MATH60142/70142 Mathematics of Business & Economics
Class test – Duration: 40 minutes

29 February 2024

Question (20 marks)

Suppose a firm's production is based on three input goods x_1, x_2, and x_3, with respective prices $w_1 > 0, w_2 > 0$, and $w_3 > 0$. The production function $f : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the firm is given via

$$f(x_1, x_2, x_3) = x_1^\alpha x_2^{\alpha/2} + 2x_3, \quad \text{with } \alpha > 0. \tag{1}$$

For parts (a) to (f), assume that the quantity of good x_3 is fixed at some non-negative value, and the firm can only vary the goods x_1 and x_2 flexibly for production.

(a) **(2 marks)** Describe the economic notions of long-run and short-run. Which scenario is described above?

(b) **(3 marks)** Calculate the Marginal Rate of Technical Substitution $\text{MRTS}(x_1, x_2)$ of f, and briefly explain the economical meaning of the MRTS.

(c) **(1 mark)** The firm wants to find the input bundle that minimizes their cost while achieving some given level of output $\tilde{y}$. Write down the minimization problem of the firm (you can assume that $\tilde{y} \geq 2x_3$).

(d) **(3 marks)** Write down the Lagrangian for the minimization problem given in part (c), and derive the first-order conditions for minimization of the Lagrangian.

(e) **(3 marks)** Compute the cost minimising input bundle as a function of $\tilde{y}, w_1, w_2$, and x_3. [*You may assume that the second order condition is satisfied.*]

(f) **(1 mark)** Compute the firm's cost function $c_S^*(w_1, w_2, w_3, x_3, y)$.

For the remaining parts (g) to (j), you can now assume that the firm can vary all three input goods x_1, x_2, x_3 flexibly.

(g) **(2 mark)** Explain briefly in words (no more than three sentences needed) how the additional flexibility of input good x_3 might change the cost function from part (f). *(No calculations needed.)*

(h) **(2 mark)** Derive a value for α such that the production function is positively homogeneous of degree $k = 1$. Justify your reasoning.

(i) **(2 mark)** For which values of α does f exhibit (I) constant, and (II) increasing returns to scale? Justify your reasoning.

(j) **(1 mark)** Provide some simplified real-world example for increasing returns to scale.