Imperial College
London

MATH97007 MATH70002

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics 2

Date: 25 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. A spherical drop of radius $a = 1$, density ρ_1 and viscosity μ_1 is surrounded by another fluid of density ρ_2 and viscosity μ_2. In the frame where the drop is at rest, the fluid at infinity moves with uniform velocity $\mathbf{U}$ such that the Reynolds number is very low. Use spherical polar coordinates (r, θ, ϕ) centred on the drop and aligned with $\mathbf{U}$. At $r = 1$, the velocity $\mathbf{u}$ and tangential stress $\sigma_{r\theta}$ are continuous, and you may assume that surface tension is large enough to keep the drop spherical, so that the normal stress balances and also the normal velocity is zero.

(a) Represent the flow inside and outside the sphere using a Stokes streamfunction ψ, such that

$$\mathbf{u} = \nabla \times \left(0,\ 0,\ \frac{\psi}{r \sin\theta}\right).$$

Write down all the boundary conditions to apply at $r = 0$, $r = 1$ and as $r \to \infty$. Just as for the solid sphere, we seek a separable solution as suggested by the boundary conditions of the form $\psi = f(r)\sin^2\theta$, where the expressions for f inside and outside the sphere are different. Note that the stress boundary condition requires $\mu(f'/r^2)'$ to be continuous at $r = 1$. Find ψ for all r and show that in $r > 1$

$$\psi = \frac{U}{4}\sin^2\theta\left[\left(\frac{\mu_1}{\mu_1+\mu_2}\right)\frac{1}{r} - \left(\frac{2\mu_2+3\mu_1}{\mu_1+\mu_2}\right)r + 2r^2\right],$$

where $U = |\mathbf{U}|$. (13 marks)

(b) When $\mu_2 \ll \mu_1$, you may assume that the sphere behaves as a solid and hence it experiences the drag $D = 6\pi a \mu_2 U$. By considering the form of the flow at large r deduce the drag for general μ_1 and μ_2. (4 marks)

(c) Suppose the drop rises or sinks because of gravity. Deduce that at low Reynolds number its speed is

$$U = \frac{2a^2 g}{3\mu_2}(\rho_2 - \rho_1)\left(\frac{\mu_1+\mu_2}{3\mu_1+2\mu_2}\right).$$

(3 marks)

[Note: You may assume that

$$\nabla \times \nabla \times \left(0,\ 0,\ \frac{f(r,\theta)}{r\sin\theta}\right) = \left(0,\ 0,\ -\frac{1}{r\sin\theta}D^2 f\right)$$

where

$$D^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right).$$

The tangential stress component on a surface of constant r is

$$\sigma_{r\theta} = \mu\left(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right).]$$

(Total: 20 marks)

2. In lectures we analysed the evolution of a 2-dimensional jet emerging through a slit. Here we consider a thin axisymmetric jet, such as might occur when one blows out a candle.

The jet flows predominantly in the z-direction and is concentrated around the symmetry axis $R = 0$ in terms of cylindrical polar coordinates $(R,\, \phi,\, z)$. Let the axisymmetric velocity be

$$\mathbf{u} = (u,\, 0,\, w) = \frac{1}{R}\left(-\frac{\partial \psi}{\partial z},\, 0,\, \frac{\partial \psi}{\partial R}\right)$$

in terms of a Stokes streamfunction $\psi(R, z)$. The axisymmetric boundary layer equations can be written

$$u\frac{\partial w}{\partial R} + w\frac{\partial w}{\partial z} = \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w}{\partial R}\right), \qquad \frac{1}{R}\frac{\partial (Ru)}{\partial R} + \frac{\partial w}{\partial z} = 0.$$

(a) Why is there no pressure gradient term in the momentum equation? (2 marks)
Show that the momentum flux

$$M = \int_0^\infty R w^2 \, dR$$

is independent of z. (5 marks)

(b) You may assume that a similarity solution

$$\psi = \nu z f(\eta) \qquad \text{where} \quad \eta = R/z$$

exists provided that $f(\eta)$ satisfies the equation

$$\eta f'' - f' + f f' = 0 \quad \text{in } 0 < \eta < \infty.$$

Integrate this equation once, explaining how conditions on $R = 0$ determine the constant of integration. (3 marks)

(c) Integrate once more to find $f(\eta)$ in terms of an unknown constant. Write down an integral condition which determines this constant in terms of M and ν. (6 marks)
Deduce that

$$f(\eta) = \frac{12 M \eta^2}{32\nu^2 + 3M\eta^2}.$$

(2 marks)

(d) Calculate the radial velocity component u for large R, and interpret the result physically. (2 marks)

(Total: 20 marks)

3. Light fluid, of density ρ_1, occupies the region $0 < y < h$ and moves to the right with velocity $(U, 0, 0)$. Heavier fluid of density ρ_2 occupies $-h < y < 0$ and moves to the left with velocity $(-U, 0, 0)$. The boundaries at $y = \pm h$ are solid, and the flow is inviscid and irrotational. The interface $y = 0$ is a free surface, and there is no surface tension. Gravity acts in the negative y-direction.

The interface is perturbed to $y = \varepsilon e^{ikx} e^{st}$, where $0 < \varepsilon \ll 1$, k is a given real wavenumber, and s is to be found. [In the usual way, the real part is implicit in expressions involving e^{ikx}.]

(a) Perturb the velocity and pressure in fluid 1 according to

$$\mathbf{u} = (U,\, 0,\, 0) + \varepsilon \nabla \phi_1, \qquad p = P_0(y) + \varepsilon p_1$$

and similarly for fluid 2, where $P_0(y)$ is the equilibrium pressure distribution. Write down all the conditions to be satisfied on the boundaries and interface. (4 marks)

(b) Solve for the perturbations, and find an equation giving the growth rate s. Show that the interface is unstable if

$$U^2 > \frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2} \frac{g \tanh kh}{4k}.$$

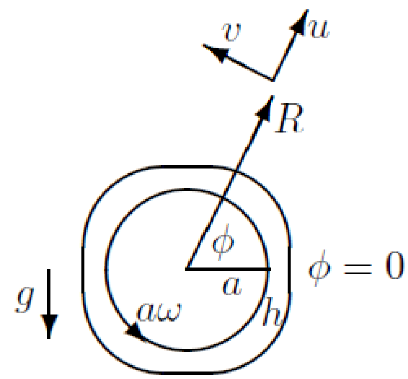
(10 marks)

(c) Find the smallest speed U such that every mode $k > 0$ is unstable.

Discuss whether or not the interface can be stable to all perturbations, bearing in mind the limitations of the model. (6 marks)

(Total: 20 marks)

4. Viscous fluid of density ρ and viscosity μ adheres in a thin layer to a horizontal cylindrical rod of radius a. The rod rotates about its axis with angular speed ω. The layer thickness is $h(\phi, t)$ in terms of polar coordinates (R, ϕ), where $\phi = 0$ is horizontal, with corresponding velocity components (u, v), as in the figure.



(a) Explain the assumptions leading to the lubrication equations in $a < R < a + h$

$$\mu v_{RR} \simeq \rho g \cos\phi, \qquad a u_R + v_\phi = 0,$$

and state the pressure distribution. Write down the boundary conditions on the solid boundary $R = a$ and the free surface $R = a + h$. [You may assume that the tangential stress is $\sigma_{R\phi} = \mu \partial v / \partial R$.] (6 marks)

(b) Show that

$$a\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial \phi} = 0, \qquad \text{where} \quad Q \equiv \int_a^{a+h} v \, dR = a\omega h - \frac{\rho g h^3 \cos\phi}{3\mu}.$$

(6 marks)

(c) Seek now a steady, smooth state, for which Q is constant and $h = h(\phi) > 0$. Express $\cos\phi$ as a function of h and deduce that there is no such state if

$$a\omega > \left(\frac{9Q^2 \rho g}{4\mu}\right)^{1/3}.$$ (6 marks)

What do you think will happen if ω exceeds this value? (2 marks)

(Total: 20 marks)

5. A body moves through fluid, either self-propelled or driven externally. Discuss the significance of the shape of the body for the resulting flow and forces upon it. (20 marks)

(Total: 20 marks)

1. **Seen for solid sphere and for dynamically negligible bubble.** sim. seen ⇓

(a) We have both inside and outside the sphere 3, A

$$D^2(D^2\psi) = 0, \qquad \text{with the boundary conditions:}$$ 2, B

(a) ψ and its derivatives must be nonsingular at $r = 0$,

(b) the flow is uniform at infinity, so $\psi \to \frac{1}{2}Ur^2\sin^2\phi$ as $r \to \infty$

(c) the radial velocity is zero at $r = a$ so that $\psi =$ constant there. Wlog, $\psi = 0$ at $r = 1$.

(d) ψ_r must be continuous at $r = 1$ and

(e) we must also have continuity of tangential stress so that

$$\mu\left(\frac{\partial(u_\phi/r)}{\partial r}\right) \qquad \text{takes the same value in fluids 1 and 2.}$$

We seek a solution with $\psi = f(r)\sin^2\phi$, and we find

$$D^2\psi = \left(f'' - \frac{2f}{r^2}\right)\sin^2\phi \equiv F(r)\sin^2\phi, \qquad \text{say, and}$$

$$D^2(D^2\psi) = \left(F'' - \frac{2F}{r^2}\right)\sin^2\phi \qquad \Longrightarrow \qquad F = A'r^2 + B'/r.$$

for some constants A' and B'. Then

$$f(r) = Ar^4 + Br + Cr^2 + D/r, \qquad \text{for different constants in both } 0 < r < 1 \text{ and } r > 1.$$

In $r > 1$ the boundary conditions at infinity require $A = 0$ and $C = \frac{1}{2}U$, whereas in $0 < r < 1$ we must have $B = D = 0$. Note that $f \sim r$ leads to $F \sim 1/r$ and the vorticity would be singular. We therefore have

$$f = \begin{cases} Ar^4 + Cr^2 & 0 < r < 1 \\ Br + \frac{1}{2}Ur^2 + \frac{D}{r} & r > 1 \end{cases}$$

At $r = 1$ we require $\psi = 0$ both inside and out and so $C = -A$ and $B + \frac{1}{2}U + D = 0$. We now have

$$f = \begin{cases} A(r^4 - r^2) & 0 < r < 1 \\ Br + \frac{1}{2}Ur^2 - \frac{B+U/2}{r} & r > 1 \end{cases}$$

Continuity of f' requires

$$2A = 2B + 3U/2.$$

Finally we need continuity of $\mu\partial(u_\phi/r)/\partial r$ which requires continuity of $\mu(f'/r^2)'$ so that

$$\mu_1(6A) = \mu_2(-2B - U - 4(B + U/2)) = (-6B - 3U)\mu_2$$

giving finally

$$A = \frac{U}{4}\left(\frac{\mu_2/4}{\mu_1 + \mu_2}\right), \qquad B = -\frac{U}{4}\left(\frac{2\mu_2 + 3\mu_1}{\mu_1 + \mu_2}\right), \qquad D = \frac{U}{4}\left(\frac{\mu_1}{\mu_1 + \mu_2}\right)$$

so that 2, A

$$\psi = \frac{U}{4}\sin^2\phi\left(\frac{\mu_2}{\mu_1 + \mu_2}\right)(r^4 - r^2) \qquad \text{in } 0 < r < 1, \text{ and}$$ 2, B

2, C

$$\psi = \frac{U}{4}\sin^2\phi\left[\left(\frac{\mu_1}{\mu_1 + \mu_2}\right)\frac{1}{r} - \left(\frac{2\mu_2 + 3\mu_1}{\mu_1 + \mu_2}\right)r + 2r^2\right] \qquad \text{in } r > 1.$$ 2, D

(b) As $\mu_1/\mu_2 \to \infty$ we recover the solid sphere solution with a sign change because of the definition of U.

$$\psi = \frac{U}{4}\sin^2\phi\left[\frac{1}{r} - 3r\right],$$

$\boxed{\text{seen} \Downarrow}$

switching to the frame where infinity is at rest. We know this flow gives rise to a drag $6\pi\mu U$. We also know that for the unforced Stokes equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \Longrightarrow \int_S \sigma_{ij} n_j \, dS = \int_{S_\infty} \sigma_{ij} n_j \, dS$$

using the divergence theorem, and so the force on any body can be calculated by the form of the flow for large r. It is only the term for which $\psi \propto r$ which contributes to this integral as then $\sigma \sim 1/r^2$. We can therefore deduce that the drag is proportional to that coefficient (the Stokeslet strength). As we know its value as $\mu_1/\mu_2 \to \infty$, we know the constant of proportionality, and hence the drag

$$D = 6\pi a\mu_2 U\left(\frac{2\mu_2 + 3\mu_1}{\mu_1 + \mu_2}\right)\frac{1}{3} = 2\pi a\mu_2 U\left(\frac{2\mu_2 + 3\mu_1}{\mu_1 + \mu_2}\right),$$

where we have re-introduced the radius a using dimensional arguments. [Not required.] $\boxed{2,\ \text{C}}$

(c) If the flow is driven by gravitational buoyancy, we know the drag force must balance the Archimedean upthrust, which is the difference between the weight of the drop and the weight of the displaced fluid. Thus $\boxed{2,\ \text{D}}$

$$\tfrac{4}{3}\pi a^3 g(\rho_2 - \rho_1) = 2\pi a\mu_2 U\left(\frac{2\mu_2 + 3\mu_1}{\mu_1 + \mu_2}\right)$$

or as required $\boxed{3,\ \text{A}}$

$$U = \tfrac{2}{3}\frac{a^2 g(\rho_2 - \rho_1)}{\mu_2}\left(\frac{\mu_1 + \mu_2}{3\mu_1 + 2\mu_2}\right)$$

2. **Seen the calculation for the 2-D jet, which also has an invariant momentum flux.** seen $\Downarrow$

The pressure does not vary across a thin layer, and so equals its value outside the layer, UU' where $U(z)$ is the slip velocity from the irrotational flow, $w \to U$ as $R \to \infty$. Here $U = 0$ and so the pressure gradient vanishes. 2, A

(a) We have

$$\frac{dM}{dz} = \frac{d}{dz}\int_0^\infty Rw^2\,dR = \int_0^\infty 2Rww_z\,dR = \int_0^\infty 2R\left(\frac{\nu}{R}(Rw_R)_R - uw_R\right)dz$$
$$= \left[2\nu Rw_R\right]_0^\infty - \Big[2Ruw\Big]_0^\infty + \int_0^\infty 2w(Ru)_R\,dR = -2\int_0^\infty Rww_z\,dR = -\frac{dM}{dz},$$

where we have used the governing equations and the condition $w \to 0$ as $R \to \infty$. It follows that $dM/dz = 0$ and hence M is a constant, just as it was for the 2-D jet. 5, B

(b) If we write $\psi = \nu z f(\eta)$ where $\eta = R/z$, we are given that the momentum equation reduces to meth seen $\Downarrow$

$$\eta f'' - f' + ff' = 0 \qquad \Longrightarrow \qquad \eta f' - 2f + \tfrac{1}{2}f^2 = A,$$

where A is a constant. Now the symmetry axis must be a streamline, so we require $f(0) = 0$. It follow that $A = 0$. 3, A

(c) We then have

$$\int \frac{df}{f(2-f/2)} = \int \frac{d\eta}{\eta} \qquad \Longrightarrow \qquad \log\eta = \int\left(\frac{1/2}{f} + \frac{1/4}{2-f/2}\right)df$$

$$\log\eta = \tfrac{1}{2}\log f - \tfrac{1}{2}\log(4-f) + c \qquad \Longrightarrow \qquad \frac{f}{4-f} = \eta^2/B$$

Thus 5, A

$$f = \frac{4\eta^2}{\eta^2 + B}, \qquad \text{for some constant } B.$$ 1, D

We must now satisfy the integral constraint. Since $w = \psi_R/R = \nu f'(\eta)/R$, we have

$$M = \int_0^\infty R\nu^2\frac{f'(\eta)^2}{R^2}dR = \nu^2\int_0^\infty \frac{(f')^2}{\eta}\,d\eta.$$

We have $f' = 8B\eta/(\eta^2+B)^2$. The integral constraint thus gives

$$\frac{M}{\nu^2} = \int_0^\infty \frac{64B^2\eta}{(\eta^2+B)^4}\,d\eta = \left[\frac{-32B^2}{3(\eta^2+B)^3}\right]_0^\infty = \frac{32}{3B}.$$

Thus $B = 32\nu^2/3M$ and

$$f = \frac{12M\eta^2}{3M\eta^2 + 32\nu^2}.$$

2, C

(d) As $R \to \infty$ we have $\eta \to \infty$ and $f \to 4$. Thus $\psi \to 4\nu z$ and $u \to -\frac{4\nu}{R}$. Thus fluid is being sucked in from infinity to feed the jet at a rate which does not vary with z. The jet is entraining fluid as it thickens downstream. Note also the decay in the radial direction is only algebraic – most boundary layers decay exponentially. 2, B

3. **SEEN the analysis for infinite layers, but not with finite h.** sim. seen $\Downarrow$

(a) (i) No normal velocity on $y = \pm h$, i.e. $\partial\phi/\partial y = 0$.

(ii) Kinematic condition on $y = \varepsilon e^{ikx+st}$ is

$$0 = \frac{D}{Dt}(y - \varepsilon e^{ikx+st}) = \varepsilon\frac{\partial\phi}{\partial y} - \varepsilon s \mp \varepsilon U ik + O(\varepsilon^2)$$

This can be evaluated on $y = 0$ with another error of $O(\varepsilon^2)$, so

$$\frac{\partial\phi}{\partial y} = (s + ikU) \qquad \text{on } y = 0_+, \qquad \frac{\partial\phi}{\partial y} = (s - ikU) \qquad \text{on } y = 0_-.$$

(iii) Dynamic condition: the pressure must be continuous at the interface. Using the time-dependent Bernoulli theorem,

$$p + \rho g y + \rho\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\rho|\mathbf{u}|^2 \qquad \text{is constant in each fluid.}$$

Now in fluid 1 $|\mathbf{u}|^2 = U^2 + 2Uik\varepsilon\phi_1(0) + O(\varepsilon^2)$. Once again, we may evaluate this on $y = 0$ to leading order in ε. So writing $E \equiv e^{ikx+st}$, we have on $y = 0$

$$p_1 + \rho_1\big(gE + s\phi_1(0) + Uik\phi_1(0)\big) = p_2 + \rho_2\big(gE + s\phi_2(0) - Uik\phi_2(0)\big) \qquad \text{and } p_1 = p_2.$$

2, A

(b) Now we have $\nabla^2\phi_1 = 0$ and $\phi_1 \propto e^{ikx}$. So the y-dependence is like $e^{\pm ky}$. We want $\partial\phi_1/\partial y = 0$ on $y = h$ so we write 2, C

$$\phi_1 = A\cosh[k(y-h)]e^{ikx+st} \qquad \text{and similarly} \qquad \phi_2 = B\cosh[k(y+h)]e^{ikx+st}.$$

Then the kinematic condition at $y = 0_+$ gives

$$kA\sinh(-kh) = s + ikU, \qquad \text{and at } y = 0_- \qquad kB\sinh(kh) = s - ikU$$

which determines A and B and hence the perturbed flow. In particular, $\phi_1(0) = A\cosh kh\, e^{ikx+st}$. The dynamic condition now gives the dispersion relation

$$\rho_1(g - (s + ikU)^2\coth(kh)/k) = \rho_2(g + (s - ikU)^2\coth(kh)/k).$$

Rearranging, we obtain the quadratic equation for s

$$(\rho_1 + \rho_2)s^2 + 2ikUs(\rho_1 - \rho_2) + (\rho_2 - \rho_1)gk\tanh kh - k^2U^2(\rho_1 + \rho_2) = 0.$$

[Check: as $h \to \infty$ we recover the known result from lecture notes for infinite layers. We can continue to check our results against the notes by replacing g with $g\tanh kh$.]

The roots of the quadratic are

$$s = -ik(\text{something real}) \pm \sqrt{(\text{something real})}/(\text{something positive}).$$

We therefore have instability iff the discriminant is positive. This happens when 4, A

$$-4k^2U^2(\rho_1 - \rho_2)^2 > 4(\rho_1 + \rho_2)\left[(\rho_2 - \rho_1)gk\tanh kh - k^2U^2(\rho_1 + \rho_2)\right].$$ 3, B

Or 3, D

$$4k^2U^2[(\rho_1 + \rho_2)^2 - (\rho_1 - \rho_2)^2] > 4(\rho_2^2 - \rho_1^2)gk\tanh kh$$

Or

$$4\rho_1\rho_2U^2 > (\rho_2^2 - \rho_1^2)gh\frac{\tanh kh}{kh}. \qquad (*)$$

(c) We calculate the maximum of the RHS as kh varies. Consider $f(x) = \tanh x - x$. $f'(x) = \mathrm{sech}^2 x - 1 < 0$ for $x > 0$. We have $f(0) = 0$ and so $f(x) < 0$ for $x > 0$ by integration. We conclude $\tanh x/x < 1$ for $x > 0$. It follows that the maximum of the RHS of our stability equation occurs at $k = 0$. Thus all modes are unstable for

$$U^2 > \frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2} \frac{gh}{4}.$$

We know $\rho_2 > \rho_1$, and the RHS of $(*)$ is positive. So the configuration is stable if $U = 0$. If $U \neq 0$, then as $k \to \infty$ $\tanh kh \to 1$ and the RHS $\to 0$. It follows that short enough waves (kh large) are always unstable. We need surface tension or possibly viscosity to stabilise these. (We know from lectures that a combination of surface tension and gravity can stabilise all modes for the infinite depth problem.) Although in practice our container has a finite width, and we might regard perturbations with a large horizontal wavelength as unphysical, the long wavelengths are not the unstable ones and so this wouldn't affect the stability greatly. $\boxed{3, \mathrm{A}}$ $\boxed{3, \mathrm{D}}$

4. **Seen a layer falling down an inclined plane** seen ⇓

(a) Lubrication theory requires $h \ll a$, and the inertia terms to be negligible compared to the viscous terms on the short length scale, $\rho U/a \ll \mu/h^2$. Here U is a typical velocity in the ϕ-direction, so $U = a\omega$. Scaling $v \sim a\omega$, $u \sim h\omega$, $R \sim a$ but $\partial/\partial R \sim 1/h$ the equations become

$$u_R + \frac{1}{a}v_\phi = 0, \qquad p_R = -\rho g \sin\phi, \qquad \frac{1}{a}p_\phi = -\rho g\cos\phi + \mu v_{RR}.$$

The boundary conditions are

$$u = 0,\ v = a\omega \qquad \text{on } R = a, \qquad p = p_0, \frac{\partial v}{\partial R} = 0 \qquad \text{on } R = a + h.$$

2, A

(b) Integrating the radial equation gives the pressure 2, B

$$p = p_0 - \rho g(R - a - h)\sin\phi.$$

2, C

Substituting this into the ϕ-equation, we see that p_ϕ is an $O(h/a)$ smaller than $\rho g\cos\phi$ and therefore sim. seen ⇓

$$v_{RR} = \frac{\rho g\cos\phi}{\mu} \qquad \Longrightarrow \qquad v = \frac{\rho g\cos\phi}{2\mu}[(R-a)^2 - 2h(R-a)] + a\omega$$

where we have imposed $v_R = 0$ on $R = a + h$ and $v = a\omega$ on $R = a$. So the flux through the layer is

$$Q \equiv \int_a^{a+h} v\, dR = a\omega h - \frac{\rho g\cos\phi}{3\mu}h^3.$$

Now integrating the incompressibility condition,

$$0 = \int_a^{a+h} a u_R\, dR + \int_a^{a+h} v_\phi\, dR = au|_{R=a+h} + \frac{\partial Q}{\partial\phi} - h_\phi v|_{R=a+h},$$

using the boundary condition $u = 0$ on $R = a$. The kinematic condition on $R - a - h = 0$ gives

$$0 = \frac{D}{Dt}(R - a - h) = u - h_t - \frac{v}{a}h_\phi \qquad \text{on } R = a + h.$$

Combining the last two equations,

$$a\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial\phi} = 0.$$

2, A

2, B

(c) In a steady state we have constant and $h(\phi)$ satisfies

1, C

$$\cos\phi = \frac{3\mu}{\rho g}\left(\frac{a\omega}{h^2} - \frac{Q}{h^3}\right)$$

1, D

The LHS takes all values in $[-1, 1]$. As h varies the RHS can take all negative values (as $h \to 0$) but it has a maximum when $Q = \frac{2}{3}a\omega h$. At this value of h, we have

$$RHS_{max} = \frac{3\mu}{\rho g}\frac{4}{27}\frac{a^3\omega^3}{Q^2}$$

If this value is > 1 then there exists a value of h for all ϕ, but otherwise there is no solution at $\phi = 0$. So we require

$$a\omega < \left(\frac{9Q^2\rho g}{4\mu}\right)^{1/3}.$$

[Note that Q is not really independent of ω – we would expect it to scale linearly. So this does indicate an upper limit to ω.]

If this condition is violated, then we have no steady, two-dimensional, thin layer solution. We might guess that the solution goes unstady, or that the assumption $h_\phi \ll a$ breaks down, or that a z-dependent instability sets in. Any plausible argument will be accepted, but the idea that centrifugal force throws the fluid off the rod will not earn full credit, as inertia can still be neglected for very viscous fluid. $\boxed{1, \mathrm{A}}$

$\boxed{2, \mathrm{B}}$

$\boxed{1, \mathrm{C}}$

$\boxed{4, \mathrm{D}}$

5. Essay question. Anything pertinent will earn credit, unless copied verbatim from notes. I would expect mention of the drag at low Re not varying too much with shape or orientation, and increasing with body size. Perhaps mention resistive force theory. At high Re, the importance of minimal cross-section in the direction of travel. Streamlined vs bluff bodies; boundary layer separation. Lift generation. etc $\boxed{20, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Fluid Dynamics 2_MATH60002 MATH97007 MATH70002	1	Some general comments on the exam: Each question was similar to a problem solved in the notes, but with key differences, ensuring that the notes couldn't simply be copied. Overall the exam was found challenging - perhaps people found it harder than I expected to move back and forth between the notes and the questions in the available time. In question 1, nobody wrote down all the boundary conditions correctly, so not surprisingly nobody finished the question. I was disappointed by the last two parts - the Archimidean upthrust on a sphere we met in lectures. We also deduced what the drag on the bubble from the must be by looking at one coefficient in the external flow, and the given formulae in the question should have made it very easy. I think people gave up on this question too soon, forgetting that the easy marks can be at the end.
Fluid Dynamics 2_MATH60002 MATH97007 MATH70002	2	The checker thought this question was too easy, but people did not seem to agree! Again, the arguments are very similar for the 2-D jet considered in lectures. In each case the momentum flux is constant and the pressure gradient UU'=0. People found integrating the ODE difficult, even though they had access to online software if they wished. Very few realised that w~ 1/R for large R means that there is a constant inward mass flux 2pi R w for the axisymmetric problem, so just as for the 2-D jet, fluid is entrained from infinity.
Fluid Dynamics 2_MATH60002 MATH97007 MATH70002	3	This question was exactly as considered in notes, except in 0<|y|<h rather than all y. Everything in the notes applies except that d(phi)/dy=0 on y=h rather than as y-> infinity. So phi ~ cosh k(y-h) rather than exp(-ky). Following through the algebra the same condition is obtained with g replaced by g tanh kh. People seemed to find it hard to identify what was the same and what required modification. The last part requires finding the maximum and minimu of tanh x/x, which decreases monotonically from 1 to 0.

Fluid Dynamics 2_MATH60002 MATH97007 MATH70002	4	The first part is the same as the thin layer sliding down the plane. Again there was very poor accuracy on the boundary conditions, with people confusing the solid boundary at r=a with the free surface at r=a+h. Very few people solved the lubrication equations to relate the mass flux to gravity. The final part was again relatively easy, but infrequently attempted. The question instructed to solve for cos(phi) and all that was required was to realise that this takes all values in [-1,1]. Maybe people were short of time by now.
Fluid Dynamics 2_MATH60002 MATH97007 MATH70002	5	The essay question was answered well, but not excessively so. My impression was that the BSc candidates performed as well as the MSc/Msci candidates on Q1-4, so maybe this is not surprising.