Matrices 101

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A **matrix** is a **rectangular array of objects**. Each object in a matrix is called an **element** of the matrix.

We always write a matrix enclosed in **round brackets**, thus:

$$\begin{pmatrix} 1 & -8 & 11 \\ 2 & 0-0.25 & \\ -4 & 6 & 9 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 8 & -2 & 9 & 15 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -6 \\ \sqrt{2} & 4 \\ -0.5 & 1-\sqrt{3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ b \end{pmatrix}$$

Because the array in a matrix is rectangular, it has **rows** and **columns**.

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \textbf{2nd row} \\ \textbf{3rd column} & \end{array} \tag{1}$$

We describe the size and shape (or **order**) of a matrix with m **rows and** n **columns** as an $m \times n$ **matrix**. Martrices with only one row or one column are called vectors. A $1 \times n$ matrix is a **row vector**, an $m \times 1$ matrix is a **column vector**.

It is also useful to be able to refer to a particular element in a matrix, and for this we use suffixes. The element in the ith row and the jth column of a matrix A is written a_{ij} (or b_{ij}, etc). Another way of putting this is that we label the elements in this order:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

Other ways you might see a matrix defined are

$$\mathsf{A} = (a_{ij})_{m \times n}$$

This tells you the order, $m \times n$....sometimes this is omitted.

Definition 0.0.1
If $\mathsf{A} = (a_{ij})_{m \times n}$ and $\mathsf{B} = (b_{ij})_{p \times q}$ then A and B are equal if and only if

1. $m = p$ and $n = q$ (i.e. they are the same order)
2. $a_{ij} = b_{ij}$ for all i, j (i.e. the corresponding elements are equal)

Notation 0.0.2
We write $M_{n \times m}(F)$ to represent the set of $n \times m$ matrices with entries in F.

Definition 0.0.3
Given $m \times n$ matrices, $\mathsf{A} = [a_{ij}]_{m \times n}$ and if $\mathsf{B} = [b_{ij}]_{m \times n}$, then the **(matrix) sum of** A **and** B is the $m \times n$ matrix $\mathsf{C} = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.

Example 0.0.4:

$$\begin{pmatrix} 2 & 1 & 8 \\ 1 & 5 & 3 \\ 0 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 6 & 1 & 8 \\ 0 & 2 & 1 \\ 5 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 16 \\ 1 & 7 & 4 \\ 5 & 9 & 4 \end{pmatrix}$$

Properties 0.0.5: Let $\mathsf{A}, \mathsf{B}, \mathsf{C}$ be matrices of the same order. Then

(i) Associative Law: $(\mathsf{A} + \mathsf{B}) + \mathsf{C} = \mathsf{A} + (\mathsf{B} + \mathsf{C})$.

(ii) $\mathsf{A} + 0 = 0 + \mathsf{A} = \mathsf{A}$ where 0 is the **null** matrix (where every entry is 0).

(iii) Additive inverse: $\mathsf{A} + (-\mathsf{A}) = (-\mathsf{A}) + \mathsf{A} = 0$ where 0 is the null matrix

(iv) Commutative law: $\mathsf{A} + \mathsf{B} = \mathsf{B} + \mathsf{A}$

Definition 0.0.6

Let $\mathsf{A} = (a_{ij})$ be any matrix, and let $\lambda \in \mathbb{R}$. Then the **scalar multiple of** A by λ, denoted by $\lambda\mathsf{A}$, is obtained by multiplying every element of A by λ. Thus if $\mathsf{A} = (a_{ij})_{m\times n}$ then $\lambda\mathsf{A} = (\lambda a_{ij})_{m\times n}$.

Example 0.0.7:

$$5\begin{pmatrix} 3 & 7 \\ 2 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 15 & 35 \\ 10 & -20 \\ -5 & 0 \end{pmatrix}$$

Properties 0.0.8: Let A, B be matrices of the same order, and let $\alpha, \beta \in \mathbb{R}$. Then

(i) $\alpha(\mathsf{A} + \mathsf{B}) = \alpha\mathsf{A} + \alpha\mathsf{B}$

(ii) $(\alpha + \beta)\mathsf{A} = \alpha\mathsf{A} + \beta\mathsf{A}$

(iii) $(\alpha\beta)\mathsf{A} = \alpha(\beta\mathsf{A})$

(iv) $(-1)\mathsf{A} = -\mathsf{A}$

Definition 0.0.9

Let $\mathsf{A} = (a_{ij})_{p\times q}$ and $\mathsf{B} = (b_{ij})_{q\times r}$. Then the matrix product of A and B, denoted by AB, is the matrix C, where

$$\mathsf{C} = (c_{ij})_{p\times r}, \qquad \text{where} \qquad c_{ij} = \sum_{k=1}^{q} a_{ik}b_{kj}$$

Example 0.0.10:

$$\begin{pmatrix} 1 & 0 & 5 \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5+5 \\ 20+6+1 \end{pmatrix} = \begin{pmatrix} 10 \\ 27 \end{pmatrix}$$