Problem Sheet 4, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed plane curve parametrised by arc length, say $\gamma(t) = (x(t), z(t))$ with $x(t) > 0$ for all t, and let $S \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating γ around the z -axis.

(a) Prove that S has area $2\pi \int_a^b x(t)dt$.

(b) Compute $\int_S K dA$, where K is Gaussian curvature.

Problem 2. Let $S \subset \mathbb{R}^3$ be a compact, connected, nonempty surface whose curvature K is everywhere positive. Prove that $\int_S K dA \geq 4\pi$. (Hint: use the Gauss map $N : S \to \mathbb{S}^2$ to compare this to an integral over a sphere.)

Problem 3. Let $\gamma : (a, b) \to \mathbb{R}^3$ be a regular curve which has no self intersections, and let S be the surface parametrised by

$$\phi(u, v) = \gamma(u) + vb(u), \quad a < u < b, -\epsilon < v < \epsilon,$$

Note: this φ is not a chart

where $b(u)$ is the binormal vector to γ at time u. Show that there is $\epsilon > 0$ so that S is a regular surface. Prove that γ is a geodesic in S.