MATH70010 Notes: Geometric Mechanics

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Text for the course MATH70010:

Geometric Mechanics I: Dynamics and Symmetry, by Darryl D Holm
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MATH70010

Should it be PA or AP? We will do both in this class!

AP = Applications of Pure maths, e.g., Noether's theorem: Lie group symmetry of Hamilton's variational principle implies conservation laws for its equations of motion.

PA = Purifications of Applied maths, e.g., Euler fluid dynamics describes geodesic flow on the manifold of smooth invertible maps acting on the domain of flow.

We will do both in this class! We'll leave it to you decide which it was!

Marks!

- ***Lecture notes*** will be available online at `http://wwwf.imperial.ac.uk/~dholm/classnotes/`
- ***Class participation:***
  Definition of class participation:
  For each lecture, answer for yourself the following two questions in complete sentences (no lists!)

  Then send it to me by email or mention it in problem sessions and try to create a dialogue.

  (i) What was this lecture about? Short original paragraph discussing topics and their relationships with each other.

  (ii) What was your take away from this lecture? What question did this lecture suggest to you that might be answerable using the material in this lecture?
- ***Problem sessions:*** Thursdays 13:00-14:00 Hux 140
- ***Office Hours:*** Flexible – on MS Teams
- ***Assessed Homework 10%***

Figure 1: ***Three assessed homeworks at 3-week intervals. The Final Exam taken mainly from these.***

- ***Final Exam 90%*** – *strongly* based on the assessed homework. Note Risk/Benefit leverage on your homework!

Class introduction

- **Classical mechanics may be visualised as a commutative diagram.**

$$
\begin{tikzcd}
& L(q,v) : TM \to \mathbb{R} \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] \arrow[dl, "\text{Lagrange variational principle}"] & & H(q,p) : T^*M \to \mathbb{R} \arrow[dl, "\text{Hamilton variational principle}"] \\
\text{Euler-Lagrange eqns} \arrow[rr, Leftrightarrow] & & \text{Hamilton's eqns} &
\end{tikzcd}
$$

Figure 2: Commutative diagram of Classical Mechanics

- ***Hamilton's principle*** (HP) [Hamilton1835] and ***Legendre transformation*** (LT) [Legendre1787]:]
- ***Noether theorem*** [Noether1918]: Each Lie symmetry of HP implies conservation of ***momentum map***, $J_\xi(q,p)$

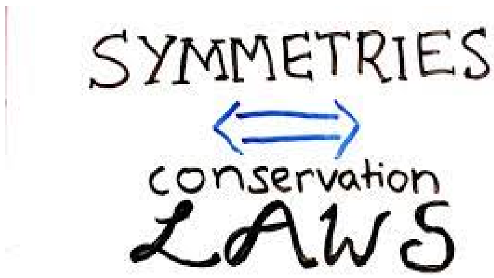

Figure 3: Noether's theorem organises classical mechanics, on both the Lagrangian and Hamiltonian sides.

- **Geometric mechanics extends equivariance to a cube of commutative diagrams.**

- See sec:GMF

What do we mean by *Geometric Mechanics*? E.g., Rigid body motion

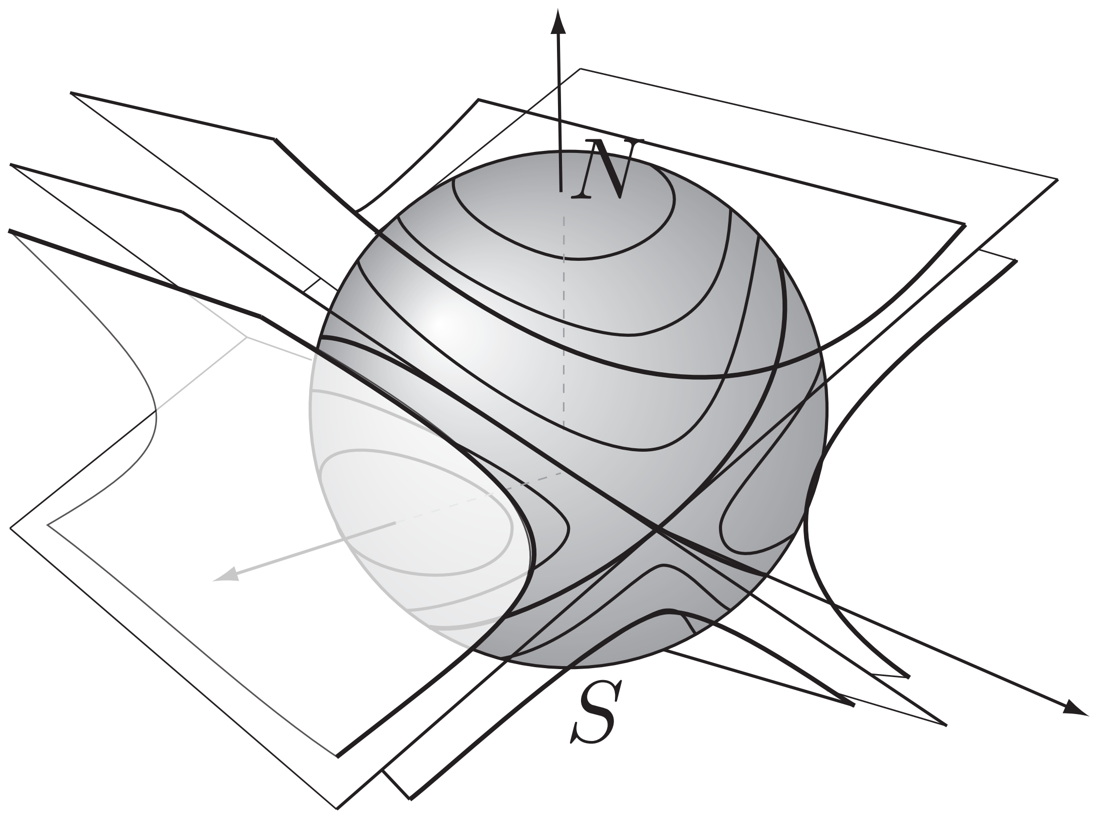

Figure 4: ***Rigid body dynamics follows the intersections of two-dimensional surfaces in $\mathbb{R}^3$ that are level sets conserved energy and body angular momentum. Both level sets are symplectic manifolds.***

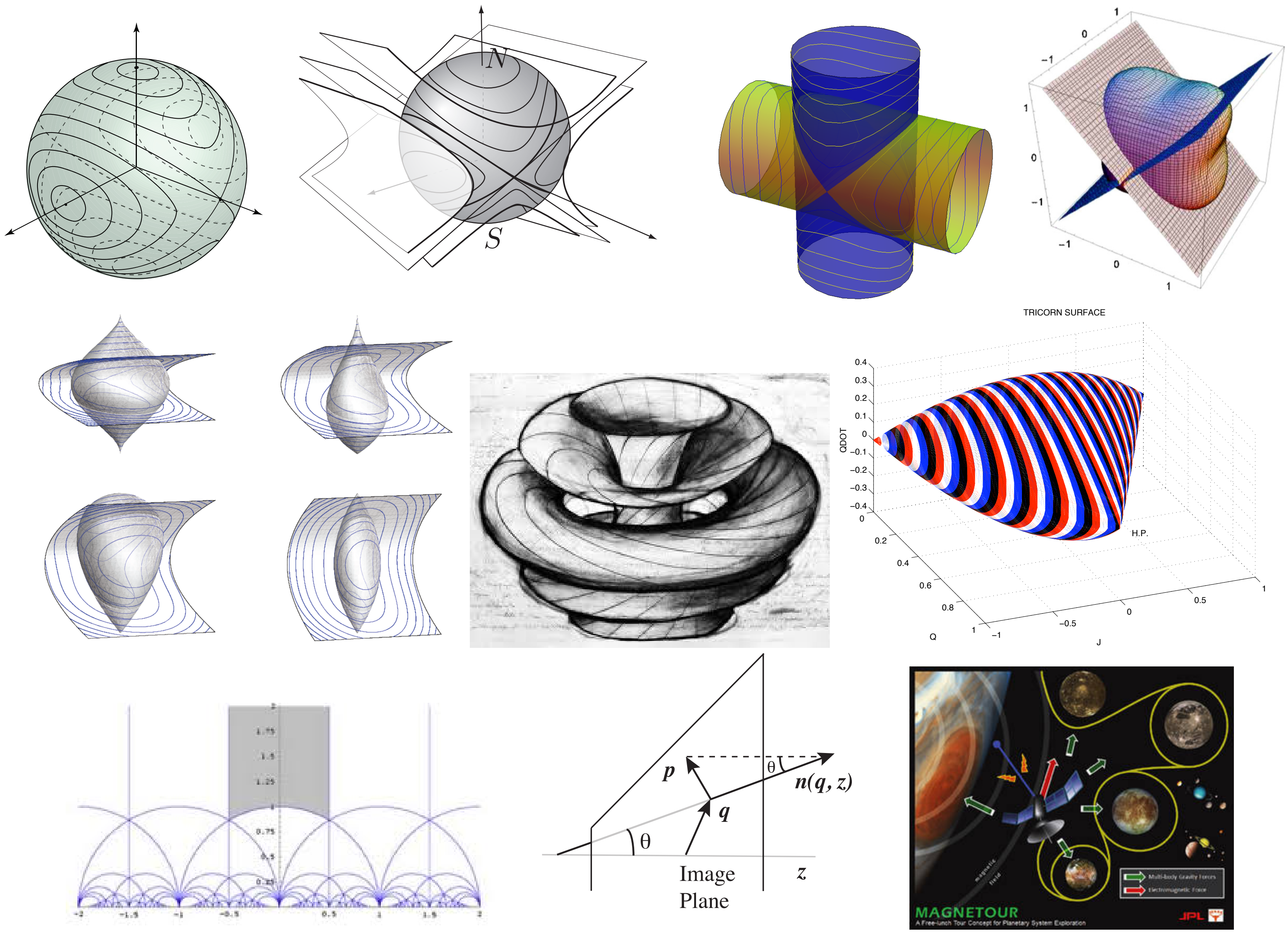

Figure 5: *Various classical mechanics examples of GM in finite dimensions can easily be investigated on the GM cube. These include both deterministic and stochastic [Ho2015] versions of the: Planar nonlinear resonant oscillator - Hopf fibration; Rigid body and heavy top; Spherical pendulum and double pendulum; Geodesic flow, e.g. on Lobachevsky space; Monodromy in molecular dynamics, e.g., H_2O and CO_2, Design of multi-moon orbiters.*

Schedule: Readings, Lecture Topics, Assessed Homework MATH70010 Spring Term 2023

Problem sessions: Tues 13:00-14:00, weeks 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Week 1 pp. 1-8: Mon Lect#1 Marking scheme, background and overview for MATH70010

9-31: For Monday Read pages 9-30 as a Review and Self-Test of the prerequisites

33-41: Mon Lect#1 GM example – Kepler's problem: Newton, Lagrange, Hamilton and Noether

Week 2 pp. 43-51: Mon Lect#2 Review of Hamiltonian dynamics on symplectic manifolds & momentum maps

Assessed Homework #1 30 Jan Due by 13 Feb 2023

Week 3 pp. 52-60: Mon Lect#3 The framework of Geometric Mechanics

Week 4 pp. 62-74: Mon Lect#4 Lie group theory review for GM

Week 5 pp. 76-81: Mon Lect#5 Reduction by symmetry on the Hamiltonian side

Assessed Homework #2 20 Feb Due by 6 Mar 2023

Week 6 pp. 83-95: Mon Lect#6 Reduction by symmetry on the Lagrangian side

Week 7 pp. 97-116: Mon Lect#7 Transformation theory for differential forms and vector fields

Week 8 pp. 118-127 : Mon Lect#8 Integral calculus formulas arising from Stokes' theorem

Week 9 pp. 129-139: Mon Lect#9 Hamilton's principle and the geometry of ideal fluid dynamics

Assessed Homework #3 13 Mar Due by 27 March 2023

Week 10 pp. 141-151: Mon Lect#10 Kelvin's circulation theorem, Lamb surfaces and linking numbers for vorticity

Week 11 pp. 153-160: Mon Lect#11 Ertel Theorem for potential vorticity and elements of geophysical fluid dynamics

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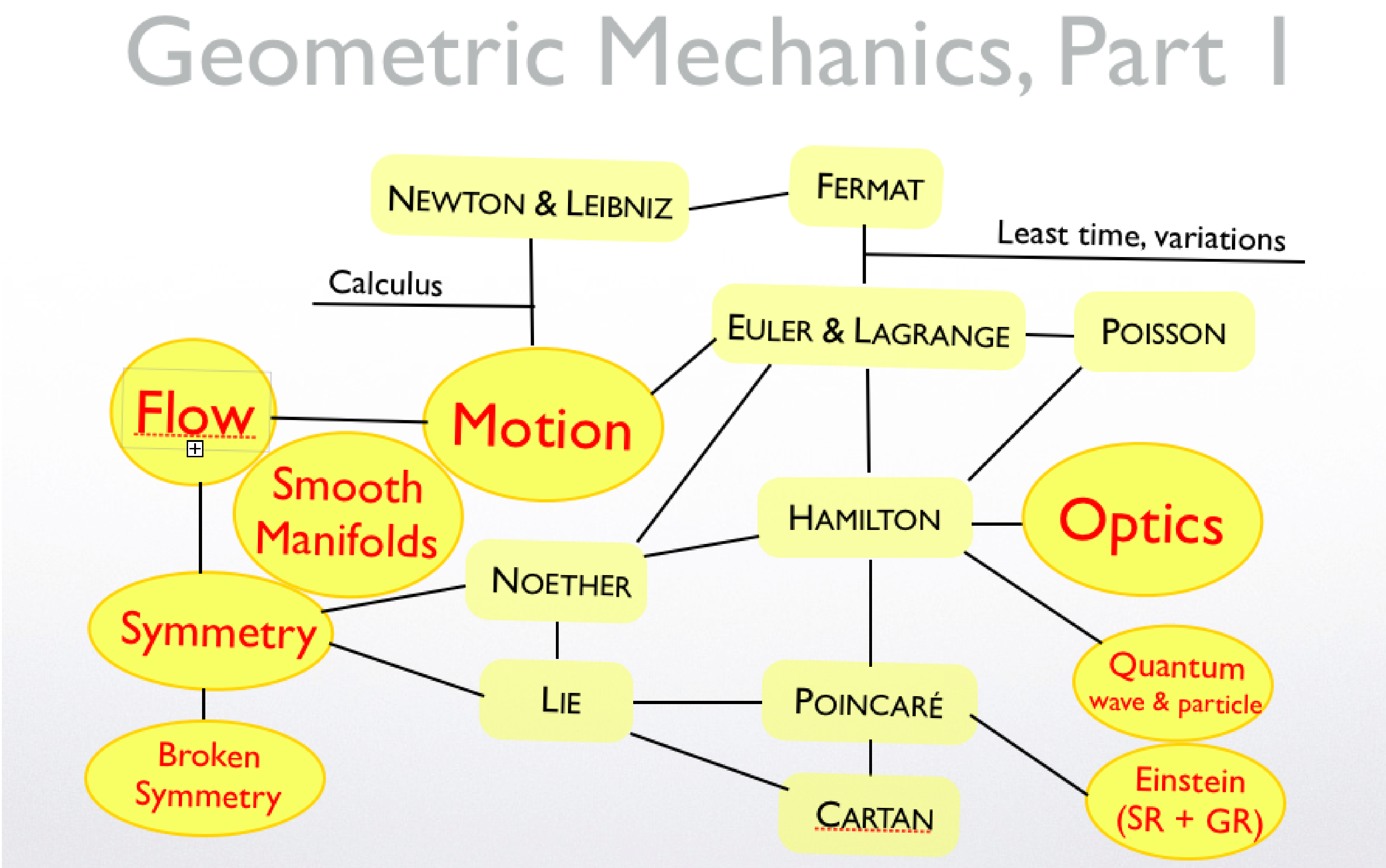


Figure 6: Geometric Mechanics has involved many great mathematicians!

Go2Kepler See sec:GMF

1 Self test: Basic elements of classical mechanics

1.1 Definitions: Space, Time, Motion, ..., Tangent space, Velocity, Motion equation

Space

$\boxed{\textbf{\textit{Space}}}$ is taken to be a smooth manifold Q with points $q \in Q$ (Positions, States, Configurations).

Let Q be a $\boxed{\textbf{\textit{smooth manifold}}}$ $\dim Q = n$. That is, Q is a smooth space that is locally $\mathbb{R}^n$. Operationally, a smooth manifold is a space on which the rules of calculus apply.

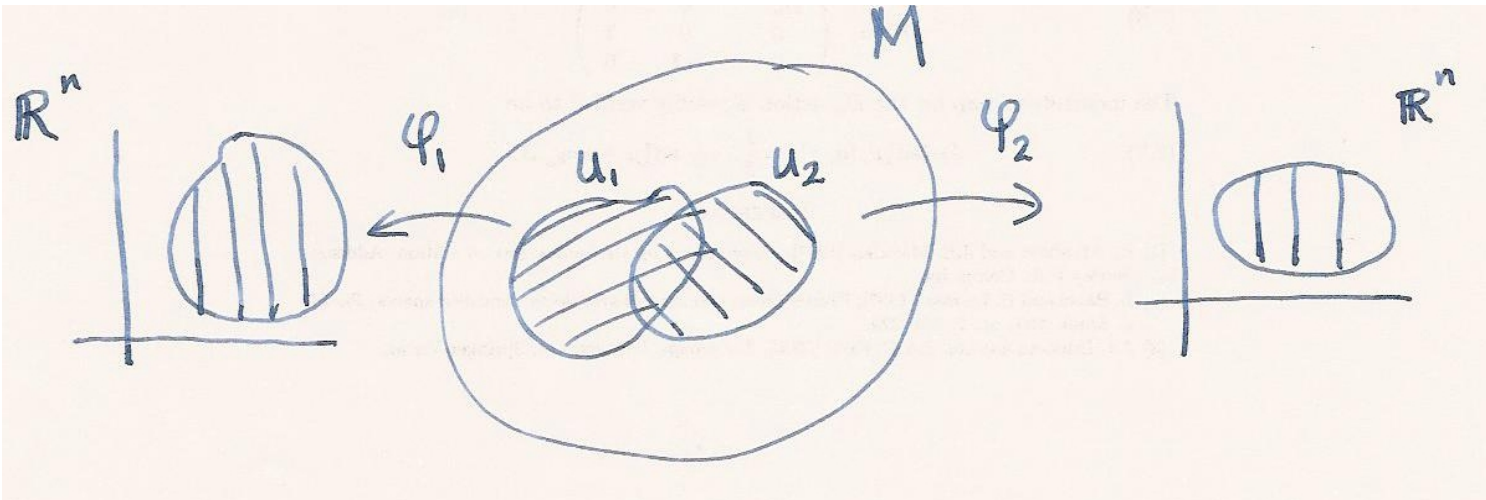


Figure 7: A manifold Q is defined by the disjoint union (or, atlas) of local charts, each of which is isomorphic to $\mathbb{R}^{\dim Q}$.

Examples of manifolds

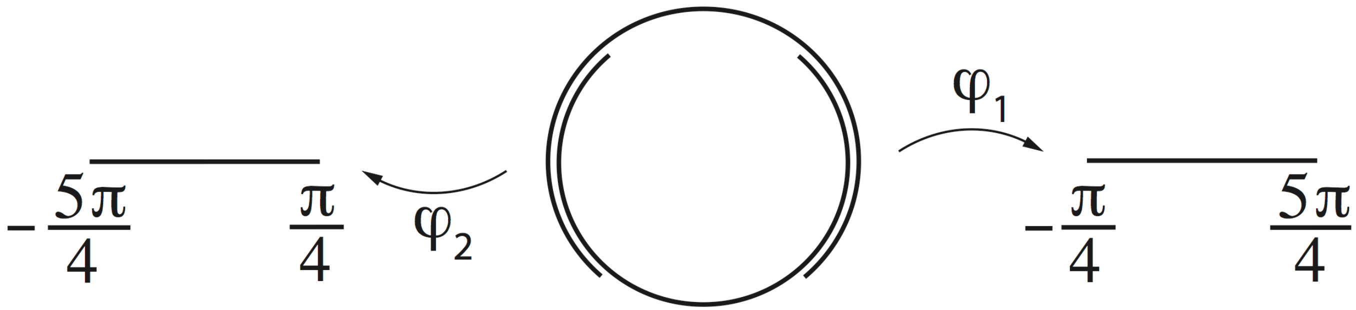


Figure 8: The circle S^1 is an example of a manifold that can be covered with two charts that are each locally $\mathbb{R}^1$.

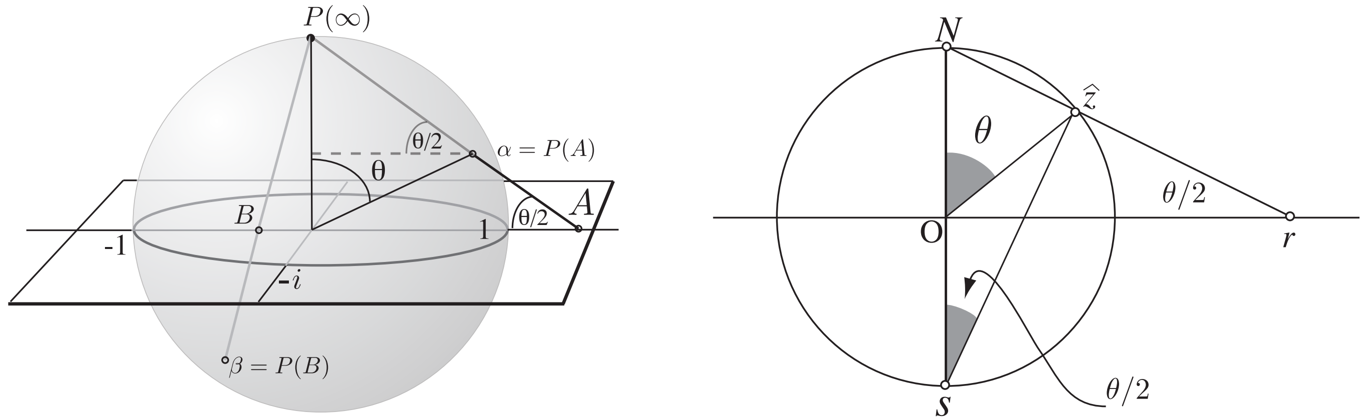


Figure 9: The Riemann map shows that the unit sphere S^2 is a manifold that can be covered with two charts that are each locally $\mathbb{R}^2$.

Exercise. Figure 9 illustrates Riemann's stereoscopic projection, which shows that the circle S^1 is a manifold which may be covered by two charts. Derive the values of the stereoscopic projections x_N and x_S from the North and South poles onto the x-axis, respectively, of a point on the circle at polar angle θ. Explain the angle $\theta/2$. How are x_N and x_S related to each other? Hint: you may use trigonometry. ★

Answer. A point on the circle at polar angle θ from the North pole has height $z = \cos\theta$. The intersection of its stereographic projection with the x-axis is found from the proportion

$$r = \frac{x_N}{1} = \frac{\sin\theta}{1-\cos\theta} = \cot(\theta/2)\,, \quad \text{provided } \cos\theta \neq 1\,.$$

The corresponding stereographic projection from the South pole in Figure 9 satisfies the proportion

$$\frac{x_S}{1} = \frac{\sin\theta}{1+\cos\theta}\,, \quad \text{provided } \cos\theta \neq -1\,.$$

Consequently, $x_S x_N = 1$, so that $x_S = 1/x_N = \tan(\theta/2)$ for $\theta \neq 0, \pi$. ▲

Remark. The manifold Q may sometimes be identified with a Lie group G. We will do this when we consider rotation and translation, for example. In this case, the configurations are obtained from the group action $G \times Q_0 \to Q$ where Q_0 is a reference configuration and the group is $G = SE(3)$ the special Euclidean Lie group of motions in three dimensions.

Time

Time is taken to be a manifold T with points $t \in T$. Usually $T = \mathbb{R}$ (for real 1D time), but we will also consider $T = \mathbb{R}^2$, and the option to let T and Q both be complex manifolds is not out of the question.

Motion

Motion is a map $\phi_t : T \to Q$, where subscript t denotes dependence on time t.

For example, when $T = \mathbb{R}$, the motion is a curve $q_t = \phi_t \circ q_0$ obtained by composition of functions.

The motion is called a $\boxed{\textbf{\textit{flow}}}$ if $\phi_{t+s} = \phi_t \circ \phi_s$, for $s, t \in \mathbb{R}$, and $\phi_0 = \mathrm{Id}$, so that $\phi_t^{-1} = \phi_{-t}$.

Note that the composition of functions is associative,

$$(\phi_t \circ \phi_s) \circ \phi_r = \phi_t \circ (\phi_s \circ \phi_r) = \phi_t \circ \phi_s \circ \phi_r = \phi_{t+s+r}\,,$$

but in general it is not commutative.

When the motion is obtained from a Lie group action $G \times Q \to Q$, then it may be identified with a map $\phi_t : T \to G$, which we may regard as a curve g_t on the Lie group G.

Thus, we should anticipate motion and mechanics on Lie group manifolds.

1.2 Curves on manifolds and their tangent spaces

The $\boxed{\textbf{\textit{tangent space}}}$ T_qQ contains vectors $v_q = \dot{q}(t) \in T_qQ$, tangent to curve $q(t) \in Q$ at point q. The coordinates are $(q, v_q) \in TQ_q$. Note, $\dim T_qQ = 2n$ and subscript q reminds us that v_q is an element of the tangent space at the point q and that on manifolds we must keep track of base points.

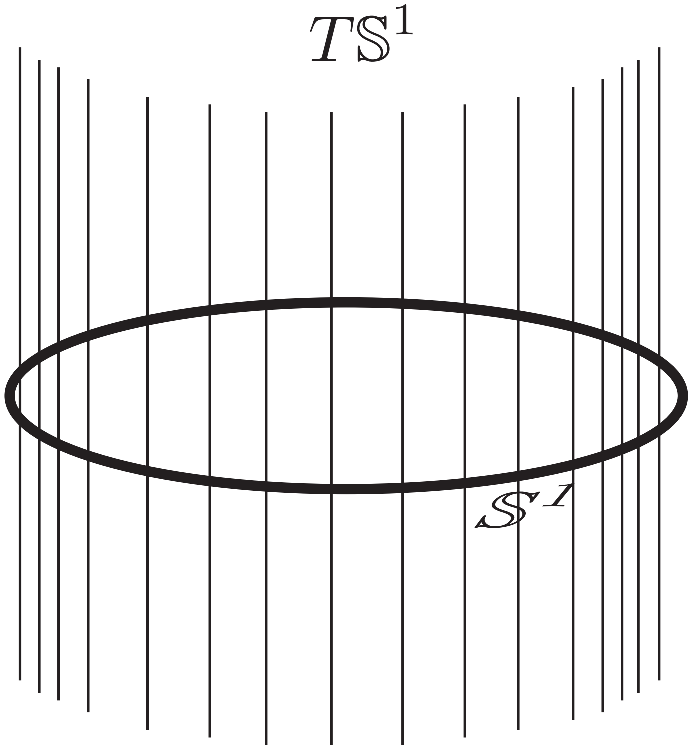


Figure 10: This is a sketch of the tangent bundle TS^1 of the circle S^1, $TS^1 = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^2 : |\mathbf{x}|^2 = 1 \text{ and } \mathbf{x} \cdot \mathbf{v} = 0\}$.

The union of tangent spaces $TQ := \cup_{q \in Q} T_qQ$ is also called the $\boxed{\textbf{\textit{tangent bundle}}}$ of the manifold Q.
The curve $q(t) \in Q$ describes the $\boxed{\textbf{\textit{motion}}}$. The curve $v = \dot{q}(t) \in T_qQ$ is the $\boxed{\textbf{\textit{tangent lift}}}$ of the curve $q(t) \in Q$.

1.3 Velocity and the Motion Equation

Velocity

The $\boxed{\textbf{\textit{tangent lift vector}}}$ $v_q = \dot{q}(t) \in T_qQ$ is the $\boxed{\textbf{\textit{velocity}}}$ along a flow $q(t)$ that describes a smooth curve in Q.

Motion Equation

The $\boxed{\textbf{\textit{motion equation}}}$ that determines the flow $q_t \in Q$ takes the form

$$\dot{q}_t = f(q_t)$$

where the map $f : q \in M \to f(q) \in T_qM$ is a prescribed $\boxed{\textbf{\textit{vector field}}}$ at $q \in Q$.

For example, if the curve $q_t = \phi_t \circ q_0$ is a flow, then

$$\dot{q}_t = \dot{\phi}_t \phi_t^{-1} \circ q_t = f(q_t)$$

so that

$$\dot{\phi}_t = f \circ \phi_t =: \phi_t^* f$$

where $\phi_t^* f$ denotes the $\boxed{\textbf{\textit{pullback}}}$ of f by ϕ_t.

1.4 Hamilton's principle $\delta S = 0$ deals with variational derivatives of functionals

Definition 1 (Functionals and variational derivatives)**.**
A functional $F[\rho]$ is defined as a map $F : \rho \in C^\infty(M) \to \mathbb{R}$.

The $\boxed{\textbf{\textit{variational derivative}}}$ *of a functional $F(\rho)$, denoted $\delta F/\delta\rho$, is defined by*

$$\delta F[\rho] := \lim_{\varepsilon\to 0}\frac{F[\rho+\varepsilon\phi]-F[\rho]}{\varepsilon} =: \frac{d}{d\varepsilon}F[\rho+\varepsilon\phi]\Big|_{\epsilon=0} = \int_\Omega \frac{\delta F}{\delta\rho}(x)\phi(x)\,dx =: \left\langle \frac{\delta F}{\delta\rho}\,,\,\phi\right\rangle \tag{1}$$

Here $\varepsilon \in \mathbb{R}$ is a real parameter, ϕ is an arbitrary smooth function and the angle brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ indicate L^2 real symmetric pairing of integrable smooth functions on the flow domain Ω.

The function $\phi(x)$ above is called the $\boxed{\textbf{\textit{variation of}}\ \rho}$ *and may be denoted as $\delta\rho := \phi(x)$.*

Since the variation is a linear operator on functionals, we can denote the functional derivative (δ) operationally as

$$\delta F[\rho] = \left\langle \frac{\delta F}{\delta\rho}\,,\,\delta\rho\right\rangle .$$

1.5 Lagrange's and Hamilton's variational principles for classical mechanics

- Define $\boxed{\textbf{\textit{kinetic energy}}}$ $KE : TM \to \mathbb{R}$, via a *Riemannian metric* $g_q(\,\cdot\,,\,\cdot\,) : TM \times TM \to \mathbb{R}$. Explicitly, $KE = \frac{1}{2} g_q(\dot q, \dot q) =: \frac{1}{2}\|\dot q\|^2$.

- Choose the $\boxed{\textbf{\textit{Lagrangian}}}$ $L : TM \to \mathbb{R}$. (For example, one could choose L to be KE.)

- $\boxed{\textbf{\textit{Hamilton's principle}}}$ is $\delta S = 0$ with $S = \int_a^b L(q, \dot q)dt$, for a family of curves $q(t, \epsilon)$ parameterised smoothly by $(t, \epsilon) \in \mathbb{R} \times \mathbb{R}$. The linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot q(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

defines the $\boxed{\textbf{\textit{variational derivative}}}$ δS of S near the identity $\epsilon = 0$. The variations in q are assumed to vanish at endpoints in time, so that $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$.

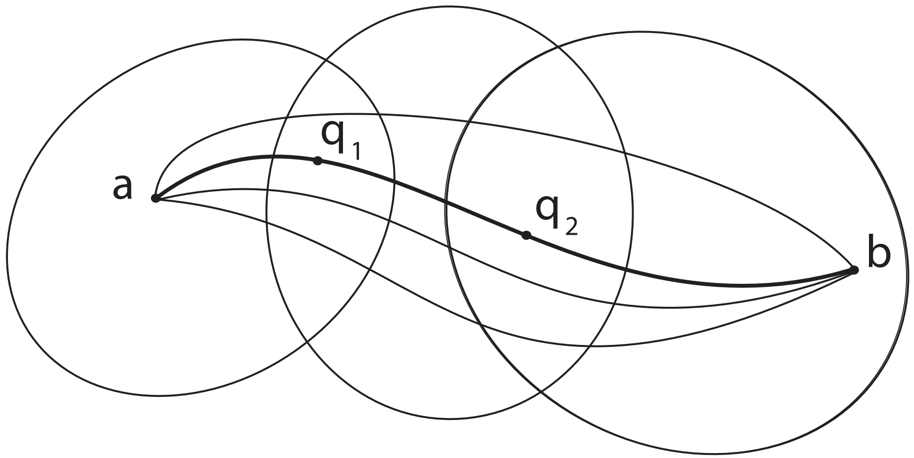


Figure 11: This is a sketch of variations of a family of curves on a manifold. Back2Outlook

2 Classical mechanics via Hamilton's principle

2.1 Euler–Lagrange equation

Theorem 2 (Hamilton 1835, Euler 1750, Lagrange 1756). *Hamilton's principle* $\delta S = 0$ *with* $S = \int_a^b L(q, \dot q)dt$ *implies the* $\boxed{\textbf{\textit{Euler–Lagrange (EL) equation:}}}$

$$\frac{d}{dt}\frac{\partial L(q,\dot q)}{\partial \dot q} = \frac{\partial L(q,\dot q)}{\partial q}\,, \quad \textit{for any differentiable } L(q,\dot q)\,.$$

$\boxed{\textbf{Proof 1}}$ Vary the curve $q(t)$ in the family $q(t,\epsilon) \in \mathcal{C}(Q)$ by using the linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot q(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

and set $\delta \frac{dq}{dt} = \frac{d}{dt}\delta q$ in the variation of the action S as

$$\delta S = \delta \int_a^b L(q,\dot q)dt = \int_a^b \delta L(q,\dot q)dt = \int_a^b \left\langle \frac{\partial L}{\partial \dot q}, \delta \dot q \right\rangle + \underbrace{\left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle}_{\text{Pairing}} dt = \int_a^b \left\langle \frac{\partial L}{\partial \dot q}, \frac{d}{dt}\delta q \right\rangle + \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle dt$$

$$= \int_a^b \left\langle \underbrace{-\frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{\partial L}{\partial q}}_{\text{EL equation}}, \delta q \right\rangle dt \; + \underbrace{\left\langle \frac{\partial L}{\partial \dot q}, \delta q \right\rangle \bigg|_a^b}_{\text{Endpoint term}\,=\,0} \qquad \square$$

$\boxed{\textbf{Proof 2}}$ Vary coordinates $(q,v)\in TQ$, subject to the constraint $v=\frac{dq}{dt}$ (tangent lift)

$$
\begin{aligned}
\delta S &= \delta\int_a^b L(q,v)+\left\langle p,\frac{dq}{dt}-v\right\rangle dt\\
&= \int_a^b \left\langle \frac{\partial L}{\partial v}-p,\delta v\right\rangle+\left\langle \frac{\partial L}{\partial q}-\frac{dp}{dt},\delta q\right\rangle+\Big\langle \delta p,\dot q-v\Big\rangle dt+\Big\langle p,\delta q\Big\rangle\Big|_a^b
\end{aligned}
$$

The three stationary conditions for $\delta S=0$ are:

$$
\delta v:\ \frac{\partial L}{\partial v}-p=0\,,\qquad \delta q:\ \frac{dp}{dt}-\frac{\partial L}{\partial q}=0\,,\qquad \delta p:\ \dot q-\frac{\delta H}{\delta p}=0
$$

After some assembly, these stationary conditions yield the Euler–Lagrange equations:

$$
\frac{\partial L}{\partial v}\Big|_{v=\dot q}=\frac{\partial L}{\partial q}\,.
$$

$\boxed{\textbf{Proof 3}}$ After the Legendre transform to the phase-space action, we vary coordinates $(q,p)\in T^*Q$,

$$
\begin{aligned}
\delta S &= \delta\int_a^b L(q,\dot q)dt \overset{LT}{=} \delta\int_a^b \Big\langle p,\dot q\Big\rangle_{TM}-H(q,p)dt\\
&= \int_a^b \Big\langle \delta p,\frac{dq}{dt}-\frac{\delta H}{\delta p}\Big\rangle_{TM}-\Big\langle \frac{dp}{dt}+\frac{\delta H}{\delta q},\delta q\Big\rangle_{TM}dt+\underbrace{\Big\langle p,\delta q\Big\rangle\Big|_a^b}_{\text{Endpoint term}\,=\,0}
\end{aligned}
$$

Stationarity $\delta S=0$ yields Hamilton's canonical equations:

$$
\delta p:\ \frac{dq}{dt}-\frac{\delta H}{\delta p}=0\,,\qquad \delta q:\ \frac{dp}{dt}+\frac{\delta H}{\delta p}=0\,.
$$

Back2Outlook

2.2 Noether's theorem

Theorem 3 (Noether's theorem). *If the Lagrangian $L(q,v)$ in the action integral $S := \int_a^b L(q,v)dt$ is* ***invariant*** *under a smooth infinitesimal transformation $\delta q = \Phi_\xi(q)$ by a 1-parameter Lie group, then the quantity*

$$J_\xi(q,p) := \langle p, \Phi_\xi(q)\rangle_{TM}$$

is a constant of the motion. That is, $J_\xi(q,p)$ is a conserved phase-space function when the equations of motion hold.

Proof. On the Lagrangian side, suppose that $L(q, \frac{dq}{dt})$ is invariant under the 1-parameter perturbation

$$q \to q_\epsilon = q + \epsilon\Phi_\xi(q)\,, \quad \text{so} \quad \delta q := \frac{dq_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = \Phi_\xi(q) \quad \text{for a real parameter } \epsilon. .$$

If $\delta S = 0$ for $\delta q = \Phi_\xi(q)$ because the Lagrangian is invariant under $q \to q_\epsilon$ and the Euler-Lagrange equation also holds, then the endpoint term yields $\langle p, \Phi_\xi(q)\rangle\Big|_a^b$ must vanish. This means that $J_\xi(q,p) := \langle p, \Phi_\xi(q)\rangle$ must be constant when evaluated on solutions of Euler-Lagrange equation. The quantity $J_\xi(q,p)$ is then said to be a *constant of the Euler-Lagrange motion.*

Likewise, the same conclusion $\langle p, \Phi_\xi(q)\rangle\Big|_a^b = 0$ follows on the Hamiltonian side, when Hamilton's canonical equations hold, and the phase space action is invariant. $\square$

Back2Outlook

2.3 What to do in solving a Lagrangian mechanics problem

1. Define the configuration manifold of a given mechanical system and find a suitable coordinate system on that manifold.
2. Find the Lagrangian and compute the Euler-Lagrange equations.
3. For simple (one-dimensional) systems, sketch phase portraits.
4. Compute energy and find how it evolves as a function of time.
5. Find the equilibria of the system and compute linear oscillations about that equilibrium by finding the normal frequencies and normal modes.
6. Find integrals of motion for a Lagrangian system using symmetries of the Lagrangian and Noether's theorem.
7. Compute the Legendre transformation from the Lagrangian to the Hamiltonian description and derive Hamilton's canonical equations.
8. Write the Poisson brackets and find the evolution equations for phase space functions of (q, p) and possibly of t.
9. Determine the equilibria of the system from critical points of the sum of the Hamiltonian and constants of the motion.
10. Determine the stability of the equilibrium solutions by taking a second variation around the equilibrium and finding whether the corresponding Hamiltonian for the linearised problem is definite in sign.
11. Reduce by a symmetry of the Lagrangian
12. Repeat from 1. above.

2.4 Example - The isoperimetric problem (what Lagrange wrote to Euler about).

The problem is to find the curve between two points (x_1, y_1) and (x_2, y_2), of specified length, that maximises the area integral $\int_{x_1}^{x_2} y(x)dx$.
In this example the length of the curve is

$$L[y] = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx, \quad \text{with} \quad y'(x) = dy/dx\,,$$

which takes the specified value $l = const$. The oriented area is

$$A[y] = \iint dx \wedge dy = \int_{x_1}^{x_2} y(x)dx.$$

We look for extrema of the modified functional

$$S[y] = \int_{x_1}^{x_2} ydx - \lambda \int_{x_1}^{x_2} (\sqrt{1 + y'^2}dx - l\,),$$

where λ is a scalar constant (Lagrange multiplier), to be determined. The Euler-Lagrange equation is

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) + 1 = 0\,. \tag{2}$$

Hence, a first integration yields $\frac{y'}{\sqrt{1+y'^2}} = -(x - x_0)/\lambda$, giving the parametric solution, after solving for y'^2,

$$x = x_0 \pm \lambda \sin(\theta), \qquad y = y_0 \pm \lambda \cos(\theta), \tag{3}$$

so $(x - x_0)^2 + (y - y_0)^2 = \lambda^2$ and the extremum is the arc of a circle of radius λ.
The variational problem satisfied by a soap bubble is analogous to the isoperimetric problem. For the soap bubble, the surface area is extremised, holding the volume integral constant. The Lagrange multiplier is the pressure, p.

2.5 Example: Hamilton's Principle for geodesics (covariant derivatives)

- **Geodesics:** When $L = KE = \frac{1}{2}g_q(\dot q, \dot q) =: \frac{1}{2}\|\dot q\|^2$, the solution $q(t)$ of the EL equations that passes from point $q(a)$ to $q(b)$ is called the *geodesic path* with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. The geodesic represents the path of shortest distance $q(a) \to q(b)$ measured by

$$ds^2 := dq^a g_{ab}(q) dq^b = g_q(\dot q, \dot q)dt^2 = \|\dot q\|^2\, dt^2$$

- **Exercise:** Compute the EL equations for a geodesic with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. That is, compute the EL equations for $L = KE = \frac{1}{2}g_q(\dot q, \dot q) =: \frac{1}{2}\|\dot q\|^2$.

- **Answer:** The KE Lagrangian is

$$L(q,\, \dot q) = \frac{1}{2}\dot q^b g_{bc}(q)\dot q^c\,.$$

Its partial derivatives are given by

$$\frac{\partial L}{\partial \dot q^a} = g_{ac}(q)\dot q^c \quad \text{and} \quad \frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial g_{bc}(q)}{\partial q^a}\dot q^b \dot q^c\,.$$

Consequently, its Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^a} - \frac{\partial L}{\partial q^a} \;=\; g_{ae}(q)\ddot q^{\,e} + \frac{\partial g_{ae}(q)}{\partial q^b}\dot q^b \dot q^e - \frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot q^b \dot q^e = 0\,.$$

Symmetrising the coefficient of the middle term and contracting with co-metric g^{ca} satisfying $g^{ca}g_{ae} = \delta^c_e$ yields

$$\boxed{\ddot q^{\,c} + \Gamma^c_{be}(q)\dot q^b \dot q^e = 0} \quad \text{with} \quad \Gamma^c_{be}(q) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(q)}{\partial q^b} + \frac{\partial g_{ab}(q)}{\partial q^e} - \frac{\partial g_{be}(q)}{\partial q^a}\right], \tag{4}$$

in which the Γ^c_{be} are called the ***Christoffel symbols*** for the Riemannian metric g_{ab}.

These Euler–Lagrange equations are the ***geodesic equations*** of a free particle moving in a Riemannian space. They are often written as

$$\ddot{q} + \nabla_{\dot{q}}\dot{q} = 0,$$

in terms of the $\boxed{\textbf{\textit{covariant derivative}}}$ $\nabla_{\dot{q}}$.

2.6 Ten examples of Hamilton's principle for Simple Mechanical Systems (with answers)

For simple mechanical systems, $L(q, \dot q) = T(\dot q) - V(q) = KE - PE$. For example,

1. Planar isotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k}{2}|\mathbf{x}|^2 \quad \Longrightarrow \quad \ddot{\mathbf{x}} = -\,\omega^2\mathbf{x}$ with $\omega^2 = k/m$

2. Planar anisotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}x_2^2 \quad \Longrightarrow \quad \ddot{x}_i = -\,\omega_i^2 x_i$ with $\omega_i^2 = k_i/m \quad i = 1, 2$

3. Planar pendulum, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x}, \dot{\mathbf{x}} \in T\mathbb{R}^2 |\, 1 - |\mathbf{x}|^2 = 0 \ \&\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_2\cdot\mathbf{x} - \frac{\mu}{2}(1 - |\mathbf{x}|^2) \quad \Longrightarrow \quad m\ddot{\mathbf{x}} = -\,mg\hat{\mathbf{e}}_2\cdot(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x}) - \,m|\dot{\mathbf{x}}|^2\mathbf{x}$, (gravity & centripetal force)

4. Planar pendulum motion lifted to a curve in $SO(2)$: $\mathbf{x}(t) = O(\theta(t))\mathbf{x}_0 \in \mathbb{R}^2\,, \quad O(\theta(t)) \in SO(2)\,, \quad |\mathbf{x_0}|^2 = R^2$, where $\mathbf{x}(0) = \mathbf{x}_0$.

   $\dot{\mathbf{x}}(t) = \dot{O}O^{-1}(t)\mathbf{x} = \dot\theta(t)\,\hat{\mathbf{e}}_3\times\mathbf{x}$ for $(\theta, \dot\theta) \in TSO(2)\,,$

   $L = \frac{m}{2}R^2\dot\theta^2 - mgR(1 - \cos\theta) \quad \Longrightarrow \quad \ddot\theta = -\omega^2\sin\theta$ with $\omega^2 = g/R$

5. Charged particle in a magnetic field, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) \quad \Longrightarrow \quad \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}}\times\mathbf{B}$ with $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$

6. Kepler problem, $(r, \dot r, \theta, \dot\theta) \in T\mathbb{R}_+ \times TS^1$: $\quad L = \frac{m}{2}\Big(\dot r^2 + r^2\dot\theta^2\Big) + \frac{GMm}{r} \quad \Longrightarrow \quad \ddot r = -\frac{GM}{r^2} + \frac{J^2}{r^3}$ with $J = r^2\dot\theta = const$

7. Free motion on a sphere, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $TS^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 : |\mathbf{x}| = 1 \ \&\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$: $L = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \frac{1}{2}\mu(1 - |\mathbf{x}|^2)$

8. Spherical pendulum (a), $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, on $TS^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 : |\mathbf{x}| = 1 \ \&\ \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \frac{1}{2}\mu(1 - |\mathbf{x}|^2)$

9. Spherical pendulum (b), setting $\mathbf{x}(t) = O(t)\mathbf{x}_0\,, \quad \dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0 \quad$ for $\quad (O,\,\dot{O}) \in TSO(3)\,$, where $\mathbf{x}_0 = \mathbf{x}(0)$ initially and

$$
\begin{aligned}
L &= \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} \\
&= \frac{m}{2}|\dot{O}(t)\mathbf{x}_0|^2 - mg\,O^T(t)\hat{\mathbf{e}}_3\cdot\mathbf{x}_0 \\
&= \frac{m}{2}|O^{-1}\dot{O}(t)\mathbf{x}_0|^2 - mg\,O^{-1}(t)\hat{\mathbf{e}}_3\cdot\mathbf{x}_0 \\
&= \frac{m}{2}|\boldsymbol{\Omega}\times\mathbf{x}_0|^2 - mg\,\boldsymbol{\Gamma}\cdot\mathbf{x}_0
\end{aligned}
$$

where $(O^{-1}\dot{O})_{ij} = -\epsilon_{ijk}\Omega^k$ and $\boldsymbol{\Gamma} = O^{-1}(t)\hat{\mathbf{e}}_3$. Rotations preserve length, so $|\mathbf{x}|^2 = \mathbf{x}_0|^2 = 1$. Set $g = 0$ for free motion on TS^2.

10. Rotating rigid body, $\widehat{\Omega} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3) \quad \ell = \frac{1}{2}\boldsymbol{\Omega}\cdot I\boldsymbol{\Omega} \quad$ with $\quad \boldsymbol{\Omega}\times = \widehat{\Omega}\,, \quad$ that is, $\quad -\epsilon_{ijk}\Omega^k = \widehat{\Omega}_{ij}$

3 Classical mechanics via Hamilton's canonical equations

3.1 Legendre transform (LT)

- $LT : (q, \dot{q}) \in TM \to (q, p) \in T^*M$ defines *momentum* p as the *fibre derivative* of L, namely

$$p := \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \in T^*M \qquad \textit{(fibre derivative)}\,.$$

The LT is *invertible* for $\dot{q} = f(q, p)$, provided the *Hessian* $\partial^2 L(q, \dot{q})/\partial\dot{q}\partial\dot{q}$ has nonzero determinant. Note, $\dim T^*M = 2n$.

- In terms of the LT, the ***Hamiltonian*** $H : T^*M \to \mathbb{R}$ is defined by

$$H(q, p) = \Big\langle p, \dot{q} \Big\rangle - L(q, \dot{q})$$

in which the expression $\langle p, \dot{q} \rangle$ in this calculation identifies a *pairing* $\langle \,\cdot\,, \,\cdot\, \rangle : T^*M \times TM \to \mathbb{R}$.
Taking the differential of this definition yields

$$\begin{aligned} dH &= \Big\langle H_p, dp \Big\rangle + \Big\langle H_q, dq \Big\rangle \\ &= \Big\langle dp, \dot{q} \Big\rangle + \Big\langle p - L_{\dot{q}}, d\dot{q} \Big\rangle - \Big\langle L_q, dq \Big\rangle \end{aligned}$$

from which Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q} \rangle - H(q, p)\, dt$ produces *Hamilton's canonical equations* on phase space T^*M,

$$H_p = \dot{q} \quad \text{and} \quad H_q = -\, L_q = -\, \dot{p}\,.$$

- Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt$ produces *Hamilton's canonical equations* on phase space T^*M,

$$H_p = \dot{q} \quad \text{and} \quad H_q = -\,L_q = -\,\dot{p}\,.$$

Exercise. Verify the previous statement. That is, compute the results of the following
Phase-space form of Hamilton's principle on T^*M, given by $\delta S = 0$
with $S = \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt$. ★

-

Answer. One computes

$$\begin{aligned}\delta S &= \delta \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt = \int_a^b \delta\langle p, \dot{q}\rangle - \delta H(q,p)\,dt \\ &= \int_a^b \Big\langle \delta p\,, \dot{q} - H_p \Big\rangle - \Big\langle \dot{p} + H_q, \delta q \Big\rangle dt + \underbrace{\Big\langle p, \delta q \Big\rangle\Big|_a^b}_{\text{Endpoint term}}\end{aligned}$$

▲

Remark 4. *We will return to the endpoint term in formulating Noether's theorem on phase space, that is, on T^*M.*

3.2 Legendre transform in simple mechanical systems – Exercise sheet (with answers)

- Legendre transform: $H(q,p) = \langle p,\, \dot q\rangle - L(q,\dot q) = T(p) + V(q) = KE + PE.$

For example,

1. Planar isotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k}{2}|\mathbf{x}|^2$
2. Planar anisotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$
3. Planar pendulum in polar coordinates, $(\theta, p_\theta) \in T^*S^1$: $\quad H = \frac{1}{2mR^2}p_\theta^2 + mgR(1-\cos\theta)$
4. Planar pendulum, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x}\in\mathbb{R}^2 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_2\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$
5. Charged particle in a magnetic field, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2 \quad \mathbf{p} := \partial L/\partial\dot{\mathbf{q}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$
6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^*\mathbb{R}_+ \times T^*S^1$: $\quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $p_\theta = r^2\dot\theta = const$
7. Free motion on a sphere, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $H = \frac{1}{2m}|\mathbf{p}|^2 - \mu(1-|\mathbf{x}|^2)$
8. Spherical pendulum (a), $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$
9. Spherical pendulum (b), $(O, \dot O) \in TSO(3)$, $\widehat{\xi} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$, $\boldsymbol{\Pi} = \partial\ell/\partial\boldsymbol{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ $\quad H = \frac12\boldsymbol{\Pi}\cdot I^{-1}\boldsymbol{\Pi} + g\,\boldsymbol{\Gamma}\cdot\mathbf{x}_0$ with $\boldsymbol{\Pi} = \frac{\partial\ell}{\partial\boldsymbol{\Omega}} = I\boldsymbol{\Omega}$. Set $g=0$ to get freely rotating rigid body motion.
10. Rotating rigid body, $\boldsymbol{\Pi} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ $\quad H = \frac12\boldsymbol{\Pi}\cdot I^{-1}\boldsymbol{\Pi}$ with $\boldsymbol{\Pi} = \frac{\partial\ell}{\partial\boldsymbol{\Omega}} = I\boldsymbol{\Omega}$.

LECTURE #1

This lecture begins by deriving Newton's equation of planetary motion in an orbital plane from Hamilton's principle. However, Noether's theorem only yields conservation of angular momentum normal to the plane of the orbit, while planetary motion is independent of the orientation of the plane in 3D. The lecture derives Kepler's three laws of planetary motion on the Hamiltonian side from transformations generated by the conserved Runge-Lenz vector lying in the plane of the orbit which leave invariant two global space-time properties of the spatially bounded Kepler orbits (with negative energy, $H < 0$) while changing a third global property. Namely, Runge-Lenz transformations map Kepler ellipses into Kepler ellipses in the same plane without any precession, by changing the shape of an elliptical orbit while leaving invariant the length of its semi-major axis and the period of its orbit.

One question is, "What is left invariant and how do the Poisson bracket relations change when $H \geq 0$ and the Kepler orbits are spatially unbounded?" Another question is, "Why does the transformation generated by the conserved Runge-Lenz vector involve global properties of the solution in both space and time?" Go2toc

3.3 The reduced Kepler problem: Newton (1686)

The ***reduced Kepler problem*** of planetary motion arises from Hamilton's principle for the Lagrangian

$$\mathcal{L}(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|) = \frac{1}{2}\big(|\dot{r}|^2 + r^2\dot{\theta}^2\big) - V(r) \quad \text{with} \quad V(r) = -\frac{\mu}{r}\,.$$

For this Lagrangian, Hamilton's Principle with $\mathbf{p} = \partial L/\partial\dot{\mathbf{r}} = \dot{\mathbf{r}}$ implies ***Newton's equation of motion***,

$$\ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = 0\,, \tag{5}$$

in which μ is a constant (called k in Figure 12) and $r = |\mathbf{r}|$ with $\mathbf{r} \in \mathbb{R}^3$. Also, $p_r = \partial L/\partial\dot{r} = \dot{r}$ and $p_\theta = \partial L/\partial\dot{\theta} = r^2\dot{\theta}$.

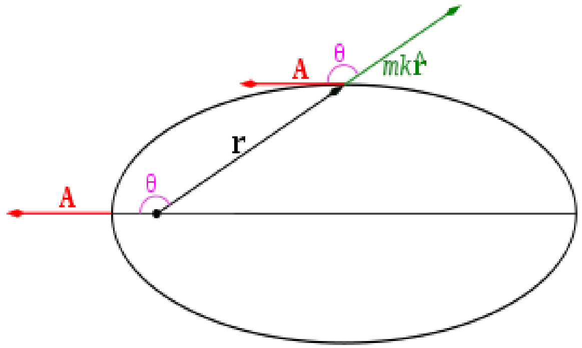


Figure 12: This is a Wiki-sketch of an elliptical Kepler orbit. In our notation, $k \to \mu$, $\mathbf{A} \to \mathbf{J}$.

Scale invariance of equation (5) under the changes $R \to s^2R$ and $T \to s^3T$ in the units of space R and time T for any constant (s) means that it admits families of solutions whose space and time scales are related by $T^2/R^3 = const$.

This is ***Kepler's third law***. Newton (1686) showed that his equation (5) implied that $T^2/a^3 = (2\pi)^2/\mu = constant$. The universality of Newton's constant μ makes celestial mechanics possible.

1. ***Conservation laws.*** The scalar (resp. vector) product of equation (5) with $\mathbf{r}$ shows conservation of the energy E and (resp.) specific angular momentum $\mathbf{L}$, given by

$$
\begin{aligned}
E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \text{ or } H(r,p_r,p_\theta) = \frac{1}{2}\Big(p_r^2 + \frac{p_\theta^2}{r^2}\Big) - \frac{\mu}{r} \quad \text{(Hamiltonian)} \\
\mathbf{L} &= \mathbf{r}\times\dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,.
\end{aligned}
$$

   Since $\mathbf{r}\cdot\mathbf{L} = 0$, the orbital motion in $\mathbb{R}^3$ takes place in a plane to which vector $\mathbf{L}$ is perpendicular.
   In this orbital plane, one may specify plane polar coordinates (r,θ) with unit vectors $(\hat{\mathbf{r}},\,\hat{\boldsymbol{\theta}})$ in the plane and $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \widehat{\mathbf{L}}$ normal to it. Hence,

$$
\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}} = r\hat{\mathbf{r}}\times(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = r^2\dot{\theta}\,\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = r^2\dot{\theta}\,\widehat{\mathbf{L}} = p_\theta\,\widehat{\mathbf{L}}\,,
$$

   so the magnitude of the angular momentum is $L = |\mathbf{L}| = r^2\dot{\theta} = p_\theta = \partial\mathcal{L}/\partial\dot{\theta}$.
   By Noether's theorem, the invariance of the Lagrangian $\mathcal{L}$ under translations in θ ($\partial\mathcal{L}/\partial\theta = 0$) implies that the magnitude $L = |\mathbf{L}| = p_\theta$ is conserved. This also follows from the Euler-Lagrange equation $dp_\theta/dt = \partial\mathcal{L}/\partial\theta = 0$.

2. The unit vectors for polar coordinates in the orbital plane are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. These vectors satisfy

$$
\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}}\times\hat{\mathbf{r}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \quad\text{and}\quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}}\times\hat{\boldsymbol{\theta}} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,,\quad \text{where}\quad \dot{\theta} = \frac{L}{r^2}\,.
$$

   Newton's equation of motion (5) for the Kepler problem may now be written equivalently using $\dot{\theta}/L = 1/r^2$ and $d\hat{\boldsymbol{\theta}}/dt = -\,\dot{\theta}\,\hat{\mathbf{r}}$, as

$$
0 = \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \ddot{\mathbf{r}} + \frac{\mu}{L}\dot{\theta}\,\hat{\mathbf{r}} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big)\,.
$$

   With $\mathbf{p} = \dot{\mathbf{r}}$, this equation implies conservation of the following vector ***in the orbital plane***:

$$
\mathbf{K} = \mathbf{p} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad (\textbf{\textit{Hamilton's vector}})\,.
$$

The cross product of the two conserved vectors $\mathbf{K}$ and $\mathbf{L}$ yields another conserved vector in the plane of motion[1]

$$\mathbf{J} = \mathbf{K} \times \mathbf{L} = \mathbf{p} \times \mathbf{L} - \mu \hat{\mathbf{r}} \quad (\textbf{\textit{Laplace–Runge–Lenz vector}})\,.$$

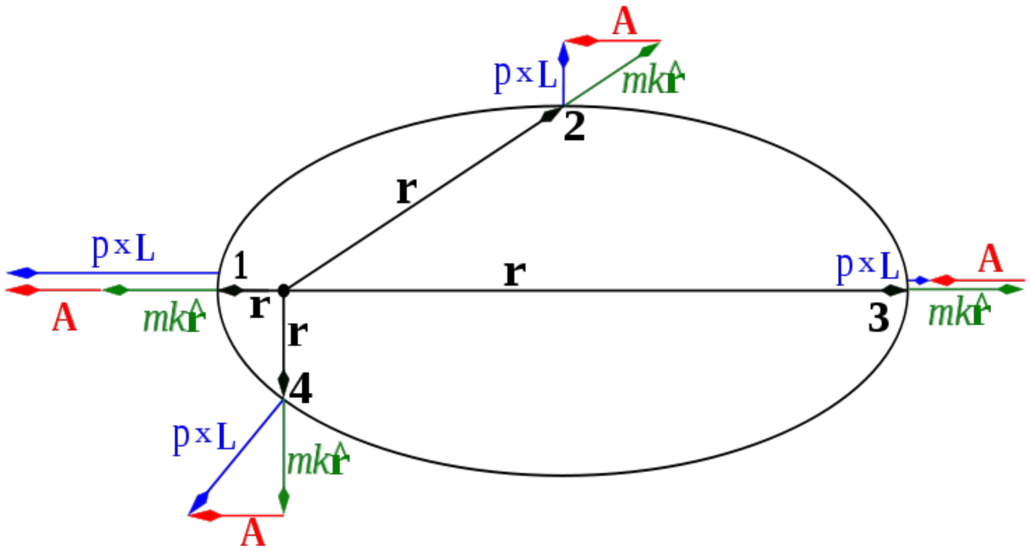


Figure 13: Wiki-sketch of the Laplace-Runge-Lenz vector (called **A**) for an elliptical Kepler orbit.

3. Note immediately that $\mathbf{J}\cdot\mathbf{L} = 0 = \mathbf{J}\cdot\mathbf{K}$, $J^2 = 2EL^2 + \mu^2 = K^2L^2$ and the dimensions of $\mathbf{J}$ are given by $[J] = [\mu] = [r]^3[t]^{-2}$, the same as ***Kepler's Third Law***! Thus, from their definitions, these conserved quantities are related by

$$K^2 = 2E + \frac{\mu^2}{L^2} = \frac{J^2}{L^2}\,, \quad \text{upon using} \quad K^2 = \Big|\mathbf{p} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big|^2 = |\mathbf{p}|^2 - \frac{2\mu}{L}\mathbf{p}\cdot\hat{\boldsymbol{\theta}} + \frac{\mu^2}{L^2} = |\mathbf{p}|^2 - \frac{2\mu}{r} + \frac{\mu^2}{L^2}\,.$$

[1]For more history, see `https://en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector`.

The last step follows since $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $\mathbf{p} = p_r\,\hat{\mathbf{r}} + p_\theta/r\,\hat{\boldsymbol{\theta}}$, so $\mathbf{p}\cdot\hat{\boldsymbol{\theta}} = p_\theta/r = r\dot{\theta} = L/r$. Likewise,

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)} \quad\Longrightarrow\quad -2E = \frac{\mu^2 - J^2}{L^2} \quad\text{and}\quad \mathbf{J}\cdot\mathbf{K}\times\mathbf{L} = K^2L^2 = J^2\,,$$

where $J^2 := |\mathbf{J}|^2$, etc. and $-2E > 0$ for bounded orbits. Hence, the motion $(\mathbf{r},\mathbf{p}) \in T^*\mathbb{R}^3 \simeq \mathbb{R}^3\times\mathbb{R}^3$ takes place in 6 dimensions on the intersections of level sets of E, $J^2 - 2EL^2 = \mu^2$ and $\mathbf{J}\cdot\mathbf{L} = 0$.

4. ***Conic section Kepler orbits.*** Orient the conserved Laplace–Runge–Lenz vector $\mathbf{J}$ in the orbital plane to point along the reference line for the measurement of the polar angle θ, say from the centre of the orbit (Sun) to the perihelion (point of nearest approach, on midsummer's day). The scalar product of $\mathbf{r}$ and $\mathbf{J}$ then yields an elegant result for the Kepler orbit in plane polar coordinates:

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L} - \mu\,\mathbf{r}/r) = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - \mu\,r\,,$$

which implies

$$r(\theta) = \frac{L^2/\mu}{1 + J/\mu\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,. \tag{6}$$

As expected, the orbit $r(\theta)$ is a ***conic section*** whose origin is at one of the two foci. This is ***Kepler's first law***.

Elliptical orbit. Let a and b be respectively the semi-major and semi-minor axes of the ellipse drawn with a string of length $2a$ attached at foci $\pm ea$. One may form two right angles with the string to discover that $(ea)^2 + b^2 = a^2$ and $l_\perp = b^2/a$, by Pythagoras' theorem and perimeter $2a$. The eccentricity vanishes ($e = 0$) for a circle and correspondingly $K = 0$ implies that $\dot{\mathbf{r}} = \mathbf{p} = \mu\,\hat{\boldsymbol{\theta}}/L = p_\theta$. The eccentricity takes values $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola.

In summary: the Laplace–Runge–Lenz vector $\mathbf{J}$ is directed from the focus of the orbit to its perihelion (point of closest approach). The eccentricity of the elliptical orbit is given by $e = J/\mu = KL/\mu = \sqrt{1 - b^2/a^2}$ and its semi-latus rectum (normal distance from the line through the foci to the orbit) is $l_\perp = L^2/\mu = b^2/a$.

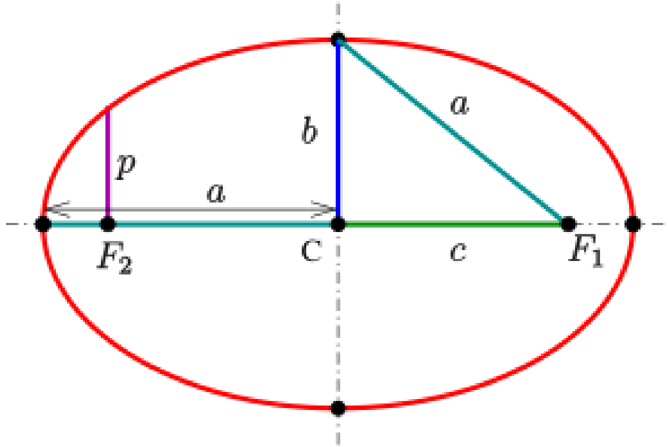


Figure 14: This is a Wiki-sketch of the parameters of an elliptical Kepler orbit with $c = ea$.

5. The conservation of $\mathbf{L}$ in $\mathbf{L}dt = \mathbf{r} \times d\mathbf{r}$ or L in $Ldt = r^2 d\theta$ shows that the constancy of magnitude $L = |\mathbf{L}|$ means the orbit sweeps out equal areas in equal times. This is ***Kepler's second law***.

   For an elliptical orbit, the integral $LT = \int_0^T Ldt = \int_0^T p_\theta dt = \int_0^{2\pi} r(\theta)^2 d\theta = 2A$ yields the period in terms of angular momentum and the area; namely, $T = 2A/L$. Hence, $4A^2/T^2 = L^2$.

6. One may use the result of part 5 and the geometric properties of ellipses to show that the period of the orbit is given by

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}\,. \tag{7}$$

   The relation $T^2/a^3 = (2\pi)^2/\mu = \textit{constant}$ comprises Kepler's third law, which reflects the scale invariance of Newton's equation. The constant μ is Newton's universal constant of gravitational attraction.

   This is a profound relation! Time and space are linked! The period and aphelion-to-perihelion distance of a planetary orbit determines the fundamental gravitational property which holds the universe together!

7. By using the Poisson bracket definition $\{r_i, p_j\} = \delta_{ij}$, one may check that the Poisson brackets amongst the components of the vectors $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{J} = \mathbf{p} \times \mathbf{L} - \mu\mathbf{r}/r$ satisfy the following relations:

$$
\begin{aligned}
\{L_i,\, L_j\} &= \epsilon_{ijk} L_k =: (L\times)_{ij} = -\,(L\times)_{ji}\,, \quad \Longrightarrow \quad \{\boldsymbol{L}\cdot\boldsymbol{\xi},\, \boldsymbol{L}\cdot\boldsymbol{\eta}\} = \boldsymbol{L}\cdot\boldsymbol{\xi}\times\boldsymbol{\eta}\,, \quad \boldsymbol{\xi} = \frac{\partial F}{\partial \boldsymbol{L}}\,, \quad \boldsymbol{\eta} = \frac{\partial H}{\partial \boldsymbol{L}}\,,\\
\{L_i,\, J_j\} &= \epsilon_{ijk} J_k\,, \quad \Longrightarrow \quad \{L_i,\, \bar{J}_j\} = \epsilon_{ijk}\bar{J}_k \quad \text{with} \quad \bar{J}_k := J_k/\sqrt{-2H}\,,\\
\{J_i,\, J_j\} &= -\,2H\epsilon_{ijk} L_k\,, \quad \Longrightarrow \quad \{\bar{J}_i,\, \bar{J}_j\} = \epsilon_{ijk} L_k \quad \text{after using} \quad \{L_i,\, H\} = 0 = \{J_i,\, H\}\,.
\end{aligned}
$$

In tabular form, this is

$$
\{(L,J)\,,\,(L,J)\} = \begin{array}{|c|cc|}
\hline
\{\,\cdot\,,\,\cdot\,\} & L & J \\
\hline
L & L\times & J\times \\
J & J\times & -2H\,L\times \\
\hline
\end{array}\,. \tag{8}
$$

Importantly, these relations imply that the Poisson bracket with J_i alters both the eccentricity and the width of an elliptical orbit, as one finds upon using $\{J^2, J_i\} = 4H(\mathbf{J}\times\mathbf{L})_i = 2H\{L^2, J_i\}$ in the following sequence,

$$
\{J_i,\, e^2\}\mu^2 = \{J_i,\, J^2\} = -\,4H\epsilon_{ijk}J_jL_k = -\,4H(\mathbf{J}\times\mathbf{L})_i = -\,2H\{L^2,\, J_i\} = -\,2H\mu\{l_\perp,\, J_i\}\,.
$$

The conservation laws $\{L^2, H\} = 0$ and $\{J_i, H\} = 0$ allow the use of formula (8) to check the properties of the previous Poisson bracket relations. In particular, two *Casimir functions* Poisson commute with any other functions $F(\mathbf{J},\mathbf{L})$ on any level surface of H, and therefore are constants of the $(\mathbf{J},\mathbf{L})$ motions. Namely,

$$
C_1 = J^2 - 2H\,L^2 \quad \text{and} \quad C_2 = \mathbf{J}\cdot\mathbf{L} \Longrightarrow \{J_i, C_a\} = 0 = \{L_i, C_a\}\,, \quad i = 1,2,3, \quad a = 1,2\,.
$$

For the Kepler problem, the level sets of the Casmirs take the physically meaningful values $C_1 = \mu^2$ and $C_2 = 0$. In summary, the motion $(\mathbf{r},\mathbf{p}) \in T^*\mathbb{R}^3 \simeq \mathbb{R}^3\times\mathbb{R}^3$ takes place in 6 dimensions on the intersections of level sets of phase space functions H, $C_1 = J^2 - 2EL^2 = \mu^2$ and $C_2 = \mathbf{J}\cdot\mathbf{L} = 0$, all ***in involution*** under Poisson brackets. The dimensions drop as $6-2-2=2$. Hence, this Hamiltonian system is *integrable*.

8. ***Proposition.*** Canonical transformations generated by $\mathbf{J}$ change both eccentricity e and width $l_\perp$ of the orbit, but in conserving a certain combination of them, *they map closed ellipses into closed ellipses*, without any precession.

   To draw this conclusion, one uses Poisson brackets to create Hamiltonian vector fields

$$X_F := \{\,\cdot\,,\,F\,\} = \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\,,$$

   whose integral curves determine the flow lines of the canonical transformations. Namely, one computes

$$\big\{J_i, \big(J^2 - 2H\,L^2\big)\big\} = (-2H\mu)\left\{J_i, \frac{e^2\mu}{-2H} + l_\perp\right\} = 0\,.$$

   Note that because of equation (7) the canonical transformations generated by the Runge-Lenz vector $\mathbf{J}$ preserve the period and the semi-major axis of the elliptical Kepler orbit, while changing its area ($A = \pi ab = LT/2$) and shape (b/a) with $b = a\sqrt{1-e^2} = \sqrt{l_\perp a}$, by altering its eccentricity ($e = |\mathbf{J}|/\mu$) and its semi-latus rectum ($l_\perp = L^2/\mu$) with $J^2/L^2 = K^2 = e^2\mu/l_\perp$, according to

$$\begin{aligned}
\{e^2, J_i\} &= \frac{1}{\mu^2}\{J^2, J_i\} = \frac{4H}{\mu^2}(\mathbf{J}\times\mathbf{L})_i = -\frac{4HL^2}{\mu^2}K_i = -\frac{4Hl_\perp}{\mu}K_i\,,\\
\{l_\perp, J_i\} &= \frac{1}{\mu}\{L^2, J_i\} = \frac{2}{\mu}(\mathbf{J}\times\mathbf{L})_i = -\frac{2L^2}{\mu}K_i = -2l_\perp K_i\,.
\end{aligned}$$

   Finally, if we define $\mathbf{M}^\pm := \sqrt{-2H}\,\mathbf{L}\pm\mathbf{J}$, with $|\mathbf{M}^+|^2 = (J^2 - 2HL^2) = \mu^2 = |\mathbf{M}^-|^2$ since $\mathbf{J}\cdot\mathbf{L} = 0$ then we find the following Poisson bracket relations for negative energy (-2H>0),

$$\{M_i^+, M_j^-\} = 0\,,\quad \{M_i^+, M_j^+\} = 2\epsilon_{ijk}M_k^+\,,\quad \{M_i^-, M_j^-\} = 2\epsilon_{ijk}M_k^-\,. \tag{9}$$

   These are Lie-Poisson brackets on the dual Lie algebra of $so(3)\times so(3) \simeq so(4)$ and $|\mathbf{M}^\pm|^2$ are Casimirs.

   Exercise. How do the Poisson bracket relations in (9) change for positive energy when $(-2H < 0)$? ★

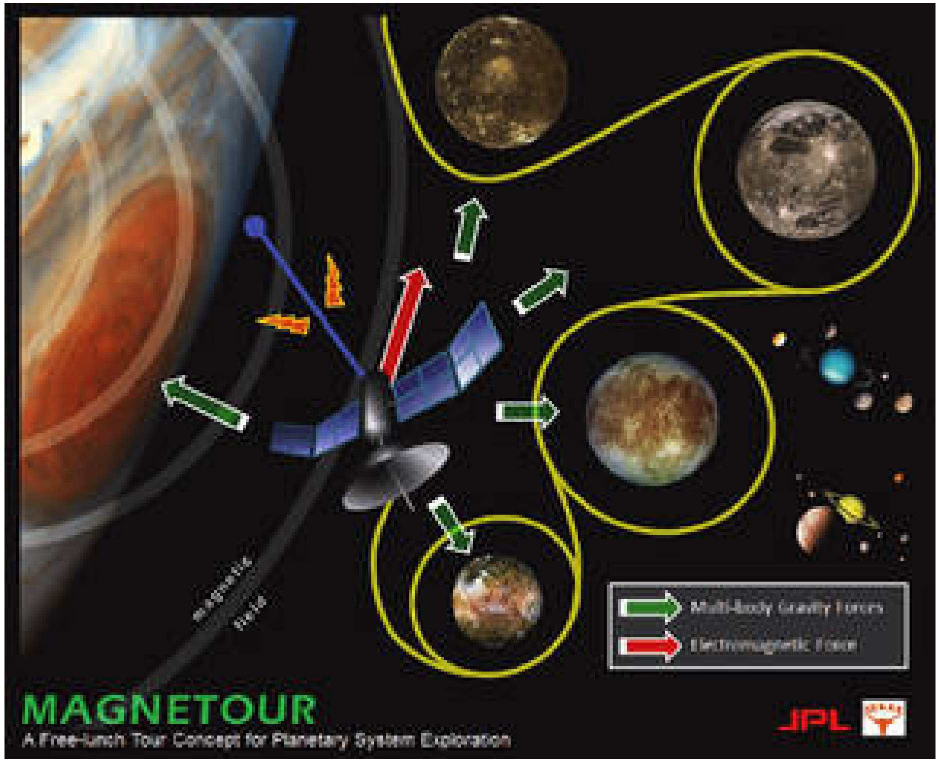


Figure 15: What is it good for? Overview of the NASA Magnetour concept.

https://www.nasa.gov/directorates/spacetech/niac/2012_Phase_I_megnetour/
https://authors.library.caltech.edu/20340/1/RoKoLoMa2003.pdf (MULTI-MOON ORBITER)
https://en.wikipedia.org/wiki/Kepler's_laws_of_planetary_motion
https://en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector

- Geometric mechanics is good for robotics and space missions (with a finite number of degrees of freedom).
- As we shall see, geometric mechanics also applies for ideal fluid dynamics!

LECTURE #2

This lecture introduces Hamiltonian dynamics in the compact notation of ***vector fields and differential forms on a symplectic manifold***, which for definiteness is taken to be T^*M, the cotangent bundle of a smooth manifold M. This is the standard notation for the operations of vector fields on differential forms in the Hamiltonian case. Go2toc

$$
\begin{tikzcd}
 & L(q,v) : TM \to \mathbb{R} \arrow[dl, "\text{Lagrange variational principle}"] \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] & & H(q,p) : T^*M \to \mathbb{R} \arrow[dl, "\textbf{\textit{Canonical Poisson brackets}}"] \\
\text{Euler-Lagrange eqns} \arrow[rr, Leftrightarrow] & & \text{Hamilton's eqns} &
\end{tikzcd}
$$

Commutative diagram of Classical Mechanics

3.4 Canonical Poisson bracket and Hamiltonian vector fields

The Hamiltonian dynamics of a phase-space function $F \in \mathcal{F}(T^*M)$ is given by

$$\frac{d}{dt}F(q,p) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F,\, H\}\,.$$

The operation $\{F,\, H\}$ is called the ***canonical Poisson bracket*** of F with H on the phase space T^*M.
The canonical Poisson bracket operation $\{\cdot\,,\,\cdot\}$ is a map among smooth real functions $\mathcal{F}(T^*M) : T^*M \to \mathbb{R}$

$$\{\cdot\,,\,\cdot\} : \mathcal{F}(T^*M) \times \mathcal{F}(T^*M) \to \mathcal{F}(T^*M)\,, \tag{10}$$

so that Hamiltonian dynamics on ***phase space*** T^*M is given by $\dot{F} = \{F\,,\,H\}$ for any $F \in \mathcal{F}(T^*M)$.

Definition 5 (Poisson bracket)**.** *A **Poisson bracket operation** $\{\cdot\,,\,\cdot\}$ is defined by its properties listed below:*

- *It is **bilinear**.*
- *It is **skew-symmetric**, $\{F\,,\,H\} = -\{H\,,\,F\}$.*
- *It satisfies the **Leibniz rule** (product rule),*

$$\{FG\,,\,H\} = \{F\,,\,H\}G + F\{G\,,\,H\}\,,$$

for the product of any two functions F and G on M.

- *It satisfies the **Jacobi identity**,*

$$\{F\,,\,\{G\,,\,H\}\} + \{G\,,\,\{H\,,\,F\}\} + \{H\,,\,\{F\,,\,G\}\} = 0\,, \tag{11}$$

for any three functions F, G and H on M.

Remark. The Leibniz rule associates Poisson brackets with differential operators acting on smooth phase-space functions $F \in \mathcal{F}(T^*M)$.

Definition 6 (Hamiltonian vector field)**.** *The differential operator or* ***Hamiltonian vector field*** $X_F \in \mathfrak{X}(T^*M)$ *generated by the canonical Poisson bracket with* $F \in \mathcal{F}(T^*M)$ *is defined as*

$$X_F := \{\,\cdot\,,\,F\,\} = \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\,. \tag{12}$$

Definition 7 (Commutator of Hamiltonian vector fields)**.** *The commutator of Hamiltonian vector fields* $X_F, X_H \in \mathfrak{X}(T^*M)$ *is defined in terms of their canonical Poisson bracket representation as,*

$$[X_F, X_H]J := (X_F X_H - X_H X_F)J = \{\{J,H\},F\} - \{\{J,F\},H\}\,.$$

Theorem 8 (Commutator of Hamiltonian vector fields)**.** *Hamiltonian vector fields defined via the Poisson bracket as* $X_H = \{\,\cdot\,,H\} \in \mathfrak{X}(T^*(M))$ *in* (12) *satisfy the* ***commutation relation*** $[\,\cdot\,,\,\cdot\,] : \mathfrak{X}(T^*M) \times \mathfrak{X}(T^*M) \to \mathfrak{X}(T^*M)$*,*

$$[X_F, X_H] = -X_{\{F,H\}}\,. \tag{13}$$

Proof. The proof of Theorem 8 follows immediately from the Jacobi identity for the canonical Poisson bracket.

$$[X_F, X_H]J = \{\{J,H\},F\} - \{\{J,F\},H\} = -\{J,\{F,H\}\} = -X_{\{F,H\}}J\,.$$

□

Remark. Theorem 8 relates the canonical Poisson bracket between phase-space functions $F, H \in C^\infty(T^*M)$ and the commutator of Hamiltonian vector fields $\mathfrak{X}(T^*M)$ by the ***anti-homomorphism*** in (13).

3.5 General remarks about Hamiltonian dynamics

We begin by recalling Hamilton's canonical equations and using them to demonstrate Poincaré's theorem for Hamiltonian flows heuristically, by a simple direct calculation. This calculation will motivate discussion of manifolds, tangent bundles, cotangent bundles, vector fields and differential forms in this lecture.

Definition 9 (Hamilton's canonical equations)**.** ***Hamilton's canonical equations*** *are written on* ***phase space****, a locally Euclidean space with pairs of coordinates denoted $(q,p) \in T^*M$. Namely,*

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}\,, \qquad \frac{dp}{dt} = -\,\frac{\partial H}{\partial q}\,, \tag{14}$$

where $\partial H/\partial q$ and $\partial H/\partial p$ are the gradients of a smooth function on phase space $H(q,p)$ called the ***Hamiltonian****. The curves in phase space $(q(t),p(t))$ satisfying Hamilton's canonical equations (14) are called* ***Hamiltonian flows****.*

Definition 10 (Symplectic two-form)**.** *The oriented area in phase-space coordinates $(q(t),p(t))$*

$$\omega = dq \wedge dp = -\,dp \wedge dq$$

is called the ***symplectic two-form****.*

Definition 11 (Sym·plec·tic)**.** *The word* ***symplectic*** *is from the Greek for plaiting, braiding or joining together.*

Definition 12 (Symplectic flows)**.** *Flows that preserve area in phase space are said to be symplectic.*

Remark 13. *The wedge product in the symplectic two-form $\omega = dq\wedge dp$ is the analogue of the cross product of vectors in the skewness $S = \mathbf{q}\times\mathbf{p}$, which is Lagrange's invariant in axisymmetric ray optics.*

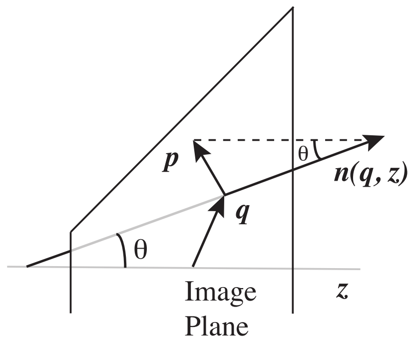


Figure 16: Construction of ***Lagrange's invariant area*** in $T^*\mathbb{R}^3$ phase space $S = \mathbf{q} \times \mathbf{p}$ in ray optics.

Theorem 14 (Poincaré's theorem)**.** *Hamiltonian flows are symplectic. That is, they preserve the oriented phase-space area $\omega = dq \wedge dp$.*

Proof. Preservation of ω may first be verified via a simple formal calculation. Namely, along the characteristic equations of the Hamiltonian vector field $(dq/dt, dp/dt) = (\dot{q}(t), \dot{p}(t)) = (H_p, -H_q)$, for a solution of Hamilton's equations for a smooth Hamiltonian function $H(q, p)$, the flow of the symplectic two-form $\omega = dq \wedge dp$ is governed by

$$
\begin{aligned}
\frac{d\omega}{dt} = \frac{d}{dt}(dq \wedge dp) &= d\dot{q} \wedge dp + dq \wedge d\dot{p} = dH_p \wedge dp - dq \wedge dH_q \\
&= (H_{pq}dq + H_{pp}dp) \wedge dp - dq \wedge (H_{qq}dq + H_{qp}dp) \\
&= H_{pq}\, dq \wedge dp - H_{qp}\, dq \wedge dp = (H_{pq} - H_{qp})\, dq \wedge dp = 0\,.
\end{aligned}
$$

The first step uses the product rule for differential forms, the second uses antisymmetry of the wedge product ($dq \wedge dp = -\, dp \wedge dq$) and the last step uses the equality of cross derivatives $H_{pq} = H_{qp}$ for smooth Hamiltonian function H. □

4 Preliminaries for exterior calculus

4.1 Manifolds and bundles

Let us review some of the fundamental concepts that have already begun to emerge in the previous chapter and cast them into the language of ***exterior calculus***.

Definition 15 (Smooth submanifold of $\mathbb{R}^{3N}$)**.** *A smooth K-dimensional submanifold M of the Euclidean space $\mathbb{R}^{3N}$ is any subset which in a neighbourhood of every point on it is a graph of a smooth mapping of $\mathbb{R}^K$ into $\mathbb{R}^{(3N-K)}$ (where $\mathbb{R}^K$ and $\mathbb{R}^{(3N-K)}$ are coordinate subspaces of $\mathbb{R}^{3N} \simeq \mathbb{R}^K \times \mathbb{R}^{(3N-K)}$).*

Definition 16 (Tangent vectors and tangent bundle)**.**
The solution $q(t) \in M$ is a curve (or trajectory) in manifold M parameterised by time in some interval $t \in (t_1\,, t_2)$.
*The **tangent vector** of the curve $q(t)$ is the velocity $\dot{q}(t)$ along the trajectory that passes though the point $q \in M$ at time t. This is written $\dot{q} \in T_qM$, where T_qM is the tangent space at position q on the manifold M.*
*The union of the tangent spaces T_qM over the configuration manifold defines the **tangent bundle** $(q, \dot{q}) \in TM$.*

Remark 17 (Tangent and cotangent bundles)**.** *The configuration manifold M has coordinates $q \in M$.*
The union of positions on M and tangent vectors (velocities) at each position comprises the tangent bundle TM.
*The union of **positions and momenta** expressed in coordinates as $(q, p) \in T^*M$, where T^*M is the cotangent bundle of the configuration space.*

*The terms **tangent bundle** and **cotangent bundle** just introduced are defined in the context of manifolds.*
In the context of examples, we can think of the tangent bundle as simply the space of positions and velocities.
In examples, we have regarded the cotangent bundle as a pair of vectors on an optical screen, or as the space of positions and canonical momenta for a system of particles.
*This lecture will formalise these intuitive definitions and make precise by using the language of **differential forms**.*

4.2 Contraction

Definition 18 (Contraction). *In exterior calculus, the operation of* ***contraction*** *denoted as* $\,\lrcorner\,$ *introduces a pairing between vector fields and differential forms. Contraction is also called* ***substitution****, or* ***insertion*** *of a vector field into a differential form. For basis elements in phase space, contraction defines* ***duality relations****,*

$$\partial_q \lrcorner\, dq = 1 = \partial_p \lrcorner\, dp\,, \quad \textit{and} \quad \partial_q \lrcorner\, dp = 0 = \partial_p \lrcorner\, dq\,, \tag{15}$$

so that differential forms are linear functions of vector fields. A ***Hamiltonian vector field****,*

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_p\partial_q - H_q\partial_p = \{\,\cdot\,,\, H\,\}\,, \tag{16}$$

satisfies the intriguing linear functional relations with the basis elements in phase space,

$$X_H \lrcorner\, dq = H_p \quad \textit{and} \quad X_H \lrcorner\, dp = -\,H_q\,. \tag{17}$$

Definition 19 (Contraction rules with higher forms). *The rule for contraction or substitution of a vector field into a differential form is to sum the substitutions of X_H over the permutations of the factors in the differential form that bring the corresponding dual basis element into its leftmost position. For example, substitution of the Hamiltonian vector field X_H into the symplectic form $\omega = dq \wedge dp$ yields*

$$X_H \lrcorner\, \omega = X_H \lrcorner\, (dq \wedge dp) = (X_H \lrcorner\, dq)\, dp - (X_H \lrcorner\, dp)\, dq\,.$$

In this example, $X_H \lrcorner\, dq = H_p$ and $X_H \lrcorner\, dp = -\,H_q$, so

$$X_H \lrcorner\, \omega = H_p dp + H_q dq = dH\,,$$

which follows from the duality relations (15).

This calculation has proved the following theorem.

Theorem 20 (Hamiltonian vector field)**.** *The Hamiltonian vector field $X_H = \{\,\cdot\,,\,H\,\}$ satisfies*

$$X_H \,\lrcorner\, \omega \;=\; dH \quad \textit{and} \quad X_H \,\lrcorner\, (X_F \,\lrcorner\, \omega) = \big\{F, H\big\} \quad \textit{with} \quad \omega = dq \,\wedge\, dp\,. \tag{18}$$

Remark 21. *The purely geometric nature of relation (18) argues for it to be taken as defining both Hamiltonian vector fields and canonical Poisson brackets.*

Lemma 22. $d^2 = 0$ *for smooth phase-space functions.*

Proof. For any smooth phase-space function $H(q,p)$, one computes

$$dH = H_q dq + H_p dp$$

and taking the second exterior derivative yields

$$\begin{aligned} d^2 H \;&=\; H_{qp}\, dp\wedge dq + H_{pq}\, dq\wedge dp \\ &=\; (H_{pq} - H_{qp})\, dq\wedge dp = 0\,. \end{aligned}$$

□

Relation (18) also implies the following.

Corollary 23. *The flow of X_H preserves the exact two-form $\omega = dq \,\wedge\, dp = d(-\,pdq)$ for any Hamiltonian H.*

Proof. Preservation of ω may be verified first by a formal calculation using (18). Along $(dq/dt, dp/dt) = (\dot q, \dot p) = (H_p, -H_q)$, for a solution of Hamilton's equations, we have

$$\begin{aligned} \frac{d\omega}{dt} \;&=\; d\dot q \,\wedge\, dp + dq \,\wedge\, d\dot p = dH_p \,\wedge\, dp - dq \,\wedge\, dH_q \\ &=\; d(H_p\, dp + H_q\, dq) = d(X_H \,\lrcorner\, \omega) = d(dH) = 0\,. \end{aligned}$$

The first step uses the product rule for differential forms and the third and last steps use the property of the exterior derivative d that $d^2 = 0$ for continuous forms. The latter is due to the equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product $dq \wedge dp = -\, dp \wedge dq$. $\square$

Definition 24 (Symplectic flow). *A flow is **symplectic** if it preserves the phase-space area or symplectic two-form, $\omega = dq \wedge dp$.*

Corollary 25 (Poincaré's theorem). *The flow of a Hamiltonian vector field is symplectic.*

Definition 26 (Canonical transformations). *A smooth invertible map g of the phase space T^*M is called a **canonical transformation** if it preserves the canonical symplectic form ω on T^*M, i.e., $g^*\omega = \omega$, where $g^*\omega$ denotes the transformation of ω under the map g.*

Remark 27. *The usage of the notation $g^*\omega$ as the transformation of ω under the map g foreshadows the fundamental idea of **pull-back**, which will be pivotal in much of our further discussions.*

Remark 28 (Criterion for a canonical transformation). *Suppose in original coordinates (p, q) the symplectic form is expressed as $\omega = dq \wedge dp$. A transformation $g : T^*M \mapsto T^*M$ written as $(Q, P) = (Q(p, q), P(p, q)$ is canonical if the direct computation shows that $dQ \wedge dP = c\, dq \wedge dp$, up to a constant factor c. (Such a constant factor c is unimportant, since it may be absorbed into the units of time in Hamilton's canonical equations.)*

Remark 29. *By Corollary 25 of Poincaré's theorem 14, the Hamiltonian phase flow g_t is a one-parameter group of canonical transformations.*

Noether's theorem for Hamilton's principle on T^*M

For infinitesimal transformations $(\delta q\,,\delta p)$ that induce variations $\delta L = \delta\big(\langle p,\dot q\rangle - H(q,p)\big)$ Hamilton's principle is

$$
\begin{aligned}
\delta S &= \delta\int_a^b \Big\langle p,\dot q\Big\rangle - H(q,p)\,dt = \int_a^b \delta\Big\langle p,\dot q\Big\rangle - \delta H(q,p)\\
&= \int_a^b \Big\langle \delta p\,,\dot q - H_p\Big\rangle - \Big\langle \dot p + H_q,\delta q\Big\rangle dt + \underbrace{\Big\langle p,\delta q\Big\rangle\Big|_a^b}_{\text{Endpoint}}
\end{aligned}
$$

Look at the endpoint term. Recall Noether's theorem from section 2.2. Noether's theorem states that the endpoint term $\big\langle p,\delta q\big\rangle$ is conserved whenever the Lagrangian admits a Lie symmetry, for example when $\delta q = \Phi(q)$ for a smooth function Φ. Consequently, the Hamiltonian vector field $X_N = \{\,\cdot\,,N\,\}$ for endpoint term $N(q,p) := \big\langle p,\Phi(q)\big\rangle$ must generate a symplectic Lie symmetry of the Hamiltonian. Hence, the relation $X_N H = \{H,N\} = 0 = -\{N,H\}$ must hold; so that ***H and N are preserved along each other's flows.***

Suppose $\delta q = \Phi_\xi(q) = -\pounds_\xi q$ for $\xi\in\mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of a Lie group G that acts transitively on the configuration manifold M by $G\times M\to M$. The statement $\delta q = -\pounds_\xi q$ means that the variations are taken along the flow lines of the Lie group G and $q(\epsilon)$ is ***pushed forward*** along finite transformations of G as $q(\epsilon) = exp(-\epsilon\xi)$.

For such infinitesimal variations by the Lie symmetry group, G, the function $\Phi_\xi(q)$ is linear in both ξ and q, so one may define another pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}} : \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ in addition to the natural pairing on T^*M. This additional pairing enables us to write the Noether endpoint term $N(q,p) := \big\langle p,\Phi(q)\big\rangle$ as

$$
N_\xi(q,p) = \langle p,-\pounds_\xi q\rangle_{T^*M} =: \langle\, p\diamond q\,,\,\xi\,\rangle_{\mathfrak{g}}
$$

The quantity $J(q,p) = p\diamond q\in\mathfrak{g}^*$ defined above in terms of two different pairings is called the ***momentum map*** for the Lie symmetry group G acting on manifold M.

The infinitesimal transformations on the coordinates $(q,p) \in T*M$ are given by the Hamiltonian vector field

$$X_{N_\xi} = \big\{ \,\cdot\,, N_\xi \,\big\} = (-\pounds_\xi q)\frac{\partial}{\partial q} + (-\pounds_\xi^T p)\frac{\partial}{\partial p}\,.$$

Remark 30. *Noether's theorem holds for invariance under a 1-parameter Lie group. Suppose the Lagrangian is invariant under a multi-parameter Lie group, but the invariance group is not Abelian, so the group transformations for the various parameters do not commute among themselves? Then what does Noether's theorem and the momentum map have to say? Answering this question will be our next endeavour.*

Exercise. Derive the expression for $J(q,p) = p \diamond q$ when $G \times M \to M$ is given by $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, with $(q,p) \in T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$. Then choose a Hamiltonian $h(J)$ and determine the equation of motion for $J(q,p) = p \diamond q$. ★

4.3 Cotangent lift and Noether's theorem on the Hamiltonian side

Suppose the variations due to the infinitesimal transformations on M take the form $\delta q = \xi_M(q)$.
Then the corresponding Hamiltonian for these infinitesimal transformations is

$$J_\xi := \Big\langle p, \xi_M(q) \Big\rangle \quad \text{so that} \quad \delta q = \frac{\partial J_\xi}{\partial p} = \xi_M(q) \quad \text{and} \quad \delta p = -\,\frac{\partial J_\xi}{\partial q} = -\,\xi_M'(q)^T p$$

The last expression is called the $\boxed{\textbf{\textit{cotangent lift}}}$ to T_q^*M of the infinitesimal transformation $q \to q_\epsilon = q + \epsilon \xi_M(q)$. The cotangent lift specifies the infinitesimal transformation of $p \in T_q^*M$, given the infinitesimal transformation of $q \in M$.

$$q \to q_\epsilon = q + \epsilon\xi_M(q) \text{ on } M \Longrightarrow (q,p) \to (q_\epsilon, p_\epsilon) = (q + \epsilon\xi_M(q),\, p -\, \epsilon\xi_M'(q)^T p) \text{ on } T_q^*M\,.$$

The time derivative of $J_\xi(q,p)$ is given by

$$\frac{d}{dt}J_\xi(q,p) = \frac{\partial J_\xi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial J_\xi}{\partial p}\frac{\partial H}{\partial q} = -\,\frac{\partial H}{\partial p}\delta p - \,\frac{\partial H}{\partial q}\delta q = -\,\delta H = \{J_\xi, H\} = -\,\{H, J_\xi\} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(q,p).$$

Corollary 31. *On the Hamiltonian side, Noether's theorem for conservation of the endpoint term* $J_\xi := \langle p, \xi_M(q)\rangle := \langle p, \delta q\rangle$ *follows from Lie symmetry of the Hamiltonian function under* $\delta H = \{H, J_\xi\} = X_{J_\xi} H = 0$.

In the previous step, we have defined the infinitesimal transformation

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} H(q,p) = \delta H = \{H, J_\xi\} = X_{J_\xi} H$$

of the phase-space Hamiltonian $H(q,p)$ under canonical transformations generated by the Hamiltonian vector field $X_{J_\xi} \in \mathfrak{X}(T^*(M))$, given by

$$X_{J_\xi} = \big\{\,\cdot\,,\ J_\xi(q,p)\big\} \quad \text{and satisfying} \quad dJ_\xi(q,p) = X_{J_\xi} \,\lrcorner\, \omega$$
$$\text{where} \quad J_\xi(q,p) := \langle p, \delta q\rangle =: \langle p, \xi_M(q)\rangle = \langle p, -\mathcal{L}_\xi q\rangle = \langle J(q,p)\,,\,\xi\rangle_{\mathfrak{g}} = \langle p \diamond q, \xi\rangle_{\mathfrak{g}}$$

is the conserved endpoint term in Noether's theorem.

Remark 32. *To summarise, the vector field* $X_{J_\xi} = \big\{\,\cdot\,,\ J_\xi(q,p)\big\}$ *arising via the Legendre transformation from Noether's theorem for a 1-parameter Lie symmetry of the Lagrangian generates a Lie symmetry of the Hamiltonian,*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} = X_{J_\xi} := \{\,\cdot\,,\ J_\xi\,\} = \frac{\partial J_\xi}{\partial p}\frac{\partial}{\partial q} - \frac{\partial J_\xi}{\partial q}\frac{\partial}{\partial p} = \xi_M(q)\frac{\partial}{\partial q} - \xi\,'(q)^T p\frac{\partial}{\partial p} = \delta q\frac{\partial}{\partial q} + \delta p\frac{\partial}{\partial p}\,.$$

LECTURE #3

The present lecture begins the discussion by formulating Noether's theorem on the Hamiltonian side of the commutative diagram of classical mechanics in terms of a ***cotangent lift momentum map***, $J(q,p) : T^*M \to \mathfrak{g}^*$.

When the Hamiltonian can be written as a function of a momentum map $J(q,p) = p \diamond q : T^*M \to \mathfrak{g}^*$, Hamilton's canonical equations for phase space functions of $(q,p) \in T^*M$ map to equations of motion on functions of $J \in \mathfrak{g}^*$. This momentum-map construction provides the important step "Hamiltonian reduction by Lie group symmetry".

The lecture ends with a discussion of the ***equivariance of geometric mechanics*** by illustrating its framework as a cube whose six faces each comprise a commutative diagram of maps.

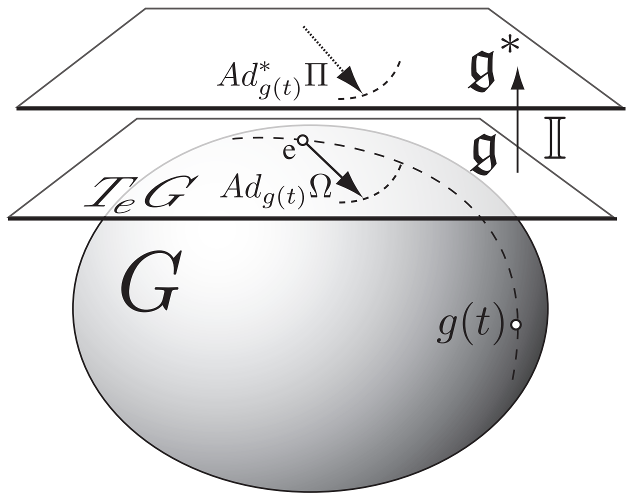
The cube of commutative diagrams is bolstered on the Lagrangian and Hamiltonian sides by Noether's theorem.

Much of the remaining work in this course will involve constructing the various faces and edges of this cube of commutative maps.
The exercise is to show explicitly how the properties of the $\diamond$-operator produce this Poisson map. Go2toc

5 Geometric Mechanics (GM) involves tangent and cotangent manifolds

GM lifts mechanics on a *manifold* M to mechanics on a *Lie group* G which *acts* (transitively) on M [Po1901].
This sentence characterising GM introduces three main concepts into classical Lagrangian and Hamiltonian mechanics:

- The configuration spaces are Manifolds.
  A manifold M is a space which admits differentiable transformations along curves (motions).
  In particular, manifolds admit the rules of calculus.
- Lie groups are groups of *transformations* which depend smoothly on a set of parameters.
  A Lie group G is a manifold and it can act on its tangent and cotangent manifolds TG and T^*G.
- A group action is a transformation of a Lie group G which takes an initial point $q_0 \in M$ in a manifold M to another one along a smooth curve $q_t \in M$, denoted $q_t = g_t q_0$, for g_t a curve in Lie group G parameterised by t. The tangent and cotangent lifts of the action of Lie group G on M determine how G acts on TM and T^*M.
- The actions (Ad & Ad*) of Lie group G on its tangent and cotangent manifolds (TG & T^*G) are central to GM.



6 Quick review of classical mechanics

1. ***Space*** is taken to be a smooth manifold M with points $q \in M$ (Positions, States, Configurations).

   Time is taken to be a manifold T with points $t \in T$. Usually $T = \mathbb{R}$

   Let M be a ***smooth manifold*** $\dim M = n$. That is, M is a smooth space that is locally $\mathbb{R}^n$.

   Operationally, a smooth manifold M is a space on which the rules of calculus apply.

   We will consider curves $q(t) \in M$.

2. Define the ***Lagrangian*** function as a smooth real-valued mapping defined on TM, the tangent space of the configuration manifold M; namely,

$$\text{Lagrangian } L(q(t), v(t)) : TM \to \mathbb{R}$$
$$\text{Action integral } S := \int_a^b L(q, v) + \left\langle p, \frac{dq}{dt} - v \right\rangle dt\,.$$

3. Embed the curve $q(t) \in M$ in a $f$amily of curves given by the one-parameter smooth map

$$q(t) \to q(t, \epsilon) \in M \quad \text{with} \quad q(t, 0) = q(t),$$

   so that $\epsilon = 0$ is the identity map.

   Define the $v$ariation operation $\delta q := dq/d\epsilon|_{\epsilon=0}$ and invoke ***Hamilton's Principle***:

$$0 = \delta S = \int_a^b \left\langle \delta p, \frac{dq}{dt} - v \right\rangle + \left\langle \frac{\partial L}{\partial v} - p\,,\, \delta v \right\rangle + \left\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}\,,\, \delta q \right\rangle dt + \langle p, \delta q \rangle \Big|_a^b$$

Remark 33. *The variations define the* ***pairing*** $\langle\,\cdot\,,\,\cdot\,\rangle : TM \times T^*M \to \mathbb{R}$, *where* $(q,v) \in TM$ *(tangent space) and* $(q,p) \in T^*M$ *(cotangent space); and* $p := \partial L/\partial v$ *is the* ***fibre derivative*** *of the Lagrangian* $L(q,v)$.

Consequently, for variations that vanish at the endpoints in time, so that $\delta q(a) = 0 = \delta q(b)$, and are otherwise arbitrary, we have the $s$tationarity conditions

$$\delta p : \frac{dq}{dt} - v = 0\,, \quad \delta v : \frac{\partial L}{\partial v} - p = 0\,, \quad \delta q : \frac{dp}{dt} - \frac{\partial L}{\partial q} = 0$$

Remark 34. *The variable* p *in the action integral is a* ***Lagrange multiplier***. *Its variation enforces the constraint* $v = dq/dt$ *known as the* ***tangent lift*** *of the curve* $q(t)$.

4. The three stationarity conditions obtained from Hamilton's Principle imply the ***Euler-Lagrange equation***.

$$\frac{\partial L}{\partial v}\Big|_{v=dq/dt} - \frac{\partial L}{\partial q} = 0\,.$$

5. Define the ***Hamiltonian*** $H(p,q) : T^*M \to \mathbb{R}$ via the ***Legendre transformation*** $LT : TM \to T^*M$,

$$H(q,p) = \langle p, v\rangle - L(q,v)\,,$$

with differential

$$\begin{aligned} dH(q,p) &= \left\langle \frac{\partial H}{\partial p}, dp \right\rangle + \left\langle \frac{\partial H}{\partial q}, dq \right\rangle \\ &= \langle v, dp\rangle - \left\langle \frac{\partial L}{\partial q}, dq \right\rangle + \left\langle p - \frac{\partial L}{\partial v}, dv \right\rangle . \end{aligned}$$

6. Introduce ***Hamilton's principle on Phase Space*** $(p, q) \in T^*M$

$$\text{Hamiltonian } H(p,q) : T^*M \to \mathbb{R}$$

$$\text{Phase Space Action } S := \int_a^b \left\langle p, \frac{dq}{dt} \right\rangle - H(q,p)dt$$

$$\text{Hamilton's Principle: } 0 = \delta S = \int_a^b \left\langle \delta p, \frac{dq}{dt} - \frac{\partial H}{\partial p} \right\rangle - \left\langle \frac{\partial H}{\partial q} + \frac{dp}{dt}\,,\, \delta q \right\rangle dt + \langle p, \delta q \rangle \Big|_a^b$$

$$\text{Hamilton's Canonical Equations: } \delta p : \frac{dq}{dt} - \frac{\partial H}{\partial p} = 0\,, \quad \delta q : \frac{dp}{dt} + \frac{\partial H}{\partial q} = 0$$

7. Prove ***Noether's theorem*** on both the Lagrangian and Hamiltonian sides.

Theorem 35 (Noether's theorem)**.** *If the action integral S is invariant under a smooth infinitesimal transformation $\delta q = \Phi_\xi(q)$, then the quantity*

$$J_\xi(q,p) := \langle p, \Phi_\xi(q) \rangle$$

is a constant of the motion. That is, $J_\xi(q,p)$ is conserved when the equations of motion hold.

Proof. On the Lagrangian side, suppose that $L(q, \frac{dq}{dt})$ is invariant under $q \to q + \epsilon \Phi_\xi(q)$.

Then, if the Euler-Lagrange equation holds, we have $\langle p, \Phi_\xi(q) \rangle \Big|_a^b = 0$, and $J_\xi(q,p)$ must be constant.

Likewise, the same conclusion $\langle p, \Phi_\xi(q) \rangle \Big|_a^b = 0$ follows on the Hamiltonian side, when Hamilton's canonical equations hold, and the phase space action is invariant. $\square$

Definition 36 (***Canonical Poisson bracket*****).** *By direct computation, one finds*

$$\frac{d}{dt} J_\xi(q,p) = \frac{\partial J_\xi}{\partial q}\frac{dq}{dt} + \frac{\partial J_\xi}{\partial p}\frac{dp}{dt} = \frac{\partial J_\xi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial J_\xi}{\partial p} =: \Big\{ J_\xi\,,\, H \Big\}$$

Consequently, Hamilton's Canonical equations may be rewritten as

$$\frac{dq}{dt} = \Big\{ q\,,\, H \Big\} \quad \text{and} \quad \frac{dp}{dt} = \Big\{ p\,,\, H \Big\}$$

Exercise. Show that the canonical Poisson brackets defined by $\{F, H\} := F_q H_p - H_q F_p$ have the following properties

1. Binary map on smooth phase space functions $\{\,\cdot\,,\,\cdot\,\} : \mathcal{C}^\infty(T^*M, \mathbb{R}) \times \mathcal{C}^\infty(T^*M, \mathbb{R}) \to \mathcal{C}^\infty(T^*M, \mathbb{R})$
2. Skew symmetric: $\{F, H\} = -\{H, F\}$, for $F, H \in \mathcal{C}^\infty(T^*M, \mathbb{R})$
3. Bilinear: $\{F, aH + bJ\} = a\{F, H\} + b\{F, J\}$, for $a, b \in \mathbb{R}$
4. Leibnitz: $\{F, HJ\} = \{F, H\}J + H\{F, J\}$
5. Jacobi: $\{F, \{H, J\}\} + \{H, \{J, F\}\} + \{J, \{F, H\}\} = 0$

   The Jacobi identity may be verified formally by writing $\{G,\, H\} = G(H) - H(G)$ symbolically. Then write

   $$\{F\,,\,\{G\,,\,H\}\,\} = F(G(H)) - F(H(G)) - G(H(F)) + H(G(F))\,.$$

   Summation over cyclic permutations then yields the result.

★

Lie groups. Consider a group of transformations which depend smoothly on a set of parameters, labelled ϵ,

$$q(t) \to q(t,\epsilon) \in M \quad \text{with} \quad q(t,0) = q(t),$$

so that $\epsilon = 0$ is the identity map. Such a group is called a ***Lie group***.
The ***infinitesimal transformation*** of q under this group is given by the tangent at the identity, denoted as

$$\delta q := \frac{dq}{d\epsilon}\Big|_{\epsilon=0}.$$

Cotangent lift. Suppose the infinitesimal transformation of q is given by $\delta q = \Phi_\xi(q)$.
This $\delta q = \Phi_\xi(q)$ lifts to the infinitesimal cotangent transformation of p by analogy with Hamilton's equations, as

$$\delta q := \frac{dq}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ q \,,\, J_\xi \Big\} = \Phi_\xi(q) \qquad \text{and} \qquad \delta p := \frac{dp}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ p \,,\, J_\xi \Big\} = -p\frac{\partial \Phi_\xi(q)}{\partial q}$$

Consequently, Noether's theorem (Conservation of J_ξ) implies (Invariance of H) under the infinitesimal transformations associated with J_ξ, and *vice* versa since, according to the canonical Poisson bracket.

$$\delta H = \frac{dH}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ H \,,\, J_\xi \Big\} = -\Big\{ J_\xi \,,\, H \Big\} = -\frac{d}{dt} J_\xi(q,p) = 0$$

On the other hand, if the Hamiltonian H depends on the variables $J_\xi(q,p)$, but even if $H(J)$ is *not* invariant under the transformations generated by $\{\,\cdot\,,\, J_\xi\}$ then we may still write ***by the chain rule***

$$\frac{d}{dt} J_\xi = \Big\{ J_\xi \,,\, H \Big\} = \Big\{ J_\xi \,,\, J_\eta \Big\} \frac{\partial H}{\partial J_\eta}$$

which yields the ***transformation law for the canonical Poisson brackets***,

$$\frac{d}{dt} F(J) = \Big\{ F \,,\, H \Big\}(J(q,p)) = \frac{\partial F}{\partial J_\xi} \Big\{ J_\xi \,,\, J_\eta \Big\} \frac{\partial H}{\partial J_\eta}$$

Closure. If the Poisson brackets of the components of J ***close among themselves***, so that

$$\Big\{ J_\xi \,,\, J_\eta \Big\} = J_\gamma C^\gamma_{\xi\eta}$$

where $C^\gamma_{\xi\eta}$ comprise a set of constants (they do for Lie algebras) then the dynamics on the J space reduces to

$$\frac{d}{dt} J_\xi = J_\gamma C^\gamma_{\xi\eta} \frac{\partial H}{\partial J_\eta} \qquad \text{and} \qquad \frac{d}{dt} F(J) = J_\gamma C^\gamma_{\xi\eta} \frac{\partial F}{\partial J_\xi} \frac{\partial H}{\partial J_\eta} =: \big\{ F \,,\, H \big\}_{LP}(J)\,,$$

in which $\{\,\cdot\,,\,\cdot\,\}_{LP}$ preserves the properties of the canonical Poisson bracket provided the constants $C^\gamma{}_{\xi\eta}$ for $\xi, \eta, \gamma = 1, 2, \dots, r$ are structure constants for a Lie algebra, where $[e_\xi, e_\eta] = e_\gamma C^\gamma{}_{\xi\eta}$.

For a Lie algebra whose structure constants are $C^\gamma_{\xi\eta}$ in the basis e_ξ with $\xi = 1, 2, \dots, r$ the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{LP}$ becomes

$$\begin{aligned} \big\{ F \,,\, H \big\}_{LP}(J) &:= J_\gamma C^\gamma_{\xi\eta} \frac{\partial F}{\partial J_\xi} \frac{\partial H}{\partial J_\eta} = J_\gamma \left[\frac{\partial F}{\partial J_\xi} e_\xi \,,\, \frac{\partial H}{\partial J_\eta} e_\eta \right]^\gamma \\ &= \left\langle J_\gamma e^\gamma \,,\, \left[\frac{\partial F}{\partial J_\xi} e_\xi \,,\, \frac{\partial H}{\partial J_\eta} e_\eta \right] \right\rangle_{\mathfrak{g}} =: \left\langle J \,,\, \left[\frac{\partial F}{\partial J} \,,\, \frac{\partial H}{\partial J} \right] \right\rangle_{\mathfrak{g}} \end{aligned}$$

where we have introduced the natural pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ defined by $\langle e_\xi, e^\gamma \rangle_{\mathfrak{g}} = \delta^\gamma_\xi$.

Remark 37 (The bracket $\{\,\cdot\,,\,\cdot\,\}_{LP}$ is Poisson)**.** *Having been introduced as a linear functional of the Lie algebra bracket $[e_\xi, e_\eta] = e_\gamma C^\gamma{}_{\xi\eta}$ for a Lie algebra, $\mathfrak{g}$, the Lie-Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{LP}$ satisfies the Jacobi identity. It is also plainly a skew, bilinear Leibnitz operator. Therefore, the bracket $\{\,\cdot\,,\,\cdot\,\}_{LP}$ is Poisson.*

Exercise. Show explicitly how the properties of the $\diamond$-operator produce this Poisson map. ★

6.1 The Geometric Mechanics Framework (GMF) of relationships for understanding dynamics

Classical mechanics may be visualised as the top face of the GMF cube.

$$
\begin{tikzcd}
L : TM \to \mathbb{R} \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] \arrow[d, "\text{Lagrange variational principle}"] & & H : T^*M \to \mathbb{R} \arrow[d, "\text{Hamilton variational principle}"] \\
\text{Euler-Lagrange eqns} \arrow[rr, Leftrightarrow] & & \text{Hamilton's eqns}
\end{tikzcd}
$$

Figure 17: Framework for Classical Mechanics

Geometric mechanics will trace through all of the corners and edges of the equivariant GMF cube.

$$
\begin{tikzcd}
 & & L : TM \to \mathbb{R} \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] \arrow[dl, "\text{Lag vp}"] \arrow[dd, "TM/G"] & & H : T^*M \to \mathbb{R} \arrow[dl, "\text{Ham vp}"'] \arrow[dd, "\text{Reduction via Momentum Map } J(q,p) \in \mathfrak{g}^* \simeq T^*M/G"] & \\
 & \text{EL eqns} \arrow[rr, Leftrightarrow] \arrow[dd, "TM/G"] & & \text{Ham eqns} \arrow[dd, "J(q,p)"] & & \\
\text{Noether's Theorem} \arrow[rr, dash] \arrow[uurr, dash, bend left] & & \ell : \mathfrak{g} \to \mathbb{R} \arrow[rr, leftrightarrow, "\textbf{LT}"] \arrow[dl, "\text{EP vp}"] & & h : \mathfrak{g}^* \to \mathbb{R} \arrow[r, dash] \arrow[dl, "\text{LP vp}"'] & \text{Noether's Theorem} \arrow[uul, dash, bend right] \\
 & \text{Euler-Poincaré eqns} \arrow[rr, Leftrightarrow] & & \text{Lie-Poisson eqns} & &
\end{tikzcd}
$$

Figure 18: Framework for Geometric Mechanics Back2:intro LagReduxSym

LECTURE #4

This lecture reviews what we need to know about Lie groups and Lie algebras in order to continue the construction of the equivariant cube of geometric mechanics. Figure 22 provides a diagrammatic view of the construction for the rigid body. The lecture also presents a worked example of reduction by symmetry on the Hamiltonian side for rigid body motion is three dimensions. The momentum map in this case is the angular momentum defined by the map $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \to \boldsymbol{q} \times \boldsymbol{p} = \boldsymbol{J} \in \mathfrak{so}(3)$.

Go2toc

7 Quick review of what we need here about Lie groups and Lie algebras

Definition 38 (Group)**.** *A **group** G is a set of elements that possesses a binary product (multiplication), $G \times G \to G$, such that the following properties hold:*

- *The product gh of g and h is associative, that is, $(gh)k = g(hk)$.*
- *An identity element exists, $e : eg = g$ and $ge = g$, for all $g \in G$.*
- *The inverse operation exists, $G \to G$, so that $gg^{-1} = g^{-1}g = e$.*

Definition 39 (Lie group)**.** *A **Lie group** is a group that depends smoothly on a set of parameters. That is, a Lie group is both a group and a smooth manifold, for which the group operation is by composition of smooth invertible operations, such as matrix multiplication, or composition of smooth invertible functions.*

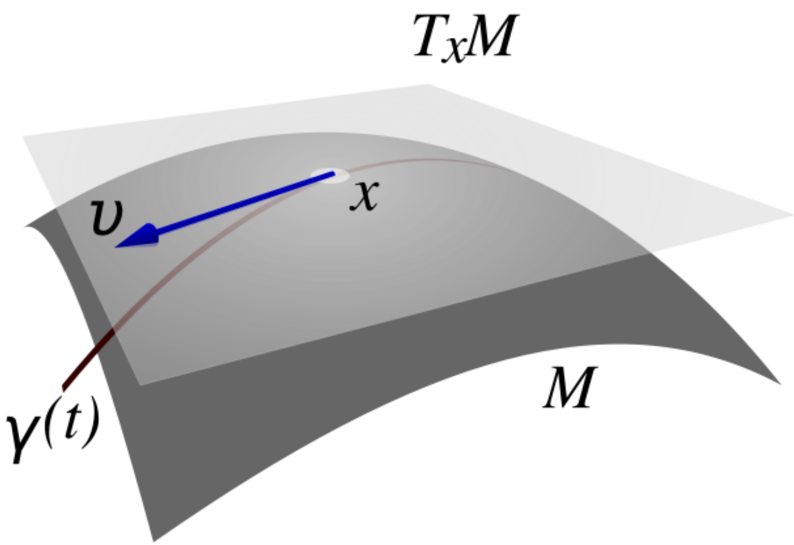


Figure 19: Being a manifold, a Lie group can have tangent spaces.

Definition 40. *A **Lie algebra** is a vector space* $\mathfrak{g}$ *together with a bilinear operation*

$$[\,\cdot\,,\,\cdot\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\,,$$

*called the **Lie bracket** for* $\mathfrak{g}$*, that satisfies the defining properties:*

- *bilinearity, e.g.,*

$$[a\mathbf{u}+b\mathbf{v}\,,\,\mathbf{w}]=a[\mathbf{u}\,,\,\mathbf{w}]+b[\mathbf{v}\,,\,\mathbf{w}]\,,$$

  for constants $(a,b)\in\mathbb{R}$ *and any vectors* $(\mathbf{u},\mathbf{v},\mathbf{w})\in\mathfrak{g}$*;*

- *skew-symmetry,*

$$[\mathbf{u}\,,\,\mathbf{w}]=-[\mathbf{w}\,,\,\mathbf{u}]\,;$$

- *Jacobi identity,*

$$[\mathbf{u}\,,\,[\mathbf{v}\,,\,\mathbf{w}]\,]+[\mathbf{v}\,,\,[\mathbf{w}\,,\,\mathbf{u}]\,]+[\mathbf{w}\,,\,[\mathbf{u}\,,\,\mathbf{v}]\,]=0\,,$$

  for all $\mathbf{u},\mathbf{v},\mathbf{w}$ *in* $\mathfrak{g}$.

7.1 Structure constants of finite-dimensional Lie algebras

Suppose $\mathfrak{g}$ is any finite-dimensional Lie algebra. The Lie bracket for any choice of basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ of $\mathfrak{g}$ must again lie in $\mathfrak{g}$. Thus, constants c_{ij}^k exist, where $i, j, k = 1, 2, \ldots, r$, called the ***structure constants*** of the Lie algebra $\mathfrak{g}$, such that

$$[\,\mathbf{e}_i\,,\,\mathbf{e}_j\,] = c_{ij}^k \mathbf{e}_k\,. \tag{19}$$

Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ form a vector basis, the structure constants in (19) determine the Lie algebra $\mathfrak{g}$ from the bilinearity of the Lie bracket. The conditions of skew-symmetry and the Jacobi identity place further constraints on the structure constants. These constraints are

- skew-symmetry
$$c_{ji}^k = -\,c_{ij}^k\,, \tag{20}$$
- Jacobi identity
$$c_{ij}^k c_{lk}^m + c_{li}^k c_{jk}^m + c_{jl}^k c_{ik}^m = 0\,. \tag{21}$$

Conversely, any set of constants c_{ij}^k that satisfy relations (20) and (21) defines a Lie algebra $\mathfrak{g}$.

Exercise. Prove that the Jacobi identity requires the relation (21). ★

Answer. The Jacobi identity involves summing three terms of the form

$$[\,\mathbf{e}_l\,,\,[\,\mathbf{e}_i\,,\,\mathbf{e}_j\,]\,] = c_{ij}^k [\,\mathbf{e}_l\,,\,\mathbf{e}_k] = c_{ij}^k c_{lk}^m \mathbf{e}_m\,.$$

Summing over the three cyclic permutations of (l, i, j) of this expression yields the required relation (21) among the structure constants for the Jacobi identity to hold. ▲

7.2 Infinitesimal vs Finite Transformation of a Lie Group

The infinitesimal transformation of a Lie group G acting on a manifold Q as $G \times Q \to Q$ is given by the linear term in the Taylor series of the finite transformation

$$\begin{aligned} q_\epsilon = \phi_\epsilon(q_0) &= q_0 + \epsilon \left[\frac{d}{d\epsilon} \phi_\epsilon(q_0) \right]_{\epsilon=0} + O(\epsilon^2) \\ &= q_0 + \epsilon \Phi(q_0) + O(\epsilon^2) \end{aligned}$$

and denoted as

$$\delta q = \frac{dq_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \Phi(q)$$

In more generality, for smooth functions $f \in C^\infty(Q)$ we have the ***pull-back relation***

$$\frac{d}{d\epsilon}({\phi_\epsilon}^* f) = {\phi_\epsilon}^* \mathcal{L}_{v_\Phi} f \, ,$$

where the vector field v_Φ generates the smooth flow ϕ_ϵ. This is the ***Lie chain rule***.
Then, evaluating at $\epsilon = 0$ gives the infinitesimal transformation known as a ***Lie derivative***

$$\frac{d}{d\epsilon}({\phi_\epsilon}^* f)\bigg|_{\epsilon=0} = \mathcal{L}_{v_\Phi} f$$

Review: Geometric Mechanics deals with group invariant variational principles (Noether)

$$
\begin{tikzcd}
\left.\begin{array}{c} \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0, \\ \big\langle \frac{\partial L}{\partial \dot q}\,,\,\delta q \big\rangle_{TM}\Big|_{t=a}^{t=b} = 0 \end{array}\right\} \arrow[d, dotted, leftrightarrow] & L(q,v) : TM \to \mathbb{R} \arrow[l, "\delta S = 0"'] \arrow[r, "\mathcal{L}T"] \arrow[d, "\text{Reduction by Lie symmetry}"] & H(q,p) : T^*M \to \mathbb{R} \arrow[r, "\delta S = 0"] \arrow[d, "{\langle p,\delta q\rangle_{TM} = \langle J(q,p),\xi\rangle_{\mathfrak{g}}\ \text{(momentum map)}}"] & \left\{\begin{array}{c} \frac{\partial H}{\partial p} = \dot q, \qquad \frac{\partial H}{\partial q} = -\dot p, \\ \Big(X_H \,\lrcorner\, (dq \wedge dp) = dH \Big) \end{array}\right. \arrow[d, dotted, leftrightarrow] \\
\begin{array}{c} \frac{d}{dt}\frac{\partial l}{\partial \xi^\alpha} = \frac{\partial l}{\partial \xi^\gamma} c^\gamma_{\alpha\beta}\xi^\beta \\ \Big(\frac{d}{dt}\frac{\partial l}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi} \Big) \end{array} & l(\xi) : \mathfrak{g} \to \mathbb{R} \arrow[l, "\delta S_{red} = 0"'] \arrow[r, leftrightarrow, "\text{Reduced } \mathcal{L}T"] & h(J) : \mathfrak{g}^* \to \mathbb{R} \arrow[r, "\delta S_{red} = 0"] & \begin{array}{c} \frac{dJ_\alpha}{dt} = \{J_\alpha, J_\beta\}\frac{\partial h}{\partial J_\beta} = J_\gamma c^\gamma_{\alpha\beta}\frac{\partial h}{\partial J_\beta} \\ \Big(\frac{dJ}{dt} = \mathrm{ad}^*_{\partial h/\partial J} J \Big) \end{array}
\end{tikzcd}
$$

Hamilton's principle (HP): $\delta S = 0$ with $S = \int_a^b L(q,v) + \langle p, \dot q - v\rangle_{TM}\, dt$ and $S = \int_a^b \langle p, \dot q\rangle_{TM} - H(q,p)dt$

Reduction by Lie symmetry TM***:*** $q_t = g_t q_0$, $\dot q_t = \dot g_t q_0$, and $L(g,v) = L(kg,kv)$, $k \in G$, set $L(e, g^{-1}v) =: l(\xi)$

Noether's theorem: Lie symmetry of HP implies conservation of $\langle \frac{\partial L}{\partial \dot q}\,,\,\delta q\rangle_{TM} = \langle p\,,\,\delta q\rangle_{TM} = \langle J(q,p), \xi\rangle_{\mathfrak{g}}$

Legendre transformation (LT): $p := \frac{\partial L}{\partial \dot q}$, $H(q,p) := \langle p, v\rangle_{TM} - L(q,v)$, and $J := \frac{\partial l}{\partial \xi}$, $h(J) := \langle J, \xi\rangle_{\mathfrak{g}} - l(\xi)$

Reduced Hamilton's principle: $S_{red} = \int_a^b l(\xi) + \langle J, g^{-1}\dot g - \xi\rangle_{\mathfrak{g}}\, dt$ and $S_{red} = \int_a^b \langle J, g^{-1}\dot g\rangle_{\mathfrak{g}} - h(J)dt$

Adjoint and co-adjoint actions: $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ GMF

7.3 AD, Ad, and ad operations for Lie algebras and groups

The notation AD, Ad, and ad follows the standard notation for the corresponding actions of a Lie group on itself, on its Lie algebra (its tangent space at the identity), the action of the Lie algebra on itself, and their dual actions.

7.3.1 ADjoint, Adjoint and adjoint for matrix Lie groups

- AD (conjugacy classes of a matrix Lie group): The map $I_g : G \to G$ given by $I_g(h) \to ghg^{-1}$ for matrix Lie group elements $g, h \in G$ is the ***inner automorphism*** associated with g. Orbits of this action are called ***conjugacy classes***.

$$\mathrm{AD} : G \times G \to G : \quad \mathrm{AD}_g h := ghg^{-1} .$$

- Differentiate $I_g(h)$ with respect to h at $h = e$ to produce the ***Adjoint operation***,

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g} : \quad \mathrm{Ad}_g \,\eta = T_e I_g \,\eta =: g\eta g^{-1} ,$$

  with $\eta = h'(0)$.

- Differentiate $\mathrm{Ad}_g \,\eta$ with respect to g at $g = e$ in the direction ξ to produce the ***adjoint operation***,

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : \quad T_e(\,\mathrm{Ad}_g \,\eta)\,\xi = [\xi, \eta] = \,\mathrm{ad}_\xi \,\eta .$$

  Explicitly, one computes the ad operation by differentiating the Ad operation directly as

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g(t)} \,\eta &= \frac{d}{dt}\Big|_{t=0} \Big(g(t)\eta g^{-1}(t) \Big) \\ &= \dot{g}(0)\eta g^{-1}(0) - g(0)\eta g^{-1}(0)\dot{g}(0)g^{-1}(0) \\ &= \xi\,\eta - \eta\,\xi = [\xi, \eta] = \,\mathrm{ad}_\xi \,\eta , \end{aligned} \tag{22}$$

where $g(0) = Id$, $\xi = \dot{g}(0)$ and the ***Lie bracket***

$$[\xi, \eta] \, : \, \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \, ,$$

is the matrix commutator for a matrix Lie algebra.

Summary figure

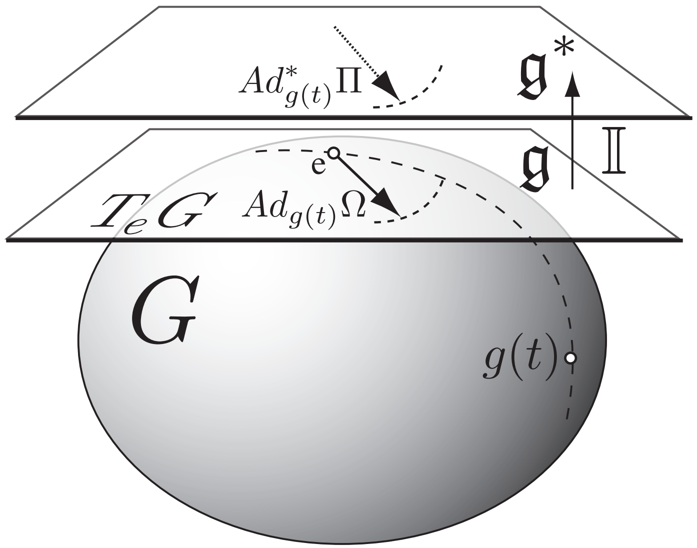


Figure 20: ***The Ad and Ad* operations of $g(t)$ act, respectively, on the Lie algebra Ad $: G \times \mathfrak{g} \to \mathfrak{g}$ and on its dual Ad* $: G \times \mathfrak{g}^* \to \mathfrak{g}^*$. For rigid bodies, Ad$^*_{g_t^{-1}}\Pi_0 = \Pi_t$ traces out the solution dynamics for $g_t \in SO(3)$ and $\mathfrak{g}^* = \mathfrak{so}^*(3)$.***

Remark 41 (Adjoint action)**.** *Composition of the Adjoint action of $G \times \mathfrak{g} \to \mathfrak{g}$ of a Lie group on its Lie algebra represents the group composition law as*

$$Ad_g\ Ad_h\eta = g(h\eta h^{-1})g^{-1} = (gh)\eta(gh)^{-1} = \ Ad_{gh}\eta\,,$$

for any $\eta \in \mathfrak{g}$.

Exercise. Verify that (note the minus sign)

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g^{-1}(t)}\,\eta = -\ \mathrm{ad}_\zeta\,\eta\,,$$

for any fixed $\eta \in \mathfrak{g}$ and $\zeta = g_t^{-1}\dot{g}_t\big|_{t=0} = \dot{g}_t\big|_{t=0}$. ★

Proposition 42 (Adjoint motion equation)**.** *Let $g(t)$ be a path in a Lie group G and $\eta(t)$ be a path in its Lie algebra $\mathfrak{g}$. Then*

$$\frac{d}{dt}Ad_{g(t)}\eta(t) = Ad_{g(t)}\left[\frac{d\eta}{dt} + ad_{\xi(t)}\eta(t)\right],$$

where $\xi(t) = g(t)^{-1}\dot{g}(t)$.

Proof. By Equation (22), for a curve $\eta(t) \in \mathfrak{g}$,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=t_0} \mathrm{Ad}_{g(t)}\,\eta(t) &= \frac{d}{dt}\Big|_{t=t_0}\Big(g(t)\eta(t)g^{-1}(t)\Big) \\
&= g(t_0)\Big(\dot{\eta}(t_0) + g^{-1}(t_0)\dot{g}(t_0)\eta(t_0) \\
&\qquad - \eta(t_0)g^{-1}(t_0)\dot{g}(t_0)\Big)g^{-1}(t_0) \\
&= \left[\mathrm{Ad}_{g(t)}\left(\frac{d\eta}{dt} + \mathrm{ad}_\xi\eta\right)\right]_{t=t_0}.
\end{aligned} \tag{23}$$

□

Exercise. (Inverse Adjoint motion relation) Verify that

$$\frac{d}{dt}\mathrm{Ad}_{g(t)^{-1}}\eta = -\mathrm{ad}_\xi \mathrm{Ad}_{g(t)^{-1}}\eta\,, \tag{24}$$

for any fixed $\eta \in \mathfrak{g}$. Note the placement of $\mathrm{Ad}_{g(t)^{-1}}$ and compare with Exercise on page 71. ★

7.3.2 Compute the coAdjoint and coadjoint operations by taking duals

The pairing

$$\big\langle \cdot\,,\,\cdot \big\rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R} \tag{25}$$

(which is assumed to be nondegenerate) between a Lie algebra $\mathfrak{g}$ and its dual vector space $\mathfrak{g}^*$ allows one to define the following dual operations:

- The ***coAdjoint operation*** of a Lie group on the dual of its Lie algebra is defined by the pairing with the Ad operation,

  $$\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^* : \quad \langle\ \mathrm{Ad}^*_g\,\mu\,,\,\eta\,\rangle := \langle\,\mu\,,\ \ \mathrm{Ad}_g\,\eta\,\rangle\,, \tag{26}$$

  for $g \in G$, $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$.

- Likewise, the ***coadjoint operation*** is defined by the pairing with the ad operation,

  $$\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* : \quad \langle\ \mathrm{ad}^*_\xi\,\mu\,,\,\eta\,\rangle := \langle\,\mu\,,\ \ \mathrm{ad}_\xi\,\eta\,\rangle\,, \tag{27}$$

  for $\mu \in \mathfrak{g}^*$ and $\xi, \eta \in \mathfrak{g}$.

Definition 43 (CoAdjoint action). *The map*

$$\Phi^* : G \times \mathfrak{g}^* \to \mathfrak{g}^* \quad \textit{given by} \quad (g, \mu) \mapsto Ad^*_{g^{-1}}\mu \tag{28}$$

*defines the **coAdjoint action** of the Lie group G on its dual Lie algebra $\mathfrak{g}^*$.*

Remark 44 (Coadjoint group action with g^{-1}). *Composition of coAdjoint operations with Φ^* reverses the order in the group composition law as*

$$Ad^*_g \; Ad^*_h = \; Ad^*_{hg} \, .$$

However, taking the inverse g^{-1} in Definition 43 of the coAdjoint action Φ^* restores the order and thereby allows it to represent the group composition law when acting on the dual Lie algebra, for then

$$Ad^*_{g^{-1}} \; Ad^*_{h^{-1}} = \; Ad^*_{h^{-1}g^{-1}} = \; Ad^*_{(gh)^{-1}} \, . \tag{29}$$

(See [MaRa1994] for further discussion of this point.)

The following proposition will be used later in the context of Euler–Poincaré reduction.

Proposition 45 (Coadjoint motion relation). *Let $g(t)$ be a path in a matrix Lie group G and let $\mu(t)$ be a path in $\mathfrak{g}^*$, the dual (under the Frobenius pairing) of the matrix Lie algebra of G. The corresponding Ad^* operation satisfies*

$$\frac{d}{dt} Ad^*_{g(t)^{-1}}\mu(t) = Ad^*_{g(t)^{-1}} \left[\frac{d\mu}{dt} - ad^*_{\xi(t)}\mu(t) \right] , \tag{30}$$

where $\xi(t) = g(t)^{-1}\dot{g}(t)$.

Proof. The Exercise on page 72 introduces the inverse Adjoint motion relation (24) for any fixed $\eta \in \mathfrak{g}$, repeated as

$$\frac{d}{dt}\mathrm{Ad}_{g(t)^{-1}}\eta = -\mathrm{ad}_{\xi(t)}\left(\mathrm{Ad}_{g(t)^{-1}}\eta\right) .$$

Relation (24) may be proven by the following computation,

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_0} \mathrm{Ad}_{g(t)^{-1}}\eta &= \frac{d}{dt}\bigg|_{t=t_0} \mathrm{Ad}_{g(t)^{-1}g(t_0)}\left(\mathrm{Ad}_{g(t_0)^{-1}}\eta\right) \\ &= -\mathrm{ad}_{\xi(t_0)}\left(\mathrm{Ad}_{g(t_0)^{-1}}\eta\right) , \end{aligned}$$

in which for the last step one recalls

$$\frac{d}{dt}\bigg|_{t=t_0} g(t)^{-1}g(t_0) = \left(-g(t_0)^{-1}\dot{g}(t_0)g(t_0)^{-1}\right) g(t_0) = -\xi(t_0) .$$

Relation (24) plays a key role in demonstrating relation (30) in the theorem, as follows. Using the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$ between the Lie algebra and its dual, one computes

$$\begin{aligned} \left\langle \frac{d}{dt}\mathrm{Ad}^*_{g(t)^{-1}}\mu(t), \eta \right\rangle &= \frac{d}{dt}\left\langle \mathrm{Ad}^*_{g(t)^{-1}}\mu(t), \eta\right\rangle \\ \text{by (26)} &= \frac{d}{dt}\left\langle \mu(t), \mathrm{Ad}_{g(t)^{-1}}\eta\right\rangle \\ &= \left\langle \frac{d\mu}{dt}, \mathrm{Ad}_{g(t)^{-1}}\eta \right\rangle + \left\langle \mu(t), \frac{d}{dt}\mathrm{Ad}_{g(t)^{-1}}\eta \right\rangle \\ \text{by (24)} &= \left\langle \frac{d\mu}{dt}, \mathrm{Ad}_{g(t)^{-1}}\eta \right\rangle + \left\langle \mu(t), -\mathrm{ad}_{\xi(t)}\left(\mathrm{Ad}_{g(t)^{-1}}\eta\right)\right\rangle \\ \text{by (27)} &= \left\langle \frac{d\mu}{dt}, \mathrm{Ad}_{g(t)^{-1}}\eta \right\rangle - \left\langle \mathrm{ad}^*_{\xi(t)}\mu(t), \mathrm{Ad}_{g(t)^{-1}}\eta \right\rangle \\ \text{by (26)} &= \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\frac{d\mu}{dt}, \eta \right\rangle - \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\mathrm{ad}^*_{\xi(t)}\mu(t), \eta \right\rangle \\ &= \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\left[\frac{d\mu}{dt} - \mathrm{ad}^*_{\xi(t)}\mu(t)\right], \eta \right\rangle . \end{aligned}$$

This concludes the proof.

□

Corollary 46. *The* ***coadjoint orbit relation***

$$\mu(t) = Ad^*_{g(t)}\mu(0) \tag{31}$$

is the solution of the ***coadjoint motion equation*** *for* $\mu(t)$,

$$\frac{d\mu}{dt} - ad^*_{\xi(t)}\mu(t) = 0\,. \tag{32}$$

Proof. Substituting Equation (32) into Equation (30) yields

$$\mathrm{Ad}^*_{g(t)^{-1}}\mu(t) = \mu(0)\,. \tag{33}$$

Operating on this equation with $\mathrm{Ad}^*_{g(t)}$ and recalling the composition rule for Ad^* from Remark 44 yields the result (31). □

Remark 47. *As it turns out, the equations in Poincaré (1901) [Po1901] for which we have been preparing describe coadjoint motion!*
Moreover, by equation (33) *in the proof, coadjoint motion implies that* $Ad^*_{g(t)^{-1}}\mu(t)$ *is a conserved quantity.*

7.4 Worked example of reduction by symmetry on the Hamiltonian side

Worked example for angular momentum.
(a) What are the conditions on $\delta q^i = \widehat{\xi}\,^i_j q^j$ where $\widehat{\xi}\,^i_j$ is a constant 3×3 matrix and $i,j = 1,2,3$, so that $\Big\{ J_\xi\,,\, J_\eta \Big\} = J_\gamma C^\gamma_{\xi\eta}$ for a set of constants $C^\gamma_{\xi\eta}$?

(b) Show that infinitesimal rotations of $\mathbb{R}^3$ are involved when $\widehat{\xi}\,^i_j$ is a constant 3×3 skew-symmetric matrix.

(c) Compute the cotangent lift formulas in the ladder of commuting diagrams above for reduction in the case that $\mathbf{q}\in\mathbb{R}^3$ and $\delta\mathbf{q} = \boldsymbol{\xi}\times\mathbf{q}$; that is, $\delta q^i = \widehat{\xi}\,^i_j q^j$ for $i,j = 1,2,3$. ★

Answer. (a) When $\widehat{\xi}\,^i_j$ is a constant matrix, we have the bilinear form,

$$
p_i \delta q^i = p_i \widehat{\xi}\,^i_k q^k \,.
$$

(b) In the case that $\widehat{\xi}$ is a 3×3 skew-symmetric matrix, we discover the ***hat map*** isomorphism $\widehat{\ }: so(3)\simeq\mathbb{R}^3$, by which the Lie algebra $so(3)$ of infinitesimal rotations in $\mathbb{R}^3$ may be represented by 3×3 skew-symmetric matrices,

$$
\widehat{\xi}\,^i_k = -\,\epsilon^i_{kj}\xi^j = -\,\widehat{\xi}\,^k_i \,, \quad \text{or} \quad \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix} .
$$

Remark 48 (Properties of the hat map for $so(3)$)**.** *The hat map arises in the infinitesimal rotations*

$$
\widehat{\xi}\,^i_k = (O^{-1}dO/ds)\,^i_k\Big|_{s=0} = -\,\epsilon^i_{kj}\xi^j \,,
$$

for $O\in SO(3)$ with $\det O = 1$ and $OO^T = Id$.
The matrix $\widehat{\xi} = O^{-1}\dot{O} = O^T\dot{O}$ is skew, since $\frac{d(O^TO)}{dt} = \frac{d(Id)}{dt} = \dot{O}^TO + O^T\dot{O} = (O^{-1}\dot{O})^T + O^{-1}\dot{O} = \widehat{\xi}^T + \widehat{\xi} = 0$.

The hat map is an isomorphism:

$$\widehat{(\,\cdot\,)} : (\mathbb{R}^3, \times) \mapsto (\mathfrak{so}(3), [\,\cdot\,,\,\cdot\,]\,)\,.$$

That is, the hat map identifies the composition of two vectors in $\mathbb{R}^3$ *using the cross product with the commutator of two skew-symmetric* 3×3 *matrices. Specifically, we write for any two vectors* $\mathbf{q}, \boldsymbol{\xi} \in \mathbb{R}^3$*,*

$$(\boldsymbol{\xi} \times \mathbf{q})^k = \epsilon^k{}_{jm} \xi^j q^m = \widehat{\xi}^k_m \, q^m\,.$$

In matrix form, we may write

$$\boldsymbol{\xi} \times \mathbf{q} = \widehat{\xi}\,\mathbf{q} \quad \textit{for all} \quad \boldsymbol{\xi},\, \mathbf{q} \in \mathbb{R}^3\,.$$

Exercise. *Verify the following formulas for* $\mathbf{p}, \mathbf{q}, \boldsymbol{\xi} \in \mathbb{R}^3$*:*

$$\begin{aligned}
(\mathbf{p} \times \mathbf{q})\widehat{\ } &= \Big[\widehat{p}, \widehat{q}\Big]\,,\\
\Big[\widehat{p}, \widehat{q}\Big]\boldsymbol{\xi} &= (\mathbf{p} \times \mathbf{q}) \times \boldsymbol{\xi}\,,\\
\mathbf{p} \cdot \mathbf{q} &= \frac{1}{2}\,\mathrm{trace}\left(\widehat{p}^T\,\widehat{q}\right) = -\,\frac{1}{2}\,\mathrm{trace}\left(\widehat{p}\,\widehat{q}\right)\,.
\end{aligned}$$

★

(c) In the case that $\mathbf{q} \in \mathbb{R}^3$ and $\delta\mathbf{q} = \boldsymbol{\xi} \times \mathbf{q} = \{\mathbf{q}, J_\xi\}$, the canonical Poisson bracket generates the cotangent lift infinitesimal transformation of the canonical momentum, $\delta\mathbf{p} = \boldsymbol{\xi} \times \mathbf{p} = \{\mathbf{p}, J_\xi\}$, and the momentum map turns out to be the familiar expression for the ***angular momentum*** of a particle in phase space $T^*\mathbb{R}^3$,

$$J_\xi = \mathbf{J} \cdot \boldsymbol{\xi} = \mathbf{p} \cdot \boldsymbol{\xi} \times \mathbf{q} = \mathbf{q} \times \mathbf{p} \cdot \boldsymbol{\xi} \Longleftrightarrow \mathbf{J} = \mathbf{q} \times \mathbf{p}$$

When we take $\boldsymbol{\xi} = \boldsymbol{e}_i$, with $i = 1, 2, 3$, as the basis of orthonormal unit vectors in $\mathbb{R}^3$, we find the Poisson bracket relations for the components with $J_i = \mathbf{J} \cdot \boldsymbol{e}_i$ to be

$$\Big\{J_j\,,\,J_k\Big\} = \Big\{\mathbf{J} \cdot \boldsymbol{e}_j\,,\,\mathbf{J} \cdot \boldsymbol{e}_k\Big\} = \mathbf{J} \cdot \boldsymbol{e}_j \times \boldsymbol{e}_k = \mathbf{J} \cdot \epsilon^i_{jk}\boldsymbol{e}_i = J_i \epsilon^i_{jk} \quad \text{with} \quad i, j, k = 1, 2, 3\,.$$

Consequently, we may verify our previous calculation for arbitrary linear transformations in this case simply in terms of vector multiplication in $\mathbb{R}^3$, as

$$\Big\{J_\xi,\,J_\eta\Big\}=\Big\{J\cdot\xi,\,J\cdot\eta\Big\}=\Big\{q\times p\cdot\xi,\,q\times p\cdot\eta\Big\}_{can}=(q\times p)\cdot(\xi\times\eta)=J\cdot(\xi\times\eta)=J_{\xi\times\eta}=J_{[\widehat{\xi},\widehat{\eta}]},$$

where the middle part of the calculation follows by expanding as

$$\Big\{q\times p\cdot\xi,\,q\times p\cdot\eta\Big\}_{can}=-\eta\cdot\{q\times p,J_\xi(q,p)\}_{can}=-\eta\cdot\xi\times(q\times p)=-J\cdot\eta\times\xi=J\cdot\xi\times\eta\,.$$

▲

The corresponding Poisson bracket is given by

$$\{F,H\}(J)=\frac{\partial F}{\partial J_\alpha}\{J_\alpha,J_\beta\}\frac{\partial H}{\partial J_\beta}=J_\gamma\epsilon^\gamma_{\alpha\beta}\frac{\partial F}{\partial J_\alpha}\frac{\partial H}{\partial J_\beta}=\mathbf{J}\cdot\frac{\partial F}{\partial\mathbf{J}}\times\frac{\partial H}{\partial\mathbf{J}}\quad\text{with}\quad\alpha,\beta,\gamma=1,2,3.$$

In particular, as one might have expected, since $\mathbf{J}\in\mathbb{R}^3$, the infinitesimal rotation of $\mathbf{J}$ generated by the Poisson bracket with $J_\xi=\mathbf{J}\cdot\boldsymbol{\xi}$ is given by

$$\delta\mathbf{J}=\{\mathbf{J}\,,\,J_\xi\}=\boldsymbol{\xi}\times\mathbf{J}$$

Upon denoting $\mathbf{J}=\mathbf{x}\in\mathbb{R}^3$ this Poisson bracket becomes

$$\{F,\,H\}=\nabla C\cdot\nabla F\times\nabla H$$

with motion equation

$$\dot{\mathbf{x}}=-\,\nabla C\times\nabla H\quad\text{where}\quad C(\mathbf{x})=\frac{1}{2}|\mathbf{x}|^2\,.$$

This means the motion takes place on $\boxed{\textbf{\textit{spheres}}}$ along intersections of level sets of C and H.

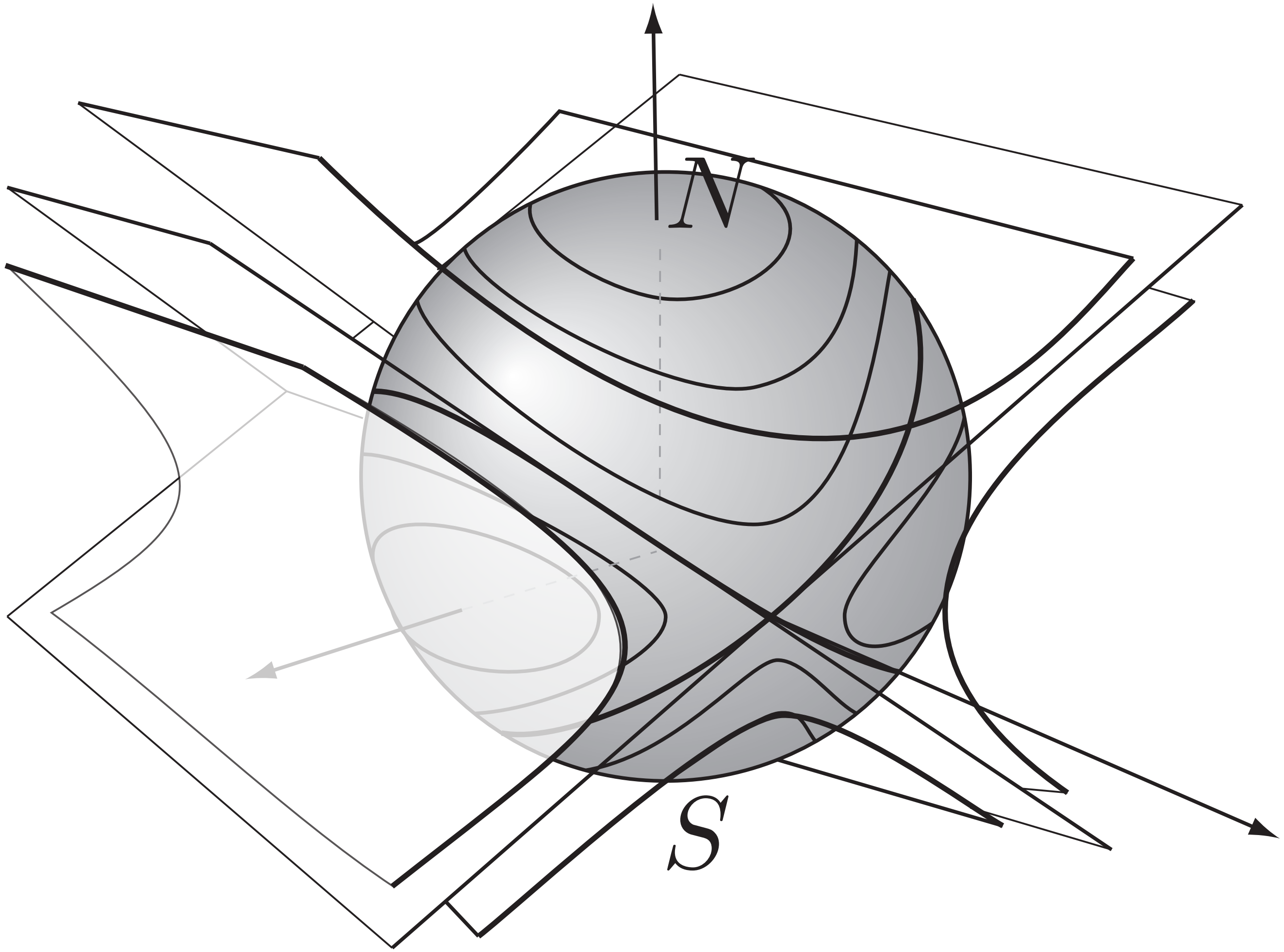


Figure 21: ***Rigid body dynamics follows the intersections of two-dimensional surfaces in $\mathbb{R}^3$ that are level sets of the conservation laws for energy and body angular momentum. Both level sets are symplectic manifolds.***

Schematic summary of Hamiltonian reduction by symmetry for the rigid body

$$\frac{dz}{dt} = \{z,\, H(z)\}_{can} \quad \dim T^*\mathbb{R}^3 = 6$$

$$\begin{tikzcd}
z = (q,p) \in T^*\mathbb{R}^3 \arrow[r] \arrow[d, "J(0) = q(0) \times p(0)"'] \arrow[dr, phantom, "\text{Equivariance}"] & T^*\mathbb{R}^3 \arrow[d, "J(t) = q(t) \times p(t)\ (\text{momap})"] \\
J \in \mathbb{R}^3 \simeq \mathfrak{so}(3)^* \arrow[r] & \mathbb{R}^3 \simeq \mathfrak{so}(3)^*
\end{tikzcd}$$

$$\frac{dJ}{dt} = \{J,\, H(J)\}_{LP} = -\, J \times \frac{\partial H}{\partial J} \quad \dim(S^2) = 2$$

Figure 22: ***The rigid-body example of reduction by symmetry on the Hamiltonian side uses the cotangent lift momentum map. GMF***

Definition 49. *The cotangent-lift momentum map* $J : T^*Q \mapsto \mathfrak{g}^*$ *is defined by the relations*

$$\begin{aligned}
J_\xi(p,q) &:= \Big\langle J(p,q),\, \xi \Big\rangle_{\mathfrak{g}} \\
&= \Big\langle (p,q),\, \xi_Q(q) \Big\rangle_{TM} \\
&= \Big\langle p,\, \pounds_\xi q \Big\rangle_{TM} \\
&= \Big\langle q \diamond p,\, \xi \Big\rangle_{\mathfrak{g}} .
\end{aligned} \tag{34}$$

Proposition 50 (Equivariant group actions)**.**

- *A group action $\Phi_g : G \times T^*Q \mapsto T^*Q$ is said to be **equivariant** if it satisfies*

$$J \circ \Phi_g = Ad^*_{g^{-1}} \circ J \,.$$

This means the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\Phi_{g(t)}"] \arrow[d, "J"'] & P \arrow[d, "J"] \\ \mathfrak{g}^* \arrow[r, "Ad^*_{g(t)^{-1}}"] & \mathfrak{g}^* \end{tikzcd}$$

- *At the tangent to the identity this implies that*

$$\frac{d}{dt}\bigg|_{t=0} J\big(\Phi_{g(t)}(z)\big) = \frac{d}{dt}\bigg|_{t=0} \big(\Phi^*_{g(t)}J(z)\big) = \frac{d}{dt}\bigg|_{t=0} Ad^*_{g^{-1}} \circ J(z) = -ad^*_\xi J(z) = -\mathcal{L}_\xi J(z)$$

*and so by the **Lie chain rule***

$$dJ(z)\cdot \xi_P(z) \;=\; -ad^*_\xi J(z)\,,$$

with $z = (p,q)$. Setting $dJ(z)\cdot\xi_P(z) = X_{J_\xi}J$ and pairing with a fixed Lie algebra element η yields the η-component:

$$\begin{aligned} \Big\langle dJ(z)\cdot \xi_P(z)\,,\,\eta\Big\rangle &= \Big\langle -\,ad^*_\xi J(z)\,,\,\eta\Big\rangle\,, \\ X_{J_\xi}J_\eta &= \Big\langle J(z)\,,\,-ad_\xi\eta\Big\rangle\,, \qquad (35) \\ \textit{Hence,}\quad \Big\{J_\eta(z)\,,\,J_\xi(z)\Big\} &= \Big\langle J(z)\,,\,\big[\eta\,,\,\xi\big]\Big\rangle \quad \Longleftarrow\,. \end{aligned}$$

- *Consequently,* ***infinitesimal equivariance implies***

$$\Big\{ \Big\langle J(p,q),\, \eta \Big\rangle\,,\, \Big\langle J(p,q),\, \xi \Big\rangle \Big\} = \Big\langle J(p,q),\, \big[\,\eta\,,\,\xi\,\big] \Big\rangle \quad \textit{or} \quad \big\{ J_\xi\,,\ J_\eta \big\} = J^{[\eta,\xi]}\,. \tag{36}$$

  This means that the map $(\mathfrak{g},\,[\,\cdot\,,\,\cdot\,] \to (C^\infty(T^*Q),\,\{\,\cdot\,,\,\cdot\,\})$ *defined by* $\xi \mapsto J_\xi$, $\xi \in \mathfrak{g}$ *is a* ***Lie algebra anti-homomorphism*** *(i.e., it preserves bracket relations up to an over-all sign).*

- *Infinitesimal equivariance implies that the momentum map*

$$J : T^*Q \mapsto \mathfrak{g}^* \quad \textit{is a } \textbf{\textit{Poisson map}}.$$

  That is, J *corresponding to left (resp., right) group action produces a* $+$ *(resp.,* $-$*) Lie–Poisson bracket on* $\mathfrak{g}^*$.

LECTURE #5

This lecture discusses examples of momentum maps that arise from the action of Lie transformation groups on various configuration spaces. It provides several worked examples of equivariant momentum maps, focusing mainly on cotangent lift momentum maps for the following Lie group actions:
(i) $SO(3)$ acting on $\mathbb{R}^3$;
(ii) $SU(2)$ acting on $\mathbb{C}^2$; and
(iii) $Sp(2, \mathbb{R})$ acting on $T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$; as well as
(iv) the general case for cotangent lift momentum maps in terms of the diamond ($\diamond$) operation.
Remark: Not all momentum maps arise from cotangent lifts. As an example, we discuss the symplectic momentum map.

Go2toc

8 Equivariant momentum maps for matrix Lie groups $SO(3)$, $SU(2)$ & $Sp(2,\mathbb{R})$

Definition 51 (Lie transformation groups).

- *A **transformation** is a one-to-one mapping of a set onto itself.*
- *A collection of transformations is called a **group**, provided*
  - *it includes the identity transformation and the inverse of each transformation;*
  - *it contains the result of the consecutive application of any two transformations; and*
  - *composition of that result with a third transformation is associative.*
- *A group is a **Lie group**, provided its transformations depend smoothly on a set of parameters.*

8.1 Momentum map for $SO(3)$ acting on $\mathbb{R}^3$

Let us briefly recall the detailed calculations in the previous lecture for $M = \mathbb{R}^3$ and $\mathfrak{g} = \mathfrak{so}(3)$. Those calculations showed that applying the Lie chain rule to the action $\mathbf{q}(t) = O^{-1}(t)\mathbf{q}(0)$ yields the infinitesimal transformation of $SO(3)$ acting on $\mathbb{R}^3$. This transformation takes the form $\xi_M(\mathbf{q}) = \widehat{\xi}\mathbf{q} = \pounds_\xi \mathbf{q} = \boldsymbol{\xi} \times \mathbf{q}$ with $\widehat{\xi} = O^{-1}O'|_{s=0} = -\widehat{\xi}^T$ after applying the hat-map isomorphism $\widehat{(\,\cdot\,)} : (\mathbb{R}^3, \times) \mapsto (\mathfrak{so}(3), [\,\cdot\,,\,\cdot\,]\,)$. In the previous lecture we also saw that

$$\Big\langle \mathbf{p},\, \pounds_\xi \mathbf{q} \Big\rangle_{TM} = \mathbf{p}\cdot\boldsymbol{\xi}\times\mathbf{q} = (\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\xi} = \Big\langle J(\mathbf{p},\mathbf{q}),\, \boldsymbol{\xi} \Big\rangle_{\mathfrak{so}(3)} = J_\xi(\mathbf{p},\mathbf{q})\,,$$

which is the Hamiltonian for an infinitesimal rotation around $\boldsymbol{\xi}$ in $\mathbb{R}^3$ generated by the Hamiltonian vector field $X_{J_\xi} = \{\,\cdot\,,\, J_\xi\,\}$.

For $\mathfrak{g} = \mathfrak{so}(3)$, the pairings $\langle\,\cdot\,,\,\cdot\,\rangle_{TM}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{so}(3)}$ may both be taken as dot products of vectors in $\mathbb{R}^3$. Consequently, the ***cotangent-lift*** momentum map $J(\mathbf{p},\mathbf{q}) = \mathbf{q}\diamond\mathbf{p} = \mathbf{q}\times\mathbf{p} \in \mathbb{R}^3$ coincides with the phase-space expression for angular momentum and the diamond $(\diamond)$ operation turns out to be $\times$, the cross product of vectors in $\mathbb{R}^3$.

8.2 Momentum map for $SU(2)$ acting on $\mathbb{C}^2$

The Lie group $SU(2)$ of complex 2×2 unitary matrices $U(s)$ with $UU^\dagger = Id$ and unit determinant acts on $\boldsymbol{a}\in\mathbb{C}^2$ by matrix multiplication as

$$\boldsymbol{a}(s) = U(s)\boldsymbol{a}(0) = \exp(is\xi)\boldsymbol{a}(0)\,,$$

in which $i\xi = U'U^{-1}|_{s=0}\in\mathfrak{su}(2)$ is a 2×2 ***traceless skew-Hermitian matrix***, as seen from the following:

$$UU^\dagger = Id \quad \text{implies} \quad U'U^\dagger + UU'^{\,\dagger} = 0 = U'U^\dagger + (U'U^\dagger)^\dagger\,.$$

Likewise, ξ alone (i.e., not multiplied by $i=\sqrt{-1}$) is a 2×2 ***traceless Hermitian matrix***.
The infinitesimal generator $\xi(\boldsymbol{a})\in\mathbb{C}^2$ may be expressed as a linear transformation,

$$\xi(\boldsymbol{a}) = \frac{d}{ds}\big[\exp(is\xi)\boldsymbol{a}\big]\Big|_{s=0} = i\xi\boldsymbol{a}\,,$$

in which the product $(\xi\boldsymbol{a})$ of the Hermitian matrix (ξ) and the two-component complex vector $(\boldsymbol{a})$ has components $\xi_{kl}a_l$, with $k,l=1,2$.
To be a cotangent lift momentum map, $J:\mathbb{C}^2\mapsto\mathfrak{su}(2)^*$ must satisfy the defining relation (34),

$$\begin{aligned}
J_\xi(\boldsymbol{a}) :=& \Big\langle J(\boldsymbol{a}),\,\xi\Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)} = \Big\langle\!\!\Big\langle \boldsymbol{a},\,\xi(\boldsymbol{a})\Big\rangle\!\!\Big\rangle_{\mathbb{C}^2} = \Big\langle\!\!\Big\langle \boldsymbol{a},\,i\xi\boldsymbol{a}\Big\rangle\!\!\Big\rangle_{\mathbb{C}^2}\\
=&\ \mathrm{Im}(a_k^*(i\xi)_{kl}a_l) = a_k^*\xi_{kl}a_l = \ \mathrm{tr}\big((\boldsymbol{a}\otimes\boldsymbol{a}^*)\xi\big) = \ \mathrm{tr}\big(Q^\dagger\xi\big)\,.
\end{aligned}$$

Being traceless, ξ has zero pairing with any multiple of the identity; so one may subtract the trace of $Q=\boldsymbol{a}\otimes\boldsymbol{a}^*$. Thus, the following traceless Hermitian quantity defines a momentum map $J:\mathbb{C}^2\mapsto\mathfrak{su}(2)^*$,

$$J(\boldsymbol{a}) = Q - \frac{1}{2}(\ \mathrm{tr}\ Q)\ \mathrm{Id} = \boldsymbol{a}\otimes\boldsymbol{a}^* - \frac{1}{2}\ \mathrm{Id}\ |\boldsymbol{a}|^2\in\mathfrak{su}(2)^*\,. \tag{37}$$

That is, J maps $\boldsymbol{a} \in \mathbb{C}^2$ to the traceless Hermitian matrix $J(\boldsymbol{a})$, which is an element of $\mathfrak{su}(2)^*$, the dual space to $\mathfrak{su}(2)$ under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{su}(2)^* \times \mathfrak{su}(2) \mapsto \mathbb{R}$ given by the trace of the matrix product,

$$\Big\langle J,\, \xi \Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)} = \ \mathrm{tr}\Big(J(\boldsymbol{a})^\dagger \xi\Big)\,, \tag{38}$$

$$\text{with}\quad J(\boldsymbol{a}) = J(\boldsymbol{a})^\dagger \in \mathfrak{su}(2)^* \quad\text{and}\quad i\xi = -(i\xi)^\dagger \in \mathfrak{su}(2)\,. \tag{39}$$

Proposition 52 (Momentum map equivariance)**.** *Let $U \in SU(2)$ and $\mathbf{a} \in \mathbb{C}^2$. The momentum map for $SU(2)$ acting on $\mathbb{C}^2$ defined by*

$$\Big\langle J(\boldsymbol{a}),\, \xi \Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)} = \Big\langle \boldsymbol{a},\, i\xi\boldsymbol{a} \Big\rangle_{\mathbb{C}^2} \tag{40}$$

is equivariant. That is,

$$U^*J(\mathbf{a}) := J(U\mathbf{a}) = Ad^*_{U^{-1}}J(\mathbf{a}).$$

Proof. Substitute $\mathrm{Ad}_{U^{-1}}\xi$ into the momentum map definition,

$$\begin{aligned}
\Big\langle \mathrm{Ad}^*_{U^{-1}}J(\mathbf{a})\,,\, \xi \Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)} &= \Big\langle J(\mathbf{a})\,,\, \mathrm{Ad}_{U^{-1}}\xi \Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)} \\
&= \Big\langle \boldsymbol{a},\, U^\dagger i\xi U\boldsymbol{a} \Big\rangle_{\mathbb{C}^2} \\
&= \Big\langle (U\mathbf{a})\,,\, i\xi(U\mathbf{a}) \Big\rangle_{\mathbb{C}^2} \\
&= \Big\langle J(U\mathbf{a})\,,\, \xi \Big\rangle_{\mathfrak{su}(2)^*\times\mathfrak{su}(2)}\,.
\end{aligned} \tag{41}$$

Therefore, $U^*J(\mathbf{a}) = J(U\mathbf{a}) = \mathrm{Ad}^*_{U^{-1}}J(\mathbf{a})$, as claimed. □

Remark 53 (Poincaré sphere momentum map)**.** *The momentum map $J(\boldsymbol{a}) : \mathbb{C}^2 \mapsto \mathfrak{su}(2)^*$ for the action of $SU(2)$ acting on $\mathbb{C}^2$ in equation (37) is a component of the map $\mathbb{C}^2 \mapsto S^2$ to the Poincaré sphere, which defines the Hopf* *fibration* $S^3 \simeq S^2 \times S^1$. *To see this, one replaces* $\xi \in \mathfrak{su}(2)$ *with the vector of Hermitian Pauli matrices* $\boldsymbol{\sigma}$ *whose components are given by*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textit{with} \quad \big[\, \sigma_i \,,\, \sigma_j \,\big] = \epsilon^i_{ij}\sigma_k \,. \tag{42}$$

This substitution yields the following formula for the momentum map

$$J(\mathbf{a}) = \frac{1}{2}\mathbf{n}\cdot\boldsymbol{\sigma} \quad \textit{in which} \quad \mathbf{n} = \ tr\,(Q\,\boldsymbol{\sigma}) = a^*_k \boldsymbol{\sigma}_{kl} a_l \,. \tag{43}$$

The quadratic S^1*-invariant quantities comprising the components of the vector* $\mathbf{n} = (n_1, n_2, n_3)$ *defined in* (43) *are given in terms of the* $\mathbb{C}^2$ *variables by*

$$n_1 + i\,n_2 = 2a_1 a_2^*, \;\; n_3 = |a_1|^2 - |a_2|^2 \,. \tag{44}$$

The sum of the squares of the three $SU(2)$*-invariant components of the momentum map* $J(\mathbf{a}) \in \mathfrak{su}(2)^*$ *yields another* $SU(2)$*-invariant quantity which comprises a sphere* $S^3 \subset \mathbb{C}^2 \simeq \mathbb{R}^4$,

$$n_1^2 + n_2^2 + n_3^2 - n_0^2 = 0 \quad \textit{with} \quad n_0^2 = (|a_1|^2 + |a_2|^2)^2 = \frac{1}{2} tr(\boldsymbol{a}\otimes\boldsymbol{a}^* = \ tr\,(Q \;\; Id)\,. \tag{45}$$

A level set of n_0^2 *comprises the Poincaré sphere* $S^2 \subset S^3$.

Exercise. Compute the Poisson brackets among the four quantities $(\mathbf{n}, n_0)$ and their finite transformations by using their definitions in terms of $\boldsymbol{a} \in \mathbb{C}^2$ and the canonical Poisson brackets $\{a^*_k, a_l\} = 2i\delta_{kl}$. ★

Answer. By a direct calculation, the canonical Poisson brackets $\{a^*_k, a_l\} = 2i\delta_{kl}$ imply

$$\{n_i,\, n_j\} = 4\epsilon_{ijk}n_k \,, \quad \{n_j,\, n_0\} = 0 \quad \text{and} \quad \{a_j,\, n_0\} = -2ia_j \,.$$

Thus, the finite transformations of $\mathbf{n}$ are $SO(3)$ rotations of the Poincaré sphere leaving n_0 invariant. The finite transformations of n_0 are S^1 phase shifts of the $\mathbb{C}^2$ variables leaving $\mathbf{n}$ invariant. Consequently, the reduced S^1-invariant $(\mathbf{n}, n_0)$ generate finite transformations that leave invariant the two components of the Hopf fibration $S^3 \simeq S^2 \times S^1$.

▲

8.3 Momentum map for $Sp(2,\mathbb{R})$ acting on $T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$

The Lie group $Sp(2,\mathbb{R})$ of symplectic real 2×2 matrices $M(s)$ acts diagonally on $\mathbf{z} = (\mathbf{q},\,\mathbf{p})^T \in T^*\mathbb{R}^2$ by matrix multiplication as

$$\mathbf{z}(s) = M(s)\mathbf{z}(0) = \exp(s\xi)\mathbf{z}(0)\,,$$

in which

$$M(s)\mathbb{J}M^T(s) = \mathcal{J}\,, \quad \text{with} \quad \mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a symplectic 2×2 matrix. The matrix tangent to the 2×2 symplectic matrix $M(s)$ at the identity $s=0$ is given by

$$\xi = \Big[M'(s)M^{-1}(s)\Big]_{s=0}\,.$$

This is a 2×2 matrix in the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$, satisfying

$$\mathbb{J}\xi + \xi^T\mathbb{J} = 0 \quad \text{so that} \quad \mathbb{J}\xi = (\mathbb{J}\xi)^T\,. \tag{46}$$

That is, for $\xi \in \mathfrak{sp}(2,\mathbb{R})$, the matrix $\mathbb{J}\xi$ is symmetric.

Exercise. Verify (46). What is the corresponding formula for $\zeta = [M^{-1}(s)M'(s)]_{s=0}$? ★

The vector field $\xi_M(\mathbf{z}) \in T\mathbb{R}^2$ may be expressed as a derivative,

$$\xi_M(\mathbf{z}) = \frac{d}{ds}\big[\exp(s\xi)\mathbf{z}\big]\Big|_{s=0} = \xi\mathbf{z}\,,$$

in which the diagonal action $(\xi\mathbf{z})$ of the Hamiltonian matrix (ξ) and the two-component real multi-vector $\mathbf{z} = (\mathbf{q},\,\mathbf{p})^T$ has components given by $(\xi_{kl}q_l, \xi_{kl}p_l)^T$, with $k,l = 1,2$. The matrix ξ is any linear combination of the following traceless constant Hamiltonian matrices, defined below.

Definition 54 (Hamiltonian matrices)**.** *The traceless constant **Hamiltonian matrices** are defined by*

$$m_1 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \; m_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \; m_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{47}$$

Their exponentiation defines the $Sp(2,\mathbb{R})$ symplectic matrices

$$e^{\tau_1 m_1} = M_1(\tau_1)\,, \quad e^{\tau_2 m_2} = M_2(\tau_2)\,, \quad e^{\tau_3 m_3} = M_3(\tau_3)\,. \tag{48}$$

and they are the right-invariant tangent vectors at their respective identity transformations,

$$m_1 = \Big[M\,'_1(\tau_1) M_1^{-1}(\tau_1) \Big]_{\tau_1=0}, \quad m_2 = \Big[M\,'_2(\tau_2) M_2^{-1}(\tau_2) \Big]_{\tau_2=0}, \quad m_3 = \Big[M\,'_3(\tau_3) M_3^{-1}(\tau_3) \Big]_{\tau_3=0}. \tag{49}$$

Exercise. Show that the Hamiltonian matrices in (47) satisfy the condition (46) required for elements of the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$. ★

Definition 55 (Momentum map $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \to sp(2,\mathbb{R})^*$)**.**
The momentum map $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \to sp(2,\mathbb{R})^*$ is defined in terms of $\mathbf{z} = (\mathbf{q},\,\mathbf{p})^T \in \mathbb{R}^2 \times \mathbb{R}^2$.by

$$\begin{aligned} \mathcal{J}^\xi(\mathbf{z}) &:= \Big\langle \mathcal{J}(\mathbf{z}),\, \xi \Big\rangle_{sp(2,\mathbb{R})^* \times sp(2,\mathbb{R})} = \Big(\mathbf{z},\, \mathbb{J}\xi\mathbf{z} \Big)_{\mathbb{R}^2 \times \mathbb{R}^2} \\ &:= z_k (\mathbb{J}\xi)_{kl} z_l = \mathbf{z}^T \cdot \mathbb{J}\xi\mathbf{z} = tr\Big((\mathbf{z} \otimes \mathbf{z}^T \mathbb{J})\xi \Big)\,. \end{aligned} \tag{50}$$

Remark 56. *The map $\mathcal{J}(\mathbf{z})$ given in (50) by*

$$\mathcal{J}(\mathbf{z}) = (\mathbf{z} \otimes \mathbf{z}^T \mathbb{J}) \in \mathfrak{sp}(2,\mathbb{R})^* \tag{51}$$

sends $\mathbf{z} = (\mathbf{q},\,\mathbf{p})^T \in \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathcal{J}(\mathbf{z}) = (\mathbf{z} \otimes \mathbf{z}^T \mathbb{J})$, which is an element of $\mathfrak{sp}(2,\mathbb{R})^*$, the dual space to $\mathfrak{sp}(2,\mathbb{R})$.

Under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{sp}(2,\mathbb{R})^*\times\mathfrak{sp}(2,\mathbb{R}) \to \mathbb{R}$ *given by the trace of the matrix product, one finds the Hamiltonian, or phase-space function,*

$$\mathcal{J}^\xi(\mathbf{z}) := \Big\langle \mathcal{J}(\mathbf{z}),\, \xi \Big\rangle = tr\big(\mathcal{J}(\mathbf{z})\,\xi\big)\,, \tag{52}$$

for $\mathcal{J}(\mathbf{z}) = (\mathbf{z}\otimes\mathbf{z}^T\mathbb{J}) \in \mathfrak{sp}(2,\mathbb{R})^*$ *and* $\xi\in\mathfrak{sp}(2,\mathbb{R})$.

Remark 57 (Momentum map to axisymmetric invariant variables)**.** *The map,* $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2\times\mathbb{R}^2 \to \mathfrak{sp}(2,\mathbb{R})^*$ *in (50) for* $Sp(2,\mathbb{R})$ *acting diagonally on* $\mathbb{R}^2\times\mathbb{R}^2$ *in Equation (51) may be expressed in matrix form as*

$$\begin{aligned} \mathcal{J} &= (\mathbf{z}\otimes\mathbf{z}^T\mathbb{J}) \\ &= 2\begin{pmatrix} \mathbf{p}\cdot\mathbf{q} & -|\mathbf{q}|^2 \\ |\mathbf{p}|^2 & -\mathbf{p}\cdot\mathbf{q} \end{pmatrix} \\ &= 2\begin{pmatrix} X_3 & -X_1 \\ X_2 & -X_3 \end{pmatrix}. \end{aligned} \tag{53}$$

Thus, the momentum map, $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2\times\mathbb{R}^2 \to \mathfrak{sp}(2,\mathbb{R})^* \simeq \mathbb{R}^3$ *in (50) produces axisymmetric invariant phase-space variables,*

$$T^*\mathbb{R}^2 \to \mathbb{R}^3 : (\mathbf{q},\mathbf{p})^T \to \mathbf{X} = (X_1, X_2, X_3)\,,$$

defined as

$$X_1 = |\mathbf{q}|^2 \ge 0\,,\quad X_2 = |\mathbf{p}|^2 \ge 0\,,\quad X_3 = \mathbf{p}\cdot\mathbf{q}\,. \tag{54}$$

Applying the momentum map $\mathcal{J}$ *to the vector of Hamiltonian matrices* $\mathbf{m} = (m_1,\, m_2,\, m_3)$ *in Equation (47) yields the individual components,*

$$\mathcal{J}\cdot\mathbf{m} = 2\mathbf{X} \quad\Longleftrightarrow\quad \mathbf{X} = \frac{1}{2}\,z_k(\mathbb{J}\mathbf{m})_{kl}z_l\,. \tag{55}$$

Thus, the map $\mathcal{J} : T^*\mathbb{R}^2 \simeq \mathbb{R}^2\times\mathbb{R}^2 \to \mathfrak{sp}(2,\mathbb{R})^*$ *recovers the components of the vector* $\mathbf{X} = (X_1,\, X_2,\, X_3)$.

Exercise. Verify Equation (55) explicitly. ★

The determinant of the momentum map expressed in matrix form in (53) yields another axisymmetric invariant,

$$S^2 = X_1X_2 - X_3^2 = |\mathbf{q}|^2|\mathbf{p}|^2 - (\mathbf{p}\cdot\mathbf{q})^2 = |\mathbf{p}\times\mathbf{q}|^2\,.$$

Level sets of the axisymmetric invariant $S^2 = X_1X_2 - X_3^2$ are hyperboloids of revolution around the $X_1 = X_2$ axis (labelled Y_1 in figure 23) in the horizontal plane, $X_3 = 0$. In the coordinates $Y_1 = (X_1{+}X_2)/2\,, Y_2 = (X_2{-}X_1)/2\,, Y_3 = X_3$, one sees that the level sets of $S^2 = Y_1^2 - Y_2^2 - Y_3^2$ comprise a ***hyperbolic onion***.

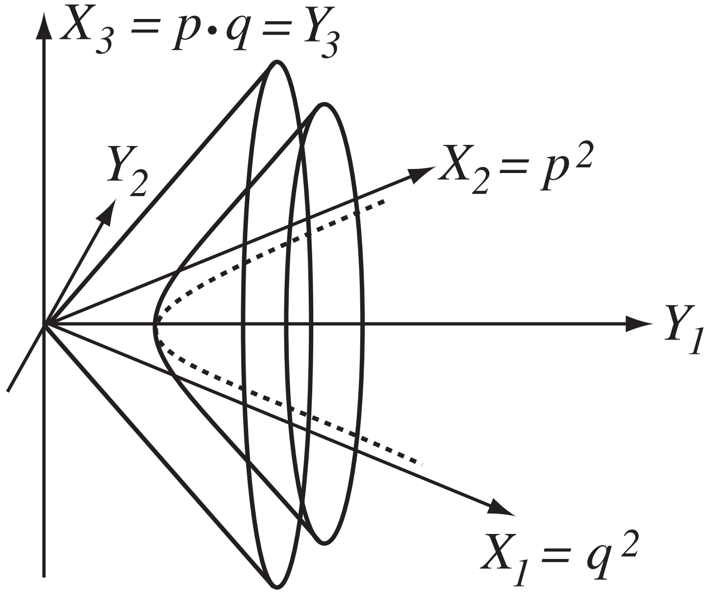


Figure 23: The axisymmetric invariants $\mathbf{X}\in\mathbb{R}^3$ evolve along the intersections of these level sets by $\dot{\mathbf{X}} = \nabla S^2 \times \nabla H$, as the level sets of the Hamiltonian knife $H = \mathit{const}$ slices through the hyperbolic onion of level sets of S^2. Being invariant under the flow of the Hamiltonian vector field $X_{S^2} = \{\,\cdot\,,\,S^2\}$, each point on any layer of the hyperbolic onion comprises an S^1 orbit in phase space, in hyperbolic analogy with the Hopf fibration.

9 Lie–Poisson brackets

Definition 58 (Lie Poisson bracket)**.** *A **Lie–Poisson bracket** is a bracket operation defined as a linear functional of a Lie algebra bracket by a real-valued pairing between a Lie algebra and its dual space.*

Remark 59 (Cotangent-lift momentum maps)**.** *The formula* (34) *determining the momentum map for the cotangent lift action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairings*

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R} \quad \textit{and} \quad \langle\,\cdot\,,\,\cdot\,\rangle_{TM} : T^*M \times TM \mapsto \mathbb{R}\,,$$

as

$$\Big\langle\, J(p,q)\,,\, \xi \,\Big\rangle_{\mathfrak{g}} = \Big\langle q \diamond p,\, \xi \Big\rangle_{\mathfrak{g}} = \Big\langle\, p\,,\, \pounds_\xi q \,\Big\rangle_{TM}, \tag{56}$$

where $(q,p) \in T^*_qM$ and $\pounds_\xi q \in T_qM$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Remark 60 (Symplectic momentum maps)**.** *Not all momentum maps arise as cotangent lifts. Momentum maps may also arise from the infinitesimal action of the Lie algebra on the phase-space manifold $\xi_{T^*M}(\mathbf{z})$ with $\mathbf{z} = (\mathbf{q},\, \mathbf{p})$ by using the pairing with the symplectic form. The formula for the momentum map is then*

$$\mathcal{J}^\xi(\mathbf{z}) = \Big\langle \mathcal{J}(\mathbf{z}),\, \xi \Big\rangle = \Big(\mathbf{z},\, \mathbb{J}\xi_{T^*M}(\mathbf{z})\Big)\,, \tag{57}$$

where $\mathbb{J}$ is the symplectic form and $(\,\cdot\,,\,\cdot\,)$ is the inner product on phase space $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ for n degrees of freedom. The transformation to axisymmetric variables in (53) is an example of a momentum map obtained from the symplectic pairing.

Proposition 61 (Equivariance of cotangent lift momentum maps)**.** *Cotangent lift momentum maps (56) are equivariant. That is,*

$$g^* J(p,q) = J(g\cdot p, g\cdot q) = Ad^*_{g^{-1}} J(p,q)\,, \tag{58}$$

where $(g\cdot p, g\cdot q)$ *denotes the cotangent lift to* T^*M *of the action of* G *on manifold* M*.*

Proof. The proof follows because $\mathrm{Ad}^*_{g^{-1}}$ is a representation of the coAdjoint action Φ^*_g of the group G on its dual Lie algebra $\mathfrak{g}^*$. This means that $\mathrm{Ad}^*_{g^{-1}}(q\diamond p) = (g\cdot q \diamond g\cdot p)$, and we have

$$\begin{aligned}\Big\langle \mathrm{Ad}^*_{g^{-1}} J(p,q)\,,\,\xi\Big\rangle_{\mathfrak{g}^*\times\mathfrak{g}} &= \Big\langle\, \mathrm{Ad}^*_{g^{-1}}(q\diamond p)\,,\,\xi\,\Big\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \\ &= \Big\langle\, g\cdot q\diamond g\cdot p\,,\,\xi\,\Big\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \\ &= \Big\langle J(g\cdot p, g\cdot q)\,,\,\xi\Big\rangle_{\mathfrak{g}^*\times\mathfrak{g}}\,.\end{aligned} \tag{59}$$

Thus, Equation (58) holds and cotangent-lift momentum maps are equivariant. □

$$\begin{tikzcd} P \arrow[r, "\Phi_{g(t)}"] \arrow[d, "J"'] & P \arrow[d, "J"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g(t)^{-1}}"] & \mathfrak{g}^* \end{tikzcd}$$

Importance of equivariance. Equivariance of a momentum map is important, because Poisson brackets among the components of an equivariant momentum map close among themselves and satisfy the Jacobi identity. That is, the following theorem holds.

Theorem 62. *Equivariant momentum maps are Poisson.*

Proof. As we know, a momentum map $J:\, P \to \mathfrak{g}^*$ is equivariant, if

$$J \circ \Phi_{g(t)} = \mathrm{Ad}^*_{g(t)^{-1}} \circ J\,,$$

for any curve $g(t) \in G$. As discussed earlier, the time derivative of the equivariance relation leads to the infinitesimal equivariance relation,

$$\{\langle J\,,\, \xi\rangle\,,\, \langle J\,,\, \eta\rangle\} = \big\langle J\,,\, [\xi,\, \eta]\big\rangle\,, \tag{60}$$

where ξ, $\eta \in \mathfrak{g}$ and $\{\,\cdot\,,\,\cdot\,\}$ denotes the Poisson bracket on the manifold P. This in turn implies that the momentum map preserves Poisson brackets in the sense that

$$\{F_1 \circ J\,,\, F_2 \circ J\} = \{F_1\,,\, F_2\}_{LP} \circ J\,, \tag{61}$$

for all $F_1, F_2 \in \mathcal{F}(\mathfrak{g}^*)$, where $\{F_1\,,\, F_2\}_{LP}$ denotes the Lie–Poisson bracket for the appropriate (left or right) action of $\mathfrak{g}$ on P. That is, equivariance implies infinitesimal equivariance, which is sufficient for the momentum map to be Poisson (in finite dimensions). □

After explaining why Hamiltonian reduction by Lie group symmetry succeeds (equivariance of the momentum map), next we will discuss Lagrangian reduction by Lie group symmetry.

Of course, Noether's theorem appears prominently in both approaches and the results on the two sides of reduction by Lie group symmetry will agree.

LECTURE #6

Reduction by symmetry on the Lagrangian side

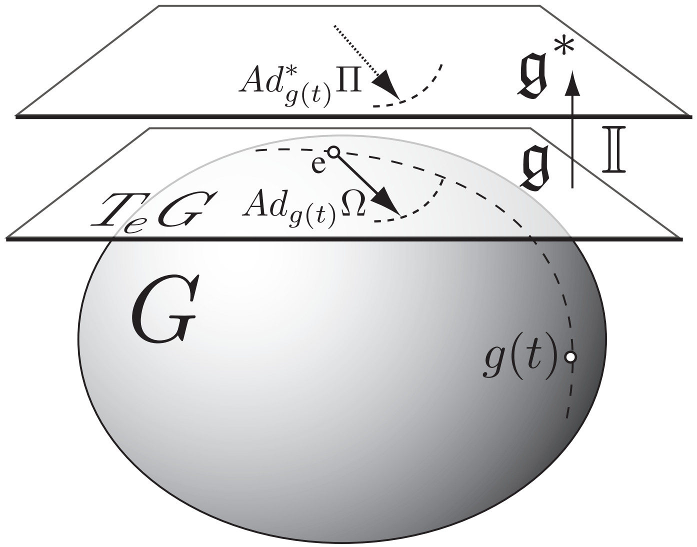


Figure 24: **The Ad and Ad* operations of Lie group $g(t)$ act, respectively, on the Lie algebra Ad : $G\times\mathfrak{g}\to\mathfrak{g}$ and on its dual Ad* : $G\times\mathfrak{g}^*\to\mathfrak{g}^*$. The tangents at the identity to the operations Ad and Ad* are ad : $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ and ad* : $\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{g}^*$.**

Go2toc

9.1 Reduction by symmetry on the Lagrangian side

GMF

1. We may lift the dynamics for the curve $q(t) \in M$ on a manifold M to a curve on a Lie group G by setting $q(t) = g(t)q(0) \in M$ for a curve $g(t) \in G$. For geometric mechanics on a Lie group G, reduce $L : TG \to \mathbb{R}$ and $H : T^*G \to \mathbb{R}$ by the action of the Lie group G on its tangent space TG and its cotangent space, T^*G, via pullbacks $TG \to T_eG \simeq \mathfrak{g}$ and $T^*G \to T_e^*G \simeq \mathfrak{g}^*$.

2. Apply Hamilton's principle on the Lagrangian side after lifting $q(t) = g(t)q(0) \in M$ for a curve $g(t) \in G$ with tangent $\dot{g}(t) = v_g \in T_gG$. Obtain
(i) the EL equation from the Lagrangian $L(g, v) : TG \to \mathbb{R}$ in the constrained Hamilton's principle
$$0 = \delta S(g, v) = \int_a^b L(g, v_g) + \langle p\,,\, \dot{g} - v_g\rangle \, dt$$
which implies as before, but now with $q(t) = g(t)q(0) \in M$
$$\delta p : \frac{dg}{dt} - v_g = 0\,, \quad \delta v_g : \frac{\partial L}{\partial v_g} - p = 0\,, \quad \delta g : \frac{dp}{dt} - \frac{\partial L}{\partial g} = 0$$
and
(ii) the reduced Euler–Poincaré (EP) dynamics on $\mathfrak{g}^*$, the dual of the Lie algebra, obtained when the Lagrangian is invariant under the action of any $k \in G$ so that
$$L(g, v_g) = L(kg, kv_g)$$
In this case, we choose $L(e, g^{-1}v_g) = \ell(\xi)$, where $\xi := g^{-1}v_g \in T_eG \simeq \mathfrak{g}$. Then we rewrite the previous constrained Hamilton's principle as the Hamilton–Pontryagin principle, given by
$$0 = \delta S(\xi, \mu, g) = \int_a^b \ell(\xi) + \langle \mu\,,\, g^{-1}\dot{g} - \xi\rangle \, dt$$

The stationarity conditions now are obtained from a side calculation which yields, with $\eta := g^{-1}\delta g$,

$$\delta(g^{-1}\dot{g}) = \frac{d\eta}{dt} + (\xi\eta - \eta\xi) := \frac{d\eta}{dt} + [\xi, \eta] =: \frac{d\eta}{dt} + \mathrm{ad}_\xi\, \eta$$

in which we use $\delta(g^{-1}) = -\, g^{-1}(\delta g)g^{-1}$ and $\delta(\dot{g}) = (\delta g)^{\dot{}}$.

By defining the *v*ariation operation $\delta q := dq/d\epsilon|_{\epsilon=0}$ and invoking the Hamilton–Pontryagin principle one finds:

$$0 = \delta S = \int_a^b \big\langle \delta\mu, g^{-1}\dot{g} - \xi \big\rangle + \left\langle \frac{\partial \ell}{\partial \xi} - \mu \,,\, \delta\xi \right\rangle + \left\langle \mu \,,\, \frac{d\eta}{dt} + \mathrm{ad}_\xi\, \eta \right\rangle \, dt + \langle \mu, \eta \rangle \Big|_a^b$$

$$\delta\mu : g^{-1}\dot{g} - \xi = 0\,, \qquad \delta\xi : \frac{\partial \ell}{\partial \xi} - \mu = 0\,, \qquad \delta g : \frac{d\mu}{dt} - \mathrm{ad}^*_\xi\, \mu = 0$$

where the operation ad^* : $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the operation ad : $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with respect to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, according to

$$\langle \mathrm{ad}^*_\xi\, \mu \,,\, \eta \rangle = \langle \mu \,,\, \mathrm{ad}_\xi\, \eta \rangle\,.$$

3. The three stationarity conditions obtained from the Hamilton–Pontryagin principle imply the ***Euler–Poincaré equation***.

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\, \frac{\partial \ell}{\partial \xi} = 0$$

GMF

4. Define the ***Hamiltonian*** $h(\mu) : \mathfrak{g}^* \to \mathbb{R}$ via the ***reduced Legendre transformation*** $LT : \mathfrak{g} \to \mathfrak{g}^*$.

$$h(\mu) = \langle \mu, \xi \rangle - \ell(\xi)$$

with differential

$$\begin{aligned} dh(\mu) &= \left\langle \frac{\partial h}{\partial \mu}, d\mu \right\rangle \\ &= \langle \xi, d\mu \rangle + \left\langle \mu - \frac{\partial \ell}{\partial \xi}, d\xi \right\rangle \end{aligned}$$

Thus, we find

$$\frac{\partial h}{\partial \mu} = \xi \quad \text{and} \quad \frac{\partial \ell}{\partial \xi} = \mu \,.$$

5. Introduce ***Hamilton's principle on*** $\mathfrak{g}^*$ ($\mu \in \mathfrak{g}^*$)

   Hamiltonian $h(\mu) : \mathfrak{g}^* \to \mathbb{R}$

   Phase Space Action $S := \int_a^b \langle \mu, g^{-1}\dot{g} \rangle - h(\mu)\, dt$

   Hamilton's Principle: $0 = \delta S = \int_a^b \left\langle \delta\mu, g^{-1}\dot{g} - \frac{\partial h}{\partial \mu} \right\rangle - \left\langle \frac{d\mu}{dt} - \mathrm{ad}^*_{g^{-1}\dot{g}}\,\mu \,,\, g^{-1}\delta g \right\rangle dt + \langle \mu, g^{-1}\delta g \rangle \Big|_a^b$

   Lie–Poisson Equations: $\delta\mu : g^{-1}\dot{g} - \frac{\partial h}{\partial \mu} = 0 \,, \quad g^{-1}\delta g : \frac{d\mu}{dt} - \mathrm{ad}^*_{g^{-1}\dot{g}}\,\mu = 0$

9.2 Rigid body – Clebsch Hamilton's principle

Review of the Clebsch Hamilton's principle for the Euler-Lagrange equations

First, before deriving the Lagrangian and Hamiltonian formulations of rigid body dynamics, let's recall our earlier derivation of the Euler-Lagrange equations from the constrained Hamilton's principle, in which we varied coordinates $(q, v) \in TQ$, subject to the constraint $v = \frac{dq}{dt}$ (tangent lift). In this case, the constrained action integral was varied according to

$$\delta S = \delta \int_a^b L(q, v) + \Big\langle p, \frac{dq}{dt} - v \Big\rangle \, dt \; = \int_a^b \Big\langle \frac{\partial L}{\partial v} - p, \delta v \Big\rangle + \Big\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}, \delta q \Big\rangle + \Big\langle \delta p, \dot{q} - v \Big\rangle dt + \Big\langle p, \delta q \Big\rangle \Big|_a^b .$$

Then we assembled the EL equation

$$\frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} = \frac{\partial L(q, \dot{q})}{\partial q}$$

from the various stationary conditions, and evaluated $\frac{\partial L}{\partial v}\Big|_{v=\dot{q}} = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$.

In our next paragraph, we are going to do the same sort of variational calculation when $Q \in SO(3)$ and derive the equations for a rigidly rotating body described by a curve in $SO(3)$, for the case $G \times M \to M$ when both $G = SO(3)$ and $M = SO(3)$. That is, $SO(3) \times SO(3) \to SO(3)$, with flow $Q(t + s) = Q(t)Q(s)$ and $Q(t - t) = Q(0) = Id$, obtained from the rotation group $SO(3)$ acting on itself.

A sketch of the computation using the hat map isomorphism $\widehat{\xi} \in so(3) \to \xi \in \mathbb{R}^3$ follows, as preparation for the Lie algebra operations in our next paragraph. Here, the constrained action integral is varied according to

$$0 = \delta S = \delta \int_a^b \ell(\Omega) + \Big\langle p, \frac{dq}{dt} - \Omega \times q \Big\rangle \, dt \;\Longrightarrow\; \frac{\partial \ell}{\partial \Omega} = q \times p \,, \quad \frac{dq}{dt} = \Omega \times q \,, \quad \frac{dp}{dt} = \Omega \times p \,, \quad \Big\langle p, \delta q \Big\rangle \Big|_a^b = 0.$$

$$\frac{d}{dt} \frac{\partial \ell}{\partial \Omega} = -\Omega \times (q \times p) = -\Omega \times \frac{\partial \ell(\Omega)}{\partial \Omega} \;\Longrightarrow\; \frac{d\Pi}{dt} = -\Omega \times \Pi \quad \text{for} \quad \Pi = \frac{\partial \ell}{\partial \Omega} \;\Longrightarrow\; \frac{d|\Pi|^2}{dt} = 0 \,.$$

Theorem 63 (Clebsch form of Hamilton's principle for the rigid body)**.**
For $Q \in SO(3)$, the Euler-Lagrange equations become Euler-Poincaré rigid-body equations in matrix commutator form,

$$\frac{d}{dt}\frac{\partial l}{\partial \widehat{\Omega}} = -\left[\widehat{\Omega},\, \frac{\partial l}{\partial \widehat{\Omega}}\right] \quad \textit{or, for} \quad \widehat{\Pi} := \frac{\partial l}{\partial \widehat{\Omega}}\,, \quad \textit{equivalently} \quad \frac{d\widehat{\Pi}}{dt} = -\widehat{\Omega}\widehat{\Pi} + \widehat{\Pi}\widehat{\Omega} = -\big[\widehat{\Omega},\, \widehat{\Pi}\big], \tag{62}$$

with (body, left-invariant) angular velocity $\widehat{\Omega} = Q^{-1}\dot{Q} = -\widehat{\Omega}^T \in \mathfrak{so}(3) = T_eSO(3)$ and body angular momentum $\widehat{\Pi} := \partial l/\partial\widehat{\Omega}$. The commutator equation (62) emerges from the constrained Hamilton's principle, $\delta S = 0$ with constrained action integral

$$S(\widehat{\Omega}, Q, P) = \int_a^b l(\widehat{\Omega}) + \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle\, dt = \int_a^b l(\widehat{\Omega}) + tr\Big(P^T\big(\dot{Q} - Q\widehat{\Omega}\big)\Big)\, dt = \int_a^b l(\widehat{\Omega}) + tr\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\, dt\,, \tag{63}$$

for $(Q,P) \in T^*SO(3)$. Stationarity ($\delta S = 0$) leads to the following variational conditions

$$\widehat{\Pi} = \frac{\delta l}{\delta \widehat{\Omega}} = \frac{1}{2}(P^TQ - Q^TP) \in \mathfrak{so}(3)^*\,, \quad \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle := tr\Big(P^T\,\big(\dot{Q} - Q\widehat{\Omega}\big)\Big) = tr\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\,,$$

and the quantities $(Q,P) \in T^*SO(3)$ satisfy the following symmetric equations,

$$\dot{Q} = Q\widehat{\Omega} \quad \textit{and} \quad \dot{P} = P\widehat{\Omega}\,, \tag{64}$$

as a result of the constraints. These equations have Lie-Poisson Hamiltonian form,

$$\frac{dF}{dt} = \Big\{F\,,\, H\Big\} = -\,\left\langle \Pi\,,\, \left[\frac{\partial F}{\partial \Pi}\,,\, \frac{\partial H}{\partial \Pi}\right]\right\rangle. \tag{65}$$

Proof. The variations of the constrained action S in (63) are given by

$$
\begin{aligned}
\delta S &= \int_a^b \Big\langle \frac{\delta l}{\delta \widehat{\Omega}} \,,\, \delta\widehat{\Omega} \Big\rangle - \Big\langle P \,,\, Q\,\delta\widehat{\Omega} \Big\rangle + \Big\langle \delta P \,,\, \dot{Q} - Q\widehat{\Omega} \Big\rangle + \Big\langle P \,,\, \delta\dot{Q} - (\delta Q)\,\widehat{\Omega} \Big\rangle \, dt \\
&= \int_a^b \Big\{ \mathrm{tr}\Big[\Big(\widehat{\Pi}^T - \frac{1}{2}(P^TQ - Q^TP) \Big) \delta\widehat{\Omega} \Big] \\
&\qquad + \mathrm{tr}\Big[\delta P^T \Big(\dot{Q} - Q\widehat{\Omega} \Big) \Big] - \mathrm{tr}\Big[\Big(\dot{P}^T + \widehat{\Omega} P^T \Big) \delta Q \Big] \Big\} dt + \Big\langle P \,,\, \delta Q \Big\rangle \Big|_a^b .
\end{aligned}
$$

Thus, stationarity of this ***implicit variational principle*** implies the following set of equations

$$
\widehat{\Pi} = \frac{\delta l}{\delta \widehat{\Omega}} = \frac{1}{2}(P^TQ - Q^TP)\,, \quad \dot{Q} = Q\widehat{\Omega} \quad \text{and} \quad \dot{P} = P\widehat{\Omega}\,. \tag{66}
$$

The commutator form of the rigid-body equations in (62) emerges from these, upon elimination of Q and P, as

$$
\begin{aligned}
\frac{d\widehat{\Pi}}{dt} &= \frac{1}{2}(\dot{P}^TQ + P^T\dot{Q} - \dot{Q}^TP - Q^T\dot{P}) \\
&= \frac{1}{2}\widehat{\Omega}(Q^TP - P^TQ) - \frac{1}{2}(P^TQ - Q^TP)\widehat{\Omega} \\
&= -\,\widehat{\Omega}\,\widehat{\Pi} + \widehat{\Pi}\,\widehat{\Omega} = -\,\big[\widehat{\Omega},\,\widehat{\Pi}\big]\,.
\end{aligned}
$$

These are Euler's equations for the rigid body on $T^*SO(3) \simeq so(3)^*$.

We Legendre transform the Lagrangian $l(\widehat{\Omega})$ to the Hamiltonian $H(\widehat{\Pi})$, as

$$H(\widehat{\Pi}) = \Big\langle \widehat{\Pi}\,,\,\widehat{\Omega} \Big\rangle - l(\widehat{\Omega}) \quad \text{with} \quad dH = \Big\langle d\widehat{\Pi}\,,\,\widehat{\Omega} \Big\rangle + \Big\langle \widehat{\Pi} - \frac{\partial l}{\partial \widehat{\Omega}}\,,\, d\widehat{\Omega} \Big\rangle .$$

Then, by using $\widehat{\Omega} = \partial H/\partial \widehat{\Pi}$ and $\widehat{\Omega}^T = -\widehat{\Omega}$ we find the following *Lie-Poisson bracket* for the Hamiltonian formulation of the rigid body dynamics,

$$\begin{aligned} \frac{dF}{dt} = \Big\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{d\widehat{\Pi}}{dt} \Big\rangle &= \Big\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\,\Big[\widehat{\Pi}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle \\ &= \mathrm{tr}\left(\frac{\partial F}{\partial \widehat{\Pi}} \Big[\widehat{\Pi}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big]^T \right) = -\,\mathrm{tr}\left(\widehat{\Pi}^T \Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \right) \\ &= -\Big\langle \widehat{\Pi}\,,\,\Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle =: \Big\{ F\,,\,H \Big\}. \end{aligned}$$

In the ad-ad* notation, with $[\xi, \eta] =: \mathrm{ad}_\xi \eta$ for $\xi, \eta \in \mathfrak{g}$, where in this case $\mathfrak{g} = \mathfrak{so}(3)$, the Lie-Poisson bracket is written as

$$\frac{dF}{dt} = -\Big\langle \widehat{\Pi}\,,\,\Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle =: \Big\langle \widehat{\Pi}\,,\,\mathrm{ad}_{\frac{\partial H}{\partial \widehat{\Pi}}} \frac{\partial F}{\partial \widehat{\Pi}} \Big\rangle =: \Big\langle \mathrm{ad}^*_{\frac{\partial H}{\partial \widehat{\Pi}}} \widehat{\Pi}\,,\,\frac{\partial F}{\partial \widehat{\Pi}} \Big\rangle .$$

From this Lie Poisson equation, one verifies that

$$\frac{d\widehat{\Pi}}{dt} = \mathrm{ad}^*_{\frac{\partial H}{\partial \widehat{\Pi}}} \widehat{\Pi} = -\Big[\frac{\partial H}{\partial \widehat{\Pi}}\,,\,\widehat{\Pi}\Big] .$$

$\square$

9.3 Hamilton-Pontryagin principle for the Rigid Body equations

The Hamilton-Pontryagin constrained variation principle is more direct than the Clebsch variational principle, although it does not reveal the momentum map associated with the reduction by symmetry of the Lagrangian.

Consider the following constrained left-invariant action integral,

$$
\begin{aligned}
0 = \delta S(\widehat{\Omega}, O, \dot{O}) &= \delta \int_a^b l(\widehat{\Omega}) + \langle \, \widehat{\Pi} \, , \, O^{-1}\dot{O} - \widehat{\Omega} \, \rangle \, dt \\
&= \int_a^b \Big\langle \, \frac{\delta l}{\delta \widehat{\Omega}} - \widehat{\Pi} \, , \, \delta\widehat{\Omega} \Big\rangle + \langle \, \delta\widehat{\Pi} \, , \, O^{-1}\dot{O} - \widehat{\Omega} \, \rangle + \langle \, \widehat{\Pi} \, , \, \delta(O^{-1}\dot{O}) \, \rangle \, dt \, ,
\end{aligned}
$$

with Frobenius pairing of skew symmetric matrices $\langle \, \widehat{\Pi} \, , \, \widehat{\Omega} \rangle = \mathrm{t}r(\widehat{\Pi}^T\widehat{\Omega})$.

Denote $\delta(\,\cdot\,) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}(\,\cdot\,) = (\,\cdot\,)'$ as well as $\widehat{\Omega} = O^{-1}\dot{O}$ and $\widehat{\Xi} = O^{-1}O'$, then compute that

$$
\begin{aligned}
(O^{-1}\dot{O})' = \widehat{\Omega}' &= -(O^{-1}O')(O^{-1}\dot{O}) + \dot{O}' = -\widehat{\Xi}\widehat{\Omega} + \dot{O}' \\
(O^{-1}O')^{\cdot} = \widehat{\Xi}^{\,\cdot} &= -(O^{-1}\dot{O})(O^{-1}O') + O'^{\,\cdot} = -\widehat{\Omega}\widehat{\Xi} + \dot{O}' \, .
\end{aligned}
$$

Subtracting these two equations yields

$$
\widehat{\Omega}' = \delta\widehat{\Omega} = \widehat{\Xi}^{\,\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega} =: \widehat{\Xi}^{\,\cdot} + \Big[\widehat{\Omega} \, , \, \widehat{\Xi}\Big] =: \widehat{\Xi}^{\,\cdot} + \mathrm{a}d_{\widehat{\Omega}} \, \widehat{\Xi} \, .
$$

Substitution then yields

$$
\int_a^b \langle \, \widehat{\Pi} \, , \, \delta(O^{-1}\dot{O}) \, \rangle \, dt = \int_a^b \langle \, \widehat{\Pi} \, , \, \frac{d\widehat{\Xi}}{dt} + \mathrm{a}d_{\widehat{\Omega}} \, \widehat{\Xi} \, \rangle \, dt + \langle \, \widehat{\Pi} \, , \, \widehat{\Xi} \, \rangle\Big|_a^b = \int_a^b \Big\langle \, - \frac{d\widehat{\Pi}}{dt} + \mathrm{a}d^*_{\widehat{\Omega}}\widehat{\Pi} \, , \, \widehat{\Xi} \, \Big\rangle \, dt + \langle \, \widehat{\Pi} \, , \, \widehat{\Xi} \, \rangle\Big|_a^b \, ,
$$

where $\mathrm{a}d_{\widehat{\Omega}}\widehat{\Xi} = [\widehat{\Omega} \, , \, \widehat{\Xi}]$ and $\mathrm{a}d^*_{\widehat{\Omega}}\widehat{\Pi} = -[\widehat{\Omega} \, , \, \widehat{\Pi}]$ via the Frobenius pairing $\langle \, \widehat{\Pi}, \mathrm{a}d_{\widehat{\Omega}} \, \widehat{\Xi} \rangle = \langle \, \mathrm{a}d^*_{\widehat{\Omega}}\widehat{\Pi}, \widehat{\Xi} \, \rangle$.

Consequently, we recover Euler's rigid body equation, $\frac{d\widehat{\Pi}}{dt} = -[\widehat{\Omega} \, , \, \widehat{\Pi}]$.

9.4 Hamilton-Pontryagin principle for the Euler–Poincaré equations

Theorem 64 (Hamilton–Pontryagin principle for the Euler–Poincaré equations)**.**
The Euler–Poincaré equation

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad^*_\xi \frac{\delta l}{\delta \xi} \tag{67}$$

on the dual Lie algebra $\mathfrak{g}^*$ is equivalent to the following variational principle,

$$\delta S(\xi, g, \dot g) = \delta \int_a^b l(\xi, g, \dot g)\, dt = 0, \tag{68}$$

for a constrained left-invariant action integral

$$S(\xi, g, \dot g) = \int_a^b l(\xi, g, \dot g)\, dt = \int_a^b \Big[l(\xi) + \langle\, \mu\,,\, (g^{-1}\dot g - \xi)\,\rangle \Big]\, dt\,.$$

Proof. The variations of S in formula (68) are given by

$$\delta S = \int_a^b \Big\langle\, \frac{\delta l}{\delta \xi} - \mu\,,\, \delta\xi\, \Big\rangle + \Big\langle\, \delta\mu\,,\, (g^{-1}\dot g - \xi)\, \Big\rangle + \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g)\, \Big\rangle\, dt\,.$$

Substituting $\delta(g^{-1}\dot g) = \dot\eta + \mathrm{ad}_\xi\, \eta$ obtained from $\delta(\dot g) = (\delta g)^{\cdot}$ with $\eta := g^{-1}\delta g$ into the last term produces

$$\begin{aligned}
\int_a^b \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g)\, \Big\rangle\, dt &= \int_a^b \Big\langle\, \mu\,,\, \dot\eta + \mathrm{ad}_\xi\, \eta\, \Big\rangle\, dt \\
&= \int_a^b \Big\langle\, -\dot\mu + \mathrm{ad}^*_\xi\, \mu\,,\, \eta\, \Big\rangle\, dt + \Big\langle\, \mu\,,\, \eta\, \Big\rangle\Big|_a^b\,,
\end{aligned}$$

where $\eta = g^{-1}\delta g$ vanishes at the endpoints in time. Thus, stationarity $\delta S = 0$ of the Hamilton–Pontryagin variational principle yields the following set of equations:

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad g^{-1}\dot g = \xi\,, \quad \dot\mu = \mathrm{ad}^*_\xi\,\mu\,. \tag{69}$$

$\square$

Legendre transformation. After the Legendre transformation to

$$h(\mu) = \langle\, \mu\,,\, \xi\,\rangle - \ell(\xi) \tag{70}$$

we have the differential relations

$$dh = \left\langle\, \frac{\partial h}{\partial \mu}\,,\, d\mu \,\right\rangle = \langle\, d\mu\,,\, \xi\,\rangle + \left\langle\, \mu - \frac{\partial l}{\partial \xi}\,,\, \delta\xi \,\right\rangle \tag{71}$$

so that $\partial h/\partial\mu = \xi$, which leads to the Hamiltonian formulation of the Hamilton–Pontryagin equations (72)

$$\dot\mu = \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,, \quad \frac{dF}{dt} = \left\langle\, \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,,\, \frac{\partial F}{\partial \mu} \,\right\rangle = \left\langle\, \mu\,,\, \mathrm{ad}_{\partial h/\partial\mu}\,\frac{\partial F}{\partial \mu} \,\right\rangle = -\left\langle\, \mu\,,\, \left[\frac{\partial F}{\partial \mu}\,,\, \frac{\partial H}{\partial \mu}\right] \,\right\rangle =: \Big\{\, F\,,\, H \Big\}\,. \tag{72}$$

Exercise. Recalculate the Hamilton-Pontryagin variational principle and derive its associated Lie-Poisson bracket for a constrained *right-invariant* action integral

$$S(\xi, g, \dot g) = \int_a^b \Big[l(\xi) + \langle\, \mu\,,\, (\dot g g^{-1} - \xi)\,\rangle \Big]\, dt\,.$$

★

Answer.

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad \dot{g}g^{-1} = \xi\,, \quad \dot{\mu} = -\,\mathrm{ad}^*_\xi\,\mu\,. \quad \text{Note the minus sign for right-invariance, cf. (72).}$$

We will need the right-invariant Euler-Poincaré and Lie-Poisson equations when we study fluid dynamics. ▲

Exercise. Suppose left-invariance of the previous action principle is broken by the presence in the Lagrangian of a parameter a_0 which transforms under the group g as $a_t = g_t^{-1}a_0$, whose definition implies that it satisfies the auxiliary equation,

$$\frac{da_t}{dt} = \frac{d(g_t^{-1}a_0)}{dt} = -(g_t^{-1}\dot{g}_t g_t^{-1})a_0 = -g_t^{-1}\dot{g}_t a_t = -\xi a_t$$

The reduced Lagrangian becomes $l(\xi) \to l(\xi, g^{-1}a_0)$ with $\xi = \dot{g}g^{-1}$ and $g \in G$. The symmetry of the Lagrangian is thus reduced from the group G to the isotropy subgroup $G_{a_0} \subset G$ which leaves a_0 invariant under left action. That is, $ga_0 = 0$ for all $g \in G_{a_0}$. In the presence of this broken symmetry, the previous constrained action integral becomes

$$S(\xi, a_0, g, \dot{g}) = \int_a^b l(\xi, g^{-1}a_0, g, \dot{g})\,dt = \int_a^b \Big[l(\xi, g^{-1}a_0) + \langle\, \mu\,,\, (g^{-1}\dot{g} - \xi)\,\rangle \Big]\,dt\,.$$

Show that the Euler–Poincaré equation (67) changes to

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\partial l}{\partial a_t} \diamond a_t \tag{73}$$

where the $\diamond$ operator notation is defined by

$$\left\langle \frac{\partial l}{\partial a_t} \diamond a_t\,,\, \eta \right\rangle = \left\langle \frac{\partial l}{\partial a_t}\,,\, -\eta a_t \right\rangle$$

★

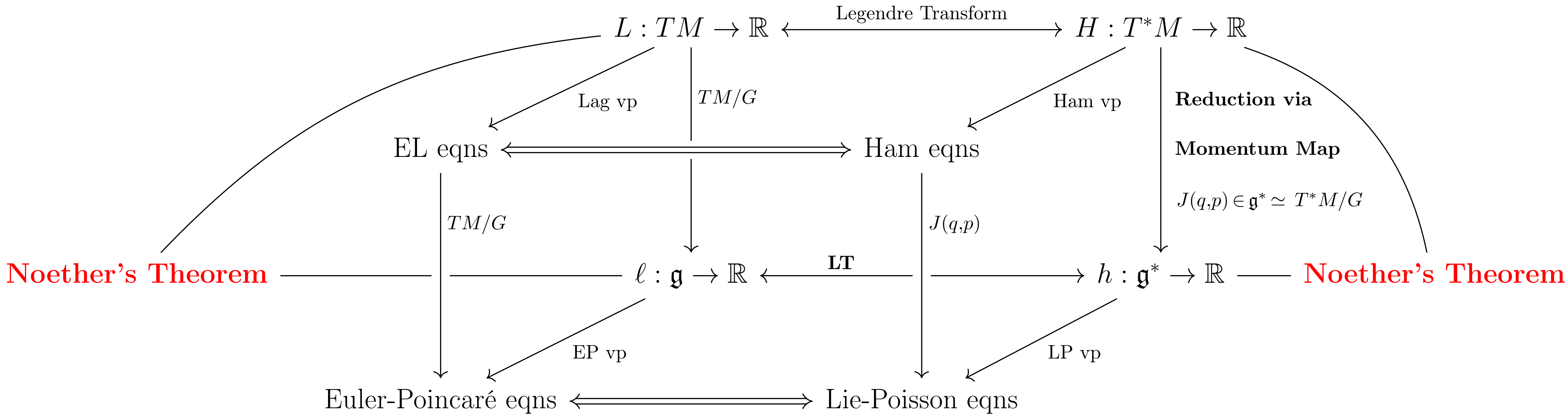


Figure 25: Framework for Geometric Mechanics

Summary for the rigid body

- Relation between left-invariant Lagrangians:

$$L(Q, \dot{Q}) = L(e, Q^{-1}\dot{Q}) = \ell(\Omega)$$

- Poisson brackets:

$$\frac{dF}{dt} = \{F, H\} = -\Pi \cdot \frac{\partial F}{\partial \Pi} \times \frac{\partial H}{\partial \Pi} = -\left\langle \Pi, \left[\frac{\partial F}{\partial \Pi}, \frac{\partial H}{\partial \Pi}\right]\right\rangle$$

$$\frac{d\Pi}{dt} = \{\Pi, H\} = \Pi \times \mathbb{I}^{-1}\Pi, \qquad (\Pi = \mathbb{I}\Omega)$$

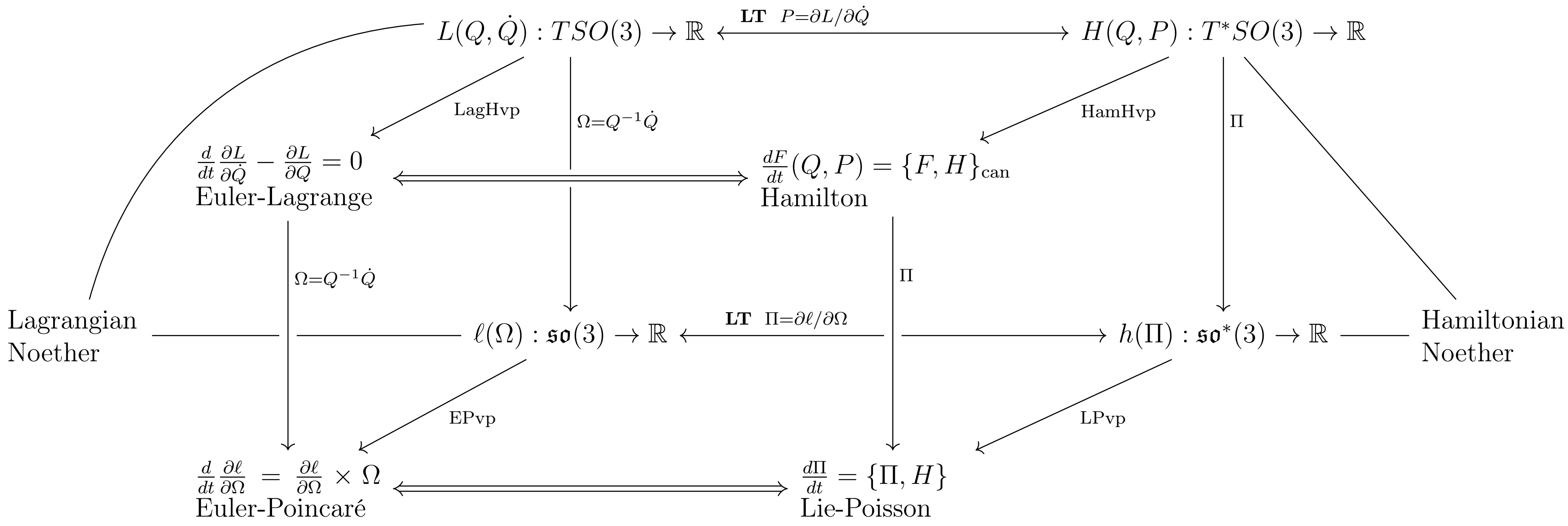


Figure 26: Rigid body dynamics

- Momentum map

  For $Q \in SO(3)$ and $\delta Q = \Phi_\xi(Q) \in T_eSO(3) = \mathfrak{so}(3)$, the Noether quantity is defined as

$$
J_\xi(P,Q) := \big\langle P, \Phi_\xi(Q)\big\rangle_{TSO(3)} = \big\langle \widehat{\Pi}, \widehat{\Xi}\big\rangle_{\mathfrak{so}(3)} \quad \text{with} \quad \widehat{\Pi} = \frac{1}{2}(P^TQ - Q^TP)\,, \quad \dot Q = Q\widehat{\Omega}
$$

Next we discuss transformation theory for differential forms, in preparation for moving geometric mechanics from finite dimensional particle dynamics to infinite dimensional continuum dynamics.

LECTURE #7

Transformation Theory for Differential Forms

Summary of natural operations on differential forms

Besides the wedge product for $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k$, three other basic operations are commonly applied to differential forms. These are ***contraction, exterior derivative and Lie derivative***.

- **Contraction $\,\lrcorner\,$ with a vector field X (aka ι_X) lowers the degree of a differential form $\alpha \in \Lambda^k$:**

$$X \,\lrcorner\, \Lambda^k \mapsto \Lambda^{k-1} \quad \text{and} \quad X \,\lrcorner\, (X \,\lrcorner\, \Lambda^k) = 0\,.$$

- **Exterior derivative d raises the degree:**

$$d\Lambda^k \mapsto \Lambda^{k+1} \quad \text{and} \quad d(d\Lambda^k) = 0\,.$$

- **Lie derivative $\pounds_X$ by vector field X preserves the degree:**

$$\pounds_X \Lambda^k \mapsto \Lambda^k\,, \quad \text{where} \quad \pounds_X \Lambda^k = \frac{d}{ds}\bigg|_{s=0} \phi_s^* \Lambda^k \quad \text{for} \quad X = \frac{d\phi_s}{ds}\bigg|_{s=0}\,,$$

  in which ϕ_s is the flow of vector field X. In analogy with fluids, one writes $\pounds_X \Lambda^k = \frac{d}{ds}\big|_{s=0} \Lambda^k(x)$ along $\frac{dx}{ds} = X$.

- **Lie derivative $\pounds_X$ also satisfies *Cartan's formula*:**

$$\pounds_X \alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha) \quad \text{for} \quad \alpha \in \Lambda^k \quad \text{Note:} \quad d(\pounds_X \alpha) = \pounds_X d\alpha\,.$$

Go2toc

10 Transformation Theory for Differential Forms

motion	linearisation	differential, d
motion equation	infinitesimal transformation	differential k-form
vector field	pull-back	wedge product, $\wedge$
diffeomorphism	push-forward	Lie derivative, $\pounds_Q$
flow	Jacobian matrix	product rule
fixed point	directional derivative	fluid dynamics
equilibrium	commutator	other flows

10.1 Motions, pull-backs, push-forwards, commutators & differentials

- A ***motion*** is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the ***motion equation***, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM\,, \tag{74}$$

  or in components

$$\dot{q}^i(t) = \frac{dq^i}{dt} = f^i(q) \quad i = 1, 2, \dots, n\,, \tag{75}$$

- The map $f : q \in M \to f(q) \in T_qM$ is a ***vector field***.

  According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, $q(t)$ exists, provided f is sufficiently smooth, which will always be assumed to hold.

- Vector fields can also be defined as ***differential operators*** that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^i(t)\frac{\partial G}{\partial q^i} = f^i(q)\frac{\partial G}{\partial q^i} \quad i = 1, 2, \ldots, n, \quad \text{(sum on repeated indices)} \tag{76}$$

  for any smooth function $G(q) : M \to \mathbb{R}$.

- To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

$$q(t) = \phi_t(q_0)\,.$$

  Because the vector field f is independent of time t, for any fixed value of t we may regard ϕ_t as mapping from M into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

  and

$$\phi_0 = \mathrm{I}d\,.$$

  Setting $s = -t$ shows that ϕ_t has a smooth inverse. A smooth mapping that has a smooth inverse is called a ***diffeomorphism***. Geometric mechanics deals with diffeomorphisms.

- The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (74) with initial condition $q(0) = q_0$ is called the ***flow*** of the vector field Q.

  A point $q^\star \in M$ at which $f(q^\star) = 0$ is called a ***fixed point*** of the flow ϕ_t, or an ***equilibrium***.

  Vice versa, the vector field f is called the ***infinitesimal transformation*** of the mapping ϕ_t, since

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_t \circ q_0) = f(q)\,.$$

That is, $f(q)$ is the ***linearisation*** of the flow map ϕ_t at the point $q \in M$.

More generally, the ***directional derivative*** of the function h along the vector field f is given by the action of a differential operator, as

$$\frac{d}{dt}\bigg|_{t=0} h \circ \phi_t = \left[\frac{\partial h}{\partial \phi_t}\frac{d}{dt}(\phi_t \circ q_0)\right]_{t=0} = \frac{\partial h}{\partial q^i}\dot{q}^i = \frac{\partial h}{\partial q^i} f^i(q) =: Qh\,.$$

- Under a smooth change of variables $q = c(r)$ the vector field Q in the expression Qh transforms as

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \mapsto \quad R = g^j(r)\frac{\partial}{\partial r^j} \quad \text{with} \quad g^j(r)\frac{\partial c^i}{\partial r^j} = f^i(c(r)) \quad \text{or} \quad g = c_r^{-1} f \circ c\,, \tag{77}$$

where $[c_r]^i_j := \partial c^i/\partial r^j$ is the ***Jacobian matrix*** of the transformation. That is, since $h(q)$ is a function of q,

$$(Qh) \circ c = R(h \circ c)$$

In coordinates, this is a change of variables obtained by substituting $q = c(r)$, as

$$(Qh) \circ c = \frac{\partial h(c(r))}{\partial r^j}\left(\left[\frac{\partial c}{\partial r}\right]^{-1}\right)^j_i f^i(c(r)) = g_j(r)\frac{\partial h((c(r))}{\partial r^j} = R(h \circ c)\,.$$

We express the transformation between the vector fields as $R =: c^*Q$ and write this relation as

$$(Qh) \circ c = R(h \circ c) =: (c^*Q)(h \circ c)\,. \tag{78}$$

The expression c^*Q is called the ***pull-back*** of the vector field Q by the map c. Two vector fields are equivalent under a map c, if one is the pull-back of the other, and fixed points are mapped into fixed points.

The inverse of the pull-back is called the ***push-forward***, $c_*Q = (c^{-1})^*Q$. It is the pull-back by the inverse map. Under the push-forward, $R = g^j(r)\frac{\partial}{\partial r^j} \to Q = f^i(q)\frac{\partial}{\partial q^i}$ with $r = c^{-1}(q)$.

- The ***commutator***

$$QR - RQ =: \big[Q,\, R\big]$$

of two vector fields Q and R defines another vector field. Indeed, if

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \text{and} \quad R = g^j(q)\frac{\partial}{\partial q^j}$$

then

$$\big[Q,\, R\big] = \left(f^i(q)\frac{\partial g^j(q)}{\partial q^i} - g^i(q)\frac{\partial f^j(q)}{\partial q^i}\right)\frac{\partial}{\partial q^j}$$

because the second-order derivative terms cancel. By the pull-back relation (78) we have

$$c^*\big[Q,\, R\big] = \big[c^*Q,\, c^*R\big] \tag{79}$$

under a change of variables defined by a smooth map, c. This means the definition of the vector field commutator is independent of the choice of coordinates. As we shall see, the ***tangent*** to the relation $c_t^*\big[Q,\, R\big] = \big[c_t^*Q,\, c_t^*R\big]$ at the identity $t = 0$ is the ***Jacobi condition*** for the vector fields to form an algebra.

- The ***differential*** of a smooth function $f : M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i}dq^i\,.$$

- Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i}dq^i\,, \quad d(f\circ\phi) = \frac{\partial f}{\partial \phi^j(q)}\frac{\partial \phi^j}{\partial q^i}dq^i = \frac{\partial f}{\partial s^j}ds^j \quad \Longrightarrow \quad d(f\circ\phi) = (df)\circ\phi \tag{80}$$

That is, the differential d commutes with the pull-back ϕ^* of a smooth transformation ϕ,

$$d(\phi^* f) = \phi^* df\,. \tag{81}$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

10.2 Wedge products

- Differential k-forms on an n-dimensional manifold are defined in terms of the differential d and the antisymmetric ***wedge product*** ($\wedge$) satisfying

$$dq^i \wedge dq^j = -\, dq^j \wedge dq^i\,, \quad \text{for} \quad i, j = 1, 2, \ldots, n \tag{82}$$

By using wedge product, any k-form $\alpha \in \Lambda^k$ on M may be written locally at a point $q \in M$ in the differential basis dq^j as

$$\alpha_m = \alpha_{i_1 \ldots i_k}(m) dq^{i_1} \wedge \cdots \wedge dq^{i_k} \in \Lambda^k\,, \quad i_1 < i_2 < \cdots < i_k\,, \tag{83}$$

where the sum over repeated indices is ordered, so that it must be taken over all i_j satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms Λ^k are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

- Pull-backs of other differential forms may be built up from their basis elements, the dq^{i_k}. By equation (81),

Theorem 65 (Pull-back of a wedge product)**.** *The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:*

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta\,. \tag{84}$$

10.3 Lie derivatives

Definition 66 (Lie derivative of a differential k-form)**.** *The **Lie derivative** of a differential k-form Λ^k by a vector field $Q \in \mathfrak{X}$ is defined by linearising its flow ϕ_t around the identity $t = 0$,*

$$\pounds_Q \Lambda^k = \frac{d}{dt}\Big|_{t=0} \phi_t^* \Lambda^k \quad \textit{maps} \quad \mathfrak{X} \times \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (84), the Lie derivative satisfies the product rule for the wedge product.

Corollary 67 (Product rule for the Lie derivative of a wedge product)**.**

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q \alpha \wedge \beta + \alpha \wedge \pounds_Q \beta \,. \tag{85}$$

- Pullbacks of vector fields lead to Lie derivative expressions, too.

Definition 68 (Lie derivative of a vector field)**.** *The **Lie derivative** of a vector field $Y \in \mathfrak{X}$ by another vector field $X \in \mathfrak{X}$ is defined by linearising the flow ϕ_t of X around the identity $t = 0$,*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y \quad \textit{maps} \quad \pounds_X \in \mathfrak{X} \mapsto \mathfrak{X} \,.$$

Theorem 69. *The Lie derivative $\pounds_X Y$ of a vector field Y by a vector field X satisfies*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y = [X,\, Y] \,, \tag{86}$$

where $[X,\, Y] = XY - YX$ is the commutator of the vector fields X and Y.

Proof. Denote the vector fields in components as

$$X = X^i(q)\frac{\partial}{\partial q^i} = \frac{d}{dt}\bigg|_{t=0}\phi_t^* \quad \text{and} \quad Y = Y^j(q)\frac{\partial}{\partial q^j}\,.$$

Then, by the pull-back relation (78) a direct computation yields, on using the matrix identity $dM^{-1} = -\,M^{-1}dMM^{-1}$,

$$\begin{aligned}
\pounds_X Y &= \frac{d}{dt}\bigg|_{t=0}\phi_t^* Y = \frac{d}{dt}\bigg|_{t=0}\left(Y^j(\phi_t q)\frac{\partial}{\partial(\phi_t q)^j}\right)\\
&= \frac{d}{dt}\bigg|_{t=0}\left(Y^j(\phi_t q)\left[\frac{\partial(\phi_t q)}{\partial q}^{-1}\right]_j^k\frac{\partial}{\partial q^k}\right)\\
&= \left(X^j\frac{\partial Y^k}{\partial q^j} - Y^j\frac{\partial X^k}{\partial q^j}\right)\frac{\partial}{\partial q^k}\\
&= [X,\,Y]\,.
\end{aligned}$$

□

Corollary 70. *The Lie derivative of the relation (79) for the pull-back of the commutator* $c_t^*\big[Y,\,Z\big] = \big[c_t^*Y,\,c_t^*Z\big]$ *yields the* ***Jacobi identity*** *condition for the vector fields to form an algebra.*

Proof. By the product rule (85) and the definition of the Lie bracket (86) we have

$$\frac{d}{dt}\bigg|_{t=0}\phi_t^*\big[Y,\,Z\big] = \big[X,\big[Y,\,Z\big]\big] = \big[[X,Y],\,Z\big] + \big[Y,\,[X,Z]\big] = \frac{d}{dt}\bigg|_{t=0}\big[\phi_t^*Y,\,\phi_t^*Z\big]$$

This is the ***Jacobi identity*** for vector fields. □

Exercise. Use the hat map and the relation $R_t(\mathbf{x}\times\mathbf{y}) = R_t\mathbf{x}\times R_t\mathbf{y}$ to show that the same argument gives the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$, when ϕ_t^* is a rotation. ★

10.4 Summary of operations on differential forms that are natural under pullback

Besides the wedge product, three basic operations are commonly applied to differential forms. These are contraction, exterior derivative and Lie derivative.

- **Contraction $\lrcorner$ with a vector field X lowers the degree:**

$$X \lrcorner \Lambda^k \mapsto \Lambda^{k-1}\,.$$

- **Exterior derivative d raises the degree:**

$$d\Lambda^k \mapsto \Lambda^{k+1}\,.$$

- **Lie derivative $\pounds_X$ by vector field X preserves the degree:**

$$\pounds_X \Lambda^k \mapsto \Lambda^k\,, \quad \text{where} \quad \pounds_X \Lambda^k = \frac{d}{dt}\bigg|_{t=0} \phi_t^* \Lambda^k\,,$$

  in which ϕ_t is the flow of the vector field X. In analogy with fluids one may write $\pounds_X \Lambda^k = \frac{d}{dt}\Lambda^k$ along $\frac{dx}{dt} = X$.

- **Lie derivative $\pounds_X$ satisfies *Cartan's formula*:** (The proof is a direct calculation.)

$$\pounds_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \quad \text{for} \quad \alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Lambda^k\,.$$

Remark 71.
Note also that Cartan's formula provides another proof that the Lie derivative commutes with the exterior derivative. That is,

$$d(\pounds_X \alpha) = \pounds_X d\alpha\,, \quad \textit{for} \quad \alpha \in \Lambda^k(M) \quad \textit{and} \quad X \in \mathfrak{X}(M)\,.$$

10.5 Examples of contraction, or interior product

Definition 72 (Contraction, or interior product)**.** *Let $\alpha \in \Lambda^k$ be a k-form on a manifold M,*

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k \,, \quad \textit{with} \quad i_1 < i_2 < \cdots < i_k \,,$$

and let $X = X^j \partial_j$ be a vector field. The contraction or interior product $X \,\lrcorner\, \alpha$ of a vector field X with a k-form α is defined by

$$X \,\lrcorner\, \alpha = X^j \alpha_{j i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k} \,. \tag{87}$$

Note that

$$\begin{aligned} X \,\lrcorner\, (Y \,\lrcorner\, \alpha) &= X^l Y^m \alpha_{m l i_3 \dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= -Y \,\lrcorner\, (X \,\lrcorner\, \alpha) \,, \end{aligned}$$

by antisymmetry of $\alpha_{m l i_3 \dots i_k}$, particularly in its first two indices.

Remark 73 (Examples of contraction)**.**

1. *A mnemonic device for keeping track of signs in contraction or substitution of a vector field into a differential form is to sum the substitutions of $X = X^j \partial_j$ over the permutations that bring the corresponding dual basis element into the leftmost position in the k-form α. For example, in two dimensions, contraction of the vector field $X = X^j \partial_j = X^1 \partial_1 + X^2 \partial_2$ into the two-form $\alpha = \alpha_{jk} dx^j \wedge dx^k$ with $\alpha_{21} = -\,\alpha_{12}$ yields*

   $$X \,\lrcorner\, \alpha = X^j \alpha_{j i_2} dx^{i_2} = X^1 \alpha_{12} dx^2 + X^2 \alpha_{21} dx^1 \,.$$

   Likewise, in three dimensions, contraction of the vector field $X = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3$ into the three-form

$\alpha = \alpha_{123} dx^1 \wedge dx^2 \wedge dx^3$ *with* $\alpha_{213} = -\alpha_{123}$, *etc. yields*

$$\begin{aligned} X \,\lrcorner\, \alpha &= X^1 \alpha_{123} dx^2 \wedge dx^3 + \textit{cyclic permutations} \\ &= X^j \alpha_{j i_2 i_3} dx^{i_2} \wedge dx^{i_3} \quad \textit{with} \quad i_2 < i_3 \,. \end{aligned}$$

2. *The rule for contraction of a vector field with a differential form develops from the relation*

$$\partial_j \,\lrcorner\, dx^k = \delta^k_j \,,$$

in the coordinate basis $e_j = \partial_j := \partial / \partial x^j$ *and its dual basis* $e^k = dx^k$. *Contraction of a vector field with a one-form yields the dot product, or inner product, between a covariant vector and a contravariant vector is given by*

$$X^j \partial_j \,\lrcorner\, v_k dx^k = v_k \delta^k_j X^j = v_j X^j \,,$$

or, in vector notation,

$$X \,\lrcorner\, \mathbf{v} \cdot d\mathbf{x} = \mathbf{v} \cdot \mathbf{X} \,.$$

This is the ***dot product of vectors*** $\mathbf{v}$ *and* $\mathbf{X}$.

3. *By the linearity of its definition (87), contraction of a vector field* X *with a differential* k*-form* α *satisfies*

$$(hX) \,\lrcorner\, \alpha = h(X \,\lrcorner\, \alpha) = X \,\lrcorner\, h\alpha \,.$$

Our previous calculations for two-forms and three-forms provide the following additional expressions for contraction of a vector field with a differential form, which may be written in vector notation as:

$$
\begin{aligned}
X \lrcorner\, \mathbf{B}\cdot d\mathbf{S} &= -\,\mathbf{X}\times\mathbf{B}\cdot d\mathbf{x}\,,\\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S}\,,\\
d(X \lrcorner\, d^{\,3}x) &= d(\mathbf{X}\cdot d\mathbf{S}) = (\mathrm{d}iv\,\mathbf{X})\,d^{\,3}x = \pounds_X\, d^{\,3}x\,.
\end{aligned}
$$

Remark 74 (Physical examples of contraction)**.**
The first of these contraction relations represents the Lorentz, or Coriolis force, when $\mathbf{X}$ *is particle velocity and* $\mathbf{B}$ *is either magnetic field, or rotation rate, respectively. The second contraction relation is the flux of the vector* $\mathbf{X}$ *through a surface element. The third is the exterior derivative of the second, thereby yielding the divergence of the vector* $\mathbf{X}$*.*

Exercise. Show that

$$X \lrcorner\, (X \lrcorner\, \mathbf{B}\cdot d\mathbf{S}) = 0$$

and

$$(X \lrcorner\, \mathbf{B}\cdot d\mathbf{S}) \wedge \mathbf{B}\cdot d\mathbf{S} = 0\,,$$

for any vector field X and two-form $\mathbf{B}\cdot d\mathbf{S}$. ★

Proposition 75 (Contracting through wedge product)**.** *Let* α *be a* k*-form and* β *be a one-form on a manifold* M *and let* $X = X^j\partial_j$ *be a vector field. Then the contraction of* X *through the wedge product* $\alpha\wedge\beta$ *satisfies*

$$X \lrcorner\, (\alpha\wedge\beta) = (X \lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \lrcorner\, \beta)\,. \tag{88}$$

Proof. The proof is a straightforward calculation using the definition of contraction. The exponent k in the factor $(-1)^k$ counts the number of exchanges needed to get the one-form β to the left most position through the k-form α. □

Proposition 76. *[Contraction is natural under pull-back]*
That is, with $m \in M$,

$$\phi^*(X(m) \,\lrcorner\, \alpha) = X(\phi(m)) \,\lrcorner\, \phi^*\alpha =: \phi^* X \,\lrcorner\, \phi^*\alpha\,. \tag{89}$$

Proof. Direct verification using the relation between pull-back of forms and push-forward of vector fields. □

Corollary 77. *By the Lie product rule* (85)*, equation* (89) *implies*

$$\pounds_X(Y \,\lrcorner\, \alpha) = [X\,,\,Y] \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)\,. \tag{90}$$

Definition 78 (Alternative notations for contraction)**.** *Besides the hook notation with $\lrcorner$, one also finds in the literature the following two alternative notations for contraction of a vector field X with k-form $\alpha \in \Lambda^k$ on a manifold M:*

$$X \,\lrcorner\, \alpha = \iota_X\alpha = \alpha(X,\ \underbrace{\cdot\,,\cdot\,,\ldots\,,\cdot}_{k-1 \ slots}\ ) \in \Lambda^{k-1}\,. \tag{91}$$

In the last alternative, one leaves a dot $(\,\cdot\,)$ in each remaining slot of the form that results after contraction. For example, contraction of the Hamiltonian vector field $X_H = \{\,\cdot\,,\,H\}$ with the symplectic two-form $\omega \in \Lambda^2$ produces the one-form

$$X_H \,\lrcorner\, \omega = \omega(X_H,\,\cdot\,) = -\,\omega(\,\cdot\,,\,X_H) = dH\,.$$

In this alternative notation, the proof of formula (89) in Proposition 76 may be written, as follows.

Proof. Since forms are multilinear maps to the real numbers, one may define the pull-back of a k-form, α, by

$$\phi^*\alpha(X_1, X_2, ...) := \alpha(\phi_*X_1, \phi_*X_2, ...)\,.$$

Therefore, we are able to use the following proof, recalling that $X^*(m) := X(\phi(m))$

$$\begin{aligned}
\phi^* X \,\lrcorner\, \phi^*\alpha(X_1, X_2, ...) &= \phi^*\alpha(\phi^* X, X_2, X_3, ...)\\
&= \alpha(\phi_*\phi^* X, \phi_* X_2, \phi_* X_3, ...)\\
&= \alpha(X, \phi_* X_2, \phi_* X_3, ...)\\
&= (X \,\lrcorner\, \alpha)(\phi_* X_2, \phi_* X_3, ...)\\
&= \phi^*(X \,\lrcorner\, \alpha)(X_2, X_3, ...)
\end{aligned}$$

Now, if we allow $X_2, X_3, \ldots$ to be arbitrary, then formula (89) in Proposition 76 follows. □

Proposition 79 (Hamiltonian vector field definitions)**.** *The following two definitions of Hamiltonian vector field X_H are equivalent.*

$$dH = X_H \,\lrcorner\, \omega \quad \textit{and} \quad X_H = \{\,\cdot\,,\, H\}$$

Proof. The symplectic Poisson bracket satisfies $\{F, H\} = \omega(X_F, X_H)$, because

$$\omega(X_F, X_H) := X_H \,\lrcorner\, X_F \,\lrcorner\, \omega = X_H \,\lrcorner\, dF = -X_F \,\lrcorner\, dH = \{F,\, H\}\,.$$

□

Remark 80.

The relation $\{F,\, H\} = \omega(X_F, X_H)$ means that the Hamiltonian vector field defined via the symplectic form coincides exactly with the Hamiltonian vector field defined using the Poisson bracket.

Exercises!

10.6 Exercises in exterior calculus operations

Vector notation for differential basis elements One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$ in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,,\\
d\mathbf{S} &= (dS_1, dS_2, dS_3)\\
&:= (dx^2 \wedge dx^3,\; dx^3 \wedge dx^1,\; dx^1 \wedge dx^2)\,,\\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,,\\
d^{\,3}x &= dVol := dx^1 \wedge dx^2 \wedge dx^3\\
&= \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k\,.
\end{aligned}
$$

Exercise. **(**Vector calculus operations) Show that contraction $\lrcorner : \mathfrak{X} \times \Lambda^k \to \Lambda^{k-1}$ of the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ with the differential basis elements $d\mathbf{x}$, $d\mathbf{S}$ and $d^{\,3}x$ recovers the following familiar operations among vectors:

$$
\begin{aligned}
X \lrcorner\, d\mathbf{x} &= \mathbf{X}\,,\\
X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,,\\
(\text{or,}\quad X \lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k)\\
Y \lrcorner\, X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,,\\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,,\\
Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,,\\
Z \lrcorner\, Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

★

Exercise. (Exterior derivatives in vector notation) Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation:

$$\begin{aligned} df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x}\,, \\ 0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j\,, \\ df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \\ &=: (\nabla f \times \nabla g)\cdot d\mathbf{S}\,, \\ df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l \\ &=: (\nabla f \cdot \nabla g \times \nabla h)\, d^{\,3}x\,. \end{aligned}$$

★

Exercise. (Vector calculus formulas) Prove the following familiar vector calculus formulas:

$$\begin{aligned} df &= \nabla f \cdot d\mathbf{x}\,, \\ d(\mathbf{v}\cdot d\mathbf{x}) &= (curl\,\mathbf{v})\cdot d\mathbf{S}\,, \\ d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{d}iv\,\mathbf{A})\, d^{\,3}x\,. \end{aligned}$$

The compatibility condition $d^2 = 0$ is written for these forms as

$$\begin{aligned} 0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (curl\,grad\,f)\cdot d\mathbf{S}\,, \\ 0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((curl\,\mathbf{v})\cdot d\mathbf{S}\big) &= (\mathrm{d}iv\,curl\,\mathbf{v})\, d^{\,3}x\,. \end{aligned}$$

The product rule is written for these forms as

$$\begin{aligned} d\big(f(\mathbf{A}\cdot d\mathbf{x})\big) &= df \wedge \mathbf{A}\cdot d\mathbf{x} + f curl\,\mathbf{A}\cdot d\mathbf{S} \\ &= \big(\nabla f \times \mathbf{A} + f curl\,\mathbf{A}\big)\cdot d\mathbf{S} \\ &= curl\,(f\mathbf{A})\cdot d\mathbf{S}\,, \end{aligned}$$

$$
\begin{aligned}
d\Big((\mathbf{A}\cdot d\mathbf{x})\wedge(\mathbf{B}\cdot d\mathbf{x})\Big) &= (curl\,\mathbf{A})\cdot d\mathbf{S}\wedge\mathbf{B}\cdot d\mathbf{x} - \mathbf{A}\cdot d\mathbf{x}\wedge(curl\,\mathbf{B})\cdot d\mathbf{S}\\
&= \Big(\mathbf{B}\cdot curl\,\mathbf{A} - \mathbf{A}\cdot curl\,\mathbf{B}\Big)d^{\,3}x\\
&= d\Big((\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}\Big)\\
&= \mathrm{d}iv(\mathbf{A}\times\mathbf{B})\,d^{\,3}x\,.
\end{aligned}
$$

These calculations yield familiar formulas from vector calculus for quantities $curl(grad)$, $\mathrm{d}iv(curl)$, $curl(f\mathbf{A})$ and $\mathrm{d}iv(\mathbf{A}\times\mathbf{B})$. ★

LECTURE #8

This lecture treats two coordinate-free discussions of:
(1) The integral calculus formulas arising from Stokes' theorem relying on key definitions; and
(2) A coordinate-free formulation of classical Hamiltonian mechanics.

Then, an array of exercises in the calculus of differential forms (with answers) are offered as preparation for the geometric mechanics of ideal fluid dynamics to be discussed later.

Go2toc

10.7 Integral calculus formulas arising from Stokes' theorem

Theorem 81 (Stokes' theorem)**.** *Suppose M is a compact oriented k-dimensional manifold with boundary ∂M and α is a smooth $(k-1)$-form on M. Then*

$$\int_M d\alpha = \oint_{\partial M} \alpha\,.$$

Exercise. (Integral calculus formulas) Show that the Stokes' theorem for the vector calculus formulas yields the following familiar results in $\mathbb{R}^3$:

- The ***fundamental theorem of calculus***, upon integrating df along a curve in $\mathbb{R}^3$ starting at point a and ending at point b:

$$\int_a^b df \;=\; \int_a^b \nabla f \cdot d\mathbf{x} = f(b) - f(a)\,.$$

- The ***classical Stokes theorem***, for a compact surface S with boundary ∂S:

$$\int_S (curl\,\mathbf{v}) \cdot d\mathbf{S} \;=\; \oint_{\partial S} \mathbf{v} \cdot d\mathbf{x}\,.$$

  (For a planar surface $S \in \mathbb{R}^2$, this is ***Green's theorem***.)

- The ***Gauss divergence theorem***, for a compact spatial domain D with boundary ∂D:

$$\int_D (\mathrm{d}iv\,\mathbf{A})\, d^{\,3}x \;=\; \oint_{\partial D} \mathbf{A} \cdot d\mathbf{S}\,.$$

★

10.8 Summary

The ***pull-back*** ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X on a smooth manifold M commutes with the exterior derivative d, wedge product $\wedge$ and contraction $\,\lrcorner\,$.

That is, for k-forms α, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$\begin{aligned} d(\phi_t^* \alpha) &= \phi_t^* d\alpha \,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^* \alpha \wedge \phi_t^* \beta \,, \\ \phi_t^*(X \,\lrcorner\, \alpha) &= \phi_t^* X \,\lrcorner\, \phi_t^* \alpha \,. \end{aligned}$$

In addition, the ***Lie derivative*** $\pounds_X \alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined either dynamically or geometrically (by Cartan's formula) as

$$\pounds_X \alpha \;=\; \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha), \tag{92}$$

in which the last equality is Cartan's geometric formula in (92) for the Lie derivative.

Definition 82. ***(The Lie chain rule)***
The tangent to the pull-back $\phi_t^*\alpha$ of a differential k-form $\alpha \in \Lambda^k$ is the pull-back of the Lie derivative of α wrt the vector field X that generates the flow, ϕ_t:

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*\big(\pounds_X\alpha\big)\,.$$

Likewise, for the push-forward, which is the pull-back by the inverse, we have

$$\frac{d}{dt}((\phi_t^{-1})^*\alpha) = -\,(\phi_t^{-1})^*\big(\pounds_X\alpha\big)\,.$$

Definition 83. ***(Advected quantity)***
An advected quantity is invariant along a flow trajectory. Hence, advected quantities satisfy

$$\alpha_0(x_0) = \alpha_t(x_t) = (\phi_t^*\alpha_t)(x_0)\,,\quad \textit{or equivalently,}\quad \alpha_t(x_t) = (\alpha_0\circ\phi_t^{-1})(x_t) = ((\phi_t)_*\alpha_0)(x_t)\,.$$

The dynamics of an advected quantity is given by the Lie chain rule as

$$\frac{d}{dt}\alpha_t(x_t) = \frac{d}{dt}(\phi_t)_*\alpha_0 = -\pounds_X\alpha_t\,.$$

The Lie chain rule implies the same advection dynamics in terms of the pull-back,

$$0 = \frac{d}{dt}\alpha_0(x_0) = \frac{d}{dt}(\phi_t^*\alpha_t)(x_0) = \phi_t^*(\partial_t + \pounds_X)\alpha_t(x_0) = (\partial_t + \pounds_X)\alpha_t(x_t)$$

Remark 84. *These formulas enable us to write a coordinate-free formulation of ideal fluid mechanics with advected quantities, by introducing the advection dynamics as a constraint on Hamilton's principle for fluids, [HMR1998].*

10.9 A coordinate-free formulation of classical Hamiltonian mechanics

A coordinate-free definition of the Poisson bracket can be formulated in terms of the operations of the ***differential*** (or exterior derivative), ***insertion*** (or contraction), and ***Lie derivative***, denoted, respectively, as[2]

$$d:\mathfrak{X}(T^*M)\times\Lambda^k\to\Lambda^{k+1}\qquad \iota:\mathfrak{X}(T^*M)\times\Lambda^k\to\Lambda^{k-1}\qquad \pounds_X\omega=d(\iota_X)+\iota_X d:\ \mathfrak{X}(T^*M)\times\Lambda^k\to\Lambda^k\,,$$

where k is an even number for the cotangent bundle T^*M of a manifold M. For example, in two dimensions, insertion of the vector field $X=X^j\partial_j=X^1\partial_1+X^2\partial_2$ into the two-form $\alpha=\alpha_{jk}dx^j\wedge dx^k$ with $\alpha_{21}=-\alpha_{12}$ yields

$$X\,\lrcorner\,\alpha=\iota_X\alpha=X^j\alpha_{ji_2}dx^{i_2}=X^1\alpha_{12}dx^2+X^2\alpha_{21}dx^1=\alpha_{12}(X^1dx^2-X^2dx^1)\,.$$

Consequently, the Poisson bracket can be defined by insertion of Hamiltonian vector fields

$$X_F:=\{\,\cdot\,,\,F\,\}\quad\text{and}\quad X_F:=\{\,\cdot\,,\,H\,\}$$

into the closed symplectic two form $\omega\in\Lambda^2(T^*M)$ with $d\omega=0$, as

$$\{F,H\}=\iota_{X_H}\big(\iota_{X_F}\omega(\,\cdot\,,\,\cdot\,)\big)=\omega(X_F,X_H)=X_H\,\lrcorner\,\big(X_F\,\lrcorner\,\omega(\,\cdot\,,\,\cdot\,)\big)\,.$$

This formula is related to the original phase space coordinates by,

$$\omega=\sum_{i=1}^2 dq_i\wedge dp_i=-\sum_{i=1}^2 dp_i\wedge dq_i\quad\text{and}\quad X_H=\{\,\cdot\,,H\}=\sum_{i=1}^2\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\,.$$

Notice that the calculation of $\omega(X_F,X_H)$ in these coordinates yields

$$\{F,H\}=\iota_{X_H}\big(\iota_{X_F}\omega(\,\cdot\,,\,\cdot\,)\big)=\iota_{X_H}\omega(\,X_F\,,\,\cdot\,)=\iota_{X_H}dF=\omega(X_F,X_H)\,.$$

[2]The operation of insertion of a vector field into a differential form is often called *substitution*, or *contraction*, as well. For example, the operation $\iota_X\alpha$ for insertion of a vector field (X) into a k-form $\alpha\in\Lambda^k(T^*M)$ may also be denoted equivalently as $X\,\lrcorner\,\alpha$, or simply as $\alpha(X)$. Although all of these notations can be used interchangeably, here we will use ι_{X_H} for insertion of a Hamiltonian vector field into a symplectic form, and apply $X\,\lrcorner\,\alpha$ for contraction in the general bases of vector fields and differential k-forms. This compact coordinate-free notation has become standard in modern mathematics.

Consequently, the dynamics along the integral curves of X_H are determined by

$$dH = \omega(X_H\,,\,\cdot\,) = \iota_{X_H}\omega\,,$$

in which the vector field X_H is inserted (ι) into the symplectic 2-form ω to create the exact 1-form dH. In the original coordinates, this is

$$dH = \iota_{(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p})}(dq \wedge dp) = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp\,.$$

One can check that $\omega(X_F, X_H) = \{F, H\}$ directly, as

$$\begin{aligned}
\frac{dF}{dt} &= \omega(X_F, X_H) \\
&= \iota_{X_H}(\iota_{X_F}\omega) = \iota_{X_H}dF \\
&= \iota_{(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p})}\Big(\frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial p}dp\Big) \\
&= \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} \\
&= \{F, H\}\,.
\end{aligned}$$

Definition 85 (Cartan's geometric definition of the Lie derivative)**.** *The coordinate-free expression*

$$\pounds_{X_F}\omega = d(\iota_{X_F}\omega) + \iota_{X_F}d\omega = d(X_F \,\lrcorner\, \omega) + X_F \,\lrcorner\, d\omega$$

is Cartan's geometric definition of the Lie derivative *of the symplectic 2-form ω with respect to the Hamiltonian vector field $X_F = \{\,\cdot\,,\, F\,\}$.*

Lemma 86. *Since the symplectic form ω is closed ($d\omega = 0$) and $\iota_{X_F}\omega = dF$ for a Hamiltonian vector field X_F, we have*

$$\pounds_{X_F}\omega = d(\iota_{X_F}\omega) = d^2F = 0\,.$$

This means the symplectic form ω is locally invariant under the Lie algebra actions of Hamiltonian vector fields.

Definition 87 (Symplectic flow)**.** *The finite transformation ϕ_ϵ generated by the left-invariant Hamiltonian vector field $X_F = \phi_\epsilon^{-1}\phi_\epsilon'|_{\epsilon=0}$ is called a **symplectic flow**.*

Theorem 88 (Symplectic flows preserve the symplectic 2-form ω)**.** *A smooth symplectic flow*

$$\phi_\epsilon^{X_F} := exp(\epsilon X_F)$$

generated by a (time-independent) Hamiltonian vector field X_F given by

$$X_F = \frac{d}{d\epsilon}\phi_\epsilon^{X_F}\Big|_{\epsilon=0} = \{\,\cdot\,,\,F\,\}\,,$$

*with $\iota_{X_F}\omega = dF$, preserves the symplectic 2-form ω under **pull-back** by the flow $\phi_\epsilon^{X_F} = exp(\epsilon X_F)$, defined as*

$$\phi_\epsilon^{X_F\,*}\omega(q,p) := \omega(\phi_\epsilon^{X_F}q, \phi_\epsilon^{X_F}p)\,.$$

Proof.

$$\frac{d}{d\epsilon}(\phi_\epsilon^{X_F\,*}\omega) = \phi_\epsilon^{X_F\,*}(\pounds_{X_F}\omega) = \phi_\epsilon^{X_F\,*}(d(\iota_{X_F}\omega) + \iota_{X_F}d\omega) = \phi_\epsilon^{X_F\,*}d(dF) = 0\,,$$

since $d\omega = 0$ and $d(\iota_{X_F}\omega) = d^2F = 0$. □

Remark. In the context of pull-back by smooth flows here, the proof uses the dynamic definition of the Lie derivative,

$$\pounds_X\omega = \frac{d}{d\epsilon}(\phi_\epsilon{}^*\omega)\Big|_{\epsilon=0} \quad \text{with} \quad X = \phi_\epsilon^{-1}\phi_\epsilon'|_{\epsilon=0}\,,$$

in the first step. In the second step, the proof uses the equivalence of the dynamic and Cartan definitions of the Lie derivative with respect to vector fields.

Exercise. Demonstrate the equivalence of the dynamic and Cartan definitions of the Lie derivative $\pounds_X$ by calculating their actions on scalar functions. How is this result related to the familiar directional derivative of a scalar function?

Exercises!

Exercise.

(a) Verify the formula $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$.

(b) Use (a) to verify $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.

(c) Use (b) to verify the Jacobi identity.

(d) Use (c) to verify that the divergence-free vector fields are closed under commutation.

(e) For a top-form α show divergence-free vector fields that

$$[X\,,\,Y] \,\lrcorner\, \alpha = d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big)\,. \tag{93}$$

(f) Write the equivalent of equation (93) as a formula in vector calculus.

★

Answer.

(a) The required formula follows immediately from the product rule in (85) for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y \,\lrcorner\, \alpha) = \phi_t^* Y \,\lrcorner\, \phi_t^* \alpha \,.$$

A direct computation using the dynamical definition of the Lie derivative $\pounds_Y \alpha = \frac{d}{dt}\big|_{t=0}(\phi_t^*\alpha)$, then yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \phi_t^* \big(Y \,\lrcorner\, \alpha\big) \;&=\; \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^* Y\Big) \,\lrcorner\, \alpha \\ &\quad + Y \,\lrcorner\, \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^* \alpha\Big) \,. \end{aligned}$$

Hence, we recognise that the desired formula is the ***product rule*** met earlier in equation (85):

$$\pounds_X (Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha) \,.$$

(b) Insert $\pounds_X Y = [X\,,\,Y]$ into the product rule formula in part (a). Then

$$[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X (Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha).$$

Now use Cartan's formula in (92)

$$\pounds_X \alpha \;=\; \frac{d}{dt}\bigg|_{t=0} (\phi_t^* \alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha),$$

to compute the required result, as

$$
\begin{aligned}
\pounds_{[X\,,Y]}\alpha &= d([X\,,\,Y] \,\lrcorner\, \alpha) + [X\,,\,Y] \,\lrcorner\, d\alpha \\
&= d\Big(\pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha) \Big) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X d\alpha) \\
&= \pounds_X d(Y \,\lrcorner\, \alpha) - d(Y \,\lrcorner\, (\pounds_X\alpha) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, d(\pounds_X\alpha) \\
&= \pounds_X(\pounds_Y\alpha) - \pounds_Y(\pounds_X\alpha)\,.
\end{aligned}
$$

Can you think of an alternative proof based on the dynamical definition of the Lie derivative?

(c) Applying part (b), ($\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$) to $\alpha = d^3x$ proves that $\pounds_{[X\,,Y]}d^3x = 0$; since both $\pounds_Y d^3x = 0 = \pounds_X d^3x$, because, e.g., $\pounds_Y d^3x = ({\rm d}ivY)\,d^3x$.

(d) As a consequence of part (b),

$$
\begin{aligned}
\pounds_{[Z\,,[X\,,Y]]}\alpha &= \pounds_Z(\pounds_X\pounds_Y - \pounds_Y\pounds_X)\alpha - (\pounds_X\pounds_Y - \pounds_Y\pounds_X)\pounds_Z\alpha \\
&= \pounds_Z\pounds_X\pounds_Y\alpha - \pounds_Z\pounds_Y\pounds_X\alpha - \pounds_X\pounds_Y\pounds_Z\alpha + \pounds_Y\pounds_X\pounds_Z\alpha\,,
\end{aligned}
$$

and summing over cyclic permutations verifies that

$$
\pounds_{[Z\,,[X\,,Y]]}\,\alpha + \pounds_{[X\,,[Y\,,Z]]}\,\alpha + \pounds_{[Y\,,[Z\,,X]]}\,\alpha = 0\,.
$$

This is the ***Jacobi identity for the Lie derivative***.

(e) Substituting the relation $\pounds_X Y = [X,\,Y]$ into the product rule above in part (b) and rearranging yields

$$
[X,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)\,, \tag{94}
$$

as required, for an arbitrary k-form α.

From formula (94), we have

$$
\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha &= \pounds_X (Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + X \,\lrcorner\, d(Y \,\lrcorner\, \alpha)\big) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y \alpha - Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
[X\,,\,Y] \,\lrcorner\, \alpha &= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + (\mathrm{d}iv\,\mathbf{Y}) X \,\lrcorner\, \alpha - (\mathrm{d}iv\,\mathbf{X}) Y \,\lrcorner\, \alpha .
\end{aligned}
\tag{95}
$$

The last two steps to obtain (95) follow, because $d\alpha = 0$ and $\pounds_X \alpha = (\mathrm{d}iv\,\mathbf{X})\,\alpha$ for a top-form α.

For divergence-free vectors $\mathbf{X}$ and $\mathbf{Y}$, the last result takes the elegant form,

$$
[X\,,\,Y] \,\lrcorner\, \alpha = d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big),
\tag{96}
$$

when $\mathrm{d}iv\,\mathbf{X} = 0 = \mathrm{d}iv\,\mathbf{Y}$.

(f) The vector calculus formula to which equation (95) is equivalent may be found by writing its left and right sides in a coordinate basis, as

$$
\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha \;&=\; (\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X})\cdot d\mathbf{S} \\
d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \;&=\; -\,curl\,(\mathbf{X}\times\mathbf{Y})\cdot d\mathbf{S} + (\mathrm{d}iv\,\mathbf{Y})\,\mathbf{X}\cdot d\mathbf{S} - (\mathrm{d}iv\,\mathbf{X})\,\mathbf{Y}\cdot d\mathbf{S}
\end{aligned}
$$

Thus, equation (95) for a top-form $\alpha\, d^n x$ is equivalent to the well-known vector calculus identity

$$
(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X}) = -\,curl\,(\mathbf{X}\times\mathbf{Y}) + (\mathrm{d}iv\,\mathbf{Y})\,\mathbf{X} - (\mathrm{d}iv\,\mathbf{X})\,\mathbf{Y}\,.
$$

▲

Exercise.

(a) Starting from

$$[u\,,\,v] \lrcorner\, \alpha = \pounds_u(v \lrcorner\, \alpha) - v \lrcorner\, (\pounds_u \alpha)$$

prove the following

$$\begin{aligned}
\pounds_u(v \lrcorner\, \alpha) - \pounds_v(u \lrcorner\, \alpha) &= 2[u,v] \lrcorner\, \alpha + v \lrcorner\, \pounds_u \alpha - u \lrcorner\, \pounds_v \alpha \\
&= [u,v] \lrcorner\, \alpha - u \lrcorner\, (v \lrcorner\, \alpha) + d(u \lrcorner\, (v \lrcorner\, \alpha))
\end{aligned}$$

(b) Evaluate the last equation for a k-form α with $k = 3, 2, 1$, in terms of vector calculus expressions.

★

Answer.

(a)

$$\begin{aligned}
[u,v] \lrcorner\, \alpha &= \pounds_u(v \lrcorner\, \alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, d(v \lrcorner\, \alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, (\pounds_v \alpha - v \lrcorner\, d\alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, \pounds_v \alpha - u \lrcorner\, (v \lrcorner\, d\alpha) - v \lrcorner\, \pounds_u \alpha \\
[u,v] \lrcorner\, \alpha + u \lrcorner\, (v \lrcorner\, \alpha) &= d(u \lrcorner\, (v \lrcorner\, \alpha)) + (u \lrcorner\, \pounds_v \alpha - v \lrcorner\, \pounds_u \alpha)
\end{aligned}$$

$$\begin{aligned}
u \lrcorner\, \pounds_v \alpha - v \lrcorner\, \pounds_u \alpha &= [u,v] \lrcorner\, \alpha + u \lrcorner\, (v \lrcorner\, \alpha) - d(u \lrcorner\, (v \lrcorner\, \alpha)) \\
v \lrcorner\, \pounds_u \alpha - u \lrcorner\, \pounds_v \alpha &= [v,u] \lrcorner\, \alpha + v \lrcorner\, (u \lrcorner\, \alpha) - d(v \lrcorner\, (u \lrcorner\, \alpha)) \\
&= -[u,v] \lrcorner\, \alpha - u \lrcorner\, (v \lrcorner\, \alpha) + d(u \lrcorner\, (v \lrcorner\, \alpha))
\end{aligned}$$

$$
\begin{aligned}
\pounds_u(v \mathbin{\lrcorner} \alpha) - \pounds_v(u \mathbin{\lrcorner} \alpha) &= 2[u,v] \mathbin{\lrcorner} \alpha + v \mathbin{\lrcorner} \pounds_u \alpha - u \mathbin{\lrcorner} \pounds_v \alpha \\
&= [u,v] \mathbin{\lrcorner} \alpha - u \mathbin{\lrcorner} (v \mathbin{\lrcorner} d\alpha) + d(u \mathbin{\lrcorner} (v \mathbin{\lrcorner} \alpha))
\end{aligned}
$$

(b) For a 3-form $\alpha = d^3x$ (top form in 3D) one again finds the vector calculus identity in the previous exercise which is antisymmetric under exchange of $\mathbf{u}$ and $\mathbf{v}$,

$$
(\mathrm{d}iv\mathbf{v})\mathbf{u} - (\mathrm{d}iv\mathbf{u})\mathbf{v} - curl(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} =: [\mathbf{u}, \mathbf{v}]
$$

For a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ in 3D one finds the vector calculus identity

$$
-\mathbf{u} \times curl(\boldsymbol{\alpha} \times \mathbf{v}) + \mathbf{v} \times curl(\boldsymbol{\alpha} \times \mathbf{u}) = \boldsymbol{\alpha} \times [\mathbf{u}, \mathbf{v}] + (\mathrm{d}iv\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v})
$$

in which we denote as in the previous vector calculus identity

$$
[\mathbf{u}, \mathbf{v}] := (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} = -\,curl(\mathbf{u} \times \mathbf{v}) - (\mathrm{d}iv\mathbf{u})\mathbf{v} + (\mathrm{d}iv\mathbf{v})\mathbf{u}
$$

After the substitution of this expression for $[\mathbf{u}, \mathbf{v}]$ obtained in the case of the 3-form $\alpha = d^3x$, one sees that the vector calculus identity for a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ has cyclic permutation symmetry

$$
\mathbf{u} \times curl(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times curl(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times curl(\mathbf{u} \times \mathbf{v}) = (\mathrm{d}iv\mathbf{u})(\mathbf{v} \times \boldsymbol{\alpha}) + (\mathrm{d}iv\mathbf{v})(\boldsymbol{\alpha} \times \mathbf{u}) + (\mathrm{d}iv\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v})
$$

Also, in the divergence-free case this reduces to

$$
curl\Big(\mathbf{u} \times curl(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times curl(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times curl(\mathbf{u} \times \mathbf{v})\Big) = 0.
$$

For a 1-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{x}$, the result turns out to be

$$
(\mathbf{u} \cdot \nabla)(\mathbf{v} \cdot \boldsymbol{\alpha}) - (\mathbf{v} \cdot \nabla)(\mathbf{u} \cdot \boldsymbol{\alpha}) = \big((\mathbf{u} \cdot \nabla)\mathbf{v}\big) \cdot \boldsymbol{\alpha} - \big((\mathbf{v} \cdot \nabla)\mathbf{u}\big) \cdot \boldsymbol{\alpha} - \mathbf{v} \cdot (\mathbf{u} \times curl\boldsymbol{\alpha})
$$

or, equivalently,

$$
\big((\mathbf{u} \cdot \nabla)\boldsymbol{\alpha}\big) \cdot \mathbf{v} - \big((\mathbf{v} \cdot \nabla)\boldsymbol{\alpha}\big) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot curl\boldsymbol{\alpha}
$$

▲

LECTURE #9

This lecture begins our discussion of the geometric mechanical properties of the Euler fluid equations for ideal incompressible flow. In particular, it derives the Euler-Poincaré representation of the Euler fluid equations. This derivation sets the Euler fluid equations onto the cube of relationships in the framework of geometric mechanics. Many of their properties then can be derived by simply following along the edges of cubic commutative diagram of relationships in geometric mechanics.

One question that could be asked is, "How general are these geometric properties of ideal fluid dynamics for other applications? For example, do these properties persist in plasma physics?" That is indeed a worthy question!

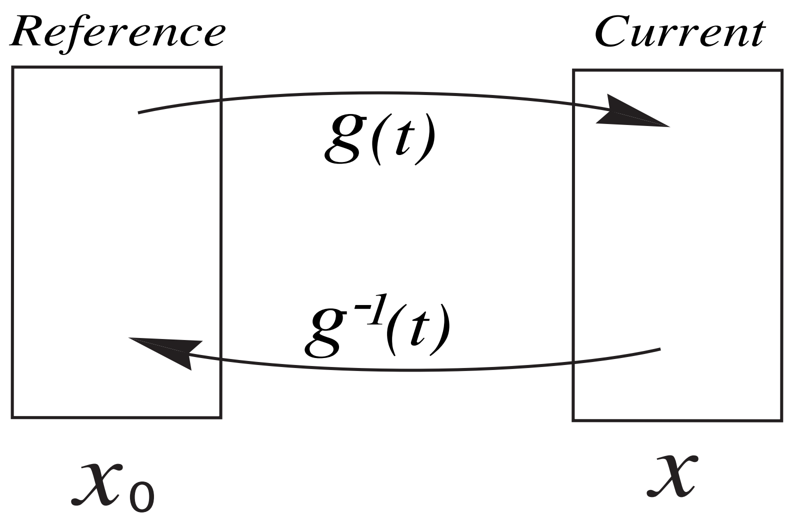
Go2toc

11 Hamilton's principle for geometry of ideal fluid dynamics

11.1 Advected quantities in ideal fluid dynamics

We regard fluid flow as a smooth invertible time-dependent transformation of initial conditions x_0 regarded as fluid labels taking values in a configuration manifold M acted on by smooth invertible maps $\mathrm{D}iff(M)$. Thus, we lift the motion of fluid parcels $x_t \in M$ with initial condition $x_0 \in M$ to the manifold of diffeomorphisms by identifying it with a time-dependent curve $g_t \in \mathrm{D}iff(M)$ with $g_0 = Id$, whose action from the left generates the motion x_t,

$$x_t = g_t x_0 \quad \text{with} \quad \dot{x}_t = \dot{g}_t x_0 = (u_t \circ g_t) x_0$$



Advected quantity. A quantity $a_t(x_t) = a_0(x_0)$ which remains invariant under the flow is said to be *advected* by the flow. In terms of the group action, advected quantities satisfy

$$a_0(x_0) = a_t(x_t) = (a_t \circ g_t)(x_0) = (g_t^* a_t)(x_0)$$

where $g_t^* a_t$ is the *pull-back* of a_t by g_t. Invariance of an advected quantity implies an evolution equation

$$0 = \frac{d}{dt} a_0(x_0) = \frac{d}{dt}(g_t^* a_t)(x_0) = g_t^*\big((\partial_t + \pounds_u)a_t\big)(x_0) = (\partial_t + \pounds_u)a_t(x_t)$$

where $\pounds_u$ denotes the Lie derivative with respect to the vector field $u = \dot{g}g^{-1}$ which generates the flow g_t. Back2Outlook

Vice versa, we have the *push-forward* relation

$$\frac{d}{dt} a_t(x_t) = \frac{d}{dt}(a_0 g_t^{-1})(x_t) = \frac{d}{dt}\big((g_t)_* a_0\big)(x_t) = -(\pounds_u a_t)(x_t)\,.$$

The previous formula will be useful in taking variations of advected quantities in Hamilton's principle, since it implies the following formula for the variation of an advected quantity, a_t at fixed t,

$$\delta a_t(x_t) = a_t'(x_t) := \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_{t,\epsilon}\Big)(x_t) = -(\pounds_{v_t} a_t)(x_t) \quad \text{where} \quad v_t = \Big(\frac{dg_{t,\epsilon}}{d\epsilon} g_{t,\epsilon}^{-1}\Big)\Big|_{\epsilon=0} = g' g^{-1}$$

where $\pounds_v$ denotes the Lie derivative with respect to the vector field $v = [g_\epsilon' g_\epsilon^{-1}]_{\epsilon=0}$ which generates the flow g_ϵ. Equality of cross derivatives in t and ϵ implies the following pair of relations

$$\begin{aligned}
(\dot{g})' \circ g^{-1} &= (u \circ g)' \circ g^{-1} = (\partial_x u) g' \circ g^{-1} + (u' \circ g) \circ g^{-1} \\
&= (\partial_x u) v + u' \\
(g')^{\cdot} \circ g^{-1} &= (\partial_x v) u + \dot{v}\,,
\end{aligned}$$

from which we conclude upon substituting $u = \dot{g}g^{-1}$ that

$$\delta(\dot{g}g^{-1}) = (\dot{g}g^{-1})' = \dot{v} + (\partial_x v)u - (\partial_x u)v = \dot{v} - [u, v] = \dot{v} - \mathrm{ad}_u v = \dot{v} - \mathrm{ad}_{\dot{g}g^{-1}} v$$

Now we are ready to compute the Euler-Poincaré equations for fluid dynamics.

11.2 Euler-Poincaré equations for fluid dynamics

We shall compute the compute the Euler-Poincaré equations for fluid dynamics using the Hamilton-Pontryagin principle,

$$
\begin{aligned}
0 = \delta S &= \delta \int_0^T \ell(u, a_0 g_t^{-1}) + \langle m\,,\, \dot{g}g^{-1} - u\rangle \, dt \\
&= \int_0^T \left\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a}\,,\, -\pounds_v a \right\rangle + \left\langle m\,,\, \partial_t v - \mathrm{ad}_{\dot{g}g^{-1}} v \right\rangle + \left\langle \delta m\,,\, \dot{g}g^{-1} - u \right\rangle dt \\
&= \int_0^T \left\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a} \diamond a - \partial_t m - \mathrm{ad}^*_{\dot{g}g^{-1}} m\,,\, v \right\rangle + \left\langle \delta m\,,\, \dot{g}g^{-1} - u \right\rangle dt + \langle m\,,\, v\rangle \Big|_0^T ,
\end{aligned}
$$

where we have used $\delta(a_0 g_t^{-1}) = -\pounds_v a$ and have defined the diamond operator $(\diamond)$ as

$$
\diamond : V^* \times V \to \mathfrak{X}^* \quad \text{defined by} \quad \left\langle \frac{\delta \ell}{\delta a} \diamond a\,,\, v \right\rangle := \left\langle \frac{\delta \ell}{\delta a}\,,\, -\pounds_v a \right\rangle
$$

and the ad^* operation as

$$
\mathrm{a}d^* : \mathfrak{X} \times \mathfrak{X}^* \to \mathfrak{X}^* \quad \text{defined by} \quad \langle \mathrm{ad}^*_u m\,,\, v\rangle = \left\langle m\,,\, \mathrm{ad}_u v \right\rangle
$$

In particular, $\mathrm{ad}^*_u m = \pounds_u m$, so that the fluid motion equation for $m = \mathbf{m}\cdot d\mathbf{x} \otimes d^3x$ and advection equations become

$$
(\partial_t + \pounds_u) m = \frac{\delta \ell}{\delta a} \diamond a \quad \text{and} \quad (\partial_t + \pounds_u) a = 0
$$

In general, fluid motion advects mass, so that $D_t(x_t) d^3x_t = D_0(x_0) d^3x_0$, which implies the continuity equation

$$
0 = (\partial_t + \pounds_u)(D_t(x_t) d^3x_t) = \big(\partial_t D + \mathrm{d}iv(D\mathbf{u})\big) d^3x
$$

Consequently, the motion equation may be rewritten as

$$\big(\partial_t + \pounds_u\big)(D^{-1}\mathbf{m}\cdot d\mathbf{x}) = \frac{1}{D}\frac{\delta \ell}{\delta a}\diamond a$$

in which $\frac{1}{D}\frac{\delta \ell}{\delta a}\diamond a$ is a 1-form. Integrating this relation around a material loop c_t moving with the fluid yields

$$\frac{d}{dt}\oint_{c_t}(D^{-1}\mathbf{m}\cdot d\mathbf{x}) = \oint_{c_t}\frac{1}{D}\frac{\delta \ell}{\delta a}\diamond a$$

This is the Kelvin-Noether theorem, which arises from relabelling symmetry of the Lagrangian fluid parcels.

11.3 Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ satisfying $div\,\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $curl\,\mathbf{R} = 2\boldsymbol{\Omega}$ are given in the form of Newton's law of force by

$$\underbrace{\partial_t\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}}\,. \tag{97}$$

> **Exercise.** Prove that Euler's equations in a rotating frame arise as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.
>
> $$0 = \delta S = \int_0^T \frac{1}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R} - p(D-1)\,d^3x\,dt$$
>
> ★

The Newton's law equation for Euler fluid motion in (97) may be rearranged into an alternative form,

$$\partial_t\,\mathbf{v} - \mathbf{u}\times\boldsymbol{\omega} + \nabla\Big(p + \frac{1}{2}|\mathbf{u}|^2\Big) = 0\,, \tag{98}$$

by denoting

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,, \quad \boldsymbol{\omega} = curl\,\mathbf{v} = curl\,\mathbf{u} + 2\boldsymbol{\Omega}\,, \tag{99}$$

and using the fundamental vector calculus identity of fluid dynamics

$$\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j = -\,\mathbf{u}\times curl\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\,. \tag{100}$$

This identity follows from equality of the dynamic and geometric definitions of the Lie derivative $\pounds_u\alpha$ of a k-form $\alpha\in\Lambda^k(M)$ by the vector field $u = \dot{g}g^{-1}$ tangent to the flow g_t on M as

$$\pounds_u\alpha \;=\; \frac{d}{dt}\Big|_{t=0}(g_t^*\alpha) = u \,\lrcorner\, d\alpha + d(u \,\lrcorner\, \alpha), \tag{101}$$

in which the last equality is Cartan's geometric formula for the Lie derivative.
For the case of the circulation 1-form $\alpha = \mathbf{v}\cdot d\mathbf{x}$, this becomes

$$\begin{aligned}
\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \big(\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x}\\
&= u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x})\\
&= u \,\lrcorner\, (curl\,\mathbf{v}\cdot d\mathbf{S}) + d(\mathbf{u}\cdot\mathbf{v})\\
&= \big(-\,\mathbf{u}\times curl\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,
\end{aligned} \tag{102}$$

and the identity (99) emerges. This identity and the calculation (102) recasts Euler's fluid motion equation into the following geometric form:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) &= (\partial_t\mathbf{v} -\,\mathbf{u}\times curl\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v}))\cdot d\mathbf{x}\\
&= -\,\nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x}\\
&= -\,d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right).
\end{aligned} \tag{103}$$

Requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u}=0$ results in a Poisson equation for pressure p, which may be written in several equivalent forms,

$$\begin{aligned} -\Delta p &= \mathrm{d}iv\big(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}\times 2\boldsymbol{\Omega}\big)\\ &= u_{i,j}u_{j,i}-\mathrm{d}iv\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big)\\ &= \mathrm{t}r\,\mathsf{S}^2-\frac{1}{2}|curl\,\mathbf{u}|^2-\mathrm{d}iv\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big)\,, \end{aligned} \tag{104}$$

where $\mathsf{S}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the ***strain-rate tensor***.
We introduce the ***Lamb vector***,

$$\boldsymbol{\ell}:=-\,\mathbf{u}\times\boldsymbol{\omega}\,, \tag{105}$$

which represents the nonlinearity in Euler's fluid equation (98). The Poisson equation (104) for pressure p may now be expressed in terms of the divergence of the Lamb vector,

$$-\,\Delta\left(p+\frac{1}{2}|\mathbf{u}|^2\right)=\mathrm{d}iv(-\,\mathbf{u}\times curl\,\mathbf{v})=\mathrm{d}iv\,\boldsymbol{\ell}\,. \tag{106}$$

Remark 89 (***Boundary conditions).***
Because the velocity $\mathbf{u}$ *must be tangent to any fixed boundary, the normal component of the motion equation must vanish. This requirement produces a Neumann condition for pressure given by*

$$\partial_n\left(p+\frac{1}{2}|\mathbf{u}|^2\right)+\hat{\mathbf{n}}\cdot\boldsymbol{\ell}=0\,, \tag{107}$$

at a fixed boundary with unit outward normal vector $\hat{\mathbf{n}}$.

Remark 90 (***Helmholtz vorticity dynamics).***
Taking the curl of the Euler fluid equation (98) yields the ***Helmholtz vorticity equation***

$$\partial_t\,\boldsymbol{\omega}-curl(\mathbf{u}\times\boldsymbol{\omega})=0\,, \tag{108}$$

whose geometrical meaning will emerge in discussing Stokes' Theorem 106 for the vorticity of a rotating fluid.

The rotation terms have now been fully integrated into both the dynamics and the boundary conditions. In this form, the ***Kelvin circulation theorem*** and the ***Stokes vorticity theorem*** will emerge naturally together as geometrical statements.

11.4 Kelvin's circulation theorem

Theorem 91 (***Kelvin's circulation theorem*).** *The Euler equations (97) preserve the circulation integral $I(t)$ defined by*

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x}\,, \tag{109}$$

where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity $\mathbf{u}$.

Proof. The dynamical definition of the Lie derivative in (101) yields the following for the time rate of change of this circulation integral:

$$\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v} \cdot d\mathbf{x}) \\
&= \oint_{c(\mathbf{u})} \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x^j}u^j + v_j\frac{\partial u^j}{\partial \mathbf{x}}\right) \cdot d\mathbf{x} \\
&= -\oint_{c(\mathbf{u})} \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v}\right) \cdot d\mathbf{x} \\
&= -\oint_{c(\mathbf{u})} d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v}\right) = 0\,.
\end{aligned} \tag{110}$$

The last step in the proof follows, because the integral of an exact differential around a closed loop vanishes. □

The exterior derivative of the Euler fluid equation in the form (103) yields Stokes' theorem, after using the commutativity of the exterior and Lie derivatives $[d,\, \pounds_{\mathbf{u}}] = 0$,

$$
\begin{aligned}
d\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= d\Big(- \mathbf{u}\times curl\mathbf{v}\cdot d\mathbf{x} + d(\mathbf{u}\cdot\mathbf{v})\Big) \\
&= \pounds_{\mathbf{u}}(curl\,\mathbf{v}\cdot d\mathbf{S}) \\
&= -\,curl\big(\mathbf{u}\times curl\,\mathbf{v}\big)\cdot d\mathbf{S} \\
&= \Big[\mathbf{u}\cdot\nabla curl\,\mathbf{v} + curl\,\mathbf{v}(\mathrm{d}iv\,\mathbf{u}) - (curl\,\mathbf{v})\cdot\nabla\mathbf{u}\Big]\cdot d\mathbf{S}\,, \\
(\text{by } \mathrm{d}iv\,\mathbf{u}=0) \quad &= \Big[\mathbf{u}\cdot\nabla curl\,\mathbf{v} - (curl\,\mathbf{v})\cdot\nabla\mathbf{u}\Big]\cdot d\mathbf{S} \\
&=: \big[\,u,\, curl\,v\,\big]\cdot d\mathbf{S}\,,
\end{aligned}
\tag{111}
$$

where $[\,u,\, curl\,v\,]$ denotes (minus) the ***Jacobi–Lie bracket*** of the vector fields u and $curl\,v$.
This calculation proves the following.

Theorem 92. *Euler's fluid equations (98) imply that*

$$
\frac{\partial\omega}{\partial t} = -\,[\,u,\,\omega\,]
\tag{112}
$$

where $[\,u,\,\omega\,]$ denotes the Jacobi–Lie bracket of the divergenceless vector fields u and $\omega := curl\,v$.

The exterior derivative of Euler's equation in its geometric form (103) is equivalent to the curl of its vector form (98). That is,

$$
d\Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(\mathbf{v}\cdot d\mathbf{x}) = \Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(curl\,\mathbf{v}\cdot d\mathbf{S}) = 0\,.
\tag{113}
$$

Hence from the calculation in (111) and the Helmholtz vorticity equation (113) we have

$$
\Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(curl\,\mathbf{v}\cdot d\mathbf{S}) = \Big(\partial_t\,\boldsymbol{\omega} - curl(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} = 0\,,
\tag{114}
$$

in which one denotes $\boldsymbol{\omega} := curl\,\mathbf{v}$. This Lie-derivative version of the Helmholtz vorticity equation may be used to prove the following form of Stokes' theorem for the Euler equations in a rotating frame.

Theorem 93. ***[Kelvin-Stokes theorem for vorticity of a rotating fluid]***

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \frac{d}{dt}\iint_{S(\mathbf{u})} curl\,\mathbf{v}\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(curl\,\mathbf{v}\cdot d\mathbf{S}) \\
&= \iint_{S(\mathbf{u})} \Big(\partial_t\,\boldsymbol{\omega} - curl\,(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} = 0\,,
\end{aligned}
\tag{115}
$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid.

LECTURE #10

This lecture continues our discussion of the equations of the ideal Euler fluid equations in the framework of geometric mechanics. It mainly discusses the symplectic nature of steady Euler flows in 3D and the dynamical preservation of their topology, particularly the linkages of their vortex field lines.

Upon recognising these geometric connections in ideal fluid dynamics, many questions arise. In particular, one can ask, "Why is this topology preserved?" Of course, the answer lies in the further developments of ideal fluid dynamics based on geometric mechanics. And those developments are still being pursued. See, for example, [ArKh2008] for some of the basic elements.

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11.5 Steady Euler solutions: Lamb surfaces, linking numbers

According to Theorem 92, Euler's fluid equations (98) imply that

$$\frac{\partial\omega}{\partial t} = -\,[\,u,\,\omega\,]\,. \tag{116}$$

Consequently, the vector fields u, ω in *steady* Euler flows, which satisfy $\partial_t\omega = 0$, also satisfy the condition necessary for the Frobenius theorem to hold – namely, that their Jacobi–Lie bracket vanishes. That is, in smooth steady, or equilibrium, solutions of Euler's fluid equations, the flows of the two divergenceless vector fields u and ω commute with each other and lie on a surface in three dimensions.
A sufficient condition for this commutation relation is that the ***Lamb vector*** $\boldsymbol{\ell} := -\,\mathbf{u}\times curl\,\mathbf{v}$ in (105) satisfies

$$\boldsymbol{\ell} := -\,\mathbf{u}\times curl\,\mathbf{v} = \nabla H(\mathbf{x})\,, \tag{117}$$

for some smooth function $H(\mathbf{x})$. This condition means that the flows of vector fields u and $curl\,v$ (which are steady flows of the Euler equations) are both confined to the same surface $H(\mathbf{x}) = const$. Such a surface is called a ***Lamb surface***.
The vectors of velocity $(\mathbf{u})$ and total vorticity $(curl\,\mathbf{v})$ for a steady Euler flow are both perpendicular to the normal vector to the Lamb surface along $\nabla H(\mathbf{x})$. That is, the Lamb surface is invariant under the flows of both vector fields, *viz*

$$\pounds_u H = \mathbf{u}\cdot\nabla H = 0 \quad\text{and}\quad \pounds_{curl\,v} H = curl\,\mathbf{v}\cdot\nabla H = 0\,. \tag{118}$$

The Lamb surface condition (117) has the following coordinate-free representation.

Theorem 94 (***Lamb surface condition***)**.** *The Lamb surface condition (117) is equivalent to the following double substitution of vector fields into the volume form,*

$$dH = u \,\lrcorner\, curl\,v \,\lrcorner\, d^3x\,. \tag{119}$$

Proof. Recall that the contraction of vector fields with forms yields the following useful formula for the surface element:

$$\nabla \lrcorner\, d^{\,3}x = d\mathbf{S}\,. \tag{120}$$

Then using results from previous exercises in vector calculus operations one finds by direct computation that

$$\begin{aligned} u \lrcorner\, curl\, v \lrcorner\, d^{\,3}x &= u \lrcorner\, (curl\,\mathbf{v}\cdot d\mathbf{S}) \\ &= -\big(\mathbf{u}\times curl\,\mathbf{v}\big)\cdot d\mathbf{x} \\ &= \nabla H\cdot d\mathbf{x} \\ &= dH\,. \end{aligned} \tag{121}$$

□

Remark 95.
Formula (121)

$$u \lrcorner\, (curl\,\mathbf{v}\cdot d\mathbf{S}) = dH$$

is to be compared with

$$X_h \lrcorner\, \omega = dH\,,$$

in the definition of a Hamiltonian vector field in Equation (18) of Theorem 20. Likewise, the stationary case of the Helmholtz vorticity equation (113), namely,

$$\pounds_{\mathbf{u}}(curl\,\mathbf{v}\cdot d\mathbf{S}) = 0\,, \tag{122}$$

is to be compared with the proof of Poincaré's theorem in Corollary 23

$$\pounds_{X_h}\omega = d(X_h \lrcorner\, \omega) = d^2H = 0\,.$$

Thus, the two-form $curl\,\mathbf{v}\cdot d\mathbf{S}$ *plays the same role for stationary Euler fluid flows as the symplectic form* $dq\wedge dp$ *plays for canonical Hamiltonian flows. We seek the corresponding symplectic coordinates.*

Definition 96. *The **Clebsch representation** of the one-form* $\mathbf{v}\cdot d\mathbf{x}$ *is defined by*

$$\mathbf{v}\cdot d\mathbf{x} = -\,\Pi\, d\Xi + d\Psi\,. \tag{123}$$

The functions Ξ, Π *and* Ψ *are called **Clebsch potentials** for the vector* $\mathbf{v}$.[3]

In terms of the Clebsch representation (123) of the one-form $\mathbf{v}\cdot d\mathbf{x}$, the total vorticity flux $curl\,\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x})$ is the exact two-form,

$$curl\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi\,. \tag{124}$$

This amounts to writing the flow lines of the *vector field* of the total vorticity $curl\,v$ as the intersections of level sets of surfaces $\Xi = const$ and $\Pi = const$. In other words,

$$curl\,\mathbf{v} = \nabla\Xi\times\nabla\Pi\,, \tag{125}$$

with the assumption that these level sets foliate $\mathbb{R}^3$. That is, one assumes that any point in $\mathbb{R}^3$ along the flow of the total vorticity vector field $curl\,v$ may be assigned to a regular intersection of these level sets. The main result of this assumption is the following theorem.

[3] The Clebsch representation is another example of a cotangent lift momentum map.

Theorem 97 (***Lamb surfaces are symplectic manifolds***)**.** *[ArKh1992, ArKh2008] The steady flow of the vector field u satisfying the symmetry relation given by the vanishing of the commutator $[\,u,\, curl\, v\,] = 0$ on a three-dimensional manifold $M \in \mathbb{R}^3$ reduces to incompressible flow on a two-dimensional symplectic manifold whose canonically conjugate coordinates $(\Xi,\, \Pi)$ are provided by the total vorticity flux*

$$
curl\, v \;\lrcorner\; d^{\,3}x = curl\, \mathbf{v}\cdot d\mathbf{S} = d\Xi \wedge d\Pi\,.
$$

The reduced flow is canonically Hamiltonian on this symplectic manifold. Furthermore, the reduced Hamiltonian is precisely the restriction of the invariant H onto the reduced phase space.

Proof. Restricting formula (121) to coordinates on a total vorticity flux surface (124) yields the exterior derivative of the Hamiltonian,

$$
\begin{aligned}
dH(\Xi,\, \Pi) &= u \;\lrcorner\; (curl\, \mathbf{v}\cdot d\mathbf{S}) \\
&= u \;\lrcorner\; (d\Xi \wedge d\Pi) \\
&= (\mathbf{u}\cdot\nabla\Xi)\, d\Pi - (\mathbf{u}\cdot\nabla\Pi)\, d\Xi \\
&=: \frac{d\Xi}{dT}\, d\Pi - \frac{d\Pi}{dT}\, d\Xi \\
&= \frac{\partial H}{\partial \Pi}\, d\Pi + \frac{\partial H}{\partial \Xi}\, d\Xi\,,
\end{aligned}
\tag{126}
$$

where $T \in \mathbb{R}$ is the time parameter along the flow lines of the steady vector field u, which carries the Lagrangian fluid parcels. On identifying corresponding terms, the steady flow of the fluid velocity $\mathbf{u}$ is found to obey the canonical Hamiltonian equations,

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\Xi) &= \pounds_u \Xi =: \frac{d\Xi}{dT} = \frac{\partial H}{\partial \Pi} = \big\{\Xi,\, H\big\}\,, \\
(\mathbf{u}\cdot\nabla\Pi) &= \pounds_u \Pi =: \frac{d\Pi}{dT} = -\,\frac{\partial H}{\partial \Xi} = \big\{\Pi,\, H\big\}\,,
\end{aligned}
\tag{127}
$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket for the symplectic form $d\Xi \wedge d\Pi$. $\square$

Corollary 98 (Invariance of the symplectic form)**.**
The vorticity flux

$$curl\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi$$

is invariant under the flow of the velocity vector field u.

Proof. By (126), one verifies

$$\pounds_u(d\Xi\wedge d\Pi) = d\big(u \,\lrcorner\, (d\Xi\wedge d\Pi)\big) = d^2H = 0\,.$$

This is the standard computation in the proof of Poincaré's theorem in Corollary 23 for the preservation of a symplectic form by a canonical transformation. Its interpretation here is that the steady Euler flows preserve the total vorticity flux, $curl\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi$. □

11.6 Conserved linking numbers in dynamics of ideal incompressible flows

Definition 99 (Helicity)**.** *The helicity $\Lambda[curl\,\mathbf{v}]$ of a divergence-free vector field $\omega := curl\,\mathbf{v}$ in a domain whose boundary conditions admit the relation $curl^{-1}\omega := \mathbf{v}$ with $div\,\mathbf{v}=0$ is defined as*

$$\Lambda[curl\,\mathbf{v}] = \int_D \mathbf{v}\cdot curl\,\mathbf{v}\,d^{\,3}x = \int_D \mathbf{v}\cdot d\mathbf{x}\wedge d(\mathbf{v}\cdot d\mathbf{x})\,. \tag{128}$$

Exercise. Can Lamb flows possess helicity? Prove it. ★

Remark 100. *The helicity of a vector field curl* **v** *measures the total linking of its field lines, or their relative winding. (For details and mathematical history, see [ArKh2008].) The idea of helicity goes back to Helmholtz and Kelvin in the 19th century. The principal feature of this concept for fluid dynamics is embodied in the following theorem.*

Theorem 101 (***Euler flows preserve helicity***)**.** *When homogeneous or periodic boundary conditions are imposed, Euler's equations for an ideal incompressible fluid flow in a rotating frame with Coriolis parameter $curl\,\mathbf{R} = 2\boldsymbol{\Omega}$*

preserves the helicity

$$\Lambda[curl\,\mathbf{v}] = \int_D \mathbf{v}\cdot curl\,\mathbf{v}\, d^{\,3}x\,, \tag{129}$$

with $\mathbf{v} = \mathbf{u} + \mathbf{R}$*, for which* $\mathbf{u}$ *is the divergenceless fluid velocity* $(div\,\mathbf{u} = 0)$ *and* $curl\,\mathbf{v} = curl\,\mathbf{u} + 2\boldsymbol{\Omega}$ *is the total vorticity.*

Proof. Rewrite the geometric form of the Euler equations (103) for rotating incompressible flow with unit mass density in terms of the circulation one-form $v := \mathbf{v}\cdot d\mathbf{x}$ as

$$\big(\partial_t + \pounds_u\big)v = -\,d\,\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) =: -\,d\,\varpi\,, \tag{130}$$

and $\pounds_u\, d^{\,3}x = 0$. Here, ϖ is an augmented pressure variable,

$$\varpi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\,. \tag{131}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ with $\mathrm{d}iv\,\mathbf{u} = 0$. Then the ***helicity density***, defined as

$$v\wedge dv = \mathbf{v}\cdot curl\,\mathbf{v}\, d^{\,3}x = \lambda\, d^{\,3}x\,, \quad \text{with} \quad \lambda = \mathbf{v}\cdot curl\,\mathbf{v}\,, \tag{132}$$

obeys the dynamics it inherits from the Euler equations,

$$\big(\partial_t + \pounds_u\big)(v\wedge dv) = -d\varpi\wedge dv - v\wedge d^2\varpi = -d(\varpi\, dv)\,, \tag{133}$$

after using $d^2\varpi = 0$ and $d^2 v = 0$. In vector form, this result may be expressed as a conservation law,

$$\big(\partial_t\lambda + \mathrm{div}\,\lambda\mathbf{u}\big)d^{\,3}x = -\,\mathrm{d}iv(\varpi\, curl\,\mathbf{v})\, d^{\,3}x\,. \tag{134}$$

Consequently, the time derivative of the integrated helicity in a domain D obeys

$$\begin{aligned}\frac{d}{dt}\Lambda[curl\,\mathbf{v}] &= \int_D \partial_t\lambda\, d^{\,3}x = -\int_D \mathrm{div}(\lambda\mathbf{u} + \varpi\, curl\,\mathbf{v})\, d^{\,3}x \\ &= -\oint_{\partial D}(\lambda\mathbf{u} + \varpi\, curl\,\mathbf{v})\,\cdot\, d\mathbf{S}\,, \end{aligned} \tag{135}$$

which vanishes when homogeneous, or periodic, or even Neumann boundary conditions are imposed on the values of $\mathbf{u}$ and $curl\,\mathbf{v}$ at the boundary ∂D. □

Remark 102.
*This result means the **helicity integral***

$$\Lambda[curl\,\mathbf{v}] = \int_D \lambda\, d^{\,3}x$$

is conserved in periodic domains, or in all of $\mathbb{R}^3$ with vanishing boundary conditions at spatial infinity. However, if either the velocity or total vorticity at the boundary possesses a nonzero normal component, then the boundary is a source *of helicity (that is, it causes winding of field lines of curl* $\mathbf{v}$*). For a fixed impervious boundary, the normal component of velocity does vanish, but no such condition is imposed on the total vorticity by the physics of fluid flow. Thus, we have the following.*

Corollary 103. *A flux of total vorticity curl* $\mathbf{v}$ *into the domain is a source of helicity.*

Exercise. Use Cartan's formula in (101) to compute $\pounds_u(v \wedge dv)$ in Equation (133). ★

Exercise. Compute the helicity for the one-form $v = \mathbf{v} \cdot d\mathbf{x}$ in the Clebsch representation (123). What does this mean for the linkage of the vortex lines that admit the Clebsch representation? ★

Theorem 104 (Diffeomorphisms preserve helicity)**.** *The helicity* $\Lambda[\xi]$ *of any divergenceless vector field* ξ *is preserved under the action on* ξ *of any volume-preserving diffeomorphism of the manifold* M *[ArKh2008].*

Remark 105 (Helicity is a topological invariant)**.**
The helicity $\Lambda[\xi]$ *is a topological invariant, not a dynamical invariant, because its invariance is independent of which diffeomorphism acts on* ξ*. This means the invariance of helicity is independent of which Hamiltonian flow produces the diffeomorphism. This is the hallmark of a Casimir function. Although it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a divergenceless vector field* ξ *into another such field with the same helicity. However, independently of any metric properties, the action of diffeomorphisms does not create or destroy linkages of the characteristic curves of divergenceless vector fields.*

LECTURE #11

This lecture lays the ground work for modelling other ideal fluid dynamics equations in the framework of geometric mechanics.

One question is, "What's next for geometric mechanics of fluid flows?" One answer is that all of the deterministic considerations of geometric mechanics also transfer to its stochastic version. The basic elements of the stochastic version of fluid dynamics in the framework of geometric mechanics are discussed in [Ho2015].

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11.7 Ertel theorem for potential vorticity

Euler–Boussinesq equations The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity $\mathbf{u}$ satisfying $div\,\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $curl\,\mathbf{R} = 2\boldsymbol{\Omega}$ are given by

$$\underbrace{\partial_t\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{-\,gb\nabla z}_{\text{buoyancy}} + \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}} \tag{136}$$

where $-g\nabla z$ is the constant downward acceleration of gravity and b is the buoyancy, a scalar function of space and time which satisfies the **advection relation**,

$$\partial_t\,b + \mathbf{u}\cdot\nabla b = 0\,. \tag{137}$$

As for Euler's equations without buoyancy, requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u} = 0$ results in a Poisson equation for pressure p,

$$-\,\Delta\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = div(-\,\mathbf{u}\times curl\,\mathbf{v}) + g\partial_z b\,, \tag{138}$$

which satisfies a Neumann boundary condition because the velocity $\mathbf{u}$ must be tangent to the boundary, where we denote

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,,\quad \boldsymbol{\omega} = curl\,\mathbf{v} = curl\,\mathbf{u} + 2\boldsymbol{\Omega}\,, \tag{139}$$

The Newton's law form of the Euler–Boussinesq equations (136) may be rearranged as

$$\partial_t\,\mathbf{v} -\,\mathbf{u}\times curl\,\mathbf{v} + gb\nabla z + \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = 0\,, \tag{140}$$

where $\mathbf{v} \equiv \mathbf{u} + \mathbf{R}$ and $\nabla\cdot\mathbf{u} = 0$.

Exercise. Prove that the Euler–Boussinesq equations in (136) emerge as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.

$$0 = \delta S = \delta \int_0^T \frac{1}{2} D|\mathbf{u}|^2 + D\mathbf{u} \cdot \mathbf{R} - Dbz - p(D-1)\, d^3x\, dt$$

★

Theorem 106. ***[Kelvin-Stokes theorem for vorticity of a stratified, rotating fluid]***

$$\begin{aligned} \frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} &= \frac{d}{dt} \iint_{S(\mathbf{u})} curl\, \mathbf{v} \cdot d\mathbf{S} \\ &= \iint_{S(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \right) (curl\, \mathbf{v} \cdot d\mathbf{S}) \\ &= \iint_{S(\mathbf{u})} \Big(\partial_t \boldsymbol{\omega} - curl\, (\mathbf{u} \times \boldsymbol{\omega}) \Big) \cdot d\mathbf{S} \\ &= \iint_{S(\mathbf{u})} \Big(- g \nabla b \times \nabla z \Big) \cdot d\mathbf{S} \,, \end{aligned} \tag{141}$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Thus, non-alignment of the gradient of buoyancy ∇b with the vertical ∇z creates circulation. Compare this result with equation (115) in the absence of stratification.

Geometrically, equation (140) may be written as

$$\big(\partial_t + \pounds_u\big) v + gbdz + d\varpi = 0 \,, \tag{142}$$

where ϖ is defined in (131). In addition, the buoyancy satisfies

$$\big(\partial_t + \pounds_u\big) b = 0 \,, \quad \text{with} \quad \pounds_u\, d^3x = 0 \,. \tag{143}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ and the circulation one-form as $v = \mathbf{v}\cdot d\mathbf{x}$. The exterior derivatives of the two equations in (142) are written as

$$\big(\partial_t + \pounds_u\big)dv = -g db\wedge dz \quad\text{and}\quad \big(\partial_t + \pounds_u\big)db = 0\,. \tag{144}$$

Consequently, one finds from the product rule for Lie derivatives (85) that

$$\big(\partial_t + \pounds_u\big)(dv\wedge db) = 0 \quad\text{or}\quad \partial_t\, q + \mathbf{u}\cdot\nabla q = 0\,, \tag{145}$$

in which the quantity

$$q = \nabla b\cdot curl\,\mathbf{v}\,, \tag{146}$$

is called ***potential vorticity*** and is abbreviated as PV. The potential vorticity is an important diagnostic for many processes in geophysical fluid dynamics. Conservation of PV on fluid parcels is called ***Ertel's theorem***.

Remark 107 (***Ertel's theorem for the vorticity vector field).***
Writing the vorticity vector field $\omega = \boldsymbol{\omega}\cdot\nabla$*, we have*

$$\big(\partial_t + \pounds_u\big)\omega = \partial_t\omega + [u,\,\omega] = g\nabla z\times\nabla b\cdot\nabla\,.$$

Thus, conservation of the potential vorticity may also be proved by the product rule, as

$$\big(\partial_t + \pounds_u\big)q = \big(\partial_t + \pounds_u\big)(\boldsymbol{\omega}\cdot\nabla b) = \big(\partial_t + \pounds_u\big)(\omega b) = \Big(\big(\partial_t + \pounds_u\big)\omega\Big)b + \omega\big(\partial_t + \pounds_u\big)b = 0\,.$$

Remark 108 (***Material derivative formulation).***
Denoting

$$\frac{D}{Dt} = \partial_t + \pounds_u \quad\textit{and}\quad \omega = \boldsymbol{\omega}\cdot\nabla$$

provides an intuitive expression of the Ertel theorem (145) that helps understand it in terms of the time derivative $\frac{D}{Dt}$ *following the flow of the fluid particles. Namely, it suggests writing in vector form*

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla) = g\nabla z\times\nabla b\cdot\nabla \quad\textit{and}\quad \frac{Db}{Dt} = 0\,,$$

so that the product rule for derivatives yields conservation of PV on fluid parcels, as

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla b) = \Big(\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla)\Big)b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = g\nabla z\times\nabla b\cdot\nabla b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = 0\,.$$

Remark 109 (***The conserved quantities associated with Ertel's theorem).***
The constancy of the scalar quantities b and q on fluid parcels implies conservation of the spatially integrated quantity,

$$C_\Phi = \int_D \Phi(b,q)\,d^{\,3}x\,, \tag{147}$$

for any smooth function Φ for which the integral exists.

Proof.

$$\begin{aligned}\frac{d}{dt}C_\Phi &= \int_D \Phi_b\partial_t b + \Phi_q\partial_t q\,d^{\,3}x = -\int_D \Phi_b\mathbf{u}\cdot\nabla b + \Phi_q\mathbf{u}\cdot\nabla q\,d^{\,3}x\\ &= -\int_D \mathbf{u}\cdot\nabla\Phi(b,q)\,d^{\,3}x = -\int_D \nabla\cdot\big(\mathbf{u}\,\Phi(b,q)\big)\,d^{\,3}x = -\oint_{\partial D}\Phi(b,q)\,\mathbf{u}\cdot\hat{\mathbf{n}}\,d\,S = 0\,,\end{aligned}$$

when the normal component of the velocity $\mathbf{u}\cdot\hat{\mathbf{n}}$ vanishes at the boundary ∂D.

□

Remark 110 (***Energy conservation).***
In addition to C_Φ, the Euler–Boussinesq fluid equations (140) also conserve the total energy

$$E = \int_D \frac{1}{2}|\mathbf{u}|^2 + bz\;d^{\,3}x\,, \tag{148}$$

which is the sum of the kinetic and potential energies.

We do not develop the Hamiltonian formulation of the three-dimensional stratified rotating fluid equations studied here. However, one may imagine that the conserved quantity C_Φ with the arbitrary function Φ would play an important role. For more explanation in the framework of Geometric Mechanics, see [Ho2011GM] and references therein. These issues will be discussed in Spring 2023 in Geometry, Symmetry and Integrable Systems.

11.8 Rotating shallow water (RSW) equations

Consider dynamics of rotating shallow water (RSW) on a two dimensional domain with horizontal planar coordinates $\mathbf{x} = (x, y)$. This RSW motion is governed by the following nondimensional equations for variables depending on $(\mathbf{x}, t)$ comprising the horizontal fluid velocity vector $\mathbf{u} = (u, v)$ and the total depth η,

$$\epsilon \frac{d}{dt}\mathbf{u} + f(\mathbf{x})\hat{\mathbf{z}} \times \mathbf{u} + \nabla h = 0\,, \qquad \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) = 0\,, \tag{149}$$

with notation

$$\frac{d}{dt} := \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \quad \text{and} \quad h := \left(\frac{\eta - B}{\epsilon \mathcal{F}} \right),$$

where $\epsilon \ll 1$ and $\mathcal{F} = O(1)$ are nondimensional constants. These equations include spatially variable Coriolis parameter $f(\mathbf{x})\hat{\mathbf{z}} = curl \mathbf{R}(\mathbf{x})$ and mean depth $B = B(\mathbf{x})$.

Exercise.

(i) Show that the RSW equations in (149) follow as Euler-Poincaré equations

$$\big(\partial_t + \pounds_u\big)\frac{1}{\eta}\frac{\delta l}{\delta u} = \frac{1}{\eta}\frac{\delta l}{\delta \eta} \diamond \eta \quad \text{and} \quad \big(\partial_t + \pounds_u\big)\big(\eta\, d^2x\big) = 0\,,$$

from Hamilton's variational principle for the following action integral.

$$0 = \delta S \quad \text{with} \quad S = \int_0^T l(\mathbf{u},\eta)dt \quad \text{and} \quad l(\mathbf{u},\eta) = \int \frac{\epsilon}{2}\eta|\mathbf{u}|^2 + \eta\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{(\eta - B(\mathbf{x}))^2}{2\epsilon\mathcal{F}}\,d^2x\,.$$

in which $\eta(\mathbf{x},t)\,d^2x$ is an advected quantity. Recall that $\diamond : V^*\times V \to \mathfrak{X}^*$ is defined by $\langle \frac{\delta \ell}{\delta a}\diamond a\,,\, v\rangle :=$ $\langle \frac{\delta \ell}{\delta a}\,,\, -\pounds_v a\rangle$ for vector field $v \in \mathfrak{X}$ and L^2 pairing $\langle\,\cdot\,,\,\cdot\,\rangle$.

(ii) Use the Euler-Poincaré equations to show that the RSW equations satisfy Kelvin's circulation theorem

$$\frac{d}{dt}\oint_{c_t} \mathbf{v}\cdot d\mathbf{x} = 0\,,$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iii) Use the Euler-Poincaré equations to show that the RSW equations satisfy

$$\big(\partial_t + \pounds_u\big)d\big(\mathbf{v}\cdot d\mathbf{x}\big) = 0\,,$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iv) Show that $d(\mathbf{v}\cdot d\mathbf{x}) = \omega\, d^2x$, with $\omega := \hat{\mathbf{z}}\cdot curl\mathbf{v}$.

(v) Use $(\partial_t + \pounds_u)(\omega\, d^2x) = 0$ obtained in the previous two parts to derive conservation of potential vorticity on fluid particles.

★

Answer.

1. The Euler-Poincaré equations are

$$\big(\partial_t + \pounds_u\big)\frac{1}{\eta}\frac{\delta l}{\delta u} = \frac{1}{\eta}\frac{\delta l}{\delta \eta}\diamond\eta \quad\text{and}\quad \big(\partial_t + \pounds_u\big)\big(\eta\, d^2x\big) = 0\,,$$

   where $\eta^{-1}\frac{\delta l}{\delta u} = (\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x}))\cdot d\mathbf{x} =: \mathbf{v}\cdot d\mathbf{x}$ and $\eta^{-1}\frac{\delta l}{\delta\eta}\diamond\eta = d(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h)$. Thus,

$$\big(\partial_t + \pounds_u\big)(\mathbf{v}\cdot d\mathbf{x}) = d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big)$$

   with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

2. Integrating the previous equation around a loop moving with the fluid produces

$$\frac{d}{dt}\oint_{c_t}\mathbf{v}\cdot d\mathbf{x} = \oint_{c_t} d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0\,,$$

   with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

3. The differential of the Euler-Poincaré equation yields with $\omega := \hat{\mathbf{z}}\cdot curl\mathbf{v}$

$$\big(\partial_t + \pounds_u\big)(\omega d^2x) = \big(\partial_t + \pounds_u\big)d(\mathbf{v}\cdot d\mathbf{x}) = d^2\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0$$

   upon commuting the differential d with the Lie derivative and using $d^2 = 0$.

4. By direct computation,

$$d(\mathbf{v}\cdot d\mathbf{x}) = v_{i,j}dx^j\wedge dx^i = v_{1,2}dx^2\wedge dx^1 + v_{2,1}dx^1\wedge dx^2 = (v_{2,1} - v_{1,2})\,d^2x = \hat{\mathbf{z}}\cdot curl\mathbf{v}\,d^2x = \omega\,d^2x$$

5. We have $(\partial_t + \pounds_u)(\omega d^2x)$ and $(\partial_t + \pounds_u)(\eta d^2x)$. Therefore, by the product rule for the evolutionary operator $(\partial_t + \pounds_u)$ we have

$$0 = (\partial_t + \pounds_u)\Big(\frac{\omega}{\eta}(\eta d^2x)\Big) = \Big((\partial_t + \pounds_u)\frac{\omega}{\eta}\Big)(\eta d^2x) + \frac{\omega}{\eta}(\partial_t + \pounds_u)(\eta d^2x)$$

Since the second term vanishes via the continuity equation, $(\partial_t + \pounds_u)(\eta d^2x)$, the first term yields

$$0 = (\partial_t + \pounds_u)\frac{\omega}{\eta} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\frac{\omega}{\eta}. \quad \text{Hence,} \quad \frac{dq}{dt} = 0\,, \quad \text{with} \quad q := \omega/\eta\,.$$

This is conservation of potential vorticity on fluid particles.

▲

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