IMPERIAL

MATH70021

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Advanced Topics in Partial Differential Equations

Date: Wednesday, May 22, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let d denote the dimension and $\alpha \in \mathbb{N}^d$ be a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$. Consider the PDE

$$\sum_{|\alpha|=k} a_\alpha[x, u, Du, \ldots, D^{k-1}u] D^\alpha u(x) + a_0[x, u, \ldots, D^{k-1}u] = 0. \tag{1}$$

Classify the above PDE as linear/semilinear/quasi-linear or fully non-linear. Give an explicit example of a semi-linear PDE. (5 marks)

(b) (i) Let $U = \mathbb{R}^d$. When do we say that the hypersurface $\Gamma = \{x_d = 0\}$ is non-characteristic for the PDE (1) at $x \in \Gamma \cap U$?

(4 marks)

(ii) Consider the PDE

$$(x - c)u_{xx} - u_y = 0, \text{ on } (x, y) \in \mathbb{R}^2, \text{ for some } c \geq 0. \tag{2}$$

Find all the characteristic surfaces (characteristic curves) for the PDE (2) . (6 marks)

(c) Consider (2) with initial data

$$\begin{cases} u(0, y) = \sin(5y) \\ u_x(0, y) = 0. \end{cases} \tag{3}$$

For what choices of $c \geq 0$ and for which $y \in \mathbb{R}$, does the PDE (2) admit an analytic solution in a neighbourhood of $(0, y)$? (5 marks)

(Total: 20 marks)

2. (a) Let $U \subset \mathbb{R}^d$ be an open and bounded domain. Assume that $u \in W^{k,p}(U)$ for $k \in \mathbb{N}$ and $p \in [1, \infty)$. Prove that there exists a sequence $(u_n)_{n \geq 0} \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_n \to u \text{ in } W^{k,p}(U) \quad \text{as } n \to \infty.$$

[You may use without proof the local smooth approximation of Sobolev functions in the interior of U. You may also use without proof that $u \in W^{k,p}(U)$, $\xi \in C^k(\overline{U})$ implies $\xi u \in W^{k,p}(U)$.] (7 marks)

(b) Let U be an open bounded domain on $\mathbb{R}^d$. State the Extension Theorem for $W^{1,p}(U)$-functions. Then prove the Gagliardo-Nirenberg-Sobolev (GNS) Inequality on a bounded domain U, assuming that the inequality holds on $\mathbb{R}^d$. Please state clearly all the theorems you use without proof. (9 marks)

(c) Let $d = 3$ and U an open bounded subset of $\mathbb{R}^3$. Prove that there is a constant C independent of u, so that

$$\|u\|^2_{L^3(\mathbb{R}^3)} \leq C\|u\|_{L^2(U)}\|u\|_{H^1(U)}$$

. (4 marks)

(Total: 20 marks)

3. (a) Consider U to be a bounded domain of $\mathbb{R}^d$ and the operator

$$Lu := -\Delta u + \sum_{i=1}^{d} \sin(4x_i) u_{x_i} + u. \tag{4}$$

Let $f \in L^2(U)$. Write down the associated bilinear form for L and define what it means for $u \in H_0^1(U)$ to be a weak solution to the Boundary Value Problem:

$$\begin{cases} Lu = f, \text{ in } U \\ u = 0, \text{ on } \partial U. \end{cases} \tag{5}$$

(7 marks)

(b) Prove the following energy estimate: Let L be uniformly elliptic with $a_{ij}, b_i, c \in L^\infty(U)$ in

$$L := -\sum_{i,j=1}^{d} (a_{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{d} b_i(x) u_{x_i} + c(x)u.$$

For $u \in H_0^1(U)$ satisfying weakly (5), there exists a constant C so that

$$\|Du\|^2_{L^2(U)} \leq C[\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}].$$

Which existence of weak solutions theorem directly applies when all $b_i \equiv 0$? Explain your answer.
(8 marks)

(c) Let L be a uniformly elliptic operator. If λ is not in the spectrum of L, $\lambda \notin \Sigma$, show that there exists a constant C so that whenever $f \in L^2(U)$ and $u \in H_0^1(U)$ is the unique weak solution to

$$\begin{cases} Lu = \lambda u + f, \text{ in } U \\ u = 0, \text{ on } \partial U, \end{cases} \tag{6}$$

it holds that

$$\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}.$$

(5 marks)

(Total: 20 marks)

4. (a) State the Interior Elliptic Regularity Theorem in an open set U on $\mathbb{R}^d$. Under which assumptions on L, can the regularity be improved?
(6 marks)

(b) Let $d = 3$ and assume ∂U is C^2. Consider the nonlinear equation

$$\begin{cases} -\Delta u + \cos(7x)u + |u|^5 = f, \text{ in } U \\ u = 0, \text{ on } \partial U. \end{cases} \tag{7}$$

Assume that $f \in L^2(U)$, is such that $\|f\|_{L^2(U)} < \varepsilon$ for ε sufficiently small.

(i) Consider the space

$$\mathcal{B}_\theta := \{u \in H^2(U) : \|u\|_{H^2(U)} \leq \theta\}$$

and the mapping $\mathcal{F} : \mathcal{B}_\theta \to \mathcal{B}_\theta$, where for $w \in H^2(U) \cap H_0^1(U)$, $\mathcal{F}(w) = v \in H^2(U) \cap H_0^1(U)$ is a solution to the linear elliptic boundary value problem

$$\begin{cases} -\Delta v + \cos(7x)v = f - |w|^5, \text{ in } U \\ v = 0, \text{ on } \partial U. \end{cases} \tag{8}$$

Use elliptic theory and regularity (without proof) to show that there exist ε, θ sufficiently small, so that the mapping $\mathcal{F}$ indeed maps $\mathcal{B}_\theta$ into $\mathcal{B}_\theta$.

Hint: You may use that when $k > \frac{d}{p}$, $W^{k,p} \hookrightarrow C^{k-\left[\frac{d}{p}\right]-1,\left[\frac{d}{p}\right]-\frac{d}{p}+1}$, where d is the dimension. (7 marks)

(ii) Prove that there exist ε, θ sufficiently small, so that the mapping $\mathcal{F} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is a contraction mapping. Then deduce that the map $\mathcal{F}$ has a fixed point and conclude that the nonlinear boundary value problem (7) has a solution. (7 marks)

(Total: 20 marks)

5. Let $U \subset \mathbb{R}^d$ open and bounded with C^1 boundary and $U_T = (0,T) \times U$, $\Sigma_t = \{t\} \times U$ for $t \in [0,T]$ and $\partial^* U_T = [0,T] \times \partial U$.

(a) Let $f \in L^2(U_T)$, $\phi \in L^2(U)$ and consider the parabolic initial boundary value problem:

$$\begin{cases} u_t + Lu = f, \text{ in } U_T \\ u = \phi, \text{ on } \Sigma_0 \\ u = 0, \text{ on } \partial^* U_T, \end{cases} \tag{9}$$

where

$$L := -\sum_{i,j=1}^{d} (a_{ij}(t,x) u_{x_i})_{x_j} + \sum_{i=1}^{d} b_i(t,x) u_{x_i} + c(t,x) u.$$

Let $a_{ij} = a_{ji}, b_i, c \in C^1(\overline{U_T})$ satisfying the uniform ellipticity condition. We say that $u \in L^2((0,T); H_0^1(U))$ is a weak solution to (9) if:

$$\int_{U_T} \left[-uv_t + \sum_{i,j=1}^{d} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \right] dxdt = \int_{\Sigma_0} \phi v dx + \int_{U_T} fudxdt,$$

holds for all $v \in H^1(U_T)$ so that $v = 0$ on $\Sigma_T \cup \partial^* U_T$.

Prove that if $u \in C^2(\overline{U_T})$ then u solves (9) in the classical sense.

(6 marks)

(b) Consider the heat equation

$$\begin{cases} u_t - \Delta u = f, \text{ on } U_T, \\ u = \phi \text{ on } \Sigma_0, \\ u = 0 \text{ on } \partial^* U_T. \end{cases} \tag{10}$$

Establish the a-priori estimate (i.e. assuming that there is a smooth solution) :

$$\|u\|^2_{L^\infty([0,T];L^2(U))} + \|u\|^2_{L^2([0,T];H^1(U))} \le C \left[\|f\|^2_{L^2(U_T)} + \|\phi\|_{L^2(U)^2} \right].$$

(8 marks)

(c) Show that, if a weak solution to the equation (9) exists, it is unique. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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XXX (Solutions)

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1. (a) It is quasilinear. The example demonstrated must be an equation which is non-linear in lower order. seen ⇓

5, A

(b) For (i): The hypersurface $\Gamma = \{x_d = 0\}$ is non-characteristic for the given PDE, at $x \in \Gamma \cap U$, if the function meth seen ⇓

$$A(x) := a_{(0,0,\dots,k)}(D^{k-1}u(x), \dots, u(x), x) \neq 0 \text{ on } \Gamma \cap U_x$$

4, A

6, B

for U_x being a neighbourhood of x.

For (ii): We are looking for curves. If Γ is a characteristic curve then by definition

$$\sum_{|\alpha|=k} a_\alpha(x) n^\alpha(x) = 0 \; \forall x \in \Gamma,$$

where n is the normal to Γ. We parametrise Γ: $x = x(t), y = y(t)$ for $t \in \mathbb{R}$. Then this curve Γ has a tangent vector parametrised by $(\dot{x}(t), \dot{y}(t))$ with corresponding normal being: $n(t) = (\dot{y}(t), -\dot{x}(t))$ for $t \in \mathbb{R}$.

Thus Γ describes a characteristic curve for the given second order PDE if (by definition) at each point it holds

$$\sum_{\alpha_1+\alpha_2=2} a_{(\alpha_1,\alpha_2)} n_1^{\alpha_1} n_2^{\alpha_2} = 0.$$

Since our PDE is second order only in the x derivative, our condition translates into

$$(x(s) - c)(\dot{y}(s))^2 = 0.$$

That means that either $\dot{y}(s) = 0$ or $x(s) = c$. So the characteristic curves are $y(s) =$ constant and $x = c$.

unseen ⇓

(c) First, as long as $c \neq 0$, the surface $\{x = 0\}$ is a non-characteristic surface. It is also analytic. The boundary conditions are analytic as well along this boundary curve. The coefficients of the PDE are also analytic. Thus we can apply the Cauchy - Kovalevskaya Theorem to find an analytic solution to the given PDE in a neighbourhood of $(0, y)$. This holds for all $y \in \mathbb{R}$. 5, C

2. (a) Proof done in the lectures. Can be found in the lecture notes. We sketch it here: Write $U = \cup_{n\geq 1} U_n$ where $U_n := \{x \in U : \text{dist}(x, \partial U) > n^{-1}\}$. Also define $V_n := U_{n+3} \setminus \overline{U}_{n+1}$ and choose $V_0 \subset\subset U$ so that $U = \cup_{n\geq 0} U_n$. Now let $\{\xi_n\}_n$ be a partition of unity subordinate to the open sets V_n (i.e. $0 \leq \xi_i \leq 1$, $\xi_i \in C_c^\infty(V_i)$, $\sum_{i\geq 1} \xi_i = 1$ on U). Now take $u \in W^{k,p}(U)$. Then $\xi_i u \in W^{k,p}(U)$ and $\text{supp}(\xi u) \subset V_i$ for all i. Fic $\delta < 0$ and choose $\epsilon_i > 0$ small enough so that $u^i := \eta_{\epsilon_i} * (\xi u)$ satisfies $\text{supp}(u^i) \subset W_i$ with $V_i \subset\subset W_i := U_{i+4} \setminus \overline{U}_i$ and $\|u^i - \xi_i u\|_{W^{k,p}(U)} = \|u^i - \xi_i u\|_{W^{k,p}(W_i)} \leq \delta 2^{-i-1}$. This we can do thanks to the interior approximation result. Now consider $v = \sum_{i\geq 0} u^i$ which is in $C^\infty(V)$ since for each open set V compactly contained in U, the sum is in fact a finite sum of smooth functions. Then $u = u \cdot 1 = \sum_i \xi_i u$ for every open set V compactly contained in U, and so

seen ⇓

7, A

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{i\geq 0} \|u^i - \xi_i u\|_{W^{k,p}(U)} \leq \delta \sum_{i\geq 0} 2^{-i-1} = \delta.$$

In the first inequality we applied triangle inequality. Now as δ does not depend on $V \subset\subset U$, we may take the supremum over all such V, to conclude that $\|v - u\|_{W^{k,p}(U)} \leq \delta$.

seen ⇓

(b) The first part of the Question concerns the statement of the Extension Theorem which can be found in the lecture notes.

3, A

6, B

For the second part: Thanks to the Extension theorem, we can find $\bar{u} \in W^{1,p}(\mathbb{R}^d)$ so that $u = \bar{u}$ a.e. in U and the following estimate holds

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(U)}.$$

We have that

$$\begin{aligned} \|u\|_{L^{p^*}(U)} &= \|\bar{u}\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^d)} \\ &\leq \|D\bar{u}\|_{L^p(\mathbb{R}^d)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^d)} \\ &\leq C'\|u\|_{W^{1,p}(U)}. \end{aligned}$$

where firstly we used that $U \subset \mathbb{R}^d$. In the next line we applied Gagliardo-Nirenberg-Sobolev Inequality and after that the definition of Sobolev norm. Finally, in the third line we applied the Extension Theorem.

unseen ⇓

(c) We apply first Hölder Inequality with $p = 4, q = 4/3$:

4, D

$$\|u\|_{L^3(U)} = \left[\int_U u^{3/2} u^{3/2} dx\right]^{1/3} \leq \left[\int_U u^6 dx\right]^{1/12} \left[\int_U u^2 dx\right]^{1/2} = \|u\|_{L^6(U)}^{1/2} \|u\|_{L^2(U)}^{1/2}.$$

Then by the Sobolev embedding we get the bound

$$\|u\|_{L^3(U)} \leq C\|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2}.$$

3. (a) The procedure is standard and in the lectures we did something very similar: Suppose that $u \in C^2(\bar{U})$ is a classical solution to the given PDE. Let $v \in C^2(\bar{U})$ with $v|_{\partial U} = 0$. Multiply the equation by v and integrate by parts so that

seen ⇓

7, A

$$\int_U \left[\nabla u \cdot \nabla v + \sum_{i=1}^{d} \sin(4x_i) u_{x_i} v + uv \right] dx - \int_{\partial U} v \nabla u \cdot n dS = \int_U f v dx.$$

The boundary term is 0 by assumption, so we have

$$B[u,v] := \int_U \left[\nabla u \cdot \nabla v + \sum_{i=1}^{d} \sin(4x_i) u_{x_i} v + uv \right] dx = (f,v)_{L^2(U)}$$

for all $v \in^2 (\bar{U})$ with $v|_{\partial U} = 0$. This expression makes sense for all $u, v \in H_0^1(U)$, so we define for $f \in L^2(U)$, $u \in H_0^1(U)$ to be a weak solution to the given PDE if

$$B[u,v] = (f,v)_{L^2(U)}, \ \forall \ v \in H_0^1(U).$$

We have thus established that a classical solution is necessarily a weak solution. Conversely, take $u \in C^2(\bar{U})$ with $u|_{\partial U} = 0$ and the above weak formulation holds. Let $v \in C^2(\bar{U})$ with $v|_{\partial U} = 0$ and undo the integration by parts to get

$$\int_U \left[-\Delta u + \sum_{i=1}^{d} \sin(4x_i) u_{x_i} + u - f \right] v dx = 0.$$

In particular this holds for all $v \in C_c^\infty(U)$. Thus $Lu - f = 0$. Hence if $u \in C^2(\bar{U})$ with $u|_{\partial U} = 0$, then u is a weak solution iff u is a classical solution.

seen ⇓

8, B

(b) This energy estimate is part of a proof we did in the lectures (The first Existence Theorem proof). We use the uniform ellipticity condition, with constant say A:

$$A \int_U |Du|^2 dx \leq \int_U \sum_{i,j=1}^{d} a_{ij}(x) u_{x_i} u_{x_j} dx. \tag{1}$$

The RHS equals to (due to the general weak formulation)

$$B[u,u] - \int_U \left[\sum_{i=1}^{d} b_i u_{x_i} u + cu^2 \right] dx \leq B[u,u] + \sum_{i=1}^{d} \|b_i\|_{L^\infty(U)} \int_U |Du||u| dx + \|c\|_{L^\infty(U)} \int_U |u|^2 dx. \tag{2}$$

Then apply Young's Inequality with $\epsilon > 0$:

$$\int_U |Du||u| dx \leq \epsilon \int_U |Du|^2 dx + \frac{1}{4\epsilon} \int_U |u|^2 dx. \tag{3}$$

Pick ϵ sufficiently small so that $\epsilon \sum_{i=1}^{d} \|b_i\|_{L^\infty(U)} < A/2$. Then altogether

$$\frac{A}{2} \|Du\|_{L^2(U)}^2 = \frac{A}{2} \int_U |Du|^2 dx \leq B[u,u] + \gamma \|u\|_{L^2(U)}^2, \tag{4}$$

for some constant $\gamma > 0$. Using the weak formulation and Cauchy-Schwarz:

$$B[u,u] = \int_U f u dx \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)} \leq \frac{1}{2} \left[\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 \right],$$

by Young's Inequality again. So up to some constant C independent of f and u we have the estimate.

Now if $b_i \equiv 0$ for all i, we immediately get $A\|Du\|^2_{L^2(U)} \leq B[u,u]$, i.e. $\gamma = 0$ in the above estimate. By Poincaré Inequality $\|u\|^2_{L^2(U)} \leq c\|Du\|^2_{L^2(U)}$ for all $u \in H^1_0(U)$, then we have $B[u,u] \geq c\|u\|_{H^1_0}$, for some constant c. Hence we get exactly the assumptions of the Lax-Milgram theorem, to deduce existence of weak solutions. $\boxed{\text{meth seen} \Downarrow}$

(c) (A similar exercise was in the exercise sheets.) If not there would exist sequences $(f_k)_k \subset L^2(U)$ and $(u_k)_k \subset H^1_0(U)$ so that $\boxed{5, \text{D}}$

$$Lu_k = \lambda u_k + f_k, \text{ in } U$$

and $u_k = 0$ on ∂U in the weak sense, but

$$\|u_k\|_{L^2(U)} > k\|f_k\|_{L^2(U)}, \quad \text{for } k = 1, \dots .$$

Without loss of generality we may assume that $\|u_k\|_{L^2(U)} = 1$. Then there should be that $f_k \to 0$ in $L^2(U)$. According to the energy estimate as in Part (b) the sequence $(u_k)_k$ is bounded in $H^1_0(U)$. Thus there exists a subsequence (by Rellich-Kondrachov) $(u_{k_j})_j \subset (u_k)_k$ so that $u_{k_j} \rightharpoonup u$ weakly in $H^1_0(U)$ and $u_{k_j} \to u$ strongly in $L^2(U)$. For every $v \in H^1_0(U)$ we have

$$B[u_k, v] + \lambda(u_k, v) = (f_k, v)_{L^2(U)}.$$

Since $f_k \to 0$ in $L^2(U)$ and u_k converges weakly to u in $H^1_0(U)$ we conclude that $B[u,v] + \lambda(u,v) = 0$. Or that u is a weak solution to

$$\begin{cases} Lu = \lambda u, & \text{in } U \\ u = 0, & \text{on } \partial U. \end{cases} \tag{5}$$

By Fredholm Alternative, since λ is not in the spectrum of L, we should have that $u = 0$. But this contradicts the fact that $\|u\|_{L^2(U)} = 1$ (We have $\|u\|_{L^2(U)} = 1$ because the convergence is strong on $L^2(U)$).

4. (a) Suppose L is uniformly elliptic on $U \subset \mathbb{R}^d$ open set. Let $a_{ij} \in C^1(U)$, $b_i, c \in L^\infty(U)$, $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is a weak solution: it holds $B[u,v] = (f,v)_{L^2(U)}$ for all $v \in H_0^1(U)$. Then $u \in H^2_{loc}(U)$ and for any $V \subset\subset U$, we have

seen ⇓

6, A

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

with C independent of f, u.

If we assume better regularity on the coefficients, we gain regularity on our solution: if $a_{ij}, b_i, c \in C^{m+1}(U)$, $m \in \mathbb{N}$ and $f \in H^m(U)$, then we deduce $u \in H^{m+2}_{loc}$.

meth seen ⇓

7, C

(b) (i) Let $\mathcal{F}$ be the map (as instructed in the statement) taking $w \in H^2(U)$ to the solution $v \in H_0^1(U)$ of the linear problem

$$\begin{cases} -\Delta v + \cos(7x)v = f - |w|^5, \text{ in } U \\ v = 0, \text{ on } \partial U. \end{cases} \tag{6}$$

We want to show that $\mathcal{F} : \mathcal{B}_\theta \to \mathcal{B}_\theta$. Take $w \in \mathcal{B}_\theta$. By the Sobolev Embedding Theorem we have that $w \in C^{0,1/2}(\bar{U})$ and there is a constant C_1 depending only on U so that

$$\|w\|_{C^{0,1/2}(\bar{U})} \le C_1 \|w\|_{H^2(U)} \le C_1 \theta. \tag{7}$$

By Hölder's Inequality we then have that

$$\||w|^5\|_{L^2(U)} \le \||w|^5\|_{L^\infty(U)} |U|^{1/2} \le |U|^{1/2} [C_1\theta]^5 := \theta^5 C_2. \tag{8}$$

(C_2 depends only on U). Moreover given $g \in L^2(U)$, the linear elliptic problem

$$\begin{cases} -\Delta v + \cos(7x)v = g, \text{ in } U \\ v = 0, \text{ on } \partial U. \end{cases} \tag{9}$$

admits a unique solution in $H_0^1(U) \cap H^2(U)$ (from elliptic regularity theory as the coefficients are smooth and the boundary is C^2) and there exists a constant C_3 so that

$$\|v\|_{H^2(U)} \le C_3 \|g\|_{L^2(U)}.$$

Now apply this result to $g = f + |w|^5$ to get

$$\|v\|_{H^2(U)} \le C_3 \|f + |w|^5\|_{L^2(U)} \le C_3(\varepsilon + C_2\theta^5). \tag{10}$$

There we also utilised the bound we assumed on $\|f\|_{L^2}$. Now we may pick θ, ε sufficiently small, e.g. $\theta^4 < (2C_2c_3)^{-1}$ and $\varepsilon < \theta(2C_3)^{-1}$. This gives us that

$$\|\mathcal{F}(w)\|_{H^2(U)} \le \theta.$$

meth seen ⇓

7, D

(ii) Take $w_1, w_2 \in \mathcal{B}_\theta$. We want to show that

$$\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{H^2(U)} \le \alpha \|w_1 - w_2\|_{H^2(U)},$$

for $\alpha < 1$. Let $u_i = \mathcal{F}(w_i)$. By linearity of the elliptic problem we have that $u := u_1 - u_2$ is the unique solution in $H_0^1(U) \cap H^2(U)$ of

$$\begin{cases} -\Delta u + \cos(7x)u = |u_2|^5 - |u_1|^5, \text{ in } U \\ u = 0, \text{ on } \partial U. \end{cases} \tag{11}$$

By elliptic regularity theory we deduce for $u_1 - u_2$

$$\|u_1 - u_2\|_{H^2(U)} = \|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{H^2(U)} \leq C_3 \||w_1|^5 - |w_2|^5\|_{L^2(U)}. \quad (12)$$

Now in order to bound the RHS, we first write

$$||w_1(x)|^5 - |w_2(x)|^5| \leq ||w_1(x)| - |w_2(x)|| \left(|w_1(x)|^4 + |w_2(x)|^4\right) \quad (13)$$
$$\leq ||w_1(x)| - |w_2(x)|| \left(\|w_1\|^4_{L^\infty(U)} + \|w_1\|^4_{L^\infty(U)}\right) \leq |w_1(x) - w_2(x)| \left(\|w_1\|^4_{L^\infty(U)} + \|w_1\|^4_{L^\infty(U)}\right) \quad (14)$$

where in the last line we applied the reverse triangle inequality. We square now both sides, we integrate over U and then we take the square root, so that eventually

$$\||w_1|^5 - |w_2|^5\|_{L^2(U)} \leq \|w_1 - w_2\|_{L^2(U)} \left(\|w_1\|^4_{L^\infty(U)} + \|w_1\|^4_{L^\infty(U)}\right) \quad (15)$$
$$\leq \|w_1 - w_2\|_{L^2(U)} [2(C_1\theta)^4] \quad (16)$$

as $w_i \in \mathcal{B}_\theta$. We go back to (12) to see that

$$\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{H^2(U)} \leq 2C_3(C_1\theta)^4 \|w_1 - w_2\|_{L^2(U)}.$$

For θ small enough, $\alpha := 2C_3(C_1\theta)^4 < 1$, we deduce that $\mathcal{F} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is a contraction.

All these allow us to conclude that $\mathcal{F}$ has a unique fixed point in $\mathcal{B}_\theta$. From the definition of $\mathcal{F}$, this implies that the original nonlinear PDE has a solution (which is unique among $\mathcal{B}_\theta$).

5. (a) Suppose $u \in C^2(\overline{U_T})$ satisfies

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$$\int_{U_T}\left[-uv_t + \sum_{i,j=1}^{d} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv\right]dxdt = \int_{\Sigma_0}\phi v dx + \int_{U_T} fudxdt,$$

holds for all $v \in H^1(U_T)$ so that $v = 0$ on $\Sigma_T \cup \partial^* U_T$. Let $v \in C_c^\infty(U_T)$. Integrating by parts and discarding boundary terms, we have that

$$\int_{U_T}\left[u_t - \sum_{i,j=1}^{d}(a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu - f\right] vdxdt = 0.$$

As this holds for all $v \in C_c^\infty(U_T)$, and as $u \in C^2(\overline{U_T})$, we conclude that $u_t + Lu = f$ holds in U_T. The condition that $u \in L^2((0,T); H_0^1(U))$ implies that $u = 0$ on $\partial^* U_T$. It remains to verify the initial condition. Consider a test function $v \in C^\infty(\overline{U_T})$ so that $v|_{\Sigma_T \cup \partial^* U_T} = 0$. Integrating by parts and using that $u_t + Lu = f$ holds in U_T:

$$\int_{\Sigma_0}(u-\phi)vdx = 0.$$

Say by contradiction that $(u - \phi) \neq 0$ on Σ_0. Then there exists $x \in U$ such that $u(0,x) - \phi(x) \neq 0$, say > 0 (wlog). Due to continuity, for any $y \in B_\delta(x)$ for some $\delta > 0$, it holds that $u(0,x) - \phi(x) > \varepsilon > 0$. Let $v \in C^\infty(B_r(0,x))$ a positive function which is 1 on $B_{r/2}(0,x)$ so that it vanishes on the boundary $\Sigma_T \cup \partial^* U_T$. Then such a v can be considered as a test function in our problem and then $\int_{\Sigma_0}(u-\phi)vdx > \varepsilon\ \mathsf{Vol}(B_{r/2}(0,x)) > 0$. So we have a contradiction. 8, C

(b) Multiply the heat equation by u:

unseen ⇓

$$\frac{1}{2}\partial_t(u^2) - \mathsf{div}(uDu) + |Du|^2 = uf.$$

Integrate now over $[0,t] \times U$ we write

$$\frac{1}{2}\int_{\Sigma_t} u(x)^2 dx - \frac{1}{2}\int_U \phi^2 dx + \int_0^t\int_U |Du|^2 dxdt = \int_0^t\int_U ufdxdt.$$

First we apply Young's to the integrand in the RHS:

$$\left|\int_0^t\int_U ufdxdt\right| \leq \epsilon\int_0^t\int_U u^2 dxdt + \frac{4}{\epsilon}\int_0^t\int_U f^2 dxdt$$

and by Poincare:

$$\int_0^t\int_U u^2 dxdt \leq C\int_0^t\int_U |Du|^2 dxdt.$$

Altogether gives

$$\int_{\Sigma_t} u(x)^2 dx + \int_0^t\int_U (u^2 + |Du|^2)dxdt \leq C\left(\int_0^t\int_U f^2 dxdt + \int_U \phi^2 dx\right).$$

Take now the supremum over $t \in [0,T]$:

$$\|u\|^2_{L^\infty([0,T];L^2(U))} + \|u\|^2_{L^2([0,T];H^1(U))} \leq C(\|f\|_{L^2(U_T)} + \|\phi\|^2_{L^2(U_T)}).$$

8, C

(c) It suffices, by linearity, to consider the case where $\psi = f = 0$ and establish then that $u = 0$. We take as a test function, for $\lambda \in \mathbb{R}$ to be chosen later,

unseen $\Downarrow$

$$v(t,x) = \int_t^T e^{-\lambda s} u(s,x) ds.$$

Then v is an admissible test function as we have that $v \in H^1(U_T)$ and $v = 0$ on $\partial^* U_T \cup \Sigma_T$ in the sense of traces. We compute $v_t = -e^{-\lambda t} u$ and we insert this into the weak formulation, to get:

$$\int_{U_T} \left\{ u^2 e^{-\lambda t} - e^{\lambda t} \sum_{ij} a_{ij} v_{tx_i} v_{x_j} - e^{\lambda t} u_t v + \sum_i b_i u_{x_i} v + (c-1) uv \right\} dxdt.$$

Then this can be rewritten as

$$\int_{U_T} \left\{ u^2 e^{-\lambda t} - \frac{1}{2}\frac{d}{dt}\left(e^{\lambda t} \sum_{ij} a_{ij} v_{x_i} v_{x_j} + e^{\lambda t} v^2 \right) + \frac{\lambda}{2} e^{\lambda t} \left(\sum_{ij} a_{ij} v_{x_i} v_{x_j} + v^2 \right) \right\}$$
$$= \int_{U_T} \left\{ -e^{\lambda t} \sum_{ij} (a_{ij})_t v_{x_i} v_{x_j} + \sum_i \left[-(b_i uv)_{x_i} + (b_i)_{x_i} uv + b_i uv_{x_i} \right] + (1-c) uv \right\} dxdt. \tag{17}$$

Which we call $A = B$ following the notation as in the lectures. Now for A, we integrate the time-derivative part and use that $v|_{\Sigma_T} = 0$:

$$A = \frac{1}{2} \int_{\Sigma_0} \left(\sum_{ij} a_{ij} v_{x_i} v_{x_j} + v^2 \right) dx + \int_{U_T} \left\{ u^2 e^{-\lambda t} + \frac{\lambda}{2} e^{\lambda t} \left(\sum_{ij} a_{ij} v_{x_i} v_{x_j} + v^2 \right) \right\} dxdt. \tag{18}$$

We continue by using the uniform ellipticity condition to get the lower bound:

$$A \geq \int_{U_T} \left(u^2 e^{-\lambda t} + \frac{\lambda}{2} e^{\lambda t} (\theta |Dv|^2 + v^2) \right) dxdt. \tag{19}$$

In order to get an upper bound on B we apply Young's inequality and also use that the coefficients are all assumed to be $C^1(\overline{U_T})$:

$$B \leq \int_{U_T} \left(\frac{1}{2} u^2 e^{-\lambda t} + \frac{C}{2} e^{\lambda t} (\theta |Dv|^2 + v^2) \right) dxdt,$$

where C is a constant made sufficiently large, in order to include the parameter θ, and it does not depend on λ. Since $A = B$, all these give

$$\int_{U_T} \left(u^2 e^{-\lambda t} + \frac{\lambda}{2} e^{\lambda t} (\theta |Dv|^2 + v^2) \right) dxdt \leq \int_{U_T} \left(\frac{u^2 e^{-\lambda t}}{2} + \frac{C e^{\lambda t}}{2} (\theta |Dv|^2 + v^2) \right) dxdt$$

or that

$$\int_{U_T} \left(\frac{1}{2} u^2 e^{-\lambda t} + \frac{\lambda - C}{2} e^{\lambda t} (\theta |Dv|^2 + v^2) \right) dxdt \leq 0.$$

But now we can choose $\lambda > C$, which immediately implies that $u = 0$.

4, D

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 28 of 12 marks

Total D marks: 20 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 0 of 20 marks