MATH50004 Differential Equations
Spring Term 2021/22
Mid-term exam on 22 February 2022

Question 1 (total: 12 points)
Consider the initial value problem

$$\dot{x} = \cos(x)\,, \qquad x(0) = 0\,,$$

and let $\{\lambda_n : \mathbb{R} \to \mathbb{R}\}_{n \in \mathbb{N}_0}$ be the Picard iterates corresponding to this initial value problem.

(i) Compute the first three Picard iterates λ_0, λ_1 and λ_2. [6 points]

(ii) Explain briefly why the assumptions of the global version of the Picard–Lindelöf theorem are satisfied for this differential equation. [2 points]

(iii) Explain briefly (but clearly) why there exists a compact interval J with 0 in its interior such that these Picard iterates converge uniformly on J to a solution $\lambda : J \to \mathbb{R}$. [4 points]
Hint. Use results from the lectures and not your computations from (i).

Question 2 (total: 12 points)
Consider the differential equation

$$\dot{x} = 3\sqrt[3]{x^2}\,,$$

whose right hand side $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t, x) = 3\sqrt[3]{x^2}$, is globally defined.

(i) Prove that there is not a unique solution satisfying the initial condition $x(0) = 0$, by identifying two different solutions to the initial value problem. [6 points]

(ii) Explain briefly why the local version of the Picard–Lindelöf theorem is applicable when the right hand side f is restricted to the domain $D = \mathbb{R} \times (0, \infty)$. [2 points]

(iii) Use the quantitative version of this theorem to find for all $(t_0, x_0) \in D$ a positive real number $h = h(t_0, x_0)$ such that a unique solution satisfying the initial condition $x(t_0) = x_0$ can be defined on $[t_0 - h, t_0 + h]$. [4 points]
Hint. For $(t_0, x_0) \in \mathbb{R}\times(0, \infty)$, fix first an appropriate neighbourhood of the form $W^{\tau,\delta}(t_0, x_0)$, and then establish the existence of constants K and M, depending only on t_0 and x_0.

Question 3 (total: 6 points)
Find an example of an autonomous one-dimensional differential equation $\dot{x} = f(x)$ that satisfies the following property: there exists a solution $\lambda : \mathbb{R} \to \mathbb{R}$ to this differential equation such that for any solution $\mu : I \to \mathbb{R}$ to this differential equation, there exists a $T \in \mathbb{R}$ with

$$\mu(t) = \lambda(T + t) \quad \text{for all } t \in I\,.$$

Provide justification why your example and the identified solution λ satisfy this property.