Mathematics Year 1, Calculus and Applications I

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Problem Sheet 1 - Solutions

1. (a) $\lim_{x\to 0} \exp\left(\frac{3x}{\tan x}\right) = \exp(3)$

   (b) $\lim_{x\to 0} \cos\left(\frac{\pi \sin x}{4x}\right) = \cos(\pi/4) = 1/\sqrt{2}$.

2. (a) $\lim_{x\to 27} \frac{x^{1/3}-3}{x-27} = \lim_{x\to 27} \frac{(x^{1/3}-1)}{(x^{1/3}-1)(x^{2/3}+3x^{1/3}+9)} = \frac{1}{27}$.

   (b) $\lim_{x\to 0} \frac{(3+x)^2-9}{x} = \lim_{x\to 0} \frac{6x+x^2}{x} = 6$.

   (c) $\lim_{x\to 1+} \frac{x(x+3)}{(x-1)(x-2)} = -4\lim_{x\to 1+} \frac{1}{(x-1)} = -\infty$

   (d) $\lim_{x\to 0+} \frac{(x^3-1)|x|}{x} = -1$

   (e) $\lim_{x\to \frac{1}{2}-} \frac{2x-1}{\sqrt{(2x-1)^2}}$. Substitute $x = \frac{1}{2} - \epsilon$ where $\epsilon > 0$, the limit becomes $\lim_{\epsilon\to 0+} \frac{-\epsilon}{\sqrt{\epsilon^2}} = -1$.

   (f) $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{ax+b} - \sqrt{ax+b/2}\right)$, $(a, b > 0)$. Rationalise,

$$
\begin{aligned}
&= \lim_{x\to\infty} \sqrt{x}\frac{(\sqrt{ax+b} - \sqrt{ax+b/2})(\sqrt{ax+b} + \sqrt{ax+b/2})}{(\sqrt{ax+b} + \sqrt{ax+b/2})} \\
&= \lim_{x\to\infty} \sqrt{x}\frac{b/2}{(\sqrt{ax+b} + \sqrt{ax+b/2})} \\
&= \lim_{x\to\infty} \sqrt{x}\frac{b/2}{\sqrt{x}(\sqrt{a+bx^{-1}} + \sqrt{a+(b/2)x^{-1}})} = \frac{b}{4\sqrt{a}}
\end{aligned}
$$

3. (a) Establish the Comparison Test 2 given in the handout, using the $\varepsilon - A$ definition of the limit.

   Solution: We are given $\lim_{x\to\infty} f(x) = 0$, hence given any $\varepsilon > 0$ there is a number $A > 0$, so that $|f(x)| < \varepsilon$ whenever $x > A$. Now using these same ε and A and since we also know that $|g(x)| \le |f(x)|$ for x large enough (we can always pick A large enough for this to hold), we have $|g(x)| < \varepsilon$ when $x > A$.

   (b) Use (a) above to find $\lim_{x\to\infty} \frac{1}{x}\sin\left(\frac{1}{x}\right)$.

   Solution: Take $g(x) = \frac{1}{x}\sin(1/x)$ and $f(x) = 1/x$. Clearly $|g(x)| \le |f(x)|$ and we know $\lim_{x\to\infty}(1/x) = 0$.

4. (a) Use the $B - \delta$ definition of limits to show that if $\lim_{x\to x_0} f(x) = \infty$ and $g(x) \ge f(x)$ for x close to x_0, $x \ne x_0$, then $\lim_{x\to x_0} g(x) = \infty$.

   Solution: For $f(x)$ we know that given any real $B > 0$, there exists a $\delta > 0$ so that $f(x) > B$ whenever $|x - x_0| < \delta$. For the same B and δ we also have $g(x) > B$ since $g(x) \ge f(x)$.

   (b) Use (a) above to show that $\lim_{x\to 1} \frac{1+\cos^2 x}{(1-x)^2} = \infty$.

   Solution: Take $f(x) = 1/(1-x)^2$ and $g(x) = (1+\cos^2 x)/(1-x)^2$, so that $g(x) \ge f(x)$.

5. (a) The given function is equal to 1 for $x > 0$, equal to -1 for $x < 0$ and equal to 1 at $x = 0$. It is not continuous at $x = 0$ because $\lim_{h\to 0+} f(h) = 1$, $\lim_{h\to 0-} f(x) = -1$ whereas $f(0) = 1$.

   (b) Graphs straight forward. Again the limit as $x \to 0+$ is -1 whereas the limit as $x \to 0-$ is $+1$, hence the function is not continuous.

   (c) The function is now

$$y = \begin{cases} x & x < 0 \\ 2x & x \geq 0 \end{cases}$$

   It is continuous and the limit exists, hence adding two functions can get rid of discontinuities.

   (d) Many examples will do. Here is one

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{1+x}{x}.$$

   Singular at $x = 0$, but $f(x) - g(x) = -1$ which is perfectly nice.

6. Can rewrite the inequality as

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - m\right| |x - x_0| \leq K(x - x_0)^2 = K|x - x_0|^2 \quad \Rightarrow$$

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - m\right| \leq K|x - x_0|$$

   Now sending $x \to x_0$ shows that by the comparison test for limits

$$\lim_{x\to x_0} \left|\frac{f(x) - f(x_0)}{x - x_0} - m\right| = 0 \quad \Rightarrow \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - m\right) = 0,$$

   giving $f'(x_0) = m$.

7. Write $x = 3 + \epsilon$ to find that we need

$$|25\epsilon + 9\epsilon^2 + \epsilon^3| < 10^{-3}.$$

   So taking $\epsilon = \pm\frac{10^{-3}}{26}$ will do, because the sum $9\epsilon^2 + \epsilon^3$ is much smaller than 10^{-5} so does not affect things.

   You can do better of course! *I don't expect you to have produced this solution but it is a good technique to learn.* Here's how, using an iterative method that you will encounter again in Numerical Analysis and elsewhere. I will take $\varepsilon > 0$ to begin with and consider the equation

$$25\epsilon + 9\epsilon^2 + \epsilon^3 = 10^{-3} \qquad \text{i.e.} \qquad \varepsilon = \frac{1}{25}\left(10^{-3} - 9\varepsilon^2 - \varepsilon^3\right) := f(\varepsilon)$$

   The last equation is of the form

$$\varepsilon = f(\varepsilon) \qquad ,$$

   and we can set up an *iteration* to produce a sequence of approximations $\varepsilon_0, \varepsilon_1, \ldots$ through

$$\varepsilon_{n+1} = f(\varepsilon_n), \qquad n \geq 0. \quad (*)$$

To get this off the ground we need a guess for ε_0. I will take it to be $\varepsilon_0 = \frac{10^{-3}}{25}$ which is almost what I guessed in the first part (the initial guess can be much cruder - try it out). Equation (*) gives me $\varepsilon_1 = f(\varepsilon_0)$, etc. Here is what I found

$$\begin{aligned}
\varepsilon_0 &= 3.999942399744000 \times 10^{-5} \\
\varepsilon_1 &= 3.999942401402887 \times 10^{-5} \\
\varepsilon_2 &= 3.999942401402839 \times 10^{-5} \\
\varepsilon_3 &= 3.999942401402839 \times 10^{-5}
\end{aligned}$$

By ε_3 I have accuracy to 16 significant figures! Anything slightly smaller than $\varepsilon = 3.999942401402839 \times 10^{-5}$ will ensure that I am less than 10^{-3} close to $x = 3$.

For completeness, here is a calculation with a wildly bad initial condition of $\varepsilon_0 = 1$ (always 16 sig figures reported):

$$\begin{aligned}
\varepsilon_0 &= 1.0 \\
\varepsilon_1 &= -0.399960000000000 \\
\varepsilon_2 &= -0.054989248499203 \\
\varepsilon_3 &= -0.001041923184214 \\
\varepsilon_4 &= 3.960922783278650 \times 10^{-5} \\
\varepsilon_5 &= 3.999943519677967 \times 10^{-5} \\
\varepsilon_6 &= 3.999942401370633 \times 10^{-5} \\
\varepsilon_7 &= 3.999942401402840 \times 10^{-5} \\
\varepsilon_8 &= 3.999942401402839 \times 10^{-5} \\
\varepsilon_9 &= 3.999942401402839 \times 10^{-5}
\end{aligned}$$

So again we *converge* to the same value as before.

Equations such as (*) are called *fixed point iterations* or iteration maps. The converged value $\varepsilon^* = \lim_{n\to\infty} \varepsilon_n$ must satisfy

$$\varepsilon^* = f(\varepsilon^*).$$

If $|f'(\varepsilon^*)| < 1$ then the iteration $\varepsilon_{n+1} = f(\varepsilon_n)$ will converge. In this particular example the function is so nice as to allow a wild initial guess. Starting with $\varepsilon_0 = 3.0$ took 11 iterations. Starting with $\varepsilon_0 = 5.0$ the iteration diverged.

8. Suppose the limit exists and call it L. Then if a_n is a sequence of non-zero numbers satisfying $\lim_{n\to\infty} a_n = 0$, we would have $\lim_{n\to\infty} f(a_n) = L$.

   Now consider $a_n = 1/n$ as such a sequence. Since each a_n is now rational we have

   $$L = \lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 1 = 1.$$

   Now take $a_n = \sqrt{2}/n$ which is now a sequence of irrational numbers. Now we have

   $$L = \lim_{n\to\infty} f(\sqrt{2}/n) = \lim_{n\to\infty} 0 = 0.$$

   Contradiction, hence the limit does not exist.