1 MATH60132/MATH70132 Mathematical Logic : Module Information.

1.1 Lecturer:

Professor David Evans

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OFFICE HOUR: Monday, 2 - 2.30pm and 4 - 4.30pm.

1.2 Coursework

Two assignments, each worth 5 percent of module.

- C/w 1: released Monday week 4 (27 January); hand-in deadline 1300 Monday week 6 (10 February)
- C/w 2: released Monday week 8 (24 February); hand-in deadline 1300 Monday week 10 (10 March)

1.3 Mastery material

Some aspects of Model Theory: details will be posted on Blackboard towards the end of the module.

1.4 Problem sheets and problem classes

There will be around 8 problem sheets and 4 problem classes. Solutions to problem sheets will be released approximately 2 weeks after the problems have been distributed.

1.5 Prerequisites

Second year algebra and analysis, together with an appetite for abstraction and proofs, will suffice. I will use basic notions from algebra (groups, rings and vector spaces) in examples. Similarly, I will use graphs as examples in some places, but you do not need to have taken the Graph Theory module (essentially we just use the definition).

1.6 Syllabus

Propositional logic: Formulas and logical validity; a formal system; soundness and completeness.

Predicate logic: First-order languages and structures; satisfaction and truth of formulas; the formal system; Gödel's completeness theorem; the compactness theorem; the Loewenheim-Skolem theorem.

Set theory: The axioms of ZF set theory; ordinals; cardinality; the Axiom of Choice.

1.7 What's not covered

I will not cover philosophical aspects of logic. I will gloss over the important aspects of logic concerned with decidability and the Gödel Incompleteness Theorems. I will not cover anything about Category Theory or Proof Theory - two other very important areas of Logic.

1.8 Past exam papers

The module has run previously with different module codes. In 2017-18 and 2018-19 it was M3/4P65; in 2019-20 it was MATH96057/ MATH97006/ MATH97171. You can find exam papers from these years on MathsCentral, along with the last two years' exam. The logic part was taught in a different style in 2019-20 and this is reflected in the exam questions. The module did not run in 2020-21 and 2021-22.

1.9 Some notes and books

The lecture notes should be fairly self-contained and you can find typed lecture notes prepared by students from previous years on the Imperial Maths Wiki. The notes this year will be very close to those from 2018-19. As we progress through the module, I will put an updated version of these notes on Blackboard.

The following books might also be of use (see the Leganto reading list for details). You might find that the notation which they use differs from that used in the lectures. You will be able to find various other lecture notes on the internet. Some will be good, others not so good.

[1] is very concise, but covers a surprising amount; [2] is friendlier, but skips some of the harder material; [4] is quite comprehensive (and also available in the original French); [3] is useful for the logic part and [5] is a very nice introduction to set theory. The book [6] is a very nice new text. If you are curious about other parts of Logic (such as recursion theory or the incompleteness theorems) this would be a good place to look. The mastery material on model theory will come from part II of [4] (but is also covered in [6]).

1. Peter Johnstone, Notes on Logic and Set Theory, Cambridge University Press.

2. Peter J. Cameron, Sets, Logic and Categories, Springer.

3. A. G. Hamilton, Logic for Mathematicians, (Cambridge University Press, 1988)

4. René Cori and Daniel Lascar ‘Mathematical Logic: a course with exercises, Parts I and II,’ (Oxford University Press, 2001).

5. K. Hrbaček and T. Jech, Introduction to Set Theory, 3rd Edition, Marcel Dekker, 1999.

6. David Marker, An invitation to Mathematical Logic, Graduate Texts in Mathematics 301, Springer, 2024.

2 What’s it about?

The module is concerned with some of the foundational issues of mathematics. In propositional and (‘first-order’) predicate logic, we analyse the way in which we reason formally about mathematical structures. In set theory, we will look at the ZFC axioms and use these to develop the notion of cardinality. These topics have applications to other areas of mathematics: first-order logic has applications via model theory and ZFC provides an essential toolkit for handling infinite objects.

2.1 Formal logic

analyses symbolically the way in which we reason formally, particularly about mathematical structures. The subject was developed in the 20th century with David Hilbert, Kurt Gödel and Alfred Tarski amongst its major early figures. Although it is highly abstract, its ideas are fundamental in theoretical computer science and artificial intelligence and its results have applications in other parts of Mathematics (via an area known as Model Theory).

The first level of the subject is propositional logic. We look at the way simple statements (propositions) can be built into more complicated ones using connectives (‘or,’ ‘and,’ ‘not’ ‘implies’) and make precise how the truth or falsity of the component statements influences the truth or falsity of the compound statement. This is done using truth tables and can be useful for testing the validity of various forms of reasoning. It provides a way of analysing deductions of the form ‘if the following statements are true: ..... ; then so is ....’. We then move on to a completely symbolic process of deduction and describe the formal deduction system for propositional logic. The statements we consider (propositional formulas) are regarded as strings of symbols and we give rules for deducing a new formula from a given collection of formulas. We want these deduction rules to have the property that anything that could be deduced using truth tables (so by considering truth or falsity of the various statements), can be deduced in this formal way, and vice versa. This is the soundness and completeness of our formal system.

The next level of the subject is first-order predicate logic. This is what is needed to analyse ‘real’ mathematics and the extra ingredient is the use of quantifiers (‘for all’ and ‘there exists’). ‘First-order’ refers to the fact that the quantifiers range only over the elements of a structure, rather than over all subsets. We introduce the notion of a first-order structure, which is general enough to include many of the algebraic objects you come across in mathematics (groups, rings, vector spaces). We then have to be precise about the expressions (formulas) which make statements about these structures, and give a precise definition of what it means for a particular formula to be true in a structure. This is quite intricate, and the clever part is in getting the definitions right, but it corresponds to ordinary mathematical usage. Once this is done, we set up a formal deduction system for first-order logic. This parallels what we did for propositional logic, but is much harder. Nevertheless, the end result is the same: the formulas which are produced by our formal deduction system (the 'theorems') are precisely the formulas which are true in all first-order structures. This is Gödel's Completeness Theorem.

The formal systems which we will use are sometimes referred to as *Hilbert-style* systems. Other formulations of propositional and first-order logic are possible. In particular, the *natural deduction* system is used in some books and lecture notes (for example, the 2019-20 lecture notes), but we will not say very much about this.

2.2 Set theory

has a dual role in mathematics. As well as providing the basic foundations and the language in which most of modern mathematics can be expressed, it also provides the means for discussing the various notions of 'sizes of infinity.' For example, although the set of natural numbers, the set of integers and the set of real numbers are all infinite, there is a very natural sense in which the first two have the same size, whereas the third is strictly bigger. This is expressed properly in the notion of cardinality. To avoid paradoxes and inconsistencies, we have to be careful about what collections of objects we allow to be called sets. This is done by the Zermelo - Fraenkel axioms, which essentially tell us how we are allowed to create new sets out of old ones. Of course, having laid down these quite rigid rules, we have to show that they are sufficiently flexible to allow us to talk about everyday objects of mathematics. There are also situations in mathematics where an extra axiom is needed: the Axiom of Choice. For example without this axiom (or rather, method of construction of a new set from given ones) we cannot show that every vector space has a basis. But it also has some slightly counterintuitive consequences, and we shall also look at some of these.

David Evans, January 2025.