Mathematical Logic

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1 Propositional logic

Lecture 1 Monday Week 2

In propositional logic we start off by looking at the way simple statements or propositions can be built into more complicated ones using connectives and make precise how the truth or falsity of the component statements influences the truth or falsity of the compound statement. This is the *semantics* of the logic and is done using truth tables. It can be useful for testing the validity of various forms of reasoning and provides a way of analysing deductions. Once we have done this, we then move on to a completely symbolic process of deduction and describe the formal deduction system for propositional logic. This is the *syntax* of the logic.The statements we consider, propositional formulas, are regarded as strings of symbols and we give rules for deducing a new formula from a given collection of formulas. We want these deduction rules to have the property that anything that could be deduced using truth tables, so by considering truth or falsity of the various statements, can be deduced in this formal way, and vice versa. This is the soundness and completeness of our formal system.

As a non-mathematical example of the type of expression or reasoning we can analyse in this logic, consider the following statement:

If Mr Jones is happy, then Mrs Jones is unhappy, and if Mrs Jones is unhappy, then Mr Jones is unhappy, so Mr Jones is unhappy.

We can represent this symbolically:

Let p be 'Mr Jones is happy' and q be 'Mrs Jones is unhappy'. Then, symbolically, the statement is

$$(((p \to q) \wedge (q \to (\neg p))) \to (\neg p)) .$$

1.1 Propositional formulas

A **proposition** is a statement that is either **True** (T) or **False** (F). The following are **truth table rules**.

Definition 1.1.1. Represent propositions symbolically using **propositional variables**.

$$p, \quad q, \quad \ldots \qquad p_1, \quad p_2, \quad \ldots$$

Combine basic propositions into others using **connectives**.

- **negation (not)**.

$$(\neg p) \text{ has value } F \qquad \Longleftrightarrow \qquad p \text{ has value } T.$$

- **conjunction (and)**.

$$(p \wedge q) \text{ has value } T \qquad \Longleftrightarrow \qquad p \text{ and } q \text{ both have value } T.$$

- **disjunction (or)**.

$$(p \vee q) \text{ has value } T \qquad \Longleftrightarrow \qquad \text{at least one of } p \text{ and } q \text{ has value } T.$$

- **implication (implies)**.

$$(p \to q) \text{ has value } F \qquad \Longleftrightarrow \qquad p \text{ has value } T \text{ and } q \text{ has value } F.$$

- **biconditional (iff)**.

$$(p \leftrightarrow q) \text{ has value } T \qquad \Longleftrightarrow \qquad p \text{ and } q \text{ have the same value.}$$

The following is a **truth table**.

p	q	$(p \wedge q)$	$(p \vee q)$	$(p \to q)$	$(p \leftrightarrow q)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Definition 1.1.2. A **propositional formula** is obtained in the following way.

- Any propositional variable is a formula.
- If ϕ and ψ are formulas, then so are

$$(\neg\phi)\,, \qquad (\phi \wedge \psi)\,, \qquad (\phi \vee \psi)\,, \qquad (\phi \to \psi)\,, \qquad (\phi \leftrightarrow \psi)\,.$$

- Any formula arises in this way.

Example.

- Formulas.

$$p_1 \qquad p_2 \qquad (\neg p_1) \qquad (p_1 \to (\neg p_2)) \qquad ((p_1 \to (\neg p_2)) \to p_2)$$

- Not formulas.

$$p_1 \wedge p_2 \quad \text{(missing brackets)} \qquad)(\neg p_1 \quad \text{(not well-formed)}$$

Remark. (1) Because of the brackets, every formula is either a propositional variable or is built from shorter formulas in a unique way. Arguments are often proved by induction, on length of the formula, or the number of connectives in the formula.
(2) Any assignment of truth values to the propositional variables in a formula ϕ determines the truth value of ϕ in a unique way.

Example. Let

$$\phi : ((p_1 \to (\neg p_2)) \to p_1)\,.$$

The truth table of ϕ is

p_1	p_2	$(\neg p_2)$	$(p_1 \to (\neg p_2))$	ϕ
T	T	F	F	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

Definition 1.1.3. Let $n \in \mathbb{N}$.

(1) A **truth function** of n variables is a function $f : \{T, F\}^n \to \{T, F\}$, where

$$\{T, F\}^n = \{(x_1, \ldots, x_n) \mid x_i \in \{T, F\}\}\,.$$

(Exercise: how many truth functions of n variables are there?)

(2) Suppose ϕ is a formula whose variables are amongst $p_1, \ldots, p_n$. We obtain a truth function

$$F_\phi : \{T, F\}^n \to \{T, F\}\,,$$

whose value at $(x_1, \ldots, x_n)$ is the truth value of ϕ when p_i has value x_i, for $i = 1, \ldots, n$, computed using the rules in 1.1.1. F_ϕ is the truth function of ϕ.

Example. So, for example, $F_\phi(T, F) = T$. The following is the truth table in its **condensed form**.

$((p_1$	$\to$	$(\neg$	$p_2))$	$\to$	$p_1)$
T	F	F	T	T	T
T	T	T	F	T	T
F	T	F	T	F	F
F	T	T	F	F	F
				ϕ	

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Example. What is the truth function of $(((p \to q) \wedge (q \to (\neg p))) \to (\neg p))$? Always T.

Definition 1.1.4.

(1) A propositional formula ϕ is a **tautology** if its truth function F_ϕ always has value T.

(2) Say that formulas ϕ, ψ are **logically equivalent (l.e.)** if and only if $(\phi \leftrightarrow \psi)$ is a tautology.

Remark 1.1.5.

1. ϕ, ψ are logically equivalent if and only if, when regarded as formulas involving the same variables, they have the same truth function. For example, $((p \to q) \wedge (q \to (\neg p)))$ is l.e. to $(\neg p)$.
2. Suppose ϕ is a formula with variables $p_1, \ldots, p_n$, and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots q_r$. For each $i \leq n$, substitute ϕ_i in place of p_i in ϕ. Then the result is a formula θ, and if ϕ is a tautology, then so is θ.

Example 1.1.6. Check that

$$(((\neg p_2) \to (\neg p_1)) \to (p_1 \to p_2))$$

is a tautology. So by 1.1.5.2, if ϕ_1 and ϕ_2 are any formulas, then

$$(((\neg \phi_2) \to (\neg \phi_1)) \to (\phi_1 \to \phi_2))$$

is a tautology.

Proof of 1.1.5.

1. Easy.
2. Prove

$$F_\theta(p_1, \ldots, p_r) = F_\phi\left(F_{\phi_1}(q_1, \ldots, q_r), \ldots, F_{\phi_n}(q_1, \ldots, q_r)\right),$$

by induction on the number of connectives in ϕ.

□

Example. Logically equivalent formulas.

$(p_1 \wedge (p_2 \wedge p_3))$ is logically equivalent to $((p_1 \wedge p_2) \wedge p_3)$.
$(p_1 \vee (p_2 \vee p_3))$ is logically equivalent to $((p_1 \vee p_2) \vee p_3)$.
$(p_1 \vee (p_2 \wedge p_3))$ is logically equivalent to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$.
$(p_1 \wedge (p_2 \vee p_3))$ is logically equivalent to $((p_1 \wedge p_2) \vee (p_1 \wedge p_3))$.
$(\neg(\neg p_1))$ is logically equivalent to p_1.
$(\neg(p_1 \wedge p_2))$ is logically equivalent to $((\neg p_1) \vee (\neg p_2))$.
$(\neg(p_1 \vee p_2))$ is logically equivalent to $((\neg p_1) \wedge (\neg p_2))$.

So we usually omit brackets in $(p_1 \wedge p_2 \wedge p_3)$ and $(p_1 \vee p_2 \vee p_3)$.

Note. By 1.1.5, we obtain, for formulas ϕ, ψ, χ,

$$(\phi \wedge (\psi \wedge \chi)) \text{ is logically equivalent to } ((\phi \wedge \psi) \wedge \chi)\,,$$

etc.

Lemma 1.1.7. There are 2^{2^n} truth functions of n variables.

Proof. A truth function is a function

$$F : \{T, F\}^n \to \{T, F\}\,.$$

$\left|\{T, F\}^n\right| = 2^n$, and for each $\overline{x} \in \{T, F\}^n$, $F(\overline{x}) \in \{T, F\}$. Hence the result. □

Definition 1.1.8. Say that a set of connectives is **adequate** if for every $n \geq 1$, every truth function of n variables is the truth function of some formula which involves only connectives from the set, and variables $p_1, \ldots, p_n$.

Theorem 1.1.9. The set $\{\neg, \wedge, \vee\}$ is adequate.

Proof. Let $G : \{T, F\}^n \to \{T, F\}$.

Case 1. $G(\overline{v}) = F$ for all $\overline{v} \in \{T, F\}^n$. Take ϕ to be

$$(p_1 \wedge (\neg p_1))\,.$$

Then $G = F_\phi$.

Case 2. List the $\overline{v} \in \{T, F\}^n$ with $G(\overline{v}) = T$ as

$$\overline{v_1}, \ldots, \overline{v_r}.$$

Write

$$\overline{v_i} = (v_{i1}, \ldots, v_{in})\,,$$

where each v_{ij} is T or F. Define

$$q_{ij} = \begin{cases} p_j & v_{ij} = T \\ (\neg p_j) & v_{ij} = F \end{cases}.$$

So

$$q_{ij} \text{ has value } T \qquad \Longleftrightarrow \qquad p_j \text{ has value } v_{ij}.$$

Let

$$\psi_i : (q_{i1} \wedge \cdots \wedge q_{in})\,.$$

Then[1]

$$F_{\psi_i}(\overline{v}) = T \qquad \Longleftrightarrow \qquad \text{each } q_{ij} \text{ has value } T \qquad \Longleftrightarrow \qquad \overline{v} = \overline{v_i}.$$

Let

$$\theta : (\psi_1 \vee \cdots \vee \psi_r)\,.$$

Then

$$F_\theta(\overline{v}) = T \qquad \Longleftrightarrow \qquad F_{\psi_i}(\overline{v}) = T \qquad \Longleftrightarrow \qquad \overline{v} = \overline{v_i}$$

for some $i \leq r$. Thus

$$F_\theta(\overline{v}) = T \qquad \Longleftrightarrow \qquad G(\overline{v}) = T,$$

that is $F_\theta = G$.

As ϕ and θ were constructed using only $\neg, \wedge, \vee$, 1.1.9 follows. □

[1] For example, if $n = 3$ and $\overline{v_1} = (T, F, T)$ then ψ_1 is $(p_1 \wedge (\neg p_2) \wedge p_3)$.

A formula θ as in case 2 is said to be in **disjunctive normal form (DNF)**.

Corollary 1.1.10. Suppose χ is a formula whose truth function is not always F. Then χ is logically equivalent to a formula in disjunctive normal form.

Proof. Take $G = F_\chi$, and apply case 2 of 1.1.9. $\square$

Example. Let
$$\chi : ((p_1 \to p_2) \to (\neg p_2)) .$$
Then
$$F_\chi(\overline{v}) = T \qquad \iff \qquad \overline{v} \in \{(T, F), (F, F)\} .$$
Thus disjunctive normal form is
$$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2))) .$$

Corollary 1.1.11. The following sets of connectives are adequate.

1. $\{\neg, \vee\}$,
2. $\{\neg, \wedge\}$, and
3. $\{\neg, \to\}$.

Proof.

1. By 1.1.9, enough to show that we can express $\wedge$ using $\neg, \vee$.
$$(p_1 \wedge p_2) \text{ is logically equivalent to } (\neg((\neg p_1) \vee (\neg p_2))) .$$
2. By 1.1.9, enough to show that we can express $\vee$ using $\neg, \wedge$.
$$(p_1 \vee p_2) \text{ is logically equivalent to } (\neg((\neg p_1) \wedge (\neg p_2))) .$$
3. By 1.1.9, enough to show that we can express $\vee$ using $\neg, \to$.
$$(p_1 \vee p_2) \text{ is logically equivalent to } ((\neg p) \to q) .$$

$\square$

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Example 1.1.12. The following are not adequate.

1. $\{\wedge, \vee\}$, and
2. $\{\neg, \leftrightarrow\}$.

Proof.

1. If ϕ is built using $\vee, \wedge$, then $F_\phi(T, \ldots, T) = T$. Proof by induction on number of connectives.
2. Exercise. Puzzle.

$\square$

Example 1.1.13. The **NOR** connective $\downarrow$ has the following truth table.

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

$(p \downarrow q)$ is logically equivalent to $((\neg p) \wedge (\neg q))$. $\{\downarrow\}$ is adequate.

- $(p \downarrow p)$ is logically equivalent to $(\neg p)$, and
- $((p \downarrow p) \downarrow (q \downarrow q))$ is logically equivalent to $(p \wedge q)$.

So, as $\{\neg, \wedge\}$ is adequate, so is $\{\downarrow\}$.

1.2 A formal system for propositional logic

Idea is to try to generate all tautologies from basic assumptions (*axioms*) using appropriate *deduction rules*.

Definition 1.2.1. Very general definition, which you can essentially ignore for now.

- A **formal deduction system** Σ has the following ingredients.
  - An **alphabet** $A \neq \emptyset$ of symbols.
  - A non-empty subset $\mathcal{F}$ of the set of all finite sequences, **strings**, of elements of A, the **formulas** of Σ.
  - A subset $\mathcal{A} \subseteq \mathcal{F}$ called the **axioms** of Σ.
  - A collection of **deduction rules**.
- A **proof** in Σ is a finite sequence of formulas in $\mathcal{F}$, $\phi_1, \ldots, \phi_n$ such that each ϕ_i is either an axiom, that is is in $\mathcal{A}$, or is obtained from $\phi_1, \ldots, \phi_{i-1}$ using one of the deduction rules.
- The last, or any, formula in a proof is a **theorem** of Σ. Write $\vdash_\Sigma \phi$ for ϕ is a theorem of Σ.

Remark.

- If $\phi \in \mathcal{A}$, then $\vdash_\Sigma \phi$.
- Should have an algorithm to test whether a string is a formula and whether it is an axiom. Then a computer can systematically generate all possible proofs in Σ, and check whether something is a proof. Say Σ is **recursive** in this case.

The following is the main example here.

Definition 1.2.2. The formal system L for propositional logic has the following.

- Alphabet.

Variables	p_1	p_2	$\ldots$
Connectives	$\neg$	$\rightarrow$	
Punctuation	(	)	

- Formulas. L-formulas defined in 1.1.2 for $\neg, \rightarrow$.
  - Any variable p_i is a formula.
  - If ϕ, ψ are formulas, then so are
  $$(\neg\phi)\,, \qquad (\phi \rightarrow \psi)\,.$$
  - Any formula arises in this way.
- Axioms. Suppose ϕ, ψ, χ are L-formulas. The following are axioms of L.

  (A1) $(\phi \rightarrow (\psi \rightarrow \phi))$,

  (A2) $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$, and

  (A3) $(((\neg\psi) \rightarrow (\neg\phi)) \rightarrow (\phi \rightarrow \psi))$.
- Deduction rule.

  (MP) **Modus Ponens**. From $\phi, (\phi \rightarrow \psi)$, deduce ψ.
- A **proof** in L is a finite sequence of L-formulas, $\phi_1, \ldots, \phi_n$ such that each ϕ_i is either an axiom, or is obtained from $\phi_1, \ldots, \phi_{i-1}$ using the deduction rule MP. The n here is the **length** of the proof.

- The last (or indeed, any) formula in a proof is a **theorem** of L. Write $\vdash_L \phi$ for ϕ is a theorem of L.

Example 1.2.3. (Theorem 0.) Suppose ϕ is an L-formula. Then $\vdash_L (\phi \to \phi)$. Here is a proof in L.

1	$(\phi \to ((\phi \to \phi) \to \phi))$	(A1)
2	$((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$	(A2)
3	$((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$	$(1, 2, \mathrm{MP})$
4	$(\phi \to (\phi \to \phi))$	(A1)
5	$(\phi \to \phi)$	$(3, 4, \mathrm{MP})$

Definition 1.2.4. Suppose Γ is a set of L-formulas.

- A **deduction from** Γ is a finite sequence of L-formulas $\phi_1, \ldots, \phi_n$ such that each ϕ_i is either an axiom, a formula in Γ, or is obtained from previous formulas $\phi_1, \ldots, \phi_{i-1}$ using the deduction rule MP. Here, n is called the **length** of the deduction.
- Write $\Gamma \vdash_L \phi$ if there is a deduction from Γ ending in ϕ. Say ϕ is a **consequence** of Γ. So $\emptyset \vdash_L \phi$ is the same as $\vdash_L \phi$.

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Remark. In the expression $\Gamma \vdash_L \phi$, the set Γ is sometimes referred to as the set of *assumptions* or *hypotheses*. Note that if $\Delta \subseteq \Gamma$ and $\Delta \vdash_L \phi$, then $\Gamma \vdash_L \phi$.

Theorem 1.2.5 (Deduction Theorem (DT))**.** Suppose Γ is a set of L-formulas and ϕ, ψ are L-formulas. Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$. Then $\Gamma \vdash_L (\phi \to \psi)$.

[Easy exercise: the converse is also true.]

Corollary 1.2.6 (Hypothetical syllogism (HS))**.** Suppose ϕ, ψ, χ are L-formulas and $\vdash_L (\phi \to \psi)$ and $\vdash_L (\psi \to \chi)$. Then $\vdash_L (\phi \to \chi)$.

Proof. Use deduction theorem with $\Gamma = \emptyset$. Show that $\{\phi\} \vdash_L \chi$. Here is a deduction of χ from ϕ.

1	$(\phi \to \psi)$	(theorem of L)
2	$(\psi \to \chi)$	(theorem of L)
3	ϕ	(assumption)
4	ψ	$(1, 3, \mathrm{MP})$
5	χ	$(2, 4, \mathrm{MP})$

Thus $\{\phi\} \vdash_L \chi$. By deduction theorem, $\emptyset \vdash_L (\phi \to \chi)$, that is $\vdash_L (\phi \to \chi)$. □

Proposition 1.2.7. Suppose ϕ, ψ are L-formulas. Then

1. $\vdash_L ((\neg\psi) \to (\psi \to \phi))$,
2. $\{(\neg\psi), \psi\} \vdash_L \phi$, and
3. $\vdash_L (((\neg\phi) \to \phi) \to \phi)$.

Proof.

1. Problem sheet 1.
2. By 1 and MP twice.
3. Suppose χ is any formula. Then
$$\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L \chi,$$
by MP and 2. Let α be any axiom and let χ be $(\neg\alpha)$. Apply deduction theorem to get
$$\{((\neg\phi) \to \phi)\} \vdash_L ((\neg\phi) \to (\neg\alpha)).$$
A3 is $\{((\neg\phi) \to (\neg\alpha))\} \vdash_L (\alpha \to \phi)$. Using this and MP, we get
$$\{((\neg\phi) \to \phi)\} \vdash_L (\alpha \to \phi).$$
As α is an axiom, we get, from MP,
$$\{((\neg\phi) \to \phi)\} \vdash_L \phi.$$
Now use deduction theorem to obtain
$$\vdash_L (((\neg\phi) \to \phi) \to \phi).$$

□

Proof of 1.2.5. Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ using a deduction of length n. Show, by induction on n, that $\Gamma \vdash_L (\phi \to \psi)$.

- Base case. $n = 1$. In this case, ψ is either an axiom, or in Γ, or is ϕ. In the first two cases, $\Gamma \vdash_L \psi$, a one line deduction. Using the A1 axiom $(\psi \to (\phi \to \psi))$ and MP, we obtain $\Gamma \vdash_L (\phi \to \psi)$. If ϕ is ψ, we have $\Gamma \vdash_L (\phi \to \phi)$, by 1.2.3. This finishes the base case.

- Inductive step. In our deduction of ψ from $\Gamma \cup \{\phi\}$, either

  Case 1. ψ is an axiom, or in Γ, or is ϕ,

  Case 2. or ψ is obtained from earlier steps using MP.

  In case 1, we argue as in the base case to get $\Gamma \vdash_L (\phi \to \psi)$. In case 2, there are formulas $\chi, (\chi \to \psi)$ earlier in the deduction. We use the inductive hypothesis to get

  $$\Gamma \vdash_L (\phi \to \chi), \qquad \Gamma \vdash_L (\phi \to (\chi \to \psi)). \tag{1}$$

  We have the A2 axiom $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$. Using (1), the A2 axiom, and MP twice, we obtain $\Gamma \vdash_L (\phi \to \chi)$, as required, completing the inductive step.

□

Lecture 5
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1.3 Soundness and completeness of L

Theorem 1.3.1 (Soundness of L)**.** Suppose ϕ is a theorem of L. Then ϕ is a tautology.

Definition 1.3.2. A **propositional valuation** v is an assignment of truth values to the propositional variables $p_1, p_2, \ldots$. So

$$v(p_i) \in \{T, F\}, \qquad i \in \mathbb{N}.$$

Note. Using the truth table rules, this assigns a truth value $v(\phi) \in \{T, F\}$ to every L-formula ϕ, satisfying $v((\neg\phi)) \neq v(\phi)$ and $v((\phi \to \psi)) = F \Leftrightarrow (v(\phi) = T$ and $v(\psi) = F)$.

Proof of 1.3.1. By induction on the length of a proof of ϕ, it is enough to show

1. every axiom is a tautology, and
2. MP preserves tautologies, that is if $\psi, (\psi \to \chi)$ are tautologies, so is χ.

1. Use truth tables, or argue as follows. For A2, suppose for a contradiction there is a valuation v with

   $$v(((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))) = F.$$

   Then
   $$v((\phi \to (\psi \to \chi))) = T, \tag{2}$$
   and
   $$v(((\phi \to \psi) \to (\phi \to \chi))) = F. \tag{3}$$
   By (3),
   $$v((\phi \to \psi)) = T, \tag{4}$$
   and
   $$v((\phi \to \chi)) = F. \tag{5}$$
   So by (5), $v(\phi) = T$ and $v(\chi) = F$. By (4), $v(\psi) = T$. This contradicts (2). (Exercise: A1 and A3)
2. If v is a valuation, and $v(\psi) = T$ and $v((\psi \to \chi)) = T$, then $v(\chi) = T$.

$\square$

Theorem 1.3.3 (Generalisation of soundness)**.** Suppose Γ is a set of formulas and ϕ a formula with $\Gamma \vdash_L \phi$. Suppose v is a valuation with $v(\Gamma) = T$ (meaning: $v(\psi) = T$ for all $\psi \in \Gamma$). Then $v(\phi) = T$.

Proof. Exercise. Same proof. $\square$

Theorem 1.3.4 (Completeness (or Adequacy) of L)**.** Suppose ϕ is a tautology, that is $v(\phi) = T$ for every valuation v. Then $\vdash_L \phi$.

Steps in proof.

- If $v(\phi) = T$ for all valuations v, want to show $\vdash_L \phi$.
- Try to prove a **Generalisation:** Suppose that, for every v with $v(\Gamma) = T$, that is $v(\psi) = T$ for all $\psi \in \Gamma$, we have $v(\phi) = T$. Then $\Gamma \vdash_L \phi$.
- Equivalently, if $\Gamma \nvdash_L \phi$, show there is a valuation v with $v(\Gamma) = T$ and $v(\phi) = F$.

Definition 1.3.5. Suppose Γ is a set of L-formulas.

(1) We say that Γ is **consistent** if there is no L-formula ϕ such that $\Gamma \vdash_L \phi$ and $\Gamma \vdash_L (\neg\phi)$.

(2) We say that Γ is **complete** if for every L-formula ϕ we have $\Gamma \vdash_L \phi$ or $\Gamma \vdash_L (\neg\phi)$.

Remark. (1) By 1.3.1 there is no L-formula ψ with $\vdash_L \psi$ and $\vdash_L (\neg\psi)$. So we say that L is consistent. More generally, if Γ is a set of formulas and there is a valuation v with $v(\Gamma) = T$, then Γ is consistent, by 1.3.3.

(2) Note that the Completeness Theorem for L DOES NOT SAY 'If $\Gamma = \emptyset$ then Γ is complete in the above sense.' [Why?] Unfortunately 'complete' is being used in two different ways here.

Proposition 1.3.6. Suppose Γ is a consistent set of L-formulas and $\Gamma \nvdash_L \phi$. Then $\Gamma \cup \{(\neg\phi)\}$ is consistent.

Proof. Suppose not. So there is some formula ψ with

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \psi, \tag{6}$$

and

$$\Gamma \cup \{(\neg\phi)\} \vdash_L (\neg\psi). \tag{7}$$

Apply deduction theorem to (7),

$$\Gamma \vdash_L ((\neg\phi) \to (\neg\psi)).$$

By A3 and MP, we obtain

$$\Gamma \vdash_L (\psi \to \phi). \tag{8}$$

By (6), (8), and MP,

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \phi.$$

By deduction theorem,

$$\Gamma \vdash_L ((\neg\phi) \to \phi). \tag{9}$$

By 1.2.7.3,

$$\vdash_L (((\neg\phi) \to \phi) \to \phi).$$

So by this, (9), and MP, $\Gamma \vdash_L \phi$. This contradicts $\Gamma \nvdash_L \phi$. $\square$

Proposition 1.3.7 (Lindenbaum lemma)**.** Suppose Γ is a consistent set of L-formulas. Then there is a consistent set of formulas $\Gamma^* \supseteq \Gamma$ such that, for every ϕ, either $\Gamma^* \vdash_L \phi$ or $\Gamma^* \vdash_L (\neg\phi)$.

Proof. The set of L-formulas is countable, so we can list the L-formulas as $\phi_0, \phi_1, \ldots$. This is countable, since the alphabet

$$\neg \quad \to \quad) \quad (\quad p_1 \quad p_2 \quad \ldots$$

is countable, and the formulas are finite sequences from this alphabet, so only countable many. Lecture 6 Wednesday Week 3

Define inductively sets of formulas

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \dots, \qquad \Gamma_0 = \Gamma, \qquad \Gamma_{n+1} = \begin{cases} \Gamma_n & \Gamma_n \vdash_L \phi_n \\ \Gamma_n \cup \{(\neg \phi_n)\} & \Gamma_n \nvdash_L \phi_n \end{cases}.$$

An easy induction using 1.3.6 shows that each Γ_i is consistent. Then let $\Gamma^* = \bigcup_{i \in \mathbb{N}} \Gamma_i$. Claim that Γ^* is consistent. If $\Gamma^* \vdash_L \phi$ and $\Gamma^* \vdash_L (\neg \phi)$, then as deductions are finite sequence of formulas,

$$\Gamma_n \vdash_L \phi, \qquad \Gamma_n \vdash_L (\neg \phi)$$

for some $n \in \mathbb{N}$, a contradiction. Let ϕ be any formula. So $\phi = \phi_n$ for some n. If $\Gamma^* \nvdash_L \phi$, then $\Gamma_n \nvdash_L \phi$. So by construction,

$$\Gamma_{n+1} \vdash_L (\neg \phi),$$

as $(\neg \phi) = (\neg \phi_n) \in \Gamma_{n+1}$. Thus $\Gamma^* \vdash_L (\neg \phi)$. □

Lemma 1.3.8. Suppose Γ^* is a set of L formulas which is consistent and complete. Then there is a valuation v such that, for every L-formula ϕ,

$$v(\phi) = T \qquad \Longleftrightarrow \qquad \Gamma^* \vdash_L \phi.$$

The proof is later.

Corollary 1.3.9. (1) Suppose Γ is a consistent set of L-formulas. Then there is a valuation v with $v(\Gamma) = T$.

(2) Suppose Δ is a set of L-formulas which is consistent and $\Delta \nvdash_L \phi$. Then there is a valuation v with $v(\Delta) = T$ and $v(\phi) = F$.

Proof. (1) By 1.3.7, there is $\Gamma^* \supseteq \Gamma$ which is still consistent, and which is complete.. By 1.3.8, there is a valuation v with

$$v(\Gamma^*) = T.$$

(2) Let

$$\Gamma = \Delta \cup \{(\neg \phi)\}.$$

By 1.3.6, Γ is consistent. In particular, $v(\Delta) = T$ and $v((\neg \phi)) = T$. So $v(\phi) = F$. □

Theorem 1.3.10 (Completeness / Adequacy Theorem for L)**.** (1) (General form) If Δ is a consistent set of L-formulas and ϕ is an L-formula such that for every valuation v with $v(\Delta) = T$ we have $v(\phi) = T$, then $\Delta \vdash_L \phi$.

(2) (The Completeness Theorem as a special case) If the L-formula ϕ is a tautology, then ϕ is a theorem of L.

Proof. (1) By 1.3.9(2).

(2) Take $\Delta = \emptyset$ in (1). □

[Exercise: in (1), the assumption that Δ is consistent is unnecessary: why?]

Proof of 1.3.8. Let Γ^* be a consistent set of L-formulas such that, for every L-formula ϕ, either $\Gamma^* \vdash_L \phi$ or $\Gamma^* \vdash_L (\neg \phi)$. Want a valuation v with $v(\phi) = T$ for all $\phi \in \Gamma^*$, that is

$$v(\phi) = T \qquad \Longleftrightarrow \qquad \Gamma^* \vdash_L \phi.$$

Note that, for each variable p_i, either $\Gamma^* \vdash_L p_i$ or $\Gamma^* \vdash_L (\neg p_i)$. So let v be the valuation with

$$v(p_i) = T \qquad \Longleftrightarrow \qquad \Gamma^* \vdash_L p_i.$$

Prove, by induction on the length of ϕ, that

$$v(\phi) = T \qquad \Longleftrightarrow \qquad \Gamma^* \vdash_L \phi.$$

- Base case. ϕ is just a propositional variable. This case is by definition of v.
- Inductive step.

  Case 1. ϕ is $(\neg\psi)$.

  $\Longrightarrow$ As v is a valuation,

$$\begin{array}{rcll} v(\phi) = T & \Longrightarrow & v(\psi) = F & \text{as } v \text{ is a valuation} \\ & \Longrightarrow & \Gamma^* \nvdash_L \psi & \text{by inductive hypothesis} \\ & \Longrightarrow & \Gamma^* \vdash_L (\neg\psi) & \text{by Lindenbaum property} \\ & \Longrightarrow & \Gamma^* \vdash_L \phi. & \end{array}$$

  $\Longleftarrow$ Conversely suppose $\Gamma^* \vdash_L \phi$.

$$\begin{array}{rcll} \Gamma^* \vdash_L \phi & \Longrightarrow & \Gamma^* \nvdash_L \psi & \text{by consistency} \\ & \Longrightarrow & v(\psi) = F & \text{by inductive hypothesis} \\ & \Longrightarrow & v((\neg\psi)) = T & \text{as } v \text{ is a valuation} \\ & \Longrightarrow & v(\phi) = T. & \end{array}$$

  Case 2. ϕ is $(\psi \to \chi)$.

  $\Longleftarrow$ Suppose $v(\phi) = F$. Show that $\Gamma^* \nvdash_L \phi$. Then

$$v(\psi) = T, \qquad v(\chi) = F.$$

  By inductive hypothesis,

$$\Gamma^* \vdash_L \psi, \qquad \Gamma^* \nvdash_L \chi.$$

  If $\Gamma^* \vdash_L \phi$, then using $\Gamma^* \vdash_L \psi$ and MP, we get $\Gamma^* \vdash_L \chi$, a contradiction. So $\Gamma^* \nvdash_L \phi$.

  $\Longrightarrow$ Suppose $\Gamma^* \nvdash_L \phi$, that is $\Gamma^* \nvdash_L (\psi \to \chi)$. Then

$$\Gamma^* \nvdash_L \chi, \tag{10}$$

  as $\vdash_L (\chi \to (\psi \to \chi))$. Also

$$\Gamma^* \nvdash_L (\neg\psi), \tag{11}$$

  as $\vdash_L ((\neg\psi) \to (\psi \to \chi))$ by 1.2.7.1. By (10), (11), and inductive hypothesis,

$$v(\chi) = F, \qquad v((\neg\psi)) = F,$$

  so $v(\psi) = T$. Thus $v(\phi) = F$, which does the inductive step.

$\square$

We did not cover the following in Lectures, but you can have a look at it.

Theorem 1.3.11 (Compactness theorem for L)**.** Suppose Γ is a set of L-formulas. The following are equivalent.

(1) There is a valuation v with $v(\Gamma) = T$.

(2) For every finite subset $\Sigma \subseteq \Gamma$, there is a valuation w with $w(\Sigma) = T$.

Proof. Clearly (1) implies (2), so suppose (1) does not hold. By 1.3.10, Γ is not consistent. As deductions are finite, and therefore only involve finitely many formulas in Γ, some finite subset Σ of Γ is inconsistent. It follows that Σ is inconsistent (by Generalised Soundness 1.3.3). $\square$

Example. Exercise. Let

$$P = \{\text{sequences of } \{T, F\}\} = \{\text{functions } f : \mathbb{N} \to \{T, F\}\}.$$

Topologise with basic open sets

$$O(a_1, \ldots, a_n) = \{\text{all sequences starting } a_1, \ldots, a_n\}$$

for $a_1, \ldots, a_n \in \{T, F\}$ and $n \in \mathbb{N}$. Use the compactness theorem to prove P is compact.

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Lecture 7 is a problem class.

Lecture 8
Tuesday
Week 4

2 First-order logic

Also known as Predicate Logic.

Plan.

- Semantics.
  - Introduce the mathematical objects, the first-order structures.
  - Introduce the formulas, the first-order languages.
- Syntax.
  - Describe a formal system.
  - Show that its theorems are precisely the formulas true in all structures. This is Gödel's completeness theorem.

2.1 Structures

Definition 2.1.1. Suppose A is a set and $n \in \mathbb{N}_{\geq 1}$.

- An n**-ary relation** on A is a subset
$$\overline{R} \subseteq A^n = \{(a_1, \dots, a_n) \mid a_i \in A\},$$
- An n**-ary function** on A is a function
$$\overline{f} : A^n \to A.$$

Example.

- Ordering $\leq$ on $\mathbb{R}$ is a binary relation on $\mathbb{R}$.
- $+$ on $\mathbb{C}$ is a binary function on $\mathbb{C}$.
- Even integers as a subset of $\mathbb{Z}$ is a unary relation on $\mathbb{Z}$.

Notation. If $\overline{R} \subseteq A^n$ is an n-ary relation and $a_1, \dots, a_n \in A$, write $\overline{R}(a_1, \dots, a_n)$ to mean $(a_1, \dots, a_n) \in \overline{R}$.

Definition 2.1.2. A **first-order structure** $\mathcal{A}$ consists of

- a non-empty set A, the **domain** of $\mathcal{A}$,
- a set of **relations** on A,
$$\left\{\overline{R_i} \subseteq A^{n_i} \mid i \in I\right\}$$
- a set of **functions** on A, and
$$\left\{\overline{f_j} : A^{m_j} \to A \mid j \in J\right\}$$
- a set of **constants**, just elements of A.
$$\{\overline{c_k} \mid k \in K\}$$

The sets I, J, K are indexing sets, which can be empty. Usually subsets of $\mathbb{N}$. The information
$$(n_i \mid i \in I), \qquad (m_j \mid j \in J), \qquad K$$
is called the **signature** of $\mathcal{A}$. Might denote the structure by
$$\begin{aligned}\mathcal{A} &= \left\langle A; \left(\overline{R_i} \mid i \in I\right), \left(\overline{f_j} \mid j \in J\right), \left(\overline{c_k} \mid k \in K\right)\right\rangle \\ &= \langle \text{domain};\ \text{relations},\ \text{functions},\ \text{constants}\rangle.\end{aligned}$$

Example 2.1.3.

- Orderings.

$$A \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}, \qquad I = \{1\}, \qquad J = \emptyset, \qquad K = \emptyset$$
$$\overline{R_1}\,(a_1, a_2) \qquad \text{to mean} \qquad a_1 < a_2$$

- Groups.

$\overline{R}$	the binary relation for equality
$\overline{m}$	the binary function for multiplication
$\overline{i}$	the unary function for inversion
$\overline{e}$	the constant for identity element

- Rings.

$\overline{R}$	the binary relation for equality
$\overline{m}$	the binary function for multiplication
$\overline{a}$	the binary function for addition
$\overline{n}$	the binary function for negation
$\overline{0}$	the constant for zero
$\overline{1}$	the constant for one

- Peano Arithmetic $\langle \mathbb{N}; \overline{R}, \overline{m}, \overline{a}, \overline{s}, \overline{0} \rangle$ where

$\overline{R}, \overline{m}, \overline{a}, \overline{0}$	as in Rings
$\overline{s}$	the successor function $\mathbb{N} \to \mathbb{N}$, $x \mapsto x + 1$

- Graphs.

$\overline{R}$	the binary relation for equality
$\overline{E}$	the binary relation for adjacency

2.2 First-order languages

Definition 2.2.1. A **first-order language** $\mathcal{L}$ has an alphabet of **symbols** of the following types.

Variables	$x_0 \quad x_1 \quad \dots$
Connectives	$\neg \quad \to$
Punctuation	$(\quad) \quad ,$
Quantifier	$\forall$
Relation symbols	$R_i, \quad i \in I$
Function symbols	$f_j, \quad j \in J$
Constant symbols	$c_k, \quad k \in K$

Here I, J, K are indexing sets, which could have $J, K = \emptyset$.

- Each R_i comes equipped with an arity n_i.
- Each f_j comes equipped with an arity m_j.

The information

$$(n_i \mid i \in I), \qquad (m_j \mid j \in J), \qquad K$$

is called the signature of $\mathcal{L}$.

An **$\mathcal{L}$-structure** is a structure

$$\mathcal{A} = \left\langle A; \left(\overline{R_i} \mid i \in I\right), \left(\overline{f_j} \mid j \in J\right), \left(\overline{c_k} \mid k \in K\right)\right\rangle$$

of the same signature as $\mathcal{L}$. There is a correspondence between the relation, function, and constant symbols of $\mathcal{L}$ and the actual relations, functions, and constants in $\mathcal{A}$, and the arities match up. This correspondence

$$R_i \leftrightsquigarrow \overline{R_i}, \qquad f_j \leftrightsquigarrow \overline{f_j}, \qquad c_k \leftrightsquigarrow \overline{c_k},$$

or $\mathcal{A}$, is called an **interpretation** of $\mathcal{L}$.

Definition 2.2.2. A **term** of $\mathcal{L}$ is defined as follows.

- Any variable is a term.
- Any constant symbol is a term.
- If f is an m-ary function symbol of $\mathcal{L}$ and $t_1, \dots, t_m$ are terms, then

  $$f(t_1, \dots, t_m)$$

  is also a term.
- Any term arises in this way.

Example. Suppose $\mathcal{L}$ has a binary function symbol f and constant symbols c_1, c_2.

- Some terms.

  $$c_1 \qquad c_2 \qquad x_1 \qquad f(c_1, x_1) \qquad f(f(c_1, x_2), c_2) \qquad f(x_1, f(f(c_1, x_2), c_2))$$

- Not terms.

  $$ffx_1 \quad \text{(not well-formed)}$$

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Definition 2.2.3. Use the previous notation.

- An **atomic formula** of $\mathcal{L}$ is of the form

  $$R(t_1, \dots, t_n),$$

  where R is an n-ary relation symbol of $\mathcal{L}$ and $t_1, \dots, t_n$ are terms.
- The formulas of $\mathcal{L}$ are defined as follows.
  - Any atomic formula is a formula.
  - If ϕ, ψ are $\mathcal{L}$-formulas, then

    $$(\neg\phi), \qquad (\phi \to \psi), \qquad (\forall x)\,\phi$$

    are $\mathcal{L}$-formulas, where x is any variable.
  - Every $\mathcal{L}$-formula arises in this way.

Example. Suppose $\mathcal{L}$ has a binary function symbol f, a unary relation symbol P, a binary relation symbol R, and constant symbols c_1, c_2.

- Some terms.

  $$x_1 \qquad c_1 \qquad f(x_1, c_1) \qquad f(f(x_1, c_1), x_2)$$

- Some atomic formulas.

  $$P(x_1) \qquad R(f(x_1, c_1), x_2)$$

- Some formulas.

  $$(\forall x_1)\,(R(f(x_1, c_1), x_2) \to P(x_1))$$

Definition 2.2.4. With the same notation, suppose $\mathcal{A}$ is an $\mathcal{L}$-structure. A **valuation** in $\mathcal{A}$ is a function v from the set of terms of $\mathcal{L}$ to A satisfying

- $v(c_k) = \overline{c_k}$, and
- if $t_1, \dots, t_m$ are terms of $\mathcal{L}$ and f is an m-ary function symbol, then

$$v(f(t_1, \dots, t_m)) = \overline{f}(v(t_1), \dots, v(t_m)),$$

where $\overline{f}$ is an interpretation of f in $\mathcal{A}$.

Lemma 2.2.5. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and $a_0, a_1, \dots \in A$. Then there is a unique valuation v, in $\mathcal{A}$, with $v(x_l) = a_l$, for all $l \in \mathbb{N}$, where the variables of $\mathcal{L}$ are $x_0, x_1, \dots$.

Proof. By induction on the length of terms. Show that, if we let

- $v(x_l) = a_l$ for all $l \in \mathbb{N}$,
- $v(c_k) = \overline{c_k}$ for all $k \in K$, and
- $v(f(t_1, \dots, t_m)) = \overline{f}(v(t_1), \dots, v(t_m))$,

then v is a well-defined valuation. □

Example. Groups. Signature, as in 2.1.3, has

- a binary relation symbol R, for equality,
- a binary function symbol m, for multiplication,
- a unary function symbol i, for inversion, and
- a constant symbol e, for identity element.

Let G be a group and $g, h \in G$. Let v be a valuation with $v(x_0) = g$ and $v(x_1) = h$. Then

$$\begin{aligned} v(m(m(x_0, x_1), i(x_0))) &= \overline{m}(v(m(x_0, x_1)), v(i(x_0))) \\ &= \overline{m}\left(\overline{m}(v(x_0), v(x_1)), \overline{i}(v(x_0))\right) \\ &= \overline{m}\left(\overline{m}(g, h), \overline{i}(g)\right) \\ &= \overline{m}\left(gh, g^{-1}\right) \\ &= ghg^{-1}. \end{aligned}$$

Definition 2.2.6. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and x_l is any variable. Suppose v, w are valuations in $\mathcal{A}$. We say v, w are x_l**-equivalent** if $v(x_m) = w(x_m)$ whenever $m \neq l$.

Definition 2.2.7. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$. Define, for an $\mathcal{L}$-formula ϕ, what is meant by v **satisfies** ϕ, in $\mathcal{A}$.

- Atomic formulas. Suppose R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$. Then
$$v \text{ satisfies the atomic formula } R(t_1, \ldots, t_n) \qquad \iff \qquad \overline{R}(v(t_1), \ldots, v(t_n)) \text{ holds in } \mathcal{A}.$$
- Suppose ϕ, ψ are $\mathcal{L}$-formulas, and we already know about valuations satisfying ϕ, ψ.
$$\begin{aligned} v \text{ satisfies } (\neg\phi) \text{ in } \mathcal{A} \quad &\iff \quad v \text{ does not satisfy } \phi \text{ in } \mathcal{A}, \\ v \text{ satisfies } (\phi \to \psi) \text{ in } \mathcal{A} \quad &\iff \quad \text{it is not the case that } v \text{ satisfies } \phi \text{ in } \mathcal{A} \\ &\qquad \text{and } v \text{ does not satisfy } \psi \text{ in } \mathcal{A}, \\ v \text{ satisfies } (\forall x_l)\,\phi \text{ in } \mathcal{A} \quad &\iff \quad \text{whenever } w \text{ is a valuation in } \mathcal{A} \text{ which is } x_l\text{-equivalent to } v, \\ &\qquad \text{then } w \text{ satisfies } \phi \text{ } \textit{in } \mathcal{A}. \end{aligned}$$

Remark. 2.2.7 does not work if we allow empty structure.

Notation. If v satisfies ϕ, write $v[\phi] = T$. If v does not satisfy ϕ, write $v[\phi] = F$. If every valuation in $\mathcal{A}$ satisfies ϕ, say that ϕ is **true** in $\mathcal{A}$ or $\mathcal{A}$ is a **model** of ϕ, and write $\mathcal{A} \vDash \phi$. If $\mathcal{A} \vDash \phi$ for every $\mathcal{L}$-structure $\mathcal{A}$, we say that ϕ is **logically valid**, and write $\vDash \phi$.

These are the analogues of tautologies in the propositional logic. Difference is that in propositional logic there is an algorithm to decide whether a given formula is a tautology. There is no such algorithm to decide whether a given $\mathcal{L}$-formula is logically valid or not. This is a consequence of Gödel's incompleteness theorem.

Example.

- Suppose $\mathcal{L}$ has a binary relation symbol R. The $\mathcal{L}$-formula
$$(R(x_1, x_2) \to (R(x_2, x_3) \to R(x_1, x_3)))$$
is true in $\mathcal{A} = \langle \mathbb{N}; R \rangle$, where R is interpreted as $<$. If not, there is a valuation v, in $\mathcal{A}$, such that v satisfies $R(x_1, x_2)$ and v does not satisfy $(R(x_2, x_3) \to R(x_1, x_3))$. So
$$v[R(x_2, x_3)] = T, \qquad v[R(x_1, x_3)] = F.$$
Let $v(x_i) = a_i \in \mathbb{N}$. So
$$a_1 < a_2, \qquad a_2 < a_3, \qquad a_1 \not< a_3.$$
As $<$ is transitive on $\mathbb{N}$, this is a contradiction.
- The same formula is not true in the structure $\mathcal{B}$ with domain $\mathbb{N}$ where we interpret $R(x_i, x_j)$ as $x_i \neq x_j$. Take a valuation in $\mathcal{B}$ with
$$v(x_1) = 1 = v(x_3), \qquad v(x_2) = 2.$$
v does not satisfy the formula in $\mathcal{B}$.

Lecture 10 Monday Week 5

Definition 2.2.8. Suppose ϕ, ψ are $\mathcal{L}$-formulas.

- $(\exists x)\,\phi$ means $(\neg(\forall x)(\neg\phi))$.
- $(\phi \vee \psi)$ means $((\neg\phi) \to \psi)$ as in propositional logic. (etc.)

Lemma 2.2.9. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and ϕ an $\mathcal{L}$-formula. Let v be a valuation in $\mathcal{A}$. Then v satisfies $(\exists x_1)\,\phi$ in $\mathcal{A}$ if and only if there is a valuation w, which is x_1-equivalent to v, such that w satisfies ϕ.

$\implies$. Suppose v satisfies $(\neg(\forall x_1)(\neg\phi))$. Using 2.2.7, v does not satisfy $(\forall x_1)(\neg\phi)$. So there is valuation w, x_1-equivalent to v, such that w does not satisfy $(\neg\phi)$. Such a w satisfies ϕ. [$\impliedby$] Exercise. $\square$

Example 2.2.10.
$$(\forall x_1)(\exists x_2)\,R(x_1, x_2)$$
is true in $\langle \mathbb{Z}; < \rangle$ and $\langle \mathbb{N}; < \rangle$, but not in $\langle \mathbb{N}; > \rangle$.

Example. Suppose ϕ is any $\mathcal{L}$-formula. Then

1. $((\exists x_1)(\forall x_2)\,\phi \to (\forall x_2)(\exists x_1)\,\phi)$ is logically valid, and
2. $((\forall x_2)(\exists x_1)\,\phi \to (\exists x_1)(\forall x_2)\,\phi)$ is not necessarily logically valid.

Proof. Exercise.

1. Argument with valuation.
2. Give an example.

□

Example. Consider the propositional formula

$$\chi : (p_1 \to (p_2 \to p_1)) .$$

Suppose $\mathcal{L}$ is a first-order language and ϕ_1, ϕ_2 are $\mathcal{L}$-formulas. Substitute ϕ_1 in place of p_1, and ϕ_2 in place of p_2, in χ. We obtain an $\mathcal{L}$-formula

$$\theta : (\phi_1 \to (\phi_2 \to \phi_1)) .$$

Check that, as χ is a tautology, θ is logically valid.

Definition 2.2.11. Suppose χ is an $\mathcal{L}$-formula involving propositional variables $p_1, \ldots, p_n$. Suppose $\mathcal{L}$ is a first-order language and $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas. A **substitution instance** of χ is obtained by replacing each p_i in χ by ϕ_i, for $i = 1, \ldots, n$.

Call the result θ.

Theorem 2.2.12.

- θ is an $\mathcal{L}$-formula, and
- if χ is a tautology, then θ is logically valid.

Proof. Take an $\mathcal{L}$-structure $\mathcal{A}$ and a valuation v in $\mathcal{A}$. Use this to define a propositional valuation w with

$$w(p_i) = v[\phi_i], \qquad i \leq n.$$

Then prove, by induction on the number of connectives in χ, that

$$w(\chi) = v[\theta] .$$

In particular, if χ is a tautology, then $v[\theta] = T$. (Exercise) For example, in the inductive step, consider

$$\chi : (\alpha \to \beta) .$$

So θ is $(\theta_1 \to \theta_2)$, where θ_1 is obtained from α and θ_2 is obtained from β. By inductive hypothesis,

$$w(\alpha) = v[\theta_1], \qquad w(\beta) = v[\theta_2] .$$

So $w(\alpha \to \beta) = v[(\theta_1 \to \theta_2)]$, etc. □

Note. Not all logically valid formulas arise in this way.

Example.

$$((\exists x_2)(\forall x_1)\,\phi \to (\forall x_1)(\exists x_2)\,\phi) .$$

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Week 5

2.3 Three topics

1. Bound and free variables

Definition 2.3.1. Suppose ϕ, ψ are $\mathcal{L}$-formulas and $(\forall x_i)\,\phi$ occurs as a subformula of ψ, that is ψ is

$$\ldots \quad (\forall x_i)\,\phi \quad \ldots .$$

- We say that ϕ is the **scope** of that quantifier $(\forall x_i)$ here in ψ. An occurrence of a variable x_j in ψ is **bound** if it is in the scope of a quantifier $(\forall x_j)$ in ψ, or it is the x_j here in $(\forall x_j)$.
- Otherwise, it is a **free** occurrence, of x_j. Variables having a free occurrence in ψ are called **free variables** of ψ.
- A formula with no free variables is called a **closed formula** or a **sentence**, of $\mathcal{L}$.

Example.

$$\psi_1 : \left(R_1 \left(\overbrace{x_1, x_2}^{\text{free}} \right) \to (\forall x_3)\, \underbrace{R_2 \left(\overbrace{x_1}^{\text{free}}, \overbrace{x_3}^{\text{bound}} \right)}_{\text{scope of } (\forall x_3)} \right)$$

$$\psi_2 : \left((\forall x_1)\, \underbrace{R_1 \left(\overbrace{x_1}^{\text{bound}}, \overbrace{x_2}^{\text{free}} \right)}_{\text{scope of } (\forall x_1)} \to R_2 \left(\overbrace{x_1, x_2}^{\text{free}} \right) \right) \qquad \psi_{2'} : (\forall x_1)\, \underbrace{\left(R_1 \left(\overbrace{x_1}^{\text{bound}}, \overbrace{x_2}^{\text{free}} \right) \to R_2 \left(\overbrace{x_1}^{\text{bound}}, \overbrace{x_2}^{\text{free}} \right) \right)}_{\text{scope of } (\forall x_1)}$$

$$\psi_3 : \left((\exists x_1)\, \underbrace{R_1 \left(\overbrace{x_1}^{\text{bound}}, \overbrace{x_2}^{\text{free}} \right)}_{\text{scope of } (\exists x_1)} \to (\forall x_2)\, \underbrace{R_2 \left(\overbrace{x_2}^{\text{bound}}, \overbrace{x_3}^{\text{free}} \right)}_{\text{scope of } (\forall x_2)} \right)$$

Definition 2.3.2. If ψ is an $\mathcal{L}$-formula with free variables amongst $x_1, \ldots, x_n$, we might write

$$\psi\,(x_1, \ldots, x_n)\,,$$

instead of ψ. If $t_1, \ldots, t_n$ are terms, by

$$\psi\,(t_1, \ldots, t_n)\,,$$

we mean the $\mathcal{L}$-formula obtained by replacing each free occurrence of x_i in ψ by t_i.

Example. Let t_1 be $f_1\,(x_1)$, t_2 be $f_2\,(x_1, x_2)$, and

$$\psi\,(x_1, x_2) : \left((\forall x_1)\, R \left(x_1, \overbrace{x_2}^{\text{free}} \right) \to (\forall x_3)\, R \left(\overbrace{x_1}^{\text{free}}, \overbrace{x_2}^{\text{free}}, x_3 \right) \right).$$

So

$$\psi\,(t_1, t_2) : ((\forall x_1)\, R_1\,(x_1, f_2\,(x_1, x_2)) \to (\forall x_3)\, R_2\,(f_1\,(x_1)\,, f_2\,(x_1, x_2)\,, x_3))\,.$$

Theorem 2.3.3. Suppose ϕ is a closed $\mathcal{L}$-formula and $\mathcal{A}$ is an $\mathcal{L}$-structure. Then either $\mathcal{A} \vDash \phi$ or $\mathcal{A} \vDash (\neg\phi)$. More generally, if ϕ has free variables amongst $x_1, \ldots, x_n$, and v, w are valuations in $\mathcal{A}$ with $v(x_i) = w(x_i)$, for $i = 1, \ldots, n$, then

$$v[\phi] = T \qquad \Longleftrightarrow \qquad w[\phi] = T.$$

Allow $n = 0$ here, for no free variables.

Proof. Note that the first statement follows from the generalisation. If ϕ has no free variables, then, for any valuations v, w in $\mathcal{A}$, they agree on the free variables of ϕ, so

$$v[\phi] = w[\phi].$$

Prove the generalisation by induction on the number of connectives and quantifiers in ϕ.

- Base case. ϕ is atomic. Let
$$\phi : R(t_1, \ldots, t_m),$$
where t_j are terms. The t_j only involve variables amongst $x_1, \ldots, x_n$. As v and w agree on these variables,
$$v(t_j) = w(t_j).$$
So
$$v[R(t_1, \ldots, t_m)] = T \qquad \Longleftrightarrow \qquad \overline{R}(v(t_1), \ldots, v(t_m)) \qquad \Longleftrightarrow \qquad w[R(t_1, \ldots, t_m)] = T.$$

- Inductive step. ϕ is $(\neg\psi)$, $(\psi \to \chi)$, or $(\forall x_i)\psi$. (Exercise: first two cases) Suppose ϕ is $(\forall x_i)\psi$. Suppose $v[\phi] = F$. By 2.2.7, there is a valuation v', x_i-equivalent to v, with $v'[\psi] = F$. The free variables of ψ are amongst $x_1, \ldots, x_n, x_i$. Let w' be the valuation x_i-equivalent to w with
$$w'(x_i) = v'(x_i).$$
Then v', w' agree on the free variables of ψ. By inductive hypothesis,
$$v'[\psi] = w'[\psi],$$
so $w'[\psi] = F$. As w' is x_i-equivalent to w, we obtain $w[(\forall x_i)\psi] = F$.

$\square$

Lecture 12 Wednesday Week 5

Remark 2.3.4. If $\mathcal{A}$ is an $\mathcal{L}$-structure and $\psi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula, whose free variables are amongst $x_1, \ldots, x_n$, and $a_1, \ldots, a_n \in A$, for domain A, then we write

$$\mathcal{A} \vDash \psi(a_1, \ldots, a_n),$$

to mean $v[\psi] = T$ for every valuation v in $\mathcal{A}$ with $v(x_i) = a_i$, for $i = 1, \ldots, n$.

Remark. By the proof of 2.3.3, this holds if $v[\psi] = T$ for some such valuation.

2. Substituting terms for variables

Example 2.3.5. A warning example, where $\mathcal{A} \vDash (\forall x_1)\phi(x_1)$, but we have term t, and a valuation v in $\mathcal{A}$, with $v[\phi(t)] = F$. Let

$$\phi(x_1) : \left(\underbrace{(\forall x_2) R\left(\overbrace{x_1}^{\text{free}}, x_2\right)}_{\text{scope of } (\forall x_2)} \to S\left(\overbrace{x_1}^{\text{free}}\right)\right).$$

Let t_1 be x_2. Then $\phi(t_1)$ is

$$((\forall x_2) R(x_2, x_2) \to S(x_2)).$$

Suppose $\mathcal{A} = \langle \mathbb{N}; \leq, = 0 \rangle$.

- Domain is $\mathbb{N} = \{0, 1, \ldots\}$,
- $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$, and
- $S(x_1)$ is interpreted as $x_1 = 0$.

So $\mathcal{A} \vDash (\forall x_1)\, \phi(x_1)$. But if we choose a valuation $v(x_2) = 1$, then $v[\phi(t_1)] = F$ in $\mathcal{A}$.

Definition 2.3.6. Let ϕ be an $\mathcal{L}$-formula, x_i a variable, and t an $\mathcal{L}$-term. We say t **is free for** x_i **in** ϕ if there is no variable x_j in t, such that x_i has a free occurrence within the scope of a quantifier $(\forall x_j)$ in ϕ.

Example. Exercise. Let $t = f(x_3, x_2, x_5)$. Let

$$\begin{aligned}
&\phi_1 : (((\forall x_2)\, R(x_2, x_4) \to K(x_1)) \to (\forall x_1)\, R(x_1, x_1)), \\
&\phi_2 : ((\forall x_2)\, (R(x_2, x_4) \to (\forall x_1)\, K(x_1)) \to (\forall x_2)\, R(x_1, x_1)).
\end{aligned}$$

For which t is t free for x_1?

Theorem 2.3.7. Suppose $\phi(x_1)$ is an $\mathcal{L}$-formula, possibly with other free variables. Let t be a term free for x_1 in ϕ, then

$$\vDash ((\forall x_1)\, \phi(x_1) \to \phi(t)).$$

In particular, if $\mathcal{A}$ is an $\mathcal{L}$-structure with $\mathcal{A} \vDash (\forall x_1)\, \phi(x_1)$, then $\mathcal{A} \vDash \phi(t)$.

Lemma 2.3.8. With this notation, suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$. Let v' be the valuation in $\mathcal{A}$ which is x_1-equivalent to v, with $v'(x_1) = v(t)$. Then

$$v'[\phi(x_1)] = T \qquad \Longleftrightarrow \qquad v[\phi(t)] = T.$$

Proof of 2.3.7. Suppose v is a valuation with $v[\phi(t)] = F$. Show that $v[(\forall x_1)\, \phi(x_1)] = F$, then

$$v[((\forall x_1)\, \phi(x_1) \to \phi(t))] = T.$$

Take v', x_1-equivalent to v, and

$$v'(x_1) = v(t).$$

Then by 2.3.8,

$$v'[\phi(x_1)] = F.$$

Thus $v[(\forall x_1)\, \phi(x_1)] = F$. □

Proof of 2.3.8 is in separate notes (not examinable).

3. Comparing structures

Definition 2.3.9. Suppose $\mathcal{L}$ is a first-order language with relation, function and constant symbols:

$$(R_i : i \in I);\ (f_j : j \in J);\ (c_k : k \in K),$$

where R_i is of arity n_i and f_j is of arity m_j.

Consider two $\mathcal{L}$-structures

$$\mathcal{A} = \langle A; (R_i^A : i \in I), (f_j^A : j \in J), c_k^A : k \in K) \rangle$$

and

$$\mathcal{B} = \langle B; (R_i^B : i \in I), (f_j^B : j \in J), c_k^B : k \in K) \rangle.$$

A function $\alpha : A \to B$ is an **isomorphism** (from $\mathcal{A}$ to $\mathcal{B}$) if α is a bijection and:

(a) $R_i^A(a_1, \ldots, a_{n_i}) \Leftrightarrow R_i^B(\alpha(a_1), \ldots, \alpha(a_{n_i}))$, for $i \in I$ and $a_1, \ldots, a_{n_i} \in A$;

(b) $\alpha(f_j^A(a_1, \ldots, a_{m_j}) = f_j^B(\alpha(a_1), \ldots, \alpha(a_{m_j}))$, for $j \in J$ and $a_1, \ldots, a_{m_j} \in A$;

(c) $\alpha(c_k^A) = c_k^B$, for $k \in K$.

Remark. If there exists such an isomorphism α from $\mathcal{A}$ to $\mathcal{B}$, then α^{-1} is also an isomorphism (from $\mathcal{B}$ to $\mathcal{A}$) and we say that $\mathcal{A}$ is isomorphic to $\mathcal{B}$, or that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Theorem 2.3.10. Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures and α is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Let v be a valuation in $\mathcal{A}$ and let w be the unique valuation in $\mathcal{B}$ with $w(x_i) = \alpha(v(x_i))$ (for all variables x_i). Then:

(i) For every term t (of $\mathcal{L}$) we have $w(t) = \alpha(v(t))$.

(ii) If ϕ is an $\mathcal{L}$-formula, then

$$v \text{ satisfies } \phi \text{ (in } \mathcal{A}) \Leftrightarrow w \text{ satisfies } \phi \text{ (in } \mathcal{B}).$$

(iii) If ϕ is a closed $\mathcal{L}$-formula, then

$$\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi.$$

Proof. (i) By induction on the length of t using 2.3.9 (b), (c).

(ii) By induction on the number of connectives and quantifiers in ϕ.

The base case is where ϕ is atomic, and this follows from (i) and 2.3.9 (a).

For the inductive step consider separately the cases where ϕ is $(\neg\psi), (\psi \to \chi)$ or $(\forall x_i)\psi$. We leave the first two cases as exercises (as usual) and consider the third.

Suppose $v[(\forall x_i)\psi] = F$. Then there is a valuation v' (in $\mathcal{A}$) which is x_i-equivalent to v and with $v'[\psi] = F$. Let w' be the valuation in $\mathcal{B}$ with $w'(x_j) = \alpha(v'(x_j))$ for all j. Then w' is x_i-equivalent to w and by induction hypothesis, $w'[\psi] = F$. Thus $w[(\forall x_i)\psi] = F$, as required. The converse direction is by symmetry (using that α^{-1} is also an isomorphism).

(iii) This follows from (ii) and the fact that α is a bijection (so every valuation in $\mathcal{B}$ arises from a valuation in $\mathcal{A}$ as in previious parts, and vice versa). $\square$

Lecture 13 Monday Week 6

Lecture 13 is a problem class.

2.4 The formal system $K_{\mathcal{L}}$

Lecture 14
Tuesday
Week 6

Definition 2.4.1. Suppose $\mathcal{L}$ is a first-order language. The formal system $K_{\mathcal{L}}$ has, as formulas, $\mathcal{L}$-formulas, and the following.

- Axioms. For $\mathcal{L}$-formulas ϕ, χ, ψ,

  (A1) $(\phi \to (\psi \to \phi))$,

  (A2) $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$,

  (A3) $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi))$,

  (K1) $((\forall x_i)\, \phi\,(x_i) \to \phi\,(t))$, where t is a term free for x_i in ϕ, and ϕ can have other free variables, and

  (K2) $((\forall x_i)\,(\phi \to \psi) \to (\phi \to (\forall x_i)\,\psi))$, if x_i is not free in ϕ.

- Deduction rules.

  (MP) Modus Ponens. From $\phi, (\phi \to \psi)$, deduce ψ.

  (Gen) **Generalisation**. From ϕ, deduce $(\forall x_i)\,\phi$.

A proof in $K_{\mathcal{L}}$ is a finite sequence of $\mathcal{L}$-formulas, each of which is an axiom, or deduced from previous formulas in the proof using a rule of deduction. A theorem of $K_{\mathcal{L}}$ is the last formula in some proof. Write $\vdash_{K_{\mathcal{L}}} \phi$ for ϕ is a theorem in $K_{\mathcal{L}}$.

Note. (1) We often write $\vdash \phi$ instead of $\vdash_{K_{\mathcal{L}}} \phi$. Some books use a different formal system, so be careful.

(2) Special case of K1: take t to be x_i.

Definition 2.4.2. Suppose Σ is a set of $\mathcal{L}$-formulas and ψ an $\mathcal{L}$-formula. A deduction of ψ from Σ is a finite sequence of formulas, ending with ψ, each of which is one of

- an axiom,
- a formula in Σ, or
- obtained from earlier formulas in the deduction using MP or Gen, with the restriction that, when Gen is applied to deduce $(\forall x_i)\phi$ from ϕ, the deduction of ϕ from Σ has not used any formulas from Σ where x_i occurs free.

Write $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ if there is a deduction from Σ to ψ and say that ψ is a consequence of Σ.

Remark 2.4.3.

- If Σ consists of closed formulas, do not need to worry about the restriction on Gen.
- Without the restriction on Gen, we would have $\phi \vdash_{K_{\mathcal{L}}} (\forall x_i)\,\phi$ and this would not be sensible if x_i is free in ϕ. (Why?)
- The way the restriction on Gen is phrased means that if $\Sigma' \subseteq \Sigma$ and $\Sigma' \vdash \psi$, then $\Sigma \vdash \psi$.

Theorem 2.4.4. Suppose ϕ is an $\mathcal{L}$-formula which is a substitution instance of a tautology in propositional logic. Then $\vdash_{K_{\mathcal{L}}} \phi$.

Example. For an $\mathcal{L}$-formula ϕ,

$$((\neg\,(\neg\phi)) \to \phi)\,,$$

as this is a substitution instance of

$$((\neg\,(\neg p_1)) \to p_1)\,.$$

Proof. There is a tautology χ with propositional variables $p_1, \ldots, p_n$ and $\mathcal{L}$-formulas $\psi_1, \ldots, \psi_n$, such that ϕ is obtained from χ by substituting ψ_i for p_i, for $i = 1, \ldots, n$. By completeness of propositional logic in 1.3.4, there is a proof in L of χ,

$$\chi_1, \ldots, \chi_r,$$

where each χ_i is a propositional formula, that is in L, and $\chi_r = \chi$. If we substitute $\psi_1, \ldots, \psi_n$ for $p_1, \ldots, p_n$ in all χ_j, we obtain a sequence of $\mathcal{L}$-formulas,

$$\phi_1, \ldots, \phi_r,$$

which is a proof of $\phi = \phi_r$ in $K_{\mathcal{L}}$. □

Theorem 2.4.5 (Soundness of $K_{\mathcal{L}}$)**.** If $\vdash_{K_{\mathcal{L}}} \phi$, then $\vDash \phi$, that is it is logically valid.

Proof. Like in the proof for L, we need to show

1. axioms are logically valid, and
2. deduction rules preserve logical validity.

1. A1, A2, A3 are substitution instances of propositional tautologies, by 2.2.1, so are logically valid by 2.4.4. K1 is logically valid by 2.3.7. K2 is $((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi))$, if x_i is not free in ϕ. Suppose we have valuation v such that $v[(\phi \to (\forall x_i)\psi)] = F$. So

$$v[\phi] = T, \qquad v[(\forall x_i)\psi] = F.$$

   So there is a valuation v', x_i-equivalent to v, with $v'[\psi] = F$. v and v' agree on all variables free in ϕ. So by 2.3.3,

$$v[\phi] = v'[\phi] = T,$$

   so $v'[(\phi \to \psi)] = F$. Thus $v[(\forall x_i)(\phi \to \psi)] = F$, so $v[\text{K2}] = T$.

2. For MP, if $\vDash \phi$ and $\vDash (\phi \to \psi)$, then $\vDash \psi$. For Gen, if $\vDash \phi$, then $\vDash (\forall x_i)\phi$. (Exercise: proof)

□

Corollary 2.4.6. (Consistency of $K_{\mathcal{L}}$) There is no $\mathcal{L}$-formula ϕ with $\vdash_{K_{\mathcal{L}}} \phi$ and $\vdash_{K_{\mathcal{L}}} (\neg\phi)$.

Example 2.4.7. (Generalised Soundness for $K_{\mathcal{L}}$) Suppose $\Sigma \vdash_{K_{\mathcal{L}}} \psi$. Then for every valuation v with $v[\sigma] = T$ for all $\sigma \in \Sigma$, we have $v[\psi] = T$.

Proof. Exercise: not trivial as you need to use the restriction on the use of Gen! □

Theorem 2.4.8 (Deduction theorem)**.** Suppose $\mathcal{L}$ is a first-order language, Σ is a set of $\mathcal{L}$-formulas, and ϕ, ψ are $\mathcal{L}$-formulas. If $\Sigma \cup \{\phi\} \vdash_{K_{\mathcal{L}}} \psi$, then $\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi)$.

Proof. Follows proof of deduction theorem for L in 1.2.5, by induction on the length of the deduction.

- Base case. One line deduction. Argue exactly as in 1.2.5. Note that $\vdash_{K_{\mathcal{L}}} (\phi \to \phi)$ by 2.4.4.
- Inductive step. Suppose ψ follows from earlier formulas in the deduction using MP or Gen. MP is exactly as in 1.2.5.

  Suppose ψ is obtained using Gen. Then ψ is $(\forall x_i)\theta$ and for some $\Delta \subseteq \Sigma \cup \{\phi\}$ with x_i not free in any formula in Δ we have $\Delta \vdash_{K_{\mathcal{L}}} \theta$.

  Case 1: Suppose $\phi \notin \Delta$. By Gen, we have $\Delta \vdash \psi$. By an axiom A1 and MP, we obtain $\Delta \vdash (\phi \to \psi)$. As $\Delta \subseteq \Sigma$ we then have $\Sigma \vdash (\phi \to \psi)$ as required.

Case 2: Suppose $\phi \in \Delta$. In particular, x_i is not free in ϕ. Let $\Sigma' = \Delta \setminus \{\phi\}$. So $\Sigma' \cup \{\phi\} \vdash \theta$ and by induction, we have

$$\Sigma' \vdash_{K_{\mathcal{L}}} (\phi \to \theta) .$$

By Gen,

$$\Sigma' \vdash_{K_{\mathcal{L}}} (\forall x_i)(\phi \to \theta) ,$$

so

$$\Sigma \vdash_{K_{\mathcal{L}}} (\forall x_i)(\phi \to \theta) .$$

By K2, as x_i is not free in ϕ

$$\Sigma \vdash_{K_{\mathcal{L}}} ((\forall x_i)(\phi \to \theta) \to (\phi \to (\forall x_i)\,\theta)) .$$

So by MP, we get $\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to (\forall x_i)\,\theta)$, which is $\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi)$.

□

Lecture 15 Wednesday Week 6

2.5 Gödel's completeness theorem

Definition 2.5.1. (1) A set Σ of $\mathcal{L}$-formulas is **consistent** if there is no formula ϕ with

$$\Sigma \vdash_{K_{\mathcal{L}}} \phi \qquad \text{and} \qquad \Sigma \vdash_{K_{\mathcal{L}}} (\neg\phi) .$$

(2) A set Σ of $\mathcal{L}$-formulas is **complete** if for every closed formula ϕ we have

$$\Sigma \vdash_{K_{\mathcal{L}}} \phi \qquad \text{or} \qquad \Sigma \vdash_{K_{\mathcal{L}}} (\neg\phi) .$$

By soundness, or 2.4.6, $\emptyset$ is consistent, so $K_{\mathcal{L}}$ is consistent.

Remark. If Σ is inconsistent, then

$$\Sigma \vdash_{K_{\mathcal{L}}} \chi,$$

for any $\mathcal{L}$-formula χ.

Recall that a closed $\mathcal{L}$-formula is one without free variables, sometimes called a sentence of $\mathcal{L}$. Show that, if Σ is a set of closed $\mathcal{L}$-formulas which is consistent, then there is an $\mathcal{L}$-structure $\mathcal{A}$ with $\mathcal{A} \vDash \sigma$ for all $\sigma \in \Sigma$. For a simplification, suppose that $\mathcal{L}$ is countable, that is the variables are $x_0, x_1, \ldots$, and there are countably many relation, function, and constant symbols. So we can enumerate the $\mathcal{L}$-formulas, or any subset thereof, as a list indexed by $\mathbb{N}$. Enumerate the closed $\mathcal{L}$-formulas as

$$\psi_0, \psi_1, \dots \in \{\psi_i : i \in \mathbb{N}\} .$$

Proposition 2.5.2. Suppose Σ is a consistent set of closed $\mathcal{L}$-formulas and ϕ is a closed $\mathcal{L}$-formula.

1. (Compare 1.3.6.) If $\Sigma \nvdash_{K_{\mathcal{L}}} \phi$, then $\Sigma \cup \{(\neg\phi)\}$ is consistent.
2. (Compare the Lindenbaum lemma in 1.3.7.) There is a consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-formulas such that, for every closed $\mathcal{L}$-formula ψ, either $\Sigma^* \vdash_{K_{\mathcal{L}}} \psi$ or $\Sigma^* \vdash_{K_{\mathcal{L}}} (\neg\psi)$.

Proof.

1. As in 1.3.6, use deduction theorem and $\vdash_{K_{\mathcal{L}}} (((\neg\phi) \to \phi) \to \phi)$.
2. Use 1 and the enumeration $\psi_0, \psi_1, \ldots$ of the closed $\mathcal{L}$-formulas.

□

The most difficult step in the proof is:

Theorem 2.5.3. (Model Existence Theorem) Suppose $\mathcal{L}$ is a countable first-order language and Σ is a consistent set of closed $\mathcal{L}$-formulas. Then there is a countable $\mathcal{L}$-structure $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$, that is $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$.

We will prove this later. Let's see how it is used to prove the Completeness Theorem.

Theorem 2.5.4. Let Σ be a set of closed $\mathcal{L}$-formulas and ϕ a closed $\mathcal{L}$-formula. If every model of Σ is a model of ϕ, then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$. (In other words: Suppose that whenever $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$, then $\mathcal{A} \models \phi$. Then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$.)

Notation. Write $\Sigma \models \phi$, if every model of Σ is a model of ϕ. Then the Theorem is saying

$$\Sigma \models \phi \qquad \Longrightarrow \qquad \Sigma \vdash_{K_{\mathcal{L}}} \phi.$$

Note that the converse of this is also true (Generalised Soundness).

Proof. May assume Σ is consistent. Otherwise, everything is a consequence of Σ. By assumption, there is no model of $\Sigma \cup \{(\neg\phi)\}$. So by 2.5.3, $\Sigma \cup \{(\neg\phi)\}$ is inconsistent. So by 2.5.2.1, $\Sigma \vdash_{K_{\mathcal{L}}} \phi$. □

Theorem 2.5.5 (Gödel's completeness theorem for $K_{\mathcal{L}}$)**.** If ϕ is an $\mathcal{L}$-formula with $\models \phi$, then ϕ is a theorem of $K_{\mathcal{L}}$, that is $\vdash_{K_{\mathcal{L}}} \phi$.

Proof. If ϕ is closed this follows from 2.5.4 with $\Sigma = \emptyset$. Suppose ϕ has free variables amongst $x_1, \ldots, x_n$, and consider the closed formula

$$\psi : (\forall x_1) \ldots (\forall x_n)\, \phi.$$

As $\models \phi$, we obtain $\models \psi$. So, by the closed case, $\vdash_{K_{\mathcal{L}}} \psi$, that is

$$\vdash_{K_{\mathcal{L}}} (\forall x_1) \ldots (\forall x_n)\, \phi. \tag{12}$$

If θ is any formula, then

$$\vdash_{K_{\mathcal{L}}} ((\forall x_i)\, \theta \to \theta)\,,$$

by the K1 axiom. So from (12) and this fact, and MP, applied n times, we obtain $\vdash_{K_{\mathcal{L}}} \phi$. □

Now we give a sketch of the proof of the Model Existence Theorem. More detailed notes can be found on the Blackboard page.

Lecture 16 Monday Week 7

Sketch of proof of 2.5.3. Series of steps. Notation is cumulative.

Step 1. Let $b_0, b_1, \ldots$ be new constant symbols. Form $\mathcal{L}^+$ by adding these to the symbols of $\mathcal{L}$. Regard Σ as a set of $\mathcal{L}^+$-formulas. Check that Σ is still consistent, in the formal system $K_{\mathcal{L}^+}$. Note that $\mathcal{L}^+$ is still a countable language.

Step 2. Adding witnesses. Claim that there is a consistent set of closed $\mathcal{L}^+$-formulas $\Sigma_\infty \supseteq \Sigma$ such that, for every $\mathcal{L}^+$-formula $\theta\,(x_i)$ with one free variable, there is some b_j with

$$\Sigma_\infty \vdash_{K_{\mathcal{L}^+}} ((\neg\,(\forall x_i)\,\theta\,(x_i)) \to (\neg\theta\,(b_j)))\,.$$

Think of $\theta\,(x_i)$ as $(\neg\chi\,(x_i))$. Then this formula is essentially

$$((\exists x_i)\,\chi\,(x_i) \to \chi\,(b_j))\,,$$

so b_j witnesses the existence of x_i satisfying χ.

Step 3. By the Lindenbaum lemma in 2.5.2, there is a consistent set $\Sigma^* \supseteq \Sigma_\infty$ of closed $\mathcal{L}^+$-formulas such that, for every closed ϕ, either $\Sigma^* \vdash_{K_{\mathcal{L}^+}} \phi$ or $\Sigma^* \vdash_{K_{\mathcal{L}^+}} (\neg\phi)$.

Step 4. Building a structure. Let

$$A = \left\{ \overline{t} \mid t \text{ is a closed term of } \mathcal{L}^{+} \right\}.$$

Note that

- a term is closed if it only involves constant symbols and function symbols, and no variables,
- use the $\bar{\cdot}$ to distinguish when we are thinking of a term as an element of A, and
- as $\mathcal{L}^{+}$ is countable, A is countable.

Make A into an $\mathcal{L}^{+}$ structure.

(a) Each constant symbol c of $\mathcal{L}^{+}$ is interpreted as $\overline{c} \in A$.

(b) Suppose R is an n-ary relation symbol. Define the relation $\overline{R} \subseteq A^n$ by

$$\left(\overline{t_1}, \ldots, \overline{t_n}\right) \in \overline{R} \qquad \Longleftrightarrow \qquad \Sigma^* \vdash_{K_{\mathcal{L}^+}} R\left(t_1, \ldots, t_n\right),$$

where $R\left(t_1, \ldots, t_n\right)$ is a closed atomic $\mathcal{L}^{+}$-formula and $t_1, \ldots, t_n$ are closed $\mathcal{L}^{+}$-terms.

(c) Suppose f is an m-ary function symbol. Define a function $\overline{f} : A^m \to A$ by

$$\overline{f}\left(\overline{t_1}, \ldots, \overline{t_m}\right) = \overline{f\left(t_1, \ldots, t_m\right)},$$

for closed terms $t_1, \ldots, t_m$.

Call this structure $\mathcal{A}$. Note that, if v is a valuation in $\mathcal{A}$ and t is a closed term, then $v(t) = \overline{t}$, by (a) and (c) here.

Step 5. Main lemma. Claim that, for every closed $\mathcal{L}^{+}$-formula ϕ,

$$\Sigma^* \vdash_{K_{\mathcal{L}^+}} \phi \qquad \Longleftrightarrow \qquad \mathcal{A} \vDash \phi. \tag{13}$$

Proof by induction on number of connectives and quantifiers in ϕ.

- Base case. ϕ is atomic, that is ϕ is $R\left(t_1, \ldots, t_n\right)$, for some closed terms t_i, and relation symbol R. (13) holds by (b) in definition of $\mathcal{A}$.
- Inductive step. Assume (13) holds for closed formulas involving fewer connectives and quantifiers.

  Case 1. ϕ is $(\neg\psi)$.
  Case 2. ϕ is $(\psi \to \chi)$.
  Case 3. ϕ is $(\forall x_i)\,\psi$.

  In cases 1 and 2, ψ, χ are closed. So (13) holds for these.

  Case 1. ϕ is $(\neg\psi)$.

$$\begin{aligned}
\mathcal{A} \vDash \phi \qquad &\Longleftrightarrow \qquad \mathcal{A} \nvDash \psi && \text{by 2.3.3} \\
&\Longleftrightarrow \qquad \Sigma^* \nvdash_{K_{\mathcal{L}^+}} \psi && \text{by inductive hypothesis (13)} \\
&\Longleftrightarrow \qquad \Sigma^* \vdash_{K_{\mathcal{L}^+}} (\neg\psi) && \text{by step 3.}
\end{aligned}$$

  Case 2. Exercise.
  Case 3. ϕ is $(\forall x_i)\,\psi$.

  Case 3a. x_i is not free in ψ. So ψ is closed and we can use inductive hypothesis.
  Case 3b. x_i is free in ψ. So $\psi(x_i)$ has a single free variable.

$\Longleftarrow$ Suppose for a contradiction that $\mathcal{A} \vDash \phi$ and $\Sigma^* \nvdash_{K_{\mathcal{L}^+}} \phi$. Then by step 3,

$$\Sigma^* \vdash_{K_{\mathcal{L}^+}} (\neg\phi).$$

By step 2,

$$\Sigma^* \vdash_{K_{\mathcal{L}^+}} ((\neg (\forall x_i)\, \psi(x_i)) \to (\neg\psi(b_j)))$$

for some constant symbol b_j. That is,

$$\Sigma^* \vdash_{K_{\mathcal{L}^+}} ((\neg\phi) \to (\neg\psi(b_j))).$$

So

$$\Sigma^* \vdash_{K_{\mathcal{L}^+}} (\neg\psi(b_j)).$$

$(\neg\psi(b_j))$ is closed and, by case 1, (13) applies. We obtain

$$\mathcal{A} \vDash (\neg\psi(b_j)). \tag{14}$$

This contradicts $\mathcal{A} \vDash (\forall x_i)\, \psi$. Take a valuation v in $\mathcal{A}$ with $v(x_i) = \overline{b_j}$, then v does not satisfy ψ, by (14).

□

Example. Exercise. Think about this where Σ consists of the group axioms. What is $\mathcal{A}$? Is it a group?

Corollary 2.5.6 (Compactness theorem for $K_{\mathcal{L}}$)**.** Suppose Σ is a set of closed $\mathcal{L}$-formulas and every finite subset of Σ has a model. Then Σ has a model.

Proof. Suppose Σ has no model. By 2.5.3, Σ is inconsistent, so there is a formula ϕ with

$$\Sigma \vdash_{K_{\mathcal{L}}} \phi, \qquad \Sigma \vdash_{K_{\mathcal{L}}} (\neg\phi).$$

Deductions only involve finitely many formulas in Σ. So there is a finite $\Sigma_0 \subseteq \Sigma$ with

$$\Sigma_0 \vdash_{K_{\mathcal{L}}} \phi, \qquad \Sigma_0 \vdash_{K_{\mathcal{L}}} (\neg\phi).$$

But then Σ_0 is inconsistent, so has no model, a contradiction. □

Lecture 17
Tuesday
Week 7

2.6 Equality

Example. In the language of groups, have a binary relation symbol $E(x_1, x_2)$ for equality $x_1 = x_2$.

Definition 2.6.1. Suppose $\mathcal{L}^E$ is a first-order language with a distinguished binary relation symbol E.

- An $\mathcal{L}^E$-structure in which E is interpreted as equality $=$ is a **normal** $\mathcal{L}^E$-structure.
- The following are the **axioms of equality**, Σ_E.
  - $(\forall x_1)\, E(x_1, x_1)$.
  - $(\forall x_1)(\forall x_2)(E(x_1, x_2) \to E(x_2, x_1))$.
  - $(\forall x_1)(\forall x_2)(\forall x_3)(E(x_1, x_2) \to (E(x_2, x_3) \to E(x_1, x_3)))$.
  - For each n-ary relation symbol R of $\mathcal{L}^E$,

$$(\forall x_1 \dots x_n)(\forall y_1 \dots y_n)((R(x_1, \dots, x_n) \wedge E(x_1, y_1) \wedge \dots \wedge E(x_n, y_n)) \to R(y_1, \dots, y_n)).$$

  - For each m-ary function symbol f of $\mathcal{L}^E$,

$$(\forall x_1 \dots x_m)(\forall y_1 \dots y_m)((E(x_1, y_1) \wedge \dots \wedge E(x_m, y_m)) \to E(f(x_1, \dots, x_m), f(y_1, \dots, y_m))).$$

Remark 2.6.2.

- If $\mathcal{A}$ is a normal $\mathcal{L}^E$-structure, then $\mathcal{A} \vDash \Sigma_E$.
- Suppose $\mathcal{A} = \left\langle A; \overline{E}, \dots \right\rangle$ is an $\mathcal{L}^E$-structure and $\mathcal{A} \vDash \Sigma_E$. Then $\overline{E}$ is an equivalence relation on A. Denote, for $a \in A$,
$$\widehat{a} = \left\{ b \in A \mid \overline{E}(a, b) \text{ holds} \right\},$$
the equivalence class of a. Let
$$\widehat{A} = \{ \widehat{a} \mid a \in A \}.$$
Make $\widehat{A}$ into an $\mathcal{L}^E$-structure $\widehat{\mathcal{A}}$.
  - If R is an n-ary relation symbol and $\widehat{a_1}, \dots, \widehat{a_n} \in \widehat{A}$, then say
  $$\overline{R}(\widehat{a_1}, \dots, \widehat{a_n}) \text{ holds in } \widehat{\mathcal{A}} \qquad \Longleftrightarrow \qquad \overline{R}(a_1, \dots, a_n) \text{ holds in } \mathcal{A}.$$
  This is well-defined by Σ_E.
  - Similarly, if f is an m-ary function symbol and $\widehat{a_1}, \dots, \widehat{a_m} \in \widehat{A}$, let
  $$\overline{f}(\widehat{a_1}, \dots, \widehat{a_m}) = \widehat{\overline{f}(a_1, \dots, a_m)}.$$
  This is also well-defined by Σ_E.
  - If c is a constant symbol, then interpret c as $\widehat{\overline{c}}$ in $\widehat{\mathcal{A}}$, where $\overline{c}$ is the interpretation in $\mathcal{A}$.

Note. In $\widehat{\mathcal{A}}$,
$$\overline{E}(\widehat{a_1}, \widehat{a_2}) \qquad \Longleftrightarrow \qquad \overline{E}(a_1, a_2) \text{ in } \mathcal{A} \qquad \Longleftrightarrow \qquad \widehat{a_1} = \widehat{a_2}.$$
So $\widehat{\mathcal{A}}$ is a normal $\mathcal{L}^E$-structure.

Lemma 2.6.3. Suppose $\mathcal{A}$ is an $\mathcal{L}^E$-structure with $\mathcal{A} \vDash \Sigma_E$. Let v be a valuation in $\mathcal{A}$. Let $\widehat{\mathcal{A}}$ be as given above. Let $\widehat{v}$ be the valuation in $\widehat{\mathcal{A}}$ with
$$\widehat{v}(x_i) = \widehat{v(x_i)}.$$
Then for every $\mathcal{L}^E$-formula ϕ,
$$\widehat{v} \text{ satisfies } \phi \text{ in } \widehat{\mathcal{A}} \qquad \Longleftrightarrow \qquad v \text{ satisfies } \phi \text{ in } \mathcal{A}.$$
In particular, if ϕ is closed, then
$$\mathcal{A} \vDash \phi \qquad \Longleftrightarrow \qquad \widehat{\mathcal{A}} \vDash \phi.$$

Note. If t is any term, then $\widehat{v}(t) = \widehat{v(t)}$, by definition of $\overline{f}$ on the structure $\widehat{\mathcal{A}}$.

Proof of 2.6.3. (Omitted from lectures; not examinable, but also not difficult.) The result 2.6.3 is proved by induction on the number of connectives and quantifiers in ϕ.

- Base case. ϕ is an atomic formula $R(t_1, \dots, t_n)$, where R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms. Then
$$\begin{aligned} v[\phi] = T \qquad & \Longleftrightarrow \qquad \overline{R}(v(t_1), \dots, v(t_n)) \text{ holds in } \mathcal{A} \\ & \Longleftrightarrow \qquad \overline{R}\left(\widehat{v(t_1)}, \dots, \widehat{v(t_n)}\right) \text{ holds in } \mathcal{A} \qquad \text{by definition of } \overline{R} \text{ in } \widehat{\mathcal{A}} \\ & \Longleftrightarrow \qquad \overline{R}(\widehat{v}(t_1), \dots, \widehat{v}(t_n)) \text{ holds in } \mathcal{A} \\ & \Longleftrightarrow \qquad \widehat{v}[\phi] = T, \end{aligned}$$
as required.
- Inductive step.

Case 1. ϕ is $(\neg\psi)$. Exercise.

Case 2. ϕ is $(\theta \to \chi)$. Exercise.

Case 3. ϕ is $(\forall x_i)\,\psi$.

$\implies$ If $v\,[(\forall x_i)\,\psi] = F$, there is a v', x_i-equivalent to v, with $v'\,[\psi] = F$. Then $\widehat{v'}$ is x_i-equivalent to $\widehat{v}$, and by the inductive hypothesis,

$$\widehat{v'}\,[\psi] = F.$$

So $\widehat{v}\,[(\forall x_i)\,\psi] = F$.

$\impliedby$ Suppose $\widehat{v}\,[(\forall x_i)\,\psi] = F$. So there is a valuation w in $\widehat{\mathcal{A}}$ which is x_i-equivalent to $\widehat{v}$ and $w\,[\psi] = F$. There is a valuation v' in $\mathcal{A}$, x_i-equivalent to v, with $\widehat{v'} = w$. We just change $v\,(x_i)$, so

$$\widehat{v'\,(x_i)} = w\,(x_i)\,.$$

Then $v'\,[\psi] = F$, by inductive hypothesis. So $v\,[(\forall x_i)\,\psi] = F$.

$\square$

Lemma 2.6.4. Suppose Δ is a set of closed $\mathcal{L}^E$-formulas. Then Δ has a **normal model**, that is a normal $\mathcal{L}^E$-structure $\mathcal{B}$ with $\mathcal{B} \vDash \sigma$ for all $\sigma \in \Delta$, if and only if $\Delta \cup \Sigma_E$ has a model.

Proof.

$\implies$ Trivial, as Σ_E holds in a normal $\mathcal{L}^E$-structure.

$\impliedby$ If $\mathcal{A} \vDash \Delta \cup \Sigma_E$, then by 2.6.3, $\widehat{\mathcal{A}} \vDash \Delta$, and $\widehat{\mathcal{A}}$ is a normal $\mathcal{L}^E$-structure.

$\square$

Theorem 2.6.5 (Compactness theorem for normal models)**.** Suppose $\mathcal{L}^E$ is a countable language with equality, and Δ is a set of closed $\mathcal{L}^E$-formulas such that every finite subset of Δ has a normal model. Then Δ has a normal model.

Proof. Every normal $\mathcal{L}^E$-structure is a model of Σ_E, so every finite subset of $\Delta \cup \Sigma_E$ has a model. By 2.5.6, $\Delta \cup \Sigma_E$ has a model $\mathcal{A}$. Then by 2.6.3 or 2.6.4, $\widehat{\mathcal{A}}$ is a normal model of Δ. $\square$

Notation. From now on, write $\mathcal{L}^=$ instead of $\mathcal{L}^E$, and $x_1 = x_2$ instead of $E\,(x_1, x_2)$, etc.

Theorem 2.6.6 (Countable downward Löwenheim-Skolem theorem)**.** Suppose $\mathcal{L}^=$ is a countable first-order language with equality, and $\mathcal{B}$ is a normal $\mathcal{L}^=$ structure. Then there is a countable normal $\mathcal{L}^=$-structure $\mathcal{A}$ such that, for every closed $\mathcal{L}^=$-formula ϕ, Lecture 18 Wednesday Week 7

$$\mathcal{B} \vDash \phi \qquad \Longleftrightarrow \qquad \mathcal{A} \vDash \phi.$$

Example. Let $\mathcal{B} = \langle \mathbb{R}; +, \cdot, 0, 1, \leq, \exp() \rangle$. What is $\mathcal{A}$?

Proof. Let

$$\Sigma = \{\text{closed } \phi \mid \mathcal{B} \vDash \phi\},$$

called the **theory** of $\mathcal{B}$. Then $\Sigma \supseteq \Sigma_E$, the axioms of equality, and Σ is consistent. By 2.5.3, Σ has a countable model $\mathcal{C}$. Then $\widehat{\mathcal{C}}$ is a countable normal model of Σ by 2.6.3. So if ϕ is closed and $\mathcal{B} \vDash \phi$, then $\widehat{\mathcal{C}} \vDash \phi$. Conversely, if ϕ is closed and $\mathcal{B} \nvDash \phi$, then $\mathcal{B} \vDash (\neg\phi)$, by 2.3.3, so $\widehat{\mathcal{C}} \vDash (\neg\phi)$, so $\widehat{\mathcal{C}} \nvDash \phi$. Take

$$\mathcal{A} = \widehat{\mathcal{C}}.$$

□

2.7 Beginning model theory: Examples and applications

Let $\mathcal{L}^=$ be a first-order language with equality and a binary relation symbol $\leq$.

Definition 2.7.1.

- A **linear order** $\mathcal{A} = \langle A; \leq_A \rangle$ is a normal model of

$$\begin{aligned}
&\phi_1 : (\forall x_1)(\forall x_2)\left(((x_1 \leq x_2) \wedge (x_2 \leq x_1)) \leftrightarrow (x_1 = x_2)\right), \\
&\phi_2 : (\forall x_1)(\forall x_2)(\forall x_3)\left(((x_1 \leq x_2) \wedge (x_2 \leq x_3)) \rightarrow (x_1 \leq x_3)\right), \\
&\phi_3 : (\forall x_1)(\forall x_2)\left((x_1 \leq x_2) \vee (x_2 \leq x_1)\right).
\end{aligned}$$

- It is **dense** if also

$$\phi_4 : (\forall x_1)(\forall x_2)(\exists x_3)\left((x_1 < x_2) \rightarrow ((x_1 < x_3) \wedge (x_3 < x_2))\right),$$

  where $(x_1 < x_2)$ is an abbreviation for $((x_1 \leq x_2) \wedge (x_1 \neq x_2))$.

- It is **without endpoints** if

$$\begin{aligned}
&\phi_5 : (\forall x_1)(\exists x_2)(x_1 < x_2), \\
&\phi_6 : (\forall x_1)(\exists x_2)(x_2 < x_1).
\end{aligned}$$

- Let

$$\Delta = \{\phi_1, \dots, \phi_6\}.$$

  - $\mathcal{Q} = \langle \mathbb{Q}; \leq \rangle$ is a normal model of Δ.
  - $\mathcal{R} = \langle \mathbb{R}; \leq \rangle$ is also a model of Δ.

Will prove the following.

Theorem 2.7.2.

1. For every closed $\mathcal{L}^=$-formula ϕ,

$$\mathcal{Q} \vDash \phi \qquad \Longleftrightarrow \qquad \mathcal{R} \vDash \phi \qquad \Longleftrightarrow \qquad \Delta \vdash_{\mathcal{L}^=} \phi.$$

2. There is an algorithm which decides, given a closed $\mathcal{L}^=$-formula ϕ, whether $\mathcal{Q} \vDash \phi$ or $\mathcal{Q} \nvDash \phi$, that is $\mathcal{Q} \vDash (\neg\phi)$, by 2.3.3.

Definition 2.7.3. (From 2.3.9 and 10)

1. Linear orders $\mathcal{A} = \langle A; \leq_A \rangle$ and $\mathcal{B} = \langle B; \leq_B \rangle$ are **isomorphic** if there is a bijection $\alpha : A \to B$ such that for all $a, a' \in A$,
$$a \leq_A a' \qquad \Longleftrightarrow \qquad \alpha(a) \leq_B \alpha(a').$$
2. If $\mathcal{A}, \mathcal{B}$ are isomorphic and ϕ is closed, then
$$\mathcal{A} \vDash \phi \qquad \Longleftrightarrow \qquad \mathcal{B} \vDash \phi.$$

Theorem 2.7.4 (Cantor)**.** If $\mathcal{A}, \mathcal{B}$ are countable dense linear orders without endpoints, then $\mathcal{A}, \mathcal{B}$ are isomorphic.

Lemma 2.7.5 (Los-Vaught test)**.** Let $\Sigma = \Sigma_E \cup \Delta$. Then for every closed $\mathcal{L}^=$-formula ϕ, we have either $\Sigma \vdash_{K_{\mathcal{L}^=}} \phi$ or $\Sigma \vdash_{K_{\mathcal{L}^=}} (\neg\phi)$. Say Σ is complete.

Proof. Suppose not. Then as Σ has a model, it is consistent, so we can use 2.5.2 to get that
$$\Sigma_1 = \Sigma \cup \{(\neg\phi)\}, \qquad \Sigma_2 = \Sigma \cup \{(\neg(\neg\phi))\}$$
are consistent. So $\Sigma \cup \{\phi\}$ is consistent. By 2.5.3, 2.6.6, it follows that Σ_1, Σ_2 have countable normal models $\mathcal{A}_1, \mathcal{A}_2$. So $\mathcal{A}_1, \mathcal{A}_2$ are countable dense linear orders without endpoints, and
$$\mathcal{A}_1 \vDash (\neg\phi), \qquad \mathcal{A}_2 \vDash \phi.$$
This contradicts 2.7.4 and 2.7.3.2. ☐

Proof of 2.7.2.

1. Show that
$$\mathcal{Q} \vDash \phi \qquad \Longleftrightarrow \qquad \Sigma \vdash_{K_{\mathcal{L}^=}} \phi.$$
   $\Longleftarrow$ As $\mathcal{Q} \vDash \Sigma$, this is 2.4.7.

   $\Longrightarrow$ If $\Sigma \nvdash_{K_{\mathcal{L}^=}} \phi$, then by 2.7.5, $\Sigma \vdash_{K_{\mathcal{L}^=}} (\neg\phi)$. So $\mathcal{Q} \vDash (\neg\phi)$, so $\mathcal{Q} \nvDash \phi$.

   Similarly,
$$\mathcal{R} \vDash \phi \qquad \Longleftrightarrow \qquad \Sigma \vdash_{K_{\mathcal{L}^=}} \phi.$$
   So
$$\mathcal{R} \vDash \phi \qquad \Longleftrightarrow \qquad \Sigma \vdash_{K_{\mathcal{L}^=}} \phi \qquad \Longleftrightarrow \qquad \mathcal{Q} \vDash \phi.$$
2. Σ is a **recursively enumerable** set of formulas. That is, we can write an algorithm to systematically generate the formulas in Σ. Note that the set of axioms for $K_{\mathcal{L}}$ is also recursively enumerable. So the set of deductions from Σ is recursively enumerable. Therefore the set of consequences of Σ is recursively enumerable. Method is to run the algorithm which generates all consequences of Σ. By 1, at some point, we will see either θ or $(\neg\theta)$. At this point, the method stops.

☐

Note.

- Depends on
  - the completeness theorem, and
  - the axioms Δ for $\mathcal{Q}$,
- Works for some other structures, but can have better algorithms.
- But, there is no such algorithm for
$$\langle \mathbb{N}; +, \cdot, 0 \rangle.$$
  This is Gödel's incompleteness theorem.