

Problem Sheet 1

1. If \mathbf{A} is a constant vector field, calculate the gradients of the following scalar fields:

- (i) $\mathbf{A} \cdot \mathbf{r}$,
- (ii) r^n ,
- (iii) $\mathbf{r} \cdot \nabla(x + y + z)$,

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.

2. If $\phi = x^2y + z^2x$ and P is the point $(1, 1, 2)$, find the directional derivative of ϕ at P in the direction $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

3. If $\phi = \phi(\mathbf{r})$ with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $x = x(t), y = y(t), z = z(t)$, show that

$$\frac{d\phi}{dt} = \mathbf{r}'(t) \cdot \nabla\phi.$$

Verify this relation for ϕ as given in Q2 and $(x, y, z) = (\cos t, \sin t, t)$. Further, if $\phi = \phi(\mathbf{g}(t))$ with $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$, show that

$$\frac{d\phi}{dt} = \mathbf{g}'(t) \cdot \nabla_{\mathbf{g}}\phi.$$

where $\nabla_{\mathbf{g}} \equiv \mathbf{i}\frac{\partial}{\partial g_1} + \mathbf{j}\frac{\partial}{\partial g_2} + \mathbf{k}\frac{\partial}{\partial g_3}$.

4. Find the equations of the tangent planes to the following surfaces at the points indicated

- (i) $x^2 + 2y^2 - z^2 - 8 = 0$ at $(1, 2, 1)$,
- (ii) $z = 3x^2y \sin(\pi x/2)$ at $x = 1, y = 1$.

5. If $\phi = xr^2$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(r)$ is an arbitrary function of $r = |\mathbf{r}|$, evaluate:

- (i) $\nabla\phi$,
- (ii) $\operatorname{div}(\phi\mathbf{r})$,
- (iii) $\operatorname{curl}(f(r)\mathbf{r})$.

6. If $\mathbf{u} = z^2\mathbf{i}$, $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\psi = |\mathbf{v}|^2$, verify the identities

- (i) $\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$,
- (ii) $\operatorname{div}(\psi\mathbf{u}) = (\nabla\psi) \cdot \mathbf{u} + \psi\operatorname{div} \mathbf{u}$.

7. Use tensor notation and the relation $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ to establish the following vector identities:

- (i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$,
- (ii) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$,
- (iii) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d})\mathbf{c} - ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})\mathbf{d}$,

8. Simplify the following expressions:

- (i) $\delta_{ij}\partial x_i/\partial x_j,$
- (ii) $\delta_{ij}\delta_{ik}x_jx_k,$
- (iii) $\delta_{ij}\partial^2\phi/\partial x_i\partial x_j,$
- (iv) $\delta_{ij}\delta_{jk}\delta_{ki},$
- (v) $\varepsilon_{ijk}\partial/\partial x_i (\partial A_k/\partial x_j).$

9. Use tensor notation to prove the following identities:

- (i) $\text{curl}(\phi \mathbf{A}) = \phi \text{curl} \mathbf{A} + \nabla\phi \times \mathbf{A},$
- (ii) $\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl} \mathbf{A} - \mathbf{A} \cdot \text{curl} \mathbf{B},$
- (iii) $\mathbf{A} \times \text{curl} \mathbf{A} = \frac{1}{2}\nabla(|\mathbf{A}|^2) - (\mathbf{A} \cdot \nabla)\mathbf{A}.$

Sheet 1 Answers

1. (i) $\mathbf{A};$ (ii) $nr^{n-2}\mathbf{r};$ (iii) $\mathbf{i} + \mathbf{j} + \mathbf{k}.$

2. $20/\sqrt{14}.$

3. $d\phi/dt = \cos^3 t - 2\cos t \sin^2 t + 2t \cos t - t^2 \sin t.$

4. (i) $x + 4y - z = 8;$ (ii) $6x + 3y - z = 6.$

5. (i) $(3x^2 + y^2 + z^2)\mathbf{i} + 2xy\mathbf{j} + 2zx\mathbf{k};$ (ii) $6xr^2;$ (iii) zero.

8. (i) 3; (ii) $r^2;$ (iii) $\partial^2\phi/\partial x_1^2 + \partial^2\phi/\partial x_2^2 + \partial^2\phi/\partial x_3^2;$ (iv) 3; (v) zero.