MATH50004 Differential Equations
Spring Term 2022/23
Mid-term exam on 23 February 2023

Question 1 (total: 9 points)

Consider the initial value problem

$$\dot{x} = t\cos(x), \qquad x(0) = 0,$$

and the Picard iterates $\{\lambda_n : \mathbb{R} \to \mathbb{R}\}_{n \in \mathbb{N}_0}$ corresponding to this initial value problem.

(i) Say what it means for a function $\lambda : I \to \mathbb{R}$ ($I \subset \mathbb{R}$ an interval) to solve this initial value problem. [2 points]

(ii) Compute the first three Picard iterates λ_0, λ_1 and λ_2. [7 points]

Question 2 (total: 15 points)

Consider an initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is function and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$.

(i) State the global version of the Picard–Lindelöf theorem to locally solve such an initial value problem. [2 points]

(ii) Show that under the conditions of this theorem, the solution to this initial value problem exists globally, i.e. for all $t \in \mathbb{R}$. [6 points]

Consider now specifically the initial value problem

$$\dot{x} = x^3(2 - x)^5, \qquad x(0) = 1.$$

(iii) Explain why the global version of the Picard–Lindelöf theorem is not applicable for this initial value problem. [3 points]

(iv) Explain why, nevertheless, the solution to this initial value problem exists globally. [4 points]

Question 3 (total: 6 points)

We consider examples of autonomous differential equations

$$\dot{x} = f(x),$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and defined on an open set $D \subset \mathbb{R}^d$. The flows of such differential equations are denoted by φ.

(i) Find an example of such a differential equation and an $x \in D$ such that

$$\bigcap_{t > 0} O^+(\varphi(t, x)) = O^+(x).$$

(ii) Find an example of such a differential equation and an $x \in D$ such that

$$\bigcap_{t > 0} O^+(\varphi(t, x)) = \emptyset.$$

(i) and (ii): [3 points] each. Justify your answer in each case.