(a) There are 15 nodes, and the graph is connected, so there is the single right null vector comprising the 15-vector with equal components, hence the rank is 14 (by the rank-nullity theorem).

2 marks

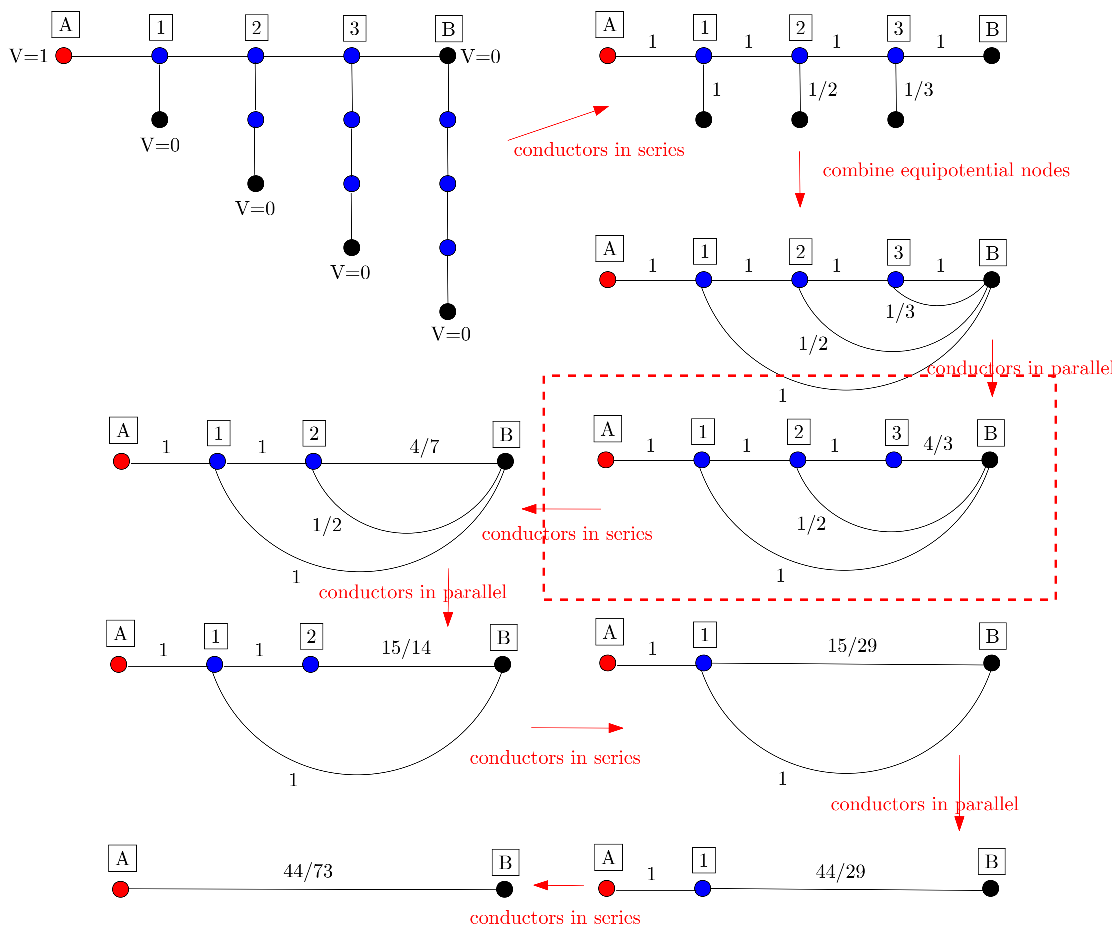
There are two ways to find the dimension of the left nullspace.

One way is to use the fact that it is equal to the number of closed loops in the graph, of which there are none (by inspection), so the dimension of the left null space is zero.

Alternatively, on counting to find there are 14 edges in the graph, the dimension of the left-null space is the difference between the number of edges (i.e., 14 edges) and the rank (which is 14, as just found above) hence the dimension of the left null space is zero (by the rank-nullity theorem).

2 marks

(b) The effective conductance can be found by the following sequence of equivalent circuits, repeatedly using the rules for two conductors in parallel ($c_1 + c_2$) and conductors in series ($c_1c_2/(c_1 + c_2)$) where c_1 and c_2 are the individual conductances:



To explain this sequence, notice, in the first equivalent circuit, all the edges connecting node $\boxed{B}$ to another grounded node can be eliminated because it will never carry any current (there is no voltage drop across these edges). And the remaining grounded nodes can be merged, *while preserving all edges*, without changing any of the currents in those edges. The remaining reductions follow from repeated use of the rules for conductors in series and in parallel.

Any other correct method is acceptable. For example, by solving directly for the potentials – as required anyway in part (c) – as done in the final **Note** below.

8 marks

(c) There are many ways to proceed. The system to solve is the usual

$$\mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

but it is not sensible to proceed with all the nodes in the original circuit. Rather, the reduced circuit shown in the dashed red box will save time, but since all the nodes $\boxed{1}$–$\boxed{3}$ are still present and have not been touched in the reductions, the potentials there can be determined from this simpler equivalent circuit. Assigning columns to nodes in the order $\boxed{1}$–$\boxed{3}$, $\boxed{A}$, $\boxed{B}$, the Laplacian for this reduced circuit is found using the usual rules to be

$$\mathbf{K} = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & \frac{5}{2} & -1 & 0 & -\frac{1}{2} \\ 0 & -1 & \frac{7}{3} & 0 & -\frac{4}{3} \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -\frac{1}{2} & -\frac{4}{3} & 0 & \frac{17}{6} \end{pmatrix} \tag{2}$$

with

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_0 \\ -f_0 \end{pmatrix}, \tag{3}$$

where f_0 is the effective conductance found in part (b) and (x, y, z) are the three voltage values at nodes $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ to be found. Using the sub-block decomposition

$$\mathbf{K} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{pmatrix}, \tag{4}$$

where

$$\mathbf{P} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & \frac{5}{2} & -1 \\ 0 & -1 & \frac{7}{3} \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -\frac{1}{2} & -\frac{4}{3} \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{17}{6} \end{pmatrix}, \tag{5}$$

the sub-system to solve for the required potentials is

$$\mathbf{P}\hat{\mathbf{x}} = -\mathbf{Q}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

Since $\mathbf{P}$ is just a 3-by-3 matrix one could, in principle, find the inverse matrix by hand using techniques learned in linear algebra. But it is easier, on noticing some zeros in $\mathbf{P}$, to solve the system directly by elimination. The first equation gives

$$3x - y = 1, \qquad x = \frac{1+y}{3} \tag{7}$$

and the third equation gives

$$-y + \frac{7z}{3} = 0, \qquad z = \frac{3y}{7} \tag{8}$$

so that the middle equation is

$$-x + \frac{5y}{2} - z = 0, \qquad \text{or} \qquad -\frac{1+y}{3} + \frac{5y}{2} - \frac{3y}{7} = 0. \tag{9}$$

This is easily solved,

$$y = \frac{14}{73}, \tag{10}$$

so the final solution, on back substitution, is

$$\boxed{x = \frac{29}{73}, \qquad y = \frac{14}{73}, \qquad z = \frac{6}{73}} \tag{11}$$

8 marks

Note: A useful check is to notice that the net current out of node $\boxed{\text{A}}$ is $1 - x = 1 - 29/73 = 44/73$ which must coincide with the effective conductance in part (b). Indeed, as already mentioned, this is an alternative to the "equivalent circuit method" given in part (b).

Total score is out of 20