3.6 Dimension

Lemma 3.6.1. *Steinitz Exchange Lemma*
Let V be a vector space over F. Take $X \subseteq V$ and suppose $u \in Span(X)$ but $u \notin Span(X \setminus \{v\})$ for some $v \in X$. Let $Y = (X \setminus \{v\}) \cup \{u\}$ (i.e., we "exchange v for u"). Then $Span(X) = Span(Y)$.

This lemma is essential to being able to define the dimension of a vector space - and relies on being able to invert elements in the field.

Exercise 3.6.2. Verify the Steinitz exchange lemma where:

- $V = \mathbb{R}^3$
- $X = \{e_1, e_2\}$
- $u = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

Theorem 3.6.3. Let V be a finite dimensional vector space over F. Let S, T be finte subsets of V. If S is LI and T spans V then $|S| \leq |T|$. That is, LI sets are at most as big as spanning sets.

Corollary 3.6.4. Let V be a finite dimensional vector space. Let S, T be bases of V, then S and T are both finite and $|S| = |T|$.

Definition 3.6.5. Let V be a finite dimensional vector space. The *dimension of V*, written $\dim V$, is the size of any basis of V.

Remark 3.6.6 *Note that we needed Corollary 3.6.4 and thus the SEL to know that the size of a basis is unique (a basis certainly isn't).*

Example 3.6.7. In PS2 you were asked to describe all the subspaces of $\mathbb{R}^3$ this becomes much easier once we know about dimensions.$\mathbb{R}^3$ is an $\mathbb{R}$ vector space of dimension 3.

As subspaces are vector spaces in their own right so they also have dimensions, and these must be less than or equal to 3:

- $\dim 3$: the only subspace of dimension 3 is $\mathbb{R}^2$
- $\dim 2$: planes going through the origin
- $\dim 1$: lines going through the

- $\dim 0$: $\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$

Lemma 3.6.8. Suppose that $\dim V = n$:

1. Any spanning set of size n is a basis.
2. Any linearly independent set of size n is a basis.
3. S is a spanning set if and only if it contains a basis (as a subset).
4. S is linearly independent if and only if it is contained in a basis (i.e. it's a subset of a basis).
5. Any subset of V of size $> n$ is linearly dependent.

Proof: Exercise.

3.7 More subspaces

Definition 3.7.1. Let V be a vector space, U and W be subspaces of V.

- The *intersection of U and W* is:

$$U \cap W = \{v \in V : v \in W \text{ and } v \in U\}$$

- The *sum of U and W* is:

$$U + W = \{u + w : u \in U, w \in W\}$$

Remark 3.7.2. $U \subseteq U + W$ and $W \subseteq U + W$. This is because $0 \in U$ and $0 \in W$, so for every $u \in U$, $u = u + 0 \in U + W$. Similarly, for every $w \in W$, $w = 0 + w \in U + W$

Example 3.7.3. Let $V = \mathbb{R}^2$ over $\mathbb{R}$, $U = Span\{(1, 0)\}$, $W = Span\{(0, 1)\}$. *Claim* $U+W = \mathbb{R}^2$.

Proof: Let $(\lambda, \mu) \in \mathbb{R}^2$ then $(\lambda, 0) \in U$, $(0, \mu) \in W$ so

$$(\lambda, \mu) = (\lambda, 0) + (0, \mu) \in U + W$$

Exercise 3.7.4. Let U and W be subspaces of V an F-vector space. Then $U + W$ and $U \cap W$ are subspaces of V.

Proposition 3.7.5. Let V be a vector space over F. Let U and W be subspaces of V, suppose additionally:

- $U = Span\{u_1, ..., u_s\}$
- $W = Span\{w_1, ..., w_r\}$

Then $U + W = Span\{u_1, ..., u_s, w_1, ..., w_r\}$.

Example 3.7.6. Let $V = \mathbb{R}^2$, let $U = Span\{(0, 1)\}$, $W = Span\{(1, 0)\}$. Then by proposition 3.7.5 we have $U + W = Span\{(0, 1), (1, 0)\} = \mathbb{R}^2$. Agrees with example 3.7.3.

Example 3.7.7. Let $V = \mathbb{R}^3$ and:
Let $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$
Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -x_1 + 2x_2 + x_3 = 0\}$
Question: Find bases for $U, W, U \cap W, U + W$.

Remark 3.7.8. A neater way of finding a basis for $U + W$ would have been to use the basis for $U \cap W$. Since $U \cap W \subset U$ we can find a basis for U containing out basis for $U \cap W$ and similarly for W. The union of these bases will be a basis for U_W.

For instance, a basis for U is $\{(1, 0, -1), (1, 2, -3)\}$, and a basis for W is $\{(1, 0, 1), (1, 2, -3)\}$, so a basis for $U + W$ is $\{(1, 0, 1), (1, 0, -1), (1, 2, -3)\}$. Note that this has three elements, and $dim(U + W) = 3$ so as this is a spanning set it must be a basis.

Theorem 3.7.9. Let V be a vector space over F, U and W subspaces of V. Then

$$dim(U + W) = dimU + dimW - dim(U \cap W).$$