MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Quiz 1

Question 1 (Differential equation under time reversal, the autonomous case).
Consider a solution $\lambda : I \to \mathbb{R}^d$ of a differential equation $\dot{x} = f(x)$, and consider the *time-reversal* of this solution, given by $\mu(t) := \lambda(-t)$, defined on the interval $-I := \{t \in \mathbb{R} : -t \in I\}$. The time-reversal is a solution to the differential equation

(a) $\dot{x} = -f(-x)$,

(b) $\dot{x} = -f(x)$,

(c) $\dot{x} = f(-x)$,

Question 2 (Differential equation under time reversal, the nonautonomous case).
Consider a solution $\lambda : I \to \mathbb{R}^d$ of a differential equation $\dot{x} = f(t, x)$, and consider the *time-reversal* of this solution, given by $\mu(t) := \lambda(-t)$, defined on the interval $-I := \{t \in \mathbb{R} : -t \in I\}$. The time-reversal is a solution to the differential equation

(a) $\dot{x} = -f(-t, -x)$,

(b) $\dot{x} = -f(|t|, x)$,

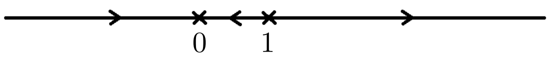
(c) $\dot{x} = -f(-t, x)$,

(d) $\dot{x} = f(-t, x)$,

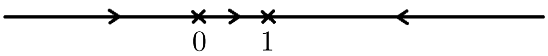
(e) $\dot{x} = -f(t, -x)$.

Question 3 (One-dimensional phase portraits).
The phase portrait of the one-dimensional differential equation $\dot{x} = x(x-1)$ is given by

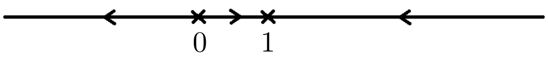
(a)



(b)



(c)



Question 4 (Constant solutions to nonautonomous differential equations).
Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and consider the differential equation

$$\dot{x} = f(t, x)\,. \tag{1}$$

Suppose that there exists an initial pair $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ such that $f(t_0, x_0) = 0$. Which of the following two statements is true?

(a) The differential equation (1) has a constant solution.

(b) The differential equation (1) does not necessarily have a constant solution.

Question 5 (Polar coordinates).

Consider the one-dimensional differential equation of order two,

$$\ddot{x} = -ax\,,$$

modelling frictionless harmonic oscillations of a spring with $a = \frac{k}{m} > 0$, where k is the spring constant and m the mass of the body attached to the spring. This differential equation corresponds to a first-order two-dimensional differential equation (see Repetition Material 1). Use polar coordinates to determine for which values of a, the angular rotation speed $\dot{\phi}$ of this two-dimensional system is constant in time. The angular rotation speed is constant in time for

(a) $a = 2\pi$,

(b) $a = 1$,

(c) for all $a > 0$.

Polar coordinates. Consider a continuous function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ and the differential equation

$$\dot{x} = f(x)\,, \tag{2}$$

and the corresponding system in *polar coordinates*, given by

$$\dot{r} = p(r,\phi)\,, \qquad \dot{\phi} = q(r,\phi)\,, \tag{3}$$

where

$$\begin{aligned} p(r,\phi) &:= f_1(r\cos\phi, r\sin\phi)\cos\phi + f_2(r\cos\phi, r\sin\phi)\sin\phi\,,\\ q(r,\phi) &:= \frac{1}{r}\big(f_2(r\cos\phi, r\sin\phi)\cos\phi - f_1(r\cos\phi, r\sin\phi)\sin\phi\big)\,. \end{aligned}$$

It is straightforward to show that if $\mu : I \to \mathbb{R}^2$ is a solution to (3), then $\lambda : I \to \mathbb{R}^2$, defined by

$$\lambda(t) := (\mu_1(t)\cos\mu_2(t), \mu_1(t)\sin\mu_2(t)) \quad \text{for all } \; t \in I\,,$$

is a solution to (2).