Imperial College London
Department of Mathematics

Question Sheet 8 - Probl. Class week 11

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

This is the final problem sheet for this module, released on Monday of week 10. Solutions will be released on Monday of week 11. Question 5 is related to the material on Dihedral Groups covered in Lecture 19 in the notes.

Question 1 (a) Write down all of the cycle shapes of the elements of S_5. For each cycle shape, calculate how many elements there are with that shape. (Check that your answers add up to $|S_5| = 120$.)

(b) How many elements of S_5 have order 2?

(c) How many subgroups of size 3 are there in the group S_5?

Question 2 What is the largest order of an element of S_8?

Question 3 Let G be a group, and let S be a subset of G. Recall that we say that S *generates* G if every element in G can be written as a product of elements of S and their inverses.

(i) Let $2 \leq k \leq n$. Show that a k-cycle $(a_1 \ldots, a_k)$ in S_n can be written as a product of $k-1$ distinct cycles of length 2. Deduce that the set of 2-cycles in S_n generates S_n.

(ii) (Harder) Let α be the n-cycle $(1234\ldots n)$ and β the 2-cycle (12). Prove that $\langle \alpha, \beta \rangle = S_n$.

[Hint: $\alpha\beta\alpha^{-1} = (23)$. Use tricks like this.]

Question 4 (a) Use the inclusion - exclusion principle to give a formula for the number of permutations in S_n which have no fixed points. Prove that the proportion of such permutations in S_n tends to $1/e$ as $n \to \infty$.

(b) Give a formula for the number of permutation in S_n which have one fixed point.

(c) A standard deck of 52 cards is shuffled at random. What (approximately) is the probability that at least one card is still in the same place after the shuffle?

Question 5 Suppose G is a group and $a, b \in G$ are of order 2. Let $c = ab$ and suppose that c has finite order $m \geq 3$.

(a) Prove that $aca = c^{-1}$ and deduce that for all $n \in \mathbb{N}$ we have $ac^n a = c^{-n}$.

(b) Show that $H = \{a^s c^t : s = 0, 1 \text{ and } 0 \leq t < m\}$ is a subgroup of G of order $2m$.

Question 6

(a) Describe the group G of rotational symmetries of a cube, saying what the possible axes of rotation are and what the possible angles of rotation are. Hence show that there are 24 such rotational symmetries.

(b) Use Question 8 from Problem Sheet 7 to give a different proof that $|G| = 24$, by thinking of G as a group of permutations on the faces of the cube.

(c) Consider one of the three pairs of opposite faces of the cube. Show that the set of rotational symmetries of the cube which send this pair of faces to itself forms a subgroup of G of order 8.