Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Mathematics Pre-arrival course

Solutions to Weekly Quiz 1 – Language of Mathematics and Real Functions

1. Which of the following are true statements:

   (a) $\forall x, y \in \mathbb{R} \quad \exists z \in \mathbb{R}(x < y \Rightarrow x < z < y)$
   (b) $\exists z \in \mathbb{N} \quad \forall x \in \mathbb{R}(z < x)$
   (c) $\exists z \in \mathbb{R} \quad \forall x, y \in \mathbb{R}(x < y \Rightarrow x < z < y)$
   (d) $\exists z \in \mathbb{Z} \quad \forall x \in \mathbb{R}(z < x)$

2. Which of the following formalises the statement: 'between any two rational numbers there is an irrational number'

   (a) $\forall x, y \in \mathbb{Q} \quad \exists z \in \mathbb{R}(z \notin \mathbb{Q} \wedge x < z < y)$
   (b) $\forall x, y \in \mathbb{Q} \quad \exists z \in \mathbb{R}((z \notin \mathbb{Q} \Rightarrow x \neq y) \Rightarrow x < z < y)$
   (c) $\exists z \in \mathbb{R} \quad \forall x, y \in \mathbb{Q}(z \notin \mathbb{Q} \wedge (x \neq y \Rightarrow (x < z < y \vee y < z < x)))$
   (d) $\forall x, y \in \mathbb{Q} \quad \exists z \in \mathbb{R}(x \neq y \Rightarrow ((x < z < y \vee y < z < x) \wedge z \notin \mathbb{Q}))$

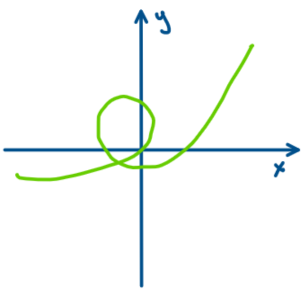
3. Which of the following are well defined functions (they must be written formally):

   (a) $f(x) = 2x$
   (b) $f : \mathbb{R} \to \mathbb{Z}$ such that $f(x) = 2x$
   (c) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (x^3 + 2x + 1)^{\frac{1}{2}}$
   (d) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ such that $f(x) = \frac{x^2+3x-1}{x-2}$

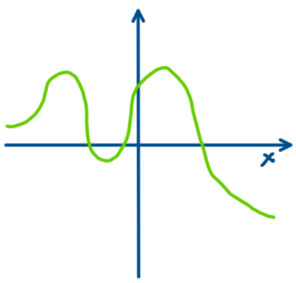
4. Find the inverse of the following function: $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 8x^3 + 2$

   (a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{2}(y - 2)^{\frac{1}{3}}$
   (b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 8x^{\frac{1}{3}} - 2$
   (c) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x^{\frac{1}{3}} - 2$
   (d) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{2}(x - 2)^{\frac{1}{3}}$

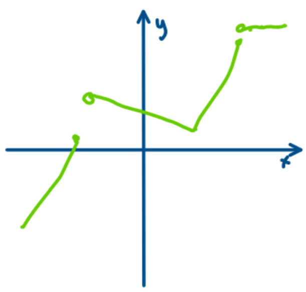
5. Which of the following could be continuous functions from $\mathbb{R}$ to $\mathbb{R}$.



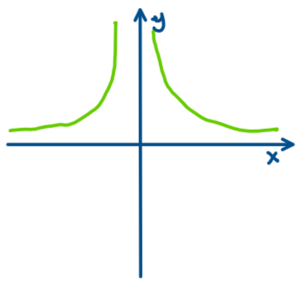
(a)



(b)



(c)



(d)

6. Which of the following are injective (one-to-one) functions:

   (a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$
   (b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x|$
   (c) $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}, \quad f(x) = x^2$
   (d) $f : \mathbb{N}^{>0} \to \mathbb{N}^{>0}, \quad f(x) = x^2 - 2x + 10$

7. Which of the following are solutions to the following differential equation:

$$\frac{d^2y}{dx^2} = -4y$$

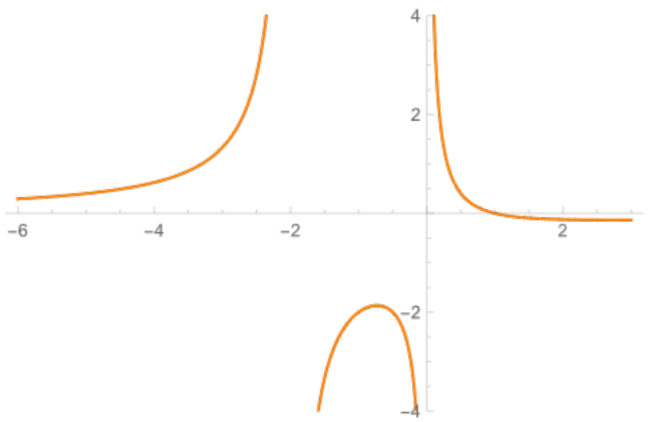
   (a) $y = \cos(x) - \sin(x)$
   (b) $y = \cos(2x) + \sin(2x)$
   (c) $y = \cos(2x)$
   (d) $y = \cos(2x) - e^{2x}$

8. Which of the following is a solution to the following initial value problem:

$$\frac{dx}{dy} = 2x + 2 \quad \text{with} \quad y(0) = 0$$

   (a) $\frac{2}{3}x^3 + 2x$
   (b) $\frac{1}{2}\log(x + 1)$
   (c) $\log(2x + 2)$
   (d) $2x^3 + 2x$

9. Which of the following functions is this a sketch of :



   (N.B.: the codomain is always the reals, the domain is also the reals with certain points removed to ensure that the function is well-defined.)

   (a) $y = \frac{-x+1}{x^2+2x}$
   (b) $y = \frac{x+1}{x^2+2x}$
   (c) $y = \frac{1}{x+2}$
   (d) $y = \frac{-1}{x^2+2x}$