Ex O3h, from Final Exam 2022 Q1(e)

(e) In principle we just need to solve the system

$$V_1^{h,g} := h(S_1 - (1+r)S_0) + g(X_1 - (1+r)X_0) \geq 0$$

and find values of h, g such that $V_1^{h,g}$ is not identically 0. We can however simplify the problem by eliminating one variable, as follows.

By plugging $t = 12$ into the above formula for $\mathcal{P}(t)$, or in eq. (2), gives that $\mathcal{P} = \mathcal{P}(12) = (5, 8)$. Thus, we see that the price 4 for the derivative X is too small to be fair; thus, to make an arbitrage one needs to *buy* the derivative (not sell it), i.e. we should take any $g > 0$, so we might as well take $g = 1$ So, we need to solve the system

$$V_1^{h,1} = h(S_1 - (1+r)S_0) + (X_1 - (1+r)X_0) \geq 0,$$

i.e.

$$\begin{cases} h(2 - \frac{5}{4} \cdot 4) + (1 - \frac{5}{4} \cdot 4) \geq 0 \\ h(4 - \frac{5}{4} \cdot 4) + (12 - \frac{5}{4} \cdot 4) \geq 0 \\ h(6 - \frac{5}{4} \cdot 4) + (8 - \frac{5}{4} \cdot 4) \geq 0 \end{cases},$$

i.e. the system

$$\begin{cases} -3h \geq 4 \\ -h \geq -7 \\ h \geq 3 \end{cases},$$

which has solution $h \in [-3, -\frac{4}{3}]$.

Question 2

(Total: 25 marks)

[default,O27]

On the sample space $\Omega = \{\omega_i\}_{i=1,\ldots,5}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period arbitrage-free market model where the bank account has zero interest rate, and there are two stocks S^1, S^2 with prices $S_0^1 = 3, \quad S_0^2 = 3$

$$S_1^1 = \begin{pmatrix} 1 & 3 & 5 & 7 & 4 \end{pmatrix}^T, \quad S_1^2 := \begin{pmatrix} 1 & 4 & 6 & 5 & 4 \end{pmatrix}^T.$$

For the *non-replicable* derivative with payoff X_1, we consider the problem of finding

$$p := \min\{x : (x,h) \in \mathbb{R}\times\mathbb{R}^2 \text{ satisfies } V_1^{x,h}(\omega_i) \geq X_1(\omega_i) \text{ for all } i\}, \tag{3}$$

the smallest initial capital p of a portfolio (x,h) super-replicating X_1 $\mathbb{P}$ a.s., where as usual we denote with $V_1^{x,h} = x + \sum_{j=1}^2 h_j(S_1^j - S_0^j)$ the wealth relative to the initial capital x and the trading strategy $h = (h_1, h_2)$; and its dual linear program, i.e.

$$d := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \mathcal{M}\}, \text{ where } \mathcal{M} := \{\mathbb{Q} \text{ proba. on } \Omega : \mathbb{E}^{\mathbb{Q}}(S_1^j - S_0^j) = 0, \ j = 1,2\} \tag{4}$$

is the set of martingale measures. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) (6 points) Is the market (B, S^1, S^2) arbitrage-free?

Solution:

a 1^{st} **solution:** $h = (h_1, h_2)$ is an arbitrage iff

$$V^h := \overline{V}_1^{0,h} := h \cdot (\overline{S}_1 - \overline{S}_0) = h_1(\overline{S}_1^1 - \overline{S}_0^1) + h_2(\overline{S}_1^2 - \overline{S}_0^2)$$

satisfies $V^h \geq 0$ a.s. and $V^h > 0$ with non-zero probability, i.e. iff h is a solution to the system of inequalities

$$\begin{cases} -2h_1 - 2h_2 \geq 0 \\ \qquad\qquad h_2 \geq 0 \\ 2h_1 + 3h_2 \geq 0 \\ 4h_1 + 2h_2 \geq 0 \\ \ \ h_1 + \ \ h_2 \geq 0 \end{cases},$$

s.t. at least one inequality is strict. The first and last inequalities are equivalent to $h_1 = -h_2$, and with this substitution the system of inequalities becomes

$$\begin{cases} h_2 \geq 0 \\ h_2 \geq 0 \\ -2h_2 \geq 0 \end{cases},$$

implying $h_2 = 0$, and thus $h_1 = 0$, and thus all inequalities hold with equality. So, there is no arbitrage.

2^{nd} **solution:** Let us prove that the market is arbitrage-free by using the FTAP. Using the usual bijection between a probability $\mathbb{Q} \sim \mathbb{P}$ and the vector $(q_i)_{i=1}^5$ with components $q_i = \mathbb{Q}(\{\omega_i\}) \in (0, \infty)$ which add up to 1, we identify any EMM $\mathbb{Q}$ with the solutions to system of linear equalities $\mathbb{Q}(\Omega) = 1, \mathbb{E}^{\mathbb{Q}}(\overline{S}^j_{n+1}) = \overline{S}^j_n, j = 1, \ldots, m$ (in the variable $\mathbb{Q}$), i.e.

$$\begin{cases} q_1 + q_2 + q_3 + q_4 + q_5 = 1 \\ q_1 + 3q_2 + 5q_3 + 7q_4 + 4q_5 = 3 \\ q_1 + 4q_2 + 6q_3 + 5q_4 + 4q_5 = 3 \end{cases},$$

such that $q_i > 0$ for all $i = 1, \ldots, 5$. Notice that it is not enough to argue that the system of linear equalities has solution, since we additionally have to check that (some of) those solutions have strictly positive components.

If we use such system to compute q_1, q_2, q_3 as functions of $q_4 = s, q_5 = t$, and solve the resulting system, we find

$$q_1 = 1 - 3t - s \tag{5}$$

$$q_2 = -1 + 7t + \frac{3}{2}s \tag{6}$$

$$q_3 = 1 - 5t - \frac{3}{2}s. \tag{7}$$

So, the set of EMM is identified with the set of vectors

$$\mathcal{M} = \{(1 - 3t - s, -1 + 7t + \frac{3}{2}s, 1 - 5t - \frac{3}{2}s, t, s) : t, s \in \mathbb{R}\} \cap (0, \infty)^5,$$

Finally, we need to impose the conditions $q_i > 0$ for $i = 1, \ldots, 5$, which leads to 5 inequalities to be satisfied by s, t, and to show that such system of 5 inequalities has at least one solution. It is obvious that $t = \frac{1}{8}, s = \frac{1}{8}$ is one such solution, which corresponds to taking

$$q = \left(\frac{1}{2}, \frac{1}{16}, \frac{3}{16}, \frac{1}{8}, \frac{1}{8}\right) = \frac{1}{16}(8, 1, 3, 2, 2).$$

So there is no arbitrage. There is no need to find the general solution; if you wanted to, you'd need to solve the system $q_1, q_2, q_3, q_4, q_5 \geq 0$ (e.g. using the FM algorithm), and then exclude the values of s, t for which at least one of the above inequalities holds with equality (i.e. at least one of the 5 equations of the form $q_i = 0$ holds).

Ex M17: from Final exam 2022, Q2(d)

(d) (S, T) is not in general Markov. It is easy to guess this, looking at the expression $T_{n+1} = T_n + 1_{\{a \le M_{n+1} \le b\}}$, which suggests that to predict T_{n+1} one needs to know also M_n, not just S_n. To build a counter-example is easy, but a tad time-consuming, since it involves quite some algebraic calculations. Pick some simple values of S, for example by taking

$$S_0 = 4, \quad u = 1/d = 2.$$

To compute the values of (S, T), it helps to draw the value of (S, Q), since its values can be calculated quickly recursively: S_{n+1} can be calculated from the value of S_n and of the $n + 1$ coin toss, without knowing the whole past, since

$$S_{n+1}(\omega_1, \dots, \omega_{n+1}) = S_n(\omega_1, \dots, \omega_n) a(\omega_{n+1}) \quad \text{for } a(H) = u, \quad a(T) = d,$$

and analogously Q_{n+1} can be calculated recursively via $Q_{n+1} = Q_n S^2_{n+1}$. Once calculated the value of (S, Q), it then becomes easy to compute the values of M (and thus of T). Notice that we only need to compute (and thus draw) the branches of the tree that will matter to us: it is clear what these will be, since to disprove the markov property I need to find a value of $k \in \mathbb{N}$ (ideally as small as possible) and two values A, B of ω_k such that $(S_k, T_k)(A) = (S_k, T_k)(B)$ and yet the two futures of T emanating from the nodes A, B are different; in particular I need $S_k(A) = S_k(B)$ and so it is natural to look at $k = 2, A = HT, B = TH$, and at the two branches emanating from these nodes.

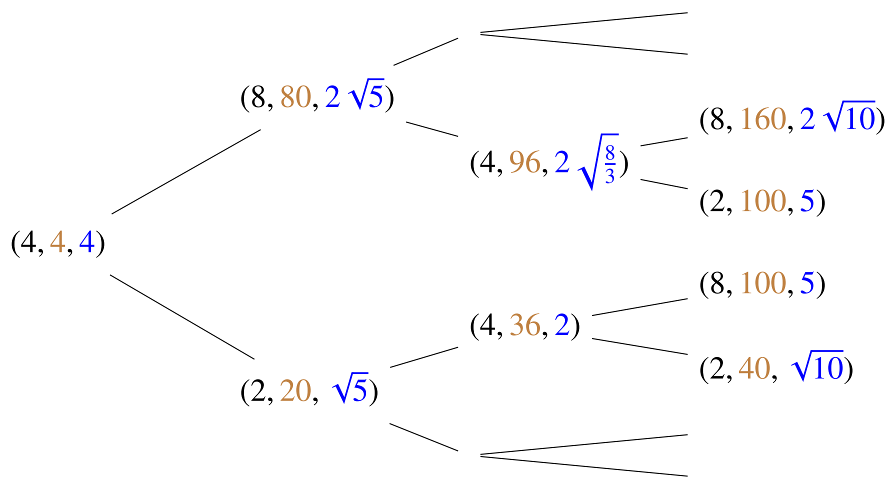

Figure 1: Partial view of the binary tree of the process (S, Q, M)

Thus, choosing

$$a = 1, \quad b = 2\sqrt{9} = 6$$

which clearly satisfy

$$a \leq 2 = \min(4, \sqrt{5}, 2, \sqrt{10}, 5), \quad \max\left(4, 2\sqrt{5}, 2\sqrt{\frac{8}{3}}, 5\right) = 5 \leq 6 = b = 2\sqrt{9} < 2\sqrt{10}$$

we find that while $(S_2, T_2)(HT) = (S_2, T_2)(TH) = (4, 3)$, we have that

$$3 = T_3(HTH) \neq T_3(HTT) = T_3(THH) = T_3(THT) = 4, \tag{20}$$

and thus for $p := \mathbb{Q}(X_n = H) = \frac{1/2}{3/2} = \frac{1}{3}$ we get that

$$\mathbb{E}_2^{\mathbb{Q}}(f(S_3, T_3))(HT) = pf(8, 3) + (1 - p)f(2, 4)$$

differs from

$$\mathbb{E}_2^{\mathbb{Q}}(f(S_3, T_3))(TH) = pf(8, 4) + (1 - p)f(2, 4)$$

for any f such that $f(8, 3) \neq f(8, 4)$. This shows that $\mathbb{E}_2(f(S_3, T_3))$ is not $\sigma(S_2, T_2)$-measurable, and so (S, T) is not Markov.

(f) Since $V_N = f_N(W_N)$ for $f_N(w) = (w - K)^+$, and W is Markov, there exist f_n such that $V_n = f_n(W_n)$ (we show this more explicitly in the next item, where we explicitly calculate f_n). In particular there exist f_n such that $V_n = f_n(S_n, W_n)$ for all n.

As for S, this is a bit of a trick question. At first sight, it looks like we cannot express V_N as a function of S_N. However in this model if $k = k(n, \omega)$ is the number of Heads in the first n coin tosses, and thus $n - k$ is the number of Tails in the first n coin tosses, then

$$S_n = S_0 + 2k - (n - k), \quad W_n = S_0 + 2k + (n - k).$$

If we solve for k the first equation then insert such k in the second equation we get that W_n is a function of S_n (more precisely $W_n = \frac{1}{3}(2S_0 + 4n + S_n)$), and thus such is also V_n, i.e. $V_n = h_n(S_n)$ for some h_n for all n. Notice however that this would not have been the case had the Y_i's had a more complicated law, i.e. if we considered a trinomial model and each Y_i's took 3 distinct values.