Mathematics Pre-arrival course

Solutions to Problem Sheet 2 – Calculus: Integration and Differentiation

Note in all problems that $\log(x)$ represents the natural logarithm of x (sometimes written $\ln(x)$).

1. Calculate the derivative of the following functions:

   (a) $y = x^{-4}$

   $$\frac{dy}{dx} = -4x^{-5}$$

   (b) $y = 4x^{-\frac{3}{2}}$

   $$\frac{dy}{dx} = -6x^{-5/2}$$

   (c) $y = \sin(3x + 1)$

   $$\frac{dy}{dx} = 3\cos(3x + 1)$$

   (d) $y = \sqrt{x^2 + 4}$

   $$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 4}}$$

   (e) $y = (x^2 + 2x - 3)\log(x)$

   $$\frac{dy}{dx} = (2x + 2)\log(x) + (x^2 + 2x - 3)\frac{1}{x} = 2(x + 1)\log(x) + \left(x + 2 - \frac{3}{x}\right)$$

   (f) $y = \sin^2(x)e^x$

   $$\frac{dy}{dx} = 2\cos(x)\sin(x)e^x + \sin^2(x)e^x = \sin(x)e^x\left[2\cos(x) + \sin(x)\right]$$

   (g) $y = \frac{3x+2}{x-1}$ with $(x \neq 1)$

   $$\frac{dy}{dx} = \frac{3(x - 1) - (3x + 2)}{(x - 1)^2} = -\frac{5}{(x - 1)^2}$$

   (h) $y = \frac{e^{\cos(x)}}{x+2}$ with $(x \neq -2)$

   $$\frac{dy}{dx} = \frac{-\sin(x)e^{\cos(x)}(x + 2) - e^{\cos(x)}}{(x + 2)^2} = \frac{-e^{\cos(x)}\left[(x + 2)\sin(x) + 1\right]}{(x + 2)^2}$$

(i) $y = \cot(\sqrt{x})$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}\csc^2(\sqrt{x}) = -\frac{\csc^2(\sqrt{x})}{2\sqrt{x}}$$

(j) $y = \log\left(\cot\left(\frac{x}{2}\right)\right)$

$$\frac{dy}{dx} = \frac{1}{\cot(x/2)}\left(\frac{1}{2}\right)(-\csc^2(x/2)) = -\frac{\csc^2(x/2)}{2\cot(x/2)} = -\frac{\sin(x/2)}{2\cos(x/2)\sin(x/2)} = -\csc(x)$$

2. Calculate an expression for $\frac{dy}{dx}$ when:

(a) $xy = y^2$

$$y + x\frac{dy}{dx} = 2y\frac{dy}{dx} \Rightarrow y = (2y - x)\frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{2y - x}}$$

(b) $\sin(xy) = y$

$$\cos(xy)\left[y + x\frac{dy}{dx}\right] = \frac{dy}{dx} \Rightarrow y\cos(xy) = \frac{dy}{dx}[1 - x\cos(xy)] \Rightarrow \boxed{\frac{dy}{dx} = \frac{y\cos(xy)}{1 - x\cos(xy)}}$$

(c) $x^2 y^2 = 1$

$$2x^y 2 + 2y\frac{dy}{dx}x^2 = 0 \Rightarrow 2x^2 y\frac{dy}{dx} = -2xy^2 \Rightarrow \boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

(d) $\log(y^3) = \frac{x}{2}\log(x - 2)$, $x > 2$, $y > 0$

$$3\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\log(x - 2) + \frac{x}{4}\frac{1}{x - 2} \Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{12}\log(x - 2) + \frac{xy}{12(x - 2)}}$$

(e) $\sqrt{xy} + x + y^2 = 0$

$$\frac{1}{2}\frac{1}{\sqrt{x}}\sqrt{y} + \sqrt{x}\frac{dy}{dx}\frac{1}{2}\frac{1}{\sqrt{y}} + 1 + \frac{dy}{dx}(2y) = 0 \Rightarrow \frac{dy}{dx}\left(\frac{1}{2}\frac{\sqrt{x}}{\sqrt{y}} + 2y\right) = \left(-\frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}} - 1\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sqrt{y/x} - 2}{\sqrt{x/y} + 4y}}$$

3. Find:

(a) $\int_1^2 (x + 3)^8 dx$

$$\int_1^2 (x + 3)^8 dx = \left[\frac{1}{9}(x + 3)^9\right]_1^2 = \frac{1}{9}\left[5^9 - 4^9\right]$$

(b) $\int (x^3 + x + 1)(3x^2 + 1)\mathrm{d}x$

We recognize that this integral is of the form

$$\int (x^3 + x + 1)(3x^2 + 1)\mathrm{d}x = \int u'(x)u(x)\mathrm{d}x$$

with $u(x) = (x^3 + x + 1)$. It can thus be directly integrated as

$$\int (x^3 + x + 1)(3x^2 + 1)\mathrm{d}x = \frac{1}{2}(x^3 + x + 1)^2 + c$$

with $c \in \mathbb{R}$ an arbitrary constant.

(c) $\int \sec^6(x) \tan(x)\mathrm{d}x$

We know that $(\sec(x))' = \sec(x)\tan(x)$, so we again recognize this integral as being of the form

$$\int \sec^6(x) \tan(x)\mathrm{d}x = \int u'(x)(u(x))^5\mathrm{d}x$$

with $u(x) = \sec(x)$. It can thus be directly integrated as

$$\int \sec^6(x) \tan(x)\mathrm{d}x = \frac{1}{6}\sec^6(x) + c$$

with $c \in \mathbb{R}$ an arbitrary constant.

(d) $\int_1^2 \frac{1}{\sqrt{4x+2}}\mathrm{d}x$

We use the following substitution $u = 4x + 2 \Rightarrow \mathrm{d}u = 4\mathrm{d}x$. The bounds of integration are changed into $x = 1 \Rightarrow u = 6$ and $x = 2 \Rightarrow u = 10$, leading to

$$\int_1^2 \frac{1}{\sqrt{4x+2}}\mathrm{d}x = \int_6^{10} \frac{1}{\sqrt{u}}\frac{\mathrm{d}u}{4} = \frac{1}{4}\left[2\sqrt{u}\right]_6^{10} = \frac{1}{4}\left(2\sqrt{10} - 2\sqrt{6}\right) = \frac{\sqrt{2}}{2}\left(\sqrt{5} - \sqrt{3}\right)$$

(e) $\int_1^2 \frac{1}{\sqrt{25-4x^2}}\mathrm{d}x$

$$\int_1^2 \frac{1}{\sqrt{25 - 4x^2}}\mathrm{d}x = \frac{1}{5}\int_1^2 \frac{1}{\sqrt{1 - 4x^2/25}}\mathrm{d}x = \frac{1}{5}\int_1^2 \frac{1}{\sqrt{1 - u^2}}\frac{5}{2}\mathrm{d}u = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{1 - u^2}}\mathrm{d}u$$

where we have used the substitution $u = 2x/5 \Rightarrow \mathrm{d}x = 5/2\mathrm{d}u$ and the bounds of integration are changed into $x = 1 \Rightarrow u = 2/5$ and $x = 2 \Rightarrow u = 4/5$. We finally recognize in $1/\sqrt{1 - u^2}$ the derivative of $\arcsin(u)$ and we obtain

$$\int_1^2 \frac{1}{\sqrt{25 - 4x^2}}\mathrm{d}x = \frac{1}{2}\left[\arcsin(x)\right]_{2/5}^{4/5} = \frac{1}{2}\left[\arcsin(4/5) - \arcsin(2/5)\right]$$

(f) $\int_0^\pi x^2 \sin(x)\mathrm{d}x$

Here, we proceed by integration by parts (twice).

$$\int_0^\pi x^2 \sin(x)\mathrm{d}x \underset{(u=x^2,\ v=-\cos(x))}{=} \left[-x^2 \cos(x)\right]_0^\pi + \int_0^\pi 2x \cos(x)\mathrm{d}x$$

$$\underset{(u=2x,\ v=\sin(x))}{=} \pi^2 + \left[2x \sin(x)\right]_0^\pi - 2\int_0^\pi \sin(x)\mathrm{d}x$$

$$= \pi^2 + \left[2x \sin(x)\right]_0^\pi - 2\int_0^\pi \sin(x)\mathrm{d}x$$

$$= \pi^2 - 2\left[\cos(x)\right]_0^\pi$$

$$= \pi^2 - 4$$

(g) $\int_1^2 \frac{1}{(x-3)(x-4)} dx$

$$\int_1^2 \frac{1}{(x-3)(x-4)} dx = \int_1^2 \left[\frac{-1}{x-3} + \frac{1}{x-4} \right] dx = \left[\log \left| \frac{x-4}{x-3} \right| \right]_1^2 = \log \left| \frac{4}{3} \right|$$

(h) $\int_1^2 (\log(x))^2 dx$

Here, we proceed by integration by parts (twice).

$$\int_1^2 (\log(x))^2 dx \underset{(u=\log(x)^2,\; v=x)}{=} \left[x (\log(x))^2 \right]_1^2 - 2 \int_1^2 \log(x) dx$$

$$\underset{(u=\log(x),\; v=x)}{=} 2\log(2)^2 - 2 \left([x \log(x)]_1^2 - \int_1^2 dx \right)$$

$$= 2\log(2)^2 - 2 \left(2\log(2) - [x]_1^2 \right)$$

$$= 2\log(2)^2 - 4\log(2) + 2(2-1)$$

$$= 2\log(2) \left[\log(2) - 2 \right] + 2$$

(i) $\int \frac{x^2}{x^3+5} dx$

$$\int \frac{x^2}{x^3+5} dx = \frac{1}{3} \int \frac{3x^2}{x^3+5} dx$$

which is of the form $\int u'(x)/u(x) dx$ with $u(x) = x^3 + 5$, so we obtain

$$\int \frac{x^2}{x^3+5} dx = \frac{1}{3} \log \left| x^3 + 5 \right| + c$$

with $c \in \mathbb{R}$ an arbitrary constant.

(j) $\int_4^5 \frac{9x^2-7x+10}{(x-3)(x+1)} dx$

$$\int_4^5 \frac{9x^2 - 7x + 10}{(x-3)(x+1)} dx = \int_4^5 \frac{9(x-3)(x+1) + 11(x+1) + 26}{(x-3)(x+1)} dx$$

$$= \int_4^5 \left(9 + \frac{11}{x-3} + \frac{13}{2} \left[\frac{1}{x-3} - \frac{1}{x+1} \right] \right) dx$$

$$= \int_4^5 \left(9 + \frac{35}{2(x-3)} - \frac{13}{2(x+1)} \right) dx$$

$$= \left[9x + \frac{35}{2} \log|x-3| - \frac{13}{2} \log|x+1| \right]_4^5$$

$$= 9 + \frac{35}{2} \log 2 - \frac{13}{2} \log 6 - \frac{35}{2} \log 1 + \frac{13}{2} \log 5$$

$$= 9 + \frac{35}{2} \log 2 - \frac{13}{2} \log 6 + \frac{13}{2} \log 5$$

(k) $\int \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)} dx$

$$\int \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx = \log |\cos(x) + \sin(x)| + c$$

with $c \in \mathbb{R}$ an arbitrary constant.

(l) $\int_0^{\pi/2} 2^{\sin(x)} \cos(x) \mathrm{d}x$

$$\int_0^{\pi/2} 2^{\sin(x)} \cos(x) \mathrm{d}x = \int_0^{\pi/2} e^{\log(2)\sin(x)} \cos(x) \mathrm{d}x$$

$$= \frac{1}{\log(2)} \left[e^{\log(2)\sin(x)} \right]_0^{\pi/2}$$

$$= \frac{1}{\log(2)} \left(e^{\log(2)} - 1 \right)$$

$$= \frac{1}{\log(2)}$$

(m) $\int_{-1}^{1} |xe^x| \mathrm{d}x$

$$\int_{-1}^{1} |xe^x| \mathrm{d}x = \int_{-1}^{0} (-xe^x) \mathrm{d}x + \int_0^1 xe^x \mathrm{d}x$$

$$= [-xe^x]_{-1}^0 + \int_{-1}^0 e^x \mathrm{d}x + [xe^x]_0^1 - \int_0^1 e^x \mathrm{d}x$$

$$= -\frac{1}{e} + [e^x]_{-1}^0 + e - [e^x]_0^1$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - (e-1)$$

$$= 2\left(1 - \frac{1}{e}\right)$$

4. Find constants a, b, c and d such that

$$\frac{ax+b}{x^2+2x+2} + \frac{cx+d}{x^2-2x+2} = \frac{1}{x^4+4}.$$

Find

$$\int_0^1 \frac{1}{x^4+4} \mathrm{d}x$$

We find the following constants

$$a = \frac{1}{8}, \quad b = \frac{1}{4}, \quad c = -\frac{1}{8}, \quad d = \frac{1}{4}$$

With this information in hand, we can write that

$$\int_0^1 \frac{1}{x^4+4} \mathrm{d}x = \frac{1}{16} \int_0^1 \left(\frac{2x+4}{x^2+2x+2} + \frac{-2x+4}{x^2-2x+2} \right) \mathrm{d}x$$

$$= \frac{1}{16} \int_0^1 \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} + \frac{-2x+2}{x^2-2x+2} + \frac{2}{(x-1)^2+1} \right] \mathrm{d}x$$

$$= \frac{1}{16} \left[\log\left|x^2+2x+2\right| + 2\arctan(x+1) - \log\left|x^2-2x+1\right| + 2\arctan(x-1) \right]_0^1$$

$$= \frac{1}{16} \left[\log(5) + 2\arctan(2) - \log(1) + 2\arctan(0) \right.$$

$$\left. - \log(2) - 2\arctan(1) + \log(2) - 2\arctan(-1) \right]$$

$$= \frac{1}{16} \log(5) + \frac{1}{8} \arctan(2)$$

5. Not until your second year will you learn how to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx,$$

but one of the following answers is correct. Which one?

(a) $-\pi$

(b) 0

(c) $\frac{\pi}{e}$

(d) $\sqrt{2}\pi$

(e) Infinity (the integral doesn't converge)

While you cannot yet compute this integral, there are still a lot of things we can say about by examination! As $\forall x \in \mathbb{R}, -1 \leq \cos(x) \leq 1$, we have

$$\int_{-\infty}^{\infty} \frac{-1}{1 + x^2} \, dx < \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx < \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$$
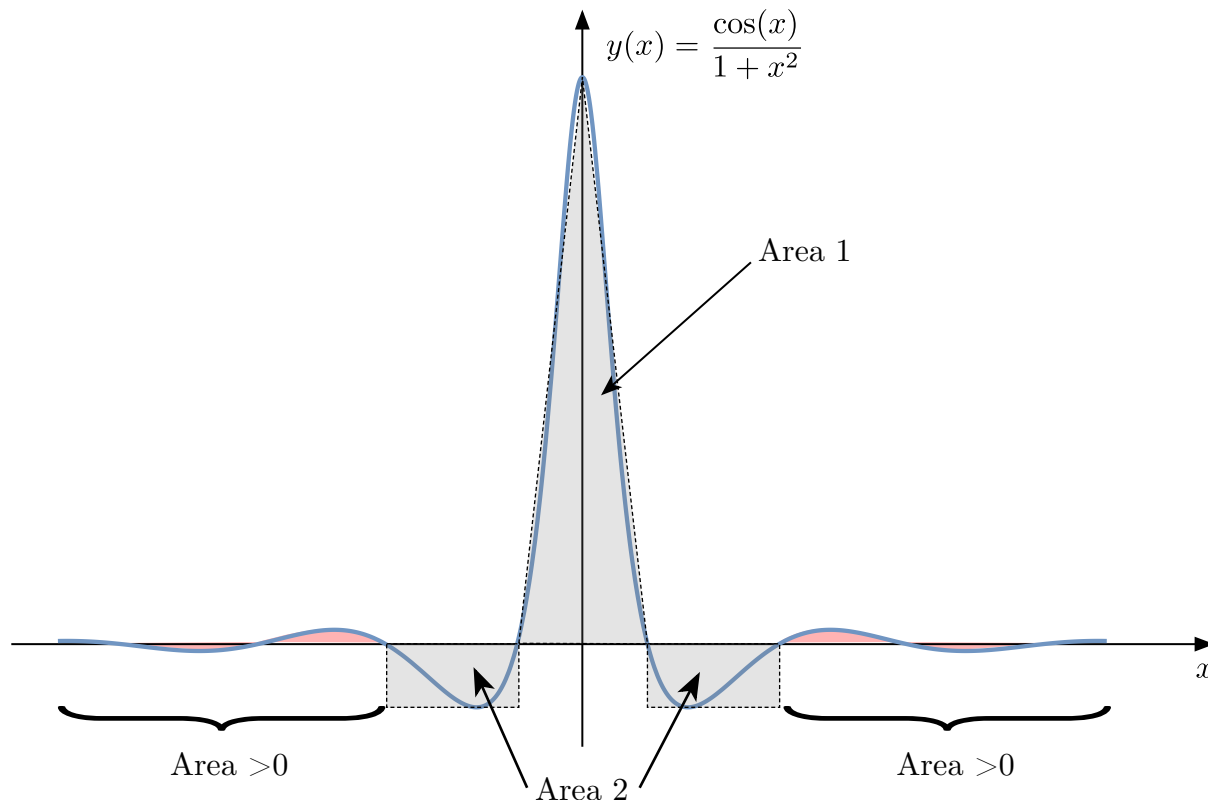
but

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = [\arctan(x)]_{-\infty}^{\infty} = \pi,$$

so we conclude that

$$-\pi < \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx < \pi$$

So we can eliminate the following propositions: $-\pi$, $\sqrt{2}\pi$ and ∞. Further, a sketch shows

and we observe that

$$\text{Area 1} - \text{Area 2} = \frac{1}{4}\pi - \frac{2\pi}{1 + \pi^2} > 0$$

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So the integral must be strictly positive and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, \mathrm{d}x = \frac{\pi}{e}$$

NB: Our argument here is not very rigorous, remember that a sketch is not a proof!

6. If $y = \sinh^{-1} x$ show that $y = \log(x + \sqrt{x^2 + 1})$.
   Given that:

$$\sinh^{-1} x - \cosh^{-1} x = \log(2),$$

show that

$$\sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = x.$$

Hence, find the value of x.

We define $y = \sinh^{-1}(x)$, we thus have

$$\sinh(y) = x \Rightarrow \frac{e^y - e^{-y}}{2} = x \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

We can solve this quadratic equation easily to obtain

$$e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1}$$

where we have kept only the positive root as this is the only one which makes sense (remember that exponential is strictly positive!) and we conclude

$$y = \log\left(x + \sqrt{x^2 + 1}\right)$$

Given that

$$\sinh^{-1} x - \cosh^{-1} x = \log(2),$$

we thus have

$$\log\left(x + \sqrt{x^2 + 1}\right) - \log\left(x + \sqrt{x^2 - 1}\right) = \log(2)$$
$$\Rightarrow \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}} = 2$$
$$\Rightarrow x + \sqrt{x^2 + 1} - 2x - 2\sqrt{x^2 - 1} = 0$$
$$\Rightarrow \sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = x$$
$$\Rightarrow (x^2 + 1) - 4\sqrt{x^2 + 1}\sqrt{x^2 - 1} + 4(x^2 - 1) = x^2$$
$$\Rightarrow 4x^2 - 3 = 4\sqrt{x^2 + 1}\sqrt{x^2 - 1}$$
$$\Rightarrow 16x^4 - 24x^2 + 9 = 16(x^2 + 1)(x^2 - 1)$$
$$\Rightarrow 16x^4 - 24x^2 + 9 = 16x^4 - 16$$
$$\Rightarrow 24x^2 = 25$$
$$\Rightarrow x = \pm\frac{5}{2\sqrt{6}}$$

but $x > 0$ for the equation to make sense, and so we conclude $x = \frac{5}{2\sqrt{6}}$.

7. Solve the equation

$$7\operatorname{sech}(x) - \tanh(x) = 5.$$

$7\operatorname{sech}(x) - \tanh(x) = 5$

$$\Rightarrow \frac{7}{\cosh(x)} - \frac{\sinh(x)}{\cosh(x)} = 5$$

$$\Rightarrow 7 - \sinh(x) = 5\cosh(x)$$

$$\Rightarrow 7 - \sinh(x) = 5\sqrt{1 + \sinh^2(x)}$$

$$\Rightarrow 49 - 14\sinh(x) + \sinh^2(x) = 25 + 25\sinh^2(x)$$

$$\Rightarrow 24\sinh^2(x) + 14\sinh(x) - 24 = 0$$

$$\Rightarrow 12\sinh^2(x) + 7\sinh(x) - 12 = 0$$

$$\Rightarrow \sinh(x) = \frac{-7 \pm \sqrt{49 + 4 \times 12 \times 12}}{2 \times 12}$$

$$\Rightarrow \sinh(x) = \frac{-7 \pm 25}{24}$$

$$\Rightarrow \sinh(x) = \frac{18}{24} \quad \text{or} \quad -\frac{32}{24}$$

$$\Rightarrow \sinh(x) = \frac{3}{4} \quad \text{or} \quad -\frac{4}{3}$$

We thus conclude that the two solutions are given by

$$x = \sinh^{-1}(3/4) = \log\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right) = \log(2)$$

or

$$x = \sinh^{-1}(-4/3) = \log\left(-\frac{4}{3} + \sqrt{\left(-\frac{4}{3}\right)^2 + 1}\right) = -\log(3)$$

8. Show that

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}).$$

Show that the area of the region defined by the inequalities $y^2 \geq x^2 - 8$ and $x^2 \geq 25y^2 - 16$ is $(72/5)\log(2)$.

Let $y = \cosh^{-1}(x)$, then $x = \cosh(y)$, which implies that

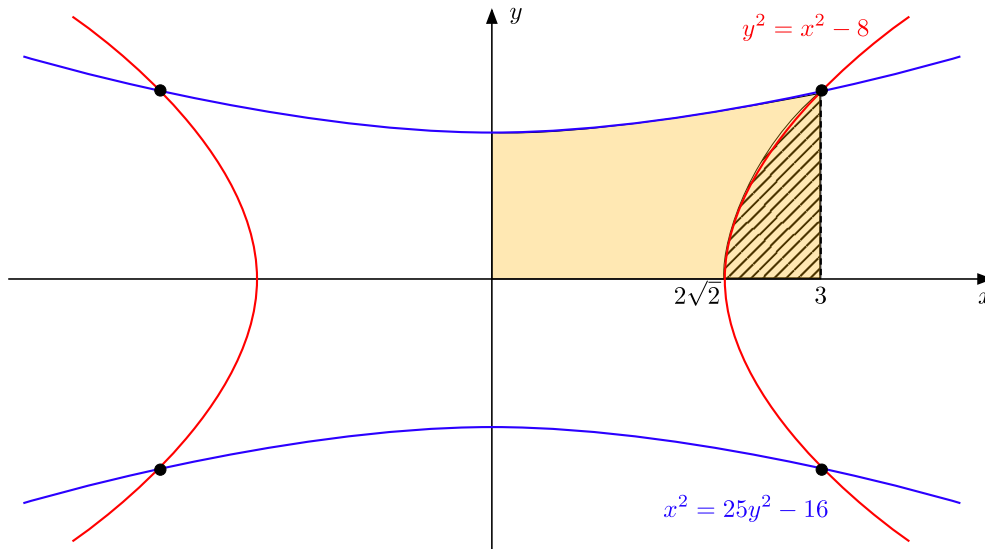$$x = \frac{e^y + e^{-y}}{2} \Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$\Rightarrow e^y = x + \sqrt{x^2 - 1}$$

$$\Rightarrow \cosh^{-1}(x) = \log\left(x + \sqrt{x^2 - 1}\right)$$

where we have taken at the third step the positive root to define $\cosh^{-1}(x)$ as the positive number whose $\cosh$ gives x.

For the second part of the question, we start by a sketch:

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We are trying to compute the area $\mathcal{A}$ of the region contained within the four branches of the hyperbolas on the above sketch. We know that

$$\mathcal{A} = 4 \times [\text{Orange Area} - \text{Shaded area}]$$

The intersection points between the red and blue parabolas are given as follows:

$$x^2 = 25(x^2 - 8) - 16 \Rightarrow 24x^2 - 216 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

and the intersection points between the right red parabolas and the x-axis is given by

$$0 = x^2 - 8 \Rightarrow x = \pm 2\sqrt{2}$$

So the area of interest can be expressed as follows:

$$\mathcal{A} = 4 \left[\int_0^3 \sqrt{\frac{x^2 + 16}{25}} \mathrm{d}x - \int_{2\sqrt{2}}^3 \sqrt{x^2 - 8} \mathrm{d}x \right]$$

$$= \frac{4}{5} \int_0^3 \sqrt{x^2 + 16} \mathrm{d}x - 4 \int_{2\sqrt{2}}^3 \sqrt{x^2 - 8} \mathrm{d}x$$

For the first integral, we use the following substitution $x = 4\sinh(u) \Rightarrow \mathrm{d}x = 4\cosh(u)\mathrm{d}u$, for this substitution the integral bounds are changed into $x = 3 \Rightarrow u = \sinh^{-1}(3/4)$ and $x = 0 \Rightarrow u = 0$. For the second integral, we use the following substitution $x = 2\sqrt{2}\cosh(v) \Rightarrow \mathrm{d}x = 2\sqrt{2}\sinh(v)\mathrm{d}v$, for this substitution the integral bounds are changed into $x = 3 \Rightarrow v = \cosh^{-1}(3/2\sqrt{2})$ and $x = 2\sqrt{2} \Rightarrow v = 0$. This leads to

$$\mathcal{A} = \frac{4}{5} \int_0^{\sinh^{-1}(3/4)} 4\cosh(u)(4\cosh(u))\mathrm{d}u - 4 \int_0^{\cosh^{-1}(3/2\sqrt{2})} 2\sqrt{2}\sinh(v)(2\sqrt{2}\sinh(v))\mathrm{d}v$$

$$= \frac{64}{5} \int_0^{\sinh^{-1}(3/4)} \frac{\cosh(2u) + 1}{2} \mathrm{d}u - 32 \int_0^{\cosh^{-1}(3/2\sqrt{2})} \frac{\cosh(2v) - 1}{2} \mathrm{d}v$$

$$= \frac{32}{5} \left[\frac{1}{2}\sinh(2u) + u \right]_0^{\sinh^{-1}(3/4)} - 16 \left[\frac{1}{2}\sinh(2v) - v \right]_0^{\cosh^{-1}(3/2\sqrt{2})}$$

and using the fact that $\sinh^{-1}(3/4) = \log(2)$ and $\cosh^{-1}(2\sqrt{2}) = \log 2/2$ (using previous results),

we obtain

$$\mathcal{A} = \frac{32}{5} \left[\frac{1}{2} \sinh(2\log(2)) + \log(2) \right] - 16 \left[\frac{1}{2} \sinh(\log(2)) - \frac{\log(2)}{2} \right]$$

$$= \frac{32}{5} \left[\frac{1}{2} \left(\frac{15}{8} \right) + \log(2) \right] - 16 \left[\frac{1}{8} (3 - 4\log(2)) \right]$$

$$= 6 + \frac{32}{5} \log(2) - 6 + 8\log(2) = \frac{72}{5} \log(2)$$