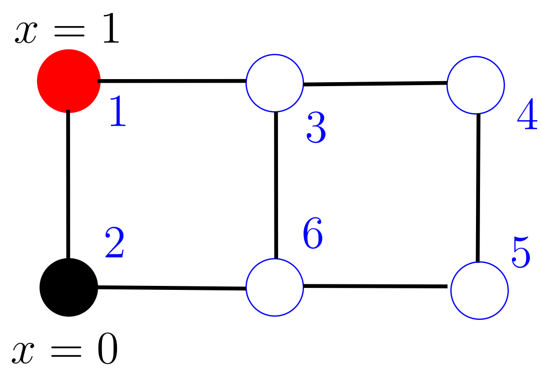
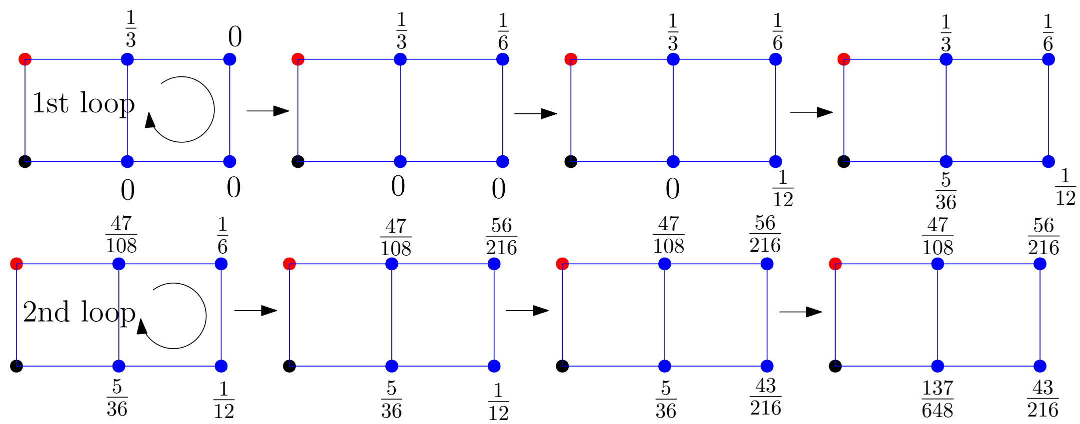
1 (a). Labelling nodes as follows:



and carrying out 2 loops of the method of relaxtion in the order 3,4,5 and then 6 we find:



At the end of the first loop we find

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{12} & \frac{5}{36} \end{bmatrix}^T. \tag{1}$$

At the end of the second loop we find

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & \frac{47}{108} & \frac{56}{216} & \frac{43}{216} & \frac{137}{648} \end{bmatrix}^T. \tag{2}$$

1 (b). The energy dissipation has the 7 terms (a sum over the edges)

$$\begin{aligned} \mathcal{E} = {} & (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 \\ & + (x_5 - x_6)^2 + (x_3 - x_6)^2 + (x_2 - x_6)^2. \end{aligned} \tag{3}$$

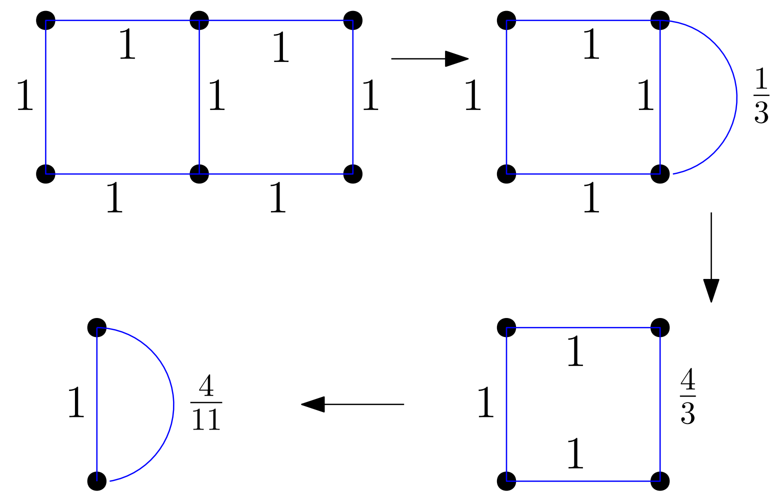
At the end of the first loop this has value

$$\mathcal{E}_1 = 1.5394. \tag{4}$$

At the end of the second loop this has value

$$\mathcal{E}_2 = 1.4485. \tag{5}$$

1 (c). We can use the formula that says the dissipation with unit voltage at one node and another grounded is given by the effective conductance and then compute the effective conductance by reducing to a sequence of "equivalent circuit" as follows:



where we have used the rule for "3 conductors in series" twice. The conductance is therefore equivalent to two conductors in parallel with conductances 1 and 4/11:

$$C_{\text{eff}} = \mathcal{E} = \frac{15}{11} \approx 1.3636. \tag{6}$$

This dissipation can also be found by first finding the exact potentials. Using the harmonicity of the potentials the equations to solve are

$$\begin{aligned} 3p_3 &= 1 + p_4 + p_6, \\ 2p_4 &= p_3 + p_5, \\ 2p_5 &= p_4 + p_6, \\ 3p_6 &= p_3 + p + 5. \end{aligned} \tag{7}$$

These equations are easily solved by hand to find

$$x_3 = \frac{7}{11}, \qquad x_4 = \frac{6}{11}, \qquad x_5 = \frac{6}{11}, \qquad x_6 = \frac{4}{11}. \tag{8}$$

These values can be substituted in (3) to find (6).

1 (d). We can see from these numerical values that

$$\mathcal{E}_1 \geq \mathcal{E}_2 \geq \mathcal{E}, \tag{9}$$

as expected in accordance with Dirichlet's Principle.

2 (a) The linear equation for the vector of potentials/voltages is

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \tag{10}$$

where

$$\mathbf{f} = [-1 \quad +1 \quad \mathbf{0}^T] = [-1 \ \hat{\mathbf{f}}^T], \qquad \hat{\mathbf{f}} = \begin{bmatrix} +1 \\ \mathbf{0} \end{bmatrix}, \tag{11}$$

and where we have grounded the node corresponding to the first entry of $\mathbf{x}$. Hence

$$\mathbf{x} = [0 \ \ \hat{\mathbf{x}}^T], \tag{12}$$

where $\hat{\mathbf{x}}$ is the set of voltages to be found and now includes the voltage at the node having unit current into it. Note that if we introduce the sub-block decomposition of $\mathbf{K}$:

$$\mathbf{K} = \begin{bmatrix} p & \mathbf{q}^T \\ \mathbf{q} & \mathbf{R} \end{bmatrix} \tag{13}$$

then we know that $\mathbf{R}$ is positive definite and invertible. It is also easy to verify that

$$\mathcal{E}_0(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} = \hat{\mathbf{x}}^T R \hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T \hat{\mathbf{f}}.$$

The equation for $\hat{\mathbf{x}}$ is

$$\mathbf{R}\hat{\mathbf{x}} = \hat{\mathbf{f}} \tag{14}$$

so that

$$\hat{\mathbf{x}} = \mathbf{R}^{-1}\hat{\mathbf{f}}. \tag{15}$$

However we can write

$$\begin{aligned} \mathcal{E}_0(\mathbf{x}) &= \hat{\mathbf{x}}^T R \hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T \hat{\mathbf{f}} = (\hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}})^T \mathbf{R} (\hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}}) - (\mathbf{R}^{-1}\hat{\mathbf{f}})^T \mathbf{R}\mathbf{R}^{-1}\hat{\mathbf{f}} \\ &= \mathbf{X}^T \mathbf{R} \mathbf{X} + \mathbf{c}, \end{aligned}$$

where $\mathbf{c}$ is a vector that is independent of $\mathbf{x}$ (and hence $\hat{\mathbf{x}}$) and

$$\mathbf{X} \equiv \hat{\mathbf{x}} - \mathbf{R}^{-1}\hat{\mathbf{f}}. \tag{16}$$

It is clear, on use of the positive definiteness of $\mathbf{R}$, that $\mathcal{E}_0(\mathbf{x})$ is minimized when $\mathbf{X} = 0$, i.e., when

$$\hat{\mathbf{x}} = \mathbf{R}^{-1}\hat{\mathbf{f}} \tag{17}$$

which is the same as (15). Hence we have shown that the set of potentials determined by Ohm's law, satisfying KCL at the interior nodes and associated with unit input current, minimizes the dissipation function $\mathcal{E}_0(\mathbf{x})$ among all possible potentials defined at the ungrounded nodes.

3 (a) Since
$$-\mathbf{A}\mathbf{x} \tag{18}$$
is precisely the m-dimensional vector of potential drops across nodes it is clear that $-(\mathbf{A}\mathbf{x})^T\mathbf{j} = \mathbf{x}^T(-\mathbf{A}^T\mathbf{j}) = \mathbf{x}^T\mathbf{f}$.

3 (b) The previous expression can be written as
$$\sum_{\text{edges k}} (x_i - x_j)j_k = j_a(x_a - x_b), \tag{19}$$
where edge k is assumed to join node i to node j with the direction taken in forming the incidence matrix is from node i to node j.

3 (c) The energy dissipation is
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges}} \frac{j_k^2}{c_k}. \tag{20}$$
Write
$$\mathbf{j} = \mathbf{w} + \mathbf{d} \tag{21}$$
where $\mathbf{w}$ is a current satisfying Ohm's law and KCL on all interior nodes; it is clear that
$$\mathbf{d} = \mathbf{j} - \mathbf{w} \tag{22}$$
is the difference between a general unit flow $\mathbf{j}$ and this unit current. Note that $\mathbf{d}$ is a flow of *zero* strength because
$$d_a = j_a - w_a = 1 - 1 = 0. \tag{23}$$
Writing d_k and w_k to be the values of $\mathbf{d}$ and $\mathbf{w}$ in edge k we can write
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{j_k^2}{c_k} = \sum_{\text{edges k}} \frac{(w_k + d_k)^2}{c_k} = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k} + 2\sum_{\text{edges k}} \frac{w_k d_k}{c_k} \tag{24}$$
But we have said that $\mathbf{w}$ is a current satisfying Ohm's law on each edge meaning that
$$w_k = c_k(x_i - x_j), \tag{25}$$
where edge k is assumed to join nodes i and j. Hence
$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k} + 2\sum_{\text{edges k}} (x_i - x_j)d_k. \tag{26}$$

But by part (a) with the relevant flow taken as **d** which has zero strength $d_a = 0$ we see that

$$\tilde{\mathcal{E}}(\mathbf{j}) = \sum_{\text{edges k}} \frac{w_k^2}{c_k} + \frac{d_k^2}{c_k}, \tag{27}$$

where the first term on the right is the dissipation of the current **w** and the second on the right is another (positive) dissipation associated with **d**. It is therefore clear that

$$\tilde{\mathcal{E}}(\mathbf{w}) \leq \tilde{\mathcal{E}}(\mathbf{j}). \tag{28}$$

Thus the current **w** satisfying Ohm's law minimizes the dissipation.

4. We know the escape probability is related to the effective conductance (of the graph viewed within the electric circuit analogy); moreover, the effective conductance is equal to the dissipation function evaluated at **x**, the vector of node voltages for the original graph satisfying Ohm's law and KCL. In this case, where the starting node of the random walk has 3 attached edges, we have

$$p_{\text{esc}} = \frac{C_{\text{eff}}}{3} = \frac{\mathcal{E}(\mathbf{x})}{3} = \frac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{3} \tag{29}$$

Since the removed edge is **not** connected to the starting node in this problem, to prove the required inequality concerning the escape probability p_{esc} it is therefore enough to prove it for the dissipation function. Next note that

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^T\mathbf{K}\mathbf{x} = \frac{1}{2}\sum_i\sum_j c_{ij}(x_i - x_j)^2, \tag{30}$$

where we have written the dissipation as a double sum over the nodes of the graph and included a factor of $1/2$ since edges are counted twice in this double sum; the constant c_{ij} denotes the conductance of the edge connecting nodes i and j. If an edge of the graph is removed then the conductance of that edge is effectively set equal to zero. If $\overline{\mathcal{E}}(\mathbf{x})$ is the dissipation function associated with this modified graph then

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^T\mathbf{K}\mathbf{x} = \frac{1}{2}\sum_i\sum_j c_{ij}(x_i - x_j)^2 \geq \overline{\mathcal{E}}(\mathbf{x}) \tag{31}$$

since, in the modified graph, a positive contribution to the dissipation in the unperturbed dissipation is set equal to zero when computing $\overline{\mathcal{E}}(\mathbf{x})$.

Suppose now that $\hat{\mathbf{x}}$ is the vector of voltages (satisfying Ohm's law and KCL) in the modified graph. Then we know by Dirichlet's principle that

$$\overline{\mathcal{E}}(\mathbf{x}) \geq \overline{\mathcal{E}}(\hat{\mathbf{x}}). \tag{32}$$

But, by the same result used above, the right hand side is precisely $\overline{p_{\text{esc}}}/3$ where $\overline{p_{\text{esc}}}$ is the escape probability for the modified graph. Therefore, we have shown

$$p_{\text{esc}} = \frac{\mathcal{E}(\mathbf{x})}{3} \geq \frac{\overline{\mathcal{E}}(\hat{\mathbf{x}})}{3} = \overline{p_{\text{esc}}} \tag{33}$$

and the escape probability can only decrease when the edge is removed.