Imperial College London
Department of Mathematics

Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Time allowed: 45 minutes. You should answer all questions.

Question 1

i) Compute the determinant of the matrix: $A = \begin{pmatrix} 1 & 3 & 4 & 10 \\ 2 & 5 & 9 & 11 \\ 6 & 8 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{R})$.

Compute the entry in the first row and column of A^{-1} (i.e., the (1,1)-entry of A^{-1}), giving a reason for your answer. (8 marks)

ii) Let V be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. The linear transformation $S : V \to V$ is defined by

$$S(a + bt + ct^2) = (a + 2c)t + (b + c)t^2$$

(for $a, b, c \in \mathbb{R}$; here t is the variable in the polynomial). Find the eigenvalues and eigenvectors of S and hence calculate $S^{100}(t)$. (12 marks)

Question 2 Suppose $n \geq 1$ is a natural number and F is a field. We say that matrices $A, B \in M_n(F)$ are *similar* (denoted by $A \sim B$) if there is an invertible matrix $P \in M_n(F)$ with $B = P^{-1}AP$.

For each of the following statements, say whether it is true or false for all matrices in $M_n(F)$. If it is true, give a short proof; if it is false, give a counterexample.

i) The relation $\sim$ is an equivalence relation on $M_n(F)$.

ii) If $A \sim B$, then $A^2 \sim B^2$.

iii) If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1A_2 \sim B_1B_2$.

iv) If $A \sim B$, then $A^T \sim B^T$.

v) If $\chi_A(x) = \chi_B(x)$, then $A \sim B$.

vi) If A, B are diagonalisable, then $A + B$ is diagonalisable.

vii) If A, B are symmetric, then AB is symmetric.

viii) If $F = \mathbb{R}$, then $A + A^T$ is diagonalisable.

(20 marks, 2.5 per part)