Question 1

Suppose that the data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are observations of the independent random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, where each random variable each follow a Poisson distribution with parameter $\lambda > 0$. Recall that the if random variable $X \sim \text{Pois}(\lambda)$, then X has the probability mass function

$$p_X(x) = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & \text{if } x \in \{0, 1, 2, \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the maximum likelihood estimate of λ given the data $\mathbf{x}$.

Question 2

Suppose you are on a gameshow and win a prize: there are two containers, labelled A and B, that are each filled with 1000 banknotes, where each banknote is either a £10 note, a £20 note or a £50 note. The prize is that you can choose to keep one of the containers. Before you choose a container, you are told the distributions of the banknotes in each of the two containers is different. Writing the sample space as $\Omega = \{10, 20, 50\}$, and considering the parameter θ as identifying the distribution, $\theta \in \{1, 2\}$, you are given the information that the two distributions are summarised by the following table, where f_θ is the probability mass function of the distribution when the parameter value is θ:

	$\omega = 10$	$\omega = 20$	$\omega = 50$
$f_1(\omega)$	0.3	0.4	0.3
$f_2(\omega)$	0.3	0.1	0.6

You do not know which container has which distribution: either A has distribution f_1 and B has distribution f_1, or vice versa, but this is unknown. However, you are allowed to sample from one of the containers before making your choice: you can pull exactly one banknote out of exactly one of the containers, and then choose to keep either that container or the other container. We plan to use maximum likelihood estimation to aid us in choosing which container to keep (and you would like to keep the container with the most money).

(a) What is the maximum likelihood estimate (MLE) for θ if you sample $\omega = 50$?

(b) What is expected amount of money in each container?

(c) If you chose to sample from container A and pulled out a £50, would you prefer to keep container A or container B? (Supposing you want to keep the container with the most money.)

(d) Again, suppose you choose to sample from container A and pull out a £50. How many times more (or less) plausible/likely is it that container A is the container with the most money?

(e) What is the MLE for θ if you sample $\omega = 20$?

(f) Suppose you choose to sample from container A and pull out a £20. Would you choose to keep container A or container B?

(g) Again, suppose you choose to sample from container A and pull out a £20. How many times more (or less) plausible/likely is it that container A is the container with the most money?

(h) What is the MLE for θ if you sample $\omega = 10$?

(i) If you chose to sample from container A and pulled out a £10, would you choose to keep container A or container B?

(j) Is the MLE always unique?

Question 3

Suppose that for every batch of lightbulbs produced in a factory an unknown proportion θ are defective. Suppose that a random sample of n lightbulbs is taken from a batch, and for $i = 1, 2, \dots, n$ let the random variable $X_i = 1$ if the ith lightbulb is defective and let $X_i = 0$ otherwise. We assume that the random variables $\mathbf{X} = (X_1, X_2, \dots, X_n)$ are independent, and we observe $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

(a) What distribution can we use to model each X_i?

(b) Given your answer in (a), write down the probability mass/density function $f(x_i|\theta)$.
Hint: the p.m.f./p.d.f. should be a polynomial in θ.

(c) Given your answer in (b), compute down the likelihood function $L(\theta|\mathbf{x})$.

(d) Compute the maximum likelihood estimate for θ, given the observations $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

(e) Write down the maximum likelihood estimator for θ, given the random variables $\mathbf{X} = (X_1, X_2, \dots, X_n)$.

Question 4

Suppose that the random variables $X_1, X_2, \dots, X_n$ are independent and identically distributed according to a uniform distribution on the closed interval $[0, \theta]$, for some parameter $\theta > 0$, where the exact value of the parameter θ is unknown. Given that $\mathbf{X} = (X_1, X_2, \dots X_n)$ is observed as $\mathbf{x} = (x_1, x_2, \dots, x_n)$, find the maximum likelihood estimator of θ by doing the following steps:

(a) Write down the probablity density function $f(x_i|\theta)$ for observation x_i, for $i = 1, 2, \dots, n$.

(b) Derive the likelihood $L(\theta|\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

(c) Identify any conditions the maximum likelihood estimate of θ must satisfy, and find $\widehat{\theta}$, the maximum likelihood esitmate of θ.

(d) Given your answer in Part (c), write down the maximum likelihood **estimator** of θ.