Classical Dynamics **Autumn Term 2024**

Problem Sheet 1

1. A particle moving in two dimensions is subject to a position dependent force $\mathbf{F} = axy\mathbf{i} + bx^2\mathbf{j}$ where a and b are constants. For what a and b is the force conservative?

2. A particle moving in two dimensions is subject to the conservative force

$$\mathbf{F} = -\lambda(x\mathbf{i} + y\mathbf{j})(x^2 + y^2 - 1),$$

   where λ is a positive constant. Determine the potential energy V. Comment on the motion of the particle for large λ.

3. A particle moving in two dimensions is subject to the force

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j}.$$

   Is the force conservative?

4. Show that a central force has the following properties

   (i) it is conservative.

   (ii) the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a particle with respect to the origin is a constant of the motion.

5. A charged particle of mass m in a magnetic field $\mathbf{B}(\mathbf{r})$ is subject to the Lorentz force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B},$$

   where q is the electric charge of the particle. Show that the kinetic energy of the charged particle is a constant of the motion.

6. The potential energy of a particle of unit mass is

$$V = \frac{1}{x^2} + x^2.$$

   Determine the frequency of small oscillations about $x = 1$.

7. The potential energy of a simple harmonic oscillator of mass m is $V = \frac{1}{2}m\omega^2x^2$ where ω is the angular frequency. Show that the average kinetic energy is equal to the average potential energy (integrate $T - V$ over a period of oscillation).

8. For the Kepler problem the force has the form $\mathbf{F} = -k\mathbf{r}/r^3$, where k is a positive constant. As this is a central force the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a constant of the motion.

   (i) The Laplace-Runge-Lenz vector, $\mathbf{A}$, is defined by

   $$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\frac{\mathbf{r}}{r}.$$

   Show that $\mathbf{A}$ is a constant of the motion.

   (ii) Show that $\mathbf{A} \cdot \mathbf{L} = 0$ and $A^2 = m^2k^2 + 2mEL^2$.

   Remark: It appears that the Kepler problem has 7 constants of the motion (the energy, E, and the components of $\mathbf{L}$ and $\mathbf{A}$). However, there are really five independent constants as $\mathbf{A} \cdot \mathbf{L} = 0$ and $A^2 = m^2k^2 + 2mEL^2$.

9. A particle of unit mass is subject to the central force

   $$\mathbf{F} = -\mu\mathbf{r},$$

   where μ is a constant. Describe the motion (distinguish the cases $\mu > 0$ and $\mu < 0$).