# Relationships between Probability Distributions

Note: The gamma distribution mentioned in this notes uses parameters shape and rate: i.e. Gamma($\alpha,\beta$) where $\alpha$ is shape, $\beta$ is rate. There is a different parametrisation using shape and scale

| DISCRETE DISTRIBUTIONS | | | | | | | |
|---|---|---|---|---|---|---|---|
| | range $\mathbb{X}$ | parameters | pmf $f_X$ | cdf $F_X$ | E[X] | Var[X] | mgf $M_X$ |
| $Bernoulli(\theta)$ | $\{0,1\}$ | $\theta \in (0,1)$ | $\theta^x(1-\theta)^{1-x}$ | | $\theta$ | $\theta(1-\theta)$ | $1-\theta+\theta e^t$ |
| $Binomial(n,\theta)$ | $\{0,1,...,n\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ | | $n\theta$ | $n\theta(1-\theta)$ | $(1-\theta+\theta e^t)^n$ |
| $Poisson(\lambda)$ | $\{0,1,2,...\}$ | $\lambda \in \mathbb{R}^+$ | $\dfrac{e^{-\lambda}\lambda^x}{x!}$ | | $\lambda$ | $\lambda$ | $\exp\{\lambda(e^t-1)\}$ |
| $Geometric(\theta)$ | $\{1,2,...\}$ | $\theta \in (0,1)$ | $(1-\theta)^{x-1}\theta$ | $1-(1-\theta)^x$ | $\dfrac{1}{\theta}$ | $\dfrac{(1-\theta)}{\theta^2}$ | $\dfrac{\theta e^t}{1-e^t(1-\theta)}$ |
| $NegBinomial(n,\theta)$ | $\{n,n+1,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$ | | $\dfrac{n}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$ |
| or | $\{0,1,2,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n+x-1}{x}\theta^n(1-\theta)^x$ | | $\dfrac{n(1-\theta)}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$ |

| CONTINUOUS DISTRIBUTIONS | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | parameters | pdf | cdf | E[X] | Var[X] | mgf |
| $Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$) | $(\alpha,\beta)$ | $\alpha < \beta \in \mathbb{R}$ | $\dfrac{1}{\beta-\alpha}$ | $\dfrac{x-\alpha}{\beta-\alpha}$ | $\dfrac{(\alpha+\beta)}{2}$ | $\dfrac{(\beta-\alpha)^2}{12}$ | $\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$ |
| $Exponential(\lambda)$ (stand. model $\lambda=1$) | $\mathbb{R}^+$ | $\lambda \in \mathbb{R}^+$ | $\lambda e^{-\lambda x}$ | $1-e^{-\lambda x}$ | $\dfrac{1}{\lambda}$ | $\dfrac{1}{\lambda^2}$ | $\left(\dfrac{\lambda}{\lambda-t}\right)$ |
| $Gamma(\alpha,\beta)$ (stand. model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta \in \mathbb{R}^+$ | $\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ | | $\dfrac{\alpha}{\beta}$ | $\dfrac{\alpha}{\beta^2}$ | $\left(\dfrac{\beta}{\beta-t}\right)^\alpha$ |
| $Weibull(\alpha,\beta)$ (stand. model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta \in \mathbb{R}^+$ | $\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$ | $1-e^{-\beta x^\alpha}$ | $\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$ | $\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$ | |
| $Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$) | $\mathbb{R}$ | $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$ | $\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$ | | $\mu$ | $\sigma^2$ | $e^{\{\mu t+\sigma^2 t^2/2\}}$ |
| $Student(\nu)$ | $\mathbb{R}$ | $\nu \in \mathbb{R}^+$ | $\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$ | | $0$ (if $\nu>1$) | $\dfrac{\nu}{\nu-2}$ (if $\nu>2$) | |
| $Pareto(\theta,\alpha)$ | $\mathbb{R}^+$ | $\theta,\alpha \in \mathbb{R}^+$ | $\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$ | $1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$ | $\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$) | $\dfrac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha>2$) | |
| $Beta(\alpha,\beta)$ | $(0,1)$ | $\alpha,\beta \in \mathbb{R}^+$ | $\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | | $\dfrac{\alpha}{\alpha+\beta}$ | $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |

Binomial distribution is sum of iid Bernoulli

$$X \sim \text{Binomial}(n, \theta) \implies X = \sum_{i=1}^{n} X_i \text{ where } X_i \sim \text{Bernoulli}(\theta)$$

Poisson distribution can be cut into n iid small fragments

$$X \sim \text{Poisson}(\lambda) \implies X = \sum_{i=1}^{n} X_i \text{ where } X_i \sim \text{Poisson}(\lambda/n), \text{ so that } \mu = E(X_i) = \lambda/n$$
$$\text{and } \sigma^2 = Var(X_i) = \lambda/n$$

Negative Binomial is sum of iid geometric distributions

$$X \sim \text{Negative Binomial}(n, \theta) \implies X = \sum_{i=1}^{n} X_i \text{ where } X_i \sim \text{Geometric}(\theta), \text{ so that }$$
$$\mu = E(X_i) = 1/\theta \text{ and } \sigma^2 = Var(X_i) = (1 - \theta)/\theta^2, \text{ and hence}$$

## Linearity of Normal distribution
if $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, then
$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
$X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

if $X \sim N(\mu, \sigma^2)$, then $kX \sim N(k\mu, k^2\sigma^2)$

## Student t-distribution
If $X_1, X_2, ..., X_n \sim N(0, \sigma^2)$ where $\sigma^2$ is unknown, then
$(\bar{x}_n - \mu) / (S/\sqrt{n}) \sim t_{n-1}$ where $\bar{x}_n$ is mean of $X_1, X_2, ..., X_n$

## Chi-squared
$\chi^2_n = \sum Z_i^2$ where $Z_i$ are iid $N(0, 1)$
Therefore, for integer k, sum of k iid $\chi^2_n$ is $\chi^2_{kn}$

## Exp and gamma
$\text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$
sum of n iid $\text{Gamma}(1, \lambda) \sim \text{Gamma}(n, \lambda)$, sum of k iid $\text{Gamma}(\lambda, 1) \sim \text{Gamma}(\lambda, k)$
so sum of n iid $\text{Exp}(\lambda)$ is $\text{Gamma}(n, \lambda)$

$\chi^2_n \sim \text{Gamma}(n/2, 1/2)$
so $\text{Exp}(1/2) = \chi^2_2$

$\text{Gamma}(a, \lambda) = \text{Gamma}(a, 1) / \lambda$

so $2\lambda$ Gamma$(n, \lambda)$ = Gamma$(n, 1/2)$ = $\chi^2_{2n}$

## Gamma and F-distribution

If $U \sim$ Gamma$(\alpha_1, \beta_1)$, $V \sim$ Gamma$(\alpha_2, \beta_2)$, then

$\alpha_2\,\beta_1 U \,/\, \alpha_1\,\beta_2 V \sim F_{2\alpha_1,\ 2\alpha_2}$

## Dirichlet Distribution :

A general result is that if for $i = 1, 2, \ldots k$, $U_i \sim \Gamma(\alpha_i, \beta)$ independent, then

$$\frac{1}{\sum_{i=1}^{k} U_i}(U_1, U_2, \ldots U_k) \sim \mathrm{DIRICHLET}(\alpha_1, \alpha_2, \ldots \alpha_k),$$