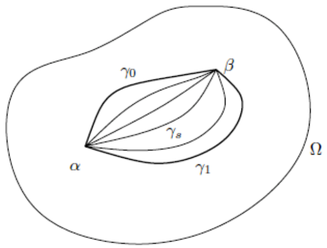
MATH50001/50017/50018 - Analysis II

Complex Analysis

Lecture 7

To remind:

In the previous lecture we introduced homotopic curves:



Theorem. If γ_0 and γ_1 are homotopic in Ω and if f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Besides, we had

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_\gamma f(w)\,dw,$$

where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z+h) - F(z) = \int_\eta f(w)\,dw,$$

where η is the line segment joining z and $z+h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

The proof is complete.

Corollary. (Cauchy-Goursat theorem)
If f is holomorphic in the simply connected open set Ω, then

$$\oint_{\gamma} f(z)\, dz = 0,$$

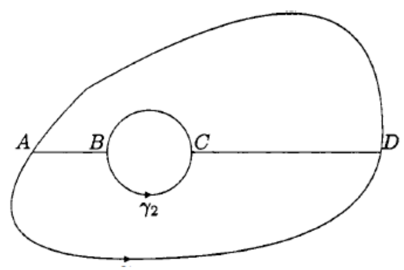
for any closed, piecewise-smooth, curve $\gamma \subset \Omega$.

1858 - 1936, French

Theorem. (Deformation Theorem)
Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2. Then

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz.$$

Proof.



Example. Let $\gamma = \{z \in \mathbb{C} : |z-1| = 2\}$. Then

$$\oint_\gamma \frac{1}{z^2-4}\,dz = \oint_\gamma \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\oint_\gamma \left(\frac{1}{z-2} - \frac{1}{z+2}\right) dz.$$

Since $1/(z+2)$ is holomorphic inside and on γ, then

$$\oint_\gamma \frac{1}{z+2}\,dz = 0.$$

On the other hand

$$\oint_\gamma \frac{1}{z-2}\,dz = \oint_{\{z:\,|z-2|=1\}} \frac{1}{z-2}\,dz = 2\pi i.$$

Therefore

$$\oint_\gamma \frac{1}{z^2-4}\,dz = i\,\frac{\pi}{2}.$$

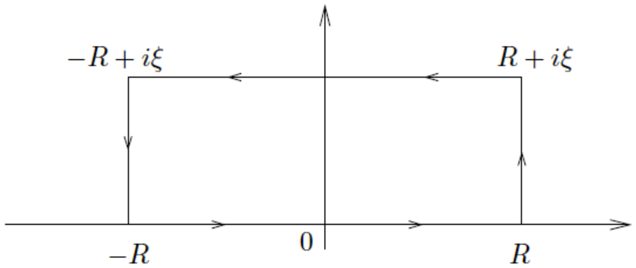
Example. We show that if $\xi \in \mathbb{R}$ then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx.$$

This gives a proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $\xi = 0$, the formula is precisely the known integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx.$$

Now suppose that $\xi > 0$, and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the contour γ_R



The contour γ_R consists of a rectangle with vertices R, $R + i\xi$, $-R + i\xi$, $-R$ and the positive counterclockwise orientation.
By the Cauchy-Goursat theorem theorem

$$\oint_{\gamma_R} f(z)dz = 0 \qquad (*)$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2}\, dx$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$\begin{aligned} |I(R)| = \left| \int_0^{\xi} f(R+iy)\, i\, dy \right| &= \left| \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, dy \right| \\ &\leq e^{-\pi R^2} \int_0^{\xi} |e^{-\pi(2iRy-y^2)}|\, dy \leq e^{-\pi R^2}\, \xi\, e^{\pi \xi^2} \to 0, \end{aligned}$$

as $R \to \infty$.

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons.

Finally, the integral over the horizontal segment on top is

$$\int_{R}^{-R} e^{-\pi(x+i\xi)^2}\,dx = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\,dx$$
$$= -\,e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}\,dx.$$

Therefore, in the limit as $R \to \infty$ we obtain that $(*)$ gives

$$0 = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i x\xi}\,dx.$$

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Proof. If z_0 is interior to γ then for any $r > 0$ such that $\gamma_r = \{z : |z - z_0| = r\}$ lying wholly inside γ, using the deformation theorem we obtain

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0}\, dz.$$

Then

$$
\begin{aligned}
\frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{z-z_0}\, dz
&= \frac{1}{2\pi i} f(z_0) \oint_{\gamma_r} \frac{1}{z-z_0}\, dz + \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz \\
&= f(z_0) + \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz.
\end{aligned}
$$

Since f is holomorphic it is continuous at z_0. Therefore for a given $\varepsilon > 0$ there is $\delta > r > 0$ such that as soon $|z - z_0| < \delta$ we have

$$
|f(z) - f(z_0)| < \varepsilon.
$$

Then, by using the ML-inequality we have

$$
\left| \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| \leq \frac{1}{2\pi} \frac{\varepsilon}{r} 2\pi r = \varepsilon.
$$

So we have proved that for any $\varepsilon > 0$

$$\left| \frac{1}{2\pi\, i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) \right| < \varepsilon$$

and hence

$$\frac{1}{2\pi\, i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0).$$

The proof is complete.

Quizzes

Question: What is the value of the integral $\int_\gamma \frac{e^{\pi z}}{z-i} dz$, where γ is the circle of radius $1/2$ centered at i, traversed in the direction such that its interior remains on the left.

Answers:

A. $2\pi i$

B. $-2\pi i$

C. 0

D. $4\pi i$

Thank you