Math40002 Analysis 1 **Unseen 3**

1. In this question you should work **from first principles**, proving any result you need.

   Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

   (a) Define what it means to say $a_n \to a \in \mathbb{R}$ as $n \to \infty$.

   (b) If $a_n = \frac{(n+1)(n+2)}{(2n-5)(n+3)}$, is $(a_n)_{n=1}^{\infty}$ convergent or not? Prove your answer carefully.

   (c) Instead of asking for the *difference $a_n - a$ to be close to 0*, we could ask for the *ratio $(a_n + M)/(a + M)$ to be close to 1* (for some $M \in \mathbb{R}$ included to avoid dividing by zero).

   So we make the following definition: $a_n \rightsquigarrow a$ if and only if there exists $M \neq -a$ such that

   $$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \quad \frac{a_n + M}{a + M} \in \big(1 - \epsilon, 1 + \epsilon\big).$$

   Prove that $a_n \rightsquigarrow a$ if and only if $a_n \to a$.

2. Show that if $a_n \to l$, and we define $b_n = \left(\sum_{k=1}^{n} a_n\right)/n$, then $b_n \to l$ too. Give an example to show that the converse does not hold.

3. Let $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. Compute $\lim_{n\to\infty} s_n$.