(4.2.7) **Example**. Suppose X is an infinite set & $|X| = \lambda \geq \omega$. Let S be the set of finite sequences of elts. of X. So

$$S = \bigcup_{n \in \omega} X^n .$$

Claim $|S| = \lambda$.

If $n \geq 1$

$$|X^n| = \underbrace{\lambda \cdot \lambda \cdot \ldots \cdot \lambda}_{n \text{ times}} = \lambda \quad (4.2.5\ ii)$$

By 4.2.6 $|S| \leq \omega \cdot \lambda = \lambda$.

Also $X \subseteq S$ so $\lambda = |X| \leq |S|$.

So $|S| = \lambda$. #.

Example: Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Suppose $X \subseteq \mathbb{R}$ spans $\mathbb{R}$: for all $r \in \mathbb{R}$ there are $s \in \mathbb{N}$ & $q_1, \dots, q_s \in \mathbb{Q}$ and $x_1, \dots, x_s \in X$ with $r = \sum_{k=1}^{s} q_k x_k$. ②

Claim: $|X| = |\mathbb{R}|$.

Let P be the set of pairs:

$$P = \left\{ \left((q_1, \dots, q_s), (x_1, \dots, x_s) \right) : q_i \in \mathbb{Q},\ x_i \in X,\ s \in \mathbb{N} \right\}$$

$$\subseteq \left(\left(\bigcup_{n \in \omega} \mathbb{Q}^n \right) \times \left(\bigcup_{n \in \omega} X^n \right) \right).$$

Thus $|P| \leq |\mathbb{Q}|\,|X|$ by 4.2.7.

$= \omega \cdot |X|$.

there is a surjection $P \to \mathbb{R}$ map the pair to $\sum_{k=1}^{s} q_k x_k$.

So $|P| \geqslant |\mathbb{R}|$.

$|X| \leq |\mathbb{R}| \leqslant |P| \leq \omega \cdot |X| = |X|$. Thus $|\mathbb{R}| = |X|$.

#.

(4.3) Zorn's Lemma.

① A partially ordered set (poset) $(A;\leq)$ satisfies

$$\forall xyz \quad x\leq y\leq z \to x\leq z$$
$$\wedge \quad (x\leq y)\wedge(y\leq x)\to(x=y)$$
$$\wedge \quad (x\leq x)$$

[Example: $A = \mathcal{P}(X)$, $\leq$ is $\subseteq$]

② A chain C in a poset $(A;\leq)$ is a subset $C\subseteq A$ st.

$$\forall x,y\in C \quad (x\leq y)\vee(y\leq x).$$

③ A upper bound of C in A is an elt. $a\in A$ st.

$$c\leq a \quad \forall c\in C.$$

Eg. if $C\subseteq \mathcal{P}(X)$ then $\bigcup C \in \mathcal{P}(X)$ is an upper bound for C. (in $(\mathcal{P}(X);\subseteq)$).

④ An elt. $z\in A$ is a maximal elt. of A if for all $x\in A$ $((x\geq z)\to(x=z))$.

(4.3.1) Zorn's Lemma. (ZL) is the statement:

Suppose $(A;\leq)$ is a non-empty poset in which every chain has an upper bound in A. Then $(A;\leq)$ has a maximal element.

(4.3.2) Thm.

(1) Assuming ZFC, ZL holds.

(2) Assuming ZF + ZL then AC holds.

(i.e. $ZF \vdash (ZL \leftrightarrow AC)$).

//

(4.3.3) Example. (Assume ZFC).

Suppose V is a vector space over a field F. Then V has a basis (over F).

=

Use ZL. Let A be the set of linearly independent subsets of V, ordered by $\subseteq$.

Claim: If $C \subseteq A$ is a chain then $\bigcup C \in A$.

Pf: Must show $\bigcup C$ is a l.i. set.

If $y_1, \ldots, y_n \in \bigcup C$ then $y_1, \ldots, y_n$ are l.i. . There are $C_1, \ldots, C_n \in C$ with $y_i \in C_i$ for $i \leq n$. As C is a chain there is $j \leq n$ with $C_i \subseteq C_j$ $\forall i \leq n$. Thus $y_1, \ldots, y_n \in C_j$; so (as $C_j \in A$) $y_1, \ldots, y_n$ are l.i. #.

By ZL: there is a maximal elt B of A. Show B is a basis of V. B is l.i., so show B spans V.

If $v \in V \setminus B$ then $B \cup \{v\}$ is not l.i. (as B is maxl. in A).

As B is l.i., v is a linear comb. of elts. of B. //. #

Pf of $\underline{AC \Rightarrow ZL}$.

$\underline{\text{Given}}$ a poset $(A; \leq)$ satisfying hypotheses of ZL.

Let $f: \mathcal{P}(A) \setminus \{\emptyset\} \to A$ be a $\underline{\text{choice function}}$.

Suppose for a contradiction that $(A; \leq)$ has no maximal elt.

Let $C \subseteq A$ be a chain in A.

By assumption there is $y \in A$ with $c \leq y$ for all $c \in C$.

As y is not maximal, there is $z \in A$ with $y < z$.

then $c \leq y < z \quad \forall c \in C$.

So $C \subset C \cup \{z\}$ is a chain.

Use transfinite recursion to define an operation G st. for ordinals α, β with $\beta < \alpha$

$$G(\alpha) \in A \quad \text{and} \quad G(\beta) < G(\alpha)$$

(when $\beta < \alpha$)

Let

$$G(\alpha) = f\Big(\underbrace{\{z \in A : z > G(\beta) \text{ for } \beta < \alpha\}}_{\text{non-empty by previous argument}}\Big)$$

So $G(0) < G(1) < \dots < G(\beta) < \dots < G(\alpha) \dots$

For every α we have an injective function $G \upharpoonright \alpha : \alpha \to A$

This contradicts Hartogs' Lemma.

#.