Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M4A36/M5A36

Ergodic Theory

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y is a topological space endowed with the Borel σ-algebra and $f : X \to Y$.

(i) (a) State the definition of measurability of f.

Answer: [1, seen]

A function f is measurable if $f^{-1}(A) \in \mathcal{A}$ for any open $A \subset Y$.

(b) Let $f : X \to [0, \infty]$ be measurable. Show that there exist simple measurable functions s_n on X such that $0 \le s_1 \le s_2 \ldots \le f$ and

$$\lim_{n\to\infty} s_n(x) = f(x).$$

Answer: [5, unseen]

For $n = 1, 2, 3, \ldots$, and for $1 \le i \le n2^n$, define

$$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)$$

and

$$F_n = f^{-1}([n, \infty])$$

and put

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n,i}} + n\chi_{F_n}$$

Clearly, $E_{n,i}$ and F_n are measurable sets and s_n is an increasing sequence of functions bounded by f. If n is large then $f(x) - s_n(x) \le 2^{-n}$ when $f(x) < \infty$ and $s_n(x) = n$ if $f(x) = \infty$.

(ii) Let $(X, \mathcal{A}, \mu)$ be a probability measure space and let $T : X \to X$ be a measure preserving map.

(a) State and prove Poincare's recurrence theorem

Answer: [2+10, seen]

Let $A \in \mathcal{A}$ such that $\mu(A) > 0$. Then, μ-almost every point $x \in A$ there exists some $n \in \mathbb{N}$ such that $T^n(x) \in A$. Consequently, there are infinitely many $k \in \mathbb{N}$ for which $T^k \in A$.

$B := \{x \in A : \ T^k(x) \notin A, \forall k \in \mathbb{N}\} = A \setminus \cup_{k\in\mathbb{N}} T^{-k}(A)$.

Then $B \in \mathcal{A}$, $T^{-k}(B) \in \mathcal{A}$ and $\mu(T^{-k}(B)) = \mu(B)$. Furthermore, $T^{-m} \cap T^{-n} = \emptyset$ for $m \ne n$. Otherwise,

$$T^m(x) \in T^m\left(T^{-m}(B) \cap T^{-n}(B)\right) = B \cap T^{m-n}$$

in contradiction to the definition of the set B. Furthermore,

$$\begin{aligned} 1 = \mu(X) &\le \mu(\cup_{k\in\mathbb{N}} T^{-k}(B)) \\ &= \sum_{k\in\mathbb{N}} \mu(T^{-k}(B)) \\ &= \sum_{k\in\mathbb{N}} \mu(B) \end{aligned}$$

which implies $\mu(B) = 0$. That is $\mu(A \setminus B) = \mu(A)$ and every point in $A \setminus B$ returns to A.
For the second assertion,

$$\tilde{B}_n := \{x \in A : T^n(x) \in A \text{ and } T^k(x) \notin A,\ k > n\},\ n \leq 1.$$

Since $T^n(\tilde{B}_n) \subset B$ thus $\mu(T^n(\tilde{B}_n)) = 0$. But, since

$$\tilde{B}_n \subset T^{-n}\left(T^n(\tilde{B}_n)\right)$$

thus

$$\mu(\tilde{B}_n) \leq \mu\left(T^{-n}\left(T^n(\tilde{B}_n)\right)\right) = 0.$$

(iii) State the Krylov-Bogolubov Theorem

Answer: [2, seen]

Let X be a compact metric space and $T : X \to X$ be a continuous map. Then there exists a T-invariant Borel probability measure on X.

2. Let $(X, \mathcal{A}, \mu)$ be a probability measure space and $T : X \to X$ a measure-preserving transformation.

(i) (a) State the definition of ergodicity of T.

Answer: **[2, seen]**

T is ergodic if for any $A \in \mathcal{A}$ such that $T^{-1}(A) = A$ either $\mu(A) = \mu(X)$ or $\mu(A) = 0$.

(b) Assume that μ is the only invariant probability measure of a map $T : X \to X$. Show that μ is ergodic.

Answer: **[5, unseen]**

Consider $A \subset X$ of positive measure and let

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}.$$

If μ is not ergodic then there is a T-invariant set A such that $0 < \mu(A) < 1$. That is μ_A and μ_{A^c} are T-invariant measures. However, $\mu_A(A) = 1$ and $\mu_{A^c}(A) = 0$ in contradiction to the uniquness of μ.

(ii) Let $T : X \to X$ be a probability preserving map and $\mathcal{G} = \sigma(\{A \in \mathcal{A} : T^{-1}(A) = A\})$.

(a) Show that any $\mathcal{G}$-measurable random variable $f : X \to \mathbb{R}$ is T-invariant.

Answer: **[5, unseen]**

Using 1.(i)(b), it is enough to validate the statement for elementary functions. Let

$$s_n = \sum_{i=1}^{k_n} c_i^{(n)} \chi_{A_i^{(n)}}$$

where $c_i^{(n)} \in \mathbb{R}$ and $A_i^{(n)} \in \mathcal{G}$. Then

$$s_n \circ T = \sum_{i=1}^{k_n} c_i^{(n)} \chi_{A_i^{(n)}} \circ T = \sum_{i=1}^{k_n} c_i^{(n)} \chi_{T^{-1}\left(A_i^{(n)}\right)} = \sum_{i=1}^{k_n} c_i^{(n)} \chi_{A_i^{(n)}} = s_n$$

because $T^{-1}(A_i^n) = A_i^n$. Thus $s_n,\ n \in \mathbb{N}$ is invariant. Since $s_n \to f$ as $n \to \infty$, we get $s_n \circ T \to f \circ T$ as $n \to \infty$. The uniquness of the limit implies the statement.

(b) Let

$$S_N(x) = \sum_{n=0}^{N-1} f(T^n(x)) \text{ and } M_N(x) = \max\{S_0(x), \ldots, S_N(x)\}$$

where $S_0 = 0$. Prove that

$$\int_{\{M_N > 0\}} f d\mu \geq 0.$$

Answer: **[5, seen]**

For $0 \leq k \leq N$ and $x \in X$ $M_N(T(x)) > S_k(T(x))$ and $f(x) + M_N(T(x)) \geq f(x) + S_k(T(x)) = S_{k+1}(x)$. Thus,

$$f(x) \geq \max\{S_1(x), \ldots, S_N(x)\} - M_N(x).$$

In addition, $\max\{S_1(x), \ldots, S_N(x)\} = M_N(x)$ on $\{M_N > 0\}$ thus

$$\begin{aligned}
\int_{\{M_N>0\}} f d\mu &\geq \int_{\{M_N>0\}} (M_N - M_n \circ T) d\mu & (1) \\
&\geq \mathrm{E}[M_N] - \int_{\{M_N>0\}} (M_N \circ T) d\mu & (2)
\end{aligned}$$

since $M_N \geq 0$. Thus, since T is measure-preserving

$$\begin{aligned}
\int_{\{M_N>0\}} (M_N \circ T) d\mu &= \int_X \chi_{\{M_N>0\}} (M_N \circ T) d\mu \\
&= \int_X \chi_{\{T(x)|M_N(x)>0\}} M_N d(T_* \mu) \\
&= \int_{\{T(x):M_N(x)>0\}} M_N d\mu
\end{aligned}$$

where $T_*\mu$ is the push-forward of μ. Thus, since $\int_B M_N d\mu$ by the non-negativity of M_N, (2) implies that

$$\int_{\{M_N>0\}} f d\mu \geq \mathrm{E}[M_N] - \int_{\{M_N>0\}} (M_N \circ T) d\mu \geq 0.$$

(c) Formulate Birkhoff's Ergodic Theorem

Answer: [3, seen]

If $f : X \to \mathbb{R}$ such that $\mathrm{E}[|f|] < \infty$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \mathrm{E}[f|\mathcal{G}](x) \ a.s.$$

where $\mathcal{G} = \sigma(\{A \in \mathcal{A} : T^{-1}(A) = A\})$.

3. (i) Let $f : [0,1] \to [0,1]$ be defined by $f(x) := |1-2x|$, which is a measurable mapping with respect to the Borel σ-algebra of $[0,1]$.

(a) Show that the Lebesgue measure is invariant with respect to f.

Answer: [2, seen similar]

The preimage of any interval $J \subset [0,1]$ is the union of two intervals, each of which has the length $\lambda(J)/2$. This proves invariance of the Lebesgue measure.

(b) Show that the Lebesgue measure is ergodic with respect to f.

Answer: [8, seen similar]

Let A be a Borel set such that $f^{-1}(A) = A$ and suppose that $\lambda(A) > 0$. For any $n \in \mathbb{N}$, the mapping f^n is also piecewise affine with a partition P_n consisting of 2^n intervals of length $1/2^n$. Lebesgue's Density Theorem implies that for all $\epsilon > 0$, there is an $n \in \mathbb{N}$ and an interval $I \in P_n$ with

$$\frac{\lambda(A \cap I)}{\lambda(I)} \geq 1 - \epsilon$$

$f^n : I \to [0,1]$ is a diffeomorphism and $f^{-n}(A) = A$ implies that $f^n(I \setminus A) = [0,1] \setminus A$. f^n also preserves the ratios of measures of sets, so

$$\frac{\lambda([0,1] \setminus A)}{\lambda([0,1])} = \frac{\lambda(f^n(I \setminus A))}{\lambda(f^n(I))} = \frac{\lambda(I \setminus A)}{\lambda(I)} \leq \epsilon \,.$$

Since ϵ is arbitrary, it follows that $\lambda(A) = 1$.

(ii) Let X be a finite set, $\mathcal{A}$ be the set of all subsets of X, and consider a mapping $g : X \to X$.

(a) Show that there exists an invariant probability measure $\mu : \mathcal{A} \to [0,1]$ with respect to g.

Answer: [4, unseen]

Let $x \in X$. Since X is finite, there exist $n, m \in \mathbb{N}$ such that $f^n(x) = f^{m+n}(x)$. Choosing m minimal with this property gives a periodic point $f^n(x)$ of primitive period m. Hence the Dirac measure concentrated on the induced periodic orbit, given by $\frac{1}{m}\sum_{i=0}^{m-1} \delta_{f^i(x)}$, is an invariant measure.

(b) State what it means for an invariant probability measure $\mu : \mathcal{A} \to [0,1]$ to be weakly mixing with respect to g.

Answer: [2, seen]

μ is weakly mixing w.r.t. g if for all $A, B \in \mathcal{A}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(A \cap g^{-n}(B)) - \mu(A)\mu(B)| = 0 \,.$$

(c) Clarify the question if any invariant probability measure with respect to an arbitrary mapping $g : X \to X$ is weakly mixing (give a proof or show that there exists a counterexample).

Answer: [4, unseen]

Let $X = \{0,1\}$ and $g(i) = i + 1 \operatorname{mod} 2$. Then $\mu := \frac{1}{2}(\delta_0 + \delta_1)$ is invariant (see solution of a) above), but this measure is not weakly mixing: Let $A = B = \{0\}$. Then $|\mu(A \cap g^{-n}(A)) - \mu(A)^2| = 1/4$ for all $n \in \mathbb{N}$, which implies the assertion.

4. We consider mappings on the circle $\mathbb{S}^1$ and the torus $\mathbb{T}^2$ which involve an irrational rotation with an angle $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

(i) Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $f(x, y) := (x + \alpha \bmod 1, x + y \bmod 1)$. Use Fourier series to show the Lebesgue measure is ergodic with respect to f (you do not need to show the fact that the Lebesgue measure is invariant with respect to f).

Answer: [10, seen similar]

Let $h \in L^2(\mathbb{T}^2)$ such that $h \circ T = h$ almost everywhere; h is given by its Fourier series

$$h(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{n,m} e^{2\pi i(nx+my)} .$$

Then

$$h(f(x, y)) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{n,m} e^{2\pi i n\alpha} e^{2\pi i(x(n+m)+my)} .$$

The uniqueness of the Fourier coefficients implies that $a_{n+m,m} = a_{n,m} e^{2\pi i n\alpha}$. Suppose that $m = 0$. Then $a_{n,0} = a_{n,0} e^{2\pi i n\alpha}$. Since α is irrational, this implies that $a_{n,0} = 0$ for all $0 \neq n \in \mathbb{Z}$. Suppose now that $m \neq 0$ and there exists $0 \neq n \in \mathbb{Z}$ with $a_{n,m} \neq 0$. Then $|a_{n+km,m}| = |a_{n,m}|$ for all $k \in \mathbb{N}$ by induction, and this proves $a_{n,m} = 0$ by Riemann–Lebesgue Lemma. This proves that all $a_{n,m}$ are zero, except $a_{0,0}$, so h is constant almost everywhere. This implies ergodicity of the Lebesgue measure.

(ii) Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be the irrational circle rotation $g(x) := e^{2\pi i\alpha} x$. Show that the Lebesgue measure is not mixing.

Answer: [10, seen] We show that the Lebesgue measure is not weakly mixing, which implies that it is not mixing. Set $A := \{e^{2\pi x} : x \in [0, \frac{1}{4}]\}$ and $B := \{e^{2\pi x} : x \in [\frac{1}{2}, \frac{3}{4}]\}$. Since the Lebesgue measure is ergodic w.r.t. $f_{-\alpha}$, Birkhoff's Ergodic Theorem implies that there exists a $y \in A$ such that $\frac{1}{n} \sum_{i=0}^{n-1} \chi_B(f_{-\alpha}^i(y)) \to \frac{1}{4}$ as $n \to \infty$. The set $\{i \in \mathbb{N}_0 : f_\alpha^{-i}(A) \cap A = \emptyset\}$ thus has density of at least $\frac{1}{4}$. This means that $|\lambda(g^{-i}(A) \cap A) - \lambda(A)^2| = \frac{1}{16}$ for all i from a set of density of at least $\frac{1}{4}$. Hence, the equivalent characterisation of weakly mixing from the lecture implies that the Lebesgue measure is not weakly mixing w.r.t. g.