Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Groups and Representations

Date: **Monday 23rd May 2016**

Time: **14.00 – 16.30**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.

- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.

- Calculators may not be used.

1. Let G be a simple group of order 60. Prove the following assertions.

   (a) G is unique up to isomorphism.

   (b) G is isomorphic to $L_2(4)$.

   (c) G is isomorphic to $L_2(5)$.

   (d) The alternating group A_6 of degree 6 contains at least 12 different subgroups isomorphic to G.

   (e) If G_1 and G_2 are subgroups in A_6, isomorphic to G, then there is an automorphism of A_6 which maps G_1 onto G_2.

   (f) The group A_6 possesses an automorphism which is not induced by conjugation by an element of S_6.

2. Prove that the groups $L_3(2)$ and $L_2(7)$ are isomorphic.

3. Let $H \cong L_3(2) \cong L_2(7)$. Let Ω be the set of 1-dimensional subspaces in a 3-dimensional $GF(2)$-space $V_3(2)$ on which H acts as $L_3(2)$. Let Δ be the set of 1-dimensional subspaces in a 2-dimensional $GF(7)$-space $V_2(7)$ on which H acts as $L_2(7)$. Let S be a subgroup of order 7 in H and let

$$\Lambda = \{Sa \mid a \in H\}$$

be the set of right cosets of S in H on which H acts by right multiplication:

$$h : Sa \mapsto S(ah).$$

For a H-set X let 2^X denote the corresponding $GF(2)$-permutation module.

(a) Show that 2^Ω is the direct sum of the 1-dimensional trivial module and an indecomposable extension of $V_3(2)^*$ by $V_3(2)$.

(b) Show that 2^Δ contains a submodule W_1 which is the trivial 1-dimensional module and a 6-dimensional submodule W_6 containing it, such that $W_6/W_1 \cong V_3(2) \oplus V_3(2)^*$.

(c) Show that 2^Λ is the direct sum of 2^Δ and two copies of the 8-dimensional Steinberg module of H, the latter being realized by (3×3)-matrices over $GF(2)$ with trace equal to zero, with H acting via conjugation as $L_3(2)$.

4. For H and Δ being as in Question 3, let Δ_1, Δ_2, Δ_3 be pairwise disjoint 8-element sets, let $\varphi_i : \Delta \to \Delta_i$ be a bijection, and let Δ_i be turned into H-set by the rule $h \cdot \varphi_i(x) = \varphi_i(h(x))$, where $h \in H$ and $x \in \Delta$, and let $i \in \{1, 2, 3\}$. Let A and B be two different 4-dimensional submodules of 2^Δ, Ξ be the disjoint union of Δ_1, Δ_2, Δ_3.

Define $C \subseteq 2^\Xi$ by the following rule:

$$C = \{\varphi_1(X) + \varphi_1(Z) + \varphi_2(Y) + \varphi_2(Z) + \varphi_3(X) + \varphi_3(Y) + \varphi_3(Z) \mid X, Y \subseteq A, Z \subseteq B\}$$

(a) What is the dimension of C?

(b) Show that $|D|$ is divisible by 4 for every $D \subseteq C$.

(c) Show that if D is a non-empty subset from C then $|D| \geq 8$.

5. Let $H = L_3(2)$ and let $V = V_3(2)$ be the corresponding vector space. Let P_1 be the stabilizer of a 1-dimensional subspace U of V, and let P_2 be the stabilizer in H of a 2-dimensional subspace W of V containing U.

(a) Show that $P_1 \cong P_2 \cong S_4$ and $P_1 \cap P_2 \cong D_8$.

(b) For $F = SL_3(\mathbb{C})$ construct a pair of injective homomorphisms $\varphi_1 : P_1 \to F$, $\varphi_2 : P_2 \to F$, such that $\varphi_1(h) = \varphi_2(h)$ for every $h \in P_1 \cap P_2$.