IMPERIAL

MATH70038

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Graph Theory

Date: Thursday, May 16, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Throughout this paper, all graphs are assumed to be finite and simple unless otherwise stated. The set $\mathbb{N}$ consists of the positive integers.

1. (a) Provide the examples requested below. For each example, give a brief justification of why it has the required properties.

   (i) Give an example of a graph G such that $|G| = 5$ and $\kappa(G) = 4$.
   (*Recall that* $\kappa(G) = \max\{k \in \mathbb{N} \mid G$ *is* k*-connected*$\}$.) (3 marks)

   (ii) Give an example of a connected graph such that $\kappa(G) < \delta(G)$.
   (*Recall that* $\delta(G) = \min\{\deg(v) \mid v \in V_G\}$.) (3 marks)

   (iii) Give an example of a graph whose automorphism group is *not abelian* (that is, *not commutative*).
   (*You may use without proof that certain groups are known to be non-abelian.*) (3 marks)

   (b) Show that if a graph G is bipartite, then it does not have any odd cycle. (4 marks)

   (c) Show that if a graph G does not have any odd cycle then it is bipartite. (7 marks)

(Total: 20 marks)

2. (a) Answer the questions below. If you are asked to provide an example, give a brief justification of why it has the required properties.

(i) Let G be a graph. We say that a matching M of G is perfect if every vertex is the endpoint of an edge in M. Give an example of a complete graph that does not have a perfect matching. (2 marks)

(ii) Give an example of a non-trivial graph (not a K_1) in which a minimal cover and a maximal matching do not have the same size. (2 marks)

(iii) Find a minimal cover of K_5, the complete graph on 5 vertices. (2 marks)

(iv) Find a perfect matching of $K_{3,3}$.
(*Recall that $K_{3,3}$ is the bipartite graph with two parts A and B, each consisting of 3 vertices, and such that every vertex in A is connected to all vertices in B.*) (2 marks)

(b) (i) Let G be a graph. Let $\sigma \in \mathrm{Aut}(G)$. Prove that if C is a minimal cover of G, then $\sigma(G)$ is also a minimal cover of G. (4 marks)

(ii) Prove that a perfect matching is maximal.
(*Recall that a matching is maximal when its cardinality is maximal among the cardinalities of all the matchings.*) (3 marks)

(c) Let $n \in \mathbb{N}$. Prove that a maximal matching of a complete graph on n vertices has cardinality $\lfloor n/2 \rfloor$. (5 marks)

(Total: 20 marks)

3. (a) For each of the following statements say if it is true or false. Justify your answer with a short proof or a counterexample. Credit will only be given for the part of your answer containing the justification. For this part of the question you may use without proof that certain graphs are known to be non-planar.

(i) A 5-colourable graph is planar. (2 marks)

(ii) A complete graph on n vertices is never planar for $n \geq 5$. (2 marks)

(iii) A disjoint union of finitely many planar graphs is planar. (2 marks)

(iv) A triangle-free graph is planar. (2 marks)

(b) Without using Wagner's theorem, prove that $K_{3,3}$ is not planar. (5 marks)

(c) Let G be a maximal planar graph of order $|G| \geq 6$. Suppose we add an edge to G and obtain a (non-planar) graph G'.

(*Recall that a graph is maximal planar when adding an edge makes it non-planar.*)

(i) Prove that if $K_{3,3}$ is a minor of G', then K_5 is a minor of G'. (3 marks)

(ii) Prove that G' has both K_5 and $K_{3,3}$ as minors. (4 marks)

(Total: 20 marks)

4. (a) Answer the questions below. If you are asked to provide an example, give a brief justification of why it has the required properties.

   (i) Prove that $R(3,3) > 5$ and that $R(4,3) \leq 10$.
   (*Hint: for this step, you may use without proof the upper bound for the Ramsey number $R(k,\ell)$ involving a binomial coefficient.*) (5 marks)

   (ii) Let G and H be graphs. Prove that $R(G,H) \leq R(|G|,|H|)$. (3 marks)

(b) Let $k, \ell \in \mathbb{N}$. Prove that, if $R(k-1,\ell)$ and $R(k,\ell-1)$ exist, then

$$R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1).$$

(5 marks)

(c) Let $k \in \mathbb{N}$ and let $s_1, \dots, s_k \in \mathbb{N}$. If it exists, we define

$$N_k(s_1, \dots, s_k) \in \mathbb{N}$$

to be the least positive integer such that, for all $n \geq N_k(s_1, \dots, s_k)$, if the edges of K_n are coloured with colours $\{1, \dots, k\}$, then there is always a complete subgraph of order s_i with all edges of colour i, for some $i \leq k$. Use that $R(s_1, s_2)$ always exists for all $s_1, s_2 \in \mathbb{N}$ to prove that $N_k(s_1, \dots, s_k)$ exists for all $k \geq 2$ and $s_1, \dots, s_k \in \mathbb{N}$.

(*Note that in this problem we allow incident edges to have the same colour.*)

Hint: Set up an induction on k where the base case is $k = 2$. Then show that, for $k \geq 3$, $N_k(s_1, \dots, s_k) \leq N_{k-1}(R(s_1, s_2), s_3, \dots, s_k))$. (7 marks)

(Total: 20 marks)

5. (a) Let G be the bipartite graph defined as follows: $V_G = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ with $a_i E_G b_i$ for $i = 1, 2, 3$ and no other edges.

   (i) Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Prove that $d(A, B) = 1/3$. (2 marks)

   (ii) Prove that (A, B) is not a $(1/3)$-regular pair. (3 marks)

(b) Let $m, n \in \mathbb{N}$ and let G be a complete bipartite graph with parts A and B of size m and n respectively (that is $G \cong K_{m,n}$).

   (i) Prove that for all $X \subseteq A$ and for all $Y \subseteq B$, $d(A, B) = 1$. (2 marks)

   (ii) Let ε be a positive real number. Prove that (A, B) is an ε-regular pair. (2 marks)

(c) Let G be a graph.

   (i) Let α, ε be positive real numbers with $\alpha \geq \varepsilon$. Suppose that (A, B) is an ε-regular pair and that $A' \subseteq A$, $B' \subseteq B$ such that

$$|A'| \geq \alpha|A|, \quad |B'| \geq \alpha|B|.$$

   Let $\varepsilon' = \max(\varepsilon/\alpha, 2\varepsilon)$. Prove that (A', B') is an ε'-regular pair. (5 marks)

   (ii) Let ε be a positive real number with $\varepsilon \leq 1/2$. Suppose $\ell, k \in \mathbb{N}$ with $k \geq 2$ and $\ell \leq 1/\varepsilon$. Suppose further that $\{V_0, V_1, \ldots, V_k\}$ is an ε-regular partition of V_G and that it is possible to further partition each V_i (for $i = 1, \ldots, k$) into ℓ pairwise disjoint subsets

$$V_i^1, \ldots, V_i^\ell$$

   (such that $V_i = V_i^1 \cup \cdots \cup V_i^\ell$) **all of the same cardinality**. Prove that

$$\{V_0, V_1^1, V_1^2, \ldots, V_1^s, \ldots, V_k^1, \ldots, V_1^2, \ldots, V_k^\ell\}$$

   is an $(\ell\varepsilon)$-regular partition. (6 marks)

(Total: 20 marks)

Module:	MATH60038/70038
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH60038/70038 Graph Theory

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes.

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 9 marks, 2(a) 8 marks, 3(a) 8 marks, 4(a) 8 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(b) 4 marks, 2(b)(i) 4 marks, 3(b) 5 marks, 4(b) 5 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
1(c) 3 marks, 2(b)(ii) 3 marks, 3(c)(i) 3 marks, 4(c) 3 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(c) 4 marks, 2(c) 5 marks, 3(c)(ii) 4 marks, 4(c) 4 marks.

Signatures are required for the final version:

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Solutions

1. (a) Below are some examples with justifications. Solutions that use less words are also accepted as long as the key points are there.

   i. An example is the complete graph on 5 vertices K_5 . Indeed if we remove $\ell < 4$ vertices, the resulting graph is $K_{5-\ell}$, which is connected. The graph cannot be 5-connected by definition, because its order is 5. Hence $\kappa(G) = 4$. (3 marks) **Cat A** seen

   ii. Let G be the bowtie graph (two copies of K_3 glued together on one vertex v). Then $\delta(G) = 2$, but removing the vertex v makes the graph disconnected, so it cannot be 2-connected. (3 marks) **Cat A** seen

   iii. The null graph on 3 vertices $\overline{K_3}$ has S_3 as its automorphism group, because all permutations of the 3 vertices are automorphism (there are no edges that can be broken). S_3 is not abelian so $\overline{K_3}$ is an example. (3 marks) **Cat A** seen

   (b) Let G be a bipartite graph with parts A and B. If a path starts in A and traverses an odd number of edges then it ends in B (or vice-versa if a path starts in B). Then it is clear that a bipartite graph does not have any odd cycles because a path cannot return to the same vertex without traversing an even number of edges.

   (4 marks) **Cat B** unseen

   (c) Conversely, assume that a graph does not have any odd cycles. A graph is bipartite if and only if every connected component is bipartite or is trivial, so we may assume that G is connected. We show the graph is bipartite by defining a partition in the following way:

   - choose a vertext $u \in V_G$.
   - define $A = \{v \in V_G \mid \text{there is an odd-length path from } u \text{ to } v\}$.
   - define $B = \{v \in V_G \mid \text{there is an even-length path from } u \text{ to } v\}$. Note that the path (u) has length 0 so $u \in B$.

   (3 marks) **Cat C**

   unseen Clearly, since G is connected $A \cup B = V_G$. Moreover, there are no odd-length cycles in G so $A \cap B = \emptyset$. Indeed, suppose $v \in V_G$ and there are two paths from u to v, one of odd length and the other of even length, say P_1 and P_2. Then there are two vertices x and y on both paths such that P_1 does not intersect P_2 between x and y and the concatenation of (the piece of) P_1 from x to y and of P_2 form y to x is a cycle of odd length. If this is not the case, then P_1 and P_2 are either both odd-length or both even-length. Clearly there cannot be an edge between two vertices in A or we get that there is both an odd-length and an even-length cycle between u and another vertex. The same is true for the vertices of B, so we are done. (4 marks) **Cat D** unseen

   (Total: 20 marks)

2. (a) i. One such example is K_3, because if an edge e belongs to a matching then the other two cannot be in the same matching as they are incident with e. Hence, if two vertices are matched, the third one cannot be.

(2 marks) **Cat A** seen

ii. K_3 is an example. Indeed a maximal matching has cardinality 1 (for the reasons discussed above), while a minimal cover needs at least two vertices. Indeed, if $\{u_1, u_2, u_3\}$ are the vertices of a complete graph of order 3 and one of them is chosen, say u_1, then we need another one to cover the edge $\{u_2, u_3\}$.

(2 marks) **Cat A** seen

iii. Let $G = K_5$. If we leave out two vertices from a subset $C \subseteq V_G$, then C is not a cover, because there is an edge between the remaining vertices that is not covered by C. Thus a minimal cover has cardinality 4. Indeed, all subsets $C \subseteq V_G$ with $|C| = 4$ contain an endpoint for all the edges of G: if $C = V_G \smallsetminus \{v\}$ then all edges incident with v have an endpoint in C (like the other edges, as they have both endpoints in C).

(2 marks) **Cat A** seen

iv. Let $G = K_{3,3}$ and $V_G = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. Where, identifying the edges with pairs of vertices for convenience, the edges of G are $\{a_i, b_j\}$ for $i, j \in \{1, 2, 3\}$. Then $M = \{\{a_i, b_i\} \mid i = 1, 2, 3\}$ is a perfect matching. (2 marks) **Cat A** seen

(b) i. We only need prove that the image of a cover is a cover, minimality will follow because σ is bijective and so $|\sigma(C)| = |C|$. For convenience we identify an edge in E_G with the pair of its endpoints in V_G. If $\sigma \in \mathrm{Aut}(G)$ and $\{u, v\} \in E_G$, then $\{\sigma(u), \sigma(v)\} \in E_G$ and all edges of G are of the form $\{\sigma(u), \sigma(v)\}$ for some $u, v \in V$ such that uE_Gv, because σ is an automorphism. (In other words $\{u, v\} \mapsto \{\sigma(u), \sigma(v)\}$ defines a bijective map $E_G \to E_G$.) It follows that if C contains an endpoint for all edges of G, then $\sigma(C)$ has the same property and so it is a cover.

(4 marks) **Cat B** seen (similar to 2023 exam)

ii. A matching M has cardinality $|M| \leq |G|/2$, because every edge has two endpoints and two edges in a matching do not share endpoints. A perfect matching contains exactly $|G|/2$ edges because no two edges in M are incident, so we need $|G|/2$ of them in order to match every vertex in G. It follows that a perfect matching has the maximum possible cardinality for a matching and so it is maximal. (3 marks) **Cat C** unseen

(c) Induction on the order of the graph. The base cases are K_1 (where the only matching has cardinality 0) and K_2 (where the maximal matching has cardinality 1). Suppose the statement is true for all complete graphs on less than n vertices and let M be a maximal matching in a complete graph of order $n + 1$. Consider an edge $e \in M$ with endpoints u and v. Then all other edges incident with u or v cannot be in M. It follows that $M' = M \smallsetminus \{e\}$ is a maximal matching of a complete graph of order $n - 1$ (if M' is not maximal then M is not maximal either).Therefore, $|M| = \lfloor (n-1)/2 \rfloor + 1$ by inductive hypothesis. We conclude because $\lfloor (n-1)/2 \rfloor + 1 = \lfloor (n+1)/2 \rfloor$.

(5 marks) **Cat D** unseen

(Total: 20 marks)

3. (a) i. False. The complete graph on 5 vertices is not planar but it is 5-colourable, because its order is 5.

(2 marks) $\mathbf{Cat\ A}$ seen

ii. True deleting $n-5$ vertices creates a non-planar minor (a K_5)

(2 marks) $\mathbf{Cat\ A}$ seen

iii. True. We may draw each planar graph separately in the plane

(2 marks) $\mathbf{Cat\ A}$ seen

iv. False $K_{3,3}$ is non-planar and it is triangle-free because it is bipartite.

(2 marks) $\mathbf{Cat\ A}$ seen

(b) $K_{3,3}$ has 6 vertices and 9 edges, so if it were planar then it would have 5 faces by Euler's formula.

(1 mark) $\mathbf{Cat\ B}$ seen (Problem Sheet 3)

As with K_5, each edge of $K_{3,3}$ borders 2 faces. Each face of $K_{3,3}$ must be bounded by at least 4 edges, as $K_{3,3}$ is bipartite and all its cycles have an even number of vertices. Therefore $2|E| \geq 4|F|$.

(3 marks) $\mathbf{Cat\ B}$ seen (Problem Sheet 3)

However, if $K_{3,3}$ is planar then $2 \times 9 \geq 4 \times 5$, which is clearly false, so $K_{3,3}$ cannot be planar.

(1 mark) $\mathbf{Cat\ B}$ seen (Problem Sheet 3)

(c) i. Assume there is a $K_{3,3}$ minor and let $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ be its vertices with edges $\{a_i, b_j\}$ for $i, j \in \{1, 2, 3\}$. Since G was maximal planar, there must have been more edges that have been deleted to create $K_{3,3}$. Namely, there must have been an edge between every pair of the a_i's and between every pair of the b_i's. This means that before $K_{3,3}$ appeared as a minor, there was a K_6 minor. We conclude, because K_5 is a minor of K_6 (obtained by deleting one vertex). (3 marks) $\mathbf{Cat\ C}$ unseen

ii. By Kuratowski-Wagner (Wagner's theorem in the old notes before 2023) G' must have either K_5 or $K_{3,3}$ as a minor. We just showed that if $K_{3,3}$ is a minor, then K_5 is also a minor. It remains to show that if K_5 is a minor, then $K_{3,3}$ is also a minor. Assume then that $|G| \geq 6$ and K_5 is a minor and let $\{v_1, \ldots, v_5\}$ be the vertices of this minor. Then, before K_5 was seen as a minor, there must have been at least another vertex v_6 and edges from v_6 to the other v_i's (again this is because G was maximal planar). To see $K_{3,3}$ as a minor from here, we could – for example – delete the edges $\{v_6, v_4\}$, $\{v_6, v_5\}$ and then all the edges between the pairs in $\{v_1, v_2, v_3\}$. This graph is bipartite with parts $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ by construction and is complete because we did not remove the edges connecting v_6 to v_1, v_2, and v_3 (while the other edges are already there from the K_5 we considered in the beginning). (4 marks) $\mathbf{Cat\ D}$ unseen

(Total: 20 marks)

4. (a) i. We need to find a graph on 5 vertices without a triangle or 3 independent vertices The cycle graph C_5 is clearly triangle-free and every two non-adjacent vertices we select contain all the rest as their neighbours, so there cannot be 3 independent vertices.

(3 marks) **Cat A** seen

The upper bound comes directly from the binomial upper bound: that is

$$R(4,3) \leq \binom{3+4-2}{4-1} = \binom{5}{3} = 10.$$

(2 marks) **Cat A** seen

ii. Proof from the notes. Clearly $G \subseteq K_{|G|}$ and $H \subseteq K_{|H|}$. If K is such that $|K| \geq R(|G|,|H|)$ then either $K_{|G|} \leq K$ or $\overline{K_{|H|}} \leq K$. In the first case $G \subseteq K$. In the second case $H \subseteq K_{|H|}$; hence, $H \subseteq K_{|H|} \subseteq \overline{K}$, from which we deduce $H \subseteq \overline{K}$.

(3 marks) **Cat A** seen

(b) Proof from the notes. We need to show that there is R such that for all graphs G with $|G| \geq R$, $K_k \leq G$ or $\overline{K_\ell} \leq G$. To this end, let $R = R(k-1,\ell) + R(k,\ell-1)$ and let $v \in V_G$. We define

$$N_G(v) = \{u \in V_G \mid uE_Gv\}.$$

We distinguish the following two cases:

i. if $|N_G(v)| \geq R(k-1,\ell)$ and $\overline{K_\ell} \not\leq G$, then there is an $A \leq N_G(v)$ such that $G[A] \cong K_{k-1}$. Since v is connected by an edge to all the vertices in $N_G(v)$, it follows that $G[A \cup \{v\}] \cong K_k$. (2 marks) **Cat B** seen

ii. If $|N_G(v)| < R(k-1,\ell)$ then $|V_G \smallsetminus N_G(v)| > R(k,\ell-1)$. Therefore,

$$|V_G \smallsetminus (N_G(v) \cup \{v\}| \geq R(k,\ell-1).$$

It follows that, if $K_k \not\leq G$, there is a $B \leq G \smallsetminus (N_G(v) \cup \{v\})$ such that $G[B] \cong \overline{K_{\ell-1}}$. Since there are no connecting edges between v and the vertices in B, we deduce that $G[B \cup \{v\}] \cong K_\ell$. (3 marks) **Cat B** seen

(c) Following the hint, the case $k = 2$ follows directly from the graph theoretic version of Ramsey's theorem. Let $n \geq R(s_1, s_2)$. We colour the edges of K_n with 1 and 2 (red and blue for convenience), then delete those coloured with 2 and define a graph G. By Ramsey's theorem this graph contains and induced K_{s_1} or an induced $\overline{K_{s_2}}$ as a subgraph. Now it is clear that this construction implies $N_2(s_1,s_2) \leq R(s_1,s_2)$ as an induced K_{s_1} is an red monochromatic K_{s_1} in K_n and an induced $\overline{K_{s_2}}$ is a blue monochromatic K_{s_2} in the original K_n.

(3 marks)**Cat C** unseen

To show that, for $k \geq 3$, $N_k(s_1,\ldots,s_k) \leq N_{k-1}(R(s_1,s_2),s_3,\ldots,s_k))$, suppose that $n \geq N_{k-1}(R(s_1,s_2),s_3,\ldots,s_k))$ and that the edges of K_n have been coloured with k colours: $1,\ldots,k$. Identifying colours 1 and 2 we get a $(k-1)$-colouring of K_n with colour $1,3,\ldots,k$. Let $\ell_1 = R(s_1,s_2)$ and $\ell_i = s_i$ for $i = 3,\ldots,k$. The assumption $n \geq N_{k-1}(R(s_1,s_2),s_3,\ldots,s_k))$ ensures by induction that there is $i \in \{1,3,\ldots,k\}$ such that K_n contains a monochromatic K_{ℓ_i} subgraph. If $i \geq 3$ we are done. If $i = 1$, the subgraph K_{ℓ_1} contains a monochromatic K_{s_1} or a monochromatic K_{s_2} by the base case.

(4 marks) **Cat D** unseen

(Total: 20 marks)

5. (a) i. We have $|A| = |B| = 3$ and there are exactly 3 edges from A to B. Hence

$$d(A, B) = 3/9 = 1/3.$$

(2 marks) seen

ii. Consider $Y = \{a_1\}$ and $B = \{b_1\}$. Then, since there is exactly one edge X to Y and both sets are singletons, we compute $d(X, Y) = 1$. As noted already $|A| = |B| = 3$; hence,

$$|X| \geq \frac{1}{3}|A|, \quad |Y| \geq \frac{1}{3}|B|.$$

However $d(X, Y) - d(A, B) = 2/3 > 1/3$. Thus the pair (A, B) is not $(1/3)$-regular.

(3 marks) seen

(b) i. Clearly there are mn edges from A to B, and these two sets have sizes m and n respectively. Hence

$$d(A, B) = mn/mn = 1.$$

(2 marks) seen

ii. For all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|B| \geq \varepsilon|B|$ the graph $G[X \cup Y]$ is complete bipartite so, by the previous point, $d(X, Y) = 1$. Now

$$|d(A, B) - d(X, Y)| = 0 \leq \varepsilon,$$

hence the pair (A, B) is ε-regular.

(2 marks) seen

(c) i. Since $\alpha \geq \varepsilon$, the inequalities $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$ imply that$|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$. Hence, if $d = d(A, B)$ and $d' = d(A', B')$,

$$|d - d'| \leq \varepsilon$$

because (A, B) is a regular pair.

(1 mark) unseen

Suppose that $X \subseteq A'$ and $Y \subseteq B'$ are such that

$$|X| \geq \varepsilon'|A'|, \quad |Y| \geq \varepsilon'|B'|.$$

Then $\varepsilon' \geq \varepsilon/\alpha$ implies that

$$|X| \geq \varepsilon'|A'| \geq \varepsilon'\alpha|A| \geq \varepsilon|A|, \quad |Y| \geq \varepsilon|A|.$$

Hence, by regularity, we get that $|d - d(X, Y)| \leq \varepsilon$. (2 marks) unseen

We conclude because

$$|d' - d(X, Y)| = |d' - d + d - d(X, Y)| \leq |d' - d| + |d - d(X, Y)| \leq 2\varepsilon \leq \varepsilon'.$$

(2 marks) unseen

ii. The case $\ell = 1$ is trivial as there is no subdivision. Assume that $\ell \geq 2$ and set $\alpha = 1/\ell$. Then the assumptions imply that $\alpha \geq \varepsilon$ and that $\varepsilon' = \max(\varepsilon/\alpha, 2\varepsilon) = \ell\varepsilon$. Hence, for all $i, j \leq k$, such that (V_i, V_j) is ε-regular, the previous point implies that

$$(V_i^{i'}, V_j^{j'})$$

is $(\ell\varepsilon)$-regular for all $i', j' \leq \ell$. (2 marks) unseen

Now, at most εk^2 of the pairs (V_i, V_j) are irregular, so at most $\varepsilon\ell^2 k^2$ of the pairs $(V_i^{i'}, V_j^{j'})$ will be irregular for $i \neq j$ (and for all $i', j' \leq \ell$). However, for $i = j$ all the pairs $(V_i^{i'}, V_i^{j'})$ (for $i', j' \leq \ell$) may be irregular. There are

$$\frac{\ell(\ell-1)}{2}k$$

of these pairs in total. (2 marks) unseen

We get at most

$$\varepsilon\ell^2 k^2 + \frac{\ell(\ell-1)}{2}k$$

irregular pairs. Since

$$\varepsilon\ell^2 k^2 + \frac{\ell(\ell-1)}{2}k \leq (\varepsilon + 1/k)\ell^2 k^2,$$

and $(\varepsilon + 1/k) \leq (\varepsilon + 1/2) \leq 2\varepsilon \leq \ell\epsilon$, we get the desired upper bound on the number of irregular pairs. (2 marks) unseen

(Total: 20 marks)

MATH60038 Graph Theory

Question	Marker's comment
1	Very good answers for this question. There was a typo in the last question, which forgot to exclude the empty and trivial graph. No students were affected by this, but I applied more generous marking criteria in any case for that.
2	Very good answers generally. In b (ii) there was some confusion about the definition of maximal matching (this is maximal cardinality, rather than maximal for inclusion) so showing that a perfect matching is maximal for inclusion doesn't really show it has maximal cardinality without saying more.
3	Good answers for a -- b. Not so good for c, where more details were required in most cases (some forgot to use that G is maximal planar, which makes the proof automatically fail as the result is false without that assumption)
4	Generally good answers a -- b (although some of the answers to b were too simplified or contained small mistakes). c was not answered well generallynbsp;

MATH70038 Graph Theory

Question	Marker's comment
1	Very good answers for this question. There was a typo in the last question, which forgot to exclude the empty and trivial graph. No students were affected by this, but I applied more generous marking criteria in any case for that.
2	Very good answers generally. In b (ii) there was some confusion about the definition of maximal matching (this is maximal cardinality, rather than maximal for inclusion) so showing that a perfect matching is maximal for inclusion doesn't really show it has maximal cardinality without saying more.
3	Good answers for a -- b. Not so good for c, where more details were required in most cases (some forgot to use that G is maximal planar, which makes the proof automatically fail as the result is false without that assumption)
4	Generally good answers a -- b (although some of the answers to b were too simplified or contained small mistakes). c was not answered well generallynbsp;
5	Generally good answers for parts a - b and c (i). Some problems counting the number of irregular pairs in c(ii).