Imperial College London

MATH97060 MATH97170

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebra 4

Date: Monday, 10 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In all questions you can use any results from lectures if you state them clearly.

In this paper R is an associative ring with 1.

1. (a) Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of left R-modules. Let M be a left R-module.

(i) For each of its three terms, determine if the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0$$

is exact at this term for all M. In each case give a proof or a counter-example.

(ii) What can you add to your answer in part (i) when C is a projective R-module?

(iii) What can you add to your answer in part (i) when A is an injective R-module?

(iv) What can you add to your answer in part (i) when M is a projective R-module?

(v) What can you add to your answer in part (i) when $R = \mathbb{Z}/5$ and B is finite?

(12 marks)

(b) Determine, with proof, which of the following abelian groups are zero and which are not:

$$\text{(i) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}), \quad \text{(ii) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}), \quad \text{(iii) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}, \mathbb{Q}), \quad \text{(iv) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}/\mathbb{Q}, \mathbb{Q}).$$

(8 marks)

(Total: 20 marks)

2. (a) Let A be a submodule of a flat R-module B. Let M be a right R-module.

(i) What can you say about the kernel of the natural map $M \otimes_R A \to M \otimes_R B$? Can this kernel be non-zero?

(ii) What can you add to your answer in part (i) when R is a principal ideal domain and M is a torsion-free R-module? (6 marks)

(b) Let I be an ideal of R such that $I \neq 0$ and $I \neq R$.

(i) Give an example of R and I such that the R-module I is free. (Justify your answer.)

(ii) Give an example of R and I such that the R-module I is both projective and injective, but not free. (Justify your answer.) (6 marks)

(c) Let $\ldots \to C_n \to C_{n-1} \to \ldots$ be an exact chain complex of free abelian groups. Let G be an abelian group. Prove that the chain complex $\ldots \to C_n \otimes_{\mathbb{Z}} G \to C_{n-1} \otimes_{\mathbb{Z}} G \to \ldots$ is exact. (8 marks)

(Total: 20 marks)

3. (a) Briefly explain how extensions of R-modules are classified in terms of Ext^1_R. (You are not asked to prove your statements.) (6 marks)

(b) Let $\mathbb{F}_{4^n}$ be the field with 4^n elements, $n \geq 1$. Write $\mathbb{F}^{\times}_{4^n}$ for the multiplicative group of $\mathbb{F}_{4^n}$. It is known that $\mathbb{F}_4$ is a subfield of $\mathbb{F}_{4^n}$, so we have a short exact sequence of abelian groups

$$1 \longrightarrow \mathbb{F}_4^{\times} \longrightarrow \mathbb{F}_{4^n}^{\times} \longrightarrow \mathbb{F}_{4^n}^{\times}/\mathbb{F}_4^{\times} \longrightarrow 1.$$

Determine, with proof, the values of n for which this sequence is split. (7 marks)

(c) Consider the extension of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/6 \longrightarrow 0, \qquad (*)$$

where the second map is multiplication by 6. Compute the middle term of the extension of $\mathbb{Z}/6$ by $\mathbb{Z}$ in the category of abelian groups given by the Baer sum of extension (*) with itself. (You are not asked to compute the maps in this extension.) (7 marks)

(Total: 20 marks)

4. Let G be a finite group and let M be a G-module.

(a) Prove that if M is finite, then $\mathrm{H}^n(G, M)$ is finite for all $n \geq 0$. (2 marks)

(b) Assume that G is a finite cyclic group and M is a trivial finite G-module. Prove that the groups $\mathrm{H}^n(G, M)$ are isomorphic for all $n \geq 1$. (6 marks)

(c) Let $G = C_m$ be the cyclic group of order $m \geq 2$. Let M be the abelian group $\mathbb{Z}^m$ on which G acts by cyclic shifts of coordinates, that is, a generator of G sends $(a_1, a_2, \ldots, a_m)$ to $(a_m, a_1, \ldots, a_{m-1})$.

(i) Compute the groups $\mathrm{H}^n(G, M)$ for all $n \geq 0$. (6 marks)

(ii) Let M_0 be the submodule of M consisting of the elements $(a_1, a_2, \ldots, a_m) \in M$ such that $\sum_{i=1}^m a_i = 0$. Compute the groups $\mathrm{H}^n(G, M_0)$ for all $n \geq 0$. (6 marks)

(Total: 20 marks)

5. (a) Suppose that G is a finite group with a normal abelian subgroup A such that the orders of A and G/A are coprime. What can you say about G? (Justify your answer.) (6 marks)

(b) Give an example of a short exact sequence of R-modules, which is not split as a sequence of R-modules, but which is split as a sequence of abelian groups. (Justify your answer.) (7 marks)

(c) The commutator of a group G is the smallest subgroup of G that contains the elements $ghg^{-1}h^{-1}$ for all $g, h \in G$. Suppose that a finite group G is equal to its commutator. Compute $\mathrm{H}^2(G, \mathbb{Z})$, where $\mathbb{Z}$ is a trivial G-module. (7 marks)

(Total: 20 marks)

In all questions you can use any results from lectures if you state them clearly.

In this paper R is an associative ring with 1.

1. (a) *Let*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of left R-modules. Let M be a left R-module.

(i) *For each of its three terms, determine if the sequence*

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C) \longrightarrow 0$$

is exact at this term for all M. In each case give a proof or a counter-example.

The sequence is always exact at $\mathrm{Hom}_R(M, A)$. Indeed, a map of R-modules $f\colon M \to A$ is zero if and only if $\alpha \circ f = 0$, because α is injecitve. (Seen, 1 mark, A)
The sequence is always exact at $\mathrm{Hom}_R(M, B)$. Indeed, for a map of R-modules $f\colon M \to B$ we have $\beta \circ f = 0$ if and only if f factors through $\alpha(A) \subset B$, in which case there is a unique map of R-modules $g\colon M \to A$ such that $f = \alpha \circ g$. (Seen, 1 mark, A)
The sequence is not always exact at $\mathrm{Hom}_R(M, C)$. Indeed, for $R = \mathbb{Z}$, $A = C = \mathbb{Z}/2$, $B = \mathbb{Z}/4$, $M = \mathbb{Z}/2$, the map $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$ is not surjective. (Seen, 1 mark, A)

(ii) *What can you add to your answer in part (i) when C is a projective R-module?*

In this case the original sequence is split, by lectures, so the sequence of Hom-groups is split exact, due to the canonical isomorphism $\mathrm{Hom}_R(M, A \oplus C) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, C)$. (Seen similar, 2 marks, A)

(iii) *What can you add to your answer in part (i) when A is an injective R-module?*

The same answer as in part (ii). (Seen similar, 2 marks, A)

(iv) *What can you add to your answer in part (i) when M is a projective R-module?*

From the definition of a projective R-module we deduce the surjectivity of $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$, so the sequence of Hom-groups is exact. (Seen, 2 marks, A)

(v) *What can you add to your answer in part (i) when $R = \mathbb{Z}/5$ and B is finite?*

5 is prime, hence $\mathbb{Z}/5$ is a field. Thus the original sequence of R-modules is a short exact sequence of finite-dimensional vector spaces over $\mathbb{Z}/5$. A finite-dimensional vector space over a field has a basis and so is a free module. Thus our short exact sequence is split, and hence the induced sequence of Hom-groups is split exact. (Seen similar, 3 marks, A)

(b) *Determine, with proof, which of the following abelian groups are zero and which are not:*

$$\text{(i) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}), \quad \text{(ii) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}), \quad \text{(iii) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}, \mathbb{Q}), \quad \text{(iv) } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}/\mathbb{Q}, \mathbb{Q}).$$

$\mathbb{Q}$ is divisible, but $\mathbb{Z}$ has no non-zero divisible elements, hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. (Seen similar, 1 mark, A)

$\mathbb{Q}/\mathbb{Z}$ is a torsion group, but $\mathbb{Q}$ has no non-zero torsion elements, hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$. (Seen similar, 1 mark, A)

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}, \mathbb{Q}) \neq 0$. Indeed, since $\mathbb{Q}$ is injective as an abelian group, the tautological embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ can be extended to a homomorphism $\mathbb{C} \to \mathbb{Q}$ which is obviously non-zero. (Unseen, 3 marks, B)

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}/\mathbb{Q}, \mathbb{Q}) \neq 0$. Indeed, let $x \in \mathbb{R}$ be an irrational number. Then the image of x in $\mathbb{R}/\mathbb{Q}$ generates an infinite cyclic group $\mathbb{Z}x$. Let $\mathbb{Z}x \to \mathbb{Q}$ be the homomorphism sending x to 1. By the injectivity of $\mathbb{Q}$ it extends to a homomorphism $\mathbb{R}/\mathbb{Q} \to \mathbb{Q}$ which is obviously non-zero. (Unseen, 3 marks, B)

(Total: 0 marks)

2. (a) *Let A be a submodule of a flat R-module B. Let M be a right R-module.*

   (i) *What can you say about the kernel of the natural map $M \otimes_R A \to M \otimes_R B$? Can this kernel be non-zero?*

   Let $C = B/A$. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to a long exact sequence of Tor-groups. Since B is flat, we have $\mathrm{Tor}_1^R(M, B) = 0$ by lectures. Thus the kernel of our map is canonically isomorphic to $\mathrm{Tor}_1^R(M, C)$. This group can be non-zero, e.g. for $R = \mathbb{Z}$, $M = C = \mathbb{Z}/2$, we have $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$. (Seen similar, 4 marks, A)

   (ii) *What can you add to your answer in part (i) when R is a principal ideal domain and M is a torsion-free R-module?*

   In this case M is flat, by lectures. Hence the map $M \otimes_R A \to M \otimes_R B$ is injective. (Seen, 2 marks, A)

   (b) *Let I be an ideal of R such that $I \neq 0$ and $I \neq R$.*

   (i) *Give an example of R and I such that the R-module I is free. (Justify your answer.)*
   $R = \mathbb{Z}$, $I = 2\mathbb{Z}$ is an example. Multiplication by 2 map $R \to I$ is an isomorphism of abelian groups. (Seen similar, 2 marks, A)

   (ii) *Give an example of R and I such that the R-module I is both projective and injective, but not free. (Justify your answer.)*

   $R = \mathbb{Z}/6$, $I = 2R \simeq \mathbb{Z}/3$ is an example. It is clear that I is not free, as 3 is not a power of 6. The free R-module R is a direct sum $\mathbb{Z}/2 \oplus \mathbb{Z}/3$, so I is projective. Using Baer's criterion we check that R is an injective R-module, hence I, as a direct summand of R, is injective too. (Seen similar, 4 marks, B.)

   (c) *Let $\ldots \to C_n \to C_{n-1} \to \ldots$ be an exact chain complex of free abelian groups. Let G be an abelian group. Prove that the chain complex $\ldots \to C_n \otimes_{\mathbb{Z}} G \to C_{n-1} \otimes_{\mathbb{Z}} G \to \ldots$ is exact.*

   Let B_n be the image of $C_{n+1} \to C_n$. Since the original complex is exact, B_n is the kernel of $C_n \to C_{n-1}$. But B_n is a subgroup of a free abelian group C_n, so it is free, by a theorem from lectures. Thus the exact sequence $0 \to B_n \to C_n \to B_{n-1} \to 0$ is split, so it remains exact after tensoring with G. It follows that the natural map $C_n \otimes_{\mathbb{Z}} G \to C_{n-1} \otimes_{\mathbb{Z}} G$ is the composition of a surjective map $C_n \otimes_{\mathbb{Z}} G \to B_{n-1} \otimes_{\mathbb{Z}} G$ with kernel $B_n \otimes_{\mathbb{Z}} G$ and the injective map $B_{n-1} \otimes_{\mathbb{Z}} G \to C_{n-1} \otimes_{\mathbb{Z}} G$. Thus the kernel of $C_n \otimes_{\mathbb{Z}} G \to C_{n-1} \otimes_{\mathbb{Z}} G$ is equal to the image of $C_{n+1} \otimes_{\mathbb{Z}} G \to C_n \otimes_{\mathbb{Z}} G$ since both groups are equal to $B_n \otimes_{\mathbb{Z}} G$. (Seen similar, 8 marks, C)

(Total: 0 marks)

3. (a) *Briefly explain how extensions of R-modules are classified in terms of Ext^1_R. (You are not asked to prove your statements.)*

An extension of R-modules C by A is a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0.$$

The long exact sequence of $\mathrm{Ext}^i_R(C, -)$ gives rise to a canonical map $\mathrm{Hom}_R(C, C) \to \mathrm{Ext}^1_R(C, A)$. One defines the class of an extension as the image of the identity map on C under this map. This gives a bijection between equivalence classes of extensions of R-modules C by A and the set $\mathrm{Ext}^1_R(C, A)$. Under this bijection, the equivalence class of split extensions corresponds to $0 \in \mathrm{Ext}^1_R(C, A)$, and the Baer sum of extensions corresponds to addition in $\mathrm{Ext}^1_R(C, A)$. (Seen, 6 marks, A)

(b) *Let $\mathbb{F}_{4^n}$ be the field with 4^n elements, $n \geq 1$. Write $\mathbb{F}^\times_{4^n}$ for the multiplicative group of $\mathbb{F}_{4^n}$. It is known that $\mathbb{F}_4$ is a subfield of $\mathbb{F}_{4^n}$, so we have a short exact sequence of abelian groups*

$$1 \longrightarrow \mathbb{F}^\times_4 \longrightarrow \mathbb{F}^\times_{4^n} \longrightarrow \mathbb{F}^\times_{4^n}/\mathbb{F}^\times_4 \longrightarrow 1.$$

Determine, with proof, the values of n for which this sequence is split.

Using that the multiplicative group of a finite field is cyclic, we rewrite this extension as

$$0 \longrightarrow \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3m \longrightarrow \mathbb{Z}/m \longrightarrow 0,$$

where $m = 4^{n-1} + \ldots + 4 + 1$. This extension is split if and only if $\mathbb{Z}/3m \cong \mathbb{Z}/3 \times \mathbb{Z}/m$ which happens precisely when 3 does not divide m. Since $4 \equiv 1 \bmod 3$, we have $m \equiv n \bmod 3$, so the extension is split if and only if n is not divisible by 3. (Unseen, 7 marks, B)

(c) *Consider the extension of abelian groups*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/6 \longrightarrow 0, \qquad (*)$$

where the second map is multiplication by 6. Compute the middle term of the extension of $\mathbb{Z}/6$ by $\mathbb{Z}$ in the category of abelian groups given by the Baer sum of extension (*) with itself. (You are not asked to compute the maps in this extension.)

The kernel of the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/6$ sending $(1, 0)$ to 1 and $(0, 1)$ to -1 is $\mathbb{Z}(1, 1) \oplus \mathbb{Z}(6, 0) = \mathbb{Z}(1, 1) \oplus \mathbb{Z}(3, -3)$. The quotient of this group by the image of $\mathbb{Z}$ under the map sending 1 to $(6, -6)$ is isomorphic to $\mathbb{Z}(1, 1) \oplus \mathbb{Z}/2(3, -3)$. Thus the desired extension is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/6 \longrightarrow 0.$$

(Unseen, 7 marks, D)

(Total: 0 marks)

4. *Let G be a finite group and let M be a G-module.*

(a) *Prove that if M is finite, then $\mathrm{H}^n(G, M)$ is finite for all $n \geq 0$.*

By lectures, $\mathrm{H}^n(G, M)$ is the n-th cohomology group of the cochain complex

$$0 \longrightarrow M \xrightarrow{d} \mathrm{Fun}(G, M) \xrightarrow{d} \ldots \xrightarrow{d} \mathrm{Fun}(G^n, M) \xrightarrow{d} \mathrm{Fun}(G^{n+1}, M) \xrightarrow{d} \ldots$$

where $\mathrm{Fun}(G^n, M)$ is the set of functions from G^n to M. In our assumptions, each term of this complex is finite, hence the cohomology groups are also finite. (Seen similar, 2 marks, A)

(b) *Assume that G is a finite cyclic group and M is a trivial finite G-module. Prove that the groups $\mathrm{H}^n(G, M)$ are isomorphic for all $n \geq 1$.*

Let $a = |G|$. By lectures, for n odd we have $\mathrm{H}^n(G, M) = \mathrm{Hom}(\mathbb{Z}/a, M) = M[a]$, whereas for even $n \geq 2$ we have $\mathrm{H}^n(G, M) = M/aM$. It is enough to prove that for any finite abelian group M the groups $M[a]$ and M/aM are isomorphic. By the structure theorem of finite abelian groups, every such group is a product of cyclic groups, so we can assume that $M \cong \mathbb{Z}/b$. But $(\mathbb{Z}/b)[a]$ and $(\mathbb{Z}/b)/(a\mathbb{Z}/b)$ are isomorphic, since both groups are isomorphic to $\mathbb{Z}/c$, where $c = \gcd(a, b)$. (Unseen, 6 marks, B)

(c) *Let $G = C_m$ be the cyclic group of order $m \geq 2$. Let M be the abelian group $\mathbb{Z}^m$ on which G acts by cyclic shifts of coordinates, that is, a generator of G sends $(a_1, a_2, \ldots, a_m)$ to $(a_m, a_1, \ldots, a_{m-1})$.*

(i) *Compute the groups $\mathrm{H}^n(G, M)$ for all $n \geq 0$.*

$\mathrm{H}^0(G, M) = M^G \cong \mathbb{Z}$ generated by $(1, \ldots, 1)$. Next, the group of 1-cocycles is $M_0 \subset M$ consisting of the elements $(a_1, a_2, \ldots, a_m) \in M$ such that $\sum_{i=1}^m a_i = 0$. The group of 1-coboundaries is generated by all cyclic shifts of $(1, -1, 0, \ldots, 0)$. The two groups are equal, so $\mathrm{H}^n(G, M) = 0$ for any odd n. If $n \geq 2$ is even, then $\mathrm{H}^n(G, M)$ is the quotient of M^G by NM, where $N \in \mathbb{Z}[G]$ is the norm element. Since $N \cdot (1, 0, \ldots, 0) = (1, \ldots, 1)$, we have $\mathrm{H}^n(G, M) = 0$ for all $n \geq 1$. (Unseen, 6 marks, B)

(ii) *Let M_0 be the submodule of M consisting of the elements $(a_1, a_2, \ldots, a_m) \in M$ such that $\sum_{i=1}^m a_i = 0$. Compute the groups $\mathrm{H}^n(G, M_0)$ for all $n \geq 0$.*

$\mathrm{H}^0(G, M_0) = M_0^G = 0$. For even $n \geq 2$ the group $\mathrm{H}^n(G, M_0)$ is a quotient of M_0^G, so this group is zero. We have an exact sequence of G-modules

$$0 \longrightarrow M_0 \longrightarrow M \longrightarrow \mathbb{Z} \longrightarrow 0$$

The associated long exact sequence of cohomology groups gives an exact sequence

$$0 \longrightarrow \mathbb{Z}(1, \ldots, 1) \longrightarrow \mathbb{Z} \longrightarrow \mathrm{H}^1(G, M_0) \longrightarrow 0$$

where the second map is given by the sum of coordinates. Hence $\mathrm{H}^n(G, M_0) \cong \mathbb{Z}/m$ for odd $n \geq 1$. (Unseen, 6 marks, C)

(Total: 0 marks)

5. (a) *Suppose that G is a finite group with a normal abelian subgroup A such that the orders of A and G/A are coprime. What can you say about G? (Justify your answer.)*

Let $B = G/A$. The action of G on A by conjugations makes A a B-module. By lectures, equivalence classes of group extensions of B by A with a given B-module structure on A are in a natural bijection with the set $\mathrm{H}^2(B, A)$. Under this bijection, the semi-direct product $A \rtimes B$ corresponds to $0 \in \mathrm{H}^2(B, A)$. By lectures, since the orders of A and B are coprime we have $\mathrm{H}^2(B, A) = 0$, hence $G \cong A \rtimes B$. (Seen, 6 marks, A)

(b) *Give an example of a short exact sequence of R-modules, which is not split as a sequence of R-modules, but which is split as a sequence of abelian groups. (Justify your answer.)*

Consider the sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\alpha} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\beta} \mathbb{Z}/2 \longrightarrow 0,$$

where $\alpha(1) = (1, 1)$ and $\beta\big((1, 0)\big) = \beta\big((0, 1)\big) = 1$. This is a split exact sequence of abelian groups (a section of β is the homomorphism sending $1 \in \mathbb{Z}/2$ to $(1, 0) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$). Let $G = \mathbb{Z}/2$ act on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ by permutation of coordinates. The maps α and β respect the action of G, where $\mathbb{Z}/2$ is a trivial G-module. The only non-zero map $\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ respecting the action of G sends 1 to $(1, 1)$, but this is not a section of β, so our sequence is not split as an exact sequence of $\mathbb{Z}[G]$-modules. (Unseen, 7 marks, D)

(c) *The commutator of a group G is the smallest subgroup of G that contains the elements $ghg^{-1}h^{-1}$ for all $g, h \in G$. Suppose that a finite group G is equal to its commutator. Compute $\mathrm{H}^2(G, \mathbb{Z})$, where $\mathbb{Z}$ is a trivial G-module.*

There is a short exact sequence of trivial G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

By lectures, we have $\mathrm{H}^n(G, \mathbb{Q}) = 0$ for $n \geq 1$. The long exact sequence of cohomology groups thus gives an isomorphism $\mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Any homomorphism $f \colon G \to \mathbb{Q}/\mathbb{Z}$ is trivial on $ghg^{-1}h^{-1}$ for all $g, h \in G$. Hence f is trivial on the commutator subgroup of G. By assumption, this implies that $f = 0$, so that $\mathrm{H}^2(G, \mathbb{Z}) = 0$. (Unseen, 7 marks, D)

(Total: 0 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97060_MATH97170	1	This question was correctly answered by almost all student.
MATH97060_MATH97170	2	Parts (a) and (b) were correctly answered by most students. Only a minority attempted to answer part (c).
MATH97060_MATH97170	3	Part (a) is essentially bookwork. Part (b) required some knowledge of finite abelian groups, but most of student got it wrong. Part (c) is a short but tricky calculation, only one correct answer here.
MATH97060_MATH97170	4	This question was OK, it was reasonably well answered.
MATH97060_MATH97170	5	Not many candidates attempted this question. I think this exam was too long. With hindsight I think that there should have been fewer questions.