Exercise 4.1. Suppose A is a symmetric $(n \times n)$ matrix. Consider the function:

$$\begin{array}{rcccl} f & : & \mathbb{R}^n & \to & \mathbb{R} \\ & & x & \mapsto & xAx^t. \end{array}$$

(a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with:

$$Df(p) = 2pA$$

(b) Find:

$$\operatorname{Hess} f(p).$$

Exercise 4.2. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : (x, y, z) = xy^2 + x^2 + xze^y.$$

(a) Compute the first and second partial derivatives. Observe the properties of the second partial derivative.

(b) Write the terms of the Taylor expansion of f at zero up to and including the second-order terms.

(c) Without computation, write the same Taylor expansion up to and including the fourth-order terms.

Exercise 4.3 (*). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{xy^3 - x^3y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

(a) Show that:

$$D_1 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{y^3 - 3x^2y}{x^2 + y^2} - \dfrac{2x\left(xy^3 - x^3y\right)}{\left(x^2 + y^2\right)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

and

$$D_2 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{3y^2x - x^3}{x^2 + y^2} - \dfrac{2y\left(xy^3 - x^3y\right)}{\left(x^2 + y^2\right)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0), \end{cases}$$

and show that these functions are both continuous at $(0, 0)$.

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

(b) Show that:

$$\lim_{t\to 0} \frac{1}{t}\left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t}\left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

(c) Conclude that both $D_2D_1f(0)$ and $D_1D_2f(0)$ exist, but that:

$$D_2D_1f(0) \neq D_1D_2f(0)$$

Exercise 4.4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = e^x \sin(y)$.

a) Compute the degree 1 and degree 2 Taylor polynomial of f near the point $(x_0, y_0) = (0, \pi/2)$ and use those to approximate the value of f at $(x_1, y_1) = (0, \pi/2 + 1/4)$. Compare your results with the values you obtain from a calculator.

b) How precise is the degree 1 approximation in the closed ball of radius 1/4 around (x_0, y_0). Find a rigorous upper bound for the approximation error.

Exercise 4.5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - xy \\ x^2 \end{pmatrix}$$

Determine the set of points in $\mathbb{R}^2$ such that f is invertible near those points, and compute the derivative of the inverse map.

Exercise 4.6. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

[Hint: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto x^3$.]

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

[Hint: Assume that f^{-1} is differentiable at $f(0)$ and apply the chain rule to $\iota = f^{-1} \circ f = f \circ f^{-1}$ to derive a contradiction.]

Exercise 4.7. The non-linear system of equations

$$\begin{aligned} e^{xy} \sin(x^2 - y^2 + x) &= 0 \\ e^{x^2+y} \cos(x^2 + y^2) &= 1 \end{aligned}$$

admits the solution $(x, y) = (0, 0)$. Prove that there exists $\varepsilon > 0$ such that for all (ξ, η) with $\xi^2 + \eta^2 < \varepsilon^2$, the perturbed system of equations

$$\begin{aligned} e^{xy} \sin(x^2 - y^2 + x) &= \xi \\ e^{x^2+y} \cos(x^2 + y^2) &= 1 + \eta \end{aligned}$$

has a solution $(x(\xi, \eta), y(\xi, \eta))$ which depends continuously on (ξ, η).

Unseen Exercise. Find the minimum of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f(x, y, z) = x^4(y^2 + x^2) + z^2 - 4z$$

Unseen Exercise. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Consider the function $f : \Omega \to \mathbb{R}^2$ given by:

$$f : (x, y) = (x \sin y, x \cos y).$$

(a) Show that f is differentiable at all $p = (\xi, \eta) \in \Omega$, with:

$$Df(p) = \begin{pmatrix} \sin \eta & \xi \cos \eta \\ \cos \eta & -\xi \sin \eta \end{pmatrix}.$$

(b) Show that $Df(p)$ is invertible for all $p \in \Omega$.

(c) Show that $f : \Omega \to \mathbb{R}^2$ is not injective. Deduce that the restriction to open sets U, V in the inverse function theorem is necessary.