Imperial College London

**Department of Mathematics**

## M3PA50 Introduction to Riemann Surfaces and Conformal Dynamics

| Question | Examiner's Comments |
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| Q 1 | Some students have forgotten to verify that the map was not the identity when showing that it was neither parabolic not hyperbolic |
| Q 2 | Some students had difficulties in showing that a particular set is a subset of the Fatou set |
| Q 3 | Was generally good |
| Q 4 | No student succeeded to get the full mark for Q 4.3 |

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| Q 5 | No student had a positive mark for Q 5.2 |

Imperial College
London

M4/5PA50

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**Introduction to Riemann Surfaces and Conformal**

Date: Tuesday, 29 May 2018

Time: 10:00 AM - 12:30 PM

Time Allowed: 2.5 hours

**This paper has 5 questions.**

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. Let $\gamma_0 \in \mathrm{Mob}(\mathbb{H})$ be defined as $\gamma_0(z) = -1/z$ and let $\beta(z) = 2z$.

   (1) State what is meant by an *elliptic Möbius* transformation.

   (2) Show that $\gamma_0$ is *elliptic*.

   (3) Show that if $\gamma \in \mathrm{Mob}(\mathbb{H})$ is elliptic and $\alpha \in \mathrm{Mob}(\mathbb{H})$ then $\alpha \circ \gamma \circ \alpha^{-1}$ is also elliptic.

   (4) Define for $n \geq 0$,
   $$\gamma_{n+1} := \beta \circ \gamma_n \circ \beta^{-1}.$$
   Show that $\gamma_n$ is elliptic for all $n \geq 1$.

2. We will study the Newton method applied to the polynomial
$$p(z) = z^2 + 1.$$
Let
$$N(z) := z - \frac{p(z)}{p'(z)}$$
be defined on $\hat{\mathbb{C}}$. We consider the dynamics of $N$ on $\hat{\mathbb{C}}$. Let $-\mathbb{H}$ denote the set of complex numbers with negative imaginary part. You may use that on the Riemann sphere no sequence of holomorphic functions can diverge uniformly locally.

   (1) Show that $-\mathbb{H}$ is biholomorphic to $\mathbb{H}$. Conclude that it is a hyperbolic Riemann surface.

   (2) Show that $N(\mathbb{H}) \subset \mathbb{H}$ and $N(-\mathbb{H}) \subset -\mathbb{H}$.

   (3) Deduce that $\mathbb{H}$ and $-\mathbb{H}$ are connected open subsets of the Fatou set $F(N)$ of $N$.

   (4) Show that $i$ and $-i$ are superattracting fixed points for $N$.

3. Let
$$\mathbb{D}^* = \mathbb{D} \setminus \{0\} = \{z : 0 < |z| < 1\}$$
be the punctured disk. Recall that the hyperbolic conformal metric on $\mathbb{D}^*$ is equal to
$$\rho(z) = \frac{1}{|z|\,|\log|z||}$$
for $z \in \mathbb{D}^*$.

   (1) Let $\pi : \mathbb{H} \to \mathbb{D}^*$ be the universal covering map whose expression is
   $$\pi(z) = e^{iz}.$$
   We define the *pull back* of the conformal metric $\rho$ by $\pi$ as the conformal metric
   $$\rho^*(z) = \rho(\pi(z))\,|\pi'(z)|,$$
   on $\mathbb{H}$.

   Show that $\rho^*(z) = \frac{1}{\operatorname{Im} z}$, for all $z \in \mathbb{H}$.

(2) Using the universal covering $\pi$ defined above, show that the radial lines of $\mathbb{D}^*$ are geodesics for the hyperbolic metric on $\mathbb{D}^*$.

(3) Let $d_{\mathbb{D}^*}$ denote the hyperbolic distance on $\mathbb{D}^*$. Let $\theta, r_0, r_1$ be such that $0 \leq \theta < 2\pi$ and $0 < r_0 \leq r_1 < 1$. Show that

$$d_{\mathbb{D}^*}\left(r_0 e^{i\theta}, r_1 e^{i\theta}\right) = \log \frac{\log r_0}{\log r_1}.$$

Remark: you can use the fact that the derivative of $-\log(-\log r)$ for $0 < r < 1$ is $\frac{-1}{r \log r}$.

4. Let $B = \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 1\}$ and consider the map $\beta : B \to B$ defined as $\beta(z) = \overline{z} + 1$. Define $M = B/\beta$, that is, the quotient of $B$ by the action of $\beta$ and $\pi : B \to M$ the associated projection.

Consider the sets

$$U = \{x + iy \in B : 0 < x < 1, -1 < y < 1\}$$

and

$$V = \{x + iy \in B : 0 \leq x < 1/4, -1 < y < 1\} \cup \{x + iy \in B : 3/4 < x \leq 1, -1 < y < 1\}.$$

(1) Prove that $U$ is an open fundamental domain for the action of $\beta$ on $M$.

(2) Let $\psi_V : V \to \mathbb{C}$ be defined as

$$\psi_V(z) = \begin{cases} x + iy, & \text{if } 0 \leq x < 1/4, \\ x - iy - 1, & \text{if } 3/4 < x \leq 1, \end{cases}$$

where $x$ and $y$ denote respectively the real and imaginary parts of $z$. We would like to define a map $\varphi_V : V/\beta \to \mathbb{C}$ by $\varphi_V(\zeta) = \psi_V(z)$ for $z \in \zeta \cap V$. Show that the map $\varphi_V$ is indeed well defined and continuous.

Remark: you can use the fact that if $W$ is an open subset of $B$ on which the projection map $\pi : B \to M$ is injective then $\pi : W \to \pi(W)$ is a homeomorphism.

(3) Define $\varphi_U : U/\beta \to \mathbb{C}$ as $\varphi_U(\zeta) = z$ where $z$ is the unique element of $U \cap \zeta$. Show that $((U, \varphi_U), (V, \varphi_V))$ is not a complex atlas for $M$.

5. A simple version of the Implicit Function Theorem for holomorphic functions can be stated as follows.

**Theorem** *Let $U_1, U_2$ be non empty open subsets of $\mathbb{C}$ and let $F : U_1 \times U_2 \to \mathbb{C}$ be a continuous map which is holomorphic in each variable (that is for all $z_1 \in U_1$, $z_2 \mapsto F(z_1, z_2)$ is holomorphic on $U_2$ and for all $z_2 \in U_2$, $z_1 \mapsto F(z_1, z_2)$ is holomorphic on $U_1$).*

*For $(z_1, z_2) \in U_1 \times U_2$, let $\frac{\partial F}{\partial z_2}(z_1, z_2)$ denote the derivative of the holomorphic map $z \mapsto F(z_1, z)$ evaluated at $z_2$.*

*Assume that there exists $(a_1, a_2) \in U_1 \times U_2$ such that $F(a_1, a_2) = 0$ and $\frac{\partial F}{\partial z_2}(a_1, a_2) \neq 0$.*

*Then there exists an open neighbourhood $W_1$ of $a_1$ in $U_1$, an open neighbourhood $W_2$ of $a_2$ in $U_2$ and a holomorphic function $\varphi : W_1 \to U_2$ such that $\varphi(a_1) = a_2$ and for all $(z_1, z_2) \in W_1 \times W_2$, $F(z_1, z_2) = 0$ if and only if $z_2 = \varphi(z_1)$.*

Consider the map $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined as $f(c, z) = z^2 + c$. For each $c \in \mathbb{C}$, we consider $f_c(z) := f(c, z)$ as a function of the variable $z \in \mathbb{C}$ that we wish to iterate.

(1) Assume that there exists $c_0 \in U$ and $z_0 \in \mathbb{C}$ such that $z_0$ is a periodic point of $f_{c_0}$ of minimal period $p$ and multiplier $\lambda = \left(f_{c_0}^p\right)'(z_0)$. Show that if $\lambda \neq 1$ then there exists a neighbourhood $W$ of $c_0$ and a neighbourhood $V$ of $z_0$ such that for all $c \in W$ the function $f_c$ has a periodic point in $V$ which depends holomorphically on $c$ and whose minimal period divides $p$.

(2) Show that, by choosing $W$ small enough, the minimal period of the periodic point of $f_c$ found in part (1) is exactly $p$.

PA 50 Solutions

**1.1.**(3 marks, seen) An elliptic Möbius transformation is a Möbius transformation that has a single fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$.

**1.2.**(3 marks, seen) The trace of $\gamma_0$ is 0 so it is elliptic.

**1.3.**(11 marks, seen) Let $\zeta$ be the unique fixed point of $\gamma$ in $\mathbb{H}$, let $\gamma' = \alpha \circ \gamma \circ \alpha^{-1}$ and let $\zeta' = \alpha(\zeta)$.
Since $\alpha \in \mathrm{Mob}(\mathbb{H})$ and $\zeta \in \mathbb{H}$ then $\zeta' = \alpha(\zeta) \in \mathbb{H}$.
We have $\gamma'(\zeta') = \alpha(\gamma(\zeta)) = \alpha(\zeta) = \zeta'$.
Hence $\zeta'$ is a fixed point of $\gamma'$. (5 marks)
Similarly if $\xi$ is a fixed point of $\gamma'$ in $\mathbb{H}$ then $\alpha^{-1}(\xi)$ is a fixed point of $\gamma$ in $\mathbb{H}$. (3 marks)
By uniqueness of the fixed point of $\gamma$ it follows that $\gamma'$ has no fixed point other than $\zeta'$ in $\mathbb{H}$.
Since $\gamma' \in \mathrm{Mob}(\mathbb{H})$ it follows that $\gamma'$ is elliptic. (3 marks)

**1.4.**(3 marks, seen) Remark that $\beta \in \mathrm{Mob}(\mathbb{H})$. (1 mark)
Proceed by induction: since $\gamma_{n+1}$ is conjugated to $\gamma_n$ in $\mathrm{Mob}(\mathbb{H})$ and $\gamma_n$ is elliptic, it follows from part 3 that $\gamma_{n+1}$ is also elliptic. (2 marks)

**2.1.**(6 marks, seen) The map $-\mathbb{H} \to \mathbb{H} : z \mapsto -z$ is a biholomorphism between $-\mathbb{H}$ and $\mathbb{H}$. (3 marks)
The surface $\mathbb{H}$ is hyperbolic hence $-\mathbb{H}$ is a hyperbolic surface. (3 marks)

**2.2.**(3 marks, seen similar) We compute that $N(z) = \frac{1}{2}(z - 1/z)$.
From a direct computation we see that $\operatorname{Im} N(z) = \frac{1}{2}\operatorname{Im}\left(z - \bar{z}/|z|^2\right) = \operatorname{Im} z\left(1 + 1/|z|^2\right)/2$.
Since $\left(1 + 1/|z|^2\right)/2 > 0$, we conclude that $\operatorname{Im} N(z) > 0$ if and only if $\operatorname{Im} z > 0$ and a similar statement holds for $< 0$.

**2.3.**(5 marks, seen similar) From the previous part it follows that for all $n \geq 1$, the restriction of $N^n$ to $\mathbb{H}$ has values in $\mathbb{H}$. (1 mark)
The Riemann surface $\mathbb{H}$ is hyperbolic. This implies that $(N^n)_n$ is normal on $\mathbb{H}$. (1 mark)
Hence $\mathbb{H} \subset F(N)$. (1 mark)
The surface $-\mathbb{H}$ is hyperbolic. It follows in a similar way as above that $(N^n)_n$ is normal on $-\mathbb{H}$ and that $-\mathbb{H} \subset F(N)$ ($F(N)$ being the Fatou set of $N$). (1 mark)
Finally $\mathbb{H}$ and $-\mathbb{H}$ are clearly connected. (1 mark)

**2.4.**(6 marks, seen) We can check directly that $N(i) = i$ and $N'(i) = 0$.
The same for $-i$.

**3.1**(5 marks, unseen) We have

$$\rho^*(z) = \rho(\pi(z))\,|\pi'(z)| \tag{1}$$
$$= \frac{|ie^{iz}|}{|e^{iz}|\,|\log|e^{iz}||}. \tag{2}$$

(2 marks)
Since $|e^{iz}| = e^{-\operatorname{Im} z}$, it follows $\rho^*(z) = \frac{1}{|-\operatorname{Im} z|}$. (2 marks)
For $z \in \mathbb{H}$, $\operatorname{Im} z > 0$, then $\rho^*(z) = \frac{1}{\operatorname{Im} z}$. (1 mark)

**3.2.**(9 marks, unseen) Let $\theta \in \mathbb{R}$. Then the image of $\{\theta + iy : y > 0\}$ by $\pi$ is the set $\left\{e^{-y}e^{i\theta} : y > 0\right\}$.
This set is the radial line from 0 at angle $\theta \mod 2\pi$ in $\mathbb{D}^*$. So $\pi$ maps vertical lines to radial lines in $\mathbb{D}^*$. (3 marks)
We know that the universal covering of a hyperbolic surface is a local isometry with respect to the hyperbolic metrics.
This implies that the image of a geodesic of $\mathbb{H}$ by $\pi$ is a geodesic of $\mathbb{D}^*$.
Indeed let $z \in \mathbb{H}$. If $z' \in \mathbb{H}$ is close enough to $z$ then $d_{\mathbb{D}^*}(\pi(z), \pi(z')) = d_{\mathbb{H}}(z, z')$.
Let $\sigma$ be a path in $\mathbb{H}$ describing the geodesic arc from $z$ to $z'$.
Since $\pi : \mathbb{H} \to \mathbb{D}^*$ is a covering map it preserves the length of paths.
Hence $\text{length}_{\mathbb{D}^*}\, \pi \circ \sigma = d_{\mathbb{D}^*}(\pi(z), \pi(z'))$
This means that $\pi \circ \sigma$ is a geodesic arc. (3 marks)
Finally vertical half lines in $\mathbb{H}$ are geodesic for the hyperbolic metric on $\mathbb{H}$. Their images by $\pi$ are the radial lines in $\mathbb{D}^*$. This means that the latter are geodesics for the hyperbolic metric on $\mathbb{D}^*$. (3 marks)

**3.3**(6 marks, seen similar) Since $z_0 = r_0 e^{i\theta}$ and $z_1 = r_1 e^{i\theta}$ are on the same radial line, we can compute the distance between them by computing the hyperbolic length of the euclidean arc $[z_0, z_1]$. (2 marks)
Let's parametrise $[z_0, z_1]$ with $\sigma(t) = te^{i\theta}$ for $t \in [r_0, r_1]$.
Using the expression of the conformal metric of $\mathbb{D}^*$ given at the beginning we can compute its length by computing the integral

$$\int_{r_0}^{r_1} \frac{\mathrm{d}t}{t\,|\log t|}. \tag{3}$$

(2 marks)

We can then use the hint to compute as follows:

$$\int_{r_0}^{r_1} \frac{\mathrm{d}\,t}{t\log t} = -\log(-\log r_1) - (-\log(-\log r_0)) = -\log\frac{\log r_0}{\log r_1}. \qquad (4)$$

(2 marks)

**4.1**(7 marks, seen similar) The set $U$ is open. (1 mark)
The orbit of $z \in B$ by the action of $\beta$ is the set of points $\{z + 2n : n \in \mathbb{Z}\} \cup \{\overline{z} + 2n + 1 : n \in \mathbb{Z}\}$.
In particular the difference of the real part of two points in the orbit is an integer (Note: it is not necessary to compute the orbit explicitely to see that). (2 marks)
But $\sup \{\operatorname{Re} z - \operatorname{Re} z' : z, z' \in U\} = 1$. It follows that any orbit intersects $U$ at most once. (2 marks)
Finally if $n$ is the largest integer $\leq \operatorname{Re} z$, $\beta^n(z) \in \overline{U}$.
Hence $U$ is an open fundamental domain. (2 marks)

**4.2.**(6 marks, seen similar) In order to show that $\varphi_V$ is well defined we only need to show that for all $y$ such that $-1 < y < 1$, $\psi_V(iy) = \psi_V(1 - iy)$.
We have $\psi_V(iy) = iy$ and $\psi_V(1 - iy) = 1 - i(-y) - 1 = \psi(iy)$. (2 marks)
The map $\varphi_V$ is clearly continuous for points of the form $\pi(x + iy)$ with $0 < x < 1/4$ or $3/4 < x < 1$. (2 marks)
From the definition of $\varphi_V$ it follows that we really only need to show that $\lim_{x \to 0^+} \psi_V(x + iy) = \lim_{x \to 1^-} \psi_V(x - iy)$ for any $y$.
This is clear from the definition of $\psi_V$ by using the same identity as for showing that $\varphi_V$ is well defined. (2 marks)

**4.3.**(7 marks, unseen) In order to have a complex atlas it is necessary that the transition maps between charts are holomorphic. (and other requirements from the definition of a complex atlas). (2 marks)
The set $W = \{z \in \mathbb{C} : 3/4 < \operatorname{Re} z < 1, -1 < \operatorname{Im} z < 1\}$ is a connected component of $\varphi_U(U \cap V)$. (1 mark)
We can compute directly and find that on $W$ we have $\varphi_V \circ \varphi_U^{-1}(z) = \overline{z} - 1$. (2 marks)
The map $z \mapsto \overline{z} - 1$ is not holomorphic. (can check the Cauchy-Riemann equation, or say this reverses the orientation or simply claim this as a prior knowledge). Hence the transition map $\varphi_V \circ \varphi_U^{-1}$ is not holomorphic. It follows that $((U, \varphi_U), (V, \varphi_V))$ cannot be a complex atlas for $M$. (2 marks)

**5.1.**(10 marks, unseen) Let $F$ be the function defined on $U \times \mathbb{C}$ by $F(c, z) = f_c^p(z) - z$.
The function $F$ is holomorphic in $c$ and $z$ because $(c, z) \mapsto f_c(z)$ is holomorphic in each variable. (2 marks)
We have $F(c_0, z_0) = 0$ by assumption. (2 marks)
We have $\frac{\partial F}{\partial z}(c_0, z_0) = \lambda - 1$. (2 marks)
Hence if $\lambda \neq 1$ it follows from the implicit function theorem that there exists $W \subset U$ a neighbourhood of $c_0$ and a holomorphic map $\varphi : W \to \mathbb{C}$ such that $\varphi(c_0) = z_0$ and for all $c \in W$, $F(c, \varphi(c)) = 0$. (2 marks)
This means that $\varphi(c)$ is a periodic point of $f_c$ of period dividing $p$. (2 marks)

**5.2.**(10 marks, unseen) Let $(z_0, z_1, \ldots, z_{p-1})$ be the periodic cycle of $f_{c_0}$ of exact period $p$ and starting at $z_0$, that is $z_j = f_{c_0}^j(z_0)$ for all $j = 0, \ldots, p-1$. Then for each $j$ there exists a neighbourhood $V_j$ of $z_j$ such that the $(V_j)_{j=0,\ldots,p-1}$ are pairwise disjoint. (3 marks)
Since the mappings $\varphi$ and $f_c^j$, $j = 0, \ldots, p-1$, are continuous we can find a neighbourhood $W'$ of $c_0$ such that $W' \subset W$ and for all $c \in W'$ and all $j = 0, \ldots, p-1$, $f_c^j(\varphi(c)) \in V_j$. (4 marks)
Since the $V_j$ are disjoint, the period of $\varphi(c)$ cannot be less than $p$. (3 marks)