2.4 The Completeness Axiom

Exercise 2.22. Show if a subset $S \subset \mathbb{R}$ has a maximum (i.e. an element $\max S \in S$ such that $\max S \geq s\ \forall s \in S$) then it is *unique*.

Show if $\max S$ exists then $-S := \{-s \colon s \in S\}$ has a minimum, $\min(-S) = -\max S$.

Exercise 2.23. What is the maximum of the interval $(0,1)$?

1. 0
2. 0.5
3. $0.\overline{9}$.
4. 1
5. Something else.
6. More than one of these.
7. It has no maximum.

Clearly we think of 1 as being some kind of substitute for $\max(0,1)$. We call it $\sup S$, the *supremum* of S.

Definition. $\emptyset \neq S \subset \mathbb{R}$ is *bounded above* if and only if

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in S,\ x \leq M.$$

Such an M is called an *upper bound* for S.

S is *bounded below* if and only if

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in S, M \leq x.$$

Such an M is called a *lower bound*.

S is *bounded* if and only if S is *bounded above* and *below*.

So for instance $S = (0,1)$ is bounded above by any $M \geq 1$.

Exercise 2.24. Show that S is bounded if and only if

$$\exists R > 0 \text{ such that } \forall x \in S,\ -R \le x \le R,$$

or equivalently

$$\exists R > 0 \text{ such that } \forall x \in S,\ |x| \le R.$$

Definition. Suppose $\emptyset \neq S \subset \mathbb{R}$ is bounded above. We say $x \in \mathbb{R}$ is a *least upper bound* of S or **supremum** of S if and only if

Remark 2.25. Once we are bounded above we can pick an upper bound, then we slide it leftwards until we first hit an element of S (so long as $S \neq \emptyset$).

It is common for people write things down $\sup S = +\infty$ if S is not bounded above and $\sup S = -\infty$ if S is empty.

We are being super careful in this class and will want $\sup S$ to be a real number if it exists, so we enforce that S is non-empty and bounded above in our definition.

Exercise 2.26. Prove such an x is *unique* if one exists. Therefore we can call it $\sup S$.

Exercise 2.27. Write down the definition *greatest lower bound* (or *infimum*) of S. We call this $\inf S$ when it exists.

Exercise 2.28. Suppose $\exists \sup S$. Then show that $\inf(-S)$ exists too, and equals

1. $\sup S$
2. $-\sup S$
3. $\inf S$
4. $-\inf S$
5. None of these

Example 2.29. $S = (0, 1)$. Then $T = \{y : y \text{ is an upper bound for } S\}$.

T has a minimum: $1 = \min T = \sup(0, 1)$. Similarly $0 = \inf(0, 1)$.

Exercise 2.30. $\sup S \in S \iff S$ has a maximum and $\max S = \sup S$.

Theorem 2.31: Completeness axiom of $\mathbb{R}$

Remark 2.32. Recall "bounded above" rules out $\sup S = +\infty$, while "nonempty" rules out $\sup S = -\infty$.

Either we can work with this as an axiom (an act of faith) not worrying about whether anyone every created a set $\mathbb{R}$ satisfying both the completeness axiom and the previous Axioms 1-12. Or we can give a construction of $\mathbb{R}$ and show it has the property that any bounded above $\emptyset \neq S \subset \mathbb{R}$ has a supremum. Next we show our construction of $\mathbb{R}$ using decimal expansions has this property. (Later, in Section 2.5, we will give another construction using Dedekind cuts.)

Proof. Without loss of generality (replacing S by $S + a := \{s + a : s \in S\}$) we may assume $S \neq \emptyset$ has a positive element $0 \leq s \in S$. This will simplify things, because positive decimals behave better than negative decimals.

S is bounded above by $R \geq 0$, say. Set $N := \lceil R \rceil \in \mathbb{N}$. So we may replace S by $S \cap [0, N]$: both are nonempty with the same upper bounds (easy exercise), so one has a sup if and only if the other one does, and the two suprema are equal.

We will create the supremum $\sup S = a_0.a_1a_2 \ldots \geq 0$ digit by digit.

Leading integer. Write each $s \in S$ as a decimal $s_0.s_1s_2s_3\ldots$ not ending in $\overline{9}$. Since $s \in [0, N]$ we see that $s_0 \in \{0, 1, \ldots, N\}$, a finite set. So the set of leading integers s_0 is finite, so has a maximum $a_0 \geq 0$.

First decimal place. So $S \cap [a_0, a_0 + 1)$ is nonempty and we may replace S by it (same easy exercise). All its elements are of the form $a_0.s_1s_2\ldots$ with $s_1 \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_1 value; call it a_1.

Second decimal place. So can replace S by $S \cap \big[a_0.a_1,\, a_0.(a_1+1)\big)$. (If $a_1 = 9$ we mean $S \cap [a_0.9,\, a_0+1)$.) Every $s \in S$ has decimal expansion $a_0.a_1 s_2 s_3 \dots$ with $s_2 \in \{0,1,\dots,9\}$ – a finite set. Thus there is a maximum s_2 value; call it a_2.

nth decimal place. Assume I've defined $a_0, \dots, a_{n-1}$ and shown that

$$S \cap \big[a_0.a_1 \dots a_{n-1},\ a_0.a_1 \dots (a_{n-1}+1)\big)$$

is nonempty and has the same upper bounds as the original S. Any element is $s = a_0.a_1 \dots a_{n-1} s_n s_{n+1} \dots$ with $s_n \in \{0,1,\dots,9\}$ – a finite set. Thus there is a maximum s_n value; call it a_n.

Upper bound. We claim $a := a_0.a_1a_2\dots$ is an upper bound for S. By the construction of a_0, every element of $s_0.s_1s_2\dots$ of S has either

- $s_0 < a_0$ ($\Longrightarrow s < a$ and we're done), or
- $s_0 = a_0$.

In the second case, by the construction of a_1, either

- $s_1 < a_1$ ($\Longrightarrow s < a$ and we're done), or
- $s_1 = a_1$.

In the second case, by the construction of $a_2 \dots$

Ok you get the idea. Either this process terminates ($\Longrightarrow s < a$) or it doesn't ($\Longrightarrow s = a$). Either way $s \le a$ so a is an upper bound.

Least upper bound. If $b < a$ is a smaller upper bound for S, suppose their decimal expansions first differ in the nth place, i.e.

$$b \;=\; a_0.a_1a_2\dots a_{n-1}b_n\dots \qquad \text{with } b_n < a_n.$$

But remember the construction of a_n above: there was an element $s \in S$ with $s = a_0.a_1a_2\dots a_{n-1}a_n\dots$ so $s > b$ ※ So a is the *least* upper bound of S.

Finally note $a_0.a_1a_2\dots$ could end in $\overline{9}$ (in fact check it will indeed do so if the set is $(0,1)$. So to consider it as a real number according to our definition we should round up the 9s. □

> **Exercise 2.33.** Apply Theorem 2.31 to $-S$ to deduce if $\emptyset \neq S \subset \mathbb{R}$ is bounded below then S has an inf.

The completeness axiom means $\mathbb{R} \supset \mathbb{Q}$ fills in all the "holes". For example:

Proposition 2.34. *There exists $0 < x \in \mathbb{R}$ such that $x^2 = 3$. We call $x =: \sqrt{3}$.*

Proof. Since this is not true in $\mathbb{Q}$ we'd better use the completeness axiom! So we need a set $S \subset \mathbb{R}$ to apply it to. Set

To prove $x^2 = 3$ we need to show $x^2 \not< 3$ and $x^2 \not> 3$ by the trichotomy axiom.

• If $x^2 < 3$ then we *expect* $(x + \epsilon)^2 < 3$ for sufficiently small $\epsilon > 0$. (This would give a contradiction: $S \ni x + \epsilon > x = \sup S$.) So let's compute

• If $x^2 > 3$ then we *expect* $(x - \epsilon)^2 \geq 3$ for *all* sufficiently small $\epsilon > 0$. (Then $x - \epsilon$ would be an upper bound for S smaller than $\sup S$.) So let's compute

□

Once it's all written out, the proof looks like a blur of symbols and formulae. Reading it will do you little good. Trying to "learn" or remember it will do you even less good. You must come up with your own proof. When you do, the formulae will all make perfect sense to you.

Close your lecture notes, and write out your own proof. Only look at this proof if you're stuck and you've struggled for more than 10 minutes.

You shouldn't be thinking about exams, but if you are, this is a typical type of proof you'll have to give in exams. But the exam will ask for a proof you've not seen before, so practise coming up with your own proofs.

Exercise 2.35. Show $\sqrt[3]{2}$ exists.

See Question sheet 2 for nth roots and even rational qth powers of positive real numbers. Next term, once you're experts on continuity, it will be easy to create xth powers of all positive real numbers $\forall x \in \mathbb{R}$.

Exercise 2.36. A student is trying to prove there exists $0 < x \in \mathbb{R}$ such that $x^2 = 2$. Since

$$S \ := \ \left\{0 < a \in \mathbb{R} : \ a^2 < 2\right\},$$

is nonempty ($1 \in S$) and bounded above by 2 (if $a \geq 2$ then $a^2 \geq 4$ so $a \notin S$) he sets $x := \sup S > 0$.
Next he gives this proof that $x^2 \not< 2$. Is any step wrong?

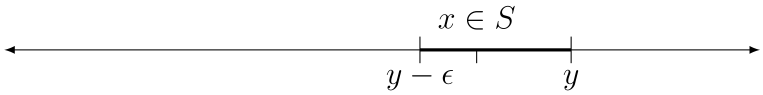
1. Suppose $x^2 < 2$, then we try to find $\epsilon > 0$ such that $(x+\epsilon)^2 < 2$.
2. Note $2 > (x+\epsilon)^2 = x^2 + 2\epsilon x + \epsilon^2 > x^2 + 2\epsilon x$ implies that $\frac{2-x^2}{2x} > \epsilon$.
3. So if we take $0 < \epsilon < \frac{2-x^2}{2x}$ then $(x+\epsilon)^2 < 2$.
4. So $x + \epsilon \in S$ but $x + \epsilon > x = \sup S$ ※
5. Nothing wrong, full marks for the Buzzard.

Exercise 2.37. Let $S = \{x \in \mathbb{Z} : x^2 < 3\}$. Then S is nonempty and bounded above. What is $\sup S$?

1. 0
2. 1

3. 2
4. 3
5. $\sqrt{3}$
6. Something else

Proposition 2.38. *Suppose $\emptyset \neq S \subset \mathbb{R}$ and y is an upper bound for S. Then $y = \sup S \iff \forall \epsilon > 0\ \exists s \in S \colon s > y - \epsilon$.*



Think of $\epsilon > 0$ as *small*, then $y - \epsilon$ is a little bit smaller than y and the condition is just saying that $y - \epsilon$ is not an upper bound for S.

Proof.

$\square$

As an aside we note that the completeness axiom implies the Archimedean axiom: that if we fix $x \in \mathbb{R}$ then $\exists N \in \mathbb{N}$ such that $N \geq x$.

Suppose for a contradiction that no such N exists, i.e. $\mathbb{N}$ is bounded above by x. Since it is also nonempty, $\exists \sup \mathbb{N} =: y$. But $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$, so

$$\forall n \in \mathbb{N},\ y \ \geq \ n+1 \iff y - 1 \ \geq \ n,$$

so $y - 1$ is a smaller upper bound for $\mathbb{N}$ ※

2.5 Alternative approach: Dedekind cuts

Intuition. Suppose we have a construction of $\mathbb{R}$, e.g. by decimals. Then to every real number $r \in \mathbb{R}$ we can associate a semi-infinite subset S_r of $\mathbb{Q}$,

$$S_r \ := \ (-\infty, r) \cap \mathbb{Q}.$$

That is S_r is the set of *all rational numbers* $< r$.

Exercise 2.39. Show $r_1 = r_2 \iff S_{r_1} = S_{r_2}$.
(Obviously $\Leftarrow$ is the important one!)

Slightly abusing the original notation, we will call the $S_r \subset \mathbb{Q}$ *Dedekind cuts*. The fantastic thing is, we only need to know about $\mathbb{Q}$ to define them. So now we forget all about $\mathbb{R}$ and pretend we only know about $\mathbb{Q}$.

Definition. We say a nonempty subset $S \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies (i) and (ii) below.

(i)

(ii)

So using only this notion from $\mathbb{Q}$ we can *construct* $\mathbb{R}$, showing that $\mathbb{Q}$ "knows" about its "completion" $\mathbb{R}$.

Definition.

$$\mathbb{R} \; := \; \big\{\text{Dedekind cuts } S \subset \mathbb{Q}\big\}.$$

(I.e. we think of identifying S_r with $r \in \mathbb{R}$.)

Exercise 2.40. Check that we can identify $\mathbb{Q} \subset \mathbb{R}$ by taking $q \in \mathbb{Q}$ to the Dedekind cut $S_q := \{s \in \mathbb{Q} : s < q\}$.

We can generalise all the usual arithmetic operations that we already have on $\mathbb{Q}$ to our newly constructed $\mathbb{R}$; eg if $S \subset \mathbb{Q}$ and $T \subset \mathbb{Q}$ are Dedekind cuts, we define

$$S + T := \{s + t \colon s \in S, \; t \in T\} \subset \mathbb{Q}.$$

Exercise 2.41. Check this is a Dedekind cut (an element of $\mathbb{R}$!) and gives the usual $+$ on $\mathbb{Q}$: i.e. $S_{q_1} + S_{q_2} = S_{q_1+q_2}$.

Similarly we can define $<$ on $\mathbb{R}$ to be just $\subsetneq$ on Dedekind cuts:

$$S < T \iff S \subsetneq T.$$

Exercise 2.42. Show a set of real numbers $A \subset \mathbb{R}$ is bounded above iff A is a set of Dedekind cuts S all contained in some fixed interval $(-\infty, N)$ for some $N \in \mathbb{N}$.

Now proving Theorem 2.31 (i.e. verifying the completeness axiom) becomes easy:

Exercise 2.43. If A is a bounded above nonempty set of Dedekind cuts, define

$$\sup A \ := \ \bigcup_{S \in A} S \ \subset \ \mathbb{Q}.$$

Show this is also a Dedekind cut (i.e. a real number!) and check it is the least upper bound of A.

For more details of this construction see for example **W. Rudin**, *"Principles of mathematical analysis"*, or the webpage http:\tinyurl.com/yjt5olv

2.6 Triangle inequalities

Theorem 2.44

For all $a, b \in \mathbb{R}$ we have

$$|a+b| \ \le \ |a| + |b|.$$

Proof. If $|a+b| > |a| + |b|$ then applying order Axiom 11 several times,

$$|a+b|^2 \ > \ (|a|+|b|)|a+b| \ > \ (|a|+|b|)^2 \ = \ |a|^2 + 2|a||b| + |b|^2.$$

But this contradicts

$$|a+b|^2 \ = \ (a+b)^2 \ = \ a^2 + 2ab + b^2 \ \le \ |a|^2 + 2|a||b| + |b|^2.$$ □

Exercise 2.45. Why is this called the triangle inequality?
Give a direct proof *without squaring* by first proving $x \le |x|$ by splitting into two cases.

There are many more which you will prove on Problem Sheet 2.

Don't try to memorise them! Understand them, and then work each one out as you need it. They're all intuitively obvious if thought about in the right way.

E.g. (d) says that if x is close to y then $|x|$ is close to $|y|$ (possibly closer – if x, y have different signs).

E.g. (e) says the distance in $\mathbb{R}$ from 0 to x is no more than the distance if we go $0 \to y \to x$ via y.

E.g. (g) says the distance from x to y is no more than the distance if we go $x \to z \to y$ via z.

For instance, let's prove (g) from (a):

They are a crucial component of the rest of this course, as there's many things close to each other where we need to estimate distance between things.

Here are some different ways of saying a, b are close to each other; get used to them.

Exercise 2.46. Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0,\ |x - a| < \epsilon \qquad (*)$$

mean for the number x? I.e. which of the following is it equivalent to?

1. x is close to a
2. $x \in (a - \epsilon, a + \epsilon)$
3. $x = a$
4. $x = a + \epsilon$
5. $x = a - \epsilon$
6. More than one of these
7. None of these