Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Computational Algebra & Geometry

Date: Friday, 22 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (i) State clearly the definition of a *Gröbner basis.* State Buchberger's Criterion for determining when a generating set $G = \{g_1, \dots, g_m\}$ for an ideal $I \subset k[x_1, \dots, x_n]$ is a Gröbner basis.

(ii) Using lex order, find a Gröbner basis for the ideal $I = (x^3 - y, x^2 - 4) \subset \mathbb{C}[x, y]$.

(iii) State the definition of a *reduced* Gröbner basis. Is your basis in (ii) reduced? If not, transform it into a reduced Gröbner basis.

(iv) Write down the points contained in $\mathbb{V}(I) \subset \mathbb{C}^2$.

2. (i) Let $A \subset \mathbb{Z}^n_{\geq 0}$ and let $I = (x^\alpha \mid \alpha \in A)$ be a *monomial ideal.* State Dickson's Lemma.

(ii) Let $>$ be a relation on $\mathbb{Z}^n_{\geq 0}$ satisfying:

(a) $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$;

(b) if $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$ then $\alpha + \gamma > \beta + \gamma$.

Prove that $>$ is a well-ordering if and only if $\alpha \geq 0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$.

(iii) Let $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ be a vector such that $u_1, \dots, u_n$ are positive and linearly independent over $\mathbb{Q}$. For $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ define $\alpha >_u \beta$ if and only if $u \cdot \alpha > u \cdot \beta$, where $\cdot$ is the usual dot-product. Prove that $>_u$ is a monomial order. Be sure to clearly indicate where you make use of the assumptions on u.

3. (i) Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal. Define the *radical* $\sqrt{I}$ of I, and state a version of the Nullstellensatz relating $\mathbb{I}(\mathbb{V}(I))$ and $\sqrt{I}$.

(ii) Recall that an ideal I is said to be *prime* if whenever $f, g \in \mathbb{C}[x_1, \dots, x_n]$ and $fg \in I$, then either $f \in I$ or $g \in I$. Prove that every prime ideal is radical. Give an example to show that the converse is not true.

(iii) Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal, $I \neq \mathbb{C}[x_1, \dots, x_n]$. Prove that

$$\sqrt{I} = \bigcap_{\substack{J \text{ prime} \\ I \subseteq J}} J.$$

(iv) Hence or otherwise conclude that $\sqrt{(x^a, y^b)} = (x, y)$ for any positive integers a and b.

4. (i) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Recall that the *l-th elimination ideal* I_l of I is given by $I_l := I \cap k[x_{l+1}, \ldots, x_n]$. State the Elimination Theorem.

(ii) Let $V \subset k^n$ be an affine variety, and let $\pi_l : k^n \to k^{n-l}$ denote the l-th projection map. Prove that $\pi_l(V) \subset \mathbb{V}(I_l)$, where $I = \mathbb{I}(V)$.

(iii) (a) Let $I = (x^2 + 3, y^2 + z^2 - 1) \subset \mathbb{C}[x, y, z]$, where you may assume that $G = \{x^2 + 3, y^2 + z^2 - 1\}$ is a lex-ordered Gröbner basis for I. Show that $\pi_1(\mathbb{V}(I)) = \mathbb{V}(I_1)$.

(b) Now suppose that $I = (x^2+3, y^2+z^2-1) \subset \mathbb{R}[x, y, z]$ (that is, we are now working over $\mathbb{R}$ rather than $\mathbb{C}$). Again you may assume that $G = \{x^2 + 3, y^2 + z^2 - 1\}$ is a lex-ordered Gröbner basis for I. Calculate $\mathbb{V}(I)$ and $\mathbb{V}(I_1)$.

(iv) Let $V \subset \mathbb{C}^n$ be an affine variety and set $I = \mathbb{I}(V)$. Recall that the Closure Theorem tells us that $\mathbb{V}(I_l)$ is the Zariski closure of $\pi_l(V) \subset \mathbb{C}^{n-l}$. Explain your answers in (iii) with respect to the Closure Theorem.