Math40003 Linear Algebra and Groups **Problem Sheet 2**
Seen A Solutions

A.1. Exercise 2.7.8: Let R_θ be the anticlockwise rotation of $\mathbb{R}^2$ around the origin through θ radians. Find the matrix representing R_θ. Example 2.7.7. may be helpful here.

Let's investigate what R_θ does to an arbitrary point in $\mathbb{R}^2$. Let's write that arbitrary point in polar coordinates, as that will work well with rotations.

$$R_\theta \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix} = \begin{pmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{pmatrix}$$

The multiple angle formulas let us simplify this:

$$\begin{pmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{pmatrix} = r \begin{pmatrix} \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta) \end{pmatrix}$$

From this format, we can guess at a matrix:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

A.2. Prove that clockwise rotation of $\mathbb{R}^2$ around $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ by 1 radian can not be represented by a matrix. You may find Definition 2.7.3 useful here.

If A is a 2×2 matrix, then $A \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ but this rotation sends $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to something else, so there is no matrix that represents this rotation.

A.3. Exercise 3.1.4: Which of the following are examples of vector spaces over $\mathbb{R}$:

(a) $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ with the usual vector addition and scalar multiplication.

(b) $V = \left\{ \begin{pmatrix} a+1 \\ 2 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ with the usual vector addition and scalar multiplication.

(c) $V = \mathbb{R}^2$ with standard addition and scalar multiplication defined to be:

$$r \odot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ rb \end{pmatrix}$$

Please look at Definition 3.1.1. for this question.

The method to show that something is a vector space is actually the same as showing that its not. You need to check that the definition of vector space holds, and if any part of it fails then it's not a vector space.

(a) This question shows that it's important to not just check A1-A8, we also need to check that addition and scalar multiplication are functions as specified by the definition. The scalar multiplication from A.3.(a) is not a function from $\mathbb{R} \times V$ to V. For example,

$$\pi . \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin V$$

So this V is not a vector space.

(b) The same issue as in (a) occurs here, the functions do not have the correct domain and co-domain for this to be a Vector Space.
Even if we were to overlook this, since the addition we're using here is the standard addition, we would have to use $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as the additive identity, but $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin V$. Therefore A3 is not true for this V, and so V is not a vector space.

(c) In this question, both addition and scalar multiplication are functions with the appropriate domain and codomain, so now we need to check A1-A8. Since the addition used in this question is the standard vector addition, we know straight away that A1-A4 are true here. However, since scalar multiplication is different, we'll need to check A5-A8 in more detail.

A5 Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in V$ and let $r \in \mathbb{R}$. Then

$$\begin{aligned} r\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= r\begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ r(v_2 + u_2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ rv_2 + ru_2 \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 \\ rv_2 \end{pmatrix} + \begin{pmatrix} 0 \\ ru_2 \end{pmatrix}\right) \\ &= \left(r\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + r\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \end{aligned}$$

Therefore A5 holds.

A6 Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ and let $r, s \in \mathbb{R}$. Then

$$\begin{aligned} (r+s)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ (r+s)v_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ rv_2 + sv_2 \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 \\ rv_2 \end{pmatrix} + \begin{pmatrix} 0 \\ sv_2 \end{pmatrix}\right) \\ &= \left(r\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + s\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \end{aligned}$$

Therefore A6 holds.

A7 Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ and let $r, s \in \mathbb{R}$. Then

$$\begin{aligned} r\left(s\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) &= r\begin{pmatrix} 0 \\ sv_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ r(sv_2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (rs)v_2 \end{pmatrix} \\ &= (rs)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$

Therefore A7 holds.

However, $1.\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, so A8 does not hold, and so this V is not a vector space.

A.4. Exercise 3.2.5: Prove that all subspaces of a vector space over F are vector spaces over F in their own right. Again, Definition 3.1.1. should be useful here.

Let V be a vector space, and let U be one of its subspaces. Like in A.3., we need to check the all vector space axioms. A1, A2, A5, A6, A7, and A8 follow automatically from the fact that $U \subseteq V$. All of these axioms say that every vector in V has a certain property, and therefore every vector in U also has this property. A3 and A4 are different.

A3 asserts that a certain vector exists, namely 0_U when we're talking about U. We know that O_V exists, but we have to make sure O_U does also. We're going to show that $O_V \in U$, and that will show that there is a zero vector in U, and therefore A3 holds for U. Let $u \in U$. Then $u+(-1)u \in U$, as U is a subspace. However, $u+(-1)u = 1u+(-1)u = (1-1)u = 0u = 0_V$, and therefore $0_V \in U$, and U satisfies A3.

A4 asserts that every vector has an inverse. V satisfies A4, but to show that U satisfies A4, we need to show that the inverse is contained in U. If $v \in U$ then $(-1).v \in U$, and therefore $-v \in U$. This shows that U satisfies A4.