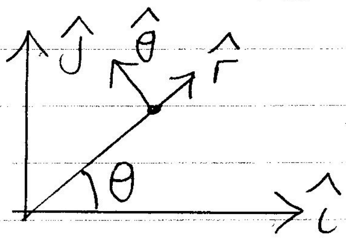
MVC Quiz 2 (2023) Answers

1. $\hat{\imath} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$\hat{\jmath} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$



$x = r\cos\theta,\ y = r\sin\theta$

$$\underline{A} = (\alpha r\cos\theta)(\hat{r}\cos\theta - \hat{\theta}\sin\theta) + (\beta r\sin\theta)(\hat{r}\sin\theta + \hat{\theta}\cos\theta) + \gamma z\hat{k}$$

$\therefore\ A_r = r(\alpha\cos^2\theta + \beta\sin^2\theta)$

$A_\theta = r(\beta - \alpha)\sin\theta\cos\theta$

$\therefore$ $\boxed{B}$ is the only correct answer.

2. In cylindrical polars

$$\operatorname{div}\underline{F} = \frac{\partial F_1}{\partial r} + \frac{F_1}{r} + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z}$$

$$= 2rz^3\sin\theta + rz^3\sin\theta + rz\cos\theta + 2r^3z\cos\theta$$

$$= 3rz^3\sin\theta + (1+2r^2)rz\cos\theta \qquad \boxed{B}$$

3. $$\operatorname{Curl}\underline{F} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ r^2z^3\sin\theta & r^3z\sin\theta & r^3z^2\cos\theta \end{vmatrix}$$

The $\hat{r}$-component is

$$\frac{1}{r}\left\{\frac{\partial}{\partial\theta}(r^3z^2\cos\theta) - \frac{\partial}{\partial z}(r^3z\sin\theta)\right\}$$

$$= -r^2z^2\sin\theta - r^2\sin\theta$$

$$= -r^2(1+z^2)\sin\theta \qquad \boxed{A}$$

4. $A_1 = \partial\Phi/\partial r = (2r + 4r^{-5})\cos^2\theta$

$A_2 = (1/r)\partial\Phi/\partial\theta = (r^{-4} - r^2)\,2\cos\theta\sin\theta/r$

$= (r^{-5} - r)\sin 2\theta \qquad \boxed{C}$

5. In spherical polars

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$\equiv (2-20r^{-6})\cos^2\theta + (4+8r^{-6})\cos^2\theta$$
$$+ \cot\theta\,(r^{-6}-1)\,2\sin\theta\cos\theta$$
$$+ (r^{-6}-1)\underbrace{2\cos 2\theta}_{(4\cos^2\theta-2)}$$

$$= \cos^2\theta\{\cancel{2}-20r^{-6}+\cancel{4}+8r^{-6}+2r^{-6}\cancel{-2}+4r^{-6}\cancel{-4}\}$$
$$+2(1-r^{-6})$$

$$= 2(1-r^{-6}) - 6r^{-6}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)$$
$$= 2-5r^{-6}-3r^{-6}\cos 2\theta \qquad \boxed{A}$$

6. $$\int_V z^4\,dV = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{b}(r\cos\theta)^4\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= 2\pi\left[\frac{r^7}{7}\right]_a^b\int_0^{\pi}\cos^4\theta\sin\theta\,d\theta$$

$$= \frac{2\pi}{7}(b^7-a^7)\left[\frac{\cos^5\theta}{5}\right]_{\pi}^{0} = \frac{4\pi}{35}(b^7-a^7) \qquad \boxed{D}$$

7. $$\int_0^{\pi/2}\int_4^5 r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$$

$$= \left[\frac{r^3}{3}\right]_4^5\left[\sin\theta-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{2}{3}(5^3-4^3) = 122/3 \qquad \boxed{C}$$

8. $|J| = \left| 1 \Big/ \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \right| = \left| 1 \Big/ \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right|$

$= 1/2.$

$\therefore\ dx\,dy = \frac{1}{2}\,du\,dv$

$\alpha x + \beta y = \alpha(u+v)/2 + \beta(u-v)/2 = \frac{1}{2}(\alpha+\beta)u + \frac{1}{2}(\alpha-\beta)v$

R is the region with $0 \leq u \leq 2$, $0 \leq v \leq 2$

$$I = \int_0^2 \int_0^2 \left\{ \frac{1}{2}(\alpha+\beta)u + \frac{1}{2}(\alpha-\beta)v \right\} \frac{1}{2}\,du\,dv$$

$$= \frac{1}{4}\left\{ (\alpha+\beta)(2)\left[\frac{u^2}{2}\right]_0^2 + (\alpha-\beta)(2)\left[\frac{v^2}{2}\right]_0^2 \right\}$$

$$= (\alpha+\beta) + (\alpha-\beta) = 2\alpha$$

$\boxed{B}$

9. S_1 is parameterized by

$x = 2\cos\theta,\ y = 2\sin\theta \quad (0 \leq \theta \leq 2\pi)$

with $0 \leq z \leq 3 + 2\cos\theta$

& $dS = 2\,dz\,d\theta$ (Since radius $= 2$)

$$\therefore \int_{S_1} (x+y+z)\,dS = \int_0^{2\pi} \int_{z=0}^{3+2\cos\theta} ((2\cos\theta + 2\sin\theta) + z)\,2\,dz\,d\theta$$

$$= \int_0^{2\pi} 2\left[2z\cos\theta + 2z\sin\theta + \frac{z^2}{2} \right]_{z=0}^{z=3+2\cos\theta} d\theta$$

$$= 2\int_0^{2\pi} 2(3+2\cos\theta)\cos\theta + 2(3+2\cos\theta)\sin\theta + \frac{1}{2}(3+2\cos\theta)^2 \, d\theta$$

$$= 8\int_0^{2\pi} \cos^2\theta\,d\theta + \int_0^{2\pi} 9 + 4\cos^2\theta\,d\theta$$ (other contributions integrate to zero)

$= 8\pi + 18\pi + 4\pi$

$= 30\pi$

$\boxed{A}$

10. S_2 is parameterized by $x = r\cos\theta,\ y = r\sin\theta,\ z = 3 + r\cos\theta$
with $0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2$

$$\underline{J} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta \end{vmatrix} = -r\hat{i} + r\hat{k}$$

$$\therefore |\underline{J}| = r\sqrt{2} \Rightarrow dS = r\sqrt{2}\, dr\, d\theta$$

$$\therefore \int_{S_2} (x+y+z)\, dS = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta + 3 + r\cos\theta) r\sqrt{2}\, dr\, d\theta$$

$$= 2\pi \int_0^2 3r\sqrt{2}\, dr$$

$$= 6\pi\sqrt{2}\left[\frac{r^2}{2}\right]_0^2 = 12\pi\sqrt{2} \qquad \boxed{B}$$