IMPERIAL

MATH50006

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Lebesgue Measure and Integration

Date: Thursday, May 16, 2024

Time: 10:00 – 12:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. For each of the following statements determine whether it is true or false. No justification is needed.

(a) Let λ denote the Lebesgue measure on $\mathbb{R}$.

(i) Each countable set $A \subset \mathbb{R}$ satisfies $\lambda(A) = 0$. (2 marks)

(ii) Each uncountable measurable set $A \subset \mathbb{R}$ satisfies $\lambda(A) > 0$. (2 marks)

(b) The function

$$f(t) = e^{-\frac{1}{t}} t^{-\frac{1}{2023}}, \quad t > 0$$

is in $L^{2024}((0,\infty))$, where $L^p(X)$ denotes the usual L^p-space on $X \subset \mathbb{R}$. (4 marks)

(c) The sequence of functions (f_n) with

$$f_n(x) = \begin{cases} n^{1/4} \sin\left(\frac{x}{n^{5/4}}\right), & \text{if } |x| \le n^{3/4} \\ 0, & \text{else.} \end{cases}$$

does not converge in measure (with respect to the Lebesgue measure on $\mathbb{R}$) as $n \to \infty$. (4 marks)

(d) Let μ denote the Lebesgue measure on $[0,1]$ and $\nu = \mu + \delta_1$, where δ_z is the Dirac measure at $z \in \mathbb{R}$, i.e. $\delta_z(A) = 1$ if $z \in A$ and 0 otherwise, for measurable A.

(i) One has that $1_{[0,1)} = 1_{[0,1]}$ ν-a.e, where 1_A is the indicator function of A. (4 marks)

(ii) One has that $\mu \ll \nu$, where $\ll$ denotes absolute continuity. (4 marks)

(Total: 20 marks)

2. (a) Determine

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(e^x)}{1+nx^2} dx.$$

(8 marks)

(b) Suppose that the functions $f_n : [0,1] \to \mathbb{R}$ are measurable for all $n \in \mathbb{N}$ and that f_n converges in measure (relative to the Lebesgue measure) to a measurable function $f : [0,1] \to \mathbb{R}$. Let

$$\Psi(g) \overset{\text{def.}}{=} \int \frac{|g|}{1+|g|} dx.$$

Does $\Psi(f_n - f)$ converge to 0 as $n \to \infty$? Justify your answer. (12 marks)

(Total: 20 marks)

3. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$ and $1 \leq p < \infty$.

(a) Show that if $f \in L^p(\mu)$,

$$\sup_{t \geq 1} t^p \mu(|f| \geq t) < \infty. \tag{Tail$_p$}$$

(6 marks)

(b) Show that if $f \in L^p(\mu)$,

$$\|f\|_{L^p(\mu)}^p = \int_0^\infty p t^{p-1} \mu(|f| \geq t) dt.$$

(8 marks)

(c) Show that if f satisfies $(\text{Tail})_q$ for some $q > p$, then $f \in L^p(\mu)$. (6 marks)

(Total: 20 marks)

4. Let λ denote the Lebesgue measure on $\mathbb{R}$ and

$$C = \bigcap_{k \geq 0} C_k,$$

where $C_0 = [0, 1]$ and C_k is obtained by removing the (open) middle third of each interval constituting C_{k-1} for all $k \geq 1$, i.e.

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc.

(a) Is C countable? Justify your answer. (3 marks)

(b) Determine $\lambda(C)$. (3 marks)

(c) For $k \geq 1$, let μ_k be the measure defined by

$$\frac{d\mu_k}{d\lambda} = \lambda(C_k)^{-1} 1_{C_k}$$

and let $f_k : [0, 1] \to [0, 1]$ be the function given by

$$f_k(x) = \mu_k((0, x]), \quad x \in [0, 1].$$

(i) Show that $f(x) = \lim_{k\to\infty} f_k(x)$ exists for all $x \in [0, 1]$ and that f is continuous on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. *Hint:* estimate $|f_{k+1} - f_k|$. (10 marks)

(ii) Show that f is differentiable almost everywhere on $[0, 1]$ with $f' = 0$. (4 marks)

(Total: 20 marks)

Module: MATH50006
Setter: Rodriguez
Checker: Krasovsky
Editor: editor
External: external
Date: April 10, 2024
Version: Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH50006 Lebesgue Measure and Integration – **SOLUTIONS**

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a)(i) 2 marks; 1(b) 4 marks; 1(d)(i) 4 marks; 1(d)(ii) 4 marks; 3(a) 6 marks; 3(c) 6 marks; 4(a) 3 marks; 4(b) 3 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(c) 4 marks; 2(a) 8 marks; 3(b) 8 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(b) 12 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(a)(ii) 2 marks; 4(c)(i) 10 marks; 4(c)(ii) 4 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Lebesgue Measure and Integration – **SOLUTIONS**

Date: ??

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Time Allowed: 2 Hours

This paper has *4 Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. For each of the following statements determine whether it is true or false. No justification is needed.

(a) Let λ denote the Lebesgue measure on $\mathbb{R}$.

(i) Each countable set $A \subset \mathbb{R}$ satisfies $\lambda(A) = 0$. (2 marks)

Solution: TRUE.

(ii) Each uncountable measurable set $A \subset \mathbb{R}$ satisfies $\lambda(A) > 0$. (2 marks)

Solution: FALSE.

(b) The function

$$f(t) = e^{-\frac{1}{t}} t^{-\frac{1}{2023}}, \quad t > 0$$

is in $L^{2024}((0,\infty))$, where $L^p(X)$ denotes the usual L^p-space on $X \subset \mathbb{R}$. (4 marks)

Solution: TRUE.

(c) The sequence of functions (f_n) with

$$f_n(x) = \begin{cases} n^{1/4} \sin\left(\frac{x}{n^{5/4}}\right), & \text{if } |x| \leq n^{3/4} \\ 0, & \text{else.} \end{cases}$$

does not converge in measure (with respect to the Lebesgue measure on $\mathbb{R}$) as $n \to \infty$. (4 marks)

Solution: FALSE.

(d) Let μ denote the Lebesgue measure on $[0,1]$ and $\nu = \mu + \delta_1$, where δ_z is the Dirac measure at $z \in \mathbb{R}$, i.e. $\delta_z(A) = 1$ if $z \in A$ and 0 otherwise, for measurable A.

(i) One has that $1_{[0,1)} = 1_{[0,1]}$ ν-a.e, where 1_A is the indicator function of A. (4 marks)

Solution: FALSE.

(ii) One has that $\mu \ll \nu$, where $\ll$ denotes absolute continuity. (4 marks)

Solution: TRUE.

(Total: 20 marks)

2. (a) Determine

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin(e^x)}{1+nx^2}dx.$$

(8 marks)

Solution: The function

$$f_n(x) = \frac{\sin(e^x)}{1+nx^2}$$

converges to 0 pointwise for all x as $n \to \infty$. Moreover, one has that

$$|f_n(x)| \leq g(x) = \frac{1}{1+x^2}$$

and g is integrable hence the limit is 0 by application of the dominated convergence theorem.

(b) Suppose that the functions $f_n : [0,1] \to \mathbb{R}$ are measurable for all $n \in \mathbb{N}$ and that f_n converges in measure (relative to the Lebesgue measure) to a measurable function $f : [0,1] \to \mathbb{R}$. Let

$$\Psi(g) \stackrel{\text{def.}}{=} \int \frac{|g|}{1+|g|}dx.$$

Does $\Psi(f_n - f)$ converge to 0 as $n \to \infty$? Justify your answer. (12 marks)

Solution: Yes, it does. We argue by contradiction. If $\Psi(f_n - f)$ does not converge to 0 as $n \to \infty$, we can find an $\varepsilon > 0$ and a subsequence $\Lambda \subset \mathbb{N}$ such that $|\Psi(f_n - f)| \geq \varepsilon$ for all $n \in \mathbb{N}$. But since (f_n) converges in measure by assumption, so does $(f_n)_{n\in\Lambda}$, and by passing to a subsequence $\Lambda' \subset \Lambda$ we can ensure that the convergence is a.e. on $[0,1]$. But then

$$\frac{|f_n - f|}{1+|f_n - f|} \to 0$$

pointwise as $(n \to \infty, n \in \Lambda')$ and since the integrand is bounded by 1, which is integrable, dominated convergence yields that $\Psi(f_n - f) \to 0$ as $(n \to \infty, n \in \Lambda')$, violating our assumption.

(Total: 20 marks)

3. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$ and $1 \leq p < \infty$.

(a) Show that if $f \in L^p(\mu)$,

$$\sup_{t \geq 1} t^p \mu(|f| \geq t) < \infty. \qquad (\mathrm{Tail}_p)$$

(6 marks)

Solution: By Markov's inequality,

$$\mu(|f| \geq t) = \mu(|f|^p \geq t^p) \leq t^{-p} \|f\|_{L^p(\mu)},$$

and the claim follows.

(b) Show that if $f \in L^p(\mu)$,

$$\|f\|^p_{L^p(\mu)} = \int_0^\infty p t^{p-1} \mu(|f| \geq t) dt.$$

. (8 marks)

Solution: The map $X \times (0, \infty) \to \mathbb{R}$, $(x, t) \mapsto pt^{p-1} 1\{|f| \geq t\}(x)$ is measurable and non-negative. Hence, Fubini applies and gives that

$$\int_0^\infty p t^{p-1} \mu(|f| \geq t) dt = \int d\mu \int_0^{|f|} p t^{p-1} dt = \int |f|^p d\mu.$$

(c) Show that if f satisfies $(\mathrm{Tail})_q$ for some $q > p$, then $f \in L^p(\mu)$. (6 marks)

Solution: Since $f_n = f_+ \wedge n$ is in $L^p(\mu)$, (b) applies and gives that

$$\|f_n\|^p_{L^p(\mu)} \leq p + \int_1^\infty p t^{p-1} \mu(|f| \geq t) dt \leq p + \int_1^\infty p t^{p-q-1} dt \leq C,$$

where we used that $|f_n| \leq |f|$, $(\mathrm{Tail})_q$ and the fact that $1 + p - q > 1$. Letting $n \to \infty$ and using monotone convergence, it follows that $f_+ \in L^p(\mu)$ and similarly $f_- \in L^p(\mu)$.

(Total: 20 marks)

4. Let λ denote the Lebesgue measure on $\mathbb{R}$ and

$$C = \bigcap_{k \geq 0} C_k,$$

where $C_0 = [0, 1]$ and C_k is obtained by removing the (open) middle third of each interval constituting C_{k-1} for all $k \geq 1$, i.e.

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1],\ C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1], \text{ etc.}$$

(a) Is C countable? Justify your answer. (3 marks)

Solution: Any point x in C can be mapped bijectively to a sequence in $\{0, 1\}^{\mathbb{N}}$, where the k-th entry of the sequence corresponds to the binary choice of subintervals in C_k stemming from a common interval in C_{k-1} in which x lies. Hence C is uncountable.

(b) Determine $\lambda(C)$. (3 marks)

Solution: One finds that $\lambda(C_k) \leq c^k$ by explicit calculation for some $c \in (0, 1)$ and hence $\lambda(C) = \lim_k \lambda(C_k) = 0$ where the decreasing limit is justified because $\lambda(C_0) < \infty$.

(c) For $k \geq 1$, let μ_k be the measure defined by

$$\frac{d\mu_k}{d\lambda} = \lambda(C_k)^{-1} 1_{C_k}$$

and let $f_k : [0, 1] \to [0, 1]$ be the function given by

$$f_k(x) = \mu_k((0, x]), \quad x \in [0, 1].$$

(i) Show that $f(x) = \lim_{n \to \infty} f_n(x)$ exists for all $x \in [0, 1]$ and that f is continuous on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. *Hint:* estimate $|f_{k+1} - f_k|$. (10 marks)

Solution: Since $f_k(0) = 1 - f_k(1) = 0$ for all k, it immediately follows that $f(0) = 0$ and $f(1) = 1$. The functions f_k are continuous and monotone for all k, it is enough to show that for all $x \in [0, 1]$,

$$|f_{k+1}(x) - f_k(x)| \leq 2^{-k}. \tag{$*$}$$

For, this implies together with the triangle inequality that $(f_k(x))$ is Cauchy for every x, hence converges towards some $f(x)$, and moreover that the convergence is uniform, whence f is continuous.

We now show $(*)$. Let R_n^k, $1 \leq n \leq 2^{k-1}$ denote the open intervals that are removed when constructing C_k from C_{k-1}, for $k \geq 1$. One observes that f_{k+1} and f_k coincide for all points in $R_n^{k'}$ for any $k' \leq k$ and $1 \leq n \leq 2^{k'}$. Let now $I = [a, b]$ denote any of the intervals constituting C_k. By construction, both f_k and f_{k+1} agree at the endpoints of I and moreover

$$f_k(b) - f_k(a) = f_{k+1}(b) - f_{k+1}(a) = 2^{-k}.$$

Since f_k and f_{k+1} are both monotone on I, $(*)$ follows.

(ii) Show that f is differentiable almost everywhere on $[0,1]$ with $f' = 0$. (4 marks)

Solution: We show that f is differentiable on $[0,1] \setminus C$, hence almost everywhere by application of (b). By construction any $x \in ([0,1] \setminus C)$ lies in a removed interval R_n^k, for some $k \geq 1$ and $1 \leq n \leq 2^{k-1}$, and

$$f = f_k$$

in a neighborhood of x. So f is locally constant, hence differentiable and $f'(x) = 0$.

(Total: 20 marks)

MATH50006 Lebesgue Measure and Integration

Question	Marker's comment
1	Mostly solved well. Some issues with 1(a)(ii) counterexample: Cantor set and 1(b), which is integrable both near 0 and infinity
2	Q2(a) solved mostly well; the majorizing function needs to be uniform in n !Q2(b) more difficult. Applying Vitali's theorem is an option but uniform integrability needs to be checked
4	Several people did not recognise the Cantor set or were not able to properly show it is uncountable. Part c was not aswered well in general. The main points missing werenbsp;- showing the sequence of the f_k's is Cauchy and converges uniformly- correctly showing that f is locally constant outside the Cantor set.