Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Model Theory

Date: Monday, 16 May 2011. Time: 2.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. We say that a graph G is a perfect matching if for every vertex v of G there is exactly one vertex w of G adjacent to v, and w is different from v. We say that a graph G has a perfect matching if there is a subgraph of G which contains every vertex of G and which is a perfect matching. Let G be a graph such that for every vertex v of G there are only finitely many vertices of G adjacent to v. Prove that the graph G has a perfect matching if and only if every finite subset of vertices of G is contained in a perfect matching in G.

2.

- Let G be a torsion-free divisible abelian group. Let A, B be two finite subsets of G of cardinality k and fix a numbering $a_1, a_2, \dots, a_k$ of the elements of A. Show that there is a numbering $b_1, b_2, \dots, b_k$ of the elements of B such that the sums $a_1 + b_1, \cdots, a_k + b_k$ are pairwise different.

- Show that there is a number $N(k) \in \mathbb{N}$ depending on k only such that for every prime $p \geq N(k)$ and for every pair A, B of subsets of $\mathbb{F}_p$ of cardinality k and for every numbering $a_1, a_2, \dots, a_k$ of the elements of A there is a numbering $b_1, b_2, \dots, b_k$ of the elements of B such that the sums $a_1 + b_1, \cdots, a_k + b_k$ are pairwise different.

3. Let $A, B \subseteq \mathbb{N}$ be the following sets:

$$A = \{p \in \mathbb{N} | \ p \text{ is a prime and } p \equiv 1 \pmod 4\}, \tag{1}$$
$$B = \{p \in \mathbb{N} | \ p \text{ is a prime and } p \equiv -1 \pmod 4\}. \tag{2}$$

Let $\mathcal{U}, \mathcal{V}$ be two non-principal ultrafilters on A and B, respectively.

- Prove that the ultraproducts $\prod_{p \in A} \mathbb{F}_p/\mathcal{U}$ and $\prod_{p \in B} \mathbb{F}_p/\mathcal{V}$ are elementarily equivalent as groups. (Here we equip $\mathbb{F}_p$ with its additive group structure.)
- Prove that the ultraproducts $\prod_{p \in A} \mathbb{F}_p/\mathcal{U}$ and $\prod_{p \in B} \mathbb{F}_p/\mathcal{V}$ are not elementarily equivalent as rings. (*Hint: you may use the fact that the congruence* $x^2 \equiv -1 \pmod p$ *has a solution for an odd prime p if and only if* $p \in A$.)

4. Let $\mathcal{L}$ be the language of graphs, containing one relation symbol E with $n_E = 2$, and let $k \geq 2$ be an integer. We say that a graph G is a k-regular forest if the following holds:

   1. the graph G is simple (no vertex is adjacent to itself),
   2. for every vertex v of G there are exactly k vertices of G adjacent to v,
   3. there is no circle of length n for every integer $n \geq 3$, that is, there is no sequence of pair-wise different vertices $v_0, v_1, \ldots, v_{n-1}$ of G such that $v_0 E v_1, v_1 E v_2, \cdots, v_{n-2} E v_{n-1}, v_{n-1} E v_0$.

   Let $\mathcal{C}$ be the class of k-regular forests.

- Show that $\mathcal{C}$ is axiomatizable by an $\mathcal{L}$-theory T.
- We say that a graph $G \in \mathcal{C}$ is a k-regular tree if it is minimal in the sense that no proper subgraph of G is a k-regular forest. Prove that up to isomorphism there is a unique a k-regular tree. (*Hint: draw a picture!*)
- Prove that the theory T is complete.