Imperial College
London

MATH70008

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamical Systems

Date: 16 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

In all your answers, you may quote results derived in the course (lecture notes inclusive of exercises) without proof, but any such results must be carefully stated or referenced.

The exam questions feature the following three maps $f_{1,2,3} : [0,1] \to [0,1]$ of the interval $[0,1]$ to itself:

$$
\begin{aligned}
f_1(x) &:= \frac{1}{2}(1+\cos(\pi x)),\\
f_2(x) &:= \begin{cases} \frac{1}{4} - \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}],\\ 4x - 2 & \text{if } x \in (\frac{1}{2}, \frac{3}{4}],\\ 4 - 4x & \text{if } x \in (\frac{3}{4}, 1]. \end{cases}\\
f_3(x) &:= \begin{cases} 4x & \text{if } x \in [0, \frac{1}{4}],\\ 2 - 4x & \text{if } x \in (\frac{1}{4}, \frac{1}{2}],\\ \frac{1}{2}x - \frac{1}{4} & \text{if } x \in (\frac{1}{2}, 1], \end{cases}
\end{aligned}
$$

Their graphs are sketched in Figure 1.

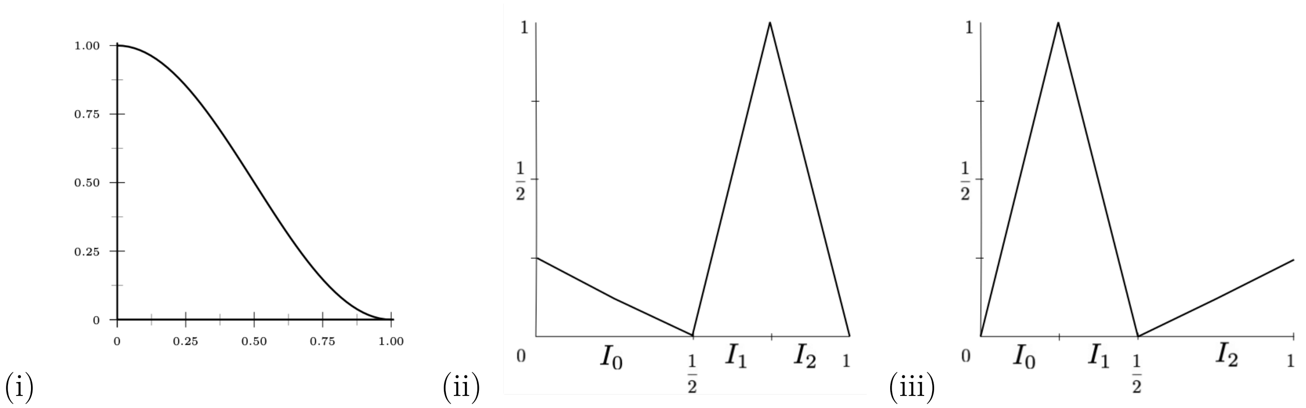

Figure 1: Graphs of the interval maps (i) f_1 (ii) f_2, and (iii) f_3. Partitions of the interval $[0,1]$, for the maps f_2 and f_3, referred to in Question 2(a)(i), are also depicted.

1. (a) Let $f : [0,1] \to [0,1]$ be continuous and consider the dynamical system generated by f.

   (i) Give the definition of the $\underline{\omega\text{-limit set}}$ $\omega(x)$ of a point $x \in [0,1]$. (4 marks)

   (ii) Define the property of <u>sensitive dependence on initial conditions</u> for f. (4 marks)

   (b) For each of the continuous interval maps f_1, f_2 as defined on Page 2:

   (i) Find all ω-limit sets that are attractors and determine whether their basin of attraction has full Lebesgue measure. (6 marks)

   (ii) Determine whether the map has sensitive dependence on initial conditions. Motivate your answer. (6 marks)

(Total: 20 marks)

2. Let $f : [0,1] \to [0,1]$ be continuous and consider the dynamical system generated by f. Let h_f be defined on Σ_3^+ as

$$h_f(\omega_0\omega_1\omega_2\ldots) := \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} f^{-i}(I_{\omega_i})}. \tag{1}$$

where $\{I_0, I_1, I_2\}$ are a set of open intervals that form a partition of $[0,1]$ in the sense that $\overline{I_0 \cup I_1 \cup I_2} = [0,1]$, and Σ_3^+ denotes the set of semi-infinite sequences $\omega_0\omega_1\omega_2\omega_3\ldots$ with $\omega_i \in \{0,1,2\}$ for all $i \in \mathbb{N}_0$.

   (a) (i) Consider the interval map f_2, as defined on Page 2 and let $I_0 = (0, \frac{1}{2})$, $I_1 = (\frac{1}{2}, \frac{3}{4})$ and $I_2 = (\frac{3}{4}, 1)$. Determine $h_{f_2}(1\overline{0})$ and find all $\omega \in \Sigma_3^+$ such that $\frac{14}{17} \in h_{f_2}(\omega)$. (6 marks)

   (ii) Consider the interval map f_3, as defined on Page 2 and let $I_0 = (0, \frac{1}{4})$, $I_1 = (\frac{1}{4}, \frac{1}{2})$ and $I_2 = (\frac{1}{2}, 1)$. Determine $h_{f_3}(1\overline{0})$ and find all $\omega \in \Sigma_3^+$ such that $\frac{1}{4} \in h_{f_3}(\omega)$. (6 marks)

   (b) (i) Define what it means for a continuous map $f : J \to J$ with $J \subset [0,1]$ to be <u>chaotic</u> (in the sense of Devaney). (4 marks)

   (ii) Find $J \subset [0,1]$ such that $f_2 : J \to J$ is chaotic. (Hint: find J so that f_2 is topologically (semi-)conjugate to shift dynamics.) (4 marks)

(Total: 20 marks)

3. Consider the interval map $f_3 : [0,1] \to [0,1]$, as defined on Page 2.

(a) Show that f_3 is topologically semi-conjugated to a shift on the topological Markov chain $\Sigma^+_{3,M}$ where

$$M := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(5 marks)

(b) The return time of an orbit of f_3 with initial condition $x \in A$ to A is defined as

$$n_A(x) := \inf\{n \in \mathbb{N} \mid f_3^n(x) \in A\}.$$

Let μ be an ergodic invariant measure of f_3. Kac's Lemma asserts that for any μ-measurable set $A \subset [0,1]$

$$\int_A n_A(x) d\mu(x) = 1.$$

(i) Consider a (μ-measurable) subset $A \subset [0,1]$, $x \in A$ and its forward orbit under f_3, $\mathcal{O}_{f_3}(x) = \{x, f_3(x), f_3^2(x), \ldots\}$. Let $n_1 := n_A(x)$ be the first return time of the orbit of x to A, $n_2 := n_A(f_3^{n_A(x)}(x))$ the subsequent first return time of the orbit of x to A, etc, then the average first return time to A along the forward orbit of x is defined as

$$\langle n_A(x) \rangle_{f_3} := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} n_i.$$

Prove that for μ-almost all initial conditions $x \in A \subset [0,1]$, $\langle n_A(x) \rangle_{f_3} = \mu(A)^{-1}$.

(5 marks)

Consider the subintervals I_0, I_1, I_2 of $[0,1]$, as in Question 2(a)(i).

(ii) Show that for Lebesgue-almost every initial condition $x \in I_i$, $i \in \{0,1,2\}$, $\langle n_{I_i}(x) \rangle_{f_3}$ is constant. Determine these average first return times for each of the intervals I_0, I_1, and I_2. (5 marks)

(iii) Show that there exists an uncountably infinite set of initial conditions x, for which the average return time from $x \in I_i$ to I_i, $i \in \{0,1,2\}$, is equal to 3. (5 marks)

(Total: 20 marks)

4. Let $f : [0,1] \to [0,1]$ be continuous and consider the dynamical system generated by f. The topological entropy of f is defined as

$$h_{\text{top}}(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \ln \operatorname{span}(n, \varepsilon, f). \tag{2}$$

(a) (i) Define $\operatorname{span}(n, \varepsilon, f)$. (4 marks)

(ii) Show that $h_{\text{top}}(f)$ is well-defined, in the sense that the limit in (2) exists. (4 marks)

(b) Determine the topological entropy of the intervals maps f_1, f_2 and f_3, as defined on Page 2. Motivate your answers. (12 marks)

(Total: 20 marks)

5. Let $f : S^1 \to S^1$ be an orientation preserving circle homeomorphism, and $F : \mathbb{R} \to \mathbb{R}$ a lift of f.

(a) (i) Show that a circle map f and its lift F are topologically semi-conjugate. (2 marks)

(ii) Give the definition of an orientation preserving circle homeomorphism. (2 marks)

Let the rotation number of an orientation preserving circle homeomorphism f be defined as

$$\rho(f) := \rho(F) \mod 1, \text{ where } \rho(F) := \lim_{n\to\infty} \frac{F^n(x)}{n}.$$

You may assume that $\rho(f)$ does not depend on x and neither on the choice of the lift F.

(b) Let f and g be orientation preserving circle homeomorphisms that are topologically conjugate by an orientation reversing circle homeomorphism h, so that $f \circ h = h \circ g$.
Show that $\rho(f) = -\rho(g) \mod 1$. (5 marks)

(c) Let $p, q \in \mathbb{N}$, with $p \leq q$ and $\gcd(p, q) = 1$. A point $x \in \mathbb{R}$ is called a p/q-periodic point of F, if $F^q(x) = x + p$. Show that $\rho(f) = \frac{p}{q}$ if and only if F has a p/q-periodic point. (5 marks)

(d) Consider the circle map $f_\omega(x) = x + \omega + \frac{1}{\pi}\sin(2\pi x) \mod 1$, with $\omega \in [0, 1)$. Show that $g : \omega \to \rho(f_\omega)$ is a *Cantor function*, i.e. that g is constant on an open and dense subset of its domain. You may use properties of rotation numbers of circle homeomorphisms, from the literature that you studied, without proof, but any such properties should be carefully stated. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH96040/MATH97065/MATH97176/MATH97285

Dynamical Systems (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. (a) (i) A point $\tilde{x} \in X$ is an ω-limit point of $x \in [0,1]$ for a continuous map $f : [0,1] \to [0,1]$ if there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} f^{n_k}(x) = \tilde{x}$. The collection of all ω-limit points of a point $x \in [0,1]$ is called the ω-limit set of x. seen $\Downarrow$ 4, A

(ii) A continuous map $f : [0,1] \to [0,1]$ with metric d has *sensitive dependence* if there exists a *sensitivity constant* $\Delta > 0$ such that for all $x \in X$ and $\varepsilon > 0$, there exists $y \in [0,1]$ with $d(x,y) < \varepsilon$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \Delta$. seen $\Downarrow$ 4, A

(b) (i) $f_1(\frac{1}{2}) = \frac{1}{2}$, so $\frac{1}{2}$ is a fixed point and $f_1^2(0) = f_1(1) = 0$ so $\{0,1\}$ is a period-2 orbit. There are no other such orbits. f_1^2 is monotonically decreasing on $[0, \frac{1}{2})$ and monotonically increasing on $(\frac{1}{2}, 1]$. Hence, since $\{0, \frac{1}{2}, 1\}$ are the only fixed points of f_1, $\lim_{n\to\infty} f_2^n(x) = 0$ if $x \in [0, \frac{1}{2})$ and $\lim_{n\to\infty} f_2^n(x) = 1$ if $x \in (\frac{1}{2}, 1]$. So $\{0,1\}$ is the unique ω-limit set of f_1 that is an attractor and its basin of attraction is $[0,1] \setminus \{\frac{1}{2}\}$, which has full Lebesgue measure. seen/sim.seen $\Downarrow$ 3, B

$f_2 : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ is a contraction and $\lim_{n\to\infty} f_2^n(x) = \frac{1}{6}$ for all $x \in [0, \frac{1}{2}]$. The set of non-escaping points from $(\frac{1}{2}, 1]$ is $N((\frac{1}{2}, 1]) := \lim_{n\to\infty} \bigcap_{i=0}^{n-1} f^{-i}((\frac{1}{2}, 1])$. As f_2 is uniformly expanding on $(\frac{1}{2}, 1]$, this set does not contain an attractor. So $\frac{1}{6}$ is the unique ω-limit set of f_2 that is an attractor and its basin of attraction is $[0,1] \setminus N((\frac{1}{2}, 1]))$. As $\mathrm{Leb}(\bigcap_{i=0}^{n-1} f_2^{-i}((\frac{1}{2}, 1])) = 2^{-(n+1)}$, it follows that $\mathrm{Leb}(N((\frac{1}{2}, 1]))) = 0$ and $\mathrm{Leb}([0,1] \setminus N((\frac{1}{2}, 1]))) = 1$. unseen $\Downarrow$ 3, D

(ii) f_1 has no sensitive dependence on initial conditions. We note that $|D(f_1^2)(x)| < 1$ for all $x \in [0, 0.1] \cup [0.9, 1]$ and that $f_2([0, 0.1]) \subset [0.9, 1]$ as well as $f_1([0.9, 1]) \subset [0, 0.1]$. Hence any pair of initial conditions $x, y \in [0, 0.1]$ satisfies for all integers $n > 1$, $d(f_1^n(x), f_1^n(y)) < cd(x,y)$ for some $c < 1$. This contradicts the existence of a sensitivity constant $\Delta > 0$. unseen $\Downarrow$ 3, C

f_2 is a contraction on $[0, \frac{1}{2}]$ which directly contradicts the existence of a sensitivity constant $\Delta > 0$. unseen $\Downarrow$ 3, B

2. (a) (i) $f_2(\overline{I_0}) \subset \overline{I_0}$ and $\overline{f_2^{-1}(I_0) \cap I_1} = [\frac{1}{2}, \frac{5}{8}]$. Hence $h_{f_2}(1\overline{0}) = [\frac{1}{2}, \frac{5}{8}]$. unseen ⇓

We have $\frac{14}{17} \in I_2$ and $f_2(\frac{14}{17}) = \frac{12}{17} \in I_1$ and $f_2(\frac{12}{17}) = \frac{14}{17}$. Hence only if $\omega = \overline{21}$ we have that $h_{f_2}(\omega) \ni \frac{14}{17}$. 3, C

unseen ⇓

(ii) f_3 is eventually piece-wise expanding on $\{I_0, I_1, I_2\}$. Indeed, f_3^2 is piecewise expanding on the first refinement $\{I_{00}, I_{01}, I_{02}, I_{10}, I_{11}, I_{12}, I_{20}\}$ since I_2 maps into I_0 and the product of the derivatives of f_3 on I_2 and I_0 is $4 \cdot \frac{1}{2} = 2 > 1$. 3, B

meth seen ⇓

This implies that $h_{f_3} : \Sigma_3^+ \to [0,1] \cup \emptyset$. First note that $h_{f_3}(\overline{0}) = 0$ (the unique fixed point of f_3 in $\overline{I_0}$). Then $x = h_{f_3}(1\overline{0})$ is the pre-image of 0 in $\overline{I_1}$: ie. $\frac{1}{2}x - \frac{1}{4} = 0 \Leftrightarrow x = \frac{1}{2}$. 3, A

sim. seen ⇓

The definition of h_f implies that if $x \in h_f(\omega)$ then $f^n(x) \in \overline{I_{\omega_n}}$ for all $n \in \mathbb{N}_0$ and that $x \in h_f(\omega)$ if $f^n(x) \in I_{\omega_n}$ for all $n \in \mathbb{N}_0$.
$\frac{1}{4}$ lies on the boundary between the partition elements I_0 and I_1. $f_3(\frac{1}{4}) = 1 \in \overline{I_3}$, $f_3(1) = 0 \in \overline{I_0}$, and $f_3(0) = 0$. Hence ω can be $13\overline{0}$ and/or $03\overline{0}\}$. It remains to verify that indeed $h_{f_3}(13\overline{0}) \ni \frac{1}{4}$ and $h_{f_3}(03\overline{0}) \ni \frac{1}{4}$. This follows from examination of the image of cylinder sets: $h_f(C_{\omega_0\omega_1\omega_2\ldots\omega_{n-1}}) := \overline{\bigcap_{i=0}^{n-1} f^{-i}(I_{\omega_i})}$. $h_{f_3}(C_{030^m})$ and $h_{f_3}(C_{130^m})$ are intervals adjacent to $\frac{1}{4}$ on the left and right, respectively, whose widths shrink exponentially with m (due to the piecewise uniform expansion of f_3 in I_0, implying convergence of both sequences of approximants to $\frac{1}{4}$. (The answer may also refer to the eventual overall piecewise expanding behaviour of f_3 and symbolic dynamics, cf. Q3(a).)

3, C

seen ⇓

(b) (i) A continuous map $f : I \to I$, $I \subset [0,1]$ is *chaotic* if it has the following three properties:

1. the periodic points of f are dense in I, ie every open set in I contains a periodic point,
2. f is topologically transitive, ie f has a dense orbit in I,
3. f has sensitive dependence on initial conditions, as defined in Q1(a)(ii).

4, A

unseen ⇓

(ii) From Q1(b)(i), as there cannot be chaotic behaviour in the basin of attraction of the attracting fixed point 0, it follows that $J \subset N(\overline{I_1 \cup I_2})$, where $N(U) := \cap_{i=0}^{n} f^{-i}(U)$ is the set of points that does not escape from U under iteration by f. Indeed, we may take $J = N(\overline{I_1 \cup I_2})$. f_2 is piecewise expanding on the partition of $\overline{I_1 \cup I_2}$ by $\{I_1, I_2\}$, and $\overline{f_2(I_i)} \supset \overline{I_j}$ for all $i, j \in \{1, 2\}$. Hence the restriction of f_2 to $N(\overline{I_2 \cup I_3})$ is topologically semi-conjugate conjugate to a full shift on 2 symbols (it is actually topologically conjugate since $N(\overline{I_1 \cup I_2})$ is totally disconnected (Cantor set)). As the latter full shift is chaotic (proved in lectures) and chaotic dynamics is preserved under topological conjugacy (or semi-conjugacy where the pre-image set of each point under the transformation has bounded cardinallity, as would be necessarilly the case here due to one-dimensionality), we obtain that $f_2|_{N(\overline{I_2 \cup I_3})}$ is indeed chaotic. 4, D

3. (a) $\overline{f_3(I_0)} = \overline{f_3(I_1)} = [0,1]$ and $\overline{f_3(I_2)} = \overline{I_0}$, and f_3 is piecewise eventually expanding on the partition $\{I_0, I_1, I_2\}$, since $|D(f_3^2)(x)| = \frac{1}{2}\cdot 4 = 2 > 1$ for all $x \in I_2 \cup f_3^{-1}(I_2)$. Hence, this is a Markov partition for f_3^2. This implies (as shown in the course) that $f_3 \circ h_{f_3} = h_{f_3} \circ \sigma_M$. meth seen ⇓

5, A

(b) (i) By application of Birkhoff's ergodic theorem we have μ-a.s. unseen ⇓

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_A(f_3^n(x))n_A(f_3^n(x)) &= \int_0^1 \mathbb{1}_A(f_3^n(x))n_A(f_3^n(x))d\mu(x)\\ &= \int_A n_A(x)d\mu(x) = 1.\end{aligned}$$

Now we need to take account that we average over the number of times we return to A:

$$< n_A(x) >_{f_3} = \frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_A(f_3^n(x))n_A(f_3^n(x))}{\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_A(f_3^n(x))} = \frac{1}{\mu(A)},$$

where the final equality holds for μ-a.a. $x \in A$. 5, B

(ii) There is a unique Markov measure $\mu_{\nu,P}$, such that $(h_{f_3})_*\mu$ is an ergodic invariant measure for f_3, that is equivalent to Lebesgue. Any property holding for $(h_{f_3})_*\mu$-almost all points, thus also holds for Lebesgue almost all points. The relevant stochastic matrix P is found by assigning transition probabilities to the allowed transitions in the topological Markov chain from Q3(a), representing the relative transfer of Lebesgue measure. The resulting transition matrix is meth seen ⇓

$$P := \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix}.$$

3, C

The unique left eigenvector ν such $\nu P = \nu$ is $\frac{1}{6}(4,1,1)$, from which we find $(h_{f_3})_*\mu(I_i) = \nu_i$. Hence the requested average return times are $\frac{3}{2}$ for I_0, 6 for I_1 and 6 for I_2. unseen ⇓

2, B

(iii) We seek Markov measures $\mu_{\nu,P}$ so that $\nu = \frac{1}{3}(1,1,1)$. The general form of P is unseen ⇓

$$P := \begin{pmatrix} p_1 & p_2 & 1-p_1-p_2 \\ q_1 & q_2 & 1-q_1-q_2 \\ 1 & 0 & 0 \end{pmatrix},$$

with $0 \le p_i, q_i \le 1$, $i = 0,1,2$. $(1,1,1)$ is a left eigenvector for eigenvalue 1 only if $p_1 = q_1 = 0$ and $q_2 = 1 - p_2$. This leaves a one-parameter (p_2) family of distinct ergodic invariant measures for the shift, corresponding also to a one-parameter familly of distinct ergodic invariant measures for f_3. The support of this measure intersects all of the partition intervals and for each distinct ergodic invariant measure there exists at least one orbit that realizes its statistics. Hence in each partition interval there are an uncountable infinity of initial conditions whose average first return time along its orbit (to the partition element from which the orbit starts) is equal to 3. 5, D

4. (a) (i) Let $f : [0,1] \to [0,1]$ be continuous with metric d. For each $n \in \mathbb{N}$, let

seen $\Downarrow$

$$d_n(x, \tilde{x}) := \max_{0 \le k \le n-1} d(f^k(x), f^k(\tilde{x})). \qquad (1)$$

A subset $A \subset X$ is (n, ε)-*spanning* if for every $x \in X$ there is $\tilde{x} \in A$ such that $d_n^X(x, \tilde{x}) < \varepsilon$. $\mathrm{span}(n, \varepsilon, f)$ is the minimum cardinality of an (n, ε)-spanning set.

4, A

(ii) $\mathrm{span}(n, \varepsilon, f)$ is finite due to compactness of $[0,1]$, Namely, as $[0,1]$ is compact wrt to the (Euclidean) metric d then so is $[0,1]$ wrt the metric d_n, since they are topologically equivalent. Thus $h_\varepsilon(f) := \limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{span}(n, \varepsilon, f)$ exists. If $\tilde{\varepsilon} < \varepsilon$ then $\mathrm{span}(n, \varepsilon, f) \le \mathrm{span}(n, \tilde{\varepsilon}, f)$ since every $(n, \tilde{\varepsilon})$-spanning set is also (n, ε)-spanning. Hence $h_\varepsilon(f) \le h_{\tilde{\varepsilon}}(f)$, and because of this monotonicity $\lim_{\varepsilon\to 0} h_\varepsilon(f)$ exists (in $\mathbb{R}_{\ge 0} \cup \{\infty\}$).

part seen $\Downarrow$

4, A

(b) f_1: as f_1 is an invertible map on the interval $[0,1]$, $h_{\mathrm{top}}(f_1) = 0$. To show this, let $B_{0,m} := \{\frac{n}{m}\}_{n=0}^m$ and $B_{k,m} := \{f^{-\ell}(\frac{n}{m} \mid n = 0, \ldots, m,\ \ell = 0, \ldots, k\}$. Then $\#B_{k,m} = (k+1)\#B_{0,m}$ and $B_{k,m}$ is (m, ε)-spanning for all $\varepsilon > \frac{1}{m} \Rightarrow \mathrm{span}(k, \frac{1}{m}, f_1) \le (k+1)\#B_{0,m} \Rightarrow \lim_{k\to\infty} \frac{1}{k} \ln \mathrm{span}(k, \frac{1}{m}, f_1) = 0$ for all $m \in \mathbb{N}$ $\Rightarrow h_{\mathrm{top}}(f_1) = 0$.

sim. seen $\Downarrow$

4, B

f_2: all points in the basin of the attraction converge exponentially fast to 0, so these do not contribute to the topological entropy. Hence the topological entropy of f_2 is equal to the topological entropy of the restriction of f_2 to the non-escape set $N([\frac{1}{2}, 1])$. It was established in (b)(ii) that this dynamics is topologically (semi-)conjugate to a full shift on two symbols. This shift dynamics has topological entropy $\ln 2$. As topological entropy is preserved under this (particular type of semi-) topological conjugacy, $h_{\mathrm{top}}(f_2) = \ln 2$.

unseen $\Downarrow$

4, D

f_3: From Q3(a), f_1 is topologically semi-conjugate to the shift on the Markov chain $\Sigma_{3,M}^+$ with connectivity matrix M. $h_{\mathrm{top}}(\sigma_M) = r(M) = 1 + \sqrt{3}$. As the semi-conjugacy to shift dynamics is at most 2-to-1 (due to the one-dimensionality of the setting), by a result in the course, the topological entropy is preserved under the semi-conjugacy so that $h_{\mathrm{top}}(f_3) = r(M) = 1 + \sqrt{3}$.

sim. seen $\Downarrow$

4, A

5. (a) (i) By definition, $F : \mathbb{R} \to \mathbb{R}$ is a lift of $f : S^1 \to S^1$, if (*) $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{R} \to S^1$ is the projection $\pi(x) = x \mod 1$, with $S^1 \simeq [0, 1)$ (or alternatively $\pi(x) = \exp(2\pi i x)$ where $S^1 \simeq \{z \in \mathbb{C} \mid |z| = 1\}$). The relation (*) is by definition a semi-conjugacy and as π is continuous, a topological semi-conjugacy. seen ⇓ 2, M

(ii) A map $f : S^1 \to S^1$ is a homeomorphism if it is continuous and invertible, with continuous inverse. A circle homeomorphism f is called orientation preserving if it has a lift F that is an increasing homeomorphism of $\mathbb{R}$. seen ⇓ 2, M

(b) If $hfh^{-1} = g$ then the lifts F, G, H of f, g, h satisfy $HFH^{-1} = G$. It suffices to show that $\rho(F) = -\rho(G)$. Without loss, suppose H is such that $H(0) \in [0, 1)$. Then, using the fact that H is decreasing so that $H(1) \in [-1, 0)$, for $x \in [0, 1)$ we have $0 = 1 - 1 < H(x) + x < H(0) + 1 < 2$. Hence, by periodicity $|H(x) + x| < 2$ and by the same argument also $|H^{-1}(x) + x| < 2$. Moreover, if $|x - y| < 2$ then $|F^n(x) - F^n(y)| < 3$, since F^n is a continuous circle homeomorphism. Combining both estimates yields $|H^{-1}F^nH(x) + F^n(x)| \leq |H^{-1}F^nH(x) + F^nH(x)| + |F^nH(x) - F^n(x)| < 2 + 3 = 5$, which implies that $\rho(G) + \rho(F) = \lim_{n\to\infty}(H^{-1}F^nH(x) + F^n(x))/n \leq \lim_{n\to\infty} 5/2 = 0$. unseen ⇓ 5, M

(c) If $F^q(x) = x + p$ then $\lim_{n\to\infty} F^n(x)/n = p/q$. It remains to be shown that $\rho(f) = p/q$ implies the existence of a p/q-periodic point for the lift F. Suppose F has no p/q-periodic point. Then for all $x \in \mathbb{R}$ we have $F^q(x) \neq x + p$. Then by the intermediate function theorem, either $F^q(x) - x > p$ or $F^q(x) - x < p$ for all $x \in \mathbb{R}$. Suppose $F^q(x) - x > p$ for all $x \in \mathbb{R}$. For the evaluation of $F^q(x) - x$, it suffices to consider $x \in [0, 1]$. By compactness $F^q(x) - x$ assumes a minimum m for $x \in [0, 1]$. By assumption $p < m$, let $\varepsilon := m - p$ so that $F^q(x) - x \leq p + \varepsilon$ for all x. Since F is increasing, $F^{2q} \geq F^q(x+p+\varepsilon) \geq x+2(p+\varepsilon)$, and similarly $F^{kq} \geq F^q(x + p + \varepsilon) \geq x + k(p + \varepsilon)$ for all $k \in \mathbb{N}$. Then $\rho(F) = \lim_{k\to\infty} \frac{F^{kq}(x)-x}{kq} > \frac{p+\varepsilon}{q} > \frac{p}{q}$. By virtually the same argument (changing the inequalities and defining m as the local maximum),we find that if $F^q(x)-x < p$ for all x, then $\rho(f) < \frac{p}{q}$. sim. seen ⇓ 5, M

(d) Note that the lift F_ω : $x \to x + \omega + \frac{1}{\pi}\sin(2\pi x)$ of f_ω has the property (**) $F_\omega(x) > F_{\tilde{\omega}}(x)$ whenever $\omega > \tilde{\omega}$. This implies that $\rho(F_\omega) \geq \rho(F_{\tilde{\omega}})$ whenever $\omega > \tilde{\omega}$ (result from notes). We observe that $F_0(0) = 0$ and $F_1(0) = 1$. Together with the observation that $\rho(F_\omega)$ is increasing, this implies that $\rho(f_0) = 0$ and $\lim_{\omega\to 1} \rho(f_\omega) = 1$. The alternative would be that $\rho(f_\omega) = 0$ for all $\omega \in [0, 1)$, but this is not the case since for ω slightly larger than $1/\pi$, f_ω does not have a fixed point. Since f_ω depends continuously on ω, so does $g(\omega) := \rho(f_\omega)$. As $g(0) = 0$ and $\lim_{\omega\to 1} g(\omega) = 1$, thus for all $p/q \in [0, 1) \cap \mathbb{Q}$ there exists $\omega \in [0, 1)$ such that $g(\omega) = p/q$. Next we need we use a result (from the notes) that asserts when (**) holds, whenever $g(\omega) = p/q \in \mathbb{Q}$, then there exists an open interval $I \subset [0, 1)$ such that $\omega \in I$ so that $g(\tilde{\omega}) = p/q$ for all $\tilde{\omega} \in I$. Finally, the notes show that condition (**) implies that if $\rho(f_\omega) \notin \mathbb{Q}$ or $\rho(f_{\tilde{\omega}}) \notin \mathbb{Q}$ then $\rho(f_\omega) > \rho(f_{\tilde{\omega}})$ if $|\omega - \tilde{\omega}| << 1$. This all implies that irrational rotational numbers only arise at isolated values of ω and that each rational rotation number is attained in an open interval. This implies that g is a Cantor function. sim. seen ⇓ 6, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60008/70008	1	No Comments Received Yet
MATH60008/70008	2	No Comments Received Yet
MATH60008/70008	3	No Comments Received Yet
MATH60008/70008	4	No Comments Received Yet
MATH70008	5	No Comments Received Yet