

≤

(1.3.1) Theorem. (Soundness of L)

Suppose ϕ is a theorem of L.
Then ϕ is a tautology.

(1.3.2) Notation

A (propositional) valuation v is
an assignment of truth values
to the prop. variables p_1, p_2, p_3, \dots
So $v(p_i) \in \{T, F\}$ (for $i \in \mathbb{N}$)

Using the truth table rules, this
assigns a truth value $v(\phi) \in \{T, F\}$

to every L-formula ϕ , satisfying

$$v((\neg \phi)) \neq v(\phi)$$

and $v((\phi \rightarrow \psi)) = F$

$$\Leftrightarrow v(\phi) = T \wedge v(\psi) = F$$

(for all fules ϕ, ψ).

Proof of 1.3.1 : By induction on
the length of a proof of ϕ it is
enough to show that

- (a) every axiom of L is a tautology
(b) MP preserves tautologies.

= (a) Use truth tables or argue as
follows. Do A2.

①

Suppose for a contradiction that v is a valuation with (2)

$$v((\phi \rightarrow (\psi \rightarrow x)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow x))) = F.$$

then $v((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow x)) = F \dots (1)$

& $v(\phi \rightarrow (\psi \rightarrow x)) = T \dots (2)$

By (1) $v((\phi \rightarrow x)) = F \dots (3)$

& $v((\phi \rightarrow \psi)) = T \dots (4)$

By (2) $v(\phi) = T \& v(x) = F$

By this and (4) $v(\psi) = T$.

this contradicts (2). $\#_{(a)}$

For A2.

Ex: Do A1 & A3. \cancel{X} .

(b) If ϕ and $(\phi \rightarrow \psi)$ are tautologies then so is ψ .

If $v(\phi) = T \& v((\phi \rightarrow \psi)) = T \quad v(\psi) = T. \#_{(b)}.$

\cancel{X} .

(1.3.3) Thm (Generalisation of soundness)

Suppose Γ is a set of L-formulas and ϕ is an L-formula. Suppose that $\Gamma \vdash_L \phi$.

Then for every valuation v with $v(\Gamma) = T$ (i.e. $v(\psi) = T$ for all $\psi \in \Gamma$)
 we have $v(\phi) = T$.

Pf: Almost same as 1.3.1. #.

(1.3.4) Thm.

(Completeness/Adequacy theorem for L).

Suppose ϕ is a tautology. Then $\vdash_L \phi$.

Generalisation

(3)

Suppose that for every valuation v with $v(\Gamma) = T$ we have $v(\phi) = T$. THEN $\Gamma \vdash_L \phi$.

Equivalently if $\Gamma \not\vdash_L \phi$ there is a valuation v with $v(\Gamma) = T$ and $v(\phi) = F$.

(1.3.6) Def ① A set Γ of L-formulas is consistent if there is no L-formula ψ such that $\Gamma \vdash_L \psi$ and $\Gamma \vdash_L (\neg \psi)$.

② Say Γ is complete if for every ψ $\Gamma \vdash_L \psi$ or $\Gamma \vdash_L (\neg \psi)$.

(4)

Remarks ① By 1.3.1 there
 is not L-fmla. ψ with
 $\vdash_L \psi$ and $\vdash_L (\neg \psi)$
 (Say L is consistent) .

More generally if there is a val.

v with $v(\Gamma) = T$ then

Γ is consistent, by 1.3.3.

② If v is a valuation and
 $\Gamma = \{\phi : v(\phi) = T\}$

then Γ is consistent &
complete .

(1.3.7) Proposition Suppose
 Γ is a consistent set of L-formulas
and $\Gamma \not\vdash_L \phi$.

Then $\Gamma \cup \{\neg\phi\}$ is
consistent.

Pf: Suppose not. So there is
a formula ψ such that

$$\Gamma \cup \{\neg\phi\} \vdash \psi \dots \textcircled{1}$$

$$+ \Gamma \cup \{\neg\phi\} \vdash \neg\psi \dots \textcircled{2}$$

Apply DT to $\textcircled{2}$

$$\Gamma \vdash (\neg\phi \rightarrow \neg\psi)$$

so by A3 + MP

$$\Gamma \vdash (\psi \rightarrow \phi) \dots \textcircled{3}$$

By $\textcircled{3}, \textcircled{1} + \text{MP}$

obtain

$$\Gamma \cup \{\neg\phi\} \vdash \phi$$

By DT

$$\Gamma \vdash ((\neg\phi) \rightarrow \phi) \dots \textcircled{5}$$

$$1.2.7(c) \vdash ((\neg\phi) \rightarrow \phi) \rightarrow \phi$$

This and $\textcircled{5}$ give

$$\Gamma \vdash \phi. \text{ Contradiction.}$$

(6)

(1.3.8) Proposition (Lindenbaum Lemma)

Suppose Γ is a consistent set of L-formulas. Then there is a consistent set of L-formulas $\Gamma^* \supseteq \Gamma$ which is complete.

Pf: The set of L-formulas is countable, so we can list the L-formulas as

$$\phi_0, \phi_1, \phi_2, \dots$$

[Why countable? Alphabet
 $\rightarrow (\) P_1 P_2 \dots$

is countable. ~~Each~~ Formulas
 are finite sequences of these.

the set of these is countable.]