Probability for Statistics
Problem Sheet 3

Questions marked (†) may require material from Thursdays lecture.

1. Suppose that X is an absolutely continuous random variable with density function given by

$$f_X(x) = 4x^3, \text{ for } 0 < x < 1,$$

and zero otherwise. Find the density functions of the following random variables:

(a) $Y = X^4$, (b) $W = e^X$, (c) $Z = \log X$, (d) $U = (X - 0.5)^2$.

2. The measured radius of a circle, R, is an absolutely continuous random variable with density function given by

$$f_R(r) = 6r(1-r), \text{ for } 0 < r < 1,$$

and zero otherwise. Find the density functions of (a) the circumference and (b) the area of the circle.

3. Suppose that X is an absolutely continuous random variable with density function given by

$$f_X(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}, \text{ for } x > 0,$$

and zero elsewhere, with α and β non-negative parameters.

   (a) Find the density function and cdf of the random variable defined by $Y = \log X$.
   (b) Find the density function of the random variable defined by $Z = \xi + \theta Y$.

4. Let X be an absolutely continuous random variable with range $\mathbb{X} = \mathbb{R}^+$, pdf f_X and cdf F_X.

   (a) Show that
   $$\mathrm{E}(X) = \int_0^\infty [1 - F_X(x)] \, \mathrm{d}x.$$
   (b) Show also that for integer $r \geq 1$,
   $$\mathrm{E}(X^r) = \int_0^\infty r x^{r-1} [1 - F_X(x)] \, \mathrm{d}x.$$
   (c) Find a similar expression for $\mathrm{E}(X^r)$ for random variables for which $\mathbb{X} = \mathbb{R}$.

5. Consider two absolutely continuous random variables X and Y such that

$$\Pr(X \leq x \text{ and } Y \leq y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right), \text{ for } x > 0 \text{ and } -\infty < y < \infty,$$

with

$$\Pr(X \leq x \text{ and } Y \leq y) = 0, \text{ for } x \leq 0.$$

Find the joint pdf, $f_{X,Y}$. Are X and Y independent ? Justify your answer.

6. (†) Suppose that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = 24xy, \text{ for } x > 0, \ y > 0, \text{ and } x + y < 1,$$

and zero otherwise. Find

(a) the marginal pdf of X, f_X,
(b) the marginal pdf of Y, f_Y,
(c) the conditional pdf of X given $Y = y$, $f_{X|Y}$,
(d) the conditional pdf of Y given $X = x$, $f_{Y|X}$,
(e) the expected value of X,
(f) the expected value of Y,
(g) the conditional expected value of X given $Y = y$, and
(h) the conditional expected value of Y given $X = x$.

[Hint: Sketch the region on which the joint density is non-zero; remember that the integrand is only non-zero for some part of the integral range.]

For discussion

7. (†) Consider two independent random variables X_1 and X_2, exponentially distributed with rate 1. Suppose we wish to consider the density function of X_1 conditional on the event $\{X_1 = X_2\}$.

(a) One way to do this is to consider the variable $Z = X_1 - X_2$, and condition on the event $Z = 0$. Find the pdf $f(x_1|z = 0)$.
(b) Alternatively, one could consider the variable $W = \frac{X_2}{X_1}$, and condition on the event $W = 1$. Find the pdf $f(x_1|w = 1)$.
(c) Comment on your answers to the two parts above. *(This is an instance of the Borel-Kolmogorov paradox.)*

8. (*Harder*) Let X_1, X_2, X_3 be independent random variables, each with the mass function

$$\Pr(X_i = x) = (1 - p_i)p_i^{x-1}, \qquad x = 1, 2, 3, \ldots$$

Show that

$$\Pr(X_1 < X_2 < X_3) = \frac{(1 - p_1)(1 - p_2)p_2 p_3^2}{(1 - p_2 p_3)(1 - p_1 p_2 p_3)}.$$