Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Introduction to Infinite Dimensional Analysis

Date: Wednesday, 27 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

Notation

$\mathbb{N}$ set of natural numbers;
$\mathbb{R}$ set of real numbers;
$\lambda_n(dx)$ the Lebesgue measure in $\mathbb{R}^n$;
Δ - Laplacian and ∇ - gradient in $\mathbb{R}^n$;

Q1.

(1.i) Define the generator L of a C_0 - semigroup on a Banach space and prove that it is densely defined and closed.

(1.ii) Which of the following operators is the generator of a Markov semigroup on a suitable space of functions ?

(1.ii.a) $E - I$, where E denotes a regular conditional expectation and I the identity operator;

(1.ii.b) $E_1 E_2 - I$, where $E_i, i = 1, 2$, denote different regular conditional expectations.

(1.iii) Which of the following operators is the generator of a symmetric semigroup on a suitable space of functions ?

(1.iii.a) $E_1 E_2 - I$, where $E_i, i = 1, 2$, denote different regular conditional expectations;

(1.iii.b) $\Delta - x \cdot \nabla$.

Q2.

(2.i) Prove that if a probability measure μ on real line satisfies Log-Sobolev Inequality, then the following exponential bound is true

$$\mu e^{tf} \leq e^{Ct^2 |||f|||^2 + t\mu f}$$

with some constant $C \in (0, \infty)$ for any $t \in \mathbb{R}$ and any real function f for which its Lipschitz semi-norm $|||f|||$ is finite.

(2.ii) Prove or disprove that a measure of the form

$$d\mu \equiv e^{-|x|^\alpha} d\lambda / \int e^{-|x|^\alpha} d\lambda$$

does not satisfy a Log-Sobolev Inequality if $\alpha < 1$.

(2.iii) Give an example of a probability measure different from $d\nu_m \equiv e^{-m|x|} d\lambda / \int e^{-m|x|} d\lambda$, $m \in (0, \infty)$, for which the Poincare Inequality holds but the Log-Sobolev Inequality fails.

Q3.

(3.i) Give the defintion of

(3.i.a) contractive,
(3.i.b) ultracontractive
(3.i.c) hypercontractive

semigroup.

(3.ii) Prove or disprove that the following semigroups are

- ultracontractive;
- hypercontractive.

(3.ii.a) $e^{-\varepsilon t}f + (1 - e^{-\varepsilon t})Ef$ in $L_p(\mu)$, $p \in [1, \infty]$, where E is a conditional expectation associated to a probability measure μ and $\varepsilon > 0$;

(3.ii.b) $\frac{1}{(2\pi t)^{n/2}} \int e^{-\frac{1}{2t}|y-x|^2} f(y)\lambda_n(dx)$ in $L_p(\lambda_n)$, $p \in [1, \infty]$;

(3.ii.c) $\int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right)\mu(dy)$, where μ denotes standard Gaussian measure on real line.

Q4.

(4.i) Let $P_t = e^{tL}$ be a diffusion semigroup given by a Laplace-Beltrami operator on a compact boundaryless Riemannian manifold of dimension $n \in \mathbb{N}$ with non-negative Ricci curvature. Prove a short time regularity estimate of the form
$|\nabla P_t f|^2 \leq \frac{C}{t^\beta} \|f\|_u^2$
with some constants $C, \beta \in (0, \infty)$ and $\|f\|_u \equiv \sup |f|$.

(4.ii) Let $L \equiv X^2 + \kappa Y + \gamma Z$ where X, Y are generators of Heisenberg group satisfying $[X, Y] = -Z$ and $\kappa, \gamma \in \mathbb{R} \setminus \{0\}$. Prove or disprove a short time regularity estimate of the form

$|XP_t f|^2 \leq \frac{C}{t^\beta} \|f\|_u^2$
with some constants $C, \beta \in (0, \infty)$ and $\|f\|_u \equiv \sup |f|$.

Hint: Study suitable time dependent quadratic form.