Midterm solutions Feb 2023

06 February 2023 18:48

Q 1 (a)

$$\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{a^2+w^2}$$ (1 marks)

set $a=1$ and using symmetry formula: (1 marks)

$$\mathcal{F}\left\{\frac{1}{1+x^2}\right\} = \frac{2\pi}{2} e^{-|-w|} = \pi e^{-|w|}$$ (2 marks)

(b)

$$\mathcal{F}\left\{\frac{df}{dx} + xf\right\} = \mathcal{F}\{\delta(x+x_0) - \delta(x-x_0)\}$$

$$iw\hat{f}(w) + i\frac{d\hat{f}(w)}{dw} = e^{iwx_0} - e^{-iwx_0} = 2i\sin wx_0$$ (2 marks)

$$\frac{d\hat{f}}{dw} + w\hat{f}(w) = 2\sin wx_0$$ linear first order ODE

$$I = e^{\int w\,dw} = e^{\frac{1}{2}w^2}$$

$$\boxed{\hat{f}(w) = c e^{-\frac{1}{2}w^2} + 2e^{-\frac{1}{2}w^2}\int e^{\frac{w^2}{2}}\sin wx_0\,dw}$$ (2 marks)

Q 2

$$\begin{cases} y' = y-1 \\ x' = x+1 \end{cases} \Rightarrow \frac{dy'}{dx'} = \frac{x'-y'}{x'+y'}$$ (1 mark) dimensionally homogenous

$$u = \frac{y'}{x'} \Rightarrow \frac{dy}{dx} = \frac{dy'}{dx'} = u + x'\frac{du}{dx'} \Rightarrow$$

$$u + x'\frac{du}{dx'} = \frac{1-u}{1+u} \Rightarrow$$ (2 mark)

$$x'\frac{du}{dx'} = \frac{1-2u-u^2}{1+u} \Rightarrow -\frac{1}{2}\int\frac{d(u^2+2u-1)}{u^2+2u-1} = \int\frac{dx'}{x'} + c$$

$$\Rightarrow -\frac{1}{2}\log|u^2+2u-1| = \log|x'| + c$$

$$\Rightarrow \quad -\tfrac{1}{2}\log|u^2+2u-1| = \log|x| + c$$

$$\Rightarrow \quad (u^2+2u-1) = c' x^{-2}$$

$$\Rightarrow \quad (y-1)^2 + 2(x+1)(y-1) - (x+1)^2 = c'$$

$$y(0)=1 \Rightarrow c' = -1 \Rightarrow \boxed{y^2 - x^2 - 4x + 2xy - 1 = 0} \quad \text{(2 marks)}$$

(1 marks)

Alternative method: $u = x+y \Rightarrow u\dfrac{du}{dx} = 2(x+1) \Rightarrow$ separable

Q3. $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda = 2$ (repeated)

step 1. $y_{CF} = c_1 e^{-2x} + c_2 x e^{-2x}$ (2 marks)

step 2. $y_{PI} = Axe^{x} + Be^{x}$ (1 mark)

$$\forall x \quad (2A + Ax + B)e^{x} + 4(A + Ax + B)e^{x} + 4(Ax + B)e^{x} = xe^{x}$$

x^1: $A + 4A + 4A = 1 \Rightarrow A = \frac{1}{9}$

x^0: $2A + 4A + 9B = 0 \Rightarrow B = -\frac{6}{9}\cdot\frac{1}{9} = -\frac{2}{27}$

$$y_{PI} = \frac{x}{9}e^{x} - \frac{2}{27}e^{x} \quad \text{(2 mark)}$$

$$y_{GS} = y_{CF} + y_{PI} \quad \text{(1 mark)}$$