Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Problem Sheet 2

1. ✦ Newton's second law tells us that the displacement $x(t)$ of the mass in a mass-spring-dashpot system satisfies the following ODE

$$m\frac{d^2x}{dt^2} = F_s + F_d$$

where m is the mass, F_s is the restoring force in the spring and F_d is the damping force from the dashpot. We can complete this equation with the following initial conditions

$$x(0) = 0, \quad \frac{dx}{dt}(0) = v_0$$

(a) Consider the case where there is no damping, $F_d = 0$, and where the spring is linear, $F_s = -kx$. What are the dimensions of the spring constant k? Nondimensionalize the resulting initial value problem. Your choice of x_c and t_c should result in no dimensionless products being left in the IVP.

(b) Now, in addition to a linear spring, suppose that we also have linear damping, i.e.

$$F_d = -c\frac{dx}{dt}$$

What are the dimensions for the damping constant c? Using the same scaling as in part (a), nondimensionalize the IVP. Your answer will normally contain a dimensionless parameter ε that measures the strength of the damping. In which limit does the system have weak damping?

2. ✦✦ When the end of a long and thin strip of paper is put into a cup of water, the water rises up into the paper due to capillarity. It can be shown that the density ρ of water along the strip satisfies a conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

where J is the flux function.

(a) What are the dimensions of J?

(b) The flux J depends on the gravitational constant g, the strip width d, the density gradient $\frac{\partial \rho}{\partial x}$, and the surface tension σ of the water. Find a dimensionally reduced form for J.

(c) What does your result in (b) reduce to if it is found that J depends linearly on the density gradient, with $J = 0$ if $\frac{\partial \rho}{\partial x} = 0$. What is the resulting differential equation?

(d) If the strip has a length h the boundary conditions are given by $\rho = \rho_0$ at $x = 0$, $J = 0$ at $x = h$. The initial condition is $\rho = 0$ at $t = 0$. With this information, and the differential equation from (c), nondimensionalize the problem for ρ in such a way that no dimensional groups appear in the final answer.

3. ✦ Consider the following first-order linear homogeneous PDE

$$\frac{\partial u}{\partial t} + e^{x+t}\frac{\partial u}{\partial x} = 0$$

Solve this equation for the following initial conditions $u(x, 0) = \phi(x)$. What if $\phi(x) = x^3$?

4. ✦ Using the method of characteristics, solve the following partial differential equations

$$\text{(a)}\quad x\frac{\partial u}{\partial t} + t^2\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = x^2, \text{ when } t = 0$$
$$\text{(b)}\quad (1+t)\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = x^5, \text{ when } t = 0$$
$$\text{(c)}\quad \cos x\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u = t^4, \text{ when } x = 0$$

5. ✦✦ Solve the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - x^2\frac{\partial u}{\partial x} = 0, \quad x > 0,\ t > 0,\\
&u(x,0) = e^{-x}, \quad x > 0\\
&u(0,t) = 1, \quad t > 0
\end{aligned}$$

6. ✦✦ Consider the following PDE

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x^2 + y^2$$

Solve this equation subject to the boundary conditions

$$u(x,y) = \begin{cases} 1 + x^2 & \text{on } y = 0 \\ 1 + y^2 & \text{on } x = 0 \end{cases}$$

7. ✦✦✦ In this problem, we consider once more time the example of a pollutant transported along a thin and long water channel. The water stream moves with speed v. We consider that diffusion is negligible in this problem. Suppose that due to biological decomposition, the pollutant decays at a rate proportional to the pollutant density.

   (a) Write the equation governing the density $u(x,t)$ of pollutant in the channel?

   (b) If the initial conditions are given by $u(x,0) = f(x)$, find the solution to this problem. This is called a *damped travelling wave*. Explain briefly what this means.

8. ✦✦✦ In this problem, we want to derive partial differential equations modelling the flow of a fluid in 1D. Consider a one-dimensional flow in a pipe of cross-sectional area A (constant), and a fixed slice of fluid in between $x = a$ and $x = b$ (where $b > a$). We denote the fluid velocity $u(x,t)$ and its density $\rho(x,t)$. The quantity $\rho(x,t)u(x,t)$ is called the momentum density. We will assume that viscous forces in the fluid can be neglected.

   (a) What are the dimensions of the momentum density?

   (b) What is the net gain in momentum of the slice?

   (c) Assuming that the only forces acting on the slice are pressure forces at the ends, deduce that

$$\frac{d}{dt}\int_a^b \rho(x,t)u(x,t)dx = (\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2) + (p(a,t) - p(b,t))$$

   where $p = p(x,t)$ is the fluid pressure.

   (d) Show that

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = -\frac{\partial p}{\partial x}$$

(e) Finally, use the conservation of mass to show that the above equation simplifies to

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = - \frac{\partial p}{\partial x}$$

This is the celebrated Euler equation for a 1D flow.

(f) Now suppose that mass is being created within the fluid such that, in the absence of fluid motion, the change in mass of a slice of fluid δx is

$$\delta m = r(x,t) A \delta x \delta t$$

in a time interval δt. Explain how the conservation laws for mass and momentum used above need to be modified to account for this mass creation.