Chapter 5: The Wiener-Hopf Method

5.1 Riemann-Hilbert Problems

Definition 5.38. Given functions $f(z)$ and $g(z)$ defined on a smooth contour γ, a (scalar) **Riemann-Hilbert problem** consists of finding an analytic function $\phi(z)$ on $\mathbb{C}/\gamma$ which remains regular as $z \to \infty$ (i.e. $\lim_{z\to\infty} \phi(z) = \text{constant}$) such that the jump condition

$$\phi_+(t) - g(t)\phi_-(t) = f(t), \quad \text{for } t \in \gamma,$$

holds (where, as before, $\phi_+(t)$ and $\phi_-(t)$ represent the limiting values of ϕ as it approaches $t \in \gamma$ from the left or right).

Remark: We solved one of these for the function $\phi(z) = \sqrt{z^2 - 1}$ along the segment of the real line from $[-1, 1]$ way back in chapter 2 when deriving the Hilbert-inversion formula.

There are a huge number of applications to such problems (see for instance Trogdon & Olver 2015) and an entire course could be dedicated to their applications and solution methods. Some classical applications include:

(1) Solving integral-differential equations on half-lines via Wiener-Hopf factorisation (the focus of this chapter).

(2) Spectral analysis of Schrödinger operators.

(3) Problems in Ideal fluid flow.

More recently, non-classical applications have arisen from integrable systems:

(4) Solutions to Painlevé equations.

(5) Random matrix eigenvalue statistics.

(6) Asymptotics of orthogonal polynomials.

(7) Solving partial differential equations like the Korteweg de Vries (KdV) equation describing shallow water waves:

$$u_t + 6uu_x + u_{xxx} = 0.$$

In our course we will focus on one application and method of solution, namely the Wiener-Hopf method to solve a Riemann-Hilbert problem that arises from applying Fourier transforms to integral-differential equations.

Before we discuss the problems we will solve, let's recap some properties of the Fourier transform; a key tool in the Wiener-Hopf method.

5.2 The Fourier transform

Definition 5.39. Suppose $f(x)$ is defined for $-\infty < x < \infty$. For $s \in \mathbb{R}$, we define the **Fourier transform**, $F(s)$ (in many texts this is denoted $\hat{f}(s)$), of $f(x)$ to be

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx. \tag{70}$$

The inversion (**inverse Fourier transform**) of (70) is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds. \tag{71}$$

Note that these definitions hold for certain 'nice' functions (for example those which are piecewise differentiable and absolutely integrable). Note also the factor of $1/2\pi$ outside the integral in (71). In some definitions this factor is moved to (70) and sometimes the signs in the exponents of the exponential terms are also switched.

Let's do an example.

Example: Let

$$f(x) = \frac{1}{x^2+a^2},$$

where $a > 0$ is real. Then we have

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+a^2}dx.$$

First, for $s = 0$:

$$F(s) = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int_{-\infty}^{\infty}\frac{dx}{1+(x/a)^2} = \frac{1}{a}\int_{-\pi/2}^{\pi/2} dt = \frac{\pi}{a},$$

where the substitution $\tan t = x/a$ was used.

Now let $s > 0$; consider

$$g(z) = \frac{e^{isz}}{z^2+a^2} = \frac{e^{isz}}{(z-ai)(z+ai)}.$$

Take γ to be $\gamma = \gamma_1 + \gamma_R$ as shown in figure 69. Then, by the residue theorem

$$\oint_\gamma g(z)dz = 2\pi i \text{Res}\{g, ai\} = \frac{\pi}{a}e^{-sa}.$$

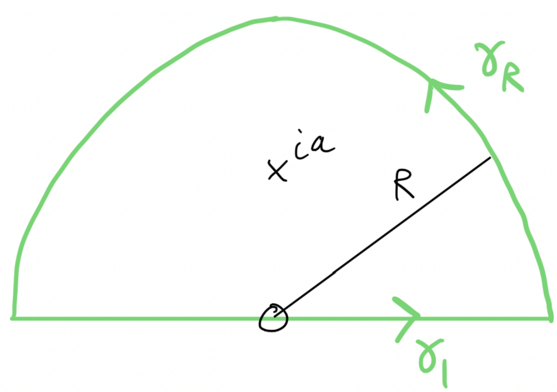

Figure 69: Contour γ.

Also, since $s > 0$, one can check that

$$\lim_{R\to\infty}\int_{\gamma_R} g(z)dz = 0.$$

Thus

$$F(s) = \frac{\pi}{a} e^{-sa}, \quad s > 0.$$

For $s < 0$, one can find that (by closing the contour in the LHP) $F(s) = \frac{\pi}{a} e^{sa}$, $s < 0$. Hence, putting everything together

$$F(s) = \frac{\pi}{a} e^{-a|s|}, \quad \forall s \in \mathbb{R}.$$

Let's check the inversion formula works:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{a} e^{-a|s|} e^{-isx} ds \\
&= \frac{1}{2a} \left(\int_{-\infty}^{0} e^{(a-ix)s} ds + \int_{0}^{\infty} e^{-(a+ix)s} ds \right) \\
&= \frac{1}{2a} \left[\frac{e^{(a-ix)s}}{a-ix} \right]_{-\infty}^{0} + \frac{1}{2a} \left[\frac{e^{-(a+ix)s}}{-(a+ix)} \right]_{0}^{\infty} \\
&= \frac{1}{2a} \left(\frac{1}{a-ix} + \frac{1}{a+ix} \right) \\
&= \frac{1}{x^2 + a^2},
\end{aligned}$$

as expected!

Fourier transforms of derivatives

Let $F_1(s)$ denote the Fourier transform of the derivative $f'(x)$ of some function $f(x)$. We have

$$\begin{aligned}
F_1(s) &= \int_{-\infty}^{\infty} f'(x) e^{isx} dx \\
&= \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} isf(x) e^{isx} dx \\
&= -isF(s),
\end{aligned}$$

provided we confine ourselves to functions which decay as $|x| \to \infty$ so that the boundary term from the integration by parts in line 2 tends to 0. In a similar manner, after conducting n integrations by parts, one can show that

$$F_n(s) = (-is)^n F(s). \tag{72}$$

This property allow one to convert differential equations for $f(x)$ to algebraic equations for $F(s)$. If we can then solve for $F(s)$, then we can retrieve $f(x)$ by the inversion formula.

Fourier transforms of convolution integrals

Let $F(s)$ and $G(s)$ be the Fourier transforms of $f(x)$ and $g(x)$ respectively. Then one can show that the Fourier transform of both

$$\int_{-\infty}^{\infty} g(y) f(x-y) dy \quad \text{and} \quad \int_{-\infty}^{\infty} f(y) g(x-y) dy,$$

is $F(s)G(s)$. That is, if

$$h(x)=\int_{-\infty}^{\infty} f(y)g(x-y)dy \quad \Rightarrow \quad H(s)=F(s)G(s). \tag{73}$$

Proof.

$$\begin{aligned}
h(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)dy \\
&= \int_{-\infty}^{\infty} f(y)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} G(s)e^{-is(x-y)}ds\right)dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)e^{isy}dy\right)G(s)e^{-isx}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)G(s)e^{-isx}ds,
\end{aligned}$$

where in the second line we replaced $g(x-y)$ by its inverse Fourier transform, in the third line the order of integration was changed and in the final line we replaced $f(y)$ by its Fourier transform. Hence

$$H(s)=F(s)G(s),$$

as required. □

Using Fourier transforms to solve integral equations

In physical applications one often encounters integral equations of the form

$$\lambda f(x)+\int_{-\infty}^{\infty} k(x-y)f(y)dy = p(x), \quad -\infty < x < \infty, \tag{74}$$

where λ is a known parameter, $k(x)$ is known (often referred to as the kernel function), $p(x)$ is known (the 'driving' term) and we wish to solve for $f(x)$.

We can convert (74) to an algebraic equation in $F(s)$. Taking Fourier transforms of both sides gives

$$\lambda F(s)+K(s)F(s)=P(s),$$

which upon re-arranging for $F(s)$ gives

$$F(s)=\frac{P(s)}{\lambda+K(s)},$$

and then, assuming the Fourier transforms of $p(x)$ and $k(x)$, namely $P(s)$ and $K(s)$, can be found, applying the inversion gives

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(s)}{\lambda+K(s)}e^{-isx}ds.$$

5.3 Half-Fourier transforms

Motivation: First let's discuss a little motivation. Suppose equation (74) is given only for some restricted interval of the real line, say $x \geq 0$. Then is it somehow possible to replicate the method in the last subsection to solve for the unknown $f(x)$. Well, sort-of, but we need the concept of a Fourier transform which is not over the whole real line, but which instead works on half-lines: a **half-Fourier transform**.

The Wiener-Hopf method will allow us to solve such equations defined on half-lines. The method may also be applied to certain differential equations when domains are bounded by half-lines or lines have different boundary conditions on two segments. Half-Fourier transforms are a key tool in the Wiener-Hopf method and we study these now.

The Right-sided (+) Fourier transform

Introduce the function

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Definition 5.40. The **Right-sided Fourier transform**, $F_+(s)$, is given by

$$F_+(s) = \int_0^\infty f_+(x) e^{isx} dx, \tag{75}$$

where now $s = s_1 + is_2 \in \mathbb{C}$ is a complex variable.

Proposition 5.41. *If $|f_+(x)| < Ae^{\alpha x}$ as $x \to \infty$, where A, $\alpha \in \mathbb{R}$ with $A > 0$, α **any** sign, then the integral in (75) is convergent for all s such that $Im\{s\} > \alpha$. Furthermore, under these conditions, $F_+(s)$ is an **analytic** function of s.*

Proof. We outline some details of the proof. For convergence note

$$\begin{aligned} \left| \int_0^\infty f_+(x) e^{isx} dx - \int_0^t f_+(x) e^{isx} dx \right| &= \left| \int_t^\infty f_+(x) e^{isx} dx \right| \\ &\leq \int_t^\infty \left| f_+(x) e^{isx} \right| dx \\ &< \int_t^\infty A e^{(\alpha + si)x} dx \to 0, \quad \text{if } s_2 > \alpha. \end{aligned}$$

For analyticity see (Titchmarsh 'the theory of functions' p.100). The main idea is that the FT of a function is analytic, so then the half-FT can be shown to be analytic too. □

Asymptotic behaviour of $F_+(s)$

Observe that

$$\begin{aligned} F_+(s) &= \int_0^\infty f_+(x) e^{isx} dx \\ &= \left[\frac{1}{is} e^{isx} f_+(x) \right]_0^\infty - \frac{1}{is} \int_0^\infty f_+'(x) e^{isx} dx, \end{aligned}$$

after an integration by parts. Now assuming $|f_+(x) < Ae^{\alpha x}$ as $x \to \infty$, where $A > 0$, $\alpha \in \mathbb{R}$ is any sign, provided $\text{Im}\{s\} > \alpha$, we have

$$\begin{aligned} &= \left(0 - \frac{f_+(0)}{is} \right) - \frac{1}{is} \int_0^\infty f_+'(x) e^{isx} dx \\ &= -\frac{f_+(0)}{is} + \frac{1}{s^2} \left(\left[f_+'(x) e^{isx} \right]_0^\infty - \int_0^\infty f_+''(x) e^{isx} dx \right), \end{aligned}$$

after a second integration by parts. Now provided $f'_+(x)$ has sufficiently nice behaviour at infinity, we have

$$F_+(s) = \frac{if_+(0)}{s} - \frac{f'_+(0)}{s^2} + O\left(\frac{1}{s^3}\right), \tag{76}$$

i.e. $F_+(s)$ **decays as** $s \to \infty$, but stronger is the precise behaviour in (76) which we will use.

Inverting the right-sided Fourier transform

Note that

$$F_+(s) = \int_{-\infty}^{\infty} f_+(x) e^{isx} dx,$$

where we have lowered the bottom limit to $-\infty$, since $f_+(x) = 0$ for $x < 0$. Now writing $s = s_1 + is_2$, we find

$$F_+(s) = \int_{-\infty}^{\infty} \left(e^{-s_2 x} f_+(x)\right) e^{is_1 x} dx.$$

Therefore $F_+(s)$ can be regarded as the **ordinary** Fourier transform of $e^{-s_2 x} f_+(x)$. Thus, provided $\mathrm{Im}\{s\} > \alpha$, we can apply the inverse formula as usual, giving

$$\begin{aligned} e^{-s_2 x} f_+(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_+(s) e^{-s_1 x} ds_1 \\ \Rightarrow f_+(x) &= \frac{e^{s_2 x}}{2\pi} \int_{-\infty}^{\infty} F_+(s) e^{-s_1 x} ds_1 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_+(s) e^{-isx} ds_1, \end{aligned}$$

where we substituted $s = s_1 + is_2$ in the final line.

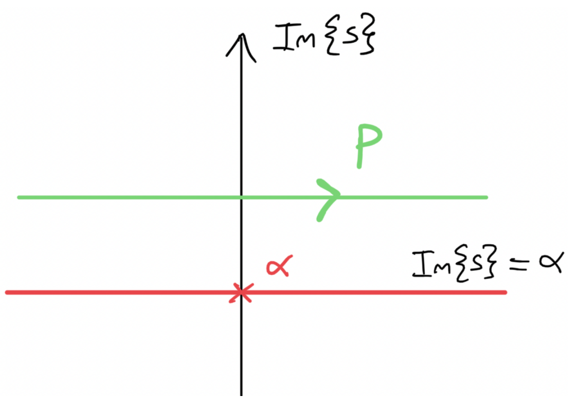

Figure 70: A suitable contour P.

Now for a constant s_2, $ds_1 = ds$, and we find

$$f_+(x) = \frac{1}{2\pi} \int_P F_+(s) e^{-isx} ds, \tag{77}$$

where P is a straight line along which $s_2 = \mathrm{Im}\{s\} = \text{constant} > \alpha$.

Let's check this inversion formula (77) gives $f_+(x) = 0$ for $x < 0$. To evaluate this contour integral, let's close P with a semi-circle above, see figure 71.

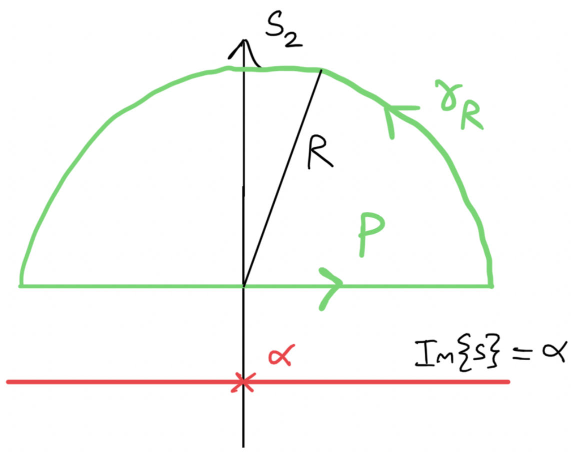

Figure 71: Close P in the half-plane above.

Let $\gamma = \gamma_R + P$. Since γ is contained in the region $\mathrm{Im}\{s\} > \alpha$, then $F_+(s)$ is analytic inside it. Also $e^{-isx} = e^{-is_1x}e^{s_2x}$, and this is also analytic in γ (as $|s| \to \infty$, $s_2 \to +\infty$, $x < 0$). Hence, by Cauchy's theorem

$$\oint_\gamma F_+(s)e^{-isx}ds = 0.$$

On γ_R, as $R \to \infty$, $s \approx Re^{i\theta}$; $\left|F_+(s)e^{-isx}\right| = e^{s_2x}|F_+(s)| \leq |F_+(s)|$, and $|F_+(s)| \to 0$ as $s \to 0$ by (76). Thus

$$\int_P F_+(s)e^{-isx}ds = 0,$$

as expected for $x < 0$.

Summary of key properties

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases} \qquad F_+(s) = \int_0^\infty f_+(x)e^{isx}dx, \quad s \in \mathbb{C}.$$

If $|f_+(x)| < Ae^{\alpha x}$ as $x \to \infty$ (where $A, \alpha \in \mathbb{R}$ with $A > 0$, α **any** sign) then $F_+(s)$ exists and is an **analytic** function of s for $\mathrm{Im}\{s\} > \alpha$.

$$F_+(s) = \frac{if_+(0)}{s} - \frac{f_+'(0)}{s^2} + O\left(\frac{1}{s^3}\right), \quad \text{as } s \to \infty.$$

Inversion:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a straight line along which $s_2 = \mathrm{Im}\{s\} =$ constant $> \alpha$.

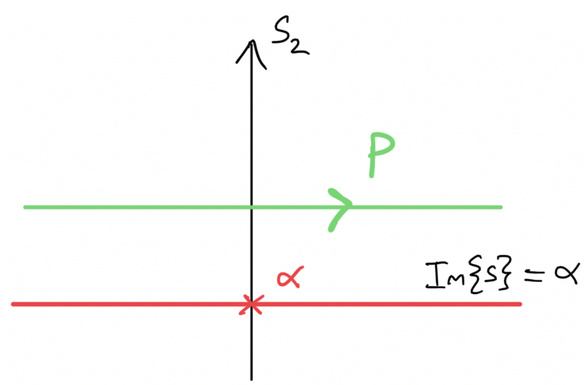

Figure 72: A suitable contour P.

The Left-sided $(-)$ Fourier transform

Now, similarly, for a function $g(x)$ specified for $x < 0$ (but not $x \geq 0$), we define

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ g(x), & \text{for } x < 0. \end{cases}$$

Definition 5.42. The **Left-sided Fourier transform**, $G_-(s)$, is given by

$$G_-(s) = \int_{-\infty}^{0} g_-(x)e^{isx}dx, \quad s \in \mathbb{C}. \tag{78}$$

We then find the similar properties

- It can be shown that if $|g_-(x)| < Be^{\beta x}$ as $x \to -\infty$ (where B, $\beta \in \mathbb{R}$ with $B > 0$, β **any** sign) then $G_-(s)$ exists and is an **analytic** function of s for $\mathrm{Im}\{s\} < \beta$.
- $G_-(s)$ decays as $|s| \to \infty$.

Inversion:

$$g_-(x) = \frac{1}{2\pi}\int_Q G_-(s)e^{-isx}ds,$$

where Q is a straight line along which $s_2 = \mathrm{Im}\{s\} =$ constant $< \beta$.

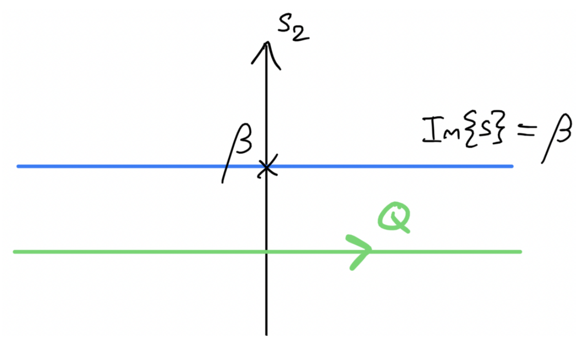

Figure 73: A suitable contour Q.

5.4 The Wiener-Hopf method

Let's return to the integral equation we want to be able to solve

$$\lambda f(x) = \int_0^\infty k(x-y)f(y)dy + p(x), \qquad x \geq 0. \tag{79}$$

Remark: Generally, one requires the kernel function $k(x)$ to have 'nice' (see shortly what we mean by this) behaviour for the Wiener-Hopf method to be applicable (in this course and in any questions given this will be the case).

Let's now outline the method by breaking it down into steps.

Step 1: Introduce unknown functions and take Fourier transforms to form the Riemann-Hilbert problem

We'd like to take a Fourier transform of (79), but we don't know how to do that directly. To circumvent this problem introduce the unknown functions

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ \int_0^\infty k(x-y)f(y)dy, & \text{for } x < 0. \end{cases}$$

Then the equation (79) can be written as

$$\int_0^\infty k(x-y)f(y)dy = \lambda f_+(x) - p_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty. \tag{80}$$

Now we have an equation (80) specified over the entire real line, so we can take its Fourier transform. On the RHS we get

$$\int_{-\infty}^{\infty} \left(\lambda f_+(x) - p_+(x) + g_-(x)\right) e^{isx} dx = \lambda F_+(s) - P_+(s) + G_-(s),$$

where $F_+(s)$ and $P_+(s)$ are the right-sided Fourier transforms of $f_+(x)$ and $p_+(x)$ respectively and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_0^\infty k(x-y)f(y)dy\right)e^{isx}dx &= \int_0^\infty f_+(y)\left(\int_{-\infty}^{\infty} k(x-y)e^{isx}dx\right)dy \\ &= \int_0^\infty f_+(y)\left(\int_{-\infty}^{\infty} k(t)e^{ist}e^{isy}dt\right)dy \\ &= \left(\int_0^\infty f_+(y)e^{isy}dy\right)\left(\int_{-\infty}^{\infty} k(t)e^{ist}dt\right) \\ &= F_+(s)\hat{K}(s),\end{aligned}$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. So, we have

$$F_+(s)\hat{K}(s) = \lambda F_+(s) - P_+(s) + G_-(s),$$

or

$$K(s)F_+(s) + G_-(s) = P_+(s), \tag{81}$$

where $K(s) = \lambda - \hat{K}(s)$.

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

If we seek solutions to the original integral equation such that $|f_+(x)| < Ae^{\alpha x}$ and $|g_-(x)| < Be^{\beta x}$ as $x \to \infty$, where A, B, α, $\beta \in \mathbb{R}$, then we know from the earlier proposition:

- $F_+(s)$ is analytic in $\mathrm{Im}\{s\} > \alpha$,
- $G_-(s)$ is analytic in $\mathrm{Im}\{s\} < \beta$.

Also, let's suppose $\alpha < \beta$ (**Note:** It turns out for exponentially decaying kernels, i.e $|k(x)| < Ce^{-\gamma|x|}$ as $x \to \infty$, that $\alpha < \beta$, i.e. a strip of analyticity exists. This is what we mean by 'nice' kernels making the method applicable.): then the regions of analyticity of $F_+(s)$ and $G_-(s)$ overlap in a strip $\Omega = \{s : \alpha < \mathrm{Im}\{s\} < \beta\}$, see figure 74.

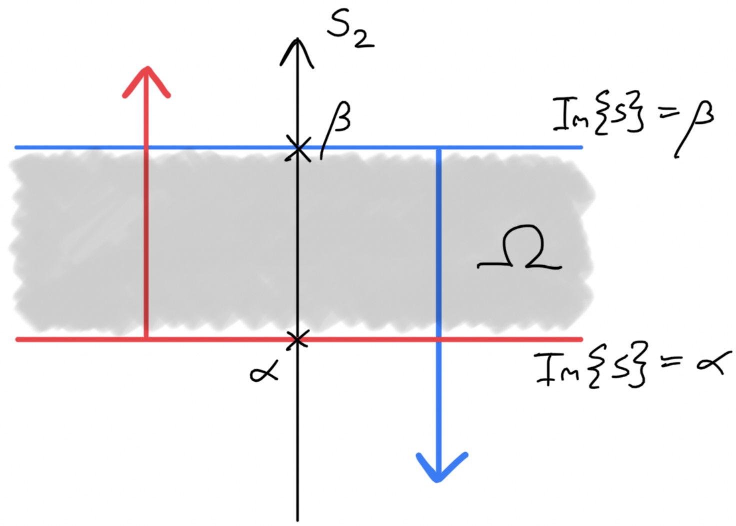

Figure 74: The strip of analyticity Ω.

Furthermore, suppose that $k(x)$ and $p(x)$ are such that $K(s)$ and $P_+(s)$ are:

(i) Analytic in Ω.

(ii) $K(s) \neq 0$ in Ω.

(iii) As $s \to \infty$ in Ω, $|K|$, $|P_+|$ are 'suitably' bounded.

Then, more generally, equation (81) and the conditions just outlined form the canonical Wiener-Hopf problem (note equation (81) and these conditions form a Riemann-Hilbert problem).

Step 3: The product decomposition

Find $K_+(s)$ and $K_-(s)$ where $K_+(s)$ is analytic in $\mathrm{Im}\{s\} > \alpha_1$, $K_-(s)$ is analytic in $\mathrm{Im}\{s\} < \beta_1$, $\alpha_1 < \beta_1$, where $\Omega_1 = \{s : \alpha_1 < \mathrm{Im}\{s\} < \beta_1\} \subseteq \Omega$, and $K(s) = K_+(s)K_-(s)$.

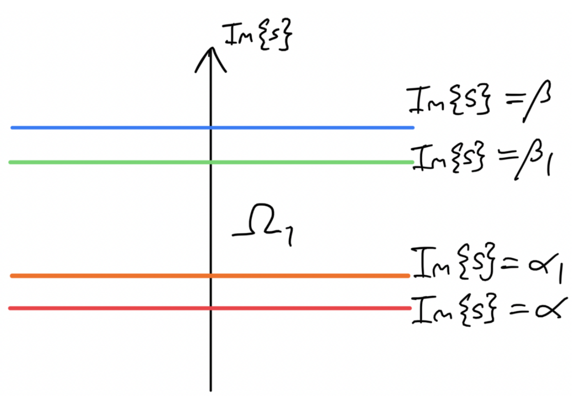

Figure 75: The new strip Ω_1.

After this, it follows that (81) can be rewritten as:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R(s), \tag{82}$$

for $s \in \Omega_1$, where

$$R(s) = \frac{P_+(s)}{K_-(s)}.$$

Step 4: The sum decomposition

Find $R_+(s)$ and $R_-(s)$ where $R_+(s)$ is analytic in $\text{Im}\{s\} > \alpha_2$, $R_-(s)$ is analytic in $\text{Im}\{s\} < \beta_2$, $\alpha_2 < \beta_2$, where $\Omega_2 = \{s : \alpha_2 < \text{Im}\{s\} < \beta_2\} \subseteq \Omega_1$, such that $R(s) = R_+(s) + R_-(s)$.

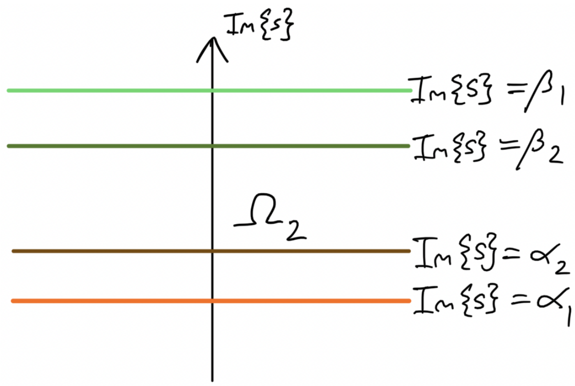

Figure 76: The new strip Ω_2.

After this, it follows that (82) can be written as:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R_+(s) + R_-(s), \qquad (83)$$

for $s \in \Omega_2$.

Step 5: Analytic Continuation

Now re-arrange (83) to get

$$K_+(s)F_+(s) - R_+(s) = R_-(s) - \frac{G_-(s)}{K_-(s)},$$

for $s \in \Omega_2$.

Here, the LHS is analytic in $\text{Im}\{s\} > \alpha_2$ (call this region $+$) and the RHS is analytic in $\text{Im}\{s\} < \beta_2$ (call this region $-$).

Since $+$ and $-$ overlap in the strip Ω_2, then the RHS is the analytic continuation of the LHS into the $-$ region, and likewise the LHS is the analytic continuation of the RHS into the $+$ region.

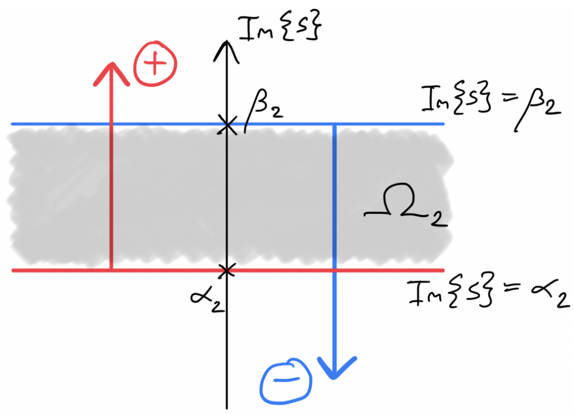

Figure 77: The regions of analyticity of the LHS and RHS, $+$ and $-$.

Hence, this means that the function

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in +, \\ R_-(s) - \frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

is **entire** (analytic everywhere in the complex s-plane).

Step 6: Behaviour at infinity

Since $K_+(s)$, $K_-(s)$, $R_+(s)$ and $R_-(s)$ are known, then so is their behaviour as $|s| \to \infty$. Furthermore, the behaviour of $F_+(s)$ and $G_-(s)$ is given by the asymptotic expansion of the relevant Fourier integral (formula (76) for $F_+(s)$). Hence we can determine the behaviour of $E(s)$ as $|s| \to \infty$. Then we can apply **Liouville's theorem**:

(a) If $E(s) \to$ constant, M, as $|s| \to \infty$, then $E(s) \equiv M$.

(b) If $E(s) \sim O(s^N)$, as $|s| \to \infty$, where $N \in \mathbb{N}$, then $E(s)$ is a polynomial of degree at most N.

Step 7: Invert

Having found $E(s)$, to find $f_+(x)$ (and also $g_-(x)$ if we wished), we invert:

$$F_+(s) = \frac{R_+(s) + E(s)}{K_+(s)} \qquad \text{and} \qquad G_-(s) = K_-(s)\left(R_-(s) - E(s)\right).$$

Then one determines $f_+(x)$ using the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a straight line contour ($\text{Im}\{s\} =$ constant) within region $+$ (similarly for $g_-(x)$ if necessary). This final integral will often need to be closed in an appropriate way and evaluated using the residue/ Cauchy's theorem.

Step 8: Check!

In non-exam conditions, for example during research problems or when doing problem sheets, it can be good practice to substitute our solution function $f(x)$ back into the original integral equation to check if our answer is correct.

5.5 The Wiener-Hopf product and sum decompositions

The key steps in the Wiener-Hopf method are the ability to decompose $K(s)$ and $R(s)$ into product and sum decompositions with the appropriate analyticity properties. In this section we justify why this can always be done (for sufficiently 'nice' kernels) and the proof will provide us with a 'constructive' method for determining these decompositions.

Nevertheless, if the decompositions can be found by inspection this often proves faster and more convenient than using the formulae we will derive here. Let's start with the sum decomposition.

Proposition 5.43 (Wiener-Hopf sum decomposition)**.** *Suppose $R(s)$ is analytic in the strip $\Omega = \{s : \alpha < Im\{s\} < \beta\}$. Then we can write $R(s) = R_+(s) + R_-(s)$, where $R_+(s)$ is analytic in $Im\{s\} > \alpha_1$ and $R_-(s)$ is analytic in $Im\{s\} < \beta_1$, for some α_1 and β_1 where $\alpha < \alpha_1 < \beta_1 < \beta$.*

Proof. Here we will further assume $R(s) \to 0$ as $|s| \to \infty$ (this condition can be relaxed, but we assume it here to make the proof simpler).

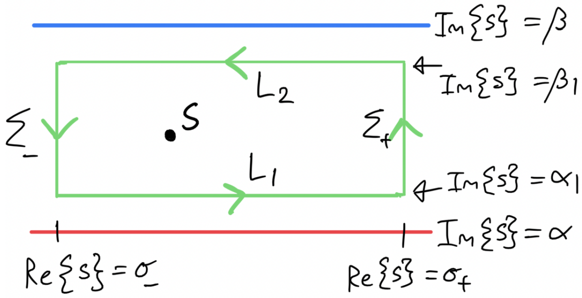

Figure 78: The region $\tilde{\Omega}$.

Let $\gamma = L_1 + L_2 + \Sigma_- + \Sigma_+$, as shown in figure 78. Let $\tilde{\Omega}$ denote the interior of γ.

We know that $R(s)$ is analytic for $s \in \tilde{\Omega}$. Hence by Cauchy's integral formula

$$R(s) = \frac{1}{2\pi i}\oint_\gamma \frac{R(t)}{t-s}dt,$$

for $s \in \tilde{\Omega}$.
Now send $\sigma_- \to -\infty$ and $\sigma_+ \to \infty$. Since $R(t) \to 0$ as $t \to \infty$, one can show that

$$\lim_{\sigma_\pm \to \pm\infty} \int_{\Sigma_\pm} \frac{R(t)}{t-s}dt = 0.$$

Hence, for $s \in \tilde{\Omega}$ (which is the strip $\alpha_1 < \text{Im}\{s\} < \beta_1$), we have

$$R(s) = \underbrace{\frac{1}{2\pi i}\int_{L_1}\frac{R(t)}{t-s}dt}_{R_+(s)} + \underbrace{\frac{1}{2\pi i}\int_{L_2}\frac{R(t)}{t-s}dt}_{R_-(s)}. \tag{84}$$

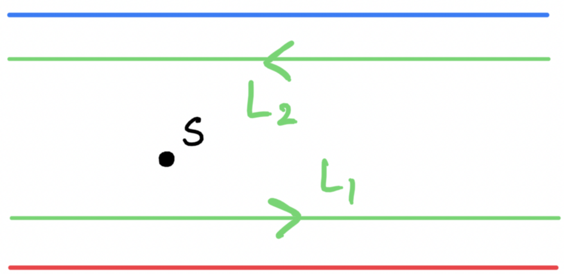

Figure 79: The region $\tilde{\Omega}$ with the sides sent to infinity.

$R_+(s)$ and $R_-(s)$ as defined above are **Cauchy-type** integrals (or Cauchy transforms; recall chapter 2) hence $R_+(s)$ is analytic in the UHP above L_1, $R_-(s)$ is analytic in the LHP below L_2 (and both analytic as $|s| \to \infty$ too). □

Proposition 5.44 (Wiener-Hopf product decomposition)**.** *Suppose $K(s)$ is analytic and non-zero in the strip $\Omega = \{s : \alpha < Im\{s\} < \beta\}$. Then we can write $K(s) = K_+(s)K_-(s)$, where $K_+(s)$ is analytic and non-zero in $Im\{s\} > \alpha_1$ and $K_-(s)$ is analytic and non-zero in $Im\{s\} < \beta_1$, for some α_1 and β_1 where $\alpha < \alpha_1 < \beta_1 < \beta$.*

Proof. Our idea is that taking logarithms of both sides, this becomes a sum decomposition for $\log K(s)$. So, take $\hat{R}(s) = \log K(s)$. Then by the sum decomposition, we get

$$\hat{R}(s) = \hat{R}_+(s) + \hat{R}_-(s),$$

where $\hat{R}_+(s)$ is analytic in $\text{Im}\{s\} > \alpha_1$ and $\hat{R}_-(s)$ is analytic in $\text{Im}\{s\} < \beta_1$, and

$$\hat{R}_+(s) = \frac{1}{2\pi i}\int_{L_1}\frac{\hat{R}(t)}{t-s}dt,$$
$$\hat{R}_-(s) = \frac{1}{2\pi i}\int_{L_2}\frac{\hat{R}(t)}{t-s}dt,$$

where $L_1 = \{t : \text{Im}\{t\} = \alpha_1\}$ and $L_2 = \{t : \text{Im}\{t\} = \beta_1\}$. Hence

$$K(s) = \exp\left\{\hat{R}(s)\right\} = \exp\left\{\hat{R}_+(s)\right\} \cdot \exp\left\{\hat{R}_-(s)\right\},$$

so we identify

$$K_\pm(s) = \exp\left\{\hat{R}_\pm(s)\right\},$$

where $\hat{R}_\pm(s)$ are as defined above. $\square$

Note: We refer to these as 'constructive' proofs, since the proofs give formulae that enable us to calculate $K_\pm(s)$ and $R_\pm(s)$ for a given problem if we cannot determine the decompositions by inspection.

5.6 Examples of finding product and sum decompositions

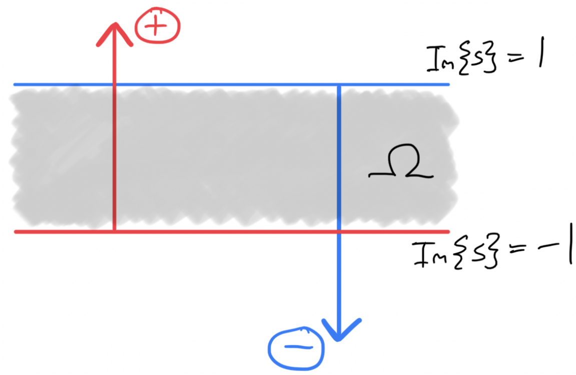

Figure 80: The strip Ω and regions + and −.

Example 1:

$$K(s) = \frac{1}{s^2+1} = \frac{1}{(s+i)(s-i)},$$

with Ω as shown in figure 80. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

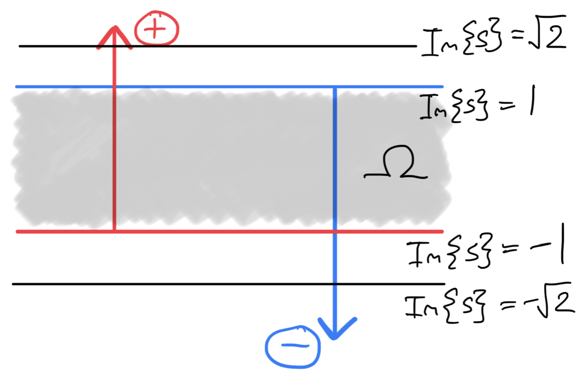

Figure 81: The strip Ω and regions + and −.

Example 2:

$$K(s) = \frac{s^2+2}{s^2+1} = \frac{(s+\sqrt{2}i)(s-\sqrt{2}i)}{(s+i)(s-i)},$$

with Ω as shown in figure 81. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{(s+\sqrt{2}i)}{s+i},$$

is **analytic** and **non-zero** in $\mathrm{Im}\{s\} > -1$, and

$$K_-(s) = \frac{(s-\sqrt{2}i)}{s-i},$$

is **analytic** and **non-zero** in $\mathrm{Im}\{s\} < 1$.

Example 3:

$$K(s) = \frac{s^2}{s^2+1} = \frac{s^2}{(s+i)(s-i)},$$

with Ω as shown in figure 82. Here $0 < \delta < 1$. We can write $K(s) = K_+(s)K_-(s)$, where

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

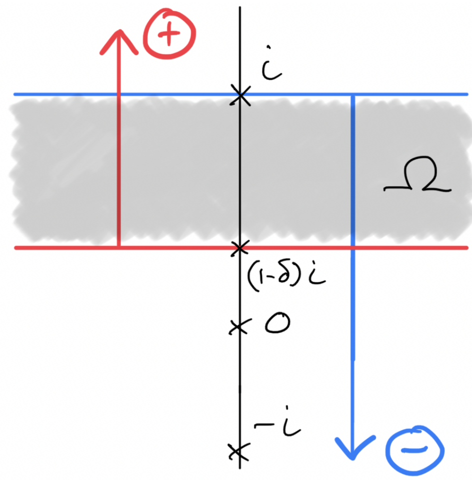

Figure 82: The strip Ω and regions $+$ and $-$.

Example 4:

$$R(s) = \frac{1}{s^2+1} = \frac{s^2}{(s+i)(s-i)},$$

with Ω as shown in figure 83. We can write $R(s) = R_+(s) + R_-(s)$, where

$$R_+(s) = \frac{i/2}{s+i}, \quad R_-(s) = \frac{-i/2}{s-i}.$$

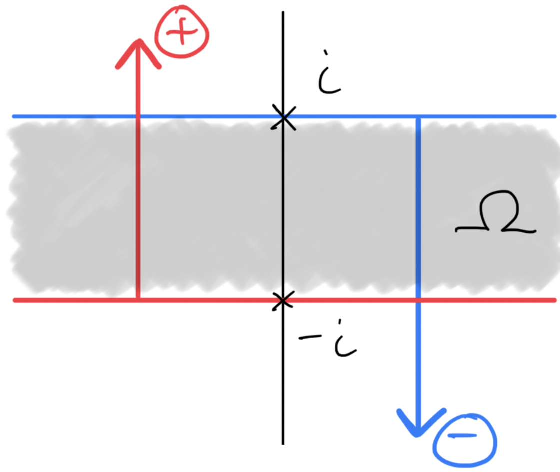

Figure 83: The strip Ω and regions $+$ and $-$.

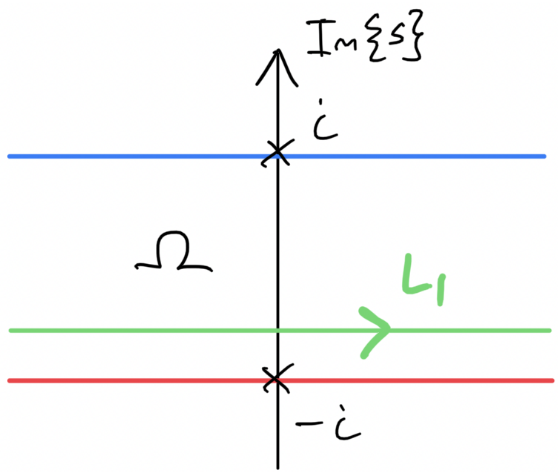

Figure 84: The contour L_1.

Let's show in this example that we can recover what we have found by inspection via the formula (84). Using this we find, for $R_+(s)$:

$$R_+(s) = \frac{1}{2\pi i}\int_{L_1} \frac{R(t)}{t-s}dt = \frac{1}{2\pi i}\int_{L_1} \frac{1}{t-s}\frac{1}{(t+i)(t-i)}dt.$$

To evaluate the integral we need to close L_1. Since the integrand $\to 0$ both above and below L_1, we can choose to close in either direction.

Let's close the contour below as shown in figure 85. As $R \to \infty$, one can check $\int_{\gamma_R} \to 0$. So, by the residue theorem

$$
\begin{aligned}
R_+(s) &= \frac{1}{2\pi i} \times -2\pi i \left(\frac{1}{-i-s} \left(\frac{-1}{2i} \right) \right) \\
&= \frac{i/2}{s+i},
\end{aligned}
$$

as found before by inspection. Rather than then calculating $R_-(s)$ via formula (84), we would simply substitute back into $R(s) = R_+(s) + R_-(s)$ to find $R_-(s)$ more simply.

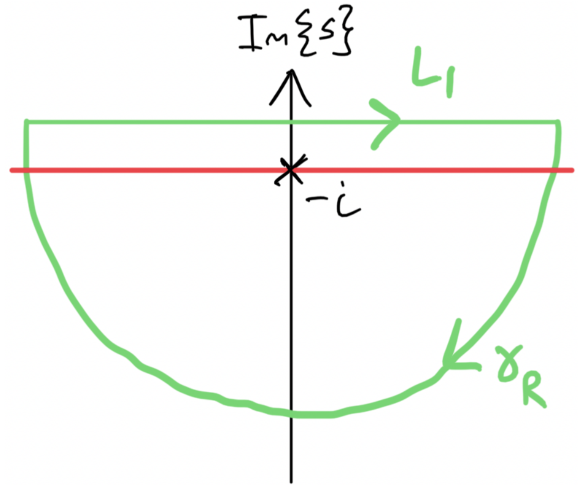

Figure 85: The closed contour γ.

5.7 Wiener-Hopf example problem 1

Let's finally do an example problem to see how everything works in practice. Let's start simple and set $p(x) = 0$. Solve

$$
f(x) = \frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy, \qquad x \geq 0,
$$

subject to the condition $f(0) = A_0$, constant.

Solution:

Step 1: Introduce functions & take Fourier transforms

Here $k(x) = \frac{1}{2} e^{-|x|}$. First, introduce

$$
f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}
$$

$$
g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ \frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy, & \text{for } x < 0. \end{cases}
$$

Then we have the equation

$$
\frac{1}{2} \int_0^\infty e^{-|x-y|} f(y) dy = f_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty.
$$

Now we can take Fourier transforms. On the RHS we get

$$
\int_{-\infty}^\infty \left(f_+(x) + g_-(x) \right) e^{isx} dx = F_+(s) + G_-(s),
$$

where $F_+(s)$ is the right-sided Fourier transforms of $f_+(x)$ and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\int_0^{\infty} e^{-|x-y|}f_+(y)dy\right)e^{isx}dx = F_+(s)\hat{K}(s),$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. Evaluating this, we find

$$\begin{aligned}
\hat{K}(s) &= \int_{-\infty}^{\infty}\frac{1}{2}e^{-|x|}e^{isx}dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{(1+is)x}dx + \frac{1}{2}\int_0^{\infty} e^{(-1+is)x}dx \\
&= \frac{1}{2}\int_0^{\infty} e^{-(1+is)x}dx + \frac{1}{2}\int_0^{\infty} e^{(-1+is)x}dx \\
&= \underbrace{\frac{1}{2}\left[\frac{e^{-(1+is)x}}{-(1+is)}\right]_0^{\infty}}_{\text{Need Im}\{s\}<1\text{ for convergence}} + \underbrace{\frac{1}{2}\left[\frac{e^{(-1+is)x}}{-1+is}\right]_0^{\infty}}_{\text{Need Im}\{s\}>-1\text{ for convergence}} \\
&= \frac{1}{2}\left(\frac{1}{1+is}+\frac{1}{1-is}\right) \\
&= \frac{1}{s^2+1}, \qquad \text{for } -1<\text{Im}\{s\}<1.
\end{aligned}$$

Plugging $s = s_1 + is_2$ into the integrands helps determine the regions of convergence for each. So, the Wiener-Hopf equation is

$$F_+(s)\hat{K}(s) = F_+(s) + G_-(s),$$

or

$$K(s)F_+(s) + G_-(s) = 0,$$

where

$$K(s) = 1 - \hat{K}(s) = \frac{s^2}{s^2+1}.$$

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

First let's consider $G_-(s)$. well

$$\begin{aligned}
g_-(x) &= \frac{1}{2}\int_0^{\infty} e^{-|x-y|}f(y)dy, \quad \text{for } x<0 \\
&= \frac{e^x}{2}\int_0^{\infty} e^{-y}f(y)dy,
\end{aligned}$$

since for $x<0$, $y>0$ we have $-|x-y| = x-y$. Thus

$$g_-(x) = Be^{1x},$$

where (provided this integral exists)

$$B = \text{constant} = \frac{1}{2}\int_0^{\infty} e^{-y}f(y)dy.$$

Hence $G_-(s)$ is analytic for $\text{Im}\{s\} < 1$.

Now consider $F_+(s)$. For the original equation to hold, the integral

$$\int_0^\infty e^{-|x-y|} f(y)dy,$$

should converge for all $x \geq 0$. Consider, as $t \to \infty$:

$$\left| \int_0^\infty e^{-|x-y|} f(y)dy - \int_0^t e^{-|x-y|} f(y)dy \right| = \left| \int_t^\infty e^{-|x-y|} f(y)dy \right|$$
$$\leq \int_t^\infty e^{-|x-y|} |f(y)|dy.$$

For a given x, as $t \to \infty$, $-|x-y| \sim -y$. So $e^{-|x-y|}|f(y)| \sim e^{-y} f(y)$. So, we have convergence if, as $y \to \infty$: $|f(y)| < Ae^{(1-\delta)y}$, for some $\delta > 0$, as then $e^{-y} f(y) \to 0$ as $y \to \infty$. Hence $F_+(s)$ is analytic for $\text{Im}\{s\} > 1-\delta$.

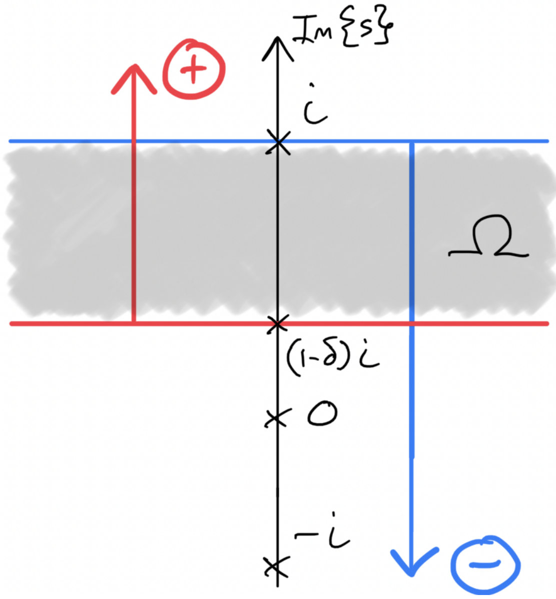

Figure 86: The strip of analyticity γ with the + and − regions indicated.

We found $K(s)$ is analytic for $-1 < \text{Im}\{s\} < 1$, but it also has a **zero** at $s = 0$, so we have ensured $s = 0$ is outside our choice of Ω.

Recall that we need to check $K(s)$ and $P_+(s)$ satisfy the analyticity and (for K) non-zero conditions to define our Ω.

Step 3: The product decomposition

We have

$$K(s) = \frac{s^2}{s^2+1} = \frac{s^2}{(s+i)(s-i)} = \underbrace{\frac{s^2}{s+i}}_{\substack{\text{analytic and}\\ \text{non-zero in +}}} \cdot \underbrace{\frac{1}{s-i}}_{\substack{\text{analytic and}\\ \text{non-zero in -}}} .$$

So we set

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

Now our equation becomes

$$K_+(s)F_+(s) = -\frac{G_-(s)}{K_-(s)}, \quad s \in \Omega.$$

Step 4: The sum decomposition

Since $p(x) = 0$, we avoid this step here: our equation is already separated into expressions analytic in +/- respectively.

Step 5: Analytic continuation

Thus

$$E(s) = \begin{cases} K_+(s)F_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

is **entire**.

Step 6: Behaviour at infinity

To determine $E(s)$, we consider its behaviour as $s \to \infty \in +$. Recall from (76), as $s \to \infty$:

$$F_+(s) = \frac{if_+(0)}{s} + O\left(\frac{1}{s^2}\right) = \frac{A_0 i}{s} + O\left(\frac{1}{s^2}\right).$$

Then

$$\begin{aligned} E(s) &= \frac{s^2}{s+i}F_+(s) \to \frac{s^2}{s+i}\left(\frac{A_0 i}{s} + O\left(\frac{1}{s^2}\right)\right), \quad \text{as } s \to \infty \\ &= A_0 i, \end{aligned}$$

a constant. Hence

$$E(s) \equiv A_0 i,$$

by Liouville's theorem.

Step 7: Invert

$$F_+(s) = \frac{A_0 i(s+i)}{s^2}, \quad G_-(s) = \frac{-A_0 i}{s-i}.$$

We obtain $f_+(x)$ from the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds = \frac{A_0 i}{2\pi}\int_P \frac{(s+i)e^{-isx}}{s^2}ds,$$

where P is some straight line contour in $+$. Recall, as shown earlier, this integral gives $f_+(x) = 0$ for $x < 0$ as expected.

For $x > 0$. Let $\gamma = P + \gamma_R$ as shown in figure (87). As $R \to \infty$ one can show $\int_{\gamma_R} \to 0$. Thus, by the residue theorem

$$f_+(x) = \frac{A_0 i}{2\pi} \times (-2\pi i) \times \text{Res}\{h(s), s = 0\},$$

where

$$h(s) = \frac{(s+i)e^{-isx}}{s^2} = \frac{(1 - isx + O(s^2))(s+i)}{s^2}$$
$$= \frac{i}{s^2} + \frac{x+1}{s} + O(1),$$

so that $\text{Res}\{h, s = 0\} = 1 + x$, giving

$$f_+(x) = A_0(1 + x), \quad x \geq 0.$$

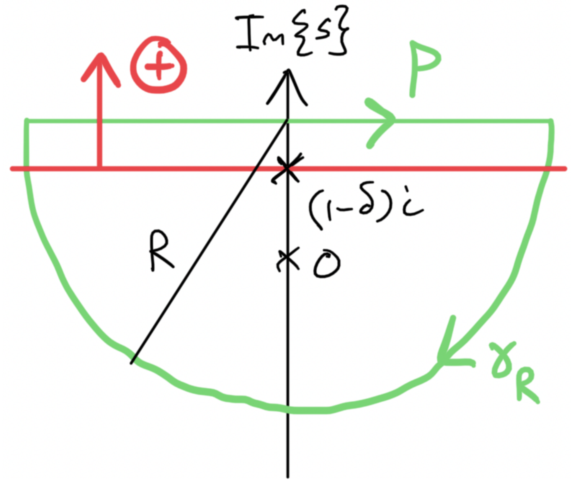

Figure 87: Closed contour γ below P.

Step 8: Check

One can check this is the correct solution by substituting back into the original equation (exercise).

Important remarks about this example

1). Let's check the behaviour of the solution as $x \to \infty$. Recall, we expect $|f(x)| < A_0 e^{(1-\delta)x}$, where $0 < \delta < 1$, i.e $1 - \delta > 0$. As $x \to \infty$, $f(x) \sim A_0 x < A_0 e^{(1-\delta)x}$, since $1 - \delta > 0$. ✓

2). Let's now take a detailed look at the $+$ and $-$ regions and decompositions. Recall, we took:

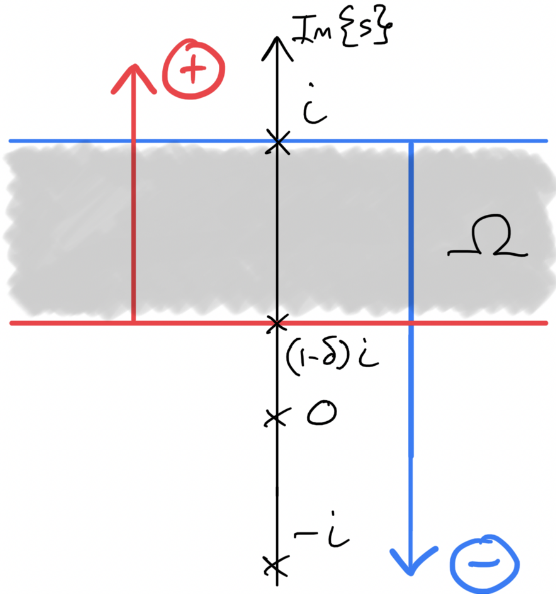

Figure 88: (a) The strip Ω we took during the problem.

The conditions we had to satisfy were:

(i) $G_-(s)$ analytic in $\text{Im}\{s\} < 1$.

(ii) $F_+(s)$ analytic in $\text{Im}\{s\} > 1 - \delta$ $(\delta > 0)$.

(iii) $K(s) = \frac{s^2}{(s+i)(s-i)}$ analytic and non-zero in Ω.

Question: What other choices for the region Ω are there? And what solutions do we find in these cases?

Well by (i), Ω must be contained in $\text{Im}\{s\} < 1$. But (ii) does not add any restrictions, as we can take δ as large as we like. (iii) means we can't have $s = 0, \pm i$ within Ω. This means there are actually two more distinctive choices for Ω:

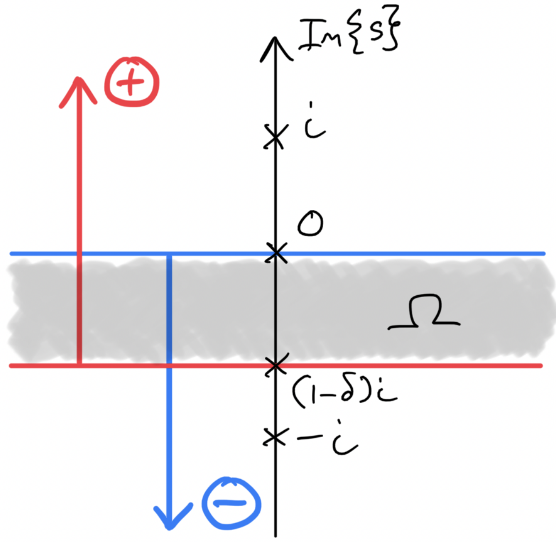

Figure 89: (b) Another possible choice for Ω.

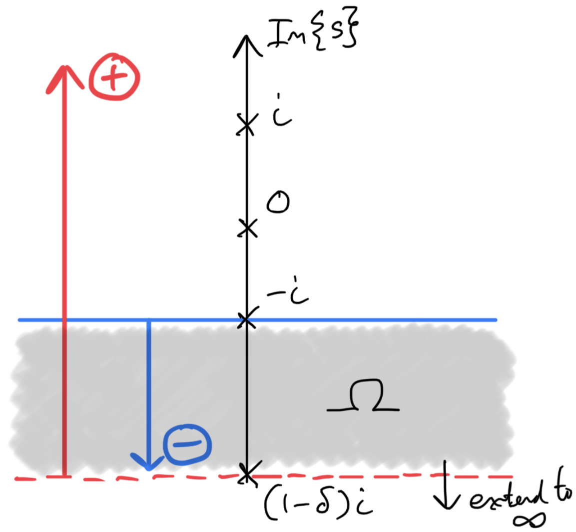

Figure 90: (c) Yet another possible choice for Ω.

(a) This was the choice we made: $\Omega = \{s : 1 - \delta < \text{Im}\{s\} < 1\}$, with $0 < \delta < 1$ (so that $1 - \delta > 0$), and

$$K_+(s) = \frac{s^2}{s+i}, \quad K_-(s) = \frac{1}{s-i}.$$

(b) In this case: $\Omega = \{s : 1 - \delta < \text{Im}\{s\} < 0\}$, with $1 < \delta < 2$, so $-1 < 1 - \delta < 0$. In this case we get

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{s^2}{s-i}.$$

Recall

$$K_+(s)F_+(s) = -\frac{G_-(s)}{K_-(s)},$$

so we get

$$E(s) = \begin{cases} K_+(s)F_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -. \end{cases}$$

By analytic continuation $E(s)$ is an entire function. To determine $E(s)$ we look at the behaviour as $s \to \infty$. We get, as $s \to \infty$ (in $+$):

$$\begin{aligned} K_+(s)F_+(s) &\to \frac{1}{s} \cdot \frac{1}{s} \to 0 \\ \Rightarrow E(s) &\equiv 0 \\ \Rightarrow F_+(s) &= 0 \\ \Rightarrow f_+(x) &= 0. \end{aligned}$$

It is trivial to check that $f(x) = 0$ is a solution of the original equation.

So why does this region for Ω not retrieve the solution found in case (a)? Recall, $|f(x)| < Ae^{(1-\delta)x}$ as $x \to \infty$. In case (a) we had $1 - \delta > 0$ and so this allowed for solutions which became infinite as $x \to \infty$ (we found $f(x) = A_0(1+x)$). But in case (b), we're restricted so that $1 - \delta < 0$ and so any solutions we find must decay as $x \to \infty$. Here we've thus retrieved the solutions of (a) which decay as $x \to \infty$, i.e those for which $A_0 = 0$, i.e $f(x) = 0$.

(c) Case (c) is similar to case (b): exercise: find $K_+(s)$ and $K_-(s)$ here and show that you find only $f(x) = 0$ again.

Important remark on this: We want to pick the region Ω that allows for the largest amount of different solutions $\to$ (a) was the correct choice here.

5.8 Wiener-Hopf example problem 2

Solve

$$2\int_0^\infty e^{-|x-y|}f(y)dy = f(x) + 2xe^{-x}, \quad x \geq 0,$$

with $f(0) = A_0$, constant.

Solution:

Step 1: Introduce functions & take Fourier transforms

Denote $k(x) = 2e^{-|x|}$, $p(x) = 2xe^{-x}$ (in the general theory we had $p(x)$ on the other side of the equation - this will lead to a sign difference here). then introduce

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

$$g_-(x) = \begin{cases} 0, & \text{for } x \geq 0, \\ 2\int_0^\infty e^{-|x-y|}f(y)dy, & \text{for } x < 0. \end{cases}$$

Then we have the equation

$$2\int_0^\infty e^{-|x-y|}f(y)dy = f_+(x) + p_+(x) + g_-(x), \quad \text{for } -\infty < x < \infty.$$

Now we can take Fourier transforms. On the RHS we get $F_+(s) + P_+(s) + G_-(s)$, where $F_+(s)$ and $P_+(s)$ are the right-sided Fourier transforms of $f_+(x)$ and $p_+(x)$ respectively and $G_-(s)$ is the left-sided Fourier transform of $g_-(x)$. On the LHS we get

$$\int_{-\infty}^\infty \left(2\int_0^\infty e^{-|x-y|}f(y)dy\right)e^{isx}dx = \hat{K}(s)F_+(s),$$

where $\hat{K}(s)$ is the **ordinary** Fourier transform of $k(x)$. Hence, we have

$$\hat{K}(s)F_+(s) = F_+(s) + P_+(s) + G_-(s).$$

Similar to the previous example (see $\hat{K}(s)$ in section 5.7), one can check that this time we find

$$\hat{K}(s) = \frac{4}{s^2+1}.$$

Now calculating $P_+(s)$:

$$\begin{aligned}
P_+(s) &= \int_0^\infty p_+(x)e^{isx}dx = 2\int_0^\infty xe^{(is-1)x}dx \\
&= 2\left[\frac{xe^{(is-1)x}}{is-1}\right]_0^\infty - 2\int_0^\infty \frac{e^{(is-1)x}}{(is-1)}dx \\
&= 0 - 2\left[\frac{e^{(is-1)x}}{(is-1)^2}\right]_0^\infty \\
&= \frac{-2}{(s+i)^2},
\end{aligned}$$

where one can check that the integrals converge provided $\text{Im}\{s\} > -1$. Hence our equation becomes

$$\left(\frac{4}{s^2+1}\right)F_+(s) = F_+(s) + G_-(s) - \frac{2}{(s+i)^2},$$

or

$$K(s)F_+(s) + G_-(s) = -P_+(s), \tag{85}$$

where

$$K(s) = 1 - \hat{K}(s) = \frac{s^2-3}{s^2+1} = \frac{(s+\sqrt{3})(s-\sqrt{3})}{(s+i)(s-i)}.$$

Step 2: Determine regions of analyticity for $F_+(s)$ and $G_-(s)$

As in the first example (section 5.7), we require $|f_+(x)| < \tilde{A}e^{(1-\delta)x}$ as $x \to \infty$, for some $\delta > 0$. Thus $F_+(s)$ is analytic in $\{s : \text{Im}\{s\} > 1-\delta\}$.

Similarly for $G_-(s)$, as in the first example, we can show $g_-(x) = Be^x$, for some constant B. Thus $G_-(s)$ is analytic for $\{s : \text{Im}\{s\} < 1\}$.

As discussed after the previous example, there are again three choices to take for Ω:

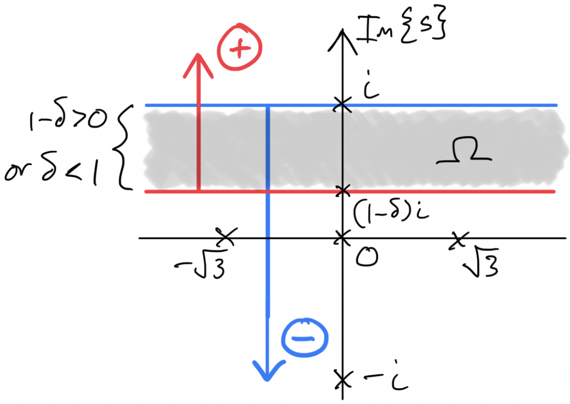

Figure 91: (a) One choice for Ω.

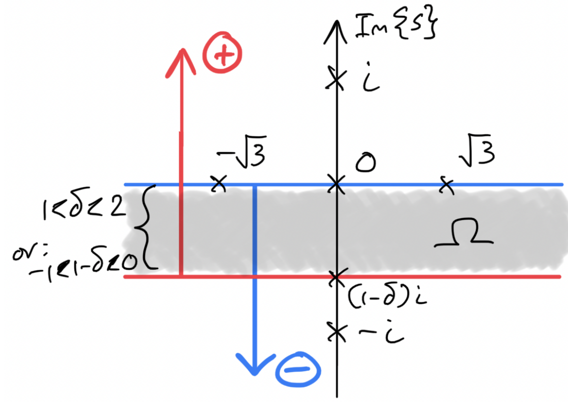

Figure 92: (b) Another possible choice for Ω.

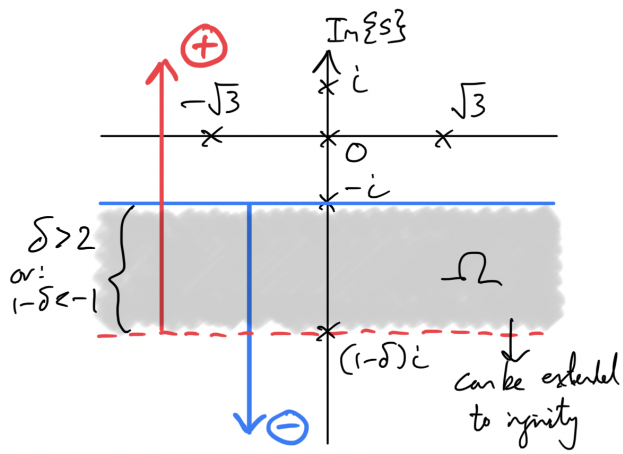

Figure 93: (c) A final possible choice for Ω.

Let's take case (a) (we'll look at regions (b) and (c) afterwards). Indeed, $K(s)$ is analytic provided $-1 < \text{Im}\{s\} < 1$ and non-zero provided $s \neq \pm\sqrt{3}$. ✓ $P_+(s)$ is analytic provided $\text{Im}\{s\} > -1$. ✓

Step 3: The product decomposition

Write $K(s) = K_+(s)K_-(s)$, where $K_+(s)$ is analytic and non-zero in $+$, $K_-(s)$ is analytic and non-zero in $-$.

$$K(s) = \frac{(s+\sqrt{3})(s-\sqrt{3})}{(s+i)(s-i)} = \underbrace{\left(\frac{(s+\sqrt{3})(s-\sqrt{3})}{s+i}\right)}_{K_+(s)}\underbrace{\left(\frac{1}{s-i}\right)}_{K_-(s)}.$$

Now we can write (85) as

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = -\frac{P_+(s)}{K_-(s)} = R(s) = \frac{2(s-i)}{(s+i)^2}. \tag{86}$$

Step 4: The sum decomposition

Writing $R(s) = R_+(s) + R_-(s)$, we have

$$R_+(s) = \frac{2(s-i)}{(s+i)^2}, \quad R_-(s) = 0,$$

since $R_+(s)$ is analytic in $+$. Unlike $K_\pm(s)$, it doesn't matter that one of $R_\pm(s)$ is zero. Thus (86) becomes

$$K_+(s)F_+(s) - R_+(s) = -\frac{G_-(s)}{K_-(s)}, \quad \text{for } s \in \Omega.$$

Step 5: Analytic continuation

Now the LHS is analytic in $+$ (except possibly infinite at ∞) and the RHS is analytic in $-$ (except possibly infinite at ∞). Since $+$ and $-$ overlap in Ω, and LHS = RHS in Ω then these are analytic continuations of one another, hence, if we define:

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in +, \\ -\frac{G_-(s)}{K_-(s)}, & s \in -, \end{cases}$$

then $E(s)$ is analytic **everywhere** in the complex s-plane, except possibly at ∞.

Step 6: Behaviour at infinity

As $s \to \infty$ in $+$:

$$\begin{aligned} K_+(s)F_+(s) - R_+(s) &\sim (s + O(1))\left(\frac{if_+(0)}{s} + O\left(\frac{1}{s^2}\right)\right) - \left(\frac{2}{s} + O\left(\frac{1}{s^2}\right)\right) \\ &= A_0 i + O(1/s) \\ &= A_0 i, \end{aligned}$$

a constant. Hence, by Liouville's theorem $E(s) \equiv A_0 i$.

Step 7: Invert

Hence we find

$$\begin{aligned} F_+(s) &= \frac{A_0 i + R_+(s)}{K_+(s)}, \quad \text{for } s \in + \\ &= \frac{i(A_0-1)(s+i)^2 + i(s^2-3)}{(s+i)(s^2-3)}. \end{aligned}$$

Finally, to get $f_+(x)$, use the inversion formula

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a horizontal line extending to ∞ in $+$. So,

$$f_+(x) = \frac{1}{2\pi}\int_P \frac{(i(A_0-1)(s+i)^2 + i(s^2-3))e^{-isx}}{(s+i)(s+\sqrt{3})(s-\sqrt{3})}ds.$$

For $x < 0$, one can check that we have $f_+(x) = 0$ as expected. For $x > 0$, close P with a semi-circle γ_R in the LHP.

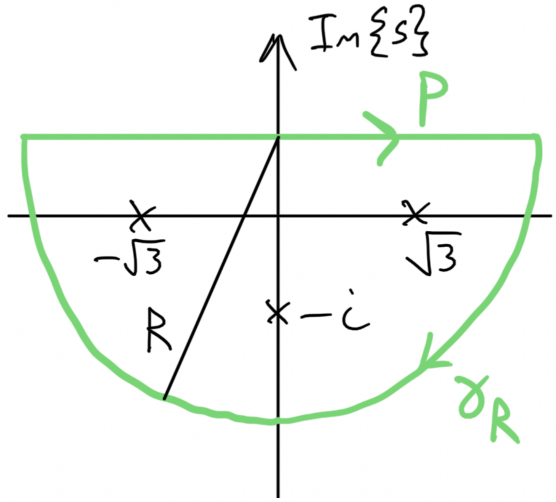

Figure 94: Closed contour γ below P.

Take $\gamma = P + \gamma_R$ as shown in figure 94. Look at the integral along γ_R: as $R \to \infty$:

$$|e^{-isx}F_+(s)| \cdot R = e^{s_2 x}|F_+(s)| \cdot R.$$

On the section of γ_R where $s_2 = \text{Im}\{s\} < 0$, then this expression tends to zero due to the exponential decay. On the sections of γ_R where $s_2 = \text{Im}\{s\} > 0$, since s_2 remains bounded there, and the lengths of these sections are finite, then since $F_+(s) \sim O(1/s)$ the contribution to the integral along these sections is also 0. Hence $\int_{\gamma_R} = 0$ as $R \to \infty$.

Hence by the residue theorem

$$f_+(x) = \frac{1}{2\pi}(-2\pi i)\left[\sum \text{residues of } e^{-isx}F_+(s) \text{ inside } \gamma\right].$$

Denoting $h(s) = e^{-isx}F_+(s)$, then $h(s)$ has poles inside γ at $s = -i, \pm\sqrt{3}$. We find

$$\begin{aligned}
\text{Res}\{h, -i\} &= ie^{-x}, \\
\text{Res}\{h, -\sqrt{3}\} &= i(A_0 - 1)\left(\frac{1 - \frac{i}{\sqrt{3}}}{2}\right)e^{i\sqrt{3}x}, \\
\text{Res}\{h, \sqrt{3}\} &= i(A_0 - 1)\left(\frac{1 + \frac{i}{\sqrt{3}}}{2}\right)e^{-i\sqrt{3}x}.
\end{aligned}$$

Hence we get

$$f(x) = e^{-x} + (A_0 - 1)\left(\cos\sqrt{3}x + \frac{1}{\sqrt{3}}\sin\sqrt{3}x\right).$$

Important remark about this example

Suppose we had opted for Ω as in (b). Then we would have

$$K_+(s) = \frac{1}{s+i}, \quad K_-(s) = \frac{s^2-3}{s^2-i}.$$

Then

$$R(s) = -\frac{P_+(s)}{K_-(s)} = \frac{2(s-i)}{(s+i)^2(s-\sqrt{3})(s+\sqrt{3})} = \frac{c_1}{s+i} + \frac{c_2}{(s+i)^2} + \frac{c_3}{s+\sqrt{3}} + \frac{c_4}{s-\sqrt{3}},$$

for some c_1, c_2, c_3 and c_4. So now:

$$R_+(s) = \frac{c_1}{s+i} + \frac{c_2}{(s+i)^2}, \quad R_-(s) = \frac{c_3}{s+\sqrt{3}} + \frac{c_4}{s-\sqrt{3}}.$$

Then we get

$$\frac{F_+(s)}{s+i} + \frac{(s-i)}{s^2-3} G_-(s) = R_+(s) + R_-(s),$$

leading to the function

$$E(s) = \begin{cases} \frac{F_+(s)}{s+i} - R_+(s), & s \in +, \\ -\frac{(s-i)}{s^2-3} G_-(s) + R_-(s), & s \in -, \end{cases}$$

is **entire**. As $s \to \infty$ we get

$$\frac{F_+(s)}{s+i} - R_+(s) \to 0,$$

hence $E(s) = 0$, by Liouville's theorem. Thus

$$F_+(s) = (s+i)R_+(s) = c_1 + \frac{c_2}{s+i},$$

and in fact one can check (exericse) that $c_1 = 0$ and $c_2 = i$, so we get

$$F_+(s) = \frac{i}{s+i}.$$

Finding $f_+(x)$ by applying the inversion formula, again closing in the LHP, one finds:

$$f(x) = e^{-x}.$$

Recall $|f(x)| < \tilde{A}e^{(1-\delta)x}$ as $x \to \infty$. We see that in case (b) we get the solutions from case (a) which decay as $x \to \infty$. This is because we now have $1 - \delta < 0$ (remember in (a): $1 - \delta > 0$, so it allowed for solutions which do **not** decay as $x \to \infty$ (we found the extra cosine and sine solution)).

Finally, in case (c), one can show that no solutions are found (this is because it requires solutions which decay faster than e^{-x} as $x \to \infty$).

5.9 Differential equations on a half-line

Suppose we want to solve

$$f''(x) - f'(x) - 2f(x) = 0, \quad x \geq 0, \tag{87}$$

with $f(0) = 1$, $\lim_{x\to\infty} f(x) = 0$ (all derivatives of f exist and are bounded as $x \to 0+$ and $x \to \infty$).

We know from our knowledge of how to solve odes that

$$f(x) = A_1 e^{-x} + A_2 e^{2x},$$

so we need to set $A_2 = 0$ and $A_1 = 1$ to satisfy our conditions. Now let's solve the equation using Fourier transforms (this is overkill for this toy example but will be useful for what's coming next)!

First, as usual, introduce

$$f_+(x) = \begin{cases} f(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Now we want to take the Fourier transform of (87), so we will need the Fourier transforms of the derivatives $f_+''(x)$ and $f_+'(x)$. Let's find them:

$$\begin{aligned} \int_{-\infty}^{\infty} f_+''(x) e^{isx} dx &= \int_0^{\infty} f_+''(x) e^{isx} dx \\ &= \underbrace{\left[f_+'(x) e^{isx}\right]_0^{\infty}}_{\substack{\text{provided } \mathrm{Im}\{s\}>0, \text{ then} \\ e^{isx} f_+'(x) \to 0 \text{ as } x \to \infty}} - is \int_0^{\infty} f_+'(x) e^{isx} dx \\ &= -f_+'(0) - is \int_0^{\infty} f_+'(x) e^{isx} dx \\ &= -f_+'(0) - is \left[f_+(x) e^{isx}\right]_0^{\infty} - s^2 \int_0^{\infty} f_+(x) e^{isx} dx. \end{aligned}$$

Hence we find

$$\int_{-\infty}^{\infty} f_+''(x) e^{isx} dx = -f_+'(0) + isf_+(0) - s^2 F_+(s), \tag{88}$$

and also

$$\int_{-\infty}^{\infty} f_+'(x) e^{isx} dx = -f_+(0) - isF_+(s). \tag{89}$$

Higher derivatives can be calculated in a similar manner. So, rewriting (87) as

$$f_+''(x) - f_+'(x) - 2f_+(x) = 0, \quad -\infty < x < \infty,$$

and taking Fourier transforms, we find

$$-f_+(0) + isf_+(0) - s^2 F_+(s) + f_+(0) + isF_+(s) - 2F_+(s) = 0,$$

now substituting for $f_+(0) = 1$ and letting $\varepsilon = f_+'(0)$ be an unknown constant

$$F_+(s) = \frac{is + 1 - \varepsilon}{s^2 - is + 2} = \frac{is + 1 - \varepsilon}{(s - 2i)(s + i)}.$$

This has an apparent singularity in the region $\mathrm{Im}\{s\} > 0$ at $s = 2i$, where we require $F_+(s)$ to be analytic! It follows that we must take

$$(is + 1 - \varepsilon)|_{s=2i} = 0 \quad \Rightarrow \quad -1 - \varepsilon = 0 \quad \Rightarrow \quad \varepsilon = -1,$$

so that

$$F_+(s) = \frac{is + 2}{(s - 2i)(s + i)} = \frac{i(s - 2i)}{(s - 2i)(s + i)} = \frac{i}{s + i},$$

which is analytic in $\text{Im}\{s\} > 0$. Then finally, we can find $f_+(x)$ using the inversion formula

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}ds,$$

where P is a horizontal line extending to ∞ in $\text{Im}\{s\} > 0$. As usual, one can check that $f_+(x) = 0$ for $x < 0$. For $x > 0$, close P in the LHP with a semi-circular contour $\gamma = P + \gamma_R$ as shown in figure 95. One can check

$$\int_{\gamma_R} \to 0 \quad \text{as } R \to \infty.$$

So, by the residue theorem

$$f_+(x) = \frac{1}{2\pi}(-2\pi i)\underbrace{\text{Res}\{F_+(s)e^{-isx}, s = -i\}}_{=ie^{-x}}$$

$$\Rightarrow f_+(x) = e^{-x},$$

as expected.

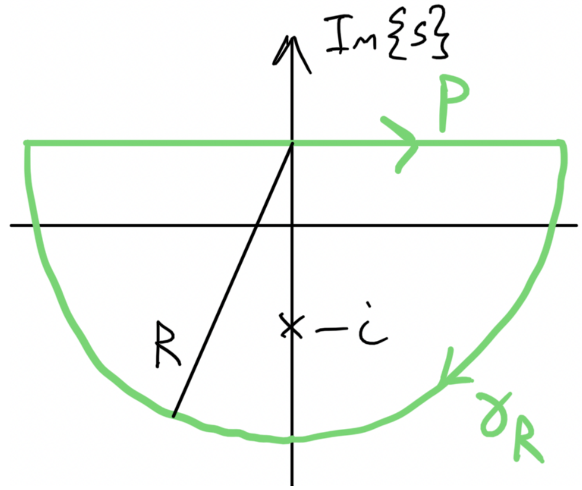

Figure 95: Closed contour γ below P.

5.10 Wiener-Hopf method to solve integral-differential equations

Now we come to the point of the 'toy' problem in the last section. What if the integral equations we have been dealing with throughout the chapter also contain derivatives of the unknown function $f(x)$ too. For example suppose

$$f''(x) + f'(x) + \lambda f(x) = \int_0^\infty k(x-y)f(y)dy, \quad \text{for } x \geq 0.$$

Well the Wiener-Hopf method can be employed in exactly the same way as we have been doing throughout the chapter, the only added difference is that we need to utilise the results (88) and (89) (or equivalent for higher derivatives) for taking Fourier transforms of derivatives when in step 1 of the method. The form of the resulting algebraic equation for $F_+(s)$ will be slightly different due to the added derivative terms (see problem sheet 5 for examples).