

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**  
**May-June 2022**

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Markov Processes**

Date: 27 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD  
WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS  
ANSWERED AND PAGE NUMBERS PER QUESTION.**

The probability space is  $(\Omega, \mathcal{F}, \mathbf{P})$  unless otherwise stated. Carefully justify your answers.

1. Let  $(x_n, n \geq 0)$  be a time homogeneous Markov process on the state space  $\mathcal{X} = \{1, \dots, 6\}$  with transition probabilities given by the stochastic matrix  $P$ :

$$P = \frac{1}{10} \begin{pmatrix} 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 3 & 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 & 0 & 6 \end{pmatrix}$$

- (a) Draw the incidence graph associated to  $P$  and classify the states  $\{1, \dots, 6\}$  into communication classes of recurrent states and transient states. (6 marks)
- (b) Compute  $\mathbf{P}(x_3 = 5 | x_1 = 3)$  and  $\mathbf{P}(x_4 = 6 | x_0 = 3)$ . (4 marks)
- (c) What are all the invariant probability measures for  $P$ ? (10 marks)

(Total: 20 marks)

2. Let  $\{\xi_n\}_{n \geq 0}$  be a sequence of independent identically distributed  $\mathcal{N}(0, 1)$  random variables and  $x_0$  is a real valued random variable independent of  $\{\xi_n\}_{n \geq 0}$ . Let  $(x_n)$  be the Markov process defined recursively by

$$x_{n+1} = \frac{x_n}{2} + \xi_n .$$

- (a) Write an expression for  $x_1$ ,  $x_2$ , and  $x_3$  in terms of  $x_0$  and the  $\xi_i$ . Write an expression for  $x_n$  in terms of  $x_0$  and the  $\xi_i$ ,  $i = 0, \dots, n - 1$ . (4 marks)
- (b) Give an expression for the transition probabilities  $P(x, \cdot)$ . (6 marks)
- (c) Using your answer to part (a), give an expression for the  $n$ -step transition probabilities  $P^n(x, \cdot)$ . (5 marks)
- (d) Give an invariant probability measure for this process and show that it is unique. (5 marks)

(Total: 20 marks)

3. Let  $G$  be a finite group with  $e$  its identity element.

- (a) Define what is meant be a time homogeneous Markov chain on  $G$  to be left invariant. Show that this is equivalent that there exists a probability measure  $\bar{\mathbf{P}}$  on  $G$  such that  $P(x_{n+1} = h | x_n = g) = \bar{\mathbf{P}}(g^{-1}h)$ . (6 marks)
- (b) Show that the normalised counting measure  $\pi(g) = 1/|G|$  is an invariant measure for every left invariant random walk on  $G$ . (5 marks)

- (c) Show that a left invariant random walk on a group  $G$  is reversible (with respect to which measure?) if and only if  $P(e, g) = P(g, e)$  holds for all  $g \in G$ . (4 marks)
- (d) We shuffle a deck of cards by first cutting the deck into two piles and then weave the two packs together. We describe below a fairly realistic model for the inverse action of shuffling. We first identify the arrangement of cards with a permutation in the permutation space  $S_{52}$ . Given a pack of cards, assign 1 or 0 to each card with equal probability. Pick out the cards marked with 1 and put them together (respecting their original order) and similarly pile the cards with 0 together (respect their order also), then put one pile over the other, this results in a shuffle of the pack. We have just described a Markov chain on  $S_{52}$ . Show that this Markov chain has a unique invariant probability measure. Describe this measure and find  $\mathbf{E}_1 T_1$  where 1 is the unit element. (5 marks)

(Total: 20 marks)

4. Let  $\{\xi_n\}_{n \geq 0}$  be a sequence of i.i.d. random variables that are distributed uniformly in  $[0, 1]$  and consider the recursion relation

$$x_{n+1} = \sqrt[3]{x_n(1 + \xi_n)}.$$

- a. Show (by producing a counterexample) that the corresponding semigroup is not strong Feller. (**Hint:** The sign of  $x_{n+1}$  is the same as the sign of  $x_n$ .) (6 marks)
- b. Using the fact that  $\frac{2x}{1+2x} \leq \sqrt[3]{x} \leq \frac{x}{4} + 1$  for  $x \geq 0$ , show that the function  $V(x) = \frac{1}{x} + x$  is a Lyapunov function for this system, provided we restrict ourselves to  $x_0 \in (0, \infty)$ . (6 marks)
- c. Find the smallest interval  $J$  such that if  $x_0 > 0$ , then  $x_n \in J$  almost surely for all  $n$  large enough. (3 marks)
- d. Show there are precisely three ergodic invariant probability measures for the chain and identify their supports. You need to justify the number of ergodic invariant measures, but you do not need to find explicit expressions for these invariant measures and you do not need to justify rigorously the claims concerning their supports. (5 marks)

(Total: 20 marks)

5. (a) Let  $(\mathcal{X}, \mathcal{B}, \mathbf{P})$  be a probability space. What does it mean to say that  $\theta : \mathcal{X} \rightarrow \mathcal{X}$  is a measure preserving transformation and that  $P$  is ergodic with respect to a measurable transformation  $\theta$ .  
(4 marks)
- (b) let  $\mathcal{X}$  be a separable complete metric space and consider the path space  $(\mathcal{X}^Z, \mathcal{B}(\mathcal{X}^Z))$  and let  $\theta : \mathcal{X}^Z \rightarrow \mathcal{X}^Z$  denote the shift operator, it maps a sequence  $(a_n)_{n \geq 0}$  to  $(a_{n+1})_{n \geq 0}$ . Recall that if  $\mu$  is an invariant probability measure for the Markov chain, we can construct a two sided Markov chain with the transition probabilities of  $(x_n)$ . This determines a probability measure  $\mathbf{P}_\mu$  on the path space  $(\mathcal{X}^X, \mathcal{B}(\mathcal{X}^Z))$  and  $\theta$  is a measure preserving map on  $(\mathcal{X}^Z, \mathcal{B}(\mathcal{X}^Z), \mathbf{P}_\pi)$ .
- (i) Let  $\mathcal{X} = \mathbf{R}$  and let  $\{\xi_n\}$  be independent real valued random variables with standard Gaussian distribution. Let  $x_{n+1} = F(x_n) + \xi_{n+1}$  for  $n \geq 0$  where  $F : \mathbf{R} \rightarrow \mathbf{R}$  is a continuous function with  $|F(x)| \leq \frac{1}{2}(1 + |x|)$ . Show that there exists an invariant probability measure  $\pi$  such that the corresponding probability distribution  $\mathbf{P}_\pi$  on  $(\mathcal{X}^Z, \mathcal{B}(\mathcal{X}^Z))$  is ergodic (with respect to  $\theta$ ).  
(12 marks)
- (ii) Suppose that  $(x_n)$  is a Markov chain with a unique invariant probability measure  $\pi$  on  $\mathcal{X}$ . If  $x_0$  is distributed as  $\pi$ , show that  $\lim_{n \rightarrow \infty} x_n(\omega)$  exists either for almost surely every  $\omega$  or on a set of measure zero.  
(4 marks)

(Total: 20 marks)

Module: MATH96062/MATH97216/MATH97220  
Setter: Li  
Checker: Krasovsky  
Editor: editor  
External: external  
Date: June 6, 2022  
Version: Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2021

MATH96062/MATH97216/MATH97220 Markov Processes

*The following information must be completed:*

**Is the paper suitable for resitting students from previous years: No ( Maybe yes or no: I have changed notation, the resit student is following the current course.)**

**Category A marks: available for basic, routine material (excluding any mastery question)  
(40 percent = 32/80 for 4 questions):**

Q1 20 marks; Q2(a) 5 marks Q2(b) 5 marks, Q3(a) first part 2 marks

**Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):**

Q2c 5 marks, Q2a-second part 4 marks; Q3b 5 marks; Q4a 6 marks,

**Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):**

Q2d 5 marks, Q3c 4 marks; Q4c 3 marks

**Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):**

3(d) 5 marks, 4(b) 6 marks, 4(d) 5 marks

*Signatures are required for the final version:*

Setter's signature

Checker's signature

Editor's signature

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TEMPORARY FRONT PAGE

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2021

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science.

Markov Processes

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

The probability space is  $(\Omega, \mathcal{F}, \mathbf{P})$  unless otherwise stated. Carefully justify your answers.

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 (**Hint:** The sign of  $x_{n+1}$  is the same as the sign of  $x_n$ .) (6 marks)
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(4 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

**ExamModuleCode    QuestionNumber    Comments for Students**

	1	
Q1		Q1: generally well done. main errors are due to negligence in computations for part c
	2	
Q2		The main difficulty is to identify the invariant measure and verify the measure to be indeed an invariant probability measure.
	3	
Q3		This is bookwork except for the last part which is an application. This is mostly well done, occasionally affected by illogical arguments.
	4	
Q4		Main problem: Details not given in part a, calculation in part b is messy.
	5	
Q5		