MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Guidance for the mid-term exam

This document gives you a guidance on how to prepare for mid-term exam.

It is very important that you understand the material from the lecture notes and the solutions to the exercises thoroughly. You may be asked about a particular definition or a result from the course, and please make sure that you state everything precisely. It is very important to be precise in your arguments, and you can use all material developed in the course (this includes the problem sheets). In particular, if you need a certain result from the course in order to justify a different statement, please refer to this result directly (by saying something like "as shown in the lectures/problem sheets..."). It goes without saying that if you are asked to provide a proof of a result that has been established in the course or on the problem sheets, you have to give the proof and are not allowed to refer to it.

The relevant material for this exam is given by Chapter 1 and 2 (lecture notes until page 37), and this includes all (non-optional) questions on the first four problem sheets.

This exam will be similar in style to the mid-term exam from the last three years, and I will give you access to these exams and the solutions, but please do draw not any conclusions about the topics covered. For this years exam, I also aim at three main questions (with sub-questions), and the third question will probably be the most difficult, even though for some of you it may be the easiest part of the exam. Both computational skills and your understanding of the fundamental parts of the course will be tested.

Some comments on the material covered in the lecture notes:

<u>Chapter 1. Introduction</u>

- §1 This introductory section focusses on basic definitions and properties of differential equations.
- §2 This section addresses issues arising when looking at initial value problems (remind yourself that there are examples where such problems have no solutions, many solutions, and solutions do not need to exist for all times). Via separation of variables, you can solve certain one-dimensional differential equations, and consequences with regard to uniqueness and non-uniqueness have been addressed also on problem sheets. I expect you to be in good command of this elementary material.
- §3 This section describes how to visualise ordinary differential equations. This is a very useful part of the course, and the ideas established here help to understand the material of this course better.

<u>Chapter 2. Existence and uniqueness</u>

- §1 Picard iterates are very fundamental to obtain local solutions to initial value problems. It motivates the general approach we use later in §4 of this chapter. In particular, you should understand why Picard iterates are important.
- §2 Lipschitz continuity is very important to establish existence and uniqueness of solutions. You should be very confident to understand the building blocks involving the mean value theorem and the mean value inequality as well as being able to apply this in practical settings.
- §3 The main result in the whole chapter on existence and uniqueness is given by the Picard–Lindelöf theorem, which was stated and proved for didactic reasons first for the globally

Lipschitz continuous case and then for locally Lipschitz right hand sides. This is an extremely important part of the course, and I expect you to understand the statement, the tools to prove it, its proof, and its consequences very well, and you need to know how to apply these results in practical settings.

§4 It is shown in this section that local solutions that have been obtained via the Picard–Lindelöf theorem can be extended to maximal solutions. Maximal solutions to initial value problems are maximal in the sense that their domain (i.e. the maximal existence interval) cannot be extended. I expect you to understand the arguments well and that you are able to check maximality of solutions of differential equations in practical examples.

§5 The concept of a general solution or a flow is a bit abstract, but clear if the results from the previous section have been understood well, since these two objects parameterise maximal solutions. Our thinking about differential equations now becomes more dynamical, in the sense that we think of how the dynamics evolves when we start at time t_0 in x_0 (for a nonautonomous differential equation) or in x_0 (for an autonomous differential equation, where the initial time does not matter). Given that we will deal with flows until the end of the course, the understanding of this part of the course is crucial.

I hope you have a smooth preparation for the mid-term exam. Good luck with the exam!