Exercise 9.1. Consider the metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$. Show that all of the intervals $(a, b]$, $[a, b)$, $[a, +\infty)$, and $(-\infty, b]$ are not compact.

Hint: For each of those intervals, you need to present an open cover of the set such which does not have a finite sub-cover.

Solution: Consider the open cover

$$\mathcal{R} = \{(a + 1/n, b + 1) \mid n \in \mathbb{N}\}$$

for $(a, b]$. As we discussed in the examples in the lectures, there is no finite sub-cover of $\mathcal{R}$ for $(a, b]$. Thus, $(a, b]$ is not compact. Similarly, the open cover $\{(a - 1, b - 1/n) \mid n \in \mathbb{N}\}$ for $[a, b)$ has no finite sub-cover.

The collection

$$\{(a - 1, n) \mid n \in \mathbb{N}\}$$

is an open cover for $[a, +\infty)$. There is no finite sub-cover of this cover for $[a, +\infty)$. Similarly, the open cover $\{(-n, b + 1) \mid n \in \mathbb{N}\}$ for $(-\infty, b]$ has no finite sub-cover.

Exercise 9.2. Show that if A and B are compact subsets of a metric space (X, d), then $A \cup B$ is a compact set.

Hint: For an arbitrary open cover for $A \cup B$, there is a finite sub-cover for A, and a finite sub-cover for B. Consider the union of those finite sub-covers.

Solution: Let $\mathcal{R}$ be an open cover for $A \cup B$. Then, in particular, $\mathcal{R}$ is an open cover for A, and an open cover for B. Since A is compact, there is a finite subset $\mathcal{R}_A \subseteq \mathcal{R}$ which is a cover for A. Similarly, there is a finite subset $\mathcal{R}_B \subseteq \mathcal{R}$ which is a cover for B. Clearly, $\mathcal{R}_A \cup \mathcal{R}_B$ is a finite cover for $A \cup B$, which is a sub-cover of $\mathcal{R}$. Hence, $A \cup B$ is compact.

Exercise 9.3. Show that the ball

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in the metric space $(\mathbb{R}^2, \mathrm{d}_2)$ is not compact.

Hint: consider an open cover of this set, by balls centred at $(0, 0)$ and the radii tending to 1 from below.

Solution: For every $n \in \mathbb{N}$, consider the set

$$U_n = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 - 1/n\},$$

and consider the collection

$$\mathcal{R} = \{U_n \mid n \in \mathbb{N}\}.$$

Clearly, this is a collection of open sets (balls in any metric space are open sets), which is a cover for the ball of radius 1 about $(0, 0)$; $B_1((0, 0))$. This cover has no finite sub-cover for $B_1((0, 0))$. Assume in the contrary that there is a finite sub-cover $\{U_{n_k} \mid k = 1, \dots, m\}$ of $\mathcal{R}$ for $B_1((0, 0))$. Let $p = \max_{1 \le k \le m} n_k$. The point (x, y) with $x^2 + y^2 = 1 - 1/(p + 1)$ belongs to $B_1((0, 0))$. But, that point does not belong to $\cup_{k=1}^m U_{n_k}$. This is a contradiction.

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Questions marked with $*$ are optional

Exercise 9.4. Let (X, d) be a metric space, and $A_1, A_2, \ldots, A_n$ be a finite number of bounded sets in X. Then $\cup_{i=1}^n A_i$ is a bounded set in X.

Hint: Consider the bounds M_i for the sets A_i, for $i = 1, \ldots, n$. From each i, choose a point $z_i \in A_i$, and add all the numbers M_i and $\mathrm{d}(z_i, z_j)$, over all i and j.

Solution: For every $k = 1, 2, \ldots, n$, A_k is bounded. Thus, for every $k = 1, 2, \ldots, n$, there is a constant M_k such that for all x_k and y_k in A_k we have $\mathrm{d}(x_k, y_k) \leq M_k$. If some A_k is empty, we can discard that set, since it does not make any difference in the union $\cup_{k=1}^n A_k$. So without loss of generality, we may assume that all A_k, for $k = 1, 2, \ldots, n$, are not empty. Thus, for each $k = 1, 2, \ldots, n$, we may choose a point $z_k \in A_k$. Define the number

$$\hat{M} = \sum_{k=1}^{n} M_i + \sum_{i=1, j=1}^{i=n, j=n} \mathrm{d}(z_i, z_j)\},$$

where the second sum runs over all pairs (i, j) with i and j in $\{1, 2, \ldots, n\}$. This is a constant (finite) real number. We claim that the distance between any two points in $\cup_{k=1}^n A_k$ is bounded from above by $\hat{M}$.

Let x and y be arbitrary points in $\cup_{k=1}^n A_k$. If x and y belong to the same set A_k, then $\mathrm{d}(x, y) \leq M_k \leq \hat{M}$. Now, assume that $x \in A_i$ and $y \in A_j$ for some i and j in $\{1, 2, \ldots, n\}$. Then, by the triangle inequality,

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, z_i) + \mathrm{d}(z_i, y) \leq \mathrm{d}(x, z_i) + \mathrm{d}(z_i, z_j) + \mathrm{d}(z_j, y_i) \leq M_i + \mathrm{d}(z_i, z_j) + M_j \leq \hat{M}.$$

Exercise 9.5. Let (X, d) be a non-empty metric space, and let $Z \subseteq X$. Show that Z is bounded if and only if there is $x \in X$ and $r \in \mathbb{R}$ such that $Z \subseteq B_r(x)$.

Hint:If Z is bounded, choose a bound M, and consider the ball $B_M(x)$, for an arbitrary $x \in A$. If A is contained in a ball of radius R, work with the bound $2R$ for the set A.

Solution: First assume that Z is bounded. This means that there is $M > 0$ such that for any x and y in Z, $\mathrm{d}(x, y) < M$. Since Z is not empty, we may choose $x \in Z$, and consider the ball $B_M(x)$. Then, by the definition of the ball, we have $Z \subseteq B_M(x)$.

Now assume that are $x \in X$ and $r > 0$ such that $Z \subseteq B_r(x)$. Then by the definition of $B_r(x)$, for any $y \in Z$, $\mathrm{d}(y, x) < r$. If s and t in Z are arbitrary points, then by triangle inequality, we have $\mathrm{d}(s, t) \leq \mathrm{d}(s, x) + \mathrm{d}(x, t) < 2r$. Therefore, Z is bounded.

Exercise 9.6. Consider the set $\mathbb{R}$ with the discrete metric d_{disc}. The set $(0, 1)$ is closed and bounded in $(\mathbb{R}, \mathrm{d}_{\text{disc}})$, but it is not compact.

Hint: Obviously, 1 provides a bound for the distance between any two points in $(0, 1)$. Use that any set in $\mathbb{R}$ with respect to the discrete metric is open, so any set is also closed (being the complement of some set in $\mathbb{R}$).

Solution: Recall that in the metric space $(\mathbb{R}, \mathrm{d}_{\text{disc}})$, every set is open. This, implies that every set is also closed (being the complement of an open set). Also, for every x and y in $\mathbb{R}$ we have $\mathrm{d}_{\text{disc}}(x, y) \leq 1$. Thus, $\mathbb{R}$ is bounded in the metric d_{disc}.

For every $x \in \mathbb{R}$, the set $\{x\}$ is open. Thus,

$$\mathcal{R} = \{\{x\} \mid x \in (0, 1)\}$$

is an (uncountable) open cover for $(0, 1)$. Obviously, it does not have a finite sub-cover for $(0, 1)$.

Exercise 9.7. Let (X, d) be a metric space, and assume that V_n, for $n \geq 1$, be a nest of non-empty closed sets in X.

(i) Show that if X is compact, then $\cap_{n\geq 1} V_n$ is not empty.

(ii) Give an example of a nest of non-empty closed sets V_n, for $n \geq 1$, in a metric space such that $\cap_{n\geq 1} V_n$ is empty.

Hint: If the intersection is empty, then we may consider the cover of X by the sets $X \setminus V_n$, for $n \geq 1$, and drive a contradiction. For the second part, think about closed sets in $(\mathbb{R}, \mathrm{d}_1)$.

Solution: (i) Assume in the contrary that $\cap_{n\geq 1} V_n = \emptyset$. Consider the collection

$$\mathcal{R} = \{X \setminus V_n \mid n \in \mathbb{N}\}.$$

Since each V_n is a closed set, each $X \setminus V_n$ is an open set. Thus, $\mathcal{R}$ is a collection of open sets. We claim that $\mathcal{R}$ is a cover for X. To see this, let $x \in X$ be arbitrary. Since $\cap_{n\in\mathbb{N}} V_n = \emptyset$, there is $n_0 \in \mathbb{N}$ such that $x \notin V_{n_0}$. This implies that $x \in X \setminus V_{n_0}$, and hence is covered by an element of $\mathcal{R}$.

Since X is compact, there must be a finite sub-cover of $\mathcal{R}$ for X. Thus, there is $m \in \mathbb{N}$ such that

$$X \subseteq \bigcup_{n=1}^{m} (X \setminus V_n).$$

Because $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$, we have $(X \setminus V_1) \subseteq (X \setminus V_2) \subseteq (X \setminus V_3) \subseteq \dots$. Therefore, by the above equation

$$X \subseteq X \setminus V_m.$$

This implies that $V_m = \emptyset$, which contradicts the hypothesis in the exercise.

(ii) For example, the sets $V_n = [n, +\infty)$, for $n = 1, 2, \dots$ are a nest of closed sets in the metric space $(\mathbb{R}, \mathrm{d}_1)$. We have $\cap_{n\geq 1} V_n = \emptyset$.

Exercise 9.8. Show that if a metric space is sequentially compact, then it is bounded.

Hint: If a set is not bounded, there are pairs of points z_n and w_n with $\mathrm{d}(z_n, w_n) \geq n$. Think about what happens if $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ converge to some points z and w, respectively. You will need to identify a subsequence, so that both sequences converge along that subsequence.

Solution: Assume in the contrary that there is a sequentially compact metric space which is not bounded. Because X is not bounded, for every $n \in \mathbb{N}$, we may choose x_n and y_n in X such that $\mathrm{d}(x_n, y_n) \geq n$.

Since X is sequentially compact, there is a subsequence of $(x_n)_{n\geq 1}$, say $(x_{n_k})_{k\geq 1}$, which converges to some point x in X. Now consider the sequence $(y_{n_k})_{k\geq 1}$ in X. Since X is sequentially compact, there is a subsequence of $(y_{n_k})_{k\geq 1}$, say $(y_{m_i})_{i\geq 1}$, which converges to some y in X. Note that, since $(y_{m_i})_{i\geq 1}$ is a subsequence of $(y_{n_k})_{k\geq 1}$, the sequence $(x_{m_i})_{i\geq 1}$ is a subsequence of $(x_{n_k})_{k\geq 1}$. In particular, $(x_{m_i})_{i\geq 1}$ converges to x and $(y_{m_i})_{i\geq 1}$ converges to y.

Since $(x_{m_i})_{i\geq 1}$ converges to x, for $\epsilon = 1$ there is $n_x \in \mathbb{N}$ such that for all $i \geq n_x$ we have $\mathrm{d}(x_{m_i}, x) \leq 1$. Similarly, since $(y_{m_i})_{i\geq 1}$ converges to y, for $\epsilon = 1$ there is $n_y \in \mathbb{N}$ such that for all $i \geq n_y$ we have $\mathrm{d}(y_{m_i}, y) \leq 1$. Then, for all $i \geq \max\{n_x, n_y\}$ we have

$$\mathrm{d}(x_{m_i}, y_{m_i}) \leq \mathrm{d}(x_{m_i}, x) + \mathrm{d}(x, y) + \mathrm{d}(y, y_{m_i}) \leq 1 + \mathrm{d}(x, y) + 1 = 2 + \mathrm{d}(x, y).$$

This contradicts $\mathrm{d}(x_{m_i}, y_{m_i}) \geq m_i$, when m_i is very large.

Exercise 9.9.* Let (X, d) be a sequentially compact metric space. Show that X is separable, that is, there is a countable dense set in X.

Hint: Fix an arbitrary $n \in \mathbb{N}$. Consider the open cover $\mathcal{R}_n = \{B_{1/n}(x) \mid x \in X\}$. Use the sequential compactness of X to conclude that there must be a finite sub-cover of $\mathcal{R}_n$ for X. Let A_n be the centres of the balls in that finite sub-cover of $\mathcal{R}_n$. Consider $A = \cup_{n\geq 1} A_n$, and show that A is countable and dense in X.

Solution: If X is the empty set, then there is nothing to prove. Below we assume that X is not empty.

Fix an arbitrary $n \in \mathbb{N}$. Consider the collection of open sets

$$\mathcal{R}_n = \{B_{1/n}(x) \mid x \in X\}.$$

We claim that there is a finite set $A_n \subset X$ such that

$$X \subseteq \bigcup_{x\in A_n} B_{1/n}(x).$$

We prove this by contradiction, so let us assume that there is no such a finite set. Choose an arbitrary $x_1 \in X$ (we can do this since X is not empty). Since $B_{1/n}(x_1)$ does not cover X, there is $x_2 \in X \setminus B_{1/n}(x_1)$. Again, since $\cup_{i=1}^{2} B_{1/n}(x_i)$ does not cover X, there is $x_3 \in X \setminus \cup_{i=1}^{2} B_{1/n}(x_i)$. By continuing this process, we can build a sequence of points $(x_i)_{i=1}^{\infty}$ such that, $x_{n+1} \notin \cup_{i=1}^{n} B_{1/n}(x_i)$. By the sequential compactness of X, $(x_i)_{i=1}^{\infty}$ has a convergent subsequence (x_{i_k}), for $k = 1, 2, 3, \dots$, which converges to some element, say, $x \in X$. By the definition of convergence, there is $k' \geq 1$ such that for all $k \geq k'$, $x_{i_k} \in B_{1/(2n)}(x)$. This implies that $d(x_{i'_k}, x_{i_{k'+1}}) < 1/n$, which cannot happen since by our choice of the sequence, $x_{i_{k'+1}} \notin \cup_{i=1}^{i_{k'+1}-1} B_{1/n}(x_i)$. This is a contradiction.

Define $A = \cup_{n\geq 1} A_n$. Since each A_n is a finite set, A is countable (it is either finite, or in a bijection with $\mathbb{N}$). Below we show that A is dense in X.

Let $y \in X$ and $\epsilon > 0$ be arbitrary. There is $n \in \mathbb{N}$ such that $1/n < \epsilon$. Since $X \subseteq \cup_{x\in A_n} B_{1/n}(x)$, there is $x \in A_n$ such that $y \in B_{1/n}(x)$. Thus, $x \in A$, and $\mathrm{d}(x, y) \leq 1/n < \epsilon$. Since $y \in X$ and $\epsilon > 0$ were arbitrary, we conclude that A is dense in X.

Exercise 9.10.* Let (X, d) be a sequentially compact metric space, and $\mathcal{R}$ be an open cover for X. Show that there is a countable sub-cover of $\mathcal{R}$ for X.

Hint: You can prove this statement in two steps. Step 1: Show that there is $n \in \mathbb{N}$ such that for every $x \in X$, $B_{1/n}(x)$ is contained in some element of $\mathcal{R}$ (assume that such n does not exist, so for every $n \in \mathbb{N}$ there is x_n such that $B_{1/n}(x_n)$ is not contained in any ball. Extract a subsequence and see what happens at the limit of that subsequence, ...). Step 2: By the previous exercise, there is a countable dense set $\{y_1, y_2, y_3, \dots\}$ in X. Let n be the number from Step 1. For each $i \in \mathbb{N}$, $B_{1/n}(y_i)$ is contained in some element $V_i \in \mathcal{R}$. Show that the collection $\{V_i \mid i \in \mathbb{N}\}$ is a countable sub-cover of $\mathcal{R}$ for X.

Solution: First we show that there is $n \in \mathbb{N}$ such that for every $x \in X$, there is $V \in \mathcal{R}$ such that $B_{1/n}(x) \subseteq V$. Assume in the contrary that such n does not exist. Then, for every $n \in \mathbb{N}$, there is $x_n \in X$ such that $B_{1/n}(x_n)$ is not contained in any set in $\mathcal{R}$. Because X is sequentially compact, there is a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ such that $(x_{n_k})_{k\geq 1}$ converges to some $x \in X$. Since $\mathcal{R}$ is an open cover for X, there is an open set $V \in \mathcal{R}$ such that $x \in V$. Since V is open, there is $\delta > 0$ such that $B_\delta(x) \subset V$. Let us choose n_k large enough so that $\mathrm{d}(x, x_{n_k}) < \delta/2$ and $1/n_k < \delta/2$. This implies that

$$B_{1/n_k}(x_{n_k}) \subseteq B_\delta(x) \subseteq V.$$

Thus, $B_{1/n_k}(x_{n_k})$ is contained in some element of $\mathcal{R}$, which is a contradiction.

Let $n \in \mathbb{N}$ be the number satisfying the property in the above paragraph. We showed in the previous exercise that there is a countable dense set of points in X, say $A = \{y_1, y_2, \dots\}$. By the previous paragraph, for every $i \geq 1$, there is $V_i \in \mathcal{R}$ such that $B_{1/n}(y_i) \subseteq V_i$. Obviously, $\mathcal{R}' = \{V_i \mid i \in \mathbb{N}\}$ is a sub-cover of $\mathcal{R}$, and is countable. Moreover, since A is dense in X, for any $x \in X$ we can find $y_k \in A$ such that $x \in B_{1/n}(y_k) \subseteq V_k$. This shows that $\mathcal{R}'$ is a cover for X.

Exercise 9.11. Let (X, d) be a compact metric space, and assume that $f : X \to X$ is a continuous map such that for all $x \in X$, we have $f(x) \neq x$. Show that there is $\delta > 0$ such that for all $x \in X$, we have $\mathrm{d}(x, f(x)) \geq \delta$.

Hint: Work with the map $x \mapsto \mathrm{d}(x, f(x))$ on the set X, and think about if this map is continuous, and what values it may take.

Solution: Define the map $h : X \to \mathbb{R}$ as

$$h(x) = \mathrm{d}(x, f(x)).$$

First we show that h is a continuous map on X. By exercise 5.8-(ii), for arbitrary x and y in X, we have

$$|h(x) - h(y)| = |\,\mathrm{d}(x, f(x)) - \mathrm{d}(y, f(y))| \leq \mathrm{d}(x, y) + \mathrm{d}(f(x), f(y)).$$

Fix an arbitrary $x \in X$. To see that h is continuous at x, fix an arbitrary $\epsilon > 0$. Because f is continuous at x, there is $\delta' > 0$ such that for every $y \in X$ satisfying $\mathrm{d}(x, y) < \delta'$ we have $\mathrm{d}(f(x), f(y)) < \epsilon/2$. Let $\delta = \min\{\delta', \epsilon/2\}$. Then, for every $y \in X$ satisfying $\mathrm{d}(x, y) < \delta$, by the above inequality we have

$$|h(x) - h(y)| \leq \mathrm{d}(x, y) + \mathrm{d}(f(x), f(y)) < \delta + \epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

By a theorem in the lectures, every continuous function on a compact set has a minimum, and its minimum is realised at some point in the domain of the function. Thus, there is $x_0 \in X$ such that h realises its minimum at x_0. That is, for all $x \in X$ we have $h(x) \geq h(x_0)$. However, by the assumption in the exercise $h(x_0) > 0$. We can define $\delta = h(x_0)$.

Unseen Exercise. Prove that if $X \subset \mathbb{R}$ is not compact, then there is a continuous map $f : X \to \mathbb{R}$ which is not bounded.

Hint: Consider two cases where X is not bounded, and X is not closed.

Solution: If $X \subseteq \mathbb{R}$ is not compact, then by the Heine-Borel theorem, either X is not bounded, or X is not closed. We show that in both of those cases such a continuous function f exists.

First assume that X is not bounded. Since the empty set is bounded, X is not empty. Then, we may choose a point $x \in X$. Define the map $f : X \to \mathbb{R}$ as $f(y) = \mathrm{d}(y, x)$. Then, f is continuous on X, since for every y and z in X, we have

$$|f(y) - f(z)| = |\,\mathrm{d}(y, x) - \mathrm{d}(z, x)| \leq \mathrm{d}(y, z).$$

Since X is not bounded, for every $n \in N$, X is not contained in $B_n(x)$. This implies that f is not bounded.

Now assume that X is not closed. Therefore, there is a sequence of points $(x_n)_{n\geq 1}$ in X which converges to some $x \in \mathbb{R}$, but $x \notin X$. Define $f : X \to \mathbb{R}$ as

$$f(y) = \frac{1}{\mathrm{d}(y, x)}.$$

The map f is continuous, since it is the composition of the continuous maps $y \mapsto \mathrm{d}(y, x)$ and the map $t \mapsto 1/t$. But, f is not bounded from above, since $f(x_n) = 1/\,\mathrm{d}(x_n, x) \to \infty$ as $n \to \infty$.