Imperial College
London

M4PA46/M5PA46

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Chaos & Fractals

Date: **Wednesday 25th May 2016**

Time: **09.30 – 12.00**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the Uniformization Theorem for arbitrary Riemann surfaces.

(b) Suppose that S is a Riemann surface whose universal covering space is $\hat{\mathbb{C}}$ and that S' is a Riemann surface whose universal covering space is either $\mathbb{C}$ or $\mathbb{D}$. Using the Maximum Modulus Principle, show that the only holomorphic maps from S to S' are the constant functions.

(c) Suppose that S and S' are hyperbolic Riemann surfaces. Give the definition of a normal family of functions $f : S \to S'$ and state Montel's Theorem in this context.

(d) Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a univalent map with $f(0) = 0$ and $|f'(0)| = 1$. Show that the image of f cannot contain the boundary of the unit disk. Hint: Consider f^{-1}.

2. (a) Define the Fatou set, Julia set and filled Julia set of a polynomial $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(b) Let $f(z) = z^2 + c, c > 1/4$ is real.

(i) Show that $J(f)$ is disjoint from the real line.

(ii) Prove that f does not have a attracting cycle in $\mathbb{C}$.

(iii) Show that each orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \ldots$, with $z_0 \in J(f)$, is uniquely determined by the sequence of signs $\epsilon_n = \mathrm{sign}(\mathrm{Im}(z_n))$. Hint: Use the Poincaré metric on $\mathbb{C} \setminus [c, \infty)$. Recall that if $\iota : S \to S'$ is an inclusion mapping between distinct hyperbolic Riemann surfaces, then ι contracts hyperbolic distances.

(iv) Prove that every orbit outside of $J(f)$ must escape to ∞. Hint: Use the fact that if $f : z \mapsto z^2 + c$ and $f^n(0) \to \infty$, as $n \to \infty$, then $J(f)$ is totally disconnected.

3. (a) Define the grand orbit of a point.

(b) Show that a rational map with degree ≥ 2 is conjugate (by a Möbius mapping) to a polynomial if and only if it has a point $\hat{p}$ whose grand orbit is $\{\hat{p}\}$.

(c) Suppose that f is a rational map with degree ≥ 2. Assume that f^k is conjugate to a polynomial. Show that f^2 is topologically conjugate to a polynomial.

(d) Must it be the case that f, from part (c), is conjugate to a polynomial?

4. Let $f_c : z \to z^2 + c$.

(a) Prove that ∞ is a super-attracting fixed point of f.

(b) Define the Green's function for f_c and show that it extends to the basin of ∞.

(c) State the Riemann Mapping Theorem, and prove the uniqueness part of the statement.

(d) Suppose that $J(f_c)$ is connected.

(i) Use the Riemann Mapping Theorem to show that there exist $\lambda > 0$ and a unique conformal isomorphism $H : \hat{\mathbb{C}} \setminus K(f) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ so that $H(z)/z \to \lambda$ as $z \to \infty$.

(ii) What is λ? Hint: You may use the fact that if $B : \hat{\mathbb{C}} \setminus K(f) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is the mapping given by Böttcher's Theorem, then $B(z) = z \prod_{n=0}^{\infty}(1 + O(f^n(z))/2^{n+1})$.

5. Suppose that f is a polynomial with degree ≥ 2 and a connected Julia set.

(a) State Böttcher's Theorem.

(b) Define the external rays and equipotentials for f. Explain why the foliation of $\mathbb{C} \setminus K(f)$ by external rays and the foliation of $\mathbb{C} \setminus K(f)$ by equipotentials are both invariant under f.

(c) Assume that $f(z) = z^2 + c$. Suppose that R is an external ray with the property that $f(R) \subset R$, show that if R accumulates on a repelling fixed point p, then it accumulates on no other point. Hint: first show that if $x \in R$ and $x_{-n} \in f^{-n}(x) \cap R$, then the length of the segment of the ray bounded by x_{-n} and $f(x_{-n})$ goes to 0 as $n \to \infty$ by using the Poincaré metric defined in $\mathbb{C} \setminus L$, where L is a simple curve that connects c with ∞. Now, make use of the linearization theorem in a neighbourhood of a repelling fixed point.