^{2\pi \mathrm{i} n/4}, \qquad n = 0, 1, 2, 3, \tag{26}$$

where any other choice of n simply repeats these four distinct roots.

It now remains to attempt to find an eigenvector corresponding to each of the fourth roots of unity. Let

$$\omega = e^{2\pi \mathrm{i}/4} = \mathrm{i}. \tag{27}$$

For $n = 0$ we have $\lambda_0 = 1$ and we need to solve the linear system

$$\mathbf{S}\mathbf{x}_0 = \lambda_0\mathbf{x}_0, \tag{28}$$

or

$$\mathbf{S}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{29}$$

In this case it is clear that

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{30}$$

For $n = 1$, $\lambda_1 = \omega$ and we need to solve the linear system

$$\mathbf{S}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \tag{31}$$

or

$$\mathbf{S} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix} = \omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{32}$$

This is readily solved to give

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}. \tag{33}$$

Similarly, for each of the two other roots of unity we find corresponding eigenvectors to be

$$\mathbf{x}_2 = \underbrace{\begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{pmatrix}}_{\lambda_2 = \omega^2}, \qquad \mathbf{x}_3 = \underbrace{\begin{pmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega^9 \end{pmatrix}}_{\lambda_3 = \omega^3}. \tag{34}$$

In summary, the eigenvalues and eigenvectors of $\mathbf{S}$ are given by $\mathbf{S}\mathbf{x}_n = \lambda_n \mathbf{x}_n$ where

$$\lambda_n = \omega^n, \qquad \mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}, \qquad n = 0, 1, 2, 3. \tag{35}$$

5 Eigenvectors of circulant matrices

The Laplacian matrix **K** given in (1) commutes with the shift matrix **S** in (2) but we do not yet know its eigenvectors or eigenvalues. However, we know from (9) that the matrices **K** and **S** commute. And it has just been shown that the 4 eigenvalues of **S** are distinct, each having a one-dimensional eigenspace. Therefore by the result in §3 on the eigenvectors of commuting matrices it can be immediately inferred that the eigenvectors of **K** are the same as those of **S**, namely, the 4 eigenvectors of **K** are

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}, \qquad n = 0, 1, 2, 3. \tag{36}$$

The eigenvalues of $\mathbf{K}$ are not expected to be the same as those of $\mathbf{S}$ but, with knowledge of the eigenvectors, these are easy to find. For $n = 0, 1, 2$ and 3 we have

$$\mathbf{K}\mathbf{x}_n = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} = (2 - \omega^n - \omega^{3n}) \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix}. \tag{37}$$

The eigenvalues of $\mathbf{K}$ are therefore

$$\begin{aligned} 2 - \omega^n - \omega^{3n} &= 2 - \omega^n - \omega^{(4-1)n} \\ &= 2 - \omega^n - (e^{2\pi \mathrm{i}/4})^{(4-1)n} \\ &= 2 - \omega^n - (e^{2\pi \mathrm{i}})e^{-2\pi \mathrm{i} n/4} \\ &= 2 - e^{2\pi \mathrm{i} n/4} - e^{-2\pi \mathrm{i} n/4} \\ &= 2 - 2\cos\left(\frac{2\pi n}{4}\right), \qquad n = 0, 1, 2, 3. \end{aligned} \tag{38}$$

Evaluating these explicitly gives the eigenvalues of $\mathbf{K}$ as

$$0, 2, 4, 2. \tag{39}$$

These are all real as expected from general linear algebra results on the eigenvalues of real symmetric matrices.

It is useful to pause and assess what we have done. We have used a symmetry of the matrix, encoded in the commutativity relation (9), to find the eigenvectors and eigenvalues of the 4-by-4 matrix $\mathbf{K}$, and *without solving* its associated characteristic equation, that is, without finding zeros of a determinant. This is especially desirable as the size of the matrix increases, as we demonstrate in the next section.

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6 N-by-N circulant matrix

Remarkably, the powerful symmetry argument just presented generalizes – in a straightforward and natural way – to any N-by-N circulant matrix. Let

$$\mathbf{C}_N = \begin{array}{cc} & \overbrace{\hphantom{-1 \quad -1 \quad -1 \quad -1 \quad \cdot \quad \cdot \quad -1 \quad -1}}^{N \text{ columns}} \\ N \text{ rows} \left\{ \vphantom{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \right. & \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 \\ & \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdot & 0 & -1 & 2 \end{pmatrix} \end{array}.$$

It can be argued, just as we did for the $N = 4$ case, that the eigenvectors of $\mathbf{C}_N$ are

$$\mathbf{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \\ \cdot \\ \cdot \\ \cdot \\ \omega^{(N-1)n} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/N}, \qquad n = 0, 1, 2, \ldots, N-1, \tag{40}$$

with corresponding eigenvalues

$$\lambda_n = 2 - 2\cos\left(\frac{2\pi n}{N}\right), \qquad n = 0, 1, 2, \ldots, N-1. \tag{41}$$

All we have to do is extend consideration to the following generalization of the 4-by-4 matrix $\mathbf{S}$ just studied to its N-by-N analogue

$$\mathbf{S} = \begin{array}{cc} & \overbrace{\hphantom{0 \quad 0 \quad 0 \quad 0 \quad \cdot \quad \cdot \quad 0 \quad 0}}^{N \text{ columns}} \\ N \text{ rows} \left\{ \vphantom{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \right. & \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & 0 & 0 \\ & \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The details are a straightforward adaptation of those already presented for $N = 4$.

While the result just stated is certainly true, one concern is that $\mathbf{C}_N$ is a real symmetric matrix so, by general results from linear algebra, it must have N linearly independent *real* eigenvectors. But not all the eigenvectors given in (40) are real. This apparent discrepancy can be clarified as follows.

First pick any integer j between 0 and N. Now consider the two choices

$$n = j, \qquad n = N - j. \tag{42}$$

From (41) we see that the two corresponding eigenvalues are

$$2 - 2\cos\left(\frac{2\pi j}{N}\right), \qquad 2 - 2\cos\left(\frac{2\pi(N-j)}{N}\right). \tag{43}$$

But since

$$2 - 2\cos\left(\frac{2\pi(N-j)}{N}\right) = 2 - 2\cos\left(2\pi - \frac{2\pi j}{N}\right) = 2 - 2\cos\left(\frac{2\pi j}{N}\right) \tag{44}$$

the two eigenvalues in (43) are the same. The eigenvector corresponding to $n = j$ is

$$\mathbf{x}_j = \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \\ \cdot \\ \cdot \\ \cdot \\ \omega^{(N-1)j} \end{pmatrix}, \tag{45}$$

while that corresponding to $n = N - j$ is

$$\mathbf{x}_{N-j} = \begin{pmatrix} 1 \\ \omega^{N-j} \\ \omega^{2(N-j)} \\ \omega^{3(N-j)} \\ \cdot \\ \cdot \\ \cdot \\ \omega^{(N-1)(N-j)} \end{pmatrix}. \tag{46}$$

But since $\omega_N^N = 1$ then

$$\mathbf{x}_{N-j} = \begin{pmatrix} 1 \\ \omega^{-j} \\ \omega^{-2j} \\ \omega^{-3j} \\ . \\ . \\ . \\ \omega^{-(N-1)j} \end{pmatrix} = \overline{\mathbf{x}_j}. \tag{47}$$

Therefore $\mathbf{x}_{N-j}$ is the complex conjugate eigenvector to $\mathbf{x}_j$ while both vectors have the same eigenvalue; they therefore sit in the same eigenspace. The linear combinations

$$\frac{1}{2}\left(\mathbf{x}_j + \mathbf{x}_{N-j}\right) = \frac{1}{2}\left(\mathbf{x}_j + \overline{\mathbf{x}_j}\right), \qquad \frac{1}{2\mathrm{i}}\left(\mathbf{x}_j - \mathbf{x}_{N-j}\right) = \frac{1}{2\mathrm{i}}\left(\mathbf{x}_j - \overline{\mathbf{x}_j}\right), \tag{48}$$

are therefore *also* eigenvectors with the same eigenvalue and both of these are real. They are just the real and imaginary parts of $\mathbf{x}_j$ since, clearly,

$$\mathbf{x}_j = \underbrace{\left[\frac{1}{2}\left(\mathbf{x}_j + \overline{\mathbf{x}_j}\right)\right]}_{\text{real part of } \mathbf{x}_\mathrm{j}} + \mathrm{i}\ \underbrace{\left[\frac{1}{2\mathrm{i}}\left(\mathbf{x}_j - \overline{\mathbf{x}_j}\right)\right]}_{\text{imaginary part of } \mathbf{x}_\mathrm{j}}\ . \tag{49}$$

Let us check this with the $N = 4$ case studied earlier. With $N = 4$ the real and imaginary parts of $\mathbf{x}_j$, for $j = 0, 1, 2, 3$, are

$$\begin{pmatrix} 1 \\ \cos(2\pi j/4) \\ \cos(4\pi j/4) \\ \cos(6\pi j/4) \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ \sin(2\pi j/4) \\ \sin(4\pi j/4) \\ \sin(6\pi j/4) \end{pmatrix}, \tag{50}$$

giving, for $j = 0$,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{51}$$

for $j = 1$,

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{52}$$

and for $j = 2$,

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{53}$$

The choices $j = 3, 4$ give nothing new – they just retrieve the $j = 1, 0$ cases respectively. The four real eigenvectors are therefore

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}. \tag{54}$$

For a general N the real and imaginary parts of $\mathbf{x}_j$ are

$$\begin{pmatrix} 1 \\ \cos(2\pi j/N) \\ \cos(4\pi j/N) \\ \cos(6\pi j/N) \\ . \\ . \\ \cos(2(N-1)\pi j/N) \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ \sin(2\pi j/N) \\ \sin(4\pi j/N) \\ \sin(6\pi j/N) \\ . \\ . \\ \sin(2(N-1)\pi j/N) \end{pmatrix}. \tag{55}$$

When N is even, the choices $j = 0$ and $j = N/2$ generate one real eigenvector each and $j = 1, \cdots, N/2 - 1$ generate 2 real eigenvectors each giving a total of $2 + 2(N/2 - 1) = N$ real eigenvectors.

When N is odd, the choice $j = 0$ gives one real eigenvector and the choices $j = 1, \cdots, (N-1)/2$ give 2 real eigenvectors each giving a total of $1 + 2(N-1)/2 = N$ real eigenvectors.

Thus this theoretical diversion into considering complex-valued vectors has greatly simplified the analysis and, in the end, produced all the real eigenvectors we need.