10.1.2 Distributions derived from the Multivariate Normal

In this section we define several distributions derived from the multivariate normal distribution. They will appear as distributions of pivotal quantities and test statistics. You should have met the central χ^2- and t-distribution previously.

Definition 24
Let $\mathbf{Z} \sim N(\boldsymbol{\mu}, I_n)$, where $\boldsymbol{\mu} \in \mathbb{R}^n$.
$\mathbf{U} = \mathbf{Z}^T\mathbf{Z} = \sum_{i=1}^n Z_i^2$ is said to have a *non-central* χ^2*-distribution* with n degrees of freedom (d.f.) and non-centrality parameter (n.c.p.)

$$\delta = \sqrt{\boldsymbol{\mu}^T\boldsymbol{\mu}}.$$

Notation: $\mathbf{U} \sim \chi_n^2(\delta)$, $\chi_n^2 = \chi_n^2(0)$.

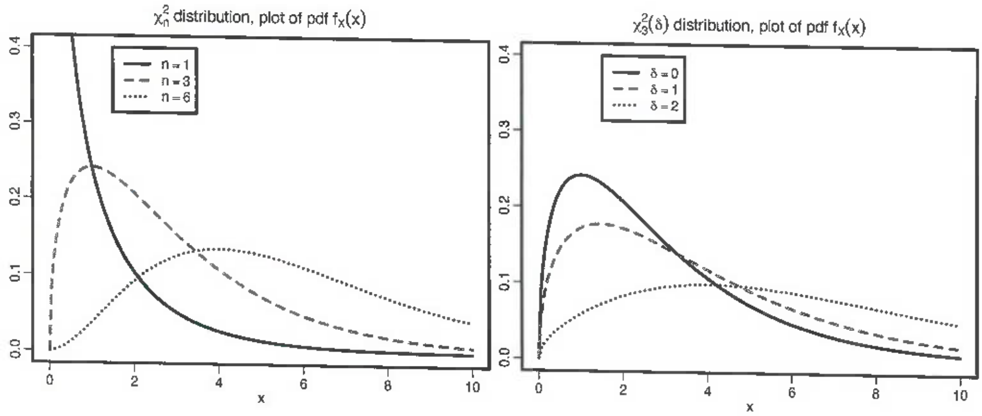
Remark In order for this to be a proper definition we need to show that the distr. of U depends on $\boldsymbol{\mu}$ only through $\boldsymbol{\mu}^T\boldsymbol{\mu}$. (One of the questions on the problem sheet asks you to prove this). One approach is to show that the moment generating function of U equals

$$M_U(t) = \frac{1}{(1-2t)^{n/2}} \exp\left(\frac{t\delta^2}{1-2t}\right)$$

The following lemma contains some properties of the χ^2-distribution.

Lemma 15
Let $U \sim \chi_n^2(\delta)$. Then $\mathrm{E}(U) = n + \delta^2$ and $\mathrm{Var}(U) = 2n + 4\delta^2$.
If $U_i \sim \chi_{n_i}^2(\delta_i)$, $i = 1, \dots, k$ and U_i's are indep. then $\sum_{i=1}^k U_i \sim \chi_{\sum n_i}^2\left(\sqrt{\sum \delta_i^2}\right)$.

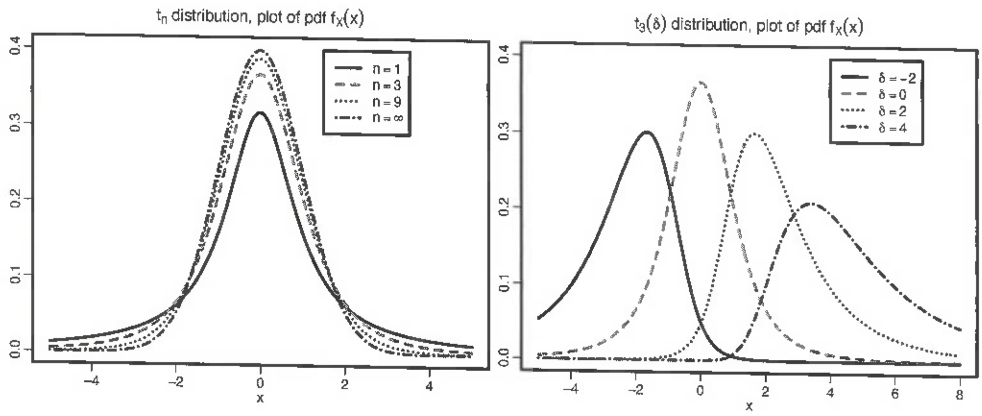
Proof: $\to$ Problem Sheet.



Definition 25
Let X and U be independent random variables with $X \sim N(\delta, 1)$ and $U \sim \chi^2_n$. The distribution of

$$Y = \frac{X}{\sqrt{U/n}}$$

is called *non-central t-distribution* with n degrees of freedom and non-centrality parameter δ.
Notation: $Y \sim t_n(\delta)$, $t_n = t_n(0)$.



Remark Convergence to the normal distribution: Suppose $Y_n \sim t_n$ for all $n \in \mathbb{N}$. Then

$$Y_n \xrightarrow{d} N(0,1) \quad (n \to \infty)$$

To see this: Let $X \sim N(0,1)$ and $U_n = \sum_{i=1}^n Z_i^2$ for $Z_1, Z_2, \dots \sim N(0,1)$ indep. Then, by the weak law of large numbers: $U_n/n \xrightarrow{\mathrm{P}} \mathrm{E}(Z_1^2) = 1$. As $x \mapsto \sqrt{x}$ is continuous,

$\frac{U_n}{n} = \frac{1}{n}\sum Z_i^2 \to E[Z_1^2] = 1$ because $Z \sim N(0,1)$

BY CONTINUOUS MAPPING THEOREM

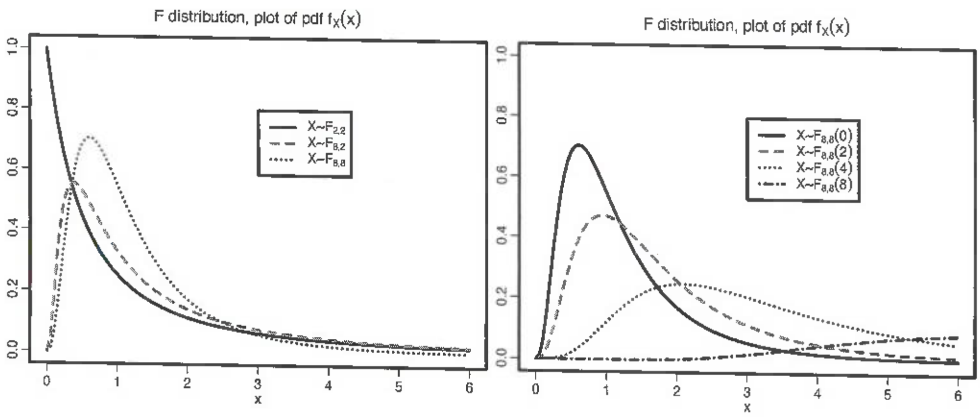
this implies $\sqrt{U_n/n} \xrightarrow{P} \sqrt{1} = 1$. Thus, by Slutsky's Lemma:

$$Y_n = \frac{X}{\sqrt{U_n/n}} \xrightarrow{d} \frac{X}{1} = X \sim N(0,1).$$

Definition 26
If $W_1 \sim \chi^2_{n_1}(\delta)$, $W_2 \sim \chi^2_{n_2}$ independently then

$$F = \frac{W_1/n_1}{W_2/n_2}$$

is said to have a *non-central* F distribution with (n_1, n_2) d.f. and n.c.p.$=\delta$.
Notation: $F \sim F_{n_1,n_2}(\delta)$, $F_{n_1,n_2} = F_{n_1,n_2}(0)$.



Remark If the n.c.p. $\delta = 0$ then the above are called *central* χ^2, *t*, *F distribution*.

10.1.3 Some Independence Results

In order to show that a random variable follows a t-distribution via the definition we need to show independence between a normal and a χ^2-distributed random variable. Similarly, to show that a random variable is F-distributed, we need to show independence of two χ^2 distributed random variables. This section provides results that help in doing this.

Lemma 16
Let $A \in \mathbb{R}^{n\times n}$ be a pos. semidefinite symmetric matrix with rank r. Then there exists $L \in \mathbb{R}^{n\times r}$ such that $\operatorname{rank} L = r$, $A = LL^T$ and $L^T L =$ diag(nonzero eigenvalues of A).

Proof (Lemma 8) $\Longrightarrow$ $\exists$ an orthogonal matrix P st

$$P^T AP = D = \text{diag(eigenvalues of } A)$$

Precisely r elements of D are positive and the others are zero. Hence, $A = PDP^T = PD^{\frac{1}{2}}D^{\frac{1}{2}}P^T = PD^{\frac{1}{2}}(PD^{\frac{1}{2}})^T$ Let L consist of the nonzero columns of $PD^{\frac{1}{2}}$. Then $A = LL^T$.

Because P is orthogonal (i.e. $P^T P = I$) we get $L^T L =$ diag(nonzero eigenvalues of A).

$$L^T L \simeq (PD^{\frac{1}{2}})^T PD^{\frac{1}{2}} = D^{\frac{1}{2}T}\underbrace{P^T P}_{I} D^{\frac{1}{2}} = D^{\frac{1}{2}T} D^{\frac{1}{2}} = D$$

Lemma 17
Let $\mathbf{X} \sim N(\boldsymbol{\mu}, I)$, $A \in \mathbb{R}^{n,n}$ pos. semidefinite symmetric and let B be a matrix such that $BA = 0$.
Then $\mathbf{X}^T A\mathbf{X}$ and $B\mathbf{X}$ are independent.

Proof Let $r = \operatorname{rank}(A)$. By Lemma 16 $\exists$ $L \in \mathbb{R}^{n\times r}$ such that $\operatorname{rank} L = r$ and $A = LL^T$.

MULTIPLY AND DIVIDE BY $L^T L$

$$\operatorname{cov}(B\mathbf{X}, L^T\mathbf{X}) = B\underbrace{\operatorname{cov}(\mathbf{X})}_{I}L = BL = BLL^T L(L^T L)^{-1} = BAL(L^T L)^{-1} = 0$$

$A = LL^T$

Thus $B\mathbf{X}$ and $L^T\mathbf{X}$ are independent (because they are jointly normally distributed).

Hence, $B\mathbf{X}$ and $\mathbf{X}^T LL^T\mathbf{X} = \mathbf{X}^T A\mathbf{X}$ are independent. $\square$

IF X AND Y ARE INDEPENDENT, THEN $f(X)$ AND $g(Y)$ ARE INDEPENDENT WHERE F AND G ARE BOREL ~~ME~~ MEASURABLE FUNCTIONS

Lemma 18
If $\mathbf{Z} \sim N(\boldsymbol{\mu}, I_n)$ and A is an $n \times n$ projection matrix of rank r, then

$$\mathbf{Z}^T A\mathbf{Z} \sim \chi_r^2(\delta) \quad \text{with } \delta^2 = \boldsymbol{\mu}^T A\boldsymbol{\mu}$$

Proof All nonzero eigenvalues of A are equal to 1.

By Lemma $\boxed{16}$ $\exists\ L \in \mathbb{R}^{n\times r}$ such that $A = LL^T$ and $L^TL = I_r$. Let $\mathbf{V} = L^T\mathbf{Z}$. Then $\mathbf{V} \sim N(L^T\boldsymbol{\mu}, \underbrace{I}_{=L^TL})$ and

$$\mathbf{Z}^TA\mathbf{Z} = \mathbf{Z}^TLL^T\mathbf{Z} = \mathbf{V}^T\mathbf{V} \sim \chi^2_r(\delta),$$

where $\delta^2 = \left(L^T\boldsymbol{\mu}\right)^T L^T\boldsymbol{\mu} = \boldsymbol{\mu}^T \underbrace{LL^T}_{=A}\boldsymbol{\mu} = \boldsymbol{\mu}^TA\boldsymbol{\mu}$.

> **Lemma 19**
> If $\mathbf{Z} \sim N(\boldsymbol{\mu}, I_n)$ and $A_1, A_2 \in \mathbb{R}^{n\times n}$ are projection matrices and $A_1A_2 = 0$ then $\mathbf{Z}^TA_1\mathbf{Z}$ and $\mathbf{Z}^TA_2\mathbf{Z}$ are independent.

Proof $\mathrm{cov}(A_1\mathbf{Z}, A_2\mathbf{Z}) = A_1 \underbrace{\mathrm{cov}(\mathbf{Z},\mathbf{Z})}_{=I} A_2^T = A_1A_2^T = 0$.

Because $A_1\mathbf{Z}$ and $A_2\mathbf{Z}$ are jointly normally distributed this shows that they are independent.

As $\mathbf{Z}^TA_i\mathbf{Z} = (A_i\mathbf{Z})^T(A_i\mathbf{Z})$ for $i = 1, 2$ (symm+ idempotent) this implies that $\mathbf{Z}^TA_1\mathbf{Z}$ and $\mathbf{Z}^TA_2\mathbf{Z}$ are independent.

This result extends to $\mathbf{Z}^TA_1\mathbf{Z}, \dots, \mathbf{Z}^TA_k\mathbf{Z}$, where $A_iA_j = 0$ $(i \neq j)$.

> **Lemma 20**
> If $A_1, \dots, A_k$ are symmetric $n\times n$ matrices such that $\sum A_i = I_n$ and if rank $A_i = r_i$ then the following are equivalent:
>
> 1. $\sum r_i = n$
> 2. $A_iA_j = 0$ for all $i \neq j$
> 3. A_i is idempotent for all $i = 1, \dots, k$.

Proof $\boxed{2}\rightarrow\boxed{3}$ $\forall j$: $A_1 + \cdots + A_k = I_n \implies A_1A_j + \cdots + A_kA_j = A_j \implies A_j^2 = A_j$.