Imperial College London

MATH70019

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Theory of Partial Differential Equations

Date: 11 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the global Cauchy problem for the Burgers equation:

$$\begin{cases} \partial_t u + u\partial_x u = 0, & \text{for } x \in \mathbb{R},\ t > 0, \\ u(0,x) = g(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 0, & \text{for } x \in (-\infty, -1], \\ -x, & \text{for } x \in (-1, 0], \\ 0, & \text{for } x \in (0, \infty). \end{cases}$$

(a) Find the unique entropy solution of the problem in the time interval $t \in (0,1)$. (12 marks)

(b) Find the unique entropy solution of the problem in the time interval $t \in (1,\infty)$. (8 marks)

(Total: 20 marks)

2. Consider the initial-boundary value problem (IBVP) for the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } t > 0,\ 0 < x < \pi, \\ u(0,x) = g(x), & \text{for } 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0, & \text{for } t \ge 0, \end{cases}$$

with $g \in C^1([0,\pi])$ with $g'(0) = g'(\pi) = 0$.

(a) Use the energy method to prove the uniqueness of classical solutions to (IBVP). (6 marks)

(b) Find the candidate solution to the (IBVP) using the separation of variables technique in terms of Fourier series of the initial data. (8 marks)

(c) Let $g(x)$ be nonnegative. Prove that the solution to (IBVP) is also nonnegative. (6 marks)

(Total: 20 marks)

3. The Duhamel principle for the wave equation.

(a) Let $v(t, x; s)$, where t is time, x is space, and s is a parameter, be a solution to the initial-value problem

$$\begin{cases} \partial_t^2 v(\cdot; s) - \partial_x^2 v(\cdot; s) = 0, & \text{in } t > 0,\ x \in \mathbb{R}, \\ v(s, x; s) = 0, \qquad \partial_t v(s, x; s) = f(s, x), & \text{for } x \in \mathbb{R}. \end{cases}$$

with $f \in C^1([0, \infty) \times \mathbb{R})$. Compute $v(t, x; s)$ in terms of f. (4 marks)

(b) Show that $u(t, x) = \int_0^t v(t, x; s)\, ds$ solves

$$\partial_t^2 u(t, x) - \partial_x^2 u(t, x) = f(t, x), \qquad \text{in } t > 0,\ x \in \mathbb{R}.$$

(5 marks)

(c) Deduce the form of solution $u(t, x)$ to the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(t, x), & \text{in } t > 0,\ x \in \mathbb{R}, \\ u(0, x) = g(x), \quad \partial_t u(0, x) = h(x), & \text{for } x \in \mathbb{R}, \end{cases}$$

for $g \in C^2(\mathbb{R}),\ h \in C^1(\mathbb{R})$. (4 marks)

(d) Find the solution $u(t, x)$ of the initial boundary-value problem:

$$\begin{cases} \partial_t^2 u(t, x) - \partial_x^2 u(t, x) = f(t, x), & \text{in } t > 0,\ x > 0, \\ u(0, x) = g(x), \quad \partial_t u(0, x) = h(x), & \text{for } x \geq 0, \\ u(t, 0) = 0, & \text{for } t \geq 0. \end{cases}$$

Hint: Extend suitably the data and use the solution from part (c). (7 marks)

(Total: 20 marks)

4. Given a smooth and bounded domain $\Omega \subset \mathbb{R}^3$, consider the Neumann problem for the Laplace equation

$$\begin{cases} \triangle u = 0, & \text{in } x \in \Omega, \\ \frac{\partial u}{\partial n} = g(x) & \text{on } x \in \partial\Omega, \end{cases}$$

where $g \in C(\partial\Omega)$.

(a) Show that if there is a classical solution of the Neumann problem, then

$$\int_{\partial\Omega} g(x) d\sigma = 0,$$

where $d\sigma$ is the surface element. (5 marks)

(b) Prove that any two classical solutions of the Neumann problem differ only by a constant. (7 marks)

(c) Prove Dirichlet's principle for the Neumann problem: among all functions $w \in C^2(\bar{\Omega})$ the quantity

$$E[w] = \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx - \int_{\partial\Omega} g w \, d\sigma$$

is the smallest for $w = u$, the solution of the Neumann problem for the Laplace equation. (8 marks)

(Total: 20 marks)

5. The following Cauchy problem for the traffic flow equation:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0, \quad \text{for } t > 0 \ , \ x \in \mathbb{R}, \\ \rho(x,0) = \begin{cases} 1 & \text{for} \quad x < 0 \\ \\ 0 & \text{for} \quad x > 0 \end{cases} \end{cases}$$

with $v(\rho) = (1-\rho)$, and $q(\rho) = \rho v(\rho)$, models the dynamics of the cars after the traffic light situated at $x = 0$ changes from red to green.

(a) Compute the unique entropy solution of the problem. (8 marks)

(b) How long does it take for a car initially located at $x_0 < 0$ to pass the traffic lights? (4 marks)

(c) What is the trajectory of the car once it starts to move? (4 marks)

(d) How many cars will get through the traffic light if the green light lasts for t^* time units? (4 marks)

(Total: 20 marks)

1. (a) The characteristic system reads as

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8, A

4, D

$$\begin{cases} \frac{dt}{ds} = 1 \quad , \qquad t(0,r) = 0 \\ \frac{dx}{ds} = z \quad , \qquad x(0,r) = r \\ \frac{dz}{ds} = 0 \quad , \quad z(0,r) = g(r) \end{cases}$$

The solution $u(t,x) = g(r)$ is thus constant on the characteristic lines $x = g(r)s + r$, $s = t$:

Plugging in for g we check that the characteristics are given by:

$$x = \begin{cases} r & \text{for} \quad r \in (-\infty, -1] \cup (0, \infty), \\ -rt + r & \text{for} \qquad r \in (-1, 0]. \end{cases}$$

Since g is discontinuous at $x = -1$ and it is increasing across the discontinuity, the rarifaction wave will appear. The characteristics emanating from point $(0, -1)$ have the equation $x = \sigma t - 1$, for $\sigma \in (0,1)$, and so we look fo solution in the form $u(t,x) = h(\sigma) = h\left(\frac{x+1}{t}\right)$. Substituting to Burgers' equation we check that h has to satisfy

$$\frac{-(x+1)}{t^2} h'\left(\frac{x+1}{t}\right) + \frac{1}{t} h\left(\frac{x+1}{t}\right) h'\left(\frac{x+1}{t}\right) = 0,$$

and so $h\left(\frac{x+1}{t}\right) = \frac{x+1}{t}$. Therefore the unique entropy solution on time interval $t \in [0,1)$ is equal to

$$u(t,x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, -1], \\ \frac{x+1}{t} & \text{for} \quad x \in (-1, t-1], \\ \frac{-x}{1-t} & \text{for} \quad x \in (t-1, 0], \\ 0 & \text{for} \quad x \in (0, \infty). \end{cases}$$

(b) Now we look at $u(1,x)$ as our new initial data, we have

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5, B

3, C

$$u(1,x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, -1], \\ x+1 & \text{for} \quad x \in (-1, 0], \\ 0 & \text{for} \quad x \in (0, \infty). \end{cases}$$

There is discontinuity at $x = 0$, and $u(1,x)$ is decreasing across this point, so the characteristics will intersect and the shock will appear. This shock will start from $t = 1$ at $x = 0$. Therefore, the entropy solution to Burgers' equation for $t > 1$ is given by

$$u(t,x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, -1], \\ \frac{x+1}{t} & \text{for} \quad x \in (-1, \sigma(t)), \\ 0 & \text{for} \quad x \in (\sigma(t), \infty). \end{cases}$$

and $\sigma(t)$ can be determined from the Rankine-Hugoniot condition:

$$\sigma'(t) = \frac{1}{2} \frac{\sigma(t) + 1}{t}, \quad \sigma(t = 1) = 0,$$

from which $\sigma(t) = \sqrt{t} - 1$.

The entropy condition $\frac{x+1}{t} > \sigma'(t) > 0$ is also satisfied as $\frac{\sqrt{t}}{t} > \frac{1}{2\sqrt{t}}$ and $\frac{1}{2\sqrt{t}} > 0$ for $t > 1$.

2. (a) Taking two solutions to the IBVP u and v, we obtain that the difference $w = u - v$ satisfies

$$\begin{cases} \partial_t w - \partial_x^2 w = 0 & \text{in } t > 0,\ 0 < x < \pi, \\ w_x(t,0) = w_x(t,\pi) = 0 & \text{for } t \geq 0, \\ w(0,x) = 0 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

meth seen $\Downarrow$

3, A

3, B

We introduce the energy

$$E(w(t)) = \frac{1}{2}\int_0^\pi w^2 dx.$$

Integrating by parts we show that

$$\frac{dE(w(t))}{dt} = \int_0^\pi w\partial_t w dx = \int_0^\pi w\partial_x^2 w dx = -\int_0^\pi (\partial_x w)^2 dx + w\partial_x w\Big|_{x=0}^{x=\pi}.$$

The boundary term vanishes due to the assumptions on the boundary data, the r.h.s. is thus nonpositive. Therefore $\frac{dE(w(t))}{dt} \leq 0$ and so

$$\frac{1}{2}\int_0^\pi (w(t,x))^2 dx = E(w(t)) \leq E(w(0)) = \frac{1}{2}\int_0^\pi (w(0,x))^2 dx = 0.$$

This means that $w(x,t) \equiv 0$, for $(x,t) \in [0,\pi] \times [0,\infty)$ meaning that $u = v$.

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8, A

(b) We look for separated variables solution of the form $u(t,x) = T(t)X(x)$, satisfying

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \in \mathbb{R}.$$

Therefore, we get that $T'(t) = -\lambda T(t)$ and $X''(x) + \lambda X(x) = 0$ for $t > 0$ and $0 < x < \pi$ with $X'(0) = X'(\pi) = 0$. We have three cases:

$\underline{\lambda = 0}$: Since $X'' = 0$, then X' is a constant, but since $X'(0) = 0$ thus $X'(x) = 0$. This in turn means that X is constant. However, since $T(t)$ is a constant as well, therefore $g(x)$ would have to be a constant, which does not have to be the case.

$\underline{\lambda > 0}$: The solutions of the ODEs are

$$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)\,.$$

The condition $X'(0) = 0$ means that $C_2 = 0$ and the condition $X'(\pi) = 0$ is equivalent to $C_1\sin(\sqrt{\lambda}\pi) = 0$. Therefore, either $\sin(\sqrt{\lambda}\pi) = 0$ or $C_1 = C_2 = 0$. The second case leads to trivial solutions, the first one leads to $\lambda = n^2$ with $n \in \mathbb{N}$. We can now solve the equation for $T(t)$ given by $T'(t) = -n^2 T(t)$ to get $T(t) = C\,e^{-n^2 t}$, with $C \in \mathbb{R}$. Summarizing, the solutions for this case are given by

$$u_n(t,x) = C_n e^{-n^2 t}\cos(nx)\,, \quad n \geq 0\,.$$

$\underline{\lambda < 0}$: The solutions of the ODEs are

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}\,.$$

The condition $X'(0) = 0$ means $C_1 - C_2 = 0$ and the condition $X'(\pi) = 0$ means $C_1 - C_2 e^{-\pi} = 0$. Therefore, $C_1 = C_2 = 0$.

The candidate solution is thus of the form:

$$u(t,x) := \sum_{n=0}^{\infty} C_n e^{-n^2 t} \cos(nx),$$

where the coefficients C_n have to match the expansion of the initial data $g(x)$, i.e.

$$g(x) = u(0,x) = \sum_{n=0}^{\infty} C_n \cos(nx),$$

and so

$$C_n = \frac{2}{\pi} \int_0^{\pi} g(x) \cos(nx)\, dx.$$

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(c) First of all, since $g(x) \geq 0$ and $\int_0^{\pi} u(t,x)dx = \int_0^{\pi} g(x)dx$, thus $u(t,x)$ cannot be negative in the whole domain $(0,\pi)$. Assume that there is a point (t^*, x^*) for which $u(t^*, x^*) < 0$. Because u is a classical solution thus there exists a subset on which the solution would be negative. Let $\Omega^* = \{x \in (0,\pi) : u(t^*, x) \leq 0\}$, and note that $\frac{\partial u}{\partial n}|_{\partial\Omega^*} \geq 0$, where n denotes outer normal vector. Thus, integrating over Ω^* we obtain 6, D

$$\frac{d}{dt} \int_{\Omega^*} u(t,x)\, dx = \int_{\partial\Omega^*} \frac{\partial u}{\partial n}\, d\sigma \geq 0.$$

Therefore

$$\int_{\Omega^*} u(t,x) \geq \int_{\Omega^*} g(x) \geq 0$$

which leads to a contradiction.

3. (a) From d'Alambert formula we can deduce that

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4, B

$$v(t,x;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ dy.$$

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(b) First note that $u \in C^2(\mathbb{R}\times[0,\infty))$, so we can compute the subsequent derivatives

5, A

$$u_t(t,x) = v(t,x;t) + \int_0^t v_t(t,x;s)\,ds = \int_0^t v_t(t,x;s)\,ds,$$

$$u_{tt}(t,x) = v_t(t,x;t) + \int_0^t v_{tt}(t,x;s)\,ds = f(t,x) + \int_0^t v_{tt}(t,x;s)\,ds.$$

Moreover, using the equation satisfied by v, we get

$$u_{xx}(t,x) = \int_0^t v_{xx}(t,x;s)\,ds = \int_0^t v_{tt}(t,x;s)\,ds.$$

Therefore,

$$u_{tt}(t,x) - u_{xx}(t,x) = f(t,x).$$

meth seen ⇓

(c) Note that the solution obtained in part (b) is equal to

4, A

$$u(t,x) = \int_0^t v(t,x;s)\,ds = \frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ dy\,ds,$$

and it satisfies the initial conditions $u(0,x) = u_t(0,x) = 0$. Since the problem is linear, the solution to the fully inhomogeneous problem is the sum of solutions of the problem with $f = 0$ given by standard d'Alambert formula, and the solution computed in part (b), therefore

$$u(t,x) = \frac{1}{2}\left[g(x+t)+g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy + \frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ dy\,ds.$$

meth seen ⇓

(d) The solution on the half line can be obtained by the method of odd extension for the data:

4, C

3, D

$$g_{odd}(x) = \begin{cases} g(x) & \text{for} \quad x \geq 0 \\ -g(-x) & \text{for} \quad x < 0 \end{cases}, \quad h_{odd}(x) = \begin{cases} h(x) & \text{for} \quad x \geq 0 \\ -h(-x) & \text{for} \quad x < 0 \end{cases}, \text{ and}$$

$$f_{odd}(t,x) = \begin{cases} f(t,x) & \text{for} \quad x \geq 0 \\ -f(t,-x) & \text{for} \quad x < 0 \end{cases}.$$

From the formula derived in point (c), we deduce that a perturbation of initial condition at point $(0,0)$ changes the solution $u(t,x)$ in the interval $(x-t, x+t)$. Therefore for $x \geq t$ the solution is not affected, and we have

$$u(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy + \frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s} f(s,y)\,dy\,ds.$$

For $0 < x < t$ we compute using the formula for the odd extension that

$$u(t,x) = \frac{g_{odd}(x+t)+g_{odd}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h_{odd}(y)\,dy + \frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s} f_{odd}(s,y)\,dy\,ds.$$

4. (a) Since $u \in C^2(\bar{\Omega})$, we apply the divergence theorem to show that

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5, B

$$\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = 0,$$

since $\Delta u = 0$ in Ω. Due to Neumann boundary condition $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$, and thus we get the desired compatibility condition.

$$\int_{\partial\Omega} g(x) d\sigma = 0,$$

(b) Multiplying the equation $\Delta u = 0$ by u and integrating by parts, we get

meth seen ⇓

4, A

3, B

$$0 = \int_\Omega u \Delta u \, dx = -\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, d\sigma.$$

Let u, v be two different solutions of the Neumann problem for the Laplace equation, then $w = u - v$ solves the problem

$$\begin{cases} \triangle w = 0, & \text{in } x \in \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } x \in \partial\Omega. \end{cases}$$

Using the above relation for $u = w$, we get

$$\int_\Omega |\nabla w|^2 \, dx = 0,$$

and thus $\nabla w = 0$ and $w = const.$ in Ω since Ω is an open connected set.

(c) Given $u, v \in C^2(\bar{\Omega})$, we compute the expression

meth seen ⇓

5, C

3, D

$$E[u+v] = E[u] + \left(\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} gv \, d\sigma \right) + \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx.$$

Taking u the solution to the original problem

$$\begin{cases} \triangle u = 0, & \text{in } x \in \Omega, \\ \frac{\partial u}{\partial n} = g(x) & \text{on } x \in \partial\Omega. \end{cases}$$

and multiplying the equation by v, we check that

$$0 = \int_\Omega v \Delta u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} vg \, d\sigma.$$

Therefore, we conclude that

$$E[u+v] = E[u] + \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx \geq E[u],$$

for all $v \in C^2(\bar{\Omega})$, from which $E[w] \geq E[u]$ for all $w \in C^2(\bar{\Omega})$.

meth seen ⇓

8, M

5. (a) The characteristic system reads as

$$\begin{cases} \frac{dt}{ds} = 1, & t(0,r) = 0, \\ \frac{dx}{ds} = 1 - 2z, & x(0,r) = r, \\ \frac{dz}{ds} = 0, & z(0,r) = \rho(r,0) := \rho_0(r). \end{cases}$$

From the third equation it follows that $z(s,r) = \rho_0(r)$, so the solution is constant on characteristics. We may thus solve the first equation to get $s = t$ and the second equation to get the equations of characteristics:

$$x = (1 - 2\rho_0(r))\, t + r.$$

Using the initial data we therefore have

$$x = \begin{cases} -t + r & \text{for} \quad r < 0 \\ t + r & \text{for} \quad r \geq 0, \end{cases}$$

and so, the solution reads

$$\rho(x,t) = \begin{cases} 1 & \text{for} \quad x < -t \\ 0 & \text{for} \quad x \geq t. \end{cases}$$

The empty space for $x \in (-t,t)$ should be filled with rarefaction wave. We look for its form by considering the "self-similar" solution of the form: $\rho\left(\frac{x}{t}\right)$, for which

$$\begin{aligned} & -\frac{x}{t^2}\rho'\left(\frac{x}{t}\right) + (1 - 2\rho\left(\frac{x}{t}\right))\rho'\left(\frac{x}{t}\right)\frac{1}{t} = 0, \\ \iff & -\xi\rho'(\xi) + (1 - 2\rho(\xi))\rho'(\xi) = 0, \quad \forall \xi = \frac{x}{t}, \\ \iff & \rho'(\xi)\,(1 - 2\rho(\xi) - \xi) = 0. \end{aligned}$$

If ρ is not a constant, then $1 - 2\rho(\xi) = \xi$ and thus

$$\rho(x,t) = \rho\left(\frac{x}{t}\right) = \frac{1}{2}\left(1 - \frac{x}{t}\right), \quad \text{for } -t < x < t.$$

meth seen ⇓

4, M

(b) The left radial of the rarefaction wave moves with the velocity -1. When this radial arrives to the point $x_0 < 0$ the car located at this point will start to move. The time it takes is thus equal to $t = -x_0$.

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4, M

(c) From the previous point we know that the car starts to move at time $t = -x_0$. The velocity of the car is given by $v = 1 - \rho$, where ρ is the density in the rarefaction wave computed in point (a), we thus have

$$x'(t) = v(x,t) = 1 - \rho(x,t) = 1 - \frac{1}{2}\left(1 - \frac{x}{t}\right) = \frac{1}{2}\left(1 + \frac{x}{t}\right).$$

Therefore, we need to solve

$$x'(t) = \frac{x}{2t} + \frac{1}{2}, \quad x\left(-x_0\right) = x_0.$$

Solving the ODE we obtain

$$x = t + C\sqrt{t}, \quad \text{and} \quad x_0 = -x_0 + C\sqrt{-x_0}.$$

Therefore, finally the trajectory of this car is:

$$x(t) = t - 2\sqrt{-x_0 t}.$$

unseen $\Downarrow$

4, M

(d) From the previous point we know that the car that will pass through the traffic light as the last one will have to have a position 0 at $t = t^*$, therefore

$$x(t^*) = t^* - 2\sqrt{-x_0 t^*} = 0 \iff x_0 = \frac{-t^*}{4}.$$

So, during the green light only the cars ahead of this one will pass, which means that there will be $|x_0| = \frac{t^*}{4}$ passing through the light in time $t = t^*$.

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60019/70019	1	No Comments Received
MATH60019/70019	2	No Comments Received
MATH60019/70019	3	No Comments Received
MATH60019/70019	4	No Comments Received
MATH70019	5	No Comments Received