Imperial College London
Department of Mathematics

Solutions to Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

You have 1h. Show the details of your computations.

1. Let $V = \mathbb{R}^3$ and $T : V \to V$ be the linear transformation defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2\,x_1 + x_3 \\ 2\,x_2 - x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

Let $E = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Let $A = [T]_E$.

a. Write down A, explaining your answer. (2 marks)

b. Compute the determinant of A and of A^2. (4 marks)

c. Find the eigenvalues of A by computing its characteristic polynomial. (2 marks)

d. Find the eigenspaces of A. (4 marks)

e. Determine whether A is diagonalisable over $\mathbb{R}$. Justify your answer. (2 marks)

f. Write e_3 as a linear combination of eigenvectors of T. Hence express $T^n(e_3)$ as a linear combination of e_1, e_2, e_3 for all $n \in \mathbb{N}$. (T^0 is the identity transformation.) (6 marks)

(Total: 20 marks)

Solution:

a. (No marks for answers that do not show the calculations.) Computing the image of the standard basis through T, we get

$$T(e_1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \text{ and } T(e_3) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(**1 mark**). Therefore

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

(**1 mark**).

b. We compute

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4 - 2 + 2 = 4$$

(**2 marks**). Since the determinant is multiplicative we obtain that $\det(A^2) = \det(A)^2 = 16$ (**2 marks**).

c. The characteristic polynomial of A (or of T) is

$$\chi_A(X) = \begin{vmatrix} X-2 & 0 & -1 \\ 0 & X-2 & 1 \\ -1 & -1 & X-1 \end{vmatrix} = ((X-1)(X-2)+1)(X-2) - X + 2$$
$$= (X-1)(X-2)^2.$$

(**1 mark**). Hence, T has two eigenvalues: 2 and 1. (**1 mark**)

d. **i.** We compute the eigenspace E_1 relative to 1. This is

$$E_1 = \ker(I_3 - A) = \ker \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \mathbf{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(**2 marks**).

ii. We compute the eigenspace E_2. This is

$$E_2 = \ker(2I_3 - A) = \ker \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} = \mathbf{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(**2 marks**).

e. The matrix is not diagonalisable because $\dim E_1 + \dim E_2 = 2 \neq 3$. (**2 marks**).

f. Let

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{ and}$$

Then $e_3 = v_1 + v_2$. (**1 mark**). For all $n \in \mathbb{N}$, we write

$$T^n(e_3) = T^n(v_1) + T^n(v_2) = v_1 + 2^n v_2$$

(**2 marks**). Since $v_1 = -e_1 + e_2 + e_3$ and $v_2 = e_1 - e_2$, we get that

$$\begin{aligned} T^n(e_3) &= -e_1 + e_2 + e_3 + 2^n(e_1 - e_2) \\ &= (2^n - 1)e_1 + (1 - 2^n)e_2 + e_3 \end{aligned}$$

(**3 marks**).

2. **a.** Let $n \in \mathbb{N} \setminus \{0\}$ and $A, B \in M_n(F)$, where F is a field. For each of the following statements, say whether it is true or false. If it is true, give a short proof; if it is false, give a counterexample.

   i. If $A = B^2$ and α is an eigenvalue of B, then α^2 is an eigenvalue of A. (2 marks)

   ii. If A and B have the same characteristic polynomial, then there is an invertible matrix $P \in M_n(F)$ such that $A = P^{-1}BP$. (3 marks)

   iii. Suppose that A and B have the same characteristic polynomial and are both diagonalisable. Then there is an invertible matrix $P \in M_n(F)$ such that $A = P^{-1}BP$. (3 marks)

   b. **i.** Show that no pair of the following vectors in $\mathbb{R}^3$ is orthogonal:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

   Show the details of your computations. (3 marks)

   ii. Use the Gram-Schmidt process applied to $\{v_1, v_2, v_3\}$ to find an orthogonal basis of $\mathbb{R}^3$ whose first vector is v_1. (5 marks)

   c. Prove the following statement. An orthogonal set of non-zero vectors in $\mathbb{R}^n$ (for $n \in \mathbb{N} \setminus \{0\}$) is linearly independent. (4 marks)

(Total: 20 marks)

Solution:

a. **i.** True. If α is an eigenvalue of B, then there is a non-zero $v \in F^n$ such that $Bv = \alpha v$ (**1 mark**). Hence $Av = B^2 v = B(Bv) = \alpha Bv = \alpha^2 v$, which means that α^2 is an eigenvalue of A (**1 mark**).

ii. False. Think of

$$I_2 \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The first has $\dim E_1 = 2$ the second has $\dim E_1 = 1$. (**3 marks**).

iii. True. Having the same characteristic polynomial implies that A and B have the same eigenvalues, say $\alpha_1, \ldots, \alpha_n \in F$. Moreover A and B are both diagonalisable, therefore there are invertible matrices $P, Q \in M_n(F)$ such that

$$P^{-1}AP = \operatorname{diag}(\alpha_1, \ldots, \alpha_n) = Q^{-1}BQ.$$

Hence $QP^{-1}APQ^{-1} = B$. (**3 marks**).

b. **i.** We compute

$$v_1 \cdot v_2 = 3 + 1 = 4, \quad v_1 \cdot v_3 = 4 + 2 = 6, \quad v_2 \cdot v_3 = 12 + 2 = 14.$$

Therefore no pair chosen among the given vectors is orthogonal (**1 mark** for each check).

ii. We complete v_1 to a basis of $\mathbb{R}^3$. We show that $\{v_1, v_2, v_3\}$ is already a basis. Indeed

$$\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 3 - 1 = 2.$$

(**1 mark**).
By the Gram-Schmidt process we define

$$w_1 = v_1 \qquad \textbf{(1 mark)}.$$

$$w_2 = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 - 2v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \textbf{(1 mark)}.$$

$$w_3 = v_3 - \frac{v_1 \cdot v_3}{v_1 \cdot v_1} v_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 = v_3 - 3v_1 - w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \textbf{(2 marks)}.$$

c. Let $v_1, \ldots, v_\ell \in \mathbb{R}^n$ be an orthogonal set. Assume

$$0 = \alpha_1 v_1 + \cdots + \alpha_\ell v_\ell,$$

for some $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$. For all $i \in \{1, \ldots, \ell\}$,

$$\begin{aligned} 0 &= v \cdot 0 = v \cdot (\alpha_1 v_1 + \cdots + \alpha_\ell v_\ell) \qquad \textbf{(2 marks)} \\ &= \alpha_i v_i \text{ because the } v_i\text{'s form an orthogonal set.} \end{aligned}$$

We deduce that $\alpha_i = 0$ for all $i \in \{1, \ldots, \ell\}$; thus, the v_i's form a linearly independent set. (**2 marks**)