Imperial College London

Department of Mathematics
Imperial College London
Academic Year 2023-2024

Partial Differential Equations in Action **MATH50008**

Coursework 1

Instructions: The deadline to submit this coursework is on **Monday 12 February at 1pm (UK time)**. The **neatness, completeness and clarity of the answers** will contribute to the final mark. You can turn in handwritten or typed solutions (for instance, using LaTeX). You should upload your answers to this coursework as a single PDF via the Turnitin Assignment called **Coursework 1** which you will find in the *Assessments* folder of our Blackboard site. On the front page of your submission, you must **not** indicate your first name or last name (as papers will be marked anonymously), but make sure that you indicate your CID. Your submission filename **must** have the following format: `MATH50008_CW1_[CID].pdf`, where [CID] is your college ID.

1. **Total: 15 Marks** The mass density $u(x,t)$ of a chemical substance in steady water in a long and thin channel is governed by the following 1D diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0,$$

where D is the diffusion coefficient. The boundary and initial conditions associated to our problem are given by

$$\begin{aligned} u(x,0) &= 0, \quad x > 0 \\ u(\infty,t) &= 0, \quad t > 0 \\ \frac{\partial u}{\partial x}(0,t) &= -q, \quad t > 0 \end{aligned}$$

where $q > 0$.

   (a) Give the physical meaning behind the initial and boundary conditions. 2 Marks

   (b) Show by using dimensional reduction and a similarity variable η (that you will define) that this partial differential equation can be reduced to the following ordinary differential equation

$$2F''(\eta) + \eta F'(\eta) - F(\eta) = 0$$

   5 Marks

   (c) Write the original boundary and initial conditions in terms of the similarity variable. 3 Marks

   (d) Show that the solution to this PDE problem is given by

$$u(x,t) = q\sqrt{\frac{4Dt}{\pi}}\left[e^{-x^2/(4Dt)} - \frac{x}{\sqrt{Dt}}\int_{x/\sqrt{4Dt}}^{\infty} e^{-\xi^2}\, d\xi\right]$$

   5 Marks

2. **Total: 10 Marks** In this question, we consider the following boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0, \quad x \in \mathbb{R},\ y \in \mathbb{R} \\ &u(x,0) = f(x), \quad x \in \mathbb{R} \end{aligned}$$

where f is a real function.

(a) Find the equation of the characteristics for this PDE. Draw them in the (x, y)-plane. $\boxed{\text{4 Marks}}$

(b) What condition do we need to impose on the function f to ensure that a solution exists. Justify your reasoning. $\boxed{\text{3 Marks}}$

(c) In which region of the (x, y)-plane is the solution uniquely determined by the boundary condition given here? Justify your answer. $\boxed{\text{3 Marks}}$

3. $\boxed{\textbf{Total: 5 Marks}}$ Solve the following boundary value problem using the method of characteristics

$$\begin{aligned}
&\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0, \quad x > 0,\ t > 0 \\
&u(x, 0) = 0, \quad x > 0 \\
&u(0, t) = \tanh(t), \quad t > 0
\end{aligned}$$

You will draw the characteristics in the (x, t)-plane and provide an explicit expression for $u(x, t)$.