IMPERIAL

MATH70052

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Hydrodynamic Stability

Date: Tuesday, May 14, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the modified Eckhaus equation,

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial x^2}=\frac{1}{R}\frac{\partial^2 u}{\partial y^2}-\frac{\partial^4 u}{\partial x^4}-\frac{b}{R},$$

which may be taken as a model for studying fluid motion and its stability, where $R>0$ and $b>0$ are real positive parameters. The boundary conditions for the sought function $u(x,y,t)$ are

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0, \quad \left.u\right|_{y=1}=0.$$

(i) Find the steady basic state $u=U(y)$.

The basic state is perturbed by a small-amplitude disturbance $u'(x,y,t)$, i.e. $u=U(y)+u'(x,y,t)$. Write down the linearised equation and boundary conditions satisfied by u'.

(3 marks)

(ii) Assume that $u'(x,y,t)$ is periodic in x, that is, $u'(x+L,y,t)=u'(x,y,t)$ for arbitrary x, y and t. Let

$$E=\frac{1}{2}\int_0^1\int_0^L\left(u'\right)^2dxdy.$$

Show that

$$\frac{dE}{dt}=\int_0^1\int_0^L\frac{d^2U}{dy^2}\Big(\frac{\partial u'}{\partial x}\Big)^2dxdy-\int_0^1\int_0^L\Big\{\frac{1}{R}\Big(\frac{\partial u'}{\partial y}\Big)^2+\Big(\frac{\partial^2 u'}{\partial x^2}\Big)^2\Big\}dxdy,$$

and comment on the stabilizing and destabilizing roles of the terms in the perturbation equation.

(4 marks)

(iii) Seek temporal normal-mode solutions of the form

$$u'=\bar{u}(y)e^{\sigma t+i\alpha x}+c.c.,$$

where $c.c.$ stands for the complex conjugate. Show that the growth rate is

$$\sigma=b\,\alpha^2-\alpha^4-(n+\frac{1}{2})^2\pi^2/R \quad (n=0,\ 1,\ 2,\ldots),$$

and find the corresponding eigenfunctions.

(5 marks)

(iv) For the case of $n=0$, show that the basic state becomes unstable for

$$R>R_c=\pi^2/b^2.$$

Sketch the neutral curve in the (α,R) parameter plane, identifying the upper and lower branches of the neutral curve in the limit $R\gg 1$.

(4 marks)

(v) Suppose that an initial perturbation of the form

$$u'(x,y,0)=g(y)e^{-x^2}$$

is introduced at $t=0$, where $g(y)$ is a function of y satisfying the boundary conditions: $\dfrac{dg}{dy}=0$ at $y=0$ and $g=0$ at $y=1$. Find the solution for $u'(x,y,t)$ for $t>0$.

[*Hint: You may use the result that the Fourier transform of* e^{-x^2} *is* $\sqrt{\pi}e^{-\alpha^2/4}$.]

(4 marks)

(Total: 20 marks)

2. Consider convection in a horizontal layer of fluid of depth d, where the temperatures at the bottom and top are $\hat{\theta}_0$ and $\hat{\theta}_1 < \hat{\theta}_0$, respectively. The entire system is rotating at a constant angular velocity Ω about an axis perpendicular to the layer, which is taken to be the $\hat{z}$-axis. The flow is described in a coordinate system $(\hat{x}, \hat{y}, \hat{z})$, which rotates about the $\hat{z}$-axis with the angular velocity Ω. The reference length, time scale and velocity are taken to be d, d^2/κ and κ/d, respectively, where κ is the heat diffusivity coefficient. The non-dimensional coordinates $\mathbf{x} = (x, y, z)$, time variable t, velocity $\mathbf{u}$, pressure p and temperature θ are introduced as follows

$$\mathbf{x} = \hat{\mathbf{x}}/d, \quad t = \hat{t}/(d^2/\kappa); \quad \mathbf{u} = \hat{\mathbf{u}}/(\kappa/d), \quad p = \hat{p}/(\hat{\rho}_0\kappa^2/d^2), \quad \theta = \hat{\theta}/(\hat{\theta}_0 - \hat{\theta}_1),$$

where the variables with a hat are dimensional, and $\hat{\rho}_0$ denotes the reference density. The layer is between $z = 0$ and 1.

When the basic state of pure conduction is perturbed, small-amplitude disturbances $(u', v', w', p', \theta')$ are governed by the linearised equations

$$\frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + Pr\nabla^2\mathbf{u}' - Ta\,\mathbf{k}\times\mathbf{u}' + RaPr\,\theta'\,\mathbf{k}, \quad \nabla\cdot\mathbf{u}' = 0, \quad \frac{\partial \theta'}{\partial t} - w' = \nabla^2\theta', \qquad (1)$$

under the Boussinesq approximation, where Ra and Pr denote the Rayleigh and Prandtl numbers, respectively, $\mathbf{k}$ is the unit vector in the vertical (i.e. z) direction, and $Ta = 2\Omega d^2/\kappa$ is the parameter representing the effect of the rotation.

(i) Deduce from the momentum and continuity equations in (1) the following equations,

$$\frac{\partial \boldsymbol{\omega}'}{\partial t} = Pr\nabla^2\boldsymbol{\omega}' + RaPr\nabla\theta'\times\mathbf{k} + Ta\frac{\partial \mathbf{u}'}{\partial z},$$

$$\frac{\partial}{\partial t}\nabla^2\mathbf{u}' = Pr\nabla^4\mathbf{u}' + RaPr\Big[\nabla^2\theta'\,\mathbf{k} - \nabla\Big(\frac{\partial\theta'}{\partial z}\Big)\Big] - Ta\frac{\partial \boldsymbol{\omega}'}{\partial z}.$$

Given that at a free boundary, $w' = 0$ and the tangential stress components vanish, show that

$$\frac{\partial \omega_3'}{\partial z} = 0.$$

[*Hint: take the curl of the momentum equations for* $\mathbf{u}'$ *to obtain the equations for the vorticity* $\boldsymbol{\omega}' = \nabla\times\mathbf{u}'$*, and then take the curl of the resulting equations. Use the relation* $\nabla\times\boldsymbol{\omega}' = -\nabla^2\mathbf{u}'$ *and the vector identities given on the next page.*]

(6 marks)

(ii) Consider the three equations for w', ω_3' (the z-component of the voricity $\boldsymbol{\omega}'$) and θ', which form a closed system. Let $(w', \omega_3', \theta') = \big(\tilde{w}(z), \tilde{\omega}_3(z), \tilde{\theta}(z)\big)f(x,y)e^{\sigma t}$. Derive the equations governing $\tilde{w}(z)$, $\tilde{\omega}_3(z)$ and $\tilde{\theta}(z)$ as well as the equation satisfied by $f(x,y)$.

(5 marks)

(iii) For the special case of $Pr = 1$, reduce the system of the three equations governing $\tilde{w}(z)$, $\tilde{\omega}_3(z)$ and $\tilde{\theta}(z)$ to a single equation of sixth order for $\tilde{w}$. Specify the boundary conditions for this equation when both the top and bottom boundaries of the fluid layer are 'free'.

(5 marks)

Question continues on the next page.

(iv) Solve the eigenvalue problem formulated in Part (iii) to obtain σ. Discuss the effects of the rotation.

(4 marks)

Vector identities Let ψ denote a scalar function, and $\mathbf{A}$ and $\mathbf{B}$ denote vectors functions. The following identities hold.

$$\nabla \times (\psi \mathbf{A}) = (\nabla \psi) \times \mathbf{A} + \psi \nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

(Total: 20 marks)

3. Consider inviscid stability of a base flow with the velocity field $\Big(0, U(r), 0\Big)$ in cylindrical polar coordinate system (r, ϕ, z) by introducing three-dimensional (non-axisymmetric) perturbations of normal-mode form, $\Big(\bar{v}(r), \bar{u}(r), \bar{w}(r), \bar{p}(r)\Big)e^{\sigma t + i(\beta z + n\phi)} + c.c.$, where $c.c.$ stands for the complex conjugate. In the inviscid limit, the eigenfunction $\Big(\bar{v}(r), \bar{u}(r), \bar{w}(r), \bar{p}(r)\Big)$ satisfies the equations,

$$\sigma\bar{u} + \frac{inU}{r}\bar{u} + \Big(\frac{dU}{dr} + \frac{U}{r}\Big)\bar{v} = -\frac{i\,n}{\rho\,r}\bar{p}, \tag{2}$$

$$\sigma\bar{v} + \frac{inU}{r}\bar{v} - \frac{2U}{r}\bar{u} = -\frac{1}{\rho}\frac{d\bar{p}}{dr}, \tag{3}$$

$$\sigma\bar{w} + \frac{inU}{r}\bar{w} = -\frac{i\beta}{\rho}\bar{p}; \tag{4}$$

$$\frac{d\bar{v}}{dr} + \frac{\bar{v}}{r} + \frac{in}{r}\bar{u} + i\beta\bar{w} = 0. \tag{5}$$

(i) Describe briefly how these equations can be reduced to a single equation of second order for $\bar{p}$. You do not have to write down the equation for $\bar{p}$ precisely at this stage.

Express the impermeability boundary condition, $\bar{v} = 0$, on a rigid cylindrical surface at $r = R$ in terms of $\bar{p}$ and its derivative.

(5 marks)

(ii) For the special case of $n = 0$, implement the procedure described in Part (i) to derive the equation satisfied by $\bar{p}$.

Suppose that the flow occupies the annular region $R_1 \leq r \leq R_2$. Using the equation for $\bar{p}$ derived above, or otherwise, prove Rayleigh's criterion: the flow is inviscidly unstable if

$$\Phi \equiv r^{-3}\frac{d}{dr}(rU)^2 < 0,$$

but stable otherwise.

(8 marks)

(iii) For the base flow $U(r) = \Omega\,r$ with Ω being a constant, which corresponds to a solid rotation, implement the procedure described in Part (i) to show that $\bar{p}$ satisfies the equation

$$\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\Big]\bar{p} = \beta^2\Big[1 + \frac{4\Omega^2}{(\sigma + in\Omega)^2}\Big]\bar{p}. \tag{6}$$

Suppose that the flow field is confined in the region $r \leq R$. Using the solution (see the hint below) and appropriate boundary condition, show that the dispersion relation is

$$\frac{2i\,n\,\Omega}{(\sigma + in\Omega)}J_n(sR) + RsJ_n'(sR) = 0,$$

where J_n is the n-th order Bessel function of the first kind, a prime denotes a derivative, and

$$s = i\beta\Big[1 + 4\Omega^2/(\sigma + in\Omega)^2\Big]^{1/2}.$$

[*Hint: You may use without proof the fact that the solution for $\bar{p}$, bounded at $r = 0$, is given by $\bar{p} = AJ_n(sr)$, where J_n is the n-th order Bessel function of the first kind, and A an arbitrary constant.*]

(7 marks)

(Total: 20 marks)

4. Linear stability of an exactly parallel shear flow with velocity profile $U(y)$ is studied by introducing small-amplitude perturbations of normal-mode form: $\Big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\Big)e^{i(\alpha x-\omega t)} + c.c.$ It follows that $\Big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\Big)$ satisfies the linearised equations,

$$i\alpha\bar{u} + \frac{d\bar{v}}{dy} = 0, \quad i\alpha(U-c)\bar{u} + \frac{dU}{dy}\bar{v} = -i\alpha\bar{p} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{u}, \tag{7}$$

$$i\alpha(U-c)\bar{v} = -\frac{d\bar{p}}{dy} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{v}, \tag{8}$$

where $c = \omega/\alpha$ is the phase speed and Re the Reynolds number. These equations will be applied to a parallel shear flow in the upper half plane $y \geq 0$.

(a) In the limit $Re \gg 1$, the equations can be reduced to the Rayleigh equation

$$(U-c)(\frac{d^2}{dy^2} - \alpha^2)\bar{v} - \frac{d^2U}{dy^2}\bar{v} = 0.$$

Suppose that the surface at $y = 0$ is 'lined' such that the boundary condition is

$$\bar{v}(0) = \Lambda\bar{v}'(0),$$

where $\Lambda{=}\Lambda_r{+}i\Lambda_i$ is a complex constant referred to as 'impedance coefficient', and a prime $'$ indicates d/dy, a derivative with respect to y.

(i) Under the assumption that $U(y)$ is a smooth function of y, show that

$$\Lambda^*\Big|\bar{v}'(0)\Big|^2 + \int_0^\infty \Big|\bar{v}'\Big|^2 dy + \int_0^\infty \Big[\alpha^2 + \frac{U''}{U-c}\Big]|\bar{v}|^2 dy = 0,$$

where $\Lambda^* = \Lambda_r - i\Lambda_i$, the complex conjugate of Λ.
Deduce that for a non-inflectional profile with $U'' < 0$, instability may arise if $\Lambda_i < 0$.

(4 marks)

(ii) Suppose that the profile $U(y)$ is modelled by

$$U(y) = \begin{cases} 1 & y > 1, \\ y & 0 < y < 1. \end{cases}$$

Solve the Rayleigh equation to derive the dispersion relation and find the phase speed c.

Show that the flow is inviscidly stable when $\Lambda = 0$, but unstable when $\Lambda_i < 0$.

[*Hint: You may use the fact that across a discontinuity of U and/or $U'(y)$ at y_d, the following (jump) relations hold,*

$$\Big[(U-c)\frac{d\bar{v}}{dy} - U'\bar{v}\Big]_{y_d^-}^{y_d^+} = 0, \quad \Big[\frac{\bar{v}}{U-c}\Big]_{y_d^-}^{y_d^+} = 0,$$

where $\Big[\ \cdot\ \Big]_{y_d^-}^{y_d^+}$ stands for the jump of the quantity across $y = y_d$.]

(7 marks)

Question continues on the next page.

(iii) For the case of $\Lambda = 0$ (usual rigid surface), find the pressure and streamwise velocity of the perturbation at $y = 0$, $\bar{p}_w = \bar{p}(0)$ and $\bar{u}(0)$. Explain why it is necessary to introduce a viscous Stokes layer at $y = 0$.

(2 marks)

(b) Now analyse the disturbance in the Stokes layer by considering the continuity equation and the momentum equations with the viscous effect included, which are given in (7)–(8).

(i) Deduce that the thickness of the Stokes layer is $O\Big((\omega Re)^{-1/2}\Big)$, and hence introduce the local coordinate $\tilde{Y} = y/(\omega Re)^{-1/2}$, and deduce that $\bar{u}$ and $\bar{v}$ expand as

$$\bar{u} = \tilde{U}(\tilde{Y}) + \ldots, \quad \bar{v} = \alpha(\omega Re)^{-1/2}\tilde{V}(\tilde{Y}) + \ldots.$$

Derive the equations that $\tilde{U}$ and $\tilde{V}$ satisfy, and specify the appropriate boundary and matching conditions.

(3 marks)

(ii) Given that $\tilde{U} = (\alpha/\omega)\bar{p}_w\Big[1 - \exp\{-(-i)^{1/2}\tilde{Y}\}\Big]$, show that

$$\tilde{V} \to -(i\alpha/\omega)\bar{p}_w\,\tilde{Y} + \tilde{V}_\infty \quad \text{as} \quad \tilde{Y} \to \infty,$$

and determine the expression for $\tilde{V}_\infty$, which is independent of $\tilde{Y}$.

Explain how $\tilde{V}_\infty$ affects the inviscid solution in the main layer, and deduce the equivalent 'impedance coefficient' Λ that the viscous motion in the Stokes layer produces.

(4 marks)

(Total: 20 marks)

5. When a boundary-layer flow is perturbed by a two-dimensional disturbance, the perturbed field is written as

$$(u, v, p) = \Big(U(x, Y),\ Re^{-1/2}V(x, Y),\ \ P\Big) + \epsilon(u',\ v',\ p')$$

in the Cartesian coordinate system (x, y), where x and y are non-dimensionalised by L, the distance to the leading edge, $Y = Re^{1/2}y$ and the Reynolds number $Re = \hat{V}_\infty L/\nu$ with $\hat{V}_\infty$ being the reference velocity and ν the kinematic viscosity. The parameter $\epsilon \ll 1$ measures the magnitude of the disturbance. The boundary layer of interest is the so-called '*wall jet of marginal separation*' type, and its velocity has the near-wall and far-field behaviours:

$$U \to \tfrac{1}{2}\lambda_2 Y^2 \quad \text{as} \quad Y \to 0; \quad U \to 0 \quad \text{as} \quad Y \to \infty,$$

where λ_2 is a function of x. The flow field $(\mathbf{u}, p) = (u, v, p)$ is governed by the two-dimensional Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}.$$

(i) Derive the linearised equations governing the perturbation u', v' and p', which are functions of x, Y and t, e.g. $u' = u'(x, Y, t)$. Indicate the terms which represent the non-parallel-flow effect, and explain Prandtl's parallel-flow approximation that leads to the Orr-Sommerfeld equation.

(4 marks)

(ii) Suppose that in the main layer (deck) where $Y = O(1)$, the solution expands as

$$(u',\ v',\ p') = \Big(\bar{u}(x, Y),\ Re^{-\frac{3}{22}}\bar{v}(x, Y),\ Re^{-\frac{3}{11}}\bar{p}(x, Y)\Big)E + c.c.,$$

where $c.c.$ stands for the complex conjugate, and

$$E = e^{i(Re^{\frac{4}{11}}\alpha x - Re^{\frac{2}{11}}\omega t)}\ .$$

Derive the equations governing $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$, and verify that they have the solution

$$\bar{u} = AU', \quad \bar{v} = -i\alpha AU, \quad \bar{p} = -\alpha^2 A\int_\infty^Y U^2 dY,$$

where $U' = \partial U/\partial Y$. Explain why it is necessary to introduce a viscous sublayer (i.e lower deck) despite the fact that the main-layer solution satisfies the no-slip condition. Is an upper layer is required?

(6 marks)

(iii) Deduce that the viscous lower deck has a width of $O(Re^{-\frac{13}{22}}L)$, and hence introduce $\tilde{y} = Re^{\frac{13}{22}}y = Re^{1/11}Y$. Deduce that the solution in this layer expands as

$$(u',\ v',\ p') = \Big(Re^{-\frac{1}{11}}\tilde{u}(x, \tilde{y}),\ Re^{-\frac{7}{22}}\tilde{v}(x, \tilde{y}),\ Re^{-\frac{3}{11}}\tilde{p}(x, \tilde{y})\Big)E + c.c.$$

Comment on the orders of magnitude of the viscous effect and the non-parallelism.

(5 marks)

(iv) Derive the equations governing $\tilde{u}$ and $\tilde{v}$. Specify the boundary and matching conditions.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

MATH70052, MATH97012

Hydrodynamic Stability (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. (i) The basic state satisfies the equation $\dfrac{d^2U}{dy^2} = b$, and so $U = c_2 + c_1 y + \frac{1}{2}by^2$. The boundary conditions at $y = 0$ and 1 require that $c_1 = 0$ and so $c_2 + \frac{1}{2}b = 0$. Hence

sim. seen ⇓

$$U(y) = \tfrac{1}{2}b(y^2 - 1).$$

The linearised equation for the disturbance:

$$\frac{\partial u'}{\partial t} + \frac{d^2U}{dy^2}\frac{\partial^2 u'}{\partial x^2} = \frac{1}{R}\frac{\partial^2 u'}{\partial y^2} - \frac{\partial^4 u'}{\partial x^4}. \tag{1}$$

The boundary conditions are

$$\frac{\partial u'}{\partial y} = 0 \quad \text{at} \quad y = 0; \qquad u' = 0 \quad \text{at} \quad y = 1.$$

3, A

(ii) Multiplying u' to both sides of the perturbation equation (1), and integrating with respect to y from 0 to 1 and with respect to x from 0 and L, we have

sim. seen ⇓

$$\frac{dE}{dt} = -I_1 + \frac{1}{R}I_2 - I_3, \tag{2}$$

where the integrals are found as

$$I_1 = \int_0^1\int_0^L \frac{d^2U}{dy^2}\left(u'\frac{\partial^2 u'}{\partial x^2}\right)dxdy = \int_0^1 \frac{d^2U}{dy^2}\left(u'\frac{\partial u'}{\partial x}\right)\Big|_{x=0}^{L} dy - \int_0^1\int_0^L \frac{d^2U}{dy^2}\left(\frac{\partial u'}{\partial x}\right)^2 dxdy$$

$$= -\int_0^1\int_0^L \frac{d^2U}{dy^2}\left(\frac{\partial^2 u'}{\partial x^2}\right)^2 dxdy,$$

$$I_2 = \int_0^1\int_0^L u'\frac{\partial^2 u'}{\partial y^2}dxdy = \int_0^L u'\frac{\partial u'}{\partial y}\Big|_{y=0}^{1} dx - \int_0^1\int_0^L \left(\frac{\partial u'}{\partial y}\right)^2 dxdy = -\int_0^1\int_0^L \left(\frac{\partial u'}{\partial y}\right)^2 dxdy,$$

$$I_3 = \int_0^1\int_0^L u'\frac{\partial^4 u'}{\partial x^4}dxdy = \int_0^1 u'\frac{\partial^3 u'}{\partial x^3}\Big|_{x=0}^{L} dy - \int_0^1\int_0^L \frac{\partial u'}{\partial x}\frac{\partial^3 u'}{\partial x^3}dxdy$$

$$= -\int_0^1 \frac{\partial u'}{\partial x}\frac{\partial^2 u'}{\partial x^2}\Big|_{x=0}^{L} dy + \int_0^1\int_0^L \left(\frac{\partial^2 u'}{\partial x^2}\right)^2 dxdy = \int_0^1\int_0^L \left(\frac{\partial^2 u'}{\partial x^2}\right)^2 dxdy;$$

in the above either the periodic condition in x or boundary conditions have been used. The expressions above indicate that $I_2 < 0$ and $-I_3 < 0$ while $(-I_1 > 0)$, and viewing these in (2) we can conclude that the two terms on the right-hand side of (1) play a stabilising role (akin to viscous diffusion (dissipation)). The second term on the left-hand side, which somewhat resembles 'shear', causes amplification of disturbances.

4, A

(iii) For disturbances of the assumed normal-mode form,

sim. seen ⇓

$$\frac{\partial}{\partial t} \to \sigma, \quad \frac{\partial}{\partial x} \to i\alpha.$$

Substitution of the normal mode disturbances into (1) yields

$$\frac{d^2\bar{u}}{dy^2} + R\big[b\,\alpha^2 - \alpha^4 - \sigma\big]\bar{u} = 0.$$

The solution is

$$\bar{u} = Ae^{\lambda y} + Be^{-\lambda y}.$$

with

$$\lambda^2 + R(b\,\alpha^2 - \alpha^4 - \sigma) = 0. \tag{3}$$

Impose the boundary conditions:

$$\frac{d\bar{u}}{dy}\bigg|_{y=0} = \lambda A - \lambda B = 0, \quad \bar{u}(1) = Ae^{\lambda} + Be^{-\lambda} = 0.$$

Non-zero solutions are possible only when $e^{2\lambda} = -1 (= e^{(2n+1)\pi i})$, i.e. when $\lambda = (n + 1/2)\pi i$ $(n = 0, 1, 2, \ldots)$. Inserting these values back into (3) gives

$$\sigma = b\,\alpha^2 - \alpha^4 - (n + \tfrac{1}{2})^2\pi^2/R,$$

and the corresponding eigenfunctions are

$$\bar{u} = \cos\big((n + \tfrac{1}{2})\pi y\big).$$

$\boxed{5, \mathrm{A}}$

(iv) The neutral curve corresponds to $\sigma = 0$:

$$b\,\alpha^2 - \alpha^4 - (n + \tfrac{1}{2})^2\pi^2/R = 0, \tag{4}$$

from which follows

$$R = \frac{(n + \frac{1}{2})^2\pi^2}{b\alpha^2 - \alpha^4}.$$

For each n, the minimum critical value of R occurs when $2b\,\alpha - 4\alpha^3 = 0$, i.e. at $\alpha = \alpha_c = \sqrt{b/2}$, and for $n = 0$,

$$R_c = \pi^2/b^2.$$

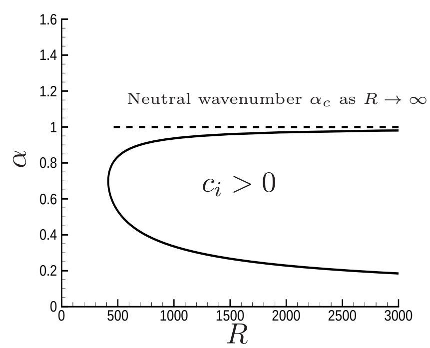

Figure 1: Sketch of the neutral curve (for illustrating qualitative features).

When $R \gg 1$, the balance of the first two terms in (4) gives

$$\alpha \to \sqrt{b} \quad \text{(the postive sign taken without losing generality)},$$

which is the asymptote of the upper branch. The balance of the first and third terms leads to the asymptote of the lower branch

$$\alpha \to \pi/(2\sqrt{b}\,)R^{-1/2}.$$

The neutral curve is shown in the sketch. $\boxed{4, \mathrm{B}}$

(v) The initial condition is Fourier transformed to $\boxed{\text{unseen} \Downarrow}$

$$\hat{u}(\alpha, y, 0) = \sqrt{\pi} e^{-\alpha^2/4} g(y).$$

Since the set of eigenfunctions, $\left\{\cos\left((n+\frac{1}{2})\pi y\right)\right\}$, is complete, we can expand $g(y)$ as

$$g(y) = \sum_n g_n \cos\big((n+\tfrac{1}{2})\pi y\big) \quad \text{with} \quad g_n = 2\int_0^1 g(y)\cos\big((n+\tfrac{1}{2})\pi y\big)\,dy.$$

Each component with a wavenumber α evolves as an eigenmode, and the disturbance as a whole has the solution

$$u'(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\pi} e^{-\alpha^2/4 + i\alpha x} \sum_n g_n \cos\big((n+\tfrac{1}{2})\pi y\big) e^{\sigma_n(\alpha)t}\,d\alpha.$$

$\boxed{4, \text{D}}$

2. (i) Taking the curl of the momentum equations, we obtain the equations for the vorticity $\boldsymbol{\omega}' = \nabla \times \mathbf{u}'$, unseen ⇓

$$\frac{\partial \boldsymbol{\omega}'}{\partial t} = Pr\nabla^2 \boldsymbol{\omega}' + RaPr\nabla\theta' \times \mathbf{k} + Ta\frac{\partial \mathbf{u}'}{\partial z}, \tag{5}$$

where use has been made of the first of the given vector identities (with $\mathbf{A} = \mathbf{k}$ so that $\nabla \times \mathbf{k} = 0$ and $\nabla \cdot \mathbf{k} = 0$), while the second implies that

$$\nabla \times (\mathbf{k} \times \mathbf{u}') = (\nabla \cdot \mathbf{u}')\mathbf{k} - (\nabla \cdot \mathbf{k})\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{k} - (\mathbf{k} \cdot \nabla)\mathbf{u}' = -\frac{\partial \mathbf{u}'}{\partial z}.$$

Taking the curl of equation (5) and noting that $\nabla \times \boldsymbol{\omega}' = -\nabla^2 \mathbf{u}'$, and using the second of the given identities, which implies

$$\nabla\times(\nabla\theta'\times\mathbf{k}) = (\nabla\cdot\mathbf{k})\nabla\theta' - (\nabla\cdot\nabla\theta')\mathbf{k} + (\mathbf{k}\cdot\nabla)\nabla\theta' - (\nabla\theta'\cdot\nabla)\mathbf{k} = -\nabla^2\theta'\mathbf{k} + \nabla(\frac{\partial \theta'}{\partial z}),$$

we arrive at the required equation

$$\frac{\partial}{\partial t}\nabla^2 \mathbf{u}' = Pr\nabla^4 \mathbf{u}' + RaPr\Big[\nabla^2\theta'\,\mathbf{k} - \nabla(\frac{\partial \theta'}{\partial z})\Big] - Ta\frac{\partial \boldsymbol{\omega}'}{\partial z}. \tag{6}$$

On a free surface, $w' = 0$ and the shear stress components $\partial u'/\partial z + \partial w'/\partial x = 0$ and $\partial v'/\partial z + \partial w'/\partial y = 0$, implying that $\partial u'/\partial z = 0$ and $\partial v'/\partial z = 0$, and hence

$$\frac{\partial \omega_3'}{\partial z} = \frac{\partial}{\partial x}(\frac{\partial v'}{\partial z}) - \frac{\partial}{\partial y}(\frac{\partial u'}{\partial z}) = 0. \tag{7}$$

6, A

(ii) The third equations in (5) and (6) read meth seen ⇓

$$\frac{\partial \omega_3'}{\partial t} = Pr\nabla^2\omega_3' + Ta\frac{\partial w'}{\partial z}, \tag{8}$$

$$\frac{\partial}{\partial t}\nabla^2 w' = Pr\nabla^4 w' + RaPr\nabla_1^2\theta' - Ta\frac{\partial \omega_3'}{\partial z}, \tag{9}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The three equations, (9), (8) and the temperature equation,

$$\frac{\partial \theta'}{\partial t} - w' = \nabla^2\theta', \tag{10}$$

form a closed system for three unknowns. Substitution of the assumed form of solutions for w' and θ' into the temperature equation, (10), leads to

$$(D^2 + \nabla_1^2 - \sigma)f\tilde{\theta} = -f\tilde{w}, \tag{11}$$

where the differential operator $D = d/dz$. Equation (11) can be rearranged into the 'variable separation' form,

$$\Big[(D^2 - \sigma)\tilde{\theta} + \tilde{w}\Big]/\tilde{\theta} = -(\nabla_1^2 f)/f.$$

Since the left-hand side is a function of z only while the right-hand side depends only on x and y, both sides should be a constant, a^2 say. Then

$$(D^2 - a^2 - \sigma)\tilde{\theta} = -\tilde{w}, \tag{12}$$

$$\nabla_1^2 f + a^2 f = 0. \tag{13}$$

Similarly, substituting the assumed form into a slightly re-arranged form of the equations for w' and ω_3', (9) and (8), we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma/Pr)\tilde{w} = a^2 Ra\,\tilde{\theta} + (Ta/Pr)D\tilde{\omega}_3, \tag{14}$$

$$(D^2 - a^2 - \sigma/Pr)\tilde{\omega}_3 = -(Ta/Pr)D\tilde{w}, \tag{15}$$

after use is made of (13). The required equations are (12), (14) and (15). $\boxed{5, \mathrm{B}}$

(iii) In order to eliminate $\tilde{\theta}$ and $\tilde{\omega}$, let the operator $(D^2 - a^2 - \sigma/Pr)$ act on (14): $\boxed{\text{unseen} \Downarrow}$

$$(D^2-a^2)(D^2-a^2-\sigma/Pr)^2\tilde{w} = a^2 Ra\,(D^2-a^2-\sigma/Pr)\tilde{\theta} + (Ta/Pr)D(D^2-a^2-\sigma/Pr)\tilde{\omega}_3,$$

which can, after making use of (15), be rearranged into

$$(D^2-a^2)(D^2-a^2-\sigma/Pr)^2\tilde{w} + (Ta/Pr)^2 D^2\tilde{w} = a^2 Ra\,(D^2-a^2-\sigma/Pr)\tilde{\theta}. \tag{16}$$

In the special case of $Pr = 1$, with the aid of (12) the equation becomes

$$(D^2 - a^2)(D^2 - a^2 - \sigma)^2\tilde{w} + Ta^2 D^2\tilde{w} = -a^2 Ra\,\tilde{w}, \tag{17}$$

which is of sixth order.

The boundary conditions at $z = 0, 1$ (free surface) are $\tilde{w} = D^2\tilde{w} = 0$, $\tilde{\theta} = 0$ and $D\tilde{\omega}_3 = 0$, where the last follows from (7). Use of the above three relations in (14) shows that

$$D^4\tilde{w} = 0 \quad \text{at} \quad z = 0, 1.$$

$\boxed{5, \mathrm{C}}$

(iv) By inspection, all *six* boundary conditions on $\tilde{w}$ can be satisfied for $\boxed{\text{unseen} \Downarrow}$

$$\tilde{w} = \sin(n\pi z),$$

substitution of which into (17) gives

$$\left[(n\pi)^2 + a^2\right]\left[(n\pi)^2 + a^2 + \sigma\right]^2 + (n\pi)^2 Ta^2 = a^2 Ra.$$

It follows that

$$\sigma = -(a^2 + n^2\pi^2) \pm \sqrt{\frac{a^2 Ra - (n\pi)^2 Ta^2}{(n\pi)^2 + a^2}}. \tag{18}$$

Growing modes are possible only for the positive sign. Neutral modes are given by $(a^2 + n^2\pi^2)^3 = a^2 Ra - (n\pi)^2 Ta^2$, that is

$$Ra = (a^2 + n^2\pi^2)^3/a^2 + (n\pi)^2 Ta^2/a^2, \tag{19}$$

which defines the neutral curve for each family. Clearly, the first family ($n = 1$) has the smallest critical Rayleigh number. As $a \to 0$, $Ra \to (\pi^4 + Ta^2)\pi^2/a^2$, while $Ra \to a^4$ as $a \to \infty$. There must exist a minimum Ra_c, the critical Rayleigh number, at a finite $a = a_c$. The expressions (18) and (19) indicate that the rotation reduces the growth rate and increases the critical Rayleigh number for the onset of instability. The rotation therefore plays a stabilising role. $\boxed{4, \mathrm{D}}$

3. (i) Consider the given equations $\boxed{\text{sim. seen} \Downarrow}$

$$\sigma\bar{u} + \frac{inU}{r}\bar{u} + \Big(\frac{dU}{dr} + \frac{U}{r}\Big)\bar{v} = -\frac{i\,n}{\rho\,r}\bar{p}, \tag{20}$$

$$\sigma\bar{v} + \frac{inU}{r}\bar{v} - \frac{2U}{r}\bar{u} = -\frac{1}{\rho}\frac{dp}{dr}, \tag{21}$$

$$\sigma\bar{w} + \frac{inU}{r}\bar{w} = -\frac{i\beta}{\rho}\bar{p}; \tag{22}$$

$$\frac{d\bar{v}}{dr} + \frac{\bar{v}}{r} + \frac{in}{r}\bar{u} + i\beta\bar{w} = 0. \tag{23}$$

In the three momentum equations, (20)–(22), the velocities $\bar{u}$, $\bar{v}$ and $\bar{w}$ are not differentiated, and so these equations are practically *simultaneous (linear) algebraic equations*, which can be solved to express the velocities in terms of the pressure and its first-order derivative. These expressions are then substituted into the continuity equation, (23), leading to a single equation for $\bar{p}$, which is of second order.

On a cylindrical surface, $\bar{v} = 0$, use of which in (20)–(21) shows

$$(\sigma + \frac{i\,nU}{r})\bar{u} = -\frac{i\,n}{\rho\,r}\bar{p}, \qquad -\frac{2U}{r}\bar{u} = -\frac{1}{\rho}\frac{d\bar{p}}{dr}.$$

Eliminating $\bar{u}$ between these two equations, we obtain

$$\Big[\frac{2i\,n\,U}{r^2(\sigma + inU/r)}\bar{p} + \frac{d\bar{p}}{dr}\Big]_{r=R} = 0. \tag{24}$$

$\boxed{5, \text{A}}$

(ii) In the special case of $n = 0$, equation (20) gives $\bar{u} = -Z\bar{v}/\sigma$, which is inserted to (21) to obtain $\boxed{\text{unseen} \Downarrow}$

$$\bar{v} = -\rho^{-1}\frac{d\bar{p}}{dr}/(\sigma + \frac{2UZ}{\sigma r}) = -\rho^{-1}\frac{d\bar{p}}{dr}/(\sigma + \Phi/\sigma),$$

where we have put

$$Z(r) = \frac{dU}{dr} + \frac{U}{r}, \qquad \Phi = 2UZ/r = r^{-3}\frac{d}{dr}(rU)^2.$$

Substituting $\bar{v}$ and $\bar{w} = -\rho^{-1}(i\beta/\sigma)\bar{p}$, which follows from (22) into the continuity equation (23), which can be recast as $\dfrac{d}{dr}(r\bar{v}) + i\beta\,r\bar{w} = 0$, we obtain

$$-\frac{d}{dr}\Big[r\frac{d\bar{p}}{dr}/(\sigma + \frac{2UZ}{\sigma r})\Big] + (\beta^2/\sigma)r\bar{p} = 0,$$

the boundary condition for which is $\frac{d\bar{p}}{dr}\big|_{r=R} = 0$. $\boxed{4, \text{B}}$

Multiplying $\sigma\bar{p}^*$ to both sides and integrating with respect to r, followed by integration by parts, we obtain

$$\int_{R_1}^{R_2}\frac{r}{1+\Phi/\sigma^2}\Big|\frac{d\bar{p}}{dr}\Big|^2 dr + \beta^2\int_{R_1}^{R_2} r|\bar{p}|^2 dr = 0,$$

which can, on noting that $1/(1+\Phi/\sigma^2) = (1+\Phi/\sigma^{*2})/|1+\Phi/\sigma^2|^2$, be rewritten as

$$\int_{R_1}^{R_2}\frac{r}{|1+\Phi/\sigma^2|^2}\Big|\frac{d\bar{p}}{dr}\Big|^2 dr + \beta^2\int_{R_1}^{R_2} r|\bar{p}|^2 dr = -(\sigma^2/|\sigma|^4)\int_{R_1}^{R_2}\frac{r\Phi}{|1+\Phi/\sigma^2|^2}\Big|\frac{d\bar{p}}{dr}\Big|^2 dr.$$

Clearly, the LHS is positive definite. When $\Phi < 0$, then $\sigma^2 > 0$, there must exist a real positive $\sigma > 0$, indicating that the flow is invisicidly unstable. When $\Phi > 0$, then $\sigma^2 < 0$, and so σ must be purely imaginary, indicating that the flow is inviscidly stable. $\boxed{4, \text{D}}$

(iii) For convenience, we introduce the notations

unseen ⇓

$$\Gamma(r) = \sigma + inU/r = \sigma + in\Omega, \quad Z(r) = \frac{dU}{dr} + \frac{U}{r} = 2\Omega.$$

From the azimuthal momentum equation (20), we have

$$\bar{u} = -\frac{1}{\Gamma}\Big[\frac{in\bar{p}}{\rho r} + Z\bar{v}\Big], \tag{25}$$

which is then substituted into the radial momentum (21) to give

$$(\Gamma + \frac{2UZ}{r\Gamma})\bar{v} = -\frac{1}{\rho}\Big[\frac{d\bar{p}}{dr} + \frac{2inU\bar{p}}{\Gamma r^2}\Big],$$

and so

$$\bar{v} = -\frac{1}{\rho}\Big[\frac{d\bar{p}}{dr} + \frac{2inU\bar{p}}{\Gamma r^2}\Big]\Big/\bar{\Gamma}, \tag{26}$$

where we have put

$$\bar{\Gamma} = \Gamma + \frac{2UZ}{r\Gamma}.$$

Inserting (26) into (25), we obtain

$$\bar{u} = -\frac{1}{\rho\Gamma}\Big\{\frac{in\bar{p}}{r} - \frac{Z}{\bar{\Gamma}}\Big[\frac{d\bar{p}}{dr} + \frac{2inU\bar{p}}{\Gamma r^2}\Big]\Big\}, \tag{27}$$

The axial momentum equation (22) gives

$$\bar{w} = -\frac{i\beta\bar{p}}{\rho\Gamma}, \tag{28}$$

The continuity equation (23) is written as $D(r\bar{v}) + in\bar{u} + i\beta r\bar{w} = 0$, into which we substitute (26), (27) and (28) to obtain

$$-\frac{D(rD\bar{p})}{\bar{\Gamma}} - \frac{2\,i\,n\Omega D\bar{p}}{\Gamma\bar{\Gamma}} + \frac{n^2\bar{p}}{\Gamma r} + \frac{inZ}{\Gamma\bar{\Gamma}}\Big[\frac{d\bar{p}}{dr} + \frac{2in\Omega\bar{p}}{\Gamma r}\Big] + \frac{r\beta^2\bar{p}}{\Gamma} = 0,$$

where use has made of the fact $U = \Omega r$ and both Γ and $\bar{\Gamma}$ are constants. Rearrangement leads to

$$-D(rD\bar{p}) + \frac{\bar{\Gamma}}{\Gamma}\Big[1 - \frac{4\Omega^2}{\Gamma\bar{\Gamma}}\Big]\frac{n^2\bar{p}}{r} + \frac{\bar{\Gamma}}{\Gamma}\beta^2 r\bar{p} = 0$$

and finally to the required equation

$$\frac{d^2\bar{p}}{dr^2} + \frac{1}{r}\frac{d\bar{p}}{dr} - \frac{n^2}{r^2}\bar{p} = \Big[1 + 4\Omega^2/(\sigma + in\Omega)^2\Big]\beta^2\bar{p},$$

which can be written into the standard Bessel equation. The solution is $\bar{p} = AJ_n(sr)$, substitution of which into the boundary condition (24) gives

$$\frac{2\,i\,n\,\Omega}{(\sigma + in\Omega)}J_n(sR) + RsJ_n'(sR) = 0, \tag{29}$$

which is the required dispersion relation.

7, C

4. (a) (i) The Rayleigh equation can be rewritten as $\boxed{\text{unseen} \Downarrow}$

$$\frac{d^2\bar{v}}{dy^2} - \Big[\alpha^2 + \frac{U''}{U-c}\Big]\bar{v} = 0.$$

Multiplying $\bar{v}^*$ to both sides and integrating (by parts), we obtain

$$\bar{v}^* D\bar{v}\Big|_0^\infty - \int_0^\infty \big|D\bar{v}\big|^2 dy - \int_0^\infty \Big[\alpha^2 + \frac{U''}{U-c}\Big]\big|\bar{v}\big|^2 dy = 0,$$

which becomes, on using the impedance boundary condition,

$$\Lambda^* \Big|D\bar{v}(0)\Big|^2 + \int_0^\infty \big|D\bar{v}\big|^2 dy + \int_0^\infty \Big[\alpha^2 + \frac{U''}{U-c}\Big]\big|\bar{v}\big|^2 dy = 0. \qquad (30)$$

On noting that $1/(U-c) = (U-c^*)/|U-c|^2$, the imaginary part of (30) reads

$$-\Lambda_i \Big|D\bar{v}(0)\Big|^2 + c_i \int_0^\infty \frac{U''}{\big|U-c\big|^2}\big|\bar{v}\big|^2 dy = 0.$$

We observe that when $U'' < 0$ and $\Lambda_i < 0$, then $c_i > 0$, indicating instability, which may arise despite the fact that the profile is non-inflectional. $\boxed{4, \text{A}}$

(ii) For the given profile, $U'' = 0$ and hence the Rayleigh equation reduces to $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{d^2\bar{v}}{dy^2} - \alpha^2 \bar{v} = 0.$$

The solution can be written as

$$\bar{v} = \begin{cases} C^+ e^{-\alpha y} & y > 1, \\ C_1 e^{-\alpha y} + C_2 e^{\alpha y} & 0 < y < 1. \end{cases}$$

Now apply the jump conditions across the discontinuity $y = 1$. First the continuity of $(U-c)\bar{v}' - U'\bar{v}$ implies

$$-\alpha(1-c)C^+ e^{-\alpha} = [-\alpha(1-c) - 1]C_1 e^{-\alpha} + [\alpha(1-c) - 1]C_2 e^{\alpha}, \qquad (31)$$

while the continuity of $\bar{v}/(U-c)$ gives

$$C^+ e^{-\alpha} = C_1 e^{-\alpha} + C_2 e^{\alpha}, \qquad (32)$$

where we have used the fact that $U' = 1$ at $y = 1^-$, and U is continuous. The boundary condition $\bar{v} = \Lambda\bar{v}'$ at $y = 0$ gives $C_1 + C_2 = \Lambda(-\alpha C_1 + \alpha C_2)$, i.e.

$$C_1 = -\frac{1-\Lambda\alpha}{1+\Lambda\alpha} C_2. \qquad (33)$$

Eliminating C^+ between (31) and (32), we obtain

$$-\alpha(1-c)\big[C_1 e^{-\alpha} + C_2 e^{\alpha}\big] = [-\alpha(1-c) - 1]C_1 e^{-\alpha} + [\alpha(1-c) - 1]C_2 e^{\alpha},$$

which simplifies to

$$C_1 e^{-\alpha} = [2\alpha(1-c) - 1]C_2 e^{\alpha}.$$

Substitution of (33) into above shows that for non-zero solutions ($c_2 \neq 0$), α and c have to satisfy the dispersion relation

$$-\frac{1-\Lambda\alpha}{1+\Lambda\alpha} e^{-\alpha} = \big[2\alpha(1-c) - 1\big]e^{\alpha},$$

from which c is determined as

$$c = 1 - \frac{1}{2\alpha}\Big[1 - \frac{1-\Lambda\alpha}{1+\Lambda\alpha}e^{-2\alpha}\Big]. \tag{34}$$

When $\Lambda = 0$, c is real and so the flow is inviscidly stable. It is worth noting that

$$c_i = -\Lambda_i e^{-2\alpha}/|1+\Lambda\alpha|^2,$$

indicating that $c_i > 0$ when $\Lambda_i < 0$. $\boxed{7, \mathrm{B}}$

(iii) For $0 \le y < 1$, the wall-normal and streamwise velocities are found as $\boxed{\text{sim. seen} \Downarrow}$

$$\bar{v} = \Big[e^{\alpha y} - \frac{1-\Lambda\alpha}{1+\Lambda\alpha}e^{-\alpha y}\Big]C_2, \quad \bar{u} = i\Big[e^{\alpha y} + \frac{1-\Lambda\alpha}{1+\Lambda\alpha}e^{-\alpha y}\Big]C_2,$$

where the second follows from the continuity equation.
The pressure can be obtained from the streamwise momentum equation,

$$\begin{aligned}\bar{p} &= -(U-c)\bar{u} + i\alpha^{-1}U'\bar{v} \\ &= -i(y-c)\Big[e^{\alpha y} + \frac{1-\Lambda\alpha}{1+\Lambda\alpha}e^{-\alpha y})\Big]C_2 + i\alpha^{-1}\Big[e^{\alpha y} - \frac{1-\Lambda\alpha}{1+\Lambda\alpha}e^{-\alpha y}\Big]C_2.\end{aligned}$$

We have

$$\bar{u}(0) = 2iC_2/(1+\Lambda\alpha), \quad \bar{p}_w = \bar{p}(0) = 2i(c + \Lambda\, C_2/(1+\Lambda\alpha).$$

For $\Lambda = 0$, $\bar{u} = 2iC_2$ and $\bar{p}_w = 2icC_2$. Clearly, $\bar{u}(0) \ne 0$, i.e. the no-slip condition is not satisfied and so a viscous (Stokes) layer is required. $\boxed{2, \mathrm{A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) (i) In the viscous (Stokes) layer, the viscous diffusion balances the unsteadiness in the streamwise momentum equation

$$i\alpha(U-c)\bar{u} + \frac{dU}{dy}\bar{v} = -i\alpha\bar{p} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{u}.$$

Let the thickness be of $O(\ell_s)$, which is now deduced. The viscous diffusion term and unsteadiness are

$$\frac{1}{Re}\frac{d^2\bar{u}}{dy^2} = O(\frac{\bar{u}}{Re\,\ell_s^2}), \quad -i(\alpha c)\bar{u} = O(\omega\bar{u}).$$

The balance of the two, $\bar{u}/(Re\,\ell_s^2) = O(\omega\bar{u})$, gives

$$\ell_s = O\big((\omega Re)^{-1/2}\big).$$

In the Stokes layer, $\bar{u}$ is expected to be of $O(C_2)$, taken to be $O(1)$. The balance in the continuity equation, $i\alpha\bar{u} = O(\bar{v}/\ell_s)$, indicates that

$$\bar{v} = O\big(\alpha(\omega Re)^{-1/2}\big),$$

which is much smaller than $\bar{u}$.
Introduce $\tilde{Y} = y/(\omega Re)^{-1/2}$, which implies that $\partial/\partial y = (\omega Re)^{1/2}\partial/\partial\tilde{Y}$. With the expansion, the continuity equation becomes

$$i\tilde{U} + \frac{d\tilde{V}}{d\tilde{Y}} = 0.$$

The pressure gradient term $-i\alpha\bar{p} = -i\alpha\bar{p}_w = -i\alpha c\bar{u}(0) = O(\omega)$. Two more terms in the momentum equation, $\alpha U\bar{u}$ and $(dU/dy)\bar{v}$, are both of $O\big(\alpha(\omega Re)^{-1/2}\big)$, which is much smaller than $\omega\bar{u}$, provided that $\alpha(\omega Re)^{-1/2} \ll \omega$. Therefore the streamwise momentum equation reduces to

$$-i\omega\tilde{U} = -i\alpha\bar{p}_w + \omega\frac{d^2\tilde{U}}{d\tilde{Y}^2}.$$

The boundary and conditions are:

$$\tilde{U} = \tilde{V} = 0 \quad \text{at} \quad \tilde{Y} = 0; \quad \tilde{U} \to \bar{u}(0) \quad \text{as} \quad \tilde{Y} \to \infty.$$

3, A

(ii) Using the given solution for $\tilde{U}$,

unseen ⇓

$$\tilde{U} = (\alpha/\omega)\bar{p}_w\Big[1 - \exp\{-(-i)^{1/2}\tilde{Y}\}\Big],$$

we integrate the continuity equation to obtain

$$\tilde{V} = -i(\alpha/\omega)\bar{p}_w\Big[\tilde{Y} - \int_0^{\tilde{Y}} \exp\{-(-i)^{1/2}\tilde{Y}\}d\tilde{Y}\Big] \to -i(\alpha/\omega)\bar{p}_w\Big[\tilde{Y} - (-i)^{-1/2}\Big],$$

as $\tilde{Y} \to \infty$. Thus $\tilde{V}_\infty = -(\alpha/\omega)\bar{p}_w(-i)^{1/2}$, which represents the displacement effect induced by the viscous motion. The unrescaled 'transpiration velocity' is $\alpha(\omega Re)^{-1/2}\tilde{V}_\infty = -(\alpha^2/\omega^{3/2})e^{-\pi i/4}\bar{p}_w Re^{-1/2}$, and it acts at the 'bottom' of the main layer. Its impact on the inviscid solution can be accounted for by replacing the boundary condition $\bar{v}(0) = 0$ by

$$\bar{v}(0) = -(\alpha^2/\omega^{3/2})e^{-\pi i/4}Re^{-1/2}\bar{p}(0) = -\omega^{-1/2}e^{\pi i/4}Re^{-1/2}\bar{v}'(0),$$

where in the last step we noted that $\bar{p}(0) = c\bar{u}(0) = -(i\alpha)^{-1}c\bar{v}'(0)$ (according to the continuity equation). The coefficient of $\bar{v}'(0)$ may be interpreted as the equivalent 'impedance coefficient' $\Lambda = -\omega^{-1/2}e^{\pi i/4}Re^{-1/2}$. As $\Lambda_i < 0$, the viscous motion plays a destabilizing role, producing a growth rate of $O(Re^{-1/2})$ when $\omega = O(1)$.

4, D

5. (i) Substituting the expression for the perturbed flow into the Navier-Stokes equations, and neglecting all nonlinear terms, we obtain the linearized equations for the disturbances, sim. seen ⇓

$$\frac{\partial u'}{\partial x} + Re^{1/2}\frac{\partial v'}{\partial Y} = 0, \tag{35}$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \underline{\frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial U}{\partial Y}v' = -\frac{\partial p'}{\partial x} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\frac{\partial^2}{\partial x^2}\Big]u', \tag{36}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + \underline{Re^{-\frac{1}{2}}\frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial Y} + \frac{\partial V}{\partial Y}v'} = -Re^{\frac{1}{2}}\frac{\partial p'}{\partial Y} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\frac{\partial^2}{\partial x^2}\Big]v', \tag{37}$$

The underlined terms represent the non-parallel-flow effects, caused by the streamwise variation of the streamwise velocity U and the presence of a normal velocity $Re^{-1/2}V$, which is associated as well with the streamwise variation of U.

Parallel-flow approximation neglects the underlined terms in (36)-(37), and treats the variation of U with x as being parametric; the latter means that at each location, the profile is 'frozen' so that normal-mode solutions may be sought. The viscous terms are retained and so the analysis leads to the usual Orr-Sommerfeld equation. 4, M

(ii) Substituting the normal-mode form into (35)-(37), and noting that the operators unseen ⇓

$$\frac{\partial}{\partial t} \to -iRe^{2/11}\omega, \quad \frac{\partial}{\partial x} \to iRe^{4/11}\alpha, \tag{38}$$

when acting on the perturbation, we obtain the equations

$$i\alpha\bar{u} + \frac{\partial \bar{v}}{\partial Y} = 0, \quad i\alpha U\bar{u} + \frac{\partial U}{\partial Y}\bar{v} = 0, \quad i\alpha U\bar{v} = -\frac{\partial \bar{p}}{\partial Y}, \tag{39}$$

with $\bar{v}$ having to satisfy the boundary condition $\bar{v} = 0$ at $Y = 0$. Elimination of $\bar{u}$ between the first two equations gives

$$U\frac{\partial \bar{v}}{\partial Y} - \frac{\partial U}{\partial Y}\bar{v} = 0,$$

which is a first-order ordinary differential equation for $\bar{v}$ and has the solution

$$\bar{v} = -i\alpha AU,$$

where A is a constant yet to be determined, and the pre-factor $-i\alpha$ is introduced for convenience. It follows that

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{p} = -\alpha^2 A\int_{\infty}^{Y} U^2 dY, \tag{40}$$

where the integration constant is taken such that $\bar{p}$ vanishes as $Y \to \infty$.
Note that for the wall jet under consideration,

$$\bar{u} \to A\lambda_2 Y, \quad \bar{v} \to -\tfrac{1}{2}i\alpha A\lambda_2 Y^2 \quad \text{as} \quad Y \to 0. \tag{41}$$

Although $\bar{u}$ does satisfy the required no-slip condition (and $\bar{v}$ satisfies impermeability condition), a viscous lower deck is still required. This is because the pressure gradient $\partial p'/\partial x$ is neglected in the streamwise momentum equation, but this term would balance the inertial term sufficiently closely to the wall, where viscous effect also comes into play.
On the other hand, since $U \to 0$ as $Y \to \infty$ for a wall jet, $(\bar{u}, \bar{v}) \to 0$ as $Y \to \infty$, and moreover $\bar{p} \to 0$. The perturbation attenuates at the outer edge of the wall jet. and so an upper layer is not required. 6, M

(iii) Let $Y = O(d)$ with $d \ll 1$ in the viscous wall layer, where $U = \frac{1}{2}\lambda_2 Y^2 = O(d^2)$. It follows that the inertia term $U\dfrac{\partial u'}{\partial x} \sim O(d^2 Re^{4/11} u')$, while the viscous diffusion $\dfrac{\partial^2 u'}{\partial Y^2} \sim O(u'/d^2)$. The pressure $p' = O(R^{-3/11})$ and its gradient $\partial p'/\partial x = O(R^{1/11})$. The balance between the three, $\boxed{\text{unseen} \Downarrow}$

$$d^2 Re^{4/11} u' \sim R^{1/11} \sim u'/d^2,$$

suggests that $u' = O(R^{-1/11})$ and $d = O(Re^{-1/11})$; the latter corresponds to $y = Re^{-1/2} Y = O(Re^{-13/22})$.

The asymptotic behaviour of the main-deck solution, (41), suggests that in the lower deck $u' = O(d) = O(R^{-1/11})$ (consistent with what was inferred above), but $v' = O(Re^{-3/22} d^2) = O(Re^{-7/22})$ as can be deduced by the matching principle. Therefore, in the lower deck, the solution should expand as

$$(u',\, v',\, w',\, p') = (Re^{-\frac{1}{11}}\tilde{u},\, Re^{-\frac{7}{22}}\tilde{v},\, Re^{-\frac{3}{11}}\tilde{p})E + c.c.$$

The local transverse variable $\tilde{y} = Re^{1/11} Y$, with which

$$\frac{\partial}{\partial Y} \to Re^{\frac{1}{11}} \frac{\partial}{\partial \tilde{y}}.$$

With the appropriate scaling identified, inspection of the terms in (36) indicates that the viscous effect appears at leading order, while the effect of non-parallelism is $O(Re^{-4/11})$ smaller, rather than $O(Re^{-1/2})$ as one might intuitively (and mistakenly) perceive based on the Orr-Sommerfeld equation. $\boxed{5,\ \mathrm{M}}$

(iv) Substituting these into (35)-(37) and using the fact that $U = Re^{-\frac{2}{11}}\frac{1}{2}(\lambda_2 \tilde{y}^2)$ as well as the relations in (38), we obtain $\boxed{\text{sim. seen} \Downarrow}$

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\tilde{y}} = 0, \tag{42}$$

$$i(\frac{1}{2}\alpha\lambda_2\tilde{y}^2 - \omega)\tilde{u} + \lambda_2\tilde{y}\tilde{v} = -i\alpha\tilde{p} + \frac{d^2\tilde{u}}{d\tilde{y}^2}, \tag{43}$$

plus $d\tilde{p}/d\tilde{y} = 0$ so that $\tilde{p}$ is a constant. By matching with the main-deck pressure, we find that

$$\tilde{p} = -\alpha^2 A \int_{\infty}^{0} U^2 dy = \alpha^2 A \int_{0}^{\infty} U^2 dy.$$

The no-slip and impermeability conditions are imposed at the wall:

$$\tilde{u} = 0, \quad \tilde{v} = 0 \quad \text{at} \quad \tilde{y} = 0.$$

Matching with the main-layer streamwise velocity requires that

$$\tilde{u} \to \lambda_2 A Y \quad \text{as} \quad \tilde{y} \to \infty.$$

$\boxed{5,\ \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70052 Hydrodynamic Stability

Question	Marker's comment
1	Overall the paper is very long and quite hard. Students' performance turns out to be more or less as expected. The workload and difficulty will be taken into account when setting the borderline marks.Question 1 is doen well by the great majority of students.
2	Marks on Question 2 are somewhat lower because it contains much unfamiliar material as well as is longer.
3	Question 3 involved some rather difficulty algebra and also a quite hard proof. Nevertheless, most did reasonablly well.
4	Again this is a long question, but it also contained some rather a subtle physical concept.
5	Question 5: this type of question might be expected, but those who had not grasped well enough to grasp the essence of the concepts, or not been skillful with algebra, would get lost or struggle with time.