IMPERIAL

MATH70045

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Applied Probability

Date: Wednesday, April 30, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

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1. (a) Consider a discrete-time, time-homogeneous Markov Chain $X = (X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3, 4, 5\}$. We denote the one-step transition matrix by $\mathbf{P} = (p_{ij})_{i,j\in E}$. Suppose that

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.3 & 0 & 0.2 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(i) Draw the transition diagram for this Markov Chain. (2 marks)

(ii) Specify the communicating classes and, for each class, determine whether it is transient, null recurrent or positive recurrent. (3 marks)

(iii) Is the chain irreducible? (1 mark)

(iv) Find all stationary distributions. (4 marks)

(v) Let $T_x = \inf\{n \geq 0 : X_n = x\}$. Calculate $\mathbb{P}(T_5 = n \mid X_0 = 1)$ for all $n \geq 1$ and $\mathbb{P}(T_3 < T_5 \mid X_0 = 1)$. (5 marks)

(b) Large icebergs break off from Greenland glaciers in a process known as calving; some icebergs then drift towards Newfoundland, Canada. We assume that such icebergs calve following a Poisson Process with rate 2/day.

(i) How many icebergs calve on average in one week? (1 mark)

(ii) What is the probability that no icebergs calve on a given day? (1 mark)

(iii) We model the mass (in kilotonnes) of each iceberg by a random variable Y with mean 150 and variance 200. Let S_{30} be the total mass of all icebergs which calved over a 30-day period and drifted towards Newfoundland. Find the mean and variance of S_{30}. (3 marks)

(Total: 20 marks)

2. (a) Consider a discrete-time, time-homogeneous Markov Chain $X = (X_n)_{n\in\mathbb{N}_0}$ on the discrete state space E. Let $\mathbf{P} = (p_{ij})_{i,j\in E}$ be the transition matrix of X.

(i) Give an expression for $\mathbb{P}[X_2 = j \mid X_0 = i]$ for all $i, j \in E$. (1 mark)

(ii) Let $Z_n = X_{2n}$ for $n \in \mathbb{N}_0$. Show that $Z = (Z_n)_{n\in\mathbb{N}_0}$ is a Markov chain. Give its transition matrix. (3 marks)

(iii) Show that if π is a stationary distribution for X, then π is also a stationary distribution for Z. (2 marks)

(iv) Construct a counter-example for the reciprocal of the previous question, *i.e.* construct a Markov Chain X and a distribution π such that π is a stationary distribution for Z, but π is not a stationary distribution for X. (4 marks)

(b) Customers arrive in a shop following a homogeneous Poisson process (N_t) with rate λ. Let $(J_n)_{n\in\mathbb{N}_0}$ be the jumping times and $(H_n)_{n\in\mathbb{N}}$ be the holding times of (N_t).
Alice and Bob have both been instructed to hand vouchers to half the customers, but they follow different strategies.

(i) Every time a customer arrives, Alice tosses a fair coin. If the coin lands on Heads, she hands them a voucher; if it lands on Tails, she does not hand them a voucher. Denote by (A_t) the corresponding process.
Explain why (A_t) is a Poisson process, and give its rate. (2 marks)

(ii) Give the distribution, mean and variance of the holding times of the process (A_t). (2 marks)

(iii) Bob hands vouchers to every other customer; that is, he hands vouchers to the 2nd, 4th, 6th... customers. Let (B_t) be the corresponding process. Give a mathematical description of (B_t) as a function of (N_t). (2 marks)

(iv) Give the distribution, mean and variance of the holding times of the process (B_t). (2 marks)

(v) Is (B_t) a Poisson process? (2 marks)

(Total: 20 marks)

3. Consider a birth-death process X with birth rates $\lambda_n = \lambda$ and $\mu_n = n\mu$. We suppose that $\lambda > 0$ and $\mu \geq 0$.

(a) Give the generator matrix of X. (2 marks)

(b) Suppose now (for this subquestion only) that $\mu = 0$, meaning that X is a birth process. Does it explode? (2 marks)

(c) From now on, we suppose that $\mu > 0$. We let $\rho = \frac{\lambda}{\mu}$.

(i) Find the stationary distribution of this process. (4 marks)

(ii) Does the process reach equilibrium? (2 marks)

(iii) Let Z be the jump chain of X. Give the transition matrix of Z and show that it is irreducible. (3 marks)

(iv) State the detailed balance condition. Then show that the stationary distribution of Z is given by

$$\pi_n = \frac{1}{2}\frac{1}{n!}\left(1 + \frac{n}{\rho}\right)\rho^n e^{-\rho}.$$

(5 marks)

(v) Explain why the distributions of question (i) and question (iv) are not identical. (2 marks)

(Total: 20 marks)

4. Let (W_t) be a standard Brownian motion. Recall that this means that $W_0 = 0$ almost surely; the increments of (W_t) are independent, stationary, and Gaussian with $W_t - W_s \sim \mathcal{N}(0, t-s)$; and the sample paths of (W_t) are almost surely continuous.

Define $Z_t = W_t - tW_1$.

(a) Show that $Cov(W_s, W_t) = \min(s,t)$. (3 marks)

(b) Fix some $t \in [0,1]$. Calculate $\mathbb{E}[Z_t]$ and $Var(Z_t)$. What is the marginal distribution of Z_t? (3 marks)

(c) Fix some $s, t \in [0,1]$. Calculate $Cov(Z_s, Z_t)$ (2 marks)

(d) Let $Z'_t = Z_{1-t}$. Show that the process (Z'_t) has the same distribution as the process (Z_t). *Hint: you may use without proof this result from the problem sheets: if (W_t) is a standard Brownian motion, then $(W_1 - W_{1-t})$ is also a standard Brownian motion.* (5 marks)

(e) Show that $\forall t \in [0,1]$, Z_t is independent of W_1. (3 marks)

(f) Show that $Y_t = (1+t)Z_{\frac{t}{1+t}}$ is a Brownian motion. (4 marks)

(Total: 20 marks)

5. This question refers to the additional reading material on "Branching processes" as described in the book by Robert Dobrow, Introduction to Stochastic Processes with R (2016), Chapter 4.

(a) What does it mean for a branching process to be supercritical? Give an example of a supercritical branching process. (2 marks)

(b) A branching process (Z_n) has offspring distribution $\mathbf{a} = \left(\frac{1}{8}, \frac{1}{8}, \frac{3}{4}\right)$ and is initialized at $Z_0 = 1$.

(i) Find the corresponding probability generating function. (2 marks)

(ii) Calculate the extinction probability. (3 marks)

(iii) Calculate $\mathbb{P}[Z_2 = 0]$. (4 marks)

(c) Let (Z_n) be a branching process, with offspring mean $\mu > 1$ and started at $Z_0 = 1$.

(i) Show that $\mathbb{E}[Z_n] = \mu^n$. (4 marks)

(ii) Compute the limit of $\mathbb{E}[Z_n/\mu^n \mid Z_n > 0]$ as $n \to \infty$. Give an interpretation. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH600045/MATH700045

Applied Probability (Solutions)

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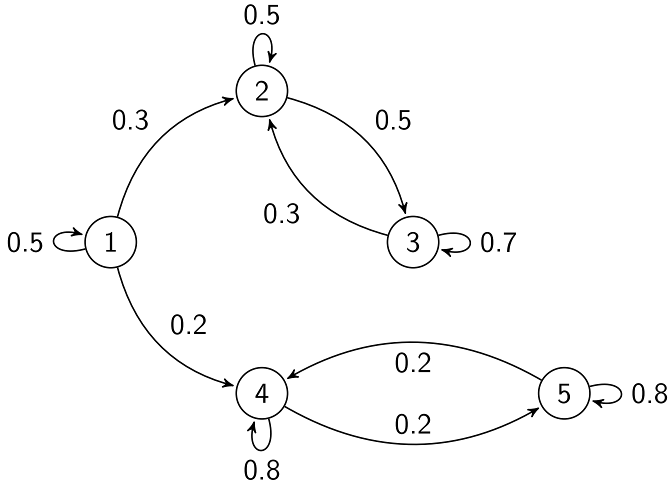
1. (a) (i) The transition diagram is sim. seen $\Downarrow$



2, A

(ii) There are 3 communicating classes:

- $C_1 = \{1\}$, transient
- $C_2 = \{2, 3\}$, positive recurrent
- $C_3 = \{4, 5\}$, positive recurrent

3, A

(iii) The chain is not irreducible, since (for example) states 2 and 4 do not communicate. 1, A

(iv) To find the stationary distributions of the entire chain, we must first find the stationary distributions of the chain restricted to each positive recurrent class. The unique stationary distribution of the chain restricted to $C_2 = \{2, 3\}$ is $(0, \pi_2, \pi_3, 0, 0)$ with

$$\begin{cases} \pi_2 + \pi_3 & = 1 \\ 0.5\pi_2 + 0.7\pi_3 & = \pi_3 \\ 0.5\pi_2 + 0.3\pi_3 & = \pi_2 \end{cases}$$

which we solve to find

$$\pi_2 = \frac{3}{8} \qquad \pi_3 = \frac{5}{8}.$$

We can proceed similarly for the chain restricted to $C_3 = \{4, 5\}$, or note that by symmetry the stationary distribution restricted to that class must be $(0, 0, 0, \frac{1}{2}, \frac{1}{2})$.

All stationary distributions for the entire chain are a linear combination of these two restricted stationary distributions, and are thus of the form

$$\left(0, \frac{3p}{8} \frac{5p}{8}, \frac{1-p}{2}, \frac{1-p}{2}\right)$$

with $p \in [0, 1]$. 4, A

(v) To reach state 5 for the first time at step n, starting from $X_0 = 1$, the chain must have

- stayed in state 1 for i steps;
- moved to state 4 at step $i + 1$;
- stayed in state 4 for $n - i - 2$ steps;
- moved to state 5 at step n

for some $0 \leq i \leq n-2$.
In other words, we have

$$\{X_0 = 1, T_5 = n\} = \bigcup_{i=0}^{n-2} \{X_0 = 1, \dots X_i = 1, X_{i+1} = 4, \dots, X_{n-1} = 4, X_n = 5\}.$$

We can thus have for $n = 1$

$$\mathbb{P}[T_5 = 1 \mid X_0 = 1] = 0$$

and for $n \geq 2$

$$\begin{aligned}
\mathbb{P}[T_5 = n \mid X_0 = 1] &= \sum_{i=0}^{n-2} \left(\frac{1}{2}\right)^i \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{n-2-i} \cdot \frac{1}{5} \\
&= \frac{4^{n-2}}{5^n} \sum_{i=0}^{n-2} \left(\frac{5}{8}\right)^i \\
&= \frac{4^{n-2}}{5^n} \cdot \frac{1 - \left(\frac{5}{8}\right)^{n-1}}{1 - \frac{5}{8}} \\
&= \frac{8}{3} \frac{4^{n-2}}{5^n} \left(1 - \left(\frac{5}{8}\right)^{n-1}\right).
\end{aligned}$$

$\boxed{3, \text{B}}$

$\boxed{\text{unseen} \Downarrow}$

To compute $\mathbb{P}[T_3 < T_5 \mid X_0 = 1]$, we recall that we have two closed communicating classes: $\{2, 3\}$ and $\{4, 5\}$. It follows that either $\{T_3 < \infty, T_5 = \infty\}$ or $\{T_3 = \infty, T_5 < \infty\}$.
Starting from $X_0 = 1$, the chain will either:

- eventually move to state 2, and then to state 3: we then have $\{T_3 < \infty, T_5 = \infty\}$ and $\{T_3 < T_5\}$;
- or eventually move to state 4, and then to state 5: we then have $\{T_3 = \infty, T_5 < \infty\}$ and $\{T_5 < T_3\}$.

The first possibility occurs with probability $\frac{0.3}{0.3+0.2} = 0.6$ and the second possibility occurs with probability $\frac{0.2}{0.3+0.2} = 0.4$. Therefore

$$\mathbb{P}[T_3 < T_5 \mid X_0 = 1] = 0.6$$

$\boxed{2, \text{C}}$

(b) (i) In 7 days, the average number of calving events is $7 \times 2 = 14$. $\boxed{\text{sim. seen} \Downarrow}$

(ii) Let N be the number of calvings in one day; then $N \sim Poisson(2)$. Therefore $\boxed{1, \text{A}}$

$$\mathbb{P}[N = 0] = e^{-2} \approx 0.135.$$

$\boxed{1, \text{A}}$

(iii) The total mass of all icebergs follows a compound Poisson process. From the lecture notes, we know that

$$\begin{aligned}
\mathbb{E}[S_{30}] &= 30 \cdot 2 \cdot \mathbb{E}[Y] = 9\,000 \\
Var(S_{30}) &= 30 \cdot 2 \cdot \left(Var(Y) + \mathbb{E}[Y]^2\right) = 1\,362\,000
\end{aligned}$$

$\boxed{3, \text{B}}$

2. (a) (i) By the Chapman-Kolmogorov equations, seen ⇓

$$\mathbb{P}[X_2 = j \mid X_0 = i] = \sum_{k \in E} p_{ik} p_{kj}.$$

This could also be written as

$$\mathbb{P}[X_2 = j \mid X_0 = i] = (\mathbf{P}^2)_{ij}$$

1, A

(ii) To show that Z is a Markov chain, we check that for $z_n, z_{n-1}, \dots, z_0 \in E$,

meth seen ⇓

$$\begin{aligned}
&\mathbb{P}[Z_n = z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots Z_0 = z_0] \\
&= \mathbb{P}[X_{2n} = z_n \mid X_{2n-2} = z_{n-1}, X_{2n-4} = z_{n-2}, \dots, X_0 = z_0] \\
&\overset{Markov}{=} \mathbb{P}[X_{2n} = z_n \mid X_{2n-2} = z_{n-1}] \\
&= \mathbb{P}[Z_n = z_n \mid Z_{n-1} = z_{n-1}]
\end{aligned}$$

This proves that Z is a Markov chain. By the previous question, we know that its transition matrix is $\mathbf{P}^2$. 3, A

(iii) Suppose that π is stationary for X. Then $\mathbf{P}\pi = \pi$. Therefore

$$(\mathbf{P}^2)\pi = \mathbf{P}(\mathbf{P}\pi) = \mathbf{P}\pi = \pi$$

and π is also stationary for Z. 2, B

unseen ⇓

(iv) For a counter-example, take $E = \{1, 2\}$ and take X to be the Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that $\mathbf{P}^2 = I_2$ the identity matrix. Therefore any distribution $\pi = (p, 1-p)$ is stationary for Z. Yet the only stationary distribution for X is the distribution $(\frac{1}{2}, \frac{1}{2})$. 4, D

(b) (i) The process (A_t) corresponds to thinning the Poisson process (N_t) with probability $\frac{1}{2}$. Therefore (A_t) is a Poisson process with rate $\frac{\lambda}{2}$. seen ⇓ 2, A

(ii) The holding times of (A_t) follow an Exponential distribution with parameter $\frac{\lambda}{2}$, mean $\frac{2}{\lambda}$ and variance $\frac{4}{\lambda^2}$. 2, A

unseen ⇓

(iii) The process (B_t) is defined by

$$B_t = \left\lfloor \frac{N_t}{2} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is the floor function.
Alternatively, we might define (B_t) by its jump times $J'_n = J_{2n}$ or by its holding times $H'_n = H_{2n-1} + H_{2n}$. 2, C

(iv) The holding times of (B_t) are $H'_n = H_{2n-1} + H_{2n}$. Since H_{2n-1} and H_{2n} are independent $Exp(\lambda)$ random variables, we conclude that $H'_n \sim Gamma(2, \lambda)$. Therefore

$$\mathbb{E}[H'_n] = \frac{2}{\lambda} \qquad Var(H'_n) = \frac{2}{\lambda^2}.$$

2, B

(v) The holding times of (B_t) are not exponentially distributed. Therefore (B_t) cannot be a Poisson process. $\boxed{2,\ \mathrm{D}}$

3. (a) The generator matrix of X is

meth seen ⇓

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & 0 & 0 & \dots \\ 0 & 2\mu & -\lambda-2\mu & \lambda & 0 & \dots \\ 0 & 0 & 3\mu & -\lambda-3\mu & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2, A

(b) From the lecture notes, we know that a birth process with rates (λ_k) explodes with probability 1 if $\sum \frac{1}{\lambda_k} < \infty$, and explodes with probability 0 otherwise.
Here, we have $\forall k, \lambda_k = \lambda$ so $\sum \frac{1}{\lambda_k} = \infty$ and the process almost surely does not explode.

2, A

(c) (i) From the lecture notes, we know that the stationary distribution b of a birth-death process with birth rates (λ_n) and death rates (μ_n) is

meth seen ⇓

$$b_n = \frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\dots\mu_n} b_0.$$

In this case, we get

$$b_n = \frac{\lambda^n}{n!\mu^n} b_0 = \frac{\rho^n}{n!} b_0.$$

The constraint that $\sum_n b_n = 1$ yields $b_0 = e^{-\rho}$ and

$$b_n = \frac{\rho^n}{n!} e^{-\rho}.$$

4, B

(ii) We have found a stationary distribution. By Theorem 6.5.6 of the lecture notes, it is the unique stationary distribution, and the process reaches equilibrium.

2, B

(iii) The transition matrix of Z is given by

sim. seen ⇓

$$p_{i,j} = \begin{cases} \frac{i\mu}{\lambda+i\mu} & \text{if } j = i-1 \\ \frac{\lambda}{\lambda+i\mu} & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lambda, \mu > 0$, all states communicate and the chain is irreducible.

3, B

seen ⇓

(iv) The detailed balance condition is that for all $i, j \in E$,

$$\pi_i p_{i,j} = \pi_j p_{j,i}$$

1, A

In this case, it is sufficient to verify detailed balance for $|j-i| = 1$, since otherwise $p_{i,j} = p_{j,i} = 0$. We verify that the distribution π given in the question is a stationary distribution for Z. To do this, we verify the detailed balance equations

meth seen ⇓

$$\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}$$

which is equivalent to verifying that

$$\frac{\pi_i}{\pi_{i+1}} = \frac{p_{i+1,i}}{p_{i,i+1}}.$$

Plugging in the value from the question we find

$$\begin{aligned}\frac{\pi_i}{\pi_{i+1}} &= \frac{\frac{1}{2i!}\left(1+\frac{i}{\rho}\right)\rho^i e^{-\rho}}{\frac{1}{2(i+1)!}\left(1+\frac{i+1}{\rho}\right)\rho^{i+1}e^{-\rho}} \\ &= \frac{(i+1)\left(1+\frac{i}{\rho}\right)}{\rho+i+1}\end{aligned}$$

and

$$\begin{aligned}\frac{p_{i+1,i}}{p_{i,i+1}} &= \frac{\frac{(i+1)\mu}{\lambda+(i+1)\mu}}{\frac{\lambda}{\lambda+i\mu}} \\ &= \frac{\frac{i+1}{\rho+i+1}}{\frac{1}{1+\frac{i}{\rho}}} \\ &= \frac{(i+1)\left(1+\frac{i}{\rho}\right)}{\rho+i+1}.\end{aligned}$$

Therefore π verifies the detailed balance condition.
To verify that π is a distribution, we note that $\forall n \in \mathbb{N}, \pi_n > 0$ and that

$$\sum_{n=0}^{\infty} \pi_n = \frac{1}{2}\sum \frac{1}{n!}\rho^n e^{-\rho} + \frac{1}{2}\sum \frac{1}{(n-1)!}\rho^{n-1}e^{-\rho} = \frac{1}{2}+\frac{1}{2} = 1.$$

Thus the vector π is a stationary distribution for Z. Since the chain is ergodic, it is the unique stationary distribution. $\boxed{4, \text{C}}$

(v) The two distributions are different because the distribution of the holding times of X depend on the state. $\boxed{2, \text{D}}$

4. (a) This is exercise 5.43 of the problem sheets. Without loss of generality, assume that $0 \le s \le t$. We find seen ⇓

$$\begin{aligned} Cov(W_s, W_t) &= \mathbb{E}[W_s W_t] - \mathbb{E}[W_s]\mathbb{E}[W_t] \\ &= \mathbb{E}[W_s W_t] \\ &= \mathbb{E}[W_s^2 - W_s^2 + W_s W_t] \\ &= \mathbb{E}[W_s^2 + W_s(W_t - W_s)] \\ &= \mathbb{E}[W_s^2] + \mathbb{E}[W_s]\mathbb{E}[W_t - W_s] \text{ by the independence of increments} \\ &= Var(W_s) + 0 \\ &= s \end{aligned}$$

and in general, $Cov(W_s, W_t) = \min(s, t)$. 3, A

(b) Following standard calculations, for $t \in [0, 1]$, meth seen ⇓

$$\begin{aligned} \mathbb{E}[Z_t] &= \mathbb{E}[W_t + tW_1] \\ &= \mathbb{E}[W_t] + t\mathbb{E}[W_1] \\ &= 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} Var(Z_t) &= Var(W_t) + t^2 Var(W_1) - 2t \cdot Cov(W_t, W_1) \\ &= t + t^2 - 2t \cdot t \\ &= t - t^2 = t(1-t) \end{aligned}$$

Since Z_t is a linear transformation of the Gaussian vector (W_t, W_1), Z_t is normally distributed. Thus marginally $Z_t \sim \mathcal{N}(0, t(1-t))$. 3, A

(c) Without loss of generality, assume that $0 \le s \le t \le 1$. We find

$$\begin{aligned} Cov(Z_s, Z_t) &= Cov(W_s - sW_1, W_t - tW_1) \\ &= Cov(W_s, W_t) - tCov(W_s, W_1) - sCov(W_1, W_t) + stVar(W_1) \\ &= s - ts - st + st \\ &= s - ts = s(1-t) \end{aligned}$$

2, A

(d) Let $W'_t = W_1 - W_{1-t}$. We know from the hint that $W'_t \stackrel{d}{=} W_t$. We write unseen ⇓

$$\begin{aligned} Z'_t &= Z_{1-t} \\ &= W_{1-t} - (1-t)W_1 \\ &\stackrel{d}{=} W'_{1-t} - (1-t)W'_1 \\ &\stackrel{d}{=} W_1 - W_t - (1-t)(W_1 - W_0) \\ &\stackrel{d}{=} tW_1 - W_t + 0 \\ &\stackrel{d}{=} -tW_1 + W_t \ = \ Z_t \end{aligned}$$

where in the last line, we have use the reflection property of Brownian motion. 5, D

(e) The vector (Z_t, W_1) is a linear transformation of the Gaussian vector (Z_t, Z_1) and is thus a Gaussian vector. To prove that Z_t and W_1 are independent, it thus suffices to prove that their covariance is 0. $\boxed{\text{meth seen} \Downarrow}$

Using question (a), we find

$$\begin{aligned} Cov(Z_t, W_1) &= Cov(W_t - tW_1, W_1) && (1) \\ &= Cov(W_t, W_1) - tCov(W_1, W_1) && (2) \\ &= t - t \cdot 1 && (3) \\ &= 0 && (4) \end{aligned}$$

and conclude that Z_t and W_1 are independent. $\boxed{3, \text{C}}$

(f) We check that (Y_t) verifies all the properties in the definition of Brownian motion. $\boxed{\text{meth seen} \Downarrow}$

1. $Y_0 = Z_0 = W_0 = 0$.
2. The sample paths $t \mapsto Y_t$ are almost surely continuous, since the sample paths $t \mapsto W_t$ are almost surely continuous.
3. The increments of Y are Gaussian since they are a linear transformation of a Gaussian vector.
4. $\mathbb{E}[Y_t] = (1+t)\mathbb{E}\left[Z_{\frac{t}{1+t}}\right] = 0$.
5. For $0 \leq s \leq t$, and using the result of question (c):

$$\begin{aligned} Cov(Y_t, Y_s) &= Cov\left((1+t)Z_{\frac{t}{1+t}}, (1+s)Z_{\frac{s}{1+s}}\right) \\ &= (1+t)(1+s)Cov\left(Z_{\frac{t}{1+t}}, Z_{\frac{s}{1+s}}\right) \\ &= (1+t)(1+s)\frac{s}{1+s}\frac{1}{1+t} \\ &= s \end{aligned}$$

Thus (Y_t) is a Brownian motion. $\boxed{4, \text{D}}$

5. (a) A branching process is supercritical if the offspring distribution has mean $\mu > 1$. For example, the process described in question (b) is supercritical. $\boxed{2, \mathrm{M}}$

(b) (i) The generating function is $G(s) = \frac{1}{8} + \frac{1}{8}s + \frac{3}{4}s^2$. $\boxed{2, \mathrm{M}}$

(ii) Let η be the extinction probability. We know that η is the smallest positive root of the equation $G(s) = s$.
It is easy to solve that equation and find that it has two roots, 1 and $\frac{1}{6}$.
Therefore the extinction probability is $\eta = \frac{1}{6}$. $\boxed{3, \mathrm{M}}$

(iii) To calculate $\mathbb{P}[Z_2 = 0]$, we can remember that

$$G^2(s) = \mathbb{E}[s^{Z_2}] = \sum_n \mathbb{P}[Z_2 = n]s^n$$

and we therefore need to calculate the constant term (the term in s^0) of $G(G(s))$. Expanding gives

$$G(G(s)) = \frac{27}{64}s^4 + \frac{9}{64}s^3 + \frac{63}{256}s^2 + \frac{5}{128}s + \frac{39}{256}$$

and so we read $\mathbb{P}[Z_2 = 0] = \frac{39}{256}$. Note that it is not necessary to get the complete expansion of $G(G(s))$: it suffices to calculate the ĉonstant term.
Alternatively, we can list all the paths to get to $Z_2 = 0$:

- No offspring at generation 1 ($Z_1 = 0$); this occurs with probability $\frac{1}{8}$.
- A single offspring at generation 1 ($Z_1 = 1$), which then has 0 offspring at generation 2; this occurs with probability $\frac{1}{8}\frac{1}{8}$.
- Two offspring at generation 2 ($Z_1 = 2$), which each have a 0 offspring at generation 2; this occurs with probability $\frac{3}{4}\frac{1}{8}\frac{1}{8}$.

Summing all these possibilities, we get $\mathbb{P}[Z_2 = 0] = \frac{39}{256}$. $\boxed{4, \mathrm{M}}$

(c) (i) We use conditional expectation to calculate

$$\begin{aligned}
\mathbb{E}[Z_n] &= \sum_{k=0}^{\infty} \mathbb{E}[Z_n \mid Z_{n-1} = k] \cdot \mathbb{P}[Z_{n-1} = k] \\
&= \sum_k \mathbb{E}\left[\sum_{i=1}^{Z_{n-1}} X_i \mid Z_{n-1} = k\right] \cdot \mathbb{P}[Z_{n-1} = k] \\
&= \sum_k \mathbb{E}\left[\sum_{i=1}^{k} X_i \mid Z_{n-1} = k\right] \cdot \mathbb{P}[Z_{n-1} = k] \\
&\overset{indep.}{=} \sum_k \mathbb{E}\left[\sum_{i=1}^{k} X_i\right] \cdot \mathbb{P}[Z_{n-1} = k] \\
&= \sum_k k\mu\mathbb{P}[Z_{n-1} = k] \\
&= \mu\mathbb{E}[Z_{n-1}]
\end{aligned}$$

Since $Z_0 = 1$, it follows immediately that $\mathbb{E}[Z_n] = \mu^n$. $\boxed{4, \mathrm{M}}$

(ii) Note that

$$\begin{aligned} 1 &= \mathbb{E}\left[\frac{Z_n}{\mu^n}\right] \\ &= \mathbb{E}\left[\frac{Z_n}{\mu^n} \mid Z_n > 0\right]\mathbb{P}[Z_n > 0] + \mathbb{E}\left[\frac{Z_n}{\mu^n} \mid Z_n = 0\right]\mathbb{P}[Z_n = 0] \\ &= \mathbb{E}\left[\frac{Z_n}{\mu^n} \mid Z_n > 0\right]\mathbb{P}[Z_n > 0] + 0 \end{aligned}$$

Therefore

$$\mathbb{E}\left[\frac{Z_n}{\mu^n} \mid Z_n > 0\right] = \frac{1}{P[Z_n > 0]} \xrightarrow[n\to\infty]{} \frac{1}{1-\eta}$$

where η is the extinction probability.
Interpretation: in the limit as $n \to \infty$, Z_n will either be at 0 (the process is extinct), or it will be a random variable with mean $\frac{\mu^n}{1-\eta}$, which grows exponentially fast with n. $\boxed{5, \mathrm{M}}$

Review of mark distribution:

Total A marks: 33 of 32 marks

Total B marks: 19 of 20 marks

Total C marks: 11 of 12 marks

Total D marks: 17 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70045 Applied Probability Marker Comments

Question 1	Most students performed well on this question. The only difficult part was 1.a.v, in which some students got lost in complex calculations instead of looking at the transition diagram to realize that the chain was almost surely end up in either {2,3} or {4,5}.
Question 2	Part 2a was quite easy for most students; only question 2.a.iv was more difficult: many students did not attempt it, but most of those who did found a good counter-example. Marks were sometimes lost because of lack of rigour in the writing of the calculations. Part 2b was more challenging. In subpart (i), a surprisingly large minority of students failed to recognize a thinned Poisson process. Subpart (iii) in particular posed difficulties, with many students proposing definitions which did not match the description, and then failing to properly analyse the process Bt in the next subparts. This part led to a high variance in the marks, with some students gaining close to full marks in a short amount of time, and others gaining very few marks. A common mistake (or misreading of the question) in subparts (ii) and (iv) was to give the distribution, mean, and variance of At and Bt, when the question asked for the holding times: the holding times were easier to calculate, and made the next questions easier to answer.
Question 3	Most students performed well on this question. In subpart 3.c.i, a substantial number of students failed to recognize that $\sum \rho^n/n! = \exp(\rho)$, and did not give a simplified formula for the stationary distribution. Subpart 3.c.iv was more difficult: almost all students were able to state the detailed balance condition; many stopped there, with others trying (and mostly failing) to compute the stationary distribution directly. It was sufficient to verify that the distribution given in the question verifies the detailed balance condition, a much easier calculation.

Question 4

Q4 was not a particularly high-scoring question. While most students managed to answer a)-c) well, many struggled with the more difficult questions d)-f).
Q4a) was a seen question and was done well by most students.

Q4b) was an easy question, mostly done well, but some students made some minor calculation errors when computing the variance.

Q4c) was also straightforward and done well by most students.

Q4d) was probably the most challenging sub-question in Q4. Many students struggled to give a rigorous proof; unfortunately, many sloppy and wrong justifications were given. Common mistakes include ignoring that the stated identities in the model solution only hold in law and not (almost) surely. Also, many students argued piecewise for the various components without taking possible dependencies into account. Other students thought that just proving the identities of the means and variances of two random variables proves equality in law. This is not generally true. For such a proof of this question, it would be necessary to additionally prove/mention that both random variables follow a Gaussian distribution (which some students did correctly).

Q4e) saw mixed attempts. Many students understood that computing the covariance and showing that it is equal to 0 is a good starting point. However, many students then argued that uncorrelated Gaussian random variables are independent. This is not necessarily true. It was important to state that, in this case, the random variables were jointly Gaussian (here, linear transformations of a Gaussian random vector) to draw this conclusion. A rather common (and surprising) mistake was that many students computed the "joint probability mass function" for the two random variables and "proved" that it factorises, although they are not discrete random variables, and such probabilities would, hence, just be equal to 0.

Q4f) was rarely answered completely correctly. Most students understood what they had to do and gave some good attempts, but often lacked some details, in particular, when establishing the independence and stationarity of the increments. It also seems that some students simply ran out of time in that question.

Question 5 5a and 5b were straightforward applications of the contents of the reading material and posed no difficulty. 5c (especially 5.c.ii) was not attempted by many students: it is more difficult, and they might have run out of time.