Analysis II, Term I

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Introduction to the module

This is a continuation of Analysis I module you had in year-one. In that module, you have learned about the real numbers, completeness, convergence of sequences and series, continuity and differentiability of functions on an interval or $\mathbb{R}$, integral of a function on an interval. Analysis II is a single module in year-two, delivered during term I and term II.

The content of Analysis II in term I has two parts. In the first part we complete the study of analysis on Euclidean spaces, by introducing the concepts of converges of sequences in higher dimensional Euclidean spaces $\mathbb{R}^n$, and the continuity and differentiability of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. In the second part of the module, we generalise these notions of analysis on Euclidean spaces into a broader setting, called metric spaces and topological spaces. That is a setting where one can define the notions of converge of sequences, completeness of spaces, continuity of maps, etc. Many theorems you have learned in the previous analysis module extends into this setting, and indeed, one can give unified proofs to all those statements at once. Many theorems find a natural form in the setting of metric spaces, and you will see that the proof you already know for a statement can be adapted to the more general setting.

Any section/subsection marked with * is not examinable, but will be valuable in future courses, especially if you take pure analysis courses in your third year and beyond. You should certainly at least read through the notes on these sections, even if you choose not to attempt the questions. I will try to indicate in lectures when I'm covering those material.

Throughout this lecture notes, the definitions are numbered successively within each chapter, that is, in Chapter 1, you will see Definition 1.1, Definition 1.2, Definition 1.3, and so on. The same numbering mechanism applies to Examples, Exercises, and Remarks in each chapter. On the other hand, the results such as lemmas, propositions, corollaries, and theorems are collectively numbered in a successive fashion. That is, in Chapter 1, you will see Proposition 1.1, Theorem 1.2, Theorem 1.3, etc.

Contents

Chapter 1

Differentiation in higher dimensions

1.1 Euclidean spaces

1.1.1 Preliminaries from analysis I

In this chapter we are going to extend some of the ideas that you saw last year (such as limits and continuity) to higher dimensions. The definitions are almost identical, so this should mostly feel like a review chapter to begin with, although some of the ideas we are going to approach from a different point of view.

Throughout these notes we frequently use the standard notations for the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the set of integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

the set of rational numbers

$$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z},\, q \in \mathbb{Z} \setminus \{0\}\},$$

and the set of real numbers $\mathbb{R}$. The set of real numbers is obtained as the *completion* of $\mathbb{Q}$. We may add, multiply and subtract elements of $\mathbb{R}$, and we can divide by elements of $\mathbb{R} \setminus \{0\}$. Note that some authors use the notation $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$, but we will omit 0 from this set.

On $\mathbb{R}$ we have a notion of ordering $\leq$, so that we may say whether a real number is greater than, less than or equal to another. Moreover, $\mathbb{R}$ satisfies the **completeness axiom**, that is, if $A \subset \mathbb{R}$ is non-empty and bounded above, then A has a least upper bound. The standard notation for the least upper bound of A is $\sup(A)$.

An important function defined on all real numbers is the **modulus function**, defined as

$$|x| := \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

This function has the following properties:

(i) for all $x \in \mathbb{R}$, we have $|x| \geq 0$, with $|x| = 0$ if and only if $x = 0$,

(ii) for all x and y in $\mathbb{R}$, $|xy| = |x|\,|y|$,

(iii) for all x and y in $\mathbb{R}$,

$$|x+y| \leq |x| + |y|\,.$$

The third property in the above list is called the **triangle inequality** for the modulus function.

1.1.2 Euclidean space of dimension n

For $n \geq 1$, the **n-dimensional Euclidean space**, denoted by $\mathbb{R}^n$, is defined as the set of ordered n-tuples $(x^1, x^2, \dots, x^n)$, where each $x^i \in \mathbb{R}$, for $i = 1, 2, \dots, n$. Each such n-tuple is denoted by a single letter $x = (x^1, x^2, \dots, x^n)$ and will be referred to as a point in $\mathbb{R}^n$. The entries x^i are called the **coordinates** of x.

One may see each element of $\mathbb{R}^n$ as a row vector with n real components, or as a column vector with n real components. We do not make this distinction (unless when a matrix is acting on the point x. When a matrix M acts on a vector with the same components as x we use Mx^t to make it clear that x is viewed as a column vector. Here ^{t} denotes the transpose operation.)

We shall try to stick to the convention of using superscripts to label components of vectors, and subscripts to label different vectors, so that $x_1, x_2 \in \mathbb{R}^n$ are two different vectors, while $x^1, x^2 \in \mathbb{R}$ are the components of one vector.

If x and y are elements of $\mathbb{R}^n$ with

$$x = \left(x^1, \dots, x^n\right), \qquad y = \left(y^1, \dots, y^n\right),$$

we can add these two elements according to

$$x + y = \left(x^1 + y^1, \dots, x^n + y^n\right).$$

Moreover, for every $\lambda \in \mathbb{R}$, we define

$$\lambda x = \left(\lambda x^1, \dots, \lambda x^n\right).$$

With these definitions, $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$.

The **inner product**,

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

is defined as

$$\big\langle (x^1, \ldots, x^n), (y^1, \ldots, y^n) \big\rangle = \sum_{i=1}^{n} x^i y^i.$$

Using the inner product, we may define the **length**, or **norm**, function

$$\|\cdot\| : \mathbb{R}^n \to [0, \infty)$$

as

$$\|x\| = \sqrt{\langle x, x\rangle} = \langle x, x\rangle^{1/2}.$$

Note that the inner product of two vectors is a real number, not a vector.

The norm function on $\mathbb{R}^n$ has the following properties:

(i) for all $x \in \mathbb{R}^n$, we have $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$,

(ii) for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \, \|x\|$,

(iii) for all x and y in $\mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\| . \tag{1.1}$$

The third property in the above list is called the **triangle inequality** for the norm on $\mathbb{R}^n$.

Remark 1.1. As we shall see later, these properties can be used in an abstract fashion to define more general "normed vector spaces". The norm gives us a useful notion of "distance" between two points, that is, the distance from x to y is given by $\|x - y\|$. Notice that if $n = 1$ we have $|\cdot| = \|\cdot\|$, and we will use either interchangeably in this case.

Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all x, y, and z in $\mathbb{R}^n$ and all $a \in \mathbb{R}$,

$$\langle x, y\rangle = \langle y, x\rangle , \qquad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle , \qquad \langle ax, y\rangle = a \langle x, y\rangle .$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y\rangle + t^2 \|y\|^2 \geq 0. \tag{1.2}$$

(c) By thinking of (1.2) as a quadratic in t, and considering its possible roots, deduce the **Cauchy-Schwartz** inequality:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| . \tag{1.3}$$

When does equality hold?

(d) Deduce the triangle inequality (1.1).

(e) Show the reverse triangle inequality:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|$$

Exercise 1.2. Suppose $x = (x^1, \dots, x^n) \in \mathbb{R}^n$.

(a) Show that:

$$\max_{k=1,\dots,n} \left|x^k\right| \leq \|x\| \,. \tag{1.4}$$

(b) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\dots,n} \left|x^k\right|. \tag{1.5}$$

1.1.3 Convergence of sequences in Euclidean spaces

Now that we have a few definitions relating to $\mathbb{R}^n$, we're ready to revisit some concepts from first year analysis and see how they can be extended to higher dimensions.

A sequence in $\mathbb{R}^n$ is an ordered list

$$x_0, x_1, x_2, \dots,$$

with each $x_i \in \mathbb{R}^n$, for $i = 0, 1, 2, \dots$. This is often written $(x_i)_{i=0}^{\infty}$, or $(x_i)_{i\in\mathbb{N}}$. A very important concept relating to sequences is convergence.

Definition 1.1. A sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ **converges** to (the vector) $x \in \mathbb{R}^n$ if the following holds: For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

We then write:

$$x_i \to x, \qquad \text{as } i \to \infty,$$

or

$$\lim_{i\to\infty} x_i = x.$$

One may compare the above definition to the one for convergence of a sequence of real numbers. Indeed, this notion is intimately related to convergence of real numbers, as stated in the next lemma.

Proposition 1.1. *The sequence of vectors $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ converges to the vector $x \in \mathbb{R}^n$ if and only if each component of x_i converges to the corresponding component of x. That is, if we write:*

$$x_i = (x_i^1, \dots, x_i^n), \quad \textit{and} \quad x = (x^1, \dots x^n),$$

then, $x_i \to x$ as $i \to \infty$ if and only if for all $k = 1, \dots n$, $x_i^k \to x^k$ as $i \to \infty$.

Proof. Let us first assume that for all $k = 1, 2, \dots, n$,

$$x_i^k \to x^k, \quad \text{as } i \to \infty.$$

Fix an arbitrary $\epsilon > 0$. Then, for each $k = 1, \dots, n$, we apply the definition of convergence of $x_i^k \to x^k$ to $\epsilon/\sqrt{n}$ to obtain $N_k \in \mathbb{N}$ such that for all $i \geq N_k$ we have

$$\left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Let $N = \max\{N_1 \dots, N_n\}$. Then, for every $i \geq N$, we have

$$\max_{k=1,\dots,n} \left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Now, recall from the inequality in (1.4) that for every $y = (y^1, y^2, \dots, y^n) \in \mathbb{R}^n$,

$$\|y\| \leq \sqrt{n} \max_{k=1,\dots,n} \left|y^k\right|,$$

so we deduce

$$\|x_i - x\| \leq \sqrt{n} \max_{k=1,\dots,n} \left|x_i^k - x^k\right| < \epsilon.$$

This establishes the result in one direction.

Now assume that

$$\lim_{i\to\infty} x_i = x.$$

Fix an arbitrary integer k with $1 \leq k \leq n$, and an arbitrary $\epsilon > 0$. We aim to show that $x_i^k \to x^k$, as $i \to \infty$. The definition of convergence of $x_i \to x$, as $i \to \infty$, with ϵ, gives us $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

Recall from Exercise 1.1, Equation (1.5) that for every $y = (y^1, y^2, \dots, y^n) \in \mathbb{R}^n$,

$$\max_{k=1,\dots,n} \left|y^k\right| \leq \|y\|.$$

In particular, for all $i \geq N$, we have

$$\left|x_i^k - x^k\right| \leq \max_{k=1,\dots,n} \left|x_i^k - x^k\right| \leq \|x_i - x\| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this shows that x_i^k converges to x^k, as $i \to \infty$. □

Exercise 1.3. Suppose that $(x_i)_{i=0}^\infty$ and $(y_i)_{i=0}^\infty$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i\to\infty} x_i = x, \quad \lim_{i\to\infty} y_i = y.$$

(a) Show that

$$x_i + y_i \to x + y \quad \text{as } i \to \infty.$$

(b) Show that

$$\langle x_i, y_i \rangle \to \langle x, y \rangle \qquad \text{as } i \to \infty,$$

deduce that

$$\|x_i\| \to \|x\| \qquad \text{as } i \to \infty.$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence in $\mathbb{R}$ with $a_i \to a$ as $i \to \infty$. Show that:

$$a_i x_i \to ax, \qquad \text{as } i \to \infty.$$

1.1.4 Open sets in Euclidean spaces

In dimension one, you are familiar with sets of the form (a, b) and $[a, b]$, i.e. the open interval and the closed interval respectively. These form natural domains for functions in dimension one, and it is fairly general to present theorems about maps in dimension one on such intervals. In higher dimensions, one may generalise these sets to sets of the from

$$\begin{aligned}&(a^1, b^1) \times (a^2, b^2) \times \cdots \times (a^n, b^n)\\ &\qquad = \left\{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid \text{ for } 1 \leq i \leq n, a^i < x^i < b^i\right\},\end{aligned}$$

or

$$\begin{aligned}&[a^1, b^1] \times [a^2, b^2] \times \cdots \times [a^n, b^n]\\ &\qquad = \left\{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid \text{ for } 1 \leq i \leq n, a^i \leq x^i \leq b^i\right\}.\end{aligned}$$

But this is very restrictive and does not capture the same level of generality of intervals in dimension one. The domains of maps in higher dimensions may appear in many forms. Due to this, we present a class of subsets of $\mathbb{R}^n$, called open sets.

For $x \in \mathbb{R}^n$ and the real number $r > 0$, the **open ball** of radius r about x is defined as the set

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

That is, $B_r(x)$ consists of all points in $\mathbb{R}^n$ which are at distance less than r from x. We sometimes denote the open ball $B_r(x)$ by $B(x, r)$. Both notations are widely used in mathematics.

Definition 1.2. A set $U \subseteq \mathbb{R}^n$ is called **open in** $\mathbb{R}^n$, if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$.

In other words, about any point in an open set we can find a small ball which is entirely contained in the set. Note that in this definition, the radius of the ball is allowed to depend on x. See Figure 1.1.4.

We may compare the above definition with the definition of open sets in $\mathbb{R}$ you saw in Analysis I. Recall that a set $I \subseteq \mathbb{R}$ is open in $\mathbb{R}$, if for every $x \in I$, there is $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq I$. This definition is consistent with the one we have given in $\mathbb{R}^n$, since in $\mathbb{R}^1$, $B_\delta(x) = (x - \delta, x + \delta)$.

Example 1.1. The ball $B_1(0)$ is open in $\mathbb{R}^n$. To see this, suppose $x \in B_1(0)$, so that $\|x\| < 1$. Let $r = (1 - \|x\|)/2$. We need to show that $B_r(x) \subseteq B_1(0)$. To that end, let $y \in B_r(x)$ be an arbitrary point. Using the triangle inequality for the norm in $\mathbb{R}^n$, we have

$$\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < r + \|x\| = \frac{1 - \|x\|}{2} + \|x\| < \frac{1 + \|x\|}{2} < 1.$$

This means that $y \in B_1(0)$.

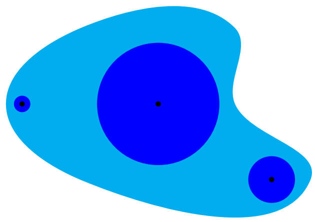

Figure 1.1: An open set in $\mathbb{R}^2$ in cyan, and some balls inside it. The radius of the ball depends on the location of the point.

Observe that in the above example, one can replace 1 with any other positive real number, and the result is still valid. That is, for every $\delta > 0$, the set $B_\delta(0)$ is open in $\mathbb{R}^n$. Similarly, one can also replace 0 with any $y \in \mathbb{R}^n$. Thus, in general, for any $y \in \mathbb{R}^n$ and any $\delta > 0$, $B_\delta(y)$ is open in $\mathbb{R}^n$.

Example 1.2. The set $A = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is not open. Clearly $y := (1, 0, \ldots, 0)$ belongs to A. On the other hand, if $r > 0$ then $z = (1 + r/2, 0, \ldots, 0)$ belongs to $B_r(y)$ but not to A, so there is no $r > 0$ such that $B_r(y) \subset A$.

Exercise 1.4. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 > 0\}$?

(d) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \forall i,\ x^i \in [0, 1)\}$?

(e) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \forall i,\ x^i \in \mathbb{Q}\}$?

Exercise 1.5. Let $(x_i)_{i=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ with $\lim_{i\to\infty} x_i = x \in \mathbb{R}^n$. Assume that there is $r > 0$ such that for all $i \geq 0$, we have $\|x_i\| < r$. Show that

$$\|x\| \leq r.$$

Exercise 1.6. (a) Show that if U_1 and U_2 are open sets in $\mathbb{R}^n$, then $U_1 \cup U_2$ and $U_1 \cap U_2$ are open in $\mathbb{R}^n$.

(b) Suppose that U_α, for α in an index set I, are open sets in $\mathbb{R}^n$.

(i) Show that the set $\bigcup_{\alpha \in I} U_\alpha$ is open in $\mathbb{R}^n$.

(ii) Give an example showing that $\bigcap_{\alpha \in I} U_\alpha$ need not be open.

Remark 1.2. It is worth noting that the notion of open sets in $\mathbb{R}^n$ relies on the length function $\|\cdot\|$ we have on $\mathbb{R}^n$. As we shall see in the next chapter, one can consider functions (called metric) with similar properties on a wide range of other sets (such as the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ or the set of all sequences in $[0, 1]$, etc). These lead to notions of open sets on such sets. We will look into this in the next chapter.

1.2 Continuity

Last year, you learned about the notion of continuity for functions from $\mathbb{R}$ (or subsets of $\mathbb{R}$) to $\mathbb{R}$. In this section we revisit those definitions and upgrade them to higher dimensions. In fact, the definitions we shall give are almost identical: the only thing that changes is that we use the appropriate "norm" for the domain and range.

1.2.1 Continuity at a point, and continuity on an open set

We start with the simple definition

Definition 1.3. Let $A \subset \mathbb{R}^n$ be an open set, and suppose $f : A \to \mathbb{R}^m$. We say that f is **continuous at** $p \in A$ if the following holds: for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $\|x - p\| < \delta$ we have

$$\|f(x) - f(p)\| < \epsilon.$$

If f is continuous at every p in A, we say f is **continuous on** A.

We can think of this as saying "f maps points in A close to p to points in $\mathbb{R}^m$ close to $f(p)$". Notice that in the definition above, the symbol $\|\cdot\|$ is playing two slightly different roles: as the norm on $\mathbb{R}^n$ and the norm on $\mathbb{R}^m$.

Remark 1.3. The words "function" and "map" are not identical. For $f : X \to Y$, we use the word "function" when the target space Y is the real numbers or the complex numbers (or in general a field). Otherwise, we use the word "map". Of course it is correct to refer to $f : X \to \mathbb{R}$ as a map, but it is uncommon to refer to $f : X \to Y$ as a function, when Y is not a set of numbers where one can not add and multiply elements. On the other hand, it is common in analysis and geometry to see expressions like, "let f be a function on X", which means that $f : X \to \mathbb{R}$ or $f : X \to \mathbb{C}$. In those cases, the target space is understood from the context.

Example 1.3. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = \|x\|$ is continuous on $\mathbb{R}^n$.

To show this, fix an arbitrary $p \in \mathbb{R}^n$. Suppose $\|x-p\| < \delta$, then by the reverse triangle inequality (see Exercise 1.1) we have:

$$|f(x) - f(p)| = \big|\, \|x\| - \|p\| \,\big| \le \|x-p\| < \delta.$$

Thus we can take $\delta = \epsilon$ and we have satisfied the criteria for continuity of f at p.

Example 1.4. Every linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

Let $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathbb{R}^n$, that is, all entries of e_j are 0 except the j-th entry which is 1. We may define the real number

$$M = \max_{j=1,\dots,n} \|\Lambda(e_j)\| .$$

We note that,

$$\begin{aligned}
\|\Lambda(x) - \Lambda(p)\| = \|\Lambda(x-p)\| &= \left\| \Lambda\Big(\sum_{j=1}^n e_j (x-p)^j \Big) \right\| \\
&= \left\| \sum_{j=1}^n (x-p)^j \Lambda(e_j) \right\| \\
&\le \sum_{j=1}^n \left\| (x-p)^j \Lambda(e_j) \right\| \\
&\le \sum_{j=1}^n \left| (x-p)^j \right| \|\Lambda(e_j)\| \\
&\le M \sum_{j=1}^n \left| (x-p)^j \right|
\end{aligned}$$

Thus, using the inequality in Equation (1.4),

$$\|\Lambda(x) - \Lambda(p)\| \le M \sum_{j=1}^n \|x-p\| = Mn \|x-p\| .$$

Thus, if we take $\delta = \epsilon/(2Mn)$, then for any x with $0 < \|x-p\| < \delta$, we have

$$\|\Lambda(x) - \Lambda(p)\| < \frac{\epsilon}{2Mn} Mn < \epsilon,$$

so Λ is continuous.

Example 1.5. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x^1, \dots, x^n) = x^1$ is continuous on $\mathbb{R}^n$.

To see this, fix an arbitrary $p \in \mathbb{R}^n$. Suppose $\|x-p\| < \delta$, then by the inequality in (1.5) we have:

$$|f(x) - f(p)| = \left| x^1 - p^1 \right| \le \max_{k=1,\dots,n} \left| x^k - p^k \right| \le \|x-p\| < \delta,$$

so we may take $\delta = \epsilon$ and we have satisfied the condition for continuity. Obviously the same argument shows that all of the coordinate maps (i.e. the map taking x to x^k) are continuous.

Theorem 1.2. *Let A be an open subset of $\mathbb{R}^n$ and B be an open subset of $\mathbb{R}^m$. Suppose $f : A \to B$ is continuous at p and $g : B \to \mathbb{R}^l$ is continuous at $f(p)$. Then $g \circ f : A \to \mathbb{R}^l$ is continuous at p.*

Proof. Fix an arbitrary $\epsilon > 0$. Since g is continuous at $f(p)$, we know that there exists $\delta_1 > 0$ such that for any $y \in B$ with $\|y - f(p)\| < \delta_1$, we have $\|g(y) - g(f(p))\| < \epsilon$. Similarly, since f is continuous at p, we know that there exists $\delta > 0$ such that for any $x \in A$ with $\|x - p\| < \delta$, we have $\|f(x) - f(p)\| < \delta_1$. Combining these two statements and taking $y = f(x)$, we deduce that if $x \in A$ with $\|x - p\| < \delta$, we have $\|g(f(x)) - g(f(p))\| < \epsilon$. $\square$

It is sometimes useful to express the continuity of a map in a slightly different way, for which we need the following definition:

Definition 1.4. Let A be an open subset of $\mathbb{R}^n$ and suppose $f : A \to \mathbb{R}^m$. For $p \in A$, we say that the **limit** of f as x tends to p is equal to $q \in \mathbb{R}^m$, if the following holds: for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $0 < \|x - p\| < \delta$ we have

$$\|f(x) - q\| < \epsilon.$$

In this case, we write

$$\lim_{x \to p} f(x) = q.$$

Note that in the above definition, we do not allow $x = p$. With this notion of a limit in hand, we can give the definition of continuity more compactly as:

"f is continuous at p, if $\lim_{x \to p} f(x) = f(p)$."

Theorem 1.3. *Suppose A is an open subset of $\mathbb{R}^n$, $p \in A$, and $f, g : A \to \mathbb{R}$ with*

$$\lim_{x \to p} f(x) = F, \qquad \lim_{x \to p} g(x) = G.$$

Then

(i) $\lim_{x \to p} (f(x) + g(x)) = F + G$,

(ii) $\lim_{x \to p} (f(x)g(x)) = FG$,

(iii) If, furthermore $G \neq 0$, then:

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{F}{G}.$$

Proof. (i) Fix an arbitrary $\epsilon > 0$. Since $\lim_{x \to p} f(x) = F$, we know that there exists $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|f(x) - F| < \frac{\epsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{2}.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Evidently $\delta > 0$. For every $x \in A$ with $0 < \|x - p\| < \delta$, by the triangle inequality, we have

$$|f(x) + g(x) - (F + G)| \leq |f(x) - F| + |g(x) - G| < \epsilon.$$

(ii) Fix an arbitrary $\epsilon > 0$, and assume without loss of generality that $\epsilon < 3$ (Why can see assume this?). Since $\lim_{x \to p} f(x) = F$, we know that there exists $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|f(x) - F| < \frac{\epsilon}{3(1 + |G|)}.$$

Similarly, there exists $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{3(1 + |F|)}.$$

To control $f(x)g(x) - FG$, we add and subtract the same terms, so that we obtain terms of the form $f(x) - F$ and $g(x) - G$. That is,

$$\begin{aligned} f(x)g(x) - FG &= f(x)g(x) - f(x) \cdot G + f(x) \cdot G - F \cdot G \\ &= f(x)(g(x) - G) + (f(x) - F) \cdot G \\ &= (f(x) - F + F)(g(x) - G) + (f(x) - F) \cdot G \\ &= (f(x) - F)(g(x) - G) + F \cdot (g(x) - G) + (f(x) - F) \cdot G \end{aligned}$$

Now, take $\delta = \min\{\delta_1, \delta_2\}$. For every $x \in A$ with $0 < \|x - p\| < \delta$, by the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - FG| &\leq |f(x) - F|\,|g(x) - G| + |F|\,|g(x) - G| + |G|\,|f(x) - F| \\ &< \frac{\epsilon^2}{9(1 + |F|)(1 + |G|)} + \frac{\epsilon\,|F|}{3(1 + |F|)} + \frac{\epsilon\,|G|}{3(1 + |G|)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

(iii) Given the previous part, it suffices to show that if $\lim_{x \to p} g(x) = G$ with $G \neq 0$, then

$$\lim_{x \to p} \frac{1}{g(x)} = \frac{1}{G}.$$

Fix an arbitrary $\epsilon > 0$. Since $\lim_{x\to p} g(x) = G$, we know that there exist $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|g(x) - G| < \frac{\epsilon\,|G|^2}{2}.$$

Also, since $G \neq 0$, $G/2 > 0$, and hence, there is $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{|G|}{2}.$$

By the triangle inequality, this implies that

$$|g(x)| = |g(x) - G + G| \geq |G| - |g(x) - G| > |G| - \frac{|G|}{2} = \frac{|G|}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. For every $x \in A$ with $0 < \|x - p\| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{G}\right| = |G - g(x)| \cdot \frac{1}{|G|} \cdot \frac{1}{|g(x)|} < \frac{\epsilon\,|G|^2}{2} \cdot \frac{1}{|G|} \cdot \frac{2}{|G|} = \epsilon.$$

This completes the proof.

□

Corollary 1.4. *Suppose A is an open set in $\mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ are continuous at $p \in A$. Then,*

(i) $f + g$ is continuous at p.

(ii) fg is continuous at p.

(iii) If, furthermore $g(p) \neq 0$, then $\dfrac{f}{g}$ is continuous at p.

Exercise 1.7. Assume that A is an open set in $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Show that $\lim_{x\to p} f(x) = F$, if and only if, for any sequence $(x_i)_{i=0}^{\infty}$ in $A \setminus \{p\}$ with $\lim_{i\to\infty} x_i = p$,

$$\lim_{i\to\infty} f(x_i) = F.$$

Exercise 1.8. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \dots, 0)$ is continuous on $\mathbb{R}$.

(b) Let A be an open set in $\mathbb{R}^n$ and $f^1, f^2, \dots, f^m$ are functions from A to $\mathbb{R}$. Consider the map $f : A \to \mathbb{R}^m$ defined as

$$f(x^1, \dots, x^n) \mapsto \left(f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n)\right).$$

Show that f is continuous at $p \in A$, if and only if, for every $k = 1, \dots, m$ the map $f^k : A \to \mathbb{R}$ is continuous at p.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x^1, x^2, \ldots, x^n) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$. Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.

With the above results, one can build many continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. For example,

$$(x^1, x^2) \mapsto \left(\sin(x^1x^2), \cos(x^2)\right),$$

$$(x^1, x^2, x^3) \mapsto \left(\frac{x^1 - x^2}{1 + (x^2)^2}, e^{x^3}\right).$$

Exercise 1.9 (*). (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open set $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

1.3 Derivative of a map of Euclidean spaces

So far, when differentiating functions, we've restricted ourselves to the situation where the function depends only on one variable. This covers lots of situations that we're interested in, but of course we often wish to consider maps of more than one variable. In this chapter we will see how the idea of differentiation can be extended to maps which send (subsets of) $\mathbb{R}^n$ to $\mathbb{R}^m$. The basic idea will be that the derivative of a map at a point p should be the "best linear approximation" to the map at p.

1.3.1 Derivative as a linear map

Before we think about how to define a derivative of a map in higher dimensions, let's first note some of the potential challenges. In one dimension, we say that f is differentiable at p if the limit

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists. If $x, p \in \mathbb{R}^n$ and $f(x), f(p) \in \mathbb{R}^m$ then we obviously have a problem: we don't even know how to make sense of 'dividing by $x - p$', and it's not clear what sort of object we should end up with.

To try and find a way through this impasse, let's just remind ourselves how the derivative is introduced in one dimension. By approximating with successive chords, we consider the tangent to the graph of f at p (see Figure 1.2). Let us think a little

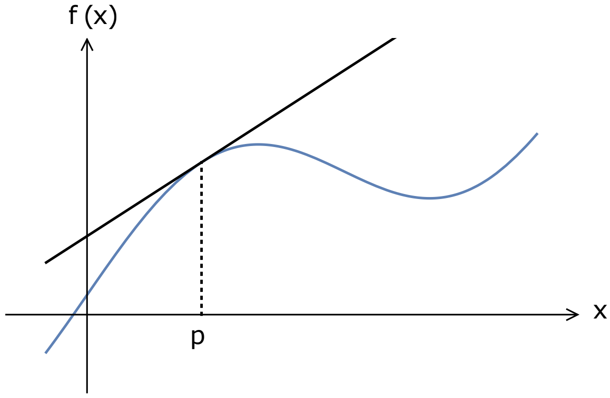

Figure 1.2: The tangent to f at p.

about how the tangent is characterised. Any (non-vertical) straight line passing through $(p, f(p))$ is the graph of the affine map

$$A_\lambda : x \mapsto \lambda(x-p) + f(p)$$

for some $\lambda \in \mathbb{R}$. Let's consider the difference between f and such an affine map

$$f(x) - A_\lambda(x) = f(x) - f(p) - \lambda(x-p).$$

In general, from the continuity of f we know that for any $\lambda \in \mathbb{R}$,

$$\lim_{x \to p} [f(x) - A_\lambda(x)] = 0. \tag{1.6}$$

However, if f is differentiable, there is a unique choice of λ that allows us to make a stronger statement. If f is differentiable, there exists a unique $\lambda \in \mathbb{R}$ such that

$$\lim_{x \to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0.$$

This is a stronger statement than (1.6) because it tells us that $f(x) - A_\lambda(x)$ is going to zero faster than $|x-p|$, as $x \to p$. We make this informal discussion more precise in the following lemma.

Lemma 1.5. *The map $f : (a, b) \to \mathbb{R}$ is differentiable at $p \in (a, b)$ if and only if there exists a map of the from $A_\lambda(x) = \lambda(x-p) + f(p)$, for some $\lambda \in \mathbb{R}$, such that*

$$\lim_{x \to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0.$$

Proof. We can re-write

$$\frac{|f(x) - f(p) - \lambda(x-p)|}{|x-p|} = \left| \frac{f(x) - f(p)}{x-p} - \lambda \right|,$$

so that

$$\lim_{x\to p}\frac{|f(x)-A_\lambda(x)|}{|x-p|}=0 \qquad \Longleftrightarrow \qquad \lim_{x\to p}\frac{f(x)-f(p)}{x-p}=\lambda.$$

The expression on the right-hand side of the above equation is the definition of differentiability of f at p. $\square$

We may rewrite

$$A_\lambda(x)=\lambda(x-p)+f(p)=\lambda x+(f(p)-\lambda p).$$

Thus, $A_\lambda:\mathbb{R}\to\mathbb{R}$ is the composition of the linear map $x\mapsto\lambda x$ and the translation $x\mapsto x+(f(p)-\lambda p)$. Such maps are called affine maps of $\mathbb{R}$. By the above lemma, the map f is differentiable at p, if it is "well approximated" by an affine map at p. We may generalise this to higher dimensions.

Since we are going to frequently apply linear an nonlinear maps to variables, to distinguish between these two cases, we shall use the notation $h[v]$ when h is a linear map and v is seen as a vector, and use $h(v)$ when h is a map and v is seen as a point in the domain of h.

Let $L(\mathbb{R}^n;\mathbb{R}^m)$ denote the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that $\Lambda:\mathbb{R}^n\to\mathbb{R}^m$ is a linear map if

$$\Lambda[x+y]=\Lambda[x]+\Lambda[y], \qquad \forall x,y\in\mathbb{R}^n,$$

$$\Lambda[ax]=a\Lambda[x], \qquad \forall a\in\mathbb{R} \text{ and } x\in\mathbb{R}^n.$$

In analogy to the statement in Lemma 1.5 we propose the following definition.

Definition 1.5. Suppose $\Omega\subset\mathbb{R}^n$ is open. The map $f:\Omega\to\mathbb{R}^m$ is **differentiable** at $p\in\Omega$, if there exists a linear map $\Lambda\in L(\mathbb{R}^n;\mathbb{R}^m)$ such that

$$\lim_{x\to p}\frac{\|f(x)-(\Lambda[x-p]+f(p))\|}{\|x-p\|}=0.$$

In this case, we write

$$Df(p):=\Lambda,$$

and call $Df(p)$ the derivative of the map f at the point p.

Note that some authors refer to the derivative of a map as **total derivative**, or **differential**. We shall refer to that as derivative.

It is often useful to have the following equivalent characterisation of differentiability in higher dimensions: $f:\Omega\to\mathbb{R}^m$ is differentiable at $p\in\Omega$ if and only if there exists $\Lambda\in L(\mathbb{R}^n;\mathbb{R}^m)$ such that

$$\lim_{h\to 0}\frac{\|f(p+h)-f(p)-\Lambda[h]\|}{\|h\|}=0.$$

Note that in the above equation, $h \to 0$ in $\mathbb{R}^n$.

Recall that using a canonical basis for $\mathbb{R}^n$ and $\mathbb{R}^m$ any linear map $\Lambda \in L(\mathbb{R}^n;\mathbb{R}^m)$ can be expressed as an $m \times n$ matrix which is called the **Jacobian** of f at p. The convention is that an $m \times n$ matrix has m rows and n columns. For the purposes of this course, we won't make a big deal of the difference between a linear map and its matrix representation with respect to the canonical basis, so will use the words derivative and Jacobian essentially indistinguishably.

Lemma 1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f : \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$, then it is continuous at p.*

Proof. Since

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - \Lambda[h]\|}{\|h\|} = 0,$$

we must have

$$\lim_{h\to 0} \|f(p+h) - f(p) - \Lambda[h]\| = 0.$$

On the other hand, since linear maps are continuous, see Example 1.4, we obtain

$$0 = \lim_{h\to 0} (f(p+h) - f(p) - \Lambda[h]) = \lim_{h\to 0} (f(p+h) - f(p)).$$

$\square$

Example 1.6. By Lemma 1.5 any function $f : (a,b) \to \mathbb{R}$ which is differentiable at p satisfies the conditions of 1.5 with $Df(p) = f'(p)$. Notice that a 1×1 matrix is simply a real number.

Example 1.7. Let $B \in L(\mathbb{R}^n;\mathbb{R}^m)$ and $V \in \mathbb{R}^m$. Then, the map $f : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$f(x) = B(x) + V$$

is differentiable at each $p \in \mathbb{R}^n$, and $Df(p) = B$. To see this, note that

$$\begin{aligned} f(p+h) - f(p) - B(h) &= (B(p+h) + V) - (B(p) + V) - B(h) \\ &= B(p) + B(h) + V - B(p) - V - B(h) = 0. \end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - B(h)\|}{\|h\|} = \lim_{h\to 0} 0 = 0.$$

Example 1.8. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \|x\|^2$$

is differentiable at each $p \in \mathbb{R}^n$, and $Df(p)$ is the linear map

$$Df(p)[h] = 2\langle p, h\rangle, \quad \forall h \in \mathbb{R}^n.$$

From the properties of the inner product in Exercise 1.1-(a), we can see that the map $h \mapsto 2\langle p, h\rangle$ is a linear map.

We note that

$$f(p+h) = \|p+h\|^2 = \langle p+h, p+h\rangle = \|p\|^2 + 2\langle p, h\rangle + \|h\|^2,$$

so that

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - 2\langle p, h\rangle\|}{\|h\|} = \lim_{h\to 0} \|h\| = 0.$$

As a matrix, we have that $Df(p) = 2p$, where p is viewed as a row vector with n components (this is in line with our convention that a $1 \times n$ matrix maps $\mathbb{R}^n$ to $\mathbb{R}^1$). So the Jacobian is a row vector for this map.

Example 1.9. Let $m \geq 1$ be an integer, and assume that for $i = 1, 2, \ldots, m$, the map $f^i : (a, b) \to \mathbb{R}$ is differentiable at $p \in (a, b)$. Then the map $f : (a, b) \to \mathbb{R}^m$ defined as

$$f(x) = \left(f^1(x), f^2(x), \ldots, f^m(x)\right),$$

is differentiable at p, and the derivative $Df(p) : \mathbb{R} \to \mathbb{R}^m$ has the matrix representation

$$Df(p) = \begin{pmatrix} (f^1)'(p) \\ \vdots \\ (f^m)'(p) \end{pmatrix}.$$

To see this, we note that

$$f(p+h) - f(p) - \begin{pmatrix} (f^1)'(p) \\ \vdots \\ (f^m)'(p) \end{pmatrix} h = \begin{pmatrix} f^1(p+h) - f^1(p) - (f^1)'(p)h \\ \vdots \\ f^m(p+h) - f^m(p) - (f^m)'(p)h \end{pmatrix}$$

so that, using the inequality in (1.5),

$$\frac{\|f(p+h) - f(p) - Df(p)[h]\|}{\|h\|} \leq \sqrt{m} \max_{j=1,\ldots,m} \frac{\left|f^j(p+h) - f^j(p) - (f^j)'(p)h\right|}{|h|}.$$

Since each f^j is differentiable at p, the left hand side of the above equation tends to 0, as $h \to 0$. And since the left hand side of the equation is non-negative, it must tend to 0, as $h \to 0$. Notice here that the expression $Df(p)[h]$ means applying the linear map $Df(p)$ to the one dimensional vector h, which gives us an element of $\mathbb{R}^m$.

Implicitly in the discussion above, we've assumed that $Df(p)$, if it exists, must be unique. Of course, this is something that we need to prove.

Theorem 1.7. *The derivative, if it exists, is unique.*

Proof. Suppose $\Omega \subset \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^m$, $p \in \Omega$ and that Λ and Λ' satisfy:

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - \Lambda[h]\|}{\|h\|} = \lim_{h \to 0} \frac{\|f(p+h) - f(p) - \Lambda'[h]\|}{\|h\|} = 0.$$

Let e be an arbitrary vector in $\mathbb{R}^n$ with $\|e\| = 1$. Then for any real number $\alpha \neq 0$ we have

$$\frac{\Lambda[\alpha e]}{\alpha} = \Lambda[e].$$

Now, let $(\alpha_j)_{j=0}^{\infty}$ be a sequence of non-zero real numbers tending to 0 as $j \to \infty$. By adding and subtracting identical terms, we see that

$$\begin{aligned}
&\|\Lambda[e] - \Lambda'[e]\| \\
&\quad = \left\| \frac{\Lambda[\alpha_j e]}{\alpha_j} - \frac{\Lambda'[\alpha_j e]}{\alpha_j} \right\| \\
&\quad = \lim_{j \to \infty} \frac{\|\Lambda[\alpha_j e] - \Lambda'[\alpha_j e]\|}{\|\alpha_j e\|} \\
&\quad = \lim_{j \to \infty} \frac{\|-f(p+\alpha_j e) + f(p) + \Lambda[\alpha_j e] + f(p+\alpha_j e) - f(p) - \Lambda'[\alpha_j e]\|}{\|\alpha_j e\|} \\
&\quad \leq \lim_{j \to \infty} \frac{\|f(p+\alpha_j e) - f(p) - \Lambda[\alpha_j e]\|}{\|\alpha_j e\|} + \lim_{j \to \infty} \frac{\|f(p+\alpha_j e) - f(p) - \Lambda'[\alpha_j e]\|}{\|\alpha_j e\|} \\
&\quad = 0.
\end{aligned}$$

For the last equality in the above equation we have used that $\alpha_j e \to 0$ as $j \to \infty$. By the above equation, for any unit vector e we have $\Lambda[e] = \Lambda'[e]$, which implies that (as linear maps) $\Lambda = \Lambda'$. $\square$

Exercise 1.10. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f(x) = x$. Show that f is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \mathrm{id},$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Exercise 1.11. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = \begin{pmatrix} 2\xi & 2\eta \end{pmatrix}.$$

Exercise 1.12. One might hope that the derivative can be calculated by finding

$$\lim_{x \to p} \frac{f(x) - f(p)}{\|x - p\|}.$$

By considering the example of Exercise 1.10 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Exercise 1.13. Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at p and

$$Dh(p) = Df(p) + Dg(p)$$

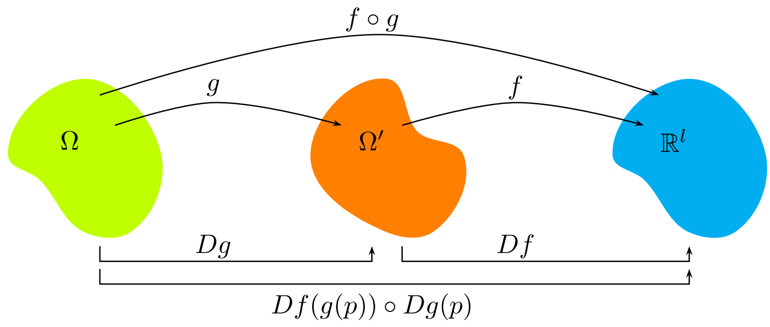

Figure 1.3: Illustration of Theorem 1.8.

1.3.2 Chain rule

In dimension one there is a simple "algorithm" which allows us to calculate the derivative of more complicated maps using the derivative of simpler ones. That algorithm is the chain rule. If $f, g : \mathbb{R} \to \mathbb{R}$, with g differentiable at p and f differentiable at $g(p)$, then $f \circ g$ is differentiable at p with

$$(f \circ g)'(p) = f'(g(p))g'(p).$$

Now, suppose that $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^l$, with g differentiable at p and f differentiable at $g(p)$. Let $h = f \circ g$. We know that $Dg(p) : \mathbb{R}^n \to \mathbb{R}^m$ and $Df(g(p)) : \mathbb{R}^m \to \mathbb{R}^l$ are linear maps, so it certainly makes sense to consider $Df(g(p)) \circ Dg(p)$, where "$\circ$" denotes the composition of linear maps (corresponding to matrix multiplication). This will be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^l$, which is the right kind of object to be $Dh(p)$. In fact, it is the case that $h = f \circ g$ is differentiable at p with

$$Dh(p) = Df(g(p)) \circ Dg(p)$$

Theorem 1.8. *Assume $\Omega \subseteq \mathbb{R}^n$ and $\Omega' \subseteq \mathbb{R}^m$ are open sets, with $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$ and $f : \Omega' \to \mathbb{R}^l$ differentiable at $g(p) \in \Omega'$. Then $h = f \circ g : \Omega \to \mathbb{R}^l$ is differentiable at p with derivative*

$$Dh(p) = Df(g(p)) \circ Dg(p).$$

(*) *Proof.* Let $g(p) = q$, $A = Dg(p)$, $B = Df(q)$. We define the map

$$\begin{aligned}
\phi(x) &= g(x) - g(p) - A(x - p), \quad \forall x \in \Omega \\
\psi(y) &= f(y) - f(q) - B(y - q), \quad \forall y \in \Omega' \\
\tau(x) &= f(g(x)) - f(g(p)) - B\left(A(x - p)\right), \quad \forall x \in \Omega.
\end{aligned}$$

By the assumptions in the theorem we know that

$$0 = \lim_{x \to p} \frac{\phi(x)}{\|x - p\|}, \tag{1.7}$$

$$0 = \lim_{y \to q} \frac{\psi(y)}{\|y - q\|}, \tag{1.8}$$

and we need to show that

$$\lim_{x \to p} \frac{\tau(x)}{\|x - p\|} = 0.$$

We may rewrite the map τ as

$$\begin{aligned}
\tau(x) &= f(g(x)) - f(g(p)) - B\left(A(x - p)\right) \\
&= f(g(x)) - f(g(p)) - B\left(g(x) - g(p) - \phi(x)\right) \\
&= f(g(x)) - f(g(p)) - B\left(g(x) - g(p)\right) + B(\phi(x)) \\
&= \psi(g(x)) + B(\phi(x)).
\end{aligned}$$

On the other hand, we recall from Example 1.4 that there is a real number M such that

$$\|A(x)\| \leq M \|x\|, \quad \forall x \in \mathbb{R}^n.$$

Since B is linear, and hence continuous by Example 1.4, we have that

$$\lim_{x \to p} \frac{B(\phi(x))}{\|x - p\|} = \lim_{x \to p} B\left(\frac{\phi(x)}{\|x - p\|}\right) = B\left(\lim_{x \to p} \frac{\phi(x)}{\|x - p\|}\right) = 0.$$

Fix an arbitrary $\epsilon > 0$. It follows from (1.8) that there exists $\delta > 0$ such that for $y \in \Omega'$ with $\|y - q\| < \delta$ we have

$$\frac{\|\psi(y)\|}{\|y - q\|} < \epsilon$$

which implies

$$\|\psi(y)\| < \epsilon \|y - q\|.$$

On the other hand, since g is continuous, there exists δ_1 such that if $x \in \Omega$ with $\|x - p\| < \delta_1$ then

$$\|g(x) - g(p)\| = \|g(x) - q\| < \delta.$$

Thus, for every $x \in \Omega$ with $\|x - p\| < \delta_1$, we have

$$\begin{aligned}
\|\psi(g(x))\| &< \epsilon \|g(x) - q\| \\
&= \epsilon \|\phi(x) + A(x - p)\| \\
&\leq \epsilon \|\phi(x)\| + \epsilon M \|x - p\|.
\end{aligned}$$

Dividing through by $\|x - p\|$ and taking the limit, we deduce that

$$\lim_{x \to p} \frac{\|\psi(g(x))\|}{\|x - p\|} \leq \epsilon M.$$

Since $\epsilon > 0$ was arbitrary, we conclude

$$\lim_{x \to p} \frac{\|\psi(g(x))\|}{\|x - p\|} = 0,$$

and we are done. $\square$

Example 1.10. Let $m \geq 1$ be an integer, and assume that for $i = 1, 2, \ldots, m$, the functions $g^i : (a, b) \to \mathbb{R}$ are differentiable at some $p \in (a, b)$. Then, the map $k : (a, b) \to \mathbb{R}$, defined as

$$k(x) = \left\| \big(g^1(x), g^2(x), \ldots, g^m(x)\big) \right\|^2$$

is differentiable at p, and its Jacobian matrix has one real entry

$$2g^1(p)(g^1)'(p) + 2g^2(p)(g^2)'(p) \cdots + 2g^m(p)(g^m)'(p).$$

We note that by Example 1.9, the map $g : (a, b) \to \mathbb{R}^m$ defined as

$$g(x) = (g^1(p), g^2(p), \ldots, g^m(p))$$

is differentiable at p with derivative

$$Dg(p) = \begin{pmatrix} (g^1)'(p) \\ \vdots \\ (g^m)'(p) \end{pmatrix}.$$

On the other hand, in Example 1.8, we saw that the map $f(x) = \|x\|^2$ is differentiable at every point in $\mathbb{R}^m$ with derivative $Df(q)[h] = 2\langle q, h\rangle$. We have $k = f \circ g$ on (a, b). Thus, by the chain rule, the map k is differentiable at p, with derivative

$$\begin{aligned} Dk(p)[h] &= Df(g(p)) \circ Dg(p)[h] \\ &= D(f(g(P)) \left[\big((g^1)'(p)h, \ldots, (g^m)'(p)h\big)\right] \\ &= 2\left\langle g(p), ((g^1)'(p)h, \ldots, (g^m)'(p)h)\right\rangle \\ &= 2\left\langle g(p), h((g^1)'(p), \ldots, (g^m)'(p))\right\rangle \\ &= 2\langle g(p), Dg(p)\rangle\, h. \end{aligned}$$

Thus, the Jacobian of k at p is the one by one matrix with real entry

$$2\langle g(p), Dg(p)\rangle = 2g^1(p)(g^1)'(p) + 2g^2(p)(g^2)'(p) \cdots + 2g^m(p)(g^m)'(p).$$

Exercise 1.14. Assume Ω and Ω' are open sets in $\mathbb{R}^n$, $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$ and $f : \Omega' \to \Omega$ differentiable at $g(p) \in \Omega'$. Moreover,

$$\begin{aligned} f \circ g(x) &= x, \qquad \forall\, x \in \Omega. \\ g \circ f(x) &= x, \qquad \forall\, x \in \Omega'. \end{aligned}$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

Exercise 1.15 (*). (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by

$$P(x, y) = xy$$

is differentiable at each point $p = (\xi, \eta) \in \mathbb{R}^2$, with Jacobian

$$DP(p) = (\eta \;\; \xi)\,.$$

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ defined as

$$Q(z) = (f(z), g(z))$$

is differentiable at q, with derivative

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

(c) Show that the map $F : \mathbb{R}^n \to \mathbb{R}$ defined as $F(z) = f(z)g(z)$, for all $z \in \mathbb{R}^n$, is differentiable at q, with derivative

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

1.4 Directional derivatives

1.4.1 Rates of change and partial derivatives

Although the definitions of differentiability in dimension one and in higher dimensions appear similar, there is a major difference which makes the latter a more difficult concept. In dimension one, to see if a map : $f(a, b) \to \mathbb{R}$ is differentiable at some $x \in (a, b)$, we only need to verify that the limit of $(f(x) - f(p)/(x - p))$ exists as $x \to p$. To verify this, we do not need to know the value of the limit beforehand, that is, the value of the limit does not appear in this ratio. However, in higher dimensions, to verify if a map $f : \Omega \to \mathbb{R}^n$ is differentiable at some $p \in \Omega$, we need to know the derivative at that point. In other words, the derivative of the map at p appears in the criteria for differentiability. For basic maps, it is possible to guess the derivative, but in general, it may not be obvious what the derivative is. See for instance the map in Example 1.8. The purpose of this section is to present a simple approach to identify a candidate for the derivative in higher dimensions.

For a function $f : (a, b) \to \mathbb{R}$, we are familiar with the idea of $f'(p)$ telling us something about the rate of change of $f(x)$ as we vary x near $p \in (a, b)$. We can connect the derivative to this sort of concept with the directional derivative. Let us suppose that we are given a function $f : \mathbb{R}^3 \to \mathbb{R}$, which is supposed to represent the temperature of some three dimensional body which is not changing in time. Suppose we start at the origin $0 \in \mathbb{R}^3$ and travel along the curve $t \mapsto vt$, for some fixed $v \in \mathbb{R}^3$, that is we move along a straight line with velocity v passing through the origin at time 0. We can record the temperature of our surroundings as a function of time, $\theta(t)$ and we will find $\theta(t) = f(vt)$. Suppose we ask what the rate of change of temperature is at $t = 0$. This will of course be $\theta'(0)$. Now, we notice that we can write:

$$\theta = f \circ V$$

where V is the linear map $V : \mathbb{R} \to \mathbb{R}^3$ given by $V(t) = vt$. Now, we can use the chain rule to calculate $\theta'(0) = D\theta(0)$ and we find:

$$\theta'(0) = D\theta(0) = Df(0) \circ DV(0).$$

Now, since V is a linear map, we have $DV(0) = v$ and we conclude:

$$\theta'(0) = Df(0)[v].$$

This gives us a nice interpretation of the derivative $Df(0)$. When we apply $Df(0)$ to a vector v, we find the rate of change of f at 0 as we travel along a line with velocity v. More generally, we can consider travelling along the line given by $V(t) = p + tv$ for some $p \in \mathbb{R}^3$. Then at $t = 0$, we are passing through the point $p \in \mathbb{R}^3$. Setting $\theta(t) = f(p + tv)$, We call the quantity:

$$\theta'(0) = D\theta(p) = Df(p)[v]$$

the **directional derivative** of f at p in the direction v. Sometimes the notation

$$\frac{\partial f}{\partial v}(p) := \lim_{t\to 0} \frac{1}{t}\left[f(p + vt) - f(p)\right] = Df(p)[v]$$

is used for the directional derivative.

Now, if we take $\{e_1, e_2, e_3\}$ to be the canonical basis vectors for $\mathbb{R}^3$, then we can write $v = v^1 e_1 + v^2 e_2 + v^3 e_3$ for $v^i \in \mathbb{R}$. Doing this, and recalling that $Df(p)$ is a linear map, we have:

$$\begin{aligned} \frac{\partial f}{\partial v}(p) &= Df(p)\left[v^1 e_1 + v^2 e_2 + v^3 e_3\right] && (1.9) \\ &= v^1 Df(p)\left[e_1\right] + v^2 Df(p)\left[e_2\right] + v^3 Df(p)\left[e_3\right] \\ &= v^1 D_1 f(p) + v^2 D_2 f(p) + v^3 D_3 f(p). && (1.10) \end{aligned}$$

In other words, we can find any directional derivative at p, provided we know the three numbers:

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p), \qquad i = 1, 2, 3.$$

called the **partial derivatives** of f at p. Equivalently, these can be defined as

$$D_i f(p) := \lim_{t\to 0} \frac{f(p + te_i) - f(p)}{t}.$$

If $f : \mathbb{R}^3 \to \mathbb{R}$, then for x, y and z in $\mathbb{R}$,

$$D_1 f(x, y, z) = \lim_{t \to 0} \frac{f(x+t, y, z) - f(x, y, z)}{t} =: \frac{\partial f}{\partial x}(x, y, z),$$

where we've introduced yet more notation. The expression $\frac{\partial f}{\partial x}$ you should think of as meaning 'differentiate f with respect to x, while treating y, z as constants. Returning to (1.10), we see that for any $v = (v^1, v^2, v^3)$, we have

$$Df(p)[v] = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix},$$

so that the Jacobian of f at p is given by

$$Df(p) = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix}.$$

To introduce even more notation, we sometimes write

$$\nabla f(p) = \begin{pmatrix} D_1 f(p) \\ D_2 f(p) \\ D_3 f(p) \end{pmatrix},$$

which is called the **gradient** of f at p, and with this notation

$$Df(p) = (\nabla f(p))^t.$$

We can extend all of these notions to more general range and domains, which leads us to the following definition.

Definition 1.6. Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$. For any vector $v \in \mathbb{R}^n$ with $\|v\| = 1$, the **directional derivative** of f at p in the direction v is given by

$$\frac{\partial f}{\partial v}(p) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} = Df(p)[v]$$

The partial derivatives of f at p are given by

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p) = \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t}, \quad i = 1, \ldots, n.$$

Notice that $f(x)$ is now a vector in $\mathbb{R}^m$, so expressions like $\lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$ have to be understood as limits in $\mathbb{R}^m$, so that $\frac{\partial f}{\partial v}(p)$ will be an $m-$dimensional column vector. That is, if

$$f(x) = (f^1(x), f^2(x), \ldots, f^m(x)),$$

then

$$D_i f(p) = \begin{pmatrix} D_i f^1(p) \\ \vdots \\ D_i f^m(p) \end{pmatrix}.$$

Theorem 1.9. *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is of the form*

$$f(x) = (f^1(x), f^2(x), \ldots, f^m(x)).$$

If f is differentiable at some $p \in \Omega$, then the Jacobian of f at p is

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & \ldots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \ldots & D_n f^m(p) \end{pmatrix}.$$

Proof. Let $\{e_i\}$ be the canonical basis for $\mathbb{R}^n$. For any $v \in \mathbb{R}^n$, we write $v = \sum_{i=1}^n v^i e_i$. Then by the linearity of $Df(p)$ we have:

$$\begin{aligned} Df(p)[v] = Df(p)\left[\sum_{i=1}^n v^i e_i\right] &= \sum_{i=1}^n v^i Df(p)\left[e_i\right] = \sum_{i=1}^n v^i D_i f(p). \\ &= \begin{pmatrix} \sum_{i=1}^n v^i D_i f^1(p) \\ \vdots \\ \sum_{i=1}^n v^i D_i f^m(p) \end{pmatrix} \\ &= \begin{pmatrix} D_1 f^1(p) & \ldots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \ldots & D_n f^m(p) \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \end{aligned}$$

□

This allows us to restate the chain rule in terms of the partial derivatives of the functions.

Corollary 1.10. *Suppose $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open sets, $g : \Omega \to \Omega'$ is differentiable at $p \in \Omega$, and $f : \Omega' \to \mathbb{R}^l$ is differentiable at $g(p)$. Then $h = f \circ g$ is differentiable at p with Jacobian*

$$Dh(p) = \begin{pmatrix} D_1 f^1(g(p)) & \ldots & D_m f^1(g(p)) \\ \vdots & \ddots & \vdots \\ D_1 f^l(g(p)) & \ldots & D_m f^l(g(p)) \end{pmatrix} \begin{pmatrix} D_1 g^1(p) & \ldots & D_n g^1(p) \\ \vdots & \ddots & \vdots \\ D_1 g^m(p) & \ldots & D_n g^m(p) \end{pmatrix}$$

In the one dimensional case, we often use the derivative to search for turning points, i.e. maxima and minima, since a differentiable function will have vanishing derivative at a local maximum or minimum. A similar result holds in the higher dimensional case.

Lemma 1.11. *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ be differentiable at each point in Ω. Suppose that f has a local maximum at $p \in \Omega$. Then:*

$$Df(p) = 0.$$

Similarly if p is a local minimum.

Proof. Pick $v \in \mathbb{R}^n$. Since Ω is open, there exists $\epsilon > 0$ such that $p + tv \in \Omega$ for $t \in (-\epsilon, \epsilon)$. Consider the function $g_v : (-\epsilon, \epsilon) \to \mathbb{R}$ defined as

$$g_v(t) = f(p + tv).$$

Since f has a local maximum at p, g_v has a local maximum at 0 and moreover, g_v is differentiable by the chain rule, so we deduce

$$0 = g_v'(0) = Df(p)[v].$$

Since v was arbitrary, we have that $Df(p) = 0$. A similar argument deals with the case where p is a minimum. $\square$

Exercise 1.16. (i) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = (x^2 + e^{x+y}, x - \log y, 2xy + 1).$$

Assuming f is differentiable at a point (x, y), what is its derivative?

(ii) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by

$$g(x, y, z) = x + y + z.$$

Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.

1.4.2 Relation between partial derivatives and differentiability

We have seen above that for a function $f : \mathbb{R}^n \to \mathbb{R}$ which is differentiable at some point p, the limits

$$D_i f(p) := \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} \tag{1.11}$$

exist for $i = 1, \ldots n$, and moreover these limits completely determine the derivative of f at p. One might hope, based on this, that in order for f to be differentiable at p it is enough to know that the partial derivatives (i.e. the limits in (1.11)) of f at p all exist. Unfortunately, this is not the case, as we show in the following example.

Example 1.11. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} 0 & x = y = 0 \\ \frac{xy}{\sqrt{x^2 + y^2}} & \text{otherwise} \end{cases}$$

See Figure 1.4 for the graph of the function f.

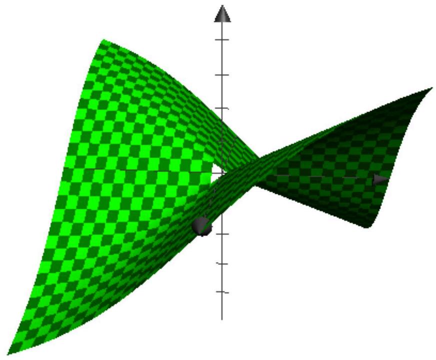

Figure 1.4: The graph of the function in Example 1.11.

First note that this function is continuous at the origin. Since $|xy| \leq \frac{1}{2}\left(x^2+y^2\right)$, we have that for $p=(x,y) \neq (0,0)$:

$$|f(p)| \leq \frac{1}{2}\sqrt{x^2+y^2},$$

so that

$$\lim_{p \to 0} f(p) = 0.$$

Now consider the partial derivatives. We have

$$D_1 f(0) = \lim_{t \to 0} \frac{1}{t}\left[f(te_1) - f(0)\right] = \lim_{t \to 0} \frac{0-0}{t} = 0$$

since $f(te_1) = 0$ for all t. Similarly, we also have

$$D_2 f(0) = \lim_{t \to 0} \frac{1}{t}\left[f(te_2) - f(0)\right] = \lim_{t \to 0} \frac{0-0}{t} = 0$$

Thus, if f is differentiable, then it must be that $Df = 0$, so all directional derivatives at 0 exist and are equal to zero. However, let $h = \frac{1}{\sqrt{2}}(1,1)$. For $t > 0$, we have

$$\frac{f(th) - f(0)}{t} = \frac{t^2/2}{t^2} = \frac{1}{2},$$

which contradicts the differentiability of f at the origin. Thus, even though the partial derivatives exist for this function, the function is not differentiable.

Away from the origin, the function is a composition of smooth functions so is differentiable. We can calculate the partial derivatives at a point $p=(x,y) \neq (0,0)$ and we find

$$D_1 f(p) = \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}},$$

and by symmetry:

$$D_2 f(p) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}.$$

We claim that the function $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by

$$g(x,y) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$$

has no limit as $p = (x,y)$ converges to $(0,0)$. To see this, let $p = (r\cos\theta, r\sin\theta)$ for some $r \in (0,\infty)$, $\theta \in [0, 2\pi)$. Then

$$g(p) = \cos^3\theta,$$

so there can be no limit as $r \to 0$, since g approaches a different value depending on which angle we approach from.

As it happens, the fact that the partial derivatives are not continuous in a neighbourhood of the origin is the only barrier to differentiability there.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$. Suppose the partial derivatives*

$$D_i f(x) := \lim_{t\to 0} \frac{f(x+te_i) - f(x)}{t}$$

exist for all $x \in \Omega$, and moreover suppose that the maps

$$x \mapsto D_i f(x)$$

are continuous at $p \in \Omega$ for all $i = 1, \dots, n$. Then f is differentiable at p.

(*) *Proof.* Since Ω is open, there exists $r > 0$ such that $B_r(p) \subset \Omega$. Suppose $h \in B_r(0)$ has components h^i, so that $h = \sum_{i=1}^n h^i e_i$. We consider

$$\begin{aligned} f(p+h) - f(p) &= f\left(p + \sum_{i=1}^n h^i e_i\right) - f(p) \\ &= f\left(p + \sum_{i=1}^n h^i e_i\right) - f\left(p + \sum_{i=1}^{n-1} h^i e_i\right) \\ &\quad + f\left(p + \sum_{i=1}^{n-1} h^i e_i\right) - f\left(p + \sum_{i=1}^{n-2} h^i e_i\right) \\ &\quad + \dots \\ &\quad + f(p + h^1 e_1) - f(p). \end{aligned}$$

Let's consider a typical line in the right hand side of the above equation, that is,

$$f\left(p + \sum_{i=1}^k h^i e_i\right) - f\left(p + \sum_{i=1}^{k-1} h^i e_i\right) = f(q + h^k e_k) - f(q),$$

where $k \in \{1, \ldots, n\}$ and $q = p + \sum_{i=1}^{k-1} h^i e_i$. Now, applying the mean value theorem to the function $g(t) = f(q + te_k)$, which is differentiable by assumption, there exists $s \in \left[-\left|h^k\right|, \left|h^k\right|\right)$ such that:

$$f(q + h^k e_k) - f(q) = h^k D_k f(q + se_k) = h^k D_k f(p + c_k),$$

where $c_k = \sum_{i=1}^{k-1} h^i e_i + se_k$. One has to consider separately the cases $h^k > 0$, $h^k < 0$ and $h^k = 0$. Now, note that since $|s| \leq \left|h^k\right|$, we have

$$\|c_k\| \leq \|h\| .$$

Putting this together, we conclude that there exists $c_1, \ldots, c_n \in \mathbb{R}^n$ with $\|c_k\| \leq \|h\|$ such that

$$f(p + h) - f(p) = \sum_{k=1}^{n} h^k D_k f(p + c_k).$$

From here we can estimate using the Cauchy-Schwartz identity

$$\begin{aligned}
\left| f(p + h) - f(p) - \sum_{k=1}^{n} h^k D_k f(p) \right| &\leq \sum_{k=1}^{n} h^k \left| D_k f(p + c_k) - D_k f(p) \right| \\
&\leq \|h\| \left(\sum_{k=1}^{n} \left| D_k f(p + c_k) - D_k f(p) \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

so that

$$\frac{\left| f(p + h) - f(p) - \sum_{k=1}^{n} h^k D_k f(p) \right|}{\|h\|} \leq \left(\sum_{k=1}^{n} \left| D_k f(p + c_k) - D_k f(p) \right|^2 \right)^{\frac{1}{2}}.$$

Now, fix $\epsilon > 0$. Since $x \mapsto D_k f(x)$ is continuous at p, for each $k = 1, \ldots, n$, there exists δ_k such that if $\|c\| < \delta_k$ we have:

$$\left| D_k f(p + c) - D_k f(p) \right| < \frac{\epsilon}{\sqrt{n}}.$$

Suppose $\|h\| < \min\{\delta_1, \ldots, \delta_n\} =: \delta$. Then as $\|c_k\| \leq \|h\|$, we deduce

$$\frac{\left| f(p + h) - f(p) - \sum_{k=1}^{n} h^k D_k f(p) \right|}{\|h\|} < \left(\sum_{k=1}^{n} \frac{\epsilon^2}{n} \right)^{\frac{1}{2}} = \epsilon.$$

As ϵ was arbitrary, we conclude that f is differentiable at p, with derivative

$$Df(p)[h] = \sum_{k=1}^{n} D_k f(p) h^k. \qquad \square$$

Exercise 1.17. Show that each of the following maps $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable

(a) $f(x, y) = x^2 + y^2 - x - xy$,

(b) $f(x, y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x, y) = x^5y^2$.

For maps $f : (a, b) \to \mathbb{R}$ we have learned that when f is differentiable at some $p \in (a, b)$, then there is a tangent line to the graph of f that passes through $(p, f(p))$ and approximates the graph of f near p. This is an intuitive picture that is only valid when we consider the graph of a function from one dimension to one dimension. By an example below, we show that this intuition should not be employed for maps of higher dimensions.

Example 1.12. Let $f : (-1, +1) \to \mathbb{R}^2$ be define as

$$f(x) = \begin{cases} (x^2, 0) & \text{if } x \geq 0 \\ (0, x^2) & \text{if } x < 0. \end{cases}$$

See Figure 1.12 for the image of the map f.

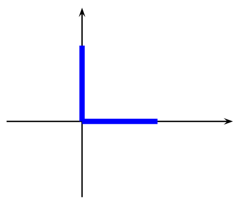

Figure 1.5: The image of the map f in Example 1.12

Clearly, f is continuous at 0 with

$$\lim_{x \to 0} f(x) = (0, 0) = f(0).$$

The map f is differentiable at 0 with derivative equal to the constant linear map $\Lambda = 0$. To see this, note that

$$\lim_{h \to 0} \frac{\|f(0 + h) - f(0) - \Lambda[h]\|}{\|h\|} = \lim_{h \to 0} \frac{\|f(h)\|}{\|h\|} = \lim_{h \to 0} \frac{h^2}{|h|} = \lim_{h \to 0} |h| = 0.$$

In fact, it is not possible to understand just by looking at the image of a map whether it is differentiable or not. As the example below shows, maps with the same image may or may not be differentiable.

Example 1.13. Define the maps k and g from $(-1, +1)$ to $\mathbb{R}^2$ as

$$k(x) = (x, x^3), \quad g(x) = (x^{1/3}, x).$$

See Figure 1.6 for the images of the maps f and g.

The maps k and g are continuous at 0 with

$$\lim_{x\to 0} k(x) = (0,0) = k(0),$$

and

$$\lim_{x\to 0} g(x) = (0,0) = g(0).$$

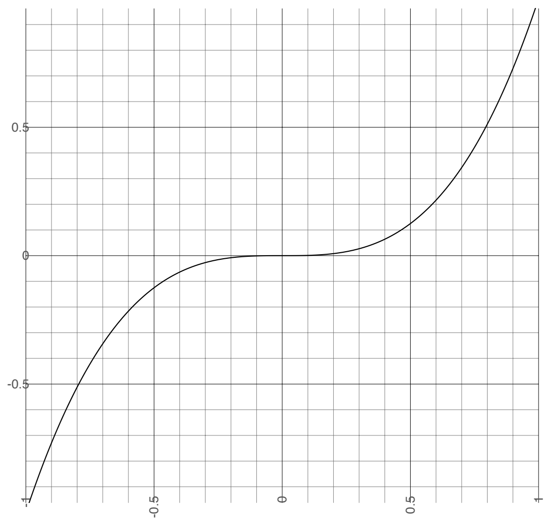

Figure 1.6: The image of the maps f and g in Example 1.13. The differentiability at $(0,0)$ depends on "how fast" we pass through the point $(0,0)$.

The maps k and g have the same image, that is, they map the interval $(-1,+1)$ to the same curve, which is the graph of the function $t \mapsto t^3$ on the interval $(-1,+1)$. However, k is differentiable at 0, but g is not differentiable at 0, as we show below.

We claim that the derivative of the map k at 0 is equal to the linear map $\Lambda(h) = (h, 0)$. To see this, note that

$$\begin{aligned}\lim_{h\to 0} \frac{\|k(0+h) - k(0) - \Lambda[h]\|}{\|h\|} &= \lim_{h\to 0} \frac{\big\|(h,h^3) - (h,0)\big\|}{\|h\|}\\ &= \lim_{h\to 0} \frac{\big\|(0,h^3)\big\|}{\|h\|}\\ &= \lim_{h\to 0} \frac{|h|^3}{|h|} = 0.\end{aligned}$$

To prove that g is not differentiable at 0, we need to show that there is no linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ which is the derivative of the map g at 0. In contrary assume that there is a linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ such that

$$\lim_{h\to 0} \frac{\|g(0+h) - g(0) - \Lambda[h]\|}{\|h\|} = 0.$$

Let $\Lambda(1) = (a, b) \in \mathbb{R}^2$, for some real constants a and b in $\mathbb{R}$. It follows that for every $h \in \mathbb{R}$ we have

$$\Lambda(h) = \Lambda(h \cdot 1) = h\Lambda(1) = h(a, b) = (ha, hb).$$

Therefore,

$$\begin{aligned} 0 = \lim_{h\to 0} \frac{\|g(0+h) - g(0) - \Lambda[h]\|}{\|h\|} &= \lim_{h\to 0} \frac{\left\|(h^{1/3} - ah, h - bh)\right\|}{|h|} \\ &= \lim_{h\to 0} \frac{\left\|h(h^{-2/3} - a, 1 - b)\right\|}{|h|} \\ &= \lim_{h\to 0} \left\|(h^{-2/3} - a, 1 - b)\right\| \\ &= \left\|\lim_{h\to 0}(h^{-2/3} - a, 1 - b)\right\| \end{aligned}$$

In the last line of the above equation we have used that $\|\cdot\|$ is a continuous function, so we may interchange the limit and the norm. Now recall that $\|y\| = 0$, if and only if $y = 0$. Thus we must have

$$\lim_{h\to 0}(h^{-2/3} - a, 1 - b) = (0, 0)$$

which implies that

$$\lim_{h\to 0} h^{-2/3} - a = 0, \qquad \text{and } \lim_{h\to 0} 1 - b = 0.$$

This is a contradiction, since for any real number a we have

$$\lim_{h\to 0} h^{-2/3} - a = \infty.$$

This contradiction shows that there is no linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ satisfying the definition of differentiability for g at 0.

Note that the value of the other limit does not lead to any contradiction, it only says that b must be equal to 1.

1.5 Higher derivatives

1.5.1 Higher derivatives as linear maps

Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f : \Omega \to \mathbb{R}^m$ is differentiable at every point $p \in \Omega$. We may think of the differential of f as a map

$$\begin{array}{rcccl} Df & : & \Omega & \to & L(\mathbb{R}^n; \mathbb{R}^m) \\ & & p & \mapsto & Df(p). \end{array}$$

Recall that every member of $L(\mathbb{R}^n; \mathbb{R}^m)$ may be expressed as an m by n matrix, using the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$. We can think of each m by n matrix as a point in $\mathbb{R}^{mn}$, for example, by

$$(a_{i,j})_{1 \le i \le m, 1 \le j \le n} \mapsto (a_{1,1}, \ldots, a_{1,n}, a_{2,1}, \ldots, a_{2,n}, \ldots, a_{m,1}, \ldots, a_{m,n}).$$

Thus, we may think of Df as a map from Ω to $\mathbb{R}^{mn}$. We can consider whether this map Df is continuous, or differentiable at a point $p \in \Omega$. If the map $Df : \Omega \to \mathbb{R}^{mn}$ is continuous, we say $f : \Omega \to \mathbb{R}^m$ is **continuously differentiable**. If $Df : \Omega \to \mathbb{R}^{mn}$ is differentiable at p, the derivative at p, denoted by $DDf(p)$, is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^{mn}$. That is,

$$DDf(p) \in L(\mathbb{R}^n; \mathbb{R}^{nm}) = L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m)).$$

Thus, $DDf(p)$ takes an n-vector to an $(m \times n)$ matrix. The above notation may appear complicated, but you have already seen some examples of maps in the right hand of the above equation. For example, the map $h \mapsto \langle h, \cdot \rangle$ is an element of $L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^1))$, that is, for every $h \in \mathbb{R}^n$, the map $u \mapsto \langle h, u \rangle$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^1$.

In terms of our definition of derivative, $DDf(p)$ is a linear map $\mathcal{L} \in L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$ such that the following holds

$$\lim_{x \to p} \frac{\|Df(x) - Df(p) - \mathcal{L}[x - p]\|}{\|x - p\|} = 0.$$

Note that in the above equation, the norm on the numerator is the norm $\|\cdot\|$ on $\mathbb{R}^{mn}$ and the norm on the denominator is $\|\cdot\|$ on $\mathbb{R}^n$.

Obviously, we can generalise this to consider a map which is $k-$times differentiable. In practice, the condition of $k-$times differentiable at a point can be difficult to establish. However, if $f : \Omega \to \mathbb{R}^m$ is k times differentiable with all those derivatives continuous, we say f is **k-times continuously differentiable**.

Assume that $f = (f^1, f^2, \ldots, f^m)$. We know from the previous results that if f is differentiable at $p \in \Omega$, the partial derivative maps

$$\begin{array}{rcccl} D_i f^j & : & \Omega & \to & \mathbb{R} \\ & & x & \mapsto & D_i f^j(x). \end{array}$$

exist for all $x \in \Omega$. If moreover, Df is differentiable at $p \in \Omega$, then the second partial derivatives

$$D_k D_i f^j(p) := \lim_{t \to 0} \frac{D_i f^j(p + te_k) - D_i f^j(p)}{t}$$

will exist.

It is easier to ask if all of the $k-$th partial derivatives exist and are continuous in a neighbourhood of p. This is a slightly stronger condition, which implies $k-$times differentiability at p by Theorem 1.12.

Example 1.14. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) = x^3 + y^3 + 5x^2 y.$$

This is differentiable at each point $p = (x, y) \in \mathbb{R}^2$, and the partial derivatives are

$$D_1 f(p) = 3x^2 + 10xy, \qquad D_2 f(p) = 3y^2 + 5x^2.$$

To find the second partial derivatives, we consider the maps

$$D_1 f(x, y) = 3x^2 + 10xy,$$

and

$$D_2 f(x, y) = 3y^2 + 5x^2,$$

and differentiate them. The second partial derivatives are thus

$$\begin{aligned} D_1 D_1 f(p) &= 6x + 10y \\ D_2 D_1 f(p) &= 10x \\ D_1 D_2 f(p) &= 10x \\ D_2 D_2 f(p) &= 6y \end{aligned}$$

Notice that

$$D_2 D_1 f(p) = D_1 D_2 f(p).$$

This is a coincidence!

1.5.2 Symmetry of mixed partial derivatives

We will state a result here, but do not give a proof. This is not the optimal result in this direction, but it is perfectly adequate for most purposes in the next section.

Theorem 1.13 (Schwartz' Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ is differentiable at every $p \in \Omega$. Suppose further that for some $i, j \in \{1, \dots, n\}$ the second partial derivatives $D_i D_j f$ and $D_j D_i f$ exist and are continuous at all $p \in \Omega$. Then, at every $p \in \Omega$,*

$$D_i D_j f(p) = D_j D_i f(p).$$

If $f : \Omega \to \mathbb{R}$, the matrix of second partial derivatives at the point p,

$$\text{Hess}\, f(p) = [D_i D_j f(p)]_{i,j=1,\dots,n}$$

is called the **Hessian** of f at p. Assuming the hypotheses on the second partial derivatives hold, Schwartz' Theorem states that the Hessian is a symmetric matrix.

Exercise 1.18. Suppose A is a symmetric $(n \times n)$ matrix. Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = xAx^t.$$

(a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with

$$Df(p) = 2pA$$

(b) Find

$$\text{Hess}\, f(p).$$

Exercise 1.19. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : (x, y, z) = xy^2 + x^2 + xze^y.$$

(i) Compute the first and second partial derivatives. Observe the properties of the second partial derivative.

(ii) Write the terms of the Taylor expansion of f at zero up to and including the second-order terms.

(iii) Without computation, write the same Taylor expansion up to and including the fourth-order terms.

Exercise 1.20 (*)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \frac{xy^3 - x^3y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Show that

$$D_1 f(x, y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2+y^2} - \frac{2x(xy^3 - x^3y)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

and

$$D_2 f(x, y) = \begin{cases} \frac{3y^2x - x^3}{x^2+y^2} - \frac{2y(xy^3 - x^3y)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that both of these functions are continuous at $(0, 0)$.

(b) Show that

$$\lim_{t\to 0} \frac{1}{t} \left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t} \left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

(c) Conclude that both $D_2 D_1 f(0)$ and $D_1 D_2 f(0)$ exist, but

$$D_2 D_1 f(0) \neq D_1 D_2 f(0)$$

1.5.3 Taylor's theorem

The differentiability of a map of higher dimensions allows us to approximate the map near a point with a linear map which is a simpler object. This has significant consequences which we discus in Section 1.6. However, when thinking of differentiabilities of higher orders, one may wonder if those lead to better approximations than the ones by a linear map, perhaps, by more complicated objects than linear maps. In terms of complexity, the next class of maps after linear ones are polynomial maps in several variables. We look into such approximations in this section.

A powerful result concerning differentiable functions of one variable is Taylor's theorem, which permits us to approximate a function in a neighbourhood of a point p by a polynomial, with an error term that goes to zero at a controlled rate as we approach p. In order to state Taylor's theorem for higher dimensions, it's useful to introduce some new notation.

When dealing with partial derivatives of high orders, the notation can get rather messy. To mitigate this, it's convenient to introduce "multi-indices". We define a multi-index α to be an element of $(\mathbb{N})^n$, i.e. an n-vector of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and

$$D^\alpha f := (D_1)^{\alpha_1} (D_2)^{\alpha_2} \cdots (D_n)^{\alpha_n} f,$$

It's convenient to also introduce, for a vector $h = (h^1, \ldots, h^n)$,

$$h^\alpha := \left(h^1\right)^{\alpha_1} \left(h^2\right)^{\alpha_2} \cdots (h^n)^{\alpha_n}$$

as well as the multi-index factorial $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$,

Theorem 1.14. *Suppose that $p \in \mathbb{R}^n$ and $f : B_r(p) \to \mathbb{R}$ is $k-$times continuously differentiable at all points $q \in B_r(p)$, for some integer $k \geq 1$. Then, for every $h \in \mathbb{R}^n$ with $\|h\| < r$, we have*

$$f(p+h) \quad = \quad \sum_{|\alpha|\leq k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p,h).$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ *with* $|\alpha| \leq k-1$ *and the remainder term is given by:*

$$R_k(p,h) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} D^\alpha f(x)$$

for some x *with* $0 < \|x-p\| < \|h\|$.

(*) *Proof.* The result follows from the one-dimensional Taylor's theorem. First, we note that there exists $\epsilon > 0$ such that $\|h\| < \frac{r}{1+\epsilon}$. Let us define the function $g : (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$ defined as

$$g(t) = f(p+th).$$

By the chain rule, this function is k-times differentiable on the interval $(-1-\epsilon, 1+\epsilon)$, and $[0,1] \subset (-1-\epsilon, 1+\epsilon)$, so by one dimensional Taylor's theorem we have

$$g(1) = g(0) + g'(0) + \frac{g''(0)}{2!} + \ldots + \frac{g^{(k-1)}(0)}{(k-1)!} + R_k,$$

where

$$R_k = \frac{g^{(k)}(\xi)}{k!},$$

for some $\xi \in (0,1)$. We will be done if we can show that for $j = 0, 1, \ldots k$ we have

$$g^{(j)}(t) = j! \sum_{|\alpha|=j} \frac{h^\alpha}{\alpha!} D^\alpha f(p+th). \tag{1.12}$$

This is certainly true for $j = 0$. Suppose it's true for some $j \geq 0$. Then we have

$$\begin{aligned} g^{(j+1)}(t) &= \sum_{l=1}^n h^l D_l \left[j! \sum_{|\alpha|=j} \frac{h^\alpha}{\alpha!} D^\alpha f \right] (p+th) \\ &= j! \sum_{l=1}^n \sum_{|\alpha|=j} \frac{h^\alpha h^l}{\alpha!} D_l D^\alpha f(p+th) \end{aligned}$$

Clearly, the right-hand side of the above equation is a sum of terms proportional to $h^\beta D^\beta f(p+th)$ where $|\beta| = j+1$. Suppose $\beta = (\beta_1, \ldots, \beta_n)$, then the coefficient of the term proportional to $h^\beta D^\beta f(p+th)$ is

$$\begin{aligned} &\frac{j!}{(\beta_1-1)!\beta_2!\cdots\beta_n!} + \frac{j!}{\beta_1!(\beta_2-1)!\cdots\beta_n!} + \ldots + \frac{j!}{\beta_1!\beta_2!\cdots(\beta_n-1)!} \\ &= \frac{(j+1)!}{\beta_1!\beta_2!\cdots\beta_n!} = \frac{(j+1)!}{\beta!}, \end{aligned}$$

by a result from combinatorics (you do not need to verify this). Thus we have

$$\begin{aligned} g^{(j+1)}(t) &= j! \sum_{l=1}^n \sum_{|\alpha|=j} \frac{h^\alpha h^l}{\alpha!} D_l D^\alpha f(p+th) \\ &= (j+1)! \sum_{|\beta|=j+1} \frac{h^\beta}{\beta!} D^\beta f(p+th) \end{aligned}$$

By induction we conclude that (1.12) holds for all $j = 0, \ldots, k$ and the result follows. □

Exercise 1.21. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = e^x \sin(y)$.

a) Compute the degree 1 and degree 2 Taylor polynomial of f near the point $(x_0, y_0) = (0, \pi/2)$ and use those to approximate the value of f at $(x_1, y_1) = (0, \pi/2 + 1/4)$. Compare your results with the values you obtain from a calculator.

b) How precise is the degree 1 approximation in the closed ball of radius 1/4 around (x_0, y_0). Find a rigorous upper bound for the approximation error.

1.6 Inverse and Implicit function theorems

1.6.1 Inverse function theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in an interval around $p \in \mathbb{R}$, with $f'(p) \neq 0$, say $f'(p) > 0$. Then there is an open interval I with $p \in I$ such that $f'(x) > 0$ for all $x \in I$. This (by the mean value theorem) implies that f is strictly monotone increasing on I and hence $f : I \to f(I)$ is bijective. In particular, there exists an inverse function $f^{-1} : f(I) \to I$. With a little work, one can establish that f^{-1} is differentiable, and moreover, by an application of the chain rule, obtain the following formula for the derivative of the inverse map,

$$f'(p) = \frac{1}{(f^{-1})'(f(p))}.$$

This result can be generalised to higher dimensions.

Theorem 1.15 (Inverse Function Theorem)**.** *Let Ω be an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ continuously differentiable on Ω, and there is $q \in \Omega$ such that $Df(q)$ invertible.*

Then, there are open sets $U \subset \Omega$ and $V \subset \mathbb{R}^n$ with $q \in U$ and $f(q) \in V$ such that

(i) $f : U \to V$ is a bijection,

(ii) $f^{-1} : V \to U$ is continuously differentiable,

(iii) for all $y \in V$,

$$Df^{-1}(y) = \left[Df(f^{-1}(y))\right]^{-1}.$$

Recall that since the Jacobian $Df(q)$ is an $n \times n$ matrix, the statement that it is invertible is equivalent to the statement that $\det Df(q) \neq 0$.

Example 1.15. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = (x + y + 5xy, y - x^2)$$

The partial derivatives of f are

$$D_1 f(x, y) = (1 + 5y, -2x), \qquad D_2 f(x, y) = (1 + 5x, 1).$$

Evidently, both of these maps are continuous from $\mathbb{R}^2$ to $\mathbb{R}^2$. Thus, by Theorem 1.12, f is differentiable at every point in $\mathbb{R}^2$. Moreover, by Theorem 1.9, the Jacobian of f at $(x, y) \in \mathbb{R}^2$ is given by the matrix

$$Df(x, y) = \begin{pmatrix} 1 + 5y & 1 + 5x \\ -2x & 1 \end{pmatrix}.$$

This is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^{2\times 2} = \mathbb{R}^4$.

We note that

$$Df(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with $\det Df(0, 0) = 1 \neq 0$, and hence $Df(0, 0)$ is invertible. By the Inverse Function Theorem, f is invertible on some neighbourhood of the origin, with

$$Df^{-1}(0, 0) = [Df(0, 0)]^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

It is worth noting that obtaining an explicit formula for the inverse map is not easy, and hence the derivative of the inverse map is out of reach using the direct approach.

Exercise 1.22. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - xy \\ x^2 \end{pmatrix}$$

Determine the set of points in $\mathbb{R}^2$ such that f is invertible near those points, and compute the derivative of the inverse map.

The Inverse Function Theorem has applications to solving systems of equations. Assume that we have n equations in n unknowns $x^1, x^2, \ldots x^n$, given in the form

$$\begin{aligned} f^1(x^1, x^2, \ldots, x^n) &= y^1, \\ f^2(x^1, x^2, \ldots, x^n) &= y^2, \\ &\vdots \\ f^n(x^1, x^2, \ldots, x^n) &= y^n. \end{aligned}$$

where $y^1, y^2, \dots y^n$ are given real numbers, and $f^1, f^2, \dots, f^n$ are some functions of $x^1, x^2, \dots, x^n$.

For arbitrary values of $x_0^1, x_0^2, \dots x_0^n$, we obtain real numbers $y_0^1, y_0^2, \dots y_0^n$ satisfying the above equation. That is, we define the values of $y_0^1, y_0^2, \dots y_0^n$ using the above functions. The Inverse Function Theorem can be used here and guarantees that if the map $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$F(x^1, x^2, \dots, x^n) = \big(f^1(x^1, x^2, \dots, x^n), f^2(x^1, x^2, \dots, x^n), \dots, f^n(x^1, x^2, \dots, x^n)\big)$$

is continuously differentiable, and DF at $(x_0^1, x_0^2, \dots, x_0^n)$ is invertible, then for all values of $y^1, y^2, \dots y^n$ sufficiently close enough to $y_0^1, y_0^2, \dots y_0^n$ the above system of equations has a unique solution. Indeed the solution is given by the inverse of the map F.

For example, by the previous example, we conclude that for a and b close to 0, the equations

$$\begin{aligned} x + y + 5xy &= 0 \\ y - x^2 &= 0 \end{aligned}$$

has unique solutions for x and y.

This is a fairly powerful statement, but the issue here is that the theorem does not immediately say how close one must have a and b to 0 in order for the solutions exist. It only says that for close enough a and b, there are solutions. However, since there is a constructive proof of the theorem, one can follow the steps in the proof, and obtain an explicit neighbourhood of $(0, 0)$ such that for all (a, b) in that neighbourhood, the solutions exit.

Let Ω and Ω' be open subsets of $\mathbb{R}^n$. We say that a map $f : \Omega \to \Omega'$ is a **C^1-diffeomorphism**, if $f : \Omega \to \Omega'$ is a bijection (i.e. injective and surjective), $f : \Omega \to \Omega'$ is continuously differentiable, and for every $x \in \Omega$, $Df(x)$ is invertible.

Example 1.16. Let Ω be an open sets in $\mathbb{R}^n$, and define $\mathcal{D}$ as the set of all C^1-diffeomorphisms from Ω to Ω. Then $\mathcal{D}$ is a group, with the operation

$$f * g = f \circ g.$$

To see this, first we show that for every f and g in $\mathcal{D}$, $f * g$ belongs to $\mathcal{D}$. So we need to show that $f \circ g$ is a C^1-diffeomorphism from Ω to Ω. We need to verify three properties for $f \circ g$.

• Since f and g belong to $\mathcal{D}$, $f : \Omega \to \Omega$ and $g : \Omega \to \Omega$ are bijections. Hence, $f \circ g : \Omega \to \Omega$ is a bijection.

• Since f and g belong to $\mathcal{D}$, they are continuously differentiable at every point in Ω. Thus, by the chain rule, the map $f \circ g : \Omega \to \Omega$ is differentiable at every point in Ω, with

$$D(f \circ g)(x) = D(f(g(x)) \circ Dg(x).$$

Thus $f \circ g$ is differentiable on Ω. Also, since the maps $y \mapsto Df(y)$ and $x \mapsto Dg(x)$ are continuous on Ω, and the composition of continuous maps is a continuous map, the above formula shows that $D(f \circ g)$ is continuous on Ω. Thus, $f \circ g$ is continuously differentiable on Ω.

• Since f and g belong to $\mathcal{D}$, both $Df(y)$ and $Dg(x)$ are invertible at all x and y in Ω. The composition of invertible matrixes is invertible. Thus, the above formula shows that $D(f * g)$ must be invertible at every point.

The associativity of the operation $*$ is obtained from the associativity of the composition operation for functions. That is, for all f, g and h in $\mathcal{D}$, we have

$$(f * g) * h = (f \circ g) \circ h = f \circ (g \circ h) = f * (g * h).$$

The identity map $\mathrm{id} : \Omega \to \Omega$ is a C^1-diffeomorphism and hence belongs to $\mathcal{D}$. It is the identity element in $\mathcal{D}$, since for every $f \in \mathcal{D}$, we have

$$f * \mathrm{id} = f \circ \mathrm{id} = f, \qquad \mathrm{id} * f = \mathrm{id} \circ f = f.$$

Finally, for every $f \in \mathcal{D}$ we need to show that f^{-1} belongs to $\mathcal{D}$. First we note that $f^{-1} : \Omega \to \Omega$ is a bijection. Since, f is continuously differentiable on Ω and $Df(x)$ is invertible, by the Inverse Function Theorem, f^{-1} is invertible on some neighbourhood of $f(x)$, and $D(f^{-1})(f(x)) = [Df(x)]^{-1}$ is invertible. This is true on a neighbourhood of $f(x)$ for every $x \in \Omega$. So, since f is surjective, this is true on a neighbourhood of every point in $f(\Omega) = \Omega$.

When $\Omega = B_1(0)$ is the open ball of radius 1 about the origin, every rotation about 0 is an element of $\mathcal{D}$. However, there are many other maps in $\mathcal{D}$. It forms a very large group, as seen, for example when $\Omega = (-1, 1)$ is the open interval in $\mathbb{R}$.

Exercise 1.23. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

Exercise 1.24. The non-linear system of equations

$$e^{xy} \sin(x^2 - y^2 + x) = 0$$
$$e^{x^2+y} \cos(x^2 + y^2) = 1$$

admits the solution $(x, y) = (0, 0)$. Prove that there exists $\varepsilon > 0$ such that for all (ξ, η) with $\xi^2 + \eta^2 < \varepsilon^2$, the perturbed system of equations

$$e^{xy} \sin(x^2 - y^2 + x) = \xi$$
$$e^{x^2+y} \cos(x^2 + y^2) = 1 + \eta$$

has a solution $(x(\xi, \eta), y(\xi, \eta))$ which depends continuously on (ξ, η).

1.6.2 Implicit Function Theorem

In the previous section, we saw that the Inverse Function Theorem has applications to systems of n equations with n unknowns. What if there are more unknowns than equations. That is for some $n > m$, we have

$$\begin{aligned} f^1(x^1, x^2, \dots, x^n) &= y^1, \\ f^2(x^1, x^2, \dots, x^n) &= y^2, \\ &\vdots \\ f^m(x^1, x^2, \dots, x^n) &= y^m. \end{aligned}$$

We look into this through a simple example. Consider the equation

$$x^2 + y^2 - 1 = 0.$$

We can consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$F(x, y) = x^2 + y^2 - 1$$

and think of the above equation as

$$F(x, y) = 0.$$

Suppose (a, b) satisfies $F(a, b) = 0$, and $a \neq 1, -1$. Then there is an open interval A containing a and an open interval B containing b with the property that for each $x \in A$ there is a unique $y \in B$ such that $F(x, y) = 0$. This permits us to define a map $g : A \to B$ by $g(x) = y$, so that $F(x, g(x)) = 0$. We can think of this as 'locally solving for y in terms of x'. If $b > 0$ then $g(x) = \sqrt{1 - x^2}$. For the problem at hand, there is in fact another number b_1 such that $F(a, b_1) = 0$. Associated to this point there is an open interval B_1 containing b_1 and a map $g_1 : A \to B_1$ such that $F(x, g_1(x)) = 0$. (If $b > 0$, then $b_1 < 0$ and $g_1(x) = -\sqrt{1 - x^2}$). Both g, g_1 are differentiable. See Figure 1.7.

In contrast when $a = \pm 1$ we must have $b = 0$ in order to have $a^2 + b^2 = 1$. Assume that $a = +1$. There are no open sets $A \subset \mathbb{R}$ containing a and $B \subset \mathbb{R}$ containing b satisfying the following property

$$\text{for every } x \in A \text{ there is a unique } y \in B \text{ satisfying } x^2 + y^2 = 1.$$

This is because, since B is open, there is $\delta > 0$ such that $(-\delta, \delta) \subset B$. Now, for every $x \in A$ close enough to $a = 1$, there are two points $\pm\sqrt{1 - x^2}$ that belong to B. Of course one might wish to rectify this problem with choosing A as an interval of the $(1-c, 1]$, and B an interval of the from $[0, \sqrt{1 - c^2})$ so that for every $x \in A$ there is a unique $y \in B$ satisfying $x^2 + y^2 = 1$. But, when we go to higher dimensions, it is not clear what is the correct analogue of the intervals of the form $[z, w)$ or $(z, w]$.

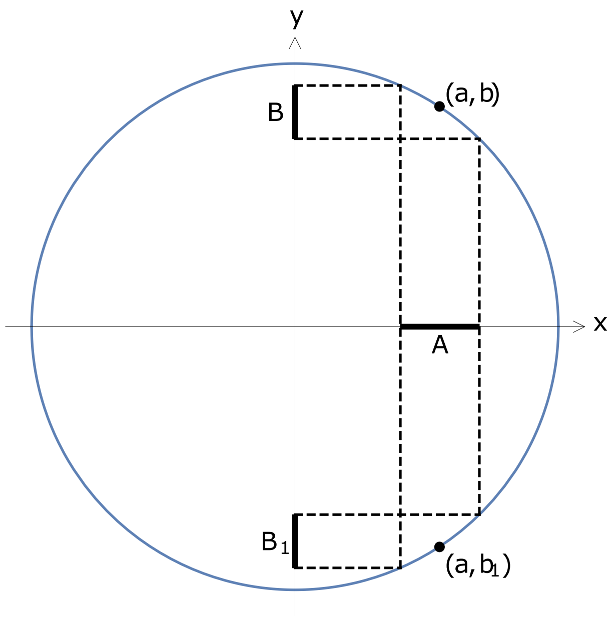

Figure 1.7: The set $x^2 + y^2 - 1 = 0$, and the intervals A, B, B_1.

The main question here is to identify the conditions on F which allows us to write the solutions of the equation $F(x, y) = 0$ as graphs of maps. The Implicit Function Theorem gives us a sufficient condition for this property to be true in a more general setting. We first state a relatively easier version of the theorem.

Theorem 1.16 (Implicit Function Theorem–low dimensional version)**.** *Assume that $\Omega \subset \mathbb{R}^2$ is open, $F : \Omega \to \mathbb{R}$ is continuously differentiable, and there is $(x', y') \in \Omega$ such that*

(i) $F(x', y') = 0$, *and*

(ii) $D_2F(x', y') \neq 0$.

Then, there are open sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ with $x' \in A$ and $y' \in B$, and a map $f : A \to B$ such that

$$(x, y) \in A \times B \text{ satisfies } F(x, y) = 0 \quad \text{iff} \quad y = f(x) \text{ for some } x \in A.$$

Moreover, the map $f : A \to B$ is continuously differentiable.

Roughly speaking, the above theorem states that for each solution x_0, y_0 of the equation

$$F(x, y) = 0,$$

the nearby solutions x, y of the above equation, look like the graph of a map from x unknown to the y unknown.

Exercise 1.25. For each of the following equations determine at which points one cannot find a function $y = f(x)$ which describes the graph in this neighbourhood. Sketch the graphs.

(a)

$$\frac{1}{3}y^3 - 2y + x = 1$$

(b)

$$x^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) - xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin(2\phi) + y^2\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) = 1,$$

where $a > 0$, $b > 0$, $0 \leq \phi \leq \pi/2$ are fixed parameters. Note the cases $a = b$, $\phi = 0$, $\phi = \pi/2$.

Exercise 1.26. Consider the equation

$$2x^2 + 4xy + y^2 = 3x + 4y$$

(a) Show that this system of equations (implicitly) defines a function $y = f(x)$ with $f(1) = 1$.

(b) Compute $f'(1)$ without knowing f explicitly.

(c) Find an explicit formula for f and check your result from b).

1.6.3 * Sketch of the proof of the Implicit Function Theorem

There is an intuitive argument which explains why the conditions in Theorem 1.16 are sufficient. With careful attention to details, one may turn this into a proof. The argument is fairly elementary, but since it is long, you may treat it as optional.

Consider a map $F : \Omega \to \mathbb{R}$ which satisfies the hypothesis in Theorem 1.16. We break the argument into several steps. Note that $D_2F(x', y') \neq 0$. Without loss of generality we may assume that $D_2F(x', y') > 0$ (the other case is similar).

Step 1. There is $\delta > 0$ such that for every $x \in [x' - \delta, x' + \delta]$ and every $y \in [y' - \delta, y' + \delta]$, we have $D_2F(x, y) > 0$.

To see this, note that since F is continuously differentiable, the map

$$(x, y) \mapsto D_2F(x, y)$$

is continuous from Ω to $\mathbb{R}$. As this function is positive at (x', y'), it must be positive on a neighbourhood of that point. Thus, there is $\delta > 0$ satisfying the property in Step 1.

Step 2. There are δ' with $0 < \delta' < \delta$ such that on the set $(x' - \delta', x' + \delta') \times \{y' - \delta\}$ we have $F < 0$, and on the set $(x' - \delta', x' + \delta') \times \{y' + \delta\}$ we have $F > 0$.

To see this, consider the map $h : [y' - \delta, y' + \delta] \to \mathbb{R}$ defined as

$$h(y) = F(x', y).$$

By the property in Step 1 we note that $h'(y) = D_2F(x', y) > 0$, for all $y \in (y' - \delta, y' + \delta)$. This implies that h is strictly increasing on the interval $(y' - \delta, y' + \delta)$. As $h(y') = 0$, we must have $h(y' - \delta) < 0$ and $h(y' + \delta) > 0$.

By the above paragraph, $F(x', y' - \delta) < 0$ and $F(x', y' + \delta) > 0$. Since F is continuous, there is $\delta' > 0$ such that F is negative on $(x' - \delta', x' + \delta') \times \{y' - \delta\}$, and is positive on $(x' - \delta', x' + \delta') \times \{y' + \delta\}$.

Step 3. For every $x \in (x' - \delta', x' + \delta')$, there is a unique $y \in (y' - \delta, y' + \delta)$ such that $F(x, y) = 0$.

Fix an arbitrary $x \in (x' - \delta', x' + \delta')$, and consider the map $g : [y' - \delta, y' + \delta] \to \mathbb{R}$ defined as

$$g(y) = F(x, y).$$

The map g is continuous on $[y' - \delta, y' + \delta]$, with $g(y') = F(x, y' - \delta) < 0$ and $g(y' + \delta) = F(x, y' + \delta) > 0$. By the intermediate value theorem, there must be $y \in [y' - \delta, y' + \delta]$ such that $g(y) = 0$. So $F(x, y) = 0$.

On the other hand, since $g'(y) = D_2F(x, y) > 0$ for all $y \in [y' - \delta, y' + \delta]$, g is strictly increasing on $[y' - \delta, y' + \delta]$. This implies that there is a unique point in $(y' - \delta, y' + \delta)$ where g becomes 0. This proves the uniqueness.

With the above argument, we can introduce $A = (x' - \delta', x' + \delta')$ and $B = (y' - \delta, y' + \delta)$.

1.6.4 The general form of the Implicit Function Theorem

There is a more general version of the Implicit Function Theorem for arbitrary dimensions.

Theorem 1.17 (Implicit Function Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$ be open sets, and $f : \Omega \times \Omega' \to \mathbb{R}^m$ be continuously differentiable on $\Omega \times \Omega'$. Suppose there is $p = (a, b) \in \Omega \times \Omega'$ such that*

(i) $f(p) = 0$, and

(ii) the $m \times m$ matrix

$$\left(D_{n+j} f^i(p)\right), \qquad 1 \leq i, j \leq m.$$

is invertible.

Then, there are open sets $A \subset \Omega$ and $B \subset \Omega'$ with $a \in A$ and $b \in B$, as well as a map $g : A \to B$ such that

$$f(x, y) = 0 \text{ for some } (x, y) \in A \times B \qquad \text{iff} \qquad y = g(x) \text{ for some } x \in A.$$

The map g is continuously differentiable.

1.6.5 * Equivalence of the two theorems

In this section we prove that the Inverse Function Theorem and the Implicit Function Theorem are equivalent.

Inverse Function Theorem implies the Implicit Function Theorem: Assume that f satisfies the assumptions in Theorem 1.17. We define a new map

$$F : \Omega \times \Omega' \to \mathbb{R}^n \times \mathbb{R}^m$$

as

$$F(x, y) = (x, f(x, y)).$$

The Jacobian of F at $p = (a, b)$ is

$$DF(p) = \left(\begin{array}{c|c} I & 0 \\ \hline N & M \end{array} \right)$$

Here I is the $n \times n$ identity matrix, M is the matrix in Theorem 1.17, and N is the $m \times n$ matrix with components:

$$\left(D_j f^i(p)\right), \qquad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Since $\det M \neq 0$, we must have $\det DF(p) \neq 0$. Note that $F(a, b) = (a, 0)$. Therefore, we can apply the Inverse Function Theorem to deduce the existence of open sets $U \subset \Omega \times \Omega'$ and $V \subset \mathbb{R}^n \times \mathbb{R}^m$ with $(a, b) \in U$, $(a, 0) \in V$ such that $F : U \to V$ has a continuously differentiable inverse $h : V \to U$. By shrinking U, if necessary, we can assume that $U = A \times B$ for some open sets $A \subset \Omega$ and $B \subset \Omega'$.

Note that the map h must be of the form $h(x, y) = (x, k(x, y))$ for some continuously differentiable map k (since F has this form). Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projection map $\pi(x, y) = y$. Then $f = \pi \circ F$. Now, by the associativity of composition of maps,

$$\begin{aligned} f(x, k(x, y)) &= f \circ h(x, y) = (\pi \circ F) \circ h(x, y) \\ &= \pi \circ (F \circ h)(x, y) = \pi(x, y) = y. \end{aligned}$$

Thus $f(x, k(x, 0)) = 0$, so we can take $g(x) = k(x, 0)$. □

Implicit Function Theorem implies the Inverse Function Theorem. Let $f : \Omega \to \mathbb{R}^n$ be the map in Theorem 1.15. Let us consider the map

$$F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$$

defined as

$$F(y, x) = y - f(x).$$

Let us also define $p = (f(q), q) \in \mathbb{R}^n \times \Omega$. We have

$$F(p) = 0.$$

We note that the matrix

$$D_{n+j}F^i(p), \qquad 1 \leq i, j \leq n$$

is $-Df(q)$. So, by the assumption in inverse function theorem, the above matrix is invertible. Therefore, by the Implicit Function Theorem, there is an open set $U \subset \mathbb{R}^n$ and $B \subset \Omega$ with $f(q) \in A$ and $q \in B$, and a map $g : A \to B$ such that

$$F(y, x) = 0 \text{ for some } (x, y) \in A \times B \qquad \text{iff} \qquad x = g(y) \text{ for some } y \in A.$$

In particular, for all $y \in A$, $F(y, g(y)) = 0$. By the definition of F, this means that $y = f(g(y))$, for all $y \in A$. The if and only in the above statement, implies that f is invertible on B, and g is the inverse of f on B. $\square$

Chapter 2

Metric and topological spaces

2.1 Metric spaces

2.1.1 Motivation and definition

The notions of modulus function on $\mathbb{R}$ and the norm function on $\mathbb{R}^n$ allow us to develop the analysis on Euclidean spaces. We would like to extend the fundamental notions of analysis, such as convergence of sequences, continuity of maps, etc, to more general settings. We have already seen that most concepts in higher dimensional Euclidean spaces are analogous to the corresponding concepts in one dimensional Euclidean space; replacing the modulus function with the norm function. Over all, all those concepts rely on a notion of "distance" on the ambient space.

We have all been using the concept of "distance" in our everyday life, for example, by asking

- how much time does it take to walk from my apartment to the maths department,
- how long does it take to travel from South Kensington tube station to Cambridge by public transport,
- how much does the cheapest public transport from South Kensington tube station to Heathrow airport cost,
- what is the distance, in kilometres, from London to Edinburgh.

What should be the correct way of defining "distance" in more general settings. From the above examples we can see that the notion of distance should be a function of two variables, that is, we give it two elements. There has been a long historical development on this question, with various properties proposed and refined. Here we present the outcome of those developments, and define what is now standard.

Definition 2.1. Let X be an arbitrary set. A **metric** on X is a function

$$\mathrm{d} : X \times X \to \mathbb{R}$$

satisfying the following three properties:

(M1) for all x and y in X we have $\mathrm{d}(x, y) \geq 0$, and $\mathrm{d}(x, y) = 0$ if and only if $x = y$;

(M2) for all x and y in X, $\mathrm{d}(x, y) = \mathrm{d}(y, x)$;

(M3) for all x, y and z in X, we have $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$.

Property M1 is called **positivity**, property M2 is called **symmetry**, and property M3 is called **triangle inequality**.

Remark 2.1. The triangle inequality in Euclidean spaces has a rather simple interpretation. That is, in any triangle, the length of each side is bounded from above by the sum of the lengths of the other two sides. In an arbitrary set, triangles may not make sense. But the interpretation still makes sense, and is the reason behind requiring condition M3. We think of $\mathrm{d}(x, y)$ as "the length of the shortest way from x to y". So the length of the shortest way from x to y should be bounded from above by the length of the shortest way from x to y passing through z. See Figure 2.1.

On the other hand, property M1 tells us that the metric "separates" points. That is, the distance between distinct points is strictly positive.

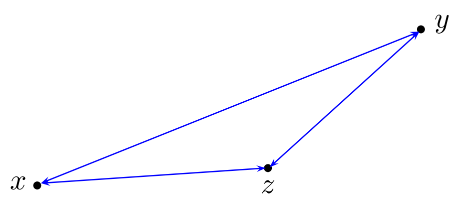

Figure 2.1: The triangle inequality.

Definition 2.2. By a **metric space** we mean a pair of a set and a metric on that set. That is often denoted as $M = (X, \mathrm{d})$, where X is a set, and $d : X \times X \to \mathbb{R}$ is a metric. We refer to M as the metric space. The elements of X are called **points**. Given two points x and y in X, the real number $\mathrm{d}(x, y)$ is called the **distance** between x and y with respect to the metric d.

In the above definition, when it is clear what metric is involved, we simply refer to $\mathrm{d}(x, y)$ as the distance between x and y.

It is customary to use the same notation for M and X, that is, the metric space $X = (X, \mathrm{d})$.

Remark 2.2. The reason that we refer to the elements of X as points, is because we would like a unified approach to all metric spaces. That is, to present statements and proofs so that it applies to a variety of settings. We understand that when $X = \mathbb{R}$, then elements of X are numbers, when $X = \mathbb{R}^n$, the elements of X are vectors, and when X is the set of all 5×5 matrices, then each element of X is a matrix. We refer to all those elements as points in X.

2.1.2 Examples of metric spaces

There are many examples of metrics. You are already familiar with some of them, although you did not use the terminology of metric spaces.

Example 2.1. Let $X = \mathbb{R}$ and $\mathrm{d}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\mathrm{d}_1(x, y) = |x - y|.$$

From the properties of the modulus function, see Section 1.1.1, we immediately see that d_1 satisfies the properties M1, M2, and M3. For example, for M2, we see that

$$\mathrm{d}_1(x, y) = |x - y| = |y - x| = \mathrm{d}_1(y, x).$$

Example 2.2. Let $X = \mathbb{R}^n$, and for $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, let

$$\mathrm{d}_2(x, y) = \|x - y\| = \left(\sum_{j=1}^{n} (x^j - y^j)^2 \right)^{1/2}.$$

By the properties of the norm function on $\mathbb{R}^n$, see Section 1.1.2, d_2 satisfies the properties M1, M2, and M3 in Definition 2.1. For example, to see property M3, we note that for every x, y, and z in $\mathbb{R}^n$, by the triangle inequality for the norm function, we have

$$\mathrm{d}_2(x, y) = \|x - y\| \leq \|x - z\| + \|z - y\| = \mathrm{d}_2(x, z) + \mathrm{d}_2(z, y).$$

The metric d_2 on $\mathbb{R}^n$ is called the **Euclidean metric** on $\mathbb{R}^n$.

Example 2.3. Let $X = \mathbb{R}^n$, and for $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, let

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j|.$$

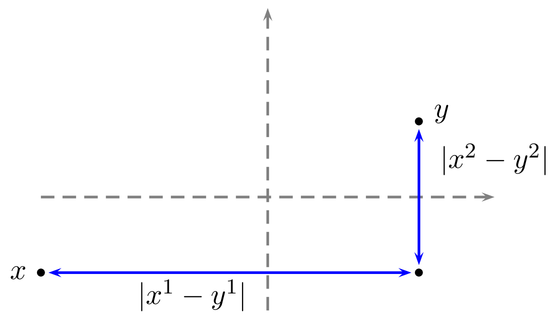

Figure 2.2: Illustration of the metric d_1 on $\mathbb{R}^2$.

We need to verify that the properties M1, M2 and M3 in Definition 2.1 hold.

M1: Fix arbitrary $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$. Since the modulus function only produces non-negative values, for every $j = 1, 2, \dots, n$, we have $|x^j - y^j| \geq 0$. Thus,

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| \geq 0.$$

On the other hand, if

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| = 0,$$

then, for all $j = 1, 2, \dots, n$, we must have $|x^j - y^j| = 0$ (because each of the numbers in the above sum is non-negative). By the first property of the modulus function, this implies that for all $j = 1, 2, \dots, n$, we have $x^j = y^j$. Hence, $x = y$.

M2: For every $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, we have

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| = \sum_{j=1}^{n} |y^j - x^j| = \mathrm{d}_1(y, x).$$

M3: For every $x = (x^1, x^2, \dots, x^n)$, $y = (y^1, y^2, \dots, y^n)$ and $z = (z^1, z^2, \dots, z^n)$ in $\mathbb{R}^n$, we have

$$\begin{aligned}
\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| &\leq \sum_{j=1}^{n} \big(|x^j - z^j| + |z^j - y^j|\big) \\
&= \sum_{j=1}^{n} |x^j - z^j| + \sum_{j=1}^{n} |z^j - y^j| \\
&= \mathrm{d}_1(x, z) + \mathrm{d}_1(z, y).
\end{aligned}$$

In the first line of the above equation we have used the triangle inequality for the modulus function n times (i.e. $|x^j - y^j| \leq |x^j - z^j| + |z^j - y^j|$, for $j = 1, 2, \dots, n$).

Intuitively, the metric d_1 on $\mathbb{R}^2$ means that we are only allowed to travel along horizontal and vertical directions to go from $x \in \mathbb{R}^2$ to $y \in \mathbb{R}^2$. See Figure 2.2.

Exercise 2.1. Let $X = \mathbb{R}^n$ and define the function $\mathrm{d}_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{d}_\infty(x, y) = \max\{|x^1 - y^1|, \dots, |x^n - y^n|\}.$$

Show that d_∞ is a metric on $\mathbb{R}^n$.

The above examples show that there can be more than one metric on $\mathbb{R}^n$. The following exercise shows that indeed, there can be many metrics on $\mathbb{R}^n$.

Exercise 2.2. Show that each of the following functions is a metric on $\mathbb{R}$:

(i) $\mathrm{d}(x, y) = |x^3 - y^3|$, (here x^3 means x raised to power 3)

(ii) $\mathrm{d}(x, y) = |e^x - e^y|$,

(iii) $\mathrm{d}(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

Which property of the maps $x \mapsto x^3$, $x \mapsto e^x$, and $x \mapsto \tan^{-1}(x)$ makes these functions a metric.

We will need the following property of the integral later on.

Lemma 2.1. *Assume that $a < b$ are real numbers, and $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f \geq 0$ on $[a, b]$, and f is not identically equal to 0. Then,*

$$\int_a^b f(t)\, dt > 0.$$

Proof. Since f is not identically equal to 0, there must be $c \in [a, b]$ such that $f(c) > 0$. Let $h = f(c)$. Since f is continuous at c, for $\epsilon = h/2 > 0$ there is $\delta > 0$ such that for all $t \in [a, b]$ with $|t - c| < \delta$, we have $|f(t) - h| \leq h/2$. This implies that for all $t \in (c - \delta, c + \delta) \cap [a, b]$, we have

$$f(t) = h + (f(t) - h) \geq h - h/2 = h/2.$$

Without loss of generality we may assume that $\delta < (b - a)/2$.

Consider the function $g : [a, b] \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} 0 & \text{if } t \notin (c - \delta, c + \delta) \cap [a, b], \\ h/2 & \text{if } t \in (c - \delta, c + \delta) \cap [a, b]. \end{cases}$$

We note that $f \geq g$ on $[a, b]$. Also, since g is only discontinuous at two points (finite number of points is ok), it is integrable on $[a, b]$. Moreover,

$$\int_a^b f(t)\, dt \geq \int_a^b g(t)\, dt \geq \delta \cdot h/2 > 0.$$

Note that since $c \in [a, b]$, the length of the interval $[c - \delta, c + \delta] \cap [a, b]$ is at least δ, with the minimum length happening when $c = a$ or $c = b$.

$\square$

Exercise 2.3. Assume that $a < b$ are real numbers, and $h : (a, b) \to (0, \infty)$ be a continuous function. For x and y in (a, b), we define

$$\mathrm{d}_h(x, y) = \int_{\min\{x,y\}}^{\max\{x,y\}} h(t)\, dt.$$

Show that d_h is a metric on (a, b).

Intuitively, in the above exercise, the function h determines the cost of travelling from x to y.

Exercise 2.4. Consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(x, y) = |x - y|^2.$$

Show that g is not a metric on $\mathbb{R}$.

Below is an example of a metric on an slightly different set.

Example 2.4. Let S^1 be the circle of radius 1 about 0 in $\mathbb{R}^2$, that is,

$$S^1 = \left\{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1 \right\}.$$

Any pair of points a and b in S^1 divides the circle S^1 into two arcs. We assume the convention that the end points of the arcs are included in the arcs (this does not make any difference when calculating the arc length). We define $\mathrm{d}(a, b)$ as the length of the shortest arc between a and b. When the points a and b are antipodal (diametrically opposite of one another), the shortest arc is not unique, but those arcs have the same length. Thus, the function $\mathrm{d} : S^1 \times s^1 \to \mathbb{R}$ is well-defined.

M1: The length of any arc is non-negative, and when the end points are distinct, the length is strictly positive.

M2: Since the shortest arc between two points does not depend on the order at which we choose the end points, M2 holds as well. When the end points lie on opposite sides, the shortest arc is not unique, but the length is unique. So in that case we have symmetry as well.

M3: Let θ_1, θ_2 and θ_3 be arbitrary points on S^1. If the points θ_1, θ_2 and θ_3 are not pairwise disjoint, then we obviously have

$$\mathrm{d}(\theta_1, \theta_3) \leq \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

That is because, if $\theta_1 = \theta_3$, the left hand side of the above inequality is 0, and the right hand side is non-negative by the definition of metric. Also, if $\theta_2 \in \{\theta_1, \theta_3\}$, the value on the left hand side also appears on the right hand side of the inequality, with the other term on the right hand side non-negative. So we may assume that the points θ_1, θ_2, and θ_3 are pairwise disjoint. Let $\ell_{i,j}$ denote the shortest arc between θ_i and θ_j, for i and j in $\{1, 2, 3\}$. We consider few cases:

(i) θ_2 belongs to $\ell_{1,3}$: Then $\ell_{1,2} \cup \ell_{2,3} = \ell_{1,3}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) = \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3) \leq \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(ii) θ_1 belongs to $\ell_{2,3}$. Then, $\ell_{1,3} \subset \ell_{2,3}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) \leq \mathrm{d}(\theta_2, \theta_3) \leq \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(iii) θ_3 belongs to $\ell_{1,2}$. Then, $\ell_{1,3} \subset \ell_{1,2}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) \leq \mathrm{d}(\theta_1, \theta_2) \leq \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(iv) neither of the cases (i)-(iii) holds. Then, $\ell_{1,2} \cup \ell_{1,3} \cup \ell_{2,3} = S^1$, and hence

$$\mathrm{d}(\theta_1, \theta_3) = \text{“length of” } \ell_{1,3} \leq \text{“length of” } (S^1 \setminus \ell_{1,3}) = \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

See Figure 2.3.

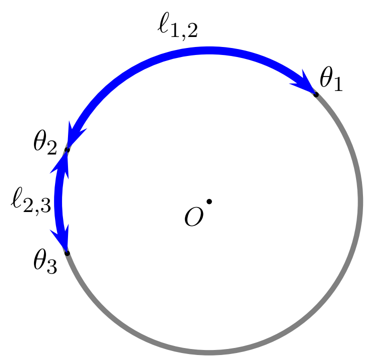

Figure 2.3: The circle of radius 1 about 0 in $\mathbb{R}^2$, and the distance of arc length.

All the examples of metrics we have seen so far are on the of real numbers and Euclidean spaces. But the purpose of giving an axiomatic definition of metric is to generalises analysis. Here are few examples of metric spaces which shows the generality of this notion.

Example 2.5. Let E be a finite set, and let $\mathscr{P}(E)$ denote the set of all subsets of E. Given $A \in \mathscr{P}(E)$, we define $\mathrm{Card}(A)$ as the number of elements in A. Also, for A and B in $\mathscr{P}(E)$, we define the symmetric difference of A and B as

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

The function $\mathrm{d}_{\text{card}} : \mathscr{P}(E) \times \mathscr{P}(E) \to \mathbb{R}$ defined as

$$\mathrm{d}_{\text{card}}(A, B) = \mathrm{Card}(A \Delta B)$$

is a metric on $\mathscr{P}(E)$.

This metric is called the Hamming metric, and plays important role in information theory and cryptography.

Although we all have an intuitive way of thinking about distances, we need to be cautious when dealing with metrics in general. The axiomatic description of the metric in Definition 2.1 captures a wide range of settings, as we discuss in the next two examples.

Example 2.6. Let X be an arbitrary non-empty set. Define, $\mathrm{d_{disc}} : X \times X \to \mathbb{R}$ as

$$\mathrm{d_{disc}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

You can see that this is a metric on X. In this metric all distinct points lie at distance 1 from each other (you may wish to imagine this for some sets). The metric $\mathrm{d_{disc}}$ is called the **discrete metric**.

Another counter intuitive example of a metric is presented in the next Exercise.

Exercise 2.5. Let $X = \mathbb{R}^2$, and define $\mathrm{d_{rail}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\mathrm{d_{rail}}(x, y) = \begin{cases} \|x - y\| & \text{if } x = ky \text{ for some } k \in \mathbb{R} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

Show that $\mathrm{d_{rail}}$ is a metric on $\mathbb{R}^2$.

This is called the British rail metric. The intuition behind this metric is that if two towns are on the same rail line, then we travel between them, but if the towns are on distinct lines, we travel via London (represented as as the origin in $\mathbb{R}^2$).

Example 2.7. We say that a sequence $(x^1, x^2, x^3, \dots)$ is bounded, if there is $M \in \mathbb{R}$ such that for all $i \geq 1$, $|x^i| \leq M$. Let X be the set of all bounded sequences, and consider the function $\mathrm{d}_\infty : X \times X \to \mathbb{R}$ defined as

$$\mathrm{d}_\infty(x, y) = \sup_{k \geq 1} |x^k - y^k|.$$

M1: Since the supremum of a collection of non-negative numbers is a non-negative number, $\mathrm{d}_\infty(x, y) \geq 0$ for all x and y in X. On the other hand, if $\mathrm{d}_\infty(x, y) = \sup_{k\geq 1} |x^k - y^k| = 0$, we must have $|x^k - y^k| = 0$ for all $k \geq 1$. Therefore, $x = y$.

M2: Evidently, since $|t| = |-t|$ for all $t \in \mathbb{R}$, we have

$$\mathrm{d}_\infty(x, y) = \sup_{k \geq 1} |x^k - y^k| = \sup_{k \geq 1} |y^k - x^k| = \mathrm{d}_\infty(y, x).$$

M3: Fix arbitrary elements of X:

$$x = (x^1, x^2, \dots,), \quad y = (y^1, y^2, \dots,), \quad z = (z^1, z^2, \dots).$$

For every $j \geq 1$, we have

$$|x^j - y^j| \leq |x^j - z^j| + |z^j - y^j| \leq \left(\sup_{k\geq 1} |x^k - z^k|\right) + \left(\sup_{k\geq 1} |z^k - y^k|\right)$$
$$= \mathrm{d}_\infty(x, z) + \mathrm{d}_\infty(z, y).$$

The right hand side of the above equation is a constant independent of j. Thus, for all $j \geq 1$, $|x^j - y^j|$ is bounded from above by that constant. Therefore, their supremum must be bounded by that constant. That is,

$$\mathrm{d}_\infty(x, y) = \sup_{j\geq 1} |x^j - y^j| \leq \mathrm{d}_\infty(x, z) + \mathrm{d}_\infty(z, y).$$

The metric space (X, d_∞) is called the l_∞ space.

Assume that a and b are real numbers with $a < b$. Define the set

$$C([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f : [a, b] \to \mathbb{R} \text{ is continuous.}\}$$

Example 2.8. For f and g in $C([a, b])$, define

$$\mathrm{d}_\infty(f, g) = \max_{a\leq t\leq b} |f(t) - g(t)|.$$

Since f and g are continuous on $[a, b]$, they are bounded so there exists k_1 and k_2 in $\mathbb{R}$ such that for all $t \in [a, b]$, $|f(t)| \leq k_1$ and $|g(t)| \leq k_2$. Therefore, $\mathrm{d}_\infty(f, g) \leq k_1 + k_2$, so d_∞ is well defined on $C([a, b])$.

As in the previous example, one can see that d_∞ is a metric on $C([a, b])$. This is called the **supremum metric**, or the **uniform metric**.

Example 2.9. For f and g in $C([a, b])$, define

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt.$$

The function d_1 is a metric on $C([a, b])$. To see this, first note that since the modulus of a continuous function is a continuous function, the integral in the above definition is defined.

M1: For every f and g in X, and every $t \in [a, b]$, $|f(t) - g(t)| \geq 0$. Thus,

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt \geq 0.$$

On the other hand, if $\mathrm{d}_1(f, g) = 0$, by Lemma 2.1, we must have $|f - g|$ is identically equal to 0. Thus, $f = g$ as functions on $[a, b]$.

M2: Since for all $t \in \mathbb{R}$, $|t| = |-t|$, we have

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt = \int_a^b |g(t) - f(t)|\, dt = \mathrm{d}_1(g, f).$$

M3: Let f, g and h be continuous functions on $[a,b]$. For all $t \in [a,b]$, by the triangle inequality for the modulus function, we have

$$|f(t)-g(t)| = \big|(f(t)-h(t))+(h(t)-g(t))\big| \leq |f(t)-h(t)|+|h(t)-g(t)|.$$

Integrating the above functions, we note that

$$\int_a^b |f(t)-g(t)|\,dt \leq \int_a^b |f(t)-h(t)|\,dt + \int_a^b |h(t)-g(t)|\,dt,$$

which gives us

$$\mathrm{d}_1(f,g) \leq \mathrm{d}_1(f,h)+\mathrm{d}_1(h,g).$$

We have already seen many examples of metric spaces. There are some ways to define new metric spaces using other metric spaces. We present two approaches below.

Definition 2.3. Let (X,d) be a metric space, and $Y \subset X$ be an arbitrary subset. Define $\mathrm{d}|_Y : Y \times Y \to \mathbb{R}$ as $d|_Y(x,y) = \mathrm{d}(x,y)$, for all x and y in Y. Clearly $\mathrm{d}|_Y$ is a metric on Y (it inherits all the properties from d). The pair $(Y,\mathrm{d}|_Y)$ is called a **metric subspace** of (X,d), and $\mathrm{d}|_Y$ is called the **induced metric** on Y from d.

Example 2.10. Consider the Euclidean metric space $(\mathbb{R},\mathrm{d}_1)$. We may restrict this metric to the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. Also, d_1 induces a metric on the set of integers $\mathbb{Z} \subset \mathbb{R}$.

Similarly, since $\mathbb{Z}^n \subset \mathbb{R}^n$ and $\mathbb{Q}^n \subset \mathbb{R}^n$, we may restrict any of the metrics d_1, d_2, and d_∞ onto those sets.

Given arbitrary sets X_1 and X_2, we define the (set-theoretical) **product** of these two sets as

$$X_1 \times X_2 = \{(x_1,x_2) \mid x_1 \in X_1, x_2 \in X_2\}.$$

That is, the set of all ordered pairs (x_1,x_2) such that $x_1 \in X_1$ and $x_2 \in X_2$.

Definition 2.4. Let (X_1,d_1) and (X_2,d_2) be two metric spaces. We may use the metrics d_1 and d_2 to define a metric on $X_1 \times X_2$. For example,

$$\mathrm{d}\left((x_1,x_2),(y_1,y_2)\right) = \max\{\mathrm{d}_1(x_1,y_1),\mathrm{d}_2(x_2,y_2)\},$$
$$\mathrm{d}\left((x_1,x_2),(y_1,y_2)\right) = \mathrm{d}_1(x_1,y_1)+\mathrm{d}_2(x_2,y_2).$$

Each of the above functions from $(X_1 \times X_2) \times (X_1 \times X_2)$ to $\mathbb{R}$ is a metric. For each of the above metrics d, the metric space $(X_1 \times X_2, \mathrm{d})$ is called a **product metric spaces**.

2.1.3 Normed vector spaces

Definition 2.5. Let V be a vector space on $\mathbb{R}$. We say that a function $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V, if the following properties are satisfied:

(N1) for every $v \in V$, $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$,

(N2) for every $v \in V$ and every $\lambda \in \mathbb{R}$, we have $\|\lambda V\| = |\lambda| \, \|v\|$,

(N3) for all u and v in V, $\|u + v\| \leq \|u\| + \|v\|$.

A **normed vector space**, is a pair of a vector space V together with a norm function on V. This is often denoted as $(V, \|\cdot\|)$.

On any vector space $(V, \|\cdot\|)$ we have a natural notion of metric coming from the norm function. We present this in the next lemma.

Lemma 2.2. *Let V be a vector space, and $\|\cdot\| : V \to \mathbb{R}$ be a norm function on V. The function $\mathrm{d}_{\|\|} : V \times V \to \mathbb{R}$, defined as*

$$\mathrm{d}_{\|\|}(u, v) = \|u - v\|$$

is a metric on V.

Proof. Property M1 comes from the property N1 of the norm function, that is,

$$\mathrm{d}_{\|\|}(v, w) = \|v - w\| \geq 0.$$

Also,

$$\mathrm{d}_{\|\|}(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w.$$

Property M2 comes from the property N2 of the norm function, since

$$\mathrm{d}_{\|\|}(w, v) = \|w - v\| = \|(-1)(v - w)\| = |-1| \, \|v - w\| = \|v - w\| = \mathrm{d}_{\|\|}(v, w).$$

Property M3 comes from the property N3 for the norm. That is because

$$\mathrm{d}_{\|\|}(v, z) = \|v - z\| \leq \|v - w\| + \|w - z\| = \mathrm{d}_{\|\|}(v, w) + \mathrm{d}_{\|\|}(w, z). \qquad \square$$

Some of the examples we already seen are normed vector spaces. For example, the distance d_2 on $\mathbb{R}^n$ comes from the norm $\|\cdot\|$ on $\mathbb{R}^n$.

Example 2.11. Let $V = \mathbb{R}^n$, and consider the functions

$$\begin{aligned} \left\|(x^1, x^2, \dots, x^n)\right\|_1 &= |x^1| + |x^2| + \cdots + |x^n|, \\ \left\|(x^1, x^2, \dots, x^n)\right\|_\infty &= \max\{|x^1|, |x^2|, \dots, |x^n|\}. \end{aligned}$$

One can easily see that these functions satisfy the three properties for the norm function. These norms induce the metrics d_1 and d_∞ on $\mathbb{R}^n$, respectively.

Assume that $a < b$ are real numbers, and let $C([a,b])$ denote the set of all continuous functions $f : [a,b] \to \mathbb{R}$. For f and g in $C([a,b])$, we define $f+g$ as the function $(f+g)(x) = f(x) + g(x)$, for all $x \in [a,b]$. Also, for $\lambda \in \mathbb{R}$, and $f \in C([a,b])$, we define $(\lambda f)(x) = \lambda f(x)$. These operations make $C([a,b])$ a vector space on $\mathbb{R}$. This vector space has infinite dimensions, since the functions $x \mapsto x$, $x \mapsto x^2$, $x \mapsto x^3$, ..., are linearly independent.

Exercise 2.6. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a,b])$:

(i)
$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

(ii)
$$\|f\|_\infty = \max_{t \in [a,b]} |f(t)|$$

(iii)
$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt \right)^{1/2}$$

Remark 2.3. The norm $\|\cdot\|_1$ on $C([a,b])$ is called the l_1-norm, $\|\cdot\|_2$ on $C([a,b])$ is called the l_2-norm, and the norm $\|\cdot\|_\infty$ on $C([a,b])$ is called the l_∞-norm, or supremum norm. The metric induced from $\|\cdot\|_1$ on $C([a,b])$ is the d_1 metric we presented in Example 2.9 and the metric induced from $\|\cdot\|_\infty$ on $C([a,b])$ is the d_∞ metric we presented in Example 2.8.

You can learn more about these spaces in the modules Lebesgue Measure and Integration, and Functional Analysis.

It is not true that every metric on a vector space comes from a norm. You can show this by the following exercise.

Exercise 2.7. Show that if V is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2\,\mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric d_{disc} on $\mathbb{R}^2$.

As we shall see in later sections, the notion of metric allows us to develop analysis on general metric spaces. It is remarkable that such a simple notion can lead to a huge volume of mathematical theory. Of all the properties of a function which makes it a metric, the triangle inequality is the non-trivial one. It is worth taking a moment to build intuition about that property. The following exercise helps you to achieve that.

Exercise 2.8. Let (X, d) be a metric space.

(i) Show that for every x, y, and z in X, we have

$$|\,\mathrm{d}(x,z)-\mathrm{d}(y,z)|\le \mathrm{d}(x,y).$$

(ii) Show that for all x, y, z and t in X, we have

$$|\,\mathrm{d}(x,y)-\mathrm{d}(z,t)|\le \mathrm{d}(x,z)+\mathrm{d}(y,t).$$

(iii) Show that for all x_1, x_2, $\dots$, x_n in X, we have

$$\mathrm{d}(x_1,x_n)\le \mathrm{d}(x_1,x_2)+\mathrm{d}(x_2,x_3)+\cdots+\mathrm{d}(x_{n-1},x_n).$$

2.1.4 Open sets in metric spaces

The notion of a metric on a set allows us to describe some geometric properties of subsets of that set. We shall discuss some of these properties in Sections 2.1.4 and 2.1.6.

Definition 2.6. Consider a metric space (X,d), a point $x\in X$, and a real number $\epsilon>0$. The **ball** of radius ϵ centred at x is the set of all points $x'\in X$ satisfying $\mathrm{d}(x,x')<\epsilon$. In other words,

$$B_\epsilon(x)=\{x'\in X \mid \mathrm{d}(x,x')<\epsilon\}.$$

This set is also referred to as **ϵ-ball** about x, or **ϵ-neighbourhood** of x. To emphasise the dependence of the ball on the metric d and the underlying space X, we may use the notation $B_\epsilon(x,X,\mathrm{d})$.

Example 2.12. We look at ϵ-balls in some of the metric spaces we introduced in the previous section.

(i) In $(\mathbb{R},\mathrm{d}_1)$, for every $a\in\mathbb{R}$ and $\epsilon>0$, we have

$$B_\epsilon(a)=\{x\in\mathbb{R}\mid \mathrm{d}_1(x,a)<\epsilon\}=\{x\in\mathbb{R}\mid |x-a|<\epsilon\}=(a-\epsilon,a+\epsilon).$$

(ii) In $(\mathbb{R}^n,\mathrm{d}_2)$, for every $a\in\mathbb{R}^n$ and $\epsilon>0$, $B_\epsilon(a)$ consists of all the points inside a hypersphere.

(iii) In $(\mathbb{R}^2,\mathrm{d}_\infty)$, for every $a=(a^1,a^2)\in\mathbb{R}^2$ and $\epsilon>0$,

$$\begin{aligned}B_\epsilon(a)&=\{(x^1,x^2)\in\mathbb{R}^2\mid \mathrm{d}_\infty((a^1,a^2),(x^1,x^2))<\epsilon\}\\&=\{(x^1,x^2)\in\mathbb{R}^2\mid \max\{|a^1-x^1|,|a^2-x^2|\}<\epsilon\}.\end{aligned}$$

This is a square with horizontal and vertical sides of lengths 2ϵ centre at a.

(iv) Let $I = [0, 1] \subset \mathbb{R}$, and d_I denote the induced metric on I from d_1 on $\mathbb{R}$. Then, in $(\mathbb{R}, \mathrm{d}_1)$, we have

$$B_1(1) = B_1(1, \mathbb{R}, \mathrm{d}_1) = \{x \in \mathbb{R} | |x - 1| < 1\} = (0, 2).$$

In (I, d_I), we have

$$\begin{aligned} B_1(1) = B_1(1, I, \mathrm{d}_I) &= \{x \in I \mid \mathrm{d}_I(x, 1) < 1\} \\ &= \{x \in [0, 1] \mid |x - 1| < 1\} \\ &= (0, 1]. \end{aligned}$$

In (I, d_I),

$$B_{1/2}(1/2) = B_{1/2}(1/2, I, \mathrm{d}_I) = \{x \in I \mid \mathrm{d}_I(x, 1/2) < 1/2\} = (0, 1).$$

(v) In $(X, \mathrm{d}_{\mathrm{disc}})$, where X is a non-empty set, and $\mathrm{d}_{\mathrm{disc}}$ is the discrete metric, for every $x \in X$ and $\epsilon > 0$ we have the following.

If $\epsilon \leq 1$, then

$$B_\epsilon(x) = \{x' \in X \mid \mathrm{d}_{\mathrm{disc}}(x, x') < \epsilon\} = \{x\}.$$

If $\epsilon > 1$,

$$B_\epsilon(x) = \{x' \in X \mid \mathrm{d}_{\mathrm{disc}}(x, x') < \epsilon\} = X.$$

(vi) In $(C([a, b]), \mathrm{d}_\infty)$, for $f \in C([a, b])$ and $\epsilon > 0$, we have

$$B_\epsilon(f) = \{g \in C([a, b]) \mid \mathrm{d}_\infty(f, g) < \epsilon\}$$
$$\{g \in C([a, b]) \mid \max_{t \in I} |f(t) - g(t)| < \epsilon\}.$$

This consists of all continuous functions $g : [a, b] \to \mathbb{R}$ such that the graph of g lies between the graphs of $f - \epsilon$ and $f + \epsilon$.

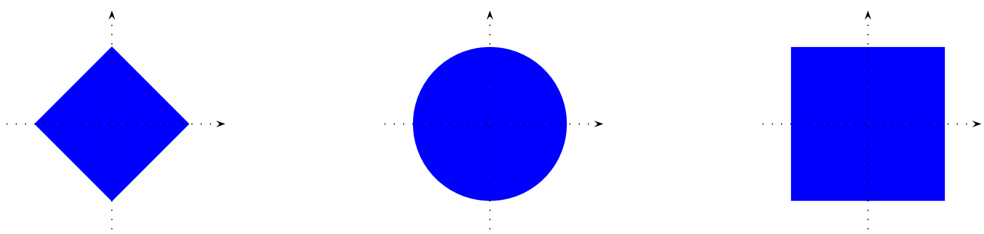

Figure 2.4: Figure on the left hand side shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_1)$, the figure in the middle shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_2)$, and the figure on the right hand side shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_\infty)$.

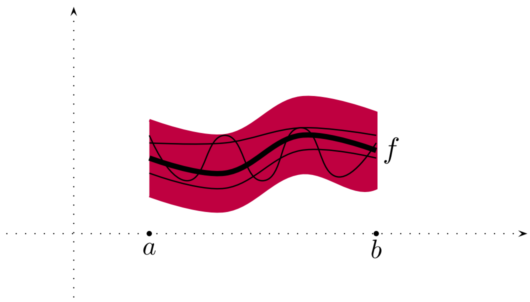

Figure 2.5: In $(C([a,b]), \mathrm{d}_\infty)$, $B_\epsilon(f)$ consists of all continuous functions on $[a,b]$ whose graphs lie in the red region. We have drawn the graphs of three functions in $B_\epsilon(f)$.

Exercise 2.9. Let (X, d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n\in\mathbb{N}} B_{1/n}(x) = \{x\}.$$

Definition 2.7. Let (X, d) be a metric space, and $U \subseteq X$. We say that U is **open** in (X, d), if for every $u \in U$, there is $\delta > 0$ such that $B_\delta(u) \subseteq U$.

Lemma 2.3. *Let (X, d) be a metric space. For every $x \in X$ and $\epsilon > 0$, the ball $B_\epsilon(x)$ is open in X.*

Proof. Fix an arbitrary $y \in B_\epsilon(x)$. Let $\delta = \epsilon - \mathrm{d}(x,y)$. Since $y \in B_\epsilon(x)$, we have $\mathrm{d}(x,y) < \epsilon$, and hence $\delta > 0$.

Let $z \in B_\delta(y)$ be an arbitrary point. By the triangle inequality of the metric,

$$\mathrm{d}(z,x) \leq \mathrm{d}(z,y) + \mathrm{d}(y,x) < \delta + (\epsilon - \delta) = \epsilon.$$

Hence, $z \in B_\epsilon(x)$. As $z \in B_\delta(y)$ was arbitrary, we conclude that $B_\delta(y) \subset B_\epsilon(x)$. As $y \in B_\epsilon(x)$ was arbitrary, we conclude that $B_\epsilon(x)$ is an open set. □

Due to the above lemma, $B_\epsilon(x)$ is also called an **open ball** of radius ϵ about x.

Lemma 2.4. *In any metric space (X, d), the empty set and the set X are open.*

Proof. To see that the empty set is open, we need to show that for every x in the empty set, there is $\delta >$ such that $B_\delta(x)$ is contained in the empty set. Since there is no such x in the empty set to begin with, for logical reasons this statement is true. So the empty set is open.

On the other hand, for every $x \in X$, we have $B_1(x) \subset X$. That is because of the definition of the ball. Thus we can take $\epsilon = 1$, in the criterion for open sets. □

Note that the definition of open set in a metric space (X, d) depends on both the metric d and the underlying set X. We make this clear in the next example.

Example 2.13. Consider the discrete metric d_{disc} on $\mathbb{R}$, that is $(\mathbb{R}, \mathrm{d}_{\text{disc}})$. In this space, any subset of $\mathbb{R}$ is open. To see that let U be an arbitrary subset of $\mathbb{R}$, and let $u \in U$ be an arbitrary point. We let $\delta = 1/2$, and note that $B_{1/2}(u) = \{u\} \subset U$. This shows that U is open. But in the metric space $(\mathbb{R}, \mathrm{d}_1)$ it is not true that every subset of $\mathbb{R}$ is open. For example, the set with single element $\{1\}$ is open in $(\mathbb{R}, \mathrm{d}_{\text{disc}})$, but it is not open in $(\mathbb{R}, \mathrm{d}_1)$.

On the other hand, let $I = [0, 1] \subset \mathbb{R}$, and let d_I be the induced metric on $[0, 1]$ from d_1 on $\mathbb{R}$. The set $[0, 1/2)$ is not open in $(\mathbb{R}, \mathrm{d}_1)$ (the definition does not hold for the point $0 \in [0, 1/2)$). But $[0, 1/2)$ is open in $([0, 1], \mathrm{d}_I)$. To show the latter property, let $x \in [0, 1/2)$. If $x \in (0, 1/2)$, we define $\delta = \min\{x, 1/2 - x\}$, and see that $\delta > 0$ and

$$B_\delta(x, I, \mathrm{d}_I) = \{x' \in [0, 1/2) \mid \mathrm{d}_I(x, x') < \delta\} = (x - \delta, x + \delta) \subset [0, 1/2).$$

If $x = 0$, we let $\delta = 1/4$, and see that

$$B_\delta(0, I, \mathrm{d}_I) = \{x' \in [0, 1/2) \mid \mathrm{d}_I(0, x') < \delta\} = [0, 1/4) \subset [0, 1/2).$$

According to the definition of open sets, this shows that $[0, 1/2)$ is open in $([0, 1], \mathrm{d}_I)$.

Lemma 2.5. *Let $X = (X, \mathrm{d})$ be a metric space. The union of any number of (finite, countable, uncountable) open sets in X is an open set in X.*

Proof. Assume that $G_\alpha \subseteq X$ is open, for all α in a set I. Let $x \in \cup_{\alpha \in I} G_\alpha$. Then there exists some $\alpha_0 \in I$ such that $x \in G_{\alpha_0}$. Since G_{α_0} is an open set, there exists $\delta > 0$ such that $B_\delta(x) \subset G_{\alpha_0}$. This implies that $B_\delta(x) \subseteq \cup_{\alpha \in I} G_\alpha$. □

Lemma 2.6. *Let $X = (X, \mathrm{d})$ be a metric space. The intersection of any finite number of open sets in X is an open set in X.*

Proof. Assume that $m \geq 1$ and $G_1, G_2, \dots, G_m$ are open sets in X. Fix an arbitrary $x \in \cap_{k=1}^m G_k$. For every $k \in \{1, 2, \dots, m\}$, $x \in G_k$. For every such k, since G_k is open, there exists $\epsilon_k > 0$ such that $B_{\epsilon_k}(x) \subset G_k$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_m\} > 0$.

By our choice of ϵ, for every $k \in \{1, 2, \dots, m\}$, $B_\epsilon(x) \subset B_{\epsilon_k}(x) \subset G_k$. Therefore, $B_\epsilon(x) \subset \cap_{k=1}^m G_k$. □

The statement in the above lemma is not necessarily true if we drop the hypothesis of finiteness. For example, as we saw in Exercise 2.9, in the metric space $(\mathbb{R}^n, \mathrm{d}_2)$, we have $\cap_{n=1}^{\infty} B_{1/n}(x) = \{x\}$. And the set $\{x\}$ is not open in $(\mathbb{R}^2, \mathrm{d}_2)$.

We have already seen that there may be many metrics on a given set. For example, we have metrics d_1, d_2, and d_∞ on $\mathbb{R}^n$. The definition of open set in a metric space depends on the metric. So a priori, for each of these metrics on $\mathbb{R}^n$, we may have different open sets. This seems to be cumbersome, but can be alleviated by the following definition.

Definition 2.8. Let d_1 and d_2 be metrics on a set X. The metrics d_1 and d_2 are called **topologically equivalent**, if the following property holds. For every $U \subseteq X$, U is open in (X, d_1) if and only if U is open in (X, d_2).

Exercise 2.10. (i) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x, y) \leq \mathrm{d}_2(x, y) \leq \sqrt{n} \cdot \mathrm{d}_\infty(x, y).$$

(ii) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x, y) \leq \mathrm{d}_1(x, y) \leq n \cdot \mathrm{d}_\infty(x, y).$$

(iii) Show/conclude that for all x and y in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}} \mathrm{d}_2(x, y) \leq \mathrm{d}_1(x, y) \leq n\, \mathrm{d}_2(x, y).$$

(iv) Conclude that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.

2.1.5 Convergence in metric spaces

Definition 2.9. Let (X, d) be a metric space, and $(x_n)_{n \geq 1}$ be a sequence of points in X. We say that the sequence $(x_n)_{n \geq 1}$ **converges** in (X, d), if there is $x \in X$ satisfying the following:

for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathrm{d}(x_n, x) < \epsilon$.

In this case, we say that x is the **limit** of the sequence $(x_n)_{n \geq 1}$, or say that the sequence $(x_n)_{n \geq 1}$ converges to x in (X, d), and write $x_n \to x$ as $n \to \infty$, or $\lim_{n \to \infty} x_n = x$.

Notice the similarly between the above definition and the definition of convergence of sequences in Euclidean spaces.

Example 2.14. In the metric space $(\mathbb{R}, \mathrm{d}_1)$ the sequence $(1/n)_{n \geq 1}$ converges. That is because $0 \in \mathbb{R}$, and for every $\epsilon > 0$ we can choose an integer $N > 1/\epsilon$, so that for all $n \geq N$ we have $\mathrm{d}_1(1/n, 0) = 1/n < \epsilon$.

Now let $I = (0,1)$, and d_I be the induced metric on I from d_1. In the metric space (I, d_I), the sequence $(1/n)_{n\geq 1}$ does not converge. That is because there is no $x \in (0,1)$ satisfying the criterion for the convergence. Assume in the contrary that there is such an $x \in (0,1)$. We choose $\epsilon = x/2 > 0$, and for every $N \in \mathbb{N}$, we choose $n \geq \max\{N, 2/x\}$. Then,

$$\mathrm{d}_I(1/n, x) = |1/n - x| = x - 1/n \geq x - x/2 = x/2 = \epsilon.$$

We say that a sequence $(x_n)_{n\geq 1}$ is **eventually constant**, if there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $x_n = x_{n_1}$.

Exercise 2.11. Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\text{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

Lemma 2.7. *Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. If the sequence $(x_n)_{n\geq 1}$ converges in (X, d), then its limit is unique.*

Proof. Let us assume that there are two points x and y in X such that the sequence $(x_n)_{n\geq 1}$ converges to. Fix an arbitrary $\epsilon > 0$. Since the sequence converges to x, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\mathrm{d}(x_n, x) < \epsilon$. Similarly, since the sequence converges to y, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $\mathrm{d}(x_n, y) < \epsilon$. Now, let $n = \max\{N_1, N_2\}$. We have

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, y) < \epsilon + \epsilon = 2\epsilon.$$

By property M1 of metrics, $\mathrm{d}(x, y) \geq 0$, and since $\epsilon > 0$ was arbitrary, the above inequality shows that $\mathrm{d}(x, y) = 0$. Then, by property M1 of the metrics, we conclude that $x = y$. $\square$

Exercise 2.12. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

As a corollary of the above exercise, we obtain the following result.

Corollary 2.8. *Let d_1 and d_2 be topologically equivalent metrics on X. Then, a sequence $(x_n)_{n\geq 1}$ in X converges in (X, d_1) if and only if it converges in (X, d_2).*

Proof. Recall that by the definition of equivalent metrics, U is open in (X, d_1) if and only if U is open in (X, d_2). The result immediately follows from the previous exercise. $\square$

2.1.6 Closed sets in metric spaces

Definition 2.10. Let (X, d) be a metric space, and $V \subseteq X$ be a set. We say that V is **closed** in (X, d), if for every sequence $(x_n)_{n\geq 1}$ in V which converges in (X, d), then the limit of $(x_n)_{n\geq 1}$ belongs to V.

When it is clear what metric is involved, we may simple say that V is closed in X. For example, when a metric is not specified on $\mathbb{R}^n$, it is assumed that it is the Euclidean metric d_2. Thus, when we say that E is closed in $\mathbb{R}^n$, we mean that E is closed in $(\mathbb{R}^n, \mathrm{d}_2)$.

Example 2.15. Consider real numbers $a < b$. The set $[a, b]$ is closed in $(\mathbb{R}^1, \mathrm{d}_1)$. That is because if $(x_n)_{n\geq}$ is a sequence in $[a, b]$ which converges to x in $\mathbb{R}$, then we have $a \leq x_n \leq b$, and hence $a \leq \lim_{n\to\infty} x_n \leq b$. This implies that $x \in [a, b]$.

The intervals (a, b) and $(a, b]$ are not closed in $(\mathbb{R}^1, \mathrm{d}_1)$. That is because

$$a + \frac{b-a}{n}, \quad n \geq 2$$

is a sequence in $(a, b]$ which converges to a in $(\mathbb{R}, \mathrm{d}_1)$, but a does not belong $(a, b]$.

On the other hand, let $I = (0, 1)$ and d_I be the induced metric on I from $(\mathbb{R}, \mathrm{d}_1)$. Then the set $V = (0, 1/2]$ is closed in $((0, 1), \mathrm{d}_I)$. To see this, assume that $(x_n)_{n\geq 1}$ is a sequence in $(0, 1/2]$ which converges in $((0, 1), \mathrm{d}_I)$. By the definition of convergence in $((0, 1), \mathrm{d}_I)$, the limit of the sequence must be in $(0, 1)$. However, since the sequence belongs to $(0, 1/2]$, its limit is at most $1/2$. Thus, the limit belongs to $(0, 1/2]$.

Exercise 2.13. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space. Then every set in X is closed.

Note that open is not the opposite of closed. If a set is not open, it does not mean that it is closed. For example, the set $(1, 2]$ is neither open or closed in $(\mathbb{R}, \mathrm{d}_1)$. There are sets that are both open and closed, as we shall see in a moment.

Theorem 2.9. *Let (X, d) be a metric space and $V \subseteq X$. Then, V is closed in (X, d) if and only if $X \setminus V$ is open in (X, d).*

Proof. First assume that V is closed. Assume in the contrary that $X \setminus V$ is not open. Then, there is $x \in X \setminus V$, such that for all $\delta > 0$, $B_\delta(x) \nsubseteq X \setminus V$. Equivalently, for all $\delta > 0$, $B_\delta(x) \cap V \neq \emptyset$. In particular, for each $n \in \mathbb{N}$, we let $\delta = 1/n$, and conclude that there is a point $x_n \in B_\delta(x) \cap V$. This process generates a sequence $(x_n)_{n\in\mathbb{N}}$ in V. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in (X, d), because $x_n \in B_\delta(x)$ implies that $\mathrm{d}(x_n, x) < 1/n$. But the limit x does not belong to V, which contradicts V is closed.

Now assume that $X \setminus V$ is open. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in V which converges to some $x \in X$. We need to show that $x \in V$. If $x \notin V$, then $x \in X \setminus V$. Then, since $X \setminus V$ is open, there is $\delta > 0$ such that $B_\delta(x) \subset X \setminus V$. On the other hand, since $(x_n)_{n\in\mathbb{N}}$ converges to x, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in B_\delta(x)$. Thus, for all $n \geq N$, $x_n \in X \setminus V$. This is a contradiction since $(x_n)_{n\in\mathbb{N}}$ is a sequence in V. □

Some authors define the notion of closed sets using the equivalence form in the above theorem. That is, a set is closed, if its complement is open. Then, they prove (as in the proof of the above theorem) that if a set is closed, it contains the limit of any convergent sequence in that set.

Lemma 2.10. *Let (X, d) be a metric space.*

(i) the intersection of any number (finite, countable or uncountable) of closed sets in (X, d) is a closed set in (X, d),

(ii) the union of any finite number of closed sets in (X, d) is a closed set in (X, d).

Proof. Let F_α, for $\alpha \in I$, be a collection of closed sets in X. By Theorem 2.9, for every $\alpha \in I$, $X \setminus F_\alpha$ is an open set. Then, by Lemma 2.5, $\cup_{\alpha\in I}(X \setminus F_\alpha)$ is open in X. Since

$$X \setminus (\cap_{\alpha\in I} F_\alpha) = \cup_{\alpha\in I}(X \setminus F_\alpha),$$

we conclude that $X \setminus (\cap_{\alpha\in I} F_\alpha)$ is open. Using Theorem 2.9 again, we conclude that $\cap_{\alpha\in I} F_\alpha$ is closed in X. This proves part (i) of the lemma.

The proof for part (ii) is similar, except that one uses Lemma 2.6 instead of Lemma 2.5. □

It is also possible to give a proof of the above lemma, directly using the definition of closed sets in Definition 2.10.