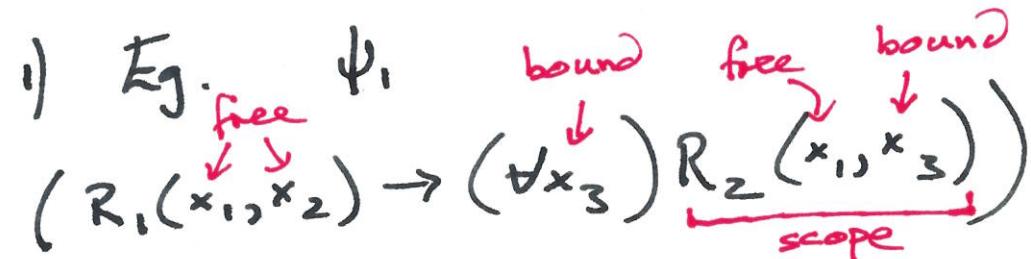


① 2.3 ① Bound + free variables.

i) Eg. ψ_1 . 

(2.3.1) Def. Suppose ϕ, ψ are L-formulas and $(\forall x_i)\phi$ occurs as a subformula of ψ
ie. ψ is $(\forall x_i)\phi$

We say that ϕ is the scope of $(\forall x_i)$ here.

An occurrence of a variable x_j in ψ is bound if it is within the scope of a quantifier

$(\forall x_j)_\alpha$ in ψ
(or it is this x_j)

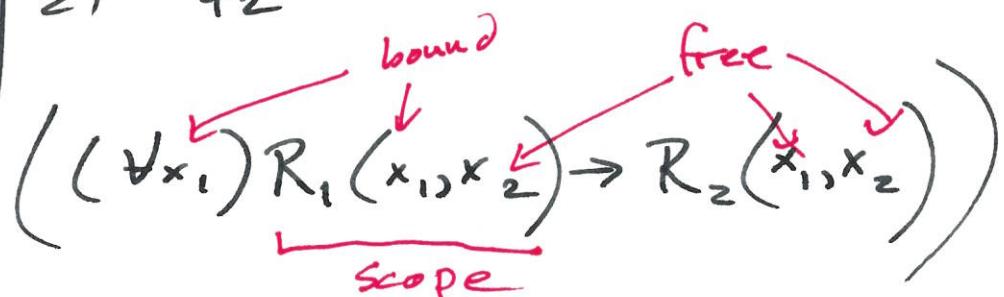
Otherwise it is a free occurrence of x_j in ψ .

Free variables in ψ : variables with a free occurrence in ψ .

= An L-formula with no free variables is called a closed formula (or a sentence).

Examples.

2) ψ_2



3) Compare with ψ_3

$$(\forall x_1)(\underbrace{R_1(x_1, x_2)}_{\text{scope}} \rightarrow R_2(x_1, \underbrace{x_2}_{\text{free variable } x_2}))$$

4) $\psi_4 ((\exists x_3) \underbrace{R_1(x_1, x_2)}_{\text{scope}} \rightarrow (\forall x_2) \underbrace{R_2(x_2, x_3)}_{\text{scope}}))$

(2.3.2) Notation If ψ is an L-functor with free variables amongst x_1, \dots, x_n

we might sometimes write $\psi(x_1, \dots, x_n)$ to show this.

If t_1, \dots, t_n are terms, then by $\psi(t_1, \dots, t_n)$ we mean
the L-functor obtained by replacing each free occurrence of x_i
in ψ by t_i (for $i = 1, \dots, n$).

(3)

Eg. $\psi(x_1, x_2)$

$$((\forall x_1) R_1(x_1, \boxed{x_2}) \rightarrow (\forall x_3) R_3(\boxed{x_1}, \boxed{x_2}, x_3))$$

t_1	$f_1(x_1)$
t_2	$f_2(x_1, x_2)$

$$\psi(t_1, t_2)$$

$$((\forall x_1) R_1(x_1, f_2(x_1, x_2)) \rightarrow (\forall x_3) R_3(f_1(x_1), f_2(x_1, x_2), \boxed{x_3}))$$

(2.3.3) Theorem.

Suppose ϕ is a closed L-formula
and \mathcal{A} is an L-structure.

then either $\mathcal{A} \models \phi$

or $\mathcal{A} \models (\neg \phi)$.

More generally if ϕ has
free variables amongst x_1, \dots, x_n
and v, w are valuations in \mathcal{A}
with $v(x_i) = w(x_i)$ for $i \leq n$

then $v[\phi] = T \Leftrightarrow w[\phi] = T$.

(allow $n=0$ here : i.e. no free
variables).

Pf: The first statement follows
from the second: if ϕ
has no free variables any
two valuations v, w in \mathcal{A}

agree on the free variables in ϕ (4)
So $v[\phi] = w[\phi]$. //

Prove the generalisation by
induction on the number of
connectives & quantifiers in ϕ .

Base case. ϕ is atomic i.e.

ϕ is $R(t_1, \dots, t_m)$
 t_1, \dots, t_m terms.

the only variables used in
 t_1, \dots, t_m are amongst x_1, \dots, x_n .

So $v(t_j) = w(t_j)$ for
 $j = 1, \dots, m$.

(see 2.2.5).

Then $v[R(t_1, \dots, t_m)] = T$

$\Leftarrow \bar{R}(v(t_1), \dots, v(t_m))$
 holds in \mathcal{A}
 $\Leftarrow \bar{R}(w(t_1), \dots, w(t_m))$
 holds
 $\Leftarrow w[\phi] = T, //$

Inductive step. ϕ is
 $(\neg\psi), (\psi \rightarrow x)$ or
 $(\forall x_i)\psi$

First two cases: Ex.
 Suppose ϕ is $(\forall x_i)\psi$
 Suppose $v[\phi] = F$
 Want to show $w[\phi] = F$.
 By Def 2.2.7 there is

a valuation v' (in \mathcal{A})
 with v' x_i -equivalent to v
 and $v'[\psi] = F$.
 The free variables of ψ
 are amongst x_1, \dots, x_n, x_i
 Let w' be the valuation
 x_i -equivalent to w with
 $w'(x_i) = v'(x_i)$.
Claim: v', w' agree on the
free variables in ψ .
 Given this the 'nd-hyp. applied)
 to ψ gives $w'[\psi] = v'[\psi] = F$.
 w' is x_i -equiv. to w
 so $w[(\forall x_i)\psi] = F$
 i.e. $w[\phi] = F$.

Pf of Claim:

Case 1 $i \leq n$.

Free vars. of ψ are amongst

x_1, \dots, x_n .

If $j \leq n$ & $j \neq i$ then

$$\begin{aligned} v'(x_j) &= v(x_j) = w(x_j) \\ &= w'(x_j) \end{aligned}$$

Also $w'(x_i) = v'(x_i)$

so v', w' agree on x_1, \dots, x_n .

Case 2 $i > n$.

Same argument works!.

~~II~~

~~III~~