Imperial College London

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Analysis 1A

Lecture 11 - Cauchy sequences

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Before Cauchy sequences, one last example that we didn't have time for last lecture!

Example 3.22

Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \leq b_n \ \forall n$ and $a_n \to a, \ b_n \to b$. Prove that $a \leq b$.

Proof

Suppose by contradiction, $a > b$. Let $\varepsilon = \frac{a-b}{2} > 0$

Since $a_n \to a$ $\exists N_1 \in \mathbb{N}$ st $n \geq N_1 \Rightarrow |a_n - a| < \varepsilon$

$b_n \to b$ $\exists N_2 \in \mathbb{N}$ st $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon$

$|a_m - a| < \frac{a-b}{2}$

Let $m = \max(N_1, N_2)$

$a_m > \underbrace{a - \left(\frac{a-b}{2}\right)}_{\frac{a+b}{2}}$

$|a_m - a| < \varepsilon \Rightarrow a_m > \frac{a+b}{2} \Rightarrow a_m > b_m$ ※

$|b_m - b| < \varepsilon \Rightarrow b_m < \frac{a+b}{2}$ ∎

$|x - y| < \varepsilon \iff x \in (y - \varepsilon, y + \varepsilon)$

The notion of Cauchy sequences gives us a way to prove convergence *without* knowing the limit.

Definition

$(a_n)_{n \geq 1}$ is called a *Cauchy* sequence if and only if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N}_{>0} \text{ such that } \forall n, m \geq N, \ |a_n - a_m| < \epsilon.$$

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Remark 3.24

$m, n \geq N$ are arbitrary in this definition. It is not enough to say that $\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See problem sheet 4.

Proposition 3.25

If $a_n \to a$ then (a_n) is Cauchy.

Pf

Let $\varepsilon > 0$, since $a_n \to a$ $\exists N$ s.t. $\forall n \geq N$ $|a_n - a| < \frac{\varepsilon}{2}$

Then $\forall n, m \geq N$

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\blacksquare$

Shown Convergent $\Rightarrow$ Cauchy

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Theorem 3.27

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Corollary 3.28

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Corollary 3.28

(a_n) Cauchy $\iff$ (a_n) convergent.

Exercise 3.29

Show this is not true in $\mathbb{Q}$: there exist Cauchy sequences (a_n) with $a_n \in \mathbb{Q}$ with no limit in $\mathbb{Q}$.

To prove Theorem 3.27 we'll want the following lemma.

Lemma 3.26

(a_n) is Cauchy $\implies$ (a_n) is bounded.

Proof Let $\varepsilon = 1$, $\exists N$ s.t. $\forall m,n \geq N$, $|a_m - a_n| < 1$

$\Rightarrow \forall n \geq N$, $|a_n - a_N| < 1 \Rightarrow |a_n| \leq |a_N| + 1$

So $\max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$ is a bound

R

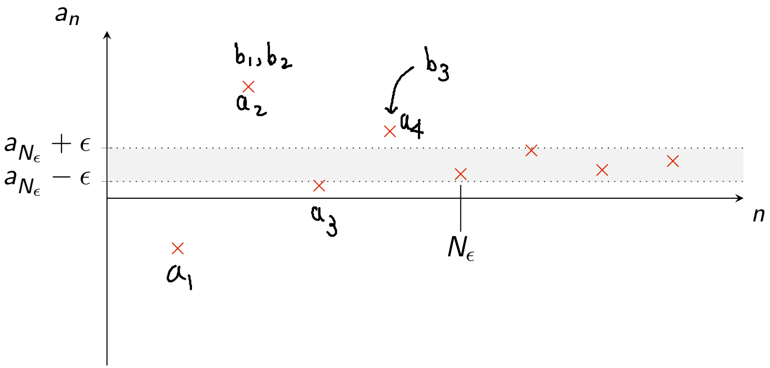
A is bounded by R if
$\forall a \in A$, $|a| \leq R$

Along with this exercise:

Exercise 3.30

If $S \subseteq \mathbb{R}$ satisfies $x < M \ \ \forall x \in S$ then $\sup S \leq M$.

For $i \in \mathbb{N}_{>0}$, set $b_i = \sup\{a_n : n \geq i\}$



Proof of Theorem 2.37

Since a_n is Cauchy, it is bounded, so we can define a new sequence $b_i = \sup\{a_n : n \geq i\}$. Since $A \subset B \Rightarrow \sup(A) \leq \sup(B)$, b_i is decreasing and since a_n is bounded, b_i is also bounded.

Let $a = \inf\{b_i : i \in \mathbb{N}_0\}$, so $b_n \downarrow a$

We claim $a_n \to a$.

Let $\varepsilon > 0$. Since a_n is Cauchy, $\exists N$ st $\forall n, m \geq N$, $|a_n - a_m| < \varepsilon/2$

$\Rightarrow \quad a_n - \varepsilon/2 < a_m < a_n + \varepsilon/2 \quad \forall m \geq N$

$\Rightarrow$ For $i \geq N \quad a_n - \varepsilon/2 < b_i \leq a_n + \varepsilon/2$

$\Rightarrow \quad a_n - \varepsilon/2 \leq a \leq a_n + \varepsilon/2 \qquad a = \inf b_i$

So $\forall n \geq N$, $a_n - \varepsilon/2 \leq a \leq a_n + \varepsilon/2 \Leftrightarrow |a_n - a| \leq \varepsilon/2 < \varepsilon$

$\boxed{\text{Therefore } a_n \to a.}$