Unseen 7

Question 1. In the following, X is a finite set and $G \leq \mathrm{Sym}(X)$.

For $x, y \in Y$, write $x \sim y$ if there exists $g \in G$ with $g(x) = y$.

(a) Prove that this is an equivalence relation on X.

The equivalence classes are of the form $\{g(x) : g \in G\}$ and are called the *orbits* of G on X.

The following formula gives a way of counting the number of orbits (it's sometimes referred to as Burnside's Lemma, or the Cauchy - Frobenius formula). If $g \in G$, let $\mathrm{Fix}(g) = \{x \in X : g(x) = x\}$, the set of fixed points of g in X.

THEOREM: The number of orbits of G on X is $\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$.

(b) Check the formula in the case where $X = \{1, 2, 3\}$ and $G = S_3$.

Before proving the formula, let's give an application. This is a typical application where we want to count a number of objects 'up to symmetry'.

(c) Each of the 6 vertices of a regular hexagon is coloured with one of 3 possible colours (R,G,B), giving a total of 3^6 possible coloured hexagons. However, some of these colourings will look the same, up to (rotational) symmetries of the hexagon (of which there are 12). How many essentially different coloured hexagons are there? Here's the start of a solution:

Let G be the group of rotational symmetries of the hexagon. Let X be the set of coloured hexagons (so this has size 3^6). Each element of G gives a permutation of X and so we can think of G as a subgroup of $\mathrm{Sym}(X)$. We want to know how many orbits there are of G on X. So for each $g \in G$ we compute how many elements of X are fixed by g. ....

Complete the solution. Note that G is the dihedral group D_{12} with 12 elements, discussed in Section 4.3 of the notes. But you should be able to do the question without looking at this: just think about what the elements of G do to the hexagon.

(d) Prove the Theorem as follows. Call the orbits $X_1, \ldots, X_t$. If $x \in X_i$ then by Question 8 on Problem Sheet 7, $|X_i| = |G|/|G_x|$, where $G_x = \{g \in G : g(x) = x\}$. Now compute the size of the set

$$\{(g, x) : g \in G, x \in X, g(x) = x\}$$

in two different ways...

Complete the proof.