Statistical Theory - Problem Sheet 5

Spring 2023. Email corrections to Kolyan Ray: kolyan.ray@ic.ac.uk

Instructions: Please attempt the non-starred questions. If you have time, attempt the starred questions (they are not necessarily more difficult).

1. Let $X_1 \dots, X_n \sim^{iid} N(0, \sigma^2)$. Find the uniformly most powerful test of size α of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2$, where $\sigma_0^2 < \sigma_1^2$.

2. Let X have density function $f_\theta(x) = \frac{\theta}{(x+\theta)^2}$ for $x > 0$, where $\theta > 0$ is an unknown parameter. Find the likelihood ratio test of size 0.05 of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ and show that the probability of a Type II error is 19/21. Would the test change if we wanted to instead test $H_0 : \theta = 1$ against $H_1' : \theta > 1$?

3. Consider the parametric models from Question 1 on Problem Sheet 2 with corresponding parameter space Θ. For all these models, derive explicit expressions for the likelihood ratio test statistic of a simple hypothesis $H_0 : \theta = \theta_0$, $\theta_0 \in \Theta$, versus $H_1 = \Theta \setminus \{\theta_0\}$.

4. Suppose we observe a single random variable $X \sim f$. Consider the two densities $f_0(x) = 2(1-x)$, $x \in [0, 1]$, and $f_1(x) = 2x$, $x \in [0, 1]$, and suppose we wish to test

$$H_0 : f = f_0, \quad \text{versus} \quad H_1 : f = f_1.$$

(a) Show that the uniformly most powerful test has critical region of the form $R = \{x : x \geq B\}$ for some constant B.

Consider now selecting B using decision theory with loss function $L(a, b) = 1\{a \neq b\}$ for $a, b \in \{0, 1\}$ (i.e. you incur loss 1 if you select the wrong hypothesis). The decision rule $\delta_B(x)$ chooses 1 ($= H_1$) if $x \geq B$ and 0 ($= H_0$) otherwise.

(b) Calculate the risks $R(\delta_B, H_0)$ and $R(\delta_B, H_1)$ as functions of B. Use this to directly find the value of B which gives the minimax test procedure.

(c) Assign a prior to $\Theta = \{H_0, H_1\}$ with $\pi(\{H_0\}) = 1 - \pi(\{H_1\}) = \nu \in [0, 1]$. Find the value of $B = B(\nu)$ that gives the Bayes test procedure (decision rule) $\delta_{B(\nu)}(x)$. Find the value of ν such that $R(\delta_{B(\nu)}, H_0) = R(\delta_{B(\nu)}, H_1)$ and hence deduce that this Bayes procedure is minimax.

5. Let $X_1, \dots, X_n$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 \in (0, \infty)$. The *Student t-statistic* is given by

$$t_n = \frac{\sqrt{n}\bar{X}_n}{S_n}, \qquad \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

Show that $t_n \to^d N(0, 1)$ as $n \to \infty$. Assuming now $EX_1 = \mu \in \mathbb{R}$, deduce an asymptotic level $1 - \alpha$ confidence interval for $\mu = EX_1$.

6. Let $X_1, \dots, X_n \sim^{iid} \text{Exp}(\theta)$. Find a uniformly most powerful size α test of $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$. Show that the power function equals

$$P\left(\chi^2_{2n} \leq \frac{\theta}{\theta_0} q_{2n}(\alpha)\right),$$

where $q_{2n}(\alpha)$ the α-quantile of the χ^2_{2n} distribution, i.e. such that $P(\chi^2_{2n} \leq q_{2n}(\alpha)) = \alpha$. By inverting this test, construct a $(1-\alpha)100\%$ confidence interval for θ.

Hint: if $Z \sim \chi^2_p$ and $c > 0$, then $cZ \sim Gamma(p/2, 1/(2c))$.

7. Suppose $X_1, \dots, X_n$ are drawn from a *Pareto distribution* with density

$$f_{\theta,\nu}(x) = \frac{\theta \nu^\theta}{x^{\theta+1}} 1_{[\nu,\infty)}(x), \qquad \theta > 0, \quad \nu > 0.$$

(a) Find the MLEs for θ and ν.

(b) Show that the likelihood ratio test of

$$H_0 : \theta = 1, \ \nu \text{ unknown} \qquad \text{versus} \qquad H_1 : \theta \neq 1, \ \nu \text{ unknown}$$

has critical region of the form $\{x = (x_1, \dots, x_n) : T(x) \leq c_1 \text{ or } T(x) \geq c_2\}$ for some $0 < c_1 < c_2$ and

$$T(X) = \log\left[\frac{\prod_{i=1}^n X_i}{(\min_i X_i)^n}\right].$$

(c)* Show that under H_0, $2T \sim \chi^2_p$, where you should determine the degrees of freedom p.