Imperial College London

MATH97007 MATH97087

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics 2

Date: 26th May 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. In the arcade game "Air Hockey", a disc hovers above a table through which air is pumped vertically. Investigate the equilibrium shown in the figure, where a circular disc of radius a rests a constant height h above the table. Except for between the disc and the table, the pressure can be assumed to be constant, $p = p_0$.

Assume an axisymmetric flow $(u(r,z),\, 0,\, w(r,z))$ in terms of cylindrical polar coordinates $(r,\, \theta,\, z)$. On the table $(z = 0)$ the radial velocity $u = 0$ while the vertical velocity $w = W_0$. The solid disc at $z = h$ is stationary, so that $u = w = 0$.

(a) The lubrication equations for the flow between disc and table are

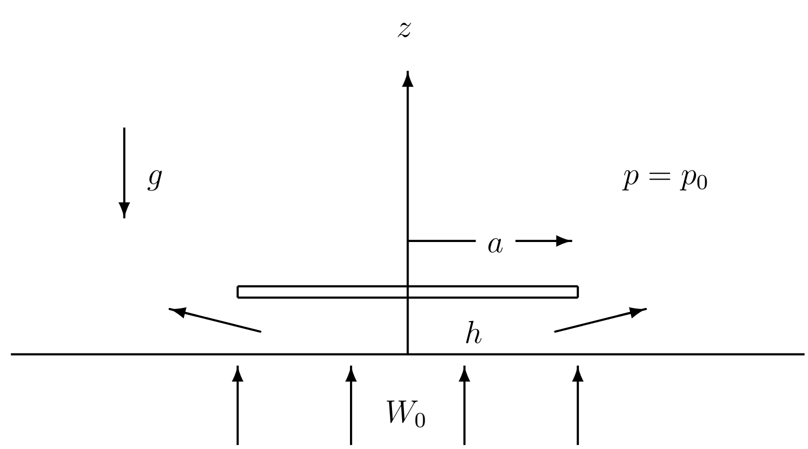
$$p_z = 0, \qquad p_r = \mu u_{zz}, \qquad (ru)_r + r w_z = 0.$$

Explain the assumptions leading to these equations. (5 marks)

(b) Find u and w in terms of the pressure field and deduce an equation for the pressure. Assuming that at $r = a$ the pressure is atmospheric find the pressure under the disc. (10 marks)

(c) Balancing the weight Mg of the disc with the net pressure force (you may neglect the viscous stress), deduce that

$$Mg = \frac{3\pi a^4 \mu W_0}{2h^3}. \qquad \text{(5 marks)}$$



(Total: 20 marks)

2. A flagellum is modelled as a solid infinite cylinder of radius a in a 2-D Stokes flow. It is placed into a simple straining flow, with velocity field

$$u = Ex, \qquad v = -Ey \qquad \text{for } x^2 + y^2 \gg a^2,$$

where E is a given constant and (u, v) are the velocity components with respect to Cartesian coordinates (x, y).

(a) Find the streamfunction $\psi_\infty(r, \theta)$ which represents the above straining motion, where r and θ are polar coordinates. (2 marks)

(b) Now seek the streamfunction, $\psi(r, \theta)$ in $r \geqslant a$, for the flow around the cylinder at $r = a$, such that $\psi \to \psi_\infty$ as $r \to \infty$.

Show that in general the 2-D Stokes equations imply that

$$\nabla^2 \omega = 0 \qquad \text{where} \qquad \omega = -\nabla^2 \psi. \qquad \text{(2 marks)}$$

Assume a separable solution $\psi = f(r)h(\theta)$ and $\omega = g(r)h(\theta)$ where $h(\theta)$ is suggested by the form of ψ_∞. Obtain ODEs for $f(r)$ and $g(r)$. (4 marks)

(c) Write down the boundary conditions and solve the problem for ψ. (8 marks)

What is the maximum value of the vorticity and where does it occur? (4 marks)

[Note: In polar coordinates the Laplacian takes the form

$$\nabla^2 X = \frac{1}{r^2}\frac{\partial^2 X}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial X}{\partial r}\right)$$

where X is any quantity.]

(Total: 20 marks)

3. In terms of cylindrical polar coordinates $(r,\, \theta,\, z)$, the unsteady vorticity equation for inviscid, two-dimensional flow $\mathbf{u} = (u,\, v,\, 0)$ takes the form

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0$$

where

$$(0,\, 0,\, \omega) = \nabla \times \mathbf{u} = \left(0,\, 0,\, \frac{1}{r}\frac{\partial (rv)}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right).$$

(a) Show that steady circular flow of the form $\mathbf{u} = (0,\, V(r),\, 0)$ satisfies the inviscid vorticity equation for any function $V(r)$. (3 marks)

(b) Circular flow takes place between the two rigid cylinders $r = R_1$ and $r = R_2$, where $R_1 < R_2$. Investigate its stability to 2-D perturbations by expanding

$$\mathbf{u} = (0,\, V(r),\, 0) + \varepsilon[u_1(r),\, v_1(r),\, 0]e^{im(\theta - \Omega t)}$$

where $0 < \varepsilon \ll 1$ and Ω is an unknown complex eigenvalue. The boundary conditions require $u_1 = 0$ on $r = R_1, R_2$.

Introducing a streamfunction, $\psi(r)$ so that $u_1 = im\psi/r$ and $v_1 = -\psi'$, express the vorticity in the form

$$\omega = Q_0(r) + \varepsilon Q_1(r)e^{im(\theta - \Omega t)}$$

and show that, neglecting terms of $O(\varepsilon^2)$,

$$\nabla^2 \psi - \frac{dQ_0}{dr}\left(\frac{\psi}{V - r\Omega}\right) = 0 \qquad \text{where} \quad \nabla^2 \psi = \frac{1}{r}(r\psi')' - \frac{m^2\psi}{r^2}.$$

(7 marks)

(c) Assuming dQ_0/dr is continuous, prove that a necessary condition for instability is that $dQ_0/dr = 0$ somewhere in the range $R_1 < r < R_2$. (10 marks)

[You may find it helpful to multiply by $r\psi^*$ where $*$ denotes the complex conjugate, as in the proof of Rayleigh's inflection point theorem.]

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(Total: 20 marks)

4. (a) Summarise the steps in the arguments leading to the boundary layer equation

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = UU'(x) + \nu \psi_{yyy}$$

starting from the steady, incompressible, Navier-Stokes equations. (6 marks)

(b) Just outside the boundary layer on a solid plate at $y = 0$, the x-component of velocity is $U = U_0 e^{cx}$ where c is a positive constant. Seek a similarity solution in the boundary layer of the form $\psi = Ae^{ax} f(\eta)$ where $\eta = Bye^{bx}$ for suitable constants A, B, a and b. Determine a and b, and choosing $A/B = \nu/c$ obtain a third order ODE for $f(\eta)$. State the appropriate boundary conditions. (12 marks)

(c) The Falkner-Skan equation, for flows at infinity $\propto x^m$, can be written in the form

$$\phi''' + \tfrac{1}{2}\phi\phi'' + \frac{m}{m+1}(1 - \phi'^2) = 0 \qquad \text{with} \quad \phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1.$$

Comment on any relation between this and your equation. (2 marks)

(Total: 20 marks)

5. Give an account of the significance of the Reynolds number, R_e, in Fluid Dynamics. In your overview you should describe the various differences in behaviour of flows at high and at low R_e. Where appropriate, illustrate each phenomenon you mention with examples, proofs or practical implications. (20 marks)

(Total: 20 marks)

Solutions to Fluid Dynamics II (CHECKED)

Seen/Unseen Guide:

1(a) SEEN, rest UNSEEN.

2(b) 1st part SEEN, rest UNSEEN (But have met flow past sphere and the insolubility of uniform flow past a cylinder)

3(a) SEEN, rest UNSEEN, but they have seen the similar proof of Rayleigh's Inflection Point Theorem.

4(a) SEEN, rest UNSEEN, but they have seen Falkner-Skan and other boundary layer similarity solutions.

5 All SEEN presumably, though if they can add things I haven't lectured if they want!

Q1(a) Lubrication theory assumes a thin gap $h \ll a$ so that derivatives in the z-direction are much greater than those in the r-direction, so that $\nabla^2\mathbf{u} \simeq \mathbf{u}_{zz}$. **[1]**

It follows from the incompressibility condition that the r-component velocity is largest, $u \sim aw/h \gg w$. **[1]**

Further, it is assumed that on the small-length scale the viscous terms dominate over inertia, so that $\nabla p \sim \mu\mathbf{u}_{zz}$. **[1]**

This gives a pressure $p \sim \mu ua/h^2$. **[1]**

As $p_r \ll p_z$ while $u_{zz} \gg w_{zz}$, a balance requires $p_z = 0$ and $p_r = \mu u_{zz}$ **[1]**

(b) Integrating the radial momentum equation, noting that $p = p(r)$, we have

$$u = \frac{p_r}{2\mu}(z^2 + Az + B) = \frac{p_r}{2\mu}(z^2 - zh)$$

imposing that $u = 0$ on $z = 0$ and $z = h$. **[2]**

Integrating the incompressibility condition with respect to z gives

$$w = -\frac{1}{2\mu r}(rp_r)_r\left(\tfrac{1}{3}z^3 - \tfrac{1}{2}z^2h + \tfrac{1}{6}h^3\right)$$

imposing that $w = 0$ on $z = h$. [Alternatively, could impose $w = W_0$ on $z = 0$ first.] Evaluating this on $z = 0$ where we know $w = W_0$ gives

$$\frac{12\mu W_0}{h^3} = -\frac{1}{r}(rp_r)_r. \qquad [\mathbf{4}]$$

Integrating with respect to r, we have

$$\frac{6\mu W_0}{h^3}r^2 + C = -rp_r.$$

At $r = 0$ we must have p_r finite, so that $C = 0$. Integrating again,

$$\frac{3\mu W_0}{h^3}r^2 + D = -p \qquad \Longrightarrow \qquad p = p_0 + \frac{3\mu W_0}{h^3}(a^2 - r^2) \qquad [\mathbf{4}]$$

imposing the pressure is atmospheric at $r = a$.

(c) The total pressure force acting on the disc is the integral of the pressure below, minus the integral of atmospheric pressure above, i.e.

$$\int_{\text{disc}} (p - p_0)r\,dr\,d\theta = \frac{3\mu W_0}{h^3}2\pi\int_0^a r(a^2 - r^2)dr = \frac{3\pi\mu W_0 a^4}{2h^3} = Mg, \qquad [\mathbf{5}]$$

as required. (Note the viscous stress is negligible – this observation is not required.)

Q2(a) If $\psi_y = u = Ex$ we have $\psi = Exy + f(x)$. Also $-\psi_x = v = -Ey$ we require $f(x)$ is a constant or zero w.l.o.g. Translating into polars $x = r\cos\theta$, $y = r\sin\theta$, so we conclude $\psi_\infty = Er^2 \sin\theta\cos\theta = \frac{1}{2}Er^2 \sin 2\theta$. **[2]**

(b) In 2-D we have $\mathbf{u} = \nabla \times (0,\, 0,\, \psi)$ and the vorticity $(0, 0, \omega) = \nabla \times \mathbf{u} = (0,\, 0,\, -\nabla^2\psi)$. Now taking the curl of the Stokes equation $\nabla p = \mu\nabla^2\mathbf{u}$ gives $\nabla^2(\nabla \times \mathbf{u}) = 0$ or $\nabla^2\omega = 0$ we conclude

$$\nabla^2\omega = 0, \qquad \omega = -\nabla^2\psi$$ **[2]**

Try a separable solution $\psi = f(r)\sin 2\theta$. Then

$$\omega = -\nabla^2\psi = \left(-f'' - \frac{1}{r}f' + \frac{4}{r^2}f\right)\sin 2\theta = g(r)\sin 2\theta,$$

say. Then our equations are

$$g'' + \frac{1}{r}g' - \frac{4}{r^2}g = 0 \qquad \text{and} \quad f'' + \frac{1}{r}f' - \frac{4}{r^2}f = -g.$$ **[4]**

(c) The g-equation is homogeneous in r, so has solutions $\propto r^n$. We require $n(n-1)+n-4 = 0$ or $n = \pm 2$. We therefore have

$$f'' + \frac{1}{r}f' - \frac{4}{r^2}f = -g = \widehat{A}r^2 + \widehat{B}r^{-2}$$

By inspection, the particular integrals are r^4 and a constant (r^0) while the complementary function is as before. So we have

$$f(r) = Cr^2 + Dr^{-2} + B + Ar^4.$$

The boundary conditions we want to impose are $f \sim \frac{1}{2}Er^2$ as $r \to \infty$, and $f(a) = f'(a) = 0$ on the solid cylinder. The far field requires $A = 0$ and $C = \frac{1}{2}E$. On $r = a$ we require $Ca^2 + D/a^2 + B = 0$ and $2Ca - 2D/a^3 = 0$. Thus $D = \frac{1}{2}Ea^4$ and $B = -Ea^2$ We conclude

$$\psi = \tfrac{1}{2}E\left(r^2 + \frac{a^4}{r^2} - 2a^2\right)\sin 2\theta = E\left(r - \frac{a}{r}\right)^2 \sin\theta\cos\theta$$ **[8]**

Only the r^0 term contributes to the vorticity, which is $\omega = -4Ea^2\sin 2\theta/r^2$. This is maximum when r is minimum and $\sin 2\theta = -1$, thus when $r = a$ and $\theta = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$. The maximum value is $\omega = 4E$. **[4]**

Q3(a) For the circular flow, $\mathbf{u} \cdot \nabla = (V/r)\frac{\partial}{\partial\theta}$. It has vorticity $(0,\, 0,\, Q_0(r))$ where $Q_0 = (1/r)(rV)'$. This is independent of t and θ. It follows that $\omega_t + \mathbf{u}\cdot\nabla\omega = 0$ for any function $V(r)$ as required. **[3]**

(b) For the perturbed flow, the vorticity is

$$\omega = Q_0(r) + \frac{\varepsilon}{r}((rv_1)' - imu_1)e^{im(\theta-\Omega t)} = Q_0(r) + \frac{\varepsilon}{r}\left(-(r\psi')' - \frac{(im)^2}{r\psi}\right)e^{im(\theta-\Omega t)}.$$

Or

$$\omega = Q_0 + \varepsilon Q_1 e^{im(\theta-\Omega t)} \qquad \text{where} \quad Q_1 = -\left(\psi'' + \frac{1}{r}\psi' - \frac{m^2}{r^2}\psi\right) = -\nabla^2\psi.$$

Substituting in the vorticity equation and neglecting terms of $O(\varepsilon^2)$, we get

$$\frac{\partial}{\partial t}\left(Q_1 e^{im(\theta-\Omega t)}\right) + \frac{V}{r}\frac{\partial}{\partial\theta}\left(Q_1 e^{im(\theta-\Omega t)}\right) + u_1 e^{im(\theta-\Omega t)}\frac{\partial Q_0}{\partial r} = 0.$$

That is

$$im\Omega\nabla^2\psi + \frac{V}{r}im(-\nabla^2\psi) + \frac{im\psi}{r}Q_0' = 0.$$

Or

$$(V - r\Omega)\,\nabla^2\psi - \frac{dQ_0}{dr}\psi = 0. \qquad \textbf{[7]}$$

(c) For instability we need $\Im m(\Omega) > 0$. In that case $V - r\Omega \neq 0$ for any r and we divide by it. Multiplying by $r\psi^*$ where * denotes the complex conjugate and integrating we have

$$\int_{R_1}^{R_2} \psi^*\left[(r\psi')' - \frac{m^2\psi}{r}\right]dr = \int_{R_1}^{R_2} \frac{dQ_0/dr}{V - r\Omega}|\psi|^2 r dr.$$

Integrating by parts and rendering the denominator real, we have

$$\left[\psi^*(r\psi')\right]_{R_1}^{R_2} + \int_{R_1}^{R_2}\left(-r|\psi'|^2 - \frac{m^2}{r}|\psi|^2\right)dr = \int_{R_1}^{R_2}\frac{V - r\Omega^*}{|V - r\Omega|^2}\frac{dQ_0}{dr}|\psi|^2\, r dr$$

Using the boundary conditions, ψ^* vanishes at the end points. The LHS is real and so taking the imaginary part of the equation we obtain

$$0 = \Im m(\Omega)\int_{R_1}^{R_2}\frac{r}{|V - r\Omega|^2}\frac{dQ_0}{dr}|\psi|^2\, r dr.$$

For instability, $\Im m(\Omega) \neq 0$ and so the integral must vanish. The integrand must therefore sometimes be positive and sometimes negative. dQ_0/dr must therefore change sign and by continuity it must therefore be zero somewhere. We conclude a necessary condition for 2-D instability is that $dQ_0/dr = 0$ somewhere in the range. **[10]**

Q4(a) We consider a solid boundary in a high Reynolds number flow. We assume there is a thin layer of thickness δ in which viscous and inertial forces balance, but outside which viscosity is negligible. To leading order we can treat the boundary as flat and use Cartesian coordinates (x, y) tangentially and normally respectively. y-derivatives will have typical magnitudes $1/\delta$ whereas x-derivatives are $O(1/L)$, say. The core flow will in general have a tangential slip velocity $U(x)$, which we take as typical of the tangential x-velocity, u. By incompressibility, the normal velocity in the layer will have a typical magnitude $v \sim U\delta/L$. The viscous terms balance inertial in the tangential momentum equation provided

$$\rho U^2/L \sim \mu U/\delta^2 \sim p/L \qquad \Longrightarrow \delta \sim (\nu L/U)^{1/2}.$$

This gives the layer thickness δ and the pressure scale. Using these scales for the normal momentum equation, one finds that nothing can balance the pressure gradient, so that $p_y = 0$. The other equations then take the form

$$u_x + v_y = 0, \qquad uu_x + vu_y = G(x) + \nu u_{yy}.$$

As the pressure gradient does not vary with y, we can use its value outside the layer, where $u \sim U(x)$ and $v = O(\delta)$, i.e. $G = UU'$. Finally, we can represent the incompressible 2-D velocity (u, v) with a streamfunction, ψ to obtain the given equation. **[6]**

(b) We have $\eta_y = Be^{bx}$ and $\eta_x = b\eta$. Thus

$$\psi_y = ABe^{(a+b)x}f', \qquad \psi_x = Ae^{ax}(af + b\eta f'), \qquad \psi_{xy} = ABe^{(a+b)x}(af + b\eta f')'$$

As $y \to \infty$ we have $\eta \to \infty$ and for $\psi_y \to U_0 e^{cx}$ we require

$$a + b = c \qquad \text{and} \quad ABf'(\infty) = U_0.$$

Now the LHS $\psi_y \psi_{xy} - \psi_x \psi_{yy}$ takes the form

$$ABe^{(a+b)x}f' ABe^{(a+b)x}[(a+b)f' + b\eta f''] - Ae^{ax}(af + b\eta f')AB^2e^{(a+2b)x}f''$$
$$= A^2B^2e^{(2a+2b)x}((a+b)f'^2 - aff'')$$

and so we have

$$A^2B^2e^{(2a+2b)x}((a+b)f'^2 - aff'') = cU_0^2e^{2cx} + \nu AB^3e^{(a+3b)x}f'''$$

For consistency we require $2a + 2b = 2c = a + 3b$, or

$$a = b = \tfrac{1}{2}c. \qquad \textbf{[4]}$$

We also choose $AB = U_0$, $A/B = \nu/c$ to give

$$f'^2 - \tfrac{1}{2}ff'' = 1 + f'''. \qquad \textbf{[7]}$$

The boundary conditions are then

$$f(0) = 0 = f'(0), \qquad f'(\infty) = 1. \qquad \textbf{[1]}$$

[Note: choosing different values of A and B will lead to slightly different ODE and boundary conditions. In that case, a transformation may be necessary to obtain the Falkner-Skan equation.]

By inspection, this ODE is the Falkner-Skan equation in the limit $m \to \infty$. This is not unreasonable, as the exponential function increases faster than any power of x. **[2]**

Q5: This is an essay question, designed to test the students' overall view of fluid dynamics. They have considerable freedom as to which topics to concentrate upon. I would expect some combination of the following:

Linearity: At low R_e forces (e.g. drag) are proportional to velocity, whereas high-R_e drag is quadratic. Flows can be superposed. For example, a cube can move in any orientation. Spatial reversibility $u \to -u$.

Energy: Viscosity is a dissipative process, but high-R_e flows may conserve energy. This could be linked, paradoxically, to the minimum dissipation theorem of low-R_e flows.

Time Reversibility: The experiments with blobs of dye returning to their original position (or not). Implications for low-R_e swimming.

Uniqueness: Low-R_e flows are (provabily) unique, high-R_e are not.

Stability: High-R_e flows may well be unstable. Concept of critical R_e. Importance of small but finite perturbations at high-R_e. Significance of multi-stable states.

Singular Inviscid Limit: Formation of thin layers as $R_e \to \infty$.

Flight: As this has not been tested on the paper, students who have prepared the subject in detail may choose to unburden their knowledge here. Arguably, the generation of lift is a high-R_e phenomenon.

Turbulence: A high-R_e phenomenon, which I have only touched upon in the course. Could link in to stability and drag considerations.

Well-posedness: The regularity of the Navier-Stokes at high-R_e is an unsolved problem. Could link to various paradoxes, e.g. low-R_e flow past a cylinder.

Drag: Links to several of the topics above.

Lubrication theory: Importance of viscosity in low-R_e regions even at high R_e.

These are just some of the possibilities. Anything relevant will be permitted. **[20]**

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionI	Comments for Students	
MATH97007 MATH97087	1	This was question was found harder than I anticipated, but the MSc/MSci did better than the BSc students. Surprisingly, few students reproduced the derivation of the lubrication equations well, especially the justification of why the pressure does not vary with z. There was also uncertainty about the area integral of the pressure over the disc - r dr d(theta)	
MATH97007 MATH97087	2	With hindsight, I should have specified the function h(theta) rather than relying on the form of psi_infinity. Those students who did not derive xy abd write it in polar coordinates correctly found it hard to continue. Ineperience in solving homogeneous differential equations was in evidence - many did not know that the solutions had to be of the form r^n. A few confused polar coordinates and spherical polar coordinates.	
MATH97007 MATH97087	3	This question was found the hardest, even though the target equation was given in the middle. Many did not appreciate that the arguments of Rayleigh's inflection point theorem require taking the imaginary part of the integrated equation (and rendering the dominator real). Questions 1 to 3 all used cylindrical polar coordinates, so that those with a sketchy grasp of the cylindrical radius may have suffered.	
MATH97007 MATH97087	4	The best answered of the 4 questions. Once more the arguments leading to p_y=0 were lacking from the start, but thereafter the similarity solution arguments were well done.	
MATH97007 MATH97087	5	Some good essays, the differences between low and high Re flows seem to have been well taken on board. It would be interesting to hear whether students liked this kind of question or not.	
		From a general point of view, this exam was perhaps harder than that of many other modules, not least because the "seen" parts were conceptual. But eher are some creditable attempts at a challenging paper.	