Classical Dynamics **Autumn Term 2022**

Answers to Test 1

1. (a) A particle of unit mass moving in one dimension is subject to the conservative force

$$F = \frac{1}{x^2} - 1.$$

(i)

$$F = \frac{1}{x^2} - 1 = -\frac{dV}{dx},$$

which integrates to

$$V = \frac{1}{x} + x,$$

ignoring a constant of integration.

(ii) Taylor expanding the force about $x = 1$

$$F = \ddot{x} = \frac{1}{x^2} - 1 = F(1) + F'(1)(x-1) + ... = -2(x-1),$$

or $\ddot{z} = -2z$ where $z = x - 1$. The angular frequency satisfies $\omega^2 = 2$ or $\omega = \sqrt{2}$.

(iii) As the force is conservative

$$L = T - V = \frac{1}{2}\dot{x}^2 - \frac{1}{x} - x.$$

[9 marks]

(b)

$$L = \dot{x}\dot{y} - xy,$$

(i) The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\dot{y} + y = 0,$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{d}{dt}\dot{x} + x = 0,$$

that is $\ddot{x} = -x$, $\ddot{y} = -y$.

(ii) These are decoupled oscillator equations with general solution

$$x = A\cos(t+\alpha), \quad y = B\cos(t+\beta),$$

where A, B, α, β are arbitrary constants.

(iii) The components of the force are $F_x = \ddot{x} = -x$ and $F_y = \ddot{y} = -y$ which derive from the potential energy $V = \frac{1}{2}(x^2+y^2)$. The force is conservative.

[8 marks]

(c) A bead of mass m moves without friction or gravity on an expanding hoop of radius $R = t$ for $t > 0$.

(i) Here the holonomic constraint is $r = t$. As there are no external forces the unconstrained Lagrangian is

$$L = T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right),$$

where r and θ are polar coordinates. Imposing the constraint gives the Lagrangian

$$L = \frac{m}{2}\left(1 + t^2\dot{\theta}^2\right),$$

or

$$L = \frac{1}{2}mt^2\dot{\theta}^2,$$

on dropping the constant part.

(ii) The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}mt^2\dot{\theta} = 0,$$

so that $t^2\dot{\theta} = C$, a constant. Accordingly,

$$d\theta = \frac{C}{t^2}dt,$$

which integrates to

$$\theta = -\frac{C}{t} + D.$$

[8 marks]

[Total: 25 marks]