Math40003 **Mid-module test 2022-23**
Linear Algebra and Groups

Throughout you may use standard (i.e. those seen in lectures/problem sheets) properties of matrix addition, multiplication and scalar multiplication.

1. Which of the following sets are subspaces of the $\mathbb{R}$-vector space $M_{n\times n}(\mathbb{R})$ with standard addition and scalar multiplication.
   Justify your answers by giving proofs or counter examples.

   (a) $D = \{A = (a_{ij}) \in M_{n\times n}(\mathbb{R}) : a_{ij} = 0 \text{ if } i \neq j\}$. (2 marks)
   (i.e. diagonal matrices). **These do form a subspace:**
   Let $A = (a_{ij}), = (b_{ij}) \in D$, $\lambda \in \mathbb{R}$ **so if** $i \neq j$ **then** $a_{ij} = b_{ij} = 0$. **Closed under addition:** $A + B = (a_{ij} + b_{ij})$ **and for** $i \neq j$, $a_{ij} + b_{ij} = 0 + 0 = 0$, **so** $A + B \in D_1$. **(1 marks)**
   Closed under scalar multiplication: $\lambda A = (\lambda a_{ij})$ **and for** $i \neq j$, $\lambda a_{ij} = \lambda 0 = 0$ **so** $\lambda A \in D$. **(1 marks)**

   (b) Let $B, C \in M_{n\times n}(\mathbb{R})$ be two specific matrices with $\lambda_1 B \neq \lambda_2 C$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$. Let G be the set generated by b and C, i.e.:

   $$G = \{A \in M_{n\times n}(\mathbb{R}) : A = \lambda_1 B + \lambda_2 C \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

   (2 marks)
   Let $A_1, A_2 \in G$, **i.e.** $A_1 = \lambda_1^1 B + \lambda_2^1 C$ $\lambda \in \mathbb{R}$ **so if** $i \neq j$ **then** $a_{ij} = b_{ij} = 0$. **Closed under addition:** $A + B = (a_{ij} + b_{ij})$ **and for** $i \neq j$, $a_{ij} + b_{ij} = 0 + 0 = 0$, **so** $A + B \in D_1$. **(1 marks)**
   Closed under scalar multiplication: $\lambda A = (\lambda a_{ij})$ **and for** $i \neq j$, $\lambda a_{ij} = \lambda 0 = 0$ **so** $\lambda A \in D_1$. **(1 marks)**

   (c) $M_{n\times n}(\mathbb{N})$ (i.e. the set of $n \times n$ matrices with entries in $\mathbb{N}$). (2 marks)
   Not closed under scalar mult. Let $Id_n \in M_{n\times n}(\mathbb{N})$ **and** $\pi \in \mathbb{R}$, **but** $\pi Id_n = (\pi Id_n) \notin M_{n\times n}(\mathbb{N})$ **(2 marks)**

   (d) $I = \{A \in M_{n\times n}(\mathbb{R}) : A^{-1} \text{ exists}\}$. (2 marks)
   (i.e. the set of invertible matrices).
   Not closed under scalar mult or addition. For example for $A \in I$, $0A = 0$ **which is not invertible thus not in** I. **(2 marks)**

   (e) $Z = \{A \in M_{n\times n}(\mathbb{R}) : AB = BA \text{ for all } B \in M_{n\times n}(\mathbb{R})\}$ (2 marks)
   (i.e. the set of matrices that commute with **all** matrices).
   Let $A_1, A_2 \in Z$, $\lambda \in \mathbb{R}$ **then**
   Closed under addition: Let $B \in M_{n\times n}(\mathbb{R})$, $(A_1 + A_2)B = A_1 B + A_2 B = BA_1 + BA_2 = B(A_1 + A_2)$, **so** $A_1 + A_2 \in Z$
   (As $A_1, A_2 \in Z$**)** **(1 marks)**

Closed under scalar mult: Let $B \in M_{n\times n}(\mathbb{R})$, $(\lambda A_1)B = \lambda(A_1B) = \lambda(BA_1) = B(\lambda A_1)$.
(As $A_1 \in Z$, Ex 2.2.6) (1 marks)

2. For any of the above that are subspaces find a basis for them and give a brief justification for why this is a basis. (10 marks)

   Let $E_{kl} = (e_{ij})$ with $e_{ij} = 1$ when $i = k$ and $j = l$ and 0 otherwise.(i.e. the matrix with entry 0 everywhere except the kl^{th} entry which is 1).

   (i) A basis for D is $E_{11}, ..., E_{nn}$. Clearly $\Sigma\lambda_i E_{ii} = 0$ iff $\lambda = 0$ so these are linearly independent. Also they if $A = (a_{ij}) \in D$ then $A = \Sigma_{i=1}^{n} a_{ii}E_{ii}$ so this set is also spanning. (3 marks)

   (ii) $\{B, C\}$ forms a basis. This is spanning by definition and linearly independent because if $\lambda_1 B + \lambda_2 C = 0$ then $\lambda_1 B = -\lambda_2 C$ which contradicts the condition on B and C given. (2 marks)

   (v) First we need to show that if $B = (b_{ij}) \in M_{n\times n}$ is the matrix where $b_{ij} = 1$ if $i = j$ and 0 otherwise then $Z = \{\lambda B \in M_{n\times n}(\mathbb{R}) : \lambda \in (R)\}$ (note $B = I_n$).

   Let $A = (a_{ij}) \in M_{n\times n}$ and $C = c_{ij} \in M_{n\times n}(\mathbb{R})$ and let $AC = D = (d_{ij})$ and $CA = F = (f_{ij})$.

   We want to show that $A \in Z$ iff $a_{ii} = \lambda$ for some $\lambda \in \mathbb{R}$ and all $i \in 1, ..., n$ and $a_{ij} = 0$ for $i \neq j$.

   Now $d_{ij} = \Sigma_{k=1}^{n} a_{ik}c_{kj}$ and $f_{ij} = \Sigma_{k=1}^{n} c_{ik}a_{kj}$
   ($\Rightarrow$) Suppose for any values of c_{ij}, $d_{ij} = f_{ij}$ then for $s \neq t$ choose $C = E_{tt}$ then we get

$$\begin{aligned} d_{st} &= \Sigma_{k=1}^{n} a_{sk}c_{kt} \\ &= a_{st} \\ f_{st} &= \Sigma_{k=1}^{n} c_{sk}a_{kt} \\ &= 0 \end{aligned}$$

   So $a_{kt} = 0$
   Now consider $C = E_{1i}$, the matrix with zero entries everywhere except $c_{1i} = 1$.
   Then

$$\begin{aligned} d_{1i} &= \Sigma_{k=1}^{n} a_{1k}c_{ki} \\ &= a_{11} \\ f_{1i} &= \Sigma_{k=1}^{n} c_{1k}a_{ki} \\ &= a_{ii} \end{aligned}$$

   So for each $i \in \{1, ..., n\}$ we have $a_{ii} = a_{11} = \lambda$.
   Thus $A = \lambda B$

($\Leftarrow$) Conversely, suppose $A = \lambda B$ then $AC = (\lambda B)C = \lambda(BC)$ and $CA = C(\lambda B) = \lambda(CB)$. So it is sufficient to prove $BC = CB$ for all matrices C, but $B = I_n$ so $BC = CB = C$. (4 marks)

Now Id_n is clearly a basis for $\{\lambda B \in M_{n\times n}(\mathbb{R}) : \lambda \in (R)\}$ as it is clearly both spanning and linearly independent. (1 mark)