BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Optimisation Mock Exam

Date: Wednesday, 11th May 2021

Time: 09:00-11:00

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) := x_2^4 - 2x_2^2 + 1 + \left(x_1^2 + x_2^2 - 1\right)^2$$

(i) Find all the stationary points of f. (5 marks)

(ii) Classify the stationary points found in i). (5 marks)

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex as well as concave function. Show that f is an affine function, that is, there exist $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$. (10 marks)

(Total: 20 marks)

2. (a) Let $f(x_1, x_2)$ be a twice-differentiable convex function in $\mathbb{R}^2$ such that $f(0,0) = f(1,0) = f(0,1) = 0$. What do you know about:

(i) $f\left(\frac{1}{2}, \frac{1}{2}\right)$? (5 marks)

(ii) $a = \frac{\partial^2 f}{\partial x_1^2}$, $b = \frac{\partial^2 f}{\partial x_2^2}$, and $c = \frac{\partial^2 f}{\partial x_1 \partial x_2}$? (5 marks)

(b) Consider the function

$$g(x_1, x_2, x_3) = 59x_1^2 + 52x_2^2 + 17x_3^2 + 80x_1x_2 - 24x_1x_3 + 8x_2x_3 + 27x_1 - 84x_2 + 20x_3\,.$$

(i) Is $g(\mathbf{x})$ convex? (5 marks)

(ii) Solve

$$\begin{aligned} \max \quad & g(x_1, x_2, x_3) \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(5 marks)

(Total: 20 marks)

3. Consider the minimization problem for $\mathbf{x} \in \mathbb{R}^n$

$$\min_{\mathbf{x} \in C} \|\mathbf{x} - \mathbf{y}\|^2 \,,$$

where $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$, with $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$. Assume that the rows of $\mathbf{A}$ are linearly independent.

(i) Determine the KKT conditions for this problem. Are these sufficient?. (6 marks)

(ii) Find the optimal solution of the problem using the KKT system. (8 marks)

(iii) Given the problem

$$\min_{(x_1, x_2) \in C} x_1^2 + 2x_2^2 - 3x_1 \,,$$

where $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$, write the explicit gradient descent iteration for this problem with a constant stepsize $t = 1$. You can help yourself using the result in part ii) (6 marks)

(Total: 20 marks)

4. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + 0.5x_2^2 + x_1 x_2 - 2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \end{aligned}$$

(i) Solve this problem using KKT conditions. Are these sufficient? (10 marks)

(ii) Find the solution of the dual problem. What is the duality gap? (10 marks)

(Total: 20 marks)

5. **Mastery question.** The dynamics

$$\dot{x}(t) = -x(t) + u(t) \qquad |u| \leq 1$$

are to be controlled so that $x(1) = 0$ while minimizing the cost

$$J = \int_0^1 |u(t)|\, dt\,.$$

Show that the control

$$u(t) = \begin{cases} 0 & 0 \leq t < 0.5 \\ -1 & 0.5 \leq t \leq 1 \end{cases}$$

satisfies Pntryagin's necessary optimality conditions for some $x(0)$.

(20 marks)

(Total: 20 marks)