Lectures on "Introduction to Geophysical Fluid Dynamics"

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• **Idea of the lectures** is to provide a relatively advanced-level course that builds up on the existing introductory-level fluid dynamics courses. The lectures target an audience of upper-level undergraduate students, graduate students, and postdocs.

• **Main topics**:

(1) *Introduction*

(2) *Governing equations*

(3) *Geostrophic dynamics*

(4) *Quasigeostrophic theory*

(5) *Ekman layer*

(6) *Rossby waves*

(7) *Linear instabilities*

(8) *Ageostrophic motions*

(9) *Transport phenomena*

(10) *Nonlinear dynamics and wave-mean flow interactions*

• **Suggested textbooks**:

(1) *Introduction to geophysical fluid dynamics* (Cushman-Roisin);

(2) *Fundamentals of geophysical fluid dynamics* (McWilliams);

(3) *Geophysical fluid dynamics* (Pedlosky);

(4) *Atmospheric and oceanic fluid dynamics* (Vallis);

(5) *Essentials of atmospheric and oceanic dynamics* (Vallis).

Motivations

• Main motivation for the recent rapid development of *Geophysical Fluid Dynamics* (GFD) is advancing our knowledge about the following very important, challenging and multidisciplinary research lines:

— *Earth system modelling*,

— *Predictive understanding of climate variability* (emerging new science!),

— *Forecast of various natural phenomena* (e.g., weather),

— *Natural hazards, environmental protection, natural resources*, etc.

What is GFD?

• Most of GFD is about dynamics of stratified and turbulent fluids on giant rotating planets (spheres).

— On smaller scales GFD becomes classical fluid dynamics with geophysical applications.

— Other planets and some astrophysical fluids (e.g., stars, galaxies) are also included in GFD.

• GFD combines applied math and theoretical physics.
It is about *mathematical representation* and *physical interpretation* of geophysical fluid motions.

• Mathematics of GFD is *heavily computational*, even relative to other branches of fluid dynamics (e.g., modelling of the ocean circulation and atmospheric clouds are the largest computational problems in the history of science).

— This is because lab experiments (i.e., analog simulations) can properly address only tiny fraction of interesting questions (e.g., small-scale waves, convection, microphysics).

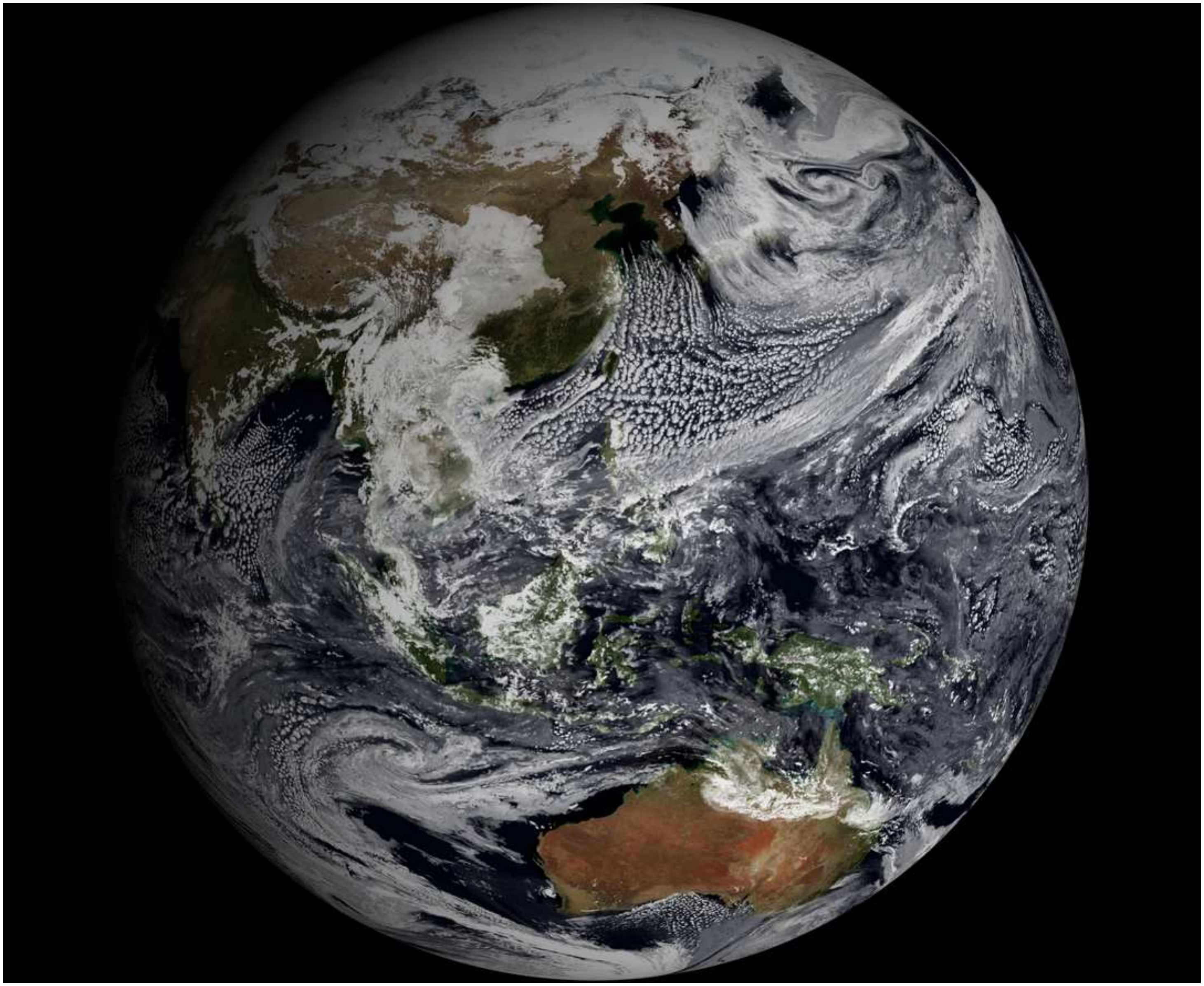
• In geophysics theoretical advances are often GFD-based rather than experiment-based, because obtaining *field measurements* is very complicated, difficult, expensive and often impossible.

Let's overview some geophysical phenomena of interest...

An image of the Earth from space:

- Earth's *atmosphere* and *oceans* are the main but not the only target of GFD

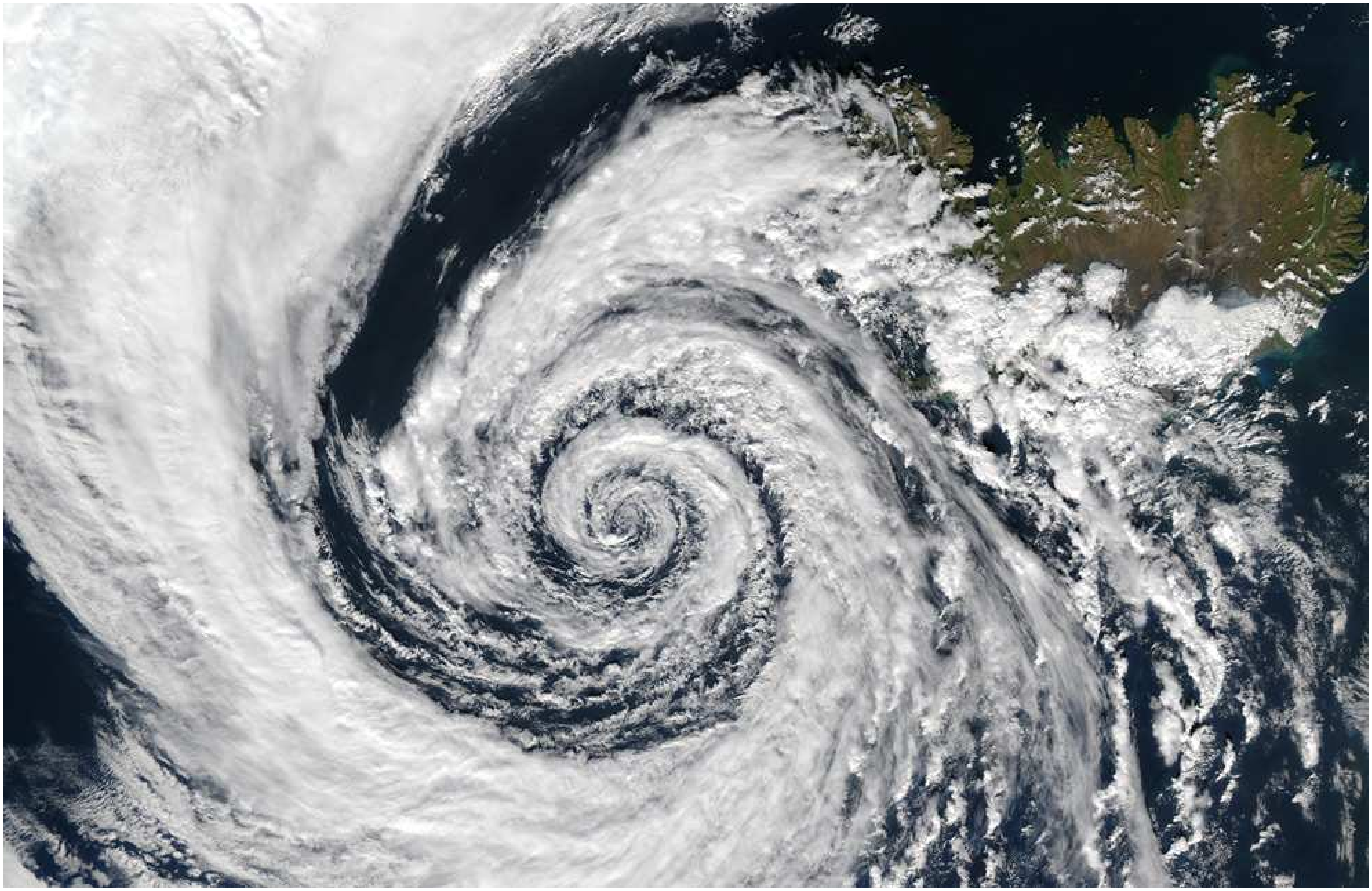
This is not an image of the Earth from space...



...but a visualized solution of the mathematical equations!

• Atmospheric *cyclones* and *anticyclones* shape up midlatitude weather.

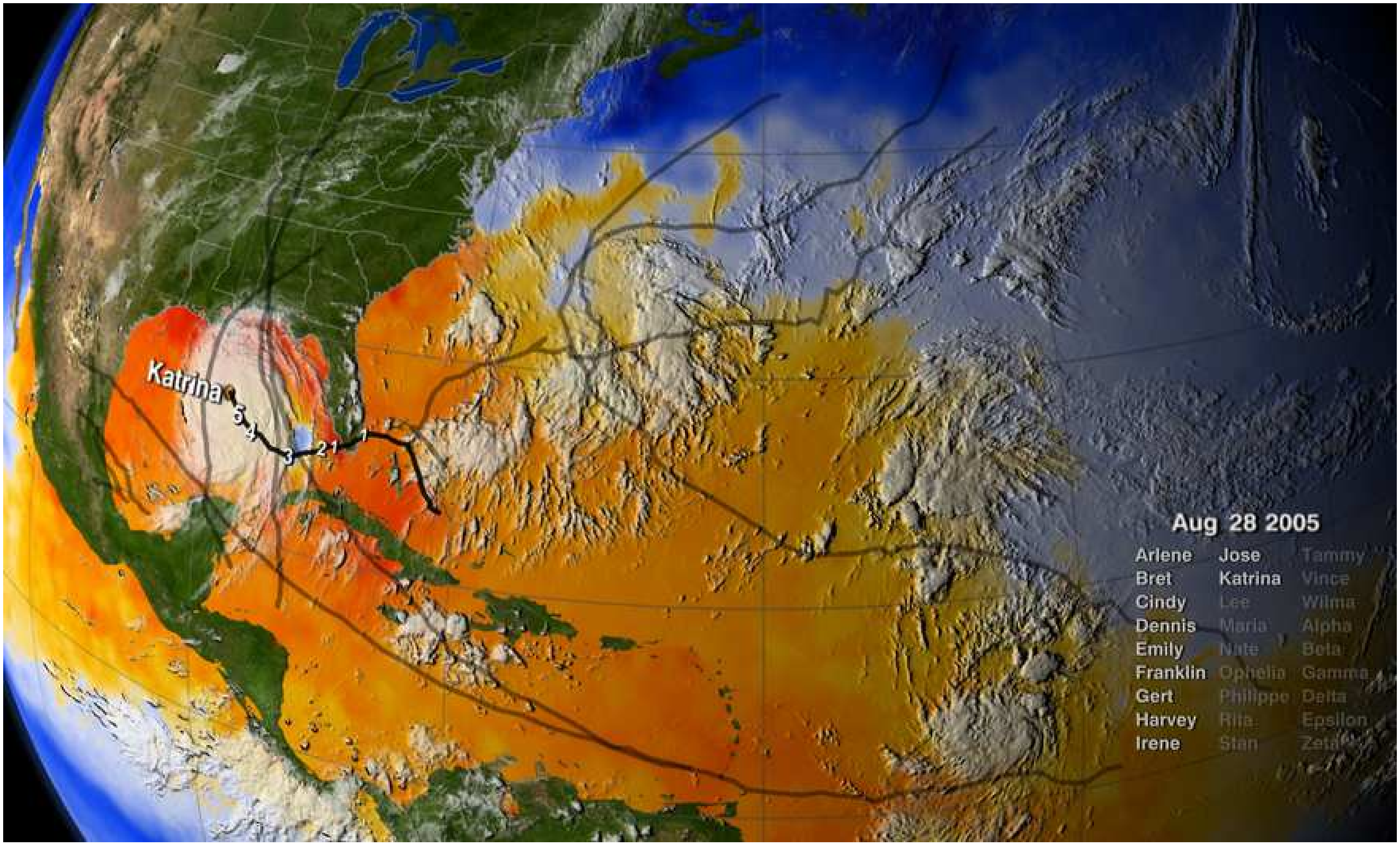
This cyclone is naturally visualized by clouds:



• Modelling *atmospheric clouds* is notoriously difficult multi-scale problem with phase transitions and chemistry involved.

• *Tropical cyclones* (hurricanes and typhoons) are a coupled ocean-atmosphere phenomenon. These are powerful storm systems characterized by low-pressure center, strong winds, heavy rain, and numerous thunderstorms.

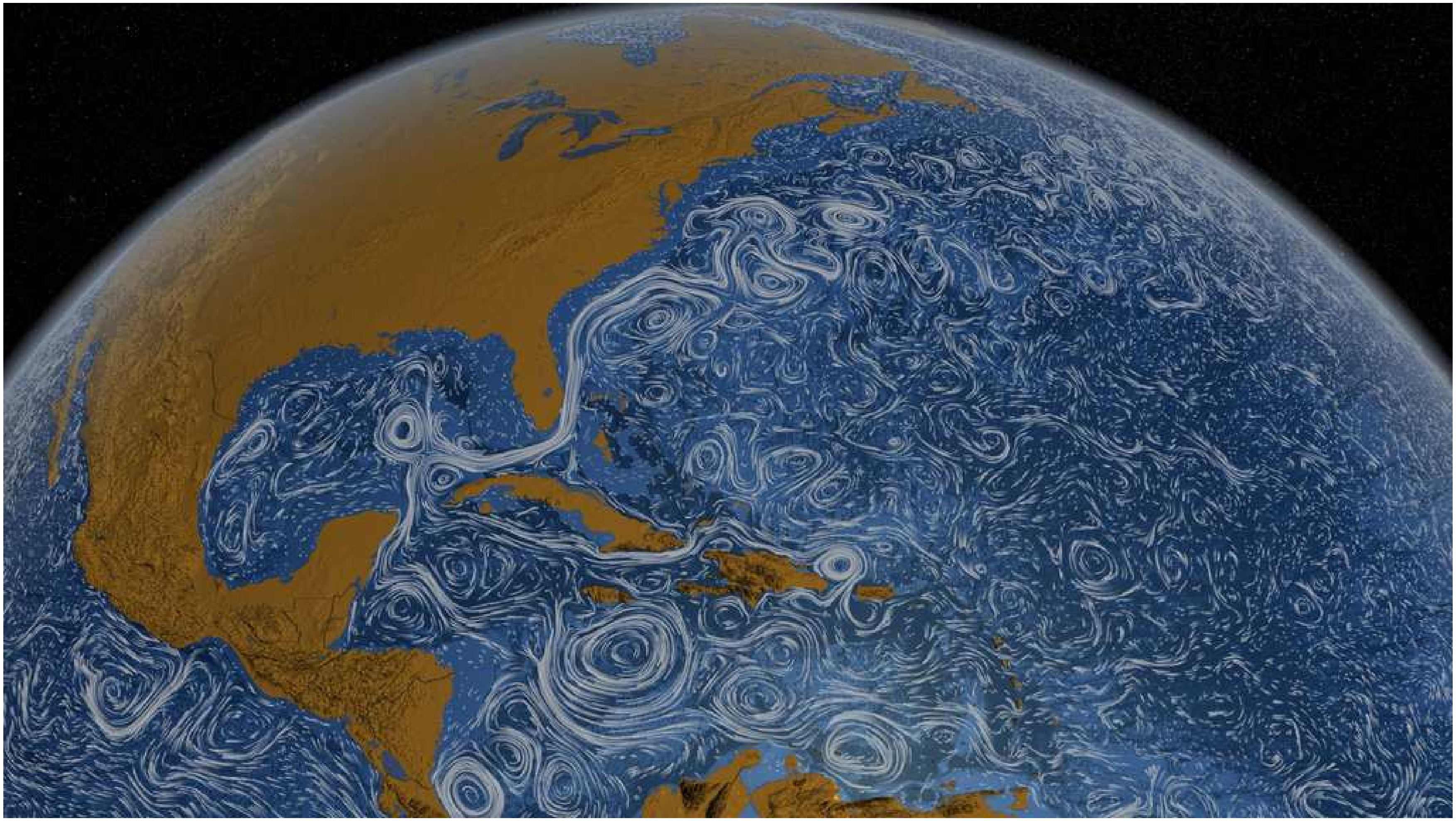
Hurricane Katrina approaching New Orleans:



• *Ocean-atmosphere coupling*: Ocean and atmosphere exchange momentum, heat, water, radiation, aerosols, and greenhouse gases.

Ocean-atmosphere interface is a very complex two-sided boundary layer:

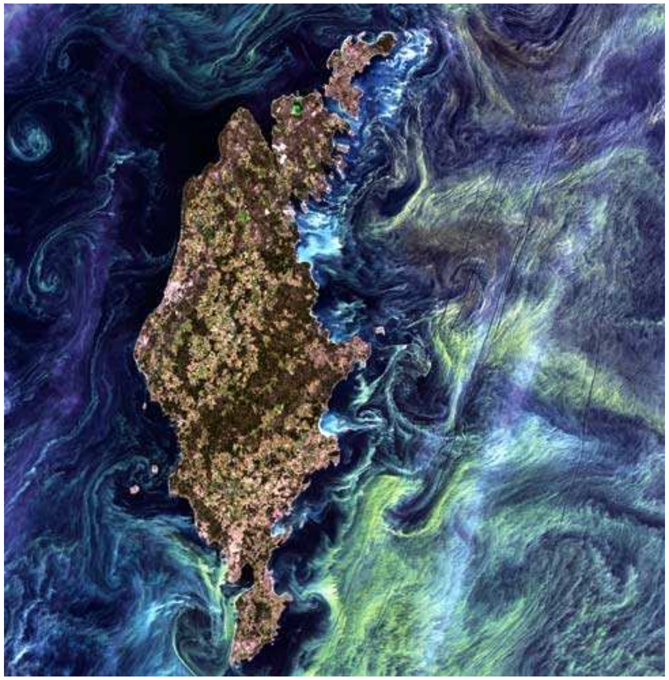
Ocean currents are full of transient mesoscale eddies:



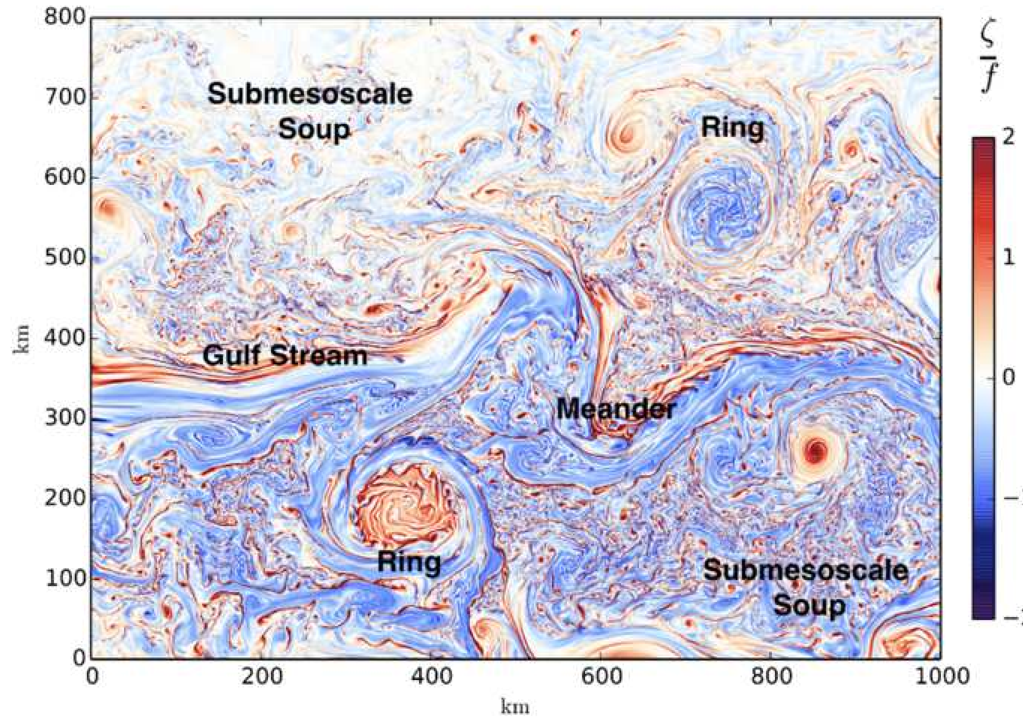
• *Mesoscale (synoptic) oceanic eddies* — also called “oceanic weather” — are dynamically similar to atmospheric cyclones and anticyclones; however, they are smaller, slower and more numerous.

• Modelling mesoscale eddies and their large-scale effects is very important (and challenging), because predictive skills of climate models crucially depend on their accurate representation.

Submesoscale eddies around island...



...and around the Gulf Stream



• *Submesoscale motions* are geostrophically and hydrostatically unbalanced, which means that they are less affected by the rotation and stratification than mesoscale eddies.

• Many submesoscale processes are steered by coasts and topography (e.g., coastal currents, upwellings, tidal mixing, lee waves).

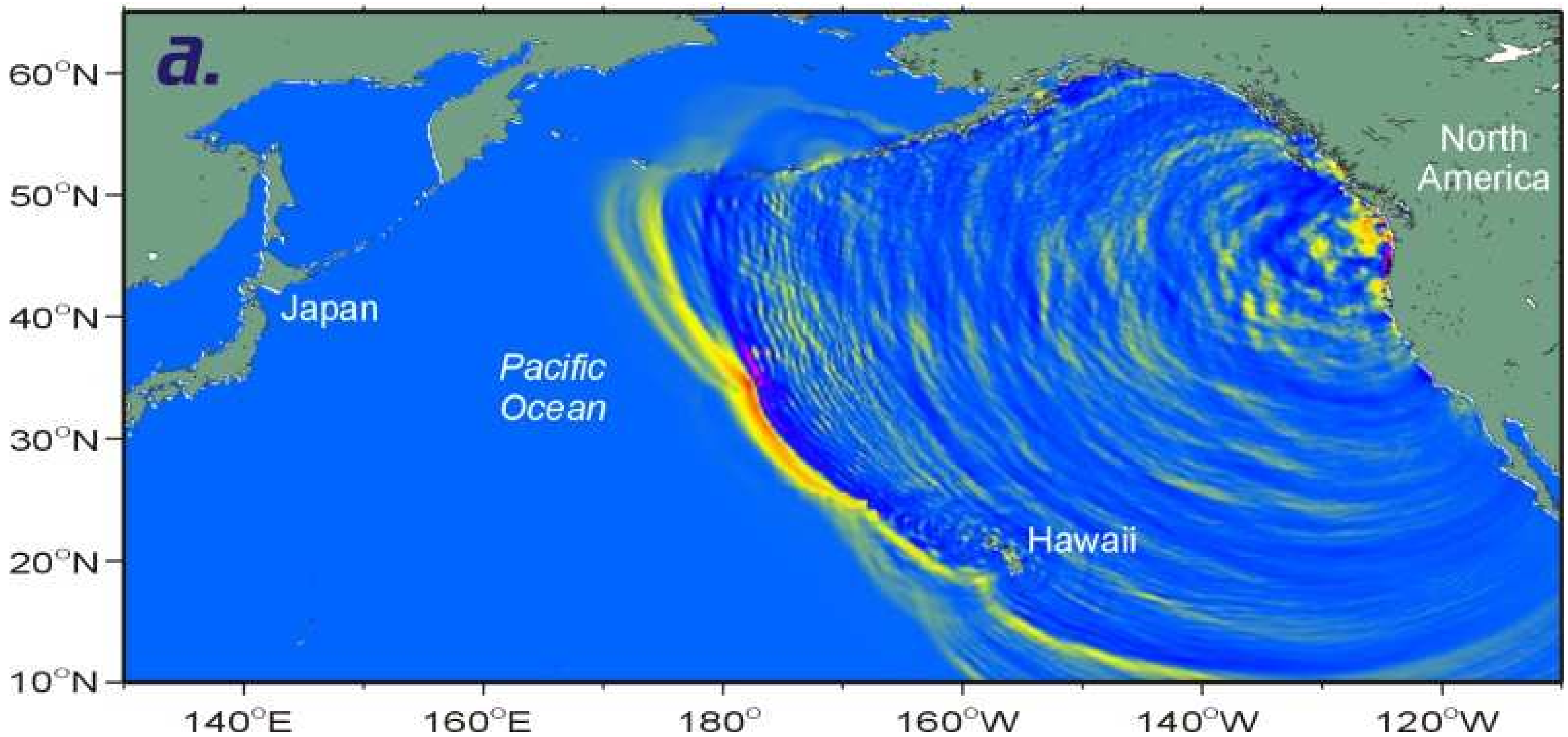
• Turbulence operates on all scales down to millimeters, but on smaller scales effects of planetary rotation and density stratification weaken, and GFD turns into classical fluid dynamics.

Breaking surface gravity wave

• GFD deals with many types of waves operating on lengthscales from centimeters to thousands of kilometers.

• Breaking *internal gravity waves* are very important for vertical mixing shaping up stratifications of geophysical fluids.

Evolution of a tsunami predicted by high-accuracy shallow-water modelling:



• *Tsunami* is specific type of surface gravity waves: long, fast and energetic. Tsunami running on coasts creates extreme danger.

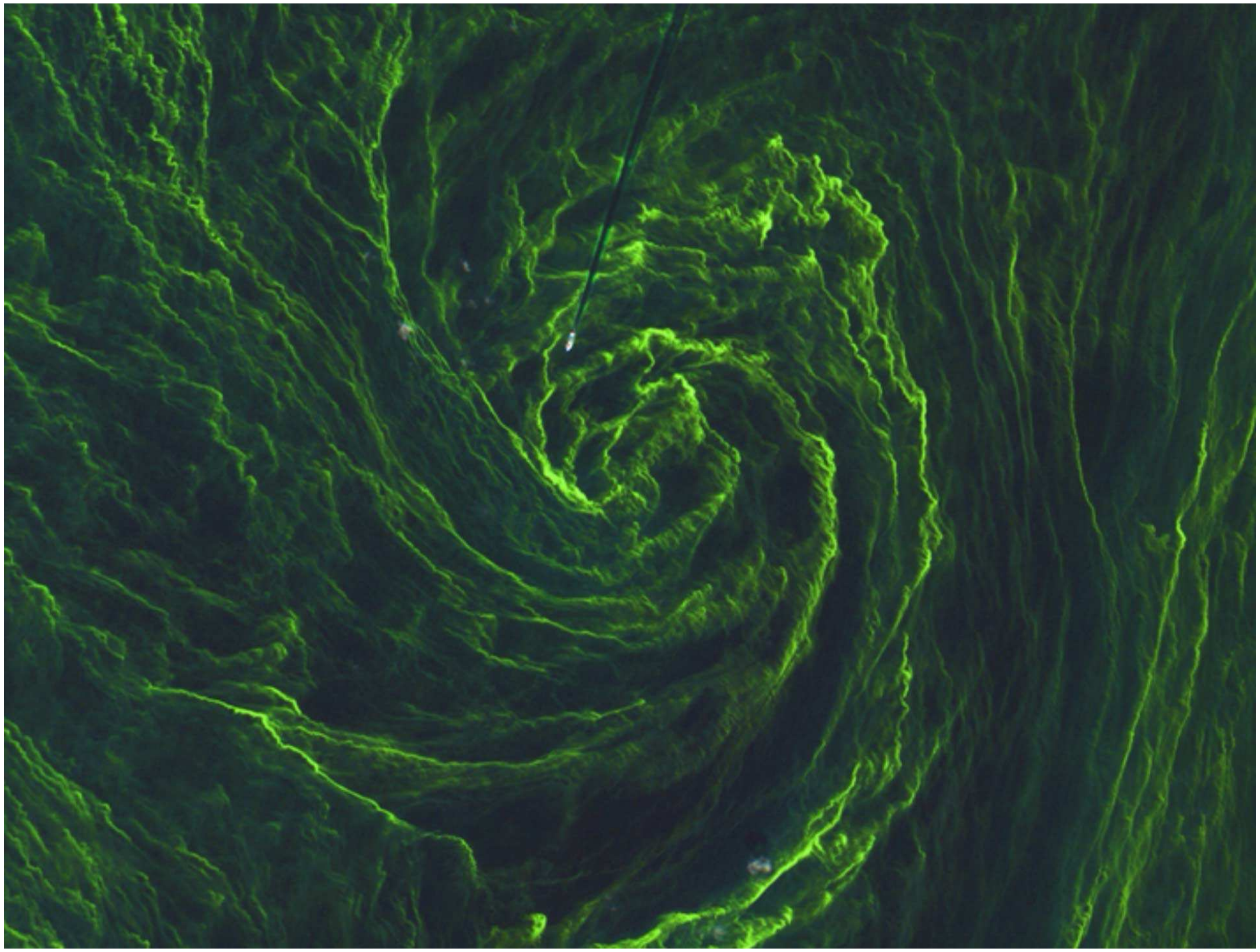
• GFD is involved in problems with formation and propagation of ice.

⟸ *Flowing glacier*

Formation of marine ice ⟹

Erupting volcano Eyjafjallajokull spewes ashes to be transported over large distances...

• GFD provides basis for modelling *turbulent material transport* of various substances and chemicals in atmospheres and oceans.

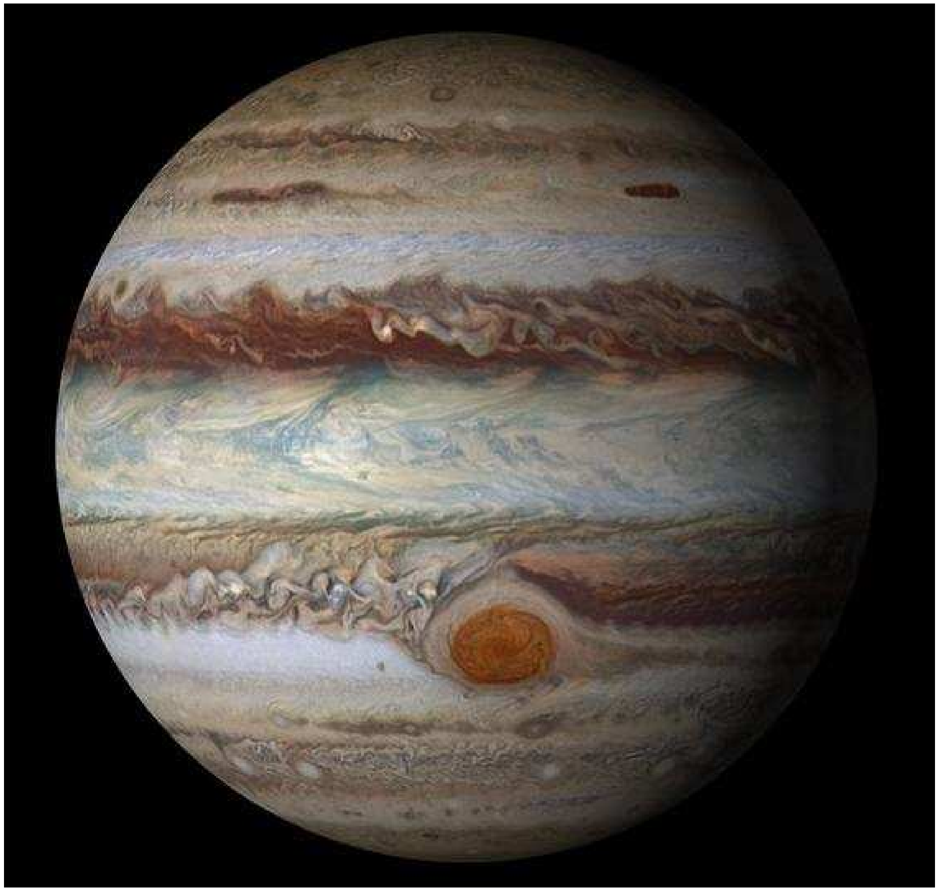


Chlorophyll concentration on the sea surface

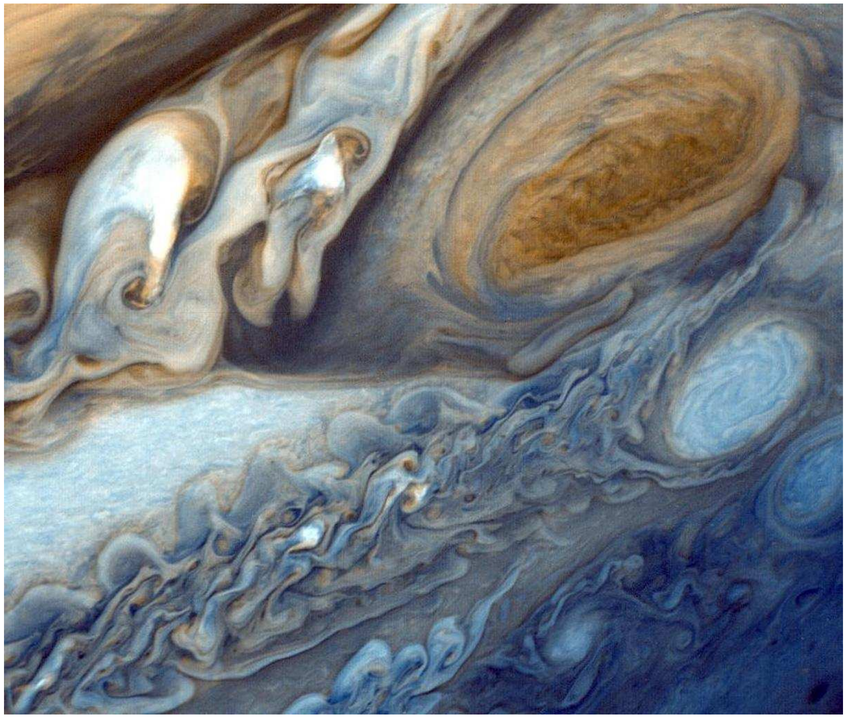
• *Biogeochemical modelling* links GFD with population biology and involves solving for concentrations of hundreds of mutually interacting species feeding on light, nutrients and each other.

• GFD applies to *atmospheres of other planets*.

Circulation of the Jupiter's weather layer:

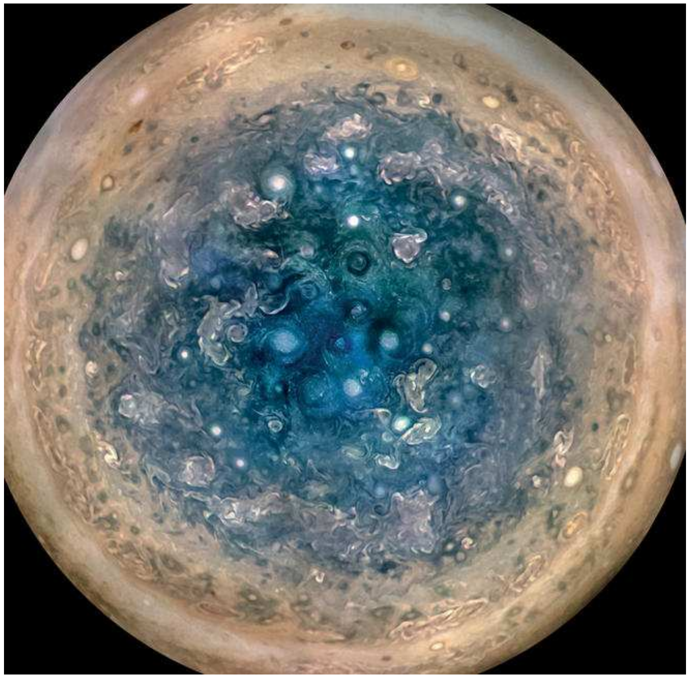


Images of Jupiter from the Cassini and Voyager missions



• Weather layer of Jupiter is characterized by multiple, alternating zonal jets, long-lived coherent vortices (e.g., Great Red Spot), waves and turbulence.

• Towards the poles jovian turbulence changes its character, as the jets fade out and give way to vortex crystals.



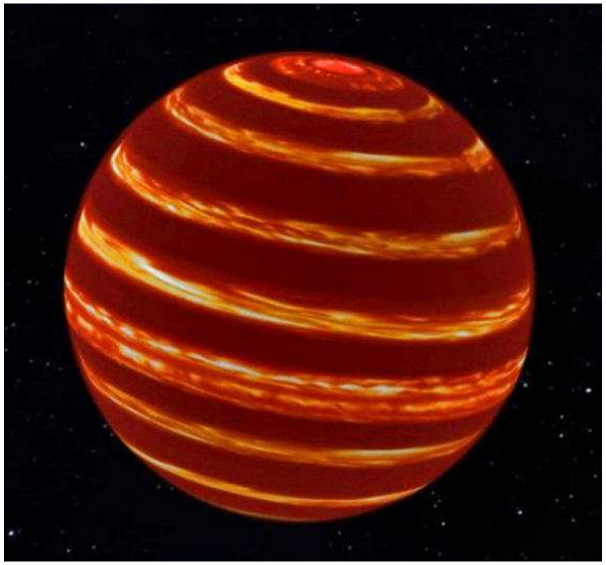
• Many physical processes shape this circulation up: thermal convection, flow instabilities, energy cascades, planetary surf zones, transport barriers, etc.

Similar jets exist on other planets, including the Earth... And not only on the planets!

Convection clouds on Jupiter (science fiction art by Andrew Stewart):

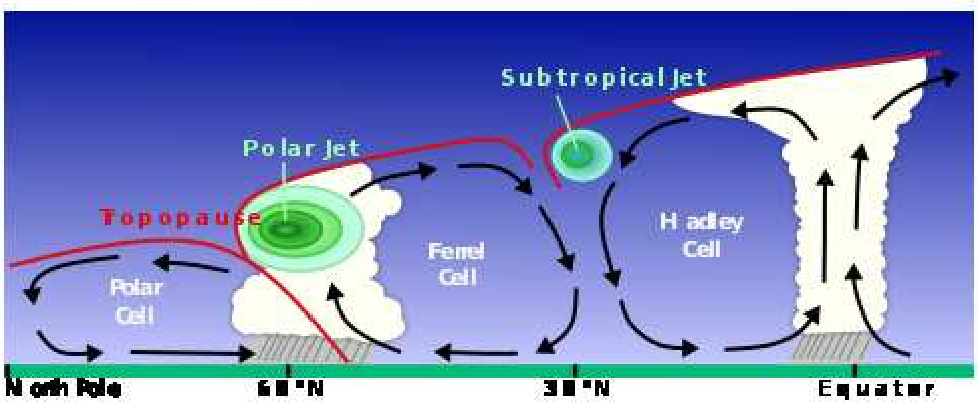
• Some theories argue that alternating jets on giant gas planets are driven by deep convective plumes that feed upscale cascade of energy.

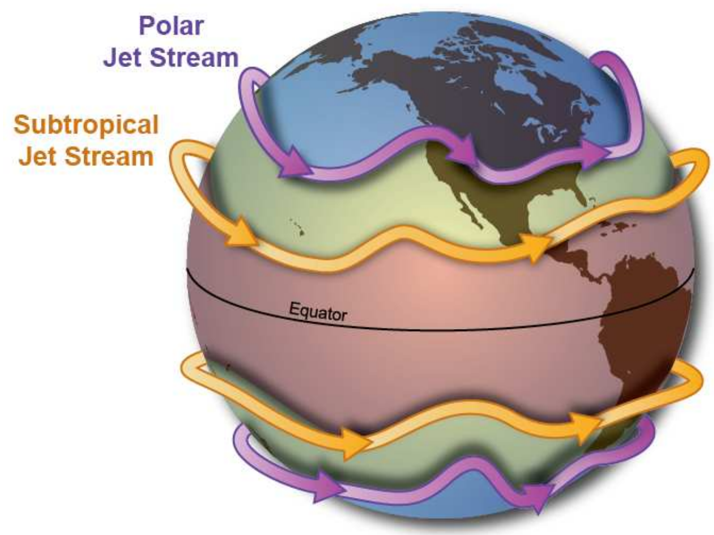
What are the other planets where alternating zonal jets also exist?



• *Brown dwarfs* are substellar objects about Jupiter size but 50 times denser

• *Earth's atmosphere has only a few jets*, for good physical reasons.

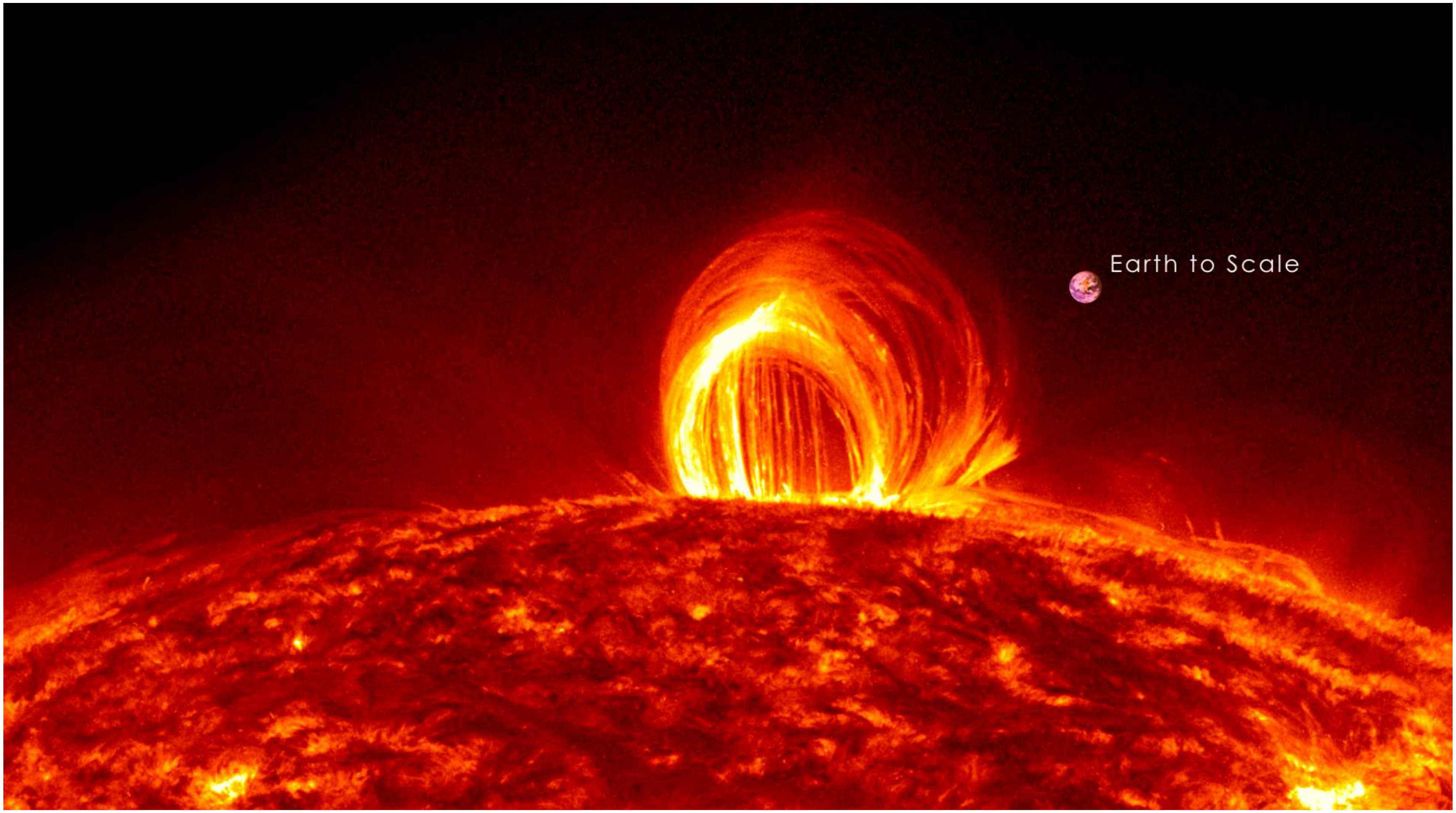




• *Earth's ocean has hundreds of (recently discovered) weak jets.*

• *MagnetoHydroDynamics (MHD)* naturally extends the realm of GFD to modelling the Sun and other stars.

Beautiful example of coronal plasma rain on the Sun:



• GFD also deals with *space weather* and *violent winds*.

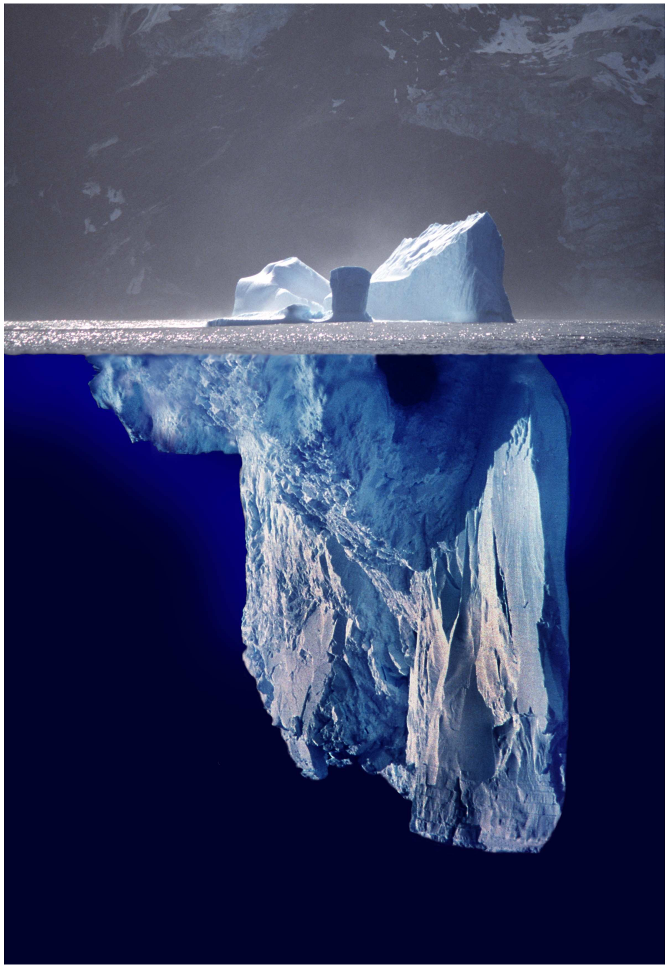
Spectacular aurora (borealis) during polar night:

Powerful tornado emerges from a funnel cloud:

• GFD also deals with *atmospheric electricity* and *motion of floating objects.*

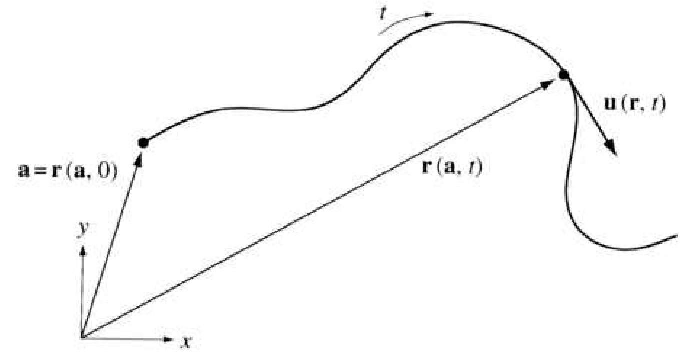
Multiple lightnings strike in a tropical thunderstorm:

Drifting iceberg near Antarctic:



And there are many other geophysical phenomena in the need of science explorers!

- **Representation of fluid flows**



Let's consider a flow consisting of *infinitesimal fluid particles*.
Each particle is characterized by its position $\mathbf{r}$ and velocity $\mathbf{u}$ vectors, which are connected by the *kinematic equation*:

$$\boxed{\frac{d\mathbf{r}(t)}{dt} = \frac{\partial \mathbf{r}(\mathbf{a},t)}{\partial t} = \mathbf{u}(\mathbf{r},t)\,, \qquad \mathbf{r}(\mathbf{a},0) = \mathbf{a}}$$

- *Trajectory* (pathline) of an individual fluid particle is "recording" of the path of this particle over some time interval. Instantaneous direction of the trajectory is determined by the corresponding instantaneous streamline.

- *Streamlines* are a family of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u} = (u, v, w)$. Streamline shows the direction a fluid element will travel in at any point in time.

A parametric representation of just one streamline (here s is coordinate along the streamline) at some moment in time is $\mathbf{X}_s(x_s, y_s, z_s)$:

$$\frac{d\mathbf{X}_s}{ds} \times \mathbf{u}(x_s, y_s, z_s) = 0 \qquad \Longrightarrow \qquad \mathbf{i}\left(w\frac{\partial y_s}{\partial s} - v\frac{\partial z_s}{\partial s}\right) - \mathbf{j}\left(w\frac{\partial x_s}{\partial s} - u\frac{\partial z_s}{\partial s}\right) + \mathbf{k}\left(v\frac{\partial x_s}{\partial s} - u\frac{\partial y_s}{\partial s}\right) = 0$$

$$\Longrightarrow \qquad \frac{dx_s}{u} = \frac{dy_s}{v} = \frac{dz_s}{w}$$

For *2D and non-divergent* flows the *velocity streamfunction* can be used to plot streamlines:

$$\mathbf{u} = -\nabla \times \psi\,, \qquad \psi = (0, 0, \psi)\,, \quad \mathbf{u} = (u, v, 0) \qquad \Longrightarrow \qquad u = -\frac{\partial \psi}{\partial y}\,, \qquad v = \frac{\partial \psi}{\partial x}$$

Note, that $\mathbf{u}\cdot\nabla\psi = 0$, hence, velocity vector $\mathbf{u}$ always points along the isolines of $\psi(x, y)$, implying that these isolines are indeed the streamlines.

- *Streakline* is the collection of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

Note: if flow is *stationary*, that is $\partial/\partial t \equiv 0$, then streamlines, streaklines and trajectories coincide.

- *Timeline (material line)* is the line formed by a set of fluid particles that were marked at the same time, creating a line or a curve that is displaced in time as the particles move.

- **Lagrangian framework**: Point of view such that fluid is described *by following fluid particles*. Interpolation problem; not optimal use of information, because evolving particles will always nonuniformly cover the fluid area.

- **Eulerian framework**: Point of view such that fluid is described *at fixed positions in space*. Nonlinearity problem.

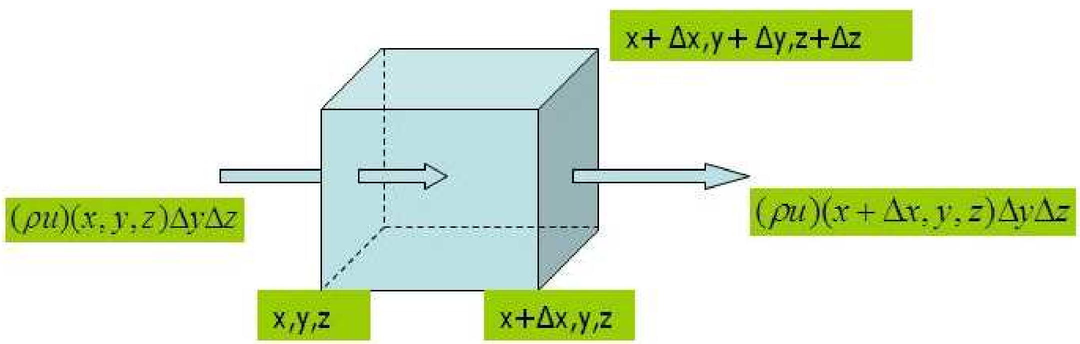
GOVERNING EQUATIONS

• *Complexity*: These equations are sufficient for finding a solution but are too complicated to solve; they are useful only as a starting point for GFD analysis.

• *Art of modelling*: Typically the governing equations are *approximated* analytically and, then, *solved approximately* (by analytical or numerical methods); one should always keep track of all main assumptions and approximations.

Continuity equation (conservation of mass)

Let us take the Eulerian view and consider a fixed infinitesimal cubic volume of fluid and flow of mass through its surface: the mass budget must state conservation of mass.



- Change in mass in x-direction: $(\rho u)(x,y,z)\Delta y\Delta z-(\rho u)(x+\Delta x,y,z)\Delta y\Delta z=-\dfrac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z$
- Total change in mass (continuity of mass): $\boxed{\dfrac{\partial\rho}{\partial t}=-\dfrac{\partial(\rho u)}{\partial x}-\dfrac{\partial(\rho v)}{\partial y}-\dfrac{\partial(\rho w)}{\partial z}=-\nabla\cdot(\rho\mathbf{v})}$

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0} \qquad or \qquad \frac{D\rho}{Dt}=-\rho\nabla\cdot\mathbf{u}\,; \qquad \boxed{\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla} \longleftarrow \textbf{material derivative}$$

Note: if fluid is incompressible (i.e., $\rho=const$), then the continuity equation is reduced to

$$\boxed{\nabla\cdot\mathbf{u}=0},$$

which is its *incompressible form*.

Material derivative is one of the most important concepts in fluid mechanics. When operating on X, it gives the rate of change of X with time *following the fluid element* and subject to a space-time dependent velocity field.

Material derivative is the fundamental link between the Eulerian $[\partial/\partial t + \mathbf{u}\cdot\nabla]$ and Lagrangian $[D/Dt]$ descriptions of changes in the fluid.

The way to see that the material derivative describes the rate of change of any property $F(t,x,y,z)$ following a fluid particle is by applying (i) the chain rule of differentiation and (ii) definition of velocity as the rate of change of particle position:

$$\frac{DF(t,x,y,z)}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial F}{\partial t} + u\,\frac{\partial F}{\partial x} + v\,\frac{\partial F}{\partial y} + w\,\frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F$$

• *Tendency term* $\partial X/\partial t$ represents the rate of change of X at a point which is fixed in space (and occupied by different fluid particles at different times). Changes of X are observed by a stand-still observer.

• *Advection term* $\mathbf{u}\cdot\nabla X$ represents changes of X due to movement with velocity $\mathbf{u}$, which is the flow supply of X to the fixed reference point. Additional advective changes of X are experienced by an observer swimming with velocity $\mathbf{u}$, even when the field of X is steady.

Material tracer equation

For any material (e.g., chemicals, aerosols, gases) *tracer concentration* τ (amount per unit mass), via similar to the continuity equation budgeting, the governing evolution equation for composition is:

$$\boxed{\frac{\partial(\rho\tau)}{\partial t} + \nabla\cdot(\rho\tau\mathbf{u}) = \rho\, S^{(\tau)}},$$

where $S^{(\tau)}$ stands for all non-conservative sources and sinks of τ (boundary sources, molecular diffusion, reaction rate, etc.). Turbulent tracer diffusion is generally added to $S^{(\tau)}$ and represented by $\nabla\cdot(\kappa\,\nabla\tau)$, where κ is diffusivity (tensor) coefficient.

Momentum equation

Consider the Newton's Second Law in a fixed frame of reference, for an infinitesimal cubic volume of fluid δV, and for some force $\mathbf{F}$ acting on the unit volume:

$$\frac{D}{Dt}(\rho \mathbf{u} \delta V) = \mathbf{F}\,\delta V \quad \Longrightarrow \quad \mathbf{u}\,\frac{D}{Dt}(\rho\delta V) + \rho\delta V\,\frac{D}{Dt}\mathbf{u} = \mathbf{F}\,\delta V \quad \Longrightarrow \quad \boxed{\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho}\mathbf{F}},$$

where the first term of the second equation is zero, because mass of the fluid element remains constant (i.e., we do not consider relativistic effects). Let us now consider different forces.

• *Pressure force* arises thermodynamically (due to internal motion of molecules) from the pressure $p(x, y, z)$ that acts perpendicularly on 6 faces of the infinitesimal cubic volume δV. Hence, the pressure force component in x is

$$F_x\,\delta V = [p(x, y, z) - p(x + \delta x, y, z)]\,\delta y\,\delta z = -\frac{\partial p}{\partial x}\,\delta V \quad \Longrightarrow \quad F_x = -\frac{\partial p}{\partial x} \quad \Longrightarrow \quad \boxed{\mathbf{F} = -\nabla p}$$

• *Frictional force* (due to internal motion of molecules and tangential stresses acting on 6 faces of the infinitesimal cubic volume) is typically approximated as $\nu\nabla^2\mathbf{u}$, where ν is the *kinematic viscosity*.

• *Body force* $\mathbf{F}_b$ is typically represented by *gravity* (e.g., downward $F_b = -g$) and electromagnetic (e.g., on the Sun) forces.

• *Coriolis force* is one of *pseudo-forces* that appear only in rotating (i.e., non-inertial!) frames of reference, which are characterized by the rotation rate given by the angular velocity vector $\mathbf{\Omega}$:

$$\boxed{\mathbf{F}_c = -2\,\mathbf{\Omega}\times\mathbf{u}}$$

(a) It acts to deflect a fluid particle at right angle to its motion; note, that only moving particles are affected.

(b) It doesn't do work on a particle, because it is perpendicular to the particle velocity.

(c) Think about motion of tossed ball on a rotating carousel, or about Foucault pendulum. Watch some YouTube movies about the Coriolis force.

(d) Physics of the Coriolis force: particle on a rotating sphere is deflected because of the conservation of angular momentum. When moving to smaller/larger latitudinal circle, the particle should be accelerated/decelerated in the latitudinal direction to conserve its angular momentum.

(e) Because of the deflecting force, moving particles will go around inertial circles that become smaller towards the planetary poles.

(f) Coriolis force is zero on the equator and acts in the opposite directions in the planetary hemispheres.

Let's derive all the pseudo-forces in rotating coordinate systems. Rates of change of general vector $\mathbf{B}$ in the inertial (fixed) and rotating (with $\mathbf{\Omega}$) frames of reference (indicated by i and r, respectively) are related as:

$$\left[\frac{d\mathbf{B}}{dt}\right]_i = \left[\frac{d\mathbf{B}}{dt}\right]_r + \mathbf{\Omega}\times\mathbf{B}$$

Apply this relationship to $\mathbf{r}$ and $\mathbf{u}_r$ and obtain

$$\left[\frac{d\mathbf{r}}{dt}\right]_i \equiv \mathbf{u}_i = \mathbf{u}_r + \mathbf{\Omega}\times\mathbf{r}\,, \qquad (*)$$

$$\left[\frac{d\mathbf{u}_r}{dt}\right]_i = \left[\frac{d\mathbf{u}_r}{dt}\right]_r + \mathbf{\Omega}\times\mathbf{u}_r\,. \qquad (**)$$

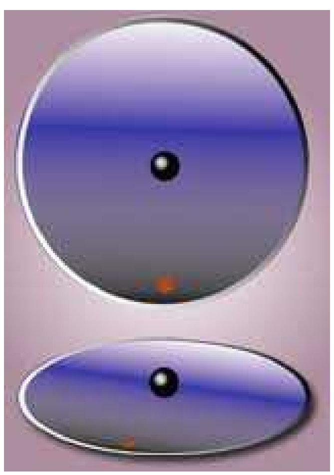
However, we need acceleration of $\mathbf{u}_i$ in the inertial frame and expressed completely in terms of $\mathbf{u}_r$ and in the rotating frame.
Let's (a) differentiate $(*)$ with respect to time, and in the inertial frame of reference; and (b) substitute $[d\mathbf{u}_r/dt]_i$ from $(**)$:

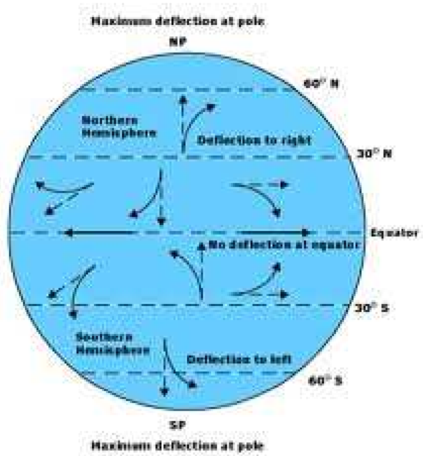
$$\left[\frac{d\mathbf{u}_i}{dt}\right]_i = \left[\frac{d\mathbf{u}_r}{dt}\right]_r + \mathbf{\Omega}\times\mathbf{u}_r + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{r} + \mathbf{\Omega}\times\left[\frac{d\mathbf{r}}{dt}\right]_i$$

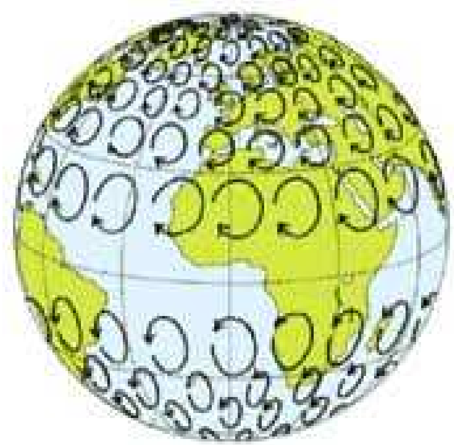
Now, we again substitute $[d\mathbf{r}/dt]_i$ from $(*)$:

$$\frac{d\mathbf{\Omega}}{dt} = 0 \qquad \Longrightarrow \qquad \boxed{\left[\frac{d\mathbf{u}_i}{dt}\right]_i = \left[\frac{d\mathbf{u}_r}{dt}\right]_r + 2\,\mathbf{\Omega}\times\mathbf{u}_r + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})}$$

The term disappearing due to the constant rate of rotation is the (minus) *Euler force*.
The last term is the (minus) *centrifugal force*, which acts both on moving and standing particles. It acts a bit like gravity but in the opposite direction, hence, it can be incorporated in the gravity force field and "be forgotten".







To summarize, the (vector) momentum equation is:

$$\boxed{\frac{D\mathbf{u}}{Dt} + 2\,\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\,\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}_b}$$

Note, that in GFD the Coriolis force is traditionally kept on the lhs of the momentum equation, to remind that it is a pseudo-force.

Equation of state $\boxed{\rho = \rho(p, T, \tau_n)}$ relates pressure p to the *state variables* — density ρ, temperature T, and chemical tracer concentrations τ_n, where $n = 1, 2, ...$ is the tracer index.
All the state variables are related to matter; therefore, the equation of state is a *constitutive equation*.

(a) Equations of state are often phenomenological and very different for different geophysical fluids (note, that the other equations are universal).

(b) The most important τ_n are *humidity* (i.e., water vapor concentration) in the atmosphere and *salinity* (i.e., concentration of diluted salt mix) in the ocean.

(c) Equation of state brings in *temperature*, which has to be determined *thermodynamically* [not part of these lectures!] from *internal energy* (i.e., energy needed to create the system), *entropy* (thermal energy not available for work), and *chemical potentials* corresponding to τ_n (energy that can be available from changes of τ_n).

(d) Example of equation of state (for sea water) involves empirically fitted coefficients of *thermal expansion* α, *saline contraction* β, and *compressibility* γ, which are all empirically determined functions of the state variables:

$$\frac{d\rho}{\rho} = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{S,p} dT + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{T,p} dS + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{T,S} dp = -\alpha\, dT + \beta\, dS + \gamma\, dp$$

Thermodynamic equation is just one more way of writing the *first law of thermodynamics*, which is an expression of the conservation of total energy. (Recall that the second law is about "arrow of time": direction of processes in isolated systems is such that the entropy only increases; in simple words, the heat doesn't go from hot to cold objects.)
The thermodynamic equation can be written for T (i.e., $DT/Dt = ...$), but in the GFD it is more convenient to write it for ρ :

$$\boxed{\frac{D\rho}{Dt} - \frac{1}{{c_s}^2}\,\frac{Dp}{Dt} = Q(\rho)},$$

where c_s is *speed of sound*, and $Q(\rho)$ is *source term* (both concepts have complicated expressions in terms of the state variables).

To summarize, we obtained (assuming one material tracer) the following **COMPLETE SET OF GOVERNING EQUATIONS**:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0 && (1)\\
\frac{D\mathbf{u}}{Dt} + 2\,\mathbf{\Omega}\times\mathbf{u} &= -\frac{1}{\rho}\,\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}_b && (2)\\
\rho &= \rho(p, T, \tau) && (3)\\
\frac{\partial(\rho\tau)}{\partial t} + \nabla\cdot(\rho\tau\mathbf{u}) &= \rho\, S^{(\tau)} && (4)\\
\frac{D\rho}{Dt} - \frac{1}{{c_s}^2}\,\frac{Dp}{Dt} &= Q(\rho) && (5)
\end{aligned}
$$

(a) Momentum equation is for the flow velocity vector, hence, it can be written as 3 equations for the (scalar) velocity components.

(b) We ended up with 7 equations and 7 unknowns (for single tracer concentration): $u,\ v,\ w,\ p,\ \rho,\ T,\ \tau.$

(c) These equations (or their approximations) are to be solved subject to some *boundary and initial conditions*.

(d) These equations are too difficult to solve not only analytically but even numerically.

(e) One remaining step that makes these equations even more difficult, is to rewrite them in the spherical coordinates which are natural for planetary fluid motions on.

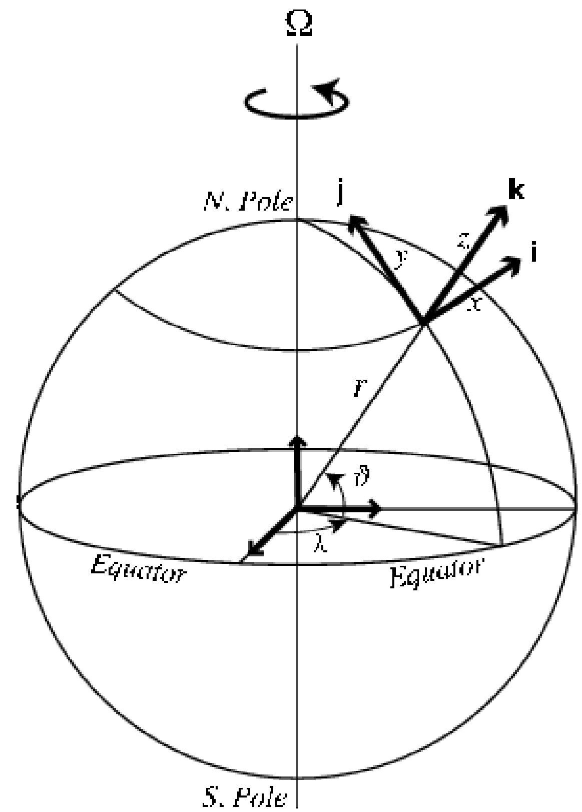
• **Spherical coordinates** are natural for GFD: longitude λ, latitude θ and altitude r.

Material derivative for a scalar quantity ϕ in spherical coordinates is:

$$\frac{D}{Dt} = \frac{\partial\phi}{\partial t} + \frac{u}{r\,\cos\theta}\frac{\partial\phi}{\partial\lambda} + \frac{v}{r}\frac{\partial\phi}{\partial\theta} + w\frac{\partial\phi}{\partial r}\,,$$

where the flow velocity in terms of the corresponding unit vectors is:

$$\mathbf{u} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w\,, \qquad (u,v,w) \equiv \left(r\cos\theta\,\frac{D\lambda}{Dt},\; r\,\frac{D\theta}{Dt},\; \frac{Dr}{Dt}\right)$$



Vector analysis provides differential operators in spherical coordinates acting on a field given by either scalar ϕ or vector $\mathbf{B} = \mathbf{i}\,B^\lambda + \mathbf{j}\,B^\theta + \mathbf{k}\,B^r$:

$$\nabla\cdot\mathbf{B} = \frac{1}{\cos\theta}\left[\frac{1}{r}\frac{\partial B^\lambda}{\partial\lambda} + \frac{1}{r}\frac{\partial(B^\theta\cos\theta)}{\partial\theta} + \frac{\cos\theta}{r^2}\frac{\partial(r^2B^r)}{\partial r}\right],$$

$$\nabla\phi = \mathbf{i}\,\frac{1}{r\cos\theta}\frac{\partial\phi}{\partial\lambda} + \mathbf{j}\,\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{k}\,\frac{\partial\phi}{\partial r}\,,$$

$$\nabla^2\phi \equiv \nabla\cdot\nabla\phi = \frac{1}{r^2\cos\theta}\left[\frac{1}{\cos\theta}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{\partial}{\partial\theta}\left(\cos\theta\,\frac{\partial\phi}{\partial\theta}\right) + \cos\theta\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right)\right],$$

$$\nabla\times\mathbf{B} = \frac{1}{r^2\cos\theta}\begin{vmatrix} \mathbf{i}\,r\cos\theta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\theta & \partial/\partial r \\ B^\lambda r\cos\theta & B^\theta r & B^r \end{vmatrix}\,,$$

$$\nabla^2\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B})\,.$$

(a) Writing down material derivative in spherical coordinates is a bit problematic, because directions of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ change when fluid element changes its location; therefore, *material derivatives of the unit vectors are not zeros*. Note, that this doesn't happen in Cartesian coordinates.

(b) Note that θ can be chosen to be *polar* rather than latitudinal angle; then, coefficients in some of the above formulas will change.

(c) GFD also uses terrain-following *sigma coordinates* or space-time varying *Lagrangian coordinates*.

• **Material derivative in spherical coordinates**:

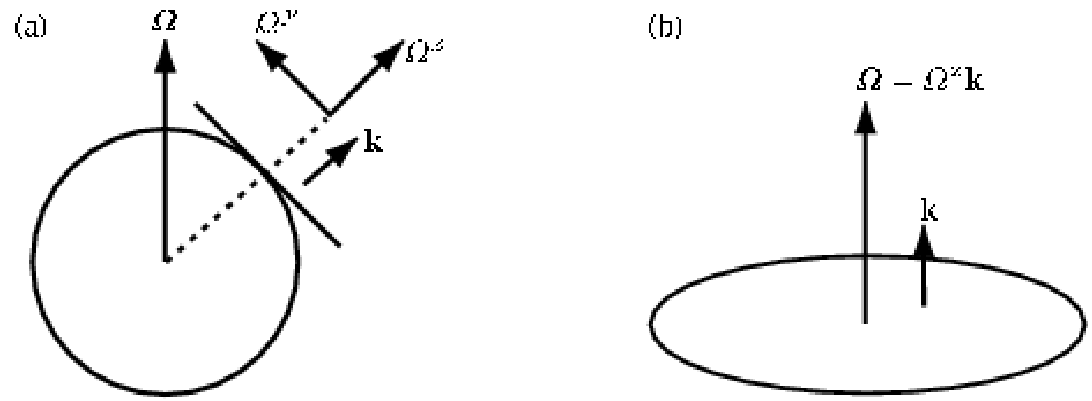
$$\frac{D\mathbf{u}}{Dt} = \frac{Du}{Dt}\mathbf{i} + \frac{Dv}{Dt}\mathbf{j} + \frac{Dw}{Dt}\mathbf{k} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} = \frac{Du}{Dt}\mathbf{i} + \frac{Dv}{Dt}\mathbf{j} + \frac{Dw}{Dt}\mathbf{k} \; + \boldsymbol{\Omega}_{flow} \times \mathbf{u}\,, \qquad (*)$$

where $\boldsymbol{\Omega}_{flow}$ is angular velocity (relative to the centre of Earth) of the unit vector corresponding to the moving element of the fluid flow:

$$\frac{D\mathbf{i}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{i}\,,$$

$$\frac{D\mathbf{j}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{j}\,,$$

$$\frac{D\mathbf{k}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{k}\,.$$



Let's find $\boldsymbol{\Omega}_{flow}$ by moving fluid particle in the direction of each unit vector and observing whether this motion generates any rotation. It is easy to see that motion in the direction of $\mathbf{i}$ makes $\boldsymbol{\Omega}_{||}$, motion in the direction of $\mathbf{j}$ makes $\boldsymbol{\Omega}_{\perp}$, and motion in the direction of $\mathbf{k}$ produces no rotation. Note (see left Figure), that $\boldsymbol{\Omega}_{||}$ is a rotation around the Earth's rotation axis, and it can be written as: $\boldsymbol{\Omega}_{||} = \Omega_{||}\,(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta)$. This rotation rate comes only from a zonally (i.e., along latitude) moving fluid element, and it can be estimated as the following:

$$u\delta t = r\cos\theta\delta\lambda \quad \rightarrow \quad \Omega_{||} \equiv \frac{\delta\lambda}{\delta t} = \frac{u}{r\cos\theta} \quad \Longrightarrow \quad \boldsymbol{\Omega}_{||} = \frac{u}{r\cos\theta}\,(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta) = \mathbf{j}\,\frac{u}{r} + \mathbf{k}\,\frac{u\tan\theta}{r}\,.$$

Note: the rotation rate vector in the perpendicular to $\boldsymbol{\Omega}$ direction is aligned with $\mathbf{i}$ and given by

$$\boldsymbol{\Omega}_{\perp} = -\mathbf{i}\,\frac{v}{r} \quad \Longrightarrow \quad \boldsymbol{\Omega}_{flow} = \boldsymbol{\Omega}_{\perp} + \boldsymbol{\Omega}_{||} = -\mathbf{i}\,\frac{v}{r} + \mathbf{j}\,\frac{u}{r} + \mathbf{k}\,\frac{u\tan\theta}{r} \quad \Longrightarrow$$

$$\frac{D\mathbf{i}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{i} = \frac{u}{r\cos\theta}\,(\mathbf{j}\sin\theta - \mathbf{k}\cos\theta)\,, \qquad \frac{D\mathbf{j}}{Dt} = -\mathbf{i}\,\frac{u}{r}\,\tan\theta - \mathbf{k}\,\frac{v}{r}\,, \qquad \frac{D\mathbf{k}}{Dt} = \mathbf{i}\,\frac{u}{r} + \mathbf{j}\,\frac{v}{r}$$

$$(*) \quad \Longrightarrow \quad \boxed{\frac{D\mathbf{u}}{Dt} = \mathbf{i}\Big(\frac{Du}{Dt} - \frac{uv\tan\theta}{r} + \frac{uw}{r}\Big) + \mathbf{j}\Big(\frac{Dv}{Dt} - \frac{u^2\tan\theta}{r} + \frac{vw}{r}\Big) + \mathbf{k}\Big(\frac{Dw}{Dt} - \frac{u^2+v^2}{r}\Big)}$$

The additional quadratic (in terms of velocity components) terms are called *metric terms*.

• *Coriolis force in the spherical coordinates* also needs to be written in terms of the unit vectors. The planetary angular velocity vector is always orthogonal to the unit vector $\mathbf{i}$ (see Figure):

$$\mathbf{\Omega} = (0,\ \Omega^y,\ \Omega^z) = (0,\ \Omega\cos\theta,\ \Omega\sin\theta)$$

However, the Coriolis force projects on all the unit vectors:

$$2\mathbf{\Omega}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2\Omega\cos\theta & 2\Omega\sin\theta \\ u & v & w \end{vmatrix} = \mathbf{i}\,(2\Omega w\cos\theta - 2\Omega v\sin\theta) + \mathbf{j}\,2\Omega u\sin\theta - \mathbf{k}\,2\Omega u\cos\theta\,.$$

By combining the metric and Coriolis terms, we obtain the spherical-coordinates governing equations (other equations are treated similarly):

$$\begin{aligned}
\frac{Du}{Dt} - \Big(2\,\Omega + \frac{u}{r\cos\theta}\Big)\,(v\sin\theta - w\cos\theta) &= -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial\lambda}\,,\\
\frac{Dv}{Dt} + \frac{wv}{r} + \Big(2\,\Omega + \frac{u}{r\cos\theta}\Big)\,u\sin\theta &= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}\,,\\
\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\,\Omega\,u\cos\theta &= -\frac{1}{\rho}\frac{\partial p}{\partial r} - g\,,\\
\frac{\partial\rho}{\partial t} + \frac{1}{r\cos\theta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{r\cos\theta}\frac{\partial(v\rho\cos\theta)}{\partial\theta} + \frac{1}{r^2}\frac{\partial(r^2 w\rho)}{\partial r} &= 0\,.
\end{aligned}$$

Metric terms are relatively small on the surface of a large planet $(r \to R_0)$ and, therefore, can be neglected for many process studies; Note, that the gravity acceleration $-g$ was included; viscous term can be also trivially added.

Local Cartesian approximation

Both for mathematical simplicity and process studies, the governing equations can be written locally for a *plane tangent to the planetary surface*. Then, the momentum equations become:

$$\frac{Du}{Dt} + 2\,(\Omega\cos\theta w - \Omega\sin\theta v) = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad \frac{Dv}{Dt} + 2\,(\Omega\sin\theta u) = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \qquad \frac{Dw}{Dt} + 2\,(-\Omega\cos\theta u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,,$$

and they can be simplified by neglecting some components of the Coriolis force:

(a) Neglect Coriolis force in the vertical momentum equation, because its effect (upward/downward deflection of fluid particles, also known as *Eotvos effect*, which can be also interpreted as change of weight of zonally moving fluid element), is small.

(b) Neglect vertical velocity in the zonal momentum equation, because the corresponding component of the Coriolis force is small relative to the other one (vertical velocity components are often small relative to the horizontal ones).

Next, we introduce the *Coriolis parameter*, which is a nonlinear function of latitude: $f \equiv 2\Omega^z = 2\Omega \sin\theta$. The following approximations are often made in GFD:

(a) *f-plane approximation*: $f = f_0$ (constant).

(b) Planetary sphericity is accounted for by *β-plane approximation*: $f(y) = f_0 + \beta y$, where β is *gradient of planetary vorticity*.

With the above inputs, the resulting *local Cartesian equations* are:

$$\boxed{\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \qquad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,, \qquad \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0}$$

These equations are to be combined with the other equations (thermodynamic, material tracer, constitutive) also written in the local Cartesian coordinates. Even this system of equations is too difficult to solve. In order to simplify it further, we have to focus on specific classes of fluid motions. Our main focus will be on *stratified incompressible flows*.

• **Stratification**. Let's think about density fields in terms of their decomposition into (a) time-dependent *dynamic anomalies* (primed) due to fluid motion and (b) background *static fields*:

$$\rho(t,x,y,z) = \rho_0 + \overline{\rho}(z) + \rho'(t,x,y,z) = \rho_s(z) + \rho'(t,x,y,z)$$

Later on static density will be represented in terms of stacked isopycnal (i.e., constant-density) and fluid layers, and dynamic density anomalies will be described by vertical deformations of these layers.

Pressure field can be also treated in terms of static and dynamic components:

$$p(t,x,y,z) = p_s(z) + p'(t,x,y,z)\,.$$

We will use symbols $[\delta\rho']$ and $[\delta p']$ to describe the corresponding dynamic scales.

With this concept of fluid stratification, we are ready to make one more important approximation (below) that will affect both thermodynamic and vertical momentum equations.

Boussinesq approximation

Boussinesq is used routinely for oceans and sometimes for atmospheres, and it invokes the following assumptions:

(1) Fluid *incompressibility*: $c_s = \infty$,

(2) *Small variations of static density*: $\overline{\rho}(z) \ll \rho_0 \quad \Longrightarrow \quad$ only $\overline{\rho}(z)$ is neglected but *not* its vertical derivative.

(3) *Anelastic approximation* (used for atmospheres) is when $\overline{\rho}(z)$ is *not neglected*.

Boussinesq approximation affects thermodynamic equation and vertical momentum equation.

• *Thermodynamic Boussinesq equation* $(D\rho/Dt = Q_\rho)$ is written for *dynamic buoyancy anomaly* b and *static buoyancy* $\overline{b}$:

$$\frac{D(\overline{b}+b)}{Dt} = Q_b\,, \qquad b(t,x,y,z) \equiv -g\frac{\rho'}{\rho_0} \qquad \overline{b}(z) \equiv -g\frac{\overline{\rho}}{\rho_0} \qquad (*)$$

where Q_b is source term proportional to $Q(\rho)$. Equation $(*)$ is often written as

$$\boxed{\frac{Db}{Dt} + N^2(z)\,w = Q_b}\,, \qquad N^2(z) \equiv \frac{d\overline{b}}{dz} \qquad (**)$$

Buoyancy frequency N measures strength of the static (background) stratification in terms of its vertical derivative, in accord with assumption (2).

NOTE: *Primitive equations* are often used in practice as approximation to $(**)$, which in the realistic general circulation models is replaced by separate material transport equations for thermodynamic variables, and, then, the buoyancy is found diagnostically from the equation of state:

$$\boxed{\frac{DT}{Dt} = Q_T\,, \qquad \frac{DS}{Dt} = Q_S\,, \qquad b = b(T,S,z)}$$

• *Vertical momentum Boussinesq equation* is written for pressure anomaly (without static pressure part):

$$p = p_s + p'\,, \qquad \rho = \rho_s + \rho'\,, \qquad -\frac{\partial p_s}{\partial z} = \rho_s g \quad (static\ \ balance)\,, \qquad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad (momentum)$$

Let's keep the static part for a while and rewrite the last equation in the Boussinesq approximation:

$$\Longrightarrow \quad (\rho_s + \rho')\,\frac{Dw}{Dt} = -\frac{\partial(p_s+p')}{\partial z} - (\rho_s+\rho')\,g \quad \Longrightarrow \quad \rho_0\,\frac{Dw}{Dt} = -\frac{\partial p'}{\partial z} - \rho'\,g \quad \Longrightarrow \quad \boxed{\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b}$$

Note, that in the vertical acceleration term $\rho_s + \rho'$ is replaced by ρ_0, in accord with approximation (2). Horizontal momentum equations are treated similarly.

To summarize, the *Boussinesq system of equations* is (we drop primes, from now on, keeping in mind that p indicates dynamic pressure anomaly):

$$\boxed{\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}\,, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,, \qquad \frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \qquad \frac{Db}{Dt} + N^2 w = Q_b}$$

• **Hydrostatic approximation**. For many fluid flows vertical acceleration is small relative to gravity acceleration, and gravity force is balanced by the vertical component of pressure gradient (we'll revisit this approximation more formally):

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \Longrightarrow \qquad \boxed{\frac{\partial p}{\partial z} = -\rho g}$$

Hydrostatic Boussinesq approximation is commonly used for many GFD phenomena.

• **Buoyancy frequency** $N(z)$ has clear physical meaning. In a continuously stratified fluid consider density difference $\delta\rho$ between a fluid particle adiabatically lifted by δz and surrounding fluid $\rho_s(z)$. Motion of the particle is determined by the buoyancy (Archimedes) force F and the Newton's second law:

$$\delta\rho = \rho_{particle} - \rho_s(z+\delta z) = \rho_s(z) - \rho_s(z+\delta z) = -\frac{\partial \rho_s}{\partial z}\,\delta z \qquad \rightarrow \qquad F = -g\,\delta\rho = g\,\frac{\partial \rho_s}{\partial z}\,\delta z$$

$$\rightarrow \qquad \rho_s\,\frac{\partial^2 \delta z}{\partial t^2} = g\,\frac{\partial \rho_s}{\partial z}\,\delta z \qquad \rightarrow \qquad \delta\ddot{z} + N^2\delta z = 0$$

(a) If $N^2 > 0$, then fluid is *statically stable* (negative N^2 describes unstable stratification prone to convective instability), and the particle will oscillate around its resting position with frequency $N(z)$ (typical periods of oscillations are $10-100$ minutes in the ocean, and about 10 times shorter in the atmosphere).

(b) In the atmosphere, which is significantly non-Boussinesq, one should take into account how density of the lifted particle changes due to the local change of pressure. Then, N^2 is reformulated with *potential density* ρ_θ, rather than density itself.

• **Rotation-dominated flows** are in the focus of GFD. Such flows are slow, in the sense that they have advective time scales longer than the planetary rotation period: $L/U \gg f^{-1}$.
Given typical observed flow speeds in the atmosphere $(U_a \sim 1-10$ m/s$)$ and ocean $(U_o \sim 0.1\,U_a)$, the length scales of rotation-dominated flows are $L_a \gg 100-1000$ km and $L_o \gg 10-100$ km. Motions on these scales constitute most of the weather and strongly influence climate and climate variability.
Rotation-dominated flows tend to be hydrostatic (to be shown later).
Later on, we will use asymptotic analysis to focus on these scales and filter out less important faster and smaller-scale motions.

• **Thin-layered framework** describes fluid in terms of stacked, vertically thin but horizontally vast layers of fluid with slightly different densities (increasing downwards) — this is rather typical situation in GFD.
Let's introduce physical scales: L and H are horizontal and vertical length scales, respectively, such that $L \gg H$; then, U and W are the horizontal and vertical velocity scales, respectively, such that $U \gg W$. From now on, we'll focus mostly on motions with such scales.
Thin-layered flows tend to be hydrostatic (to be shown later).
Later on, we will formulate models that describe fluid in terms of properly scaled, vertically thin but horizontally vast fluid layers.

Summary:
We considered the following sequence of simplified approximations:

Governing Equations (spherical coordinates) $\rightarrow$ *Local Cartesian* $\rightarrow$ *Boussinesq* $\rightarrow$ *Hydrostatic Boussinesq*.

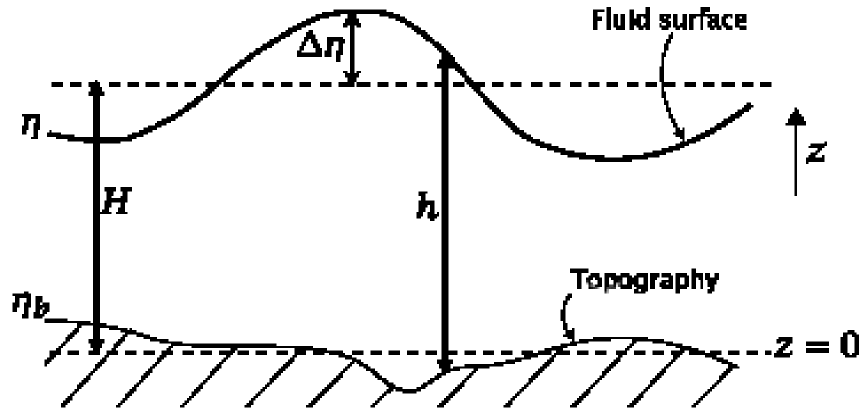
Lost by going *Local Cartesian*: some effects of rotation and sphericity.

Lost by going *Boussinesq*: compressible motions (i.e., acoustics, shocks, bubbles), strong stratifications (e.g., inner Jupiter).

Lost by going *Hydrostatic Boussinesq*: large vertical accelerations (e.g., convection, breaking gravity waves, Kelvin-Helmholtz instability, density currents, double diffusion, tornadoes).

In what follows we consider the simplest relevant thin-layered model, which is locally Cartesian, Boussinesq and hydrostatic, and try to focus on its rotation-dominated flow component...

BALANCED DYNAMICS



Shallow-water model — our starting point — describes motion of a horizontal fluid layer with variable thickness, $h(t, x, y)$. Density is a constant ρ_0 and vertical acceleration is neglected (hydrostatic approximation), hence:

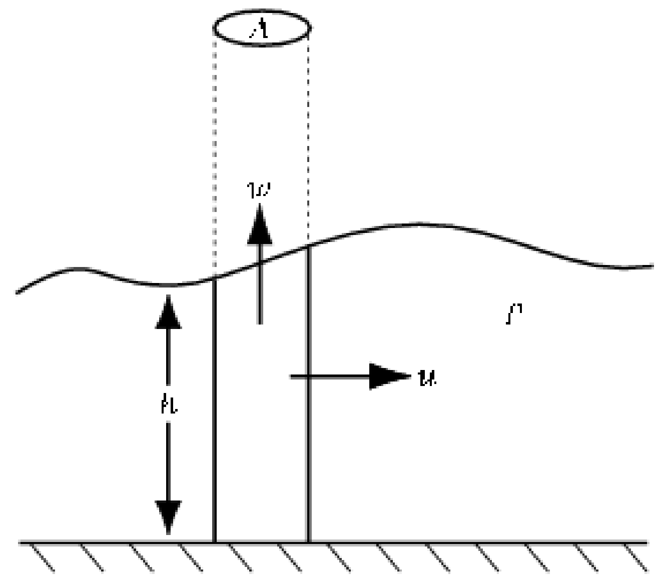
$$\frac{\partial p}{\partial z} = -\rho_0 g \qquad \rightarrow \qquad p(t, x, y, z) = \rho_0 g\,[h(t, x, y) - z]\,,$$

where we took into account that $p = 0$ at $z = h(t, x, y)$.
Note, that horizontal pressure gradient is independent of z; hence, u and v are also independent of z, that is, *fluid moves in columns*.

In local Cartesian coordinates horizontal momentum equations are:

$$\boxed{\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -g\,\frac{\partial h}{\partial x}\,, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} = -g\,\frac{\partial h}{\partial y}}\,,$$

where $\displaystyle \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\,.$



Continuity equation is needed to close the system, so let us derive it from the first principles. Recall that horizontal velocity does not depend on z and consider mass budget of a fluid column.
The *horizontal* mass convergence (see earlier derivation of the continuity equation) into a fixed-radius fluid column is (apply divergence theorem):

$$M = -\int_S \rho_0\,\mathbf{u}\cdot d\mathbf{S} = -\oint \rho_0 h\,\mathbf{u}\cdot\mathbf{n}\,dl = -\int_A \nabla\cdot(\rho_0 h\,\mathbf{u})\,dA\,,$$

and this must be balanced by the local increase of the mass due to increasing height of fluid column:

$$M = \frac{d}{dt}\int \rho_0\,dV = \frac{d}{dt}\int_A \rho_0 h\,dA = \int_A \rho_0\,\frac{\partial h}{\partial t}\,dA \qquad \Longrightarrow \qquad \frac{\partial h}{\partial t} = -\nabla\cdot(h\mathbf{u}) \qquad \Longrightarrow \qquad \boxed{\frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0}$$

(a) Note that the above *shallow-water continuity equation* can be obtained from the original one by transformation $\rho \rightarrow h$, hence, h can be treated as density of compressible fluid.

(b) It can be also obtained by integrating 3D incompressible continuity equation $\nabla\cdot\mathbf{u} + \partial w/\partial z = 0$, which yields vertical velocity component linear in z, and by using kinematic boundary conditions (see later): $w(h) = Dh/Dt$, $w(0) = 0$.

Relative vorticity of 2D flow is defined as:

$$\zeta = \left[\nabla \times \mathbf{u}\right]_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

where $\zeta > 0$ is counterclockwise *cyclonic* motion, and $\zeta < 0$ is clockwise *anticyclonic* motion.
Note that relative vorticity describes rotation of fluid particles, rather than circular motions of fluid that can be irrotational.

• **Vorticity equation** is obtained by taking curl of the momentum (vector) equation (i.e., taking y-derivative of the first equation and subtracting it from the x-derivative of the second equation). Remember to differentiate advection term of the material derivative; note that curl of the pressure gradient term is automatically zero.
The resulting *vorticity equation* is:

$$\frac{D\zeta}{Dt} + \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right](\zeta + f) + v\,\frac{df}{dy} = 0$$

By using velocity divergence from the shallow-water continuity equation we obtain single material conservation equation:

$$\frac{D\zeta}{Dt} - \frac{1}{h}(\zeta + f)\,\frac{Dh}{Dt} + v\,\frac{df}{dy} = 0 \qquad \Longrightarrow \qquad \frac{1}{h}\,\frac{D(\zeta + f)}{Dt} - \frac{1}{h^2}(\zeta + f)\,\frac{Dh}{Dt} = 0 \qquad \Longrightarrow \qquad \frac{D}{Dt}\left[\frac{\zeta + f}{h}\right] = 0\,.$$

Potential vorticity (PV) material conservation law:

$$\boxed{\frac{Dq}{Dt} = 0\,, \qquad q \equiv \frac{\zeta + f}{h}}$$

(a) This is very powerful statement that reduces dynamical description of fluid motion to solving for evolution of materially conserved, *scalar* quantity. Analogy with electric charge and field: PV can be viewed as active tracer that changes its own, induced velocity field.

(b) For each fluid column, conservation of PV constrains and mutually connects changes of ζ, $f(y)$, and h, where changes of the latter can be interpreted as stretching/squeezing of moving fluid columns.

(c) *PV inversion problem*: Under certain conditions (e.g., when flow is rotation-dominated and hydrostatic) flow solution can be determined entirely from evolving PV. For example, when $h = H = \text{const}$ the inversion is trivial.

(d) The above PV conservation law can be derived for many layers and continuous stratification.

(e) More general formulation of PV is referred to as *Ertel PV*:

$$q = -g\,(\zeta + f)\,\partial\theta/\partial p\,,$$

where θ is potential density.

Rossby number is ratio of scalings for material derivative (i.e., horizontal acceleration) and Coriolis forcing:

$$\boxed{\epsilon = \frac{U^2/L}{fU} = \frac{U}{fL}}$$

For rotation-dominated motions: $\boxed{\epsilon \ll 1}$.
More conventional notation for Rossby number is Ro, but we'll use ϵ to emphasize its smallness and apply the ϵ-asymptotic expansion.

Given smallness of ϵ, we can expand the governing equations in terms of the *geostrophic* (leading-order terms) and *ageostrophic* (ϵ-order terms) motions:

$$\mathbf{u} = \mathbf{u}_g + \epsilon\, \mathbf{u}_a + o(\epsilon^2)\,, \qquad p' = p'_g + \epsilon\, p'_a + o(\epsilon^2)\,, \qquad \rho' = \rho'_g + \epsilon\, \rho'_a + o(\epsilon^2)\,.$$

Rossby number expansion
The goal is to be able to predict strong geostrophic motions — this requires taking into account weak ageostrophic motions. Let's consider β-plane, focus on relatively slow *mesoscale motions*, and express velocity scale via ϵ:

$$T = \frac{L}{U} = \frac{L}{\epsilon f_0 L} = \frac{1}{\epsilon f_0}\,, \qquad L/R_0 \sim \epsilon \quad \Longrightarrow \quad [\beta y] \sim \frac{f_0}{R_0} L \sim \epsilon f_0\,.$$

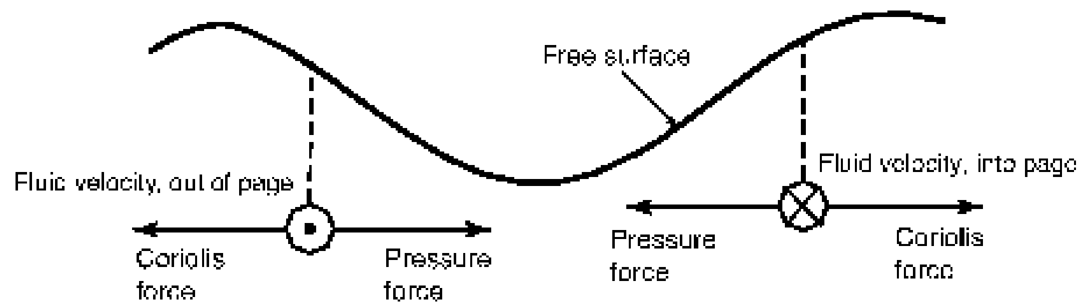
Consider ϵ-expansion of the horizontal momentum equations:

$$\begin{array}{cccccccc}
\dfrac{Du_g}{Dt} & - f_0\,(v_g + \epsilon v_a) & - \beta y\, v_g & + \epsilon^2[...] & = & -\dfrac{1}{\rho_0}\dfrac{\partial p_g}{\partial x} & - \dfrac{\epsilon}{\rho_0}\dfrac{\partial p_a}{\partial x} \\
\dfrac{Dv_g}{Dt} & + f_0\,(u_g + \epsilon u_a) & + \beta y\, u_g & + \epsilon^2[...] & = & -\dfrac{1}{\rho_0}\dfrac{\partial p_g}{\partial y} & - \dfrac{\epsilon}{\rho_0}\dfrac{\partial p_a}{\partial y} \\
\epsilon f_0 U & f_0 U & \epsilon f_0 U & \epsilon^2 f_0 U & & [p']/(\rho_0 L) & \epsilon\,[p']/(\rho_0 L)
\end{array}$$

Note that Coriolis force can be balanced only by pressure gradient term — this is called *geostrophic balance*.

Geostrophic balance is obtained from the horizontal momentum equations at the leading order:

$$\boxed{f_0 v_g = \frac{1}{\rho_0}\frac{\partial p_g}{\partial x}\,, \qquad f_0 u_g = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y}}$$



(a) Proper scaling for pressure must be

$$[p'] \sim \rho_0 f_0 U L\,.$$

(b) Counterintuitive dynamics: Induced local pressure anomaly results in a circular flow around it, rather than in a classical fluid flow response along the pressure gradient.

(c) It follows from the geostrophic balance that $\mathbf{u_g}$ is nondivergent: $\dfrac{\partial u_g}{\partial x} + \dfrac{\partial v_g}{\partial y} = 0$ (see later that $w_g = 0$).

(d) Geostrophic flow is 2D and nondivergent, hence, it can be described by a velocity streamfunction; note that pressure in the geostrophic balance acts as streamfunction in disguise!

(e) Geostrophic balance is diagnostic rather than prognostic equation, hence, it can not be used for predictions of any temporal evolution. Therefore, the next order of the ϵ-expansion is needed to determine the flow evolution (see later).

(f) Geostrophically balanced flows are also hydrostatically balanced (see below).

Hydrostatic balance

Vertical acceleration is typically small for large-scale geophysical motions, because they are thin-layered and rotation-dominated. Let's prove this formally:

$$\frac{Dw}{Dt} = -\frac{1}{\rho_s+\rho_g}\frac{\partial(p_s+p_g)}{\partial z} - g\,, \qquad \frac{Dw}{Dt} \sim 0\,, \qquad \frac{\partial p_s}{\partial z} = -\rho_s g \qquad \Longrightarrow \qquad \boxed{\frac{\partial p_g}{\partial z} = -\rho_g g} \qquad (*)$$

Use the corresponding scalings $W = UH/L$, $T = L/U$, $[p'] = \rho_0 f_0 U L$, $U = \epsilon f_0 L$ to identify the *validity bound* for the leading-order hydrostatic balance:

$$\frac{Dw}{Dt} \ll \frac{1}{\rho_0}\frac{\partial p_g}{\partial z} \qquad \Longrightarrow \qquad \frac{HU^2}{L^2} \ll \frac{\rho_0 f_0 U L}{\rho_0 H} \qquad \Longrightarrow \qquad \boxed{\epsilon\left(\frac{H}{L}\right)^2 \ll 1}$$

If the last inequality is true, then vertical acceleration can be neglected — this situation of *hydrostatic balance* routinely happens for large-scale geophysical flows.

• **Scaling for geostrophic density anomaly**
From the hydrostatic balance for geostrophic flow and the geostrophic scaling for pressure $[p']$, we find scaling for geostrophic dynamic density anomaly ρ_g :

$$[\rho_g] \equiv [\rho'] \sim \frac{[p']}{gH} = \frac{\rho_0 f_0 UL}{gH} = \rho_0\, \epsilon\, \frac{f_0^2 L^2}{gH} = \rho_0\, \epsilon\, F\,, \qquad F \equiv \frac{f_0^2 L^2}{gH} = \Big(\frac{L}{L_d}\Big)^2, \qquad L_d \equiv \frac{\sqrt{gH}}{f_0} \sim O(10^4\ km)\,,$$

where L_d is the *external deformation length scale*, and F is *Froude number* (it can be also written as ratio of characteristic flow velocity to the fastest wave velocity).

For many geophysical scales of interest: $F \ll 1$, therefore, it is safe to assume that

$$F \sim \epsilon \qquad \Longrightarrow \qquad \boxed{[\rho_g] = \rho_0\, \epsilon^2}$$

Thus, ubiquitous and powerful, double-balanced (geostrophic and hydrostatic) motions correspond to *nearly flat isopycnals*.

• **Continuity for ageostrophic flow**
Let's now turn attention to the continuity equation and also ϵ-expand it:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0\,, \qquad \rho = \rho_s + \rho_g, \quad u = u_g + \epsilon\, u_a, \quad v = v_g + \epsilon\, v_a, \quad w = w_g + \epsilon\, w_a \qquad \rightarrow$$

$$\frac{\partial \rho_g}{\partial t} + (\rho_s + \rho_g)\Big(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\Big) + u_g \frac{\partial \rho_g}{\partial x} + v_g \frac{\partial \rho_g}{\partial y} + \epsilon \rho_s \Big(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\Big) + \epsilon^2 [...] + \frac{\partial}{\partial z}(w_g \rho_s + \epsilon w_a \rho_s + w_g \rho_g + \epsilon w_a \rho_g) = 0$$

Use $\dfrac{\partial u_g}{\partial x} + \dfrac{\partial v_g}{\partial y} = 0$ and $\rho_g \sim \epsilon^2$ to obtain at the leading order: $\dfrac{\partial (w_g \rho_s)}{\partial z} = 0 \quad \longrightarrow \quad w_g\, \rho_s = const$

Because of the BCs, somewhere in the water column $w_g(z)$ has to be zero $\Longrightarrow \boxed{w_g = 0}, \quad w = \epsilon\, w_a, \quad [w] = W = \epsilon\, U\, \dfrac{H}{L}$

At the next order of the ϵ-expansion we recover the continuity equation for ageostrophic flow component:

$$\boxed{\frac{\partial (w_a \rho_s)}{\partial z} + \rho_s \Big(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\Big) = 0}\,.$$

Let's keep this in mind and use it in the derivation of geostrophic vorticity equation.

• **Geostrophic (absolute) vorticity equation** is obtained by going to the next order of ϵ in the shallow-water momentum equations:

$$\frac{D_g u_g}{Dt} - (\epsilon f_0 v_a + v_g \beta y) = -\epsilon \, \frac{1}{\rho_s} \frac{\partial p_a}{\partial x} \,, \qquad \frac{D_g v_g}{Dt} + (\epsilon f_0 u_a + u_g \beta y) = -\epsilon \, \frac{1}{\rho_s} \frac{\partial p_a}{\partial y} \,, \qquad \frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \,.$$

(i) Take curl of the above equations (i.e., subtract y-derivative of the first equation from x-derivative of the second equation) and mind complexity of the material derivative;

(ii) Use nondivergence of the geostrophic velocity;

(iii) Use continuity equation for ageostrophic flow to replace horizontal ageostrophic velocity divergence.

Thus, we obtain the *geostrophic vorticity equation*:

$$\boxed{\frac{D_g \zeta_g}{Dt} + \beta v_g = \frac{D_g}{Dt} \left[\zeta_g + \beta y \right] = \epsilon \, \frac{f_0}{\rho_s} \frac{\partial (\rho_s w_a)}{\partial z}} \,, \qquad \zeta_g \equiv \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

(a) This looks almost as PV material conservation law, but unfortunately it is not the one, because of the rhs term. Can the rhs be absorbed under the material derivative, so that PV conservation law is recovered?

(b) Evolution of absolute vorticity $\zeta_g + \beta y$ is determined by *divergence of the vertical mass flux* due to tiny vertical velocity. This is physical process of *squeezing or stretching isopycnals*; it is the *form drag* mechanism (discussed below).

(c) If ρ_s is constant within a layer (i.e., thin-layered framework), then, it cancels out from the rhs, and we are left with the vertical component of velocity divergence.

(d) Note that, although vertical velocity is tiny, its divergence is at the leading order of the absolute vorticity equation. Can this divergence be determined from the leading-order geostrophic fields?

(e) Yes! *Quasigeostrophic theory* expresses this divergence in terms of vertical movement of isopycnals, then, it relates this movement to geostrophic (dynamic) pressure, which turns to be geostrophic streamfunction in disguise.

(f) On the other hand, evolution of absolute vorticity produces squeezing and stretching deformations, which induce motions in the neighbouring isopycnal layers.

• **Form drag** is horizonatal pressure-gradient force due to varying isopycnal-layer thickness, which in turn can arise due to isopycnal squeezing and stretching.

Geostrophic motions are very efficient in terms of redistributing horizontal momentum vertically, through the form drag mechanism.

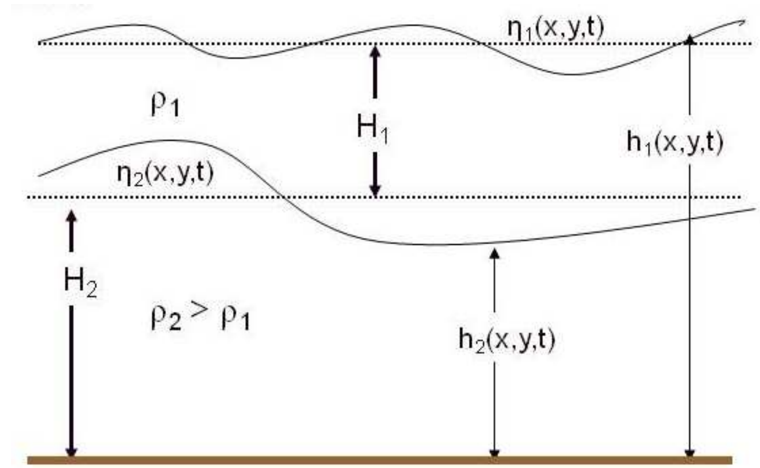
Let's consider a constant-density fluid layer confined by two interfaces, $h_1(x)$ and $h_2(x)$, and periodic in zonal direction with period L ; let's also assume that situation is 2D (homogeneous in meridional direction).
Zonal pressure-gradient force acting on a volume of fluid is obtained by integration over the domain:

$$F_x = -\frac{1}{L}\int_0^L\int_{h_2}^{h_1}\frac{\partial p}{\partial x}\,dx\,dz = -\frac{1}{L}\int_0^L\Big[\frac{\partial p}{\partial x}\,z\Big]_{h_2}^{h_1}dx = -\overline{h_1\,\frac{\partial p_1}{\partial x}} + \overline{h_2\,\frac{\partial p_2}{\partial x}} = \overline{p_1\,\frac{\partial h_1}{\partial x}} - \overline{p_2\,\frac{\partial h_2}{\partial x}}\,,$$

where p_1 and p_2 are pressures on the interfaces; $\partial p/\partial x$ does not depend on vertical position within a layer; and overline denotes zonal averaging (which is zero for x-derivatives due to the periodicity).
Note that force F_x acting on fluid is zero, if both boundaries are flat. This statement can be reversed: if isopycnal boundaries of a fluid layer are deformed (e.g., by squeezing or stretching), the layer can be accelerated or decelerated by the corresponding *form drag* pressure force.

Thus, if a geostrophic motion in some isopycnal layer squeezes or stretches it, the underlying layer is also deformed, and the resulting pressure-gradient force accelerates fluid in the underlying layer.

QUASIGEOSTROPHIC THEORY



Two-layer shallow-water model is a natural extension of the single-layer shallow-water model. It illuminates effects of isopycnal deformations on the geostrophic vorticity. This model can be straightforwardly extended to many isopycnal (i.e., constant-density) layers, thus, producing the family of *isopycnal models*.

The model assumes geostrophic and hydrostatic balances, and usual Boussinesq treatment of density:

$$\Delta\rho \equiv \rho_2 - \rho_1 \ll \rho_1, \rho_2 \,, \qquad \rho_1 \approx \rho_2 \approx \rho \,.$$

All notations are introduced on the sketch.
The layer thicknesses and pressures consist of the static and dynamic components:

$$h_1(t,x,y) = H_1 + H_2 + \eta_1(t,x,y) \,, \qquad h_2(t,x,y) = H_2 + \eta_2(t,x,y) \,,$$

$$p_1 = \rho_1 g(H_1 + H_2 - z) + p_1'(t,x,y) \,, \qquad p_2 = \rho_1 g H_1 + \rho_2 g(H_2 - z) + p_2'(t,x,y) \,,$$

Here, the shallow-water dynamic pressure anomalies are independent of z, as we have seen, and the static pressures were obtained as shown below.

Let's integrate out static pressure in the top layer:

$$P_1 = -\int_0^z \rho g dz = -\rho_2 g H_2 - \rho_1 g(z - H_2) + C_1$$

Since $P_1(z = H_1 + H_2) = 0$, we obtain $C_1 = \rho_1 g H_1 + \rho_2 g H_2$ and find:

$$P_1 = \rho_1 g(H_1 + H_2 - z) \,.$$

Similarly, in the deep layer:

$$P_2 = -\int_0^z \rho_2 g dz = -\rho_2 g z + C_2$$

Since $P_2(z = H_2) = P_1(z = H_2)$, we obtain $C_2 = \rho_1 g H_1 + \rho_2 g H_2$ and find:

$$P_2 = \rho_1 g H_1 + \rho_2 g(H_2 - z)$$

• *Continuity boundary condition for pressure* is just a component of the continuity boundary condition for stress tensor (sometimes, this involves surface tension); here, it allows to relate dynamic pressure anomalies and isopycnal deformations.

In the two-layer model this boundary condition is equivalent to saying that:
(a) pressure at the upper surface must be zero (more generally, it must be equal to the atmospheric pressure),
(b) pressure on the internal interface must be continuous, i.e., $p_1 = p_2 = P$.

Note, that in the absence of motion $(p_1' = p_2' = 0)$ both of these conditions are automatically satisfied for the static pressure component:

$$p_1|_{z=H_1+H_2} = 0\,, \qquad p_1|_{z=H_2} = p_2|_{z=H_2} = \rho_1 g H_1\,.$$

In the presence of motion, the upper-surface pressure continuity statement $p_1|_{z=\eta_1+H_1+H_2} = 0$ translates into

$$\boxed{p_1'(t,x,y) = \rho_1 g \eta_1(t,x,y)}\,.$$

On the internal interface, the pressure continuity statement is:

$$P = p_1|_{z=\eta_2+H_2} = \rho_1 g(H_1-\eta_2)+p_1'\,, \qquad P = p_2|_{z=\eta_2+H_2} = \rho_1 g H_1 - \rho_2 g\eta_2 + p_2' \quad \Longrightarrow \quad p_2'(t,x,y) = p_1'(t,x,y) + g\Delta\rho\,\eta_2(t,x,y)$$

Thus, by using expression for the upper-layer pressure, we obtain:

$$\boxed{p_2'(t,x,y) = \rho_1 g \eta_1(t,x,y) + g\Delta\rho\,\eta_2(t,x,y)}$$

• *Geostrophy* at the leading order links horizontal velocities and slopes of the isopycnals (interfaces) in the upper and deep layers:

$$\boxed{-f_0 v_1 = -g\,\frac{\partial\eta_1}{\partial x}\,, \qquad f_0 u_1 = -g\,\frac{\partial\eta_1}{\partial y}}\,; \qquad -f_0 v_2 = -g\,\frac{\rho_1}{\rho_2}\frac{\partial\eta_1}{\partial x} - g\,\frac{\Delta\rho}{\rho_2}\frac{\partial\eta_2}{\partial x}\,, \qquad f_0 u_2 = -g\,\frac{\rho_1}{\rho_2}\frac{\partial\eta_1}{\partial y} - g\,\frac{\Delta\rho}{\rho_2}\frac{\partial\eta_2}{\partial y}$$

Next, we recall that $\rho_1 \approx \rho_2 \approx \rho$ (Boussinesq argument), introduce the *reduced gravity* $g' \equiv g\,\Delta\rho/\rho$, and, thus, simplify the second-layer equations:

$$\boxed{-f_0 v_2 = -g\,\frac{\partial\eta_1}{\partial x} - g'\,\frac{\partial\eta_2}{\partial x}\,, \qquad f_0 u_2 = -g\,\frac{\partial\eta_1}{\partial y} - g'\,\frac{\partial\eta_2}{\partial y}}$$

• *Geostrophic vorticity equations*.

Now, let's take a look at the full system of the *two-layer shallow-water equations*:

$$\frac{Du_1}{Dt} - fv_1 = -g\,\frac{\partial\eta_1}{\partial x}\,, \qquad \frac{Dv_1}{Dt} + fu_1 = -g\,\frac{\partial\eta_1}{\partial y}\,, \qquad \frac{\partial(h_1-h_2)}{\partial t} + \nabla\cdot((h_1-h_2)\mathbf{u}_1) = 0\,,$$

$$\frac{Du_2}{Dt} - fv_2 = -g\,\frac{\partial\eta_1}{\partial x} - g'\frac{\partial\eta_2}{\partial x}\,, \qquad \frac{Dv_2}{Dt} + fu_2 = -g\,\frac{\partial\eta_1}{\partial y} - g'\frac{\partial\eta_2}{\partial y}\,, \qquad \frac{\partial h_2}{\partial t} + \nabla\cdot(h_2\mathbf{u}_2) = 0\,.$$

As we have argued, at the leading order the momentum equations are geostrophic. At the ϵ-order, we can formulate the layer-wise vorticity equations with the additional rhs terms responsible for vertical deformations. For this purpose:

(a) Expand the momentum equations in terms of ϵ,

(b) take curl of the momentum equations $(\partial(2)/\partial x - \partial(1)/\partial y)$,

(c) replace divergence of the horizontal ageostrophic velocity (u_a, v_a) with the vertical divergence of w_a.

The resulting geostrophic vorticity equations are:

$$\boxed{\frac{D_n\zeta_n}{Dt} + \beta v_n = f_0\,\frac{\partial w_n}{\partial z}}\,, \qquad \frac{D_n}{Dt} = \frac{\partial}{\partial t} + u_n\frac{\partial}{\partial x} + v_n\frac{\partial}{\partial y}\,, \qquad \zeta_n \equiv \frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y}\,, \qquad n = 1, 2$$

Within each layer horizontal velocity does not depend on z, therefore, vertical integrations of the vorticity equations across each layer yield (here, we assume nearly flat isopycnals everywhere by replacing $h_1 - h_2 \approx H_1$ and $h_2 \approx H_2$ on the lhs):

$$H_1\left(\frac{D_1\zeta_1}{Dt} + \beta v_1\right) = f_0\left(w_1(h_1) - w_1(h_2)\right)\,, \qquad H_2\left(\frac{D_2\zeta_2}{Dt} + \beta v_2\right) = f_0\,w_2(h_2)\,, \qquad (*)$$

Thus, we extended the assumption of nearly flat isopycnals to everywhere, beyond the scale of motions. Note, that in (*) we took $w_2(bottom) = 0$, but this is true only for the flat bottom (along topographic slopes vertical velocity can be non-zero, as only normal-to-boundary velocity component vanishes).

• *Vertical movement of isopycnals in terms of pressure* can be obtained, and this step closes the equations.
For that we use *kinematic boundary condition*, which comes from considering fluid elements on a fluid interface or surface, such that the vertical coordinates of these elements are given by $z = h(t, x, y)$.
Next, let's consider function $F(t, x, y, z) = h(t, x, y) - z$, and acknowledge, that it is always zero for a fluid elements sitting on the interface or surface; hence, its material derivative is zero:

$$\frac{DF}{Dt} = 0 = \frac{Dh}{Dt} - w\,\frac{\partial z}{\partial z} \qquad \rightarrow \qquad \boxed{w = \frac{Dh}{Dt}}$$

By combining the kinematic boundary condition with the Boussinesq argument $(\rho_1 \approx \rho_2 \approx \rho)$, we obtain:

$$w_n(h_n) = \frac{D_n h_n}{Dt} = \frac{D_n \eta_n}{Dt} \qquad \Longrightarrow \qquad w_1(h_1) = \frac{1}{\rho g}\,\frac{D_1 p_1'}{Dt}\,, \qquad w_{1,2}(h_2) = \frac{1}{\Delta\rho g}\,\frac{D_{1,2}(p_2' - p_1')}{Dt} \qquad (**)$$

• *Pressure is streamfunction in disguise*.
In each layer geostrophic velocity streamfunction is linearly related to dynamic pressure anomaly, as follows from the geostrophic momentum balance:

$$f_0 v_n = \frac{1}{\rho}\,\frac{\partial p_n'}{\partial x}\,, \qquad f_0 u_n = -\frac{1}{\rho}\,\frac{\partial p_n'}{\partial y} \qquad \Longrightarrow \qquad \boxed{\psi_n = \frac{1}{f_0 \rho}\,p_n'}\,, \qquad u_n = -\frac{\partial \psi_n}{\partial y}\,, \quad v_n = \frac{\partial \psi_n}{\partial x} \qquad (***)$$

Relative vorticity ζ is always conveniently expressed in terms of ψ : $\boxed{\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi}$

Let's now combine $(*)$, $(**)$ and $(***)$ to obtain the fully closed equations predicting evolution of the leading-order streamfunction.

Two-layer quasigeostrophic (QG) model

$$\frac{D_1\zeta_1}{Dt} + \beta v_1 - \frac{f_0^2}{gH_1}\left(\frac{\rho}{\Delta\rho}\frac{D_1}{Dt}(\psi_1-\psi_2) + \frac{D_1\psi_1}{Dt}\right) = 0\,,$$

$$\frac{D_2\zeta_2}{Dt} + \beta v_2 \qquad - \frac{f_0^2}{gH_2}\frac{\rho}{\Delta\rho}\frac{D_2}{Dt}(\psi_2-\psi_1) \qquad = 0$$

(a) Note that $\Delta\rho \ll \rho$, therefore the last term of the first equation is neglected (i.e., the *rigid-lid approximation* is taken; it states that the surface elevation is much smaller than the internal interface displacement).

(b) Familiar *reduced gravity* is $g' \equiv g\Delta\rho/\rho$, and *stratification parameters* are defined as $\boxed{S_1 = \frac{f_0^2}{g'H_1}\,, \quad S_2 = \frac{f_0^2}{g'H_2}}$.

(c) Dimensionally, $[S_1] \sim [S_2] \sim L^{-2} \rightarrow$ QG (i.e., double-balanced) motion of stratified fluid operates on the *internal deformation* scales:

$$R_1 = 1/\sqrt{S_1}\,, \qquad R_2 = 1/\sqrt{S_2}\,,$$

which are $O(100km)$ in the ocean and about 10 times larger in the atmosphere.
Note: $R_n \ll L_d = f_0^2/gH$, because $g' \ll g$.

With the above information taken into account, we obtain the final set of *two-layer QG PV equations*:

$$\boxed{\frac{D_1}{Dt}\left[\nabla^2\psi_1 - S_1(\psi_1-\psi_2)\right] + \beta v_1 = 0\,, \qquad \frac{D_2}{Dt}\left[\nabla^2\psi_2 - S_2(\psi_2-\psi_1)\right] + \beta v_2 = 0}$$

Potential vorticity anomalies are defined as:

$$\boxed{q_1 = \nabla^2\psi_1 - S_1(\psi_1-\psi_2), \qquad q_2 = \nabla^2\psi_2 - S_2(\psi_2-\psi_1)}$$

Note: These expressions for the PV anomalies can be obtained by linearization of the full shallow-water PV (without proof).

• *Potential vorticity (PV) material conservation law*.

(Absolute) PV is defined as $\boxed{\Pi_1 = q_1 + f = q_1 + f_0 + \beta y, \qquad \Pi_2 = q_2 + f = q_2 + f_0 + \beta y}$.

(a) PV is materially conserved quantity: $\boxed{\frac{D_n \Pi_n}{Dt} = \frac{\partial \Pi_n}{\partial t} + \frac{\partial \psi_n}{\partial x}\frac{\partial \Pi_n}{\partial y} - \frac{\partial \psi_n}{\partial y}\frac{\partial \Pi_n}{\partial x} = 0\,, \qquad n = 1, 2}$

(b) PV can be considered as a “charge” advected by the flow; but this is *active* charge, as it defines the flow itself.

(c) *PV inversion* brings in intrinsic and important spatial nonlocality of the velocity field around “elementary charge” of PV:

$$\boxed{\Pi_1 = \nabla^2\psi_1 - S_1\left(\psi_1 - \psi_2\right) + \beta y + f_0\,, \qquad \Pi_2 = \nabla^2\psi_2 - S_2\left(\psi_2 - \psi_1\right) + \beta y + f_0}$$

(d) PV consists of of relative vorticity, density anomaly (resulting from isopycnal displacement), and planetary vorticity.

• *Continuous stratification* yields (without derivation) similar PV conservation law and PV inversion formula for the geostrophic fields:

$$\psi = \frac{1}{f_0 \rho}\,p'\,, \qquad u = -\frac{\partial \psi}{\partial y}\,, \qquad v = \frac{\partial \psi}{\partial x}\,, \qquad \rho = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}\,, \qquad N^2(z) = -\frac{g}{\rho_s}\frac{d\rho_s}{dz}$$

$$\boxed{\frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Pi}{\partial x} = 0\,, \qquad \Pi = \nabla^2\psi + f_0^2\,\frac{\partial}{\partial z}\left(\frac{1}{N^2(z)}\frac{\partial \psi}{\partial z}\right) + f_0 + \beta y}$$

Note, that density anomalies are now described by vertical derivative of velocity streamfunction, rather than by deformation of interface η that is related to (vertical) difference between the streamfunction values above and below it.

• *Boundary conditions* for QG equations.

(a) On lateral solid boundaries there is always *no-normal-flow* condition: $\psi = C(t)$.

(b) The other lateral boundary conditions are *no-slip*: $\frac{\partial \psi}{\partial n} = 0\,,$ *free-slip*: $\frac{\partial^2 \psi}{\partial n^2} = 0\,,$ *partial-slip*: $\frac{\partial^2 \psi}{\partial n^2} + \frac{1}{\alpha}\frac{\partial \psi}{\partial n} = 0\,,$ *periodic*, *double-periodic*, etc.

(c) There are also *integral constraints* on mass and momentum. For example, we can require that basin-averaged density anomaly integrates to zero in each layer:

$$\iint \rho\, dx\, dy = 0 \qquad \rightarrow \qquad \iint \frac{\partial \psi}{\partial z}\, dx\, dy = 0\,.$$

• *Ageostrophic circulation* (of the ϵ-order) can be obtained with further efforts, and even *diagnostically*.
For example, vertical ageostrophic velocity is equal to material derivative of pressure, which is known from QG solution:

$$w_1(h_1) = \frac{1}{\rho g}\frac{D_1 p_1'}{Dt}, \qquad w_1(h_2) = \frac{1}{\Delta\rho g}\frac{D_1(p_2' - p_1')}{Dt}$$

Summary on QG PV models:

(a) *Midlatitude theory*: QG framework does not work at the equator, where $f = 0$.

(b) *Vertical control*: Nearly horizontal geostrophic motions are determined by vertical stratification, vertical component of ζ, and vertical isopycnal stretching.

(c) *Four main assumptions* that have been made:
(i) Rossby number ϵ is small (hence, the expansion focuses on mesoscales);
(ii) β-plane approximation and small meridional variations of Coriolis parameter;
(iii) isopycnals are nearly flat $([\delta\rho'] \sim \epsilon F \rho_0 \sim \epsilon^2 \rho_0)$ *everywhere*;
(iv) hydrostatic Boussinesq balance.

• *Planetary-geostrophic equations* (extra material) can be similarly derived for small-Rossby-number motions on scales that are much larger than internal deformation scale R and for large meridional variations of Coriolis parameter.

Let's start from the full shallow-water equations,

$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x}, \qquad \frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y}, \qquad \frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0,$$

and consider $F = L^2/R^2 \sim \epsilon^{-1} \gg 1$.

Then, let's assume that, for large scales of motion, fluid height variations ($[\delta\rho'] \sim \epsilon F \rho_0$) are as large as the mean height of fluid:

$$h = H\,(1 + \epsilon F \eta) = H\,(1 + \eta).$$

Asymptotic expansions $\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \ldots$, and $\eta = \eta_0 + \epsilon\eta_1 + \ldots$ yield:

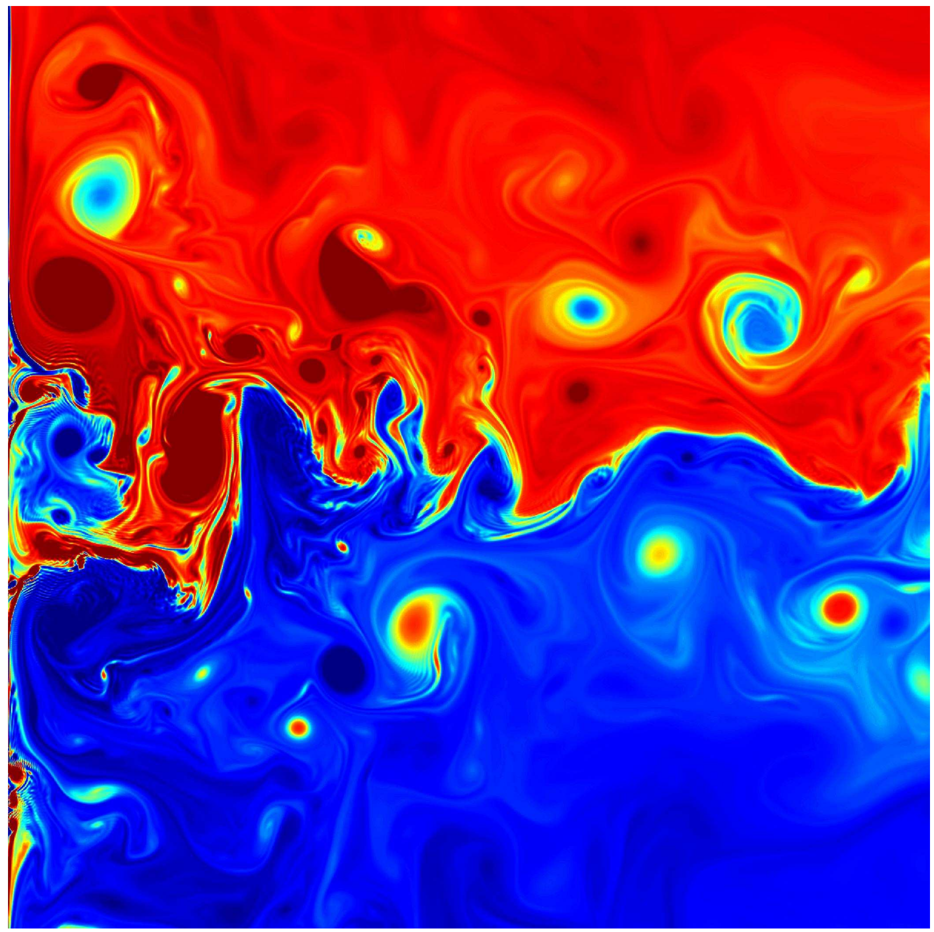
$$\epsilon\Big[\frac{\partial u_0}{\partial t} + \mathbf{u}_0\nabla u_0 - fv_1\Big] - fv_0 = -gH\frac{\partial \eta_0}{\partial x} - \epsilon gH\frac{\partial \eta_1}{\partial x} + O(\epsilon^2), \qquad \ldots\ldots, \qquad \epsilon F\Big[\frac{\partial \eta_0}{\partial t} + \mathbf{u}_0\cdot\nabla\eta_0\Big] + (1 + \epsilon F\eta_0)\nabla\cdot\mathbf{u}_0 = 0.$$

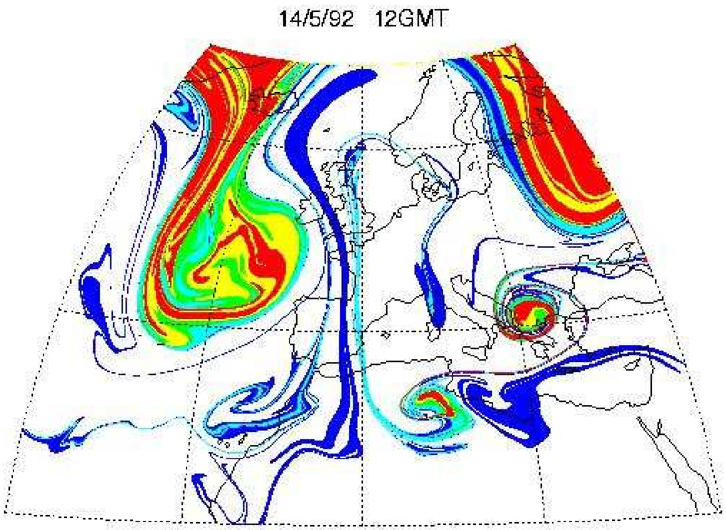
Thus, only geostrophic balance is retained in the momentum equation, and all terms are retained in the continuity equation, and the resulting set of equations is:

$$\boxed{-fv = -g\frac{\partial h}{\partial x}, \qquad fu = -g\frac{\partial h}{\partial y}, \qquad \frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0}$$

$\Longleftarrow$ Vortex street behind obstacle

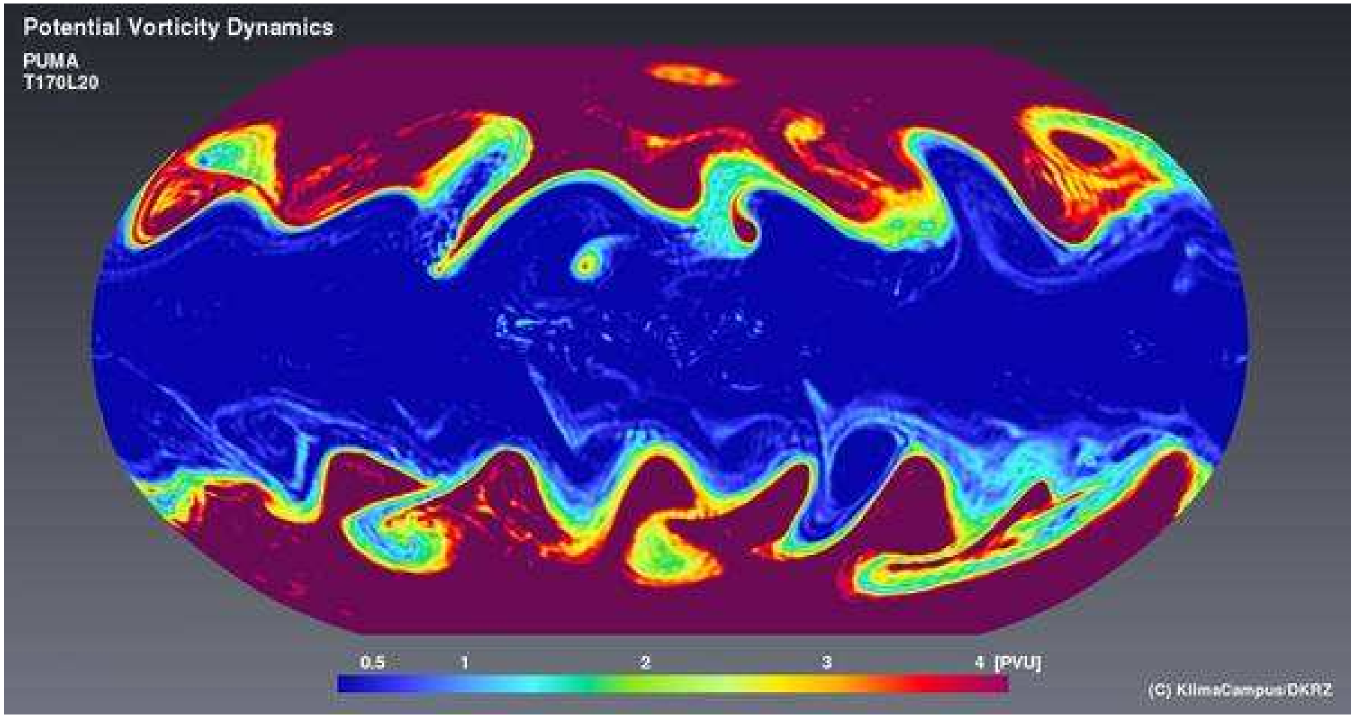
Meandering oceanic current $\Longrightarrow$

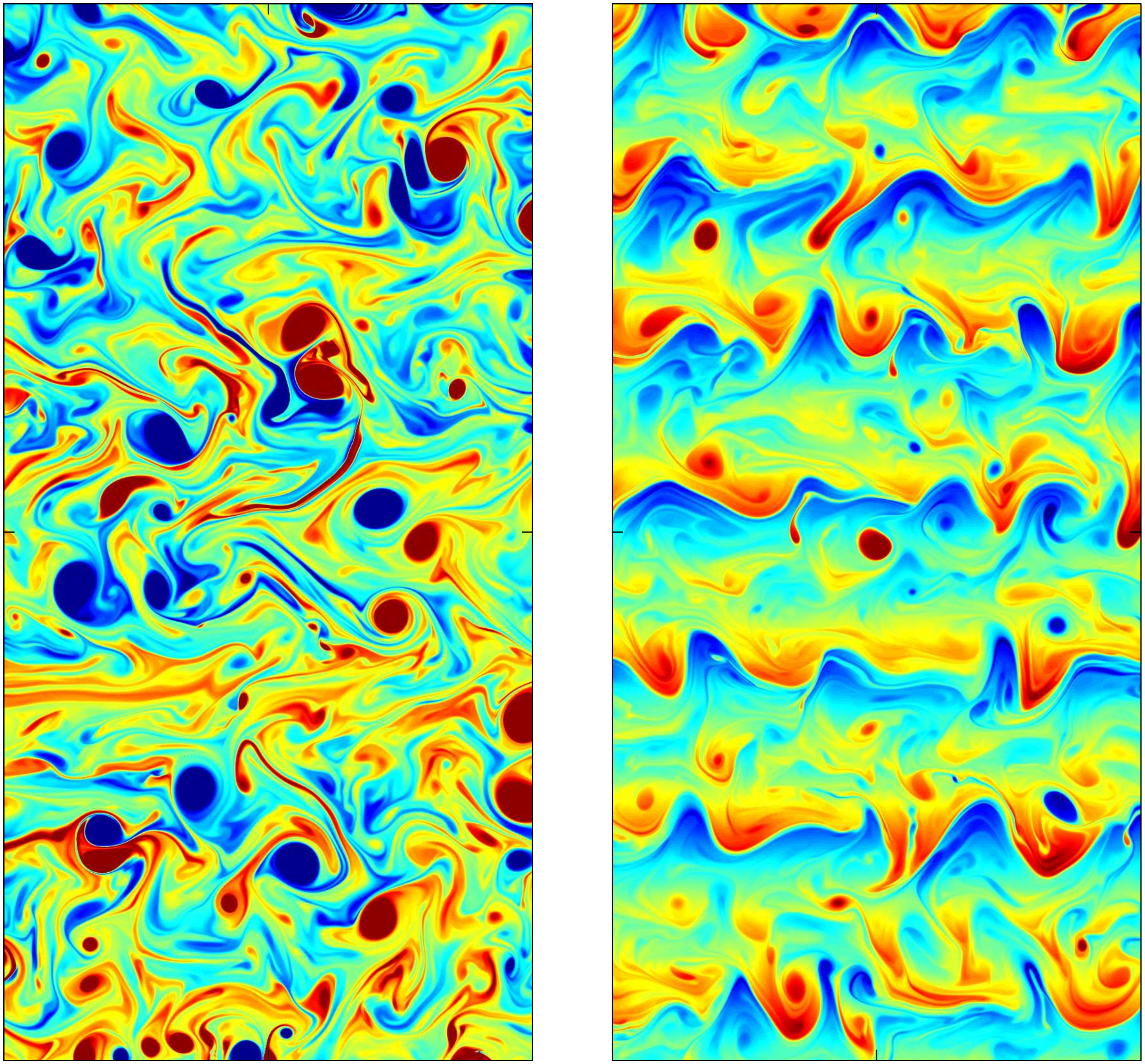




$\Longleftarrow$ Observed atmospheric PV

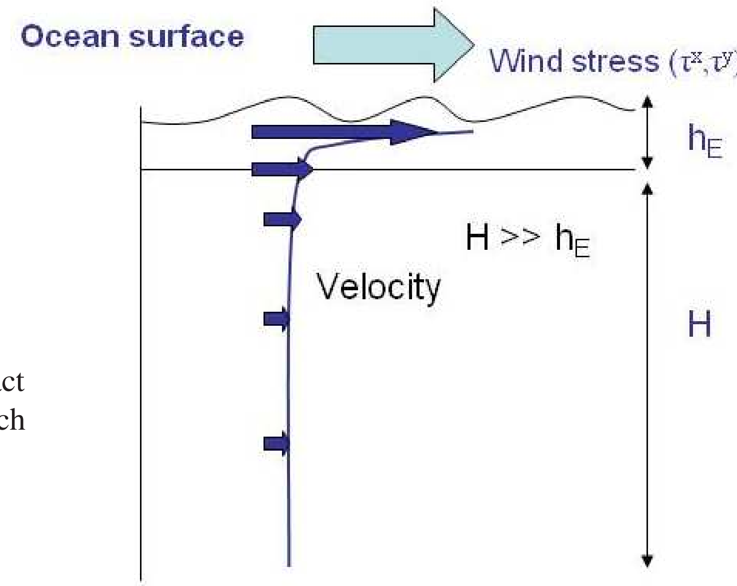
Atmospheric PV from a model $\Longrightarrow$





Solutions of geostrophic turbulence (*PV snapshots*)

EKMAN LAYERS



Ekman surface boundary layer

Boundary layers are governed by physical processes very different from those in the interior. Non-geostrophic effects at the free-surface or rigid-bottom boundary layers are responsible for transferring momentum from *wind stress* or *bottom stress* to the interior (large-scale) geostrophic currents.

Let's consider *Ekman layer* at the ocean surface:

(a) Horizontal momentum is transferred down by vertical turbulent flux (its exact form is unknown due to complexity of many physical processes involved), which is commonly modelled by *vertical viscosity* (i.e., diffusion of momentum) with constant turbulent viscosity coefficient:

$$\overline{w'\frac{\partial \mathbf{u}'}{\partial z}} = A_v \frac{\partial^2 \overline{\mathbf{u}}}{\partial z^2},$$

where overbar and prime indicate the time mean and fluctuating flow components, respectively.
Note that vertical viscosity must be balanced by some other term containing velocity, because momentum diffusion creates flow velocity, and at the leading order only Coriolis force contains velocity.

(b) Consider *boundary layer correction*, so that $\mathbf{u} = \mathbf{u}_g + \mathbf{u}_E$ in the thin layer with depth h_E :

$$-f_0(v_g + v_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + A_v \frac{\partial^2 u_E}{\partial z^2}, \qquad f_0(u_g + u_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} + A_v \frac{\partial^2 v_E}{\partial z^2}.$$

The *Ekman balance* is $\boxed{-f_0 v_E = A_v \frac{\partial^2 u_E}{\partial z^2}, \qquad f_0 u_E = A_v \frac{\partial^2 v_E}{\partial z^2}}$ $\qquad (*)$

To make the viscous term important in the balance, the *Ekman layer thickness* must be $h_E \sim [A_v/f_0]^{1/2}$, therefore, let's define:

$$\boxed{h_E = [2A_v/f_0]^{1/2}}.$$

Typical values of h_E are ~ 1 km in the atmosphere and ~ 50 m in the ocean.

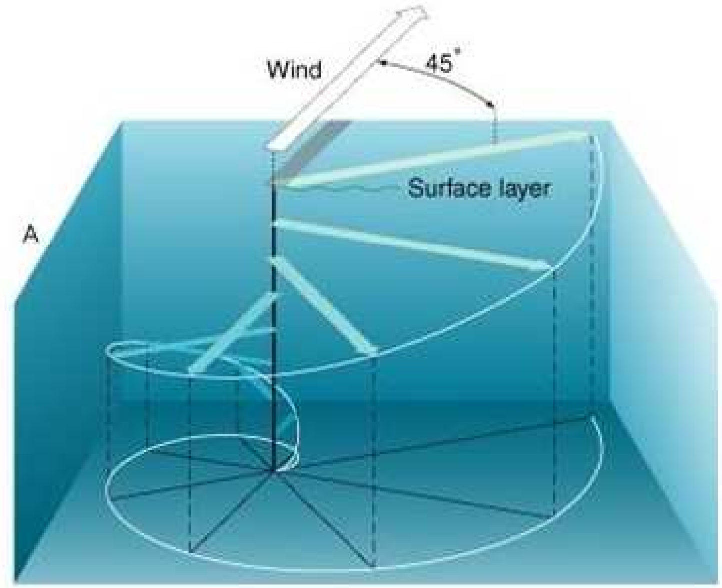
(c) If *Ekman number*,

$$Ek \equiv \left(\frac{h_E}{H}\right)^2 = \frac{2A_v}{f_0 H^2},$$

is small, i.e., $Ek \ll 1$, then, the boundary layer correction can be matched to the interior geostrophic solution.

(d) *Boundary conditions* for the Ekman flow correction are: zero at the bottom of the boundary layer and the stress condition at the upper surface:

$$A_v\,\frac{\partial u_E}{\partial z}=\frac{1}{\rho_0}\,\tau^x\,,\qquad A_v\,\frac{\partial v_E}{\partial z}=\frac{1}{\rho_0}\,\tau^y \qquad (**)$$



Let's look for solution of $(*)$ and $(**)$ in the form:

$$u_E=e^{z/h_E}\left[C_1\cos\left(\frac{z}{h_E}\right)+C_2\sin\left(\frac{z}{h_E}\right)\right],$$

$$v_E=e^{z/h_E}\left[C_3\cos\left(\frac{z}{h_E}\right)+C_4\sin\left(\frac{z}{h_E}\right)\right],$$

and obtain the *Ekman spiral* solution:

$$u_E=\frac{\sqrt{2}}{\rho_0 f_0 h_E}\,e^{z/h_E}\left[\tau^x\cos\left(\frac{z}{h_E}-\frac{\pi}{4}\right)-\tau^y\sin\left(\frac{z}{h_E}-\frac{\pi}{4}\right)\right],$$

$$v_E=\frac{\sqrt{2}}{\rho_0 f_0 h_E}\,e^{z/h_E}\left[\tau^x\sin\left(\frac{z}{h_E}-\frac{\pi}{4}\right)+\tau^y\cos\left(\frac{z}{h_E}-\frac{\pi}{4}\right)\right].$$

• *Ekman pumping*.
Vertically integrated, horizontal *Ekman transport* $\mathbf{U}_E=\int\mathbf{u}_E\,dz$ can be divergent. It satisfies:

$$-f_0V_E=A_v\left[\frac{\partial u_E}{\partial z}\Big|_{top}-\frac{\partial u_E}{\partial z}\Big|_{bottom}\right]=\frac{1}{\rho_0}\,\tau^x\,,$$

$$f_0U_E=A_v\left[\frac{\partial v_E}{\partial z}\Big|_{top}-\frac{\partial v_E}{\partial z}\Big|_{bottom}\right]=\frac{1}{\rho_0}\,\tau^y\,.$$

The bottom stress terms vanish due to the exponential decay of the boundary layer solution. In order to obtain vertical Ekman velocity at the bottom of the Ekman layer, let's integrate the continuity equation

$$-\left(w_E\Big|_{top}-w_E\Big|_{bottom}\right)=w\Big|_{bottom}\equiv w_E=\frac{\partial U_E}{\partial x}+\frac{\partial V_E}{\partial y}+\frac{\partial}{\partial x}\int u_g\,dz+\frac{\partial}{\partial y}\int v_g\,dz\,.$$

Recall the non-divergence of the geostrophic velocity and use the above-derived integrated Ekman transport components to obtain

$$w_E=\frac{\partial U_E}{\partial x}+\frac{\partial V_E}{\partial y}+\int\left(\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}\right)dz=\frac{\partial U_E}{\partial x}+\frac{\partial V_E}{\partial y}=\frac{1}{f_0\rho_0}\,\nabla\times\tau$$

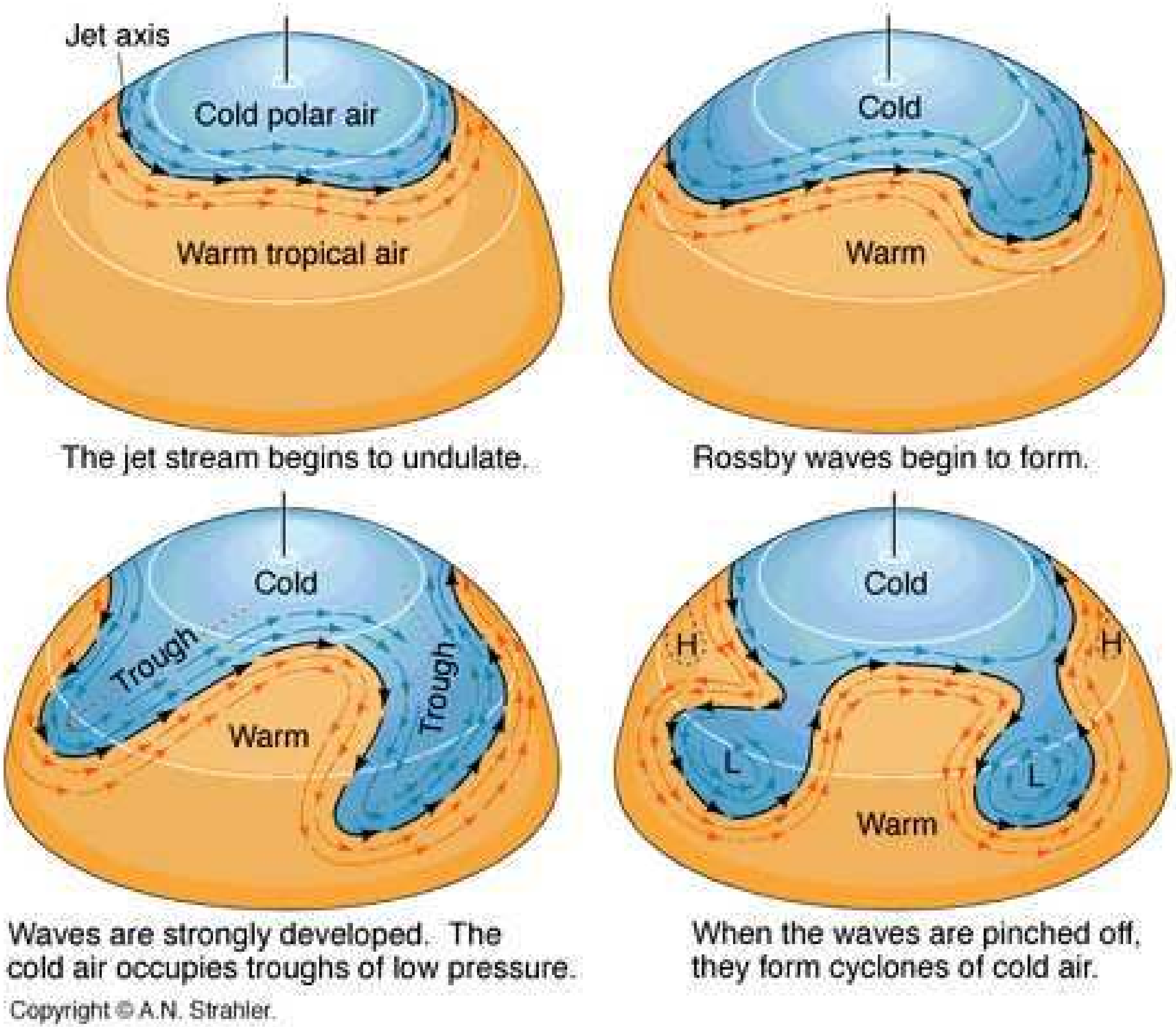
Thus, the Ekman pumping can be found from the wind curl: $\boxed{w_E = \frac{1}{f_0 \rho_0} \nabla \times \tau}$

Conclusion: Ekman pumping w_E provides external forcing for the interior geostrophic motions by vertically squeezing or stretching isopycnal layers; it can be viewed as transmission of an external stress into the geostrophic forcing.

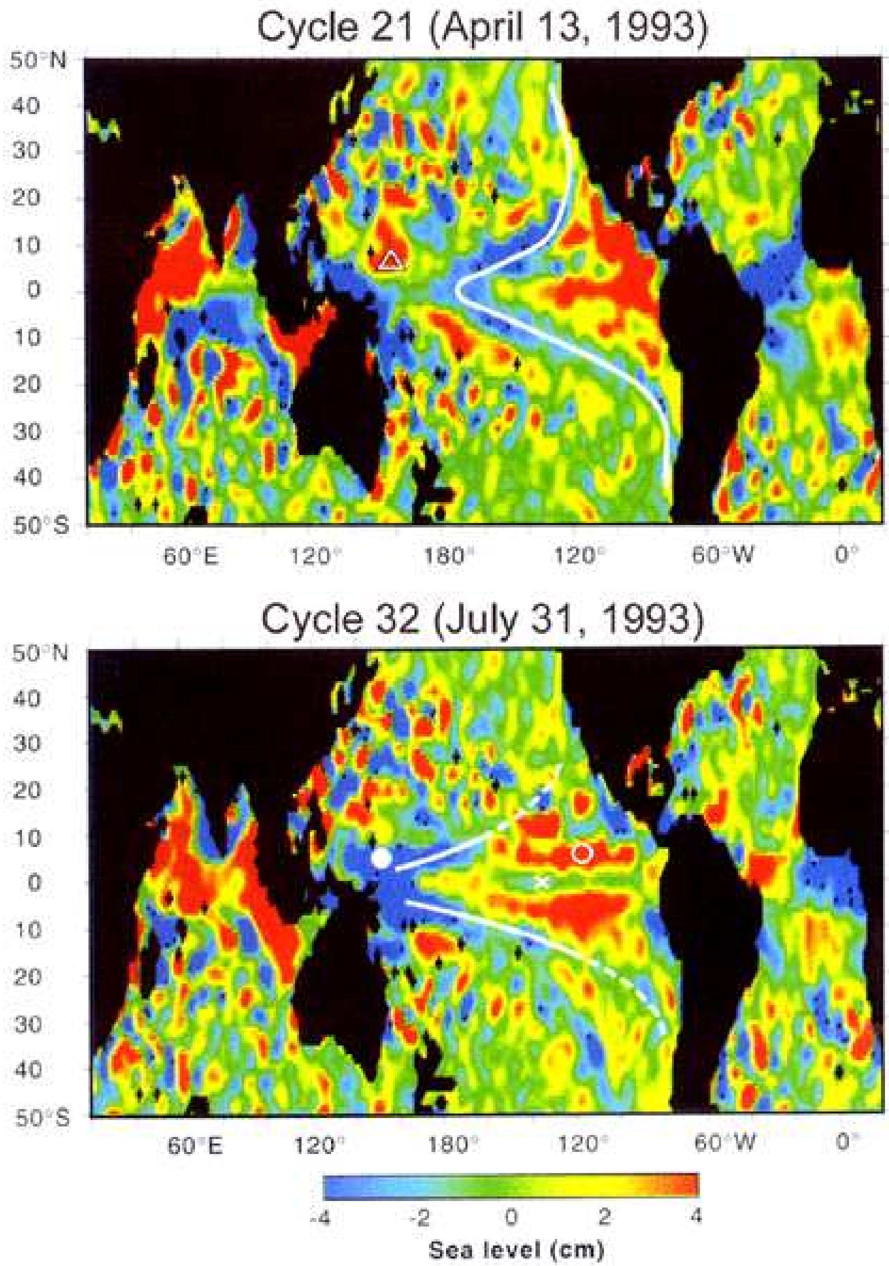
• **Ekman bottom boundary layer** can be solved for in a similar way (see Practical Problems).

ROSSBY WAVES

• In the broad sense, *Rossby wave* is inertial wave propagating on the background PV gradient. First discovered in the Earth's atmosphere.

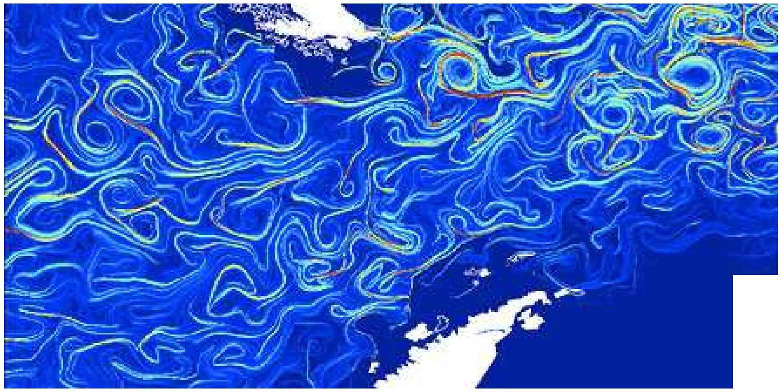


• *Oceanic* Rossby waves are more difficult to observe (e.g., from altimetry, in situ measurements)

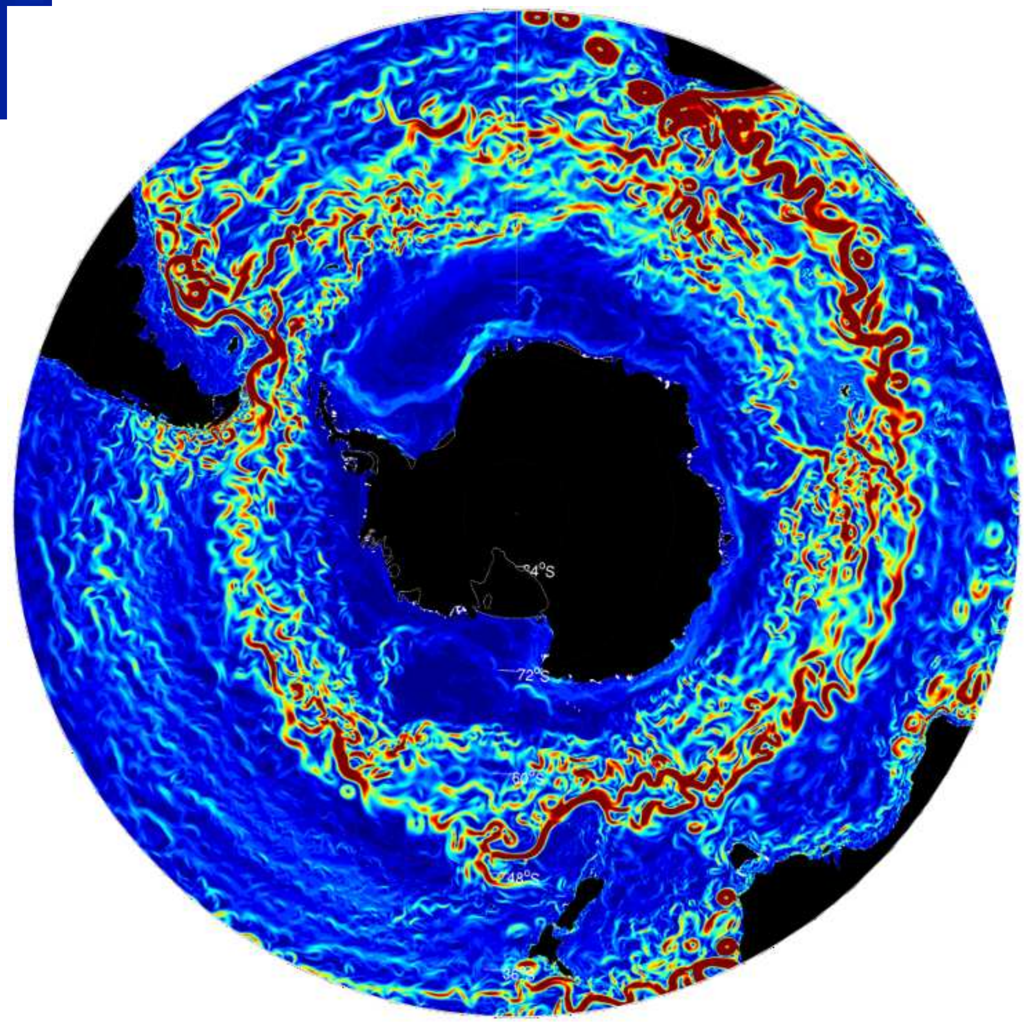


• Sea surface height anomalies propagating to the west are signatures of baroclinic Rossby waves.

• To what extent transient flow anomalies can be characterized as waves rather than isolated coherent vortices remains unclear.



⟸ Visualization of oceanic eddies/waves by virtual tracer



Flow speed from a high resolution computation shows many eddies/waves ⟹

Many properties of the flow fluctuations can be interpreted in terms of linear Rossby waves.

General properties of waves

(a) Waves provide interaction mechanism which is both long-range and fast relative to flow advection.

(b) Waves are observed as periodic propagating (or standing) patterns, e.g., $\psi = Re\{A\,\exp[i(kx + ly + mz - \omega t + \phi)]\}$, characterized by *amplitude*, *wavenumbers*, *frequency*, and *phase*.
Wavevector is defined as the ordered set of wavenumbers: $\mathbf{K} = (k, l, m)$.

(c) *Dispersion relation* comes from the dynamics and relates frequency and wavenumbers, and, thus, yields phase speeds and group velocity.

(d) *Phase speeds* along the axes of coordinates are rates at which intersections of the phase lines with each axis propagate along this axis:

$$C_p^{(x)} = \frac{\omega}{k}\,, \qquad C_p^{(y)} = \frac{\omega}{l}\,, \qquad C_p^{(z)} = \frac{\omega}{m}\,;$$

these speeds do not form a vector (note that phase speed along an axis increases with decreasing projection of $\mathbf{K}$ on this axis).

(e) *Fundamental phase speed* $C_p = \omega/K$, where $K = |\mathbf{K}|$, is defined along the wavevector. This is natural, because waves described by complex exponential functions have instantaneous phase lines perpendicular to $\mathbf{K}$.
Fundamental phase velocity (vector) is defined as

$$\mathbf{C_p} = \frac{\omega}{|\mathbf{K}|}\,\frac{\mathbf{K}}{|\mathbf{K}|} = \frac{\omega}{K^2}\,\mathbf{K}\,.$$

(f) *Group velocity* (vector) is defined as

$$\mathbf{C_g} = \left(\frac{\partial\omega}{\partial k}\,, \frac{\partial\omega}{\partial l}\,, \frac{\partial\omega}{\partial m}\right).$$

(g) *Propagation directions*: phase propagates in the direction of $\mathbf{K}$; energy (hence, information!) propagates at some angle to $\mathbf{K}$.

(h) If frequency $\omega = \omega(x, y, z)$ is spatially inhomogeneous, then trajectory traced by the group velocity is called *ray*, and the path of waves is found by *ray tracing* methods.

• **Mechanism of Rossby waves**.

Consider the simplest 1.5-layer (a.k.a. the *equivalent barotropic*) QG PV model, which is obtained by considering $H_2 \to \infty$ in the two-layer QG PV model:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Pi}{\partial x} = 0\,, \qquad \Pi = \nabla^2\psi - \frac{1}{R^2}\,\psi + \beta y\,,$$

where $R^{-2} = S_1$ is the stratification parameter written in terms of the inverse length scale parameter R.
By introducing the Jacobian operator $J(A,B) = A_x B_y - A_y B_x$, the corresponding *equivalent-barotropic equation* can be written as

$$\boxed{\frac{\partial}{\partial t}\Big(\nabla^2\psi - \frac{1}{R^2}\,\psi\Big) + J\Big(\psi, \nabla^2\psi - \frac{1}{R^2}\,\psi\Big) + \beta\,\frac{\partial \psi}{\partial x} = 0}\,. \qquad (*)$$

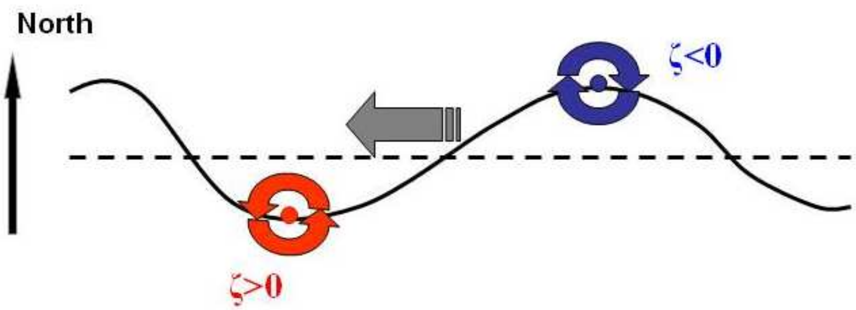
Note, that in the limit $R \to \infty$ the dynamics becomes purely 2D and deformations of the layer thickness become infinitesimal; this is equivalent to $g' \to \infty$.

We are interested in *small-amplitude* flow disturbances around the *state of rest*; the corresponding linearized equation $(*)$ is

$$\frac{\partial}{\partial t}\Big(\nabla^2\psi - \frac{1}{R^2}\,\psi\Big) + \beta\,\frac{\partial \psi}{\partial x} = 0$$

$$\to \qquad \psi \sim e^{i(kx+ly-\omega t)} \qquad \to$$

$$-i\omega\Big(-k^2 - l^2 - \frac{1}{R^2}\Big) + i\beta k = 0$$



Thus, the resulting *Rossby waves dispersion relation* is:

$$\boxed{\omega = \frac{-\beta k}{k^2 + l^2 + R^{-2}}}\,.$$

Plot dispersion relation, discuss zonal, phase and group speeds...

Consider a timeline in the fluid at rest, then, perturb it (see Figure): the resulting westward propagation of Rossby waves is due to the β-effect and material PV conservation.

• **Energy equation**. Multiply the equivalent-barotropic equation by $-\psi$ and use the identity,

$$-\psi\nabla^2\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\frac{(\nabla\psi)^2}{2} - \nabla\cdot\psi\nabla\frac{\partial\psi}{\partial t}\,,$$

to obtain the (mechanical) *energy equation*:

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{S} = 0}\,,\qquad E = \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] + \frac{1}{2R^2}\,\psi^2\,,\qquad \mathbf{S} = -\left(\psi\,\frac{\partial^2\psi}{\partial x\partial t} + \frac{\beta}{2}\,\psi^2,\;\; \psi\,\frac{\partial^2\psi}{\partial y\partial t}\right),$$

where E is energy (density), consisting of the kinetic (first term) and potential (second term) components; and $\mathbf{S}$ is energy flux (vector).

(a) It can be shown (see Practical Problems), that the mean energy $\langle E\rangle$ of a wave packet propagates according to:

$$\frac{\partial\langle E\rangle}{\partial t} + \mathbf{C}_g\cdot\nabla\langle E\rangle = 0\,.$$

(b) The energy equation for the corresponding *nonlinear* equivalent-barotropic equation is derived similarly; its energy flux vector is

$$\mathbf{S} = -\left(\psi\,\frac{\partial^2\psi}{\partial x\partial t} + \frac{\beta}{2}\,\psi^2 + \frac{\psi^2}{2}\,\nabla^2\frac{\partial\psi}{\partial y}\,,\quad \psi\,\frac{\partial^2\psi}{\partial y\partial t} - \frac{\psi^2}{2}\,\nabla^2\frac{\partial\psi}{\partial x}\right).$$

• **Background-flow effects**. Consider small-amplitude flow disturbances around some *background flow* given by its streamfunction $\Psi(x,y,z)$. What happens with the dispersion relation and, hence, with the waves?
To simplify the problem, let's stay with the 1.5-layer QG PV model, consider uniform, zonal background flow $\Psi = -Uy$, and substitute:

$$\psi \to -Uy + \psi\,,\qquad \Pi \to \left(\beta + \frac{U}{R^2}\right)y + \nabla^2\psi - \frac{1}{R^2}\,\psi\,.$$

The linearized dynamics and dispersion relation become:

$$\boxed{\left(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}\right)\left(\nabla^2\psi - \frac{1}{R^2}\,\psi\right) + \frac{\partial\psi}{\partial x}\left(\beta + \frac{U}{R^2}\right) = 0} \quad\to\quad \psi \sim e^{i(kx+ly-\omega t)} \quad\to\quad \boxed{\omega = kU - \frac{k\,(\beta + UR^{-2})}{k^2 + l^2 + R^{-2}}}$$

(a) In the dispersion relation, the first term kU is *Doppler shift*, which is due to advection of wave by the background flow;

(b) The second term contains effect of the *altered background PV*;

(c) There are also corresponding changes in the group velocity;

(d) Complicated 2D and 3D background flows profoundly influence Rossby waves properties, but the corresponding dispersion relations are difficult to obtain.

Two-layer Rossby waves

Consider now the two-layer QG PV equations linearized around the *state of rest*:

$$\frac{\partial}{\partial t}\Big[\nabla^2\psi_1 - \frac{1}{R_1^2}(\psi_1-\psi_2)\Big] + \beta\,\frac{\partial\psi_1}{\partial x} = 0\,, \qquad \frac{\partial}{\partial t}\Big[\nabla^2\psi_2 - \frac{1}{R_2^2}(\psi_2-\psi_1)\Big] + \beta\,\frac{\partial\psi_2}{\partial x} = 0\,, \qquad R_1^2 = \frac{g'H_1}{f_0^2}\,, \quad R_2^2 = \frac{g'H_2}{f_0^2}$$

Diagonalization of the dynamics: the governing equations can be decoupled from each other by a linear transformation of variables from the layer-wise streamfunctions to the streamfunctions of the *vertical modes*. The diagonalizing layers-to-modes transformation and its inverse (modes-to-layers) transformation are *linear* operations.

Barotropic mode ϕ_1 and *first baroclinic mode* ϕ_2 are defined as:

$$\boxed{\phi_1 \equiv \psi_1\,\frac{H_1}{H_1+H_2} + \psi_2\,\frac{H_2}{H_1+H_2}\,, \qquad \phi_2 \equiv \psi_1 - \psi_2\,.}$$

These modes represent separate (i.e., governed by different dispersion relations) families of Rossby waves:

$$\boxed{\frac{\partial}{\partial t}\,\nabla^2\phi_1 + \beta\,\frac{\partial\phi_1}{\partial x} = 0} \quad \rightarrow \quad \boxed{\omega_1 = -\frac{\beta k}{k^2+l^2}}$$

$$\boxed{\frac{\partial}{\partial t}\Big[\nabla^2\phi_2 - \frac{1}{R_D^2}\,\phi_2\Big] + \beta\,\frac{\partial\phi_2}{\partial x} = 0}, \qquad R_D \equiv \Big[\frac{1}{R_1^2} + \frac{1}{R_2^2}\Big]^{-1/2} \quad \rightarrow \quad \boxed{\omega_2 = -\frac{\beta k}{k^2+l^2+R_D^{-2}}}$$

where R_D is referred to as the *first baroclinic Rossby radius*.

(a) The (pure) barotropic mode can be written in terms of layers as:

$$\psi_1 = \psi_2 = \phi_1\,,$$

therefore, it is vertically uniform and actually describes vertically averaged flow. Barotropic waves are fast (typical periods are several days in the ocean and 10 times faster in the atmosphere); their dispersion relation does not depend on stratification.

(b) The (pure) baroclinic mode can be written in terms of layers as:

$$\psi_1 = \phi_2\,\frac{H_2}{H_1+H_2}\,, \qquad \psi_2 = -\phi_2\,\frac{H_1}{H_1+H_2} \qquad \rightarrow \qquad \psi_2 = -\frac{H_1}{H_2}\,\psi_1\,.$$

therefore, it changes sign vertically, and its vertical integral iz sero.

Baroclinic waves are slow (typical periods are several months in the ocean and 10 times faster in the atmosphere); they can be viewed as propagating anomalies of the pycnocline (thermocline), because the streamfunction has large vertical derivative (hence, there is large density anomaly).

Continuously stratified Rossby waves

Continuously stratified model is a natural extension of the isopycnal model with large number of layers.
The corresponding linearized QG PV dynamics is (without proof):

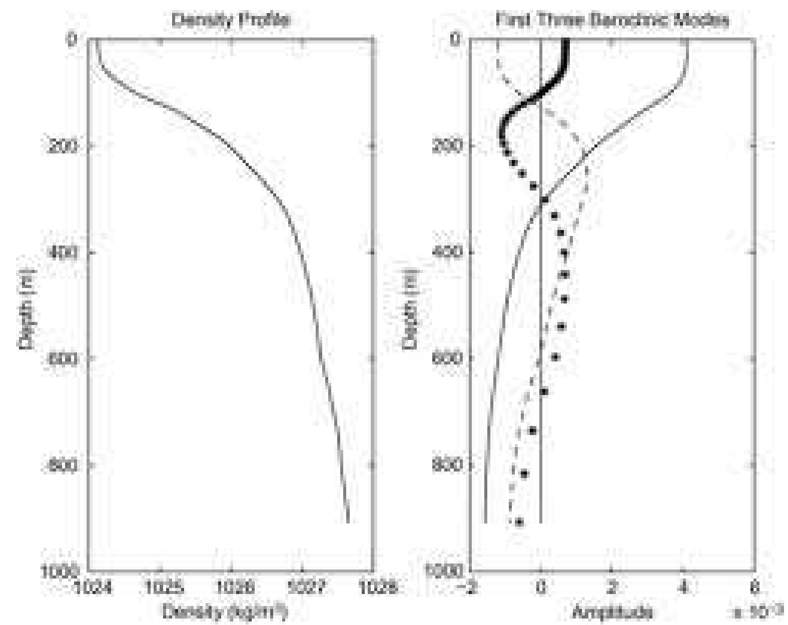
$$\boxed{\frac{\partial}{\partial t}\Big[\nabla^2\psi+\frac{f_0^2}{\rho_s}\frac{\partial}{\partial z}\Big(\frac{\rho_s}{N^2(z)}\frac{\partial\psi}{\partial z}\Big)\Big]+\beta\,\frac{\partial\psi}{\partial x}=0}$$

$$\rightarrow \qquad \psi\sim\Phi(z)\,e^{i(kx+ly-\omega t)} \qquad \rightarrow$$

$$\frac{f_0^2}{\rho_s}\frac{d}{dz}\Big(\frac{\rho_s}{N^2(z)}\frac{d\Phi(z)}{dz}\Big)=\Big(k^2+l^2+\frac{k\beta}{\omega}\Big)\,\Phi(z)\equiv\lambda\,\Phi(z) \qquad (*)$$

Boundary conditions at the top and bottom are to be specified, e.g., by imposing zero density anomalies:

$$\rho\sim\frac{d\Phi(z)}{dz}\Big|_{z=0,-H}=0\,. \qquad (**)$$



Combination of $(*)$ and $(**)$ is an *eigenvalue problem* that can be solved for *discrete spectrum of eigenvalues and eigenmodes*.

(a) *Eigenvalues* λ_n yield dispersion relations $\omega_n=\omega_n(k,l)$, and the corresponding eigenmodes $\phi_n(z)$ are the *vertical normal modes*, like the familiar barotropic and first baroclinic modes in the two-layer case.

(b) The Figure illustrates the first, second and third baroclinic modes for realistic ocean stratification.

(c) The corresponding n-th baroclinic *Rossby deformation radius* $R_D^{(n)}\equiv\lambda_n^{-1/2}$ characterizes horizontal length scale of the n-th vertical mode. The higher is the mode, the more oscillatory it is in vertical, and the slower its propagation.

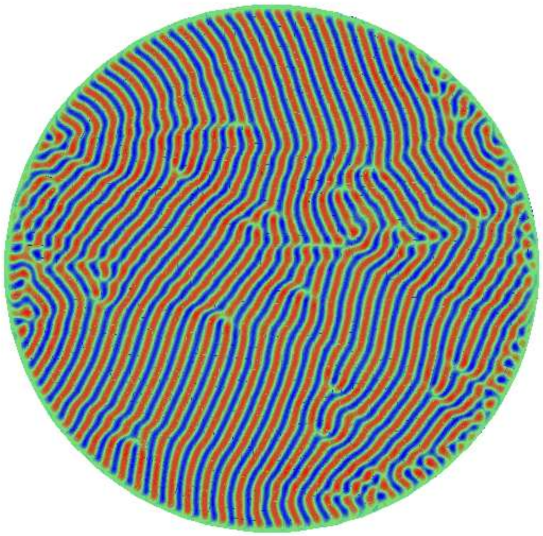
(d) The (zeroth) barotropic mode has $R_D^{(0)}=\infty$ and $\lambda_0=0$.

(e) The *first Rossby deformation radius* $R_D^{(1)}$ is the most important fundamental length scale of geostrophic turbulence; it sets length scale of mesoscale (synoptic) eddies.

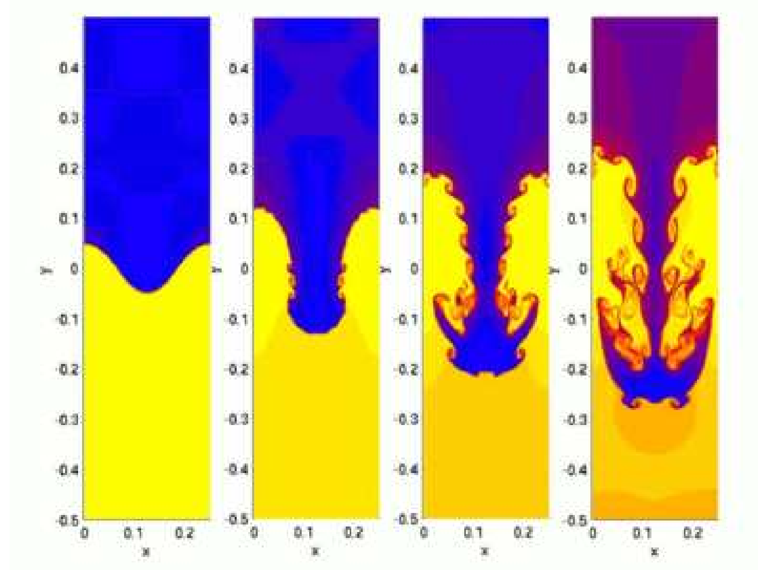
LINEAR INSTABILITIES

• *Linear stability analysis* is the first step toward understanding turbulent flows. Sometimes it can predict some patterns and properties of flow fluctuations.

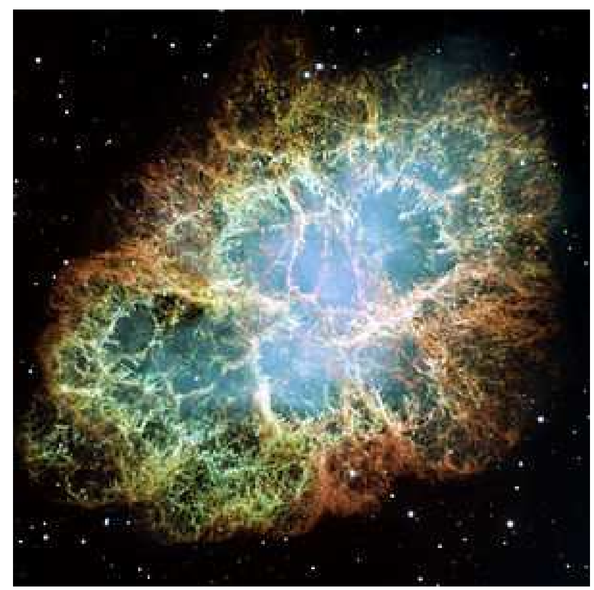
CONVECTIVE ROLLS



CONVECTIVE PLUME



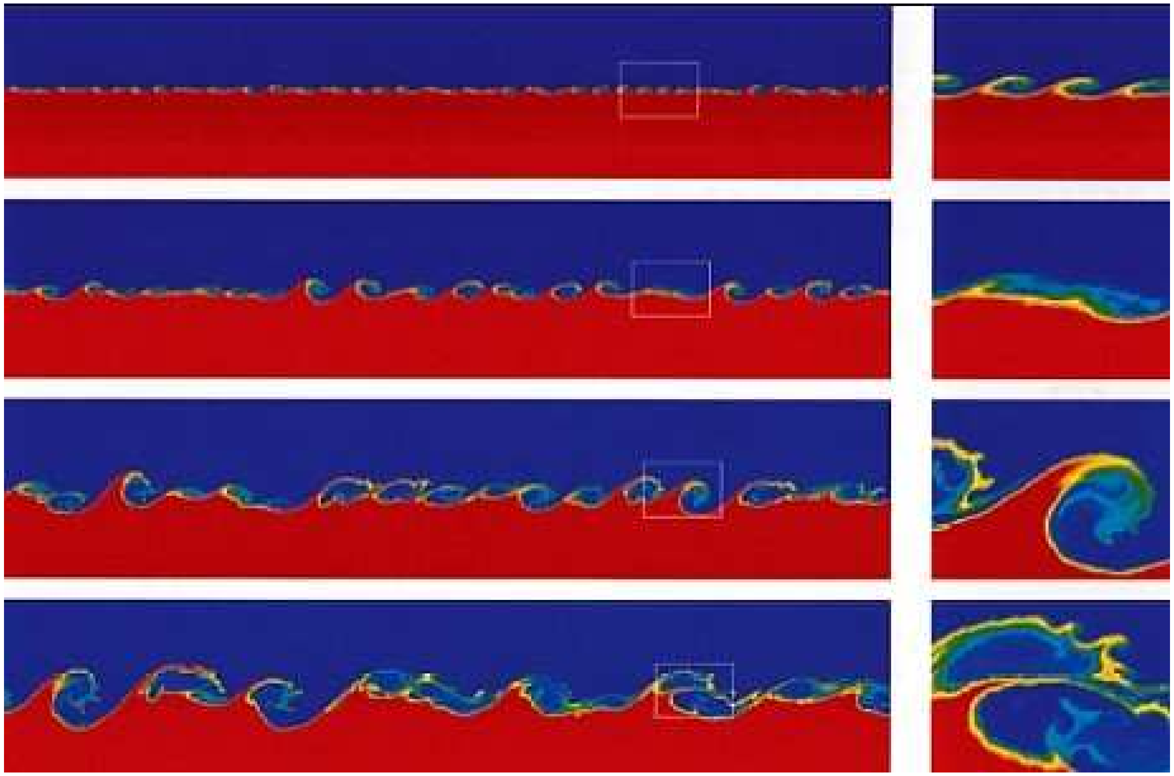
SUPERNOVA REMNANTS



These Figures illustrate different regimes of thermal convection.
Linear stability analysis is very useful for simple flows (convective rolls), somewhat useful for intermediate-complexity flows (convective plumes), and completely useless in highly developed turbulence.

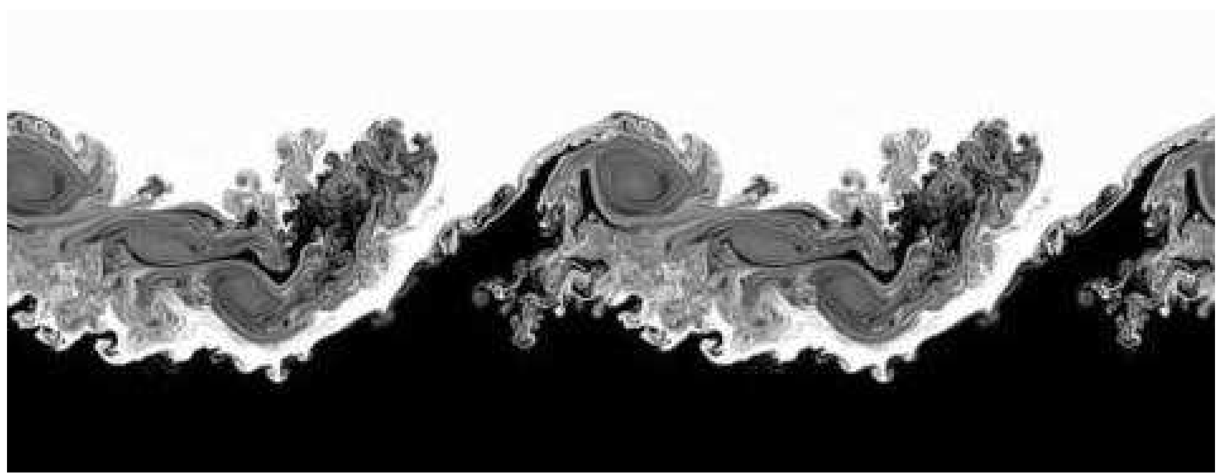
• *Small-amplitude behaviours* can be predicted by linear stability analysis very well, and some of the linear predictions carry on to turbulent flows.

• *Nonlinear effects* become increasingly more important in more complex turbulent flows.

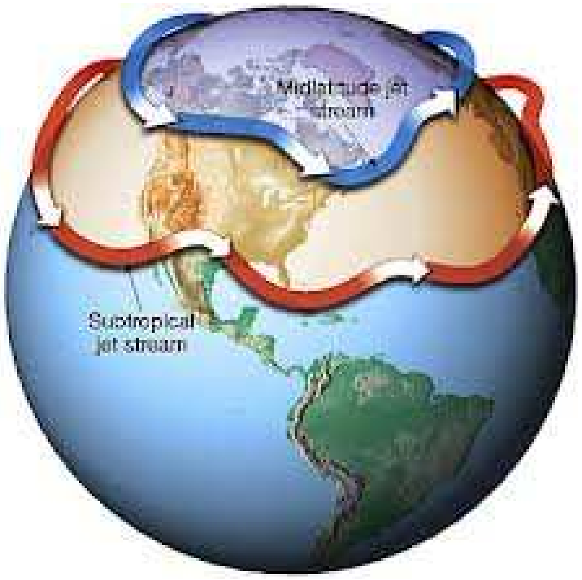


Shear instability occurs on flows with sheared velocity

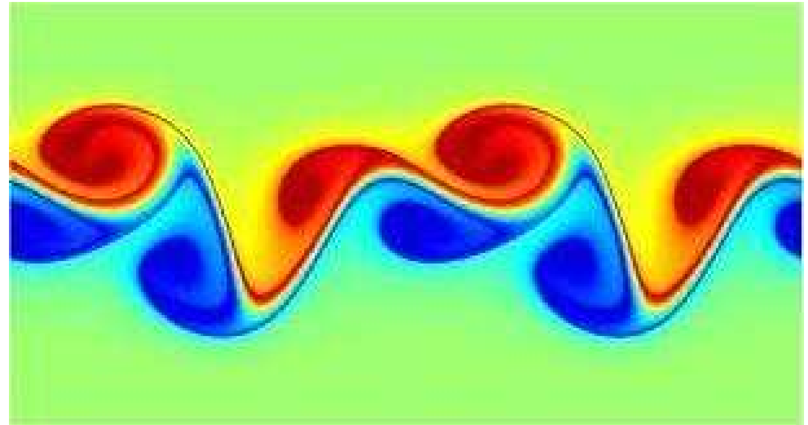
Eventually, there is nonlinear evolution leading to substantial stirring and eventual molecular mixing of material and vorticity



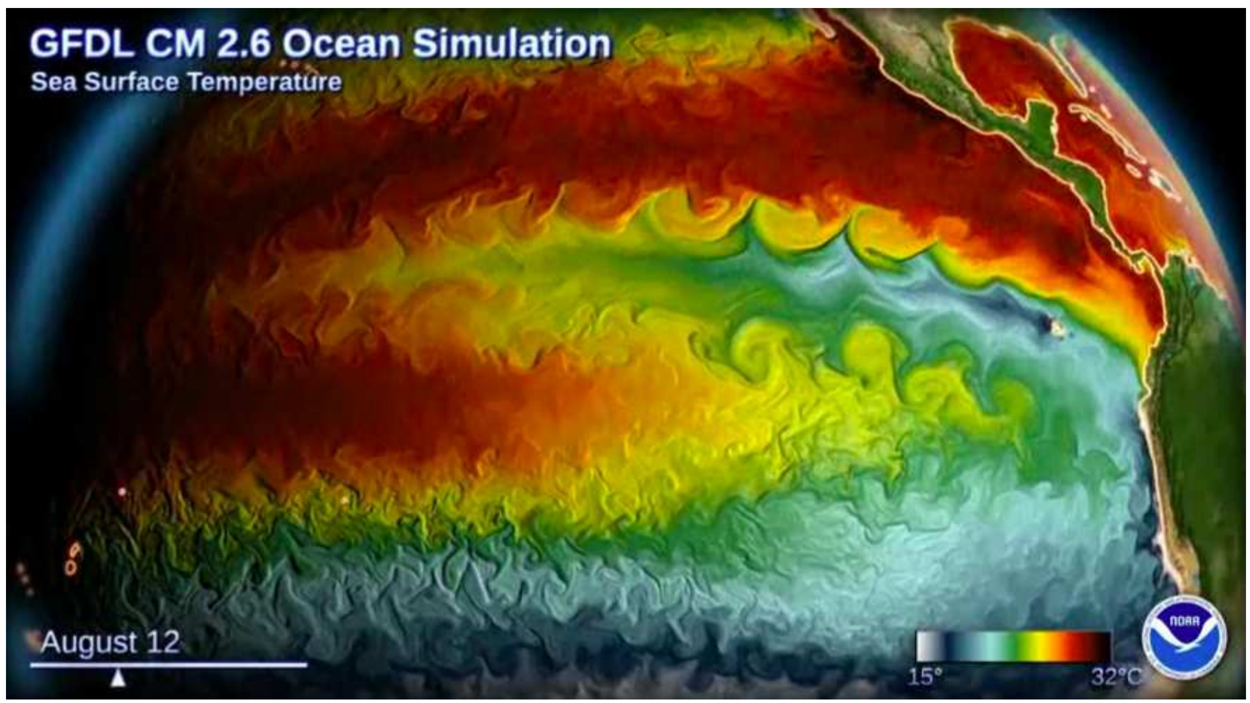
Instabilities of jet streams



Developed instabilities of idealized jet



Tropical instability waves



Barotropic instability is *horizontal-shear instability* of geophysical flows.
What is *necessary condition* for this instability?
Let's consider 1.5-layer QG PV model configured in a zonal channel $(-L<y<+L)$ and linearized around some zonally uniform and meridionally sheared background flow $U(y)$:

$$\left(\frac{\partial}{\partial t}+U(y)\,\frac{\partial}{\partial x}\right)\left[\nabla^2\psi-\frac{1}{R^2}\,\psi\right]+\frac{\partial\psi}{\partial x}\,\frac{d\Pi}{dy}=0\,,\qquad \frac{d\Pi}{dy}=\beta-\frac{d^2U}{dy^2}+\frac{U}{R^2}\,,$$

where Π is the background potential vorticity. Let's look for the usual wave solution:

$$\psi\sim\phi(y)\,e^{ik(x-ct)},\quad c=c_r+i\,\frac{\omega_i}{k}\qquad\rightarrow\qquad (U-c)\Big(-k^2\phi+\phi_{yy}-\frac{1}{R^2}\,\phi\Big)+\phi\left(\beta-U_{yy}+\frac{U}{R^2}\right)=0$$

$$\rightarrow\qquad \boxed{\phi_{yy}-\phi\left(k^2+\frac{1}{R^2}\right)+\phi\,\frac{d\Pi/dy}{U-c}=0}\,.$$

Multiply the last equation by (complex conjugated) ϕ^*, integrate it in y using the simple identity:

$$\phi^*\phi_{yy}=\frac{\partial}{\partial y}\big(\phi^*\phi_y\big)-\phi_y^*\phi_y\,,$$

and take into account that the integral of the y-derivative is zero, because of the boundary conditions on the channel sides:

$$\phi(-L)=\phi(L)=0\,.$$

The resulting integrated equation,

$$\int_{-L}^{L}\!\left(\left|\frac{d\phi}{dy}\right|^2+|\phi|^2\left(k^2+\frac{1}{R^2}\right)\right)dy-\int_{-L}^{L}\!|\phi|^2\,\frac{d\Pi/dy}{U-c}\,dy=0\,,$$

can be written so, that its first integral $[...]$ is real, and the second integral is complex, so that:

$$\rightarrow\qquad [...]+i\,\frac{\omega_i}{k}\int_{-L}^{L}\!|\phi|^2\,\frac{d\Pi/dy}{|U-c|^2}\,dy=0\,.$$

If the last integral is non-zero, then, *necessarily*: $\omega_i=0$, and the normal mode $\phi(y)$ is *neutral*; this results in the following theorem.

Necessary condition for barotropic instability states that ω_i can be nonzero (hence, instability has to occur for $\omega_i>0$), only if the above integral is zero, hence, *ONLY IF* the background PV gradient $d\Pi/dy$ changes sign somewhere in the domain.

Note: this is equivalent to existence of inflection point in the velocity profile in the case of $\beta=0$ and pure 2D dynamics.
The necessary condition is also true for non-zonal parallel flows.

Baroclinic instability is *vertical-shear instability* of geophysical flows.
What is the *necessary condition* for this instability?
Consider a channel with vertically and meridionally sheared but zonally uniform background flow $U(y,z)$; and apply the continuously stratified QG PV model:

$$\Pi = \beta y - \frac{\partial U}{\partial y} - \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\int U(y,z)\,dy\right], \qquad \frac{\partial \Pi}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}\right],$$

where Π is the background potential vorticity. The linearized PV equation is:

$$\boxed{\left(\frac{\partial}{\partial t} + U(y,z)\frac{\partial}{\partial x}\right)\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} = 0} \qquad (*)$$

Conservation of density (sum of dynamic density anomaly and background density) on material particles can be written as (first, in the full form; then, in the *linearized* form):

$$\frac{D_g\,\rho}{Dt} = \frac{D_g\,(\rho_g + \rho_b)}{Dt} = 0 \quad \rightarrow \quad \frac{\partial \rho_g}{\partial t} + \frac{\partial \rho_b}{\partial t} + (U+u)\frac{\partial \rho_g}{\partial x} + (U+u)\frac{\partial \rho_b}{\partial x} + v\frac{\partial(\rho_b+\rho_g)}{\partial y} + w\frac{\partial(\rho_b+\rho_g)}{\partial z} = 0\,.$$

By linearizing out the quadratic terms and taking into account that the background density is stationary and x-independent, we obtain *linearized conservation of density* (i.e., linearized thermodynamic equation for Boussinesq fluid):

$$\boxed{\frac{\partial \rho_g}{\partial t} + U\frac{\partial \rho_g}{\partial x} + v\frac{\partial \rho_b}{\partial y} + w\frac{\partial \rho_b}{\partial z} = 0}\,.$$

Consider this equation on the bottom and top rigid boundaries, hence $w = 0$:

$$\frac{\partial \rho_g}{\partial t} + U\frac{\partial \rho_g}{\partial x} + v\frac{\partial \rho_b}{\partial y} = 0 \qquad \text{at} \qquad z = 0, H\,.$$

Then, in the continuously stratified fluid with background flow, this statement translates into:

$$\rho_g = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}\,, \qquad \rho_b = -\frac{\rho_0 f_0}{g}\frac{\partial}{\partial z}\int(-U)dy$$

$$\Longrightarrow \qquad \boxed{\frac{\partial^2 \psi}{\partial t \partial z} + U\frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial \psi}{\partial x}\frac{\partial U}{\partial z} = 0} \qquad (**)$$

With usual wave solution $\psi \sim \phi(y,z)\,e^{ik(x-ct)}$, the linearized PV equation $(*)$ and boundary conditions $(**)$ become:

$$\boxed{\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \phi}{\partial z}\right) - k^2\phi + \frac{1}{U-c}\frac{\partial \Pi}{\partial y}\phi = 0\,; \qquad (U-c)\frac{\partial \phi}{\partial z} - \frac{\partial U}{\partial z}\phi = 0 \quad \text{at} \quad z = 0, H}$$

Let's multiply the above equation by ϕ^* and integrate over z and y. Vertical integration of the second term involves the boundary conditions:

$$\int_0^H \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \phi}{\partial z}\right)\phi^*\, dz = -\int_0^H \frac{f_0^2}{N^2}\frac{1}{2}\frac{\partial |\phi|^2}{\partial z}\, dz + \left[\frac{f_0^2}{N^2}\frac{\partial \phi}{\partial z}\phi^*\right]_0^H = ... + \left[\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}\frac{|\phi|^2}{U-c}\right]_0^H$$

Taking the above into account, full integration of the ϕ^*-multiplied equation for ϕ yields the following imaginary part equal to zero:

$$\frac{\omega_i}{k}\int_{-L}^{L}\left(\int_0^H \frac{\partial \Pi}{\partial y}\frac{|\phi|^2}{|U-c|^2}\, dz + \left[\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}\frac{|\phi|^2}{|U-c|^2}\right]_0^H\right) dy = 0$$

In the common situation: $\dfrac{\partial U}{\partial z} = 0$ at $z = 0, H$ $\Longrightarrow$ *necessary condition for baroclinic instability* is that $\dfrac{\partial \Pi(y,z)}{\partial y}$ *changes sign at some depth*.

In practice, vertical change of the PV gradient sign always indicates baroclinic instability.

Eady model

This is classical, continuously stratified model of baroclinic instability in atmosphere (Eric Eady was PhD graduate from ICL).

Eady model assumes:

(i) f-plane $(\beta = 0)$,
(ii) linear stratification: $N(z) = const$,
(iii) constant vertical shear: $U(z) = U_0\, z/H$,
(iv) rigid boundaries at $z = 0, H$.

NOTE: Background PV is zero, hence, the necessary condition for baroclinic instability is satisfied.

The linearized continuously stratified QG PV equation and boundary conditions are:

$$\left(\frac{\partial}{\partial t} + \frac{zU_0}{H}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2\psi}{\partial z^2}\right] = 0\,; \qquad \frac{\partial^2\psi}{\partial t\partial z} + \frac{zU_0}{H}\frac{\partial^2\psi}{\partial x\partial z} - \frac{U_0}{H}\frac{\partial\psi}{\partial x} = 0 \quad \text{at} \quad z = 0, H\,.$$

Look for the wave-like solution in horizontal plane to obtain the vertical-structure equation and the corresponding boundary conditions:

$$\psi \sim \phi(z)\, e^{i(k(x-ct)+ly)} \quad \rightarrow \quad \boxed{\left(\frac{zU_0}{H} - c\right)\left[\frac{f_0^2}{N^2}\frac{d^2\phi}{dz^2} - (k^2+l^2)\,\phi\right] = 0\,; \qquad \left(\frac{zU_0}{H} - c\right)\frac{d\phi}{dz} - \frac{U_0}{H}\,\phi = 0 \quad \text{at} \quad z = 0, H} \qquad (*)$$

For $c \neq U_0\,\frac{z}{H}$, we obtain linear ODE with characteristic vertical scale H/μ :

$$H^2\frac{d^2\phi}{dz^2} - \mu^2\,\phi = 0\,, \qquad \mu \equiv \frac{NH}{f_0}\sqrt{k^2+l^2} = R_D^{(1)}\sqrt{k^2+l^2}\,.$$

Look for solution of the above ODE in the exponential form $\phi(z) = A\cosh(\mu z/H) + B\sinh(\mu z/H)$, substitute it in the top and bottom boundary conditions $(*)$ and obtain 2 linear equations for A and B that yield:

$$B = -A\,\frac{U_0}{\mu c}\,, \qquad c^2 - U_0 c + U_0^2\left(\frac{1}{\mu}\coth\mu - \frac{1}{\mu^2}\right) = 0 \quad \rightarrow \quad \boxed{c = \frac{U_0}{2} \pm \frac{U_0}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{1/2}}$$

The second bracket under the square root is always positive, hence, the normal modes grow $(\omega_i > 0)$ if μ satisfies:

$$\boxed{\frac{\mu}{2} < \coth\frac{\mu}{2}}$$

which is the region to the left of the dashed curve (see Figure below).

(a) The maximum growth rate occurs at $\mu = 1.61$, and it is estimated to be $0.31\,U_0/R_D^{(1)}$. Its inverse is *Eady time scale*.

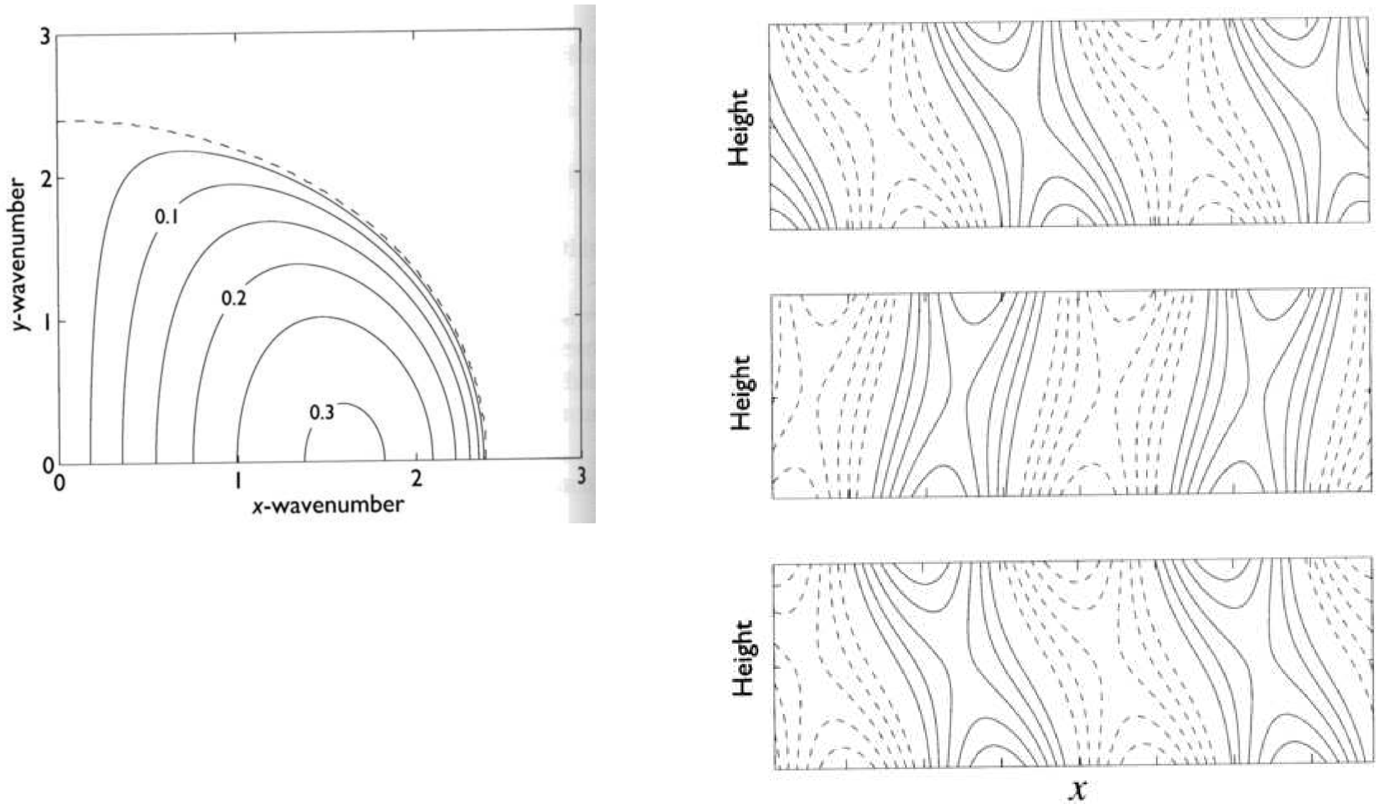
(b) For any k the most unstable wave has $l = 0$; and this wave is characterized by $k_{crit} = 1.6/R_D^{(1)}$. This yields *Eady length scale* $L_{crit} \approx 4R_D^{(1)}$.

NOTE: Both of the corresponding time and length scales are consistent with the observed synoptic scale variability.

(c) Eady solution can be interpreted as a pair of phase-locked edge waves (upper panel: ϕ, middle panel: $\rho = \partial\phi/\partial z$, and bottom panel: $v = \partial\phi/\partial x$).

(d) Assumptions of the Eady model are quite unrealistic, as well as the absense of PV gradients, but nevertheless it is a good starting point for analyses and one of the classical models illustrating the baroclinic instability mechanism.

Figure illustrating Eady's solution in terms of its growth rate and the phase-locked edge waves:

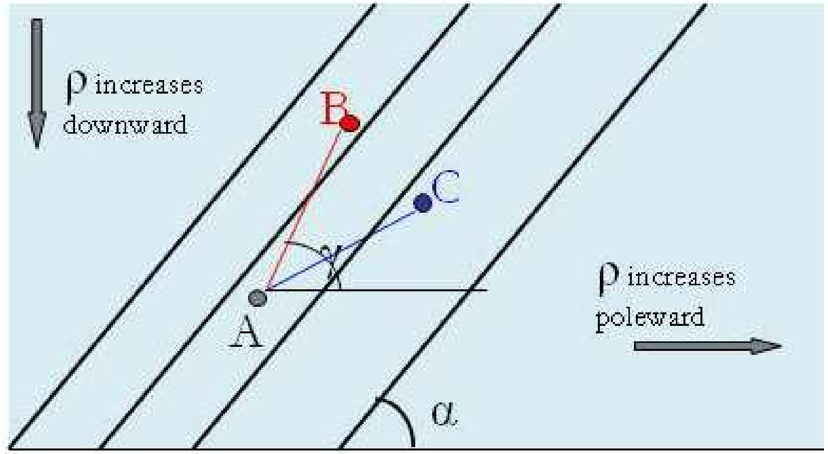


• **Phillips model** is the other classical model of the baroclinic instability mechanism.
It describes two-layer fluid with uniform background zonal velocities U_1 and U_2, and with the β-effect (see Problem Sheet). In this situation background PV gradient is *nonzero*, thus, making the set-up more realistic. New outcomes from solving this problem are:

(a) *Stabilizing effect of* β: Phillips model has *critical shear*, $U_1 - U_2 \sim \beta R_D^2$.

(b) *Westward flows are less stable*: If the upper layer is thinner than the deep layer (ocean-like situation), then the eastward critical shear is larger than the westward one.

Mechanism of baroclinic instability



Baroclinic instability, illustrated by the *Eady* and *Phillips models*, feeds geostrophic turbulence (i.e., synoptic scale variability in the atmosphere and dynamically similar mesoscale eddies in the ocean), and, therefore, it is fundamentally important.

(a) *Available potential energy (APE)* is part of potential energy that can be released as a result of complete isopycnal flattening. Baroclinic instability converts APE of the large-scale background flow into *eddy kinetic energy (EKE)*.

Figure to the right: Consider a fluid particle, initially positioned at A, that migrates to either B or C. If it moves along levels of constant pressure (in QG: streamfunction), then no work is done on the particle $\Longrightarrow$ full mechanical energy of the particle remains unchanged. However, its APE can be converted into EKE, and the other way around.

(b) Consider the following exchanges of fluid particles:

$A \longleftrightarrow B$ leads to *accumulation* of APE (the heavier particle goes "up", and the lighter particle goes "down"),

$A \longleftrightarrow C$ leads, on the opposite, to *release* of APE.

That is, if $\alpha > \gamma$ (steep tilt of isopycnals, relative to tilt of pressure isolines), then APE is released into EKE. This is a situation of the *positive baroclinicity*:

$$\boxed{\nabla p \times \nabla \rho > 0},$$

which implies that the above vector product points out of the Figure, i.e., in positive zonal direction.
This situation routinely happens in geophysical fluids because of the prevailing *thermal winds*.

Thermal wind situation is a consequence of double, geostrophic and hydrostatic balance:

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \qquad f_0 u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \qquad \frac{\partial p}{\partial z} = -\rho g \qquad \Longrightarrow \qquad \boxed{\frac{\partial u}{\partial z} = \frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial y}, \qquad \frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f_0}\frac{\partial \rho}{\partial x}}$$

Consider typical atmospheric thermal wind situation with $\partial p/\partial z < 0$ and $u > 0$; and prove that it is baroclinically unstable (i.e., corresponds to positive baroclinicity):

$$\frac{\partial u}{\partial z} > 0 \quad \text{and} \quad u > 0 \qquad \Longrightarrow \qquad \frac{\partial p}{\partial y} < 0 \quad \text{and} \quad \frac{\partial \rho}{\partial y} > 0 \qquad \Longrightarrow \qquad \boxed{\nabla p \times \nabla \rho = \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y} > 0}.$$

Energetics of barotropically and baroclinically unstable flows

Can we quantify amounts of APE and KE transferred from an unstable flow to the growing perturbations?

In the continuously stratified QG PV model, the kinetic and available potential energy densities of flow perturbations are:

$$K(t,x,y,z) = \frac{|\nabla\psi|^2}{2}\,, \qquad P(t,x,y,z) = \frac{1}{2}\,\frac{f_0^2}{N^2}\Big(\frac{\partial\psi}{\partial z}\Big)^2$$

Let's consider the continuously stratified QG PV equation linearized around some background zonal flow $U(y,z)$:

$$\Big(\frac{\partial}{\partial t} + U(y,z)\,\frac{\partial}{\partial x}\Big)\Big[\nabla^2\psi + \frac{\partial}{\partial z}\Big(\frac{f_0^2}{N^2}\,\frac{\partial\psi}{\partial z}\Big)\Big] + \frac{\partial\psi}{\partial x}\,\frac{\partial\Pi}{\partial y} = 0 \qquad (*)$$

Energy equation is obtained by multiplying $(*)$ with $-\psi$ and, then, by mathematical manipulation (see above QG energetics):

$$\frac{\partial}{\partial t}(K+P) + \nabla\cdot\mathbf{S} - \frac{\partial}{\partial z}\Big[\psi\,\frac{f_0^2}{N^2}\Big(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}\Big)\frac{\partial\psi}{\partial z}\Big] = \frac{\partial\psi}{\partial x}\,\frac{\partial\psi}{\partial y}\,\frac{\partial U}{\partial y} + \frac{\partial\psi}{\partial x}\,\frac{\partial\psi}{\partial z}\,\frac{f_0^2}{N^2}\,\frac{\partial U}{\partial z} \qquad (**)$$

Vertical energy flux is in square brackets on the rhs, and it is due to the form drag arising from isopycnal deformations.

Horizontal energy flux:

$$\mathbf{S} = -\psi\Big(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}\Big)\nabla\psi + \Big[-\frac{\partial\Pi}{\partial y}\,\frac{\psi^2}{2} + U\,(K+P) + \psi\,\frac{\partial\psi}{\partial y}\,\frac{\partial U}{\partial y} + \frac{f_0^2}{N^2}\,\psi\,\frac{\partial\psi}{\partial z}\,\frac{\partial U}{\partial z}\,, \quad 0\Big]$$

Integration of $(**)$ over the domain removes both horizontal and vertical flux divergences, and the total energy equation is obtained:

$$\boxed{\frac{\partial}{\partial t}\iiint(K+P)\,dV = \iiint\frac{\partial\psi}{\partial x}\,\frac{\partial\psi}{\partial y}\,\frac{\partial U}{\partial y}\,dV + \iiint\frac{\partial\psi}{\partial x}\,\frac{\partial\psi}{\partial z}\,\frac{f_0^2}{N^2}\,\frac{\partial U}{\partial z}\,dV} \qquad (***)$$

Energy conversion terms on the rhs of $(***)$ have clear physical interpretations:

(a) *Reynolds-stress energy conversion* term can be written as integral of $-u'v'\,\dfrac{\partial U}{\partial y}$, where primes remind that we deal with the flow fluctuations around $U(y,z)$.

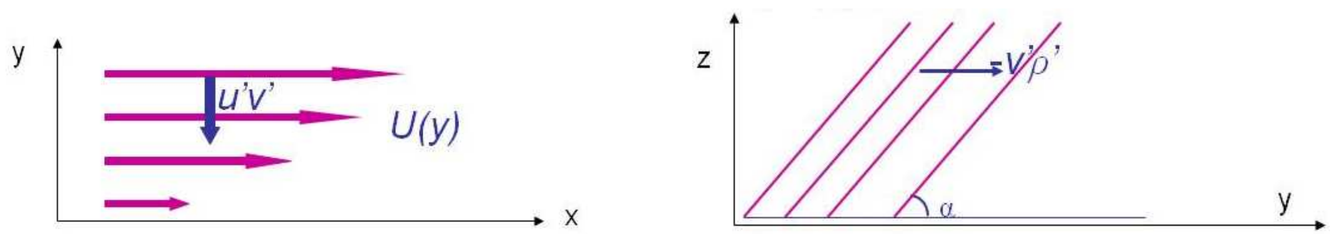
This conversion is positive (and associated with barotropic instability of horizontally sheared flow), if the *Reynolds stress* $u'v'$ acts against the velocity shear (see left panel of Figure below), that is, $u'v' < 0$. In this case the background flow feeds growing instabilities at the rate given by the energy conversion.

(b) *Form-stress energy conversion* term involves the *form stress* $v'\rho'$. The integrand can be rewritten using *thermal wind* relations and

$$\frac{\partial\psi}{\partial z} = -\frac{\rho' g}{\rho_0 f_0}\,, \qquad N^2 = -\frac{g}{\rho_0}\,\frac{d\overline{\rho}}{dz}\,, \qquad \frac{d\overline{\rho}}{dz} < 0\ :$$

$$v'\left(-\frac{\rho' g}{\rho_0 f_0}\right)\frac{f_0^2}{N^2}\left(\frac{g}{\rho_0 f_0}\frac{\partial\overline{\rho}}{\partial y}\right) = v'\rho'\,\frac{g}{\rho_0}\left[\frac{\partial\overline{\rho}}{\partial y}\Big/\frac{d\overline{\rho}}{dz}\right] = \frac{g}{\rho_0}\,v'\rho'\,[-\frac{dz}{dy}] = \frac{g}{\rho_0}\,v'\rho'\,[-\tan\alpha] \approx \frac{g}{\rho_0}\,v'\rho'\,[-\alpha] \sim -v'\rho'$$

This conversion term is positive (and associated with baroclinic instability), if the form stress is negative: $v'\rho'$. This implies flattening of tilted isopycnals (right panel of Figure below shows $-v'\rho'$ and isopycnals; the situation has negative density anomalies moving northward).



AGEOSTROPHIC MOTIONS

(a) Geostrophy filters out all types of (relatively fast) waves, which are important for many geophysical processes.

(b) Geostrophy doesn't work near the equator (where: $f = 0$), because the Coriolis force becomes too small.

Let's consider, first, *gravity waves* and, then, *equatorial waves*, that are both important ageostrophic fluid motions.

Linearized shallow-water model.

Let's consider a layer of fluid with constant density, f-plane approximation, and deviations of the free surface η :

$$\frac{\partial u}{\partial t} - f_0 v = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v}{\partial t} + f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad p = -\rho_0 g\,(z-\eta)\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

The last equation can be vertically integrated, using the linearized kinematic boundary condition on the free surface:

$$w(z=h) = \frac{\partial \eta}{\partial t} \qquad \rightarrow \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,; \qquad\qquad (*)$$

alternatively this equation can be obtained by linearization of the shallow-water continuity equation.

Take *curl of the momentum equations*, substitute the velocity divergence taken from $(*)$ into the Coriolis term and obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{f_0}{H}\,\frac{\partial \eta}{\partial t} = 0 \qquad\qquad (**)$$

Take *divergence of the momentum equations*, substitute the velocity divergence taken from $(*)$ in the tendency term and obtain:

$$\frac{1}{H}\,\frac{\partial^2 \eta}{\partial t^2} + f_0\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - g\nabla^2\eta = 0 \qquad\qquad (***)$$

By differentiating $(***)$ with respect to time and by substituting vorticity from $(**)$, we obtain:

$$\frac{\partial}{\partial t}\left[\nabla^2\eta - \frac{1}{c_0^2}\,\frac{\partial^2\eta}{\partial t^2} - \frac{f_0^2}{c_0^2}\,\eta\right] = 0\,, \qquad c_0^2 \equiv gH$$

Let's integrate this equation in time and choose the integration constant so, that $\eta = 0$ is a solution.
The resulting *free-surface evolution equation* is also known as the *Klein-Gordon equation*:

$$\boxed{\nabla^2\eta - \frac{1}{c_0^2}\,\frac{\partial^2\eta}{\partial t^2} - \frac{f_0^2}{c_0^2}\,\eta = 0} \qquad\qquad (****)$$

This equation needs lateral boundary conditions, which are to be obtained from the velocity boundary conditions.

Velocity-component equations. Let's take the u-momentum equation, differentiate it with respect to time, and add it to the v-momentum equation multiplied by f_0 ; similarly, let's take time derivative of the v-momentum equation and subtract from it the u-momentum equation multiplied by f_0 :

$$\boxed{\frac{\partial^2 u}{\partial t^2} + f_0^2 u = -g\left(\frac{\partial^2 \eta}{\partial x \partial t} + f_0 \frac{\partial \eta}{\partial y}\right), \qquad \frac{\partial^2 v}{\partial t^2} + f_0^2 v = -g\left(\frac{\partial^2 \eta}{\partial y \partial t} - f_0 \frac{\partial \eta}{\partial x}\right)}.$$

Let's consider solid boundary at $x=0$ (ocean west coast). On the boundary: $u = 0$, therefore, the *free-surface boundary condition* is:

$$\boxed{\frac{\partial^2 \eta}{\partial x \partial t} + f_0 \frac{\partial \eta}{\partial y} = 0 \quad \text{at} \quad x = 0}.$$

Let's now look for the wave solution $\eta = \tilde{\eta}(x)\, e^{i(ly-\omega t)}$ of both $(***\!*)$ and the above boundary condition:

$$\frac{d^2\tilde{\eta}}{dx^2} + \left[\frac{\omega^2}{c_0^2} - \frac{f_0^2}{c_0^2} - l^2\right]\tilde{\eta} = 0\,, \qquad -\frac{\omega}{f_0}\frac{d\tilde{\eta}}{dx}(0) + l\,\tilde{\eta}(0) = 0\,.$$

The main equation can be written as:

$$\frac{d^2\tilde{\eta}}{dx^2} = \lambda^2\tilde{\eta}\,, \qquad \text{where (dispersion relation):} \quad \lambda^2 = -\frac{\omega^2}{c_0^2} + \frac{f_0^2}{c_0^2} + l^2 \qquad \rightarrow \qquad \tilde{\eta} = e^{-\lambda x}$$

It supports solutions that are either oscillatory (imaginary λ) or decaying (real λ) in x. Let's consider them separately.

• **Poincare (gravity-inertial) waves** are the *oscillatory* solutions in x :

$$\lambda = ik\,, \quad \boxed{\tilde{\eta} = A\cos kx + B\sin kx}, \qquad x = 0: \;\; A = B\,\frac{k\omega}{l f_0}\,, \qquad \boxed{\omega^2 = f_0^2 + c_0^2\,(k^2 + l^2)}$$

(a) Dispersion relation of these dispersive waves can be visualized by paraboloids with cut-off frequency of f_0.

(b) These are *very fast surface gravity waves*: For wavelength ~ 1000 km and $H \sim 5$ km, the phase speed is $c_0 = \sqrt{gH} \sim 300$ m s^{-1} (compare this tsunami-like speed to the slow speed ~ 0.2 m s^{-1} for the oceanic baroclinic Rossby wave).

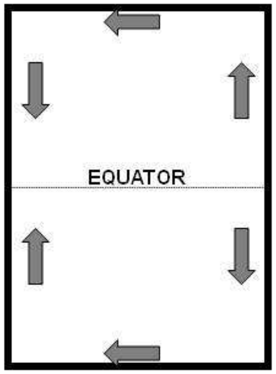
(c) In the *long-wave limit*: $\omega = f_0$. These waves are called the *inertial oscillations*; they are characterized by circular motions (see Problem Sheet).

(d) In the *short-wave limit*, the effects of rotation vanish, and these are the common (nondispersive) *non-rotating shallow-water surface gravity* waves (note their difference from the deep-ocean waves considered in the Problem Sheet!).

(e) Poincare waves are *isotropic*: their propagation properties are the same in any direction (in the flat-bottom f-plane case that we considered).

• **Kelvin waves** are *exponentially decaying* solutions (i.e., *edge waves*!); on the western (eastern) boundary they correspond to different signs of k (let's take $k > 0$) :

$$\lambda = k \quad (= -k)\,, \qquad \tilde{\eta} = Ae^{-kx} \quad (= Ae^{kx})\,, \qquad x = 0:\ k = -\frac{f_0 l}{\omega} \quad \left(= \frac{f_0 l}{\omega}\right) \qquad (*)$$



In the northern hemisphere, positive k at the western wall implies $C_p^{(y)} = \omega/l < 0,$ hence the Kelvin wave propagates to the south. Thus, the meridional phase speed is northward at the eastern wall and southward at the western wall, that is, the coast is always to the right of the Kelvin wave propagation direction. Note, that f_0 changes sign in the southern hemisphere, and this modifies the Kelvin wave so, that it has the coast always to the left (see Figure).

With $(*)$ used to get rid of $k,$ the Kelvin wave dispersion relation becomes:

$$\boxed{(\omega^2 - f_0^2)\left(1 - \frac{c_0^2}{\omega^2}\,l^2\right) = 0}\,.$$

Its first root, $\omega = \pm f_0,$ is just another class of *inertial oscillations*.

Its second root corresponds to the (nondispersive) *Kelvin wave* exponentially decaying away from the boundary:

$$\omega = \mp c_0 l\,, \quad k = \pm\frac{f_0}{c_0} \qquad \Longrightarrow \qquad \boxed{\eta = Ae^{\pm x f_0/c_0}\, e^{i(ly \mp c_0 lt)}}$$

Substitute this into the rhs of the normal-to-boundary velocity equation, and discover that this velocity component is zero everywhere:

$$\frac{\partial^2 u}{\partial t^2} + f_0^2 u = -g\left(\frac{\partial^2 \eta}{\partial x \partial t} + f_0\,\frac{\partial \eta}{\partial y}\right) = 0 \qquad \to \qquad u = Ae^{if_0 t} \qquad \to \qquad A = 0 \qquad \Longrightarrow \qquad \boxed{u = 0}\,, \qquad (*)$$

because at the boundary it is always true that $u(t, 0, y) = 0.$ Note, that this equation has oscillatory solutions, but they are not allowed by the boundary condition.

Because of $(*),$ the along-wall velocity component of the Kelvin wave is in the geostrophic balance:

$$\frac{\partial u}{\partial t} - f_0\, v = -g\,\frac{\partial \eta}{\partial x} \qquad \Longrightarrow \qquad \boxed{-f_0\, v = -g\,\frac{\partial \eta}{\partial x}}\,,$$

hence, Kelvin wave is a boundary-trapped hybrid that is simultaneously *ageostrophic* (gravity) and *geostrophic* wave.

(a) There are Kelvin waves running around islands (in the proper direction); they are often phase-locked to tides.

(b) Kelvin waves can be further subdivided into the barotropic and baroclinic modes.

• **Geostrophic adjustment** is a powerful and ubiquitous process, in which fluid in an initially unbalanced state by radiating gravity waves naturally evolves toward a state of geostrophic balance.
Let's focus on the *linearized shallow-water* dynamics, which contains both geostrophically balanced and unbalanced motions:

$$\frac{\partial u}{\partial t} - f_0 v = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v}{\partial t} + f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,,$$

and consider a manifestly unbalanced initial state: discontinuity in free-surface height.
In non-rotating flow any initial disturbance will be radiated away by the gravity waves, characterized by phase speed $c_0 = \sqrt{gH}$, and the final state will be the *state of rest*. In rotating fluid there is geostrophic balance that can trap the fluid in it, because it has absolutely no time dependence!

Effect of rotation is crucial for geostrophic adjustment, because:

(a) PV conservation provides a powerful constraint on the fluid evolution;

(b) There is fully adjusted steady state which is not the state of rest.

Let's start with the corresponding PV description of the dynamics:

$$\frac{\partial \Pi}{\partial t} + \mathbf{u}\cdot\nabla\Pi = 0\,, \qquad \Pi = \frac{\zeta + f_0}{h} = \frac{\zeta + f_0}{H + \eta} = \frac{(\zeta + f_0)/H}{1 + \eta/H}\,,$$

and *linearize both PV and its conservation law*:

$$\Pi_{LIN} \approx \frac{1}{H}\,(\zeta + f_0)\Big(1 - \frac{\eta}{H}\Big) \approx \frac{1}{H}\,\Big(\zeta + f_0 - \frac{f_0\eta}{H}\Big) \qquad \Longrightarrow \qquad \boxed{q = \zeta - f_0\,\frac{\eta}{H}\,, \qquad \frac{\partial q}{\partial t} = 0}$$

where q is the linearized PV anomaly. Note that q remains locally unchanged.

Let's consider a discontinuity in fluid height: $\eta(x, 0) = +\eta_0\,,\quad x < 0\,; \qquad \eta(x, 0) = -\eta_0\,,\quad x > 0\,.$

The initial distribution of the linearized PV anomaly is:

$$q(x, y, 0) = -f_0\,\frac{\eta_0}{H}\,, \quad x < 0\,; \qquad q(x, y, 0) = +f_0\,\frac{\eta_0}{H}\,, \quad x > 0\,.$$

During the *geostrophic adjustment* process, the height discontinuity will become smeared out into a slope by radiating gravity waves; through the geostrophic balance this slope must maintain a geostrophic flow current that will necessarily emerge during the adjustment process.
First, let's introduce the final-state geostrophic flow streamfunction:

$$f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad f_0 v = g\,\frac{\partial \eta}{\partial x} \qquad \rightarrow \qquad \Psi \equiv \frac{g\eta}{f_0}\,.$$

Since PV is conserved on the fluid particles, the particles are only redistributed along the y-axis (this is based on physical reasoning; alternative argument comes from the symmetry of the problem). The final steady state is the solution of the equation described by monotonically changing $\Psi \sim \eta$ and sharp jet concentrated along this slope:

$$\zeta - f_0\,\frac{\eta}{H} = q(x,y) \quad \Longrightarrow \quad \left(\nabla^2 - \frac{1}{R_D^2}\right)\Psi = q(x,y)\,, \qquad R_D = \frac{\sqrt{gH}}{f_0} \quad \Longrightarrow \quad \frac{\partial^2\Psi}{\partial x^2} - \frac{1}{R_D^2}\,\Psi = \frac{f_0\eta_0}{H}\,sign(x)$$

$$\Longrightarrow \qquad \Psi = -\frac{g\eta_0}{f_0}\,(1 - e^{-x/R_D})\,, \qquad x > 0\,; \qquad \Psi = +\frac{g\eta_0}{f_0}\,(1 - e^{+x/R_D})\,, \qquad x < 0$$

$$\Longrightarrow \qquad \boxed{u = 0\,, \qquad v = -\frac{g\eta_0}{f_0 R_D}\,e^{-|x|/R_D}\,, \qquad \eta = \frac{f_0}{g}\,\Psi}$$

(a) PV constrains adjustment within the deformation radius from the initial disturbance.

(b) Excessive initial energy (which can be estimated; see Problem Sheet) is radiated away by gravity waves. The underlying processes which transfer energy from (initially) unbalanced flows to gravity waves remain poorly understood.

• **Equatorial waves** are special class of linear waves populating the equatorial zone.
Let's assume the *equatorial β-plane* and write the momentum, continuity, and PV equations (and recall that $c_0 = gH$):

$$\frac{\partial u}{\partial t} - \beta y v = -g\,\frac{\partial \eta}{\partial x} \qquad \times\left[-\frac{\beta y}{c_0^2}\,\frac{\partial}{\partial t}\right] \quad \rightarrow \quad -\frac{\beta y}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\beta^2 y^2}{c_0^2}\,\frac{\partial v}{\partial t} = \frac{g\beta y}{c_0^2}\,\frac{\partial^2 \eta}{\partial x \partial t} = \frac{\beta y}{H}\,\frac{\partial^2 \eta}{\partial x \partial t} \qquad (*)$$

$$\frac{\partial v}{\partial t} + \beta y u = -g\,\frac{\partial \eta}{\partial y} \qquad \times\left[\frac{1}{c_0^2}\,\frac{\partial^2}{\partial t^2}\right] \quad \rightarrow \quad \frac{1}{c_0^2}\,\frac{\partial^3 v}{\partial t^3} + \frac{\beta y}{c_0^2}\frac{\partial^2 u}{\partial t^2} = -\frac{g}{c_0^2}\,\frac{\partial^3 \eta}{\partial y \partial t^2} = -\frac{1}{H}\,\frac{\partial^3 \eta}{\partial y \partial t^2} \qquad (**)$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad \times\left[-\frac{1}{H}\,\frac{\partial^2}{\partial y \partial t}\right] \quad \rightarrow \quad -\frac{1}{H}\,\frac{\partial^3 \eta}{\partial y \partial t^2} - \frac{\partial^2}{\partial y \partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (***)$$

$$\frac{\partial}{\partial t}\left(\zeta - \frac{\beta y}{H}\,\eta\right) + \beta v = 0 \qquad \times\left[-\frac{\partial}{\partial x}\right] \quad \rightarrow \quad -\frac{\partial^2}{\partial x \partial t}\left(\zeta - \frac{\beta y}{H}\,\eta\right) - \beta\frac{\partial v}{\partial x} = 0 \qquad (****)$$

Add up $(*)$ and $(**)$, and use $(***)$ and $(****)$ to get rid of η :

$$\frac{1}{c_0^2}\,\frac{\partial^3 v}{\partial t^3} + \frac{\beta^2 y^2}{c_0^2}\,\frac{\partial v}{\partial t} = \frac{\partial^2 \zeta}{\partial x \partial t} + \beta\,\frac{\partial v}{\partial x} + \frac{\partial^2}{\partial y \partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Substitute $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ to obtain the *meridional-velocity equation*:

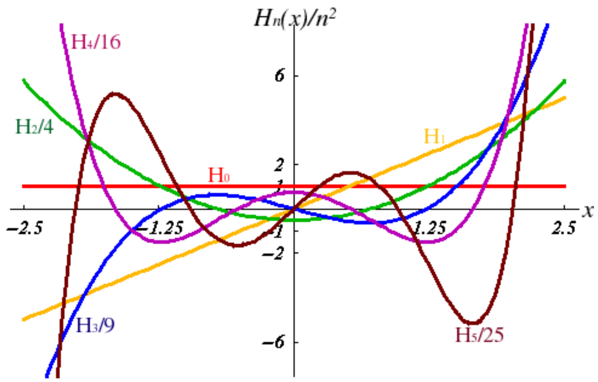
$$\boxed{\frac{\partial}{\partial t}\left[\frac{1}{c_0^2}\left(\frac{\partial^2 v}{\partial t^2} + (\beta y)^2\, v\right) - \nabla^2 v\right] - \beta\,\frac{\partial v}{\partial x} = 0}$$

Let's look for the wave solution:

$$v = \tilde{v}(y)\, e^{i(kx - \omega t)} \qquad \Longrightarrow$$

$$\frac{d^2 \tilde{v}}{dy^2} + \tilde{v}\left[\frac{\omega^2}{c_0^2} - k^2 - \frac{(\beta y)^2}{c_0^2} - \frac{\beta k}{\omega}\right] = 0 \qquad (\bullet)$$

Solutions of this inhomogeneous ODE are symmetric around the equator and are given by the set of *Hermite polynomials* H_n, which multiply the exponential:



$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48x^2 + 12\\
H_5(x) &= 32x^5 - 160x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x
\end{aligned}$$

$$\tilde{v}_n(y) = A_n\, H_n\!\left(\frac{y}{L_{eq}}\right)\, \exp\left[-\frac{1}{2}\left(\frac{y}{L_{eq}}\right)^2\right],$$

where $L_{eq} = \sqrt{c_0/\beta}$ is called the *equatorial barotropic radius of deformation* (~ 3000 km; the equatorial baroclinic deformation radii are much shorter and can be obtained by considering a multi-layer problem and projecting it on the vertical modes).

Let's obtain the dispersion relation by recalling the following recurrence relations for the Hermite polynomials:

$$H'_n = 2nH_{n-1}\,, \qquad H'_{n-1} = 2yH_{n-1} - H_n\,,$$

and by considering $v_n = H_n \exp[-y^2/2]$:

$$v'_n = (H'_n - yH_n)\, e^{-y^2/2} = (2nH_{n-1} - yH_n)\, e^{-y^2/2}\,, \qquad v''_n = \Big(2nH'_{n-1} - H_n - yH'_n - y(2nH_{n-1} - yH_n)\Big)\, e^{-y^2/2} = -(2n+1-y^2)H_n\, e^{-y^2/2}$$

$$\Longrightarrow \quad v''_n + (2n+1-y^2)\, v_n = 0 \qquad (\bullet\bullet)$$

Now, let's consider $(\bullet)$ and nondimensionalize y by L_{eq} :

$$L_{eq}^{-2}\,\frac{d^2\tilde{v}}{dy^2} + \tilde{v}\left[\left(\frac{\omega^2}{c_0^2} - k^2 - \frac{\beta k}{\omega}\right) - \frac{y^2}{L_{eq}^2}\right] = 0 \qquad \rightarrow \qquad \frac{d^2\tilde{v}}{dy^2} + \tilde{v}\left[L_{eq}^2\left(\frac{\omega^2}{c_0^2} - k^2 - \frac{\beta k}{\omega}\right) - y^2\right] = 0$$

By comparing the last equation with $(\bullet\bullet)$, we obtain the resulting *dispersion relation for equatorial waves*:

$$\boxed{\omega_n^2 = c_0^2\left(k^2 + \frac{(2n+1)}{L_{eq}^2}\right) + \frac{\beta k}{\omega_n}\, c_0^2}$$

Let's now analyze this dispersion relation by considering its frequency limits and effects of lateral boundaries:

(a) If ω_n is large, then: $\quad \omega_n^2 = c_0^2\left(k^2 + \dfrac{(2n+1)}{L_{eq}^2}\right)$.

This is identical to the dispersion relation for midlatitude *Poincare waves*, if we take $f_0 = 0$ and $l = \sqrt{2n+1}/L_{eq}$.

(b) If ω_n is small, then: $\quad \omega_n = -\dfrac{\beta k}{k^2 + (2n+1)/L_{eq}^2}$.

This is identical to the dispersion relation for midlatitude *Rossby waves*, if we take $l = \sqrt{2n+1}/L_{eq}$.

(c) *Mixed Rossby-gravity (Yanai) wave* corresponds to $n = 0$. It behaves like Rossby/gravity wave for low/high frequencies.

(d) *Equatorial Kelvin wave* is the edge wave for which equator plays role of solid bondary.
Let's take $v = 0$, and use $(*)$, $(***)$, and $(****)$:

$$\frac{\partial u}{\partial t} = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial \eta}{\partial t} + H\,\frac{\partial u}{\partial x} = 0\,, \qquad (\star)$$

$$-\frac{\partial^2 u}{\partial t\,\partial y} + \beta y\,\frac{\partial u}{\partial x} = 0 \qquad (\star\star)$$

From $(\star)$ we obtain the zonal-velocity equation and its *D'Alembert solution*:

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \qquad u = A\,G_-(x - c_0 t, y) + B\,G_+(x + c_0 t, y)\,,$$

and notice, that this solution has to satisfy the PV constraint $(\star\star)$. Substitute the D'Alembert solution in $(\star\star)$, introduce pair of propagating-wave variables $\xi = x \pm c_0 t$, and recall that $L_{eq} = \sqrt{c_0/\beta}$:

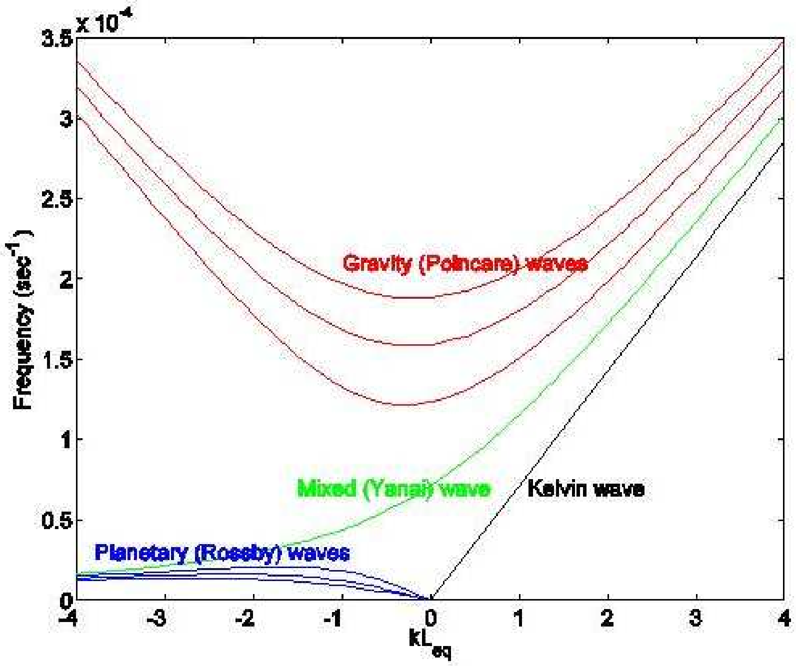
$$\frac{\partial}{\partial \xi}\Big(-c_0 \frac{\partial G_-}{\partial y} - \beta y\, G_-\Big) = 0\,, \quad \frac{\partial}{\partial \xi}\Big(c_0 \frac{\partial G_+}{\partial y} - \beta y\, G_+\Big) = 0 \quad \rightarrow \quad -c_0 \frac{\partial G_-}{\partial y} - \beta y\, G_- = 0\,, \quad c_0 \frac{\partial G_+}{\partial y} - \beta y\, G_+ = 0$$

$$G_- = A_-\, e^{-\frac{1}{2}(y/L_{eq})^2}\, F_-(\xi)\,, \quad G_+ = A_+\, e^{\frac{1}{2}(y/L_{eq})^2}\, F_+(\xi) \quad \rightarrow \quad G_- = A_-\, e^{-\frac{1}{2}(y/L_{eq})^2}\, F_-(x - c_0 t)\,, \quad G_+ = A_+\, e^{\frac{1}{2}(y/L_{eq})^2}\, F_+(x + c_0 t)$$

Only G_- remains finite away from the equator, hence, $A_+ = 0$, and Kelvin wave given by G_- *propagates only to the east*.

Vertical modes: In continuously stratified case, the flow solution can be split in a set of vertical baroclinic modes. Each baroclinic mode has its own Poincare, Rossby, Yanai and Kelvin waves and dispersion relations.

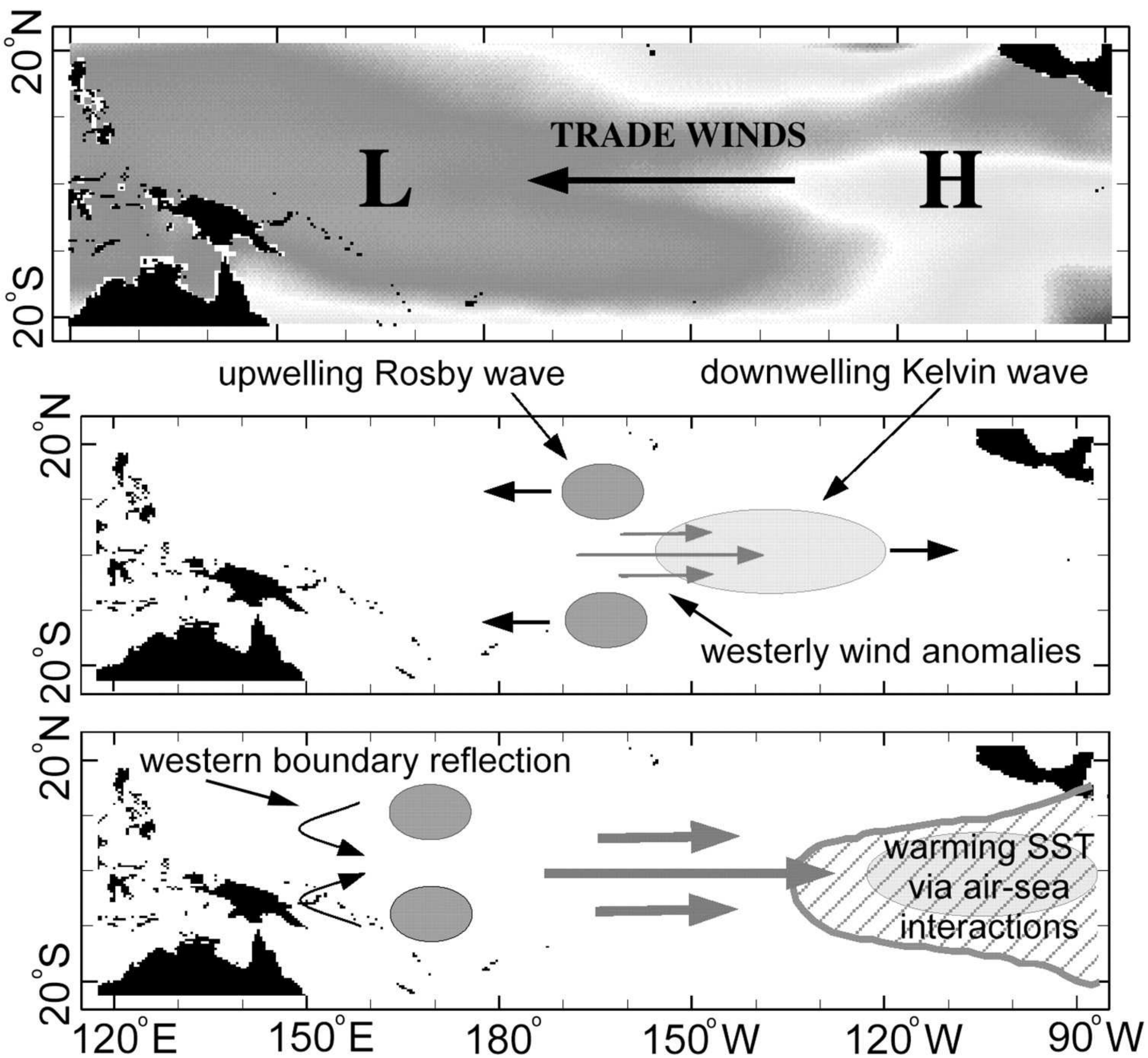
Note: Equatorial waves play key role in the global, coupled ocean-atmosphere phenomenon called ENSO (see later).



Dispersion relation diagram for equatorial waves

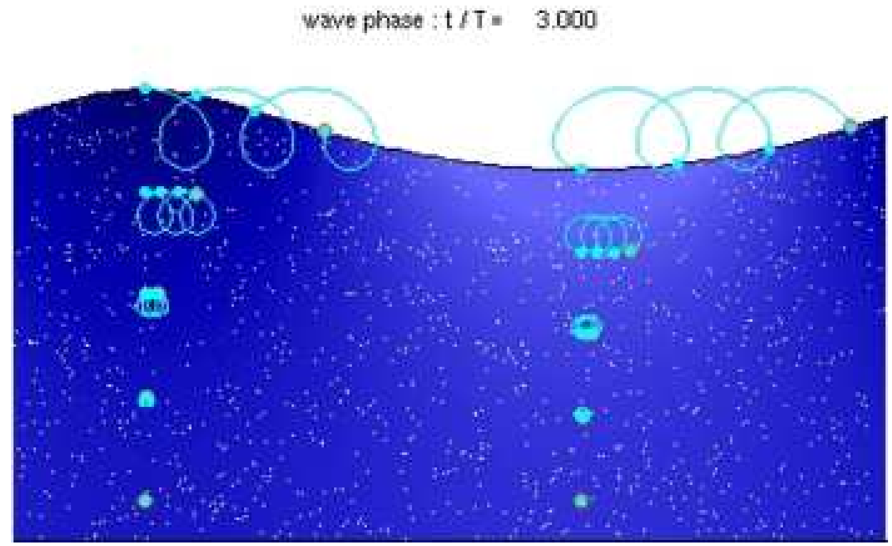
Schematic of El Nino Southern Oscillation (ENSO) “delayed oscillator” mechanism

El Nino and La Nina occur interannually causing extreme floods and droughts in many regions of the world.



- Normal state is perturbed; weakening of trade winds
- “Warm” Kelvin wave radiates to the east and “cold” Rossby wave radiates to the west (*their basin-crossing times are about 70 and 220 days*).
- When Kelvin wave reaches the boundary, it warms the upper ocean and “*El Nino*” phenomenon occurs.
- “Cold” Rossby wave reflects from the western boundary as “cold” Kelvin wave; then, it propagates to the east, terminates *El Nino*, and initiates “*La Nina*” event.

MATERIAL TRANSPORT PHENOMENA



Stokes drift

This is nonlinear phenomenon that illustrates the difference between *average Lagrangian velocity* (i.e., velocity estimated following fluid particles) and *average Eulerian velocity* (i.e., velocity estimated at fixed spatial positions).

Essential physics: Stokes drift may occur only when the flow is both *time-dependent* and *spatially inhomogeneous*.

Let's consider the text-book example of deep-water linear gravity waves (see Figure and Problem Sheet) and derive the *Stokes drift velocity*.

Lagrangian motion of a fluid particle is described by kinematics:

$$\mathbf{x} = \xi(\mathbf{a}, t)\,, \qquad \frac{\partial \xi}{\partial t} = \mathbf{u}(\xi, t)\,, \qquad \xi(\mathbf{a}, 0) = \mathbf{a}\,,$$

where $\mathbf{u}$ is the Eulerian velocity (at a fixed position), and $\partial\xi/\partial t$ is the Lagrangian velocity (found along the particle trajectory). Let's compare time averages of these velocities (denoted by overlines) and assume they are not the same (i.e., time averages along a trajectory and at a point do not have to coincide):

$$\overline{\mathbf{u}}_E = \overline{\mathbf{u}(\mathbf{x}, t)}\,, \qquad \overline{\mathbf{u}}_L = \overline{\frac{\partial \xi(\mathbf{a}, t)}{\partial t}} = \overline{\mathbf{u}(\xi(\mathbf{a}, t), t)} \qquad \rightarrow \qquad \mathbf{u}_S = \overline{\mathbf{u}}_L - \overline{\mathbf{u}}_E\,,$$

where Stokes drift velocity is the difference between the Lagrangian and Eulerian average velocities.

Let's now consider a sinusoidal plane wave on the free surface of fluid: $\eta = A\cos(kx - \omega t)$. The corresponding interior flow solution (see Problem Sheet) is given in terms of the velocity potential ϕ, which is harmonic (i.e., $\nabla^2\phi = 0$); and the corresponding (nonlinear) dispersion relation of the deep-water waves:

$$\phi = A\,\frac{\omega}{k}\,e^{kz}\sin(kx - \omega t)\,, \qquad \omega^2 = gk\,.$$

Let's focus on the horizontal ξ_x and vertical ξ_z components of the Lagrangian position vector ξ and write down Lagrangian velocity components:

$$\frac{\partial \xi_x}{\partial t} = \frac{\partial \phi}{\partial x}\,, \qquad \frac{\partial \xi_z}{\partial t} = \frac{\partial \phi}{\partial z}\,.$$

Let's integrate Lagrangian trajectory near some point $\mathbf{x} = (x, z)$. Within the linear theory this yields:

$$\boxed{\xi_x = x + \int \frac{\partial \phi}{\partial x}\,dt = x - A\,e^{kz}\sin(kx - \omega t)\,, \qquad \xi_z = z + \int \frac{\partial \phi}{\partial z}\,dt = z + A\,e^{kz}\cos(kx - \omega t)\,.}$$

The central idea is to calculate *Lagrangian velocity on trajectory* by Taylor-expanding the Eulerian velocity field around the reference position $\mathbf{x}$. We focus only on x-direction (here, direction of wave propagation):

$$\overline{u}_S = \overline{u(\xi,t)} - \overline{u(\mathbf{x},t)} = \overline{\left[u(\mathbf{x},t) + (\xi_x - x)\,\frac{\partial u(\mathbf{x},t)}{\partial x} + (\xi_z - z)\,\frac{\partial u(\mathbf{x},t)}{\partial z} + ...\right]} - \overline{u(\mathbf{x},t)}$$

$$\approx \overline{(\xi_x - x)\,\frac{\partial^2 \phi(\mathbf{x},t)}{\partial x^2}} + \overline{(\xi_z - z)\,\frac{\partial^2 \phi(\mathbf{x},t)}{\partial x \partial z}} = ...$$

$$= \overline{[-A\,e^{kz}\sin(kx-\omega t)]\,[-\omega k A\,e^{kz}\sin(kx-\omega t)]} + \overline{[A\,e^{kz}\cos(kx-\omega t)]\,[\omega k A\,e^{kz}\cos(kx-\omega t)]}$$

$$= \omega k A^2 e^{2kz}\overline{[\sin^2(kx-\omega t) + \cos^2(kx-\omega t)]} = \omega k A^2 e^{2kz} \qquad \Longrightarrow \qquad \boxed{\overline{u}_S = \frac{4\pi^2 A^2}{\lambda T}\, e^{4\pi z/\lambda}}$$

(a) Stokes drift speed $\overline{u}_S$ is a nonlinear (quadratic) quantity in terms of the wave amplitude A.

(b) Stokes drift decays exponentially with depth and depends on frequency and wavenumber of the flow fluctuations.

(c) *Darwin drift* (permanent displacement of mass after the passage of a body through a fluid) is a related phenomenon.

Homogeneous turbulent diffusion

This is a theory for describing dispersion of passive tracer (or Lagrangian particles) by spatially homogeneous and stationary turbulence; let's also for simplicity assume that the turbulence is isotropic.

Take C as passive tracer concentration, and $\mathbf{u}$ as turbulent velocity field.
Let's consider large-scale (coarse-grained) quantities: passive tracer concentration $\overline{C}$ and velocity field $\overline{\mathbf{u}}$; so that the corresponding small-scale (turbulent) fluctuations are C' and $\mathbf{u}'$.
Let's assume the complete *scale separation* between the large and small scales (i.e., $\overline{C'} = 0$ and $\overline{\mathbf{u}'} = 0$) and coarse-grain the advection-diffusion tracer equation by taking its time average:

$$\frac{\partial C}{\partial t} + \mathbf{u}\cdot\nabla C = \text{molecular diffusion } + \text{sources/sinks} \quad \rightarrow \quad \frac{\partial(\overline{C}+C')}{\partial t} + (\overline{\mathbf{u}}+\mathbf{u}')\cdot\nabla(\overline{C}+C') = \ldots$$

$$\rightarrow \quad \frac{\partial \overline{C}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\overline{C} = -\overline{\mathbf{u}'\cdot\nabla C'} + \ldots$$

Can we find a simple mathematical model (parameterization, closure) for the turbulent stress term on the rhs?

Lagrangian point of view on turbulent diffusion. For this purpose let's consider dispersion (i.e., spreading) of an *ensemble of Lagrangian particles*. Concentration of the particles is equivalent to C, and displacement of each particle from its initial position is given by the integral of its Lagrangian velocity:

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{u}_L(t')\, dt'$$

Standard functions characterizing evolution of the Lagrangian particles ensemble are *single-particle dispersion* $D(t)$ and *Lagrangian velocity autocorrelation function* $R(\tau)$. These functions are obtained by *ensemble averaging* (i.e., over many flow realizations), as indicated by angle brackets:

$$\boxed{D(t) \equiv \left\langle (\mathbf{x}(t) - \mathbf{x}(0))^2 \right\rangle}, \qquad \boxed{R(t-t') \equiv \frac{\left\langle \mathbf{u}_L(t)\cdot\mathbf{u}_L(t') \right\rangle}{\left\langle u^2 \right\rangle}}$$

These functions are mathematically connected with each other. Notice, that

$$\int_0^t R(t'-t)\, dt' = \left\langle \left[\mathbf{x}(t') - \mathbf{x}(0)\right]_0^t \frac{\mathbf{u}_L(t)}{u^2} \right\rangle,$$

therefore:

$$\frac{d}{dt} D(t) = 2\left\langle \left[\mathbf{x}(t) - \mathbf{x}(0)\right] \mathbf{u}_L(t) \right\rangle = 2\left\langle u^2 \right\rangle \int_0^t R(t'-t)\, dt' = 2\left\langle u^2 \right\rangle \int_{-t}^0 R(\tau)\, d\tau = 2\left\langle u^2 \right\rangle \int_0^t R(\tau)\, d\tau\,.$$

$$\rightarrow \qquad \boxed{\frac{dD}{dt} = 2\left\langle u^2 \right\rangle \int_0^t R(\tau)\,d\tau} \qquad\qquad (*)$$

Next, recall the formula for differentiation under integral sign,

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt \qquad \Longrightarrow \qquad \frac{d}{dx}\,F(x) = f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\,f(x,t)\,dt\,,$$

and find: $\boxed{D(t) = 2\left\langle u^2 \right\rangle \int_0^t (t-\tau)\,R(\tau)\,d\tau}$ $\qquad (**)$

Prove the above formula by differentiating it and, eventually, obtain $(*)$:

$$\frac{dD}{dt} = 2\left\langle u^2 \right\rangle \Big((t-t)\,R(t) - 0 + \int_0^t R(\tau)\,d\tau \Big)$$

Asymptotic limits: Let's consider the short- and long-time asymptotic limits of $D(t)$ by focusing on $(*)$:

(a) *Ballistic limit*: $t \rightarrow 0$.
Then, $\tau \approx 0, \quad R(\tau) \approx 1 \qquad \Longrightarrow \qquad \boxed{D \sim t^2}$

(b) *Diffusive limit*: $t \rightarrow \infty$.
Introduce *Lagrangian decorrelation time*: $T_L = \int_0^\infty R(\tau)\,d\tau$.

$$\Longrightarrow \qquad \frac{dD}{dt}\Big|_\infty = 2\,T_L\left\langle u^2 \right\rangle \qquad \Longrightarrow \qquad \boxed{D \sim t}$$

In the diffusive limit the area occupied by particles (or passive tracer) grows linearly in time, as in the molecular *diffusion process* with the *eddy diffusivity* equal to:

$$\boxed{\kappa = \left\langle u^2 \right\rangle T_L}$$

Let's prove the diffusion equation analogy by considering the one-dimensional diffusion equation and by focusing on the mean-square displacement of the tracer concentration (it is equivalent to the single-particle dispersion!):

$$\frac{\partial C}{\partial t} = \kappa\,\frac{\partial^2 C}{\partial x^2}\,, \qquad\qquad D(t) \equiv \Big[\int_{-\infty}^{\infty} x^2\,C\,dx\Big]\Big[\int_{-\infty}^{\infty} C\,dx\Big]^{-1}$$

Let's differentiate $D(t)$ and replace tendency term by rhs of the diffusion equation:

$$\frac{\partial D}{\partial t} \sim \frac{\partial}{\partial t}\int_{-\infty}^{\infty} x^2\, C\, dx = \kappa \int_{-\infty}^{\infty} x^2\, \frac{\partial^2 C}{\partial x^2}\, dx = (by\ parts) = 2\kappa \int_{-\infty}^{\infty} C\, dx = 2\kappa$$

Thus, in the diffusion process analogy, the tracer-containing area grows linearly in time.

NOTE: the same diffusion process analogy in 2D and 3D cases yields 4κ and 6κ on the rhs, respectively.

NONLINEAR DYNAMICS AND WAVE-MEAN FLOW INTERACTIONS

Nonlinear flow interactions become fundamentally important when growing flow instabilities reach significant amplitude and become finite-amplitude *nonlinear eddies and currents*.

• *Weakly nonlinear analysis* can predict slowly evolving amplitude of nearly monochromatic nonlinear waves through derivation of an *amplitude equation*.

• *Dynamical systems* framework (bifurcations, attractors, etc.) can be useful for describing *transition to turbulence*.

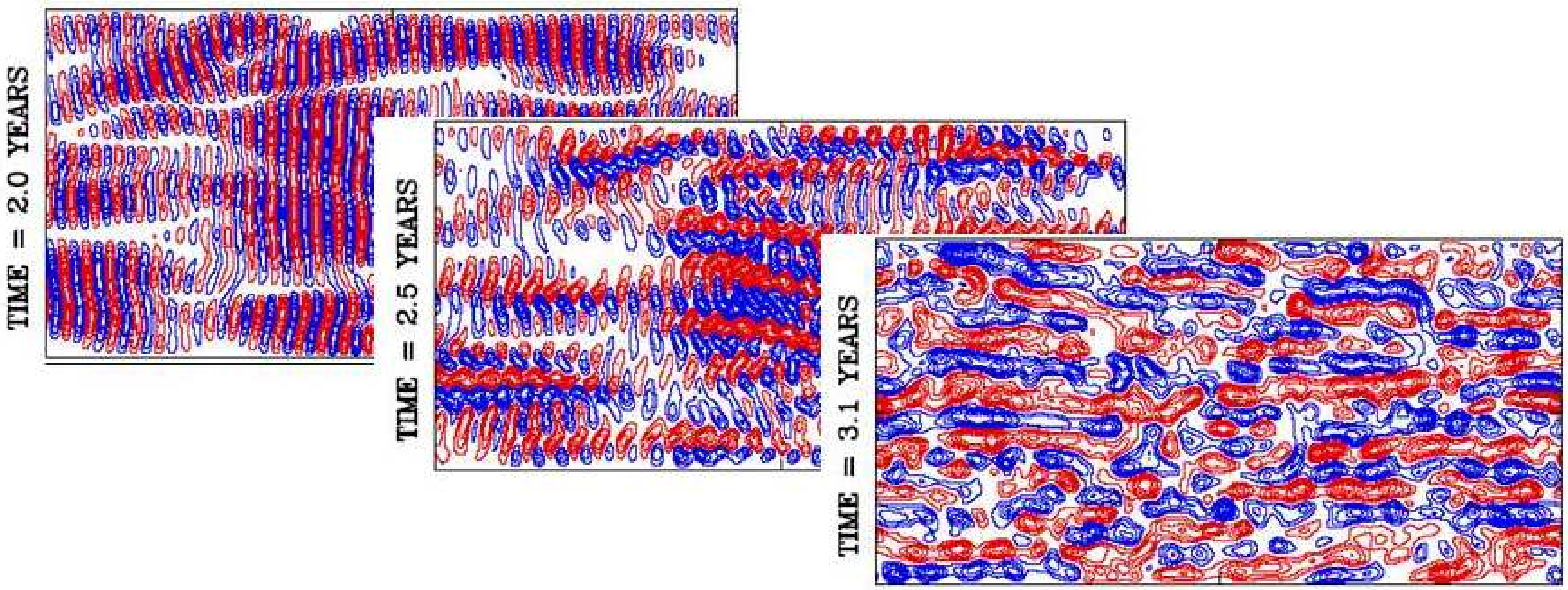
• *Exact analytic solutions* of nonlinear flows are known (e.g., solitary waves), but remain simple and exceptional.

• *Statistical wave turbulence* framework (resonant triads, kinetic equations, etc.) can be useful, when the underlying linear dynamics is relatively simple and wave coherency is weak.

• *Stochastic modelling* of turbulence is an emerging field, but it is poorly constrained by physics.

• *Numerical modelling* is presently the most useful (in terms of the new knowledge!) approach for theoretical analysis of nonlinear flows, but under the relaxed scientific standards it can be intoxicating and detrimental.

Illustration: Stages of nonlinear evolution of the growing instabilities in the Phillips model



• **Turbulence modelling** is the process of construction and use of a model aiming to predict effects of broadly defined spatio-temporally complex nonlinear flow dynamics, which is referred to as fluid *"turbulence"*.

• **Closure problem** is a dream (or a modern alchemy?) to predict *coarse-grained flow* evolution by expressing important dynamical effects of *unresolved flow features* in terms of the coarse-grained flow fields.

Let's consider some velocity field consisting of coarse-grained (i.e., large-scale obtained by some spatio-temporal filtering) and fluctuation (i.e., small-scale) components:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \,, \qquad \overline{\mathbf{u}'} = 0 \,.$$

Let's assume the following toy dynamics:

$$(*) \qquad \frac{du}{dt} + uu + Au = 0 \qquad \rightarrow \qquad \frac{d\overline{u}}{dt} + \overline{uu} + A\overline{u} = 0$$

To close the equation for $\overline{u}$, let's obtain the equation for $\overline{uu} = \overline{u}\,\overline{u} + \overline{u'u'}$ by multiplying $(*)$ with u and by coarse-graining:

$$\frac{1}{2}\frac{d\overline{uu}}{dt} + \overline{uuu} + A\overline{uu} = 0$$

What are we going to do with the cubic term? An equation determining it will contain a quartic term $\overline{uuuu}$, and so on...
Let's imagine a magic *"philosopher's stone"* relationship that makes the closure:

$$\overline{uuuu} = \alpha\,\overline{uu}\,\overline{uu} + \beta\,\overline{uuu}$$

Many theoreticians are looking for various "philosopher's stone" relationships that will be laughed at a century from now, but by doing this a great deal of physical knowledge is obtained and many mathematical instruments are developed.

• **Reynolds Decomposition**.
Common example of coarse-graining, referred to as *Reynolds decomposition*, is separation of a turbulent flow into the time-mean and fluctuation (i.e., "eddy") components:

$$\mathbf{u}(t,\mathbf{x}) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(t,\mathbf{x}) \,, \qquad p(t,\mathbf{x}) = \overline{p}(\mathbf{x}) + p'(t,\mathbf{x}) \,, \qquad \rho(t,\mathbf{x}) = \overline{\rho}(\mathbf{x}) + \rho'(t,\mathbf{x}) \,.$$

For example, let's apply the Reynolds decomposition to the x-momentum equation and, then, average this equation over time (as denoted by overline):

$$\frac{\partial\overline{u}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\,\overline{u} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x} - \nabla\cdot\overline{\mathbf{u}'u'} = -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x} \; -\frac{\partial}{\partial x}\overline{u'u'} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} \,.$$

The last group of terms is the first component of divergence of the nonlinear *Reynolds stress tensor* : $\boxed{\mathbf{T}_{ij} = \overline{u'_i u'_j}}$.

(a) Components of nonlinear stress $\overline{\mathbf{u}'\phi'}$ are usually called *eddy flux* components of ϕ. (In the above example $\phi = u_1$.) Divergence of an eddy flux can be interpreted as internally and nonlinearly generated *eddy forcing* exerted on the coarse-grained flow.

(b) It is very tempting to assume that nonlinear stress can be related to the corresponding time-mean (large-scale) gradient, for example:

$$\boxed{\overline{u'\phi'} = -\nu \, \frac{\partial \overline{\phi}}{\partial x}} .$$

This *flux-gradient assumption* is often called *eddy diffusion* or *eddy viscosity* (closure). Note, that this flux-gradient relation is exactly true for real viscous stress (but only in Newtonian fluids!) arising due to molecular dynamics.

(c) The flux-gradient assumption is common in models and theories, but it is often either inaccurate or fundamentally wrong, because fluid dynamics is different from molecular dynamics.

(d) Turbulent QG PV dynamics can be also coarse-grained to yield diverging eddy fluxes, because ϕ can stand for PV. Since PV anomalies consist of the relative-vorticity and buoyancy parts, the PV eddy flux $\overline{\mathbf{u}'q'}$ can be straightforwardly split into the Reynolds stress (i.e., eddy vorticity flux) and form stress (i.e., eddy buoyancy flux) components, which describe different physics.

• Parameterization of unresolved eddies

The above coarse-graining approach can be extended beyond the Reynolds decomposition into the time mean and fluctuations by decomposing flow into some large-scale and slowly evolving component and the small-scale residual eddies. For example, consider the equivalent-barotropic model with eddy viscosity replacing nonlinear stresses:

$$\Pi = \nabla^2\psi - \frac{1}{R^2}\,\psi + \beta y\,, \qquad \frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Pi}{\partial x} = \nu\nabla^2\zeta = \nu\nabla^4\psi\,,$$

here it is assumed that the model solves for the large-scale flow, and the viscous term represents effects of unresolved eddies. Let us interpret this viscosity.

(a) Molecular viscosity of water is $\sim 10^{-6}$ m^2 s^{-1}, but typical values of ν used in geophysical models are 100–1000 m^2 s^{-1}. What do these numbers imply? Typical viscosities (in m^2 s^{-1}): honey ~ 0.005, peanut butter ~ 0.25, basaltic lava ~ 1000.
In simple words, oceans in modern theories and models are made of basaltic lava rather than water...
(Similar analogy holds for the atmosphere; although kinematic viscosity is about 20 times larger in the air.)

(b) *Reynolds number* measures relative importance of nonlinear and viscous terms (*Peclet number* is similar but for diffusion term):

$$Re = \frac{U^2/L^2}{\nu U/L^3} = \frac{UL}{\nu}\,, \qquad Pe = \frac{UL}{\kappa}$$

Modern general circulation models strive to achieve larger and larger Re (and Pe) by progressively resolving smaller scales, and by employing better numerical algorithms and faster supercomputers.

• **Triad interactions in turbulence**: mechanism of nonlinear interactions that transfers energy between scales.

Let's consider a double-periodic domain with the following forced and dissipative 2D dynamics:

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta\,, \qquad \zeta = \nabla^2 \psi\,. \qquad (*)$$

All flow fields can be expanded in Fourier series (summation is over all negative and positive wavenumbers):

$$\psi(x,y,t) = \sum_k \tilde{\psi}(\mathbf{k},t)\, e^{i\mathbf{kx}}\,, \qquad \zeta(x,y,t) = \sum_k \tilde{\zeta}(\mathbf{k},t)\, e^{i\mathbf{kx}}\,, \qquad \mathbf{k} = \mathbf{i}k_1 + \mathbf{j}k_2\,, \qquad \tilde{\zeta} = -K^2 \tilde{\psi}\,, \qquad K^2 = k_1^2 + k_2^2\,.$$

Substituting the Fourier expansions in $(*)$ yields

$$-\frac{\partial}{\partial t} \sum_{\mathbf{k}} K^2\, \tilde{\psi}(\mathbf{k},t)\, e^{i\mathbf{kx}} = \Big[\sum_{\mathbf{p}} p_1 \tilde{\psi}(\mathbf{p},t)\, e^{i\mathbf{px}}\Big]\Big[\sum_{\mathbf{q}} q_2 \tilde{\zeta}(\mathbf{q},t)\, e^{i\mathbf{qx}}\Big] - \Big[\sum_{\mathbf{p}} p_2 \tilde{\psi}(\mathbf{p},t)\, e^{i\mathbf{px}}\Big]\Big[\sum_{\mathbf{q}} q_1 \tilde{\zeta}(\mathbf{q},t)\, e^{i\mathbf{qx}}\Big]$$

$$+ \sum_{\mathbf{k}} \tilde{F}(\mathbf{k},t)\, e^{i\mathbf{kx}} + \nu \sum_{\mathbf{k}} K^4\, \tilde{\psi}(\mathbf{k},t)\, e^{i\mathbf{kx}}\,,$$

where $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$ are 2D wavevectors.

Wavevector evolution equation is obtained for each spectral coefficient $\tilde{\psi}(\mathbf{k},t)$ by multiplying the last equation with $\exp(-i\mathbf{kx})$, by integrating over the domain, using $Q^2 = q_1^2 + q_2^2$, and by noting that the Fourier modes are orthogonal:

$$\int e^{i\mathbf{px}} e^{i\mathbf{qx}}\, dA = L^2 \delta(\mathbf{p}+\mathbf{q}) \qquad \rightarrow$$

$$\boxed{\frac{\partial}{\partial t}\, \tilde{\psi}(\mathbf{k},t) = \sum_{\mathbf{p},\mathbf{q}} \frac{-Q^2}{-K^2}\, (p_1 q_2 - p_2 q_1)\, \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\, \tilde{\psi}(\mathbf{p},t)\, \tilde{\psi}(\mathbf{q},t) + \frac{1}{-K^2}\, \tilde{F}(\mathbf{k},t) - \nu K^2 \tilde{\psi}(\mathbf{k},t)} \qquad (**)$$

This can be reformulated for evolution of the complex amplitude $|\tilde{\psi}(\mathbf{k},t)|$ by multipling the equation with the complex conjugate spectral coefficient $\tilde{\psi}^*(\mathbf{k},t)$. Note, that there are as many equations $(**)$ involved, as wavevectors $\mathbf{k}$ considered.

Interaction coefficient weighs the nonlinear term according to the dynamics, and it is nonzero only for the interacting wavevector triads that must satisfy: $\mathbf{p}+\mathbf{q}=\mathbf{k}$, because of the δ-function involved.

Hermitian (conjugate) symmetry property (i.e., $\tilde{\psi}$ is Hermitian function) states that

$$\tilde{\psi}(k_1, k_2, t) = \tilde{\psi}^*(-k_1, -k_2, t)\,,$$

because ψ is real function.

Some properties of the triad interactions:

(a) *Redistribution of spectral energy density.*
Suppose, there are initially only two Fourier modes, with wavevectors $\mathbf{p}$ and $\mathbf{q}$, and with the Fourier coefficients $\tilde{\psi}(\mathbf{p},t)$ and $\tilde{\psi}(\mathbf{q},t)$.
Due to the conjugate symmetry, these modes must have their conjugate-symmetric partners at $-\mathbf{p}$ and $-\mathbf{q}$, which are described by the Fourier coefficients $\tilde{\psi}^*(-\mathbf{p},t)$ and $\tilde{\psi}^*(-\mathbf{q},t)$; thus, the initial combination of the "two modes" are actually the "four modes" organized in 2 conjugate-symmetric pairs. Nonlinear interactions involving the initial 2 pairs will generate 2 more pairs,

$$\mathbf{k} = \mathbf{p} + \mathbf{q}\,, \qquad \mathbf{l} = -\mathbf{p} - \mathbf{q}\,,$$

$$\mathbf{m} = \mathbf{p} - \mathbf{q}\,, \qquad \mathbf{n} = -\mathbf{p} + \mathbf{q}\,,$$

and the subsequent nonlinear generation of the new wavevectors will continue to infinity.

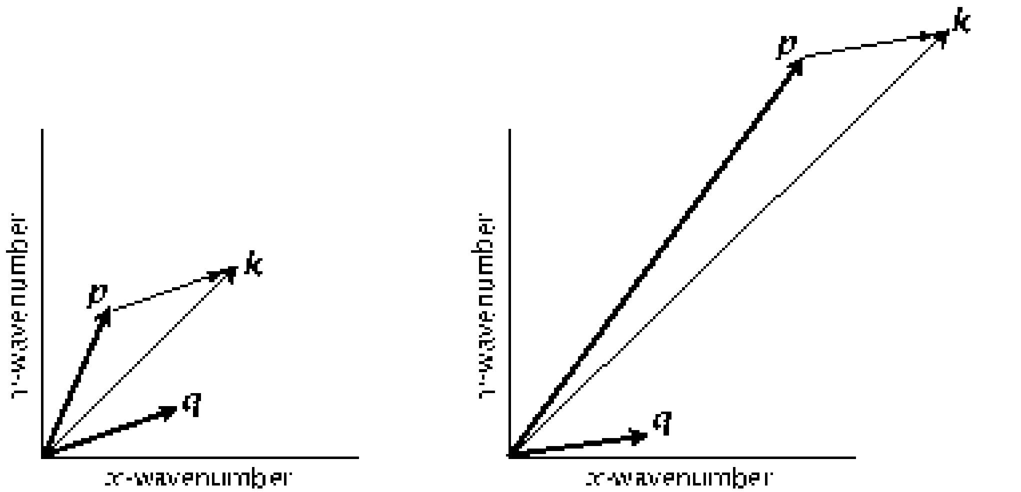


Fig. B.1 Two interacting triads, each with $k = p + q$. On the left, a local triad with $k \sim p \sim q$. On the right, a non-local triad with $k \sim p \gg q$.

(b) Nonlinear triad interactions are called *local* $(k \sim p \sim q)$ or *non-local* $(k \sim p \ll q)$, depending on the differences between the involved scales (see Figure).

(c) *Cascades* in turbulence are energy transfers between scales based on *local* interactions.

(d) *Fourier spectral descriptions* are popular, because the modes are simple and orthogonal, and in spatially homogeneous situations (only!) they even satisfy the linearized dynamics. Other spectral descriptions are possible and can be even more useful.

(e) *Fourier expansion in time* allows to talk about nonlinear interactions of individual waves rather than wavevectors. If phases of these waves are approximately random, then the problem can be approached by *wave turbulence* theory; if the phases are coherent, as typical in 2D turbulence, then people talk about *coherent structures*.

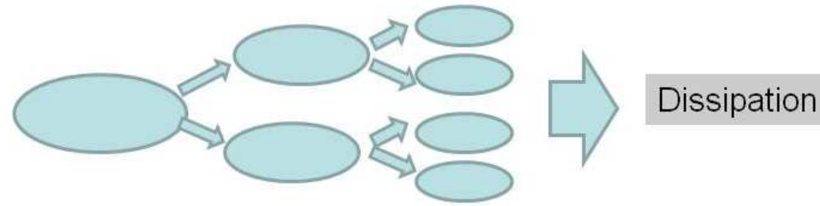
• **Homogeneous and stationary, non-rotating 3D turbulence**.

This idealized turbulence is characterized by *energy transfers* from the larger to smaller scales.

These transfers can involve both local and nonlocal interactions; however, *forward energy cascade* is a popular concept (conjecture) stating that energy is transferred only between similar scales (i.e., locally) and cascades from larger to smaller scales.

Forward energy cascade assumes the following:

(a) At large length scales there is some energy input (e.g., due to instabilities of large-scale flow), all dissipation happens on short length scales, and on the intermediate length scales the turbulence is controlled by *conservation of energy*.



(b) Dissipation acts on very short length scales, such that fluid motion is characterized by $Re \leq 1$. These are scales on which cascading energy is drained out. Within the cascade energy input to each scale/wavenumber is equal to energy output from it.

(c) Turbulence within the cascade is characterized by *self-similarity*, i.e., everything is structurally similar at each scale/wavenumber.

Let's consider:
isotropic wavenumber, k,
energy spectral density, $E(k)$, and
energy input rate, ϵ.

Energy within a spectral interval is $E(k)\delta k$.

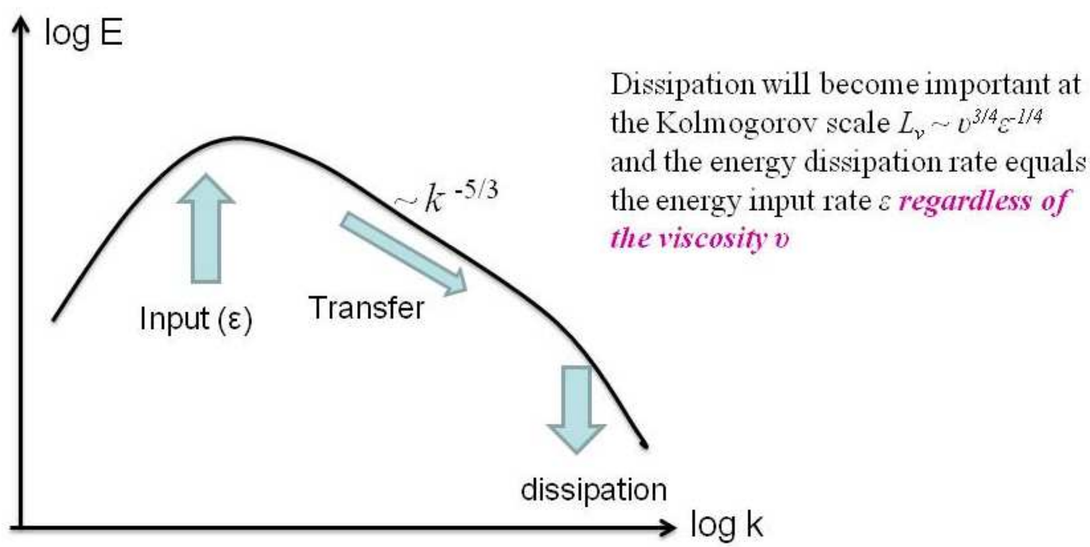
The physical dimensions are:

$$[k] = \frac{1}{L}, \quad [E] = LU^2 = \frac{L^3}{T^2}, \quad [\epsilon] = \frac{U^2}{T} = \frac{L^2}{T^3}$$

Advective *velocity scale* and *time scale* are:

$$v_k = [kE(k)]^{1/2},$$

$$\tau_k = (kv_k)^{-1} = [k^3 E(k)]^{-1/2}.$$



In the assumed *inertial spectral range* the kinetic energy is conserved; it is neither produced nor dissipated. Energy input in and output from each spectral interval, on the one hand, is ϵ, and, on the other hand, should scale with v_k and τ_k only:

$$\boxed{\epsilon \sim \frac{v_k^2}{\tau_k}} = \frac{kE(k)}{\tau_k} = k^{5/2}E(k)^{3/2} \quad \Longrightarrow \quad \boxed{E(k) \sim \epsilon^{2/3}k^{-5/3}}$$ *Kolmogorov "minus-five-thirds" law*

Kolmogorov law is robust, within $\pm 2\%$ deviations, but similarly argued predictions for the higher-order moments deviate from statistical measurements because of *intermittency* associated with relatively frequent large velocities and the corresponding energy dissipation bursts.

Kolmogorov (dissipative) length scale L_{visc} is the smallest scale in fluid dynamics. It can be obtained by equating the advective time scale τ_k and the viscous time scale $\tau_{visc} = [k^2\nu]^{-1}$ for the corresponding isotropic wavenumber k_{visc} :

$$\tau_k = k^{-3/2}\,E^{-1/2} \sim \epsilon^{-1/3}\,k^{-2/3} \quad \rightarrow \quad \tau_k = \tau_{visc} \quad \Longrightarrow \quad k_{visc} \sim \epsilon^{1/4}\,\nu^{-3/4} \quad \Longrightarrow \quad \frac{1}{k_{visc}} \equiv \boxed{L_{visc} \sim \epsilon^{-1/4}\,\nu^{3/4}}$$

Alternatively, we can find this power law scaling (and many others!) purely from the dimensional analysis:

$$k_{visc} \sim L_{visc}^{-1} \sim \epsilon^{\alpha}\,\nu^{\beta} \sim \frac{L^{2\alpha}}{T^{3\alpha}}\,\frac{L^{2\beta}}{T^{\beta}} \quad \Longrightarrow \quad 2\alpha + 2\beta = -1\,, \quad 3\alpha + \beta = 0 \quad \rightarrow \quad \alpha = \frac{1}{4}\,, \quad \beta = -\frac{3}{4}$$

• **2D homogeneous turbulence** is controlled by conservation of not only energy but also *enstrophy* $Z = \zeta^2$, which is the other useful quadratic scalar. Consider enstrophy dynamics:

$$\frac{\partial}{\partial t}\,\zeta^2 = 2\zeta\,\frac{\partial \zeta}{\partial t} = -2\zeta\,\mathbf{u}\cdot\nabla\zeta = -\mathbf{u}\cdot\nabla\zeta^2 = -\nabla\cdot(\mathbf{u}\,\zeta^2) + \zeta^2\nabla\cdot\mathbf{u}\,, \qquad (*)$$

where the second step involves the material conservation law for ζ.
The rhs in $(*)$ vanishes, because we assume nondivergent flow and periodic boundaries, i.e., $\mathbf{u}\cdot d\mathbf{S} = 0$, therefore:

$$\frac{\partial}{\partial t}\int_A \zeta^2\,dA = \int_A \frac{\partial}{\partial t}\,\zeta^2\,dA = -\int_A \nabla\cdot(\mathbf{u}\,\zeta^2)\,dA = -\int_S \mathbf{u}\,\zeta^2\,d\mathbf{S} = 0 \quad \Longrightarrow \quad \textit{conservation of enstrophy}$$

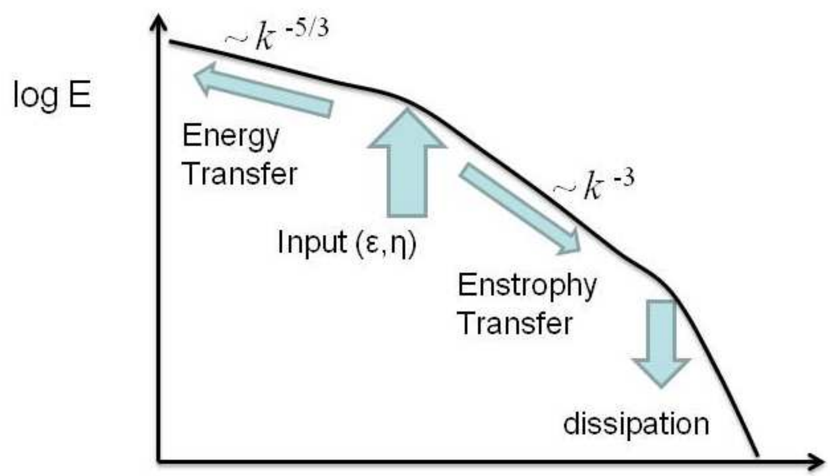
Homogeneous 2D turbulence is characterized by the following:

(a) Energy is transferred to *larger* scales (hence, *inverse energy cascade* concept is valid) and ultimately removed by some other physical processes; the Kolmogorov spectrum $E(k) \sim k^{-5/3}$ is preserved.

(b) Enstrophy is transferred to *smaller* scales (i.e., there is *forward enstrophy cascade*) and ultimately removed by viscous dissipation.

(c) Upscale energy transfer occurs often through *2D vortex mergers*.

(d) Downscale *enstrophy cascade* occurs often through irreversible process of *stretching, filamentation and stirring of relative vorticity*.



To obtain its spectral law, the enstrophy cascade can be treated similarly to the energy cascade. Let's assume that *enstrophy input rate* η produces enstrophy that cascades through the inertial spectral range to the dissipation-dominated scales:

Now, let's recall that the advective scales are $\tau_k = k^{-3/2} E(k)^{-1/2}\,,\ \ v_k = [kE(k)]^{1/2}$

$$\Longrightarrow \qquad \eta \sim \frac{\zeta_k^2}{\tau_k} = \frac{(k\,v_k)^2}{\tau_k} = \frac{k^3 E(k)}{\tau_k} = k^{9/2} E(k)^{3/2} \qquad \Longrightarrow \qquad \boxed{E(k) \sim \eta^{2/3} k^{-3}} \qquad (**)$$

Let's now use $(**)$ to ged rid of $E(k)$ $\quad \Longrightarrow \quad \tau_k = \eta^{-1/3}$

Equate this to the viscous time scale to obtain the *dissipative length scale for enstrophy*:

$$\tau_{visc} \sim [k^2 \nu]^{-1} = \eta^{-1/3} \qquad \rightarrow \qquad k_{visc} \sim \eta^{1/6} \nu^{-1/2} \qquad \rightarrow \qquad \frac{1}{k_{visc}} \equiv \boxed{L_{visc} \sim \eta^{-1/6} \nu^{1/2}}$$

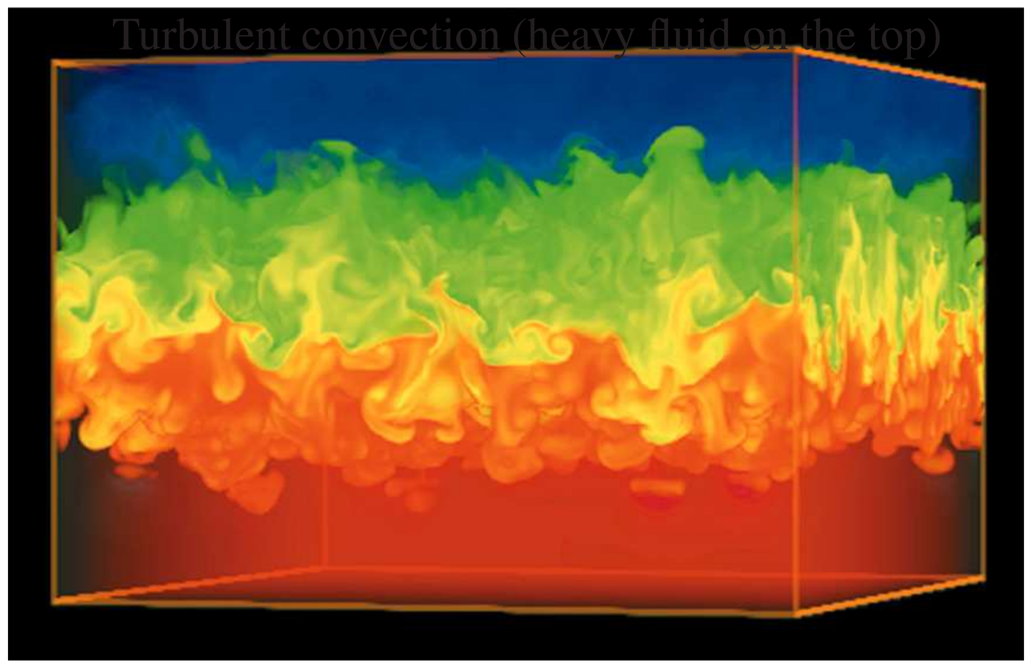
Instead of engaging into detailed analysis of 2D vortex mergers, let's consider an alternative explanation of the energy transfer to larger scales... Vorticity is conserved, but it is also being stretched and filamented (e.g., consider a circular patch of vorticity that evolves and becomes elongated as a spaghetti). The corresponding streamfunction is obtained by the *vorticity inversion* $\nabla^2 \psi = \zeta$, therefore, its length scale will be controlled by the elongated vorticity scale, hence, the streamfunction scale will keep increasing. Therefore, the total kinetic energy will become dominated by larger scales.

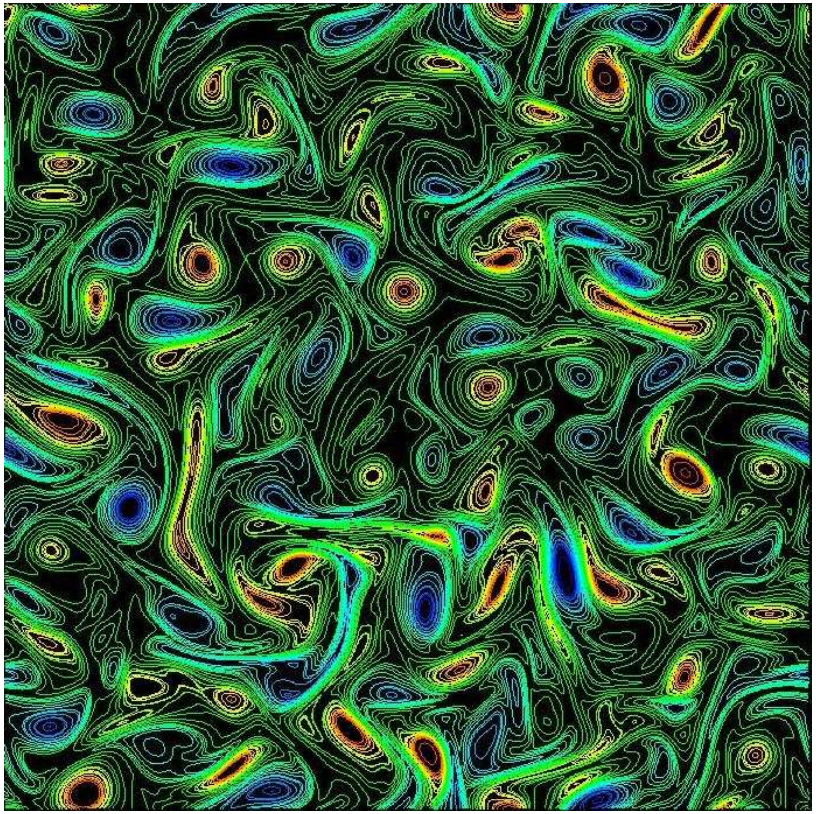
• **Effects of rotation and stratification on 3D turbulence** are such, that they suppress vertical motions, and, therefore, create and maintain *quasi-2D turbulence*.

The β-effect or other horizontal inhomogeneities of background PV make quasi-2D turbulence *anisotropic*. Example of anisotropic phenomenon is emergence of *multiple alternating jets* (e.g., zonal bands in the atmosphere of Jupiter). Length scales controlling widths of the multiple jets are *Rhines scale* $L_R = (U/\beta)^{1/2}$ (here, U is characteristic eddy velocity scale) and baroclinic Rossby radius R_D.

⟸ When people research *homogeneous 3D turbulence*, they usually deal with this kind of solutions...
(*shown are isolines of vertical relative vorticity component*)

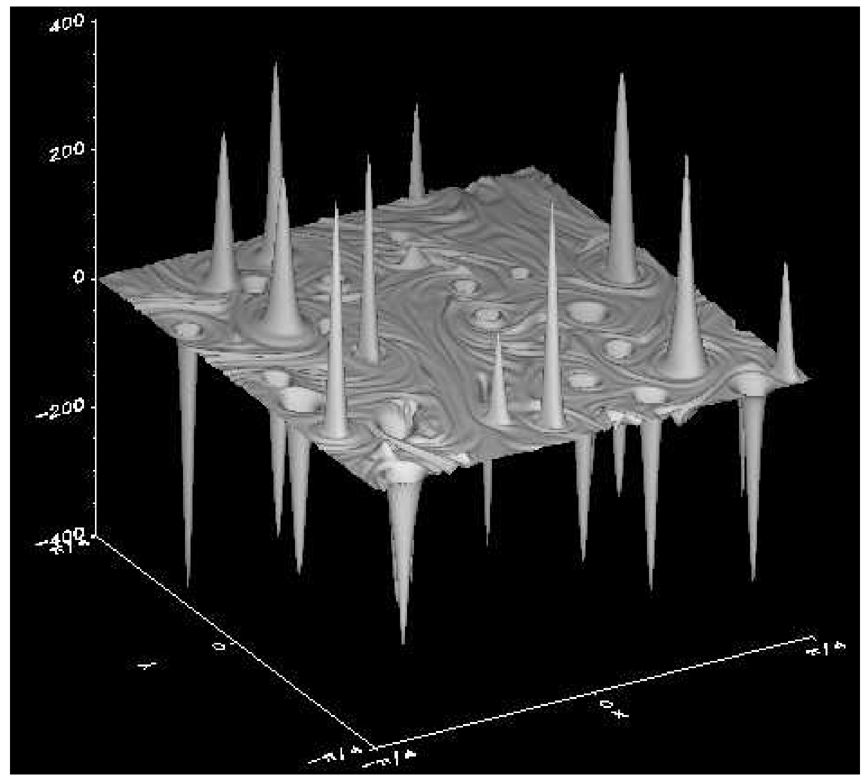
There are many types of *inhomogeneous 3D turbulence*, characterized by some broken spatial symmetries ⟹



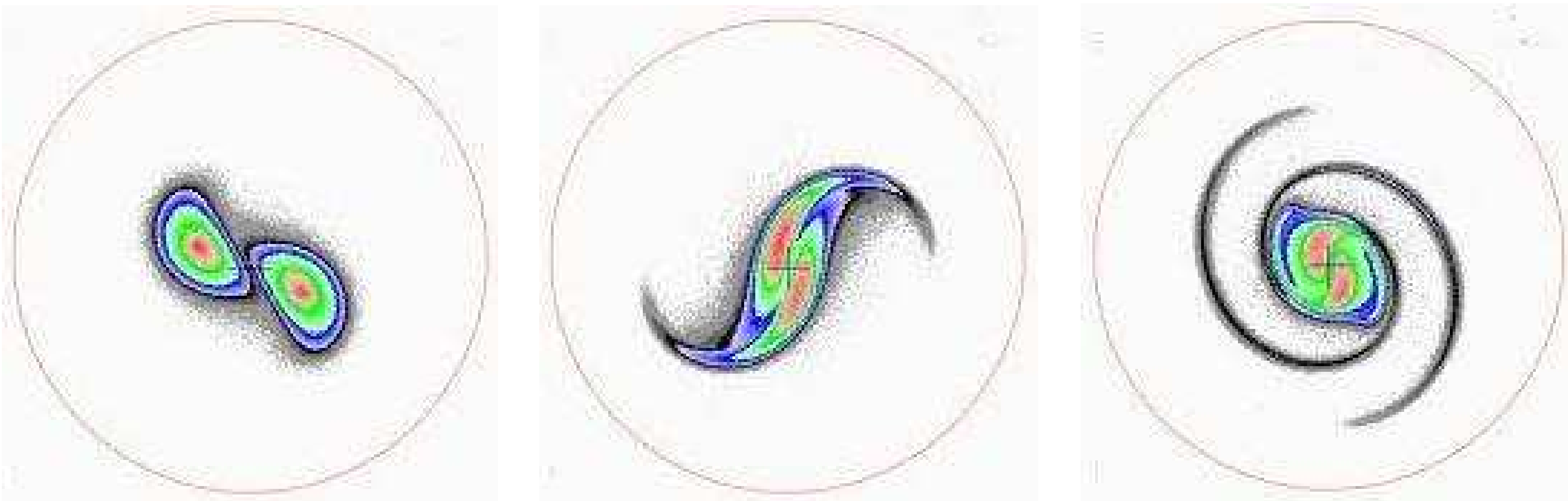


⟸ 2D turbulence is characterized by interacting and long-living *coherent vortices*

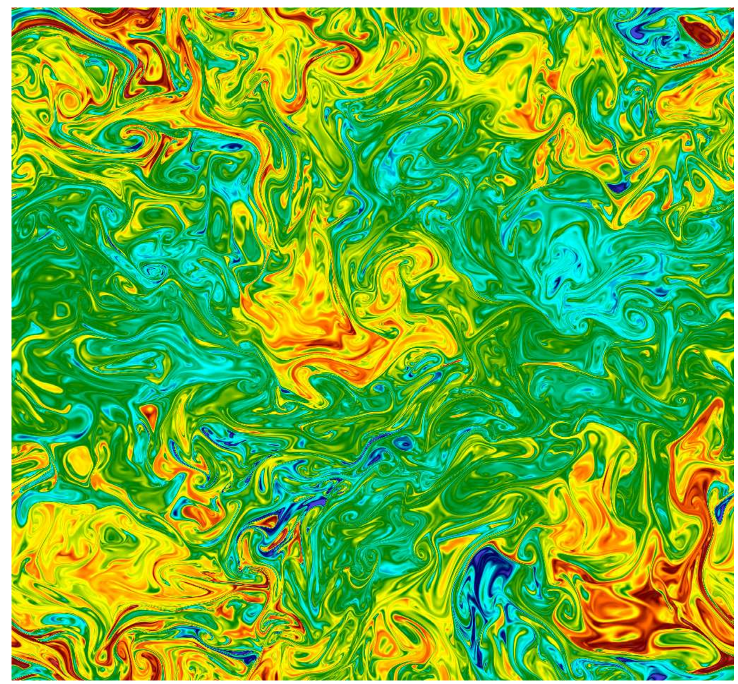
These vortices are materially conserved vorticity extrema ⟹



Merger of two same-sign vortices (*snapshots show different stages in time*)



Chaotic advection of material tracer



In 2D turbulence:

- *Inverse energy cascade* occurs through mechanism of *vortex mergers*.
- *Forward enstrophy cascade* occurs through mechanism of irreversible *filamentation and stirring of vorticity anomalies*.

• **Transformed Eulerian Mean** (TEM) is a useful transformation of the equations of motion (for predominantly zonal eddying flows, like atmospheric storm track or oceanic Circumpolar Current). TEM framework:

(a) eliminates eddy fluxes in the thermodynamic equation,
(b) in a simple form collects all eddy fluxes in the zonal momentum equation,
(c) highlights the role of eddy PV flux.

Let's start with the Boussinesq system of equations,

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F\,, \qquad \frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,, \qquad \frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - b\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \qquad \frac{Db}{Dt} + N^2 w = Q_b\,,$$

assume geostrophic and ageostrophic velocities and focus on the ϵ-order terms in the zonal momentum and thermodynamic equations:

$$\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - f_0 v_a = F\,, \qquad \frac{\partial b}{\partial t} + u_g\frac{\partial b}{\partial x} + v_g\frac{\partial b}{\partial y} + N^2 w_a = Q_b\,.$$

These equations can be rewritten in the flux divergence form:

$$\frac{\partial u_g}{\partial t} + \frac{\partial u_g u_g}{\partial x} + \frac{\partial v_g u_g}{\partial y} - f_0 v_a = F\,, \qquad \frac{\partial b}{\partial t} + \frac{\partial u_g b}{\partial x} + \frac{\partial v_g b}{\partial y} + N^2 w_a = Q_b\,.$$

Next, assume conceptual model of eddies evolving on zonally symmetric mean flow and feeding back on this flow. Separate eddies from the mean flow by applying zonal x-averaging (denoted by overline; $\overline{f'} = 0$):

$$u_g = \overline{u_g(t,y,z)} + u_g'(t,x,y,z)\,, \quad v_g = v_g'(t,x,y,z) \quad \rightarrow \quad \boxed{\frac{\partial \overline{u}_g}{\partial t} = f_0\overline{v}_a - \frac{\partial}{\partial y}\overline{u_g' v_g'} + \overline{F}} \qquad (*)$$

Note, that zonal integration of any $\partial(flux)/\partial x$ term yields zero, because of the zonal symmetry.
Similar decomposition of the buoyancy yields:

$$b = \overline{b(t,y,z)} + b'(t,x,y,z) \quad \rightarrow \quad \boxed{\frac{\partial \overline{b}}{\partial t} = -N^2\overline{w}_a - \frac{\partial}{\partial y}\overline{v_g' b'} + \overline{Q_b}} \qquad (**)$$

Equations $(*)$ and $(**)$ are *coupled* by the *thermal wind* relations, and because of this coupling, effects of the momentum and heat fluxes cannot be clearly separated from each other — this is a fundamental nature of the geostrophic turbulence.
Progress can be made by recognizing that $\overline{v}_a$ and $\overline{w}_a$ are related by mass conservation (i.e., non-divergent 2D field). Hence, we can define ageostrophic meridional streamfunction, ψ_a, such that

$$\overline{v}_a = -\frac{\partial \psi_a}{\partial z}\,, \qquad \overline{w}_a = \frac{\partial \psi_a}{\partial y}\,.$$

Meridional eddy buoyancy flux can be easily incorporated in ψ_a, and we can define the *residual mean meridional streamfunction*,

$$\boxed{\psi^* \equiv \psi_a + \frac{1}{N^2}\,\overline{v'_g b'}} \qquad \Longrightarrow \qquad \overline{v}^{\,*} = -\frac{\partial \psi^*}{\partial z} = \overline{v}_a - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\,\overline{v'_g b'}\right), \qquad \overline{w}^{\,*} = \frac{\partial \psi^*}{\partial y} = \overline{w}_a + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\,\overline{v'_g b'}\right),$$

that by construction describes non-divergent 2D flow $(\overline{v}^{\,*}, \overline{w}^{\,*})$.

(a) Thus, ψ^* combines the (ageostrophic) *Eulerian mean* circulation with the *eddy-induced* (Lagrangian) circulation.
The eddy-induced circulation can be understood as a *Stokes drift* phenomenon.

(b) These circulations tend to compensate each other, hence, mean zonal flow feels their *residual* effect.

With the definition of ψ^*, the momentum equation $(*)$ can be written as

$$\frac{\partial \overline{u}_g}{\partial t} = f_0 \overline{v}^{\,*} - \frac{\partial}{\partial y}\,\overline{u'_g v'_g} + \frac{\partial}{\partial z}\,\frac{f_0}{N^2}\,\overline{v'_g b'} + \overline{F} = f_0 \overline{v}^{\,*} + \nabla_{yz}\!\cdot\mathbf{E} + \overline{F}\,, \qquad \mathbf{E} \equiv \left(0\,,\ \ -\overline{u'_g v'_g}\,,\ \ \frac{f_0}{N^2}\,\overline{v'_g b'}\,\right),$$

where we introduced the *Eliassen-Palm flux* $\mathbf{E}$.

Next, let's take into account that $\nabla_{yz}\!\cdot\mathbf{E} = \overline{v'_g q'_g}$ (see Problem Sheet), and obtain the *Transformed Eulerian Mean (TEM)* equations:

$$\boxed{\frac{\partial \overline{u}_g}{\partial t} = f_0 \overline{v}^{\,*} + \overline{v'_g q'_g} + \overline{F}\,, \qquad \frac{\partial \overline{b}}{\partial t} = -N^2\,\overline{w}^{\,*} + \overline{Q_b}\,, \qquad \frac{\partial \overline{v}^{\,*}}{\partial y} + \frac{\partial \overline{w}^{\,*}}{\partial z} = 0\,, \qquad f_0\,\frac{\partial \overline{u}_g}{\partial z} = -\frac{\partial \overline{b}}{\partial y}} \qquad (***)$$

where the last equation is just the thermal wind balance.

Let's eliminate the left-hand sides from the first two equations by differentiating them with respect to z and y, respectively. The outcome is equal by the last equation from (***), and the resulting diagnostic equation is

$$-f_0^2\,\frac{\partial v^*}{\partial z} + N^2\,\frac{\partial w^*}{\partial y} = f_0\,\frac{\partial}{\partial z}\overline{v'_g q'_g} + f_0\,\frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{Q_b}}{\partial y}\,.$$

Now we can take into account definition of ψ^* and obtain the final diagnostic equation:

$$\boxed{f_0^2\,\frac{\partial^2 \psi^*}{\partial z^2} + N^2\,\frac{\partial^2 \psi^*}{\partial y^2} = f_0\,\frac{\partial}{\partial z}\overline{v'_g q'_g} + f_0\,\frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{Q_b}}{\partial y}} \qquad (****)$$

(a) If we know the eddy PV flux, the TEM equations allow us to solve for the complete circulation pattern. This can be done by solving the elliptic problem (****) for ψ^*, at every time (step).

(b) Eddy PV flux still has to be found dynamically, but the theory allows for many dynamical insights.

(c) The TEM framework can be extended to non-QG flows.

(d) *Non-Acceleration Theorem* states that under certain conditions eddies (or waves) have no net effect on the zonally averaged flow. Let's prove it by considering zonally averaged QG PV equation (with a non-conservative rhs $\overline{D}$):

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial \overline{v'q'}}{\partial y} = \overline{D}\,, \qquad\qquad \overline{q} = \frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{\psi}}{\partial z}\right) + \beta y\,.$$

Let's differentiate $(\partial/\partial y)$ the QG PV equation:

$$\frac{\partial^2}{\partial t \partial y}\left[\frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{\psi}}{\partial z}\right)\right] = -\frac{\partial^2}{\partial y^2}\,\overline{v'q'} + \frac{\partial \overline{D}}{\partial y}\,,$$

and recall that

$$\overline{v'q'} = \overline{v_g'q_g'} = \nabla_{yz}\cdot\mathbf{E} \qquad \rightarrow \qquad \left[\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2(\nabla_{yz}\cdot\mathbf{E})}{\partial y^2} - \frac{\partial \overline{D}}{\partial y}$$

Theorem: If there is no eddy PV flux (i.e., Eliassen-Palm flux is non-divergent) in stationary and conservative situation, then the flow can not get accelerated $(\partial\overline{u}/\partial t = 0)$, because the "Eulerian mean" and "eddy-induced" circulations completely cancel each other.