

*Correction to Q 4(ii)*

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2012

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science.

## Statistical Theory I

Date: Monday, 21 May 2012. Time: 2.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (i) For a parametric family of probability density functions  $\{f(x|\theta)\}$ , define
  - (a) *total efficient score*  $U_{\bullet}(\theta)$ ,
  - (b) *total Fisher information*  $I_{\bullet}(\theta)$ .
- (ii) Let  $X_1, X_2, \dots, X_n$  be independent identically distributed random variables having the probability density function

$$f(x|\theta) = c(\theta) x^{\theta} (1-x)^{1-\theta} \quad (0 \leq x \leq 1),$$

where, for unknown parameter  $\theta$  ( $0 < \theta < 1$ ),  $c(\theta)$  makes the probability density function integrate to 1.

Let  $\xi(\theta) = \frac{d}{d\theta} \ln c(\theta)$ .

- (a) Find  $U_{\bullet}(\theta)$ .  
From  $U_{\bullet}(\theta)$  identify the unbiased estimate  $\hat{\xi}$  of  $\xi(\theta)$ .  
Find  $I_{\bullet}(\theta)$ .
  - (b) Find  $U_{\bullet}(\xi)$ .  
Obtain the variance of  $\hat{\xi}$  in terms of  $\xi(\theta)$ .
  - (c) Explain why there is no unbiased estimator  $\hat{c}$  of  $c(\theta)$  having a variance which is the Cramér-Rao lower bound.
2. (i) (a) Explain what is meant by *complete* and *sufficient* when describing a *complete sufficient statistic*.  
(b) Explain briefly why these properties are important in statistical theory.
  - (ii) Let  $X_1, X_2, \dots, X_n$  be independent identically distributed random variables from the delayed exponential distribution having the probability density function

$$f(x|\theta) = \begin{cases} e^{\theta-x} & (x > \theta), \\ 0 & (\text{otherwise}), \end{cases}$$

where  $\theta$  is an unknown parameter.

- (a) Why does this probability density function not belong to the Exponential Family?
- (b) Show that  $T = \min(X_1, X_2, \dots, X_n)$  is sufficient for  $\theta$ .
- (c) Find the distribution of  $T$ .
- (d) Show that the distribution of  $T$  is complete.
- (e) Obtain the unique minimum variance unbiased estimator  $\hat{\theta}$  for  $\theta$ .
- (f) Obtain the variance of  $\hat{\theta}$ .

3. (i) What is meant by an *unbiased test of size  $\alpha$*  ( $0 < \alpha < 1$ )?
- (ii) Let  $x_1, x_2, \dots, x_n$  be a random sample from *Normal* ( $0, \theta^2$ ), where parameter  $\theta$  ( $\theta > 0$ ) is unknown.
- Show that there is a uniformly most powerful test of  $H_0 : \theta = \theta_0$  against  $H_1 : \theta < \theta_0$ , where  $\theta_0$  is specified, and find the explicit form of the test.
  - Find the power function of the test in (a), expressing it in terms of a standard distribution.
  - Show that the test in (a) is biased for  $H_1 : \theta \neq \theta_0$ .
- Give your reasoning throughout.*
4. (i) (a) Given data  $x \in \mathbb{X}$  from a known probability distribution having unknown parameter  $\theta \in \Theta$ , define a  $100(1 - \alpha)\%$  confidence set  $\Psi(x)$  for  $\theta$ .
- (b) Illustrate (a) by considering a size  $\alpha$  test for each  $\theta_0 \in \Theta$  of the hypothesis  $H_0 : \theta = \theta_0$  against  $H_1 : \theta \neq \theta_0$ , and the relationship between the acceptance set  $\bar{R}(\theta_0)$  of values of  $x$  and the confidence set  $\Psi(x)$ .
- (c) What is meant by a *best confidence set*?
- (ii) Let  $x_1, x_2, \dots, x_n$  be a random sample from the distribution having the probability density function
- $$f(x|\theta) = \theta x^{\theta-1} \quad (0 < \theta < 1),$$
- where  $\theta > 0$ . The prior probability density function for  $\theta$ ,  $\pi(\theta)$ , is *Exponential*( $\lambda$ ), where  $\lambda$  ( $\lambda > 0$ ) is known.
- Find the posterior probability density function, and identify the distribution.
  - Find the posterior mean (Bayes mean).
  - Find the posterior mode (Bayes MLE).
- You may wish to use that Gamma( $\nu, \xi$ ) has expectation  $\nu/\xi$ , and variance  $\nu/\xi^2$ .*
- Give your reasoning throughout.*

