Network Science
Spring 2024
Problem sheet 2

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1. Consider the following model for a growing undirected graph. Initial state: 2 nodes connected by 1 link. Iteration 1: Replace the link between the node pair with two new nodes and four new links arranged so that the two original nodes are only connected by 2 "fully distinct" length-2 paths. Here, two paths are "fully distinct" if they have zero links in common. Iteration $i+1$: Apply the process for iteration 2 to each linked node pair in the graph at iteration i. So, for each linked pair of nodes in the graph at iteration i, remove the link and replace it with two new nodes and four new links so that the two 'old' nodes are connected by 2 new fully distinct length 2 paths.

   (a) Draw the graph after iteration 1 (it should have 4 nodes and 4 links) and iteration 2.

   (b) Find expressions for the number of links and nodes after i iterations

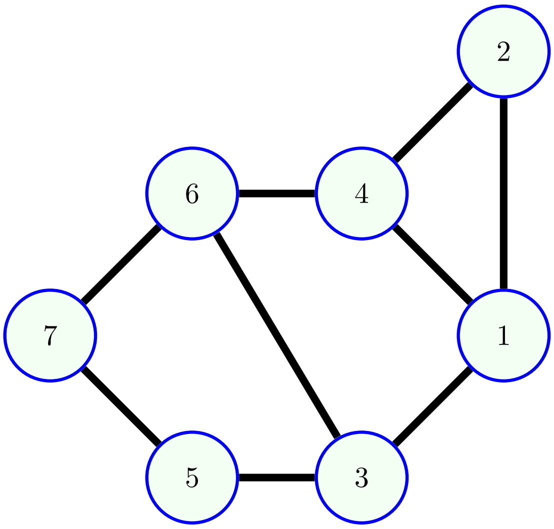
   (c) How does the diameter depend on the number of nodes in the graph? Compare your result to rectangular lattice graphs and Cayley trees.

2. Consider a simple connected graph where each node has the same degree, k. Show that the eigenvector centrality is the same for each node. Is the Katz centrality for each node the same?

3. Consider the matrix $\mathbf{G}$ used to compute the PageRank centrality for graphs where each node has at least one out-link. Let $\mathbf{x}$ be an eigenvector of $\mathbf{G}$ corresponding to eigenvalue $\lambda = 1$ and let $\mathbf{x}$ contain at least two elements with differing sign.

   (a) Show that $\sum_{i=1}^{N} |x_i| < \sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij} |x_j|$, and explain why this implies that such an eigenvector cannot exist

   (b) Show that $x_i \geq 0$ for all i implies that $x_i > 0$ for all i (assuming $\mathbf{x}$ is non-trivial).



4. Using the cosine similarity, determine which pairs of nodes in the graph above are the most similar (Attempt after lecture 3)