Imperial College London

**MATH97060 MATH70063**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Algebra 4**

|  |
|---|
| Date: 31 May 2022 |
| Time: 09:00 – 11:30 (BST) |
| Time Allowed: 2:30 hours |
| Upload Time Allowed: 30 minutes |
| **This paper has 5 Questions.** |
| Candidates should start their solutions to each question on a new sheet of paper. |
| Each sheet of paper should have your CID, Question Number and Page Number on the top. |
| Only use 1 side of the paper. |
| Allow margins for marking. |
| Any required additional material(s) will be provided. |
| Credit will be given for all questions attempted. |
| Each question carries equal weight. |

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

In all questions you can use any results from lectures if you state them clearly.

In this paper $R$ is an associative ring with $1$.

1. (a) Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of left $R$-modules. Prove that if $M$ is a projective $R$-module, then the following sequence of abelian groups is exact:

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}_R(M, B) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, C) \longrightarrow 0. \tag{2}$$

Here $\alpha_*$ sends a map of $R$-modules $f\colon M \to A$ to $\alpha \circ f$, and similarly for $\beta_*$. (6 marks)

(b) Let $M$ be a left $R$-module. Prove that if (2) is exact for *any* short exact sequence (1), then $M$ is projective. (2 marks)

(c) Let $R = \mathbb{Z}$ and let (1) be a *non-split* short exact sequence of abelian groups. Suppose that $M$ is an abelian group such that (2) is exact. Does it follow that the abelian group $M$ is projective? Give a proof or a counterexample. (6 marks)

(d) Let $D, E, F$ be left $R$-modules. Let $\phi\colon D \to E$ and $\psi\colon E \to F$ be maps of $R$-modules. Suppose that for any left $R$-module $M$ the following sequence of abelian groups is exact:

$$\mathrm{Hom}_R(M, D) \xrightarrow{\phi_*} \mathrm{Hom}_R(M, E) \xrightarrow{\psi_*} \mathrm{Hom}_R(M, F).$$

Prove that the sequence $D \xrightarrow{\phi} E \xrightarrow{\psi} F$ is exact. (6 marks)

(Total: 20 marks)

2. (a) In this question $R$ is a commutative ring. Let $M$ be an $R$-module. Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of $R$-modules such that $B$ is projective.

(i) Prove that for any $n \geq 1$ there is an isomorphism

$$\mathrm{Ext}^n_R(A, M) \cong \mathrm{Ext}^{n+1}_R(C, M).$$

(4 marks)

(ii) Determine the kernel of the map $A \otimes_R M \to B \otimes_R M$ induced by $\alpha$ in terms of the functor $\mathrm{Tor}^R_n(-,-)$. (4 marks)

(b) Determine the abelian groups

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}), \quad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}), \quad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/4), \quad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}), \quad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}), \quad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/4),$$

explaining your reasoning. (6 marks)

(c) Prove that there exists a non-zero homomorphism of abelian groups $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$. (6 marks)

(Total: 20 marks)

3. (a) Let $k$ be a field and let $R = k[x_1, \dots, x_n]$ be the ring of polynomials in $n$ variables with coefficients in $k$. Consider the ideal $I = (x_1, \dots, x_n) \subset R$. For $n = 1$ and $n = 2$ determine whether the $R$-module $I$ is (i) injective, (ii) projective, (iii) flat. (12 marks)

   You are asked to justify your answers.

   (b) List all extensions of $\mathbb{Z}/6$ by $\mathbb{Z}/6$ in the category of abelian groups, up to equivalence of extensions. Describe the maps in each extension. (No proof is required.) (8 marks)

(Total: 20 marks)

4. (a) Consider the short exact sequence of abelian groups, where the second arrow is multiplication by 7:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/7 \longrightarrow 0.$$

   Determine the terms of the attached long exact sequence of $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/7, -)$, where $n \geq 0$. Which maps in this sequence are isomorphisms? (You are asked to justify your answer.)

   (6 marks)

   (b) Let $A, B, C$ be abelian groups. Construct a natural map

$$\mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, C) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}}(B, C)),$$

   and prove that it is an isomorphism. (14 marks)

(Total: 20 marks)

5. In this question, $S_n$ is the symmetric group on $n$ elements, $n \geq 2$. The sign of a permutation $\pi \in S_n$ is denoted by $\mathrm{sgn}(\pi) \in \{\pm 1\}$, so that $\mathrm{sgn}\big((ij)\big) = -1$ for each $i$ and $j$ such that $1 \leq i < j \leq n$. The alternating group $A_n$ is the subgroup of $S_n$ consisting of the permutations $\pi \in S_n$ such that $\mathrm{sgn}(\pi) = 1$.

   (a) For $n = 3, 4, 5$ determine the cohomology group $\mathrm{H}^2(A_n, \mathbb{Z})$, where $\mathbb{Z}$ is a trivial $A_n$-module.

   (7 marks)

   (b) Determine $\mathrm{H}^1(S_5, M)$, where $M = \mathbb{Z}$ is an $S_5$-module such that a permutation $\pi \in S_5$ sends $1 \in \mathbb{Z}$ to $\mathrm{sgn}(\pi)$.

   (8 marks)

   (c) Briefly explain how to construct an exact sequence of groups

$$1 \longrightarrow (\mathbb{Z}/2) \times (\mathbb{Z}/2) \longrightarrow S_4 \longrightarrow S_3 \longrightarrow 1.$$

   Is $S_4$ isomorphic to a semi-direct product of $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$ and $S_3$? (5 marks)

   In all parts you are asked to justify your answers.

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

# XXX

# Alebra IV (Solutions)

| Setter's signature | Checker's signature | Editor's signature |
|---|---|---|
| ................ | ................ | ................ |

1. (a) The exactness at $\mathrm{Hom}_R(M,A)$ is a direct consequence of the injectivity of $\alpha$. The exactness at $\mathrm{Hom}_R(M,B)$ is proved as follows. Let $f\colon M\to B$ be a map of $R$-modules such that $\beta\circ f=0$. Then $f$ factors through $\mathrm{Ker}(\beta)=\mathrm{Im}(\alpha)\cong A$, since $\alpha$ is injective. The exactness at $\mathrm{Hom}_R(M,C)$ is a restatement of the projectivity of $M$. seen ⇓ 6, A

(b) The assumption implies that for any surjective map of $R$-modules $\beta\colon B\to C$ any map of $R$-modules $f\colon M\to C$ lifts to a map of $R$-modules $g\colon M\to B$ so that $\beta\circ g=f$. Thus $M$ is a projective $R$-module. seen ⇓ 2, A

(c) Let (1) be any non-split short exact sequence of finite abelian groups whose orders are annihilated by a power of 2. Let $M=\mathbb{Z}/3$. Then all the terms of (2) are zero, so it is a short exact sequence. By a result from lectures, projective abelian groups are free abelian groups, but $\mathbb{Z}/3$ is clearly not free (e.g., because it is finite). unseen ⇓ 6, B

(d) Take $M=D$. Then $\mathrm{id}_D\in\mathrm{Hom}_R(D,D)$ goes to $\psi\circ\phi\in\mathrm{Hom}(D,F)$, hence $\psi\circ\phi=0$. Thus $\mathrm{Im}(\phi)\subset\mathrm{Ker}(\psi)$. Let us prove that this inclusion is an equality. Take $M=\mathrm{Ker}(\psi)$ and let $i\colon\mathrm{Ker}(\psi)\to E$ be the natural inclusion. Since $\psi\circ i=0$, there exists a map of $R$-modules $j\colon\mathrm{Ker}(\psi)\to D$ such that $\phi\circ j=i$. This implies that $\mathrm{Ker}(\psi)\subset\mathrm{Im}(\phi)$, so our sequence is exact. unseen ⇓ 6, C

2. (a) (i) The long exact sequence of $\mathrm{Ext}^n_R(-,M)$-groups can be written as follows: meth seen ⇓

$$\ldots\to\mathrm{Ext}^n_R(B,M)\to\mathrm{Ext}^n_R(A,M)\to\mathrm{Ext}^{n+1}_R(C,M)\to\mathrm{Ext}^{n+1}_R(B,M)\to\ldots$$

By assumption, $B$ is projective. By lectures, this implies that $\mathrm{Ext}^n_R(B,M)=0$ for all $n\geq 1$, hence the desired isomorphism follows from this long exact sequence. 4, A

(ii) Since $B$ is projective, we have $\mathrm{Tor}_1^R(B,M)=0$. Then the long exact sequence of $\mathrm{Tor}_n^R(-,M)$-groups gives an exact sequence

$$0\to\mathrm{Tor}_1^R(C,M)\to A\otimes_R M\to B\otimes_R M.$$

Thus the kernel of $A\otimes_R M\to B\otimes_R M$ is isomorphic to $\mathrm{Tor}_1^R(C,M)$. 4, B

(b) The abelian group $\mathbb{Z}$ is free, hence projective, thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\cong\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})=0$. seen/sim.seen ⇓
The abelian group $\mathbb{Q}$ is divisible, hence injective, thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})=0$.
The abelian group $\mathbb{Q}$ is flat, thus $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})=0$. By lectures, we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}/4)\cong(\mathbb{Z}/4)/4=\mathbb{Z}/4$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}/4)\cong(\mathbb{Z}/4)[4]=\mathbb{Z}/4$. 6, A

(c) Let $x$ be an irrational real number. Then the cyclic subgroup of $\mathbb{R}/\mathbb{Z}$ generated by $x$ is a subgroup of $\mathbb{R}/\mathbb{Z}$ isomorphic to $\mathbb{Z}$. Consider an injective homomorphism $\mathbb{Z}\to\mathbb{R}$. Since $\mathbb{R}$ is divisible, it is an injective abelian group, and thus $\mathbb{Z}\to\mathbb{R}$ extends to a homomorphism $\mathbb{R}/\mathbb{Z}\to\mathbb{R}$. It is visibly non-zero. 6, C

3. (a) (i) Note that $R$ is an integral domain. The ideal $I$, considered as an $R$-module, is not divisible (for example, $x_1$ cannot be written as the product of $x_1$ and an element of $I$). By a result from lectures, $I$ is not injective.

unseen ⇓

3, A

(ii) and (iii) If $n = 1$, then $I$ is a principal ideal of an integral domain $R$, hence $I$ is free, and so is projective, hence flat.

unseen ⇓

3, A

Now let $n = 2$. In a problem sheet we constructed a free resolution of the quotient module $k = R/I$:

seen ⇓

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \xrightarrow{\epsilon} k \longrightarrow 0,$$

6, B

where the first map sends $1 \in R$ to $(-x_2, x_1)$, and the second map sends $(r_1, r_2)$ to $x_1 r_1 + x_2 r_2$. Tensoring with $k$ we obtain a sequence with zero maps. Computing homology groups we get $\mathrm{Tor}_2^R(k, k) \cong k$. This implies $\mathrm{Tor}_1^R(I, k) \cong k$. By a result from lectures, $I$ is not a flat $R$-module, hence not projective.

meth seen ⇓

(b) We have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/6) \cong \mathbb{Z}/6$, so there are six pairwise non-equivalent extensions. A split extension corresponds to the zero element in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/6)$. We also have

$$0 \to \mathbb{Z}/6 \to \mathbb{Z}/36 \to \mathbb{Z}/6 \to 0,$$

where the second arrow sends $1$ to $6$ or $-6$, and the third arrow is the quotient of $\mathbb{Z}/36$ by the unique subgroup of size $6$. Next we have

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to \mathbb{Z}/4 \oplus (\mathbb{Z}/3)^{\oplus 2} \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to 0,$$

where $1 \in \mathbb{Z}/2$ goes to $2 \in \mathbb{Z}/4$ and $1 \in \mathbb{Z}/3$ goes to $(1, 0) \in (\mathbb{Z}/3)^{\oplus 2}$; the third arrow is the quotient map. Finally, there are also the extensions

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to (\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/9 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to 0,$$

where $1 \in \mathbb{Z}/2$ goes to $(1, 0) \in (\mathbb{Z}/2)^{\oplus 2}$ and $1 \in \mathbb{Z}/3$ goes to $3$ or $6 \in \mathbb{Z}/9$; the third arrow is the quotient map.

8, D

4. (a) By a result from lectures, $\mathrm{Tor}_n^{\mathbb{Z}}(-,-) = 0$ for $n \geq 2$. Since $\mathbb{Z}$ is a free, hence projective, abelian group, we have $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/7, \mathbb{Z}) = 0$. By another result from lectures, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/7, \mathbb{Z}/7) \cong \mathbb{Z}/7$. Thus the long exact sequence of $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/7, -)$ unseen ⇓ 2, A

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/7, \mathbb{Z}/7) \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/7 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/7 \to \mathbb{Z}/7 \otimes_{\mathbb{Z}} \mathbb{Z}/7 \to 0,$$

can be written as follows:

$$0 \longrightarrow \mathbb{Z}/7 \xrightarrow{\sim} \mathbb{Z}/7 \xrightarrow{[0]} \mathbb{Z}/7 \xrightarrow{\sim} \mathbb{Z}/7 \longrightarrow 0.$$

The maps marked with a tilde are isomorphisms, as follows from the exactness of the sequence. 4, B

(b) Suppose that we are given a homomorphism $f\colon A \otimes_{\mathbb{Z}} B \to C$. We associate to $a \in A$ the map $\phi_a\colon B \to C$ as follows: $\phi_a$ sends $b \in B$ to $f(a \otimes b) \in C$. Since tensor product is additive in the second argument and $f$ is a homomorphism, $\phi_a$ is a homomorphism, i.e. $\phi_a \in \mathrm{Hom}_{\mathbb{Z}}(B, C)$. Using that tensor product is additive in the first argument and $f$ is a homomorphism, we obtain that $\phi_{a_1+a_2} = \phi_{a_1} + \phi_{a_2}$. This means that the function $a \mapsto \phi_a$ is a homomorphism, i.e. an element of $\mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}}(B, C))$. Thus we have constructed a map meth seen ⇓ 6, A

$$\Phi\colon \mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} B, C) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}}(B, C)).$$

The map $\Phi$ is visibly linear in $f$, so is a homomorphism.

Let us prove that $\Phi$ is an isomorphism.

If $f \neq 0$, then there exist $a \in A$ and $b \in B$ such that $f(a \otimes b) \neq 0$. This implies that $\phi_a(b) = f(a \otimes b) \neq 0$, hence $\Phi(f) \neq 0$. Thus $\Phi$ is injective.

Let $g\colon A \to \mathrm{Hom}_{\mathbb{Z}}(B, C)$ be a homomorphism. Consider the map $F\colon A \times B \to C$ sending $(a, b)$ to the value of $g(a) \in \mathrm{Hom}_{\mathbb{Z}}(B, C)$ at $b$. Since $g(a)$ is a homomorphism, $F$ is additive in the second argument. Since $g$ is a homomorphism, $F$ is additive in the first argument. By the universal property of tensor product, $F$ factors through a homomorphism $f\colon A \otimes B \to C$. Then $\Phi(f) = g$, which proves that $\Phi$ is surjective. 8, D

5. (a) Using the exact sequence seen/sim.seen ⇓

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

and the fact that $\mathrm{H}^n(G, \mathbb{Q}) = 0$ for $n \geq 1$ and any finite group $G$ (from lectures), the long exact sequence of group cohomology gives isomorphisms $\mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, where the last isomorphism follows from the triviality of the $G$-module $\mathbb{Q}/\mathbb{Z}$. 5, M The group $A_3$ is isomorphic to $\mathbb{Z}/3$, hence $\mathrm{Hom}(A_3, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3$. The group $A_4$ is non-abelian. Its commutator is a normal subgroup $H \cong \mathbb{Z}/2 \times \mathbb{Z}/2$; we have $A_4/H \cong \mathbb{Z}/3$. Thus $\mathrm{Hom}(A_4, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3$. 2, M Finally, $A_5$ is simple group, hence it has no non-trivial homomorphisms to $\mathbb{Q}/\mathbb{Z}$, thus $\mathrm{Hom}(A_5, \mathbb{Q}/\mathbb{Z}) = 0$. seen/sim.seen ⇓

(b) The inflation-restriction sequence for the subgroup $A_5 \subset S_5$ is the exact sequence

$$0 \to \mathrm{H}^1(\mathbb{Z}/2, M^{A_5}) \to \mathrm{H}^1(S_5, M) \to \mathrm{H}^1(A_5, M),$$

where we took into account that $S_5/A_5 \cong \mathbb{Z}/2$. The action of $A_5$ on $M$ is trivial, so that $M \cong \mathbb{Z}$ as $A_5$-modules. Thus $\mathrm{H}^1(A_5, M) \cong \mathrm{Hom}(A_5, \mathbb{Z}) = 0$, because $A_5$ is finite and $\mathbb{Z}$ is torsion-free. It remains to compute $\mathrm{H}^1(\mathbb{Z}/2, M^{A_5})$. The generator of $\mathbb{Z}/2$ acts on $M^{A_5} \cong \mathbb{Z}$ as $-1$. Using the formula for the odd degree cohomology of a cyclic group (from lectures), we obtain $\mathrm{H}^1(\mathbb{Z}/2, M^{A_5}) \cong \mathbb{Z}/2$. Thus $\mathrm{H}^1(S_5, M) \cong \mathbb{Z}/2$. 8, M

(c) Let $H \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ be the subgroup of $S_4$ whose non-trivial elements are $(12)(34)$, $(13)(24)$, $(14)(23)$. unseen ⇓ Then $S_4/H \cong S_3$. (The group $S_4$ acts on the 3-element set of partitions of $\{1, 2, 3, 4\}$ into a disjoint union of two 2-element subsets. This gives a surjective homomorphism $S_4 \to S_3$ with kernel $H$.) Thus we obtain the desired exact sequence. We can realise $S_3$ inside $S_4$, for example, as the group of permutations of $\{2, 3, 4\}$. Thus the exact seuence is split, so that $S_4$ isomorphic to the semi-direct product $H \rtimes S_3$. 5, M

**Review of mark distribution:**

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

**If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.**
**Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.**

| ExamModuleCode | QuestionNumber | Comments for Students |
|---|---|---|
| Algebra 4_MATH97060 MATH70063 | 1 | This was an easy question. Most students answered it very well. |
| Algebra 4_MATH97060 MATH70063 | 2 | Another easy question. |
| Algebra 4_MATH97060 MATH70063 | 3 | This question was harder. Only a couple of students gave a complete answer to part (b) |
| Algebra 4_MATH97060 MATH70063 | 4 | Part (b) is essentially bookwork, but takes a long time to do in full. On the whole, this was a question of about average difficulty |
| Algebra 4_MATH97060 MATH70063 | 5 | Questions on group cohomology are usually not very well answered, perhaps because this comes late in the module. |