IMPERIAL

MATH70006

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Complex Analysis

Date: Wednesday, May 14, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Consider the complex-valued function

$$\phi(z) = \frac{e^z - 1 - z}{z^2}.$$

(a) Show that $\phi(z)$ has a removable singularity at $z = 0$. (3 marks)

(b) Give a reason why $\phi(x)$ will be inaccurate when expressed in standard precision arithmetic, for small values of $x \in \mathbb{R}$. (2 marks)

(c) Using Cauchy's integral formula, the circular contour $\gamma = \{z : |z| = 1\}$, and the trapezium rule, derive the following approximation to $\phi(x)$ for $x \in \mathbb{R}$ with $|x| < 1$:

$$\phi_N(x) = \frac{1}{N}\sum_{n=1}^{N} \frac{\exp(e^{i\theta_n}) - 1 - e^{i\theta_n}}{e^{i\theta_n} - x} e^{-i\theta_n}, \quad \text{where } \theta_n = \frac{2\pi n}{N} \text{ for } n = 1, \ldots, N.$$

(6 marks)

(d) Recall that the trapezium rule approximation $I_N[f]$ to an integral

$$I[f] = \int_0^{2\pi} f(z)\mathrm{d}z$$

converges like

$$|I[f] - I_N[f]| \leq \frac{4\pi M}{e^{aN} - 1},$$

if the function f is 2π-periodic and analytic in the strip $|\mathrm{Im}\{z\}| < a$ for some $a > 0$, where $M = \max_{|\mathrm{Im}\{z\}|<a} |f(z)|$. Hence show that for $x \in \mathbb{R}$ with $|x| < \epsilon < 1$,

$$|\phi(x) - \phi_N(x)| < \frac{2\left(\exp(e^a) + 1 + e^a\right) e^a}{(e^{-a} - \epsilon)(e^{aN} - 1)} \quad \text{for any } a < \log(1/\epsilon).$$

(9 marks)

(Total: 20 marks)

2. (a) Use contour integration to show that for $x \in (-1, 1)$,

$$\fint_{-1}^{1} \frac{t^2\sqrt{1-t^2}}{t-x}\mathrm{d}t = \pi x\left(\frac{1}{2} - x^2\right).$$

(12 marks)

(b) Hence solve

$$\frac{1}{\pi}\int_{-1}^{1} \log|t-x| f(t)\mathrm{d}t = \frac{-x^3}{3} \quad \text{for } x \in (-1, 1)$$

for $f(x)$. You may use without proof the Hilbert inversion formula

$$F(x) = \frac{-1}{\pi\sqrt{1-x^2}}\fint_{-1}^{1} \frac{\sqrt{1-t^2}g(t)}{t-x}\mathrm{d}t + \frac{A}{\sqrt{1-x^2}}$$

for the solution of the singular integral equation

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{F(t)}{t-x}\mathrm{d}t = g(x)$$

for $x \in (-1, 1)$, where A denotes an arbitrary constant. (8 marks)

(Total: 20 marks)

3. (a) The gamma function $\Gamma(z)$ is defined for $\mathrm{Re}\{z\} > 0$ by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ and is analytic in this region.

   (i) Show that $\Gamma(z+1) = z\Gamma(z)$ at least for $\mathrm{Re}\{z\} > 0$. (2 marks)

   (ii) Hence show that for $\mathrm{Re}\{z\} \leq 0$, $\Gamma(z)$ is analytic except for simple poles whose positions and residues you are to determine. *(You may ignore the behaviour of $\Gamma(z)$ at infinity.)* (9 marks)

(b) The beta function $B(z,w)$ is defined by $B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t$ for $\mathrm{Re}\{z\}, \mathrm{Re}\{w\} > 0$.

   (i) For $\mathrm{Re}\{z\} > 0$, by making use of a trigonometric substitution for t, derive the identity

$$B\left(z, \frac{1}{2}\right) = 2^{2z-1}B(z,z).$$

(5 marks)

   (ii) Hence show that,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

   You may use without proof the identities $B(z,w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ and $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$. (4 marks)

(Total: 20 marks)

4. $f(x)$ satisfies the equation

$$3\int_0^\infty e^{-2|x-t|}f(t)\mathrm{d}t = 3f(x) + \frac{\mathrm{d}f(x)}{\mathrm{d}x} \qquad \text{for } x \geq 0,$$

with $f(0) = 1$, and is bounded by a non-zero constant as $x \to \infty$.

(a) Using the Wiener-Hopf method, determine the right-sided transform $F_+(s) \equiv \int_0^\infty f(x)e^{\mathrm{i}sx}\mathrm{d}x$ as a rational function of s for $\mathrm{Im}\{s\} > \alpha$ for some α. You may use without proof the fact that the (ordinary) Fourier transform of $k(x) = ce^{-\gamma|x|}$ where $\gamma > 0$ and $c \in \mathbb{R}$, is $\hat{K}(s) = \int_{-\infty}^\infty k(x)e^{\mathrm{i}sx}\mathrm{d}x = \dfrac{2c\gamma}{s^2+\gamma^2}$ for $|\mathrm{Im}\{s\}| < \gamma$.

(15 marks)

(b) Hence show that for $x \geq 0$,

$$f(x) = \frac{1}{2}\left(1 + e^{-4x}\right).$$

(5 marks)

(Total: 20 marks)

5. In this question,

$$I = \int_{-\infty}^{\infty} \frac{x e^{\mathrm{i}x^2}}{x^2+1} \mathrm{d}x.$$

(a) Explain why

$$I_R := \int_{-R}^{R} \frac{x e^{\mathrm{i}x^2}}{x^2+1} \mathrm{d}x = \int_{\gamma} \frac{z e^{\mathrm{i}z^2}}{z^2+1} \mathrm{d}z,$$

where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ with

$$\begin{aligned}
\gamma_1 &= \{Re^{\mathrm{i}\theta} : \theta \in [-\pi, -3\pi/4]\}, \\
\gamma_2 &= \{re^{\mathrm{i}\pi/4} : r \in [-R, R]\}, \\
\gamma_3 &= \{Re^{\mathrm{i}\theta} : \theta \in [0, \pi/4]\},
\end{aligned}$$

and integration along γ_1 is from $z = -R$ to $z = Re^{-3\pi\mathrm{i}/4}$, integration along γ_2 is from $z = Re^{-3\pi\mathrm{i}/4}$ to $z = Re^{\pi\mathrm{i}/4}$, and integration along γ_3 is from $z = Re^{\pi\mathrm{i}/4}$ to $z = R$.

(2 marks)

(b) Show that

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{i}re^{-r^2}}{\mathrm{i}r^2+1} \mathrm{d}r.$$

Hint 1: You may assume without proof that

$$\sin y \geq \frac{2y}{\pi} \quad \text{for } y \in [0, \pi/2].$$

Hint 2: You may assume without proof that

$$\lim_{R\to\infty} \int_{\gamma_1} \frac{ze^{iz^2}}{z^2+1} \mathrm{d}z = \lim_{R\to\infty} \int_{\gamma_3} \frac{ze^{iz^2}}{z^2+1} \mathrm{d}z.$$

(8 marks)

(c) Now let

$$f(z) = \frac{\mathrm{i}ze^{-z^2}}{\mathrm{i}z^2+1},$$

so that $I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x$. The (unbounded) trapezium rule approximation gives, for some mesh-width $h > 0$,

$$I \approx I_h = h \sum_{n=-\infty}^{\infty} f(hn).$$

You may assume that $f(z)$ is such that

$$I - I_h = \sum_{\pm} \int_{-\infty \pm \mathrm{i}a'}^{\infty \pm \mathrm{i}a'} \frac{f(z)}{1 - e^{\mp 2\pi \mathrm{i}z/h}} \mathrm{d}z,$$

for any $a' \in (-1/\sqrt{2}, 1/\sqrt{2})$.

Continued on next page.

(i) For $\hat{a} > 1/\sqrt{2}$, show that

$$|I - I_h| = O(e^{\hat{a}^2 - 2\pi\hat{a}/h}) \quad \text{as } h \to 0.$$

Hint 1: You may assume without proof that for $b', \hat{b} > 0$ and $\xi \in \mathbb{R}$,

$$\int_{\xi \pm \mathrm{i}b'}^{\xi \pm \mathrm{i}\hat{b}} \frac{f(z)}{1 - e^{\mp 2\pi \mathrm{i} z/h}} \mathrm{d}z \to 0 \quad \text{as } |\xi| \to \infty.$$

Hint 2: You may assume without proof that

$$\int_{-\infty \pm \hat{a}\mathrm{i}}^{\infty \pm \hat{a}\mathrm{i}} |f(z)| \, \mathrm{d}z = O(e^{\hat{a}^2}).$$

(6 marks)

(ii) Show that the optimal convergence rate is given by

$$|I - I_h| = O(e^{-\pi^2/h^2}) \quad \text{as } h \to 0.$$

(4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60006/MATH70006

Applied Complex Analysis (Solutions)

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1. (a) Write

$$\phi(z) = \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 - z}{z^2} = \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}. \tag{1}$$

Then, since the first two terms of the expansion of the exponential are cancelled in the numerator, and we can divide through for $z \neq 0$, we have a locally convergent Taylor expansion at $z = 0$ and thus a removable singularity. $\boxed{3,\ \mathrm{A}}$

(b) For small x, we have $e^x \approx 1$. The operation $e^x - 1$ in the numerator will only store information up to 10^{-16} in size, which will result in lost information/digits when $z = x$ is subtracted. $\boxed{2,\ \mathrm{B}}$

(c) Cauchy's integral formula gives, for $x \in \mathbb{R}$ with $|x| < 1$,

$$\phi(x) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\phi(z)}{z - x} dz.$$

$\boxed{2,\ \mathrm{A}}$

Setting $z = e^{i\theta}$, we have $dz = ie^{i\theta} d\theta$ and

$$\phi(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\phi(e^{i\theta})}{e^{i\theta} - x} e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(e^{i\theta}) - 1 - e^{i\theta}}{e^{i\theta} - x} e^{-i\theta} d\theta. \tag{2}$$

$\boxed{2,\ \mathrm{A}}$

Now approximating with the trapezium rule, with $\theta_n = 2\pi n/N$ and weights $2\pi/N$, gives

$$\phi(x) \approx \phi_N(x) = \frac{1}{N} \sum_{n=1}^{N} \frac{\exp(e^{i\theta_n}) - 1 - e^{i\theta_n}}{e^{i\theta_n} - x} e^{-i\theta_n}.$$

$\boxed{2,\ \mathrm{B}}$

(d) We will apply the result given in the question to the integral expression for $\phi(x)$ that is given by (2), that is, with θ replacing z so that we henceforth consider θ as a complex variable, and the integrand

$$f(\theta) = \frac{1}{2\pi} \left(\frac{\exp(e^{i\theta}) - 1 - e^{i\theta}}{e^{i\theta} - x} \right) e^{-i\theta}. \tag{3}$$

Evidently, $f(\theta)$ is 2π-periodic.

First, determine the constant a that defines the strip of analyticity of $f(\theta)$. The latter is singular when $e^{i\theta} = x$, i.e., when

$$\theta = i \log |1/x| + \arg x. \tag{4}$$

$\boxed{1,\ \mathrm{A}}$

Thus for $|x| < \epsilon < 1$, $f(\theta)$ is analytic in the strip $\{\theta || \mathrm{Im}\{\theta\}| < a\}$ for any a with $0 < a < \log(1/\epsilon) < \log |1/x|$. $\boxed{1,\ \mathrm{B}}$

Next, we estimate M. Considering first terms in the numerator of the above expression for $f(\theta)$, by the triangle inequality we have

$$\left| \exp(e^{i\theta}) - 1 - e^{i\theta} \right| \leq \left| \exp(e^{i\theta}) \right| + 1 + \left| e^{i\theta} \right|. \tag{5}$$

And since $e^{i\theta} = e^{-\mathrm{Im}\{\theta\}} e^{i\mathrm{Re}\{\theta\}}$, we have $\left|e^{i\theta}\right| = e^{-\mathrm{Im}\{\theta\}}$ and so for $|\mathrm{Im}\{\theta\}| < a$,

$$e^{-a} < \left|e^{i\theta}\right| < e^{a}, \tag{6}$$

and hence also

$$|e^{-i\theta}| < e^{a}. \tag{7}$$

Furthermore, still with $|\mathrm{Im}\{\theta\}| < a$, since $\mathrm{Re}\{e^{i\theta}\} = e^{-\mathrm{Im}\{\theta\}} \cos\left(\mathrm{Re}\{\theta\}\right) \leq e^{-\mathrm{Im}\{\theta\}} < e^{a}$ (where the last inequality follows from the above), we also have

$$\left|\exp(e^{i\theta})\right| = \left|\exp\left(\mathrm{Re}\{e^{i\theta}\}\right)\right| < \exp(e^{a}). \tag{8}$$

For the denominator, by the reverse triangle inequality we have

$$\begin{aligned} |e^{i\theta} - x| &\geq \left||e^{i\theta}| - |x|\right| \\ &= |e^{i\theta}| - |x| \\ &\geq |e^{i\theta}| - \epsilon \\ &> e^{-a} - \epsilon, \end{aligned} \tag{9}$$

where the second line follows from the facts that $\left|e^{i\theta}\right| > e^{-a}$ (as stated by (6)) and $-a > \log|x|$ which give $\left|e^{i\theta}\right| > |x|$, and the last line also follows from (6). $\boxed{4, \mathrm{D}}$

Now, combining (5)–(9) gives

$$M < \frac{1}{2\pi} \frac{\left(\exp(e^{a}) + 1 + e^{a}\right) e^{a}}{e^{-a} - \epsilon}. \tag{10}$$

Substituting this into the given result gives

$$|\phi(x) - \phi_N(x)| < \frac{2\left(\exp(e^{a}) + 1 + e^{a}\right) e^{a}}{(e^{-a} - \epsilon)(e^{aN} - 1)} \tag{11}$$

for any $a < \log(1/\epsilon)$. $\boxed{3, \mathrm{C}}$

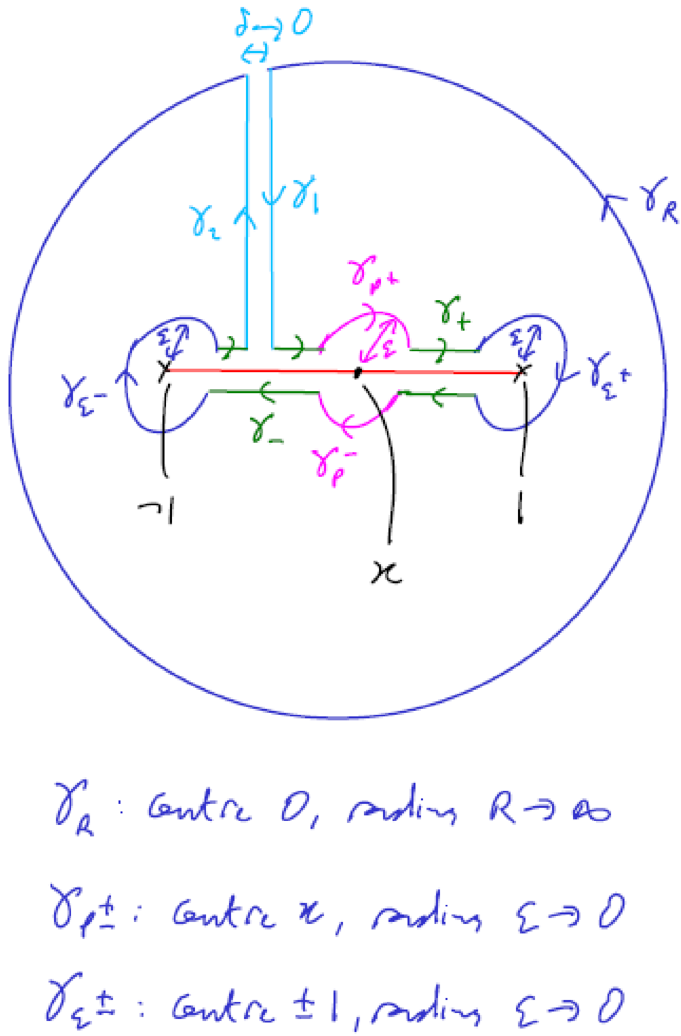

Figure 1: Contour of integration Γ for question 2(a).

2. (a) For $x \in (-1, 1)$, consider the integral $H(x) = \displaystyle\int_\Gamma h(z)dz$ where $h(z) = \dfrac{z^2\sqrt{z^2-1}}{z-x}$, where we take the branch of $\sqrt{z^2-1}$ with a branch cut along $(-1, 1)$ and which behaves like z as $|z| \to \infty$, and Γ is the closed contour shown in Figure 1, consisting of the sections γ_R, $\gamma_\pm$, $\gamma_{p\pm}$, $\gamma_{\epsilon\pm}$ and $\gamma_{1,2}$. γ_R is a circle of radius R centred on the origin, while $\gamma_{p\pm}$ are semi-circles of radius ϵ centred on x. Also $\gamma_{\epsilon\pm}$ are circles of radius ϵ centred on ± 1, and $\gamma_{1,2}$ are separated by a gap of width δ. We consider the limit as $R \to \infty$, $\epsilon \to 0$ and $\delta \to 0$. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{1, \text{A}}$

$h(z)$ is analytic inside Γ and so, by Cauchy's theorem, $H(x) = 0$. $\boxed{1, \text{A}}$

Now evaluate the integrals along the separate sections of Γ. First, those along γ_1 and γ_2 cancel one another. Next, for $z \in \gamma_{\epsilon+}$, we have $z = 1 + \epsilon e^{i\theta}$ and thus $|h(z)| \sim \mathcal{O}(\epsilon^{\frac{1}{2}})$. It follows that the integral along $\gamma_{\epsilon+}$ is zero. Similarly, one may deduce that the integral along $\gamma_{\epsilon-}$ is also zero. $\boxed{1, \text{A}}$

Next, for our choice of branch we have, for $z \in \gamma_\pm$, $\sqrt{z^2-1} = \pm i\sqrt{1-x^2}$, and hence

$$\int_{\gamma_+} h(z)dz = \int_{\gamma_-} h(z)dz = \fint_{-1}^{1} \frac{it^2\sqrt{1-t^2}}{t-x}dt, \qquad (12)$$

taking into account that we integrate along γ_- from right to left. $\boxed{3, \text{B}}$

Furthermore, on γ_{p+} we have $z = x + \epsilon e^{\mathrm{i}\theta}$ where θ goes from π to 0, while on γ_{p-}, $z = x + \epsilon e^{\mathrm{i}\theta}$ where θ goes from 0 to $-\pi$. Then, again keeping in mind our choice of branch, it follows that

$$\int_{\gamma_{p+}} h(z)dz \to \int_{\pi}^{0} \frac{\mathrm{i}x^2\sqrt{1-x^2}}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta} d\theta = \pi x^2 \sqrt{1-x^2} \tag{13}$$

and

$$\int_{\gamma_{p-}} h(z)dz \to \int_{0}^{-\pi} \frac{-\mathrm{i}x^2\sqrt{1-x^2}}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta} d\theta = -\pi x^2 \sqrt{1-x^2}. \tag{14}$$

$\boxed{2, \text{A}}$

Finally, keeping in mind the fact that our chosen branch of $\sqrt{z^2-1}$ behaves like z as $|z| \to \infty$, one finds that for $z \in \gamma_R$,

$$h(z) \sim z^2\left(1 - \frac{1}{2z^2} + \mathcal{O}\left(\frac{1}{z^4}\right)\right)\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \frac{x^3}{z^3} + \mathcal{O}\left(\frac{1}{z^4}\right)\right). \tag{15}$$

One may identify the coefficient of $\frac{1}{z}$ in this expansion as $x(x^2 - \frac{1}{2})$. Hence it follows that

$$\int_{\gamma_R} h(z)dz = 2\pi\mathrm{i}x\left(x^2 - \frac{1}{2}\right). \tag{16}$$

$\boxed{3, \text{D}}$

Thus, combining the above, one arrives at

$$H(x) = 2\mathrm{i}\fint_{-1}^{1} \frac{t^2\sqrt{1-t^2}}{t-x} dt + 2\pi\mathrm{i}x\left(x^2 - \frac{1}{2}\right) = 0. \tag{17}$$

Hence for $x \in (-1, 1)$,

$$\fint_{-1}^{1} \frac{t^2\sqrt{1-t^2}}{t-x} dt = \pi x\left(\frac{1}{2} - x^2\right). \tag{18}$$

$\boxed{1, \text{A}}$

(b) Differentiating the given integral equation with respect to x, one arrives at $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{f(t)}{t-x} dt = x^2, \quad x \in (-1, 1). \tag{19}$$

$\boxed{2, \text{B}}$

Then, the inversion formula gives the solution for $f(x)$, for $x \in (-1, 1)$, as

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{t^2\sqrt{1-t^2}}{t-x} dt + \frac{A}{\sqrt{1-x^2}}, \tag{20}$$

where A is a constant. Then, using the result from part (a), one finds

$$f(x) = \frac{x(x^2 - \frac{1}{2})}{\sqrt{1-x^2}} + \frac{A}{\sqrt{1-x^2}}, \quad x \in (-1, 1). \tag{21}$$

$\boxed{1, \text{A}}$

We may determine A by substituting (21) back into the original integral equation. Doing so and taking $x = 0$, gives

$$\int_{-1}^{1} \frac{t(t^2 - \frac{1}{2})\log|t|}{\sqrt{1-t^2}} dt + A\int_{-1}^{1} \frac{\log|t|}{\sqrt{1-t^2}} dt = 0. \tag{22}$$

$\boxed{3, \text{C}}$

However, the first integral on the left hand side must be zero (since its integrand is an odd function of t), but the second must be non-zero (since its integrand is an even function of t which does not change sign). Hence A must equal 0. $\boxed{1, \text{D}}$

Thus the final solution for $f(x)$, for $x \in (-1, 1)$, is

$$f(x) = \frac{x(x^2 - \frac{1}{2})}{\sqrt{1 - x^2}}. \tag{23}$$

$\boxed{1, \text{A}}$

3. (a) (i) For $\mathrm{Re}(z) > 0$, starting from the given definition of $\Gamma(z)$, we have, by integrating by parts, seen ⇓

$$\Gamma(z) = \frac{1}{z}\left([t^z e^{-t}]_{t=0}^{\infty} + \int_0^\infty t^z e^{-t} dt\right) = \frac{1}{z}\int_0^\infty t^z e^{-t} dt = \frac{\Gamma(z+1)}{z}. \quad (24)$$

2, A

(ii) From the given fact that $\Gamma(z)$ is analytic for $\mathrm{Re}(z) > 0$, it follows that $\Gamma(z+1)$ is analytic for $\mathrm{Re}(z) > -1$. It then follows using the identity from part (ai) and analytic continuation that $\Gamma(z) = \Gamma(z+1)/z$ for $\mathrm{Re}(z) > -1$ and that $\Gamma(z)$ is analytic for $\mathrm{Re}(z) > -1$ except possibly at $z = 0$. seen ⇓

4, A

It is easily shown from the definition given in the question statement that $\Gamma(1) = 1$, and hence, at $z = 0$ $\Gamma(z)$ has a simple pole of residue 1. 2, A

Repeating these arguments one finds that, for $n \in \mathbb{Z}, \geq 0$,

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{\prod_{k=0}^{n}(z+k)}, \quad \text{for } \mathrm{Re}(z) > -(n+1), \quad (25)$$

and hence that $\Gamma(z)$ is analytic everywhere for except for simple poles at $z = -n$ for $n \in \mathbb{Z}, \geq 0$. Furthermore, the residue at $z = -n$ is given by

$$\frac{\Gamma(-n+n+1)}{\prod_{k=0}^{n-1}(-n+k)} = \frac{\Gamma(1)}{(-1)^n(n!)} = \frac{(-1)^n}{n!}. \quad (26)$$

3, B

(b) (i) From the definition of $B(z,w)$, sim. seen ⇓

$$B\left(z,\frac{1}{2}\right) = \int_0^1 t^{z-1}(1-t)^{-1/2}\mathrm{d}t. \quad (27)$$

and

$$B(z,z) = \int_0^1 t^{z-1}(1-t)^{z-1}\mathrm{d}t. \quad (28)$$

Let $t = \sin^2\theta$. Then $1 - t = \cos^2\theta$ and $\mathrm{d}t = 2\sin\theta\cos\theta\mathrm{d}\theta$. It follows that 1, D

$$B\left(z,\frac{1}{2}\right) = 2\int_0^\pi (\sin\theta)^{2z-1}\mathrm{d}\theta \quad (29)$$

and 1, C

$$\begin{aligned} B(z,z) &= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2z-1} d\theta \\ &= 2^{2-2z}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^{2z-1} d\theta \\ &= 2^{1-2z}\int_0^{\pi} (\sin\phi)^{2z-1} d\phi \\ &= 2^{2-2z}\int_0^{\frac{\pi}{2}} (\sin\phi)^{2z-1} d\phi. \end{aligned} \quad (30)$$

The result follows on combining (29) and (30), one arrives at

$$B\left(z,\frac{1}{2}\right) = 2^{2z-1}B(z,z) \quad (31)$$

3, D

(ii) It follows from the identity

unseen ⇓

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{32}$$

and the result of the previous part that

$$\frac{\Gamma(z)\Gamma(\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2^{2z-1}\frac{(\Gamma(z))^2}{\Gamma(2z)}, \tag{33}$$

or

$$\Gamma(2z) = \frac{2^{2z-1}}{\Gamma(\frac{1}{2})}\Gamma(z)\Gamma(z+\frac{1}{2}). \tag{34}$$

Finally, using the identity

2, C

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{35}$$

one may deduce that $\Gamma(\frac{1}{2})^2 = \pi$, or $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hence the required result follows from (34).

2, B

4. (a) Let

$$k(x) = 3e^{-2|x|}. \tag{36}$$

sim. seen ⇓

Also, introduce

$$f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0; \end{cases} \tag{37}$$

and

$$g_-(x) = \begin{cases} 0, & x \geq 0 \\ 3\displaystyle\int_0^\infty e^{-2|x-t|} f(t)\mathrm{d}t, & x < 0. \end{cases} \tag{38}$$

Then we can extend the given equation to the following:

$$\int_{-\infty}^{\infty} k(x-t) f_+(t)\mathrm{d}t = 3f_+(x) + f_+'(x) + g_-(x), \quad -\infty < x < \infty. \tag{39}$$

Taking the Fourier transform (using the convolution theorem) gives

1, A

$$\hat{K}(s)F_+(s) = 3F_+(s) + F_+^{(1)}(s) + G_-(s), \tag{40}$$

where $\hat{K}(s)$ denotes the Fourier transform of $k(x)$, $F_+(s)$ denotes the right-sided transform of $f(x)$, $F_+^{(1)}(s)$ denotes the right-sided transform of $f'(x)$, and $G_-(s)$ denotes the left-sided transform of $g(x)$. From the fact provided in the question statement, $\hat{K}(s) = \dfrac{12}{s^2+4}$.

1, A

By integration by parts, one can show that, provided $\mathrm{Im}\{s\} > 0$,

$$F_+^{(1)}(s) = -f(0) - \mathrm{i}sF_+(s) = -1 - \mathrm{i}sF_+(s), \tag{41}$$

where we have made use of the facts that $f(0) = 1$ and that $f(x)$ is bounded by a non-zero constant as $x \to \infty$.

1, B

Then (40) can be written as

$$K(s)F_+(s) = G_-(s) - 1, \tag{42}$$

where

$$K(s) = \hat{K}(s) + \mathrm{i}s - 3 = \frac{12}{s^2+4} + \mathrm{i}s - 3 = \frac{\mathrm{i}s(s-\mathrm{i})(s+4\mathrm{i})}{(s+2\mathrm{i})(s-2\mathrm{i})}. \tag{43}$$

1, A

We now specify the strip of analyticity $\{s|\alpha < \mathrm{Im}\{s\} < \beta\}$. $|f(x)|$ will be bounded by $Ae^{\alpha x}$ as $x \to \infty$, for a constant A. Since we seek a solution for $f(x)$ that is bounded by a non-zero constant in this limit, we take $\alpha > 0$. Also, the strip should not contain any zeros or poles of $K(s)$. We thus take $0 < \alpha < \beta < 1$. We denote this strip by Ω. **Note to marker**: *other choices of the strip that allow for growth of $f(x)$ as $x \to \infty$ are acceptable, although ultimately, the student must discount any terms that grow in this limit from their final solution for $f(x)$. For example, $1 < \alpha < \beta < 2$ would be acceptable. Other choices for the strip might force $f(x)$ to decay to zero as $x \to \infty$ (e.g., $-2 < \alpha < \beta < 0$) and so are incorrect.*

2, C

We now decompose $K(s)$ as

$$K(s) = K_+(s)K_-(s) \tag{44}$$

where

$$K_+(s) = \frac{s(s+4\mathrm{i})}{(s+2\mathrm{i})}, \quad K_-(s) = \frac{\mathrm{i}(s-\mathrm{i})}{(s-2\mathrm{i})}, \tag{45}$$

so that $K_+(s)$ is analytic and non-zero in the region where $\mathrm{Im}\{s\} > \alpha$, and likewise for $K_-(s)$ in the region $\mathrm{Im}\{s\} < \beta$. $\boxed{2, \mathrm{D}}$

Now, for $s \in \Omega$, we have

$$K_+(s)F_+(s) = \frac{G_-(s)}{K_-(s)} + R(s) \tag{46}$$

where

$$R(s) = \frac{-1}{K_-(s)} = \frac{(s-2\mathrm{i})}{\mathrm{i}(s-\mathrm{i})}. \tag{47}$$

$\boxed{1, \mathrm{A}}$

We now decompose $R(s)$ as

$$R(s) = R_+(s) + R_-(s) \tag{48}$$

where

$$R_+(s) = 0, \quad R_-(s) = R(s). \tag{49}$$

$\boxed{2, \mathrm{D}}$

Now, for $s \in \Omega$, we have

$$K_+(s)F_+(s) = \frac{G_-(s)}{K_-(s)} + R_-(s). \tag{50}$$

$\boxed{1, \mathrm{A}}$

Then we can identify an entire function $E(s)$ for which

$$E(s) = F_+(s)K_+(s) \quad \text{for } s \in \oplus. \tag{51}$$

where $\oplus$ is the region where $\mathrm{Im}\{s\} > \alpha$. $\boxed{1, \mathrm{A}}$

To determine $E(s)$ consider its behaviour as $|s| \to \infty$ in $\oplus$. Note that in this limit we have (as follows by integration by parts of the formula for $F_+(s)$)

$$F_+(s) \sim \frac{\mathrm{i}f(0)}{s} + \mathcal{O}\left(\frac{1}{s^2}\right), \tag{52}$$

where we recall that $f(0) = 1$. It follows that as $|s| \to \infty$ in $\oplus$ we have $E(s) \to \mathrm{i}$. Hence, by Liouville's Theorem, $E(s) \equiv \mathrm{i}$ for all s. $\boxed{1, \mathrm{C}}$

It follows that, for $s \in \oplus$, we have

$$F_+(s)K_+(s) = \mathrm{i}. \tag{53}$$

Rearranging gives, for $s \in \oplus$,

$$F_+(s) = \frac{\mathrm{i}(s+2\mathrm{i})}{s(s+4\mathrm{i})}. \tag{54}$$

$\boxed{1, \mathrm{B}}$

Note to marker: *If the student made a different choice for their strip of analyticity (see the remark above) then they will arrive at a different expression for $F_+(s)$.*

$\boxed{\text{sim. seen} \Downarrow}$

(b) Finally, to retrieve $f(x)$, apply the inversion formula. We have, for $x \geq 0$,

$$f(x) = \frac{1}{2\pi}\int_L H(s)\mathrm{d}s \tag{55}$$

where

$$H(s) = \frac{\mathrm{i}(s+2\mathrm{i})}{s(s+4\mathrm{i})}e^{-\mathrm{i}sx} \tag{56}$$

and L is a horizontal line in the region $\oplus$, and we integrate along L from left to right. $\boxed{1, \mathrm{A}}$

One can evaluate this integral by considering the integral of $H(s)$ around the closed contour that consists of the union of L and a semi-circle γ_R below L, with radius R where $R \to \infty$, and applying the residue theorem. The integral of $H(s)$ along γ_R tends to 0 in this limit. $\boxed{1, \mathrm{A}}$

And, $H(s)$ has poles below L at $s = 0, -4\mathrm{i}$. One finds that the residues of $H(s)$ at $s = 0, -4\mathrm{i}$ are $\mathrm{i}/2$ and $\mathrm{i}e^{-4x}/2$, respectively. Hence one finds that for $x \geq 0$,

$$f(x) = \frac{1}{2}\left(1 + e^{-4x}\right). \tag{57}$$

$\boxed{3, \mathrm{B}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks

5. (a) The two integrals share the same endpoints, and it is possible to deform $[-R, R]$ to γ without crossing the singularities of the integrand at $\pm i$. Thus the two integrals are equal by Cauchy's Theorem. $\boxed{2,\ \mathrm{M}}$

(b) First we show that the integral along γ_3 tends to zero as $R \to \infty$.

$$\begin{aligned}\int_{\gamma_3} \frac{ze^{iz^2}}{z^2+1}dz &= \int_{\pi/4}^{0} \frac{Re^{i\theta}\exp\left(i(Re^{i\theta})^2\right)Rie^{i\theta}}{(Re^{i\theta})^2+1}d\theta \\ &= \int_{\pi/4}^{0} \frac{\exp\left(iR^2e^{2i\theta}\right)R^2ie^{2i\theta}}{R^2e^{2i\theta}+1}d\theta \\ &= \int_{\pi/4}^{0} \frac{\exp\left(iR^2\left[\cos(2\theta)+i\sin(2\theta)\right]\right)}{R^2e^{2i\theta}+1}R^2ie^{2i\theta}d\theta.\end{aligned}$$

Now, for $R > 1$,

$$\left|\int_{\gamma_3} \frac{ze^{iz^2}}{z^2+1}dz\right| \leq \int_{\pi/4}^{0} \frac{\exp\left(-R^2\sin(2\theta)\right)}{R^2-1}R^2 d\theta,$$

where we have bounded the denominator of the integrand using the reverse triangle inequality. Thus, using the first hint given in the question statement, $\boxed{4,\ \mathrm{M}}$

$$\begin{aligned}\left|\int_{\gamma_3} \frac{ze^{iz^2}}{z^2+1}dz\right| &\leq \frac{R^2}{R^2-1}\int_{\pi/4}^{0} e^{-4R^2\theta/\pi}d\theta \\ &= \frac{R^2}{R^2-1}\left[\frac{e^{-4R^2\theta/\pi}}{-4R^2/\pi}\right]_{\theta=\pi/4}^{\theta=0} \\ &= \frac{\pi(e^{-R^2}-1)}{4(R^2-1)} \\ &\to 0 \quad \text{as } R \to \infty.\end{aligned}$$

Hence

$$\lim_{R\to\infty}\int_{\gamma_3} \frac{ze^{iz^2}}{z^2+1}dz = 0.$$

$\boxed{2,\ \mathrm{M}}$

Then, by the second hint,

$$\lim_{R\to\infty}\int_{\gamma_1} \frac{ze^{iz^2}}{z^2+1}dz = 0.$$

Thus we are left with

$$I = \lim_{R\to\infty}\int_{\gamma_2} \frac{ze^{iz^2}}{z^2+1}dz.$$

Now setting $z = re^{i\pi/4}$, we have $dz = e^{i\pi/4}dr$, and then

$$I = \int_{-\infty}^{\infty} \frac{e^{i\pi/4}re^{i(e^{i\pi/4}r)^2}}{(e^{i\pi/4}r)^2+1}e^{i\pi/4}dr = i\int_{-\infty}^{\infty} \frac{re^{-r^2}}{ir^2+1}dr.$$

$\boxed{2,\ \mathrm{M}}$

(c) (i) Starting from

$$I_h - I = -\sum_{\pm}\int_{-\infty\pm a'\mathrm{i}}^{\infty\pm a'\mathrm{i}} \sigma_{\mp}(z)dz, \tag{58}$$

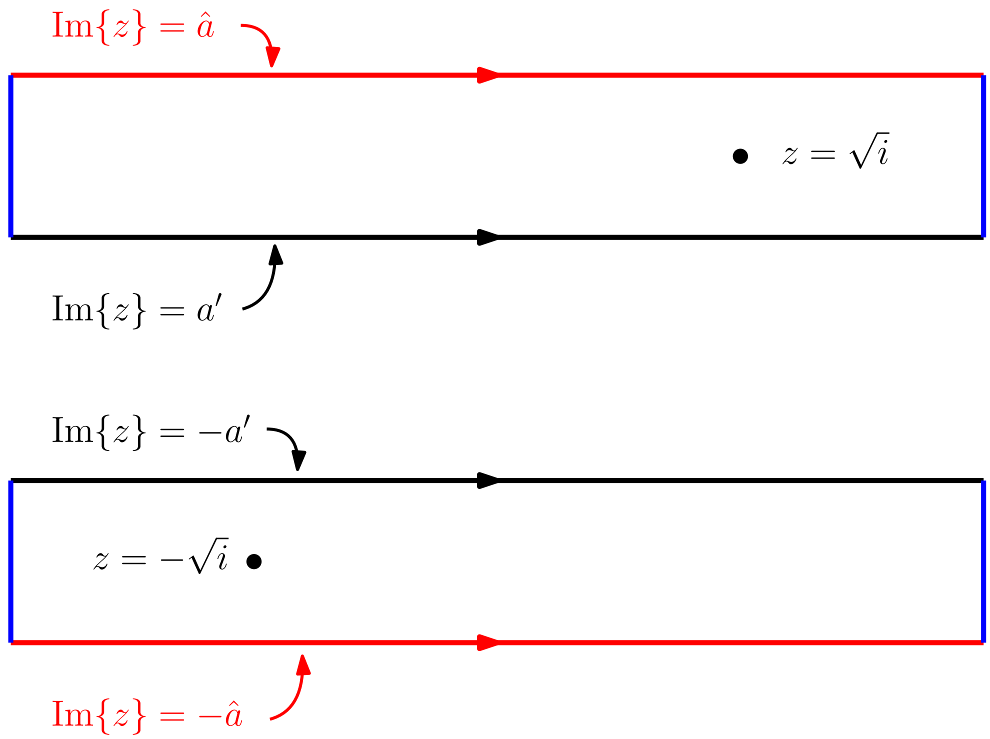

Figure 2: Deforming the contours of integration, as per question 5(c)(i).

for $a' < 1/\sqrt{2}$, where we define

$$\sigma_{\mp}(z) = \frac{f(z)}{1 - e^{\mp 2\pi i z/h}} = \frac{ize^{-z^2}}{(iz^2+1)(1 - e^{\mp 2\pi i z/h})}, \tag{59}$$

we can deform both of the contours away from the real line by increasing a' to $\hat{a} > 1/\sqrt{2}$, collecting residues as we cross the poles at $\pm\sqrt{i} = \pm e^{i\pi/4} = \pm(1+i)/\sqrt{2}$ - see figure 2. The contribution from the vertical components in this deformation is zero, as is implied by the first hint in the question statement. Then, by the residue theorem, we find

$$I_h - I = -\sum_{\pm}\left(\int_{-\infty\pm\hat{a}i}^{\infty\pm\hat{a}i} \sigma_{\mp}(z)dz \pm 2\pi i \mathrm{Res}\left(\sigma_{\mp}(z), \pm\sqrt{i}\right)\right). \tag{60}$$

$\boxed{2,\ \mathrm{M}}$

Writing

$$\sigma_{\mp}(z) = \frac{ize^{-z^2}}{i(z-\sqrt{i})(z+\sqrt{i})(1 - e^{\mp 2\pi i z/h})}, \tag{61}$$

we identify

$$\mathrm{Res}\left(\sigma_{\mp}(z), \pm\sqrt{i}\right) = \frac{\pm i\sqrt{i}e^{-i}}{(\pm 2i\sqrt{i})(1 - e^{-2\pi i\sqrt{i}/h})} = \frac{e^{-i}}{2(1 - e^{-2\pi i\sqrt{i}/h})}. \tag{62}$$

$\boxed{2,\ \mathrm{M}}$

These residues are equal and so will cancel out in (60). Bounding the integrals that remain:

$$\left|\int_{-\infty\pm\hat{a}i}^{\infty\pm\hat{a}i} \sigma(z)dz\right| \le \frac{1}{e^{2\pi\hat{a}/h} - 1}\int_{-\infty\pm\hat{a}i}^{\infty\pm\hat{a}i}\left|\frac{ize^{-z^2}}{iz^2+1}\right| dz = O(e^{\hat{a}^2 - 2\pi\hat{a}/h}), \tag{63}$$

where we have used the fact that, with $\mathrm{Im}\{z\} = \pm\hat{a}$ and $h \to 0$,

$$\begin{aligned} \left|1 - e^{\mp 2\pi i z/h}\right| &\geq \left|1 - \left|e^{\mp 2\pi i z/h}\right|\right| \\ &= \left|1 - \left|e^{2\pi \hat{a}/h}\right|\right| \\ &= e^{2\pi\hat{a}/h} - 1, \end{aligned} \tag{64}$$

and also used the second hint. Thus we arrive at

$$|I - I_h| = O(e^{\hat{a}^2 - 2\pi\hat{a}/h}) \quad \text{as } h \to 0. \tag{65}$$

$\boxed{2, \mathrm{M}}$

(ii) We want to choose $\hat{a}$ to maximise this rate. We have

$$\frac{d}{d\hat{a}}[\hat{a}^2 - 2\pi\hat{a}/h] = 0 \implies \hat{a} = \pi/h.$$

Evidently this gives a local minimum of $\hat{a}^2 - 2\pi\hat{a}/h$. Thus

$$|I - I_h| = O(\exp(-\pi^2/h^2)).$$

$\boxed{4, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70006 Applied Complex Analysis Markers Comments

Question 1	A fairly common error was to mistake the constant M for the maximum of the original function \phi(z) rather than the integrand of the integral with respect to \theta (part (d)).
Question 2	Perhaps the part of this question that required most care was accounting for and determining the residue at infinity of the integrand (part (a)).
Question 3	Determining the residues of the poles of \Gamma(z) (part (aii) and identifying a suitable trigonometric substitution (part (bi)) were the most problematic issues candidates had with this question.
Question 4	Most students chose to work with the uppermost possible strip of analyticity, namely that with 1 \Im(s) 2. Doing so, one ultimately ends up with a term in the final solution for f(x) that behaves like e^x as x tends to infinity. To remove this, and hence up with the required bounded (at infinity) solution for f(x), one must assume a value for f'(0). (The latter appears in the expansion of the right-sided transform F_+(s) as s tends to infinity. One can check that this value of f'(0) is that of the final, bounded solution one thus finds.) However, one can arrive at the solution for f(x) with less work if one chooses the strip of analyticity with 0 \Im(s) 1.
Question 5	Almost all students struggled with all parts of this question, with the exception of part (cii) and the determination of the integral with respect to r as part of part (b).