Imperial College London
Department of Mathematics

Solutions to Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2021/22

You have 1h. You should attempt all questions.

1. Let $V = \mathbb{R}^3$ and $T : V \to V$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_2 \\ -x_1 - x_2 + x_3 \end{pmatrix}.$$

Let $E = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Let $A = [T]_E$.

a. Write down A, explaining your answer. (2 marks)

b. Is the matrix A orthogonal? Explain your answer. (2 marks)

c. Find the eigenvectors of T. (6 marks)

d. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix. (1 mark)

e. Write e_1 as a linear combination of the eigenvectors of T, then express $T^n(e_1)$ as a linear combination of e_1, e_2, e_3 for all $n \in \mathbb{N}$. (T^0 is the identity transformation.) (4 marks)

f. Find a matrix $C \in M_3(\mathbb{R})$ such that $C^2 = A$. (5 marks)

(Total: 20 marks)

Solution:

a. (No marks for answers that do not show the calculations.) Computing the image of the standard basis through T, we get

$$T(e_1) = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(**1 mark**). Therefore

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

(**1 mark**).

b. Let $(b_{ij}) = AA^T$. We check that $b_{12} = 1$. Thus the matrix is not orthogonal. (**1 mark for recognising one has to check $AA^T = I_3$, 1 for the rest**).

c. The characteristic polynomial of A (or of T) is

$$\chi_A(X) = \begin{vmatrix} X-2 & -1 & 0 \\ 0 & X-1 & 0 \\ 1 & 1 & X-1 \end{vmatrix} = (X-2)(X-1)^2.$$

Hence, T has two eigenvalues: 2 and 1. (**1 mark**)

i. We compute the eigenspace E_1 relative to 1. This is

$$E_1 = \ker(I_3 - A) = \ker \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \mathbf{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(**3 marks**).

ii. We compute the eigenspace E_2. This is

$$E_2 = \ker(2I_3 = A) = \ker \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(**2 marks**). The eigenvectors of T with eigenvalue 1 are $E_1 \setminus \{0\}$, and the eigenvectors of T with eigenvalue 2 are $E_2 \setminus \{0\}$.

d. (**1 mark**) (This has multiple solutions, accept all the correct ones) We write a matrix P that has a basis of eigenvectors down its columns. For example, we could use the one we just found, which gives

$$P = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(This can also be explained by writing $P = {}_E[id]_B$ where B is a basis of eigenvectors for A.)

e. Let

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then $e_1 = -v_1 + v_2 - v_3$, thus $T^n(e_1) = T^n(-v_1 + v_2) - T^n(v_3) = v_1 + v_2 - 2^n v_3 = 2^n e_1 + (1 - 2^n)e_3$.

f. Let $R = diag(1, 1, \sqrt{2})$ (**1 mark**). Then we may choose $C = PRP^{-1}$. (Indeed $C^2 = PRP^{-1}PRP^{-1} = PR^2P^{-1} = A$). We need to compute P^{-1}. We have seen that $e_1 = -v_1 + v_2 - v_3$, we see further that $e_2 = -2v_1 + v_2 - v_3$ and $e_3 = -v_1 + v_2$. Hence

$$P^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

(**2 marks**). Therefore

$$C = PRP^{-1} = \begin{pmatrix} \sqrt{2} & \sqrt{2} - 1 & 0 \\ 0 & 1 & 0 \\ -\sqrt{2} + 1 & -\sqrt{2} + 1 & 1 \end{pmatrix}.$$

(**2 marks**).

2. **a.** Let $n \in \mathbb{N} \setminus \{0\}$. Let $A_n = (a_{ij}) \in M_n(\mathbb{Q})$ be defined by

$$a_{ij} = \begin{cases} \binom{i-1}{j-1} & j \leq i \\ 0 & j > i. \end{cases}$$

(Recall that $\binom{r}{0} = 1$ for all $r \in \mathbb{N}$ (including $r = 0$)).

i. Write down A_n for $n = 1, 2, 3$. (2 marks)

ii. Compute $\det(A_n)$ for all $n \in \mathbb{N} \setminus \{0\}$. (2 marks)

iii. Find the eigenvectors of A_n for all $n \in \mathbb{N} \setminus \{0\}$. (3 marks)

iv. For which values of n is A_n diagonalisable? Explain your answer. (2 marks)

b. For each of the following statements, say whether it is true or false. If it is true, give a short proof; if it is false, give a counterexample.

i. Let $n \in \mathbb{N} \setminus \{0\}$ and let V be an n-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. Assume $\chi_T(X) = (X - \lambda)^n$ for some $\lambda \in F$. Then T is diagonalisable if and only if $T = \lambda \cdot id$, where $id : V \to V$ is the identity transformation. (3 marks)

ii. Let $n \in \mathbb{N} \setminus \{0\}$ and $A = (a_{ij}) \in M_n(\mathbb{Q})$ be an upper-triangular matrix (that is $a_{ij} = 0$ for $j < i$). Assume that A is invertible. Then A^{-1} is upper-triangular.

(3 marks)

iii. Let $n \in \mathbb{N} \setminus \{0\}$ and $A \in M_n(\mathbb{R})$. Then there is a real symmetric matrix $A_s \in M_n(\mathbb{R})$ and a matrix $A_a \in M_n(\mathbb{R})$, such that $A_a^T = -A_a$ and

$$A = A_s + A_a.$$

(2 marks)

iv. Let $n \in \mathbb{N} \setminus \{0\}$ and F be a field. Let $A, B \in M_n(F)$. If A and B are diagonalisable, then there is an invertible $P \in M_n(F)$ such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

(3 marks)

(Total: 20 marks)

Solution:

a. **i.**

$$A_1 = (1), \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

A_n is the matrix in which the entries on and below the diagonal form a Pascal triangle.

ii. By definition A_n is lower-triangular, thus its determinant is the product of its diagonal elements (**1 mark**) For $i \in \{1, \dots, n\}$,

$$a_{ii} = \binom{i-1}{i-1} = 1;$$

therefore $\det A_n = 1$. (**2 marks**)

iii. The characteristic polynomial of A_n is $(X-1)^n$ because A_n is lower-triangular with 1 along the diagonal. The only eigenvalue is 1. The eigenspace is

$$E_1 = \ker \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & -2 & 0 & c\dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & \binom{n-1}{n-2} & 0 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

Indeed, it is clear that $(0, \dots, 0, 1)^T \in E_1$. Conversely, the matrix

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & -2 & 0 & c\dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & \binom{n-1}{n-2} & 0 \end{pmatrix}$$

has rank $n-1$ because the first $n-1$ columns are linearly independent. (It is not required but if you want to justify also this last thing: let $c_1, \dots, c_n$ be the columns of M_1. If

$$\sum_{i=0}^{n-1} \alpha_i c_i = 0,$$

then $\alpha_1 = 0$, this in turn implies that $\alpha_2 = 0$ etc. until $\alpha_{n-1} = 0$.)

iv. The matrix is not diagonalisable unless $n = 1$. This is because A_n is diagonalisable if and only if there is basis of $\mathbb{Q}^n$ consisting of eigenvectors for A_n. We just saw that the only eigenspace E_1 has dimension 1, thus A_n is diagonalisable if and only if $n = 1$.

b. **i.** True. Clearly $T = \lambda \cdot id$ is diagonalisable. Conversely if T has characteristic polynomial $(X-\lambda)^n$ and is diagonalisable, then there is a basis of eigenvectors $v_1, \dots, v_n$, all with eigenvalue λ. If $v \in V$, there are $\alpha_1, \dots, \alpha_n \in F$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$; therefore $T(v) = \lambda \sum_{i=1}^n \alpha_i v_i = \lambda v$.

ii. True. The quickest way to see this is with Cramer's rule. Let $(b_{ij}) = B = A^{-1}$. By Cramer's rule, for $i, j \in \{1, \dots, n\}$, the entry b_{ji} is a multiple of $\det(A_{ij})$, where A_{ij} is the submatrix obtained from A removing row i and column j. We want to show that $b_{ji} = 0$ for $i < j$. If $i < j$, then A_{ij} is upper-triangular (this can be seen using the formula in Problem Sheet 0). Moreover the i-th entry on the diagonal of A_{ij} is $a_{i+1,i} = 0$. This shows that $\det(A_{ij}) = 0$.

iii. True. We observe that

$$A = \frac{1}{2}(A + A^T - A^T + A) = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

We define

$$A_s = \frac{1}{2}(A + A^T) \text{ and } A_a = \frac{1}{2}(A - A^T)$$

Clearly $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ and $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A + A^T)$, so A_a and A_s have the required properties.

iv. False. If there is a matrix P that works for both A and B, then A and B commute. Thus any pair of non-commuting diagonalisable matrices is a good counter-example. For instance

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To see that if such P exists then A and B commute, we write

$$P^{-1}AP = D_1, \quad P^{-1}BP = D_2$$

for two diagonal matrices D_1 and D_2. Since D_1 and D_2 commute,

$$P^{-1}ABP = P^{-1}APP^{-1}BP = D_1D_2 = D_2D_1 = P^{-1}BPP^{-1}AP = P^{-1}BAP.$$

Hence $P^{-1}ABP = P^{-1}BAP$. Multiplying the last equality by P on the left and P^{-1} on the right, we obtain $AB = BA$.