

Exercise 6.1. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a, b])$:

(i)

$$\|f\|_1 = \int_a^b |f(t)| dt$$

(ii)

$$\|f\|_\infty = \max_{t \in [a, b]} |f(t)|$$

(iii)

$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$$

Exercise 6.2. Show that if V is a vector space, and $\|\cdot\| : V \rightarrow \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $d_{\|\cdot\|}(0, 2v) = 2 d_{\|\cdot\|}(0, v)$. Conclude that there is no norm function on \mathbb{R}^2 which induced the discrete metric d_{disc} on \mathbb{R}^2 .

Exercise 6.3. Let (X, d) be a metric space.

(i) Show that for every x, y , and z in X , we have

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

(ii) Show that for all x, y, z and t in X , we have

$$|d(x, y) - d(z, t)| \leq d(x, z) + d(y, t).$$

(iii) Show that for all x_1, x_2, \dots, x_n in X , we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n).$$

Exercise 6.4. Let (X, d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By an example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n \in \mathbb{N}} B_{1/n}(x) = \{x\}.$$

Exercise 6.5. (i) Show that for all x and y in \mathbb{R}^n , we have

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n} \cdot d_\infty(x, y).$$

(ii) Show that for all x and y in \mathbb{R}^n , we have

$$d_\infty(x, y) \leq d_1(x, y) \leq n \cdot d_\infty(x, y).$$

(iii) Show/conclude that for all x and y in \mathbb{R}^n , we have

$$\frac{1}{\sqrt{n}} d_2(x, y) \leq d_1(x, y) \leq n d_2(x, y).$$

(iv) Conclude that the metrics d_1 , d_2 and d_∞ on \mathbb{R}^n are topologically equivalent.

Exercise 6.6. Let (X, d_{disc}) be a discrete metric space, and $(x_n)_{n \geq 1}$ be a sequence in X . Then, $(x_n)_{n \geq 1}$ converges in (X, d_{disc}) if and only if the sequence $(x_n)_{n \geq 1}$ is eventually constant.

Exercise 6.7. Let (X, d) be a metric space, and $(x_n)_{n \geq 1}$ be a sequence in X . Prove that the sequence $(x_n)_{n \geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

Exercise 6.8. Let (X, d_{disc}) be a discrete metric space. Show that every set in X is closed.