Imperial College London

MATH70059

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Differential Topology

Date: 25 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let M be a smooth manifold. Show that every exact p-form on M is closed. (4 marks)

(b) Let S^n denote the n-dimensional sphere. Give a basis for each of the groups $H^p_{dR}(S^n)$ (you do not need to compute the groups themselves).

(4 marks)

(c) Let ω be a smooth one-form on M. Show that ω is exact if

$$\int_\gamma \omega = 0$$

for every piecewise smooth map $\gamma : S^1 \to M$. You may use the fact that γ is a smooth 1-cycle, without proof. (4 marks)

(d) Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with compact support. Suppose in addition that each of the components V^i is nonnegative, and that the divergence of V is zero, i.e.,

$$\operatorname{div}(V) = \frac{\partial V^1}{\partial x^1} + \frac{\partial V^2}{\partial x^2} + \frac{\partial V^3}{\partial x^3} = 0.$$

Show that V vanishes identically on $\mathbb{R}^3$. (8 marks)

(Total: 20 marks)

2. (a) Determine each of the cohomology groups of $T^3 = S^1 \times S^1 \times S^1$. (8 marks)

(b) Let M and N be smooth manifolds and consider a smooth map $F : M \to N$. Show that the pullback

$$F^* : \Omega^p(N) \to \Omega^p(M)$$

descends to a homomorphism

$$F^* : H^p_{dR}(N) \to H^p_{dR}(M).$$

Conclude that $H^p_{dR}(M) \cong H^p_{dR}(N)$ in case F is a diffeomorphism. (5 marks)

(c) Let M be a noncompact oriented smooth manifold of dimension $n \geq 1$. Are the groups $H^n_{dR}(M)$ and $H^n_c(M)$ necessarily isomorphic? Justify your answer. (2 marks)

(d) Let M and N be smooth manifolds of dimension k and l respectively. Suppose k and l are both at least one, and let $n = k + l$. Show that the product $M \times N$ is not diffeomorphic to the n-dimensional sphere. (5 marks)

(Total: 20 marks)

3. (a) Let M be an oriented smooth manifold. What is the relationship between the groups $H^p_{dR}(M)$ and $H^{n-p}_c(M)^*$? (2 marks)

(b) Give an example of two smooth manifolds M and N such that $H^p_{dR}(M) \cong H^p_{dR}(N)$ for every integer p, but M and N are not homeomorphic. (3 marks)

(c) (i) Define the Betti numbers and the Euler characteristic of a compact smooth manifold. (2 marks)

(ii) Compute the Euler characteristic of a compact oriented smooth manifold of dimension five. (2 marks)

(d) Let M be a connected compact smooth manifold of dimension 3. Suppose that $\pi_1(M)$ is finite. Determine the de Rham cohomology groups of M. (3 marks)

(e) Let

$$M = \mathbb{R}^2 \setminus \{(z, 0) : z = 0, 1, 2, \dots\}.$$

Show that $H^1_{dR}(M)$ is not finite-dimensional. (8 marks)

(Total: 20 marks)

4. (a) Let $M \subset \mathbb{R}^{10}$ be a smooth submanifold of dimension 2. Show that there is a constant vector field on $\mathbb{R}^{10}$ whose restriction to the normal space N_pM is nonzero for every $p \in M$. (5 marks)

(b) Let M and N be smooth manifolds, both of dimension at least one, and consider a smooth map $F : M \to N$.

(i) Is it necessarily true that, for every $p \in N$, the preimage $F^{-1}(p)$ is a smooth submanifold of M? Justify your answer. (2 marks)

(ii) Let U be an open subset of N such that $U \subset F(M)$. Is it possible that every point in U is a critical value of F? (3 marks)

(iii) Suppose $N = \mathbb{R}$ and that M is compact. Suppose $p \in \mathbb{R}$ is a regular value of F. Show that the preimage $F^{-1}(p)$ is a finite set. (3 marks)

(c) Let M be a compact smooth manifold of dimension $n \geq 1$. Show that there is a smooth n-form ω on M such that the zero-set $\{x \in M : \omega_x = 0\}$ is a smooth submanifold of dimension $n - 1$. (7 marks)

(Total: 20 marks)

5. Let M be a smooth manifold. Consider a vector bundle $\pi : E \to M$ of rank k.

(a) Show that $H^p_{dR}(E) \cong H^p_{dR}(M)$ for each integer p. (5 marks)

(b) Suppose $\pi_1(M) = 0$ and let $\gamma : S^1 \to M$ be a smooth map. Show that the pullback bundle $\gamma^{-1}E$ is trivial. (7 marks)

(c) One says that E is orientable if it can be given a system of local trivialisations

$$\{\varphi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha)\}$$

such that the sets U_α cover M and each of the transition functions $g_{\alpha\beta}$ satisfies

$$\det g_{\alpha\beta}(x) > 0$$

for every $x \in U_\alpha \cap U_\beta$. Show that every vector bundle over S^n is orientable when $n \geq 2$. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH70059

Differential Topology (Solutions)

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1. (a) A p-form ω is exact if it can be written as $\omega = d\eta$ for some $(p-1)$-form η. Computing locally, we may choose coordinates and express $\eta = \eta_I dx^I$, where I ranges over all multiindices $(i_1, \ldots, i_{p-1})$ such that $i_1 < \cdots < i_{p-1}$. We then have seen ⇓

$$\omega = \frac{\partial \eta_I}{\partial x^i} dx^i \wedge dx^I, \qquad d\omega = \frac{\partial^2 \eta_I}{\partial x^i \partial x^j} dx^i \wedge dx^j \wedge dx^I.$$

Since $\frac{\partial^2 \eta_I}{\partial x^i \partial x^j}$ is symmetric in i and j, and $dx^i \wedge dx^j$ is antisymmetric, we conclude that $d\omega = 0$. So ω is closed. 5, A

(b) Recall that $H^p(S^n) = 0$ for p not equal to 0 or n. Since S^n is connected, $H^0(S^n) \cong \mathbb{R}$, and a basis is given by $[f]$ where f is any nonzero constant function. Since S^n is compact and orientable $H^n(S^n) \cong \mathbb{R}$ and a basis is given by $[\omega]$, where ω is any orientation form. seen ⇓ 4, A

(c) Recall that the de Rham homorphism $\mathscr{I} : H^1_{dR}(M) \to H^1(M, \mathbb{R})$ acts by unseen ⇓

$$\mathscr{I}[\omega][c] = \int_{\tilde{c}} \omega,$$

where $\tilde{c}$ is any smooth 1-cycle homologous to c. Since γ is a smooth 1-cycle we conclude that $\mathscr{I}[\omega] = 0$. But by the de Rham theorem $\mathscr{I}$ is an isomorphism, so $[\omega] = 0$, which is to say that ω is exact. 4, B

(d) We define a compactly supported two-form unseen ⇓

$$\omega = V_3 dx^1 \wedge dx^2 - V_2 dx^1 \wedge dx^3 + V_1 dx^2 \wedge dx^3$$

and compute

$$d\omega = \operatorname{div}(V) dx^1 \wedge dx^2 \wedge dx^3 = 0,$$

so ω is closed. By the Poincaré lemma for compactly supported forms, there is a compactly supported smooth 1-form η such that $d\eta = \omega$. Let c be a constant and consider the plane $P := \{x \in \mathbb{R}^3 : x^3 = c\}$. Let B be the open ball of radius r about the origin in P. By Stokes' theorem,

$$\int_B \omega = \int_B d\eta = \int_{\partial B} \eta.$$

But η is compactly supported, so if r is chosen large enough this gives

$$0 = \int_B \omega = \int_{\mathbb{R}^2} V_3(x^1, x^2, c)\, dx^1 dx^2.$$

Since V_3 is nonnegative it follows that V_3 vanishes identically on P. But c was arbitrary, so V_3 vanishes on $\mathbb{R}^3$. Similar arguments show that V_1 and V_2 also vanish on $\mathbb{R}^3$. 8, D

2. (a) We first observe that $H^0(T^2) \cong \mathbb{R}$ and $H^2(T^2) \cong \mathbb{R}$, since T^2 is compact, connected and orientable. By the Künneth formula meth seen $\Downarrow$

$$H^1(T^2) \cong H^0(S^1) \otimes H^1(S^1) \oplus H^1(S^1) \otimes H^0(S^1) \cong \mathbb{R}^2.$$

Since T^3 is compact, connected and orientable, $H^0(T^3) \cong \mathbb{R}$ and $H^3(T^3) \cong \mathbb{R}$. The Künneth formula gives

$$H^1(T^3) \cong H^0(T^2) \otimes H^1(S^1) \oplus H^1(T^2) \otimes H^0(S^1) \cong \mathbb{R}^3$$

and

$$H^2(T^3) \cong H^0(T^2) \otimes H^2(S^1) \oplus H^1(T^2) \otimes H^1(S^1) \oplus H^2(T^2) \otimes H^0(S^1) \cong \mathbb{R}^3.$$

8, B

(b) Since F^* commutes with d, it maps closed forms to closed forms and exact forms to exact forms. In particular, if ω and $\tilde{\omega}$ differ by an exact form, then so do $F^*\omega$ and $F^*\tilde{\omega}$. Therefore, we get a well defined map $F^* : H^p_{dR}(N) \to H^p_{dR}(M)$ by setting $F^*[\omega] := [F^*\omega]$. This map is linear and hence a homomorphism. In case F is a diffeomorphism, it has a smooth inverse G, and $F^* \circ G^*$ is the identity on $H^p_{dR}(M)$ since seen $\Downarrow$

$$[\omega] = \mathrm{id_M}^*[\omega] = (G \circ F)^*[\omega] = F^*G^*[\omega].$$

Similarly $G^* \circ F^*$ is the identity on $H^p_{dR}(N)$, so F^* is an isomorphism. 5, B

(c) No. If M is noncompact and orientable then seen $\Downarrow$

$$H^n_{dR}(M) = 0, \qquad H^n_c(M) \cong \mathbb{R}.$$

2, A

(d) We proceed by contradiction. Suppose $M \times N$ is diffeomorphic to S^n. Then any orientation form on S^n can be pulled back by a diffeomorphism and restricted to each factor to yield orientation forms on M and N. So we see that M and N must be orientable. We also know that $M \times N$ is connected, since S^n is connected. It follows that M and N are both connected as well. We conclude that $H^0_{dR}(M) \cong \mathbb{R}$ and $H^l_{dR}(N) \cong \mathbb{R}$. The Künneth formula tells us that unseen $\Downarrow$

$$H^l(M \times N) \cong H^0_{dR}(M) \otimes H^l_{dR}(N) \oplus \ldots \cong \mathbb{R} \oplus \ldots,$$

so the dimension of $H^l(M \times N)$ is at least one. But $1 < l < n$, so by assumption

$$0 = H^l(S^n) \cong H^l(M \times N),$$

so this is a contradiction. 5, C

3. (a) Poincaré duality asserts that

$$H^p_{dR}(M) \cong H^{n-p}_c(M)^*$$

for every oriented smooth manifold M.

seen ⇓ 2, A

(b) Any two manifolds which are homotopy equivalent have isomorphic de Rham cohomology groups. So for example

$$H^p(S^1) \cong H^p(\mathbb{R}^2 \setminus \{0\})$$

for every p, but S^1 and $\mathbb{R}^2 \setminus \{0\}$ cannot be diffeomorphic since they are of dimension 1 and 2 respectively.

seen ⇓ 2, A

(c) (i) For a compact smooth manifold M each of the groups $H^p_{dR}(M)$ is finite dimensional. The p-th Betti number of M is

$$b_p = \dim(H^p_{dR}(M)).$$

The Euler characteristic of M is

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p.$$

seen ⇓ 2, A

(ii) Let M be a compact oriented 5-manifold. By Poincaré duality we have $b_p = b_{5-p}$ for each p, and hence

$$\chi(M) = (b_0 - b_5) - (b_1 - b_4) + (b_2 - b_3) = 0.$$

2, A

(d) Since M is connected we have $H^0_{dR}(M) \cong \mathbb{R}$, and since M is compact and orientable we have $H^3_{dR}(M) \cong \mathbb{R}$. Recall that

$$H^1_{dR}(M) \cong \mathrm{Hom}(\pi_1(M), \mathbb{R}).$$

Since $\pi_1(M)$ is finite, any homomorphism from $\pi_1(M)$ to $\mathbb{R}$ must have image a finite subgroup of $\mathbb{R}$, and hence is zero. So we have $H^1_{dR}(M) = 0$. By Poincaré duality, $H^2_{dR}(M) = 0$ as well.

meth seen ⇓ 3, B

(e) We claim that $H^1_{dR}(M)$ is not finite-dimensional. To see this we apply the Mayer–Vietoris theorem. Let

$$U = \{(x, y) : x < 3/4\}, \qquad V = \{(x, y) : x > 1/4\}.$$

Then $M = U \cup V$, $U \cap V$ is diffeomorphic to $\mathbb{R}^2$, U is homotopy equivalent to S^1, and V is diffeomorphic to M. By the Mayer–Vietoris theorem,we have an exact sequence

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V).$$

Using the fact that U, V and $U \cap V$ are all connected, $H^1(U) \cong \mathbb{R}$, $V \cong M$, and $H^1(U \cap V) = 0$, we obtain an exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(M) \to \mathbb{R} \oplus H^1(M) \to 0.$$

We read off that the map $H^1(M) \to \mathbb{R} \oplus H^1(M)$ is both injective and surjective. But no such map exists if $H^1_{dR}(M)$ is finite-dimensional, so this proves the claim.

unseen ⇓ 8, D

4. (a) Let $F : TM \to \mathbb{RP}^9$ be the map sending (x, v) to $[v]$, where $[v]$ is the equivalence class of nonzero vectors parallel to v. Then F is smooth, and since the dimension of TM is strictly less than that of $\mathbb{RP}^9$, $F(TM)$ has measure 0 in $\mathbb{RP}^9$. In particular, the complement of $F(TM)$ is dense in $\mathbb{RP}^9$. Let $[v_0]$ be any point in $\mathbb{RP}^9 \setminus F(TM)$. By construction, the vector field $p \mapsto v_0(p)$ is never tangent to M, so its normal component is always nonzero. meth seen ⇓ 5, A

(b) (i) Let $M = \mathbb{R}^2$, $N = \mathbb{R}$, and consider the smooth map $f(x, y) = x^2 - y^2$. Then $f^{-1}(0)$ is the union of the x and y axes, which is not a submanifold. seen ⇓ 2, A

(ii) Sard's theorem asserts that the set of critical values of F has measure 0 in N. No open subset of N has measure 0, so it is not possible that every point in U is a critical value. 3, A

(iii) By the regular value theorem, $F^{-1}(p)$ is a submanifold of M of dimension 0. That is, $F^{-1}(p)$ is a set of at most countably many points q_i, and each q_i is the zero-set of a chart. In particular, $F^{-1}(p)$ has no accumulation points, so since M is compact, $F^{-1}(p)$ must be finite. 3, A

(c) Let $\{U_i\}_{i=1}^N$ be a finite open cover of M with the property that, for each i, there is a chart (V_i, φ_i) such that $\overline{U}_i \subset V_i$. Let ψ_i be a bump function which equals one in U_i and vanishes outside of V_i. Since V_i admits coordinates x^k, we can define ω_i to be $\psi_i dx^1$ in V_i and zero elsewhere. It follows that ω_i is nonzero in U_i, and since the sets U_i cover M, at least one of the forms ω_i is nonzero at each point of M. Now define $F : M \times \mathbb{R}^N \to \Omega^n(M)$ by meth seen ⇓

$$F(x, s) := (x, s_i \omega_i|_x),$$

where we sum over i. The differential of F acts on $(v, t) \in T_x M \oplus T_s \mathbb{R}^N$ by

$$dF_{(x,s)}(v, t) = (v, t_i \omega_i|_x),$$

and hence is surjective for every (x, s), because at least one of forms $\omega_i|_x$ is always nonzero. In particular, F is transverse to the zero-section of $\Omega^n(M)$. By the parametric transversality theorem, $x \mapsto F(x, s)$ is transverse to the zero-section for almost every $s \in \mathbb{R}^N$. For any $\tilde{s}$ with this property, $x \mapsto F(x, \tilde{s})$ is a smooth section of $\Omega^n(M)$ whose zero-set is a submanifold of codimension one. 7, C

5. (a) Let $s : M \to E$ be the zero-section of E, i.e. $s(x) \equiv 0$. Then $s(M)$ is a submanifold of E which is diffeomorphic to M. We define a map $F : E \times [0,1] \to E$ by choosing a system of trivialisations $\{\varphi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha)\}$ and requiring that

$$F(p,t) = \varphi_\alpha(t \cdot \varphi_\alpha^{-1}(p)), \qquad t \cdot (x,v) := (x, tv)$$

whenever $p \in \pi^{-1}(U_\alpha)$. Since all of the transition maps associated with this system are linear, F is well defined. Clearly F is a deformation retraction of E onto $s(M)$, so we have

$$H^p_{dR}(E) \cong H^p_{dR}(s(M)) \cong H^p_{dR}(M).$$

$\boxed{\text{5, M}}$

(b) Since $\pi_1(M) = 0$ there is a point $p \in M$ and a smooth homotopy $\gamma_t : S^1 \to M$, $t \in [0,1]$, such that γ_0 agrees with γ and $\gamma_1(S^1) = p$. By the homotopy property of vector bundles, $\gamma_0^{-1}E$ and $\gamma_1^{-1}E$ are isomorphic. But $\gamma_1^{-1}E$ is trivial—to see this, choose a basis for $\pi^{-1}(p)$, pull back by γ_1 and apply Part (a). So $\gamma_0^{-1}E$ is trivial. $\boxed{\text{7, M}}$

(c) Let p and s be the north and south poles of S^n, respectively. We know that $S^n \setminus \{p\}$ and $S^n \setminus \{s\}$ are contractible, so by the homotopy property of vector bundles, the restricted bundles $\pi : E \to S^n \setminus \{p\}$ and $\pi : E \to S^n \setminus \{s\}$ are both trivial. In particular, by Part (a), we can choose a system of local trivialisations

$$\varphi_p : S^n \setminus \{p\} \times \mathbb{R}^k \to \pi^{-1}(S^n \setminus \{p\}), \qquad \varphi_s : S^n \setminus \{s\} \times \mathbb{R}^k \to \pi^{-1}(S^n \setminus \{s\}).$$

We only have one transition map, defined in $S^n \setminus \{p,q\}$, which is connected when $n \geq 2$. So its determinant is everywhere nonzero, and hence either positive or negative. If it is negative then modify φ_s by composing with $(x,v) \mapsto (x, Tv)$, where T is a linear transformation with $\det T = -1$. Then the new transition map has positive determinant, so E is orientable. $\boxed{\text{8, M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70059	1	No Comments Received
MATH70059	2	No Comments Received
MATH70059	3	No Comments Received
MATH70059	4	No Comments Received
MATH70059	5	No Comments Received