Parts

(i) *Similar seen as exercise*

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}\,\theta} e^{-\frac{1}{2}\left(\frac{x-\theta}{\theta}\right)^2}$$

$$\ln f = \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \ln\theta - \frac{1}{2}\left(\frac{x^2}{\theta^2} - \frac{2x\theta}{\theta^2} + 1\right)$$

$$= \underbrace{\left\{\ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\right\}}_{\alpha(x)} \underbrace{- \ln\theta}_{+\ \beta(\theta)} - \frac{1}{2}\frac{x^2}{\theta^2} + \frac{x}{\theta}$$

$$\alpha(x) + \underbrace{\beta(\theta)}_{-\ln\theta} + \underbrace{\tau_1(x)}_{-\frac{1}{2}x^2}\frac{1}{\theta^2} + \underbrace{\tau_2(x)}_{x}\frac{1}{\theta}$$

$\Rightarrow$ Exponential Family with suff stats $\left(\Sigma(-\frac{1}{2}x_i^2), \Sigma x_i\right)$ or $\left(\Sigma x_i^2, \Sigma x_i\right)$ — 2

$$\frac{\partial \ln f}{\partial\theta} = -\frac{1}{\theta} - \frac{1}{2}x^2\left(-\frac{2}{\theta^3}\right) + x\left(-\frac{1}{\theta^2}\right)$$

$$U(\theta) = \frac{X^2}{\theta^3} - \frac{X}{\theta^2} - \frac{1}{\theta} \quad \text{so} \quad U_\cdot(\theta) = \frac{\Sigma X_i^2}{\theta^3} - \frac{\Sigma X_i}{\theta^2} - \frac{n}{\theta}$$ — 2

$$\frac{\partial U}{\partial\theta} = -\frac{3X^2}{\theta^4} + \frac{2X}{\theta^3} + \frac{1}{\theta^2}$$

$$E(X) = \theta, \quad E(X^2) = var(X) + \{E(X)\}^2 = \theta^2 + \theta^2 = 2\theta^2$$

$$I(\theta) = -E\left(\frac{\partial U}{\partial\theta}\right) = \frac{3}{\theta^4}(2\theta^2) - \frac{2\theta}{\theta^3} - \frac{1}{\theta^2} = \frac{3}{\theta^2} \quad \text{so} \quad I_\cdot(\theta) = \frac{3n}{\theta^2}$$ — 2

(ii) *bookwork*

If s and a are together sufficient for θ, where $\dim(s,a) > \dim(\theta)$ and $\dim(s) \geq \dim(\theta)$, and where the sampling distribution of $a|\theta$ does not depend on θ, then a is an ancillary statistic. — 1

$T = X - Y$ is $N(0, \sigma^2)$ where σ^2 is known ~~and~~ so does not depend on θ. — 2

(iii) (a) *Unseen*

$$f(\underset{\sim}{x}|\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\Sigma(x_i^2 - 2\theta x_i + \theta^2)}$$

$$= \underbrace{\left\{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\Sigma x_i^2}\right\}}_{h(\underset{\sim}{x})} \underbrace{\left\{e^{\theta\Sigma x_i - \frac{n}{2}\theta^2}\right\}}_{\cancel{g(\bar{x},\theta)}}$$

$$\underbrace{e^{n\theta(\bar{x} - \frac{1}{2}\theta^2)}}_{g(\bar{x},\theta)}$$

So $\bar{x}$ is suff by Neyman Factorisation Thm — 2

Parts

(iii)(b) $N(\theta,1)$ is a one-parameter exponential family, so $\bar{x}$ is a complete minimal sufficient statistic — 1

(c) $\cdot\ \bar{X}$ is $N(\theta,\frac{1}{n})$

$$E(e^{\beta Y})=E(e^{\mu\beta+\frac{1}{2}\sigma^2\beta^2}) \text{ with } \beta=1, \mu=\theta, \sigma^2=1/n$$

$$\Rightarrow E(e^{\bar{X}})=e^{\theta+\frac{1}{2n}}$$

so $E(Z)=E(e^{\bar{X}-\frac{1}{2n}})=e^{-\frac{1}{2n}}E(e^{\bar{X}})=e^{\theta}$

so Z is unbiased for θ — 2

(d) By the Lehmann-Scheffé theorem, any function of a complete sufficient statistic is a MVUEstimator of its expectation, uniformly since it holds whatever the value of θ. This is true for Z. — 1

(iv)(a)

$$f(\underset{\sim}{x}|\theta)=\left(\frac{1}{2\theta}\right)^n H(x_{min}>-\theta \wedge x_{max}<\theta)\,H(\theta>1)$$

$$=\left(\frac{1}{2\theta}\right)^n H(\theta>t) \quad \text{where } t=\max(-x_{min}, x_{max}, 1)$$

$$\pi(\theta|\underset{\sim}{x}) \propto \frac{1}{\theta^{n+2}}H(\theta>t)$$

$$\int_t^\infty \frac{1}{\theta^{n+2}}\,d\theta=\left[-\frac{1}{(n+1)\theta^{n+1}}\right]_t^\infty=\frac{1}{(n+1)t^{n+1}}$$

So $\pi(\theta|\underset{\sim}{x})=(n+1)t^{n+1}\cdot\frac{1}{\theta^{n+2}}H(\theta>t)$

Unseen. Similar set as exercise on material of last lecture in course.

(b) The Bayes Decision under squared error loss is

$$d(\underset{\sim}{x})=E_{\Theta|\underset{\sim}{x}}(\Theta)$$

$$=(n+1)t^{n+1}\int_t^\infty \theta\frac{1}{\theta^{n+2}}d\theta=\frac{(n+1)t^{n+1}}{nt^n}=\frac{n+1}{n}t$$

Bookwork in last lecture — 2

EXAMINATION QUESTIONS/SOLUTIONS 2009-10

Course: M3S1/M4S1

Question 2 — Marks & seen/unseen

Parts

(a) Unseen

$$f(\underset{\sim}{x}\mid\theta) = \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!}\left(\tfrac{1}{4}(2+\theta)\right)^{x_1}\left(\tfrac{1}{4}(1-\theta)\right)^{x_2}\left(\tfrac{1}{4}(1-\theta)\right)^{x_3}\left(\tfrac{1}{4}\theta\right)^{x_4}$$

$$\propto (2+\theta)^{x_1}(1-\theta)^{x_2+x_3}\,\theta^{x_4} \quad \text{where } x_1+x_2+x_3+x_4=n \qquad 2$$

$$\ln f = \text{constant} + x_1\ln(2+\theta) + (x_2+x_3)\ln(1-\theta) + x_4\ln\theta$$

$$\frac{\partial \ln f}{\partial\theta} = \frac{x_1}{2+\theta} - \frac{x_2+x_3}{1-\theta} + \frac{x_4}{\theta} \qquad 1$$

Cannot be written as $c(\theta)\{T(\underset{\sim}{x})-\theta\}$ — 2

so there is no unbiased estimator $T(\underset{\sim}{x})$ of θ having a variance at the CRLB.

(b)

$$\frac{\partial^2\ln f}{\partial\theta^2} = -\frac{x_1}{(2+\theta)^2} - \frac{x_2+x_3}{(1-\theta)^2} - \frac{x_4}{\theta^2}$$

X_1 is Binomial$(n, \frac{1}{4}(2+\theta))$ so $E(X_1) = \frac{n}{4}(2+\theta)$

Similarly $E(X_2) = E(X_3) = \frac{n}{4}(1-\theta)$, $E(X_4) = \frac{n}{4}\theta$ — (*) — 3

$$I_{X}(\theta) = E\left(-\frac{\partial^2\ln f(\underset{\sim}{X}\mid\theta)}{\partial\theta^2}\right) = \frac{n}{4}\left\{\frac{1}{2+\theta} + \frac{2}{1-\theta} + \frac{1}{\theta}\right\}$$

$$= \frac{n}{2}\,\frac{(1+2\theta)}{(2+\theta)(1-\theta)\theta} \qquad 2$$

$$\text{CRLB} = \frac{1}{I(\theta)} = \frac{2}{n}\,\frac{(2+\theta)(1-\theta)\theta}{1+2\theta} \qquad 2$$

(c) By (*) $T(\underset{\sim}{X}) = \frac{4}{n}X_4$ has $E(T(\underset{\sim}{X})) = \theta$

Since X_4 is Binomial$(n, \frac{1}{4}\theta)$, $var(X_4) = n\frac{1}{4}\theta(1-\frac{1}{4}\theta) = \frac{n}{16}\theta(4-\theta)$

$$var(T(\underset{\sim}{X})) = var\left(\tfrac{4}{n}X_4\right) = \frac{16}{n^2}\cdot\frac{n}{16}\theta(4-\theta) = \frac{1}{n}\theta(4-\theta) \qquad 4$$

$$\text{Efficiency}(T(\underset{\sim}{x})) = \frac{\text{CRLB}}{var(T(\underset{\sim}{X}))} = \frac{2(2+\theta)(1-\theta)\cancel{\theta}}{(1+2\theta)(4-\theta)} \qquad 2$$

(d)

θ	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Eff$(T(\underset{\sim}{x}))$	1	$\frac{3}{5}$	$\frac{5}{14}$	$\frac{11}{65}$	0
		(0·6)	(0·357)	(0·169)	

2

Setter's initials: RC — Checker's initials: AY — Page number: 3

	EXAMINATION QUESTIONS/SOLUTIONS 2009-10	Course M3S1/M4S1
Question 3		Marks & seen/unseen

Parts

(i) For $H_0: \theta = \theta_0$, a simple hypothesis, against $H_1: \theta \in \Theta_1$, a composite hypothesis, with $\alpha = P(\text{reject } H_0 \mid \theta = \theta_0)$, the power of the test is $P(\text{reject } H_0 \mid H_1)$. — bookwork

For a particular $\theta_1 \in \Theta_1$, we have most powerful test of size α based on the likelihood ratio. If the critical region does not depend on θ_1 the test is uniformly most powerful. — 2

(ii)(a) — unseen

$$\lim_{\theta \to 0} f(x \mid \theta) = \lim_{\theta \to 0} \frac{\theta e^{\theta x}}{e^{\theta} - 1} = \lim_{\theta \to 0} \frac{e^{\theta x} + \theta x e^{\theta x}}{e^{\theta}} \quad (0<x<1)$$

by l'Hôpital's Rule.

$$= 1$$

i.e. Uniform$(0,1)$ — 2

(b) For $H_0: \theta = 0$ against $H_1: \theta = \theta_1 > 0$, the Neyman-Pearson Lemma gives the MP test to be reject H_0 if $\frac{f(\underset{\sim}{x} \mid H_1)}{f(\underset{\sim}{x} \mid H_0)}$ is too large i.e. if

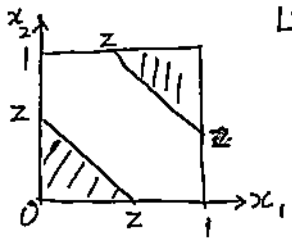
$$\frac{\prod_i \left\{ \frac{\theta_1 e^{\theta_1 x_i}}{e^{\theta_1} - 1} \right\}}{\prod_i 1} \geqslant c_\alpha$$

$$\text{i.e. } e^{\theta \sum x_i} \geqslant c_{1\alpha} \quad \text{i.e. } \sum x_i \geqslant k_\alpha$$

where k_α is such that $P(\sum x_i > k_\alpha \mid \theta = 0) = \alpha$. — 4

This does not depend on θ_1 so the test is UMP (uniformly over all θ_1), i.e. all $\theta \geqslant 0$ — 2

(c) Under H_0, $Z = X_1 + X_2$ where X_1 & X_2 are iid Uniform$(0,1)$



Looking at areas of Δs in $(0,1)\times(0,1)$

$$P(Z \leq z) = \begin{cases} \frac{1}{2} z^2 & (0 < z \leq 1) \\ 1 - \frac{1}{2}(2-z)^2 & (1 \leq z \leq 2) \end{cases}$$

$$\text{so } P(Z > z) = \begin{cases} 1 - \frac{1}{2} z^2 & (0 \leq z \leq 1) \\ \frac{1}{2}(2-z)^2 & (1 \leq z < 2) \end{cases}$$

$$f_Z(z) = \begin{cases} z & (0 < z \leq 1) \\ 2 - z & (1 < z \leq 2) \end{cases}$$

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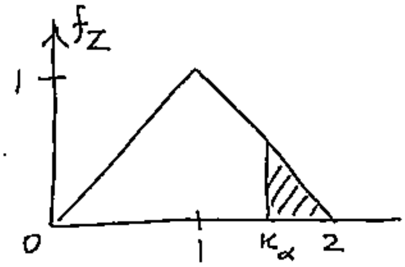
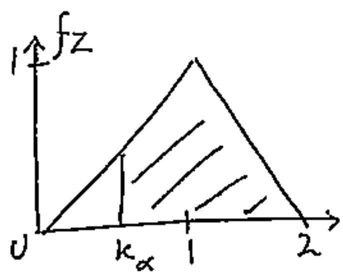
Parts

(ii)(c) (ctd)



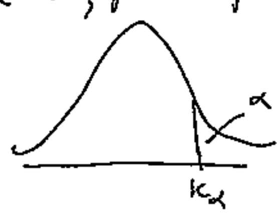
For $k_\alpha > 1$, $\alpha < 0.5$
and $\alpha = \frac{1}{2}(2-k_\alpha)^2$
so $k_\alpha = 2 - \sqrt{2\alpha}$



For $k_\alpha < 1$, $\alpha > 0.5$
and $\alpha = 1 - \frac{1}{2}k_\alpha^2$
so $k_\alpha = \sqrt{2(1-\alpha)}$

6

(d) When n is large, ΣX_i is $N\left(\frac{n}{2}, \frac{n}{12}\right)$ approximately by CLT
(since, for Uniform(0,1), $E(X) = \frac{1}{2}$, $var(X) = \frac{1}{12}$)



so $\alpha = 1 - \Phi\left(\frac{k_\alpha - n/2}{\sqrt{\frac{n}{12}}}\right)$

so $\frac{k_\alpha - \frac{n}{2}}{\sqrt{\frac{n}{12}}} = \Phi^{-1}(1-\alpha)$

and $k_\alpha = \frac{n}{2} + \sqrt{\frac{n}{12}}\,\Phi^{-1}(1-\alpha)$.

4

	EXAMINATION QUESTIONS/SOLUTIONS 2009-10	Course
Question 4		Marks & seen/unseen

Parts

(i) The Delta Method: Suppose that $\sqrt{n}(X_n-\theta) \xrightarrow{dist} Y$ as $n\to\infty$ and $\varphi(x)$ is such that — bookwork

$$\varphi(x) = \varphi(\theta) + (x-\theta)\{\varphi'(\theta)+o(1)\} \quad \text{as } x\to\theta$$

then $\sqrt{n}\{\varphi(X_n)-\varphi(\theta)\} = \sqrt{n}(X_n-\theta)\{\varphi'(\theta)+o_p(1)\}$ as $n\to\infty$

Then $\dfrac{\sqrt{n}}{\varphi'(\theta)}\{\varphi(X_n)-\varphi(\theta)\} \xrightarrow{dist} Y$ as $n\to\infty$

If the Central Limit Theorem holds for $\{X_n\}$, Y is normally distributed. — 4

(ii)(a) $\sqrt{n}(e^{-\bar{X}_n}-e^{-\lambda}) = \sqrt{n}(\bar{X}_n-\lambda)\{-e^{-\lambda}+o_p(1)\}$ — Unseen

$$Y_n = \frac{\Sigma X_i - n\lambda}{\sqrt{n\lambda}} = \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sqrt{\lambda}} \xrightarrow{dist} N(0,1) \text{ by CLT}$$

where we have used that ΣX_i is Poisson$(n\theta)$

so $\sqrt{n}(\hat{\theta}_n-\theta) \sim (-\sqrt{\lambda}e^{-\lambda})Y_n \xrightarrow{dist} N(0,(-\sqrt{\lambda}e^{-\lambda})^2)$

$= N(0, \lambda e^{-2\lambda})$ — 6

(b) Let Z_n = no. of zeros is Binomial$(n, e^{-\lambda})$ ($p_0 = e^{-\lambda}$ for Poisson(λ))

$$W_n = \frac{Z_n - ne^{-\lambda}}{\sqrt{ne^{-\lambda}(1-e^{-\lambda})}} = \frac{\sqrt{n}\left(\frac{Z_n}{n}-e^{-\lambda}\right)}{\sqrt{e^{-\lambda}(1-e^{-\lambda})}} \xrightarrow{distr} N(0,1)$$

by CLT (deMoivre) (each zero is a Bernoulli trial)

$$\sqrt{n}(\tilde{\theta}_n-\theta) = \sqrt{e^{-\lambda}(1-e^{-\lambda})}\,W_n \xrightarrow{distr} N(0, e^{-\lambda}(1-e^{-\lambda}))$$ — 6

(c) $$\text{ARE}(\hat{\theta}_n, \tilde{\theta}_n) = \frac{e^{-\lambda}(1-e^{-\lambda})}{\lambda e^{-2\lambda}} = \frac{e^{\lambda}-1}{\lambda}.$$ — 2

$$\frac{e^{\lambda}-1}{\lambda} = \frac{1}{\lambda}\left\{\left(1+\lambda+\frac{\lambda^2}{2!}+\dots\right)-1\right\} = 1+\frac{\lambda}{2}+\dots$$ — 2

≈ 1 for λ small.

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