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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2021

MATH96062/MATH97216/MATH97220 Markov Processes

The following information must be completed:

Is the paper suitable for resitting students from previous years: No (Maybe yes or no: I have changed notation, the resit student is following the current course.)

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
Q1 20 marks; Q2(a) 5 marks Q2(b) 5 marks, Q3(a) first part 2 marks

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
Q2c 5 marks, Q2a-second part 4 marks; Q3b 5 marks; Q4a 6 marks,

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
Q2d 5 marks, Q3c 4 marks; Q4c 3 marks

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
3(d) 5 marks, 4(b) 6 marks, 4(d) 5 marks

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

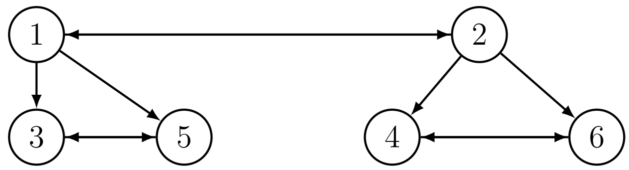
Markov Processes

Date: ??
Time: ??
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. **Solution.**

   - Incidence graph



   - The communication class $[1] = \{1, 2\}$ is transient. The classes $[3] = \{3, 5\}$ and $[4] = \{4, 6\}$ are recurrent.

   b. The matrix P restricted to the indices 3 and 5 is given by $\tilde{P} = \frac{1}{10}\begin{pmatrix} .4 & 6 \\ 5 & 5 \end{pmatrix}$. Therefore $\tilde{P}^2 = \frac{1}{100}\begin{pmatrix} 46 & 54 \\ 45 & 55 \end{pmatrix}$, and so $\mathbf{P}(x_3 = 5 \,|\, x_1 = 3) = \frac{27}{50}$. Since it is impossible to go from 3 to 6, one has $\mathbf{P}(x_4 = 6 \,|\, x_0 = 3) = 0$.

   c. The PF vector for $\tilde{P}$ is given by $(5/11, 6/11)$. The PF vector for P restricted to indices 4 and 6 is given by $(4/9, 5/9)$. Therefore, the set of all invariant measures is given by

$$\frac{t}{11}\begin{pmatrix} 0 & 0 & 5 & 0 & 6 & 0 \end{pmatrix} + \frac{1-t}{9}\begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 5 \end{pmatrix}, \quad t \in [0, 1] .$$

(Total: 0 marks)

2. **Solution.**

   a. One has $x_1 = \frac{x_0}{2} + \xi_0$, $x_2 = \frac{x_0}{4} + \frac{\xi_0}{2} + \xi_1$, and $x_3 = \frac{x_0}{8} + \frac{\xi_0}{4} + \frac{\xi_1}{2} + \xi_2$.
   The general expression is given by

$$x_n = \frac{x_0}{2^n} + \sum_{k=0}^{n-1} \frac{\xi_k}{2^{n-1-k}} .$$

(4 marks)

   b. One has $P(x, \cdot\,) = \mathbf{P}(x_n \in \cdot | x_{n-1} = x) = \mathbf{P}(\frac{x_{n-1}}{2} + \xi \in \cdot | x_{n-1} = x) = \mathcal{N}(x/2, 1)$. We have used the independence of ξ_n from x_{n-1}, since by (a), the latter is a deterministic function of $x_0, \xi_1, \ldots, \xi_{n-1}$, all of which independent of ξ_n. (6 marks)

   c. By part (a), x_n is a a Gaussian random variable, computing its mean and variance one shows that the n-step transition probabilities are given by $P^n(x, \cdot\,) = \mathbf{P}(x_n \in \cdot | x_0 = x) = \mathcal{N}(x/2^n, 2(1 - 2^{-n}))$. (5 marks)

   d. Taking the limit $n \to \infty$ in the previous expression, we find that $\mathcal{N}(0, 2)$ is an invariant measure for the system. Since it is of the form $x_{n+1} = F(x_n, \xi_n)$ with $|F(x, \xi) - F(y, \xi)| \leq \frac{1}{2}|x - y|$, the invariant measure must be unique. (5 marks)

(Total: 20 marks)

3. **Solution:** (a) A time homogeneous Markov chain on a finite group G is left invariant if the left translations preserve the matrix P: i.e. $P_{gh_1,gh_2} = P_{h_1,h_2}$ for any $g, h_1, h_2 \in G$. (2 marks)

First assume that there exists $\bar{\mathbf{P}}$ probability on G such that $P_{gh} = \bar{\mathbf{P}}(g^{-1}h)$. Let $g, h_1, h_2 \in G$ then

$$P_{gh_1,gh_2} = \tilde{\mathbf{P}}((gh_1)^{-1}gh_2) = \bar{\mathbf{P}}(h_1^{-1}h_2) = P_{h_1,h_2} ,$$

showing the left invariance.

Now assume that the left invariance holds true. Define $\bar{\mathbf{P}}(g) = P_{eg}$ where $e \in G$ is an identity element. Let $g, h \in G$ then using the left invariance:

$$P_{gh} = P_{e,g^{-1}h} = \bar{\mathbf{P}}(g^{-1}h) ,$$

thus the other implication follows. (4 marks)

(b) It is enough to show that $\mu = (1, 1, \ldots, 1) \in \mathbf{R}^{|G|}$ satisfies $\mu P = \mu$ since by normalising we have $\pi = |G|^{-1}\mu$. Let $g \in G$ then using left invariance

$$\sum_{h\in G} P_{hg} = \sum_{h\in G} P_{e,h^{-1}g} = \sum_{i\in G} P_{e,i} = 1 ,$$

where in the second equality we used the change of variables $i = h^{-1}g$ and in the last equality we used stochasticity of P. This indeed shows that $(\mu P)(i) = \sum_j \mu(j)P_{ji} = \sum_j P_{ji} = 1$ as required.

(5 marks)

(c) By above the measure π such that $\pi(g) = 1/|G|$ for all $g \in G$ is an invariant probability measure for a random walk on G. Let P be a stochastic matrix for this random walk. Now the random walk is reversible if and only if $\pi(h)P_{hg} = \pi(g)P_{gh}$, $\forall g, h \in G$, since $\pi(h) = \pi(g) = 1/|G|$ this is equivalent to say that $P_{gh} = P_{hg}$, $\forall g, h \in G \iff P_{e,g^{-1}h} = P_{e,h^{-1}}g$ for all $g, h \iff P_{e,g} = P_{g,e}$, $\forall g \in G$.

(4 marks)

(d) This is a left invariant Markov chain. Let h be the action assigning a pack of cards a permutation and g a permutation. Then $(hx)g = h(xg)$. Let P_{xy} denote the transition probability from a permutation x to a permutation y. Set $\Gamma(x) = P_{e,x}$ then $P_{x,y} = \Gamma(yx^{-1})$. The Markov chain is left invariant.

A left invariant Markov chain on a finite group has the normalised uniform measure as its invariant probability measure. The Markov chain is also irreducible, since very a positive (although it is tiny) probability the pack of cards remain in the same position. Since the chain is irreducible, it has only this one invariant measure. And $\mathbf{E}_1 T_1 = \frac{1}{\pi(1)}$= number of permutations. (5 marks)

(Total: 20 marks)

3. **Solution.**

a. Define f by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

From the hint, it follows that $T_\star f = f$, so that $T_\star$ cannot be strong Feller. (6 marks)

b. Set $V(x) = \frac{1}{x} + x$. Using the inequalities provided, one has

$$V(x_{n+1}) = \frac{1}{\sqrt[3]{x_n(1+\xi_n)}} + \sqrt[3]{x_n(1+\xi_n)} \leq \frac{1}{\sqrt[3]{x_n}} + 2\sqrt[3]{x_n} \leq \frac{1}{2x} + \frac{x}{2} + 3\;.$$

Taking expectations, shows that $\mathbf{E} V(x_{n+1}\,|\,x_n) \leq V(x_n)/2 + 3$, and thus V is a Lyapunov function for the system. (6 marks)

c. $J = [1, \sqrt{2}]$ since $1 = \sqrt[3]{1 \cdot (1+0)}$ and $\sqrt{2} = \sqrt[3]{\sqrt{2} \cdot (1+1)}$. (3 marks)

d. The measure δ_0 is an invariant measure. It is automatically ergodic because it is a δ-measure. From the fact that V is a Lyapunov function, it follows that there must be at least one invariant measure μ_+ with $\mu_+(\mathbf{R}_+) = 1$. By symmetry, every such measure has a counterpart μ_- with $\mu_-(\mathbf{R}_-) = 1$. The support of μ_+ must be included in $[1, \sqrt{2}]$ since, for almost every sequence $\{\xi_n\}$, one eventually has $x_n \in [1, \sqrt{2}]$. Write $F(x,\xi) = \sqrt[3]{x(1+\xi)}$. Then, one has

$$\Big|\frac{\partial F}{\partial x}\Big| = \frac{1}{3}\sqrt[3]{\frac{1+\xi}{x^2}} \leq \frac{\sqrt[3]{2}}{3} x^{-2/3} \leq \frac{\sqrt[3]{2}}{3} < 1\;,$$

for $\xi \in [0,1]$ and $x > 1$. Therefore, $\mathbf{E}\Big|F(x,\xi_1) - F(y,\xi_1)\Big| \leq \frac{\sqrt[3]{2}}{3} x^{-2/3}|x-y|$ there exists a unique invariant measure with support in $[1,\sqrt{2}]$. To sum up, there are three ergodic invariant measures, δ_0, μ_+, and μ_-, with respective supports $\{0\}$, $[1,\sqrt{2}]$, and $[-\sqrt{2}, -1]$. (5 marks)

(Total: 20 marks)

5. **Solution.** (a) $\theta\colon \Omega \to \Omega$ is a measure preserving map if $\mathbf{P}(\theta^{-1}(A)) = \mathbf{P}(A)$ for every $A \in \mathcal{F}$ (i.e. $\theta_* \mathbf{P} = \mathbf{P}$). We say $\mathbf{P}$ is ergodic or θ is ergodic if any θ-invariant set has either measure 0 or measure 1. (4 marks)

(b) (i) Let $V(x) = x^2$, then

$$TV(x) = \mathbf{E}(x_{n+1}^2 | x_n = x) = (F(x)^2) + 2F(x)\mathbf{E}(\xi_{n+1}) + \mathbf{E}(\xi_{n+1}^2) \le \frac{1}{2}(1 + |x|^2)$$

Since V is not identically infinity and $\{y : V(y) \le a\}$ is compact following from it is a continuous function, it is a Lyapunov function. (4 marks)

Also $Tf(x) = \mathbf{E}(f(x_{n+1})|x_n = x) = \int_{\mathbf{R}} f(F(x) + y)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|y|^2} dy$. If f is bounded continuous, so is Tf by the dominated convergence theorem, and Feller property holds.

By the Lyapunov test, it has an invariant probability measure. (4 marks)

We use the structure theorem: given a time homogeneous transition probability P, every invariant probability measure is a convex combination of ergodic invariant probability measures. If there is an invariant probability measure, an ergodic invariant probability measure exists by the structure theorem, denote it by π and $\mathbf{P}_\pi$ the measure on the path space. Then θ is a measure preserving transformation and $\mathbf{P}_\pi$ is ergodic w.r.t. θ. (4 marks)

(ii) Let X_n be the canonical process on $(\mathcal{X}^Z, \mathcal{B}(\mathcal{X}^Z), \mathbf{P}_\pi)$ so that $X_n(\omega) = \omega(n)$ where ω is a sequence. By uniqueness of the invariant measure, $\mathbf{P}_\pi$ is ergodic. Let $B = \{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\}$. It is an invariant set w.r.t. θ. Thus $\mathbf{P}_\pi(B) \in \{0, 1\}$. (4 marks)

(Total: 20 marks)