MATH50011 Statistical Modelling 1
Midterm Examination

1. Let $X_1, ..., X_n$ be a random sample, where X_1 has density $f_\theta(x) = \theta(x+1)^{-(\theta+1)}$ for $x \geq 0$ and unknown parameter $\theta > 0$. Denote by F_θ the cumulative distribution function of X_1. All the regularity conditions are satisfied in this case.

   (a) Show that $F_\theta(X_1) \sim Uniform(0,1)$. (2 marks)

   (b) Consider the random variable $Z = -2\log(1 - F_\theta(X_1))$. Show that $Z \sim \chi^2_2$. *(Hint: recall that the density of a χ^2_2 is given by $f(z) = \frac{1}{2}e^{-\frac{z}{2}}$ for $z \geq 0$ and $f(z) = 0$ for $z < 0$)* (2 marks)

   (c) Using the random sample and the result in point (b) construct a 95% confidence interval for θ. (For this question you do not need to write the critical values of the pivotal distribution explicitely). (4 marks)

   (d) Compute the MLE for θ. Is the computed MLE an unbiased estimator of θ? (4 marks)

   (e) Using the result in point (d) build an (approximate) rejection region for the α level test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$, for some $\theta_0 \in (0, \infty)$. (2 marks)

2. Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be two random samples with $E[X_1] = E[Y_1] = 0$ and unknown variances. Assume that all moments exist.

   (a) Show that $\frac{1}{n}\sum_{i=1}^n X_i^2$ is an asymptotically normal estimator for $Var(X_1)$. (2 marks)

   (b) Build an (approximate) rejection region for the α level test for $H_0 : (Var(X_1), Var(Y_1)) = (\theta_1, \theta_2)$ vs $H_1 : (Var(X_1), Var(Y_1)) \neq (\theta_1, \theta_2)$, for some $\theta_1, \theta_2 \in [0, \infty)$. (2 marks)

   (c) How would your answer in point (b) change if we assume that $X_1, ..., X_n, Y_1, ..., Y_n$ are all independent? (2 marks)

(Total 20 marks)