Math40003 Linear Algebra and Groups **Problem Sheet 5**

Seen A

A.1. Using Procedure 3.8.3, Calculate the row and column ranks of the following matrices

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$

(a) In row reduced echelon form, this matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There are 2 non-zero rows, so the row-rank of this matrix is 2. If we do the same to the transpose of the matrix, we get the same result, and therefore the column-rank is also 2.

(b) In row reduced echelon form, this matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There are 2 non-zero rows, so the row-rank of this matrix is 2. If we do the same to the transpose of the matrix, we get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

and therefore the column-rank is also 2.

A.2. Which of the following functions $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are linear transformations (Definition 4.1.1.)?

(a) $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 + x_2)$

This is a linear transformation. Let's first check that it preserves addition. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$.

$$\begin{aligned} T(x_1, x_2, x_3) + T(y_1, y_2, y_3) &= (x_1 + x_2 - x_3, 2x_1 + x_2) + (y_1 + y_2 - y_3, 2y_1 + y_2) \\ &= ((x_1 + y_1) + (x_2 + y_2) - (x_3 + y + 3), \\ &\quad\; 2(x_1 + y_1) + (x_2 + y_2)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \end{aligned}$$

Now let's check that it preserves scalar multiplication. Let $(x_1, x_2, x_3) \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$.

$$\begin{aligned} \alpha T(x_1, x_2, x_3) &= \alpha(x_1 + x_2 - x_3, 2x_1 + x_2) \\ &= (\alpha x_1 + \alpha x_2 - \alpha x_3, 2\alpha x_1 + \alpha x_2) \\ &= T(\alpha x_1, \alpha x_2, \alpha x_3) \end{aligned}$$

Since T preserves both addition and scalar multiplication, it is a linear transformation.

(b) $T(x_1, x_2, x_3) = (0, \sqrt{2}x_3)$ This is a linear transformation. Let's first check that it preserves addition. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$.

$$\begin{aligned} T(x_1, x_2, x_3) + T(y_1, y_2, y_3) &= (0, \sqrt{2}x_3) + (0, \sqrt{2}y_3) \\ &= (0, \sqrt{2}(x_3 + y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \end{aligned}$$

Now let's check that it preserves scalar multiplication. Let $(x_1, x_2, x_3) \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$.

$$\begin{aligned} \alpha T(x_1, x_2, x_3) &= \alpha(0, \sqrt{2}x_3) \\ &= (0, \sqrt{2}\alpha x_3) \\ &= T(\alpha x_1, \alpha x_2, \alpha x_3) \end{aligned}$$

Since T preserves both addition and scalar multiplication, it is a linear transformation.

(c) $T(x_1, x_2, x_3) = (x_1 x_2, x_3)$ This is not a linear transformation. There are a lot of counterexamples to preserving addition and preserving scalar multiplication, but I think this is the simplest:

$$\begin{aligned} T(1,1,0) + T(1,1,0) &= (2,0) \\ T(2,2,0) &= (4,0) \end{aligned}$$

(d) $T(x_1, x_2, x_3) = (0, 0)$ This is the simplest of all! Since $T(x_1, x_2, x_3) = T(y_1, y_2, y_3)$ for all (x_1, x_2, x_3) and (y_1, y_2, y_3), both preservation properties hold immediately, and therefore T is a linear transformation.

A.3. For the first of these questions, try working out what a linear transformation that does the required mappings would do to the standard basis elements.

(a) Find a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which sends $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$. If our T is linear, and it maps $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$, then it must map $(0, 1) = (1, 1) + (-1)(1, 0)$ to

$$(1, 1, 0) - (1, 0, -1) = (0, 1, 1)$$

Therefore a linear transformation with the desired properties is

$$\begin{aligned} T(x, y) &= x(1,1,0) + y(0,1,1) \\ &= (x, x+y, y) \end{aligned}$$

(b) Find two different linear transformations $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which send $(1, 1, 0)$ to $(1, 1)$ and $(0, 1, 1)$ to $(0, 1)$. The functions

$$\begin{aligned} T(x, y, z) &= (x, y) \\ S(x, y, z) &= (x - z, z - x) \end{aligned}$$

both have the desired properties.