Network Science
Spring 2024
Review exercises

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1. Consider a random graph model for networks consisting of N_b blue nodes and N_r red nodes. The probability of a link being placed between two nodes that have the same color is p, and q is the probability of a link being placed between a blue node and a red node.

   (a) What is the probability that a blue node will have k links to other blue nodes?

   (b) What is the expected degree for a red node?

   (c) What is the expected number of links connecting blue and red nodes?

   (d) Consider the subnetwork consisting of red nodes and links connecting red nodes. State a tight lower bound for p which ensures that this subnetwork is connected with high probability.

   (e) Say that $p > log(N_r)/N_r$ and $N_r = N_b$. Show that for any q with $0 < q \leq 1$ that the network will be connected with high probability (you may state results from lecture as needed).

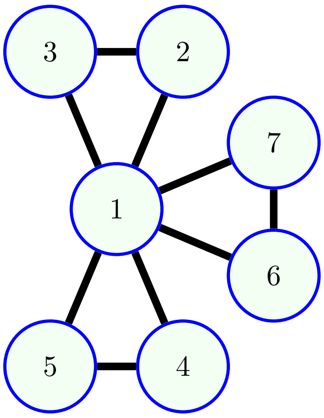
2. Consider the following model for an evolving network with a fixed number of nodes. Initially, at $t = 0$, the network consists of N nodes and zero links. Each iteration a link is placed between two nodes, node u and node v. Node u is chosen uniformly at random from the N nodes in the network. Node v is chosen based on the node degrees in the graph. Say that the degree of node j after iteration t is $k_j(t)$. Then, the probability that node j is selected as node v during iteration $t+1$ is $\frac{a+k_j(t)}{c(t)}$ where a is a constant which is given, and c is a variable that you will be asked to determine. Note that v may be the same as u, and the network may contain self-loops and multiedges.

   (a) Determine $c(t)$.

   (b) How many distinct graphs with zero self-loops can be generated after the second iteration? Here graphs should be considered distinct if their adjacency matrices are not identical.

   (c) What is the expected number of multiedges in the graph after the second iteration? Provide your answer in terms of N and a.

(d) Derive a master equation relating $\langle N_k(t+1)\rangle$ to $\langle N_k(t)\rangle$, $\langle N_{k-1}(t)\rangle$, and $\langle N_{k-2}(t)\rangle$. Your final equation should be presented in terms of N, c, a, k, $\langle N_k(t)\rangle$, $\langle N_{k-1}(t)\rangle$, and $\langle N_{k-2}(t)\rangle$.



3. Consider the graph shown above.

   (a) What is the diameter of the graph?

   (b) What is the average clustering coefficient?

   (c) What is the global clustering coefficient?

   (d) What is the probability of generating this particular graph with the G_{Np} model?

   (e) Say that this graph was generated with the configuration model.

      i. What is the expected number of links between nodes 1 and 2?

      ii. What is the probability of node 2 not having a multiedge?

   (f) Show, using the cosine similarity, that nodes 7 and 6 are more similar than than any pair of nodes that includes node 1.

   (g) The eigenvector centrality of node 7 has been set to 1. What is the eigenvector centrality of all other nodes in the graph?

   (h) The graph shown can be viewed as three triangles with a shared vertex. Now consider a graph consisting of n triangles with a shared vertex.

      i. What is the average clustering coefficient in the limit $n \to \infty$?

      ii. Say that a random walk starts at node 1 and let π_l be the probability that the walker is at node 1 after l steps. Derive a recurrence relation that relates π_l to π_{l-1}. Apply the limit $l \to \infty$ and find the stationary probability π_∞. Explain how the numerical result is is related to the graph structure.

4. The governing equations for the network-SI model on a simple connected N-node graph with adjacency matrix, $\mathbf{A}$, are,

$$\frac{d\langle x_i\rangle}{dt} = \beta\sum_{l=1}^{N} A_{il}\langle(1-x_i)x_l\rangle\,,\; i=1,2,...,N$$

(a) First consider the **naive** network-SI model on a graph with 2 nodes connected by one link. Show that solutions to this model will synchronize.

(b) Now consider the naive network-SI model on a complete graph. Find the solution, $\langle x_i(t)\rangle$, $i = 1,2,...,N$, $t>0$ when the (non-trivial) initial condition satisfies, $\langle x_1(t=0)\rangle = \langle x_2(t=0)\rangle = ... = \langle x_N(t=0)\rangle$.

(c) For a general simple graph, derive a system of linear ODEs which govern the dynamics of small perturbations to a state where all nodes are infectious. Let ϵ characterize the magnitude of the perturbation, and assume that $\epsilon \ll 1$. Describe the dynamics of these perturbations in the limit $t \to \infty$.

(d) Next consider a cycle graph with N nodes and N links.

i. Draw the graph when $N = 4$ with nodes numbered from 1 to 4, and give its adjacency matrix.

ii. Let ω be a complex number with $\omega^N = 1$, and let $\mathbf{v}_\omega = [1 \;\; \omega \;\; \omega^2 \;\; ... \;\; \omega^{N-1}]^T$. Show that $(\omega+\omega^{-1})$ is an eigenvalue of a cycle graph with N nodes with eigenvector $\mathbf{v}_\omega$.

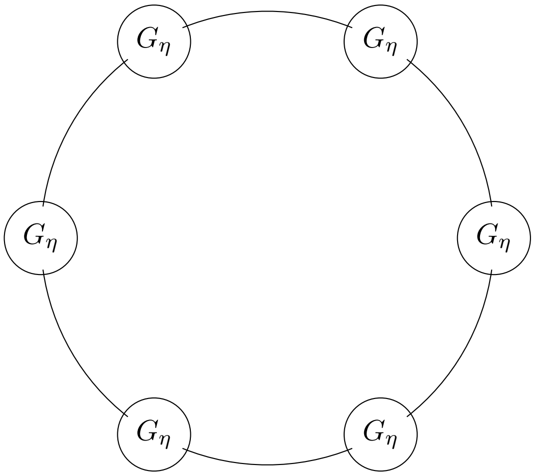
iii. Consider small perturbations to a state where all nodes are infection-free. As in part (d), let ϵ characterize the magnitude of the perturbation, and assume that $\epsilon \ll 1$. Provide the general solution for the linearized dynamics of these perturbations on a N-node cycle graph in terms of the eigenvectors and eigenvalues.

iv. Assume that N is even. Provide an initial condition such that the solution at $t = 1$ is of the form $[a \;\; -a \;\; a \;\; -a \;\; ... \;\; a \;\; -a]^T$ where a is a positive constant.

(e) The dynamics of the third moment, $\langle x_i x_j x_l\rangle$, are governed by a system of ODEs of the form,

$$\frac{d\langle x_i x_j x_l\rangle}{dt} = \beta\sum_{m=1}^{N}\left[A_{lm}\langle x_i x_j(1-x_l)x_m\rangle + term\ 2 + term\ 3\right].$$

Determine what *term* 2 and *term* 3 should be, and provide concise interpretations of what they represent.

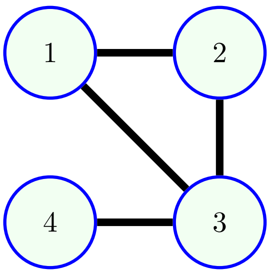


5. (a) Let G_η correspond to a complete graph with η nodes. Consider a network consisting of a ring of n of these graphs with neighboring G_η graphs connected by a single link. The figure shows an example of such a network with $n = 6$.

   i. What is the modularity of the "intuitive" partition where each G_η is assigned to its own community?

   ii. Now consider the partition where neighboring pairs of G_η's are each placed in one of $n/2$ communities. Provide a condition in the form $n < f(\eta)$ which ensures that a modularity maximization method will not prefer this partition to the intuitive partition from part (a).

   (b) Show that the largest eigenvalue for the modularity matrix for a general simple graph is non-negative.



   (c) The Laplacian matrix for the graph shown above has the following eigenvalues and eigenvectors: $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1, \lambda_4 = 0$,

$$\mathbf{v}_1 = [-0.28867513, -0.28867513, 0.8660254, -0.28867513]^T$$

$$\mathbf{v}_2 = [7.07106781e-01, -7.07106781e-01, 0, 0]^T$$

$$\mathbf{v}_3 = [-4.08248290e-01, -4.08248290e-01, 0, 8.16496581e-01]$$

$$\mathbf{v}_4 = ?$$

   i. Determine $\mathbf{v}_4$. The magnitude of the vector should be 2.

   ii. Explain if Laplacian graph partitioning will place nodes 1 and 2 in the same group