Tensor Calculus and General Relativity **Spring Term 2023**

Abridged Answers to Problem Sheet 2

1. Have to check that matrices, Λ, obeying $\Lambda^T \eta \Lambda = \eta$ satisfy the group properties associativity, closure, identity, inverse

   Matrix multiplication is always associative

   closure: suppose $\Lambda_1^T \eta \Lambda_1 = \eta$ and $\Lambda_2^T \eta \Lambda_2 = \eta$. From the first equation we have $\Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \Lambda_2^T \eta \Lambda_2$. This is η using the first equation. As $\Lambda_2^T \Lambda_1^T = (\Lambda_1 \Lambda_2)^T$ it follows that $(\Lambda_1 \Lambda_2)^T \eta \Lambda_1 \Lambda_2 = \eta$ as required.

   identity: $I^T \eta I = \eta$.

   inverse: taking the determinant of $\Lambda^T \eta \Lambda = \eta$ yields det$(\Lambda^T \Lambda) = 1$ so that det$\Lambda = \pm 1$ and so Λ (and Λ^T) is an invertible matrix. $\Lambda^T \eta \Lambda = \eta$ implies $(\Lambda^T)^{-1} \Lambda^T \eta \Lambda \Lambda^{-1} = (\Lambda^T)^{-1} \eta \Lambda^{-1}$. The left hand side is η and using $(\Lambda^T)^{-1} = (\Lambda^{-1})^T$ one has $(\Lambda^{-1})^T \eta \Lambda^{-1} = \eta$.

2. four-momentum of atom at rest $p_a = (mc, 0, 0, 0)$

   four momentum of photon $p_b = (E/c, E/c, 0, 0)$

   By conservation of four-momentum

   four momentum of recoiling atom $p_c = (mc + E/c, E/c, 0, 0)$

   To compute mass of recoiling atom $(m^*)^2 c^2 = p_c \cdot p_c = (mc + E/c)^2 - (E/c)^2 = m^2 c^2 + 2mE$ hence the result.

3. $u = \gamma(c, \mathbf{v})$ therefore

   $$a = \frac{du}{d\tau} = \frac{d\gamma}{d\tau}(c, \mathbf{v}) + \gamma \left(0, \frac{d\mathbf{v}}{dt}\frac{dt}{d\tau}\right).$$

   Now

   $$\frac{d}{d\tau}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \frac{d}{d\tau}\frac{v^2}{c^2} = \frac{1}{2}\gamma^3 \frac{dt}{d\tau}\frac{2\mathbf{v}\cdot\mathbf{a}}{c^2} = \frac{\gamma^4 \mathbf{v}\cdot\mathbf{a}}{c^2}$$

   Accordingly

   $$a = \left(\frac{\gamma^4 \mathbf{v}\cdot\mathbf{a}}{c}, \frac{\gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{c^2} + \gamma^2 \mathbf{a}\right)$$

   Remark: one can check that this satisfies $u \cdot a = 0$.