Imperial College London
MATH 50004 Multivariable Calculus
Mid-Term Examination Date: 11th November 2021
SOLUTIONS

Question One Solution

(i)

$$[\nabla(\mathbf{A}\cdot\mathbf{B})]_i = \frac{\partial}{\partial x_i}(A_jB_j) = A_j\frac{\partial B_j}{\partial x_i} + B_j\frac{\partial A_j}{\partial x_i}. \text{ [\textbf{1 mark}]} \tag{1}$$

$$-[\text{curl}\,(\mathbf{A}\times\mathbf{B})+\mathbf{B}\times\text{curl}\,\mathbf{A}+\mathbf{A}\times\text{curl}\,\mathbf{B}]_i = -\varepsilon_{ijk}\left\{\frac{\partial}{\partial x_j}[\mathbf{A}\times\mathbf{B}]_k + B_j[\text{curl}\,\mathbf{A}]_k + A_j[\text{curl}\,\mathbf{B}]_k\right\} [\textbf{1 mark}]$$

The RHS above can then be expressed as

$$\begin{aligned}
&-\varepsilon_{ijk}\varepsilon_{klm}\left\{\frac{\partial}{\partial x_j}(A_lB_m) + B_j\frac{\partial}{\partial x_l}A_m + A_j\frac{\partial}{\partial x_l}B_m\right\} \text{ [\textbf{1 mark}]}\\
&= (\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm})\left\{\frac{\partial}{\partial x_j}(A_lB_m) + B_j\frac{\partial A_m}{\partial x_l} + A_j\frac{\partial B_m}{\partial x_l}\right\} \text{ [\textbf{1 mark}]}\\
&= \frac{\partial}{\partial x_j}(A_jB_i) + B_j\frac{\partial A_i}{\partial x_j} + A_j\frac{\partial B_i}{\partial x_j} - \frac{\partial}{\partial x_j}(A_iB_j) - B_j\frac{\partial A_j}{\partial x_i} - A_j\frac{\partial B_j}{\partial x_i} \text{ [\textbf{1 mark}]}\\
&= A_j\frac{\partial B_i}{\partial x_j} + B_i\frac{\partial A_j}{\partial x_j} + B_j\frac{\partial A_i}{\partial x_j} + A_j\frac{\partial B_i}{\partial x_j} - A_i\frac{\partial B_j}{\partial x_j} - B_j\frac{\partial A_i}{\partial x_j} - B_j\frac{\partial A_j}{\partial x_i} - A_j\frac{\partial B_j}{\partial x_i}
\end{aligned}$$

The 2nd and 5th terms vanish because **A** and **B** are solenoidal, the 3rd and 6th terms cancel and the 1st and 4th terms double up, leaving

$$2A_j\frac{\partial B_i}{\partial x_j} - B_j\frac{\partial A_j}{\partial x_i} - A_j\frac{\partial B_j}{\partial x_i} \text{ [\textbf{3 marks}]}$$

Adding this expression to (1) we are left with

$$2A_j\frac{\partial B_i}{\partial x_j} \text{ [\textbf{1 mark}]}$$

and hence the expression given in the question simplifies to

$$2(\mathbf{A}\cdot\nabla)\mathbf{B}. \text{ [\textbf{1 mark}]}$$

(ii) Given $\mathbf{A} = y^2\mathbf{i} + x\mathbf{j}$ then

$$\text{div}\,\mathbf{A} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(x) = 0. \text{ [\textbf{1 mark}]}$$

Given $\mathbf{B} = x\mathbf{i} - y\mathbf{j}$ then

$$\text{div}\,\mathbf{B} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) = 1 - 1 = 0. \text{ [\textbf{1 mark}]}$$

$$\text{curl}\,\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & -y & 0 \end{vmatrix} = \mathbf{0}$$

Thus:

$$\mathbf{A} \times \text{curl}\,\mathbf{B} = \mathbf{0}. \;\; [\mathbf{1\ mark}]$$

We also have

$$\text{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & 0 \end{vmatrix} = (1-2y)\mathbf{k}$$

$$\Rightarrow \mathbf{B} \times \text{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & -y & 0 \\ 0 & 0 & 1-2y \end{vmatrix} = (2y^2-y)\mathbf{i} + (2xy-x)\mathbf{j} \;\; [\mathbf{2\ marks}],$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ y^2 & x & 0 \\ x & -y & 0 \end{vmatrix} = -(y^3+x^2)\mathbf{k}$$

$$\Rightarrow \text{curl}\,(\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & -y^3-x^2 \end{vmatrix} = -3y^2\mathbf{i} + 2x\mathbf{j}. \;\; [\mathbf{2\ marks}]$$

Finally we need

$$\mathbf{A} \cdot \mathbf{B} = xy^2 - xy,$$

so that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y})(xy^2 - xy) = (y^2-y)\mathbf{i} + (2xy-x)\mathbf{j}. \;\; [\mathbf{1\ mark}]$$

Putting all the results together:

$$\begin{aligned} \nabla(\mathbf{A} \cdot \mathbf{B}) - \text{curl}\,(\mathbf{A} \times \mathbf{B}) - \mathbf{B} \times \text{curl}\,\mathbf{A} - \mathbf{A} \times \text{curl}\,\mathbf{B} \\ = (y^2-y)\mathbf{i} + (2xy-x)\mathbf{j} + 3y^2\mathbf{i} - 2x\mathbf{j} - (2y^2-y)\mathbf{i} - (2xy-x)\mathbf{j} \\ = 2y^2\mathbf{i} - 2x\mathbf{j}. \;\; [\mathbf{1\ mark}] \end{aligned}$$

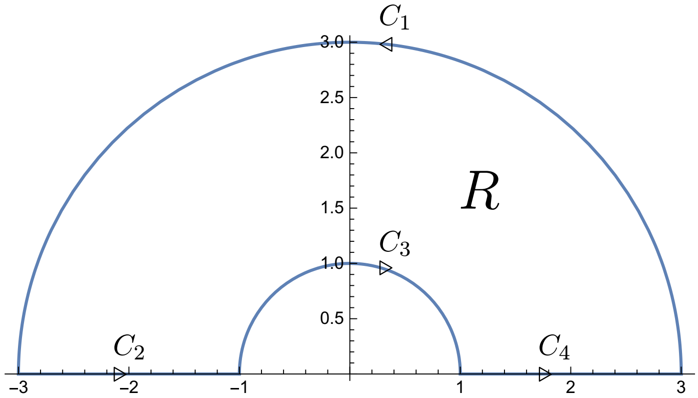
Check:

$$2(\mathbf{A} \cdot \nabla)\mathbf{B} = 2(y^2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})(x\mathbf{i} - y\mathbf{j}) = 2(y^2\mathbf{i} - x\mathbf{j}),$$

as expected [**1 mark**].

Question Two Solution

(i) [**3 marks**]



(ii) We have

$$\Gamma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (2x - y^3 + 1)\ dx - xy\ dy. \ [\mathbf{1\ mark}]$$

We split C up into 4 sections as shown in the sketch above.
On C_1 we have $x = 3\cos\theta, y = 3\sin\theta$ with θ starting from 0 and ending at π so that

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi (6\cos\theta - 27\sin^3\theta + 1)(-3\sin\theta) - (9\cos\theta\sin\theta)(3\cos\theta)\ d\theta \\ &= \left[-18\frac{\sin^2\theta}{2} + 3\cos\theta + 27\frac{\cos^3\theta}{3}\right]_0^\pi + 81\int_0^\pi \sin^4\theta\ d\theta \\ &= -24 + 81(3\pi/8), \ \ [\mathbf{3\ marks}]\end{aligned}$$

making use of the trigonometric identity to evaluate the final integral.
On C_3 we have $x = \cos\theta, y = \sin\theta$ with θ starting from π and ending at 0 so that

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_\pi^0 (2\cos\theta - \sin^3\theta + 1)(-\sin\theta) - (\cos\theta\sin\theta)(\cos\theta)\ d\theta \\ &= \left[-2\frac{\sin^2\theta}{2} + \cos\theta + \frac{\cos^3\theta}{3}\right]_\pi^0 + \int_\pi^0 \sin^4\theta\ d\theta \\ &= 8/3 - 3\pi/8. \ [\mathbf{3\ marks}]\end{aligned}$$

On C_2 we have $y = 0$ with x starting at -3 and ending at -1, so that

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-3}^{-1} (2x + 1)\ dx = [x^2 + x]_{-3}^{-1} = -6. \ [\mathbf{2\ marks}]$$

On C_4 we have $y = 0$ with x starting at 1 and ending at 3, so that

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (2x + 1)\ dx = [x^2 + x]_1^3 = 10. \ [\mathbf{2\ marks}]$$

Adding the 4 contributions together gives us

$$\Gamma = -24 + 243\pi/8 + 8/3 - 3\pi/8 - 6 + 10 = -52/3 + 30\pi, \ \ [\mathbf{1\ mark}]$$

as required.

(iii) Green's theorem states that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

In this case $F_1 = 2x - y^3 + 1, F_2 = -xy$, and so the appropriate double integral is

$$\int_R (-y + 3y^2)\ dx\, dy.\ [\mathbf{2\ marks}]$$

Switching to polar coordinates, this is

$$\int_0^\pi \int_1^3 (-r \sin\theta + 3r^2 \sin^2\theta)\, r\, dr\, d\theta = \left[\frac{r^3}{3}\right]_1^3 [\cos\theta]_0^\pi + \left[\frac{3r^4}{4}\right]_1^3 \frac{\pi}{2} = -\frac{52}{3} + 30\pi,\ \ [\mathbf{3\ marks}]$$

in agreement with part (ii).