

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
May 2024

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

## Applied Complex Analysis

Date: Wednesday, May 15, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer All Questions in 1 Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. (a) Define the Cauchy and Hilbert transforms for a complex function  $f(z)$  over a smooth contour  $\gamma \in \mathbb{C}$ . (2 marks)
- (b) Write down the Plemelj formulae satisfied by  $f(z)$  on the path  $\gamma$ . (2 marks)
- (c) Verify that the Plemelj formulae hold in the case where  $f(z) = z$  and  $\gamma = [-1, 1]$ . [Note: you may use the fact that the multi-valued function

$$\log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2),$$

where  $r_1 = |z-1|$ ,  $r_2 = |z+1|$ ,  $\theta_1 = \arg\{z-1\}$ ,  $\theta_2 = \arg\{z+1\}$ , and that we can make this function single-valued by introducing a branch-cut along  $[-1, 1]$  and restricting  $-\pi \leq \theta_1, \theta_2 \leq \pi$ .] (9 marks)

- (d) The function  $f(x)$ , integrable over the interval  $[-1, 1]$ , satisfies the integral equation

$$\frac{1}{\pi} \int_{-1}^1 f(t) \log|t-x| dt = \sqrt{1-x^2}, \quad -1 < x < 1.$$

Determine  $f(x)$ , expressing the final result in terms of the integral

$$I_0 = \frac{2}{\pi^2} \int_0^1 \frac{\log t}{\sqrt{1-t^2}} \left[ t \log\left(\frac{1-t}{1+t}\right) \right] dt.$$

You may use the result from lectures that

$$\int_{-1}^1 \frac{\log|t|}{\sqrt{1-t^2}} dt = -\pi \log 2.$$

(7 marks)

[The Hilbert inversion formula

$$f(x) = -\frac{1}{\pi \sqrt{1-x^2}} \text{P.V.} \int_{-1}^1 \frac{g(t) \sqrt{1-t^2}}{t-x} dt + \frac{A}{\sqrt{1-x^2}},$$

for the singular integral equation

$$\frac{1}{\pi} \text{P.V.} \int_{-1}^1 \frac{f(t)}{t-x} dt = g(x), \quad -1 < x < 1,$$

may be quoted without proof. Here  $A$  is an arbitrary constant and  $\text{P.V.}$  represents the principal value integral.]

(Total: 20 marks)

2. The Gamma function  $\Gamma(z)$  is defined for  $\operatorname{Re}\{z\} > 0$  by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

(a) (i) Show that for  $\operatorname{Re}\{z\} > 0$  we can write

$$\Gamma(z) = 2 \int_0^\infty x^{2z-1} e^{-x^2} dx,$$

and hence that, for  $\lambda \in [0, 1]$

$$\Gamma(\lambda)\Gamma(1-\lambda) = 4 \int_0^\infty \int_0^\infty \left(\frac{x}{y}\right)^{2\lambda-1} e^{-(x^2+y^2)} dx dy.$$

(3 marks)

(ii) Further, by introducing polar coordinates, show that

$$\Gamma(\lambda)\Gamma(1-\lambda) = 2 \int_0^\infty \frac{u^{2\lambda-1}}{1+u^2} du.$$

(4 marks)

(iii) Hence show that, for all  $z$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

(9 marks)

For  $|z| < 1$  and  $c$  not equal to zero or a negative integer, the hypergeometric series  $F(a, b; c; z)$  may be represented as

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n, \quad \text{where } (a)_n = \prod_{k=0}^{n-1} (a+k),$$

is the Pochhammer symbol.

(b) Show that

$$F(1, 2; 1; z) = \frac{1}{(1-z)^2}.$$

(4 marks)

(Total: 20 marks)

3. (a) Give a heuristic argument that if a mapping  $\zeta = f(z)$  from the complex  $z$ -plane to the complex  $\zeta$ -plane is **conformal** at a point  $z_0$ , then it preserves the angle between any two arcs passing through  $z_0$ . (5 marks)

Figure 1 shows the unbounded region, extending to infinity in the upper-half  $z$ -plane, exterior to a semi-circular protrusion of unit radius on the real axis, centred at the origin. The region is shaded in grey and labelled  $A$  in the figure.

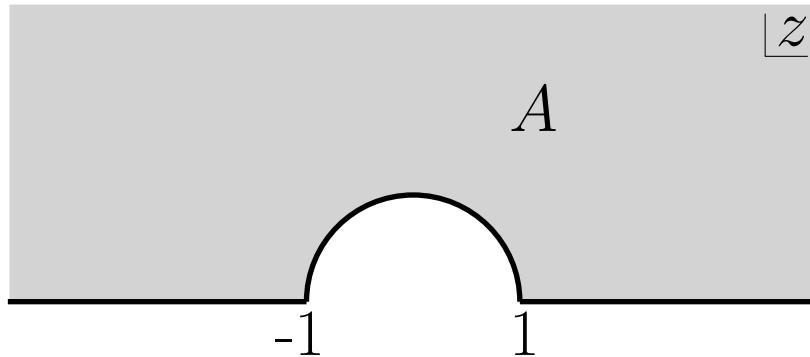


Figure 1: The region exterior to a semi-circular ‘bump’ on the real axis.

- (b) Show that the transformation  $\zeta = z + \frac{1}{z}$  maps region  $A$  to the upper-half  $\zeta$ -plane. (5 marks)

A company that manufactures radiator devices wishes to construct a heater whose shape is to follow the boundary of the shaded region in figure 1. The straight boundaries along the real axis are to be maintained at a constant temperature  $T_1$ , with the curved semi-circular boundary kept at a constant temperature  $T_2$ . The shaded region  $A$  represents the ambient medium to be heated within which the company wishes to determine the steady-state temperature distribution.

- (c) Assuming that everywhere inside region  $A$  the temperature,  $T(x, y)$ , satisfies Laplace's equation  $\nabla^2 T = 0$ , show that

$$T(r, \theta) = \frac{(T_2 - T_1)}{\pi} \left[ \tan^{-1} \left( \frac{(r^2 - 1) \sin \theta}{(r^2 + 1) \cos \theta - 2r} \right) - \tan^{-1} \left( \frac{(r^2 - 1) \sin \theta}{(r^2 + 1) \cos \theta + 2r} \right) \right] + T_1,$$

where  $r$  and  $\theta$  are the polar variables of points inside  $A$ , i.e.  $z = re^{i\theta}$  with  $r > 1$  and  $0 < \theta < \pi$ . (Note: in this question you may apply the inverse tangent function without concern over the details involved in selecting the appropriate branch cuts to ensure the function is single-valued.)

(10 marks)

(Total: 20 marks)

4. The function  $f(x)$  satisfies the integral equation

$$f(x) + \frac{5}{4} \int_0^\infty f(y) e^{-2|x-y|} dy = \frac{9}{2},$$

for  $x \geq 0$ .

- (a) Using the Wiener-Hopf method, and taking the strip of analyticity to be  $\{s : \alpha < \operatorname{Im}\{s\} < \beta\}$ , for values  $\alpha, \beta > 0$  which you should define carefully, show that for  $\operatorname{Im}\{s\} > \alpha$  the right-sided Fourier transform  $F_+(s) \equiv \int_0^\infty f(x) e^{isx} dx$  is given by

$$F_+(s) = \frac{3i(s+2i)}{s(s+3i)}.$$

(16 marks)

- (b) Hence show that for  $x \geq 0$

$$f(x) = 2 + e^{-3x}.$$

(4 marks)

(Total: 20 marks)

5. Consider two-dimensional Stokes flow in the complex  $z$ -plane with streamfunction  $\psi$ , which can be represented as

$$\psi = \operatorname{Im}\{\bar{z}f(z) + g(z)\},$$

where  $f(z)$  and  $g(z)$  are analytic functions known as the Goursat functions and the bar represents the complex conjugate. The velocity field of the flow,  $(u, v)$ , is known to satisfy

$$u - iv = -\overline{f'(z)} + \bar{z}f'(z) + g'(z),$$

where the dash represents the derivative with respect to the function argument.

- (a) Find the Goursat functions describing a stagnation point flow over the plane with velocity field given by  $(u, v) = (kx, -ky)$ , where  $k > 0 \in \mathbb{R}$ . (4 marks)

Consider now an infinitely thin rod of length 2 centred at the point  $z = i$  making an angle of 0 with the real axis (i.e aligned horizontally) positioned in the same stagnation point flow over the  $z$ -plane as discussed in part (a). A schematic of the setup is shown in figure 2.

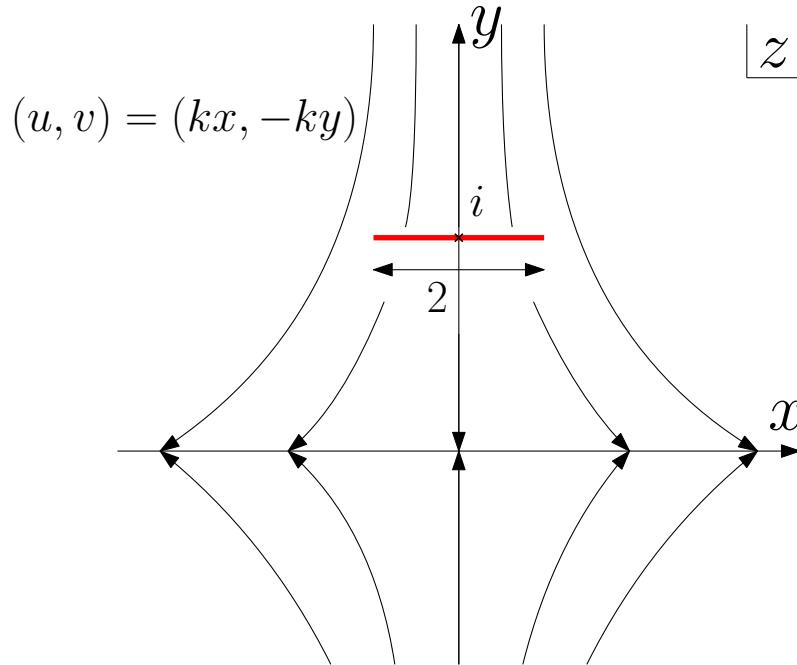


Figure 2: A thin rod of length 2 in Stokes flow with a background stagnation point flow in the  $z$ -plane.

- (b) Briefly explain why, or show that, the conformal mapping

$$z(\zeta) = i + \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right),$$

maps from the interior of the unit disc in the complex  $\zeta$ -plane, to the region exterior to the rod in the  $z$ -plane. (2 marks)

We now represent the Goursat functions in the form  $F(\zeta) = f(z(\zeta))$  and  $G(\zeta) = g'(z(\zeta))$  and proceed to work in the unit disc in the complex  $\zeta$ -plane.

- (c) The rod moves according to the effects of its induced velocity and solid body rotation. This means that  $u - iv = \bar{U} - i\Omega(\bar{z} + i)$  on the rod. Here  $U = U_{\text{rod}} + iV_{\text{rod}}$  where  $U_{\text{rod}}, V_{\text{rod}} \in \mathbb{R}$  represent the horizontal and vertical velocity of the rod respectively and  $\Omega \in \mathbb{R}$  represents the rod's angular velocity. These are unknown quantities that we will determine. Show that this boundary condition can be written in the form

$$-\bar{F}\left(\frac{1}{\zeta}\right) + \frac{\zeta(\zeta^2 - 2i\zeta + 1)}{(\zeta^2 - 1)} F'(\zeta) + G(\zeta) = \bar{U} - \frac{i\Omega}{2} \left(\zeta + \frac{1}{\zeta}\right),$$

in terms of  $\zeta$ . (4 marks)

- (d) Given that  $F(\zeta)$  takes the form

$$F(\zeta) = \frac{A}{\zeta} + B + C\zeta,$$

for unknown constants  $A, B$  and  $C \in \mathbb{C}$ , show that  $G(\zeta)$  must take the form

$$G(\zeta) = \left(-A + \bar{C} - \frac{i\Omega}{2}\right) \frac{1}{\zeta} + (\bar{B} + \bar{U} + 2iA) + \left(\bar{A} - 2A + C - \frac{i\Omega}{2}\right) \zeta + O(\zeta^2),$$

as  $\zeta \rightarrow 0$ . (3 marks)

- (e) Far away from the rod, the fluid moves according to the background stagnation point flow. Noting this; deduce that

$$A = \frac{1}{2}, \quad B = i, \quad C = \frac{k + 1 - i\Omega}{2} \quad \text{and} \quad U = -ki.$$

(3 marks)

The torque acting on the rod can be calculated via the formula

$$\text{Torque} = -2\text{Re} \left\{ \oint_{|\zeta|=1} G(\zeta) z'(\zeta) d\zeta \right\},$$

where  $\text{Re}\{\}$  denotes the real part of the expression within the parenthesis and the integration is taken in the positive (anti-clockwise) orientation.

- (f) Given that there is **no-torque** on the rod (i.e. Torque = 0), deduce the value of  $\Omega$ . (3 marks)
- (g) Describe the instantaneous motion of the rod. (1 mark)

(Total: 20 marks)

	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts	(a). The Cauchy transform of $f(z)$ over $\gamma \in \mathbb{C}$ is:	$\left. \begin{array}{l} \\ \end{array} \right\} 1 \boxed{A}$ seen
	$C(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi.$	
	The Hilbert transform of $f(z)$ over $\gamma \in \mathbb{C}$ is:	$\left. \begin{array}{l} \\ \end{array} \right\} 1 \boxed{A}$ seen
	$H(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi.$	
(b).	Plancherel formulae:	$\left. \begin{array}{l} \\ \end{array} \right\} 2 \boxed{A}$ seen
	$C_+(z) + C_-(z) = 2H(z),$	
	$C_+(z) - C_-(z) = f(z),$	
	for $z$ not on end point of $\gamma$ . Here $C_{\pm}(z_0)$ represents the limiting value of $C(z)$ as $z \rightarrow z_0$ from the left/right of $\gamma$ (going in the direction of integration).	
(c).	When $f(z) = z$ , $\gamma = [-1, 1]$ , then:	<span style="border: 1px solid red; padding: 2px;">Part (c) seen example <math>f(z) = z</math> in lectures</span>
	$C(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\xi}{\xi - z} d\xi,$	(1)
	but $\frac{\xi}{\xi - z} = \frac{\xi - z}{\xi - z} + \frac{z}{\xi - z} = 1 + \frac{z}{\xi - z},$	$\left. \begin{array}{l} \\ \end{array} \right\} 1 \boxed{A}$ seen similar
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts (c). (continued).	<p>so that:</p> $C(z) = \frac{1}{2\pi i} \int_{-1}^1 \left( 1 + \frac{z}{\xi - z} \right) d\xi$ $= \frac{1}{2\pi i} \left( [\xi]_{-1}^1 + z [\log(\xi - z)]_{-1}^1 \right)$ $= \frac{1}{\pi i} + \frac{1}{2\pi i} z \log \left( \frac{z-1}{z+1} \right).$ $= \frac{1}{\pi i} + \frac{1}{2\pi i} z \left( \log \left( \frac{\gamma_1}{\gamma_2} \right) + i(\theta_1 - \theta_2) \right),$	<p style="text-align: right;">A</p> <p style="text-align: right;">1 seen similar</p> <p style="text-align: right;">1 use result</p>
	<p>using the given fact. Now, taking <math>x \in \gamma</math> (not an end point), we have: <math>\gamma_1 = 1-x</math>, <math>\gamma_2 = 1+x</math>, and as we approach <math>x</math> from <u>above</u> (left): <math>\theta_1 = \pi</math>, <math>\theta_2 = 0</math>, but as we approach <math>x</math> from <u>below</u> (left): <math>\theta_1 = -\pi</math>, <math>\theta_2 = 0</math>, which gives:</p> $C_+(x) = \frac{1}{\pi i} + \frac{1}{2\pi i} x \log \left( \frac{1-x}{1+x} \right) + \frac{x}{2},$ $C_-(x) = \frac{1}{\pi i} + \frac{1}{2\pi i} x \log \left( \frac{1-x}{1+x} \right) - \frac{x}{2}.$ <p>Now, calculating the Hilbert transform:</p> $H(x) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\xi}{\xi - x} d\xi$	<p style="text-align: right;">C</p> <p style="text-align: right;">3 seen similar</p> <p style="text-align: right;">?</p>

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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts (c). (continued).	$= \frac{1}{2\pi i} \left( \int_{-1}^1 d\zeta + x \int_{-1}^1 \frac{1}{\zeta-x} d\zeta \right), \text{ using } \textcircled{1}$ <p style="text-align: center;"><math>\therefore</math></p> $= \log\left(\frac{1-x}{1+x}\right) \text{ from lectures, or calculate.}$ $= \frac{1}{\pi i} + \frac{1}{2\pi i} x \log\left(\frac{1-x}{1+x}\right).$ <p>Hence, for <math>x \in \gamma</math> (not an end point) :</p> $C_+(x) + C_-(x) = \frac{2}{\pi i} + \frac{1}{\pi i} x \log\left(\frac{1-x}{1+x}\right) = 2H(x)$ $C_+(x) - C_-(x) = x = f(x) \quad \checkmark$	<span style="color: red; border: 1px solid red; padding: 2px;">B</span> <span style="color: red; border: 1px solid red; padding: 2px;">2</span> seen similar
(d).	Differentiate both sides w.r.t $x$ :	<span style="color: red; border: 1px solid red; padding: 2px;">A</span> <span style="color: red; border: 1px solid red; padding: 2px;">1</span> seen similar
	$\frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{t-x} dt = (-1) \cdot \cancel{\frac{1}{2}} \cdot \frac{1}{2} \cdot (-2x) \cdot (1-x^2)^{-\frac{1}{2}}$ $= \frac{x}{\sqrt{1-x^2}}.$ <p>Apply Hilbert inversion:</p> $f(x) = \frac{-1}{\pi \sqrt{1-x^2}} \int_{-1}^1 \frac{t}{\cancel{(t-x)}} dt + \frac{A}{\sqrt{1-x^2}}$	<span style="color: red; border: 1px solid red; padding: 2px;">B</span> <span style="color: red; border: 1px solid red; padding: 2px;">2</span> seen similar
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts (d). (continued.)	$\Rightarrow f(x) = \frac{-1}{\pi \sqrt{1-x^2}} \left( 2 + x \log \left( \frac{1-x}{1+x} \right) \right) + \frac{A}{\sqrt{1-x^2}},$ <p style="text-align: right;">from (c).</p> $\Rightarrow f(x) = \frac{\left( A\pi - 2 - x \log \left( \frac{1-x}{1+x} \right) \right)}{\pi \sqrt{1-x^2}}. \quad (2)$	<span style="border: 1px solid red; padding: 2px;">D</span> <span style="color: red; font-size: 2em; vertical-align: middle;">1</span> unseen
	<p>To determine A, plug (2) into original equation:</p> $\frac{1}{\pi} \int_{-1}^1 \frac{\left( A\pi - 2 - x \log \left( \frac{1-t}{1+t} \right) \right)}{\pi \sqrt{1-t^2}} \log  t-x  dt = \sqrt{1-x^2},$ <p>and set <math>x=0</math>:</p> $\begin{aligned} & \frac{A}{\pi} \int_{-1}^1 \frac{\log  t }{\sqrt{1-t^2}} dt - \frac{2}{\pi^2} \int_{-1}^1 \frac{\log  t }{\sqrt{1-t^2}} dt \\ & - \frac{1}{\pi^2} \int_{-1}^1 \frac{\log  t }{\sqrt{1-t^2}} \left( t \log \left( \frac{1-t}{1+t} \right) \right) dt = 1 \end{aligned}$ $\Rightarrow \left( \frac{A}{\pi} - \frac{2}{\pi^2} \right) (-\pi \log 2) - I_0 = 1, \text{ since the third integral above is even so can be written as } I_0.$	<span style="border: 1px solid red; padding: 2px;">D</span> <span style="color: red; font-size: 2em; vertical-align: middle;">1</span> seen similar

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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question <u>1</u>	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen
Parts (d). (continued.)	$\Rightarrow -A \log 2 + \frac{2}{\pi} \log 2 - I_0 = 1$ $\Rightarrow A = \frac{1+I_0 - \frac{2}{\pi} \log 2}{-\log 2} = \underline{\underline{\frac{2}{\pi}}} - \underline{\underline{\frac{1+I_0}{\log 2}}}$ <p>Hence:</p> $f(x) = \frac{-\left(\frac{(1+I_0)\pi}{\log 2} + x \log\left(\frac{1-x}{1+x}\right)\right)}{\pi \sqrt{1-x^2}}$	 <span style="color:red;">D</span> <span style="color:red;">2</span> <span style="color:red;">unseen</span> <span style="color:red;">Q1</span> <span style="color:red;">Total 20</span>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 2	Topic SPECIAL FUNCTIONS	Marks& seen/unseen
Parts		
(a). (i).	$T(z) = \int_0^\infty t^{z-1} e^{-t} dt$ $= 2 \int_0^\infty x^{2z-1} e^{-x^2} dx, \text{ upon substituting } t=x^2, dt=2dx.$ <p>Then, double integrating: <math>\lambda \in [0,1]</math></p> $T(\lambda) T(1-\lambda) = 4 \iint_0^\infty x^{2\lambda-1} y^{1-2\lambda} e^{-(x^2+y^2)} dx dy$ $= 4 \int_0^\infty \int_0^\infty \left(\frac{x}{y}\right)^{2\lambda-1} e^{-(x^2+y^2)} dx dy.$	<span style="color:red; font-size:2em;">}</span> 2 <span style="border: 1px solid red; padding: 2px;">A</span> <span style="color:red; font-size:2em;">}</span> seen
(ii).	<p>Introduce polar coordinates: <math>x = r \cos \phi,</math>  <math>y = r \sin \phi,</math>  <math>dx dy = r dr d\phi,</math></p> <p>then:</p> $T(\lambda) T(1-\lambda) = 4 \int_0^{\frac{\pi}{2}} (\cot \phi)^{2\lambda-1} \left( \underbrace{\left( \int_0^\infty r e^{-r^2} dr \right)}_{= \frac{1}{2}} \right) d\phi$ <p>Now substitute <math>u = \cot \phi,</math> so:</p> $\frac{du}{d\phi} = -(1+u^2).$	<span style="color:red; font-size:2em;">}</span> 1 <span style="border: 1px solid red; padding: 2px;">A</span> <span style="color:red; font-size:2em;">}</span> seen
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 2	Topic    SPECIAL FUNCTIONS	Marks& seen/unseen
Parts (a). (ii). (continued.)	<p>Then: <math>T(\lambda)T(1-\lambda) = 2 \int_0^\infty \frac{u^{2\lambda-1}}{1+u^2} du.</math></p> <p>(iii). Since <math>0 &lt; \lambda &lt; 1</math> the remaining integral has a multi-valued function <math>z^{2\lambda-1}</math> in the integrand. Hence, we introduce the contour:</p>	<p style="text-align: right;"><math>\left. \begin{matrix} 2 \\ \text{seen} \end{matrix} \right\} \text{A}</math></p>
	<p>Let <math>\gamma = \gamma_R + \gamma_E + \gamma_1 + \gamma_2</math> and consider: <math>\oint q(z) dz</math>,</p> <p>where <math>q(z) = \frac{z^{2\lambda-1}}{1+z^2} = \frac{z^{2\lambda-1}}{(z+i)(z-i)}.</math></p> <p>By the residue theorem:</p> $\begin{aligned} \oint_{\gamma} q(z) dz &= 2\pi i (\operatorname{Res}\{q, i\} + \operatorname{Res}\{q, -i\}) \\ &= 2\pi i \left( \frac{(e^{i\frac{\pi}{2}})^{2\lambda-1}}{2i} + \frac{(e^{-i\frac{\pi}{2}})^{2\lambda-1}}{-2i} \right) \end{aligned}$	<p style="text-align: right;"><math>\left. \begin{matrix} 1 \\ \text{seen} \end{matrix} \right\} \text{A}</math></p> <p style="text-align: right;"><math>\left. \begin{matrix} 2 \\ \text{locate polys} \\ \text{branch cut} \end{matrix} \right\} \text{B}</math></p> <p style="text-align: right;"><math>\left. \begin{matrix} \text{seen} \\ \text{similar} \end{matrix} \right\}</math></p>

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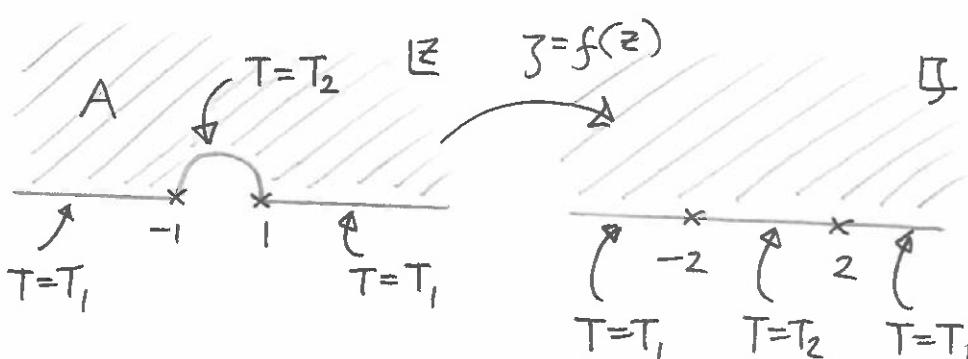
	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 2	Topic SPECIAL FUNCTIONS	Marks& seen/unseen
Parts (a). (iii). (continued)	$= \pi (e^{\pi i \lambda}(-i) - e^{3\pi i \lambda}(i))$ $= -\pi i (e^{\pi i \lambda} + e^{3\pi i \lambda}).$ <p>The contours <math>\int_{\gamma_R}</math> and <math>\int_{\gamma_\epsilon} \rightarrow 0</math> as <math>R \rightarrow \infty</math> and <math>\epsilon \rightarrow 0</math> respectively.</p> <p>Now, letting <math>I = \int_0^\infty \frac{u^{2\lambda-1}}{1+u^2} du</math>, then:</p> $\int_{\gamma_1} q(z) dz = I, \text{ and}$ $\int_{\gamma_2} q(z) dz = \int_{\infty}^0 \frac{(ue^{2\pi i})^{2\lambda-1}}{u^2+1} du = -e^{4\pi i \lambda} \int_0^\infty \frac{u^{2\lambda-1}}{1+u^2} du$ $= -e^{4\pi i \lambda} I.$	1 [B] - residue at i 1 [C] - residue at -i seen similar
	Hence, putting everything together: $T(\lambda)T(1-\lambda) = \frac{2(-\pi i (e^{\pi i \lambda} + e^{3\pi i \lambda}))}{1-e^{4\pi i \lambda}} \times \frac{e^{-2\pi i \lambda}}{e^{-2\pi i \lambda}}$ $= \frac{-2\pi i (e^{-\pi i \lambda} + e^{\pi i \lambda})}{e^{-2\pi i \lambda} - e^{2\pi i \lambda}}$	2 [D] seen similar
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 2	Topic SPECIAL FUNCTIONS	Marks& seen/unseen
Parts (a). (iii). (continued.)	$= \frac{2\pi \cos(\pi\lambda)}{\sin(2\pi\lambda)}$ $= \frac{2\pi \cos(\pi\lambda)}{2\cos(\pi\lambda)\sin(\pi\lambda)}$ $= \frac{\pi}{\sin(\pi\lambda)}$	<span style="border: 1px solid black; padding: 2px;">C</span> <span style="color: red;">1</span> unseen
(b).	<p>Hence, by <u>analytic continuation</u> to all of <math>z</math>, we deduce the result:</p> $T(z)T(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \forall z.$ $F(1, 2; 1; z) = \sum_{n=0}^{\infty} \frac{(1)_n (2)_n}{n! (1)_n} z^n, \quad \text{for }  z  < 1$ <p>Now <math>(2)_n = \prod_{k=0}^{n-1} (2+k) = 2 \times 3 \times \dots \times (n+1) = (n+1)!</math></p> <p>So we have:</p> $F(1, 2; 1; z) = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} z^n, \quad \text{for }  z  < 1$ $= \sum_{n=0}^{\infty} (n+1) z^n, \quad \text{for }  z  < 1.$	<span style="border: 1px solid black; padding: 2px;">C</span> <span style="color: red;">1</span> seen
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 2	Topic <b>SPECIAL FUNCTIONS</b>	Marks& seen/unseen
Parts (b). (continued).	<p>But note that <math>(1-z)^{-1} = 1+z+z^2+\dots</math>, for <math> z &lt;1</math>, so differentiating both sides w.r.t <math>z</math>:</p> $(1-z)^{-2} = 1+2z+3z^2+\dots, \text{ for }  z <1$ $= \sum_{n=0}^{\infty} (n+1) z^n, \text{ for }  z <1.$ <p>Hence: <math>F(1, 2; 1; z) = \frac{1}{(1-z)^2}</math>, which holds for all <math>z</math> by analytic continuation (<math>z \neq 1</math>).</p>	<span style="border: 1px solid red; padding: 2px;">D</span> 2 unseen <span style="color: red; font-size: 1.5em;">Q2 Total: 20</span>
	Setter's initials <b>SJB</b>	Checker's initials
		Page number <b>10</b>

	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts	<p>(a). Suppose <math>f(z)</math> is <u>analytic</u> <u>conformal</u> at <math>z_0</math>. This means that <math>f(z)</math> is <u>analytic</u> at <math>z_0</math> and that <math>f'(z_0) \neq 0</math>. Thus, we have:</p> $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$ <p>or, for <u>small</u> <math>\Delta z</math>, approximately:</p> $\Delta z f'(z_0) \approx f(z_0 + \Delta z) - f(z_0),$ <p>and denoting <math>z = z_0 + \Delta z</math> gives:</p> $z = f(z) \approx \underbrace{f'(z_0)}_a z + \underbrace{(f(z_0) - f'(z_0)z_0)}_b,$ <p>so that, locally, <math>z = f(z)</math> behaves like the linear mapping <math>z = az + b</math> (<math>a, b \in \mathbb{C}</math>), where <math>a = f'(z_0) \neq 0</math>. Since <math>f(z)</math> is conformal so that <math>f'(z_0) \neq 0</math>.</p> <p>We argue from lectures that linear mappings preserve the angle between any two arcs passing through <math>z_0</math>, but this is straightforward to show.</p> <p>This <math>z = f(z)</math> conformal at <math>z_0 \Rightarrow</math> angles through <math>z_0</math> preserved.</p>	<p style="text-align: right;">2 A -deg. of conformal seen</p> <p style="text-align: right;">2 A seen</p> <p style="text-align: right;">1 A seen</p>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (b).	<p>Denote <math>z = re^{i\theta}</math>. Then we have:</p> $\begin{aligned} \bar{z} &= z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= r\cos\theta + \frac{1}{r}\cos\theta + i\sin\theta - \frac{i}{r}\sin\theta \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$ <p>On ①: <math>\theta = 0, r = x \geq 1</math></p> $\Rightarrow \bar{z} = x + \frac{1}{x}, x \geq 1$ . This is clearly part of the positive real axis (one can show $x + \frac{1}{x} \geq 2$ as shown in the diagram). <p>On ②: <math>r = 1, 0 \leq \theta \leq \pi</math></p> $\Rightarrow \bar{z} = 2\cos\theta, 0 \leq \theta \leq \pi$ $\Rightarrow \bar{z} \in [-2, 2]$ , so maps to this section of the real axis, as shown in the diagram. <p>Similarly to ①, on ③: <math>\theta = \pi, r = x \geq 1</math></p> $\Rightarrow \bar{z} = -x - \frac{1}{x},$ so is negative the section ①. <p>Lastly, check a point: <math>z = 2i \Rightarrow \bar{z} = 2i + \frac{1}{2i} = \frac{3}{2}i</math>, so indeed we have the <math>UH\bar{z}</math>-plane.</p>	<p>unseen</p> <p>1 <span style="border: 1px solid red; padding: 2px;">B</span></p> <p><math>x + \frac{1}{x} \geq 2</math></p> <p>3 <span style="border: 1px solid red; padding: 2px;">A</span></p> <p>seen similar</p> <p>seen similar</p> <p>1 <span style="border: 1px solid red; padding: 2px;">B</span></p> <p>1 - check point</p>
	Setter's initials SJB	Checker's initials
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (c).		<span style="color: red; font-size: 2em;">}</span> <span style="border: 1px solid black; padding: 2px;">B</span> <span style="color: red;">2</span> Seen similar
	<p>First, employ the conformal transformation <math>w = z + \frac{1}{z}</math> to map region A to the UH <math>w</math>-plane. The boundary conditions on each section map over according to the diagram above. Now, working in the <math>w</math>-plane, we make the following ansatz for the complex potential <math>w(z) = W(w(z)) = T(x, y) + i\varphi(x, y)</math>:</p> $w(z) = -a \operatorname{arg}(z-2) - b \operatorname{arg}(z+2) + c,$ <p>where <math>a, b, c \in \mathbb{R}</math>. So then:</p> $T = \operatorname{Re}\{W(z)\} = a \cdot \operatorname{arg}\{z-2\} + b \cdot \operatorname{arg}\{z+2\} + c.$ <p>Now we use the boundary conditions:</p> <ul style="list-style-type: none"> <li>• <math>T=T_1</math>, when <math>\operatorname{arg}\{z-2\} = \operatorname{arg}\{z+2\} = \pi</math></li> </ul> $\Rightarrow \pi(a+b) + c = T_1 \quad (1)$	<span style="color: red; font-size: 2em;">}</span> <span style="border: 1px solid black; padding: 2px;">D</span> <span style="color: red;">2</span> unseen

Setter's initials  
SJB

Checker's initials

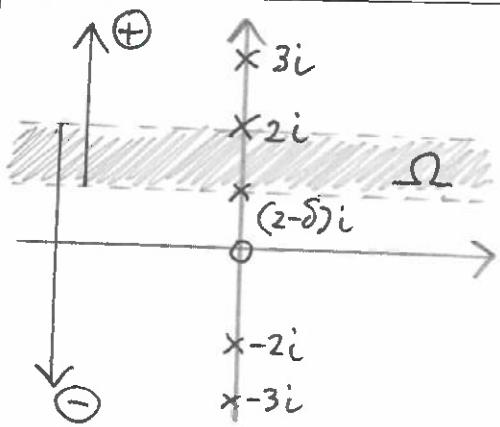
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (c). (continued.)	<p>• <math>T = T_1</math> when <math>\arg\{z-2\} = \arg\{z+2\} = 0</math></p> $\Rightarrow c = T_1$ , so ① gives: $a = -b$ ②	C 3 unseen
	<p>• <math>T = T_2</math> when <math>\arg\{z-2\} = \pi</math>, <math>\arg\{z+2\} = 0</math></p> $\Rightarrow a\pi + T_1 = T_2 \Rightarrow a = \frac{T_2 - T_1}{\pi}$ , <p>so using ②: <math>b = \frac{T_1 - T_2}{\pi}</math>.</p> <p>Hence we have:</p> $w(z) = -\left(\frac{T_2 - T_1}{\pi}\right)z \log(z-2) + \left(\frac{T_2 - T_1}{\pi}\right)z \log(z+2) + T_1,$ <p>thus:</p> $T(x, y) = \operatorname{Re}\{w(z)\}$ $= \left(\frac{T_2 - T_1}{\pi}\right) [\arg\{z-2\} - \arg\{z+2\}] + T_1,$ <p>but: <math>z-2 = z + \frac{1}{z} - 2 = re^{i\theta} + \frac{1}{r}e^{-i\theta} - 2</math></p> $= \left(r + \frac{1}{r}\right) \cos\theta - 2 + i\left(r - \frac{1}{r}\right) \sin\theta$ $\Rightarrow \arg\{z-2\} = \tan^{-1}\left(\frac{\left(r - \frac{1}{r}\right) \sin\theta}{\left(r + \frac{1}{r}\right) \cos\theta - 2}\right),$	D 2 unseen
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen
Parts (c). (Continued).	<p>and similarly: <math>\arg \{j+2\} = \tan^{-1} \left( \frac{(r-\frac{1}{r})\sin\theta}{(r+\frac{1}{r})\cos\theta + 2} \right)</math></p> <p>Thus:</p> $T(r,\theta) = \left( \frac{T_2 - T_1}{\pi r} \right) \left[ \tan^{-1} \left( \frac{(r^2-1)\sin\theta}{(r^2+1)\cos\theta - 2r} \right) \right. \\ \left. - \tan^{-1} \left( \frac{(r^2-1)\sin\theta}{(r^2+1)\cos\theta + 2r} \right) \right] + T_1,$ <p>as required upon multiplying by <math>\frac{\pi}{r}</math> within each <math>\tan^{-1}</math> function.</p>	<span style="color:red; font-size:2em;">1</span> <span style="color:red; border:1px solid red; padding:2px;">B</span> <span style="color:red; margin-left:10px;">1</span> <span style="color:red; margin-left:10px;">seen similar</span> <span style="color:red; font-size:2em; position: absolute; top: 40%; left: 50%;">Q3 Total 20</span>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen
Parts (a).	<p>Let <math>k(x) = \frac{5}{4} e^{-2 x }</math> and <math>p(x) = \frac{9}{2}</math>. Introduce:</p> $f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$ $p_+(x) = \begin{cases} p(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$ $g_-(x) = \begin{cases} 0, & x \geq 0 \\ \int_0^x f(y) k(x-y) dy, & x < 0. \end{cases}$ <p>Then, we have the equation:</p> $\frac{5}{4} \int_0^\infty f(y) e^{-2 x-y } dy = -f_+(x) + p_+(x) + g_-(x), \quad (1)$ <p>valid for <math>-\infty &lt; x &lt; \infty</math>. Taking the Fourier transform of both sides of (1) gives:</p> $\hat{k}(s) F_+(s) = -F_+(s) + P_+(s) + G_-(s), \text{ or}$ $K(s) F_+(s) + G_-(s) = -P_+(s), \quad (2)$ <p>where <math>K(s) = -\hat{k}(s) - 1</math> and where <math>F_+(s)</math> denotes the right-sided FT of <math>f_+(x)</math>, <math>P_+(s)</math> the right-sided FT of <math>p_+(x)</math>, <math>G_-(s)</math> the left-sided FT of <math>g_-(x)</math> and <math>\hat{k}(s)</math> the ordinary FT of <math>k(x)</math>.</p>	<span style="color:red">1</span> <span style="border: 1px solid red; padding: 2px;">A</span> <span style="color:red">1</span> seen
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen
Parts (a). (continued).	<p>One can either show, or use the result from problem sheets, that:</p> $\hat{K}(s) = \frac{5}{s^2 + 4}$ <p>For <math>P_+(s)</math> we calculate:</p> $P_+(s) = \frac{q}{2} \int_0^\infty e^{isx} dx = \frac{q}{2} \left[ \frac{e^{isx}}{is} \right]_0^\infty = -\frac{q}{2is} = \frac{qi}{2s}$ <p>It follows that:</p> $K(s) = -\hat{K}(s) - i = -\frac{(s^2 + 4) - s}{s^2 + 4} = -\frac{(s^2 + 9)}{(s^2 + 4)}$ <ul style="list-style-type: none"> <li>We require, from lectures: <math> f_+(x)  &lt; Ae^{(2-\delta)x}</math> as <math>x \rightarrow \infty</math>, for some <math>\delta &gt; 0</math> (A constant).</li> <li><math>\Rightarrow F_+(s)</math> analytic in <math>\{s : \operatorname{Im}\{s\} &gt; 2 - \delta\}</math>.</li> <li>Similarly, for <math>G_-(s)</math>, from lectures we can show that <math>g_-(x) = Be^{2x}</math> (B constant).</li> <li><math>\Rightarrow G(s)</math> analytic in <math>\{s : \operatorname{Im}\{s\} &lt; 2\}</math>.</li> </ul> <p>We thus take <math>\oplus</math> and <math>\ominus</math> regions defined to be:</p> $\oplus = \{s : \operatorname{Im}\{s\} > 2 - \delta\}, \ominus = \{s : \operatorname{Im}\{s\} < 2\},$ <p>where <math>0 &lt; \delta &lt; 2</math>. Hence the strip of analyticity where <math>\oplus</math> and <math>\ominus</math> overlap is the region: <math>2 - \delta &lt; \operatorname{Im}\{s\} &lt; 2</math>.</p>	<p style="text-align: right;"><span style="border: 1px solid red; padding: 2px;">A</span></p> <p style="text-align: right;">1 seen</p> <p style="text-align: right;"><span style="border: 1px solid red; padding: 2px;">P+ A</span></p> <p style="text-align: right;">1 seen similar</p> <p style="text-align: right;"><span style="border: 1px solid red; padding: 2px;">K A</span></p> <p style="text-align: right;">1 seen similar</p> <p style="text-align: right;"><span style="border: 1px solid red; padding: 2px;">α/β A</span></p> <p style="text-align: right;">2 seen similar</p>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen
Parts (a). (continued).	 <p><math>K(s)</math> analytic provided  <math>s \neq \pm 2i</math> ✓</p> <p><math>K(s)</math> non-zero provided  <math>s \neq \pm 3i</math> ✓</p> <p><math>P_+(s)</math> analytic provided  <math>s \neq 0</math> ✓</p>	
	<p>We decompose <math>K(s)</math> as <math>K(s) = K_+(s)K_-(s)</math>, where:</p> $K_+(s) = \frac{(s+3i)}{(s+2i)}, \quad K_-(s) = -\frac{(s-3i)}{(s-2i)},$ <p>then ② gives:</p> $K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = -\frac{P_+(s)}{K_-(s)} = \frac{9i(s-2i)}{2s(s-3i)} = R(s)$ <p>Now observe that we can write <math>R(s) = R_+(s) + R_-(s)</math>, where:</p> $R_+(s) = \frac{3i}{s}, \quad R_-(s) = \frac{3i}{2(s-3i)},$ <p>the constants sound from a partial fraction decomposition.</p> <p>Now we have:</p> $K_+(s)F_+(s) - R_+(s) = -\frac{G_-(s) + R_-(s)}{K_-(s)} \quad \text{analytic in } \mathbb{C} \setminus \Omega$	<p><math>\{</math> <math>K</math> decomp. <span style="border: 1px solid red; padding: 2px;">D</span></p> <p><math>\{</math> <span style="color: red;">2</span> seen similar</p> <p><math>\{</math> <math>R</math> decomp. <span style="border: 1px solid red; padding: 2px;">D</span></p> <p><math>\{</math> <span style="color: red;">2</span> seen similar</p>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 4	Topic WIENER-HOPF METHOD	Marks& seen/unseen
Parts (a). (continued.)	<p>Since <math>\oplus, \ominus</math> overlap in <math>\Omega</math>, then:</p> $E(s) = \begin{cases} F_+(s)K_+(s) - R_+(s), & s \in \oplus \\ -\frac{G_-(s)}{K_-(s)} + R_-(s), & s \in \ominus, \text{ is entire.} \end{cases}$ <p>Consider now <math>s \rightarrow \infty</math> in <math>\oplus</math>:</p> $F_+(s)K_+(s) - R_+(s) \sim \left[ \underbrace{\frac{i f(0)}{s} + O\left(\frac{1}{s^2}\right)}_{\text{from lectures}} \right] \cdot \left( 1 + O\left(\frac{1}{s}\right) \right) - O\left(\frac{1}{s}\right)$ $\sim O\left(\frac{1}{s}\right) \rightarrow 0 \text{ as } s \rightarrow \infty.$ <p>Hence, by Liouville's theorem: <math>E(s) \equiv 0</math> for all <math>s</math>.</p> <p>Therefore: <math>F_+(s)K_+(s) - R_+(s) = 0</math></p> $\Rightarrow F_+(s) = \frac{R_+(s)}{K_+(s)} = \frac{3i(s+2i)}{s(s+3i)},$ <p style="text-align: right;">as required.</p>	<p><span style="color: red;">B</span></p> <p><span style="color: red;">1</span> <math>E(s)</math> entire Seen</p> <p><span style="color: red;">expand.</span></p> <p><span style="color: red;">2</span> Seen Similar</p> <p><span style="color: red;">C</span></p> <p><span style="color: red;">1</span> Liouville. Seen</p> <p><span style="color: red;">A</span></p> <p><span style="color: red;">1</span> result. Seen Similar</p>
(b).	<p>To retrieve <math>f_+(x)</math>, we apply the inversion formula:</p> $f_+(x) = \frac{1}{2\pi} \int_P F_+(s) e^{-isx} ds,$ <p>where <math>P</math> is a horizontal line in the <math>\oplus</math> region.</p>	<p><span style="color: red;">B</span></p> <p><span style="color: red;">1</span> Seen</p>

	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 4	Topic <u>WIENER-HOPF METHOD</u>	Marks& seen/unseen
Parts (b). (continued).	<ul style="list-style-type: none"> <li>For <math>x &lt; 0</math>, from lectures; <math>f_+(x) = 0</math> as expected.</li> <li>For <math>x &gt; 0</math>, we close <math>P</math> with a semi-circle <math>\gamma_R</math> below <math>P</math> of radius <math>R</math> and take <math>R \rightarrow \infty</math>.</li> </ul> <p>Denoting <math>\gamma = P + \gamma_R</math>, we have by the <u>residue theorem</u>:</p> $\oint_{\gamma} \frac{3i(s+2i)}{s(s+3i)} e^{-isx} ds = -2\pi i (\text{Res}_{s=0} + \text{Res}_{s=-3i}) \quad \because \text{clockwise integration.}$ $= -2\pi i \left( \frac{3i \cdot 2i}{3i} \cdot e^0 + \frac{3i \cdot (-i)}{-3i} \cdot e^{-3x} \right) \quad \text{both simple poles}$ $= 4\pi + 2\pi e^{-3x}.$ <p>Now in the limit as <math>R \rightarrow \infty</math>, <math>\int_{\gamma_R} \rightarrow 0</math> (from lectures), hence:</p> $f_+(x) = \frac{1}{2\pi} (4\pi + 2\pi e^{-3x})$ $\Rightarrow f(x) = 2 + e^{-3x}, \quad x > 0.$	<div style="display: flex; align-items: center;"> <span style="margin-right: 10px;">B</span> <span style="margin-right: 10px;">2</span> <span>seen similar</span> </div> <div style="display: flex; align-items: center; margin-top: 20px;"> <span style="margin-right: 10px;">B</span> <span style="margin-right: 10px;">1</span> <span>seen similar</span> </div> <div style="margin-top: 20px; text-align: right;"> <span style="font-size: 2em; color: red;">Q4 Total 20</span> </div>
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 5	Topic COMPLEX METHODS FOR BIHARMONIC EQUATION	Marks& seen/unseen
Parts (a).	<p>We have <math>(u, v) = (kx, -ky)</math>, so:</p> $u - iv = kx + ky i = -\overline{f(z)} + \overline{z} f'(z) + g'(z)$ $\Rightarrow kz = -\overline{f(z)} + \overline{z} f'(z) + g'(z), \text{ so, comparing}$ <p>both sides; we must have:</p> $f'(z) = \text{constant} = A \in \mathbb{C}$ $g'(z) = kz$ $\Rightarrow A\overline{z} = (\overline{A}\overline{z} + \overline{B}), \text{ upon integrating } f'(z), \text{ so}$ <p>we conclude <math>B=0</math> and <math>A=1</math>, giving:</p> $f(z) = z, g'(z) = kz$	$\left. \begin{array}{l} \\ \\ \end{array} \right\} 1 \text{ seen}$ $\left. \begin{array}{l} \\ \\ \end{array} \right\} 2 \text{ seen}$ $\left. \begin{array}{l} \\ \\ \end{array} \right\} 1 \text{ seen}$
(b).	<p>(Note: a real constant may be added to each and this is also correct – it is the degree of freedom we have prescribing background pressures).</p> <p>The mapping <math>z(\bar{z}) = \frac{1}{2}(\bar{z} + \frac{1}{\bar{z}})</math> is the Joukowski mapping studied in lectures. This maps the <u>interior</u> of the <u>unit <math>\bar{z}</math>-disc</u> to the region around a slit from <math>-1</math> to <math>1</math> on the positive real axis in the <math>z</math>-plane.</p> <p>Adding <math>i</math> to this mapping then simply translates the slit so that it is centred at <math>z=i</math> in the <math>z</math>-plane, as required.</p>	$\left. \begin{array}{l} \\ \\ \end{array} \right\} 1 \text{ seen}$ $\left. \begin{array}{l} \\ \\ \end{array} \right\} 1 \text{ seen}$
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 5	Topic COMPLEX METHODS FOR BIHARMONIC EQUATION	Marks& seen/unseen
Parts (c).	<p>we have:</p> $u - i\nu = \bar{f}(\bar{z}) + \bar{z}f'(z) + g'(z) = \bar{U} - i\Omega(\bar{z} + i),$ <p>writing this in terms of <math>\zeta</math>:</p> $-\bar{F}(\zeta) + \bar{z}(\zeta) \frac{d}{dz} F(\zeta) + G(\zeta) = \bar{U} - i\Omega(\bar{z}(\zeta) + i),$ <p>on the rod.</p> <p>Now we have:</p> $z(\zeta) = i + \frac{1}{2}(\zeta + \frac{1}{\zeta}), \text{ so } \bar{z}(\zeta) = -i + \frac{1}{2}(\bar{\zeta} + \frac{1}{\bar{\zeta}}).$ <p>But, on the rod (which is the <u>boundary</u> of the unit <math>\zeta</math>-disc), we have <math>\bar{\zeta} = \frac{1}{\zeta}</math>, so this means:</p> $\bar{z}(\zeta) = -i + \frac{1}{2}(\zeta + \frac{1}{\zeta}).$ <p>Now <math>\bar{z}'(\zeta) = \frac{1}{2}(1 - \frac{1}{\zeta^2})</math>; so this means that:</p> $\begin{aligned} \bar{z}(\zeta) \frac{d}{dz} &= \left(-i + \frac{1}{2}(\zeta + \frac{1}{\zeta})\right) \cdot \frac{2}{(1 - \frac{1}{\zeta^2})} \\ &= \zeta(\zeta^2 - 2i\zeta + 1). \end{aligned}$ <p>Putting everything together:</p> $-\bar{F}\left(\frac{1}{\zeta}\right) + \zeta(\zeta^2 - 2i\zeta + 1) F'(\zeta) + G(\zeta) = \bar{U} - \frac{i\Omega}{2}\left(\zeta + \frac{1}{\zeta}\right)$ <p>as required.</p>	<p style="color: red;">1</p> <p>seen similar</p> <p style="color: red;">1</p> <p>seen similar</p> <p style="color: red;">2</p> <p>seen similar</p>
	Setter's initials <b>SJB</b>	Checker's initials 
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	EXAMINATION SOLUTION 23 - 24	Course: Applied Complex Analysis
Question 5	Topic COMPLEX METHODS FOR BIHARMONIC EQUATION	Marks& seen/unseen
Parts	<p>(d). we plug <math>F(\zeta) = \frac{A}{\zeta} + B + C\zeta</math> into the boundary condition from part (c):</p> $-\left(\bar{A}\zeta + \bar{B} + \frac{\bar{C}}{\zeta}\right) + \frac{\zeta(\zeta^2 - 2i\zeta + 1)}{(\zeta^2 - 1)} \left[ C - \frac{A}{\zeta^2} \right] + G(\zeta)$ $= \bar{U} - \frac{i\Omega}{2} \left( \zeta + \frac{1}{\zeta} \right)$ $= \frac{1}{\zeta^2 - 1} = -(1 - \zeta^2)^{-1}$ $= -[1 + \zeta^2 + \zeta^4 + O(\zeta^6)]$ <p style="text-align: center;"><i>as <math>\zeta \rightarrow 0</math></i></p> <p><math>\Rightarrow</math></p> $G(\zeta) = \frac{1}{\zeta} \left( -\bar{C} - A - \frac{i\Omega}{2} \right) + (\bar{B} + 2Ai + \bar{U})$ $+ \left( \bar{A} - 2A + C - \frac{i\Omega}{2} \right) \zeta + O(\zeta^2),$ <p style="text-align: center;"><i>as <math>\zeta \rightarrow 0</math></i></p>	<span style="color: red; border: 1px solid red; padding: 2px;">1</span> <span style="color: red; border: 1px solid red; padding: 2px;">unseen</span> <span style="color: red; border: 1px solid red; padding: 2px;">1</span> <span style="color: red; border: 1px solid red; padding: 2px;">unseen</span> <span style="color: red; border: 1px solid red; padding: 2px;">1</span> <span style="color: red; border: 1px solid red; padding: 2px;">unseen</span>
(e).	<p>In the far-field, we know:</p> $f(z) \sim z, \text{ as } z \rightarrow \infty \quad (\text{from (a).})$ $g'(z) \sim kz, \text{ as } z \rightarrow \infty \quad (\text{from (a).})$ <p>In terms of <math>\zeta</math>, these become:</p> $F(\zeta) \sim \left( i + \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right) \right) \text{ as } \zeta \rightarrow 0$ $\sim \underline{\frac{1}{2} \frac{1}{\zeta} + i} \text{ as } \zeta \rightarrow 0$ $G(\zeta) \sim \underline{\frac{k}{2} \frac{1}{\zeta} + ki} \text{ as } \zeta \rightarrow 0$	<span style="color: red; border: 1px solid red; padding: 2px;">1</span> <span style="color: red; border: 1px solid red; padding: 2px;">unseen</span> <span style="color: red; border: 1px solid red; padding: 2px;">1</span> <span style="color: red; border: 1px solid red; padding: 2px;">unseen</span>
	Setter's initials <u>SJB</u>	Checker's initials
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Parts (e). (continued).	<p>Comparing these forms with <math>F(\zeta) = \frac{A}{\zeta} + B + C\zeta</math> and the form of <math>G(\zeta)</math> from (d), leads to:</p> $A = \frac{1}{2}, B = i \quad (\text{comparing } F(\zeta))$ <p>and:</p> $\left(-\frac{1}{2} + \bar{C} - \frac{i\Omega}{2}\right) = \frac{k}{2} \Rightarrow C = \frac{k+1-i\Omega i}{2}$ $(-i + \bar{U} + i) = ki \Rightarrow U = -ki$ <p>(comparing <math>G(\zeta)</math>).</p>	1 unseen
(f).	$\oint_{ \zeta =1} G(\zeta) z'(\zeta) d\zeta = \oint_{ \zeta =1} \left( \frac{k-1}{2\zeta} + ki + \left(\frac{k}{2} - i\Omega i\right)\zeta + \dots \right) d\zeta$ $\times \frac{1}{2} \left(1 - \frac{1}{\zeta^2}\right) d\zeta$ $= \oint_{ \zeta =1} \left( \dots + \left(\frac{k}{4} - \frac{k}{4} + \frac{\Omega i}{2}\right) \frac{1}{\zeta} + \dots \right) d\zeta$ $= 2\pi i \cdot \left(\frac{i\Omega}{2}\right) = -\Omega\pi, \text{ by the residue theorem.}$ <p>Hence, torque = 0 <math>\Rightarrow -2 \cdot \operatorname{Re}\{-\Omega\pi\} = 0</math></p> $\Rightarrow \underline{\Omega = 0}$	2 unseen
(g).	$U = U_{\text{rod}} + iV_{\text{rod}} = -ki \Rightarrow \boxed{U_{\text{rod}} = 0, V_{\text{rod}} = -k, \Omega = 0}$ <p>i.e. The rod moves vertically down at speed <math>k</math> and doesn't rotate.</p>	1 unseen 1 unseen 1 Qstdn: 20
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