Proof of the continuity of $u(\underline{x})$ of the preceding theorem — beyond the scope of this course.

<u>Some additional notes:</u>

a) $\underline{x} \prec \underline{x}' \iff \underline{x} \precsim \underline{x}'$ and $\underline{x}' \not\precsim \underline{x}$

$\iff u(\underline{x}) \leq u(\underline{x}')$ and $u(\underline{x}') \not\leq u(\underline{x})$

$\iff u(\underline{x}) < u(\underline{x}')$

---

a) 'Non-satiation' means that the consumer is <u>never</u> satisfied in the sense that no matter what bundle of goods they have, there is another very similar bundle that they (strictly) prefer.

<u>Claim</u>: Let $\succsim$ be a strong monotonic preference relation over $\mathbb{R}^n_{\geq 0}$. Then $\succsim$ is locally nonsatiated.

<u>Proof</u>: Fix some $\varepsilon > 0$. Consider arbitrary $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and let $\underline{e} = (1, \dots, 1)$ $(\in \mathbb{R}^n_{\geq 0})$. For any $\lambda > 0$, we also have $\underline{x} + \lambda \underline{e} \in \mathbb{R}^n_{\geq 0}$. Since clearly $\underline{x} + \lambda \underline{e} > \underline{x}$ then $\underline{x} + \lambda \underline{e} \succ \underline{x}$ by (strong) monotonicity. Now consider the following metric:

$$d(\underline{x} + \lambda \underline{e}, \underline{x}) = \|\underline{x} + \lambda \underline{e} - \underline{x}\| = \lambda \|\underline{e}\| = \lambda \sqrt{n}.$$

Then for $\lambda < \frac{\varepsilon}{\sqrt{n}}$, $d(\underline{x} + \lambda \underline{e}, \underline{x}) < \varepsilon$ yet $\underline{x} + \lambda \underline{e} \succ \underline{x}$. //

However, the converse is not always true.

Properties of a utility function

If the underlying preferences are complete, transitive, continuous and (strictly) monotone, the corresponding utility function will be continuous and (strictly) monotone.

Check:

Consider a preference relation with utility function u.

Weak monotonicity $\Rightarrow$

$$\underline{x} \leq \underline{x}' \Rightarrow \underline{x} \precsim \underline{x}' \Rightarrow u(\underline{x}) \leq u(\underline{x}')$$

Strong monotonicity $\Rightarrow$

$$\underline{x} \leq \underline{x}' \text{ and } \underline{x} \neq \underline{x}' \Rightarrow \underline{x} \prec \underline{x}' \Rightarrow u(\underline{x}) < u(\underline{x}')$$

Some sources also discuss the strict monotonicity of preference relations, which they define as:

$$\underline{x} \leq \underline{x}' \Rightarrow \underline{x} \precsim \underline{x}', \text{ while } \underline{x} \ll \underline{x}' \Rightarrow \underline{x} \prec \underline{x}'.$$

If the preferences are (strictly) convex, the utility function is (strictly) quasi-concave.

Recall the definition of convexity for a preference relation:

$$\forall \underline{x}, \underline{x}', \underline{x}'' \in X \text{ with } \underline{x} \precsim \underline{x}' \text{ and } \underline{x} \precsim \underline{x}'',$$

$$\underline{x} \precsim t\underline{x}' + (1-t)\underline{x}'' \quad \forall t \in [0,1]. \quad (*)$$

Then, if $\precsim$ is convex and u is its associated utility function, then $\forall \underline{x}', \underline{x}'' \in X$, assuming without loss of generality (by completeness) that $\underline{x}' \precsim \underline{x}''$, then $\forall t \in [0,1]$,

$$\underline{x}' \precsim t\underline{x}' + (1-t)\underline{x}'' \quad \text{(take } \underline{x} = \underline{x}' \text{ in } (*)\text{)}$$

$$\Rightarrow \quad u(t\underline{x}' + (1-t)\underline{x}'') \geq u(\underline{x}') \geq \min\{u(\underline{x}'), u(\underline{x}'')\}$$

$\Rightarrow$ u is quasi-concave

Similarly for strict convexity/quasi-concavity.

Substitution in demand

Suppose the availability of good i drops, such that x_i must decrease. In order to preserve the same level of utility in their overall consumption bundle, consumers will want to compensate by replacing with a separate good. By how much should the consumer alter x_j such that the utility remains constant?

This is analogous to the problem of technical substitution.

Indeed, we define the ***marginal rate of substitution*** (MRS) to be the rate of change of good j with respect to the change in good i:

$$MRS_{i,j}(\underline{x}) = \frac{-\partial u(\underline{x})/\partial x_i}{\partial u(\underline{x})/\partial x_j}$$

where we also define

$$MU_i(\underline{x}) = \frac{\partial u(\underline{x})}{\partial x_i}$$

to be the ***marginal utility*** with respect to good i.

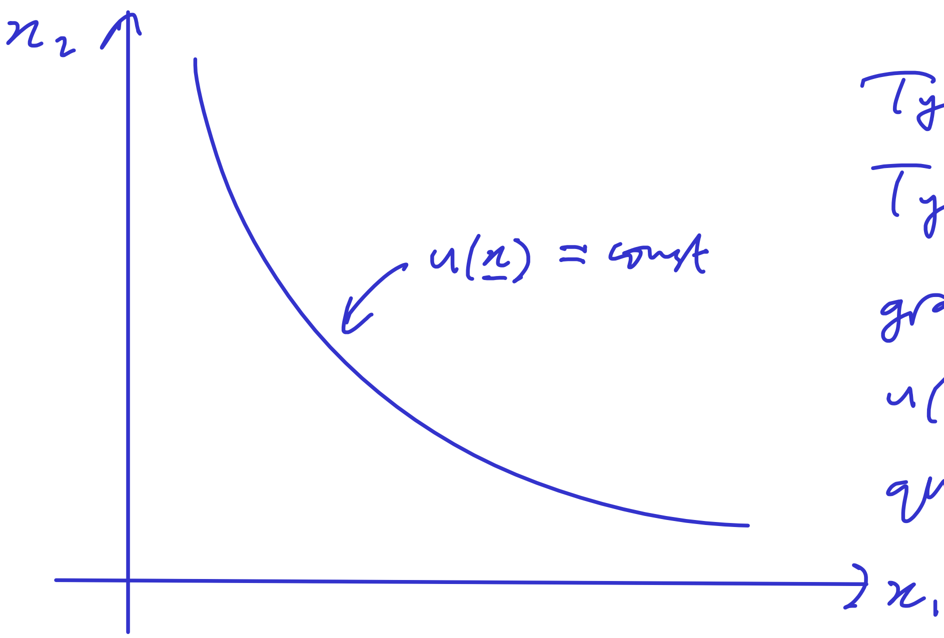
The MRS is, of course, the consumer-side analogue to the MRTS (marginal rate of technical substitution). One can check that the $MRS_{i,j}$ is indeed invariant under a strictly monotonic transformation of the utilities.

$$\frac{\partial g(u(\underline{x}))}{\partial x_i} = g'(u(\underline{x})) \cdot \frac{\partial u(\underline{x})}{\partial x_i}$$

So $MRS_{i,j}$ is independent of the choice of utility function. ($g'(u(\underline{x}))$ is cancelled from $(\partial g(u(\underline{x}))/\partial x_1)/((\partial g(u(\underline{x}))/\partial x_2)$.)

Note that g should be a strictly increasing function so that $g(u(\underline{x}))$ is indeed a utility function.

Just as it is often useful to consider a graphical representation of a firm's economic and technological capabilities, it can be useful to graphically represent consumer preferences. As a demand-side analogue to the isoquant, we define the ***indifference curve*** to be a level set of the utility function:



Typical indifference curve. Typically, it has negative gradient (MRS), and $u(\underline{x})$ is monotonic and quasi-concave.

<u>Budget Restraints, Utility Maximisation and Demand</u>

In practice, consumers can't simply pick their most preferred bundle – ∃ budget restraints. A fundamental assumption underlying consumer-side economic analysis is that the consumer will choose to purchase the most preferred consumption bundle from the set of all *affordable* bundles.

Represent the set of all affordable bundles by the **budget set**:

$$\mathcal{B} \equiv \mathcal{B}_{\underline{p},m} = \{\underline{x} \in X : \underline{p}\,\underline{x}^T \leq m\}$$

where p_i is the price per unit of the i^{th} good that the consumer must pay, and m is their budget.

At the heart of consumer choice, then, is the problem of finding the most preferred bundle $\underline{x} \in B$.

This is the problem of finding

$$\underset{\underline{x} \in B_{\underline{p},m}}{\operatorname{argmax}} \{u(\underline{x})\}$$

A solution to this problem will exist so long as u is continuous and B is closed and bounded...
and provided $p_i > 0 \ \forall i$ (c.f. existence of cost-minimising bundle $\underline{x}(\underline{w}, y)$) – details omitted.

Denote the constrained utility-maximising bundle $\underline{x}^* \in B$:

- $\underline{x}^*$ will be independent under a strictly increasing transformation of utility function.
- $\underline{x}^*$ will, in general, be dependant both on prices $\underline{p}$ and on the budget m.
- $\underline{x}^*$ is homogeneous of degree zero *jointly* in prices and budget.

i.e. $$\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \quad \forall t > 0.$$

(This is to be expected – $\underline{p}$ and m are being multiplied by the <u>same</u> non-zero factor.)

How can we find $\underline{x}^*$?

Kuhn–Tucker conditions – these are necessary conditions on $\underline{x}^*$ to be a solution of this constrained optimisation problem which has <u>inequality</u> constraints. These generalise Lagrange's conditions for the solution of an optimisation problem with equality constraints.

If we make some reasonable regularity assumptions about the consumer's preference ordering $\preccurlyeq$, we can simplify our constrained optimisation problem.

Assume *local nonsatiation* and suppose $\underline{x}^* = \underset{\underline{x}\in B}{\operatorname{argmax}}\, u(\underline{x})$:

- If $\underline{p}\,\underline{x}^{*\mathsf{T}} < m$, that means $\underline{x}^*$ is in the interior of B, then there would exist some $\underline{x}$, close enough to $\underline{x}^*$, such that both $\underline{p}\,\underline{x}^{\mathsf{T}} < m$ and (by nonsatiation) $\underline{x} \succ \underline{x}^*$.
- This would imply that $\underline{x}^*$did not maximize $u(x)$, and so we have a contradiction.

$$\Rightarrow \underline{p}\,\underline{x}^{*\mathsf{T}} \nless m$$
$$\Rightarrow \underline{p}\,\underline{x}^{*\mathsf{T}} = m.$$

So under these assumption, $\underline{x}^*$ costs the consumer all of their budget. But it means that we need only seek $\underset{\underline{x}\in\partial B}{\operatorname{argmax}} \{u(\underline{x})\}$, which allows us to use the Lagrangian approach again.

Some economists call the fact that utilities are maximised only if people spend all their money **Walras' Law.**

Example: Consider the consumer with utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \quad , \quad 0 < \alpha < 1$$

(a Cobb-Douglas utility function. This corresponds to a strongly monotonic and hence locally non-satiated preference:

$$\underline{x} \geq \underline{x}' \text{ and } \underline{x} \neq \underline{x}' \Rightarrow u(\underline{x}) > u(\underline{x}') \Rightarrow \underline{x} \succ \underline{x}')$$

We seek $\underset{\underline{x}\in X}{\operatorname{argmax}}\, u(\underline{x})$ such that $\underline{p}\,\underline{x}^{\mathsf{T}} = m$.

Introduce Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1 x_1 + p_2 x_2 - m)$$

F.O.C.: $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow p_1 x_1 + p_2 x_2 = m$ ①

$\dfrac{\partial \mathcal{L}}{\partial x_i} = 0$, $i = 1, 2$

$\Rightarrow \quad \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1$ ②

$(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2$ ③

② ÷ ③ $\Rightarrow \quad \dfrac{\alpha}{1-\alpha} \dfrac{x_2}{x_1} = \dfrac{p_1}{p_2}$

$\Rightarrow \quad x_2 = \dfrac{(1-\alpha)}{\alpha} \left(\dfrac{p_1}{p_2}\right) x_1$

Then ① $\Rightarrow \quad \left(1 + \dfrac{(1-\alpha)}{\alpha}\right) p_1 x_1 = m$

$\Rightarrow \quad x_1^*(p_1, p_2, m) = \dfrac{\alpha m}{p_1}$

$x_2^*(p_1, p_2, m) = (1-\alpha) \dfrac{m}{p_2}$

Notice $\quad MRS = -\dfrac{\partial u/\partial x_1}{\partial u/\partial x_2}$

$$= -\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}$$

$$= -\frac{\alpha}{1-\alpha}\frac{x_2}{x_1}$$

$$= -\frac{p_1}{p_2} \quad \text{at } (x_1^*, x_2^*)$$

gradient of $p_1 x_1 + p_2 x_2 = \text{const.}$

Check:

- $\underline{x}^*$ is independent of a strictly increasing transformation of $u(\underline{x})$, since such a transformation will not change the MRS. In particular, one might consider

$$\hat{u}(\underline{x}) = \log(u(\underline{x})) = \alpha \log x_1 + (1-\alpha)\log x_2$$

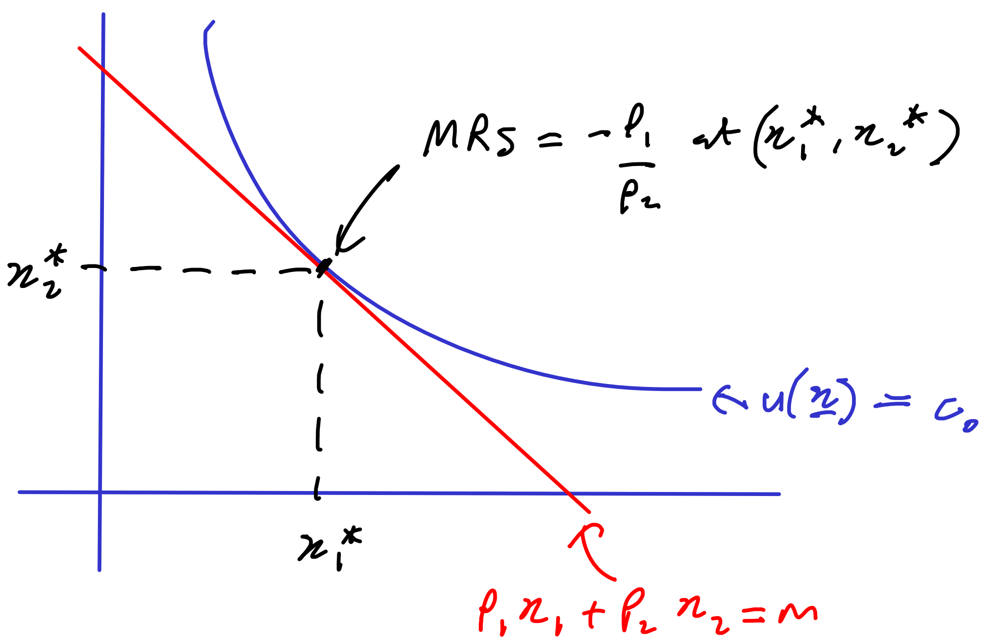
- $\underline{x}^*$ is homogeneous of degree 0 jointly in $\underline{p}$ and m:

$$\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \quad \forall t > 0.$$

Note also that, for $i=1,2$, $x_i^*(\underline{p},m)$ increases as m increases (while p_i is held fixed), and decreases as p_i increases (while m is held fixed), as one might expect.

A note on second-order conditions for the Lagrangian:

Can we derive a condition on $\underline{x}^*(\underline{p},m)$ to guarantee it gives a maximum of $u(\underline{x})$ rather than a minimum?



Consider the change in $u(\underline{x})$ as $\underline{x}$ moves along the line $p_1 x_1 + p_2 x_2 = m$. Let $\underline{x}(t)$ denote a point on this line, Take $t = x_1$. Now, considering u as a function of the single variable t, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt}$$

$$\hookrightarrow \quad x_2 = \frac{m - p_1 x_1}{p_2}$$

$$= \left(\frac{\partial}{\partial x_1} - \frac{p_1}{p_2}\frac{\partial}{\partial x_2}\right) u$$

So

$$\frac{d^2 u}{dt^2} = \left(\frac{\partial}{\partial x_1} - \frac{p_1}{p_2}\frac{\partial}{\partial x_2}\right)^2 u$$

$$= \underbrace{\frac{\partial^2 u}{\partial x_1^2} + \frac{p_1^2}{p_2^2}\frac{\partial^2 u}{\partial x_2^2} - 2\frac{p_1}{p_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}}_{= \text{(t), say}}$$

Now recall that

$$L(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m)$$

$$\Rightarrow \quad \frac{\partial^2 L}{\partial \lambda^2} = 0, \quad \frac{\partial L}{\partial \lambda \partial x_i} = -p_i \quad \text{for } i = 1, 2$$

$$\frac{\partial^2 L}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}, \quad \frac{\partial^2 L}{\partial x_1 x_2} = \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

The Hessian matrix of L is therefore

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 \\ -p_2 & \partial^2 u/\partial x_1 \partial x_2 & \partial^2 u/\partial x_2^2 \end{pmatrix} = H_L(\underline{x}, \underline{p}),$$

say.

And the determinant of this is

$$|H_L(\underline{x}, \underline{p})| = -p_1\left(-p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_1 \frac{\partial^2 u}{\partial x_2^2}\right)$$

$$-p_2\left(-p_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_2 \frac{\partial^2 u}{\partial x_1^2}\right)$$

$$= 2 p_1 p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - p_1^2 \frac{\partial^2 u}{\partial x_2^2} - p_2^2 \frac{\partial^2 u}{\partial x_1^2}$$

$$= -p_2^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{p_1^2}{p_2^2} \frac{\partial^2 u}{\partial x_2^2} - 2 \frac{p_1}{p_2} \frac{\partial^2 u}{\partial x_1 \partial x_2}\right)$$

$$= -p_2^2 . \dagger$$

So a necessary condition for $\underline{x}^*$ to give a maximum of $u(\underline{x})$ (subject to the constraint $\underline{p}\,\underline{x}^T = m$) is that

$$|H_L(\underline{x}, \underline{p})| \geq 0 .$$

Now note that with $\hat{\underline{h}} = (p_2, -p_1)$ (note that this is parallel to the line $p_1 x_1 + p_2 x_2 = m$) and

$$(D^2 u)_{i,j} \equiv \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i,j = 1,2$$

we have

$$\hat{\underline{h}}\, D^2 u(\underline{x})\, \hat{\underline{h}}^T$$

$$= (p_2 \quad -p_1) \begin{pmatrix} p_2 \dfrac{\partial^2 u}{\partial x_1^2} - p_1 \dfrac{\partial^2 u}{\partial x_1 \partial x_2} \\ p_2 \dfrac{\partial^2 u}{\partial x_1 \partial x_2} - p_1 \dfrac{\partial^2 u}{\partial x_2^2} \end{pmatrix}$$

$$= p_2^2 \frac{\partial^2 u}{\partial x_1^2} - 2 p_1 p_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + p_1^2 \frac{\partial^2 u}{\partial x_2^2}$$

$$= -|H_u(\underline{x}, \underline{p})|$$

So we can rephrase the above as follows. A necessary condition for $\underline{x}^*$ to maximize $u(\underline{x})$ subject to the constraint $\underline{p}\,\underline{x}^* = m$ is that

$$\underline{h}\, D^2 u(\underline{x}^*)\, \underline{h}^T \leq 0$$

for all vectors $\underline{h} \in \mathbb{R}^n$ such that

$$D\underline{u}(\underline{x}^*) . \underline{h}^T = 0 .$$

(i.e., all vectors $\underline{h}$ parallel to $\hat{\underline{h}} = (p_2, -p_1)$; the latter is equivalent to our F.O.C. on $\underline{x}^*$).

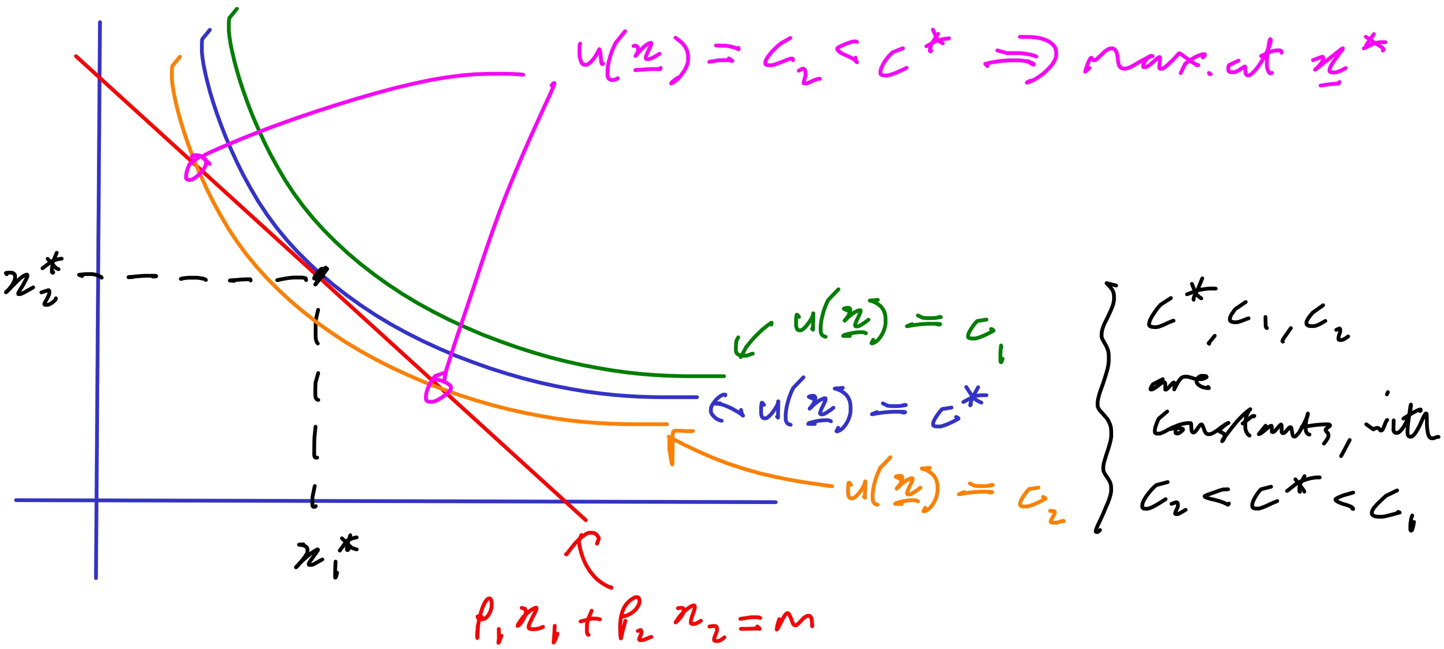
That is, $D^2\underline{u}(\underline{x}^*)$ should be negative semi-definite with respect to all such $\underline{h}$. A sufficient condition is that $D^2\underline{u}(\underline{x}^*)$ should be negative definite with respect to all such non-zero $\underline{h}$. (Compare this to the conditions for a maximum of $u(\underline{x})$ subject to <u>no</u> constraints on $\underline{x}$. Recall that a Taylor series expansion of $u(\underline{x})$ about $\underline{x}^*$ gives, for $\underline{x}$ local to $\underline{x}^*$:

$$u(\underline{x}) - u(\underline{x}^*) = \frac{1}{2}(\underline{x} - \underline{x}^*) D^2 u(\underline{x}^*)(\underline{x} - \underline{x}^*)^T .)$$

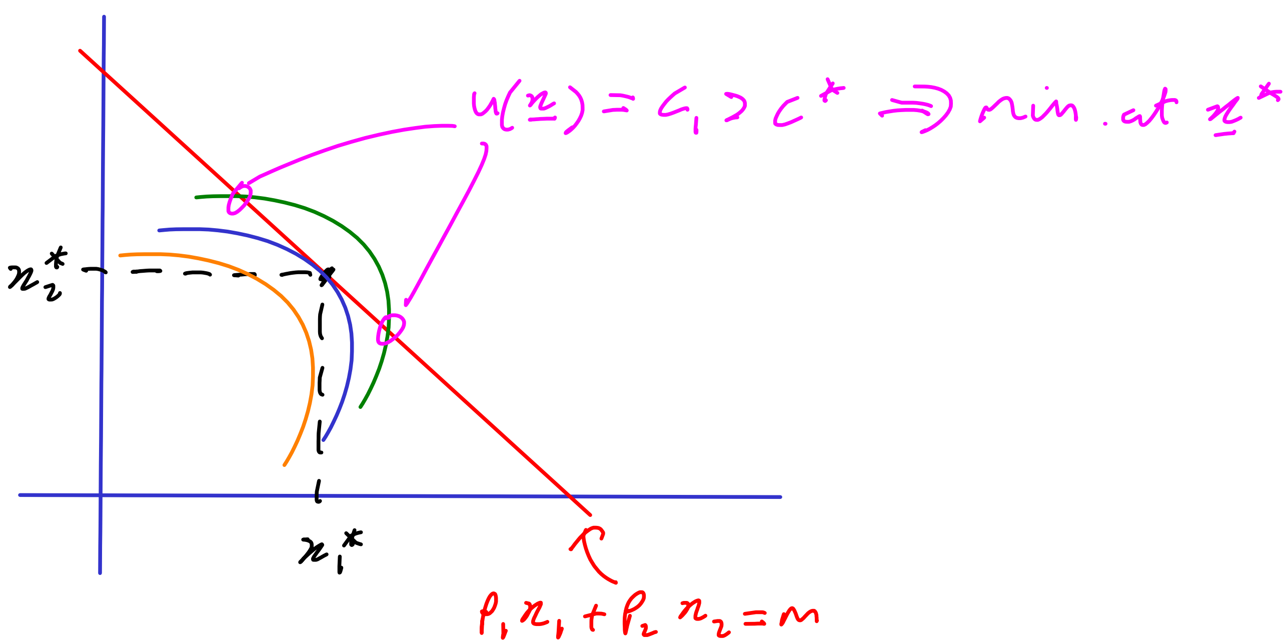
Note that $H_h(\underline{x}, \underline{p})$ is referred to as the <u>bordered</u> Hessian (here the Hessian refers to $D^2 u(\underline{x})$).

For $u(\underline{x})$ monotonic (as is usually the case) the above condition on $\underline{x}^*$ is equivalent to $u(\underline{x})$ being quasi-concave at least local to $\underline{x}^*$:

i.e.,



rather than



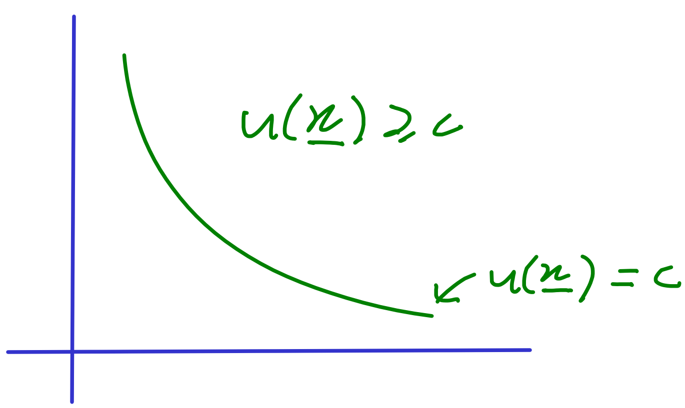
Details omitted here.

The choice of the consumption bundle that maximises the consumer's constrained utility function will be exactly the bundle that the consumer demands; this is unsurprisingly referred to as the ***demanded bundle*** or ***demand function***,

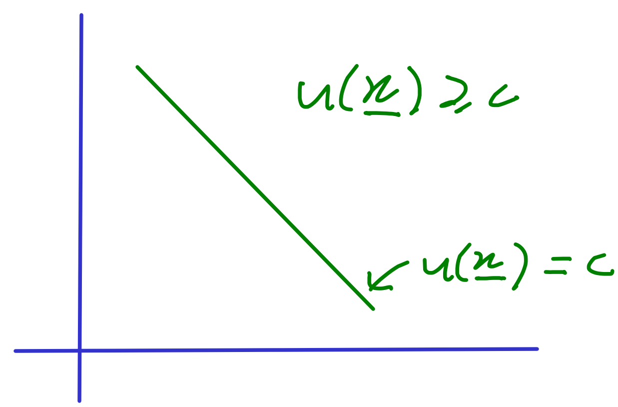
$$\underline{x}^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \qquad \underline{x}^*\left(\underline{p}, m\right) = \underset{\underline{x} \in \partial B}{\operatorname{argmax}}\, u\left(\underline{x}\right).$$

We call this the Marshallian demand function (or, uncompensated demand function).

Note that we have discussed the existence of the argmax. However, it is *per se* not clear whether the argmax is unique, that means, whether the maximum is attained at a single point over ∂B. This can be guaranteed if the underlying preferences are strictly convex (and prices are strictly positive), i.e. if $u(\underline{x})$ is strictly quasi-concave, or equivalently, the sets $\{\underline{x} \mid u(\underline{x}) \geq c\}$ for constant c, are strictly convex (so their boundaries do not contain any straight sections). Eg.



not



Moreover, the function $\underline{x}^*(\underline{p}, m)$ is homogeneous of degree 0 in $(\underline{p}, m)$.

ie. $\underline{x}^*(t\underline{p}, tm) = \underline{x}^*(\underline{p}, m) \quad \forall t > 0.$

We also note that, faced with a set of goods with prices $\underline{p}$, the maximum utility achievable with a given budget m is known as the ***indirect utility function***:

$$v : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad v\left(\underline{p}, m\right) = u\left(\underline{x}^*\left(\underline{p}, m\right)\right)$$