Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Mathematics Pre-arrival course

Problem Sheet 4 – Complex Numbers

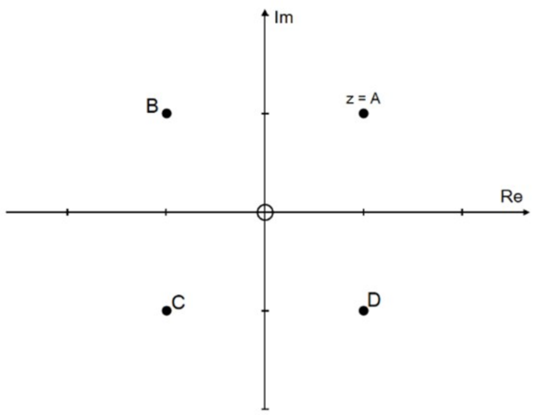
The starred questions on this problem sheet are for you to think about — we will not be giving solutions to them in the pre-arrival course. Instead these will form the basis of discussion in your first *MATH40001/MATH40009 - Introduction to University Mathematics* session once you arrive at Imperial.

1. Let $z = 1 + 2i$ and $w = 3 - 4i$. Find:
   (a) $z + w$
   (b) zw
   (c) $z^2 + w^*$
   (d) $\frac{w}{z}$

2. Let $z = -3 + 4i$
   (a) Sketch z in the complex plane.
   (b) Calculate the modulus and argument of z.
   (c) Sketch z^* in the complex plane.
   (d) Find $z \cdot z^*$.

   Note: some books/courses use $\bar{z}$ for the complex conjugate of z.

3. Given the complex number z represented by the point A on the Argand diagram below which point represents:
   (a) z^*
   (b) iz



4. Express the following in modulus-argument form:

   (a) $z_1 = \sqrt{3} + i$

   (b) $z_2 = -1 - \sqrt{3}i$

5. Given that $\arg(z) = \frac{\pi}{4}$, find:

   (a) $\arg(iz)$

   (b) $\arg(-z)$

6. Find all the complex roots of $x^3 - 1 = 0$. Do the same for $x^4 - 1 = 0$, try and sketch these roots on the complex plane. Can you guess where the roots of $x^n - 1 = 0$ will be located in the complex plane?

7. By considering the real and imaginary parts of $(e^{i\theta})^3$, derive the triple angle formulae for $\sin$ and $\cos$:

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$
$$\cos(3\theta) = 3\sin\theta - 4\sin^3\theta$$

8. On the complex plane, find:

   (a) all the points z such that $|z + 1 - 3i| = 1$.

   (b) all the points z such that $|z - 2| = |z + i|$.

   (c) all the points z such that $\arg\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$

9. ★ Prove the *conjugate root theorem*: for any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ with real coefficients $a_0 \ldots a_n$, if z is a root of P, then so is $\bar{z}$.

10. ★ Show that if $|z| = 1$, then

$$\operatorname{Im} \frac{z}{(z+1)^2} = 0.$$

   Is there a nice geometric interpretation of this equation? Find all the points on the complex plane such that $\operatorname{Im} \frac{z}{(z+1)^2} = 0$ — there are more of them than just the ones on the unit circle.