## 3.3 Subsequences

**Definition.** A *subsequence* of $(a_n)$ is a new sequence $b_i = a_{n(i)}$ $\forall i \in \mathbb{N}_{>0}$, where $n(1) < n(2) < \cdots < n(i) < \ldots$ $\forall i$.

Formally $n(\,\cdot\,)$ is a function $\mathbb{N}_{>0} \to \mathbb{N}_{>0}$ sending $i \mapsto n(i)$ which is strictly monotonically increasing. "Just go down the sequence faster, missing some terms out".

> **Exercise 3.32.** Prove this implies $n(i) \geq i$ by induction.

> **Example 3.33.** $a_n = (-1)^n$ has subsequences:
>
> - $b_n = a_{2n}$, so $b_n = 1$ $\forall n \implies b_n \to 1$.
> - $c_n = a_{2n+1}$, so $c_n = -1$ $\forall n \implies c_n \to -1$.
> - $d_n = a_{3n}$, so $d_n = (-1)^n$ $(= a_n)$ doesn't converge.
> - $e_n = a_{n+17}$, so $e_n = (-1)^{n+1}$ $(= -a_n)$ doesn't converge.

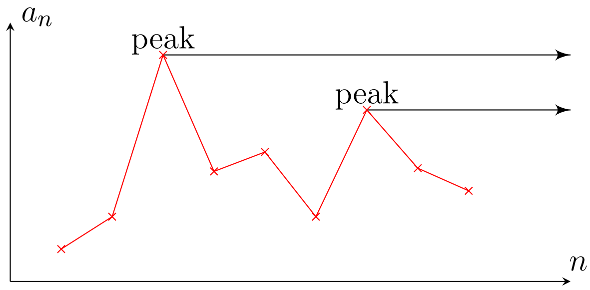
Next we work up to the following technical-sounding but vitally important:

**Theorem 3.34: Bolzano-Weierstrass**

If $(a_n)$ is a *bounded* sequence of real numbers then it has a *convergent subsequence.*

*Remark* 3.35. Of course it will have *many* convergent subsequences, and they may converge to different limits; think of $a_n = (-1)^n$ for instance.

*Cheap proof.* Use "peak points" of $(a_n)$:



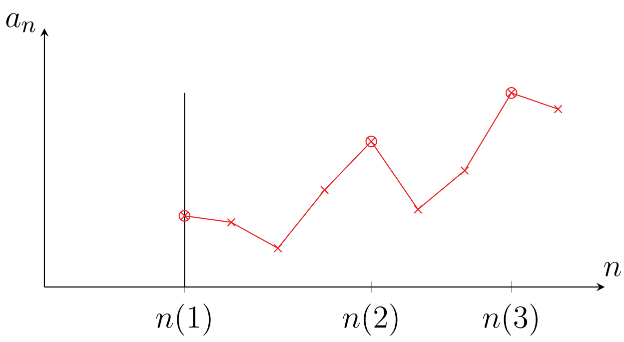

We say that $j$ is a *peak point* if and only if $a_k < a_j$ $\forall k > j$. Either

1. $(a_n)$ has a finite number of peak points, or

2. $(a_n)$ has an infinite number of peak points.

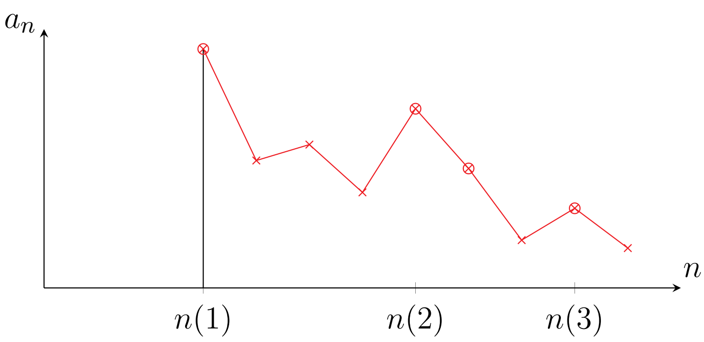
**Case 1:** We go beyond the (finitely many) peak points: pick $n(1) \geq \max\{j_1, \dots, j_k\}$ where $\{j_1, \dots, j_k\}$ are the peak points.

$a_{n(1)}$ is not a peak point $\implies \exists\, n(2) > n(1)$ such that $a_{n(2)} \geq a_{n(1)}$.
$a_{n(2)}$ not a peak point $\implies \exists\, n(3) > n(2)$ such that $a_{n(3)} \geq a_{n(2)}$.
Recursively no peak points beyond $n(1) \implies$ we get $n(i) > n(i-1) > \cdots > n(1)$ such that $a_{n(i)} \geq a_{n(i-1)}\ \forall i$.




So $a_{n(i)}$ is a monotonically increasing subsequence of $a_n$.
$(a_n)_{n\geq 1}$ bounded $\implies (a_{n(i)})_{i\geq 1}$ is bounded $\implies a_{n(i)} \uparrow \sup\{a_{n(i)} : i \in \mathbb{N}_{>0}\}$ by Theorem 3.21.

**Case 2:** There are infinitely many peak points, so we may call them $n(1), n(2), \dots$ where $n(1) < n(2) < \dots$. Then we choose our sequence to be $a_{n(i)}$.




Now $n(i+1) > n(i)$ and $a_{n(i)}$ is a peak point $\implies a_{n(i+1)} \leq a_{n(i)}$. Thus the subsequence $(a_{n(i)})_{i\geq 1}$ is monotonically decreasing and bounded $\implies$ convergent (to $\inf\{a_{n(i)} : i \in \mathbb{N}_{>0}\}$). $\square$

> **Exercise 3.36.** Give an example of an unbounded sequence with a convergent subsequence.

> **Exercise 3.37.** Given an example, with proof, of a sequence for which every subsequence is divergent.

> **Exercise 3.38.** Give an example of an unbounded sequence that has at least three convergent subsequences that converge to three different limits.

**Proposition 3.39.** *If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.*

*Proof.* We are told

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}_{>0} \text{ such that } \forall n \geq N,\ |a_n - a| < \epsilon. \qquad (*)$$

But $\forall i \geq N$, then $n(i) \geq i \geq N$, so by $(*)$, $|a_{n(i)} - a| < \epsilon$. $\square$

This gives us another proof that $(-1)^n$ is not convergent, because if $(-1)^n \to a$, then by Proposition 3.39, $(-1)^{2n} \to a$ and $(-1)^{2n+1} \to a \implies a = 1$ and $a = -1$ ※

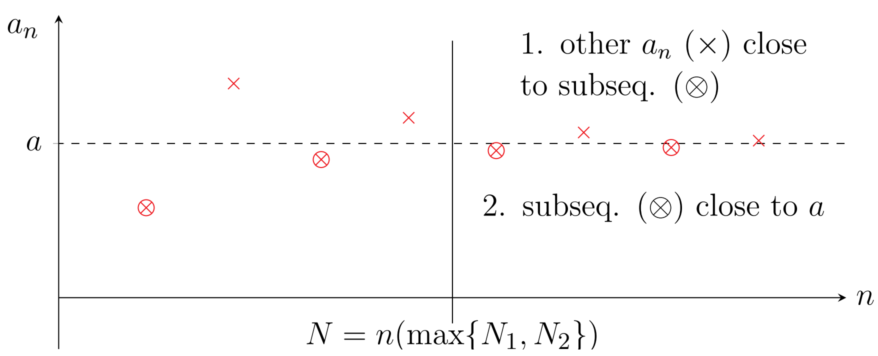
## Bolzano-Weierstrass $\implies$ the Cauchy theorem

We also get another proof of "Cauchy $\implies$ convergence" using Bolzano-Weierstrass.

*Proof #2 of Cauchy $\implies$ Convergence.* We know from Lemma 3.26 that $a_n$ is bounded (by $\max\big\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\big\}$ remember). So by Bolzano-Weierstrass, $\exists$ a convergent subsequence $(a_{n(i)})_{i \geq 1}$ such that $a_{n(i)} \to a$ as $i \to \infty$ for some $a \in \mathbb{R}$. So fix $\epsilon > 0$. We have:

(1) $\exists N_1$ such that $\forall n, m \geq N_1$, $|a_n - a_m| < \epsilon$ (Cauchy)

(2) $\exists N_2$ such that $\forall i \geq N_2$, $|a_{n(i)} - a| < \epsilon$ (convergent subsequence)



Set $N = n(\max\{N_1, N_2\}) \geq \max\{N_1, N_2\} \geq N_1$. Then $\forall n \geq N$ we have

$$\begin{aligned} |a_n - a| &= |(a_n - a_N) + (a_N - a)| \\ &\leq |a_n - a_N| + |a_N - a| \\ &< \epsilon + \epsilon \ = \ 2\epsilon, \end{aligned}$$

the first $< \epsilon$ being by the Cauchy property (1) and the second $< \epsilon$ being from the convergence of the subsequence property (2) (since $a_N$ is in the subsequence). $\square$

Above, we used the following lemma.

**Lemma 3.40.** *Fix $c > 0$. Then $a_n \to a$ if and only if*

$$\boxed{\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < c\epsilon \qquad (*)}$$

*Proof.* $\implies$. Fix $\epsilon > 0$ and let $\epsilon' := c\epsilon$. Then by the definition of convergence applied to $\epsilon' > 0$ we find

$$\exists N \in \mathbb{N} : \ n \geq N \implies |a_n - a| < \epsilon',$$

which is $(*)$.

$\impliedby$. Fix $\epsilon > 0$. Set $\epsilon' = \epsilon/c > 0$. Then $(*)$ applied to $\epsilon' > 0$ implies

$$\exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < c\epsilon' = \epsilon. \qquad \square$$

*Warning.* Do not let $c$ depend on $\epsilon$ (nor $N$! or $n$).
E.g. if we let $c = \epsilon^{-1}$ then $(*)$ becomes $\forall \epsilon > 0$, $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N$, $|a_n - a| < 1$. This is not a good definition of convergence; for instance it would say that the sequence $a_n = 1 \ \forall n$ converges to $\frac{3}{2}$!

## Bolzano-Weierstrass $\impliedby$ the Cauchy theorem

We can also go the other way round: the Cauchy theorem $\implies$ Bolzano-Weierstrass.

*Proof 2 of Bolzano-Weierstrass.* Take a bounded sequence $(a_n)$. We want to find a Cauchy subsequence, which will therefore be convergent.
Since $a_n \in [-R, R] \ \forall n$, repeatedly subdivide to make this interval smaller. Then either

1. $a_n \in [-R, 0]$ for infinitely many $n$ or
2. $a_n \in [0, R]$ for infinitely many $n$, (or both).

Pick one of these intervals which contain $a_n$ for infinitely many $n$, call it $[A_1, B_1]$ of length $R$.

Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ with contain $a_n$ for infinitely many $a_n$s in it, with length $R/2$. Etc.

Recursively we get a sequence of intervals $[A_n, B_n]$ of length $2^{1-n}R$ which are nested – i.e. $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$ – with each containing $a_n$ for infinitely many $n$.

Now we use a *diagonal argument*.
Choose $n(1)$ so that $a_{n(1)} \in [A_1, B_1]$.
Choose $n(2) > n(1)$ so that $a_{n(2)} \in [A_2, B_2]$. (Recall there are infinitely many $a_n$ in each $[A_k, B_k]$, so we can do this.)
Recursively choose $n(k+1) > n(k)$ so that $a_{n(k+1)} \in [A_{k+1}, B_{k+1}]$.

**Claim:** the subsequence $a_{n(i)}$ is convergent.

Fix $\epsilon > 0$. Take $N > \frac{2R}{\epsilon}$, so that $2^{1-N}R < 2N^{-1}R < \epsilon$. Then $\forall i, j \geq N$ we have $n(i) \geq i \geq N$ and $n(j) \geq j \geq N$, so $a_{n(i)}$, $a_{n(j)} \in [A_N, B_N]$ and

$$|a_{n(i)} - a_{n(j)}| \;<\; 2^{1-N}R \;<\; \epsilon.$$

Therefore $(a_{n(i)})$ is Cauchy and so convergent. $\square$

**Definition.** We say $a_n \to +\infty$ if and only if

$$\forall R > 0\ \exists N \in \mathbb{N} \text{ such that } a_n > R\ \forall n \geq N.$$

*Remark* 3.41. Recall this is not the same as (but it does imply) $a_n$ being divergent!

**Exercise 3.42.** Suppose $a_n > 0\ \forall n$. Show $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$.

# 4 Series

*Maths is not a spectator sport. How well you do comes down solely to the time you spend* **doing** *maths.*

- *Richard Thomas, annually*

**Definition.** An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots\,,$$

where $(a_i)_{i\geq 1}$ is a sequence.

For now, it is **not** a real number. It is just a formal expression. We could write $\sum_{n=1}^{\infty} n$, for instance, without worrying about convergence (just as we write $a_n = n$ without worrying about convergence).

### Partial sums

Given a sequence $a_n$ we get a series (formal expression!) $\sum_{n=1}^{\infty} a_n$ and another sequence of **partial sums**

$$s_n := \sum_{i=1}^{n} a_i. \qquad (*)$$

Recall in Exercise 3.2 you proved that $a_n$ and $s_n$ determine each other – they are equivalent information. In other words, the sequence $(a_n)$ determines the sequence $(s_n)$ by $(*)$, and conversely we can recover $(a_n)$ from the $(s_n)$ by

$$a_n = s_n - s_{n-1}.$$

## 4.1 Convergence of Series

**Definition.** We say that the series $\sum a_n$ "converges to $A \in \mathbb{R}$" if and only if the sequence of partial sums converges to $A$:

$$\sum_{n=1}^{\infty} a_n = A \iff s_n \longrightarrow A.$$

We often write $A$ as $\sum_{n=1}^{\infty} a_n$. In other words, if $\sum_{n=1}^{\infty} a_n$ converges (to $A$) then we use the same notation to denote the real number $A$.

We can obviously let the sum be from $n = 0$, or over $n$ even, or ...

**Example 4.1.** Consider $a_n = x^n,\ n \geq 0$, so that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

The partial sums are

$$s_n \;=\; \sum_{i=0}^{n} x^i \;=\; 1 + x + \cdots + x^n.$$

Therefore

$$xs_n \;=\; x + \cdots + x^n + x^{n+1},$$

so

$$s_n - xs_n \;=\; 1 - x^{n+1},$$

giving

$$s_n \;=\; \begin{cases} \dfrac{1-x^{n+1}}{1-x} & x \neq 1, \\ n+1 & x = 1. \end{cases}$$

So for $|x| < 1$, we see that

$$s_n \;=\; \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \;\longrightarrow\; \frac{1}{1-x} \quad \text{as } n \to \infty.$$

(Recall from the question sheet that $r^n \to 0$ if $|r| < 1$.)

So $(s_n)$ is convergent and we can finally say $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$.

For $|x| \geq 1$, $a_n = x^n \not\to 0$ as $n \to \infty$. So $\sum a_n = \sum x^n$ is *not* a real number (does not converge) by the next result.

**Theorem 4.2**

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$.

*Proof.* $s_n - s_{n-1} = a_n$. If $s_n \to A$ then $s_{n-1} \to A$ (exercise!). So by the algebra of limits $a_n$ is convergent and $a_n \to A - A = 0$. □

*Proof from first principles.* Fix $\epsilon > 0$. Since $s_n \to A$,

$$\exists N \in \mathbb{N}_{>0} \text{ such that } \forall n \geq N,\ |s_n - A| < \epsilon$$

so that

$$|a_n| \;=\; |s_n - s_{n-1}| \;=\; |(s_n - A) - (s_{n-1} - A)| \;\leq\; |s_n - A| + |s_{n-1} - A|$$

which is $< \epsilon + \epsilon$ for $n-1 \geq N$. So $\forall n \geq N+1$, $|a_n| < 2\epsilon$. $\square$

*Remark* 4.3. Converse is *not* true. E.g. $a_n = \frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ is *not* convergent.

**Example 4.4.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

*Proof.* Uses a slight trick. Arrange the partial sum as follows:

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \dots &= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \dots + \frac{1}{7}\right) \\ &\quad + \left(\frac{1}{8} + \dots + \frac{1}{15}\right) + \left(\frac{1}{16} + \dots + \frac{1}{31}\right) + \dots \end{aligned}$$

We can bound the $k$th bracketed term from below:

$$\left(\frac{1}{2^k} + \dots + \frac{1}{(2^{k+1}-1)}\right) > \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}} = \frac{2^k}{2^{k+1}} = \frac{1}{2}.$$

In particular then

$$s_{2^{k+1}-1} > 1 + \underbrace{\frac{1}{2} + \dots + \frac{1}{2}}_{k \text{ terms}} = 1 + \frac{k}{2}$$

is arbitrarily large. But if $s_n$ converged, it would be bounded: $|s_n| \leq C \ \forall n$. So we get the contradiction (to the Archimedean property) $1 + \frac{k}{2} \leq C \ \forall k \in \mathbb{N}$. $\square$

**Example 4.5.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

*Proof.* (Using a trick; we will give another proof soon.) First show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, using $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \dots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \longrightarrow 1 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent to 1.

So now compare the partial sums $\sigma_n$ of $\sum \frac{1}{n^2}$ to those of $\sum \frac{1}{n(n+1)} = 1$.

$$\begin{aligned} \sigma_n &= \sum_{i=1}^{n} \frac{1}{i^2} = 1 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2} \\ &\leq 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} \\ &= 1 + s_{n-1}. \end{aligned}$$

$s_{n-1} \uparrow 1$ because $\frac{1}{n(n+1)} > 0$. So $s_{n-1} < 1 \ \forall n \implies \sigma_n < 2 \implies$ bounded above monotonic increasing sequence $\implies \sigma_n$ is convergent $\implies \sum \frac{1}{n^2}$ is convergent. $\square$