Imperial College
London

MATH70054

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Introduction to Stochastic Differential Equations

Date: 1 June 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1.

(a) (i) Let $W(t)$ be a standard one dimensional Brownian motion. Let $c > 0$ arbitrary and define the stochastic processes

$$X(t) = \frac{1}{\sqrt{c}} W(ct),$$

and

$$Y(t) = W(c+t) - W(c).$$

Show that $\{X(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ and $\{Y(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ in law.

(ii) Let $\mathbf{W}_t$ be a standard $d-$dimensional Brownian motion and let Q be an orthogonal $d \times d$ real matrix. Show that $\mathbf{B}_t = Q\mathbf{W}_t$ is also a Brownian motion.

(7 marks)

(b) Let B_t be a one-dimensional stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $B_0 = 0$ a.s. Show that B_t is a standard Brownian motion if and only if it is a Markov process with semigroup:

$$P_0 = I, \quad (P_t f)(x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\, dy.$$

(13 marks)

(Total: 20 marks)

2. Let W_t be a standard one dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t^W$ denote the natural filtration of W_t.

(a) Calculate directly from the definition the Itô and Stratonovich stochastic integrals

$$\int_0^t W_s \, dW_s \quad \text{and} \quad \int_0^t W_s \circ dW_s.$$

(10 marks)

(b) Let $f(t, \omega) : [0, +\infty) \times \Omega \to \mathbb{R}$ such that f is $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{B}$ denotes the Borel $\sigma-$algebra on $[0, +\infty)$, that $f(t, \omega)$ is $\mathcal{F}_t^W-$adapted, and that $\mathbb{E} \int_0^T f^2(t, \omega) \, dt < +\infty$ for every $T > 0$. Give conditions on the integrand $f(s; \omega)$ so that the Itô and Stratonovich integrals are equal:

$$\int_0^t f(s; \omega) \, dW_s = \int_0^t f(s; \omega) \circ dW_s.$$

(4 marks)

(c) For $n = 0, 1, \ldots$, define

$$h_n(x, t) := \frac{(-t)^n}{n!} e^{x^2/2t} \frac{d^n}{dx^n} \left(e^{-x^2/2t} \right).$$

(i) Calculate $h_n(W_t, t)$ for $n = 0, 1, 2, 3$.

(ii) Use Itô's formula to show that

$$\int_0^t h_2(W_s, s) \, dW_s = h_3(W_t, t).$$

(6 marks)

(Total: 20 marks)

3.

(a) Let W_t denote a one dimensional Brownian motion and let $h(t) \in L^2[0,T]$ deterministic. Define

$$Y(t,\omega) = \exp\left\{\int_0^t h(t)\, dW_t(\omega) - \frac{1}{2}\int_0^t h^2(t)\, dt\right\}.$$

(i) Obtain a stochastic differential equation for $Y(t,\omega)$.

(ii) Show that $Y(t,\omega)$ is a square integrable martingale with respect to the filtration generated by the Brownian motion W_t.

(6 marks)

(b) Solve the Itô SDE

$$dX_t = -\frac{1}{2}e^{-2X_t}\, dt + e^{-X_t}\, dW_t, \quad X_0 = x > 0.$$

(4 marks)

(c) Consider the one-dimensional Itô stochastic differential equation

$$dX_t = (-a_0 - a_1 X_t)\, dt + \sqrt{b_0 + b_1 X_t + b_2 X_t^2}\, dW_t, \quad X_0 = x, \tag{1}$$

where $a_i,\, b_i > 0,\ i = 0, 1, 2$.

(i) Write down the generator and the backward and forward (Fokker-Planck) Kolmogorov equations for X_t. (2 marks)

(ii) Assume that X_0 is a random variable with probability density $\rho_0(x)$ that has finite moments of all orders. Use the forward Kolmogorov equation or Itô's formula to derive a system of differential equations for the moments of X_t. (8 marks)

(Total: 20 marks)

4. Let W_t be a standard one-dimensional Brownian motion, and let X_t be solution of the stochastic differential equation

$$\mathrm{d}X_t = -\alpha X_t \, dt + \sqrt{2\sigma} \, dW_t, \tag{2}$$

where $\alpha, \sigma > 0$ are constants and the initial condition $X(0) = x \in \mathbb{R}$ is deterministic.

(a) Solve (2) explicitly. (3 marks)

(b) Show that the solution of (2) can be written in the form

$$X_t = e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t} W_{\left(\frac{1}{2\alpha}(e^{2\alpha t} - 1)\right)}. \tag{3}$$

(3 marks)

(c) Study the limit of X_t as $t \to +\infty$. (4 marks)

(d) Use the explicit formula for the solution of the SDE (2) to obtain a formula for the Markov semigroup $(P_t f)(x) = \mathbb{E}(f(X_t) | X_0 = x)$ for all continuous, bounded functions f. Assume that f is differentiable. Use the formula for the Markov semigroup to show that

$$\frac{d}{dx}(P_t f)(x) = e^{-\alpha t} \left(P_t \frac{df}{dx} \right)(x).$$

(10 marks)

(Total: 20 marks)

5. Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary and let $f : D \to [f_{min}, f_{max}]$, $f_{min} > 0$ be a smooth scalar function. Consider the Itô stochastic differential equation

$$dX_t = \nabla f(X_t)\, dt + \sqrt{2f(X_t)}\, dW_t, \quad X_0 = x, \tag{4}$$

with reflecting boundary conditions.

(a) Write down the generator and the backward and forward (Fokker-Planck) Kolmogorov equations for the process X_t. Show that the Fokker-Planck equation can be written as a continuity equation and obtain a formula for the probability flux.

(4 marks)

(b) Show that the process X_t is ergodic and compute the invariant measure.

(4 marks)

(c) Show that the law of the process converges exponentially fast in $L^2(D)$ to the unique invariant measure.

(12 marks)

(Total: 20 marks)

Introduction to SDEs and Diffusion Processes. Solutions May 2023.

1. (a) (i) $X(t)$ has a.s. continuous paths and it is Gaussian, since $W(ct)$ is Gaussian for all $c > 0$. Furthermore, it is mean zero and its covariance function is

$$\mathbb{E}(X(t)X(s)) = \frac{1}{c}\min(ct,\, cs) = \min(t,s).$$

Consequently, $X(t)$ is a mean zero Gaussian process with a.s. continuous paths and covariance function $\min(t,s)$. Hence it is a Brownian motion.
Similarly,

$$\begin{aligned}\mathbb{E}(Y(t)Y(s)) &= \mathbb{E}(W(c+t)W(c+s)) - \mathbb{E}(W(c+t)W(c)) - \mathbb{E}(W(c)W(c+s)) + \mathbb{E}W(c)^2 \\ &= \min(t+c, c+s) - c = \min(t,s).\end{aligned}$$

(ii) (There are several different proofs of this result) We check that $\mathbf{B}_t$ and $\mathbf{W}_t$ have the same finite dimensional distributions:

$$\begin{aligned}&\mathbb{P}[B_{t_1} \in F_1, \dots, B_{t_k} \in F_k] = \mathbb{P}[W_{t_1} \in Q^T F_1, \dots, W_{t_k} \in Q^T F_k] \\ &= \int_{Q^T F_1 \cdots \times Q^T F_k} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2)\dots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \dots dx_k \\ &= \int_{F_1 \cdots \times F_k} p(t_1, 0, y_1)p(t_2 - t_1, y_1, y_2)\dots p(t_k - t_{k-1}, y_{k-1}, y_k)\, dy_1 \dots dy_k \\ &= \mathbb{P}[W_{t_1} \in F_1, \dots, W_{t_k} \in F_k].\end{aligned}$$

In the above we have made the change of variables $y_j = Qx_j$ and we have used the fact that, since Q is an orthogonal transformation, $|Qx_j - Qx_{j-1}|^2 = |x_j - x_{j-1}|^2$.
[7] MARKS –A

(b) Let $W_t,\ t \geq 0$ be a Markov processed defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with semigroup

$$P_0 = I, \quad (P_t f)(x) = \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\, dy,\ t > 0,\ x \in \mathbb{R}.$$

Thanks to the Markov property, and denoting by $\mathcal{F}_t$ the natural filtration of $W_t,\ t \geq 0$, we have for $s, t \geq 0, \lambda \in \mathbb{R}$,

$$\mathbb{E}\left(e^{i\lambda W_{t+s}}\Big|\mathcal{F}_s\right) = \int_{\mathbb{R}} e^{i\lambda(W_s+y)}\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\, dy.$$

This implies

$$\mathbb{E}\left(e^{i\lambda(W_{t+s}-W_s)}\Big|\mathcal{F}_s\right) = \int_{\mathbb{R}} e^{i\lambda y}\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\, dy = e^{-\frac{1}{2}\lambda^2 t}.$$

In particular, the increments of $W_t,\ t \geq 0$ are stationary and independent. Now for $\lambda_1, \dots \lambda_n \in \mathbb{R}$, $0 < t_1 < \cdots < t_n$,

$$\begin{aligned}\mathbb{E}\left(e^{i\sum_{k=1}^n \lambda_k(W_{t_{k+1}} - W_{t_k})}\right) &= \prod_{k=1}^n \mathbb{E}\left(e^{i\lambda_k(W_{t_{k+1}} - W_{t_k}}\right) \\ &= \prod_{k=1}^n \mathbb{E}\left(e^{i\lambda_k(W_{t_{k+1}-t_k}}\right) \\ &= \mathbb{E}\left(e^{-\frac{1}{2}\sum_{k=1}^n \lambda_k^2(t_{k+1}-t_k)}\right).\end{aligned}$$

Hence, W_t is a standard Brownian motion.

Conversely, let W_t be a standard Brownian motion with natural filtration $\mathcal{F}_t$. Let f be a bounded Borel function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $t,\, s \geq 0$. The Markov property of Brownian motion follows from the fact that Brownian motion has independent increments:

$$\mathbb{E}(f(W_{t+s})|\mathcal{F}_s) = \mathbb{E}(f(W_{t+s} - W_s + W_s)|\mathcal{F}_s) = \mathbb{E}(f(W_{t+s})|W_s).$$

For $x \in \mathbb{R}$, we have

$$\mathbb{E}(f(W_{t+s})|W_s = x) = \mathbb{E}(f(W_{t+s} - W_s + W_s)|W_s = x) = \mathbb{E}(f(X_t + x)),$$

where $X_t \sim \mathcal{N}(0,t)$ is a random varialbe independent from W_t. Therefore, the Markov semigroup, applied to a measurable function f is given by

$$\mathbb{E}(f(W_{t+s})|W_s = x) = \int_{\mathbb{R}} f(x+y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}\, dy$$

with $P_0 = I$.

[13] MARKS –D

2. (a) We introduce a uniform partition in the interval $[0,t]$. We set

$$\phi_n(s,\omega) = \sum W_j(\omega)\chi_{[t_j,t_{j+1}]}(s),$$

where $W_j = W_{t_j}$ and χ_A denotes the characteristic function of the set A. Then

$$\begin{aligned}
\mathbb{E}\left[\int_0^t (\phi_n - W_s)^2\, ds\right] &= \mathbb{E}\left[\sum_j \int_{t_j}^{t_{j+1}} (W_j - W_s)^2\, ds\right] \\
&= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j)\, ds \\
&= \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0 \quad \text{as } \ \Delta t = 0.
\end{aligned}$$

Consequently,

$$\int_0^t W_s\, dW_s = \lim_{\Delta t} \int_0^t \phi_n\, dW_s = \lim_{\Delta t \to 0} \sum_j W_j\, \Delta W_j,$$

in $L^2(\mathbb{P})$. Now,

$$\begin{aligned}
\Delta (W_j)^2 &= W_{j+1}^2 - W_j^2 = (W_{j+1} - W_j)^2 + 2W_j(W_{j+1}) \\
&= (\Delta W_j)^2 + 2W_j \Delta W_j.
\end{aligned}$$

Consequently,

$$W_t^2 = \sum_j \Delta(W_j^2) = \sum (\Delta W_j)^2 + 2\sum_j W_j \Delta W_j.$$

Thus:

$$\sum_j W_j \Delta W_j = \frac{1}{2} W_t^2 - \frac{1}{2} \sum_j (\Delta W_j)^2. \tag{1}$$

Since $\sum_j (\Delta W_j)^2 \to t$ in $L^2(\mathbb{P})$ as $\Delta t_j \to 0$, we conclude that

$$\int_0^t W_s \, dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t.$$

We now calculate the Stratonovich stochastic integral that we define as the $L^2(\mathbb{P})-$limit of the midpoint rule:

$$\int_0^t W_s \circ dW_s = \lim_{\Delta t \to 0} \sum_j \frac{W_{j+1} + W_j}{2} \Delta W_j.$$

We do similar calculations to the one that we did in order to derive (1), in particular we use the fact that the term on the right hand side is a telescopic sum:

$$\begin{aligned}
\sum_j \frac{W_{j+1} + W_j}{2} \Delta W_j &= \frac{1}{2} \left(\sum_j W_{j+1} \Delta W_j + \sum_j W_{j+1} \Delta W_j \right) \\
&= \frac{1}{2} \sum_j \left(W_{j+1} \Delta W_j + W_j \Delta W_j \right) \\
&= \frac{1}{2} \sum_j \left(W_{j+1}^2 - W_{j+1} W_j + W_j W_{j+1} - W_j^2 \right) \\
&= \frac{1}{2} \sum_j \left(W_{j+1}^2 - W_j^2 \right) = \frac{1}{2} W_{j+1}^2,
\end{aligned}$$

from which we deduce that

$$\int_0^t W_s \circ dW_s = \frac{1}{2} W_t^2.$$

[10] MARKS –B

(b) In order for the Itô and Stratonovich integrals to be the same, the integrand has to be more regular than Brownian motion. In particular, it has to be Hölder continuous with exponent $\frac{1}{2} + \epsilon/2$, for $\epsilon > 0$: there exist positive constants $C < +\infty$, ϵ such that

$$\mathbb{E} |f(t, \cdot) - f(s, \cdot)|^2 \leq C |t - s|^{1+\epsilon},$$

for $s, t \in [0, T]$.

[4] MARKS–A

(c) We caculate

$$h_0(W_t, t) = 1, \; h_1(W_t, t) = W_t, \; h_2(W_t, t) = \frac{W_t^2}{2} - \frac{t}{2}, \quad h_3(W_t, t) = \frac{W_t^3}{6} - \frac{t W_t}{2}.$$

Brownian motion is a diffusion processes with generator $\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2}$. Let $f(x, t) = \frac{1}{3} x^3 - tx$. We apply Itô's formula to $Y_t = f(W_t, t)$ to obtain

$$\begin{aligned}
df(W_t) &= (\partial_t f)(W_t, t) \, dt + (\partial_x f)(W_t, t) \, dW_t + \frac{1}{2} (\partial_x^2 f)(W_t, t) \, (dW_t)^2 \\
&= -W_t \, dt + (W_t^2 - t) \, dW_t + W_t \, dt.
\end{aligned}$$

Consequently,

$$h_3(W_t, t) = \int_0^t h_2(W_s, s)\, dW_s.$$

[6] MARKS –A

3. (a) (i) Consider the Itô process

$$dZ_t = -\frac{1}{2}h^2(t)\, dt + h(t)\, dW_t.$$

The generator of this process is

$$\mathcal{L} = -\frac{1}{2}h^2(t)\frac{d}{dx} + \frac{1}{2}h^2(t)\frac{d^2}{dx^2}.$$

The process $Y(t, \omega)$ can be written in the form

$$Y(t, \omega) = e^{Z_t}$$

We use Itô's formula, noting that

$$\mathcal{L}e^x = -\frac{1}{2}h^2(t)\frac{d}{dx}e^x + \frac{1}{2}h^2(t)\frac{d^2}{dx^2}e^x = 0,$$

to calculate

$$\begin{aligned} dY_t &= (\mathcal{L}e^{Z_t})\, dt + \frac{d}{dx}e^{Z_t}h(t)\, dW_t \\ &= Y_t h(t) dW_t. \end{aligned}$$

Consequently,

$$dY_t = Y_t h(t) dW_t, \quad Y_0 = 1.$$

(ii) The result then follows from the martingale representation theorem.
[6] MARKS –C

(b) The generator of the process is

$$\mathcal{L} = -\frac{1}{2}e^{-2x}\partial_x + \frac{1}{2}e^{-2x}\partial_x^2.$$

We apply Itô's formula to the function $f(x) = e^x$, $Y_t = f(X_t)$:

$$dY_t = (\mathcal{L}f)(X_t)\, dt + (\partial_x f)(X_t)e^{-X_t}\, dW_t = dW_t.$$

Therefore:

$$X_t = \ln\left(W_t + e^x\right).$$

[4] MARKS –A

(c) (i) The generator is

$$\mathcal{L} = (-a_0 - a_1 x)\partial_x + \frac{1}{2}(b_0 + b_1 x + b_2 x^2)\partial_x^2.$$

The Fokker-Planck operator is the $L^2(\mathbb{R})-$adjoint of the generator:

$$\mathcal{L}^* \rho = \partial_x \left((a_0 + a_1 x)\rho + \frac{1}{2}\partial_x((b_0 + b_1 x + b_2 x^2)\rho) \right).$$

The backward and forward Kolmogorov equations are

$$\partial_t u = \mathcal{L}u, \quad u(x,t) = \mathbb{E}(f(X_t)|X_0 = x), \quad u(x,0) = f(x)$$

and

$$\partial_t \rho = \mathcal{L}^* \rho, \quad \rho(x,0) = \rho_0(x).$$

[2] MARKS –A

(ii) The nth moment M_n is defined as

$$M_n(t) = \int x^n \rho(x,t)\, dx, \quad n = 0, 1, \ldots$$

We multiply the Fokker-Planck equation by x^n, integrate over $\mathbb{R}$ and integrate by parts on the right hand side of the equation to obtain

$$\begin{aligned} \dot{M}_n &= -n \int (a_0 + a_1 x) x^{n-1} \rho\, dx + \frac{1}{2} n(n-1) \int (b_0 + b_1 x + b_2 x^2) x^{n-2} \rho\, dx \\ &= \left(-a_1 n + \frac{1}{2} b_2 n(n-1) \right) M_n + \left(-n a_0 + \frac{1}{2} b_1 n(n-1) \right) M_{n-1} + \frac{1}{2} b_0 n(n-1) M_{n-2}. \end{aligned}$$

The initial conditions are $M_n(0) = \int_{\mathbb{R}} x^n \rho_0(x)\, dx$,
$n = 0, 1, \ldots$.
[8] MARKS –A

4. (a) We use the variation of constants formula to obtain the solution of the SDE:

$$X_t = e^{-\alpha t} x + \sqrt{2\sigma} \int_0^t e^{-\alpha(t-s)}\, dW_s.$$

[3] MARKS –A

(b) We use the fact that, for a function $\xi(t)$ that satisfies the standard assumptions, there exists a Brownian motion B_t such that

$$\int_0^t \xi_t\, dW_t = B\left(\int_0^t \xi_s^2\, ds \right).$$

We write the solution of the OU SDE in the form

$$\begin{aligned}
X_t &= e^{-\alpha t}x + \sqrt{2\sigma}\int_0^t e^{-\alpha(t-s)}\,dW_s \\
&= e^{-\alpha t}x + \sqrt{2\sigma}e^{-\alpha t}\int_0^t e^{\alpha s}\,dW_s \\
&= e^{-\alpha t}x + \sqrt{2\sigma}e^{-\alpha t}\int_0^t e^{\alpha s}\,dW_s \\
&= e^{-\alpha t}x + \sqrt{2\sigma}e^{-\alpha t}\,W\left(\int_0^t e^{2\alpha s}\,ds\right) \\
&= e^{-\alpha t}x + \sqrt{2\sigma}e^{-\alpha t}\,W\left(\frac{1}{2\alpha}\left(e^{2\alpha t}-1\right)\right).
\end{aligned}$$

[3] MARKS –D

(c) The process X_t is Gaussian, since it can be written as a linear combination of Gaussian random variables (Riemann approximation of the stochastic integral and Gaussianity of Brownian motion). The mean and variance have the limits $\lim_{t\to+\infty}\mathbb{E}X_t = 0$ and $\lim_{t\to+\infty}\mathrm{Var}(X_t) = \frac{\sigma}{\alpha}$. We conclude that in the limit as $t\to+\infty$, X_t converges to the Gaussian random variable $\mathcal{N}(0,\frac{\sigma}{\alpha})$.

[4] MARKS –A

(d) From the solution of the SDE, we deduce that

$$X_t \sim \mathcal{N}\left(e^{-\alpha t}x, \frac{\sigma}{\alpha}\left(1-e^{-2\alpha t}\right)\right).$$

Denote by $\gamma_{\mu,\sigma^2}(y)$ the Gaussian function with mean μ and variance σ^2. The law of the process X_t starting at the deterministic point x is $\gamma_{e^{-\alpha t}x,\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}(y)$. Consequently,

$$\begin{aligned}
(P_t f)(x) &= \int f(y)\gamma_{e^{-\alpha t}x,\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}(y)\,dy \\
&= \frac{1}{\sqrt{2\pi\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}}\int f(y)e^{-\frac{|y-e^{-\alpha t}x|^2}{2\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}}\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int f\left(e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}z\right)e^{-\frac{z^2}{2}}\,dz \\
&= \int f\left(e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}z\right)\gamma(z)\,dz,
\end{aligned}$$

where $\gamma(z)$ denotes that standard Gaussian $\gamma_{0,1}(z)$ and we have made the change of variables

$$z = \frac{y-e^{-\alpha t}x}{\sqrt{\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}}.$$

Assume now that f is differentiable. We can interchange the derivative with respect to x and the integral to calculate, using the notation $h(x,t;z) = e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}z$

$$\begin{aligned}
\frac{d}{dx}(P_t f)(x) &= \frac{d}{dx}\int f\left(e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1-e^{-2\alpha t})}z\right)\gamma(z)\,dz \\
&= \int f'(h(x,t;z))\partial_x h(x,t;z)\gamma(z)\,dz \\
&= e^{-\alpha t}\int f'(h(x,t;z))\gamma(z)\,dz = e^{-\alpha t}\left(P_t \frac{df}{dx}\right)(x).
\end{aligned}$$

[10] MARKS –B

5. (a) The generator is

$$\mathcal{L} = \nabla f \cdot \nabla + f\Delta.$$

Its adjoint is

$$\begin{aligned}
\mathcal{L}^*\rho &= \nabla\cdot\big(-\nabla f\rho + \nabla(f\rho)\big) \\
&= \nabla\cdot\big(f\nabla\rho\big).
\end{aligned}$$

The forward and backward Kolmogorov equations are

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \quad u(x,0) = f(x),$$

where $u(x,t) = \mathbb{E}(f(X_t)|X_0 = x)$ and

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^*\rho, \quad \rho(x,0) = \rho_0(x),$$

where $X_t \sim \rho$. The equations are posed in the domain D. The boundary conditions for the Fokker-Planck equation are no flux:

$$J\cdot\boldsymbol{\eta} = 0, \quad J = -f\nabla\rho,$$

where η denotes the outward normal unit vector. Since $f > 0$, this leads to Neumann boundary conditions. The Fokker Planck equation can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla\cdot J = 0, \quad \text{in } D\times[0,+\infty), \tag{2a}$$

$$\nabla\rho\cdot\boldsymbol{\eta} = 0, \quad \text{on } \partial D\times[0,+\infty), \tag{2b}$$

$$\rho(x,0) = \rho_0(x), \quad \text{on } D\times\{t=0\}. \tag{2c}$$

The flux is given by $J = -f\nabla\rho$. **[4] MARKS**

(b) We need to show that ρ_∞ is the unique normalizable solution of the stationary Fokker-Planck equation

$$\nabla\cdot\big(f\nabla\rho\big) = 0, \tag{3}$$

together with the boundary condition $\nabla\rho\cdot\boldsymbol{\eta} = 0$. We readily check the normalized uniform distribution is a solution

$$\rho_\infty = \frac{1}{|D|}, \tag{4}$$

where $|D|$ denotes the volume of the domain D. To show that it is unique, assume that there are two solutions ρ_1 and ρ_2 and let $\xi := \rho_1 - \rho_2$. We take the difference of the two stationary Fokker-Planck equations, multiply by ξ and integrate over the domain to obtain

$$\int_D \xi \nabla \cdot (f \nabla \xi) \, dx = 0.$$

We integrate by parts and use the boundary conditions to obtain

$$\int_D f |\nabla \xi|^2 \, dx = 0.$$

Since $f > 0$ in the domain, we deduce that $|\nabla \xi|^2 =$ from which we conclude that ρ_1 and ρ_2 differ by constant. Since the invariant distribution is normalized, we conclude that the unique stationary state is (4).

[4] MARKS

(c) Let $\rho(x,t)$ denote the solution of the Fokker-Planck equation and let $h := \rho - \rho_\infty$. The function h satisfies the Fokker-Planck equation with the same no-flux boundary conditions:

$$\frac{\partial h}{\partial t} = \nabla \cdot (f \nabla h).$$

We multiply the equation by h and integrate over D to obtain

$$\frac{1}{2}\frac{d}{dt}\int_D h^2 \, dx = \int_D h \nabla \cdot (f \nabla h) \, dx = -\int_D f |\nabla h|^2 \, dx.$$

We use the fact that $f \in [f_{min}, f_{max}]$ to deduce that

$$-\int_D f |\nabla h|^2 \, dx \leq -f_{min} \int_D |\nabla h|^2 \, dx.$$

Therefore:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_D h^2 \, dx &\leq -f_{min} \int_D |\nabla h|^2 \, dx \\ &\leq -C \int_D h^2 \, dx, \end{aligned}$$

where we have used Poincaré's inequality, since $h = \rho(t,x) - \frac{1}{|D|}$ is a mean zero function and we have Neumann boundary conditions. We now use Gronwall's inequality to deduce

$$\int_D \left| \rho(t,x) - \frac{1}{|D|} \right|^2 \, dx \leq e^{-Ct} \int_D \left| \rho_0(x) - \frac{1}{|D|} \right|^2 \, dx,$$

which shows exponentially fast convergence to the uniform distribution in L^2. **[12] MARKS**

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70054	1	No Comments Received
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