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Partial Differential Equations in Action **MATH50008**

Summary of resolution techniques

In this short document, we summarize the general resolution techniques seen throughout the semester. This quick summary is non-exhaustive and students should refer to the lecture notes for a more complete picture of the methods as well as examples of application. But the hope is that this will provide a useful quick access resource to consult while attempting problems during your revisions.

The method of characteristics

Method of characteristics (General)

Consider the following **first-order quasilinear partial differential equation**

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u)$$

subject to the boundary condition that $u(x,y)$ by u_0 (a known function) on a curve γ in the (x,y)-plane. The method of characteristics follows the following steps:

S1. Parametrize the initial curve as:

$$\Gamma : \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \end{cases}$$

S2. Write the characteristic equations:

$$\frac{dx}{ds} = A(x,y,u), \quad \frac{dy}{ds} = B(x,y,u) \quad \text{and} \quad \frac{du}{ds} = C(x,y,u), \quad (\star)$$

S3. Find the characteristic curves by solving the ODEs $(\star)$ subject to the initial conditions: $x(0) = x_0(t)$, $y(0) = y_0(t)$ to obtain $u(0) = u_0(t)$. You obtain solutions in terms of the parameters s and t:

$$\begin{cases} x = X(s,t) \\ y = Y(s,t) \\ u = U(s,t) \end{cases}$$

Note that the curves for $t = \text{constant}$ give the characteristics.

S4. Eliminate the parameters s and t by inverting the above equations and express (s,t) in terms of (x,y) as in

$$t = T(x,y) \quad \text{and} \quad s = S(x,y)$$

S5. The solution is then given by

$$u(x,y) = U(S(x,y), T(x,y))$$

For semilinear PDEs or for homogeneous quasilinear PDE, if the initial curve is not too intricate (e.g. straight lines or simple enough function forms), it is equivalent to solve combinations of the ODEs $(\star)$

and it is common to write the characteristic equations as:

$$\frac{dy}{dx} = \frac{B(x,y,u)}{A(x,y,u)} \quad \text{and} \quad \frac{du}{dx} = \frac{C(x,y,u)}{A(x,y,u)} \quad \textbf{or} \quad \frac{dx}{dy} = \frac{A(x,y,u)}{B(x,y,u)} \quad \text{and} \quad \frac{du}{dy} = \frac{C(x,y,u)}{B(x,y,u)}$$

Method of characteristics for transport equations

In Chapter 2, we put a particular focus on **transport equations**, i.e. first-order quasilinear problems of the form:

$$\begin{aligned} &\frac{\partial u}{\partial t} + c(x,t,u)\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ &u(x,0) = u_0(x) \end{aligned}$$

Here, the curve γ is the $t = 0$ axis. This problem can be linear with constant coefficient if $c = \text{constant}$, linear with variable coefficient if $c(x,t)$. The special case where $c(x,t,u) = u$ is known as the inviscid Burger's equation. This type of problems lends itself well to the following method of characteristics:

S1. Write the characteristic equations as

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = c(x,t,u), \quad x(0) = \xi$$

where $\xi \in \mathbb{R}$ is a real parameter for the characteristics.

S2. Obtain the equation of the characteristics by solving for $x(t)$ for a given ξ, draw the diagram of characteristics. The set of characteristics is obtained by varying ξ.

S3. Check for signatures of shocks (characteristics crossing), rarefaction fans (region of the (x,t)-plane not covered by characteristics).

S4. An implicit solution is obtain by writing that $u = u_0(\xi) = \text{constant}$ along the characteristic $x = x(t)$ which depends on ξ.

S5. Eliminate ξ to obtain an explicit solution in the form $u = f(x,t)$, complete the solution by dealing with rarefaction fans if needed; your solution is valid potentially up to the time a shock forms.

S6. (Optional) If a shock forms, apply the Rankine-Hugoniot condition (or another shock fitting method) to for the shock path and extend your solution after the shock has formed.

Note that we have seen in this module that the method of characteristics can fail. This happens in the following cases:

- ☐ The characteristic curves intersect the initial curve more than once;
- ☐ The characteristic curves intersect one another at a point in the domain (i.e. a shock forms);
- ☐ The initial curve Γ corresponds exactly to a characteristic curve.

Remark. *In this module, we have only used the method of characteristics to solve first-order PDEs but note that it can be extended to solve systems of first-order PDEs and more generally, any hyperbolic PDE which are readily transformed into sets of first-order PDEs (similarly to what you have seen for ODEs). While we have only seen the method applied to linear, semilinear and quasilinear equations in this module, it can be adapted to deal with fully nonlinear first-order PDEs.*

The method of separation of variables

Method of separation of variables

For **second-order linear PDEs in two independent variables (e.g. 1D heat and wave equations or 2D Laplace problems) on finite domains**, we have used the following method:

S1. Check that the PDE is linear and homogeneous. The method can be extended to non-homogeneous problems but this is outside the scope of what we have seen in this module.

S2. Check that the boundary conditions are of the right form. This depends on the problem:

- ☐ If you have initial conditions (e.g. heat/wave equations), check that all the BCs are linear and homogeneous.
- ☐ If you have no initial conditions (e.g. Laplace's equation), check that all but one of the BCs (at most) are linear and homogeneous.
- ☐ In some cases (such as Laplace's equation solved on a disk), a BC will take the form of requiring that the solution remains finite, just ensure this boundary condition is met.

S3. If the boundary conditions are not in the right form, use the subtraction/superposition principle to get to one or more problems with BCs in the right form, e.g.:

- ☐ If you have initial conditions, first ignore them and find the steady-state solution which respects the boundary conditions $U(x)$. Use the subtraction principle to transform your problem with homogeneous boundary conditions for $v(x,t) = u(x,t) - U(x)$.
- ☐ If you only have boundary conditions (Laplace problem), then use the superposition principle to write the problem as a superposition of multiple problems with only one nonhomogeneous boundary condition.

S4. Write the solution as a product of two function each as a function of only one of the variables in the problem, e.g. $u(x,t) = X(x)T(t)$ or $u(x,y) = X(x)Y(y)$.

S5. Plug the separated solution into the PDE, separate variables and introduce a separation constant λ, hence producing two ODEs.

S6. Plug the separated solution into the homogeneous boundary conditions to obtain boundary conditions for one of the ODEs.

S7. One of the ODEs is thus a boundary value problem, say the ODE for X. Solve this problem to determine the eigenvalues λ_n and eigenfunctions $X_n(x)$ for the problem (i.e. the values that the separation constant can take and the associated solutions). Note that this can be very difficult depending on the geometry, this is one of the limitation of this method.

S8. Solve the second ordinary differential equation using any remaining homogeneous boundary conditions to simplify the solution if possible to obtain functions $T_n(t)$ or $Y_n(y)$.

S9. Use the Principle of Superposition to write down a solution to the PDE that will satisfy the PDE itself and the homogeneous boundary conditions:

$$u(x,t) = \sum_n A_n X_n(x) T_n(t) \quad \text{or} \quad u(x,y) = \sum_n A_n X_n(x) Y_n(y)$$

S10. Impose the remaining conditions (i.e. the initial condition(s) or a single nonhomogeneous boundary condition) and use the orthogonality of the eigenfunctions $X_n(x)$ to find the coefficients A_n.

Remark. *A few remarks about the method of separation of variables:*

- ☐ *Note that each of the partial differential equations only involved two variables. The method can often be extended out to more than two variables, but the work in those problems can be quite involved and so we didn't cover any of that here.*

- ☐ *Note that in most of our examples the eigenfunctions $X_n(x)$ ended up being sines and/or cosines. However, as we have seen in Problem Sheet 6, this is not always the case! You can assume that granted the right conditions on the boundary value problem, we can ensure that the eigenfunctions will always be orthogonal and that for two integers $n \neq m$,*

$$\int_{\mathcal{I}} X_n(x) X_m(x) dx = 0$$

where I is the interval of definition of our problem, i.e. $\mathcal{I} = [0, L]$, $\mathcal{I} = [a, b]$, $\mathcal{I} = [-\pi, \pi]$ etc.

The integral transforms method

Integral transforms method

For **second-order linear PDEs in two independent variables (e.g. 1D heat and wave equations or 2D Laplace problems) on infinite or semi-infinite domains**, we have used a method based on Fourier transforms. It follows the following step:

S1. Take a spatial Fourier transform of the PDE and the associated initial and boundary conditions. If the spatial domain of definition of the PDE is infinite (e.g., $-\infty < x, \infty$), use a Fourier transform. If the spatial domain of definition of the PDE is semi-infinite (e.g., $0 < x, \infty$), use a Fourier cosine or sine transform. The choice of Fourier cosine or sine transform should be guided by the boundary condition (e.g. in $x = 0$):

- ☐ If you are provided with a Dirichlet BC, then use a Fourier sine transform.
- ☐ If you are provided with a Neumann BC, then use a Fourier cosine transform.

This is due to the fact that the Fourier cosine (resp., sine) transform of the second order derivatives of a function f involve the knowlegde of $f'(0)$ (resp., $f(0)$) (see Appendix A of the Lecture Notes).

S2. In Fourier space, the problem reduces to an ODE problem: either an initial value problem (for the heat/wave equation) or a boundary value problem (for the Laplace equation). Solve this ODE problem, to obtain the solution in Fourier space.

S3. Apply an inverse Fourier transform to the solution to obtain the final solution in real-space.

Remark. *One of the clear limitations of the integral transform methods is to know whether we are able to inverse Fourier transform the solution at the last step. This can be a very arduous step!*

D'Alembert's solution

Consider the following initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathcal{I}, \quad t > 0$$
$$u(x,0) = f_1(x), \quad x \in \mathcal{I}$$
$$\frac{\partial u}{\partial t}(x,0) = f_2(x), \quad x \in \mathcal{I}$$

where $\mathcal{I}$ can be a finite interval or infinite. D'alembert's solution for this problem is

$$\boxed{u(x,t) = \frac{1}{2}\left[f_1(x-ct) + f_1(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi}$$

Green's function solution

Green's function solution

For **Laplace and Poisson problems**

$$\nabla^2\phi = f(\mathbf{r}), \quad \mathbf{r} \in V$$

through some volume V and subject to Dirichlet boundary conditions $\psi(\mathbf{r}) = g(\mathbf{r})$ on ∂V, we can write the solution as

$$\psi(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r}) dV + \int_{\partial V} g(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_0) dS$$

where $G(\mathbf{r}, \mathbf{r}_0)$ is the Green's function for the Laplacian, i.e. the solution of the following problem

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{in } V$$
$$G = 0 \quad \text{on } \partial V$$

Solving this problem thus boils down to being able to compute the appropriate Green's function $G(\mathbf{r}, \mathbf{r}_0)$. For our Poisson problem, a method to find the Green's function follows these steps:

S1. Compute the so-called "free space" Green's function for your linear differential operator, i.e. the function that satisfies $\nabla^2 G = \delta(\mathbf{r})$ and tends to 0 as $r \to 0$;

S2. Shift the origin to obtain the Green's function satisfying $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$, this Green's function does not yet respect the homogeneous Dirichlet boundary condition;

S3. Use the method of images to find the modified Green's function which respects the homogeneous Dirichlet boundary condition.

If we wish to take V to be "all space" and the boundary condition to be such that $\psi \to 0$ as $|\mathbf{r}| \to \infty$, then one simply uses the free-space Green's function for G and the solution is then given by

$$\psi(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r}) dV$$

Remark. *Note that Green's function solutions are much more general than what we have had the time to see in this module. First, a Green's function solution can be derived for Neumann problems, in which case the Green's function is defined with a homogeneous Neumann boundary condition. It can also be used for the heat or wave equation; in the case of the heat equation, we have seen in the lectures that the heat kernel is the Green's function of the linear differential operator $\mathcal{L} = \partial/\partial t - \nabla^2$.*

Similarity solution

Similarity solution

Similarity solutions to PDEs are solutions which depend on certain groupings of the independent variables, rather than on each variable separately. Finding these groupings of independent variables may come from dimensional analysis for instance or from invariance of the problem under space and time transformations.
For a PDE in two independent variables (x, y) and one dependent variable u, we follow the following steps:

S1. we consider the following dilation transformation:

$$\{\tilde{x} = ax, \tilde{y} = a^{\beta} y, \tilde{u} = a^{\gamma} u\}$$

where a is a positive real constant.

S2. Under this transformation, the derivatives transform as follows:

$$\begin{cases} \partial_x = a\partial_{\tilde{x}} \\ \partial_{xx} = a^2 \partial_{\tilde{x}\tilde{x}} \\ \partial_y = a^{\beta} \partial_{\tilde{y}} \\ \partial_{yy} = a^{2\beta} \partial_{\tilde{y}\tilde{y}} \\ \dots \end{cases}$$

S3. Re-inject these in the original PDE to obtain a PDE in terms of $\tilde{x}$, $\tilde{y}$ and $\tilde{u}$. Do the same for the initial and/or boundary conditions.

S4. Find the values of the exponents β and γ such that the PDE problem is left invariant under this transformation, i.e. the values of β and γ such that we obtain the same equation and associated conditions, only expressed in tilde quantities.

S5. If that's possible then if u is solution of the original equation, then so is $\tilde{u}(\tilde{x}, \tilde{y}) = a^{\gamma} u(ax, a^{\beta} y)$. If our PDE problem is invariant under the above transformation, we know that a similarity solution is of the form

$$u(x, y) = y^{\gamma/\beta} f(\eta) \quad \text{with } \eta = xy^{-1/\beta}$$

S6. Re-inject this ansatz in the original PDE to obtain an ODE for $f(\eta)$ and translate the initial/boundary conditions into conditions for the ODE problem.

S7. If possible, solve the ODE problem to obtain the final solution $u(x, y)$ to the original PDE problem.

Remark. *While this method seems very general, it is not always possible to find similarity solutions to all PDE problems. Further, while it may be possible to highlight a transformation which will reduce our PDE problem to an ODE problem, the ensuing ODE problems may be very nasty to solve (highly nonlinear for instance)!*