ADVANCED TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

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1.1: INTRODUCTION

- **Ordinary Differential Equation:** $y'(t) = F(t, y(t))$ (1) F is a given function, $F: I \times U \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$, $U \subseteq \mathbb{R}^n$. We look for $y: J \subseteq I \to \mathbb{R}^n$ which solves (1) pointwise.

- **Cauchy-Lipschitz Thm:** $\begin{cases} y'(t) = f(t, y(t)) & (2) \\ y(t_0) = y_0, \ (t_0, y_0) \in I \times U \end{cases}$ Assume F cts at (t_0, y_0) and is locally Lipschitz in y uniformly in t. Then there exists a unique local solution $y(t)$ to (2), i.e. defined on some $J \subseteq I$

- **Partial Differential Equation:** y is a function of several variables. For instance $y = y(x_1, x_2, \dots, x_n)$ or $y = y(t, x)$. A PDE is a relation between the partial derivatives of f.

  → **Notation:** Let $\alpha \in \mathbb{N}^m$, $\alpha = (\alpha_1, \dots, \alpha_m)$. We denote by $\partial^\alpha f$ the partial derivative $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \dots \partial^{\alpha_m}_{x_m}}$ where $|\alpha| = \alpha_1 + \dots + \alpha_m$.

  $\alpha! = \alpha_1! \cdots \alpha_m!$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$.

  For instance, $n = 3$, $\alpha = (1, 0, 1)$ $(|\alpha| = 2)$; $\partial^\alpha f = \frac{\partial^2 f}{\partial x_1 \partial x_3}$

- **DEF:** A PDE is a relation $F(x, y, \dots, \partial^\alpha y, \dots) = 0$.

- **Ex:** Let $U \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$ (U open, connected).

  Δ; Laplacian $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$

  1) **Poisson's Eqn:** Let $g: U \to \mathbb{R}$ be given. Find $f: U \to \mathbb{R}$ such that: $-\Delta f = g$ in U.
  2) **Heat Eqn:** Let $g: I \times U \to \mathbb{R}$ be given. Find $f: J \times U \to \mathbb{R}$ $(J \subseteq I)$ such that $\partial_t f - \Delta f = g$ in $J \times U$.
  3) **Wave Eqn:** Let $g: I \times U \to \mathbb{R}$ be given. Find $f: J \times U \to \mathbb{R}$ $(J \subseteq I)$ such that $\partial_t^2 f - \Delta f = g$ in $J \times U$
  4) **Schrödinger Eqn:** Let $V: I \times U \to \mathbb{R}$ be given. Find $f: J \times U \to \mathbb{R}$ $(J \subseteq I)$ such that $i \partial_t f + \Delta f = Vf$ in $J \times U$

  Linear operators; if f_1, f_2 are solns then $af_1 + bf_2$ is too.

  → **Remark:** In general, necessary to add:
  - initial conditions (in time-evolution eqns)
  - boundary conditions (if U bdd)
  - growth conditions (if U unbdd)

  5) **Burger's Eqn:** Find $f: (t, x) \subseteq \mathbb{R}_+ \times \mathbb{R}$ such that $\partial_t f + f \partial_x f = 0$
  6) **Navier-Stokes Eqn:** Find $\underline{u} = (u_1, u_2, u_3): I \times U \to \mathbb{R}^3$ and $p: I \times U \to \mathbb{R}^3$ such that $\begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \nabla) \underline{u} - \nu \Delta \underline{u} + \nabla p = f \\ \operatorname{div} \underline{u} = 0 \end{cases}$

  $\underline{u}$ = velocity of fluid
  p = pressure
  $\operatorname{div} \underline{f} = \sum_{i=1}^n \partial_{x_i} f_i$
  $(n = 3, \operatorname{div} f = \partial_{x_1} f_1 + \partial_{x_2} f_2 + \partial_{x_3} f_3)$

- **Discussion:** What is PDE theory all about? In general, finding an explicit solution to a PDE is usually out of reach except for some specific examples.

  AIM: Study the well-posedness of the PDEs in the Hadamard sense: **Existence, Uniqueness, Stability w.r.t. the data**

  We can divide PDEs into the following classes:
  1) Elliptic (e.g. Poisson)
  2) Parabolic (e.g. Heat)
  3) Hyperbolic (e.g. Wave)
  4) Dispersive (e.g. Schrödinger)

  "PDEs within the same class generally enjoy similar propeties."

- **Variational Approach: Motivation:** We consider the following Poisson's problem: Let $f \in C([a,b])$ be given, find a function $u: [a,b] \to \mathbb{R}$ such that $\begin{cases} -u''(x) + u(x) = f(x) & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases}$ (P)

A classical (a.k.a strong) solution of (P) is a $C^2([a,b]) \cap C([a,b])$ function satisfying (P) in the usual sense.

Let us consider $\varphi \in C^1([a,b])$ with $\varphi(a) = \varphi(b) = 0$. Multiply eqn by φ and IBP:

$$\int_a^b -u''\varphi \, dx + \int_a^b u \cdot \varphi \, dx = \int_a^b f \cdot \varphi \, dx \quad \text{and} \quad \int_a^b -u'' \cdot \varphi \, dx = \cancel{u'\varphi \Big|_{x=a}^{x=b}} + \int_a^b u'\varphi' dx \implies \int_a^b u'\varphi' dx + \int_a^b u \cdot \varphi \, dx = \int_a^b f \cdot \varphi \, dx \quad (WF)$$

(WF) makes sense if $u, u' \in L^1(a,b)$ where u' is the weak derivative of u and $f \in L^1(a,b)$ ⇝ A function u satisfying (WF) is a weak solution.

Step A: Sobolev Spaces
Step B: Existence & uniqueness of weak solns. by variational approach
Step C: Regularity Theory (weak soln. is actually a strong one).

1.2: TEST FUNCTION SPACE

- **Goal:** define $f'(x)$ without asking for differentiability but only very mild regularity such as $f \in L^1_{loc}$ ($L^1_{loc} = L^1$ on any compact subset)
  - i) let $f \in L^1_{loc}(\mathbb{R})$, $T_f : \varphi \in C^\infty_c(\mathbb{R}) \longrightarrow \int_{\mathbb{R}} f\varphi \, dx$. T_f is a linear, cts operator on $C^\infty_c(\mathbb{R})$.

    If $f_1, f_2 \in L^1_{loc}(\mathbb{R})$ such that $\int_{\mathbb{R}} f_1 \varphi \, dx = \int_{\mathbb{R}} f_2 \varphi \, dx \ \ \forall \varphi \in C^\infty_c(\mathbb{R})$ then $f_1 = f_2$ a.e.
  - ii) Let $f \in C^1(\mathbb{R})$, $\varphi \in C^\infty_c(\mathbb{R})$. Follows from IBP that $\int_{\mathbb{R}} f'\varphi \, dx = -\int_{\mathbb{R}} f\varphi' dx \ \ \forall \varphi \in C^\infty_c(\mathbb{R})$. Observe RHS well-defined provided $f \in L^1_{loc}(\mathbb{R})$.

    Given $f \in L^1_{loc}(\mathbb{R})$ we define the linear form $T_f : \varphi \in C^\infty_c(\mathbb{R}) \longrightarrow -\int_{\mathbb{R}} f\varphi' dx$

- **DEF:** Given a function f we define $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R}^m : f(x) \neq 0\}}$. If $\Omega \subset \mathbb{R}^m$, we have $\mathrm{supp} f = \Omega \cap \overline{\{x \in \mathbb{R}^m : f(x) \neq 0\}}$

- **DEF:** Define $D(\mathbb{R}^m)$ as $C^\infty_c(\mathbb{R}^m)$, i.e. $\forall \varphi \in D(\mathbb{R}^m)$ $\exists R > 0$: $|x| > R \Rightarrow \varphi(x) = 0$
  - ↳ **Ex:** $\varphi(x) = \begin{cases} c \cdot \exp\{\frac{-1}{1-|x|^2}\}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$ where $c = \left(\int_{B_1(0)} \exp\left(\frac{-1}{1-|x|^2}\right) dx\right)^{-1}$. Follows by defn that $\int_{\mathbb{R}^m} \varphi(x) \, dx = 1$.

- **DEF:** Given f, g in $\mathbb{R}^m$ we define the convolution between f and g as $(f * g)(x) = \int_{\mathbb{R}^m} f(x-y)g(y) = \int_{\mathbb{R}^m} g(x-y)f(y) \, dy$.
  - ↳ **THM:** (YOUNG) Assume $f \in L^p(\mathbb{R}^m)$, $g \in L^q(\mathbb{R}^m)$, $p, q \in [1, \infty)$. Then $f * g \in L^r(\mathbb{R}^m)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $\|f * g\|_{L^r(\mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^m)} \|g\|_{L^q(\mathbb{R}^m)}$

- **PROP:** Assume $f \in L^1(\mathbb{R}^m)$, $g \in L^p(\mathbb{R}^m)$, $p \in [1, \infty]$. $\mathrm{supp}(f * g) \subset \overline{\mathrm{supp} f + \mathrm{supp} g}$ $\qquad \boxed{A + B = \{a + b : a \in A, b \in B\}}$
  - → Given φ in the example above, for any $\rho \in \mathbb{R}_+$, we define the mollifier $\varphi_\rho(x) = \rho^m \varphi(\rho x)$, $\forall x \in \mathbb{R}^m$
    - i) $\int_{\mathbb{R}^m} \varphi_\rho(x) \, dx = 1$
    - ii) $\mathrm{supp}(\varphi_\rho) = B_{\frac{1}{\rho}}(0)$

- **PROP:** We have the following results:
  - i) $\forall f \in C^0(\mathbb{R}^m)$, $f_\rho = f * \varphi_\rho \in C^\infty(\mathbb{R}^m)$ and $f_\rho \longrightarrow f$ as $\rho \to \infty$ uniformly on every compact set in $\mathbb{R}^m$, i.e. $\forall K \subset \mathbb{R}^m$ compact $\sup_{x \in K} |f_\rho(x) - f(x)| \longrightarrow 0$ as $\rho \to \infty$.
  - ii) $\forall f \in C^\infty(\mathbb{R}^m)$, $f_\rho = f * \varphi_\rho \in C^\infty(\mathbb{R}^m)$ and $\forall \alpha \in \mathbb{N}^\alpha$, $\partial^\alpha f_\rho \to \partial^\alpha f$ as $\rho \to \infty$ uniformly on every compact set in $\mathbb{R}^m$.
  - iii) $\forall f \in L^p(\mathbb{R}^m)$, $1 \leq p < \infty$, then $f_\rho = f * \varphi_\rho \in C^\infty(\mathbb{R}^m)$ and $f_\rho \longrightarrow f$ in $L^p(\mathbb{R}^m)$ as $\rho \to \infty$. ↶ doesn't hold if $p = +\infty$!

- **Pf:** $(f_\rho - f)(x) = (f * \varphi_\rho - f)(x) = \int_{\mathbb{R}^m} f(x-y)\varphi(\rho y)\rho^m dx - f(x) \cdot 1 = \int_{\mathbb{R}^m} \rho^m \varphi(\rho y)(f(x-y) - f(x)) \, dy$. Let K be a compact set of $\mathbb{R}^m$, and $x \in K$.
  Let $\rho_0 > 0$ and suppose that $\rho > \rho_0$. $\mathrm{Supp}\varphi_\rho = B_{\frac{1}{\rho}}(0) \subseteq B_{\frac{1}{\rho_0}}(0)$. Thus $x - y \in \overbrace{K + B_{\frac{1}{\rho_0}}(0)}^{K'}$ which is a compact set.
  On K', f is unif. cts. So $\forall \varepsilon > 0$ $\exists \eta > 0$: $|y| < \eta \Rightarrow |f(x-y) - f(x)| < \varepsilon$ $\forall x \in K$. We compute $|f_\rho(x) - f(x)|$
  $\leq \overbrace{\int_{|y|<\eta} \rho^m \varphi(\rho y)|f(x-y) - f(x)| \, dy}^{I_1} + \overbrace{\int_{|y| \geq \eta} \rho^m \varphi(\rho y)|f(x-y) - f(x)| \, dy}^{I_2}$. In order to estimate I we use the uniform continuity, namely
  $I_1 \leq \varepsilon \underbrace{\int_{|y|<\eta} \rho^m \varphi(\rho y)}_{\leq 1} \leq \varepsilon$. To estimate I_2, $I_2 \leq 2\|f\|_{L^\infty(\mathbb{R}^m)} \int_{|y| \geq \eta} \rho^m \varphi(\rho y)|f(x) - f(y)| \, dy \overset{z = \rho y}{\leq} 2\|f\|_{L^\infty} \int_{|z| \geq \rho\eta} \varphi(z) \, dz$. But, observe if $|z| \geq 1$, $\varphi(z) = 0$. So $I_2 = 0$ if $\rho > \frac{1}{\eta}$. Thus $\forall x \in K$ we have $|f_\rho(x) - f(x)| < \varepsilon$ provided $\rho > \max\{\rho_0, \frac{1}{\eta}\}$. This shows $f * \varphi_\rho \to f$ as $\rho \to \infty$ unif. on K.
  - ii) follows from using: $\partial^\alpha(f * \varphi_\rho) = \partial^\alpha f * \varphi_\rho$.
  - iii) We recall $C^0_c(\mathbb{R}^m)$ dense in $L^p(\mathbb{R}^m)$. Given $f \in L^p(\mathbb{R}^m)$ $\exists f_1 \in C^0_c(\mathbb{R}^m)$ such that $\|f - f_1\|_{L^p(\mathbb{R}^m)} < \varepsilon$.
    Consider $f_{1,\rho} = f_1 * \varphi_\rho$. From (i) we get $f_1 * \varphi_\rho \longrightarrow f_1$ unif. on every compact subset of $\mathbb{R}^m$ as $\rho \to \infty$.
    $\mathrm{Supp}(f_{1,\rho}) \subset \overline{\mathrm{supp}(f_1) + B_1(0)}$ which is a compact subset $\Rightarrow \|f_1 * \varphi_\rho - f_1\|_{L^p} \to 0$ as $\rho \to \infty$. We now have
    $f * \varphi_\rho - f = [\varphi_\rho * (f - f_1)] + [(\varphi_\rho * f_1) - f_1] + [f_1 - f]$. Thus we conclude $\|f * \varphi_\rho - f\|_{L^p} \leq 2\|f_1 - f\|_{L^p} + \|\varphi_\rho * f_1 - f_1\|_{L^p} \leq 3\varepsilon$ □

- **COR:**
  - i) $D(\mathbb{R}^m)$ dense in $C(\mathbb{R}^m)$ for the topology of the unif. convergence of all compact sets.
  - ii) $D(\mathbb{R}^m)$ is dense in $C^\infty(\mathbb{R}^m)$ for the topology of the unif. convergence and all of its derivatives on all compact sets.
  - iii) $D(\mathbb{R}^m)$ is dense in $L^p(\mathbb{R}^m)$, $1 \leq p < \infty$.

- **DEF:** Let $\{f_n\}_{n \geq 1}$ be a sqnce of functions of $D(\mathbb{R}^m)$. Then $f_n \xrightarrow{D} f$ iff
  - i) $\exists K$ compact set : $\mathrm{supp} f_n \subset K$ $\ \forall n \in \mathbb{N}$
  - ii) $\forall \alpha \in \mathbb{N}^\alpha$, $\partial^\alpha f_n \rightrightarrows \partial^\alpha f$ on K.

2.1: DISTRIBUTIONS, DEFINITIONS & EXAMPLES

- **DEF:** Let $\{f_n\}$ be a sqnce of functions in $D(\mathbb{R}^n)$. Then $f_n \xrightarrow{D} f$ iff: $\boxed{\begin{array}{l} 1)\ \exists K \subset \mathbb{R}^m \text{ compact set}: \operatorname{supp} f_n \subseteq K,\ \forall m \in \mathbb{N}. \\ 2)\ \forall \alpha \in \mathbb{N}^m,\ \partial^\alpha f_n \longrightarrow \partial^\alpha f \text{ uniformly on } K. \end{array}}$

- **DEF:** A distribution in $\mathbb{R}^n$ is a linear and continuous form on $D(\mathbb{R}^n)$, i.e. a linear map $T: \varphi \in D(\mathbb{R}^n) \longrightarrow \langle T, \varphi \rangle \in \mathbb{R}$ such that, $\forall$ sequence $\{\varphi_n\} \subset D(\mathbb{R}^n)$, $\varphi_n \xrightarrow{D} \varphi \Rightarrow \langle T, \varphi_k \rangle \longrightarrow \langle T, \varphi \rangle$

  → Space of distributions $:= D'(\mathbb{R}^n)$

  → Let T_1, T_2 be two distributions such that $\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle$ $\forall \varphi \in D(\mathbb{R}^n)$. Then $T_1 = T_2$.

- **Ex:**(1) Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then we define $T_f: \varphi \in D(\mathbb{R}^n) \longrightarrow \langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$.

  We observe T_f is well-defined $\forall \varphi \in D(\mathbb{R}^n)$ since $\langle T_f, \varphi \rangle = \int_{\operatorname{supp}\varphi} f(x)\varphi(x)\,dx \in \mathbb{R}$.

  T_f is a distribution: i) T_f linear ✓ ii) Assume $\varphi_k \to \varphi$ in $D(\mathbb{R}^n)$. We have $|\langle T_f, \varphi \rangle - \langle T_f, \varphi_n \rangle| = \left|\int_{\mathbb{R}^m} f(x)(\varphi(x) - \varphi_n(x))\,dx\right|$

  $\leq \int_K |f(x)||\varphi(x) - \varphi_n(x)|\,dx \leq \underbrace{\|\varphi - \varphi_n\|_{L^\infty(\mathbb{R}^n)}}_{\to 0} \cdot \underbrace{\int_K |f(x)|\,dx}_{<+\infty} \longrightarrow 0$ So $L^1_{loc}(\mathbb{R}^n)$ is a subspace of $D'(\mathbb{R}^n)$.

  Notation: we'll write f and T_f to refer to the distribution.

- **Ex:**(2) Dirac delta at x_0. $\delta_{x_0}: \varphi \in D(\mathbb{R}^n) \longrightarrow \langle \delta_{x_0}, \varphi \rangle = \varphi(x_0)$. δ_{x_0} is a distribution. Indeed if $\varphi_n \xrightarrow{D} \varphi \Rightarrow \varphi_n(x_0) \to \varphi(x_0)$.

  Can δ_{x_0} be represented as some element of $L^1_{loc}(\mathbb{R}^n)$? Assume by contradiction that there exists $f \in L^1_{loc}(\mathbb{R}^n)$ s.t. $\int f(x)\varphi(x)\,dx = \varphi(x_0)$ $\forall \varphi \in D(\mathbb{R}^n)$. Consider compact set $M \subseteq \mathbb{R}^n$ s.t. $x_0 \notin M$. Then, $\int_M f(x)\varphi(x)\,dx = 0$ $\forall \varphi \in D(M)$ $\Rightarrow f(x) = 0$ a.e. in M. As M arbitrary this means $f(x) = 0$ a.e. in $\mathbb{R}^n \Rightarrow \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = 0 \neq \varphi(0)$.

- **Ex:**(3) $f(x) = \frac{1}{x} \notin L^1_{loc}(\mathbb{R})$. We define the following form: $\text{p.v.}\left(\frac{1}{x}\right): \varphi \in D(\mathbb{R}) \longrightarrow \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x}\,dx$.

  By Taylor's Formula, $\varphi(x) = \varphi(0) + x\theta(x)$, where $\theta(x) \in C^\infty(\mathbb{R})$.

  So, $\langle \text{p.v.}\left(\frac{1}{x}\right), \varphi \rangle = \lim_{\varepsilon \to 0} \left\{ \int_{\mathbb{R} \setminus [-1,1]} \frac{\varphi(x)}{x}\,dx + \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x}\,dx \right\} = \int_{\mathbb{R} \setminus [-1,1]} \frac{\varphi(x)}{x}\,dx + \lim_{\varepsilon \to 0} \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(0)}{x} + \theta(x)\,dx = \int_{\mathbb{R} \setminus [-1,1]} \frac{\varphi(x)}{x}\,dx + \lim_{\varepsilon \to 0} \left\{ \cancelto{0}{\int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(0)}{x}\,dx} + \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \theta(x)\,dx \right\}$

  $= \int_{\mathbb{R} \setminus [-1,1]} \frac{\varphi(x)}{x}\,dx + \lim_{\varepsilon \to 0} \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \theta(x)\,dx.$

  → $\text{p.v.}\left(\frac{1}{x}\right)$ is a distribution

  → $\text{p.v.}\left(\frac{1}{x}\right)$ cannot be represented as an $L^1_{loc}(\mathbb{R})$ function f.

2.2: CONVERGENCE AND DIFFERENTIABILITY OF DISTRIBUTIONS

- **DEF:** Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of distributions in $D'(\mathbb{R}^n)$. $T_n \xrightarrow{D'(\mathbb{R}^n)} T$ if and only if $\langle T_n, \varphi \rangle \xrightarrow{n \to \infty} \langle T, \varphi \rangle$ $\forall \varphi \in D(\mathbb{R}^n)$.

  This topology is Hausdorff, i.e. if the limit exists it is unique.

- **Ex:**(4) 1) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of $L^1_{loc}(\mathbb{R}^n)$ functions and let $f \in L^1_{loc}(\mathbb{R}^n)$ be s.t. $f_n|_K \to f|_K$ in $L^1(K)$, for any compact set K in $\mathbb{R}^n$. Then we have $T_{f_n} \to T_f$ in $D'(\mathbb{R}^n)$.

  2) $\varphi(x) = \begin{cases} c \cdot \exp\left(-\frac{1}{1-|x|^2}\right), & |x| \leq 1 \\ 0, & \text{o.w} \end{cases}$ where c is s.t. $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. $\forall p \in \mathbb{R}$, $\varphi_p(x) = p^n \varphi(px)$, $x \in \mathbb{R}^n$.

  It follows that $T_{\varphi_p} \xrightarrow{D'} \delta_0$.

- <u>Motivation for differentiability:</u> $u \in C^1(\mathbb{R}^n)$, $\varphi \in D(\mathbb{R}^n)$. $\int_{\mathbb{R}^n} \partial_{x_i} u(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} u(x)\,\underbrace{\partial_{x_i}\varphi(x)}_{\in D'(\mathbb{R}^n)}\,dx$

  → RHS well-defined if $u \in L^1_{loc}(\mathbb{R}^n)$.

  → RHS well-defined if u "replaced" by a distribution T.

- **DEF:** Let $T \in D'(\mathbb{R}^n)$. We define $\frac{\partial T}{\partial x_i}$, $i = 1, \dots, n$ as the distribution given by

  $$\langle \frac{\partial T}{\partial x_i}, \varphi \rangle = -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle, \quad \forall \varphi \in D(\mathbb{R}^n)$$

  $\forall \alpha \in \mathbb{N}^\alpha$, we define $\partial^\alpha T$ as

  $$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle, \quad \forall \varphi \in D(\mathbb{R}^n).$$

  → Distributions are infinitely differentiable and their derivatives commute.

  → If $T_n \xrightarrow{D'} T$ then $\partial^\alpha T_n \xrightarrow{D'} \partial^\alpha T$ $\forall \alpha$ multi-index.

- **Ex:** [5] 1) $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$. Question: H' ?

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int H(x)\varphi'(x)\,dx = -\int_0^{+\infty} \varphi'(x)\,dx = -[\varphi(x)]_0^{+\infty} = \varphi(0) = \langle \delta, \varphi \rangle, \quad \forall \varphi \in D(\mathbb{R}^n).$$

2) $\partial^\alpha \delta_{x_0}$? $\quad \langle \partial^\alpha \delta_{x_0}, \varphi \rangle = (-1)^{|\alpha|} \langle \delta_{x_0}, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x_0) \quad \forall \varphi \in D(\mathbb{R}^n)$.

3) $f \in C^k(\mathbb{R}^n)$, $\alpha \in \mathbb{N}^\alpha$. Then we have $T_{\partial^\alpha f} = \partial^\alpha T_f$. Indeed, for any $\varphi \in D(\mathbb{R}^n)$:

$$\langle \partial^\alpha T_f, \varphi \rangle = (-1)^{|\alpha|} \langle T_f, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\partial^\alpha \varphi\,dx \overset{f \in C^k}{=} \int_{\mathbb{R}^n} \partial^\alpha f(x)\varphi(x)\,dx = \langle T_{\partial^\alpha f}, \varphi \rangle.$$

- **DEF:** Let $\varphi \in D(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.
  1) The translation of φ by h is: $\tau_h \varphi(x) = \varphi(x+h)$, $x \in \mathbb{R}^n$.
  2) The dilation of φ by λ is $H_\lambda \varphi(x) = \varphi(\lambda x)$, $x \in \mathbb{R}^n$.

→ Let's assume $f \in L^1_{loc}(\mathbb{R}^n)$, $h \in \mathbb{R}^n$. For any $f \in D(\mathbb{R}^n)$ we compute $\int_{\mathbb{R}^n} \tau_h f(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x+h)\varphi(x)\,dx$
$(y = x+h,\ dy = dx) = \int_{\mathbb{R}^n} f(y)\varphi(y-h)\,dy = \int_{\mathbb{R}^n} f(x)\underbrace{\varphi(x-h)}_{\in D(\mathbb{R}^n)}\,dx$.

- **DEF:** Let $T \in D'(\mathbb{R}^n)$. The translation of T by h is the distribution defined by $\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle$, $\forall \varphi \in D(\mathbb{R}^n)$.

- **DEF:** Let $T \in D'(\mathbb{R}^n)$. The dilation of T by a factor $\lambda \in \mathbb{R}$ is the distribution given by $\langle H_\lambda T, \varphi \rangle = \frac{1}{\lambda^n} \langle T, H_{\frac{1}{\lambda}} \varphi \rangle$, $\forall \varphi \in D(\mathbb{R}^n)$.

2.3: SUPPORT & CONVOLUTION OF DISTRIBUTIONS

- **DEF:** Let $\varphi \in D(\mathbb{R}^n)$. Then $\operatorname{supp}\varphi := \overline{\{x \in \mathbb{R}^n : \varphi(x) \neq 0\}}$.

- **DEF:** Let $T \in D'(\mathbb{R}^n)$. Let U be the largest open set in $\mathbb{R}^n$ such that $\forall \varphi \in D(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subseteq U \Rightarrow \langle T, \varphi \rangle = 0$.
Then $\mathbb{R}^n \setminus U$ is called the support of T, denoted $\operatorname{supp}(T)$.

- **Ex:** $T = \delta_0$. $\forall \varphi \in D(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subseteq \mathbb{R}^n \setminus \{0\}$ we have that $\langle \delta_0, \varphi \rangle = \varphi(0) = 0 \Rightarrow \operatorname{supp}\delta_0 = \{0\}$.

- **Convolution; motivation:** Let $u, v \in L^1(\mathbb{R}^n)$, $\varphi \in D(\mathbb{R}^n) \Rightarrow u * v \in L^1(\mathbb{R}^n)$. $\langle u*v, \varphi \rangle = \int_{\mathbb{R}^n} (u*v)(x)\varphi(y)\,dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y-x)v(x)\,dx\,\varphi(y)\,dy$
$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)v(x)\varphi(y+x)\,dx\,dy$

→ Even if $\varphi \in D(\mathbb{R}^n)$, $(x,y) \mapsto \varphi(x+y)$ doesn't have compact support.
→ If we can write $\varphi(x+y)$ like $\Psi(y)\tilde{\Psi}(x)$, above duality becomes $\int u(y)\Psi(y)\,dy \cdot \int v(x)\tilde{\Psi}(x)\,dx = \langle u, \Psi \rangle \langle v, \tilde{\Psi} \rangle$

- **DEF:** Let $T \in D'(\mathbb{R}^{n_1})$, $S \in D'(\mathbb{R}^{n_2})$. The tensor product $S \otimes T$ is the distribution of $D'(\mathbb{R}^{n_1+n_2})$ s.t. $\langle S \otimes T, \varphi \otimes \Psi \rangle = \langle S, \varphi \rangle \langle T, \Psi \rangle$
where $(\varphi \otimes \Psi)(x_1, x_2) = \varphi(x_1)\Psi(x_2)$.
→ By Stone-Weierstrass Thm, the set of fns $\varphi \otimes \Psi$ with $\varphi \in D(\mathbb{R}^{n_1})$, $\Psi \in D(\mathbb{R}^{n_2})$ is dense in $D(\mathbb{R}^{n_1+n_2})$.

- **DEF:** Let $T \in D'(\mathbb{R}^n)$, $S \in D'(\mathbb{R}^n)$ s.t. T, S have compact support. Then, $T * S \in D'(\mathbb{R}^n)$ s.t. $\langle T*S, \varphi \rangle = \langle T \otimes S, \varphi(x+y) \rangle$, $\forall \varphi \in D(\mathbb{R}^n)$. We observe that $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \varphi(x+y)$ has no compact support. So, $\varphi(x+y) \notin D(\mathbb{R}^{2n})$.
In order to define properly the above duality, we consider $\Psi \in D(\mathbb{R}^n)$ such that $x \in \operatorname{Supp}(T) \cup \operatorname{Supp}(S) \Rightarrow \Psi(x) = 1$
Then, $\langle T \otimes S, \varphi(x+y) \rangle = \langle T \otimes S, \varphi(x+y)\Psi(x)\Psi(y) \rangle$. Now we have $\varphi(x+y)\Psi(x)\Psi(y) \in D(\mathbb{R}^{2n}) \Rightarrow$ duality well-defined!
→ $T*S = S*T$
→ We say T and S have "adopted" support if either T or S have compact support.

- **Prop:** Let $T, S \in \mathcal{D}'(\mathbb{R}^n)$ with support adopted for the convolution. Let $\alpha \in \mathbb{N}^n$. Then, we have
  1) $\partial^\alpha (T * S) = (\partial^\alpha T) * S = T * (\partial^\alpha S)$
  2) If $S \in C^\infty(\mathbb{R}^n)$ then $T * S \in C^\infty(\mathbb{R}^n)$ and we have the following formula:
     $T*S(x) = \langle T, y \mapsto S(x-y)\rangle$. $\quad \partial^\alpha (T*S)(x) = \langle T, y \mapsto \partial^\alpha S(x-y)\rangle$
  3) Let φ_p be the sqnce of mollifiers introduced in the prev. lectures. Then $T * \varphi_p \in C^\infty(\mathbb{R}^n)$ such that $T*\varphi_p \xrightarrow{p\to\infty} T$ in $\mathcal{D}'(\mathbb{R}^n)$
  4) $\mathcal{D}(\mathbb{R}^n)$ dense in $\mathcal{D}'(\mathbb{R}^n)$.

- **Ex:** Let $u \in C^\infty(\mathbb{R}^n)$. $\delta_0 * u(x) = \langle \delta_0, u(x-y)\rangle = u(x) \Rightarrow \delta_0 * u = u$.

3.1: SOBOLEV SPACES: DEFINITION, EXAMPLES AND $H^1(a,b)$

- **DEF:** Let Ω be open, ctd set in $\mathbb{R}^n$, $1 \le p < +\infty$. The Sobolev Space $W^{1,p}(\Omega)$ is the set of functions in $L^p(\Omega)$ such that the first distributional derivatives belong to $L^p(\Omega)$: $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ \forall i = 1, \dots, n\}$

$$\text{Norm in } W^{1,p}(\Omega) = \begin{cases} \left(\|u\|_{L^p}^p + \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \|u\|_{L^\infty} + \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^\infty}, & p = \infty. \end{cases}$$

- **THM:** 1) For $1 \le p \le +\infty$, $W^{1,p}(\Omega)$ is a Banach space
  2) For $1 < p < +\infty$, $W^{1,p}(\Omega)$ is reflexive.
  3) For $1 \le p < +\infty$, $W^{1,p}(\Omega)$ is separable.
  4) For $p = 2$, $W^{1,2}(\Omega) = H^1(\Omega)$ is a Hilbert space with $(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$.

- **EXAMPLES:** 1) If $u \in C^1(\Omega) \cap L^p(\Omega)$, $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, $i = 1, \dots, n$ (where $\frac{\partial u}{\partial x_i}$ is classical), then $u \in W^{1,p}(\Omega)$ AND DISTRIBUTIONAL DERIVATIVES COINCIDE WITH CLASSICAL ONES. If Ω bdd, $C^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$, any $1 \le p \le +\infty$

  2) $\Omega = B_1(0)$ (open unit ball) in $\mathbb{R}^n$. We consider $u(x) = |x|^{-\alpha}$

  Question: Find $\alpha > 0$, m, p such that $u \in W^{1,p}(\Omega)$.

  → $u \in L^p(\Omega)$? $\int_{B_1(0)} \frac{1}{|x|^{\alpha p}} dx = C_n \int_0^1 \frac{1}{r^{\alpha p}} r^{m-1} dr$ which is finite provided $\alpha p - m + 1 < 1 \iff \alpha p < m$.

  → Distributional derivatives of u? Since u is smooth away from 0 we have
  $\partial_{x_i} u(x) = \frac{-\alpha x_i}{|x|^{\alpha+2}}$, $x \ne 0$ and $|\nabla u(x)| = \frac{\alpha}{|x|^{\alpha+1}}$, $x \ne 0$. We want to show that, $\forall \varphi \in \mathcal{D}(\Omega)$, we have $\int_\Omega u \partial_{x_i} \varphi\, dx = -\int_\Omega \underbrace{\partial_{x_i} u}_{= -\frac{\alpha x_i}{|x|^{\alpha+2}}} \cdot \varphi\, dx$ (i.e. want to check if $\partial_{x_i} u$, ∇u coincide with distr. deriv).

  Then, we compute $\int_{\Omega \setminus B_\varepsilon(0)} u \partial_{x_i} \varphi\, dx = -\int_{\Omega \setminus B_\varepsilon(0)} \partial_{x_i} u \cdot \varphi\, dx + \int_{\partial B_\varepsilon(0)} u \varphi n_i\, d\sigma$

  i^{th} component of normal vector.

  wts this term $\to 0$

  First, we observe that if $\alpha + 1 < n$ then $\|\nabla u\|_{L^1(\Omega)} < +\infty$

  Indeed, $\|\nabla u\|_{L^1(\Omega)} = \int_{B_1(0)} \frac{\alpha}{|x|^{\alpha+1}} dx = C_n \int_0^1 \frac{\alpha}{r^{\alpha+1}} r^{n-1} dr = C_n \int_0^1 \frac{\alpha}{r^{\alpha+2-n}} dr < +\infty$ iff $\alpha + 2 - n < 1$ iff $\alpha + 1 < n$ //

  As a consequence we have that $-\int_{\Omega \setminus B_\varepsilon(0)} \partial_{x_i} u \varphi\, dx \xrightarrow{DCT} -\int_\Omega \partial_{x_i} u \varphi\, dx$.

  Also we have $\int_{\Omega \setminus B_\varepsilon(0)} u \partial_{x_i} \varphi\, dx \to \int_\Omega u \partial_{x_i} \varphi\, dx$.

  u const. on ∂B_ε

  We observe that $\left|\int_{\partial B_\varepsilon(0)} u \varphi \nu^i d\sigma\right| \le \|\varphi\|_{L^\infty(\Omega)} \int_{\partial B_\varepsilon(0)} \varepsilon^{-\alpha} d\sigma \le C \varepsilon^{m-1-\alpha} \longrightarrow 0$ provided $m - 1 - \alpha > 0 \iff \alpha + 1 < m$.

  $\Rightarrow \int_\Omega u \partial_{x_i} \varphi\, dx = -\int_\Omega \partial_{x_i} u \varphi\, dx$

  → $\nabla u \in W^{1,p}$?
  $\|\nabla u\|_{L^p(\Omega)} = \int_{B_1(0)} \frac{\alpha^p}{|x|^{(\alpha+1)p}} dx = C_n \alpha^p \int_0^1 \frac{1}{r^{(\alpha+1)p}} r^{n-1} dr < +\infty$ if $(\alpha+1)p - m + 1 < 1 \iff (\alpha+1)p < n$
  $\iff \alpha < \frac{n}{p} - 1 = \frac{n-p}{p}$.

$$\boxed{\begin{array}{l} u \in W^{1,p}(\Omega) \text{ for } \alpha < \frac{n-p}{p} \\ u \notin W^{1,p}(\Omega) \text{ for } p \ge n. \end{array}}$$

Remark: u unbounded at $x = 0$!

3) $\Omega = B_1(0)$, $(r_k)_{k=1}^{\infty}$ be a countable dense subset in Ω. $u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha}$ $\Rightarrow u \in W^{1,p}(\Omega)$ iff $0 < \alpha < \frac{n-p}{p}$.
but u is unbdd in any open subset of Ω.

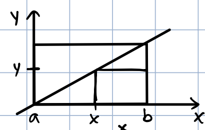
- **THM:** Let $u \in H^1(a,b)$. Then $u \in C^0([a,b])$ and $u(x) = u(a) + \int_a^x u'(y)\,dy$, $\|u\|_{C^0([a,b])} \leq C(a,b) \|u\|_{H^1(a,b)}$.
  ↳ **Remark:** $H^1(a,b) \hookrightarrow C^0([a,b])$.
  ↳ **PF:** Observe by Hölder ineq, $u' \in L^2(a,b) \Rightarrow u' \in L^1(a,b)$. We define $v(x) = \int_a^x u'(y)\,dy \overset{\text{measure theory?}}{\Rightarrow} v$ cts on $[a,b]$

In addition we have $\|v\|_{C^0([a,b])} = \sup_{x \in [a,b]} |v(x)| \leq \int_a^b 1 \cdot |u'(y)|\,dy \leq \left(\int_a^b 1^2 dy\right)^{\frac{1}{2}} \left(\int_a^b |u'(y)|^2 dy\right)^{\frac{1}{2}} = \sqrt{b-a} \cdot \|u'\|_{L^2(a,b)}$.

Now v-its distributional deriv. of v is u'. Indeed we have $\langle v', \varphi \rangle = -\langle v, \varphi' \rangle$ $\forall \varphi \in \mathcal{D}(a,b)$.

159357 $= -\int_a^b v(x)\varphi'(x)\,dx = -\int_a^b \left(\int_a^x u'(y)\,dy\right)\varphi'(x)\,dx \overset{\text{Fubini}}{=} -\int_a^b \left(\int_y^b \varphi'(x)\,dx\right) u'(y)\,dy = -\int_a^b (\underbrace{\varphi(b)}_{=0} - \varphi(y)) u'(y)\,dy$

$= \int_a^b u'(y)\,dy \Rightarrow \langle v', \varphi \rangle = \int_a^b u'(y)\varphi(y)\,dy \Rightarrow v' \in L^2(a,b) \Rightarrow v \in H^1(a,b)$.



Now we consider the distribution $T = u - v$. We know $T' = u' - v' = u' - u' = 0 \Rightarrow T = c$.

$\Rightarrow u = v + c \Rightarrow u \in C^0([a,b]) \Rightarrow$ by defn of v, we have $c = u(a)$ and so $u(x) = u(a) + \int_a^x u'(y)\,dy$ (*).

We are left to show that $\exists$ universal constant $C > 0$ s.t. $\|u\|_{C^0([a,b])} \leq C \|u\|_{H^1(a,b)}$.

From (*) we have $\|u\|_{C^0([a,b])} \leq |u(a)| + \int_a^b |u'(y)|\,dy \leq |u(a)| + \sqrt{b-a}\, \|u'\|_{L^2(a,b)}$. (1)

Using (*) once more, $|u(a)| \leq |u(x)| + \int_a^b |u'(y)|\,dy$. Integrating w.r.t x, $(b-a)|u(a)| \leq \int_a^b |u(x)|\,dx + (b-a)\int_a^b |u'(y)|\,dy$.

$\Rightarrow |u(a)| \leq \frac{1}{\sqrt{b-a}} \|u\|_{L^2(a,b)} + \sqrt{b-a}\, \|u'\|_{L^2(a,b)}$. (2) Collecting (1) and (2) together, we arrive at

$$\|u\|_{C^0([a,b])} \leq \frac{1}{\sqrt{b-a}} \|u\|_{L^2(a,b)} + 2\sqrt{b-a}\, \|u'\|_{L^2(a,b)} \leq \left(\frac{1}{\sqrt{b-a}} + 2\sqrt{b-a}\right) \|u\|_{H^1(a,b)} \quad \blacksquare$$

3.2: TRACES IN $\mathbb{R}^n_+$

- PDEs on a bdd domain need boundary conditions to be solved. (Ω, $\partial\Omega$, u?)

If Ω is a smooth bdd domain, its boundary $\partial\Omega$ is a smooth manifold in $n-1$ dim, and so it has zero measure as a set in $\mathbb{R}^n$ $\Rightarrow$ the restriction of a function $u \in L^2(\Omega)$ to $\partial\Omega$ does not make any sense.

However, $H^1(\Omega)$ is more regular than $L^2(\Omega)$! AIM: Define the value of $u \in H^1(\Omega)$ on $\partial\Omega$.

- **Remarks:** 1) If $n=1$, $H^1(a,b) \hookrightarrow C([a,b])$ $\Rightarrow u(x)$ well-defined $\forall x \in [a,b]$. If $n > 1$, $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$.
  2) The trace cannot be defined for any open subset of $\mathbb{R}^n$, but we'll need some assumptions on $\partial\Omega$.

→ STRATEGY: 1) Smooth functions til the boundary are dense in $H^1(\Omega)$
2) Show that the trace is a linear, cts operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

We'll discuss first the case $\Omega = \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

- **THM:** $\mathcal{D}(\mathbb{R}^n)$ dense in $H^1(\mathbb{R}^n)$.

- **DEF:** $\mathcal{D}(\overline{\mathbb{R}^n_+}) = \{\varphi|_{\overline{\mathbb{R}^n_+}}$ where $\varphi \in \mathcal{D}(\mathbb{R}^n)\}$ (restriction of $\mathcal{D}(\mathbb{R}^n)$ into $\overline{\mathbb{R}^n_+}$).
  ↳ **Remark:** Functions in $\mathcal{D}(\mathbb{R}^n_+)$ vanish at $x_n = 0$, instead function in $\mathcal{D}(\overline{\mathbb{R}^n_+})$ may not be zero on $x_n = 0$.

- **THM:** $\mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $H^1(\mathbb{R}^n_+)$.
  ↳ **Remark:** this fails in $\mathcal{D}(\mathbb{R}^n_+)$.

Need the following Lemma to prove above thm:

- **LEMMA (EXTENSION LEMMA):** There exists an extension operator $\mathcal{E}: W^{1,p}(\mathbb{R}^n_+) \longrightarrow W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$ such that
  1) $\mathcal{E}$ linear,
  2) $\forall u \in W^{1,p}(\mathbb{R}^n_+)$: $\mathcal{E}u|_{\mathbb{R}^n_+} = u$,
  3) $\|\mathcal{E}u\|_{W^{1,p}(\mathbb{R}^n)} \leq 2^{\frac{1}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n_+)}$

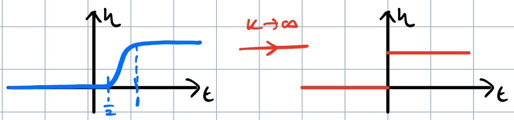
  - **REMARK:** If we consider the trivial extension $u \in H^1(0,\infty)$ with $u(a) = a \neq 0$, $\mathcal{E}u(x) = \begin{cases} u(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\Rightarrow (\mathcal{E}u)' = u' + a\delta_0$
  But $\delta_0 \notin L^2 \Rightarrow \mathcal{E}u \notin H^1(\mathbb{R})$!

• **PF:** We consider the extension by reflection $\mathcal{E}u(x',x_n) = \begin{cases} u(x',x_n), & x_n>0 \\ u(x',-x_n), & x_n<0 \end{cases}$, $x' = (x_1,\dots,x_{n-1})$.

We claim that: 1) $\frac{\partial \mathcal{E}u}{\partial x_i} = \mathcal{E}\left(\frac{\partial u}{\partial x_i}\right)$, $i=1,\dots,n-1$

2) $\frac{\partial \mathcal{E}u}{\partial x_n} = \left(\frac{\partial u}{\partial x_n}\right)^*$ where $f^*(x',x_n) = \begin{cases} f(x',x_n), & x_n>0 \\ -f(x',-x_n), & x_n<0 \end{cases}$.

Let's consider the sqnce $(\eta_k)_{k\in\mathbb{N}}$ s.t. $\eta_k \in C^\infty(\mathbb{R})$ and $\eta_k = \eta(kt)$, where $\eta(t)\in C^\infty(\mathbb{R})$: $\eta(t) = \begin{cases} 0, & t<\frac12 \\ 1, & t>1 \end{cases}$



1) $\forall \varphi \in D(\mathbb{R}^n)$, $i=1,\dots,n-1$ we have $\left\langle \frac{\partial \mathcal{E}u}{\partial x_i}, \varphi \right\rangle = -\int_{\mathbb{R}^n} \mathcal{E}u \frac{\partial \varphi}{\partial x_i} dx$

$= -\int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_i} dx - \int_{\mathbb{R}^n_-} u(x',-x_n) \frac{\partial \varphi}{\partial x_i} dx$

$= -\int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_i} dx - \int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_i}(x',-x_n) dx = -\int_{\mathbb{R}^n_+} u(x) \cdot \frac{\partial}{\partial x_i}(\varphi(x',x_n) + \varphi(x',-x_n)) dx$. We define $\Psi(x',x_n) = \varphi(x',x_n) + \varphi(x',-x_n)$. We observe that $\Psi \notin C_c^\infty(\mathbb{R}^n_+)$. However, $\eta_k(x_n)\Psi(x',x_n) \in C_c^\infty(\mathbb{R}^n_+)$. Thus, we have $\int_{\mathbb{R}^n_+} u \frac{\partial}{\partial x_i}(\eta_k \Psi) dx = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \eta_k \Psi dx$.
Also, $\frac{\partial}{\partial x_i}(\eta_k\Psi) = \eta_k \frac{\partial \Psi}{\partial x_i}$, and so $\int_{\mathbb{R}^n_+} u \eta_k \frac{\partial \Psi}{\partial x_i} dx = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \eta_k \Psi dx$.

By DCT, as $k\to+\infty$, $\int_{\mathbb{R}^n_+} u \frac{\partial \Psi}{\partial x_i} dx = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \Psi dx$. But we have $\int_{\mathbb{R}^n_+} u \frac{\partial \varphi}{\partial x_i} dx = \int_{\mathbb{R}^n_+} \mathcal{E}u \frac{\partial \varphi}{\partial x_i} dx$ and similarly
$\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \Psi dx = \int_{\mathbb{R}^n} \mathcal{E}\left(\frac{\partial u}{\partial x_i}\right) \Psi dx = \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \Psi(x',x_n) dx + \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} \varphi(x',-x_n) dx$

$= \int_{\mathbb{R}^n_-} \frac{\partial u}{\partial x_i}(x',-x_n)\varphi(x) dx$

2) $\forall \varphi \in D(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} \mathcal{E}u \frac{\partial \varphi}{\partial x_n} dx = \int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_n} dx + \int_{\mathbb{R}^n_-} u(x',-x_n) \frac{\partial \varphi}{\partial x_n} dx = \int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_n} dx + \int_{\mathbb{R}^n_+} u(x) \frac{\partial \varphi}{\partial x_n}(x',-x_n) dx$

$= \int_{\mathbb{R}^n_+} u(x) \left(\frac{\partial \varphi}{\partial x_n}(x',-x_n) + \frac{\partial \varphi}{\partial x_n}(x',-x_n) \right) dx$. Define $\mathcal{X}(x',x_n) = \varphi(x',x_n) - \varphi(x',-x_n)$

$= \int_{\mathbb{R}^n_+} u(x) \frac{\partial}{\partial x_n}(\mathcal{X}(x',x_n)) dx$. Since $\mathcal{X}(x',0) = 0 \Rightarrow |\mathcal{X}(x',x_n)| \le M|x_n|$. We notice that $\eta_k \mathcal{X} \in D(\mathbb{R}^n_+)$.

Thus, $\int_{\mathbb{R}^n_+} u \frac{\partial}{\partial x_n}(\eta_k \mathcal{X}) dx = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_n} \eta_k \mathcal{X} dx$ (★) But, $\frac{\partial}{\partial x_n}(\eta_k \mathcal{X}) = \eta_k \frac{\partial \mathcal{X}}{\partial x_n} + k\eta'(kx_n)\mathcal{X}$.

CLAIM: $\int_{\mathbb{R}^n_+} k u \eta'(kx_n) \mathcal{X} dx \longrightarrow 0$

Indeed, we have $\left| \int_{\mathbb{R}^n_+} k u \eta'(kx_n) \mathcal{X} dx \right| \le kM \int_{0<x_n<\frac1k} |u(x)| \underbrace{|\eta'(kx_n)|}_{\le C} |x_n| dx \le kMC \int_{0<x_n<\frac1k} |u(x)| \underbrace{|x_n|}_{\le \frac1k} dx \le MC \int_{0<x_n<\frac1k} |u(x)| dx \longrightarrow 0$ as $k\to\infty$.

Passing to the limit as $k\to\infty$ in (★) we get:

$$\int_{\mathbb{R}^n_+} u \frac{\partial \mathcal{X}}{\partial x_n} dx = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_n} \mathcal{X} dx \iff \int_{\mathbb{R}^n} \mathcal{E}u \frac{\partial \varphi}{\partial x_n} dx = -\int_{\mathbb{R}^n} \left(\frac{\partial u}{\partial x_n}\right)^* \varphi dx \quad \square$$

Now we can prove **THM:** $D(\overline{\mathbb{R}^n_+})$ is dense in $H^1(\mathbb{R}^n_+)$.

• **PF:** Let $u \in H^1(\mathbb{R}^n_+) \Rightarrow \exists\, \mathcal{E}u \in H^1(\mathbb{R}^n) \Rightarrow \exists \varphi_n \in D(\mathbb{R}^n) : \varphi_n \longrightarrow \mathcal{E}u$ in $H^1(\mathbb{R}^n)$. $\varphi_n|_{\mathbb{R}^n_+} \in D(\overline{\mathbb{R}^n_+})$

$\|u-\varphi_n\|^2_{H^1(\mathbb{R}^n_+)} \le \|\mathcal{E}u - \varphi_n\|^2_{L^2(\mathbb{R}^n)} + \|\nabla \mathcal{E}u - \nabla\varphi_n\|^2_{L^2(\mathbb{R}^n)} \longrightarrow 0$ ∎

• **THM (TRACE MAP):** Let γ be the restriction operator $\gamma : D(\overline{\mathbb{R}^n_+}) \longrightarrow L^2(\mathbb{R}^{n-1})$, $u \longmapsto u(x_1,\dots,x_{n-1},0)$. Then, γ is continuous from $D(\overline{\mathbb{R}^n_+})$ endowed with the $\|\cdot\|_{H^1(\mathbb{R}^n_+)}$ into $L^2(\mathbb{R}^{n-1})$ endowed with the $\|\cdot\|_{L^2(\mathbb{R}^{n-1})}$, i.e. $\exists C>0$ such that $\|\gamma u\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{H^1(\mathbb{R}^n_+)}$, $\forall u \in D(\overline{\mathbb{R}^n_+})$. Therefore, γ extends continuously to a unique linear & cts map $\gamma : H^1(\mathbb{R}^n_+) \longrightarrow L^2(\mathbb{R}^{n-1})$ called the TRACE MAP.

→ **REMARK:** Trace map isn't surjective from $H^1(\mathbb{R}^n_+)$ to $L^2(\mathbb{R}^{n-1})$. We'll see that $\gamma(H^1(\mathbb{R}^n_+)) = H^{\frac12}(\mathbb{R}^{n-1})$.

4.1: TRACES IN $\mathbb{R}^N_+$ (II)

• We showed there exists a linear cts operator $\gamma : H^1(\mathbb{R}^n_+) \longrightarrow L^2(\mathbb{R}^{n-1})$ such that

1) $\forall u \in D(\overline{\mathbb{R}^n_+})$: $\gamma(u) = u|_{\mathbb{R}^{n-1}}$ $\left(x = (x',x_n),\ x' = (x_1,\dots,x_{n-1}),\ u|_{\mathbb{R}^{n-1}} = u(x',0)\right)$

2) $\exists C>0$: $\|\gamma(u)\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{H^1(\mathbb{R}^n_+)}$ for $u \in H^1(\mathbb{R}^n_+)$.

However $\gamma(H^1(\mathbb{R}^n_+)) \ne L^2(\mathbb{R}^{n-1})$. In other words there exists at least one fn $g \in L^2(\mathbb{R}^{n-1})$ s.t. $g \ne \gamma(u)$ for some $u \in H^1(\mathbb{R}^n_+)$

Qn: What's the image of the trace operator γ acting on $H^1(\mathbb{R}^n_+)$?

SOBOLEV SPACES IN $\mathbb{R}^N$

• Let $u \in H^1(\mathbb{R}^n)$. This means that $\|u\|_{H^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u|^2 + |\nabla u|^2 dx\right)^{\frac12} < +\infty$. We introduce the Fourier Transform of u as
$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)} u(x) dx$ and its inversion formula $\mathcal{F}^{-1}(u)(x) = u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi)} \hat{u}(\xi) d\xi$.
By Plancherel Thm, we have $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $\|\hat{u}\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|u\|^2_{L^2(\mathbb{R}^n)}$ (1)

- We recall that $\widehat{\partial_{x_i} u}(\xi) = -i\xi_i \hat{u}(\xi) \Rightarrow |\nabla u| \in L^2(\mathbb{R}^n) \overset{(P)}{\Longleftrightarrow} \sqrt{\xi_1^2+\dots+\xi_n^2}\,|\hat{u}| \in L^2(\mathbb{R}^n)$

In other words we have $\|u\|_{H^1(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\left(\int_{\mathbb{R}^n} (1+|\xi|^2)\,|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$.

We generalise the norm above for any $s \in \mathbb{R}^+$:

$$\|u\|_{H^s(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

TRACE SPACES

- **THM:** The trace operator γ is a linear, cts map from $H^1(\mathbb{R}^n_+)$ to $H^{\frac{1}{2}}(\mathbb{R}^{n-1})$.
  1) $\gamma(H^1(\mathbb{R}^n_+)) = H^{\frac{1}{2}}(\mathbb{R}^{n-1})$
  2) $\exists C_{tr} > 0 : \|\gamma(u)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C_{tr}\|u\|_{H^1(\mathbb{R}^n_+)}$

  - **PF:** i) First prove that $\gamma(H^1(\mathbb{R}^n_+)) \subseteq H^{\frac{1}{2}}(\mathbb{R}^{n-1})$. Let $u \in \mathcal{D}(\overline{\mathbb{R}^n_+})$. Extend u by even reflection in $\mathbb{R}^n$. We call it $\tilde{u}$ that satisfies $\|\tilde{u}\|_{H^1(\mathbb{R}^n)} \leq \sqrt{2}\,\|u\|_{H^1(\mathbb{R}^n_+)}$.

    Notation: $u(x) = u(x', x_n)$, $x' = (x_1, \dots, x_{n-1})$, $\xi = (\xi', \xi_n)$ where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Define $g(x') = u(x', 0) = \tilde{u}(x', 0)$.

    We want to compute the norm in $H^{\frac{1}{2}}(\mathbb{R}^{n-1})$ of g. Need Fourier Trans. of g. Observe that:

    $\tilde{u}(x', x_n) = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_n\xi_n}\hat{u}(\xi', \xi_n)\,d\xi_n\right)d\xi' \underset{\tilde{u}=u}{\overset{x_n=0}{\Longrightarrow}} g(x') = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\underbrace{\left(\frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}(\xi', \xi_n)\,d\xi_n\right)}_{\hat{g}(\xi')}d\xi'$

    $\Rightarrow \hat{g}(\xi') = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}(\xi', \xi_n)\,d\xi_n$. We're now ready to compute:

    $$\|g\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} = \frac{1}{(2\pi)^{n-1}}\left(\int_{\mathbb{R}^{n-1}}(1+|\xi'|^2)^{\frac{1}{2}}|\hat{g}(\xi')|^2 d\xi'\right)^{\frac{1}{2}} = \frac{1}{(2\pi)^{n-1}}\left(\int_{\mathbb{R}^{n-1}}(1+|\xi'|^2)^{\frac{1}{2}}\Big|\frac{1}{2\pi}\overbrace{\int_{\mathbb{R}}\hat{u}(\xi', \xi_n)\,d\xi_n}^{(I)}\Big|^2 d\xi'\right)^{\frac{1}{2}} = (\star)$$

    We have

    $$I = \int_{\mathbb{R}}\underbrace{(1+|\xi|^2)^{-\frac{1}{2}}}_{f}\underbrace{(1+|\xi|^2)^{\frac{1}{2}}\hat{u}(\xi', \xi_n)}_{h}\,d\xi_n \overset{C.S}{\leq} \Big(\underbrace{\int_{\mathbb{R}}(1+|\xi|^2)^{-1}d\xi_n}_{\int_{\mathbb{R}}\frac{1}{(1+|\xi'|^2+|\xi_n|^2)}d\xi_n}\Big)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(1+|\xi|^2)\,|\hat{u}(\xi', \xi_n)|^2 d\xi_n\right)^{\frac{1}{2}} \leq$$

    (Recall $(a>0)$ $\int_{\mathbb{R}}\frac{1}{a^2+t^2}dt = \frac{1}{a}\arctan\left(\frac{t}{a}\right)\Big|_{t=-\infty}^{t=\infty} = \frac{\pi}{a}$) $\leq \left(\frac{\pi}{(1+|\xi'|^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(1+|\xi|^2)\,|\hat{u}(\xi', \xi_n)|^2 d\xi_n\right)^{\frac{1}{2}}$

    By using the above estimate in $(\star)$ we infer that

    $$(\star) \leq \frac{1}{(2\pi)^{n-1}}\left(\int_{\mathbb{R}^{n-1}}\cancel{(1+|\xi'|^2)^{\frac{1}{2}}}\left(\frac{1}{2\pi}\right)^2\frac{\pi}{\cancel{(1+|\xi'|^2)^{\frac{1}{2}}}}\int_{\mathbb{R}}(1+|\xi|^2)|\hat{u}(\xi', \xi_n)|^2 d\xi_n\,d\xi'\right)^{\frac{1}{2}} \leq \frac{1}{(2\pi)^{n-1}}\left(\frac{\pi}{(2\pi)^2}\int_{\mathbb{R}^n}(1+|\xi|^2)|\hat{u}(\xi', \xi_n)|^2 d\xi\right)^{\frac{1}{2}}$$

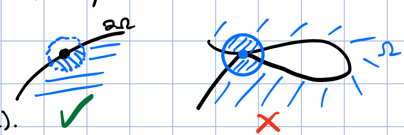
    $$= \frac{\sqrt{\pi}}{(2\pi)^{n-1}}\frac{1}{2\pi}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)|\hat{u}(\xi', \xi_n)|^2 d\xi\right)^{\frac{1}{2}} = \sqrt{\pi}\,\|\tilde{u}\|_{H^1(\mathbb{R}^n)} \leq \sqrt{2\pi}\,\|\tilde{u}\|_{H^1(\mathbb{R}^n_+)} \Rightarrow \|\gamma(u)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^1(\mathbb{R}^n_+)} \quad \forall u \in \mathcal{D}(\overline{\mathbb{R}^n_+}).$$

    By density of $\mathcal{D}(\overline{\mathbb{R}^n_+})$ into $H^1(\mathbb{R}^n_+)$ we simply obtain $\|\gamma(u)\|_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^1(\mathbb{R}^n_+)}$, $\forall u \in H^1(\mathbb{R}^n_+)$

    ii) Need to show $H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \subseteq \gamma(H^1(\mathbb{R}^n_+))$. We consider $g \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$ and we define $u(x', x_n) = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{-(1+|\xi'|)x_n}\hat{g}(\xi')e^{i\xi'\cdot x'}d\xi'$.
    It possible to show that $u \in H^1(\mathbb{R}^n_+)$. Moreover, observe that $g(x') = u(x', 0) \Rightarrow H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \subseteq \gamma(H^1(\mathbb{R}^n_+))$ ∎

4.2 : TRACE OPERATOR FOR BDD LIPSCHITZ DOMAINS

- The theory we developed for $\mathbb{R}^n_+$ can be extended to bdd domains in $\mathbb{R}^n$ with Lipschitz boundary such that "Ω is locally on one side of $\partial\Omega$"

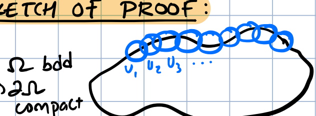


- **THM:** Let Ω be bdd domain in $\mathbb{R}^n$ with Lipschitz boundary. Then, $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$.
  ↳ **Remark:** Not true if $\mathcal{D}(\overline{\Omega})$ replaced by $\mathcal{D}(\Omega)$. Indeed we have $\overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = H^1_0(\Omega)$.

  This THM follows from extension lemma.

- **LEMMA:** Let Ω be bdd domain in $\mathbb{R}^n$ with Lipschitz boundary. Then there exists an extension operator $\mathcal{E} : H^1(\Omega) \longrightarrow H^1(\mathbb{R}^n)$ which is linear, cts in $H^1(\Omega)$, i.e.
  1) $\forall u \in H^1(\Omega) : \mathcal{E}u|_{\Omega} = u$
  2) $\exists C_{\mathcal{E}} > 0 : \|\mathcal{E}u\|_{H^1(\mathbb{R}^n)} \leq C_{\mathcal{E}}\|u\|_{H^1(\Omega)}$, $\forall u \in H^1(\Omega)$.

  → **SKETCH OF PROOF:**

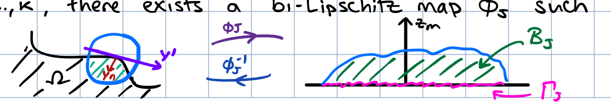


$\exists\{U_j\}$ open finite covering of $\partial\Omega$. Set $U_0 = \Omega \setminus \bigcup_{j=1}^{k} U_j$

1) LOCALIZATION: there exists a "PARTITION OF UNITY" subordinate to $U_0, U_1, \dots, U_k$ namely $k+1$ functions such that 1) $\forall J = 0, \dots, k$, $\Psi_J \in C^\infty_0(U_J)$, $0 \leq \Psi_J \leq 1$.
2) $\sum_{J=0}^{k}\Psi_J(x) = 1 \quad \forall x \in \Omega$

Let $u \in H^1(\Omega)$. We write $u = u\sum_{j=0}^{k}\Psi_J = \sum_{j=0}^{k}\underbrace{u\Psi_J}_{u_J}$. Possible to show $u_J \in H^1(\Omega)$ such that $\|u_J\|_{H^1(\Omega)} \leq C_J\|u\|_{H^1(\Omega)}$. $(*)$ Moreover, $\mathrm{supp}(u_J) \subseteq U_J$.

2) REDUCTION TO HALF-SPACE. Since $\partial\Omega$ is Lipschitz, for any $J = 0, \dots, k$, there exists a bi-Lipschitz map ϕ_J such that $\phi_J(U_J \cap \Omega) = B_J \subseteq \mathbb{R}^n_+$, B_J open, $\phi_J(U_J \cap \partial\Omega) = \Gamma_J \subseteq \partial\mathbb{R}^n_+$



• **PF:** We define $w_J = u_J \circ \phi_J^{-1}$ such that 1) w_J has compact support in $B_J \cap T_J$
2) $w_J \in H^1(\mathbb{R}^n_+)$: $\|w_J\|_{H^1(\mathbb{R}^n_+)} \leq C_{\phi_J} \|u_J\|_{H^1(\Omega)}$ (2*)

Any of these fns w_J can be extended on $\mathbb{R}^n$. We call it $\mathcal{E}(w_J)$ which satisfies $\|\mathcal{E}(w_J)\|_{H^1(\mathbb{R}^n)} \leq C \|w_J\|_{H^1(\mathbb{R}^n_+)}$ (3*)
Then it follows from (1)*–(3)* that $\|\mathcal{E}w_J\|_{H^1(\mathbb{R}^n)} \leq C \|u\|_{H^1(\Omega)}$. To conclude one can define $\mathcal{E}(u)$ by means of $\mathcal{E}(w_J)$, $J = 0, \dots, k$ by using the map ϕ_J, $J = 0, \dots, k$. $\boxtimes$

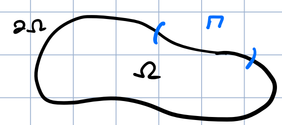
Thanks to the extension and density results we have the following trace thm.

• **THM:** Let Ω be a bdd domain in $\mathbb{R}^n$ with Lipschitz boundary. There exists a linear cts map $\gamma: H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$ s.t.:
1) $\forall u \in \mathcal{D}(\bar{\Omega})$: $\gamma(u) = u|_{\partial\Omega}$
2) $\exists C_{tr} > 0$: $\|\gamma(u)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_{tr} \|u\|_{H^1(\Omega)}$, $\forall u \in H^1(\Omega)$ where $H^{1/2}(\partial\Omega)$ is a Hilbert space, $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$

As a consequence of the trace thm we have: $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf \{ \|u\|_{H^1(\Omega)} : \gamma(u) = g \text{ on } \partial\Omega \}$

• **COR:** Let Ω be as above. $\forall u \in H^1(\Omega)$, $\underline{v} \in H^1(\Omega; \mathbb{R}^n)$ the following IBP formula holds:
$$\int_\Omega \nabla u \cdot \underline{v} \, dx = - \int_\Omega u \, (\operatorname{div} \underline{v}) \, dx + \int_{\partial\Omega} \gamma(u) \, \underline{\gamma(v)} \cdot \underline{n} \, d\sigma.$$

→ Let Γ be an open subset of $\partial\Omega$



QUESTION: how do we define the trace on Γ?

• **THM:** Let Ω be a Lipschitz domain and let Γ be an open subset of $\partial\Omega$. Then there exists a linear, cts operator $\gamma_\Gamma : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\Gamma)$ such that:
1) $\forall u \in \mathcal{D}(\bar{\Omega})$: $\gamma_\Gamma(u) = u|_\Gamma$
2) $\exists C_\Gamma > 0$: $\|\gamma_\Gamma(u)\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_\Gamma \|u\|_{H^1(\Omega)}$, $\forall u \in H^1(\Omega)$.
3) $\operatorname{Ker}(\gamma_\Gamma) = H^1_{0,\Gamma}(\Omega) = \{ u \in H^1(\Omega) : \gamma_\Gamma(u) = 0 \}$

What about $W^{1,p}$?

• **THM:** Let Ω be a Lipschitz domain, $1 \leq p < \infty$. There exists a linear cts map $\gamma: W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that
1) $\forall u \in \mathcal{D}(\bar{\Omega})$: $\gamma(u) = u|_{\partial\Omega}$
2) $\exists C_p > 0$: $\|\gamma(u)\|_{L^p(\partial\Omega)} \leq C_p \|u\|_{W^{1,p}(\Omega)}$ $\forall u \in W^{1,p}(\Omega)$.

REMARKS:
1) $\gamma(W^{1,p}(\Omega))$ is not $L^p(\partial\Omega)$ but its possible to show $\gamma: W^{1,p}(\Omega) \longrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ is a linear, cts and surjective map.
2) If $p = \infty$ then $W^{1,\infty}(\Omega)$ consists of Lipschitz functions. In particular, $W^{1,\infty}(\Omega) \hookrightarrow C(\bar{\Omega})$. Then the trace map corresponds to the classical restriction.

4.3: COMPACTNESS

• Motivation: In the study of PDEs a common strategy to find a solution is the following:
1) We construct a sequence of functions u_n solving a problem "similar" to the original PDE (usually much simpler). The sequence u_n is called approximation.
2) We show by exploiting the specific nature of the PDE that $\{u_n\}$ is bdd in some function spaces (e.g. Sobolev Spaces). This is called the "energy method".
3) We prove that $\{u_n\}$ converges to u, which is the soln to the original problem.

Step (3) requires the study of compactness properties of Sobolev Spaces.

• **DEF:** Let X be a metric space. The set $E \subset X$ is compact if for any sequence $\{x_n\} \subset E$, there exists a subsequence $\{x_{n_j}\}$ s.t. $\{x_{n_j}\}_{j \geq 1}$ converges in E.

CTS FUNCTIONS

• **ARZELA-ASCOLI:** Let Ω be a bdd domain in $\mathbb{R}^n$, $E \subset C(\bar{\Omega})$. Then E is precompact iff
1) E is bdd in $C(\bar{\Omega})$, i.e. $\exists M > 0$ s.t: $\|u\|_{C(\bar{\Omega})} \leq M$, $\forall u \in E$.
2) E is equicontinuous, i.e. $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ s.t. $|h| < \delta \Rightarrow \|u(\cdot + h) - u(\cdot)\|_{C(\bar{\Omega})} < \varepsilon$ $\forall u \in C(\bar{\Omega})$.

- **COR:** Let Ω be bdd domain in $\mathbb{R}^n$. Let $E \subset C^{0,\alpha}(\bar\Omega)$. Assume that there exists $M>0$ s.t $\|u\|_{C^{0,\alpha}(\bar\Omega)} \le M$, $\forall u \in E$.
  Then E is precompact in $C(\bar\Omega)$.
  - **PF:** Recall $\|u\|_{C^{0,\alpha}(\bar\Omega)} = \|u\|_{C(\bar\Omega)} + \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le M$. Immediately follows that: i) $\|u\|_{C(\bar\Omega)} \le M \quad \forall u \in E$
    ii) $\forall h \in \mathbb{R}^n$: $|u(x+h)-u(x)| \le M|h|^\alpha$
    $\Rightarrow$ equicontinuity

L^p SPACES

- **RIESZ-FRECHET-KOLMOGOROV:** Let Ω be bdd set in $\mathbb{R}^n$, $E \subset L^p$, $1 \le p < \infty$.
  Then, E is precompact in $L^p(\Omega)$ iff
  1) E bdd in $L^p(\Omega)$
  2) E equicts in $L^p(\Omega)$, i.e. $\lim_{|h|\to 0} \|u(\cdot+h)-u(\cdot)\|_{L^p(\Omega)} = 0$ uniformly.

SOBOLEV SPACES

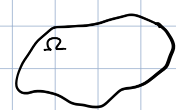
1) Since $\|u\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)}$, it follows that $H^1(\Omega) \hookrightarrow L^2(\Omega)$
2) H^1 is a Hilbert space (so it is reflexive). If $\{u_n\}$ bdd in $H^1(\Omega)$ then there exists u in $H^1(\Omega)$ s.t $u_n \to u$ weakly in $H^1(\Omega)$. $\int_\Omega u_n v\,dx + \int_\Omega \nabla u_n \cdot \nabla v\,dx \longrightarrow \int_\Omega u \cdot v\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx$
   $\hookrightarrow$ Actually, can say more if $\partial\Omega$ Lipschitz.

- **THM (RELLICH):** Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. Then $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$
  This means any bdd sqnce in H^1 has a subsqnce converging strongly in $L^2(\Omega)$.

PR. CLASS

$\hookrightarrow$ **REMARK:** If we replace $H^1(\Omega)$ with $H^1_0(\Omega)$ then we only need Ω bdd.

$\hookrightarrow$ **PF:** Let $u \in \mathcal{D}(\mathbb{R}^n)$. We have $u(x+h)-u(x) = \int_0^1 \frac{d}{dt} u(x+th)\,dt \overset{\text{chain rule}}{=} \int_0^1 \nabla u(x+th)\cdot h\,dt$ which implies that
$|u(x+h)-u(x)|^2 = |\int_0^1 \nabla u(x+th)\cdot h\,dt|^2 \le |\int_0^1 \nabla u(x+th)\cdot h\,dt|^2 \le (\int_0^1 |\nabla u(x+th)|^2 dt)(\int_0^1 |h|^2 dt) = |h|^2 \int_0^1 |\nabla u(x+th)|^2 dt$
Now integrate over $\mathbb{R}^n$: $\int_{\mathbb{R}^n} |u(x+h)-u(x)|^2 dx \le |h|^2 \int_{\mathbb{R}^n}\int_0^1 |\nabla u(x+th)|^2 dt\,dx \overset{\text{Fubini}}{=} |h|^2 \int_0^1 \underbrace{\int_{\mathbb{R}^n} |\nabla u(x+th)|^2 dx}_{=\|\nabla u\|^2_{L^2(\mathbb{R}^n)}} dt$
$= |h|^2 \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \int_0^1 dt = |h|^2 \|\nabla u\|^2_{L^2(\mathbb{R}^n)}$. We proved that $\forall h \in \mathbb{R}^n$,
(∗) $\|u(\cdot+h)-u(\cdot)\|_{L^2(\mathbb{R}^n)} \le |h|\,\|\nabla u\|_{L^2(\mathbb{R}^n)}$, $\forall u \in \mathcal{D}(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ it follows that
$\forall h \in \mathbb{R}^n$, (∗) holds $\forall u \in H^1(\mathbb{R}^n)$. $\{u_n\} \in \mathcal{D}(\mathbb{R}^n)$: $u_n \xrightarrow{H^1} u$. $u_n - u_m$, $T_h u = u(\cdot+h) - u$.
$\|T u_n - T u_m\| \le |h|\,\underbrace{\|\nabla(u_n-u_m)\|_{L^2}}_{\to 0}$ then pass to limit.



Next, lets consider a bdd set $E \subset H^1(\Omega)$. This means $\exists M>0$: $\|u\|_{H^1(\Omega)} \le M \quad \forall u \in E$.
In particular E bdd in $L^2(\Omega)$. Since Ω Lipschitz we can extend any function of E on $\mathbb{R}^n$. More precisely for any $u \in E$ there exists $\tilde u$ in $H^1(\mathbb{R}^n)$ such that:
1) $\tilde u|_\Omega = u$, 2) $\|\tilde u\|_{H^1(\mathbb{R}^n)} \le C\|u\|_{H^1(\Omega)}$. We denote by $\tilde E$ the set of extensions.
By (∗) we have $\|\tilde u(\cdot+h) - \tilde u(\cdot)\|_{L^2(\mathbb{R}^n)} \le |h| C \|u\|_{H^1(\Omega)} \le CM|h|$, $\forall \tilde u \in \tilde E$.
In particular for $\Omega \subset \Omega' \subset \mathbb{R}^n$ we get $\|\tilde u(\cdot+h) - \tilde u(\cdot)\|_{L^2(\Omega')} \le CM|h|$, $\forall \tilde u \in \tilde E$. We conclude that $\tilde E$ is precompact in $L^2(\Omega')$ which $\Rightarrow$ E precompact in $L^2(\Omega)$.

- **QN:** Does a fn in $W^{1,p}(\Omega)$ belong to other function spaces?
  $\to$ **Motivation:** for instance in 1D, $H^1(a,b) \hookrightarrow C([a,b]) \hookrightarrow L^p(a,b) \quad \forall p \in [1,\infty)$
  In two dimensions, $f(x) = \left(\log\left(\frac{1}{|x|}\right)\right)^{\frac{1}{k}}$ belongs to $H^1(B_R(0))$ with $R<1$, if $k < \frac{1}{2}$.
  But $f \notin L^\infty(B_R(0))$. On the other hand, $f \in L^p(B_R(0)) \quad \forall p \in [1,\infty)$.

  To answer above Qn, need to consider 3 different cases: 1) $1 \le p < n$
  2) $p = n$
  3) $n < p \le \infty$

GAGLIARDO-SOBOLEV-NIRENBURG-INEQUALITY

- Want to study whether or not the following ineq. holds: $\underbrace{\|u\|_{L^q(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)}}_{(SI)}$, C universal constant, $1 \le q < \infty$, $u \in C^\infty_c(\mathbb{R}^n)$
  Let's assume (SI) is true for any $u \in \mathcal{D}(\mathbb{R}^n)$. We define
  $u_\lambda(x) = u(\lambda x)$, $x \in \mathbb{R}^n$, $\lambda > 0$. By (SI) we have $\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}$.
  We observe $\|u_\lambda\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u_\lambda(x)|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx \underset{dy=\lambda^n dx}{\overset{y=\lambda x}{=}} \frac{1}{\lambda^n}\int_{\mathbb{R}^n} |u(y)|^q dy = \frac{1}{\lambda^n}\|u\|^q_{L^q(\mathbb{R}^n)}$
  and $\|\nabla u_\lambda\|^p_{L^p} = \int_{\mathbb{R}^n} |\nabla u_\lambda|^p dx = \int_{\mathbb{R}^n} |\nabla u(\lambda x)|^p dx = \lambda^p \int_{\mathbb{R}^n} |\nabla u(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n}\int_{\mathbb{R}^n} |\nabla u(y)|^p dy = \lambda^{p-n}\|\nabla u\|_{L^p(\mathbb{R}^n)}$.
  From (∗) we deduce that $\frac{1}{\lambda^{\frac{n}{q}}}\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{\frac{p-n}{p}}\|\nabla u\|_{L^p(\mathbb{R}^n)} \Rightarrow \|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|\nabla u\|_{L^p(\mathbb{R}^n)}$
  Letting $\lambda \to 0$ or $\lambda \to \infty$ we obtain a contradiction unless $1 - \frac{n}{p} + \frac{n}{q} = 0 \Rightarrow \frac{n}{q} = \frac{n}{p} - 1 \Rightarrow \frac{1}{q} = \frac{n-p}{pn} \Rightarrow q = \frac{np}{n-p}$.

P R O B L E M C L A S S

- **DEF:** For any $1 \le p < n$ the Sobolev conjugate of p is $p^* = \frac{np}{n-p}$. We note that $p^* > p$.

- **THM:** Let $1 \le p < n$. There exists a constant depending on p and n such that $\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)}$, $\forall u \in C_c^1(\mathbb{R}^n)$.
  - **PF:** Since u has compact support, $u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, x_2, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$ and so $|u(x)| \le \int_{-\infty}^{x_i} |\nabla u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i$. Then, $|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{x_i} |\nabla u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}}$
$\le \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}}$

1) $\underline{n=2}$: $|u(x)|^{\frac{2}{2-1}} \le \left(\int_{-\infty}^{+\infty} |\nabla u(y_1, x_2)|\, dy_1\right)^{\frac{1}{2-1}} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, y_2)|\, dy_2\right)^{\frac{1}{2-1}}$. Integrating w.r.t x_1, we find

$$\int_{\mathbb{R}} |u(x)|^2 dx_1 \le \left(\int_{-\infty}^{\infty} |\nabla u(y_1, x_2)|\, dy_1\right) \underbrace{\left(\int_{\mathbb{R}^2} |\nabla u(x_1, y_2)|\, dy_2\, dx_1\right)}_{= \|\nabla u\|_{L^1(\mathbb{R}^2)}}$$

Integrating w.r.t x_2, $\int_{\mathbb{R}^2} |u(x)|^2 dx_1 dx_2 \le \|\nabla u\|_{L^1(\mathbb{R}^2)} \overbrace{\left(\int_{\mathbb{R}^2} |\nabla u(y_1, x_2)|\, dy_1\, dx_2\right)}^{= \|\nabla u\|_{L^1(\mathbb{R}^2)}} \Rightarrow \|u\|^2_{L^2(\mathbb{R}^2)} \le \|\nabla u\|_{L^1(\mathbb{R}^2)}$

2) $\underline{n=3}$: $|u(x)|^{\frac{3}{2}} \le \left(\int_{\mathbb{R}} |\nabla u(y_1, x_2, x_3)|\, dy_1\right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{R}} |\nabla u(x_1, y_2, x_3)|\, dy_2\right)^{\frac{1}{2}}}_{I} \underbrace{\left(\int_{\mathbb{R}} |\nabla u(x_1, x_2, y_3)|\, dy_3\right)^{\frac{1}{2}}}_{J}$

Integrating w.r.t x_1, we find

$$\int_{\mathbb{R}} |u(x)|^{\frac{3}{2}} dx_1 \le \left(\int_{\mathbb{R}} |\nabla u(y_1, x_2, x_3)|\, dy_1\right)^{\frac{1}{2}} \int_{\mathbb{R}} I^{\frac{1}{2}} J^{\frac{1}{2}} dx_1$$

$$\le \left(\int_{\mathbb{R}} |\nabla u(y_1, x_2, x_3)|\, dy_1\right)^{\frac{1}{2}} \times \Big(\overbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, y_2, x_3)|\, dy_2\, dx_1}^{L}\Big)^{\frac{1}{2}} \times \Big(\overbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, x_2, y_3)|\, dy_3}^{Z}\Big)^{\frac{1}{2}}$$

Integrating w.r.t x_2, $\int_{\mathbb{R}^2} |u(x)|^{\frac{3}{2}} dx_1 dx_2 \le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, y_2, x_3)|\, dy_2\, dx_1\right)^{\frac{1}{2}} \times \int_{\mathbb{R}} L^{\frac{1}{2}} Z^{\frac{1}{2}} dx_2$

$$\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, y_2, x_3)|\, dy_2\, dx_1\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{R}^2} |\nabla u(y_1, x_2, x_3)|\, dy_1\, dx_2\right)^{\frac{1}{2}} \times \|\nabla u\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}$$

Integrating w.r.t x_3, $\int_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}} dx \le \|\nabla u\|^{3/2}_{L^1(\mathbb{R}^3)} \Rightarrow \|u\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le \|\nabla u\|_{L^1(\mathbb{R}^3)}$

Possible to show that: $\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|\nabla u\|_{L^1(\mathbb{R}^n)}$ (I) SOBOLEV INEQ (SI) FOR $p=1$

Next we define $v = |u|^{\gamma}$ for γ to be chosen.

We infer from (I) that $\left(\int_{\mathbb{R}^n} |u|^{\gamma \cdot \frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |\nabla(u^{\gamma}(x))|\, dx \le \int_{\mathbb{R}^n} \gamma u^{\gamma-1} |\nabla u|\, dx$

$\frac{1}{p} \rightsquigarrow p' = \frac{p}{p-1}$ Hö $\le \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \underbrace{\left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}}_{\|\nabla u\|_{L^p(\mathbb{R}^n)}}$. We choose γ s.t. $(\gamma-1)\frac{p}{p-1} = \gamma \frac{n}{n-1}$. $\rightsquigarrow (n-1)(\gamma-1)p = n(p-1)\gamma$

$\Rightarrow \gamma = \frac{p(n-1)}{n-p}$.

We notice that $\gamma \frac{n}{n-1} = \frac{p\cancel{(n-1)}}{n-p} \cdot \frac{n}{\cancel{(n-1)}} = \frac{np}{n-p} = p^*$. So we arrive at:

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\right)^{\frac{n-1}{n}} \le \gamma \left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \|u\|^{\frac{p^*(n-1)}{n}}_{L^{p^*}(\mathbb{R}^n)} \le \gamma \|u\|^{\frac{p^*(p-1)}{p}}_{L^{p^*}(\mathbb{R}^n)} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

But notice $\frac{p^*(n-1)}{n} - \frac{p^*(p-1)}{p} = \frac{p(n-1)}{n-p} - \frac{n(p-1)}{n-p} = \frac{np-p-np+n}{n-p} = \frac{n-p}{n-p} = 1$

We end up with

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le \gamma \|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \square$$

→ What happens in a bdd domain?

- **THM:** Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. Assume $1 \le p < n$, $u \in W^{1,p}(\Omega)$. Then, $u \in L^{p^*}(\Omega)$. Moreover, we have $\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$ where C is a universal constant depending on n, p, Ω.

- **REMARKS:**
  1) In a bdd domain, $W^{1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ if $1 \le q < p$
  2) $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \le q < p^*$ (Ω bdd, Lipschitz)

  For instance if $n=3, p=2$ then $W^{1,2}(\Omega) = H^1(\Omega) \hookrightarrow L^6(\Omega)$
  $(p^* = 6)$ $\Rightarrow H^1(\Omega) \hookrightarrow L^p(\Omega)$ $\forall p \in [1, 6]$

5.1: SOBOLEV INEQUALITIES, COMPACTNESS & DUAL SPACES

- Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. Qn: $W^{1,p}(\Omega) \subset L^q(\Omega)$?
  1) Case $1 \le p < n$: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $\forall q \in [1, p^*]$ where $p^* = \frac{np}{n-p}$ $\left(\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}\right)$
  2) Limiting case $p=n$: The Sobolev conjugate p^* is ∞! Might expect $W^{1,n}(\Omega) \hookrightarrow L^\infty(\Omega)$. However this is false since $f(x) = \left(\log \frac{1}{|x|}\right)^k$ defined in $\Omega = B_{\frac{1}{2}}(0) \subset \mathbb{R}^2$ belongs to $H^1(\Omega)$ but is not bdd!

- **THM:** Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. We have $W^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1,\infty)$, namely for any $q \in [1,\infty)$, there exists a constant C depending on q, n and Ω, such that $\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,n}(\Omega)}$, $\forall u \in W^{1,n}(\Omega)$.

  3) Case $n < p \le \infty$: We'll see that if $u \in W^{1,p}(\Omega)$ then u is in fact Hölder continuous.

- **THM (MORREY'S INEQ):** Assume $n < p \le \infty$. Then there exists a constant C, depending on p and n such that $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$ for any $u \in C_c^1(\mathbb{R}^n)$ for any $u \in C_c^1(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$.

  → As a consequence of this ineq we have:

- **THM:** Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. Assume $n < p \le \infty$ and $u \in W^{1,p}(\Omega)$. Then $u \in C^{0,\gamma}(\bar{\Omega})$ where $\gamma = 1 - \frac{n}{p}$, such that $\|u\|_{C^{0,\gamma}(\bar{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$ where C is a universal constant which only depends on p, n, Ω.

COMPACTNESS (PART 2)

- We proved $W^{1,2}(\Omega) = H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$.
- **THM:** Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. We have the following compact injections:
  1) $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, $\forall q \in [1, p^*)$, if $p < n$.
  2) $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, $\forall q \in [1,\infty)$, if $p = n$.
  3) $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} C(\bar{\Omega})$, if $p > n$.

  Injection $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ never compact.
  $W^{1,p} \hookrightarrow L^{p^*}$ but $W^{1,p} \overset{c}{\not\hookrightarrow} L^{p^*}$

DUAL SPACES

- We define $(W^{1,p}(\Omega))'$ as the dual space of $W^{1,p}(\Omega)$, namely the set of linear cts functionals on $W^{1,p}(\Omega)$. In a particular case we can provide an explicit representation of the element in the dual of a Sobolev space.

- **DEF:** $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ with norm $\|F\|_{H^{-1}(\Omega)} = \sup\{|\langle F, v\rangle| : v \in H_0^1(\Omega), \|u\|_{H_0^1} \le 1\}$.
  ↳ We observe that since $\mathcal{D}(\Omega)$ dense in $H_0^1(\Omega)$, $H^{-1}(\Omega)$ is a space of distributions.
  ↳ This means the following facts:
  1) If $F \in H^{-1}(\Omega)$ then restriction of F on $\mathcal{D}(\Omega)$ is a distribution.
  2) Restriction of F on $\mathcal{D}(\Omega)$ is unique, i.e. $F, G \in H^{-1}(\Omega)$, $\langle F, \varphi\rangle = \langle G, \varphi\rangle$ $\forall \varphi \in \mathcal{D}(\Omega) \Rightarrow F = G$ in $H^{-1}(\Omega)$.

- **THM:** $H^{-1}(\Omega)$ is the set of distributions F such that $F = f_0 + \operatorname{div} \underline{f}$ where $f_0 \in L^2(\Omega)$, $\underline{f} = (f_1, \dots, f_n) \in L^2(\Omega, \mathbb{R}^n)$. In addition we have $\|F\|_{H^{-1}(\Omega)} \le C(\|f_0\|_{L^2(\Omega)} + \|\underline{f}\|_{L^2(\Omega;\mathbb{R}^n)})$ for universal const. C.

- **Ex:** 1) If $n = 1$, $\delta_0 \in H^{-1}(-a,a)$
  If $n > 1$, $\delta_0 \notin H^{-1}(\Omega)$.
  2) Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$. Then, χ_Ω (char. fn of Ω) belongs to $L^2(\mathbb{R}^n)$ and $F = \nabla\chi_\Omega \in H^{-1}(\mathbb{R}^n)$.
  Indeed, we have $\langle \nabla\chi_\Omega, \varphi\rangle = -\int_{\mathbb{R}^n} \chi_\Omega \operatorname{div}\varphi \, dx = -\int_\Omega \operatorname{div}\varphi \, dx \overset{\text{IBP}}{=} -\int_{\partial\Omega} \varphi \cdot n \, d\sigma$

  → **REMARK:** $H^{-1}(\Omega) \ne (H^1(\Omega))'$. Indeed, $(H^1(\Omega))'$ is not a space of distributions, namely the restriction of $F \in (H^1(\Omega))'$ into $\mathcal{D}(\Omega)$ is not uniquely defined.

5.2: ELLIPTIC PROBLEMS IN BOUNDED DOMAINS

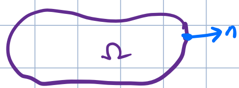
Elliptic problems in bdd domains

- Let Ω be bdd ctd open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Notation: we say Ω is a "domain".
- Ex:[1] Let $f:\Omega\to\mathbb{R}$, $g:\partial\Omega\to\mathbb{R}$. Find a function $u:\Omega\to\mathbb{R}$ which is a solution to:

DIRICHLET PROBLEM
$$\begin{cases} -\Delta u+\alpha u=f & \text{in } \Omega \\ u=g & \text{on } \partial\Omega \end{cases}$$
• $\alpha\in\mathbb{R}$ $\quad g=0$: "homogeneous case" $\quad g\neq 0$: "inhomogeneous case" $\quad \Delta u=\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}$

- Ex:[2] Let $f:\Omega\to\mathbb{R}$, $g:\partial\Omega\to\mathbb{R}$, $\alpha\in\mathbb{R}$ be given. Find a function $u:\Omega\to\mathbb{R}$ which solves

NEUMANN PROBLEM
$$\begin{cases} -\Delta u+\alpha u=f & \text{in } \Omega \\ \frac{\partial u}{\partial n}=g & \text{on } \partial\Omega \end{cases}$$
where $\frac{\partial u}{\partial n}=\nabla_x u\cdot n=\sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\,n_i(x)$ for $x\in\partial\Omega$ (normal derivative), $n(x)$ the outward normal vector at $x\in\partial\Omega$.



$\partial\Omega$ is Lipschitz $\Downarrow$ n exists a.e.

- Ex:[3] Robin (oblique boundary condition) problem

Let $f:\Omega\to\mathbb{R}$, $g:\partial\Omega\to\mathbb{R}$, $\beta:\partial\Omega\to\mathbb{R}$ ($\beta(x)\geq 0$ a.e. on $\partial\Omega$). Find a function $u:\Omega\to\mathbb{R}$ which is a solution to
$$\begin{cases} -\Delta u+\alpha u=f & \text{in } \Omega \\ \frac{\partial u}{\partial n}+\beta u=g & \text{on } \partial\Omega \end{cases}$$
• $\alpha\in\mathbb{R}$

- Ex:[4] Mixed boundary conditions.

Let $\partial\Omega=\Gamma_0\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\phi$, $\mu(\Gamma_0)>0$ and $\mu(\Gamma_1)>0$ where μ is the Lebesgue measure on $\partial\Omega$.
Let $f:\Omega\to\mathbb{R}$, $g_0:\Gamma_0\to\mathbb{R}$, $g_1:\Gamma_1\to\mathbb{R}$ be given. Let $\alpha\in\mathbb{R}$. Find a function $u:\Omega\to\mathbb{R}$ which is a solution to
$$\begin{cases} -\Delta u+\alpha u=f & \text{in } \Omega \\ u=g_0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial n}=g_1 & \text{on } \Gamma_1 \end{cases}$$

- Ex:[5] Mixed boundary condition Elliptic problem

Let $(a_{ij})_{i,j=1,\dots,n}$ and a_0 be functions $\Omega\to\mathbb{R}$ such that $a_{ij}\in L^\infty(\Omega)$, $a_0\in L^\infty(\Omega)$ satisfying
1) $\exists K>0$ such that $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j\geq K|\xi|^2$, $\forall \xi\in\mathbb{R}^n$, a.e. in Ω.
2) $\exists K_0>0$ such that $a_0(x)\geq K_0$, a.e in Ω.

} strong elliptic conditions

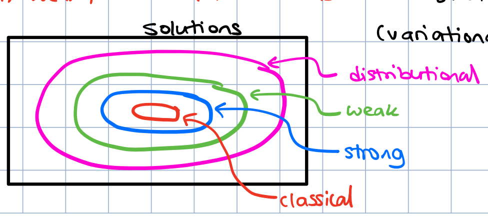
Let $f:\Omega\to\mathbb{R}$, $g_0:\Gamma_0\to\mathbb{R}$, $g_1:\Gamma_1\to\mathbb{R}$, $\alpha\in\mathbb{R}$. Find a function $u:\Omega\to\mathbb{R}$ which is a solution to
$$\begin{cases} -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a_0u=f & \text{in } \Omega \\ u=g_0 & \text{on } \Gamma_0 \\ \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}n_i=g_1 & \text{on } \Gamma_1 \end{cases}$$
We define the matrix A such that $A_{ij}=a_{ij}$. Rewrite the eqn as $-\operatorname{div}(A\nabla u)+a_0u=f$ in Ω.
$\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}n_i$ is called canonical derivative of u at $x\in\partial\Omega$

- AIM: Prove existence, uniqueness and stability of "solution" to the above problems provided that the data $f, g_0, g_1, \alpha, \beta$ satisfy some suitable assumptions.

- QN: What does "solution" mean?

Let us consider the Poisson problem with homogeneous Dirichlet boundary conditions. $\begin{cases} -\Delta u=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$

1) Classical solutions: u and its derivatives are continuous + $-\Delta u=f$ is satisfied at every point $x\in\Omega$.
2) Strong solutions: $H^2(\Omega)=\{u\in H^1(\Omega): \partial^\alpha u\in L^2(\Omega),\ \forall|\alpha|=2\}$
$u\in H^2$ + $(-\Delta u=f)$ is solved a.e in Ω + "$u=0$ on $\partial\Omega$" in sense of trace.
3) Distributional solutions: $u\in L^1_{loc}(\Omega)$ + $(-\Delta u=f)$ is solved in distributional sense, i.e. $-\int_\Omega u\Delta v\,dx=\int_\Omega fv\,dx$ $\forall v\in\mathcal{D}(\Omega)$.
4) Weak / Variational solutions: $u\in H^1_0(\Omega)$ + $(-\Delta u=f)$ is solved in weak sense $\int_\Omega \nabla u\cdot\nabla v\,dx=\int_\Omega f\cdot v\,dx$ $\forall v\in H^1_0(\Omega)$ (variational formulation)



OVERVIEW CLASS: POISSON EQN IN BDD DOMAINS

The general tool we will use is the following result from functional analysis:

- **THM:** Let X be a Hilbert space endowed with scalar prod $(\cdot,\cdot)$ and associated norm $\|\cdot\|$. Assume
  1) $a: X\times X \to \mathbb{R}$, $(u,v)\mapsto a(u,v)$ such that i) a is bilinear
     ii) a is cts: i.e. $\exists M>0$ s.t. $|a(u,v)| \le M\|u\|\|v\|$ $\qquad \|\cdot\| = \|\cdot\|_X$
     iii) a is coercive, i.e. there exists $\alpha>0$ s.t. $a(u,u) \ge \alpha\|u\|^2$
  2) L is a linear functional on X ($L\in X'$)

  Then, the problem $a(u,v) = L(v)$, $\forall v\in X$ has a unique solution $u\in X$. Moreover, $\|u\|_X \le \frac{1}{\alpha}\|L\|_{X'}$

- **Ex:** We consider the problem $\begin{cases} -\Delta u + au = f & \text{in } \Omega, \quad (a>0) \\ u = g & \text{on } \partial\Omega \end{cases}$ $\quad f:\Omega\to\mathbb{R}$, $g:\partial\Omega\to\mathbb{R}$ given.

  DIRICHLET PROBLEM

  $g=0$ "homogeneous case"

  $g\neq 0$ "inhomogeneous case"

First goal: find correct variational formulation of the above problem.

Let's assume $u\in H^2(\Omega)$ which is a strong solution to (D) (namely $-\Delta u + u = f$ holds a.e in Ω)

We consider a test fn $v\in H^1_0$. We multiply the equation by v and integrate over Ω. We obtain: $\int_\Omega -\Delta u\cdot v\,dx + a\int_\Omega u\cdot v\,dx = \int_\Omega f\cdot v\,dx$

1) Do we want to choose a test function which gives rise to a weak formulation which includes all boundary conditions?

IBP FORMULA:

if $A\in H^1(\Omega,\mathbb{R}^n)$, $v\in H^1(\Omega)$ then $\int_\Omega \operatorname{div} A\, v\,dx = -\int_\Omega A\cdot\nabla v\,dx + \int_{\partial\Omega} A\cdot n\, v\,d\sigma$

Use this with $A=\nabla u\in H^1(\Omega;\mathbb{R}^n)$ (as $u\in H^2$). So: $-\int_\Omega \Delta u\cdot v\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx - \underbrace{\int_{\partial\Omega} \overbrace{\nabla u\cdot n}^{\frac{\partial u}{\partial n}}\, v\,d\sigma}_{=0}$

want v to vanish on $\partial\Omega$!

this will be 0.

$= 0$ since $\gamma(v)=0$

We can rewrite (*) as:

$$\int_\Omega \nabla u\cdot\nabla v\,dx + a\int_\Omega u\cdot v\,dx = \int_\Omega f\cdot v\,dx \qquad (*)$$

If $g=0$ (homogeneous case) then we can look for $u\in H^1_0(\Omega)$ so apply Lax-Milgram with $X=H^1_0(\Omega)$
$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx + a\int_\Omega u\cdot v\,dx$, $L(v) = \int_\Omega f\cdot v\,dx$.

If $g\neq 0$, we assume that $g\in H^{\frac12}(\partial\Omega)$. Then there exists $\tilde g\in H^1(\Omega)$ such that $\gamma(\tilde g) = g$ on $\partial\Omega$ and $\|\tilde g\|_{H^1(\Omega)} \le C\|g\|_{H^{\frac12}(\partial\Omega)}$. Now we define $\tilde u = u - \tilde g$. Indeed, $\gamma(\tilde u) = \gamma(u) - \gamma(\tilde g) = g - g = 0$.
$\Rightarrow$ We can look for $\tilde u\in H^1_0(\Omega)$

Go back to (*); taking $u = \tilde u + \tilde g$: $\int_\Omega \nabla\tilde u\cdot\nabla v\,dx + \int_\Omega \nabla\tilde g\cdot\nabla v\,dx + a\int_\Omega \tilde u\cdot v\,dx + a\int_\Omega \tilde g\cdot v\,dx = \int_\Omega f v\,dx$
which is equivalent to

$$\int_\Omega \nabla\tilde u\cdot\nabla v\,dx + a\int_\Omega \tilde u\cdot v\,dx = \int_\Omega f\cdot v\,dx - \int_\Omega \nabla\tilde g\cdot\nabla v\,dx - a\int_\Omega \tilde g\, v\,dx$$

To apply Lax-Milgram thm we consider $a(\tilde u,v) = \int_\Omega \nabla\tilde u\cdot\nabla v\,dx + a\int_\Omega \tilde u\cdot v\,dx$,
$L(v) = \int_\Omega f\cdot v\,dx - \int_\Omega \nabla\tilde g\cdot\nabla v\,dx - a\int_\Omega \tilde g\cdot v\,dx$

Variational formulation: $\begin{cases} u = \tilde u + \tilde g, \text{ where } \tilde u\in H^1_0(\Omega),\ \gamma(\tilde g) = g \\ a(\tilde u, v) = L(v) \quad \forall v\in H^1_0(\Omega) \end{cases}$

Let $X = H^1_0(\Omega)$. We observe that
1) $|a(\tilde u,v)| \le \left|\int_\Omega \nabla\tilde u\cdot\nabla v\,dx\right| + \left|a\int_\Omega \tilde u\, v\,dx\right| \le \|\nabla\tilde u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + a\|\tilde u\|_{L^2}\|v\|_{L^2} \le (1+a)\|\tilde u\|_{H^1_0(\Omega)}\|\tilde v\|_{H^1_0(\Omega)}$
2) $a(u,u) = \int_\Omega |\nabla u|^2 dx + a\int_\Omega |u|^2 dx \ge \min\{1,a\}\left(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}\right) = \min\{1,a\}\|u\|^2_{H^1_0(\Omega)}$.
3) $L(v)$ cts functional: Let $v\in H^1_0(\Omega)$, $\|v\|_{H^1_0(\Omega)} \le 1$.
   $|L(v)| \le \left|\int_\Omega f\cdot v\,dx - \int_\Omega \nabla\tilde g\cdot\nabla v\,dx - a\int_\Omega \tilde g\cdot v\,dx\right| \le \|f\|_{L^2}\cancelto{\le 1}{\|v\|_{L^2}} + \|\nabla\tilde g\|_{L^2}\cancelto{\le 1}{\|\nabla v\|_{L^2}} + a\|\tilde g\|_{L^2}\cancelto{\le 1}{\|v\|_{L^2}}$
   $\le \|f\|_{L^2(\Omega)} + (1+a)\|\tilde g\|_{H^1(\Omega)} \Rightarrow \|L\|_{H^{-1}} \le \|f\|_{L^2(\Omega)} + (1+a)\|\tilde g\|_{H^1(\Omega)}$ $\qquad \|L\|_{X'} = \sup_{\substack{v\in X\\ \|v\|_X = 1}} |L(x)|$

We've now proved the following problem:

- **PROP:** Let $f\in L^2(\Omega)$, $g\in H^{\frac12}(\partial\Omega)$. Then the Dirichlet problem (P) has a unique weak/variational soln. $u\in H^1(\Omega)$ s.t. $\int_\Omega \nabla u\cdot\nabla v\,dx + a\int_\Omega u\cdot v\,dx = \int_\Omega f v\,dx$ $\forall v\in H^1_0(\Omega)$ and $\gamma(u) = g$ on $\partial\Omega$. Moreover, we have $\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac12}(\partial\Omega)}\right)$ (★) for some universal constant $C>0$.

Proof of (★): We have $|L(v)| \le \|f\|_{L^2(\Omega)} + (1+a)\|\tilde g\|_{H^1(\Omega)}$, where $\tilde g \in H^1(\Omega)$: $\gamma(\tilde g) = g$ on $\partial\Omega$. By Lax-Milgram it follows that $\|\tilde u\|_{H^1(\Omega)} \le \frac{1}{\min\{1,a\}}\left(\|f\|_{L^2(\Omega)} + (1+a)\|\tilde g\|_{H^1(\Omega)}\right)$. As a consequence, since $u = \tilde u + \tilde g$,
$\|u\|_{H^1(\Omega)} \le \frac{1}{\min\{1,a\}}\left(\|f\|_{L^2(\Omega)} + (1+a)\|\tilde g\|_{H^1(\Omega)}\right) + \|\tilde g\|_{H^1(\Omega)}$ (E)

We observe u is unique. Indeed, let $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$ be two solutions of the weak formulation, i.e. $\int_\Omega \nabla u_i \cdot \nabla v\,dx + a\int_\Omega u_i \cdot v\,dx = \int_\Omega f\cdot v\,dx$ $\forall v \in H^1_0(\Omega)$ and $\gamma(u_i) = g$ for $i = 1,2$.
Then, the difference $w = u_1 - u_2$ solves: $\int_\Omega \nabla w\cdot\nabla v\,dx + a\int_\Omega w\cdot v\,dx = 0$ $\forall v \in H^1_0(\Omega)$.
By Lax-Milgram, $w = 0 \Rightarrow u_1 = u_2$. So u does not depend explicitly on choice of $\tilde g$.
Now let us recall: $\|g\|_{H^{\frac12}(\partial\Omega)} = \inf\{\|\tilde g\|_{H^1(\Omega)} : \tilde g \in H^1(\Omega), \gamma(\tilde g) = g\}$. Taking the infimum over all $\tilde g \in H^1(\Omega)$ such that $\gamma(\tilde g) = g$, we get:
$\|u\|_{H^1(\Omega)} \le \frac{1}{\min\{1,a\}}\left(\|f\|_{L^2(\Omega)} + (1+a)\|g\|_{H^{\frac12}(\partial\Omega)}\right) + \|g\|_{H^{\frac12}(\partial\Omega)}$ which gives the desired conclusion ┘

→ The weak soln. u is also a distributional solution to (P). Indeed, since $\mathcal{D}(\Omega) \subset H^1_0(\Omega)$ we can rewrite this variational formulation as: $\langle -\Delta u + au, v\rangle = \langle f, v\rangle$ $\forall v \in \mathcal{D}(\Omega)$ $\Rightarrow$ $-\Delta u + au = f$ in $\mathcal{D}'(\Omega)$
Now notice that $au \in L^2(\Omega)$, $f \in L^2(\Omega)$ $\Rightarrow$ $-\Delta u \in L^2(\Omega)$ $\xRightarrow[\text{SEE LATER}]{\text{REG. THM}}$ $u \in H^2(\Omega)$ provided g is "more regular" + $-\Delta u + au = f$ holds a.e. in Ω.

- Possible extensions: 1) $a = 0$? Poincare useful for coercivity.
  2) What if $f \in H^{-1}(\Omega)$? Here, $L(v) = \langle f, v\rangle_{H^{-1}\times H^1_0} - \int \nabla\tilde g\cdot\nabla v\,dx - a\int \tilde g\cdot v\,dx$
  $\Rightarrow |L(v)| \le \|f\|_{H^{-1}}\|v\|_{H^1} + \|\nabla\tilde g\|_{L^2}\|\nabla v\|_{L^2} + a\|\tilde g\|_{L^2}\|v\|_{L^2} \le (\|f\|_{H^{-1}} + (1+a)\|\tilde g\|_{H^1})$

- Ex: NEUMANN PROBLEM
$$\begin{cases} -\Delta u + au = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega \end{cases} \quad (N)$$
We claim the variational formulation for (N) is the following:
Find $u \in H^1(\Omega)$ such that $a(u,v) = L_N(v)$, $\forall v \in H^1(\Omega)$ where
$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx + a\int_\Omega u\cdot v\,dx$, $L_N(v) = \int_\Omega f\cdot v\,dx + \int_{\partial\Omega} g\cdot v\,d\sigma$ (WN)

- PROP: Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $a > 0$. Then there exists a unique weak soln $u \in H^1(\Omega)$ to (N) such that $a(u,v) = L_N(v)$ $\forall v \in H^1(\Omega)$. Moreover, we have $\|u\|_{H^1(\Omega)} \le \frac{1}{\min\{a,1\}}\left(\|f\|_{L^2(\Omega)} + C_\gamma\|g\|_{L^2(\partial\Omega)}\right)$

  - PF: Since $a > 0$ then the bilinear form a is cts & coercive (see above). Let us take $v \in H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \le 1$. We compute $|L_N(v)| \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \le \|f\|_{L^2(\Omega)}\cancel{\|v\|_{L^2(\Omega)}}^{\le 1} + \|g\|_{L^2(\partial\Omega)}C_\gamma\cancel{\|v\|_{H^1(\Omega)}}^{\le 1}$
  $\Rightarrow \|L_N\|_{(H^1(\Omega))'} \le \|f\|_{L^2(\Omega)} + C_\gamma\|g\|_{L^2(\partial\Omega)}$. The conclusion follows from Lax-Milgram theorem. □

- Why is (WN) the correct variational formulation for (N)? If u is "more regular" and solves (WN) does u satisfy the boundary condition? For any $v \in \mathcal{D}(\Omega)$ we obtain
$a(u,v) = \int_\Omega fv\,dx + \int_{\partial\Omega} g\underbrace{v}_{0}\,d\sigma$ compact supp $\Rightarrow$ $\langle -\Delta u + au, v\rangle = \langle f, v\rangle$ $\forall v \in \mathcal{D}(\Omega)$. Since $au \in L^2$, $f \in L^2$, $-\Delta u \in L^2(\Omega)$.
$\Rightarrow$ $-\Delta u + au = f$ is true a.e. in Ω. $\Rightarrow$ The Poisson equation is recovered from (WN).

2) We w.t.s if u is regular we can recover the boundary condition from (WN). Observe:
$$\int_\Omega \nabla u\cdot\nabla v\,dx \overset{u\in H^2(\Omega)}{=} \int_\Omega -\Delta u\cdot v\,dx + \int_{\partial\Omega}\nabla u\cdot n\,v\,d\sigma \quad \forall v \in H^1(\Omega)$$
and so $\cancel{\int_\Omega -\Delta u\,v\,dx}_{1} + \int_{\partial\Omega}\nabla u\cdot n\,v\,d\sigma + \cancel{a\int_\Omega u\cdot v\,dx}_{2} = \cancel{\int_\Omega fv\,dx}_{3} + \int_{\partial\Omega} gv\,d\sigma$ $(\forall v \in H^1)$
Thanks to part (1), we infer:
$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - g\right)v\,d\sigma = 0,\ \forall v \in H^1(\Omega) \iff \int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - g\right)v\,d\sigma = 0,\ \forall v \in H^{\frac12}(\partial\Omega)$$
Possible to show $H^{\frac12}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$ which gives
$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - g\right)v\,d\sigma = 0,\ \forall v \in L^2(\partial\Omega) \Rightarrow \frac{\partial u}{\partial n} = g \text{ a.e. on } \partial\Omega$$

6.1: POISSON EQN: ROBIN/OBLIQUE CONDITIONS

- We study the problem $\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = g & \text{on } \partial\Omega \end{cases}$ (R) $f: \Omega \to \mathbb{R}$, $g, \beta: \partial\Omega \to \mathbb{R}$ are given functions.

Variational formulation: let $u \in H^2(\Omega)$ be a solution to $-\Delta u + u = f$ in Ω (holds a.e) with $f \in L^2(\Omega)$. Moreover, we also assume $\frac{\partial u}{\partial n} + \beta u = g$ holds a.e on $\partial\Omega$. Correct space for test fn?
As in the Neumann case, we take a test function $v \in H^1(\Omega)$. Now multiplying (R) by v & integrating on Ω, we obtain: $\int_\Omega -\Delta u \cdot v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx$, $v \in H^1(\Omega)$. By IBP, we have:

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} (\nabla u \cdot n) v \, d\sigma + \int_\Omega uv \, dx = \int_\Omega fv \, dx, \quad \forall v \in H^1(\Omega)$$

We observe that $\nabla u \cdot n = \frac{\partial u}{\partial n} \overset{(R)}{=} g - \beta u$ on $\partial\Omega$. Therefore, we find:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta u v \, d\sigma + \int_\Omega uv \, dx = \int_{\partial\Omega} fv \, dx + \int_{\partial\Omega} gv \, dx \quad \forall v \in H^1(\Omega).$$

In other words, the variational formulation of (R): find $u \in H^1(\Omega)$ s.t. $a_R(u,v) = L_R(v)$, $\forall v \in H^1(\Omega)$ (WR)
where $a_R(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx + \int_{\partial\Omega} \beta uv \, d\sigma$, $L_R(v) = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, dx$

Conversely, it is possible to show that if $u \in H^1(\Omega)$ is a weak solution to (WR) and $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $\beta \in L^\infty(\partial\Omega)$ s.t. $\beta \geq 0$ a.e on $\partial\Omega$, then u is a strong solution.

Indeed, first of all, taking $v \in \mathcal{D}(\Omega)$ we can prove $-\Delta u + u = f$ holds in $\mathcal{D}'(\Omega)$. Since $f \in L^2(\Omega)$, $u \in L^2(\Omega)$ then $\Delta u \in L^2(\Omega)$ and $-\Delta u + u = f$ holds a.e. in Ω. Secondly if $u \in H^2(\Omega)$ (this comes from the regularity theory), then using IBP we have

$$\cancel{\int_\Omega -\Delta u \, v \, dx} + \int_{\partial\Omega} \nabla u \cdot n \, v \, d\sigma + \cancel{\int_\Omega uv \, dx} + \int_{\partial\Omega} \beta u \cdot v \, d\sigma = \cancel{\int_\Omega fv \, dx} + \int_{\partial\Omega} gv \, d\sigma \quad \forall v \in H^1(\Omega)$$

$$\Rightarrow \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} + \beta u - g \right) v \, d\sigma = 0 \quad \forall v \in H^1(\Omega)$$

[Why need to require $g \in H^{\frac{1}{2}}(\partial\Omega)$? If $u \in H^2(\Omega)$ then $\nabla u \in H^1(\Omega; \mathbb{R}^n)$. So the normal trace $\frac{\partial u}{\partial n} \in H^{\frac{1}{2}}(\partial\Omega) \Rightarrow g$ must necessarily be in $H^{\frac{1}{2}}(\partial\Omega)$]

By density of $L^2(\partial\Omega)$ in $H^{\frac{1}{2}}(\partial\Omega)$ one can deduce $\frac{\partial u}{\partial n} + \beta u = g$ on $\partial\Omega$
Now we prove well-posedness of (WR):

- **Prop:** Let f, g, β be given such that $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $\beta \in L^\infty(\partial\Omega)$: $\beta \geq 0$ a.e. on $\partial\Omega$. Then, (WR) has a unique solution $u \in H^1(\Omega)$ such that $\|u\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)})$ for some universal constant positive C. (doesn't depend on soln.)
  - **Pf:** Let $X = H^1(\Omega)$. Consider the bilinear form $a_R(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u \cdot v \, dx + \int_{\partial\Omega} \underbrace{\beta}_{L^\infty} \underbrace{u \cdot v}_{\in L^2(\partial\Omega)} \, d\sigma$.
    $a_R(\cdot,\cdot)$ is well-defined on $H^1(\Omega) \times H^1(\Omega)$ thanks to the trace theorem.
    1) a_R cts on $H^1(\Omega) \times H^1(\Omega)$: $|a_R(u,v)| \leq |\int_\Omega \nabla u \cdot \nabla v \, dx| + |\int_\Omega uv \, dx| + |\int_{\partial\Omega} \beta uv \, d\sigma|$ (Hö)
    $\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|\beta\|_{L^\infty(\partial\Omega)} \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}$
    $\leq 2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|\beta\|_{L^\infty(\partial\Omega)} C_\gamma^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$
    $\leq (2 + \|\beta\|_{L^\infty(\partial\Omega)} C_\gamma^2) \|u\|_{H^1(\Omega)}^2 \|v\|_{H^1(\Omega)}^2$
    2) a_R coercive:
    $a_R(u,u) = \underbrace{\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx}_{\|u\|^2_{H^1(\Omega)}} + \underbrace{\int_{\partial\Omega} \beta u^2 d\sigma}_{\geq 0}$
    $\geq \|u\|^2_{H^1(\Omega)}$ //
    3) L_R cts in $H^1(\Omega)$: $|L_R(v)| \leq |\int_\Omega fv \, dx| + |\int_{\partial\Omega} gv \, dx| \overset{C\cdot S}{\leq} \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}$
    $\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} C_\gamma \|v\|_{H^1(\Omega)} \leq (\|f\|_{L^2(\Omega)} + C_\gamma \|g\|_{L^2(\partial\Omega)}) \|v\|_{H^1(\Omega)}$

    Thanks to Lax-Milgram there exists a unique solution $u \in H^1(\Omega)$ such that $a_R(u,v) = L_R(v)$ $\forall v \in H^1(\Omega)$.
    In addition, $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)} + C_\gamma \|g\|_{L^2(\partial\Omega)}$ (same const. as in coercivity). □

· **REMARKS:** 1) Only required $g \in L^2(\partial\Omega)$ instead of $g \in H^{\frac{1}{2}}(\partial\Omega)$ as for "strong solutions"

2) Existence & uniqueness of a weak solution holds if $f \in (H^1(\Omega))'$ and ($\langle \cdot,\cdot \rangle$ duality in $H^1(\Omega)$)
$L_R(v) = \langle f, v \rangle + \int_{\partial\Omega} g v \, d\sigma$. This is a more general assumption. Indeed if $n=3$ by Sobolev inequality we have $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Therefore if $f \in L^{\frac{6}{5}}(\Omega)$ then

$$\int_\Omega fv\,dx \overset{(H)}{\leq} \|f\|_{L^{\frac{6}{5}}(\Omega)} \|v\|_{L^6(\Omega)} \leq C_S \|f\|_{L^{\frac{6}{5}}(\Omega)} \|v\|_{H^1(\Omega)}.$$

We showed that, if $f \in L^{\frac{6}{5}}(\Omega)$ then $v \mapsto \int_\Omega fv\,dx$ is a linear cts functional in $H^1(\Omega)$.

NOTICE THAT $L^2(\Omega) \hookrightarrow L^{\frac{6}{5}}(\Omega)$!

6.2: REGULARITY THEORY FOR POISSON EQN

QUALITATIVE PROPERTIES OF WEAK SOLNS TO LINEAR ELLIPTIC PROBLEMS IN BDD DOMAINS (PART I)

REGULARITY THEORY

· We address the question as to the weak solution to $-\Delta u + u = f$ in Ω is in fact more regular than $H^1(\Omega)$. (for instance, $u \in H^2(\Omega)$).

Motivation: the case of $\mathbb{R}^n$; We consider the eqn $-\Delta u + u = f$ in $\mathbb{R}^n$ (E)

Let us recall $\widehat{\partial_{x_i} u}(\xi) = i\xi_i \hat{u}(\xi)$, $\widehat{\partial_{x_i}\partial_{x_j} u}(\xi) = -\xi_i\xi_j\hat{u}(\xi)$ $\Longrightarrow$ $\widehat{\Delta u}(\xi) = |\xi|^2 \hat{u}(\xi)$

Applying F.T to E, we obtain: $(1+|\xi|^2)\hat{u}(\xi) = \hat{f}(\xi)$ in $\mathbb{R}^n$ $\Rightarrow$ $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{1+|\xi|^2}$

Plancherel Thm: $\|\hat{v}\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|v\|^2_{L^2(\mathbb{R}^n)}$. So we compute:

$$\|\partial_{x_i}\partial_{x_j} u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \|\widehat{\partial_{x_i}\partial_{x_j} u}\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \xi_i^2\xi_j^2 |\hat{u}(\xi)|^2 d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \xi_i^2\xi_j^2 \frac{|\hat{f}(\xi)|^2}{(1+|\xi|^2)^2} d\xi$$

$(2\xi_i\xi_j \leq 1+|\xi|^2 \ \forall i,j=1,\dots,n)$ $\leq \frac{1}{(2\pi)^n}\frac{1}{n}\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = \frac{1}{n}\|f\|^2_{L^2(\mathbb{R}^n)}$. So we've shown $f \in L^2(\mathbb{R}^n) \Longrightarrow u \in H^2(\mathbb{R}^n)$.

Actually, since $-\Delta(\partial_{x_i} u) + \partial_{x_i} u = \partial_{x_i} f$, it is also possible to show that $f \in H^1(\mathbb{R}^n) \Rightarrow f \in H^3(\mathbb{R}^n)$

In general, $f \in H^k(\mathbb{R}^n) \Rightarrow f \in H^{k+2}(\mathbb{R}^n)$

What happens if Ω = bounded domain?

$\boxed{\text{I}}$ **DIRICHLET CASE (HOMOGENEOUS)** $\rightsquigarrow$ **THM:** Let Ω be bdd domain with C^2-boundary. Assume $\alpha \geq 0$ and $f \in L^2(\Omega)$.

$$\begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (D)$$

Then the weak soln. u to (D) belongs to $H^2(\Omega)$. Moreover $\exists\, C_D$ const. depending on Ω and α s.t. $\|u\|_{H^2(\Omega)} \leq C_D(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})$

REM$^{(1)}$: C_D universal; doesn't depend on u.

REM$^{(2)}$: this thm can be generalised to the non-homogeneous case ($u \neq 0$ on $\partial\Omega$)

$\boxed{\text{I}}$ **NEUMANN/ROBIN CASE** $\rightsquigarrow$ **THM:** Let Ω be bdd domain with C^2-boundary. Assume $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $\alpha > 0$, $\beta \geq 0$. Then the weak solution u to (NR) belongs to $H^2(\Omega)$. Moreover $\exists C_{NR}$ universal constant depending only on α, β such that

$$\begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = g & \text{on } \partial\Omega \end{cases} \quad (NR)$$

$$\|u\|_{H^2(\Omega)} \leq C_{NR}(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)})$$

These theorems can be generalised to higher order Sobolev Spaces

· **DEF:** For $k \in \mathbb{N}$, the Sobolev Space $H^k(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega),\ \forall |\alpha| \leq k\}$ with norm $\|u\|_{H^k(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{|\alpha|\leq k} \|\partial^\alpha u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$

↳ **THM:** Let Ω be bdd domain in $\mathbb{R}^n$ with C^{k+2} boundary, for $k \in \mathbb{N}$. Let $f \in H^k(\Omega)$. We have the following:

1) (NON-HOMOGENEOUS DIRICHLET) If $g \in H^{k+\frac{3}{2}}(\partial\Omega)$, then $u \in H^{k+2}(\Omega)$

2) (NEUMANN/ROBIN) If $g \in H^{k+\frac{1}{2}}(\partial\Omega)$, then $u \in H^{k+2}(\Omega)$.

Moreover there exist two positive constants $\overline{C^k_D}$ and $\overline{C^k_{NR}}$ depending only on Ω, α, β such that

$$\|u_D\|_{H^{k+2}(\Omega)} \leq \overline{C^k_D}(\|u_D\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)} + \|g\|_{H^{k+\frac{3}{2}}(\partial\Omega)})$$

$$\|u_{NR}\|_{H^{k+2}(\Omega)} \leq \overline{C^k_{NR}}(\|u_{NR}\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)} + \|g\|_{H^{k+\frac{1}{2}}(\partial\Omega)})$$

where u_D, u_{NR} are weak solns to (D), (NR) resp.

→ **REMARKS:** Main reference is the work by Agmon-Douglas-Nirenberg (1964)

As a consequence of above theorem, it is possible to show that Ω is a C^∞ domain, $f \in C^\infty(\overline{\Omega})$, $g \in C^\infty(\Omega)$ $\Longrightarrow u \in C^\infty(\overline{\Omega})$.

6.3: MAXIMUM PRINCIPLE

QUALITATIVE PROPERTIES OF WEAK SOL'NS TO LINEAR ELLIPTIC PROBLEMS IN BDD DOMAINS (PART 2)

MAXIMUM PRINCIPLE

- We consider the Dirichlet problem: $\begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ (D)

where Ω bdd Lipschitz domain. Problem (D) has a unique weak solution $u \in H^1(\Omega)$ such that $\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx \quad \forall v \in H^1_0(\Omega)$ and $\gamma(u) = g$ on $\partial\Omega$ provided $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$.

We now discuss the Stampacchio maximum principle.

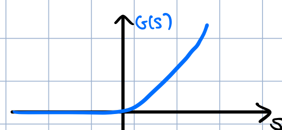
- **THM:** Let u be the weak sol'n to (D) with $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Then we have
$$\min\{\operatorname{essinf}_{\partial\Omega} g, \operatorname{essinf}_{\Omega} f\} \leq u(x) \leq \max\{\operatorname{esssup}_{\partial\Omega} g, \operatorname{esssup}_{\Omega} f\} \quad \text{for a.e. } x \in \Omega.$$

In order to prove Stampacchio's thm we need a preliminary result about the composition of functions in Sobolev spaces

- **PROP:** Let $G \in C^1(\mathbb{R})$ be such that $G(0) = 0$ and $|G'(s)| \leq M$ $\forall s \in \mathbb{R}$, for some constant M. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then,
  1) $G \circ u \in W^{1,p}(\Omega)$
  2) $\frac{\partial}{\partial x_i}(G \circ u) = (G' \circ u) \cdot \frac{\partial u}{\partial x_i}$, $i = 1, \dots, n$ in distributional sense.
  3) $\gamma(G \circ u) = G \circ \gamma(u)$ on $\partial\Omega$.

PF: We use the truncation method. Let $G: \mathbb{R} \to \mathbb{R}$ be a function in $C^1(\mathbb{R})$ such that:

1) $|G'(s)| \leq M$, $\forall s \in \mathbb{R}$
2) $G = 0$ $\forall s \in (-\infty, 0]$
3) G is strictly increasing on $(0, \infty)$



"GROWS LINEARLY AT ∞"

Let $K = \max\{\operatorname{esssup}_{\partial\Omega} g, \operatorname{esssup}_{\Omega} f\}$. We assume $K < \infty$. We choose the test function $v = G(u-K)$. Thanks to the above proposition we have that $v \in H^1(\Omega)$ (since $u - K \in H^1(\Omega)$). Furthermore we claim $u \in H^1_0(\Omega)$. Indeed, $\gamma(v) = \gamma(G \circ (u-K)) = G \circ \gamma(u-K)$. But, $\gamma(u) = g$ and $g \leq K$ for a.e. $x \in \partial\Omega$. This means $\gamma(v) = 0$. Taking v as test function in the weak-formulation, we obtain
$$\int_\Omega \nabla u \cdot \nabla(G(u-K)) \, dx + \int_\Omega u \cdot G(u-K) \, dx = \int_\Omega f \, G(u-K) \, dx.$$
We observe $\nabla(G(u-K)) = G'(u-K)\nabla(u-K) = G'(u-K)\nabla u$

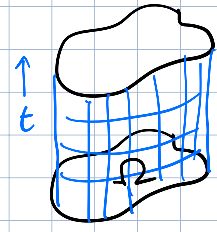
which gives $\int_\Omega G'(u-K)|\nabla u|^2 dx + \int_\Omega u G(u-K) \, dx = \int_\Omega f G(u-K) \, dx$. We subtract $K \int_\Omega G(u(x)-K) \, dx$ to both sides of the above equality and we find
$$\underbrace{\int_\Omega \underbrace{G'(u-K)}_{\geq 0} |\nabla u|^2 dx}_{\geq 0} + \int_\Omega (u-K) G(u-K) \, dx = \int_\Omega \underbrace{(f-K)}_{\leq 0} \underbrace{G(u-K)}_{\geq 0} dx$$
Therefore, we conclude that
$\int_\Omega (u-K) G(u-K) \, dx \leq 0$. But, the function $sG(s) \geq 0$. This implies $(u-K)G(u-K) = 0$ a.e. which gives that $u - K \leq 0$ a.e in Ω. A similar method shows $u(x) - \tilde{K} \geq 0$ a.e. where $\tilde{K} = \min\{\operatorname{essinf}_{\partial\Omega} g, \operatorname{essinf}_{\Omega} f\}$ ∎

- **REMARK:** The Stampacchio's truncation method also works for weak solutions to the homogeneous Neumann $\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$ and to general elliptic problem in divergence form $\begin{cases} -\partial x_i(a_{ij} \partial x_j u) + a_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

where a_{ij}, a_0 are bdd f'ns in Ω which satisfy the "strong elliptic problem"

OVERVIEW CLASS 6: EVOLUTION EQUATIONS & TIME DEPENDENT SPACES

- Let Ω be bdd smooth domain in $\mathbb{R}^n$. We want to study the following PDEs:



1) The Cauchy-Dirichlet for the Heat eqn:
We look for a solution $u:\Omega\times[0,\infty)$ such that
$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega\times(0,\infty) \\ u(\cdot,t)|_{\partial\Omega} = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega \end{cases}$$

2) The Cauchy-Dirichlet problem for the Wave equation
We look for a function $u:\Omega\times[0,\infty)$ such that
$$\begin{cases} \partial_{tt} u - \Delta u = 0 & \text{in } \Omega\times(0,\infty) \\ u(\cdot,t)|_{\partial\Omega} = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \Omega \\ \partial_t u(\cdot,0) = u_1 & \text{in } \Omega \end{cases}$$
"DIRICHLET CONDITION"

$f:[0,b]\to\mathbb{R}$, $x\mapsto f(x)$ ← Instead function space; $L^2, H^1, \dots$

$$\boxed{\begin{cases} Au=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \quad f\in L^2 \Rightarrow \exists!\, u\in D(A)}$$

- Let us introduce the operator $A=-\Delta$ with Dirichlet boundary conditions: $A: D(A)=H^2(\Omega)\cap H^1_0(\Omega)\longrightarrow L^2(\Omega)$
From now on we'll separate the role of time & space. We will consider the following viewpoint:
for any $t\geq 0$ we consider $u(\cdot,t)$ as an element in a function space. In other words, we consider $u:[0,T]\to X$, $t\mapsto u(\cdot,t)$. In measure theory we studied functions such that $f:[a,b]\to\mathbb{R}$, $x\mapsto f(x)$.
We are replacing $\mathbb{R}$ with a general Banach space X. Assuming this new viewpoint, we can "rewrite" $\partial_t u - \Delta u = 0$ as the following equation: $\frac{d}{dt}u + Au = 0$. This is an ODE in a suitable Banach space X. This approach can also be used for the wave equation $\partial_{tt}u - \Delta u = 0$.
We introduce $U=\begin{pmatrix} u \\ v\end{pmatrix}$. $\frac{d}{dt}U = \begin{pmatrix} du/dt \\ dv/dt\end{pmatrix} = \begin{pmatrix} v \\ \Delta u\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix}}_{B}\begin{pmatrix} u \\ v\end{pmatrix} \Rightarrow \frac{d}{dt}U = BU.$

For this reason we introduce the function spaces:

- **DEF:** Let X be a Banach space. The space $C([0,T];X)$ is the set of functions which are continuous from $[0,T]$ to X. We define the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]}\|u(t)\|_X$
For $m\in\mathbb{Z}$ the space $C^m([0,T];X)$ is the set of functions u that are m-times ctsly diff-ble from $[0,T]\to X$. We define the norm $\|u\|_{C^m([0,T];X)} = \max_{t\in[0,T]}\sum_{j=0}^{m}\|\frac{\partial^j}{\partial t^j}u\|_X$

- **PROP:** Let X be Banach space. $C^m([0,T];X)$ is a Banach space $\forall m\in\mathbb{N}$
QN: Is this functional framework sufficient to study variational solutions to evolution equations?
Let u be a smooth solution of the Heat Eqn. We consider a test function $v\in H^1_0(\Omega)$. We now multiply the eqn by v and integrating over Ω we have $\int_\Omega \partial_t u\cdot v\,dx - \int_\Omega \Delta u\cdot v\,dx = 0$.
Integrating by parts in 2^{nd} term, we find $\int_\Omega \partial_t u\, v\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx = 0$ for any $t>0$. Choosing $v=u(t)$ (abuse of notation $v(\cdot)=u(\cdot,t)$) we obtain $\int_\Omega \partial_t u\cdot u\,dx + \int_\Omega |\nabla u|^2 dx = 0$. Since u smooth, we can rewrite it as $\frac{d}{dt}\int_\Omega \frac{1}{2}u^2 dx + \int_\Omega |\nabla u|^2 dx = 0$. Integrating this eqn $\frac{1}{2}\partial_t u^2$ from 0 to t we arrive at
$$\int_\Omega \tfrac{1}{2}u^2(t)\,dx + \int_0^t\int_\Omega |\nabla u(s)|^2 dx\,ds = \int_\Omega \tfrac{1}{2}u^2(0)\,dx \qquad \text{(ENERGY IDENTITY FOR HEAT EQN)}$$
$$\Rightarrow \tfrac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \boxed{\int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}\,ds} = \tfrac{1}{2}\|u(0)\|^2_{L^2(\Omega)}$$

- AIM: Define a functional framework such that $\int_0^T \|u(s)\|^p_X\,ds < +\infty$, where X is a suitable Banach space. (eg $X=L^2(\Omega)$)
Let X be a Banach space.

- **DEF:** A simple function $s:[0,T]\to X$ has the form $s(t)=\sum_{j=1}^{N}\chi_{E_j}(t)\,u_j$ where $u_1,\dots,u_N\in X$, E_j are disjoint, measurable subsets of $[0,T]$.

- **DEF:** A function $u:[0,T]\to X$ is measurable if there exists a sequence of simple functions $\{s_k\}$ s.t. $\|s_k(t)-u(t)\|_X\to 0$ as $k\to\infty$ a.e in $[0,T]$.

- **PROP:** A function is measurable iff the function $t\mapsto\langle u(t),v\rangle$ is Lebesgue-measurable for any $v\in X'$.

- We define the notion of integral w.r.t time for simple functions in the following way: $\int_0^1 s_k(t)\,dt = \sum_{j=1}^N |E_j| u_j$.

- **DEF:** A measurable function $u:[0,T]\to X$ is summable if there exists a sqnce $\{s_k\}_{k\geq 1}$ of simple fns s.t. $\int_0^T \|s_k(t)-f(t)\|_X\,dt \to 0$ as $k\to\infty$. In this way we define $\int_0^T f(t)\,dt = \lim_{k\to\infty}\int_0^T s_k(t)\,dt$

  Notice that $\left\|\int_0^T s_k(t)-s_h(t)\,dt\right\|_V \overset{\text{check}}{\leq} \int_0^T \|s_k(t)-s_h(t)\|_V\,dt \leq \int_0^T \|s_k(t)-u(t)\|_X\,dt + \int_0^T \|s_h(t)-u(t)\|_X\,dt \longrightarrow 0$
  $\Rightarrow \int_0^T s_k(t)\,dt$ is Cauchy in X.

- **THM (BOCHNER):** A function $u:[0,T]\to X$ is summable iff $t\mapsto \|u(t)\|_X$ is Lebesgue summable. Moreover, we have
  1) $\left\|\int_0^t u(s)\,ds\right\|_X \leq \int_0^t \|u(s)\|_X\,ds$
  2) For any $v\in X'$, $\langle \int_0^t u(s)\,ds, v\rangle = \int_0^t \langle u(s), v\rangle\,ds$

  If X is Hilbert the latter reads as: for any $v\in X$ $\left(\int_0^t u(s)\,ds, v\right)_X = \int_0^t (u(s),v)\,ds$.

TIME-DEPENDENT SOBOLEV SPACES

- Let X be Banach.
- **DEF:** The space $L^p(0,T;X)$ consists of measurable functions $u:[0,T]\to X$ such that $\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(s)\|_X^p\,ds\right)^{\frac{1}{p}} < \infty$
  $\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|u(s)\|_X < \infty$, $p=\infty$.

- **PROP:** $L^p(0,T;X)$ is a Banach space for any p.
- **PROP:** If X is Hilbert space and $p=2$ then $L^2(0,T;X)$ is a Hilbert space endowed with the inner product $(u,v)_{L^2(0,T;X)} = \int_0^T (u(s),v(s))_X\,ds$

- In order to introduce Sobolev spaces wrt time we need to introduce the notion of distributional derivative for functions with values in a Banach space.

- **DEF:** Let $u\in L^1(0,T;X)$. A function $w\in L^1(0,T;X)$ is said to be the distributional derivative of u (denoted by $\dot u$ or $\partial_t u$) if one of the following equivalent conditions hold:
  1) $\exists c\in X$: $u(t) = c + \int_0^t w(s)\,ds$ a.e. in $[0,T]$
  2) $\int_0^T w(t)\varphi(t)\,dt \overset{\text{in } X}{=} -\int_0^T u(t)\varphi'(t)\,dt \quad \forall \varphi\in D(0,T)$
  3) $\int_0^T \langle w(t), v\rangle \eta(t)\,dt = -\int_0^T \langle u(t), v\rangle \eta'(t)\,dt \quad \forall v\in X', \eta\in D(0,T)$.

  If X Hilbert, (3) becomes $\int_0^T (w(t),v)\eta(t)\,dt = -\int_0^T (u(t),v)\eta'(t)\,dt$ for any $v\in X$, $\eta\in D(0,T)$

- **DEF:** The space $W^{1,p}(0,T;X)$ is the set of measurable functions $u:[0,T]\to X$ such that $\|u\|_{W^{1,p}(0,T;X)} = \left(\int_0^T \|u(s)\|_X^p\,ds + \int_0^T \|\dot u(s)\|_X^p\,ds\right)^{\frac{1}{p}} < +\infty$ if $1\leq p<\infty$, $\|u\|_{W^{1,\infty}(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]}\left(\|u(t)\|_X + \|\dot u(t)\|_X\right) < \infty$ if $p=\infty$.

- **PROP:** 1) $W^{1,p}(0,T;X)$ is a Banach space, $p\in[1,\infty]$
  2) $p=2$, $X=$ Hilbert space. $W^{1,2}(0,T;X)$ is a Hilbert space endowed with the inner product $(u,v)_{W^{1,2}(0,T;X)} = \int_0^T (u(s),v(s))_X + (\dot u(s),\dot v(s))_X\,ds$ Notation: $W^{1,2}(0,T;X) = H^1(0,T;X)$

- We observe $H^1(0,T;X)$ is 1D w.r.t time. Indeed, we have the following theorem:
- **THM:** Let X be Banach. Then holds $H^1(0,T;X)\hookrightarrow C([0,T];X)$. Moreover there exists a constant $C=C(t)$ such that $\|u\|_{C([0,T];X)} \leq C(T)\|u\|_{H^1(0,T;X)}$, $\forall u\in H^1(0,T;X)$.

7.1: SPECTRAL DECOMPOSITION OF COMPACT OPERATORS

- Motivation: case of $\mathbb{R}^n$:

Let A be a $n\times n$ matrix, $A\in M^{n\times n}$, let $\lambda\in\mathbb{C}$. Consider the eqn $A\underline{x}-\lambda\underline{x}=\underline{b}$, namely $(A-\lambda I)\underline{x}=\underline{b}$

We have the following dichotomy:

1) $\forall \underline{b}\in\mathbb{R}^n$, $\exists!$ solution $x\in\mathbb{R}^n$

2) $\exists \underline{u}\in\mathbb{R}^n$, $u\neq\underline{0}$ s.t. $A\underline{u}=\lambda\underline{u}$

In the latter case λ is an eigenvalue of A and u is the corresponding eigenvector.

- the set of eigenvalues of A is called the spectrum of A, denoted $\sigma_p(A)$
- If $\lambda\notin\sigma_p(A)$ then $(A-\lambda I)^{-1}$ exists. $\rho(A)=\mathbb{C}\setminus\sigma_p(A)=$ "resolvent of A"

A particularly interesting case is when A is symmetric, $A=A^T$: the eigenvalues of A are real and there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors $u_1,\dots,u_n$ of A.

Moreover we have the following spectral decomposition of A: $A=\lambda_1\underbrace{u_1u_1^T}_{n\times n}+\lambda_2\underbrace{u_2u_2^T}_{n\times n}+\dots+\lambda_n u_n u_n^T$

Our goal is to generalise the above picture to operators on infinite-dimensional Hilbert spaces.

Why? One of the motivations is the method of separation of variables for evolution equations.

Assume we want to solve the heat equation $\begin{cases}\partial_t u-\Delta u=0 & \text{in } \Omega\times(0,\infty)\\ u(x,t)|_{t=0}=u_0(x) & \text{in } \Omega\\ u(x,t)|_{\partial\Omega}=0 & \text{for } t>0\end{cases}$ $(x,t)\in\Omega\times(0,\infty)$, $u=u(x,t)$

A natural strategy is to try separation of variables. We look for a soln of the form $u(x,t)=v(x)\alpha(t)$. Substituting this form into the eqn, we find $v(x)\alpha'(t)-\alpha(t)\Delta v(x)=0$. Dividing by $v(x)\alpha(t)$, $\frac{\alpha'(t)}{\alpha(t)}=\frac{\Delta v(x)}{v(x)}=-\lambda\in\mathbb{R}$. These relations are equivalent to the following problems: $\alpha'(t)+\lambda\alpha(t)=0$ and $\begin{cases}-\Delta v=\lambda v & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega\end{cases}$ (DP).

The couple (λ,v) with $v\neq0$ is called eigenvalue/eigenfunction of the $-\Delta v$ with Dirichlet boundary condition (also called Dirichlet operator $-\Delta$)

The original heat eqn can be solved if the following happens:

1) there exists a sqnce eigenvalues λ_n with corresponding eigenfunctions v_n. For each v_n we have $\alpha_n(t)=e^{-\lambda_n(t)}$

2) the initial condition u_0 can be "written" as $u_0(x)=\sum_{n=1}^{\infty}u_0^n v_n(x)$. ($\in\mathbb{R}$)

In such a case the soln is $u(x,t)=\sum_{n=1}^{\infty}u_0^n e^{-\lambda_n t}v_n(x)$

Condition (2) requires the set of eigenfunctions of $-\Delta$ with Dirichlet boundary conditions is a base for the whole Hilbert space (In our case $L^2(\Omega)$ or $H_0^1(\Omega)$)

For this reason we need to study the spectrum and resolvent sets of operators $T\in\mathcal{L}(H)$. For simplicity we will only consider the real case ($\lambda\in\mathbb{R}$) (although the natural space should be $\mathbb{C}$)

- DEF: Let H be a Hilbert space. $T\in\mathcal{L}(H)$.

1) The resolvent set of T, $\rho(T)$ is the set of real numbers λ s.t. $T-\lambda I$ is injective & surjective.

2) The spectrum of T is $\sigma(T)=\mathbb{R}\setminus\rho(T)$.

If $\dim H<+\infty$, the spectrum only consists of eigenvalues. If $\dim H=+\infty$ there are 3 cases:

1) $T-\lambda I$ is not injective $\Rightarrow$ $Tv=\lambda v$ has some non-trivial solution v. In this case λ is an eigenvalue and v is an eigenvector. $\sigma_p(T)=$ point spectrum of $T=$ set of eigenvalues of T

2) $T-\lambda I$ is injective but not surjective

2.1) $Im(T-\lambda I)$ ($=R(T-\lambda I)=$ range of $T-\lambda I$). is dense in H and $(T-\lambda I)^{-1}$ is unbdd.

$\lambda\in\sigma_c(T)=$ cts spectrum of T

2.2) $Im(T-\lambda I)$ is not dense in H. $\lambda\in\sigma_R(T)=$ residual spectrum of T.

In conclusion we have $\sigma(T)=\sigma_p(T)\cup\sigma_c(T)\cup\sigma_R(T)$

- In general the spectrum of a linear & cts operator T seems difficult to characterise. However if T is compact, the Fredholm alternative asserts that any non-zero element of $\sigma(T)$ is an eigenvalue. Moreover we have the following fundamental result:

- **THM (SPECTRAL DECOMPOSITION OF COMPACT OPERATORS):** Let H be a separable Hilbert space, $T \in \mathcal{L}(H)$ compact self-adjoint.

  i) $0 \in \sigma(T)$, $\sigma(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$

  ii) $\sigma_p(T)\setminus\{0\}$ is finite or it consists of a sequence $\lambda_n \to 0$. Besides, for any $\lambda \in \sigma_p(T)\setminus\{0\}$, $\dim(\mathrm{Ker}(T-\lambda I)) < \infty$.

  iii) H has an ONB $\{v_n\}$ consisting of eigenfunctions of T.

  ·) The sequence of eigenvalues can be ordered with their multiplicity s.t. $|\lambda_1| \geq |\lambda_2| \geq \dots \to 0$

  ·) $u \in H$: $u = \sum_{j=1}^{\infty}(u,v_n)v_n$. $Tu = \sum_{j=1}^{\infty}(u,v_n)Tv_n = \sum_{j=1}^{\infty}(u,v_n)\lambda_n v_n$.

7.2: EIGENVALUE PROBLEM FOR LINEAR ELLIPTIC OPERATORS

- Dirichlet problem: we study the following eigenvalue problem: find $\lambda \in \mathbb{R}$ such that $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ admits a non-zero soln.

- **THM:** Let Ω be a bdd Lipschitz domain in $\mathbb{R}^n$. There exists a sqnce of eigenvalues $\{\lambda_n\} \subset \mathbb{R}$, $\lambda_n > 0$ for any $n \in \mathbb{N}$ such that $\lambda_n \to \infty$ as $n \to \infty$ and a sqnce of corresponding eigenfunctions $\{e_n\} \subset L^2(\Omega)$ s.t:

  1) $\{e_n\}$ is an ONB in $L^2(\Omega)$ and an orthogonal basis in $H^1_0(\Omega)$

  2) $\int_\Omega \nabla e_n \cdot \nabla v \, dx = \lambda_n \int_\Omega e_n v \, dx$, $\forall v \in H^1_0(\Omega)$

  If Ω is a C^2-domain then $e_n \in H^2(\Omega)$ for any $n \in \mathbb{N}$, and $-\Delta e_n = \lambda_n e_n$ a.e in Ω. Furthermore if Ω is a C^∞-domain then $e_n \in C^\infty(\bar{\Omega})$.

- **PF:** Given $f \in L^2(\Omega)$ we consider the map T that associates the weak solution to $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

  Then $T: f \in L^2(\Omega) \mapsto u$. We'll write $Tf = u$, which is the soln to $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$, $\forall v \in H^1_0(\Omega)$.

  Is T well-defined? Yes!

  Let us define $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $u,v \in H^1_0(\Omega)$, $L(v) = \int_\Omega f v \, dx$, $v \in H^1_0(\Omega)$.

  1) a coercive in $H^1_0(\Omega)$: recalling the Poincare inequality: $\|u\|_{L^2(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}$, $\forall v \in H^1_0(\Omega)$

  we have $a(u,u) = \|\nabla u\|^2_{L^2(\Omega)} = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} \geq \frac{1}{2}\cdot\frac{1}{C_p}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} \geq \min\{\frac{1}{2C_p}, \frac{1}{2}\}\|u\|^2_{H^1_0}$

  2) a is cts in $H^1_0(\Omega) \times H^1_0(\Omega)$ ✓

  3) L is cts in $H^1_0(\Omega)$ ✓

  Thanks to Lax-Milgram thm there exists a unique soln $u \in H^1_0(\Omega)$ solving (W). Then, T is a well-defined single valued map $T: L^2(\Omega) \to H^1_0(\Omega) \subset L^2(\Omega)$.

  T1) T linear since (W) is linear

  T2) T cts; indeed by L-M thm, $\|u\|_{H^1_0(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}$

  But, $u = Tf$ and $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ then $\|u\|_{L^2(\Omega)} \leq \frac{C}{\alpha}\|f\|_{L^2(\Omega)}$ for some universal constant C.

  T3) T is cts from $L^2(\Omega)$ to $H^1_0(\Omega)$. Thus by Rellich thm we infer that T is compact in $L^2(\Omega)$.

  T4) $\mathrm{Ker}(T) = \{0\}$. Indeed if $u = Tf = 0$ then by the weak formulation $\int_\Omega f \cdot v \, dx = 0$ $\forall v \in H^1_0(\Omega)$.

  In particular $\int_\Omega f v \, dx = 0$ $\forall v \in \mathcal{D}(\Omega) \implies f = 0$.

  T5) T is self-adjoint, i.e. $(Tf, g) = (f, Tg)$ $\forall f, g \in L^2(\Omega)$. We define $u = Tf$, $v = Tg$. We recall that

  $$\left.\begin{array}{lll} \int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx & \forall w \in H^1_0(\Omega) & (1) \\ \int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Omega g w \, dx & \forall w \in H^1_0(\Omega) & (2) \end{array}\right]$$

  Now we choose $w = v$ in (1), $w = u$ in (2) and obtain:

  $$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$
  $$\int_\Omega \nabla v \cdot \nabla u \, dx = \int_\Omega g u \, dx$$

  Therefore we have $\int_\Omega f v \, dx = \int_\Omega g u \, dx$, i.e. $(f, Tg) = (Tf, g)$.

  T6) T is positive. $(Tf, f) = (u, f) \underset{\substack{w=u \\ \text{in (1)}}}{=} \int_\Omega \nabla u \cdot \nabla u \, dx = \|\nabla u\|^2_{L^2(\Omega)} \geq \alpha \|u\|^2_{H^1(\Omega)}$

Thanks to the spectral decomposition of compact self-adjoint operators, there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions e_n of T, namely

1) $(e_n, e_m) = \delta_{nm} = \begin{cases} 1, & n=m \\ 0, & o.w. \end{cases}$. In particular, $\|e_n\|_{L^2(\Omega)} = 1$.
2) $Te_n = \mu_n e_n$, where $\mu_n > 0$, $\mu_n \to 0$ as $n \to \infty$.
   - $\mu_n > 0$ since $\underset{\mu_n (e_n, e_n) = \mu_n}{\underset{\parallel}{(Te_n, e_n)}} = \|\nabla e_n\|^2_{L^2(\Omega)}$, $\mu_n \neq 0$ as $\operatorname{Ker} T = \{0\}$

By defn of T we have $\int_\Omega \nabla(\mu_n e_n) \cdot \nabla v \, dx = \int_\Omega e_n v \, dx \quad \forall v \in H_0^1(\Omega)$ which implies $\int_\Omega \nabla e_n \nabla v \, dx = \frac{1}{\mu_n} \int_\Omega e_n v \, dx \quad \forall v \in H_0^1(\Omega)$

Let us now set $\lambda_n = \frac{1}{\mu_n}$. We deduce that

1) $\lambda_n \to \infty$ as $n \to \infty$
2) $e_n \in H_0^1(\Omega)$ solves $\int_\Omega \nabla e_n \cdot \nabla v \, dx = \lambda_n \int_\Omega e_n \cdot v \, dx \quad \forall v \in H_0^1(\Omega)$ which is the variational formulation of $\begin{cases} -\Delta e_n = \lambda_n e_n & \text{in } \Omega \\ e_n = 0 & \text{on } \partial\Omega \end{cases}$
3) e_n is an ONB in $H_0^1(\Omega)$. $\int_\Omega \nabla e_n \cdot \nabla e_m \, dx = \underset{\neq 0}{\lambda_n} \int_\Omega e_n \cdot e_m \, dx = 0$ if $n \neq m \Rightarrow$ If $n \neq m$, $e_n \perp e_m$ in $H_0^1(\Omega)$.
   If $n = m$, $\|\nabla e_n\|^2_{L^2(\Omega)} = \lambda_n$.

In order to show $\{e_n\}$ is a base in $H_0^1(\Omega)$ we assume that $(e_n, v)_{H_0^1(\Omega)} = 0 \quad \forall n \in \mathbb{N}$.
This implies $\int_\Omega \nabla e_n \cdot \nabla v \, dx + \int_\Omega e_n \cdot v \, dx = 0 \Rightarrow \lambda_n \int_\Omega e_n \cdot v \, dx + \int_\Omega e_n \cdot v \, dx = 0$. Namely, $(\lambda_n + 1) \int_\Omega e_n \cdot v \, dx = 0$
Since $\{e_n\}$ is a base in $L^2(\Omega)$ then $v = 0 \Rightarrow \{e_n\}$ is a base in $H_0^1(\Omega)$.

- If Ω is a C^2-domain, since $\lambda_n e_n \in L^2(\Omega)$, by the regularity theory of the Dirichlet problem, we infer that $u \in H^2(\Omega)$. In particular we have that $-\Delta e_n = \lambda_n e_n$ in Ω.
- If Ω is a C^∞-domain, we can use a bootstrap argument. Indeed, $e_n \in H^2(\Omega)$ implies that $-\Delta e_n \in H^2(\Omega)$. By regularity we have $e_n \in H^4(\Omega)$. Applying this argument recursively, we find that $e_n \in H^k(\Omega) \; \forall k \in \mathbb{N}$
  $\Rightarrow e_n \in C^\infty(\overline{\Omega})$. ▨

The General Elliptic Case

The above result can be generalised to linear elliptic problems. We have the following abstract result:

- **THM**: Let V, H be two Hilbert spaces, such that H is separable, V dense in H and $V \overset{c}{\hookrightarrow} H$.
  Assume a is a bilinear form in V that is cts, symmetric and weakly coercive in V, i.e. $\exists \lambda_0 \geq 0$, $\alpha > 0$ such that $a(u,u) + \lambda_0 \|u\|^2_H \geq \alpha \|u\|^2_V$, $\forall v \in V$. Then:
  a) There exists a sequence $\{\lambda_n\} \subset (-\lambda_0, \infty)$, with $\lambda_n \to \infty$ as $n \to \infty$ and a sqnce of functions $\{u_n\} \subset V$ such that $a(u_n, v) = \lambda_n (u_n, v)$.
  b) The sequence $\{u_n\}$ is an ONB of H and an orthogonal base of V w.r.t $((u,v))_V = a(u,v) + \lambda_0 (u,v)$

8.1: Variational Formulation of the Heat Eqn

- Let Ω be a bdd Lipschitz domain in $\mathbb{R}^n$. We study the Cauchy-Dirichlet problem of the heat eqn:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,T) \quad (H) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

Here, $f = f(x,t)$ and $u_0 = u_0(x)$ are given, and $T > 0$.
Notation: $u : [0,T] \to X$, where X Banach or Hilbert space.

AIM: Determine the variational formulation and the functional framework of (H).

We proceed as for the Poisson's equation. Let us assume u is a "smooth" soln to (H). We will not assume u is a "strong" soln at this point as in the elliptic case since the notion of "strong" soln to (H) is more delicate. But we'll discuss it later on...

Now we consider a test function $v \in H_0^1(\Omega)$. This choice is motivated by the boundary condition as in the Elliptic case.

Multiplying $(H)_1$ by v and integrating over Ω and using IBP we get:

$$\int_\Omega \partial_t u(t) \cdot v \, dx + \overset{(IBP)}{\int_\Omega \nabla u(t) \cdot \nabla v \, dx} - \int_{\partial\Omega} \nabla u(t) \cdot n \, \overset{=0\,;\, v \in H_0^1(\Omega)}{\cancel{v}} \, d\sigma = \int_\Omega f(t) \cdot v \, dx$$

namely, $$\int_\Omega \partial_t u(t) \cdot v \, dx + \int_\Omega \nabla u(t) \cdot \nabla v \, dx = \int_\Omega f(t) \cdot v \, dx, \quad \forall v \in H_0^1(\Omega). \qquad (WF-P)$$

The second term on the LHS is well-defined if $u \in L^p(0,T;H_0^1(\Omega))$. What is the most natural choice of p?

•) Take $p=2$, so $L^2(0,T;H_0^1(\Omega))$ is a Hilbert space! why not $L^2(0,T;H^1(\Omega))$?

•) Taking $v=u(t)$, we find $\int_\Omega \partial_t u(t)u(t)\,dx + \int_\Omega |\nabla u(t)|^2\,dx = \int_\Omega f(t)u(t)\,dx$, which is equivalent to $\frac{d}{dt}\frac{1}{2}\int_\Omega |u(t)|^2 dx + \|\nabla u(t)\|^2_{L^2(\Omega)} = \int_\Omega f(t)u(t)\,dx$. Integrating this equality in time between 0 and T, we get

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}\,ds = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t (f(s),u(s))_{L^2}\,ds$$

In order to control the last term on RHS, we recall the following tools:

1) Poincare ineq: $\|u\|_{L^2(\Omega)} \leq C_p\|\nabla u\|_{L^2(\Omega)}$ $\forall u \in H_0^1(\Omega)$

2) Young's ineq: $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ $\forall \varepsilon>0$.

Then we obtain the following estimate: $\left|\int_0^t (f(s),u(s))_{L^2}\,ds\right| \leq \int_0^t |(f(s),u(s))_{L^2}|\,ds \leq \int_0^t \|f(s)\|_{L^2(\Omega)}\|u(s)\|_{L^2(\Omega)}$

$$\overset{(1)}{\leq} \int_0^t C_p \|f(s)\|_{L^2(\Omega)}\|\nabla u(s)\|_{L^2(\Omega)}\,ds \underset{\varepsilon=\frac{1}{2}}{\overset{(2)}{\leq}} \int_0^t \frac{1}{2}\|\nabla u(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \frac{1}{2}C_p^2\|f(s)\|^2_{L^2(\Omega)}\,ds$$

Therefore we find $\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2\,ds \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \frac{1}{2}\|\nabla u(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \frac{1}{2}C_p^2\|f(s)\|^2_{L^2(\Omega)}\,ds$

which is equivalent to:

$$\cancel{\tfrac{1}{2}}\|u(t)\|^2_{L^2(\Omega)} + \cancel{\tfrac{1}{2}}\int_0^t \|\nabla u(s)\|^2_{L^2}\,ds \leq \cancel{\tfrac{1}{2}}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \cancel{\tfrac{1}{2}}C_p^2\|f(s)\|^2_{L^2(\Omega)}\,ds$$

We arrive at:

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2}\,ds \leq \|u_0\|^2_{L^2(\Omega)} + C_p^2\int_0^t \|f(s)\|^2_{L^2(\Omega)}\,ds$$

Taking the essential supremum w.r.t t between 0 and T we infer:

$$\underbrace{\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|^2_{L^2(\Omega)} + \int_0^T \|\nabla u(s)\|^2_{L^2(\Omega)}}_{\text{norms of soln. } u} \leq \underbrace{\|u_0\|^2_{L^2(\Omega)} + C_p^2\int_0^T \|f(s)\|^2_{L^2(\Omega)}\,ds}_{\text{norms of data}}$$

$\Rightarrow$ if $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ then we expect the solution u to be s.t.

$$\boxed{u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))}$$

REMARK: We will study another way to estimate $\int_0^T (f(s),u(s))\,ds$ without Poincare's ineq.

$\rightsquigarrow$ Gronwall Lemma.

• What is the function space for $\partial_t u$ $(=\dot{u})$?

As for the ODEs, the regularity of the time derivative is recovered from the equation.

More precisely, assuming $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, we need to understand the meaning of $\int_\Omega \partial_t u(t)\,v\,dx$ in the weak formulation (WF-P):

$$(\partial_t u(t), v) + (\nabla u(t), \nabla v) = (f(t), v) \quad \forall v \in H_0^1(\Omega) \qquad ((\cdot,\cdot) = (\cdot,\cdot)_{L^2})$$

(need $\nabla v \in L^2$ $\Rightarrow v \in H^1$)

This term is now considered as an L^2 inner prod since we assumed u smooth.

If $u \in L^2(0,T;H_0^1(\Omega))$ then the map $v \mapsto A_u(v) = (\nabla u(t), \nabla v)$ defines a functional $A_u \in L^2(0,T;H^{-1}(\Omega))$

Indeed we have $\|A_u\|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \|A_u(s)\|^2_{H^{-1}(\Omega)}\,ds = \int_0^T \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|_{H_0^1}\leq 1}} |(\nabla u(s),\nabla v)|^2\,ds$

$$\leq \int_0^T \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|\leq 1}} \|\nabla u(s)\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)}\,ds \leq \int_0^T \|\nabla u(s)\|^2_{L^2(\Omega)}\,ds = \|u\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

Thus by comparison ($\partial_t u + Au = f$ in H^{-1}) we expect that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ and the first term on LHS of (WF-P) must be interpreted in duality sense, namely $\langle \partial_t u(t), v\rangle_{H^{-1}\times H_0^1}$

In conclusion, the weak formulation of (H) is: given $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, find the solution $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that

1) $\langle \partial_t u(t), v\rangle_{H^{-1}\times H_0^1} + (\nabla u(t), \nabla v) = (f(t), v)$ $\forall v \in H_0^1(\Omega)$ for a.e $t \in [0,T]$.

2) $u(0) = u_0$

• **REMARK:** (1) can be rewritten in another equivalent way. First of all, we claim that
$\langle \partial_t u(t), v\rangle_{H^{-1}\times H_0^1} = \frac{d}{dt}(u(t),v)_{L^2}$ in $\mathcal{D}'(0,T)$
Indeed, $\forall v\in H_0^1(\Omega)$ we have $t\mapsto (u(t),v)_{L^2}$, $t\mapsto \langle\partial_t u(t), v\rangle_{H^{-1}\times H_0^1}$ belong to $L^2(0,T)$
In addition, since $u(t)\in H_0^1(\Omega)$ a.e, $\langle u(t), v\rangle_{H^{-1}\times H_0^1} = (u(t),v)_{L^2}$
Therefore by defn of time derivative, for any $\varphi\in\mathcal{D}(0,T)$ we have

$$\int_0^T \langle\partial_t u(t), v\rangle_{H^{-1}\times H_0^1}\varphi(t)\,dt = -\int_0^T \langle u(t), v\rangle_{H^{-1}\times H_0^1}\varphi'(t)\,dt = -\int_0^T (u(t),v)_{L^2}\varphi'(t)\,dt$$
$$= \int_0^T \frac{d}{dt}(u(t),v)_{L^2}\varphi(t)\,dt$$

$\Rightarrow$ (1) is equivalent to:
$\frac{d}{dt}(u(t),v)_{L^2} + (\nabla u(t),\nabla v)_{L^2} = (f(t),v)_{L^2}$ $\forall v\in H_0^1(\Omega)$ in sense of distribution.

What is the meaning of $u(0)=u_0$? This issue is solved by the following general result:

• **THM:** Let V, H be two Hilbert spaces s.t. H separable, V dense in H, $V\hookrightarrow H$. We define
$H^1(0,T;V,V') = \{u\in L^2(0,T;V) : \partial_t u\in L^2(0,T;V')\}$ endowed with norm

$$\|u\|_{H^1(0,T;V,V')} = \left(\int_0^T \|u(t)\|_V^2\,dt + \int_0^T\|\partial_t u(t)\|_{V'}^2\,dt\right)^{\frac12}.$$ The following properties hold:

1) $H^1(0,T;V,V')$ is a Hilbert space
2) $C^\infty(0,T;V)$ is dense in $H^1(0,T;V,V')$
3) $H^1(0,T;V,V')\hookrightarrow C([0,T];H)$. Moreover, there exists $C=C(T)$ such that
$\|u\|_{C([0,T];H)} \leq C(T)\,\|u\|_{H^1(0,T;V,V')}$
4) $u,v\in H^1(0,T;V,V')$. We have the IBP formula:

$$\int_s^T \langle\partial_t u(t), v(t)\rangle_{V'\times V} + \langle\partial_t v(t), u(t)\rangle_{V'\times V}\,dt = (u(T),v(T))_H - (u(s),v(s))_H \quad \forall\, 0\leq s\leq T$$

In particular if $u=v$ in (4), we find: $\frac{d}{dt}\frac12\|u(t)\|_H^2 = \langle\partial_t u(t), u(t)\rangle_{V'\times V}$ a.e in $[0,T]$

The claim (3) implies $u(0)=u_0$ is well-defined with $H=L^2(\Omega)$, $V=H_0^1(\Omega)$, $V'=H^{-1}(\Omega)$
for heat eqn and $\|u(t)-u_0\|_{L^2(\Omega)}\longrightarrow 0$ if $t\to 0^+$.

OVERVIEW CLASS 7 (03-03-22)

• Comments on weak formulation of heat eqn:
·) Given $u\in L^2(0,T;H_0^1(\Omega))$, we consider $v\mapsto\langle\nabla u(t),\nabla v\rangle_{L^2(\Omega)} = \langle Au(t), v\rangle_{H^{-1}\times H_0^1}$
If u is smooth we can write $(\partial_t u, v) + (\nabla u(t),\nabla v) = (f(t),v)$, $\forall v\in H_0^1(\Omega)$ a.e in $[0,T]$.
$\Rightarrow$ $Au(t)\in H^{-1}(\Omega)$, a.e in $[0,T]$ $\Rightarrow$ $\partial_t u(t)\in H^{-1}(\Omega)$ a.e. in $[0,T]$. Therefore, the correct weak formulation is $\langle\partial_t u(t), v\rangle_{H^{-1}\times H_0^1} + (\nabla u(t),\nabla v) = (f(t),v)$, $\forall v\in H_0^1(\Omega)$ a.e. in $[0,T]$
Another way to justify it is the following:
$u\in L^2(0,T;H_0^1(\Omega))$, $\partial_t u(t) = \Delta u(t) + f(t) = \operatorname{div}(\underbrace{\nabla u(t)}_{\in L^2(0,T;L^2(\Omega))}) + f(t)$ $\overset{cw2}{\Rightarrow}$ $\partial_t u(t)\in H^{-1}(\Omega)$ for a.e. $t\in[0,T]$

8.2 : WELL-POSEDNESS OF THE HEAT EQN

- Let Ω be a bdd Lipschitz domain in $\mathbb{R}^n$. We consider the Cauchy-Dirichlet problem for the heat eqn:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,T) \quad (H) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

where $f = f(x,t)$, $u(0) = u_0$ given.

Weak formulation: A function $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ is a weak solution to (H) if:

1) $\langle \partial_t u(t), v \rangle_{H^{-1} \times H_0^1} + (\nabla u(t), \nabla v) = (f(t), v)$, $\forall v \in H_0^1(\Omega)$ for a.e. $t \in [0,T]$
2) $u(0) = u_0$

Remark: The weak problem (1) is equivalent to the following formulations:

a) For any $v \in L^2(0,T; H_0^1(\Omega))$, $\int_0^T \langle \partial_t u(t), v(t) \rangle_{H^{-1} \times H_0^1} dt + \int_0^T (\nabla u(t), \nabla v(t))\, dt = \int_0^T (f(t), v(t))\, dt$
b) For any $v \in L^2(0,T; H_0^1(\Omega))$,

$$\langle \partial_t u(t), v(t) \rangle_{H^{-1} \times H_0^1} + (\nabla u(t), \nabla v(t)) = (f(t), v(t)) \text{ for a.e. } t \in [0,T]$$

- THM: Let $f \in L^2(0,T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ to (H).

Proof is based on Faedo-Galerkin method which consists of the following steps:

1) Find a sequence of functions $\{v_n\}$ which is an orthonormal basis in $L^2(\Omega)$ and an orthogonal base in $H_0^1(\Omega)$ $\rightsquigarrow$ $u_0 = \sum_{n=1}^{\infty} (u_0, v_n) v_n$ (This converges in $L^2(\Omega)$)

2) We construct the finite-dim space $V_n = \text{span}\{v_1, \dots, v_n\}$. We observe that $V_n \subset V_{n+1}$, $\overline{\bigcup V_n}^{H^1} = H_0^1(\Omega)$

Fix $m \in \mathbb{N}$, we look for a solution $u_m(t) = \sum_{k=1}^{m} c_k(t) v_k$ of the approximated problem.

$$\begin{cases} (\partial_t u_m(t), v_s) + (\nabla u_m(t), v_s) = (f(t), v_s) & \text{for any } s = 1, \dots, m, \text{ for a.e. } t \in [0,T] \\ u_m(0) = g_m & \text{where } g_m = \sum_{k=1}^{m} (u_0, v_k) v_k. \end{cases}$$

- REMARKS: 1) In the approximated problem the test function belongs to V_m (not in full space $H_0^1(\Omega)$)
2) $\partial_t u_m(t) = \sum_{k=1}^{m} \dot{c}_k(t) v_k$ $\Rightarrow$ For any t, $\partial_t u_m(t) \in V_n$ $\Rightarrow$ $\langle \partial_t u_m(t), v_s \rangle_{H^{-1} \times H_0^1} = (\partial_t u_m(t), v_s)_{L^2}$

The function u_m is called Galerkin approximation.

3) We perform Energy Estimates: the sequence of approximated solutions (Galerkin approximations) is bounded in $H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$.

$\Rightarrow$ (compactness) there exist a subsqnce $\{u_{m_j}\}$ which "converges" to $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$

4) We prove that u is the soln to the original weak formulation
5) We show u is unique.

9.1: GALERKIN METHOD: EXISTENCE OF GALERKIN APPROXIMATION

- AIM. Given $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$, find a function $u \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ such that

  1) $\langle \partial_t u(t), v\rangle_* + (\nabla u(t), \nabla v) = (f(t), v)$, $\forall v \in H_0^1(\Omega)$

  2) $u(0) = u_0$

  Notation: $\langle \cdot,\cdot \rangle_{H^{-1}\times H_0^1} = \langle \cdot,\cdot\rangle_*$

- REMARK FROM FUNCTIONAL ANALYSIS: $V \subset H \subset V'$

  Let H be a separable Hilbert space endowed with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Assume that V is a Hilbert space endowed with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$ such that:

  1) V is a linear dense subspace of H

  2) $V \hookrightarrow H$: $\exists c>0$: $\|u\|_H \le c\|u\|_V$, $\forall u \in V$.

  By Riesz Thm, we identify H with its dual H'. Thus we have $V \subset H \simeq H'$. Now we consider the canonical map $T: H \to V'$ that is simply the restriction to V of functionals defined on H. This is, for any $u \in H$ $\quad \langle Tu, v\rangle_{V'\times V} = (u,v)_H$, $\forall v \in V$.

  •) Since $|(u,v)_H| \le \|u\|_H \|v\|_H \le c\|u\|_H\|v\|_V$, it follows $Tu \in V'$ and $\|Tu\|_{V'} \le c\|u\|_H$.

  •) If $Tu = 0 \Rightarrow (u,v)_H = 0$ $\forall v \in H$. Since V dense in H we have $(u,v)_H = 0$ $\forall v \in H \Rightarrow u = 0 \Rightarrow T$ injective.

  We can write that $V \subset H \simeq H' \subset V'$ where the injections are continuous and dense. H is called pivot space.
  As a consequence, if $u \in H$, $\langle u, v\rangle_{V'\times V} = (u,v)_H$ $\forall v \in V$.

GALERKIN METHOD

1) Functional setting: $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$

   There exists a sequence of eigenvalues $\{\lambda_n\} \subset \mathbb{R}$ s.t. $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \to \infty$ and a sequence of eigenfunctions $\{v_n\} \subset H_0^1(\Omega)$ such that:

   1) $\{v_n\}$ is an ONB of $L^2(\Omega)$ $\quad (v_n, v_m)_{L^2} = \delta_{nm}$, $\forall n,m \in \mathbb{N}$

   2) $(\nabla v_n, \nabla w) = \lambda_n (v_n, w)$ $\forall w \in H_0^1(\Omega)$

   3) $\{v_n\}$ is an orthogonal base in $H_0^1(\Omega)$, i.e. $(\nabla v_n, \nabla v_m) + (v_n, v_m) = (\lambda_n + 1)\delta_{nm}$ $\forall n,m \in \mathbb{N}$

   (1) $\Rightarrow$ $u_0 = \sum_{k=1}^{\infty}(u_0, v_k)v_k$, $(u_0, v_k) = \int_\Omega u_0 \cdot v_k \, dx$

2) Approximation of the forcing term f

   Since $C([0,T];L^2(\Omega))$ is dense in $L^2(0,T;L^2(\Omega))$, it follows there exists a sequence $\{f_n\} \subseteq C([0,T];L^2(\Omega))$ such that $f_n \to f$ in $L^2(0,T;L^2(\Omega))$.

3) Existence of the Galerkin approximation

   $\forall m \in \mathbb{N}$, we introduce the finite-dim space $V_M = \text{span}\{v_1, \dots, v_m\}$. Notice that

   •) $V_m \subset V_{m+1}$

   •) $\overline{\cup V_m}^{H^1(\Omega)} = H_0^1(\Omega)$

   For any fixed $m \in \mathbb{N}$, we look for: $u_m(t) = \sum_{k=1}^{m} c_k^m(t) v_k$, $G_m = \sum_{k=1}^{m}(u_0, v_k)v_k$ such that:

   $$\left.\begin{aligned} &1)\ (\partial_t u_m(t), v) + (\nabla u_m(t), \nabla v) = (f_m(t), v), \quad \forall v \in V_m \text{ for a.e. } t \in [0,T]. \\ &2)\ u_m(0) = G_m. \end{aligned}\right\} (AP)$$

   This is called the approximated problem.

   •) since $\partial_t u_m(t) = \sum_{k=1}^{m} \dot{c}_k^m(t) v_k$, it follows $\partial_t u_m(t) \in V_m$. Therefore we can write $(\partial_t u_m(t), v)$ instead of $\langle \partial_t u_m(t), v\rangle_*$.

   •) by defn. of V_m, (1) in (AP) is equivalent to $(\partial_t u_m(t), v_s) + (\nabla u_m(t), \nabla v_s) = (f_m(t), v_s)$ $\forall s = 1, \dots, m$.

   By orthogonality we observe that $(\partial_t u_m(t), v_s) = \left(\sum_{k=1}^{m} \dot{c}_k^m(t) v_k, v_s\right) = \sum_{k=1}^{m}(\dot{c}_k^m(t) v_k, v_s) = \dot{c}_s^m(t)$
   and $(\nabla u_m(t), \nabla v_s) = \left(\sum_{k=1}^{m} c_k^m(t) \nabla v_k, \nabla v_s\right) = \sum_{k=1}^{m} c_k^m(t)(\nabla v_k, \nabla v_s) = c_s^m(t)\lambda_s(v_s, v_s) = \lambda_s c_s^m(t)$.
   Thus (1) in (AP) is equivalent to: $\dot{c}_s^m(t) + \lambda_s c_s^m(t) = (f_m(t), v_s)$, $\forall s = 1, \dots, m$.

It is easily seen by exercise that $t \mapsto (f_m(t), v_s) \in C([0,T])$ —

Moreover, since we want $u_m(0) = g_m$, meaning $\sum_{k=1}^{m} c_k^m(0) v_k = \sum_{k=1}^{m} (u_0, v_k) v_k$, we have $c_s^m(0) = (u_0, v_s)$ $\forall s = 1, \dots, m$

Therefore, we found a system of m decoupled linear ODEs:

$$\begin{cases} \dot{c}_s^m(t) + \lambda_s c_s^m(t) = (f_m(t), v_s) & , s = 1, \dots, m \\ c_s^m(0) = (u_0, v_s) & \forall s = 1, \dots, m \end{cases}$$

Notice that setting $C^m = (c_1^m, \dots, c_m^m)$, $F^m = ((f_m, v_1), \dots, (f_m, v_m))$, $A = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$, $G^m = ((u_0, v_1), \dots, (u_0, v_m))$

This is equivalent to the system:

$$\begin{cases} \dot{C}^m(t) + A C^m(t) = F^m(t) \\ C^m(0) = G^m \end{cases}$$

$$\begin{pmatrix} \dot{c}_1^m(t) \\ \vdots \\ \dot{c}_m^m(t) \end{pmatrix} + \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_m \end{pmatrix} \begin{pmatrix} c_1^m(t) \\ \vdots \\ c_m^m(t) \end{pmatrix} = \begin{pmatrix} (f_m, v_1) \\ \vdots \\ (f_m, v_m) \end{pmatrix}$$

By the classical theory of ODEs, for any $s = 1, \dots, m$ there exists a unique solution $c_s^m \in C^1([0,T])$ defined by $c_s^m(t) = (u_0, v_s) e^{-\lambda_s t} + e^{-\lambda_s t} \int_0^t (f_m(\tau), v_s) e^{-\lambda_s \tau} d\tau$

As a consequence, for any $m \in \mathbb{N}$ there exists $u_m(t) = \sum_{k=1}^{m} c_k^m(t) v_k \in C^1([0,T]; V_m)$ which solves

1) $(\partial_t u_m(t), v) + (\nabla u_m(t), \nabla v) = (f_m(t), v)$ $\forall v \in V_m$ for a.e. $t \in [0,T]$
2) $u_m(0) = g_m = \sum_{k=1}^{m} (u_0, v_k) v_k$

The sequence $\{u_m\}$ is called Faedo-Galerkin approximation

9.2: GALERKIN METHOD: ENERGY ESTIMATES

- In the previous step, we proved the existence of the Galerkin approximation: for any $m \in \mathbb{N}$ there exists $u_m \in C^1([0,T]; V_m)$ s.t.:

  1) $(\partial_t u_m(t), v) + (\nabla u_m(t), \nabla v) = (f_m(t), v)$ $\forall v \in V_m$ for any $t \in [0,T]$
  2) $u_m(0) = \sum_{k=1}^{m} (u_0, v_k) v_k$

  where $f_m \in C([0,T]; L^2(\Omega))$, $f_m \longrightarrow f$ in $L^2(0,T; L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $V_m = \mathrm{span}\{v_1, \dots, v_m\}$.

- AIM: Determine estimates of the norm of u_m in the space $H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ which are independent of the parameter m, namely we want to show that there exists $C > 0$ which are indep. of m, s.t.: $\|u_m\|_{H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))} \leq C$ $\forall m \in \mathbb{N}$

Why? $H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ is a separable Hilbert space $\Rightarrow$ bdd sequences are weakly compact, i.e. $\exists \{u_{m_j}\} \subset H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ such that $u_{m_j} \longrightarrow u$ weakly in $H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$

solution candidate

We will proceed with the "energy estimates" as we control $\int_\Omega \frac{1}{2} u_m^2(t)\, dx$ which is the energy of the heat equation. In order to do so, we need to introduce the following fundamental tool:

- **LEMMA (GRONWALL)**: Let F, G, H be cts functions in $[0,T]$ such that H is non-negative and G non-decreasing. If they satisfy:

$$F(t) \leq G(t) + \int_0^T F(s) H(s)\, ds, \quad \forall t \in [0,T] \quad \text{then} \quad F(t) \leq G(t) e^{\int_0^t H(s) ds} \quad \forall t \in [0,T]$$

  - PF: Define $R(s) = \int_0^s F(\tau) H(\tau) d\tau$. For any $s \in [0,T]$ we can compute $R'(s) = F(s) H(s) \leq G(s) H(s) + H(s) \int_0^s F(\tau) H(\tau) d\tau = G(s) H(s) + H(s) R(s)$. Multiplying both sides by $e^{-\int_0^s H(\tau) d\tau}$ we have:

$$R'(s) e^{-\int_0^s H(\tau) d\tau} \leq G(s) H(s) e^{-\int_0^s H(\tau) d\tau} + H(s) R(s) e^{-\int_0^s H(\tau) d\tau}$$

We observe that $R'(s) e^{-\int_0^s H(\tau) d\tau} = \frac{d}{ds}\left(R(s) e^{-\int_0^s H(\tau) d\tau}\right) + R(s) H(s) e^{-\int_0^s H(s) ds}$

This implies $\frac{d}{ds}\left(R(s) e^{-\int_0^s H(\tau) d\tau}\right) \leq G(s) H(s) e^{-\int_0^s H(\tau) d\tau}$. Integrating between 0 and t,

$R(t) e^{-\int_0^t H(\tau) d\tau} - \cancel{R(0)}^{0} \leq \int_0^t G(s) H(s) e^{-\int_0^s H(\tau) d\tau} ds$. We can rewrite it as:

$\int_0^t F(s) H(s)\, ds \leq \int_0^t G(s) H(s) e^{\int_s^t H(\tau) d\tau} ds$. Since G non-decreasing, $H \geq 0$ we deduce that

$$\int_0^t F(s) H(s)\, ds \leq G(t) \int_0^t \underbrace{H(s) e^{\int_s^t H(\tau) d\tau}}_{-\frac{d}{ds}\left(e^{\int_s^t H(\tau) d\tau}\right)} ds = -G(t) \int_0^t \frac{d}{ds}\left(e^{\int_s^t H(\tau) d\tau}\right) ds = -G(t)\left[1 - e^{\int_0^t H(\tau) d\tau}\right] = G(t) e^{\int_0^t H(\tau) d\tau} - G(t)$$

Therefore we conclude that $F(t) \le G(t) + G(t)e^{\int_0^t H(\tau)d\tau} - G(t) = G(t)e^{\int_0^t H(\tau)d\tau}$, $\forall t\in[0,T]$. ■

ENERGY ESTIMATES

- $(\partial_t u_m(t), v_s) + (\nabla u_m(t), \nabla v_s) = (f_m(t), v_s)$ (1) $\forall s = 1,\dots,m$, $\forall t\in[0,T]$

We recall that $u_m(t) = \sum_{s=1}^m c_s^m(t) v_s \in V_m$. Multiplying (1) by $c_s^m(t)$ for any $s=1,\dots,m$, we have:
$(\partial_t u_m(t), c_s^m(t) v_s) + (\nabla u_m(t), c_s^m(t)\nabla v_s) = (f_m(t), c_s^m(t) v_s)$. Summing over s, we find
$(\partial_t u_m(t), \sum_{s=1}^m c_s^m(t) v_s) + (\nabla u_m(t), \sum_{s=1}^m c_s^m(t)\nabla v_s) = (f_m(t), \sum_{s=1}^m c_s^m(t) v_s)$, namely
$(\partial_t u_m(t), u_m(t)) + (\nabla u_m(t), \nabla u_m(t)) = (f_m(t), u_m(t))$

<u>REMARK</u>: notice the analogy with the Lax-Milgram thm; indeed $\forall t$ fixed $u_m(t)\in V_m$ which is the same space of the test function. For instance, $u_m^3(t) \notin V_m$!

Observing that $(\partial_t u_m(t), u_m(t)) = \int_\Omega \partial_t u_m(t) u_m(t)\,dx = \int_\Omega \partial_t\left(\frac12 u_m^2(t)\right)dx = \frac{d}{dt}\int_\Omega \frac12 u_m^2(t)\,dx$, we find
$\frac12\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} = (f_m(t), u_m(t))$. By Young's inequality ($ab \le \frac12 a^2 + \frac12 b^2$), we have
$\frac12\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} \overset{CS+Y}{\le} \frac12\|f_m(t)\|^2_{L^2(\Omega)} + \frac12\|u_m(t)\|^2_{L^2(\Omega)}$. Integrating from 0 to T,

$$\frac12\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_m(t)\|^2_{L^2(\Omega)}dt \le \frac12\int_0^t \|f_m(s)\|^2_{L^2(\Omega)}ds + \frac12\int_0^t\|u_m(s)\|^2_{L^2(\Omega)}ds$$

Recalling that $u_m(0) = \sum_{k=1}^m (u_0, v_k)v_k$, we have by orthogonality $\|u_m(0)\|^2_{L^2(\Omega)} = \sum_{k=1}^m |(u_0,v_k)|^2 \le \sum_{k=1}^\infty |(u_0,v_k)|^2 = \|u_0\|^2_{L^2}$
Moreover since $f_m \to f$ in $L^2(0,T;L^2(\Omega))$, w.l.o.g we may assume that $\|f_m\|_{L^2(0,T;L^2(\Omega))} \le 2\|f\|_{L^2(0,T;L^2(\Omega))}$

Therefore we conclude $\overset{\forall t\in[0,T]}{\frac12}\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t\|\nabla u_m(s)\|^2_{L^2(\Omega)}ds \le \frac12\|u_0\|^2_{L^2(\Omega)} + \frac12\int_0^T 2\|f(s)\|^2_{L^2(\Omega)}ds + \underbrace{\int_0^t \frac12\|u_m(s)\|^2_{L^2(\Omega)}ds}_{\text{At (*) we bound this term}}$ (2)
Setting $F(t) = \frac12\|u_m(t)\|^2_{L^2(\Omega)}$, $G(t) = \frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds$, $H(t) = 1$,
An application of the Gronwall Lemma gives the following estimate:

$$\frac12\|u_m(t)\|^2_{L^2(\Omega)} \le \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)e^{\int_0^t 1\,ds} = \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)e^t \quad (3)$$

By inserting (3) into (2) we also find: (*)

$$\begin{aligned}\frac12\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t\|\nabla u_m(s)\|^2_{L^2(\Omega)}ds &\le \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)\Big(1 + \overbrace{\int_0^t e^s ds}^{e^t-1}\Big)\\ &= \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)e^t, \quad \forall t\in[0,T] \quad (4)\end{aligned}$$

Important remark: The RHS in (4) is indep. of m!
We now proceed with the estimate of $\partial_t u_m(t)\in H^{-1}(\Omega)$. This is done by comparison and by exploiting orthogonality of the eigenfunctions in $H_0^1(\Omega)$.

Let $v\in H_0^1(\Omega)$, we write $v = w + z$ where $w\in V_m$, $z\in V_m^\perp$ ($\{v_k\}$ is an orthogonal base of $H_0^1(\Omega)$)
In addition, we have $\|w\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)}$.
We compute $\langle\partial_t u_m(t), v\rangle_* = (\partial_t u_m(t), v)$ $(\partial_t u_m(t)\in L^2(\Omega))$ $= (\partial_t u_m(t), w) + \cancel{(\partial_t u_m(t), z)}$ $\in V_m$ $=0$ $\in V_m^\perp$
(AP) $= -(\nabla u_m(t), \nabla w) + (f_m(t), w)$. Then by C.S,

$$\begin{aligned}|\langle\partial_t u_m(t), v\rangle_*| &\le \|\nabla u_m(t)\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} + \|f_m(t)\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \le (\|\nabla u_m(t)\|_{L^2} + \|f_m(t)\|_{L^2})\|w\|_{H^1}\\ &\le (\|\nabla u_m(t)\|_{L^2} + \|f_m(t)\|_{L^2})\|v\|_{H^1(\Omega)}\end{aligned}$$

By defn of norm in $H^{-1}(\Omega)$, we deduce $\|\partial_t u_m(t)\|_{H^{-1}(\Omega)} \le \|\nabla u_m(t)\|_{L^2} + \|f_m(t)\|_{L^2}$
Taking the square & integrating between 0 and t, and using the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$ we arrive at

$$\int_0^t\|\partial_t u_m(s)\|^2_{H^{-1}(\Omega)}ds \le 2\int_0^t\|\nabla u_m(s)\|^2_{L^2(\Omega)}ds + 2\int_0^t\|f_m(s)\|^2_{L^2(\Omega)}ds$$

Exploiting (4) we conclude $\int_0^t\|\partial_t u_m(s)\|^2_{H^{-1}(\Omega)}ds \le 2e^t\left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right) + 4\int_0^T\|f(s)\|^2_{L^2(\Omega)}ds$ $\forall t\in[0,T]$ (5)
Remark: RHS of (5) is independent of m!

Overview Class Week 8

Robin problem with negative boundary coefficient

- Let Ω be a bdd, smooth domain in $\mathbb{R}^n$. We consider the eigenvalue problem for the Laplace operator with Robin boundary condition: $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial n} - \alpha u = 0 & \text{on } \partial\Omega \end{cases}$ where $\alpha > 0$.

First of all we introduce the weak formulation. Taking $v \in H^1(\Omega)$, multiplying the eqn by v and IBP:

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \underbrace{\nabla u \cdot n}_{\alpha u} \, v \, d\sigma = \lambda \int_\Omega u v \, dx \implies \int_\Omega \nabla u \cdot \nabla v \, dx - \alpha \int_{\partial\Omega} uv \, d\sigma = \lambda \int_\Omega uv \, dx \quad \forall v \in H^1(\Omega)$$

meant in sense of trace; γ_u, γ_v

REMARK: Keep in mind that u, v on $\partial\Omega$ are meant in sense of traces so $\int_{\partial\Omega} uv \, d\sigma = \int_{\partial\Omega} \gamma u \cdot \gamma v \, d\sigma$

The weak formulation can be rewritten as the eigenvalue problem: find $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$ such that
$a(u,v) = \lambda (u,v) \quad \forall v \in H^1(\Omega)$ where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx - \alpha \int_{\partial\Omega} uv \, d\sigma$

$a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$

- a cts: $|a(u,v)| \leq |\int_\Omega \nabla u \cdot \nabla v \, dx| + \alpha |\int_{\partial\Omega} u \cdot v \, d\sigma| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \alpha \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}$
  $\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \alpha C_\gamma^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$
  $\leq (1 + \alpha C_\gamma^2) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$
  (TRACE THM: $\|u\|_{L^2(\partial\Omega)} \leq C_\gamma \|u\|_{H^1(\Omega)}$)

- a weakly coercive? weakly coercive: $a(u,u) + \lambda \|u\|_{L^2}^2 \geq \lambda_0 \|u\|_{H^1}^2$

Robin condition w/ +ve boundary coeff:
$a(u,v) = \int_\Omega \nabla u \nabla v \, dx + \alpha \int_{\partial\Omega} uv \, d\sigma$
$\hookrightarrow a(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 + \underbrace{\alpha \int_{\partial\Omega} u^2 d\sigma}_{\geq 0}$
$\geq \|\nabla u\|_{L^2(\Omega)}^2$
$\implies a(u,u) + \lambda \|u\|_{L^2(\Omega)}^2 \geq \min\{1, \lambda\} (\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)$
$\therefore$ a weakly coercive for any $\lambda > 0$ $= \min\{1, \lambda\} \|u\|_{H^1(\Omega)}^2$

We can actually improve the above inequality of $a(u,u)$ from below using the following generalised Poincare inequality:
$\|u\|_{L^2(\Omega)} \leq \tilde{C}_P (\|\nabla u\|_{L^2(\Omega)} + \|\gamma u\|_{L^2(\partial\Omega)}), \quad u \in H^1(\Omega)$.

Thanks to this, we have
$a(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 + \alpha \|u\|_{L^2(\partial\Omega)}^2 = \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \alpha \|u\|_{L^2(\partial\Omega)}^2$ ("γu")
$\geq \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \min\{\frac{1}{2}, \alpha\} (\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2)$
$\geq \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\min\{\frac{1}{2}, \alpha\} (\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)})^2$ $\quad (a+b)^2 \leq 2a^2 + 2b^2$
$\geq \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\min\{\frac{1}{2}, \alpha\} \frac{1}{\tilde{C}_P^2} \|u\|_{L^2(\Omega)}^2$
$\geq \min\{\frac{1}{2}, \frac{1}{2\tilde{C}_P^2} \min\{\frac{1}{2}, \alpha\}\} \|u\|_{H^1(\Omega)}$ $\implies$ a coercive in $H^1(\Omega) \times H^1(\Omega)$!

We consider $a(u,u) = \int_\Omega \nabla u \cdot \nabla u \, dx - \alpha \int_{\partial\Omega} u^2 \, d\sigma$

1) Find a condition on α such that a is weakly coercive. weakly coercive: $a(u,u) + \lambda \|u\|_{L^2}^2 \geq \lambda_0 \|u\|_{H^1}^2$

Thanks to Trace thm, $\|u\|_{L^2(\partial\Omega)}^2 \leq C_\gamma \|u\|_{H^1(\Omega)}^2 = C_\gamma^2 \|\nabla u\|_{L^2(\Omega)}^2 + C_\gamma^2 \|u\|_{L^2(\Omega)}^2$. Then,
$a(u,u) \geq \|\nabla u\|_{L^2(\Omega)}^2 - \alpha C_\gamma^2 \|\nabla u\|_{L^2(\Omega)}^2 - \alpha C_\gamma^2 \|u\|_{L^2(\Omega)}^2 = \underbrace{(1 - \alpha C_\gamma^2)}_{>0} \|\nabla u\|_{L^2(\Omega)}^2 - \alpha C_\gamma^2 \|u\|_{L^2(\Omega)}^2$
If $1 - \alpha C_\gamma^2 > 0$, namely $\alpha < \frac{1}{C_\gamma^2}$, we obtain $a(u,u) \geq \omega \|\nabla u\|_{L^2(\Omega)}^2 - \alpha C_\gamma^2 \|u\|_{L^2(\Omega)}^2$
where $\omega = 1 - \alpha C_\gamma^2 > 0$. Therefore, we conclude
$a(u,u) + \lambda \|u\|_{L^2(\Omega)}^2 \geq \omega \|\nabla u\|_{L^2(\Omega)}^2 + (\lambda - \alpha C_\gamma^2) \|u\|_{L^2(\Omega)}^2 \geq \min\{\omega, \lambda - \alpha C_\gamma^2\} \|u\|_{H^1(\Omega)}^2$.
$\implies$ a is weakly coercive if $\alpha < \frac{1}{C_\gamma^2}$.

a weakly coercive if $\exists \lambda \geq 0$, $\lambda_0 > 0$ s.t. $a(u,u) + \lambda \|u\|_{L^2(\Omega)}^2 \geq \|u\|_{H^1(\Omega)}^2$

2) Question: is it possible to show a is weakly coercive without a condition on α?
Notice that we used that $\|u\|_{L^2(\partial\Omega)} \leq C_P \|u\|_{H^1(\Omega)}$ which is not sharp!
Indeed by Trace Thm we know that $\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \tilde{C}_\gamma \|u\|_{H^1(\Omega)}$
There might be a way to pay less than the full norm in $H^1(\Omega)$ to control $\|u\|_{L^2(\partial\Omega)}$.

"INTERPOLATION INEQUALITY"
LEMMA: there exists a constant $\overline{C}_\gamma$ s.t. $\|u\|_{L^2(\partial\Omega)} \leq \overline{C}_\gamma \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{H^1(\Omega)}^{\frac{1}{2}}$, $\forall u \in H^1(\Omega)$

$(a+b)^{\frac12} \le a^{\frac12}+b^{\frac12}$ for $a,b>0$

$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}$

Let us first use the above inequality to study the weak coercivity of a.

$\|u\|^2_{L^2(\partial\Omega)} \le \bar{C}_\gamma^2 \|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} = \bar{C}_\gamma^2 \|u\|_{L^2(\Omega)} \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac12} \le \bar{C}_\gamma^2 \|u\|_{L^2(\Omega)}\left(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right)$

$\le \bar{C}_\gamma^2 \|u\|^2_{L^2(\Omega)} + \bar{C}_\gamma^2 \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}$.

Now we deduce $a(u,u) = \|\nabla u\|^2_{L^2(\Omega)} - \alpha\|u\|^2_{L^2(\partial\Omega)} \ge \|\nabla u\|^2_{L^2(\Omega)} - \alpha\bar{C}_\gamma^2\|u\|^2_{L^2(\Omega)} - \underbrace{\alpha\bar{C}_\gamma^2\|u\|_{L^2}}_{b}\underbrace{\|\nabla u\|_{L^2}}_{a}$ $\quad \varepsilon = \frac12$

$\therefore\ a(u,u) \ge \|\nabla u\|^2_{L^2(\Omega)} - \alpha\bar{C}_\gamma^2\|u\|^2_{L^2(\Omega)} - \frac12\|\nabla u\|^2_{L^2(\Omega)} - \frac12\alpha^2\bar{C}_\gamma^4\|u\|^2_{L^2(\Omega)}$

$= \frac12\|\nabla u\|^2_{L^2(\Omega)} - \left(\alpha\bar{C}_\gamma^2 + \frac12\alpha^2\bar{C}_\gamma^4\right)\|u\|^2_{L^2(\Omega)}$

$\Rightarrow$ Fix $\lambda = \frac12 + \alpha\bar{C}_\gamma^2 + \frac12\alpha^2\bar{C}_\gamma^4$, we conclude $a(u,u) + \lambda\|u\|^2_{L^2(\Omega)} \ge \frac12\|u\|^2_{H^1(\Omega)}$

$\Rightarrow$ a weakly coercive for $\alpha>0$.

Proof of Lemma: Let $\Omega = \mathbb{R}^n_+$. We consider $u\in\mathcal{D}(\overline{\mathbb{R}^n_+})$ ($\mathcal{D}(\overline{\mathbb{R}^n_+})$ dense in $H^1(\Omega)$)

Notation: $(x_1,\dots,x_{n-1},x_n) = (x',x_n)$.

$|u(x',0)|^2 = -\int_0^\infty \frac{\partial}{\partial x_n}(|u(x',x_n)|^2)\,dx_n = -2\int_0^\infty u(x',x_n)\frac{\partial u}{\partial x_n}(x',x_n)\,dx_n$

$\overset{C.S}{\le} 2\left(\int_0^\infty u^2(x',x_n)dx_n\right)^{\frac12}\left(\int_0^\infty\left|\frac{\partial u}{\partial x_n}(x',x_n)\right|^2dx_n\right)^{\frac12}$.

Then we have $\|u\|^2_{L^2(\partial\Omega)} = \int_{\mathbb{R}^{n-1}}|u(x',0)|^2dx' \le 2\int_{\mathbb{R}^{n-1}}\underbrace{\left(\int_0^\infty u^2(x',x_n)dx_n\right)^{\frac12}}_{f(x')}\underbrace{\left(\int_0^\infty\left|\frac{\partial u}{\partial x_n}(x',x_n)\right|dx_n\right)^{\frac12}}_{g(x')}$

$\overset{C.S}{\le} 2\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty u^2(x',x_n)\,dx_n dx'\right)^{\frac12}\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty\left|\frac{\partial u}{\partial x_n}(x',x_n)\right|^2dx_n dx'\right)^{\frac12}$

$= 2\|u\|_{L^2(\mathbb{R}^n_+)}\left\|\frac{\partial u}{\partial x_n}\right\|_{L^2(\mathbb{R}^n_+)} \le 2\|u\|_{L^2(\mathbb{R}^n_+)}\|\nabla u\|_{L^2(\mathbb{R}^n_+)}$

$\Rightarrow \|u\|_{L^2(\mathbb{R}^{n-1})} \le \sqrt2\,\|u\|^{\frac12}_{L^2(\mathbb{R}^n_+)}\|\nabla u\|^{\frac12}_{L^2(\mathbb{R}^n_+)}$.

So by density, $\|u\|_{L^2(\mathbb{R}^{n-1})} \le \sqrt2\,\|u\|^{\frac12}_{L^2(\mathbb{R}^n_+)}\|\nabla u\|^{\frac12}_{L^2(\mathbb{R}^n_+)}$, $\forall u\in H^1(\mathbb{R}^n_+)$

As for the trace theorem we can use the extension lemma to prove that $\|u\|_{L^2(\partial\Omega)} \le C\|u\|^{1/2}_{L^2(\partial\Omega)}\|u\|^{1/2}_{H^1(\Omega)}$, $\forall u\in H^1(\Omega)$ for any bdd Lipschitz domain in $\mathbb{R}^n$. ∎

Interpolation inequality: If $X\subset Y\subset Z$, then one might study if $\|u\|_Y \le C\|u\|_X^\beta\|u\|_Z^{1-\beta}$ $\forall u\in X$, $\beta\in(0,1)$

A very simple but useful example: let $u\in H^2(\Omega)\cap H^1_0(\Omega)$. We have

$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega\nabla u\cdot\nabla u\,dx \overset{IBP}{=} -\int_\Omega\Delta u\cdot u\,dx + \int_{\partial\Omega}\nabla u\cdot n\,\underbrace{u}_{=0;\ (u=0 \text{ on } \partial\Omega)}\,d\sigma \le \|\Delta u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le \|u\|_{H^2(\Omega)}\|u\|_{L^2(\Omega)}$. We conclude

$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{H^2(\Omega)} + \|u\|_{L^2(\Omega)} \le 2\|u\|_{H^2(\Omega)}\|u\|_{L^2(\Omega)}$ $(H^2(\Omega)\hookrightarrow L^2(\Omega))$

$\Rightarrow \|u\|_{H^1(\Omega)} \le \sqrt2\,\|u\|^{\frac12}_{H^2(\Omega)}\|u\|^{\frac12}_{L^2(\Omega)}$ $\forall u\in H^2(\Omega)\cap H^1_0(\Omega)$

10.1: GALERKIN METHOD: PASSAGE TO THE LIMIT

- In the previous steps (existence of the Galerkin approximation and energy estimates), we showed the existence of the Galerkin approximation: for any $n\in\mathbb{N}$, there exists $u_m\in C^1([0,T];V_m)$ such that:

  1) $(\partial_t u_m(t),v) + (\nabla u_m(t),\nabla v) = (f_m(t),v)$ $\forall v\in V_m$ for any $t\in[0,T]$

  2) $u_m(0) = \sum_{k=1}^m (u_0,v_k)v_k$

  where $f_m\in C([0,T];L^2(\Omega))$, $f_m\to f$ in $L^2(0,T;L^2(\Omega))$, $u_0\in L^2(\Omega)$, $V_m = \operatorname{span}\{v_1,\dots,v_m\}$, $\overline{\bigcup V_m}^{H^1(\Omega)} = H^1_0(\Omega)$

  Moreover for any $m\in\mathbb{N}$ the Galerkin solution satisfies:

  E1) $\frac12\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}ds \le \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)e^T$, $\forall t\in[0,T]$

  E2) $\int_0^t\|\partial_t u_m(s)\|^2_{H^{-1}(\Omega)}ds \le 2e^t\left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right) + 4\int_0^t\|f(s)\|^2_{L^2(\Omega)}ds$, $\forall t\in[0,T]$

- As a consequence, we infer that

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|u_m(t)\|_{L^2(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + 2\int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)^{\frac12}e^{\frac T2} \quad (1)$$

$$\left(\int_0^T\|\nabla u_m(s)\|^2_{L^2(\Omega)}ds\right)^{\frac12} \le \left(\frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)^{\frac12}e^{\frac T2} \quad (2)$$

$$\left(\int_0^T\|\partial_t u_m(s)\|^2_{H^{-1}(\Omega)}ds\right)^{\frac12} \le 2e^{\frac T2}\left(\frac12\|u_0\|^2_{L^2(\Omega)} + 2\int_0^T\|f(s)\|^2_{L^2(\Omega)}ds\right)^{\frac12} \quad (3)$$

Therefore we deduce that $\{u_m\}$ is bdd in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$

$\{\partial_t u_m\}$ is bdd in $L^2(0,T;H^{-1}(\Omega))$

We can conclude that $\{u_m\}$ is bdd in $H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ $H^1(0,T;V,V')=\{u\in L^2(0,T;V): \partial_t u\in L^2(0,T;V')\}$

i.e. $u\in L^2(0,T;H^1_0(\Omega))$, $\partial_t u\in L^2(0,T;H^{-1}(\Omega))$

Since $L^2(0,T;H^1_0(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$ are separable Hilbert spaces and $L^2(0,T;H^1_0(\Omega))\hookrightarrow L^2(0,T;L^2(\Omega))$ it follows there exists a subsequence u_{m_j} such that:

$$u_{m_j}\rightharpoonup u \text{ weakly in } L^2(0,T;L^2(\Omega)) \quad (C1)$$
$$\nabla u_{m_j}\rightharpoonup \nabla u \text{ weakly in } L^2(0,T;L^2(\Omega)) \quad (C2)$$
$$\partial_t u_{m_j}\rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;H^{-1}(\Omega)) \quad (C3)$$

- $L^\infty(0,T;L^2(\Omega))$ is the dual of $L^1(0,T;L^2(\Omega))$ which is separable. Thus by Banach-Alaoglu thm we can deduce $u_m\rightharpoonup u$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$.

- In principle, we may have $\nabla u_{m_j}\rightharpoonup \xi$ weakly in $L^2(0,T;L^2(\Omega))$. But $\forall \varphi\in D(\Omega;\mathbb{R}^d)$, $\psi\in D(0,T)$ we have

$$\begin{array}{ccc} \int_0^T(\nabla u_{m_j}(t),\varphi)\psi(t)\,dt & = & -\int_0^T(u_{m_j}(t),\operatorname{div}\varphi)\psi(t)\,dt \\ \downarrow & & \downarrow (C1) \\ \int_0^T(\xi(t),\varphi)\psi(t)\,dt & & -\int_0^T(u(t),\operatorname{div}\varphi)\psi(t)\,dt \end{array} \implies \xi=\nabla u \text{ in distributional sense.}$$

By the Riesz representation theorem the above properties are equivalent to:

$$\int_0^T(u_{m_j}(t),v(t))\,dt \longrightarrow \int_0^T(u(t),v(t))\,dt, \quad \forall v\in L^2(0,T;L^2(\Omega))$$
$$\int_0^T(\nabla u_{m_j}(t),\nabla v(t))\,dt \longrightarrow \int_0^T(\nabla u(t),\nabla v(t))\,dt, \quad \forall v\in L^2(0,T;H^1_0(\Omega))$$
$$\int_0^T\langle\partial_t u_{m_j}(t),v(t)\rangle_*\,dt \longrightarrow \int_0^T\langle\partial_t u(t),v(t)\rangle_*\,dt, \quad \forall v\in L^2(0,T;H^1_0(\Omega))$$

(Here we used $(L^2(0,T;X))'=L^2(0,T;X')$)

Let us now recall the weak formulation satisfied by the Galerkin approximation: $\forall m\in\mathbb{N}$, we have

$$\langle\partial_t u_m(t),v_s\rangle_* + (\nabla u_m(t),\nabla v_s) = (f_m(t),v_s), \quad \forall s=1,\dots,m \quad \text{(WF-A)}$$

Let us take $\varphi\in D(0,T)$. We multiply (WF-A) by φ and integrate in time between 0 and T:

$$\int_0^T\langle\partial_t u_m(t),v_s\rangle_*\varphi(t)\,dt + \int_0^T(\nabla u_m(t),\nabla v_s)\varphi(t)\,dt = \int_0^T(f_m(t),v_s)\varphi(t)\,dt \qquad (\bigstar)$$

We now observe that $\|v_s\varphi\|_{L^2(0,T;H^1_0(\Omega))} = \left(\int_0^T\|v_s\varphi(t)\|^2_{H^1_0(\Omega)}\right)^{\frac12} = \left(\int_0^T\|v_s\|^2_{H^1_0(\Omega)}\varphi(t)^2\,dt\right)^{\frac12} = \|v_s\|_{H^1_0(\Omega)}\|\varphi\|_{L^2(0,T)}<\infty$ $\forall s\in\mathbb{N}$

Recalling $f_m\longrightarrow f$ strongly in $L^2(0,T;L^2(\Omega))$ we also deduce that

$$\left|\int_0^T(f_m(t),v_s)\varphi(t)\,dt - \int_0^T(f(t),v_s)\varphi(t)\,dt\right| = \left|\int_0^T(f_m(t)-f(t),v_s)\varphi(t)\,dt\right| \le \int_0^T|(f_m(t)-f(t),v_s)||\varphi(t)|\,dt$$
$$\overset{C.S}{\le}\int_0^T\|f_m(t)-f(t)\|_{L^2(\Omega)}\|v_s\|_{L^2(\Omega)}|\varphi(t)|\,dt = \|v_s\|_{L^2(\Omega)}\int_0^T\|f_m(t)-f(t)\|_{L^2(\Omega)}|\varphi(t)|\,dt \le \|v_s\|_{L^2(\Omega)}\|f_m-f\|_{L^2(0,T;L^2(\Omega))}\|\varphi\|_{L^2(0,T)}\longrightarrow 0$$

Next we fix $s\in\{1,\dots,m_j\}$ and we pass to the limit as $m_j\to\infty$ in $(\bigstar)$. We obtain:

$$\int_0^T\langle\partial_t u(t),v_s\rangle_*\varphi(t)\,dt + \int_0^T(\nabla u(t),\nabla v_s)\varphi(t)\,dt = \int_0^1(f(t),v_s)\varphi(t)\,dt$$

Since the above equality holds for any s fixed and $\overline{\bigcup V_m}^{H^1_0(\Omega)} = H^1_0(\Omega)$ we conclude that

$$\int_0^T\langle\partial_t u(t),v\rangle_*\varphi(t)\,dt + \int_0^T(\nabla u(t),\nabla v)\varphi(t)\,dt = \int_0^T(f(t),v)\varphi(t)\,dt \quad \forall v\in H^1_0(\Omega), \ \forall\varphi\in D(0,T)$$

Then, rewriting the above equality as: $\int_0^T\left[\langle\partial_t u(t),v\rangle_* + (\nabla u(t),\nabla v) - (f(t),v)\right]\varphi(t)\,dt = 0$ for any $\varphi\in D(0,T)$, we infer $\langle\partial_t u(t),v\rangle_* + (\nabla u(t),\nabla v) = (f(t),v)$ $\forall v\in H^1_0(\Omega)$ for a.e. $t\in[0,T]$

(WF)

In order to conclude u is a weak solution to the heat equation we need to show u attains u_0 at $t=0$. First of all, we observe that $u\in H^1(0,T;H^1(\Omega),H^{-1}(\Omega))\hookrightarrow C([0,T];L^2(\Omega))$, therefore $u(0)$ is well-defined. We now need to show $u(0)=u_0$.

Let us consider $\varphi \in C^1[0,T]$ s.t. $\varphi(0)=1, \varphi(T)=0$. Multiplying (WF) by φ and integrating over $[0,T]$, $\int_0^T \langle \partial_t u(t), v\rangle \varphi(t)\,dt + \int_0^T (\nabla u(t), \nabla v)\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt$. Since u and $v\varphi(\cdot) \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$, we can integrate by parts in the first term on LHS obtaining:

$$-\int_0^T \overset{(u(t),v)}{\overset{\shortparallel}{\langle u(t), v\rangle}} \dot\varphi(t)\,dt + (u(T),v)\cancel{\varphi(T)}^{=0} - (u(0),v)\cancel{\varphi(0)}^{=1} + \int_0^T (\nabla u(t), \nabla v)\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt$$

which is equivalent to $-\int_0^T (u(t),v)\dot\varphi(t)\,dt - (u(0),v) + \int_0^T (\nabla u(t), \nabla v)\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt$ (I1)

Similarly, recalling that by (WF-A), $\langle \partial_t u_{m_j}(t), v\rangle_* + (\nabla u_{m_j}(t), \nabla v) = (f_m(t), v)$, $\forall v \in V_{m_j}$, multiplying by φ and integrating over $[0,T]$ we find

$$-\int_0^T (u_{m_j}(t),v)\dot\varphi(t)\,dt + \cancel{(u_{m_j}(T),v)\varphi(T)}^{=0} - \underbrace{(u_{m_j}(0),v)}_{\shortparallel \atop \sum_{k=0}^{m_j}(u_0,v_k)v_k}\overset{=1}{\varphi(0)} + \int_0^T (\nabla u_{m_j}(t), v)\varphi(t)\,dt = \int_0^T (f_{m_j}(t),v)\varphi(t)\,dt$$

namely $-\int_0^T (u_{m_j}(t),v)\dot\varphi(t)\,dt - (u_{m_j}(0),v) + \int_0^T (\nabla u_{m_j}(t), v)\varphi(t)\,dt = \int_0^T (f_{m_j}(t),v)\varphi(t)\,dt$ (I2)

Passing to the limit as $m_j \to +\infty$ in (I2) as before we obtain

$$-\int_0^T (u(t),v)\dot\varphi(t)\,dt - \lim_{m_j\to\infty}(u_{m_j}(0),v) + \int_0^T (\nabla u(t), v)\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt$$

Notice that since $u_0 \in L^2(\Omega) \iff \sum_{k=1}^{\infty} |(u_0,v_k)|^2 < \infty$, $\lim_{m_j\to+\infty}(u_{m_j}(0),v) = \lim_{m_j\to+\infty}\left(\sum_{k=1}^{m_j}(u_0,v_k)v_k, v\right)$
$= \lim_{m_j\to+\infty}\sum_{k=1}^{m_j}(u_0,v_k)(v_k,v) = \sum_{k=1}^{\infty}(u_0,v_k)(v_k,v) = (u_0,v)$

This implies that $-\int_0^T (u(t),v)\dot\varphi(t)\,dt - (u_0,v) + \int_0^T (\nabla u(t), \nabla v)\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt$ (I3)

Computing (I3)−(I1) we arrive at $(u(0),v) = (u_0,v)$, $\forall v \in H_0^1(\Omega)$. But since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ we conclude that $(u(0),v) = (u_0,v)$, $\forall v \in L^2(\Omega)$, which gives $u(0)=u_0$.

10.2: UNIQUENESS AND ABSTRACT PARABOLIC PROBLEMS

- Existence of weak solution to the heat eqn: For any $u_0 \in L^2(\Omega)$, $f \in L^2(0,T; L^2(\Omega))$, there exists $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ such that:
  1) $\langle \partial_t u(t), v\rangle_* + (\nabla u(t), v) = (f(t),v) \quad \forall v \in H_0^1(\Omega)$
  2) $u(0) = u_0$

We recall that (1) is equivalent to $\langle \partial_t u(t), v(t)\rangle_* + (\nabla u(t), \nabla v(t)) = (f(t), v(t)) \quad \forall v \in L^2(0,T; H_0^1(\Omega))$ for a.e. $t\in[0,T]$ (WF)

UNIQUENESS OF WEAK SOLUTION

- Let us assume $u_1, u_2 \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ are two weak solutions to the heat equation. This is, for $i=1,2$,
  A) $\langle \partial_t u_i(t), v(t)\rangle_* + (\nabla u_i(t), \nabla v(t)) = (f(t), v(t)) \quad \forall v \in L^2(0,T; H_0^1(\Omega))$, a.e. $t\in[0,T]$
  B) $u_i(0) = u_0$

We define $w = u_1 - u_2 \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ which solves $\langle \partial_t w(t), v(t)\rangle_* + (\nabla w(t), \nabla v(t)) = 0 \quad \forall v \in L^2(0,T; H_0^1(\Omega))$ a.e. $t\in[0,T]$

Taking $v = w$, we obtain: $\langle \partial_t w(t), w(t)\rangle_* + (\nabla w(t), \nabla w(t)) = 0$

Recalling that $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = \langle \partial_t u(t), u(t)\rangle_* \quad \forall u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$,

we find $\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \|\nabla w(t)\|^2_{L^2(\Omega)} = 0$. Integrating this equality from 0 to T where $T\in[0,T]$ we get

$$\frac{1}{2}\|w(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|w(0)\|^2_{L^2(\Omega)} + \underbrace{\int_0^T \|\nabla w(t)\|^2_{L^2(\Omega)}}_{\geq 0} = 0 \implies \|w(T)\|^2_{L^2(\Omega)} \leq \|w(0)\|^2_{L^2(\Omega)}$$

But $w(0) = u_1(0) - u_2(0) = u_0 - u_0 = 0$. Thus conclude $\|w(T)\|_{L^2(\Omega)} = 0$, $\forall T\in[0,T] \implies u_1(t) = u_2(t)$ for any $t\in[0,T]$

ENERGY ESTIMATES FOR THE WEAK SOLUTION

- Choosing $v = u$ in (WF), we find $\langle \partial_t u(t), u(t)\rangle_* + (\nabla u(t), \nabla u(t)) = (f(t), u(t))$ for a.e. $t \in [0,T]$

By the chain rule in $H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$ we have $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} = (f(t), u(t))$

Integrating this equality between 0 to τ, $\tau \in [0,T]$, and using Cauchy-Schwarz:

$$\frac{1}{2}\|u(\tau)\|^2_{L^2(\Omega)} + \int_0^\tau \|\nabla u(t)\|^2_{L^2(\Omega)} = \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^\tau (f(t), u(t))\, dt \qquad (\square)$$

$$\leq \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^\tau \|f(t)\|_{L^2(\Omega)} \|u(t)\|_{L^2(\Omega)}\, dt$$

$(ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2)$

$$\leq \underbrace{\frac{1}{2}\|u(0)\|^2_{L^2(\Omega)}}_{\text{const.}} + \underbrace{\int_0^\tau \frac{1}{2}\|f(t)\|^2_{L^2(\Omega)}\, dt}_{\text{non-decreasing}} + \int_0^\tau \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}\, dt$$

Thanks to Gronwall Lema, we get $\frac{1}{2}\|u(\tau)\|^2_{L^2(\Omega)} \leq \left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt\right) e^\tau \quad \forall \tau \in [0,T] \qquad (1)$

By exploiting (1) in $(\square)$ we also deduce

$$\int_0^\tau \|\nabla u(t)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt + \int_0^\tau \left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega)}\, ds\right) e^t\, dt$$

$$\leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt + \left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt\right) \overbrace{\int_0^\tau e^t\, dt}^{= e^\tau - 1}$$

$$= \left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt\right) e^\tau, \quad \forall \tau \in [0,T] \qquad (2)$$

Going back to the weak formulation we recall that

$$|\langle \partial_t u(t), v\rangle_*| \leq |(\nabla u(t), \nabla v)| + |(f(t), v)| \leq \left(\|\nabla u(t)\|_{L^2(\Omega)} + \|f(t)\|_{L^2(\Omega)}\right)\|v\|_{H^1_0(\Omega)}$$

for any $v \in H^1_0(\Omega)$. This gives $\int_0^\tau \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} dt \leq \int_0^\tau \left(\|\nabla u(t)\|_{L^2(\Omega)} + \|f(t)\|_{L^2(\Omega)}\right)^2 dt \leq \int_0^\tau 2\|\nabla u(t)\|^2_{L^2(\Omega)} + 2\|f(t)\|^2_{L^2(\Omega)}\, dt$

$$\overset{(2)}{\leq} \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt\right) e^\tau + 2\int_0^\tau \|f(t)\|^2_{L^2(\Omega)}\, dt \qquad (3)$$

ABSTRACT PARABOLIC PROBLEM

- The Galerkin method used to show the existence of weak solution for the Cauchy-Dirichlet problem of the heat equation can be employed to prove the existence of weak solutions to a wide class of parabolic problems. The following result is the generalisation of the Lax-Milgram theorem to linear parabolic systems.

- **THM:** Let V, H be two separable Hilbert spaces such that $V \hookrightarrow H$, V dense in H. Assume that a is a bilinear form $a: V \times V \to \mathbb{R}$ such that:
  - i) there exists $M > 0$ such that $|a(u,v)| \leq M\|u\|_V \|v\|_V \quad \forall u, v \in V$
  - ii) there exist $\lambda \geq 0$, $\alpha > 0$ such that $a(u,u) + \lambda\|u\|^2_H \geq \alpha\|u\|^2_V$, $\forall u \in V$.

  For any $u_0 \in H$ and $f \in L^2(0,T; V')$, there exists a unique $u \in H^1(0,T; V, V')$ such that
  1) $\langle \partial_t u(t), v\rangle_* + a(u(t), v) = \langle f(t), v\rangle_*$, $\forall v \in V$, for a.e. $t \in [0,T]$
  2) $u(0) = u_0$

OVERVIEW CLASS 10 - Properties of heat eqn

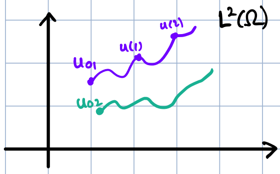
Well-posedness of weak solutions:

Let Ω be bdd Lipschitz domain in $\mathbb{R}^n$ and $T > 0$. Given $u_0 \in L^2(\Omega)$, $f \in L^2(0,T; L^2(\Omega))$, there exists a unique weak solution $u \in H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$ such that:

1) $\langle \partial_t u(t), v\rangle_* + (\nabla u(t), \nabla v) = (f(t), v) \quad \forall v \in H^1_0(\Omega)$ for a.e. $t \in [0,T]$
2) $u(0) = u_0$

- REMARKS:
  - A) (1) is equivalent to $\langle \partial_t u(t), v(t)\rangle_* + (\nabla u(t), \nabla v(t)) = (f(t), v) \quad \forall v \in L^2(0,T; H^1_0(\Omega))$ a.e. $t \in [0,T]$
  - B) $u \in H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \hookrightarrow C([0,T]; L^2(\Omega)) \implies u(0)$ is well-defined.
  - C) The assumption on the forcing term f can be generalised to $f \in L^2(0,T; H^{-1}(\Omega))$

CONTINUOUS DEPENDENCE W.R.T DATA



Let $u_{01}, u_{02} \in L^2(\Omega)$, $f_1, f_2 \in L^2(0,T; L^2(\Omega))$. We consider the weak solutions $u_1(t)$ and $u_2(t)$ to the heat eqn corresponding to (u_{01}, f_1) and (u_{02}, f_2) resp.

Question: If $u_{01} \simeq u_{02}$ and $f_1 \approx f_2$, is it possible to show $u_1(t) \sim u_2(t)$?

We want to prove that there exists a constant $C(T) > 0$ such that $\|u_1 - u_2\|_X \leq C(T)\left(\|u_{01} - u_{02}\|_{L^2(\Omega)} + \|f_1 - f_2\|_{L^2(0,T;L^2(\Omega))}\right)$ for some suitable norm $\|\cdot\|_X$.

We define $w(t) = u_1(t) - u_2(t) \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ which solves the following problem:

i) $\langle \partial_t w(t), v(t)\rangle_* + (\nabla w(t), v(t)) = (f_1(t) - f_2(t), v(t))$ $\quad \forall v \in L^2(0,T; H_0^1(\Omega))$ a.e. $t \in [0,T]$

ii) $w(0) = u_{01} - u_{02}$

Choosing $v = w$, we find $\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \|\nabla w(t)\|^2_{L^2(\Omega)} = (f_1(t) - f_2(t), w(t))$

Integrating from 0 to $t \in [0,T]$, we obtain:

$$\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|w(0)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\Omega)}\,ds = \int_0^t (f_1(s) - f_2(s), w(s))\,ds$$

Poincare

$$\leq \int_0^t \|f_1(s) - f_2(s)\|_{L^2(\Omega)} \|w(s)\|_{L^2(\Omega)}\,ds$$

$$\overset{\text{Youngs}}{\leq} \int_0^t \frac{1}{2}\|f_1(s) - f_2(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \frac{1}{2}\|w(s)\|^2_{L^2(\Omega)}\,ds$$

namely $\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\Omega)}\,ds \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \frac{1}{2}\|f_1(s) - f_2(s)\|^2\,ds + \int_0^t \frac{1}{2}\|w(s)\|^2\,ds$

By Gronwall, we deduce:

$$\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} \leq \left(\frac{1}{2}\|w_0\|^2_{L^2(\Omega)} + \int_0^t \frac{1}{2}\|f_1(s) - f_2(s)\|^2_{L^2(\Omega)}\,ds\right) e^{\overset{=t}{\int_0^t ds}}$$

In other words, we found

$$\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leq \left(\|u_{01} - u_{02}\|^2_{L^2(\Omega)} + \int_0^t \|f_1(s) - f_2(s)\|^2_{L^2(\Omega)}\right) e^t, \quad \forall t \in [0,T]$$

which implies

$(a^2+b^2)^{\frac{1}{2}} \leq a + b$

$$\underbrace{\|u_1(t) - u_2(t)\|_{C([0,T];L^2(\Omega))}}_{\times} \leq \left(\|u_{01} - u_{02}\|_{L^2(\Omega)} + \|f_1 - f_2\|_{L^2(0,T;L^2(\Omega))}\right) e^{\frac{T}{2}}$$

An interesting question is: what happens for $t \to \infty$?

Assume $f \in L^2(0,\infty; L^2(\Omega))$ or $f \in L^2(\Omega)$ (const.), we have a global solution $u : [0,\infty) \to L^2(\Omega)$.

Is it true that $u(t) \longrightarrow u^*$ as $t \to +\infty$? An answer is provided by the theory of attractors for PDES in infinite-dim. spaces.

REGULARITY FOR HEAT EQN

apparently not needed?

- AIM: Suppose u_0, f are more "regular", we want to show that the weak solution u is in fact more "regular".

- Let us consider the Dirichlet problem $\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ C^2-domain | For any $f \in L^2(\Omega)$ there exists a unique weak soln. s.t. $\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$

Moreover by the regularity theory we know $u \in H^2(\Omega)$ and there exists a constant $\tilde{C}$ s.t. $\|u\|_{H^2(\Omega)} \leq \tilde{C}\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) = \tilde{C}\left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$ $(*)$ On the other hand by Poincare, we have $\|u\|^2_{L^2(\Omega)} \leq C_P^2 \|\nabla u\|^2_{L^2(\Omega)} = C_P^2 \int_\Omega \nabla u \cdot \nabla u\,dx = -C_P^2 \int_\Omega \Delta u \cdot u\,dx \underset{\substack{u=0 \\ \partial\Omega}}{\leq} C_P^2 \|\Delta u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$

$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ $\quad \leq \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2} C_P^4 \|\Delta u\|^2_{L^2(\Omega)} \Rightarrow \frac{1}{2}\|u\|^2_{L^2(\Omega)} \leq \frac{1}{2} C_P^4 \|\Delta u\|^2_{L^2(\Omega)} \Rightarrow \|u\|_{L^2(\Omega)} \leq C_P^2 \|\Delta u\|_{L^2(\Omega)}$

By exploiting the last inequality in $(*)$ we obtain:

$$\|u\|_{H^2(\Omega)} \leq C^* \|\Delta u\|_{L^2(\Omega)} \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega)$$

Next we consider the Galerkin approximation: for any $m \in \mathbb{N}$ we have

$u_m(t) = \sum_{k=1}^m c_k^m(t) v_k$ which solves
- $(\partial_t u_m(t), v_s) + (\nabla u_m(t), \nabla v_s) = (f_m(t), v_s)$, $s = 1, \dots, m$
- $u_m(0) = \sum_{k=1}^m (u_0, v_k) v_k$

$c_1^m \neq c_1^{m+1}$!

Ω is C^2 domain $\Rightarrow \{v_k\} \subset H^2(\Omega)$. We can rewrite the problem as

$(\partial_t u_m(t), v_s) - (\Delta u_m(t), v_s) = (f_m(t), v_s) \quad \forall s = 1, \dots, m.$ $(\boxed{})$

Main observation: $\Delta u_m(t) = \sum_{k=1}^{m} C_k^m(t) \Delta v_k = -\sum_{k=1}^{m} C_k^m(t) \lambda_k v_k = -\lambda_k \sum_{k=1}^{m} C_k^m(t) v_k \in V_m$! i.e. can take it as a test fn

Multiplying $(\boxed{})$ by $-C_s^m(t)\lambda_s$, we find

$$(\partial_t u_m(t), -C_s^m(t)\lambda_s v_s) + (\Delta u_m(t), C_s^m(t)\lambda_s v_s) = (f_m(t), -C_s^m(t)\lambda_s v_s) \quad \forall s = 1, \dots, m$$

Summing over s, we obtain:

$$\underbrace{(\partial_t u_m(t), -\Delta u_m(t))}_{= \int_\Omega \nabla \partial_t u_m(t) \cdot \nabla u_m(t) dt = \int_\Omega \partial_t \left(\frac{1}{2} |\nabla u_m(t)|^2\right) dx} + \|\Delta u_m(t)\|^2_{L^2(\Omega)} = (f_m(t), -\Delta u_m(t))$$

$\therefore$ we have
$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\Delta u_m(t)\|^2_{L^2(\Omega)} = (f_m(t), -\Delta u_m(t)) \leq \|f_m(t)\|_{L^2(\Omega)} \|\Delta u_m\|_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\|f_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta u_m\|^2_{L^2(\Omega)}$$

$$\Rightarrow \frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\Delta u_m\|^2_{L^2(\Omega)} \leq \|f_m(t)\|^2_{L^2(\Omega)}$$

Integrating from 0 to t,
$$\|\nabla u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta u_m(s)\|^2_{L^2(\Omega)} ds \leq \underbrace{\|\nabla u_m(0)\|^2_{L^2(\Omega)}}_{\leq \|\nabla u_0\|^2_{L^2(\Omega)}} + \underbrace{\int_0^t \|f_m(t)\|^2_{L^2(\Omega)} ds}_{\leq 2\int_0^t \|f(s)\|^2_{L^2(\Omega)} ds}$$ "w.l.o.g"

$$\|\nabla u_m(0)\|^2_{L^2(\Omega)} = \sum_{k=1}^{m} |(u_0, v_k)|^2 \|\nabla v_k\|^2_{L^2(\Omega)} = \sum_{k=1}^{m} |(u_0, v_k)|^2 \lambda_k \underbrace{\|v_k\|^2_{L^2(\Omega)}}_{=1} = \sum_{k=1}^{m} |(u_0, v_k)|^2 \lambda_k \leq \|\nabla u_0\|^2_{L^2(\Omega)}$$

$\Rightarrow \|\nabla u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta u_m(s)\|^2_{L^2(\Omega)} ds \leq K(\|\nabla u_0\|_{L^2} \|f\|_{L^2(0,T;L^2(\Omega))})$ for any $t \in [0,T]$

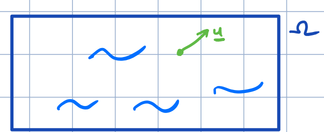
$\Rightarrow u_m$ is bdd in $L^\infty(0,T; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \Rightarrow u \in L^\infty(0,T; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$ provided that $u_0 \in H_0^1(\Omega)$, $f \in L^2(0,T; L^2(\Omega))$

II.1: NAVIER STOKES EQNS (I)

- One of the most important model arising from physics describes the motion of viscous & incompressible fluid. This is a system of PDEs introduced by C.L. Navier (1822) and G.G. Stokes (1850).

$\underline{u} = (u_1, u_2, u_3) : \Omega \times [0,T] \to \mathbb{R}^3$ velocity field

$p : \Omega \times (0,T) \to \mathbb{R}$ pressure



4 unknowns

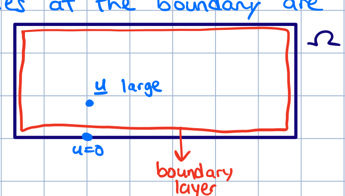
The Navier-Stokes equations reads as follows:
$$\begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \nabla)\underline{u} - \nu \Delta \underline{u} + \nabla p = \underline{f} & & (1) \\ \operatorname{div} \underline{u} = 0 & \text{in } \Omega \times (0,T) & (2) \\ \underline{u} = 0 & \text{on } \partial\Omega \times (0,T) & (3) \\ \underline{u}(0) = u_0 & \text{in } \Omega & (4) \end{cases}$$

- (1) are called the momentum equations
  $\partial_t u_i + u_j \partial_j u_i - \nu \Delta u_i + \partial_i p = f_i$, $i = 1,2,3$
  where we used Einstein notation ($u_j \partial_j u_i = \sum_{j=1}^{3} u_j \partial_j u_i$)

- (1)-(2) are also called the incompressible Navier-Stokes eqns since the density ρ of the fluid is assumed const.
  Remark: if ρ not constant, then we study the compressible Navier-Stokes eqns.
- The incompressibility constraint $\operatorname{div} \underline{u} = 0$ (namely $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$) means the fluid occupies the same volume for all times.

- The viscous term $-\nu \Delta u$ accounts for the dissipation of energy generated by the friction of fluid particles.
  REYNOLDS NUMBER $\frac{UL}{\nu} = Re$ U = characteristic velocity, L = characteristic spatial dimension
  If Re small, flow is laminar. If Re is large, flow is turbulent and the trajectory of each fluid is chaotic.
  Case $\nu = 0$ corresponds to the Euler equations.

- The boundary condition $\underline{u} = 0$ is called no-slip condition meaning particles at the boundary are stuck there forever. Since u can be very large in the bulk, this generates a small layer of large velocity gradient close to the boundary.
  This condition $\underline{u} = 0$ is much stronger than $\underline{u} \cdot \underline{n} = 0$ which means no penetrability (that is used in the case $\nu = 0$)



Mathematical theory of the Navier–Stokes equations

- Global existence of weak Leroy–Hopf solutions (Leroy 1934, Hopf 1951)
- Uniqueness: i) Yes, for subclasses of Leroy–Hopf solutions (e.g. self-similar solution or solns which satisfy the Serrin criteria)
  ii) No for certain forcing terms $\underline{f}$ in $L^1(0,T;L^2(\mathbb{R}^3))$ $(\Omega=\mathbb{R}^3)$ (2022)

- Regularity: Assume $\underline{f}$, $\underline{u}_0$ are regular, namely $\underline{f}\in C^\infty(\Omega\times(0,T))$ and $\underline{u}_0\in C^\infty(\Omega)$.
  Is the soln. $\underline{u}$ in $C^\infty(\Omega\times(0,T])$ for any $T>0$? $\rightsquigarrow$ open question (MILLENIUM PRIZE)

Why should a global weak solution exist?

Assume that $(\underline{u},\underline{p})$ is a smooth solution to NSE. Taking the inner product between (1) and $\underline{u}$, integrating on Ω:

$$\int_\Omega \partial_t\underline{u}\cdot\underline{u}\,dx + \int_\Omega(\underline{u}\cdot\nabla)\underline{u}\cdot\underline{u}\,dx - \nu\int_\Omega\Delta\underline{u}\cdot\underline{u}\,dx + \int_\Omega\nabla p\cdot\underline{u}\,dx = \int_\Omega \underline{f}\cdot\underline{u}\,dx$$

Notice that $\int_\Omega\partial_t\underline{u}\cdot\underline{u}\,dx = \int_\Omega\partial_t u_1u_1+\partial_t u_2u_2+\partial_t u_3u_3\,dx = \int_\Omega\partial_t(\frac{1}{2}u_1^2)+\partial_t(\frac{1}{2}u_2^2)+\partial_t(\frac{1}{2}u_3^2)\,dx = \int_\Omega\partial_t(\frac{1}{2}|u|^2)\,dx = \frac{d}{dt}\underbrace{\int_\Omega\frac{1}{2}|u|^2dx}_{\text{Kinetic Energy}}$.

and

$$\int_\Omega(\underline{u}\cdot\nabla)\underline{u}\cdot\underline{u}\,dx = \int_\Omega u_j\partial_j u_iu_i\,dx = \int_\Omega u_j\partial_j(\tfrac{1}{2}|u|^2)\,dx = \int_\Omega\underline{u}\cdot\nabla(\tfrac{1}{2}|u|^2)\,dx \overset{\text{Greens}}{=} \int_\Omega -\cancelto{0\,(2)}{\operatorname{div}\underline{u}}\,\tfrac{1}{2}|u|^2dx + \int_{\partial\Omega}\tfrac{1}{2}|u|^2\cancelto{0\,(3)}{\underline{u}\cdot\underline{n}}\,d\sigma = 0$$

The contribution to the energy evolution of the non-linear term is 0.

Moreover, we observe that:

$$-\int_\Omega\nu\Delta\underline{u}\cdot\underline{u}\,dx = -\int_\Omega\nu\,\Delta u_iu_i\,dx \overset{\text{IBP}}{=} \nu\int_\Omega\nabla u_i\cdot\nabla u_i\,dx - \int_{\partial\Omega}\nu(\cancelto{0\,(3)}{\nabla u_i\cdot n})u_i\,d\sigma = \nu\int_\Omega\sum_{i=1}^3|\nabla u_i|^2dx = \nu\int_\Omega|\nabla\underline{u}|^2dx = \nu\|\nabla\underline{u}\|^2_{L^2(\Omega)}$$

and

$$\int_\Omega\nabla p\cdot u\,dx = -\int_\Omega p\cdot\cancelto{0\,(2)}{\operatorname{div}\underline{u}}\,dx + \int_{\partial\Omega}p\,\cancelto{0\,(3)}{\underline{u}\cdot\underline{n}}\,d\sigma = 0$$

Therefore, we deduce the energy equation:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega|u|^2dx + \nu\int_\Omega|\nabla\underline{u}|^2dx = \int_\Omega\underline{f}\cdot\underline{u}\,dx \qquad (E)$$

(E) is very similar to the energy eqn of the heat eqn.

From a Gronwall type argument, we expect $u_i\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, $\forall T>0$, $i=1,2,3$

DIVERGENCE-FREE SOBOLEV SPACES

- Let us consider from now on Ω bdd Lipschitz domain in $\mathbb{R}^3$ (same is valid in $\mathbb{R}^2$).
  AIM: Include $\operatorname{div}\underline{u}=0$ in the functional setting.

- Let us introduce the space $C^\infty_{c,\sigma}(\Omega)=\{\underline{v}\in[C_c^\infty(\Omega)]^3 : \operatorname{div}\underline{v}=0\}$. We define $L^2_\sigma=\overline{C^\infty_{c,\sigma}(\Omega)}^{[L^2(\Omega)]^3}$, $H^1_\sigma=\overline{C^\infty_{c,\sigma}(\Omega)}^{[H_0^1(\Omega)]^3}$
  It is possible to show $L^2_\sigma=\{\underline{v}\in[L^2(\Omega)]^3 : \operatorname{div}\underline{v}=0 \text{ in } \Omega,\ \underline{v}\cdot\underline{n}=0 \text{ on } \partial\Omega\}$
  $H^1_\sigma=\{\underline{v}\in[H^1(\Omega)]^3 : \operatorname{div}\underline{v}=0 \text{ in } \Omega,\ \underline{v}=0 \text{ on } \partial\Omega\}$
  - For any $v\in L^2_\sigma$, $\operatorname{div}\underline{v}=0$ in Ω is meant in weak sense, namely $\int_\Omega\underline{v}\cdot\nabla\psi\,dx=0$ $\forall\psi\in H^1(\Omega)$.
  - What is the meaning of $\underline{v}\cdot\underline{n}=0$ on $\partial\Omega$ for $\underline{v}\in L^2_\sigma$, which is not in $H^1(\Omega)$?
    Idea: Green formula:

$$\int_{\partial\Omega}(\underline{v}\cdot\underline{n})\varphi\,d\sigma = \int_\Omega\operatorname{div}\underline{v}\,\varphi\,dx + \int_\Omega\underline{v}\cdot\nabla\varphi\,dx$$

- **LEMMA**: If $\underline{v}\in E(\Omega)=\{\underline{v}\in[L^2(\Omega)]^3 : \operatorname{div}\underline{v}\in L^2(\Omega)\}$ then the normal component of $\underline{v}$ is well-defined on $\partial\Omega$ as a linear functional on $H^{\frac{1}{2}}(\partial\Omega)$, i.e. $\underline{v}\cdot\underline{n}\in(H^{\frac{1}{2}}(\partial\Omega))'$. The formula holds:

$$\langle\underline{v}\cdot\underline{n},\varphi\rangle = \int_\Omega\operatorname{div}\underline{v}\,\varphi\,dx + \int_\Omega\underline{v}\cdot\nabla\varphi\,dx,\quad\forall\varphi\in H^1(\Omega)$$

  In particular, if $v\in L^2_\sigma$, $\langle\underline{v}\cdot\underline{n},\varphi\rangle=0$, $\forall\varphi\in H^1(\Omega)$

Another important result:

- THM (HELMHOLTZ–WEYL DECOMPOSITION): $[L^2(\Omega)]^3 = L^2_\sigma\oplus G(\Omega)$ where $G(\Omega)=\{\underline{w}\in[L^2(\Omega)]^3 : \underline{w}=\nabla g \text{ for } g\in H^1(\Omega)\}$ (orthogonal complement)
  ↳ As a consequence we can define the so-called Leroy projector:

$$P: [L^2(\Omega)]^3\longrightarrow L^2_\sigma,\quad P(\underline{v})=\underline{h}\quad\text{where}\quad\underline{v}=\underline{h}+\nabla\underline{g}$$

  - Let $\underline{v}\in[H^m(\Omega)]^3$ $(m=1,2)$, then $P\underline{v}\in[H^m(\Omega)]^3\cap L^2_\sigma$, $\|P\underline{v}\|_{[H^m(\Omega)]^3}\le C_m\|\underline{v}\|_{[H^m(\Omega)]^3}$
    But, $\underline{v}\in H^1_\sigma$ then $P\underline{v}\notin H^1_\sigma$
  - P is self-adjoint, i.e. $(P(\underline{v}),\underline{u})=(\underline{v},P\underline{u})$ for any $v,u\in[L^2(\Omega)]^3$

II.2: NAVIER STOKES EQNS (II)

$$\begin{cases} \partial_t \underline{u} + (\underline{u}\cdot\nabla)\underline{u} - \nu\Delta\underline{u} + \nabla p = \underline{f} & & (1) \\ \operatorname{div}\underline{u} = 0 & \text{in } \Omega\times(0,T) & (2) \\ \underline{u} = 0 & \text{on } \partial\Omega\times(0,T) & (3) \\ \underline{u}(0) = u_0 & \text{in } \Omega & (4) \end{cases}$$

AIM: Introduce the weak formulation of (NSE)

- The main observation due to Leray is that we can erase the pressure in the weak formulation which will be recovered later on once the velocity field is known. To be more precise, taking the inner product between (NSE), and $\underline{v} \in H^1_\sigma$ we find

$$\int_\Omega \partial_t u(t)\cdot\underline{v}\,dx + \int_\Omega (\underline{u}(t)\cdot\nabla)\underline{u}(t)\,v\,dx - \nu\int_\Omega \Delta u\cdot\underline{v}\,dx + \int_\Omega \nabla p\cdot\underline{v}\,dx = \int_\Omega f\cdot v\,dx$$

We observe $\int_\Omega \nabla p\cdot\underline{v}\,dx = -\int_\Omega p\,\cancelto{0}{\operatorname{div}\underline{v}}\,dx + \int_{\partial\Omega} p\,\underline{v}\cdot\cancelto{0}{\underline{n}}\,dx = 0$ and

$$-\int_\Omega \Delta u\cdot v\,dx = -\int_\Omega \Delta u_i v_i\,dx = \int_\Omega \nabla u_i\cdot\nabla v_i\,dx - \int_{\partial\Omega}(\nabla u_i\cdot \cancelto{0}{n})v_i\,d\sigma = \int_\Omega \nabla u_i\cdot\nabla v_i\,dx =: \int_\Omega \nabla u:\nabla v\,dx$$

Thanks to this eq. we find: $\int_\Omega \partial_t u(t)\cdot\underline{v}\,dx + b(u(t),u(t),v) + a(u(t),v) = (f(t),v)$, $\forall v\in H^1_\sigma$.
where $b(u,v,w) = \int_\Omega (\underline{u}\cdot\nabla)\underline{v}\cdot\underline{w}\,dx$, $\forall \underline{u},\underline{v},w \in H^1_\sigma$, $a(\underline{u},\underline{v}) = \int_\Omega \nabla\underline{u}:\nabla\underline{v}\,dx$, $\forall u,v\in H^1_\sigma$.

<u>Question</u>: assume $u\in L^2(0,T;H^1_\sigma)$ as we expect from energy eqn, is $b(u(t),u(t),v)$ well-defined as a function in $L^p(0,T)$ for some $p\geq 1$?

$$|b(\underline{u}(t),\underline{u}(t),\underline{v})| = \Big|\int_\Omega (\overset{L^3}{\underline{u}(t)}\cdot\nabla)\overset{L^2}{\underline{u}(t)}\cdot\overset{L^6}{\underline{v}}\,dx\Big| \overset{\text{Hö}}{\underset{\frac13+\frac12+\frac16=1}{\leq}} \|\underline{u}(t)\|_{L^3(\Omega)}\|\nabla\underline{u}(t)\|_{L^2(\Omega)}\|v\|_{L^6(\Omega)}$$

3D Sobolev embedding: $H^1(\Omega)\hookrightarrow L^6(\Omega)$

$$\leq C\|\underline{u}(t)\|_{L^6(\Omega)}\|\nabla\underline{u}(t)\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} \leq C\underbrace{\|u(t)\|^2_{H^1(\Omega)}}_{\in L^1(0,T)}\underbrace{\|v\|_{H^1(\Omega)}}_{<\infty}$$

$\Rightarrow |b(u(t),u(t),v)| \in L^1(0,T)$
By comparison, we expect $\partial_t u \in L^1(0,T;(H^1_\sigma)')$.

- We are ready to state the weak formulation of (NSE):
A function $\underline{u}\in L^2(0,T;H^1_\sigma)$ such that $\partial_t\underline{u}\in L^1(0,T;(H^1_\sigma)')$ is a weak solution to (NSE) if
  a) $\langle\partial_t\underline{u},\underline{v}\rangle_* + b(\underline{u}(t),\underline{u}(t),\underline{v}) + a(\underline{u}(t),\underline{v}) = (\underline{f},\underline{v})$ $\forall v\in H^1_\sigma$ for a.e. $t\in[0,T]$
  b) $\underline{u}(0) = u_0$

(WF-NS)

Notice (a) is equivalent to the functional equation:
$$\partial_t u + P((u\cdot\nabla)u) - \nu P(\Delta u) = Pf \qquad \text{where } P \equiv \text{Leray projector}.$$

- **THM (LERAY-HOPF)**: Let Ω be bdd Lipschitz domain in $\mathbb{R}^3$. Let $u_0\in L^2_\sigma$, $f\in L^2(0,T;[L^2(\Omega)]^3)$. Then there exists $(\underline{u},\underline{p})$ defined on $\Omega\times(0,\infty)$ such that:
  i) $\underline{u}\in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1_\sigma)$ $\quad\forall T>0$
  $\partial_t\underline{u}\in L^{\frac43}(0,T;(H^1_\sigma)')$
  $p\in W^{-1,\infty}(0,T;L^2_{(0)}(\Omega))$

  ii) u satisfies (a)-(b) in (WF-NSE)
  iii) It satisfies the energy inequality:

$$\frac12\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \nu\|\nabla\underline{u}(t)\|^2_{L^2(\Omega)}\,ds \leq \frac12\|u_0\|^2_{L^2(\Omega)} + \int_0^t (\underline{f},\underline{u}(s))\,ds, \quad \forall t>0$$

- REMARKS: 1) $p \in W^{-1,\infty}(0,T; L^2_{(0)}(\Omega))$ means $p = \frac{\partial \pi}{\partial t}$ in distributional sense where $\pi \in L^\infty(0,T; L^2_{(0)}(\Omega))$ with $\int_\Omega \pi(t)\,dx = 0$. In particular, pressure is recovered from the WF-a above thanks to the following result:

- **THM (DERHAM):** Let Ω be bdd Lipschitz domain in $\mathbb{R}^3$, let $\underline{F} \in [H^{-1}(\Omega)]^3$ such that $\langle \underline{F}, \underline{\varphi} \rangle = 0 \ \forall \underline{\varphi} \in [\mathcal{D}(\Omega)]^3 : \operatorname{div} \underline{\varphi} = 0$. Then $\underline{F} = \nabla p$ where $p \in L^2_{(0)}(\Omega)$.

2) Due to the regularity of $\partial_t \underline{u}$ which is only $L^{\frac{4}{3}}$ in time, we cannot obtain $u \in C([0,T]; L^2_\sigma)$. However it is possible to show $\underline{u} \in C([0,T]; (H^1_\sigma)')$ and so $\underline{u}(0) = \underline{u}_0$ makes sense.

3) Leray-Hopf's thm implies the existence but not uniqueness.

- In order to study the existence of a weak-soln we need to first consider Stoke's problem:

$$\begin{cases} -\nu\Delta \underline{u} + \nabla p = f & \text{in } \Omega \\ \operatorname{div} \underline{u} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

and its eigenvalue problem:

$$\begin{cases} -\nu\Delta \underline{u} + \nabla p = \lambda \underline{u} & \text{in } \Omega \\ \operatorname{div} \underline{u} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

Following the same approach as above, we deduce the weak formulation of the eigenvalue problem: "FIND $u \in H^1_\sigma : \nu a(\underline{u}, \underline{v}) = \lambda (\underline{u}, \underline{v}) \quad \forall u, v \in H^1_\sigma$" (EP)

where $a(u,v) = \int_\Omega \nabla \underline{u} \cdot \nabla \underline{v}\, dx$

REMARK: The pressure does not appear in (EP).

As in the Dirichlet problem for the Poisson eqn, the bilinear form a is cts, coercive and symmetric in $H^1_\sigma \times H^1_\sigma$. Therefore, we have:

1) There exists a sqnce $\{\lambda_k\}$ of +ve eigenvalues s.t. $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \to \infty$ and a sqnce $\{v_k\} \subset H^1_\sigma$ of corresponding eigenfunctions solving (EP).

2) $\{v_k\}$ is an ONB of L^2_σ and an orthogonal base of H^1_σ.

OVERVIEW CLASS 11

AIM: Let Ω be a bdd Lipschitz domain in $\mathbb{R}^3$ and $T>0$. Given $u_0 \in L^2_\sigma$, $\underline{f} \in L^2(0,T; [L^2(\Omega)]^3)$, find a vector field $\underline{u} \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$, $\partial_t \underline{u} \in L^{\frac{4}{3}}(0,T; (H^1_\sigma)')$ such that:

A) $\langle \partial_t \underline{u}(t), v \rangle_* + b(\underline{u}(t), \underline{u}(t), \underline{v}) + a(\underline{u}(t), \underline{v}) = (\underline{f}(t), \underline{v}) \quad \forall v \in H^1_\sigma$, a.e. $t \in (0,T)$.

B) $u(0) = u_0$ $\quad (\nu = 1)$

Proof is based on the Galerkin method.

1) Existence of the Galerkin approximation

Let $\{v_k\}_{k\in\mathbb{N}}$ be the eigenfunctions of the Stokes problem, which is an ONB in L^2_σ and an orthogonal base in H^1_σ. In particular we have $a(\underline{v_k}, \underline{w}) = \lambda_k (\underline{v_k}, \underline{w})$, $\forall w \in H^1_\sigma$. For any $n, m \in \mathbb{N}$ we define the finite dimensional space $V_m = \operatorname{span}\{v_1, \dots, v_m\}$. Recall $\overline{\bigcup_m V_m}^{H^1_\sigma} = H^1_\sigma$. We look for a function $\underline{u}_m(t) = \sum_{k=1}^m c_k^m(t) \underline{v_k}$ which solves

(AP) $(\partial_t \underline{u}_m(t), \underline{v_s}) + b(\underline{u}_m(t), \underline{u}_m(t), \underline{v_s}) + a(\underline{u}_m(t), \underline{v_s}) = (\underline{f_m}(t), \underline{v_s}) \quad \forall s = 1, \dots, m$ for any $t \in (0,T)$

where $\{f_m\} \subset C([0,T]; L^2_\sigma)$ s.t. $f_m \to f$ in $L^2(0,T; L^2_\sigma)$: $\|\underline{f}_m\|_{L^2(0,T;L^2_\sigma)} \le 2\|\underline{f}\|_{L^2(0,T;L^2_\sigma)}$

(BP) $\underline{u}_m(0) = \sum_{k=1}^m (u_0, \underline{v_k}) \underline{v_k}$

Similar to the case of the heat equation, we observe that

$$(\partial_t \underline{u}_m(t), \underline{v}_s) = \sum_{k=1}^{m} \dot{c}_k^m(t)(\underline{v}_k, \underline{v}_s) = \dot{c}_s^m(t)$$

and

$$a(\underline{u}_m(t), \underline{v}_s) = \sum_{k=1}^{m} c_k^m(t)\, a(\underline{v}_k, \underline{v}_s) = \sum_{k=1}^{m} c_k^m(t)\lambda_k(\underline{v}_k, \underline{v}_s) = c_s^m(t)\lambda_s$$

Moreover we have:

$$b(\underline{u}_m(t), \underline{u}_m(t), \underline{v}_s) = \int_\Omega (\underline{u}_m(t)\cdot\nabla)\underline{u}_m(t)\cdot \underline{v}_s\,dx = \sum_{k=1}^{m}\sum_{j=1}^{m} c_k^m(t)\, c_j^m(t)\int_\Omega (\underline{v}_k\cdot\nabla)\underline{v}_j\cdot\underline{v}_s\,dx$$

Recalling $[H^1(\Omega)]^3 \hookrightarrow [L^6(\Omega)]^3$ (in 3D) we infer that

$$\left|\int_\Omega (\underline{v}_k\cdot\nabla)\underline{v}_j\cdot\underline{v}_s\,dx\right| \le \|\underline{v}_k\|_{L^3(\Omega)}\|\nabla\underline{v}_j\|_{L^2(\Omega)}\|\underline{v}_s\|_{L^6(\Omega)} \le C(\Omega)\|\underline{v}_k\|_{L^6(\Omega)}\|\nabla\underline{v}_j\|_{L^2(\Omega)}\|\underline{v}_s\|_{L^6(\Omega)}$$
$$\le C(\Omega)C_S^2\|\underline{v}_k\|_{H^1(\Omega)}\|\nabla\underline{v}_j\|_{L^2(\Omega)}\|\underline{v}_s\|_{H^1(\Omega)} < +\infty$$

Summing up, (AP) is equivalent to:

$$\dot{c}_s^m(t) + \lambda_s c_s^m(t) + \sum_{k=1}^{m}\sum_{j=1}^{m} c_k^m(t)c_j^m(t)\, b(\underline{v}_k, \underline{v}_j, \underline{v}_s) = (\underline{f}_m(t), \underline{v}_s) \quad \forall s = 1, \dots, m$$

Setting $\underline{C}^m(t) = (c_1^m(t), \dots, c_m^m(t))$, $\underline{G}^m = ((u_0, \underline{v}_1), \dots, (u_0, \underline{v}_m))$

and $\underline{F}: \mathbb{R}^m \times [0,T] \longrightarrow \mathbb{R}^m$, $(\underline{F}(y,t))_s = -\lambda_s y_s - \sum_{i=1}^{m}\sum_{j=1}^{m} y_k y_j\, b(\underline{v}_k, \underline{v}_j, \underline{v}_s) - (\underline{f}_m(t), \underline{v}_s)$

$\forall x \in \mathbb{R}^n, \exists \delta : \forall y \in \mathbb{R}^m \; |y-x| < \delta$ and L
$|f(x) - f(y)| \le L|x-y|$

We observe F is locally Lipschitz cts w.r.t y uniformly in t and cts w.r.t both components.

We deduce (AP)-(BP) is equivalent to the following non-linear system of ODEs:

$$\begin{cases} \underline{\dot{C}}^m(t) = \underline{F}(\underline{C}^m, t) & (P_m) \\ \underline{C}^m(0) = \underline{G}^m \end{cases}$$

Thanks to the Cauchy-Lipschitz thm for non-linear system of ODEs, $\forall m \in \mathbb{N}$ there exists $T_m > 0$ and $\underline{C}^m \in C^1([0,T_m])$ that solves (P_m). Because of non-linearity we don't have a solution for all time.
As a consequence, there exists a unique $\underline{u}_m \in C^1([0,T_m]; V_m)$ solving (AP)-(BP).

2) Energy estimates

Multiplying each $(AP)_s$ by $c_s^m(t)$ and summing over s from 1 to n, we find

$$\frac{d}{dt}\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} + \underbrace{b(\underline{u}_m(t), \underline{u}_m(t), \underline{u}_m(t))}_{=0;\ \text{property of trilinear term}} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} = (\underline{f}_m(t), \underline{u}_m(t))$$

Indeed, $b(\underline{u}, \underline{u}, \underline{u}) = \int_\Omega u_j \partial_j u_i u_i\,dx = \int_\Omega u_j \partial_j\left(\frac{1}{2}|u|^2\right)dx \overset{IBP}{=} -\int_\Omega \underbrace{\partial_j u_j}_{\mathrm{div}\,\underline{u}=0} \frac{1}{2}|u|^2\,dx = 0$

Moreover by Poincaré inequality we have $|(f_m(t), u_m(t))| \le \|f_m(t)\|_{L^2(\Omega)}\|u_m(t)\|_{L^2(\Omega)} \le \|f_m(t)\|_{L^2(\Omega)} C_p \|\nabla u_m(t)\|_{L^2}$
$\le \frac{1}{2}\|\nabla u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}C_p^2\|f_m(t)\|^2_{L^2(\Omega)}$. Therefore, we find

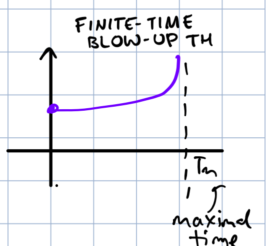
$$\frac{d}{dt}\|\underline{u}_m(t)\|^2_{L^2(\Omega)} + \|\nabla\underline{u}_m(t)\|^2_{L^2(\Omega)} \le C_p^2\|\underline{f}_m(t)\|^2_{L^2(\Omega)} \quad \text{for any } t \in [0, T_m)$$

Integrating in time from 0 to $t \in [0, T_m)$, we have

$$\|\underline{u}_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\underline{u}_m(s)\|^2_{L^2(\Omega)}ds \le \|\underline{u}_m(0)\|^2_{L^2(\Omega)} + \int_0^t C_p^2\|f_m(s)\|^2_{L^2(\Omega)}ds$$
$$\overset{O-B}{\le} \|u_0\|^2_{L^2(\Omega)} + 2C_p^2\int_0^T \|f(s)\|^2_{L^2(\Omega)}ds$$

This means that $\sup_{t\in[0,T_m]}\|u_m(t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + 2C_p^2\int_0^T\|f(s)\|^2_{L^2(\Omega)}ds =: C_1(u_0, T, f)$

Similarly $\int_0^{T_m}\|\nabla u_m(s)\|^2_{L^2(\Omega)}ds \le C_1(u_0, T, f)$



Notice $\|u_m(t)\|^2_{L^2(\Omega)} = \sum_{k=1}^{n} |c_k^m(t)|^2 \leq C_1$ for any $t \in [0,T_m]$.
Therefore by the finite-time blow up theorem for ODEs, we infer that $\underline{c}^m$ is defined on $[0,T]$. In particular, $\underline{c}^m \in C^1([0,T])$ implying that $\underline{u}_m \in C^1([0,T]; V_m)$.
Moreover we have

$$\sup_{t\in[0,T]} \|u_m(t)\|^2_{L^2(\Omega)} \leq C_1(\underline{u}_0, T, f) \quad \& \quad \int_0^T \|\nabla \underline{u}_m(s)\|^2_{L^2(\Omega)} ds \leq C_1(\underline{u}_0, T, \underline{f})$$

We are now ready to find an estimate of the time derivative. We need the following interpolation inequality:

$$\|u\|_{L^3(\Omega)} \leq C \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) \qquad \text{Try to prove} \qquad \text{(GN)} \quad \text{(3D)}$$

Better than Sobolev embedding: $\|u\|_{L^3} \leq C\|u\|_{H^1}$

For any $v \in H^1_\sigma$ we write $v = w + z$, $w \in V_m$, $z \in V_m^\perp$.

$$|\langle \partial_t u_m(t), v\rangle_*| = |(\partial_t \underline{u}_m(t), \underline{v})| = |(\partial_t \underline{u}_m(t), \underline{w})| = |-b(\underline{u}_m(t), \underline{u}_m(t), \underline{w}) - a(\underline{u}_m(t), \underline{u}_m(t), \underline{w}) - (\underline{f}_m(t), \underline{w})|$$
$$\leq \|\underline{u}_m(t)\|_{L^3(\Omega)} \|\nabla \underline{u}_m(t)\|_{L^2(\Omega)} \|\underline{w}\|_{L^6(\Omega)} + \|\nabla u_m(t)\|_{L^2(\Omega)} \|\nabla \underline{w}\|_{L^2(\Omega)} + \|\underline{f}_m(t)\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}$$
$$\leq (\|\underline{u}_m(t)\|_{L^3(\Omega)} \|\nabla \underline{u}_m(t)\|_{L^2(\Omega)} + \|\nabla u_m(t)\|_{L^2(\Omega)} + \|\underline{f}_m(t)\|_{L^2(\Omega)}) \|w\|_{H^1_\sigma}$$
$$\leq (\|\underline{u}_m(t)\|_{L^3(\Omega)} \|\nabla \underline{u}_m(t)\|_{L^2(\Omega)} + \|\nabla u_m(t)\|_{L^2(\Omega)} + \|\underline{f}_m(t)\|_{L^2(\Omega)}) \|v\|_{H^1_\sigma}$$

Then we have: $\int_0^T \|\partial_t \underline{u}_m(s)\|^{\frac{4}{3}}_{(H^1_\sigma)'} ds \leq C\int_0^T \|\underline{u}_m(s)\|^{\frac{4}{3}}_{L^3(\Omega)} \|\nabla \underline{u}_m(s)\|^{\frac{4}{3}}_{L^2(\Omega)} ds + \int_0^T \|\nabla u_m(s)\|^{4/3}_{L^2(\Omega)} ds + \int_0^T \|f_m(t)\|^{4/3}_{L^2(\Omega)} ds$

$$\overset{(G\text{-}N)}{\leq} C\int_0^T \|\underline{u}_m(s)\|^{\frac{2}{3}}_{L^2(\Omega)} \|\nabla u_m(s)\|^{\frac{2}{3}+\frac{4}{3}}_{L^2(\Omega)} ds + C(T)\int_0^T \|\nabla u_m(s)\|^2_{L^2(\Omega)} ds + C(T)\int_0^T \|f_m(s)\|^2_{L^2(\Omega)} ds$$

$\|\underline{u}_m(s)\|^{\frac{4}{3}}_{L^3(\Omega)} \|\nabla \underline{u}_m(s)\|^{\frac{4}{3}}_{L^2(\Omega)}$

$\leq \|u_m(s)\|^{\frac{2}{3}}_{L^2} \|u_m(s)\|^{\frac{2}{3}}_{H^1} \|\nabla u_m\|^{\frac{4}{3}}_{L^2}$

$= \|u_m\|^{\frac{2}{3}}_{L^2} (\|u_m\|^2_{L^2} + \|\nabla u_m\|^2_{L^2})^{\frac{1}{3}} \|\nabla u_m\|^{\frac{4}{3}}_{L^2}$

$\leq C \|u_m\|^{\frac{2}{3}}_{L^2} \|\nabla u_m\|^2_{L^2}$

$$\leq C\cdot C_1 \int_0^T \|\nabla \underline{u}_m(s)\|^2_{L^2(\Omega)} ds + C(T)\int_0^T \|\nabla \underline{u}_m(s)\|^2_{L^2(\Omega)} ds + 2C(T)\int_0^T \|f(s)\|^2_{L^2(\Omega)} ds$$
$$\leq C\cdot C_1^2 + C(T)C_1 + 2C(T)\int_0^T \|f(s)\|^2_{L^2(\Omega)} ds =: C_2(T, \underline{u}_0, \underline{f})$$

Summing up, we found that $\underline{u}_m$ is bounded in $L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$
$\partial_t \underline{u}_m$ is bounded in $L^{\frac{4}{3}}(0,T; (H^1_\sigma)')$

3) Passage to the limit
$\forall \varphi \in \mathcal{D}(0,T)$, $\int_0^T \langle \partial_t \underline{u}_m(t), v_s\rangle \varphi(t)\, dt + \int_0^T b(u_m(t), u_m(t), v_s)\varphi(t)\, dt + \int_0^T a(\underline{u}_m(t), \underline{v}_s)\varphi(t)\, dt = \int_0^T (f_m(t), v_s)\varphi(t)\, dt$

First of all, $\int_0^T (f_m(t), v_s)\varphi(t)\, dt \longrightarrow \int_0^T (f(t), \underline{v}_s)\varphi(t)\, dt$ (Same as Heat eqn!)
Since $L^2(0,T; H^1_\sigma)$ is a Hilbert space, there exists a subsqnce u_{m_j} s.t. $u_{m_j} \longrightarrow u$ weakly in $L^2(0,T; H^1_\sigma)$ "balls are weakly compact"
This implies that $\int_0^T a(\underline{u}_m(t), \underline{v}_s)\varphi(t)\, dt \longrightarrow \int_0^T a(\underline{u}(t), \underline{v}_s)\varphi(t)\, dt$ $\quad \nabla u_{m_j} \overset{w}{\longrightarrow} \nabla u_{m_j}$
Indeed $v_s(\cdot)\varphi(t) \in L^2(0,T; H^1_\sigma) \Longrightarrow$ belongs to the dual $\Rightarrow$ result follows.
Let us recall $(L^p(0,T;X))'$ (X Banach) $= L^{p'}(0,T,X')$, $\frac{1}{p}+\frac{1}{p'}=1$. Then we infer that $\partial_t u_{m_j} \longrightarrow \partial_t \underline{u}$ weakly in $L^{\frac{4}{3}}(0,T; (H^1_\sigma)')$ which means

$$\int_0^T \langle \partial_t \underline{u}_{m_j}(t), w(t)\rangle_* \, dt \longrightarrow \int_0^T \langle \partial_t u(t), w(t)\rangle_* \, dt \quad \text{for any } w \in L^4(0,T; H^1_\sigma).$$

But since $v_s\varphi \in L^\infty(0,T; H^1_\sigma)$ we obtain $\int_0^T \langle \partial_t u_{m_j}(t), \underline{v}_s\rangle_* \varphi(t)\, dt \longrightarrow \int_0^T \langle \partial_t u(t), \underline{v}_s\rangle \varphi(t)\, dt$
We are left to handle the non-linear term.

Since $H^1_\sigma \hookrightarrow [L^6(\Omega)]^3$ it is enough to show $u_{m_j}\cdot \nabla u_{m_j} \longrightarrow (u\cdot\nabla)\underline{u}$ weakly in $L^q(0,T; L^{\frac{6}{5}}(\Omega))$ some $q>1$.
Using the above estimates it is possible to show that $u_{m_j}\cdot \nabla u_{m_j}$ is bdd in $L^{\frac{4}{3}}(0,T; L^{\frac{6}{5}}(\Omega))$
Therefore up to a subsequence we have $(\underline{u}_{m_j}\cdot\nabla)\underline{u}_{m_j} \longrightarrow g$ weakly in $L^{\frac{4}{3}}(0,T; L^{\frac{6}{5}}(\Omega))$.
We need to show $g = (\underline{u}\cdot\nabla)\underline{u}$.

- **THM (AUBIN-LIONS):** Let X_0, X, X_1 be three Banach spaces s.t. $X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1$. Then, the set $E = \{\underline{u} \in L^p(0,T;X_0),\ \partial_t \underline{u} \in L^q(0,T;X_1)\}$ with $1 \le p,q < \infty$ is compactly embedded in $L^p(0,T;X)$.

We infer from the Aubin-Lions thm that $\underline{u}_{m_J} \longrightarrow \underline{u}$ in $L^2(0,T;L^p(\Omega))$, $\forall p<6$ $\quad (H^1 \overset{c}{\hookrightarrow} L^p,\ p<6)$

This is sufficient to show that $(\underline{u}_{m_J}\cdot\nabla)\underline{u}_{m_J} \longrightarrow (\underline{u}\cdot\nabla)\underline{u}$ weakly in $L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega))$.

MASTERY MATERIAL I

- **BANACH FIXED POINT THEOREM:** Assume $A: X \to X$ (X Banach) is a nonlinear mapping and assume

  (1) $\|A[u]-A[\tilde{u}]\| \le \gamma \|u-\tilde{u}\|$ $(u,\tilde{u} \in X)$ for some constant $\gamma < 1$.

  Then A has a unique fixed point

  A is a strict contraction if (1) holds

- **Ex:** Reaction diffusion eqns.

$$(2)\begin{cases} \underline{u}_t - \Delta \underline{u} = f(\underline{u}) & \text{in } U_T \\ \underline{u} = 0 & \text{on } \partial U \times [0,T] \\ \underline{u} = \underline{g} & \text{on } U \times \{t=0\} \end{cases}$$

Here, $\underline{u} = (u^1,\dots,u^m)$, $\underline{g} = (g^1,\dots,g^m)$, $U_T = U \times (0,T]$. Suppose $g \in H_0^1(U;\mathbb{R}^m)$ $\quad U$ open, bdd, ∂U smooth.

- We suppose $f: \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous. This hypothesis in particular implies $|f(z)| \overset{(4)}{\le} C(1+|z|)$ for each $z \in \mathbb{R}^m$ and some constant C.
- We say a function $u \in L^2(0,T;H_0^1(U;\mathbb{R}^m))$, with $u' \in L^2(0,T;H^{-1}(U;\mathbb{R}^m))$ $(*)$ is a weak solution of (2) provided: $\langle u',v\rangle + B[u,v] = (f(u),v)$ a.e. $0 \le t \le T$ for each $v \in H_0^1(\Omega;\mathbb{R}^m)$ and $u(0) = g$.
- $\langle\cdot,\cdot\rangle$ denotes the pairing of $H^{-1}(U;\mathbb{R}^m)$ and $H_0^1(U;\mathbb{R}^m)$, $B[,]$ is the bilinear form associated with $-\Delta$ in $H_0^1(\Omega;\mathbb{R}^m)$ and $(,)$ denotes the inner product in $L^2(U;\mathbb{R}^m)$.

  Also, $\|u\|_{H_0^1(U;\mathbb{R}^m)} = \left(\int_U |Du|^2 dx\right)^{\frac{1}{2}}$
- $(*)$ implies $u \in C([0,T];L^2(U;\mathbb{R}^m))$

- Claim: there exists unique solution of (2)

1. We apply Banach's theorem in the space $X = C([0,T];L^2(U;\mathbb{R}^m))$ with norm $\|v\| = \max_{0\le t\le T} \|v(t)\|_{L^2(U;\mathbb{R}^m)}$

   Let A be defined as follows:

   Given a function $u \in X$, set $h(t) = f(u(t))$ $(0 \le t \le T)$. In light of (4) we see $h \in L^2(0,T;L^2(U;\mathbb{R}^m))$. Consequently, previous theory asserts that:
   $$(8)\begin{cases} w_t - \Delta w = h & \text{in } U_T \\ w = 0 & \text{on } \partial U \times [0,T] \\ w = g & \text{on } U \times \{t=0\} \end{cases}$$
   has a unique weak solution

   (9) $w \in L^2(0,T;H_0^1(U;\mathbb{R}^m))$ with $w' \in L^2(0,T;H^{-1}(U;\mathbb{R}^m))$.

   Thus $w \in X$ satisfies:
   $$(10)\quad \langle w',v\rangle + B[w,v] = (h,v) \quad \text{a.e. } 0 \le t \le T$$
   for each $v \in H_0^1(U;\mathbb{R}^m)$, and $w(0) = g$. Define $A: X \to X$ by setting $A[u] = w$.

2. We now claim: If $T>0$ small enough then A is a strict contraction (11)

   To prove this, choose $u,\tilde{u} \in X$, define $w = A[u]$, $\tilde{w} = A[\tilde{u}]$ as above. Consequently w verifies (10) for $h = f(u)$, $\tilde{w}$ satisfies (10) for $\tilde{h} := f(\tilde{u})$. We calculate:

$$\left.\begin{aligned} (10)^*:&\ \langle w',w\rangle + B[w,w] = (h,w) \\ (10)^{**}:&\ \langle \tilde{w}',\tilde{w}\rangle + B[\tilde{w},\tilde{w}] = (h,\tilde{w}) \end{aligned}\right\} (10)^* - (10)^{**}: \quad \langle (w-\tilde{w})', w-\tilde{w}\rangle + B[w-\tilde{w},w-\tilde{w}] = (w-\tilde{w}, h-\tilde{h})$$

$$\frac{d}{dt}\|w(t)-\tilde{w}(t)\|^2_{L^2(U;\mathbb{R}^m)} + 2\|w(t)-\tilde{w}(t)\|^2_{H_0^1(U;\mathbb{R}^m)} = 2(w-\tilde{w},h-\tilde{h})$$

$$2(w(t)-\tilde{w}(t), h(t)-\tilde{h}(t)) \le 2\|w(t)-\tilde{w}(t)\|_{L^2(U;\mathbb{R}^m)}\|h(t)-\tilde{h}(t)\|_{L^2(U;\mathbb{R}^m)} \le \frac{1}{\varepsilon}\|f(u(t))-f(\tilde{u}(t))\|^2_{L^2(U;\mathbb{R}^m)} + \varepsilon\|w(t)-\tilde{w}(t)\|^2_{L^2(U;\mathbb{R}^m)}$$

$$\overset{\text{Poincare}}{\le} \varepsilon C\|w(t)-\tilde{w}(t)\|_{H_0^1(U;\mathbb{R}^m)} + \frac{1}{\varepsilon}\|f(u(t))-f(\tilde{u}(t))\|^2_{L^2(U;\mathbb{R}^m)}$$

Taking ε small enough, we deduce $\frac{d}{dt}\|w-\tilde{w}\|^2_{L^2(U;\mathbb{R}^m)} \le C\|f(u(t))-f(\tilde{u}(t))\|^2_{L^2(U;\mathbb{R}^m)} \overset{f\text{ Lipschitz}}{\le} C\|u(t)-\tilde{u}(t)\|^2_{L^2(U;\mathbb{R}^m)}$

Consequently, $\|w(s)-\tilde{w}(s)\|^2_{L^2(U;\mathbb{R}^m)} \leq C\int_0^s \|u(t)-\tilde{u}(t)\|^2_{L^2(U;\mathbb{R}^m)}\,dt \leq CT\|u-\tilde{u}\|^2_{L^2(0,T;U;\mathbb{R}^m)}$ (13)
Thus $\|A[u]-A[\tilde{u}]\| \leq (CT)^{\frac{1}{2}}\|u-\tilde{u}\|$. Thus A is a strict contraction provided $T>0$ is so small that $(CT)^{\frac{1}{2}} = \gamma < 1$.

3. Given any $T>0$ we select $T_1>0$ so small that $(CT_1)^{\frac{1}{2}} < 1$. We can apply Banach's Fixed point thm to find a weak solution u of (2) existing on $[0,T_1]$. Since $u(t) \in H_0^1(U;\mathbb{R}^m)$ for a.e. $0 \leq t \leq T_1$ we can upon redefining T_1 if necessary assume $u(T_1) \in H_0^1(U;\mathbb{R}^m)$. We can repeat the above argument to extend our soln. to $[T_1, 2T_1]$. Continuing, after finitely many steps we get a weak soln. on $[0,T]$.

4. Uniqueness: suppose $u, \tilde{u}$ are two weak solutions of (2). Then from (13) we get $w=u$, $\tilde{w}=\tilde{u}$, whence $\|u(s)-\tilde{u}(s)\|^2_{L^2(U;\mathbb{R}^m)} \leq C\int_0^s \|u(t)-\tilde{u}(t)\|^2_{L^2(U;\mathbb{R}^m)}\,dt$ for $0 \leq s \leq T$.
Using Gronwall: ($G\equiv 0$, $H \equiv 1$)
[Let F,G,H be cts functions in $[0,T]$ such that H is non-negative and G non-decreasing.
If they satisfy:
$$F(t) \leq G(t) + \int_0^T F(s)H(s)\,ds, \ \forall t\in[0,T] \quad \text{then} \quad F(t) \leq G(t)e^{\int_0^t H(s)ds} \ \forall t \in [0,T]$$]
we get $u \equiv \tilde{u}$ □

- **Schauder's Fixed point theorem:** Suppose $K \subset X$ compact & convex and assume also $A: K \to K$ is continuous. Then A has a fixed point in K.

- **DEF:** A nonlinear mapping $A: X \to X$ is called compact provided for each bdd sequence $\{u_k\}_{k=1}^\infty$ the sequence $\{A[u_k]\}_{k=1}^\infty$ is precompact; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ s.t. $\{A[u_{k_j}]\}_{k=1}^\infty$ converges in X.

- **Schaefer's Fixed point thm:** Suppose $A: X \to X$ is a continuous and compact mapping. Assume further that the set $\{u \in X : u = \lambda A[u] \text{ for some } 0 \leq \lambda \leq 1\}$ is bdd. Then A has a fixed pt.

- **Ex:** Quasilinear Elliptic PDE. (21) $\begin{cases} -\Delta u + b(Du) + \mu u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ where U bdd, ∂U smooth.

We assume $b: \mathbb{R}^n \to \mathbb{R}$ is smooth, Lipschitz continuous and $\therefore$ satisfies the growth condition
(22) $|b(p)| \leq C(|p|+1)$ for some constant C and all $p \in \mathbb{R}^n$

- **THM:** If $\mu > 0$ sufficiently large, there exists a function $u \in H^2(U) \cap H_0^1(U)$ solving (21).
↳ **PF:**
1. Given $u \in H_0^1(U)$, set $f \overset{(23)}{:=} -b(Du)$. Owing to (22), we see $f \in L^2(U)$. Now let $w \in H_0^1(U)$ be the unique weak soln of: $\begin{cases} -\Delta w + \mu w = f & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$ (24).

By regularity theory we additionally know $w \in H^2(U)$ with the estimate $\|w\|_{H^2(U)} \leq C\|f\|_{L^2(U)}$. (25)
Let us henceforth write $A[u] = w$, whenever w is derived from u via (23).
In light of (22) and (25) we get the estimate $\|A[u]\|_{H^2(U)} \leq C(\|u\|_{H_0^1(U)} + 1)$ (26)

2. We now assert $A: H_0^1(U) \to H_0^1(U)$ is continuous and compact. Indeed, if $u_k \overset{(27)}{\to} u$ in $H_0^1(U)$, then in view of (26), $\sup_k \|w_k\|_{H^2(U)} < +\infty$ for $w_k = A[u_k]$. So there is a subsqnce $\{w_{k_j}\}_{j=1}^\infty$ and a function $w \in H_0^1(U)$ with $w_{k_j} \to w$ in $H_0^1(U)$ (29).
Now:
$$\int_U Dw_{k_j}\cdot Dv + \mu w_{k_j} v\,dx = -\int_U b(Du_{k_j})v\,dx \quad \forall v \in H_0^1(U).$$

Using (22), (27), (29) we get $\int_U Dw\cdot Dv + \mu w v\,dx = -\int_V b(Du)v\,dx$ for each $v\in H_0^1(U)$.
Thus $w = A[u]$.
(27) implies $A[u_k] \longrightarrow A[u]$ in $H_0^1(U) \Rightarrow A$ is cts. A similar argument shows A is compact since if $\{u_k\}_{k=1}^{\infty}$ is bdd in $H_0^1(U)$, (22) says $\{A[u_k]\}_{k=1}^{\infty}$ is bdd in $H^2(U)$ and $\therefore$ posesses a strongly convergent subsqnce in $H_0^1(U)$.

3. Finally need to show that if μ is large enough, the set $\{u\in H_0^1(U) : u=\lambda A[u],\ 0\le\lambda\le 1\}$ is bounded in $H_0^1(U)$. So assume $u\in H_0^1(U)$,
$u=\lambda A[u]$ $(0<\lambda\le 1)$. Then $\frac{u}{\lambda} = A[u]$ or in other words $u\in H^2(U)\cap H_0^1(U)$ and $\because$
$-\Delta u + \mu u = \lambda b(Du)$ a.e. in U.
Multiply this by u and integrate over U:

$$\int_U |Du|^2 + \mu|u|^2\,dx = -\int_U \lambda b(Du)u\,dx \le \int_U C(|Du|^2+1)|u|\,dx \le \frac{1}{2}\int_U |Du|^2 dx + C\int_U |u|^2+1\,dx$$

So if $\mu>0$ is sufficiently large, we get $\|u\|_{H_0^1(\Omega)} \le C$, C doesn't depend on λ.

4. Applying Schaefer's Fixed point thm in $X = H_0^1(U)$, we conclude A has a fixed point $u\in H_0^1(U)\cap H^2(U)$, which in turn solves our semilinear PDE (21) $\square$

MASTERY MATERIAL II

Ch.18: An abstract Elliptic problem

- $(V, \langle\cdot,\cdot\rangle, \|\cdot\|_V)$ be a real Hilbert space. Assume we have an elliptic operator $A\in L(V,V^*)$ associated to a cts bilinear form a on V. Thus, $\langle Av, v\rangle \ge \alpha\|v\|_V^2$ $\forall v\in V$ some $\alpha>0$.
Let $B: V\to V^*$ be a cts compact map satisfying: $\langle B(v), v\rangle \ge -k - \beta\|v\|_V^2$, $\forall v\in V$, some $k\ge 0$, $0\le\beta<\alpha$.
Then: For a given $g\in V^*$, $Au + B(u) = g$ admits a possibly non-unique soln.
i.e. $a(u,v) + \langle B(u), v\rangle = \langle g, v\rangle$ $\forall v\in V$.

- Ex: $\Omega\subset\mathbb{R}^3$ bdd domain with smooth boundary $\partial\Omega$. Consider the non-linear elliptic boundary value problem:
$$\begin{cases} -\Delta u + B(u) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (18.4)$$
$L^{\frac{np}{n-p}} \subset$
where $B(u) = |u|^{p-1}u$ with $p\in[1,5)$.

Setting $V = H_0^1(\Omega)$, recall 3D sobolev embeddings: $H_0^1(\Omega) \overset{c}{\subset} L^{p+1}(\Omega)$, $L^{\frac{p+1}{p}}(\Omega) \overset{c}{\subset} H^{-1}$ (18.5)
Recall: $[L^{p+1}(\Omega)]^* = L^{\frac{p+1}{p}}(\Omega)$.

Take $g\in V^*$ and observe: $\|B(u)\|_{L^{\frac{p+1}{p}}(\Omega)} = \left(\int_\Omega |u|^{\frac{(p+1)(p-1)}{p}}|u|^{\frac{p+1}{p}}dx\right)^{\frac{p}{p+1}} = \left(\int_\Omega |u|^{p+1}dx\right)^{\frac{p}{p+1}} = \|u\|^p_{L^{p+1}(\Omega)}$ (18.6)
Let $A = -\Delta$ with associated bilinear form $a(\cdot,\cdot) = \langle\cdot,\cdot\rangle_V$.
We call $u\in V$ a weak solution if $\langle u,\varphi\rangle_V + \langle B(u),\varphi\rangle = \langle g,\varphi\rangle$ $\forall\varphi\in V$

Need to show $u\mapsto B[u]$ meets hypotheses above. Clearly $\langle B(v), v\rangle \ge -k-\beta\|v\|_V^2$ holds for $k=\beta=0$. Using (18.5)-(18.6) we learn B carries bounded sets of V into bounded sets of $L^{\frac{p+1}{p}}(\Omega)$ and so relatively compact in V'. To prove continuity of B, for $u=u(x)$ and $v=v(x)$ in V we write $B(u)-B(v) = w\int_0^1 b(\lambda u + (1-\lambda)v)\,d\lambda$ where $w = u-v$, $b(s) = \frac{d}{ds}B(s) = p|s|^{p-1}$. Then:

Hö: $\{p, \frac{p}{p-1}\}$

$\|B(u)-B(v)\|_{L^{\frac{p+1}{p}}(\Omega)} \le C\left(\int_\Omega |w|^{\frac{p+1}{p}}\left(|u|^{\frac{(p+1)(p-1)}{p}} + |v|^{\frac{(p+1)(p-1)}{p}}\right)dx\right)^{\frac{p}{p+1}} \le C\|w\|_{L^{p+1}}\left(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}}\right)$
Using previous embeddings conclude: $\|B(u)-B(v)\|_{V^*} \le C\|w\|_V\left(\|u\|_V^{p-1} + \|v\|_V^{p-1}\right)$ $\therefore u\mapsto B(u)$ cts.
Using above result conclude there is a weak solution.

Uniqueness: Let u, v be two solutions to (18.4). Then $w = u - v$ solves:

$\langle w, \varphi \rangle_V + \langle B(u) - B(v), \varphi \rangle = 0 \quad \forall \varphi \in V.$

Choosing $\varphi = w$: $\|w\|_V^2 + \langle B(u) - B(v), w \rangle = 0$.

But in this case, $B(u) - B(v) = \ell w$, where $\ell = p \int_0^1 |\lambda u + (1-\lambda) v|^{p-1} d\lambda \geq 0$

So: $0 = \|w\|_V^2 + \langle B(u) - B(v), w \rangle \geq \|w\|_V^2 \Rightarrow w = 0$.

Ch.19: Semilinear Evolution Equations

- Cauchy problem in X: $\begin{cases} x'(t) = Tx(t) + f(t, x(t)), & 0 < t < t_0 \\ x(0) = x_0 \in X \end{cases}$ (19.2)

- THM: For every $x_0 \in X$ the Cauchy problem (19.2) has a unique mild solution. Moreover the map $x_0 \mapsto x(t)$ is Lipschitz cts from X into $C([0, t_0]; X)$.

- Ex: $\Omega \subset \mathbb{R}^3$ bdd domain with smooth boundary $\partial\Omega$. Consider the semilinear problem in the unknown $u = u(x,t) : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$:

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. The equation is supplemented with initial condition: $u(x,0) = u_0(x)$ for a given $u_0 \in L^2(\Omega)$. This problem can be phrased in abstract form (19.2) by setting $X = L^2(\Omega)$ and $T = -A_0$ where A_0 is the Laplace-Dirichlet operator on X defined at chapter 17, and viewing f as a Lipschitz cts function from X to X. Indeed it is possible to show such a T is the infinitesimal generator of a strongly continuous semigroup $S(t)$ acting on the Hilbert space X. Then we meet the hypotheses of Thm 19.3 so we have a unique mild solution defined for all times.

Ch.20: Abstract parabolic problem

- (V, H, V^*) Hilbert triple; V separable, $V \overset{c}{\hookrightarrow} H$. $A : V \to V^*$ is an elliptic operator associated to a cts coercive symmetric bilinear form on V (can be assumed to be $\langle \cdot, \cdot \rangle_V$). Let A_0 be the unbounded linear operator generated by A on H of domain $W = \{u \in V : Au \in H\}$. W is Hilbert, $\|\cdot\|_W = \|A_0 \cdot\|_H$

- Semilinear Cauchy problem:

$$\begin{cases} \partial_t u(t) + A_0 u(t) = f(u(t)), & 0 < t < t_0 \\ u(0) = \zeta \in V \end{cases} \quad (20.1)$$

DEF: Let $f : W \to H$. Soln to (20.1) is a function $u \in X_{t_0}$ with $\partial_t u \in L^2(0, t_0; H)$, $u(0) = \zeta$ that solves (20.1) a.e.

- THM 20.1: Suppose nonlinearity f satisfies $f(0) = 0$ and for every $u, v \in W$,

$$\|f(u) - f(v)\|_H \leq \|u - v\|_V (1 + \|u\|_W + \|v\|_W)^{1-\varepsilon} Q_0 (\|u\|_V + \|v\|_V) \quad \text{for some } \varepsilon \in (0, 1] \text{ and}$$

increasing +ve function Q_0. Then for every $R \geq 0$ $\exists t_0 = t_0(R) > 0$ s.t. the Cauchy problem (20.1) admits a unique solution $u \in X_{t_0}$ for every initial datum $\zeta \in V$ with $\|\zeta\|_V \leq R$.

- Ex: $\Omega \subset \mathbb{R}^3$ bdd domain with smooth boundary $\partial\Omega$. We consider the semilinear parabolic problem in the unknown $u = u(x,t) : \bar{\Omega} \times [0, t_0] \to \mathbb{R}$:

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \Omega \times (0, t_0] \\ u = 0 & \text{on } \partial\Omega \times (0, t_0] \end{cases}$$

where $f(u) = \pm |u|^{p-1} u$ with $p \in [1, 5)$. This problem is supplemented with initial condition $u(x,0) = \zeta(x)$ for a given $\zeta \in H_0^1(\Omega)$. This problem can be given in abstract form (20.1) by defining the spaces $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $W = H^2(\Omega) \cap H_0^1(\Omega)$ and $-\Delta = A_0$ with $\mathrm{dom}(A_0) = W$, that is the Laplace Dirichlet operator on H defined at ch.17. Now need to show non-linearity of f complies with hypotheses of thm 20.1 Set $w = u - v$, $\varphi(s) = \pm p |s|^{p-1}$, write:

$$f(u) - f(v) = w \int_0^1 \varphi(\lambda u + (1-\lambda) v) \, d\lambda.$$

Review class: elliptic problem in divergence form

· Let Ω be bdd domain in $\mathbb{R}^3$. Consider the elliptic problem: $\begin{cases} Lu = f + \operatorname{div}\underline{f} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ (EP)

where $Lu = -\operatorname{div}(A(x)\nabla u - \underline{b}(x)u) + \underline{c}(x)\cdot\nabla u + a(x)u$. Here,

"uniformly elliptic operator"

- ·) $A = (a_{ij}(x))_{ij}$: $a_{ij} \in C(\bar\Omega)$ $\forall i,j$, $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$ a.e. $x \in \bar\Omega$, where $\alpha > 0$. $|a_{ij}(x)| \leq M$, a.e in Ω
- ·) $\underline{b}, \underline{c} \in [C(\Omega)]^3$, $a \in C(\Omega)$
- ·) $f \in L^2(\Omega)$, $\underline{f} \in [L^2(\Omega)]^3$

First let $g = 0$.

AIM: Find conditions on $\underline{b}$, $\underline{c}$ and $\underline{a}$ that ensure the existence of a weak solution to (EP)

·) Firstly notice $F = f + \operatorname{div}\underline{f} \in H^{-1}(\Omega)$ and $\|F\| \leq C(\|f\|_{L^2(\Omega)} + \|\underline{f}\|_{L^2(\Omega;\mathbb{R}^n)})$.

·) Now introduce the weak formulation associated to (EP). Assume u is a smooth solution to (EP) and multiply by $v \in C_c^\infty(\Omega)$ (dense in $H_0^1(\Omega)$) and IBP:

$$\int_\Omega (-\operatorname{div}(A(x)\nabla u - \underline{b}(x)u) + \underline{c}(x)\cdot\nabla u + a(x)u)\cdot v\,dx = \int_\Omega (f + \operatorname{div}\underline{f})\cdot v\,dx$$

$$\Downarrow$$

$$\int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx - \underbrace{\int_{\partial\Omega}(A(x)\nabla u - \underline{b}(x)u)\cdot n\, v\,d\sigma}_{=0\,;\, v\in C_c^\infty(\Omega)} + \int_\Omega \underline{c}(x)\nabla u\cdot v\,dx + \int_\Omega a(x)u\cdot v\,dx = \int_\Omega f v\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx + \underbrace{\int_{\partial\Omega}\underline{f}\cdot n\,v\,d\sigma}_{=0\,;\,v\in C_c^\infty}$$

Let $B: H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ be defined as $B(u,v) := \int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla u\cdot v\,dx + \int_\Omega a(x)u\cdot v\,dx$

and $L(v) := \int_\Omega f v\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx$. Then, (WF) is: find $u \in H_0^1(\Omega)$ such that $B(u,v) = L(v)$ $\forall v \in H_0^1(\Omega)$

i) B bilinear: ez.

ii) B cts: $|B(u,v)| \leq |\int_\Omega A(x)\nabla u\nabla v\,dx| + \int_\Omega |\underline{b}(x)||u||\nabla v|\,dx + \int_\Omega |\underline{c}(x)||\nabla u\cdot v|\,dx + \int_\Omega |a(x)||u||v|\,dx$

① : $|\int_\Omega A(x)\nabla u\nabla v\,dx| = |\sum_{i,j}\int_\Omega a_{ij}(x)\partial_j u\,\partial_j v\,dx| \leq |\sum_{i,j}\int_\Omega M\,\partial_j u\,\partial_j v\,dx| \leq n^2 M\int_\Omega |\partial_j u||\partial_j v|\,dx \leq n^2M\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}$
$= n^2 M\|u\|_{H_0^1}\|v\|_{H_0^1}$ //

② $\int_\Omega |b(x)||u||\nabla v|\,dx \leq \|b\|_{L^\infty}\|u\|_{L^2}\|\nabla v\|_{L^2}$

③ $\int_\Omega |\underline{c}(x)||\nabla u\cdot v|\,dx \leq \|c\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2}$

④ $\int_\Omega |a(x)||u||v|\,dx \leq \|a\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}$

So $|B(u,v)| \leq n^2M\|u\|_{H_0^1}\|v\|_{H_0^1} + \|b\|_{L^\infty}\|u\|_{L^2}\|\nabla v\|_{L^2} + \|c\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2} + \|a\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}$
$\leq n^2M\|u\|_{H_0^1}\|v\|_{H_0^1} + C_p\|b\|_{L^\infty}\|u\|_{H_0^1}\|v\|_{H_0^1} + C_p\|c\|_{L^\infty}\|u\|_{H_0^1}\|v\|_{H_0^1} + C_p^2\|a\|_{L^\infty}\|u\|_{H_0^1}\|v\|_{H_0^1}$
$\leq \max\{n^2M, C_p\|b\|_{L^\infty}, C_p\|c\|_{L^\infty}, C_p^2\|a\|_{L^\infty}\}\|u\|_{H_0^1}\|v\|_{H_0^1} \equiv \chi\|u\|_{H_0^1}\|v\|_{H_0^1}$ //

iii) B coercive:
$$|B(u,u)| = \int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla u\,dx + \int_\Omega \underline{c}(x)\nabla u\cdot u\,dx + \int_\Omega a(x)u\cdot u\,dx$$
$$= \int_\Omega A(x)|\nabla u|^2dx - \int_\Omega b(x)u\cdot\nabla u\,dx + \int_\Omega \underline{c}(x)\nabla u\cdot u\,dx + \int_\Omega a(x)u^2dx$$
$$\geq \int_\Omega \alpha|\nabla u|^2dx - \int_\Omega \nabla(\tfrac{1}{2}u^2)b(x)\,dx + \int_\Omega \nabla(\tfrac{1}{2}u^2)\underline{c}(x)\,dx + \int_\Omega a(x)u^2dx$$
$$\geq \alpha\int_\Omega |\nabla u|^2dx + \int_\Omega \tfrac{1}{2}u^2\operatorname{div}(\underline{b}(x))\,dx - \int_\Omega \tfrac{1}{2}u^2\operatorname{div}(\underline{c}(x))\,dx + \int_\Omega a(x)u^2dx$$
$$\geq \alpha\|u\|_{H_0^1}^2 + \underbrace{\int_\Omega\left(\operatorname{div}\left(\tfrac{1}{2}\underline{b}(x) - \tfrac{1}{2}\underline{c}(x)\right) + a(x)\right)u^2\,dx}_{\text{need } \geq 0.}$$

"$\geq \alpha\|u\|_{H_0^1}^2$" ?

so provided $\operatorname{div}\left(\tfrac{1}{2}\underline{b}(x) - \tfrac{1}{2}\underline{c}(x)\right) + a(x) \geq 0$, B is coercive and by L-M, unique solution $u \in H_0^1$ exists satisfying $\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}(\|f\|_{C(\Omega)} + \|\underline{f}\|_{L^2(\Omega;\mathbb{R}^n)})$

Now assume $g \neq 0$; $g \in H^{\frac{1}{2}}(\partial\Omega)$. By trace theory there exists $\tilde{g} \in H^1(\Omega)$ satisfying $\gamma(\tilde{g}) = g$ on $\partial\Omega$.

We look for a solution $u = \tilde{u} + \tilde{g}$ where $\tilde{u} \in H_0^1(\Omega)$, $\tilde{g} \in H^1(\Omega)$. Using

$$\int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla u \cdot v\,dx + \int_\Omega a(x)u \cdot v\,dx = \int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx$$

$u = \tilde{u} + \tilde{g}$:
$$\int_\Omega (A(x)\nabla\tilde{u} + A(x)\nabla\tilde{g})\nabla v\,dx - \int_\Omega \underline{b}(x)\tilde{u}\nabla v\,dx - \int_\Omega \underline{b}(x)\tilde{g}\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla\tilde{u}\cdot v\,dx + \int_\Omega \underline{c}(x)\nabla\tilde{g}\cdot v\,dx + \int_\Omega a(x)\tilde{u}\cdot v\,dx + \int_\Omega a(x)\tilde{g}v\,dx = L(v).$$

$$\underbrace{\int_\Omega A(x)\nabla\tilde{u}\nabla v\,dx - \int_\Omega \underline{b}(x)\tilde{u}\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla\tilde{u}\cdot v\,dx + \int_\Omega a(x)\tilde{u}v\,dx}_{A(\tilde{u},v)} = \underbrace{\int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx - \int_\Omega A(x)\nabla\tilde{g}\nabla v\,dx + \int_\Omega \underline{b}(x)\tilde{g}\nabla v\,dx - \int_\Omega \underline{c}(x)\nabla\tilde{g}\cdot v\,dx - \int_\Omega a(x)\tilde{g}v\,dx + \int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx}_{L(v)}$$

So Variational formulation: $\begin{cases} u = \tilde{u} + \tilde{g}, \text{ where } \tilde{u} \in H_0^1(\Omega), \ \gamma(\tilde{g}) = g \\ A(\tilde{u}, v) = L(v) \quad \forall v \in H_0^1(\Omega) \end{cases}$

Use Lax-Milgram:

i) A Billinear: ez

ii) A cts: $|A(u,v)| \leq \gamma \|u\|\|v\|$ by before.

iii) A coercive: $A(u,u) \geq \alpha \|u\|_{H_0^1}^2$ by before

iv) L cts functional: $|L(v)| \leq (\cdot)\|v\|_{H_0^1}$

Next, consider: $\begin{cases} Lu = f + \operatorname{div}\underline{\tilde{f}} & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = g & \text{on } \partial\Omega \end{cases}$ (EP-R)

Assume u is a smooth soln to (EP-R), multiply by $v \in H^1(\Omega)$ and integrate:

$$\int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx - \int_{\partial\Omega} (A(x)\nabla u - \underline{b}(x)u)\cdot n\,v\,d\sigma + \int_\Omega \underline{c}(x)\nabla u \cdot v\,dx + \int_\Omega a(x)u\cdot v\,dx = \int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx + \int_{\partial\Omega} \underline{f}\cdot n\,v\,d\sigma$$

$$\int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla u\cdot v\,dx + \int_\Omega a(x)u\cdot v\,dx - \int_{\partial\Omega} A(x)(g - \beta u)v\,d\sigma + \int_{\partial\Omega} \underline{b}(x)\gamma(u)\cdot n\,v\,d\sigma = \int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx + \int_{\partial\Omega} \underline{f}\cdot n\,v\,d\sigma$$

let $B: H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ be defined as:

$$B(u,v) = \int_\Omega (A(x)\nabla u - \underline{b}(x)u)\nabla v\,dx + \int_\Omega \underline{c}(x)\nabla u\cdot v\,dx + \int_\Omega a(x)u\cdot v\,dx + \int_{\partial\Omega} \underline{b}(x)\gamma(u)\cdot n\,v\,d\sigma + \int_{\partial\Omega} A(x)\,\beta u v\,d\sigma$$

and

$$L(v) = \int_\Omega fv\,dx - \int_\Omega \underline{f}\cdot\nabla v\,dx + \int_{\partial\Omega} \underline{f}\cdot n\,v\,d\sigma + \int_{\partial\Omega} A(x) g v\,d\sigma$$

i) $\underline{B \text{ cts}}$: $|B(u,v)| \leq \gamma\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \|\underline{b}\|_{L^\infty(\partial\Omega)}\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} + n^2 M\|u\|_{L^2(\partial\Omega)}\|\beta\|_{L^\infty(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}$

recalling $\|u\|_{L^2(\partial\Omega)} \leq C_\gamma \|u\|_{H^1(\Omega)}$ $\forall u \in H^1(\Omega)$, we get:

$|B(u,v)| \leq (\gamma + \|\underline{b}\|_{L^\infty(\partial\Omega)}C_\gamma^2 + n^2 M C_\gamma^2\|\beta\|_{L^\infty(\partial\Omega)})\|u\|_{H^1}\|v\|_{H^1}$ //

ii) $\underline{B \text{ coercive}}$: $B(u,u) = A(u,u) + \underbrace{\int_{\partial\Omega} \underline{b}(x)\gamma(u)\cdot n\,u\,d\sigma}_{\geq 0} + \underbrace{\int_{\partial\Omega} A(x)\,\beta u u\,d\sigma}_{\geq 0} \geq \alpha\|u\|_{H_0^1}^2 = \alpha\|\nabla u\|_{L^2}^2 = \alpha(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2) \overset{\text{Poincare}}{\geq} \alpha(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2C_p}\|u\|_{L^2}^2) \geq \min\{\frac{\alpha}{2}, \frac{\alpha}{2C_p}\}\|u\|_{H^1}^2$

provided $\underline{b}(x) \geq 0$ on $\partial\Omega$.

iii) $\underline{L \in (H^1(\Omega))'}$: let $v \in H^1(\Omega)$. $|L(v)| \leq \|f\|_{L^2}\|v\|_{L^2} + \|\underline{f}\|_{L^2}\|\nabla v\|_{L^2} + \|\underline{f}\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} + n^2 M\|g\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}$