1. (a) The velocity $u(x,t)$ of a fluid on the interval $0 < x < L$ obeys the viscous Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

where $u(0,t) = 0$, $u(L,t) = 0$ and $u(x,0) = u_0 x(L-x)/L^2$. You can assume that u_0 and ν are positive real constants.

(i) What are the dimensions of u_0 and ν? (2 marks)

(ii) Assuming a weak nonlinearity, nondimensionalize this problem. (4 marks)

(b) We now consider the inviscid Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to initial conditions

$$u(x,0) = \begin{cases} 2, & \text{for } x \leq -1, \\ 1-x, & \text{for } -1 < x \leq 0, \\ 1, & \text{for } x > 0. \end{cases}$$

(i) Sketch the initial conditions. Construct the characteristics diagram in the (x,t)-plane by sketching the characteristics emanating from each of the intervals $x \leq -1$, $-1 < x \leq 0$ and $x > 0$. Show that a shock forms at $t = 1$. (5 marks)

(ii) Find an explicit solution valid for $0 < t \leq 1$ and sketch it for $t = 1/2$ and $t = 1$. (4 marks)

(iii) Find an explicit solution after the shock has formed, sketch an amended diagram of characteristics showing the shock path and sketch the solution at $t = 2$. (5 marks)

(Total: 20 marks)

2. (a) Consider the following PDE

$$y^2\frac{\partial u}{\partial x} + x^2\frac{\partial u}{\partial y} = 2xy^2$$

(i) Find the general solution to this PDE. (4 marks)

(ii) Show that the particular solution to this PDE if you are given that $u(0,y) = \exp(y^3)$ and $u(x,0) = x^2 + \exp(-x^3)$ is

$$u(x,y) = x^2 + \exp(y^3 - x^3)$$

(3 marks)

(iii) What is the particular solution to this PDE if now you are given that $u(0,y) = -y^6$ and $u(x,0) = x^2 - x^6$? (3 marks)

(b) A flat hammer hits an infinite horizontal string. The vertical displacement of the string is the solution to the following initial value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \\
&u(x,0) = 0 \\
&\frac{\partial u}{\partial t}(x,0) = \phi(x)
\end{aligned}$$

with

$$\phi(x) = \begin{cases} 1, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

(i) Give explicit solutions for $t = 0$, $t = 1/2$, $t = 1$ and $t > 1$. (7 marks)

(ii) Sketch on the same graph the vertical displacement of the string for $t = 0$, $t = 1/2$, $t = 1$, $t = 3/2$, $t = 2$ and $t = 5$. (3 marks)

(Total: 20 marks)

3. (a) Find the regions of the (x, y)-plane where the equation

$$(1+x)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} + x^2\frac{\partial u}{\partial x} - xy^2\frac{\partial u}{\partial y} + (1-y)u = 0$$

is parabolic, hyperbolic or elliptic. Sketch these regions. (5 marks)

(b) Consider a perfectly elastic and flexible string of length L with linear density ρ under uniform tension τ. The string is stretched horizontally and fixed at both ends. Initially, it is vertically displaced at its mid-point by an amount A. At $t = 0$, the string is released.

(i) Show that the vertical displacement in the string is given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8A(-1)^n}{\pi^2(2n+1)^2} \sin\left[\frac{(2n+1)\pi x}{L}\right] \cos\left[\frac{(2n+1)\pi c t}{L}\right]$$

where $c = \sqrt{\tau/\rho}$ is the wavespeed. (7 marks)

(ii) Show that the kinetic energy of the string when it passes through its rest position is given by

$$K = \frac{2A^2\rho c^2}{L}$$

You may use the following result $\sum_{n=0}^{\infty} 1/(2n+1)^2 = \pi^2/8$. What is the work done in first displacing the string (no formal proof required)? (8 marks)

(Total: 20 marks)

4. (a) In $\mathbb{R}^3$, the Klein-Gordon equation governs the quantum mechanical wavefunction $\psi(\mathbf{r})$ of a relativistic spinless particle with mass m. It reads

$$\nabla^2\psi - m^2\psi = 0$$

Show that the solution for the scalar field $\psi(\mathbf{r})$ in any volume V bounded by a surface S is unique if either Dirichlet or Neumann conditions are specified on the surface S. (5 marks)

(b) Now we wish to use Green's functions to find solutions to the non-homogeneous Klein-Gordon equation with a density of charge $\rho(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3$.

(i) By applying the divergence theorem to the volume integral

$$\int_V \Big[u(\nabla^2 - m^2)v - v(\nabla^2 - m^2)u\Big]\, dV$$

(where u and v are twice differentiable scalar fields) obtain a Green's function expression for the solution ψ to

$$\nabla^2\psi - m^2\psi = \rho(\mathbf{r})$$

in a bounded volume V and which takes the value $\psi(\mathbf{r}) = f(\mathbf{r})$ on S, the boundary of V. The Green's function, $G(\mathbf{r}, \mathbf{r}_0)$ to be used satisfies the following equation

$$\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$$

and vanishes when $\mathbf{r} \in S$. (6 marks)

(ii) When $V = \mathbb{R}^3$, the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ can be written as $G(\eta) = g(\eta)/\eta$, where $\eta = |\mathbf{r} - \mathbf{r}_0|$ and $g(\eta)$ is bounded as $\eta \to \infty$. Show that $G(\eta)$ is a solution to the following ODE

$$\eta G'' + 2G' - m^2\eta G = 0$$

Thus, conclude that

$$G(\eta) = -\frac{e^{-m\eta}}{4\pi\eta}$$

(6 marks)

(iii) Finally, find $\psi(\mathbf{r})$ in the half-space $x > 0$ in the case of a unique positive charge located at $\mathbf{r}_1$ such that $\psi = 0$ both on the plane $x = 0$ and as $r \to \infty$. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH50008

PDEs in Action (Solutions)

Setter's signature	Checker's signature	Editor's signature
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1. (a) (i) Based on dimensional homogeneity considerations, we write that the dimensions are given as follows: $\boxed{\text{seen} \Downarrow}$

$$[u_0] = LT^{-1}$$
$$[\nu] = L^2T^{-1}$$

$\boxed{2, \text{A}}$

(ii) To nondimensionalize the problem, we write $\boxed{\text{meth seen} \Downarrow}$

$$x = x_c\xi, \quad t = t_c\tau \quad \text{and} \quad u = u_c v$$

where x_c is a characteristic length, t_c is a characteristic time and u_c is a characteristic velocity. One of the parameters in the problem is u_0 which is a velocity so a natural choice is $u_c = u_0$. We nondimensionalize the equations as follows

$$\frac{u_0}{t_c}\frac{\partial v}{\partial \tau} + \frac{u_0^2}{x_c}v\frac{\partial v}{\partial \xi} = \frac{\nu u_0}{x_c^2}\frac{\partial^2 v}{\partial \xi^2}$$

which we rewrite

$$\frac{\partial v}{\partial \tau} + \Pi_1 v\frac{\partial v}{\partial \xi} = \Pi_2\frac{\partial^2 v}{\partial \xi^2}$$

and the initial and boundary conditions can be nondimensionalized as follows:

$$v(0,\tau) = v(\Pi_3,\tau) = 0 \quad \text{and} \quad v(\xi,0) = \Pi_3\xi - \Pi_3^2\xi^2$$

where we have defined the 3 dimensionless groups

$$\Pi_1 = \frac{u_0 t_c}{x_c}, \quad \Pi_2 = \frac{\nu t_c}{x_c^2}, \quad \Pi_3 = \frac{x_c}{L}$$

We need to determine two quantities x_c and t_c, so following the rules established in lecture:

- we pick the Π group appearing in the boundary/initial conditions and set $\Pi_3 = 1 \Rightarrow x_c = L$
- we pick the Π group which would appear in the reduced problem; here, we assumed a weak nonlinearity so the reduced problem would be obtained when $\Pi_1 \to 0$ and the only Π group appearing in the reduced problem is then Π_2. We set $\Pi_2 = 1 \to t_c = L^2/\nu$.

Finally, we find that our problem reads in dimensionless form

$$\frac{\partial v}{\partial \tau} + \varepsilon v\frac{\partial v}{\partial \xi} = \frac{\partial^2 v}{\partial \xi^2}$$
$$v(0,\tau) = 0, \quad v(1,\tau) = 0, \quad v(\xi,0) = \xi(1-\xi)$$

where $\varepsilon = u_0L/\nu$ is indeed a dimensionless small parameter when the viscous term dominates (i.e. when the nonlinear term is weak). $\boxed{4, \text{A}}$

(b) (i) The initial conditions are shown in Fig. 1 $\boxed{\text{sim. seen} \Downarrow}$

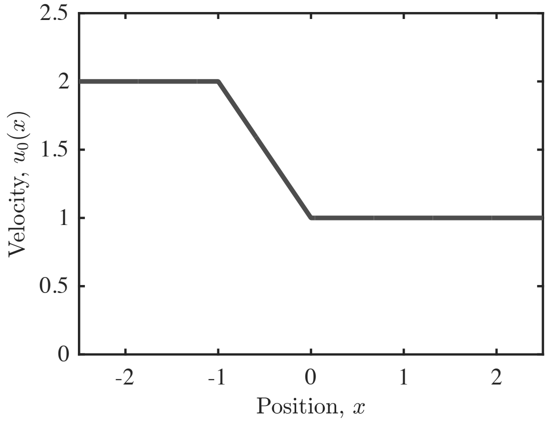

Figure 1: Initial conditions

$\boxed{1, \text{A}}$

The characteristics satisfy the following equation

$$\frac{dx}{dt} = u$$

As u is constant along the characteristics in this homogeneous problem, they are straight lines, whose slope depends on where they cross the x-axis, i.e. the slope is determined by the initial condition. We denote $x(0) = \xi$. Given the initial condition, we find that the equation of the characteristics is given by

$$\begin{cases} \text{For } \xi < -1, & u = 2 \quad \text{on} \quad x(t) = 2t + \xi \\ \text{For } -1 < \xi < 0, & u = 1 - \xi \quad \text{on} \quad x(t) = (1 - \xi)t + \xi \\ \text{For } \xi < 0, & u = 1 \quad \text{on} \quad x(t) = t + \xi \end{cases}$$

The diagram of characteristics is shown in Fig. 2

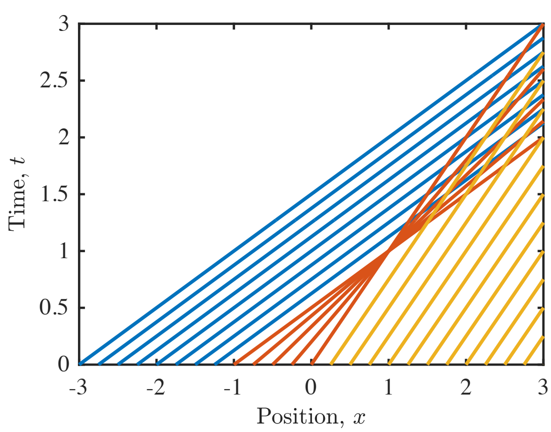

Figure 2: Diagram of characteristics.

$\boxed{3, \text{A}}$

At $t = 1$, all characteristics emanating from the interval $0 < \xi < 1$ cross at $x = 1$. This can be seen from the diagram of characteristics but also from the equation of the characteristics in this interval $x(t) = (1 - \xi)t + \xi$. $\boxed{1, \text{A}}$

(ii) We know that the solution is constant along the characteristics, so we can write

sim. seen ⇓

- For $\xi < -1$, we had $u = 2$ on $x(t) = 2t + \xi$ which we can invert and find $u(x,t) = 2$ for $x - 2t < -1$;
- For $\xi > 0$, we had $u = 1$ on $x(t) = t + \xi$ which we can invert and find $u(x,t) = 1$ for $x - t > 0$;
- For $-1 < \xi < 0$, we had $u = 1 - \xi$ on $x(t) = (1-\xi)t + \xi$, i.e. on

$$\xi = \frac{x-t}{1-t}$$

which leads to

$$u = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}$$

for $-1 < (x-t)/(1-t) < 0$. The solution is finally given by

$$u(x,t) = \begin{cases} 2, & \text{for } x < 2t-1 \\ (1-x)/(1-t), & \text{for } 2t-1 < x < t \\ 1, & \text{for } x > t \end{cases}$$

2, A

The solutions for $t = 1/2$ and $t = 1$ are shown in Fig. 3

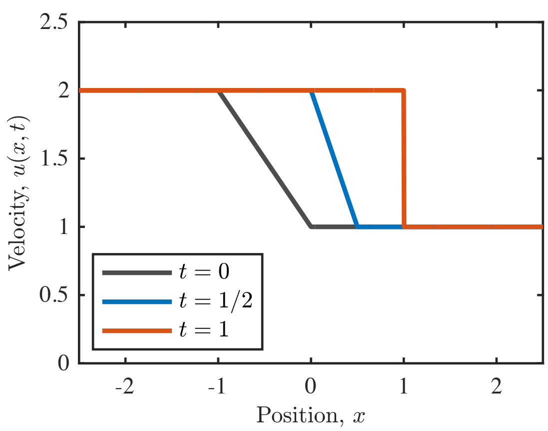

Figure 3: Solution for $0 < t \leq 1$

2, A

sim. seen ⇓

(iii) At $t = 1$, a shock form at $x = 1$. Rewriting the inviscid Burger's equation in conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

we find that the Rankine-Hugoniot jump condition gives us that the position of the shock $s(t)$ obeys the following equation

$$\frac{ds}{dt} = \frac{[u^2/2]}{[u]}$$

where $[\cdots]$ is the usual jump notation, i.e. $[u] = u_{\text{behind}} - u_{\text{ahead}}$.

From our solution for $0 < t \leq 1$, we find that

- behind the shock, $u = 2$;
- ahead of the shock, $u = 1$.

So the Rankine-Hugoniot condition reads

$$\frac{ds}{dt} = \frac{1}{2}\frac{4-1}{2-1} = \frac{3}{2}$$

We can solve this ODE subject to the initial condition $s(1) = 1$ to get

$$s(t) = \frac{3}{2}t - 1/2$$

$\boxed{2, \text{A}}$

The final diagram of characteristics with the shock path in red is shown in Fig. 4

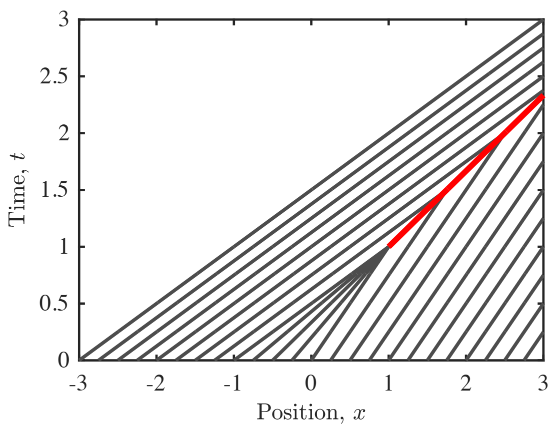

Figure 4: Diagram of characteristics with shockpath.

$\boxed{2, \text{A}}$

At $t = 2$, the shock is located at $x = 5/2$. The solution at $t = 2$ is shown in Fig. 5

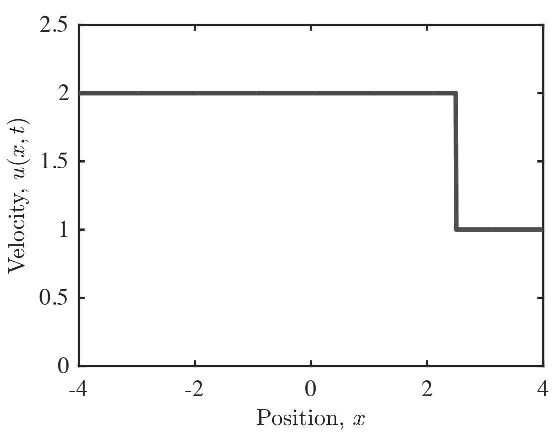

Figure 5: Solution at $t = 2$.

$\boxed{1, \text{A}}$

2. (a) (i) We consider the PDE $\boxed{\text{sim. seen} \Downarrow}$

$$y^2 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 2xy^2$$

Using the method of characteristics, we write that the general solution to this problem obeys the following equation

$$\frac{du}{dx} = 2x$$

on the characteristics whose equations are

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

This equation is separable and we can easily integrate it to find that the equations of the characteristics are given by

$$x^3 - y^3 = c_1$$

where c_1 is an arbitrary constant. Further, integrating the equation for u leads to $u = x^2 + c_2$, where c_2 is also an arbitrary constant.
As we have seen, the constants c_1 and c_2 can thought as being related through $c_2 = f(c_1)$ where f is an arbitrary function, i.e. that the general solution to this PDE reads

$$u(x, y) = x^2 + f(x^3 - y^3)$$

where f is an arbitrary function to be determined. $\boxed{4, \text{B}}$

(ii) In this first case, we consider the boundary conditions

$$u(0, y) = \exp(y^3) \Rightarrow f(-y^3) = \exp(y^3)$$
$$u(x, 0) = x^2 + \exp(-x^3) \Rightarrow f(x^3) = \exp(-x^3)$$

So we conclude that the particular solution is here given by

$$u(x, y) = x^2 + \exp(y^3 - x^3)$$

$\boxed{3, \text{A}}$

(iii) In this second case, we consider the boundary conditions

$$u(0, y) = -y^6 \Rightarrow f(-y^3) = -y^6 \Rightarrow f(t) = -t^2, \text{where } t = -y^3$$
$$u(x, 0) = x^2 - x^6 \Rightarrow x^2 + f(x^3) = x^2 - x^6 \Rightarrow f(t) = -t^2, \text{where } t = x^3$$

So we conclude that the particular solution is here given by

$$u(x, y) = x^2 - (y^3 - x^3)^2 = x^2 - x^6 + 2x^3y^3 - y^6$$

$\boxed{3, \text{A}}$

(b) (i) When first reading the problem, one may try to address it using a Fourier transform method, but this leads to quite complicated algebra. Instead, it is easier to use d'Alembert solution for the wave equation which gives here $\boxed{\text{unseen} \Downarrow}$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(\xi)d\xi$$

with $c = 1$. What is this integral? So for a fixed time t, one integrate ϕ over the interval $[x-t, x+t]$; as ϕ is constant and equal to 1 over the interval $[-1, 1]$ which is fixed, then the value of $u(x,t)$ depends on the overlap of the intervals $[x-t, x+t]$ and $[-1, 1]$, namely we write

$$u(x,t) = \frac{1}{2}\big|[-1,1] \cap [x-t, x+t]\big|$$

where $|\cdot|$ denotes the length of the interval.

Let us write explicit solutions:

- For $t = 0$:
  Note that in this case, $\forall x \in \mathbb{R}, \big|[-1,1]\big| > \big|[x,x]\big|$. Clearly, the overlap between the two intervals is always of length 0 and we have

  $$u(x,0) = 0, \quad -\infty < x < \infty$$

  which is consistent with the initial conditions provided.
- For $t = 1/2$:
  Note that in this case, $\forall x \in \mathbb{R}, \big|[-1,1]\big| > \big|[x-1/2, x+1/2]\big|$, so the value of $u(x, 1/2)$ can be at most the length of $\big|[x-1/2, x+1/2]\big|/2 = 1/2$. Notice that if $x < -3/2$, the two intervals do not overlap and so $u(x,t) = 0$. A similar case can be made for $x > 3/2$. Further, in the range $-1/2 < x < 1/2$, the interval $[x-1/2, x+1/2] \subset [-1,1]$ and so we conclude that $u(x,1/2) = \big|[x-1/2, x+1/2]\big|/2 = 1/2$. Finally, as x varies in the range $[-3/2, -1/2]$, $u(x, 1/2)$ varies linearly from its value in $x = -3/2$ to its value in $x = -1/2$. Combining all these elements, we conclude that

  $$u(x,1/2) = \begin{cases} 0, & |x| > 3/2 \\ 1/2, & |x| < 1/2 \\ (x+3/2)/2, & -3/2 \le x \le -1/2 \\ (3/2-x)/2, & 1/2 \le x \le 3/2 \end{cases}$$

- For $t = 1$:
  In this case, we have that: $\forall x \in \mathbb{R}, \big|[-1,1]\big| = \big|[x-1, x+1]\big|$ Similarly, to what we did above, we easily conclude that

  $$\forall |x| > 2, u(x,1) = \frac{1}{2}\big|[-1,1] \cap [x-1, x+1]\big| = 0$$

For $x = 0$, the 2 intervals coincide and so $u(x,1) = 1$ and this is the only point where this happens. Finally, on the intervals $[-2,0]$ and $[0,2]$, u varies linearly from its value on the left of the interval to its value on the right of the interval. We conclude that

$$u(x,1) = \begin{cases} 0, & |x| > 2 \\ (x+2)/2, & -2 \le x \le 0 \\ (2-x)/2, & 0 < x \le 2 \end{cases}$$

· For $t > 1$:
In this case, we have that: $\forall x \in \mathbb{R}, \big|[-1,1]\big| < \big|[x-1,x+1]\big|$. By the same argument, we find that

$$u(x,t) = \begin{cases} 0, & |x| > 1+t \\ 1, & |x| < 1-t \\ [x+(t+1)]/2, & -(t+1) \le x \le -(t-1) \\ [(t+1)-x]/2, & t-1 \le x \le t+1 \end{cases}$$

$\boxed{7, \text{D}}$

(ii) In Fig. 6, we show the string displacement for the required times. $\boxed{\text{unseen} \Downarrow}$

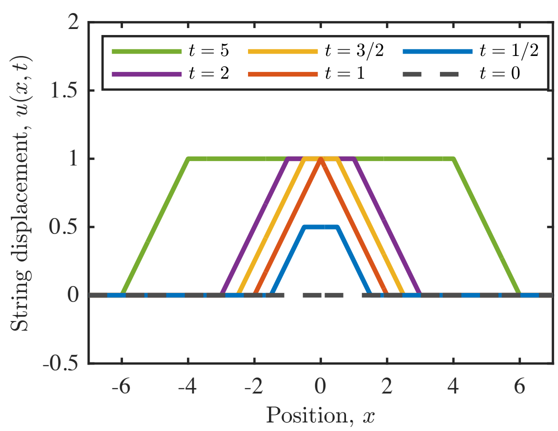

Figure 6: Solution to 2. (b) (iii).

$\boxed{3, \text{D}}$

3. (a) This is a linear second-order PDE in two independent variables (x, y) of the form $\boxed{\text{sim. seen} \Downarrow}$

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = 0$$

To determine the regions of the (x, y)-plane in which the PDE is parabolic/elliptic/hyperbolic, one needs to examine the sign of the quantity

$$\mathcal{D} = b(x,y)^2 - 4a(x,y)c(x,y)$$

with $a(x,y) = 1 + x$, $b(x,y) = 2xy$ and $c(x,y) = 2y^2$. We obtain

$$\mathcal{D} = 4x^2y^2 - 8(1+x)y^2$$

and

$$\mathcal{D} = 0 \Rightarrow y = 0 \text{ or } x^2 - 2x - 2 = 0 \Rightarrow y = 0 \text{ or } x = 1 \pm \sqrt{3}$$

So we conclude that:

* The equation is parabolic on the lines $y = 0$, $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$;
* The equation is elliptic in the region where $1 - \sqrt{3} < x < 1 + \sqrt{3}$, as long as $y \neq 0$;
* The equation is hyperbolic in the regions where $x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$, as long as $y \neq 0$.

$\boxed{3, \mathrm{A}}$

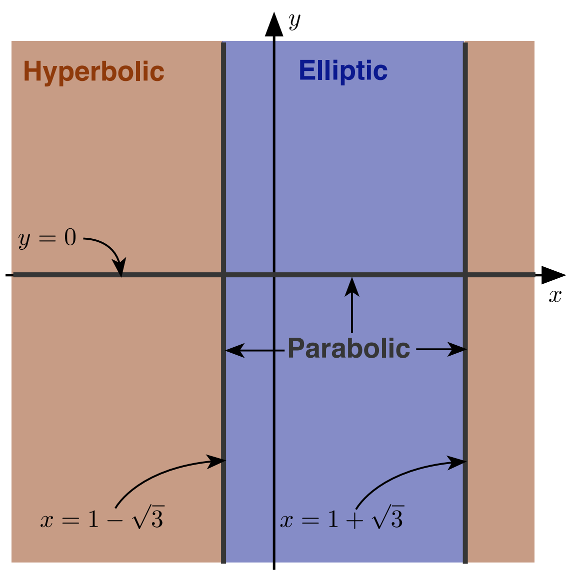

Figure 7: Solution to 3. (a).

$\boxed{2, \mathrm{A}}$

(b) (i) The vertical displacement of the string obeys the 1D wave equation $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and initial conditions

$$u(x,0) = \begin{cases} 2Ax/L, & 0 < x \leq L/2 \\ 2A(L-x)/L, & L/2 < x < L \end{cases} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0$$

$\boxed{1, \text{A}}$

In this problem on a finite domain, we seek separated solutions of the form $u(x,t) = X(x)T(t)$. By substituting this in the wave equation, we have

$$XT'' = c^2 X'' T$$

which means that there exists a separation constant λ^2 such that

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2$$

where the sign of the separation constant is dictated by the fact that we require solutions which are periodic in space. (One could also check all cases individually and show that nontrivial solutions only emerge for a negative separation constant due to the homogeneous Dirichlet boundary conditions).
So we obtain a set of two ODEs. Let us first turn our attention to that for $X(x)$: the general solution to $X'' + \lambda^2 X = 0$ is given by

$$X(x) = B\cos\lambda x + C\sin\lambda x$$

As we are looking for nontrivial solutions, the original boundary conditions impose

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$$

We impose these boundary conditions and conclude that

$$X(0) = 0 \Rightarrow B = 0$$
$$X(L) = 0 \Rightarrow \sin\lambda L = 0 \Rightarrow \lambda = n\pi/L$$

with n an integer.
Now back to the temporal equation $T'' + c^2\lambda^2 T = 0$, which has for general solution

$$T(t) = D\cos c\lambda t + E\sin c\lambda t$$

Since the initial conditions are such that $\partial u/\partial t(x,0) = 0$, we require that $T'(t) = 0$ and conclude that $E = 0$.

Combining these, we obtain the family of solutions

$$u_n(x,t) = C_n D_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

Finally, we conclude that the general solution of the original PDE is given by the following superposition

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

(Note that the sum only runs over the positive integers as the negative integers terms can be absorbed in the sum by defining $\alpha_n = C_n D_n - C_{-n} D_{-n}$). $\boxed{3, \mathrm{B}}$

To determine the coefficients α_n, we make use of the initial conditions, which are given by

$$u(x,0) = f(x) = \begin{cases} 2Ax/L, & 0 < x \leq L/2 \\ 2A(L-x)/L, & L/2 < x < L \end{cases}$$

Imposing this initial condition, we get that

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)$$

This expression is the half-range Fourier sine series for $f(x)$ and so the coefficients are given by

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We write

$$\begin{aligned}
\frac{\alpha_n L}{2} &= \int_0^{L/2} \frac{2A}{L} x \sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L} \frac{2A}{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2A}{L}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + 2A\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right)dx - \frac{2A}{L}\int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

We proceed to integration by part and write

$$\begin{aligned}
I_1 &= \int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{L}{n\pi}\left[-x\cos\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx \\
&= -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} \\
&= -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\
&= -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} \\
&= -\frac{L}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]
\end{aligned}
$$

and finally

$$
\begin{aligned}
I_3 &= \int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{L}{n\pi}\left[-x\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n\pi}(-1)^n + \frac{L^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} \\
&= \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n\pi}(-1)^n - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

Assembling all these terms, we write that

$$
\frac{\alpha_n L}{2} = \frac{2A}{L}\left(I_1 + LI_2 - I_3\right) = \frac{2A}{L}\frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
$$

So we conclude that

$$
\alpha_n = \begin{cases} 8A(-1)^m/((2m+1)^2\pi^2), & \text{if } n = 2m+1 \text{ (odd)} \\ 0, & \text{if } n = 2m \text{ (even)} \end{cases}
$$

So finally, we conclude that the solution can be written

$$
u(x,t) = \sum_{m=0}^{\infty} \frac{8A(-1)^m}{\pi^2(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right)
$$

3, B

meth seen ⇓

(ii) As the total energy is conserved in this system (no friction), when the string passes through its rest position, it is not deformed and so the elastic potential energy stored in the string is minimal. So we are looking here to compute the maximum kinetic energy. By definition, the time-dependent kinetic energy of the string is given by

$$
K(t) = \int_0^L \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 dx
$$

where

$$
\frac{\partial u}{\partial t} = -\sum_{n=0}^{\infty} \frac{8Ac(-1)^n}{\pi L(2n+1)}\sin\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{(2n+1)\pi ct}{L}\right)
$$

2, B

Now recall that the normal modes are mutually orthogonal, i.e. here

unseen $\Downarrow$

$$\forall n \neq m, \int_0^L \sin\left(\frac{(2n+1)\pi x}{L}\right) \sin\left(\frac{(2m+1)\pi x}{L}\right) dx = 0$$

So we can write the total time-dependent kinetic energy as a sum over the kinetic energy carried by each independent normal mode, i.e.

$$K(t) = \sum_{n=0}^{\infty} K_{2n+1}(t)$$

with

$$K_{2n+1}(t) = \int_0^L \frac{1}{2}\rho \frac{64A^2c^2}{\pi^2 L^2 (2n+1)^2} \sin^2\left(\frac{(2n+1)\pi x}{L}\right) \sin^2\left(\frac{(2n+1)\pi ct}{L}\right) dx$$

which is maximum when the time-dependent sine function is maximized. Finally, we obtain that

$$\begin{aligned} K_{2n+1} &= \int_0^L \frac{1}{2}\rho \frac{64A^2c^2}{\pi^2 L^2 (2n+1)^2} \sin^2\left(\frac{(2n+1)\pi x}{L}\right) dx \\ &= \frac{32A^2\rho c^2}{\pi^2 L^2 (2n+1)^2} \int_0^L \sin^2\left(\frac{(2n+1)\pi x}{L}\right) dx \\ &= \frac{32A^2\rho c^2}{\pi^2 L^2 (2n+1)^2} \frac{L}{2} \end{aligned}$$

We conclude that the total kinetic energy when the string passes through its rest position is

$$K = \sum_{n=0}^{\infty} K_{2n+1} = \frac{16A^2\rho c^2}{\pi^2 L} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{2A^2\rho c^2}{L}$$

4, C

unseen $\Downarrow$

As there is no friction in the string, the total energy is conserved; the work done in initially displacing the string is the total energy of the string as the string is released with no velocity at $t = 0$. We conclude that it is equal to the kinetic energy calculated above and

$$W = \frac{2A^2\rho c^2}{L}$$

2, C

4. (a) We consider the Klein-Gordon equation $\boxed{\text{sim. seen} \Downarrow}$

$$\nabla^2\phi - m^2\phi = 0$$

subject to **either** Dirichlet boundary conditions ($\phi = f$) or Neumann boundary conditions ($\partial\phi/\partial n = g$) on S. Suppose that we have two solutions to this Dirichlet or Neumann problem ϕ_1 and ϕ_2. We form the difference $\psi = \phi_1 - \phi_2$.
By definition, ψ is solution to the following PDE

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = m^2\phi_1 - m^2\phi_2 = m^2\psi$$

(This needs to be verified or one needs to quote that the differential operator $\mathcal{L} = \nabla^2 - m^2$ is linear.) Further, ψ respects either Dirichlet boundary conditions

$$\psi = f - f = 0 \quad \text{on } S$$

or the Neumann boundary conditions

$$\frac{\partial\psi}{\partial n} = g - g = 0 \quad \text{on } S$$

By applying Green's first identity with twice the same scalar field ψ, we obtain

$$\int_S \psi\frac{\partial\psi}{\partial n}dS = \int_V \psi\nabla^2\psi dV + \int_V |\nabla\psi|^2 \, dV$$

and whichever boundary conditions are used, we can clearly see that the LHS vanishes and using the fact that ψ is solution to the Klein-Gordon equation, we write that

$$\int_V \left(m\psi^2 + |\nabla\psi|^2\right) dV = 0$$

As both terms in the integrand are positive, we conclude that each of them must be equal to zero for the volume integral to vanish. In particular, we see that $\psi^2 = 0 \Rightarrow \psi = 0$ throughout volume V. So we conclude that throughout the volume V, $\phi_1 = \phi_2$ and the solution is unique. $\boxed{5, \text{B}}$

$\boxed{\text{meth seen} \Downarrow}$

(b) (i) We apply the divergence theorem for the following volume integral (where u and v are general scalar fields)

$$\begin{aligned}\int_V \left[u(\nabla^2 - m^2)v - v(\nabla^2 - m^2)u\right] dV &= \int_V \left[u\nabla^2 v - v\nabla^2 u\right] dV \\ &= \int_V \nabla \cdot [u\nabla v - v\nabla u] \, dV \\ &= \int_S [u\nabla v - v\nabla u] \cdot \widehat{\mathbf{n}} dS\end{aligned}$$

where $\widehat{\mathbf{n}}$ is a unit vector normal to the surface S in the outward direction. $\boxed{3, \text{C}}$

Let us apply this result to (1) $u = \phi(\mathbf{r})$, where $\phi(\mathbf{r})$ is a solution to the following Klein-Gordon equation $\nabla^2\phi - m^2\phi = \rho(\mathbf{r})$ subject to the boundary condition $\phi(\mathbf{r}) = f(\mathbf{r})$ on S and (2) $v = G(\mathbf{r}, \mathbf{r}_0)$ the Green's function for the operator $\mathcal{L} = \nabla^2 - m^2$, i.e. $\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$ and $G(\mathbf{r}, \mathbf{r}_0) = 0$ on S. Then, we obtain

$$\int_V [\phi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})] \, dV = \int_S [f(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}_0) - 0] \cdot \hat{\mathbf{n}} dS$$

which finally leads to

$$\phi(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r}) dV + \int_S f(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_0) dS$$

where we have used the sifting property of the Dirac delta function. 3, C

unseen ⇓

(ii) The Green's function we just defined is solution to the following PDE

$$\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$$

To determine the Green's function, we integrate this equation over a sphere of radius η centered on $\mathbf{r}_0$, which gives

$$\int_V \nabla^2 G dV - \int_V m^2 G dV = \int_V \delta(\mathbf{r} - \mathbf{r}_0) dV \Rightarrow \int_S \nabla G \cdot \hat{\mathbf{n}} dS - m^2 \int_V G dV = 1$$

where we made use of the divergence theorem. By assumption, $G(\mathbf{r}, \mathbf{r}_0) = G(\eta)$ where $\eta = |\mathbf{r} - \mathbf{r}_0|$. So moving the origin to $\mathbf{r}_0$ and using spherical coordinates (η, θ, ϕ), the first term on the LHS reads

$$\int_S \nabla G \cdot \hat{\mathbf{n}} dS = \int_0^{2\pi} \int_0^{\pi} \frac{dG}{d\eta} \eta^2 \sin\theta d\theta d\phi = 4\pi\eta^2 \frac{dG}{d\eta}$$

The second term can be written

$$-\int_V m^2 G dV = -m^2 \int_0^{\eta} \int_0^{2\pi} \int_0^{\pi} G(\eta')\eta'^2 \sin\theta d\eta' d\theta d\phi = -4\pi m^2 \int_0^{\eta} \eta'^2 G(\eta') d\eta'$$

which leads to

$$4\pi\eta^2 \frac{dG}{d\eta} - 4\pi m^2 \int_0^{\eta} \eta'^2 G(\eta') d\eta' = 1$$

By differentiating this equation with respect to η, we obtain

$$4\pi\eta^2 G'' + 8\pi\eta G' - 4\pi m^2 \eta^2 G = 0$$

where the prime denotes a derivative with respect to η. We conclude that G is solution to the following ODE

$$\eta G'' + 2G' - m^2 \eta G = 0$$

2, D

Now by assumption, we have that $G(\eta) = g(\eta)/\eta$ so the derivative read

$$G'(\eta) = -\frac{g}{\eta^2} + \frac{g'}{\eta}$$
$$G''(\eta) = \frac{2g}{\eta^3} - \frac{2g'}{\eta^2} + \frac{g''}{\eta}$$

Reinjecting this in the previous equation, we obtain that g is solution to

$$g'' - m^2 g = 0$$

which has for general solution

$$g(\eta) = A\exp(-m\eta) + B\exp(m\eta)$$

but as g is bounded as $\eta \to \infty$, we require that $B = 0$.
To find the value of A, recall that

$$4\pi\eta^2 G' - 4\pi m^2 \int_0^\eta \eta'^2 G(\eta')d\eta' = 1$$

which means

$$4\pi\eta^2\left(-\frac{Ae^{-m\eta}}{\eta^2} - \frac{Ame^{-m\eta}}{\eta}\right) - 4\pi m^2 A\int_0^\eta \eta' e^{-m\eta'}d\eta' = 1$$

but we know that

$$\int_0^\eta \eta' e^{-m\eta'}d\eta' = -\frac{1}{m}\left[\eta' e^{-m\eta'}\right]_0^\eta + \frac{1}{m}\int_0^\eta e^{-m\eta'}d\eta'$$
$$= -\frac{1}{m}\eta e^{-m\eta} - \frac{1}{m^2}\left(e^{-m\eta} - 1\right)$$

So we conclude that

$$4\pi A\left(-e^{-m\eta} - m\eta e^{-m\eta} + m\eta e^{-m\eta} + e^{-m\eta} - 1\right) = 1 \Rightarrow A = -\frac{1}{4\pi}$$

and we finally obtain

$$G(\eta) = -\frac{e^{-m\eta}}{4\pi\eta}$$

or equivalently

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{e^{-m|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|}$$

$\boxed{4, \mathrm{D}}$

(iii) In this final part, we want to solve the Klein-Gordon problem with $\rho(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_1)$, i.e. in the case of a unit positive charge located in $\mathbf{r}_1 = (x_1, y_1, z_1)$. We wish to find a solution ψ which takes on the value $\psi = 0$ on the $x = 0$ plane. The Green's function we just derived does not respect this boundary condition. Using the method of images, we add a unit negative charge at the mirror image of $\mathbf{r}_1$ with respect to the plane $x = 0$, i.e. at location $\mathbf{r}_1' = (-x_1, y_1, z_1)$. The solution to this problem is then given in the $x > 0$ region by

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$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\left(\frac{e^{-m|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|} - \frac{e^{-m|\mathbf{r}-\mathbf{r}_1'|}}{|\mathbf{r}-\mathbf{r}_1'|}\right)$$

or else

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\left[\frac{e^{-m\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}} - \frac{e^{-m\sqrt{(x+x_1)^2+(y-y_1)^2+(z-z_1)^2}}}{\sqrt{(x+x_1)^2+(y-y_1)^2+(z-z_1)^2}}\right]$$

3, B

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks