Imperial College London
Department of Mathematics

Unseen 8

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

Question 1. Suppose G is a group. We say that $g, k \in G$ are *conjugate* if there exists $h \in G$ with $k = hgh^{-1}$.

(a) Prove that conjugacy is an equivalence relation on G.

The equivalence classes here are called the *conjugacy classes* in G. We will now determine the conjugacy classes in the symmetric group S_n.

Suppose $g, k \in S_n$ have the same disjoint cycle shape. We can define a bijection h of $\{1, \ldots, n\}$ which sends a cycle of g to a cycle of k simply by writing the disjoint cycle forms of g, k above each other (including fixed points) and sending the top row to the bottom row. For example in S_8, suppose:
$g = (1462)(357)(8)$ and
$k = (3571)(284)(6)$. Then let

$$h = \begin{pmatrix} 1 & 4 & 6 & 2 & 3 & 5 & 7 & 8 \\ 3 & 5 & 7 & 1 & 2 & 8 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 5 & 8 & 7 & 4 & 6 \end{pmatrix}.$$

(b) In the above example, check that $hgh^{-1} = k$. Turn this into a general argument: first show that $kh(x) = hg(x)$ (for all $x \in \{1, \ldots, n\}$).

Deduce that $g, k \in S_n$ are conjugate in S_n if and only if g, k have the same disjoint cycle shape.

Question 2. A subgroup H of a group G is a *normal subgroup* if for all $g \in G$, we have $gH = Hg$.

(a) Suppose $H \leq G$. Show that the following are equivalent:

(i) H is a normal subgroup of G;

(ii) for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$;

(iii) H is a union of conjugacy classes in G.

(b) Find a normal subgroup of order 4 in S_4.

(c) Find the sizes of the conjugacy classes in S_5. Using this, together with Lagrange's theorem, prove that a normal subgroup of S_5 has order 1, 60 or 120. Is there an example in each case here?