IMPERIAL

MATH70056

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebraic Geometry

Date: Thursday, May 9, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

As in the lectures and in the lecture notes, we always assume that k is an algebraically closed field: in particular, one can use the Nullstellensatz. As in the lectures and in the lecture notes, for $I \subseteq k[X_1, ..., X_n]$, $\mathcal{V}(I) \subseteq \mathbb{A}^n$ is the zero set of I, and, for $V \subseteq \mathbb{A}^n$, $\mathcal{I}(V) \subseteq k[X_1, ..., X_n]$ is the ideal of polynomials vanishing on V.

1. Affine varieties and Zariski topology

   (a) (i) What does it mean for a topological space to be *irreducible*? (A, seen) (2 marks)

   (ii) Prove that an affine variety V is irreducible if and only if its ring of functions $k[V]$ is an integral domain. (A, seen) (6 marks)

   (b) Describe the irreducible components of each of the following affine algebraic sets (with proof):

   (i) $V = \mathcal{V}(X^2 - YZ, XZ - X) \subseteq \mathbb{A}^3$ (A, method seen) (6 marks)

   (ii) $V = \mathcal{V}(XYZ, X^2 + Y^2 + Z^2 - 1) \subseteq \mathbb{A}^3$,(give the answers when the characteristic of k is $\neq 2$ and when it is 2). (B, method seen) (6 marks)

(Total: 20 marks)

2. Regular and rational maps

   (a) Show that, for V an affine variety, and $f \in k[V]$ a regular function, $D(f) = V - \mathcal{V}(f)$ (the open subset where $f \neq 0$) is isomorphic to an affine variety. (A, seen) (4 marks)

   (b) Notice that, even if regular maps are always continuous for the Zariski topology, a map which is continuous for the Zariski topology is not always regular.

   Consider that k is of characteristic $p > 0$ (recall that k is assumed to be algebraically closed), and consider the Frobenius map

$$\begin{aligned} F :& \mathbb{A}^1 \to \mathbb{A}^1 \\ & t \mapsto t^p \end{aligned}$$

   (i) Prove that F is a regular bijective map. (D, unseen) (4 marks)

   (ii) Prove that the inverse of F is continuous for the Zariski topology. (C, unseen) (6 marks)

   (iii) Prove that the inverse of F is not regular (hence F is not an isomorphism). (B, unseen) (6 marks)

(Total: 20 marks)

3. Projective varieties

We consider here the group $Aut(\mathbb{P}^n)$ of isomorphisms of $\mathbb{P}^n$ with itself, and the group $Bir(\mathbb{P}^n)$ of birational equivalences of $\mathbb{P}^n$ with itself.

We recall that any regular map $\mathbb{P}^n \to \mathbb{P}^m$ is globally defined by homogeneous polynomials $f_1, ..., f_m$, and that, if another family $g_1, ..., g_m$ defines ϕ globally, there is $\lambda \in k - \{0\}$ such that $g_i = \lambda f_i$ for all i.

(a) Prove that $Aut(\mathbb{P}^n)$ can be identified naturally with $PGl_{n+1}(k)$, the quotient of the group $Gl_{n+1}(k)$ of invertible $(n+1)\times(n+1)$ matrices, divided by the subgroup $\{\lambda Id : \lambda \in k - \{0\}\}$. (C, unseen) (6 marks)

(b) (i) Prove that any rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is regular. (B, seen) (4 marks)

(ii) Deduce that $Bir(\mathbb{P}^1) = Aut(\mathbb{P}^1)$. (A, unseen) (2 marks)

(c) Consider the rational map:

$$\begin{aligned} \phi : &\mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \\ &[x : y : z] \mapsto [yz : zx : xy] \end{aligned}$$

(i) Give the domain of ϕ, and justify your answer. (B, similar seen) (4 marks)

(ii) Show that ϕ is a birational equivalence, and give its inverse. (A, similar seen) (4 marks)

(Total: 20 marks)

4. Dimension theory

As in question $3)a)$, we consider the vector space $V_{n,d}$ of polynomials of degree d in $k[X_0, ..., X_n]$, and the associated projective space $P_{n,d}$. We denote by $Z_{i_0,...,i_n}$ the projective coordinate of $P_{n,d}$ corresponding to the monomial $X_0^{i_0}...X_n^{i_n}$ (with $i_0 + ... + i_n = d$). We write $N = \dim P_{n,d}$.

(a) (i) Recall and explain the formula for N in terms of n, d. (A, seen) (5 marks)

(ii) Prove that, given $N = \dim P_{n,d}$ points in $\mathbb{P}^n$, there is at least one hypersurface of degree d of $\mathbb{P}^n$ passing through those points. (A, similar seen) (3 marks)

(b) Consider the closed subset:

$$\mathcal{X} = \{([f], x_1, ..., x_N) \subset P_{n,d} \times (\mathbb{P}^n)^N : f(x_i) = 0\ \forall i\}$$

(i) Consider the first projection $p_1 : \mathcal{X} \to P_{n,d}$. Show that p_1 is surjective, and give the dimensions of the irreducible components of the fibers. Deduce from the fibre dimension theorem the dimensions of the irreducible components of $\mathcal{X}$. (D, unseen) (6 marks)

(ii) Consider the second projection $p_2 : \mathcal{X} \to (\mathbb{P}^n)^N$. Deduce from the fiber dimension theorem that there is a nonempty open subset $U \subset (\mathbb{P}^n)^N$ such that $p_2^{-1}(u)$ is a single point for $u \in U$ (which means that there is one and only one hypersurface of degree d containing those N points). (D, unseen) (6 marks)

(Total: 20 marks)

5. Mastery material: tangent space and singular points

For $f \in k[X_1, ..., X_n]$, we denote as usual by $df_x : k^n \to k$ the differential of f at $x \in \mathbb{A}^n$.

(a) For $V \subset \mathbb{A}^n$ and $x \in V$, recall the definition of T_xV, the tangent space of V at x, and prove that, if $f_1, ..., f_r$ generates $\mathcal{I}(V)$, then:

$$T_xV = \bigcap_{i=1}^{r} \ker d(f_i)_x \subseteq k^n$$

(5 marks)

(b) Consider a hypersurface $V \subsetneq \mathbb{A}^n$, and $g \in k[X_1, ..., X_n]$ generating $\mathcal{I}(V)$.

(i) Prove that $x \in V$ is a singular point of V if and only if $dg_x = 0$. (4 marks)

(ii) Prove that, given $f(X) \in k[X]$ a polynomial of degree 3, the polynomial $g(X,Y) = Y^2 - f(X) \in k[X,Y]$ is irreducible (hence V is irreducible too). (5 marks)

(iii) Give the singular points of C in terms of f, and find under which condition C is smooth.

(6 marks)

(Total: 20 marks)

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2024

MATH70056 Algebraic Geometry

The following information must be completed:

This paper is suitable for students from previous years.

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1)a)i): 2, 1)a)ii): 6, 1)b)i): 6, 2)a)i): 2, 3)b)i: 2, 3)b)iii): 2, 3)c)ii): 4, 4)a)i): 5 , 4)a)ii): 3

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1)b)ii): 6, 2)a)iii): 3, 2)b)iii): 3, 3)b)ii): 4, 3)c)i): 4

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2)a)ii): 4, 2)b)ii): 4, 3)a): 4

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
2)b)i): 4, 4)b)i): 6, 4)b)ii): 6

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Algebraic Geometry

Date: Thursday, 9th May 2024
Time: 14:00 – 16:30
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

As in the lectures, we always assume that k is an algebraically closed field: in particular, one can use the Nullstellensatz. As in the lectures, for $I \subseteq k[X_1, ..., X_n]$, $\mathcal{V}(I) \subseteq \mathbb{A}^n$ is the zero set of I, and, for $V \subseteq \mathbb{A}^n$, $\mathcal{I}(V) \subseteq k[X_1, ..., X_n]$ is the ideal of polynomials vanishing on V.

1. *Affine varieties and Zariski topology*

(a) (i) *What does it mean for a topological space to be irreducible? (A, seen)* (2 marks)

A topological space V is irreducible if it cannot be written as the union of two proper closed subsets.

(ii) *Prove that an affine variety V is irreducible if and only if its ring of functions $k[V]$ is an integral domain. (A, seen)* (6 marks)

Suppose that V is irreducible. Consider two elements $f, g \in k[V]$ such that $fg = 0$. Then $V = \mathcal{V}(f) \cup \mathcal{V}(g)$, and $\mathcal{V}(f)$ and $\mathcal{V}(g)$ are closed subsets of V: by irreduciblility, one of them must be equal to V, hence by the Nullstellensatz $f = 0$ or $g = 0$.

Suppose that $k[V]$ is an integral domain. Consider a decomposition $V = V_1 \cup V_2$ into closed subsets. Then $\mathcal{I}(V_1) \cap \mathcal{I}(V_2) = \{0\}$. Because $k[V]$ is an integral domain, it implies that $\mathcal{I}(V_i) = 0$ for some i, and then, form the Nullstellensatz, that V_1 or V_2 is proper.

(b) *Describe the irreducible components of each of the following affine algebraic sets (with proof):*

(i) V=$\mathcal{V}(X^2 - YZ, XZ - X) \subseteq \mathbb{A}^3$ *(A, method seen)* (6 marks)

$$\begin{aligned} V &= \mathcal{V}(X^2 - YZ, X(Z-1)) = \mathcal{V}(X^2 - YZ, X) \cup \mathcal{V}(X^2 - YZ, Z - 1) \\ &= \mathcal{V}(YZ, X) \cup \mathcal{V}(X^2 - Y, Z - 1) \\ &= \mathcal{V}(Y, X) \cup \mathcal{V}(Z, X) \cup \mathcal{V}(X^2 - Y, Z - 1) \end{aligned} \tag{1}$$

V is the union of the Z axis $\mathcal{V}(Y, X)$ (a line), the Y axis $\mathcal{V}(Z, X)$ (a line) and the parabola $\mathcal{V}(X^2 - Y, Z - 1)$.

(ii) $V = \mathcal{V}(XYZ, X^2 + Y^2 + Z^2 - 1) \subseteq \mathbb{A}^3$,*(give the answers when the characteristic of k is $\neq 2$ and when it is 2). (B, method seen)* (6 marks)

$$\begin{aligned} V &= \mathcal{V}(X, X^2 + Y^2 + Z^2 - 1) \cup \mathcal{V}(Y, X^2 + Y^2 + Z^2 - 1) \cup \mathcal{V}(Z, X^2 + Y^2 + Z^2 - 1) \\ &= \mathcal{V}(X, Y^2 + Z^2 - 1) \cup \mathcal{V}(Y, X^2 + Z^2 - 1) \cup \mathcal{V}(Z, X^2 + Y^2 - 1) \end{aligned} \tag{2}$$

V is the union of three circles placed in the planes YZ, XZ and XY. If the characteristic of k is not two, each of these circles is irreducible. If the characteristic of k is two, $X^2 + Y^2 - 1 = (X + Y + 1)^2$, hence:

$$V = \mathcal{V}(X, Y + Z + 1) \cup \mathcal{V}(Y, X + Z + 1) \cup \mathcal{V}(Z, X + Y + 1) \tag{3}$$

hence V is the union of three lines.

(Total: 20 marks)

2. *Regular and rational maps*

(a) *Show that, for V an affine variety, and $f \in k[V]$ a regular function, $D(f) = V - \mathcal{V}(f)$ (the open subset where $f \neq 0$) is isomorphic to an affine variety. (A, seen)* (4 marks)

Denote by T the coordinate of $\mathbb{A}^1$, and consider the closed subset W of $V \times \mathbb{A}^1$ defined by the polynomial $fT - 1 \in k[V][T] = k[V \times \mathbb{A}^1]$. Then W is affine, $k[W] = k[V][T]/(fT-1) = k[V]_f$. The regular map $U \to W$ given by $(Id_V, 1/f)$, and its inverse given by the projection from $V \times \mathbb{A}^1$ to V gives an isomorphism $U \simeq W$.

(b) *Notice that, even if regular maps are always continuous for the Zariski topology, a map which is continuous for the Zariski topology is not always regular.*

Consider that k is of characteristic $p > 0$ (recall that k is assumed to be algebraically closed), and consider the Frobenius map:

$$\begin{aligned} F :& \mathbb{A}^1 \to \mathbb{A}^1 \\ & t \mapsto t^p \end{aligned} \tag{4}$$

(i) *Prove that F is a regular bijective map.* (4 marks)

F is regular because it is given by a polynomial. Because k is algebraically closed, F is surjective. We have $X^p - t^p = (X-t)^p$, hence all the p-th roots of any element of k are equal, then F is injective.

(ii) *Prove that the inverse of F is continuous for the Zariski topology.* (6 marks)

As before, the closed subsets of $\mathbb{A}^1$ are $\mathbb{A}^1$ itself and finite sets, hence any bijection is continuous for the Zariski topology: the inverse of F is then continuous for the Zariski topology.

(iii) *Prove that the inverse of F is not regular (hence F is not an isomorphism).* (6 marks)

If the inverse of F were regular, because $\mathbb{A}^1$ is affine, it would be globally given by a polynomial $g(T)$ such that $g(T^p) = T$, which is impossible for degree reasons.

(Total: 20 marks)

3. *Projective varieties*

We consider here the group $Aut(\mathbb{P}^n)$ of isomorphisms of $\mathbb{P}^n$ with itself, and the group $Bir(\mathbb{P}^n)$ of birational equivalences of $\mathbb{P}^n$ with itself.

We recall that any regular map $\mathbb{P}^n \to \mathbb{P}^m$ is globally defined by homogeneous polynomials $f_1, ..., f_m$, and that, if another family $g_1, ..., g_m$ defines ϕ globally, there is $\lambda \in k - \{0\}$ such that $g_i = \lambda f_i$ for all i.

(a) *Prove that $Aut(\mathbb{P}^n)$ can be identified naturally with $PGl_{n+1}(k)$, the quotient of the group $Gl_{n+1}(k)$ of invertible $(n+1)\times(n+1)$ matrices, divided by the subgroup $\{\lambda Id : \lambda \in k-\{0\}\}$. (C, unseen)* (6 marks)

Consider an isomorphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$: it is given by homogeneous polynomials $f_0, ..., f_n \in k[X_0, ..., X_n]$ of the same degree d. Its inverse ψ will be given by homogeneous polynomials $g_0, ..., g_n \in k[X_0, ..., X_n]$ of the same degree e. The composition:

$$\psi \circ \phi : [x_0, ..., x_n] \mapsto [g_0(f_0, ..., f_n) : ... : g_n(f_0, ..., f_n)] \tag{5}$$

is equal to the identity, described globally by the homogeneous polynomials $X_0, ..., X_n$. Then there is $\lambda \in k - \{0\}$ such that $g_i(f_0, ..., f_n) = \lambda X_i$: in particular, the $g_i(f_1, ..., f_n)$ are of degree $d \times e = 1$, hence $d = 1$ and $e = 1$. It means that $f_0, ..., f_n$ (resp $g_0, ..., g_n$) are linear functions $k^{n+1} \to k$, which fit into a $(n+1) \times (n+1)$ matrix $F = (f_0, ..., f_n)$ (resp $G = (g_0, ..., g_n)$), and that $FG = \lambda Id$, with $\lambda \neq 0$. In particular, $F \in Gl_{n+1}(k)$. F is defined from ϕ only up to a constant λ, hence we have defined an injective map $Aut(\mathbb{P}^n) \to PGl_{n+1}(k)$.

Conversely, given any $F = (f_1, ..., f_n) \in M_{(n+1),(n+1)}(k)$, it defines a regular map $\mathbb{P}^n \to \mathbb{P}^n$ of degree 1, with an inverse defined by F^{-1} when $F \in Gl_{n+1}(k)$, hence the injective map $Aut(\mathbb{P}^n) \to PGl_{n+1}(k)$ defined above is surjective.

(b) (i) *Prove that any rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is regular.* (4 marks)

Consider a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$. It suffice to show that it is regular on an open cover, hence it suffice to show that any rational map $\phi : \mathbb{A}^1 \dashrightarrow \mathbb{P}^n$ is regular. Consider $f_0, ..., f_n \in k[X]$ such that $x \mapsto [f_0(x) : ... : f_n(x)]$ is a representative of ϕ. We can divide $f_1, ..., f_n$ by their biggest common divisor h, writing $f_i = f_i' h$. Then, on the locus where the f_i' does not all vanish, $x \mapsto [f_0'(x)h(x) : ... : f_n'(x)h(x)]$ is equivalent to $x \mapsto [f_0'(x) : ... : f_n'(x)]$, which then also represent ϕ. But $f_0', ..., f_n \in k[X]$ are mutually prime, hence they have no common zero (we really use the fact that we are in one dimension here), hence $x \mapsto [f_0'(x) : ... : f_n'(x)]$ is defined on $\mathbb{A}^1$, hence ϕ is regular.

(ii) *Deduce that $Bir(\mathbb{P}^1) = Aut(\mathbb{P}^1)$.* (2 marks)

If $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ are birational inverse, the two of them are regular from the last question, hence ϕ is an isomorphism. Hence $Aut(\mathbb{P}^1) = Bir(\mathbb{P}^1)$.

(c) *Consider the rational map:*

$$\begin{aligned} \phi :& \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \\ & [x : y : z] \mapsto [yz : zx : xy] \end{aligned} \tag{6}$$

(i) *Give the domain of ϕ.* (4 marks)

ϕ is a rational map, a priori defined on $\mathbb{P}^2 - \{[1:0:0],[0:1:0],[0:0:1]\}$. Suppose by contradiction that ϕ is regular at $[1:0:0]$. For $t \neq 0$, we have $\phi([1:t:0]) = [0:0:1]$ (resp $\phi([1:0:t]) = [0:1:0]$), hence, because ϕ is regular, one must have $\phi[1:0:0] = [0:0:1]$ (resp $\phi[1:0:0] = [0:1:0]$), giving a contradiction. Hence ϕ is not regular at $[1:0:0]$, and by symmetry it is not regular at $[0:1:0],[0:0:1]$. It gives:

$$dom(\phi) = \mathbb{P}^2 - \{[1:0:0],[0:1:0],[0:0:1]\} \tag{7}$$

(ii) *Show that ϕ is a birational equivalence, and give its inverse.* (4 marks)

We have that:

$$\phi \circ \phi : [x:y:z] \mapsto \phi[yz:xz:xy] = [x(xyz):y(xyz):z(xyz)] = [x:y:z] \tag{8}$$

hence ϕ is dominant, and birational, with birational inverse ϕ itself.

(Total: 20 marks)

4. *Dimension theory*

As in question $3)a)$, we consider the vector space $V_{n,d}$ of polynomials of degree d in $k[X_0,...,X_n]$, and the associated projective space $P_{n,d}$. We denote by $Z_{i_0,...,i_n}$ the projective coordinate of $P_{n,d}$ corresponding to the monomial $X_0^{i_0}...X_n^{i_n}$ (with $i_0+...+i_n = d$). We write $N = \dim P_{n,d}$.

(a) (i) *Recall and explain the formula for N in terms of n,d. (B, seen)* (5 marks)

N is equal to the number of monomials of degree d in $n+1$ variables minus 1: $N = \binom{n+d}{n} - 1$. Indeed, the number of monomials of degree d in $n+1$ is the number of decomposition $d = d_0 + ... + d_n$. one can encode each such decomposition as a sequence of d dots and n bars, with d_0 dots, one bar, d_1 dots, one bar, and so on. The number of such sequences is the number of choice for the position of the n bars in the $d+n$ elements, hence $\binom{n+d}{n}$.

(ii) *Prove that, given $N = \dim P_{n,d}$ points in $\mathbb{P}^n$, there is at least one hypersurface of degree d of $\mathbb{P}^n$ passing through those points. (A, similar seen)* (3 marks)

Consider a point $x \in \mathbb{P}^n$: up to changing the coordinates, one can assume that $x = [1:0:...:0]$. Given $[f] \in P_{n,d}$, the condition for $f(x) = 0$ is then that the coefficient of X_0^d vanishes in f. Hence, the set $\{[f] \in P_{n,d} : f(x) = 0\}$ is a hyperplane in $P_{n,d}$. If we choose N points $x_1,...,x_N$, the set:

$$\{[f] \in P_{n,d} : f(x_1) = ... = f(x_N) = 0\} \tag{9}$$

is the intersection of N hyperplane in $P_{n,d} = \mathbb{P}^N$, hence it is a nonempty linear subspace of $P_{n,d}$ (in particular, it is irreducible).

(b) *Consider the closed subset:*

$$\mathcal{X} = \{([f], x_1, ..., x_N) \subset P_{n,d} \times (\mathbb{P}^n)^N : f(x_i) = 0 \,\forall i\} \tag{10}$$

(i) *Consider the first projection $p_1 : \mathcal{X} \to P_{n,d}$. Show that it is surjective, and give the dimensions of the irreducible components of the fibers. Deduce the dimensions of the irreducible components of $\mathcal{X}$. (D, unseen)* (6 marks)

For $[f] \in P_{n,d}$, one has the fiber $p_1^{-1}([f]) = \{[f]\} \times (\mathcal{V}(f))^N \simeq (\mathcal{V}(f))^N$. $\mathcal{V}(f)$ is nonempty, and each of its irreducible components are of dimension $n-1$. Hence each fiber $p_1^{-1}([f])$ is nonempty (hence p_2 is surjective) and each irreducible component of the fiber has dimension $N(n-1)$. By the fiber dimension theorem, each irreducible component of $\mathcal{X}$ has dimension $\dim P_{n,d} + N(n-1) = Nn$.

(ii) *Consider the second projection $p_2 : \mathcal{X} \to (\mathbb{P}^n)^N$. Deduce from the fiber dimension theorem that there is a nonempty open subset $U \subset (\mathbb{P}^n)^N$ such that $p_2^{-1}(u)$ is a single point for $u \in U$ (which means that there is one and only one hypersurface of degree d containing those N points). (D, unseen)* (6 marks)

For $(x_1, ..., x_N) \in (\mathbb{P}^n)^N$, the fiber $p_2^{-1}(x_1, ..., x_N) \subseteq P_{n,d}$ is the set of $[f] \in P_{n,d}$ such that $\mathcal{V}(f)$ contains $x_1 ..., x_N$.
From question $a)ii)$, each fiber is a nonempty linear subspace (hence irreducible), and then p_2 is surjective. $(\mathbb{P}^n)^N$ is of dimension nN, and each irreducible components of $\mathcal{X}$ have dimension nN: from the fiber dimension theorem, the set of points such that the fiber has dimension 0 (equivalently, is a single point) is open in $(\mathbb{P}^n)^N$.

(Total: 20 marks)

5. *Mastery material: tangent space and singular points*

For $f \in k[X_1, ..., X_n]$, we denote as usual by $df_x : k^n \to k$ the differential of f at $x \in \mathbb{A}^n$.

(a) *For $V \subset \mathbb{A}^n$ and $x \in V$, recall the definition of $T_x V$, the tangent space of V at x, an prove that, if $f_1, ..., f_r$ generates $\mathcal{I}(V)$, then:*

$$T_x V = \bigcap_{i=1}^{r} \ker d(f_i)_x \subseteq k^n \tag{11}$$

(5 marks)

By definition:

$$T_x V = \bigcap_{f \in \mathcal{I}(V)} \ker df_x \subseteq k^n \tag{12}$$

Suppose that $f_1, ..., f_r$ generates $\mathcal{I}(V)$. We have trivially:

$$T_x V \subseteq \bigcap_{i=1}^{r} \ker d(f_i)_x \subseteq k^n \tag{13}$$

Consider $a \in \bigcap_{i=1}^{r} \ker d(f_i)_x$. Consider $f \in \mathcal{I}(V)$: because $f_1, ..., f_r$ generates the ideal, there are $g_1, ..., g_r \in k[X_1, ..., X_n]$ such that $f = \sum_{i=1}^{r} g_i f_i$. Then:

$$df_x(a) = \sum_{i=1}^{r} d(g_i f_i)_x(a) = \sum_{i=1}^{r} (g_i(x) d(f_i)_x(x) + f_i(x) d(g_i)_x(a)) = 0 \tag{14}$$

the last equality holding because $f_i \in \mathcal{I}(V)$, hence $f_i(x) = 0$, and $a \in \ker d(f_i)_x$. Hence $a \in \ker df_x$ for any $f \in \mathcal{I}(V)$, hence $a \in T_x V$, which shows the reverse inclusion.

(b) *Consider a hypersurface $V \subsetneq \mathbb{A}^n$, and $g \in k[X_1, ..., X_n]$ generating $\mathcal{I}(V)$.*

(i) *Prove that $x \in V$ is a singular point of V if and only if $dg_x = 0$.* (4 marks)

Each irreducible component of V is of dimension $n-1$, then for each $x \in V$, $\dim_x V = n - 1$. $T_x V = \ker dg_x \subseteq k^n$, hence it is either of dimension n (if $dg_x = 0$) or of dimension $n-1$ (if $dg_x \neq 0$). V is singular at x if and only if $\dim T_x V \neq \dim_x V$, hence if and only if $dg_x = 0$.

(ii) *Prove that, given $f(X) \in k[X]$ a polynomial of degree 3, the polynomial $g(X, Y) = Y^2 - f(X) \in k[X, Y]$ is irreducible, hence defines a nondegenerate cubic curve C of $\mathbb{A}^2$.* (4 marks)

Consider a nontrivial factorization of $g(X, Y) = Y^2 - f(X)$: the dominant coefficient of g as a polynomial in Y is constant in $f(X)$, hence such a factorization must be of the form $Y^2 - f(X) = (Y - h(X))(Y - h(X))$. There is no term of degree 1 in Y in g, hence one must have $h' = -h$, and then $f(X) = h(X)^2$: but f is of degree 3, so this is impossible. Hence g is irreducible.

(iii) *Give the singular points of C in terms of f, and find at which condition C is smooth.* (7 marks)

We have:

$$\frac{\partial(Y^2 - f(X))}{X} = -f'(X) \quad \frac{\partial(Y^2 - f(X))}{Y} = 2Y \tag{15}$$

Hence $dg_{(x,y)} = 0$ if and only if $y = 0$ and $f'(x) = 0$. A singular point (x, y) of V satisfies then $y = 0, f(x) = 0$ and $g(x, y) = -f(x) = 0$. Hence $SingV$ is the set of points $(x, 0)$ such that x is at least a double root of f. In particular, C is smooth if and only if f has simple roots.

(Total: 20 marks)