Imperial College London
Department of Mathematics

Solutions to Blackboard quiz 4

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

You should enter your answers on Blackboard by 1pm UK time on Wednesday 15 March 2023. The test is worth 1.5 percent of the marks for the module.

(A) (The following text refers to Questions 1 - 7) In each of the following, decide whether the given statement is true or false. Throughout, $(G, .)$ is a group with identity e.

Qu 1: If G is a group and H, K are two subgroups such that

$$HK := \{hk \mid h \in H,\, k \in K\} = \{kh \mid h \in H,\, k \in K\} =: KH,$$

then HK is a subgroup of G.

Solution: True. Clearly, HK is closed under taking inverses, because if $hk \in HK$, then $k^{-1}h^{-1} \in KH = HK$. Moreover, if $h'k' \in HK$, then there are $h'' \in H$ and $k'' \in K$ such that

$$hkh'k' = hh''k''k' \in HK.$$

Qu 2: Let H, K be finite subgroups of G of order a and b respectively and such that $H \cap K = \{e\}$. Then $|HK| = ab$.

Solution: True. The set HK consists of products of the form hk with $h \in H$ and $k \in K$; thus, clearly $|HK| \leq ab$. If $h_1, h_2 \in H$ and $k_1, k_2 \in K$ are such that $h_1k_1 = h_2k_2$, then $h_2^{-1}h_1 = k_2k_1^{-1}$. Since, in the last equality, the left-hand side is in H and the right-hand side is in K, we deduce that $h_2^{-1}h_1 = k_2k_1^{-1} = e$; hence, by the uniqueness of inverses, $h_1 = h_2$ and $k_1 = k_2$. This implies that $|HK| \geq ab$, and hence equality holds.

Qu 3: Let p be a prime. Let H, K be finite subgroups of G of order p. Then either $H \cap K = \{e\}$ or $H = K$.

Solution: True. This is because $H \cap K$ is a subgroup of H, hence it may only have order dividing the order of H.

Qu 4: Let $\phi : G \to G$ be a homomorphism. If $g \in G$ has infinite order, then so does $\phi(g)$.

Solution: False. Clearly this is false if ϕ is the homomorphism that sends everything to e: $\phi(g) = e$ for all $g \in G$. (Check this is a homomorphism).

Qu 5: Let p be a prime. If $|G| = p$ then, G is cyclic.

Solution: True. If G has order p and $g \in G$, then $o(g) \mid p$. Hence it is either 1 or p. We deduce that every $g \neq e$ has order p in G.

Qu 6: Let G be finite. Then there is $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of S_n.

Solution: True. Let n be the order of G. For all $g \in G$ we define

$$\lambda_g : G \to G \text{ by } h \mapsto gh.$$

We claim that λ_g is a bijection for all $g \in G$. Indeed, for all $k \in G$, $k = g(g^{-1}k) = \lambda_g(g^{-1}k)$. Thus λ_g is surjective. It is also injective, because if $\lambda_g(h) = \lambda_g(k)$ for some $h, k \in G$, we deduce that

$$gh = gk; \text{ hence } h = k \text{ by multiplying the last identity by } g^{-1} \text{ on the left.}$$

The rule $g \mapsto \lambda_g$ defines a map

$$\Lambda : G \to \mathrm{Sym}(G).$$

We show that Λ is an injective homomorphism. This will conclude the proof as then the image of Λ will be a subgroup of $\mathrm{Sym}(G)$ isomorphic to G and $\mathrm{Sym}(G)$ is isomorphic to S_n.

We prove that Λ is a homomorphism. Let $g, g' \in G$. Then for all $h \in G$,

$$\lambda_{gg'}(h) = (gg')h = g(g'h) = g\lambda_{g'}(h) = \lambda_g(\lambda_{g'}(h)) = \lambda_g \circ \lambda_{g'}(h).$$

This shows that Λ is a homomorphism because the product in $\mathrm{Sym}(G)$ is the composition of functions. It remains to show that the map is injective. To this end, let $g, g' \in G$ be such that $gh = g'h$ for all $h \in G$. Then, multiplying on the right by h^{-1} we get $g = g'$; thus Λ is injective.

Qu 7: Let $n \in \mathbb{N}$ with $n > 1$. Suppose that all the abelian group of order n are cyclic. Then n is prime.

Solution: False. All the abelian groups of order 6 are cyclic. We show that all abelian groups of order 6 contain an element of order 6.

Let $g \in G \setminus \{e\}$. If $o(g) = 6$ we have found an element of order 6 and there is nothing more to do. If $o(g) \neq 6$, then either $o(g) = 2$ or $o(g) = 3$. Let $H = \langle g \rangle$. Assume $o(g) = 2$ and let $k \in G \setminus H$. If $o(k) = 6$ then we are done, otherwise $o(k) = 3$ and gk has order 6. Indeed, let $K = \langle k \rangle$. If $o(k) = 2$, then $H \cap K = \{e\}$ and HK is a subgroup of order 4 inside a subgroup of order 6, which is impossible. A similar argument applies to the case $o(g) = 3$.

(B) (The following text refers to Questions 8 - 10) For the following choices of groups G, H say whether

i) there is an injective homomorphism $G \to H$,

ii) there is a surjective homomorphism $G \to H$,

iii) there is an isomorphism $G \to H$,

iv) none of the above.

You will need to get all the answers right in order to get the mark for the question.

Qu 8: $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R}) : a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}$ with the matrix multiplication and $H = \mathbb{C}^\times$ (with the multiplication of complex numbers).

Solution: The map $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + ib$ is an injective homomorphism, but there cannot be a surjective one because the images of the elements of G have norm 1.

Qu 9: $G = (\mathbb{Z}, +)$ and H the Klein four-group.

Solution: None of the above. $\mathbb{Z}$ is cyclic so in order to define a homomorphism it suffices to specify the image of 1. (Indeed if $1 \mapsto g$, then we define a map $\varphi : \mathbb{Z} \to H$ by $\varphi(m) = g^m$. It is easy to check that this is a homomorphism and that all homomorphisms $\mathbb{Z} \to H$ arise this way.) None of the choices for g gives rise to a surjective homomorphism because the image of $\mathbb{Z}$ is a cyclic group and H is not cyclic. Clearly there cannot be an injective homomorphism $\mathbb{Z} \to H$ as H is finite and $\mathbb{Z}$ is infinite.