Applied Complex Analysis - Lecture Ten

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The trapezium rule

Quadrature rules

A quadrature rule approximates an integral via a weighted sum of samples.

$$I = \int_a^b f(x)dx \approx I_N = \sum_{n=1}^{N} f(x_n)w_n,$$

- This is an N-point quadrature rule
- The nodes/points/samples are x_n for $n = 1, \ldots, N$
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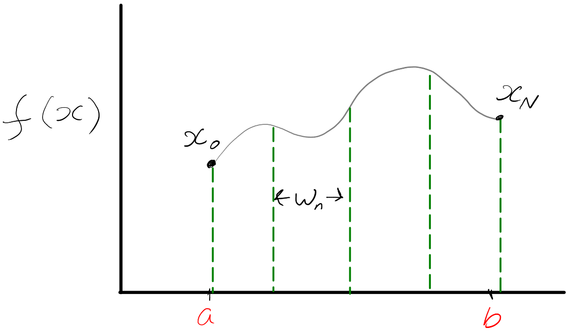
The trapezium rule

- The $(N+1)$-point trapezium rule is a quadrature rule with nodes $x_j = a + (b-a)j/N$ for $j = 0, \ldots, N$ and weights $w_0 = w_N = (b-a)/(2N)$ and $w_j = (b-a)/N$ for $j = 1, \ldots, N-1$.
- Physically, it can be interpreted as approximating the area under the curve by N trapeziums.



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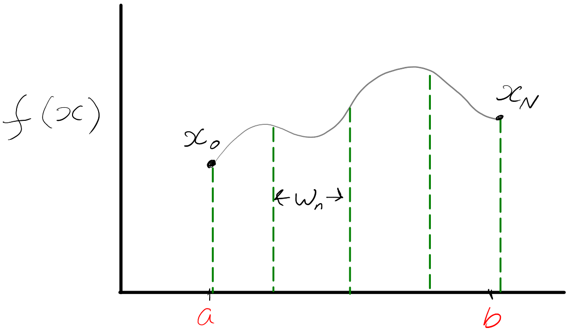

Quadratic error

Thm: The $(N+1)$ trapezium rule converges like $O(N^{-2})$ for $C^2[a,b]$ functions.

- Proof sketch
- This can be extended to functions with Lipschitz continuous derivatives
- Similar result holds for the midpoint rule
- Numerical example

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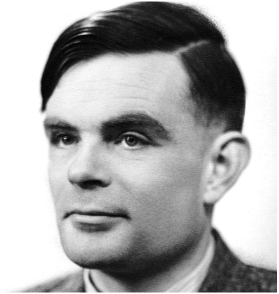

- Exponential convergence of trapezium rule initially observed by mathematicians in the previous century
- In the case where f is periodic, the 0th and Nth points coincide
- Closed contour integrals, once parametrised, are periodic
- The proof(s) we use for exponential convergence use residue calculus!

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Trapezium rule on the unit circle

$$I = f(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} dz$$

- Changing variables $z = e^{i\theta}$, for $\theta \in [0, 2\pi]$

$$I = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

- Trapezium rule approx:

$$I \approx I_N = \frac{1}{N} \sum_{j=1}^{N} f(z_j), \quad \text{where } z_j = e^{2\pi i j/N}, \quad \text{for } j = 1, \ldots, N$$

- **Thm**: Suppose f is analytic and satisfies $|f(z)| < M$ inside the complex disk $|z| < r$ for some $r > 1$. Then

$$|I - I_N| \leq \frac{M}{r^N - 1} = O(r^{-N}) \quad \text{as } N \to \infty.$$

- Proof

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$$I = \int_0^{2\pi} f(\theta) d\theta \approx I_N = \frac{1}{N} \sum_{n=1}^{N} f(\theta_n), \quad \theta_n = 2\pi n/N,$$

if f is 2π-periodic and analytic in the strip $S_a := \{\theta : -a < \mathrm{Im}\theta < a\}$ for $a > 0$, then

$$|I - I_N| \leq \frac{4\pi \sup_{\theta \in S_a} |f(\theta|}{e^{aN} - 1}.$$

- Convergence of real integral depends on complex behavior
- More general class: may be applied to non-circular contour integrals $\oint_\gamma f$, where f is analytic in an annulus containing γ
- Requires the parametrisation of γ to be analytic
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$$f(z) = \sum_{j=-N}^{N-2} a_n (z - z_0)^n,$$

for z in some annulus D, then an N-point trapezium rule I_N can exactly approximate

$$I = \oint_\gamma f(z) dz,$$

where γ is a closed anti-clockwise-oriented contour in D.

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