Mock Exam 2: Question 4

April 26, 2022

Consider the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{m} \|\mathbf{A}_i\mathbf{x}+\mathbf{b}_i\|$$

where $\mathbf{A}_i \in \mathbb{R}^{k_i\times n}, \mathbf{b}_i \in \mathbb{R}^{k_i}, i = 1, 2, \ldots, m$.

i Using auxiliary variables $\mathbf{y}_i = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i$, write an equivalent constrained optimization problem, derive its Lagrangian, and the dual objective function.

ii Derive an explicit formulation of the dual problem.

Solutions

i) Using the suggested auxiliary variables, the equivalent formulation reads

$$\begin{array}{ll} \min_{\mathbf{x},\mathbf{y}_i} & \sum_{i=1}^{m} \|\mathbf{y}_i\| \\ \text{s.t.} & \mathbf{y}_i = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \quad i = 1, 2, \ldots, m \end{array}$$

Associating a Lagrange multiplier vector $\lambda_i \in \mathbb{R}^{k_i}$ with the i th set of constraints $\mathbf{y}_i = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i$, we obtain the following Lagrangian:

$$\begin{aligned} L(\mathbf{x}, \lambda_1, \lambda_2, \ldots, \lambda_m) &= \sum_{i=1}^{m} \|\mathbf{y}_i\| + \sum_{i=1}^{m} \lambda_i^\top (\mathbf{y}_i - \mathbf{A}_i\mathbf{x} - \mathbf{b}_i) \\ &= \sum_{i=1}^{m} \left[\|\mathbf{y}_i\| + \lambda_i^\top \mathbf{y}_i\right] - \left(\sum_{i=1}^{m} \mathbf{A}_i^\top \lambda_i\right)^\top \mathbf{x} - \sum_{i=1}^{m} \mathbf{b}_i^\top \lambda_i \end{aligned}$$

By the separability of the Lagrangian with respect to $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$, it follows that the dual objective function is given by

$$q(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{i=1}^{m} \min_{\mathbf{y}_i\in\mathbb{R}^{k_i}} \left[\|y_i\| + \lambda_i^\top \mathbf{y}_i\right] + \min_{\mathbf{x}\in\mathbb{R}^n} \left[-\left(\sum_{i=1}^{m} \mathbf{A}_i^\top \lambda_i\right)^\top \mathbf{x}\right] - \sum_{i=1}^{m} \mathbf{b}_i^\top \lambda_i \,.$$

ii) Observing the terms in the dual objective, it is clear that

$$\min_{x\in\mathbb{R}^n} \left[-\left(\sum_{i=1}^{m} \mathbf{A}_i^\top \lambda_i\right)^\top \mathbf{x}\right] = \begin{cases} 0, & \sum_{i=1}^{m} \mathbf{A}_i^\top \lambda_i = 0, \\ -\infty & \text{else.} \end{cases}$$

In addition, for any $i = 1, \ldots, m$ we have

$$\min_{y_i \in \mathbb{R}^{k_i}} \left[\| y_i \| + \lambda_i^\top \mathbf{y}_i \right] = \begin{cases} 0, & \| \lambda_i \| \leq 1, \\ -\infty & \text{else} \end{cases} .$$

Combining these expressions, the dual objective function is

$$q\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right) = \begin{cases} -\sum_{i=1}^m \lambda_i^\top \mathbf{b}_i, & \| \lambda_i \| \leq 1, i = 1, 2, \ldots, m, \sum_{i=1}^m \mathbf{A}_i^\top \lambda_i = 0, \\ -\infty & \text{else.} \end{cases}$$

The dual problem is therefore

$$\begin{array}{ll} \max & -\sum_{i=1}^m \mathbf{b}_i^\top \lambda_i \\ \text{s.t.} & \sum_{i=1}^m \mathbf{A}_i^\top \lambda_i = 0, \\ & \| \lambda_i \| \leq 1, \quad i = 1, 2, \ldots, m \end{array}$$