Imperial College
London

MATH70039

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Representation Theory

Date: 11 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

All vector spaces are over the complex numbers. You may freely use results from lectures, course notes, coursework, and problem sheets. You may also use the statements of previous parts of a question in your solution to later parts, without having solved the previous parts. The dihedral group D_n is the one acting on an n-gon, with $|D_n| = 2n$.

1. (a) Define *an irreducible representation* of a group G. (1 mark)

   (b) For every $n \geq 1$, give an example of an irreducible representation of dimension n of any finite group of your choosing (depending on n). You do not need to prove it is irreducible. (2 marks)

   (c) For every finite group G, give an example of a faithful representation. (2 marks)

   (d) Let $G = C_6$ with generator g. Let G act on the regular hexagon with generator g acting by 60° rotation. Let X be the set of vertices of the regular hexagon. Decompose $\mathbb{C}[X]$ explicitly into irreducible representations. (5 marks)

   (e) Now let $G = D_6$ acting on the set X of vertices of the regular hexagon. Find the decomposition of $\mathbb{C}[X]$ into irreducible representations, explicitly. (5 marks)

   (f) Let V and W be two non-isomorphic irreducible finite-dimensional representations of a group G. Find all irreducible subrepresentations of $V \oplus V \oplus W$. (5 marks)

(Total: 20 marks)

2. (a) Let V and W be irreducible finite-dimensional representations of a group G. Suppose that $V \otimes W$ contains a one-dimensional subrepresentation L. Prove that $W \cong V^* \otimes L$.
(Hint: use linear algebra to see that $\mathrm{Hom}(L, V \otimes W) \cong \mathrm{Hom}(L \otimes V^*, W)$.*)* (3 marks)

(b) Let $\theta : G \to \mathbb{C}^\times$ be a one-dimensional matrix representation of a finite group G, and let (V, ρ) be a finite-dimensional representation. Show that the map $T : V \to V$, $T(v) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\rho(g)$ is a G-linear projection, whose image is isomorphic to some number of copies of the dual representation of θ. (5 marks)

(c) Let $G = (D_4 \times Q_8)/N$ where $N = \langle (x^2, -1) \rangle$ for $x \in D_4$ the 90° rotation and $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ the quaternionic group (with $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j$). Note that $|G| = 32$.

(i) Show that there is a four-dimensional irreducible representation of G of the form $\mathbb{C}^2 \otimes \mathbb{C}^2$ with D_4 acting on the first factor and Q_8 on the second.
You may use the fact that, if (V, ρ_V) *is an irreducible representation of a finite group* K *and* (W, ρ_W) *is an irreducible representation of a finite group* H*, then* $(V \otimes W, \rho')$ *is an irreducible representation of* $K \times H$*, with* $\rho'(k, h) := \rho_V(k) \otimes \rho_W(h)$. (3 marks)

(ii) Show that the abelianisation of G, $G_{\mathsf{ab}} := G/[G, G]$, is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$. (Recall that the abelianisations of D_4 and of Q_8 are both isomorphic to $C_2 \times C_2$.)
(2 marks)

(iii) Give the dimensions of the irreducible representations of G up to isomorphism, and describe these representations in terms of the preceding parts. (2 marks)

(d) Let G be any group and let $H \leq G$ with H an abelian group and $[G : H] = n$. Let V be an irreducible finite-dimensional representation of G. Prove that $\dim V \leq n$.
(5 marks)

(Total: 20 marks)

3. Let $\zeta := e^{\pi i/3}$, a primitive sixth root of unity. Consider a finite group G with seven conjugacy classes and a representation (V, ρ) with the character:

1	ζ	ζ^2	ζ^3	ζ^4	ζ^5	1

We will complete and use the character table from this information, in parts as follows:

(a) Prove that $\dim V = 1$ and find the minimum $m \geq 1$ such that $\rho(g)^m = I$ for all $g \in G$. (2 marks)

(b) Deduce from the preceding part that $G_{\mathsf{ab}} := G/[G,G] \cong C_6$ and find the characters of all one-dimensional representations. (4 marks)

(c) Using the answer to (b), show that there is an irreducible representation of dimension a multiple of 6. (2 marks)

(d) Using the fact that the dimension of an irreducible representation divides $|G|$, prove that the irreducible representation in (c) has dimension equal to 6. (1 mark)

(e) Using the preceding parts, find $|G|$ and give the complete character table of G. (4 marks)

(f) In this part, we will use a different finite group, which has the following character table:

1	1	1	1	1	1
1	1	1	-1	1	-1
1	1	1	1	-1	-1
1	1	1	-1	-1	1
2	-2	2	0	0	0
8	0	-1	0	0	0

Find the centre and all normal subgroups of this group in terms of unions of conjugacy classes. (7 marks)

(Total: 20 marks)

4. (a) Define *a semisimple module* over an algebra A. (1 mark)

(b) Let A be an algebra. Suppose that $V = V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}$ is a finite-dimensional A-module with $(V_1, \rho_{V_1}), \ldots, (V_m, \rho_{V_m})$ non-isomorphic simple modules. Show that there exists a surjective A-linear map $A \to V$ if $r_i \leq \dim V_i$ for all i. (3 marks)

(c) Now prove the converse to the previous part: with A, V as there, if there exists a surjective A-linear map $T : A \to V$, then $r_i \leq \dim V_i$ for all i.
(Hint: Show that the kernel of T includes the intersection of the kernels of the $\rho_{V_i} : A \to \mathrm{End}(V_i)$.) (6 marks)

(d) Let G be a finite group. Consider the element $z = \frac{1}{|[G,G]|} \sum_{g \in [G,G]} g \in \mathbb{C}[G]$. Show that, for every finite-dimensional $\mathbb{C}[G]$-module (V, ρ), $\rho(z)$ is the projection to the sum of all one-dimensional subrepresentations of V. Conclude that $\rho(z)$ is zero if V is irreducible of dimension greater than one. (5 marks)

(e) Let A be an algebra, and let (V, ρ_V) be a finite-dimensional representation. Recall that there are non-isomorphic simple matrix representations $\rho_i : A \to \mathrm{Mat}_{n_i}(\mathbb{C})$ and integers $r_i \geq 1$ such that $\chi_V = r_1 \chi_{\rho_1} + \cdots + r_m \chi_{\rho_m}$. Prove that there is a basis for V in which $\rho_V(A)$ consists only of block upper-triangular matrices with r_i diagonal blocks of the form ρ_i for all $1 \leq i \leq m$. (5 marks)

(Total: 20 marks)

5. (a) Let (V, ρ_V) be an irreducible representation of a group G (not necessarily finite), and let $H < G$ be a subgroup.

   (i) Prove using Frobenius reciprocity that, if W is any nonzero quotient module of $\operatorname{Res}^G_H V$, then V is isomorphic to a submodule of $\operatorname{coInd}^G_H W$. (4 marks)

   (ii) Conclude that, if the irreducible representations of H have dimension at most m, then the irreducible representations of G have dimension at most $m[G : H]$. (2 marks)

(b) For each of the following representations V of G, find a subgroup $H \leq G$ and a one-dimensional representation W of H such that $V \cong \operatorname{coInd}^G_H W$.

   (i) Let X be a transitive G-set (i.e., $G \times X \to X$ is an action and $X = G \cdot x$ for some $x \in X$), and let $V = \mathbb{C}[X]$. (2 marks)

   (ii) Let $G = D_{2n}$, let X be the set of vertices, and let $V \subset \mathbb{C}[X]$ be the set of labellings of the $2n$ vertices such that opposite labels are equal. (3 marks)

   (iii) Let $G = A_4$, and let V be the three-dimensional irreducible representation. (4 marks)

(c) Let $N \lhd G$ be a normal subgroup of finite index (but G need not be finite). Let (V, ρ) be a finite-dimensional representation of N.

   (i) Show (using Mackey's formula) that $\chi_{\operatorname{coInd}^G_N V}|_N = \sum_{gN \in G/N} \chi_{\rho \circ \operatorname{Ad}_g}$. Here $\operatorname{Ad}_g : N \to N$ is the automorphism given by $\operatorname{Ad}_g(n) = gng^{-1}$. (2 marks)

   (ii) Now show (without using Maschke's theorem, which does not apply if G is infinite) that $\rho_{\operatorname{Res}^G_N \operatorname{coInd}^G_N V} \cong \bigoplus_{gN \in G/N} \rho \circ \operatorname{Ad}_g$.
   (Hint: Use the description of the coinduced representation in terms of functions on cosets of N.) (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH60039/70039

Group Representation Theory (Solutions)

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All vector spaces are over the complex numbers. You may freely use results from lectures, course notes, coursework, and problem sheets. You may also use the statements of previous parts of a question in your solution to later parts, without having solved the previous parts. The dihedral group D_n is the one acting on an n-gon, with $|D_n| = 2n$. seen ⇓

1. (a) *Define an irreducible representation of a group G.*

   This is a pair (V, ρ) where $\rho : G \to \mathrm{GL}(V)$ is a homomorphism and there is no proper nonzero subspace $W \subset V$ with $\rho(g)(W) \subseteq W$. 1, A

   (b) *For every $n \geq 1$, give an example of an irreducible representation of dimension n of any finite group of your choosing (depending on n). You do not need to prove it is irreducible.* seen ⇓

   This can be given by the reflection representation of S_{n+1}. 2, A

   (c) *For every finite group G, give an example of a faithful representation.* seen ⇓

   This is given by the left regular representation $(\mathbb{C}[G], \rho_L)$. It is faithful because $\rho_L(g)(e) = g$, which is distinct for all distinct $g \in G$. 2, A

   (d) *Let $G = C_6$ with generator g. Let G act on the regular hexagon with generator g acting by 60° rotation. Let X be the set of vertices of the regular hexagon. Decompose $\mathbb{C}[X]$ explicitly into irreducible representations.* sim. seen ⇓

   Let $\zeta := e^{\pi i/3}$, a primitive sixth root of unity. For $m \in \mathbb{Z}$, beginning with a fixed vertex and going counterclockwise, we can take the vector $v_m := (1, \zeta^m, \zeta^{2m}, \ldots, \zeta^{5m})$. This spans a subrepresentation, where g acts by eigenvalue ζ^{-m}. These are linearly independent for $0 \leq m \leq 5$ by linear algebra, since ζ^{-m} are all distinct for these values. So this gives a decomposition of $\mathbb{C}[X]$ as a direct sum of six one-dimensional representations of G. (Actually, $\mathbb{C}[X]$ is isomorphic to the left regular representation, which can be seen directly by mapping $a_0 e + a_1 g + \cdots + a_5 g^5$ to $(a_0, \ldots, a_5)$.) 5, A

   (e) *Now let $G = D_6$ acting on the set X of vertices of the regular hexagon. Find the decomposition of $\mathbb{C}[X]$ into irreducible representations, explicitly.* unseen ⇓

   The group G is generated by C_6 and a single reflection. As every subrepresentation under D_6 contains an irreducible C_6 representation, we conclude that the irreducible subrepresentations of $\mathbb{C}[X]$ are generated by an irreducible C_6 representation and its image under a single reflection. This yields $\mathrm{Span}(v_m, v_{-m})$. These are one-dimensional for $v_m = v_{-m}$, when $m = 0$ or 3. They are two-dimensional for $m = 1, 2$. This yields the decomposition $\mathbb{C}[X] = \langle v_0 \rangle \oplus \langle v_3 \rangle \oplus \langle v_1, v_5 \rangle \oplus \langle v_2, v_4 \rangle$ into irreducible subrepresentations. 5, B

   (f) *Let V and W be two non-isomorphic irreducible finite-dimensional representations of a group G. Find all irreducible subrepresentations of $V \oplus V \oplus W$.* sim. seen ⇓

   As seen in lectures, an irreducible subrepresentation must be isomorphic to either V or to W, by Schur's Lemma. The G-linear injections of V into $V \oplus V \oplus W$ must be of the form $v \mapsto (\lambda v, \mu v, 0)$ again by Schur's Lemma, and the G-linear injections of W of the form $w \mapsto (0, 0, \lambda w)$. The images of these are of the form $\{(\lambda v, \mu v, 0) : v \in V\}$ for at least one of λ, μ nonzero, and $\{(0, 0, w) : w \in W\}$. Note that the first spaces only depend on the ratio $[\lambda : \mu]$, i.e., do not change if we simultaneously scale λ and μ (so are parameterised by complex projective space, or the Riemann sphere, but this need not be mentioned). 5, C

2. (a) *Let V and W be irreducible finite-dimensional representations of a group G. Suppose that $V \otimes W$ contains a one-dimensional subrepresentation L. Prove that $W \cong V^* \otimes L$.* meth seen ⇓

(Hint: use linear algebra to see that $\mathrm{Hom}(L, V \otimes W) \cong \mathrm{Hom}(L \otimes V^*, W)$.)

By linear algebra, we have $\mathrm{Hom}(L, V \otimes W) \cong L^* \otimes V \otimes W \cong L^* \otimes (V^*)^* \otimes W \cong (L \otimes V^*)^* \otimes W \cong \mathrm{Hom}(L \otimes V^*, W)$ These are G-linear isomorphisms. If L is a subrepresentation of $V \otimes W$, the isomorphisms give a nonzero G-linear homomorphism $L \otimes V^* \to W$. Since L is one-dimensional, $L \otimes V^*$ is irreducible (this was mentioned in lectures, but it is because being closed under $\rho_{V^*}(g)$ and closed under $\lambda_g \rho_{V^*}(g)$, $\lambda_g \neq 0$, are equivalent). 3, A

(b) *Let $\theta : G \to \mathbb{C}^\times$ be a one-dimensional matrix representation of a finite group G, and let (V, ρ) be a finite-dimensional representation. Show that the map $T : V \to V$, $T(v) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\rho(g)$ is a G-linear projection, whose image is isomorphic to some number of copies of the dual representation of θ.* seen ⇓

To see this, note first that the dual representation to θ is isomorphic to $\theta^{-1}(g) := \theta(g)^{-1}$, since θ is one-dimensional (so that the transpose of a one-by-one matrix is the same matrix back). Next we have that $\rho(h)T(v) = \frac{1}{|G|} \sum_{g \in G} \theta(g)\rho(hg) = \frac{1}{|G|} \sum_{g' \in G} \theta(h^{-1}g')\rho(g') = \theta(h)^{-1}T(v)$. Similarly if $\rho(h)v = \theta(h)^{-1}v$ we get $Tv = v$. So we have that the image of $T(v)$ is the set of all vectors v such that $\rho(g)v = \theta(g)^{-1}v$ for all g, and T is the identity on this set. That means T is a linear projection. Finally, $T \circ \rho(h)v = \theta(h)^{-1}v$ by the same reasoning, so T is G-linear as well. 5, B

(c) *Let $G = (D_4 \times Q_8)/N$ where $N = \langle (x^2, -1) \rangle$ for $x \in D_4$ the 90° rotation and $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ the quaternionic group (with $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j$). Note that $|G| = 32$.* meth seen ⇓

(i) *Show that there is a four-dimensional irreducible representation of G of the form $\mathbb{C}^2 \otimes \mathbb{C}^2$ with D_4 acting on the first factor and Q_8 on the second.*

You may use the fact that, if (V, ρ_V) is an irreducible representation of a finite group K and (W, ρ_W) is an irreducible representation of a finite group H, then $(V \otimes W, \rho')$ is an irreducible representation of $K \times H$, with $\rho'(k, h) := \rho_V(k) \otimes \rho_W(h)$.

By the comment, the tensor product of the two-dimensional irreducible representations of D_4 and of Q_8 defines an irreducible representation of $D_4 \times Q_8$. We can see that N acts trivially because the nontrivial element acts by $-I \otimes -I = I$. So this defines a representation of G. Since it acts by the same transformations as $D_4 \times Q_8$, it must also be irreducible, as desired.

3, B

(ii) *Show that the abelianisation of G, $G_{ab} := G/[G, G]$, is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$. (Recall that the abelianisations of D_4 and of Q_8 are both isomorphic to $C_2 \times C_2$.)*

We have a map $G \to C_2 \times C_2 \times C_2 \times C_2$ coming from the abelianisation maps $D_4 \to C_2 \times C_2$ and $Q_8 \to C_2 \times C_2$, since these maps are trivial on the centres $\langle x^2 \rangle$ and $\langle -1 \rangle$. This map is surjective since the abelianisation maps are surjective. By the universal property of the abelianisation, the abelianisation maps surjectively to $C_2 \times C_2 \times C_2 \times C_2$. Hence the abelianisation has size 16 or 32. The latter is impossible since G itself is not commutative (it contains both D_4 and Q_8). So the abelianisation is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$.

2, A

(iii) *Give the dimensions of the irreducible representations of G up to isomorphism, and describe these representations in terms of the preceding parts.*

By the preceding we can count a four dimensional irreducible representation and sixteen one-dimensional representations, namely the characters of C_2^4 (defined by independently sending the four generators to ± 1). Since $4^2 + 16 = 32$, these must be all of the irreducible representations.

2, A

(d) *Let G be any group, and let $H \leq G$ with H an abelian group and $[G : H] = n$. Let V be an irreducible finite-dimensional representation of G. Prove that $\dim V \leq n$.*

unseen ⇓

If we restrict V to H, there must be an irreducible H-subrepresentation W (by induction on $\dim V$). This subrepresentation is finite-dimensional, hence one-dimensional by Schur's Lemma. Now, the span of $\rho(g)(W)$ must be a subrepresentation of V, hence it equals V by irreducibility. Now for every $gH \in G/H$, we have $\rho(g)W = \rho(gh)W$ for all $h \in H$. So if $\{g_i\}$ are representatives of the left cosets of H in G, then $\{\rho(g_i)W\} = V$. Each $\rho(g_i)W$ has dimension one as W does, and the number of these is $[G : H]$. So the overall dimension of V is at most $[G : H]$.

5, D

3. *Let $\zeta := e^{\pi i/3}$, a primitive sixth root of unity. Consider a finite group G with seven conjugacy classes and a representation (V, ρ) with the character:*

1	ζ	ζ^2	ζ^3	ζ^4	ζ^5	1

We will complete and use the character table from this information, in parts as follows: meth seen ⇓

(a) *Prove that $\dim V = 1$ and find the minimum $m \geq 1$ such that $\rho(g)^m = I$ for all $g \in G$.*

As all the positive values are 1, the value of the character on the identity must be one and so $\dim V = 1$. Next since all values are sixth roots of unity, some of which are primitive, the minimum power to get all values one is six. Thus $m = 6$ is the minimum value (as ρ has dimension one, so powers of the character value are the same thing as powers of ρ itself). 2, A

(b) *Deduce from the preceding part that $G_{ab} := G/[G,G] \cong C_6$ and find the characters of all one-dimensional representations.* meth seen ⇓

The fact that ρ has order six means that $6 \mid G_{\mathsf{ab}}$, since the image of ρ has order six. But $|G_{\mathsf{ab}}|$ equals the number of one-dimensional representations up to isomorphism; by the table we see that there can be at most seven of them. So $|G_{\mathsf{ab}}| = 6$. Since it is abelian, this forces it to be isomorphic to C_6 [or we can use the general fact that there is a noncanonical isomorphism $G_{\mathsf{ab}} \cong \mathrm{Hom}(G, \mathbb{C}^\times)$ whenever G_{ab} is finite]. The one-dimensional representations given as powers of ρ are all distinct, and there are six of them, so they must be all of the one-dimensional representations. 4, B

(c) *Using the answer to (b), show that there is an irreducible representation of dimension a multiple of 6.* unseen ⇓

We have that $n^2 + 6 = |G|$ by the sum of squares formula. On the other hand, $G_{\mathsf{ab}} \cong C_6$ is a quotient of G. So $6 \mid n^2 + 6$, and we get $6 \mid n^2$, which implies $6 \mid n$. 2, C

(d) *Using the fact that the dimension of an irreducible representation divides $|G|$, prove that the irreducible representation in (c) has dimension equal to 6.* seen ⇓

Using that $n \mid |G| = 6 + n^2$, we have $n \mid 6$ as well. 1, A

(e) *Using the preceding parts, find $|G|$ and give the complete character table of G.* meth seen ⇓

We showed already that $n = 6$, so that $|G| = 6 + 6^2 = 42$. Now the left regular representation has character $(42, 0, 0, 0, 0, 0, 0)$, and it equals the sum of the six one-dimensional characters with six times the missing six-dimensional representation. So we can compute the remaining character as

6	0	0	0	0	0	-1

The first six lines are the six powers of ρ. Put together we have:

1	1	1	1	1	1	1
1	ζ	ζ^2	ζ^3	ζ^4	ζ^5	1
1	ζ^2	ζ^4	1	ζ^2	ζ^4	1
1	ζ^3	1	ζ^3	1	ζ^3	1
1	ζ^4	ζ^2	1	ζ^4	ζ^2	1
1	ζ^5	ζ^4	ζ^3	ζ^2	ζ	1
6	0	0	0	0	0	-1

4, B

(f) *In this part, we will use a different finite group, which has the following character table:* meth seen ⇓

1	1	1	1	1	1
1	1	1	-1	1	-1
1	1	1	1	-1	-1
1	1	1	-1	-1	1
2	-2	2	0	0	0
8	0	-1	0	0	0

Find the centre and all normal subgroups of this group in terms of unions of conjugacy classes.

Call the group in question H. The centre of H is given by the columns which always have absolute value equal to that of the first column (as G is finite). This is just the first column, so the centre is trivial. Next, the normal subgroups are intersections of kernels of representations. In turn, kernels mean unions of conjugacy classes with positive values equal to the value for the trivial representation. In these terms the kernels are given as the following unions of conjugacy classes. For $I \subseteq \{1, 2, \ldots, 6\}$, let $\mathcal{C}_I$ be the union of the i-th columns for $i \in I$. The kernels of the irreps are then, in order: H, $\mathcal{C}_{\{1,2,3,5\}}$, $\mathcal{C}_{\{1,2,3,4\}}$, $\mathcal{C}_{\{1,2,3,6\}}$, $\mathcal{C}_{\{1,3\}}$, and $\{e\}$. Intersecting these yields also $\mathcal{C}_{\{1,2,3\}}$, a total of seven normal subgroups. $\boxed{7, \mathrm{A}}$

4. (a) *Define a semisimple module over an algebra* A.

This is a module which is a direct sum of finitely many simple modules.

seen ⇓

1, A

(b) *Let* A *be an algebra. Suppose that* $V = V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}$ *is a finite-dimensional* A*-module with* $(V_1, \rho_{V_1}), \ldots, (V_m, \rho_{V_m})$ *non-isomorphic simple modules. Show that there exists a surjective* A*-linear map* $A \to V$ *if* $r_i \leq \dim V_i$ *for all* i.

sim. seen ⇓

We know that there is a surjective A-linear map $A \to \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m)$, by the density theorem. Since $\mathrm{End}(V_i) \cong V_i^{\dim V_i}$, under the hypotheses, there is a surjective A-linear map $\mathrm{End}(V_i) \to V_i^{r_i}$. Thus we obtain a surjective A-linear map $A \to V$.

3, A

(c) *Now prove the converse to the previous part: with* A, V *as there, if there exists a surjective* A*-linear map* $T : A \to V$*, then* $r_i \leq \dim V_i$ *for all* i.
(Hint: Show that the kernel of T *includes the intersection of the kernels of the* $\rho_{V_i} : A \to \mathrm{End}(V_i)$*.)*

unseen ⇓

Following the hint, if $a \in A$ is in the kernel of all the ρ_{V_i}, then $T(a) = T(a \cdot 1) = \rho_V(a)T(1) = 0$, as V is a direct sum of copies of the V_i. So the kernel of T includes the intersection $K := \bigcap \ker(\rho_{V_i})$ of these kernels. On the other hand, there is a surjective A-linear map $A \to \bigoplus_{i=1}^m \mathrm{End}(V_i)$, with kernel K. So the first isomorphism thoerem implies that $A/K \cong \bigoplus_{i=1}^m \mathrm{End}(V_i)$. On the other hand $A/K' \cong V$ for $K' \supseteq K$ (again the first isomorphism theorem). By the third isomorphism theorem, we then have that $\bigoplus_{i=1}^m \mathrm{End}(V_i)/(K'/K) \cong V$. So we have a surjective homomorphism $\bigoplus V_i^{\dim V_i} \to V$. As we proved in lectures, in this case we can restrict to a summand $\bigoplus V_i^{s_i}$ which maps isomorphically to V (with $s_i \leq \dim V_i$ here): see the next paragraph for more details. This means that $r_i = s_i$ for all i, for example by linear independence of characters of nonisomorphic simple modules (or by Schur's Lemma and induction).

Note: in the above paragraph we used the following general fact proved in lectures: Suppose that $T : W := \bigoplus_{i \in I} V_i \twoheadrightarrow V$ is a surjection with V_i simple. Then there exists a subset $J \subseteq I$ such that, for $V_J := \bigoplus_{j \in J} V_j$, the map $T|_{V_J} : V_J \to V$ is an isomorphism (see Proposition 4.5.5 of the notes, applied to $U = \ker T$). If a student does not recall this, there are other ways to proceed. For instance, $\ker(T) \subseteq W$ is semisimple and has a semisimple complement U, so that $W = \ker(T) \oplus U$, with U mapping isomorphically to V. We can write $\ker(T)$ and U as direct sums of simple modules. Then, W is written as a sum of simple modules, such that a subset of these maps isomorphically to V. Finally we conclude the desired statement because we also proved in lectures that any two decompositions into simple modules have isomorphic summands up to reordering.

6, D

(d) *Let* G *be a finite group. Consider the element* $z = \frac{1}{|[G,G]|} \sum_{g \in [G,G]} g \in \mathbb{C}[G]$*. Show that, for every finite-dimensional* $\mathbb{C}[G]$*-module* (V, ρ)*,* $\rho(z)$ *is the projection to the sum of all one-dimensional subrepresentations of* V*. Conclude that* $\rho(z)$ *is zero if* V *is irreducible of dimension greater than one.*

meth seen ⇓

The projections to the one-dimensional representations (which are zero on all nonisomorphic irreducible representations) are of the form $|G|^{-1} \sum \theta(g) g$ for θ a one-dimensional matrix representation. Writing $G_{\mathsf{ab}} \cong C_{n_1} \times \cdots \times C_{n_k}$, the one-dimensional representations are of the form $\theta_1^{r_1} \cdots \theta_k^{r_k}$ for θ_i of order n_i, a representation of the factor C_{n_i} which is trivial on other factors. Adding all of these projections, and applying the formula that the sum of all n_i-th roots of unity is zero unless n_i=1, and using that $|[G,G]||G_{\mathsf{ab}}| = |G|$, we obtain that the sum of the projections is the desired element z. $\boxed{5, \text{C}}$

(e) *Let A be an algebra, and let (V, ρ_V) be a finite-dimensional representation. Recall that there are nonisomorphic simple matrix representations $\rho_i : A \to \mathrm{Mat}_{n_i}(\mathbb{C})$ and integers $r_i \geq 1$ such that $\chi_V = r_1\chi_{\rho_1} + \cdots + r_m\chi_{\rho_m}$. Prove that there is a basis for V in which $\rho_V(A)$ consists only of block upper-triangular matrices with r_i diagonal blocks of the form ρ_i for all $1 \leq i \leq m$.*

We can prove this by induction on the dimension of V. For the inductive step, find a simple submodule W of V and take a basis of it. Then replace V by V/W; by induction it has a basis in which ρ_V is block upper triangular as desired (note also that $\chi_V = \chi_W + \chi_{V/W}$ shows that r_i decreases for the unique i such that $V_i \cong W$). Putting the basis of W first we get the desired form, since W is a submodule, and the blocks corresponding to two basis elements of V/W have the same form as before. $\boxed{5, \text{D}}$

5. (a) *Let (V, ρ_V) be an irreducible representation of a group G (not necessarily finite), and let $H < G$ be a subgroup.* $\boxed{\text{meth seen} \Downarrow}$

(i) *Prove using Frobenius reciprocity that, if W is any nonzero quotient module of $\mathrm{Res}^G_H V$, then V is isomorphic to a submodule of $\mathrm{coInd}^G_H W$.*

We have $\mathrm{Hom}_G(V, \mathrm{coInd}^G_H W) \cong \mathrm{Hom}_H(\mathrm{Res}^G_H V, W)$. Under the assumptions, the target is nonzero; hence so is the source. As V is irreducible, the nonzero A-linear map $V \to \mathrm{coInd}^G_H V$ is injective. By the first isomorphism theorem we conclude that V is isomorphic to a submodule of $\mathrm{coInd}^G_H V$. $\boxed{4, \mathrm{M}}$

(ii) *Conclude that, if the irreducible representations of H have dimension at most m, then the irreducible representations of G have dimension at most $m[G : H]$.*

The dimension of $\mathrm{coInd}^G_H W$ is $(\dim W)[G : H]$. Since dimension cannot decrease under an injective linear map, the previous part then shows the desired statement. $\boxed{2, \mathrm{M}}$

(b) *For each of the following representations V of G, find a subgroup $H \leq G$ and a one-dimensional representation W of H such that $V \cong \mathrm{coInd}^G_H W$.* $\boxed{\text{seen} \Downarrow}$

(i) *Let X be a transitive G-set (i.e., $G \times X \to X$ is an action and $X = G \cdot x$ for some $x \in X$), and let $V = \mathbb{C}[X]$.*

Here we just take $H = G_x$, the stabiliser of x in G, and let $W = \mathbb{C}$ be the trivial representation. Then the isomorphism $\mathbb{C}[X] \to \mathrm{coInd}^G_{G_x} \mathbb{C}$ is given by $gx \mapsto \delta_{G_x g^{-1}}$, the function which is one on the right coset $G_x g^{-1}$ and zero on other right cosets. This is G-linear since $ghx \mapsto \delta_{G_x h^{-1} g^{-1}} = \rho_{\mathrm{coInd}^G_{G_x}}(g) \cdot \delta_{G_x h^{-1}}$. It is also a linear isomorphism since a basis for the source is given by X and for the target by $\delta_{G_x g}$, with $G_x g \in H \setminus G \cong X$. $\boxed{2, \mathrm{M}}$

(ii) *Let $G = D_{2n}$, let X be the set of vertices, and let $V \subset \mathbb{C}[X]$ be the set of labelings of the $2n$ vertices such that opposite labels are equal.* $\boxed{\text{sim. seen} \Downarrow}$

Here we can take H to be the subgroup generated by two perpendicular reflections through vertices (they exist since we are looking at an even-sided regular polygon). We again take $W = \mathbb{C}$ to be the trivial representation. This reduces to the previous part if we note that $V \cong \mathbb{C}[X']$ with X' the set of pairs of opposite vertices, which is a transitive G-set with a stabiliser equal to H. $\boxed{3, \mathrm{M}}$

(iii) *Let $G = A_4$, and let V be the three-dimensional irreducible representation.* $\boxed{\text{unseen} \Downarrow}$

We can let H be the subgroup of elements $(ab)(cd)$ for a, b, c, d distinct along with the identity ($H \cong C_2 \times C_2$, and it is normal). Viewing V as the reflection representation of S_4, we can restrict it to H, and the character we get of H is $(3, -1, -1, -1)$ (all elements of H are their own conjugacy class, and the trace for any nonidentity element is -1, the trace on the conjugacy class $[(12)(34)] \subseteq A_4$ of V). This restriction to H is the sum of the three nontrivial one-dimensional characters of H. So by (a).(i), the coinduced representation of any of these three contains a subrepresentation isomorphic to V. But the dimension of any of these coinduced representations is $[A_4 : H] = 3$, so any of these coinduced representations is itself isomorphic to V. $\boxed{4, \mathrm{M}}$

(c) *Let $N \lhd G$ be a normal subgroup of finite index (but G need not be finite). Let (V, ρ) be a finite-dimensional representation of N.*

$\boxed{\text{seen} \Downarrow}$

(i) *Show (using Mackey's formula) that* $\chi_{\mathrm{coInd}_N^G V}|_N = \sum_{gN \in G/N} \chi_{\rho \circ \mathrm{Ad}_g}$. *Here* $\mathrm{Ad}_g : N \to N$ *is the automorphism given by* $\mathrm{Ad}_g(n) = gng^{-1}$.

This is just a rewriting of the formula: the formula says that $\chi(n) = \sum_{gN \in G/N} \chi(g^{-1}ng)$, which clearly matches the given formula. $\boxed{2, \mathrm{M}}$

(ii) *Now show (without using Maschke's theorem, which does not apply if* G *is infinite) that* $\rho_{\mathrm{Res}_N^G \mathrm{coInd}_N^G V} \cong \bigoplus_{gN \in G/N} \rho \circ \mathrm{Ad}_g$. $\boxed{\text{unseen} \Downarrow}$

(Hint: Use the description of the coinduced representation in terms of functions on cosets of N*.)*

On each subspace $V(Ng)$, the representation $\mathrm{Res}_N^G \mathrm{coInd}_N^G V$ of N has the action: $\rho_{\mathrm{Res}_N^G \mathrm{coInd}_N^G V}(n)(f)(g) = f(gn) = f(gng^{-1}g) = \rho_V(gng^{-1})f(g)$. So identifying $V(Ng) \cong V$ as vector spaces via $f \mapsto f(g)$, we get that the action of N on $V(Ng)$ matches that of $\rho \circ \mathrm{Ad}_g$. Since V is a direct sum of all the $V(Ng)$, we get the desired result. $\boxed{3, \mathrm{M}}$

Review of mark distribution:

Total A marks: 31 of 32 marks
Total B marks: 21 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60039/70039	1	Parts (a)--(c) were straightforward for most students. Some students had difficulty with (e), especially if the answer to part (d) was not as explicit as in the solutions. Part (f) was challenging for most students, requiring a use of Schur's Lemma that was not explained in full detail in lectures.
MATH60039/70039	2	This was the next most challenging question after Question 4. Many students had difficulty with the tensor product isomorphisms required for (a), or with seeing how it implies the desired statement. Parts (b) and (c) were a bit more doable, except for, in (c), realising how to relate representations of a product of groups with a quotient of such. Part (d) was very difficult for most students.
MATH60039/70039	3	This was well-done, and demonstrated that students continue to develop a strong background in dealing with character tables of finite groups.
MATH60039/70039	4	This was the most challenging question on the exam. This unit, in addition to being the last of the course, covers several deep ideas and further develops multiple ideas from the preceding material of the course. Part 4c was very challenging and many students did not have much time for 4e.
MATH70039	5	This question went better than in past years for the mastery question on this paper. Part (a) was accessible to most students; for part (b) several students found it difficult to identify the correct subgroups required. Part (c).(i) was quite doable but (c).(ii) was difficult, applying the explicit description of the coinduced representation.