IMPERIAL

MATH70061

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Commutative Algebra

Date: Monday, May 19, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

In the questions below, any results from the course can be used as long as you clearly state them and you are not explicitly asked to prove them.

1. Let R be a commutative ring.

   (a) Let $I, J \subset R$ be prime ideals. Prove or give a counterexample:

      (i) the intersection $I \cap J$ is a prime ideal. (2 marks)

      (ii) the sum $I + J$ is a prime ideal. (2 marks)

   (b) Let $a \in R$ be a unit and let $x \in R$ be a nilpotent element. Show that $a + x$ is a unit. (4 marks)

   (c) Show that if $\mathfrak{p} \subset R$ is prime, then there exists $\mathfrak{q} \subset \mathfrak{p}$ minimal prime ideal. (4 marks)

   (d) Let $S = R[x]$ and let
$$f(x) = \sum_{i=0}^{n} a_i x^i \in S,$$
   with $a_0, \ldots, a_n \in R$. Show that if $f(x)$ is a unit in S then a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent in R. (4 marks)

   (e) Let $S = R[x]$. Show that the nilradical $\mathcal{N}(S)$ of S coincides with the Jacobson radical $\mathcal{J}(S)$ of S. (4 marks)

(Total: 20 marks)

2. Let R be a commutative ring.

   (a) Prove or give a counterexample:

      (i) Assume that $R[x]$ is Noetherian. The ring R is also Noetherian. (4 marks)

      (ii) Assume that R is Noetherian and $R \subset S$ is a finite generated extension over R. Then S is Noetherian. (6 marks)

   (b) Show that if R is Noetherian, then the formal power series ring $R[[x]]$ is also Noetherian. (7 marks)

   (c) Show that if R is Noetherian then there exists a positive integer m such that if $\mathcal{N}(R)$ is the nilradical of R then $\mathcal{N}(R)^m = (0)$. (3 marks)

(Total: 20 marks)

3. (a) Let R be an Artin ring.

   (i) Show that R has only a finite number of maximal ideals. (4 marks)

   (ii) Show that every prime ideal of R is maximal. (4 marks)

   (iii) Show that the nilradical of R is equal to the Jacobson radical of R. (2 marks)

   (b) Let R be a Noetherian ring.

   (i) Show that every ideal of R is a finite intersection of irreducible ideals. (5 marks)

   (ii) Show that every irreducible ideal of R is primary. (5 marks)

(Total: 20 marks)

4. (a) Show that if $(R, \mathfrak{m})$ is a DVR then

   (i) there exists $t \in R$ such that $\mathfrak{m} = (t)$. (3 marks)

   (ii) Let $I, J \subset R$ be ideals. Show that $I \subset J$ or $J \subset I$. (3 marks)

   (b) Let $(R, \mathfrak{m})$ be a local ring. Show that if R is Noetherian and every non-zero ideal of R is a power of $\mathfrak{m}$ then $(R, \mathfrak{m})$ is a DVR. (4 marks)

   (c) Let R be an integral domain. Show that if R is normal then $R_{\mathfrak{p}}$ is normal for any prime ideal $\mathfrak{p} \subset R$. (5 marks)

   (d) Let R be a normal Noetherian integral domain and let $\mathfrak{p} \subset R$ be a minimal nonzero prime ideal of R. Show that $R_{\mathfrak{p}}$ is a DVR. (5 marks)

(Total: 20 marks)

5. (a) Let K be a field. First, state the definition of a *discrete valuation* on K. Then, explain how such a valuation naturally induces a norm on the field K. (4 marks)

(b) Let p be a prime number and let $f(x) = x^{p-1} - 1$. Show that, for every non-zero $\overline{\xi} \in \mathbb{Z}/p\mathbb{Z}$, there exists a unique lift $\xi \in \mathbb{Z}_p$ such that $f(\xi) = 0$. (4 marks)

(c) Let k be a field. Show that the completion of the discrete valuation ring $k[x]_{(x)}$ is isomorphic to the formal power series ring $k[[x]]$. (6 marks)

(d) Let p be a prime number. Show that $\mathbb{Z}[[x]]/(x-p)$ is isomorphic to $\mathbb{Z}_p$. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70061

Commutative Algebra (Solutions)

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1. (a) (i) Let $R = \mathbb{Z}$ and let $I = (2)$ and $J = (3)$. Then I and J are prime ideals but $I \cap J = (6)$ is not a prime ideal. unseen $\Downarrow$

2, A

(ii) As above, choose $R = \mathbb{Z}$ and let $I = (2)$ and $J = (3)$. Then I and J are prime ideals but $I + J = (1) = R$. Thus, since $I + J$ is not proper, it follows that $I + J$ is not prime. 2, A

(b) Let n be a positive integer such that $x^n = 0$. Then let sim. seen $\Downarrow$

$$z := a^{-1} \cdot \sum_{i=0}^{n-1} \left(-x \cdot a^{-1}\right)^i = a^{-1} - x \cdot a^{-2} + x^2 \cdot a^{-3} + \cdots + a^{-1} \cdot (-x \cdot a^{-1})^{n-1}.$$

It is easy to check that z is the inverse of $a + x$. 4, A

(c) Let $\mathcal{P}$ be the set of prime ideals contained in $\mathfrak{p}$ and consider the relation $\leq$ in $\mathcal{P}$ given by unseen $\Downarrow$

$$\mathfrak{p}_1 \leq \mathfrak{p}_2 \qquad \text{if and only if} \qquad \mathfrak{p}_2 \subset \mathfrak{p}_1.$$

Then $\mathcal{P}$ is partially ordered and by Zorn's lemma, it is enough to show that any totally ordered subset T of $\mathcal{P}$ admits an upper bound.

Indeed, if T is a totally ordered subset of $\mathcal{P}$ then

$$\mathfrak{q} := \bigcap_{J \in T} J \subset \mathfrak{p}$$

is a prime ideal and, in particular, it is contained in $\mathcal{P}$. Thus, it is an upper bound of T. 4, C

(d) Let unseen $\Downarrow$

$$g(x) = \sum_{i=0}^{m} b_i x^i \in S$$

be the inverse of $f(x)$. Then $a_0 \cdot b_0 = 1$ and, therefore, a_0 and b_0 are units. We want to show that, for all $r = 0, \dots, m$, we have that $a_n^{r+1} \cdot b_{m-r} = 0$. We proceed by induction on r. Since $f \cdot g = 1$ we have that $a_n \cdot b_m = 0$. Thus, the case $r = 0$ follows. Assume now that $r > 0$. Since $f \cdot g = 1$, we also have that

$$\sum_{i=0}^{r} a_{n-i} \cdot b_{m-r+i} = 0.$$

Thus, by induction, we get

$$a_n^{r+1} \cdot b_{m-r} = -\sum_{i=1}^{r} a_{n-i} \cdot a_n^r \cdot b_{m-r+i} = 0,$$

as claimed. In particular, $a_n^{m+1} \cdot b_0 = 0$ and since b_0 is a unit, it follows that a_n is nilpotent. By Ex. (b), since f is a unit and $a_n x^n$ is nilpotent, it follows that $\sum_{i=0}^{n-1} a_i x^i$ is also a unit. Thus, proceeding as above, we obtain that $a_1, \dots, a_n$ are nilpotent in R.

Alternative method: Assume that $\mathfrak{p} \subset R$ is a prime ideal. Since $R/\mathfrak{p}$ is an integral domain and f is a unit, it follows that the image $\overline{f}$ of f in $R/\mathfrak{p}[x]$ is a constant. Thus $a_1, \dots, a_n \in \mathfrak{p}$. Since the nilradical of R is the intersection of all the prime ideals of R, it follows that $a_1, \dots, a_n$ are nilpotent. 4, C

(e) Since $\mathcal{J}(S)$ is the intersection of all the maximal ideals of S and $\mathcal{N}(S)$ is the intersection of all the prime ideals of S, it follows immediately that $\mathcal{N}(S) \subset \mathcal{J}(S)$. Let $f = \sum_{i=0}^{n} a_i x^i \in \mathcal{J}(S)$. We showed that $1 - gf$ is a unit for any $g \in S$. In particular, if $g = x$ then $1 - xf$ is a unit. The previous exercise then implies that $a_0, \ldots, a_n$ are nilpotent in R. Thus, there exists a positive integer m such that $a_i^m = 0$ for all $i = 0, \ldots, n$. It then follows that $f^{(n+1)m} = 0$. Thus, $f \in \mathcal{N}(S)$. 4, D

unseen $\Downarrow$

2. (a) (i) Yes. R is Noetherian. Indeed, if $I = (x) \subset R[x]$ then R is isomorphic to $R[x]/I$. Since R is Noetherian, it follows that $R[x]/I$ is also Noetherian and, therefore, R is Noetherian as well. unseen $\Downarrow$

4, A

(ii) Yes. Let $z_1, \ldots, z_n \in S$ be generators over R and let I be the kernel of the induced map unseen $\Downarrow$

$$R[x_1, \ldots, x_n] \to S \qquad x_i \mapsto z_i.$$

In particular, S is isomorphism to $R[x_1, \ldots, x_n]/I$. By Hilbert basis theorem, we know that $R[x_1, \ldots, x_n]$ is Noetherian. Thus, $S \simeq R[x_1, \ldots, x_n]/I$ is also Noetherian. 6, B

(b) Let $J \subset R[[x]]$ be a non-zero ideal. For any $n \geq 0$, we define unseen $\Downarrow$

$$I_n := \{r \in R \mid \exists f \in J \text{ such that } f(x) = rx^n + \text{higher order terms}\}.$$

Then

$$I_0 \subset I_1 \subset I_2 \subset \ldots$$

is an increasing chain of ideals in R. Indeed, if $f \in J$ then $xf \in J$.

Since R is Noetherian, there exists $n \geq 0$ such that $I_i = I_n$ for any $i \geq n$. Moreover, since R is Noetherian, it also follows that each I_i is finitely generated. In particular, we may write

$$I_i = (a_{i,1}, \ldots, a_{i,\ell_i}) \qquad \text{where } a_{i,1}, \ldots, a_{i,\ell_i} \in I_i$$

for any $i \leq n$. Thus, for each $i = 0, \ldots, n$ and $j = 1, \ldots, \ell_i$, there exists $f_{i,j} \in J$ such that

$$f_{i,j} = a_{i,j}x^i + \text{higher order terms}.$$

We want to show that J is generated by the elements $f_{i,j}$, for $i = 0, \ldots, n$ and $j = 1, \ldots, \ell_i$.

Let $f \in J$. First, we may find $b_{i,j} \in R$, with $i = 0, \ldots, n$ and $j = 1, \ldots, \ell_i$, such that

$$f - \sum b_{i,j} f_{i,j} \in (x^{n+1}) \cap J.$$

Next, we may find $c_{1,i} \in R$ such that

$$f - \sum b_{i,j} f_{i,j} - \sum c_{1,i} x f_{n,i} \in (x^{n+2}) \cap J.$$

Similarly, we may find $c_{2,i} \in R$ such that

$$f - \sum b_{i,j} f_{i,j} - \sum c_{1,i} x f_{n,i} - \sum c_{2,i} x^2 f_{n,i} \in (x^{n+3}) \cap J.$$

Continuing this way, we may find $c_{d,i} \in R$, such that

$$f = \sum_{i=0}^{n} \sum_{j=1}^{\ell_i} b_{i,j} f_{i,j} + \sum_{i=1}^{\ell_n} \left(\sum_{d \geq 1} c_{d,i} x^d \right) f_{n,i}.$$

Thus, f is contained in the ideal generated by $f_{i,j}$, as claimed. 7, D

(c) Since R is Noetherian, we have that $\mathcal{N}(R) = \sqrt{(0)}$ is finitely generated. Let $x_1, \ldots, x_k$ be generators. Then $x_i^{n_i} = 0$ for some positive integer n_i, for any $i = 1, \ldots, k$. Let seen $\Downarrow$

$$m := \sum_{i=1}^{k} (n_i - 1) + 1.$$

Then $\mathcal{N}(R)^m$ is generated by the products $x_1^{r_1} \cdot \cdots \cdot x_k^{r_k}$ where $r_1, \ldots, r_k$ are non-negative integers such that $\sum_{i=1}^{k} r_i = m$. Thus, $r_i \geq n_i$ for some $i = 1, \ldots, k$ and, therefore, $x_1^{r_1} \cdot \cdots \cdot x_k^{r_k} = 0$. It follows that $\mathcal{N}(R)^m = 0$. 3, A

3. (a) (i) Consider the set $\boxed{\text{seen} \Downarrow}$

$$\mathcal{P} := \{\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k \mid \mathfrak{m}_i \subset R \text{ is a maximal ideal } \}$$

given by all the finite intersection of maximal ideals. As in Exercise 1 (c), we consider the ordering on $\mathcal{P}$ given by the inverse inclusion. Since R is Artin, it follows that any totally ordered subset of $\mathcal{P}$ has an upper bound and by Zorn's lemma, it follows that $\mathcal{P}$ admits a maximal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$, which is minimal with respect to the inclusion.
Let $\mathfrak{m} \subset R$ be a maximal ideal. Then

$$\mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k,$$

which implies that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k \subset \mathfrak{m}$. Assume by contradiction that $\mathfrak{m}_i \neq \mathfrak{m}$ for all $i = 1, \ldots, m$. Then there exists $x_i \in \mathfrak{m}_i \setminus \mathfrak{m}$ for all $i = 1, \ldots, m$ and, therefore

$$x_1 \cdot \cdots \cdot x_m \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m \subset \mathfrak{m},$$

which contradicts the fact that $\mathfrak{m}$ is maximal and therefore prime. Thus, $\mathfrak{m} = \mathfrak{m}_i$ for some $i = 1, \ldots, m$ and our claim follows. $\boxed{4, \mathrm{A}}$

(ii) Let $\mathfrak{p} \subset R$ be a prime ideal. Then $S := R/\mathfrak{p}$ is an integral domain. It is easy to see that it is an Artin ring. $\boxed{\text{seen} \Downarrow}$
It is enough to show that S is a field. Let $0 \neq x \in S$. Then there exists a positive integer n such that $(x^n) = (x^{n+1})$ and hence $x^n = x^{n+1} \cdot y$ for some $y \in S$. Since S is an integral domain and $x \neq 0$, it follows that $xy = 1$. Thus, x has an inverse in S, as claimed. $\boxed{4, \mathrm{B}}$

(iii) Recall that the nilradical $\mathcal{N}(R)$ is the intersection of all the prime ideals of R. By the previous exercise, we have that every prime ideal is maximal and, therefore, $\mathcal{N}(R)$ is the intersection of all the maximal ideals of R. In particular, it coincides with the Jacobson radical $\mathcal{J}(R)$ of R. $\boxed{\text{seen} \Downarrow}$ $\boxed{2, \mathrm{A}}$

(b) (i) Let $I \subset R$ be an ideal. Consider the set $\boxed{\text{seen} \Downarrow}$

$$P := \{J \subset R \text{ ideal } \mid J \text{ is not a finite intersection of irreducible ideals } \}.$$

Assume by contradiction that P is not empty. Then since R is Noetherian, Zorn's lemma implies that P admits a maximal element J_0 with respect to the inclusion. In particular, J_0 is not irreducible and there exist ideals J_1 and J_2 such that

$$J_0 = J_1 \cap J_2 \qquad \text{and} \qquad J_0 \neq J_i \quad i = 1, 2.$$

Thus, by maximality, $J_i \notin P$ and therefore it is a finite intersection of irreducible ideals, for $i = 1, 2$. But then also J_0 is a finite intersection of irreducible ideals, a contradiction. $\boxed{5, \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(ii) Let $I \subset R$ be an irreducible ideal.
Let $S := R/I$. Since I is irreducible, it follows easily that $(0) \subset S$ is an irreducible ideal. It is enough to show that every zero divisor x of S is nilpotent. Let $y \neq 0$ such that $x \cdot y = 0$ and let $J_i := \operatorname{ann}(x^i)$ for all $i \geq 1$. Then

$$J_1 \subset J_2 \subset \ldots$$

is an ascending chain of ideals and by ACC, there exists a positive integer n such that $J_i = J_n$ for all $i \geq n$.
We want to show that

$$(x^n) \cap (y) = (0). \tag{1}$$

Indeed, let $r \in (x^n) \cap (y)$. Since $r \in (y)$ we have that $r \cdot x = 0$ and since $r \in (x^n)$ we have that $r = s \cdot x^n$ for some $s \in S$ and hence $s \cdot x^{n+1} = r \cdot x = 0$. Thus, $s \in J_{n+1} = J_n$, which implies that $r = s \cdot x^n = 0$ and (1) follows.
Since $(0) \subset S$ is irreducible and $(y) \neq (0)$, it follows that $(x^n) = (0)$, i.e. $x^n = 0$. Therefore, x is nilpotent and our claim follows. $\boxed{5, \text{B}}$

4. (a) (i) Let $\nu\colon K\setminus\{0\}\to\mathbb{Z}$ be a discrete valuation, where K is the field of fractions of R. Since ν is surjective, there exists $t\in K$ such that $\nu(t)=1$. If $x\in\mathfrak{m}$ then $\nu(x)=n\in\mathbb{Z}_{>0}$. Let

$$u:=\frac{x}{t^n}\in k.$$

Then $\nu(u)=\nu(ut^n)-\nu(t^n)=0$ and, therefore, $u\in R$. Thus, $x=ut^n\in(t)$.

seen ⇓ 3, A

(ii) Let $I\subset R$ be a non-zero ideal and let

$$n=\min\{\nu(x)\mid 0\neq x\in I\}.$$

Then, if $0\neq x\in I$, it follows as in (i) that $x=u\cdot t^q$ where u is a unit and $q=\nu(x)\geq n$. Thus, $x\in(t^n)$ and, therefore, we have $I=(t^n)$.
Similarly, if J is non-zero then we have that $J=(t^m)$ for some non-negative integer m. Thus, we have that $I\subset J$ (if $n\leq m$) or $J\subset I$ (if $n\geq m$).

sim. seen ⇓ 3, A

(b) By a Theorem from Lectures, it is enough to show that there exists $t\in R$ such that $\mathfrak{m}=(t)$. By Nakayama's Lemma, we know that $\mathfrak{m}^2\neq\mathfrak{m}$. Thus, there exists $t\in\mathfrak{m}$ such that $t\notin\mathfrak{m}^2$. By assumption, there exists a positive integer r such that $(t)=\mathfrak{m}^r$ and since $t\notin\mathfrak{m}^2$, it follows that $r=1$. Thus, our claim follows.

unseen ⇓ 4, C

(c) Let $x\in K$ be an integral element over $R_\mathfrak{p}$. Then there exist

$$\frac{b_0}{c_0},\dots,\frac{b_{n-1}}{c_{n-1}}\in R_\mathfrak{p}$$

such that $c_0,\dots,c_{n-1}\notin\mathfrak{p}$ and

$$x^n+\frac{b_{n-1}}{c_{n-1}}\cdot x^{n-1}+\cdots+\frac{b_0}{c_0}=0.$$

Then if $y=c_0\cdot\dots\cdot c_{n-1}\cdot x$, we have that

$$y^n+a_{n-1}\cdot y^{n-1}+\cdots+a_0=0$$

for some $a_0,\dots,a_n\in R$. Since R is normal, it follows that $y\in R$. Since $\mathfrak{p}$ is prime, we have that $c:=c_0\cdot\dots\cdot c_{n-1}\notin\mathfrak{p}$. Thus, $x=\frac{y}{c}\in R_\mathfrak{p}$.

seen ⇓ 5, B

(d) By the previous exercise, we have that $R_\mathfrak{p}$ is normal. Let

$$\varphi\colon R\to R_\mathfrak{p}\qquad x\mapsto\frac{x}{1}.$$

unseen ⇓

Then, if

$$I_0\subset I_1\subset I_2\dots$$

is an ascending chain of ideals of $R_\mathfrak{p}$, it follows that

$$\varphi^{-1}(I_0)\subset\varphi^{-1}(I_1)\subset\varphi^{-1}(I_2)\dots$$

is an ascending chain of ideals of R. Since R is Noetherian, it satisfies ACC and, therefore, there exists $n\geq 0$ such that $\varphi^{-1}(I_i)=\varphi^{-1}(I_n)$ for all $i\geq n$. For all $i\geq 0$, we have that

$$I_i=\varphi(\varphi^{-1}(I_i))R_\mathfrak{p}$$

and, therefore,

$$I_0\subset I_1\subset I_2\dots$$

satisfices ACC. Thus $R_{\mathfrak{p}}$ is Noetherian.

Let $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ then $\mathfrak{m}$ is the only maximal ideal in $R_{\mathfrak{p}}$. Thus $(R_{\mathfrak{p}}, \mathfrak{m})$ is a local ring. We showed that there exists a bijection between $\mathrm{Spec}(R_{\mathfrak{p}})$ and the set of prime ideals $\mathfrak{q} \subset R$ such that $\mathfrak{q} \subset \mathfrak{p}$. Since $\mathfrak{p}$ is a minimal nonzero prime ideal, it follows that $\mathrm{Spec}(R_{\mathfrak{p}}) = \{0, \mathfrak{m}\}$. By a Theorem of Lectures, it follows that $(R_{\mathfrak{p}}, \mathfrak{m})$ is a DVR. $\boxed{5, \mathrm{D}}$

5. (a) Let K be a field. A discrete valuation of K is a surjective map seen ⇓

$$\nu\colon K\setminus\{0\}\to\mathbb{Z}$$

such that, for all $x,y\in K\setminus\{0\}$, we have

1. $\nu(xy)=\nu(x)+\nu(y)$;
2. $\nu(x\pm y)\geq\min\{\nu(x),\nu(y)\}$

Let $r\in\mathbb{R}$ with $r>1$. We set

$$|x|_\nu := r^{-\nu(x)},\text{ for } x\in K.$$

Then $|\cdot|: K\to\mathbb{R}_{\geq 0}$ is a norm on K. 4, M

seen ⇓

(b) Let $0\neq\bar{\xi}\in\mathbb{Z}/p\mathbb{Z}$ and let ξ_0 be an lift of $\bar{\xi}$ in $\mathbb{Z}_p$. Since

$$\bar{f}(X)=X^{p-1}-1=\prod_{0\neq x\in\mathbb{Z}/p\mathbb{Z}}(X-x),$$

we have that

$$\bar{f}(\bar{\xi})=0\qquad\text{and}\qquad\bar{f}'(\bar{\xi})=(p-1)\bar{\xi}^{p-2}\neq 0\quad\text{in }\mathbb{Z}/p\mathbb{Z}.$$

Hence, by Hensel's Theorem, f has a root ξ in $\mathbb{Z}_p$ such that $|\xi-\xi_0|<1$ and therefore, ξ is also a lift of $\bar{\xi}$. 4, M

(c) Let $R=k[x]_{(x)}$, let K be the field of fraction of R and let $\nu\colon K\to\mathbb{Z}\cup\{\infty\}$ be the induced valuation. For any $f(x)=\sum_{i=0}^\infty a_ix^i\in k[[x]]$, we denote $f_n(x)=\sum_{i=0}^n a_ix^i\in k[x]$. Then $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in K. The completion $\hat{K}_\nu$ of K is given by $\hat{K}_\nu:=\mathcal{C}_\nu/\mathcal{I}_\nu$ where $\mathcal{C}_\nu$ is the ring of Cauchy sequences in K and $\mathcal{I}_\nu$ is the maximal ideal defined by the Cauchy sequences whose limit is zero. unseen ⇓

Thus, we have a ring homomorphism

$$\Phi\colon k[[x]]\to\hat{K}_\nu\qquad f(x)\mapsto(f_n(x))_{n\geq 1}.$$

First note that, since $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in R, the image of Φ is contained in $\hat{R}_\nu$. Let $\mathfrak{m}:=(x)k[x]_{(x)}$ be the unique maximal ideal in R. Let $\bar{x}$ be the image of x in R. Then $\bar{x}$ is a uniformiser and we have an isomorphism $R/\mathfrak{m}\simeq k$. Thus, every element of the completion of $(R,\mathfrak{m})$ can be written uniquely as $\sum_{i=0}^\infty a_i\bar{x}^i$ where $a_i\in k\subset R$.

Thus, Φ induces an isomorphism $k[[x]]\to\hat{R}_\nu$. 6, M

(d) Recall that if $\nu_p\colon\mathbb{Q}\to\mathbb{Z}\cup\{\infty\}$ is the valuation defined by p, then p is a uniformiser and, therefore, every element of $\mathbb{Z}_p$ can be written uniquely as $\sum_{i=0}^\infty a_ip^i$ where $a_i\in\{0,\dots,p-1\}$. unseen ⇓

Consider the function

$$\Phi\colon\mathbb{Z}[[x]]\to\mathbb{Z}_p\qquad f(x)\mapsto f(p).$$

More precisely, if $f(x)=\sum_{i\geq 0}b_ix^i$ and $b_i=\sum_{j\geq 0}a_{i,j}p^j$ then

$$\Phi(f(x))=\sum_{k\geq 0}\left(\sum_{i+j=k}a_{i,j}\right)p^k.$$

Then Φ is a surjective ring homomorphism. We want to show that, for any $f \in \mathbb{Z}[[x]]$ we have that $\Phi(f) = 0$ if and only if there exists $g(x) \in \mathbb{Z}[[x]]$ such that $f(x) = g(x)(x - p)$, i.e. if and only if $f(x) \in (x - p)$. This immediately implies the desired isomorphism.

The claim is equivalent in showing that $\Phi(f) = 0$ if and only if

$$f(x) = \sum_{i \geq 1} (c_{i-1} - pc_i)x^i - pc_0$$

for some $c_0, c_1, \dots \in \mathbb{Z}$.

Let $f(x) = \sum_{i \geq 0} b_i x^i$ and assume that $\Phi(f) = 0$. Then, since $\sum_{i \geq n+1} b_i p^i$ is divisible by p^{n+1}, since $\Phi(f) = 0$ we have that $\sum_{i=0}^{n} b_i p^i$ is also divisible by p^{n+1}. Let

$$c_n := \frac{\sum_{i=0}^{n} b_i p^i}{p^{n+1}}.$$

Then, it is easy to check by induction on $n \geq 1$ that $b_{n+1} = c_n - pc_{n-1}$. Thus,

$$f(x) = \sum_{i \geq 1} (c_{i-1} - pc_i)x^i - pc_0$$

as claimed. The other implication is immediate. Thus, our claim follows. $\boxed{6, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70061 Commutative Algebra Markers Comments

Question 1	Overall good. Many students found (d) hard.
Question 2	Overall good. Many students found (b) hard.
Question 3	Overall good
Question 4	Overall good.
Question 5	Overall good. Many students found (d) to be hard.