Network Science
Spring 2024
Problem class 5 solutions

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1. Let $\mathbf{L}$ be the Laplacian matrix for a simple graph.

   (a) Show that $\mathbf{z}$ is always an eigenvector $\mathbf{L}$. What is the corresponding eigenvalue?

   (b) Let λ_1 be the most positive eigenvalue of $\mathbf{L}$. Show that $\lambda_1 \leq 2k_{max}$ where k_{max} is the largest degree in the graph which corresponds to $\mathbf{L}$.

   (c) In this exercise, you will explore a connection between the 2-D diffusion equation and the graph diffusion equation.

      i. Consider the 5-node graph shown above on the left. What is the graph diffusion equation for node 1?:

      ii. Now re-interpret the graph as 5 points in the x-y plane as shown above on the right, and consider the 2d-diffusion equation applied to $f(x, y)$. The 2-d diffusion equation is,
      $$\partial f/\partial t = \alpha\left(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2\right).$$
      It can be shown using Taylor series expansions that,
      $$d^2 f/dx^2|_{x0} \approx (1/h^2)\left(f(x_0 + h) - 2f(x_0) + f(x_0 - h)\right).$$
      Using approximations of this form for the second derivatives in the 2-D diffusion equation, obtain an expression for $\partial f/\partial t|_{(x_0,y_0)}$ which is comparable with your result for (a).