Assessed Coursework 2

You may discuss these problems with other students, but you must write up your own solutions.

Problem 1. Let $C : (-\infty, +\infty) \to \mathbb{R}^2$ be a regular curve with no self-intersections, and consider the surface

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in C\left((-\infty, +\infty)\right)\right\}.$$

Show that there is an open set $I \subseteq \mathbb{R}$ and an isometry from the set $\{(x, y) \in \mathbb{R}^2 \mid x \in I, y \in \mathbb{R}\}$ onto S, and determine the Gaussian curvature of S at each point.

Problem 2. Let $S \subset \mathbb{R}^3$ be a regular surface, and $\phi : U \to S$ be a chart. Assume that there is a smooth function $\lambda : U \to \mathbb{R}$ such that the first fundamental form of S at each point $\phi(u, v)$ is

$$\begin{pmatrix} e^{\lambda(u,v)} & 0 \\ 0 & e^{\lambda(u,v)} \end{pmatrix}.$$

(Such coordinates are called isothermal.)

(a) Show that the Christoffel symbols satisfy

$$\Gamma^1_{11} = \Gamma^2_{12} = \lambda_u/2, \quad \Gamma^1_{22} = -\lambda_u/2, \quad \Gamma^1_{12} = \Gamma^2_{22} = \lambda_v/2, \quad \Gamma^2_{11} = -\lambda_v/2.$$

(b) Show that the Gaussian curvature K on $\phi(U)$ satisfies

$$\Delta\lambda + 2Ke^{\lambda} = 0,$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ is the Laplacian.

Problem 3. Let S be the unit sphere in $\mathbb{R}^3$. Using the map

$$\phi(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)),$$

compute the Christoffel symbols $\Gamma^k_{i,j}$, for $i, j, k = 1, 2$, at each point in $S \setminus \{\pm(0, 0, 1)\}$.

Problem 4. Let S_1, S_2 be regular surfaces in $\mathbb{R}^3$, and assume that the maps

$$\phi(u, v) = (u\cos(v), u\sin(v), \log u), \quad \psi(u, v) = (u\cos(v), u\sin(v), v)$$

are charts for S_1 and S_2, respectively, for (u, v) in some open set with $u > 0$.

(a) Show that the Gaussian curvature of S_1 at $\phi(u, v)$ is equal to the Gaussian curvature of S_2 at $\psi(u, v)$.

(b) Show that the map $F : S_1 \to S_2$ defined as $F(\phi(u, v)) = \psi(u, v)$, that is, $F = \psi \circ \phi^{-1}$, is not a local isometry.