Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Solutions to Revision Problem Sheet

1. Here, we consider the following problem

$$3x\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad y > 0$$
$$u(x,0) = 1, \quad x > 0 \quad ; \quad u(0,y) = 0, \quad y > 0$$

We will try to obtain a similarity solution to this problem. We consider the transformation:

$$\{\tilde{x} = ax, \tilde{y} = a^\beta y, \tilde{u} = a^\gamma u\}$$

where a is a positive real constant. The idea is here to find β and γ such that our PDE problem is left unchanged under the transformation.

(a) Under the above transformation, the derivatives are transformed as follows:

$$\frac{\partial}{\partial y} = a^\beta \frac{\partial}{\partial \tilde{y}}$$
$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial \tilde{x}} \Rightarrow \frac{\partial^2}{\partial x^2} = a^2 \frac{\partial^2}{\partial \tilde{x}^2}$$

Reinjecting this in the PDE, we obtain

$$3x\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow 3\tilde{x}a^{-1}a^{-\gamma}a^\beta \frac{\partial \tilde{u}}{\partial \tilde{y}} - a^2 a^{-\gamma}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0$$
$$\Rightarrow 3a^{\beta-3}\tilde{x}\frac{\partial \tilde{u}}{\partial \tilde{y}} - \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0$$

and we can see that the PDE is left invariant under this transformation if $\beta = 3$. Note that this does not give us any condition on γ.

(b) What about the value of γ? For this, we should consider the boundary conditions. In particular, we know that $u(x,0) = 1$, so under the transformation this BC reads $a^{-\gamma}\tilde{u} = 1$. And we see that this boundary condition will only be left invariant if $\gamma = 0$ (remember that a is an arbitrary positive constant).

(c) We can then write our similarity ansatz as

$$u(x,y) = y^{\gamma/\beta} f(\eta), \quad \text{with} \quad \eta = xy^{-1/\beta}$$

which here gives us

$$u(x,y) = f(\eta), \quad \text{with} \quad \eta = x/y^{1/3}$$

Reinjecting this ansatz in the PDE, we can obtain an ODE for f! Let's do it:

$$3x\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow 3x\frac{\partial \eta}{\partial y}f'(\eta) - \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}f'(\eta)\right) = 0$$

with

$$\frac{\partial \eta}{\partial y} = -\frac{x}{3y^{4/3}} \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \frac{1}{y^{1/3}}$$

which leads to

$$-\frac{x^2}{y^{4/3}}f'(\eta)-\frac{1}{y^{2/3}}f''(\eta)=0\Rightarrow -\underbrace{\frac{x^2}{y^{2/3}}}_{\eta^2}f'(\eta)-f''(\eta)=0$$

Finally, we obtain the following ODE for $f(\eta)$

$$\eta^2 f'(\eta)+f''(\eta)=0$$

What about the boundary conditions? When $x=0$, $\eta=0$ so naturally, we have $u(0,y)=0\Rightarrow f(0)=0$.
Similarly, when $y\to 0$, $\eta\to\infty$, then we have $u(x,0)=1\Rightarrow \lim_{\eta\to\infty}f(\eta)=1$.

(d) Let's solve this ODE! We can separate it and integrate it as follows

$$\frac{f''}{f'}=-\eta^2\Rightarrow \ln f'=-\frac{\eta^3}{3}+C\Rightarrow f'(\eta)=Ae^{-\eta^3/3}$$

A further integration leads to

$$f(\eta)=A\int_0^\eta e^{-s^3/3}ds+B$$

where A and B are integration constants to be determined. In particular, when $\eta\to 0$, we have

$$0=f(0)=A\int_0^0 e^{-s^3/3}ds+B\Rightarrow B=0$$

and when $\eta\to\infty$, we have

$$1=\lim_{\eta\to\infty}f(\eta)=A\int_0^\infty e^{-s^3/3}ds\Rightarrow A=\left[\int_0^\infty e^{-s^3/3}ds\right]^{-1}=\frac{3^{2/3}}{\Gamma(1/3)}$$

where $\Gamma(x)$ is the Euler gamma function. We finally write that

$$u(x,y)=\frac{3^{2/3}}{\Gamma(1/3)}\int_0^{x/y^{1/3}}e^{-s^3/3}ds$$

2. Here, we consider the Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,\quad -\infty<x<\infty,\quad t>0,$$

subject to the initial condition

$$u(x,0)=u_0(x)=\begin{cases}0, & x<0\\ 1-x, & 0<x<1\\ 0, & x>1\end{cases}$$

(a) On Figure 1, we show the initial conditions. The method of characteristics gives us here:

$$\frac{du}{dt}=0\quad\text{on}\quad\frac{dx}{dt}=u,\quad x(0)=\xi$$

with ξ a parameter. Here, we see that u is thus constant along the characteristics, so the characteristics carry in the (x,t)-plane the information about the initial conditions! We get in particular

$$u=u_0(\xi)\quad\text{on}\quad x=u_0(\xi)t+\xi$$

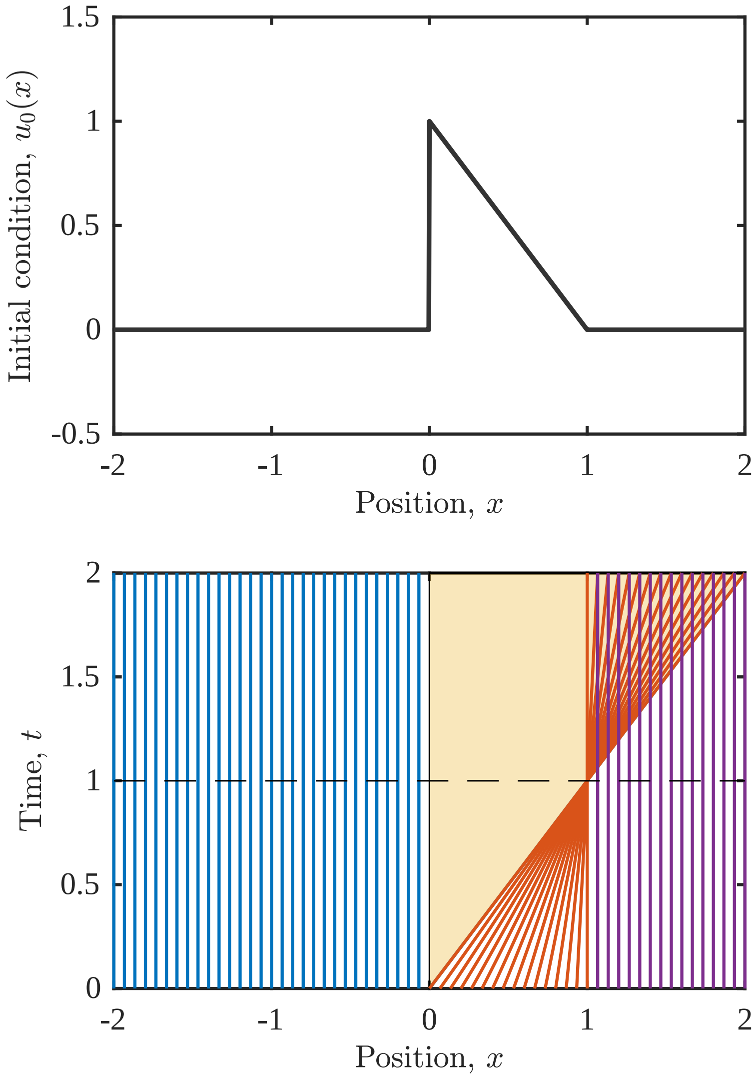

Figure 1: Initial conditions (top) and diagram of characteristics (bottom)

Based on the initial conditions, we ocnsider 3 regimes:

$$\begin{cases} \text{I} — \xi < 0 : & x = \xi \\ \text{II} — 0 < \xi < 1 : & x = (1-\xi)t + \xi \\ \text{III} — \xi > 1 : & x = \xi \end{cases}$$

The diagram of characteristics is given in Figure 1.
We see in particular that:

- ☐ The characteristics from region II all cross in $x = 1$ for $t = 1$ leading to shock formation (so we know that our diagram of characteristics is only valid up to $t = 1$ for further times, it will need to be amended). In this region, the equation of the characteristics is $x = (1-\xi)t + \xi$, we can easily see that $x = 1$ at $t = 1$ for all $0 < \xi < 1$.
- ☐ Further the characteristics do not cover the yellow region leading to a rarefaction fan!

(b) The explicit solution for $t < 1$ can easily be obtained; indeed, remember that u is constant along the characteristics.

- ☐ In Region I, we have $\xi < 0$ and so we obtain that $u = 0$ for $x < 0$.
- ☐ In Region III, we have $\xi > 1$ and so we obtain that $u = 0$ for $x > 1$.

- ☐ In Region II, we have $0 < \xi < 1$ and $u = u_0(\xi) = 1 - \xi$ on the curves $x = (1-\xi)t + \xi$. We can eliminate ξ to obtain an explicit solution, for this write: $\xi = (x-t)/(1-t)$ by inverting the equation of the characteristics. So we obtain

  $$u = 1 - \xi = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}$$

  Over what range? Let's go back to

  $$0 < \xi < 1 \Rightarrow 0 < \frac{x-t}{1-t} < 1 \Rightarrow 0 < x - t < 1 - t \Rightarrow t < x < 1$$

- ☐ At this point for a given time t, we have defined our solution profiles over the following ranges: $x < 0$, $t < x < 1$ and $x > 1$. What about $0 < x < t$? This is the fan region. In this region, we know that our solution profile goes linearly from the value the solution takes on the left of the fan to the value the solution takes on the right of the fan, i.e. from 0 in $x = 0$ to 1 in $x = t$. This leads to the following solution $u = x/t$.

We conclude that the explicit solution for $u(x,t)$ before the shock forms is given by

$$u(x,t) = \begin{cases} 0, & x < 0 \\ x/t, & 0 < x < t \\ (1-x)/(1-t), & t < x < 1 \\ 0, & x > 1 \end{cases}$$

(c) Solution for $t > 1$? This is after the shock has formed so shock fitting will be required. When $t \to 1^-$, we can see that the region $t < x < 1$ disappears, it corresponds to the red characteristics crossing in Figure 1!

Denoting $s(t)$ the shock position, the Rankine-Hugoniot condition gives us

$$\frac{ds}{dt} = \frac{[q(u)]}{[u]} = \frac{[u^2/2]}{u} = \frac{1}{2}(u_- + u_+)$$

where we have used the fact that the flux for Burgers' equation is given by $q(u) = u^2/2$ (to see this, just write the Burgers' equation in its conservation law form).

$$\begin{cases} \text{Ahead of the shock:} & u_+ = 0 \quad \text{(from region III)} \\ \text{Behind the shock:} & u_- = s/t \quad \text{(from fan region)} \end{cases}$$

So we obtain that

$$\frac{ds}{dt} = \frac{1}{2}\frac{s}{t}$$

but what are the initial conditions for this ODE? As the shock forms in $x = 1$ at $t = 1$, then we know that $s(1) = 1$. By integration of the ODE, we get $s(t) = At^{1/2}$ and using the initial condition, we conclude that

$$s(t) = \sqrt{t}$$

The amended diagram of characteristics is given on Figure 2.

So we conclude that the solution for $t > 1$ is given by

$$u(x,t) = \begin{cases} 0, & x < 0 \\ x/t, & 0 < x < \sqrt{t} \\ 0, & x > \sqrt{t} \end{cases}$$

(d) Finally, the shock strength is given by $u_- - u_+ = s/t - 0 = 1/\sqrt{t}$. So the shock strength is equal to 3 when $t = 9$. When $t = 9$, we have

$$u(x,9) = \begin{cases} 0, & x < 0 \\ x/9, & 0 < x < 3 \\ 0, & x > 3 \end{cases}$$

This solution is shown in Figure 3.

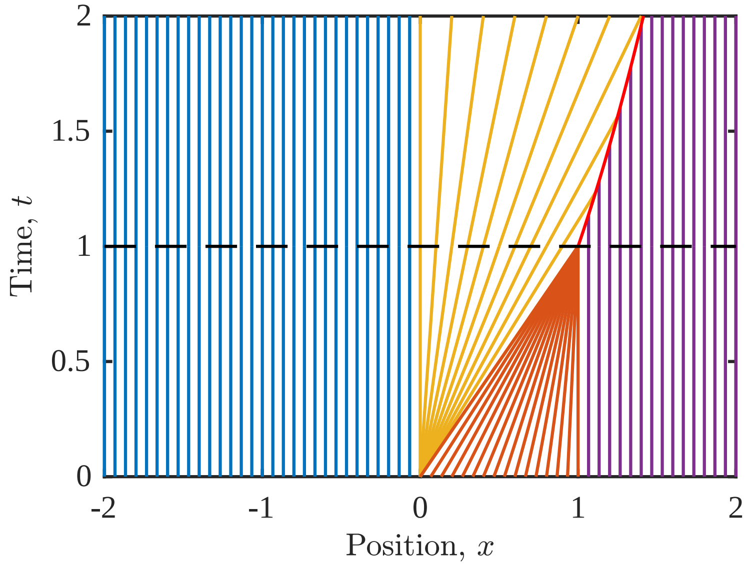

Figure 2: Amended diagram of characteristics with shock path in red.

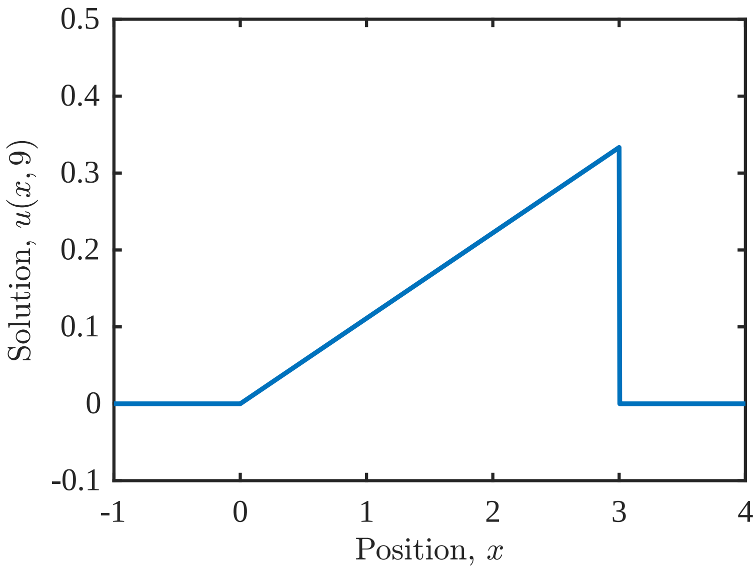

Figure 3: Solution profile for shock strength equal to 3

3. Here, we try to solve a nonhomogeneous diffusion equation on an infinite domain. We will be looking at using a Fourier transform method. Suppose that we denote $\widehat{G}(\omega, t)$ the FT of G with respect to x, then

$$\begin{aligned}\frac{\partial \widehat{G}}{\partial t} + \omega^2 \widehat{G} &= \mathcal{F}\left\{\delta(x - x_0)\right\}\delta(t - t_0) \\ &= \delta(t - t_0)\int_{-\infty}^{+\infty} \delta(x - x_0) e^{-i\omega x} dx \\ &= \delta(t - t_0) e^{-i\omega x_0} \quad \text{(by the sifting property of the delta function)}\end{aligned}$$

We can integrate this first-order ODE using the integrating factor $e^{\omega^2 t}$. Indeed, we have

$$\frac{\partial}{\partial t}\left[\widehat{G} e^{\omega^2 t}\right] = \delta(t - t_0) e^{-i\omega x_0} e^{\omega^2 t}$$

By integration, we obtain

$$\widehat{G} e^{\omega^2 t} = \int_0^t \delta(s - t_0) e^{-i\omega x_0} e^{\omega^2 s} ds + C$$

where C is an integration constant. Now the initial conditions are imposing that $G = 0$ at $t = 0$ which implies that $\widehat{G} = 0$ when $t = 0$ and so we conclude that $C = 0$.

$$\widehat{G} e^{\omega^2 t} = \int_0^t \delta(s - t_0) e^{-i\omega x_0} e^{\omega^2 s} ds$$

We have two cases to deal with:

Case 1: if $t < t_0$, then the integral on the RHS will always be zero because of the delta function.

Case 2: if $t \geq t_0$, then the integral is

$$\int_0^t \delta(s - t_0) e^{-i\omega x_0} e^{\omega^2 s} ds = e^{\omega^2 t_0} e^{-i\omega x_0}$$

We conclude that the Fourier representation of the solution is

$$\widehat{G}(\omega, t) = \begin{cases} 0, & t \leq t_0 \\ e^{\omega^2 (t_0 - t) - i\omega x_0}, & t > t_0 \end{cases}$$

Now by definition of the inverse Fourier transform, we have

$$G(x, t) = \begin{cases} 0, & t \leq t_0 \\ (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{\omega^2 (t_0 - t) - i\omega x_0} e^{i\omega x} d\omega, & t > t_0 \end{cases}$$

For $t > t_0$, we can rewrite the solution as

$$G(x, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\omega^2 (t - t_0) + i\omega(x - x_0)} d\omega$$

A natural idea to compute this integral is to complete the square in the exponential; let's define $\omega' = \omega - i\frac{(x - x_0)}{2(t - t_0)}$, we have $d\omega = d\omega'$ and

$$-\omega^2 (t - t_0) + i\omega(x - x_0) = -(t - t_0)\omega'^2 - \frac{(x - x_0)^2}{4(t - t_0)}$$

so we have

$$G(x, t) = \frac{1}{2\pi}\exp\left(-\frac{(x - x_0)^2}{4(t - t_0)}\right)\int_{-\infty}^{+\infty} e^{-\omega'^2 (t - t_0)} d\omega$$

Here, we recognize the Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

(note that in an exam setting, you would be ask to prove the result as part of the question or you will be given this formula.)

Using this result, we finally conclude that

$$G(x,t) = \begin{cases} 0, & t \leq t_0 \\ \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t-t_0}} \exp\left(-\frac{(x-x_0)^2}{4(t-t_0)}\right), & t > t_0 \end{cases}$$

4. Here, we are trying to solve an IVP for the wave equation on an infinite domain. You would naturally think about two possible strategies: (1) Fourier method or (2) D'Alembert's solution. Solving it using a Fourier method leads to complicated algebra. Let's look at using d'Alembert's solution to the wave equation given by

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi) d\xi$$

Here, we have $g(x) \equiv 0$ and so the solution reduces to

$$u(x,t) = \frac{1}{2}\left[f(x-2t) + f(x+2t)\right]$$

Now remember that

$$f(x) = \begin{cases} 0, & |x| \geq 1 \\ x^2 - x^4, & |x| < 1 \end{cases}$$

Note that $f(x)$ changes form based on the value of $|x|$. We need to break down our solution:

(A) If $|x+2t| \geq 1$ and $|x-2t| \geq 1$, then we have

$$\begin{cases} f(x+2t) = 0 \\ f(x-2t) = 0 \end{cases}$$

so we have $u(x,t) = 0$.

(B) If $|x+2t| < 1$ and $|x-2t| \geq 1$, then we have

$$\begin{cases} f(x+2t) = (x+2t)^2 - (x+2t)^4 \\ f(x-2t) = 0 \end{cases}$$

so we have $u(x,t) = [(x+2t)^2 - (x+2t)^4]/2$.

(C) If $|x+2t| \geq 1$ and $|x-2t| < 1$, then we have

$$\begin{cases} f(x+2t) = 0 \\ f(x-2t) = (x-2t)^2 - (x-2t)^4 \end{cases}$$

so we have $u(x,t) = [(x-2t)^2 - (x-2t)^4]/2$.

(D) If $|x+2t| < 1$ and $|x-2t| < 1$, then we have

$$u(x,t) = \frac{1}{2}[(x+2t)^2 + (x-2t)^2 - (x+2t)^4 - (x-2t)^4].$$

You can draw up a diagram to understand what the solution looks like in the (x,t)-plane.

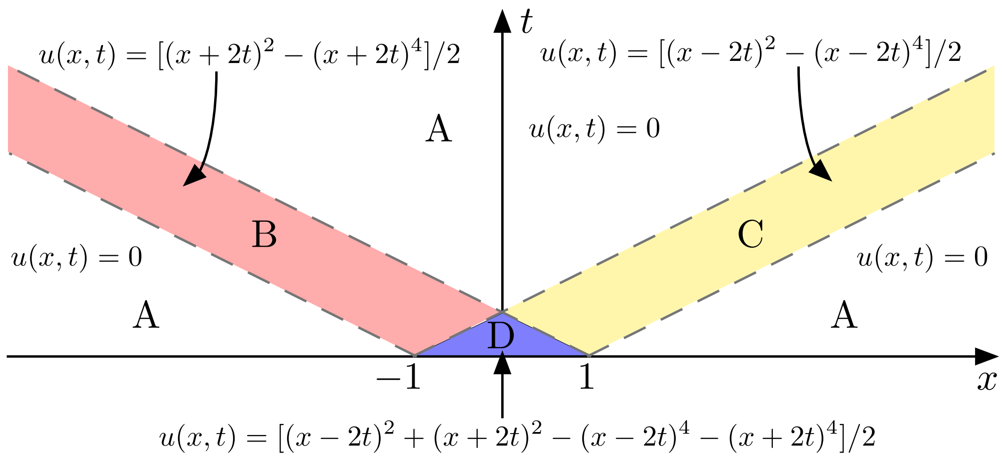

Figure 4: Diagram for the solution to Q4.

5. Here, we work in 2D.

   (a) We are looking for solutions to
   $$\nabla^2 G_f = \delta(\mathbf{r})$$
   We look for radially symmetric solutions $G_f = G_f(r)$, with $r = |\mathbf{r}|$. We thus work in plane polar coordinates. The Laplacian in this case reduces to
   $$G_f'' + \frac{1}{r}G_f' = \delta(r)$$
   where primes denote a derivative with respect to r. Let's first suppose that $r \neq 0$, we have
   $$(rG_f')' = 0 \Rightarrow G_f' = \frac{A}{r} \Rightarrow G_f = A\ln(r) + B$$
   where A and B are separation constants. As the Green's function is always defined up to a constant, we set $B = 0$ for simplicity.
   How do we find A? G_f is solution to $\nabla^2 G_f = \delta(r)$, let's integrate both sides over a disc $\mathcal{D}$ of radius R centered on the origin.
   $$\int_{\mathcal{D}} \nabla^2 G_f dxdy = \int_{\mathcal{D}} \delta(r)dxdy = 1$$
   as the disc contains the origin. But
   $$\int_{\mathcal{D}} \nabla^2 G_f dxdy = \int_{\mathcal{D}} \nabla\cdot(\nabla G_f)dxdy = \oint_{\mathcal{C}} \nabla G_f \cdot \hat{\mathbf{n}}ds = \oint_{\mathcal{C}} \nabla G_f \cdot \hat{\mathbf{r}}ds$$
   and so we have that
   $$1 = \int_{\mathcal{D}} \left(\frac{\partial G_f}{\partial r}\right)_{r=R} ds = \frac{A}{R}\oint_{\mathcal{C}} ds = \frac{A}{R}2\pi R \Rightarrow A = \frac{1}{2\pi}$$
   So the 2D free-space Green's function for the Laplacian reads
   $$G_f(\mathbf{r}) = \frac{1}{2\pi}\ln|\mathbf{r}|$$

   (b) We now want to solve $\nabla^2 H = \delta(\mathbf{r} - \mathbf{r}_0)$ in the upper-half plane $y > 0$ and $x \in \mathbb{R}$ with the BC: $\frac{\partial H}{\partial y} = 0$ on $y = 0$. Let's denote $\mathbf{r}_0 = (x_0, y_0)$. A solution to $\nabla^2 H = \delta(\mathbf{r} - \mathbf{r}_0)$ is $H_1 = \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_0|$. The solution H_1 does not respect the boundary condition! Let's

introduce an image point at $\mathbf{r}_0' = (x_0, y_0)$. The function $H_2 = \frac{B}{2\pi}\ln|\mathbf{r}-\mathbf{r}_0'|$ is such that $\nabla^2 H = 0$ throughout the upper-half plane R as $\mathbf{r} \neq \mathbf{r}_0'$ in region R.
We thus conclude that $H = H_1 + H_2$ satisfies $\nabla^2 H = \delta(\mathbf{r}-\mathbf{r}_0)$ throughout the upper-half plane (by the superposition principle).
Now on $y = 0$, we wish to have $\frac{\partial H}{\partial y} = 0$. But

$$\begin{aligned}\frac{\partial H}{\partial y} &= \frac{1}{2\pi}\frac{\partial}{\partial y}\left[\ln\left[(x-x_0)^2+(y-y_0)^2\right]^{1/2} + B\ln\left[(x-x_0)^2+(y+y_0)^2\right]^{1/2}\right] \\ &= \frac{1}{2\pi}\left[\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} + \frac{B(y+y_0)}{(x-x_0)^2+(y-y_0)^2}\right]\end{aligned}$$

When $y = 0$ to obtain

$$\left.\frac{\partial H}{\partial y}\right| y = 0 = \frac{1}{2\pi}\left[\frac{-y_0+By_0}{(x-x_0)^2+(y-y_0)^2}\right] = 0 \Rightarrow B = 1$$

Via the method of images, we finally obtained the following solution

$$H == \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}_0| + \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}_0'|$$

6. In this problem, we consider the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$

(a) Assume that the Schrödinger equation admits separated solutions of the form $\psi(x,t) = \tau(t)\phi(x)$, subbing in the equation, we obtain

$$i\hbar\tau'\phi = -\frac{\hbar^2}{2m}\phi''\tau + V\tau\phi \Rightarrow \frac{i\hbar\tau'}{\tau} = \left[-\frac{\hbar^2}{2m}\frac{\phi''}{\phi} + V\right]$$

so we conclude that there exists $E \in \mathbb{R}$ such that

$$\tau' = -\frac{iE}{\hbar}\tau \Rightarrow \tau(t) = Ae^{-i\frac{E}{\hbar}t}$$

without loss of generality, we can set $A = 1$ as any constant can be absorbed in the prefactors of the space dependent function in the product solution. Further, we have

$$\frac{\hbar^2}{2m}\phi'' = (V(x) - E)\phi$$

where by homogeneity E is an energy (as $V(x)$ is an energy).

(b) Assuming that

$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le L \\ V_0, & \text{if } x < 0 \text{ or } x > L \end{cases}$$

We need to split our problem and we have

$$\begin{cases} \frac{\hbar^2}{2m}\phi'' = -E\phi, & x \in [0, L] \\ \frac{\hbar^2}{2m}\phi'' = (V_0 - E)\phi, & \text{otherwise} \end{cases}$$

with $V_0 - E > 0$. This leads to the following physically relevant solution:

$$\phi(x) = \begin{cases} C\exp(\sqrt{2m(V_0-E)}/\hbar x), & x < 0 \\ A\cos(\sqrt{2mE}/\hbar x) + B\sin(\sqrt{2mE}/\hbar x), & 0 \le x \le L \\ D\exp(-\sqrt{2m(V_0-E)}/\hbar x), & x > L \end{cases}$$

where A, B, C and D are integration constants to be obtained by imposing that the solution must be continously differentiable at the boundary points $x = 0$ and $x = L$. To obtain this solution, we also impose that the solution must remain bounded in $x = \pm\infty$. As sketch of the solution for some value of E is given below.

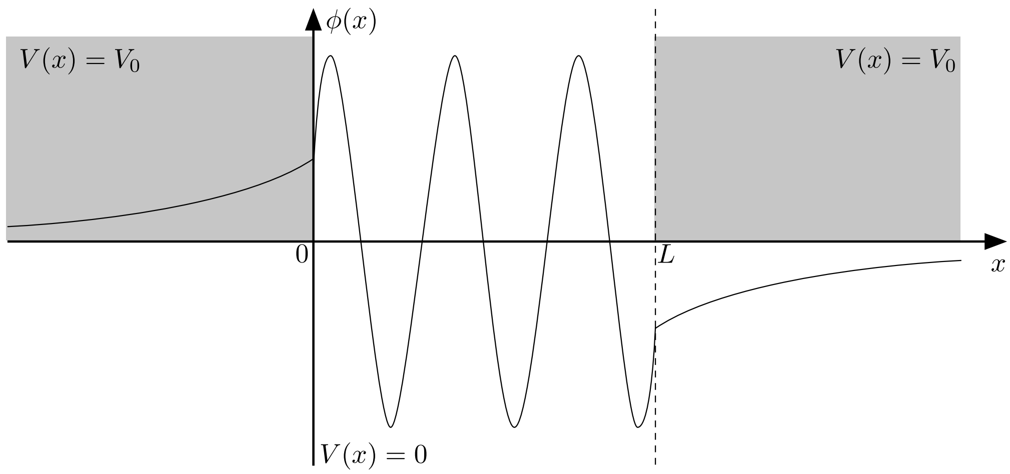

Figure 5: Diagram for the solution to Q6.

(c) Recall that the probability to observe the quantum particle at location x at time t is given by $|\psi(x,t)|^2$. Provided the solution we obtained in the previous question, we showed that even though $E < V_0$ (and so would not allow a classical particle to escape for the square potential well with height V_0) here as long as $C \neq 0$ or $D \neq 0$ the probability to find the quantum particle outside of the interval $[0, L]$ is non-zero. It is thus quantumly possible for the particle to escape from the potential well, this is called quantum tunnelling.

(d) Finally, imposing that the solution must be continuously differentiable in $x = 0$ and $x = L$ leads to

$$\phi(0) = C = A$$

$$\phi(L) = De^{-\sqrt{\frac{2m}{\hbar}(V_0-E)}L} = A\cos\left(\sqrt{\frac{2m}{\hbar}}EL\right) + B\sin\left(\sqrt{\frac{2m}{\hbar}}EL\right)$$

$$\phi'(0) = C\sqrt{\frac{2m}{\hbar}(V_0 - E)} = B\sqrt{\frac{2m}{\hbar}E}$$

$$\phi'(L) = -D\sqrt{\frac{2m}{\hbar}(V_0 - E)}e^{-\sqrt{\frac{2m}{\hbar}(V_0-E)}L} = \left[-A\sin\left(\sqrt{\frac{2m}{\hbar}}EL\right) + B\cos\left(\sqrt{\frac{2m}{\hbar}}EL\right)\right]\sqrt{\frac{2m}{\hbar}E}$$

Set $A = C = B\sqrt{\frac{E}{V_0-E}}$, then we conclude that E must satisfy the following equation

$$\tan\sqrt{\frac{2m}{\hbar}}EL = \frac{2\sqrt{\frac{E}{V_0-E}}}{\frac{E}{V_0-E} - 1}$$

Solutions only exist for E solution to the above transcendental equation, of which there is an infinite but discrete set! Those values represent the possible values of the energy of the quantum particle, we call this quantization of energy.