

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: 22 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) What is a regular curve in \mathbb{R}^n ? What is a reparametrisation of a regular curve in \mathbb{R}^n ? (4 marks)
- (b) Show that the length of a regular curve in \mathbb{R}^n is independent of its parametrisation. (6 marks)

- (c) Find an arc-length reparametrisation of the curve

$$\gamma(t) = (\cos(2t), 7t, \sin(2t)), \quad t \in [1, 5].$$

(5 marks)

- (d) Calculate the curvature of the curve

$$\eta(t) = (t^3, t), \quad t \in \mathbb{R},$$

at each point on the curve.

(5 marks)

(Total: 20 marks)

2. Consider the set

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid 3x^2 + y^2 = e^z\}.$$

- (a) Show that C is a regular surface in \mathbb{R}^3 . (5 marks)
- (b) Find the tangent plane of C at $(0, 1, 0)$. (4 marks)

Consider the regular surface

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

- (c) Show that the map
- $$F(x, y, z) = (e^{-z/2}\sqrt{3}x, e^{-z/2}y, z)$$
- is a smooth map from C to D . (6 marks)
- (d) Find the derivative of F at point $(0, 1, 0)$. (5 marks)

(Total: 20 marks)

3. (a) Define the first fundamental form of a surface at a point on the surface. (2 marks)
- (b) Define the Gauss map of a regular orientable surface in \mathbb{R}^3 . (3 marks)
- (c) Prove that the Gauss map of any regular orientable surface in \mathbb{R}^3 is smooth. (4 marks)
- (d) Define the second fundamental form of a surface at a point on the surface. (4 marks)
- (e) Assume that $S \subset \mathbb{R}^3$ is a regular surface, and at some point $p \in S$, the tangent plane of S at p , $T_p S$, is generated by the vectors $v_1 = (0, 1/\sqrt{2}, 1/\sqrt{2})$ and $v_2 = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, and the matrix of the second fundamental form at p in the basis $\{v_1, v_2\}$ is given by

$$A_p = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Find the principal curvatures of S at p . (7 marks)

(Total: 20 marks)

4. (a) What does it mean for a regular curve on a regular surface in \mathbb{R}^3 to be a geodesic? (4 marks)
- (b) Prove that the regular curve $\gamma(\theta) = (\sin(\theta), \cos(\theta), 0)$, $\theta \in \mathbb{R}$, is a geodesic on the regular surface

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 + z^2\}.$$

(6 marks)

- (c) State the Gauss-Bonnet Theorem (no need to give a proof). (4 marks)
- (d) Let $S \subset \mathbb{R}^3$ be a compact, connected surface without boundary which is not diffeomorphic to a sphere. Prove that S contains points where the Gaussian curvature is negative and zero. (6 marks)

(Total: 20 marks)

5. (a) Is the set

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)^2 = z^2\}$$

a regular surface in \mathbb{R}^3 ? Justify your answer.

(5 marks)

(b) Assume that C is a regular surface in \mathbb{R}^3 such that for all $(x, y, z) \in C$ we have $z \geq 0$, and the set $\{(x, y, z) \in C \mid z = 0\}$ is a regular curve γ . Is it true that the Gaussian curvature of C at every point on γ is 0? Justify your answer.

(5 marks)

(c) Consider the regular surface with boundary

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, |z| \leq 1\},$$

and the open set

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 < 1/4\}.$$

Calculate the Euler characteristic of the regular surface with boundary $E \setminus U$. (5 marks)

(d) Let $S \subset \mathbb{R}^3$ be a regular surface such that its Gaussian curvature $K \leq 0$ at all points. Assume that S is diffeomorphic to a plane, and $\gamma : (a, b) \rightarrow S$ is a geodesic parametrised by arc length. Prove that γ is injective. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60032/70032/97049

Geometry of Curves and Surfaces (Solutions)

Setter's signature

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Editor's signature

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1. (a) A regular curve in \mathbb{R}^n is the image of a smooth map $\phi : [a, b] \rightarrow \mathbb{R}^n$ such that for all $t \in [a, b]$, we have $|\phi'(t)| \neq 0$. (2 marks)

seen ↓

Given a regular curve $\phi : [a, b] \rightarrow \mathbb{R}^n$, and a smooth function $f : [c, d] \rightarrow [a, b]$ with $|f'(x)| \neq 0$ for all $x \in [c, d]$ and $f(\{c, d\}) = \{a, b\}$, the curve

$$\phi \circ f : [c, d] \rightarrow \mathbb{R}^n, \quad t \mapsto \phi(f(t))$$

is called a reparametrisation of ϕ . (2 marks)

4, A

- (b) Let $\phi : [a, b] \rightarrow \mathbb{R}^n$ be a regular curve, and $\phi \circ f : [c, d] \rightarrow \mathbb{R}^n$ be a reparametrisation of ϕ by $f : [c, d] \rightarrow [a, b]$.

seen ↓

Since $f'(s) \neq 0$, for all $s \in [c, d]$, and f' is continuous on $[c, d]$, either $f' > 0$ on $[c, d]$ or $f' < 0$ on $[c, d]$. Without loss of generality, let us assume that $f' > 0$ on $[c, d]$. Then, $f(c) = a$ and $f(d) = b$.

Let us write $\phi(t) = (x_1(t), x_2(t), \dots, x_n(t))$. We have

$$\begin{aligned} |\psi'(s)| &= |(\phi \circ f)'(s)| \\ &= |((x_1 \circ f)'(s), (x_2 \circ f)'(s), \dots, (x_n \circ f)'(s))| \\ &= f'(s) |\phi'(f(s))|. \end{aligned}$$

Therefore, by the change of variable formula, we obtain

$$\begin{aligned} \ell(\psi([c, d])) &= \int_c^d |\psi'(s)| ds = \int_c^d |\phi'(f(s))| f'(s) ds = \int_{f(c)}^{f(d)} |\phi'(t)| dt \\ &= \int_a^b |\phi'(t)| dt = \ell(\phi([a, b])). \end{aligned}$$

6, A

- (c) We have

$$\phi'(t) = (-2 \sin(2t), 7, 2 \cos(2t)),$$

sim. seen ↓

and hence

$$|\phi'(t)| = (4 \sin^2(2t) + 7^2 + 4 \cos^2(2t))^{1/2} = \sqrt{54}.$$

For $t \in [1, 5]$, consider the map

$$h(t) = \ell(\phi([1, t])) = \int_1^t |\phi'(u)| du = \sqrt{54}(t - 1).$$

(2 pt up to here)

We have $h^{-1}(s) = s/\sqrt{54} + 1$, with $s \in [0, 4/\sqrt{54}]$. (1 pt)

By a theorem in the lectures, the map

$$\phi \circ h^{-1}(s) = (\cos(2s/\sqrt{54} + 2), 7s/\sqrt{54} + 7, \sin(2s/\sqrt{54} + 2))$$

is parametrised by arc length. (2 pt)

5, B

- (d) By a theorem in the lectures the signed curvature of C is given by

meth seen ↓

$$\kappa(\gamma(t)) = \frac{\langle \gamma''(t), n(t) \rangle}{|\gamma'(t)|^2},$$

where $n(t)$ is a unit normal to the curve γ at $\gamma(t)$. (1 pt)

We have

$$\gamma'(t) = (3t^2, 1), \quad \gamma''(t) = (6t, 0), \quad n(t) = \frac{(-1, 3t^2)}{|(-1, 3t^2)|} = \frac{(-1, 3t^2)}{(1 + 9t^4)^{1/2}}.$$

Thus,

$$\kappa(\gamma(t)) = \frac{-6t}{(1 + 9t^4)^{3/2}}.$$

(2 pt)

Since the curvature of a curve is the absolute value of the signed curvature, the curvature of C at $\gamma(t)$ is

$$k(\gamma(t)) = \frac{6|t|}{(1 + 9t^4)^{3/2}}.$$

(1 pt)

5, C

2. (a) Consider the map $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined as

meth seen ↓

$$G(x, y, z) = 3x^2 + y^2 - e^z.$$

This is a smooth function on \mathbb{R}^3 , and we have $G^{-1}(0) = C$. We note that

$$\nabla G(x, y, z) = (6x, 2y, -e^z).$$

Then, for all $(x, y, z) \in \mathbb{R}^3$, $\nabla G(x, y, z) \neq 0$ due to the last coordinate. This shows that C is a regular level set. By a result in the lectures, every regular level set is a regular surface.

5, B

- (b) By a result in the lectures, the tangent plane to C at p , $T_p C$, is the set of all vectors which are orthogonal to $\nabla G(p)$. We have

meth seen ↓

$$\nabla G(0, 1, 0) = (0, 2, -e^0) = (0, 1, -1).$$

Equivalently, the tangent plane to C at $(0, 1, 0)$, is the set of all $(x, y, z) \in \mathbb{R}^3$ such that $y - z = 0$.

4, B

- (c) Let (x, y, z) be an arbitrary point in C , so that $3x^2 + y^2 = e^z$. Then, $F(x, y, z)$ belongs to D because

$$(e^{-z/2}x\sqrt{3})^2 + (e^{-z/2}y)^2 = e^z(3x^2 + y^2) = 1.$$

In order to show that F is smooth, we need to show that for every chart $\phi : U \rightarrow C$, the map $F \circ \phi : U \rightarrow \mathbb{R}^3$ is a smooth map. The map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has continuous partial derivatives of all orders, so it is smooth on all of \mathbb{R}^3 . Let $\phi : U \rightarrow C$ be an arbitrary chart for S . By definition of charts, ϕ is smooth as a map from U to \mathbb{R}^3 . Therefore, by the chain rule, the map $F \circ \phi : U \rightarrow \mathbb{R}^3$ is a smooth map.

6, C

- (d) We have

$$\begin{aligned} D_x F(x, y, z) &= (e^{-z/2}\sqrt{3}, 0, 0), & D_y F(x, y, z) &= (0, e^{-z/2}, 0), \\ D_z F(x, y, z) &= (-e^{-z/2}x\sqrt{3}/2, -e^{-z/2}y/2, 1). \end{aligned}$$

Therefore,

$$\begin{aligned} D_x F(0, 1, 0) &= (\sqrt{3}, 0, 0), & D_y F(0, 1, 0) &= (0, 1, 0), \\ D_z F(0, 1, 0) &= (0, -1/2, 1). \end{aligned}$$

In order to identify $DF_{(0,1,0)} : T_{(0,1,0)}C \rightarrow T_{F(0,1,0)}D$, it is enough to find the value of this map on a basis for the tangent plane $T_{(0,1,0)}C$. By part (b), we may choose a basis for $T_p C$, say

$$\{v_1 = (0, 1, 1), v_2 = (1, 1, 1)\}.$$

Then,

$$DF_{(0,1,0)}(v_1) = D_y F(0, 1, 0) + D_z F(0, 1, 0) = (0, 1/2, 1),$$

$$DF_{(0,1,0)}(v_2) = D_x F(0, 1, 0) + D_y F(0, 1, 0) + D_z F(0, 1, 0) = (\sqrt{3}, 1/2, 1).$$

5, B

3. (a) Let $S \subset \mathbb{R}^3$ be a regular surface, and $p \in S$. The first fundamental form at p is the bilinear map

seen ↓

$$g : T_p S \times T_p S \rightarrow \mathbb{R},$$

defined as $g(v, w) = \langle v, w \rangle$. In other words, the inner product on \mathbb{R}^3 is restricted to the tangent plane of S at p .

2, A

- (b) For a regular surface $S \subset \mathbb{R}^3$, the tangent plane at each point in S is a two dimensional subspace of \mathbb{R}^3 . It follows that at each point in S , there are two unit normal vectors to the tangent plane. The Gauss map is a continuous choice of a unit normal vector to S . As discussed in the lecture, we can always make such a choice when the surface is orientable.

seen ↓

- (c) In any chart $\phi : U \rightarrow S$ for S , with $\phi(q) = p$ we know that ϕ_u and ϕ_v at q span $T_p S$. Then, the unit normal to the surface S in the chart ϕ is one of the vectors

3, A

seen ↓

$$N(\phi(q)) = \frac{\pm \phi_u(q) \times \phi_v(q)}{|\phi_u(q) \times \phi_v(q)|}. \quad (1)$$

Since ϕ is smooth, we conclude that N is a smooth map.

4, A

- (d) For an orientable regular surface $S \subset \mathbb{R}^3$, with Gauss map N and $p \in S$, the second fundamental form of S at p is the map

seen ↓

$$A_p : T_p S \times T_p S \rightarrow \mathbb{R}, \quad A_p(X, Y) = -\langle X, dN_p(Y) \rangle.$$

4, A

- (e) Let $\lambda_1 \leq \lambda_2$ denote the principal curvatures of S at p . By a theorem in the lectures, we have

unseen ↓

$$\lambda_1 = \min\{A_p(v, v) \mid v \in T_p S, |v| = 1\},$$

$$\lambda_2 = \max\{A_p(v, v) \mid v \in T_p S, |v| = 1\}.$$

Since v_1 and v_2 span $T_p S$, any $v \in T_p S$ can be written as a linear combination of v_1 and v_2 . Also, since v_1 and v_2 are orthonormal, if $|v| = 1$, $v = \cos(t)v_1 + \sin(t)v_2$, for some $t \in \mathbb{R}$. By the bi-linearity of A_p , we have

$$\begin{aligned} A_p(v, v) &= A_p(\cos(t)v_1 + \sin(t)v_2, \cos(t)v_1 + \sin(t)v_2) \\ &= (\cos(t), \sin(t)) A (\cos(t), \sin(t))^t, \end{aligned}$$

where t denotes the transpose operation. Using the matrix in the question, we obtain

$$A_p(v, v) = 2\cos^2(t) + 2\sin(t)\cos(t) + 2\sin^2(t) = 2 + 2\sin(t)\cos(t).$$

We need to find the maximum and minimum of the above function, for $t \in \mathbb{R}$. Let $g(t)$ denote the above function. We have

$$g'(t) = 2\cos^2(t) - 2\sin^2(t),$$

which is 0 iff $\sin^2(t) = \cos^2(t)$ which implies that $t = \pi/4 + 2k\pi$ or $t = 3\pi/4 + 2k\pi$. At those values, $A_p(v, v)$ has values 1 and 3. Thus, the principal curvatures of 1 and 3.

7, D

4. (a) (There are several ways to define this, and one does not have to present all.) Let $\gamma : [a, b] \rightarrow D$ be a regular curve, and N be a unit normal vector on D . Then, γ' is tangent to D and hence is orthogonal to N , so if γ is parametrised by arc-length, then

seen ↓

$$\{\gamma', N \times \gamma', N\}$$

is an orthonormal basis for \mathbb{R}^3 . Recall that the curvature vector $\vec{k} = \gamma''(t)$ is orthogonal to γ' at $\gamma(t)$. Then,

$$\begin{aligned}\vec{k} &= \langle \vec{k}, N \rangle N + \langle \vec{k}, N \times \gamma' \rangle N \times \gamma' \\ &= k_n N + k_g (N \times \gamma')\end{aligned}$$

The number

$$k_g = \langle \vec{k}, N \times \gamma' \rangle$$

is called the geodesic curvature of γ .

If $\gamma : [a, b] \rightarrow D$ is a regular curve with $k_g \equiv 0$ on γ , then γ is called a geodesic.

4, A

- (b) First we note that D is a regular level set. Let $F(x, y, z) = x^2 + y^2 - 1 - z^2$ on \mathbb{R}^3 . We have $\nabla F(x, y, z) = (2x, 2y, -2z)$, which becomes $(0, 0, 0)$ only when $x = y = z = 0$. However, the point $(0, 0, 0)$ does not belong to D . Therefore, ∇F is non-zero near D and D is a regular level set.

seen ↓

For any regular level set, the gauss map is given by

$$N(x, y, z) = \frac{\nabla F(x, y, z)}{|\nabla F(x, y, z)|} = \frac{(2x, 2y, -2z)}{(2(x^2 + y^2 + z^2))^{1/2}} = \frac{(x, y, -z)}{\sqrt{1 + 2z^2}}.$$

On the other hand, we have

$$\gamma'(\theta) = (\cos(\theta), -\sin(\theta), 0),$$

which has a unit size, and hence γ is parametrised by arc length. We also have

$$\gamma''(\theta) = (-\sin(\theta), -\cos(\theta), 0).$$

and

$$N(\gamma(\theta)) \times \gamma'(\theta) = (0, 0, -1).$$

Thus, for every $\theta \in \mathbb{R}$, the geodesic curvature

$$k_g(\gamma(t)) = \langle \gamma''(\theta), N(\gamma(\theta)) \times \gamma'(\theta) \rangle = 0.$$

This shows that γ is a geodesic on S .

6, C

- (c) Theorem: Let $S \subset \mathbb{R}^3$ be a compact and orientable regular surface, possibly with boundary. Then

$$\int_{\partial S} k_g ds + \int_S K dA = 2\pi\chi(S),$$

where ∂S is parametrised by arc-length and is oriented positively. Moreover, if $\partial S = \emptyset$, we have

$$\int_S K dA = 2\pi\chi(S).$$

Here $\chi(S)$ is the Euler characteristic of the surface S .

4, A

- (d) We have seen in the lectures that on any compact surface there is a point where the Gaussian curvature is positive. Let p be such a point. By continuity of the Gaussian curvature, $K > 0$ on an open neighbourhood $V \subset S$ of p . If the curvature of S is nonnegative at all points of S , then by Gauss-Bonnet we have

seen ↓

$$2\pi\chi(S) = \int_S K dA \geq \int_V K dA > 0.$$

Thus, $\chi(S) > 0$, which implies that S is diffeomorphic to a sphere. But this is a contradiction, so there is $q \in S$ such that $K(q) < 0$. Since S is connected, it is also path connected. Let us choose a path from p to q in S . Since K is continuous along that path, the intermediate value theorem implies that there is a point on the path where K becomes 0.

6, D

5. (a) No, it is not. We note that A consists of two sets

meth seen ↓

$$A_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}, \quad A_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = -z\}.$$

Each of A_1 and A_2 is a regular surface since each of them is the graph of a smooth function. However, their union is not a regular surface.

By a result in the lectures, any regular surface in \mathbb{R}^3 is locally the graph of a smooth function of either (x, y) , (y, z) , or (x, z) . The set A near the point $(0, 0, 0)$ is not the graph of any such function. It cannot be the graph of a function of (x, z) because for any (x, z) near $(0, 0)$ there are two points (x', y', z') and $(x', -y', z')$ near $(0, 0, 0)$ in A . For the same reason it cannot be the graph of a function of (y, z) and (x, y) .

There is a unique function, $g(x, y) = (x, y, \sqrt{|y|})$, which maps a neighbourhood of $(0, 0)$ in \mathbb{R}^2 into W . However, this function is not smooth at $(0, 0)$; it is not even differentiable at $(0, 0)$.

5, M

- (b) Let w be an arbitrary point in the xy -plane which also belongs to C . By the hypothesis, there is a regular curve $\gamma : (-\epsilon, +\epsilon) \rightarrow C \cap xy\text{-plane}$ such that $\gamma(0) = w$. Also, by the hypothesis, the unit normal vector to the plane at $\gamma(t)$, $N(\gamma(t))$, is constant over t (this is either $(0, 0, 1)$ or $(0, 0, -1)$). Differentiating $N(\gamma(t))$ with respect to t , we obtain

unseen ↓

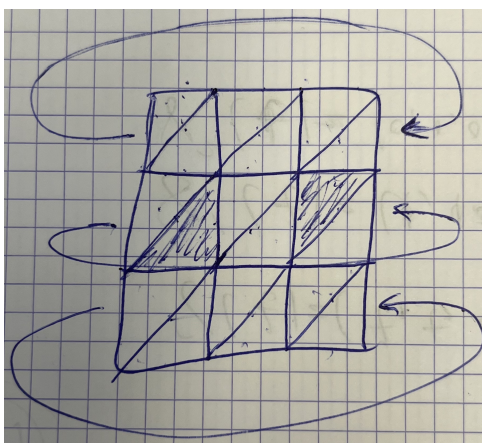
$$0 = (N(\gamma(t)))'(0) = dN_{\gamma(0)}(\gamma'(0)).$$

Thus, the map $dN_w : T_w C \rightarrow T_w C$ is not invertible, it maps a non-zero vector to 0. This implies that $\det dN_p : T_w C \rightarrow T_w C$ is zero. By definition, this determinant is the Gaussian curvature of C at w .

5, M

- (c) We may triangulate the set $E \setminus U$ as follows:

meth seen ↓



In this triangulation, the number of faces $F = 16$, the number of edges $E = 30$, and the number of vertexes $V = 12$. Then,

$$\chi(E \setminus G) = V - E + F = 12 - 30 + 16 = -2.$$

5, M

- (d) Suppose $\gamma(t)$ is not injective, so that (up to shifting t by a constant, and changing the direction) we have $\gamma(0) = \gamma(L)$, for some $L > 0$. Since γ is continuous the set

$$\{t \in [0, L] \mid \gamma(t) = \gamma(0)\}$$

is closed in $[0, L]$. This set contains 0 as an isolated point, since $\gamma'(0) \neq 0$, and it contains L . It follows that there is the smallest positive element t_0 in that set. These imply that $\gamma([0, t_0])$ is a simple closed geodesic (it may not be regular).

Since S is diffeomorphic to a plane, by the Jordan curve theorem, $\gamma([0, t_0])$ bounds a region, say D , which is homeomorphic to an open ball in \mathbb{R}^2 . Applying Gauss-Bonnet theorem to D , we get

$$\int_{\gamma([0, t_0])} k_g ds + \int_D K dA + \Theta = 2\pi,$$

where Θ is the exterior angle between $\gamma'(t_0)$ and $\gamma'(0)$. The integral over $\gamma([0, t_0])$ is zero, since the curve is a geodesic. The integral over K is non-positive, since $K \leq 0$ by assumption. We also know that $-\pi \leq \Theta \leq \pi$ by our convention. Thus, the left hand side of the above equation is bonded from above by π . This is a contradiction.

5, M

Review of mark distribution:

Total A marks: 31 of 32 marks

Total B marks: 19 of 20 marks

Total C marks: 17 of 12 marks

Total D marks: 13 of 16 marks

Total marks: 100 of 100 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60032/70032	1	This is a routine problem, done well by most.
MATH60032/70032	2	Standard problem, where I saw a wide range of solutions presented by students.
MATH60032/70032	3	The difficult part is (e), and the students did well in this, presented three different approaches to this problem.
MATH60032/70032	4	In the Gauss-Bonnet Theorem, the orientation of the surface and its boundary is an important aspect, which has been missed in many solutions.
MATH70032	5	Part b was the most difficult to deal with, which is done only by very few. Most students are comfortable with triangulation, and calculating the Euler characteristic.