Imperial College London

MATH70009

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Bifurcation Theory

Date: 24 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the map

$$\begin{cases} \bar{x} = y, \\ \bar{y} = a - bx - y^2. \end{cases}$$

with two parameters a and b.

(a) How many fixed points does this map have at most? (4 marks)

(b) Find the bifurcation curves in the (a, b)-plane corresponding to the following bifurcations of the fixed points.

(i) Saddle-node bifurcation. (5 marks)

(ii) Period-doubling bifurcation. (5 marks)

(iii) Neimark-Sacker bifurcations. (6 marks)

(Total: 20 marks)

2. (a) Consider the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x - 6x^2 + 2xy. \end{cases}$$

Find the first Lyapunov coefficient corresponding to the zero equilibrium. (8 marks)

(b) Consider the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + \varepsilon y^2 + x^3. \end{cases}$$

(i) Find the first and second Lyapunov coefficients for the zero equilibrium at $\varepsilon = 0$. (8 marks)

(ii) Describe the dynamics near the zero equilibrium for small $\varepsilon \neq 0$. (4 marks)

(Total: 20 marks)

3. Consider the system

$$\begin{cases} \frac{dx}{dt} = y + 1, \\ \frac{dy}{dt} = \varepsilon x + x^2 - y - 1. \end{cases}$$

(a) Find the equilibrium at $\varepsilon = 0$, and show that there is a coordinate transformation $(x, y) \mapsto (v, w)$ taking the system to the form

$$\begin{cases} \frac{dv}{dt} = -v - \varepsilon(v + w) - (v + w)^2, \\ \frac{dw}{dt} = \varepsilon(v + w) + (v + w)^2. \end{cases}$$

(4 marks)

(b) (i) Write down the Taylor expansion for the center manifold up to quadratic terms in coordinate and parameter. (8 marks)

(ii) Find the restriction of the system to the center manifold. (2 marks)

(c) Describe the dynamics near the equilibrium found in (a) for small ε. (6 marks)

(Total: 20 marks)

4. Consider the map

$$\bar{x} = f(x) = ax - 2ax^3,$$

where $a \geq 0$ is a parameter.

(a) Find all fixed points of this map, and the parameter values for which they have $+1$ or -1 as their multipliers. (4 marks)

(b) For each of the fixed points, find the set of parameter values for which it is stable.

(8 marks)

(c) Let O be a fixed point in $(0, +\infty)$ as found in (a) with multiplier -1 at $a = a^*$. Show that a period-2 orbit emanates from O as a increases slightly beyond a^*. Until which value of a does this period-2 orbit exist? Explain your answer. (8 marks)

(Total: 20 marks)

5. (Mastery question) Consider the map

$$\begin{cases} \bar{x} = y + x^3, \\ \bar{y} = ax + by, \end{cases}$$

with two parameters a and b.

(a) Find the curve of parameter values for which the fixed point at $(0,0)$ undergoes a Neimark-Sacker bifurcation. Which of these parameter values correspond to strong resonances.

(6 marks)

(b) For $b \in (-1,0) \cup (0,2)$, find the first Lyapunov coefficient of the Neimark-Sacker bifurcation as a function of parameters. (7 marks)

(c) Consider any point (a,b) on the bifurcation curve found in (a) with $b \in (-1,0) \cup (0,2)$. How to change a to obtain invariant curves from the Neimark-Sacker bifurcation? How many invariant curves can be born? What are their stabilities? (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60009/MATH70009

Bifurcation Theory (Solutions)

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1. (a) Fixed points of the map are solutions of

$\boxed{\text{sim. seen} \Downarrow}$

$$\begin{cases} x = y, \\ y = a - bx - y^2. \end{cases}$$

Substituting the first equation into the second one, yields

$$y^2 + (1+b)y - a = 0. \tag{1}$$

Thus, there can be at most two fixed points.

$\boxed{4, \text{A}}$

(b) (i) The Jacobian matrix of the system is

$\boxed{\text{sim. seen} \Downarrow}$

$$J = \begin{pmatrix} 0 & 1 \\ -b & -2y \end{pmatrix}.$$

A saddle-node bifurcation corresponds to a $+1$ multiplier, which is equivalent to $\det(J - I) = 0$ where I is the 2 by 2 identity matrix. Solving the equation gives

$$1 + b + 2y = 0. \tag{2}$$

Thus, fixed points having a $+1$ multiplier correspond to the solutions of the system consisting of (1) and (2). Eliminating y in these equations gives the sought bifurcation curve

$$s_1 = \{(a,b) \mid a = -\frac{(1+b)^2}{4}\}.$$

$\boxed{5, \text{A}}$

(ii) Similarly, having -1 as a multiplier means $\det(J + I) = 0$, or,

$$1 + b - 2y = 0.$$

Combining this equation with (1) gives the bifurcation curve for the period-doubling bifurcation:

$$s_2 = \{(a,b) \mid a = \frac{3(1+b)^2}{4}\}.$$

$\boxed{5, \text{A}}$

(iii) At the critical moment, the multipliers are $\lambda_{1,2} = e^{\pm i\theta}$ with $\theta \in (0, \pi)$. The necessary condition is

$$\det J = \lambda_1 \lambda_2 = 1,$$

which gives

$$1 - b = 0. \tag{3}$$

$\boxed{2, \text{C}}$

Next, one needs to exclude the case of $\lambda_1 = \lambda_2^{-1}$ being real. That is, the characteristic equation of J: $\lambda^2 + 2y\lambda + b = 0$ has no real roots, which by (3) amounts to $4y^2 - 4 < 0$ or $|y| < 1$. We find y from (1) as

$$y = -1 \pm \sqrt{1+a}.$$

Then, $|y| < 1$ is equivalent to

$$-1 < -1 - \sqrt{1+a} < 1 \quad \text{or} \quad -1 < -1 + \sqrt{1+a} < 1.$$

The first inequality cannot be satisfied since $\sqrt{1+a} \geqslant 0$ (y must be real). The second inequality implies $-1 < a < 3$, which together with (3) gives the bifurcation curve

$$s_3 = \{(a,b) \mid -1 < a < 3, b = 1\}.$$

$\boxed{4, \mathrm{C}}$

2. (a) We take the linearization matrix to the diagonal form by letting

sim. seen ⇓

$$z = x - iy, \quad z^* = x + iy \quad \text{with the inverse} \quad x = \frac{z+z^*}{2}, \quad y = i\frac{z-z^*}{2}.$$

1, A

We get

$$\frac{dz}{dt} = \frac{dx}{dt} - i\frac{dy}{dt} = iz + \frac{3}{2}i(z+z^*)^2 + \frac{1}{2}(z^2 - (z^*)^2) = iz + (\frac{1}{2} + \frac{3i}{2})z^2 + 3izz^* + (-\frac{1}{2} + \frac{3i}{2})(z^*)^2.$$

3, A

We use the formula

$$L_1 = -\frac{1}{\omega} Im(ab),$$

where ω, a, b are the coefficients in front of the terms $iz, z^2, (z^*)^2$, respectively. Thus, $L_1 = -2/3$

4, A

meth seen ⇓

(b) (i) At $\varepsilon = 0$ the system is Hamiltonian with the energy

$$H = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

2, B

So, orbits lie in level sets of H. Since there is no other equilibrium in a small neighborhood of 0, all orbits except for 0 are one-dimensional curves. As a result, they coincide with the level sets, since orbits can not intersect. By the formula for H the level sets are topological circles around 0, so the orbits are all periodic.

4, B

This implies that all Lyapunov coefficients are zero, and particularly the first two are zero.

2, B

(ii) The linearization matrix at 0 is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for all ε. This means that the eigenvalues of the zero equilibrium remain on the imaginary axis for all ε. Hence, we cannot apply the Hopf theorem. Instead, we find that at small $\varepsilon \neq 0$ the Hamiltonian is given by

$$H = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} - \varepsilon x y^2,$$

whose level sets are still topological circles. By the same reason as in (i), all orbits near 0 are periodic.

4, A

3. (a) The equilibrium at $\varepsilon = 0$ is $(0, -1)$. Taking $y^{new} = y + 1$ and still denoting y^{new} by y, yields $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \varepsilon x + x^2 - y. \end{cases}$$

Now the equilibrium is $(0, 0)$. $\boxed{1, \text{B}}$

The linearization matrix at (0,0) with $\varepsilon = 0$ is $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. It has eigenvalues 0 and -1 with eigenvectors $(1, 0)$ and $(1, -1)$, respectively. So, it can be made diagonal by the coordinate transformation

$$v = -y, \quad w = x + y \quad \text{with the inverse} \quad x = v + w, \quad y = -v.$$

Then, the system becomes

$$\begin{cases} \frac{dv}{dt} = -\frac{dy}{dt} = -\varepsilon x - x^2 + y = -v - \varepsilon(v + w) - (v + w)^2, \\ \frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = y + \varepsilon x + x^2 - y = \varepsilon(v + w) + (v + w)^2. \end{cases}$$

$\boxed{3, \text{B}}$

(b) (i) We need to find the center manifold with a parameter. So, adding ε as a new central variable gives the extended system $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{cases} \frac{dv}{dt} = -v - \varepsilon(v + w) - (v + w)^2, \\ \frac{dw}{dt} = \varepsilon(v + w) + (v + w)^2, \\ \frac{d\varepsilon}{dt} = 0. \end{cases} \tag{4}$$

$\boxed{2, \text{B}}$

The center manifold of this system has the form

$$v = \varphi(x, \varepsilon) = aw^2 + bw\varepsilon + c\varepsilon^2 + O(|w|^3 + |\varepsilon|^3),$$

for some constants a, b, c. Denote $O_i = O(|w|^i + |\varepsilon|^i)$. From this formula and the equations for dw/dt and $d\varepsilon/dt$ in (4) we get

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial\varphi}{dw}\frac{dw}{dt} + \frac{\partial\varphi}{d\varepsilon}\frac{d\varepsilon}{dt} \\ &= 2aw(\varepsilon(v + w) + (v + w)^2) + b\varepsilon(\varepsilon(v + w) + (v + w)^2) + O_3 \\ &= 2aw(\varepsilon(w + O_2) + (w + O_2)^2) + b\varepsilon(\varepsilon(w + O_2) + (w + O_2)^2) + O_3 \\ &= O_3 \end{aligned}$$

From the equation for dv/dt in (4), we get

$$\begin{aligned} \frac{dv}{dt} &= -aw^2 - bw\varepsilon - c\varepsilon^2 + O_3 - \varepsilon(w + O_2) - (w + O_2)^2 \\ &= (-a - 1)w^2 - (b + 1)w\varepsilon - c\varepsilon^2 + O_3 \end{aligned}$$

Comparing the quadratic terms in the above two equations, we find

$$a = -1, \qquad b = -1, \qquad c = 0.$$

Hence the center manifold is given by

$$v = -\varepsilon w - w^2 + O(|w|^3 + |\varepsilon|^3). \tag{5}$$

$\boxed{6, \text{B}}$

(ii) Substituting (5) into the dw/dt equation in (4) the restricted system on the center manifold

$$\frac{dw}{dt} = \varepsilon(1-\varepsilon)w + ((1-\varepsilon)^2 - \varepsilon)w^2 + O(w^3) =: f(w,\varepsilon).$$

2, A

(c) At $\varepsilon = 0$, we have

meth seen ⇓

$$\frac{dw}{dt} = w^2 + O(w^3).$$

The first non-zero Lyapunov coefficient is $\ell_2 = 1 > 0$, so the equilibrium 0 is semi-stable: stable from left and unstable from right.

1, C

To investigate the bifurcation, let us find the control parameter. Consider the equation

$$\frac{\partial f(w,\varepsilon)}{\partial w} = \varepsilon(1-\varepsilon) + 2((1-\varepsilon)^2 - \varepsilon)w + O(w^2) = 0.$$

Since $(0,0)$ is a special solution and

$$\frac{\partial}{\partial w}\left(\frac{\partial f(w,\varepsilon)}{\partial w}\right)\bigg|_{0,0} = 2\ell_2 = 2 \neq 0,$$

using Implicit Function Theorem, we get the solution

$$w = w^*(\varepsilon) = -\frac{\varepsilon(1-\varepsilon)}{2((1-\varepsilon)^2 - \varepsilon)} + O(\varepsilon^2) = -\frac{1}{2}\varepsilon + O(\varepsilon^2),$$

for all ε close to 0, since both $(1-\varepsilon)$ and $(1-\varepsilon)^2 - \varepsilon$ are close to 1.

The control parameter is then $\mu = f(w^*(\varepsilon),\varepsilon) = -\dfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)$.

2, C

For a saddle-node bifurcation, the equilibrium decomposes into two if $\mu\ell_2 < 0$ and disappears if $\mu\ell_2 > 0$. In our case, $\mu < 0$ for all non-zero ε and $\ell_2 > 0$, so the semi-stable equilibrium decomposes into two equilibria for all small non-zero ε. Due to the stability of 0 at $\varepsilon = 0$, of these two new equilibria, the left one is stable and the right one is unstable.

2, C

After return to the full system, a contraction transverse to W^c should be added. The right one is now a saddle (so still unstable), while the left one remains stable.

1, C

4. (a) Solving $x = ax - 2ax^3$ gives the non-negative fixed points $O_1 : x = 0$ and $O_{2,3} : x = \pm\sqrt{\frac{a-1}{2a}}$, where the former exists for all parameter values and the latter exists for $a > 1$. sim. seen ⇓ 2, A

The multiplier is given by $\lambda = f'(x) = a(1 - 6x^2)$. Thus, O_1 has multiplier $+1$ at $a = 1$ and $O_{2,3}$ have multiplier -1 at $a = 2$. 2, A

(b) The fixed points are stable when $|\lambda| < 1$. So, O_1 is stable when $a < 1$ and $O_{2,3}$ are stable when $a < 2$. 2, D

We need to find the stability at $|\lambda| = 1$, i.e., $a = 1$ for O_1 and $a = 2$ for $O_{2,3}$. O_1 is stable if the sign of the first non-zero Lyapunov coefficient is negative, which is the same as the sign of $f'''(O_1) = -12a = -12 < 0$. Thus O_1 is stable for $a \leqslant 1$. 3, D

For $O_{2,3}$, we need to check the sign of the Schwarzian derivative S at O_2. We have

$$S = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \frac{-12}{1 - 6x^2} - \frac{3}{2}\left(\frac{-12}{1 - 6x^2}\right)^2.$$

At $a = 2$, $O_{2,3}$ are given by $x = \pm 1/2$, so $S < 0$ and they are stable for $a \leqslant 2$. 3, D

(c) The positive fixed point is O_2. An period-2 orbit is born from O_2 when it changes stability, namely, when $a > 2$ is close to 2. 1, D

When we increase a further, this period-2 orbit can disappear only if the orbit has a multiplier $+1$. 1, D

Denote the orbit by (x_1, x_2) with $x_1 \neq x_2$. The conditions for being a period-2 orbit with a multiplier $+1$ are

$$x_1 = f(x_2), \qquad x_2 = f(x_1), \qquad (f^2)'(x_1) = f'(x_1)f'(x_2) = 1.$$

In our case they are

$$x_1 = a(x_2 - 2x_2^3), \qquad x_2 = a(x_1 - 2x_1^3), \qquad a^2(1 - 6x_1^2)(1 - 6x_2^2) = 1. \quad (6)$$

2, D

Taking sum and difference of the first two equations give

$$x_1 + x_2 = a(x_1 + x_2)(1 - 2x_1^2 + 2x_1x_2 - 2x_2^2) \Rightarrow a = \frac{1}{1 - 2x_1^2 + 2x_1x_2 - 2x_2^2}$$

and

$$x_1 - x_2 = a(x_1 - x_2)(-1 + 2x_1^2 + 2x_1x_2 + 2x_2^2) \Rightarrow a = \frac{1}{-1 + 2x_1^2 + 2x_1x_2 + 2x_2^2}.$$

Thus, we obtain $x_1^2 + x_2^2 = \frac{1}{2}$. Divide the product of the first two equations in (6) and divide it by the third one, we get

$$\frac{x_1x_2}{1} = \frac{x_1x_2(1 - 2x_2^2)(1 - 2x_1^2)}{(1 - 6x_1^2)(1 - 6x_2^2)},$$

which implies $x_1^2x_2^2 = \frac{1}{8}(x_1^2 + x_2^2) = \frac{1}{16}$. One sees that the only possible solution to this equation is $x_1^2 = x_2^2$. Note that this orbit is born from O_2, so it is bounded by $O_1 : x = 0$ and hence x_1 and x_2 must be positive. It follows that we have $x_1 = x_2$, contradicting the condition $x_1 \neq x_2$. Thus, the period-2 orbit born from O_2 persist for all $a > 2$. 4, D

5. (a) The Jocobian matrix is unseen $\Downarrow$

$$J = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}.$$

The fixed point 0 undergoes a Neimark-Sacker bifurcation if and only if

1. $\det J|_0 = 1 \Rightarrow a = -1$, and 1, M
2. the characteristic equation $|J_0 - \lambda I| = \lambda^2 - b\lambda - a = 0$ has non-real solutions at $a = -1$, that is, $b^2 - 4 < 0 \Rightarrow |b| < 2$. 2, M

So, the bifurcation curve is $\{a = -1, -2 < b < 2\}$.

The strong resonances correspond to $\omega = 2\pi/3$ and $\omega = \pi/2$, where ω is the argument of the multipliers of the fixed point. 1, M

The eigenvalues at $a = 0$ are

$$\lambda = \frac{b - \sqrt{b^2 - 4}}{2} = \frac{b}{2} - i\frac{\sqrt{4 - b^2}}{2} = e^{-i\omega},$$

and λ^* (the complex conjugate of λ) with

$$\tan\omega = \frac{\sqrt{4 - b^2}}{b} \tag{7}$$

Thus, $\omega = 2\pi/3$ at $b = -1$ and $\omega = \pi/2$ at $b = 0$. 2, M

(b) The eigenvector corresponding to λ is $w = \begin{pmatrix} 1 \\ \frac{b}{2} \end{pmatrix} + i\begin{pmatrix} 0 \\ -\frac{\sqrt{4-b^2}}{2} \end{pmatrix}$. Denote $Q = \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & -\frac{\sqrt{4-b^2}}{2} \end{pmatrix}$ with $Q^{-1} = \begin{pmatrix} 1 & \frac{b}{\sqrt{4-b^2}} \\ 0 & -\frac{2}{\sqrt{4-b^2}} \end{pmatrix}$. Applying the coordinate transformation $(u, v)^T = Q^{-1}(x, y)^T$, we obtain unseen $\Downarrow$

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^3 \\ 0 \end{pmatrix}.$$

2, M

Further taking $z = u + iv$ and $z^* = u - iv$ leads to

$$\bar{z} = ze^{i\omega} + (\frac{z + z^*}{2})^3.$$

Since there is no strong resonance for $b \notin \{-1, 0\}$, $z^2 z^*$ is the only non-resonant term. Thus, there exists normal form coordinates where the map assumes the form

$$\bar{z} = ze^{i\omega} + \frac{1}{8}z^2 z^* = e^{i\omega}(z + \frac{1}{8}(\cos\omega - i\sin\omega)z^2 z^*). \tag{8}$$

3, M

This along with (7) gives the first Lyapunov coefficient as

$$L_1 = \frac{1}{8}\cos\omega = \frac{1}{8}\cos\arctan\frac{\sqrt{4 - b^2}}{b}.$$

2, M

(c) Recall that the control parameter of the Neimark-Sacker bifurcation is the deviation of the modulus of λ from 1, which is $|\lambda| - 1 = \sqrt{\det J} - 1 = \sqrt{-a} - 1$. Hence, (8) with parameter a takes the form unseen $\Downarrow$

$$\bar{z} = e^{i\omega}(z(\sqrt{-a} - 1) + \frac{1}{8}(\cos\omega - i\sin\omega)z^2 z^*),$$

where ω also depends on a. $\boxed{3, \mathrm{M}}$

For $b \in (-1, 0) \cup (0, 2)$, we do not have strong resonances so one invariant curve can be born when $L_1(\sqrt{-a} - 1) < 0$.

For $b \in (-1, 0)$, ω changes from $2\pi/3$ to $\pi/2$ by (7), which implies that $\cos\omega$ and hence L_1 is negative. Thus, a stable invariant curve is born when a decreases slightly from -1. $\boxed{2, \mathrm{M}}$

For $b \in (0, 2)$, ω changes from $\pi/2$ to 0 by (7), which implies that $\cos\omega$ and hence L_1 is positive. Thus, an unstable invariant curve is born when a increases slightly from -1. $\boxed{2, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60009/70009	1	No Comments Received
MATH60009/70009	2	No Comments Received
MATH60009/70009	3	No Comments Received
MATH60009/70009	4	No Comments Received
MATH60009/70009	5	No Comments Received