MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Quiz 5

Question 1. Correct answer: (a).

In Section 3 of Chapter 3, we considered all phase portraits in Jordan normal form, and only in I.(C4) a periodic orbit can occur. Here, one sees that all non-trival orbits are periodic. That this carries over to the orignal system follows from $e^{At} = Te^{Jt}T^{-1}$ (see also Exercise 22).

Question 2. Correct answer: (b).

This is the stable focus with $a = -1$ and $b = 1$ from page 47. Clearly all non-zero solutions are unbounded.

Question 3. Correct answer: (b).

Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then the two matrix exponential functions read as $e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ and $e^{Bt} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}$. Now $A + B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and we have $e^{(A+B)t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$, which is clearly not the same as $e^{At}e^{Bt}$.

Question 4. Correct answer: (a).

We have $\dot{\lambda}(t) = \dot{\mu}(t) - \dot{\nu}(t) = A\mu(t) + g(t) - A\nu(t) - h(t) = A(\mu(t) - \nu(t)) + g(t) - h(t) = A\lambda(t) + g(t) - h(t)$.

Question 5. Correct answer: (a).

Consider $X(t) = \left(e^{At}\right)^\top e^{At}$ for all $t \in \mathbb{R}$. The correct answer for Question 4 from Quiz 4 implies that $X(t) = e^{A^\top t}e^{At}$ for all $t \in \mathbb{R}$. This means that $\dot{X}(t) = A^\top e^{A^\top t}e^{At} + e^{A^\top t}Ae^{At} = e^{A^\top t}A^\top e^{At} + e^{A^\top t}Ae^{At} = e^{A^\top t}(A^\top + A)e^{At} = 0$. This implies that the matrix-valued function X is constant. We obviously have $X(0) = \mathrm{Id}_d$, and thus $X(t) = \mathrm{Id}_d$ for all $t \in \mathbb{R}$. This implies that e^{At} is orthogonal.