Math40002 Analysis 1   Unseen 2

1. Say $S \subset \mathbb{R}$ is a set of real numbers, with the property that $\forall s \in S, \exists t \in S,\ s < t$. Can S be bounded above?

2. Say $S \subset \mathbb{R}$ is a set of real numbers, with the property that $\forall n \in \mathbb{N}, \exists t \in S,\ n < t$. Can S be bounded above.

3. (a) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above. Assume:

$$\forall a \in A,\ \exists b \in B \text{ such that } a \leq b.$$

   Prove $\sup A \leq \sup B$.

   (b) Prove that if $A \subseteq B \subset \mathbb{R}$ then $\sup A \leq \sup B$.

   (c) Let $A, B \subset \mathbb{R}$ be subsets of $\mathbb{R}$ which are bounded above. Assume:

$$\forall a \in A, \exists b \in B \text{ such that } a \geq b.$$

   Prove $\inf A \geq \inf B$.

   (d) Prove that if $A \subseteq B \subset \mathbb{R}$ then $\inf A \geq \inf B$.

4. *Nested Intervals Theorem:* For all $n \in \mathbb{N}$, assume that the closed intervals $I_n = [a_n, b_n] \subset \mathbb{R}$ satisfy the inclusions $I_{n+1} \subset I_n$.

   (a) Show that $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

   (b) Does the result hold if we allow the intervals to be open?