

## Question Sheet 1 - Probl. Class week 3

MATH40003 Linear Algebra and Groups

Term 2, 2022/23

Some of the questions are meant to be easier than others. If you can do the easy ones without much effort, skip to the harder ones. Solutions will be released on Monday of week 3 after the problem class.

**Question 1** Compute the determinants of the following matrices, assuming that the entries are from the field  $\mathbb{R}$ . Which matrices are invertible (and for which  $a$ )?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

**Question 2** (i) Factorise  $\det \begin{pmatrix} x & y & x \\ y & x & x \\ x & x & y \end{pmatrix}$ . (ii) Solve  $\det \begin{pmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{pmatrix} = 0$  for  $t \in \mathbb{R}$ .

**Question 3** Calculate the determinant of the matrix  $\begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$ .

How would you check your answer?

**Question 4** For a real number  $\alpha$  define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix} \in M_4(\mathbb{R}).$$

- (a) Find the determinant of  $A(\alpha)$ .
- (b) Find a value  $\alpha_0$  of  $\alpha$  such that the system  $A(\alpha_0)x = 0$  has a nonzero solution for  $x \in \mathbb{R}^4$ .
- (c) Prove that when  $\alpha < \alpha_0$ , there is no real  $4 \times 4$  matrix  $B$  such that  $B^2 = A(\alpha)$ .

**Question 5** Let  $A, B \in M_n(F)$  (where  $F$  is an arbitrary field). Prove that  $|A| = 0$  or  $|B| = 0$  if and only if  $|AB| = 0$ .

*Note: you may NOT assume the result  $|AB| = |A||B|$  from lectures as this question is supposed to be part of the proof of that result. You MAY assume the result that says a matrix is invertible iff it has nonzero determinant.*

**Question 6** Let  $B_n$  be the  $n \times n$  matrix

$$\begin{pmatrix} -2 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & \cdots & 0 & 0 & 0 \\ & & & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

- (a) Prove that if  $n \geq 4$ , then  $|B_n| = 8|B_{n-3}|$ .
- (b) Prove that  $|B_n| = 0$  if  $n = 3k - 1$  (where  $k$  is a positive integer).
- (c) Find  $|B_n|$  if  $n = 3k$  or  $3k + 1$ .

**Question 7** The *trace*  $\text{tr}(A)$  of an  $n \times n$  matrix  $A = (a_{ij})$  over a field  $F$  is defined to be the sum of the diagonal entries of  $A$ :  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ . Prove that, for  $n \times n$  matrices  $A, B$  (over  $F$ ) we have: (a)  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ ; (b)  $\text{tr}(AB) = \text{tr}(BA)$ .

Suppose  $V$  is an  $n$ -dimensional  $F$ -vector-space with a basis  $C$  and  $T : V \rightarrow V$  is a linear transformation. We define the trace of  $T$  to be  $\text{tr}([T]_C)$ . Prove that this does not depend on the choice of basis  $C$ .

**Question 8** If  $A$  is an  $n \times n$  matrix over a field  $F$ , the *characteristic polynomial* of  $A$  is  $\det(xI_n - A)$ . Prove that this is a polynomial of degree  $n$  over  $F$  and: (i) the coefficient of  $x^n$  is 1; (ii) the constant term is  $(-1)^n \det(A)$ ; (iii) the coefficient of  $x^{n-1}$  is  $-\text{tr}(A)$ .

**Question 9** Let  $A$  be the  $n \times n$  matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the  $a_i$  are in the field  $F$ . Prove that the characteristic polynomial of  $A$  is  $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ . (*Hint*: Try induction.)