

## 22 The Method of Characteristics

Recall earlier notes (repeated here) : Most physical systems are governed by second order PDEs. In this course we discuss **Finite Difference Methods (FDMs)** for solving such equations. We want our algorithms to be able to reproduce the physics and so to begin with, we must understand the physical background. We consider the equation for  $u(x, y)$

$$a u_{xx} + b u_{xy} + c u_{yy} = f. \quad (22.1)$$

This equation is called **quasi-linear** provided the functions  $a$ ,  $b$ ,  $c$  and  $f$  do not depend on  $u_{xx}$ ,  $u_{xy}$  or  $u_{yy}$ , namely that the highest order derivatives occur linearly. They may, however depend on  $x$ ,  $y$ ,  $u$ ,  $u_x$  and  $u_y$ , so that (22.1) is not necessarily linear – for example it is perfectly allowable in the discussion that follows that

$$f = d u_x + e u_y + h u + g, \quad (22.2)$$

provided  $d$ ,  $e$ ,  $h$ ,  $g$  functions retain the *quasi-linear* requirement.

Suppose we know  $u$ ,  $u_x$  and  $u_y$  along some curve  $\Gamma$  in  $(x, y)$ -space. From a point  $P$  on  $\Gamma$  we move a small vector displacement  $(dx, dy)$  to a new point  $Q$  not on  $\Gamma$ , as shown in Fig. 22.1. Under what circumstances can we determine uniquely the values of  $u$ ,  $u_x$  and  $u_y$  at  $Q$ ? We denote the change in these variables by  $du$ ,  $d(u_x)$  and  $d(u_y)$ . Then by the chain rule for partial derivatives  $du = u_x dx + u_y dy$ , which is known because  $u_x$  and  $u_y$  are known along  $\Gamma$ . Similarly,

$$\left. \begin{aligned} d(u_x) &= u_{xx} dx + u_{xy} dy \\ d(u_y) &= u_{xy} dx + u_{yy} dy \end{aligned} \right\}. \quad (22.3)$$

We combine (22.1) and (22.3) in matrix form:

$$\begin{pmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix}, \quad (22.4)$$

$a$ ,  $b$  and  $c$  are known locally because  $u$ ,  $u_x$  and  $u_y$  are known, and so the  $3 \times 3$  matrix is known. Equation (22.4) will have a unique solution for  $u_{xx}$ ,  $u_{xy}$  and  $u_{yy}$  unless the determinant of that matrix vanishes, that is unless

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = a(dy)^2 - b dx dy + c(dx)^2 = 0. \quad (22.5)$$

If (22.5) holds, the equation (22.4) will have either no solution or infinitely many solutions. The condition for solutions to exist, which we will use later in the course, is that

$$a \frac{d(u_x)}{dx} + c \frac{d(u_y)}{dy} = f. \quad (22.6)$$

This is surprising. If we can choose a direction  $(dx, dy)$  which satisfies (22.5) we have the possibility that the second derivatives  $u_{xx}$  etc. may not be uniquely defined. In other words, the solution may have discontinuities across the line  $PQ$ , with  $u_{xx}$  taking different values on each side.

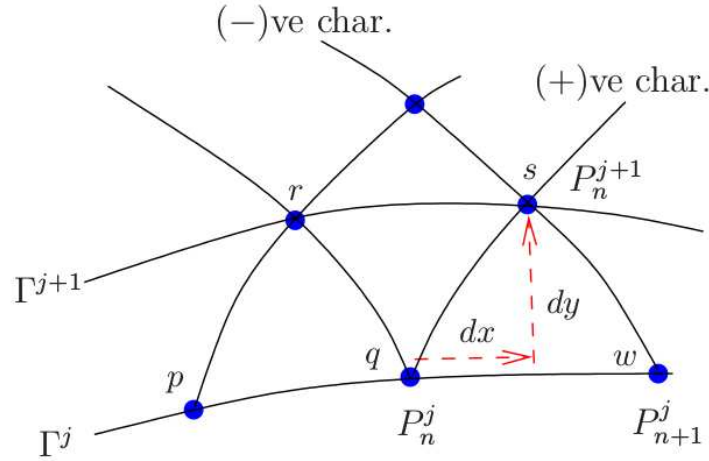


Figure 22.1: Problem description.

It is very important to know whether our solution can have this property. Equation (22.5) is called the **Characteristic equation** of (22.1). It is a quadratic in  $dy/dx$  with solution

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (22.7)$$

We can use equations (22.7) and (22.6) to find the solution. Equation (22.7) defines two families of characteristics, one with the + sign and one with the - sign. Starting from a given point  $(x, y)$  we can step (22.7) using some ODE-solver to find a new point

$$(x + dx, y + dy)$$

which lies on the characteristic. Note that characteristics from the same family are approximately parallel – if ever they cross, then a **shock** develops and the solution becomes singular. However the two families have different gradients and so characteristics from different families meet all the time. So if we have a set of points  $P_n^j$  on some curve  $\Gamma^j$  at some “time”,  $j$ , we can define points  $P_n^{j+1}$  to be the intersection of the (+)ve characteristic from  $P_n^j$  and the (-)ve characteristic from  $P_{n+1}^j$  as in the diagram, thus defining a new curve  $\Gamma^{j+1}$ .

Now assume  $u, u_x$  and  $u_y$  are known on  $\Gamma^j$ . This means that  $u$  is known after an infinitesimal step  $(dx, dy)$  by

$$du = u_x dx + u_y dy,$$

but  $u_x$  and  $u_y$  need to be found. The key is to use the condition (22.7) along the characteristics, relating how much  $u_x$  and  $u_y$  change  $j$ . As we have two equations in two unknowns we can from the known values at  $P_n^j$  and  $P_{n+1}^j$  easily determine  $u_x$  and  $u_y$  at the new point  $P_n^{j+1}$  and hence we can advance the region where the solution is known to a larger region, bounded by  $\Gamma^{j+1}$ . This defines the method of characteristics.

As an example suppose the roots of (22.7) are real and distinct, and can be denoted by

$$\frac{dy}{dx} = F, \quad \text{and} \quad \frac{dy}{dx} = G, \quad (22.8)$$

where initial values of  $u$ ,  $u_x$  and  $u_y$  are known on  $\Gamma^j$ . As a first approximation, with reference to Fig. 22.1, we may regard  $p - r$  and  $q - r$  as straight lines of slopes  $F_p$  and  $G_q$ . Then (22.8) can be approximated by

$$y_r - y_p = F_p (x_r - x_p), \text{ and } y_r - y_q = G_q (x_r - x_q), \quad (22.9)$$

giving two equations for the two unknowns  $(x_r, y_r)$ . Next equation (22.1) can be used to show that the differentials  $P = u_x$  and  $Q = u_y$  are related by the following **compatibility** relationship

$$a \frac{dy}{dx} dP + c dQ - f dy = 0, \quad (22.10)$$

hence

$$a F dP + c dQ - f dy = 0, \text{ and } a G dP + c dQ - f dy = 0. \quad (22.11)$$

We approximate the first expression along  $p - r$  by the equation

$$a_p F_p (P_r - P_p) + c_p (Q_r - Q_p) - f_p (y_r - y_p) = 0, \quad (22.12)$$

and the second expression along  $q - r$  thus

$$a_q G_q (P_r - P_q) + c_q (Q_r - Q_q) - f_q (y_r - y_q) = 0. \quad (22.13)$$

Once  $(x_r, y_r)$  have been evaluated from (22.9), these are two equations for unknowns  $(P_r, Q_r)$ . The value of  $u$  on  $r$  can then be obtained from

$$du = P dx + Q dy, \quad (22.14)$$

on replacing values of  $(P, Q)$  along  $p - r$  by

$$u_r - u_p = \frac{1}{2}(P_p + P_r)(x_r - x_p) + \frac{1}{2}(Q_p + Q_r)(y_r - y_p). \quad (22.15)$$

Solution of the above gives us a first approximation to  $u_r$ . We next improve upon this by replacing expressions (22.9) with new improved estimates of  $(x_r, y_r)$ , using

$$y_r - y_p = \frac{1}{2}(F_p + F_r)(x_r - x_p), \text{ and } y_r - y_q = \frac{1}{2}(G_q + G_r)(x_r - x_q), \quad (22.16)$$

and equations (22.12)–(22.13) become

$$\left. \begin{aligned} \frac{1}{4}(a_p + a_r)(F_p + F_r)(P_r - P_p) + \frac{1}{2}(c_p + c_r)(Q_r - Q_p) - \frac{1}{2}(f_p + f_r)(y_r - y_p) &= 0, \\ \frac{1}{4}(a_q + a_r)(G_q + G_r)(P_r - P_q) + \frac{1}{2}(c_q + c_r)(Q_r - Q_q) - \frac{1}{2}(f_q + f_r)(y_r - y_q) &= 0. \end{aligned} \right\}, \quad (22.17)$$

An improved value for  $u_r$  then follows from (22.15). Repetition of the above steps, through iteration and correction thus provides  $u_r$  to sufficient accuracy. The number of iterations can be minimised by making  $q$  and  $p$  close to each other.

An advantage of the method is that by explicitly following characteristics we are modelling the physics well. If the solution does have discontinuities, we expect to follow them accurately. A disadvantage is that it does not generalise easily to 3D, when the characteristic curves become cones. Also, we can no longer have a regular grid – the equation decides

where to place the points  $P_n^j$ , and they may bunch together. One could argue, that grid points are being carried to areas where they are needed, but there is the danger of thinning of points in some regions.

This leads us on to a more fundamental question for this course : we have always tried to use a regular grid, which affords us reliable 2-nd order accuracy. Should we relax this in some cases? We need points in order to resolve rapid variation. If we have a solution like

$$u = e^{-10x+x^2},$$

we might want to have more points near  $x = 0$  than near  $x = 1$ . To put it another way, if we use a small enough step-length to resolve the fast variation, we are being unnecessarily wasteful of computing power elsewhere. Suppose we introduce a new coordinate,  $\xi = f(x)$ . Then we can rewrite the PDE in terms of the new variable  $\xi$  and then use central differences in  $\xi$ , maintaining 2-nd order accuracy. We may clutter up the PDE with derivatives of  $f$  (which may be large in some regions), but this nevertheless gives us a methodical means of redistributing points to where we think they are needed – see notes on grid mappings.

## 23 Operator Splitting – Fractional Step Methods

If we have a parabolic-like equation and discretise the spatial derivatives, we have a set of differential equations of the form

$$u_t = \mathcal{L}u, \text{ where } \mathcal{L} \text{ is an operator, perhaps a matrix.}$$

Using a Taylor series, we have

$$\frac{u^{j+1} - u^j}{k} = u_t + \frac{1}{2}k u_{tt} + \dots \quad (23.1)$$

Assuming the operator  $\mathcal{L}$  is time-independent, then using the Lax-Wendroff idea, we can write

$$u_{tt} = \mathcal{L}u_t = \mathcal{L}(\mathcal{L}u) \equiv \mathcal{L}^2u$$

and then

$$u^{j+1} = u^j + k\mathcal{L}u^j + \frac{1}{2}k^2\mathcal{L}^2u^j + \dots = (\mathbf{I} + k\mathcal{L} + \frac{1}{2}k^2\mathcal{L}^2 + \dots)u^j. \quad (23.2)$$

Here  $\mathbf{I}$  denotes the identity operator – we can think of it as “1”. Defining the exponential operator

$$\exp(k\mathcal{L}) \equiv (\mathbf{I} + k\mathcal{L} + \frac{1}{2}k^2\mathcal{L}^2 + \frac{1}{6}k^3\mathcal{L}^3 + \dots),$$

then the exact solution of (23.1) can be written

$$u^{j+1} = \exp(k\mathcal{L}) u^j. \quad (23.3)$$

But we will only be concerned with the  $O(k^2)$  terms below.

Often we will be dealing with equations with qualitatively different terms (e.g. advection, diffusion, nonlinear forcing) which, drawing on our previous experience, we may wish to deal with individually in different manners. For example, perhaps one part we feel should be dealt with explicitly using Lax-Wendroff, and another implicitly using multi-grid. Can we treat the two parts differently without sacrificing the overall accuracy of the scheme? This is the idea behind **Operator Splitting**. Recall the lecture on the ADI method, where for the 2D diffusion equation we alternately treated the Laplacian explicitly in the x-direction and implicitly in the y-direction and then vice versa. Can we do the same thing in general?

Suppose we have two processes which we represent by spatial discretisations  $\mathcal{L}$  and  $\mathcal{M}$

$$u_t = \mathcal{L}u + \mathcal{M}u \quad (23.4)$$

which has the solution

$$u^{j+1} = u^j + k(\mathcal{L} + \mathcal{M})u^j + \frac{1}{2}k^2(\mathcal{L} + \mathcal{M})^2u^j + \dots \quad (23.5)$$

Note this is a schematic representation – the operators could involve iteration or implicit methods, but ultimately we represent the operation in this manner. Suppose we alternate solving

$$u_t = \mathcal{L}u \text{ and } u_t = \mathcal{M}u$$

by our favourite methods, solving

$$\left. \begin{aligned} \hat{u} &= (\mathbf{I} + k\mathcal{L} + \frac{1}{2}k^2\mathcal{L}^2)u^j, \text{ and then} \\ u^{j+1} &= (\mathbf{I} + k\mathcal{M} + \frac{1}{2}k^2\mathcal{M}^2)\hat{u} \end{aligned} \right\} \quad (23.6)$$

We then have

$$\begin{aligned} u^{j+1} &= (\mathbf{I} + k\mathcal{M} + \frac{1}{2}k^2\mathcal{M}^2)(\mathbf{I} + k\mathcal{L} + \frac{1}{2}k^2\mathcal{L}^2)u^j, \\ &= u^j + k(\mathcal{L} + \mathcal{M})u^j + \frac{1}{2}k^2(\mathcal{L}^2 + \mathcal{M}^2 + 2\mathcal{M}\mathcal{L})u^j. \end{aligned} \quad (23.7)$$

Is this the same as (23.5)? Only if  $\mathcal{M}\mathcal{L} = \mathcal{L}\mathcal{M}$  so that the two operators commute. This will not usually be the case, so that the simple splitting method has an error of  $O(k)$ , even if  $\mathcal{L}$  and  $\mathcal{M}$  are implemented in an  $O(k^2)$  manner.

With a little more work, we can achieve 2nd order accuracy. We could, for example, now go back to  $u^j$  and do an  $\mathcal{L}$  step and then an  $\mathcal{M}$  step upon it. Averaging the two resulting estimates for  $u^{j+1}$ , we will have the correct  $O(k^2)$  term, namely

$$\frac{1}{2}(\mathcal{L}^2 + \mathcal{M}^2 + \mathcal{L}\mathcal{M} + \mathcal{M}\mathcal{L}).$$

More subtly, and more efficiently, we could adopt the time-symmetric scheme of taking a half-step with  $\mathcal{L}$ , a full step with  $\mathcal{M}$  and then a half-step with  $\mathcal{L}$ , a process known as **Strang splitting**. Thus

$$\begin{aligned} \hat{u} &= (\mathbf{I} + \frac{1}{2}k\mathcal{L} + \frac{1}{2}(\frac{1}{2}k)^2\mathcal{L}^2)u^j, \\ u^* &= (\mathbf{I} + k\mathcal{M} + \frac{1}{2}k^2\mathcal{M}^2)\hat{u} \\ u^{j+1} &= (\mathbf{I} + \frac{1}{2}k\mathcal{L} + \frac{1}{2}(\frac{1}{2}k)^2\mathcal{L}^2)u^*. \end{aligned} \quad (23.8)$$

The final estimate is  $u^{j+1} = \mathcal{P}u$  where, neglecting terms of  $O(k^3)$ ,

$$\begin{aligned} \mathcal{P} &= (\mathbf{I} + \frac{1}{2}k\mathcal{L} + \frac{1}{8}k^2\mathcal{L}^2)(\mathbf{I} + k\mathcal{M} + \frac{1}{2}k^2\mathcal{M}^2)(\mathbf{I} + \frac{1}{2}k\mathcal{L} + \frac{1}{8}k^2\mathcal{L}^2) \\ &= \mathbf{I} + k(\frac{1}{2}\mathcal{L} + \mathcal{M} + \frac{1}{2}\mathcal{L}) + k^2(\frac{1}{8}\mathcal{L}^2 + \frac{1}{2}\mathcal{M}^2 + \frac{1}{8}\mathcal{L}^2 + \frac{1}{2}\mathcal{L}\mathcal{M} + \frac{1}{2}\mathcal{M}\mathcal{L} + \frac{1}{4}\mathcal{L}^2) \\ &= \mathbf{I} + k(\mathcal{L} + \mathcal{M}) + \frac{1}{2}k^2(\mathcal{L}^2 + \mathcal{M}^2 + \mathcal{L}\mathcal{M} + \mathcal{M}\mathcal{L}) \end{aligned} \quad (23.9)$$

which agrees with the estimate (23.5). The method (23.8) uses two calls to the “ $\mathcal{L}$ -routine” and one to the “ $\mathcal{M}$ -routine”. Other things being equal, we would choose  $\mathcal{M}$  to be the more time-expensive routine.

But wait! We are not just doing one step, we are taking several. Our 1st time step ends with a half-step in  $\mathcal{L}$ , and the second begins with a half step in  $\mathcal{L}$  – we could combine those and take a full step with  $\mathcal{L}$ . As a check we note that taking two half-steps involves

$$(\mathbf{I} + \frac{1}{2}k\mathcal{L} + \frac{1}{8}k^2\mathcal{L}^2)^2 = \mathbf{I} + k\mathcal{L} + (\frac{2}{8} + \frac{1}{4})k^2\mathcal{L}^2 = \mathbf{I} + k\mathcal{L} + \frac{1}{2}k^2\mathcal{L}^2,$$

which is the same as one complete  $k$ -step. To put it another way, we have shown that operating alternately with  $\mathcal{L}$  and  $\mathcal{M}$  is the same as  $(\mathcal{L} + \mathcal{M})$ . Clearly the operator  $\frac{1}{2}\mathcal{L}$  commutes with itself. Thus we can preserve  $O(k^2)$  accuracy by beginning with a half-step with  $\mathcal{L}$ , then alternating full-steps with  $\mathcal{M}$  and then  $\mathcal{L}$ , ending with a half-step in  $\mathcal{L}$ .

## 24 The Navier-Stokes equations in two-dimensions

An important application of many of the techniques of this course is to the equations of Fluid Dynamics. The motion of an incompressible Newtonian fluid is defined by its velocity  $\mathbf{u}(\mathbf{x}, t)$  and pressure  $p(\mathbf{x}, t)$  which satisfy the equations

$$\nabla \cdot \mathbf{u} = 0, \quad (24.1)$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}, \quad (24.2)$$

where  $\mathbf{F}$  is a body force such as gravity  $g$  (say). These equations are tremendously important with very wide application. Here  $\nu$  is a constant parameter, the inverse of the Reynolds number ( $Re$ ) if the variables have been made non-dimensional. It measures the relative strength of diffusion and advection. The character of the equations is very different if  $\nu$  is large or small because  $\nu$  multiplies the highest derivative in the equation.

Here, we shall only consider two-dimensional (2D) flows, of the form

$$\mathbf{u} = (u(x, y, t), v(x, y, t), 0), \text{ and } p = p(x, y, t) \quad (24.3)$$

Flows of this form can be represented by a single scalar function,  $\psi(x, y, t)$ . The incompressibility condition (24.1) can be satisfied by writing

$$\mathbf{u} = \nabla \times (0, 0, \psi), \text{ and } u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (24.4)$$

Then the vorticity,

$$\nabla \times \mathbf{u} = (0, 0, \omega), \text{ where } \omega = -\nabla^2 \psi. \quad (24.5)$$

Now taking the curl of (24.2), we can eliminate the pressure field. The resulting vorticity equation only has a  $z$ -component, which is

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + \mathbf{G}, \quad (24.6)$$

where

$$\mathbf{G} = \tilde{z} \cdot \nabla \times \mathbf{F}. \quad (24.7)$$

We could if we wish substitute for  $\omega$  to obtain a single equation in  $\psi$

We see that  $\nu$  measures the relative importance of diffusion to advection. We also note that  $\nu$  multiplies the highest derivative in the equation, so we require an extra boundary condition when  $\nu \neq 0$ .

What boundary conditions shall we impose? As we have not included an external force, we will only get motion if it is driven by a moving boundary. A typical problem is the Driven Cavity flow. Suppose we have a rectangle,  $0 < x < 1$  and  $0 > y > -L$ . On the three walls  $x = 0$ ,  $x = 1$  and  $y = -L$  we impose that the velocity is zero,  $\psi_x = \psi_y = 0$ . On the top wall  $y = 0$ , we impose a constant horizontal flow, so that  $\psi_y = 1$  on  $y = 0$ , but otherwise  $\psi$ ,  $\psi_x$  and  $\psi_y$  are all zero on the boundary. We could envisage ourselves caught in a storm. We dig ourselves a hole to sit in to shelter from the driving wind. But does the flow penetrate into the hole?

## 24.1 Nearly inviscid flow, $\nu \rightarrow 0$ , High Reynolds Number

If  $\nu$  is small, we are nearly at the inviscid Euler flow we considered last lecture, where  $\omega$  was advected around, being stretched but not dissipated. If anything, adding a small amount of diffusion would probably make the flow better behaved, by preventing high gradients from developing. If  $\nu$  is small, we might consider using an explicit representation of the diffusion, as the stability constraint  $\nu k/h^2 < 1/2$  might not be too severe.

## 24.2 Stokes flow: $\nu \rightarrow \infty$ , Low Reynolds Number

If the flow is very viscous, or slow, the parameter  $\nu \gg 1$ . In these circumstances the equations become

$$\nabla^2 \omega = 0, \quad \nabla^2 \psi = -\omega \quad (24.8)$$

This type of flow is known as Stokes flow or creeping flow which is of importance in many applications, such as the flow in **MEMS** (micro-electromechanical systems), flow in polymers and material processing, and flow in and around micro-biological systems. Inertial effects in these systems can often be ignored leading to a simplified set of equations. How might we solve this equation numerically? Those two Laplacians look very tempting. Can we devise a multigrid code? Effectively, we are solving the **biharmonic equation**

$$\nabla^2(\nabla^2 \psi) = 0$$

We have a slight difficulty. We are missing a boundary condition on  $\omega$  but instead we have two conditions on  $\psi$ . How can we nevertheless use two multigrid processes?

Suppose we knew  $\omega$ . Then we could clearly calculate  $\psi$  using the Dirichlet boundary condition  $\psi = 0$  only. The solution would not in general satisfy the Neumann boundary condition. However, if we could use it and the newly found  $\psi$  to define a boundary condition on  $\omega$ , we would have the basis of an iterative scheme.

Suppose at  $x = 0$  we have  $\psi = 0$  and  $\psi_x = f(y)$ . Thus one step away from the wall\*, we have

$$\psi(h) = h\psi_x + \frac{1}{2}h^2\psi_{xx}(0).$$

Now,

$$-\omega = \psi_{xx} + \psi_{yy} = \psi_{xx} \quad \text{on } x = 0. \quad (24.9)$$

We can therefore identify

$$\omega(0, y) = \frac{2f(y)}{h} - \frac{2\psi(h, y)}{h^2} + O(h), \quad (24.10)$$

So we can envisage an iterative scheme where we use  $\psi$  to find  $\omega$  and then  $\omega$  to find  $\psi$ .

In place of (24.10) we could derive an  $O(h^2)$  approximation using two points.

$$\left. \begin{aligned} \psi(h) &= h\psi' + \frac{h^2}{2}\psi'' + \frac{h^3}{3!}\psi''' + O(h^4) \\ \psi(2h) &= 2h\psi' + 2h^2\psi'' + \frac{4h^3}{3}\psi''' + O(h^4) \end{aligned} \right\}, \quad (24.11)$$

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\*It follows, since  $\psi = 0$  for all  $y$ , then  $\psi_y = 0$  and hence  $\psi_{yy} = 0$ .

Eliminating  $\psi'''$  we have

$$8\psi(h) - \psi(2h) = 6h\psi' + 2h^2\psi'' + O(h^4),$$

and so

$$\omega(0, y) = \frac{3f(y)}{h} - \frac{8\psi(h, y) - \psi(2h, y)}{2h^2} + O(h^2). \quad (24.12)$$

Using such schemes we can solve Dirichlet problems for  $\omega$  and  $\psi$  in turn.

## 24.3 The general case – intermediate Reynolds number

If both advection and diffusion are important, we would like somehow to marry our ideas for dealing with each process. Ideally, we would like to treat the advection explicitly but the diffusion implicitly, hopefully using a multigrid algorithm. One idea we might try is an **Operator Splitting** approach. In the first part of the algorithm we advect  $\omega$  ignoring the diffusion, while in the 2nd part we diffuse the new  $\omega$ .

## 24.4 Rayleigh-Benard Convection

An interesting fluid dynamical is known as **Convection**. Convection occurs because fluids expand when heated. Their density decreases and they become buoyant, in accordance with Archimedes's principle. If  $T$  denotes the excess temperature The body force

$$\mathbf{F} = -\beta T \mathbf{g}$$

where  $\mathbf{g}$  is the gravitational acceleration and  $\beta$  is the coefficient of expansion. Gravity can then drive fluid motion, which then advects heat around. Suitably non-dimensionalised, the governing equations for the temperature  $T$  and velocity  $\mathbf{u}$  are

$$\left. \begin{aligned} T_t + \mathbf{u} \cdot \nabla T &= \nabla^2 T \\ P^{-1}(\omega_t + \mathbf{u} \cdot \nabla \omega) &= R T_x \nabla^2 \omega \\ \omega &= -\nabla^2 \psi \end{aligned} \right\}, \quad (24.13)$$

There are two parameters in the problem: The Prandtl number,  $P = \nu/\kappa$  is the ratio of the fluid kinematic viscosity to the thermal diffusivity ( $\kappa$ ). The Rayleigh number  $R$  is a measure of the thermal driving force. In deriving (24.13), it is assumed that the density decreases linearly with temperature, and that the fluid speed is much less than the speed of sound, permitting the Boussinesq approximation. Gravity acts in the  $y$ -direction.

The problem is known as Rayleigh-Benard convection. The equations above model a variety of flow with industrial relevance such as ventilation of rooms, flow in solar energy collectors, crystal growth in liquids, cooling of electronic parts or flow in a heat exchanger. For example, a typical example would be that of applying uniform heat to the bottom of a box (or the computational domain) while keeping the top and side walls at a constant and lower temperature. For small temperature gradients, measured by the Rayleigh number  $R$ , the velocity field is zero and the thermal equilibrium is given by the conductive state. There is a purely conductive solution, with  $\mathbf{u} = 0$  and  $T = T(y)$ . What we expect to happen is that for  $R \leq R_c$  the only solution is the conductive solution, but for  $R > R_c$  this solution is

unstable, and a flow (convection) develops. We want to investigate the critical value of  $R$ , and the convection patterns formed for different values of  $P, R$ . As the temperature gradient, or Rayleigh number, is increased beyond a critical value, a circular motion sets in that transports hot fluid from the lower wall in exchange for cooler fluid from the top wall. After a transient period, a steady number of rolls appear. As the Rayleigh number is increased further, the initially two rolls split and four rolls appear, with additional bifurcations observed on increasingly higher Rayleigh numbers.

Our solution plan is to use operator splitting. If  $T, \psi$  and  $\omega$  are known at time  $t$ , then we :

1. Advect  $T$
2. Diffuse  $T$
3. Advect  $\omega$
4. Diffuse  $\omega$
5. Find new  $\psi$  and hence new velocity field
6. Derive new boundary condition on  $\omega$  from new  $\psi$
7. Repeat.

The equations (24.13) written out explicitly, for a slightly more generalised form are :

$$\left. \begin{aligned} T_t + u T_x + v T_y &= T_{xx} + T_{yy} \\ \omega_t + u \omega_x + v \omega_y &= P(\omega_{xx} + \omega_{yy}) + R P [T_x \cos(\phi) + T_y \sin(\phi)] \\ \psi_{xx} + \psi_{yy} &= -\omega \end{aligned} \right\}, \quad (24.14)$$

with  $\phi$  representing an inclination angle, with  $(R, P)$  defined as follows :

$$R = \frac{g\beta D^3(T_h - T_c)}{\nu\kappa}, \quad P = \frac{\nu}{\kappa} \quad (24.15)$$

where  $D$  is a characteristic box size,  $\beta$  the thermal expansion coefficient,  $(T_c, T_h)$  temperatures of the cold and hot wall respectively,  $\kappa$  is the thermal diffusivity and  $g$  the gravity.

## 24.5 Molten cores of planets and moons

The final topic, consists of the solution of the Convection Equations in an annulus. This is intended as a simplified (two-dimensional) model of thermally-driven convection between two spheres. This problem is important in the centre of the earth, whose outer core consists of a spherical shell of liquid iron. It also occurs in the recently discovered underground oceans in the moons of the outer planets. We hope our 2-D annular model will capture many of the important features of the problem.

Convection occurs because fluids expand when heated. Their density decreases and they become buoyant. Gravity can then drive fluid motion, which then advects heat around.

Suitably non-dimensionalised, the governing equations for the temperature  $T$ , pressure  $p$  and velocity  $\mathbf{u}$  are

$$\left. \begin{aligned} T_t + \mathbf{u} \cdot \nabla T &= \nabla^2 T \\ P^{-1}(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\nabla p - R \hat{g} T + \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\}, \quad (24.16)$$

There are two parameters in the problem: The Prandtl number,  $P$  is the ratio of the fluid kinematic viscosity to the thermal diffusivity. The Rayleigh number  $R$  is a measure of the thermal driving force, as defined earlier. In deriving (24.16), it is assumed that the density decreases linearly with temperature, and that the fluid speed is much less than the speed of sound, permitting the Boussinesq approximation. Gravity acts in the  $\hat{g}$ -direction. In two-dimensions, we may once again eliminate the pressure by taking the curl. The vorticity  $\nabla \times \mathbf{u}$  then has a single component  $\omega$ , related to a stream-function  $\psi$  by

$$-\omega = \nabla^2 \psi. \quad (24.17)$$

In our annular model, we will consider flow in the domain  $1 < r < b$  in terms of polar coordinates  $(r, \theta)$ . The temperature boundary conditions are  $T = 1$  on  $r = 1$  and  $T = 0$  on  $r = b$ . Gravity acts in the negative  $\hat{r}$ -direction. The  $r$ - and  $\theta$ -components of the velocity are respectively  $\psi_\theta/r$  and  $-\psi_r$ . The equations for  $\psi(r, \theta, t)$  and  $T(r, \theta, t)$  reduce to

$$\left. \begin{aligned} rT_t + \psi_\theta T_r - \psi_r T_\theta &= (rT_r)_r + \frac{1}{r} T_{\theta\theta} \\ P^{-1}(r\omega_t + \psi_\theta \omega_r - \psi_r \omega_\theta) &= -RT_\theta + (r\omega_r)_r + \frac{1}{r} \omega_{\theta\theta} \end{aligned} \right\}, \quad (24.18)$$

The problem obviously is very similar to the Rayleigh-Bénard convection considered above. Here too, there is a purely conductive solution, with  $\mathbf{u} = 0$  and  $T = T(r)$ . What we expect to happen is that for  $R \leq R_c$  the only solution is the conductive solution, but for  $R > R_c$  this solution is unstable, and a flow (*convection*) develops. We want to investigate the critical value of  $R$ , and the convection patterns formed for different values of  $P$ ,  $R$  and maybe  $b$ .