Imperial College London

MATH97223 MATH70011

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Classical Dynamics

Date: 10 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

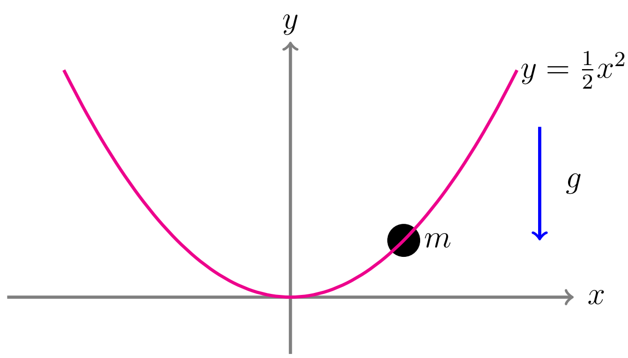
1. (a) (i) What is meant by a conservative force? (2 marks)

(ii) A particle moves in the plane subject to the force

$$\mathbf{F} = y\mathbf{i} + (ay^2 + bx)\mathbf{j},$$

where a and b are constants. For what a and b is the force conservative? (3 marks)

(b) A bead of mass m moves without friction on a parabola shaped wire. The parabola is defined by $y = \frac{1}{2}x^2$ and the bead is subject to a downward gravitational force as shown in the diagram below (g is the acceleration due to gravity).



Obtain the equation of motion for the bead. (8 marks)

(c) Determine the canonical transformation defined by the type-2 generating function

$$F_2(q, P) = qe^P.$$

Find a type-1 generating function for this canonical transformation.

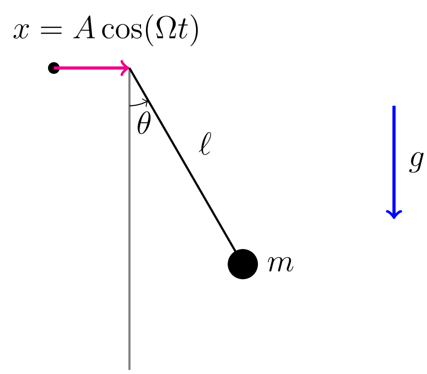
$$\left[\text{ definition of type-2 generating function:} \quad p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}. \right]$$

. (7 marks)

(Total: 20 marks)

2. A pendulum is suspended from a horizontally oscillating support as shown in the diagram below.

The generalised coordinate θ is the angle of the limb to the vertical. The x coordinate of the support point is $A\cos(\Omega t)$ where the constants A and Ω are the amplitude and angular frequency of the support, respectively. ℓ is the length of the limb (assumed to be massless), g is the acceleration due to gravity and m is the mass of the pendulum bob.



(a) Show that the kinetic energy of the pendulum bob is

$$T = \frac{1}{2}mA^2\Omega^2\sin^2(\Omega t) + \frac{1}{2}m\ell^2\dot{\theta}^2 - mA\ell\Omega\dot{\theta}\sin(\Omega t)\cos\theta.$$

(6 marks)

(b) Show that a suitable Lagrangian for the oscillating pendulum is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mA\ell\Omega\dot{\theta}\sin(\Omega t)\cos\theta + mg\ell\cos\theta.$$

Hint: adding a function of t to the Lagrangian does not change the equation of motion.

(4 marks)

(c) Obtain the equation of motion (simplify your answer if possible). (5 marks)

(d) For small oscillations of the bob, the following Lagrangian may be used

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mA\ell\Omega\dot{\theta}\sin(\Omega t) - \frac{1}{2}mg\ell\theta^2.$$

Solve the equation of motion deriving from this Lagrangian (assume $\Omega^2 \neq g/\ell$). (5 marks)

(Total: 20 marks)

3. The motion of a particle of unit mass in the plane is governed by the Hamiltonian

$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{\cos\theta}{r^2},$$

where r and θ are polar coordinates.

(a) Write down (four) Hamilton's equations. (5 marks)

(b) Show that the Hamilton-Jacobi equation is completely separable. (7 marks)

(c) The Hamiltonian H is a constant of the motion. Using your analysis from part (b), or otherwise, identify a second constant of the motion. Hence, or otherwise, determine $r(t)$.

Hint: do not solve Hamilton's equations directly - exploit the two conserved quantities to obtain a first order ODE for $r(t)$. (8 marks)

(Total: 20 marks)

4. (a) Show that the rate of change of a function $A(q_i, p_i)$ is

$$\frac{dA}{dt} = \{A, H\},$$

where $H = H(q_i, p_i)$ is the Hamiltonian. Here q_i and p_i are the coordinates and momenta of phase space. (5 marks)

(b) The Hamiltonian of a freely rotating rigid body can be written in the form

$$H = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2).$$

I_1, I_2, and I_3 are principal moments of inertia. ω_1, ω_2 and ω_3 are the components of the angular velocity with respect to the (moving) principal axes. The ω_i are functions of the 6 phase space variables and satisfy the Poisson bracket relations

$$\{\omega_1, \omega_2\} = -\frac{I_3}{I_1 I_2}\omega_3, \quad \{\omega_2, \omega_3\} = -\frac{I_1}{I_2 I_3}\omega_1, \quad \{\omega_3, \omega_1\} = -\frac{I_2}{I_3 I_1}\omega_2.$$

Derive the Euler equation

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3.$$

(7 marks)

(c) The complete set of Euler's equations read

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2.$$

Suppose that

$$I_1 = 2, \quad I_2 = 3, \quad I_3 = 4,$$

$\omega_3 \approx \Omega$ where Ω is a constant, and that ω_1 and ω_2 are small ($|\omega_1| \ll \Omega$, $|\omega_2| \ll \Omega$). Use Euler's equations to find the form of $\omega_1(t)$ and $\omega_2(t)$. (8 marks)

(Total: 20 marks)

5. The following Lagrangian describes the motion of a simple pendulum mounted on a freely rotating turntable

$$L = \frac{1}{2}h\dot{\phi}^2 + \frac{1}{2}\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + g\ell\cos\theta.$$

The generalised coordinates θ and ϕ specify the angle of the pendulum to the vertical and the orientation of the turntable, respectively. h, ℓ and g are positive constants. Consider a Routhian treatment using the velocity $\dot{\theta}$ and the momentum p_ϕ. Here the Routhian, $R(\theta, \phi, \dot{\theta}, p_\phi)$, is the Legendre transform of $L(\theta, \phi, \dot{\theta}, \dot{\phi})$ with respect to $\dot{\phi}$.

(a) Show that the Routhian has the form

$$R = U(\theta, p_\phi) - \frac{1}{2}\ell^2\dot{\theta}^2.$$

Use the Routhian to obtain the equations of motion. (10 marks)

(b) Determine the frequency of small oscillations about the equilibrium at $\theta = 0$. (5 marks)

(c) Identify the equilibrium points for $g = 0$. Comment on their stability. (5 marks)

(Total: 20 marks)

Answers to 2021-2022 Examination

1. (a) (i) A conservative force derives from a potential energy function V. For a single particle $\mathbf{F} = -\nabla V$ where $V = V(\mathbf{r}, t)$.

(2 marks, bookwork, A)

(ii) $\mathbf{F} = y\mathbf{i} + (ay^2 + bx)\mathbf{j} = -\nabla V$ if the force is conservative. This gives

$$y = -\frac{\partial V}{\partial x}, \quad ay^2 + bx = -\frac{\partial V}{\partial y}.$$

Integrating the first equation gives $V = -xy + f(y)$. Inserting this into the second equation yields $ay^2 + bx = x - f'(y)$ which implies that $b = 1$ and a is arbitrary. Alternatively, use $\nabla \times \mathbf{F} = 0$.

(3 marks, seen similar, A)

(b) $y = \frac{1}{2}x^2$ so that $\dot{y} = x\dot{x}$. The kinetic energy of the bead is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(1 + x^2)\dot{x}^2$.

The potential energy of the bead is $V = mgy = \frac{1}{2}mgx^2$.

A Lagrangian is

$$L = T - V = \frac{1}{2}m(1 + x^2)\dot{x}^2 - \frac{1}{2}mgx^2.$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}m(1 + x^2)\dot{x} - mx\dot{x}^2 + mgx,$$

or

$$(1 + x^2)\ddot{x} + x\dot{x}^2 + gx = 0.$$

(8 marks, seen similar, A)

(c)

$$p = \frac{\partial F_2}{\partial q} = e^P, \quad Q = \frac{\partial F_2}{\partial P} = qe^P,$$

or

$$Q = qp, \quad P = \log p.$$

To determine the type-1 generating function write p and P as functions of q and Q:

$$p = \frac{Q}{q} = \frac{\partial F}{\partial q}, \quad P = \log\frac{Q}{q} = -\frac{\partial F}{\partial Q}.$$

The first equation integrates to $F = Q \log q + f(Q)$ which is consistent with the second equation if $f(Q) = -Q \log Q + Q$. Accordingly

$$F(q, Q) = Q \left(\log \frac{q}{Q} + 1 \right).$$

(7 marks, seen similar, C)

(Total: 20 marks)

2. (a) The x and y coordinates of the pendulum bob are

$$X = A\cos(\Omega t) + \ell \sin\theta, \quad Y = -\ell\cos\theta.$$

The components of the velocity are

$$\dot{X} = -A\Omega\sin(\Omega t) + \ell\dot{\theta}\cos\theta, \quad \dot{Y} = \ell\dot{\theta}\sin\theta.$$

The kinetic energy is

$$\begin{aligned} T = \frac{m}{2}(\dot{X}^2+\dot{Y}^2) &= \frac{m}{2}\Big[\Big(A^2\Omega^2\sin^2(\Omega t) + \ell^2\dot{\theta}^2\cos^2\theta - 2A\Omega\ell\sin(\Omega t)\dot{\theta}\cos\theta\Big) \\ &\qquad + \ell^2\dot{\theta}^2\sin^2\theta\Big] \\ &= \frac{m}{2}\Big[A^2\Omega^2\sin^2(\Omega t) + \ell^2\dot{\theta}^2 - 2A\Omega\ell\sin(\Omega t)\dot{\theta}\cos\theta\Big] \end{aligned}$$

(6 marks, seen similar, A)

(b) The potential energy is $V = mgY = -mg\ell\cos\theta$. A suitable Lagrangian is

$$L = T - V = \frac{m}{2}\Big[A^2\Omega^2\sin^2(\Omega t) + \ell^2\dot{\theta}^2 - 2A\Omega\ell\sin(\Omega t)\dot{\theta}\Big] + mg\ell\cos\theta.$$

The first term can be discarded as it does not affect the Euler-Lagrange equation (although it depends on t it does not depend on θ or $\dot{\theta}$).

(4 marks, seen similar, A)

(c) The Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} &= \frac{d}{dt}\Big[m\ell^2\dot{\theta} - mA\Omega\ell\sin(\Omega t)\cos\theta\Big] \\ &\quad - mA\Omega\ell\sin(\Omega t)\dot{\theta}\sin\theta + mg\ell\sin\theta = 0, \end{aligned}$$

which simplifies to

$$\ddot{\theta} - \frac{A\Omega^2}{\ell}\cos(\Omega t)\cos\theta + \frac{g}{\ell}\sin\theta = 0.$$

(5 marks, seen similar, B)

(d) In this approximation the equation of motion is

$$\ddot{\theta} - \frac{A\Omega^2}{\ell}\cos(\Omega t) + \frac{g}{\ell}\theta = 0,$$

or

$$\ddot{\theta} + \frac{g}{\ell}\theta = \frac{A\Omega^2}{\ell}\cos(\Omega t),$$

with general solution

$$\theta = C\cos\left(\sqrt{\frac{g}{\ell}}t + \alpha\right) + \frac{A\Omega^2}{g - \Omega^2\ell}\cos(\Omega t).$$

(5 marks, seen similar, C)

(Total: 20 marks)

3. (a) Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = p_r, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2},$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2 + 2\cos\theta}{r^3}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{\sin\theta}{r^2}.$$

(5 marks, seen similar, A)

(b) The Hamilton-Jacobi equation is

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\cos\theta}{r^2} + \frac{\partial S}{\partial t} = 0.$$

Writing $S = W_r(r) + W_\theta(\theta) - \alpha_1 t$, yields

$$\frac{1}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\cos\theta}{r^2} - \alpha_1 = 0.$$

Multiply by r^2 and separate

$$\frac{r^2}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 - \alpha_1 r^2 = -\frac{1}{2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 - \cos\theta.$$

As the LHS is a function of r only and the RHS is a function of θ only they each must be a constant which we call $-\alpha_2$.

(7 marks, seen similar, B)

(c) $\alpha_2 = \frac{p_\theta^2}{2} + \cos\theta$, is a second constant of the motion. The total energy is

$$\alpha_1 = \frac{p_r^2}{2} + \frac{\alpha_2}{r^2}.$$

As $p_r = \dot{r}$

$$\frac{\dot{r}^2}{2} + \frac{\alpha_2}{r^2} = \alpha_1,$$

so that

$$dr = \pm\sqrt{2\alpha_1 - \frac{2\alpha_2}{r^2}}dt,$$

or

$$\frac{r\ dr}{\sqrt{2\alpha_1 r^2 - 2\alpha_2}} = \pm dt,$$

which integrates to

$$\frac{1}{2\alpha_1}\sqrt{2\alpha_1 r^2 - 2\alpha_2} = \pm(t + C).$$

This can also be written in the form

$$r^2 = 2\alpha_1(t + C)^2 + \frac{\alpha_2}{\alpha_1}.$$

(8 marks, unseen, D)

(Total: 20 marks)

4. (a) Using the chain rule

$$\frac{dA}{dt} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right) = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial A}{\partial p_i} \cdot - \frac{\partial H}{\partial q_i} \right),$$

using Hamilton's equations. Hence

$$\frac{dA}{dt} = \{A, H\}.$$

(5 marks, bookwork, A)

(b)

$$\begin{aligned} \dot{\omega}_1 = \{\omega_1, H\} &= \frac{1}{2}I_2\{\omega_1, \omega_2^2\} + \frac{1}{2}I_3\{\omega_1, \omega_3^2\} \\ &= I_2\omega_2\{\omega_1, \omega_2\} + I_3\omega_3\{\omega_1, \omega_3\} \\ &= -\frac{I_3}{I_1}\omega_2\omega_3 + \frac{I_2}{I_1}\omega_3\omega_2, \end{aligned}$$

hence $I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$.

(7 marks, seen similar, B)

(c) Inserting $I_1 = 2$, $I_2 = 3$, $I_3 = 4$, $\omega_3 = \Omega$ into Euler's equation gives

$$2\dot{\omega}_1 = -\Omega\omega_2, \quad 3\dot{\omega}_2 = 2\Omega\omega_1.$$

Accordingly, $2\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\frac{2}{3}\Omega^2\omega_1$ or $\ddot{\omega}_1 = -\frac{1}{3}\Omega^2\omega_1$ which is an oscillator equation with general solution

$$\omega_1 = A\cos\left(\frac{1}{\sqrt{3}}\Omega t + \beta\right),$$

where A and β are constants. Here

$$\omega_2 = -\frac{2}{\Omega}\dot{\omega}_1 = \frac{2A}{\sqrt{3}}\sin\left(\frac{1}{\sqrt{3}}\Omega t + \beta\right).$$

(8 marks, seen similar, D)

(Total: 20 marks)

5. (a) $R = p_\phi \dot\phi - L$ where $\dot\phi$ is eliminated through $p_\phi = (h + \ell^2 \sin^2\theta)\dot\phi$

$$R = p_\phi \dot\phi - \frac{1}{2}(h + \ell^2 \sin^2\theta)\dot\phi^2 - \frac{1}{2}\ell^2\dot\theta^2 - g\ell\cos\theta$$

$$= \frac{p_\phi^2}{2(h + \ell^2\sin^2\theta)} - \frac{1}{2}\ell^2\dot\theta^2 - g\ell\cos\theta,$$

or

$$R = U(\theta, p_\phi) - \frac{1}{2}\ell^2\dot\theta^2,$$

where

$$U(\theta, p_\phi) = \frac{p_\phi^2}{2(h + \ell^2\sin^2\theta)} - g\ell\cos\theta.$$

The Routhian equations of motion are

$$\dot\phi = \frac{\partial R}{\partial p_\phi} = \frac{p_\phi}{h + \ell^2\sin^2\theta}, \quad \dot p_\phi = -\frac{\partial R}{\partial \phi} = 0,$$

and

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot\theta}\right) - \frac{\partial R}{\partial \theta} = -\ell^2\ddot\theta - \frac{\partial U}{\partial \theta} = 0.$$

(10 marks)

(b)

$$\ell^2\theta = -\frac{\partial U}{\partial \theta} = \sin\theta\left[-g\ell + \frac{p_\phi^2\cos\theta}{(h + \ell^2\sin^2\theta)^2}\right] \approx -\theta\left[g\ell - \frac{p_\phi^2}{h^2}\right],$$

or $\ddot\theta \approx -\omega^2\theta$ with angular frequency

$$\omega = \left(\frac{g}{\ell} - \frac{p_\phi^2}{h^2\ell^2}\right)^{1/2}.$$

(5 marks)

(c) With $g = 0$

$$U = \frac{p_\phi^2}{2(h + \ell^2\sin^2\theta)},$$

Which has local maxima at $\theta = 0$ and $\theta = \pi$ (unstable equilibria) and a minimum at $\theta = \pi/2$ (stable equilibrium).

(5 marks)

(Total: 20 marks)

Category A

1(a)(b) 13 marks, 2(a)(b) 10 marks, 3(a) 5 marks, 4(a) 5 marks

Total: **33/80**

Category B

2(c) 5 marks, 3(b) 7 marks, 4(b) 7 marks

Total: **19/80**

Category C

1(c) 7 marks, 2(d) 5 marks

Total: **12/80**

Category D

3 (c) 8 marks, 4(c) 8 marks

Total: **16/80**

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60011/MATH70011/MATH97223	1	This is a straighforward question comprising three short problems which was well answered.
MATH60011/MATH70011/MATH97223	2	This question about a pendulum with an oscillating support was well answered. The hint in 4(b) was rather generous but without it students may have wasted a lot of time attempting to justify the given result!
MATH60011/MATH70011/MATH97223	3	This question using the Hamilton-Jacobi equation was more challenging than the first two. Most students did not express the second constant as a function of the phase space variables.
MATH60011/MATH70011/MATH97223	4	This question on applying Poisson brackets in rigid body calculations was mostly well answered.
MATH70011/MATH97223	5	The Mastery question on the Routhian approach appeared to be well calibrated as the average marks were a few points lower than for the first four questions.