Imperial College London
Department of Mathematics

Solutions to Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

You should answer all questions. Time allowed: 40 minutes.

Question 1

(a) Suppose F is a field and $n \in \mathbb{N}$. Suppose $A = (a_{ij}) \in M_n(F)$ is such that $a_{ij} = 0$ if $j > n - i + 1$.

(i) In the case $n = 3$, prove that $\det(A) = -a_{13}a_{22}a_{31}$. (3 marks)

(ii) What is $\det(A)$ in the cases $n = 5$ and $n = 6$? Explain your answer. (3 marks)

(iii) Write down an expression for $\det(A)$ for general n (you need not prove your statement). (2 marks)

(b) Suppose V is a vector space over $\mathbb{R}$ with a basis B consisting of vectors v_1, v_2. Suppose $T : V \to V$ is the linear map with

$$T(v_1) = -10v_1 - 6v_2 \text{ and } T(v_2) = 18v_1 + 11v_2.$$

(i) Write down the matrix $[T]_B$. (1 mark)

(ii) Find the eigenvectors of T (show the details of your calculation). (6 marks)

(iii) Express v_1 as a linear combination of eigenvectors of T and hence write down an expression for $T^{50}(v_1)$ as a linear combination of v_1 and v_2. (5 marks)

Solution:

(a) (i) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$. Interchanging rows 1 and 3 we obtain $\det(A) = -\det \begin{pmatrix} a_{31} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$. This is lower triangular, hence the result. (3 marks)

(ii) In case $n = 5$, a similar argument works: we bring the matrix into lower triangular form by interchanging rows 1 and 5 and rows 2 and 4. We obtain $\det(A) = a_{15}a_{24}a_{33}a_{42}a_{51}$. For the $n = 6$ case, we interchange 3 pairs of rows, so $\det(A) = -a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}$. (3 marks)

(iii) $\det(A) = (-1)^k a_{1n}a_{2,n-1} \dots a_{n-1,2}a_{n1}$, where k is the integer part of $n/2$. (2 marks)

[Alternative method: expansion along last row or column and induction. Note that upper triangular matrices appeared in the lectures, but the same argument about why the determinant of such a matrix is the product of the diagonal entries also works with lower triangular matrices.]

(b) (i) $[T]_B = \begin{pmatrix} -10 & 18 \\ -6 & 11 \end{pmatrix}$. (1 mark)

(ii) First, find the eigenvalues and eigenvectors of the above matrix. The characteristic polynomial is:

$$\det \begin{pmatrix} x+10 & -18 \\ 6 & x-11 \end{pmatrix} = x^2 - x - 2 = (x-2)(x+1).$$

So the eigenvalues are -1 and 2. Corresponding eigenvectors of the matrix are then (all non-zero scalar multiples of) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, respectively.

The eigenvectors of T are therefore all non-zero scalar multiples of $w_1 = 2v_1 + v_2$ (with eigenvalue -1), and all non-zero scalar multiples of $w_2 = 3v_1 + 2v_2$ (with eigenvalue 2).

(6 marks, lose 2 marks if only the evectors of the matrix are given)

(iii) As $v_1 = 2w_1 - w_2$ we have

$$T^{50}(v_1) = 2T^{50}(w_1) - T^{50}(w_2) = 2w_1 - 2^{50}w_2 = (4 - 3 \cdot 2^{50})v_1 + (2 - 2^{51})v_2.$$

(5 marks)

Question 2

(a) (7 marks) Suppose V is a vector space over $\mathbb{R}$ and $T : V \to V$ is a linear map. Suppose $v_1, v_2, v_3 \in V$ are eigenvectors of T with eigenvalues $1, 2, 3$ respectively. Without quoting a result from your notes, prove that v_1, v_2, v_3 are linearly independent.

If you had been allowed to quote a result from your notes, what would it have said?

(b) (13 marks, 2 or 3 per part) For each of the following statements, say whether it is true or false. If it is true, give a short proof; if it is false, give a counterexample.

(i) Suppose A is a 3×3 matrix of even integers. Then there is no matrix of integers B with $AB = I_3$.

(ii) There is a matrix $A \in M_2(\mathbb{R})$ with $A \neq I_2$ and $A^{17} = I_2$.

(iii) If $A, B \in M_2(\mathbb{R})$ are diagonalisable over $\mathbb{R}$, then so is AB.

(iv) The matrix $\begin{pmatrix} 0 & i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \in M_3(\mathbb{C})$ is diagonalisable over $\mathbb{C}$.

(v) If $A \in M_n(\mathbb{C})$ and the characteristic polynomial of A has no repeated roots in $\mathbb{C}$, then A is diagonalisable over $\mathbb{C}$.

Solution:

(a) Suppose $a, b, c \in \mathbb{R}$ and $av_1 + bv_2 + cv_3 = 0$. Apply T twice to this and use the fact that v_1, v_2, v_3 are eigenvectors. We obtain:

$av_1 + bv_2 + cv_3 = 0$
$av_1 + 2bv_2 + 3cv_3 = 0$
$av_1 + 4bv_2 + 9cv_3 = 0.$

Subtracting the first from the second and third, we obtain $bv_2 + 2cv_3 = 0$ and $3bv_2 + 8cv_3 = 0$. This then gives $2cv_3 = 0$. As $v_3 \neq 0$ (because it is an eigenvector), we obtain $c = 0$. It then follows that $b = 0$ and $a = 0$, as required. (6 marks)

We could have used the general theorem which states that if V is any vector space, $T : V \to V$ is a linear map and $v_1, \ldots, v_r$ are eigenvectors of T with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then $v_1, \ldots, v_r$ are linearly independent. (1 mark)

(b) No marks without a reason. The better the explanation, the more marks are obtained.

(i) TRUE: We can take a factor of 2 out of each row of A and still have a matrix of integers. So $\det(A)$ is an integer divisible by 8. But if $AB = I_3$ then $\det(A)\det(B) = 1$ and this is an equation in integers, which is impossible. (3 marks)

(ii) TRUE: consider the matrix $A = \begin{pmatrix} \cos(2\pi/17) & -\sin(2\pi/17) \\ \sin(2\pi/17) & \cos(2\pi/17) \end{pmatrix}$ which represents anti-clockwise rotation about 0 through an angle $2\pi/17$. (2 marks)

(iii) FALSE: Take $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Both A, B are diagonalisable over $\mathbb{R}$ (as B has two real eigenvalues ± 1). But AB has no real eigenvalues, so is not diagonalisable over $\mathbb{R}$. (Geometrically: A, B are reflections and AB is a rotation.) (3 marks)

(iv) TRUE: This is iB where B is a real, symmetric matrix. As B is diagonalisable (over $\mathbb{R}$, and therefore over $\mathbb{C}$), so is iB. (3 marks)

[Do not accept: this is a symmetric matrix so is diagonalisable.]

(v) TRUE: The characteristic polynomial is a product of linear factors over $\mathbb{C}$ (by the Fundamental Theorem of Algebra), so if there are no repeated roots, it has n distinct roots. Hence there are n distinct eigenvalues of A in $\mathbb{C}$ and therefore (by the theorem in the notes quoted in (a)), A is diagonalisable. (2 marks)