Introduction to Quantum Mechanics – Problem sheet 2

1. **A quantum wave function**

Our system is a quantum particle in one dimension described by the wave function

$$\psi(x,t_0) = Nx\,\mathrm{e}^{-\frac{(x-x_0)^2}{2}},$$

with real constants $N > 0$, and $x_0 > 0$.

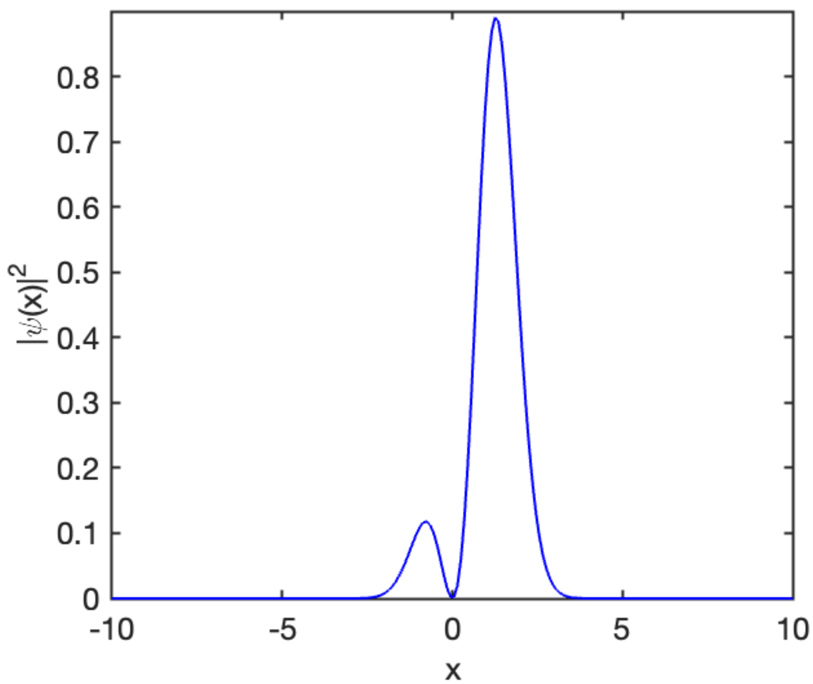

Figure 1: A plot of $|\psi(x)|^2$ for $x_0 = 0.5$ and $N = 1$.

(a)

(b) The most probable location is the absolute maximum of the probability density. We first look for extrema of the probability density, where it holds

$$\frac{d}{dx}|\psi(x)|^2 = 0.$$

This gives

$$N^2(-2(x-x_0)x^2 + 2x)e^{-(x-x_0)^2} = 0,$$

which we solve to obtain

$$x = 0, \frac{1}{2}(x_0 \pm \sqrt{4+x_0^2}).$$

For the value $x_0 = 0.5$ the maximum at $x = \frac{1+\sqrt{17}}{16}$ is the absolute maximum, consistent with the plot in part (a).

2. A quantum wave function, probabilities and energies

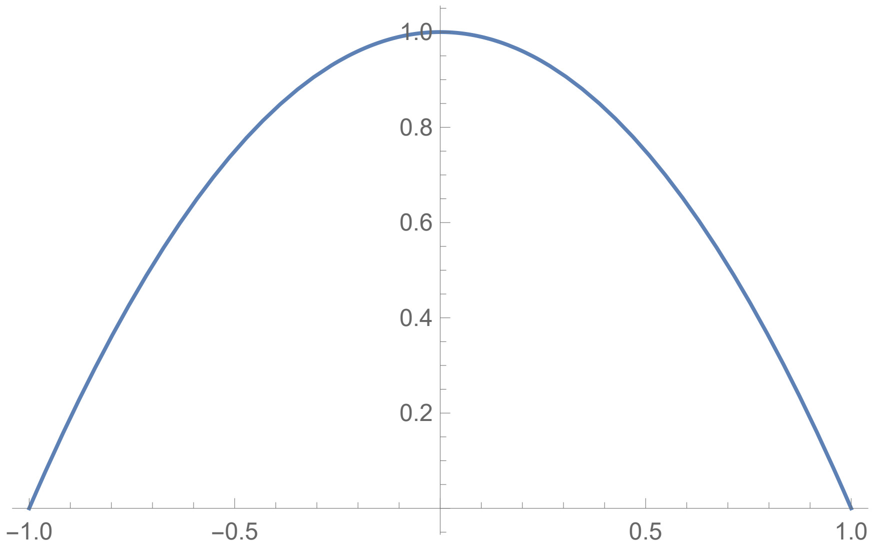

Figure 2: Plot of $\psi(x)$ for $a = b = 1$

(a)

(b) We have

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$\begin{aligned}\int_{-\infty}^{\infty} |\psi(x)|^2 dx &= \int_{-a}^{a} b^2 \left(a^4 - 2a^2x^2 + x^4\right) dx \\ &= b^2 \left[a^4 x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right]_{-a}^{a} \\ &= b^2 \left[2a^5 - \frac{4}{3}a^5 + \frac{2}{5}a^5\right] \\ &= \frac{16}{15}a^5 b^2\end{aligned}$$

Therefore

$$b = \frac{\sqrt{15}}{4} a^{-\frac{5}{2}}.$$

(c) Calculate the probability of finding the particle between $-a/2$ and $-a/4$.

$$\begin{aligned}\int_{-\frac{a}{2}}^{-\frac{a}{4}} |\psi(x)|^2 dx &= \frac{15}{16}a^{-5} \left[a^4 x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right]_{-\frac{a}{2}}^{-\frac{a}{4}} \\ &= \frac{2813}{16384}\end{aligned}$$

(d) We have

$$\frac{\partial^2}{\partial x^2}\psi(x) = -2b$$

The expectation of the Hamiltonian is given by

$$
\begin{aligned}
\langle H\rangle &= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\psi^*(x)\psi''(x)dx \\
&= \frac{\hbar^2}{2m}\int_{-a}^{a}2b^2(a^2-x^2)dx \\
&= \frac{\hbar^2 b^2}{m}\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{a} \\
&= \frac{\hbar^2}{m}\frac{5}{4}a^{-2}
\end{aligned}
$$

4. Stationary states and complex energies

Consider the two stationary states

$$
\begin{aligned}
\psi_A(x,t) = e^{-iE_At}\psi_A(x,0) \\
\psi_B(x,t) = e^{-iE_Bt}\psi_B(x,0),
\end{aligned}
$$

where $E_A \neq E_B$.

(a) Calculate the probability densities of ψ_A, ψ_B, $\psi_A+\psi_B$ and $\psi_A-\psi_B$.

$$
|\psi_A(x,t)|^2 = e^{-iE_At}e^{iE_At}|\psi_A(x,0)|^2 = |\psi_A(x,0)|^2
$$

$$
|\psi_B(x,t)|^2 = |\psi_B(x,0)|^2
$$

$$
\begin{aligned}
|\psi_A(x,t)+\psi_B(x,t)|^2 &= (\psi_A(x,t)+\psi_B(x,t))(\psi_A(x,t)^*+\psi_B(x,t)^*) \\
&= |\psi_A(x,0)|^2+|\psi_B(x,0)|^2+2\mathsf{Re}(e^{i(E_A-E_B)t}\psi_A(x,0)\psi_B(x,0)^*) \\
&= |\psi_A(x,0)|^2+|\psi_B(x,0)|^2+2\cos((E_A-E_B)t)\mathsf{Re}(\psi_A(x,0)\psi_B(x,0)^*)
\end{aligned}
$$

$$
\begin{aligned}
|\psi_A(x,t)-\psi_B(x,t)|^2 &= (\psi_A(x,t)-\psi_B(x,t))(\psi_A(x,t)^*-\psi_B(x,t)^*) \\
&= |\psi_A(x,0)|^2+|\psi_B(x,0)|^2-2\mathsf{Re}(e^{i(E_A-E_B)t}\psi_A(x,0)\psi_B(x,0)^*) \\
&= |\psi_A(x,0)|^2+|\psi_B(x,0)|^2-2\cos((E_A-E_B)t)\mathsf{Re}(\psi_A(x,0)\psi_B(x,0)^*)
\end{aligned}
$$

(b) We have

$$
\psi_A(x,t) = e^{(-iE_R+E_I)t}\psi_A(x,0).
$$

The Probability density of $\psi_A(x,t)$ is therefore given by

$$
|\psi_A(x,t)|^2 = e^{2E_It}|\psi_A(x,0)|^2.
$$

We see that the norm of the state, that is the overall probability is time dependent. In particular the probability decays exponentially with time for $E_I<0$ and growth exponentially with time for $E_I>0$. While a growth of the overall probability is a little strange and it is difficult to imagine physical situations corresponding to this, an exponential decay of the overall probability arises in some familiar situations, such as nuclear decay.