Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Problem Sheet 0

This first problem sheet was designed as a revision problem sheet; it was designed to help you work through concepts which are important for this module but were covered in other modules. In this sheet, you will work through problems relating to solving ODEs, Fourier series and Fourier transforms. We finish with some general questions about PDEs.
Note that the problem sheets have been designed to contain some easier problems (direct applications of the lecture material) and some harder problems (meant to make you think more deeply). As such, a number of stars has been included in front of each question as a guide for how hard each problem is. Hopefully, it will be useful in guiding your work on during the term and during revisions!

1. ✦ Obtain the general solution of

$$\frac{d^2y}{dx^2} = 5e^{2x}$$

2. ✦ Find the general solutions of the following equations

(a) $\dfrac{dy}{dx} = 3ye^x$

(b) $\dfrac{dy}{dx} = \dfrac{e^{-x}}{y}$

(c) $x\dfrac{dy}{dx} = \tan(y)$

(d) $\dfrac{dy}{dx} = 4yx$

3. ✦✦ Using a well-chosen substitution, find the general solution of the equation

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x} + 1$$

4. ✦✦ Use an appropriate integrating factor to find the general solutions to:

(a) $x\dfrac{dy}{dx} + 2xy = xe^{-2x}$

(b) $\dfrac{dy}{dx} - \tan(x)y = 1$

5. ✦ Find the general solution to the following homogeneous second-order linear equations with constant coefficients:

(a) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 0$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 0$

(c) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$

6. ✦✦ Find:

(a) the general solution of $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 3x^2$

(b) the particular solution of $y'' + y' - 12y = 4e^{2x}$, satisfying $y(0) = 7$ and $y'(0) = 0$.

7. ✦ Find the Fourier series of period 2π which represent the following functions on the interval $-\pi < x < \pi$:

   (a) $f(x) = x$
   (b) $f(x) = x^2$
   (c) $f(x) = \sinh(x)$

   In each case, sketch the function represented by the Fourier series in the range $-3\pi < x < 3\pi$ and find the value to which the Fourier series converges at $x = \pi$.

8. ✦✦ Obtain the Fourier expansion

$$\cos\alpha x = \frac{\sin(\alpha\pi)}{\alpha\pi} + \sum_{n=1}^{\infty}(-1)^n \frac{2\alpha\sin\alpha\pi}{\pi(\alpha^2 - n^2)}\cos nx, \quad |x| \leq \pi$$

   when α is not an integer. What happens to the terms of the series when $\alpha \to m$, an integer?

9. ✦✦ Show that the Fourier series representation on $(-\pi, \pi)$ of the function

$$f(x) = \begin{cases} 1 + (x/\pi), & -\pi < x < 0 \\ 1 - (x/\pi), & 0 \leq x \leq \pi \end{cases}$$

   is given by

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}$$

   Deduce that $\sum_{k=0}^{\infty} 1/(2k+1)^2 = \pi^2/8$.

10. ✦✦✦ For the function $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$, derive the Fourier half-range sine and cosine expansions. Use these expansions to find

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$
    (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$
    (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}$

11. ✦✦ Show that the half-range Fourier sine series for $f(x) = 1 + (x/L),\ 0 < x < L$, is:

$$\sum_{n=1}^{\infty}\frac{2}{n\pi}(1 - 2(-1)^n)\sin\left(\frac{n\pi x}{L}\right)$$

12. ✦✦ Let $a > 0$, find the Fourier transforms of the following functions:

    (a) $f(x) = \exp(-a|x|)$
    (b) $f(x) = \mathrm{sgn}(x)\exp(-a|x|)$, where $\mathrm{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$
    (c) $f(x) = 2a/(a^2 + x^2)$
    (d) $f(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise
    (e) $f(x) = \sin(ax)/(\pi x)$ [Hint: what is the Fourier transform of a rectangular pulse?]

    By applying the inversion formula to the transforms obtained for (a) and (d), establish the following results:

$$\int_0^{\infty}\frac{\cos x}{x^2 + a^2}dx = \frac{\pi e^{-a}}{2a}, \text{ if } a > 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty}\frac{\sin x - x\cos x}{x^3}dx = \frac{\pi}{2}$$

13. ✦✦ Consider the function given by

$$f(x) = \begin{cases} 2d - |x|, & \text{for } |x| \leq 2d, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\widehat{f}(\omega) = (2/\omega)^2 \sin^2(\omega d)$.

Use the energy theorem to demonstrate that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^4 dx = \frac{2\pi}{3}$$

14. ✦✦ Show that the Fourier transform of $\exp(-cx)H(x)$, where H is the Heaviside function and c is a positive constant, is given by $1/(c + i\omega)$. Use the convolution theorem to find the inverse Fourier transform of

$$\frac{1}{(a + i\omega)(b + i\omega)}$$

where $a > b > 0$.

15. ✦ Write down:

   (a) the most general linear first-order PDE in two variables;
   (b) the most general linear second-order PDE in two variables;
   (c) the most general semilinear first-order PDE in two variables.

16. ✦ Given the following PDEs: find their order, discuss whether they are linear, semilinear, quasilinear or fully nonlinear. In the case of linear equations, state whether they are homogeneous or not.

   (a) $x\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

   (b) $\dfrac{\partial^2 u}{\partial x^2} + u\dfrac{\partial u}{\partial y} = xyu$

   (c) $\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial u}{\partial t} = \sin(u)$

   (d) $\left(\dfrac{\partial u}{\partial t}\right)^2 + \dfrac{\partial^3 u}{\partial x^3} = 0$

   (e) $\left(\dfrac{\partial u}{\partial t}\right)^2 + \left(\dfrac{\partial u}{\partial x}\right)^3 = 0$

   (f) $x^3\dfrac{\partial u}{\partial x} - u^3\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial t^2} = x^5 + t^4$

17. ✦ We define the operator $\mathcal{L}$ by

$$\mathcal{L}u(x, y) = a(x, y)\frac{\partial^2 u}{\partial x^2} + b(x, y)\frac{\partial^2 u}{\partial y^2} + c(x, y)\frac{\partial^2 u}{\partial x \partial y}$$

Show that $\mathcal{L}$ is a linear differential operator.