Sheet 3 Solutions

1. Applying the divergence theorem to $\phi\mathbf{A}$ we have

$$\int_V \nabla\cdot(\phi\mathbf{A})\,dV = \int_S \phi\mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = 0$$

(since $\phi = 0$ on the surface S). We also have

$$\operatorname{div}(\phi\mathbf{A}) = \nabla\phi\cdot\mathbf{A} + \phi\operatorname{div}\mathbf{A},$$

Hence result. If $\mathbf{A}$ is solenoidal throughout V then this means that $\operatorname{div}\mathbf{A} = 0$ throughout V and hence $\int_V \phi\operatorname{div}\mathbf{A}\,dV = 0$. It therefore follows that

$$\int_V \mathbf{A}\cdot\nabla\phi\,dV = 0,$$

as required.
In two dimensions the divergence theorem applied to $\phi\mathbf{A}$ is

$$\int_R \operatorname{div}(\phi\mathbf{A})\,dx\,dy = \oint_C \phi\mathbf{A}\cdot\widehat{\mathbf{n}}\,ds = 0$$

since $\phi = 0$ on C. Then using

$$\operatorname{div}(\phi\mathbf{A}) = \nabla\phi\cdot\mathbf{A} + \phi\operatorname{div}\mathbf{A}$$

again, we can establish the given result.

2. By the divergence theorem

$$\int_S \mathbf{r}\cdot\widehat{\mathbf{n}}\,dS = \int_V \nabla\cdot\mathbf{r}\,dV = 3\int_V dV = 3V,$$

where V is the volume enclosed by S.

3. By the divergence theorem

$$\int_S \frac{\mathbf{r}\cdot\widehat{\mathbf{n}}}{r^2}\,dS = \int_V \nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\,dV.$$

Now

$$\begin{aligned}
\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right) &= \nabla\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)\\
&= \frac{3}{x^2+y^2+z^2} - \frac{2x^2}{(x^2+y^2+z^2)^2} - \frac{2y^2}{(x^2+y^2+z^2)^2} - \frac{2z^2}{(x^2+y^2+z^2)^2}\\
&= \frac{1}{x^2+y^2+z^2} = \frac{1}{r^2}.
\end{aligned}$$

Thus

$$\int_S \frac{\mathbf{r}\cdot\widehat{\mathbf{n}}}{r^2}\,dS = \int_V \frac{dV}{r^2},$$

as required.

4. (i) Let $\mathbf{A} = \phi(x,y,z)\,\mathbf{i}$, and suppose that S is a surface with outward normal $\widehat{\mathbf{n}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$. If we apply the divergence theorem we obtain

$$\int_S l\phi\,dS = \int_\tau \frac{\partial \phi}{\partial x}\,d\tau.$$

Similarly, by considering $\mathbf{A} = \phi\,\mathbf{j}$, and $\mathbf{A} = \phi\,\mathbf{k}$ we obtain

$$\int_S m\phi\,dS = \int_\tau \frac{\partial \phi}{\partial y}\,d\tau, \quad \int_S n\phi\,dS = \int_\tau \frac{\partial \phi}{\partial z}\,d\tau.$$

By multiplying the first equation by $\mathbf{i}$, the second by $\mathbf{j}$, the third by $\mathbf{k}$ and then adding we get

$$\int_S \widehat{\mathbf{n}}\,\phi\,dS = \int_\tau \nabla\phi\,d\tau.$$

(ii) Expanding out the LHS we have

$$\begin{aligned}\int_S \widehat{\mathbf{n}} \times \mathbf{A}\,dS &= \int_S \mathbf{i}(mA_3 - nA_2) - \mathbf{j}(lA_3 - nA_1) + \mathbf{k}(lA_2 - mA_1)\,dS \\ &= \int_\tau \mathbf{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \mathbf{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \mathbf{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\,d\tau,\end{aligned}$$

where we have made use of the results obtained in (i). The integrand above is equal to curl $\mathbf{A}$, and hence we have

$$\int_S \widehat{\mathbf{n}} \times \mathbf{A}\,dS = \int_\tau \operatorname{curl}\mathbf{A}\,d\tau.$$

5. Firstly if $\mathbf{A} = x\mathbf{i}$ then $\nabla\cdot\mathbf{A} = 1$ and so

$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} dx\,dy\,dz = (2a)^3 = 8a^3.$$

Now we turn to the surface integral. We need to evaluate $\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS$. Now, two faces of the cube have normals in the $\pm\mathbf{i}$ directions, call these faces S_{x+} and S_{x-}. The other faces of the cube have normals in the $\pm\mathbf{j}, \pm\mathbf{k}$ directions and so $\mathbf{A}\cdot\widehat{\mathbf{n}} = 0$ on these faces for this choice of $\mathbf{A}$. Therefore

$$\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = \int_{Sx+} (x\mathbf{i})\cdot\mathbf{i}\,dy\,dz + \int_{Sx-} (x\mathbf{i})\cdot(-\mathbf{i})\,dy\,dz.$$

But on S_{x+} we have $x = a$ and hence $\mathbf{A} = a\mathbf{i}$, while on S_{x-} we have $x = -a$ and $\mathbf{A} = -a\mathbf{i}$. Therefore these integrals simplify to

$$\int_{-a}^{a}\int_{-a}^{a} a\,dydz + \int_{-a}^{a}\int_{-a}^{a} a\,dydz = 8a^3,$$

which agrees with the value computed for the volume integral.

6. There are two parts to the closed surface. The first one is the surface of the cone (S_1 say), which is given by $\phi = z^2 - x^2 - y^2 = 0$. A unit normal to S_1 is therefore given by

$$\begin{aligned}\pm\frac{\nabla\phi}{|\nabla\phi|} &= \pm\frac{-2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \pm\frac{-2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{8z^2}} \\ &= \pm\frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{2}z}.\end{aligned}$$

For the divergence theorem we need the outward normal to the cone, which is obtained by taking the minus sign above, so that

$$\widehat{\mathbf{n}} = \frac{x\mathbf{i} + y\mathbf{j} - z\mathbf{k}}{\sqrt{2}z}.$$

Thus we have $\mathbf{A}\cdot\widehat{\mathbf{n}} = [x(x+y) + y(y-x-z) - z(z-y)]/\sqrt{2}z = (x^2+y^2-z^2)/\sqrt{2}z = 0$ on the surface of the cone. So in this case

$$\int_{S_1} \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = 0.$$

Now we need to consider the surface integral over the flat cap at $z = 1$. Call this the surface S_2. Here the outward unit normal is simply $\widehat{\mathbf{n}} = \mathbf{k}$ and so $\mathbf{A} \cdot \widehat{\mathbf{n}} = z - y = 1 - y$ on S_2 and so

$$\int_{S_2} \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_{x^2+y^2 \le 1} (1 - y)\, dx\, dy = \int_0^{2\pi} \int_0^1 (1 - r\sin\theta)\, r\, dr\, d\theta = \pi,$$

since S_2 is a circle of radius 1. Thus the total contribution from the surface integrals is

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \pi.$$

Now we turn to the volume integral. This is straightforward since $\operatorname{div}\mathbf{A} = \partial(x + y)/\partial x + \partial(y - x - z)/\partial y + \partial(z - y)/\partial z = 3$, and so

$$\int_V \operatorname{div}\mathbf{A}\, dV = 3V,$$

where V is the volume of a cone of height 1 and radius 1. Thus $V = (1/3)\pi$ and the volume integral is equal to π. The divergence theorem is therefore verified.

7. In this case the closed curve (γ say) that forms the rim of the ellipsoid is the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the plane $z = 0$. Then by Stokes theorem the given surface integral is equal to $\oint_\gamma \mathbf{A} \cdot d\mathbf{r}$. Now in this case

$$\begin{aligned}
\mathbf{A} \cdot d\mathbf{r} &= ((y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}).(dx\mathbf{i} + dy\mathbf{j}) \\
&= (y - z)dx + (z - x)dy \\
&= ydx - xdy,
\end{aligned}$$

since we are in the plane $z = 0$. On the ellipse $x = a\cos\theta, y = b\sin\theta$, and so

$$\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = \int_0^{2\pi} ab(-\sin^2\theta - \cos^2\theta)\, d\theta = -2\pi ab,$$

where we have traversed γ in an anti-clockwise manner. (Using the right-hand screw rule, this means that the normal to the ellipsoid should point to positive values of z, i.e. $\widehat{\mathbf{n}} \cdot \mathbf{k} > 0$, which is consistent with the direction given in the question).

8. First we compute

$$\operatorname{curl}\mathbf{A} = \mathbf{i}(-yz + yz) - \mathbf{j}(0) + \mathbf{k}(0 - (-1)) = \mathbf{k}.$$

The unit normal to the hemisphere is $\nabla(x^2 + y^2 + z^2)/\left|\nabla(x^2 + y^2 + z^2)\right| = \pm(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/a$. We are free to choose either sign for the normal: let's take the positive root so that the normal points in the direction of increasing z. We therefore have that

$$(\nabla \times \mathbf{A}) \cdot \widehat{\mathbf{n}} = z/a = \cos\theta,$$

using the parametrization given in the question. An element of surface is also given as $dS = a^2 \sin\theta\, d\theta\, d\phi$. Thus

$$\begin{aligned}
\int_S (\nabla \times \mathbf{A}) \cdot \widehat{\mathbf{n}}\, dS &= \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin\theta\cos\theta\, d\theta\, d\phi \\
&= \int_0^{2\pi} a^2[(\sin^2\theta)/2]_0^{\pi/2} d\phi = \pi a^2.
\end{aligned}$$

Now we need to work out the other side of Stokes theorem which is the line integral $\oint_\gamma \mathbf{A} \cdot d\mathbf{r}$. Firstly γ is in the plane $z = 0$ and so for this integral we have

$$\mathbf{A} = (3x - y, 0, 0), \quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$$

and so

$$\mathbf{A} \cdot d\mathbf{r} = (3x - y)dx.$$

Since γ is a circle of radius a we can use plane polar coordinates

$$\begin{aligned} x &= a\cos\phi, y = a\sin\phi \\ &\Rightarrow \mathbf{A}\cdot d\mathbf{r} = -(3a\cos\phi - a\sin\phi)\, a\sin\phi\, d\phi. \end{aligned}$$

We therefore have that

$$\begin{aligned} \oint_\gamma \mathbf{A}\cdot d\mathbf{r} &= \int_0^{2\pi} a^2(\sin^2\phi - 3\cos\phi\sin\phi)\, d\phi \\ &= a^2(\pi - 3[\sin^2\phi]_0^{2\pi}) = \pi a^2, \end{aligned}$$

and the theorem is verified. Note that we travelled in an anti-clockwise direction around γ as viewed from above: this is in accordance with the right hand rule and our earlier choice for the direction of $\widehat{\mathbf{n}}$.

9. In this case the boundary curve γ is the circle around the top of the cone and has the equation $x^2 + y^2 = 9$ in the plane $z = 3$. We will calculate the line integral first as this is the easy bit.

$$\begin{aligned} I &= \oint \mathbf{A}\cdot d\mathbf{r} = \oint_\gamma (-y\mathbf{i} + x\mathbf{j} - xyz\mathbf{k})\cdot(dx\mathbf{i} + dy\mathbf{j}) \\ &= \oint_\gamma -ydx + xdy. \end{aligned}$$

Since γ is the circle $x^2 + y^2 = 9$ we can parametrize using plane polars and write $x = 3\cos\theta, y = 3\sin\theta$ with $0 \le \theta \le 2\pi$. If we traverse γ in an anti-clockwise manner we then have

$$I = \int_0^{2\pi} (-3\sin\theta)^2 + (3\cos\theta)^2 d\theta = 18\pi.$$

Now we need to work out the surface integral $\int_S \mathrm{curl}\mathbf{A}\cdot\widehat{\mathbf{n}}\, dS$, where S is the cone surface. First we calculate

$$\mathrm{curl}\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & -xyz \end{vmatrix} = -xz\mathbf{i} + yz\mathbf{j} + 2\mathbf{k}.$$

Next we need the unit normal to the cone. Here we need to be careful about the direction of the normal. Recall that in Stokes theorem the direction we traverse the curve and the direction of the normal to the surface are related by a right-hand screw rule. We decided to traverse γ in an anti-clockwise manner: using the screw rule this means that the normal should point in the direction of increasing z (which means in this case that the normal points into the cone). In question 6 we showed that the unit normals to the cone are given by

$$\pm\frac{-xi - yj + zk}{\sqrt{2}z}$$

and so in this case we should choose

$$\widehat{\mathbf{n}} = \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{2}z}.$$

It follows that on S we have

$$\mathrm{curl}\mathbf{A}\cdot\widehat{\mathbf{n}} = \frac{x^2z - y^2z + 2z}{\sqrt{2}z} = \frac{1}{\sqrt{2}}(x^2 - y^2 + 2).$$

We have to integrate this over the surface of the cone. To do this we can use the projection theorem to project it onto the plane $z = 3$, .where the projected shape Σ say, is the circle $x^2 + y^2 = 9$. This gives

$$\begin{aligned} \int_S (\mathrm{curl}\mathbf{A}\cdot\widehat{\mathbf{n}})\, dS &= \int_\Sigma \frac{1}{\sqrt{2}}(x^2 - y^2 + 2)\frac{dxdy}{|\widehat{\mathbf{n}}\cdot\mathbf{k}|} \\ &= \int_\Sigma (x^2 - y^2 + 2)\, dx\, dy. \end{aligned}$$

Since Σ is the area inside a circle of radius 3 we can parametrize using $x = r\cos\theta, y = r\sin\theta$ with $0 \le r \le 3, 0 \le \theta \le 2\pi$ and $dxdy = rdrd\theta$. The integral becomes

$$\begin{aligned}
&\int_0^{2\pi}\int_0^3 (r^2\cos^2\theta - r^2\sin^2\theta + 2)rdrd\theta \\
= &\int_0^{2\pi} \frac{3^4}{4}\cos^2\theta - \frac{3^4}{4}\sin^2\theta + 9\,d\theta \\
= &\frac{3^4}{4}\pi - \frac{3^4}{4}\pi + 18\pi \\
= &18\pi.
\end{aligned}$$

The answer therefore agrees with that computed by the line integral, and Stokes theorem is verified.

10. If we start with the surface integral we can proceed as in Q9 (with the normal again pointing into the cone) to show that

$$\begin{aligned}
\int_S (\text{curl}\mathbf{A}\cdot\widehat{\mathbf{n}})\,dS &= \int_\Sigma \frac{1}{\sqrt{2}}(x^2 - y^2 + 2)\frac{dxdy}{|\widehat{\mathbf{n}}\cdot\mathbf{k}|} \\
&= \int_\Sigma (x^2 - y^2 + 2)\,dx\,dy.
\end{aligned}$$

This time the projection on $z = 0$ is the annulus $2 \le r \le 3$ and so the integral becomes

$$\begin{aligned}
&\int_0^{2\pi}\int_2^3 (r^2\cos^2\theta - r^2\sin^2\theta + 2)rdrd\theta \\
= &\int_0^{2\pi} \frac{(3^4 - 2^4)}{4}\cos^2\theta - \frac{(3^4 - 2^4)}{4}\sin^2\theta + (3^2 - 2^2)\,d\theta \\
= &\frac{(3^4 - 2^4)}{4}\pi - \frac{(3^4 - 2^4)}{4}\pi + 10\pi \\
= &10\pi
\end{aligned}$$

(or we could spot that $\cos^2\theta - \sin^2\theta \equiv \cos 2\theta$ and so the first 2 terms combined integrate to zero).
We now have 2 path integrals to compute with the one around the top of the cone being as in Q9: $x^2 + y^2 = 9; z = 3$. Applying the right hand rule when there is more than one boundary can be a bit problematical. The following generalisation is easier to use in these circumstances: stand on the boundary with your head in the direction of the normal and keep the surface to your left. This results (as seen from above) in traversing the top boundary counter-clockwise (as in Q9) and the lower boundary clockwise. Denoting the top boundary by γ_1:

$$I = \oint_{\gamma_1} \mathbf{A}\cdot d\mathbf{r} = \cdots = 18\pi.$$

Now we turn to the lower boundary $x^2 + y^2 = 4; z = 2$ which we denote by γ_2. As explained above, this boundary needs to be traversed clockwise to be consistent with the direction of the normal and so we write $x = 2\cos\theta, y = 2\sin\theta$ with θ starting at 2π and ending at zero. Thus:

$$I = \oint_{\gamma_2} \mathbf{A}\cdot d\mathbf{r} = \int_{2\pi}^0 (-2\sin\theta)^2 + (2\cos\theta)^2\,d\theta = -8\pi.$$

Therefore

$$I = \oint_\gamma \mathbf{A}\cdot d\mathbf{r} = \oint_{\gamma_1} \mathbf{A}\cdot d\mathbf{r} + \oint_{\gamma_2} \mathbf{A}\cdot d\mathbf{r} = 18\pi - 8\pi = 10\pi.$$

This is the same value as that for the surface integral computed above and so Stokes theorem is verified.