CHAPTER 1: INTRODUCTION TO PDES

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ABSTRACT. These notes follow closely the book of S. Salsa [1].

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1. WHAT IS A PARTIAL DIFFERENTIAL EQUATION?

A partial differential equation (PDE) is a relation of the type

$$F(x_1,\dots,x_d,u,\partial_{x_1}u,\dots,\partial_{x_d}u,\partial_{x_1x_1}u,\partial_{x_1x_2}u,\dots\partial_{x_dx_d}u,\partial_{x_1x_1x_1}u,\dots)=0 \tag{1.1}$$

where the unknown $u=u(x_1,\dots,x_d)$ is a function of d variables and $\partial_{x_j}u,\partial_{x_ix_j}u,\dots$ are its partial derivatives. The highest order of differentiation occurring in the equation is the *order of the equation*.

A first important distinction is between linear and nonlinear equations. Equation (1.1) is linear if F is *linear* with respect to u and all its derivatives, otherwise it is *nonlinear*. A second distinction concerns the types of nonlinearity. We distinguish:

- Semilinear equations where F is nonlinear only with respect to u but is linear with respect to all its derivatives;
- Quasilinear equations where F is linear with respect to the highest order derivatives of u;
- Fully nonlinear equations where F is nonlinear with respect to the highest order derivatives of u.

The theory of linear equations can be considered sufficiently well developed and consolidated, at least for what concerns the most important questions. On the contrary, the non linearities present such a rich variety of aspects and complications that a general theory does not appear to be conceivable. The existing results and the new investigations focus on more or less specific cases, especially interesting in the applied sciences.

1.1. **Linear PDEs and examples.**

We start with linear equations. First of all, we recall that an operator L acting on functions u,v is linear if

$$L(u+v)=L(u)+L(v), \qquad \text{for all functions } u,v$$
$$L(cu)=cL(u), \qquad \text{for all functions } u \text{ and } c\in\mathbb{R}.$$

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Partial derivatives are the most important examples of linear operators that we will look at. For instance, one can check that the operator $L = \partial_{xx}$, defined as

$$L(u) = \partial_{xx} u, \qquad \text{for any } u \in C^2(\mathbb{R}),\ x \in \mathbb{R},$$

it is indeed a linear operator. In fact, any linear combination of partial derivatives give rise to another linear operator. The linearity property does not depend on the variables $x_1, \dots, x_d$, meaning that an operator as $L = x^2 \partial_x$ is also linear. A general linear differential operator acting on functions $u : \mathbb{R} \to \mathbb{R}$ can be written as

$$L = \sum_{i=0}^{m} f_i(x) \partial_x^{(i)}, \qquad x \in \mathbb{R},$$

where $f_i : \mathbb{R} \to \mathbb{R}$ are given functions, m is the order of the operator and $\partial_x^{(0)} = 1$, $\partial_x^{(1)} = \partial_x$, $\partial_x^{(2)} = \partial_{xx}$ and so on. This general definition can be extended to higher dimensions introducing more indices and sums, but we avoid writing a precise definition since we will see some concrete examples afterwards. An important property of linear equations is given in the following theorem.

Theorem 1.1 (Superposition principle). *Let $u_1, \dots u_k$ be solutions to the same* ***linear*** *PDE. Then also $u = c_1 u_1 + \dots + c_k u_k$, with $c_1, \dots, c_k \in \mathbb{R}$, is a solution.*

Proof. Since $u_1, \dots, u_k$ are solutions to the same PDE, there exists a linear differential operator L such that

$$L(u_1) = 0, \dots, L(u_k) = 0.$$

We now have to prove that $L(u) = 0$. But this follows by the definition of u and the linearity of L, namely

$$L(u) = L(c_1 u_1 + \dots c_k u_k) = c_1 L(u_1) + \dots +, c_k L(u_k) = 0.$$

□

The superposition principle is extremely useful. Indeed, for certain PDEs it is enough to find just one solution, called the *fundamental solution*, and then all other solutions are constructed from the fundamental one.

We now show examples of linear equations that are crucial and their theory constitutes a starting point for many other equations.

Transport equation. It is a first order equation of the type

$$\partial_t u + \boldsymbol{v} \cdot \nabla u = 0. \tag{1.2}$$

It describes for instance the transport of a solid polluting substance along a channel; here u is the concentration of the substance and $\boldsymbol{v}$ is the stream speed. We will consider it in Chapter 2.

Diffusion or heat equation. It is a second order equation of the type

$$\partial_t u - \kappa \Delta u = 0, \tag{1.3}$$

where $\Delta = \partial_{x_1 x_1} + \dots \partial_{x_d x_d}$ is the Laplace operator. It describes the conduction of heat through a homogeneous and isotropic medium; u is the temperature and $\kappa > 0$ encodes the thermal properties of the material. Chapter 3 is devoted to the heat equation and its variants.

Wave equation. It is a second order equation of the type

$$\partial_{tt} u - c^2 \Delta u = 0. \tag{1.4}$$

It describes for instance the propagation of transversal waves of small amplitude in a perfectly elastic chord (e.g. of a guitar) if $d = 1$, or membrane (e.g. of a drum) if $d = 2$. If $d = 3$ it governs the propagation of electromagnetic waves in vacuum or of small amplitude sound waves. Here u may represent the wave amplitude and c is the propagation speed. We will take a look at this equation in Chapter 4.

Laplace or potential equation. It is a second order equation of the type

$$\Delta u = 0, \tag{1.5}$$

where $u = u(\boldsymbol{x})$. The diffusion and the wave equations model evolution phenomena. The Laplace equation describes the corresponding steady state, in which the solution does not depend on time anymore. Together with its non-homogeneous version $\Delta u = f$, called Poisson equation, it plays an important role in electrostatics as well. Chapter 5 is devoted to these equations.

1.2. **Examples of nonlinear PDEs.** Let us list now some examples of nonlinear equations.

Burgers equation. It is a first order equation, quasilinear equation of the type

$$\partial_t u + u\partial_x u = 0, \qquad x \in \mathbb{R}. \tag{1.6}$$

It governs a one dimensional flux of a non viscous fluid but it is used to model traffic dynamics as well. It can undergo interesting phenomena such as shock formation. We will discuss these topics in Chapter 2.

Navier-Stokes equations. They are three quasilinear scalar equations of second order and one linear equation of first order:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \nu\Delta \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases} \tag{1.7}$$

where $\boldsymbol{u} = (u_1(t, \boldsymbol{x}), u_2(t, \boldsymbol{x}), u_3(t, \boldsymbol{x}))$, $p = p(t, \boldsymbol{x})$ and $\boldsymbol{x} \in \mathbb{R}^3$. This equation governs the motion of a viscous, homogeneous and incompressible fluid. Here u is the fluid velocity, p its pressure and ν is the kinematic viscosity, given by the ratio between the fluid viscosity and its density. The term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ represents the inertial acceleration due to fluid transport. We will briefly meet the Navier-Stokes equations in Chapter 5.

2. Tools from calculus

We specify some of the notions we will constantly use throughout the lecture notes and recall some basic facts about sets, functions. We also state a few important results that will turn out to be useful later.

2.1. **Smooth and Lipschitz domains.** By a domain we mean an open connected set. Domains are usually denoted by the letter Ω. We will need to distinguish the domains Ω in $\mathbb{R}^d$ according to the degree of smoothness of their boundary. A point is in the boundary of Ω if any ball $B_r(\boldsymbol{x})$ of radius $r > 0$ and centered at $\boldsymbol{x}$ contains points of Ω and of its complement $\mathbb{R}^d \setminus \Omega$. The set of boundary points of Ω, the boundary of Ω, is denoted by $\partial\Omega$.

Definition 2.1. We say that Ω is a C^1-domain if for every point $\boldsymbol{x} \in \partial\Omega$, there exist a system of coordinates $y_1, y_2, \dots, y_{d-1}, y_d = (\boldsymbol{y}', y_d)$ with origin at $\boldsymbol{x}$, a ball $B(\boldsymbol{x})$ centered at $\boldsymbol{x}$ and a function φ defined in a neighborhood $\mathcal{N} \subset \mathbb{R}^{d-1}$ of $\boldsymbol{y}' = \boldsymbol{0}'$, such that

$$\varphi \in C^1(\mathcal{N}), \qquad \varphi(\boldsymbol{0}') = 0 \tag{2.1}$$

and

1. $\partial\Omega \cap B(\boldsymbol{x}) = \{(\boldsymbol{y}', y_d) : \ y_d = \varphi(\boldsymbol{y}'),\ \boldsymbol{y}' \in \mathcal{N}\}$,
2. $\Omega \cap B(\boldsymbol{x}) = \{(\boldsymbol{y}', y_d) : \ y_d > \varphi(\boldsymbol{y}'),\ \boldsymbol{y}' \in \mathcal{N}\}$.

The first condition expresses the fact that $\partial\Omega$ locally coincides with the graph of a C^1-function. The second one requires that Ω be locally placed on one side of its boundary (see Figure 1).

The boundary of a C^1-domain does not have corners or edges and for every point $\boldsymbol{p} \in \partial\Omega$, a tangent straight line ($d = 2$) or plane ($d = 3$) or hyperplane ($d > 3$) is well defined, together with the outward and inward normal unit vectors. Moreover these vectors vary continuously on $\partial\Omega$. The pairs $(\varphi, \mathcal{N})$ appearing in the above definition are called local charts. If they are all C^k-functions, for some $k \geq 1$, Ω is said to be a C^k-domain. If Ω is a C^k-domain for every $k \geq 1$, it is said to be a C^∞-domain. These are the domains we consider *smooth domains*.

In a great number of applications the relevant domains are rectangles, prisms, cones, cylinders or unions of them. Very important are polygonal domains obtained by triangulation procedures of smooth

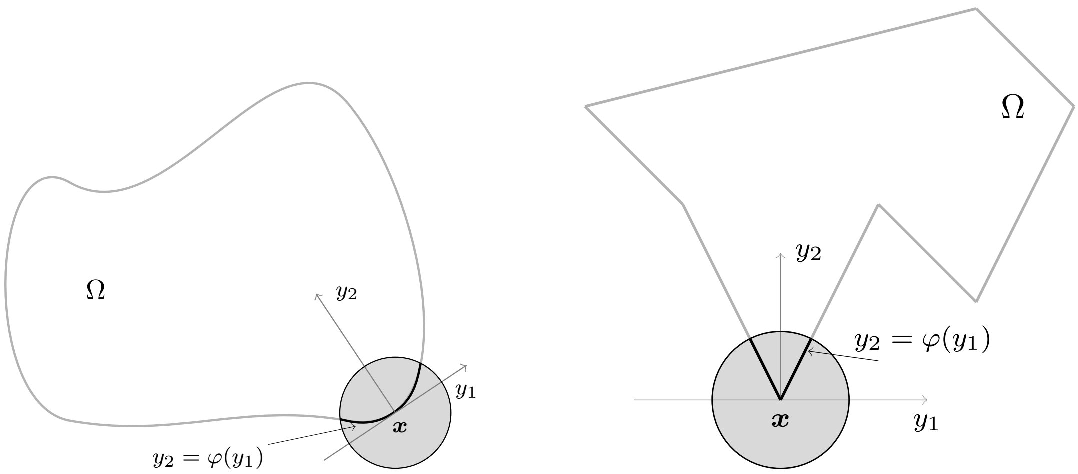

FIGURE 1. A C^1-domain and a Lipschitz domain

domains, for numerical approximations. These types of domains belong to the class of *Lipschitz domains*, whose boundary is locally described by the graph of a Lipschitz function. Recall that $\varphi : \Omega \to \mathbb{R}$ is Lipschitz if there exists $L > 0$ such that

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|, \tag{2.2}$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. The number L is called the Lipschitz constant of φ. Roughly speaking, a function is Lipschitz in Ω if the increment quotients in every direction are bounded. In fact, Lipschitz functions are differentiable at all points of their domain with the exception of a negligible set of points. Typical real Lipschitz functions in $\mathbb{R}^d$ are $\varphi(\boldsymbol{x}) = |\boldsymbol{x}|$. We say that a domain is Lipschitz if in Definition 2.1 the functions φ are Lipschitz (see again Figure 1).

2.2. **Integration by parts formulas.** Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. For vector fields

$$\boldsymbol{F} = (F_1, F_2, \ldots, F_d) : \Omega \to \mathbb{R}^d, \tag{2.3}$$

with $\boldsymbol{F} \in C^1(\overline{\Omega})$, the Gauss divergence formula holds:

$$\int_\Omega \nabla \cdot \boldsymbol{F}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{F}(\boldsymbol{\sigma}) \cdot \boldsymbol{n}(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma}, \tag{2.4}$$

where $\nabla \cdot \boldsymbol{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \ldots + \partial_{x_d} F_d$ is the divergence of $\boldsymbol{F}$, $\boldsymbol{n}$ denotes the *outward* normal unit vector to $\partial\Omega$, and $\mathrm{d}\boldsymbol{\sigma}$ is the "surface" measure on $\partial\Omega$, locally given in terms of local charts by

$$\mathrm{d}\boldsymbol{\sigma} = \sqrt{1 + |\nabla\varphi(\boldsymbol{y}')|^2} \mathrm{d}\boldsymbol{y}'. \tag{2.5}$$

A number of useful identities can be derived from (2.4). Applying (2.4) to $v\boldsymbol{F}$, with $v \in C^1(\overline{\Omega})$ a scalar function, and recalling the identity

$$\nabla \cdot (v\boldsymbol{F}) = v\nabla \cdot \boldsymbol{F} + \nabla v \cdot \boldsymbol{F}, \tag{2.6}$$

we obtain the following integration by parts formula

$$\int_\Omega v(\boldsymbol{x}) \nabla \cdot \boldsymbol{F}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} v(\boldsymbol{\sigma}) \boldsymbol{F}(\boldsymbol{\sigma}) \cdot \boldsymbol{n}(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} - \int_\Omega \nabla v(\boldsymbol{x}) \cdot \boldsymbol{F}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{2.7}$$

Choosing $\boldsymbol{F} = \nabla u$, for a scalar function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, since $\nabla \cdot \nabla u = \Delta u$ and $\nabla u \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} u$, the following Green's identity follows

$$\int_\Omega v(\boldsymbol{x}) \Delta u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} v(\boldsymbol{\sigma}) \partial_{\boldsymbol{n}} u(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} - \int_\Omega \nabla v(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{2.8}$$

In particular, the choice $v = 1$ yields

$$\int_\Omega \Delta u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \partial_{\boldsymbol{n}} u(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}. \tag{2.9}$$

If also $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, interchanging the roles of u and v in (2.8) and subtracting, we derive a second Green's identity

$$\int_\Omega [v(\boldsymbol{x})\Delta u(\boldsymbol{x}) - u(\boldsymbol{x})\Delta v(\boldsymbol{x})]\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} [v(\boldsymbol{\sigma})\partial_{\boldsymbol{n}} u(\boldsymbol{\sigma}) - u(\boldsymbol{\sigma})\partial_{\boldsymbol{n}} v(\boldsymbol{\sigma})]\,\mathrm{d}\boldsymbol{\sigma}. \tag{2.10}$$

All the above formulas hold for Lipschitz domains as well. In fact, the Rademacher theorem implies that at every point of the boundary of a Lipschitz domain, with the exception of a set of points of surface measure zero, there is a well-defined tangent plane. This is enough for extending the formulas (2.7), (2.8) and (2.10) to Lipchitz domains.

2.3. **Three important theorems.** We list here three important results that we will need later. The first one is the classical existence and uniqueness theorem for ordinary differential equations.

Theorem 2.2 (Cauchy-Lipschitz existence and uniqueness)**.** *Fix $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$, $a, b > 0$, and define the parallelepiped $R = \{(t, y)\colon t_0 \le t \le t_0 + a,\ |y - y_0| \le b\}$. We consider the ordinary differential equation*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{2.11}$$

where f is continuous on R (with maximum equaling $M > 0$), and uniformly Lipschitz continuous w.r.t. y. Then (2.11) has a unique solution $y(t)$ defined on $[t_0, t_0 + T]$, where $T = \min\{a, b/M\}$.

We then proceed to the inverse function theorem. For a vector field $\boldsymbol{F} \in C^1(\Omega)$ as in (2.3), we denote by $D\boldsymbol{F}(\boldsymbol{x})$ the differential of $\boldsymbol{F}$ at a point $\boldsymbol{x} \in \Omega$, identified with the $d \times d$ Jacobian matrix $[D\boldsymbol{F}]_{ij} = \partial_{x_j} F_i$, with $i, j = 1, \dots d$. The inverse function theorem states, roughly speaking, that a continuously differentiable vector field $\boldsymbol{F}$ is invertible in a neighborhood of any point $\boldsymbol{x} \in \Omega$ at which the linear transformation $D\boldsymbol{F}(\boldsymbol{x})$ is invertible.

Theorem 2.3. *Suppose $\boldsymbol{F} \in C^1(\Omega)$, $D\boldsymbol{F}(\boldsymbol{a})$ is invertible at some $\boldsymbol{p} \in \Omega$, and $\boldsymbol{q} = \boldsymbol{F}(\boldsymbol{p})$. Then*

(a) *there exist open sets U, V in $\mathbb{R}^d$ such that $\boldsymbol{p} \in U$, $\boldsymbol{q} \in V$, $\boldsymbol{F}$ is one-to-one on U and $\boldsymbol{F}(U) = V$;*
(b) *if $\boldsymbol{G}$ is the inverse of $\boldsymbol{F}$ (which exists, by (a)), defined in V by*

$$\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{x})) = \boldsymbol{x} \qquad (\boldsymbol{x} \in U), \tag{2.12}$$

then $\boldsymbol{G} \in C^1(V)$.

Writing the equation $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$ in component form, we arrive at the following interpretation of the conclusion of the above theorem: The system of d equations

$$y_i = F_i(x_1, \dots, x_d) \qquad (1 \le i \le d) \tag{2.13}$$

can be solved for $x_1, \dots, x_d$ in the terms of $y_1, \dots, y_d$, if we restrict $\boldsymbol{x}$ and $\boldsymbol{y}$ to small enough neighborhoods of $\boldsymbol{p}$ and $\boldsymbol{q}$; the solutions are unique and continuously differentiable.

Lastly, we state the implicit function theorem. If f is a continuously differential real function in the plane, then the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighborhood of any point (p, q) at which $f(p, q) = 0$ and $\partial_y f(p, q) \neq 0$. Likewise, one can solve for x in terms of y near (p, q) if $\partial_x f(p, q) \neq 0$. For a simple example which illustrate the need for assuming $\partial_y f(p, q) \neq 0$ consider $f(x, y) = x^2 + y^2 - 1$. The preceding very informal statement is the simplest case of the so-called implicit function theorem.

Let $\boldsymbol{F} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function. We think of $\mathbb{R}^{n+m}$ as the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ and we write a point of this product as $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, \dots, x_n, y_1, \dots, y_m)$. Starting from the given function $\boldsymbol{F}$, our goal is to construct a function $\boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^m$ whose graph $(\boldsymbol{x}, \boldsymbol{G}(\boldsymbol{x}))$ is precisely the set of all $(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$.

As noted above, this may not always be possible. We will therefore fix a point $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^{n+m}$ which satisfies $\boldsymbol{F}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{0}$, and we will ask for a $\boldsymbol{G}$ that works near the point $(\boldsymbol{p}, \boldsymbol{q})$. In other words, we want an open set $U \subset \mathbb{R}^n$ containing $\boldsymbol{p}$, an open set $V \subset \mathbb{R}^m$ containing $\boldsymbol{q}$, and a function $\boldsymbol{G} : U \to V$ such that the graph of $\boldsymbol{G}$ satisfies the relation $\boldsymbol{F} = \boldsymbol{0}$ on $U \times V$, and that no other points within $U \times V$ do so. For notational purposes, we indicate by $D\boldsymbol{F} = [D_{\boldsymbol{x}}\boldsymbol{F}|D_{\boldsymbol{y}}\boldsymbol{F}]$ the $m \times (n+m)$ Jacobian matrix of $\boldsymbol{F}$ (in all the variables), obtained by putting together the Jacobians in each of the variables, defined by $[D_{\boldsymbol{x}}\boldsymbol{F}]_{ij} = \partial_{x_j} F_i$ and $[D_{\boldsymbol{y}}\boldsymbol{F}]_{ij} = \partial_{y_j} F_i$. Notice that these are $m \times n$ and $m \times m$ matrices, respectively.

Theorem 2.4. *Suppose $\boldsymbol{F} \in C^1(\Omega)$, with $\Omega \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ such that $\boldsymbol{F}(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{0}$ for some point $(\boldsymbol{p},\boldsymbol{q}) \in \Omega$. Assume that $D_{\boldsymbol{y}}\boldsymbol{F}(\boldsymbol{p},\boldsymbol{q})$ is invertible. Then there exist open set $V \subset \mathbb{R}^{n+m}$ and $U \subset \mathbb{R}^n$, with $(\boldsymbol{p},\boldsymbol{q}) \in V$ and $\boldsymbol{p} \in U$, having the following property. To every $\boldsymbol{x} \in U$ corresponds a unique $\boldsymbol{y}$ such that*

$$(\boldsymbol{x},\boldsymbol{y}) \in V \qquad \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0}. \tag{2.14}$$

If this $\boldsymbol{y}$ is defined to be $\boldsymbol{G}(\boldsymbol{x})$, then $\boldsymbol{G} \in C^1(U)$ into $\mathbb{R}^m$, $\boldsymbol{G}(\boldsymbol{p}) = \boldsymbol{q}$,

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{G}(\boldsymbol{x})) = \boldsymbol{0}, \qquad \boldsymbol{x} \in U, \tag{2.15}$$

and

$$D\boldsymbol{G}(\boldsymbol{x}) = -\left(D_{\boldsymbol{y}}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{G}(\boldsymbol{x}))\right)^{-1} D_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{G}(\boldsymbol{x})), \qquad \boldsymbol{x} \in U. \tag{2.16}$$

3. Problems

Problem 1. For the PDEs below, find their order and check their linearity.

a. $\partial_t u + \partial_x u = -3u$,

b. $\partial_t u + \partial_{xxx} u - 6u\partial_x u = 0$,

c. $\partial_t u + \Delta u + \Delta^2 u + \dfrac{1}{2}|\nabla u|^2 = 0$,

d. $\partial_t u + u\partial_x u = \partial_{xx} u$,

e. $\partial_t u = \nabla \cdot (u^2 \nabla u)$,

f. $\nabla \cdot \left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0$.

Problem 2. Consider the heat equation $\partial_t u = \partial_{xx} u$ for $x \in \mathbb{R}$ and $t > 0$. We say that u is a classical solution if $u \in C^2((0,\infty) \times \mathbb{R})$ and satisfies the heat equation for every $x \in \mathbb{R}$ and $t > 0$. Verify that the functions

$$u(t,x) = t + \frac{1}{2}x^2, \qquad v(t,x) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}$$

are classical solutions to the heat equation.

Problem 3. Compute the surface area of a hemisphere of radius 1, by identifying it with the graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$, with $(x,y) \in \Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$.

Problem 4. Let $\boldsymbol{F}(x,y,z) = (2xy^2, 2x^2y, (x^2+y^2)z^2)$, and consider the cylinder $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 4, 0 \le z \le 2\}$. Compute the flux of $\boldsymbol{F}$ through $\partial\Omega$, namely

$$\int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{\sigma},$$

using the Gauss divergence formula.

Problem 5. Let $\boldsymbol{F}(x,y,z) = (x + yz, y - xz, z - \mathrm{e}^x \sin y)$, and consider $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 4, x^2+y^2 \ge 1\}$. Compute the flux of $\boldsymbol{F}$ through $\partial\Omega$, using the Gauss divergence formula.

Problem 6. Consider the initial value problem for the ordinary differential equation

$$y'(t) = \sqrt{|y(t)|}, \qquad y(0) = 0.$$

Show that there exist infinitely many solutions to this problem.

Problem 7. Let f be a non-negative $C^1([0,T])$ function and $h \in C([0,T])$ (not necessarily non-negative) be such that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \le h(t) f(t), \qquad \forall t \in [0,T].$$

Show that

$$f(t) \le f(0) \exp\left(\int_0^t h(s) \mathrm{d}s \right).$$

Problem 8. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a C^1-function. Show that f is not injective. (*Hint: if* $(\partial_x f)(\boldsymbol{p})\neq 0$ *for some* $\boldsymbol{p}\in\mathbb{R}^2$*, define* $\boldsymbol{F}:\mathbb{R}^2\to\mathbb{R}^2$ *as* $\boldsymbol{F}(x,y)=(f(x,y),y)$ *and use the inverse function theorem.*)

Problem 9. Consider the set

$$H=\left\{(a,b,c,d,e)\in\mathbb{R}^5:\ \text{the polynomial } ax^4+bx^3+cx^2+dx+e \text{ has at least one real root}\right\}.$$

Prove that there exists a neighborhood U of the point $\boldsymbol{p}=(1,2,-4,3,-2)$ such that $U\subset H$. Note that $\boldsymbol{p}\in H$ since the corresponding polynomial has a root at $x=1$.

Problem 10. Let $\Omega\subset\mathbb{R}^d$ be a bounded domain, $u\in C^1(\Omega)$ and $v\in C(\Omega)$. Given $\varphi\in C^\infty(\Omega)$ we say that it is a C^∞*-compactly supported* function in Ω, and we write $\varphi\in C_c^\infty(\Omega)$, if $\varphi\equiv 0$ outside of a compact set in Ω (in particular, $\varphi|_{\partial\Omega}\equiv 0$). Assume that for all $\varphi\in C_c^\infty(\Omega)$ the identity

$$\int_\Omega u(\boldsymbol{x})\partial_{x_i}\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=-\int_\Omega v(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

holds for some $i\in\{1,\dots,d\}$. Then, show that $v=\partial_{x_i}u$.

Problem 11. Let $\Omega\subset\mathbb{R}^d$ be a bounded smooth domain, and consider the Dirichlet problem for the Laplace equation, that is

$$\begin{cases}\Delta u=0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases}$$

Show that any solution $u\in C^2(\overline{\Omega})$ of the above problem has to be identically zero.

Problem 12. Let $\Omega\subset\mathbb{R}^d$ be a bounded smooth domain, $f\in C(\overline{\Omega})$ and consider the Neumann problem for the Laplace equation, that is

$$\begin{cases}\Delta u=f, & \text{in } \Omega,\\ \partial_{\boldsymbol{n}}u=0, & \text{on } \partial\Omega.\end{cases}$$

Show that if there is a solution $u\in C^2(\overline{\Omega})$ of the above problem, then necessarily

$$\int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0\,.$$

Problem 13. Let $\Omega\subset\mathbb{R}^d$ be a bounded smooth domain and $u,w\in C_c^1(\Omega)$ (see Problem 10 for the definition of compactly supported functions). Consider a vector field $\boldsymbol{v}=(v_1,\dots,v_d):\Omega\to\mathbb{R}^d$ such that $\boldsymbol{v}\in C^1(\Omega)$ and $\nabla\cdot\boldsymbol{v}=0$. Prove that

$$\int_\Omega(\boldsymbol{v}(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x}))\,w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=-\int_\Omega(\boldsymbol{v}(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x}))\,u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.1}$$

Assume now that $u(t,\cdot)\in C_c^1(\Omega)$ is a solution to (1.2) with the vector field $\boldsymbol{v}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}=0.$$

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CHAPTER 2: FIRST ORDER EQUATIONS

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ABSTRACT. These notes follow closely the book of S. Salsa [1].

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1. LINEAR TRANSPORT EQUATIONS

We start by consider the simplest possible first order partial differential equation. Consider a fluid, water, say, flowing at a constant rate $a > 0$ along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Calling $\rho(t, x)$ its concentration at position x and at time t, the amount of pollutant in the interval $[0, b]$ at the time t is

$$M = \int_0^b \rho(t, x)\mathrm{d}x. \tag{1.1}$$

At the later time $t + h$, the same molecules of pollutant have moved to the right by $a \cdot h$. Hence

$$M = \int_0^b \rho(t, x)\mathrm{d}x = \int_{ah}^{b+ah} \rho(t + h, x)\mathrm{d}x. \tag{1.2}$$

Differentiating with respect to b, we get

$$\rho(t, b) = \rho(t + h, b + ah). \tag{1.3}$$

Differentiating with respect to h and putting $h = 0$, we get

$$0 = \partial_t\rho + a\partial_x\rho, \tag{1.4}$$

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which is the first PDE that we will study.

1.1. **Pure transport.** Consider the *pure transport* equation
$$\partial_t \rho + a\partial_x \rho = 0. \tag{1.5}$$
Introducing the vector
$$\boldsymbol{a} = \begin{pmatrix} a \\ 1 \end{pmatrix}, \tag{1.6}$$
we immediately see from (1.5) that
$$\nabla\rho \cdot \boldsymbol{a} = a\partial_x \rho + \partial_t \rho = 0. \tag{1.7}$$
Therefore $\nabla\rho$ and $\boldsymbol{a}$ are orthogonal. However, $\nabla\rho$ is also orthogonal to the level lines of ρ, along which ρ is constant. Therefore the level lines of ρ are the straight lines parallel to $\boldsymbol{a}$, of equation
$$x = at + x_0, \tag{1.8}$$
for an arbitrary $x_0 \in \mathbb{R}$. These straight lines are called *characteristics* (see Figure 1).

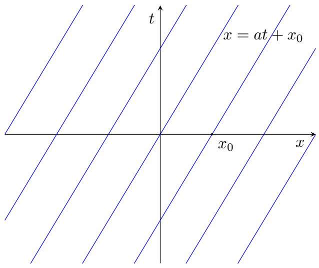


FIGURE 1. The characteristic lines in the (x, t) plane.

Computing ρ along the characteristics and using the transport equation (1.5), we rightfully find
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = (\partial_t \rho + a\partial_x \rho)(t, at + x_0) = 0. \tag{1.9}$$
Hence, ρ is constant along the characteristics. In particular, given an initial condition
$$\rho(0, x) = g(x), \qquad x \in \mathbb{R}, \tag{1.10}$$
for an assigned function g, the above calculation tells us that
$$\rho(t, at + x_0) = \rho(0, x_0) = g(x_0). \tag{1.11}$$
Using (1.8), or simply changing variables, we can then write the solution to the (1.5) with initial condition (1.10) as
$$\rho(t, x) = g(x - at). \tag{1.12}$$
The solution (1.12) represents a *traveling wave*, moving with speed a in the positive x-direction (see Figure 2).

1.2. **Distributed source.** Given $f = f(t, x)$, consider the initial value problem
$$\begin{cases} \partial_t \rho + a\partial_x \rho = f(t, x), & t > 0, \ x \in \mathbb{R}, \\ \rho(0, x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1.13}$$
The function f represents the intensity of an external distributed source along the channel. Using the characteristics and arguing as in (1.9), we find
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = f(t, at + x_0). \tag{1.14}$$
Integrating on $(0, t)$, we find
$$\rho(t, at + x_0) = g(x_0) + \int_0^t f(s, as + x_0)\mathrm{d}s. \tag{1.15}$$

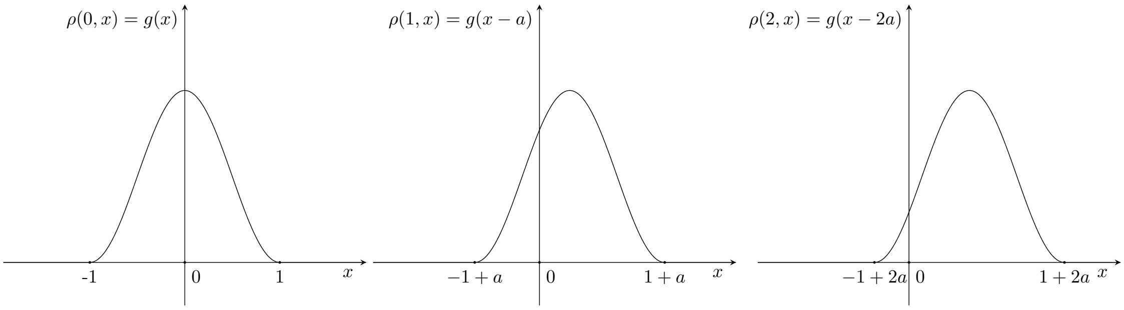


FIGURE 2. The solution of the pure transport equation, at times $t = 0, 1, 2$. It is simply a translation of the initial datum, to the right since $a > 0$.

Therefore, the solution to (1.13) is

$$\rho(t,x) = g(x-at) + \int_0^t f(s, x - a(t-s))\mathrm{d}s. \tag{1.16}$$

1.3. **Damped traveling waves.** Suppose that the density ρ decays at the rate $-\delta\rho$, for some $\delta > 0$. This could be due to biological decomposition if the density ρ models the concentration of a chemical substance in water. Without external sources, the mathematical model to study becomes

$$\begin{cases} \partial_t \rho + a\partial_x \rho = -\delta\rho, & t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1.17}$$

Again, from an analogous computation as in (1.9), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = -\delta\rho(t, at + x_0). \tag{1.18}$$

This is exactly an ODE of the type $y' = -\delta y$, where $y(t) = \rho(t, at + x_0)$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\delta t}\rho(t, at + x_0)\right] = 0, \tag{1.19}$$

and integrating we obtain

$$\mathrm{e}^{\delta t}\rho(t, at + x_0) = g(x_0). \tag{1.20}$$

Thus, the solution to (1.17) is

$$\rho(t,x) = \mathrm{e}^{-\delta t} g(x - at). \tag{1.21}$$

Notice that this a traveling wave like (1.12), except that the amplitude decays in time (if, for example, g is a bounded function). For this reason, it is called damped traveling wave (see Figure 3).

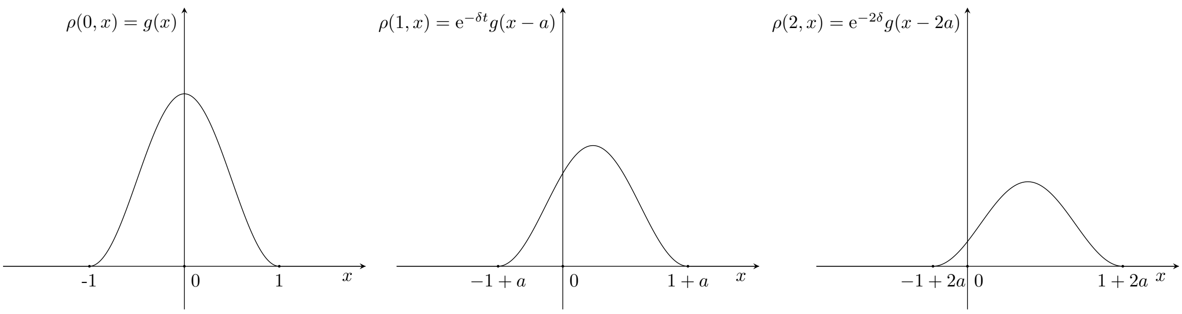


FIGURE 3. The solution of (1.17), at times $t = 0, 1, 2$. It is still a translation of the initial datum, but with decreasing amplitude.

2. The method of characteristics

The examples presented in the previous section are particular cases of equations of the form

$$a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u), \tag{2.1}$$

for $x, y \in \Omega \subset \mathbb{R}^2$ and with $a, b, c : \mathbb{R}^3 \to \mathbb{R}$ continuously differentiable functions (where we renamed the time-variable as y). Notice that this is a first order quasilinear equation. The ideas involved are generalizations of what we have seen in Section 1. As before, solutions to (2.1) can be constructed by taking advantage of the following geometric interpretation: let (x_0, y_0, z_0) be a point on the graph of u, namely, $z_0 = u(x_0, y_0)$. The tangent plane to the graph u at (x_0, y_0, z_0) has equation

$$\partial_x u(x_0, y_0)(x - x_0) + \partial_y u(x_0, y_0)(y - y_0) - (z - z_0) = 0. \tag{2.2}$$

Hence, the vector

$$\boldsymbol{n}_0 = (\partial_x u(x_0, y_0), \partial_y u(x_0, y_0), -1) \tag{2.3}$$

is normal to the plane defined by (2.2), but also normal to the vector

$$\boldsymbol{v}_0 = (a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0)), \tag{2.4}$$

by (2.1). Thus, $\boldsymbol{v}_0$ is tangent to the graph of u. In other words, (2.1) says that, at every point (x, y, z), the graph of any solution is tangent to the vector field

$$\boldsymbol{v} = (a(x, y, z), b(x, y, z), c(x, y, z)). \tag{2.5}$$

In this case we say that the graph of a solution is an integral surface of the vector field $\boldsymbol{v}$. The idea is to construct integral surfaces of $\boldsymbol{v}$ as union of curves tangent to $\boldsymbol{v}$ at every point. Let's say we parametrize a curve on the graph of u as $(x(s), y(s), z(s))$, where

$$z(s) = u(x(s), y(s)). \tag{2.6}$$

As noticed above, we know that the tangent vector of this curve must be $\boldsymbol{v}$. This means that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x, y, z), \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = b(x, y, z), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = c(x, y, z). \tag{2.7}$$

The curves $(x(s), y(s), z(s))$ are called *characteristics*, and we remark again that $z(s)$ gives the value of u along a characteristic, see (2.6). Indeed, this can also verified by differentiating (2.6) and using (2.1) with (2.7) to get

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}s} &= \partial_x u(x(s), y(s))\frac{\mathrm{d}x}{\mathrm{d}s} + \partial_y u(x(s), y(s))\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= a(x(s), y(s), z(s))\partial_x u(x(s), y(s)) + b(x(s), y(s), z(s))\partial_y u(x(s), y(s)) \\ &= c(x(s), y(s), z(s)).\end{aligned} \tag{2.8}$$

Thus, along a characteristic the partial differential equation (2.1) degenerates into an ordinary differential equation.

2.1. **The Cauchy problem for first order quasilinear equations.** Let $I \subset \mathbb{R}$ be an interval containing 0, and $\gamma : I \to \mathbb{R}^2$ be a smooth curve in the (x, y)-plane, parametrized as

$$I \ni \tau \mapsto \gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)) \in \mathbb{R}^2. \tag{2.9}$$

Our goal is to find a solution to the Cauchy problem

$$\begin{cases} a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u), & (x, y) \in \mathbb{R}^2, \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau), & \tau \in I. \end{cases} \tag{2.10}$$

We assume that γ_1, γ_2 and g are continuously differentiable in I. The data are often assigned in the form of initial values, where $\gamma_1(\tau) = \tau$ and $\gamma_2(\tau) = 0$, so that

$$u(\tau, 0) = g(\tau) \tag{2.11}$$

and y plays the role of time (compare with (1.13)), but it does not always have to be this way. Writing down the characteristic system (2.7), we look at the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x,y,z), \qquad x(0)=\gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x,y,z), \qquad y(0)=\gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x,y,z), \qquad z(0)=g(\tau), \end{aligned} \tag{2.12}$$

where $\tau \in I$ is a parameter. Invoking the Cauchy-Lipschitz theorem for ODEs, (2.12) has a unique solution in a neighborhood of $s=0$, for every $\tau \in I$, which we denote by

$$x = X(s,\tau), \qquad y = Y(s,\tau), \qquad z = Z(s,\tau). \tag{2.13}$$

We now need to check that the above (X,Y,Z) define a function $u(x,y)$, so that this u is a solution to (2.12). To check invertibility, let us think in a neighborhood of $s=0$, and for a fixed $\tau_0 \in I$ set

$$X(0,\tau_0)=\gamma_1(\tau_0)=x_0, \qquad Y(0,\tau_0)=\gamma_2(\tau_0)=y_0, \qquad Z(0,\tau_0)=g(\tau_0)=z_0. \tag{2.14}$$

The goal is to invert the first two equations in (2.13). In particular, if we call $\boldsymbol{x}=(x,y)$, $\boldsymbol{s}=(s,\tau)$ and $\boldsymbol{F}=(X,Y)$ we have the system

$$\boldsymbol{x}=\boldsymbol{F}(\boldsymbol{s}).$$

We then want to find a function $\boldsymbol{G}=(S,T)$ of class C^1 such that $\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{s}))=\boldsymbol{s}$, namely $s=S(x,y)$ and $\tau=T(x,y)$. If we can find $\boldsymbol{G}$, from the third equation $z=Z(s,\tau)$, reasoning as in (2.6), we get

$$z = Z(S(x,y),T(x,y)) = u(x,y) \tag{2.15}$$

and therefore we solved the problem (2.10). To find the inverse, first notice that

$$S(x_0,y_0)=0, \qquad T(x_0,y_0)=\tau_0, \tag{2.16}$$

so denote $\boldsymbol{s_0}=(0,\tau_0)$. Then, from the inverse function theorem, we know that $\boldsymbol{G}$ exists in a neighbourhood of (x_0,y_0) if

$$|D\boldsymbol{F}(\boldsymbol{s_0})| = \begin{vmatrix} \partial_s X(0,\tau_0) & \partial_\tau X(0,\tau_0) \\ \partial_s Y(0,\tau_0) & \partial_\tau Y(0,\tau_0) \end{vmatrix} \neq 0. \tag{2.17}$$

From (2.12) and (2.13), this is equivalent to

$$\begin{vmatrix} a(x_0,y_0,z_0) & \gamma_1'(\tau_0) \\ b(x_0,y_0,z_0) & \gamma_2'(\tau_0) \end{vmatrix} = a(x_0,y_0,z_0)\gamma_2'(\tau_0) - b(x_0,y_0,z_0)\gamma_1'(\tau_0) \neq 0. \tag{2.18}$$

The above means that the vectors $(a(x_0,y_0,z_0), b(x_0,y_0,z_0))$ and $(\gamma_1'(\tau_0),\gamma_2'(\tau_0))$ are not parallel. Therefore, if condition (2.18) holds, then (2.15) is a well defined function of class C^1. A precise statement of what we have just discussed is contained in the following result.

Theorem 2.1. *Let a,b,c be C^1-functions in a neighborhood of $(x_0,y_0,u(x_0,y_0))$ and assume that γ_1,γ_2,g are C^1-functions in I. If* (2.18) *holds, then, in a neighborhood of (x_0,y_0), there exists a unique C^1-solution $u=u(x,y)$ of the Cauchy problem* (2.10)*. Moreover, u is defined by the parametric equations* (2.13)*.*

Let us summarize the steps to solve the Cauchy problem (2.10).

Step 1. Determine the solution

$$x = X(s,\tau), \qquad y = Y(s,\tau), \qquad z = Z(s,\tau) \tag{2.19}$$

of the characteristic system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x,y,z), \qquad x(0)=\gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x,y,z), \qquad y(0)=\gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x,y,z), \qquad z(0)=g(\tau). \end{aligned} \tag{2.20}$$

Step 2. For each $\tau_0 \in I$, check that

$$\begin{vmatrix} \partial_s X(0,\tau_0) & \partial_s Y(0,\tau_0) \\ \gamma_1'(\tau_0) & \gamma_2'(\tau_0) \end{vmatrix} \neq 0. \tag{2.21}$$

2.2. **The Cauchy problem for first order linear equations.** In the special case in which a, b and c do not depend on u, (2.10) becomes the linear equation

$$\begin{cases} a(x,y)\partial_x u + b(x,y)\partial_y u = c(x,y), & (x,y) \in \mathbb{R}^2, \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau), & \tau \in I. \end{cases} \tag{2.22}$$

The characteristic system (2.20) then becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x,y), \qquad x(0) = \gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x,y), \qquad y(0) = \gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x,y), \qquad z(0) = g(\tau). \end{aligned} \tag{2.23}$$

It is important to notice that once we solve for x, y, then z is simply obtained by integration. In other words, the equation for z is *decoupled* from the other two. Thus, once we determine

$$x = X(s,\tau), \qquad y = Y(s,\tau), \tag{2.24}$$

then

$$z = Z(s,\tau) = g(\tau) + \int_0^s c(X(\sigma,\tau), Y(\sigma,\tau))\mathrm{d}\sigma. \tag{2.25}$$

Thus

$$u(X(s,\tau), Y(s,\tau)) = g(\tau) + \int_0^s c(X(\sigma,\tau), Y(\sigma,\tau))\mathrm{d}\sigma. \tag{2.26}$$

which gives a well defined function of class C^1. At this point, one still has to check (2.21) and invert (2.24).

Example 2.2. We want to solve the Cauchy problem

$$\begin{cases} -y\partial_x u + x\partial_y u = 4xy, & (x,y) \in \mathbb{R}^2, \\ u(x,0) = g(x), & x > 0. \end{cases} \tag{2.27}$$

In this case, $\gamma(\tau) = (\tau, 0)$, for $\tau \in (0,\infty)$. First, we have to solve the characteristic system, which in this case reads

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -y, \qquad x(0) = \tau, \tag{2.28}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = x, \qquad y(0) = 0, \tag{2.29}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 4xy, \qquad z(0) = g(\tau). \tag{2.30}$$

From the first two equations we obtain

$$x = X(s,\tau) = \tau\cos(s), \qquad y = Y(s,\tau) = \tau\sin(s), \tag{2.31}$$

so that from (2.26) we obtain

$$u(X(s,\tau), Y(s,\tau)) = Z(s,\tau) = g(\tau) + 4\tau^2 \int_0^s \cos(\sigma)\sin(\sigma)\mathrm{d}\sigma = g(\tau) + 2\tau^2\sin^2(s). \tag{2.32}$$

Now, (2.31) is a familiar change of coordinate, the polar coordinates! One can easily check (2.21) and thus (since $x > 0$)

$$s = S(x,y) = \arctan\left(\frac{y}{x}\right), \qquad \tau = T(x,y) = \sqrt{x^2 + y^2}. \tag{2.33}$$

From (2.32) we then find that

$$u(x,y) = g\left(\sqrt{x^2+y^2}\right) + 2y^2 \tag{2.34}$$

is the unique solution to (2.27).

3. Scalar conservation laws

The nonlinear one-dimensional generalization of (1.5) is the *scalar conservation law*

$$\partial_t \rho + \partial_x [q(\rho)] = 0, \tag{3.1}$$

where $q(\rho)$ is a nonlinear function. If q is differentiable, the above equation can be equivalently written as

$$\partial_t \rho + q'(\rho)\partial_x \rho = 0. \tag{3.2}$$

Both formulations are useful. We can naturally put (3.2) in the framework analyzed in the previous section (see (2.10)) to study the Cauchy problem

$$\begin{cases} \partial_t \rho + q'(\rho)\partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

In particular, from (2.20) we can write down the characteristic system as

$$\frac{\mathrm{d}x}{\mathrm{d}s} = q'(z), \qquad x(0) = \tau, \tag{3.4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad t(0) = 0, \tag{3.5}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 0, \qquad z(0) = g(\tau). \tag{3.6}$$

Solving the system gives

$$x = X(s,\tau) = \tau + q'(g(\tau))s, \qquad t = T(s,\tau) = s, \qquad z = Z(s,\tau) = g(\tau). \tag{3.7}$$

In particular, we easily obtain the formula

$$x = \tau + q'(g(\tau))t. \tag{3.8}$$

Thus, the characteristics are straight lines with slope $q'(g(\tau))$. Different values of τ give, in general, different values of the slope. We are thus facing again a problem of *invertibility* of the coordinate system. A typical situation in which this is problematic is depicted in Figure 4. For certain initial data, it may well happen that two characteristics intersect, hence producing a discontinuity in the solution.

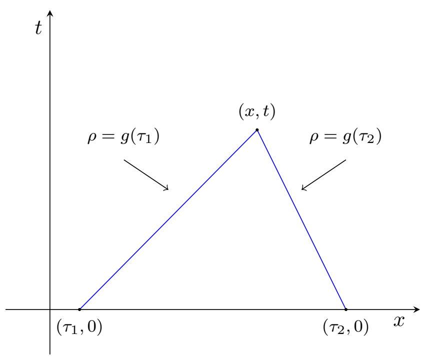


FIGURE 4. Intersection of characteristics. At the point (x,t), the solution ρ has a discontinuity due to the intersection of characteristics.

The purpose now is to understand how to deal with this and other issues that can arise in conservation laws.

3.1. **Existence of classical solutions.** We begin by giving a precise definition of solutions to (3.22), commonly referred to as classical

Definition 3.1. Let $T > 0$ be given. A *classical solution* ρ to (3.3) is a function in $C^1([0,T) \times \mathbb{R})$ such that $\rho(0,x) = g(x)$ and that satisfies (3.3) for every $(t,x) \in (0,T) \times \mathbb{R}$. If $T = \infty$, then ρ is a global-in-time classical solution.

The following is an existence and uniqueness theorem for classical solutions to (3.3).

Theorem 3.2. *Suppose that $q \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, and assume that there exists $M > 0$ such that*

$$\sup_{r\in\mathbb{R}} |q''(r)| \leq M, \qquad \sup_{r\in\mathbb{R}} |g'(r)| \leq M. \tag{3.9}$$

Then there exists $T = T(M) > 0$ such that there exists a unique classical solution ρ to (3.3) defined on $[0,T] \times \mathbb{R}$. If $q \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, and further

$$q''(r)g'(s) \geq 0, \qquad \forall r,s \in \mathbb{R}, \tag{3.10}$$

then the solution is global in time, namely, it is defined for $(t,x) \in [0,\infty) \times \mathbb{R}$.

Proof. From (3.8), we have that the solution to (3.3) is given by

$$\rho(t,x) = g\left(x - q'(g(\tau))t\right). \tag{3.11}$$

Since $\rho(t,x) = g(\tau)$ along the characteristic based at $(0,\tau)$, from (3.11) we obtain that ρ is implicitly defined by the equation

$$G(t,x,\rho) = \rho - g\left(x - q'(\rho)t\right) = 0. \tag{3.12}$$

Since g and ρ are regular enough, the Implicit Function Theorem implies that equation (3.12) defines ρ as a function of (t,x), as long as

$$\partial_\rho G(t,x,\rho) = 1 + tq''(\rho)g'\left(x - q'(\rho)t\right) \neq 0. \tag{3.13}$$

Now, if (3.9) holds, then the above condition holds whenever $t < M^{-2}$, proving the first part of the theorem. If furthermore (3.10) is satisfied, then (3.13) is satisfied for every $t \geq 0$, thereby concluding the proof.

□

Example 3.3 (Burgers equation)**.** The simplest nonlinearity that one can think of is a quadratic one, when

$$q(\rho) = \frac{\rho^2}{2}. \tag{3.14}$$

From (3.2), this gives the Cauchy problem for the so called *Burgers equation*

$$\begin{cases} \partial_t \rho + \rho\partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3.15}$$

We claim that a global classical solution exists if and only if g is increasing. Since $q(\rho) = \rho^2/2$ in this case, if g is increasing then (3.10) is satisfied, and therefore Theorem 3.2 guarantees that we have a global solution. On the other hand, suppose that g is not increasing. Then there are $\tau_1 < \tau_2$ such that $g(\tau_1) > g(\tau_2)$. From (3.8), we have the two characteristics

$$x = \tau_1 + g(\tau_1)t, \qquad x = \tau_2 + g(\tau_2)t, \tag{3.16}$$

which intersect at

$$t = \frac{\tau_2 - \tau_1}{g(\tau_1) - g(\tau_2)} > 0, \tag{3.17}$$

which is equivalent to the fact the solution has a discontinuity, and hence cannot be a classical solution. Notice that Theorem 3.2 still guarantees the existence of a classical solution for small times (less than some $T > 0$). In particular, from (3.12), for $t < T$ we know that

$$\rho = g\left(x - \rho t\right). \tag{3.18}$$

Differentiating with respect to x we find

$$\partial_x \rho = g'(x-\rho t)(1-t\partial_x \rho) \qquad \Rightarrow \qquad \partial_x \rho = \frac{g'(x-\rho t)}{1+g'(x-\rho t)\,t}. \tag{3.19}$$

Now, assume that $g' < 0$ is bounded below and attains its minimum at some point $\tau_s \in \mathbb{R}$. Let

$$t_s = -[g'(\tau_s)]^{-1} > 0, \qquad x_s = \tau_s + g(\tau_s)t_s. \tag{3.20}$$

By construction, (t_s, x_s) is a point on the characteristic $x = \tau_s + g(\tau_s)t$ emanating from $(0, \tau_s)$. From (3.19), we then take a limit as $(t, x) \to (t_s, x_s)$ *on the characteristic* $x = \tau_s + g(\tau_s)t$ and find

$$\lim_{(t,x)\to(t_s,x_s)} \partial_x \rho(t,x) = \lim_{\tau\to\tau_s} \frac{g'(\tau)}{1-g'(\tau)\,[g'(\tau_s)]^{-1}} = \infty. \tag{3.21}$$

Hence (t_s, x_s) corresponds to the time and location of a *shock*, at which the solution ceases to be a C^1 function.

3.2. **Weak solutions.** We have seen that the method of characteristics is not sufficient, in general, to determine the solution of an initial value problem for all times $t > 0$. This is a common theme in PDEs. If we are too greedy about the regularity requirements in the definition of a solution to a problem, we may not be able to find one for all times. We introduce a more flexible definition of solutions for the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x [q(\rho)] = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3.22}$$

as follows. Let v be a smooth function in $[0,\infty) \times \mathbb{R}$, with compact support. We call v a *test function*. Multiply the differential equation by v and integrate on $[0,\infty) \times \mathbb{R}$, to get

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x [q(\rho)])\, v \mathrm{d}x\mathrm{d}t = 0. \tag{3.23}$$

An integration by parts in both t and x and the fact that v has compact support yields

$$\int_0^\infty \int_{\mathbb{R}} (\rho \partial_t v + q(\rho)\partial_x v)\, \mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}} g(x)v(0,x)\mathrm{d}x. \tag{3.24}$$

We have obtained an integral equation, valid for every test function v, in which *no derivative* on ρ appears. On the other hand, suppose that a smooth function ρ satisfies (3.24) for every test function v. Integrating by parts in the reverse order, we arrive to the equation

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x [q(\rho)])\, v \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} [g(x) - \rho(0,x)]\, v(0,x)\mathrm{d}x = 0. \tag{3.25}$$

Choosing v vanishing for $t = 0$, the second integral is zero and the arbitrariness of v implies that ρ satisfies the differential equation in (3.22). Choosing now v non vanishing for $t = 0$, and using that $\partial_t \rho + \partial_x [q(\rho)] = 0$ we get that $\rho(0,x) = g(x)$, and therefore ρ is the classical solution to (3.22). It is then natural to introduce the following notion of solution.

Definition 3.4. A function ρ, bounded in $[0,\infty) \times \mathbb{R}$ is a *weak solution* to (3.22) if equation (3.24) holds for every test function v in $[0,\infty) \times \mathbb{R}$ with compact support.

The only requirement on ρ is being a bounded function, so in particular it is allowed for ρ to be discontinuous. This will help us dealing with shocks. However, a possible drawback in enlarging the class of solutions is that we may lose their uniqueness.

Example 3.5 (Rarefaction waves for Burgers)**.** Consider the Burgers equation (3.15) with initial datum

$$g(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{3.26}$$

The characteristics are the straight lines

$$x = \tau + g(\tau)t. \tag{3.27}$$

Therefore, $\rho(t,x) \equiv 0$ for $x \le 0$, and $\rho(t,x) \equiv 1$ for $x > t$. Since the region $S = \{(t,x) : 0 < x \le t\}$ is not covered by the characteristics, we connect the states 0 and 1 through a *rarefaction wave* (see Figure 5a). This mean that we seek a solution $\rho(t,x) = h(x/t)$, for some function h to be determined. Using (3.15), we can write an equation for h as

$$-\frac{x}{t^2}h' + \frac{1}{t}hh' = 0 \qquad \Rightarrow \qquad h(x/t) = \frac{x}{t}. \tag{3.28}$$

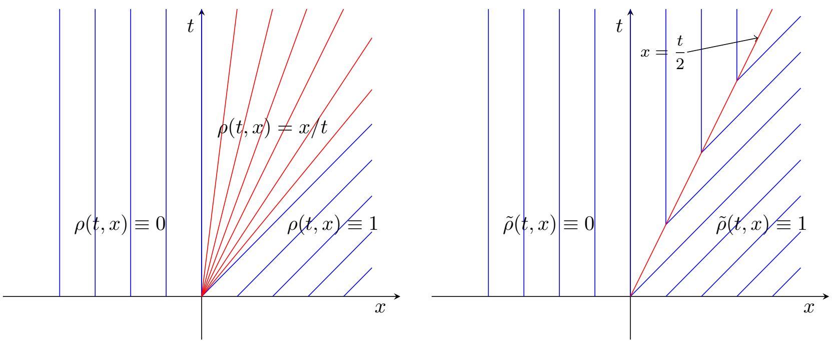
Thus, the weak solution constructed can be written as

$$\rho(t,x) = \begin{cases} 0, & x \le 0, \\ x/t, & 0 < x < t, \\ 1, & x \ge t. \end{cases} \tag{3.29}$$

However, ρ is not the unique weak solution, but there exists also a shock wave solution (see Figure 5b). It is not hard to check that

$$\tilde{\rho}(t,x) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2, \end{cases} \tag{3.30}$$

is another weak solution. As we shall see, this shock wave has to be considered not physically acceptable. What happens is that we have more than one way to "fill in" the region S that is not covered by the characteristics, as depicted in Figure 5.



(A) The rarefaction wave solution ρ (B) The shock wave solution $\tilde{\rho}$

FIGURE 5. An example of non-uniqueness of solutions to the Burgers equation with initial datum (3.26).

3.3. **The Rankine-Hugoniot condition.** The appearance of shocks is hidden in the definition of weak solutions. Consider an open set V, contained in the half-plane $t > 0$, partitioned into two disjoint domains V_+ and V_- by a smooth (shock) curve Γ of equation $x = \sigma(t)$, as in Figure 6.

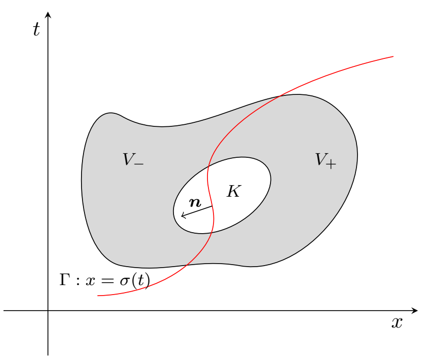


FIGURE 6. A shock curve dividing a domain V

Now, assuming that ρ is a classical solution on both sides of Γ, with continuous derivatives up to Γ, we take a test v function supported in a compact set $K \subset V$ that intersects Γ. Since $v(0,x) = 0$, we compute from the definition of weak solution in (3.24) that

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} (\rho \partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t \\ &= \int_{V_+} (\rho \partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t + \int_{V_-} (\rho \partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.31}$$

Integrating by parts and observing that $v = 0$ on $\partial V_+ \setminus \Gamma$, we have

$$\begin{aligned} \int_{V_+} (\rho \partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t &= -\int_{V_+} (\partial_t \rho + \partial_x q(\rho))\, v\mathrm{d}x\mathrm{d}t + \int_\Gamma (\rho_+ n_2 + q(\rho_+)n_1)\, v\mathrm{d}\gamma \\ &= \int_\Gamma (\rho_+ n_2 + q(\rho_+)n_1)\, v\mathrm{d}\gamma, \end{aligned} \tag{3.32}$$

where ρ_+ denotes the value of ρ on Γ from the V_+ side, $\boldsymbol{n} = (n_1, n_2)$ is the outward unit normal vector on ∂V_+, and $\mathrm{d}\gamma$ denotes the arc length on Γ. Similarly, since $\boldsymbol{n}$ is inward with respect to V_-, we find

$$\int_{V_-} (\rho \partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t = -\int_\Gamma (\rho_- n_2 + q(\rho_-)n_1)\, v\mathrm{d}\gamma, \tag{3.33}$$

where ρ_- denotes the value of ρ on Γ from the V_- side. Thus, plugging (3.32)-(3.33) back into (3.31) and using that v is arbitrary, we find

$$(\rho_+ - \rho_-)n_2 + (q(\rho_+) - q(\rho_-))n_1 = 0, \qquad \text{on } \Gamma. \tag{3.34}$$

Of course, if ρ is continuous on Γ, the above is automatically satisfied since $\rho_+ = \rho_-$. Otherwise, since Γ is given by the equation $x = \sigma(t)$, writing explicitly that

$$\boldsymbol{n} = (n_1, n_2) = \frac{1}{\sqrt{1 + |\sigma'(t)|^2}}(-1, \sigma'(t)), \tag{3.35}$$

we find that

$$\sigma' = \frac{q(\rho_+(t,\sigma)) - q(\rho_-(t,\sigma))}{\rho_+(t,\sigma) - \rho_-(t,\sigma)}, \tag{3.36}$$

known as the *Rankine-Hugoniot condition* for the shock Γ. Therefore, to find the equation of a shock, one in general has to solve a nonlinear ODE of the type above. In general, functions constructed by connecting classical solutions and rarefaction waves in a continuous way are weak solutions. The same is true for shock waves satisfying the Rankine-Hugoniot condition. However, the shock wave solution of Example 3.5 does satisfy the Rankine-Hugoniot condition (double-check this!), and so uniqueness of solutions is not guaranteed even if we require (3.36) to hold. We now look at another example.

Example 3.6 (Shock for Burgers)**.** Consider the Burgers equation (3.15) with initial datum

$$g(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{3.37}$$

As in Example 3.5, we know that the characteristics are straight line (see Figure 7), but they start intersecting right away at the point $(t,x) = (0,0)$, forming a shock. Since $\rho_+ \equiv 0$ to the right of the shock and $\rho_- \equiv 1$ to the left, the Rankine-Hugoniot condition (3.36) tells us that

$$\sigma' = \frac{1}{2}, \qquad \sigma(0) = 0 \qquad \Rightarrow \qquad x = \sigma(t) = \frac{t}{2}. \tag{3.38}$$

The initial condition for the shock reflects the fact that characteristics intersect starting at the point $(t,x) = (0,0)$. Hence a solution is given by

$$\rho(t,x) = \begin{cases} 1, & x < t/2, \\ 0, & x > t/2. \end{cases} \tag{3.39}$$

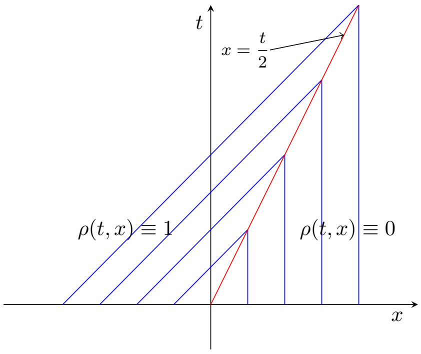


FIGURE 7. Characteristics for the Burgers equation with initial datum (3.37) and the corresponding shock line.

Notice that the shock line is the same as the one found in Example 3.5 for $\tilde{\rho}$. However, the important difference is that here characteristics go *into* a shock, while in Figure 5b characteristics are emanating *from* a shock.

3.4. **The entropy condition.** As we saw in Example 3.5, weak solutions to conservation laws satisfying the Rankine-Hugoniot condition may not be unique. To recover uniqueness, we introduce the following concept to select the "physically relevant" solution among the possible weak solutions.

Assume that we are given a weak solution ρ to the conservation law (3.3), and suppose that ρ is discontinuous along a shock Γ of equation $x = \sigma(t)$, with left and right limits ρ_- and ρ_+, respectively (see Section 3.3). The shock is called *entropic* if

$$q'(\rho_+(t,\sigma(t))) < \sigma'(t) < q'(\rho_-(t,\sigma(t))), \tag{3.40}$$

for every t for which the shock is defined. In words, the slope of a shock curve is less than the slope of the left-characteristics and greater than the slope of the right-characteristics. Roughly, the characteristics hit forward in time the shock line, so that it is not possible to go back in time along characteristics and hit a shock line (see Figure 7), expressing a sort of irreversibility after a shock. The above considerations lead us to select the entropy solutions as the only physically meaningful ones. On the other hand, if the characteristics hit a shock curve backward in time, the shock wave is to be considered non-physical (see Figure 5b). We state the following theorem, without proof.

Theorem 3.7. *If $q \in C^2(\mathbb{R})$ is convex (or concave) and g is bounded, there exists a unique entropy solution of the problem*

$$\begin{cases} \partial_t \rho + \partial_x [q(\rho)] = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3.41}$$

3.5. **Traffic dynamics.** An intense traffic on a highway can be considered as a fluid flow and described by means of macroscopic variables such as the density of cars ρ, their average speed v and their flux q. These three functions are linked by the simple convection relation

$$q = v\rho. \tag{3.42}$$

To construct a model for the evolution of ρ, we assume that there is only one lane and overtaking is not allowed, there are no exit or entrance gates, and the average speed is not constant and depends on the density alone, namely $v = v(\rho)$. Since we expect that the speed decreases as the density increases, we can assume that $v'(\rho) \le 0$. Moreover, we may think that there is a maximum velocity $v_m > 0$, given by a speed limit, and that traffic slows down and stops at the maximum density $\rho_m > 0$. The simplest model consistent with the above considerations gives

$$v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right) \qquad \Rightarrow \qquad q(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right). \tag{3.43}$$

Since

$$q'(\rho) = v_m\left(1 - \frac{2\rho}{\rho_m}\right), \qquad q''(\rho) = -\frac{2v_m}{\rho_m} < 0, \tag{3.44}$$

so q is strictly concave. Writing down the conservation law (3.41) explicitly, we find

$$\begin{cases} \partial_t\rho + v_m\left(1 - \dfrac{2\rho}{\rho_m}\right)\partial_x\rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3.45}$$

The associated characteristics are given as in (3.8) by the straight lines

$$x = \tau + q'(g(\tau))t, \qquad \tau \in \mathbb{R}. \tag{3.46}$$

We now analyze a few possible situations.

Example 3.8 (The green light problem)**.** Suppose that traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. Accordingly, the initial density profile is

$$g(x) = \begin{cases} \rho_m, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{3.47}$$

At time $t = 0$ the traffic light turns green and we want to describe the car flow evolution for $t > 0$. At the beginning, only the cars nearer to the light start moving while most remain standing. Since

$$q'(g(\tau)) = \begin{cases} -v_m, & \tau \le 0, \\ v_m, & \tau > 0, \end{cases} \tag{3.48}$$

and the characteristics (see Figure 8) are the straight lines

$$x = \tau - v_m t, \qquad \tau \le 0, \tag{3.49}$$

$$x = \tau + v_m t, \qquad \tau > 0. \tag{3.50}$$

We are therefore in the presence of a rarefaction wave. We therefore look for a solution of the form $\rho(t,x) = h(x/t)$, where, from (3.45), h needs to satisfy

$$-\frac{x}{t^2}h' + \frac{1}{t}v_m\left(1 - \frac{2h}{\rho_m}\right)h' = 0 \qquad \Rightarrow \qquad h(x/t) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right). \tag{3.51}$$

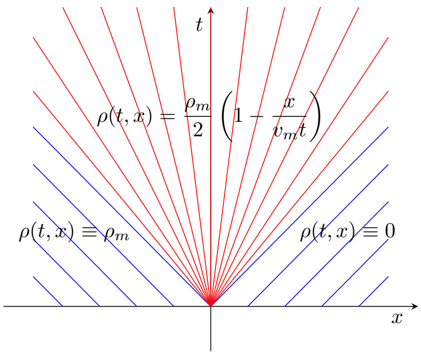


FIGURE 8. The rarefaction wave in the green light problem.

Therefore, the unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \rho_m, & x \le -v_m t, \\ \dfrac{\rho_m}{2}\left(1 - \dfrac{x}{v_m t}\right), & -v_m t < x < v_m t, \\ 0, & x \ge v_m t. \end{cases} \tag{3.52}$$

Example 3.9 (Traffic jam ahead)**.** Suppose that the initial density profile is

$$g(x) = \begin{cases} \rho_m/8, & x \leq 0, \\ \rho_m, & x > 0. \end{cases} \tag{3.53}$$

For $x > 0$, the density is maximal and therefore the traffic is bumper-to-bumper. The cars on the left move with speed $v = \frac{7}{8}v_m$ so that we expect congestion propagating back into the traffic. The characteristics are

$$x = \tau + \frac{3}{4}v_m t, \qquad \tau \leq 0, \tag{3.54}$$

$$x = \tau - v_m t, \qquad \tau > 0. \tag{3.55}$$

Since they intersect (see Figure 9), we are in presence of a shock. From the Rankine-Hugoniot condition (3.36), we find

$$\sigma' = -\frac{v_m}{8}, \qquad \sigma(0) = 0 \qquad \Rightarrow \qquad x = \sigma(t) = -\frac{v_m}{8}t. \tag{3.56}$$

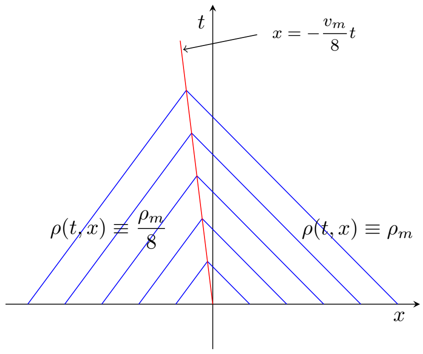


FIGURE 9. The shock wave in the traffic jam problem.

Note that the slope of the shock is negative: the shock propagates back with speed $-\frac{1}{8v_m}$, as it is revealed by the braking of the cars, slowing down because of a traffic jam ahead. The unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \frac{\rho_m}{8}, & x < -\frac{v_m}{8}t, \\ \rho_m, & x > -\frac{v_m}{8}t. \end{cases} \tag{3.57}$$

4. THE CONTINUITY EQUATION

A possible multi-dimensional generalization of (1.5) is as follows. Let $\Omega \subset \mathbb{R}^d$ be a smooth domain, and let $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x}) : \mathbb{R} \times \Omega \to \mathbb{R}^d$ a smooth vector field. Consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, & t > 0,\ \boldsymbol{x} \in \Omega, \\ \rho(0,\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{4.1}$$

for a given regular initial datum. This is called *continuity* equation in the physics literature, and describes the evolution of ρ from the *Eulerian* point of view, namely in which the unknowns are measured at a stationary position $(t,\boldsymbol{x})$ in space time. Notice that we are just considering the linear problem here. In fact, the multi-dimensional version of (3.1) is much more complicated and its study is still an active are of research nowadays. Using the ideas developed in the first two sections, we would like to find a solution to (4.1). As before, one should think as ρ to be a density that is transported by a fluid with velocity $\boldsymbol{u}$.

4.1. **Particle trajectories.** We may first consider the change in a quantity as experienced by a particle that is traveling with the fluid. This is the *Lagrangian* description. Given an initial configuration of particles, labeled by $\boldsymbol{a} \in \Omega$, the unknowns are the *particle trajectories* at times $t > 0$:

$$\boldsymbol{X}(t, \boldsymbol{a}) = (X_1, \ldots, X_d)(t, \boldsymbol{a})\colon [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d. \tag{4.2}$$

The map

$$\boldsymbol{X}(t, \cdot)\colon \Omega \to \Omega, \qquad \boldsymbol{a} \mapsto \boldsymbol{X}(t, \boldsymbol{a}) \tag{4.3}$$

is called the *flow map* associated to the velocity field $\boldsymbol{u}$. Under suitable conditions this map is an (volume-preserving) isomorphism of Ω. We denote the inverse of $\boldsymbol{X}$ by

$$\boldsymbol{A}(t, \boldsymbol{x}) = (A_1, \ldots, A_d)(t, \boldsymbol{x})\colon [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \tag{4.4}$$

which obeys

$$\boldsymbol{A}(t, \boldsymbol{X}(t, \boldsymbol{a})) = \boldsymbol{a}, \qquad \boldsymbol{X}(t, \boldsymbol{A}(t, \boldsymbol{x})) = \boldsymbol{x} \tag{4.5}$$

for all $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^d$. The map $\boldsymbol{A}$ is sometimes called the *back-to-labels* map, and we sometimes write $\boldsymbol{A} = \boldsymbol{X}^{-1}$.

The connection between the Eulerian and the Lagrangian description of fluid flow is given by the fact that the particle trajectories $\boldsymbol{X}$ move on the integral curves of the velocity field $\boldsymbol{u}$, that is they obey the ODE

$$\partial_t \boldsymbol{X}(t, \boldsymbol{a}) = \boldsymbol{u}(t, \boldsymbol{X}(t, \boldsymbol{a})) \tag{4.6}$$

with initial conditions

$$\boldsymbol{X}(0, \boldsymbol{a}) = \boldsymbol{a}. \tag{4.7}$$

Geometrically, this means that the velocity field $\boldsymbol{u}$, evaluated at position $(t, x) = (t, \boldsymbol{X}(t, \boldsymbol{a}))$, is tangent to the curve described by the motion of the particle $\boldsymbol{a}$, namely $\{\boldsymbol{X}(\tau, \boldsymbol{a})\}_{\tau \in I}$, at the point $\boldsymbol{X}(t, \boldsymbol{a})$.

The instantaneous rate of change of any function f with respect to time t, at the fixed position $\boldsymbol{x}$, is the partial time derivative $\partial_t f$. However, in many cases in fluid mechanics it is more natural to measure the rate of change along the flow, denoted by the *convective derivative*

$$\partial_t \left(f(t, \boldsymbol{X}(t, \boldsymbol{a})) \right) = (D_t f)(t, \boldsymbol{X}(t, \boldsymbol{a})). \tag{4.8}$$

From the chain rule and (4.6), assuming all of the functions involved are C^1, we observe that

$$D_t f = \partial_t f + \boldsymbol{u} \cdot \nabla f, \tag{4.9}$$

which is precisely measuring the change in time of f as experienced by a particle moving along the integral curves of $\boldsymbol{u}$. Notice that (4.1) can be rewritten as

$$\begin{cases} D_t \rho = -\rho \nabla \cdot \boldsymbol{u}, & t > 0,\ \boldsymbol{x} \in \Omega, \\ \rho(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{4.10}$$

The quantity $\nabla \cdot \boldsymbol{u}$ plays an important role, as we shall see in the next sections.

It is useful to get comfortable with the change of variables given by $\boldsymbol{X}$. Let V be a volume element in the fluid, and denote by

$$V(t) = \boldsymbol{X}(t, V) = \{\boldsymbol{X}(t, \boldsymbol{a})\colon \boldsymbol{a} \in V\}. \tag{4.11}$$

Recall the change of variables formula

$$\int_{V(t)} f(t, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_V f(t, \boldsymbol{X}(t, \boldsymbol{a})) \det(\nabla_{\boldsymbol{a}} \boldsymbol{X})(t, \boldsymbol{a}) \mathrm{d}\boldsymbol{a} = \int_V f(t, \boldsymbol{X}(t, \boldsymbol{a})) J(t, \boldsymbol{a}) \mathrm{d}\boldsymbol{a}, \tag{4.12}$$

where we have denoted by $J(t, \boldsymbol{a})$ the determinant of Jacobian $\nabla_{\boldsymbol{a}} \boldsymbol{X}$ associated to the map $\boldsymbol{a} \mapsto \boldsymbol{X}(t, \boldsymbol{a})$, that is

$$J(t, \boldsymbol{a}) = \det(\nabla_{\boldsymbol{a}} \boldsymbol{X})(t, \boldsymbol{a}). \tag{4.13}$$

One of the most useful properties of J is stated in the following lemma.

Lemma 4.1. *Assume that the vector field $\boldsymbol{u}$ is C^1, let $\boldsymbol{X}$ be defined by (4.6)–(4.7), and J be given by (4.13). Then we have*

$$\partial_t J(t, \boldsymbol{a}) = J(t, \boldsymbol{a})(\nabla \cdot \boldsymbol{u}(t, \boldsymbol{X}(t, \boldsymbol{a}))) \tag{4.14}$$

pointwise in $(t, \boldsymbol{a})$.

A consequence of Lemma 4.1 is that we may compute the rate of change of the average of a quantity f over a domain $V(t)$ transported by the fluid.

Lemma 4.2. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a C^1 function, and assume that the velocity field $\boldsymbol{u}$ defining the flow map $\boldsymbol{X}(t, \cdot)$ is also C^1. Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} f(t, \boldsymbol{x})\mathrm{d}\boldsymbol{x}\right) = \int_{V(t)} \left(\partial_t f + \nabla \cdot (f\boldsymbol{u})\right)(t, \boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{4.15}$$

for every $t > 0$ and every fluid element V.

Proof of Lemma 4.2. By using the change of variables formula, the convective derivative identity (4.8), and Lemma 4.1 we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} f(t, \boldsymbol{x})\mathrm{d}\boldsymbol{x}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V} f(t, \boldsymbol{X}(t, \boldsymbol{a}))J(t, \boldsymbol{a})\mathrm{d}\boldsymbol{a}\right) \\
&= \int_V (D_t f)(t, X(t, \boldsymbol{a}))J(t, \boldsymbol{a})\mathrm{d}\boldsymbol{a} + \int_V f(t, \boldsymbol{X}(t, \boldsymbol{a}))\partial_t J(t, \boldsymbol{a})\mathrm{d}\boldsymbol{a} \\
&= \int_V (\partial_t f + \boldsymbol{u} \cdot \nabla f + (\nabla \cdot \boldsymbol{u})f)(t, \boldsymbol{X}(t, \boldsymbol{a}))J(t, \boldsymbol{a})\mathrm{d}\boldsymbol{a} \\
&= \int_V (\partial_t f + \nabla \cdot (f\boldsymbol{u}))(t, \boldsymbol{X}(t, \boldsymbol{a}))J(t, \boldsymbol{a})\mathrm{d}\boldsymbol{a} \\
&= \int_{V(t)} (\partial_t f + \nabla \cdot (f\boldsymbol{u}))(t, \boldsymbol{x})\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{4.16}$$

which concludes the proof. □

Remark 4.3 (Conservation of mass)**.** Let V be a volume element in the fluid. If we think of the solution ρ to (4.1) as the density of a concentration, its total mass in this volume element is given by

$$m(t, V) = \int_V \rho(t, \boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{4.17}$$

Setting $f = \rho$ in Lemma 4.2 and using (4.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t, V(t)) = 0, \tag{4.18}$$

namely, *mass is neither created nor destroyed* in a volume element moving with the fluid.

4.2. **Incompressible flows.** We say that the velocity field $\boldsymbol{u}$ is *incompressible* if the flow map $\boldsymbol{X}(t, \cdot)$ is *volume-preserving*, meaning that

$$|V| = |V(t)| \tag{4.19}$$

for all $V \subset \Omega$, and all $t \geq 0$. Here and throughout we denote by $|V|$ the Lebesgue measure of a set V. As it turns out, assumption (4.19) is equivalent to

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.20}$$

namely that the velocity field is *divergence free*. This is a consequence of Lemma 4.2.

Proposition 4.4. *The velocity field $\boldsymbol{u}$ is incompressible if and only if $\nabla \cdot \boldsymbol{u} \equiv 0$ for every $\boldsymbol{x} \in \Omega$ and every $t \geq 0$.*

Proof. Take $f = 1$ in Lemma 4.2. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|V(t)| = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} 1\mathrm{d}\boldsymbol{x}\right) = \int_{V(t)} (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{4.21}$$

for every $t \geq 0$ and every open set $V \subset \Omega$. If $\nabla\cdot\boldsymbol{u} \equiv 0$, then $|V(t)|$ does not change in time, and therefore it is equal to its initial value $|V|$. On the other hand, if $|V(t)|$ is constant in time, then

$$\int_{V(t)} (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \tag{4.22}$$

for every $t \geq 0$ and every open set $V \subset \Omega$. Since $\boldsymbol{X}(t,\cdot)$ is a bijection, it follows that

$$\int_{W} (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \tag{4.23}$$

for every open set $W \subset \Omega$ and all $t \geq 0$, from which $\nabla\cdot\boldsymbol{u} \equiv 0$ immediately follows. □

Another important property of incompressible flows follows from Lemma 4.1.

Corollary 4.5. *Under the assumption of Lemma 4.1, further assume that $\boldsymbol{u}$ is divergence free, i.e. that* (4.20) *holds. Then we have that*

$$J(t,\boldsymbol{a}) = 1 \tag{4.24}$$

for all $\boldsymbol{a} \in \Omega$ and $t > 0$.

Proof of Corollary 4.5. From (4.7) it follows that $J(0,\cdot) = \det(I) = 1$ identically. Solving the differential equation (4.14) we arrive at

$$J(t,\boldsymbol{a}) = J(0,\boldsymbol{a})\exp\left(\int_0^t (\nabla\cdot\boldsymbol{u})(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right) = J(0,\boldsymbol{a}) = 1 \tag{4.25}$$

by using that $\nabla\cdot\boldsymbol{u} = 0$ identically. □

Now, the solution to the continuity equation (4.1) is easily computed if we take into account the equivalent formulation (4.10), the incompressibility condition (4.20) and (4.8). Indeed,

$$D_t\rho = 0 \qquad \Rightarrow \qquad \rho(t,\boldsymbol{X}(t,\boldsymbol{a})) = \rho(0,\boldsymbol{X}(0,\boldsymbol{a})) = \rho(0,\boldsymbol{a}) = g(\boldsymbol{a}), \tag{4.26}$$

for every $\boldsymbol{a} \in \Omega$. Hence,

$$\rho(t,\boldsymbol{x}) = g(\boldsymbol{A}(t,\boldsymbol{x})). \tag{4.27}$$

It is clear at this point the knowing when the map $\boldsymbol{X}$ is invertible (with inverse $\boldsymbol{A}$) is crucial to be able to define such solution. A possible condition for this to happen is that $\boldsymbol{u}$ is C^1.

4.3. **Compressible flows.** If $\nabla\cdot\boldsymbol{u} \neq 0$, then again from (4.8) and (4.10) we have

$$\partial_t\left(\rho(t,\boldsymbol{X}(t,\boldsymbol{a}))\right) = -\rho(t,\boldsymbol{X}(t,\boldsymbol{a}))\nabla\cdot\boldsymbol{u}(t,\boldsymbol{X}(t,\boldsymbol{a})). \tag{4.28}$$

Hence,

$$\begin{aligned}\rho(t,\boldsymbol{X}(t,\boldsymbol{a})) &= \rho(0,\boldsymbol{X}(0,\boldsymbol{a}))\exp\left[-\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right]\\ &= g(\boldsymbol{a})\exp\left[-\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right]\end{aligned} \tag{4.29}$$

From Lemma 4.1, we have that

$$J(t,\boldsymbol{a}) = \exp\left[\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right], \tag{4.30}$$

we can write

$$\rho(t,\boldsymbol{X}(t,\boldsymbol{a})) = \frac{g(\boldsymbol{a})}{J(t,\boldsymbol{a})}, \tag{4.31}$$

and therefore

$$\rho(t, \boldsymbol{x}) = \frac{g(\boldsymbol{A}(t, \boldsymbol{x}))}{J(t, \boldsymbol{A}(t, \boldsymbol{x}))}. \tag{4.32}$$

Once again, this formula reduces to (4.27) in the incompressible case.

5. Problems

Problem 1. Find all solutions for the first order PDE with constant coefficients:

$$a\partial_x u + b\partial_y u + cu = d$$

with $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Problem 2. Solve the Cauchy problem

$$\begin{cases} \partial_x u + \partial_y u = u, & (x, y) \in \mathbb{R}^2, \\ u(x, 0) = \cos x, & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Give the maximal domain of existence of the classical solution $u(x, y)$.

Problem 3. Solve the Cauchy problem

$$\begin{cases} \partial_x u + 3y^{2/3}\partial_y u = 2, & (x, y) \in \mathbb{R}^2, \\ u(x, 1) = 1 + x, & x \in \mathbb{R}. \end{cases} \tag{5.2}$$

Give the maximal domain of existence of the classical solution $u(x, y)$.

Problem 4. Solve the Cauchy problem

$$\begin{cases} \partial_x u + x\partial_y u = y, & (x, y) \in \mathbb{R}^2, \\ u(0, y) = \cos(y), & y \in \mathbb{R}. \end{cases} \tag{5.3}$$

Give the maximal domain of existence of the classical solution $u(x, y)$.

Problem 5. Solve the Cauchy problem

$$\begin{cases} u\partial_x u + y\partial_y u = x, & (x, y) \in \mathbb{R}^2, \\ u(x, 1) = 2x, & x \in \mathbb{R}. \end{cases} \tag{5.4}$$

Give the maximal domain of existence of the classical solution $u(x, y)$.

Problem 6. Solve the Cauchy problem

$$\begin{cases} x\partial_x u + \partial_y u = y, & (x, y) \in \mathbb{R}^2, \\ u(x, 0) = x^2, & x \in \mathbb{R}. \end{cases} \tag{5.5}$$

Give the maximal domain of existence of the classical solution $u(x, y)$.

Problem 7. Prove Lemma 4.1 in dimension two.

Problem 8. Let $a \geq 1$ be a fixed parameter, $\Omega \subset \mathbb{R}^2$ a smooth bounded domain, and $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field such that $\operatorname{div}(\boldsymbol{u}) = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the outward unit normal to the boundary of Ω. Consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = -\rho^a, & (t, \boldsymbol{x}) \in (0, \infty) \times \Omega, \\ \rho(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{5.6}$$

In what follows, assume that g is a continuously differentiable function on Ω.

a. Assume that g is positive. Using the method of characteristics, find the solution ρ of the problem and deduce that ρ is positive, global in time, bounded above by $\sup_{\boldsymbol{x} \in \Omega} |g(\boldsymbol{x})|$, and $\lim\limits_{t \to \infty} \sup\limits_{\boldsymbol{x} \in \Omega} |\rho(t, \boldsymbol{x})| = 0$. How does the rate of convergence to 0 depend on a?
b. Assume that g is positive. Use the energy method to prove the uniqueness of the solution found in part a.
c. Find a value of $a \geq 1$ and an initial datum g for which the solution to (5.6) is not global in time. Find the maximum time of existence of this solution.

Problem 9. Generalize the method of characteristics to higher dimensions and find the solution $u = u(x, y, z)$, $u : \mathbb{R}^3 \to \mathbb{R}$ to the Cauchy problem

$$\begin{cases} \partial_x u + x \partial_y u - \partial_z u = u, & (x, y, z) \in \mathbb{R}^3 \\ u(x, y, 1) = x + y, & (x, y) \in \mathbb{R}^2. \end{cases} \tag{5.7}$$

Problem 10. Study the global Cauchy problem for Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = \alpha x, & x \in \mathbb{R}, \end{cases} \tag{5.8}$$

with $\alpha \in \mathbb{R}$. Discuss the maximal time of existence of the classical solution depending on α.

Problem 11. Consider the the global Cauchy problem

$$\begin{cases} \partial_t \rho + 6(1 - 2\rho) \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = g(x), & x \in \mathbb{R}, \end{cases} \qquad g(x) = \begin{cases} 1/3, & x \leq 0, \\ 1/3 + 5x/12, & 0 < x < 1, \\ 3/4, & x \geq 1. \end{cases} \tag{5.9}$$

Draw the characteristic lines, compute the shocks and find the unique entropy solution of the problem.

Problem 12. Study the global Cauchy problem for Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = g(x), & x \in \mathbb{R}, \end{cases} \qquad g(x) = \begin{cases} 0, & x \leq 0, \\ 2x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases} \tag{5.10}$$

Draw the characteristic lines, compute the shocks and find the unique entropy solution of the problem.

Problem 13. Study the global Cauchy problem for Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = g(x), & x \in \mathbb{R}, \end{cases} \qquad g(x) = \begin{cases} 0, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases} \tag{5.11}$$

Draw the characteristic lines, compute the shocks and find the unique entropy solution of the problem.

Problem 14. Study the global Cauchy problem for Burgers equation

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = g(x), & x \in \mathbb{R}, \end{cases} \qquad g(x) = \begin{cases} 1, & x < -1, \\ -1/2, & -1 \leq x \leq 1, \\ -1, & x > 1. \end{cases} \tag{5.12}$$

Draw the characteristic lines, compute the shocks and find the unique entropy solution of the problem.

Problem 15. Consider the the global Cauchy problem for the green light problem as in Example 3.8. Compute the trajectory of a car initially located at $x_0 < 0$ and the time it takes to pass by $x = 0$.

Problem 16. Consider the the global Cauchy problem for the traffic jam problem as in Example 3.9. How much time does it take to a car initially located at $x_0 < 0$ to hit the traffic jam and stop?

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CHAPTER 3: THE DIFFUSION EQUATION

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ABSTRACT. These notes follow in part the book of S. Salsa [1].

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1. THE DIFFUSION EQUATION

The diffusion equation is the linear second order partial differential equation

$$\partial_t u - \kappa \Delta u = f, \tag{1.1}$$

where $u = u(t, \boldsymbol{x})$, $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ is the space variable, t a time variable, κ a positive constant called the diffusion coefficient, and Δ denotes the Laplace operator

$$\Delta = \sum_{i=1}^{n} \partial_{x_i x_i}. \tag{1.2}$$

When $f = 0$, the equation is said to be homogeneous and in this case the superposition principle holds: if u and v are solutions of (1.1) and a, b are real numbers, $au + bv$ also is a solution of (1.1).

1.1. **Heat conduction.** A common example of diffusion is given by heat conduction in a solid body. We assume that the body is homogeneous and isotropic, with constant mass density ρ, and that it can receive energy from an external source (for instance, from an electrical current or a chemical reaction or from external absorption/radiation). Denote by r the time rate per unit mass at which heat is supplied by the external source. Since heat is a form of energy, it is natural to use the law of conservation of energy, that we can formulate in the following way:

> Let V be an arbitrary control volume inside the body. The time rate of change of thermal energy in V equals the net flux of heat through the boundary ∂V of V, due to the conduction, plus the time rate at which heat is supplied by the external sources.

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If we denote by $e = e(t, \boldsymbol{x})$ the thermal energy per unit mass, the rate of change in time of the total quantity of thermal energy inside V is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho e \mathrm{d}\boldsymbol{x} = \int_V \rho \partial_t e \mathrm{d}\boldsymbol{x}. \tag{1.3}$$

Denote by $\boldsymbol{q}$ the heat flux vector, which specifies the heat flow direction and the magnitude of the rate of flow across a unit area. More precisely, if $\mathrm{d}\sigma$ is an area element contained in ∂V with outer unit normal $\boldsymbol{n}$, then $\boldsymbol{q} \cdot \boldsymbol{n} \mathrm{d}\sigma$ is the energy flow rate through $\mathrm{d}\sigma$ and therefore the total inner heat flux through ∂V is given by

$$-\int_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n}\, \mathrm{d}\sigma = -\int_V \nabla \cdot \boldsymbol{q}\, \mathrm{d}\boldsymbol{x}. \tag{1.4}$$

Finally, since the contribution due to the external source is given by $\int_V \rho r \mathrm{d}\boldsymbol{x}$, conservation of energy requires

$$\int_V \rho \partial_t e \mathrm{d}\boldsymbol{x} = -\int_V \nabla \cdot \boldsymbol{q}\, \mathrm{d}\boldsymbol{x} + \int_V \rho r \mathrm{d}\boldsymbol{x}. \tag{1.5}$$

The arbitrariness of V allows us to convert the integral equation (2.3) into the point-wise relation

$$\rho \partial_t e = -\nabla \cdot \boldsymbol{q} + \rho r, \tag{1.6}$$

that constitutes a basic law of heat conduction. However, e and $\boldsymbol{q}$ are unknown and we need additional information through constitutive relations for these quantities.

Fourier law. The heat flux is a linear function of the temperature gradient, that is

$$\boldsymbol{q} = -\kappa_F \nabla u, \tag{1.7}$$

where $\kappa_F > 0$ is a positive constant.

Thermal energy is a linear function of temperature. Namely,

$$e = c_0 u, \tag{1.8}$$

where $c_0 > 0$ denotes the specific heat (at constant volume) of the material. In many cases of interest c_0 can be considered constant, and we shall do so here.

Using (1.7) and (1.8) in (1.6), we arrive at

$$\partial_t u = \frac{\kappa_F}{\rho c_0}\Delta u + \frac{r}{c_0}, \tag{1.9}$$

which is the diffusion equation (1.1) with $\kappa = \kappa_F/(\rho c_0)$ and $f = r/c_0$.

2. The diffusion equation in bounded domains

Suppose we want to determine the evolution of the temperature in a heat conducting body that occupies a bounded domain $\Omega \subset \mathbb{R}^d$, during an interval of time $[0, T]$. Under the hypotheses of Section 1.6, the temperature is a function $u = u(t, \boldsymbol{x})$ that satisfies the heat equation (1.1) in the space-time cylinder

$$Q_T = (0, T) \times \Omega. \tag{2.1}$$

To select a unique solution we have to prescribe first of all the initial distribution

$$u(0, \boldsymbol{x}) = g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \overline{\Omega}, \tag{2.2}$$

where $\overline{\Omega} = \Omega \cup \partial\Omega$ denotes the closure of Ω. The control of the interaction of the body with the surroundings is modeled through suitable conditions on $\partial\Omega$. The most common ones are the following.

Dirichlet condition. The temperature is kept at a prescribed level on $\partial\Omega$; this amounts to assigning

$$u(t, \boldsymbol{\sigma}) = h(t, \boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \quad t \in (0, T], \tag{2.3}$$

for some assigned function h.

Neumann condition. The heat flux through $\partial\Omega$ is assigned. To model this condition, we assume that the boundary $\partial\Omega$ is a smooth curve or surface, having a tangent line or plane at every point with outward unit vector $\boldsymbol{n}$. From Fourier law (1.7) we have that the heat flux is given by $\boldsymbol{q} = -\kappa_F \nabla$, so that the *inward heat flux* is

$$-\boldsymbol{q} \cdot \boldsymbol{n} = \kappa_F \nabla u \cdot \boldsymbol{n} = \kappa_F \partial_{\boldsymbol{n}} u. \tag{2.4}$$

Thus the Neumann condition reads

$$\partial_{\boldsymbol{n}} u(t, \boldsymbol{\sigma}) = h(t, \boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \quad t \in (0, T], \tag{2.5}$$

for some assigned function h.

Robin condition. Let the surroundings be kept at temperature U, and assume that the the inward heat flux through $\partial\Omega$ depends linearly on the difference $U - u$, so that

$$-\boldsymbol{q} \cdot \boldsymbol{n} = \gamma(U - u), \tag{2.6}$$

for some $\gamma > 0$. Then from Fourier law (1.7) we obtain

$$\partial_{\boldsymbol{n}} u(t, \boldsymbol{\sigma}) + \alpha u(t, \boldsymbol{\sigma}) = h(t, \boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \quad t \in (0, T], \tag{2.7}$$

for some assigned function h and for $\alpha = \gamma/\kappa_F$.

The space-time cylinder Q_T. The diffusion equation requires conditions only on parts of the boundary of the space-time cylinder Q_T in (2.1). In what follows, the symbol ∂Q_T will refer to the so-called *parabolic boundary* of Q_T, namely the union

$$\partial Q_T = \left(\{t = 0\} \times \overline{\Omega}\right) \cup \left((0, T] \times \partial\Omega\right). \tag{2.8}$$

This reflects that fact that no final condition (at $t = T$) is required (see Figure 1).

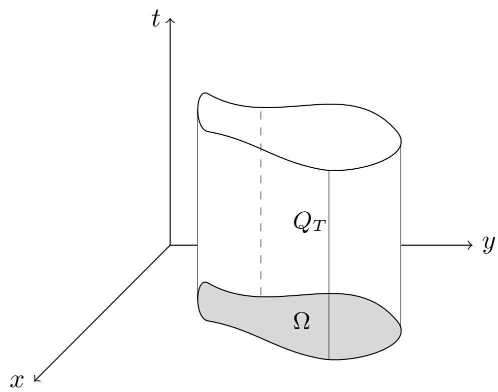

FIGURE 1. The space-time cylinder Q_T and its parabolic boundary ∂Q_T.

2.1. **One-dimensional Dirichlet problem.** We consider here the diffusion equation (1.1) posed on the one-dimensional domain $\Omega = (0, L)$, with $L > 0$, and with homogeneous Dirichlet boundary conditions. Namely, we consider the initial boundary value problem (IBVP)

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in (0, T), \\ u(0, x) = g(x), & x \in [0, L]. \end{cases} \tag{2.9}$$

The main idea is to exploit the linear nature of the problem constructing the solution by superposition of simpler solutions of the form $w(t)v(x)$, in which the variables t and x appear in separated form. To begin the procedure, let us neglect the initial condition $u(0, x) = g(x)$, and let us plug in the function $u(t, x) = w(t)v(x)$ in the first equation of (2.9). We find

$$w'(t)v(x) - \kappa w(t)v''(x) = 0 \qquad \Rightarrow \qquad \frac{1}{\kappa}\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)}. \tag{2.10}$$

Now, the left hand side in (2.10) is a function of t only, while the right hand side is a function of x only and the equality must hold for every $t > 0$ and every $x \in (0, L)$. This is possible only when both sides are equal to a common constant λ, say. Hence we have

$$v'' - \lambda v = 0, \qquad \text{with } v(0) = v(L) = 0, \tag{2.11}$$

and

$$w' - \lambda \kappa w = 0. \tag{2.12}$$

We consider the three different cases below.

◇ Case 1: If $\lambda = 0$, then $v(x) = A + Bx$ for some constants A, B. However, the boundary conditions imply $A = B = 0$.

◇ Case 2: If $\lambda > 0$, say $\lambda = \mu^2 > 0$, then $v(x) = A\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}$ for some constants A, B, and again the boundary conditions imply $A = B = 0$.

◇ Case 3: If $\lambda < 0$, say $\lambda = -\mu^2 < 0$, then

$$v(x) = A \cos \mu x + B \sin \mu x. \tag{2.13}$$

From the boundary conditions we obtain

$$v(0) = A = 0, \qquad v(L) = A \cos \mu L + B \sin \mu L = 0, \tag{2.14}$$

from which

$$A = 0, \quad B = \text{arbitrary}, \quad \mu = \mu_n = \frac{\pi n}{L}, \quad n = 1, 2, \ldots \tag{2.15}$$

Thus, only in Case 3 we find non-trivial solutions

$$v_n(x) = B \sin \frac{\pi n x}{L}. \tag{2.16}$$

In this context, (2.11) is called an eigenvalue problem; the special values $\lambda_n = -\mu_n^2$ are the eigenvalues and the solutions v_n are the corresponding eigenfunctions.

With $\lambda = -\mu_n^2$, the general solution to (2.12) is

$$w_n(t) = B\mathrm{e}^{-\kappa \mu_n^2 t}, \qquad B = \text{arbitrary}. \tag{2.17}$$

Putting together (2.16) and (2.17), we obtain the family

$$u_n(t, x) = \mathrm{e}^{-\kappa \frac{n^2 \pi^2}{L^2} t} \sin \frac{\pi n x}{L}. \tag{2.18}$$

Although the solutions u_n satisfy the homogeneous Dirichlet conditions, they do not match, in general, the initial condition $u(0, x) = g(x)$. As we already mentioned, we try to construct the correct solution superposing the u_m by setting

$$u(t, x) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa \frac{n^2 \pi^2}{L^2} t} \sin \frac{\pi n x}{L}. \tag{2.19}$$

There are two questions that come to mind.

Q1. Can we ensure that the initial condition is satisfied? Namely do there exist coefficients B_n such that

$$g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{L}\ ? \tag{2.20}$$

Q2. And furthermore, it is true that any *finite* linear combination of the u_n's is a solution to the heat equation with Dirichlet boundary conditions, because of the superposition principle. However, can we be sure about an infinite sum?

Both questions are related to Fourier series, see Section 2.5. For **Q1**, let us assume further that $g \in C^1([0,L])$ and $g(0) = g(L) = 0$, namely, also the initial datum satisfied the boundary conditions. We extend g to $[-L,L]$ by its odd extenstion, namely we define a new function $\tilde{g}$ such that $\tilde{g}(x) = g(x)$ for $x \geq 0$ and $\tilde{g}(x) = -g(-x)$ for $x < 0$. Notice that $\tilde{g} \in C^1([-L,L])$ and $\tilde{g}(-L) = \tilde{g}(L) = 0$. Then (2.68) guarantees that its Fourier coefficients are given by

$$B_n = \widetilde{B}_n = \frac{1}{L}\int_{-L}^{L} \tilde{g}(x) \sin\frac{\pi n x}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{\pi n x}{L}, \qquad n \geq 1. \tag{2.21}$$

In particular,

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{L}, \tag{2.22}$$

and the series converges uniformly by Theorem 2.7. Moreover, from (2.19) we deduce that

$$u(t,x) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa\frac{n^2\pi^2}{L^2}t} \sin\frac{\pi n x}{L}, \qquad B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{\pi n x}{L}. \tag{2.23}$$

Regarding **Q2**, if $t \geq t_0 > 0$, the fast convergence rate of the exponential in (2.23) as $n \to \infty$ allows to differentiate term-wise (to any order), and in particular (recall (2.18))

$$\partial_t u - \kappa\partial_{xx}u = \sum_{n=1}^{\infty} B_n\left[\partial_t u_n - \kappa \partial_{xx}u_n\right] = 0. \tag{2.24}$$

We have proved the following result.

Theorem 2.1. *Let $\kappa > 0$ be a constant and $g \in C^1([0,L])$, with $g(0) = g(L) = 0$. The function*

$$u(t,x) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa\frac{n^2\pi^2}{L^2}t} \sin\frac{\pi n x}{L}, \qquad B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{\pi n x}{L}. \tag{2.25}$$

is of class $C([0,\infty) \times [0,L]) \cap C^\infty((0,\infty)\times[0,L])$ and solves the Cauchy problem

$$\begin{cases} \partial_t u - \kappa\partial_{xx}u = 0, & (t,x) \in (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0, & t \in (0,T), \\ u(0,x) = g(x), & x \in [0,L]. \end{cases} \tag{2.26}$$

This is an example of an existence theorem. We will prove that the solution found is the unique one in Section 2.3.

2.2. **One-dimensional Neumann problem.** If instead of the homogeneous Dirichlet boundary conditions in (2.9) we take homogeneous Neumann conditions, we need to study the problem

$$\begin{cases} \partial_t u - \kappa\partial_{xx}u = 0, & (t,x) \in (0,T)\times(0,L), \\ \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,T), \\ u(0,x) = g(x), & x \in [0,L]. \end{cases} \tag{2.27}$$

Arguing as for (2.10), we end up with studying the problems

$$v'' - \lambda v = 0, \qquad \text{with } v'(0) = v'(L) = 0, \tag{2.28}$$

and

$$w' - \lambda\kappa w = 0. \tag{2.29}$$

As before, we consider the three different cases below.

◇ Case 1: If $\lambda = 0$, then $v(x) = A + Bx$ for some constants A, B. The boundary conditions imply $B = 0$, and therefore $v(x) = A$, with A an arbitrary constant, is a possible solution.

◇ Case 2: If $\lambda > 0$, say $\lambda = \mu^2 > 0$, then $v(x) = A\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}$ for some constants A, B, and again the boundary conditions imply $A = B = 0$.

◇ Case 3: If $\lambda < 0$, say $\lambda = -\mu^2 < 0$, then

$$v(x) = A\cos\mu x + B\sin\mu x. \tag{2.30}$$

From the boundary conditions we obtain

$$v'(0) = B = 0, \qquad v'(L) = -A\mu\sin\mu L + B\mu\cos\mu L = 0, \tag{2.31}$$

from which

$$A = \text{arbitrary}, \quad B = 0, \quad \mu = \mu_n = \frac{\pi n}{L}, \quad n = 1, 2, \dots \tag{2.32}$$

Thus we find non-trivial solutions

$$v_n(x) = A\cos\frac{\pi n x}{L}, \qquad n = 0, 1, \dots \tag{2.33}$$

The general solution to (2.29) is

$$w_n(t) = C\mathrm{e}^{-\kappa\mu_n^2 t}, \qquad C = \text{arbitrary}. \tag{2.34}$$

Putting together (2.33) and (2.34), we obtain the family

$$u_n(t,x) = \mathrm{e}^{-\kappa\frac{n^2\pi^2}{L^2}t}\cos\frac{\pi n x}{L}, \qquad n = 0, 1, \dots \tag{2.35}$$

In the Neumann case, the solutions is therefore constructed as

$$u(t,x) = \sum_{n=0}^{\infty} A_n\mathrm{e}^{-\kappa\frac{n^2\pi^2}{L^2}t}\cos\frac{\pi n x}{L}. \tag{2.36}$$

Notice that here the sum starts at $n = 0$, unlike in (2.19), to take into account nonzero constant solutions. Assuming the g is continuously differentiable and such that $g'(0) = g'(L) = 0$, by using an even extension we find from (2.70) that

$$g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{\pi n x}{L}, \qquad A_n = \frac{2}{L}\int_0^L g(x)\cos\frac{\pi n x}{L}, \tag{2.37}$$

with the series being uniformly convergent by Theorem 2.7. As before, we can prove the following theorem.

Theorem 2.2. *Let $\kappa > 0$ be a constant and $g \in C^1([0,L])$, with $g'(0) = g'(L) = 0$. The function*

$$u(t,x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\mathrm{e}^{-\kappa\frac{n^2\pi^2}{L^2}t}\cos\frac{\pi n x}{L}. \tag{2.38}$$

is of class $C([0,\infty)\times[0,L]) \cap C^\infty((0,\infty)\times[0,L])$ and solves the Cauchy problem

$$\begin{cases} \partial_t u - \kappa\partial_{xx}u = 0, & (t,x) \in (0,T)\times(0,L), \\ \partial_x u(t,0) = \partial_x u(t,L) = 0, & t \in (0,T), \\ u(0,x) = g(x), & x \in [0,L]. \end{cases} \tag{2.39}$$

It is now time to consider the question of uniqueness of solutions.

2.3. **Uniqueness by energy methods.** We have seen in the previous sections that we can construct a solution to the one-dimensional diffusion equation with either Dirichlet or Neumann boundary conditions. The question of their uniqueness can be addressed in greater generality for the diffusion equation

$$\begin{cases} \partial_t u - \kappa\Delta u = f, & (t,\boldsymbol{x}) \in Q_T, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{2.40}$$

where $\Omega \subset \mathbb{R}^d$ is bounded domain with sufficiently smooth boundary $\partial\Omega$. For this section, we can assume either Dirichlet (2.3) or Neumann (2.5) boundary conditions. Assume that u_1, u_2 are two solutions to (2.40) that are C^2 in $\boldsymbol{x}$ and C^1 in t. The difference $w = u_1 - u_2$ satisfies the homogeneous problem

$$\begin{cases} \partial_t w - \kappa\Delta w = 0, & (t,\boldsymbol{x}) \in Q_T, \\ w(0,\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{2.41}$$

with *homogeneous* boundary conditions (of either type). Multiplying the above equation by w and integrating on Ω we find

$$\int_\Omega \partial_t w w \,\mathrm{d}\boldsymbol{x} = \kappa \int_\Omega \Delta w w \,\mathrm{d}\boldsymbol{x}. \tag{2.42}$$

Now, on the one hand

$$\int_\Omega \partial_t w w \,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w|^2 \,\mathrm{d}\boldsymbol{x}, \tag{2.43}$$

and, on the other hand, using integration by parts and the boundary conditions, we have

$$\kappa \int_\Omega \Delta w w \,\mathrm{d}\boldsymbol{x} = -\kappa \int_\Omega |\nabla w|^2 \,\mathrm{d}\boldsymbol{x} + \kappa \int_{\partial\Omega} \partial_{\boldsymbol{n}} w w \,\mathrm{d}\sigma = -\kappa \int_\Omega |\nabla w|^2 \,\mathrm{d}\boldsymbol{x}. \tag{2.44}$$

Thus, letting

$$E(t) = \frac{1}{2}\int_\Omega |w(t,\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}, \tag{2.45}$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\kappa \int_\Omega |\nabla w|^2 \,\mathrm{d}\boldsymbol{x} \leq 0 \qquad \Rightarrow \qquad E(t) \leq E(0) = 0. \tag{2.46}$$

Here, $E(0) = 0$ thanks to the initial condition $w(0,\boldsymbol{x}) = 0$ in Ω. In turn, $u_1 = u_2$, as we wanted. We have proven the following result.

Theorem 2.3. *Let Ω be a bounded domain with smooth boundary. The diffusion equation with Dirichlet or Neumann boundary conditions has at most one solution belonging in the class $C^1_t C^2_x(\overline{Q}_T)$.*

2.4. **Maximum principles.** The fact that heat flows from higher to lower temperature regions implies that a solution of the homogeneous heat equation attains its maximum and minimum values on ∂Q_T. This result is known as the (weak) *maximum principle*.

Theorem 2.4 (Weak maximum principle). *Let $w \in C^1_t C^2_x(Q_T) \cap C(\overline{Q}_T)$ be a function such that*

$$\partial_t w - \kappa \Delta w = q \leq 0, \qquad \text{in } Q_T. \tag{2.47}$$

Then w attains its maximum on ∂Q_T, namely

$$\max_{\overline{Q}_T} w = \max_{\partial Q_T} w. \tag{2.48}$$

In particular, if w is negative on ∂Q_T, then is negative in all Q_T.

Proof. We split the proof into two steps.

STEP 1. Let $\varepsilon > 0$ be such that $T - \varepsilon > 0$. We prove that

$$\max_{\overline{Q}_{T-\varepsilon}} w \leq \max_{\partial Q_T} w + \varepsilon T. \tag{2.49}$$

Let $u = w - \varepsilon t$. Then

$$\partial_t u - \kappa \Delta u = q - \varepsilon < 0. \tag{2.50}$$

We claim that the maximum of u on $\overline{Q}_{T-\varepsilon}$ occurs on $\partial Q_{T-\varepsilon}$. Towards a contradiction, assume that there exists a point $(t_0, \boldsymbol{x}_0)$ with $\boldsymbol{x}_0 \in \Omega$ and $t_0 \in (0, T-\varepsilon]$ where u attains its maximum on $\overline{Q}_{T-\varepsilon}$. Then $\Delta u(t_0,\boldsymbol{x}_0) \leq 0$ and either $\partial_t u(t_0,\boldsymbol{x}_0) = 0$ if $t_0 \in (0, T-\varepsilon)$ or $\partial_t u(t_0,\boldsymbol{x}_0) \geq 0$ if $t_0 = T - \varepsilon$. In both cases,

$$\partial_t u(t_0,\boldsymbol{x}_0) - \kappa \Delta u(t_0,\boldsymbol{x}_0) \geq 0, \tag{2.51}$$

contradicting (2.50). Hence

$$\max_{\overline{Q}_{T-\varepsilon}} u \leq \max_{\partial Q_T} u \leq \max_{\partial Q_T} w, \tag{2.52}$$

since $u \leq w$. On the other hand, $w \leq u + \varepsilon T$, and therefore

$$\max_{\overline{Q}_{T-\varepsilon}} w \leq \max_{\overline{Q}_{T-\varepsilon}} u + \varepsilon T \leq \max_{\partial Q_T} w + \varepsilon T, \tag{2.53}$$

which is (2.49).

STEP 2. Since w is continuous in $\overline{Q}_T$, we deduce that

$$\lim_{\varepsilon\to 0}\max_{\overline{Q}_{T-\varepsilon}} w = \max_{\overline{Q}_T} w. \tag{2.54}$$

Hence, letting $\varepsilon \to 0$ in (2.49) gives the conclusion. The proof is over. □

Theorem 2.4 is a version of the so called weak maximum principle, weak because this result says nothing about the possibility that a solution achieves its maximum or minimum at an interior point as well. However, there are some important consequences.

Maximum and minimum for the diffusion equation. If

$$\partial_t u - \kappa\Delta u = 0, \qquad \text{in } Q_T, \tag{2.55}$$

then u attains its maximum and minimum on ∂Q_T. In particular

$$\min_{\partial Q_T} u \le u(t,\boldsymbol{x}) \le \max_{\partial Q_T} u, \qquad \forall (t,\boldsymbol{x}) \in \overline{Q}_T. \tag{2.56}$$

Note that this is mostly useful when Dirichlet boundary conditions are assigned.

Comparison and stability. If u_1 and u_2 satisfy

$$\partial_t u_1 - \kappa\Delta u_1 = f_1, \qquad \partial_t u_2 - \kappa\Delta u_2 = f_2, \tag{2.57}$$

respectively, then we have the following.

- if $u_1 \ge u_2$ on ∂Q_T and $f_1 \ge f_2$ in Q_T, then $u_1 \ge u_2$ in all Q_T.
- the stability estimate

$$\max_{\overline{Q}_T} |u_1 - u_2| \le \max_{\partial Q_T} |u_1 - u_2| + T\max_{\overline{Q}_T} |f_1 - f_2|. \tag{2.58}$$

holds true. In particular the initial-Dirichlet problem has at most one solution that, moreover, depends continuously on the data.

2.5. **Fourier series.** A Fourier series is a way of representing a periodic function as a (possibly infinite) sum of sine and cosine functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a $2T$-periodic function and assume that it can be expanded in a trigonometric series as follows:

$$f(x) = F + \sum_{k=1}^{\infty} a_k \cos\frac{\pi k x}{T} + \sum_{k=1}^{\infty} b_k \sin\frac{\pi k x}{T}. \tag{2.59}$$

How do f and the coefficients F, a_k, b_k relate? We take advantage of the orthogonality relations

$$\begin{aligned}
&\int_{-T}^{T} \cos\frac{\pi k x}{T}\cos\frac{\pi n x}{T}\,\mathrm{d}x = 0, \qquad \forall k \ne n,\\
&\int_{-T}^{T} \sin\frac{\pi k x}{T}\sin\frac{\pi n x}{T}\,\mathrm{d}x = 0, \qquad \forall k \ne n,\\
&\int_{-T}^{T} \cos\frac{\pi k x}{T}\sin\frac{\pi n x}{T}\,\mathrm{d}x = 0, \qquad \forall k, n \ge 0,
\end{aligned} \tag{2.60}$$

and

$$\int_{-T}^{T} \cos^2\frac{\pi k x}{T}\,\mathrm{d}x = \int_{-T}^{T} \sin^2\frac{\pi k x}{T}\,\mathrm{d}x = T. \tag{2.61}$$

Assuming that (2.59) converges uniformly, in $\mathbb{R}$, we multiply (2.59) by $\cos\frac{\pi k x}{T}$ for $k \ge 1$, integrate term by term over $(-T, T)$, and use (2.60) to obtain

$$\int_{-T}^{T} f(x)\cos\frac{\pi k x}{T}\,\mathrm{d}x = T a_k \qquad \Rightarrow \qquad a_k = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{\pi k x}{T}\,\mathrm{d}x. \tag{2.62}$$

For $k = 0$, we obtain

$$\int_{-T}^{T} f(x)\mathrm{d}x = 2FT \tag{2.63}$$

or, setting $F = a_0/2$,

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\mathrm{d}x, \tag{2.64}$$

coherent with (2.62). Similarly

$$\int_{-T}^{T} f(x)\sin\frac{\pi kx}{T}\mathrm{d}x = Tb_k \qquad \Rightarrow \qquad b_k = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{\pi kx}{T}\mathrm{d}x. \tag{2.65}$$

Thus, if u has the uniformly convergent expansion (2.59), the coefficients a_k, b_k (with $a_0 = 2F$) must be given by the formulas (2.62) and (2.65). This motivates the following definition.

Definition 2.5. The *Fourier series* of a periodic function $f : \mathbb{R} \to \mathbb{R}$, periodic of period $2T > 0$ is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{\pi kx}{T} + \sum_{k=1}^{\infty} b_k \sin\frac{\pi kx}{T} \tag{2.66}$$

for some set of *Fourier coefficients* a_k and b_k defined by the integrals

$$a_k = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{\pi kx}{T}\mathrm{d}x, \qquad b_k = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{\pi kx}{T}\mathrm{d}x. \tag{2.67}$$

There are a few important properties to keep in mind.

Odd and even functions. If f is an odd function, i.e.$f(-x) = -f(x)$, we have $a_k = 0$ for every $k \geq 0$, while

$$b_k = \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{\pi kx}{T}\mathrm{d}x. \tag{2.68}$$

Thus, if f is odd, its Fourier series is a sine Fouries series:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{\pi kx}{T}. \tag{2.69}$$

Similarly, if f is an even function, i.e.$f(-x) = f(x)$, then $b_k = 0$ for every $k \geq 1$ and

$$a_k = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{\pi kx}{T}\mathrm{d}x, \tag{2.70}$$

so that its Fourier series is a cosine Fouries series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{\pi kx}{T}. \tag{2.71}$$

Fourier coefficients and derivatives. Let $f \in C^1(\mathbb{R})$ be $2T$-periodic. Then we may compute the Fourier coefficients a'_k and b'_k of f'. We have, integrating by parts, for $k \geq 1$

$$a'_k = \frac{1}{T}\int_{-T}^{T} f'(x)\cos\frac{\pi kx}{T}\mathrm{d}x = \frac{\pi k}{T^2}\int_{-T}^{T} f(x)\sin\frac{\pi kx}{T}\mathrm{d}x = \frac{\pi k}{T}b_k, \tag{2.72}$$

and similarly

$$b'_k = \frac{1}{T}\int_{-T}^{T} f'(x)\sin\frac{\pi kx}{T}\mathrm{d}x = -\frac{\pi k}{T^2}\int_{-T}^{T} f(x)\cos\frac{\pi kx}{T}\mathrm{d}x = -\frac{\pi k}{T}a_k. \tag{2.73}$$

Informally, derivation corresponds to multiplication (by k) on the Fourier side.

Convergence of Fourier series. The convergence of a Fourier series is a rather delicate matter, so we give here a few results (see Figure 2). Of course, a series of functions such as (2.66) can converge in many different ways (point-wise, uniformly...). Perhaps, the most natural type of convergence for Fourier series in the L^2 convergence (least square convergence). For $N \geq 1$, let

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos \frac{\pi k x}{T} + \sum_{k=1}^{N} b_k \sin \frac{\pi k x}{T} \tag{2.74}$$

be the N-partial sum of the Fourier series of f.

Theorem 2.6. *Assume that f is a square integrable function on $(-T, T)$, that is*

$$\int_{-T}^{T} |f(x)|^2 \mathrm{d}x < \infty. \tag{2.75}$$

Then

$$\lim_{N\to\infty} \int_{-T}^{T} |S_N(x) - f(x)|^2 \, \mathrm{d}x = 0. \tag{2.76}$$

Moreover, the Parseval relation holds

$$\frac{1}{T} \int_{-T}^{T} |f(x)|^2 \mathrm{d}x = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left[|a_k|^2 + |b_k|^2 \right], \tag{2.77}$$

and (Riemann-Lebesgue)

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0. \tag{2.78}$$

An example of convergence in L^2 is given in Figure 2 below.

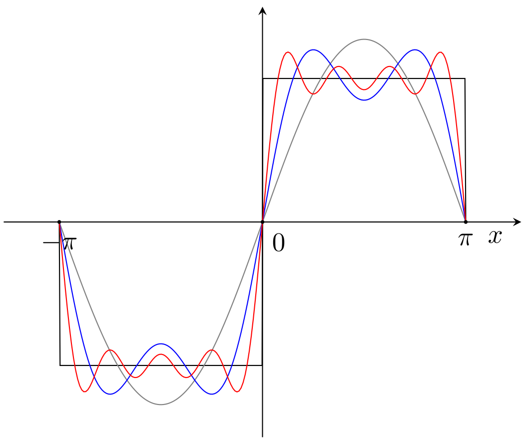

FIGURE 2. Convergence of a (sine) Fourier series. The series converges in L^2, but not uniformly (see the endpoints).

A simple criterion of uniform convergence is provided by the Weierstrass test. Since

$$\left| a_k \cos \frac{\pi k x}{T} + b_k \sin \frac{\pi k x}{T} \right| \leq |a_k| + |b_k|, \tag{2.79}$$

we deduce the following criterion.

Theorem 2.7. *If $f \in C^1[-T, T]$, then the numerical series*

$$\sum_{k=1}^{\infty} |a_k|, \qquad \sum_{k=1}^{\infty} |b_k| \tag{2.80}$$

are convergent. In particular, the Fourier series of f is uniformly convergent in $\mathbb{R}$, with sum f.

3. The diffusion equation in the whole space

In this section, we analyze the homogeneous diffusion equation posed on the whole space $\mathbb{R}^d$

$$\begin{cases} \partial_t u - \kappa \Delta u = f, & (t, \boldsymbol{x}) \in (0,T) \times \mathbb{R}^d, \\ u(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

When $f = 0$, is important to analyze the various symmetries the a solution to the diffusion equation has. So for now, consider u to solve

$$\partial_t u - \kappa \Delta u = 0, \quad (t, \boldsymbol{x}) \in (0,T) \times \mathbb{R}^d. \tag{3.2}$$

We then have the following properties.

Time reversal. The function

$$v(t, \boldsymbol{x}) = u(-t, \boldsymbol{x}) \tag{3.3}$$

obtained by the change of variable $t \mapsto -t$ is a solution of the backward equation $\partial_t v + \kappa \Delta v = 0$. The non-invariance of (3.2) with respect to a change of sign in time is an aspect of time irreversibility.

Space and time translations invariance. For $\boldsymbol{y} \in \mathbb{R}^d$ and $s \in \mathbb{R}$ fixed, the function

$$v(t, \boldsymbol{x}) = u(t - s, \boldsymbol{x} - \boldsymbol{y}) \tag{3.4}$$

is still a solution of (3.2).

Parabolic dilations. For $a > 0$ fixed, the function

$$v(t, \boldsymbol{x}) = u(a^2 t, a\boldsymbol{x}) \tag{3.5}$$

is still a solution of (3.2). In particular, the expression $\boldsymbol{x}/\sqrt{\kappa t}$ is invariant under the change of coordinates $(t, \boldsymbol{x}) \mapsto (a^2 t, a\boldsymbol{x})$.

3.1. The fundamental solution.

In order to solve (3.1), we start by constructing a special solution to (3.2) which complies with the symmetries the we derived above. We will start from the one-dimensional case $d = 1$. As we saw, $x/\sqrt{\kappa t}$ is invariant under the change of coordinates $(t, \boldsymbol{x}) \mapsto (a^2 t, ax)$, and so it makes sense to look for solutions of the form

$$u^\star(t, x) = \frac{1}{\sqrt{\kappa t}} U\left(\frac{x}{\sqrt{\kappa t}}\right), \tag{3.6}$$

where U is a function of a single variable (say, ξ), of total unit mass and decaying as $\xi \to \pm\infty$. The pre-factor $1/\sqrt{\kappa t}$ can be guessed by dimensional analysis. Otherwise, one can simply multiply by $t^{-\alpha}$ the above expression and then check that $\alpha = 1/2$ is the correct exponent. A direct computation shows that

$$\partial_t u^\star = -\frac{1}{2\sqrt{\kappa t^3}} \left[U + \frac{x}{\sqrt{\kappa t}} U'\right] \tag{3.7}$$

and

$$\partial_{xx} u^\star = \frac{1}{(\kappa t)^{3/2}} U''. \tag{3.8}$$

Therefore, setting $\xi = x/\sqrt{\kappa t}$, we find

$$0 = \partial_t u^\star - \kappa \partial_{xx} u^\star = -\frac{1}{\sqrt{\kappa t^3}} \left[\frac{1}{2} U + \frac{x}{2\sqrt{\kappa t}} U' + U''\right] \qquad \Rightarrow \qquad U'' + \frac{\xi}{2} U' + \frac{1}{2} U = 0. \tag{3.9}$$

Now, since the above equation is invariant with respect to the change of variables $\xi \mapsto -\xi$, we look for even solutions (such that $U(\xi) = U(-\xi)$) on the half-line $(0, \infty)$, with $U'(0) = 0$ and $U(+\infty) = 0$. In turn, we can re-write the equation as

$$\left(U' + \frac{\xi}{2} U\right)' = 0 \qquad \Rightarrow \qquad U' + \frac{\xi}{2} U = 0, \tag{3.10}$$

where we used that $U'(0) = 0$. Integrating once more, we arrive at

$$U(\xi) = c_0 \mathrm{e}^{-\xi^2/4}, \tag{3.11}$$

for an arbitrary constant $c_0 \in \mathbb{R}$. This constant is chosen so that U has total unit mass, that is

$$\int_{\mathbb{R}} \mathrm{e}^{-\xi^2/4} = 2\int_{\mathbb{R}} \mathrm{e}^{-z^2} = 2\sqrt{\pi} \qquad \Rightarrow \qquad c_0 = \frac{1}{\sqrt{4\pi}}. \tag{3.12}$$

Going back to the original variables, we have found the following solution of (3.2)

$$u^\star(t,x) = \frac{1}{\sqrt{4\pi\kappa t}}\mathrm{e}^{-\frac{x^2}{4\kappa t}}, \qquad x \in \mathbb{R},\ t > 0, \tag{3.13}$$

positive, even in x, and such that

$$\int_{\mathbb{R}} u^\star(t,x)\mathrm{d}x = 1, \qquad \forall t > 0. \tag{3.14}$$

This gives a family of Gaussians, parametrized with time, and it is natural to think of a normal probability density. The generalization to higher dimension $d \geq 2$ is completely analogous and gives rise to the following definition.

Definition 3.1. The function

$$\Gamma_\kappa(t,\boldsymbol{x}) = \frac{1}{(4\pi\kappa t)^{d/2}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{4\kappa t}}, \qquad \boldsymbol{x} \in \mathbb{R}^d,\ t > 0, \tag{3.15}$$

is called the fundamental solution (heat kernel) of the heat equation (3.2).

3.2. **The Dirac distribution.** It is worthwhile to examine the behavior of the fundamental solution (again, we restrict to $d = 1$ for the moment). It is not hard to see that

$$\lim_{t\to 0^+} \Gamma_\kappa(t,x) = 0, \qquad \forall x \neq 0, \tag{3.16}$$

while

$$\lim_{t\to 0^+} \Gamma_\kappa(t,0) = +\infty. \tag{3.17}$$

If we interpret Γ_κ as a probability density, this implies that when $t \to 0^+$, the fundamental solution tends to concentrate mass around the origin; eventually, the whole probability mass is concentrated at $x = 0$. The limiting density distribution can be mathematically modeled by the so called Dirac distribution (or measure) at the origin, denoted by the symbol δ_0 or simply by δ. The Dirac distribution is not a function in the usual sense of Analysis; if it were, it should satisfy

$$\delta(x) = \begin{cases} +\infty, & x = 0, \\ 0, & x \neq 0, \end{cases} \qquad \int_{\mathbb{R}} \delta(x)\mathrm{d}x = 1, \tag{3.18}$$

clearly incompatible with any concept of classical function or integral. A rigorous definition of the Dirac measure requires the theory of generalized functions or distributions. Here we restrict ourselves to some heuristic considerations. Let

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{3.19}$$

be the characteristic function of the interval $[0,\infty)$, known as the Heaviside function. Observe that

$$I_\varepsilon(x) = \frac{H(x+\varepsilon) - H(x-\varepsilon)}{2\varepsilon} = \begin{cases} \dfrac{1}{2\varepsilon}, & x \in [-\varepsilon,\varepsilon], \\ 0, & \text{else}. \end{cases} \tag{3.20}$$

Then

$$\int_{\mathbb{R}} I_\varepsilon(x)\mathrm{d}x = 1, \tag{3.21}$$

so I_ε can be interpreted as a unit impulse of extent 2ε, as depicted in Figure 3.

Moreover

$$\lim_{\varepsilon\to 0} I_\varepsilon(x) = \begin{cases} +\infty, & x = 0, \\ 0, & x \neq 0. \end{cases} \tag{3.22}$$

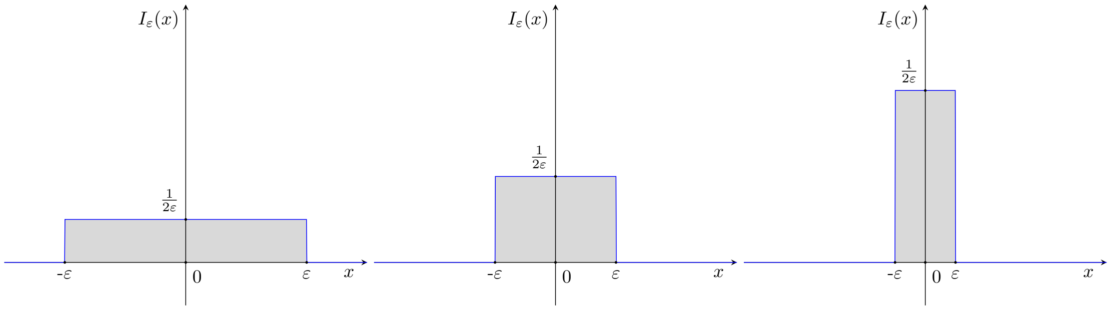

FIGURE 3. Approximation of the Dirac measure, as $\varepsilon \to 0$.

Finally, if $v = v(x)$ is a smooth function with compact support (a test function), then

$$\int_{\mathbb{R}} I_\varepsilon(x)v(x)\mathrm{d}x = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} v(x)\mathrm{d}x \to v(0), \qquad \text{as } \varepsilon \to 0. \tag{3.23}$$

In particular, I_ε tends to a mathematical object that has precisely the formal features of the Dirac distribution at the origin. Furthermore, (3.23) suggests to identify this object through its action on test functions.

Definition 3.2. We call Dirac measure at the origin the generalized function, denoted by δ_0, that acts on a test function v as

$$\langle \delta_0, v \rangle = v(0). \tag{3.24}$$

The brackets $\langle \cdot, \cdot \rangle$ should be read as “the action of δ_0 on v”.

Equation (3.24) is often written in the form

$$\int_{\mathbb{R}} \delta(x)v(x)\mathrm{d}x = v(0). \tag{3.25}$$

where the integral symbol is purely formal. With the notion of Dirac measure at hand, we can say that the fundamental solution Γ_κ satisfies the initial condition

$$\Gamma_\kappa(0, x) = \delta(x). \tag{3.26}$$

More generally, we denote by δ_y the Dirac measure at y, defined through the formula

$$\langle \delta_y, v \rangle = v(y), \tag{3.27}$$

or

$$\int_{\mathbb{R}} \delta(x-y)v(x)\mathrm{d}x = v(y). \tag{3.28}$$

Then, by translation invariance, the fundamental solution $\Gamma_\kappa(t, x-y)$ is a solution of the diffusion equation, that satisfies the initial condition

$$\Gamma_\kappa(0, x-y) = \delta(x-y). \tag{3.29}$$

We can think of the fundamental solution as a unit source solution: $\Gamma_\kappa(t, x)$ gives the concentration at the point x at time t, generated by the diffusion of a unit mass concentrated at the origin at the initial time $t = 0$. Initially Γ_κ is zero outside the origin. As soon as $t > 0$, Γ_κ becomes positive everywhere: this amounts to saying that the unit mass diffuses instantaneously all over the x-axis and therefore with *infinite speed* of propagation.

In the multidimensional case, nothing really changes: in particular, Definition 3.2 can be interpreted in any dimension. For fixed $\boldsymbol{y}$, the fundamental solution $\Gamma_\kappa(t, \boldsymbol{x}-\boldsymbol{y})$ is the unique solution of the global Cauchy problem

$$\begin{cases} \partial_t u - \kappa\Delta u = 0, & (t, \boldsymbol{x}) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, \boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{y}), & \boldsymbol{x} \in \mathbb{R}^d, \end{cases} \tag{3.30}$$

with total mass 1.

3.3. **The homogeneous problem.** We are now ready to prove existence of solutions to the homogeneous diffusion equation

$$\begin{cases} \partial_t u - \kappa\Delta u = 0, & (t,\boldsymbol{x}) \in (0,T)\times\mathbb{R}^d, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^d, \end{cases} \tag{3.31}$$

under some assumptions on the initial datum.

Theorem 3.3. *Let $g \in C_b(\mathbb{R}^d)$, i.e. a contitnuous bounded function. Then*

$$u(t,\boldsymbol{x}) = \int_{\mathbb{R}^d} \Gamma_\kappa(t,\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{(4\pi\kappa t)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\kappa t}} g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{3.32}$$

defines an infinite differentiable function on $(0,\infty)\times\mathbb{R}^d$ such that

$$\partial_t u - \kappa\Delta u = 0, \tag{3.33}$$

and

$$\lim_{t\to 0^+} u(t,\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \forall \boldsymbol{x}\in\mathbb{R}^d. \tag{3.34}$$

Moreover, (3.32) is the unique solution of the Cauchy problem.

Proof. For simplicity, we treat the one-dimensional case $d = 1$, and we skip the uniqueness part. First, notice that by a change of variables

$$u(t,x) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{4\kappa t}} g(y)\mathrm{d}y = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|\xi|^2}{4}} g\left(x+\xi\sqrt{\kappa t}\right)\mathrm{d}\xi. \tag{3.35}$$

Hence, since g is bounded,

$$|u(t,x)| \le \frac{\sup|g|}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|\xi|^2}{4}}\mathrm{d}\xi \le \sup|g|, \tag{3.36}$$

so that the integral is absolutely convergent. If we now take a space derivative, we formally have

$$\partial_x u(t,x) = \int_{\mathbb{R}} \partial_x\Gamma_\kappa(t,x-y)g(y)\mathrm{d}y. \tag{3.37}$$

To justify the differentiation under the integral sign, we can argue as follows. Since

$$\partial_x\Gamma_\kappa(t,x-y) = -\frac{2\pi(x-y)}{(4\pi\kappa t)^{3/2}}\mathrm{e}^{-\frac{|x-y|^2}{4\kappa t}}, \tag{3.38}$$

the integral

$$-\int_{\mathbb{R}} \frac{2\pi(x-y)}{(4\pi\kappa t)^{3/2}}\mathrm{e}^{-\frac{|x-y|^2}{4\kappa t}} g(y)\mathrm{d}y = \frac{1}{4\sqrt{\pi\kappa t}}\int_{\mathbb{R}} \xi\mathrm{e}^{-\frac{|\xi|^2}{4}} g\left(x+\xi\sqrt{\kappa t}\right)\mathrm{d}\xi \tag{3.39}$$

is also absolutely convergent. Thus (3.37) is justified and moreover

$$\sup_{x\in\mathbb{R}}|\partial_x u(t,x)| \le \frac{c_1}{\sqrt{t}}\sup_{x\in\mathbb{R}}|g(x)|, \qquad \forall t>0. \tag{3.40}$$

It is not hard to see that this argument carries over to derivatives of all orders, and therefore u is infinitely differentiable whenever we stay away from $t = 0$. Moreover, if $t > 0$ we have

$$\partial_t u - \kappa\partial_{xx}u = \int_{\mathbb{R}} \left[\partial_t - \kappa\partial_{xx}\right]\Gamma_\kappa(t,x-y)g(y)\mathrm{d}y = 0, \tag{3.41}$$

by translation invariance and the fact that Γ_κ is a solution of the heat equation. It remains to show (3.34). Notice that

$$u(t,x) - g(x) = \int_{\mathbb{R}} \Gamma_\kappa(t,x-y)\left[g(y)-g(x)\right]\mathrm{d}y = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|\xi|^2}{4}}\left[g\left(x+\xi\sqrt{\kappa t}\right) - g(x)\right]\mathrm{d}\xi. \tag{3.42}$$

Fix $\varepsilon > 0$. Since g is continuous, there exists $\eta > 0$ such that $|g(x) - g(y)| < \varepsilon/2$ whenever $|x - y| < \eta$. Hence

$$\frac{1}{\sqrt{4\pi}}\left|\int_{|\xi|<\frac{\eta}{\sqrt{\kappa t}}} \mathrm{e}^{-\frac{|\xi|^2}{4}}\left[g\left(x+\xi\sqrt{\kappa t}\right)-g(x)\right]\mathrm{d}\xi\right| \le \frac{\varepsilon}{2}\frac{1}{\sqrt{4\pi}}\int_{|\xi|<\frac{\eta}{\sqrt{\kappa t}}}\mathrm{e}^{-\frac{|\xi|^2}{4}}\mathrm{d}\xi \le \frac{\varepsilon}{2}. \tag{3.43}$$

On the other hand, taking t small enough we find

$$\frac{1}{\sqrt{4\pi}}\left|\int_{|\xi|\ge\frac{\eta}{\sqrt{\kappa t}}} \mathrm{e}^{-\frac{|\xi|^2}{4}}\left[g\left(x+\xi\sqrt{\kappa t}\right)-g(x)\right]\mathrm{d}\xi\right| \le \frac{2\sup|g|}{\sqrt{4\pi}}\int_{|\xi|\ge\frac{\eta}{\sqrt{\kappa t}}}\mathrm{e}^{-\frac{|\xi|^2}{4}}\mathrm{d}\xi \le \frac{\varepsilon}{2}. \tag{3.44}$$

Putting the two above together, we deduce that for every $\varepsilon > 0$ there exists t_ε small enough such that

$$|u(t,x) - g(x)| \le \varepsilon, \qquad \forall t \le t_\varepsilon, \tag{3.45}$$

which is precisely (3.34). The proof is over. □

Notice that Theorem 3.3 tells us that the heat equation has a *smoothing* effect: as soon as $t > 0$, the initial datum is regularized and the solution becomes infinitely differentiable. Moreover, (3.32) again highlights the infinite speed of propagation effect: if g is positive and compactly supported, u is *everywhere* positive for any $t > 0$.

3.4. **Duhamel method for the non-homogeneous problem.** We are now ready to solve the full non-homogeneous problem (3.1), via the so-called Duhamel's principle. This resembles the way one would solve the ordinary differential equation

$$z'(t) + az(t) = h(t), \qquad z(0) = z_0, \tag{3.46}$$

with $a \in \mathbb{R}$ and h an assigned forcing term. This can be re-written as

$$\left(\mathrm{e}^{at}z\right)' = \mathrm{e}^{at}h(t), \tag{3.47}$$

implying that

$$z(t) = \mathrm{e}^{-at}z_0 + \int_0^t \mathrm{e}^{-a(t-s)}h(s)\mathrm{d}s, \tag{3.48}$$

Now, the above formula (3.48) consists of the solution of the *homogeneous* problem $\mathrm{e}^{-at}z_0$ plus a part due to the forcing term. If we think a bit more abstractly, we can define a family $\{S(t)\}_{t\ge0}$ of solution operators, parametrized by $t \ge 0$, that to each initial condition $z_0 \in \mathbb{R}$ assigns the corresponding solution to the (linear) homogeneous version of (3.46). In symbols,

$$S(t) : \mathbb{R} \to \mathbb{R}, \qquad z_0 \mapsto S(t)z_0 = z(t). \tag{3.49}$$

In the particular case of (3.46), $S(t)z_0 = \mathrm{e}^{-at}z_0$. However, it is important to understand that, in general, $S(t)$ *acts* on initial conditions, and the symbol $S(t)z_0$ means "$S(t)$ applied to z_0", not necessarily interpreted as multiplication. Formula (3.48) can then be re-written abstractly as

$$z(t) = S(t)z_0 + \int_0^t S(t-s)h(s)\mathrm{d}s. \tag{3.50}$$

In the above (3.50), the knowledge of the solution operator for the homogeneous problem allows us to solve the non-homogeneous problem as well. This is Duhamel's principle.

In the case of the heat equation, we know from Theorem 3.3 that given an initial datum $g \in C_b(\mathbb{R}^d)$, the unique solution is given by (3.32), namely

$$u(t,\boldsymbol{x}) = \int_{\mathbb{R}^d} \Gamma_\kappa(t, \boldsymbol{x} - \boldsymbol{y})g(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{3.51}$$

Moreover, $u(t,\cdot) \in C_b(\mathbb{R}^d)$ for every $t > 0$ (and more in fact). In terms of solution operators, we can therefore define

$$S(t) : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d), \qquad g \mapsto S(t)g = u(t,\cdot). \tag{3.52}$$

In this case, $S(t)$ is not a multiplication, but a convolution with the fundamental solution. Notice also that $S(0)$ is the identity. In analogy with (3.50), we can educatedly guess that the solution to the non-homogeneous diffusion equation (3.1) is

$$u(t,\boldsymbol{x}) = \int_{\mathbb{R}^d} \Gamma_\kappa(t,\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_0^t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}s. \tag{3.53}$$

This is in fact the case.

Theorem 3.4. *Let $g \in C_b(\mathbb{R}^d)$, and $f \in C_b^2([0,\infty)\times\mathbb{R}^d)$. Then (3.53) is the unique solution to the non-homogeneous diffusion equation (3.1).*

Proof. We simply check that (3.53) satisfies the non-homogeneous diffusion equation (3.1). Since

$$\int_{\mathbb{R}^d} \Gamma_\kappa(t,\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{3.54}$$

solves the homogeneous heat equation, we know that

$$(\partial_t - \kappa\Delta)\int_{\mathbb{R}^d} \Gamma_\kappa(t,\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 0. \tag{3.55}$$

Differentiating in time the second term and using $\Gamma_\kappa(t,\boldsymbol{x}-\boldsymbol{y})$ solves (3.30), we find

$$\partial_t \int_0^t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}s = f(t,\boldsymbol{x}) + \int_0^t \partial_t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}s. \tag{3.56}$$

However,

$$\int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{3.57}$$

is the solution of the homogeneous (time-translated) heat equation with initial datum $f(s,\boldsymbol{x})$, where s can be regarded as a parameter. In particular,

$$(\partial_t - \kappa\Delta)\int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 0. \tag{3.58}$$

Hence, from (3.56) we find

$$\begin{aligned}(\partial_t - \kappa\Delta)&\int_0^t \int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}s \\ &= f(t,\boldsymbol{x}) + \int_0^t (\partial_t - \kappa\Delta)\int_{\mathbb{R}^d} \Gamma_\kappa(t-s,\boldsymbol{x}-\boldsymbol{y})f(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}s = f(t,\boldsymbol{x}).\end{aligned} \tag{3.59}$$

Putting this together with (3.55) shows that (3.53) solves non-homogeneous diffusion equation (3.1), and therefore concludes the proof. □

4. Problems

Problem 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic C^1 function of period $2T>0$, such that

$$\int_{-T}^{T} f(x)\mathrm{d}x = 0. \tag{4.1}$$

Using Fourier series, prove the *Poincaré inequality*

$$\int_{-T}^{T} |f(x)|^2\mathrm{d}x \le \frac{T^2}{\pi^2}\int_{-T}^{T} |f'(x)|^2\mathrm{d}x. \tag{4.2}$$

Show that equality holds if and only if $f(x) = a\cos\frac{\pi x}{T} + b\sin\frac{\pi x}{T}$ for some $a,b\in\mathbb{R}$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period $2T > 0$, that is n times continuously differentiable (namely, f can be differentiated n times and all its first n derivatives are continuous). Show that its Fourier coefficients (as in Definition 2.5) satisfy the bound

$$|a_k| + |b_k| \leq \frac{C}{k^n}, \qquad \forall k \geq 1, \tag{4.3}$$

for some constant $C > 0$.

Problem 3. Using Theorem 2.6, prove Theorem 2.7.

Problem 4. Let $g \in C^1([0,\pi])$ such that $g(0) = g(\pi) = 0$. Consider the initial-boundary value problem

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0,\infty), \\ u(0,x) = g(x), & x \in [0,\pi]. \end{cases} \tag{4.4}$$

Prove that

$$\lim_{t\to\infty} \sup_{x\in[0,\pi]} |u(t,x)| = 0. \tag{4.5}$$

Assume that u_1 is the solution to the Cauchy problem (4.12) with initial datum g_1, and u_2 is the solution to the Cauchy problem (4.12) with initial datum g_2. Show that

$$\sup_{(t,x)\in[0,\infty)\times[0,\pi]} |u_1(t,x) - u_2(t,x)| \leq \sup_{x\in[0,\pi]} |g_1(x) - g_2(x)|. \tag{4.6}$$

Problem 5. Let $\kappa > 0$ be a constant and $g \in C^1([0,\pi])$ such that $g'(0) = g'(\pi) = 0$. Consider the initial-boundary value problem

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,\pi), \\ \partial_x u(t,0) = \partial_x u(t,\pi) = 0, & t \in (0,\infty), \\ u(0,x) = g(x), & x \in [0,\pi]. \end{cases} \tag{4.7}$$

a. Show that the integral $\displaystyle\int_0^\pi u(t,x)\mathrm{d}x$ is constant in time. Assuming that $u(t,x) \to U$ in a suitable sense as $t \to \infty$, where U is a constant, argue that $U = \displaystyle\frac{1}{\pi}\int_0^\pi g(x)\mathrm{d}x$.
b. Assume that u is C^1 on $[0,\pi] \times [0,\infty)$. Show that

$$\lim_{t\to\infty} \int_0^\pi (u(t,x) - U)^2 \,\mathrm{d}x = 0. \tag{4.8}$$

You do not need to use the explicit solution of the equation, but you can use the Poincaré inequality (4.2).
c. Using the explicit solution in Fourier series of the equation, prove that

$$\lim_{t\to\infty} \sup_{x\in[0,\pi]} |u(t,x) - U| = 0. \tag{4.9}$$

Problem 6. Solve the following initial boundary value problem

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty), \\ u(0,x) = \sin \pi x + 2 \sin 2\pi x, & x \in [0,1]. \end{cases} \tag{4.10}$$

Problem 7. Solve the following initial boundary value problem

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0,\infty), \\ u(0,x) = x(\pi - x), & x \in [0,1]. \end{cases} \tag{4.11}$$

Problem 8. Let u be a solution to

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,1), \\ u(t,0) = 2t\mathrm{e}^{1-t}, \quad u(t,1) = 1 - \cos \pi t, & t \in (0,\infty), \\ u(0,x) = \sin \pi x, & x \in [0,1], \end{cases} \tag{4.12}$$

that is continuous on the closure of the half-strip $S = (0,\infty) \times (0,1)$.

a. Prove that u is non-negative.
b. Determine an upper bound for u at the point $(t_0, x_0) = (1/8, 1/2)$.

Problem 9. Consider the classical solution to the initial boundary-value problem for the heat equation:

$$\begin{cases} \partial_t u - \kappa \partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty), \\ u(0,x) = 4x(1-x), & x \in [0,1]. \end{cases} \tag{4.13}$$

a. Show that $0 \leq u(t,x) \leq 1$ for all $(t,x) \in (0,\infty) \times [0,1]$.
b. Show that $u(t,x) = u(t,1-x)$ for all $(t,x) \in (0,\infty) \times [0,1]$.
c. Use the energy method to show that

$$E(t) = \int_0^1 |u(t,x)|^2 \mathrm{d}x \tag{4.14}$$

is a decreasing function of time.

Problem 10. For any $b \in \mathbb{R}$, solve the Cauchy problem

$$\begin{cases} \partial_t u - \partial_{xx} u = bu, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{4.15}$$

where $g \in C_b(\mathbb{R})$.

Problem 11. For any $b \in \mathbb{R}$, solve the Cauchy problem

$$\begin{cases} \partial_t u - \partial_{xx} u = b\partial_x u, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{4.16}$$

where $g \in C_b(\mathbb{R})$.

Problem 12. The objective of this exercise is to prove the uniqueness of bounded solutions to the the homogeneous diffusion equation posed on the whole space $\mathbb{R}^d$. Assume that u solves

$$\begin{cases} \partial_t u - \kappa \Delta u = 0, & (t,\boldsymbol{x}) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^d, \end{cases} \tag{4.17}$$

and that there exists a constant $M > 0$ such that

$$\sup_{(t,\boldsymbol{x}) \in (0,\infty) \times \mathbb{R}^d} |u(t,\boldsymbol{x})| \leq M. \tag{4.18}$$

a. Given $T, R > 0$, define

$$w(t,\boldsymbol{x}) = \frac{dM}{R^2} \left(\frac{|\boldsymbol{x}|^2}{d} + 2\kappa t \right), \qquad (t,\boldsymbol{x}) \in [0,T] \times B_R(\mathbf{0}). \tag{4.19}$$

Check that $\partial_t w - \kappa \Delta w = 0$.
b. Use the maximum principle to show that $|u| \leq w$ in $[0,T] \times B_R(\mathbf{0})$, for any $T > 0$. Deduce that $u = 0$.
c. Use the previous result to show the uniqueness of bounded solutions for the Cauchy problem of the heat equation with bounded initial datum g.

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CHAPTER 4: THE WAVE EQUATION

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ABSTRACT. These notes follow in part the book of S. Salsa [1].

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1. TRANSVERSAL WAVES IN A STRING

We derive a classical model for the small transversal vibrations of a tightly stretched horizontal string (e.g. a string of a guitar). We assume the following hypotheses:

1. Vibrations of the string have small amplitude. This entails that the changes in the slope of the string from the horizontal equilibrium position are very small.
2. Each point of the string undergoes vertical displacements only. Horizontal displacements can be neglected, according to 1.
3. The vertical displacement of a point depends on time and on its position on the string. If we denote by u the vertical displacement of a point located at x when the string is at rest, then we have $u = u(t, x)$ and, according to 1, $|\partial_x u(t, x)| \ll 1$.
4. The string is perfectly flexible. This means that it offers no resistance to bending. In particular, the stress at any point on the string can be modeled by a tangential force $\boldsymbol{T}$ of magnitude τ, called tension. Figure 1 shows how the forces due to the tension acts at the end points of a small segment of the string.
5. Friction is negligible.

Under the above assumptions, the equation of motion of the string can be derived from conservation of mass and Newton law. Let $\rho_0 = \rho_0(x)$ be the linear density of the string at rest and $\rho = \rho(t, x)$ be its density at time t. Consider an arbitrary part of the string between x and $x + \Delta x$ and denote by Δs the corresponding length element at time t. Then, conservation of mass yields

$$\rho_0(x)\Delta x = \rho(t, x)\Delta s \tag{1.1}$$

To write Newton law of motion we have to determine the forces acting on our small piece of string. Since the motion is vertical, the horizontal forces have to balance. On the other hand they come from

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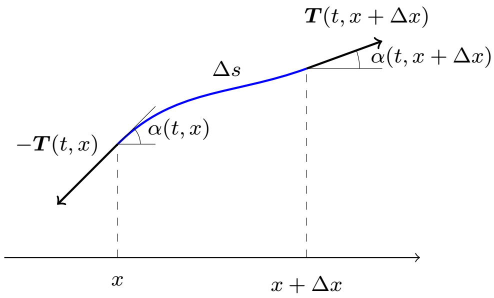


FIGURE 1. Tension at the end points of a small segment of a string.

the tension only, so that if $\tau(t,x)$ denotes the magnitude of the tension at x at time t, we can write (see Figure 1)

$$\tau(t, x+\Delta x)\cos\alpha(t, x+\Delta x) - \tau(t,x)\cos\alpha(t,x) = 0. \tag{1.2}$$

Dividing by Δx and letting $\Delta x \to 0$, we obtain

$$\partial_x\left[\tau(t,x)\cos\alpha(t,x)\right] = 0, \tag{1.3}$$

from which

$$\tau(t,x)\cos\alpha(t,x) = \tau_0(t), \tag{1.4}$$

where $\tau_0(t)$ is positive, since it is the magnitude of a force. Therefore, the (scalar) vertical component of the force acting on our small piece of string, due to the tension, is

$$\tau_{vert}(t, x+\Delta x) - \tau_{vert}(t,x) = \tau_0(t)\left[\partial_x u(t, x+\Delta x) - \partial_x u(t,x)\right]. \tag{1.5}$$

Denote by $f(t,x)$ the magnitude of the (vertical) body forces per unit mass. Then, using (1.1), the magnitude of the body forces acting on the string segment is given by

$$\int_x^{x+\Delta x} \rho(t,y) f(t,y)\mathrm{d}y = \int_x^{x+\Delta x} \rho_0(y) f(t,y)\mathrm{d}y \tag{1.6}$$

Thus, using (1.1) again and observing that $\partial_{tt}u$ is the (scalar) vertical acceleration, Newton law gives

$$\int_x^{x+\Delta x} \rho_0(y)\partial_{tt}u(t,y)\mathrm{d}y = \tau_0(t)\left[\partial_x u(t,x+\Delta x) - \partial_x u(t,x)\right] + \int_x^{x+\Delta x} \rho_0(y) f(t,y)\mathrm{d}y. \tag{1.7}$$

Dividing by Δx and letting $\Delta x \to 0$, we obtain the equation

$$\partial_{tt}u - c^2(t,x)\partial_{xx}u = f, \tag{1.8}$$

where $c^2 = \tau_0(t)/\rho_0(x)$. If the string is homogeneous then ρ_0 is constant. If moreover it is perfectly elastic (for instance, guitar and violin strings are nearly homogeneous, perfectly flexible and elastic), then τ_0 is constant as well, since the horizontal tension is nearly the same as for the string at rest, in the horizontal position.

1.1. **Conservation of energy.** Suppose that a perfectly flexible and elastic string has length L at rest, in the horizontal position. We may identify its initial position with the segment $[0, L]$ on the x axis. Since $\partial_t u(t,x)$ is the vertical velocity of the point at x, the expression

$$E_{cin}(t) = \frac{1}{2}\int_0^L \rho_0 |\partial_t u(t,x)|^2 \mathrm{d}x \tag{1.9}$$

represents the total kinetic energy during the vibrations. The string stores potential energy too, due to the work of elastic forces, given by

$$E_{pot}(t) = \frac{1}{2}\int_0^L \tau_0 |\partial_x u(t,x)|^2 \mathrm{d}x \tag{1.10}$$

From (1.9) and (1.10) we find, for the total energy

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_0|\partial_t u(t,x)|^2 + \tau_0|\partial_x u(t,x)|^2\right] \mathrm{d}x. \tag{1.11}$$

Let us compute the variation of E. Taking the time derivative under the integral, we find (remember that ρ_0 and τ_0 are constants),

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \int_0^L \left[\rho_0 \partial_t u \partial_{tt} u + \tau_0 \partial_x u \partial_{tx} u\right] \mathrm{d}x \tag{1.12}$$

By an integration by parts we get

$$\int_0^L \tau_0 \partial_x u \partial_{tx} u \mathrm{d}x = \tau_0 \left[\partial_x u(t,L)\partial_t u(t,L) - \partial_x u(t,0)\partial_t u(t,0)\right] - \tau_0 \int_0^L \tau_0 \partial_{xx} u \partial_t u \mathrm{d}x, \tag{1.13}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \int_0^L \left[\rho_0 \partial_{tt} u - \tau_0 \partial_{xx} u\right] \partial_t u \mathrm{d}x + \tau_0 \left[\partial_x u(t,L)\partial_t u(t,L) - \partial_x u(t,0)\partial_t u(t,0)\right]. \tag{1.14}$$

Using (1.8), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \int_0^L \rho_0 f \partial_t u \mathrm{d}x + \tau_0 \left[\partial_x u(t,L)\partial_t u(t,L) - \partial_x u(t,0)\partial_t u(t,0)\right]. \tag{1.15}$$

In particular, if $f = 0$ and u is constant at the end points 0 and L (therefore $\partial_t u(t,L) = \partial_t u(t,0) = 0$) we deduce $\frac{\mathrm{d}}{\mathrm{d}t}E = 0$. This implies

$$E(t) = E(0), \qquad \forall t \geq 0, \tag{1.16}$$

which expresses the *conservation of energy*.

2. The one-dimensional wave equation

Equation (1.8) is called the one-dimensional wave equation. The coefficient c has the dimensions of a speed and in fact, we will shortly see that it represents the wave propagation speed along the string. When $f = 0$, the equation is homogeneous and the superposition principle holds: if u_1 and u_2 are solutions of

$$\partial_{tt} u - c^2 \partial_{xx} u = 0, \tag{2.1}$$

and a, b are (real or complex) scalars, then $au_1 + bu_2$ is a solution as well.

Suppose we are considering the space-time region $0 < x < L$, $0 < t < T$. By analogy with the Cauchy problem for second order ordinary differential equations, the second order time derivative in (2.1) suggests that not only the initial profile of the string but the initial velocity has to be assigned as well. Thus, our initial (or Cauchy) data are

$$u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), \qquad x \in [0,L]. \tag{2.2}$$

The boundary data are formally similar to those for the heat equation.

Dirichlet condition. Dirichlet data describe the displacement of the end points of the string:

$$u(t,0) = a(t), \qquad u(t,L) = b(t), \qquad t \in (0,T]. \tag{2.3}$$

If $a(t) = b(t) \equiv 0$, both ends are fixed, with zero displacement.

Neumann condition. Neumann data describe the applied (scalar) vertical tension at the end points. As in the derivation of the wave equation, we may model this tension by $\tau_0 \partial_x u$ so that the Neumann conditions take the form

$$\partial_x u(t,0) = a(t), \qquad \partial_x u(t,L) = b(t), \qquad t \in (0,T]. \tag{2.4}$$

In the special case of homogeneous data, $a(t) = b(t) \equiv 0$, both ends of the string are attached to a frictionless sleeve and are free to move vertically.

Robin condition. Robin data describe a linear elastic attachment at the end points. One way to realize this type of boundary condition is to attach an end point to a linear spring whose other end is fixed. This translates into assigning

$$\tau_0 \partial_x u(t,0) = k u(t,0), \qquad \tau_0 \partial_x u(t,L) = -k u(t,L), \tag{2.5}$$

where $k > 0$ is the elastic constant of the spring.

Although physically unrealistic, it turns out that the solution of the Cauchy problem posed on the whole line, where $x \in \mathbb{R}$, is of fundamental importance. We shall solve it in Section 2.2. Under reasonable assumptions on the data, the above problems are well posed. In the next section we use separation of variables to show it for a Cauchy-Dirichlet problem.

2.1. **Separation of variables.** Suppose that the vibration of a guitar string is modeled by the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(t,0) = u(t,L) = 0, & t \in (0,T), \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), & x \in [0,L]. \end{cases} \tag{2.6}$$

where c^2 is constant. As for the heat equation, we aim at constructing the solution by superposition of simpler solutions of the form $w(t)v(x)$, in which the variables t and x appear in separated form. Plug in $u(t,x) = w(t)v(x)$ in the first equation of (2.6) and find

$$v'' - \lambda v = 0, \qquad \text{with } v(0) = v(L) = 0, \tag{2.7}$$

and

$$w'' - \lambda c^2 w = 0. \tag{2.8}$$

We consider the three different cases below.

$\diamond$ Case 1: If $\lambda = 0$, then $v(x) = A + Bx$ for some constants A, B. However, the boundary conditions imply $A = B = 0$.

$\diamond$ Case 2: If $\lambda > 0$, say $\lambda = \mu^2 > 0$, then $v(x) = A\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}$ for some constants A, B, and again the boundary conditions imply $A = B = 0$.

$\diamond$ Case 3: If $\lambda < 0$, say $\lambda = -\mu^2 < 0$, then

$$v(x) = A\cos\mu x + B\sin\mu x. \tag{2.9}$$

From the boundary conditions we obtain

$$v(0) = A = 0, \qquad v(L) = A\cos\mu L + B\sin\mu L = 0, \tag{2.10}$$

from which

$$A = 0, \quad B = \text{arbitrary}, \quad \mu = \mu_n = \frac{\pi n}{L}, \quad n = 1, 2, \dots \tag{2.11}$$

Thus, only in Case 3 we find non-trivial solutions

$$v_n(x) = B\sin\frac{\pi n x}{L}. \tag{2.12}$$

In this context, (2.7) is called an eigenvalue problem; the special values $\lambda_n = -\mu_n^2$ are the eigenvalues and the solutions v_n are the corresponding eigenfunctions.

With $\lambda = -\mu_n^2$, the general solution to (2.8) is

$$w_n(t) = C_n\cos(\mu_n ct) + D_n\sin(\mu_n ct), \qquad C_n, D_n = \text{arbitrary}. \tag{2.13}$$

Putting together (2.12) and (2.13), we obtain the family

$$u_n(t,x) = [a_n\cos(\mu_n ct) + b_n\sin(\mu_n ct)]\sin\mu_n x, \qquad n \in \mathbb{N}, \tag{2.14}$$

where a_b and b_n are arbitrary constants.

The function u_n is called the nth-normal mode of vibration or nth–harmonic. The first harmonic and its frequency $1/2L$, the lowest possible, are said to be *fundamental*. All the other frequencies are integer multiples of the fundamental one. Because of this reason it seems that a guitar string produces good quality tones, pleasant to the ear (this is not so, for instance, for a vibrating membrane like a drum).

If the initial conditions are

$$u(0,x) = a_n \sin \mu_n x, \qquad \partial_t u(0,x) = c\mu_n b_n \sin \mu_n x, \tag{2.15}$$

then the solution of our problem is exactly u_n and the string vibrates at its nth–mode. In general, the solution is constructed by superposing the harmonics u_n through the formula

$$u(t,x) = \sum_{n=1}^{\infty} [a_n \cos(\mu_n ct) + b_n \sin(\mu_n ct)] \sin \mu_n x, \tag{2.16}$$

where the coefficients a_n and b_n have to be chosen such that, as required by (2.6), the initial conditions

$$u(0,x) = \sum_{n=1}^{\infty} a_n \sin \mu_n x = g(x) \tag{2.17}$$

and

$$\partial_t u(0,x) = \sum_{n=1}^{\infty} c\mu_n b_n \sin \mu_n x = h(x) \tag{2.18}$$

are satisfied, for $0 \le x \le L$. This is a question of Fourier series again. Looking at (2.17) and (2.18), it is natural to assume that both g and h have an expansion in Fourier sine series in the interval $[0,L]$. Let

$$\hat{g}_n = \frac{2}{L}\int_0^L g(x) \sin \mu_n x \mathrm{d}x, \qquad \hat{h}_n = \frac{2}{L}\int_0^L h(x) \sin \mu_n x \mathrm{d}x, \tag{2.19}$$

be the Fourier sine coefficients of g and h. If we choose

$$a_n = \hat{g}_n, \qquad b_n = \frac{\hat{h}_n}{\mu_n c}, \tag{2.20}$$

then (2.16) becomes

$$u(t,x) = \sum_{n=1}^{\infty} \left[\hat{g}_n \cos(\mu_n ct) + \frac{\hat{h}_n}{\mu_n c} \sin(\mu_n ct) \right] \sin \mu_n x, \tag{2.21}$$

and satisfies (2.17) and (2.18). Although every u_n is a smooth solution of the wave equation, in principle (2.21) is only a formal solution, unless we may differentiate term by term twice with respect to both x and t. This is possible if $\hat{g}_n$ and $\hat{h}_n$ vanish sufficiently fast as $n \to \infty$.

To show that (2.21) is the unique solution of problem (2.6), we use conservation of energy. Let u and v be solutions of (2.6). Then $w = u - v$ is a solution of the same problem with zero initial and boundary data. We want to show that $w \equiv 0$. Formula (1.11) for the total energy of w, namely

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_0 |\partial_t w(t,x)|^2 + \tau_0 |\partial_x w(t,x)|^2 \right] \mathrm{d}x \tag{2.22}$$

gives (see (1.15))

$$\frac{\mathrm{d}}{\mathrm{d}t} E = 0, \qquad \forall t \ge 0, \tag{2.23}$$

since $f = 0$ and $\partial_t w(t,L) = \partial_t w(t,0) = 0$, whence $E(t) = E(0) = 0$ for every $t \ge 0$. Therefore, $\partial_t w = \partial_x w = 0$, so that w is constant, and since $w(0,x) = 0$, we conclude that $w(t,x) = 0$ for every $t \ge 0$, as we wanted. We have proved the following result.

Theorem 2.1. *Let $g \in C^4([0,L])$, $h \in C^3([0,L])$, with $g(0) = g(L) = g''(0) = g''(L) = 0$ and $h(0) = h(L) = 0$. The function*

$$u(t,x) = \sum_{n=1}^{\infty} \left[\hat{g}_n \cos(\mu_n ct) + \frac{\hat{h}_n}{\mu_n c} \sin(\mu_n ct) \right] \sin \mu_n x, \tag{2.24}$$

is the unique solution of the Cauchy problem (2.6).

2.2. **The d'Alembert formula.** In this section we establish the celebrated formula of d'Alembert for the solution of the following global Cauchy problem

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{2.25}$$

To find the solution, we first factorize the wave equation as

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0. \tag{2.26}$$

Now let

$$v = \partial_t u + c\partial_x u. \tag{2.27}$$

Then v solves the linear transport equation

$$\partial_t v - c\partial_x v = 0 \qquad \Rightarrow \qquad v(t,x) = \psi(x+ct), \tag{2.28}$$

where ψ is a differentiable arbitrary function. From (2.27) we have

$$\partial_t u + c\partial_x u = \psi(x+ct) \tag{2.29}$$

so that from [Ch2, Section 1.2] we get

$$u(t,x) = \varphi(x-ct) + \int_0^t \psi(x - c(t-s) + cs)\mathrm{d}s, \tag{2.30}$$

where φ is another arbitrary differentiable function. Letting $x - ct + 2cs = y$, we find

$$u(t,x) = \varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\mathrm{d}y. \tag{2.31}$$

To determine ψ and ψ we impose the initial conditions

$$g(x) = u(0,x) = \varphi(x) \tag{2.32}$$

and

$$h(x) = \partial_t u(0,x) = \psi(x) - c\varphi'(x) \qquad \Rightarrow \qquad \psi(x) = h(x) + cg'(x). \tag{2.33}$$

Inserting (2.32) and (2.33) into (2.31) we get

$$u(t,x) = g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left[h(y) + cg'(y)\right]\mathrm{d}y, \tag{2.34}$$

which becomes the *d'Alembert formula*

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\mathrm{d}y. \tag{2.35}$$

If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, formula (2.35) defines a C^2-solution in the half-plane $[0,\infty)\times\mathbb{R}$. On the other hand, a C^2-solution u in $[0,\infty)\times\mathbb{R}$ has to be given by (2.35), just because of the procedure we have used to solve the Cauchy problem. Thus the solution is unique. Observe however, that no regularizing effect takes place here: the solution u remains no more than C^2 for any $t > 0$. Thus, there is a striking difference with diffusion phenomena, governed by the heat equation.

Rearranging the terms in (2.35), we may write u in the form

$$u(t,x) = F(x+ct) + G(x-ct), \tag{2.36}$$

with

$$F(x+ct) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(y)\mathrm{d}y, \qquad G(x-ct) = \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_{x-ct}^0 h(y)\mathrm{d}y. \tag{2.37}$$

This gives u as a superposition of two progressive waves moving at constant speed c in the negative and positive x-direction, respectively. The two terms in (2.36) are respectively constant along the two families of straight lines given by

$$x + ct = const, \qquad x - ct = const. \tag{2.38}$$

These lines are called characteristics and carry important information. Notice that they are precisely the characteristic lines of the two terms in the factorization (2.26).

From d'Alembert formula it follows that the value of u at the point (t,x) depends on the values of g at the points $x-ct$ and $x+ct$ and on the values of h over the whole interval $[x-ct, x+ct]$. This interval is called *domain of dependence* of (t,x), as depicted in Figure 2. From a different perspective,

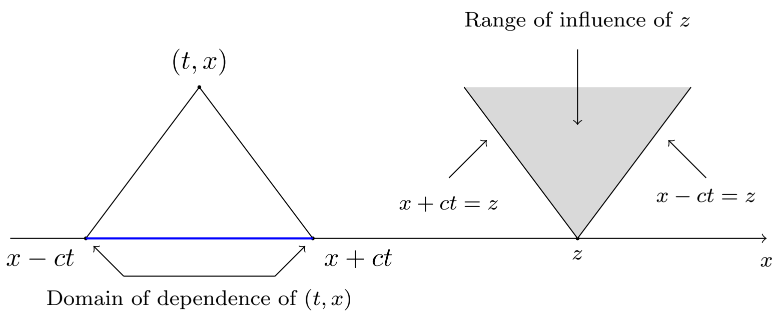


FIGURE 2. Domain of dependence and range of influence.

the values of g and h at a point z affect the value of u at the points (t,x) in the sector

$$z-ct \le x \le z+ct, \tag{2.39}$$

which is called *range of influence* of z (Figure 2). This entails that a disturbance initially localized at z is not felt at a point x until time

$$t = \frac{|x-z|}{c}. \tag{2.40}$$

In particular, contrary to the heat equation, there is *finite speed of propagation* of a signal.

Remark 2.2. Observe that d'Alembert formula makes perfect sense even for g continuous and h bounded. The question is in which sense the resulting function satisfies the wave equation, since, in principle, it is not even differentiable, only continuous. There are several ways to weaken the notion of solution to include this case; for instance, we can mimic what we did for conservation laws.

Example 2.3. Consider the problem

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0, & (t,x)\in(0,\infty)\times\mathbb{R}, \\ u(0,x) = g(x), \quad \partial_t u(0,x) = 0, & x\in\mathbb{R}, \end{cases} \tag{2.41}$$

where $g(x) = 1$ if $|x| < a$ and $g(x) = 0$ if $|x| > a$, for a given $a > 0$. Although g is not even continuous, we can still compute a formal (i.e. not rigorous) solution given by the d'Alembert formula (2.35). Indeed we have

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right]. \tag{2.42}$$

We then need to distinguish the regions in the plane where $|x\pm ct| < a$ or $|x\pm ct| > a$. The possible cases are described below (see the corresponding regions in Figure 3 starting from the right):

- $x > a+ct$. Then also $x > a-ct$, so $g(x-ct) = g(x+ct) = 0$ and $u(t,x) = 0$;
- $\max\{a-ct, -a+ct\} < x < a+ct$. Here $g(x-ct) = 1$ and $g(x+ct) = 0$, so $u(t,x) = 1/2$;
- $\min\{a-ct, -a+ct\} < x < \max\{a-ct, -a+ct\}$. Here $g(x-ct) = g(x+ct) = 1$ and $u(t,x) = 1$;
- $-a+ct < x < a-ct$. Here $g(x-ct) = g(x+ct) = 1$ and $u(t,x) = 1$;
- $a-ct < x < -a+ct$. Here $g(x-ct) = g(x+ct) = 0$ and $u(t,x) = 0$;
- $-a-ct < x < \min\{a-ct, -a+ct\}$. Here $g(x-ct) = 0$ and $g(x+ct) = 1$, so $u(t,x) = 1/2$;
- $x < -a-ct$. Then also $x < -a+ct$, so $g(x-ct) = g(x+ct) = 0$ and $u(t,x) = 0$.

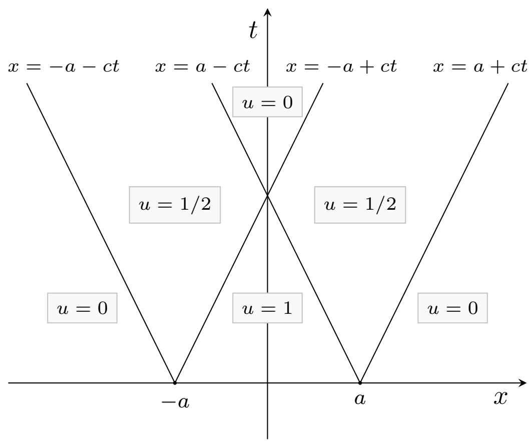


FIGURE 3. The solution of the initial-value problem (2.41).

3. THE MULTI-DIMENSIONAL WAVE EQUATION

In this section we consider the global Cauchy problem for the homogeneous wave equation

$$\begin{cases} \partial_{tt}u - c^2\Delta u = 0, & (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^d, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^d, \end{cases} \tag{3.1}$$

where $d = 2, 3$. Our purpose here is to show that the solution u exists and to find an explicit formula for it, in terms of the data g and h.

3.1. **Fundamental solution and strong Huygens principle.** In the three-dimensional case $d = 3$, first we need a lemma that reduces the problem to the case $g = 0$ (and which actually holds in any dimension). Denote by w_h the solution of the problem

$$\begin{cases} \partial_{tt}w - c^2\Delta w = 0, & (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^d, \\ w(0,\boldsymbol{x}) = 0, \quad \partial_t w(0,\boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^d, \end{cases} \tag{3.2}$$

Lemma 3.1. *If w_g has continuous third-order partials, then $v = \partial_t w_g$ solves the problem*

$$\begin{cases} \partial_{tt}w - c^2\Delta w = 0, & (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^d, \\ w(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t w(0,\boldsymbol{x}) = 0, & \boldsymbol{x}\in\mathbb{R}^d. \end{cases} \tag{3.3}$$

Therefore the solution of (3.1) *is given by*

$$u = \partial_t w_g + w_h. \tag{3.4}$$

Proof. Le $v = \partial_t w_g$. Differentiating the wave equation with respect to t we have

$$0 = \partial_t(\partial_{tt}w_g - c^2\Delta w_g) = (\partial_{tt} - c^2\Delta)\partial_t w_g = \partial_{tt}v - c^2\Delta v. \tag{3.5}$$

Moreover,

$$v(0,\boldsymbol{x}) = \partial_t w_g(0,\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \partial_t v(0,\boldsymbol{x}) = \partial_{tt}w_g(0,\boldsymbol{x}) = c^2\Delta w_g(0,\boldsymbol{x}) = 0 \tag{3.6}$$

Thus, v is a solution of (3.3) and $u = v + w_h$ is the solution of (3.1). □

The lemma shows that, once the solution of (3.2) is determined, the solution of the complete problem (3.1) is given by (3.4).

Therefore, we focus on the solution of (3.2), first with a special h, given by the three-dimensional Dirac measure at $\boldsymbol{y}$, $\delta(\boldsymbol{x}-\boldsymbol{y})$. For example, in the case of sound waves, this initial data models a sudden change of the air density, concentrated at a point $\boldsymbol{y}$. If w represents the density variation with respect to a static atmosphere, then w solves the problem

$$\begin{cases} \partial_{tt}w - c^2\Delta w = 0, & (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^3, \\ w(0,\boldsymbol{x}) = 0, \quad \partial_t w(0,\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{y}), & \boldsymbol{x}\in\mathbb{R}^3. \end{cases} \tag{3.7}$$

The solution of (3.7), which we denote by $K(t, \boldsymbol{x}, \boldsymbol{y})$, is called *fundamental solution* of the three-dimensional wave equation. To solve (3.7) we use the heat equation, approximating the Dirac measure with the fundamental solution of the three-dimensional diffusion equation (with $\kappa = 1$). Indeed, we know that

$$\Gamma(\varepsilon, \boldsymbol{x}-\boldsymbol{y}) = \frac{1}{(4\pi\varepsilon)^{3/2}} \mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\varepsilon}} \to \delta(\boldsymbol{x}-\boldsymbol{y}), \qquad \text{as} \quad \varepsilon \to 0. \tag{3.8}$$

Denote by w_ε the solution of (3.7) with $\delta(\boldsymbol{x}-\boldsymbol{y})$ replaced by $\Gamma(\varepsilon, \boldsymbol{x}-\boldsymbol{y})$. Since $\Gamma(\varepsilon, \boldsymbol{x}-\boldsymbol{y})$ is radially symmetric with pole at $\boldsymbol{y}$, we expect that w_ε shares the same type of symmetry and is a spherical wave of the form

$$w_\varepsilon = w_\varepsilon(t, r), \qquad r = |\boldsymbol{x}-\boldsymbol{y}| \tag{3.9}$$

Now, passing to spherical coordinates in the wave equation[1] we find

$$\partial_{tt} w_\varepsilon - c^2 \left(\partial_{rr} + \frac{2}{r}\partial_r\right) w_\varepsilon = 0 \qquad \Leftrightarrow \qquad \partial_{tt}(r w_\varepsilon) - c^2 \partial_{rr}(r w_\varepsilon) = 0. \tag{3.10}$$

In particular, $r w_\varepsilon$ satisfies the one-dimensional wave equation, so that in view of (2.36) we can write

$$w_\varepsilon(t, r) = \frac{F(r+ct)}{r} + \frac{G(r-ct)}{r}, \tag{3.11}$$

for functions F, G to be determined as follows. The initial conditions require

$$F(r) + G(r) = 0, \qquad c(F'(r) - G'(r)) = r\Gamma(\varepsilon, r). \tag{3.12}$$

or, equivalently,

$$F(r) = -G(r), \qquad G'(r) = -\frac{r\Gamma(\varepsilon, r)}{2c}. \tag{3.13}$$

Integrating the second relation yields

$$G(r) = \frac{1}{2c(4\pi\varepsilon)^{3/2}} \int_0^r s\mathrm{e}^{-\frac{s^2}{4\varepsilon}} \mathrm{d}s = \frac{1}{4\pi c(4\pi\varepsilon)^{1/2}} \left[\mathrm{e}^{-\frac{r^2}{4\varepsilon}} - 1\right], \tag{3.14}$$

and finally

$$w_\varepsilon(t, r) = \frac{1}{4\pi c(4\pi\varepsilon)^{1/2} r} \left[\mathrm{e}^{-\frac{(r-ct)^2}{4\varepsilon}} - \mathrm{e}^{-\frac{(r+ct)^2}{4\varepsilon}}\right] \tag{3.15}$$

Now observe that the function

$$\Gamma(\varepsilon, r) = \frac{1}{(4\pi\varepsilon)^{1/2}} \mathrm{e}^{-\frac{r^2}{4\varepsilon}} \tag{3.16}$$

is the fundamental solution of the one-dimensional diffusion equation with $x = r$ and $t = \varepsilon$. Letting $\varepsilon \to 0$ in (3.15) we find

$$w_\varepsilon(t, r) \to \frac{1}{4\pi c r} \left[\delta(r-ct) - \delta(r+ct)\right] \tag{3.17}$$

Since $r + ct > 0$ for every $t > 0$, we deduce that there is no contribution from $\delta(r+ct)$ and therefore we conclude that

$$K(t, \boldsymbol{x}, \boldsymbol{y}) = \frac{\delta(r-ct)}{4\pi c r}, \qquad r = |\boldsymbol{x}-\boldsymbol{y}|. \tag{3.18}$$

Thus, the fundamental solution is an outgoing traveling wave, initially concentrated at $\boldsymbol{y}$ and thereafter on

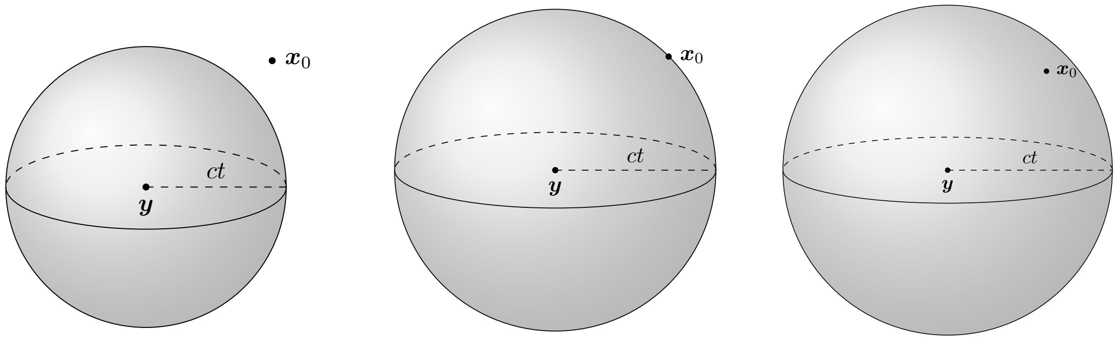
$$\partial B_{ct}(\boldsymbol{y}) = \{\boldsymbol{x} : |\boldsymbol{x}-\boldsymbol{y}| = ct\}. \tag{3.19}$$

The union of the surfaces $\partial B_{ct}(\boldsymbol{y})$ is called the support of K and coincides with the boundary of the *forward space-time cone*, with vertex at $(0, \boldsymbol{y})$ and opening $\theta = \arctan c$, given by

$$C^*_{0,\boldsymbol{y}} = \{(t, \boldsymbol{x}) : |\boldsymbol{x}-\boldsymbol{y}| \le ct, t > 0\}. \tag{3.20}$$

[1] In dimension $d \ge 2$, the Laplacian of a radial function $u = u(r)$ is $\partial_{rr} u + \frac{d-1}{r}\partial_r u$.

In the terminology of Section 2.2, $\partial C^*_{0,\boldsymbol{y}}$ constitutes the range of influence of the point $\boldsymbol{y}$. The fact that the range of influence of the point $\boldsymbol{y}$ is only the boundary of the forward cone and not the full cone has important consequences on the nature of the disturbances governed by the three-dimensional wave equation. The most striking phenomenon is that a perturbation generated at time $t = 0$ by a point source placed at $\boldsymbol{y}$ is felt at the point $\boldsymbol{x}_0$ *only at time* $t_0 = |\boldsymbol{x}_0 - \boldsymbol{y}|/c$. This is known as strong Huygens principle and explains why sharp signals are propagated from a point source (Figure 4). We will shortly see that this is not the case in two dimensions.



(A) $ct < |\boldsymbol{x}_0 - \boldsymbol{y}|$ (B) $ct = |\boldsymbol{x}_0 - \boldsymbol{y}|$ (C) $ct > |\boldsymbol{x}_0 - \boldsymbol{y}|$

FIGURE 4. Huygens principle

3.2. **The Kirchhoff formula.** Using the fundamental solution, we may derive a formula for the solution of (3.2) with a general h. Since

$$h(\boldsymbol{x}) = \int_{\mathbb{R}^3} \delta(\boldsymbol{x} - \boldsymbol{y}) h(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{3.21}$$

we may see h as a superposition of impulses $\delta(\boldsymbol{x} - \boldsymbol{y})h(\boldsymbol{y})$ localized at $\boldsymbol{y}$, of strength $h(\boldsymbol{y})$. Accordingly, the solution of (3.2) is given by the superposition of the corresponding solutions $K(t, \boldsymbol{x}, \boldsymbol{y})h(\boldsymbol{y})$, that is

$$\begin{aligned} w_h(t, \boldsymbol{x}) &= \int_{\mathbb{R}^3} K(t, \boldsymbol{x}, \boldsymbol{y}) h(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^3} \frac{\delta(|\boldsymbol{x} - \boldsymbol{y}| - ct)}{4\pi c |\boldsymbol{x} - \boldsymbol{y}|} h(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &= \int_0^\infty \frac{\delta(r - ct)}{4\pi c r} \mathrm{d}r \int_{\partial B_r(\boldsymbol{x})} h(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\boldsymbol{x})} h(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma}, \end{aligned} \tag{3.22}$$

where we have used the formula

$$\int_0^\infty \delta(r - ct) f(r) \mathrm{d}r = f(ct). \tag{3.23}$$

Lemma 3.1 and the above intuitive argument lead to the following theorem.

Theorem 3.2 (Kirchhoff formula)**.** *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then*

$$u(t, \boldsymbol{x}) = \partial_t \left[\frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\boldsymbol{x})} g(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} \right] + \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\boldsymbol{x})} h(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} \tag{3.24}$$

is the unique solution $u \in C^2([0, \infty) \times \mathbb{R}^3)$ of the wave equation (3.1).

Proof. Letting $\boldsymbol{\sigma} = \boldsymbol{x} + ct\boldsymbol{\sigma}'$, where $\boldsymbol{\sigma}' \in \partial B_1(\boldsymbol{0})$, we have $\mathrm{d}\boldsymbol{\sigma} = c^2 t^2 \mathrm{d}\boldsymbol{\sigma}'$ and we may write

$$w_g(t, \boldsymbol{x}) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\boldsymbol{x})} g(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} = \frac{t}{4\pi} \int_{\partial B_1(\boldsymbol{0})} g(\boldsymbol{x} + ct\boldsymbol{\sigma}') \mathrm{d}\boldsymbol{\sigma}' \tag{3.25}$$

Since $g \in C^3(\mathbb{R}^3)$, this formula shows that w_g satisfies the hypotheses of Lemma 3.1. Therefore it is enough to check that

$$w_h(t, \boldsymbol{x}) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\boldsymbol{x})} h(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} = \frac{t}{4\pi} \int_{\partial B_1(\boldsymbol{0})} h(\boldsymbol{x} + ct\boldsymbol{\sigma}') \mathrm{d}\boldsymbol{\sigma}' \tag{3.26}$$

solves problem (3.2). We have

$$\partial_t w_h(t,\boldsymbol{x}) = \frac{1}{4\pi}\int_{\partial B_1(\mathbf{0})} h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\mathrm{d}\boldsymbol{\sigma}' + \frac{ct}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\cdot\boldsymbol{\sigma}'\mathrm{d}\boldsymbol{\sigma}'. \tag{3.27}$$

Thus,

$$w_h(0,\boldsymbol{x}) = 0, \qquad \partial_t w_h(0,\boldsymbol{x}) = h(\boldsymbol{x}). \tag{3.28}$$

Moreover, by Gauss formula, we may write

$$\begin{aligned}\frac{ct}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\cdot\boldsymbol{\sigma}'\mathrm{d}\boldsymbol{\sigma}' &= \frac{1}{4\pi ct}\int_{\partial B_{ct}(\boldsymbol{x})} \partial_{\boldsymbol{n}} h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = \frac{1}{4\pi ct}\int_{B_{ct}(\boldsymbol{x})} \Delta h(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi ct}\int_0^{ct}\mathrm{d}r\int_{\partial B_r(\boldsymbol{x})} \Delta h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}\end{aligned} \tag{3.29}$$

whence from (3.27)

$$\begin{aligned}\partial_{tt} w_h(t,\boldsymbol{x}) &= \frac{c}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\cdot\boldsymbol{\sigma}'\mathrm{d}\boldsymbol{\sigma}'\\ &\quad - \frac{c}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\cdot\boldsymbol{\sigma}'\mathrm{d}\boldsymbol{\sigma}' + \frac{1}{4\pi t}\int_{\partial B_{ct}(\boldsymbol{x})} \Delta h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}\\ &= \frac{1}{4\pi t}\int_{\partial B_{ct}(\boldsymbol{x})} \Delta h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}.\end{aligned} \tag{3.30}$$

On the other hand, from (3.26) we have

$$\Delta w_h(t,\boldsymbol{x}) = \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} \Delta h(\boldsymbol{x}+ct\boldsymbol{\sigma}')\mathrm{d}\boldsymbol{\sigma}' = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\boldsymbol{x})} \Delta h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}, \tag{3.31}$$

and therefore $\partial_{tt} w_h - c^2\Delta w_h = 0$, concluding the proof. □

Using the calculations in the proof of the above theorem, we may write the Kirchhoff formula in the form

$$u(t,\boldsymbol{x}) = \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\boldsymbol{x})} \left[g(\boldsymbol{\sigma}) + \nabla g(\boldsymbol{\sigma})\cdot(\boldsymbol{\sigma}-\boldsymbol{x}) + th(\boldsymbol{\sigma})\right]\mathrm{d}\boldsymbol{\sigma}. \tag{3.32}$$

The presence of the gradient of g in (3.32) suggests that, unlike the one-dimensional case, the solution u may be more irregular than the data. Indeed, if $g \in C^k(\mathbb{R}^3)$ and $h \in C^{k-1}(\mathbb{R}^3)$, $k \geq 2$, then we can only guarantee that u is C^{k-1} and $\partial_t u$ is C^{k-2} at a later time. Formula (3.32) makes perfect sense also for $g \in C^1(\mathbb{R}^3)$ and h bounded. Clearly, under these weaker hypotheses, (3.32) satisfies the wave equation in an appropriate weak sense (analogous at what observed in Remark 2.2).

According to (3.32), $u(t,\boldsymbol{x})$ depends upon the data g and h only on the surface $\partial B_{ct}(\boldsymbol{x})$, which therefore coincides with the domain of dependence for $(t,\boldsymbol{x})$. Assume that the support of g and h is the compact set D. Then $u(t,\boldsymbol{x})$ is different from zero only for $t_{min} < t < t_{max}$ where t_{min} and t_{max} are the first and the last time t such that $D\cap\partial B_{ct}(\boldsymbol{x}) \neq \emptyset$. In other words, a disturbance, initially localized inside D, starts affecting the point $\boldsymbol{x}$ at time t_{min} and ceases to affect it after time t_{max}. This is another way to express the strong Huygens principle.

3.3. **The 2d wave equation.** The solution of the Cauchy problem in two dimensions can be obtained from Kirchhoff formula, using the so called *Hadamard method of descent*. Consider first the problem

$$\begin{cases}\partial_{tt} w - c^2\Delta w = 0, & (t,\boldsymbol{x})\in(0,\infty)\times\mathbb{R}^2,\\ w(0,\boldsymbol{x}) = 0, \quad \partial_t w(0,\boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^2.\end{cases} \tag{3.33}$$

The key idea is to "immerse" the two-dimensional problem (3.33) in a three-dimensional setting. More precisely, write points in $\mathbb{R}^3$ as $(\boldsymbol{x}, x_3)$ and set $h(\boldsymbol{x}, x_3) = h(\boldsymbol{x})$. The solution U of the three-dimensional problem is given by Kirchhoff formula (see (3.24))

$$U(t,\boldsymbol{x},x_3) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\boldsymbol{x},x_3)} h(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}. \tag{3.34}$$

We claim that, since h does not depend on x_3, U is independent of x_3 as well, and therefore the solution of (3.33) is given by (3.34) with, say, $x_3 = 0$. To prove the claim, note that the spherical surface $\partial B_{ct}(\boldsymbol{x}, x_3)$ is a union of the two hemispheres whose equation are

$$y_3 = F_{\pm}(y_1, y_2) = x_3 \pm \sqrt{c^2t^2 - r^2}, \qquad r^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2. \tag{3.35}$$

On both hemispheres we have

$$\mathrm{d}\boldsymbol{\sigma} = \sqrt{1 + |\nabla F_{\pm}|^2}\mathrm{d}y_1\mathrm{d}y_2 = \sqrt{1 + \frac{r^2}{c^2t^2 - r^2}}\mathrm{d}y_1\mathrm{d}y_2 = \frac{ct}{\sqrt{c^2t^2 - r^2}}\mathrm{d}y_1\mathrm{d}y_2, \tag{3.36}$$

so that we may write ($\mathrm{d}\boldsymbol{y} = \mathrm{d}y_1\mathrm{d}y_2$)

$$U(t, \boldsymbol{x}, x_3) = \frac{1}{2\pi c}\int_{B_{ct}(\boldsymbol{x})} \frac{h(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y}, \tag{3.37}$$

and U is independent of x_r as claimed. From the above calculations and recalling Lemma 3.1 we deduce the following theorem.

Theorem 3.3 (Poisson formula). *Let $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then*

$$u(t, \boldsymbol{x}) = \frac{1}{2\pi c}\left[\partial_t \int_{B_{ct}(\boldsymbol{x})} \frac{g(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y} + \int_{B_{ct}(\boldsymbol{x})} \frac{h(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y}\right] \tag{3.38}$$

is the unique solution $u \in C^2([0, \infty) \times \mathbb{R}^2)$ of the problem

$$\begin{cases} \partial_{tt}u - c^2\Delta u = 0, & (t, \boldsymbol{x}) \in (0, \infty) \times \mathbb{R}^2, \\ u(0, \boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t u(0, \boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^2. \end{cases} \tag{3.39}$$

Also Poisson formula can be written in a somewhat more explicit form. Indeed, letting $\boldsymbol{y} - \boldsymbol{x} = ct\boldsymbol{z}$, we have

$$\mathrm{d}\boldsymbol{y} = c^2t^2\mathrm{d}\boldsymbol{z}, \qquad |\boldsymbol{x} - \boldsymbol{y}|^2 = c^2t^2|\boldsymbol{z}|^2, \tag{3.40}$$

whence

$$\int_{B_{ct}(\boldsymbol{x})} \frac{g(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y} = ct\int_{B_1(\boldsymbol{0})} \frac{g(\boldsymbol{x} + ct\boldsymbol{z})}{\sqrt{1 - |\boldsymbol{z}|^2}}\mathrm{d}\boldsymbol{z}. \tag{3.41}$$

Then

$$\partial_t \int_{B_{ct}(\boldsymbol{x})} \frac{g(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y} = c\int_{B_1(\boldsymbol{0})} \frac{g(\boldsymbol{x} + ct\boldsymbol{z})}{\sqrt{1 - |\boldsymbol{z}|^2}}\mathrm{d}\boldsymbol{z} + c^2t\int_{B_1(\boldsymbol{0})} \frac{\nabla g(\boldsymbol{x} + ct\boldsymbol{z}) \cdot \boldsymbol{z}}{\sqrt{1 - |\boldsymbol{z}|^2}}\mathrm{d}\boldsymbol{z} \tag{3.42}$$

and, going back to the original variables, we obtain

$$u(t, \boldsymbol{x}) = \frac{1}{2\pi ct}\int_{B_{ct}(\boldsymbol{x})} \frac{g(\boldsymbol{y}) + \nabla g(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x}) + th(\boldsymbol{y})}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{y}|^2}}\mathrm{d}\boldsymbol{y}. \tag{3.43}$$

Poisson formula displays an important difference with respect to its three-dimensional analogue, Kirchhoff formula. In fact the domain of dependence for the point $(t, \boldsymbol{x})$ is given by the full circle $B_{ct}(\boldsymbol{x}) = \{\boldsymbol{y} : |\boldsymbol{x} - \boldsymbol{y}| < ct\}$. This entails that a disturbance, initially localized at $\boldsymbol{\xi}$, starts affecting the point $\boldsymbol{x}$ at time $t_{min} = |\boldsymbol{x} - \boldsymbol{\xi}|/c$. However, this effect does not vanish for $t > t_{min}$, since $\boldsymbol{\xi}$ still belongs to the circle $B_{ct}(\boldsymbol{x})$ after t_{min}. It is the phenomenon one may observe by placing a cork on still water and dropping a stone not too far away. The cork remains undisturbed until it is reached by the wave front but its oscillations persist thereafter. Thus, sharp signals do not exist in dimension two and the strong Huygens principle does not hold.

4. Problems

Problem 1. Solve the following initial boundary value problem

$$\begin{cases} \partial_{tt} u - 4\partial_{xx} u = 0, & (t,x) \in (0,T) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0,T), \\ u(0,x) = \sin x + \frac{1}{2}\sin 2x, & x \in [0,\pi], \\ \partial_t u(0,x) = \sin 3x, & x \in [0,\pi]. \end{cases} \tag{4.1}$$

Problem 2. Solve the following initial boundary value problem

$$\begin{cases} \partial_{tt} u - 4\partial_{xx} u = 0, & (t,x) \in (0,T) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in (0,T), \\ u(0,x) = \sin 5x, & x \in [0,\pi], \\ \partial_t u(0,x) = 0, & x \in [0,\pi]. \end{cases} \tag{4.2}$$

Problem 3. Consider the one-dimensional Cauchy problem for the wave equation on $\mathbb{R}$:

$$\begin{cases} \partial_{tt} u - c^2\partial_{xx} u = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = g(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{4.3}$$

Here, g, h are C^2 functions on $\mathbb{R}$ that vanish outside the interval $[a,b]$. Define

$$K(t) = \frac{1}{2}\int_{\mathbb{R}} |\partial_t u(t,x)|^2 \mathrm{d}x, \qquad P(t) = \frac{c^2}{2}\int_{\mathbb{R}} |u_x(t,x)|^2 \mathrm{d}x. \tag{4.4}$$

Show that $K(t) = P(t)$ for every $t > \dfrac{b-a}{2c}$.

Problem 4. Similar to Example 2.3, solve the problem

$$\begin{cases} \partial_{tt} u - c^2\partial_{xx} u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = 0, \quad \partial_t u(0,x) = h(x), & x \in \mathbb{R}, \end{cases} \tag{4.5}$$

where $h(x) = 1$ if $|x| < a$ and $h(x) = 0$ if $|x| > a$, for a given $a > 0$.

Problem 5. Consider the problem on the half-line

$$\begin{cases} \partial_{tt} u - c^2\partial_{xx} u = 0, & (t,x) \in (0,\infty) \times (0,\infty), \\ u(t,0) = 0, & t \in (0,\infty), \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), & x \geq 0, \end{cases} \tag{4.6}$$

where g, h are regular functions with $g(0) = 0$. Extend suitably the initial data to $\mathbb{R}$ and use d'Alembert formula to write a representation formula for the solution.

Problem 6. Given a smooth bounded domain $\Omega \subset \mathbb{R}^d$, consider the weakly damped wave equation

$$\begin{cases} \partial_{tt} u + \partial_t u - \Delta u = 0, & (t,\boldsymbol{x}) \in (0,\infty) \times \Omega, \\ u(t,\boldsymbol{\sigma}) = 0, & (t,\boldsymbol{\sigma}) \in (0,\infty) \times \partial\Omega, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{4.7}$$

with homogeneous Dirichlet boundary conditions. You may assume the following Poincaré inequality: there exists a constant $C_0 > 0$ such that

$$\int_\Omega |u(t,\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \leq C_0 \int_\Omega |\nabla u(t,\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}, \tag{4.8}$$

for every $t > 0$. Show that for all $\varepsilon > 0$ small enough, the modified energy

$$K(t) = \int_\Omega \left[|\partial_t u(t,\boldsymbol{x})|^2 + 2\varepsilon u(t,\boldsymbol{x})\partial_t u(t,\boldsymbol{x}) + \varepsilon |u(t,\boldsymbol{x})|^2 + |\nabla u(t,\boldsymbol{x})|^2 \right] \mathrm{d}\boldsymbol{x} \tag{4.9}$$

satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}K + \varepsilon K \leq 0. \tag{4.10}$$

Conclude that the energy

$$E(t) = \frac{1}{2}\int_\Omega \left[|\partial_t u(t,\boldsymbol{x})|^2 + |\nabla u(t,\boldsymbol{x})|^2\right] \mathrm{d}\boldsymbol{x} \tag{4.11}$$

decays exponential fast to 0 as $t \to \infty$.

Problem 7. Consider the weakly damped wave equation

$$\begin{cases} \partial_{tt}u + \kappa\partial_t u - c^2\Delta u = 0, & (t,\boldsymbol{x}) \in (0,\infty)\times\mathbb{R}^2, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x}) = h(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^2, \end{cases} \tag{4.12}$$

on the whole plane, with $\kappa \geq 0$.

1. Determine $\alpha \in \mathbb{R}$ so that
$$v(t,\boldsymbol{x}) = \mathrm{e}^{\alpha t}u(t,\boldsymbol{x}) \tag{4.13}$$
solves an equation without first-order term (but with zero-order one) on $\{t > 0\} \times \mathbb{R}^2$.
2. Find $\beta \in \mathbb{R}$ so that
$$w(t,x,y,z) = w(t,\boldsymbol{x},z) = \mathrm{e}^{\beta z}v(t,\boldsymbol{x}) \tag{4.14}$$
solves an equation with second-order terms only, on $\{t > 0\} \times \mathbb{R}^3$.
3. Determine the solution u of the original problem.

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CHAPTER 5: THE LAPLACE EQUATION

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ABSTRACT. These notes follow in part the book of S. Salsa [1].

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1. THE LAPLACE EQUATION

The Laplace equation $\Delta u = 0$ occurs frequently in applied sciences, in particular in the study of the steady state phenomena. Its solutions are called harmonic functions. For instance, the velocity potential of a homogeneous fluid is a harmonic function.

Slightly more generally, Poisson equation $\Delta u = f$ plays an important role in the theory of conservative fields (electrical, magnetic, gravitational, ...) where the vector field is derived from the gradient of a potential. For example, let $\boldsymbol{E}$ be a force field due to a distribution of electric charges in a domain $\Omega \subset \mathbb{R}^3$. Then, in standard units, $\nabla \cdot \boldsymbol{E} = 4\pi\rho$, where ρ represents the density of the charge distribution. When a potential u exists such that $\nabla u = -\boldsymbol{E}$, then $\Delta u = \nabla \cdot \nabla u = -4\pi\rho$, which is Poisson equation.

1.1. Uniqueness.

Consider the Poisson equation

$$\Delta u = f, \qquad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain. The well posed problems associated with equation (1.1) are the stationary counterparts of the corresponding problems for the diffusion or the wave equation. Clearly here there is no initial condition. On the boundary $\partial\Omega$ we may assign various boundary conditions, such as the *Dirichlet condition*

$$u(\boldsymbol{\sigma}) = h(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \tag{1.2}$$

for some assigned function h, the *Neumann condition*

$$\partial_{\boldsymbol{n}} u(\boldsymbol{\sigma}) = h(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \tag{1.3}$$

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where $\boldsymbol{n}$ is the outer unit normal to $\partial\Omega$, or the *Robin condition*

$$\partial_{\boldsymbol{n}} u(\boldsymbol{\sigma}) + \alpha u(\boldsymbol{\sigma}) = h(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in \partial\Omega, \tag{1.4}$$

for some $\alpha > 0$. Using integration by parts we can prove the following uniqueness result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain. Then there exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of (1.1) , satisfying on $\partial\Omega$ one of the conditions (1.2) or (1.4). In the case of the Neumann condition (1.3), two solutions differ by a constant.*

Proof. Let u and v be solutions of the same problem, sharing the same boundary data, and let $w = u - v$. Then w is harmonic and satisfies homogeneous boundary conditions (one among (1.2)-(1.4)). We then have

$$0 = \int_\Omega w\Delta w = \int_{\partial\Omega} w\partial_{\boldsymbol{n}} w - \int_\Omega |\nabla w|^2 \qquad \Rightarrow \qquad \int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w\partial_{\boldsymbol{n}} w. \tag{1.5}$$

If Dirichlet or Neumann conditions hold, we have

$$\int_{\partial\Omega} w\partial_{\boldsymbol{n}} w = 0. \tag{1.6}$$

When a Robin condition holds

$$\int_{\partial\Omega} w\partial_{\boldsymbol{n}} w = -\alpha \int_{\partial\Omega} |w| \leq 0 \tag{1.7}$$

In any case we obtain that

$$\int_\Omega |\nabla w|^2 \leq 0 \tag{1.8}$$

From this we infer $\nabla w = 0$ and therefore $w = u - v = const$. This concludes the proof in the case of Neumann condition. In the other cases, the constant must be zero, hence $u = v$, from which the proof is concluded. □

Remark 1.2. Consider the Neumann problem $\Delta u = f$ in Ω, $\partial_{\boldsymbol{n}}$ on $\partial\Omega$. Integrating the equation on Ω we find

$$\int_\Omega f = \int_\Omega \Delta u = \int_{\partial\Omega} \partial_{\boldsymbol{n}} u = \int_{\partial\Omega} h. \tag{1.9}$$

This relation appears as a compatibility condition on the data f and h, that has necessarily to be satisfied in order for the Neumann problem to admit a solution. Thus, when having to solve a Neumann problem, the first thing to do is to check the validity of (1.9). If it does not hold, the problem does not have any solution.

2. Harmonic functions

We want to establish some fundamental properties of harmonic functions. To be precise, we say that a function u is *harmonic* in a domain $\Omega \subset \mathbb{R}^d$ if $u \in C^2(\Omega)$ and

$$\Delta u = 0, \qquad \text{in } \Omega. \tag{2.1}$$

2.1. Mean value properties.

Harmonic functions inherit a mean value property of the following kind: the value at the center of any ball $B \Subset \Omega$, i.e. compactly contained in Ω, equals the average of the values on the boundary ∂B. Actually, something more is true. In what follows, $B_R(\boldsymbol{x})$ denotes the ball of radius $R > 0$ centered at some point $\boldsymbol{x} \in \Omega$.

Theorem 2.1. *Let u be harmonic in $\Omega \subset \mathbb{R}^d$. Then, for any ball $B_R(\boldsymbol{x}) \Subset \Omega$ the following mean value formulas hold:*

$$u(\boldsymbol{x}) = \frac{d}{\omega_d R^d} \int_{B_R(\boldsymbol{x})} u(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{2.2}$$

$$u(\boldsymbol{x}) = \frac{1}{\omega_d R^{d-1}} \int_{\partial B_R(\boldsymbol{x})} u(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma}, \tag{2.3}$$

where ω_d is the measure of unit sphere $\partial B_1(\mathbf{0})$.

Proof. For simplicity, we restrict to the two-dimensional case $d = 2$, for which $\omega_2 = 2\pi$ is the perimeter of the unit circle. Let us start from the second formula. For $r < R$ define

$$\phi(r) = \frac{1}{2\pi r} \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} \tag{2.4}$$

Perform the change of variables $\boldsymbol{\sigma} = \boldsymbol{x} + r\boldsymbol{\sigma}'$. Then $\boldsymbol{\sigma}' \in \partial B_1(\mathbf{0})$, $\mathrm{d}\boldsymbol{\sigma} = r\mathrm{d}\boldsymbol{\sigma}'$ and

$$\phi(r) = \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} u(\boldsymbol{x} + r\boldsymbol{\sigma}') \mathrm{d}\boldsymbol{\sigma}'. \tag{2.5}$$

Let $v(\boldsymbol{y}) = u(\boldsymbol{x} + r\boldsymbol{y})$ and observe that

$$\nabla v(\boldsymbol{y}) = r \nabla u(\boldsymbol{x} + r\boldsymbol{y}), \qquad \Delta v(\boldsymbol{y}) = r^2 \Delta u(\boldsymbol{x} + r\boldsymbol{y}). \tag{2.6}$$

Then we have

$$\begin{aligned} \phi'(r) &= \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} \partial_r u(\boldsymbol{x} + r\boldsymbol{\sigma}') \mathrm{d}\boldsymbol{\sigma}' = \frac{1}{2\pi} \int_{\partial B_1(\mathbf{0})} \nabla u(\boldsymbol{x} + r\boldsymbol{\sigma}') \cdot \boldsymbol{\sigma}' \mathrm{d}\boldsymbol{\sigma}' \\ &= \frac{1}{2\pi r} \int_{\partial B_1(\mathbf{0})} \nabla v(\boldsymbol{\sigma}') \cdot \boldsymbol{\sigma}' \mathrm{d}\boldsymbol{\sigma}' = \frac{1}{2\pi r} \int_{B_1(\mathbf{0})} \Delta v(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &= \frac{r}{2\pi} \int_{B_1(\mathbf{0})} \Delta u(\boldsymbol{x} + r\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = 0. \end{aligned} \tag{2.7}$$

Thus, ϕ is constant and since $\phi(r) \to u(\boldsymbol{x})$ as $r \to 0$, we get (4.2). To obtain (4.1), let $R = r$ in (4.2), multiply by r and integrate both sides between 0 and R. We find

$$\frac{R^2}{2} u(\boldsymbol{x}) = \frac{1}{2\pi} \int_0^R \left[\int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma} \right] \mathrm{d}r = \frac{1}{2\pi} \int_{B_R(\boldsymbol{x})} u(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{2.8}$$

from which (4.1) follows. □

Even more significant is a converse of Theorem 2.1. We say that a continuous function u satisfies the mean value property in Ω if (4.1) or (4.2) holds for any ball $B_R(\boldsymbol{x})$. It turns out that if u is continuous and possesses the mean value property in a domain Ω, then u is harmonic in Ω. Thus we obtain a characterization of harmonic functions through a mean value property. As a by product we deduce that every harmonic function in a domain Ω is continuously differentiable of any order in Ω, that is, it belongs to $C^\infty(\Omega)$. Notice that this is not a trivial fact since it involves mixed derivatives.

Theorem 2.2. *Let $u \in C(\Omega)$. If u satisfies the mean value property, then $u \in C^\infty(\Omega)$ and it is harmonic in Ω.*

We will not prove this theorem here.

2.2. **Maximum principles.** A function satisfying the mean value property in a domain Ω cannot attain its maximum or minimum at an interior point of Ω, unless it is constant. In case Ω is bounded and u (non constant) is continuous up to the boundary of Ω, it follows that u attains both its maximum and minimum *only on* $\partial\Omega$. This result expresses a maximum principle that we state precisely in the following theorem.

Theorem 2.3. *Let $u \in C(\Omega)$, $\Omega \subset \mathbb{R}^d$. If u has the mean value property and attains its maximum or minimum at $\boldsymbol{p} \in \Omega$, then u is constant. In particular, if Ω is bounded and $u \in C(\overline{\Omega})$ is not constant, then, for every $\boldsymbol{x} \in \Omega$,*

$$u(\boldsymbol{x}) < \max_{\partial\Omega} u \qquad \textit{and} \qquad u(\boldsymbol{x}) > \min_{\partial\Omega} u. \tag{2.9}$$

Proof. Once more, we prove this for the two-dimensional case $d = 2$ and only in the case that $\boldsymbol{p} \in \Omega$ is a minimum point of u, as the argument for the maximum is the same. We have

$$m := u(\boldsymbol{p}) \le u(\boldsymbol{y}), \qquad \forall \boldsymbol{y} \in \Omega. \tag{2.10}$$

We want to show that $u \equiv m$ in Ω. Let $\boldsymbol{q}$ be another arbitrary point in Ω. Since Ω is connected, it is possible to find a finite sequence of circles $B(\boldsymbol{x}_j) \Subset \Omega$, $j = 0, \ldots, N$, such that (see Figure 1) $\boldsymbol{x}_j \in B(\boldsymbol{x}_{j-1})$, $j = 1, \ldots, N$ and $\boldsymbol{x}_0 = \boldsymbol{p}$ and $\boldsymbol{x}_N = \boldsymbol{q}$.

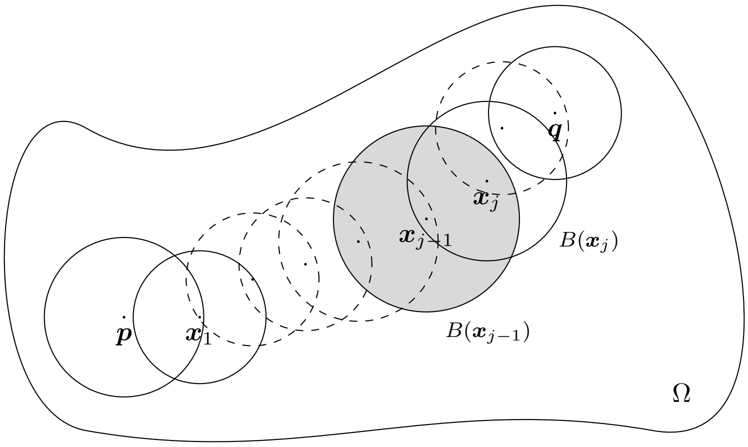

FIGURE 1. A sequence of overlapping circles connecting the points $\boldsymbol{p}$ and $\boldsymbol{q}$

The mean value property gives

$$u(\boldsymbol{p}) = \frac{1}{|B(\boldsymbol{p})|}\int_{B(\boldsymbol{p})} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{2.11}$$

Suppose there exists $\boldsymbol{z} \in B(\boldsymbol{p})$ such that $u(z) > m$. Then, given a circle $B_r(\boldsymbol{z}) \subset B(\boldsymbol{p})$ we can write

$$m = \frac{1}{|B(\boldsymbol{p})|}\int_{B(\boldsymbol{p})} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{|B(\boldsymbol{p})|}\left[\int_{B(\boldsymbol{p})\setminus B_r(\boldsymbol{z})} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{B_r(\boldsymbol{z})} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right]. \tag{2.12}$$

Since $u(\boldsymbol{y}) \geq m$ for every $\boldsymbol{y}$ and, by the mean value again,

$$\int_{B_r(\boldsymbol{z})} u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = |B_r(\boldsymbol{z})|u(\boldsymbol{z}) > m|B_r(\boldsymbol{z})| \tag{2.13}$$

continuing from (2.11) we obtain

$$m > \frac{1}{|B(\boldsymbol{p})|}\left[m|B(\boldsymbol{p}) \setminus B_r(\boldsymbol{z})| + m|B_r(\boldsymbol{z})|\right] = m, \tag{2.14}$$

giving a contradiction. Thus it must be that $u \equiv m$ in $B(\boldsymbol{p})$ and in particular $u(\boldsymbol{x}_1) = m$. We repeat now the same argument with $\boldsymbol{x}_1$ in place of $\boldsymbol{p}$ to show that $u \equiv m$ in $B(\boldsymbol{x}_1)$ and in particular $u(\boldsymbol{x}_2) = m$. Iterating the procedure we eventually deduce that $u(x_N) = u(\boldsymbol{q}) = m$. Since $\boldsymbol{q}$ is an arbitrary point of Ω, we conclude that $u \equiv m$ in Ω, as we wanted. □

An important consequence of the maximum principle is the following corollary.

Corollary 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $h \in C(\partial\Omega)$. The problem*

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = h, & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

has at most a solution $u_h \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, let u_{h_1} and u_{h_2} be the solutions corresponding to the data $h_1, h_2 \in C(\partial\Omega)$. Then

a. *(Comparison) If $h_1 \geq h_2$ on $\partial\Omega$ and $h_1 \neq h_2$, then*

$$u_{h_1} > u_{h_2}, \qquad \text{in } \Omega. \tag{2.16}$$

b. *(Stability) The estimate*

$$|u_{h_1}(\boldsymbol{x}) - u_{h_2}(\boldsymbol{x})| \leq \max_{\partial\Omega} |h_1 - h_2| \tag{2.17}$$

holds for every $\boldsymbol{x} \in \Omega$.

Proof. We first show (2.16) and (2.17). Let $w = u_{h_1} - u_{h_2}$. Then w is harmonic and $w = h_1 - h_2 \geq 0$ on $\partial\Omega$. Since $h_1 \neq h_2$, w is not constant and from Theorem 2.3 we have

$$w(\boldsymbol{x}) > \min_{\partial\Omega}(h_1 - h_2) \geq 0 \tag{2.18}$$

for every $\boldsymbol{x} \in \Omega$. This is (2.16). To prove (2.17), apply Theorem 2.3 to w and $-w$ to find

$$\pm w(\boldsymbol{x}) \leq \max_{\partial\Omega} |h_1 - h_2|, \tag{2.19}$$

for every $\boldsymbol{x} \in \Omega$, which is equivalent to (2.17). Now, if $h_1 = h_2$, (2.17) implies $w = u_{h_1} - u_{h_2} = 0$, so that the Dirichlet problem (2.15) has at most one solution. □

Inequality (2.17) is a stability estimate. Indeed, suppose h is known within an absolute error less than $\varepsilon > 0$, or, in other words, suppose h_1 is an approximation of h and $\max_{\partial\Omega} |h - h_1| < \varepsilon$; then (2.17) gives

$$\max_{\Omega} |u_h - u_{h_1}| < \varepsilon \tag{2.20}$$

so that the approximate solution is known within the same absolute error.

2.3. **The Dirichlet problem in a circle and Poisson formula.** To prove the existence of a solution to one of the boundary value problems we considered so far is not an elementary task. However, in special cases, elementary methods, like separation of variables, work. One example is the solution of the Dirichlet problem on a rectangle, by means of Fourier series. Another is the Dirichlet problem in a circle. Precisely, let $B_R = B_R(\boldsymbol{p})$ be the circle of radius $R > 0$ centered at $\boldsymbol{p} \in \mathbb{R}^2$ and $h \in C(\partial B_R)$ Then the following holds true.

Theorem 2.5. *The unique solution* $u \in C^2(B_R) \cap C(\overline{B_R})$ *of the problem*

$$\begin{cases} \Delta u = 0, & \text{in } B_R, \\ u = h, & \text{on } \partial B_R, \end{cases} \tag{2.21}$$

is given by the Poisson formula

$$u(\boldsymbol{x}) = \frac{R^2 - |\boldsymbol{x} - \boldsymbol{p}|^2}{2\pi R} \int_{\partial B_R} \frac{h(\boldsymbol{\sigma})}{|\boldsymbol{x} - \boldsymbol{\sigma}|^2} \mathrm{d}\boldsymbol{\sigma}. \tag{2.22}$$

In particular, $u \in C^\infty(B_R)$.

The proof of this is not hard and uses Fourier series, but it is long. We do not prove it here. For later use, (2.22) can be also written in polar coordinates as

$$u(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\alpha)}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} \mathrm{d}\alpha, \tag{2.23}$$

using that $\boldsymbol{\sigma} = R(\cos\alpha, \sin\alpha)$ and $\mathrm{d}\boldsymbol{\sigma} = R\mathrm{d}\alpha$.

Theorem 2.5 has an appropriate extension in any number of dimensions. When $B_R = B_R(\boldsymbol{p})$ is a d-dimensional ball, the solution of the Dirichlet problem (2.21) is given by

$$u(\boldsymbol{x}) = \frac{R^2 - |\boldsymbol{x} - \boldsymbol{p}|^2}{\omega_d R} \int_{\partial B_R} \frac{h(\boldsymbol{\sigma})}{|\boldsymbol{x} - \boldsymbol{\sigma}|^d} \mathrm{d}\boldsymbol{\sigma}. \tag{2.24}$$

2.4. **Harnack inequality and Liouville theorem.** From the mean value and Poisson's formulas we deduce another maximum principle, known as Harnack inequality.

Theorem 2.6 (Harnack inequality)**.** *Let u be harmonic and nonnegative in the ball $B_R = B_R(\mathbf{0}) \subset \mathbb{R}^d$. Then*

$$\frac{R^{d-2}(R-|\boldsymbol{x}|)}{(R+|\boldsymbol{x}|)^{d-1}}u(\mathbf{0}) \leq u(\boldsymbol{x}) \leq \frac{R^{d-2}(R+|\boldsymbol{x}|)}{(R-|\boldsymbol{x}|)^{d-1}}u(\mathbf{0}), \tag{2.25}$$

for any $\boldsymbol{x} \in B_R$.

Proof. We prove it in the three-dimensional case $d = 3$. From Poisson formula (2.24),

$$u(\boldsymbol{x}) = \frac{R^2 - |\boldsymbol{x}|^2}{\omega_3 R}\int_{\partial B_R}\frac{u(\boldsymbol{\sigma})}{|\boldsymbol{x}-\boldsymbol{\sigma}|^3}\mathrm{d}\boldsymbol{\sigma}. \tag{2.26}$$

Observe that $R - |\boldsymbol{x}| \leq |\boldsymbol{\sigma} - \boldsymbol{x}| \leq R + |\boldsymbol{x}|$ and $R^2 - |\boldsymbol{x}|^2 = (R - |\boldsymbol{x}|)(R + |\boldsymbol{x}|)$. Then, by the mean value property,

$$u(\boldsymbol{x}) \leq \frac{R+|\boldsymbol{x}|}{(R-|\boldsymbol{x}|)^2}\frac{1}{4\pi R}\int_{\partial B_R}u(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = \frac{R(R+|\boldsymbol{x}|)}{(R-|\boldsymbol{x}|)^2}u(\mathbf{0}). \tag{2.27}$$

Analogously,

$$u(\boldsymbol{x}) \geq \frac{R-|\boldsymbol{x}|}{(R+|\boldsymbol{x}|)^2}\frac{1}{4\pi R}\int_{\partial B_R}u(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = \frac{R(R-|\boldsymbol{x}|)}{(R+|\boldsymbol{x}|)^2}u(\mathbf{0}). \tag{2.28}$$

The proof is over. □

Harnack inequality has an important consequence: the only harmonic functions in $\mathbb{R}^d$ bounded from below or above are the constant functions.

Corollary 2.7 (Liouville theorem)**.** *If u is harmonic in $\mathbb{R}^d$ and $u(\boldsymbol{x}) \geq M$, then u is constant.*

Proof. The function $w = u - M$ is harmonic in $\mathbb{R}^d$ and nonnegative. Fix $\boldsymbol{x} \in \mathbb{R}^d$ and choose $R > |\boldsymbol{x}|$; Harnack inequality gives

$$\frac{R^{d-2}(R-|\boldsymbol{x}|)}{(R+|\boldsymbol{x}|)^{d-1}}w(\mathbf{0}) \leq w(\boldsymbol{x}) \leq \frac{R^{d-2}(R+|\boldsymbol{x}|)}{(R-|\boldsymbol{x}|)^{d-1}}w(\mathbf{0}). \tag{2.29}$$

Letting $R \to \infty$ above we get

$$w(\mathbf{0}) \leq w(\boldsymbol{x}) \leq w(\mathbf{0}). \tag{2.30}$$

whence $w(\boldsymbol{x}) = w(\mathbf{0})$. Since $\boldsymbol{x}$ is arbitrary we conclude that w, and therefore also u, is constant. □

2.5. **Calculus of variations.** Given a smooth bounded domain $\Omega \subset \mathbb{R}^d$, a solution to the Poisson problem (1.1) with various boundary conditions can be characterized in terms of a minimization problem. We consider a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = h, & \text{on } \partial\Omega, \end{cases} \tag{2.31}$$

for a given, regular enough boundary datum h. For any function $w \in C^2(\Omega) \cap C(\overline{\Omega})$, define the energy

$$E[w] = \frac{1}{2}\int_\Omega |\nabla w(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}. \tag{2.32}$$

The above functional is called the *Dirichlet energy* of u. We claim that the unique solution u to (2.31) has the property that

$$E[u] \leq E[w], \qquad \forall w \in C^2(\Omega) \cap C(\overline{\Omega}), \quad \text{such that } w = h \text{ on } \partial\Omega. \tag{2.33}$$

In other words, u is the minimizer of the Dirichlet energy among all functions with the same regularity that satisfy the boundary condition. Notice that

$$\begin{aligned} E[w] &= \frac{1}{2}\int_\Omega |\nabla w(\boldsymbol{x}) - \nabla u(\boldsymbol{x}) + \nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_\Omega |\nabla w(\boldsymbol{x}) - \nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla(w(\boldsymbol{x}) - u(\boldsymbol{x}))\cdot\nabla u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + E[u]. \end{aligned} \tag{2.34}$$

Moreover, using that $\Delta u = 0$ and $u - w = 0$ on $\partial\Omega$, integration by parts yield

$$\begin{aligned} &\int_\Omega \nabla(w(\boldsymbol{x}) - u(\boldsymbol{x}))\cdot\nabla u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &\qquad = -\int_\Omega (w(\boldsymbol{x}) - u(\boldsymbol{x}))\Delta u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} (w(\boldsymbol{\sigma}) - u(\boldsymbol{\sigma}))\partial_{\boldsymbol{n}} u(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = 0. \end{aligned} \tag{2.35}$$

Therefore,

$$E[w] = \frac{1}{2}\int_\Omega |\nabla w(\boldsymbol{x}) - \nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} + E[u] \geq E[u], \tag{2.36}$$

as we wanted to show.

The converse of the above claim is also true: if $u \in C^2(\Omega)\cap C(\overline{\Omega})$ with $u = h$ on $\partial\Omega$ minimizes the Dirichlet energy, then it solves (2.31). To show this, let $w \in C^2(\Omega)\cap C(\overline{\Omega})$ be such that $w = 0$ on $\partial\Omega$, and consider the function $\phi = \phi(t) : [-1, 1] \to [0, \infty)$ defined by

$$\phi(t) = E[u + tw] = E[u] + t\int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + t^2 E[w]. \tag{2.37}$$

Since u is a minimizer, ϕ attains its minimum at $t = 0$, where $\phi'(0) = 0$. Now,

$$\phi'(t) = \int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + 2tE[w] \qquad \Rightarrow \qquad \phi'(0) = \int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.38}$$

Therefore,

$$\int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \qquad \forall w \in C^2(\Omega)\cap C(\overline{\Omega}). \tag{2.39}$$

Now, since $w = 0$ on $\partial\Omega$, an integration by parts gives

$$0 = \int_\Omega \nabla u(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_\Omega \Delta u(\boldsymbol{x}) w(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.40}$$

Since the above holds for every w, we get that $\Delta u = 0$ in Ω. To summarize we have the following result.

Theorem 2.8. *A function $u \in C^2(\Omega)\cap C(\overline{\Omega})$ solves (2.31) if and only if it minimizes the Dirichlet energy $E[u]$ in (2.32) among all function in $C^2(\Omega)\cap C(\overline{\Omega})$ with boundary value h.*

Remark 2.9. If instead of (2.31) we consider the more general problem

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = h, & \text{on } \partial\Omega, \end{cases} \tag{2.41}$$

the same argument holds true for the more general energy

$$E[w] = \frac{1}{2}\int_\Omega |\nabla w(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} + \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.42}$$

The equation (2.41) is called the *Euler-Lagrange* equation associated to the Dirichlet energy E. Similar results hold for Neumann boundary conditions as well.

3. Fundamental solution and Newtonian potential

We derive here another solution formula for the Laplace equation in the whole space $\mathbb{R}^d$. It involves various types of potentials, constructed using a special function, called the fundamental solution of the Laplace operator.

3.1. **The fundamental solution.** As we did for the diffusion equation, let us look at the invariance properties characterizing the operator Δ: the invariances by translations and by rotations. Let $u = u(\boldsymbol{x})$ be harmonic in $\mathbb{R}^d$. Invariance by translations means that the function $v(\boldsymbol{x}) = u(\boldsymbol{x} - \boldsymbol{y})$, for each fixed $\boldsymbol{y} \in \mathbb{R}^d$, is also harmonic, as it is immediate to check. Invariance by rotations means that, given a rotation in $\mathbb{R}^d$, represented by an orthogonal matrix M (i.e. $M^T = M^{-1}$), also $v(\boldsymbol{x}) = u(M\boldsymbol{x})$ is harmonic in $\mathbb{R}^d$. A typical rotation invariant quantity is the distance function from a point, for instance from the origin, that is $r = |\boldsymbol{x}|$. Thus, let us look for radially symmetric harmonic functions $u = u(r)$. Starting in dimension $d = 2$, the Laplace operator is polar coordinates is

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}. \tag{3.1}$$

Thus we look to solve the one-dimensional problem

$$\partial_{rr}u + \frac{1}{r}\partial_r u = 0 \qquad \Rightarrow \qquad u(r) = c_0 + c_1 \log r. \tag{3.2}$$

In dimension $d \geq 3$, notice that

$$r^2 = |\boldsymbol{x}|^2 = \sum_{i=1}^{d} x_i^2 \qquad \Rightarrow \qquad \partial_{x_i} r = \frac{x_i}{r}. \tag{3.3}$$

Hence the chain rule entails

$$\partial_{x_i} u = \partial_r u(r)\frac{x_i}{r}, \qquad \partial_{x_i x_i} u = \partial_{rr} u(r)\frac{x_i^2}{r^2} + \partial_r u(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right). \tag{3.4}$$

Therefore, the Laplace equation for radial functions in dimension $d \geq 3$ is

$$\partial_{rr}u + \frac{d-1}{r}\partial_r u = 0 \qquad \Rightarrow \qquad u(r) = c_0 + c_1 \frac{1}{r^{d-2}}. \tag{3.5}$$

The constants are chosen as $c_0 = 0$ in all dimensions $d \geq 2$, $c_1 = -\frac{1}{2\pi}$ for $d = 2$ and $c_1 = \frac{1}{(d-2)\omega_d}$ for $d \geq 3$. We then have the following definition.

Definition 3.1. The function

$$\Phi(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{2\pi}\log|\boldsymbol{x}|, & d = 2, \\ \dfrac{1}{(d-2)\omega_d}\dfrac{1}{|\boldsymbol{x}|^{d-2}}, & d \geq 3 \end{cases} \tag{3.6}$$

is called the fundamental solution of the Laplace operator Δ.

The above choice of the constant c_1 is made in order to have

$$-\Delta\Phi(x) = \delta(x), \tag{3.7}$$

where δ denotes the Dirac measure at $\boldsymbol{x} = \boldsymbol{0}$. The physical meaning of Φ is remarkable: if $d = 3$, in standard units, $4\pi\Phi$ represents the electrostatic potential due to a unitary charge located at the origin and vanishing at infinity. Clearly, if the origin is replaced by a point $\boldsymbol{y}$, the corresponding potential is $\Phi(\boldsymbol{x} - \boldsymbol{y})$ and

$$-\Delta\Phi(\boldsymbol{x} - \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{3.8}$$

Recall that the above identity has to be understood by the action on test functions.

3.2. **The newtonian potential in three-dimensions.** The convolution between a given function f and the fundamental solution Φ is called the Newtonian potential of f. In dimension $d = 3$ it is written as

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}\mathrm{d}\boldsymbol{y} \tag{3.9}$$

Formally, from (3.8), if f is smooth and has compact support one can write

$$-\Delta u(\boldsymbol{x}) = -\int_{\mathbb{R}^3} \Delta\Phi(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^3} \delta(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = f(\boldsymbol{x}). \tag{3.10}$$

The above formula is true under certain assumptions on f. Clearly, u is not the only solution of $-\Delta u = f$, since $u + c$, c constant, is a solution as well. However, the Newtonian potential is the only solution vanishing at infinity. All this is stated precisely in the theorem below, where, for simplicity, we assume $f \in C^2(\mathbb{R}^3)$ with compact support.

Theorem 3.2. *Let $f \in C^2(\mathbb{R}^3)$ with compact support. Let u be the Newtonian potential of f, defined by (3.9). Then, u is the only solution in $\mathbb{R}^3$ of*

$$-\Delta u = f, \tag{3.11}$$

such that $u \in C^2(\mathbb{R}^3)$ vanishes at infinity.

Proof. The uniqueness part follows from Liouville theorem (Corollary 2.7). Let $v \in C^2(\mathbb{R}^3)$ be another solution to (3.11), vanishing at infinity. Then $u - v$ is a bounded harmonic function in all $\mathbb{R}^3$ and therefore is constant. Since it vanishes at infinity it must be zero; thus $u - v$.

To show that (3.9) belongs to $C^2(\mathbb{R}^3)$ and satisfies (3.11), observe that we can write (3.9) in the alternative form

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} \tag{3.12}$$

Since $1/|\boldsymbol{y}|$ is integrable near the origin and f is zero outside a compact set, we can take first and second order derivatives under the integral sign to get

$$\partial_{x_j x_k} u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y}) \partial_{x_j x_k} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{3.13}$$

Since $\partial_{x_j x_k} f \in C(\mathbb{R}^3)$, formula (3.13) shows that also $\partial_{x_j x_k} u$ is continuous and therefore $u \in C^2(\mathbb{R}^3)$. It remains to prove (3.11). Since $\Delta_{\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{y}) = \Delta_{\boldsymbol{y}} f(\boldsymbol{x}-\boldsymbol{y})$, from (3.13), we have

$$\Delta u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y})\Delta_{\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y})\Delta_{\boldsymbol{y}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{3.14}$$

We want to integrate by parts, but since Φ has a singularity at $\boldsymbol{y} = \boldsymbol{0}$, we have first to isolate the origin, by choosing a small ball $B_\varepsilon = B_\varepsilon(\boldsymbol{0})$ and writing

$$\Delta u(\boldsymbol{x}) = \int_{B_\varepsilon} \Phi(\boldsymbol{y})\Delta_{\boldsymbol{y}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathbb{R}^3\setminus B_\varepsilon} \Phi(\boldsymbol{y})\Delta_{\boldsymbol{y}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} = I_\varepsilon + J_\varepsilon. \tag{3.15}$$

On the one hand,

$$|I_\varepsilon| \le \frac{\max|\Delta f|}{4\pi}\int_{B_\varepsilon}\frac{1}{|\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} = \max|\Delta f|\int_0^\varepsilon r\mathrm{d}r = \frac{\max|\Delta f|}{2}\varepsilon^2. \tag{3.16}$$

In particular, $I_\varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand, keeping in mind that f vanishes outside a compact set, we can integrate J_ε by parts (twice), obtaining

$$\begin{aligned} J_\varepsilon &= -\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon} \nabla_{\boldsymbol{\sigma}} f(\boldsymbol{x}-\boldsymbol{\sigma})\cdot\boldsymbol{n}(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} - \int_{\mathbb{R}^3\setminus B_\varepsilon}\nabla_{\boldsymbol{y}}\Phi(\boldsymbol{y})\nabla_{\boldsymbol{y}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ &= -\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon} \nabla_{\boldsymbol{\sigma}} f(\boldsymbol{x}-\boldsymbol{\sigma})\cdot\boldsymbol{n}(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} + \int_{\partial B_\varepsilon}\nabla\Phi(\boldsymbol{\sigma})\cdot\boldsymbol{n}(\boldsymbol{\sigma}) f(\boldsymbol{x}-\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}, \end{aligned} \tag{3.17}$$

where $\boldsymbol{n}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}/\varepsilon$ is the outward unit normal to ∂B_ε (hence the switch of signs...) and we used that $\Delta\Phi$ is harmonic in $\mathbb{R}^3 \setminus B_\varepsilon$. We have

$$\frac{1}{4\pi\varepsilon}\left|\int_{\partial B_\varepsilon}\nabla_{\boldsymbol{\sigma}} f(\boldsymbol{x}-\boldsymbol{\sigma})\cdot\boldsymbol{n}(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}\right| \le \frac{\max|\nabla f|}{4\pi}\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.18}$$

Moreover, $\nabla_{\boldsymbol{y}}\Phi(\boldsymbol{y}) = -\boldsymbol{y}/|\boldsymbol{y}|^3$ and hence

$$\int_{\partial B_\varepsilon}\nabla_{\boldsymbol{\sigma}}\Phi(\boldsymbol{\sigma})\cdot\boldsymbol{n}(\boldsymbol{\sigma}) f(\boldsymbol{x}-\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} = -\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon} f(\boldsymbol{x}-\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma} \to -f(\boldsymbol{x}) \qquad \text{as } \varepsilon \to 0. \tag{3.19}$$

Thus, $J_\varepsilon \to f(\boldsymbol{x})$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ in (3.15) we get $-\Delta u = f$, as we wanted. □

It is worth mentioning that there is nothing special about $d = 3$, and various versions of Theorem 3.2 hold for any $d \ge 2$. We will not discuss them here.

3.3. **A div-curl system.** Using the properties of the Newtonian potential we can solve the following two problems, that appear in several applications e.g. to linear elasticity, fluid dynamics or electrostatics.

Problem 1. Precisely, given a scalar f and a vector field $\boldsymbol{\omega}$, we want to find a vector field $\boldsymbol{u}$ such that

$$\begin{cases} \nabla \cdot \boldsymbol{u} = f, \\ \nabla \times \boldsymbol{u} = \boldsymbol{\omega}. \end{cases} \tag{3.20}$$

Essentially, we want to reconstruct a vector field $\boldsymbol{u}$ in $\mathbb{R}^3$ from the knowledge of its divergence and curl. We assume that $\boldsymbol{u}$ has continuous second derivatives and vanishes at infinity, as it is required in most applications.

Problem 2. Given a vector field $\boldsymbol{u}$ in $\mathbb{R}^3$, we want to find φ and a vector field $\boldsymbol{\psi}$ such that

$$\boldsymbol{u} = \nabla\varphi + \nabla \times \boldsymbol{\psi} \tag{3.21}$$

The above is called *Helmholtz decomposition* formula. It decomposes $\boldsymbol{u}$ into the sum of a divergence free vector field and a curl free vector field.

Solution to Problem 1. First of all observe that, since $\nabla \cdot (\nabla \times \boldsymbol{u}) = 0$, a necessary condition for the existence of a solution is $\nabla \cdot \boldsymbol{\omega} = 0$. Let us check uniqueness. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are solutions sharing the same data f and $\boldsymbol{\omega}$, their difference $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ vanishes at infinity and satisfies

$$\nabla \cdot \boldsymbol{w} = 0, \qquad \nabla \times \boldsymbol{w} = \mathbf{0}. \tag{3.22}$$

From $\nabla \times \boldsymbol{w} = \mathbf{0}$ we infer the existence of a scalar function U such that $\nabla U = \boldsymbol{w}$. Taking the divergence of this equation and using that $\boldsymbol{w}$ is divergence free, we deduce that $\Delta U = 0$. Thus U is harmonic. Hence its derivatives, i.e. the components v_j of $\boldsymbol{w}$, are bounded harmonic functions in $\mathbb{R}^3$. Liouville theorem implies that each v_j is constant and therefore identically zero since it vanishes at infinity. We conclude that, under the stated assumptions, the solution of Problem 1 is unique.

To find $\boldsymbol{u}$, split it into $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{z}$ and look for $\boldsymbol{v}$ and $\boldsymbol{z}$ such that

$$\nabla \cdot \boldsymbol{z} = 0, \qquad \nabla \times \boldsymbol{z} = \boldsymbol{\omega}, \tag{3.23}$$

$$\nabla \cdot \boldsymbol{v} = f, \qquad \nabla \times \boldsymbol{v} = \mathbf{0}. \tag{3.24}$$

As before, from $\nabla \times \boldsymbol{v} = \mathbf{0}$ we infer the existence of a scalar function ϕ such that $\nabla\phi = \boldsymbol{v}$, while $\nabla \cdot \boldsymbol{v} = f$ implies that $\Delta\phi = f$. We have seen that, under suitable hypotheses on f, ϕ is given by the Newtonian potential of f, that is

$$\phi(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}, \tag{3.25}$$

and $\boldsymbol{v} = \nabla\phi$. To find $\boldsymbol{z}$, recall the identity

$$\nabla \times \nabla \times \boldsymbol{z} = \nabla(\nabla \cdot \boldsymbol{z}) - \Delta\boldsymbol{z}. \tag{3.26}$$

Since $\nabla \cdot \boldsymbol{z} = 0$, we get

$$-\Delta\boldsymbol{z} = \nabla \times \nabla \times \boldsymbol{z} = \nabla \times \boldsymbol{\omega}, \tag{3.27}$$

so that

$$\boldsymbol{z}(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}. \tag{3.28}$$

Let us summarize the conclusions in the next theorem, also specifying the hypotheses on f and $\boldsymbol{\omega}$.

Theorem 3.3. *Let $f \in C^2(\mathbb{R}^3)$ and $\boldsymbol{\omega} \in C^3(\mathbb{R}^3)$ with compact support be given, with $\nabla \cdot \boldsymbol{\omega} = 0$. Then, the unique solution vanishing at infinity of the system*

$$\begin{cases} \nabla \cdot \boldsymbol{u} = f, \\ \nabla \times \boldsymbol{u} = \boldsymbol{\omega}, \end{cases} \tag{3.29}$$

is given by the vector field

$$\boldsymbol{u}(\boldsymbol{x}) = -\frac{1}{4\pi}\nabla\int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}. \tag{3.30}$$

Somewhat informally, one can write $\boldsymbol{u} = \nabla\Delta^{-1} f - \nabla \times \Delta^{-1}\boldsymbol{\omega}$.

Solution to Problem 2. If $\boldsymbol{u}$, $\nabla \cdot \boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ satisfy the hypotheses of Theorem 3.3, we can write

$$\boldsymbol{u}(\boldsymbol{x}) = -\frac{1}{4\pi}\nabla \int_{\mathbb{R}^3} \frac{\nabla \cdot \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \nabla \times \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}. \tag{3.31}$$

Since $\boldsymbol{u}$ is rapidly vanishing at infinity, we have

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \nabla \times \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} = \frac{1}{4\pi}\nabla \times \int_{\mathbb{R}^3} \frac{\nabla \times \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}. \tag{3.32}$$

We conclude that

$$\boldsymbol{u} = \nabla\varphi + \nabla \times \boldsymbol{\psi} \tag{3.33}$$

where

$$\varphi(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y} \tag{3.34}$$

and

$$\boldsymbol{\psi}(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \boldsymbol{u}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}. \tag{3.35}$$

3.4. **Application to fluid dynamics.** Consider the three dimensional flow of an incompressible fluid of constant unit density and viscosity $\nu > 0$, subject to a conservative external force $\boldsymbol{F} = \nabla f$, like gravity . If $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ denotes the velocity field and $p = p(t, \boldsymbol{x})$ is the hydrostatic pressure, the laws of conservation of mass and linear momentum give for $\boldsymbol{u}$ and p the celebrated Navier-Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \nabla f, \\ \nabla\cdot\boldsymbol{u} = 0. \end{cases} \tag{3.36}$$

We look for solution subject to a given initial condition

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3. \tag{3.37}$$

where $\boldsymbol{g}$ is also divergence free. In general, the system is extremely difficult to solve. In the case of slow flow, for instance due to high viscosity, the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ becomes negligible, compared for instance to $\nu\Delta$, and (3.36) simplifies to the linear equation Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \nabla f, \\ \nabla\cdot\boldsymbol{u} = 0. \end{cases} \tag{3.38}$$

It is possible to find an explicit formula for the solution of (3.38) by writing everything in terms of the *vorticity* $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. In fact, taking the curl of the first equation in (3.38) and using that $\nabla \times (-\nabla p + \nu\Delta\boldsymbol{u} + \nabla f) = \nu\Delta\boldsymbol{\omega}$, we have to solve

$$\begin{cases} \partial_t \boldsymbol{\omega} = \nu\Delta\boldsymbol{\omega}, \\ \boldsymbol{\omega}(0, \boldsymbol{x}) = \nabla \times \boldsymbol{g}(\boldsymbol{x}). \end{cases} \tag{3.39}$$

This is a global Cauchy problem for the heat equation. If $\boldsymbol{g} \in C^2(\mathbb{R}^3)$ and $\nabla \times \boldsymbol{g}$ is bounded, we have

$$\boldsymbol{\omega}(t, \boldsymbol{x}) = \frac{1}{(4\pi\nu t)^{3/2}}\int_{\mathbb{R}^3} \mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu t}} \nabla \times \boldsymbol{g}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{3.40}$$

Moreover, for $t > 0$, we can take the divergence operator under the integral in the above formula and deduce that $\nabla\cdot\boldsymbol{\omega} = 0$. Therefore, if $\nabla\times\boldsymbol{g}$ has compact support, we can recover $\boldsymbol{u}$ by solving the system

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \nabla\times\boldsymbol{u} = \boldsymbol{\omega} \tag{3.41}$$

according to formula (3.30) with $f = 0$. Finally to find the pressure, from (3.38) we have

$$\nabla p = -\partial_t\boldsymbol{u} + \nu\Delta\boldsymbol{u} + \nabla f. \tag{3.42}$$

Since $\partial_t\boldsymbol{\omega} = \nu\Delta\boldsymbol{\omega}$, the right hand side has zero curl; hence the above can be solved and determines p up to an additive constant.

4. Problems

Problem 1. A function $w \in C^2(\mathbb{R}^d)$ is called subharmonic if $\Delta w \geq 0$ in $\mathbb{R}^d$. Show that if w is subharmonic, then for any $R > 0$ and any $\boldsymbol{x} \in \mathbb{R}^d$,

$$w(\boldsymbol{x}) \leq \frac{d}{\omega_d R^d} \int_{B_R(\boldsymbol{x})} w(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{4.1}$$

$$w(\boldsymbol{x}) \leq \frac{1}{\omega_d R^{d-1}} \int_{\partial B_R(\boldsymbol{x})} w(\boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma}, \tag{4.2}$$

where ω_d is the measure of unit sphere $\partial B_1(\mathbf{0})$.

Problem 2. Show that if $u \in C^2(\mathbb{R}^d)$ is harmonic and $F \in C^2(\mathbb{R})$ is convex, then the function $w = F(u)$ is subharmonic.

Problem 3. Assume that $u \in C^2(\mathbb{R}^d)$ is harmonic in $\mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} |u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = M < \infty. \tag{4.3}$$

Prove that u is identically zero.

Problem 4. Assume that $u \in C^2(\mathbb{R}^d)$ is harmonic in $\mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} |\nabla u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = M < \infty. \tag{4.4}$$

Prove that u is constant.

Problem 5. Suppose that u is a harmonic function in the disk $B_2(\mathbf{0}) \subset \mathbb{R}^2$ and that $u = 3\sin(2\theta) + 1$ for $r = 2$. Without finding the solution, answer the following questions:

a. Find the maximum value of u on $\overline{B}_2(\mathbf{0})$;
b. Calculate the value of u at the origin.

Problem 6. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and consider the problem

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ \partial_{\boldsymbol{n}} u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $f \in C^2(\Omega)$.

a. Show that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves (4.5) if and only if u is the minimizer of

$$E[u] = \int_\Omega \left[\frac{1}{2}|\nabla u(\boldsymbol{x})|^2 + u(\boldsymbol{x}) f(\boldsymbol{x})\right] \mathrm{d}\boldsymbol{x}; \tag{4.6}$$

b. Assume further that $\displaystyle\int_\Omega f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \neq 0$. Prove that $E[u]$ has no minimum, and therefore (4.5) has no solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

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