IMPERIAL

MATH70015

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Quantum Mechanics I

Date: Thursday, May 8, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. The principles of quantum mechanics

Consider a system on the Hilbert space $\mathbb{C}^3$ and a Hamiltonian $\hat{H}$ represented by the matrix

$$\hat{H} = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & 2E_0 & 0 \\ 0 & 0 & 3E_0 \end{pmatrix},$$

with $E_0 \in \mathbb{R}$. Let another observable $\hat{A}$ be described by the matrix

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Calculate the eigenvalues and a set of normalised eigenvectors of $\hat{A}$.

(4 marks)

(b) Assume the system is in the state described by the vector $\psi = \frac{1}{\sqrt{2}}(1, 0, 1)^T$.

(i) With what probability does a measurement of the energy yield which outcome?

(3 marks)

(ii) With what probability does a measurement of the observable $\hat{A}$ yield which outcome?

(4 marks)

(c) Assume that at time $t = 0$ a measurement of the observable $\hat{A}$ yields the outcome -2. What is the probability that a subsequent measurement of $\hat{A}$ at time $t > 0$ yields the same result?

(4 marks)

(d) Write down a normalised state χ in which the probability to obtain the result -2 in a measurement of $\hat{A}$ is zero, and for which the energy uncertainty is as large as possible. Do these two properties determine the expectation value of $\hat{A}$ in the state χ?

(5 marks)

(Total: 20 marks)

2. **Position and momentum representation**

Consider a quantum particle in one dimension, with position and momentum operators $\hat{q}$ and $\hat{p}$ fulfilling the commutation relation $[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{I}$. The eigenstates $|q\rangle$ and $|p\rangle$ of the position and momentum operators with eigenvalues $q, p \in \mathbb{R}$, ($\hat{q}|q\rangle = q|q\rangle$, and $\hat{p}|p\rangle = p|p\rangle$), fulfil the generalised orthonormality conditions $\langle q'|q\rangle = \delta(q-q')$, and $\langle p'|p\rangle = \delta(p-p')$ and form resolutions of the identity, i.e.,

$$\int_{-\infty}^{\infty} |q\rangle\langle q|\mathrm{d}q = \hat{I} = \int_{-\infty}^{\infty} |p\rangle\langle p|\mathrm{d}p.$$

It holds

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}.$$

The position and momentum representations $\phi(q)$ and $\tilde{\phi}(p)$ of a state $|\phi\rangle$ are given by

$$\phi(q) = \langle q|\phi\rangle, \quad \text{and} \quad \tilde{\phi}(p) = \langle p|\phi\rangle,$$

respectively, and are related to each other via

$$\phi(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\tilde{\phi}(p)\mathrm{d}p.$$

(a) Show that $\langle q|\hat{p}^2|\phi\rangle = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\phi(q)$.

(4 marks)

(b) (i) Show by induction that for integer $n \geq 1$ it holds $[\hat{p}, \hat{q}^n] = -\mathrm{i}n\hbar\hat{q}^{n-1}$.

(4 marks)

(ii) Using the Taylor expansion of the operator exponential, show that

$$[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] = \hbar\mathrm{e}^{\mathrm{i}\hat{q}}.$$

(4 marks)

(c) Using the results from (b) show that $\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle$ is an eigenstate of $\hat{p}$ with eigenvalue $p+\hbar$, i.e., show that $\hat{p}\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle = (p+\hbar)\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle$. (Similarly, it holds $\hat{p}\mathrm{e}^{-\mathrm{i}\hat{q}}|p\rangle = (p-\hbar)\mathrm{e}^{-\mathrm{i}\hat{q}}|p\rangle$, but you do not need to show this.)

(3 marks)

(d) Consider the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \cos(\hat{q}).$$

Express the eigenvalue equation $\hat{H}|\phi\rangle = E|\phi\rangle$

(i) in position representation in terms of $\phi(q)$.

(2 marks)

(ii) in momentum representation, in terms of $\tilde{\phi}(p)$.

(3 marks)

(Total: 20 marks)

3. **Angular Momentum**

Consider the three components $\hat{J}_1,\ \hat{J}_2,\ \hat{J}_3$ of the angular momentum operator in three dimensions, fulfilling the commutation relations

$$[\hat{J}_j, \hat{J}_k] = \mathrm{i}\hbar\epsilon_{jkl}\hat{J}_l,$$

and the total angular momentum operator defined as $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, which commutes with all three angular momentum components $\hat{J}_k$. The ladder operators $\hat{J}_\pm$ are defined as

$$\hat{J}_\pm = \hat{J}_x \pm \mathrm{i}\hat{J}_y.$$

(a) Verify the commutation relations $[\hat{J}_3, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$, and $[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_3$.

(4 marks)

(b) Let $|\beta, m\rangle$ be a joint eigenvector of $\hat{J}^2$ and $\hat{J}_3$, such that

$$\hat{J}^2|\beta, m\rangle = \hbar^2\beta|\beta, m\rangle, \quad \text{and} \quad \hat{J}_3|\beta, m\rangle = \hbar m|\beta, m\rangle.$$

(i) Show that $m^2 \leq \beta$ by considering the expectation value $\langle\beta, m|\hat{J}^2|\beta, m\rangle$.

(5 marks)

(ii) Show that $\hat{J}_+|\beta, m\rangle$ is either an eigenvector of $\hat{J}_3$ corresponding to the eigenvalue $\hbar(m+1)$ or the zero vector, and $\hat{J}_-|\beta, m\rangle$ is either an eigenvector of $\hat{J}_3$ corresponding to the eigenvalue $\hbar(m-1)$ or the zero vector.

(4 marks)

From the above, we can conclude that the possible eigenvalues of $\hat{J}^2$ are given by $\hbar^2 j(j+1)$ with $2j \in \mathbb{N}$, and for each given value of j the eigenvalues of $\hat{J}_3$ are given by $\hbar m$ with m running in integer steps from $-j$ to j. We can then label the states by j and m such that

$$\hat{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle, \quad \text{and} \quad \hat{J}_3|j, m\rangle = \hbar m|j, m\rangle,$$

and furthermore we have

$$\hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle.$$

(c) Let us consider a system with Hamiltonian $\hat{J}_3$, which at time $t = 0$ is prepared in the state $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|j=1, m=1\rangle + \mathrm{i}|j=1, m=-1\rangle)$. Calculate the expectation value of $\hat{J}_1$

(i) At time $t = 0$.

(3 marks)

(ii) At later times $t > 0$.

(4 marks)

(Total: 20 marks)

4. A quantum wave function

Consider a quantum particle, the state of which is described by the wave function

$$\psi(x) = A\mathrm{e}^{-(x-5)^2},$$

with a real and positive normalisation constant A.

(a) Write an expression for the probability to find the particle in the interval $[0, a]$, where a is a positive constant, in terms of the wave function $\psi(x)$. You do not need to calculate the probability.

(2 marks)

(b) Sketch the absolute value of the wave function as a function of x.

(2 marks)

(c) State whether it is more likely to find the particle at a position with $x \leq 0$ or at a position $x \geq 0$. Provide a reason for your answer.

(2 marks)

(d) Find the value of A such that the wave function $\psi(x)$ is normalised to one.
Hint: You may use that $\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$, for $\mathrm{Re}(a) > 0$.

(5 marks)

(e) What is the expectation value $\langle \hat{x} \rangle$ of the position?

(2 marks)

(f) Assume that the wave function $\psi(x)$ above is the eigenfunction of a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ belonging to the eigenenergy E_0. Find the potential $V(x)$.

(7 marks)

(Total: 20 marks)

5. **Mastery - Coherent states and Husimi distribution for a $su(2)$ system**

Consider an angular momentum system with basis states $|j,m\rangle$ defined such that

$$\hat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \quad \text{and} \quad \hat{J}_3|j,m\rangle = \hbar m|j,m\rangle.$$

Specifically, we will consider the case $j = 1$ here and label the basis states by m as $|1\rangle$, $|0\rangle$, and $|-1\rangle$.

In analogy to the coherent states of the harmonic oscillator one can define angular momentum coherent states as

$$|\zeta\rangle = N(\zeta)\mathrm{e}^{\zeta\hat{J}_+/\hbar}|-1\rangle,$$

where the complex variable ζ parameterises the spherical phase space. It can be related to spherical coordinates via the parameterisation $\zeta = -\mathrm{e}^{-\mathrm{i}\phi}\tan(\frac{\theta}{2})$. $N(\zeta)$ is a real-valued positive normalisation factor.

(a) Use the Taylor expansion of the exponential and $\hat{J}_+|m\rangle = \hbar\sqrt{2-m(m+1)}|m+1\rangle$ to express the coherent states $|\zeta\rangle$ in the standard basis $|1\rangle$, $|0\rangle$, and $|-1\rangle$.

(5 marks)

(b) Verify that the normalisation constant is given by $N(\zeta) = \cos^2(\frac{\theta}{2})$.

(5 marks)

(c) We can then define a Husimi distribution on the sphere as

$$Q(\theta,\phi) = \frac{1}{4\pi}|\langle\zeta|\psi\rangle|^2.$$

Calculate the Husimi distribution

(i) of the state $|\psi\rangle = |0\rangle$;

(4 marks)

(ii) of the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|-1\rangle + |1\rangle)$.

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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MATH60015/70015

Quantum Mechanics 1 (Solutions)

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1. **The principles of quantum mechanics**

Consider a system on the Hilbert space $\mathbb{C}^3$ and a Hamiltonian $\hat{H}$ represented by the matrix

$$\hat{H} = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & 2E_0 & 0 \\ 0 & 0 & 3E_0 \end{pmatrix},$$

with $E_0 \in \mathbb{R}$. Let another observable $\hat{A}$ be described by the matrix

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Calculate the eigenvalues and a set of normalised eigenvectors of $\hat{A}$.

(Seen similar)

$\hat{A}$ is already block diagonal with one 1×1 block with eigenvalue

$$\lambda = -2,$$

where the corresponding eigenvector of $\hat{A}$ is given by

$$\phi_{-2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The remaining block of $\hat{A}$ is the Pauli matrix $\boxed{4, \mathrm{A}}$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and we either remember or calculate the eigenvalues to be

$$\lambda = \pm 1.$$

The corresponding eigenvectors (up to a phase factor) of $\hat{A}$ are the found from the eigenvalue equation as

$$\phi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \\ 0 \end{pmatrix}.$$

(b) Assume the system is in the state described by the vector $\psi = \frac{1}{\sqrt{2}}(1,0,1)^T$.

(i) With what probability does a measurement of the energy yield which outcome?

(Seen similar)

The probability to measure an eigenvalue of an observable is given by the squared modulus of the projection of the state onto the corresponding eigenspace of the observable. The eigenvectors of the Hamiltonian are the three basis vectors, thus the probabilities are found as the absolute value squares of the components of the vector ϕ as $\boxed{3, \mathrm{A}}$

$$P(E_0) = \frac{1}{2}, \quad P(2E_0) = 0, \quad P(3E_0) = \frac{1}{2}.$$

(ii) With what probability does a measurement of the observable $\hat{A}$ yield which outcome?

(Seen similar)

We express ψ in the eigenbasis of $\hat{A}$ as

$$\psi = \frac{1}{\sqrt{2}}(\phi_{-2} + \frac{1}{\sqrt{2}}(\phi_+ + \phi_-)) = \frac{1}{2}\phi_+ + \frac{1}{2}\phi_- + \frac{1}{\sqrt{2}}\phi_{-2}.$$

We then read of the probabilities for the different measurement outcomes of $\hat{A}$ as $\boxed{4, \mathrm{B}}$

$$P(-2) = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}, \quad P(\pm 1) = |\frac{1}{2}|^2 = \frac{1}{4}.$$

(c) Assume that at time $t = 0$ a measurement of the observable $\hat{A}$ yields the outcome -2. What is the probability that a subsequent measurement of $\hat{A}$ at time $t > 0$ yields the same result?

(Seen similar)

The state directly after the measurement is the eigenvector of $\hat{A}$ belonging to the eigenvalue -2, i.e. $\boxed{2, \mathrm{C}}$

$$\psi(t = 0) = \phi_{-2}.$$

This is also an eigenstate of the Hamiltonian, and thus the state changes only by a phase factor in the time evolution. Therefore, at later times the probability to find the eigenvalue -2 in a measurement of $\hat{A}$ is still given by $P(-2) = 1$. $\boxed{2, \mathrm{B}}$

(d) Write down a normalised state χ in which the probability to obtain the result -2 in a measurement of $\hat{A}$ is zero, and for which the energy uncertainty is as large as possible. Do these two properties determine the expectation value of $\hat{A}$ in the state χ?

(unseen)

Given that the probability to obtain the outcome -2 in a measurement of $\hat{A}$ is zero, we have

$$\chi \propto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Now to maximise the uncertainty of the energy in this state it has to hold that $|x|^2 = |y|^2$, and for the state to be normalised we have $|x|^2 + |y|^2 = 1$. Thus we find that the states fulfilling the requirement (up to an overall phase) can be written as $\boxed{3, \text{D}}$

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\theta} \\ 0 \end{pmatrix}, \quad \text{with} \quad \theta \in \mathbb{R}.$$

This does not uniquely specify the expectation value of $\hat{A}$. For example, we have $\langle \hat{A} \rangle = 1$ for $\theta = 0$ and $\langle \hat{A} \rangle = -1$ for $\theta = \pi$. $\boxed{2, \text{D}}$

2. Position and momentum representation

Consider a quantum particle in one dimension, with position and momentum operators $\hat{q}$ and $\hat{p}$ fulfilling the commutation relation $[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{I}$. The eigenstates $|q\rangle$ and $|p\rangle$ of the position and momentum operators with eigenvalues $q, p \in \mathbb{R}$, ($\hat{q}|q\rangle = q|q\rangle$, and $\hat{p}|p\rangle = p|p\rangle$), fulfil the generalised orthonormality conditions $\langle q'|q\rangle = \delta(q - q')$, and $\langle p'|p\rangle = \delta(p - p')$ and form resolutions of the identity, i.e.,

$$\int_{-\infty}^{\infty} |q\rangle\langle q|\mathrm{d}q = \hat{I} = \int_{-\infty}^{\infty} |p\rangle\langle p|\mathrm{d}p.$$

It holds

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}.$$

The position and momentum representations $\phi(q)$ and $\tilde{\phi}(p)$ of a state $|\phi\rangle$ are given by

$$\phi(q) = \langle q|\phi\rangle, \quad \text{and} \quad \tilde{\phi}(p) = \langle p|\phi\rangle,$$

respectively, and are related to each other via

$$\phi(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq} \tilde{\phi}(p)\mathrm{d}p.$$

(a) Show that

$$\langle q|\hat{p}^2|\phi\rangle = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}q^2}\phi(q).$$

(Seen similar)

We insert a resolution of the identity $\hat{I} = \int_{-\infty}^{\infty} |p\rangle\langle p|\mathrm{d}p$ to write $\boxed{2, \mathrm{B}}$

$$\langle q|\hat{p}^2|\phi\rangle = \int_{-\infty}^{\infty} \langle q|\hat{p}^2|p\rangle\langle p|\phi\rangle\mathrm{d}p$$

Now using that $\hat{p}^2|p\rangle = p\hat{p}|p\rangle = p^2|p\rangle$, and inserting $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}$ and $\tilde{\phi}(p) = \langle p|\phi\rangle$ we find

$$\langle q|\hat{p}^2|\phi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^2 \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\tilde{\phi}(p)\mathrm{d}p$$

Now we notice that $p^2\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq} = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}$, and thus $\boxed{2, \mathrm{C}}$

$$\langle q|\hat{p}^2|\phi\rangle = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\tilde{\phi}(p)\mathrm{d}p\right) = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\phi(q). \qquad \square$$

(b) (i) Show by induction that for integer $n \geq 1$ it holds $[\hat{p}, \hat{q}^n] = -\mathrm{i}n\hbar\hat{q}^{n-1}$.

(seen similar)

We know the statement holds for $n = 1$. Let us now assume we have proven it for n. We have

$$[\hat{p}, \hat{q}^{n+1}] = \hat{q}[\hat{p}, \hat{q}^n] + [\hat{p}, \hat{q}]\hat{q}^n.$$

Using the induction assumption we thus have $\boxed{4, \mathrm{A}}$

$$[\hat{p}, \hat{q}^{n+1}] = -\mathrm{i}n\hbar\hat{q}\hat{q}^{n-1} - \mathrm{i}\hbar\hat{q}^n = -\mathrm{i}(n+1)\hbar\hat{q}^n. \qquad \square$$

(ii) Using the Taylor expansion of the operator exponential, show that

$$[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] = \hbar\mathrm{e}^{\mathrm{i}\hat{q}}.$$

(unseen)

Using $\boxed{2, \mathrm{A}}$

$$\mathrm{e}^{\mathrm{i}\hat{q}} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\hat{q}^n,$$

we have

$$[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}[\hat{p}, \hat{q}^n].$$

Now using the result from part (b)(i) we find $\boxed{2, \mathrm{C}}$

$$[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}(-\mathrm{i}n\hbar\hat{q}^{n-1}) = \hbar\sum_{n=0}^{\infty} \frac{\mathrm{i}^{n-1}}{n!}n\hat{q}^{n-1}.$$

Re-labelling the sum we have

$$[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] = \hbar\sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\hat{q}^n = \hbar\mathrm{e}^{\mathrm{i}\hat{q}}. \qquad \square$$

(c) Using the results from (b) show that $\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle$ is an eigenstate of $\hat{p}$ with eigenvalue $p+\hbar$, i.e., show that $\hat{p}\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle = (p+\hbar)\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle$. (Similarly, it holds $\hat{p}\mathrm{e}^{-\mathrm{i}\hat{q}}|p\rangle = (p-\hbar)\mathrm{e}^{-\mathrm{i}\hat{q}}|p\rangle$, but you do not need to show this.)

(unseen)

We have $\boxed{3, \mathrm{B}}$

$$\hat{p}\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle = \left([\hat{p}, \mathrm{e}^{\mathrm{i}\hat{q}}] + \mathrm{e}^{\mathrm{i}\hat{q}}\hat{p}\right)|p\rangle$$

using the result from (b) we thus find

$$\hat{p}\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle = \left(\hbar\mathrm{e}^{\mathrm{i}\hat{q}} + \mathrm{e}^{\mathrm{i}\hat{q}}\hat{p}\right)|p\rangle = (p+\hbar)\mathrm{e}^{\mathrm{i}\hat{q}}|p\rangle,$$

(d) Consider the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \cos(\hat{q}).$$

Express the eigenvalue equation $\hat{H}|\phi\rangle = E|\phi\rangle$

(i) in position representation in terms of $\phi(q)$.

(seen smiliar)

We have

$$\langle q|\hat{H}|\phi\rangle = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}q^2}\phi(q) + \cos(q)\phi(q),$$

and thus, in position representation the eigenvalue equation reads $\boxed{2, \mathrm{A}}$

$$-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}q^2}\phi(q) + \cos(q)\phi(q) = E\phi(x).$$

(ii) in momentum representation, in terms of $\tilde{\phi}(p)$.

(unseen)

Using the result from part (c) we have

$$\langle p|\mathrm{e}^{\mathrm{i}\hat{q}}|\phi\rangle = \tilde{\phi}(p+\hbar)$$

and $\boxed{3, \mathrm{D}}$

$$\langle p|\mathrm{e}^{-\mathrm{i}\hat{q}}|\phi\rangle = \tilde{\phi}(p-\hbar),$$

that is,

$$\langle p|\cos(\hat{q})|\phi\rangle = \frac{1}{2}\left(\tilde{\phi}(p-\hbar) + \tilde{\phi}(p+\hbar)\right),$$

and therefore the eigenvalue equation in momentum representation reads

$$\frac{p^2}{2}\tilde{\phi}(p) + \frac{1}{2}\left(\tilde{\phi}(p-\hbar) + \tilde{\phi}(p+\hbar)\right) = E\tilde{\phi}(p).$$

3. **Angular Momentum**

Consider the three components $\hat{J}_1$, $\hat{J}_2$, $\hat{J}_3$ of the angular momentum operator in three dimensions, fulfilling the commutation relations

$$[\hat{J}_j, \hat{J}_k] = \mathrm{i}\hbar\epsilon_{jkl}\hat{J}_l,$$

and the total angular momentum operator defined as $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, which commutes with all three angular momentum components $\hat{J}_k$. The ladder operators $\hat{J}_\pm$ are defined as

$$\hat{J}_\pm = \hat{J}_x \pm \mathrm{i}\hat{J}_y.$$

(a) Verify the commutation relations

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_3.$$

(Seen and straight forward)

Solution:

We have $\boxed{4, \mathrm{A}}$

$$[\hat{J}_3, \hat{J}_+] = [\hat{J}_3, \hat{J}_x] + \mathrm{i}[\hat{J}_3, \hat{J}_y] = \mathrm{i}\hbar\hat{J}_y + \hbar\hat{J}_x = \hbar\hat{J}_+$$

similarly,

$$[\hat{J}_3, \hat{J}_-] = \mathrm{i}\hbar\hat{J}_y - \hbar\hat{J}_x = -\hbar\hat{J}_-,$$

and

$$[\hat{J}_+, \hat{J}_-] = [\hat{J}_x + \mathrm{i}\hat{J}_y, \hat{J}_x - \mathrm{i}\hat{J}_y] = -\mathrm{i}[\hat{J}_x, \hat{J}_y] + \mathrm{i}[\hat{J}_y, \hat{J}_x] = 2\hbar\hat{J}_3.$$

(b) Let $|\beta, m\rangle$ be a joint eigenvector of $\hat{J}^2$ and $\hat{J}_3$, such that

$$\hat{J}^2|\beta, m\rangle = \hbar^2\beta|\beta, m\rangle, \quad \text{and} \quad \hat{J}_3|\beta, m\rangle = \hbar m|\beta, m\rangle.$$

(i) Show that $m^2 \leq \beta$ by considering the expectation value $\langle\beta, m|\hat{J}^2|\beta, m\rangle$.

(Seen)

Solution:

By definition we have $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_3^2$, and thus $\boxed{3, \mathrm{A}}$

$$\langle\beta, m|\hat{J}^2|\beta, m\rangle = \langle\beta, m|\hat{J}_x^2|\beta, m\rangle + \langle\beta, m|\hat{J}_y^2|\beta, m\rangle + \langle\beta, m|\hat{J}_3^2|\beta, m\rangle$$

We have

$$\langle\beta, m|\hat{J}_j^2|\beta, m\rangle = ||\hat{J}_j|\beta, m\rangle||^2,$$

which has to be positive. For $\hat{J}_3$ we specifically have $\boxed{2, \mathrm{B}}$

$$\langle\beta, m|\hat{J}_3^2|\beta, m\rangle = \hbar^2 m^2.$$

Thus, we have

$$\langle\beta, m|\hat{J}^2|\beta, m\rangle \geq \hbar^2 m^2.$$

At the same time we have

$$\langle\beta, m|\hat{J}^2|\beta, m\rangle = \hbar^2\beta,$$

and thus we deduce

$$\beta \geq m^2.$$

(ii) Show that $\hat{J}_+|\beta, m\rangle$ is either an eigenvector of $\hat{J}_3$ corresponding to the eigenvalue $\hbar(m+1)$ or the zero vector, and $\hat{J}_-|\beta, m\rangle$ is either an eigenvector of $\hat{J}_3$ corresponding to the eigenvalue $\hbar(m-1)$ or the zero vector.

(Seen)

Solution:

Let us consider $\hat{J}_3\hat{J}_+|\beta, m\rangle$: $\boxed{2, \mathrm{A}}$

$$\begin{aligned}\hat{J}_3\hat{J}_+|\beta, m\rangle &= (\hat{J}_3\hat{J}_+ - \hat{J}_+\hat{J}_3 + \hat{J}_+\hat{J}_3)|\beta, m\rangle \\ &= ([\hat{J}_3, \hat{J}_+] + \hat{J}_+\hat{J}_3)|\beta, m\rangle \\ &= (\hbar\hat{J}_+ + \hbar m\hat{J}_+)|\beta, m\rangle \\ &= \hbar(m+1)\hat{J}_+|\beta, m\rangle.\end{aligned}$$ $\boxed{2, \mathrm{B}}$

Thus, the vector $\hat{J}_+|\beta, m\rangle$ is either the zero vector, or it is an eigenvector of $\hat{J}_3$ with eigenvalue $\hbar(m+1)$. A similar calculation yields

$$\hat{J}_3\hat{J}_-|\beta, m\rangle = \hbar(m-1)\hat{J}_-|\beta, m\rangle.$$

Thus, the vector $\hat{J}_-|\beta, m\rangle$ is either the zero vector, or it is an eigenvector of $\hat{J}_3$ with eigenvalue $\hbar(m-1)$.

From the above, we can conclude that the possible eigenvalues of $\hat{J}^2$ are given by $\hbar^2 j(j+1)$ with $2j \in \mathbb{N}$, and for each given value of j the eigenvalues of $\hat{J}_3$ are given by $\hbar m$ with m running in integer steps from $-j$ to j. We can then label the states by j and m such that

$$\hat{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle, \quad \text{and} \quad \hat{J}_3|j, m\rangle = \hbar m|j, m\rangle,$$

and furthermore we have

$$\hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle.$$

(c) Let us consider a system with Hamiltonian $\hat{J}_3$, which at time $t = 0$ is prepared in the state $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|j=1, m=1\rangle + \mathrm{i}|j=1, m=-1\rangle)$. Calculate the expectation value of $\hat{J}_1$

(i) At time $t = 0$.

(Unseen) $\boxed{3, \mathrm{C}}$

We have $\hat{J}_1 = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$, and thus

$$\langle\psi_0|\hat{J}_1|\psi_0\rangle = \frac{1}{2}\left(\langle\psi_0|\hat{J}_+|\psi_0\rangle + \langle\psi_0|\hat{J}_-|\psi_0\rangle\right)$$

since $\hat{J}_-$ is the adjoint of $\hat{J}_+$ we only need to calculate $\langle\psi_0|\hat{J}_+|\psi_0\rangle$ and can then use $\langle\psi_0|\hat{J}_-|\psi_0\rangle = \langle\psi_0|\hat{J}_+|\psi_0\rangle^*$. We have

$$\hat{J}_+|\psi(t=0)\rangle = \mathrm{i}\hbar|j=1, m=0\rangle$$

Since the angular momentum states are orthonormal, we thus find

$$\langle\psi_0|\hat{J}_+|\psi_0\rangle = \frac{\mathrm{i}}{\sqrt{2}}\hbar,$$

and thus

$$\langle\psi_0|\hat{J}_1|\psi_0\rangle = 0.$$

(ii) At later times $t > 0$.

(Unseen)

Solution:

The time evolved state is given by $\boxed{4, \text{D}}$

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{J}_3 t/\hbar}|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}t}|1,1\rangle + \mathrm{i}\mathrm{e}^{\mathrm{i}t}|1,-1\rangle\right)$$

We then calculate

$$\langle\psi_0|\hat{J}_+|\psi_0\rangle = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\sqrt{2}}\hbar,$$

and thus

$$\langle\psi_0|\hat{J}_-|\psi_0\rangle = \frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}t}}{\sqrt{2}}\hbar,$$

therefore

$$\langle\psi_0|\hat{J}_1|\psi_0\rangle = -\frac{\hbar}{\sqrt{2}}\sin(t).$$

4. **A quantum wave function**

Consider a quantum particle, the state of which is described by the wave function

$$\psi(x) = A\mathrm{e}^{-(x-5)^2},$$

with a real and positive normalisation constant A.

(a) Write an expression for the probability to find the particle in the interval $[0, a]$, where a is a positive constant, in terms of the wave function $\psi(x)$. You do not need to calculate the probability.

(Seen) $\boxed{2, \mathrm{A}}$

The probability is given by $P(x \in [0,a]) = \int_0^a |\psi(x)|^2 \mathrm{d}x$, assuming that $\psi(x)$ is normalised to one.

(b) Sketch the absolute value of the wave function as a function of x.

(Seen similar) $\boxed{2, \mathrm{A}}$

Full marks for a sketch of this Gaussian shape with maximum at $x = 5$, and an asymptotic decay to zero at x to plus and minus infinity, symmetric around $x = 5$.

(c) Is it more likely to find the particle at a position with $x \leq 0$ or at a position $x \geq 0$. Provide a reason for your answer.

(Seen similar) $\boxed{2, \mathrm{B}}$

The wavefunction is symmetric around the single peak at $x = 5$, thus it is more likely to find the particle at a positive position.

(d) Find the value of A such that the wave function $\psi(x)$ is normalised to one.
Hint: You may use that $\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$, for $\mathrm{Re}(a) > 0$.

(method seen) $\boxed{2, \mathrm{A}}$

We have

$$\begin{aligned} ||\psi||^2 &= A^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-2(x-5)^2} \mathrm{d}x \\ &= A^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-2y^2} \mathrm{d}y. \end{aligned}$$

where we have made the substitution $y = x - 5$. $\boxed{3, \mathrm{B}}$
Using the integral provided we find

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-2y^2} \mathrm{d}y = \sqrt{\frac{\pi}{2}},$$

and thus

$$||\psi||^2 = A^2 \sqrt{\frac{\pi}{2}}.$$

That is, $\psi(x)$ is normalised to one for $A = \left(\frac{2}{\pi}\right)^{1/4}$.

(e) What is the expectation value $\langle \hat{x} \rangle$ of the position?

(Seen similar)

The wave function is symmetric around a single maximum, thus this is the expectation value

$$\langle \hat{x} \rangle = 5.$$

2, A

(f) Assume that the wave function $\psi(x)$ above is the eigenfunction of a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ belonging to the eigenenergy E_0. Find the potential $V(x)$.

(unseen)

Let $|\psi\rangle$ the vector with position representation $\psi(x) = \langle x|\psi\rangle$. For $\psi(x)$ be an eigenfunction of $\hat{H}$ with energy E_0 it has to hold

$$\langle x|\hat{H}|\psi\rangle = \langle x|E_0|\psi\rangle,$$

or explicitly

4, D

$$-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V(x)\psi(x) = E_0\psi(x).$$

We calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = -2A(x-5)\mathrm{e}^{-(x-5)^2},$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = A(-2+4(x-5)^2)\mathrm{e}^{-(x-5)^2} = (-2+4(x-5)^2)\psi(x),$$

and insert this into the Schrödinger equation in position representation to find

3, C

$$-\frac{\hbar^2}{2m}(-2+4(x-5)^2)\psi(x) + V(x)\psi(x) = E_0\psi(x),$$

and thus we conclude that

$$V(x) = E_0 - \frac{\hbar^2}{m} + \frac{2\hbar^2}{m}(x-5)^2.$$

5. **Mastery - Coherent states and Husimi distribution for a $su(2)$ system**

Consider an angular momentum system with basis states $|j,m\rangle$ defined such that

$$\hat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \quad \text{and} \quad \hat{J}_3|j,m\rangle = \hbar m|j,m\rangle.$$

Specifically, we will consider the case $j=1$ here and label the basis states by m as $|1\rangle$, $|0\rangle$, and $|-1\rangle$.

In analogy to the coherent states of the harmonic oscillator one can define angular momentum coherent states as

$$|\zeta\rangle = N(\zeta)\mathrm{e}^{\zeta\hat{J}_+/\hbar}|-1\rangle,$$

where the complex variable ζ parameterises the spherical phase space. It can be related to spherical coordinates via the parameterisation $\zeta = -\mathrm{e}^{-\mathrm{i}\phi}\tan(\frac{\theta}{2})$. $N(\zeta)$ is a real-valued positive normalisation factor.

(a) Use the Taylor expansion of the exponential and $\hat{J}_+|m\rangle = \hbar\sqrt{2-m(m+1)}|m+1\rangle$ to express the coherent states $|\zeta\rangle$ in the standard basis $|1\rangle$, $|0\rangle$, and $|-1\rangle$.

Taylor expanding the exponential we have

$$|\zeta\rangle = N(\zeta)\sum_{n=0}^{\infty}\frac{\zeta^n}{n!\hbar^n}\hat{J}_+^n|-1\rangle.$$

We have $\boxed{5, \mathrm{M}}$

$$\hat{J}_+|-1\rangle = \hbar\sqrt{2}|0\rangle, \quad \hat{J}_+|0\rangle = \hbar\sqrt{2}|1\rangle, \quad \text{and} \quad \hat{J}_+|1\rangle = 0.$$

Thus we find

$$|\zeta\rangle = N(\zeta)\left(|-1\rangle + \sqrt{2}\zeta|0\rangle + \zeta^2|1\rangle\right).$$

(b) Verify that the normalisation constant is given by $N(\zeta) = \cos^2(\frac{\theta}{2})$.

We calculate the normalisation of a coherent state $|\zeta\rangle$ as $\boxed{5, \mathrm{M}}$

$$\begin{aligned}\langle\zeta|\zeta\rangle &= N^2(\zeta)\left(\langle -1| + \sqrt{2}\zeta^*\langle 0| + \zeta^2\langle 1|\right)\left(|-1\rangle + \sqrt{2}\zeta|0\rangle + \zeta^2|1\rangle\right)\\ &= N^2(\zeta)\left(1 + 2|\zeta|^2 + |\zeta|^4\right)\end{aligned}$$

We have

$$|\zeta|^2 = \tan^2(\theta/2) = \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)},$$

and thus

$$1 + 2|\zeta|^2 + |\zeta|^4 = \frac{\cos^4(\theta/2) + 2\cos^2(\theta/2)\sin^2(\theta/2) + \sin^4(\theta/2)}{\cos^4(\theta/2)} = \frac{1}{\cos^4(\theta/2)}.$$

Thus we have

$$\langle\zeta|\zeta\rangle = \frac{N^2(\zeta)}{\cos^4(\theta/2)},$$

and we find $|\zeta\rangle$ to be normalised to one for

$$N(\zeta) = \cos^2(\theta/2). \qquad \square$$

(c) We can then define a Husimi distribution on the sphere as

$$Q(\theta, \phi) = \frac{1}{4\pi}|\langle\zeta|\psi\rangle|^2.$$

Calculate the Husimi distribution

(i) of the state $|\psi\rangle = |0\rangle$;
We have by definition $\boxed{4, \mathrm{M}}$

$$Q(\theta, \phi) = \frac{1}{4\pi}|\langle\zeta|0\rangle|^2.$$

From (a) we deduce

$$\langle\zeta|0\rangle = \sqrt{2}N(\zeta)\zeta^*,$$

and thus

$$Q(\theta, \phi) = \frac{1}{2\pi}N^2(\theta)|\zeta|^2 = \frac{1}{2\pi}\cos^4(\theta/2)\tan^2(\theta/2) = \frac{1}{2\pi}\cos^2(\theta/2)\sin^2(\theta/2) = \frac{1}{8\pi}\sin^2(\theta).$$

(ii) of the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|-1\rangle + |1\rangle)$.
We have by definition

$$Q(\theta, \phi) = \frac{1}{8\pi}\left|\langle\zeta|-1\rangle + \langle\zeta|1\rangle\right|^2.$$

From (a) we deduce $\boxed{6, \mathrm{M}}$

$$\langle\zeta|-1\rangle = N(\zeta), \quad \text{and} \quad \langle\zeta|-1\rangle = N(\zeta)(\zeta^*)^2$$

and thus

$$\begin{aligned} Q(\theta, \phi) &= \frac{1}{8\pi}N^2(\theta)|1 + (\zeta^*)^2|^2 \\ &= \frac{1}{8\pi}\left(\cos^4(\theta/2) - 2\cos(2\phi)\cos^2(\theta/2)\sin^2(\theta/2) + \sin^4(\theta/2)\right). \end{aligned}$$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70015 Quantum Mechanics I Markers Comments

Question 1

Parts (a)-(c) did not come as a surprise to anyone as there are similar questions in most years, and there were very few problems here. Part (d) caused a bit of overthinking with many. I had hoped that without a lengthy calculation it should be clear that the uncertainty of H is largest when the two contributing eigenstates have the same probability, but few realised this before calculating. Nevertheless there were a number of correct solutions.

Question 2

In part a) I unfortunately had not made it completely clear that you were not supposed to use the position representation of the momentum operator, but to actually derive it. I gave one mark for those who just used it. Part (b) caused hardly any problems for anyone, I was very strict about the limits of the sum, and several people lost a mark there. Part d) was perceived as harder as I had anticipated, I only saw a handful of correct answers.

Question 3

Few problems in parts a) and b), the commutators are fiddly, and some people lost time not remembering the commutator rules. Apologies also, if anyone was confused by the typo (I used the indices x and y instead of 1 and 2 in the definition of J+-. Anyone who didn't give a good reason why J_j^2 are positive lost a mark.
Part c was a little harder for many. It would have sufficed to calculate J+ here, as J- is the conjugate of this and J1 is the real part. A common mistake was to think that J+|1,-1 was proportional to |1,1, not |1,0. That mistake lead to two marks being lost in part c(i), but I did not double penalise this and gave full marks in part (c)(ii) if it repeated the same mistake but contained now further mistakes.

Question 4

This was a very easy question that caused few problems. The most common issue I have seen was with the sketch of the probability amplitude, where many people depicted a wave function with a tip at x=5, rather than a smooth gaussian profile.

Question 5

The question went very well for the vast majority. Common mistakes included forgetting a complex conjugation in the inner product.