ANALYSIS 1

Fall 2024

Lecturer: Marco Guaraco.

Office hours: ~~Friday 5:00 PM HXLY 440~~
~~(starting on Jan 19th).~~

Problem Sessions: ~~Tuesday 2PM HXLY 340, 341/342~~

ASSESMENT and FEEDBACK

Please check BB and Syllabus.

TOPICS

(see the detailed schedule on the Syllabus / BB)

PART I - CONTINUITY

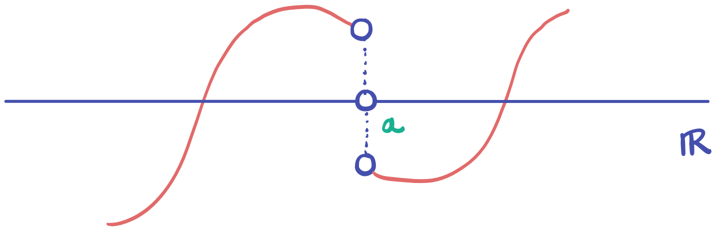
PART II - THE DERIVATIVE

PART III - THE INTEGRAL

PART IV - TAYLOR'S THEOREM & SERIES

What is Calculus/Analysis about?

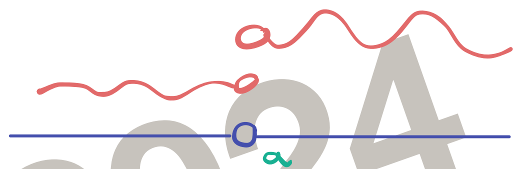
- Behavior of a function near a point.



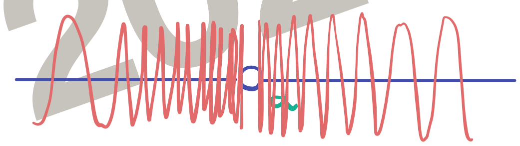
$$f : \mathbb{R} \setminus \{a\} \longrightarrow \mathbb{R}$$

Questions. What is f doing near a?

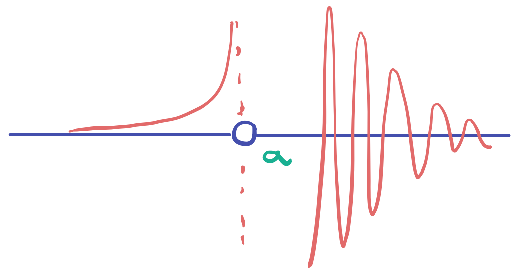
Is it jumping?



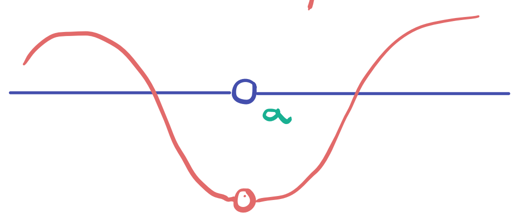
Is it oscillating?



Is it going to $\pm\infty$?



Is it converging to a value?



Are these formal mathematical notions?

We are talking about real numbers $\mathbb{R}$

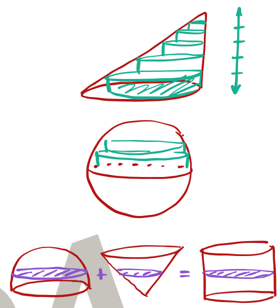
$\{ \mathbb{R}, \pm, \times, \leq, |\cdot|,$ sup and inf. $\}$

There is no "near", "jumping", "converging" or "oscillating"!

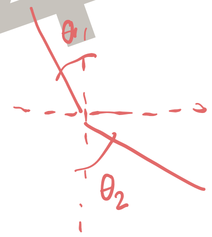
Nowadays, part of the role of Calculus / Analysis is translating these intuitive notions into the language of the real numbers

However, this was not always the case

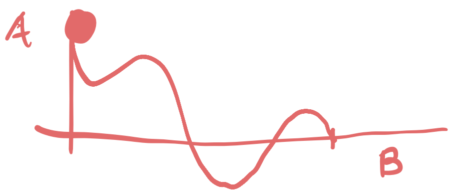
- The ancient Greeks computed the volumes of cones and spheres using arguments that require the notion of integral.



- Fermat showed that Snell's law of refraction follow from the fact that the derivative $f'(x)$ a minimum of f is 0.



- Bernoulli, Newton, Leibnitz, L'hôpital worked with integrals, derivatives in multiple dimensions before having a notion of limit!



- Cauchy famously assumed that the limit of a sequence of continuous functions is also a continuous function
A statement you will easily proof is false.

- Riemann famously assumed the existence of the minimum of a positive continuous function on an open set. You will also be able to show this is false easily!

Part of the problem was that the basic notions were intuitive but ambiguous.. So it was impo_ssible to prove theorems with sufficient rigour.

It was not until Weierstrass that Analysis took the form it has today. Building on works by Cauchy, Bolzano and others he came up with a notion of limit that allows us to reduce ambiguities to a workable minimum.

It goes as follows...

LIMITS

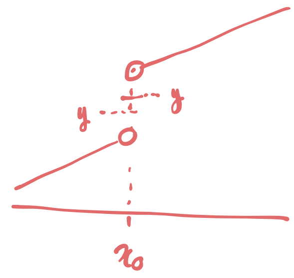
Let
$$\begin{cases} f : (a,b) \setminus \{x_0\} \longrightarrow \mathbb{R} \\ a < x_0 < b \\ y \in \mathbb{R} \end{cases}$$

we want to define what it means:

"$f(x)$ converges to y as $x \to x_0$"

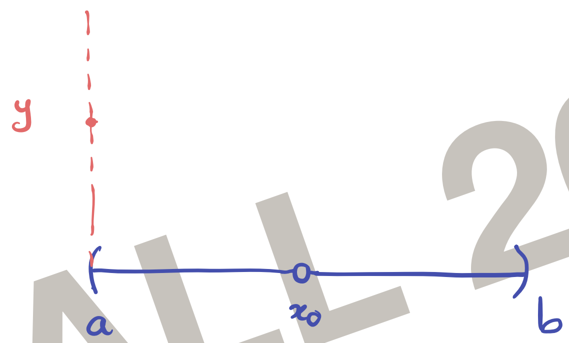
$$\lim_{x \to x_0} f(x) = y$$

Attempt: it means that the values of f get closer to y, the closer x is to x_0



Is this what we want?

Can you give me a counter-example?



In this example we see that the values of $f(x)$ are getting closer to y but they also stay far away from y no matter how close x is to x_0.

Therefore, our 1st attempt is incorrect!

But we learned something: we have to

avoid staying far away!

Being far away is a statement about distances. In $\mathbb{R}$, distances are measured by $|\cdot|$:

distance between x and x_0 is $|x - x_0|$

distance between $f(x)$ and y is $|f(x) - y|$

We could read:

"$|x - x_0| = d$" as "x is at a distance d from x_0"

"$|x - x_0| < d$" as "x is at a distance less than d from x_0".

"$x \in (x_0 - d, x_0 + d)$"
"$x_0 \in (x - d, x + d)$"

Since we want to avoid $f(x)$ being far from y, this is the same as saying:

$|f(x) - y|$ is arbitrarily small IF x is close enough to x_0.

DEFINITION: Given $a < x_0 < b$ and

$f : (a,b) \setminus \{x_0\} \longrightarrow \mathbb{R}$ we define

$$\lim_{x \to x_0} f(x) = y$$

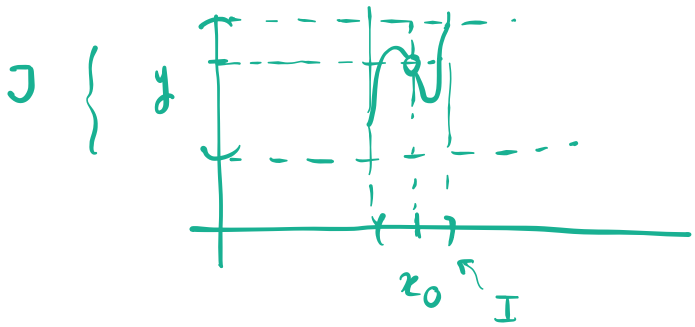
as

For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$

such that $|x - x_0| < \delta$ and $x \neq x_0$

implies $|f(x) - y| < \varepsilon$

Definition: We say that $f(x)$ converges to y as x goes to x_0, if for any open interval $J \subset \mathbb{R}$ containing y, there exists an open interval $I \subset \mathbb{R}$, containing x_0, such that $f(I \setminus \{x_0\}) \subset J$.



EXAMPLE: Prove that $\lim_{x \to x_0} x = x_0$.

Proof:

EXAMPLE: Prove that $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$, as long as $x_0 \neq 0$.

Idea: Let $\varepsilon > 0$ be arbitrary, but fixed. We must produce $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{x_0} \right| < \varepsilon.$$

Let $\delta > 0$, be a positive number to be chosen later. Then

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x_0 - x}{x x_0} \right| \leq \frac{\delta}{|x x_0|}$$

we cannot choose δ as $\varepsilon = \dfrac{\delta}{|x x_0|}$,

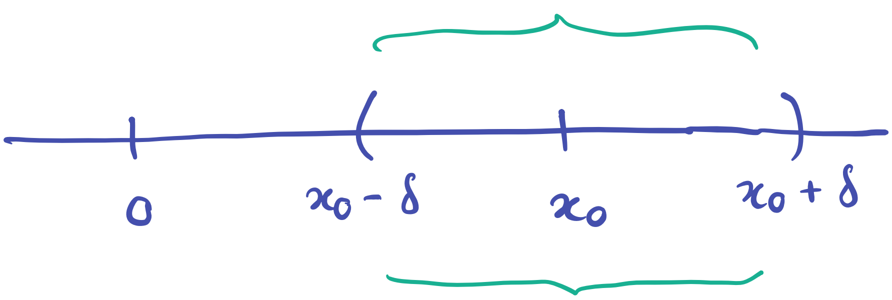
because δ cannot depend on a variable x!

To continue with the inequality we need to bound $\dfrac{1}{|x|}$ from above by some $c > 0$

i.e. $\dfrac{1}{|x|} < c \iff \dfrac{1}{c} \leq |x|$

All we need is to keep $|x|$ far from 0 by a fixed amount.

x lives here $\leftrightarrow$ $|x - x_0| < \delta$



It is enough if this interval does not touch 0.

Perhaps we can restrict our δ to be $\delta > 0$

and smaller than distance$(x_0, 0) = |x_0|$ by a fixed amount.

Let's see if that works.

If $|x - x_0| < \delta < \dfrac{|x_0|}{2}$ it should be true from the picture that $|x| > |x_0|/2$.

Of course, a picture is not a proof.

However

$$
\begin{aligned}
|x_0| &\leq |x_0 - x| + |x| && (\text{triangle ineq}) \\
|a+b| &\leq |a| + |b| \\
&\leq \delta + |x| && (|x_0 - x| < \delta) \\
&\leq |x_0|/2 + |x| && (\delta < |x_0|/2)
\end{aligned}
$$

$$\Rightarrow \quad |x_0|/2 \leq |x|$$

which is what we wanted

Now we are ready to write the formal proof.

EXAMPLE: Prove that $\displaystyle\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$, as long as $x_0 \neq 0$.

Proof: Let $\delta \in (0, |x_0|/2)$ to be choosen later.

Note that $|x - x_0| < \delta$ implies that $|x| \geq |x_0|/2$

In fact, by the triangle inequality

$$|x_0| \leq |x_0 - x| + |x|$$

since $|x_0 - x| < \delta < |x_0|/2$ we obtain by transitivity, that

$$|x_0| \leq |x_0|/2 + |x|.$$

substracting $|x_0|/2$ to both sides we get.

$$|x| \geq |x_0|/2. \Rightarrow 1/|x| \leq \frac{2}{|x_0|}$$

Now, note that.

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x_0 - x|}{|x_0 \cdot x|}$$

since $|x_0 - x| < \delta$ and $1/|x| \leq 2/|x_0|$

this implies

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| \leq \frac{2\delta}{|x_0|^2}$$

If $\delta < \varepsilon \cdot \frac{|x_0|^2}{2}$ this would imply our claim. Therefore, it is enough to choose

$$\delta \in (0, |x_0|/2) \cap (0, \varepsilon|x_0|^2/2)$$

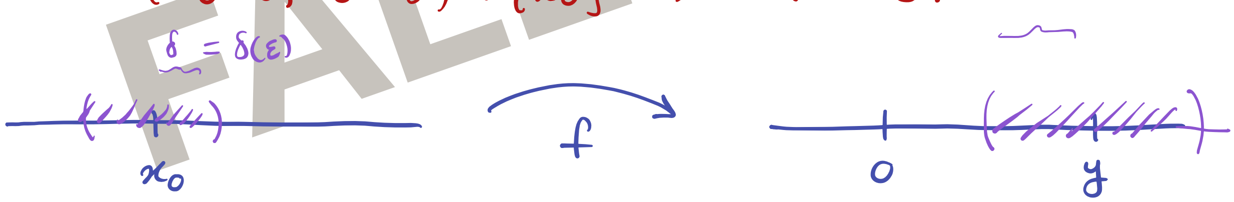
i.e. any

$$0 < \delta < \min\left\{ \frac{|x_0|}{2}, \frac{\varepsilon|x_0|^2}{2} \right\}$$

Proposition If $\lim_{x \to x_0} f(x) = y > 0$ then there exists $\delta > 0$, such that

$$x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \Rightarrow f(x) > 0.$$

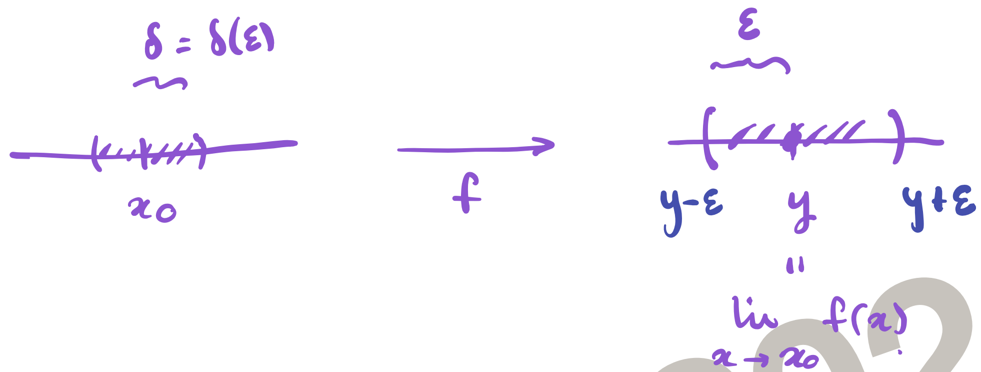
$y/2 = \varepsilon$



Idea for Proof:

Choose $\varepsilon = y/2$ and appy the defintion of $\lim_{x \to x_0} f(x) = y$. Complete the proof as an exercise.

Proposition: If $\lim_{x \to x_0} f(x)$ exists then there is an open interal I containing x_0, such that f is bounded in $I \setminus \{x_0\}$.



Proof: Choose any $\varepsilon > 0$. Let $y \in \mathbb{R}$ be $y = \lim_{x \to x_0} f(x)$ that exists by assumption. In particular, $\exists\ \delta > 0$ such that

$$\begin{cases} |x - x_0| < \delta \\ x \neq x_0 \end{cases} \Rightarrow \quad |f(x) - y| < \varepsilon.$$

$$\Rightarrow \quad -\varepsilon < f(x) - y < \varepsilon$$

$$\Rightarrow \quad y - \varepsilon < f(x) < y + \varepsilon$$

It is enough to choose

$$I = (x_0 - \delta, x_0 + \delta) \qquad \blacksquare$$

Proposition (uniqueness of limits):

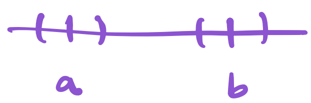
If $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} f(x) = b$

then $a = b$.

Proof: Fix an $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = a$, there exists a $\delta_1 > 0$, such that

$$\begin{cases} |x - x_0| < \delta_1 \\ x \neq x_0 \end{cases} \Rightarrow |f(x) - a| < \varepsilon$$

Similarly, since $\lim_{x \to x_0} f(x) = b$, the exists a $\delta_2 > 0$, such that



$$\begin{cases} |x - x_0| < \delta_2 \\ x \neq x_0 \end{cases} \Rightarrow |f(x) - b| < \varepsilon.$$

$$\begin{aligned} |a - b| &= |a - f(x) + f(x) - b| \\ &\leq |a - f(x)| + |f(x) - b| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

If $\delta = \min\{\delta_1, \delta_2\}$. Therefore $|a - b| < 2\varepsilon$ for any $\varepsilon > 0 \Rightarrow |a - b| = 0 \Rightarrow a = b$ ∎

Proposition: (Squeeze theorem)

Let f, g and h be functions defined on

$$(a, b) \setminus \{x_0\}$$

where $a < x_0 < b$. Assume for all $x \in (a,b) \setminus \{x_0\}$

$$f(x) \leq g(x) \leq h(x).$$

If the limits $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} h(x)$ exist and coincide, then $\lim_{x \to x_0} g(x)$ exists and

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x).$$

Proof: Fix an $\varepsilon > 0$. Let $y = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$ where $y \in \mathbb{R}$, which exists by assumption. Then, there are $\delta_1, \delta_2 > 0$ such that

$$\begin{cases} |x - x_0| < \delta_1 \\ x \neq x_0 \end{cases} \Longrightarrow \quad |f(x) - y| < \varepsilon$$

$$\begin{cases} |x - x_0| < \delta_2 \\ x \neq x_0 \end{cases} \Longrightarrow \quad |h(x) - y| < \varepsilon$$

Therefore, if $\delta = \min(\delta_1, \delta_2)$ we have $\delta > 0$ and

$$\begin{cases} |x - x_0| < \delta \\ x \neq x_0 \end{cases} \Longrightarrow \quad \begin{array}{c} |f(x) - y| < \varepsilon \\ \text{and} \\ |h(x) - y| < \varepsilon \end{array}$$

In particular, since $f(x) \leq g(x) \leq h(x)$

we have:

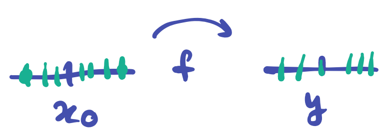
$$-\varepsilon < f(x) - y \leq g(x) - y \leq h(x) - y < \varepsilon$$

$$\Rightarrow \quad |g(x) - y| < \varepsilon$$

EXERCISE: (Limits of functions and sequences.)

- Show that $\lim_{n \to \infty} x_n = x$ iff for every open interval $I \subset \mathbb{R}$ containing x, the sequence x_n is eventually contained in I.

- Show that $\lim_{x \to x_0} f(x) = y$ iff for every sequence $x_n \in \mathbb{R} \setminus \{x_0\}$ such that $x_n \to x_0$ we also have $f(x_n) \to y$.



Proposition: Assume $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ both exist then:

i) $\lim_{x \to x_0} f(x) + g(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$

ii) $\lim_{x\to x_0} f(x) \times g(x) = \lim_{x\to x_0} f(x) \times \lim_{x\to x_0} g(x)$

iii) If $\lim_{x\to x_0} g(x) \neq 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

Proof: i) and ii) are an exercise for you prove it directly from the definition first. Then, prove it using sequential continuity.

We will prove iii) now.

Assume $a = \lim_{x\to x_0} f(x)$ and $b = \lim_{x\to x_0} g(x)$.

First we look at

$$\left| \frac{f(x)}{g(x)} - \frac{a}{b} \right| = \left| \frac{f(x)\,b - a\,g(x)}{g(x)\cdot b} \right|$$

$$= \frac{1}{|g(x)|} \cdot \frac{1}{|b|} \cdot \left| f(x)b - f(x)g(x) + f(x)g(x) - a\,g(x) \right|$$

$$\leq \frac{1}{|g(x)|} \cdot \frac{1}{|b|} \left(\underbrace{|f(x)|}\, \underbrace{|b - g(x)|} + \underbrace{|g(x)|}\, \underbrace{|f(x) - a|} \right)$$

Let $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$.

we are assuming $b \neq 0$. In particular

There exists $\delta_1 > 0$ such that

$$\begin{cases} |x - x_0| < \delta_1 \\ x \neq x_0 \end{cases} \Rightarrow |g(x) - b| < |b|/2$$

By the triangle inequality, for these x

$$|b| \leq |g(x) - b| + |g(x)|$$

$$< |b|/2 + |g(x)|$$

that is :

$$\begin{cases} |x - x_0| < \delta_1 \\ x \neq x_0 \end{cases} \Rightarrow |g(x)| > \frac{|b|}{2}$$

$$\Rightarrow \frac{1}{|g(x)|} \leq \frac{2}{|b|}.$$

In addition, there exists $\delta_2 > 0$ such that

$$\begin{cases} |x-x_0| < \delta_2 \\ x \neq x_0 \end{cases} \Rightarrow |f(x)-a| \leq 1$$

$$\Rightarrow -1+a \leq f(x) \leq a+1$$

$$\Rightarrow |f(x)| \leq |a|+1$$

Similarly, there exists $\delta_3 > 0$ such that

$$\begin{cases} |x-x_0| < \delta_3 \\ x \neq x_0 \end{cases} \Rightarrow |g(x)| \leq |b|+1$$

Finally, given $\varepsilon > 0$ there are

$\delta_4 > 0$ and $\delta_5 > 0$,

such that

$$\begin{cases} |x-x_0| < \delta_4 \\ x \neq x_0 \end{cases} \Rightarrow |f(x)-a| < \varepsilon$$

$$\begin{cases} |x-x_0| < \delta_5 \\ x \neq x_0 \end{cases} \Rightarrow |g(x)-b| < \varepsilon$$

Going back to our original computation

$$\left| \frac{f(x)}{g(x)} - \frac{a}{b} \right| \leq$$

$$\leq \frac{1}{|g(x)|} \cdot \frac{1}{|b|} \Big(|f(x)|\,|b - g(x)| + |g(x)|\,|f(x) - a| \Big)$$

Choosing $\delta = \min(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) > 0$ we have, for $|x - x_0| < \delta$. and $x \neq x_0$:

$$\left| \frac{f(x)}{g(x)} - \frac{a}{b} \right| \leq \frac{2}{|b|} \cdot \frac{1}{|b|} \Big((|a|+1)\,\varepsilon + (|b|+1) \cdot \varepsilon \Big)$$

$$\leq \varepsilon \times \frac{2}{|b|^2} \big(|a| + |b| + 2 \big)$$

$$= C \cdot \varepsilon$$

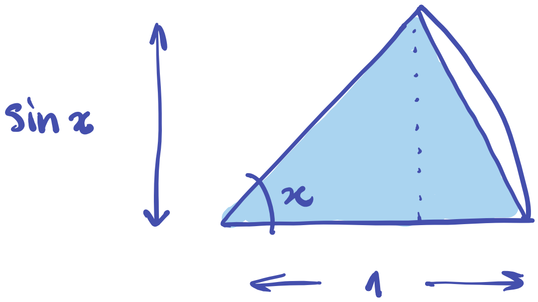
where C is a fixed positive constant.

This finishes the proof.

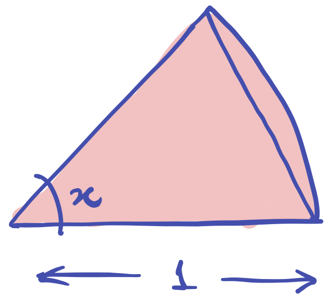
Exercise: Show that

$$\begin{cases} \lim\limits_{x \to 0} \sin(x) = 0 \\ \\ \lim\limits_{x \to 0} \cos(x) = 1 \end{cases}$$

<u>Idea</u> We only care about small values of x. Assume $x \in [0, \pi/2]$.



$\leq$



$$\frac{\sin x}{2} = \frac{\sin x \cdot 1}{2} \leq \left(\frac{x}{2\pi}\right) \pi \cdot 1^2 = \frac{x}{2}$$

Conclude that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$.

You can then use the squeeze theorem.

For the limit of $\cos(x)$ try using the formula for $\cos\left(\frac{x}{2} + \frac{x}{2}\right)$.

Lecture 3 - CONTINUITY

Definition: We say that $f: I \to \mathbb{R}$ is **continuous at** $a \in I$ if

an interval

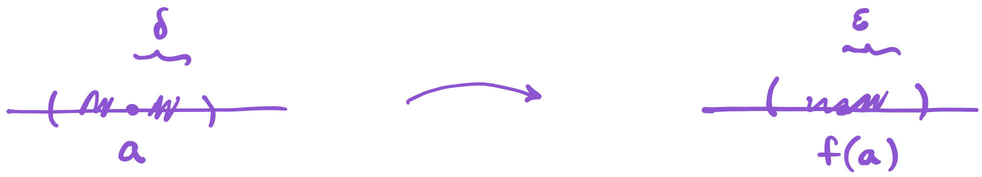
$$\lim_{x \to a} f(x) = f(a)$$

Remark: Continuity is a "local" concept. This means it describes the behaviour of a function near a point.

We can understand this better if we translate the definition into what it actually means:

$f : I \longrightarrow \mathbb{R}$ is continuous at a if:

$$\left\{ \begin{array}{c} \text{For every } \varepsilon > 0, \text{ there exists } \delta = \delta(\varepsilon) > 0 \\ \text{such that} \\ |x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon \end{array} \right\}$$



<u>Example:</u> $f(x) = x \sin(1/x)$.

Is f continuous at 0? We need to have a value for $f(0)$. so that this quest makes sense:

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

What should be the value of a?

Computing $\lim_{x \to 0} x \sin(1/x)$:

Since $|\sin(1/x)| \leq 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$; then

$$|x \sin(1/x)| \leq |x| \quad \forall x \neq 0.$$

Then, given $\varepsilon > 0$.

$$|x| < \varepsilon \implies |x \sin(1/x)| < \varepsilon$$

$$|x - 0| < \varepsilon \implies |x \sin(1/x) - 0| < \varepsilon$$

therefore

$$\lim_{x \to 0} x \sin(1/x) = 0$$

We conclude that.

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

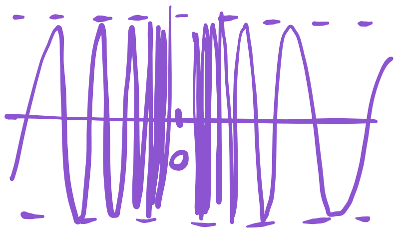
is a continuous function.

Example: Can we extend $\sin(1/x)$ to $x = 0$ so that the extension is continuous?

In other words; is there an $a \in \mathbb{R}$ such that.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

is continuous?



From the drawing it seems that $\lim_{x \to 0} f(x)$ does not exist.

We must prove it!

Remember that if $\lim_{x \to a} f(a) = L$ then,

$$\lim_{n \to \infty} x_n = a \implies \lim_{n \to \infty} f(x_n) = L$$

We have two sequences

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \longrightarrow \sin\left(\frac{1}{x_n}\right) = 1 \quad \forall n.$$

$$y_n = \frac{1}{2\pi n} \longrightarrow \sin\left(\frac{1}{y_n}\right) = 0 \quad \forall n$$

So we have

$x_n \longrightarrow 0$ with $\lim_{n \to \infty} f(x_n) = 1$

$y_n \longrightarrow 0$ with $\lim_{n \to \infty} f(y_n) = 0$

Therefore $\lim\limits_{x \to 0} \sin(1/x)$ does not exist.

Therefore $\lim\limits_{x \to 0} \sin(1/x)$ does not exist.

Exercise Prove that $\lim\limits_{x \to 0} 1/x$ does not exist.

Theorem If f and g are continuous at 0, then

(i) $f+g$ is continuous at a.
(ii) $f \cdot g$ is continuous at a.

If $g(a) \neq 0$ then

(iii) f/g is continuous at a.

Proof.

(i) Since $\lim\limits_{x \to a} f(x) = f(a)$ and $\lim\limits_{x \to a} g(x) = g(a)$

then

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a)$$

where the first "=" holds because the individual limits exist. and the second "=" is the defintion of continuity.

(ii) Exercise.

(iii) We want to show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$$

assuming that $g(a) \neq 0$ and f and g are continuous at a. Since $\frac{f(x)}{g(x)} = f(x) \times \frac{1}{g(x)}$

then, if we show that

$$\lim_{x \to a} 1/g(x) = 1/g(a)$$

the result would follow from part (ii).

First we must show that $|g(x)|$ is bounded away from zero. near a. From continuity

$$\exists\, \delta_1 > 0 \text{ s.t. } |x-a| < \delta_1 \Rightarrow |g(x) - g(a)| < \frac{|g(a)|}{2}$$

By the triangle inequality.

$$\frac{|g(a)|}{2} > |g(x) - g(a)| \geq |g(a)| - |g(x)|$$

$\uparrow$ triangle ineq.

$$\Rightarrow \quad |g(x)| \geq |g(a)|/2 > 0.$$

In other words, $\exists\, \delta_1 > 0$ s.t.

$$|x - a| < \delta_1 \Rightarrow |g(x)| \geq |g(a)|/2 > 0.$$

Now, we look at

$$\left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| = \frac{|g(x) - g(a)|}{|g(x)||g(a)|}$$

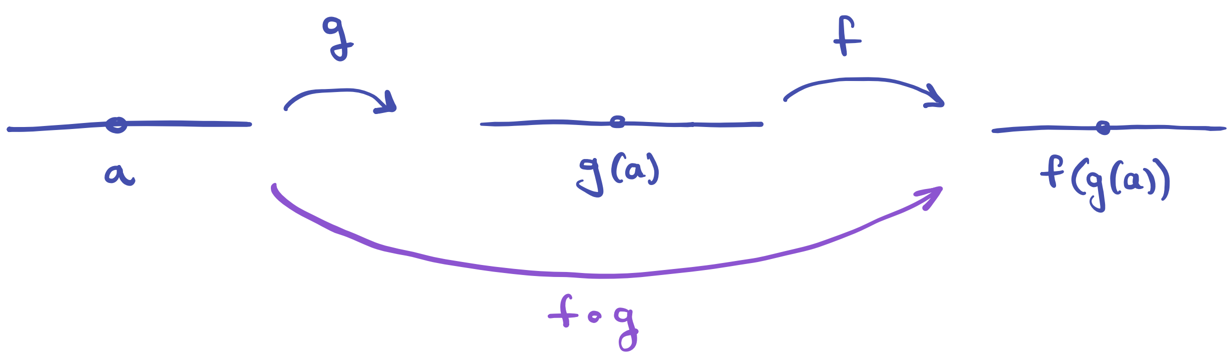
Let $\delta \in (0, \delta_1)$, then

$$\left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| \leq \frac{2}{|g(a)|^2} |g(x) - g(a)|.$$ Now

you complete the proof.

∎

Composition of continuous functions



Theorem: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof: Let $\varepsilon > 0$.

Since f is continuous at $g(a)$ then $\exists\ \delta_1 > 0$ s.t. if $|y - g(a)| < \delta_1$ then

$$|f(y) - f(g(a))| < \varepsilon.$$

Since g is continuous at a, then $\exists\ \delta_2 > 0$ s.t. if $|x - a| < \delta_2$ then

$$|g(x) - g(a)| < \delta_1$$

Therefore

$$|x - a| < \delta_2 \Rightarrow |g(x) - g(a)| < \delta_1$$

$$\Rightarrow |f(g(x)) - f(g(a))| < \varepsilon.$$

The power of continuity will be more evident when we consider functions which are continuous at every point

Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous at every point of the closed interval $[a,b]$.

Intermediate Value Theorem: f attains all the values between $f(a)$ and $f(b)$.

Extreme Value Theorem: f attains its maximum in $[a,b]$.

For now, try the following exercise:

Exercise: If f is continuous at a and $f(a) > 0$ then $f(x) > 0$ for all x close to a.

Exercise: Show that $\sin(x)$ is continuous for every $a \in \mathbb{R}$.

<u>Hint</u>: Show first that

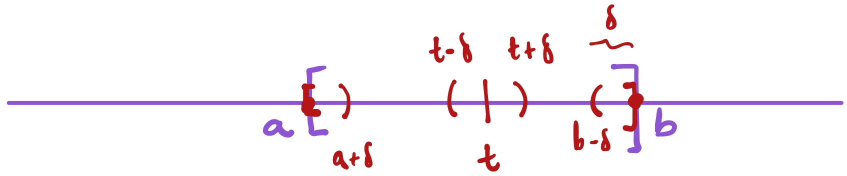
$$\lim_{x \to a} \sin(x) = \lim_{h \to 0} \sin(a+h)$$

then try the formula:

$$\sin(a+h) = \sin(h)\cos(a) + \sin(a)\cos(h).$$

If $a \leq b$ and $f : [a,b] \longrightarrow \mathbb{R}$ we defined what it means for f to be continuous at x where $a < c < b$.

But what if we want to talk about continuity at $x = a$ or $x = b$?



Definition We say that <u>$f(x)$ tends to L as x approaches a from the right</u> if

For all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

In this case we write:

$$\lim_{x \to a^+} f(x) = L$$

Exercise: Define $\lim_{x \to a^-} f(x) = L$ i.e.

$f(x)$ tends to L as x approaches a from the left

Exercise: Show that $\lim_{x \to a} f(x)$ exist if and only if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist a coincide.

Now we can define continuity on a closed interval:

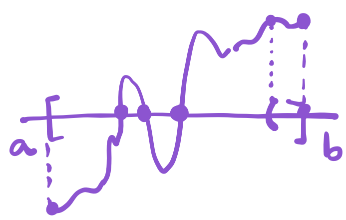
Definition We say that $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ if:

- $\lim_{x \to t} f(x) = f(t)$, $\forall t \in (a,b)$
- $\lim_{x \to a^+} f(x) = f(a)$
- $\lim_{x \to b^-} f(x) = f(b)$

In other words:

f is continuou at every $t \in (a,b)$ and continuou at a and b using one-sided limits

From the local condition of continuity we can derive global properties:



Bolzano's Theorem:

Let f be continuous on $[a,b]$ and assume

$$f(a) < 0 < f(b)$$

Then, there exists $x \in (a,b)$ such that

$$f(x) = 0$$

Idea: We should find a zero the first time f stops being negative.

Proof:

Let $A = \{ x \in [a,b] : f < 0 \text{ on } [a,x] \}$

(i) A is non-empty: $a \in A$, since $[a,a] = \{a\}$ and $f(a) < 0$. But also, from continuity there is $\delta > 0$ such that $f(x) < 0$ $\forall x \in [a, a+\delta)$.

so $a + \delta/2 \in A$.

(ii) A is bounded from above: In fact b is an upper bound for A. Moreover, since $f(b) > 0$ and f is continuous at b there exists a $\delta > 0$ s.t. $f(x) > 0$ for all $x \in (b-\delta, b]$. In particular

$$A \cap (b-\delta, b] = \emptyset$$

$\Rightarrow$ $b - \delta$ is an upper bound for A.

Because (i) and (ii) we can apply the axiom of the supremum to A:

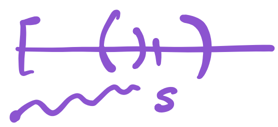
Let $s = \sup A$. Our conjecture is

$$\begin{cases} a < s < b \\ f(s) = 0. \end{cases}$$ ← Check this. Exercise!

Well, by trichotomy it can only happen

$$f(s) < 0 \quad \text{or} \quad f(s) > 0 \quad \text{or} \quad f(s) = 0.$$

Let's rule out the first two:



<u>$f(s) < 0$ cannot happen:</u> If it did, then by continuity, $\exists\ \delta > 0 \Rightarrow f < 0$ in $(s-\delta, s+\delta)$

Since $s = \sup A$, $\exists\ h \in (s-\delta, s]$ such that $h \in A$. Then $[a, h] \cup (s-\delta, s+\delta) = [a, s+\delta)$

$$f(x) < 0 \quad \forall\ x \in [a, s+\delta/2]$$

$$\Rightarrow \quad s + \delta/2 \in A$$

But $s = \sup A$ and $s + \delta/2 > s$ so this is a contradiction.

$\underline{f(s) > 0 \text{ cannot happen}}$: If it did,

$\exists\ \delta > 0$ s.t

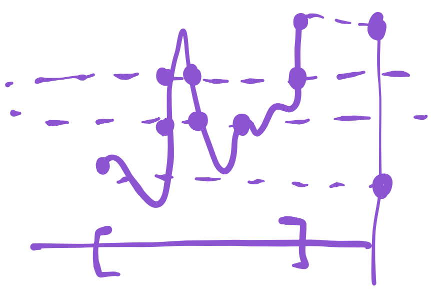
$$f(x) > 0 \quad \forall\, x \in (s-\delta, s+\delta)$$

In particular ; $(s-\delta, s+\delta) \cap A = \emptyset$

Therefore s cannot be $\sup A$.

which is a contradiction.

Therefore $f(s) = 0$.



$\blacksquare$.

Corollary (Intermediate Value Theorem).

Let g be continuous on $[a,b]$ with $g(a) < g(b)$. Then, $\forall y \in (g(a), g(b))$, there is $x \in (a,b)$ such that $g(x) = y$.

Idea: Apply Bolzano's theorem to

$$f(x) = g(x) - y$$

Note that f is continuous and

$$\begin{cases} f(a) = g(a) - y < 0 \\ f(b) = g(b) - y > 0 \end{cases}$$

Proof: Exercise!

Corollary Every odd degree polynomial has a real root.

Idea: $x^3 + ax^2 + bx + c$; the leading term for large values of x

is x^3, so when x is large we would expect that:

$$\operatorname{sign}(x^3) = \operatorname{sign}(x^3 + ax^2 + bx + c)$$

In fact, given $\alpha \in \mathbb{R}_{>0}$ we have

$$|x|^n \underset{\substack{\Updownarrow \\ |x| \geq \alpha}}{\geq} \alpha |x|^{n-1} \underbrace{\geq \alpha |x|^{n-2} \geq \dots \geq \alpha |x| \geq \alpha}_{\substack{\Updownarrow \\ |x| \geq 1}}$$

Therefore if $|x| \geq 3 \cdot \max\{|a|, |b|, |c|, 1\}$

$$\Rightarrow \quad |x|^3 \geq |a||x|^2 + |b||x| + |c|.$$

This implies

$$\operatorname{sign}(x^3 + ax^2 + bx + c) = \operatorname{sign}(x^3)$$

as long as

$$|x| \geq 3 \cdot \max\{|a|, |b|, |c|, 1\}.$$

Use this idea to proof the Corollary.

Open and Closed sets

Definition: A set $A \subset \mathbb{R}$ is called open if for every $a \in A$ there exists a $\delta > 0$ such that $(a-\delta, a+\delta) \subset A$.

Definition: A set $B \subset \mathbb{R}$ is called closed if its complement $A = \mathbb{R} \setminus B$ is open.

Remark: The empty set is open and closed.

Exercise: Show that a set B is closed iff

$$\left.\begin{cases} \{x_n\} \subset B \\ \lim_{n\to\infty} x_n \text{ exists} \end{cases}\right. \Rightarrow \lim_{n\to\infty} x_n \in B$$

In the exercise above assuming $\lim_{n\to\infty} x_n$ exists is crucial, even if we allow passing to sub-sequences:

Exercise: Give an example of a closed set B and a sequence $\{x_n\} \subset B$ such that no subsequence of $\{x_n\}$ is convergent.

Definition A set $K \subset \mathbb{R}$ is <u>compact</u> iff every sequence $\{x_n\} \subset K$ contains a convergent subsequence $\{x_{n_i}\}$ and $\lim_{i \to \infty} x_{n_i} \in K$.

Exercise: Let $K \subset \mathbb{R}$, show that

K is compact $\iff$ K is closed and bounded

Hint: $(\Rightarrow)$ closed follows from the definition and boundedness from the previous exercise

$(\Leftarrow)$ Use Bolzano-Weierstrass.

THE EXTREME VALUE THEOREM

Theorem: Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$. Then, f is bounded from above in $[a,b]$.

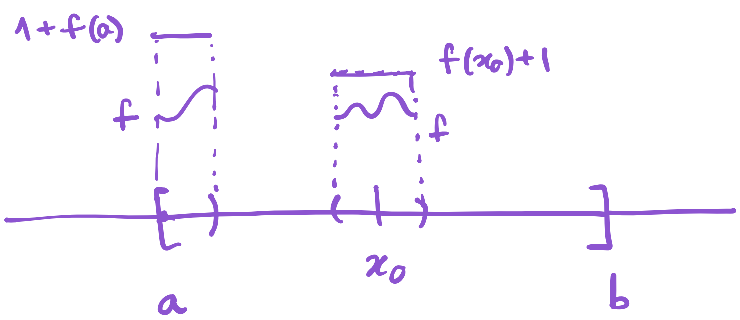
Remark Remember that <u>f is bounded from above on a set X</u> iff there exists an $M \in \mathbb{R}$ such that $f(x) \leq M$, $\forall x \in X$.

Proof: We opt for a similar strategy.

First, we know that continuous functions are, in some sense, locally bounded:

If f is continuous at x_0, then there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < 1$

In particular, $f(x) \leq f(x_0) + 1$, $\forall x \in (x_0 - \delta, x_0 + \delta)$



Idea: if f is bounded from above in $[a, a+\delta)$

when is the last x for which $[a,x]$ is bounded?

Let $A = \{x \in [a,b] : f \text{ is } \underset{\text{from above}}{\text{bounded}} \text{ in } [a,x]\}$

We want to show that $b \in A$!

i) <u>A is not empty</u>: $a \in A$ but also $a + \delta/2 \in A$ for some $\delta/2$.

ii) <u>A is bounded</u>: follows from $A \subset [a,b]$.

After verifying i) + ii) we can say that A has a supremum. Let $s = \sup A$.

As before $A \subset [a,b] \implies s \in [a,b]$ Prove this! EXERCISE

<u>Claim</u>: $s = b$.

<u>Proof of Claim:</u> Assume the claim is false

Then $s < b$. Since f is continuous at s

there exists $M_1 \in \mathbb{R}$ and $\delta > 0$ such that

$$(*) \quad f(x) \leq M_1 \qquad \text{for all } x \in (s-\delta, s+\delta)$$

Moreover, since $s = \sup A$, there exists $h \in \mathbb{R}$ such that

$$\begin{cases} h \in A \\ s - \delta < h \leq s. \quad (***) \end{cases}$$

$h \in A \Rightarrow f$ is bounded from above in $[a,h]$

$\Rightarrow \exists\, M_2 \in \mathbb{R}$ s.t. $f(x) \leq M_2 \ \forall x \in [a,h]$.
$(**)$

From $(*)$ and $(**)$ we conclude that

$$f(x) \leq \operatorname{Max}(M_1, M_2) \qquad \forall x \in [a,h] \cup (s-\delta, s+\delta)$$

but from $(***)$ $\quad [a,h] \cup (s-\delta, s+\delta) = [a, s+\delta)$

Therefore:

$$f(x) \leq \operatorname{Max}(M_1, M_2) \quad \forall x \in [a, s+\delta)$$

$$\Rightarrow \quad f(x) \leq \operatorname{Max}(M_1, M_2) \quad \forall x \in [a, s+\delta/2]$$

$$\Rightarrow \quad s + \delta/2 \in A$$

Since $s + \delta/2 > s = \sup A$ this is a contradiction.

Finally, note that $b \in A$. In fact, there are δ arbitrarily small such that $b-\delta \in A$. Joining with an interval around b, we complete the proof. ∎

<u>Corollary</u> (The Extreme Value Theorem):
Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$. Then, there exists $x \in [a,b]$ such that

$$f(x) \geq f(t)$$

for all $t \in [a,b]$.

<u>Proof</u>: By the previous theorem the non-empty set

$$F = \{ f(x) : x \in [a,b] \}$$

is bounded from above.

Let $m = \sup F$. Since m is the supremum of F there exists a sequence $y_n \in F$ such that $y_n \longrightarrow m$. However, since $y_n \in F$ for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $y_n = f(x_n)$.

We would like to use continuity to conclude that $x_n \to x \in [a,b]$ implies

$$f(x) = \lim_{n\to\infty} f(x_n) = m$$

However, we only know that the sequence $f(x_n)$ converges. In principle the sequence x_n might not converge.

How to deal with this?

Since $a \le x_n \le b$, we know by Bolzano-Weierstrass that there is a subsequence

x_{n_k} which is convergent. Let

$$x = \lim_{k\to\infty} x_{n_k}$$

Now, $f(x_{n_k})$ is a subsequence of $f(x_n)$.

From last semester we know that a sub_ sequence of a convergent sequence converges to the same limit. In particular we have

$$\begin{cases} x_{n_k} \longrightarrow x \in [a,b] \\ f(x_{n_k}) \longrightarrow m = \sup\{f(t) : t \in [a,b]\} \end{cases}$$

By continuity we have

$$f(x) = \lim_{k \to \infty} f(x_{n_k}) = m.$$

$\blacksquare$

Exercise: Show that $f : [a,b] \longrightarrow \mathbb{R}$ attains a minimum in $[a,b]$.

Exercise: Construct $f : (a,b) \longrightarrow \mathbb{R}$ that does not satisfy the extreme value property.

Exercise: Show that if $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and X is compact then f attains a maximum in X

UNIFORM CONTINUITY

Let $f : (a,b) \longrightarrow \mathbb{R}$ be continuous on (a,b)

Fix two points: $x < y \in (a,b)$

Choose $\varepsilon > 0$.

Then:

f continuous at x: $\exists\, \delta_1 > 0$ s.t $|t - x| < \delta_1 \Rightarrow |f(t) - f(x)| < \varepsilon$

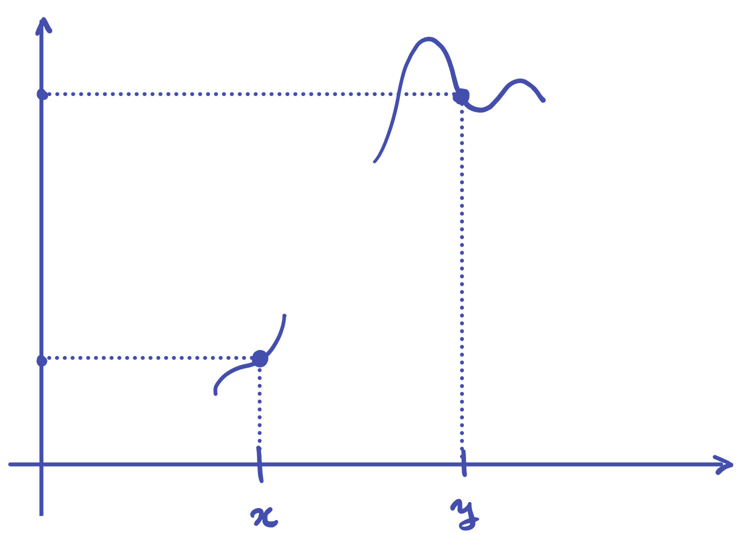
f continuous at y: $\exists\, \delta_2 > 0$ s.t $|t - y| < \delta_2 \Rightarrow |f(t) - f(y)| < \varepsilon$

<u>Question</u>: Can we assume $\delta_1 = \delta_2$?

Well, we can use $\delta = \min(\delta_1, \delta_2)$

In other words, for a given $\varepsilon > 0$, our choice of δ is <u>uniform</u> on x and y

As usual you can understand it geometrically:



This generalises to any finite number of points. But what if we have an infinite number of them?

<u>Definition</u>: We say that $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ is <u>uniformly continuous on X</u> if

$\forall \varepsilon > 0$, there exists a $\delta > 0$ s.t. for all $x, y \in X$

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

<u>EXAMPLE</u>: $f(x) = \frac{1}{x}$ continuous on $(0,1)$ but it is not uniformly continuous on $(0,1)$.

<u>Proof</u>: Consider $x_n = \frac{1}{n}$.

Then $|x_n - x_{n+1}| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$

which is arbitrarily small, but

$|f(x_n) - f(x_{n+1})| = |n - (n+1)| = 1$ is

not arbitrarilly small.

EXERCISE*: Let $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ be continuous

show that

i) If $X = [a,b]$ then f is uniformly continuous on X.

ii) If X is compact then f is uniformly continuous on X.

Hint: Assume f is not uniformly continuous

Show first there exists $\varepsilon > 0$ and sequences x_n and y_n such that

$$\begin{cases} |x_n - y_n| \longrightarrow 0 \\ |f(x_n) - f(y_n)| \geq \varepsilon > 0 \end{cases}$$

Since X is compact you can pass to a convergent subsequence. Obtain a contradiction from this.

MORE EXERCISES

- Show that the function $f(x) = |x|$ is continuous for all $x \in \mathbb{R}$.

- Show that $f(x) = \sqrt{x}$ is continuous for all $x \in [0, \infty)$.

(*) Let f be continuous on $[0,1]$. Assume $f(x)$ is in $[0,1]$ for each $x \in [0,1]$. Prove that there exist $t \in [0,1]$ such that $f(t) = t$.

(*) We say that $\lim_{x \to a} f(x) = \infty$ if for all $N > 0$, there exists a $\delta > 0$ such that $f(x) > N$ for all $|x - a| < \delta$.

Show that if $f:(a,b) \to \mathbb{R}$ is continuous on (a,b) and $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to b^-} f(x) = \infty$, then f attains a minimum on (a,b).

(*) Let $p(x)$ be a polynomial. Show that $|p(x)|$ attains a minimum in $\mathbb{R}$.

- Let $A \subset \mathbb{R}$ be a non-empty closed set. Show that for any $x \in \mathbb{R}$ there exists a point $a \in A$ which is closest to x.

(*) Prove that there does not exist a continuous function $f:\mathbb{R} \longrightarrow \mathbb{R}$ which takes on every value exactly twice.

- For which of the following values of α is the function $f(x) = x^\alpha$ uniformly continuous on $[0,\infty)$. $\alpha = 1/3, 1/2, 2, 3$.

- Find a function continuous and bounded in $(0,1]$ which is not uniformly conti_ nuous on $(0,1]$.

- Find a function that is continuous and bounded on $[0,\infty)$ which is not uniformly continuous on $[0,\infty)$.

THE DERIVATIVE

The concept of derivative appears when we try to formalise the following ideas:

- instant velocity (instant rate of change).
- a line tangent to the graph of a function.

EXAMPLE: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ represents the movement of a particle on the real line, i.e.,

$$t \longmapsto f(t)$$

time $= t$ position at time t

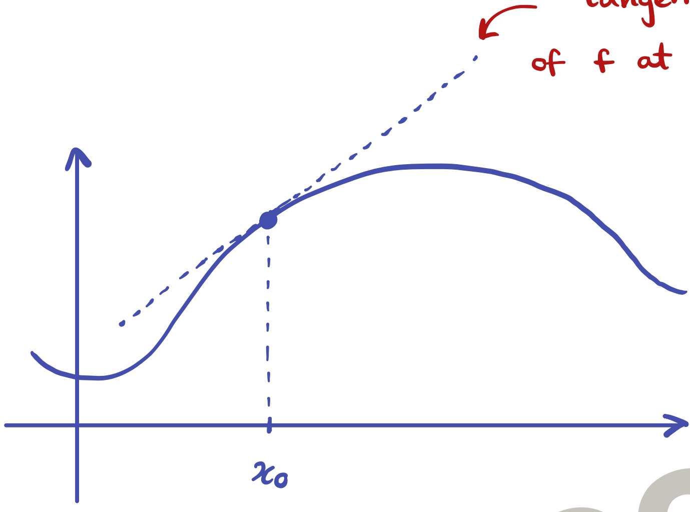
The going from $f(t_0)$ to $f(t)$ takes the particle $t - t_0$ time. So the average velocity of the particle on the interval $[t_0, t]$ is

$$\frac{f(t) - f(t_0)}{t - t_0}$$

If we want to talk about <u>instant</u> velocity at the time t_0 then it makes sense to ask:

$$\text{does } \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \text{ exists?}$$

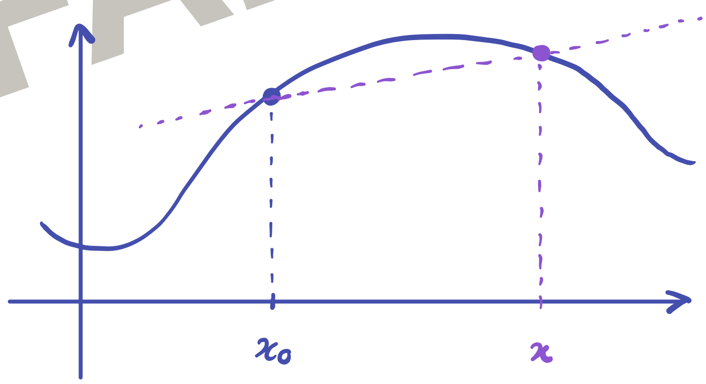
EXAMPLE: We can always think of a function in terms of its graph



tangent to the graph of f at the point $(x_0, f(x_0))$.

Should be the graph of a linear function and a translation $t \longmapsto \alpha t + \beta$

What equation does the tangent line satisfies?



$$t \longmapsto \frac{f(x) - f(x_0)}{x - x_0}(t - x_0) + f(x_0)$$

This is the equation of the line passing through the points

$$(x_0, f(x_0)) \quad \text{and} \quad (x, f(x))$$

As $x \to x_0$, we would expect this line to converge (in some sense) to the tangent line to the curve.

Notice

$$\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\text{only this term depends on } x!}(t-t_0) + f(x_0)$$

So, again:

does $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists?

If it does then we would expect the tangent line at x_0 to have equation:

$$t \longmapsto \left(\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}\right)(t-x_0) + f(x_0)$$

<u>Definition</u> We say that f is differentiable at x_0 if $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. In

this case we use the notation:

$$f'(x_0) = \frac{df}{dx}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

and we call $f'(x_0)$ the derivative of f at x_0.

EXERCISE: Show that f is differentiable at x if and only if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists and in that case:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In other words we can use as our definition of derivative!

Differentiability is a stronger condition than continuity.

Theorem: If f is differentiable at $a \in \mathbb{R}$ then f is continuous at a.

Proof: The statement f is continuous at a, is:

$$\lim_{h\to 0} f(a+h) = f(a)$$

which is what we want to show.

This is the same as:

$$\lim_{h\to 0} f(a+h) - f(a) = 0$$ Why? EXERCISE

To prove this, note that, for $h \neq 0$

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h$$

Now, f differentiable at a, means that the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Since $\lim_{h\to 0} h$ also exists and its equal to 0, then we can use the algebra of limits. and conclude:

$$\lim_{h\to 0} f(a+h)-f(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h\to 0} h$$

$$= f'(a) \times 0$$

$$= 0. \qquad \blacksquare$$

EXAMPLES

Constant functions. $f(x) = \alpha$. $\forall x \in \mathbb{R}$ then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\alpha-\alpha}{h} = 0.$$

Identity: $f(x) = x \ \forall x \in \mathbb{R}$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1$$

Monomials: Let $f(x) = x^n$ for $n \in \mathbb{N}$.

$$f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$$

$$= \lim_{y\to x} \frac{y^n - x^n}{y-x}$$

$$= \lim_{y\to x} \frac{(y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \dots + x^{n-1})}{(y-x)}$$

$$= \lim_{y \to x} \underbrace{y^{n-1} + y^{n-2} \cdot x + y^{n-3} x^2 + \dots + x^{n-1}}_{n \text{ elements}}$$

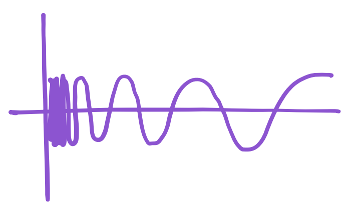
We use that $y \longmapsto y^k$ is continuous $\forall k \in \mathbb{N}$

$$= n \cdot x^{n-1}, \text{ i.e.}$$

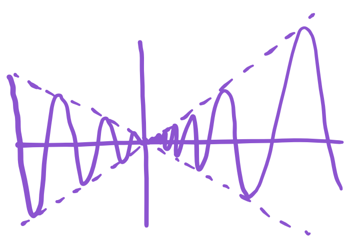
$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

EXAMPLE: Are the following functions differentiable at $x = 0$?

A) $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$



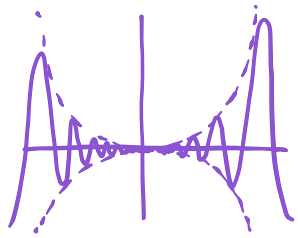
B) $f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$



for $x \neq 0$.

$$\frac{x \cdot \sin\left(\frac{1}{x}\right) - 0}{x} = \sin\left(\frac{1}{x}\right)$$ which does not have a limit. as $x \to 0$.

c) $$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



In this case $$\frac{x^2 \sin(1/x) - 0}{x} = x \sin(1/x)$$

so

$$f'(0) = \lim_{x \to 0} x \sin(1/x) = 0.$$

EXERCISE: Show that $\sin(x)$ is differentiable at every $x \in \mathbb{R}$ and $\frac{d}{dx}\sin(x) = \cos(x)$.

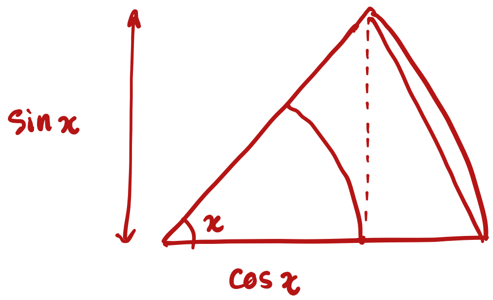
Hint: $$\frac{\sin(x+h) - \sin(x)}{h} =$$

$$= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\,\frac{\sin(h)}{h}$$

Now, show $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$

For the first, use the following geometric argument.



$$\frac{x}{2\pi}\pi \cos x^2 \leq \frac{\sin x \cdot 1}{2} \leq \frac{x}{2\pi}\pi$$

$$\cos x^2 \leq \frac{\sin x}{x} \leq 1$$

For the second, use $\cos(h)^2 + \sin(h)^2 = 1$.

EXERCISE: Show that $\cos(x)$ is differentiable at every x and $\frac{d}{dx}\cos(x) = -\sin(x)$.

HIGHER ORDER DERIVATIVES

- Let $f : (a,b) \longrightarrow \mathbb{R}$ be differentiable at every point $x \in (a,b)$. In this situation

$$x \longmapsto f'(x) = \frac{df}{dx}(x)$$

is a new function $f' : (a,b) \longrightarrow \mathbb{R}$ and we can ask whether f' is itself differentiable at a point $x \in (a,b)$.

- If $\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ exists for all $x \in (a,b)$ we say that <u>f is twice differentiable on (a,b)</u>.

and we denote

$$f''(x) = \frac{d^2}{dx^2} f(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

- We say that a function <u>f is k times differentiable on (a,b)</u> if this process can be repeated k times and denote

$$f^{(k)}(x) = \frac{d^k}{dx^k} f(x).$$

PROPERTIES OF THE DERIVATIVE

<u>Proposition</u>: If f and g are differentiable at $x \in \mathbb{R}$ then $f+g$ is differentiable at x and

$$(f+g)'(x) = f'(x) + g'(x)$$

<u>Proof</u>: EXERCISE.

<u>Proposition</u>: (Leibniz rule or Product rule).

If f and g are differentiable at $x \in \mathbb{R}$ then $f \times g$ is differentiable at x and

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Proof: COMPLETE THE FOLLOWING COMPUTATION:

$$\frac{(f\cdot g)(x+h)-(f\cdot g)(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{f(x+h)\,g(x+h) \qquad\qquad\qquad\qquad - f(x)g(x)}{h}$$

$$= f(x+h)\,\frac{g(x+h)-g(x)}{h} + g(x)\,\frac{f(x+h)-f(x)}{h}$$

use that f is continuous

use that g is differentiable

use that f is differentiable

Example: Monomials, again!

$$\frac{d}{dx}(x^n) =$$

$$=$$

$$=$$

Prove by induction using the Leibniz rule that

$$\frac{d}{dx}x^n = n x^{n-1}.$$

Exercise: Compute the derivative of a generic polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Proposition: Assume g is differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$. Then the function

$$x \longmapsto \frac{1}{g(x)}$$

is differentiable at a and

$$\left(\frac{d}{dx}\frac{1}{g}\right)(a) = -\frac{g'(a)}{g(a)^2}$$

Proof: We want to determine whether

$$\lim_{h \to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h}$$

exists. First of all, by continuity $g(a+h) \neq 0$ for small values of h. In particular, for such values we can write:

$$\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} =$$

$$=$$

Now, the first term goes to by assumption.

The second term converges to by continuity since $g(a) \neq 0$. Finally, the third term is constant and by the algebra of limits we obtain

$$\frac{d}{dx}\frac{1}{g(x)}\Bigg|_{x=a} =$$

the notation $F(x)\Big|_{x=a}$ means "F evaluated at a"

it is often used with derivatives to indicate we first differentiate and then evaluate.

<u>Remark</u>: Note that in several of these proofs we are using continuity!

Now we can prove a general formula for the derivative of a quotient $\frac{f(x)}{g(x)}$.

Corollary If f and g are differentiable at a with $g(a) \neq 0$ then

Leibniz rule.

$$\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=a} =$$

$$=$$

Corollary: If c is a constant and f is differentiable at a, then

$$\left.\frac{d}{dx} c f(x)\right|_{x=a} = c f'(a).$$

Remark Let $I = [a,b]$. We denote by

$$C^0(I) := \{ f : [a,b] \to \mathbb{R} : f \text{ is continuous on } [a,b] \}$$

$$C^1(I) := \{ f : [a,b] \to \mathbb{R} : f' \text{ is continuous on } [a,b] \}$$

Then $\frac{d}{dx} : C^1(I) \longrightarrow C^0(I)$

and is a linear map!

In fact, both $C^0(I)$ and $C^1(I)$ are Real vector spaces:

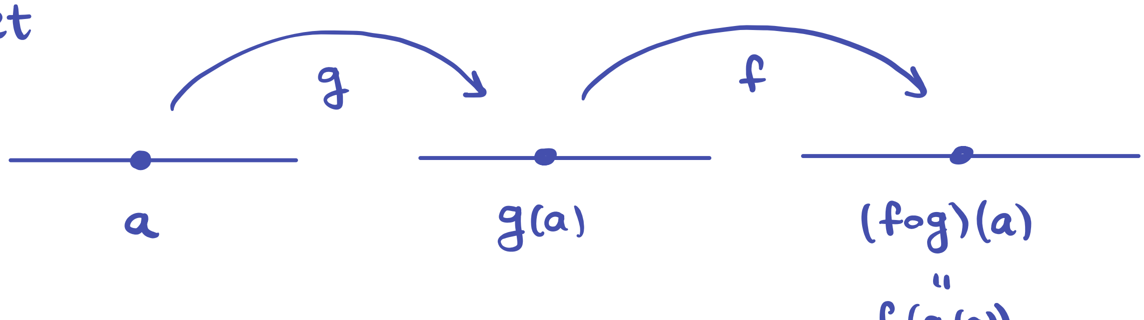
- $(f+g)(x) :=$
- $(cf)(x) :=$

The derivative satisfies:

$$\begin{cases} \frac{d}{dx}(f+g) = \\ \\ \frac{d}{dx}(cf) = \end{cases}$$

Now we ask: What is the relation between differentiability and composition?

Let



$f(g(a))$

where we assume that

•

•

Is $f \circ g$ differentiable at a?

and if so, what is $(f\circ g)'(a)$?

As always we go back to the definition.

$$\frac{(f\circ g)(a+h) - (f\circ g)(a)}{h} =$$

$$= \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \times \frac{g(a+h) - g(a)}{h}$$

However, this informal computation,

does gives us some insight: if $(f\circ g)'(a)$

exists a good guess for it is

$$f'(g(a)) \cdot g'(a)$$

What can go wrong?

$$\frac{(f\circ g)(a+h)-(f\circ g)(a)}{h} = \overbrace{\frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}}^{(*)} \times \frac{g(a+h)-g(a)}{h}$$

If we want to take limits we must show that $(*)$ converges to $f'(g(a))$.

However $(*)$ might not even be defined for some values of h!

On the other hand if $g(a+h) = g(a)$ then

$$\frac{(f\circ g)(a+h)-(f\circ g)(a)}{h} = 0.$$

so perhaps in that case we can define $(*)$ to be what we think it should be:

Let ϕ be defined as:

$$\phi(h) = \begin{cases} \dfrac{(f\circ g)(a+h)-(f\circ g)(a)}{g(a+h)-g(a)}, & \text{if } g(a+h) \neq g(a) \end{cases}$$

$$\begin{cases} \quad f'(g(a)) & , \text{if } g(a+h) = g(a) \end{cases}$$

Then the formula

$$\frac{(f\circ g)(a+h) - (f\circ g)(a)}{h} = \phi(h) \cdot \frac{g(a+h) - g(a)}{h} \quad (**)$$

is now true for all $h \neq 0$.

Moreover, proving $(f\circ g)'(a) = f'(g(a)) \cdot g'(a)$ is now equivalent to showing that

$$\lim_{h\to 0} \phi(h) = f'(g(a)) = \phi(0)$$

<u>Lemma:</u> $\phi(h)$ as defined above is continuous at 0.

<u>Proof</u>: Fix $\varepsilon > 0$.

<u>f differentiable at $g(a)$ implies:</u>

$\exists\, \delta' > 0$ such that

$$0 < |k| < \delta' \Rightarrow \left| \frac{f(g(a)+k) - f(g(a))}{k} - f'(g(a)) \right| < \varepsilon \qquad (1)$$

g continuous at a implies:

$\exists\, \delta > 0$ such that

$$|h| < \delta \Rightarrow |g(a+h) - g(a)| < \delta' \qquad (2)$$

When we choose h with $|h| < \delta$ two things can happen:

Case 1: If $g(a+h) - g(a) = 0$ then

$$\phi(h) = f'(g(a))$$

in particular $\underbrace{|\phi(h) - f'(g(a))|}_{=0} < \varepsilon.$

Case 2: If $g(a+h) - g(a) \neq 0$, then

$$\phi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}$$

$$= \frac{f\big(g(a) + \overbrace{(g(a+h) - g(a))}^{k}\big) - f(g(a))}{\underbrace{g(a+h) - g(a)}_{k}}$$

$$= \frac{f(g(a)+k) - f(g(a))}{k}$$

Since $|h| < \delta$ by (2) $\Rightarrow$ $|k| < \delta'$

by (1) $\Rightarrow$ $|\phi(h) - f'(g(a))| < \varepsilon$.

<u>Conclusion</u>: We have found $\delta > 0$ such that

$$|h| < \delta \quad \Rightarrow \quad |\phi(h) - f'(g(a))| < \varepsilon$$

But $\varepsilon > 0$ is arbitraty $\Rightarrow$ $\lim_{h \to 0} \phi(h) = f'(g(a))$.

For the exponential $f'(x) = f(x)$:

$$g'(f(x)) = \frac{1}{f(x)}$$

substituting $y = f(x)$ we obtain

$$g'(y) = \frac{1}{y}$$

i.e. $$\frac{d}{dy} \log(y) = \frac{1}{y}.$$

Remark: later on we will define $\log(x)$ using the notion of integrals. We will then define e^x as the inverse of $\log$ and show that it satisfies $(*)$.

We obtain the following important corallary.

Theorem (The Chain Rule): If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

Proof: Take $\lim\limits_{h \to 0}$ in formula $(**)$ $\blacksquare$

Remark: Another way of writing the result:

If $g \in C^1(I)$, $f \in C^1(J)$ and $g(I) \subset J$

then $\begin{cases} (f \circ g) \in C^1(I) \\ \text{and} \\ (f \circ g)' = f'(g) \cdot g' \end{cases}$

The Derivative of Elementary Functions

- Polynomials. COVERED
- Trigonometric functions.

  COVERED assuming $\sin(x)$ and $\cos(x)$ satisfy the usual trigonometric identities. We will see such functions actually exist later on when we talk about curves on the plane.

- Exponential.

  COVERED in the problem sheet, we prove $f(x) = e^x$ is the unique function satisfying

$$(*) \begin{cases} \cdot\ f(x+y) = f(x) \cdot f(y) \\ \cdot\ f'(0) = 1. \end{cases}$$

we also show there that the exponential satisfies

$$\begin{cases} f'(x) = f(x). \\ f : \mathbb{R} \longrightarrow (0,+\infty) \\ f \text{ is monotone increasing} \\ f \text{ is bijective} \end{cases}$$

- <u>Logarithm:</u> We define $x \longmapsto \log(x)$ as the inverse of $x \longmapsto e^x$.

To simplify the notation let us write

$$\begin{cases} f(x) = e^x \\ g(x) = \log(x) \end{cases}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "f"] & (0,+\infty) \arrow[l, bend left, "g"] \end{tikzcd}$$

f being the inverse of g means:

$$\begin{cases} f \circ g(y) = y & \forall y \in (0,+\infty) \\ g \circ f(x) = x & \forall x \in \mathbb{R} \end{cases}$$

We will prove that if $f'(x) \neq 0$ then its inverse g is differentiable at $y = f(x)$

By the chain rule, this implies:

$$g \circ f(x) = x \Rightarrow (g \circ f)'(x) = 1$$
$$\Rightarrow g'(f(x)) \cdot f'(x) = 1$$
$$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

The significance of the Derivative

1. Local Comparison: If $f'(c) > 0$, then

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c) > 0$$

This implies that there exists $\delta > 0$ such that

$$h \in (-\delta, \delta) \setminus \{0\} \Rightarrow \frac{f(c+h) - f(c)}{h} > 0$$

We obtain an inequality whose sign depends on whether δ is positive or negative

For $h \in (-\delta, 0)$:

$$\frac{f(c+h) - f(c)}{h} > 0 \Rightarrow$$

$$\Rightarrow$$

For $h \in (0, \delta)$:

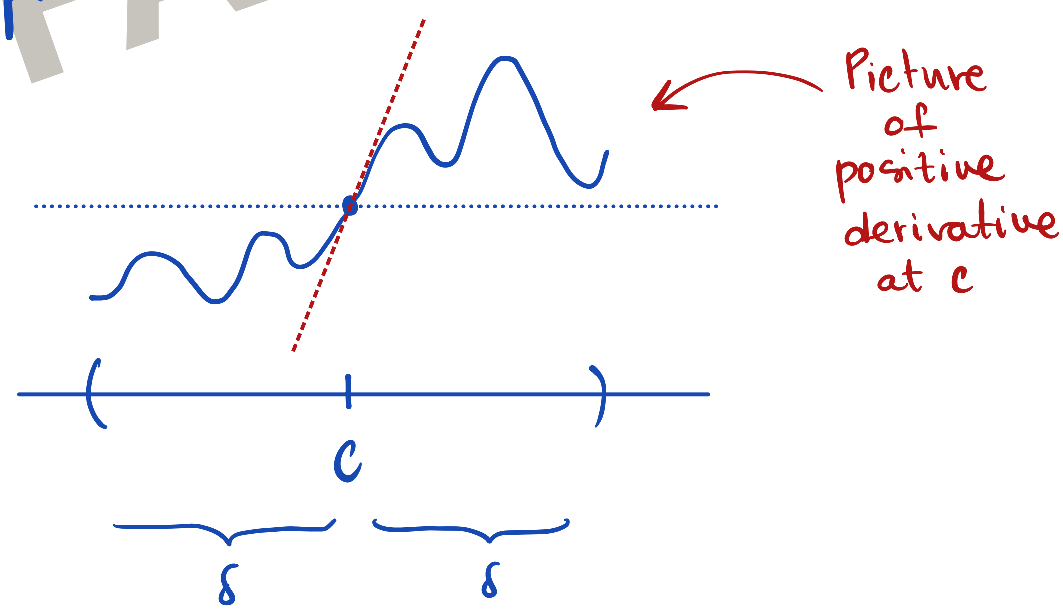
$$\frac{f(c+h) - f(c)}{h} > 0 \quad \Longrightarrow$$

$$\Longrightarrow$$

Putting both together we have

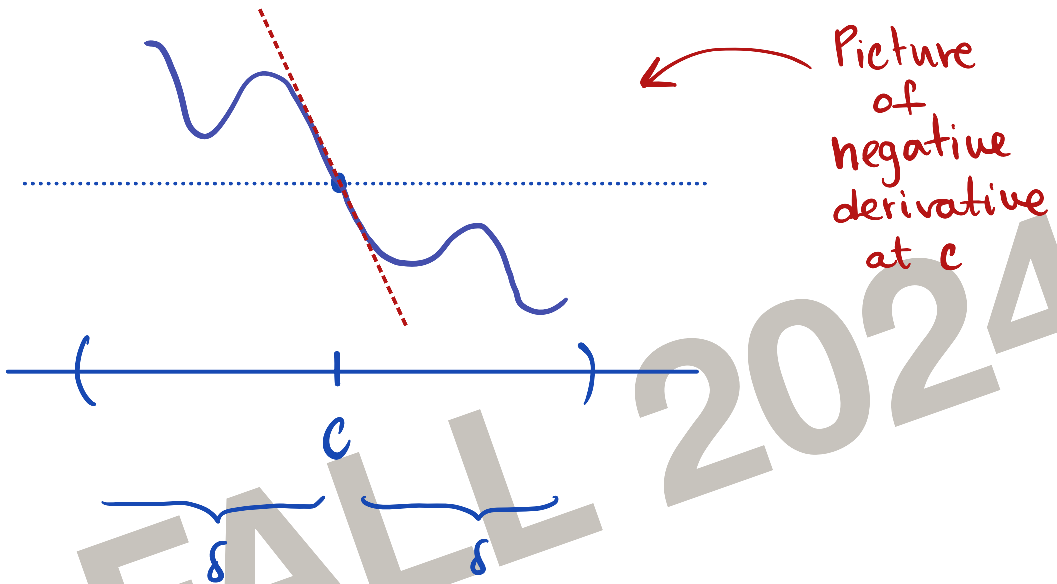
$$\Bigg\} (*)$$

In a picture:



Remark: Note that $(*)$ does not imply monotonicity on $(c-\delta, c+\delta)$ but it tell us that we can compare the values of $f(c)$ with the values of f near c.

Exercise: What can we say when $f'(c) < 0$?



Exercise: What about the reciprocal? Show that if $f(c) < f(y)$ for all $y \in (c, c+\delta)$, then

$$\lim_{y \to c^+} \frac{f(y) - f(c)}{y - c} \geq 0$$

whenever the limit exists. What are examples of the limit not existing and of the limit being $= 0$?

2. The derivative can indicate global growth:

Proposition:

If
- f is differentiable on (a,b) and
- $f'(t) > 0$ for all $t \in (a,b)$.

then, f is monotone increasing on (a,b).

Proof: Fix $x \in (a,b)$. We want to show that $f(x) < f(y)$, for all $y \in (x,y)$.

We can argue by contradiction:

Assume that there is $y \in (x,b)$ such that $f(x) \geq f(y)$. A natural question is: which the smallest y with these properties?

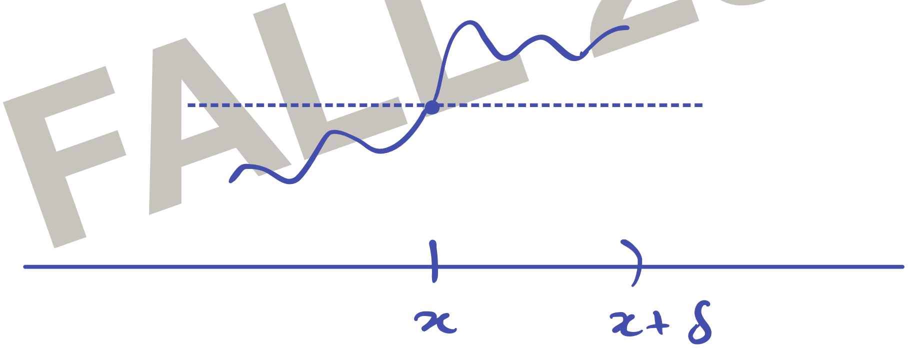
This calls for using an infimum of a set.

Let $A = \{ y \in (x, b) : f(y) \leq f(x) \}$

A is bounded: $A \subset (x, b)$.

A in non-empty: by assumption.

Let $\alpha = \inf A$. Note that $x \leq \alpha$, because $A \subset (x, b)$. First, note that $f'(x) > 0$ together with the argument from 1.) implies that $\exists \, \delta > 0$ such that $f(y) > f(x)$ for all $y \in (x, x+\delta)$.



This implies $x + \delta$ is a l wer bond for A

Therfore

$$x + \delta \leq \alpha.$$

what is the value of $f(\alpha)$?

Since $f'(\alpha)$ exists then f is continuous at α. Moreover since $\alpha = \inf A$, ...

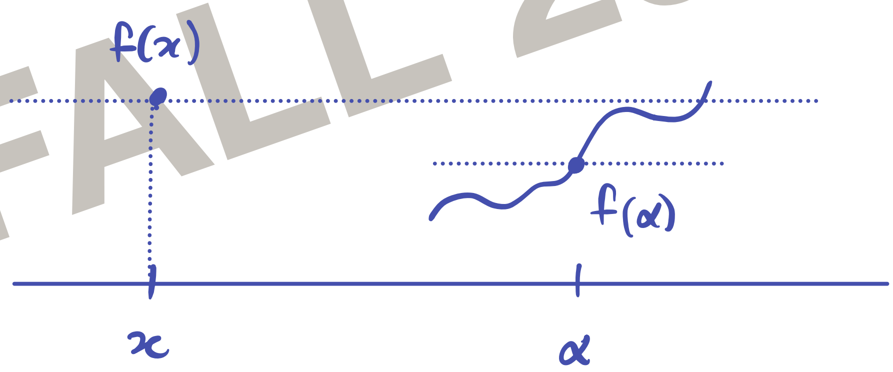
at α. Moreover since $\alpha = \inf A$, there exists a sequence $y_n \in A$ such that

$$\lim_{n \to \infty} y_n = \alpha .$$

By continuity, and since $f(y_n) \leq f(x)$ we have

$$f(\alpha) = \lim_{n \to \infty} f(y_n) \leq f(x)$$

Notice how this should lead to a contradiction since $f'(\alpha) > 0$.



The same argument as before gives $\delta' > 0$ such that $f(y) < f(\alpha)$ for all $y \in (\alpha - \delta', \alpha)$.

Choose $y \in (x, \alpha) \cap (\alpha - \delta', \alpha)$. Then $y \in A$ but also $y < \alpha = \inf \alpha$. This is a contradiction.

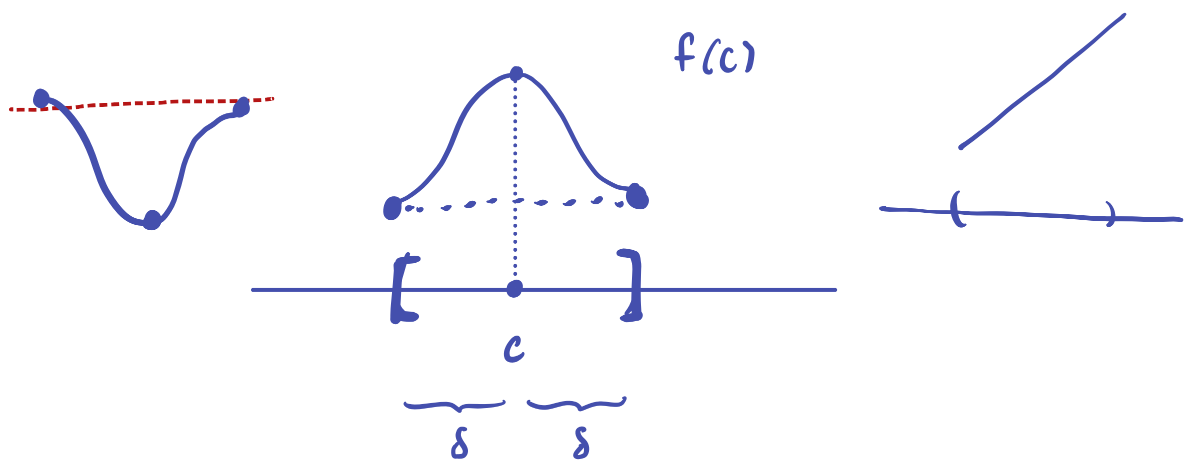
$\blacksquare$

Exercise: Repeat the proof for the case $f'(t) < 0 \ \forall\, t \in (a,b)$. What is the conclusion then? What can we say if

$$f'(t) \geq 0 \quad \forall\, t \in (a,b)?$$

Exercise: What about the reciprocal? Does f monotone increasing on (a,b) implies that $f'(t) > 0$ for all $t \in (a,b)$? If not, what is the example?

3. The derivative is helpful when we search for maxima and minima



We say that <u>f attains a local maximum at c</u>

if $f(c) \geq f(x)$, for all $x \in (c-\delta, c+\delta)$

This inequality must mean something for $f'(c)$

$$f(x) - f(c) \leq 0, \quad \text{for all } x \in (c-\delta, c+\delta)$$

Implies

- $\dfrac{f(x) - f(c)}{x - c} \leq 0$ for all $x > c$

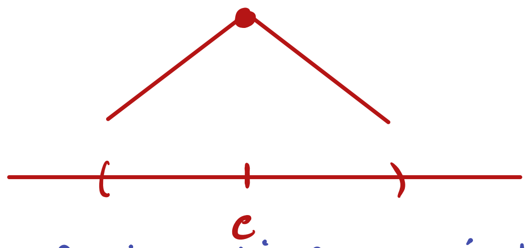
- $\dfrac{f(x) - f(c)}{x - c} \geq 0$ for all $x < c$

If the one sided limits of $\dfrac{f(x) - f(c)}{x - c}$

exist as $x \to c$, then:

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \geq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Remark:



However, if f is differentiable at c then:

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \geq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

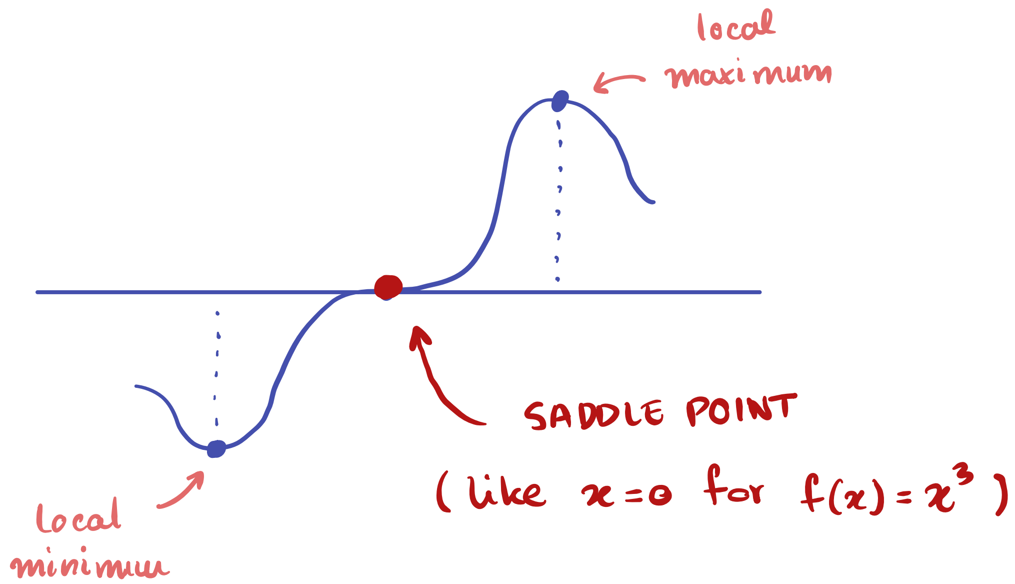
$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c), \qquad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'(c)$$

i.e. $\boxed{f'(c) = 0}$

we just proved the following theorem.

Fermat's Theorem: If $f : [a,b] \longrightarrow \mathbb{R}$ attains an interior local maximum (or minimum) at c, i.e. $a < t < b$. and f is differentiable at c then $f'(c) = 0$.

Remark: Although maxima and minima of differentiable functions satisfy $f' = 0$ the converse is not true.



<u>Definition</u> : c is a <u>critical point</u> of f if

$$f'(c) = 0.$$

$$\left.\begin{array}{l}\text{MAXIMUM}\\ \text{MINIMUM}\\ \text{SADDLE}\end{array}\right\}\ \text{critical points}$$

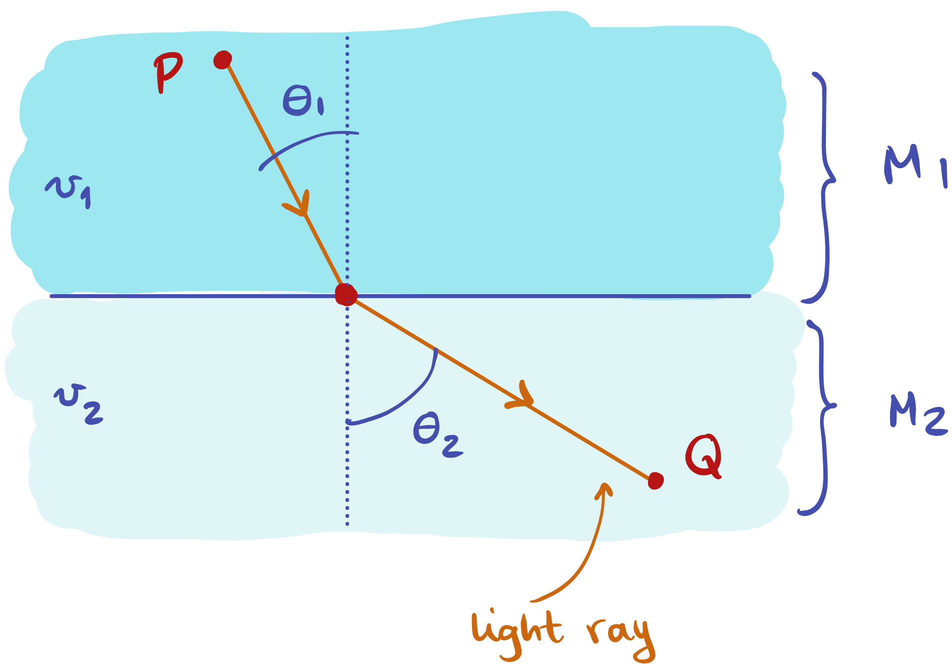
Solutions of { physical systems / optimisation problems } are usually critical points of functions. (If you are curious look for "Least action principle" and "gradient descent methods in machine learning.")

If we are searching for maxima or minima of a differentiable function f we can narrow our search to just those points which satisfy $f'(c) = 0$. Then we usually need to do something else in order to determine if the points we found are maxima, minima (local or global) or saddle.

Let us study an example of significant historical relevance:

The Law of light refraction

Snell's Law: Let M_1 and M_2 two media separated by a straight line. Assume that the speed of light is v_1 in M_1 and v_2 in M_2



then

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}}$$

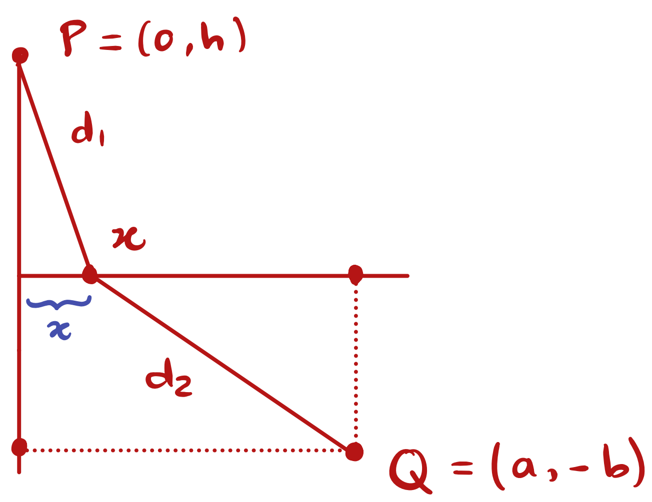
Remark: Fermat derived Snell's law from the following general principle:

Fermat's Principle: The path light takes to travel between two points P and Q is the path of least time.

Exercise: Show that Fermat's Principle implies Snell's law.

Hint: use coordinates

On the figure a, b and h are fixed. Write d_1 and d_2 as a function of x



Then use that $v_1 \cdot t_1 = d_1$ and $v_2 \cdot t_2 = d_2$ to write the total time

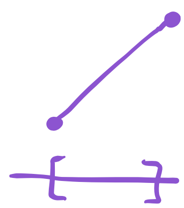
$$T = t_1 + t_2$$

as a function of x. Now find its minimum!

EXISTENCE OF CRITICAL POINTS

Let $f : [a,b] \longrightarrow \mathbb{R}$ continuous on $[a,b]$. and differentiable on (a,b). We proved that continuous functions on compact sets always attain a maximu or minima.

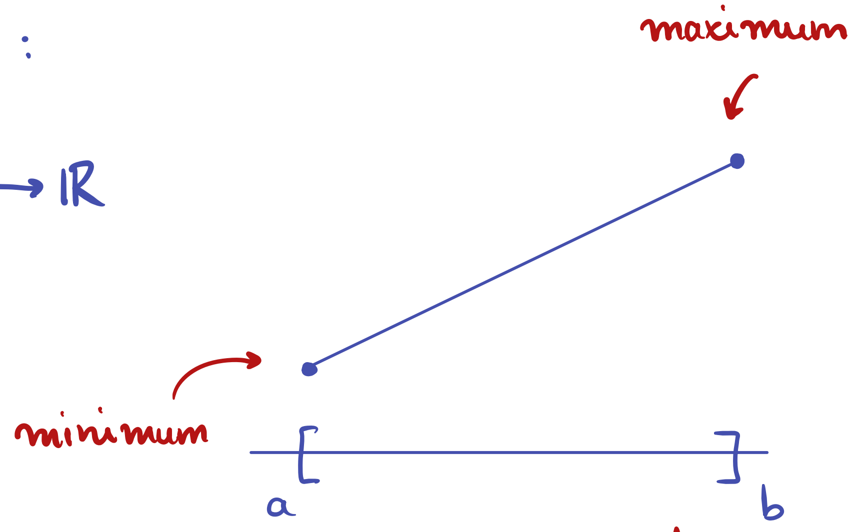
Could we use this fact to show that f admilts a critical point on $[a,b]$?



Well, we have to be careful. Local minima and maxima are critical points when the point is interior.

For example:

$$\begin{cases} f:[a,b] \longrightarrow \mathbb{R} \\ f(x) = x \end{cases}$$

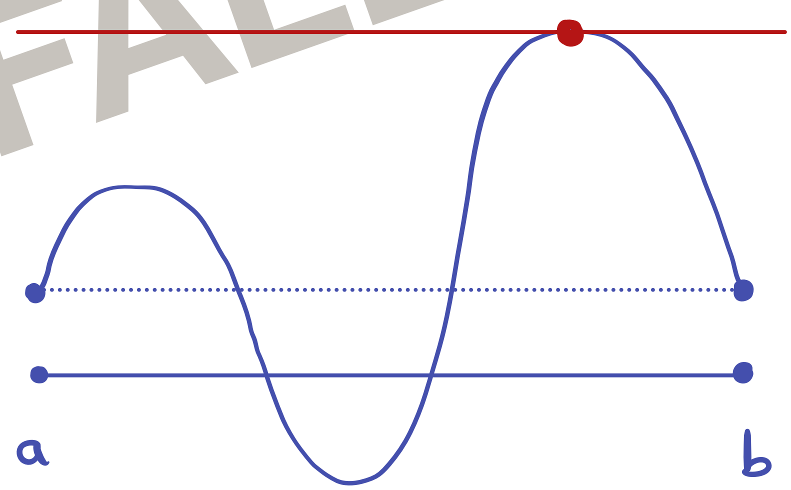


but $f'(x) = 1$ so no critical points!

However in several situations we can guarantee the existence of critical points.

<u>critical points</u> <u>when $f(a) = f(b)$</u>:

move this line until it touches for the 1st time.



The formalisation of this idea is known as:

<u>Rolle's Theorem</u>: Let $f:[a,b] \longrightarrow \mathbb{R}$, continuous on $[a,b]$ and differentiable on (a,b).

If $f(a) = f(b)$, then there exists $t \in (a,b)$ such that $f'(t) = 0$

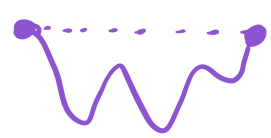
Proof: Since f is continuous on $[a,b]$ there exists a point $t \in [a,b]$ such that

$$f(t) = \max_{x \in [a,b]} f(x)$$

- If $a < t < b$ then $f'(t) = 0$ by Fermat's theorem, in which case we are done.
- If $t = a$ or $t = b$ then $f(a) = f(b) = \max_{x \in [a,b]} f(x)$

So we look at the minimum. Let $s \in [a,b]$ such that

$$f(s) = \min_{x \in [a,b]} f(x)$$



then,

- If $a < s < b$, Fermat's theorem implies $f'(s) = 0$.
- If $s = a$ or $s = b$, then

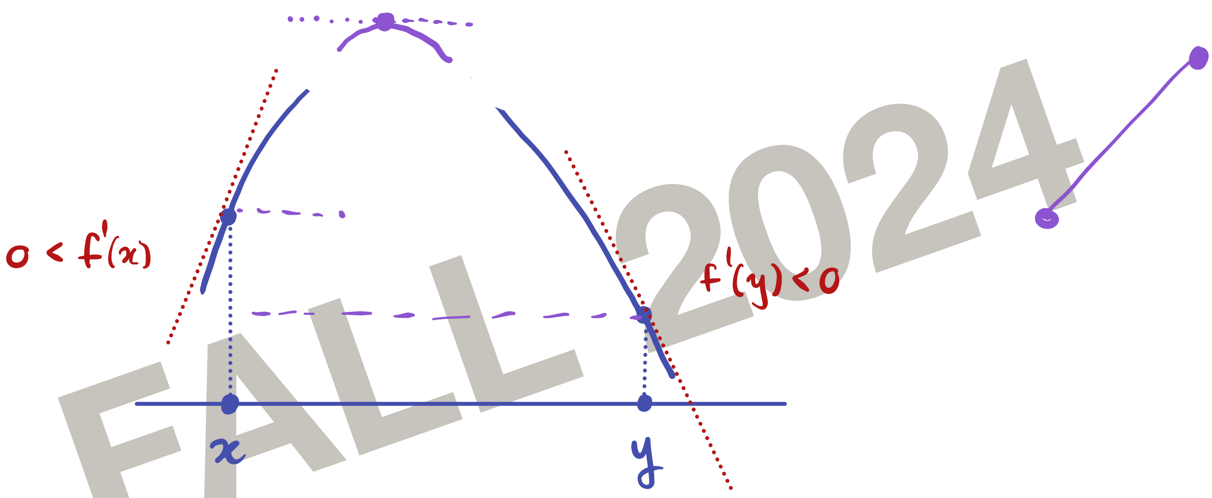
$$\min_{x \in [a,b]} f(x) = f(a) = f(b) = \max_{x \in [a,b]} f(x)$$

it follows that $\forall x \in [a,b]$ we have

$$f(x) = f(a) \qquad f(a) \leq f(x) \leq f(a)$$

so f is constant $\Rightarrow$ $f'(t)$ $\forall t \in [a,b]$. $\blacksquare$

Critical points when $f'(x) > 0 > f'(y)$:



In this situation neither x or y can be the maximum.

Proposition Let $f : (a,b) \longrightarrow \mathbb{R}$ be differentiable on (a,b). Assume $x < y \in (a,b)$ are such that $f'(x) > 0 > f'(y)$. Then there is $t \in (x,y)$ such that $f'(t) = 0$.

Proof By local comparison there exists a $\delta > 0$ such that $f(s) > f(x)$ for all $s \in (x, x+\delta)$. In particular

$$f(x) < \max_{s \in [x,y]} f(s)$$

Similarly,

$$f(y) < \max_{s \in [x,y]} f(s)$$

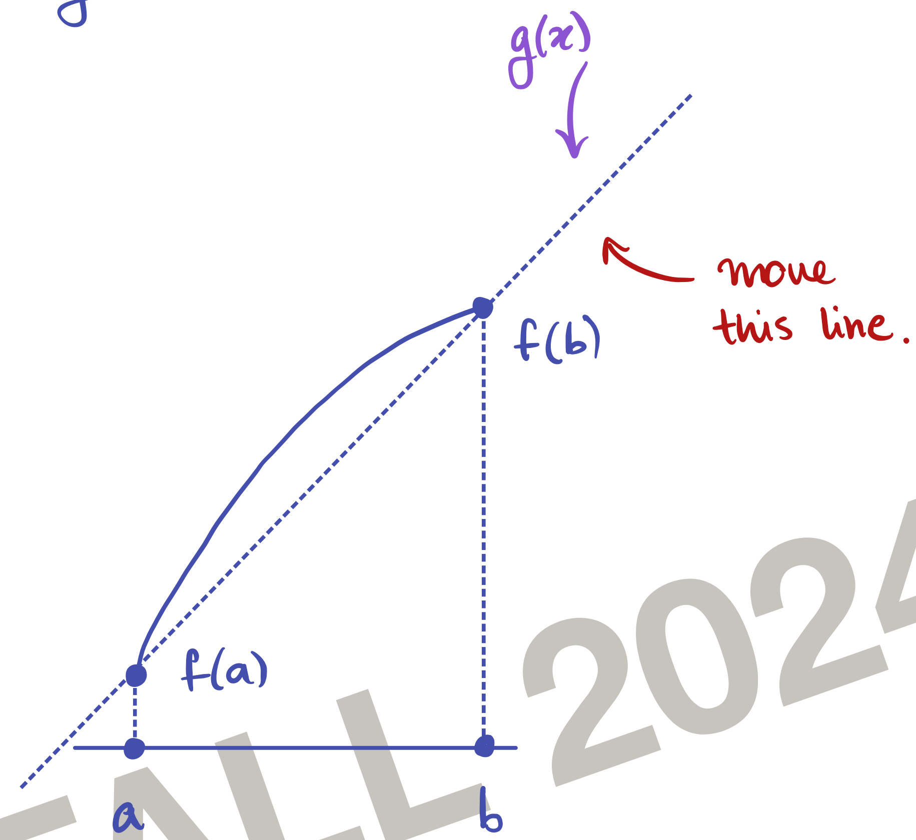
Therefore, the point $t \in [x,y]$ such that

$$f(t) = \max_{s \in [x,y]} f(s),$$

which exists because f is continuous, must be an interior point! Therefore, by Fermat's theorem $f'(t) = 0$.

TWO IMPORTANT APPLICATIONS

Even when $f(a) \neq f(b)$ we can always say something about the derivative:



The formalisation of this idea is

Cauchy's Mean Value Theorem:
Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there is $t \in (a,b)$ such that $f'(t) = \dfrac{f(b) - f(a)}{b - a}$.

Proof: The idea is that we can substract the function whose graph is the straight line, bringing the situation to one where we can apply Rolle's theorem.

Let $$g(t) = \frac{f(b) - f(a)}{b - a} \cdot (t - a) + f(a)$$

Then
$$\begin{cases} g \text{ is continuous on } [a,b] \\ g \text{ is differentiable on } (a,b) \\ g(a) = f(a) \\ g(b) = f(b) \end{cases}$$

Therefore the function $f - g$ satisfies the hypothesis of Rolle's theorem

$$\begin{cases} (f-g) \text{ is continuous on } [a,b] \\ (f-g) \text{ is differentiable on } (a,b) \\ (f-g)(a) = (f-g)(b) = 0 \end{cases}$$

By Rolle's theorem, there exists $t \in (a,b)$ such that $(f-g)'(t) = 0$

i.e. $$f'(t) = g'(t) = \frac{f(b)-f(a)}{b-a}$$

which is what we wanted to show. $\blacksquare$

This gives us another proof of:

<u>Corollary</u>: if $f'(t) > 0$ on (a,b) then f is strictly monotone increasing on (a,b).

<u>Proof</u>: Choose $x < y \in (a,b)$ and check you can apply Cauchy's MVT. Complete the proof!

Another application of the results from the previous section is

<u>The Intermediate Value Theorem for Derivatives</u>

Let $f : (a,b) \longrightarrow \mathbb{R}$ be differentiable on (a,b). Then, $f'(t)$ satisfies the intermediate value property, i.e.

$\forall x < y$ and any c
between $f'(x)$ and $f'(y)$
$\exists\, t \in (x,y)$ with $f'(t) = c$.

<u>Remark</u> This is true even when f' is not continuous!

<u>Proof:</u> Assume wlog that $f'(x) < f'(y)$.
Then $c \in (f'(x), f'(y))$.
Let $g(s) = c \cdot s - f(s) \quad \forall s \in (a,b)$.

$$\Rightarrow \quad \begin{aligned} g'(x) &= c - f'(x) > 0 \\ g'(y) &= c - f'(y) < 0 \end{aligned}$$

we prove that in this situation g must have a critical point $t \in (x,y)$.

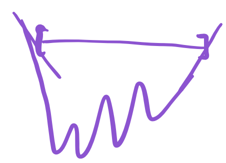
i.e.
$$\begin{aligned} 0 &= g'(t) \\ &= c - f'(t) \end{aligned}$$

therefore $f'(t) = c$ for $t \in (x,y)$ $\blacksquare$

<u>Corollary</u> Show that $f'(t) \neq 0 \ \forall t \in (a,b)$ implies that f is strictly monotone on (a,b).

<u>Proof</u> : There is a one line proof using the tools we have learned. Exercise!

<u>Review Problems</u>



1) Prove that in the following situations f has a global minimum:

a) $f : (a,b) \longrightarrow \mathbb{R}$, f differentiable on (a,b)

with $\lim\limits_{x \to a^+} f'(x) < 0 < \lim\limits_{x \to b^-} f'(x)$.

$\exists \, \delta > 0$ s.t. $f'(a+h) < 0$ for all $0 < h < \delta$. since $\lim\limits_{x \to a^+} f'(x) < 0$

In particular f is monotone decreasing on $(a, a+\delta)$. Similarly, we can assume f is monotone increasing on $(b-\delta, b)$. Then the minimum of f on $[a+\frac{\delta}{2}, b-\frac{\delta}{2}]$ is the global minimum of f on (a,b). Since $f(x) > f(a+\frac{\delta}{2}) > f(a+\delta)$ $\forall$ $x < a+\frac{\delta}{2} < a+\delta$. and we can argue similarly for $f(b-\delta/2)$.

b) $f:(a,b) \longrightarrow \mathbb{R}$, f differentiable on (a,b)

with $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = +\infty$.

$\forall R$; $\exists$ $\delta > 0$, $f(x) > R$ $\forall x \in (a, a+\delta] \cup [b-\delta, b)$.

Then, let $R = f(x_0)$, for some $x_0 \in (a,b)$. Then

$\min_{(a,b)} f = \min_{[a+\delta, b-\delta]} f$ which exists by the extreme value property.

c) $f:(a,b) \longrightarrow \mathbb{R}$, f differentiable on (a,b)

with $\lim_{x\to a^+} f'(x) < 0$ and $\lim_{x\to b^+} f(x) = +\infty$.

Practice!

2) Prove that in the following situations f has a global minimum:

a) $f:\mathbb{R} \longrightarrow \mathbb{R}$, f continuous on $\mathbb{R}$ and

$$\lim_{x\to -\infty} f(x) = \lim_{x\to +\infty} f(x) = +\infty.$$

From $\lim_{|x|\to\infty} f(x) = +\infty$ we know, $\exists R > 0$ such that $f(x) > f(0)$ for all $|x| > R$. Since f is continuous on $\mathbb{R}$ by the EVT there

is $x_0 \in [-2R, 2R]$ such that $f(x_0) \leq f(x)$ $\forall x \in [-2R, 2R]$.
If $|y| > 2R$ then, $f(y) > f(0) \geq f(x_0)$. Combining both it follows that x_0 is a global minimum for f.

b) $f: \mathbb{R} \to \mathbb{R}$, $\exists$ $R > 0$ such that

$f'(x) < 0 < f'(y)$ for all $x < -R < R < y$

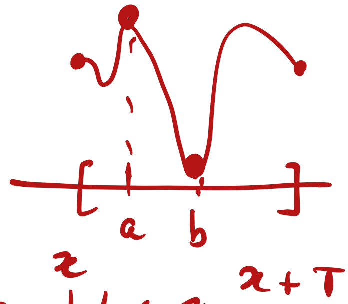
Practice!

3) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Show that if f has a strict local maximum and f is unbounded from above, then f admitts a local minimum.

Since f is continuous then it is bounded in $[-R, R]$. It follows that f must be unbounded on $\{x : |x| > R\}$ for all $R > 0$. Let x_0 be a strict local maximum of f. Let $R > |x_0|$. Then, there is $|y_0| > R$ such that. $f(y_0) > f(x_0)$. WLOG we can assume $x_0 < y_0$. We claim f admits a local minimum on (x_0, y_0). It is enough to show that the minimum of f in $[x_0, y_0]$ is interior. This must be true since $f(y_0) > f(x_0) > f(x_0 + h)$ for some $h \in (0, y_0 - x_0)$ small.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that if f is periodic then it admitts infinctly many global maxima and global minima.

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$



Claim:

$f(a+kT)$ are all global maxima $\forall k \in \mathbb{Z}$

$f(b+kT)$ are " " mima $\forall k \in \mathbb{Z}$

5) Prove the equivalence between sequential continuity and the definition of continuity using ε and δ.

Sequential Continuity $\Rightarrow$ (ε, δ) continuity: By contradiction.

Assume that f is not continuous at x_0. Then, there is $\varepsilon_0 > 0$ and $x_n \in \mathbb{R}$ such that $|x_n - x_0| < 1/n$ and $|f(x_n) - f(x_0)| > \varepsilon_0$.
In particular $\lim_{n\to\infty} x_n = x_0$ but $\{f(x_n)\}$ does not converge to $f(x_0)$.
This contradicts sequential continuity.

(ε, δ) continuity $\Rightarrow$ sequential continuity:

Practice. No need to use contradiction here.

6) Show that the following functions are <u>not</u> uniformly continuous on their domains

a) $f : (0, +\infty) \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$.

Choose $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$. Then $|x_n - y_n| < \frac{1}{n(n+1)}$ which is arbitrarily small but $|f(x_n) - f(y_n)| = 1$.

b) $f : (0, +\infty) \longrightarrow \mathbb{R}$, $f(x) = \sin\left(\frac{1}{x}\right)$.

Choose $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$ then $|x_n - y_n| \leq \frac{1}{\pi n}$ which is arbitrarily small , but $|f(x_n) - f(y_n)| = 1$.

c) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

If $x \in \mathbb{R}$ and $y = x + \delta$, for $\delta > 0$, then $|x - y| = \delta$ and $|f(x) - f(y)| = |(x+\delta)^2 - x^2| = \delta\,|2x + \delta| > \delta |x|$. Choosing $\delta = \frac{1}{n}$ and $x = n$. We see that f is not uniformly continuous. on $\mathbb{R}$.

<u>Hint:</u> Before you attempt a), b) or c) think about what you are trying to prove in general.

What does it mean for $f: A \to \mathbb{R}$ to <u>not</u> be uniformly continuous on A? Can we express it in terms of sequences?

7) Show that $f(x) = \sin(1/x)$ is uniformly continuous on $[c, +\infty)$ for all $c > 0$.

Idea: since $\lim_{x \to \infty} \sin(1/x) = 0$. the function fluctuates less as x becoms larger.

<u>Answer</u>: Fix $\varepsilon > 0$. $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0 \Rightarrow \exists R > 0$ s.t. $\left|\sin\left(\frac{1}{x}\right)\right| < \varepsilon/2$ if $x \in (R, +\infty)$. Since $\sin\left(\frac{1}{x}\right)$ is continuous on $[c, 2R]$, then $\exists\, \delta > 0$ s.t. $|x-y| < \delta \Rightarrow \left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{y}\right)\right| < \varepsilon$ as long as $x, y \in [c, 2R]$.

Let. $\delta' = \min(\delta, R) > 0$ then $|x-y| < \delta' \Rightarrow x, y \in [c, 2R]$ or $x, y \in [R, \infty)$

In both cases $\left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{y}\right)\right| < \varepsilon$.

8) Let $A \subset \mathbb{R}$ be a set. Prove the following are equivalent:

i) A is closed

ii) If $\{x_n\} \subset A$ and $\lim_{n\to\infty} x_n = L$ then $L \in A$.

$i \Rightarrow ii$: By contradiction. Assume for some $\{x_n\} \subset A$ we have $\lim_{n\to\infty} x_n = L \in \mathbb{R} \setminus A = B$. Since B is open, there is $\delta > 0$ such that $(L-\delta, L+\delta) \subset B$. In particular, $x_n \notin (L-\delta, L+\delta)$. This implies $|x_n - L| \geq \delta$ for all $n \in \mathbb{N}$. Contradicting $x_n \to L$.

$ii \Rightarrow i$: Practice!

APPLICATIONS OF THE MEAN VALUE THEOREM

THE EXTENDED MEAN VALUE THEOREM: If f and g are continuous on $[a,b]$ and differentiable on (a,b), then there exists $x \in (a,b)$ such that

$$\{f(b) - f(a)\} \times g'(x) = \{g(b) - g(a)\} \times f'(x)$$

Remark: The formula above comes from the expression

$$\frac{f'(x)}{g'(x)} = \frac{\dfrac{f(b)-f(a)}{b-a}}{\dfrac{g(b)-g(a)}{b-a}}$$

which is easier to remember. However we can only write this when $g'(x) \neq 0$ and $g(b) \neq g(a)$

For example, choosing $g(x) = x$ we recover the usual mean value theorem.

<u>Proof</u>: $h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))$

is continuous on $[a,b]$ and differentiable on (a,b).

Moreover,

$$\begin{cases} h(a) = f(a)g(b) - \cancel{f(a)g(a)} - g(a)f(b) + \cancel{g(a)f(a)} \\ h(b) = \cancel{f(b)g(a)} - f(b)g(a) - g\cancel{(b)f(b)} + g(b)f(a) \end{cases}$$

so $h(a) = h(b)$. and the result follows from Rolle's theorem.

The expression $$\frac{f'(x)}{g'(x)} = \frac{\dfrac{f(b)-f(a)}{b-a}}{\dfrac{g(b)-g(a)}{b-a}}$$

is what will lead us to L'Hôpital's Rule.

which helps with the evaluation of limits of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ when

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

L'Hôpital's Rule: Assume that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

If $\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$

also exists and both limits are equal.

Proof: When we say "$\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists"

we are assuming several things:

$\exists\, \delta > 0$, such that $\underline{\text{for all } x \in (a-\delta, a+\delta) \setminus \{a\}}$

i) $f'(x)$ and $g'(x)$ both exists

ii) $g'(x) \neq 0$.

Note that i) + ii)
do not need
to hold at $x = a$.

In fact, f and g do not need to be

defined at a for "$\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists" to make sense. Since we are assuming

$$\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$$

it makes sense to set $f(a) = g(a) = 0$ (where we redefine f and g if necessary). because now we can say f and g are continuous on

$$(a-\delta, a+\delta)$$

including a, therefore we can use Rolle's theorem and the extended mean value theorem on closed intervals contained in $(a-\delta, a+\delta)$.

<u>Claim</u>: $g(x) \neq 0$ for all $x \in (a-\delta, a+\delta) \setminus \{a\}$.

Assume $a < x < a+\delta$. If $g(x) = g(a)$ then by Rolle's theorem there would be $y \in (a, x)$ with $g'(y) = 0$, contradicting (iii). The proof for $a - \delta < x < a$ is similar. This proves the claim.

Now, since $g(x) \neq 0$ for all $x \neq a$, by the Extended Mean Value Theorem, we have that there exists α_x such that.

$$\frac{f'(\alpha_x)}{g'(\alpha_x)} = \frac{\frac{f(x) - f(0)}{x - 0}}{\frac{g(x) - g(0)}{x - 0}} = \frac{f(x)}{g(x)}$$

and $|\alpha_x - a| < |x - a|$. Given $\varepsilon > 0$ there exists $\delta' > 0$ such that

$$|y - a| < \delta \Rightarrow \left| \frac{f'(y)}{g'(y)} - L \right| < \varepsilon$$

where $L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. Therefore.

is we choose $|x - a| < \delta$ we have

$$|x - a| < \delta \Rightarrow |\alpha_x - a| < \delta$$

$$\Rightarrow \left| \frac{f'(\alpha_x)}{g'(\alpha_x)} - L \right| < \varepsilon$$

$$\Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon .$$

which is what we wanted to show.

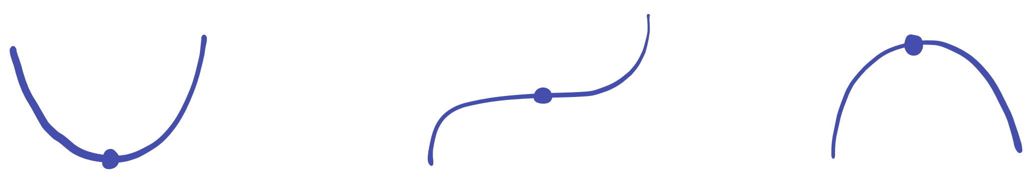
SECOND DERIVATIVES

When we say "$f''(x)$ exists", we are assuming

i) $\displaystyle\lim_{y \to x} \frac{f'(y) - f'(x)}{y - x}$ exists

ii) $f'(y)$ exists for all $y \in (x-\delta, x+\delta)$

The second derivatives can help us determine the local geometry of f near a critical point.



Theorem:

Let a be a critical point of f, i.e. $f'(a) = 0$.

- If $f''(a) > 0$ then f has a local minimum at a.
- If $f''(a) < 0$ then f has a local maximum at a.

Proof: We prove the case $f''(a) > 0$. The case $f''(a) < 0$ is similar.

Let $g(x) = f'(x)$. Then $\begin{cases} g(a) = 0 \\ g'(a) > 0. \end{cases}$

by local comparison.

$$\begin{cases} g(a+h) > 0 & \text{for small } h > 0 \\ g(a+h) < 0 & \text{for small } h < 0 \end{cases}$$

$$\Rightarrow \begin{cases} f'(a+h) > 0 & \text{for small } h > 0 \\ f'(a+h) < 0 & \text{for small } h < 0 \end{cases}$$

By the MVT, it follows that, for a small $\delta > 0$ f is strictly monotone decreasing on $[a-\delta, a]$ and strictly monotone increasing on $[a, a+\delta]$. ∎

What about the reciprocal?

Example: $f(x) = x^8$. It has a minimum at $x = 0$ but $f''(0) = 0$.

Theorem: Assume $f''(a)$ exists and f has a local minimu at a. Then $f''(a) \geq 0$.

Proof: $f''(a)$ exists $\Rightarrow$ f differentiable at a.

f has a minmu at a $\Rightarrow$ $f'(a) = 0$. If $f''(a) < 0$ then f would also have a local maximu at a. Meaning f would be constant and $f''(a) = 0$, contradicting $f''(a) < 0$. Therefore $f''(a) \geq 0$.

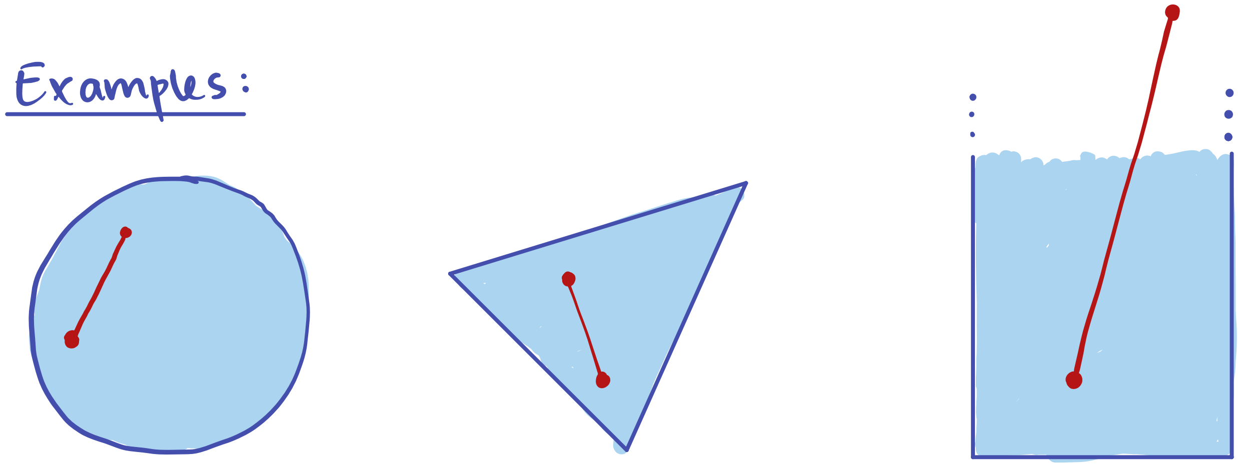
$\blacksquare$

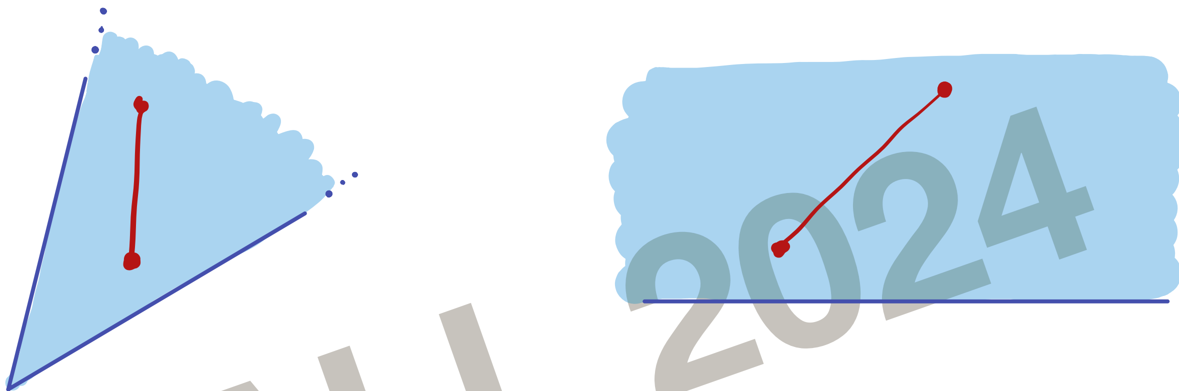
What if $f''(x) > 0$ on a whole interval?

CONVEXITY

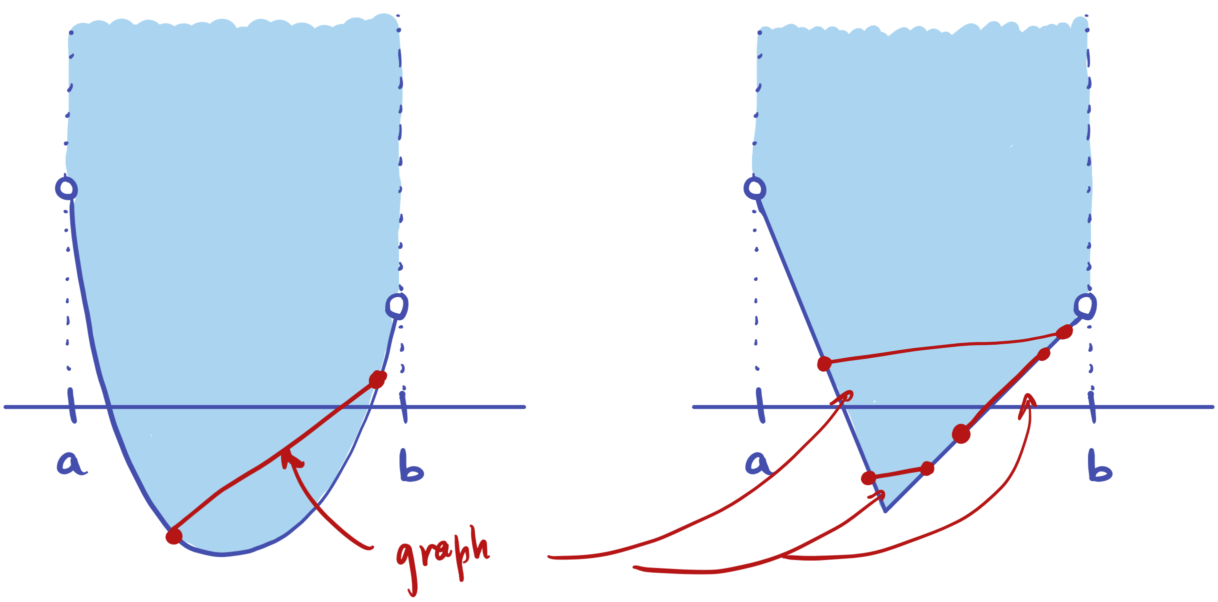
A set $U \subset \mathbb{R}^2$ is called <u>convex</u> if for any two points $p, q \in U$, the straigth line segment joining p and q is also contained in U.

Examples:

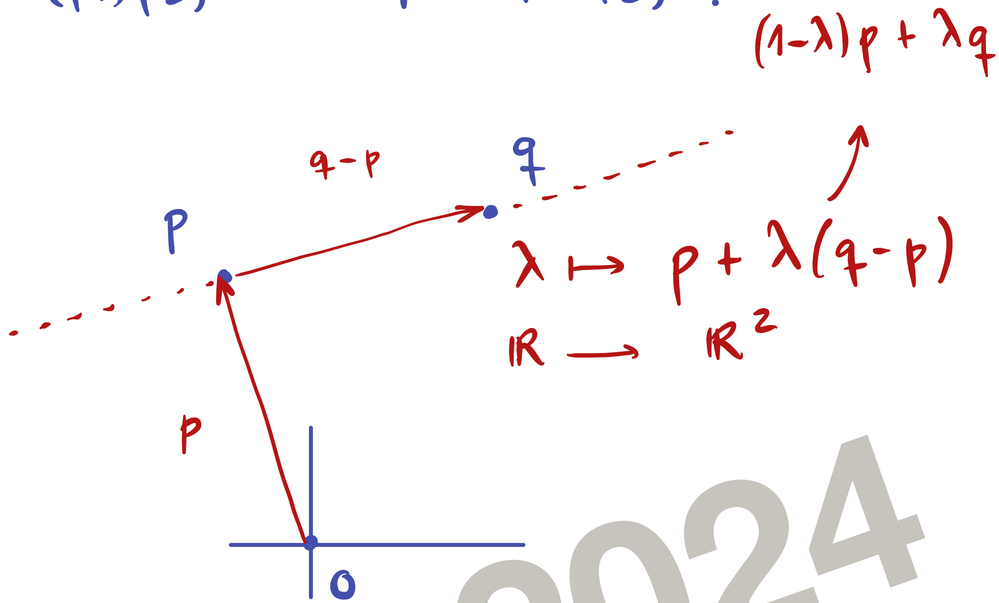




Definition: We say that a function $f:(a,b)\to\mathbb{R}$ is <u>convex</u> if the region above its graph is convex.



How do we describe the line segment joining two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$?



i.e. we can parametrise the segment as

$$\begin{cases} \gamma : [0,1] \longrightarrow \mathbb{R}^2 \\ \gamma(t) = (1-t)\,p + t\,q \end{cases}$$

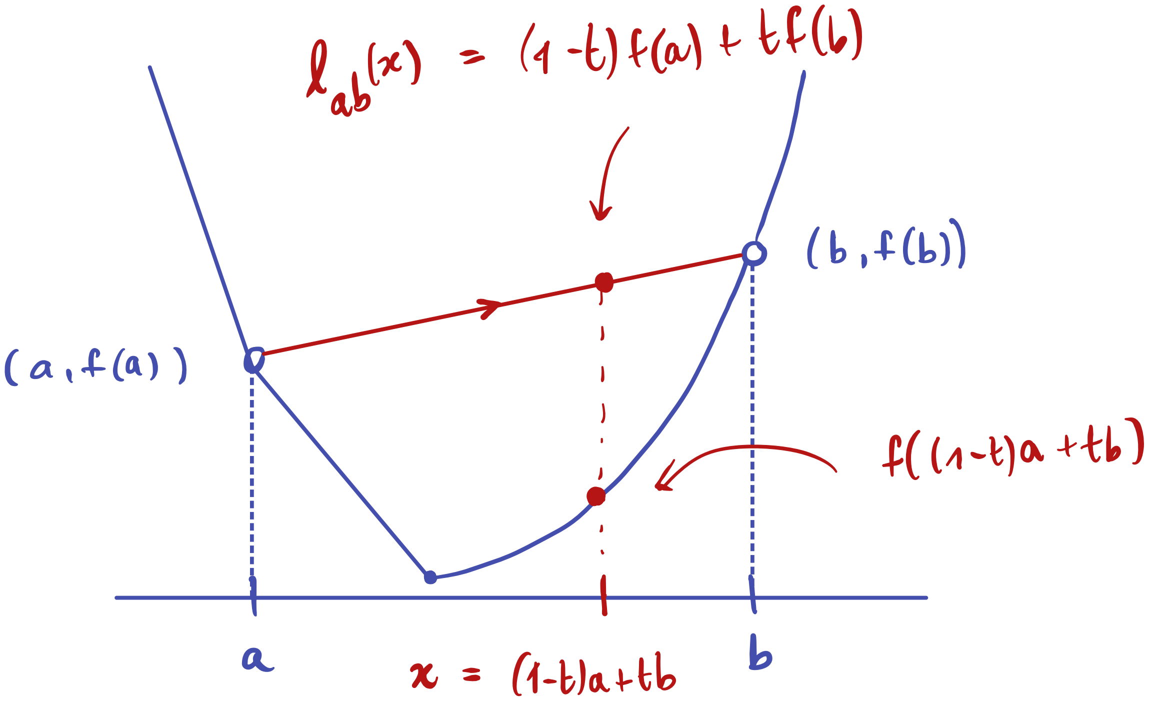
In coordinates:

$$\gamma(t) = (1-t)(p_1, p_2) + t\,(q_1, q_2)$$

$$= \big((1-t)p_1 + t q_1 \,,\; (1-t)p_2 + t q_2 \big)$$

What does this formula gives us when

we apply it to the case of a convex function?



Convexity of f on I translates to:

$$(*) \begin{cases} \text{For all } a < b \in I \text{ and } t \in [0,1], \text{ we have:} \\ f((1-t)a + tb) \leq (1-t)f(a) + tf(b) \end{cases}$$

Equivalently we can describe the segment as the graph of a linear function:

$$\ell_{ab}(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Note that ℓ_{ab} is linear and $\begin{cases} \ell_{ab}(a) = f(a) \\ \ell_{ab}(b) = f(b) \end{cases}$

Then convexity of f on I translates to:

(**) For all $a < x < b \in I$, we have $f(x) \leq \ell_{ab}(x)$

i.e.

$$f(x) \leq \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Exercise: Check that (*) and (**) are equivalent.

Definition we say that f is strictly convex on I if: for all $a, b \in I$ and $t \in (0,1)$ we have

$$f((1-t)a + tb) < (1-t)f(a) + t f(b)$$

or equivalently if: for all $a < x < b \in I$ we have

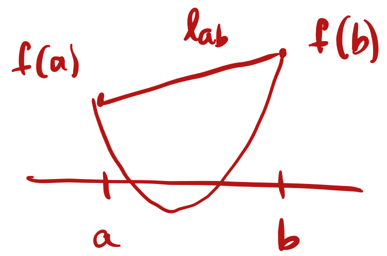
$$f(x) < \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

There is a simple criterium that implies strict convexity:

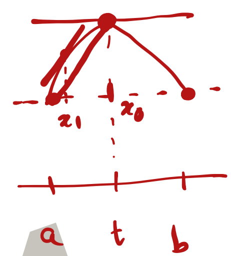
Theorem: If f is differentiable on I and

f' is strictly monotone increasing, then f is strictly convex.

<u>Proof</u>: We want to show that $\forall\, a < x < b \in I$ we have $f(x) < l_{ab}(x)$.



<u>Case $f(a) = f(b)$</u>: In this case



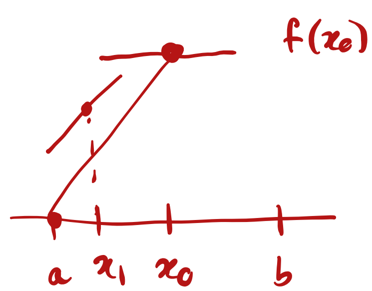
$$l_{ab}(x) = \text{constant} = f(a) = f(b)$$

To proceed by contradiction, let's assume this does not happen, i.e.

$$f(x) \geq f(a) = f(b) \text{ for some } x \in (a,b)$$

We conclude that f attains an interior maximum

$$x_0 \in (a,b).$$



At this point $f'(x_0) = 0$.

On the other hand, we can apply the MVT on $[a, x_0]$ to obtain the existence of

$$x_1 \in (a, x_0)$$

such that

$$f'(x_1) = \frac{f(x_0) - f(a)}{x_0 - a} \geq 0$$

Since $x_1 < x_0$ but $f'(x_1) \geq 0 = f'(x_0)$

this contradicts that f' is strictly increasing

this concludes the case $f(a) = f(b)$.

<u>General case:</u> let

$$g(x) = f(x) - \ell_{ab}(x)$$

$$= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

constant!

monotone

Then
$$\begin{cases} g(a) = 0 \\ g(b) = 0 \\ g'(x) = f'(x) - \left(\frac{f(b) - f(a)}{b - a}\right) \end{cases}$$

monotone

In particular, we can apply the previous case to g and conclude.

$$g(x) < 0 \quad \forall x \in (a, b)$$

$$\Rightarrow \quad f(x) < \ell_{ab}(x) \quad \forall x \in (a, b).$$

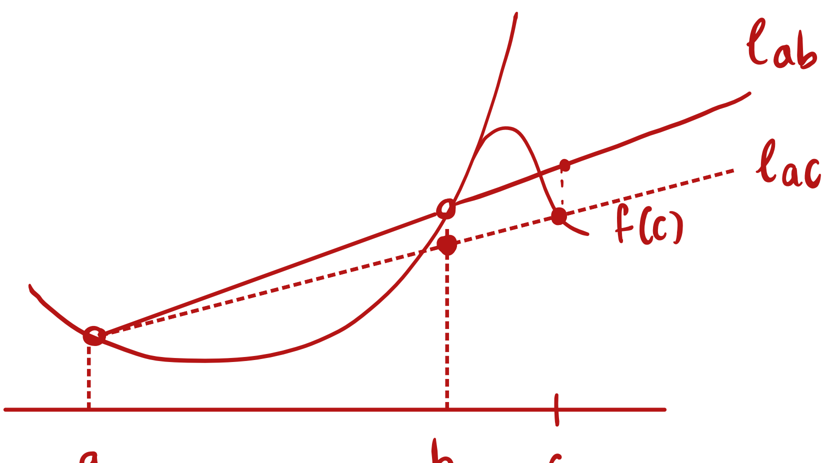
Remark: Part of the relevance of convex functions comes from the following facts:

Exercise:

i) If f is convex on I then f is continuous on I

ii) If f is strictly convex on I then f has at most one critical point on I, which is a global minimum.

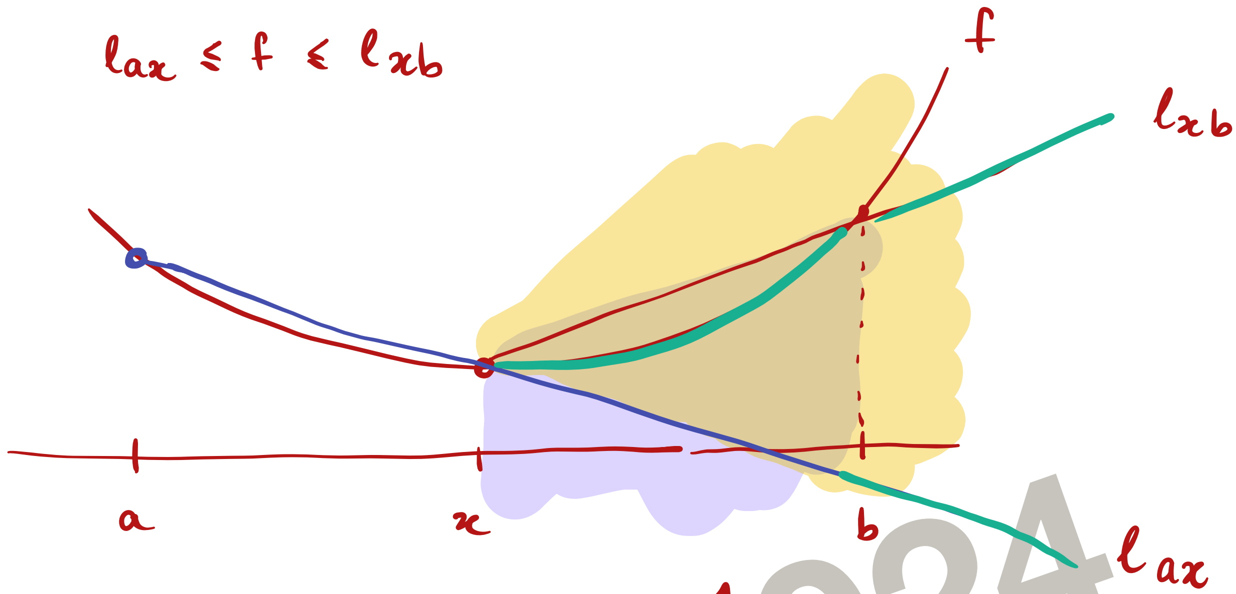
Discussion:



$$\boxed{\begin{array}{c} \text{Assume } f(c) < l_{ab}(c) \\ \Downarrow \\ f(b) > l_{ac}(b) \end{array}}$$

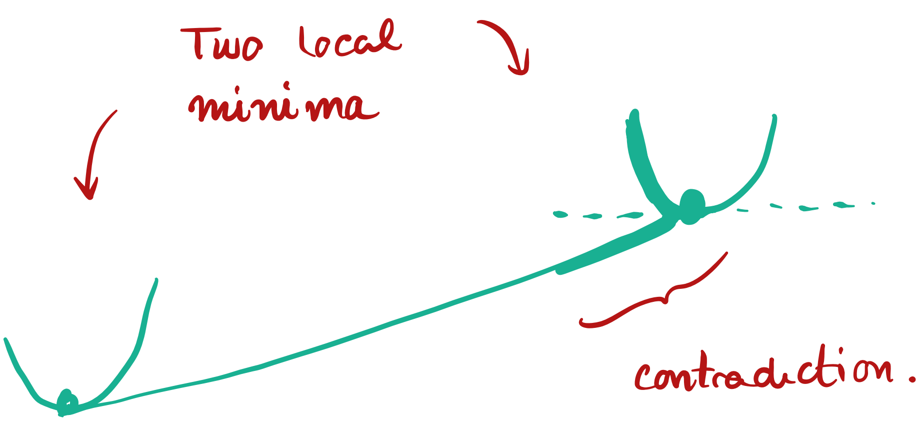
After you prove the formula above we see that on $[x, b)$ we must have

$$l_{ax} \le f \le l_{xb}$$



This should lead you to the proof of continuity

To rule out multiple critical points you can study case by case, e.g.



Curves on the Plane and the Inverse Function Theorem

Definition: A *curve* is a map $c : I \longrightarrow \mathbb{R}^2$ where I is an open interval.

Remark: Note that $c(t) = (u(t), v(t))$, where $u : I \longrightarrow \mathbb{R}$ and $v : I \longrightarrow \mathbb{R}$. The idea is that each coordinate is a map from $\mathbb{R}$ to $\mathbb{R}$.

Definition: A curve $c(t) = (u(t), v(t))$ is called

continuous if and only if u & v are continuous

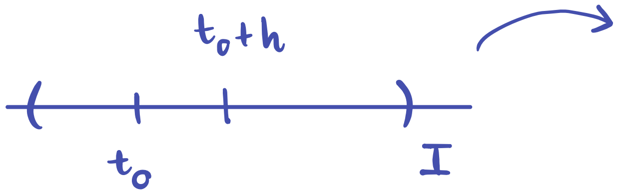
and

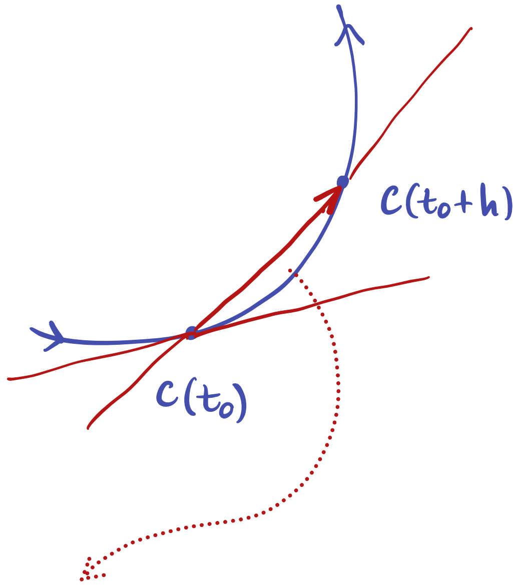
differentiable if and only if u & v are differentiable

Definition: If $c : I \longrightarrow \mathbb{R}^2$ is a differentiable curve then its derivative is the map.

$$\frac{dc}{dt}(t) = c'(t) = (u'(t), v'(t)).$$

Geometric significance:





$$c(t_0+h) - c(t_0) = \big(u(t_0+h), v(t_0+h)\big) - \big(u(t_0), v(t_0)\big)$$

$$= \big(u(t_0+h) - u(t_0),\ v(t_0+h) - v(t_0)\big)$$

Therefore:

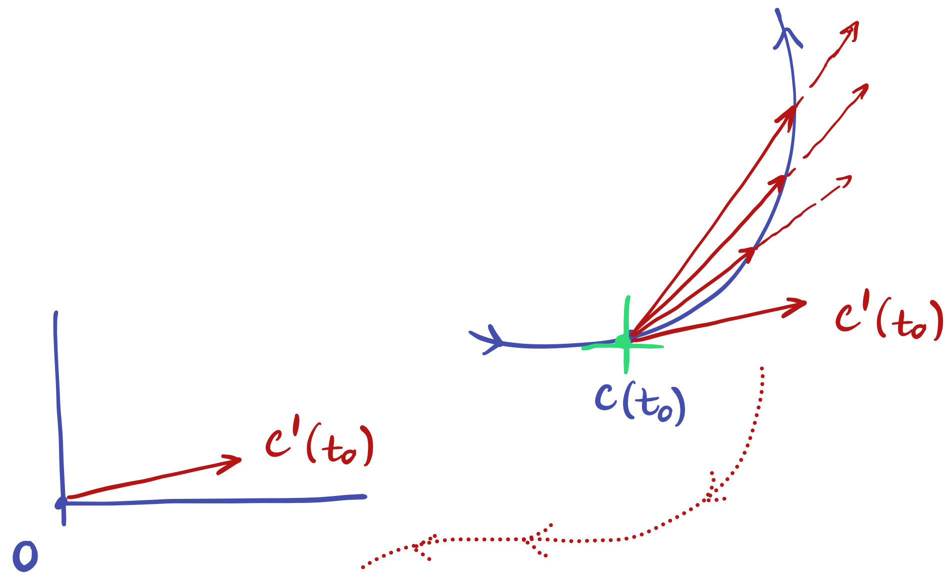
$$\frac{1}{h}\left\{c(t_0+h) - c(t_0)\right\} = \left(\frac{u(t_0+h) - u(t_0)}{h},\ \frac{v(t_0+h) - v(t_0)}{h}\right)$$

$$\downarrow \quad \text{as } h \to 0 \quad \downarrow$$

$$= \big(u'(t_0),\ v'(t_0)\big)$$

$$= c'(t_0)$$

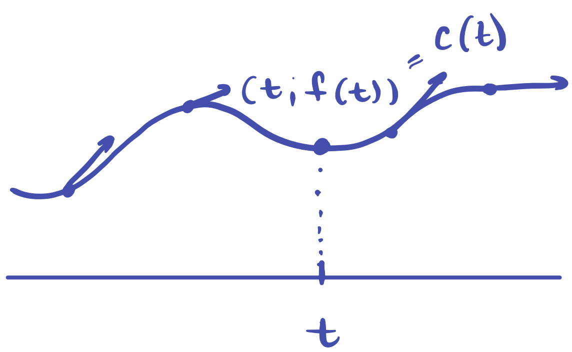
<u>Remark</u>: We think of $c'(t)$ as a vector with base point on $c(t_0)$ even though it is not.



Examples:

The graph of a function: If $f : I \to \mathbb{R}$ then $c(t) = (t, f(t))$ is a curve whose image is the graph of f. If f is differentiable then

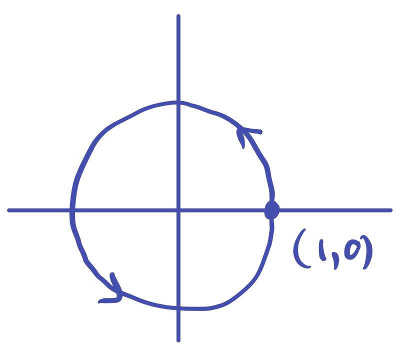
$$c'(t) = (1, f'(t))$$



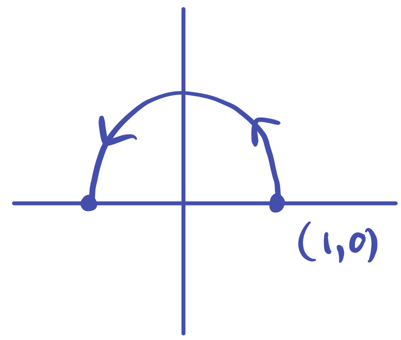
$$c : I \longrightarrow \mathbb{R}^2$$

The circle:

$$\begin{cases} c : \mathbb{R} \longrightarrow \mathbb{R}^2 \\ c(t) = (\cos(t), \sin(t)) \end{cases}$$



$$\begin{cases}\tilde{c} : [-1,1] \longrightarrow \mathbb{R}^2 \\ \tilde{c}(t) = \left(-t, \sqrt{1-t^2}\right)\end{cases}$$



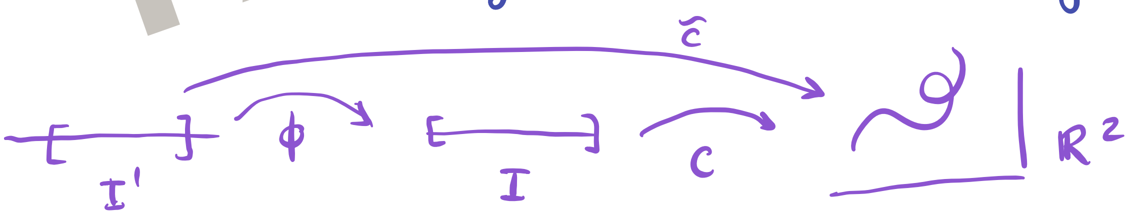
$\tilde{c}$ on $[-1,1]$ and c on $[0,\pi]$

describe the "same" curve

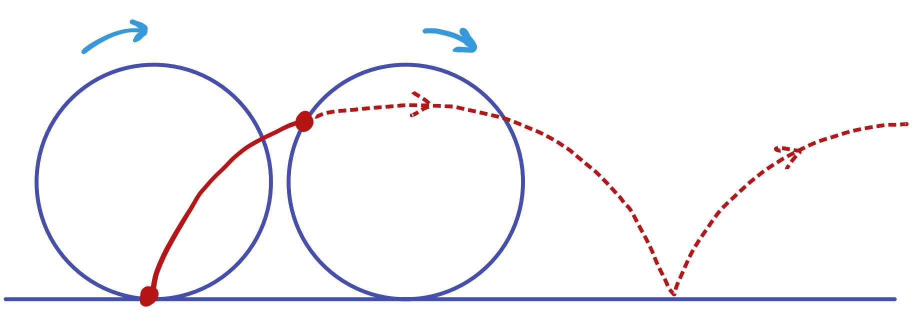
<u>Definition</u> : We say that $\tilde{c} : I' \longrightarrow \mathbb{R}$ is a <u>reparametrisation</u> of $c : I \to \mathbb{R}$ if there is a map $\phi : I' \longrightarrow I$ such that

$$\tilde{c} = c \circ \phi$$

and ϕ is strictly monotone increasing.



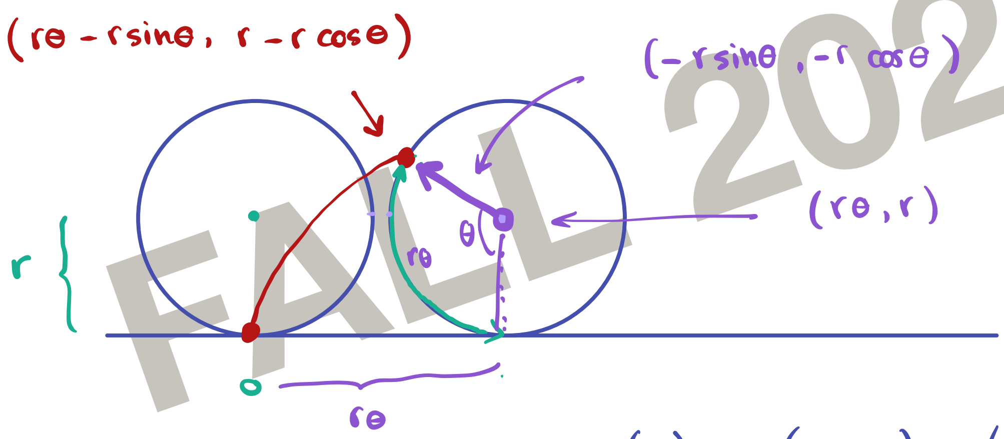
<u>The cycloid</u>: Is the curve described by a point on a rolling wheel (with no sliding)



The cycloid is a very "famous" curve because it has very interesting mechanical properties.

ESPECIAL LECTURE ON WEDNESDAY 2PM

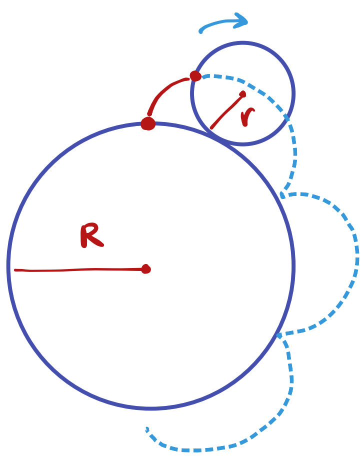
From light rays to soap bubbles: stories from the calculus of variations
CLORE 1 MAR 2PM



$$c(\theta) = (r\theta, r) + (-r\sin\theta, -r\cos\theta)$$

$$= r(\theta - \sin\theta, 1 - \cos\theta)$$

<u>Exercise</u> (The epicycloid)



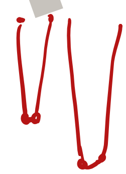
Compute the formula for the epicycloid in terms or r, R and the rotation angle. Using the same strategy.

<u>Exercise</u> *:

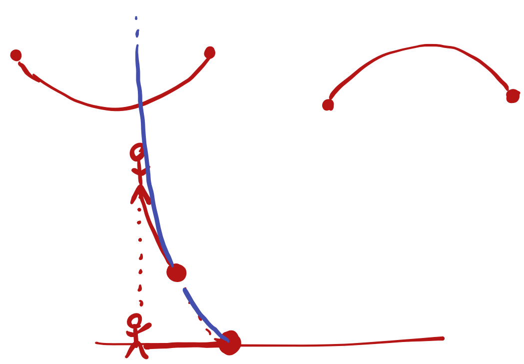
i) The epicycloid closes if and only if $\frac{R}{r} \in \mathbb{Q}$.

ii) If $\frac{R}{r} \notin \mathbb{Q}$ then the epicycloid touches the circle with radius R on a dense set!

Other curves worth mentioning are:



- The catenary
- The tractrix.

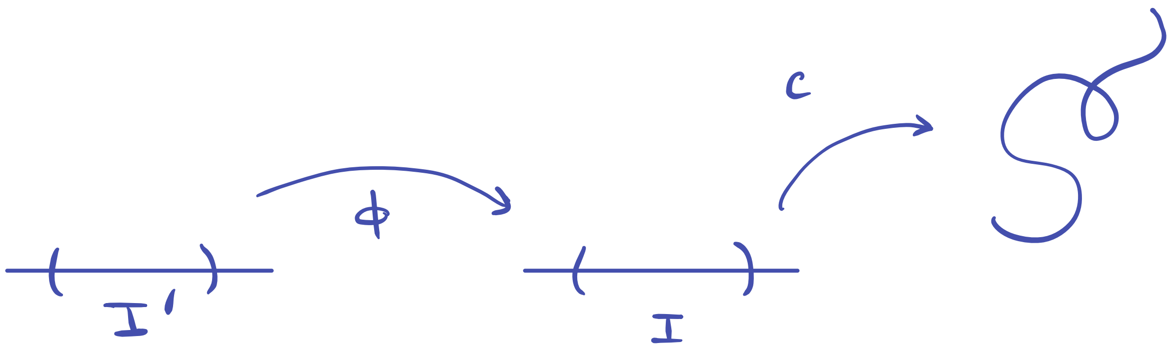


<u>Definition</u> We say that a differentiable curve $c : I \longrightarrow \mathbb{R}$ is <u>parametrised with unit speed</u> if $|c'(t)| = 1$, $\forall t \in I$.

i.e.

$$1 = |c'(t)| = \left((u'(t))^2 + (v'(t))^2\right)^{1/2}.$$

Question: When can a curve be reparametrised to have unit speed?



We will answer this question using an important theorem:

The Inverse Function Theorem: Let $f : I \longrightarrow \mathbb{R}$ be differentiable on I with $f' : I \rightarrow \mathbb{R}$ continuous. Assume $f'(a) \neq 0$. Then there exist open intervals $a \in J \subset I$ and $J' \subset \mathbb{R}$

i) f is a bijection between J and $J' = f(J)$.

$f(f^{-1}(x)) = x$

$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

$(f^{-1})'(x) = \frac{1}{f'(f(x))}$

ii) $f^{-1} : J' \longrightarrow J$ is differentiable and

$$\left(\frac{d}{dy} f^{-1}\right)(y) = \frac{1}{f'(f^{-1}(y))} .$$

Moreover, J can be choosen to be the largest $J \subset I$ such that :

$$a \in J \quad \text{and}$$
$$f'(t) \neq 0 \quad \text{for all } t \in J .$$

We begin with the following general question:

<u>Q</u>: Say $f : I \to J$ is a bijective function between open intervals. which properties of f are shared by its inverse function f^{-1} ?

<u>Remark</u>: Remember f^{-1} is defined as

Given $y \in J$, $f^{-1}(y) = x$ where x is the only point $x \in I$ such that $f(x) = y$.

This makes sense only because f is bijective.

Q. We can refine the question.

- Is f^{-1} continuous when f is continuous?
- Is f^{-1} differentiable when f is differentiable?

Theorem: If $f : I \longrightarrow J$ is bijective and continuous on I then f is strictly monotone.

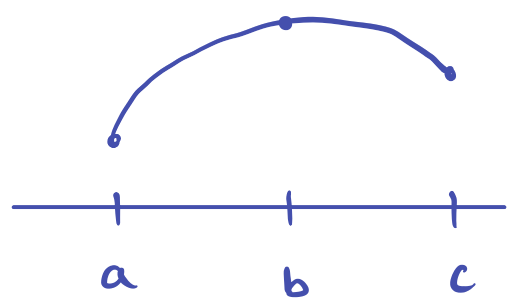
Proof:

Claim 1: Given $a < b < c \in I$ then either

(1) $f(a) < f(b) < f(c)$, or

(2) $f(a) > f(b) > f(c)$.

Proof of Claim 1: Assume $f(a) < f(c)$. If $f(b) > f(c)$, then we have the situation



In other words, $f(c) \in [f(a), f(b)]$ and by the intermediate value theorem we find $x \in [a,b]$ with $f(x) = f(c)$, contradicting the injectivity of f.

<u>Claim 2</u>: Given $a<b<c<d \in I$ then either

(1) $f(a) < f(b) < f(c) < f(d)$ or

(2) $f(a) > f(b) > f(c) > f(d)$

<u>Proof of Claim 2</u>: Apply claim 1 to $a<b<c$ then to $b<c<d$.

<u>Claim 3</u>: f is monotone.

<u>Proof of Claim 3</u>: Take any $a<b \in I$. then given any other two $x<y \in I$ we have one of the following.

$$\begin{cases} x < y < a < b \\ x < a < y < b \\ a < x < y < b \\ a < x < b < y \\ a < b < x < y \end{cases}$$

but in all cases we can apply claim 2. and conclude.

$$f(a) < f(b) \Rightarrow f(x) < f(y) \quad \text{or}$$

$$f(a) > f(b) \Rightarrow f(x) > f(y). \qquad \blacksquare$$

We are ready to answer our first question:

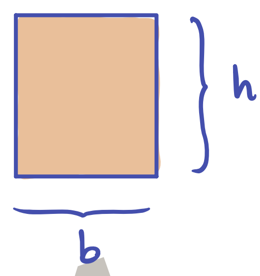
Theorem: If $f : I \longrightarrow J$ is bijective and continuous on I then f^{-1} is continuous on J.

Proof:

The Integral

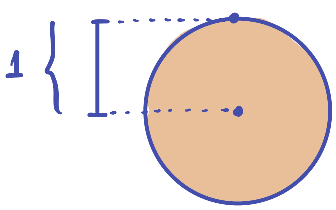
The concept of Area is a simple and intuitive one yet formalising presents difficult challenges.

In the case of a rectangle we define its area as: $b \times h =$



Then we are used to say something along the lines of: "the area of a region is the number of squares of length 1 that fit into the region."

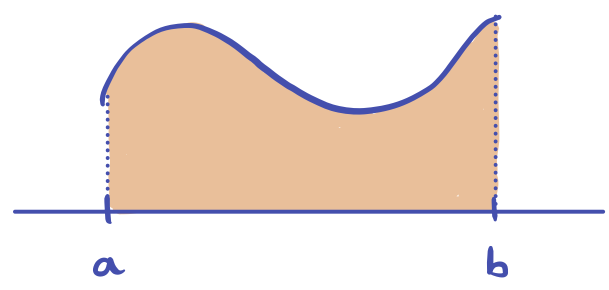
For example:



Area $= \pi$ so

π squares of side 1 ?

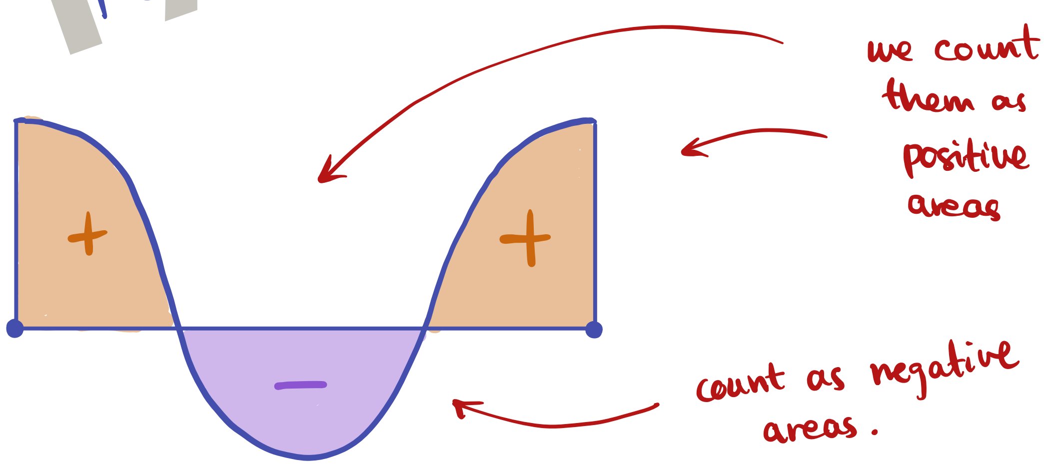
The notion of integral formalises the idea of area under the graph of a function:



The word "under" needs to be carefully explained:

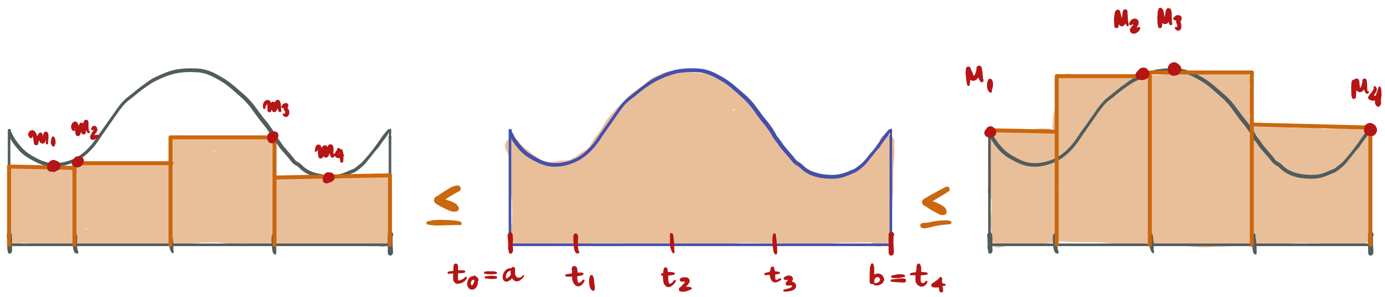
the types of regions the integral considers are regions between the graph of a function and the x-axis. Moreover, areas in the lower half plane are considered negative:

For example:



The idea behind the notion of integral is comparing the areas of rectangles containing the region with the areas of rectangles contained in the region.

The idea is in the following figure:



The areas of the rectangles containing the region are:

$$S = M_1(t_1-t_0) + M_2(t_2-t_1) + M_3(t_3-t_2) + M_4(t_4-t_3)$$

The areas of the rectangles contained in the region are:

$$s = m_1(t_1-t_0) + m_2(t_2-t_1) + m_3(t_3-t_2) + m_4(t_4-t_3)$$

The area of the region under the graph should satisfy:

$$s \leq \text{Area} \leq S$$

regardless of the subdivision.

We now formalise these ideas:

Definition: A partition of $[a,b]$ is a finite collection of points contained in $[a,b]$ and containing both a and b, i.e. a partition is

$$a = t_0 < t_1 < t_2 < \dots < t_n = b.$$

We usually denote a partition P as

$$P : a = t_0 < t_1 < \dots < t_n = b$$

or $$P = \{t_0, t_1, \dots, t_n\}$$

<u>Definition</u>: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Given a partition P of $[a,b]$ let:

$$\begin{cases} m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \} \\ M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \} \end{cases}$$

We define:

- The <u>lower sum of f for $\overbrace{P}^{\{a=t_0<t_1<\dots<t_n=b\}}$</u> as

$$L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$$

- The <u>upper sum of f for P</u> as

$$U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$$

It follows immediately from the definition that for any partition P of $[a,b]$:

since $m_i \leq M_i$

$$L(f,P) \leq U(f,P).$$

However, what is less obvious is the following:

Lemma: Given P and Q partitions of $[a,b]$ we have

$$L(f,P) \leq U(f,Q).$$

Proof:

Claim 1: If $P \subseteq P'$ then

$$\underbrace{L(f,P) \leq L(f,P')}_{} \leq U(f,P') \leq U(f,P)$$

Proof of Claim 1: We prove $L(f,P) \leq L(f,P')$. $(*)$

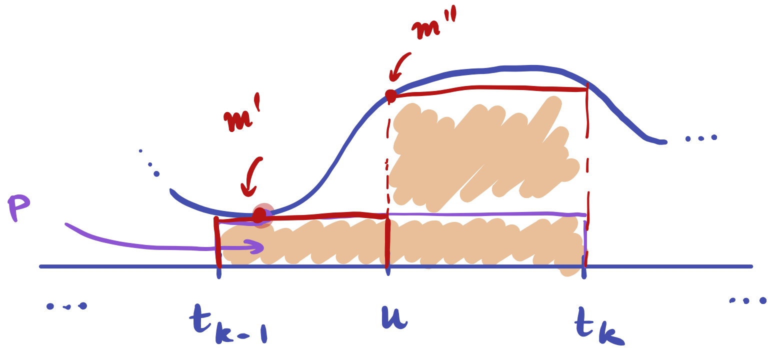
Note that there are portitions.

$$P = P_1 \subseteq P_2 \subseteq \dots \subseteq P_\alpha = P'$$

such that P_{k+1} contains exactly one more point than P. In particular, it is enough to prove $(*)$ for this case:

$$\begin{cases} P = \{t_0, \dots, t_n\} \\ P' = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\} \end{cases}$$

new point.



Let $\quad m' = \inf\{f(x) : x \in [t_{k-1}, u]\}$

$\quad\quad m'' = \inf\{f(x) : x \in [u, t_k]\}$

Note that $\quad$ (remember $m_k = \inf\{f(x) : x \in [t_{k-1}, t_k]\}$)

$$[t_{k-1}, u] \subset [t_{k-1}, t_k] \Rightarrow m' \geq m_k$$

$$[u, t_k] \subset [t_{k-1}, t_k] \Rightarrow m'' \geq m_k$$

Therefore:

$$\begin{aligned} L(f,P') - L(f,P) &= m'(u - t_{k-1}) + m''(t_k - u) - m_k(t_k - t_{k-1}) \\ &\geq m_k(u - t_{k-1}) + m_k(t_k - u) - m_k(t_k - t_{k-1}) \\ &\geq 0. \end{aligned}$$

The proof of $U(f,P') \leq U(f,P)$ is similar.

This proves the Claim. To prove the lemma let P and Q be arbitrary partitions of $[a,b]$. There is a partition that refines both P and Q. Let $R = P \cup Q$. By the claim we have:

$$\underline{\underline{L(f,P)}} \leq L(f,R) \leq U(f,R) \leq \underline{\underline{U(f,Q)}}$$

$\blacksquare$

We have shown that the set of all lower sums lies below the set of all upper sums.

In particular:

$$\sup\left\{L(f,P) : \underset{\text{of } [a,b]}{P \text{ partition}}\right\} \leq \inf\left\{U(f,P) : \underset{\text{of } [a,b]}{P \text{ partition}}\right\}$$

In general both sets might be far appart. and in that case we say the function is <u>not</u> <u>integrable</u>:

<u>Exercise</u>: show that $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$ is not integrable.

<u>Definition</u>: we say that a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ is integrable if.

$$\sup\left\{L(f,P) : \underset{\text{of } [a,b]}{P \text{ partition}}\right\} = \inf\left\{U(f,P) : \underset{\text{of } [a,b]}{P \text{ partition}}\right\}$$

and we denote this number as $\displaystyle\int_a^b f(x)\,dx$.

COURSEWORK 3

(2 Problems due on Friday 17 March)

Problem 1: Let A and B be non-empty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$ and that the equality holds if and only if for all $\varepsilon > 0$, there are $a \in A$ and $b \in B$ such that $b - a < \varepsilon$.

Problem 2: Using lower and upper sums, show that the function $t \longmapsto t^2$ is integrable on $[0, x]$ for all $x > 0$ and that $\int_0^x t^2 \, dt = \frac{x^3}{3}$.

Problem 3: Using lower and upper sums, show that if f is integrable on $[a,b]$ then $|f|$ is also integrable on $[a,b]$ and
$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx .$$

The first exercise above gives us another way of talking about integrability:

Theorem: If f is bounded on $[a,b]$ then f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that
$$U(f,P) - L(f,P) < \varepsilon .$$

Proof: $A = \{ L(f,P) : P \text{ part of } [a,b] \}$

$B = \{ U(f,\tilde{P}) : \tilde{P} \text{ " " " } \}$

By the exercise, $\forall \varepsilon>0$, $\exists$ P and $\tilde{P}$ s.t. $U(f,\tilde{P}) - L(f,P) < \varepsilon$.
Let $P' = P \cup \tilde{P}$, last time we showed $U(f,P') \leq U(f,\tilde{P})$
and $L(f,P') \geq L(f,P) \Rightarrow U(f,P') - L(f,P') < \varepsilon$. ∎

Now we can prove our first criterium for integrability:

<u>Theorem</u>: If f is continuous on $[a,b]$ then it is integrable on $[a,b]$.

<u>Proof</u>: It is enough to show that given $\varepsilon>0$, $\exists$ P partition of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$. Let

$$P: a = t_0 < t_1 < \ldots < t_n = b$$

$$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} = f(x_i) \qquad x_i, X_i \in$$
$$M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\} = f(X_i) \qquad [t_{i-1}, t_i]$$

because f is cont.

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

Similarly, because f is cont. on $[a,b]$, $\forall \varepsilon>0$, $\exists$ $\delta>0$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$, $\forall x,y \in [a,b]$.

Then choose P so that $t_{i+1} - t_i < \delta$ $\forall i = 0, \ldots, n-1$

$$= \sum_{i=1}^{n} (f(X_i) - f(x_i))(t_i - t_{i-1})$$

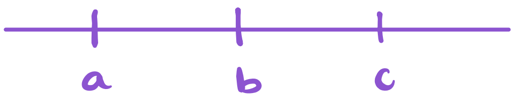
but $|X_i - x_i| \leq |t_i - t_{i-1}| \leq \delta$

$$\leq \sum_{i=1}^{n} \varepsilon (t_i - t_{i-1}) = \varepsilon \cdot (b-a).$$

∎

The following three facts are useful:

1) f is integrable on $[a,c]$ $\iff$ f is integrable on $[a,b]$ AND f is integrable on $[b,c]$



2) f and g are integrable on $[a,b]$ $\implies$ $f+g$ are integrable on $[a,b]$

3) f is integrable on $[a,b]$ anc $c \in \mathbb{R}$ $\implies$ $c \cdot f$ is integrable on $[a,b]$ $\leftarrow$ Exercise.

We now prove them:

Additivity of the domain

Theorem: Let $a<b<c$. Then:

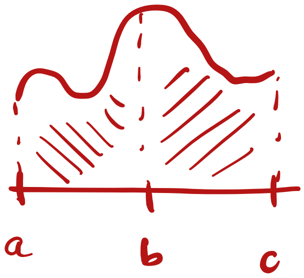
f is integrable on $[a,c]$ $\iff$ f is integrable on $[a,b]$ AND f is integrable on $[b,c]$

Moreover, in such a case:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Proof: Assume f is int. on $[a,b]$ and on $[b,c]$.

Given $\varepsilon>0$, $\exists$ P' partition of $[a,b]$ and



P'' part. of $[b,c]$ s.t.

$$\begin{cases} U(f,P') - L(f,P') \leq \varepsilon \\ U(f,P'') - L(f,P'') \leq \varepsilon \end{cases}$$

Now let $P = P' \cup P''$ this is now a part of $[a,c]$

Moreover

$$U(f,P) = U(f,P') + U(f,P'')$$
$$L(f,P) = L(f,P') + L(f,P'')$$

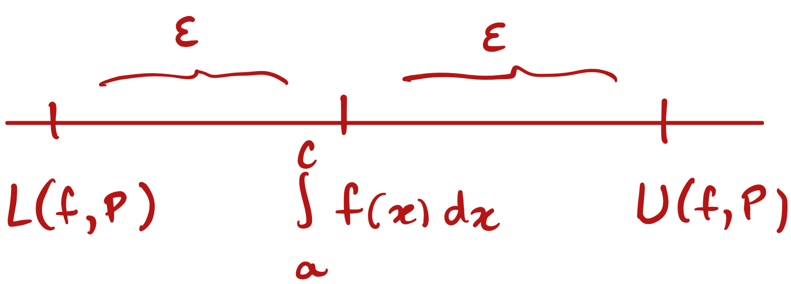
$\Rightarrow$ $U(f,P) - L(f,P) \leq 2\varepsilon$. so f is integrable on $[a,c]$. To determine the value note

$$U(f,P) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)$$

$$= \underbrace{\left(U(f,P') - \int_a^b f(x)\,dx\right)} + \underbrace{\left(U(f,P'') - \int_b^c f(x)\,dx\right)}.$$

and we can select P' and P'' so that these terms are arbitrarily small.

Now assume f is integrable on $[a,c]$. Let P be a partition of $[a,c]$ such that:



w.l.o.g we can assume $c \in P$ so that P splits as $P = P' \cup P''$ where P' is a partition of $[a,b]$ and P'' a partition of $[b,c]$.

As before:

$$U(f,P) = U(f,P') + U(f,P'')$$
$$L(f,P) = L(f,P') + L(f,P'')$$

so $\quad 2\varepsilon \geq U(f,P) - L(f,P)$

$$= \big(U(f,P') - L(f,P')\big) - \big(U(f,P'') - L(f,P'')\big)$$
$$\geq 0$$

$$\Longrightarrow \quad \begin{cases} 0 \leq U(f,P') - L(f;P') \leq 2\varepsilon \\ 0 \leq U(f,P'') - L(f;P'') \leq 2\varepsilon \end{cases}$$

which implies that f is integrable on both $[a,b]$ and $[b,c]$. The equality of the integrals follows similarly.

$\blacksquare$

Theorem: f and g are integrable on $[a,b]$ $\Longrightarrow$ $f + g$ are integrable on $[a,b]$

Moreover, in such a case:

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof: W.l.o.g we can assume, by taking unions of partitions if necessary, that $\exists$ P of $[a,b]$ s.t.

$$\begin{cases} U(f,P) - \varepsilon \leq \int_a^b f(x)\,dx \leq L(f,P) + \varepsilon \\ L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P) \end{cases} \qquad (*)$$

$$\begin{cases} U(g,P) - \varepsilon \leq \int_a^b g(x)\,dx \leq L(g,P) + \varepsilon \\ L(g,P) \leq \int_a^b g(x)\,dx \leq U(g,P) \end{cases}$$

What we want to understand is

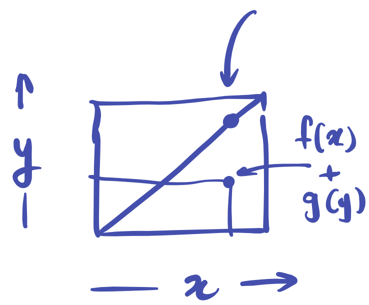
$$U(f+g,P) - L(f+g,P).$$

Let
$$\begin{cases} M_i = \sup\{f+g : [t_{i-1},t_i]\} \\ m_i = \inf\{f+g : [t_{i-1},t_i]\} \end{cases}$$

$$\begin{cases} M_i' = \sup\{f : [t_{i-1},t_i]\} \\ m_i' = \inf\{f : [t_{i-1},t_i]\} \end{cases}$$

$$\begin{cases} M_i'' = \sup\{g : [t_{i-1},t_i]\} \\ m_i'' = \inf\{g : [t_{i-1},t_i]\} \end{cases}$$

$(f+g)(x) = f(x) + g(x)$



How do we compare M_i with M_i' and M_i''?

$$M_i' + M_i'' \geq M_i$$
$$m_i' + m_i'' \leq m_i$$

Exercise: Use this to finish the proof.

Solution: The inequalities imply:

$$L(f,P)+L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P)+U(g,P)$$

which by (*) implies $U(f+g,P) - L(f+g,P) \leq 2\varepsilon$

so $f+g$ is integrable, and

$$\left| \int_a^b (f+g)(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| \leq 2\varepsilon.$$

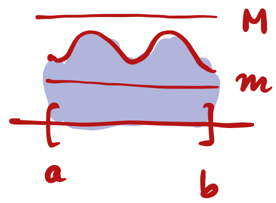
If we have bounds for f on $[a,b]$ and f is integrable then we can give bounds for $\int_a^b f(x)\,dx$:

Theorem: Let f be integrable on $[a,b]$ and assume $m \leq f(x) \leq M \;\; \forall x \in [a,b]$.

Then:

$$m\cdot(b-a) \leq \int_a^b f(x)\,dx \leq M\cdot(b-a)$$

Proof: Geometrically



$$m\cdot(b-a) \leq \int_a^b f(t)\,dt \leq M\cdot(b-a)$$

$$L(f,P) \geq m\cdot(b-a)$$

true for any P partition of $[a,b]$

$$\begin{aligned} U(f,P) &= \sum_i M_i\cdot(t_{i+1}-t_i) \\ &\leq \sum_i M\cdot(t_{i+1}-t_i) \\ &= M\cdot(b-a) \end{aligned}$$

$$m\cdot(b-a) \leq L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P) \leq M\cdot(b-a)$$

In fact, something more general is true

Theorem: If $f(x) \leq g(x)$ $\forall x \in [a,b]$ and both functions are integrable, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Monotonicity of the integrand

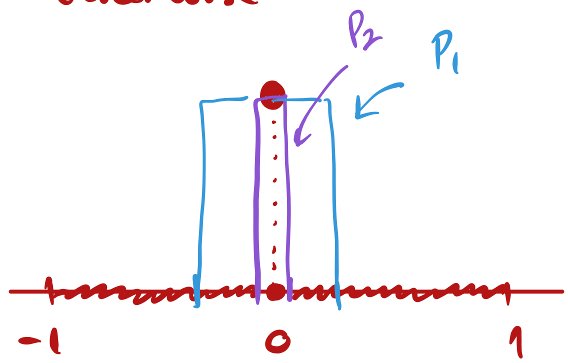
Proof: Exercise!

Question: If $f \geq 0$ and f is integrable does

$$\int_a^b f(x)\,dx = 0 \implies f(x) = 0 \;\; \forall x \in [a,b]\,?$$

Answer: No. For example.

Exercise: Let $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$ Show that $\int_{-1}^{1} f(x)\,dx = 0.$



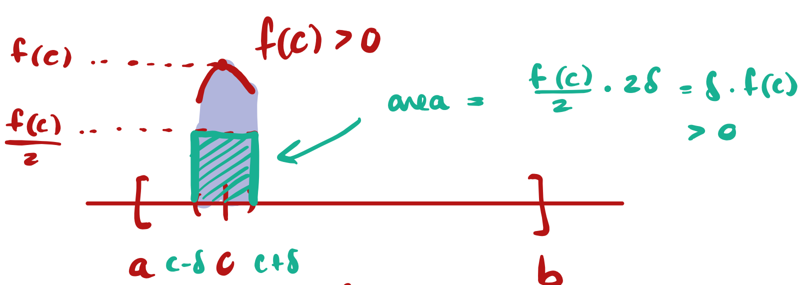
Remark If $f = g$ except perhaps at a finite number of points, their $\int$ are the same.

$$0 = \int_a^b f - g = \int_a^b f - \int_a^b g$$

($f-g$ is) 0 except at finitely many points

Question: What if f is continuous? Let $f \geq 0$ be continuous on $[a,b]$. Does

$$\int_a^b f(x)\,dx = 0 \implies f(x) = 0 \ \forall x \in [a,b]?$$



$$\int_a^b f(x)\,dx = \underbrace{\int_a^{c-\delta} f(x)\,dx}_{\geq 0} + \underbrace{\int_{c-\delta}^{c+\delta} f(x)\,dx}_{\geq \frac{f(c)}{2}\cdot 2\delta} + \underbrace{\int_{c+\delta}^{b} f(x)\,dx}_{\geq 0} \geq \delta \cdot f(c) > 0.$$

Exercise: Show that the answer to the last question is "yes".

New functions using the integral

We have seen that if f is integrable on $[a,b]$ then it is also integrable on $[a,x]$, $\forall x \in [a,b]$. Therefore, we can define a new function

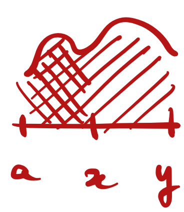
$$F(x) = \int_a^x f(t)\,dt$$

we would like to understand what properties does F has.

Theorem: If f is integrable on $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$.

Proof: We want to compare $F(x)$ and $F(y)$ when x and y are close to each other

w.l.o.g assume $a \leq x < y \leq b$. then



$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(x)\,dt$$

$$= \int_x^y f(t)\,dt.$$

Now, by definition the function f is bounded:

$$|f| \leq M \quad , \text{ for some } M \in \mathbb{R}_{\geq 0}.$$

then.

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right|$$

$$\leq \int_x^y |f(t)|\,dt$$

$$\leq M \cdot |y - x|.$$

$$\left| \frac{F(y) - F(x)}{y - x} \right| \leq M$$

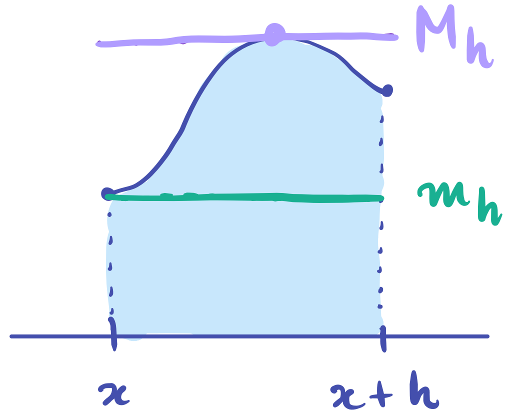
in particular F is continuous (in fact Lipschitz).

■

Lipschitz functions are "almost" differentiable.

it should be possible to say more:

$$m_h \cdot h \leq \int_x^{x+h} f(t)\,dt \leq M_h \cdot h$$



$$\Rightarrow \quad m_h \cdot h \leq F(x+h) - F(x) \leq M_h \cdot h$$

$$\Rightarrow \quad m_h \le \frac{F(x+h)-F(x)}{h} \le M_h$$

If f is continuous at x

then $\lim\limits_{h\to 0} M_h = \lim\limits_{h\to 0} m_h = f(x)$.

We have (almost) shown:

1st version

<u>Theorem</u> (The fundamental theorem of Calculus).

Let f be integrable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$.

If f is continuous at $c \in [a,b]$ then F is differentiable at c and $F'(c) = f(c)$.

<u>Exercise</u>: Complete the proof! Note that we have shown that if f is continuous at c then

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(c)$$

so you only need to check the limit from the left.

Corollary: If f is continuous on $[a,b]$ and $f = g'$ for some function g, then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

Proof Let $F(x) = \int_a^x f(t)\,dt$. Then

$$F'(x) = f(x) = g'(x)$$

Therefore $F - g$ is a constant function.

because $(F-g)'(x) = F'(x) - g'(x) = 0$

$$F(b) - g(b) = \cancel{F(a)}^{\,0} - g(a) \Rightarrow \int_a^b f(x)\,dx = g(b) - g(a)$$

$\int_a^a f \equiv 0$

∎

Remark: What if we vary the lower limit of integration

$$x \longmapsto \int_x^b f(t)\,dt$$

Well we know

$$\int_x^b f(t)\,dt = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt$$

Therefore

$$\frac{d}{dx}\int_x^b f(t)\,dt\Big|_{x=c} = -f(c)$$

when f is continuous at c

<u>Remark</u> Because of this, for $x<b$ we can define

$$\int_b^x f(t)\,dt := -\int_x^b f(t)\,dt.$$

and still have

$$\frac{d}{dx}\int_b^x f(t)\,dt\Big|_{x=c} = -(-f(c)) = f(c).$$

<u>Example</u>: we know that $f_k(x) = x^k$ satisfies

$$f_k'(x) = k\cdot x^{k-1} \quad \text{for all } k\in\mathbb{Z}.$$

In particular $x^{k-1} = \frac{1}{k}f_k'(x)$ for $k\neq 0$.

$$\int_a^b x^{k-1}\,dx = \int_a^b \frac{1}{k} f_k'(x)\,dx \qquad (\text{for } 0<a<b).$$

$$= \frac{1}{k}\left(f_k(b) - f_k(a)\right) = \frac{1}{k}\left(b^k - a^k\right).$$

Definition: We say that F is a primitive for f if $F' = f$.

The example above does not give us a primitive for $\frac{1}{x}$. We will soon study this function.

But first let us deduce some useful general formulas to deal with integrals.

In the corollary above:

$$\int_a^b g'(t)\,dt = g(b) - g(a)$$

we assumed that g' was continuous. It turns out this is not necessary

2nd version

Theorem (The Fundamental Theorem of Calculus):

If g' is integrable on $[a,b]$ then

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

<u>Proof</u>: Let $P: t_0 = a < t_1 < \dots < t_n = b$ be any partition of $[a,b]$. Then by the MVT there exist a point $x_i \in [t_{i-1}, t_i]$ s.t.

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1})$$

Since :

$$m_i(t_i - t_{i-1}) \le \overbrace{g'(x_i)(t_i - t_{i-1})}^{g(t_i) - g(t_{i-1})} \le M_i(t_i - t_{i-1})$$

$$m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})$$

$$\Rightarrow \quad L(g',P) \le g(b) - g(a) \le U(g',P)$$

$$\Rightarrow \quad \int_a^b g'(x)\,dx = g(b) - g(a)$$

$\blacksquare$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Some useful formulas :

<u>Exercise</u>: (integration by parts).

Prove that if f', g' are continuous on $[a,b]$ then :

$$\int_a^b f'(x)g(x)\,dx = -\int_a^b f(x)g'(x)\,dx + f(a)g(a) - f(b)g(b)$$

<u>Exercise</u>: (Change of variables formula).

Let $\phi : [a,b] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$.

Assume f and ϕ' are continuous everywhere.

prove that

$F(\phi(b))$

$$\int_a^b (f\circ\phi)(t)\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(t)\,dt$$

$G(b)$

Idea: $\quad F(x) = \displaystyle\int_{\phi(a)}^{x} f(t)\,dt$

$$G(x) = \int_a^x (f\circ\phi)(t)\,\phi'(t)\,dt$$

$$G'(x) = (f\circ\phi)(x)\,\phi'(x)$$

$$(F\circ\phi)'(x) \underset{\text{chain rule}}{=} F'(\phi(x))\cdot\phi'(x) = f(\phi(x))\cdot\phi'(x)$$

$$G - F\circ\phi = \text{constant} = G(a) - F\circ\phi(a) = 0 \qquad \blacksquare$$

<u>Example</u>: The function $f : (0,+\infty) \to (0,+\infty)$

given by
$$f(t) = \frac{1}{t}$$

is continuous on $(0,+\infty)$.

The function

$$F(x) = \int_1^x \frac{1}{t}\,dt$$

is a primitive for $\frac{1}{t}$.

what properties does this primitive have?

I). F is strictly monotone increasing

$\frac{1}{t}$ is cont on $(0,+\infty)$ $\Rightarrow$ $F'(x) = \frac{1}{x} > 0 \quad \forall\, x \in (0,+\infty)$

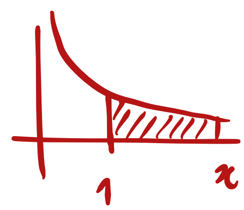
Fund T. of C.

$\Downarrow$

F is monotone increasing

II). $\lim\limits_{x \to +\infty} F(x) = +\infty$.

$F(x) = \int_1^x \frac{1}{t}\,dt$



$P_n : 1 < 2 < \dots < n$

$$F(n) = \int_1^n \frac{1}{t}\,dt \geq L(f, P_n) = \underbrace{\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}_{} \longrightarrow \infty \text{ as } n \to \infty$$

III). If $a, b > 1$ then $F(a\cdot b) = F(a) + F(b)$.

$$F(a\cdot b) = \int_1^{a\cdot b} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{a\cdot b} \frac{1}{t}\,dt$$

$$= F(a) + \int_1^{b} \frac{1}{\cancel{a}\cdot t}\cdot \cancel{a}\,dt = F(a) + F(b).$$

change of v.

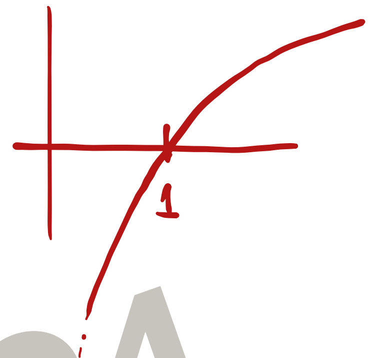
IV) If $a > 1$ then $F(1/a) = -F(a)$.

$\downarrow$

Exercise.

$$\int_1^{1/a} \frac{1}{t}\,dt \quad \ldots$$

V) $\lim_{x \to 0^+} F(x) = -\infty$.



$$F\left(\tfrac{1}{n}\right) = -F(n) \longrightarrow -\infty$$

+

use that F is strictly monotone.

VI) let $G : \mathbb{R} \to (0, +\infty)$ be the inverse of F.

then

$$\begin{cases} G'(0) = 1 \\ G(a+b) = G(a) \cdot G(b) \end{cases}$$

<u>Definition</u>: We define, for $x > 0$,

$$\log(x) := \int_1^x \frac{dt}{t}$$

and $\exp(y) := \log^{-1}(y)$ for $y \in \mathbb{R}$.

Taylor's Theorem

The principle behind the following theorems is that polynomials are easier to evaluate than, say, exp, log, sin, cos, etc.

For example, let:

$$P(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$$

then:

$$P(a) = c_0 = 0! \cdot c_0$$

$$P'(a) = c_1 = 1! \, c_1$$

$$P''(a) = 2 \cdot c_2 = 2! \, c_2$$

$$P^{(k)}(a) = k! \cdot c_k \qquad \text{for } 0 \leq k \leq n$$

$$P^{(n)}(a) = n! \, c_n$$

$$P^{(n+k)}(a) = 0 \qquad \text{for } n+k \geq n.$$

This formula allows us to construct a polynomial whose derivatives at a are known, we just set

$$c_k = \frac{P^{(k)}(a)}{k!}$$

<u>Question</u>: If f is a function such that $f(a), f'(a), f''(a), \dots, f^{(n)}(a)$ all exist, how does it compare with the polynomial that has the same derivatives up to order n.

<u>Definition</u> Given f as above, its <u>Taylor polynomial of order n at a</u> is

$$P_{n,a}(x) := f(a) + f'(a)(x-a) + \dots$$

$$\dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

has the same derivatives as f up to order n.

in a closed form:

$$\boxed{P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

<u>Definition</u>: The difference

$$R(x) = f(x) - P_{n,a}(x)$$

is called <u>the reminder of f at a</u>.

We can rephrase our original question to

<u>Question</u>: What type of function is the reminder?

<u>Remark</u>: By construction $f^{(k)}(a) = P_{n,a}^{(k)}(a)$ for all $k = 0, 1, \dots, n$. Therefore $R^{(k)}(a) = 0$ for the same values of k.

<u>Taylor's theorem</u>: Assume $f', f'', \dots, f^{(n+1)}$ are defined on $[a,x]$. Then there exists $\theta \in (a,x)$ such that

$$R(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-a)^{n+1}$$

- $\theta = \theta(x)$ it depends on x
- there is no general formula for θ.

In other words

$$f(x) = P_{n,a}(x) + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-a)^{n+1}$$

This is called the "Lagrange form" of the reminder.

<u>Proof</u>: The idea is that we can apply Cauchy's mean value theorem, similarly to what you did in the problem sheets.

Let $G(x) = (x-a)^{n+1}$. Note that both R and G

satisfy.

$$\begin{cases} R^{(k)}(a) = 0 \\ G^{(k)}(a) = 0 \end{cases} \qquad \text{for all} \quad k = 0, 1, \dots, n$$

Therefore applying CMVT several times:

$$\frac{R(x)}{G(x)} = \frac{R(x) - R(a)}{G(x) - G(a)}$$

$$\frac{f(x) - P_{n,a}(x)}{(x-a)^{n+1}}$$

$$= \frac{R'(\theta_1)}{G'(\theta_1)} \qquad (\text{for } a < \theta_1 < x)$$

$$= \frac{R'(\theta_1) - R'(a)}{G'(\theta_1) - G'(a)}$$

$$= \frac{R^{(2)}(\theta_2)}{G^{(2)}(\theta_2)} \qquad (\text{for } a < \theta_2 < x)$$

$$\vdots$$

$$= \frac{R^{(n+1)}(\theta_n)}{G^{(n+1)}(\theta_n)} \qquad (\text{for } a < \theta_{n+1} < x)$$

$$= \frac{f^{(n+1)}(\theta)}{(n+1)!} \qquad (\text{for } a < \overset{\theta_{n+1}}{\overset{\|}{\theta}} < x).$$

$\blacksquare$

"little o notation"

<u>Definition</u> We say that $f \in o(|x|^k)$ as $x \to a$ if.

$$\lim_{x \to a} \frac{f(x)}{|x-a|^k} = 0$$

Remark:

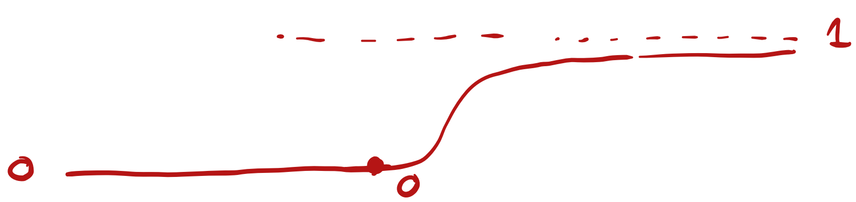
If $f^{(n+1)}$ is bounded, Taylor's theorem tell us that $f - P_{n,a} = R \in o(|x|^n)$.

Remark: However, there are important functions such that $P_{n,a} \equiv 0 \ \forall n \in \mathbb{N}$ without f being zero.

Can you think of an example?

Example:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$



The constant θ in Taylor's theorem depends on x and in general is unknown. Because of this one is often interested in other forms for the reminder:

Taylor's Theorem (Integral reminder)

Assume $f, f', \dots, f^{n+1}$ are all continuous near a

Then:

$$R(x) = \int_a^x f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt$$

In other words:

$$f(x) = P_{n,a}(x) + \int_a^x f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt$$

Proof:

$(n=0)$

$$f(x) = P_{0,a}(x) + \int_a^x f'(t)\,dt$$

$$= f(a) + \int_a^x f'(t)\,dt$$

is just the FT of Calculus.

$(n=1)$

$$f(x) = P_{1,a}(x) + \int_a^x f''(t)(x-t)\,dt$$

$$= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt$$

$$= f(a) + f'(a)(x-a) + \underbrace{\int_a^x \left(\big(f'(t)(x-t)\big)' - f'(t)(x-t)' \right) dt}$$

$$\to \; f'(t)(x-t)\Big|_{t=a}^{x} - \int_a^x f'(t)(-1)\,dt$$

$$= \cancel{f(a)} + \cancel{f'(a)(x-a)} + f'(x)\cancel{(x-x)} - \cancel{f'(a)(x-a)} + f(x) - \cancel{f(a)}$$

<u>Exercise</u>: Prove by induction that:

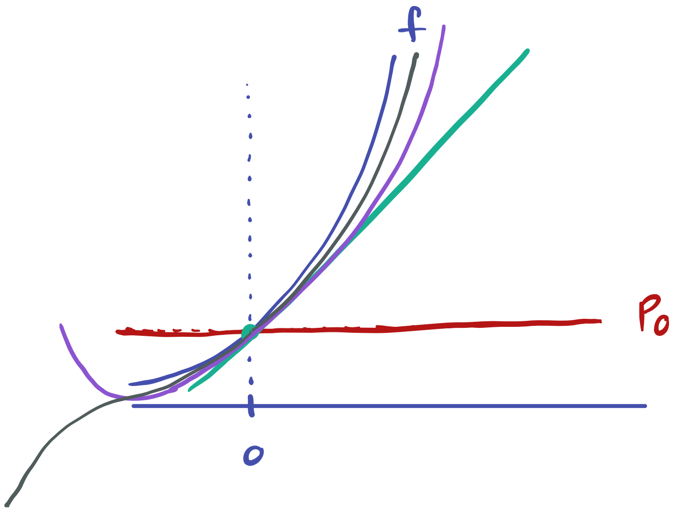
$$f(x) = P_{n,a}(x) + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!}\, dt$$

<u>Convergence of functions</u>

Let $f(x) = e^x$. The coefficients of the Taylor polynomials of f around $x=0$ are given by:

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

Therefore :



$P_0 = 1$

$P_1 = 1 + x$

$P_2 = 1 + x + \frac{x^2}{2}$

$P_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

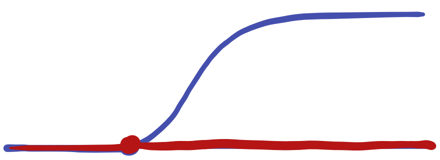
$\vdots$

It seems that $P_n(x)$ is approaching $f(x)$.

Can we always expect such behavior ?

No :

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



$$c_n = \frac{f^{(n)}(0)}{n!} = 0$$

$P_n(x) = 0$

$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$

When can we expect the Taylor Polynomials to converge to the function? ← These are called Analytic functions

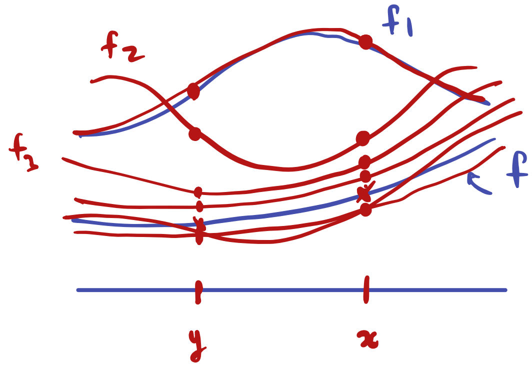
<u>Remark</u>: As our example shows this is a difficult question in general, but we can still say a few things:

If f can be extended to a complex differentiable function (holomorphic) then the Taylor polynomials converge to f.

Imagine $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions

$$f_n : I \longrightarrow \mathbb{R}$$

What does it mean for the sequence f_n to converge to a function f?



Moreover, if the functions f_n have a desirable property (e.g. they are continuous, differentiable, integrable), does f retains such properties?

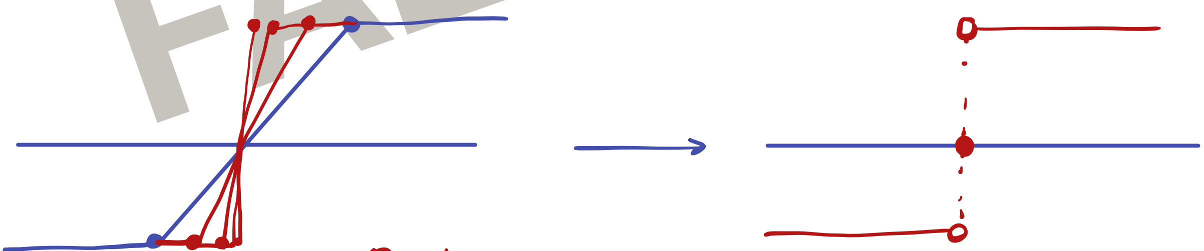
Definition: We say a sequence of functions

$$f_n : I \longrightarrow \mathbb{R}$$

converges pointwise to f if for every $x \in I$

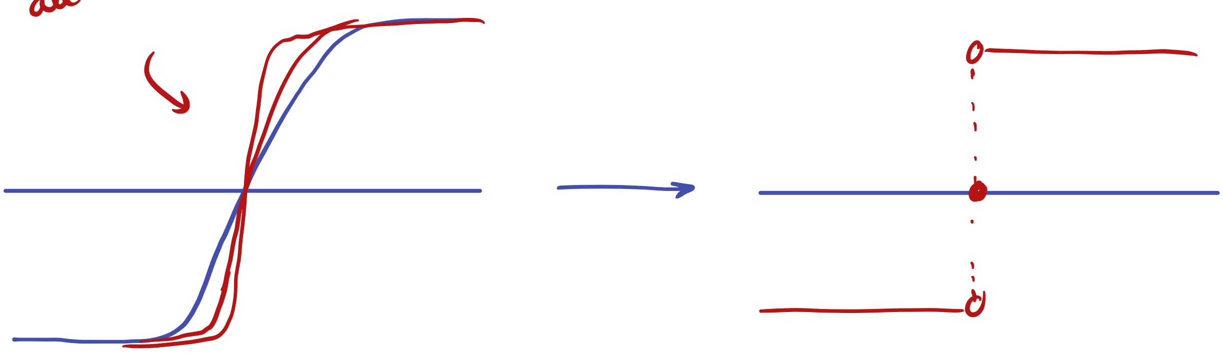
$$\lim_{n \to \infty} f_n(x) = f(x).$$

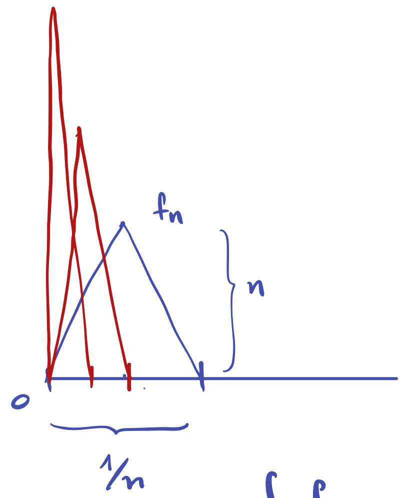
Example:



Pointwise convergence does not preserve continuity

all differentiable





In this case, the pointwise limit is

$$\lim_{n\to\infty} f_n(x) = f(x) = 0$$

However $\int f_n \not\to \int f = 0$

$$\int f_n = \tfrac{1}{2}$$

<u>Remark</u> : Pointwise convergence was defined as

$$f(x) = \lim_{n\to\infty} f_n(x)$$

which is the same as :

for all $\varepsilon > 0$, and for all $x \in I$, there exists N such that if $n > N$, then $\lvert f(x) - f_n(x) \rvert < \varepsilon$.

Note that, in this case $N = N(\varepsilon, x)$ depends on both ε and x.

It turns out, one condition for f to preserve some of the properties of f_n is when N is independent of x :

Definition We say that **f_n converges uniformly to f** if:

$$\boxed{\text{for all } \varepsilon>0, \text{ there exists } N \text{ such that, for all } x\in I, \text{ if } n>N, \text{ then } |f(x)-f_n(x)|<\varepsilon.}$$

Proposition: If $f_n \to f$ uniformly on $[a,b]$ and f_n, f are all integrable, then

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n$$

Proof: Pick $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $n>N$ implies

$$|f_n(x)-f(x)|<\varepsilon \quad \forall\, x\in I.$$

Then, if $n>N$.

$$\left|\int_a^b f(x) - \int_a^b f_n(x)\,dx\right| \leq \int_a^b |f_n(x)-f(x)|\,dx$$

$$\leq \varepsilon\cdot(b-a)$$

which is arbitrarily small. ∎

Similarly, uniform convergence also preserves continuity:

Proposition: If $f_n \to f$ uniformly on $[a,b]$ and f_n are continuous on $[a,b]$ then f is continuous on $[a,b]$.

Proof: Pick $x, y \in [a,b]$.

$$|f(x)-f(y)|$$

$$= |f(x) \overbrace{- f_n(x) + f_n(x)}^{=0} \overbrace{- f_n(y) + f_n(y)}^{=0} - f(y)|$$

$$\leq |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|$$

If n is large then $|f(z)-f_n(z)| < \varepsilon \quad \forall z \in [a,b]$

and since f_n is continuous $\Rightarrow$ unif. continuous.

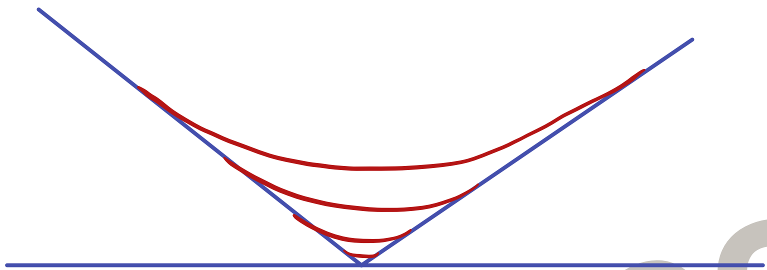
$\exists\, \delta > 0$, s.t $|x-y| < \delta \implies |f_n(x)-f_n(y)| < \varepsilon$

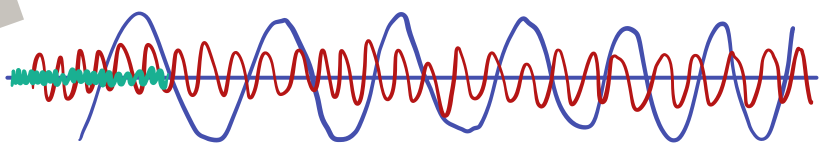
We showed

If $|x-y| < \delta$ and $n > N$, then $|f(x)-f(y)| \leq 3\varepsilon$.

What about differentiability?

Examples





However, we proves integrals behave well and the FT of Calculus relates integrals and derivatives:

Proposition: Assume $f_n \to f$ pointwise, $f_n' \to g$ uniformly the f_n' are integrable and g is continuous, then:

f is differentiable and $f'(x) = \lim_{n\to\infty} f_n'(x)$.

$g = \lim_{n\to\infty} f_n'$

<u>Proof</u>:

$$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n'$$

$$= \lim_{n\to\infty} f_n(x) - f_n(a)$$

$$= f(x) - f(a)$$

Since g is continuous it follows that

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x)$$

Therefore:

$$\frac{d}{dx} f(x) = g(x) = \lim_{n\to\infty} f_n'(x).$$

<u>Corollary</u>: Let $\sum_{n=1}^{\infty} f_n \longrightarrow f$ uniformly on $[a,b]$

(1). If each f_n is continuous $\Rightarrow$ f is continuous

(2) If f and f_n are integrable $\Rightarrow$ $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$.

Moreover, if $\sum_{n=1}^{\infty} f_n \to f$ pointwise, f_n' are integrable and $\sum_{n=1}^{\infty} f_n'$ converges uniformly to some continuous function, then

$$(3) \qquad f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad \forall\, x \in [a,b]$$

Analysis I - 2nd Term

Revision Lecture

We start with a rough map of the content:

LIMITS

(ε,δ) definition
or
sequential definition

CONTINUITY $\xLeftarrow{\text{local}}$ DIFFERENTIABILITY

DIFFERENTIABILITY $\downarrow$ Critical Points, Convexity, Taylor's Theorem

CONTINUITY $\xRightarrow{\text{global}}$ INTEGRABILITY

INTEGRABILITY: uses suprema and infima of approx
or
(ε, P) definition

The Fundamental Theorem of Calculus.

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition says?
- What examples do I know that do <u>NOT</u> satisfy the definition?
- How do I prove such examples do <u>NOT</u> satisfy the definition?

Revising Theorems

Many theorems are presented with a conditional structure i.e.,

$$A \Rightarrow B$$

hypothesis   conclusion

Sometimes $A = \underbrace{A_1 \text{ and } A_2 \text{ and } \dots \text{ and } A_n}$

splits into several hypotheses.

With this in mind the following questions might be useful:

- What does the theorem say?
- Why is it true?
- Is it an equivalence? i.e. is it also true that $B \Rightarrow A$?
- If it is not an equivalence, what is a counterexample?
- Why is every hypothesis necessary? e.g. is there a counterexample if we do not assume H_1 but assume $H_2, H_3, \dots, H_n$?
- Where is each H_i used in the proof?
- Why is the result useful / important?

Example: The Definition of limit of a function

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition say?
- What examples do I know that do <u>NOT</u> satisfy the definition?
- How do I prove such examples do <u>NOT</u> satisfy the definition?

- $\lim_{x\to a} f(x) = L$ means $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.
- $\lim_{x\to +\infty} f(x) = L$ means $\forall \varepsilon>0$, $\exists S>0$ s.t. $x>S \Rightarrow |f(x)-L|<\varepsilon$.
- $\lim_{x\to a} f(x) = +\infty$ ...
- Equivalent definitions exist using open intervals.

Negations:

"$\lim_{x\to a} f(x) \neq L$" means. $\exists \varepsilon_0>0$ such that $\forall \delta>0$ there is $\underbrace{x \in (a-\delta, a+\delta)\setminus\{a\}}_{\exists x,\ \text{s.t. } 0<|x-a|<\delta}$ such that $|f(x)-L| \geq \varepsilon_0$.

"$\lim_{x\to a} f(x)$ does not exist" means ... complete the definition

Examples:

$f(x) = \frac{1}{|x|}$ then $\lim_{x\to 0} f(x) = +\infty$ ← Prove it.

$f(x) = \sin(\frac{1}{x})$ then $\lim_{x\to 0} f(x)$ does not exist. ← Prove it (use the sequential definition).

Example: The definition of continuity.

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition says?
- What examples do I know that do **NOT** satisfy the definition?
- How do I prove such examples do **NOT** satisfy the definition?

Negation:

"f is not continuous at a."

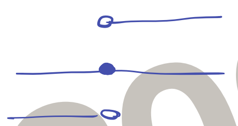
means

$\exists\, \varepsilon_0 > 0,\ \forall\, \delta > 0$ there is x satisfying

$|x - a| < \delta$ and $|f(x) - f(a)| > \varepsilon_0$

Examples:

Step function $f(x) = \mathrm{sgn}(x)$



$$= \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

To prove it is not continuous the sequential continuity def gives a shorter argument.

Example: Differentiability $\Rightarrow$ Continuity

Revising Theorems

- What does the theorem say?
- Why is it true?
- Is it an equivalence? i.e. is it also true that $B \Rightarrow A$?
- If it is not an equivalence, what is a counterexample?
- Why is every hypothesis necessary? e.g. is there a counterexample if we do not assume H_1 but assume $H_2, H_3, \dots, H_n$?
- Where is each H_i used in the proof?
- Why is the result useful / important?

Proof:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L$$

Idea:

$$|f(x) - f(a)| = \underbrace{\left|\frac{f(x) - f(a)}{x - a}\right|}_{\text{is bounded as } x \to a} \cdot \underbrace{|x - a|}_{\text{goes to zero as } x \to a}$$

It is NOT an equivalence because $f(x) = |x|$ is continuous at $x = 0$ but not differentiable.

(Prove it!).

Example: Continuity $\Rightarrow$ Integrability.

Revising Theorems

- What does the theorem say?
- Why is it true?
- Is it an equivalence? i.e. is it also true that $B \Rightarrow A$?
- If it is not an equivalence, what is a counterexample?
- Why is every hypothesis necessary? e.g. is there a counterexample if we do not assume H_1 but assume $H_2, H_3, \dots, H_n$?
- Where is each H_i used in the proof?
- Why is the result useful / important?

If f is continuous on $[a,b]$ then f is integrable on $[a,b]$.

Tools in the proof

- Uniform Continuity
- Criteria for integrability (using ε and P).

Example: The definition of uniform continuity.

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition says?
- What examples do I know that do NOT satisfy the definition?
- How do I prove such examples do NOT satisfy the definition?

Def: $f : X \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0$ there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Negation:

$\exists\, \varepsilon_0 > 0$, st. $\forall \delta > 0$ there are $x, y \in X$ satisfying

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| > \varepsilon_0.$$

Example: The definition of integrability.

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition says?
- What examples do I know that do NOT satisfy the definition?
- How do I prove such examples do NOT satisfy the definition?

Example: The definition of critical point.

Revising Definitions

- What does it say?
- Are there helpful equivalent definitions?
- What picture should I have in mind?
- Is the concept local or global?
- What examples do I know that satisfy the definition?
- How do I prove such examples satisfy the definition?
- What does the negation of the definition says?
- What examples do I know that do NOT satisfy the definition?
- How do I prove such examples do NOT satisfy the definition?

Example: $f''(a) > 0$ and $f'(a) = 0 \Rightarrow$ a is a local minimum for f

Revising Theorems

- What does the theorem say?
- Why is it true?
- Is it an equivalence? i.e. is it also true that $B \Rightarrow A$?
- If it is not an equivalence, what is a counterexample?
- Why is every hypothesis necessary? e.g. is there a counterexample if we do not assume H_1 but assume $H_2, H_3, \dots, H_n$?
- Where is each H_i used in the proof?
- Why is the result useful / important?

Why are the hypothesis necessary?

1. Is "$f'(a) = 0 \Rightarrow$ a is a local minimum" true?

NO. $f(x) = x^3$. $f'(0) = 0$ but it is not a minimum.

2. Is "$f''(a) > 0 \Rightarrow$ a is a local min" true?

NO: $f(x) = x^2$ because $f''(x) = 2 > 0$ everywhere but not every point is a minimum. $f''(1) > 0$.

Example : The fundamental theorem of Calculus

Revising Theorems

- What does the theorem say?
- Why is it true?
- Is it an equivalence? i.e. is it also true that $B \Rightarrow A$?
- If it is not an equivalence, what is a counterexample?
- Why is every hypothesis necessary? e.g. is there a counterexample if we do not assume H_1 but assume $H_2, H_3, \ldots, H_n$?
- Where is each H_i used in the proof?
- Why is the result useful / important?