MATH97035 MATH97143

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Representation Theory

Date: Tuesday, 4 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

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All vector spaces are over the complex numbers.

1. (a) Let $\rho : C_n = \{1, g, \ldots, g^{n-1}\} \to \mathrm{GL}_3(\mathbb{C})$ be a representation where $\rho(g^j)$ is the counter-clockwise rotation matrix by $\frac{2\pi j}{n}$ about the z-axis, and let $n \geq 3$. Find *all* subrepresentations. How many are there? (5 marks)

   (b) Let (V, ρ_V) be a finite-dimensional representation of S_3. Suppose that there is no common nonzero eigenvector of all the operators $\rho_V(\sigma)$ for $\sigma \in S_3$. Show that V is even-dimensional. (5 marks)

   (c) Let (V, ρ_V) be a finite-dimensional representation of a group G and $W \subseteq V$ be a subrepresentation.

      (i) Let G be finite. Show that there exists a surjective G-linear map $V \to W$. (3 marks)

      (ii) Again let G be finite. Show that there exists a subrepresentation of V isomorphic to V/W. (3 marks)

      (iii) Give a counterexample (G, V, W) to both (i) and (ii) when the group G is infinite: explain why (i) and (ii) both fail. (4 marks)

(Total: 20 marks)

2. (a) Let $G = S_4$ and let (V, ρ_V) be a two-dimensional irreducible representation. Show that $(V, \rho_V|_{A_4})$ is a decomposable representation of A_4 and find, up to isomorphism, its irreducible summands. (5 marks)

   (b) Let $G = A_4$ and (V, ρ_V) be the (three-dimensional) reflection representation of S_4 restricted to A_4. Let (W, ρ_W) be any finite-dimensional nonzero representation of A_4. Show that $(V \otimes W, \rho_{V \otimes W})$ contains a subrepresentation isomorphic to V. (5 marks)

   (c) Let G be a finite group. Show that G is abelian if and only if the (left) regular representation $\mathbb{C}[G]$ has finitely many subrepresentations $W \subseteq \mathbb{C}[G]$ (different subrepresentations are allowed to be isomorphic). (5 marks)

   (d) Let G be a simple finite group and V a nontrivial irreducible representation of dimension 2. Show that there exists an irreducible representation of dimension 3.
   (Hint: consider $\mathrm{Hom}(V, V)$.) (5 marks)

(Total: 20 marks)

3. The goal of this question is to complete the following partial character table for a group G of order 216. There are ten conjugacy classes with sizes indicated. Let $\omega = e^{2\pi i/3}$. You must justify each step.

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
size	1	9	8	12	12	24	24	36	36	54
χ_V	2	-2	2	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	ω^2	ω	0

(a) Make a table of the correct size, fill in the first row (the representation you always have), and put χ_V on the fifth line. (2 marks)

(b) Explain why V^* is another irreducible representation, and give its character. Put this on the sixth line. (2 marks)

(c) Show that $\mathrm{Hom}(V, V)$ contains a trivial one-dimensional subrepresentation U, and compute the character of the quotient $\mathrm{Hom}(V, V)/U$. Show that this character is irreducible. Put this on the seventh line. (4 marks)

(d) Prove that $V \otimes V$ decomposes as a direct sum of the three-dimensional representation you just found and a nontrivial one-dimensional representation, call it $\mathbb{C}_\omega$, with the following character:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
$\chi_{\mathbb{C}_\omega}$	1	1	1	ω^2	ω	ω^2	ω	ω	ω^2	1

Put this character on line two. (4 marks)

(e) By tensor product or other means, fill in lines three and four, with one and two dimensional representations, respectively. (3 marks)

(f) We now give you the following additional partial line:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
χ_W	8	?	?	2	2	-1	-1	?	0	0

Finish the character table.
(Hint: tensor with one-dimensional representations. To fill in the question marks, employ the regular character and orthogonality.) (5 marks)

(Total: 20 marks)

4. Let A be an associative algebra.

(a) Let $A = \mathbb{C}[Q_8]$, for $Q_8 = \{e, z, \pm i, \pm j, \pm k\}$ the quaternionic group of order eight: $i^2 = j^2 = k^2 = z$, and $ij = k, ki = j, jk = i$, with e the identity and z acting to flip signs: $z(\pm i) = \mp i, z(\pm j) = \mp j$, and $z(\pm k) = \mp k$. **We use the notation** e, z in order to avoid confusion with scalars (as we are using the group algebra).

Recall the character table of Q_8:

	$\{e\}$	$\{z\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
Size of class:	1	1	2	2	2
$\rho_{+,+}$	1	1	1	1	1
$\rho_{-,+}$	1	1	-1	1	-1
$\rho_{+,-}$	1	1	1	-1	-1
$\rho_{-,-}$	1	1	-1	-1	1
$\chi_{\mathbb{C}^2}$	2	-2	0	0	0

(i) Find a nonzero element $a \in Z(A)$ such that $\widetilde{\rho}_V(a) = 0$ if (V, ρ_V) is a one-dimensional representation, with $\widetilde{\rho}_V : A \to \mathrm{End}(V)$ the linear extension. (3 marks)

(ii) Show that such an element a as in part (i) is unique up to scaling. (3 marks)

(iii) Using the preceding parts, find the maximum subrepresentation W of the left regular representation $(\mathbb{C}[Q_8], \rho_L)$ in which the operators $\rho_L(g) \in \mathrm{End}(W)$ do not all have a common nonzero eigenvector. (4 marks)

(b) Now let A be arbitrary. Suppose that A has a module (V, ρ_V) of dimension four, but no modules of dimension one or three. Let $B := \rho_V(A)$ be the image of A in $\mathrm{End}(V)$. Recall that V is therefore also a B-module, via the inclusion $B \subseteq \mathrm{End}(V)$.

(i) Show that $\dim B \geq 4$. (3 marks)

(ii) Prove that, if V decomposes as a direct sum of two isomorphic simple modules, then we have equality in (i). (3 marks)

(iii) Prove that, if $\dim B$ is not a power of two, then V is indecomposable as an A-module but not simple. (4 marks)

(Total: 20 marks)

5. (a) Let $(\mathbb{C}^2, \rho)$ be the reflection and rotation representation of the dihedral group $G = D_n$. Let $H = C_n < D_n$ be the subgroup of rotations.

   (i) Decompose $\mathrm{Res}^G_H(\mathbb{C}^2, \rho)$ explicitly into irreducible subrepresentations of H. (2 marks)

   (ii) Find all $\rho' : H \to \mathrm{GL}_1(\mathbb{C})$ such that $(\mathbb{C}^2, \rho) \cong \mathrm{coInd}^G_H(\mathbb{C}, \rho')$. (4 marks)

   (iii) Recall that the irreducible two-dimensional representations of G are $(\mathbb{C}^2, \rho \circ \psi_j)$ where $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $\psi_j : G \to G$ is the map that multiplies all angles by j, i.e., $\psi_j(x) = x^j$ for x a rotation, and $\psi_j(y) = y$ for y the reflection about the x-axis. Using part (ii), for each j, find all ρ' such that $\mathrm{coInd}^G_H(\mathbb{C}, \rho') \cong (\mathbb{C}^2, \rho \circ \psi_j)$. (3 marks)

(b) Now let G be a group (not necessarily finite) and H an abelian subgroup of index two. Let $\rho_{-1} : G \to \mathbb{C}^\times$ be the one-dimensional representation with $\rho_{-1}(h) = I = (1)$ for $h \in H$ and $\rho_{-1}(g) = -I = (-1)$ if $g \notin H$. Let (V, ρ_V) be an irreducible representation of G.

   (i) Let (V, ρ_V) be a finite-dimensional irreducible representation of G of dimension greater than one. Show that $V \cong \mathrm{coInd}^G_H W$ for some one-dimensional representation W of H, and therefore that V is two-dimensional. (4 marks)

   (ii) Use Mackey's formula to show that, in the situation of (i), $V \cong V \otimes \rho_{-1}$. (4 marks)

   (iii) Now suppose that G is finite, and that (V, ρ_V) is a one-dimensional representation of G. Prove that $V \oplus (V \otimes \rho_{-1})$ is isomorphic to a coinduced representation. (Hint: consider $W := \mathrm{Res}^G_H V$.) (3 marks)

(Total: 20 marks)

Solutions: Group Representation Theory, 2021 exam

Question 1.

Part 1(a). *Let $\rho : C_n = \{1, g, \ldots, g^{n-1}\} \to \mathrm{GL}_3(\mathbb{C})$ be a representation where $\rho(g^j)$ is the counter-clockwise rotation matrix by $\frac{2\pi j}{n}$ about the z-axis, and let $n \geq 3$. Find* all *subrepresentations. How many are there?*

5, B

In this case the eigenvalues of $\rho(g)$ are $e^{\pm\frac{2\pi i}{n}}$ and 1, which are all distinct, so that we get three distinct one-dimensional representations by diagonalising these. By results from lectures, in this case the only subrepresentations are sums of these, so we get $2^3 = 8$ different subrepresentations, including the zero and everything.

Part 1(b). *Let (V, ρ_V) be a finite-dimensional representation of S_3. Suppose that there is no common nonzero eigenvector of all the operators $\rho_V(\sigma)$ for $\sigma \in S_3$. Show that V is even-dimensional.*

5, A

The hypothesis equivalently means that V has no one-dimensional subrepresentations. By Maschke's theorem V is nonetheless the direct sum of irreducible representations, and for S_3 the only irreducible representation of dimension bigger than one is the reflection representation, of dimension two. Thus V is a direct sum of copies of these and hence it is even-dimensional.

Part 1(c). *Let (V, ρ_V) be a finite-dimensional representation of a group G and $W \subseteq V$ be a subrepresentation.*

Subpart 1(c)(i). *Let G be finite. Show that there exists a surjective G-linear map $V \to W$.*

3, A

By Maschke's theorem there is a complement to W, call it U, and then $V = W \oplus U$ shows that $V/U \cong W$, so we get the surjective map by composing the quotient morphism with this isomorphism.

Subpart 1(c)(ii). *Again let G be finite. Show that there exists a subrepresentation of V isomorphic to V/W.*

3, A

Similarly, we take the same decomposition as before, and now compose the inclusion $U \to V$ with the isomorphism $U \cong V/W$.

Subpart 1(c)(iii). *Give a counterexample (G, V, W) to both (i) and (ii) when the group G is infinite: explain why (i) and (ii) both fail.*

4, C

Let G be the group of all invertible upper-triangular matrices of size 2×2, acting on $V = \mathbb{C}^2$ via left multiplication. This has a unique subrepresentation W given by scalar multiples of the unique eigenvector $(1, 0)^t$. The quotient representation V/W is not isomorphic to W: indeed $\chi_W \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a$ whereas $\chi_{V/W} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = c$. Thus the unique one-dimensional quotient representation and the unique one-dimensional subrepresentation are non-isomorphic. In particular there is no surjective G-linear map $V \to W$, as all surjective G-linear maps with one-dimensional target have image isomorphic to V/W. Similarly there is no injective G-linear map $V/W \to V$.

Question 2.

Part 2(a). *Let $G = S_4$ and let (V, ρ_V) be a two-dimensional irreducible representation. Show that $(V, \rho_V|_{A_4})$ is a decomposable representation of A_4 and find, up to isomorphism, its irreducible summands.*

5, A

Since A_4 has no irreducible two-dimensional representation, the first statement is a consequence of Maschke's theorem.

For the second, taking the character of V and the nontrivial one-dimensional characters $\mathbb{C}_\omega, \mathbb{C}_{\omega^2}$ of A_4, one sees directly that $\chi_V|_{A_4} = \chi_{\mathbb{C}_\omega} + \chi_{\mathbb{C}_{\omega^2}}$.

Alternative proof of the second statement: Note that $V \cong V^*$ as this is the unique two-dimensional irreducible representation of S_4. But $\mathbb{C}_\omega^* \cong \mathbb{C}_{\omega^2}$, and they are not isomorphic. Since $(V, \rho_V|_{A_4})$ is also isomorphic to its dual, it must either be trivial or the direct sum of both $\mathbb{C}_\omega$ and $\mathbb{C}_{\omega^2}$. But $\rho_V|_{A_4}$ is not trivial: otherwise ρ_V would itself have kernel A_4, and then would give an irreducible two-dimensional representation of $S_4/A_4 \cong C_2$, but C_2 has no two-dimensional irreducible representation as it is abelian.

Part 2(b). *Let $G = A_4$ and (V, ρ_V) be the (three-dimensional) reflection representation of S_4 restricted to A_4. Let (W, ρ_W) be any finite-dimensional nonzero representation of A_4. Show that $(V \otimes W, \rho_{V\otimes W})$ contains a subrepresentation isomorphic to V.*

5, C

Purely computational proof: We simply compute $\langle \chi_{V\otimes W}, \chi_V \rangle = \chi_W |\chi_V|^2 = \frac{1}{12}(9\chi_W(e) + 3\chi_W((12)(34)))$, which is positive for any character since $\chi_W(e) = \dim W \geq |\chi_W(g)|$ for all g.

Alternative proof: Being the unique irreducible of dimension three, $V \otimes W \cong V$ for every one-dimensional representation W. We can check that $V \otimes V$ contains a subrepresentation isomorphic to V—this is easily done using the character table, but let us give an alternative argument: the multiplicity of each one-dimensional representation W in $V \otimes V$ is $\dim \mathrm{Hom}_{A_4}(W, V \otimes V) \cong \dim \mathrm{Hom}_{A_4}(W \otimes V^*, V) \cong \dim_{A_4}(V, V) = 1$, since $W \otimes V^*$ is irreducible of dimension three, so that the multiplicity of V in $V \otimes V$ must be two for the dimensions to add up correctly. So if we tensor V with any irreducible representation, V is isomorphic to a subrepresentation of the result. For any representation W, $V \otimes W$ contains $V \otimes W'$ for some irreducible subrepresentation W' of W. Thus we get the desired result.

More conceptual proof: Note that (V, ρ_V) is the unique irreducible representation of A_4 of dimension greater than one, up to isomorphism. The one-dimensional representations all kill $[A_4, A_4] \neq \{e\}$. Now for every nonzero W, $V \otimes W$ is still faithful, by the following lemma:

Lemma: Let G be a group with trivial centre and V a faithful representation. Then for every nonzero W, $V \otimes W$ is also faithful.

Proof: Since G has trivial centre, $\rho_V(g)$ is not a scalar multiple of the identity for any g. Then there is $v \in V$ which is not an eigenvector for $\rho_V(g)$. Thus $v \otimes w$ is also not an eigenvector of $\rho_{V\otimes W}(g)$ for any nonzero w. As a result $\rho_{V\otimes W}(g)$ is also not a scalar multiple of the identity. In particular $\rho_{V\otimes W}(g)$ is not the identity, so $V \otimes W$ is faithful.

So in the case at hand, $V \otimes W$ must decompose by Maschke's theorem into irreducible representations which do not all kill $[A_4, A_4]$, so one of the summands must be isomorphic to (V, ρ_V).

Part 2(c). *Let G be a finite group. Show that G is abelian if and only if the (left) regular representation $\mathbb{C}[G]$ has finitely many subrepresentations $W \subseteq \mathbb{C}[G]$ (different subrepresentations are allowed to be isomorphic).*

5, D

By results from lectures, there are finitely many subrepresentations of a semisimple representation V (which includes all finite-dimensional representations by Maschke's theorem) if and only if all irreducible summands are non-isomorphic. That is, the multiplicity of each irreducible representation in V must be at most one. In the group algebra $\mathbb{C}[G]$ of a finite group, all irreducible representations occur with multiplicity equal to their dimension. Thus this multiplicity is (at most) one if and only if all irreducible representations are one-dimensional. In this case $[G, G]$ is in the kernel of all irreducible representations, hence of all representations by Maschke's theorem. But $\mathbb{C}[G]$ is faithful, so $[G, G]$ is trivial and G is abelian.

Part 2(d). *Let G be a simple finite group and V a nontrivial irreducible representation of dimension 2. Show that there exists an irreducible representation of dimension 3. (Hint: consider* $\mathrm{Hom}(V, V)$*.)*

5, C

Remark: Actually the assumption can not be satisfied: no simple finite group has an irreducible representation of dimension two. This does not affect the validity of the problem or solution.

The representation $\mathrm{Hom}(V, V)$ contains a one-dimensional trivial subrepresentation $\mathbb{C} \cdot I$. So the quotient $\mathrm{Hom}(V, V)/\mathbb{C} \cdot I$ is three-dimensional. If it is not irreducible then by Maschke's theorem it has a one-dimensional subrepresentation. Since G is simple this must be trivial. But in this case by Maschke's theorem again, the multiplicity of the trivial one-dimensional representation in $\mathrm{Hom}(V, V)$ is at least two. However this multiplicity is $\dim \mathrm{Hom}(V, V)^G = \dim \mathrm{Hom}_G(V, V)$ which is one by Schur's Lemma.

Question 3. *The goal of this question is to complete, with justification, the following partial character table for a group G of order 216. There are ten conjugacy classes with sizes indicated. Let $\omega = e^{2\pi i/3}$. You must justify each step.*

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
size	*1*	*9*	*8*	*12*	*12*	*24*	*24*	*36*	*36*	*54*
χ_V	*2*	*-2*	*2*	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	ω^2	ω	*0*

Part 3(a). *Make a table of the correct size, fill in the first row (the representation you always have), and put χ_V on the fifth line.*

2, A

To save space we won't continually reprint the table here, but it should be a ten-by-ten table as there are ten conjugacy classes. The first row is the all ones vector.

Part 3(b). *Explain why V^* is another irreducible representation, and give its character. Put this on the sixth line.*

2, A

We showed that the dual of a finite-dimensional irreducible representation is irreducible (it follows since the double dual is the original representation back which is irreducible, so a subrepresentation of V^* would produce a quotient representation of V and hence also a subrepresentation; one can also prove this using the character of the dual which is the complex conjugate of the original character, so will still have inner product equal to one with itself). The character is the complex conjugate.

Part 3(c). *Show that* $\mathrm{Hom}(V,V)$ *contains a trivial one-dimensional subrepresentation* U*, and compute the character of the quotient* $\mathrm{Hom}(V,V)/U$*. Show that this character is irreducible. Put this on the seventh line.*

4, B

The subspace $U := \mathbb{C}I_W$ is a trivial one-dimensional subrepresentation because $\rho_W(g) \circ I_W \circ \rho_W(g^{-1}) = I_W$ for all $g \in G$. We have $\chi_{\mathrm{Hom}(V,V)/U} = \chi_V \overline{\chi_V} - 1$, as $\chi_U = 1$ is the constant function. Now we can check it is irreducible by the standard formula: $3^2 + 9(3^2) + 8(3^3) + 54 = 216$.

Part 3(d). *Prove that* $V \otimes V$ *decomposes as a direct sum of the three-dimensional representation you just found and a nontrivial one-dimensional representation, call it* $\mathbb{C}_\omega$*, with the following character:*

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
$\chi_{\mathbb{C}_\omega}$	*1*	*1*	*1*	ω^2	ω	ω^2	ω	ω	ω^2	*1*

Put this character on line two.

4, B

Let $(\mathbb{C}^3, \rho_{\mathbb{C}^3})$ be the representation we just found with character $\chi_{\mathbb{C}^3}$. Note that $\chi_{V \otimes V} = \chi_V^2$, which we can easily compute. Next, $\langle \chi_{V \otimes V}, \chi_{\mathbb{C}^3} \rangle = 1$ and hence $(\mathbb{C}^3, \rho_{\mathbb{C}^3})$ is isomorphic to a summand of $V \otimes V$, with complementary representation of dimension one (this complement is actually unique since it is not isomorphic to the three-dimensional subrepresentation). The character of the resulting one-dimensional subrepresentation is $\chi_V^2 - \chi_{\mathbb{C}^3}$, the desired one-dimensional character (or equivalently, matrix representation).

Alternative proof: Note that $\chi_V \chi_{\mathbb{C}^3} = \chi_{V^*} \chi_{\mathbb{C}^3}$ as the non-real values of χ_V are all on conjugacy classes where $\chi_{\mathbb{C}^3}$ is zero. Thus $V \otimes \mathbb{C}^3 \cong V^* \otimes \mathbb{C}^3$. Also $\mathbb{C}^3$ is isomorphic to its dual as it has a real character (or because $\mathbb{C}^3$ is the unique three-dimensional summand of a representation $V \otimes V^*$ which is isomorphic to its dual). Then the multiplicity spaces $\mathrm{Hom}_G(\mathbb{C}^3, V \otimes V) \cong ((\mathbb{C}^3)^* \otimes V \otimes V)^G$ and $\mathrm{Hom}_G(\mathbb{C}^3, V^* \otimes V) \cong ((\mathbb{C}^3)^* \otimes V \otimes V^*)^G$ are isomorphic, and hence the former is one-dimensional. We conclude by the last line of the previous proof.

Part 3(e). *By tensor product or other means, fill in lines three and four, with one and two dimensional representations, respectively.*

3, A

We simply take $\mathbb{C}_\omega^*$ and $V \otimes \mathbb{C}_\omega$. These are irreducible since they are the tensor products of irreducible representations with one-dimensional representations.

Part 3(f). *We now give you the following additional partial line:*

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
size	*1*	*9*	*8*	*12*	*12*	*24*	*24*	*36*	*36*	*54*
χ_W	*8*	*?*	*?*	2	2	-1	-1	*?*	*0*	*0*

Finish the character table.
(Hint: tensor with one-dimensional representations. To fill in the question marks, employ the regular character and orthogonality.)

5, D

For each question mark, we can put in a variable. Note that we can tensor this representation by $\mathbb{C}_\omega$ and $\mathbb{C}_\omega^*$ and get irreducible ones, and we see they have different characters. Thus we get the following table:

set	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$
size	1	9	8	12	12	24	24	36	36	54
$\chi_{\mathbb{C}}$	1	1	1	1	1	1	1	1	1	1
$\chi_{\mathbb{C}_\omega}$	1	1	1	ω^2	ω	ω^2	ω	ω	ω^2	1
$\chi_{\mathbb{C}_\omega^*}$	1	1	1	ω	ω^2	ω	ω^2	ω^2	ω	1
$\chi_{V\otimes\mathbb{C}_\omega}$	2	-2	2	-1	-1	-1	-1	1	1	0
χ_V	2	-2	2	$-\omega$	$-\omega^2$	$-\omega$	$-\omega^2$	ω^2	ω	0
χ_{V^*}	2	-2	2	$-\omega^2$	$-\omega$	$-\omega^2$	$-\omega$	ω	ω^2	0
$\chi_{\mathbb{C}^3}$	3	3	3	0	0	0	0	0	0	-1
χ_W	8	a	b	2	2	-1	-1	c	0	0
$\chi_{W\otimes\mathbb{C}_\omega}$	8	a	b	$2\omega^2$	2ω	$-\omega^2$	$-\omega$	ωc	0	0
$\chi_{W\otimes\mathbb{C}_\omega^*}$	8	a	b	2ω	$2\omega^2$	$-\omega$	$-\omega^2$	$\omega^2 c$	0	0

We can sum these up times dimensions to get the regular representation. This produces $24a = 0$ and $24b = -24$, so $a = b = 0$. Using orthogonality with the trivial representation, or orthonormality of the row, we get $c = 0$.

Question 4. *Let A be an associative algebra.*

Part 4(a). *Let $A = \mathbb{C}[Q_8]$, for $Q_8 = \{e, z, \pm i, \pm j, \pm k\}$ the quaternionic group of order eight: $i^2 = j^2 = k^2 = z$, and $ij = k, ki = j, jk = i$, with e the identity and z acting to flip signs: $z(\pm i) = \mp i, z(\pm j) = \mp j$, and $z(\pm k) = \mp k$.* ***We use the notation*** *e, z in order to avoid confusion with scalars (as we are using the group algebra).*

Recall the character table of Q_8:

	$\{e\}$	$\{z\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
Size of class:	1	1	2	2	2
$\rho_{+,+}$	1	1	1	1	1
$\rho_{-,+}$	1	1	-1	1	-1
$\rho_{+,-}$	1	1	1	-1	-1
$\rho_{-,-}$	1	1	-1	-1	1
$\chi_{\mathbb{C}^2}$	2	-2	0	0	0

Subpart 4(a)(i). *Find a nonzero element $a \in Z(A)$ such that $\widetilde{\rho}_V(a) = 0$ if (V, ρ_V) is a one-dimensional representation, with $\widetilde{\rho}_V : A \to \mathrm{End}(V)$ the linear extension.* 3, A

In lectures it was shown that the following element is zero on all representations except the two-dimensional irreducible $\mathbb{C}^2$, where it acts by the identity:

$$\frac{2}{8}\sum_{g\in Q_8} \overline{\chi_V(g)}g = \frac{1}{4}(2e - 2z) = \frac{1}{2}(e - z).$$

Any nonzero multiple of this is also acceptable.

Subpart 4(a)(ii). *Show that such an element z as in part (i) is unique up to scaling.* 3, B

We proved in lectures that $Z(A) \cong \bigoplus_i \mathbb{C}I_{V_i}$ for V_i the irreducible representations of Q_8. Now Q_8 has only one irreducible representation which is not of dimension two. Thus the elements in $\bigoplus_i \mathbb{C}I_{V_i}$ which act by zero on all one-dimensional representations are precisely the multiples of I_V for V the two-dimensional irreducible.

Subpart 4(a)(iii). *Using the preceding parts, find the maximum subrepresentation W of the left regular representation $(\mathbb{C}[Q_8], \rho_L)$ in which the operators $\rho_L(g) \in \mathrm{End}(W)$ do not all have a common nonzero eigenvector.*

$\boxed{\text{4, D}}$

Note that such a representation cannot have a one-dimensional subrepresentation, so it must be a direct sum of representations isomorphic to the two-dimensional irreducible one. But the sum of all of these is precisely the image of the element found in part (a), which must then be the maximal representation sought. This image on the regular representation is easy to compute, and we get: $\{a - az + b(i) - b(-i) + c(j) - c(-j) + dk - d(-k) \mid a, b, c, d \in \mathbb{C}\}$.

Part 4(b). *Now let A be arbitrary. Suppose that A has a module (V, ρ_V) of dimension four, but no modules of dimension one or three. Let $B := \rho_V(A)$ be the image of A in* $\mathrm{End}(V)$*. Recall that V is therefore also a B-module, via the inclusion $B \subseteq \mathrm{End}(V)$.*

Subpart 4(b)(i). *Show that* $\dim B \geq 4$.

$\boxed{\text{3, A}}$

This is because there must be at least one simple submodule, call it W, of V. By assumption $\dim W \geq 2$. Now an A-submodule of V is the same thing as a B-submodule. Thus W is also a simple B-module. By a result from lectures, $B \to \mathrm{End}(W)$ is surjective. Taking dimensions produces the result.

Subpart 4(b)(ii). *Prove that, if V decomposes as a direct sum of two isomorphic simple modules, then we have equality in (i).*

$\boxed{\text{3, B}}$

Write $V = W \oplus W$. In this case, the map $A \to \mathrm{End}(V)$ has the form $\rho_V(a)(w, w) = (\rho_W(a)w, \rho_W(a)w)$. In other words, in a basis of W repeated twice, B is block-diagonal with the two blocks identical. Thus $B \cong \mathrm{End}(W)$ and $\dim B = 4$.

Subpart 4(b)(iii). *Prove that, if $\dim B$ is not a power of two, then V is indecomposable as an A-module but not simple.*

$\boxed{\text{4, D}}$

By our assumptions there can be no A-module of dimension one or three: the former would be simple hence is ruled out, whereas the latter would have to have a simple two-dimensional submodule, and the quotient would be one-dimensional, a contradiction. Thus the same holds for B, as every B-module is also an A-module. Since $\dim B = 16$, and $B \to \mathrm{End}(W)$ is surjective for every simple B-module, every simple B-module must have dimension two or four. Thus if B is semisimple, by Artin–Wedderburn, it must have dimension $4, 8$, or 16, i.e., a power of two.

Now, if V is decomposable, it must be a direct sum of two simple modules of dimension two, hence semisimple. Since $B \subseteq \mathrm{End}(V) \cong V^4$, B is also semisimple as an A-module and hence as a B-module. So B is a semisimple algebra. We then conclude by the preceding paragraph.

Question 5.

Part 5(a). *Let $(\mathbb{C}^2, \rho)$ be the reflection and rotation representation of the dihedral group $G = D_n$. Let $H = C_n < D_n$ be the subgroup of rotations.*

Subpart 5(a)(i). *Decompose $\mathrm{Res}^G_H(\mathbb{C}^2, \rho)$ explicitly into irreducible subrepresentations of H.*

$\boxed{\text{2, M}}$

The eigenvectors of the rotation matrices are $(1, -i)^t$ and $(1, i)^t$, and the eigenvalues are $e^{\pm i\theta}$ where θ is the angle of rotation. If we use the generator of H which is the counterclockwise rotation by $2\pi/n$, then the representations are therefore $\rho_\zeta, \rho_{\zeta^{-1}}$ for $\zeta = e^{2\pi i/n}$.

Subpart 5(a)(ii). *Find all $\rho' : H \to \mathrm{GL}_1(\mathbb{C})$ such that $(\mathbb{C}^2, \rho) \cong \mathrm{coInd}_H^G(\mathbb{C}, \rho')$.*

4, M

By Frobenius reciprocity, $\mathrm{Hom}_H(\mathrm{Res}_H^G \mathbb{C}^2, V) \cong \mathrm{Hom}_G(\mathbb{C}^2, \mathrm{coInd}_H^G V)$. Now a G-linear map $\mathbb{C}^2 \to \mathrm{coInd}_H^G V$ must be either zero or an injection, because $\mathbb{C}^2$ is irreducible and so the kernel can only be zero or everything. Now for $\dim V = 1$ we get $\dim \mathrm{coInd}_H^G V = 2$, so $\mathrm{Hom}_G(\mathbb{C}^2, \mathrm{coInd}_H^G V) \neq 0$ if and only if $\mathbb{C}^2 \cong \mathrm{coInd}_H^G V$. Thus this isomorphism holds if and only if $\mathrm{Hom}_H(\mathrm{Res}_H^G \mathbb{C}^2, V) \neq 0$. By the preceding solution, for $V = (\mathbb{C}, \rho')$, this holds if and only if $\rho' = \rho_\zeta$ or $\rho_{\zeta^{-1}}$, with ζ as there.

Subpart 5(a)(iii). *Recall that the irreducible two-dimensional representations of G are $(\mathbb{C}^2, \rho \circ \psi_j)$ where $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $\psi_j : G \to G$ is the map that multiplies all angles by j, i.e., $\psi_j(x) = x^j$ for x a rotation, and $\psi_j(y) = y$ for y the reflection about the x-axis. For each j, find all ρ' such that $\mathrm{coInd}_H^G(\mathbb{C}, \rho') \cong (\mathbb{C}^2, \rho \circ \psi_j)$.*

3, M

We can apply the preceding: ψ_j preserves the rotational subgroup H and we get that $\mathrm{Hom}_H(\mathrm{Res}_H^G(\mathbb{C}^2, \rho \circ \psi_j), (\mathbb{C}, \rho')) \neq 0$ if and only if $\rho' = \rho_{\zeta^{\pm 1}} \circ \psi_j = \rho_{\zeta^{\pm j}}$. For j such that $(\mathbb{C}^2, \rho \circ \psi_j)$ is irreducible we get as before that it is isomorphic to $\mathrm{coInd}_H^G(\mathbb{C}, \rho')$ if and only if $\rho' = \rho_{\zeta^{\pm j}}$.

Part 5(b). *Now let G be a group (not necessarily finite) and H an abelian subgroup of index two. Let $\rho_{-1} : G \to \mathbb{C}^\times$ be the one-dimensional representation with $\rho_{-1}(h) = I = (1)$ for $h \in H$ and $\rho_{-1}(g) = -I = (-1)$ if $g \notin H$. Let (V, ρ_V) be an irreducible representation of G.*

Subpart 5(b)(i). *Let (V, ρ_V) be a finite-dimensional irreducible representation of G of dimension greater than one. Show that $V \cong \mathrm{coInd}_H^G W$ for some one-dimensional representation W of H, and therefore that V is two-dimensional.*

4, M

Let W be an irreducible quotient representation of $\mathrm{Res}_H^G V$. Then $0 \neq \mathrm{Hom}_H(\mathrm{Res}_H^G V, W) \cong \mathrm{Hom}_G(V, \mathrm{coInd}_H^G W)$, the isomorphism by Frobenius reciprocity. Take a nonzero homomorphism $V \to \mathrm{coInd}_H^G W$. Since V is irreducible, this must be injective (a variant of Schur's Lemma). Thus it is isomorphic to a subrepresentation of $\mathrm{coInd}_H^G W$. Now, as every irreducible finite-dimensional representation of H is one-dimensional (as it is abelian), the target has dimension two. Thus this is an isomorphism and $\dim V = 2$.

Subpart 5(b)(ii). *Use Mackey's formula to show that, in the situation of (i), $V \cong V \otimes \rho_{-1}$.*

4, M

Mackey's formula says:

$$\chi_{\mathrm{coInd}_H^G W}(g) = \sum_{Hg' \mid Hg'g \cap Hg' \neq \emptyset} \chi_(g' g (g')^{-1}).$$

In the case that H is normal, such as when it has index two, this becomes $\chi_{\mathrm{coInd}_H^G W} = [H : G]\chi_W$, extending χ_W to a function on G which is zero on elements not in H. Now $\chi_W \rho_{-1} = \chi_W$ because of this property. Now V is irreducible, hence also $V \otimes \rho_{-1}$. By results from lectures, irreducible characters are linearly independent *even if the group is infinite.* Thus we get the desired isomorphism.

Subpart 5(b)(iii). *Now suppose that G is finite, and that (V, ρ_V) is a one-dimensional representation of G. Prove that $V \oplus (V \otimes \rho_{-1})$ is isomorphic to a coinduced representation. (Hint: consider $W := \mathrm{Res}_H^G V$.)*

3, M

We give below a proof without requiring G be finite. If G is finite then one may alternatively use Mackey's formula which gives that $\chi_{\mathrm{coInd}_H^G W} = 2\chi_W$, extended by zero to all of G, which easily gives the statement.

By Frobenius reciprocity, $\mathrm{Hom}_G(V, \mathrm{coInd}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$, and this is $\mathbb{C}I_W$ by Schur's Lemma. Thus there is a nonzero homomorphism $V \to \mathrm{coInd}_H^G W$. We can do the same thing with $V \otimes \rho_{-1}$, since the restriction to H is unchanged. Note that $V \ncong V \otimes \rho_{-1}$, since for any $g \notin H$, we have $\rho_V(g)$ nonzero since this is in $\mathbb{C}^\times \cong \mathrm{GL}_1(\mathbb{C})$, and $\rho_{-1}\rho_V(g) = -\rho_V(g) \neq \rho_V(g)$. Then by results from lectures, the only proper nonzero subrepresentations of $V \oplus (V \otimes \rho_{-1})$ are V and $V \otimes \rho_{-1}$. As these are neither in the kernel of the sum map $V \oplus (V \otimes \rho_{-1}) \to \mathrm{coInd}_H^G W$, the homomorphism is injective, and hence an isomorphism.

Total A marks: 29 of 32 marks
Total B marks: 19 of 20 marks
Total C marks: 14 of 12 marks
Total D marks: 18 of 16 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96026 MATH97035 MATH97143	1	This question was relatively easy, particularly parts (b) and (c)(i),(ii). On the other hand, (a) was not that easy due to needing both to diagonalise the rotation matrix and to remember that if all the irreducible summands in some decomposition are nonisomorphic, then they are the only irreducible subrepresentations. Part 1c(iii) was one of the hardest of the exam: the common example of Z acting via upper-triangular matrices with ones on the diagonal actually has both the nontrivial subrepresentation and quotient representation isomorphic to the trivial one, so this is not actually a counterexample (though it received partial marks).
MATH96026 MATH97035 MATH97143	2	This was a difficult question as is typically the case. Many students succeeded in part (a), and part (c) was doable by those remembering the decomposition of the regular representation and the fact that there are finitely many subrepresentations if and only if some decomposition has all nonisomorphic summands. Part (b) was difficult due to having to deal with tensoring a general representation, or the reflection one, with the reflection one---time may have been a factor as well. Part (d) was difficult due to few students noticing that the multiplicity of the trivial representation in End(V) is one by Schur's Lemma for V irreducible. (It's worth noting that actually the hypotheses of (d) can never be satisfied, although this does not affect the validity of the problem or solution.)
MATH96026 MATH97035 MATH97143	3	This was the easiest for the students, due to them all preparing carefully to handle a question of exactly this type. There were many perfect or near perfect marks. Still I think this is a credit to the students, as this problem cannot objectively be considered easy.

MATH96026 MATH97035 MATH97143	4	This was the hardest question, probably due to the students being weak on this somewhat disjoint unit of the module on algebras. This is in spite of a very similar question to 4(a) appearing in the revision session. I note that most students confused the centre of the group with the centre of the group algebra. For part (b), relatively few students remembered the basic statement that if V is a simple module, then the algebra maps surjectively to End(V), and the generalistion to a direct sum of nonisomorphic simple modules. For 4(b)(ii) few students realised that A -> End(V) + End(V) lands in the diagonal copy of End(V).
MATH96026 MATH97035 MATH97143	5	For the few students taking the mastery portion of the exam, this was difficult, although not as difficult as problem 4. I think they had little time and energy left to tackle it.