Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M3S15/M4S15

Monte Carlo Methods in Financial Engineering

Date: Monday, 2nd June 2008 Time: 10 am – 12 pm

Answer all questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. In the Black Scholes model the stock price at time T is

$$S_T = s \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right)$$

where $s, r, \sigma > 0$ are the fixed usual parameters for the initial stock price, interest rate, and volatility, and (W_T) is a Brownian motion under the risk-neutral measure P.

(a) By the principle of risk neutral pricing, the no-arbitrage price $C_0 = C(s, K, \sigma, r, \tau)$ of a call option at time $t = 0$ with time to maturity $\tau = T - t = T$ and strike K, equals

$$C_0 = \exp(-Tr) E[(S_T - K)^+] .$$

Derive the Black-Scholes formula, that is an explicit function C for the call price of the given parameters (K, σ, r), the time to maturity τ and the current stock price $S_t = s$ at time t, involving the cumulative distribution function (cdf) Φ of the standard normal.

(b) Suppose you can simulate i.i.d. standard normal random variables $Z_1, Z_2, \ldots$. Describe the Monte Carlo algorithm to compute, based on a sample of size n, (i) an estimate $\widehat{X}$ for the call option's price C_0 at time $t = 0$, and (ii) an estimate $\widehat{\sigma}$ for the standard deviation of $\widehat{X}$.

(c) Using results from (b),

(i) state how one can compute an (approximate) 95%-confidence interval for the option price.

(ii) determine the order of decrease of the standard deviation of $\widehat{X}$ as a function of the sample size.

(iii) determine by what factor you would need to increase the sample size in order to halve the length of the 95%-confidence interval?

(d) Apply Itô's formula to derive the stochastic differential equation (SDE) that is satisfied by the price process S_t and by the log-price process $X_t := \log(S_t)$. Then, write down the Euler schemes for a numerical approximation of the SDE for (S_t) and for (X_t), repectively, along a discrete time grid $0 = t_0 < t_1 < \cdots < t_n = T$.

Explain which of the two schemes you would use to simulate paths of the stock price process when computing option price estimates by Monte Carlo. Give an example for an option whose Monte-Carlo pricing would require simulation of stock price paths at many $(n >> 1)$ time points.

2. Let $(S_t)_{t\geq 0}$ be the price process of a stock that pays no dividends, and let r denote the riskfree interest rate (as usual, continuously compounded) for both borrowing and lending.

(a) Consider a forward contract that is settled in cash. What is the payoff at maturity T for a long position in the forward contract whose delivery price F_t was agreed at time $t < T$? What is the profit from entering a long forward position at time t and holding it until T ? What is the cashflow at time t from entering a long forward position? In which situation does a loss occur from entering a short forward position at time t and holding it until T?

(b) Demonstrate that there is a unique no-arbitrage forward price F_t, in the sense that any other forward price would imply that there exists arbitrage opportunities.

(c) Assume that the market is free of arbitrage, and suppose there are (European) call and put options traded with maturity T of the same strike K.

(i) Derive an equation relating the call and put prices C_t and P_t at time t to the forward price F_t.

(ii) Now assume furthermore that the call price equals the put price, i.e. $C_t = P_t$. What can you conclude about the relation between K and F_t?

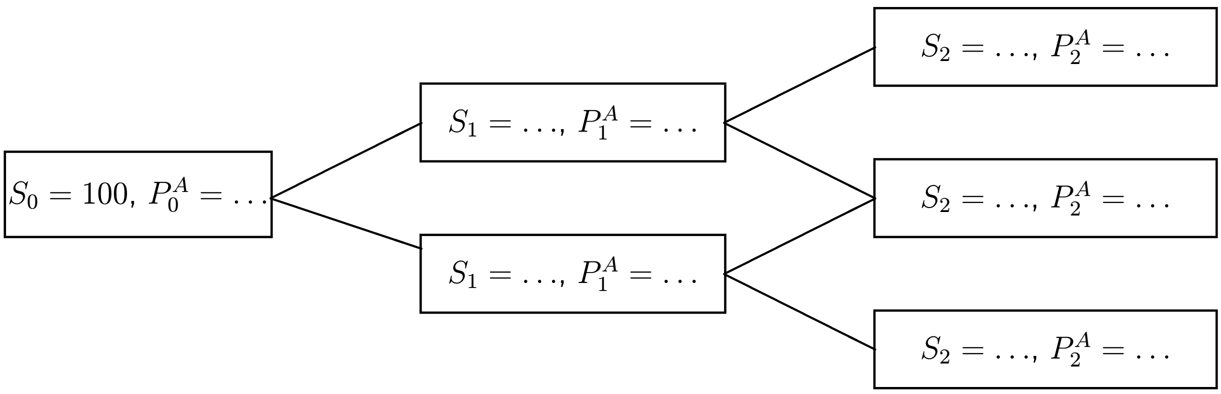
(iii) If the spot price at $t = 6$ is $S_t = 60$, the delivery time is $T = 10$ and the riskfree interest rate is $r = 0.04$, what is the forward price at t?

(iv) Let $t < u < T$. Suppose the value of a long forward contract, entered at time t, is negative at time u. What does this imply about the relation of S_u, F_t and F_u? Compute the forward's value if $u = 9$, $S_u = 55$ and other parameters are as in (iii).

3. Consider the binomial model for a financial market on a finite probability space $(\Omega, \mathcal{F}, P)$ with T periods of length 1 each. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables under the real world probability $\widetilde{P}$ with $\widetilde{p} := \widetilde{P}[Y_k = 1.2] = 1 - \widetilde{P}[Y_k = 0.8] = \frac{1}{2}$ and let the stock price process S be given by $S_0 = 100$ and $S_k = S_{k-1}Y_k$ for $k = 1, 2, \ldots$. The interest rate r is such that $\exp(r) = 1.1$. The filtration $(\mathcal{F}_k)_{k=0,1,2,\ldots}$ modelling the information flow is given by $\mathcal{F}_k = \sigma(Y_i \,|\, i \leq k) = \sigma(S_i \,|\, i \leq k)$.

(a) Compute the transition probabilities p (and $1-p$) for up (respectively down) moves of the stock price under the *risk neutral measure* P. Define what it means that a process (M_k) is a martingale under the risk neutral probability measure P with respect to $(\mathcal{F}_k)$. Then show that the discounted stock price $M_k := S_k \exp(-r\,k)$ is a martingale under the risk neutral measure P.

(b) Compute the price (P^A_k) of the American put option with strike $K = 90$ and maturity $T = 2$ on the stock at all nodes of the binomial tree. Explain the method of computation, and record your results about the evolution of the American option price in a sketched diagram of the tree, similar to the diagram below, detailing which prices the stock and the American option are taking on each tree node.



(c) In order to hedge the American option, how many shares of the stock and how much money in the bank account should one hold at time $k = 0$, that is at the initial tree node? State whether it is optimal to exercise or to continue holding the option at time $k = 1$ if the stock price has gone down. Justify your statement by using the notion of the *continuation value*.

4. This Question is about variance reduction using *control variates*.

(a) To apply the method, consider the Black Scholes model where the stock price evolves as

$$S_t = s\,\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

As usual, $s, r, \sigma > 0$ denote the fixed parameters for the initial stock price, interest rate, and volatility, and (W_t) is a Brownian motion under the risk-neutral measure P. You want to compute the price of an arithmetic Asian put option with strike $K > 0$ that pays $H^a := \left(K - \frac{1}{n}\sum_{k=1}^{n} S_{t_k}\right)^+$ at maturity T for a given discrete time grid $t_k = kT/n$, $k = 1, \ldots, n$.

(i) Show how the price of a geometric Asian call with payoff $H^g := \left(K - ((\prod_{k=1} S_{t_k})^{1/n}\right)^+$ can be computed explicitly.
[Hint. You can use without proof $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+2)}{6}$, and the Black-Scholes formula for the European put price $P(S_0, K, \sigma, r, T) := \exp(-rT)E[(K - S_T)^+] = -S_0\,\Phi(-d_+) + \exp(-rT)K\,\Phi(-d_-)$ with $d_\pm := (\log(S_t/K) + (r \pm \sigma^2/2)T)/(\sigma\sqrt{T})$.]

(ii) Suppose you have sampled $j = 1, \ldots, N$ paths of the stock price, with $S_{k,j}$ denoting the value of the jth path at time t_k. State the Monte-Carlo estimate for the arithmetic Asian put price that uses as control variate H^g.
[Note. You are not required to detail how to simulate the paths here.]

(b) This part is about how to further improve a given control variate by optimal linear scaling. Suppose you wish to estimate an expectation $E[X]$ by simulation, but already know $E[Y]$ for a related random variable Y that you want to use as a control variate. Assume $\mathrm{Var}[Y] > 0$ and $\mathrm{Var}[Y], \mathrm{Var}[X] < \infty$. You can simulate independent identically distributed random vectors $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$.

(i) Find $b^* \in \mathbb{R}$ such that the variance $\mathrm{Var}[X - b(Y - E[Y])]$ is minimized.

(ii) State the standard Monte Carlo estimator for $E[X]$ and the control-variate based estimator for $E[X]$ (for given b), based on a sample (X_i, Y_i) with $i = 1, \ldots, N$.

(iii) Find the variance estimates for the standard and for the control-variate Monte Carlo estimate for $E[X]$. Find the theoretical variances of the standard and of the control-variate Monte Carlo estimates, the latter involes b. Derive the factor by which the optimal control-variate estimate with $b = b^*$ reduces the variance of the standard estimate.

5. (a) State the three properties that define the (standard) Brownian motion (W_t) process.

Parts (b) and (c) below are relevant for the Brownian-bridge construction of discretely sampled Brownian paths.

(b) Let $u < s < t < T$. Suppose you have sampled for a Brownian path (W_u, W_s, W_t, W_T) already, and want to refine your simulation to also include the additional time $t' := (s+t)/2$ while keeping all previously sampled path values. At your disposal you have an independent normal random variable X of mean zero and variance c^2. Compute $c > 0$ such that with

$$W_{t'} := (W_s + W_t)/2 + X \, ,$$

the random vector $(W_u, W_s, W_{t'}, W_t, W_T)$ has the multivariate normal distribution of a Brownian path that is discretely sampled at times $u < s < t' < t < T$.

(c) Iterate the argument from the previous part to explain how to refine a simulation of a Brownian path from time grid $t_k = kT/2^n$ with $k = 0, 1, \ldots, 2^n$, $n \in \mathbb{N}_0$, to the finer grid $t'_i = iT/2^{n+1}$ with $i = 0, 1, \ldots, 2^{n+1}$, that has half of the previous time step length, by using a sequence of independent normal random variables with zero mean and suitable variances, which you should specify.

(d) Apply Ito's formula to $\cos W_t$ in order to specify the integrands in an expression of the form

$$\cos(W_t) = \cos(W_0) + \int_0^t \ldots ds + \int_0^t \ldots dW_s \, ,$$

and use this to compute the expectation $E[\cos W_t]$.