Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 7 with solutions

*You should prepare starred question, marked by * to discuss with your personal tutor.*

1. * Find $\partial u/\partial x$ and $\partial u/\partial y$ for the following functions of two real variables:

$$\begin{aligned} &\text{(a)} \quad u = x^3 + 3xy - y^2 \\ &\text{(b)} \quad u = e^{xy}\sin x \end{aligned}$$

In each case:

(i) write the expression for du

$$\text{(a)} \quad du = (3x^2 + 3y)dx + (3x - 2y)dy.$$
$$\text{(b)} \quad du = (y\sin x + \cos x)e^{xy}dx + x\sin x e^{xy}dy.$$

(ii) verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

$$\text{(a)} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 3.$$

$$\text{(b)} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = (\sin x + xy\sin x + x\cos x)e^{xy}.$$

2. The following are a few examples of the application of: the total differential, the chain rule, and the implicit function.

(a) Using partial derivatives, find du/dt when

$$u(x, y) = \frac{x - y}{x + y} \quad \text{with} \quad x = e^{ct},\ y = e^{-ct}.$$

Check your answer otherwise.

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{2y}{(x+y)^2}ce^{ct} - \frac{2x}{(x+y)^2}\left(-ce^{-ct}\right) \\ &= \frac{4c}{(e^{ct} + e^{-ct})^2} = \frac{c}{\cosh^2(ct)}. \end{aligned}$$

To check this, we could write u explicitly as function of t. We have

$$u = \tanh(ct) \quad \Rightarrow \quad \frac{du}{dt} = \frac{c}{\cosh^2(ct)}.$$

(b) Consider

$$f(x,y) = x^2 + 3y^3 \quad \text{with} \quad x = s+t,\ y = 2s-t.$$

Use the chain rule to obtain $\partial f/\partial t$ and $\partial f/\partial s$ and check your answer by direct substitution.

Using chain rule we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= (2x)(1) + (9y^2)(-1) = 2(s+t) - 9(2s-t)^2.\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= (2x)(1) + (9y^2)(2) = 2(s+t) + 18(2s-t)^2.\end{aligned}$$

Alternatively, we can explicity write f as a function of s and t and take partial derivatives.

(c) Consider

$$u(x,y) = xy \quad \text{and} \quad \sin y + xy - x^3 = 0.$$

Find du/dx.

Using total derivatives we have:

$$du = ydx + xdy \quad \text{and} \quad (y - 3x^2)dx + (x + \cos y)dy = 0.$$

Substituting for dy from the second equation into the first equation we obtain:

$$dy = ydx - \frac{x(y-3x^2)}{(x+\cos y)}dx \quad \Rightarrow \quad \frac{du}{dx} = \frac{y\cos y + 3x^3}{x + \cos y}.$$

(d) The temperature in a region of space is given by the formula

$$f(\mathbf{x}) = f(x,y,z) = kx^2(y-z),$$

where k is a positive constant. An insect flies along a trajectory $\mathbf{x}(t) = (x(t), y(t), z(t)) = (t,t,2t)$. Find the rate of change of the temperature along its path.

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\\ &= 2kx(y-z)(1) + kx^2(1) - kx^2(2) = -3kt^2.\end{aligned}$$

(e) (The following is a classic result in Thermodynamics. Do not get flustered by the notation. Stick to the mathematical formulation to prove the result.)

The equation of state of a gas is usually given by an implicit relation $f(p,V,T)=0$ between the pressure p, the volume V, and the temperature T. Show that:

$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{\left(\frac{\partial f}{\partial V}\right)_{p,T}}{\left(\frac{\partial f}{\partial p}\right)_{V,T}},$$

and obtain similar expressions for $(\partial V/\partial T)_p$ and $(\partial T/\partial p)_V$. Hence derive the identity:

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1,$$

which is known as the reciprocity theorem.

$$df = \left(\frac{\partial f}{\partial p}\right)_{V,T} dp + \left(\frac{\partial f}{\partial V}\right)_{p,T} dV + \left(\frac{\partial f}{\partial T}\right)_{p,V} dT = 0.$$

For the first one, we solve for dp:

$$dp = \frac{-\left(\frac{\partial f}{\partial V}\right)_{p,T}}{\left(\frac{\partial f}{\partial p}\right)_{V,T}} dV + \frac{-\left(\frac{\partial f}{\partial T}\right)_{p,V}}{\left(\frac{\partial f}{\partial p}\right)_{V,T}} dT.$$

But we also know that

$$dp = \left(\frac{\partial p}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial T}\right)_V dT.$$

So we can identify the first result by equating the coeficients of dV in the above two equations.
Using a similar proof by solving for dV and dT we can show that:

$$\left(\frac{\partial V}{\partial T}\right)_p = -\frac{\left(\frac{\partial f}{\partial T}\right)_{p,V}}{\left(\frac{\partial f}{\partial V}\right)_{p,T}} \quad \text{and} \quad \left(\frac{\partial T}{\partial p}\right)_V = -\frac{\left(\frac{\partial f}{\partial p}\right)_{V,T}}{\left(\frac{\partial f}{\partial T}\right)_{p,V}}.$$

Multiplying these three results we obtain the following as required:

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1,$$

3. 'Projectile man' needs to estimate bounds on the accuracy of his landing place for his next stunt. He knows that the horizontal range R of a projectile is given by:

$$R = \frac{U^2 \sin 2\alpha}{g},$$

where U is the projectile initial speed, α is the angle of elevation and g is the gravitational acceleration. If U and α are each known to $\pm 0.2\%$ (and g can be considered to be known exactly), find the % accuracy bounds for R when:

$$\text{(a) } \alpha = 25^\circ, \quad \text{(b) } \alpha = 65^\circ, \quad \text{(c) } \alpha = 45^\circ.$$

In our formula for R only U and α can change independently but g is a constant. So we have to fitst order:

$$\delta R \simeq \frac{2U \sin 2\alpha}{g} \delta U + \frac{2U^2 \cos 2\alpha}{g} \delta\alpha.$$

We are given that $|\delta U/U| \leq 0.002$ and $|\delta\alpha/\alpha| \leq 0.002$. Now, by dividing both sides of the equation above by R we obtain:

$$\left(\frac{\delta R}{R}\right) \simeq 2\left(\frac{\delta U}{U}\right) + 2\alpha \cot 2\alpha \left(\frac{\delta\alpha}{\alpha}\right).$$

Now we have for (a):

$$\alpha = 25^\circ = \frac{25\pi}{180} \text{ radians} \quad \Rightarrow \quad \left(\frac{\delta R}{R}\right) \simeq 2\left(\frac{\delta U}{U}\right) + 0.732\left(\frac{\delta\alpha}{\alpha}\right).$$

Worst case in this case happens if δU and $\delta\alpha$ have the same sign. So we have $|\delta R/R| \leq 0.55\%$. We have for (b):

$$\alpha = 65^\circ = \frac{65\pi}{180} \text{ radians} \quad \Rightarrow \quad \left(\frac{\delta R}{R}\right) \simeq 2\left(\frac{\delta U}{U}\right) - 1.904\left(\frac{\delta\alpha}{\alpha}\right).$$

Worst case in this case happens if δU and $\delta\alpha$ have opposite sign. So we get $|\delta R/R| \leq 0.78\%$. And we have for (c):

$$\alpha = 45^\circ = \frac{65\pi}{180} \text{ radians} \quad \Rightarrow \quad \left(\frac{\delta R}{R}\right) \simeq 2\left(\frac{\delta U}{U}\right).$$

In this case the error does not depend on $\delta\alpha$. So we get $|\delta R/R| \leq 0.4\%$. Note that all of these estimates are to first order.

4. * The cost P of a computer depends on the required CPU c and memory storage s according to the relation:

$$P = kc^2 s^3,$$

where k is some positive constant. Estimate the percentage change in cost if c and s are increased and decreased by 1%, respectively.

$$\delta P \simeq 2kcs^3 \delta c + 3kc^2 s^2 \delta s \quad \Rightarrow \quad \frac{\delta P}{P} \simeq 2\frac{\delta c}{c} + 3\frac{\delta s}{s}.$$

Then given that $\delta c/c = +0.01$ and $\delta s/s = -0.01$, we have $\delta P/P = -0.01$. Change in the cost is then $\simeq 1\%$ decrease to the first order.

5. The following are a couple of examples to practise the Taylor expansion of functions of two variables:

   (a) Find the Taylor expansion up to quadratic terms for $f(x, y) = \ln(1 + x + 2y)$ about the point $(x_0, y_0) = (2, 1)$. Use your result to estimate the value of $\ln(5+h+2k)$ when $h = 0.2$ and $k = -0.05$ and compare your estimate to the 'true' value.

   Using the Taylor series for functions of two variables and denoting partial derivative of f with respect to x by f_x and so on, we have:

$$f(2+h, 1+k) = f(2, 1) + hf_x(2, 1) + kf_y(2, 1) + \frac{1}{2}\left(h^2 f_{xx}(2, 1) + 2hk f_{xy}(2, 1) + k^2 f_{yy}(2, 1)\right) + \cdots$$

So we obtain:

$$\ln(5+h+2k) = \ln 5 + \frac{h}{5} + \frac{2k}{5} - \frac{h^2}{50} - \frac{2hk}{25} - \frac{2k^2}{25} + \cdots$$

Now for $h = 0.2$ and $k = -0.05$ we have

$$\ln(5.1) \simeq= \ln 5 + 0.4 - 0.02 - 0.0002 = 1.629237$$

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The true answer is $\ln(5.1) = 1.6292405$.

(b) Find the Taylor expansion up to third-order terms for $f(x, y) = (x + 2y)\cos(2x + y)$ about the point $(x_0, y_0) = (0, 0)$ and compare the result in this case to an expansion based on the cosine function of one variable.

Using the notation introduced in part (a) for partial derivatives, we have

$$f_x = \cos(2x+y) - 2(x+2y)\sin(2x+y)$$

$$f_y = 2\cos(2x+y) - (x+2y)\sin(2x+y)$$
$$f_{xx} = -4\sin(2x+y) - 4(x+2y)\cos(2x+y)$$
$$f_{yy} = -4\sin(2x+y) - (x+2y)\cos(2x+y)$$
$$f_{xy} = f_{yx} = -5\sin(2x+y) - 2(x+2y)\cos(2x+y)$$
$$f_{xxx} = -12\cos(2x+y) + 8(x+2y)\sin(2x+y)$$
$$f_{yyy} = -6\cos(2x+y) + (x+2y)\sin(2x+y)$$
$$f_{yxx} = -12\cos(2x+y) + 4(x+2y)\sin(2x+y)$$
$$f_{xyy} = -9\cos(2x+y) + 2(x+2y)\sin(2x+y)$$

When $x = y = 0$, we have $f = 0$, $f_x = 1$, $f_y = 2$, $f_{xx} = f_{yy} = f_{xy} = f_{yx} = 0$, $f_{xxx} = -12$, $f_{yyy} = -6$, $f_{yxx} = -12$, $f_{xyy} = -9$. So we obtain:

$$f(x,y) = 0 + (x+2y) + 0/2! + (-12x^3 - 36x^2y - 27xy^2 - 6y^3)/3! + \cdots$$

Naturally this is the same as what one obtains using the expansion of cosine function of one variable.

$$f(x,y) = (x+2y)(1 - \frac{1}{2!}(2x+y)^2 + \cdots)$$