Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3/4/5 S4

Applied Probability

Date: Thursday, 10th May 2012 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Suppose $\{N_t\}_{t\geq 0}$ is a counting process. Give a formal definition, if it is to be a homogeneous Poisson process of rate $\lambda > 0$.

(b) Let $\{N_t\}_{t\geq 0}$ denote a homogeneous Poisson process of rate $\lambda > 0$. Let X_1 denote the corresponding first inter–arrival time. Show that X_1 is exponentially distributed with parameter λ.

(c) Let $\{N_t\}_{t\geq 0}$ be a homogeneous Poisson process of rate $\lambda > 0$. Let $0 < t_1 < t_2 < t_3$ and $x_1 \leq x_2 \leq x_3$ for $x_1, x_2, x_3 \in \{0, 1, 2, \dots\}$. Show that

$$\mathbb{P}(N_{t_1} = x_1, N_{t_2} = x_2, N_{t_3} = x_3) = \exp(-\lambda t_3)\lambda^{x_3}\frac{t_1^{x_1}(t_2 - t_1)^{x_2 - x_1}(t_3 - t_2)^{x_3 - x_2}}{x_1!(x_2 - x_1)!(x_3 - x_2)!}.$$

(d) You go on a summer holiday. Your airline has two check–in desks at the airport. The service times are independent and exponentially distributed with parameter $\lambda > 0$. When the check–in starts at 10am, you get to the check–in area with two other tourists. You are generous and let the two other tourists proceed to be served. You will then be served by the next available check–in desk. What is the probability that, of the three tourists, you will be the last to leave the check–in area?
Hint: There is a clever (and quick!) solution to the problem. Alternatively, express the event in question in terms of three independent random variables which are exponentially distributed with parameter λ.

2. (a) Suppose $\{N_t\}_{t\geq 0}$ is a counting process. Give a formal definition, if it is to be a non–homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$.

For (b)-(d) assume that $N = \{N_t\}_{t\geq 0}$ is a non–homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$.

(b) Let $p_n(t) = \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \dots\}$. Show that the forward equations are given by

$$\begin{aligned}\frac{dp_0(t)}{dt} &= -\lambda(t)p_0(t),\\ \frac{dp_n(t)}{dt} &= -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t), \qquad \text{for } n \geq 1.\end{aligned}$$

(c) Compute $p_0(t)$ and $p_1(t)$.
Hint: Recall that a one–dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t)f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g and initial condition $f(0) = C$ has solution

$$f(t) = \frac{\int_0^t g(u)M(u)du + C}{M(t)},$$

where M is the integrating factor

$$M(t) = \exp\left\{\int_0^t \alpha(u)du\right\}.$$

(d) Let X_1, X_2 denote the first two inter–arrival times of N, i.e. X_1 is the time from 0 to the first event, and X_2 is the time from the first event to the second event. Define $m(t) := \int_0^t \lambda(s)ds$.

(i) Derive the probability density function of X_1;

(ii) Derive the conditional probability density function of $X_2|X_1$;

(iii) Hence, find the marginal probability density function of X_2, leaving your answer as an integral.

3. (a) Let $\{X_n\}_{n\in\{0,1,2,\dots\}}$ denote a discrete–time stochastic process taking values in a state space $E \subseteq \mathbb{Z}$. Under which condition is $\{X_n\}_{n\in\{0,1,2,\dots\}}$ a Markov chain on E?

(b) Let X_n be the minimum observation obtained in the first of n rolls of a fair die (for $n \in \{1,2,3,\dots\}$). Show that $\{X_n\}_{n\in\{1,2,3,\dots\}}$ is a Markov chain, and give the transition probabilities.

(c) Consider a Markov chain with state space $E = \{1,2,3,4,5\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

(i) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(ii) Is the Markov chain irreducible?

(iii) Let $\boldsymbol{\pi}$ denote a 5–dimensional row vector. Which conditions have to be satisfied in order for $\boldsymbol{\pi}$ to be a stationary distribution of the Markov chain?

(iv) Find a stationary distribution for the Markov chain.

(v) Decide whether or not the stationary distribution is unique and justify your answer.

4. (a) Let $\{X_t\}_{t\geq 0}$ denote a continuous–time stochastic process taking values in a state space $E \subseteq \mathbb{Z}$. Under which condition is $\{X_t\}_{t\geq 0}$ a Markov chain on E?

(b) Let $X = \{X_t\}_{t\geq 0}$ denote a continuous–time Markov chain on the state space $E = \{0, 1, \ldots\}$. Suppose X is a birth–death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$ satisfying

$$\lambda_i, \geq 0 \quad \mu_i \geq 0, \text{ for all } i \in E, \quad \mu_0 = 0.$$

State the infinitesimal transition probabilities

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n)$$

for $n, m \in \{0, 1, 2, \ldots\}$ and $t \geq 0, \delta > 0$.

(c) What is the generator $\mathbf{G}$ of the birth–death process defined in (b)?

(d) Let $N = \{N_t\}_{t\geq 0}$ denote a pure birth process with $N_0 = 0$ and

$$\mathbb{P}(\text{one event happens in } (t, t+\delta] | N_t \text{ is odd}) = \alpha\delta + o(\delta),$$
$$\mathbb{P}(\text{one event happens in } (t, t+\delta] | N_t \text{ is even}) = \beta\delta + o(\delta),$$

for $t \geq 0,\ \delta, \alpha, \beta > 0$.

(i) Derive the forward equations for $p_n(t) := \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \ldots\}$.

(ii) Find the following probabilities:

$$P_e(t) := \mathbb{P}(N_t \text{ is even}), \qquad P_o(t) := \mathbb{P}(N_t \text{ is odd}).$$

Hint 1: Derive the following differential equations

$$P_e'(t) = \alpha P_o(t) - \beta P_e(t), \qquad P_o'(t) = -\alpha P_o(t) + \beta P_e(t),$$

and solve them by using the identity $P_e(t) + P_o(t) = 1$.

Hint 2: Recall that a one–dimensional ordinary differential equation

$$\frac{df(t)}{dt} + \alpha(t) f(t) = g(t), \qquad t \geq 0$$

with continuous functions α, g *and initial condition* $f(0) = C$ *has solution*

$$f(t) = \frac{\int_0^t g(u)M(u)du + C}{M(t)},$$

where M *is the integrating factor*

$$M(t) = \exp\left\{\int_0^t \alpha(u)du\right\}.$$