IMPERIAL

MATH70020

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Function Spaces and Applications

Date: Wednesday, April 30, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. For a positive sequence (w_n), and for $p \in [1,\infty]$, we define the weighted ℓ^p norm as

$$\|(u_n)\|_{\ell^p_w} = \|(w_n u_n)\|_{\ell^p} = \left[\sum_{n=1}^{\infty} w_n^p |u_n|^p\right]^{1/p},$$

and we define ℓ^p_w as the set of sequences whose ℓ^p_w norm is finite.

(a) Show that ℓ_p^w endowed with its norm is a Banach space (we saw in class the case $w = 1$ which is the classical ℓ^p space). (5 marks)

(b) Show that the identity map $\mathrm{Id} : (u_n) \mapsto (u_n)$ is compact from ℓ^p_w to ℓ^p if $w_n \to \infty$ as $n \to \infty$. (10 marks)

(c) Show that the identity map is bounded from ℓ^∞_w to ℓ^1 if and only if w_n^{-1} is in ℓ^1. (5 marks)

(Total: 20 marks)

2. Show that the following linear maps are bounded and compute their norms

(a) The application T_1 from ℓ^2 to ℓ^2 which maps $(x_n)_{n\in\mathbb{N}}$ to $(x_{n+1})_{n\in\mathbb{N}}$. (5 marks)

(b) The application T_2 from $L^2(0,1)$ to $\mathbb{R}$ which maps $f(x)$ to $\int_0^1 x^2 f(x)\,dx$. (5 marks)

(c) The application T_3 from $L^\infty(0,1)$ to $\mathbb{R}$ which maps $f(x)$ to $\int_0^1 x^2 f(x)\,dx$. (5 marks)

(d) The application T_4 from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ which maps $f(x)$ to $(f * \mathbf{1}_{[0,1]})(x)$. (5 marks)

(Total: 20 marks)

3. (a) State the Hahn-Banach theorem in a Banach space. (5 marks)

(b) If ℓ is a bounded linear form defined on a dense subset D of a Banach space B, show that it admits a unique extension to all of B with the same norm. (5 marks)

(c) Turning to the case of a Hilbert space H, we want to show that a bounded linear form ℓ with norm 1 defined on a closed linear subset $C \subset H$ admits a unique extension to H with norm 1. Under these hypotheses, and with the help of the projection theorem in Hilbert spaces (but without using the Hahn-Banach theorem), find an extension of ℓ to H with norm 1. (5 marks)

(d) Using the Riesz representation theorem, show that this extension is unique. (5 marks)

(Total: 20 marks)

4. For k a continuous function on $[0,1]^2$, we consider the operator

$$[Tf](x) = \int_0^1 k(x,y) f(y)\,dy.$$

mapping real functions on $[0,1]$ to real functions on $[0,1]$.

(a) Show that it maps boundedly the set of continuous functions $\mathcal{C}([0,1])$ to itself. (5 marks)

(b) Show that this operator maps compactly $\mathcal{C}([0,1])$ to itself. (5 marks)

(c) What is the norm of this operator on $\mathcal{C}([0,1])$? (5 marks)

(d) Find a weaker condition than k continuous which still guarantees that T is bounded on $\mathcal{C}([0,1])$ (and justify that this weaker condition is indeed sufficient)! (5 marks)

(Total: 20 marks)

5. Let H be a separable Hilbert space; it is convenient to choose a Hilbert basis (e_n). We let S be the unit sphere

$$S = \{x \in H,\ \|x\| = 1\} = \{x = \sum_{n=1}^{\infty} x_n e_n,\ \sum |x_n|^2 = 1\}.$$

(a) Show that S is closed in H.

(5 marks)

(b) We want to find the weak closure $\widehat{S}$ of S in H, namely the set of $x \in H$ such that: there exists a sequence (x_n) in S converging weakly to x in H ($x_n \rightharpoonup x$). Show that $\widehat{S}$ is contined in $\overline{B(0,1)}$ (closed unit ball). (5 marks)

(c) Show that $\overline{B(0,1)}$ is contained in $\widehat{S}$. (5 marks)

(d) Show that a sequence (x_n) in H converges strongly to x if and only if it converges weakly to x and $\|x_n\|$ also converges to $\|x\|$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

60020 / 70020

Functions spaces and applications (Solutions)

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1. (a) From the identity $\|(u_n)\|_{\ell^p_w} = \|(w_n u_n)\|_{\ell^p}$, and from the fact that $\|\cdot\|_{\ell^p}$ is a norm as seen is class, we deduce that $\|\cdot\|_{\ell^p_w}$ is also a norm. To show that ℓ^p_w is complete, we pick a Cauchy sequence $(u_n^{(k)})$ in ℓ^p_w. This means that $(w_n u_n^{(k)})$ is Cauchy in ℓ^p, and hence has a limit (v_n). Setting $u_n = w_n^{-1} u_n$, we claim that it is the limit of $(u_n^{(k)})$ as $k \to \infty$ in ℓ^p_w. Indeed, we have

$$\|(u_n^{(k)}) - (u_n)\|_{\ell^p_w} = \|(w_n u_n^{(k)}) - (w_n u_n)\|_{\ell^p} = \|(w_n u_n^{(k)}) - (v_n)\|_{\ell^p} \overset{k\to\infty}{\longrightarrow} 0.$$

$\boxed{5,\ \mathrm{A}}$

(b) Consider a bounded sequence $(u_n^{(k)})$ in (ℓ^p_w); we aim at showing that $(u_n^{(k)})$ admits a convergent subsequence in ℓ^p.

For any n, $(u_n^{(k)})$ is a bounded sequence. Therefore, we can extract iteratively (in n) subsequences such that $(u_n^{(k)})$ converges (in k). By a diagonal argument, we obtain a subsequence, still denoted $(u_n^{(k)})$, and a sequence (u_n) such that: for any n, $(u_n^{(k)})$ converges to u_n as $k \to \infty$.

First, we check that $(u_n) \in \ell^p_w$, which is contained in ℓ^p since $\inf_n |w_n| > 0$: indeed, for any N,

$$\sum_{n=1}^{N} |w_n u_n|^p = \sum_{n=1}^{N} \lim_{k\to\infty} |w_n u_n^{(k)}|^p \le \lim_{k\to\infty} \sum_{n=1}^{N} |w_n u_n^{(k)}|^p \le \lim_{k\to\infty} \|(u_n^{(k)})\|^p_{\ell^p_w} < \infty.$$

Letting $N \to \infty$ on the left-hand side, we obtain that $(u_n) \in \ell^p$.

Next, we show that $(u_n^{(k)})$ converges to (u_n) in ℓ^p as $k \to \infty$. Fix $\epsilon \in (0,1)$ and choose successively $N \in \mathbb{N}$ such that $|w_n| > \epsilon^{-1}$ if $n > N$; and then $K \in \mathbb{N}$ such that $\sum_{n=1}^{N} |w_n(u_n^{(k)} - u_n)|^p < \epsilon$. Then

$$\begin{aligned}
\|(u_n) - (u_n^{(k)})\|^p_{\ell^p} &= \sum_{n=1}^{N} |u_n - u_n^{(k)}|^p + \sum_{n=N+1}^{\infty} |u_n - u_n^{(k)}|^p \\
&\le \sum_{n=1}^{N} |u_n - u_n^{(k)}|^p + 2^p \sum_{n=N}^{\infty} |u_n|^p + 2^p \sum_{n=N+1}^{\infty} |u_n^{(k)}|^p \\
&\le \sum_{n=1}^{N} |u_n - u_n^{(k)}|^p + 2^p \epsilon^p \sum_{n=N+1}^{\infty} |w_n u_n|^p + 2^p \epsilon^p \sum_{n=N+1}^{\infty} |w_n u_n^{(k)}|^p \\
&\le \epsilon + 2^p \epsilon^p \sup_n \|(u_n)\|_{\ell^p_w} + 2^p \epsilon^p \|(u_n)\|_{\ell^p_w} \le C\epsilon,
\end{aligned}$$

for an appropriately chosen constant C. This gives the desired statement. $\boxed{5,\ \mathrm{B}}$

$\boxed{4,\ \mathrm{C}}$

(c) If $(w_n^{-1}) \in \ell^1$, then

$\boxed{1,\ \mathrm{D}}$

$$\sum_{n=1}^{\infty} |u_n| \le \sum w_n^{-1} w_n |u_n| \le \|(w_n^{-1})\|_{\ell^1} \|(u_n)\|_{\ell^\infty_w}.$$

Conversely, if the identity map is bounded from ℓ^∞_w to ℓ^1, then w_n^{-1}, which belongs to ℓ^∞_w, must also belong to ℓ^1. $\boxed{5,\ \mathrm{A}}$

2. (a) It is clear that T_1 has norm less than 1 since

$$\sum_{n=1}^{\infty} |x_{n+1}|^2 \leq \sum_{n=1}^{\infty} |x_n|^2.$$

Furthermore, if $(x_n) = (0, 1, 0, 0, 0, 0, \dots)$, it is easy to check that $\|T(x_n)\|_{\ell^2} = \|(x_n)\|_{\ell^2} = 1$, which shows that T_1 has norm greater than 1. Therefore, T_1 has norm 1. $\boxed{5, \text{A}}$

(b) By the Cauchy-Schwarz inequality, since an elementary computation shows that $\|x^2\|_{L^2(0,1)} = \frac{1}{\sqrt{5}}$,

$$|T_2 f| \leq \|x^2\|_{L^2} \|f\|_{L^2} \leq \frac{1}{\sqrt{5}} \|f\|_{L^2}$$

which shows that $\|T_2\| \leq \frac{1}{\sqrt{5}}$. Furthermore,

$$|T_2 x^2| = \|x^2\|_{L^2}^2 = \frac{1}{5},$$

which shows that $\|T_2\| \geq \frac{1}{\sqrt{5}}$. Overall, T_2 has norm $\frac{1}{\sqrt{5}}$. $\boxed{5, \text{B}}$

(c) By the Hölder inequality, since an elementary computation shows that $\|x^2\|_{L^1(0,1)} = \frac{1}{3}$,

$$|T_3 f| \leq \|x^2\|_{L^1} \|f\|_{L^\infty} \leq \frac{1}{3} \|f\|_{L^2},$$

whis shows that $\|T_3\| \leq \frac{1}{3}$. Furthermore,

$$|T_3 1| = \|x^2\|_{L^1} = \frac{1}{3},$$

which shows that $\|T_3\| \geq \frac{1}{3}$. Overall, T_3 has norm $\frac{1}{3}$. $\boxed{5, \text{B}}$

(d) We can estimate

$$\|T_4 f\|_{L^1(\mathbb{R})} = \int \left| \int f(x-y) \mathbf{1}_{[0,1](y)} \, dy \right| dx \leq \int |f(x-y)| \mathbf{1}_{[0,1](y)} \, dy \, dx \leq \|f\|_{L^1},$$

which shows that $\|T_4\| \leq 1$. Conversely, the inequalities above become equalities if $f \geq 0$. This shows that $\|T_4\| = 1$. $\boxed{5, \text{B}}$

3. (a) Let F be a (linear) subspace of a Banach space B, and ℓ be a bounded linear form on F with norm M. Then ℓ can be extended to a bounded linear form on B with norm M. $\boxed{5, \text{A}}$

(b) Given ℓ, we aim at finding a bounded extension $\widetilde{\ell}$ to B. If $x \in B$, it is the limit of a sequence (x_n) of D by density of D. Since (x_n) is Cauchy, so it $(\ell(x_n))$ by boundedness of ℓ. Therefore, we can define $\widetilde{\ell}(x)$ by

$$\widetilde{\ell}(x) = \lim_{n\to\infty} \ell(x_n). \tag{1}$$

By a similar argument, one checks that this definition is independent of the sequence (x_n) converging to x. It is obvious that ℓ is linear; its boundedness is not hard to check:

$$\text{for any } n, \quad \|\ell(x_n)\| \le \|\ell\| \cdot \|x_n\|.$$

Passing to the limit gives

$$\|\widetilde{\ell}(x)\| \le \|\ell\| \cdot \|x\|.$$

There remains to see that this extension is unique; but any extension should satisfy the identity (1), and therefore it is the only possible extension. $\boxed{5, \text{A}}$

(c) By the orthogonal projection theorem in Banach spaces, there exists a projection operator P onto C such that, for any x, $Px \in C$ and $x - Px \in C^\perp$. As a consequence, P is the identity on C and for any $x \in H$, $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$, which implies $\|Px\| \le \|x\|$. We now define the extension $\widetilde{\ell}$ of ℓ by

$$\widetilde{\ell}(x) = \ell(P(x)).$$

Since P is the identity on C, the forms ℓ and $\widetilde{\ell}$ agree on C. Furthermore, the norm of $\widetilde{\ell}$ is M since, if $x \in H$,

$$\|\widetilde{\ell}(x)\| = \|\ell(P(x))\| \le \|\ell\| \cdot \|P\| \cdot \|x\| \le M\|x\|.$$

$\boxed{5, \text{D}}$

(d) Since C is a closed subspace of H, it is a Hilbert space in its own right. Therefore, by the Riesz representation theorem, there exists $z \in C$ such that

$$\ell(x) = (z\,,\, x) \qquad \text{for any } x \in C.$$

Similarly, if $\widetilde{\ell}$ is an extension of ℓ, there exists $Z \in H$ such that

$$\widetilde{\ell}(x) = (Z\,,\, x) \qquad \text{for any } x \in H.$$

Since ℓ and $\widetilde{\ell}$ agree on C, we have necessarily that $PZ = z$. Furthermore, since both forms have the same norm,

$$\|z\| = \|\ell\| = \|\widetilde{\ell}\| = \|Z\|.$$

In other words, we have $\|PZ\| = \|Z\|$, which is only possible if $Z = z$. Therefore, the extension defined in the previous question is the only one. $\boxed{5, \text{D}}$

4. (a) To show that Tf is continuous if f is, we let $\epsilon > 0$ and choose δ such that

$$\sup_y |k(x,y) - k(x',y)| < \epsilon \qquad \text{if } |x - x'| < \delta$$

(which is possible by uniform continuity of k). Then, if $|x - x'| < \delta$

$$\begin{aligned} |[Tf](x) - [Tf](x')| &= \left| \int_0^1 (k(x,y) - k(x',y)) f(y)\, dy \right| \le \sup_y |k(x,y) - k(x',y)| \|f\|_{L^\infty} \\ &\le \epsilon \|f\|_{L^\infty}, \end{aligned}$$

which shows that T is continuous. $\boxed{5, \mathrm{A}}$

(b) First, it is easily proved that

$$\|Tf\|_{L^\infty} \le C \|f\|_{L^\infty}$$

for an appropriate constant C.
Second, we saw in the previous question that

$$|[Tf](x) - [Tf](x')| \le \epsilon \|f\|_{L^\infty} \qquad \text{if } |x - x'| < \delta,$$

with $\delta = \delta(\epsilon)$ chosen as above. This shows equicontinuity of the Tf if $\|f\|_\infty \le 1$. Thanks to these two estimates, we deduce by the Arzela-Ascoli theorem that T is compact from $\mathcal{C}([0,1])$ to itself. $\boxed{5, \mathrm{C}}$

(c) On the one hand, we have that

$$\|Tf\|_\infty = \sup_x \left| \int_0^1 k(x,y) f(y)\, dy \right| \le \underbrace{\sup_x \int_0^1 |k(x,y)|\, dy}_{M} \cdot \|f\|_\infty.$$

On the other hand, if we choose x_0 such that $M = \int_0^1 |k(x_0,y)|\, dy$ (which is possible by continuity of k) and $f(y) = f_0(y) = \operatorname{sign} k(x_0,y)$, then $f_0 \in L^\infty$ with $\|f_0\| = 1$ while

$$\|Tf_0\|_\infty \ge Tf_0(x_0) = \int k(x_0,y) \operatorname{sign} k(x_0,y)\, dy = \int |k(x_0,y)|\, dy = M.$$

This shows that $\|T\| = M$... except that our test function f_0 is not continuous, but only bounded! However, it can be approximated by continuous functions f_n such that

$$\|f_n\|_\infty = 1, \qquad f_n \xrightarrow{L^1} f_0 \text{ as } n \to \infty$$

It is then easy to check that $\|Tf_n\|_\infty \to M$, which implies that $\|T\| = M$. $\boxed{3, \mathrm{C}}$

(d) The above arguments are left unchanged if we only assume that $\boxed{2, \mathrm{A}}$

$$\sup_y \int |k(x,y) - k(x',y)|\, dy \longrightarrow 0 \qquad \text{if } |x - x'| \to 0.$$

This is a weaker condition than continuity. $\boxed{5, \mathrm{D}}$

5. (a) This is a consequence of the continuity of the map $x \to \|x\|$ from H to $\mathbb{R}$. $\boxed{5, \text{M}}$

(b) Indeed, assume that $x_n \rightharpoonup x$ with $x_n \in S$, $x \in H$. Then for any $y \in H$, by the Cauchy-Schwarz inequality

$$|(x_n, y)| \leq \|x_n\|\|y\| \leq \|y\|.$$

Letting $n \to \infty$ results in the inequality

$$|(x, y)| \leq \|y\|.$$

Choosing $y = x$ gives $\|x\| \leq 1$, in other words $x \in \overline{B(0,1)}$. $\boxed{5, \text{M}}$

(c) We want to show that any element $z \in \overline{B(0,1)}$ is the weak limit of elements of S. We start with the case where z can be written

$$z = \sum_{n=1}^{N} z^k e_k, \qquad \text{with } N \in \mathbb{N}, \tag{2}$$

and we let $\lambda = \|z\| \leq 1$. We define then

$$x_n = z + \sqrt{1-\lambda^2} e_{N+n}.$$

Then it is easy to check that $\|x_n\| = 1$ and $x_n \rightharpoonup x$.

Considering now a more general z, it can be approximated by elements of the form (2), which in turn are weak limits of (x_n) in S as we saw above. This gives the desired result. $\boxed{5, \text{M}}$

(d) It is clear that, if x_n converges strongly to x, then it also converges weakly, and furthermore $\|x_n\|$ converges to $\|x\|$.

To prove the converse, let us assume that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then we can expand

$$(x_n - x\,,\, x_n - x) = \|x_n\|^2 - 2(x_n\,,\, x) + \|x\|^2 \longrightarrow 0$$

since $(x_n, x) \to \|x\|^2$ and $\|x_n\|^2 \to \|x\|^2$. $\boxed{5, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH60020 Function Spaces and Applications Marker Comments

Question 1	The first question proved difficult for most students, but most did well for the two last questions.
Question 2	Most students did well for this question.
Question 3	Most students did well for questions (a) and (b), but only a few could answer questions (c) and (d).
Question 4	Most students did well, except for the last question.