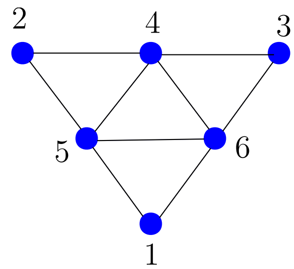
Symmetry, eigenvectors/eigenvalues and circulant matrices

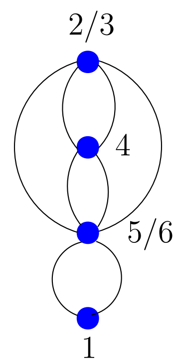
1. Consider the electric circuit with 6 nodes shown in the figure:



All edges have unit conductance. Suppose that node 1 is set to unit voltage and nodes 2 and 3 are grounded. It is required to find the effective conductance of this circuit.

(a) Write down a linear system involving the 6-by-6 Laplacian of the graph which can be solved to find the effective conductance.

(b) Using invoking symmetry, argue why we can alternatively find the effective conductance of the following "equivalent" circuit:



Write down a linear system involving the 4-by-4 Laplacian of *this* graph which can be solved to find the effective conductance.

(c) By introducing an appropriate symmetry matrix $\mathbf{S}$, and following the example set out in lectures, prove that the effective conductances found using the linear systems in parts (a) and (b) are the same.

2. Consider the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Verify that $\mathbf{S}^6 = \mathbf{I}$ where $\mathbf{I}$ is the 6-by-6 identity.

(b) Can you use this fact to find the eigenvalues and eigenvectors of $\mathbf{S}$?

3. In lectures it was shown that the set of n-dimensional complex-valued vectors

$$\mathbf{x}_k = A_k \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{pmatrix}, \qquad \omega = e^{2\pi \mathrm{i}/n}, \qquad k = 0, 1, \cdots, n-1,$$

are eigenvectors of *any* n-by-n circulant matrix.

(a) Confirm that if $k \neq m$ then
$$\overline{\mathbf{x}}_m^T \mathbf{x}_k = 0,$$
where $\overline{\mathbf{x}}_m$ denotes the complex conjugate of $\mathbf{x}_m$.

(b) Determine the normalization parameter A_k that ensures
$$\overline{\mathbf{x}}_k^T \mathbf{x}_k = 1, \qquad k = 0, 1, \cdots, n-1.$$

4. In question 9 of Problem Sheet 1 the Laplacian matrix associated with a complete graph with n nodes is considered. In grounding one of the nodes the $(n-1)$-by-$(n-1)$ matrix

$$\mathbf{K}_0 = \begin{bmatrix} n-1 & -1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdots & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}.$$

is obtained from it as a *reduced* Laplacian. This matrix is now invertible. You were asked on Problem Sheet 1 to find $\mathbf{K}_0^{-1}$ by computing it explicitly for small values of n and, by inspection, guessing the form of $\mathbf{K}_0^{-1}$ for general values of n.

Notice, however, that $\mathbf{K}_0$ is a *circulant matrix*. Use the result of question 3 above to determine $\mathbf{K}_0^{-1}$ for any $n \geq 2$ in a different way using the known eigenvectors and eigenvalues of $\mathbf{K}_0$.

5. We know, from the lectures, that eigenvectors of the n-by-n matrix

$$\mathbf{K}_n \equiv \begin{bmatrix} 2 & -1 & 0 & . & . & 0 \\ -1 & 2 & -1 & . & . & 0 \\ 0 & -1 & 2 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -1 & 2 & -1 \\ 0 & 0 & . & . & -1 & 2 \end{bmatrix}$$

are

$$\mathbf{\Phi}_k = A_k \begin{pmatrix} \sin(k\pi/(n+1)) \\ \sin(2k\pi/(n+1)) \\ . \\ . \\ \sin(nk\pi/(n+1)) \end{pmatrix}, \qquad k = 1, 2, \cdots, n.$$

(a) Show that in order that each of these vectors satisfies ${\mathbf{\Phi}_k}^T \mathbf{\Phi}_k = 1$ for $k = 1, \cdots, n$ then we must choose the normalization parameter A_k to be

$$A_k = \sqrt{\frac{2}{n+1}}, \qquad \text{for all } k = 1, \cdots, n.$$

(b) Verify by direct calculation that the set

$$\mathbf{\Phi}_k = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin(k\pi/(n+1)) \\ \sin(2k\pi/(n+1)) \\ . \\ . \\ \sin(nk\pi/(n+1)) \end{pmatrix}, \qquad k = 1, 2, \cdots, n$$

is then an *orthonormal* set of eigenvectors.

6. Let n be an even integer. Show that *half* of the eigenvectors of K_n can be found by considering the circulant matrix C_{n+1}.

7. Compute the Fourier sine series of

$$1 - \frac{x}{\pi}$$

and confirm that it agrees with the $n \to \infty$ result obtained in lectures of $n+1$ masses attached to a wall at one end and pulled with unit force at the other.

8. Find the eigenvectors and eigenvalues of the matrix

$$\tilde{\mathbf{K}}_n \equiv \begin{bmatrix} a & -b & 0 & . & . & 0 \\ -b & a & -b & . & . & 0 \\ 0 & -b & a & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -b & a & -b \\ 0 & 0 & . & . & -b & a \end{bmatrix},$$

where a and b are real constants.