Mathematics Year 1, Calculus and Applications I

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Problem Sheet 7

1. The following functions are defined on the interval $[0, \pi]$. In each case (i) find the even and odd extensions of the given functions on $[-\pi, \pi]$ and extend them periodically with period 2π on the real line; (ii) sketch these over the interval $-4\pi < x < 4\pi$ making sure you include the assumed values of the function at any discontinuities; (iii) find the Fourier series for both even and odd extensions and state whether the convergence of the series is uniform or not. [You can state theorems without proof.]

$$f(x) = \cos x, \qquad f(x) = x^2, \qquad f(x) = e^x, \qquad f(x) = e^x - 1.$$

By inspecting your sketches, which of the Fourier series can be differentiated term-by-term to yield the Fourier series of new functions? Explain using theorems without proofs.

2. Obtain the Fourier series of the function $f(x) = \pi x$ on the interval $0 \le x \le 1$ as a sine series and a cosine series (extend the function appropriately and note that the interval is $2-$periodic not $2\pi-$periodic).

3. (a) Sketch the function $f(x) = |\sin x|$ defined on $-\pi \le x \le \pi$, and show that its Fourier series is given by

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

   (b) What value does the Fourier series converge to at $x = 0, \pi, -\pi$?

   (c) Use the series result to show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

   (d) Use your results to also show that

$$\sum_{n=1}^{\infty} \frac{1}{4(2n-1)^2 - 1} = \frac{1}{4\cdot 1} + \frac{1}{4\cdot 3^2 - 1} + \frac{1}{4\cdot 5^2 - 1} + \ldots = \frac{\pi}{8}$$

4. (a) Consider the function $f(x) = x\cos x$ on $-\pi < x < \pi$. Sketch the function. Is it even or odd?

   (b) Find the Fourier series of $f(x)$ extended periodically over the whole of the real line. What values does the series converge to at $x = -\pi, +\pi$?

   (b) Now introduce the function $\phi(x) = x$ on $-\pi < x < \pi$. Write down the Fourier series for $\phi(x)$ (extended periodically on the real line) and hence show that the Fourier series of $\chi(x) := x(1 + \cos x)$ (extended periodically on the real line) is given by

$$\chi(x) = \frac{3}{2}\sin x + 2\left(\frac{\sin 2x}{1\cdot 2\cdot 3} - \frac{\sin 3x}{2\cdot 3\cdot 4} + \frac{\sin 4x}{3\cdot 4\cdot 5} + \ldots\right) \tag{1}$$

   (c) What values do you expect the Fourier series of $\chi(x)$ to converge to at the end points $x = -\pi$ and $x = \pi$? Is the periodic extension of χ continuous at the end points? Is the convergence uniform or not?

(d) Does the periodically extended function $\chi(x)$ have continuous derivatives of any order on the closed interval $[-\pi, \pi]$ (clearly the problematic points are the end points, so you may find it useful to carry out a local one-sided Taylor series expansion).

By considering the Fourier series (1) can you think of a series comparison test that would establish its absolute convergence for all $x \in [-\pi, \pi]$?

5. Consider the function $f(x) = \cos \alpha x$ for $-\pi < x < \pi$, where α is not an integer.

(a) Show that the Fourier series of $f(x) = \cos \alpha x$ is

$$\cos \alpha x = \frac{2\alpha \sin \alpha\pi}{\pi} \left(\frac{1}{2\alpha^2} - \frac{\cos x}{\alpha^2 - 1^2} + \frac{\cos 2x}{\alpha^2 - 2^2} + \dots \right) \tag{2}$$

(b) Confirm that the periodic extension of the function remains continuous at $x = \pm\pi$. Hence, select $x = \pi$ in (2) to show that the following expression holds

$$\cot \pi x = \frac{2x}{\pi} \left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \dots \right). \tag{3}$$

This expression resolves $\cot \pi x$ into partial fractions!

(c) Re-write (3) in the form

$$\pi \left(\cot \pi x - \frac{1}{\pi x} \right) = -2x \left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \dots \right), \tag{4}$$

and take x to lie in the interval $0 \le x \le \beta < 1$. Show that the series (4) converges uniformly in the given interval and can therefore be integrated term-by-term (consider the *nth* term and bound its absolute value by the term of a known convergent series).

(d) Integrate (4) from 0 to x and show that (careful with improper integrals at $x = 0$)

$$\log \left(\frac{\sin \pi x}{\pi x} \right) = \lim_{n\to\infty} \log \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2} \right). \tag{5}$$

(e) Show that (5) is equivalent to (exponentiate both sides)

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2} \right) \left(1 - \frac{x^2}{2^2} \right) \left(1 - \frac{x^2}{3^2} \right) \dots$$

Show how your expression above can be used to produce the so-called *Wallis's* product

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \dots$$

6. (*You may have never seen a partial differential equation but you have learned plenty to be able to solve the following Calculus problem.*)

The evolution of the wave amplitude $u(x,t)$ in a nonlinear system is given by[1]

$$u_t + uu_x = u + \varepsilon u_{xx}, \tag{6}$$

where $\varepsilon > 0$ and subscripts denote partial derivatives, e.g. $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc. The wave amplitude is a function of time t and a single spatial variable x. In addition, the motion is spatially periodic, that is

$$u(x+2\pi,t) = u(x,t), \qquad x \in [-\pi,\pi].$$

Define the L^2-norm (or "energy" norm) of a function $f(x,t)$ by

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x,t)dx\right)^{1/2}.$$

(i) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u\,u_t\,dx = \frac{1}{2}\frac{d}{dt}\|u\|^2.$$

(ii) By multiplying (6) by $u(x,t)$ and integrating over $-\pi \le x \le \pi$, show that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \|u\|^2 - \varepsilon\|u_x\|^2.$$

(iii) Use Parseval's Theorem to find an upper bound of $\|u\|^2 - \varepsilon\|u_x\|^2$ involving $\|u\|^2$, and hence show that when $\varepsilon > 1$ then $u(x,t) \to 0$ as $t \to \infty$ starting from fairly arbitrary initial conditions $u(x,0) = u_0(x)$.

[1] This equation is called the Burgers-Sivashinsky equation that has been analysed by J. Goodman 1994 *Stability of the Kuramoto-Sivashinsky and related systems*, Communications on Pure and Applied Mathematics, Vol. XLVII, 293–306.