Geometry of curves and surfaces

Davoud Cheraghi

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Contents

Introduction

These lecture notes are prepared in Winter 2021 for the module Geometry of Curves and Surfaces at Imperial College London. This is a slightly revised version of the lecture notes by Prof Tom Coates and Dr Stergios Antonakoudis delivered in the previous years.

1 Regular curves in Euclidean spaces

We will mostly work in the Euclidean spaces of dimension one, two, and three; using the standard notation

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid \forall i, x_i \in \mathbb{R}\}, \quad n \geq 1.$$

The main tool here is the inner product, denoted by $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, define as

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i.$$

Using the inner product we may define the length, or the norm, of a vector $x \in \mathbb{R}^n$ as

$$|x| = \sqrt{\langle x, x \rangle} = \sqrt{\textstyle\sum_{i=1}^n x_i^2}.$$

A **parametrised curve** in $\mathbb{R}^n$ is a smooth map $\phi : [a, b] \to \mathbb{R}^n$. The parametrised curve ϕ is called **regular**, if for all $t \in [a, b]$, we have $|\phi'(t)| \neq 0$. Recall that a map from $[a, b]$ to $\mathbb{R}^n$ is called smooth, if for each component of the map, the partial derivatives of all orders exist and are continuous.

In this module we will only consider regular curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. One may think of a regular curve as the trajectory of a particle in $\mathbb{R}^2$ or $\mathbb{R}^3$ which is always in motion.

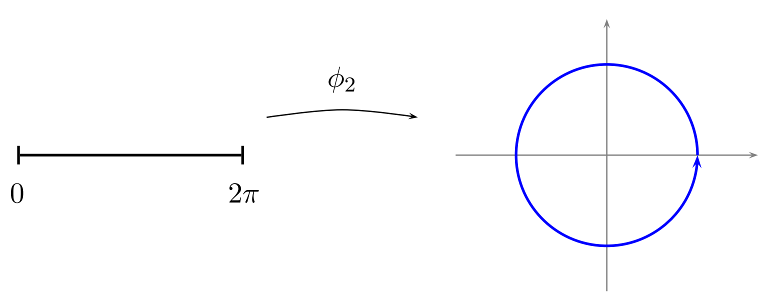
Here are some examples of regular curves:

(i) A straight line segment

$$\phi_1 : [0, 1] \to \mathbb{R}^2, \quad \phi_1(t) = (2t - 1, 3t + 2);$$

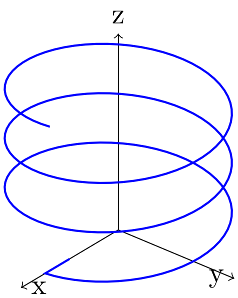
(ii) The circle of radius r centred at the origin

$$\phi_2 : [0, 2\pi] \to \mathbb{R}^2, \quad \phi_2(\theta) = (r \cos\theta, r \sin\theta);$$



(iii) A helix,

$$\phi_3 : [0, 6\pi] \to \mathbb{R}^3, \quad \phi_3(\theta) = (\cos\theta, \sin\theta, \theta).$$



However, the map

$$\phi_4 : [-8, 8] \to \mathbb{R}^2, \quad \phi_4(t) = (t, |t|)$$

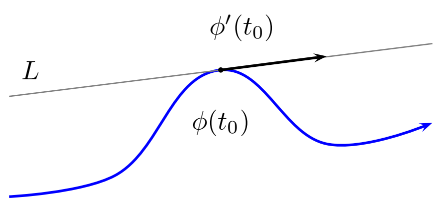
is not a regular curve, since it is not smooth (it is not even C^1). Also, the map

$$\phi_5 : [-1, 1] \to \mathbb{R}^2, \quad \phi_5(t) = (0, t^2)$$

is not regular, since $|\phi_5'(0)| = |(0, 0)| = 0$.

We use regularity to excluded constant maps, such as $\phi(t) = (1, 1, 1)$, and more importantly, to make sure that there is a well-defined tangent line at every point on the curve. More precisely, the tangent line to the regular curve ϕ at $\phi(t_0)$ is given by

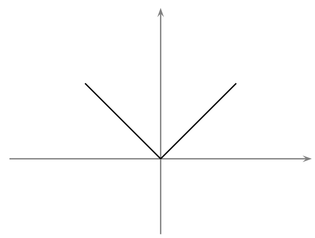
$$L = \{\phi(t_0) + \phi'(t_0)s \mid s \in \mathbb{R}\}.$$



Note that if $\alpha : (-\epsilon, +\epsilon) \to \mathbb{R}^2$ is a smooth map, one does not necessarily have a tangent line to α at $\alpha(0)$. For example, if $\alpha : (-1, +1) \to \mathbb{R}^3$ is defined as

$$\alpha(t) = \begin{cases} (e^{-1/t^2}, e^{-1/t^2}, 0) & t > 0, \\ (0, 0, 0) & t = 0, \\ (-e^{-1/t^2}, e^{-1/t^2}, 0) & t < 0, \end{cases}$$

then there is no tangent line at $\alpha(0)$, but the map is smooth. However, as we discussed before, if $\alpha'(0) \neq 0$ there is a tangent line to α at $\alpha(0)$. In this case, $\alpha'(0)$ is the zero vector. See the figure below.



For the purpose of geometry, we are interested in the image of the regular curve ϕ, that is, $\phi([a,b])$, which we call a **regular curve**. From this point of view, a curve can be written in many different ways. For instance, the curve ϕ_1 presented above, is the same as the curve

$$\phi_6 : [0,2] \to \mathbb{R}^2, \quad \phi_6(t) = (t^3/4 - 1, 3t^3/8 + 2).$$

Check that the two curves have the same image. Similarly, the set

$$\{(x,y) \in \mathbb{R}^2 \mid x \in [-2,2], y = 3\sqrt{1 - x^2/4}\},$$

is a regular curve, since this set can be written as the image of the regular map

$$\phi : [0,\pi] \to \mathbb{R}^2, \phi(t) = (2\cos(t), 3\sin(t)).$$

Check that this is a regular curve. However, the set

$$\{(x,y) \in \mathbb{R}^2 \mid x \in [-1,+1], y = |x|\}$$

is not a regular curve, since it cannot be the image of a regular map.

Given a regular curve $\phi : [a,b] \to \mathbb{R}^n$, and a smooth function $f : [c,d] \to [a,b]$ with $|f'(x)| \neq 0$ for all $x \in [c,d]$ and $f(\{c,d\}) = \{a,b\}$, the curve

$$\phi \circ f : [c,d] \to \mathbb{R}^n, \quad t \mapsto \phi(f(t))$$

is called a **reparametrisation** of ϕ. The curve $\phi \circ f$ is regular. That is because, by the chain rule, the composition of two smooth maps is a smooth map, and for all $t \in [c,d]$,

$$(\phi \circ f)'(t) = \phi'(f(t))f'(t) \neq 0.$$

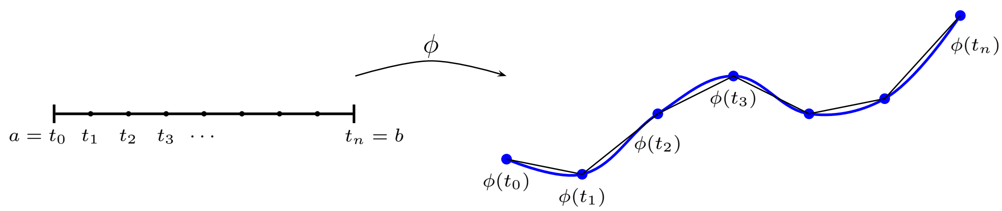
We are interested in the properties of regular curves which remain invariant under reparametrisations (i.e. features which do not depend on the parametrisation). The first example of such a notion is the length of a regular curve.

The **length** of a regular curve $\phi : [a,b] \to \mathbb{R}^n$ is defined as

$$\ell(\phi) = \int_a^b |\phi'(t)| dt.$$

Note that as the partition becomes finer and finer we have

$$\textstyle\sum_{i=0}^{n-1} |\phi'(t_i)||t_{i+1} - t_i| \to \int_a^b |\phi'(t)|\, dt,$$



and for each i

$$|\phi'(t_i)||t_{i+1} - t_i| \simeq |\phi(t_{i+1}) - \phi(t_i)|.$$

For instance, the length of the curve ϕ_1 presented above is

$$\ell(\phi_1) = \int_0^1 |\phi_1'(t)|dt = \int_0^1 |(2,3)|dt = \int_0^1 \sqrt{13}dt = \sqrt{13}.$$

The length of the curve ϕ_3 is

$$\ell(\phi_3) = \int_0^{6\pi} |\phi_3'(\theta)|d\theta = \int_0^{6\pi} |(-\sin\theta, \cos\theta, 1)|d\theta = \int_0^{6\pi} \sqrt{2}\, d\theta = 6\sqrt{2}\pi.$$

Lemma 1.1. *The length of a regular curve is invariant under reparametrisations.*

Proof. Let $\phi : [a, b] \to \mathbb{R}^n$ be a regular curve, and $f : [c, d] \to [a, b]$ be a smooth function with $f'(s) \neq 0$ for all $s \in [c, d]$, and $f(\{c, d\}) = \{a, b\}$. Then $\psi = \phi \circ f : [c, d] \to \mathbb{R}^n$ is another parametrisation of $\phi([a, b])$. We need to show that the length of $\phi([a, b])$ is the same as the length of $\psi([c, d])$.

Since $f'(s) \neq 0$, for all $s \in [c, d]$, and f' is continuous on $[c, d]$, either $f' > 0$ on $[c, d]$ or $f' < 0$ on $[c, d]$. Without loss of generality, let us assume that $f' > 0$ on $[c, d]$. Then, $f(c) = a$ and $f(d) = b$.

Let us write $\phi(t) = (x_1(t), x_2(t), \dots, x_n(t))$. We have

$$\begin{aligned}
|\psi'(s)| &= \big|(\phi \circ f)'(s)\big| \\
&= \big|\big((x_1 \circ f)'(s), (x_2 \circ f)'(s), \dots, (x_n \circ f)'(s)\big)\big| \\
&= \big|\big(x_1'(f(s))f'(s), x_2'(f(s))f'(s), \dots, , x_n'(f(s))f'(s)\big)\big| \\
&= |f'(s)|\,\big|\big(x_1'(f(s)), x_2'(f(s)), \dots, x_n'(f(s))\big)\big| \\
&= f'(s)|\phi'(f(s))|.
\end{aligned}$$

Therefore, by the change of variable formula, we obtain

$$\begin{aligned}
\ell(\psi([c, d])) = \int_c^d |\psi'(s)|ds &= \int_c^d |\phi'(f(s))|f'(s)\, ds \\
&= \int_{f(c)}^{f(d)} |\phi'(t)|dt \\
&= \int_a^b |\phi'(t)|dt \\
&= \ell(\phi([a, b])).
\end{aligned}$$

$\square$

Given a parametrised curve $\phi : [a, b] \to \mathbb{R}^n$, there are infinitely many possible parametrisations of $\phi([a, b])$. We wish to get rid of all those choices, and have a best possible one, by requiring that $|\phi'(t)| = 1$ for all t. With that choice, we will have

$$\ell(\phi([a, b])) = \int_a^b |\phi'(t)|\,dt = \int_a^b 1\,dt = b - a.$$

This is called an **arc-length parametrisation**. This can be always achieved, as we show in the next lemma.

Lemma 1.2. *Any regular curve $\phi : [a, b] \to \mathbb{R}^n$ can be reparametrised by arc-length.*

Proof. Let us fix an arbitrary regular curve $\phi : [a, b] \to \mathbb{R}^n$. We need to find an interval $[c, d]$ and a smooth function $f : [c, d] \to [a, b]$ with $f(\{c, d\}) = \{a, b\}$ such that $\psi = \phi \circ f : [c, d] \to \mathbb{R}^n$ is a regular curve parametrised by arc-length.

It is convenient to find the inverse of f (we still do not know the domain of f). Let $h : [a, b] \to \mathbb{R}$ be the inverse of f. Then, for every $t \in [a, b]$, we must have

$$\int_{h(a)}^{h(t)} |\psi'(u)|\,du = \int_{h(a)}^{h(t)} 1\,du = h(t) - h(a).$$

On the other hand,

$$\int_{h(a)}^{h(t)} |\psi'(u)|\,du = \ell(\psi([h(a), h(t)])) = \ell(\phi([a, t])) = \int_a^t |\phi'(u)|\,du.$$

Therefore, for all $t \in [a, b]$, we have

$$h(t) = \int_a^t |\phi'(u)|\,du + h(a).$$

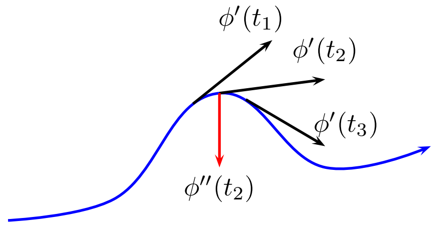
We need to find a function $h : [a, b] \to \mathbb{R}$ which satisfies the above equation. We are free to choose $c = h(a)$ as we wish, so we may let $c = h(a) = 0$. Then, since for every $t \in [a, b]$, $\phi'(t)$ exists and is a continuous function of t, $h(t)$ is a well-defined function. We have $h(b) = \ell(\phi([a, b]))$.

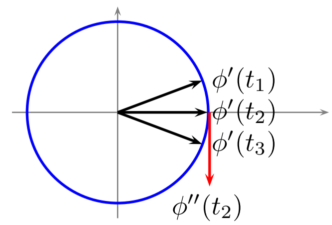
The equation for h implies that $h'(t) = |\phi'(t)|$, and hence h is a smooth function with non-zero derivative at every point. In particular, the inverse of h is defined, is smooth, and has non-zero derivative at every point. $\square$

2 Curvature of a regular curve

Let $\phi : [0, L] \to \mathbb{R}^n$ be a regular curve which is parametrised by arc-length. The **(geodesic) curvature** of ϕ at the point $\phi(t)$ is defined as

$$k(t) = |\phi''(t)|.$$





The **curvature vector** at $\phi(t)$ is defined as

$$\vec{k}(t) = \phi''(t).$$

We choose a parametrisation by arc-length so that the curvature only depends on the curve $\phi([a, b])$. Indeed, if $\psi = \phi \circ f$ is another parametrisation of ϕ by arc-length (for example, changing the direction), then

$$1 = |\psi'(t)| = |\phi'(f(t))||f'(t)| = |f'(t)|.$$

Thus, $f'(t) \equiv \pm 1$ on the domain of f. This implies that $f(t) = \pm t + C$, for some constant $C \in \mathbb{R}$. Then, $\psi'(t) = \pm\phi'(\pm t + C)$, and hence $\psi''(t) = \phi''(\pm t + C)$. This implies that both $k(t)$ and $\vec{k}(t)$ are independent of the parametrisation of ϕ by arc-length.

Proposition 2.1. *Let $\phi : [a, b] \to \mathbb{R}^n$ be a regular curve. Then, the curvature $k(t) = 0$ for all $t \in [a, b]$ if and only if $\phi([a, b])$ is a straight line.*

Proof. We note that $k \equiv 0$ iff $|\phi''| \equiv 0$ iff $\phi'' \equiv 0$ iff $\phi' \equiv a$ for some $a \in \mathbb{R}^n$ iff $\phi(t) = ta + b$ for some $b \in \mathbb{R}^n$. □

Note that when calculating the curvature, it is crucial that the curve is parametrised by arc-length. For instance, we can consider the following parametrisation of a straight line in the plane:

$$\phi(t) = (0, e^t), \quad 0 \le t \le 1.$$

Then, $\phi''(t) = (0, e^t) \neq 0$, while, as we just saw in the previous example, the curvature must be zero.

Proposition 2.2. *For any regular curve $\phi : [a, b] \to \mathbb{R}^n$, and every $t \in [a, b]$, the curvature vector $\vec{k}(t)$ is perpendicular to the tangent line to the curve $\phi([a, b])$ at $\phi(t)$.*

Proof. The tangent line to the curve $\phi([a, b])$ at the point $\phi(t)$ is generated by the vector $\phi'(t)$. Note that for every $t \in [a, b]$,

$$\langle \phi'(t), \phi'(t) \rangle = |\phi'(t)|^2 = 1.$$

Differentiating with respect to t, we obtain

$$\frac{d}{dt}\langle \phi'(t), \phi'(t) \rangle = \langle \phi''(t), \phi'(t) \rangle + \langle \phi'(t), \phi''(t) \rangle = 2\langle \vec{k}(t), \phi'(t) \rangle = 0.$$

This shows that $\vec{k}(t)$ is perpendicular to $\phi'(t)$. □

Let us calculate the curvature of a circle of radius R centred at the origin. We can parametrise this curve by

$$\phi(t) = (R\cos(t), R\sin(t)), \quad 0 \leq t \leq 2\pi.$$

But, this curve is not parametrised by arc-length since

$$|\phi'(t)| = |(-R\sin(t), R\cos(t))| = R.$$

To find the parametrisation by arc-length we note that

$$h(t) = \int_0^t |\phi'(s)|\, ds = \int_0^t R\, ds = Rt.$$

Thus we may re-parametrise the curve by the smooth function $f(t) = h^{-1}(t) = t/R$. That is,

$$\psi(t) = \phi \circ f(t) = \big(R\cos(t/R), R\sin(t/R)\big), \quad t \in [0, 2\pi R].$$

Then, for all $t \in [0, 2\pi R]$,

$$\psi'(t) = \big(-\sin(t/R), \cos(t/R)\big), \quad \vec{k}(t) = \psi''(t) = \big(-R^{-1}\cos(t/R), -R^{-1}\sin(t/R)\big).$$

Therefore, $k(t) = |\psi''(t)| = 1/R$.

3 Curves in $\mathbb{R}^3$, Frenet frames

Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parametrised by arc-length. Then, we define the three vectors:

(i) the **unit tangent vector** $T(t) = \phi'(t)$ to the curve at $\phi(t)$.

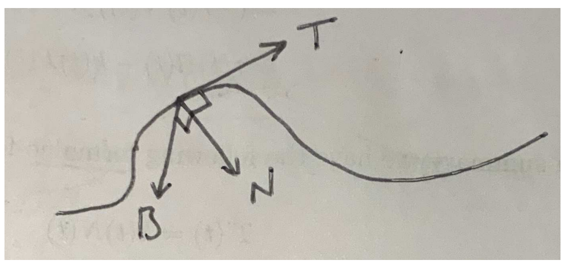
(ii) Assuming that $T'(t) \neq 0$, for all $t \in [a, b]$, we define

$$N(t) = \frac{T'(t)}{|T'(t)|},$$

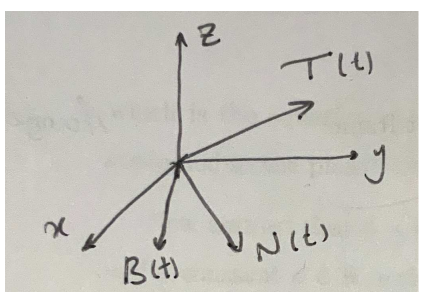
which is called the **principle normal vector** to the curve at $\phi(t)$. Recall that by Proposition 2.2, $T'(t) = \phi''(t) = \vec{k}(t)$ is orthogonal to $T(t)$. Thus, $N(t)$ is orthonormal to $T(t)$.

(iii) the **binormal vector** $B(t) = T(t) \times N(t)$ at $\phi(t)$ [1].

The triple (T, N, B) is a positively oriented orthonormal bases of $\mathbb{R}^3$ at $\phi(t)$, which is called the **Frenet frame**.



We can study how the Frenet frame changes over time:



(i) We have

$$T'(t) = \phi''(t) = \vec{k}(t) = k(t)N(t).$$

[1]We shall use the notation $u \times v$ to denote the cross product of the vectors u and v in $\mathbb{R}^3$. It is obtained from the right hand rule, and its size is equal to the area of the parallelogram spanned by u and v.

(ii) $B'(t)$ is orthogonal to both $B(t)$ and $T(t)$. That is because, using $|B(t)| = 1$, we have

$$0 = \frac{d}{dt}\langle B(t), B(t)\rangle = 2\langle B'(t), B(t)\rangle.$$

Also,

$$\begin{aligned} B'(t) &= (T(t) \times N(t))' \\ &= T'(t) \times N(t) + T(t) \times N'(t) \\ &= k(t)N(t) \times N(t) + T(t) \times N'(t) \\ &= T(t) \times N'(t). \end{aligned}$$

Therefore, there is a smooth function $\tau : [a, b] \to \mathbb{R}$ called the **torsion** of ϕ, such that

$$B'(t) = -\tau(t)N(t).$$

(iii) We can also calculate $N'(t)$ using the formula $N(t) = B(t) \times T(t)$, as follows

$$\begin{aligned} N'(t) &= B'(t) \times T(t) + B(t) \times T'(t) \\ &= (-\tau(t)N(t)) \times T(t) + B(t) \times (k(t)N(t)) \\ &= \tau(t)B(t) - k(t)T(t). \end{aligned}$$

In summary, we have the following formulae for the Frenet frame:

$$\begin{aligned} T'(t) &= k(t)N(t) \\ B'(t) &= -\tau(t)N(t) \\ N'(t) &= \tau(t)B(t) - k(t)T(t). \end{aligned}$$

It is convenient to present the above differential equations in the matrix form

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{1}$$

In other words, the evolution of the Frenet frame over time depends only on the curvature k, the torsion τ, and the frame itself!

Note that we need the condition $\phi'' \neq 0$, in order to define the Frenet frame.

Intuitively, the curvature of a curve measures the failure of a curve to be straight, and the torsion measures the failure of the curve to be planar.

Proposition 3.1. *Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parametrised by arc-length, and for all $t \in [a, b]$ we have $\phi''(t) \neq 0$. Then, ϕ is contained in a plane, if and only if, $\tau(t) \equiv 0$.*

Proof. First assume that $\tau \equiv 0$. Then, $B' = -\tau N \equiv 0$, and hence $B(t) = \vec{C}$ is a constant. We have

$$\frac{d}{dt}\langle \phi(t), \vec{C} \rangle = \langle \phi'(t), \vec{C} \rangle = \langle T(t), B(t) \rangle = 0.$$

Therefore, there is a constant $d \in \mathbb{R}$ such that

$$\langle \phi(t), \vec{C} \rangle = d.$$

This implies that $\phi(t)$ belongs to the set

$$\{(x, y, z) \in R^3 \mid \langle (x, y, z), \vec{C} \rangle = d\},$$

which is the equation of a plane perpendicular to the vector $\vec{C} = B(t)$. Thus, ϕ is contained in the plane spanned by T and N.

Now assume that ϕ is contained in a plane, that is , there are a unit vector $v \in \mathbb{R}^3$ and a constant $d \in \mathbb{R}$ such that $\langle \phi(t), \vec{v} \rangle = d$. Differentiating this equation, we obtain $\langle \phi'(t), \vec{v} \rangle = 0$, and hence

$$\langle T(t), \vec{v} \rangle = 0.$$

Differentiating the above equation, we obtain $\langle T'(t), \vec{v} \rangle = 0$ and hence $\langle k(t)N(t), \vec{v} \rangle = 0$. Since $k(t) = |\phi''(t)| \neq 0$, we must have

$$\langle N(t), \vec{v} \rangle = 0.$$

Therefore, $\vec{v}$ is orthogonal to both $T(t)$ and $N(t)$. Since $\vec{v}$ has unit size, we must have $B(t) \equiv \pm\vec{v}$. Thus, $0 = B'(t) = -\tau(t)N(t)$, and hence $\tau \equiv 0$. □

For an example, the curve

$$\phi(t) = (\cos(t), \sin(t), 0), \quad 0 \leq t \leq 2\pi,$$

is regular and is parametrised by arc-length. We have

$$T(t) = \phi'(t) = (-\sin(t), \cos(t), 0),$$

and

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{(-\cos(t), -\sin(t), 0)}{|(-\cos(t), -\sin(t), 0)|} = (-\cos(t), -\sin(t), 0),$$

and

$$B(t) = T(t) \times N(t) = (0, 0, 1).$$

Therefore, the curvature is given by

$$k(t) = |T'(t)| = 1.$$

We have $0 = B'(t) = -\tau(t)N(t)$, and since $N(t) \neq 0$, we must have $\tau(t) \equiv 0$. Check the Frenet frame equation (1) holds.

Let us also look at the helix

$$\phi(t) = \big(\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2}), \quad t \in \mathbb{R}.$$

First we check that this is parametrised by arc-length

$$|\phi'(t)| = \left|\left(\frac{-1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)\right| = 1.$$

Then,

$$T(t) = \left(\frac{-1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)$$

and

$$T'(t) = \left(\frac{-1}{2}\cos(\frac{t}{\sqrt{2}}), \frac{-1}{2}\sin(\frac{t}{\sqrt{2}}), 0\right)$$

These imply that $k(t) = |T'(t)| = 1/2$ and

$$N(t) = \frac{T'(t)}{|T'(t)|} = \left(-\cos(\frac{t}{\sqrt{2}}), -\sin(\frac{t}{\sqrt{2}}), 0\right).$$

On the other hand

$$B(t) = T(t) \times N(t) = \left(\frac{1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \frac{-1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)$$

and hence the relation $B'(t) = -\tau(t)N(t)$ gives us

$$\left(\frac{1}{2}\cos(\frac{t}{\sqrt{2}}), \frac{1}{2}\sin(\frac{t}{\sqrt{2}}), 0\right) = -\tau(t)\left(-\cos(\frac{t}{\sqrt{2}}), -\sin(\frac{t}{\sqrt{2}}), 0\right).$$

Therefore, $\tau(t) \equiv 1/2$.

In summary we have

$$\begin{aligned} T(t) &= \left(\frac{-1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right) \\ N(t) &= \frac{T'(t)}{|T'(t)|} = \left(-\cos(\frac{t}{\sqrt{2}}), -\sin(\frac{t}{\sqrt{2}}), 0\right) \\ B(t) &= \left(\frac{1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \frac{-1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right) \end{aligned}$$

with

$$k(t) = 1/2, \quad \tau(t) = 1/2.$$

We can check the Frenet equation (1) holds.

Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parametrised by arc-length, and $\phi''(t) \neq 0$ for all t. Then, the Frenet frame (T, N, B) with $T = \phi'$, $N = \phi''/|\phi''|$, $B = T \times N$ satisfies the Frenet equations

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We are now ready to prove the fundamental theorem of the local theory of curves.

Theorem 3.2. *Assume that $k, \tau : [a, b] \to \mathbb{R}$ are smooth functions with $k > 0$. There exists a regular curve $\phi : [a, b] \to \mathbb{R}^3$, which is parametrised by arc-length, and has curvature k and torsion τ.*

Moreover, ϕ is unique up to rigid motions of $\mathbb{R}^3$, that is, if $\psi : [a, b] \to \mathbb{R}^3$ is another such curve, there are $g \in SO(3)$ and $\vec{c} \in \mathbb{R}^3$ such that $\psi = g \circ \phi + \vec{c}$.

Proof of the uniqueness. Let us first show that the rigid motions preserve arc-length, curvature, and the torsion. Assume that $\phi : [a, b] \to \mathbb{R}^3$ is a regular curve. Fix an arbitrary $g \in SO(3)$ and $\vec{c} \in \mathbb{R}^3$, and define $\mu : [a, b] \to \mathbb{R}^3$ as $\mu = g \circ \phi + \vec{c}$. Then, by the chain rule,

$$|\mu'| = |g(\phi')| = |\phi'|, \quad k_\mu = |\mu''| = |(g \circ \phi)''| = |g(\phi'')| = |\phi''| = k_\phi.$$

Thus, μ is parametrised by arc-length and has the same curvature. Now assume that $\phi''(t) \neq 0$ for all $t \in [a, b]$, so that the torsion τ_ϕ is defined. We have

$$T_\mu = g(T_\phi), \quad N_\mu = g(N_\phi)$$

which imply that

$$B_\mu = T_\mu \times N_\mu = g(T_\phi) \times g(N_\phi) = g(T_\phi \times N_\phi) = g(B_\phi).$$

The torsion τ_μ satisfies the relation $B'_\mu = -\tau_\mu N_\mu$. Thus, we have $(g(B_\phi))' = -\tau_\mu(g(N_\phi))$, and hence

$$g(B'_\phi) = g(-\tau_\mu N_\phi).$$

Since g is invertible, we get $B'_\phi = -\tau_\mu N_\phi$. However, this implies that $\tau_\mu = \tau_\phi$, since τ_ϕ is the unique function satisfying $B'_\phi = -\tau_\phi N_\phi$.

Now assume that there are two regular curves $\phi, \psi : [a, b] \to \mathbb{R}^3$ which are parametrised by arc length and have curvature k and torsion τ. There is a rigid motion of $\mathbb{R}^3$, which maps the curves $T_\psi(a)$, $N_\psi(a)$ and $B_\psi(a)$ to $T_\phi(a)$, $N_\phi(a)$ and $B_\phi(a)$, respectively. Without loss of generality let us assume that these vectors are pairwise identical, and try to show that the two curves only differ by a translation of $\mathbb{R}^3$.

By calculation, we note that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(&\langle T_\phi - T_\psi, T_\phi - T_\psi\rangle + \langle N_\phi - N_\psi, N_\phi - N_\psi\rangle + \langle B_\phi - B_\psi, B_\phi - B_\psi\rangle \Big) \\
&= \left\langle T_\phi - T_\psi, T'_\phi - T'_\psi \right\rangle + \left\langle N_\phi - N_\psi, N'_\phi - N'_\psi \right\rangle + \left\langle B_\phi - B_\psi, B'_\phi - B'_\psi \right\rangle \\
&= k \langle T_\phi - T_\psi, N_\phi - N_\psi\rangle + (\tau \langle N_\phi - N_\psi, B_\phi - B_\psi\rangle - k \langle N_\phi - N_\psi, T_\phi - T_\psi\rangle) \\
&\qquad - \tau \langle B_\phi - B_\psi, N_\phi - N_\psi\rangle \\
&= 0
\end{aligned}$$

Thus, the expression in the parenthesis must be constant. At $t = a$, the expression is equal to 0, so the constant must be 0. We conclude that $T_\phi - T_\psi \equiv 0$, $N_\phi - N_\psi \equiv 0$ and $B_\phi - B_\psi \equiv 0$. Then,

$$\phi(t) = \phi(a) + \int_a^t T_\phi(s)\,ds = \phi(a) + \int_a^t T_\psi(s)\,ds = \phi(a) - \psi(a) + \psi(t).$$

Thus, $\phi \equiv \psi + v$, for the constant vector $v = \phi(a) - \psi(a)$. □

Proof of the existence. Given $k > 0$ and τ, we can pick any positively oriented orthonormal basis (T_a, N_a, B_a) for $\mathbb{R}^3$, and use the existence theorem for the linear differential equation to find $T, N, B : [a, b] \to \mathbb{R}^3$ satisfying

$$\begin{pmatrix} T(a) \\ N(a) \\ B(a) \end{pmatrix} = \begin{pmatrix} T_a \\ N_a \\ B_a \end{pmatrix}, \quad \begin{pmatrix} T'(t) \\ N'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}.$$

We need to check that $(T(t), N(t), B(t))$ is an orthonormal basis for $\mathbb{R}^3$ for all $t \in [a, b]$. We need to show that these vectors are pairwise orthogonal, and have unit size. It is convenient to simultaneously deal with all these conditions in a matrix form. That is, define

$$M = \begin{pmatrix} | & | & | \\ T & N & B \\ | & | & | \end{pmatrix}$$

and require that $M^T M = I$, where ^{T} denotes the transpose operation and I denotes the 3×3 identity matrix. The notation in the above matrix means that the first column of M is the vector T, the second column is the vector N, and the third column is the vector B. We note that

$$M' = \begin{pmatrix} | & | & | \\ T' & N' & B' \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ T & N & B \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} = M \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Then,

$$\frac{d}{dt}\left(M^T M\right) = (M')^T M + M^T M' = \begin{pmatrix} 0 & +k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} M^T M + M^T M \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Note that at $t = a$ we have $M^T M = I$.

The linear system

$$\frac{d}{dt} A = \begin{pmatrix} 0 & +k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} A + A \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

with the initial condition $A(a) = I$ has a solution $A(t) = I$. Thus, by the uniqueness of the solution of such equations, we must have $A(t) = I$. Applying this argument to the matrix $A = M^T M$, and using the initial condition $M(a)^T M(a) = I$, we conclude that $M(t)^T M(t) = I$, for all $t \in [a, b]$. □

As an application of the above theorem we show the following result.

Corollary 3.3. *If a regular curve in $\mathbb{R}^3$ has 0 torsion and constant curvature $c > 0$, then it must be contained in a circle of radius $1/c$.*

Proof. We have already shown that the curve

$$\phi(t) = \big(R\cos(t/R), R\sin(t/R), 0\big), \quad 0 \le t \le 2\pi\mathbb{R}$$

has curvature $k(t) \equiv 1/R$, and torsion $\tau(t) \equiv 0$ since ϕ lies in the xy-plane. We can choose $R = 1/c$, and apply the fundamental theorem of the local theory of curves to conclude that the given curve is equal to ϕ on some interval $[a, b]$, modulo rigid motions of $\mathbb{R}^3$. □

4 Curves in $\mathbb{R}^2$, winding number, and the total curvature

In this section we focus on regular curves in $\mathbb{R}^2$. Let

$$\phi(t) = (x(t), y(t)) \quad t \in [a, b]$$

be a regular curve (not necessarily parametrised by arc-length). Then, $\phi'(t) = (x'(t), y'(t))$ is tangent to ϕ at $\phi(t)$, so

$$n(t) = \frac{(-y'(t), x'(t))}{|\phi'(t)|} \tag{2}$$

is a unit vector which is orthogonal to the tangent vector $\phi'(t)$. Note that $(\phi'(t), n(t))$ forms a positively oriented orthonormal basis for $\mathbb{R}^2$, that is, $n(t)$ is obtained from $\phi'(t)$ by a 90 degree counter-clockwise rotation. If ϕ is parametrised by arc-length, then we defined the **signed curvature**

$$\kappa(t) = \langle n(t), \phi''(t)\rangle.$$

Remark 4.1. This is consistent with our earlier notations $k(t) = \langle N(t), \phi''(t)\rangle$, except that here $n(t) = \pm N(t)$ and hence $\kappa(t)$ may be negative or positive. We need to do this in order to derive a remarkable relation for closed curves in Theorem 4.4. It is worth noting that in this definition, when the direction of the curve changes, the sign of κ changes, while its absolute value remains the same. See the remark after Theorem 4.4. It is the absolute value of κ which has a geometric meaning. See Problem 4 in Problem Sheet 1.

For a curve parametrised by arc-length, we have

$$\kappa(t) = \langle \phi''(t), n(t)\rangle = \langle \phi''(t), (-y'(t), x'(t))\rangle = x'(t)y''(t) - y'(t)x''(t). \tag{3}$$

Proposition 4.2. *For any regular curve $\phi : [a, b] \to \mathbb{R}^2$, we have*

$$\kappa(t) = \frac{\langle \phi''(t), n(t)\rangle}{|\phi'(t)|^2}.$$

Proof. Let $\psi(s) = \phi \circ f(s)$ be a reparametrisation of ϕ in the same direction by the arc-length, that is $f' > 0$ and $|\psi'| \equiv 1$. Write

$$\psi(s) = (x(f(s), y(f(s))).$$

Recall that $f = h^{-1}$ where

$$h(t) = \int_a^t |\phi'(s)|\, ds.$$

Then,

$$f'(s) = \frac{1}{h'(f(s))} = \frac{1}{|\phi'(f(s))|}.$$

Note that since $\phi(f(s)) = \psi(s)$ and both ϕ and ψ are parametrised in the same direction, we have $n_\phi(f(s)) = n_\psi(s)$. By Equation (2), and since ψ is parametrised by arc-length, we have

$$\begin{aligned} n_\phi(f(s)) = n_\psi(s) &= \frac{(-(y(f(s)))', (x(f(s)))')}{|\psi'(s)|} \\ &= (-(y(f(s)))', (x(f(s)))') = f'(s)(-y'(f(s)), x'(f(s))). \end{aligned}$$

On the other hand, since ψ is a reparametrisation of ϕ with $\psi(s) = \phi(f(s))$, we have $k_\phi(f(s)) = k_\psi(s)$. Thus, $|\kappa_\psi(s)| = k_\psi(s) = k_\phi(f(s)) = |\kappa_\phi(f(s))|$. Since ϕ and ψ have the same direction, we conclude that $\kappa_\psi(s) = \kappa_\phi(f(s))$. Therefore, by Equation (3), we have

$$\begin{aligned} \kappa_\phi(f(s)) = \kappa_\psi(s) &= (x(f(s)))'(y(f(s)))'' - (y(f(s)))'(x(f(s)))'' \\ &= \left(x'(f(s))y''(f(s)) - y'(f(s))x''(f(s))\right)\left(f'(s)\right)^3 \\ &= \langle \phi''(f(s)), (-y'(f(s)), x'(f(s))\rangle \left(f'(s)\right)^3 \\ &= \frac{\langle \phi''(f(s)), n_\phi(f(s))\rangle}{\left(\phi'(f(s))\right)^2}. \end{aligned}$$

Letting $t = f(s)$, the above equation gives the desired formula in the proposition. ☐

Example 4.3. Let $\phi : [a, b] \to \mathbb{R}^2$ be the graph of a smooth function $f : [a, b] \to \mathbb{R}$, that is

$$\phi(t) = (t, f(t)), \quad t \in [a, b].$$

Then,

$$\phi'(t) = (1, f'(t)), \quad |\phi'(t)|^2 = 1 + (f'(t))^2,$$

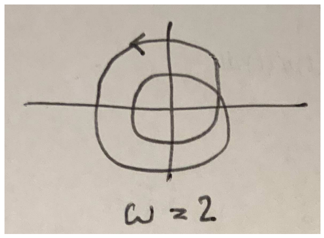
and

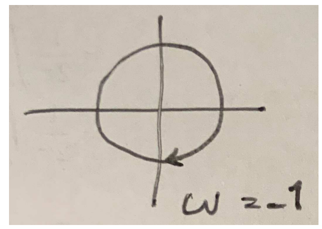
$$n(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}, \quad \phi''(t) = (0, f''(t)).$$

Therefore, by the formula in Proposition 4.2, we obtain

$$\kappa(t) = \frac{f''(t)}{(1 + (f'(t))^2)^{3/2}}.$$

We now suppose that $\phi : [a, b] \to \mathbb{R}^2$ is a smooth closed curve, that is, $\phi(a) = \phi(b)$ and $\phi^{(k)}(a) = \phi^{(k)}(b)$, for all $k \geq 1$. The **winding number** (or the rotation number) of ϕ about a point $p \in \mathbb{R}^2$ is the number of times the curve ϕ rotates around p in a counter-clockwise fashion. By applying a translation of the plane, we can always move the point p (and the curve ϕ) so that we consider the winding number of ϕ about the origin. We use the notation $w(\phi)$ to denote the winding number of ϕ about the origin.





It is convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, via $(x, y) \leftrightarrow z = x + iy$. By writing $\phi(t) = x(t) + iy(t)$, we know from complex analysis that

$$\begin{aligned}
w(\phi) = \frac{1}{2\pi i}\int_{\phi([a,b])}\frac{dz}{z} &= \frac{1}{2\pi i}\int_a^b \frac{x'(t)+iy'(t)}{x(t)+iy(t))}dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{(x'(t)+iy'(t))(x(t)-iy(t))}{x^2(t)+y^2(t)}dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{x'(t)x(t)+y(t)y'(t)+i(x(t)y'(t)-y(t)x'(t))}{x^2(t)+y^2(t)}dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{x'(t)x(t)+y(t)y'(t)}{x^2(t)+y^2(t)}dt + \frac{1}{2\pi}\int_a^b \frac{x(t)y'(t)-y(t)x'(t)}{x^2(t)+y^2(t)}dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\ln\sqrt{x^2(t)+y^2(t)}dt + \frac{1}{2\pi}\int_a^b \frac{x(t)y'(t)-y(t)x'(t)}{x^2(t)+y^2(t)}dt
\end{aligned}$$

Since $\phi(a) = \phi(b)$, the first integral on the right hand side of the above equation is

$$\frac{1}{2\pi i}\left[\ln|\phi(b)| - \ln|\phi(a)|\right] = 0.$$

Therefore,

$$w(\phi) = \frac{1}{2\pi}\int_a^b \frac{x(t)y'(t)-y(t)x'(t)}{x^2(t)+y^2(t)}dt = \frac{1}{2\pi}\int_a^b \frac{\langle\phi(t),(y'(t),-x'(t))\rangle}{|\phi(t)|^2}\,dt. \tag{4}$$

Note that if $\phi : [a, b] \to \mathbb{R}^2$ is a smooth closed curve, the map $T(t) = \phi'(t) : [a, b] \to \mathbb{R}^2$ is also a smooth closed curve.

Theorem 4.4. *If $\phi : [a, b] \to \mathbb{R}^2$ is a closed regular curve parametrised by arc-length, the winding number of the curve $T(t) = \phi'(t)$ satisfies the following relation*

$$w(T) = \frac{1}{2\pi}\int_a^b \kappa(t)\,dt.$$

Remark 4.5. In the above theorem, if the direction of the curve ϕ is changed, the left hand side of the formula changes sign (why), so must the right hand side of the formula. This is a simple explanation that it is required to have a signed curvature. In reality, the closed curve may consist of many parts with positive and negative signed curvatures, which cancel out to give an integer multiple of 2π.

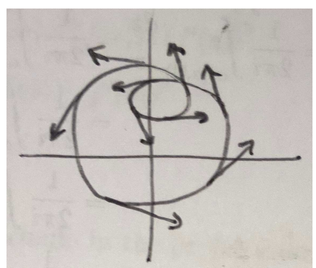
Proof. Applying the formula in Equation (4), and using (3), we obtain

$$\begin{aligned} w(T) = w(\phi') &= \frac{1}{2\pi}\int_a^b \frac{\langle \phi'(t), (y''(t), -x''(t))\rangle}{|\phi'(t)|^2}\, dt \\ &= \frac{1}{2\pi}\int_a^b x'(t)y''(t) - x''(t)y'(t)\, dt \\ &= \frac{1}{2\pi}\int_a^b \kappa(t)\, dt. \end{aligned}$$

□

The winding number of $w(T)$ is called the **index** or the **turning** number of ϕ, and is denoted $\mathrm{Ind}(\phi)$. Thus,

$$\mathrm{Ind}(\phi) = \frac{1}{2\pi}\int_a^b \kappa(t)\, dt.$$



The index of a curve measures the number of times the unit tangent vector rotates around the origin as we travel along the curve ϕ. This number does not change if we perturb ϕ slightly, but the curvature may change a lot by such small perturbations. The theorem states that the integral of the curvature remains invariant under perturbations of the curve.

5 Surfaces in $\mathbb{R}^3$

A **regular surface** is a set $S \subset \mathbb{R}^3$ such that for all points $p \in S$ there are

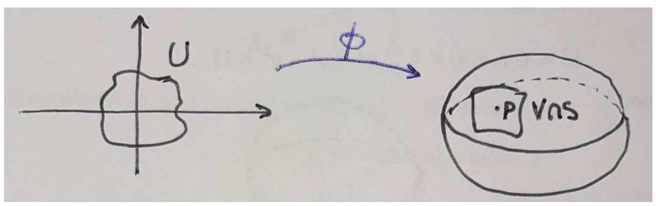
(i) an open neighbourhood $V \subseteq \mathbb{R}^3$ of p,

(ii) an open set $U \subseteq \mathbb{R}^2$,

(iii) a smooth map $\phi : U \to \mathbb{R}^3$ such that

  1) $\phi(U) = V \cap S$,
  2) $\phi : U \to V \cap S$ is a homeomorphism,
  3) for all $q \in U$, the derivative of ϕ at q, $d\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The pair (ϕ, U) is called a **chart** or a **local parametrisation** of S at p.



Conditions (1) and (2) ensure that S locally "looks like" $\mathbb{R}^2$ near any point. Condition (3) ensures "regularity" of S near p.

Let us write

$$\phi(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U.$$

Then,

$$d\phi_q = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix}$$

is injective, if and only if the columns

$$\frac{\partial \phi}{\partial u}(q) \quad \text{and} \quad \frac{\partial \phi}{\partial v}(q)$$

are linearly independent vectors in $\mathbb{R}^3$.

For $q = (u_0, v_0) \in U$, the curves

$$u \mapsto \phi(u, v_0), \quad v \mapsto \phi(u_0, v)$$

lie on S, and have tangent vectors

$$\frac{\partial \phi}{\partial u}(u_0, v_0) \quad \text{and} \quad \frac{\partial \phi}{\partial v}(u_0, v_0).$$

Thus, $d\phi_q$ is injective if and only if the above tangent vectors span a plane in $\mathbb{R}^3$.

Example 5.1. 1) The xy-plane in $\mathbb{R}^3$ is a regular surface. We only need a single chart to cover this surface:

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3, \quad \phi(u,v) = (u,v,0).$$

2) For any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, the graph

$$\Gamma_f = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{R}, y \in \mathbb{R}, z = f(x,y)\}$$

is a regular surface. We use a chart $\phi : \mathbb{R}^2 \to \mathbb{R}^3$, $\phi(u,v) = (u,v,f(u,v))$. To check the regularity, we see that

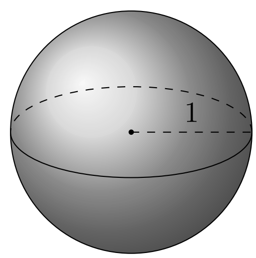
$$\frac{\partial \phi}{\partial u} = (1, 0, \partial_u f(u,v)) \quad \text{and} \quad \frac{\partial \phi}{\partial v} = (0, 1, \partial_v f(u,v))$$

are always linearly independent vectors.

3) The unit sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.



A single chart is not enough to cover this surface (why), but we can cover S with two charts:

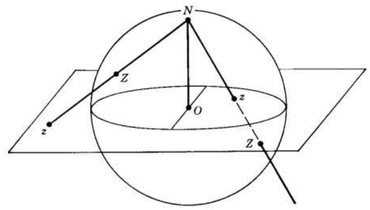
$$\phi_N : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\}, \qquad \phi_S : \mathbb{R}^2 \to S^2 \setminus \{(0,0,+1)\}$$

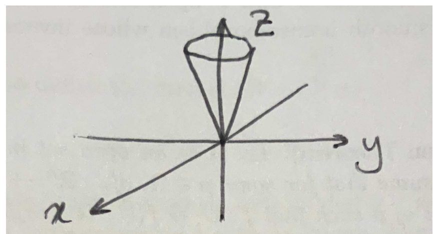
which are defined using the stereographic projections for each. For example, for $(u,v) \in \mathbb{R}^2$, we define $\phi_S(u,v)$ as the unique point of $S^2 \setminus \{(0,0,+1)\}$ on the line from $(u,v,-1)$ to $(0,0,1)$. One can see that

$$\phi_S(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4} \right).$$

The following items present cases where the definition is not satisfied.

1) P is the xy-plane in $\mathbb{R}^3$, with the chart $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $\phi(u,v) = (u^3, v^3, 0)$. Here P is a regular surface, but ϕ is not a chart since $d\phi_{(0,0)} \equiv 0$.





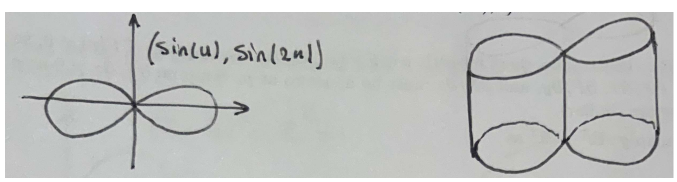
2) The set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, z \geq 0\},$$

which is a cone in $\mathbb{R}^3$. Here there are no smooth charts at the cone $(0, 0, 0)$.

3) Consider the map $\phi : (0, 2\pi) \times (-1, +1) \to \mathbb{R}^3$, defined as

$$\phi(u, v) = (\sin(u), \sin(2u), v)\,.$$



Here, ϕ is smooth and injective, and $d\phi_q$ is injective at every q in $(0, 2\pi) \times (-1, +1)$, but ϕ is not a homeomorphism onto its image.

One naturally wishes to find a convenient way to produce a large class of examples of regular surfaces. We do this by the following proposition.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set, $F : \Omega \to \mathbb{R}$ be a smooth function, and $c \in \mathbb{R}$. Assume that for every p in the set*

$$S = \{(x, y, z) \in \Omega \mid F(x, y, z) = c\} = F^{-1}(c)$$

the gradient $\nabla F(p) \neq 0$. Then, S is a regular surface.

A set S is called a **regular level set**, if there is an open set $\Omega \subset \mathbb{R}^3$, a smooth function $F : \Omega \to \mathbb{R}$ and a value $c \in \mathbb{R}$ such that $S = F^{-1}(c)$, and for all $p \in S$, $\nabla F(p) \neq 0$.

Using the above proposition, one can easily see that the sphere $x^2 + y^2 + z^2 = 1$ is a regular surface. One can take $F(x, y, z) = x^2 + y^2 + z^2$ with the value $c = 1$, and note that $\nabla F(x, y, z) = (2x, 2y, 2z)$ is non-zero at every point where $x^2 + y^2 + z^2 = 1$. Similarly, one can see that the xy-plane is a regular surface. How? Of course, many non-trivial examples can be produced, as we shall look at once we have a proof.

In order to prove the above proposition we need to use the Inverse Function Theorem.

For open sets U and V in $\mathbb{R}^n$, for some $n \geq 1$, a map $f : U \to V$ is called a *diffeomorphism*, if $f : U \to V$ is a homeomorphism, and both maps $f : U \to V$ and $f^{-1} : V \to U$ are C^1. More generally, we say that $f : U \to V$ is a C^k*-diffeomorphism*, for $k \in \mathbb{N} \cup \{\infty\}$, if $f : U \to V$ is a homeomorphism, and both maps $f : U \to V$ and $f^{-1} : V \to U$ are C^k. A C^∞-diffeomorphism is also called a smooth diffeomorphism. A key result concerning C^k maps is the following.

Theorem 5.3 (Inverse Function Theorem)**.** *Let Ω be an open set in $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ be a C^k map, for some $k \in \mathbb{N} \cup \{\infty\}$. Assume that for some $p \in \Omega$, $df_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, there is a neighbourhood $U \subset \Omega$ of p such that $f : U \to f(U)$ is a C^k-diffeomorphism.*

We are now ready to prove the proposition.

Proof of Proposition 5.2. We need to construct a chart for each $p \in F^{-1}(c)$. By the assumption, $\nabla F(p) \neq 0$, so at least one of $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$ must be non-zero at p. Suppose $\partial F/\partial z \neq 0$ at p (the other cases are similar).

Define the map $g : \Omega \to \mathbb{R}^3$ as

$$g(x, y, z) = (x, y, F(x, y, z)).$$

Then, g is a smooth map on Ω, with derivative

$$dg_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x}(p) & \frac{\partial F}{\partial y}(p) & \frac{\partial F}{\partial z}(p) \end{pmatrix}.$$

Thus, the Jacobian of g at p is $\partial F/\partial z(p)$, which is non-zero by our assumption. Therefore, dg_p is invertible and the Inverse Function Theorem may be applied here. Thus, there is a neighbourhood U of p in Ω such that $g : U \to g(U)$ is a diffeomorphism.

Consider the set

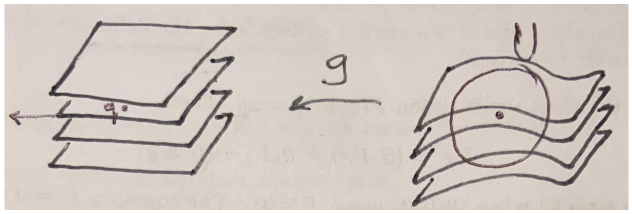
$$W = \{(u, v) \in \mathbb{R}^2 \mid (u, v, c) \in g(U)\}.$$

The set W is the projection of the intersection of the plane $z = c$ with $g(U)$. Since $g(U)$ is an open set in $\mathbb{R}^3$, W is open in $\mathbb{R}^2$.

Define the map $\phi : W \to U$ as

$$\phi(u, v) = g^{-1}(u, v, c).$$

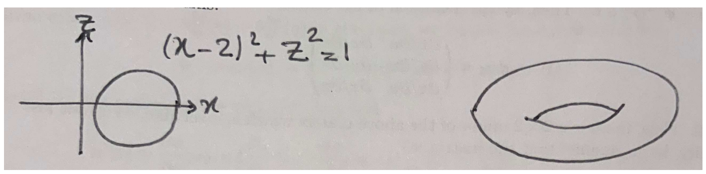
Recall that $p \in U$. The inverse function theorem tells us that g^{-1} is a smooth homeomorphism. In particular, ϕ is a smooth homeomorphism onto its image. Moreover, $d\phi_q$ is injective at all q in the domain of ϕ. Thus, ϕ is a chart for $S = F^{-1}(c)$ at p. $\square$



Example 5.4. Let S be the torus given in the cylindrical coordinates by

$$(r-2)^2 + z^2 = 1.$$

One can see that the above set is obtained from revolving the circle $(x-2)^2 + z^2 = 1$ in the xz-plane about the z axis.



We may consider the function $F : \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \neq 0\} \to \mathbb{R}$ defined as

$$F(x,y,z) = \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2,$$

and note that

$$\nabla F(x,y,z) = \left(2\left(\sqrt{x^2+y^2}-2\right)\left(\frac{x}{x^2+y^2}\right), 2\left(\sqrt{x^2+y^2}-2\right)\left(\frac{y}{x^2+y^2}\right), 2z\right).$$

Therefore, if $\nabla F(x,y,z) = (0,0,0)$, then $z = 0$ and either $\sqrt{x^2+y^2} = 2$ or $x = y = 0$. The latter point with $x = y = z = 0$ does not belong to the domain of F. When $z = 0$ and $\sqrt{x^2+y^2} = 2$ we have $F(x,y,z) = 0$. These imply that $\nabla F \neq 0$ at all points on the set $F^{-1}(1)$. By the definition of the Cylindrical coordinates, $S = F^{-1}(1)$. Therefore, S is a regular surface.

If we wanted to show that S is a regular surface using the definition of the chart, we need more than one chart to cover S.

Example 5.5. Consider the function $F(x,y,z) = zy$. Then,

$$\nabla F = (\partial_x F, \partial_y F, \partial_z F) = (0, z, y)$$

and so $\nabla F(p) = (0,0,0)$ when $(0,0,0) = p \in F^{-1}(0)$. The equation $F = 0$ consists of the two planes $z = 0$ and $y = 0$, which is not a regular surface (why). This shows that the condition in the above proposition is necessary.

In the proof of Proposition 5.2, we showed that any regular level set is locally the graph of a function. For the purpose of computations, it is useful to know that every regular surface is locally the graph of some function. We prove this result in the next statement.

Proposition 5.6. *Let S be a regular surface, and $p \in S$. There is a neighbourhood $V \subseteq S$ of p such that V is the graph of a smooth function of the from $z = f(x, y)$, $y = f(x, z)$, or $x = f(y, z)$.*

Proof. Let $\phi : U \to S$ be a chart at p, where $U \subset \mathbb{R}^2$ is an open set. Let us write

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)) .$$

Also let $q = \phi^{-1}(p) \in U$. Then, by the definition of the charts,

$$d\phi_q = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{pmatrix}$$

has rank 2. Thus, there is a 2×2 minor of the above matrix which is invertible. Without loss of generality, let us assume that the matrix

$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

is invertible.

Define the map $g : U \to \mathbb{R}^2$ as $g(u, v) = (x(u, v), y(u, v))$. We may apply the Inverse Function Theorem to g at q and obtain a neighbourhood $V \subset U$ of q such that $g : V \to g(V)$ is a smooth diffeomorphism.

Let $W = \phi(V)$ which is a neighbourhood of $p = \phi(q)$ in S. The set W is the graph of the function $f : g(V) \to \mathbb{R}$ defined as the composition

$$(x, y) \xrightarrow{g^{-1}} (u, v) \mapsto z(u, v),$$

that is, $f = z \circ g^{-1}$, which is smooth by the chain rule. $\square$

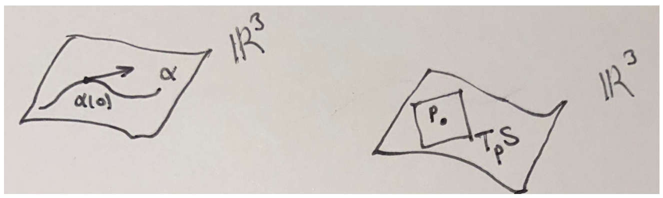
6 Tangent vectors and tangent planes

Given a regular surface $S \subset \mathbb{R}^3$ and $p \in S$, a **tangent vector** to S at p is a vector of the form $\alpha'(0)$, where $\alpha : (-\epsilon, +\epsilon) \to S$ is a smooth map with $\epsilon > 0$ and $\alpha(0) = p$. Here, when we say $\alpha : (-\epsilon, +\epsilon) \to S$ is smooth, we mean that α, when regarded as a map from $(-\epsilon, +\epsilon)$ into $\mathbb{R}^3$, is a smooth map.

The **tangent plane** of S at p is define as

$$T_pS = \{\alpha'(0) \mid \alpha : (-\varepsilon, \varepsilon) \to S \text{ is a smooth map with } \epsilon > 0 \text{ and } \alpha(0) = p\}.$$

In other words, T_pS is the collection of all tangent vectors to S at p. Intuitively, $T_p(S)$ is a plane in $\mathbb{R}^3$, as we shall discuss below.



The advantage of the above definition is that it does not depend on the charts defining S. However, the disadvantage is that it is impossible to compute the tangent plane using that definition. For instance if we want to compute the tangent plane to $S = \{z = x^2 + y^2\}$ at $p = (1, 1, 2)$ then it seems tricky. Thus the following lemma is clearly helpful.

Theorem 6.1. *If $S \subset \mathbb{R}^3$ is a regular surface with $p \in S$, and $\phi : U \to S$ is a chart with $\phi(q) = p$, then*

$$d\phi_q(\mathbb{R}^2) = \operatorname{span}\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\} = T_pS.$$

Proof. The set $d\phi_q(\mathbb{R}^2)$ is spanned by the columns of

$$d\phi_q = \begin{pmatrix} | & | \\ \frac{\partial \phi}{\partial u}(q) & \frac{\partial \phi}{\partial v}(q) \\ | & | \end{pmatrix}.$$

Given any vector $v = a\frac{\partial \phi}{\partial u}(q) + b\frac{\partial \phi}{\partial v}(q)$ in this span, for a and b in $\mathbb{R}$, we let $q = (u_0, v_0)$ and define

$$\alpha : (-\epsilon, +\epsilon) \to S, \qquad \alpha(t) = \phi(u_0 + at, v_0 + bt).$$

Clearly, α is a smooth map with $\alpha(0) = p$ and

$$\alpha'(0) = \frac{\partial \phi}{\partial u}(u_0, v_0)\frac{\partial u}{\partial t}(0) + \frac{\partial \phi}{\partial v}(u_0, v_0)\frac{\partial v}{\partial t}(0) = \frac{\partial \phi}{\partial u}(q)a + \frac{\partial \phi}{\partial v}(q)b = v.$$

Therefore, $v \in T_pS$. This argument shows that

$$\operatorname{span}\left\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right\} \subseteq T_pS.$$

We need to show that the right hand side of the above equation is contained in the left hand side. For that direction, we use that S is the graph of a smooth function near p.

By Proposition 5.6, near p, S is the graph of a smooth function. Without loss of generality, let us assume that it is given by the equation $z = f(x, y)$ (the other cases are similar). Therefore, $\phi : U \to S$ is of the form

$$\phi(u, v) = (x(u, v), y(u, v), f\,(x(u, v), y(u, v)))\,.$$

Let $q = \phi^{-1}(p)$ and $q = (u_0, v_0)$.

Define the map $F : U \times (-\epsilon, +\epsilon) \to \mathbb{R}^3$ as

$$F(u, v, t) = \Big(x(u, v), y(u, v), f\big(x(u, v), y(u, v)\big) + t\Big).$$

Then, $F(u_0, v_0, 0) = p$ and

$$dF_{(u_0,v_0,0)} = \begin{pmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ f_x x_u + f_y + y_u & f_x x_v + f_y y_v & 1 \end{pmatrix}.$$

Since $d\phi_q$ is invertible, the matrix

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

is invertible (why?). This implies that $dF_{(u_0,v_0,0)}$ is invertible as well (has a non-zero determinant). We apply the Inverse Function Theorem to the map F at $(u_0, v_0, 0)$, and obtain a smooth homeomorphism $G = F^{-1}$ defined near p. Therefore, there are smooth functions $u(x, y)$ and $v(x, y)$ such that $G(x, y, f(x, y)) = (u(x, y), v(x, y), 0)$.

Now assume that $\alpha'(0)$ is an arbitrary tangent vector in T_pS, where $\alpha : (-\epsilon, +\epsilon) \to S$ is a smooth map with $\alpha(0) = p$. Let $\alpha(t) = (x(t), y(t), f(x(t), y(t)))$. Then,

$$\alpha(t) = \phi\big(u(x(t), y(t)), v(x(t), y(t))\big).$$

Thus, by the chain rule,

$$\alpha'(0) = \frac{\partial \phi}{\partial u}(q)\frac{\partial u}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q)\frac{\partial v}{\partial t}(0).$$

This shows that $\alpha'(0)$ belongs to the span of $\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\}$, that is $d\phi_q(\mathbb{R}^2)$. □

Note that we imagine the tangent plane as a plane in $\mathbb{R}^3$ which approximates the surface S. However, just like tangent lines to curves, the tangent plane to a surface passes through the origin. If the tangent plane is translated in $\mathbb{R}^3$ by p, then it becomes a plane approximating the surface S. This is also evident in the following example.

Example 6.2. We may use the formula in Theorem 6.1 to identify the tangent plane to the regular surface S, $x^2+y^2+z^2=1$, at $p=(0,0,1)$. We use the chart

$$\phi : \{(u,v) \mid u^2+v^2<1\} \to \{(x,y,z) \mid x^2+y^2+z^2=1\}, \quad \phi(u,v) = \big(u, v, \sqrt{1-u^2-v^2}\big).$$

Then, $\phi(0,0)=(0,0,1)=p$, and we have

$$\begin{aligned} T_pS &= \operatorname{Span}\Big\{\frac{\partial \phi}{\partial u}(0,0), \frac{\partial \phi}{\partial v}(0,0)\Big\} \\ &= \operatorname{span}\Big\{\big(1,0,\frac{-u}{\sqrt{1-u^2-v^2}})\big|_{(u,v)=(0,0)}\big), \big(1,0,\frac{-v}{\sqrt{1-u^2-v^2}})\big|_{(u,v)=(0,0)}\big)\Big\} \\ &= \operatorname{span}\big\{(1,0,0),(0,1,0)\big\} \\ &= \{(x,y,z)\in \mathbb{R}^3 \mid z=0\}. \end{aligned}$$

Because many regular surfaces are given by the regular level sets of smooth functions, it is worth to consider how the tangent plane is identified in those cases. We do that in the next proposition.

Proposition 6.3. *Let $F:\mathbb{R}^3\to\mathbb{R}$ be a smooth function so that $\nabla F\neq 0$ on $S=F^{-1}(0)$. Then for any $p\in S$,*

$$T_pS = \{v\in\mathbb{R}^3 \mid \langle v, \nabla F(p)\rangle = 0\} = \big(\nabla F(p)\big)^{\perp}.$$

Proof. Let $\alpha:(-\epsilon,+\epsilon)\to S$ be a regular curve with $\alpha(0)=p$. Then, $F(\alpha(t))=0$, for all $t\in(-\epsilon,+\epsilon)$. Let us write $\alpha(t)=(x(t),y(t),z(t))$. By the chain rule, we have

$$0=\frac{d}{dt}\Big(F(\alpha(t)\Big) = \frac{\partial F}{\partial x}(p)\frac{\partial x}{\partial t}(0)+\frac{\partial F}{\partial y}(p)\frac{\partial y}{\partial t}(0)+\frac{\partial F}{\partial z}(p)\frac{\partial z}{\partial t}(0) = \langle \nabla F(p), \alpha'(0)\rangle.$$

This shows that T_pS is a subset of $\big(\nabla F(p)\big)^{\perp}$. On the other hand, since both T_pS and $\big(\nabla F(p)\big)^{\perp}$ are two dimensional, we must have $T_pS=\big(\nabla F(p)\big)^{\perp}$. ☐

Example 6.4. Let S be the parabola $z=x^2+y^2$ in $\mathbb{R}^3$. We may employ Proposition 6.3 to find the tangent plane to S at the point $p=(1,3,10)$. We consider the smooth function $f(x,y,z)=x^2+y^2-z$, so that $F^{-1}(0)=S$. We have $\nabla F(x,y,z)=(2x,2y,-1)$, and hence $\nabla F(1,3,10)=(2,6,-1)$. The plane orthogonal to this vector is given by $2x+6y-z=0$.

7 Smooth maps on surfaces

Let $S_1, S_2\subset\mathbb{R}^3$ be regular surfaces.

(i) A map $F:S_1\to\mathbb{R}^n$ is called **smooth** if for every chart $\phi:U\to S_1$ the composition map $F\circ\phi:U\to\mathbb{R}^n$ is smooth.

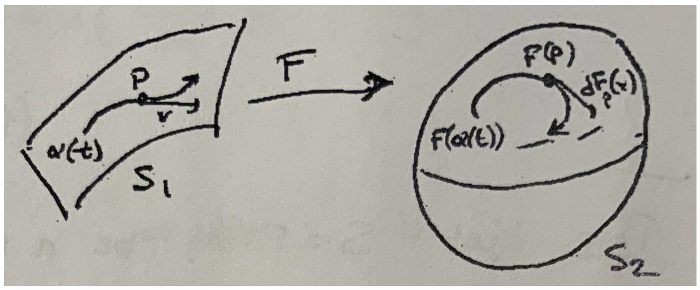
(ii) A map $F:S_1\to S_2$ is called smooth if it is smooth when viewed as a map from S_1 to $\mathbb{R}^3$.

The **differential** of a smooth map $F : S_1 \to S_2$ at a point $p \in S_1$ is the map

$$dF_p : T_pS_1 \to T_{F(p)}S_2$$

defined as follows. Let $v \in T_pS$ be an arbitrary element. There is a smooth map $\alpha : (-\epsilon, +\epsilon) \to S_1$ satisfying $\alpha(0) = p$ and $v = \alpha'(0)$. Then, $\beta = F \circ \alpha : (-\epsilon, +\epsilon) \to S_2$ is a smooth map satisfying $\beta(0) = F(p)$. We define

$$dF_p(v) = \beta'(0).$$



Proposition 7.1. *The definition of the differential dF_p given above is independent of the choice of α.*

Proof. Let $\alpha_1, \alpha_2 : (-\epsilon, +\epsilon) \to S_1$ be smooth maps with $\alpha_1(0) = \alpha_2(0) = p$ and $\alpha_1'(0) = \alpha_2'(0)$. Recall from the proof of Theorem 6.1 that given a chart $\phi : U \to S_1$ for S_1 at p, there are smooth maps (curves) $t \mapsto (u_1(t), v_1(t))$ and $t \mapsto (u_2(t), v_2(t))$ such that

$$\alpha_1(t) = \phi(u_1(t), v_1(t)), \qquad \alpha_2(t) = \phi(u_2(t), v_2(t)).$$

Let $\phi^{-1}(p) = (q_1, q_2)$. Then, we have

$$\begin{aligned} \alpha_1'(0) &= \frac{\partial \phi}{\partial u}(q_1, q_2)\frac{\partial u_1}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q_1, q_2)\frac{\partial v_1}{\partial t}(0), \\ \alpha_2'(0) &= \frac{\partial \phi}{\partial u}(q_1, q_2)\frac{\partial u_2}{\partial t}(0) + \frac{\partial \phi}{\partial v}(q_1, q_2)\frac{\partial v_2}{\partial t}(0). \end{aligned}$$

Since $\alpha_1'(0) = \alpha_2'(0)$, and the vectors $\frac{\partial \phi}{\partial u}(q_1, q_2)$ and $\frac{\partial \phi}{\partial v}(q_1, q_2)$ are linearly independent, we must have

$$u_1'(0) = u_2'(0), \qquad v_1'(0) = v_2'(0).$$

In other words, the curves (u_1, v_1) and (u_2, v_2) have the same tangent vectors at $t = 0$.

On the other hand, we have

$$\begin{aligned} dF_p(\alpha_i'(0)) &= (F \circ \alpha_i)'(0) \\ &= \frac{d}{dt}(F \circ \phi(u_i(t), v_i(t))\Big|_{t=0} \\ &= \frac{\partial (F \circ \phi)}{\partial u}(q_1, q_2)\frac{\partial u_i}{\partial t}(0) + \frac{\partial (F \circ \phi)}{\partial v}(q_1, q_2)\frac{\partial v_i}{\partial t}(0). \end{aligned}$$

In the above equation, we have used that $F \circ \phi$ is a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

It follows that $dF_p(\alpha_1'(0)) = dF_p(\alpha_2'(0))$, and hence dF_p is independent of α. $\square$

Proposition 7.2. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces, $p \in S_1$ and $F : S_1 \to S_2$ be a smooth map. The differential $dF_p : T_pS_1 \to T_{F(p)}S_2$ is a linear map.*

Proof. Let $v = \alpha_1'(0)$ and $w = \alpha_2'(0)$ be arbitrary vectors in T_pS_1 with α_1 and α_2 defined on $(-\epsilon, +\epsilon)$. Choose a chart $\phi : U \to S_1$ for S_1 at p. As in the previous proof, there are smooth functions u_1, v_1, u_2, and v_2 such that

$$\alpha_i(t) = \phi(u_i(t), v_i(t)), \qquad \text{for } i = 1, 2.$$

Without loss of generality (just pre-compose ϕ with a translation) we may assume that $\phi(0,0) = p$.

Let λ be an arbitrary real number and define the smooth map

$$\alpha_3(t) = \phi\big(u_1(t) + \lambda u_2(t), v_1(t) + \lambda v_2(t)\big), \quad t \in (-\epsilon, +\epsilon).$$

We have $\alpha_3(0) = \phi(0,0) = p$.

$$\begin{aligned}
\alpha_3'(0) &= \frac{\partial \phi}{\partial u}(0,0)\left(u_1'(0) + \lambda u_2'(0)\right) + \frac{\partial \phi}{\partial v}(0,0)\left(v_1'(0) + \lambda v_2'(0)\right) \\
&= \left(\frac{\partial \phi}{\partial u}(0,0)u_1'(0) + \frac{\partial \phi}{\partial v}(0,0)v_1'(0)\right) + \lambda\left(\frac{\partial \phi}{\partial u}(0,0)u_2'(0) + \frac{\partial \phi}{\partial v}(0,0)v_2'(0)\right) \\
&= \alpha_1'(0) + \lambda \alpha_2'(0) \\
&= v + \lambda w.
\end{aligned}$$

Then,

$$\begin{aligned}
dF_p\left(v + \lambda w\right) &= dF_p\left(\alpha_3'(0)\right) \\
&= \frac{d}{dt}\left(F \circ \phi\big(u_1(t) + \lambda u_2(t), v_1(t) + \lambda v_2(t)\big)\right)\Big|_{t=0} \\
&= \frac{\partial\left(F \circ \phi\right)}{\partial u}(0,0)\left(u_1'(0) + \lambda u_2'(0)\right) + \frac{\partial\left(F \circ \phi\right)}{\partial v}(0,0)\left(v_1'(0) + \lambda v_2'(0)\right) \\
&= \left(\frac{\partial(F \circ \phi)}{\partial u}(0,0)u_1'(0) + \frac{\partial\left(F \circ \phi\right)}{\partial v}(0,0)v_1'(0)\right) \\
&\qquad + \lambda\left(\frac{\partial\left(F \circ \phi\right)}{\partial u}(0,0)u_2'(0) + \frac{\partial\left(F \circ \phi\right)}{\partial v}(0,0)v_2'(0)\right).
\end{aligned} \tag{5}$$

We calculate each of the parentheses on the right hand side of the above equation as follows. Using the chain rule,

$$\begin{aligned}
\left(\frac{\partial(F \circ \phi)}{\partial u}(0,0)u_1'(0) + \frac{\partial\left(F \circ \phi\right)}{\partial v}(0,0)v_1'(0)\right) &= \frac{d}{dt}\left(F \circ \phi(u_1(t), v_1(t))\right)\Big|_{t=0} \\
&= dF_p\left(\frac{d}{dt}\phi(u_1(t), v_1(t))\Big|_{t=0}\right) \\
&= dF_p(\alpha_1'(0)) = dF_p(v).
\end{aligned} \tag{6}$$

Similarly, we have

$$\left(\frac{\partial (F\circ\phi)}{\partial u}(0,0)u_2'(0) + \frac{\partial\,(F\circ\phi)}{\partial v}(0,0)v_2'(0)\right) = dF_p(\alpha_2'(0)) = dF_p(w).$$

Combining the above equations, we obtained the desired relation

$$dF_p\,(v+\lambda w) = dF_p(v) + \lambda dF_p(w). \qquad \square$$

Given a smooth function $f : S \to \mathbb{R}$, we can define the **differential** of f at $p \in S$ as follows. If $\alpha'(0)$ is an element of T_pS, we let

$$df_p(\alpha'(0)) = \frac{d}{dt}(f(\alpha(t)))\Big|_{t=0}.$$

As an exercise, you can show that the above definition is well-defined, that is, it does not dependent on the choice of α.

Example 7.3. Let S be the unit sphere in $\mathbb{R}^3$, $x^2+y^2+z^2=1$, and $p=(0,1,0)\in S$. Define $f : S \to \mathbb{R}$ as $f(x,y,z)=z$. We aim to compute T_pS and $df_p : T_pS \to \mathbb{R}$.

We have $S = F^{-1}(1)$ where $F(x,y,z) = z^2+y^2+z^2$. Then, $T_pS = (\nabla F(p))^\perp = (0,2,0)^\perp$ is spanned by $(1,0,0)$ and $(0,0,1)$. Near p we have a chart $\phi : U \to \mathbb{R}^3$ defined as

$$\phi(u,v) = (u, \sqrt{1-u^2-v^2}, v)$$

where $U = \{(u,v)\in\mathbb{R}^2 \mid u^2+v^2<1\}$. We have $f\circ\phi(u,v)=v$. So,

$$df_p\left(\frac{\partial\phi}{\partial u}(0,0)\right) = \frac{\partial(f\circ\phi)}{\partial u}(0,0) = 0,$$

and hence $df_p(1,0,0)=0$. Similarly,

$$df_p\left(\frac{\partial\phi}{\partial v}(0,0)\right) = \frac{\partial(f\circ\phi)}{\partial v}(0,0) = 1,$$

and hence $df_p(0,0,1)=1$.

We have one more useful statement about differentials.

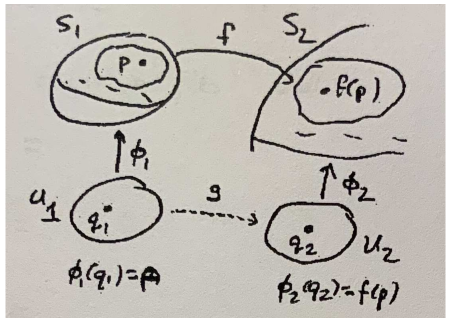
Proposition 7.4. *Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces, $f : S_1 \to S_2$ be a smooth map, and $p \in S^1$. If $df_p : T_pS_1 \to T_{f(p)}S_2$ is invertible, there is a neighbourhood $V \subset S_1$ of p such that $f : V \to f(V)$ is a diffeomorphism.*

Proof. This is an application of the Inverse Function Theorem. So we only sketch the argument. Choose charts $\phi_1 : U \to S_1$ for S_1 at p, and $\phi_2 : U_2 \to S_2$ for S_2 at $f(p)$. Let $q_1 = \phi^{-1}(p)$ and $q_2 = \phi^{-1}(f(p))$.

We consider the map

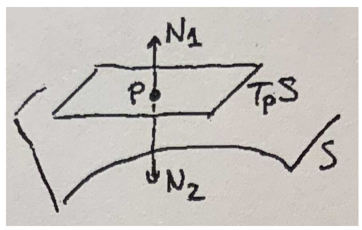
$$g = \phi_2^{-1}\circ f\circ\phi_1 : U_1 \to U_2$$

and note that $g(q_1)=q_2$. Check that g is smooth and its differential $dg_q : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible (using the hypothesis on df_p). By the Inverse Function Theorem for maps of $\mathbb{R}^2$, there is an open neighbourhood v_1 of q_1 in U_1 such that $g : V_1 \to g(V_1)$ is a diffeomorphism. It follows that f is a diffeomorphism from $\phi(V_1)$ onto its image. $\square$



8 Normal vectors, and the Gauss map

For a regular surface $S \subset \mathbb{R}^3$, the tangent plane at teach point in S is a two dimensional subspace of $\mathbb{R}^3$. It follows that at each point in S, there are two unit normal vectors.



When $S = F^{-1}(c)$ is a regular level set of a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, we have a natural choice for the normal vector; T_pS is the orthogonal complement of $\nabla F(p) \neq 0$. Thus, we define the unit normal vector to S at p as

$$N(p) = \frac{\nabla F(p)}{|\nabla F(p)|}.$$

This is not a canonic choice of $N(p)$ for S, since one may also consider the smooth function $-F$ with the value $-c$. The crucial point here is that this defines the normal vector $N(p)$ on all of S which depends continuously on $p \in S$

In general, given a chart $\phi : U \to S$ for S at p with $\phi^{-1}(p) = q$, we know that $\partial\phi/\partial u$ and $\partial\phi/\partial v$ at q span T_pS. We define

$$N(p) = \frac{\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)}{\left|\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)\right|}. \tag{7}$$

This can always be done locally (in a chart near p), but since this choice depends on the chart, it may not be possible to combine these to obtained a continuous normal vector on all of S.

A regular surface S is called **orientable** if there is a continuous choice of unit normal vector $N(p)$ for $p \in S$.

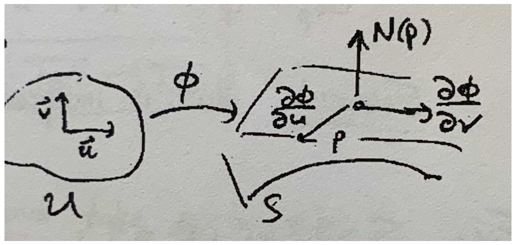


Figure 0.1: The Möbius band is obtained from identifying the two vertical edges of the square $[0,1] \times [0,1]$ in the opposite directions.

It is not true that every surface in $\mathbb{R}^3$ is orientable. For example, on a Möbius band it is not possible to make a globally defined continuous normal vector.

Since for each point $p \in S$, the unit vector $N(p)$ belongs to $\mathbb{R}^3$, we may view N as a continuous map from S to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The map

$$N : S \to \mathbb{S}^2$$

is called the **Gauss map** of S.

Remark 8.1. Note that if a surface is regular, and there is a continuous choice of $N : S \to \mathbb{S}^2$, then the map $N : S \to \mathbb{R}^3$ is smooth. That is because, in every chart, the map N or $-N$ is given by the formula in Equation (7).

Example 8.2. Let $S = \mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Then, $S = F^{-1}(1)$ where $F(x,y,z) = x^2 + y^2 + z^2$, and hence we have

$$N(x,y,z) = \frac{\nabla F(x,y,z)}{|\nabla F(x,y,z)|} = \frac{(2x,2y,2z)}{\sqrt{4(x^2+y^2+z^2)}} = (x,y,z).$$

Alternatively, we can see that near the north pole $(0,0,1)$ we have the chart

$$\phi(u,v) = (u, v, \sqrt{1-u^2-v^2}),$$

and hence

$$\frac{\partial \phi}{\partial u} = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \quad \frac{\partial \phi}{\partial v} = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right).$$

Therefore,

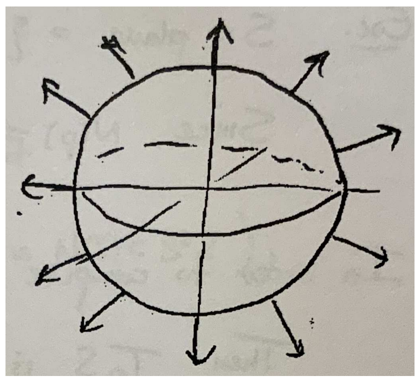
$$\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right).$$

and then

$$\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right| = \frac{1}{\sqrt{1-u^2-v^2}}$$

which gives us

$$N(\phi(u,v)) = \frac{\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}}{\left|\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}\right|} = (u, v, \sqrt{1-u^2-v^2}) = \phi(u,v).$$



Example 8.3. Let S be the plane $\{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$. This is a regular level set for the function $F(x,y,z) = ax + by + cz$, and hence

$$N(x,y,z) = \frac{\nabla F(x,y,z)}{|\nabla F(x,y,z)|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}.$$

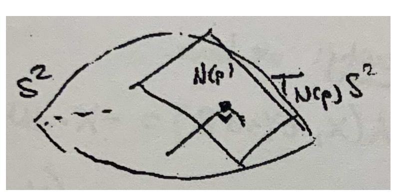
In this case the Gauss map is a constant vector.

The derivative of the Gauss map $N : S \to \mathbb{S}^2$ at p is a linear map denoted by

$$dN_p : T_pS \to T_{N(p)}\mathbb{S}^2.$$

The tangent plane $T_{N(p)}\mathbb{S}^2$ consists of all vectors orthogonal to the vector $N(p)$, which is the same as the tangent plane T_pS. Using this identification, we can write

$$dN_p : T_pS \to T_pS.$$



There is an alternative way to see that dN_p maps T_pS to T_pS. Let $\alpha'(0)$ be an arbitrary vector in T_pS. Since N is a unit normal vector, we must have $\langle N(\alpha(t)), N(\alpha(t))\rangle = 1$. Thus, by differentiating this relation at $t = 0$, we obtain

$$\big\langle dN_p(\alpha'(0)), N_p(\alpha(0))\big\rangle = \left\langle \frac{d}{dt}N_p(\alpha(t))\Big|_{t=0}, N_p(\alpha(0))\right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle N_p(\alpha(t)), N_p(\alpha(t))\right\rangle\Big|_{t=0} = 0.$$

This shows that $dN_p(\alpha'(0))$ is orthogonal to $N(\alpha(0))$, and hence $dN_p(\alpha'(0))$ belongs to T_pS.

Example 8.4. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$ be the sphere of radius $r > 0$ in $\mathbb{R}^3$, and fix an arbitrary $p = (x, y, z) \in S$. Then, $S = F^{-1}(r^2)$, where $F(x, y, z) = x^2 + y^2 + z^2$. At every $(x, y, z) \in S$, we have

$$N(x, y, z) = \frac{\nabla F(x, y, z}{|\nabla F(x, y, z)|} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right).$$

If $\alpha : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\alpha(0) = p$, then $N(\alpha(t)) = \alpha(t)/r$. Therefore,

$$dN_p(\alpha'(0)) = \frac{d}{dt}N(\alpha(t))\Big|_{t=0} = \frac{d}{dt}\frac{\alpha(t)}{r}\Big|_{t=0} = \frac{1}{r}\alpha'(0).$$

In other words,

$$dN_p = \frac{1}{r}\,\mathrm{id} : T_pS \to T_pS.$$

This looks like curvature, which we try to express in terms of the Gauss map.

Example 8.5. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$ be a plane in $\mathbb{R}^3$. We saw that $N(p)$ is a constant map, so we have $dN_p = 0$ for all p in S.

We aim to present a general formula for dN_p in terms of charts. To this end, let $\phi : U \to S$ be a chart for S at $p \in S$, with $q = \phi^{-1}(p)$. We have seen that the vectors $\frac{\partial\phi}{\partial u}(q)$ and $\frac{\partial\phi}{\partial v}(q)$ form a basis for T_pS. Therefore, it suffices to compute dN_p on these vectors.

Proposition 8.6. *Let S be a regular surface in $\mathbb{R}^3$, and $\phi : U \to S$ be a chart for S at $\phi(q) = p \in S$. We have*

$$dN_p\left(\frac{\partial\phi}{\partial u}(q)\right) = \frac{\partial(N\circ\phi)}{\partial u}(q), \qquad dN_p\left(\frac{\partial\phi}{\partial v}(q)\right) = \frac{\partial(N\circ\phi)}{\partial v}(q).$$

Proof. These follow from the chain rule. See Equation (6) □

9 Second fundamental form of a surface

Let S be an orientable regular surface in $\mathbb{R}^3$, and $p \in S$. We define the **second fundamental form** of S at p as the map

$$A_p : T_pS \times T_pS \to \mathbb{R}, \qquad A_p(X, Y) = -\langle X, dN_p(Y)\rangle.$$

For example, when S is the sphere of radius r, we saw that $dN_p(Y) = Y/r$. Therefore, $A_p(X, Y) = -r^{-1}\langle X, Y\rangle$.

Proposition 9.1. *For any regular surface $S \subset \mathbb{R}^3$, and every $p \in S$, the second fundamental form A_p is a symmetric bilinear form.*

Recall that for vector spaces V and W, a map $T : V \times W \to R$ is called bilinear, if for all $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $\lambda \in \mathbb{R}$ we have

$$T(v_1, w_1 + \lambda w_2) = T(v_1, w_1) + \lambda T(v_1, w_2), \quad T(v_1 + \lambda v_2, w_1) = T(v_1, w_1) + \lambda T(v_2, w_1).$$

The map $T : V \times V \to \mathbb{R}$ is called symmetric, if for all $v_1, v_2 \in V$ we have $T(v_1, v_2) = T(v_2, v_1)$.

Proof. To see that A_p is bilinear, we note that

$$\begin{aligned} A_p(X, \lambda_1 Y + \lambda_2 Z) &= -\langle X, dNp(\lambda_1 Y + \lambda_2 Z)\rangle \\ &= -\langle X, \lambda_1 dN_p(Y) + \lambda_2 dN_p(Z)\rangle \\ &= \lambda_1\langle -X, dN_p(Y)\rangle - \lambda_2\langle X, dN_p(Z)\rangle. \\ &= \lambda_1 A_p(X, Y) + \lambda_2 A_p(X, Z). \end{aligned}$$

and

$$\begin{aligned} A_p(\lambda_1 X + \lambda_2 Y, Z) &= -\langle \lambda_1 X + \lambda_2 Y, dN_p(Z)\rangle \\ &= -\lambda_1\langle X, dNp(Z)\rangle - \lambda_2\langle Y, dN_p(Z)\rangle \\ &= \lambda_1 A_p(X, Z) + \lambda_2 A_p(Y, Z). \end{aligned}$$

To see that A_p is symmetric, let us choose a chart $\phi : U \to S$ for S at p. Define $q = \phi^{-1}(p)$. First we show that

$$A_p\left(\frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q)\right) = A_p\left(\frac{\partial\phi}{\partial v}(q), \frac{\partial\phi}{\partial u}(q)\right).$$

By the definition of A_p, we have

$$A_p\left(\frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q)\right) = -\left\langle \frac{\partial\phi}{\partial u}(q), dN_p\left(\frac{\partial\phi}{\partial v}(q)\right)\right\rangle = -\left\langle \frac{\partial\phi}{\partial u}(q), \frac{\partial(N\circ\phi)}{\partial v}(q)\right\rangle$$

Since $N \circ \phi$ maps into vectors normal to S and $\partial\phi/\partial u$ belongs to the tangent space of S, we have

$$\left\langle \frac{\partial\phi}{\partial u}, N\circ\phi\right\rangle = 0,$$

we may apply $\partial/\partial v$ to get

$$\left\langle \frac{\partial \phi}{\partial u}, \frac{\partial (N \circ \phi)}{\partial v} \right\rangle + \left\langle \frac{\partial}{\partial v}\frac{\partial \phi}{\partial u}, N \circ \phi \right\rangle = 0.$$

Combining with the above equation we obtain

$$A_p\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) = \left\langle \frac{\partial^2 \phi}{\partial v \partial u}(q), N \circ \phi(q) \right\rangle.$$

In a similar fashion, we also obtain

$$A_p\left(\frac{\partial \phi}{\partial v}(q), \frac{\partial \phi}{\partial u}(q)\right) = \left\langle \frac{\partial^2 \phi}{\partial u \partial v}(q), N \circ \phi(q) \right\rangle.$$

Because ϕ is smooth, the partial derivatives commute, and we conclude the desired relation.

Finally, because $\frac{\partial \phi}{\partial u}(q)$ and $\frac{\partial \phi}{\partial v}(q)$ form a basis for T_pS, and A_p is bilinear, it follows that A_p is symmetric. $\square$

10 Curvature of a surface

Assume that S is an orientable regular surface, $p \in S$, and $N : S \to \mathbb{S}^2$ is a Gauss map. We aim to study the variations of N along different directions in T_pS. It is clear that if we know dN_p then we know A_p. The converse is also true, because if $\{v, w\}$ is an orthonormal basis for T_pS, then for every $X \in T_pS$,

$$dN_p(X) = \langle dN_p(X), v\rangle v + \langle dN_p(X), w\rangle w = -A_p(v, X)v - A_p(w, X)w.$$

Let $\{v, w\}$ be an orthonormal basis for T_pS, and write $dN_p(v) = a_1 v + a_2 w$ and $dN_p(w) = b_1 V + b_2 w$, for some real numbers a_1, a_2, b_1, b_2. By the symmetry of A_p, we have

$$a_2 = \langle w, dN_p(v)\rangle = -A_p(w, v) = -A_p(v, w) = \langle v, dN_p(w)\rangle = b_1.$$

It follows that the matrix representation of dN_p in the basis $\{v, w\}$ is symmetric. By the spectral theorem for symmetric matrices, dN_p is diagonalisable, and has real eigenvalues whose eigenvectors are orthonormal to each other. In other words, T_pS has an orthonormal basis X_1 and X_2 such that

$$dN_p(X_1) = -\lambda_1 X_1, \qquad dN_p(X_2) = -\lambda_2 X_2.$$

In terms of A_p, these imply that

$$\begin{aligned} A_p(X_1, X_1) &= -\langle X_1, dN_p(X_1)\rangle = -\langle X_1, -\lambda_1 X_1\rangle = \lambda_1, \\ A_p(X_2, X_2) &= -\langle X_2, dN_p(X_2)\rangle = -\langle X_2, -\lambda_2 X_2\rangle = \lambda_2, \\ A_p(X_1, X_2) &= -\langle X_1, dN_p(X_2)\rangle = -\langle X_1, -\lambda_2 X_2\rangle = 0. \end{aligned}$$

The tangent vectors X_1 and X_2 in T_pS are called the **principle directions** at p, and the real numbers λ_1 and λ_2 are called the **principle curvatures**.

Lemma 10.1. *If S is a regular surface, $p \in S$, and $\lambda_1(p) \le \lambda_2(p)$ are the principle curvatures at p, then*

$$\lambda_1(p) = \min\{A_p(X,X) \mid X \in T_pS, |X| = 1\},$$
$$\lambda_2(p) = \max\{A_p(X,X) \mid X \in T_pS, |X| = 1\}.$$

Proof. Any $X \in T_pS$ with $|X| = 1$ can be written as $X = c_1X_1 + c_2X_2$ with $c_1^2 + c_2^2 = 1$. Then, as A_p is bilinear,

$$A_p(X,X) = A_p(c_1X_1 + c_2X_2, c_1X_1 + c_2X_2) = c_1^2A_p(X_1,X_1) + c_2^2A_p(X_2,X_2).$$

Therefore,

$$A_p(X,X) = c_1^2\lambda_1 + c_2^2\lambda_2 \ge c_1^2\lambda_1 + c_2^2\lambda_1 = \lambda_1,$$

and

$$A_p(X,X) = c_1^2\lambda_1 + c_2^2\lambda_2 \le c_1^2\lambda_2 + c_2^2\lambda_2 = \lambda_2.$$

We already saw that λ_1 and λ_2 are realised for $X = X_1$ and $X = X_2$, respectively. □

Example 10.2. Let S be the sphere of radius $r > 0$. We saw before that at every point $p \in S$ we have $dN_p = (1/r)\,\mathrm{id}$. This implies that $A_p(X,Y) = -\langle X,Y\rangle/r$, and hence both principle curvatures are $-1/r$.

If we choose the inward-pointing Gauss map $N(p) = -p/r$, then $dN_p = (-1/r)\,\mathrm{id}$ and we get $\lambda_1 = \lambda_2 = 1/r$. In other words, the principle curvatures depend on the orientation.

Example 10.3. If S is a connected surface, and $\lambda_1(p) = \lambda_2(p) = 0$ for all $p \in S$, then dN_p is the zero map for all $p \in S$. Then $N : S \to \mathbb{S}^2$ is a constant, say $N \equiv \vec{v}$, where $\vec{v} \in \mathbb{S}^2$. Then, S must lie in a plane orthogonal to $\vec{v}$.

Indeed, if we pick a chart $\phi : U \to S$ for S at p, such that $U \subset \mathbb{R}^2$ is connected, for any point $p' \in \phi(U)$, we can find a smooth path $\gamma : [0,1] \to U$ with $\gamma(0) = \phi^{-1}(p)$ and $\gamma(1) = \phi^{-1}(p')$. Then, $\phi \circ \gamma : [0,1] \to S$ is a smooth path from p to p'. We have $\frac{d}{dt}(\phi\circ\gamma)(t) \in T_{\phi(\gamma(t))}S$ is orthogonal to $\vec{v}$, and so

$$\frac{d}{dt}\langle \phi(\gamma(t)), \vec{v}\rangle = \langle \frac{d}{dt}(\phi(\gamma(t)), \vec{v}\rangle = 0.$$

Thus, $\langle \phi(\gamma(t)), \vec{v}\rangle$ is constant, and hence $\langle p', \vec{v}\rangle = \langle p, \vec{v}\rangle$ for all $p' \in \phi(U)$. Because S is connected, we conclude that $\langle p', \vec{v}\rangle = \langle p, \vec{v}\rangle$, for all $p' \in S$.

Let $c : (-\epsilon, +\epsilon) \to S$ be a regular curve in S, with $c(0) = p$. When we view the curve c in $\mathbb{R}^3$, the curvature of c at p, $\vec{k}(0)$, is a vector in $\mathbb{R}^3$, which measures how c turns in $\mathbb{R}^3$. We may decompose $\vec{k}(0)$ into two component; the component tangent to S at p and the component perpendicular to S at p. We have

$$\vec{k}(0) = \langle \vec{k}(0), N(p)\rangle N(p) + \vec{k}_{\mathrm{tang}}(0),$$

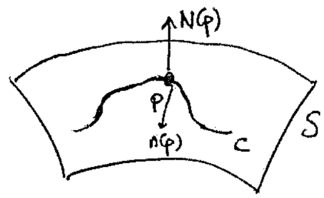
where $\langle \vec{k}(0), N(p)\rangle N(p)$ is the component perpendicular to $T_p(S)$, and $\vec{k}_{\text{tang}}(0)$ is the component in T_pS. The perpendicular component measures the curving of S at p in the direction of $c'(0)$. The **normal curvature** of S at p in the direction $c'(0)$ is defined as

$$k_n(0) = \langle \vec{k}(0), N(p)\rangle,$$

which is the length of the perpendicular component. We note that if θ is the angle between $N(p)$ and $\vec{k}(0)/k(0)$, then

$$k_n(0) = k(0)\cos(\theta).$$

There is a convenient formula for the normal curve in terms of the second fundamental form A_p.



Theorem 10.4. *Let S be an orientable regular surface in $\mathbb{R}^3$, $p \in S$, and $v \in T_pS$ with $\|v\| = 1$. Then, the normal curvature of S at p in the direction v satisfies*

$$k_n(p) = A_p(v, v).$$

Proof. Let $c : (-\epsilon, +\epsilon) \to S$ be a regular curve parametrised by arc length such that $c(0) = p$ and $c'(0) = v$. For every $t \in (-\epsilon, +\epsilon)$ we have $\langle c'(t), N(c(t))\rangle = 0$. Differentiating this relation with respect to t, at $t = 0$ we obtain

$$\begin{aligned} 0 &= \langle c''(0), N(c(0))\rangle + \langle c'(0), dN_{c(0)}(c'(0))\rangle \\ &= \langle \vec{k}(0), N(p)\rangle + \langle v, dN_p(v)\rangle \\ &= k_n(p) - A_p(v, v). \end{aligned}$$

□

Combining the above theorem with Lemma 10.1, we obtain the following result.

Corollary 10.5. *For any orientable regular surface S and every $p \in S$, the principal curvatures $\lambda_1(p)$ and $\lambda_2(p)$ are the minimum and maximum values of the normal curvature at p along all directions in T_pS, respectively.*

A point $p \in S$ is called **umbilical**, if the principal curvatures λ_1 and λ_2 at p are equal.

Proposition 10.6. *Let S be a connected regular surface which is orientable. If every point in S is umbilical, then either S is contained in a plane, or S is contained in a sphere.*

Proof. Let $\phi : U \to S$ be a chart for S. By the assumption there is a function $\lambda : S \to \mathbb{R}$ such that $\lambda_1(q) = \lambda_2(q) = \lambda(q)$ for all $q \in S$. For every $q \in S$, using the orthonormal basis on T_qS by the eigenvectors of $dN_p : T_qS \to T_qS$, we have $dN_q = -\lambda(q)\,\mathrm{id}$. We have

$$\frac{\partial(N \circ \phi)}{\partial u} = dN_\phi\left(\frac{\partial \phi}{\partial u}\right) = -(\lambda \circ \phi)\frac{\partial \phi}{\partial u} \tag{8}$$

and similarly,

$$\frac{\partial(N \circ \phi)}{\partial v} = dN_\phi\left(\frac{\partial \phi}{\partial v}\right) = -(\lambda \circ \phi)\frac{\partial \phi}{\partial v}. \tag{9}$$

These imply that

$$\begin{aligned}
\frac{\partial^2(N \circ \phi)}{\partial v \partial u} &= \frac{\partial}{\partial v}\left(-(\lambda \circ \phi)\frac{\partial \phi}{\partial u}\right) = \frac{-\partial(\lambda \circ \phi)}{\partial v}\frac{\partial \phi}{\partial u} - (\lambda \circ \phi)\frac{\partial^2 \phi}{\partial v \partial u}, \\
\frac{\partial^2(N \circ \phi)}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(-(\lambda \circ \phi)\frac{\partial \phi}{\partial v}\right) = -\frac{\partial(\lambda \circ \phi)}{\partial u}\frac{\partial \phi}{\partial v} - (\lambda \circ \phi)\frac{\partial^2 \phi}{\partial u \partial v}.
\end{aligned}$$

Because $N \circ \phi$ is a smooth map, the left hand side of the above equations are equal, and since ϕ is smooth, we also have $\frac{\partial^2 \phi}{\partial v \partial u} = \frac{\partial^2 \phi}{\partial u \partial v}$. Therefore,

$$\frac{\partial(\lambda \circ \phi)}{\partial v}\frac{\partial \phi}{\partial u} = \frac{\partial(\lambda \circ \phi)}{\partial u}\frac{\partial \phi}{\partial v}.$$

Since $\partial \phi / \partial u$ and $\partial \phi / \partial v$ are linearly independent, we must have

$$\frac{\partial(\lambda \circ \phi)}{\partial v} = \frac{\partial(\lambda \circ \phi)}{\partial u} = 0.$$

This means that $\lambda \circ \phi : U \to \mathbb{R}$ is constant. Because S is connected, and is covered by such charts, we conclude that λ is a constant on S, say $\lambda \equiv \lambda_0$.

If $\lambda_0 = 0$, we have already seen in Example 10.3 that S must be contained in a plane.

If $\lambda_0 \neq 0$, without loss of generality (by replacing N by $-N$) we may assume that $\lambda_0 > 0$. From Equations (8) and (9), we get

$$\frac{\partial}{\partial u}\left(\phi + \frac{1}{\lambda_0}N \circ \phi\right) = 0, \quad \frac{\partial}{\partial v}\left(\phi + \frac{1}{\lambda_0}N \circ \phi\right) = 0.$$

This implies that $\phi + \frac{1}{\lambda_0}N \circ \phi$ is a constant on U, and hence on S. Let us choose $\vec{c}_0 \in \mathbb{R}^3$ such that $\phi + \frac{1}{\lambda_0}N \circ \phi \equiv \vec{c}_0$. Then,

$$|\phi - \vec{c}_0| = \left|\frac{1}{\lambda_0}N \circ \phi\right| = \frac{1}{\lambda_0},$$

and hence $\phi(U)$ is contained in the sphere of radius $1/\lambda_0$ about $\vec{c}_0$. Since S is connected, we conclude that all of S is contained in the sphere of radius $1/\lambda_0$ about $\vec{c}_0$. ∎

Let S be a regular surface, and $p \in S$, with principle curvatures $\lambda_1(p)$ and $\lambda_2(p)$. The **Gaussian curvature** of S at p is

$$K(p) = \lambda_1(p)\lambda_2(p) = \det(dN_p)$$

and the **mean curvature** of S at p is defined as

$$H(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2} = \frac{-1}{2}\,\mathrm{tr}(dN_p),$$

where tr denotes the trace of a matrix.

Remark 10.7. If we change the orientation of S (that is changing N to $-N$), the sign of λ_1 and λ_2 change, but the product $\lambda_1\lambda_2 = K$ does not change. Thus the Gaussian curvature is independent of the choice of orientation. However, the mean curvature changes sign if we change the orientation.

Example 10.8. A sphere of radius $r > 0$ has $K = 1/r^2$ at all points, and $H = \pm 1/r$ (with the sign depending on the choice of orientation). For a plane, we have $K = H = 0$ at all points.

11 Elliptic, hyperbolic, and parabolic points on a surface

We are interested in understanding the local shape of a regular surface in terms of its principle curvatures. To this end let us use a local chart $\phi : U \to S$ at $p \in S$, with $\phi(0,0) = p$. Fix arbitrary $c, d \in \mathbb{R}$, and define the curve $\gamma(t) = \phi(ct, dt)$, for t in some small neighbourhood $(-\epsilon, +\epsilon)$. Since

$$\frac{d}{dt}\langle \gamma'(t), N(\gamma(t))\rangle = \frac{d}{dt}0 = 0, \text{ for } t \in (-\epsilon, +\epsilon)$$

we have

$$\langle \gamma''(t), N(\gamma(t))\rangle + \langle \gamma'(t), dN_{\gamma(t)}(\gamma'(t))\rangle = 0$$

which is equivalent to

$$A_{\phi(\gamma(t))}(\gamma'(t), \gamma'(t)) = \langle \gamma''(t), N(\gamma(t))\rangle.$$

On the other hand,

$$\gamma'(t) = c\frac{\partial \phi}{\partial u}(ct, dt) + d\frac{\partial \phi}{\partial v}(ct, dt), \quad \gamma''(t) = c^2\frac{\partial^2 \phi}{\partial u^2}(ct, dt) + 2cd\frac{\partial^2 \phi}{\partial u \partial v}(ct, dt) + d^2\frac{\partial^2 \phi}{\partial v^2}(ct, dt).$$

Combining the above, at $t = 0$, we have

$$\begin{aligned} A_p &\left(c\frac{\partial \phi}{\partial u}(0,0) + d\frac{\partial \phi}{\partial v}(0,0), c\frac{\partial \phi}{\partial u}(0,0) + d\frac{\partial \phi}{\partial v}(0,0) \right) \\ &= \left\langle c^2\frac{\partial^2 \phi}{\partial u^2}(0,0) + 2cd\frac{\partial^2 \phi}{\partial u \partial v}(0,0) + d^2\frac{\partial^2 \phi}{\partial v^2}(0,0), N(p) \right\rangle. \end{aligned} \tag{10}$$

Proposition 11.1. *Let $S \subset \mathbb{R}^3$ be a regular surface, and $p \in S$. We have the following:*

- *If $K(p) > 0$, there is a neighbourhood $V \subset \mathbb{R}^3$ of p such that $S \cap V$ lies on the same side of $p + T_pS$,*
- *If $K(p) < 0$, on any neighbourhood $V \subset \mathbb{R}^3$ of p, $S \cap V$ meets both sides of $p + T_pS$.*

Proof. Let $\phi : U \to S$ be a chart for S at p, with $\phi(0,0) = p$. By the Taylor's theorem, near 0, we have

$$\begin{aligned} \phi(u,v) = \phi(0,0) &+ \left(\frac{\partial \phi}{\partial u}(0,0)u + \frac{\partial \phi}{\partial v}(0,0)v \right) \\ &+ \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2 \phi}{\partial u \partial v}(0,0)uv + \frac{\partial^2 \phi}{\partial v^2}(0,0)v^2 \right) + R(u,v), \end{aligned}$$

where

$$\lim_{(u,v)\to(0,0)} \frac{R(u,v)}{u^2+v^2} = 0.$$

Since the degree one term in the Taylor series expansion belongs to T_pS, we have

$$\begin{aligned}&\langle \phi(u,v)-p, N(p)\rangle\\ &\qquad = \frac{1}{2}\left\langle \frac{\partial^2\phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0,0)uv + \frac{\partial^2\phi}{\partial v^2}(0,0)v^2, N(p)\right\rangle + \langle R(u,v), N(p)\rangle\\ &\qquad = \frac{1}{2}A_p\left(\frac{\partial\phi}{\partial u}(0,0)u + \frac{\partial\phi}{\partial v}(0,0)v, \frac{\partial\phi}{\partial u}(0,0)u + \frac{\partial\phi}{\partial v}(0,0)v\right) + \langle R(u,v), N(p)\rangle.\end{aligned}$$

First assume that $K(p) > 0$. Without loss of generality we may assume that $\lambda_2(p) \geq \lambda_1(p) > 0$ (otherwise change N to $-N$). Then, by Lemma (10.1), for every $w \in T_p(s)$ we have $A_p(w,w) \geq \lambda_1(p)|w|^2$. Combining with the above equation, we conclude that for small values of u and v, $\langle \phi(u,v)-p, N(p)\rangle \geq 0$, which gives us the desired property in the first item.

If $K(p) < 0$, then $\lambda_1(p)$ and $\lambda_2(p)$ have different signs. Then, when w is one of the principle directions, then $A_p(w,w) = \lambda_1(p)|w|^2$ and $A_p(w,w) = \lambda_2(p)|w|^2$ takes both positive and negative values. This shows that $\langle \phi(u,v)-p, N(p)\rangle$ takes both values in any neighbourhood of $(0,0)$. $\square$

The above calculations suggest a simple interpretation of the principle curvatures of a surface at each point. Suppose by the change of coordinates in U and a rigid motion in the range $\mathbb{R}^3$, we have $\phi(0,0) = (0,0,0)$, and

$$\frac{\partial\phi}{\partial u}(0,0) = (1,0,0), \quad \frac{\partial\phi}{\partial v}(0,0) = (0,1,0)$$

are the principle directions of S at p. Then, we have a Taylor series

$$\phi(u,v) = (u,v,0) + \frac{1}{2}\left(\frac{\partial^2\phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0,0)uv + \frac{\partial^2\phi}{\partial v^2}(0,0)v^2\right) + \dots.$$

We would like to identify the last coordinate of the quadratic term. To that end, we note that

$$N(0,0,0) = \frac{\frac{\partial\phi}{\partial u}(0,0) \times \frac{\partial\phi}{\partial v}(0,0)}{\left|\frac{\partial\phi}{\partial u}(0,0) \times \frac{\partial\phi}{\partial v}(0,0)\right|} = (1,0,0)\times(0,1,0) = (0,0,1).$$

Then, by Equation (10), we have

$$\left\langle \frac{\partial^2\phi}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0,0)uv + \frac{\partial^2\phi}{\partial v^2}(0,0)v^2, N(0,0,0)\right\rangle = A_{(0,0,0)}\big((u,v,0),(u,v,0)\big).$$

The left hand side of the above equation gives us the last coordinate of the quadratic term, while the right hand side of the above equation, using the principle directions $(1,0,0)$ and $(0,1,0)$, becomes

$$A_{(0,0,0)}\big((u,v,0),(u,v,0)\big) = A_{(0,0,0)}\big(u(1,0,0)+v(0,1,0), u(1,0,0)+v(0,1,0)\big) = \lambda_1 u^2 + \lambda_2 v^2,$$

where λ_1 and λ_2 are the principle curvatures in the directions $(1,0,0)$ and $(0,1,0)$, respectively.

Combining the above equation, we get

$$\phi(u,v) = \left(u, v, \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2\right) + \text{ higher order terms.}$$

In other words, S is approximated near $(0,0,0)$ by the graph of the function

$$f(x,y) = \frac{1}{2}(\lambda_1 x^2 + \lambda_2 y^2).$$

Based on this, we give the following classification of points on a surface:

(i) $p \in S$ is called **elliptic** if $K(p) > 0$ (then λ_1 and λ_2 have the same sign),

(ii) $p \in S$ is called **hyperbolic** if $K(p) < 0$ (then λ_1 and λ_2 have opposite sign),

(iii) $p \in S$ is called **parabolic** if $K(p) = 0$ and $H(p) \neq 0$ (only one of λ_1 and λ_2 is zero),

(iv) $p \in S$ is called **planar** if $K(p) = H(p) = 0$ (both λ_1 and λ_2 are 0).

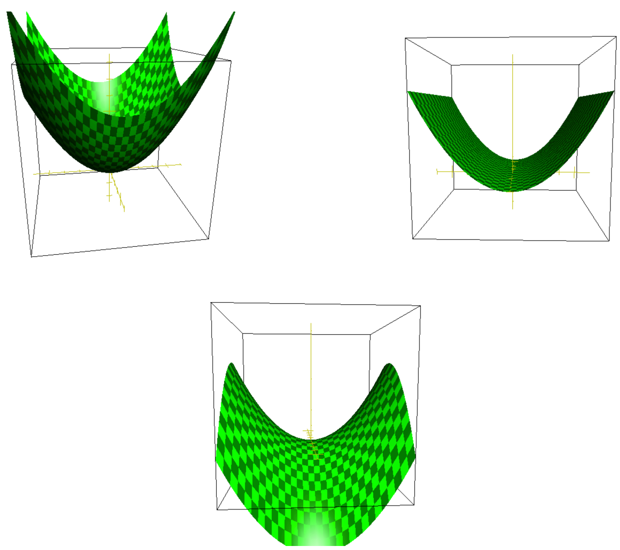


Figure 0.2: In a clockwise fashion starting from 1pm, origin is a parabolic, hyperbolic, and elliptic point, respectively.

12 Gaussian and mean curvatures in charts

Proposition 12.1. *Let $\phi : U \to S$ be a chart, and define the 2×2 matrices*

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} \qquad A = \begin{pmatrix} A_{\phi(u,v)}(\phi_u, \phi_u) & A_{\phi(u,v)}(\phi_u, \phi_v) \\ A_{\phi(u,v)}(\phi_v, \phi_u) & A_{\phi(u,v)}(\phi_v, \phi_v) \end{pmatrix}.$$

If we let $\sigma = g^{-1}A$, we have

$$K(\phi(u,v)) = \det(\sigma) = \frac{\det(A)}{\det(g)}, \qquad H(\phi(u,v)) = \frac{1}{2}\operatorname{tr}(\sigma).$$

Proof. If the partial derivatives ϕ_u and ϕ_v are the principle directions, then the formulae in the proposition are immediate. Below we discuss the general case.

Let us fix a basis for T_pS, and express $dN_{\phi(u,v)}$, ϕ_u, and ϕ_v in that basis, so that

$$A = \begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} \begin{pmatrix} | & | \\ -dN_{\phi(u,v)}(\phi_u) & -dN_{\phi(u,v)}(\phi_v) \\ | & | \end{pmatrix} = \begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} \Big(- dN_{\phi(u,v)} \Big) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}$$

Therefore,

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} \det \Big(- dN_{\phi(u,v)} \Big) \det \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \\ &= \lambda_1 \lambda_2 \det \left(\begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \right) \\ &= K \det(g). \end{aligned}$$

Alternatively, we may rewrite

$$\begin{aligned} A &= \begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}^{-1} \Big(- dN_{\phi(u,v)} \Big) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \\ &= g \cdot \left(\begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}^{-1} \Big(- dN_{\phi(u,v)} \Big) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix} \right). \end{aligned}$$

On the last line of the above equation g is multiplied by a matrix which is conjugate to $-dN_p$. Therefore, $-dN_{\phi(u,v)}$ is conjugate to the matrix $\sigma = g^{-1}A$, and hence

$$H = \frac{1}{2}\operatorname{tr}(\sigma) = \frac{\lambda_1 + \lambda_2}{2}.$$

□

Recall that

$$\frac{\partial}{\partial u} \left\langle \frac{\partial \phi}{\partial v}, N \right\rangle = \frac{\partial}{\partial u}(0) = 0,$$

and hence

$$\left\langle \frac{\partial^2 \phi}{\partial u \partial v}, N \right\rangle = - \left\langle \frac{\partial \phi}{\partial v}, dN\left(\frac{\partial \phi}{\partial u}\right) \right\rangle.$$

It follows that the matrix A in Proposition 12.1 satisfies

$$A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix}. \tag{11}$$

Example 12.2. We compute the curvature of the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}.$$

We have a chart $\phi : \mathbb{R}^2 \to S$ given by

$$\phi(u, v) = (u, v, u^2 - v^2).$$

Then,

$$\phi_u = (1, 0, 2u), \quad \phi_v = (0, 1, -2v)$$

and hence

$$g = \begin{pmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{pmatrix}.$$

To calculate A we note that

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-2u, 2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}$$

and

$$\phi_{uu} = (0, 0, 2), \quad \phi_{uv} = (0, 0, 0), \quad \phi_{vv} = (0, 0, -2).$$

Then, by Equation (11), we get

$$A = \begin{pmatrix} \frac{2}{\sqrt{1+4u^2+4v^2}} & 0 \\ 0 & \frac{-2}{\sqrt{1+4u^2+4v^2}} \end{pmatrix}.$$

Now,

$$\begin{aligned} \sigma = g^{-1} A &= \left(\frac{1}{1 + 4u^2 + 4v^2} \begin{pmatrix} 1 + 4v^2 & 4uv \\ 4uv & 1 + 4u^2 \end{pmatrix} \right) \left(\frac{2}{\sqrt{1 + 4u^2 + 4v^2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ &= \frac{2}{(1 + 4u^2 + 4v^2)^{3/2}} \begin{pmatrix} 1 + 4v^2 & -4uv \\ 4uv & -(1 + 4u^2) \end{pmatrix} \end{aligned}$$

Then,

$$K(\phi(u, v)) = \det(\sigma) = \frac{-4}{(1 + 4u^2 + 4v^2)^2},$$

$$H(\phi(u, v)) = \frac{-1}{2} \operatorname{tr}(\sigma) = \frac{4(v^2 - u^2)}{(1 + 4u^2 + 4v^2)^{3/2}}.$$

13 First fundamental form

Let $S \subset \mathbb{R}^3$ be a regular surface, and $p \in S$. The **first fundamental form/metric** at p is the bilinear map

$$g : T_pS \times T_pS \to \mathbb{R},$$

defined as

$$g(v, w) = \langle v, w\rangle.$$

In other words, the inner product on $\mathbb{R}^3$ is restricted to the tangent plane of S at p. If $\phi : U \to S$ is a chart for S at p, then T_pS is spanned by $\partial\phi/\partial u$ and $\partial\phi/\partial v$ at p. In this basis, g has the from

$$g\left(a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v}, c\frac{\partial\phi}{\partial u} + d\frac{\partial\phi}{\partial v}\rangle\right) = ac\left\langle \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial u}\right\rangle + (ad+bc)\left\langle \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right\rangle + bd\left\langle \frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial v}\right\rangle.$$

In a matrix form, the bilinear map g can be represented by the matrix

$$\begin{pmatrix} \langle\phi_u, \phi_u\rangle & \langle\phi_u, \phi_v\rangle \\ \langle\phi_v, \phi_u\rangle & \langle\phi_v, \phi_v\rangle \end{pmatrix}.$$

This is the same matrix that we called g in using $\sigma = g^{-1}A$ in Proposition 12.1. This first fundamental form/metric determines properties such as arc length completely. That is, let $\alpha : [a, b] \to S$ be a path contained in the image of a chart $\phi(U)$. Then, there are functions u and v such that $\alpha(t) = \phi(u(t), v(t))$, for $t \in [a, b]$. Then, α has length

$$\begin{aligned} \ell(\alpha([a, b])) = \int_a^b |\alpha'(t)|\, dt &= \int_a^b \left|\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial t}\right| dt \\ &= \int_a^b \sqrt{\begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \langle\phi_u, \phi_u\rangle & \langle\phi_u, \phi_v\rangle \\ \langle\phi_v, \phi_u\rangle & \langle\phi_v, \phi_v\rangle \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}}\, dt. \end{aligned}$$

Given regular surfaces S_1 and S_2, we say that a smooth map $F : S_1 \to S_2$ is a **local isometry**, if it preserves the first fundamental form, that is, for all $p \in S_1$ and all $X, Y \in T_pS_1$, we have

$$\langle dF_p(X), dF_p(Y)\rangle = \langle X, Y\rangle.$$

The map F is called an **isometry**, if it is a local isometry, and it is also a bijective map from S_1 to S_2.

Note that if F is a local isometry from S_1 to S_2, then $dF_p : T_pS_1 \to T_{F(p)}S_2$ is a bijective map. That is because, if $dF_p(v) = 0$, then $|v|^2 = \langle v, v\rangle = \langle dF_p(v), dF_p(v)\rangle = 0$, and hence $v = 0$. On the other hand, since T_pS_1 and $T_{F(p)}S_2$ have the same dimension, any injective map is also surjective. we have seen before in Proposition 7.4 that this makes F a local diffeomorphism near p. That is, there is a neighbourhood $U \subset S_1$ of p such that $F : U \to F(U)$ is a diffeomorphism.

Lemma 13.1. *Let S_1 and S_2 be regular surfaces, and $F : S_1 \to S_2$ be a smooth map. Then, F is a local isometry, if and only if, it preserves the lengths of curves, that is, for every smooth map $\alpha : [a, b] \to S_1$ we have*

$$\ell(\alpha([a, b])) = \ell(F \circ \alpha([a, b])).$$

Proof. Let us first assume that F is a local isometry. Using $(F \circ \alpha)'(t) = dF_{\alpha(t)}(\alpha'(t))$, we have

$$\begin{aligned}
\ell\left(F \circ \alpha([a, b])\right) = \int_a^b \left|(F \circ \alpha)'(t)\right| \, dt &= \int_a^b \left|dF_{\alpha(t)}\left(\alpha'(t)\right)\right| \, dt \\
&= \int_a^b \sqrt{\left\langle dF_{\alpha(t)}\left(\alpha'(t)\right), dF_{\alpha(t)}\left(a'(t)\right)\right\rangle} \, dt \\
&= \int_a^b \sqrt{\langle \alpha'(t), a'(t)\rangle} \, dt = \int_a^b \left|\alpha'(t)\right| \, dt = \ell(\alpha([a, b])).
\end{aligned}$$

In the above equation, to go from the second line to the third line, we have used that F is a local isometry.

Now assume that F preserves the lengths of smooth curves. Fix an arbitrary $p \in S_1$ and an arbitrary $v \in T_pS_1$. Choose an smooth curve $\alpha : (-\epsilon, +\epsilon) \to S_1$ satisfying $\alpha(0) = p$ and $\alpha'(0) = v$. Fix an arbitrary $\delta \in (0, \epsilon)$. By the assumption, for every $t \in [-\delta, \epsilon)$ we have

$$\ell(\alpha([-\delta, t]) = \ell(F \circ \alpha([-\delta, t])).$$

This implies that

$$\frac{d}{dt} \int_{-\delta}^t \left|\alpha'(s)\right| ds \bigg|_{t=0} = \frac{d}{dt} \int_{-\delta}^t \left|(F \circ \alpha)'(s)\right| ds \bigg|_{t=0}.$$

Hence, $|\alpha'(0)| = |(F \circ \alpha)'(0)|$, or $|v| = |dF_p(v)|$. Since p and v were chosen arbitrarily, we conclude that

$$|v| = |dF_p(v)|, \qquad \forall p \in S_1, \forall v \in T_pS_1.$$

Now, let X an Y be arbitrary vectors in T_pS_1. Using,

$$\langle X + Y, X + Y\rangle = \langle X, X\rangle + 2\langle X, Y\rangle + \langle Y, Y\rangle,$$

and by the above equation, we have

$$\begin{aligned}
\langle X, Y\rangle &= \frac{1}{2}\left(|X + Y|^2 - |X|^2 - |Y|^2\right) \\
&= \frac{1}{2}\left(|dF_p(X + Y)|^2 - |dF_p(X)|^2 - |dF_p(Y)|^2\right) \\
&= \frac{1}{2}\left(|dF_p(X) + dF_p(Y)|^2 - |dF_p(X)|^2 - |dF_p(Y)|^2\right) \\
&= \langle dF_p(X), dF_p(Y)\rangle.
\end{aligned}$$

By definition, this shows that F is a local isometry. □

Example 13.2. Let $S \subset \mathbb{R}^3$ be a regular surface, and $B \in \mathrm{SO}(3)$. Define the surface

$$S' = B(S) = \{Bv \mid v \in S\}$$

and the map

$$F : S \to S', \quad F(v) = Bv.$$

For any $p \in S$ and $v \in T_pS$, we have $dF_p(v) = Bv$, and hence, for every $v, w \in T_pS$, we have

$$\langle dF_p(v), dF_p(w)\rangle = \langle Bv, Bw\rangle = (Bv)^T(Bw) = v^T B^T BW = v^T w = \langle v, w\rangle.$$

In the above equation, we have used that $B^T = B^{-1}$. Thus, F is a local isometry, and since it is a bijective map (B is invertible), F is an isometry.

Example 13.3. Consider the regular surfaces

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}, \quad S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\},$$

and the map $F : S_1 \to S_2$ defined as

$$F(x, y, 0) = (\cos(x), \sin(x), y).$$

At an arbitrary point $p = (x_1, y_1, 0)$, a tangent vector $v = (x, y, 0)$ can be written as $v = \alpha'(0)$ where $\alpha(t) = (x_1 + tx, y_1 + ty, 0)$. Then, $\langle v, v\rangle = x^2 + y^2$, and we compute

$$\begin{aligned} dF_p(v) = \frac{d}{dt}(F \circ \alpha)\Big|_{t=0} &= \frac{d}{dt}(\cos(x_1 + tx), \sin(x_1 + tx), y_1 + ty)\Big|_{t=0} \\ &= (-x\sin(x_1 + tx), x\cos(x_1 + tx), y)\Big|_{t=0} \\ &= (-x\sin(x_1), x\cos(x_1), y)\,. \end{aligned}$$

Therefore,

$$\langle dF_p(v), dF_p(v)\rangle = x^2\sin^2(x_1) + x^2\cos^2(x_1) + y^2 = x^2 + y^2.$$

This shows that dF_p preserves the length of the tangent vectors. As in the proof of Lemma 13.1, using

$$\langle X + Y, X + Y\rangle = \langle X, X\rangle + 2\langle X, Y\rangle + \langle Y, Y\rangle,$$

we conclude that F is a local isometry. Note that here F is not an isometry since it is not a bijection.

14 Christoffel symbols

Let $S \subset \mathbb{R}^3$ be a regular surface, and $\phi : U \to S$ be a chart for S. We have defined two fundamental forms in the chart ϕ as

$$g = \left(\left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \right\rangle \right)_{i,j=1,2}, \qquad A = \left(\left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_j}, N \right\rangle \right)_{i,j=1,2},$$

where (x_1, x_2) is the coordinate of a point in U. Recall that $\partial\phi/\partial x_i$ and $\partial\phi/\partial x_j$ are linearly independent and span $T_{\phi(\cdot)}S$. As $N(\phi(\cdot))$ is orthogonal to those two vectors, the collection $\{\partial\phi/\partial x_i, \partial\phi/\partial x_j, N\}$ forms a basis for $\mathbb{R}^3$.

The **Christoffel symbols** $\Gamma^k_{i,j}$, for $i, j, k = 1, 2$, are defined according to

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \Gamma^1_{ij} \frac{\partial \phi}{\partial x_1} + \Gamma^2_{ij} \frac{\partial \phi}{\partial x_2} + A_{ij} N. \tag{12}$$

Note that

$$\left\langle N, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right\rangle = A_{ij} = A_{\phi(\cdot)} \left(\frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \right)$$

as suggested by the notation. Also, since $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$, we have

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \quad \forall i, j, k = 1, 2. \tag{13}$$

Proposition 14.1. *For any regular surface in $\mathbb{R}^3$, the Christoffel symbols are completely determined by the first fundamental form $g = (g_{ij})_{i,j=1,2}$ and its partial derivatives.*

Proof. By Equation (12), for $k = 1, 2$, we get

$$\begin{aligned} \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \frac{\partial \phi}{\partial x_k} \right\rangle &= \Gamma^1_{ij} \left\langle \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_k} \right\rangle + \Gamma^2_{ij} \left\langle \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_k} \right\rangle + A_{i,j} \left\langle N, \frac{\partial \phi}{\partial x_k} \right\rangle \\ &= \Gamma^1_{ij} g_{1k} + \Gamma^2_{ij} g_{2k}. \end{aligned} \tag{14}$$

On the other hand, for $i, j, k \in \{1, 2\}$, we have

$$\frac{\partial}{\partial x_i} \left\langle \frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial x_k} \right\rangle = \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \frac{\partial \phi}{\partial x_k} \right\rangle + \left\langle \frac{\partial \phi}{\partial x_j}, \frac{\partial^2 \phi}{\partial x_i \partial x_k} \right\rangle. \tag{15}$$

If we let $j = k$ in Equation (15), and then use Equation (14), we obtain

$$\frac{\partial}{\partial x_i} (g_{jj}) = 2 \left(\Gamma^1_{ij} g_{1j} + \Gamma^2_{ij} g_{2j} \right) \tag{16}$$

If we assume $i = j \neq k$ in Equation (15), we get

$$\begin{aligned} \frac{\partial}{\partial x_i} (g_{ik}) &= \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_i}, \frac{\partial \phi}{\partial x_k} \right\rangle + \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial^2 \phi}{\partial x_i \partial x_k} \right\rangle \\ &= \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_i}, \frac{\partial \phi}{\partial x_k} \right\rangle + \frac{1}{2} \frac{\partial}{\partial x_k} \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle \\ &= \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_i}, \frac{\partial \phi}{\partial x_k} \right\rangle + \frac{1}{2} \frac{\partial}{\partial x_k} (g_{ii}). \end{aligned} \tag{17}$$

Combining Equation (17) with Equation (14), we obtain

$$\Gamma_{ii}^1 g_{1k} + \Gamma_{ii}^2 g_{2k} = \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_i}, \frac{\partial \phi}{\partial x_k} \right\rangle = \frac{\partial}{\partial x_i}(g_{ik}) - \frac{1}{2}\frac{\partial}{\partial x_k}(g_{ii}) \quad (i \neq k). \tag{18}$$

Using Equations (16) and (18), we can consider the following three cases:

$$\begin{cases} i=j=k=1 & \implies \Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{21} = \frac{1}{2}\frac{\partial g_{11}}{\partial x_1} \\ i=j=1, k=2 & \implies \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22} = \frac{\partial g_{12}}{\partial x_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x_2} \end{cases} \iff g\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{\partial g_{11}}{\partial x_1} \\ \frac{\partial g_{12}}{\partial x_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x_2} \end{pmatrix}$$

$$\begin{cases} i=2, j=k=1 & \implies \Gamma_{21}^1 g_{11} + \Gamma_{21}^2 g_{21} = \frac{1}{2}\frac{\partial g_{11}}{\partial x_2} \\ i=1, j=k=2 & \implies \Gamma_{12}^1 g_{12} + \Gamma_{12}^2 g_{22} = \frac{1}{2}\frac{\partial g_{22}}{\partial x_1} \end{cases} \iff g\begin{pmatrix} \Gamma_{21}^1 \\ \Gamma_{21}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{\partial g_{11}}{\partial x_2} \\ \frac{1}{2}\frac{\partial g_{22}}{\partial x_1} \end{pmatrix}$$

$$\begin{cases} i=j=2, k=1 \implies \Gamma_{22}^1 g_{11} + \Gamma_{22}^2 g_{21} = \frac{\partial g_{21}}{\partial x_2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x_1} \\ i=j=k=2 \implies \Gamma_{22}^1 g_{12} + \Gamma_{22}^2 g_{22} = \frac{1}{2}\frac{\partial g_{22}}{\partial x_2} \end{cases} \iff g\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial g_{21}}{\partial x_2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x_1} \\ \frac{1}{2}\frac{\partial g_{22}}{\partial x_2} \end{pmatrix}$$

Since the matrix g is invertible (why), we can solve each of the matrix equations on the right hand sides for

$$\begin{pmatrix} \Gamma_{i,j}^1 \\ \Gamma_{i,j}^2 \end{pmatrix},$$

and the solution depends only on g and its partial derivatives. □

Example 14.2. We may parametrise the xy-plane in $\mathbb{R}^3$ by $\phi(u,v) = (u,v,0)$. Then,

$$\frac{\partial^2 \phi}{\partial u^2} = \frac{\partial^2 \phi}{\partial u \partial v} = \frac{\partial^2 \phi}{\partial v^2} = 0.$$

It follows that all Christoffel symbols are 0.

Example 14.3. We may parametrise the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ by $\psi : \mathbb{R}^2 \to \mathbb{R}^3$, defined as

$$\psi(u,v) = (\cos(u), \sin(u), v).$$

Then, we have a local isometry from the xy-plane to this cylinder given by $F(x,y,0) = (\cos(x), \sin(x), y)$, and $\psi = F \circ \phi$, where ϕ is the map in the previous example. The Christoffel symbols (which are determined by the metric) must be zero.

Alternatively, one can directly calculate the first and second partial derivatives of ψ and use Equation (12) to find the coefficients Γ_{ij}^k.

15 Theorema Egregium

Gauss in 1827 showed that the Gaussian curvature is an intrinsic quantity, i.e., it is invariant under isometries. This is a cornerstone in Geometry.

Theorem 15.1 (Theorema Egregium)**.** *The Gaussian curvature is an intrinsic quantity, i.e., it only depends on the first fundamental form.*

Remark 15.2. The mean curvature H is not an intrinsic notion because, for instance, the plane is locally isometric to a cylinder but the mean curvatures are different (for the plane $H = 0$ and for the cylinder of radius r, $H = 1/r$). It is remarkable that the trace of $-dN$ is not intrinsic while the determinant of $-dN$ is. Note that K is defined as the determinant of $g^{-1}A$, where g has intrinsic information only, while A has information about how the surface lies in space. Nonetheless, the determinant only has intrinsic information, i.e., can be computed knowing g only.

Proof. Let $\phi : U \to S$ be an arbitrary chart for S. Since partial derivatives of ϕ commute,

$$\frac{\partial}{\partial x_2}\left(\frac{\partial^2 \phi}{\partial x_1 \partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)$$

which in terms of the Christoffel symbols becomes

$$\frac{\partial}{\partial x_2}\left(\Gamma^1_{11}\frac{\partial \phi}{\partial x_1} + \Gamma^2_{11}\frac{\partial \phi}{\partial x_2} + A_{11}N\right) = \frac{\partial}{\partial x_1}\left(\Gamma^1_{12}\frac{\partial \phi}{\partial x_1} + \Gamma^2_{12}\frac{\partial \phi}{\partial x_2} + A_{12}N\right).$$

Let us use the notation $u = x_1$ and $v = x_2$, and expand the above derivatives to obtain

$$\begin{aligned}\left(\Gamma^1_{11}\phi_{uv} + (\Gamma^1_{11})_v\phi_u\right) + \left(\Gamma^2_{11}\phi_{vv} + (\Gamma^2_{11})_v\phi_v\right) + ((A_{11})_vN + A_{11}N_v)& \\ = \left(\Gamma^1_{12}\phi_{uu} + (\Gamma^1_{12})_u\phi_u\right) + \left(\Gamma^2_{12}\phi_{vu} + (\Gamma^2_{12})_u\phi_v\right) + ((A_{12})_uN + A_{12}N_u)&.\end{aligned} \tag{19}$$

Both sides of the above equation is a vector in $\mathbb{R}^3$, which at each point is a linear combination of the vectors ϕ_u, ϕ_v and N. Since these vectors are linearly independent, we can compare the coefficients of ϕ_u, ϕ_v and N on each side of the equation. Let us calculate the coefficients of ϕ_u on both sides of the equation. We have

$$\begin{aligned}\phi_{uv} &= \frac{\partial^2 \phi}{\partial x_2 \partial x_1} = \Gamma^1_{21}\phi_u + \Gamma^2_{21}\phi_v + A_{12}N\\ \phi_{vv} &= \frac{\partial^2 \phi}{\partial x_2 \partial x_2} = \Gamma^1_{22}\phi_u + \Gamma^2_{22}\phi_v + A_{22}N\\ \phi_{uu} &= \frac{\partial^2 \phi}{\partial x_1 \partial x_1} = \Gamma^1_{11}\phi_u + \Gamma^2_{11}\phi_v + A_{11}N\end{aligned}$$

On the other hand, let

$$N_u = a\phi_u + b\phi_v, \qquad N_v = c\phi_u + d\phi_v,$$

for some real numbers a, b, c and d. We need to find a and c. To that end, we note that

$$-A_{11} = -A(\phi_u, \phi_u) = \langle \phi_u, dN(\phi_u)\rangle = \left\langle \phi_u, \frac{\partial N \circ \phi}{\partial u} \right\rangle = \langle \phi_u, N_u \rangle = a g_{11} + b g_{12},$$
$$-A_{21} = -A(\phi_v, \phi_u) = \langle \phi_v, dN(\phi_u)\rangle = \left\langle \phi_v, \frac{\partial N \circ \phi}{\partial u} \right\rangle = \langle \phi_v, N_u \rangle = a g_{21} + b g_{22},$$
$$-A_{12} = -A(\phi_u, \phi_v) = \langle \phi_u, dN(\phi_v)\rangle = \left\langle \phi_u, \frac{\partial N \circ \phi}{\partial v} \right\rangle = \langle \phi_u, N_v \rangle = c g_{11} + d g_{12},$$
$$-A_{22} = -A(\phi_v, \phi_v) = \langle \phi_v, dN(\phi_v)\rangle = \left\langle \phi_v, \frac{\partial N \circ \phi}{\partial v} \right\rangle = \langle \phi_v, N_v \rangle = c g_{12} + d g_{22}.$$

We may rewrite the above four equations in the matrix form

$$-A = g \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

This implies that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -g^{-1} A = \left(\frac{-1}{g_{11} g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \right) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Therefore,

$$a = \frac{g_{12} A_{21} - g_{22} A_{11}}{g_{11} g_{22} - g_{12}^2}, \quad c = \frac{g_{12} A_{22} - g_{22} A_{12}}{g_{11} g_{22} - g_{12}^2}.$$

Now, comparing the coefficients of ϕ_u in Equation (19) we note that

$$\Gamma_{11}^1 \Gamma_{21}^1 + \left(\Gamma_{11}^1\right)_v + \Gamma_{11}^2 \Gamma_{22}^1 + A_{11} \left(\frac{g_{12} A_{22} - g_{22} A_{12}}{\det(g)} \right)$$
$$= \Gamma_{12}^1 \Gamma_{11}^1 + \left(\Gamma_{12}^1\right)_u + \Gamma_{12}^2 \Gamma_{12}^1 + A_{12} \left(\frac{g_{12} A_{21} - g_{22} A_{11}}{\det(g)} \right).$$

Using the symmetry in Equation (13), the above relation may be simplified to

$$\left(\Gamma_{11}^1\right)_v - \left(\Gamma_{12}^1\right)_u = \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 + \frac{(A_{12} A_{21} - A_{11} A_{22})\, g_{12}}{\det(g)}.$$

Using $K = (\det A)/(\det g)$, this give us

$$K g_{12} = \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 + \left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v. \tag{20}$$

Similarly, if we compare the coefficients of ϕ_v in Equation (19), we obtain

$$\Gamma_{11}^1 \Gamma_{21}^2 + \Gamma_{11}^2 \Gamma_{22}^2 + \left(\Gamma_{11}^2\right)_v + A_{11} \left(\frac{g_{21} A_{12} - g_{11} A_{22}}{\det(g)} \right)$$
$$= \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 + \left(\Gamma_{12}^2\right)_u + A_{12} \left(\frac{g_{21} A_{11} - g_{11} A_{21}}{\det(g)} \right).$$

This results in

$$Kg_{11} = \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + \left(\Gamma^2_{11}\right)_v - \left(\Gamma^2_{12}\right)_u . \tag{21}$$

Let X and Y denote the right hand sides of the Equations (20) and (21), respectively, so that $Kg_{12} = X$ and $Kg_{11} = Y$. This gives us,

$$K(g_{11}g_{22} - g_{12}^2) = Yg_{22} - Xg_{12},$$

and hence

$$K = \frac{Yg_{22} - Xg_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Since $\det(g) = g_{11}g_{22} - g_{12}^2$ is non-zero at every point, the above equation is well-defined.

Recall from Proposition 14.1 that the Christoffel symbols only depend on g and its derivative. Thus, X and Y only depend on g and its derivatives. Then, by the above equation K depends only on g and its derivatives. □

As an immediate corollary of Theorem 15.1 we obtain the following statements.

Corollary 15.3. *Assume that S_1 and S_2 are regular surfaces in $\mathbb{R}^3$ with Gaussian curvatures K_1 and K_2, respectively. If $F : S_1 \to S_2$ is a local isometry, then $K_2 \circ F = K_1$.*

Corollary 15.4. *There is no local isometry from a plane to a sphere.*

Proof. We have already seen that for a plane in $\mathbb{R}^3$ we have $K \equiv 0$, and for a sphere of radius r we have $K \equiv 1/r^2$. □

The above corollary can be rephrased as one cannot map the Earth onto a flat sheet of paper without distorting distances.

In fact, using a statement in the problem sheets, we can replace the sphere in the above corollary with any compact surface. We have seen that on any compact surface S, there is a point $p \in S$ such that $K(p) > 0$. Therefore, any local isometry from a plane to S must avoid p.

16 Surfaces of constant Gaussian curvature

Let us consider the Gaussian curvature as a function from a surface to the set of real numbers. We can ask what kind of behaviour this function may have. In general, this is a difficult question, so we look at simple cases. For example, can a compact surface have a constant curvature K? If $K \leq 0$ then the answer is no, since we saw in the problem sheets that on any compact surface there is a point where the Gaussian curvature is positive.

On the other hand, when $K = 0$, we already saw that there are several surfaces with zero Gaussian curvature which are not identical via rigid motions (i.e. they are not identical). For example, any plane and any cylinder has constant Gaussian curvature 0.

Theorem 16.1. *If S is a compact and connected regular surface which has a constant positive Gaussian curvature, S must be a sphere.*

Proof. Let $\lambda_1(x) \leq \lambda_2(x)$ be the principle curvatures at each $x \in S$. We aim to show that for all $x \in S$, we have $\lambda_1(x) = \lambda_2(x)$. Once we establish this, then every point in S is umbilical. By Proposition 10.6, we conclude that S is either contained in a sphere or contained in a plane. However, since S is compact, it may not be contained in a plane (why?), so it must be contained in a sphere. However, since S is compact, it must be equal to all of the sphere (why?). Below we show that $\lambda_1 = \lambda_2$ on S.

By the compactness of S, there is $p \in S$ such that

$$\lambda_2(p) = \max_{x \in S} \lambda_2(x).$$

Because $\lambda_1(x)\lambda_2(x) = K > 0$, we conclude that

$$\lambda_1(p) = \min_{x \in S} \lambda_1(x).$$

By applying a rigid motion of $\mathbb{R}^3$ and a translation, we may assume that $p = (0,0,0)$ and the principle directions at p are $X_1 = (1,0,0)$ and $X_2 = (0,1,0)$. Indeed, we saw in Section 11 that near p, S is the graph of a function of the form

$$F(u,v) = \frac{\lambda_1(p)u^2 + \lambda_2(p)v^2}{2} + \text{higher order terms.} \tag{22}$$

In particular, we have $F(0,0) = F_u(0,0) = F_v(0,0) = 0$. Using the form of the surface S as the graph of such a function, we aim to look at the curvature of S at points near p.

Let us consider a chart $\phi : U \to S$ of the form

$$\phi(u,v) = (u, v, F(u,v))$$

near p. We have

$$\phi_u = (1, 0, F_u), \quad \phi_v = (0, 1, F_v),$$
$$\phi_{uu} = (0,0,F_{uu}), \quad \phi_{uv} = (0,0,F_{uv}) \quad \phi_{vv} = (0,0,F_{vv}).$$

The unit normal vector to S near p is

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-F_u, -F_v, 1)}{\sqrt{1 + |\nabla F|^2}}.$$

Therefore, the second fundamental form is

$$A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix} = \frac{1}{\sqrt{1 + |\nabla F|^2}} \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix}.$$

At each $(u,v) \in U$, the vectors ϕ_u and ϕ_v belong to $T_{\phi(u,v)}S$. Let us normalise these vectors to

$$E_1(u,v) = \frac{\phi_u}{|\phi_u|} = \frac{(1, 0, F_u(u,v))}{\sqrt{1 + F_u^2(u,v)}}, \quad E_2(u,v) = \frac{\phi_v}{|\phi_v|} = \frac{(0, 1, F_v(u,v))}{\sqrt{1 + F_v^2(u,v)}}.$$

Then, consider the functions

$$
\begin{aligned}
h_1(t) = A_{\phi(0,t)}(E_1(0,t), E_1(0,t)) &= \frac{1}{1+F_u^2(0,t)} A_{\phi(0,t)}(\phi_u(0,t), \phi_u(0,t)) \\
&= \frac{1}{1+F_u^2(0,t)} \frac{F_{uu}(0,t)}{\sqrt{1+|\nabla F(0,t)|^2}},
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
h_2(t) = A_{\phi(t,0)}(E_2(t,0), E_2(t,0)) &= \frac{1}{1+F_v^2(t,0)} A_{\phi(t,0)}(\phi_v(t,0), \phi_v(t,0)) \\
&= \frac{1}{1+F_v^2(t,0)} \frac{F_{vv}(t,0)}{\sqrt{1+|\nabla F(t,0)|^2}}.
\end{aligned} \tag{24}
$$

Since, λ_1 is minimised at p and λ_2 is maximised at p, by Lemma 10.1, we have

$$
\lambda_1(p) \le \lambda_1((\phi(0,t)) = \min_{\{X \in T_{\phi(0,t)}, |X|=1\}} A_{\phi(0,t)}(X,X) \le A_{\phi(0,t)}(E_1(0,t), E_1(0,t)) = h_1(t),
$$

$$
\lambda_2(p) \ge \lambda_2(\phi(t,0)) = \max_{\{X \in T_{\phi(t,0)}, |X|=1\}} A_{\phi(t,0)}(X,X) \ge A_{\phi(t,0)}(E_2(t,0), E_2(t,0)) = h_2(t).
$$

These show that h_1 has a local minimum at $t = 0$, and h_2 has a local maximum at $t = 0$. In particular, we must have $h_1''(0) \ge 0$ and $h_2''(0) \le 0$, so

$$
h_1''(0) - h_2''(0) \ge 0. \tag{25}
$$

We may directly calculate $h_1''(0) - h_2''(0)$ using Equations (23) and (24). By Equation (22), we have

$$
\nabla F(u,v) = (\lambda_1(p)u, \lambda_2(p)v) + O(|u|^2 + |v|^2),
$$

and hence

$$
|\nabla F(u,v)|^2 = \lambda_1^2(p)u^2 + \lambda_2^2(p)v^2 + O(|u|^3 + |v|^3).
$$

Thus, by Equations (23), and using $(1+x)^{1/2} = 1 - x/2 + O(x^2)$, we obtain

$$
\begin{aligned}
h_1(t) &= \frac{1}{1+O(t^4)} \frac{F_{uu}(0,t)}{\sqrt{1+\lambda_2^2(p)t^2 + O(t^3)}} \\
&= \left(1 - O(t^4)\right)\left(1 - \frac{1}{2}\lambda_2^2(p)t^2 + O(t^3)\right) F_{uu}(0,t) \\
&= \left(1 - \frac{1}{2}\lambda_2^2(p)t^2\right) F_{uu}(0,t) + O(t^3).
\end{aligned} \tag{26}
$$

Similarly, for h_2 we have

$$
h_2(t) = \left(1 - \frac{1}{2}\lambda_1^2(p)t^2\right) F_{vv}(t,0) + O(t^3). \tag{27}
$$

Then,

$$\begin{aligned} h_1'(t) - h_2'(t) = &\left(\left(1 - \frac{\lambda_2^2(p)t^2}{2}\right) F_{uuv}(0,t) - \lambda_2^2(p) t F_{uu}(0,t) + O(t^2) \right) \\ &- \left(\left(1 - \frac{\lambda_1^2(p)t^2}{2}\right) F_{vvu}(t,0) - \lambda_1^2(p) t F_{vv}(t,0) + O(t^2) \right). \end{aligned}$$

and hence

$$\begin{aligned} &h_1''(t) - h_2''(t) \\ &\quad = \left(\left(1 - \frac{\lambda_2^2(p)t^2}{2}\right) F_{uuvv}(0,t) - 2\lambda_2^2(p) t F_{uuv}(0,t) - \lambda_2^2(p) F_{uu}(0,t) + O(t) \right) \\ &\qquad - \left(\left(1 - \frac{\lambda_1^2(p)t^2}{2}\right) F_{vvuu}(0,t) - 2\lambda_1^2(p) t F_{vvu}(0,t) - \lambda_1^2(p) F_{vv}(0,t) + O(t) \right). \end{aligned}$$

Thus, at $t = 0$, we obtain

$$\begin{aligned} h_1''(0) - h_2''(0) &= \lambda_1^2(p) F_{vv}(0,0) - \lambda_2^2(p) F_{uu}(0,0) \\ &= \lambda_1^2(p)\lambda_2(p) - \lambda_2^2(p)\lambda_1(p) \\ &= \lambda_1(p)\lambda_2(p)(\lambda_1(p) - \lambda_2(p)) \\ &= K(p)(\lambda_1(p) - \lambda_2(p)). \end{aligned}$$

Now, combining the above equation with the inequality in (25), we must have $K(p)(\lambda_1(p) - \lambda_2(p)) \geq 0$, and since $K(p) = K > 0$, we get $\lambda_1(p) - \lambda_2(p) \geq 0$. Since, $\lambda_1(p) \leq \lambda_2(p)$ by our assumption, we must have

$$\lambda_1(p) = \lambda_2(p).$$

Because λ_1 attains its minimum at p and λ_2 attains its maximum at p, for every $x \in S$ we have

$$\lambda_1(p) \leq \lambda_1(x) \leq \lambda_2(x) \leq \lambda_2(p).$$

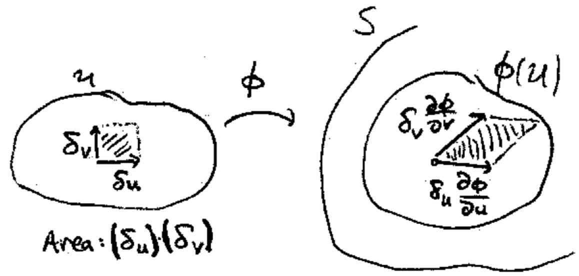
Therefore, $\lambda_1(x) = \lambda_2(x)$, for all $x \in S$. □

17 Area of a surface

Let $S \subset \mathbb{R}^3$ be a regular surface, and $\phi : U \to S$ be a chart. A small rectangle in U with sides $(\delta_u, 0)$ and $(0, \delta_v)$ has area $\delta_u \delta_v$. If δ_u and δ_v are small positive numbers, then the image of this rectangle in $\phi(U) \subset S$ is nearly a parallelogram with sides $\delta_u \phi_u$ and $\delta_v \phi_v$. Then, the area of that parallelogram is approximately

$$|\delta_u \phi_u \times \delta_v \phi_v| = \delta_u \delta_v |\phi_u \times \phi_v|.$$

If $D \subset U$ is compact, we define the **area** of $\phi(D)$ as



$$\text{area}(\phi(D)) = \int_D |\phi_u \times \phi_v| \; dudv.$$

Proposition 17.1. *Let S be a regular surface, $\phi : U \to S$ be a chart for S, and $D \subset U$ be a compact set. The area of $\phi(D)$ does not depend on the choice of the coordinate chart ϕ, but only on the set $\phi(D)$.*

Proof. Let us consider arbitrary charts $\phi : U \to S$ and $\psi : U' \to S$, as well as compact sets $D \subset U$ and $D' \subset U'$ such that $\phi(D) = \psi(D')$. Consider the map

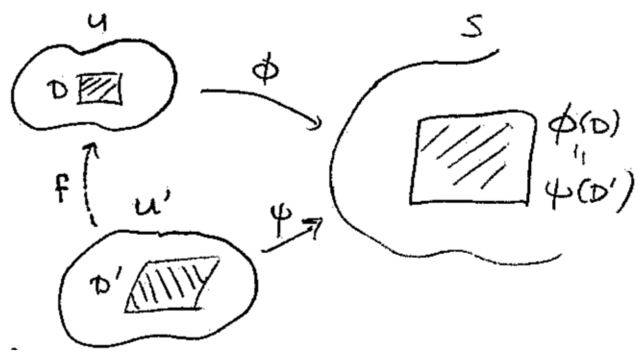
$$f = \phi^{-1} \circ \psi : U' \to U,$$

and assume that in coordinates (u, v) on U', it is defined as

$$f(u, v) = (x(u, v), y(u, v)).$$

Then, $\psi = \phi \circ f$, and hence

$$\begin{aligned} \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} &= \left(\frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} \right) \times \left(\frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} \right) \\ &= \left(\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right) \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial u} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right). \end{aligned}$$



Therefore, by a change of coordinates in the integral, we have

$$\begin{aligned}\text{area}\big(\psi(D')\big) = \int_{D'} \left|\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v}\right| dudv &= \int_{D'} \left|\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}\right| \det \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} dudv \\ &= \int_{D} \left|\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y}\right| dxdy = \text{Area}(\phi(D)).\end{aligned}$$

In the above equation we have assumed that f is orientation preserving, that is the determinant in the integral is non-negative. Otherwise, one needs to consider the modulus of the determinant. ☐

Lemma 17.2. *Let S be a regular surface, $\phi : U \to S$ be a chart for S, and $D \subset U$ be compact set. We have*

$$\text{area}(\phi(D)) = \int_D \sqrt{\det(g)}\, dudv,$$

where g denotes the matrix of the first fundamental form of S in the chart ϕ.

Proof. Let v and w be arbitrary vectors in $\mathbb{R}^3$, with v non-zero. Then, v is orthogonal to the vector $w - v\langle v, w\rangle/|v^2|$, and hence, we have

$$\begin{aligned}|v \times w|^2 = \left|v \times \left(w - \frac{\langle v, w\rangle}{|v|^2} v\right)\right|^2 &= |v|^2 \left|w - \frac{\langle v, w\rangle}{|v|^2} v\right|^2 \\ &= |v|^2 \left(|w|^2 - 2\frac{\langle v, w\rangle}{|v|^2}\langle v, w\rangle + \left(\frac{\langle v, w\rangle}{|v|^2}\right)^2 |v|^2\right) = |v|^2|w|^2 - \langle v, w\rangle^2.\end{aligned}$$

Using the above formula for $v = \phi_u$ and $w = \phi_v$, we obtain

$$|\phi_u \times \phi_v|^2 = |\phi_u|^2|\phi_v|^2 - \langle \phi_u, \phi_v\rangle^2 = g_{11}g_{22} - g_{1,2}^2.$$ ☐

If $f : S \to \mathbb{R}$ is a smooth function on S, $\phi : U \to S$ is a chart, $D \subset U$ is compact, we define the **integral of f on $\phi(D)$** as

$$\int_{\phi(D)} f\, dA = \int_D f \circ \phi\, |\phi_u \times \phi_v|\; dudv = \int_D f \circ \phi \sqrt{\det(g)}\, dudv.$$

Again, this does not depend on ϕ (the proof is the same).

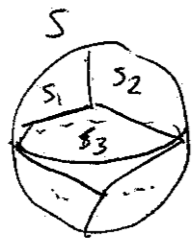
To integrate a function over all of S, we divide S into pieces:

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

such that the following hold:

- for all distinct pairs i, j, we have $S_i \cap S_j \subset \partial S_i \cap \partial S_j$,
- for each i there are a chart $\phi_i : U_i \to S$ and a compact set $D_i \subset U_i$ such that $S_i = \phi_i(D_i)$
- for every i, ∂D_i has zero area.

We are assuming that S is compact, so such a finite partition always exists. Then, we define



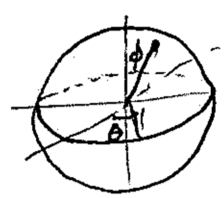
$$\int_S f\, dA = \sum_{i=1}^{k} \int_{S_i} f\, dA = \sum_{i=1}^{k} \int_{D_i} (f \circ \phi_i) \left| \frac{\partial \phi_i}{\partial u} \times \frac{\partial \phi_i}{\partial v} \right| dudv.$$

One can see that this is well-defined and does not depend on the decomposition.

Example 17.3. We can use the above formula to calculate the surface area of the unit sphere in $\mathbb{R}^3$. We may consider the chart on $\mathbb{S}^2$ minus some small neighbourhoods of the poles, defined as

$$\Psi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi), \qquad 0 \le \theta \le 2\pi, \epsilon < \phi < \pi - \epsilon,$$

for some small positive epsilon.



Then,

$$\frac{\partial \Psi}{\partial \theta}(\theta, \phi) = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$\frac{\partial \Psi}{\partial \phi}(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

and hence

$$\left(\frac{\partial \Psi}{\partial \theta} \times \frac{\partial \Psi}{\partial \phi}\right)(\theta, \phi) = \left(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\right)$$

and hence,

$$\left|\frac{\partial \Psi}{\partial \theta} \times \frac{\partial \Psi}{\partial \phi}\right|(\theta, \phi) = \sin\phi.$$

Then,

$$\begin{aligned}
\text{area}\left(\mathbb{S}^2 \setminus \epsilon - \text{neighbourhood of poles}\right) &= \int_\epsilon^{\pi-\epsilon} \int_0^{2\pi} \sin\phi \, d\theta d\phi \\
&= 2\pi \int_\epsilon^{\pi-\epsilon} \sin\phi \, d\phi \\
&= 2\pi(2\cos(\epsilon)).
\end{aligned}$$

As $\epsilon \to 0$, the area of the ϵ-neighbourhoods of the poles tends to 0, and we obtain

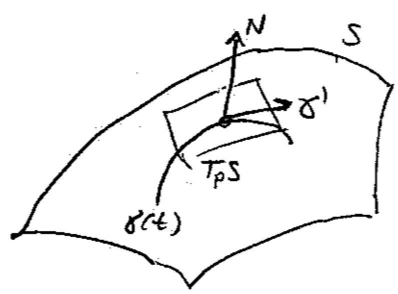
$$\text{area}\left(\mathbb{S}^2\right) = 4\pi.$$

18 Geodesics, and the geodesic curvature

Let $\gamma : [a, b] \to S \subset \mathbb{R}^3$ be a regular curve in a regular surface S, and N be a unit normal vector on S. Then, γ' is tangent to S and is orthogonal to N, so if γ is parametrised by arc-length, then

$$\left\{\gamma', N \times \gamma', N\right\}$$

is an orthonormal basis for $\mathbb{R}^3$.



Recall that the curvature vector $\vec{k} = \gamma''(t)$ is orthogonal to γ' at $\gamma(t)$. Then,

$$\begin{aligned}
\vec{k} &= \langle \vec{k}, N \rangle N + \langle \vec{k}, N \times \gamma' \rangle N \times \gamma' \\
&= k_n N + k_g (N \times \gamma')
\end{aligned}$$

In the above equation, k_n is the normal curvature of γ at $\gamma(t)$.

The number

$$k_g = \langle \vec{k}, N \times \gamma' \rangle$$

is called the **geodesic curvature** of γ. Note that in defining this, we have assume that γ is parametrised by arc-length.

If γ has curvature k, then

$$|\vec{k}|^2 = \left| k_n N + k_g \left(N \times \gamma' \right) \right|^2 = k_n^2 + k_g^2.$$

That is because, N and $N \times \gamma'$ are orthogonal and have unit size. In other words,

$$k^2 = k_n^2 + k_g^2.$$

If $k_g = 0$ then $\vec{k} = k_n N$, which means that $\gamma(t)$ curves only in directions orthogonal to S. If $\gamma : [a, b] \to S$ has $k_g \equiv 0$, then γ is called a **geodesic**.

Example 18.1. Let S be the xy-plane in $\mathbb{R}^3$, $N = (0, 0, 1)$, and $\gamma(t) = (x(t), y(t), 0)$ is parametrised by arc-length, that is, $x'^2 + y'^2 \equiv 1$. Then,

$$k_n(t) = \langle \gamma''(t), N(t) \rangle = \big\langle (x'', y'', 0), (0, 0, 1) \big\rangle = 0,$$

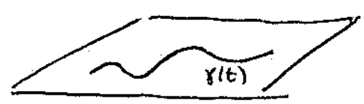
and hence

$$\vec{k} = k_g (N \times \gamma')$$

and hence

$$\big(x'', y'', 0 \big) = k_g (-y', x', 0).$$

Thus, $k_g = 0$ if $x'' = y'' = 0$. In other words, the geodesics in S are precisely the straight lines.

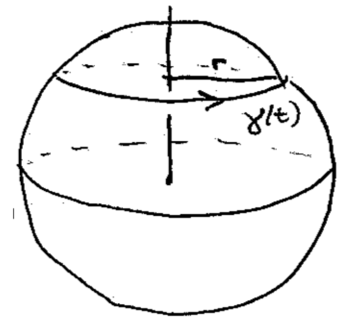


Example 18.2. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, and let γ be a circle of radius r parallel to the xy-plane, that is,

$$\gamma(t) = \left(r \cos(t/r), r \sin(t/r), \sqrt{1 - r^2} \right)$$

This curve is parametrised by are length. We have

$$\begin{aligned} N \times \gamma'(t) &= \left(r \cos(t/r), r \sin(t/r), \sqrt{1 - r^2} \right) \times (-\sin(t/r), \cos(t/r), 0) \\ &= \left(-\sqrt{1 - r^2} \cos(t/r), -\sqrt{1 - r^2} \sin(t/r), r \right) \end{aligned}$$



and

$$\gamma''(t) = \left(-\frac{1}{r}\cos\left(\frac{t}{r}\right), -\frac{1}{r}\sin\left(\frac{t}{r}\right), 0\right)$$

Then

$$k_g(t) = \big\langle \gamma''(t), N(\gamma(t)) \times \gamma'(t) \big\rangle = \frac{\sqrt{1-r^2}}{r}.$$

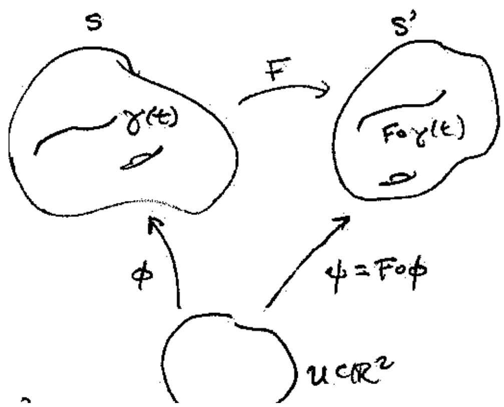
Thus γ is a geodesic ($k_g = 0$) precisely when $r = 1$, i.e. γ is a great circle (equator).

Proposition 18.3. *Any local isometry of regular surfaces in $\mathbb{R}^3$ maps geodesics to geodesics.*

Proof. Let $\gamma : [a, b] \to S$ be a geodesic parametrised by are-length, and $F : S \to S'$ be a local isometry. From

$$\vec{k}(t) = k_n N(\gamma(t)) + k_g (N(\gamma(t)) \times \gamma'(t))$$

we see that γ is a geodesic iff $\gamma''(t)$ is a multiple of $N(\gamma(t))$. In any chart $\phi : U \to S$, that is



equivalent to

$$\langle \gamma'', \phi_u \rangle = \langle \gamma'', \phi_v \rangle = 0.$$

Let us write $\gamma(t) = \phi(u(t), v(t))$. Then,

$$\gamma'(t) = \phi_u(u(t), v(t))u'(t) + \phi_v(u(t), v(t))v'(t),$$

and hence

$$\begin{aligned}\gamma''(t) = &\left(\phi_{uu}(u(t),v(t))u'(t) + \phi_{uv}(u(t),v(t))v'(t)\right)u'(t)\\ &+ \left(\phi_{vu}(u(t),v(t))u'(t) + \phi_{vv}(u(t),v(t))v'(t)\right)v'(t)\\ &+ \phi_u(u(t),v(t))u''(t) + \phi_v(u(t),v(t))v''(t).\end{aligned}$$

Therefore,

$$\begin{aligned}0 = \langle \gamma'', \phi_u\rangle &= \langle \phi_u, \phi_{uu}\rangle (u')^2 + 2\langle \phi_u, \phi_{uv}\rangle \left(u'v'\right) + \langle \phi_u, \phi_{vv}\rangle \left(v'\right)^2 + g_{11}u'' + g_{12}v''\\ &= \frac{1}{2}\frac{\partial}{\partial u}\langle \phi_u, \phi_u\rangle \left(u'\right)^2 + \frac{\partial}{\partial v}\langle \phi_u, \phi_u\rangle \left(u'v'\right) + \left(\frac{\partial}{\partial v}\langle \phi_u, \phi_v\rangle - \frac{1}{2}\frac{\partial}{\partial u}\langle \phi_v, \phi_v\rangle\right)\left(v'\right)^2\\ &\qquad + g_{11}u'' + g_{12}v''\\ &= \frac{1}{2}\frac{\partial}{\partial u}g_{11}(u')^2 + \frac{\partial}{\partial v}g_{11}(u'v') + \left(\frac{\partial}{\partial v}g_{12} - \frac{1}{2}\frac{\partial}{\partial u}g_{22}\right)\left(v'\right)^2 + g_{11}u'' + g_{12}v''.\end{aligned}$$

There is a similar relation for $\langle \gamma'', \phi_v\rangle$. These are determined by u, v and the first fundamental form g.

Since F is a local isometry, it preserves g with respect to the chart $\psi = F \circ \phi$, and $F(\gamma(t)) = \psi(u(t), v(t))$ implies that

$$\langle (F\circ\gamma)'', \psi_u\rangle = \langle \gamma'', \phi_u\rangle = 0$$

and

$$\langle (F\circ\gamma)'', \psi_v\rangle = \langle \gamma'', \phi_v\rangle = 0.$$

□

Exercise: For any curve $\gamma(t) = \phi(u(t), v(t))$, use the relations

$$\begin{aligned}\langle \gamma''(t), \phi_u(u(t),v(t))\rangle &= \left(\langle \gamma'(t), \phi_u(u(t),v(t))\rangle\right)' - \langle \gamma'(t), (\phi_u(u(t),v(t)))'\rangle\\ \langle \gamma''(t), \phi_v(u(t),v(t))\rangle &= \left(\langle \gamma'(t), \phi_v(u(t),v(t))\rangle\right)' - \langle \gamma'(t), (\phi_v(u(t),v(t)))'\rangle\end{aligned}$$

to derive the geodesic equations

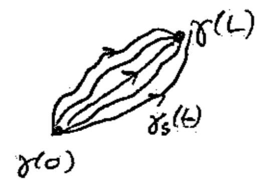
$$\begin{aligned}\left(g_{11}u' + g_{12}v'\right)' &= \frac{1}{2}\Big((g_{11})_u (u')^2 + 2(g_{12})_u (u'v') + (g_{22})_u (v')^2\Big),\\ \left(g_{21}u' + g_{22}v'\right)' &= \frac{1}{2}\Big((g_{11})_v (u')^2 + 2(g_{12})_v (u'v') + (g_{22})_v (v')^2\Big).\end{aligned}$$

19 Geodesics minimise arc length

Let S be a regular surface in $\mathbb{R}^3$, and $\gamma : [0, L] \to S$ be a smooth map. A **variation** of the curve γ is a smooth map from $[0, L] \times [-\epsilon, +\epsilon]$ to S, say

$$(t, s) \mapsto \gamma_s(t)$$

such that $\gamma_0 \equiv \gamma$, and for all $s \in [-\epsilon, +\epsilon]$ we have $\gamma_s(0) = \gamma(0)$ and $\gamma_s(L) = \gamma(L)$.



Proposition 19.1. *For any regular surface $S \subset \mathbb{R}^3$, if $\gamma : [0, L] \to S$ is parametrised by arc-length and is the shortest regular curve from $\gamma(0)$ to $\gamma(L)$, then γ is a geodesic.*

Proof. Let γ_s be a variation of γ. Because $s \mapsto \ell(\gamma_s)$ is minimised at $s = 0$, the derivative of this map must be zero at $s = 0$. Let us look at this derivative in more details.

$$\frac{d}{ds}\ell(\gamma_s)\Big|_{s=0} = \frac{d}{ds}\int_0^L \left|\gamma_s'(t)\right| \, dt\Big|_{s=0} = \frac{d}{ds}\int_0^L \sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\, dt\Big|_{s=0}$$

Since the map $(t, s) \mapsto \sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}$ is smooth in s and t, we can move the derivative inside the integral, and obtain

$$\frac{d}{ds}\int_0^L \sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\, dt\Big|_{s=0} = \int_0^L \frac{d}{ds}\sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\Big|_{s=0} dt = \int_0^L \frac{\frac{d}{ds}\langle\gamma_s'(t), \gamma_s'(t)\rangle\Big|_{s=0}}{2\sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\Big|_{s=0}}\, dt.$$

Because γ is parametrised by arc-length, we have $\sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\Big|_{s=0} \equiv 1$, we can continue our calculations to get

$$\begin{aligned}
\int_0^L \frac{\frac{d}{ds}\langle\gamma_s'(t), \gamma_s'(t)\rangle\Big|_{s=0}}{2\sqrt{\langle\gamma_s'(t), \gamma_s'(t)\rangle}\Big|_{s=0}}\, dt &= \int_0^L \frac{1}{2}\frac{d}{ds}\left(\gamma_s'(t), \gamma_s'(t)\right)\Big|_{s=0} dt \\
&= \int_0^L \left\langle \frac{d}{ds}\left(\frac{d\gamma_s(t)}{dt}\right)_{s=0}, \frac{d\gamma_0(t)}{dt}\right\rangle dt \\
&= \int_0^L \left\langle \frac{d}{dt}\left(\frac{d\gamma_s(t)}{ds}\right)_{s=0}, \frac{d\gamma_0(t)}{dt}\right\rangle dt.
\end{aligned}$$

Now, we can use integration by parts, to calculate the above integral. That is,

$$\begin{aligned}
&\int_0^L \left\langle \frac{d}{dt}\left(\frac{d\gamma_s(t)}{ds}\right)_{s=0}, \frac{d\gamma_0(t)}{dt}\right\rangle dt \\
&\qquad = \left[\left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, \frac{d\gamma_0(t)}{dt}\right\rangle\right]_{t=0}^{t=L} - \int_0^L \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, \gamma_0''(t)\right\rangle dt.
\end{aligned}$$

Now, since $\gamma_s(0) = \gamma(0)$ and $\gamma_s(L) = \gamma(L)$, $\frac{d\gamma_s(t)}{ds}\Big|_{s=0}$ is equal to zero at $t = 0$ and $t = L$. Also, we have

$$\gamma_0'' \equiv \gamma'' = \big(k_n(t)N(\gamma(t)) + k_g(t)(N(\gamma(t)) \times \gamma'(t)\big).$$

Therefore, the right hand side of the above equation becomes,

$$
\begin{aligned}
-\int_0^L \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, \gamma''(t) \right\rangle dt \\
&= -\int_0^L \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, \big(k_n(t)N(\gamma(t)) + k_g(t)(N(\gamma(t)) \times \gamma'(t))\big) \right\rangle dt
\end{aligned}
$$

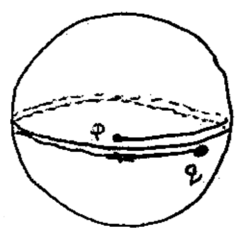
Now, we note that $\frac{d\gamma_s(t)}{ds}\Big|_{s=0}$ belongs to the tangent space of S at $\gamma(t)$, and hence the inner product with $N(\gamma(t))$ must be 0. Then, overall, we end of up

$$\frac{d}{ds}\ell(\gamma_s)\Big|_{s=0} = -\int_0^L k_g(t) \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, N(\gamma(t)) \times \gamma'(t) \right\rangle dt.$$

The above relation holds for any variation γ_s of γ. If $k_g(t) \neq 0$ for some $t \in [0, L]$, we can find a variation γ_s which is constant (in s) away from a small neighbourhood of t, and $\frac{d\gamma_s(t)}{ds}\Big|_{(s,t)=(0,t)}$ is in the direction of $N \times \gamma'$. That implies that the right hand side of the above equation is non-zero, which is a contradiction. Therefore, we must have $k_g(t) = 0$ for all $t \in [0, L]$. □

In general, we cannot explicitly describe geodesics on an arbitrary surface, even if they exist.

Note that geodesics may not minimise the arc-length globally. For example on the unit sphere in $\mathbb{R}^3$, a curve which wraps around the equator several times is a geodesic, but it is not the shortest path from p to q.



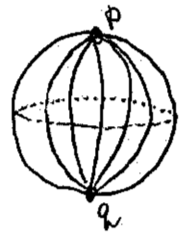
On the other hand, geodesics between arbitrary points on a surface may not exist. For example, if we remove the point $(0, 0)$ from the xy-plane, then there is no geodesic between the points $p = (-1, 0)$ and $q = (1, 0)$. Also, when a geodesic between a pair of points exist, it may not be unique.

Example 19.2. Let S be the cylinder $x^2 + y^2 = 1$. We have a local isometry F from the xy-plane to S defined as

$$F(x, y) = (\cos(x), \sin(x), y).$$

We know that geodesics on the xy-plane are straight lines. Since isometries preserve geodesics, the geodesics on S are precisely the curves of the form

$$\gamma(t) = F(at + b, ct + d) = (\cos(at + b), \sin(at + b), ct + d).$$



where $a^2 + c^2 = 1$, which guarantees that γ is parametrised by arc-length.

20 Local version of Gauss-Bonnet

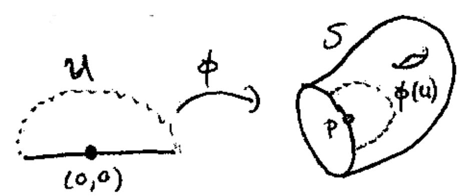
A **regular surface with boundary** is a set $S \subset \mathbb{R}^3$ such that for every $p \in S$, either

(i) there is a chart $\phi : U \to S$ for S at p, or

(ii) there is an open neighbourhood $U \subset \mathbb{R}^2$ of $(0,0)$, an open neighbourhood $V \subset \mathbb{R}^3$ of p, and a smooth map $\phi : U \to V$ satisfying the following:

- $\phi(0,0) = p$,
- $\phi : \{(x,y) \in U \mid y \geq 0\} \to V \cap S$ is a homeomorphism,
- for every $q \in U$, $d\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

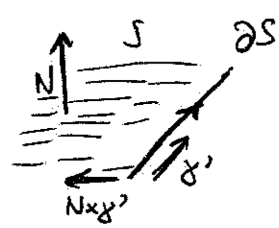
In other words, a regular surface with boundary, is a regular surface, except that at some points on S, we have charts defined on a half disk with a boundary curve. We refer to the map ϕ in each of the above cases as a chart for S. See figure below.



In the above definition, any point satisfying item (i) is called an **interior point** of S, and any point satisfying item (ii) is called a **boundary point** of S. The set of all points satisfying item (ii) is called the **boundary** of S and is denoted by ∂S. It is easy to see that ∂S is a union of regular curves.

For any regular surface with boundary, we may define the tangent space at each point of S using the same approach we employed for regular surfaces. That is, the collection of all vectors $\alpha'(0)$ such that either $\alpha : [0, \epsilon) \to S$, or $\alpha : (-\epsilon, 0] \to S$, is a smooth map with $\alpha(0) = p$. We still have $T_pS = d\phi_{(0,0)}(\mathbb{R}^2)$, even when p is a boundary point. Thus, we can define unit normal vectors at boundary points as well.

Any orientation for S induces a unique orientation on ∂S. That is, for a choice of a unit normal vector $N : S \to \mathbb{R}^3$, and a parametrisation $\gamma : [a,b] \to \partial S$, we say that γ is positively oriented, if $N \times \gamma'$ points into S. In other words, S is to the left of ∂S. In this case, we say



∂S is **positively oriented** (i.e. the orientation of S is consistent with the orientation of ∂S).

We are now ready to state a local version of a remarkable result in geometry. Below, the notation $\overline{U}$ refers to the (topological) closure of a set U.

Theorem 20.1 (Local version of Gauss-Bonnet)**.** *Let S be a regular surface with boundary, and assume that $\phi : U \subset \mathbb{R}^2 \to S$ is a chart such that*

(i) ϕ is smooth on a neighbourhood of $\overline{U}$, and $\overline{U}$ is diffeomorphic to a closed disc,

(ii) $S = \phi(\overline{U})$ and $\partial S = \phi(\partial U)$.

Then

$$\int_{\partial S} k_g \, ds + \int_S K \, dA = 2\pi,$$

where ∂S is positively oriented.

One may compare the formula in the above theorem to the one in 4.4. That is, when the surface lies in a plane, the curvature is 0, and the formula in the above theorem implies the result about the plane curves.

The proof of the above theorem is rather long, but the basic idea is that we use the coordinate ϕ to bring the problem to the Euclidean setting, where we use the Green's integral formula to relate the integral on a region to an integral on the boundary of that region.

Proof. Let us consider the maps $E_1 : \overline{U} \to \mathbb{R}^3$ and $E_2 : \overline{U} \to \mathbb{R}^3$ defined as

$$E_1(u, v) = \frac{\phi_u(u, v)}{|\phi_u(u, v)|}, \qquad E_2(u, v) = N(\phi(u, v)) \times E_1(u, v).$$

Then, for every $(u, v) \in \overline{U}$, $E_1(u, v)$ and $E_2(u, v)$ form a basis for the tangent plane $T_{\phi(u,v)}S$. Also, $\{E_1, E_2, N\}$ forms an orthonormal basis for $\mathbb{R}^3$ at each point. Note that $N = E_1 \times E_2$.

Let $\gamma : [0, L] \to \partial S$ be a parametrisation of ∂S by arc-length, and with positive orientation. There is a regular curve $\sigma : [0, L] \to \partial U$ such that $\gamma = \phi \circ \sigma$.

There is a smooth function $\theta : [0, L] \to \mathbb{R}$ such that

$$\gamma'(t) = \cos \theta(t) E_1(\sigma(t)) + \sin \theta(t) E_2(\sigma(t)) \in T_{\gamma(t)}S. \tag{28}$$

We break the proof into the following 3 steps:

F1: for every $t \in [0, L]$ we have

$$k_g(\gamma(t)) = \theta'(t) - \langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \rangle,$$

F2: we have

$$\int_0^L \langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \rangle dt = \int_U (K \circ \phi) |\phi_u \times \phi_v| \, du dv.$$

F3: we have

$$\int_0^L \theta'(s) ds = 2\pi,$$

Putting (F1), (F2), and (F3) together, we obtain

$$\begin{aligned}
\int_{\partial S} k_g\, ds + \int_S K\, dA &= \int_0^L k_g(\gamma(t))\, dt + \int_U (K\circ\phi)|\phi_u \times \phi_v|\, dudv \\
&= \int_0^L \theta'(t)\, dt - \int_0^L \langle E_1\circ\sigma(t), (E_2\circ\sigma)'(t)\rangle\, dt + \int_U (K\circ\phi)|\phi_u\times\phi_v|\, dudv \\
&= \int_0^L \theta'(t)\, dt \\
&= 2\pi.
\end{aligned}$$

Below we prove the features F1, F2, and F3.

Proof of F1: By differentiating Equation (28) with respect to t, we obtain

$$\gamma''(t) = \theta'(t)\Big(-\sin\theta(t)E_1(\sigma(t)) + \cos\theta(t)E_2(\sigma(t))\Big) + \cos\theta(t)(E_1\circ\sigma)'(t) + \sin\theta(t)(E_2\circ\sigma)'(t).$$

We also have

$$N(\gamma)\times\gamma' = N(\gamma)\times(\cos\theta(E_1\circ\sigma) + \sin\theta(E_2\circ\sigma)) = -\sin\theta(E_1\circ\sigma) + \cos\theta(E_2\circ\sigma).$$

Differentiating the relations $\langle E_1\circ\sigma, E_1\circ\sigma\rangle = \langle E_2\circ\sigma, E_2\circ\sigma\rangle = 1$ and $\langle E_1\circ\sigma, E_2\circ\sigma\rangle = 0$ with respect to t, we obtain the relations

$$\langle E_1\circ\sigma, (E_1\circ\sigma)'\rangle = \langle E_2\circ\sigma, (E_2\circ\sigma)'\rangle = 0 \qquad \langle E_1\circ\sigma, (E_2\circ\sigma)'\rangle = -\langle (E_1\circ\sigma)', E_2\circ\sigma\rangle.$$

Therefore,

$$\begin{aligned}
k_g(\gamma(t)) &= \big\langle \gamma''(t), N(\gamma(t))\times\gamma'(t)\big\rangle \\
&= \big\langle \gamma''(t), -\sin\theta(t)E_1\circ\sigma(t) + \cos\theta(t)E_2\circ\sigma(t)\big\rangle \\
&= \theta'(t) \\
&\quad + \big\langle \cos\theta(t)(E_1\circ\sigma)'(t) + \sin\theta(t)(E_2\circ\sigma)'(t), -\sin\theta(t)E_1\circ\sigma(t) + \cos\theta(t)E_2\circ\sigma(t)\big\rangle \\
&= \theta'(t) + \cos^2\theta(t)\langle (E_1\circ\sigma)'(t), E_2\circ\sigma(t)\rangle - \sin^2\theta(t)\langle (E_2\circ\sigma)'(t), E_1\circ\sigma(t)\rangle \\
&= \theta'(t) - (\cos^2\theta + \sin^2\theta)\langle (E_2\circ\sigma)'(t), E_1\circ\sigma(t)\rangle \\
&= \theta'(t) - \big\langle E_1\circ\sigma(t), (E_2\circ\sigma)'(t)\big\rangle\,.
\end{aligned}$$

Proof of F2: We have

$$\left\langle E_1, \frac{\partial E_1}{\partial u}\right\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle E_1, E_1\rangle = 0,$$

and similarly,

$$\left\langle E_1, \frac{\partial E_1}{\partial v}\right\rangle = \left\langle E_2, \frac{\partial E_2}{\partial u}\right\rangle = \left\langle E_2, \frac{\partial E_2}{\partial v}\right\rangle = 0.$$

We also have

$$\frac{\partial}{\partial u}\langle E_1, N\circ\phi\rangle = \left\langle \frac{\partial E_1}{\partial u}, N\circ\phi\right\rangle + \left\langle E_1, \frac{\partial N\circ\phi}{\partial u}\right\rangle = 0.$$

Therefore, there is a real function $a : U \to \mathbb{R}$ such that

$$\begin{aligned}
\frac{\partial E_1}{\partial u} &= aE_2 + \left\langle \frac{\partial E_1}{\partial u}, N \circ \phi \right\rangle N \circ \phi \\
&= aE_2 - \left\langle E_1, \frac{\partial N \circ \phi}{\partial u} \right\rangle N \circ \phi \\
&= aE_2 - \left\langle E_1, dN_\phi \left(\frac{\partial \phi}{\partial u} \right) \right\rangle N \circ \phi \\
&= aE_2 + A \left(E_1, \frac{\partial \phi}{\partial u} \right) N \circ \phi.
\end{aligned}$$

Similarly, there are real functions b, c, and d, such that

$$\begin{aligned}
\frac{\partial E_1}{\partial v} &= bE_2 + A \left(E_1, \frac{\partial \phi}{\partial v} \right) N \circ \phi \\
\frac{\partial E_2}{\partial u} &= cE_1 + A \left(E_2, \frac{\partial \phi}{\partial u} \right) N \circ \phi \\
\frac{\partial E_2}{\partial v} &= dE_1 + A \left(E_2, \frac{\partial \phi}{\partial v} \right) N \circ \phi
\end{aligned}$$

Let us also choose the real functions $c_{i,j}$, for $i, j = 1, 2$, such that

$$\frac{\partial \phi}{\partial u} = c_{11} E_1 + c_{12} E_2, \quad \frac{\partial \phi}{\partial v} = c_{21} E_1 + c_{22} E_2.$$

Combining the above relations, at every (u, v) in U we have

$$\begin{aligned}
&\left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle - \left\langle \frac{\partial E_2}{\partial u}, \frac{\partial E_1}{\partial v} \right\rangle \\
&\qquad = A \left(E_1, \frac{\partial \phi}{\partial u} \right) A \left(E_2, \frac{\partial \phi}{\partial v} \right) - A \left(E_2, \frac{\partial \phi}{\partial u} \right) A \left(E_1, \frac{\partial \phi}{\partial v} \right) \\
&\qquad = \left(c_{11} A(E_1, E_1) + c_{12} A(E_1, E_2) \right) \left(c_{21} A(E_2, E_1) + c_{22} A(E_2, E_2) \right) \\
&\qquad\qquad - \left(c_{11} A(E_2, E_1) + c_{12} A(E_2, E_2) \right) \left(c_{21} A(E_1, E_1) + c_{22} A(E_1, E_2) \right) \\
&\qquad = \left(c_{11} c_{22} - c_{12} c_{21} \right) \left(A(E_1, E_1) A(E_2, E_2) - A(E_1, E_2) A(E_2, E_1) \right) \\
&\qquad = \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \det(A) \\
&\qquad = |\phi_u \times \phi_v| \, K \circ \phi.
\end{aligned} \tag{29}$$

In the above equations we have used that the first fundamental form g in the basis E_1 and E_2 is the identity matrix, so its determinant is equal to $+1$. Note that under a change of basis, the matrix of the second fundamental form changes to $P^t A P$, and the matrix of the first fundamental form changes to $P^t g P$, where t denotes the transpose operation. This implies that the quantity $\det A / \det g$ is independent of the choice of the basis.

Now let us write $\sigma(t) = (u(t), v(t))$, where $u(t)$ and $v(t)$ are some smooth functions in t. This gives us

$$\frac{\partial}{\partial t}(E_2 \circ \sigma)(t) = \frac{\partial}{\partial t}(E_2(u(t), v(t)) = \frac{\partial E_2}{\partial u}(u(t), v(t))u'(t) + \frac{\partial E_2}{\partial v}(u(t), v(t))v'(t).$$

We may now calculate the integral

$$\begin{aligned}
\int_0^L &\langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t)\rangle \, dt \\
&= \int_0^L \left\langle E_1(u(t), v(t)), \frac{\partial E_2}{\partial u}(u(t), v(t)) \right\rangle u'(t) \, dt \\
&\qquad + \int_0^L \left\langle E_1(u(t), v(t)), \frac{\partial E_2}{\partial v}(u(t), v(t)) \right\rangle v'(t) \, dt \\
&= \int_{\partial U} \left\langle E_1(u,v), \frac{\partial E_2}{\partial u}(u,v) \right\rangle du + \int_{\partial U} \left\langle E_1(u,v), \frac{\partial E_2}{\partial v}(u,v) \right\rangle dv \\
&= \int_U \frac{\partial}{\partial u}\left(\left\langle E_1(u,v), \frac{\partial E_2}{\partial v}(u,v) \right\rangle\right) - \frac{\partial}{\partial v}\left(\left\langle E_1(u,v), \frac{\partial E_2}{\partial u}(u,v) \right\rangle\right) dudv \\
&= \int_U \left(\left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle + \left\langle E_1, \frac{\partial^2 E_2}{\partial u \partial v} \right\rangle\right) - \left(\left\langle \frac{\partial E_1}{\partial v}, \frac{\partial E_2}{\partial v} \right\rangle + \left\langle E_1, \frac{\partial^2 E_2}{\partial v \partial u} \right\rangle\right) dudv \\
&= \int_U \left\langle \frac{\partial E_1}{\partial u}, \frac{\partial E_2}{\partial v} \right\rangle - \left\langle \frac{\partial E_1}{\partial v}, \frac{\partial E_2}{\partial v} \right\rangle dudv \\
&= \int_U |\phi_u \times \phi_v| K \circ \phi \, dudv.
\end{aligned}$$

In the above equation, for the third equality we have used the Green's integral formula, and for the last equality we have used Equation (29). This completes the proof of property F2.

Proof of F3: Evidently, we have

$$\int_0^L \theta'(s) \, ds = \theta(L) - \theta(0). \tag{30}$$

Recall that γ is a closed curve, which implies that $\gamma'(0) = \gamma'(L)$. Thus, by Equation (28), $\theta(L) - \theta(0)$ is a multiple of 2π. We need to show that it is equal to 2π.

By the hypothesis, there is a diffeomorphism $\Psi : \overline{B(0,1)} \to \overline{U}$. Using Ψ, we may define a continuous family of regular curves $\sigma_s : [0, L] \to \overline{U}$, for $s \in [0,1]$, such that $\sigma_0 \equiv \sigma$ and σ_1 is a circle of some small radius ϵ in U. For example, we may let σ_s be the restriction of Ψ to the circle of radius $1 - cs$, for some suitable constant c. Now we consider the continuous family of regular curves $\gamma_s = \phi \circ \sigma_s$. This is a continuous perturbation of the curve γ to some small closed curve in S.

The left hand side of Equation (30) depends continuously on the curve γ_s (this requires defining the family of functions θ_s). Since the value of the integral is an integer multiple of

2π for all curves γ_s, the value of the integral is the same for the curve γ_1, which is a small closed curve in S.

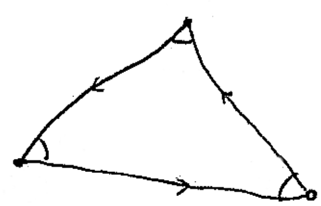
Since the curve σ_1 is very small, the map E_1 on σ_1 is nearly constant (since E is smooth). Note that $\theta_1(t)$ is the angle between $\gamma_1'(t)$ and $E_1 \circ \sigma_1(t)$. Now, since γ_1 turns around a small circle in a counter-clockwise fashion, and $E_1 \circ \sigma$ is nearly constant on that circle, the change in the angle is 2π. $\square$

21 Gauss-Bonnet for triangles

In this section we make a slight generalisation of the local Gauss-Bonnet theorem. It is an important step towards the proof of the general form of the Gauss-Bonnet theorem.

A **curvilinear triangle** in $\mathbb{R}^2$ is a continuous map $\beta : \mathbb{R} \to \mathbb{R}^2$ such that for all $t \in \mathbb{R}$ we have $\beta(t+3) = \beta(t)$, and for some t_0, t_1, t_2 satisfying $t_0 < t_1 < t_2 < t_0 + 3$ the following hold:

- β is regular on the intervals (t_0, t_1), (t_1, t_2), and $(t_2, t_0 + 3)$,
- $\beta : [t_0, t_0 + 3) \to \mathbb{R}^2$ is injective,
- the one-sided derivatives
$$\beta'(t_i^-) = \lim_{t \to t_i^-} \beta'(t), \qquad \beta'(t_i^+) = \lim_{t \to t_i^+} \beta'(t)$$
exists for $i = 0, 1, 2$,
- each collection $\{\beta'(t_0^+), \beta'(t_3^-)\}$, $\{\beta'(t_1^+), \beta'(t_1^-)\}$, and $\{\beta'(t_2^+), \beta'(t_2^-)\}$ is linearly independent.

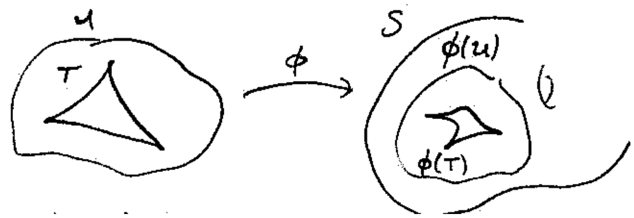


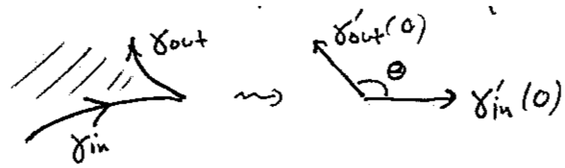
Remark 21.1. It is possible to relax the last condition in the above list, and include degenerate triangles which have only two sides, that is, the curve is regular at one of the vertexes (or the angle is 0, as defined below). We can also consider curvilinear polygons (n-gons), but it does not bring anything new since those may be considered as union of curvilinear triangles. We shall come back to this in Examples 24.3.

In the above definition, the points $\beta(t_i)$, for $i = 0, 1, 2$, are called the **vertices** of the curvilinear triangle β, and the regular curves $\beta(t_0, t_1)$, $\beta(t_1, t_2)$, and $\beta(t_2, t_0 + 3)$ are called the **edges** of the curvilinear triangle β.

Let $S \subset \mathbb{R}^3$ be a regular surface, $\phi : U \to S$ be a chart, N be a unit normal vector to S, and $T \subset U$ be a curvilinear triangle. We say that $\phi(T)$ is a curvilinear triangle in S, where the edges of $\phi(T)$ are the images of the edges of T, and the vertexes of $\phi(T)$ are the images of the vertexes of T, by the map ϕ.

For each vertex of the curvilinear triangle $\phi(T) \subset S$, the adjacent edges may be parametrised as
$$\gamma_{in} : (-\epsilon, 0] \to S, \qquad \gamma_{out} : [0, \epsilon) \to S,$$

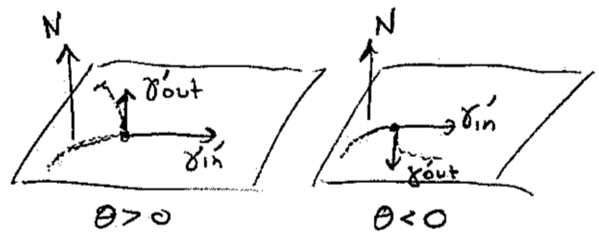




and their tangent vectors meet at an angle θ, with $-\pi < \theta < \pi$, that is,

$$\cos(\theta) = \frac{\langle \gamma'_{in}(0), \gamma'_{out}(0)\rangle}{|\gamma'_{in}(0)||\gamma'_{out}(0)|}.$$

The angle θ is selected using the following conventions:

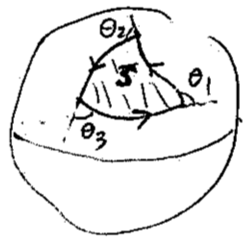


- $\theta > 0$ if $(\gamma'_{in}(0), \gamma'_{out}(0), N)$ is a positive basis for $\mathbb{R}^3$,
- $\theta < 0$ if $(\gamma'_{in}(0), \gamma'_{out}(0), N)$ is a negative basis for $\mathbb{R}^3$.

The number θ is called the **exterior angle** at this vertex, and the number $\pi - \theta$ is called the **interior angle** at that vertex.

Theorem 21.2 (Triangular version of Gauss-Bonnet). *Let $S' \subset \mathbb{R}^3$ be a regular surface, and S be a curvilinear triangle in S'. If ∂S has edges γ_i, for $i = 1, 2, 3$, parametrised by arc-length, which meet at exterior angles θ_i, for $i = 1, 2, 3$, then*

$$\sum_{i=1}^{3} \int_{\gamma_i} k_g ds + \sum_{i=1}^{3} \theta_i + \int_S K\, dA = 2\pi.$$



Proof. The proof is essentially the same as the one for the local version. As in that proof, for each γ_i, the geodesic curvature satisfies $k_g = \theta'(t) - \langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \rangle$. Moreover, property F2 carries over since Green's Theorem holds for triangular domains, i.e.,

$$\int_{\partial S} \langle E_1 \circ \sigma(t), (E_2 \circ \sigma)'(t) \rangle dt = \int_S (K \circ \phi) |\phi_u \times \phi_v| \, dudv.$$

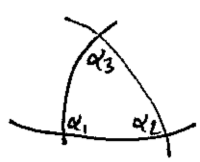
The only difference is in the property F3. We need to see that

$$\int_{L_0}^{L_1} \theta'(t)\, dt + \int_{L_1}^{L_2} \theta'(t)\, dt + \int_{L_2}^{L_3} \theta'(t)\, dt = 2\pi - \sum_{i=1}^{3} \theta_i,$$

where $\gamma_i : [L_{i-1}, L_i] \to S$ is a parametrisation of the curve γ_i by arc length. In this case, the tangent vectors to γ_i along ∂S still rotate by 2π, but that includes jumps of θ_1, θ_2, and θ_3 at the three vertexes. So $\int \theta'(t)\, dt$ does not include those jumps. $\square$

The triangular version of Gauss Bonnet gives us the following classical interpretation of Gaussian curvature. Let's consider a triangle T inside a surface S where the sides of T are geodesics, i.e., they have $k_g = 0$. We define the interior angles $\alpha_i = \pi - \theta_i$. Then the theorem gives us

$$\int_T K\, dA = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$



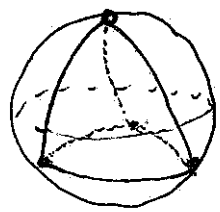
- If S is a plane, $\sum_{i=1}^3 \alpha_i = \pi$, which is a familiar formula (Thales' Theorem),
- If S is a unit sphere, $\sum_{i=1}^3 \alpha_i - \pi = \int_T 1\, dA = \text{area}(T)$,
- If S has negative Gaussian curvature, then $\sum_{i=1}^3 \alpha_i - \pi = \int_T K\, dA < 0$, so $\sum_{i=1}^3 \alpha_i < \pi$.

22 Triangulation of surfaces and the Euler characteristic

In this section we introduce an important topological invariant for surfaces. The arguments here are topological, and do not depend on how the surface is embedded in $\mathbb{R}^3$.

Given a compact regular surface $S \subset \mathbb{R}^3$, a **triangulation** of S is a partition of S into curvilinear triangles $\{T_1, \dots, T_n\}$ such that

- $S = \cup_{i=1}^n T_i$,
- for every distinct i and j, if $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a vertex or an edge,
- for any edge of the triangulation which lies in the interior of S, there are exactly two distinct triangles sharing this edge, and for any edge of the triangulation which lies on ∂S there is a unique triangle of the partition containing that edge.



For a triangulation $\cup_{i=1}^n T_i$ of S, the **Euler characteristic** is defined as

$$\chi\left(\cup_{i=1}^n T_i\right) = V - E + F,$$

where F is the number of faces (i.e. triangles), E is the number of distinct edges, and V is the number of distinct vertices in the triangulation.

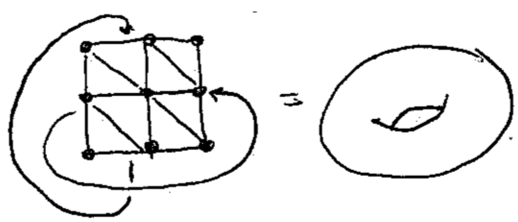
By a classical theorem in topology, every compact surface admits a triangulation. Moreover, for a compact surface S, the Euler characteristic does not depend on the choice of the triangulation for S. That is, given two triangulations of the same surface, the Euler characteristic is the same. It is customary to denote the **Euler characteristic** of S by $\chi(S)$.

For example, the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is homeomorphic to a tetrahedron as in the following figure thus,

$$\chi(\mathbb{S}^2) = 4 - 6 + 4 = 2.$$

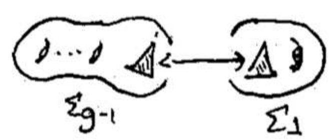
For a torus, we may use the triangulation as in the following figure and note that $V = 4$, $E = 12$, and $F = 8$. Therefore,

$$\chi(\mathbb{T}^2) = 4 - 12 + 8 = 0.$$



In general, a surface of genus g, Σ_g, is obtained from attaching g toruses together as in the following figure. We may calculate the Euler characteristic of Σ_g by an inductive argument. Consider arbitrary triangulations of Σ_{g-1} and Σ_1, remove a face from each of these, and

glue these remaining surfaces along their boundary (which is a circle). With this process, we



obtained a surface of genus g.

We note that

$$\begin{aligned} V_{\Sigma_g} &= V_{\Sigma_{g-1}} + V_{\Sigma_1} - 3, \\ E_{\Sigma_g} &= E_{\Sigma_{g-1}} + E_{\Sigma_1} - 3, \\ F_{\Sigma_g} &= F_{\Sigma_{g-1}} + F_{\Sigma_1} - 2. \end{aligned}$$

Therefore,

$$\chi(\Sigma_g) = \chi(\Sigma_{g-1}) + \chi(\Sigma_1) - 2.$$

Then, by induction, we can see that

$$\chi(\Sigma_g) = 2 - 2g.$$

If $S = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ is a disc, then $\chi(S) = 1$. If $S = \{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq 1\}$ is a cylinder with boundary, then $\chi(S) = 0$.

If S and S' are two surfaces which are homeomorphic, i.e., there is a homeomorphism $F : S \to S'$, then $\chi(S) = \chi(S')$.

The following is a classical result in topology. The proof is not long, but it is not in the scope of this module.

Theorem 22.1 (Classification of closed surfaces)**.** *Let S be a compact, connected, orientable surface without boundary. Then, S is homeomorphic to some Σ_g, determined by $\chi(S) = 2-2g$.*

23 Gauss-Bonnet theorem

Now we are ready to state the most general form of the Gauss-Bonnet Theorem.

Theorem 23.1 (Gauss-Bonnet Theorem)**.** *Let $S \subset \mathbb{R}^3$ be a compact and orientable regular surface, possibly with boundary. Then*

$$\int_{\partial S} k_g \, ds + \int_S K \, dA = 2\pi\chi(S),$$

where ∂S is parametrised by arc-length and is oriented positively.

Moreover, if $\partial S = \emptyset$, we have

$$\int_S K \, dA = 2\pi\chi(S).$$

In the above theorem, when $\partial S = \emptyset$, then the Gauss-Bonnet theorem is the surface analogue of Theorem 4.4. In both relations, we integrate a geometric quantity (the left-hand side) to obtain a topological quantity (the right-hand side).

Proof. Consider a triangulation $\{T_i\}_{i=1}^n$ of S, and note that by making each triangle very small, we can assume that for each triangle T_i there is a chart $\phi : U \to S$ such that $T_i \subset \phi(U)$. Moreover, by making small adjustments, if necessary, we may assume that the triangles in the partition are curvilinear.

Denote the edges of each triangle T_i by E_{ij}, for $j = 1, 2, 3$, and the exterior angles at each vertex by θ_{ij}, $j = 1, 2, 3$. Orient ∂T_i positively for all i. Applying the triangular version of the Gauss-Bonnet theorem, Theorem 21.2, for each i we obtain,

$$\int_{T_i} K \, dA + \int_{\partial T_i} k_g \, ds = \sum_{j=1}^{3} \theta_{ij} - 2\pi = \sum_{j=1}^{3} \alpha_{ij} - \pi.$$

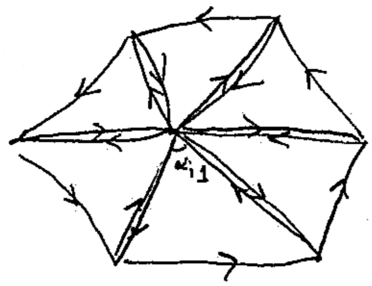
In the above relation, α_{ij} denote the corresponding internal angles.

We sum the above formula over all $i = 1, 2, \dots, n$.

$$\sum_{i=1}^{n} \int_{T_i} K \, dA + \sum_{i=1}^{n} \int_{\partial T_i} k_g \, ds = \sum_{i=1}^{n} \sum_{j=1}^{3} \alpha_{ij} - \sum_{i=1}^{n} \pi.$$

Since each edge of the triangulation which lies in the interior of S appears two times in the total sum, and those terms have opposite signs, they cancel out in the sum. So the sum of the integrals over ∂T_i boils down to the integral of k_g along the boundary of S.

On the other hand, the sums of the interior angles $\Sigma_{i,j}\alpha_{ij}$ contributes by 2π at each interior vertex, and by π at each vertex on the boundary.



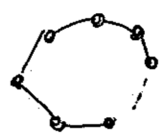
Combining the above arguments, we obtain

$$\int_{\partial S} k_g\, ds + \int_S K\, dA = (2\pi V - \pi(\#\text{boundary vertices})) - \pi F. \tag{31}$$

Now, note that

$$\begin{aligned} 3F &= 2(\#\text{ interior edges}) + (\#\text{ boundary edges}) \\ &= 2E - (\#\text{ boundary edges}) \\ &= 2E - (\#\text{ bondary vertices}) \end{aligned}$$

For the last equality in the above equation, one can see that along each boundary component of S, the number of edges is equal to the number of vertices. Thus, the right hand side of



Equation (31) becomes

$$2\pi(V + F) - \pi(3F + (\#\text{ boundary vertices})) = 2\pi(V + F) - \pi(2E) = 2\pi\chi(S).$$

This completes the proof of the theorem. □

24 Applications of Gauss-Bonnet Theorem

We now derive some basic consequences of the Gauss-Bonnet theorem.

Example 24.1. If S is a compact connected regular surface without boundary, and has Gaussian curvature $K \geq 0$ on S, then S is homeomorphic to a sphere. Stating differently, we can say that if S is a surface without boundary and has genus $g > 0$, then the Gaussian curvature of S must be negative at some point on S.

To see this, we note that by Gauss-Bonnet theorem,

$$0 \leq \int_S K\, dA = 2\pi\chi(S) = 2\pi(2 - 2g).$$

Thus, either $g = 0$ which implies that S is homeomorphic to a sphere, or $g = 1$ which implies that S is homeomorphic to a torus. However, if $g = 1$, then we must have $\int_S K\, dA = 0$. By the hypothesis $K \geq 0$, that implies that $K \equiv 0$. On the other hand, from the problem sheets, the Gaussian curvature is strictly positive at some point on any compact surface.

Example 24.2. Let S be a compact connected regular surface without boundary, and $K > 0$ on S. If γ_1 and γ_2 are simple (i.e. have no self intersection) closed geodesics on S, then $\gamma_1 \cap \gamma_2 \neq \emptyset$.

Assume in the contrary that $\gamma_1 \cap \gamma_2 = \emptyset$. By the previous example, S is homeomorphic to a sphere. Recall that by the Jordan curve Theorem, any simple closed curve on the sphere divides the sphere into two connected components (you may have seen the plane version of this theorem, but this version follows from that one). It follows that γ_1 and γ_2 bound a region Σ on the sphere, that is, $\partial\Sigma = \gamma_1 \cup \gamma_2$. We now apply the Gauss-Bonnet theorem on Σ,

$$\int_{\partial\Sigma} k_g\, ds + \int_\Sigma K\, dA = 2\pi\chi(\Sigma).$$

Since γ_1 and γ_2 are geodesics, the above equation implies that

$$0 < \int_\Sigma K\, dA = 2\pi\chi(\Sigma).$$

As Σ is homeomorphic to the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq 1\}$, we have $\chi(\Sigma) = 0$, which is a contradiction.

Example 24.3. Let S be a regular surface which is diffeomorphic to a disk, and satisfies $K \leq 0$. Assume that γ_1 and γ_2 are two geodesics in S which are not part of the same geodesic, and $\gamma_1(0) = \gamma_2(0) = p \in S$. Then, $\gamma_1 \cap \gamma_2 = \{p\}$, in other words, the two geodesics meet at most at one point.

Assume in the contrary that the two curves meet at some other point $q \in S$. By changing the direction of the curves, if necessary, we may assume that $\gamma_1(l) = \gamma_2(s) = q$, for some $l, s > 0$. Moreover, we assume that l is the smallest positive number satisfying this property. Set $a = \gamma_1([0, l])$ and $b = \gamma_2([0, s])$. Let Σ denote the region bounded by a and b, that is, $\partial\Sigma = a \cup b$. The set Σ must be diffeomorphic to a disk, and hence $\chi(\Sigma) = 1$.

Let θ_1 and θ_2 be the exterior angles at p and q, respectively. Note that Σ is a triangle with vertices p, q (because $\partial\Sigma$ is smooth outside p and q, the other vertex will have exterior angle 0 and so we do not worry about it). Hence we obtain from the triangular version of Gauss-Bonnet that

$$\int_{a\cup b} k_g ds + \theta_1 + \theta_2 + \int_\Sigma K dA = 2\pi \implies \int_\Sigma K dA = 2\pi - \theta_1 - \theta_2.$$

The angles θ_1 and θ_2 are strictly smaller than π. That is, because, if any of the two angles is equal to π, then $\gamma_1 = \gamma_2$. Therefore, $\int_\Sigma K\, dA > 0$. This contradicts $K \leq 0$ on S.