IMPERIAL

**MATH70016**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**Special Relativity and Electromagnetism**

Date: Thursday, May 2, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer All Questions in 1 Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. **Lorentz transforms**

In the following the spatial origins of frames $K$ and $K'$ are chosen so that $(0,0,0,0)$ refers to the same event in both frames. All axes are parallel at all time. The origin of frame $K'$ moves with velocity $V$ relative to frame $K$ along its $x$-axis. Dashed variables refer to measurements in the dashed frame $K'$, undashed variables to measurements in the undashed frame $K$.

To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$. The transformation law from frame $K'$ to frame $K$ for contravariant vectotrs may then we written as

$$x^i = \hat{\mathcal{L}}^i_{\ j} x'^j \quad \text{with} \quad \hat{\mathcal{L}}^i_{\ j} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and that for covariant vectors as

$$x_i = \mathcal{L}_i^{\ j} x'_j \quad \text{with} \quad \mathcal{L}_i^{\ j} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Lorentz transform the four vectors

$$x^i = (ct, q, r, s) \quad \text{and} \quad x_i = (ct, q, r, s)$$

from frame $K$ to frame $K'$. (4 marks)

(b) Lorentz transform the rank-2 tensors

$$A^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_i^{\ j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(4 marks)

(c) In frame $K$ a flash of light departs at time $\tau_0 = 0$ from the spatial origin, arriving at time $\tau$ at detectors at $\mathbf{r}_1 = (\ell, 0, 0)$ and $\mathbf{r}_2 = (0, \ell, 0)$.

(i) Express $\ell$ in terms of $\tau$. (2 marks)

(ii) Write down the four vectors of the three events $K$, namely emission of the flash, detection at $\mathbf{r}_1$ and detection at $\mathbf{r}_2$. (2 marks)

(iii) Transform the three four-vectors to frame $K'$. (2 marks)

(iv) Determine $\mathbf{r}'_1$ and $\mathbf{r}'_2$, the locations of the detectors as seen in frame $K'$. (2 marks)

(v) Determine $\tau'_1$ and $\tau'_2$ the times of detection of the flash as seen in frame $K'$. (2 marks)

(vi) Show that the resulting speeds of the flash, $|\mathbf{r}'_1|/\tau'_1$ and $|\mathbf{r}'_2|/\tau'_2$ respectively, are $c$. (2 marks)

(Total: 20 marks)

2. **Particle decay**

A single particle with rest mass $M$ at rest in frame $K$ decays into two particles with rest masses $m_1$ and $m_2$ respectively, travelling away with velocities $\mathbf{w}_1 = (w_{1x}, w_{1y}, w_{1z})$ and $\mathbf{w}_2 = (w_{2x}, w_{2y}, w_{2z})$ respectively.

To ease notation, use $\gamma_1 = 1/\sqrt{1 - w_1^2/c^2}$ and $\gamma_2 = 1/\sqrt{1 - w_2^2/c^2}$.

(a) (i) State energy and momentum conservation that governs the decay in terms of the energy-momentum four-vectors. (2 marks)

(ii) Show that $M \geq m_1 + m_2$. (2 marks)

(b) We now change the observer frame to $K'$. The spatial origins of frames $K$ and $K'$ are chosen so that $(0,0,0,0)$ refers to the same event in both frames. All axes are parallel at all time. The origin of frame $K'$ moves with velocity $V$ relative to frame $K$ along its $x$-axis. Dashed variables refer to measurements in the dashed frame $K'$, undashed variables to measurements in the undashed frame $K$.

The velocities of the two daughter particles as measured in frame $K'$ are denoted $\mathbf{w}'_1$ and $\mathbf{w}'_2$. To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$, $\gamma'_1 = 1/\sqrt{1 - w_1'^2/c^2}$, $\gamma'_2 = 1/\sqrt{1 - w_2'^2/c^2}$, $\beta = V/c$ and $\mathbf{V} = (V, 0, 0)$.

(i) Derive the velocity addition theorem, using the infinitesimals

$$\begin{aligned} c\,\mathrm{d}t' &= (c\,\mathrm{d}t - \mathrm{d}x\beta)\gamma \\ \mathrm{d}x' &= (\mathrm{d}x - c\,\mathrm{d}t\beta)\gamma \\ \mathrm{d}y' &= \mathrm{d}y \\ \mathrm{d}z' &= \mathrm{d}z \end{aligned}$$

for all three components of a velocity. (2 marks)

(ii) Using the velocity addition theorem, calculate $\mathbf{w}'_1$ and $\mathbf{w}'_2$ of the two daughter particles in frame $K'$.

*Hint: To avoid sign mistakes, verify that* $\mathbf{w}_1 = (V, 0, 0)$ *results in* $\mathbf{w}'_1 = \mathbf{0}$. (2 marks)

(iii) State energy and momentum conservation that governs the decay in terms of the energy-momentum four-vectors based on the velocities $\mathbf{V}$, $\mathbf{w}'_1$ and $\mathbf{w}'_2$. (3 marks)

(iv) Transform from frame $K$ to frame $K'$ the energy-momentum four-vectors as obtained in Part (a.i). (3 marks)

(v) Show that the following two sets of energy-momentum four-vectors in frame $K'$ are identical: 1) Those obtained directly from the transformed velocities in Part (b.iii) and 2) those obtained by transforming the four-vectors in $K$ in Part (b.iv).

*Hint: You can use* $\gamma'_1 = \gamma_1\gamma(1 - w_{1x}V/c^2)$ *and* $\gamma'_2 = \gamma_2\gamma(1 - w_{2x}V/c^2)$ *without proof.* (3 marks)

(vi) Based on the energy-momentum four-vectors in $K'$, show that $M \geq m_1 + m_2$. (3 marks)

(Total: 20 marks)

## 3. Charge in an electric field

A charge $e$ with rest mass $m$ moves within an observer frame initially along the $x$-axis with velocity $\mathbf{v}_0 = v_0\mathbf{e}_x$ within an electric field $\mathbf{E} = E\mathbf{e}_z$ that is parallel to the $z$-axis everywhere, and that is constant in time and uniform in space. The magnetic field vanishes everywhere and at all times, $\mathbf{H} \equiv \mathbf{0}$. In the following, we aim to determine the velocity $\mathbf{v}(t)$ as a function of time $t$ with initial condition $\mathbf{v}(0) = \mathbf{v}_0$.

To ease notation, use $\gamma(t) = 1/\sqrt{1 - v(t)^2/c^2}$ and $\beta(t) = v(t)/c$.

(a) Determine the momentum $\mathbf{p}(t)$ using the Lorentz force $e\mathbf{E} + e\mathbf{v} \times \mathbf{H}/c$. (5 marks)

(b) Express the particle's momentum $\mathbf{p}(t)$ in terms of its (kinetic, total) energy $\mathcal{E}(t) = mc^2\gamma(t)$ and its velocity $\mathbf{v}(t)$. (5 marks)

(c) Making use of the results in parts (a) and (b), find $\mathbf{v}(t)$ and express it in terms of the initial conditions, the rest mass $m$, electric field strength $E$ and time $t$.
*Hint:* $\mathcal{E}^2 = m^2c^4 + p^2c^2$ *with* $p = |\mathbf{p}|$. (5 marks)

(d) Calculate $\mathbf{H}'$ in the initial frame of reference of the particle, $i.e.$ at time $t = 0$.
*Hint: The electromagnetic field tensor has the form*

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}$$

*and draws on*

$$\hat{\mathcal{L}}^i_j = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

*to be transformed from frame* $K$ *to* $K'$*.* (5 marks)

(Total: 20 marks)

4. **Gauge and the wave equation**

(a) Gauge invariance of the four potential implies that the electromagnetic field tensor

$$F^{ij} = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}$$

does not change under shifts of the four potential by a four gradient,

$$A^j \quad \longrightarrow \quad A'^j = A^j + \frac{\partial f}{\partial x_j}$$

Confirm this by explicit calculation. (4 marks)

(b) State monochromatic plane-wave solution of the wave equation in four-vector form,

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x^i} A^j = 0 \ .$$

(4 marks)

(c) Using $k^i = (k, \mathbf{k})$ and $x^i = (ct, \mathbf{r})$ the instantaneous phase of a plane wave may be written as $k^i x_i$. Here, $k = 2\pi/\lambda$ is the wavenumber, with $\lambda$ the wavelength, and the vector $\mathbf{k} = k\mathbf{n}$ encodes the direction of propagation $\mathbf{n}$ of the wave.

(i) For $x^i = (0, 0, 0, 0)$ the projection $k^i x_i$ vanishes. Find the plane of four vectors $y^i$, where $k^i y_i = -2\pi$ at time $t = 0$, for a wave propagating along the $x$-axis, so that $\mathbf{n} = (1, 0, 0)$. (4 marks)

(ii) Transform $k^i$ to $k'^i$ from the original frame of reference $K$ to an arbitrary alternative frame $K'$ moving with speed $V$ along the $x$-axis. (4 marks)

(iii) Transforming $x^i = (0, 0, 0, 0)$ and $y^i$ into the new frame of reference, say $x'^i$ and $y'^i$, show that $k'^i x'_i = 0$ and $k'^i y'_i = 2\pi$. (4 marks)

(Total: 20 marks)

## 5. Potentials by charges and currents

(a) Consider a system of charges $e_a$ at rest at locations $\mathbf{r}_a \in \mathbb{R}^3$ with $a = 1, 2, \ldots, N$. What is their scalar potential $\phi(\mathbf{r})$ at position $\mathbf{r}$ large compared to any of the $\mathbf{r}_a$, to first order in their dipole moment? Do not assume $\sum_{a=1}^{N} e_a = 0$. (5 marks)

(b) Assuming that $\partial_t \mathbf{E} = \mathbf{0}$ and using the gauge $\nabla \cdot \mathbf{A} = 0$ show that the vector potential obeys

$$\triangle \mathbf{A}(\mathbf{r}) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r})$$

with space dependent current $\mathbf{j}(\mathbf{r})$. (5 marks)

(i) Show that $\mathbf{A}(\mathbf{r})$ can be expressed in terms of $\mathbf{j}(\mathbf{r})$ as

$$\mathbf{A}(\mathbf{r}) = \int \mathrm{d}^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{c|\mathbf{r} - \mathbf{r}'|}$$

using the Green function of the Poisson equation. (5 marks)

(ii) Expand the integrand of $\mathbf{A}(\mathbf{r})$ in the integral above assuming $|\mathbf{r}'| \ll |\mathbf{r}|$ to leading order using $\int \mathrm{d}^3 r' \mathbf{j}(\mathbf{r}') = \mathbf{0}$ and calculate the resulting magnetic field. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

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# MATH60016

# Special Relativity and Electromagnetism (Solutions)

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1. **Lorentz transforms**

seen $\Downarrow$

(a) This is a matter of straight-forward Lorentz transforming to $\boxed{x'^i = \big((ct - q\beta)\gamma, (ct\beta - q)\gamma, 0, 0\big)}$ and $\boxed{x'_i = \big((ct + q\beta)\gamma, (ct\beta + q)\gamma, 0, 0\big)}$.

4, A

(b)

sim. seen $\Downarrow$

$$\boxed{A'^{ij} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+\beta^2)\gamma^2 & -2\beta\gamma^2 & 0 & 0 \\ -2\beta\gamma^2 & (1+\beta^2)\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

and

$$\boxed{B'^{j}_{i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

as this is a well-known Lorentz-invariant mixed tensor of rank 2.

4, B

sim. seen $\Downarrow$

(c) (i) $\boxed{\ell = \tau c}$ as the flash travels with the speed of light.

2, A

sim. seen $\Downarrow$

(ii) $\boxed{(0, 0, 0, 0)}$, $\boxed{(\ell, \ell, 0, 0)}$ and $\boxed{(\ell, 0, \ell, 0)}$ by inspection and $\ell = c\tau$.

2, A

sim. seen $\Downarrow$

(iii) $\boxed{(0, 0, 0, 0)}$, $\boxed{(\ell(1-\beta)\gamma, \ell(1-\beta)\gamma, 0, 0)}$ and $\boxed{(\ell\gamma, -\ell\beta\gamma, \ell, 0)}$ by standard transformation.

2, C

sim. seen $\Downarrow$

(iv) $\boxed{\mathbf{r}'_1 = (\ell(1-\beta)\gamma, 0, 0)}$ and $\boxed{\mathbf{r}'_2 = (-\ell\beta\gamma, \ell, 0)}$ reading off from (iii).

2, A

sim. seen $\Downarrow$

(v) $\boxed{\tau'_1 = \ell(1-\beta)\gamma/c}$ and $\boxed{\tau'_2 = \ell\gamma/c}$ reading off from (iii).

2, A

sim. seen $\Downarrow$

(vi) $r'_1 = \ell(1-\beta)\gamma$ so that $\boxed{r'_1/\tau'_1 = c}$ and $r'_2 = \ell\sqrt{1+\beta^2\gamma^2} = \ell\gamma$ using Pythagoras and $1 + \beta^2\gamma^2 = \gamma^2$, so that $\boxed{r'_2/\tau'_2 = c}$.

2, B

2. **Lorentz transforms**

sim. seen ⇓

(a.i) The energy-momentum four-vectors can be written down straight from the description of the process

$$\boxed{(Mc, \mathbf{0}) = (m_1\gamma_1 c, m_1\gamma_1 \mathbf{w}_1) + (m_2\gamma_2 c, m_2\gamma_2 \mathbf{w}_2)}$$

2, A

(a.ii) This follows immediately from energy conservation and $\gamma_{1,2} \geq 1$,

seen ⇓

$$\boxed{M = m_1\gamma_1 + m_2\gamma_2 \geq m_1 + m_2}$$

2, B

(b.i) This can be done by inspecting the infinitesimals,

seen ⇓

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = c\frac{(\,\mathrm{d}x - c\,\mathrm{d}t\beta)\gamma}{(c\,\mathrm{d}t - \,\mathrm{d}x\beta)\gamma} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - v}{1 - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{v}{c^2}}$$

and similar, which finally produces

$$\boxed{\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - v}{1 - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{v}{c^2}}}$$

$$\boxed{\frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}\gamma^{-1}}{1 - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{v}{c^2}}}$$

$$\boxed{\frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}z}{\mathrm{d}t}\gamma^{-1}}{1 - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{v}{c^2}}}$$

2, C

(b.ii) Calculating the velocities is merely a matter of using the equations above

sim. seen ⇓

$$\boxed{w'_{1x} = \frac{w_{1x} - v}{1 - w_{1x}v/c^2}}$$

$$\boxed{w'_{1y} = \frac{w_{1y}/\gamma}{1 - w_{1x}v/c^2}}$$

$$\boxed{w'_{1z} = \frac{w_{1z}/\gamma}{1 - w_{1x}v/c^2}},$$

and similar for $\mathbf{w}_2$. For $w_{1x} = v$ this produces correctly $w'_{1x} = 0$.

2, B

(b.iii) In the new frame of reference, the energy-momentum four-vectors obey

sim. seen ⇓

$$\boxed{(M\gamma c, -M\gamma\mathbf{V}) = (m_1\gamma'_1 c, m_1\gamma'_1\mathbf{w}'_1) + (m_2\gamma'_2 c, m_2\gamma'_2\mathbf{w}'_2)}$$

where the sign in the momentum component of on the left is a matter of inspection.

3, A

(b.iv) The energy-momentum four-vectors of (a.1) transform as follows

unseen ⇓

$$\boxed{(Mc, \mathbf{0}) \;\rightarrow\; (M\gamma c, -M\gamma\mathbf{V})}$$

and

$$\boxed{(m_1\gamma_1 c, m_1\gamma_1\mathbf{w}_1) \;\rightarrow\; \left(m_1\gamma_1(c - w_{1x}\beta)\gamma, m_1\gamma_1(w_{1x} - V)\gamma, m_1\gamma_1 w_{1y}, m_1\gamma_1 w_{1z}\right)},$$

and correspondingly for the second particle.

3, B

(b.v) The energy-momentum four-vector $(M\gamma c, -M\gamma\mathbf{V})$ of the initial particle is obtained in identical form in both (b.iii) and (b.iv). What remains to be shown in that $(m_1\gamma_1' c, m_1\gamma_1'\mathbf{w}_1')$ is identical to $\big(m_1\gamma_1(c - w_{1x}\beta)\gamma, m_1\gamma_1(w_{1x} - V)\gamma, m_1\gamma_1 w_{1y}, m_1\gamma_1 w_{1z}\big)$. Starting with the latter, unseen ⇓

$$\begin{aligned}
&\boxed{m_1\gamma_1\Big((c - w_{1x}\beta)\gamma, (w_{1x} - V)\gamma, w_{1y}, w_{1z}\Big)} \\
&= m_1\gamma_1\left(1 - \frac{w_{1x}V}{c^2}\right)\gamma\Big(c, \frac{w_{1x} - V}{1 - \frac{w_{1x}V}{c^2}}, \frac{w_{1y}\gamma^{-1}}{1 - \frac{w_{1x}V}{c^2}}, \frac{w_{1z}\gamma^{-1}}{1 - \frac{w_{1x}V}{c^2}}\Big) \\
&= \boxed{m_1\gamma_1'(c, \mathbf{w}_1')}
\end{aligned}$$

as required. The last identity draws on the hint as suggested. The same arguments apply to the second particle. 3, D

sim. seen ⇓

(b.vi) The energy-momentum four-vectors look much less messy in the form (b.iii), but their further analysis requires a detailed treatment of $\gamma$, $\gamma_{1,2}'$ and $\mathbf{w}_{1,2}'$. Using instead the form in (b.iv) gives

$$\begin{aligned}
&(M\gamma c, -M\gamma\mathbf{V}) \\
&\qquad = \big(m_1\gamma_1(c - w_{1x}\beta)\gamma, m_1\gamma_1(w_{1x} - V)\gamma, m_1\gamma_1 w_{1y}, m_1\gamma_1 w_{1z}\big) \\
&\qquad\qquad + \big(m_2\gamma_2(c - w_{2x}\beta)\gamma, m_2\gamma_2(w_{2x} - V)\gamma, m_2\gamma_2 w_{2y}, m_2\gamma_2 w_{2z}\big)
\end{aligned}$$

which produces

$$\begin{aligned}
M &= m_1\gamma_1(c - w_{1x}\beta) + m_2\gamma_2(c - w_{2x}\beta) \\
-MV &= m_1\gamma_1(w_{1x} - V) + m_2\gamma_2(w_{2x} - V)
\end{aligned}$$

while the $y$ and the $z$-component of the momentum conservation produces just $w_{1y} + w_{2y} = w_{1z} + w_{2z} = 0$. The first equation can be simplified to

$$M = m_1\gamma_1 + m_2\gamma_2 - \left(m_1\gamma_1\frac{w_{1x}V}{c^2} + m_2\gamma_2\frac{w_{2x}V}{c^2}\right)$$

which can be further simplified using the second equation in the form

$$m_1\gamma_1 w_{1x} + m_2\gamma_2 w_{2x} = V(m_1\gamma_1 + m_2\gamma_2 - M)$$

which gives

$$\boxed{M\left(1 - \frac{V^2}{c^2}\right) = \Big(m_1\gamma_1 + m_2\gamma_2\Big)\left(1 - \frac{V^2}{c^2}\right)}$$

which is easily simplified by cancelling $1 - V^2/c^2$ on both sides. Because $\gamma_{1,2} \geq 1$ the statement follows. 3, D

3. **Charge in an electric field**

sim. seen $\Downarrow$

(a) The Lorentz force gives immediately $\dot{p}_x(t) = 0$, $\dot{p}_y(t) = 0$ and $\dot{p}_z(t) = eE$. Given the initial conditions, it follows that $\boxed{p_x(t) = mv_0\gamma_0}$ with $\gamma_0 = \gamma(0)$, $\boxed{p_y(t) = 0}$ and $\boxed{p_z(t) = eEt}$.

5, B

seen $\Downarrow$

(b) From $\mathbf{p} = m\gamma\mathbf{v}$ and $\mathcal{E} = m\gamma c^2$ follows that $\boxed{\mathbf{p}(t) = \mathcal{E}(t)\mathbf{v}(t)/c^2}$.

5, A

unseen $\Downarrow$

(c) From (a) it follows that $p^2 = (mv_0\gamma_0)^2 + (eEt)^2$, so that the energy is $\mathcal{E}(t) = \sqrt{m^2c^4 + (mv_0\gamma_0 c)^2 + (eEtc)^2}$. From $\mathbf{v}(t) = \mathbf{p}(t)c^2/\mathcal{E}(t)$ the components of $\mathbf{v}(t)$ follow

$$\boxed{v_x(t) = c\frac{mv_0\gamma_0}{\sqrt{m^2c^2 + (eEt)^2 + (mv_0\gamma_0)^2}}}$$

$$\boxed{v_y(t) = 0}$$

$$\boxed{v_z(t) = c\frac{eEt}{\sqrt{m^2c^2 + (eEt)^2 + (mv_0\gamma_0)^2}}}$$

5, C

sim. seen $\Downarrow$

(d) The three components of $\mathbf{H}'$ can be read off $F'^{ij}$:

$$\boxed{H'_x = F'^{32} = \hat{\mathcal{L}}^2_i\hat{\mathcal{L}}^3_j F^{ij} = H_x}$$

as $\hat{\mathcal{L}}^2_i$ is non-zero only for $i = 2$ and similar for $\hat{\mathcal{L}}^3_j$.
Similarly

$$\boxed{H'_y = F'^{13} = \hat{\mathcal{L}}^1_i\hat{\mathcal{L}}^3_j F^{ij} = \hat{\mathcal{L}}^1_i F^{i3} = -\beta\gamma F^{03} + \gamma F^{13} = \beta\gamma E_z + \gamma H_y}$$

and

$$\boxed{H'_z = F'^{21} = \hat{\mathcal{L}}^2_i\hat{\mathcal{L}}^1_j F^{ij} = \hat{\mathcal{L}}^1_j F^{2j} = -\beta\gamma F^{20} + \gamma F^{21} = -\beta\gamma E_y + \gamma H_z}$$

Because the only non-vanishing field component is $E_z = E$, it follows that

$$\boxed{\mathbf{H}' = (0, \beta\gamma E, 0)}$$

5, D

4. (a) This follows from direct evaluation $\boxed{\text{unseen} \Downarrow}$

$$\boxed{\frac{\partial A'^j}{\partial x_i} - \frac{\partial A'^i}{\partial x_j} = \frac{\partial A^j}{\partial x_i} + \frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial A^i}{\partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}}$$

$\boxed{4, \text{B}}$

$\boxed{\text{seen} \Downarrow}$

(b) One possible solution is of the form discussed in the lectures,

$$\boxed{A^j(x^i) = B^j \exp(\mathring{\imath} k^i x_i)}$$

with fixed wave four-vector $k^i$. $\boxed{4, \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(c.i) Given the direction of propagatio, the wave four-vector is $k^i = (2\pi/\lambda, 2\pi/\lambda, 0, 0)$. The locus of $k^i y_i = -2\pi$ for $\boxed{y^i = (0, \mathbf{y})}$ thus has $\boxed{\mathbf{y} = (\lambda, a, b)}$ with arbitrary $a$ and $b$. $\boxed{4, \text{C}}$

$\boxed{\text{unseen} \Downarrow}$

(c.ii) The wave four-vector transforms like any other four vector to $\boxed{k'^i = (2\pi/\lambda)\Big((1-\beta)\gamma, (1-\beta)\gamma, 0, 0\Big)}$. $\boxed{4, \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(c.iii) The four-vector $(0,0,0,0)$ transforms to $(0,0,0,0)$ in $K'$. The plane $y^i = (0, \lambda, a, b)$ transforms to $y'^i = (-\beta\lambda\gamma, \lambda\gamma, a, b)$. The projection $\boxed{k'^i x'_i = 0}$ is trivial, while

$$\boxed{k'^i y'_i = (2\pi/\lambda)(1-\beta)\gamma\Big(-\beta\lambda\gamma - \lambda\gamma\Big) = -2\pi(1-\beta)\gamma^2(\beta+1) = -2\pi}$$

using $\gamma^2 = 1/(1-\beta^2)$. $\boxed{4, \text{D}}$

5. (a) The total potential is

unseen $\Downarrow$

$$\phi(\mathbf{r}) = \sum_a \frac{e_a}{|\mathbf{r} - \mathbf{r}_a|}$$

and the expansion of $|\mathbf{r} - \mathbf{r}_a|^{-1}$ is given by

$$\frac{1}{|\mathbf{r} - \mathbf{r}_a|} = \frac{1}{r} + \frac{\mathbf{r}_a \cdot \mathbf{r}}{r^3} + \dots$$

so that

$$\boxed{\phi(\mathbf{r}) = \frac{\sum_a e_a}{r} + \mathbf{d} \cdot \frac{\mathbf{r}}{r^3} + \dots}$$

using $\mathbf{d} = \sum_a e_a \mathbf{r}_a$, Eq. (40.3) in LL2. 5, A

(b) Using

unseen $\Downarrow$

$$\nabla \times \mathbf{H} = \frac{1}{c}\partial_t \mathbf{E} + \frac{4\pi}{c}\mathbf{j}$$

with $\partial_t \mathbf{E}$ vanishing by assumption and

$$\mathbf{H} = \nabla \times \mathbf{A}$$

it follows that $4\pi\mathbf{j}/c = \nabla \times \nabla \times \mathbf{A}$. Since Eq. (46.5),

$$\nabla \times \nabla \times \mathbf{A} = -\triangle \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

but $\nabla \cdot \mathbf{A} = 0$, the statement follows

$$\boxed{\frac{4\pi}{c}\mathbf{j} = -\triangle \mathbf{A}}$$

5, B

(c.i) Based on Eq. (36.9)

unseen $\Downarrow$

$$\triangle \frac{1}{r} = -4\pi\delta(\mathbf{r})$$

it follows that

$$\mathbf{B}(\mathbf{r}) = \int \mathrm{d}^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

obeys $\triangle \mathbf{B} = -4\pi \mathbf{j}(\mathbf{r})$, so that

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathrm{d}^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}}$$

as required, Eq. (43.5). 5, C

(c.ii) Expanding again $1/|\mathbf{r} - \mathbf{r}'|$ produces in principle the same terms as in the electric case, (a). However, $\int \mathrm{d}^3 r' \mathbf{j}(\mathbf{r}') = \mathbf{0}$, so that

unseen $\Downarrow$

$$\boxed{A(\mathbf{r}) = \int \mathrm{d}^3 r' \mathbf{j}(\mathbf{r}') \frac{\mathbf{r}' \cdot \mathbf{r}}{r^3} + \dots}$$

To calculate the resulting magnetic field, we need $\nabla \times \mathbf{A}$ and therefore

$$\nabla \times \left(\mathbf{j}(\mathbf{r}')\frac{\mathbf{r}' \cdot \mathbf{r}}{r^3}\right) = \mathbf{j}(\mathbf{r}') \times \nabla \frac{\mathbf{r}' \cdot \mathbf{r}}{r^3} = \mathbf{j}(\mathbf{r}') \times \left(\frac{\mathbf{r}'}{r^3} - 3\mathbf{r}' \cdot \mathbf{r}\frac{\mathbf{r}}{r^5}\right)$$

so that

$$\boxed{\mathbf{H} = \int \mathrm{d}^3 r' \mathbf{j}(\mathbf{r}') \times \left(\frac{\mathbf{r}'}{r^3} - 3\mathbf{r}' \cdot \mathbf{r}\frac{\mathbf{r}}{r^5}\right) + \cdots} \, .$$

5, D

**Review of mark distribution:**
Total A marks: 30 of 32 marks
Total B marks: 22 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks

# MATH60016 Special Relativity & Electromagnetism

| Question | Marker's comment |
|---|---|
| 1 | This was an easy question. There were high scores throughout. nbsp;I marked it allowing for both signs in the transformation. Some students had difficulties reading off the physical observables from the four-vectors. |
| 2 | This was a difficult question, in particular the last two parts.Some students were confused about deriving the velocity addition theorem for the y and the z component. Using the hint, many were able to find the answer to b.v. The last question was the hardest one. Most students did not consider the momentum conservation and rather tried to come up with a simple argument about the ratios of the gammas. Some were confused about the sign of 1-w_{1,2x}V/c^2.I had discussed a similar question in the lectures and had added a note about it on blackboard. The constraints on the rest masses cannot depend on the observer frame. |
| 3 | This question was generally answered every well. It was similar to problems discussed in classes. Some students overlooked the initial conditions and some struggled with transforming a tensor in the last part. |
| 4 | Easy first part, with most material just a matter of book work. Some students overlooked that there is a whole plane with the same phase. Demonstrating that the phase is invariant under transforms was essentially a confirmation of the invariance of Lorentz scalars.nbsp; |

# MATH70016 Special Relativity & Electromagnetism

| Question | Marker's comment |
|---|---|
| 1 | See MATH60016 |
| 2 | See MATH60016 |
| 3 | See MATH60016 |
| 4 | See MATH60016 |
| 5 | This was a reasonably difficult question with some material straight from the book and some extensions to it. The first two questions were pretty straight-forward, but some students overlooked the need to expand to first order in the dipole moment and left only the leading order term. Question b.i was a matter of knowing the (structure of the) Green function, which plays a prominent role throughout the mastery material. I thought it was useful to juxtapose the desired integral with the differential equation in the previous question. The last question required some careful algebra. |