Imperial College London

Department of Mathematics
Imperial College London
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Part I – Solutions to Problem Sheet 1: Logic and sets

1. (a) We can just check all the cases.

P	Q	$P \vee Q$	$Q \vee P$	$(P \vee Q) \implies (Q \vee P)$
false	false	false	false	true
false	true	true	true	true
true	false	true	true	true
true	true	true	true	true

The fifth column is always `true`, hence $P \vee Q \implies Q \vee P$ is always true. In fact the third and fourth columns are identical, so the stronger statement $P \vee Q \iff Q \vee P$ is true.

(b) It is not. For $\implies$ to be symmetric, we must have $(P \implies Q) \implies (Q \implies P)$ being true whatever P and Q are. But if P is false and Q is true, then $P \implies Q$ is true, $Q \implies P$ is false, and $(P \implies Q) \implies (Q \implies P)$ is then false.

(c) It is. We can just check by drawing a truth table like in part (a).

2. We know that if Q is false then R is false. So $\neg Q \implies \neg R$ is true. Hence $R \implies Q$ is true (the contrapositive). We also know that if Q is true then P is true, so $Q \implies P$ is true. In particular we know $R \implies Q$ and $Q \implies P$ are both true. Hence $R \implies P$ is also true ("by transitivity of $\implies$", if you want to sound clever).

3. It is possible to find examples – for example we could have P being false, Q being anything, and R being true.

4. If all of the P_n are false, then we're done! So let's say that at least one of them is true. Let's say P_M is true, for some integer M. Let's prove a lemma before we start on this question.

Lemma.

(a) For all $n \in \mathbb{Z}$, $P_n \implies P_{n+1}$.

(b) For all $n \in \mathbb{Z}$, $P_n \implies P_{n-1}$.

Proof.

(a) From the assumptions, $P_n \implies P_{n+8} \implies P_{n+16} \implies P_{n+13} \implies P_{n+10} \implies P_{n+7} \implies P_{n+4} \implies P_{n+1}$.

(b) From the assumptions, $P_n \implies P_{n+8} \implies P_{n+5} \implies P_{n+2} \implies P_{n-1}$.

□

We now solve the problem. Recall that we're assuming that P_M is true. I claim that P_{M+d} is true for all non-negative integers d. We prove this by induction on d. The base case $d = 0$ is true by assumption, and for the inductive step we need to show that if P_{M+e} is true for some non-negative integer e then P_{M+e+1} is also true, but this follows immediately from part (a) of the lemma.

We next claim that P_{M-d} is true for all non-negative integers d. Again we prove this by induction, the base case is true by assumption and the inductive step is part (b) of the lemma.

Because every integer n is of the form $M + d$ for some $d \geq 0$ or $d \leq 0$, we have proved P_n for all integers n, and we're done.

5. Here's how I would do it. To prove that two sets are equal, it suffices to prove that they have the same elements (this is the axiom of "set extensionality" if you like fancy words). So we have to prove that for every t, $t \in X \cup Y \iff t \in Y \cup X$. By definition of $\cup$, this is the same thing as proving that $(t \in X \vee t \in Y) \iff (t \in Y \vee t \in X)$. Let P be the proposition $t \in X$ and Q be the proposition $t \in Y$. Then we have to prove $P \vee Q \iff Q \vee P$, and we can do this by drawing the truth table (or using Q1a twice, once for each direction).

6. (a) This is true. You could draw a "Venn diagram" and shade the two regions. Alternatively you could say that by set extensionality we need to show that for any $t \in \Omega$, we have $t \in A \cup (B \cap C) \iff t \in (A \cup B) \cap (A \cup C)$, so if P is the statement $t \in A$, Q is the statement $t \in B$ and R is the statement $t \in C$, then we have to prove $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$ which we can do by drawing a truth table.

   Do you understand why the "Venn diagram" proof and the "truth table" proof are both the same piece of mathematics, but being expressed in two different ways?

   (b) This is also true, and can be proved in the same way as the previous part.

   (c) This is false. If you draw the Venn diagram, you will see that the issue is that if $x \in A$ but $x \notin C$ then $x \notin (A \cup B) \cap C$ but $x \in A \cup (B \cap C)$. So for an explicit counterexample set $A = \{37\}$ and let B and C both be the empty set. Then $(A \cup B) \cap C$ is empty (because C is empty) but $A \cup (B \cap C)$ is nonempty (because A is nonempty).

7. We start by observing that B is the set $\{-1, 0, 1\}$; indeed if $x \geq 2$ is an integer then $x^2 \geq 4 > 3$ and if $x \leq -2$ then $x^2 = (-x)^2 > 3$ (although I would be happy if people said this was "obvious").

   (a) $\frac{1}{2} \in A \cap B$ is false because $\frac{1}{2} \notin B$ as $\frac{1}{2}$ isn't an integer.

   (b) $\frac{1}{2} \in A \cup B$ is true, because it is easily checked that $\frac{1}{2} \in A$.

   (c) $A \subseteq C$ is false, because $x = \frac{3}{2}$ satisfies $x^2 = 9/4 < 3$ but $x^3 = 27/8 > 3$, so $x \in A$ but $x \notin C$, and thus $A \nsubseteq C$.

   (d) $B \subseteq C$ is true. Because $B = \{-1, 0, 1\}$ all we have to do is to check that these elements are in C, which is easy to do.

   (e) $C \subseteq A \cup B$ is not true, because $-100 \in C$ (as its cube is less than zero) but its square is bigger than 3 so it is not in A or B.

   (f) $(A \cap B) \cup C = (A \cup B) \cap C$ is also not true. Again let's try $x = -100$. This number is in C so it's clearly in the left hand side, but we just checked it's not in $A \cup B$ so it's not in the right hand side.

8. (a)

$$\begin{aligned} \neg(\forall x \in X, P(x) \wedge \neg Q(x)) &\iff \exists x \in X, \neg(P(x) \wedge \neg Q(x)) \\ &\iff \exists x \in X, \neg P(x) \vee Q(x) \end{aligned}$$

   with the first equivalence coming from "$\neg\forall = \exists\neg$", and the second one being one of de Morgan's laws (and we also use that $\neg\neg Q(x) \iff Q(x)$).

   Note that we could even rewrite $\neg P(x) \vee Q(x)$ as $P(x) \implies Q(x)$.

   (b)

$$\begin{aligned} \neg(\exists x \in X, (\neg P(x)) \wedge Q(x)) &\iff \forall x \in X, \neg(\neg P(x) \wedge Q(x)) \\ &\iff \forall x \in X, P(x) \vee \neg Q(x) \end{aligned}$$

   (c)

$$\begin{aligned} \neg(\forall x \in X, \exists y \in Y, R(x, y)) &\iff \exists x \in X, \neg(\exists y \in Y, R(x, y)) \\ &\iff \exists x \in X, \forall y \in Y, \neg R(x, y) \end{aligned}$$

Finally, the logical negation of the statement that f is continuous at x is $\exists \epsilon \in \mathbb{R}_{>0}, \forall \delta \in \mathbb{R}_{>0}, \exists y \in \mathbb{R}, \neg(|y - x| < \delta \implies |f(y) - f(x)| < \epsilon)$, or the logically equivalent but typically more useful $\exists \epsilon \in \mathbb{R}_{>0}, \forall \delta \in \mathbb{R}_{>0}, \exists y \in \mathbb{R}, |y - x| < \delta \wedge |f(y) - f(x)| \geq \epsilon$.

9. (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 2$ is true, and here's the proof. Let $x \in \mathbb{R}$ be arbitrary. We need to prove that there exists $y \in \mathbb{R}$ such that $x + y = 2$. Clearly $y = 2 - x$ works.

   (b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 2$ is false. Imagine trying to prove it in the same way. First we need to choose a value for $y \in \mathbb{R}$, but this time x comes later on in so we can't make it depend on x. After that we need things to work out for all x, with y fixed, and clearly this is too much. The logical negation of the statement is $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \neq 2$, and this is not so hard to prove. Let $y \in \mathbb{R}$ be arbitrary. Now set $x = 3 - y$. Then $x + y = 3$, and $3 \neq 2$, so we have proved the negation and hence the original proposition is false.

10. (a) The statement is false, and we can show this by proving its logical negation. Its logical negation is $\forall x \in \varnothing, 2 + 2 \neq 5$, which is not hard to prove. Indeed, let $x \in \varnothing$ be arbitrary. We want to deduce that $2 + 2 \neq 5$, but this is easy to prove because $2 + 2 = 4$ and $4 \neq 5$, so we are done.

    (b) $\forall x \in \varnothing, 2 + 2 = 5$: this statement is true. Let $x \in \varnothing$ be arbitrary. We now want to prove that $2 + 2 = 5$, and we do it by contradiction. Assume that $2 + 2 \neq 5$; we seek a contradiction, and if we can find one then we're done. But look – we have an element x of the empty set! That can't happen, and this contradiction finishes the proof.