IMPERIAL

MATH70003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Geophysical Fluid Dynamics

Date: Tuesday, April 30, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Linearized shallow-water dynamics

Consider the linearized shallow-water model describing a single layer of fluid on the f-plane with the Coriolis parameter f_0. The fluid has constant density ρ_0 and the rest depth H. Use conventional notations for velocity (u, v, w), deviation of the free surface η, and pressure p.

(a) (4 marks)

Write down horizontal momentum equations, continuity equation, and vertically integrated hydrostatic balance, which relates p and η.

(b) (2 marks)

Eliminate w from the continuity equation (by vertical integration and use of the kinematic boundary condition); write down the resulting prognostic equation for η.

(c) (4 marks)

Manipulate the momentum equations to obtain velocity equations of the form

$$\frac{\partial^2 u}{\partial t^2} + f_0^2 u = \ldots\ldots, \qquad \frac{\partial^2 v}{\partial t^2} + f_0^2 v = \ldots\ldots,$$

and find their right hand sides as differential operators acting on η.

(d) (8 marks)

By taking the curl and divergence of the momentum equations obtain an equation for η.

(e) (2 marks)

In the equation for η, look for wave-like solutions in an unbounded domain and obtain the dispersion relation that describes Poincare waves.

(Total: 20 marks)

2. Rossby waves on the mean flow

Consider situation that you are given some remote-sensing data about newly discovered rotating exoplanet with stratified and turbulent atmosphere. The data shows that in the middle latitudes the time-mean zonal winds are uniform in the lower atmosphere and negligible otherwise.

With this information, you decided to model the winds with a periodic zonal channel, represent the fluid by two isopycnal layers, and assume constant time-mean zonal velocity U_2 in the bottom layer. The rest of motion are wind fluctuations. Assume also material conservation of the layer-wise potential vorticities and β-plane, so that:

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) + \beta\,\frac{\partial \psi_i}{\partial x} = 0\,, \qquad q_1 = \nabla^2\psi_1 - S_1\,(\psi_1 - \psi_2)\,, \qquad q_2 = \nabla^2\psi_2 - S_2\,(\psi_2 - \psi_1)\,,$$

where $J(,)$ is Jacobian, $S_{1,2}$ are stratification parameters, and i-index runs from top to bottom.

Your job is to predict planetary wave characteristics (hence, the type of weather) in this atmosphere.

(a) (4 marks)

Assume that the total flow can be decomposed into the time-mean (overbared) and fluctuation (primed) components: $\psi_i = \overline{\psi_i} + \psi_i'\,,\quad q_i = \overline{q_i} + q_i'\,,\quad i = 1, 2$. Express $\overline{\psi_i}$ and $\overline{q_i}$ via parameters of the problem and find their first spatial derivatives.

(b) (2 marks)

Prove that the given time-mean flow is a solution of the governing equations.

(c) (4 marks)

For each fluid layer apply the flow decomposition and write down all 8 components of the corresponding Jacobian $J(\psi_i, q_i)$. Explain why some of these components are equal to zero. Write down the nonlinear terms.

(d) (4 marks)

Write down the final linearized potential-vorticity equations for each layer and explain briefly meanings of the involved terms.

(e) (4 marks)

Look for the linear wave solutions $\psi_i' = \tilde{\psi}_i \exp\left(kx + ly - \omega t\right)$ of the linearized equations and obtain the corresponding system of algebraic equations. For these equations, write down the 4 coefficients of the matrix multiplying $(\tilde{\psi}_1, \tilde{\psi}_2)^T$.

(f) (2 marks)

Explain how the dispersion relation can be obtained. For $S_1 = S_2 = S$ and for $U_2 = 0$, find the dispersion for the barotropic mode, $\psi = \psi_1 + \psi_2$, and sketch it.

(Total: 20 marks)

3. Linear instability of sheared flow

Consider 1.5-layer QG PV model configured in a zonal channel $(-\infty < x < \infty;\ \ 0 \le y \le 1)$. Assume there is a rigid lid on the top of the active layer, a rotating β-plane, and that the fluid is inviscid. Let the stratification parameter be $S = R^{-2}$, where R is the deformation radius. Assume that the channel contains background flow given by zonal velocity with meridional profile $U(y) = U_0\, y\,(1-y)$, such that $U_0 > 0$ (eastward flow).

(a) (6 marks)

Write down expressions for the background velocity streamfunction $\Psi(y)$ and potential vorticity $\Pi(y)$, and briefly explain the meaning of each term in Π. Write down expression for the meridional gradient of $\Pi(y)$.

(b) (8 marks)

Write down the expression for the potential vorticity (PV) anomaly that corresponds to fluctuations in the background flow, and explain the meaning of each term. Write down the governing equation describing material conservation law for the full PV, and explain the meaning of each term. Linearize the governing equation around the background flow.

(c) (6 marks)

What is the necessary condition for linear instability of a zonal background flow? Is the background flow $U(y)$ considered in this problem stable or unstable?

(Total: 20 marks)

4. 3D homogeneous turbulence

Consider the case of homogeneous and stationary 3D turbulence without rotation. Assume that there exists an inertial spectral interval of wavenumbers, $k_\epsilon < k < k_{visc}$, which is self-smilar for all wavenumbers k, that is, the spectral energy density $E(k)$ depends only on k and the energy input rate ϵ. Assume that the energy is injected by an external forcing at about k_ϵ and removed by the dissipation operator at about k_{visc}.

(a) (2 marks)

Find and write down physical dimensions of k, $E(k)$ and ϵ, denoting them as $[k]$, $[E]$ and $[\epsilon]$, respectively. Write down expression for the total amount of energy contained in the inertial spectral interval.

(b) (2 marks)

Using dimensionality arguments, find and write down mathematical expressions for the k-dependent velocity v_k and time τ_k scales in the inertial spectral range.

(c) (6 marks)

Using dimensionality arguments, assume conservation of kinetic energy (i.e., energy input is equal to the energy output for each spectral interval) and derive how $E(k)$ scales with ϵ and k.

(d) (6 marks)

Assume that the dissipation operator is biharmonic, that is, given by

$$\nu \nabla^4 \mathbf{u}\,,$$

where ν is the viscosity coefficient. Derive how the dissipative wavenumber k_{visc} scales with ϵ and ν. What are the corresponding scaling laws for the viscous length scale L_{visc}? Find the dissipative time scale τ_k.

(e) (4 marks)

Assume that the dissipation operator is frictional, that is, given by

$$-\kappa \mathbf{u}\,,$$

where κ is the friction coefficient. How does k_{visc} scale with ϵ and ν in this case?

(Total: 20 marks)

5. Homogeneous turbulent dispersion and diffusion

Imagine that in the far future you are hired by the UK Space Agency to explore synoptic turbulence in the atmospheres of two exoplanets. Advanced automatic spaceships dropped ensembles of trackable Lagrangian floats (i.e., infinitesimal particles) in the turbulent exoplanetary atmospheres and remotely sensed their trajectories for a long time. On-board computers processed the Lagrangian velocity data and estimated that the corresponding ensemble-averaged velocity variances σ are the same on both planets, but the corresponding ensemble-averaged velocity autocorrelation functions are very different and can be approximated as

$$R_1(\tau) = (1-\tau)\, e^{-\tau}\,, \qquad |\tau| < 1\,; \qquad\qquad R_1(\tau) = 0\,, \qquad |\tau| \geq 1\,;$$

$$R_2(\tau) = (1-\tau)\, e^{-\tau}\,;$$

where τ is time delay measured in appropriate units, and subscript indicates the exoplanet.

Your job is to answer the following set of questions and, eventually, provide theoretical estimates of the turbulent diffusivities for your colleagues in the atmosphere modelling team:

(a) (4 marks)

How can velocity of a particle be determined from its trajectory? Write down the relation between position and velocity of a Lagrangian particle. For the rest of the assignment, assume that atmospheric turbulence is isotropic and spatially homogeneous. What does this mean? Define the Lagrangian single-particle dispersion $D(t)$ and autocorrelation $R(\tau)$ functions and describe in words what kind of information they provide. Sketch both $R_1(\tau)$ and $R_2(\tau)$.

(b) (6 marks)

Derive the integro-differential equation connecting the rate of change of $D(t)$ with $R(\tau)$.

(c) (5 marks)

Estimate for both exoplanets: $\lim\limits_{t\to 0} D(t)\,.$

What does it tell us?

(d) (5 marks)

Estimate for both exoplanets their diffusive asymptotics: $\lim\limits_{t\to\infty} D(t)\,,$

and, thus, find the corresponding turbulent diffusivities. What do they describe?

Hint: $\displaystyle\int_0^t (1-x)\, e^{-x}\, dx = t\, e^{-t}$

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

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M3A28, M4A28, M5A28

Introduction to Geophysical Fluid Mechanics (Solutions)

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1. (a) $\boxed{\text{seen} \Downarrow}$

$$\frac{\partial u}{\partial t} - f_0 v = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v}{\partial t} + f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad p = -\rho_0 g\,(z - \eta)\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$\boxed{4, \text{A}}$

(b) Integrate continuity equation using $w(z = H + \eta) = \partial\eta/\partial t$: $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (*)$$

$\boxed{2, \text{A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(c) Take u-momentum equation, differentiate it with respect to time, and add it to v-momentum equation multiplied by f_0; and, similarly, obtain the other equation:

$$\frac{\partial^2 u}{\partial t^2} + f_0^2 u = -g\left(\frac{\partial^2 \eta}{\partial x \partial t} + f_0\,\frac{\partial \eta}{\partial y}\right), \qquad \frac{\partial^2 v}{\partial t^2} + f_0^2 v = -g\left(\frac{\partial^2 \eta}{\partial y \partial t} - f_0\,\frac{\partial \eta}{\partial x}\right)$$

$\boxed{4, \text{A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(d) Curl of the momentum equations with the divergence taken from (*) substituted in the Coriolis term yields:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{f_0}{H}\,\frac{\partial \eta}{\partial t} = 0$$

Divergence of the momentum equations with the divergence taken from (*) and substituted in the tendency term yields:

$$\frac{1}{H}\,\frac{\partial^2 \eta}{\partial t^2} + f_0\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - g\nabla^2\eta = 0$$

From the above two equations obtain:

$$\frac{\partial}{\partial t}\left[\nabla^2\eta - \frac{1}{c_0^2}\,\frac{\partial^2 \eta}{\partial t^2} - \frac{f_0^2}{c_0^2}\,\eta\right] = 0\,, \qquad c_0^2 \equiv gH \qquad \Longrightarrow \qquad \nabla^2\eta - \frac{1}{c_0^2}\,\frac{\partial^2 \eta}{\partial t^2} - \frac{f_0^2}{c_0^2}\,\eta = 0$$

$\boxed{8, \text{C}}$

$\boxed{\text{seen} \Downarrow}$

(e) The last equation yields its dispersion relation:

$$\eta \sim e^{i(kx + ly - \omega t)} \qquad \rightarrow \qquad \omega^2 = f_0^2 + c_0^2\,(k^2 + l^2)$$

$\boxed{2, \text{B}}$

2. (a) $\boxed{\text{sim. seen} \Downarrow}$

$$\overline{\psi_1}=0\,,\quad \overline{\psi_2}=-U_2y\,,\quad \frac{\partial\overline{\psi_i}}{\partial x}=0\,,\quad \frac{\partial\overline{\psi_1}}{\partial y}=0\,,\quad \frac{\partial\overline{\psi_2}}{\partial y}=-U_2\,,\quad \nabla^2\overline{\psi_i}=0$$

$$\overline{q_1}=S_1\overline{\psi_2}\,,\quad \overline{q_2}=-S_2\overline{\psi_2}\,,\quad \frac{\partial\overline{q_i}}{\partial x}=0\,,\quad \frac{\partial\overline{q_1}}{\partial y}=-S_1U_2\,,\quad \frac{\partial\overline{q_2}}{\partial y}=S_2U_2$$

$\boxed{\text{4, A}}$

(b) All terms of the mean equations contain $\partial/\partial x$, hence they vanish. The tendency term acting on the time-mean flow vanishes because it is steady state. $\boxed{\text{sim. seen} \Downarrow}$

$\boxed{\text{2, A}}$

(c) Time-mean/time-mean interaction terms vanish, because they contain $\partial/\partial x$; fluctuation/fluctuation terms are quadratic in terms of the primed fields; half of the time-mean/fluctuation terms vanish, because they involve $\partial/\partial x$ of the mean state. The term involving $\partial\overline{\psi_1}/\partial y$ is zero, because there is no time-mean flow in the top layer. The nolninear terms are $\boxed{\text{unseen} \Downarrow}$

$$J(\psi_1',\nabla^2\psi_1'-S_1(\psi_1'-\psi_2')) \qquad \text{and} \qquad J(\psi_2',\nabla^2\psi_2'-S_2(\psi_2'-\psi_1'))$$

$\boxed{\text{4, A}}$

$\boxed{\text{unseen} \Downarrow}$

(d) The linearized equations are

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1'-S_1\psi_1'+S_1\psi_2'\right]+\frac{\partial\psi_1'}{\partial x}(-U_2S_1)+\beta\,\frac{\partial\psi_1'}{\partial x}=0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2'+S_2\psi_1'-S_2\psi_2'\right]+U_2\,\frac{\partial\nabla^2\psi_2'}{\partial x}+U_2S_2\,\frac{\partial\psi_1'}{\partial x}\quad -U_2S_2\,\frac{\partial\psi_2'}{\partial x}+\frac{\partial\psi_2'}{\partial x}(U_2S_2)\quad +\beta\,\frac{\partial\psi_2'}{\partial x}=0$$

Tendency terms describe local rate of change of PV; beta-terms describe meridional advection of planetary PV; the remaining terms are advection by mean flow of fluctuation PV components (in deep layer only), followed by advection of mean PV by fluctuations (in both layers). Note that two terms in the last equation cancel out. $\boxed{\text{4, C}}$

$\boxed{\text{unseen} \Downarrow}$

(e) Let's denote $K=k^2+l^2$, then the elements of the matrix are

$$A_{11}=-\omega(K+S_1)-k(\beta-U_2S_1)\,,\qquad A_{12}=\omega S_1\,,$$

$$A_{21}=\omega S_2-kU_2S_2\,,\qquad A_{22}=-\omega(K+S_2)-k(\beta-KU_2)$$

$\boxed{\text{4, B}}$

$\boxed{\text{unseen} \Downarrow}$

(f) For nontrivial solution to exist, determinant of the matrix must vanish — this sets the dispersion relation between wavenumbers and frequency. The barotropic-mode dispersion relation is the classical one for the Rossby waves. $\boxed{\text{2, A}}$

3. (a) Background velocity streamfunction is

unseen ⇓

$$\Psi(y) = \int -U(y)\,dy = \frac{U_0}{3}\,y^3 - \frac{U_0}{2}\,y^2\,.$$

Background PV consists of the planetary vorticity, background velocity curl and background isopycnal deformation:

$$\Pi(y) = \beta y - \frac{dU}{dy} - S\,\Psi = \beta y + U_0\,(2y-1) - S\left(\frac{U_0}{3}\,y^3 - \frac{U_0}{2}\,y^2\right).$$

The meridional PV gradient is

$$\frac{d\Pi}{dy} = \beta + 2U_0 - SU_0\,(y^2 - y)\,.$$

6, A

(b) Potential vorticity anomaly is

sim. seen ⇓

$$q = \nabla^2\psi - S\,\psi\,,$$

where the first term is relative vorticity anomaly, and the second term is isopycnal deformation.

Material conservation law for PV involves tendency and advective terms:

$$\frac{D}{Dt}(\Pi + q) = \left[\frac{\partial}{\partial t} + \left(U(y) - \frac{\partial\psi}{\partial y}\right)\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\,\frac{\partial}{\partial y}\right](\Pi + q) = 0$$

The linearized material conservation law for PV can be obtained as

$$\left(\frac{\partial}{\partial t} + U(y)\,\frac{\partial}{\partial x}\right)\left[\nabla^2\psi - S\,\psi\right] + \frac{\partial\psi}{\partial x}\,\frac{d\Pi}{dy} = 0\,,$$

because both the nonlinear terms and x-derivatives of Π drop out.

8, B

unseen ⇓

(c) The necessary condition for instability to occur, that is, for existence of exponentially growing solutions, is that $d\Pi/dy$ changes sign somewhere in the channel (i.e., for some $0 \le Y \le 1$):

$$\frac{d\Pi}{dy} = \beta + 2U_0 - SU_0\,(Y^2 - Y) = 0 \qquad \to \qquad Y = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4(\beta + 2U_0)}{SU_0}}$$

Only one of the roots is positive, and it has to be ≤ 1 :

$$\sqrt{1 + \frac{4(\beta + 2U_0)}{SU_0}} \le 1 \qquad \to \qquad \frac{4(\beta + 2U_0)}{SU_0} \le 0\,,$$

and this never happens for $U_0 > 0$, therefore, the flow is STABLE.

6, D

4. (a) $\boxed{\text{seen} \Downarrow}$

$$[k] = \frac{1}{L}\,, \qquad [E] = LU^2 = \frac{L^3}{T^2}\,, \qquad [\epsilon] = \frac{U^2}{T} = \frac{L^2}{T^3}$$

$$E_{tot} = \int_{k_\epsilon}^{k_{visc}} E(k)\, dk$$

$\boxed{2, \text{A}}$

(b) $\boxed{\text{seen} \Downarrow}$

$$v_k = [kE(k)]^{1/2}\,, \qquad \tau_k = (kv_k)^{-1} = [k^3 E(k)]^{-1/2}\,.$$

$\boxed{2, \text{A}}$

(c) $\boxed{\text{seen} \Downarrow}$

$$\epsilon \sim \frac{v_k^2}{\tau_k} = \frac{kE(k)}{\tau_k} = k^{5/2} E(k)^{3/2} \qquad \Longrightarrow \qquad E(k) \sim \epsilon^{2/3} k^{-5/3}$$

$\boxed{6, \text{B}}$

(d) Note that $[\nu] = L^4/T$, therefore: $\boxed{\text{unseen} \Downarrow}$

$$k_{visc} \sim \epsilon^\alpha \, \nu^\beta \sim \frac{L^{2\alpha}}{T^{3\alpha}} \frac{L^{4\beta}}{T^\beta} \qquad \to \qquad 2\alpha{+}4\beta = -1\,, \quad 3\alpha{+}\beta = 0 \qquad \to \qquad \alpha = \frac{1}{10}\,, \quad \beta = -\frac{3}{10}$$

$$\to \qquad k_{visc} \sim \epsilon^{1/10}\, \nu^{-3/10}$$

Note that $L_{visc} = 1/k_{visc}$.

$$\tau_{visc} = k_{visc}^{-4}\, \nu^{-1}$$

$\boxed{6, \text{D}}$

(e) Note that $[\kappa] = 1/T$, therefore: $\boxed{\text{unseen} \Downarrow}$

$$k_{visc} \sim \epsilon^\alpha \, \nu^\beta \sim \frac{L^{2\alpha}}{T^{3\alpha}} \frac{1}{T^\beta} \qquad \to \qquad \alpha = -\frac{1}{2}\,, \quad \beta = \frac{3}{2} \qquad \to \qquad \epsilon^{-1/2}\, \kappa^{3/2}$$

$\boxed{4, \text{D}}$

5. (a) Displacement of each particle is given by the integral of its Lagrangian velocity: $\boxed{\text{seen} \Downarrow}$

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{u}_L(t')\, dt' \,.$$

Lagrangian velocity can be estimated by the differentiation along the trajectory. Isotropicity and spatial homogeneity mean that particle statistics does not depend on direction and geographical location. Single-particle dispersion and Lagrangian velocity autocorrelation function are

$$D(t) = \left\langle (\mathbf{x}(t){-}\mathbf{x}(0))^2 \right\rangle , \qquad R(\tau) = \frac{\left\langle \mathbf{u}_L(t)\cdot\mathbf{u}_L(t-\tau) \right\rangle}{\left\langle u^2 \right\rangle} = \frac{1}{\sigma} \left\langle \mathbf{u}_L(t)\mathbf{u}_L(t{-}\tau) \right\rangle ,$$

where angular brackets denote ensemble averaging. On the first exoplanet $R_1(\tau)$ monotonically decays to zero, whereas on the second exoplanet $R_2(\tau)$ changes sign while decaying to zero. $\boxed{4,\ \mathrm{M}}$

(b) $\boxed{\text{seen} \Downarrow}$

$$\int_0^t R(t'-t)\, dt' = \frac{1}{\sigma} \left\langle \left[\mathbf{x}(t') - \mathbf{x}(0)\right]_0^t \mathbf{u}_L(t) \right\rangle \qquad \rightarrow$$

$$\frac{d}{dt} D(t) = 2 \left\langle [\mathbf{x}(t){-}\mathbf{x}(0)]\, \mathbf{u}_L(t) \right\rangle = 2\,\sigma \int_0^t R(t'{-}t)\, dt' = 2\,\sigma \int_{-t}^0 R(\tau)\, d\tau = 2\,\sigma \int_0^t R(\tau)\, d\tau$$

Recall the formula for differentiation under integral sign,

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\, dt \qquad \Longrightarrow \qquad \frac{d}{dx} F(x) = f(x, b(x))\, b'(x) {-} f(x, a(x))\, a'(x) {+} \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt \,,$$

$$\rightarrow \qquad D(t) = 2\,\sigma \int_0^t (t-\tau)\, R(\tau)\, d\tau$$

$\boxed{6,\ \mathrm{M}}$

(c) For both exoplanets, small t implies $\boxed{\text{unseen} \Downarrow}$

$$\int_0^t (1{-}\tau)\, e^{-\tau}\, d\tau = t\, e^{-t} \qquad \rightarrow \qquad \frac{d}{dt} D(t) \approx 2\,\sigma\, t \qquad \rightarrow \qquad D(t) \sim t^2 \quad for \quad t \to 0$$

Spreading of particles is very fast (ballistic). $\boxed{5,\ \mathrm{M}}$

(d) Lagrangian decorrelation time is $\boxed{\text{unseen} \Downarrow}$

$$T = \int_0^\infty R(\tau)\, d\tau \qquad \rightarrow \qquad T_1 = \int_0^1 (1{-}\tau)\, e^{-\tau}\, d\tau = \frac{1}{e} \,, \qquad T_2 = \int_0^\infty (1{-}\tau)\, e^{-\tau}\, d\tau = 0$$

$$\left.\frac{dD_1}{dt}\right|_\infty = 2\,\frac{\sigma}{e} \qquad \rightarrow \qquad \kappa_1 = \frac{\sigma}{e}$$

$$\left.\frac{dD_2}{dt}\right|_\infty = 0 \qquad \rightarrow \qquad \kappa_2 = 0$$

On the first exoplanet, there is diffusive (i.e., linear in time) spreading of particles in the long-time limit, but there is no such long-time spreading at all on the second exoplanet and dispersion saturates. $\boxed{5,\ \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

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Question	Marker's comment
1	Fairly standard question. Students did fairly well, as expected.
2	Fairly standard question. Students did fairly well, as expected.
3	Fairly standard question. Students did fairly well, as expected.
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