1 Vector Calculus

1.1 Preliminary ideas and some revision of vectors

1.1.1 The Einstein summation convention

In any product of terms, if we have a repeated suffix, then that quantity is considered to be summed over (from 1 to 3, since we will usually be working in three dimensions). For example

$$a_i x_i \text{ is shorthand for } \sum_{i=1}^{3} a_i x_i.$$

1.1.2 The Kronecker delta

This is the quantity δ_{ij} and is defined such that

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Example

$$\delta_{ij} a_j = $$

$$= $$

Note that the left-hand-side had two different subscripts, while the right-hand-side ends up with only one subscript - this is known as a **contraction**.

1.1.3 The permutation symbol

This is the quantity ε_{ijk}, defined as

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two of } i, j, k \text{ are the same;} \\ 1, & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3; \\ -1, & \text{if } i, j, k \text{ is an acyclic permutation of } 1, 2, 3. \end{cases}$$

For example

$$\varepsilon_{123} = \qquad , \ \varepsilon_{321} = \qquad , \ \varepsilon_{133} = \qquad .$$

We can show, by considering the various cases, that the Kronecker delta and the permutation symbol are connected by the formula

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

(I will put a proof on blackboard). The quantities δ_{ij} and ε_{ijk} are known as **tensors**.
Exercise: Show this can be rewritten in the alternative form

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

1.1.4 Vector product

Recall that this is the multiplication of two vectors which results in a third vector, perpendicular to the first two. It can be written in the form of a determinant as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then the two vectors are parallel. Recall that $(\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a})$. If we just consider the first component of this vector we can write this as

$$\begin{aligned} a_2 b_3 - a_3 b_2 &= \\ \\ &= \end{aligned}$$

since $\varepsilon_{123} = 1, \varepsilon_{132} = -1$, and $\varepsilon_{1ij} = 0$ for all other i and j. In general we can write the ith component of $a \times b$ as

$$[a \times b]_i =$$

1.1.5 Scalar product

This is defined as

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \\ \\ &= \end{aligned}$$

using the summation convention. Recall that if $\mathbf{a} \cdot \mathbf{b} = 0$ then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

1.1.6 Triple scalar product

This is the quantity

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \\ \\ &= \end{aligned}$$

If this quantity is zero then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar. A useful property of the triple scalar product is that the dot and cross can be swapped without changing the answer, provided the order of the vectors remains unchanged, i.e.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

1.1.7 Triple vector product

This is defined as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

Since $\mathbf{b} \times \mathbf{c}$ is a vector normal to the plane of $\mathbf{b}$ and $\mathbf{c}$, and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is normal to $\mathbf{b} \times \mathbf{c}$, it follows that the triple vector product must lie in the plane of $\mathbf{b}$ and $\mathbf{c}$. In component notation

$$\begin{aligned} [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

and so we conclude that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

which confirms explicitly that the triple vector product indeed lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

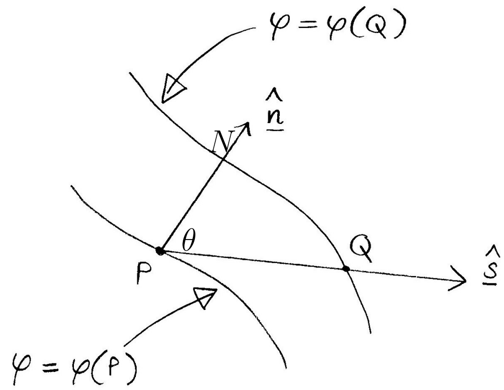


Figure 1: The surface $\phi =$ constant through two neighbouring points.

1.2 Gradient

Let ϕ be a differentiable scalar function of position in three dimensions. If P is a general point, ϕ will depend on the position of P, so we may write $\phi = \phi(P)$. The position of P is defined by reference to a coordinate system e.g. if we consider Cartesian coordinates, then P depends on (x, y, z) and hence $\phi = \phi(x, y, z)$, while if we consider cylindrical polar coordinates (r, θ, z) then $\phi = \phi(r, \theta, z)$.

The equation $\phi =$ constant defines a surface in three dimensions. Varying the constant, we can define a family of surfaces called 'level surfaces' or 'equi-ϕ surfaces'. For example, if ϕ represents pressure, then $\phi =$ constant defines a family of surfaces over which the pressure is constant. The surface through a **specific point** P is $\phi = \phi(P)$. Let Q be a neighbouring point. (See figure 1). The equation of the level surface through Q is $\phi = \phi(Q)$. We draw the normal to $\phi = \phi(P)$ at P. Suppose that it intersects $\phi = \phi(Q)$ at the point N. Since N is on $\phi = \phi(Q)$ we have $\phi(N) = \phi(Q)$. Let s denote the length along PQ and let n denote the length along PN. Introduce unit vectors $\widehat{\mathbf{s}}$ and $\widehat{\mathbf{n}}$ in those directions. We define $\partial\phi/\partial s$ to be the **directional derivative** of ϕ in the direction $\widehat{\mathbf{s}}$:

$$\frac{\partial \phi}{\partial s} =$$

$$=$$

$$=$$

$$=$$

$$=$$

Since $\cos\theta \leq 1$, the maximum directional derivative at P occurs along the normal to $\phi = \phi(P)$ at P.

The vector $\widehat{\mathbf{n}}\,\partial\phi/\partial n$ is called the **gradient** of ϕ at P. We write it as $\operatorname{grad}\phi$ or $\nabla\phi$. The operator grad or ∇ is known as the **vector gradient operator**. We have

$$\frac{\partial\phi}{\partial s} = \widehat{\mathbf{s}}\cdot\nabla\phi.$$

1.2.1 Cartesian components of $\nabla\phi$

If $\nabla\phi = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ then $\mathbf{i}\cdot\nabla\phi = A_1$. But, by definition, $\mathbf{i}\cdot\nabla\phi = \partial\phi/\partial x$. Hence $A_1 = \partial\phi/\partial x$. Similarly we find $A_2 = \partial\phi/\partial y, A_3 = \partial\phi/\partial z$ and so we have the result:

$$\nabla\phi =$$

Example

If $\phi = axy^2 + byz + cx^3z^2$, where a, b, c are constants, find $\nabla\phi$. Also find the directional derivative of ϕ at the point $(1, 4, 2)$ in the direction towards the point $(2, 0, -1)$.

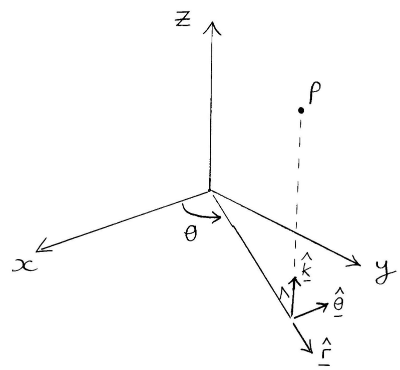


Figure 2: Sketch showing a point P represented by Cartesian coordinates (x, y, z) and cylindrical polar coordinates (r, θ, z).

1.2.2 Cylindrical polar components of $\nabla\phi$

The set-up is as shown in figure 2. We write $\nabla\phi = A_1\widehat{\mathbf{r}} + A_2\widehat{\theta} + A_3\mathbf{k}$. Then it follows that

$$\begin{aligned} A_1 &= \widehat{\mathbf{r}}\cdot\nabla\phi \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

Similarly, we find

$$\begin{aligned} A_2 &= \widehat{\theta}\cdot\nabla\phi \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

and $A_3 = \mathbf{k}\cdot\nabla\phi = \partial\phi/\partial z$. Hence

$$\nabla\phi = \widehat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \frac{\widehat{\theta}}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{k}\frac{\partial\phi}{\partial z}.$$

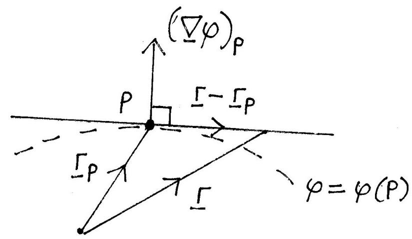


Figure 3: The tangent plane to a surface.

1.2.3 Equation of a tangent plane to $\phi = \phi(P)$

We have that $(\nabla\phi)_P$ is normal to $\phi = \phi(P)$ at P. The equation of the tangent plane is therefore

$$(\mathbf{r} - \mathbf{r}_P) \cdot (\nabla\phi)_P = 0,$$

i.e.

$$\left(\frac{\partial\phi}{\partial x}\right)_P (x - x_P) + \left(\frac{\partial\phi}{\partial y}\right)_P (y - y_P) + \left(\frac{\partial\phi}{\partial z}\right)_P (z - z_P) = 0.$$

Example

Find the tangent plane to the surface

$$z = e^{-(x^2+y^2)^{1/2}}$$

at the point $x = -1, y = 0$.

1.3 Divergence and Curl

In this section we will assume that $\mathbf{A}$ is a vector function of position in three dimensions, with continuous first partial derivatives.

Since ∇ is a vector operator, we can define formally a scalar product $\nabla \cdot \mathbf{A}$. This is called the **divergence** of the vector $\mathbf{A}$. We can also define the vector product $\nabla \times \mathbf{A}$, which is called the **curl** of $\mathbf{A}$. So to summarize we have

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A}, \quad \operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A}.$$

1.3.1 Cartesian form

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \\ &= \\ \operatorname{curl} \mathbf{A} &= \\ &= \end{aligned}$$

Note that these simple forms for div and curl arise because $\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}$ are **constant** vectors: this is not so in other coordinate systems.

Examples

(a) If

$$\mathbf{A} = (y^2 \cos x + z^3)\mathbf{i} + (2y \sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k},$$

find div$\mathbf{A}$ and curl $\mathbf{A}$.

(b) Find div $\mathbf{u}$ and curl $\mathbf{u}$ when (i) $\mathbf{u} = \mathbf{r}$; (ii) $\mathbf{u} = \omega \times \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\omega = \Omega\mathbf{k}$ with Ω constant.

1.4 Operations with the gradient operator

1.4.1 Important sum and product formulae

Note that ∇ is a linear operator, and so:

$$\begin{aligned}
\text{(i)}\,\nabla(\phi_1+\phi_2) &= \nabla\phi_1+\nabla\phi_2,\\
\text{(ii)}\,\operatorname{div}(\mathbf{A}+\mathbf{B}) &= \operatorname{div}\mathbf{A}+\operatorname{div}\mathbf{B},\\
\text{(iii)}\,\operatorname{curl}(\mathbf{A}+\mathbf{B}) &= \operatorname{curl}\mathbf{A}+\operatorname{curl}\mathbf{B}.
\end{aligned}$$

The proofs of these results follow immediately from the definition of ∇.

Other key results are:

$$\begin{aligned}
\text{(iv)}\,\nabla(\phi\psi) &= \phi\nabla\psi+\psi\nabla\phi,\\
\text{(v)}\,\operatorname{div}(\phi\mathbf{A}) &= \phi\operatorname{div}\mathbf{A}+\nabla\phi\cdot\mathbf{A}.
\end{aligned}$$

Proof of (v)

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{A}) &= \\
&= \\
&=
\end{aligned}$$

In writing out these proofs it is easier to use the **summation convention** that we introduced earlier. Rather than write (x,y,z) for Cartesian components, we write (x_1,x_2,x_3) and in place of $(\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}\,)$ we write $(\widehat{\mathbf{e}}_1,\widehat{\mathbf{e}}_2,\widehat{\mathbf{e}}_3)$. Then we saw earlier that

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_iB_i,\\
\mathbf{A}\times\mathbf{B} &= \varepsilon_{ijk}\widehat{\mathbf{e}}_iA_jB_k
\end{aligned}$$

Also recall the useful result that

$$\varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}.$$

Thus, under the summation convention:

$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \\
[\nabla\phi]_i &= \\
[\operatorname{curl}\mathbf{A}]_i &=
\end{aligned}$$

where $[\ ]_i$ indicates the ith component. Using this approach, the proof of (v) takes the form

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{A}) &= \\
&=
\end{aligned}$$

Other important results are:

$$\begin{aligned}
\text{(vi)}\,\mathrm{curl}\,(\phi\mathbf{A}) &= \phi\,\mathrm{curl}\,\mathbf{A} + \nabla\phi \times \mathbf{A},\\
\text{(vii)}\,\mathrm{div}(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B},\\
\text{(viii)}\,\mathrm{curl}\,(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}\,\mathrm{div}\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\,\mathrm{div}\mathbf{B},\\
\text{(ix)}\,\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A} + \mathbf{A}\times\mathrm{curl}\,\mathbf{B}.
\end{aligned}$$

Example

Prove relation (ix) above. If we work on the RHS we can write

$$[(\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A} + \mathbf{A}\times\mathrm{curl}\mathbf{B}]_i$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

as required.

Note: In the following sections we will assume that our scalar and vector functions possess continuous second derivatives.

1.4.2 The divergence of a gradient: the Laplacian

Consider the operation

$$\begin{aligned} \mathrm{div}(\nabla\phi) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \\ &= \\ &\equiv \end{aligned}$$

This is to be read as '**del squared** ϕ' or the **Laplacian** of ϕ. The operator ∇^2 is known as the Laplacian operator. We also define the Laplacian of a vector as

$$\nabla^2\mathbf{A} \equiv \frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2}$$

in Cartesian coordinates, and the equation $\nabla^2\phi = 0$ is known as **Laplace's equation**.

Example

If $\phi = x^2 + y^2$, find $\nabla^2\phi$.

1.4.3 The curl of a gradient

Consider the operation

$$\mathrm{curl}\,(\nabla\phi) =$$

(This result can also be established by using tensor notation).

Example

Consider $\phi = axy^2 + byz + cx^3z^2$ and show explicitly that $\mathrm{curl}\,\nabla\phi = 0$.

1.4.4 The divergence of a curl

This is also always zero, as can be seen from the following argument:

$$\operatorname{div}(\operatorname{curl}\mathbf{A}) =$$

Example

Verify that $\operatorname{div}(\operatorname{curl}\mathbf{A}) = 0$ for the quantity $\mathbf{A} = y\,e^x\mathbf{i} + (x^2 + z)\mathbf{j} + y^3\cos(zx)\mathbf{k}$.

1.4.5 The curl of a curl

This is the vector quantity

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A}).$$

Using tensor notation and the summation convention we can show that

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A}) = \nabla(\mathrm{div}\mathbf{A}) - \nabla^2\mathbf{A}.$$

Proof

Exercise

Calculate $\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A}), \nabla(\mathrm{div}\mathbf{A})$ and $\nabla^2\mathbf{A}$ for $\mathbf{A} = y\,e^x\mathbf{i} + (x^2+z)\mathbf{j} + y^3\cos(zx)\mathbf{k}$.

1.4.6 Scalar and vector fields

If, at each point of a region V of space, a scalar function ϕ is defined, we say that ϕ is a **scalar field** over the region V. Similarly, if a vector function $\mathbf{A}$ is also defined at all points of V, then $\mathbf{A}$ is a vector field over V. If curl $\mathbf{A} = \mathbf{0}$ we say that A is an **irrotational** vector field. If div $\mathbf{A} = 0$ we say $\mathbf{A}$ is a **solenoidal** vector field. An obvious example of a vector field is the position vector $\mathbf{r}$ of a point in space. In three dimensions:

$$
\begin{aligned}
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,, \\
\\
\operatorname{div}\mathbf{r} &= \\
\\
\operatorname{curl}\mathbf{r} &= \\
\\
|\mathbf{r}| &= r = (x^2 + y^2 + z^2)^{1/2} \\
\nabla r &= \nabla(x^2 + y^2 + z^2)^{1/2} \\
\\
&= \\
\\
&= \\
\\
&= \\
\\
&=
\end{aligned}
$$

Example

Find

$$\nabla^2(1/r).$$

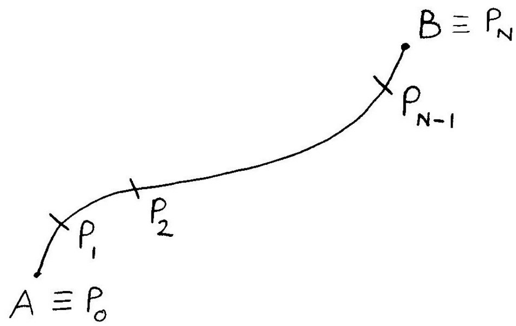


Figure 4: A curve γ joining A to B and divided into N sections.

1.5 Path Integrals

1.5.1 Definition

Consider a curve γ (not necessarily in the plane, and not necessarily smooth) joining the points A and B. (See figure 4). Suppose that the curve is divided into N sections: $AP_1, P_1P_2, \dots, P_{N-1}B$. Let $AP_1 = \delta s_1, P_1P_2 = \delta s_2, \dots, P_{N-1}B = \delta s_N$. Next, suppose a function f is defined along this curve γ. We compute the sum

$$f_1\delta s_1 + f_2\delta s_2 + \cdots + f_N\delta s_N,$$

where $f_n = f(P_n)$. On increasing N indefinitely, while letting the maximum $\delta s_n \to 0$, the resulting limit of the sum, if it exists, is called the **path integral of f along γ**, and we write:

$$\int_\gamma f\, ds =$$

The function f may be a scalar or a vector.

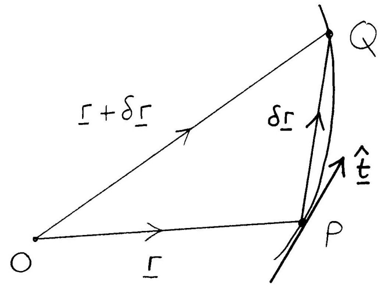


Figure 5: Diagram showing the tangent vector at a point P.

1.5.2 Path element

See figure 5. Let δs represent the arc PQ and suppose that the vector $\overrightarrow{PQ} = \delta\mathbf{r}$. We define the **tangent vector**

$$\widehat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \lim_{\delta s\to 0}\frac{\delta\mathbf{r}}{\delta s},$$

and the **path element**

$$d\mathbf{r} = \widehat{\mathbf{t}}\,ds.$$

Note that $\widehat{\mathbf{t}}$ has length unity because $|\delta\mathbf{r}| \to \delta s$ as $\delta s \to 0$. We can then define the quantity

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} =$$

1.5.3 Conservative forces

Consider the special case where we have a vector $\mathbf{F}$ of the form

$$\mathbf{F} = \nabla\phi$$

with ϕ a differentiable scalar function. Consider the integral (with γ defined as in figure 3):

$$\begin{aligned}
\int_\gamma \mathbf{F}\cdot d\mathbf{r} &= \\
&= \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

We note that the result is **independent of the path** γ joining A to B. In particular, if γ is a closed curve (i.e. $B \equiv A$), then we have $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$, where we put a circle on the integral to denote the path is closed. We sometimes refer to such an integral as the **circulation** of $\mathbf{F}$ around γ.

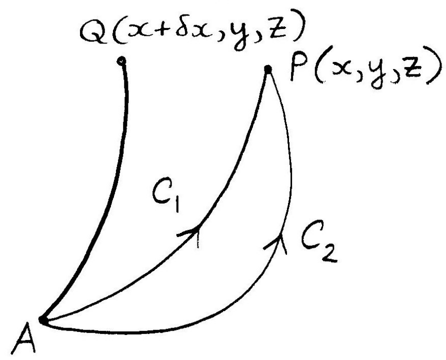


Figure 6: Two curves joining A to P. Q is a neighbouring point.

If a vector field $\mathbf{F}$ has the property that $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$ for **any** closed curve γ, we say that $\mathbf{F}$ is a **conservative field**. Thus, if $\mathbf{F} = \nabla\phi$, then $\mathbf{F}$ is conservative. Conversely, if $\mathbf{F}$ is conservative we can always find a differentiable scalar function ϕ such that $\mathbf{F} = \nabla\phi$. The function ϕ is called the **potential** of the field $\mathbf{F}$.

Proof of this last part

See figure 6. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Since we know that $\mathbf{F}$ is conservative it must be the case that $\int_A^P \mathbf{F}\cdot d\mathbf{r}$ is independent of the path from A to P and hence

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r},$$

where C_1 and C_2 are any two curves drawn from A to P. Suppose that the point A is fixed. Then

$$\begin{aligned}\int_A^P \mathbf{F}\cdot d\mathbf{r} &= G(P), \text{ say}\\ &= G(x,y,z)\end{aligned}$$

Let Q be the point $(x+\delta x, y, z)$ and let P be the point (x,y,z). Consider the quantity

$$\begin{aligned}G(x+\delta x,y,z) - G(x,y,z) &\equiv \\ \\ &=\end{aligned}$$

But we can choose the path from P to Q so that only x varies, in which case $d\mathbf{r} = \mathbf{i}\,dx$. Thus

$$G(x+\delta x,y,z) - G(x,y,z) =$$

and hence

$$\begin{aligned}\frac{\partial G}{\partial x} &= \\ &= \\ &= \end{aligned}$$

Similarly we can show that

$$F_2 = \frac{\partial G}{\partial y}, \quad F_3 = \frac{\partial G}{\partial z}.$$

Thus, if $\mathbf{F}$ is conservative then a scalar function (G in this case) can be found such that $\mathbf{F} = \nabla G$.

Example

For the vector field

$$\mathbf{F} = (3x^2 + yz)\mathbf{i} + (6y^2 + xz)\mathbf{j} + (12z^2 + xy)\mathbf{k}$$

find a scalar function $\phi(x, y, z)$ such that $\mathbf{F} = \nabla\phi$. Hence calculate $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ where $A = (0, 0, 0)$ and $B = (1, 1, 1)$.

1.5.4 Practical evaluation of path integrals

Suppose we wish to evaluate

$$I = \int_\gamma \mathbf{F} \cdot d\mathbf{r}$$

explicitly, where $\mathbf{F}$ is a known function of (x, y, z) and γ is some known curve joining the points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

Along γ we can write

Here, t is a parameter that takes us along γ with $x(t_0) = x_0, x(t_1) = x_1$ and similarly for y and z. Then we can write

$$d\mathbf{r} =$$

and hence, with $\mathbf{F} = F_1(t)\mathbf{i} + F_2(t)\mathbf{j} + F_3(t)\mathbf{k}$:

$$I = \int_\gamma \mathbf{F} \cdot d\mathbf{r} =$$

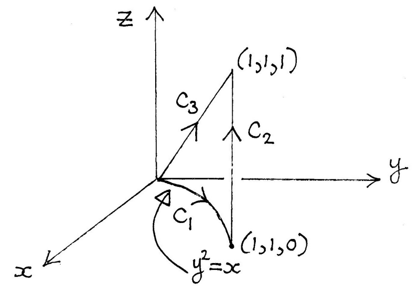


Figure 7: The integration path for this example.

Example (see figure 7)

Evaluate

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} \text{ with } \mathbf{F} = yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$$

when γ joins $(0,0,0)$ to $(1,1,1)$ along

(i) $C_1 + C_2$ with C_1 the curve $x = y^2, z = 0$ from $(0,0,0)$ to $(1,1,0)$ and C_2 is the straight line joining $(1,1,0)$ to $(1,1,1)$;

(ii) C_3 is the straight line joining $(0,0,0)$ to $(1,1,1)$.

1.6 Surface integrals

1.6.1 Definition

To define a surface integral of $f = f(P)$ over a surface S, we divide S into elements of area $\delta S_1, \delta S_2, \ldots, \delta S_N$. Let $f_1, f_2, \ldots, f_N$ be the values of f at typical points $P_1, P_2, \ldots, P_N$ of $\delta S_1, \delta S_2, \ldots, \delta S_N$ respectively. We calculate the quantity

$$\sum_{n=1}^{N} f_n \delta S_n.$$

We now let $N \to \infty$, $\max \delta S_n \to 0$. The resulting limit, if it exists, is called the **surface integral of** f **over** S, and we write it as

$$\int_S f \, dS =$$

As with the line integral, the function f may be a vector or a scalar.

1.6.2 Types of surfaces

Closed surface: this divides three-dimensional space into two non-connected regions - an interior region and an exterior region;

Convex surface: this is a surface which is crossed by a straight line at most twice;

Open surface: this does not divide space into two non-connected regions - it has a rim which can be represented by a closed curve. (A closed surface can be thought of as the sum of two open surfaces).

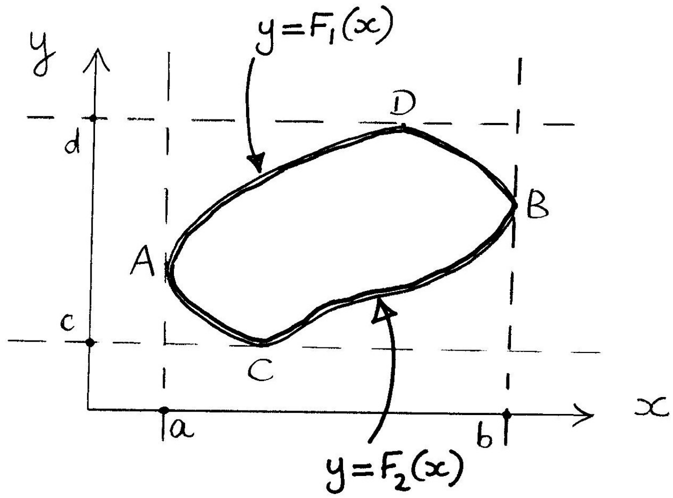


Figure 8: Diagram to illustrate the evaluation of surface integrals.

1.6.3 Evaluation of surface integrals for plane surfaces in the $x-y$ plane

An **areal element** dS is an 'infinitesimally small' element of area of a surface. Even for curved surfaces it can be thought of as approximately plane. The **vector areal element** $d\mathbf{S}$ is the vector $\widehat{\mathbf{n}}\,dS$ where $\widehat{\mathbf{n}}$ is the unit vector normal to dS. For plane surfaces dS can be expressed in Cartesian coordinates (x, y) since we may choose the surface to lie in the plane $z = 0$. Thus we can write $dS = dx\,dy$. (See figure 8).

Let the rectangle $x = a, b$ and $y = c, d$ circumscribe S. We will assume for simplicity that S is convex. (If it isn't then we split S up into convex sub-regions). Let the equation of the boundary of S be denoted by

$$y = \begin{cases} F_1(x) & \text{upper half } ADB \\ F_2(x) & \text{lower half } ACB \end{cases}.$$

(n.b. we need to ensure these are single-valued functions, which they will be if S is convex). Then

$$S =$$

$$=$$

If $f(x, y)$ is any function of position:

$$\int_S f\,dS =$$

In some situations it may be more convenient to do the $x-$integration first. If we want to do this we need to write the boundaries in terms of functions of y instead of x. In this case let the boundary be described by

$$x = \begin{cases} G_1(y) & \text{right half } CBD \\ G_2(y) & \text{left half } CAD \end{cases} .$$

Then

$$S =$$

$$=$$

and

$$\int_S f \, dS =$$

1.6.4 Example

Find the area of the circle $x^2 + y^2 = a^2$.

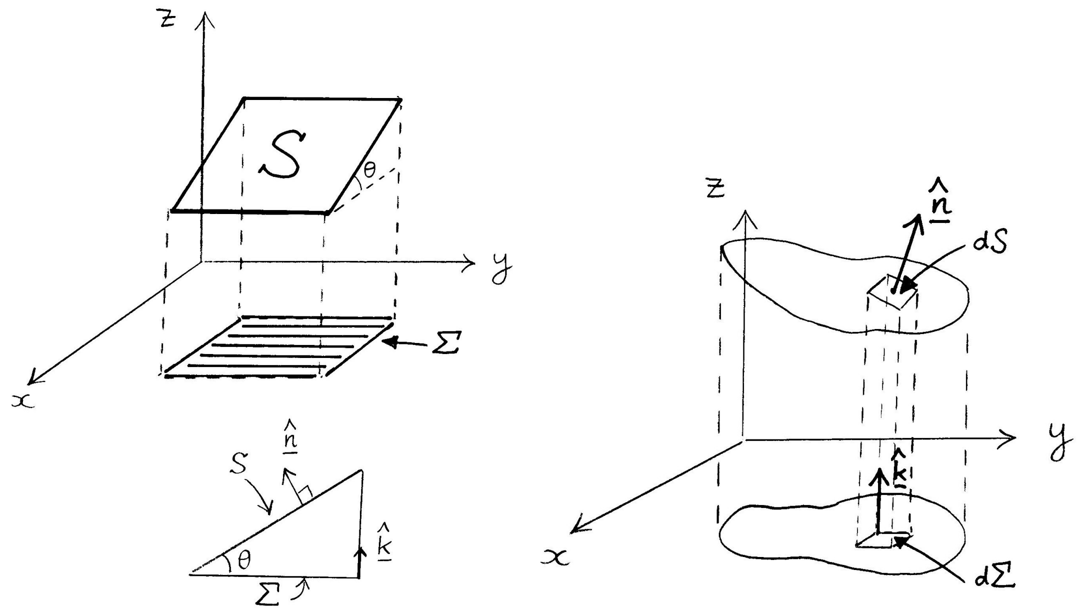


Figure 9: Left: The projection of a plane area S onto the $x-y$ plane. Right: The projection of a curved surface S onto the $x-y$ plane.

1.6.5 Projection of an area onto a plane

Consider first a plane area S (left hand diagram in figure 9). Suppose Σ is the projected area in the $x-y$ plane. Then $\Sigma = S\cos\theta$, where $\cos\theta = |\widehat{\mathbf{n}}\cdot\mathbf{k}\,|$.

Now consider a curved surface. (Right hand diagram in figure 9). If we consider an areal element dS then this will be effectively plane, and so

$$dS =$$

1.6.6 The projection theorem

Let P denote a general point of a surface S which at no point is orthogonal to the direction $\mathbf{k}$. Then:

$$\int_S f(P)\, dS = \int_\Sigma f(P)\, \frac{dx\, dy}{|\widehat{\mathbf{n}} \cdot \mathbf{k}\,|},$$

where Σ is the projection of S onto the plane $z = 0$, and $\widehat{\mathbf{n}}$ is normal to S.

Proof

$$\int_S f(P)\, dS \;=$$

$$=$$

where $\varepsilon_r \to 0$ as $\delta S_r \to 0$. (Here $\widehat{\mathbf{n}}_r$ is the unit vector normal to S at P_r and $\delta\Sigma_r$ is the projection of δS_r onto the plane $z = 0$. It therefore follows that

$$\int_S f(P)\, dS =$$

as required. Note that $f(P)$ is evaluated at $P(x, y, z)$ on S in **both integrals**.

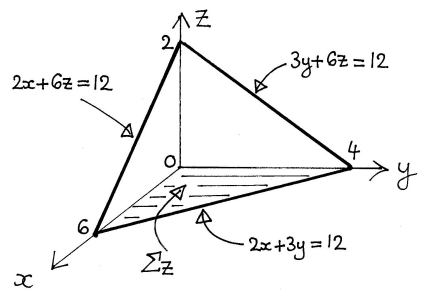
If, for example, the equation of S is $z = \phi(x, y)$ then the theorem gives

$$\int_S f(x, y, z)\, dS =$$

Alternatively, we may choose to project the surface onto $x = 0$ or $y = 0$ to give:

$$\int_S f(P)\, dS =$$

where Σ_x is the projection of S onto $x = 0$ and Σ_y is the projection of S onto $y = 0$.



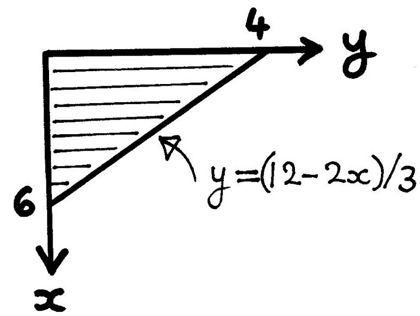


Figure 10: Left: The plane $2x + 3y + 6z = 12$ and its projection onto the $x - y$ plane. Right: The projected region Σ_z viewed from above.

Example of using the projection theorem

Evaluate

$$\int_S (y + 2z - 2)\, dS$$

where S is the part of the plane $2x + 3y + 6z = 12$ in the first octant $(x, y, z \geq 0)$, by projecting onto the plane $z = 0$.

1.7 Volume Integrals

1.7.1 Definition

Consider a volume τ and split it up into N subregions $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$. Let $P_1, P_2, \ldots, P_N$ be typical points of $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$.

Consider the sum

Now let $N \to \infty, \max \delta\tau_i \to 0$. If this sum tends to a limit we call it the volume integral of f over τ and write this as

$$\int_\tau f \, d\tau.$$

The function f may be a vector or a scalar.

1.7.2 Volume element

In Cartesian coordinates the volume element

$$d\tau = dx \, dy \, dz.$$

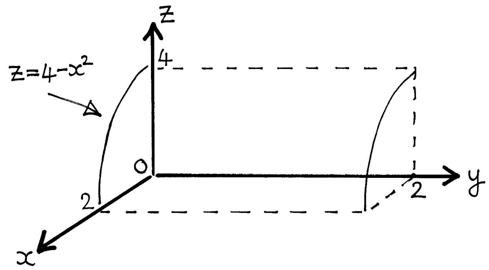


Figure 11: The volume τ for the example.

Example

Evaluate

$$\int_{\tau} (2x + y)\, d\tau$$

when τ is the volume enclosed by the parabolic cylinder $z = 4 - x^2$ and the planes $x = y = z = 0$ and $y = 2$.

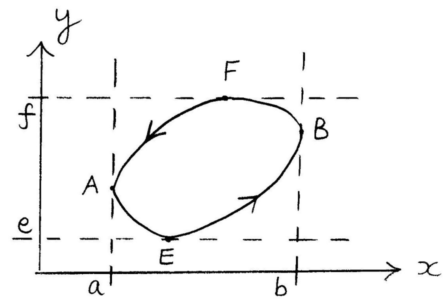


Figure 12: Diagram for proof of Green's theorem.

1.8 Results relating line, surface and volume integrals

1.8.1 Green's theorem in the plane

Suppose R is a closed plane region bounded by a simple plane closed convex curve in the $x-y$ plane. Let L, M be continuous functions of x, y having continuous derivatives throughout R. Then:

$$\oint_C (L\,dx + M\,dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy,$$

where C is the boundary of R described in the counter-clockwise (positive) sense.

Proof. We draw a rectangle formed by the tangent lines $x = a, b$ and $y = e, f$ (figure 12). This rectangle circumscribes C. Let $x = X_1(y), x = X_2(y)$ be the equations of EAF and EBF respectively. We then can write

$$\int_R \frac{\partial M}{\partial x}\,dx\,dy \quad =$$

Now, let the equations of AEB and AFB be $y = Y_1(x), y = Y_2(x)$. Then

$$\int_R \frac{\partial L}{\partial y}\,dx\,dy \quad =$$

1.8.2 Vector forms of Green's Theorem

(i) (2D Stokes Theorem). Let $\mathbf{F} = L\mathbf{i} + M\mathbf{j}$, and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$. Then

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathbf{k}.$$

Over the region R we can write $dx\,dy = dS$. Thus using Green's theorem:

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_R \mathbf{k}\cdot\operatorname{curl}\mathbf{F}\,dS \\ &= \int_R \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.\end{aligned}$$

This result can be generalized to three dimensions (see **Stokes theorem** later).

(ii)(Divergence theorem in 2D). This time let $\mathbf{F} = M\,\mathbf{i} - L\,\mathbf{j}$. Then

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

and so Green's theorem can be rewritten as

$$\int_R \operatorname{div}\mathbf{F}\,dx\,dy = \oint_C F_1\,dy - F_2\,dx.$$

Now it can be shown (exercise) that

$$\widehat{\mathbf{n}}\,ds = (dy\,\mathbf{i} - dx\,\mathbf{j})$$

where s is arclength along C, and $\widehat{\mathbf{n}}$ is the unit normal to C. Therefore we can rewrite Green's theorem as

$$\int_R \operatorname{div}\mathbf{F}\,dx\,dy = \oint_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds.$$

This result also turns out to be true in three dimensions, where it is known as the **Divergence Theorem**.

Example

Show that the area enclosed by a simple closed curve with boundary C can be expressed as

$$\frac{1}{2}\oint_C x\,dy - y\,dx.$$

Use this result to calculate the area of an ellipse.

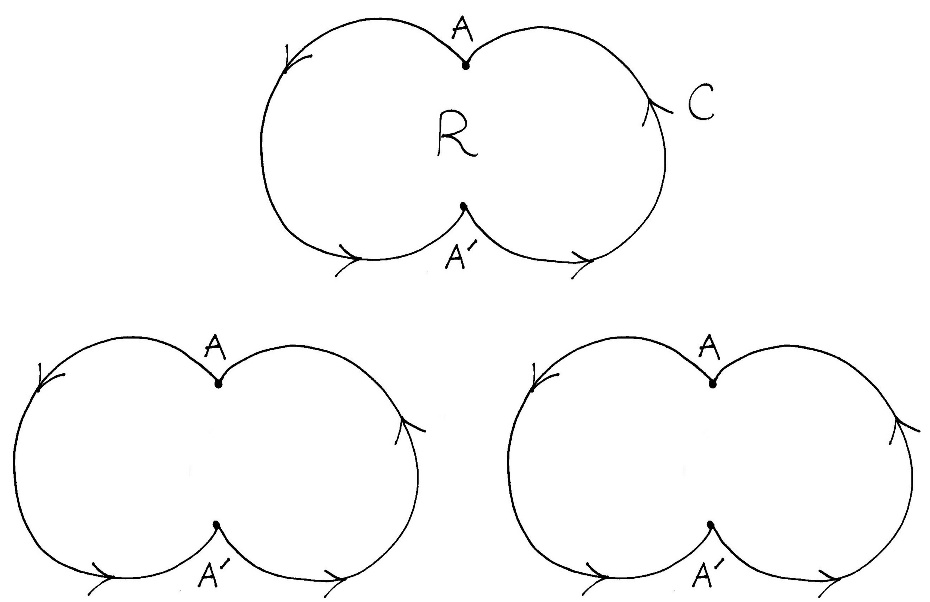


Figure 13: A non-convex boundary.

1.8.3 Extensions of Green's theorem in the plane

Green's theorem is true for more complicated geometries than that assumed in the proof given above. e.g. if C is not convex, but has the shape given in figure 13. We can join the points A, A' so as to form 2 (or more) simple convex closed curves C_1, C_2 enclosing R_1, R_2 where $R_1 + R_2 = R$. Then:

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \quad =$$

$$=$$

Now

$$\oint_{C_1} \quad =$$

$$\oint_{C_2} \quad =$$

and so

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} =$$

We see therefore that the theorem still holds.

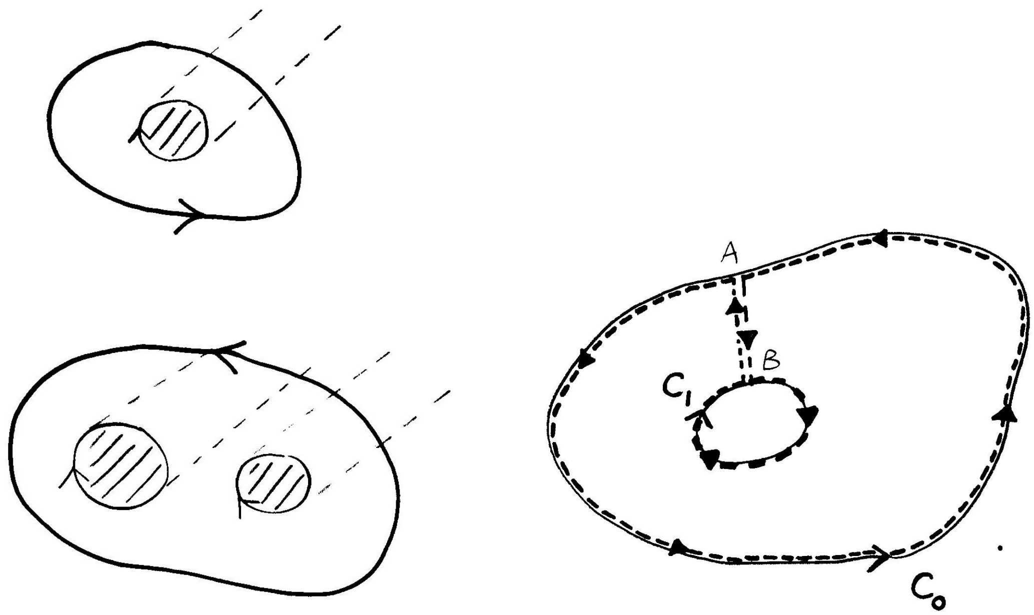


Figure 14: Left: Examples of doubly- and triply-connected regions. Right: Green's theorem in a multiply-connected region.

1.8.4 Green's theorem in multiply-connected regions

A region R is said to be **simply-connected** if any closed curve drawn in R can be shrunk to a point without leaving R. If we restrict ourselves to two dimensions then any region with a hole in it is not simply-connected (left-hand picture in figure 14). A region which is not simply-connected is said to be **multiply-connected**.

If R is multiply-connected, Green's theorem is still true provided C is now interpreted as the entire (outer and inner) boundary, with C described so that the region R is always on the left (right hand picture in figure 14).

For example if we have a doubly-connected region, we describe the outer boundary C_0 in an anti-clockwise fashion and the inner boundary C_1 clockwise. We can then join the point A on C_0 to the point B on C_1 by the line AB. This line then divides R in such a way that it is a simply connected region bounded by the closed curve $C_0 + AB + C_1 + BA$. Then, by Green's theorem:

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} =$$

and therefore it follows that

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} =$$

where $C = C_0 + C_1$.

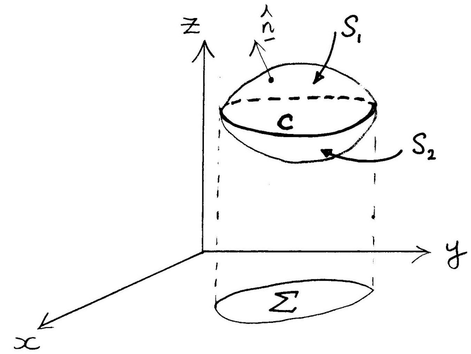


Figure 15: Diagram for the proof of the divergence theorem.

1.8.5 Flux

If S is a surface then the flux of $\mathbf{A}$ across S is defined as

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS.$$

If S is a closed surface then, by convention, we always draw the unit normal $\widehat{\mathbf{n}}$ **out** of S.

1.8.6 The divergence theorem

If τ is the volume enclosed by a closed surface S with unit outward normal $\widehat{\mathbf{n}}$ and $\mathbf{A}$ is a vector field with continuous derivatives throughout τ, then:

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_\tau \mathrm{div}\mathbf{A}\, d\tau.$$

Proof

We will assume that S is convex and that τ is simply connected, with no interior boundaries. Let $\mathbf{A} = (A_1, A_2, A_3)$ and $\widehat{\mathbf{n}} = (l, m, n)$. We have to prove that

$$\int_S (lA_1 + mA_2 + nA_3)\, dS =$$

Project S onto the plane $z = 0$ (figure 15). The cylinder with normal cross-section Σ and generators parallel to the $z-$axis circumscribes S and it touches S along the curve C which divides S into two open surfaces, S_1 (upper) and S_2 (lower). Both S_1 and S_2 have projection Σ in the plane $z = 0$. Suppose the equations of S_1 and S_2 are $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Then:

$$\int_\tau \frac{\partial A_3}{\partial z}\, dx\, dy\, dz \quad =$$

$$=$$

Now, using the projection theorem:

$$\int_{S_1} nA_3 \, dS \quad = $$

$$= $$

Similarly:

$$\int_{S_2} nA_3 \, dS \quad = $$

$$= $$

Thus:

$$\int_S nA_3 \, dS = $$

and therefore

$$\int_\tau \frac{\partial A_3}{\partial z} \, d\tau = $$

Similarly, by projecting onto the planes $x = 0$ and $y = 0$:

$$\int_\tau \frac{\partial A_1}{\partial x} \, d\tau = $$

and

$$\int_\tau \frac{\partial A_2}{\partial y} \, d\tau = $$

and hence

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}} \, dS = $$

as required.

Note that the surface S need not necessarily be smooth - it could be, for example, a cube or a tetrahedron.

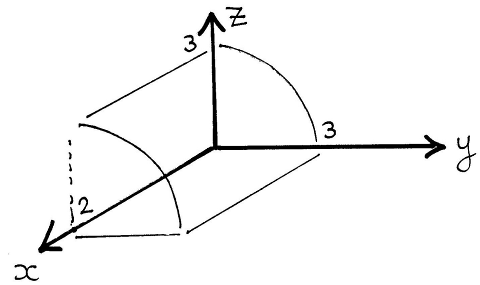


Figure 16: The surface S in the example.

Example

Evaluate

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS \text{ if } \mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k},$$

and S is the surface of the region in the first octant bounded by $y^2 + z^2 = 9, x = 2$ and $x = y = z = 0$.

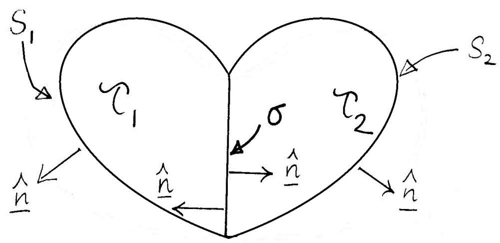


Figure 17: The divergence theorem applied to a non-convex surface.

1.8.7 The divergence theorem in more-complicated geometries

(i) Non-convex surfaces

If the surface is non-convex, the divergence theorem still holds. This can be established for a given surface by slicing the enclosed volume into sub-volumes, the boundaries of which can be described by single-valued functions of $(x, y), (y, z)$ and (x, z).

As an example consider the volume with 'heart-shaped' cross-section in figure 17. In this case the non-convex surface S can be divided by a surface σ into two parts S_1 and S_2 which, together with σ, form convex surfaces $S_1 + \sigma, S_2 + \sigma$ (figure 17). We can then apply the divergence theorem to $S_1 + \sigma, S_2 + \sigma$ with τ_1, τ_2 being the respective enclosed volumes, where $\tau_1 + \tau_2 = \tau$. On adding the results, the surface integrals over σ cancel out, and since $S = S_1 + S_2$ we have

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_\tau \mathrm{div}\mathbf{A}\, d\tau$$

as before.

More complicated geometries require further slicing. For example, for the case of a torus see `https://www.math.uci.edu/ ndonalds/math2e/16-9divergence.pdf` .

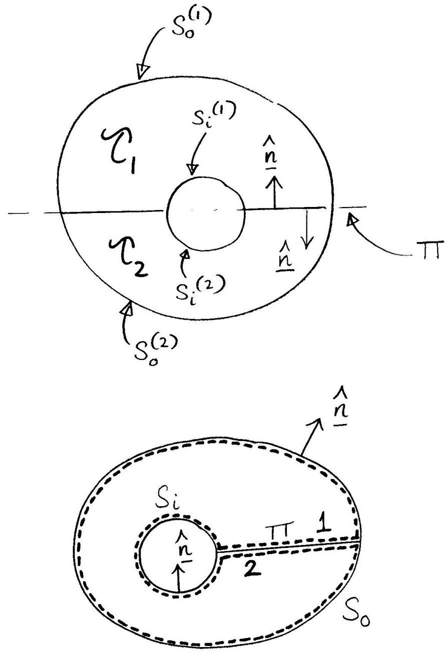


Figure 18: Diagrams for the proof of the divergence theorem in (top): a simply-connected domain; (bottom): a multiply-connected region.

(ii) A region with internal boundaries

(a) Simply-connected regions (top diagram in figure 18)

For example this could be the space between concentric spheres. Suppose we have an interior surface S_i and outer surface S_o. Draw a plane Π that cuts both S_o and S_i. This divides S_o into two open surfaces $S_o^{(1)}, S_o^{(2)}$. S_i is similarly divided into $S_i^{(1)}, S_i^{(2)}$. We then apply the divergence theorem to the volume τ_1 which is bounded by the closed surface $S_o^{(1)} + S_i^{(1)} + \Pi$, and we then apply the divergence theorem to the volume τ_2 which is bounded by $S_o^{(2)} + S_i^{(2)} + \Pi$. We add these results together. The contributions over Π cancel, leaving the result:

$$\int_{S_o+S_i} \mathbf{A}\cdot\widehat{\mathbf{n}}\, dS =$$

with the normal to S_i drawn inwards, i.e. out of τ.

(b) Multiply-connected regions (bottom diagram in figure 18)

For example this could be the region between two cylinders. Again let S_o and S_i be the outer and inner surfaces, linked by the plane Π. Label the two sides of the plane 1 and 2. Consider the surface

This is closed and encloses a simply-connected region τ. We then apply the divergence theorem to τ. The contributions along the two sides of Π cancel, giving

$$\int_{S_o+S_i} \mathbf{A}\cdot\widehat{\mathbf{n}}\, dS = \int_\tau \mathrm{div}\mathbf{A}\, d\tau.$$

1.8.8 Green's identities in 3D

Let ϕ and ψ be two scalar fields with continuous second derivatives. Consider the quantity

$$\mathbf{A} = \phi \nabla \psi.$$

It follows that

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \\ \widehat{\mathbf{n}} \cdot \mathbf{A} &= \end{aligned}$$

Applying the divergence theorem we obtain

$$\int_S \left\{ \phi \frac{\partial \psi}{\partial n} \right\} dS = \tag{1}$$

which is known as **Green's first identity**. Interchanging ϕ and ψ we have

$$\int_S \left\{ \psi \frac{\partial \phi}{\partial n} \right\} dS = \tag{2}$$

Subtracting (2) from (1) we obtain

$$\int_S \left\{ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right\} dS =$$

which is known as **Green's second identity**. These identities are very useful when constructing solutions to partial differential equations (see for example 'PDEs in action' in term 2).

1.8.9 Green's identities in 2D

If we use the divergence theorem in 2D derived in the first section of the notes:

$$\int_R \operatorname{div} \mathbf{F} \, dx \, dy = \oint_C \mathbf{F} \cdot \widehat{\mathbf{n}} \, ds.$$

then we can calculate down the corresponding Green identities. These are

$$\oint_C \phi \frac{\partial \psi}{\partial n} \, ds = \int_R \left[\phi \nabla^2 \psi + (\nabla \psi) \cdot (\nabla \phi) \right] \, dx \, dy$$

and

$$\oint_C \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \, ds = \int_R \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] \, dx \, dy.$$

These formulae are the generalisation of integration by parts to two dimensions.

1.8.10 Gauss' flux theorem

Let S be a closed surface with outward unit normal $\widehat{\mathbf{n}}$, and let O be the origin of the coordinate system. Then:

$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \begin{cases} 0, & \text{if } O \text{ is exterior to } S \\ 4\pi, & \text{if } O \text{ is interior to } S. \end{cases}$$

Proof

First suppose O is exterior to S and that S encloses a volume τ. Then we have $r \neq 0$ throughout τ. Applying the divergence theorem:

$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS =$$

But

$$\operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) =$$

Hence we have that

$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \int_\tau \operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right)\,d\tau = 0,$$

as required.

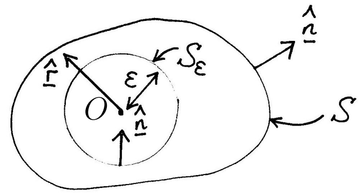


Figure 19: Diagram for the proof of Gauss theorem with O interior to S.

Now suppose O is interior to S (figure 19). We surround O with a small sphere radius ε, with surface S_ε, lying entirely within S. We consider the volume τ_ε enclosed between S and S_ε Then, applying the divergence theorem and proceeding as above we have

$$\int_{S+S_\varepsilon} \frac{\widehat{\mathbf{n}} \cdot \mathbf{r}}{r^3}\, dS =$$

Breaking up the surface integral into two parts:

$$0 = \int_{S+S_\varepsilon} \frac{\widehat{\mathbf{n}} \cdot \mathbf{r}}{r^3}\, dS =$$

However (since $r = \varepsilon$ on S_ε):

$$\int_{S_\varepsilon} \frac{\widehat{\mathbf{r}} \cdot \mathbf{r}}{r^3}\, dS =$$

Thus it follows that

$$\int_{S} \frac{\widehat{\mathbf{n}} \cdot \mathbf{r}}{r^3}\, dS =$$

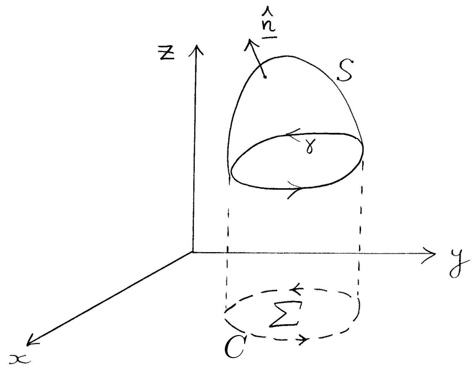


Figure 20: Diagram for the proof of Stokes' theorem.

1.8.11 Stokes theorem

Suppose S is an **open** surface with a simple closed curve γ forming its boundary, and let $\mathbf{A}$ be a vector field with continuous partial derivatives. Then:

$$\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{A} \cdot \widehat{\mathbf{n}} \, dS,$$

where the direction of the unit normal to S and the sense of γ are related by a right-hand rule (i.e. $\widehat{\mathbf{n}}$ is in the direction a right-handed screw moves when turned in the direction of γ).

Proof

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Consider

$$\operatorname{curl}(A_1\mathbf{i}) =$$

Then we have

$$\int_S [\operatorname{curl}(A_1\mathbf{i})] \cdot \widehat{\mathbf{n}} \, dS =$$

If we now project onto the $x-y$ plane, S becomes Σ say, and γ becomes C (figure 20). Let the equation of S be $z = f(x, y)$. Then we have

$$\widehat{\mathbf{n}} = \frac{\nabla(z - f(x, y))}{|\nabla(z - f(x, y))|} =$$

Therefore, on S :

$$\mathbf{j} \cdot \widehat{\mathbf{n}} =$$

Thus:

$$\begin{aligned}\int_S [\text{curl}\,(A_1\mathbf{i})] \cdot \widehat{\mathbf{n}}\, dS &= \\ &= \\ &= \\ &= \end{aligned}$$

with the last line following by using Green's theorem. However on γ we have $z = f$ and so

$$\oint_C A_1(x, y, f)\, dx =$$

We have therefore established that

$$\int_S (\text{curl}\, A_1\mathbf{i}\,) \cdot \widehat{\mathbf{n}}\, dS =$$

In a similar way we can show that

$$\int_S (\text{curl}\, A_2\mathbf{j}\,) \cdot \widehat{\mathbf{n}}\, dS =$$

and

$$\int_S (\text{curl}\, A_3\mathbf{k}\,) \cdot \widehat{\mathbf{n}}\, dS =$$

and so the theorem is proved by adding all three results together.

Note that although S must be open, it is not necessarily smooth. For example it could be in the shape of a box without a lid.

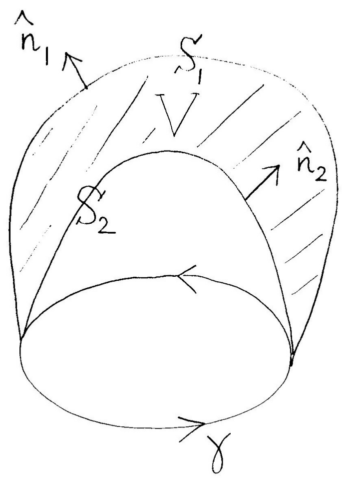


Figure 21: Two different open surfaces, both with the closed curve γ as boundary.

The theorem is actually true for **any** open surface with γ as boundary. To see this consider figure 21. The normal to S_1 is $\widehat{\mathbf{n}}_\mathbf{1}$ and to S_2 is $\widehat{\mathbf{n}}_\mathbf{2}$. The surface $S_1 + S_2$ is closed: let it enclose a volume V. Applying the divergence theorem to curl $\mathbf{A}$ over this region gives

$$\int_{S_1+S_2} \operatorname{curl}\mathbf{A}\cdot\widehat{\mathbf{n}}\,dS =$$

In the divergence theorem the normal must always point out of V and hence

$$0 = \int_{S_1+S_2} \operatorname{curl}\mathbf{A}\cdot\widehat{\mathbf{n}}\,dS =$$

implying that

Theorem

A necessary and sufficient condition that $\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = 0$ for any simple closed curve γ is that curl $\mathbf{A} = 0$ throughout the region in which γ is drawn (assuming $\mathbf{A}$ is continuously differentiable and the region is simply-connected).

Proof

We already know that if $\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = 0$ then there exists a potential ϕ such that $\mathbf{A} = \nabla\phi$. Therefore we see that curl $\mathbf{A} = 0$ since the curl of a gradient is always zero.

Conversely, if curl $\mathbf{A} = 0$ then by Stokes' theorem we have $\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = 0$ for any simple closed curve γ.

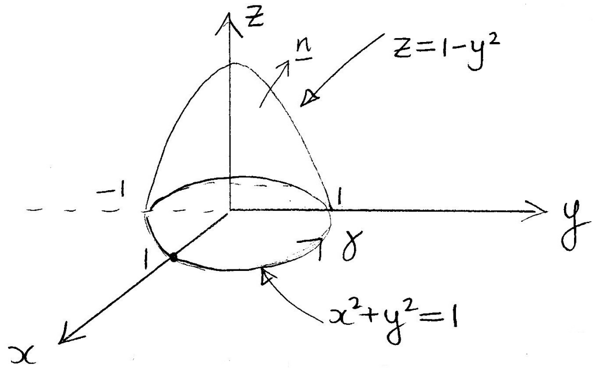


Figure 22: The parabolic surface $z = 1 - x^2 - y^2$ with $z \geq 0$.

Example

Verify Stokes theorem for the vector field $\mathbf{A} = (y, z, x)$ and the surface S given by $z = 1 - x^2 - y^2$ with $z \geq 0$.

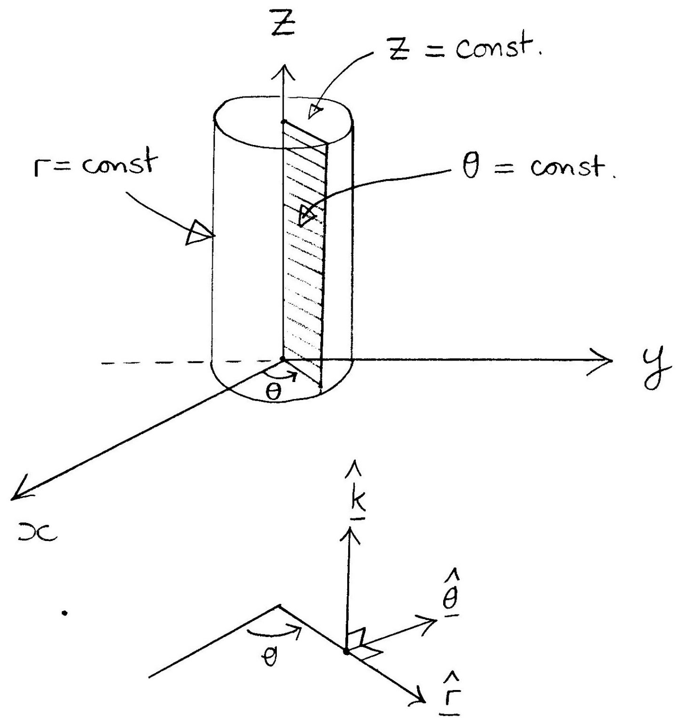


Figure 23: The surfaces $r=$constant, $\theta=$constant, $z=$constant, for the cylindrical polar coordinate system, and the orientation of the unit vectors.

1.9 Curvilinear coordinates

1.9.1 Introduction & definition

Often it is more convenient, depending on the geometry of the problem under consideration, to use coordinates other than Cartesians. An example is cylindrical polar coordinates (r, θ, z) which are related to Cartesian coordinates by

from which we can deduce that

The equation r = constant therefore defines a family of circular cylinders with axes along the z-axis, while the equation θ = constant defines a family of planes, as does the equation z = constant (figure 23). Cylindrical polar coordinates are an example of **curvilinear coordinates**. The unit vectors $\widehat{\mathbf{r}}, \widehat{\theta}, \widehat{\mathbf{k}}$ at any point P are perpendicular to the surfaces r = constant, θ = constant, z = constant through P in the directions of increasing r, θ, z. Note that the direction of the unit vectors $\widehat{\mathbf{r}}, \widehat{\theta}$ vary from point to point, unlike the corresponding Cartesian unit vectors.

More generally now, let us suppose that our Cartesian coordinates $(x, y, z) \equiv (x_1, x_2, x_3)$ can be expressed as single-valued differentiable functions of the new coordinates (u_1, u_2, u_3), i.e.

We would like to know what the conditions are under which we can invert these expressions and write the u_i as single-valued differentiable functions of the x_i. First let's differentiate the above expression with respect to x_j :

Writing this out for each i and j we have the matrix equation

$$\begin{pmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \partial x_1/\partial u_3 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \partial x_2/\partial u_3 \\ \partial x_3/\partial u_1 & \partial x_3/\partial u_2 & \partial x_3/\partial u_3 \end{pmatrix} \begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{pmatrix} = I,$$

where I is the identity matrix. We can express this more succinctly as

where $J(x_u)$ is the **Jacobian matrix** for the (x_1, x_2, x_3) system and $J(u_x)$ is the corresponding Jacobian for (u_1, u_2, u_3). We therefore see that $J(u_x)$ exists (i.e. the u_i are differentiable functions of the x_i provided $(J(x_u))^{-1}$ exists, i.e. we require

It turns out that this condition is sufficient to guarantee that our transformation can be inverted. More precisely, the **inverse function theorem** states that around any point where $\det(J(x_u))$ is nonzero, there exists a neighbourhood in which the u_i can be expressed as single-valued differentiable functions of the x_i. There is more on this theorem in the Differential Equations course next term.

Note also that the result $J(x_u)J(u_x) = I$ implies that

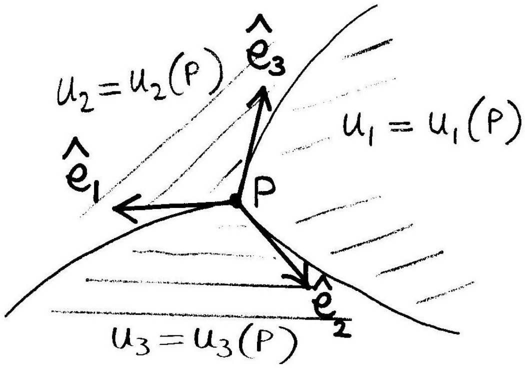
a useful result that we will exploit later when we consider the transformation of integrals. From now on we will assume we are in a region where $\det(J(x_u)) \neq 0$ and so our transformations can indeed be inverted.

Example

Consider cylindrical polar coordinates (r, θ, z) again. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{pmatrix} =$$

and so the determinant is equal to $r(\cos^2\theta + \sin^2\theta) = r$. So provided $r \neq 0$, the transformation can be inverted.



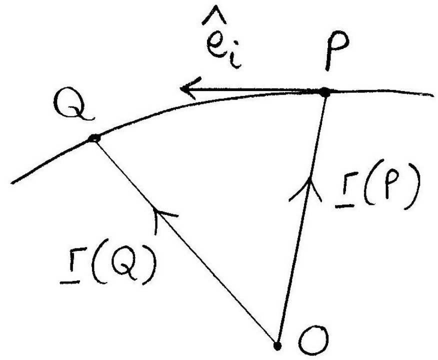


Figure 24: Left: the intersection of the surfaces $u_i = u_i(P)$; right: P and Q are points on a curve along which only one component u_i varies.

Given that we can now write $u_i = u_i(x_1, x_2, x_3)$, the equations $u_1 =$ constant, $u_2 =$ constant, $u_3 =$ constant define three families of surfaces, and (u_1, u_2, u_3) is said to be a **curvilinear coordinate system**. Through each point $P(x, y, z)$ there passes one member of each family. Let $(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)$ be unit vectors at P in the directions normal to $u_1 = u_1(P), u_2 = u_2(P), u_3 = u_3(P)$ respectively, such that u_1, u_2, u_3 increase in the directions $\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3$. Clearly we must have

$$\widehat{\mathbf{a}}_i =$$

If $(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)$ are mutually orthogonal, the coordinate system is said to be an **orthogonal curvilinear coordinate system**.

The surfaces $u_2 = u_2(P)$ and $u_3 = u_3(P)$ intersect in a curve, along which only u_1 varies. Let $\widehat{\mathbf{e}}_1$ be the unit vector tangential to the curve at P. Let $\widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$ be unit vectors tangential to curves along which only u_2, u_3 vary. For an orthogonal system we must have $\widehat{\mathbf{e}}_i = \widehat{\mathbf{a}}_i$ (left diagram in figure 24). Let Q be a neighbouring point to P on the curve along which only u_i varies (right diagram of figure 24). We have

$$\frac{\partial \mathbf{r}}{\partial u_i} =$$

where we have defined $h_i = |\partial \mathbf{r}/\partial u_i|$. The quantities h_i are often known as the **length scales** for the coordinate system.

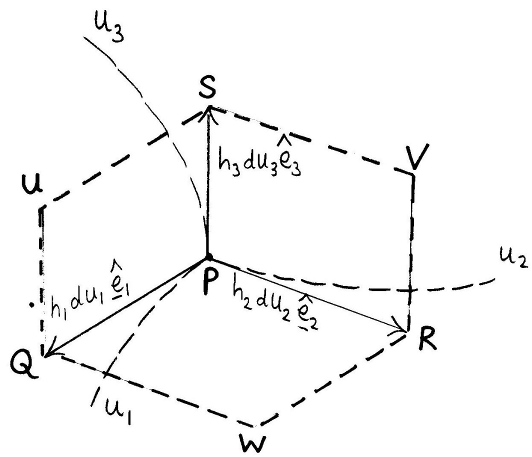


Figure 25: A volume element in an orthogonal curvilinear coordinate system.

1.9.2 Path element

Since $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$, the **path element** $d\mathbf{r}$ is given by

$$d\mathbf{r} \quad =$$

If the system is orthogonal then it follows that

$$(ds)^2 =$$

In what follows we will assume we have an orthogonal system so that

$$\widehat{\mathbf{e}}_i = \widehat{\mathbf{a}}_i =$$

In particular, path elements along curves of intersection of u_i surfaces have lengths $h_1 du_1, h_2 du_2, h_3 du_3$ respectively.

1.9.3 Volume element

Since the volume element is approximately rectangular (figure 25) we can take

$$d\tau \quad =$$

1.9.4 Surface element

Also from figure 25, by looking at the areas of the faces of the volume element, we can see that the surface element for a surface with u_1 constant is

$$dS =$$

and similarly for $u_2 =$ constant, $u_3 =$ constant.

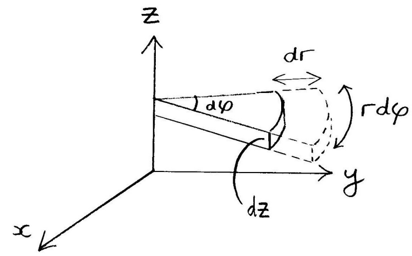


Figure 26: An element of volume in cylindrical polar coordinates.

1.9.5 Properties of various orthogonal coordinate systems

(i) Cartesian coordinates (x, y, z)

$$d\tau \quad = \qquad\qquad\qquad d\mathbf{r} =$$

$$(ds)^2 \quad =$$

and so $h_1 = h_2 = h_3 = 1$ in this case.

(ii) Cylindrical polar coordinates (r, ϕ, z)

See figure 26. The coordinates are related to Cartesians by

To show that this is an orthogonal system we calculate

$\partial\mathbf{r}/\partial r \quad =$

$\partial\mathbf{r}/\partial\phi \quad =$

$\partial\mathbf{r}/\partial z \quad =$

Orthogonality then follows from the fact that

The lengthscales are

and so the elements of length and volume are

The surface elements can also be calculated, e.g. an element of the surface along which r is constant (i.e. a cylinder) is

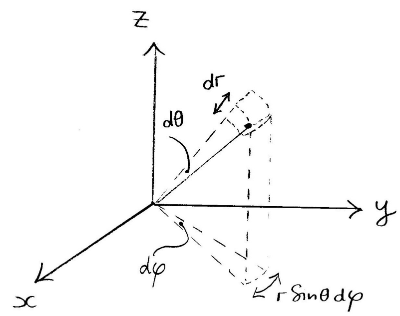


Figure 27: An element of volume in spherical polar coordinates.

(iii) Spherical polar coordinates (r, θ, ϕ)

See figure 27. In this case the relationship between the coordinates is

Then

$\partial\mathbf{r}/\partial r \;\; =$

$\partial\mathbf{r}/\partial\theta \;\; =$

$\partial\mathbf{r}/\partial\phi \;\; =$

It can then be seen that

$(\partial\mathbf{r}/\partial r)\cdot(\partial\mathbf{r}/\partial\theta) =$

Similarly:

and so the system is orthogonal. Then

$h_1 \;\; =$

$h_2 \;\; =$

$h_3 \;\; =$

(We have assumed here that $\sin\theta > 0$, which is OK since the range of θ is 0 to π). The volume element is

Also, an element of the surface $r =$ constant $= a$ (i.e. a sphere of radius a) is:

Example

Find the volume and surface area of a sphere of radius a, and also find the surface area of a cap of the sphere that subtends an angle 2α at the centre of the sphere.

1.9.6 Gradient in orthogonal curvilinear coordinates

Let

$$\nabla\Phi = \lambda_1\widehat{\mathbf{e}}_1 + \lambda_2\widehat{\mathbf{e}}_2 + \lambda_3\widehat{\mathbf{e}}_3$$

in a general coordinate system, where $\lambda_1, \lambda_2, \lambda_3$ are to be found. Recall that the element of length is given by

$$d\mathbf{r} =$$

Now

$$d\Phi =$$

$$=$$

$$=$$

But, using our expressions for $\nabla\Phi$ and $d\mathbf{r}$ above:

$$(\nabla\Phi)\cdot d\mathbf{r} =$$

and so we see that

$$h_i\lambda_i =$$

Thus we have the result that

$$\nabla\Phi =$$

This result now allows us to write down ∇ easily for other coordinate systems.

(i) Cylindrical polars (r, ϕ, z)

Recall that $h_1 = 1, h_2 = r, h_3 = 1$. Thus

$$\nabla =$$

(ii) Spherical polars (r, θ, ϕ)

We have $h_1 = 1, h_2 = r, h_3 = r\sin\theta$, and so

$$\nabla =$$

1.9.7 Expressions for unit vectors

From the expression for ∇ we have just derived it is easy to see that:

$$\widehat{\mathbf{e}}_i =$$

Alternatively, since the unit vectors are orthogonal, if we know two unit vectors we can find the third from the relation

$$\widehat{\mathbf{e}}_1 =$$

and similarly for the other components, by permuting in a cyclic fashion.

1.9.8 Divergence in orthogonal curvilinear coordinates

Suppose we have a vector field

$$\mathbf{A} = A_1\widehat{\mathbf{e}}_1 + A_2\widehat{\mathbf{e}}_2 + A_3\widehat{\mathbf{e}}_3.$$

First consider

$$\nabla \cdot (A_1\widehat{\mathbf{e}}_1) =$$

$$=$$

using the results established just above. Also we know that

$$\nabla \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot \operatorname{curl} \mathbf{B} - \mathbf{B} \cdot \operatorname{curl} \mathbf{C},$$

and so it follows that

$$\nabla \cdot (\nabla u_2 \times \nabla u_3) =$$

since the curl of a gradient is always zero. Thus we are left with

$$\nabla \cdot (A_1\widehat{\mathbf{e}}_1) =$$

We can proceed in a similar fashion for the other components, and establish that

$\nabla \cdot \mathbf{A} =$

It is now easy to write down div in other coordinate systems.

(i) Cylindrical polars (r, ϕ, z)

Recall that $h_1 = 1, h_2 = r, h_3 = 1$. Thus using the above formula:

$$\nabla \cdot \mathbf{A} =$$

$$=$$

(ii) Spherical polars (r, θ, ϕ)

We have $h_1 = 1, h_2 = r, h_3 = r\sin\theta$. Hence

$\nabla \cdot \mathbf{A} =$

1.9.9 Curl in orthogonal curvilinear coordinates

Again just consider the curl of the first component of $\mathbf{A}$:

$\nabla \times (A_1 \widehat{\mathbf{e}}_1) =$

$=$

$=$

$=$

$=$

(since $\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_1 = 0$, $\widehat{\mathbf{e}}_2 \times \widehat{\mathbf{e}}_1 = -\widehat{\mathbf{e}}_3$, $\widehat{\mathbf{e}}_3 \times \widehat{\mathbf{e}}_1 = \widehat{\mathbf{e}}_2$). We can obviously find curl $(A_2\widehat{\mathbf{e}}_2)$ and curl $(A_3\widehat{\mathbf{e}}_3)$ in a similar way. These can be shown to be

$$\begin{aligned}
\nabla \times (A_2\widehat{\mathbf{e}}_2) &= \frac{\widehat{\mathbf{e}}_3}{h_2 h_1}\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\widehat{\mathbf{e}}_1}{h_2 h_3}\frac{\partial}{\partial u_3}(h_2A_2), \\
\nabla \times (A_3\widehat{\mathbf{e}}_3) &= \frac{\widehat{\mathbf{e}}_1}{h_3 h_2}\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\widehat{\mathbf{e}}_2}{h_3 h_1}\frac{\partial}{\partial u_1}(h_3A_3).
\end{aligned}$$

Adding the three contributions together, we find we can write this in the form of a determinant as

$\operatorname{curl} \mathbf{A} =$

in which form it is probably easiest remembered. It's then straightforward to write down curl in various orthogonal coordinate systems.

(i) Cylindrical polars

$$\operatorname{curl} \mathbf{A} = \frac{1}{r}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\phi} & \widehat{\mathbf{k}} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_1 & rA_2 & A_3 \end{vmatrix}.$$

(ii) Spherical polars

$$\operatorname{curl} \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\theta} & r\sin\theta\,\widehat{\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_1 & rA_2 & r\sin\theta A_3 \end{vmatrix}.$$

1.9.10 The Laplacian in orthogonal curvilinear coordinates

From the formulae already established for grad and div, we can see that

$$\begin{aligned}\nabla^2\Phi &= \nabla\cdot(\nabla\Phi)\\ &=\end{aligned}$$

This formula can then be used to calculate the Laplacian for various coordinate systems.

(i) Cylindrical polars (r, ϕ, z)

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}.\end{aligned}$$

(ii) Spherical polars (r, θ, ϕ)

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.\end{aligned}$$

1.9.11 Alternative definitions for grad, div, curl (not examinable)

Let τ be a region enclosed by a surface S and let P be a general point of τ. We established earlier that

$$\int_\tau \nabla\phi\, d\tau = \int_S \widehat{\mathbf{n}}\,\phi\, dS.$$

This result is a consequence of the divergence theorem (see problem sheet). It follows that

$$\int_\tau \mathbf{i}\cdot\nabla\phi\, d\tau = \int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi\, dS.$$

Now the left-hand-side above can be written as $\tau\{\overline{\mathbf{i}\cdot\nabla\phi}\}$ where the bar denotes the mean value of this quantity over τ. Since we are assuming that ϕ has continuous derivatives throughout τ, we can write

$$\{\overline{\mathbf{i}\cdot\nabla\phi}\} = \{\mathbf{i}\cdot\nabla\phi\}_Q$$

for some point Q of τ. Thus we have that

$$\{\mathbf{i}\cdot\nabla\phi\}_Q = \frac{1}{\tau}\int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi\, dS.$$

Now let $\tau \to 0$ about P. Then $P \to Q$ and we have that at any point P of τ :

$$\mathbf{i}\cdot\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi\, dS.$$

Similar results can be established for $\mathbf{j}\cdot\nabla\phi$ and $\mathbf{k}\cdot\nabla\phi$. Taken together, these imply that

$$\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S \widehat{\mathbf{n}}\,\phi\, dS.$$

This can be regarded as an alternative way of defining $\nabla\phi$, rather than defining it as $(\partial\phi/\partial x)\mathbf{i} + (\partial\phi/\partial y)\mathbf{j} + (\partial\phi/\partial z)\mathbf{k}$.

We can similarly establish that

$$\begin{aligned} \mathrm{div}\mathbf{A} &= \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\mathbf{n}}\cdot\mathbf{A})\, dS, \\ \mathrm{curl}\,\mathbf{A} &= \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\mathbf{n}}\times\mathbf{A})\, dS, \end{aligned}$$

which are alternative definitions of the divergence and curl, and are clearly independent of the choice of coordinates, which is one of the advantages of this approach. In particular we can see that the divergence is a measure of the flux of a quantity.

Equivalence of definitions

Let's show that the definition of divergence given here is consistent with the curvilinear formula given earlier. Consider $\delta\tau$ to be the volume of a curvilinear volume element located at the point P, with edges of length $h_1\delta u_1, h_2\delta u_2, h_3\delta u_3$, and unit vectors aligned as shown in the picture (figure 28). The volume of the element $\delta\tau \simeq h_1h_2h_3\delta u_1\delta u_2\delta u_3$. We start with our definition

$$\mathrm{div}\mathbf{A} = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\mathbf{n}}\cdot\mathbf{A})\, dS,$$

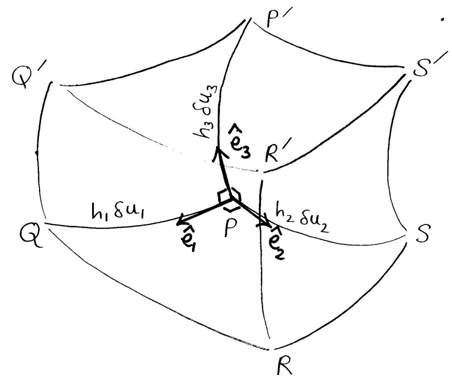


Figure 28: A curvilinear volume element.

and aim to compute explicitly the right-hand-side. This involves calculating the contributions to $\int_S$ arising from the six faces of the volume element. If we start with the contribution from the face $PP'S'S$, this is:

$$-(A_1h_2h_3)_P\,\delta u_2\delta u_3 + \text{ higher order terms.}$$

The contribution from the face $QQ'R'R$ is

$$(A_1h_2h_3)_Q\,\delta u_2\delta u_3 + \text{h.o.t.} \;=\left[(A_1h_2h_3)+\frac{\partial}{\partial u_1}(A_1h_2h_3)\delta u_1\right]_P\delta u_2\delta u_3 + \text{h.o.t.},$$

using a Taylor series expansion. Adding together the contributions from these two faces we get

$$\left[\frac{\partial}{\partial u_1}(A_1h_2h_3)\right]_P\delta u_1\delta u_2\delta u_3 + \text{ h.o.t.}$$

Similarly, the sum of the contributions from the faces $PSRQ, P'S'R'Q'$ is

$$\left[\frac{\partial}{\partial u_3}(A_3h_1h_2)\right]_P\delta u_1\delta u_2\delta u_3 + \text{ h.o.t.},$$

while the combined contributions from $PQQ'P', SRR'S'$ is

$$\left[\frac{\partial}{\partial u_2}(A_2h_3h_1)\right]_P\delta u_1\delta u_2\delta u_3 + \text{ h.o.t..}$$

If we then let $\delta\tau \to 0$ we have that

$$\lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\int_S \widehat{\mathbf{n}}\cdot\mathbf{A}\,dS = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_3h_1)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right\},$$

and so we can see that the integral expression for div$\mathbf{A}$ is consistent with the formula in curvilinear coordinates derived earlier.

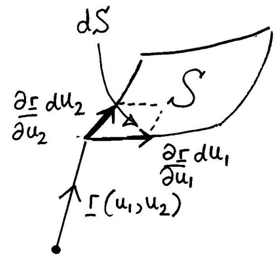


Figure 29: A surface S parameterized by u_1 and u_2.

1.10 Changes of variable in surface integration

Suppose we have a surface S which is parameterized by the quantities u_1, u_2. We can therefore write that on S:

$$x = \qquad\qquad , \quad y = \qquad\qquad , \quad z = \qquad\qquad .$$

[For example, if S is the surface of a sphere of unit radius we have $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$ and so we can take $u_1 = \theta$, $u_2 = \phi$.]

We can consider the surface to be comprised of arbitrarily small parallelograms whose sides are obtained by keeping either u_1 or u_2 constant: see figure 29, i.e.

$$\begin{aligned} dS &= \text{Area of parallelogram with sides } \frac{\partial \mathbf{r}}{\partial u_1} du_1 \text{ and } \frac{\partial \mathbf{r}}{\partial u_2} du_2 \\ &= |\mathbf{J}|\, du_1\, du_2, \end{aligned}$$

where the **vector Jacobian J** is given by $\mathbf{J} =$

This result is particularly useful when using a substitution in a surface integral, as we can write

$$\int_S f(x, y, z)\, dS =$$

where $F(u_1, u_2) = f(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$.

If S is a region R in the $x - y$ plane, (i.e. $z = 0$ on R), the result reduces to

$$\int_R f(x, y)\, dx\, dy =$$

where $J(x_u)$ is the Jacobian matrix we met earlier, i.e.

$$J(x_u) =$$

Note that since $dx\, dy = |\det(J(x_u))|\ du_1 du_2$ it follows that $du_1 du_2 = (1/|\det(J(x_u))|) dx\, dy$, and hence

$$1/|\det(J(x_u))| =$$

which is a result we found earlier by a different method. These formulae apply for both orthogonal and non-orthogonal transformations.

Suppose a surface is described by $z = f(x, y)$. Then $u_1 = x$, $u_2 = y$ and $\mathbf{r} = (x, y, f(x, y))$. It follows that

$$\frac{\partial \mathbf{r}}{\partial u_1} = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial u_2} = \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

so then

$$\left|\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}\right| =$$

$$=$$

$$=$$

$$=$$

Therefore the area of surface is

$$\int_\Sigma \sqrt{(1 + |\nabla f|^2)}\, dx\, dy,$$

where Σ is the projection of S onto the $x - y$ plane. We will use this expression in the next section.

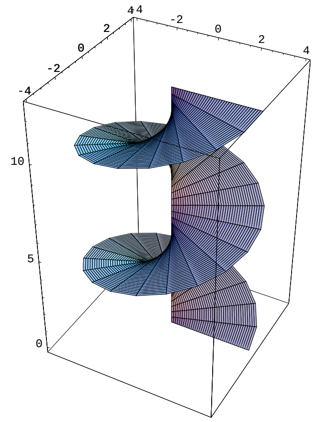


Figure 30: A section of a helicoid.

Example

Evaluate the integral

$$\int_S \sqrt{(1 + x^2 + y^2)}\, dS$$

where S is the surface of the helicoid (shown in figure 30):

$$x = u\cos v, \quad y = u\sin v, \quad z = v,$$

with $0 \leqslant u \leqslant 4$ and $0 \leqslant v \leqslant 4\pi$.