Imperial College London
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Unseen 3

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

Definition 1. Let $T : V \to V$ be a linear transformation. A subspace $W \subseteq V$ is *T-invariant* if $T(W) \subseteq W$.

1. Let F be a field, let V be an F-vector space, let $T : V \to V$ be a linear transformation and let $\lambda \in F$. Prove that $W := \{v \in V \mid T(v) = \lambda v\}$ is a T-invariant subspace.

2. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation. Let $0 < k < n$.

   (a) Prove that there is a k-dimensional T-invariant subspace if and only if there is some basis $\mathcal{E}$ of V and matrices $A \in M_{k\times k}(F), B \in M_{(n-k)\times(n-k)}(F), C \in M_{k\times(n-k)(F)}$ such that $[T]_{\mathcal{E}} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

   (b) Prove that there are T-invariant subspaces W_1, W_2 such that $V = W_1 + W_2$, $W_1 \cap W_2 = \{0\}$, and $\dim(W_1) = k$ if and only if there is some basis $\mathcal{E}$ of V and matrices $A \in M_{k\times k}(F), B \in M_{(n-k)\times(n-k)}(F)$ such that $[T]_{\mathcal{E}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Definition 2. (i) A matrix $A \in M_n(F)$ is *upper triangular* if $a_{i,j} = 0$ for all $i < j$, i.e.

$$A = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & * \end{pmatrix}$$

(ii) A field F is *matrix-triangularisable* (**note: this is not standard terminology**) if for all $n \in \mathbb{N}$, for all $A \in M_n(F)$ there is some invertible matrix $P \in M_n(F)$ and upper triangular $B \in M_n(F)$ such that $A = P^{-1}BP$.

(iii) A field F is *algebraically closed* if for every non-constant polynomial $p(x) \in F[x]$, there is some $a \in F$ such that $p(a) = 0$.

3. Prove that $\mathbb{R}$ is not matrix-triangularisable.

4. Prove that if a field F is matrix-triangularisable, then it is algebraically closed.

   hint: *use Question 9 from Problem Sheet 1 (term 2).*

Theorem 1 (The Fundamental Theorem of Algebra)**.** $\mathbb{C}$ *is algebraically closed.*

5. Prove that $\mathbb{C}$ is matrix-triangularisable.
6. Prove that a field F is matrix-triangularisable if and only if F is algebraically closed.