Mathematical Logic (MATH6/70132;P65)
Problem Sheet 8

[0] Using AC, prove that if $f : C \to D$ is a surjective function, then there is an injective function $g : D \to C$ with $f(g(d)) = d$ for all $d \in D$. Does this statement imply AC (given the ZF axioms)?

Work in ZFC unless otherwise stated.

[1] (i) Suppose A is a set of cardinality λ and $\kappa \leq \lambda$ is a cardinal. Show that A has a subset B with $|B| = \kappa$.

(ii) Prove that ω is equinumerous with a proper subset of itself.

(iii) Suppose X is any set. Prove that X is infinite if and only if X is equinumerous with a proper subset of itself.
(Hint: use question 2, sheet 7 for one direction.)

[2] (i) Suppose A, B, C are sets. Give a bijection between $A^{B\times C}$ and $(A^B)^C$.

(ii) Using the Fundamental Theorem of Cardinal Arithmetic, show that if A, B are sets with A infinite and $2 \leq |B| \leq |A|$, then $|B^A| = |\mathcal{P}(A)| = |2^A|$.
[Hint: Use the idea of Question 4(b) on Problem sheet 6.]

[3] Using Zorn's Lemma (or otherwise), prove the following.

(i) Suppose $(A; \leq_1)$ is any partially ordered set. Prove that there is a linearly ordered set $(A; \leq_2)$ with the property that for all $a, a' \in A$ we have $a \leq_1 a'$ implies $a \leq_2 a'$.

(ii) Let R be any (commutative) ring with identity element and $I \subset R$ be a proper ideal of R. Then there is a maximal proper ideal J of R with $I \subseteq J \subset R$.

(iii) Suppose G is a non-trivial group with an element g whose conjugates generate G. Prove that G has a maximal proper normal subgroup. Is this necessarily true without assuming the existence of such an element g?

[4] In this question, assume ZF. We will show that Zorn's Lemma implies the Axiom of Choice: that is, $ZF \vdash (ZL \to AC)$.

Suppose X is a set of non-empty sets. By a *partial choice function* on X, with domain $Y \subseteq X$, we mean a function $f : Y \to \bigcup X$ with $f(y) \in y$ for all $y \in Y$. We let A be the set of all partial choice functions on X and we order these by inclusion $\subseteq$.

(i) Suppose $C \subseteq A$ is a chain in A. Prove that $\bigcup C \in A$.

(ii) Show that if the domain of $f \in A$ is not equal to X, then f is not maximal in A.

(iii) Deduce that if Zorn's Lemma holds, then there is a function $g : X \to \bigcup X$ with $g(x) \in x$ for all $x \in X$.

[5] Suppose κ is a cardinal with $\kappa > |\mathbb{R}|$. Prove that there is a vector space V over $\mathbb{R}$ with $|V| = \kappa$. (You could use the Löwenheim - Skolem Theorem here, but it's probably also instructive to try to do this directly.) Prove that a basis of V has cardinality κ.

Prove that if V_1, V_2 are $\mathbb{R}$-vector spaces with $|V_1| = |V_2| > |\mathbb{R}|$ then there is a bijective linear map $T : V_1 \to V_2$ (i.e. V_1, V_2 are isomorphic).

[6] Let A be a non-empty set. A set F of subsets of A is called a *filter* on A if it satisfies the first three of the following properties. If it satisfies all four, it is called an *ultrafilter* on A.

UF1 $\emptyset \notin F$;

UF2 if $x \in F$ and $x \subseteq y \subseteq A$, then $y \in F$;

UF3 if $x, y \in F$ then $x \cap y \in F$;

UF4 if x is any subset of A then either x or its complement $A \setminus x$ is in F.

(i) (Nothing to do with Zorn's Lemma) Suppose A is a finite set and F an ultrafilter on A. Show that there exists $a \in A$ such that $F = \{x \subseteq A : a \in x\}$.

(ii) Show that if A is an infinite set the the set of subsets whose complements are finite forms a filter on A.

(iii) Show that if F_0 is a filter on A then the set of filters which contain it is a poset (under inclusion) which satisfies the hypotheses of Zorn's Lemma.

(iv) Show that a maximal filter satisfies (UF4).

(v) Let F be a maximal filter containing the filter in (ii). Show that F does not contain any finite set.