Imperial College London
Department of Mathematics

Question Sheet 5 - Probl. Class week 8

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

Problem sheet released on Monday of week 6. All questions can be attempted before the problem class on Monday of week 8. Solutions will be released after the last problem class on Monday of week 8.

Question 1 Let S be the two-element set $\{a, b\}$. Show that there are precisely 16 distinct binary operations on S. How many of them make S a group?

Find a formula for the total number of binary operations on a set of n elements.

Question 2 Prove that multiplication of complex numbers is associative.

Question 3 Which of the following are groups?

(a) The set of all complex numbers z such that $|z| = 1$, with the usual complex multiplication.

(b) The set $\{x \in \mathbb{R} \mid x \geq 0\}$, with the operation $x * y = \max(x, y)$.

(c) The set $\mathbb{C} \setminus \{0\}$, with the operation $a * b = |a| \cdot b$.

(d) The set of all rational numbers with odd denominators, with the usual addition.

(e) The set $\{a, b\}$, where $a \neq b$, with the binary operation $*$ given by

$$a * a = a,\ b * b = b,\ a * b = b,\ b * a = b.$$

(f) The set $\{a, b\}$, with $a \neq b$, with the binary operation $*$ given by

$$a * a = a,\ b * b = a,\ a * b = b,\ b * a = b.$$

(g) The set $\mathbb{R}^3$, with the binary operation $v * w = v \times w$ (the vector product).

(h) The set $\mathbb{R}^3$, with the usual vector addition.

Question 4 Let S be the set of all real numbers except -1. For $a, b \in S$ define

$$a * b = ab + a + b.$$

Show that $(S, *)$ is a group. (Note: you need to check the closure axiom.)

Question 5 Let G be a group, and let $a, b, c \in G$. Prove the following facts.

(a) If $ab = ac$ then $b = c$.

(b) The equation $axb = c$ has a unique solution for $x \in G$.

(c) $(a^{-1})^{-1} = a$.

(d) $(ab)^{-1} = b^{-1}a^{-1}$.

Question 6 Let G be a group, and let e be the identity of G. Suppose that $x * x = e$ for all $x \in G$. Show that $y * z = z * y$ for all $y, z \in G$. Can you find infinitely many examples of groups G with the property that $x * x = e$ for all $x \in G$?

Question 7 (i) (Harder) Suppose X is a non-empty set and α, β are permutations of X with the property that any element of X moved by α is fixed by β and any element of X moved by β is fixed by α, i.e. for all $x \in X$:

$$(\alpha(x) \neq x \Rightarrow \beta(x) = x) \text{ and } (\beta(x) \neq x \Rightarrow \alpha(x) = x).$$

Prove that $\alpha \circ \beta = \beta \circ \alpha$. [Hint: consider $\alpha(\beta(x))$ in the cases where x is moved by β and where it is moved by α; note that if x is moved by β, then so is $\beta(x)$.]

(ii) Suppose $(G, \cdot)$ is a group and $g, h \in G$ are such that $gh = hg$. Show that for all $r \in \mathbb{N}$ we have $(gh)^r = g^r h^r$. Give an example of $g, h \in S_3$ (the symmetric group on $\{1, 2, 3\}$) where $(gh)^2 \neq g^2 h^2$.

Question 8 Which of the following subsets H are subgroups of the given group G?

(a) $G = (\mathbb{Z}, +)$, $H = \{n \in \mathbb{Z} \mid n \equiv 0 \bmod 37\}$.

(b) $G = \mathrm{GL}(2, \mathbb{C})$, $H = \{A \in G \mid A^2 = I\}$.

(c) $G = \mathrm{GL}(2, \mathbb{R})$, $H = \{A \in G \mid \det(A) = 1\}$.

(d) $G = S_n$, $H = \{g \in G \mid g(1) = 1\}$ (for $n \in \mathbb{N}$).

(e) $G = S_n$, $H = \{g \in G \mid g(1) = 2\}$ (for $n \geq 2$).

(f) $G = S_n$, H is the set of all permutations $g \in G$ such that $g(i) - g(j) \equiv i - j \bmod n$ for all $i, j \in \{1, \ldots, n\}$.

Question 9 Prove the following statements.

(a) Every cyclic group is abelian.

(b) The group S_n is *not* abelian, unless $n < 3$.