Math40002 Analysis 1 — Problem Sheet 5

1. Fix $x > 0$. Prove $(1+x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$. Deduce that $(1+x)^{-n} \to 0$. Deduce that if $r \in (0,1)$ then $r^n \to 0$, and if $r \in (1,\infty)$ then $r^n \to \infty$.

2. Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. In lectures we proved that if $L < 1$ then $a_n \to 0$.

   (a) Prove that if $L > 1$ then $|a_n| \to \infty$.

   (b) Give an example with $|a_{n+1}/a_n| < 1\ \forall n$ but $a_n \not\to 0$.

   Give (without proof) examples where $L = 1$ and

   (i) $a_n \to 0$, (iii) a_n divergent and bounded,

   (ii) $a_n \to a \neq 0$, (iv) $a_n \to \infty$.

3. Let $(a_n)_{n\geq 1}$ be a sequence of *strictly positive* real numbers.
   Give an example such that $(1/a_n)_{n\geq 1}$ is unbounded.
   Suppose that $a_n \to a \neq 0$. Prove *from first principles* that $(1/a_n)_{n\geq 1}$ is bounded.

4.† Fix $r \in (0, 1/8)$. Define $(a_n)_{n\geq 1}$ by $a_1 := 1$ and $a_{n+1} = ra_n^2 + 1$.

   (a) Show that $a_{n+1} - a_n = r(a_n + a_{n-1})(a_n - a_{n-1})$.

   (b) Show that if
   $$0 < a_j < 2 \qquad \forall\, j \leq n, \tag{1}$$
   then
   $$|a_{n+1} - a_n| \;<\; (4r)^n/4. \tag{2}$$

   (c) Deduce that if (1) holds, then $a_{n+1} < r/(1-4r) + 1$.

   (d) Conclude that (1) holds for $j = n+1$ too, and so $\forall j$ by induction.

   (e) Using (2) deduce $|a_m - a_n| < (4r)^n/4(1-4r)$ for $m \geq n$.

   (f) Deduce a_n is Cauchy. What does it converge to?

5.* Show that *any* sequence of real numbers $(a_n)_{n\geq 0}$ has a subsequence which either converges, or tends to ∞, or tends to $-\infty$.

6. At home Professor Papageorgiou has made a fully realistic mathematical model of a dart board. It is a copy of the unit interval $[0,1]$ in a frictionless vacuum. He throws a countably infinite number of darts at it, the nth landing at $a_n \in [0,1]$.

   He then makes a small dot $(x - \epsilon_x,\, x + \epsilon_x)$ around each point $x \in [0,1]$ with his pen. Prove that however small he makes each dot, at least one of them will contain an infinite number of darts $a_n \in [0,1]$.

   What if he only makes dots around each dart $a_n \in [0,1]$?

7. Let $(a_n)_{n\geq 1}$ be the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$

   (i) Give (without proof) a subsequence of $(a_n)_{n\geq 1}$ which converges to $\ell = 0$, and one which converges to $\ell = 1$.

   (ii) Given any $\ell \in (0,1)$, give (with proof) a subsequence convergent to ℓ.

8. A student is learning about Cauchy sequences, and thinks they have a brilliant proof that allows them to precisely identify the limit of a Cauchy sequence straight from the Cauchy condition. The student gives their proof below, is it correct?

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } n, m \geq N \;\Rightarrow\; |a_n - a_m| < \epsilon$$

$$\Rightarrow \; \forall n \geq N \quad |a_n - a_N| < \epsilon$$

$$\Rightarrow \; a_n \to a_N \;\; \text{as} \;\; n \to \infty.$$

Starred questions * *are good to prepare to discuss at your Problem Class.*
Questions marked † are slightly harder (closer to exam standard), but good for you.