Imperial College
London

MATH97018 MATH70014

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Mathematical Biology

Date: 07 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. A bioreactor is used to grow algae in a laboratory. A constant supply of nutrients is provided to the algae. The algae consume the nutrients, resulting in an increase in their population size. The algae are harvested by continuously extracting the liquid mixture from the reactor. A model for this bioreactor is given by

$$\frac{da}{dt} = a\left(\frac{\alpha n}{\beta + n} - D\right)$$
$$\frac{dn}{dt} = D(n_0 - n) - \frac{\alpha n a}{\gamma(\beta + n)}$$

where a is the concentration of algae and n is the concentration of nutrients. The constant n_0 is the concentration of the nutrient supply and D is the constant harvesting rate per capita. The parameters α, β and γ are positive constants.

(a) Let $x = a/(\gamma n_0)$. Introduce suitable nondimensional variables C and τ to transform the system to

$$\frac{dx}{d\tau} = x\left(\frac{mC}{K + C} - 1\right)$$
$$\frac{dC}{d\tau} = 1 - C - \frac{mCx}{K + C}.$$

Determine expressions for the nondimensional parameters m and K. *Hint: The nondimensionalisations for a and n are slightly different.* (5 marks)

(b) Show that the steady states of the nondimensional system are $(x, C) = (0, 1)$ and $(x, C) = (1 - c_0, c_0)$, providing also an expression for c_0. You may assume $m > 1$. (4 marks)

(c) Assess the stability of the steady states and determine how it depends on the quantity $K/(m + 1)$. *Hint: For $(1 - c_0, c_0)$, stability is most easily analysed using the trace and determinant of the Jacobian and leaving the entries in terms of c_0.* (6 marks)

(d) Provide the *dimensional* expression for $F(D)$, the harvest at steady state $(1 - c_0, c_0)$. Determine the quadratic equation in D whose solution provides the D that maximises the yield of the reactor, i.e. the algae concentration being harvested. (5 marks)

(Total: 20 marks)

2. Recall the Fitzhugh-Nagumo model,

$$\frac{dv}{dt} = f(v) - w + I_a,$$
$$\frac{dw}{dt} = bv - \gamma w, \tag{1}$$

for neuron dynamics, where b and γ are positive constants, and I_a is a constant.

(a) First, consider the case where $I_a = 0$ and $b = 0$ and $f(v) = v(a - v)(v - 1)$ with $0 < a < 1$. Spatial dynamics are included such that the system becomes

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) - w,$$
$$\frac{\partial w}{\partial t} = -\gamma w,$$

where $x \in (-\infty, \infty)$. Assume a travelling wave solution such that $v(x, t) = V(\xi)$ and $w(x, t) = W(\xi)$ where $\xi = x - ct$ with wave speed $c \geq 0$.

(i) Show that $W = 0$ if we require $W \to 0$ as $\xi \to \infty$. (3 marks)

(ii) Introduce the variable $U = dV/d\xi$. Find the fixed points, (V^*, U^*) of the resulting system and examine how their stability and type (node, spiral, etc.) change with c, assuming $c > 0$. (4 marks)

(iii) One possible exact travelling wave solution is

$$V(\xi) = \left[1 + e^{\xi/\sqrt{2}}\right]^{-1}.$$

Assuming that $0 < a < 1/2$, find the wavespeed, c, in terms of a. *Hint:* Notice that $dV/d\xi = V(V - 1)/\sqrt{2}$. (5 marks)

(b) Consider again the Fitzhugh-Nagumo model (1), but now with $f(v)$ given by the following piecewise-linear function,

$$f(v) = \begin{cases} -v, & v \leq a/2, \\ v - a, & a/2 < v \leq (1 + a)/2, \\ 1 - v, & (1 + a)/2 < v. \end{cases}$$

where $0 < a < 1$.

(i) What range of values must b/γ have in order for a stable limit-cycle to be possible? (4 marks)

(ii) Supposing that the condition on b/γ is met, for what range of I_a can there be a stable limit-cycle? (4 marks)

(Total: 20 marks)

3. Let $X(t)$ be the random variable for population size and $p_i(t) = \text{Prob}\{X(t) = i\}$. Consider the stochastic process with infinitesimal transition probabilities

$$\begin{aligned} p_{i+j,i}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = i+j | X(t) = i\} \\ &= \begin{cases} \mu i \Delta t + o(\Delta t), & j = -1 \\ \nu \Delta t + o(\Delta t), & j = 1 \\ 1 - (\mu i + \nu)\Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1, 0, 1. \end{cases} \end{aligned}$$

where μ, and ν are positive constants. Take the initial population to be $X(0) = N$.

(a) What do the constants μ and ν represent? Provide the deterministic ODE model for the population dynamics corresponding to this process. (3 marks)

(b) Derive the forward Kolmogorov equations for $p_i(t)$ for $i \neq 0$ and $i = 0$. (3 marks)

(c) Show that the partial differential equation for the probability generating function, $\mathcal{P}(z,t)$, is

$$\frac{\partial \mathcal{P}}{\partial t} + \mu(z-1)\frac{\partial \mathcal{P}}{\partial z} = \nu(z-1)\mathcal{P} \tag{2}$$

with initial condition $\mathcal{P}(z,0) = z^N$. *Hint:* Recall that $\mathcal{P}(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i$. (4 marks)

(d) From the partial differential equation for $\mathcal{P}(z,t)$, find the ODE for the mean population size, $m(t) = \sum_{i=0}^{\infty} i p_i(t) = \left.\frac{\partial \mathcal{P}}{\partial z}\right|_{z=1}$. Is it the same as the deterministic ODE model? (5 marks)

(e) Using the method of characteristics, solve (2) for $\mathcal{P}(z,t)$. (5 marks)

(Total: 20 marks)

4. Suppose the morphogens U and V involved in the following set of chemical reactions provide the number of segments a nematode (worm) might have at any time during its development:

$$U + V \xrightarrow{1} 5U$$
$$2U \xrightarrow{2} U$$
$$V \xrightarrow{1} 2V$$
$$2V \xrightarrow{1} 3V$$

(a) The morphogen concentrations $u = [U]$ and $v = [V]$ evolve according to

$$\frac{du}{dt} = f(u,v)$$
$$\frac{dv}{dt} = g(u,v).$$

Using the law of mass action, determine the functions $f(u,v)$ and $g(u,v)$ appearing in the system of differential equations. (6 marks)

(b) Find the nontrivial steady state of the system and show that it is stable. (4 marks)

(c) Consider now the reaction diffusion equations

$$\frac{\partial u}{\partial t} = f(u,v) + D\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = g(u,v) + \frac{\partial^2 u}{\partial x^2}$$

in one spatial dimension with $x \in [0, L]$. Consider no-flux boundary conditions at $x = 0$ and $x = L$. The functions $f(u,v)$ and $g(u,v)$ are identical to those in Part (a) and $D > 0$.

(i) What inequality must D satisfy for a Turing instability to develop? (6 marks)

(ii) Show that in the limit $D \to \infty$, the range of unstable wave numbers is $4/D + O(D^{-2}) \leq k^2 \leq 1 - 8/D + O(D^{-2})$. *Hint: For* $0 < \epsilon \ll 1$, $\sqrt{1+\epsilon} = 1 + \epsilon/2 + O(\epsilon^2)$. (4 marks)

(Total: 20 marks)

5. A certain species of bacteria can produce a chemical to which other nearby bacteria can respond. A model for the bacteria-chemical system is

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \chi \boldsymbol{\nabla} \cdot (n \boldsymbol{\nabla} c)$$
$$\frac{\partial c}{\partial t} = d\nabla^2 c - hc + \alpha n.$$

where n is the concentration of bacteria, c is the chemical concentration, and D, d, h, α, and χ are positive constants.

(a) Define chemotaxis and indicate the term in the model that captures this behaviour. Are the bacteria attracted or repelled by the chemical? (3 marks)

(b) Why is it appropriate to model the motion of a population of bacteria, such as *E. coli*, using diffusion? (2 marks)

(c) Write the model in the case of only one spatial dimension, x, and subject to no-flux conditions at $x = 0$ and $x = L$. (2 marks)

(d) Find the spatially homogeneous steady-states, (n_0, c_0). You may leave c_0 in terms of n_0. What happens if c is perturbed from its steady-state value? What if n is perturbed instead? (3 marks)

(e) Linearise the system about a steady-state seeking solutions of the form

$$n(x,t) = n_0 + \epsilon N_k(t)\cos(kx)$$
$$c(x,t) = c_0 + \epsilon C_k(t)\cos(kx)$$

where k is the wavenumber and $0 < \epsilon \ll 1$. Provide the expression for k and the differential equations for $N_k(t)$ and $C_k(t)$. (6 marks)

(f) Determine the range of unstable wavenumbers. (4 marks)

(Total: 20 marks)

(a)

$$\frac{da}{dt} = a\left(\frac{\alpha n}{\beta + n} - D\right)$$

SEEN SIMILAR

$$\frac{dn}{dt} = D(n_0 - n) - \frac{\alpha n a}{\gamma(\beta + n)}$$

LET

$$n = n_0 C, \quad a = \gamma n_0 x$$

AND $t = \tau / D$

UPON SUBSTITUTING THESE INTO THE SYSTEM WE HAVE

$$\frac{dx}{d\tau} = x\left(\frac{m C}{K + C} - 1\right)$$

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	$$\frac{dc}{d\tau} = 1 - c - \frac{mcx}{K+c}$$ WHERE $m = \frac{\alpha}{D}$ AND $K = \beta/n_0$.	5B
(b)	STEADY STATES $$0 = x\left(\frac{mc}{K+c} - 1\right) \quad (*)$$ $$0 = 1 - c - \frac{mcx}{K+c}$$ FROM $(*)$ WE HAVE $$x = 0 \quad \text{OR} \quad c = \frac{K}{m-1}$$	SEEN SIMILAR
	Setter's initials	Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	FOR $x=0$, WE SEE THAT $C=1$. FOR $C=\frac{K}{m-1}$ WE HAVE THAT $$x = 1-\frac{K}{m-1}$$ THUS, $(0,1)$ AND $(1-c_0, c_0)$ WITH $c_0=\frac{K}{m-1}$ ARE THE FIXED POINTS.	4A
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		
(c)	STABILITY	SEEN SIMILAR

$$J(0,1) = \begin{bmatrix} \frac{m}{K+1} - 1 & 0 \\ -\frac{m}{K+1} & -1 \end{bmatrix}$$

IF $\frac{m}{K+1} < 1$, STABLE NODE

IF $\frac{m}{K+1} > 1$, SADDLE (UNSTABLE)

FOR $(1-c_0, c_0)$

WE HAVE

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		

$J(1-c_0, c_0)$

$$= \begin{bmatrix} 0 & \dfrac{mk(1-c_0)}{(k+c_0)^2} \\ -1 & -1-\dfrac{mk(1-c_0)}{(k+c_0)^2} \end{bmatrix}$$

$$\det(J) = \frac{mk(1-c_0)}{(k+c_0)^2}$$

> 0, IF $1 > c_0$

$\mathrm{TRACE}(J) = -1 - \det(J)$

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	Thus, $(1-c_0, c_0)$ is STABLE WHEN $$c_0 = \frac{K}{m-1} < 1.$$	6 A UNSEEN
(d)	$$F(D) = D a$$ $$= D \gamma n_0 (1 - c_0)$$ $$= \frac{\gamma n_0 \alpha - D \gamma n_0 - \beta D \gamma}{\frac{\alpha}{D} - 1}$$ $$= \frac{\gamma n_0 \alpha D - D^2 \gamma n_0 - \beta D^2 \gamma}{\alpha - D}$$	
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	$$F'(D) = \frac{rn_0\alpha - 2Drn_0 - 2\beta Dr}{\alpha - D} + \frac{rn_0 D\alpha - D^2 rn_0 - \beta D^2 r}{\{\alpha - D\}^2}$$ REARRANGING WE SEE THAT $F'(D) = 0$ WHEN $$D^2\left(1 + \frac{\beta}{n_0}\right) - 2\alpha\left(1 + \frac{\beta}{n_0}\right)D + \alpha^2 = 0.$$	5 C
	Setter's initials	Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question 2		Marks & seen/unseen
Parts		
(a) (i)	$v(x,t) = V(\xi)$ $w(x,t) = W(\xi)$ WHERE $\xi = x - ct$. THUS, $\frac{\partial v}{\partial t} = -cV', \quad \frac{\partial w}{\partial t} = -cW'$ $\frac{\partial^2 v}{\partial x^2} = V''$ THE SYSTEM THEN BECOMES $-cV' = f(V) + V'' - W \quad (*)$ $-cW' = -\gamma W. \quad (**)$	SEEN SIMILAR
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EXAMINATION SOLUTIONS 2021-22

Course

Question

Marks & seen/unseen

Parts

(**) HAS SOLUTION

$$W(\xi) = Ae^{\gamma\xi/c}$$

AS $W \to 0$ AS $\xi \to \infty$, WE MUST HAVE $A = 0$.

THUS, $W = 0$. 3 — 2A, 1B

SEEN SIMILAR

(a)(ii) LET $V' = U$

SO WE HAVE

$$V' = U$$
$$U' = -cU - f(V)$$

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	THE STEADY STATES ARE $(0,0)$, $(0,a)$, $(0,1)$. THE JACOBIAN IS GIVEN BY $$J = \begin{bmatrix} 0 & 1 \\ -f'(v^*) & -c \end{bmatrix}$$ AND HAS E-VALS $$\lambda = \frac{-c \pm \sqrt{c^2 - 4f'(v^*)}}{2}$$	
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	SINCE $f'(0) = -a < 0$ $f'(a) = a(1-a) > 0$ $f'(1) = -(1-a) < 0.$ WE HAVE: FOR $v^* = 0$ $\lambda = \frac{-c \pm \sqrt{c^2+4a}}{2}$ SADDLE FOR $c>0$. FOR $v^* = 1$ $\lambda = -c \pm \sqrt{c^2+4(1-a)}$ SADDLE FOR $c>0$.	
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	FOR $v^* = a$ $\lambda = \frac{-c \pm \sqrt{c^2 - 4a(1-a)}}{2}$ $c < 2\sqrt{a(1-a)}$ STABLE SPIRAL $c > 2\sqrt{a(1-a)}$ STABLE NODE.	4 B
(a) (iii)	FOR THIS SOLUTION WE SEE THAT $V' = \frac{1}{\sqrt{2}} V(V-1)$	UNSEEN
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	DIFFERENTIATING AGAIN, WE SEE THAT $$V'' = \frac{V(V-1)}{2}[2V-1]$$ THUS FROM (*) WE SEE THAT $$-c\frac{V(V-1)}{\sqrt{2}}$$ $$= V(1-V)(V-a)$$ $$+ \frac{V(V-1)}{2}[2V-1]$$	
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	SOLVING FOR c GIVES $$c = \sqrt{2}\left(\frac{1}{2} - a\right).$$	5 C
(b) (i)	THE NULLCLINES ARE $$w = \frac{b}{\gamma} v$$ AND $$w = f(v) + I_a.$$ THE JACOBIAN IS $$J = \begin{bmatrix} f'(v) & -1 \\ b & -\gamma \end{bmatrix}$$	UNSEEN
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	THE P-B THEOREM REQUIRES THAT THE FIXED POINT BE UNSTABLE. WE REQUIRE THEN $\mathrm{TRACE}(J) = f'(v) - \gamma > 0$ $\mathrm{DET}(J) = \gamma f'(v) + b > 0$ ACCORDINGLY, THE FIXED POINT MUST BE WHERE $f' = 1$ AND WE THEN HAVE: $1 - \gamma > 0 \Rightarrow 1 > \gamma$ AND $b - \gamma > 0 \Rightarrow b > \gamma.$	4 D
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts (b) (ii)	FOR THE NULLCLINES TO INTERSECT WHERE $f'=1$, WE MUST HAVE $$\frac{b}{\gamma}\left(\frac{a}{2}\right) < -\frac{a}{2} + I_a$$ AND $$\frac{b}{\gamma}\left(\frac{1+a}{2}\right) > \frac{1}{2} - \frac{a}{2} + I_a$$ THESE CONDITIONS GIVE $$\frac{a}{2}\left(\frac{b}{\gamma}+1\right) < I_a < \frac{a}{2}\left(\frac{b}{\gamma}+1\right) + \frac{1}{2}\left(\frac{b}{\gamma}-1\right).$$	4 D
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	EXAMINATION SOLUTIONS 2021-22	Course
Question 3		Marks & seen/unseen
Parts		
(a)	μ: DEATH RATE PER CAPITA	SEEN SIMILAR
	ν: IMMIGRATION RATE	
	$$\frac{dn}{dt} = -\mu n + \nu$$	3 A
(b)	$$\frac{dp_i}{dt} = \mu(i+1)p_{i+1} + \nu p_{i-1} - (\mu i + \nu) p_i$$	SEEN SIMILAR
	$$\frac{dp_0}{dt} = \mu p_1 - \nu p_0$$	3 A
(c)	$$\frac{d}{dt}\sum_i p_i z^i = \sum_{i=0}^{\infty} \mu (i+1) p_{i+1} z^i + \nu \sum_{i=0}^{\infty} p_{i-1} z^i$$	SEEN SIMILAR
	Setter's initials Checker's initials	Page number

	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	$$- \sum_{i=0}^{\infty} (\mu i + \nu) p_i z^i$$ $$\frac{\partial P}{\partial t} = \sum_{i=0}^{\infty} \mu i p_i z^{i-1} + \nu \sum_{i=0}^{\infty} p_i z^{i+1} - z \sum_{i=0}^{\infty} \mu i z^{i-1} p_i - \nu \sum_{i=0}^{\infty} p_i z^i$$ $$\frac{\partial P}{\partial t} = \mu \frac{\partial P}{\partial z} + \nu z P - z\mu \frac{\partial P}{\partial z} - \nu P$$ $$= \mu(1-z)\frac{\partial P}{\partial z} + \nu(z-1)P.$$	4B
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		
(d)	$$\frac{\partial}{\partial t}\left(\frac{\partial P}{\partial z}\right) = \mu(z-1)\frac{\partial^2 P}{\partial z^2} - \mu\frac{\partial P}{\partial z} + \nu(z-1)\frac{\partial P}{\partial z} + \nu P.$$ EVALUATING AT $z=1$ AND WRITING $m = \left.\frac{\partial P}{\partial z}\right\rvert_{z=1}$ GIVES $$\frac{dm}{dt} = -\mu m + \nu$$ SAME AS DETERMINISTIC MODEL.	UNSEEN 5 (2C, 3D)
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		
(e)	INTRODUCE τ AND s SUCH THAT $t(s,\tau)$, $z(s,\tau)$ AND $P(s,\tau)$ WHERE $$\frac{dP}{d\tau} = \nu(z-1)P \quad (*)$$ $$\frac{dt}{d\tau} = 1, \quad \frac{dz}{d\tau} = \mu(z-1) \quad (**)$$ $\Downarrow$ $t = \tau + C_1$ EQ. $(**)$ HAS SOLUTION $$z = 1 - C_2 e^{\mu\tau}.$$ SINCE WE HAVE THE	SEEN SIMILAR.
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	DATA AT $t=0$, WE HAVE $C_1 = 0$ AND $z = s$. THUS, $s = 1 - C_2$ AND $C_2 = 1 - s$. THIS GIVES $s = 1 + (z-1) e^{-\mu\tau}$. USING THIS IN (*) WE HAVE $\frac{dP}{d\tau} = \nu (s-1) e^{\mu\tau} P$	
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	WHICH HAS SOLUTION $P = K e^{\nu/\mu \, e^{\mu\tau}(s-1)}$ SINCE AT $\tau = 0$, $P = s^N$, WE HAVE $K = s^N e^{-\nu/\mu (s-1)}$ THUS $P = s^N e^{-\frac{\nu}{\mu}(s-1)[e^{\mu\tau}-1]}$ OK $P(z,t) = [1 + (z-1)e^{-\mu t}]^N \times e^{\frac{\nu}{\mu}(z-1)[1-e^{-\mu t}]}$.	5 D
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	EXAMINATION SOLUTIONS 2021-22	Course
Question 4		Marks & seen/unseen

Parts

(a) THE LAW OF MASS ACTION GIVES

SEEN SIMILAR

$$\frac{du}{dt} = 4uv - 2u^2$$

3 B

$$\frac{dv}{dt} = -uv + v + v^2$$

3 B

(b) STEADY STATES GIVEN BY

SEEN SIMILAR

$$0 = u[4v - 2u]$$

$$0 = v[-u + 1 + v]$$

STEADY STATES ARE

$$(0,0), \quad \cancel{(0,-1)}, \quad \boxed{(2,1)}$$

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	STABILITY OF $(2,1)$	

$$\frac{\partial f}{\partial u} = 4v - 4u$$

$$\frac{\partial f}{\partial v} = 4u$$

$$\frac{\partial g}{\partial u} = -v$$

$$\frac{\partial g}{\partial v} = -u + 1 + 2v$$

$$J(2,1) = \begin{bmatrix} -4 & 8 \\ -1 & 1 \end{bmatrix}$$

TRACE $(J) = -3 < 0$

DET $(J) = 4 > 0$ STABLE. 4 A

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts (c) (i)	REQUIRE THAT 1. $J_{22} D_1 + J_{11} D_2$ $= D - 4 > 0$ $\Rightarrow \boxed{D > 4}$ 2. $(J_{22} D_1 + J_{11} D_2)^2$ $- 4 D_1 D_2 \, \mathrm{DET}(J)$ $= (D-4)^2 - 16D > 0.$ EXPANDING AND REARRANGING GIVES $\boxed{D > 12 + 8\sqrt{2}}$	SEEN SIMILAR 6A
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EXAMINATION SOLUTIONS 2021-22

Course

Question

Marks & seen/unseen

Parts

UNSEEN

(b) (ii) THE UNSTABLE WAVE NUMBERS ARE $k_-^2 < k^2 < k_+^2$ WHERE

$$k_\pm^2 = \frac{(D-4) \pm \sqrt{(D-4)^2 - 16D}}{2D}$$

$$= \frac{1}{2} - \frac{2}{D} \pm \sqrt{\frac{D^2 - 8D + 16 - 16D}{4D^2}}$$

$$= \frac{1}{2} - \frac{2}{D} \pm \frac{1}{2}\left[1 - \frac{24}{D} + \frac{16}{D^2}\right]^{1/2}$$

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		

SINCE

$$\sqrt{1-\epsilon} = 1 - \tfrac{1}{2}\epsilon + O(\epsilon^2),$$

$$k_\pm^2 = \frac{1}{2} - \frac{2}{D} \pm \frac{1}{2}\left(1 - \frac{12}{D}\right) + O(D^{-2})$$

$$= \frac{1}{2} - \frac{2}{D} \pm \frac{1}{2} \mp \frac{6}{D}$$

$$k_+^2 = 1 - \frac{8}{D}$$

$$k_-^2 = \frac{4}{D}$$

4A

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	EXAMINATION SOLUTIONS 2021-22	Course
Question 5		Marks & seen/unseen
Parts		ALL UNSEEN
(a)	CHEMOTAXIS IS THE ABILITY OF AN ORGANISM TO MOVE EITHER TOWARDS OR AWAY FROM A CHEMICAL CONCENTRATION.	
	IN THE MODEL, THE TERM IS: $$-\chi \nabla \cdot (n \nabla c)$$	
	AND IT IS ATTRACTIVE.	3
(b)	THESE BACTERIA DISPLAY "RUN AND TUMBLE" BEHAVIOUR, ALLOWING	
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	THEM TO EXECUTE A RANDOM WALK THROUGH SPACE, HENCE DIFFUSE.	2
(c)	$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - \chi\frac{\partial}{\partial x}\left(n\frac{\partial c}{\partial x}\right)$$ $$\frac{\partial c}{\partial t} = d\frac{\partial^2 c}{\partial x^2} - hc + \alpha n$$ WITH $\frac{\partial n}{\partial x} = \frac{\partial c}{\partial x} = 0$ AT $x=0$ AND $x=L$.	2
(d)	REMOVING THE SPATIAL DERIVATIVES, WE SEE THAT.	
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	$$\frac{dn}{dt} = 0$$ $$\frac{dc}{dt} = -hc + \alpha n$$ THUS, THE STEADY STATE IS n_0 FOR ANY $n_0 > 0$ AND $c_0 = \frac{\alpha}{h} n_0$. ANY PERTURBATIONS IN C WILL DECAY AWAY WHILE IF n IS PERTURBED, IT	
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	WILL REMAIN AT THIS NEW VALUE AND C WILL ADJUST ACCORDINGLY.	3
(e)	$n = n_0 + \epsilon N_k(t) \cos kx$ $c = \frac{\alpha}{h} n_0 + \epsilon C_k(t) \cos kx$ $k = \frac{\pi n}{L}, \quad n \in \mathbb{Z}$ INSERTING THE TRIAL SOLUTION INTO THE EQUATIONS AND NEGLECTING TERM $O(\epsilon^2)$, WE SEE	
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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts		

$$\frac{dN_k}{dt} = -k^2 D N_k + \chi n_0 k^2 C_k$$

$$\frac{dC_k}{dt} = -(k^2 d + h) C_k + \alpha N_k$$

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(f) THE JACOBIAN IS

$$J = \begin{bmatrix} -k^2 D & \chi n_0 k^2 \\ \alpha & -(k^2 d + h) \end{bmatrix}$$

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	EXAMINATION SOLUTIONS 2021-22	Course
Question		Marks & seen/unseen
Parts	$\mathrm{trace}(J) = -\left[k^2D + k^2d + h\right] < 0.$ $\det(J) = k^2D(k^2d+h) - \chi n_0 k^2 \alpha.$ SINCE WE REQUIRE $\det(J) < 0$ FOR INSTABILITY, WE SEE THAT $0 < k^2 < \dfrac{\chi n_0 \alpha - hD}{dD}$	4
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If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
Mathematical Biology_MATH60014 MATH97018 MATH70014	1	Most students answered point a and amost all answered point b correctly
Mathematical Biology_MATH60014 MATH97018 MATH70014	2	In general the students found this question challenging. Students faired better on part a, though in many cases, students had the sign in front of f(v) in the reaction diffusion equation incorrect. Results for part b were generally quite low, though there were some notable exceptions where students understood that the slope of the function needed to be evaluated in part i, and the nullcline intersection gave the correct result in part ii.
Mathematical Biology_MATH60014 MATH97018 MATH70014	3	The answer was a bit long. Points a and b were in general done correctly, for point c too it was clear how to do it but required more attention to the changes of indices. Point d created some confusion but several students managed to address it correctly. Point e was the most challenging, it was carried out correctly only up to a certain point, the most common mistake being in finding the right dependence of the probability P on z correctly inserting the relation between z and s.
Mathematical Biology_MATH60014 MATH97018 MATH70014	4	In general the students did very well on this question. Most errors were related to wrongly applying the conditions for Turning instability, or Taylor expanding correctly the expression for the range of wave numbers.
Mathematical Biology_MATH60014 MATH97018 MATH70014	5	The results on Q5 were varied, but generally okay. Many students did will on the first parts of the question, but began to struggle with obtaining the system of equations of C_k and N_k, though many students were able to obtain some terms. The final part regarding the wave numbers posed the most difficulty.