

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024**

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fourier Analysis and Theory of Distributions

Date: Tuesday, April 30, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let $f \in L_1[-\pi, \pi]$ have Fourier series $\sum_{-\infty}^{\infty} c_k e^{ikx}$.
 - (i) Do the Fourier series necessarily converge at all $x \in [-\pi, \pi]$? (No justification needed) (2 marks)
 - (ii) What can be said about the limit of the sequence c_1, c_2, \dots ? (No justification needed) (3 marks)
 - (iii) Let n be a fixed positive integer. Assume that $|c_k| < 1/|k|$ for all k such that $|k| > n$. Does it follow that $f \in L_2[-\pi, \pi]$? Justify your answer. (5 marks)
- (b) Find the Fourier expansion of $f(x) = |x|$, $x \in [-\pi, \pi]$, and use it to compute the value of the series $\sum_{k=1}^{\infty} 1/k^2$. Justify your calculation. (10 marks)

2. (a) State the definition of the Fourier transform of a function in $L_1[-\infty, \infty]$ and the definition of the inverse Fourier transform. (6 marks)
- (b) Let $f \in L_1[-\infty, \infty]$ and such that $|f(x)| < e^{-(\log|x|)^2}$ for all real x such that $|x| > 10$. Is the 10'th derivative of the Fourier transform of f a differentiable function? Justify your answer. (7 marks)
- (c) Find the inverse Laplace transform of the function $\Phi(p) = \frac{1}{p^2}$ by contour integration. (7 marks)

3. (a) State what it means for a functional on a normed linear space to be continuous. (5 marks)
- (b) Show that
 - (i) In any normed linear space, the strong convergence of a sequence implies its weak convergence. (5 marks)
 - (ii) In a finite-dimensional space, the weak convergence of a sequence implies its strong convergence. (5 marks)
- (c) Let E be a topological linear space, and let E^* its adjoint (the space of continuous linear functionals on E). Show that the weak-* topology on E^* is not stronger than the weak topology on E^* . (5 marks)

4. (a) (i) State the definitions of the space of Schwarz functions and the space of tempered distributions. (5 marks)
- (ii) State the definition of the Fourier transform of a tempered distribution. (5 marks)
- (b) Find the Fourier transform of the tempered distribution $f(x) = x^{10}$. (10 marks)

5. Prove the existence of a continuous periodic function whose Fourier series diverge at a point following the steps below: Consider the (Banach) space E of continuous functions on the unit circle with the sup-norm.
- (a) Let c_k be the Fourier coefficients of $f \in E$. Fix any positive integer n . Show that the linear functional $S_n(f) = \sum_{k=-n}^n c_k$ is continuous. (4 marks)
 - (b) Show that the norms of S_n , $n = 1, 2, \dots$ are not uniformly bounded. (8 marks)
 - (c) Conclude that there exists $f \in E$ such that the sequence $S_n(f)$ is unbounded. (8 marks)

Fourier analysis
and distribution theory
exam 2024 solutions

①

1a i No [2 marks]

1a ii $C_n \rightarrow 0, n \rightarrow \infty$ [3 marks]

1a iii If $|c_n| < \frac{1}{|n|}, n > n$

$$\text{then } \sum_{|k| > n} |c_k|^2 < \sum_{|k| > n} \frac{1}{k^2} < \infty$$

Therefore, $f \in L_2[-\pi, \pi]$ by Riesz thm.
[5 marks]

1b. We have, by evenness of $|x|$,

$$|x| = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \text{ at any } x \in [-\pi, \pi]$$

since both right and left derivatives of $|x|$ exist at each point.

(2)

$$a_0 = \int_0^{\pi} x dx = \frac{\pi^2}{2}$$

$$a_n = \int_0^{\pi} x \cos nx dx =$$

$$= \frac{1}{n} \int_0^{\pi} x d \sin nx$$

$$= -\frac{1}{n} \int_0^{\pi} \sin nx dx = \frac{1}{n^2} \cos nx \Big|_0^{\pi}$$

$$= \frac{1}{n^2} ((-1)^n - 1)$$

Therefore,

$$f(x) = \frac{\pi^2}{2} - 4 \sum_{p=0}^{\infty} \frac{\cos(2p+1)x}{(2p+1)^2}$$

At $x=0$ we have $\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} + \sum_{p=1}^{\infty} \frac{1}{(2p)^2}$$

$$\Rightarrow \frac{3}{4} S = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}$$

$$\Rightarrow S = \frac{\pi^2}{6}$$

[10 marks]

2a. Let $f \in L_1(-\infty, \infty)$

The Fourier transform

$$g(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt.$$

The inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda x} d\lambda$$

if f satisfies the Dini condition at x

[6 marks]

2b.

Note that with $y = \log|x|$, $|x| > 10$,

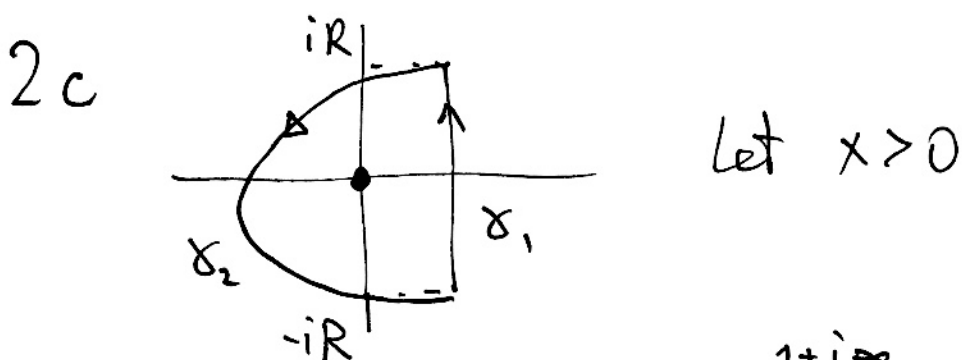
$$\begin{aligned} |x^p f(x)| &\leq |x|^p e^{-\log^2|x|} = \\ &= e^{py} e^{-y^2} \rightarrow 0, \quad |x| \rightarrow \infty \end{aligned}$$

for any p . Take $p > 2$, $k = p - 2$

Then $|x^k f| \leq \frac{C_k}{x^2}$, $|x| > 10$,

so that $x^k f(x) \in L_1(-\infty, \infty)$

Therefore $g(\lambda)$ is k -times differentiable. Take $k \geq 11$.
[7 marks]



By definition $f(x) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{px}}{p^2} dp$

The residue at zero is x since

$$\frac{e^{px}}{p^2} = \frac{1 + px + O(p^2)}{p^2} = \frac{1}{p^2} + \frac{x}{p} + O(1), \quad p \rightarrow 0$$

Thus

$$2\pi i x = \int_{\gamma_1 \cup \gamma_2} \frac{e^{px}}{p^2} dp = \int_{\gamma_2} + \int_{\gamma_1}$$

$$\int_{\gamma_1} \rightarrow 2\pi i f(x), \quad \int_{\gamma_2} \rightarrow 0 \quad \text{by Jordan's lemma as } R \rightarrow \infty \text{ since } x > 0.$$

$$\Rightarrow f(x) = x, \quad x > 0. \quad [7 \text{ marks}]$$

(5)

3a. A functional f on E is called continuous if for any $x_0 \in E$, $\varepsilon > 0$, there is a neighbourhood U_0 of x_0 s.t. $|f(x) - f(x_0)| < \varepsilon$, $x \in U_0$. [5 marks]

3bi Let $\|x_n - x\| \rightarrow 0$, $x_n, x \in E$,

E is normed linear space.

Let $f \in E^*$. Then $|f(x_n) - f(x)| =$

$$= |f(x_n - x)| \leq \|f\| \cdot \|x_n - x\| \rightarrow 0.$$

[5 marks]

Therefore, $x_n \xrightarrow{w} x$.

3bii Let $x_k \xrightarrow{w} x$, $x_k = \sum_{j=1}^n x_k^{(j)} e_j$,

$x = \sum_{j=1}^n x^{(j)} e_j$, where $e_j, 1, \dots, n$ is a basis of E . Since the inner product is a continuous linear functional,

$x_k^{(j)} \rightarrow x^{(j)}$ for any $j = 1, \dots, n$.

$$\Rightarrow \|x_k - x\|^2 = \sum_{j=1}^n (x_k^{(j)} - x^{(j)})^2 \rightarrow 0$$

[5 marks]

3c. The defining system of neighbourhoods
of zero of the weak* topology

⑥

$$U = \{ f \in E^* : |f(x_k)| < \varepsilon, k=1, \dots, n \}$$

But $f(x_k)$ is a functional

$x_k \in E^{**}$ applied to f .

Thus U is an open set of the form

$$\{ f \in E^* : |\psi_k(f)| < \varepsilon, k=1, \dots, n \}$$

in the weak topology on E^* ,

$$\psi_k \in E^{**}.$$

[5 marks]

4ai. S_∞ is a set of functions on \mathbb{R} which are infinitely differentiable such that for any $p, q = 0, 1, \dots$ there are constants C s.t.

$$|x^p f^{(q)}(x)| < C(p, q, f) \quad \forall x \in \mathbb{R}.$$

The space of tempered distributions is $(S_\infty)^*$. [5 marks]

4a ii. The Fourier transform g of $f \in (S_\infty)^*$ is given by

$$(g, \varphi) = (f, F[\varphi]), \quad \varphi \in S_\infty.$$

[5 marks]

$$4b. (F[x^{10}], \varphi) = (x^{10}, F[\varphi]) =$$

$$= \int_{-\infty}^{\infty} x^{10} F[\varphi] dx = (-i)^{10} \int_{-\infty}^{\infty} F[\varphi^{(10)}] dx$$

$$= (-1)^5 2\pi \varphi^{(10)}(0). \text{ Thus } F[x^{10}] = -2\pi \delta^{(10)}$$

[10 marks]

5a.

$$|S_n(f)| \leq (2n+1) \max_{|e^{ix}|=1} |f(x)| \\ = (2n+1) \|f\|_E$$

Thus $\|S_n\| \leq 2n+1$. [4 marks]

5b. Consider continuous functions ψ_n
 s.t. $\|\psi_n\|_E \leq 1$, $\psi_n(x) = \text{sgn}(D_n(x))$
 except in small intervals around points
 of discontinuity of $D_n(x)$, where D_n
 is the Dirichlet kernel. If the sum
 of the length of these intervals is less than $\frac{\varepsilon}{2n}$,

$$\|S_n\| \geq \|S_n(\psi_n)\| = \left| \int_{-\pi}^{\pi} \psi_n(x) D_n(x) dx \right| > \\ > \int_{-\pi}^{\pi} |D_n(x)| dx - \varepsilon ,$$

$$\text{but } \int_{-\pi}^{\pi} |D_n(x)| dx \geq \frac{1}{100} \log n .$$

[8 marks]

9

5c. If the sequence $S_n(f)$ were bounded for any $f \in E$ then since E is Banach, the norms $\|S_n\|$ would be uniformly bounded.

(by thm. on p. 94 and remark on p. 90 of the lecture notes § - particular case of uniform boundedness thm).

Thus $\exists f \in E$ s.t. $S_n(f)$ is unbounded.

In other words, Fourier series of f do not converge at $x=0$.

MATH60030 Fourier Analysis & Theory of Distributions

Question Marker's comment

- 1 Generally well done
- 2 Some students had problems computing inverse Laplace transform
- 3 There are different correct arguments for 3c.
- 4 Generally well done

MATH70030 Fourier Analysis & Theory of Distributions

Question Marker's comment

1 For Q1-4 see the non-master exam comments

5 mixed results