MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 3

Exercise 11.

Obviously, this differential equation has the trivial solution (i.e. solution that constantly zero). To find a non-zero solution that satisfies the initial condition $x(0) = 0$, use separation of variables, but consider initial conditions of the form $x(t_0) = 0$ with $t_0 > 0$, and note that you can concatenate solutions. For (iii), note that non-uniqueness only occurs for solutions that are zero at some point (why?). Note then that from the initial value -1, one needs some time to move into 0.

Exercise 12.

Solutions for (i) and (ii) can be found via separation of variables (see also Example 1.8). Maximality of solutions is shown using Theorem 2.17, by proving that, in case of a finite boundary of the maximal existence interval, the solution converges either to the boundary or is unbounded. Continuation of the solution is then not possible and maximality is proven. For (iii), use estimates on the right hand side to determine the limits for $t \to \pm\infty$. For (iv), analyse the solution identity and use Theorem 2.17.

Exercise 13.

Show first that $\tau, \delta > 0$ exists such that $W^{\tau,\delta}(t_0, x_0) \subset D$ and (2.13) and (2.14) hold. Then consider smaller neighbourhoods $W^{\frac{\tau}{2},\frac{\delta}{2}}(\tilde{t}_0, \tilde{x}_0)$ for initial pairs $(\tilde{t}_0, \tilde{x}_0) \in W^{\frac{\tau}{2},\frac{\delta}{2}}(t_0, x_0)$.

Exercise 14.

Let $\lambda : I_{max}(0, x_0) \to \mathbb{R}^d$ be a solution to the initial value problem and determine how fast $\alpha(t) := \|\lambda(t)\|^2$, the norm of this solution, grows. The inequality provided gives information on that, and see whether Example 2.18 can be of help here, together with Exercise 4.

Exercise 15.

Show that all assumptions of Schauder's fixed point theorem are fulfilled for the mapping $P : S \to X$. The proof that $P(S)$ is relatively compact requires the Arzelà–Ascoli theorem. It may be of help to have a look at the proof of the local version of the Picard–Lindelöf theorem, given in *Extra Material 1*.