Imperial College
London

MATH97062 MATH97173

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Functional Analysis

Date: Monday, 17 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Prove that the space of polynomials on the unit interval $[0, 1]$ with the supremum norm is a linear metric space. (6 marks)

   (b) Prove or disprove that the space in (a) is :

      (i) separable; (5 marks)

      (ii) complete. (4 marks)

   (c) Prove or disprove that the space l_4 is a Hilbert space. (5 marks)

(Total: 20 marks)

2. (a) Let X be a proper closed subspace of the space l_p, $p \in (1, \infty)$. For $\tilde{p} > p$, consider the norm of $l_{\tilde{p}}$ and of l_p restricted to X and assume they are well defined on X. Prove that, if the space X is finite dimensional, then these norms are equivalent. By example show that if the space X is infinite dimensional, then the two norms on X are not necessarily equivalent. (7 marks)

   (b) Prove that a unit ball in the linear metric space dual to the space $(\mathcal{C}([0, 1]), \|\cdot\|_u)$ of continuous functions with supremum norm $\|\cdot\|_u$, is infinite dimensional and not separable. (6 marks)

   (c) Consider the following linear space

$$W \equiv \{(x_j \in \mathbb{R})_{j \in \mathbb{N}} : x_3 = 0; \qquad \sum_{j \in \mathbb{N}} |x_j|^2 < \infty\}$$

with a norm

$$\|x\| = \left(\sum_j |x_j|^2 \right)^{\frac{1}{2}}.$$

Let z be a bounded linear functional on this space. Find a Hahn-Banach extension of this functional to l_2 which has the same norm as original functional z on the space W. (7 marks)

(Total: 20 marks)

3. (a) Let $H_n(x)$, $n \in \mathbb{N}$, be Hermite polynomials associated to the measure

$$d\mu \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d\lambda.$$

For what values of $\xi \in \mathbb{R}$ the following operator is contractive in $\mathbb{L}_2(\mathbb{R}, \mu)$

$$T_n f(x) = H_k(x) + K_\xi f(x) \equiv H_k(x) + \xi H_n(x) \int H_n(y) f(y) \mu(dy) \qquad (*)$$

with $k \neq n$. (5 marks)

(b) Prove that for $\xi = 1$ the operator in (*) is not strictly contractive but nevertheless has a fixed point. (5 marks)

(c) Let $(p_j)_{j \in \mathbb{N}}$ be a sequence of distinct prime numbers. Prove that the operator defined as follows

$$Kf \equiv \sum_{n \in \mathbb{N}} \frac{1}{p_n} H_n(x) \int H_n(y) f(y) \mu(dy)$$

is compact in $\mathbb{L}_2(\mathbb{R}, \mu)$. (10 marks)

Hint: The k-th prime number asymptotically is of order

$$p_k \sim k \log(k).$$

(Total: 20 marks)

4. (a) Prove that the following subspace

$$V \equiv \{f \in \mathcal{C}([0,1]) : \exists c \in \mathbb{R}\ \forall x \in [1/3, 2/3]\ f(x) = c\}$$

is a closed linear subspace in the space of continuous functions $\mathcal{C}([0,1])$ with supremum norm $\|\cdot\|_u$. (5 marks)

(b) Prove or disprove that V is nowhere dense in $(\mathcal{C}([0,1]), \|\cdot\|_u)$. (6 marks)

(c) For $p \in (1, \infty)$ and a sequence $\rho \equiv (\rho_j \in (0, \infty))_{j \in \mathbb{N}}$ satisfying $\sum_{j \in \mathbb{N}} \rho_j = 1$, let

$$l_p(\rho) \equiv \{(x_n \in \mathbb{R})_{n \in \mathbb{N}} : \sum_j |x_j|^p \rho_j < \infty\}.$$

Prove that, if for some $(z_j \in \mathbb{R})_{j \in \mathbb{N}}$ we have

$$\forall x \in l_p(\rho) \qquad \sum_j z_j x_j \rho_j < \infty,$$

then there exists uncountably many $q \in (1, \infty)$ such that

$$\sum_j |z_j|^q \rho_j < \infty.$$

(9 marks)

(Total: 20 marks)

5. (a) Explain what does it mean that a Banach space $(X, \|\cdot\|_X)$ is compactly embeded into another Banach space $(Y, \|\cdot\|_Y)$

(4 marks)

(b) Define $W_{k,p}(\Omega)$ space and show that it is a separable Banach space.

(6 marks)

(c) Let $\Omega = [0,1]^d \subset \mathbb{R}^d$, $d \geq 3$, and for any $x \in \Omega$ define

$$Tf(x) \equiv \int_\Omega e^{-\frac{(y-x)^2}{2}} f(y)\lambda(dy)$$

for any $f : \Omega \to \mathbb{R}$ for which the right hand side is well defined. Using Sobolev inequality or otherwise, show that the following set

$$\{Tf : \|f\|_2 \leq 2\}$$

is compact in $\mathbb{L}_2(\Omega)$.

(10 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MMATH96037/MATH97062/MATH97173

Functional Analysis (Solutions)

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1. (a) In the space of polynomials we are given natural addition of vectors and multiplication by a number as for functions as follows. For

sim. seen $\Downarrow$

$$P^{(j)}(t) = \sum_{k\geq 0} \alpha_k^{(j)} t^k$$

with a convention that $\exists n \in \mathbb{N}\ \forall k > n \qquad \alpha_k^{(j)} = 0$, we have

$$\left(P^{(1)} + P^{(2)}\right)(t) = \sum_k (\alpha_k^{(1)} + \alpha_k^{(2)}) t^k$$

and

$$\left(\lambda \cdot P^{(1)}\right)(t) = \sum_k (\lambda \alpha_k^{(1)}) t^k$$

One needs to show that this operations are continuous. Since the metric

$$\rho(v, v') \equiv \|v \ominus v'\|$$

is given by the norm, the continuity is immediate by the following general argument

$$\|v \oplus w \ominus v' \oplus w'\| \leq \|v \ominus v'\| + \|w \ominus w'\|$$

for any vectors v, v', w, w', using commutativity of addition of vectors and Minkowski inequality. Similarly for multiplication of a vector by a number, we have

$$\begin{aligned}\|\lambda \odot v \ominus \lambda' \odot v'\| &= \|(\lambda - \lambda') \odot v \oplus \lambda' \odot (v \ominus v')\| \leq |\lambda - \lambda'| \cdot \|v\| + |\lambda'| \cdot \|v \ominus v'\| \\ &\leq \max\{\|v\|, |\lambda'|\} \left(|\lambda - \lambda'| + \|v \ominus v'\|\right)\end{aligned}$$

6, A

(b)

unseen $\Downarrow$

(i) The space is separable. This is because the countable set of monomials $X_n \equiv t^n$, $n \in \mathbb{Z}^+$ is the Hamel basis of the space, (by definition of polynomials), and we can approximate coefficients by rational numbers, that is for any $\varepsilon > 0$ and any polynomial $P(t) = \sum_{k=0,..,n} \alpha_k t^k$ and rationals $q_k \in \mathbb{Q}$, $k = 0, .., n$, we have

$$\| \sum_{k=0,..,n} \alpha_k t^k - \sum_{k=0,..,n} q_k t^k \| \leq \sum_{k=0,..,n} |\alpha_k - q_k|$$

and choosing $\max_{k=0,..,n} |\alpha_k - q_k| < \frac{1}{n}\varepsilon$ we get

$$\|P(t) - \sum_{k=0,..,n} q_k t^k\| < \varepsilon.$$

1, A

(ii) The space is not complete. For example a sequence

4, B

unseen $\Downarrow$

$$P^{(n)} \equiv \sum_{k=0}^{n} \frac{t^n}{2^n}$$

is Chauchy with respect to the sup norm on $[0, 1]$ and the sequence converges in the sup norm to a bounded continuous function $(1 - \frac{t}{2})^{-1}$.

4, A

(c) Since the norm of a Hilbert space by definition is given by a scalar product, it satisfies the following parallelogram identity unseen $\Downarrow$

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

For the l_4 norm, consider two different vectors which have coordinates all equal to zero but one which is equal to 1. Then we have

$$\|v+w\|^2 + \|v-w\|^2 = 2(2)^{\frac{1}{4}}$$

and

$$2\|v\|^2 + 2\|w\|^2 = 4.$$

Since these numbers are different, the parallelogram identity fails and so l_4 is not a Hilbert space. $\boxed{5, \mathrm{D}}$

2. (a) If the space X has dimension $n \in \mathbb{N}$, we can choose a basis $e_j,\ j = 1,..,n$ and for any vector v in the space we have a representation

sim. seen ⇓

$$v = \sum_{j=1,..,n} v_j e_j.$$

Then, we have

$$\|v\|_p \leq \max_{j=1,..,n} \|e_j\|_p \sum_{j=1,..,n} |v_j|$$

On the other hand, the function

$$(v_j)_{j=1,..,n} \to \| \sum_{j=1,..,n} v_j e_j \|$$

attains its minimum on the closed bounded set defined by the normalisation condition $\sum_{j=1,..,n} |v_j| = 1$. This yields

$$\|v\|_p \geq m_p \sum_{j=1,..,n} |v_j|.$$

with some $m_p \in (0, \infty)$.

Since similar property holds with the $\tilde{p}$ norms, we conclude that the norms are equivalent.

4, A

Next we give examples of infinite dimensional proper closed subspaces on which both norms are well defined, but not equivalent.

unseen ⇓

Let

$$V_j \equiv \{(z_k)_{k \in \mathbb{N}} \in l_p : z_j = 0\}$$

for any $j \in \mathbb{N}$. V_j are proper closed subspaces of l_p and infinite dimensional (with a canonical basis $(\delta_{k\cdot}), k \neq j$, respectively). It contains truncated sequences from the space $l_{\tilde{p}}$ of the form

$$(z_k \chi_L(k))_{k \in \mathbb{N}}$$

where $\chi_L(k) = 1$ for $k \leq L$ and zero otherwise. One can choose a sequence which is Cauchy with respect to $l_{\tilde{p}}$ norm for $\tilde{p} > p$, but not convergent with respect to l_p norm. Thus the norms are not equivalent.

3, C

(b) Using Hahn-Banach theorem one can show that the dual space of $\mathcal{C}([0,1])$ is given by Rieman-Stielties integrals

meth seen ⇓

$$\mathcal{C}([0,1]) \ni f \to \int f dF$$

associated to functions F of finite variation on $[0,1]$. Consider a set of functions $\chi_x \equiv \chi_{[x,1]}$, which for $x \in (0,1)$ are equal to zero on $[0,x)$ and one otherwise. This is an uncountable set of linearly independent functions with the property that total variation norm of a difference of any of two such functions is equal to one. This implies that the dual space cannot be separable.

3, A

3, B

(c) Since the subspace W is closed, it is a Hilbert space in itself. Then, by Riesz representation theorem there exists a unique vector $u_z \in W$ such that $\boxed{2, \text{A}}$

$$z(v) = \langle u_z, v\rangle$$

and

$$\|z\| = \|u_z\|.$$

By Hahn-Banach theorem, there exists a functional $\tilde{z}$ on l_2 such that its restriction to W coincides with z and its norm satisfies $\|\tilde{z}\| = \|z\|$. Again invoking Riesz representation theorem on l_2, we can find a unique vector $\tilde{u}_z \in l_2$ representing $\tilde{z}$ and such that $\|\tilde{u}_z\| = \|\tilde{z}\| = \|z\| = \|u_z\|$. Since by our assumption $\boxed{3, \text{B}}$

$$\langle u_z, w\rangle = \langle \tilde{u}_z, w\rangle$$

for all $w \in W$, the vector $\tilde{u}_z - u_z$ is orthogonal to W and in particular to u_z. By parallelogram identity we have

$$\|\tilde{u}_z + u_z\|^2 + \|\tilde{u}_z - u_z\|^2 = 2\|u_z\|^2 + 2\|\tilde{u}_z\|^2$$

and hence, using $\tilde{u}_z - u_z \perp u_z$, we get

$$\begin{aligned} \|\tilde{u}_z + u_z\|^2 + \|\tilde{u}_z - u_z\|^2 &= \|\tilde{u}_z - u_z + 2u_z\|^2 + \|\tilde{u}_z - u_z\|^2 \\ &= 2\|\tilde{u}_z - u_z\|^2 + 4\|u_z\|^2. \end{aligned}$$

This together with parallelogram identity implies

$$\|\tilde{u}_z - u_z\|^2 = \|\tilde{u}_z\| - \|u_z\| = 0.$$

Thus we conclude that

$$\tilde{u}_z = u_z.$$

$\boxed{2, \text{C}}$

3. (a) We have

unseen ⇓

$$\|T_n f(x) - T_n g\|^2 = \int |\xi|^2 H_n(x)^2 \left| \int H_n(y)\,(f(y) - g(y))\,\mu(dy) \right|^2 \mu(dx).$$

and hence

$$\|T_n f - T_n g\| = |\xi| \cdot \|f - g\|.$$

Thus the operator is strictly contractive for $|\xi| < 1$.

7, A

(b) To show that for $|\xi| = 1$ it is only weakly contractive (or to show that it is not contractive at all for $|\xi| > 1$), consider $f = H_n$ and $g = \lambda H_n$ with $\lambda \in \mathbb{R}$. Using orthogonality of Hermite polynomials we can show that

unseen ⇓

$$T_n(\alpha H_k + \beta H_n) = H_k + \beta \xi H_n.$$

Hence we get the following condition for a fixed point

$$\alpha = 1, \qquad \beta = \beta\xi.$$

If $|\xi| < 1$, the only solution is $\alpha = 1, \qquad \beta = 0$, while for $|\xi| = 1$ one gets infinitely solutions with $\alpha = 1, \qquad \beta \in \mathbb{R}$.

2, B

(c) For $n \in \mathbb{N}$ define

3, C

unseen ⇓

$$K_N f \equiv \sum_{n=1}^{n} \frac{1}{p_n} H_n(x) \int H_n(y) f(y) \mu(dy).$$

We have

$$\|K_N f\|^2 = \sum_{n=1}^{n} \frac{1}{p_n^2} \left| \int H_n(y) f(y) \mu(dy) \right|^2 \leq \sum_{n=1}^{n} \frac{1}{p_n^2} \|f\|^2$$

Thus K_N is bounded. This is a finite rank operator mapping l_2 into a finite dimensional subspace spanned by $H_n,\ n \leq N$. Since a bounded operator with finite dimensional range is compact, each K_n is a compact operator. We can show that the sequence $K_N,\ N \in \mathbb{N}$ of compact operators converges in Hilbert-Schmidt norm to K as follows. First of all we have

$$\|K_N - K\|_{H-S}^2 = \sum_{n \geq N+1} |\langle H_n, (K_N - K) H_n|^2$$

$$= \sum_{n \geq N+1} \frac{1}{p_n^2}.$$

It is known that the k-th prime number asymptotically is of order

$$p_k \sim k \log(k),$$

and from basic series analysis one knows that

$$\sum_{k>1} \frac{1}{k^2 (\log(k))^2} < \infty.$$

This implies that

$$\|K_N - K\|_{H-S}^2 \underset{N \to \infty}{\to} 0.$$

Since the Hilbert-Schmidt norm dominates the operator norm, so $K_N \to K$ in the operator norm as well. This together with the fact that K_N's are compact implies that K is a compact operator.

2, B

6, D

4. (a) By definition of the subspace

unseen ⇓

$$V \equiv \{f \in \mathcal{C}([0,1]) : \exists c \in \mathbb{R}\ \forall x \in [1/3,2/3]\ f(x) = c\}$$

if $f_n \in V,\ n \in \mathbb{N}$, converges to a function f in the supremum norm, then it converges pointwise uniformly on a compact interval $[0,1]$. Hence the limit $f(x) \equiv \lim_{n\to\infty} f_n(x)$ is well defined and it is a continuous function. Moreover for any $x \in [1/3,2/3]$ we have $f_n(x) = c_n$ and the sequence of numbers c_n, $n \in \mathbb{N}$, is convergent to a number $c \in \mathbb{R}$, the same for all $x \in [1/3,2/3]$. Hence $f \in V$ and so the subspace V is closed.

5, A

(b) We need to show that V does not contain any open ball of $(\mathcal{C}([0,1]), \|\cdot\|_u)$, i.e. that any set

unseen ⇓

$$B_{f,\varepsilon} \equiv \{g \in \mathcal{C}([0,1]) : \|g - f\| < \varepsilon\}$$

contains functions which do not belong to V. To this end consider a function

2, B

$$g(x) = \begin{cases} f(x), & \textit{if } x \in [0,1/3] \cup [2/3,1] \\ 3\varepsilon(x - 1/3) + f(1/3) & \textit{if } x \in [1/3,1/2] \\ 3\varepsilon(1/2 - x) + f(2/3) + \varepsilon/2 & \textit{if } x \in [1/3,1/2] \end{cases}$$

By our definition of the function g, we have

2, C

$$\|g-f\| = \sup_{x\in[0,1]} |g(x)-f(x)| = \sup_{x\in[1/3,2/3]} |g(x)-f(x)| = |g(1/2)-f(1/2)| = \varepsilon/2.$$

Thus $g \in B_{f,\varepsilon}$ and $g \notin V$. Since $\varepsilon \in (0,\infty)$ was arbitrary, we conclude that the closed set V does not contain any open ball and so its interior is empty.

2, B

(c) The proof is based on use of Banach-Steinhaus theorem: Let X and Y be a Banach space and a normed space, respectively. If a family of operators $T_\alpha : X \to Y$, $\alpha \in \mathbb{I}$, for some index set $\mathbb{I}$, satisfies

meth seen ⇓

$$\forall x \in X \exists c_x \in (0,\infty) \qquad \sup_{\alpha\in\mathbb{I}} \|T_\alpha x\|_Y \le c_x,$$

then

$$\sup_{\alpha\in\mathbb{I}} \|T_\alpha\| < \infty.$$

We assume that for some $(z_j \in \mathbb{R})_{j\in\mathbb{N}}$ we have

$$\forall x \in l_p(\rho) \qquad \sum_j z_j x_j \rho_j < \infty.$$

Consider a family of functionals $(f_n)_{n\in\mathbb{N}}$

$$f_n(x) \equiv \sum_{j=1}^n z_j x_j \rho_j.$$

Using Hölder inequality, with $\frac{1}{q} + \frac{1}{p} = 1$, we have

$$|f_n(x)| \le \left(\sum_{j=1}^n |z_j|^q \rho_j \right)^{\frac{1}{q}} \|x\|_{l_p(\rho)},$$

and thus all operators f_n are bounded from the Banach space $l_p(\rho)$ to $\mathbb{R}$ (which with the modulus $|\cdot|$ form a normed space). Moreover

$$\|f_n\| \leq \left(\sum_{j=1}^{n} |z_j|^q \rho_j\right)^{\frac{1}{q}}.$$

In fact one can choose x so that the equality is attained. By our assumption of convergence of the series $\boxed{2, \mathrm{C}}$

$$\sum_j z_j x_j \rho_j$$

we have

$$\forall x \in l_p \quad \exists c_x \in (0, \infty) \qquad \sup_n |f_n(x)| < c_x$$

Hence by Banach-Steinhaus theorem

$$\sup_n \sup_{\|x\|_{l_p(\rho)}} |f_n(x)| < \infty.$$

That is using our expression for the norm $\|f_n\|$ we have $\boxed{5, \mathrm{D}}$

$$\sup_n \left(\sum_{j=1}^{n} |z_j|^q \rho_j\right)^{\frac{1}{q}} < \infty,$$

which implies that $z \in l_q(\rho)$. Finally since ρ is normalised, we can use Hölder inequality

$$\|z\|_{l_{\tilde{q}}(\rho)} \leq \|z\|_{l_q(\rho)}$$

for all $\tilde{q} \in [1, q]$. $\boxed{2, \mathrm{B}}$

5. (a) Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces such that $X \subset Y$ and seen $\Downarrow$

$$\exists C \in (0,\infty) \quad \forall x \in X \qquad \|x\|_Y \le C\|x\|_X$$

4, M

the identity map from X to Y is a compact operator (i.e. maps bounded sets in X to sets in Y which after closure are compact). Then we say that the Banach space $(X, \|\cdot\|_X)$ is compactly embeded into the Banach space $(Y, \|\cdot\|_Y)$.

seen $\Downarrow$

(b) For a nonnegative integer $m \in \mathbb{Z}^+$ and $p \in [1,\infty]$, and for an open set $\Omega \subset \mathbb{R}^N$, the space $W_{m,p}(\Omega)$ is by definition the space of (equivalence classes of) functions $f \in \mathbb{L}_p(\Omega)$ such that $\nabla^\alpha f \in \mathbb{L}_p(\Omega)$ for all derivations ∇^α of length $|\alpha| \le m$, that is for any $\varphi \in \mathcal{C}^\infty$ compactly supported in Ω we have 6, M

$$\int f \nabla^\alpha \varphi d\lambda = (-1)^{|\alpha|} \int g\varphi d\lambda$$

and $g \in \mathbb{L}_p(\Omega)$. It is a normed space equipped with the norm

$$\|f\| = \sum_{|\alpha| \le m} \|\nabla^\alpha f\|_{\mathbb{L}_p(\Omega)}$$

or equivalent norm, for $p \in [1,\infty)$

$$\|f\|_{m,p} = \left(\sum_{|\alpha| \le m} \int_\Omega |\nabla^\alpha u| d\lambda \right)^{\frac{1}{p}}$$

while for $p = \infty$

$$\|u\|_{m,p} = \max_{|\alpha| \le m} \|\nabla^\alpha u\|_{\mathbb{L}_\infty(\Omega)}$$

Using similar arguments as in case of $\mathbb{L}_p(\Omega)$ based on absolutely convergent series, one can show that $W_{m,p}(\Omega)$ are Banach spaces. To prove the separability it is enough to observe that the set of m times continuously differentiable functions with bounded derivatives $\mathcal{C}_b^m(\Omega)$ is dense in $W_{m,p}(\Omega)$ and use the fact proved in the course that $\mathcal{C}_b^m(\Omega)$ is separable.

unseen $\Downarrow$

(c) First of all we note that

$$\nabla Tf(x) \equiv \int_\Omega (x-y) e^{-\frac{(y-x)^2}{2}} f(y)\lambda(dy)$$

and hence for any $r, s \in (0,\infty)$

$$\begin{aligned}\|\nabla Tf\|_p &= \left(\int_\Omega \left| \int_\Omega (x-y) e^{-\frac{(y-x)^2}{2}} f(y)\lambda(dy) \right|^p \lambda(dx) \right)^{\frac{1}{p}} \\ &\le |\Omega| \| |w| e^{-\frac{(w)^2}{2}} \|_s \cdot \|f\|_r\end{aligned}$$

Hence for $f \in \mathbb{L}_r$, we have $Tf \in W_{1,p}$, for any $p \le r$ Using this, in particular with $2 \le r$, together with the Relich-Kodrachov compact embedding theorem we conclude that

$$\{Tf : \|f\|_2 \le 2\}$$

is compact in $\mathbb{L}_2(\Omega)$. 10, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96037 Functional Analysis	1	1st question was well attended with many full scores.
	2	Generaly good level
	3	Generaly good level
	4	Generaly good level
MATH97062/MATH97173 Functional Analysis	5	For the 4th year the 5th question was not that well done in particular its last part (which was the hardest one)."