MATH60005/70005: Optimisation (Autumn 24-25)

Additional Exercises

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1. a) Find and classify all the stationary points of

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 1\,.$$

   b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

      i) $f(\mathbf{x}) = \log\left(\sum_{i=1}^{k} e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)$, where $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

      iii) $f(\mathbf{x}) = \|\mathbf{x}\|^4$.

2. a) Consider the function

$$f(\mathbf{x}) = f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2\,, \qquad \mathbf{x} \in \mathbb{R}^2\,,$$

   with

$$\begin{aligned} f_1(\mathbf{x}) &= -13 + x_1 + ((5 - x_2)x_2 - 2)x_2\,,\\ f_2(\mathbf{x}) &= -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2\,. \end{aligned}$$

   Knowing that there exists an $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = 0$, find a minimizer for this function, and discuss whether it is local or global.

   b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

      i) $f(\mathbf{x}) = \sum_{i=1}^{n} x_i \ln(x_i) - \left(\sum_{i=1}^{n} x_i\right) \ln\left(\sum_{i=1}^{n} x_i\right)$ over $\mathbb{R}^n_{++}$

      ii) $f(\mathbf{x}) = \sqrt{\mathbf{x}^T Q\mathbf{x} + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix.

3. a) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x_1, x_2) := x_2^4 - 2x_2^2 + 1 + \left(x_1^2 + x_2^2 - 1\right)^2$$

i) Find all the stationary points of f.

ii) Classify the stationary points found in i).

b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex as well as concave function. Show that f is an affine function, that is, there exist $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$.

4. a) Let $f(x_1, x_2)$ be a twice-differentiable convex function in $\mathbb{R}^2$ such that $f(0, 0) = f(1, 0) = f(0, 1) = 0$. What do you know about:

i) $f\left(\frac{1}{2}, \frac{1}{2}\right)$?

ii) $a = \frac{\partial^2 f}{\partial x_1^2}$, $b = \frac{\partial^2 f}{\partial x_2^2}$, and $c = \frac{\partial^2 f}{\partial x_1 \partial x_2}$?

b) Consider the function

$$g(x_1, x_2, x_3) = 59x_1^2 + 52x_2^2 + 17x_3^2 + 80x_1x_2 - 24x_1x_3 + 8x_2x_3 + 27x_1 - 84x_2 + 20x_3\,.$$

i) Is $g(\mathbf{x})$ convex?

ii) Solve

$$\begin{array}{ll} \max & g(x_1, x_2, x_3) \\ \text{s.t.} & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

5. a) Given the function

$$f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$$

i) Determine its stationary points.

ii) Classify the stationary points found in i).

b) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{g} \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which define the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top H \mathbf{x} + \mathbf{g}^\top \mathbf{x}$. Does there exists a vector $\mathbf{u} \in \mathbb{R}^3$ such that $f(t\mathbf{u}) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct $\mathbf{u}$.

6. a) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

   i) $f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$

   ii) $h(\mathbf{x}) = \left(\|\mathbf{x}\|^2 + 1\right)^2$, $\mathbf{x}$ in $\mathbb{R}^n$.

   b) Consider the problem

$$
\begin{array}{lll}
\text{(P)} & \min & f(\mathbf{x}) \\
 & \text{s.t.} & g(\mathbf{x}) \le 0 \\
 & & \mathbf{x} \in X
\end{array}
$$

   where f, g are convex an $X \subseteq \mathbb{R}^n$ is convex. Suppose $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$
\begin{array}{ll}
\min & f(\mathbf{x}) \\
\text{s.t.} & \mathbf{x} \in X
\end{array}
$$

7. Consider the constrained minimization problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \Delta_n\},$$

   where f is a continuously differentiable function over Δ_n. Show that $\mathbf{x}^* \in \Delta_n$ is a stationary point of this problem if and only if there exists $\mu \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) \begin{cases} = \mu, & x_i^* > 0\,, \\ \ge \mu\,, & x_i^* = 0\,. \end{cases}.$$

8. Consider the problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\,,\ \mathbf{e}^\top \mathbf{x} = 1\},$$

   where f is a continuously differentiable function. Show that $\mathbf{x}^*$ is a stationary point of this problem if and only if

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \ldots = \frac{\partial f}{\partial x_n}(\mathbf{x}^*)\,.$$

9. Find the optimal solution of the problem

$$
\begin{array}{ll}
\max\limits_{\mathbf{x} \in \mathbb{R}^3} & 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - 3x_2 + 4x_3 \\
\text{s.t.} & x_1 + x_2 + x_3 = 1 \\
 & x_1, x_2, x_3 \ge 0
\end{array}
$$

10. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \geq 4 \\ & x_3 \leq 1 \end{aligned}$$

   i) Write down the KKT conditions.
   ii) Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions.
   iii) Find the optimal solution of the problem using the KKT system.

11. Consider the problem

$$\begin{aligned} \min \quad & -x_1 x_2 x_3 \\ \text{s.t.} \quad & x_1 + 3x_2 + 6x_3 \leq 48 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (i) Write the KKT conditions for the problem.
   (ii) Find the optimal solution of the problem.

12. Consider the minimization problem for $\mathbf{x} \in \mathbb{R}^n$

$$\min_{\mathbf{x} \in C} \|\mathbf{x} - \mathbf{y}\|^2,$$

where $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$, with $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$. Assume that the rows of $\mathbf{A}$ are linearly independent.

   i) Determine the KKT conditions for this problem. Are these sufficient?.
   ii) Find the optimal solution of the problem using the KKT system.
   iii) Given the problem

$$\min_{(x_1, x_2) \in C} x_1^2 + 2x_2^2 - 3x_1,$$

   where $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$, write the explicit gradient descent iteration for this problem with a constant stepsize $t = 1$. You can help yourself using the result in part ii)

13. Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   i) Is the problem convex?.
   ii) Find all KKT points of the problem.
   iii) Find the optimal solution of the problem.

14. Consider the minimization

$$\begin{array}{ll} \min & x_1 - 4x_2 + x_3^4 \\ \text{s.t.} & x_1 + x_2 + x_3^2 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

   i) Formulate the dual problem.

   ii) Solve the dual problem.

15. Consider the problem

$$\begin{array}{ll} \min & x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array}$$

   i) Solve this problem using KKT conditions. Are these sufficient?

   ii) Find the solution of the dual problem. What is the duality gap?

16. Consider the problem

$$\begin{array}{ll} \min & x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \leq 1 \\ & x_3 \leq 1 \end{array}$$

   i) Is the problem convex?

   ii) Using KKT conditions, find an optimal solution of the primal problem.

   iii) Find a dual problem and solve it.

17. **Mastery question.** Consider the problem

$$\begin{array}{ll} \underset{u(\cdot)}{\text{minimize}} & \frac{1}{2}(x(T))^2 \\ \text{subject to} & \dot{x}(t) = u(t) \\ & x(0) = x_0 \text{ given} \\ & u(t) \in [-1, 1], \text{ for all } t \in \mathbb{R} \end{array}$$

   i) Using the PMP, find an expression for the optimal control as a feedback law.

   ii) Find an explicit expression for the optimal value function of the problem.

18. **Mastery question.** We consider a set of N robots with state (position,velocity) $(x_i(t), v_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ interacting under second-order dynamics of the form

$$\frac{dx_i}{dt} = v_i \,, \tag{1}$$

$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) + u_i(t) \,, \tag{2}$$

$$x_i(0) = x_0 \,, \quad v_i(0) = v_0 \,, \qquad i = 1, \ldots, N \,, \tag{3}$$

where $u_i(t) \in \{u : \mathbb{R}_+ \longrightarrow \mathbb{R}^2\}$ correspond to control signals for each robot, and $a(r)$ is a communication kernel of the type

$$a(r) = \frac{1}{(1+r^2)} \,.$$

Our goal is to drive the system to consensus, that is, to converge towards a configuration in which

$$v_i = \bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j \quad \text{for all } i.$$

For this, we write a finite horizon control problem of the form

$$\min_{\mathbf{u}(\cdot)} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \left(\|\bar{v} - v_j\|^2 + \gamma\|u_j\|^2\right) \, dt \,,$$

with $\gamma > 0$, subject to the dynamics (1)-(3).

Write the necessary optimality conditions for this problem, giving an explicit expression of the optimal control as a function of the adjoint variable.

19. **Mastery question.** The dynamics

$$\dot{x}(t) = -x(t) + u(t) \qquad |u| \leq 1$$

are to be controlled so that $x(1) = 0$ while minimizing the cost

$$J = \int_0^1 |u(t)| \, dt \,.$$

Show that the control

$$u(t) = \begin{cases} 0 & 0 \leq t < 0.5 \\ -1 & 0.5 \leq t \leq 1 \end{cases}$$

satisfies Pontryagin's necessary optimality conditions for some $x(0)$.

20. **Mastery question.** A community living around a lake wants to maximize the yield of fish taken out of the lake. The amount of fish at a certain time is denoted x. The growth rate of the fish is kx and fish is captured with a rate ux where u is the control variable, which is assumed to satisfy $0 \leq u \leq u_{\max}$. The dynamics of the fish population is then given by

$$\dot{x} = (k - u)x, \quad x(0) = x_o$$

Here $k > 0$ and $x_o > 0$. The total amount of fish obtained during a time period T is

$$J = \int_0^T uxdt$$

i) Derive the necessary conditions given by the PMP for the problem of maximizing J.

ii) Show that the necessary conditions are satisfied by a bang-bang control, that is, it only takes boundary values of the constraint set. How many switching times are there?

iii) Determine an equation for calculating the switching time(s).