IMPERIAL

MATH70019

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Theory of Partial Differential Equations

Date: Friday, May 31, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the Cauchy problem given by the following scalar conservation law,

$$\begin{cases} \partial_t u + \partial_x q(u) &= 0, \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) &= g(x), \quad \text{for } x \in \mathbb{R}. \end{cases}$$

Let the flux be $q(u) = u \log(u)$, and the initial datum

$$g(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ e^{-1}, & \text{if } x > 0. \end{cases}$$

(a) Are there shocks? If so, compute the shock curve. (8 marks)

(b) Draw the characteristics and find the unique entropy solution. (12 marks)

(Total: 20 marks)

2. Let $\kappa > 0$ and consider the diffusion equation in $\mathbb{R}^3$,

$$\begin{cases} \partial_t u - \kappa \Delta u &= 0, \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ u(0,x) &= u_0(x), \quad \text{for } x \in \mathbb{R}^3. \end{cases}$$

(a) Use the energy functional

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |u(t,x)|^2 \mathrm{d}x$$

to prove that the Cauchy problem has at most one solution in the class of smooth functions, that are rapidly decaying at infinity with rapidly decaying derivatives.

(8 marks)

(b) Let $B_1 = \{x \in \mathbb{R}^3 : |x| < 1\}$ be the ball of radius 1 centred at the origin, and assume that initially u has the form

$$u_0(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 0 & \text{if } x \notin B_1. \end{cases}$$

Find an explicit expression for the solution to the Cauchy problem and show that $|u(t,x)| \to 0$ as $t \to \infty$ for all $x \in \mathbb{R}^3$.

(8 marks)

(c) What is the regularity of the solution if $t = 0$? And if $t > 0$?
What is the support of the solution if $t = 0$? And if $t > 0$? (4 marks)

(Total: 20 marks)

3. Let $c \in \mathbb{R}$, and assume that $\Phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ smooth satisfies the one dimensional wave equation

$$\partial_{tt}\Phi - c^2 \partial_{xx}\Phi = 0,$$

for all $t > 0$ and $x \in \mathbb{R}$.

(a) Prove that $u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ which is radially symmetric and defined by

$$u(t, \mathbf{x}) = \frac{\Phi(t, |\mathbf{x}|)}{|\mathbf{x}|}$$

satisfies the three dimensional wave equation[1] for all $t > 0$ and $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$. (6 marks)

(b) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth odd function. Deduce from Part (a) that u defined as

$$u(t, \mathbf{x}) = \frac{\varphi(|\mathbf{x}| - ct) + \varphi(|\mathbf{x}| + ct)}{|\mathbf{x}|},$$

satisfies the three dimensional wave equation for all $t > 0$ and $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$. (6 marks)

(c) Using Part (b), find a radially symmetric smooth solution $u(t, |\mathbf{x}|)$ to the Cauchy problem

$$\begin{cases} \partial_{tt}u - c^2 \Delta u &= 0, \\ u(0, \mathbf{x}) &= |\mathbf{x}|^2, \\ \partial_t u(0, \mathbf{x}) &= 0, \end{cases}$$

for $t > 0$ and $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$. Is the smooth solution well-defined also for $\mathbf{x} = 0$? (8 marks)

(Total: 20 marks)

[1] Recall that the Laplace operator in 3D spherical coordinates is given by

$$\Delta u = \partial_{rr}u + \frac{2}{r}\partial_r u + \frac{1}{r^2 \tan\theta}\partial_\theta u + \frac{1}{r^2}\partial_{\theta\theta}u + \frac{1}{r^2 \sin^2\theta}\partial_{\phi\phi}u.$$

4. Let $Q = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ be the two dimensional unit square. Let $f \in C^4([0, 1])$ with $f(0) = f(1) = 0$. Consider the Laplace equation in the unit square

$$\Delta u = 0 \quad \text{in } Q,$$

with boundary conditions

$$\begin{cases} u(0, y) = u(1, y) = 0, & y \in (0, 1), \\ u(x, 0) = 0, & x \in (0, 1), \\ u(x, 1) = f(x), & x \in (0, 1). \end{cases}$$

(a) Find a candidate solution to the Cauchy problem using the separation of variables technique in terms of Fourier series of the initial datum.

(10 marks)

(b) Use the maximum principle to prove that the solution from Part (a) is the unique classical solution to the Cauchy problem. (5 marks)

(c) Prove that the (unique) solution u is a minimiser of the Dirichlet energy functional

$$E[w] = \frac{1}{2} \int_Q |\nabla w|^2 dxdy$$

among all functions $w \in C^2(Q) \cap C(\overline{Q})$ that satisfy the same boundary conditions on ∂Q. (5 marks)

(Total: 20 marks)

5. *The Lamb–Oseen vortex.* Given $\nu > 0$, we look for a scalar function $\omega(t, \mathbf{x})$ and a vector field $\mathbf{u}(t, \mathbf{x})$ that satisfy the two dimensional Navier-Stokes equation in vorticity formulation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega, \quad \text{with } t > 0,\ \mathbf{x} = (x, y) \in \mathbb{R}^2.$$

In addition, $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$ are related via $\omega(t, \mathbf{x}) = \partial_x u_2(t, \mathbf{x}) - \partial_y u_1(t, \mathbf{x})$.

(a) Let δ_0 be a Dirac distribution at the origin in $\mathbb{R}^2$. Find a solution to the diffusion equation

$$\partial_t \omega - \nu \Delta \omega = 0,$$

with initial condition $\omega(0, \mathbf{x}) = \delta_0(\mathbf{x})$. What is the regularity of the solution $\omega(t, \mathbf{x})$ for $t > 0$?

(6 marks)

(b) Consider the vector field $\mathbf{u}$ given by

$$\mathbf{u}(t, \mathbf{x}) = \frac{\mathbf{x}^\perp}{2\pi |\mathbf{x}|^2} \left(1 - e^{-\frac{|\mathbf{x}|^2}{4\nu t}}\right),$$

where $\mathbf{x}^\perp = (-y, x)$. Prove that $\omega(t, \mathbf{x}) = \partial_x u_2(t, \mathbf{x}) - \partial_y u_1(t, \mathbf{x})$. (6 marks)

(c) Prove that $\mathbf{u}(t, \mathbf{x})$ and $\nabla \omega(t, \mathbf{x})$ are perpendicular for all $t > 0$ and all $x \in \mathbb{R}^2$, and conclude that $(\omega, \mathbf{u})$ is a solution to the two dimensional Navier-Stokes equation. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60019/MATH70019

Theory of PDEs (Solutions)

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1. (a) First of all, we compute the derivative of the flux $\partial_x q(u) = (\log(u) + 1)\partial_x u$. For the initial datum we define the curve $\gamma(\tau) = (0, \tau)$ such that $u(\gamma(\tau)) = g(\tau)$. Then, the characteristics satisfy the ODE

meth seen ⇓

6, A

2, B

$$\begin{cases} \frac{dt}{ds} = 1, & t(0,\tau) = 0, \\ \\ \frac{dx}{ds} = \log(z) + 1, & x(0,\tau) = \tau, \\ \\ \frac{dz}{ds} = 0, & z(0,\tau) = g(\tau). \end{cases}$$

The solution to this coupled ODE is given by $t(s,\tau) = s$, $z(s,\tau) = g(\tau)$, and $x(s,\tau) = (\log g(\tau) + 1)s + \tau$. The equation of the characteristics is given by

$$x = (\log g(\tau) + 1)t + \tau.$$

Now we substitute the value of the initial datum $g(\tau)$, that is defined piecewise.

* If $\tau \leq 0$, then $u(t,x) = g(\tau) = 1$ and $x = t + \tau$.
* If $\tau > 0$, then $u(t,x) = g(\tau) = e^{-1}$ and $x = \tau$.

Therefore, yes, there are shocks: for any $\tau_1 \leq 0$ and any $\tau_2 > 0$ there exists $t_1 \geq 0$ such that $\tau_1 + t_1 = \tau_2$. In order to determine the equation of the shock curve we use Rankine-Hugoniot condition, since we are looking for a weak solution. If the shock curve is parametrised by $x = \sigma(t)$, we find that it must satisfy the relation

$$\sigma'(t) = \frac{q(u_+(t,\sigma(t))) - q(u_-(t,\sigma(t)))}{u_+(t,\sigma(t)) - u_-(t,\sigma(t))},$$

where $u_+ = e^{-1}$ and $u_- = 1$. Hence, we obtain that the shock curve is a straight line arising from the origin and given by

$$\sigma(t) = \frac{t}{e-1}.$$

(b) Thus, the solution we found is given by

meth seen ⇓

4, A

6, B

2, D

$$u(t,x) = \begin{cases} 1, & \text{if } x \leq \sigma(t), \\ e^{-1}, & \text{if } x > \sigma(t). \end{cases}$$

Finally, we must check that this solution satisfies the entropy condition, that is given by

$$q'(u_+(t,\sigma(t))) < \sigma'(t) < q'(u_-(t,\sigma(t))),$$

where $q'(u) = \log(u) + 1$. We obtain that it is indeed satisfied since $q'(u_+(t,\sigma(t))) = 0$, $q'(u_-(t,\sigma(t))) = 1$ and $\sigma'(t) = (e-1)^{-1} \in (0,1)$. All in all, we found the unique entropy solution since the flux q is a convex function and since the initial datum g is bounded.

2. (a) Suppose that there are two classical solutions $u, v \in C^\infty((0,\infty)\times\mathbb{R}^3)$ to the same Cauchy problem, then $w = u - v \in C^\infty((0,\infty)\times\mathbb{R}^3)$ satisfies the diffusion equation with initial datum $w(0,x) = 0$. Computing the time derivative of the energy functional of w, we obtain

part seen ⇓

2, A

2, B

4, D

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{\mathbb{R}^3} w(t,x)\partial_t w(t,x)dx = \kappa\int_{\mathbb{R}^3} w(t,x)\Delta w(t,x)dx\\ &= \kappa\lim_{r\to\infty}\int_{\partial B_r(0)} w(t,x)\nabla w(t,x)\cdot\mathbf{n}(x)d\gamma(x) - \kappa\int_{\mathbb{R}^3}|\nabla w(t,x)|^2dx\end{aligned}$$

Using the assumption that both $|u(t,x)|\to 0$ and $|v(t,x)|\to 0$ as $|x|\to\infty$ fast enough for all $t\geq 0$, we find that the boundary integral satisfies

$$\begin{aligned}\left|\int_{\partial B_r(0)} w(t,x)\nabla w(t,x)\cdot\mathbf{n}(x)d\gamma(x)\right| &\leq |\partial B_r(0)|\sup_{|x|=r}\left(|w(t,x)||\nabla w(t,x)|\right)\\ &= 4\pi r^2\sup_{|x|=r}\left(|w(t,x)||\nabla w(t,x)|\right)\\ &\to 0 \quad \text{as } r\to\infty.\end{aligned}$$

Hence, the boundary term vanishes, and the derivative of the energy functional satisfies

$$\frac{d}{dt}E(t) = -\kappa\int_{\mathbb{R}^3}|\nabla w(t,x)|^2dx \leq 0.$$

However, $E(t)\geq 0$ and $E(0) = 0$, therefore $E(t) = 0$ for all $t\geq 0$. Since w is C^1 in t and C^2 in x, there holds that $u(t,x) = v(t,x)$ for all $t\geq 0$ and $x\in\mathbb{R}^3$.

part seen ⇓

2, A

6, C

(b) An explicit solution to this problem is found through a convolution with the fundamental solution to the heat equation,

$$u(t,x) = (\Gamma(t,\cdot)*u_0)(x) = \frac{1}{(4\pi\kappa t)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4\kappa t}}u_0(y)dy.$$

Using the definition of the initial datum we obtain

$$u(t,x) = \frac{1}{(4\pi\kappa t)^{3/2}}\int_{B_1} e^{-\frac{|x-y|^2}{4\kappa t}}dy.$$

Moreover, with the explicit expression of the solution in hand, and taking absolute value, we find the upper bound

$$|u(t,x)| \leq \frac{1}{(4\pi\kappa t)^{3/2}}\int_{B_1}\left|e^{-\frac{|x-y|^2}{4\kappa t}}\right|dy \leq \frac{1}{(4\pi\kappa t)^{3/2}}|B_1| = \frac{|B_1|}{(4\pi\kappa t)^{3/2}}.$$

All in all we obtain,

$$|u(t,x)| \leq \frac{|B_1|}{(4\pi\kappa t)^{3/2}} \to 0$$

as $t\to\infty$.

sim. seen ⇓

$\boxed{4, \text{A}}$

(c) Initially ($t = 0$), the solution is defined by the constant (and strictly positive) value $3/(4\pi)$ if $x \in B_1$, and 0 otherwise. Therefore the solution is bounded but not continuous. In addition it is clear that it is supported in the ball $B_1 \subset \mathbb{R}^3$.

For any positive time $t > 0$, the solution is given by

$$u(t,x) = \frac{3}{(4\pi)^{5/2}(\kappa t)^{3/2}} \int_{B_1} e^{-\frac{|x-y|^2}{4\kappa t}} \, dy,$$

that is a C^k function for every $k \in \mathbb{N}$ in both variables t and x, namely we say that u is C^∞ in $(0,\infty) \times \mathbb{R}^3$. Moreover, this function is strictly positive for any $t > 0$ and $x \in \mathbb{R}^3$, thus the support of the solution is the full space $\mathbb{R}^3$.

3. (a) From now on we will write $r = |\mathbf{x}|$. Then taking derivatives we find, meth seen ⇓

3, A

3, B

$$\partial_{tt}u(t,r) = \frac{1}{r}\partial_{tt}\Phi(t,r),$$

$$\partial_r u(t,r) = \frac{1}{r}\partial_r\Phi(t,r) - \frac{1}{r^2}\Phi(t,r),$$

$$\partial_{rr}u(t,r) = \frac{1}{r}\partial_{rr}\Phi(t,r) - \frac{2}{r^2}\partial_r\Phi(t,r) + \frac{2}{r^3}\Phi(t,r).$$

Now, using the definition of the Laplacian in spherical coordinates and taking into account that u is radially symmetric,

$$\Delta u(t,r) = \partial_{rr}u(t,r) + \frac{2}{r}\partial_r u(t,r) = \frac{1}{r}\partial_{rr}\Phi(t,r),$$

and therefore

$$\partial_{tt}u(t,r) - c^2\Delta u(t,r) = \frac{1}{r}\partial_{tt}\Phi(t,r) - \frac{c^2}{r}\partial_{rr}\Phi(t,r) = 0$$

for all $t > 0$ and all $r > 0$ since $\Phi(t,r)$ satisfies the one-dimensional wave equation. meth seen ⇓

(b) Given a smooth function φ of one real variable, we know from Part (a) that

3, A

3, B

$$u(t,r) = \frac{\varphi(r-ct) + \varphi(r+ct)}{r},$$

satisfies the three dimensional wave equation provided that

$$\Phi(t,r) = \varphi(r-ct) + \varphi(r+ct)$$

satisfies the one dimensional wave equation. This is indeed the situation since

$$\partial_{tt}\Phi(t,r) = c^2(\varphi''(r-ct) + \varphi''(r+ct)),$$

$$\partial_{rr}\Phi(t,r) = \varphi''(r-ct) + \varphi''(r+ct),$$

hence $\partial_{tt}\Phi(t,r) - c^2\partial_{rr}\Phi(t,r) = 0$. unseen ⇓

(c) We look for a solution to the Cauchy problem of the form

8, D

$$u(t,r) = \frac{\varphi(r-ct) + \varphi(r+ct)}{r},$$

since we already know from Part (b) that it satisfies the three dimensional wave equation. We see that the homogeneous initial condition for $\partial_r u(0,r)$ are trivially satisfied. For the remaining initial condition notice that

$$u(0,r) = \frac{2\varphi(r)}{r} = r^2,$$

hence we find the odd and smooth function

$$\varphi(r) = \frac{r^3}{2}.$$

Putting everything together, we obtain the solution

$$u(t,r) = \frac{(r-ct)^2 + (r+ct)^2}{2} = r^2 + (ct)^2$$

for all $t > 0$ and all $r > 0$. The extension to $r = 0$ in a smooth way is trivial.

4. (a) We are looking for a separated variables solution to the Laplace equation in Q. We propose hence a solution of the form $u(x,y) = v(x)w(y)$, so computing the Laplacian of u we find

meth seen $\Downarrow$

$\boxed{2, \text{A}}$

$\boxed{6, \text{C}}$

$\boxed{2, \text{D}}$

$$v''(x)w(y) + v(x)w''(y) = 0.$$

Then, there must exist a constant $\lambda \in \mathbb{R}$ such that

$$v''(y) = \lambda v(y), \quad w''(y) = -\lambda w(y).$$

If $\lambda = 0$, we can solve the ODEs with an affine equation, that is, for some constants $a, b, c, d \in \mathbb{R}$,

$$v(x) = ax + b, \quad w(y) = cy + d.$$

However, imposing $u(x,1) = f(x)$, there yields that unless f is linear in x, there is no solution for $\lambda = 0$. Notice that $f \not\equiv 0$ cannot be linear since $f(0) = f(1) = 0$. If $\lambda = \mu^2 > 0$, we find that $v'' = \mu^2 v$ yields a solution of the form

$$v(x) = ae^{-\mu x} + be^{\mu x},$$

where $a, b \in \mathbb{R}$ are to be chosen. Imposing the boundary conditions $v(0) = 0$ and $v(1) = 0$, we obtain that the constants must satisfy $a = -b$, and $a(e^{-\mu} - e^{\mu}) = 0$. This is impossible for $a, \mu \neq 0$. Finally, for $\lambda = -\mu^2 < 0$, we find that $v'' = -\mu^2 v$ yields a solution of the form

$$v(x) = a\cos(\mu x) + b\sin(\mu x).$$

Then, $v(0) = v(1) = 0$ yields that the constants must be given by $a = 0$, and $b\sin(\mu) = 0$, that is, we find a solution for every $n \in \mathbb{N}$ with $\mu = n\pi$. In addition, regarding the ODE for w with $\lambda = -n^2\pi^2$ we find a solution to the ODE for v in terms of exponential functions. All in all, putting everything together, we find via separated variables solutions of the form

$$u_n(x,y) = b_n(e^{n\pi y} - e^{-n\pi y})\sin(n\pi x) = 2b_n \sinh(n\pi y)\sin(n\pi x)$$

for any $n \in \mathbb{N}$. In general, using the superposition principle we can, formally for now, look for solutions in the form of infinite series

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi y)\sin(n\pi x). \tag{1}$$

The last boundary condition yields the condition

$$f(x) = \sum_{n=1}^{\infty} b_n \sinh(n\pi)\sin(n\pi x),$$

which must be ensured by the coefficients b_n. In order to do so we write f in terms of its Fourier series, the convergence of the series comes from the regularity, since $f \in C^4$ and $\partial_{xx} f \in C^2$. In addition, since $f(0) = 0$ we can consider its odd extension, so that the Fourier series will be given exclusively in terms of sines,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x),$$

with coefficients

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)dx.$$

To sum up, we find that the solution to the Cauchy problem is given by (1) where the b_n coefficients must satisfy

$$b_n = \frac{2}{\sinh(n\pi)}\int_0^1 f(x)\sin(n\pi x)dx$$

for all $n \in \mathbb{N}$. $\boxed{\text{sim. seen} \Downarrow}$

(b) Assume that there exist two solutions $u \not\equiv v$ to the Cauchy problem with same boundary conditions. This means that $w = u - v$ is a harmonic function and satisfies $\boxed{2, \text{A}}$ $\boxed{3, \text{B}}$

$$\begin{cases} \Delta w &= 0 \quad \text{in } Q, \\ w &= 0 \quad \text{on } \partial Q. \end{cases}$$

If we assume that w is not constant, then by the maximum principle for harmonic functions, for every $(x, y) \in Q$

$$\min_{\partial Q} u < u(x, y) < \max_{\partial Q} u.$$

However $\min_{\partial Q} u = \max_{\partial Q} u = 0$, which yields a contradiction coming from the fact that we were assuming w to be not constant. Hence, it is constant and equal to zero, namely $u \equiv v$. $\boxed{\text{seen} \Downarrow}$

(c) To prove that u is a minimiser of the Dirichlet energy consider a general $C^2(Q) \cap C(\overline{Q})$ function of the form $w = u + tv$, where $v \in C^2(Q) \cap C(\overline{Q})$ satisfies $v = 0$ on ∂Q, and $t \in \mathbb{R}$. Then, $\boxed{4, \text{A}}$ $\boxed{1, \text{B}}$

$$E[w] = E[u + tv] = E[u] + t^2 E[v] + t\int_Q \nabla u \cdot \nabla v dxdy$$

Using the boundary conditions for v we can integrate by parts in Q to obtain,

$$\int_Q \nabla u \cdot \nabla v dxdy = -\int_Q v\Delta u dxdy = 0$$

since u is harmonic. Hence $E[u + tv] = E[u] + t^2 E[v] \geq E[u]$ for all $t \in \mathbb{R}$ and $v \in C^2(Q) \cap C(\overline{Q})$ with $v = 0$ on ∂Q. This implies that $E[w] \geq E[u]$ for all admissible w, so u is a minimiser of the Dirichlet energy.

5. (a) The fundamental solution to the heat equation in $\mathbb{R}^2$ is the unique solution to the Cauchy problem

seen ⇓

6, M

$$\partial_t \omega - \nu \Delta \omega = 0, \quad \omega(0, \mathbf{x}) = \delta_0(\mathbf{x}),$$

that vanishes at infinity, and it is given by

$$\omega(t, \mathbf{x}) = \frac{1}{4\pi\nu t} e^{-\frac{|\mathbf{x}|^2}{4\nu t}}.$$

This function is smooth for all $\mathbf{x} \in \mathbb{R}^2$ and all $t > 0$.

sim. seen ⇓

6, M

(b) For this part we omit the dependence on t in the definition of the functions. Let us write the vector field $\mathbf{u}(\mathbf{x}) = \mathbf{x}^\perp F(|\mathbf{x}|)$, where

$$F(|\mathbf{x}|) = \frac{1}{2\pi|\mathbf{x}|^2} \left(1 - e^{-\frac{|\mathbf{x}|^2}{4\nu t}}\right).$$

Then $\mathbf{u} = (u_1, u_2)$ is given by $u_1(\mathbf{x}) = -yF(|\mathbf{x}|)$ and $u_2(\mathbf{x}) = xF(|\mathbf{x}|)$. A direct computation yields that

$$\partial_x u_2(\mathbf{x}) - \partial_y u_1(\mathbf{x}) = 2F(|\mathbf{x}|) + |\mathbf{x}|G(|\mathbf{x}|),$$

where

$$G(|\mathbf{x}|) = \frac{1}{4\pi\nu t|\mathbf{x}|} e^{-\frac{|\mathbf{x}|^2}{4\nu t}} - \frac{1}{\pi|\mathbf{x}|^3} \left(1 - e^{-\frac{|\mathbf{x}|^2}{4\nu t}}\right).$$

Namely, $|\mathbf{x}|G(|\mathbf{x}|) = \omega(\mathbf{x}) - 2F(|\mathbf{x}|)$, so we obtain the desired result.

unseen ⇓

8, M

(c) Notice that if $\mathbf{u}$ and $\nabla\omega$ are perpendicular for all $t > 0$ and $\mathbf{x} \in \mathbb{R}^2$, then $\mathbf{u} \cdot \nabla\omega = 0$ for all $t > 0$ and $\mathbf{x} \in \mathbb{R}^2$. This means that

$$\partial_t \omega + \mathbf{u} \cdot \nabla\omega - \nu\Delta\omega = \partial_t \omega - \nu\Delta\omega,$$

and it vanishes due to Part (a). Computing the gradient of ω we obtain,

$$\nabla\omega(t, \mathbf{x}) = -\frac{\mathbf{x}}{8\pi\nu^2 t^2} e^{-\frac{|\mathbf{x}|^2}{4\nu t}},$$

hence $\mathbf{u}$ is a vector in the direction of $\mathbf{x}^\perp = (-y, x)$, and $\nabla\omega$ is a vector in the direction of $\mathbf{x} = (x, y)$. These two vectors are perpendicular $\mathbf{x}^\perp \cdot \mathbf{x} = 0$, and thus we obtain that the pair $(\mathbf{u}, \omega)$ solves the two dimensional Navier Stokes equation in vorticity formulation with initial condition $\omega(0, \mathbf{x}) = \delta_0(\mathbf{x})$. This initial configuration in terms of a Dirac distribution is called *point vortex*. The solution that we found for any $t > 0$ is called *Lamb–Oseen vortex*.

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks