Applied Complex Analysis - Lecture Four

Andrew Gibbs

January 2025

Derivatives via Cauchy's Integral formula

Let $f(z)$ be analytic inside and on a closed anti-clockwise path γ bounding a simply-connected region D. Then for any z within D:

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

- Proved
- This implies that analytic function derivatives decay exponentially: $|f^{(n)}(z)| \leq n! M / r^n$.

Derivatives via Cauchy's Integral formula

Let $f(z)$ be analytic inside and on a closed anti-clockwise path γ bounding a simply-connected region D. Then for any z within D:

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

- Proved
- This implies that analytic function derivatives decay exponentially: $|f^{(n)}(z)| \leq n! M / r^n$.

Derivatives via Cauchy's Integral formula

Let $f(z)$ be analytic inside and on a closed anti-clockwise path γ bounding a simply-connected region D. Then for any z within D:

$$\frac{d^n}{dz^n}f(z) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

- Proved
- This implies that analytic function derivatives decay exponentially: $|f^{(n)}(z)| \leq n!M/r^n$.

Some more theorems

The maximum modulus principle states that a function f, analytic in $D \subset \mathbb{C}$, takes its maximal absolute value $|f(z)|$ on the boundary of D.

- Implies that stationary points are saddle points
- Let's just look at some plots to convince ourselves...

$\Longrightarrow$ *Louville's Theorem*: If a function is entire and bounded everywhere in $\mathbb{C}$, then it must be constant.

$\Longrightarrow$ *The fundamental Theorem of algebra:* Every non-constant polynomial must have a root in the complex plane.

Some more theorems

The maximum modulus principle states that a function f, analytic in $D \subset \mathbb{C}$, takes its maximal absolute value $|f(z)|$ on the boundary of D.

- Implies that stationary points are saddle points
- Let's just look at some plots to convince ourselves...

$\Longrightarrow$ *Louville's Theorem*: If a function is entire and bounded everywhere in $\mathbb{C}$, then it must be constant.

$\Longrightarrow$ *The fundamental Theorem of algebra:* Every non-constant polynomial must have a root in the complex plane.

Some more theorems

The maximum modulus principle states that a function f, analytic in $D \subset \mathbb{C}$, takes its maximal absolute value $|f(z)|$ on the boundary of D.

- Implies that stationary points are saddle points
- Let's just look at some plots to convince ourselves...

$\Longrightarrow$ *Louville's Theorem*: If a function is entire and bounded everywhere in $\mathbb{C}$, then it must be constant.

$\Longrightarrow$ *The fundamental Theorem of algebra:* Every non-constant polynomial must have a root in the complex plane.

Some more theorems

The maximum modulus principle states that a function f, analytic in $D \subset \mathbb{C}$, takes its maximal absolute value $|f(z)|$ on the boundary of D.

- Implies that stationary points are saddle points
- Let's just look at some plots to convince ourselves...

$\Longrightarrow$ *Louville's Theorem*: If a function is entire and bounded everywhere in $\mathbb{C}$, then it must be constant.

$\Longrightarrow$ *The fundamental Theorem of algebra:* Every non-constant polynomial must have a root in the complex plane.

Taylor series

Suppose $f(z)$ is analytic in $|z - z_0| \leq R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

- Always converges for complex analytic functions - in contrast to real analytic functions, e.g. $f(z) = e^{-1/x^2}$.
- Proof
- By combining with earlier results, this implies exponential convergence of Taylor polynomials in a smaller disk with $r < R$.

Taylor series

Suppose $f(z)$ is analytic in $|z - z_0| \leq R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

- Always converges for complex analytic functions - in contrast to real analytic functions, e.g. $f(z) = e^{-1/x^2}$.
- Proof
- By combining with earlier results, this implies exponential convergence of Taylor polynomials in a smaller disk with $r < R$.

Taylor series

Suppose $f(z)$ is analytic in $|z - z_0| \leq R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

- Always converges for complex analytic functions - in contrast to real analytic functions, e.g. $f(z) = e^{-1/x^2}$.
- Proof
- By combining with earlier results, this implies exponential convergence of Taylor polynomials in a smaller disk with $r < R$.

Laurent series

Suppose $f(z)$ is analytic in the annular region $r < |z - z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
$$= \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

- Proof
- We see exponential convergence of Laurent polynomials, for similar reasons to the Taylor case.
- We have touched on the concept of *rational approximation*, a current hot topic in approximation theory.

Laurent series

Suppose $f(z)$ is analytic in the annular region $r < |z - z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
$$= \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

- Proof
- We see exponential convergence of Laurent polynomials, for similar reasons to the Taylor case.
- We have touched on the concept of *rational approximation*, a current hot topic in approximation theory.

Laurent series

Suppose $f(z)$ is analytic in the annular region $r < |z - z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
$$= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

- Proof
- We see exponential convergence of Laurent polynomials, for similar reasons to the Taylor case.
- We have touched on the concept of *rational approximation*, a current hot topic in approximation theory.

Zeros and Singularities of Complex Functions

Zeros and Singularities of Complex Functions

- We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(z_0) \neq 0$.
- Thm: A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
- Proof
- A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.
- A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Zeros and Singularities of Complex Functions

- We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(z_0) \neq 0$.
- Thm: A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
- Proof
- A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.
- A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Zeros and Singularities of Complex Functions

- We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(z_0) \neq 0$.
- Thm: A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
- Proof
- A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.
- A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Zeros and Singularities of Complex Functions

- We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(z_0) \neq 0$.
- Thm: A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
- Proof
- A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.
- A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Zeros and Singularities of Complex Functions

- We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(z_0) \neq 0$.
- Thm: A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.
- Proof
- A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.
- A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for $0 < |z-z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z-z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Residue Theory

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) =$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) =$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \mathrm{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\mathrm{Res}(f, 0) =$
- for $f = \frac{1}{z^2}$, we have $\mathrm{Res}(f, 0) =$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\mathrm{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\mathrm{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \mathrm{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\mathrm{Res}(f, 0) =$
- for $f = \frac{1}{z^2}$, we have $\mathrm{Res}(f, 0) =$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\mathrm{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\mathrm{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) =$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) =$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) = 0$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \mathrm{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\mathrm{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\mathrm{Res}(f, 0) = 0$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\mathrm{Res}(f, 0) =$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\mathrm{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) = 0$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) = 0$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) = 0$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) = 0$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) =$

Residue Theory

The coefficient a_{-1} in the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) = 1$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) = 0$
- for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\text{Res}(f, 0) = 0$
- for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\text{Res}(f, 0) = 1$

Connecting residues to closed contour integrals

Thm: Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (the radius of convergence), then

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z) dz.$$

Proof

Residue Theorem Let $f(z)$ be analytic in some $\mathcal{D} \setminus \{z_1, z_2, \ldots z_n\}$ bounded by a closed path γ, where $z_1, z_2, \ldots z_n$ are poles or essential singularities lying inside γ. Then

$$\oint_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

Proof

Connecting residues to closed contour integrals

Thm: Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (the radius of convergence), then

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z) dz.$$

Proof

Residue Theorem Let $f(z)$ be analytic in some $\mathcal{D} \setminus \{z_1, z_2, \ldots z_n\}$ bounded by a closed path γ, where $z_1, z_2, \ldots z_n$ are poles or essential singularities lying inside γ. Then

$$\oint_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

Proof

Connecting residues to closed contour integrals

Thm: Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (the radius of convergence), then

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z) dz.$$

Proof

Residue Theorem Let $f(z)$ be analytic in some $\mathcal{D} \setminus \{z_1, z_2, \ldots z_n\}$ bounded by a closed path γ, where $z_1, z_2, \ldots z_n$ are poles or essential singularities lying inside γ. Then

$$\oint_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

Proof

Connecting residues to closed contour integrals

Thm: Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (the radius of convergence), then

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z)dz.$$

Proof

Residue Theorem Let $f(z)$ be analytic in some $\mathcal{D} \setminus \{z_1, z_2, \dots z_n\}$ bounded by a closed path γ, where $z_1, z_2, \dots z_n$ are poles or essential singularities lying inside γ. Then

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

Proof

Ways to compute residues

1. For

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + \cdots \, ,$$

so that $f(z)$ has a pole of order m at z_0,

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)].$$

2. For

$$f(z) = \frac{A(z)}{(z-z_0)^m},$$

where $A(z)$ is analytic at $z = z_0$ (and that $A(z_0) \neq 0$),

$$\text{Res}(f, z_0) = \frac{A^{(m-1)}(z_0)}{(m-1)!}.$$

Ways to compute residues

1. For

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + \cdots ,$$

so that $f(z)$ has a pole of order m at z_0,

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)].$$

2. For

$$f(z) = \frac{A(z)}{(z-z_0)^m},$$

where $A(z)$ is analytic at $z = z_0$ (and that $A(z_0) \neq 0$),

$$\text{Res}(f, z_0) = \frac{A^{(m-1)}(z_0)}{(m-1)!}.$$

Ways to compute residues (continued)

3. If $f(z)$ contains a simple pole (pole of order $m = 1$) and $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0 and B has a simple zero at z_0 $(m = 1)$, with $A(z_0) \neq 0$, then

$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}.$$

Example: Residues and contour integrals of

$$f(z) = \frac{1}{1 + z^4}$$

Ways to compute residues (continued)

3. If $f(z)$ contains a simple pole (pole of order $m = 1$) and $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0 and B has a simple zero at z_0 $(m = 1)$, with $A(z_0) \neq 0$, then

$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}.$$

Example: Residues and contour integrals of

$$f(z) = \frac{1}{1 + z^4}$$

Analytic Continuation

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

Using contour deformation to evaluate

$$\int_{-\infty}^{\infty} f(z)dz,$$

where f has poles.

(Some) applications

- Statistics, e.g. Cauchy-Lorentz distribution
- Fourier and (inverse) Laplace transforms
- Potential flow theory, poles represent sinks

(Some) applications

- Statistics, e.g. Cauchy-Lorentz distribution
- Fourier and (inverse) Laplace transforms
- Potential flow theory, poles represent sinks

(Some) applications

- Statistics, e.g. Cauchy-Lorentz distribution
- Fourier and (inverse) Laplace transforms
- Potential flow theory, poles represent sinks

Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$

- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad k > 0.$$

- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1.$$