Calculus and Applications (MATH40004): Part II

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Outline of the (second part of the) course

Part I: Fourier Transforms

1. Definitions
   - Fourier Transforms as the limit of Fourier Series
   - Exponential, cosine and sine transforms
2. Elementary properties
3. Convolution theorem
4. Energy theorem
5. Dirac delta function

Outline of the (second part of the) course

Part II: Introduction to Ordinary Differential Equations (ODEs)

1. Definitions and notations
2. Solutions for 1st order ODEs and some 2nd order ODEs
3. Linear ODEs: Mostly with constant coefficients
4. Systems of linear ODEs with constant coefficients
5. Qualitative analysis of linear ODEs
   - Phase plane analysis
   - Stability of systems
6. Qualitative analysis of nonlinear ODEs
   - Bifurcation analysis

Outline of the (second part of the) course

Part III: Introduction to Multivariate Calculus

1. General properties of functions of several variables.
   - Definitions, partial derivatives, total differentiation, Taylor expansion, change of variables
2. Applications of differential calculus.
   - Sketching, exact ODEs, PDEs

Fourier Fransforms

Motivation

Last term, we saw that *Fourier series* allows us to represent a given function, defined over a finite range of the independent variable, in terms of sine and cosine waves of different amplitudes and frequencies. *Fourier Transforms* are the natural extension of Fourier series for functions defined over $\mathbb{R}$.

Applications

- Solving ordinary and partial differential equtions
- Various applications in science and engineering, particularly in the context of signal processing.

Fourier Series (Reminder)

Exponential form:

Using the exponential form the Fourier series for a function $f(x)$ defined over the interval $[-L, L]$ can be represented as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, |x| < L,$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx, \quad n = 0, \pm 1, \pm 2, \cdots .$$

By defining angular frequency and frequency difference as the following we can rewrite the Fourier series in the new notation

Fourier Transform

This result is extended for a function $f(x)$ defined on $\mathbb{R}$ by taking the limit of $L \to \infty$ from the Fourier series. Using the angular frequency notation from before and replacing sum with integral using the Reimann sum, we obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(s) e^{-i\omega s}\, ds \right\} e^{i\omega x}\, d\omega.$$

The function $\hat{f}(\omega)$ (also denoted as $\mathcal{F}\{f(x)\}$) is known as Fourier transform of $f(x)$, which is analogous to the Fourier coefficients in a Fourier series. The relation above between $f(x)$ and $\hat{f}(\omega)$ is also known as inverse Fourier transform.

In order to evaluate the integrals above, a necessary condition is that $f(x)$ and its transform decay at $\pm\infty$. Using the Dirac delta function this restriction can be overcome as seen later.

Fourier Transform

Example

Find the Fourier transform of the rectangular wave

$$f(x) = \begin{cases} 1, & \text{if } |x| < d, \\ 0, & \text{if } |x| > d. \end{cases}$$

Fourier Transform

Fourier cosine and sine transforms (1)

We can exploit the symmetry to define transforms over the range $[0, \infty)$. First, if we suppose that $f(x)$ is even about $x = 0$, we have

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\,dx \\ &= 2\int_{0}^{\infty} f(x)\cos\omega x\,dx.\end{aligned}$$

We define Fourier cosine transfrom of $f(x)$ to be

$$\hat{f}_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx.$$

Thus, for an even function $f(x)$ we have

Fourier Transform

Fourier cosine and sine transforms (2)

Using the inversion formula for the regular transform and exploiting the evenness of $\hat{f}_c(\omega)$ we can obtain the inversion formula for the Fourier cosine transform:

In a similar way, by considering $f(x)$ to be odd about $x = 0$, we can define a Fourier sine transform and derive the corresponding inversion formula. We obtain the pair of expressions:

$$\hat{f}_s(\omega) = \int_0^\infty f(x) \sin \omega x \, dx$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, dw.$$

Fourier Transform

Quiz

Find the Fourier cosine transform of the function $f(x) = e^{-a|x|}$, where a is a positive constant.

Properties of Fourier transforms

(i)

The Fourier and inverse Fourier transforms are linear, and so

where a and b are constants and $\mathcal{F}^{-1}$ denotes the inverse transform.

(ii)

If $a > 0$:

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\hat{f}(\frac{\omega}{a}).$$

Proof. Starting on the LHS, and making the substitution $s = ax$:

Properties of Fourier transforms

(iii)

In a similar way we can establish that $\mathcal{F}\{f(-x)\} = \hat{f}(-\omega)$.

(iv)

The transform of a shifted function can be calculated as follows (using $s = x - x_0$):

(v)

A similar result, but this time involving a shift in transform space:

Properties of Fourier transforms

(vi) Symmetry formula

The following result is very useful. Suppose the Fourier transform of $f(x)$ is $\hat{f}(\omega)$; change the variable ω to x; then

$$\mathcal{F}\{\hat{f}(x)\} = 2\pi f(-\omega).$$

Proof. Starting with the inversion formula we have

Properties of Fourier transforms

The following results are particularly useful when applying Fourier transforms to differential equations.

(vii)
$$\mathcal{F}\{d^n f/dx^n\} = (i\omega)^n \hat{f}(\omega).$$

(viii)
$$\mathcal{F}\{xf(x)\} = i\hat{f}'(\omega).$$

Proof.

(ix)
$$\begin{aligned}
&\text{(a) } \mathcal{F}_c\{f'(x)\} = -f(0) + \omega \hat{f}_s(\omega),\\
&\text{(b) } \mathcal{F}_s\{f'(x)\} = -\omega \hat{f}_c(\omega),\\
&\text{(c) } \mathcal{F}_c\{f''(x)\} = -f'(0) - \omega^2 \hat{f}_c(\omega),\\
&\text{(d) } \mathcal{F}_s\{f''(x)\} = \omega f(0) - \omega^2 \hat{f}_s(\omega).
\end{aligned}$$

Properties of Fourier transforms

(x)

If $f(x)$ is a complex-valued function and $[f(x)]^*$ is its complex conjugate, then

$$\mathcal{F}\{[f(x)]^*\} = [\hat{f}(-\omega)]^*.$$

Proof. We have that

$$\hat{f}(-\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx$$

and so by taking complex conjugate from both sides, it follows that

Properties of Fourier transforms

Convolution theorem for Fourier transforms

We define the convolution of two functions $f(x)$, $g(x)$, defined over $(-\infty, \infty)$, as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\, du.$$

An important result is the so-called convolution theorem:

$$\mathcal{F}\{f * g\} = \hat{f}(\omega)\hat{g}(\omega).$$

Proof. We start on the LHS, change the order of integration and then use the substitution $s = x - u$ at fixed u:

$$\begin{aligned}
&\int_{x=-\infty}^{\infty} \left\{ \int_{u=-\infty}^{\infty} f(x-u)g(u)\, du \right\} e^{-i\omega x}\, dx \\
&= \int_{u=-\infty}^{\infty} g(u) \left\{ \int_{x=-\infty}^{\infty} f(x-u)e^{-i\omega x}\, dx \right\} du \\
&= \int_{u=-\infty}^{\infty} g(u) \left\{ \int_{s=-\infty}^{\infty} f(s)e^{-i\omega(s+u)}\, ds \right\} du \\
&= \left(\int_{-\infty}^{\infty} g(u)e^{-i\omega u}\, du \right) \left(\int_{-\infty}^{\infty} f(s)e^{-i\omega s}\, ds \right) = \hat{g}(\omega)\hat{f}(\omega).
\end{aligned}$$

Properties of Fourier transforms

Example

Find the inverse Fourier transform of the function

$$\frac{1}{(4+\omega^2)(9+\omega^2)}$$

setting

$$\hat{f}(\omega) = 1/(4+\omega^2),\ \hat{g}(\omega) = 1/(9+\omega^2),$$

we have (from the last quiz) that

Fourier Transform

Quiz

Find the Fourier transform of the function $f(x) = \cos 2x/(9 + x^2)$.

Properties of Fourier transforms

The energy theorem

This is the analogous result to Parseval's theorem for Fourier series. It states that if $f(x)$ is a real-valued function, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2\,d\omega=\int_{-\infty}^{\infty}[f(x)]^2\,dx.$$

Proof. Properties (iii) and (x) of the Fourier transforms give $\mathcal{F}\{[f(-x)]^*\}=[\hat{f}(\omega)]^*$. Since we are assuming f to be real, this simplifies to $\mathcal{F}\{f(-x)\}=[\hat{f}(\omega)]^*$. If we now use the convolution theorem with $\hat{g}(\omega)=[\hat{f}(\omega)]^*$, we have

$$\mathcal{F}\{f(x)*f(-x)\}=\hat{f}(\omega)[\hat{f}(\omega)]^*=\left|\hat{f}(\omega)\right|^2.$$

Using the defintion of convolution and the inverse transform we have

In particular, setting $x=0$, we obtain the required result:

$$\int_{-\infty}^{\infty}[f(u)]^2du=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2\,d\omega.$$

The Dirac delta-function

Before we define the Dirac delta-function, we need to be aware of the following theorem.

Mean-value theorem for integrals

If $g(x)$ is continuous on $[a, b]$ then

$$\int_a^b g(x)\, dx = (b - a)g(\bar{x})$$

for at least one $\bar{x}$ with $a \leq \bar{x} \leq b$. The proof follows from the regular mean-value theorem for G say, by defining $g = G'$. Geometrically this means that the area under the curve is equivalent to that of a rectangle with length equal to the interval of integration.

The Dirac delta-function

Definition of the Dirac delta-function (impulse function)

Consider the following step-function:

$$f_k(x) = \begin{cases} k/2, & \text{if } |x| < 1/k, \\ 0, & \text{if } |x| > 1/k. \end{cases}$$

Clearly we can see that an important property of this function is that

$$\int_{-\infty}^{\infty} f_k(x)dx = 1.$$

As k increases, $f_k(x)$ gets taller and thinner. We define the delta function to be

$$\delta(x) = \lim_{k\to\infty} f_k(x),$$

although, of course, this limit doesn't exist in the ususal mathematical sense. Effectively $\delta(x)$ is infinite at $x = 0$ and zero at all other values of x. The key property however, is that its integral (area under the curve) is one.

The Dirac delta-function

Sifting property of the delta function

The delta function is most useful in how it interacts with other functions. Consider

$$\int_{-\infty}^{\infty} g(x)\delta(x)\,dx,$$

where $g(x)$ is a continuous function defined over $(-\infty, \infty)$. Using our definition of the delta-function we can rewrite this as

for some $\bar{x}$ in $[-1/k, 1/k]$, using the mean-value theorem for integrals. Clearly, as $k \to \infty$, we must have $\bar{x} \to 0$. The expression above simplifies to

We have therefore established that for any continuous function g:

$$\int_{-\infty}^{\infty} g(x)\delta(x)\,dx = g(0).$$

The Dirac delta-function

The sifting property can easily be generalized to

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)\,dx = g(a).$$

Example

Find the Fourier transform of $\delta(x)$.

From this we can deduce that the inverse Fourier transform of 1 is $\delta(x)$. From this last result, and using the inversion formula, we see that an alternative representation of the delta function is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\omega x}\,d\omega,$$

with the $\pm$ arising from the observation that $\delta(x)$ is an even function of x about $x=0$.

The Dirac delta-function

If we are prepared to work in terms of delta-functions, we can now take the Fourier transforms of functions that do not decay as $x \to \pm\infty$.

Example

Find the Fourier transform of $\cos \omega_0 x$.

Fourier Transform

Quiz

A convolution theorem holds for the inverse functions:

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}\hat{f}(\omega) * \hat{g}(\omega).$$

Prove this result using properties of delta functions (Hint: Start from the RHS and write it in terms of an integral.)

Intro to ODEs - Definitions and Notations

Before defining ODEs, we review some concepts:

Functions of one variable (ordinary)

Ordinary refers to dealing with functions of one independent variable.

Differentiability up to order k

Equation

Intro to ODEs - Definitions and Notations

Definition. Ordinary Differential Equation (ODE)

$$G(x, f(x), \frac{df}{dx}, ..., \frac{d^k f}{dx^k}) = 0$$

Order of the ODE is the order of the highest derivative: k.
Degree of the ODE is the power of highest derivative (when fractional powers have been removed).
The ODE is called *linear* if G is a linear function of $f(x)$ and its derivatives.

Example

$$\frac{d^2 f}{dx^2} = 5\left[1 + \left(\frac{df}{dx}\right)^2\right]^{\frac{1}{3}}$$

Intro to ODEs - Definitions and Notations

- Implicit form of an ODE:

- Explicit form of an ODE:

Solving an ODE is the task of finding $f(x)$ such that

is satisfied over the domain of x (*e.g.* $\mathbb{R}$).

Intro to ODEs - Definitions and Notations

Examples

ODEs appear naturally in many areas of sciences and humanities.

1. Mechanics: Second Newton Law (A one page introduction)

   Kinematics is a branch of mechanics that describes the motion of points (objects) without considering the forces that cause them to move.

   In one dimension $x(t)$ denotes the position of a particle at time t. Then $\frac{dx}{dt} = \dot{x} = v$ is defined as the velocity of the particle and $\frac{d^2x}{dt^2} = \ddot{x} = a$. This can be generalised to higher dimensions.

Intro to ODEs - Definitions and Notations

Examples

Dynamics is the branch of mechanics concerned with the study of *forces* and their effects on motion. Isaac Newton defined the fundamental physical laws which govern dynamics in physics:

- **First law** an object not acted upon by any force either remains at rest or continues to move at a constant velocity
- **Second law** the vector sum of the forces F on an object is equal to the mass m of that object multiplied by its acceleration a: $F = ma$.
- **Third law** when one body exerts a force on a second body, the second body exerts a force equal in magnitude and opposite in direction on the first body.

2. Population dynamics: Malthus (1798)

Intro to ODEs - Definitions and Notations

Examples

3. Population dynamics: Logistic Growth (Verhulst, 1845)

4. Radius of curvature
   Given radius of curvature $R(x, y)$ find equation for the curve $y(x)$.

Intro to ODEs - Definitions and Notations

$f_{PI}(x)$ is called a *Particular Integral* or Particular Solution of an ODE such that

is satisfied over the domain $x \in \mathbb{R}$.

f_{GS} is called a *General Solution* of an ODE of the order k, if

is a general family of solutions that fulfil the ODE. The parameters $\{c_i\}_{i=1}^{k}$ are the constants of integration and are usually fixed by initial or boundary conditions.

Intro to ODEs - Definitions and Notations

Example

Object moving with constant speed v:

$$\frac{dx}{dt} = v$$

If we are also told that $x(t = 0) = x_0$, we have

Intro to ODEs - Definitions and Notations

Quiz: Are the following ODEs linear or nonlinear? What is the degree and order?

$$\frac{d^3y}{dx^3} = x^2y + 1$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = xy$$

$$\frac{dy}{dx} = sin(x)y$$

$$\frac{dy}{dx} = 5xy - \frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = y\frac{dy}{dx}$$

Intro to ODEs - First Order ODEs

- Implicit form:

$$G(t, x, \frac{dx}{dt}) = 0$$

- Explicit form:

$$\frac{dx}{dt} = F(x, t)$$

1. Separable First Order ODEs

$$\frac{dx}{dt} = F_1(x)F_2(t)$$

2. Linear First Order ODEs

$$\frac{dy}{dx} + p(x)y = q(x)$$

Solution by finding an integrating factor (IF). We look for $I(x)$ such that:

Then, we have

Intro to ODEs - First Order ODEs

Integrating factors must fulfil:

$$\frac{d(Iy)}{dx} = I\frac{dy}{dx} + Ipy$$

So, we have the following for the General Solution:

Intro to ODEs - First Order ODEs

The other types of First Order ODEs that can be solved are based on transformations or change of variables.

3. Dimensionally Homogeneous

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Intro to ODEs - First Order ODEs

There are other examples of transformations can turn specific ODEs into separable or linear. Some are classic:

4. Bernoulli ODEs

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

Intro to ODEs - First Order ODEs

Quiz

Are the following first ODEs linear (A), separable (B), Homogenous (C) or Bernoulli (D)?

$$x\frac{dy}{dx} - y = x^2 + 1$$

$$\frac{dy}{dx} = \frac{\sin(x)y^3}{1+x}$$

$$\frac{dy}{dx} - xy = \frac{x^3}{y^2}$$

$$e^x\frac{dy}{dx} - \cos(x)y = 0$$

$$\frac{dy}{dx} = \frac{x+y+2}{2x+y+3}$$

Intro to ODEs - Second Order ODEs

- Implicit form:

$$G(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

- Explicit form:

$$\frac{d^2y}{dx^2} = F(x, y, \frac{dy}{dx})$$

This is common in Mechanics, Newton's second law with independent variable t. Difficult to solve for general F but there are some special cases that can be solved as described in the following.

Intro to ODEs - Second Order ODEs

1. F only depends on x

$$\frac{d^2y}{dx^2} = F(x)$$

Intro to ODEs - Second Order ODEs

2. F only depends on x and $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = F(x, \frac{dy}{dx})$$

Intro to ODEs - Second Order ODEs

3. F only depends on y

$$\frac{d^2y}{dx^2} = F(y) \qquad \text{let } u = \frac{dy}{dx} \quad \Longrightarrow \quad \frac{du}{dx} = F(y)$$

Intro to ODEs - Second Order ODEs

Example: Mechanics Harmonic Oscillator

Hooke's law states if $x(t)$ is displacement relative to an ideal spring relaxed position, the spring force is: $F = -kx$ Using second Newton Law we have: $ma = F \quad \Longrightarrow \quad m\frac{d^2x}{dt^2} = -kx$

Intro to ODEs - Second Order ODEs

$$u = \frac{dx}{dt} = \pm\sqrt{\frac{2E - kx^2}{m}} \quad \Longrightarrow \quad \int \frac{dx}{\pm\sqrt{\frac{2E-kx^2}{m}}} = \int dt$$

Intro to ODEs - Second Order ODEs

4. F only depends on y and $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = F(y, \frac{dy}{dx})$$

let $u = \frac{dy}{dx} \implies \frac{du}{dx} = F(y, u)$. So we have

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy} = \frac{d}{dy}\left(\frac{1}{2}u^2\right).$$

Therefore we have the following first order ODE for $u(y)$ to solve

$$\frac{d}{dy}\left(\frac{1}{2}u^2\right) = F(y, u).$$

Given $u_{GS}(y; c_1)$ being a general solution for the above ODE, we have the following first order ODE for $y(x)$:

$$\frac{dy}{dx} = u_{GS}(y; c_1).$$

Intro to ODEs - Second Order ODEs

Quiz: Find the family of curves with constant radius of curvature R.

$$R(x, y) = \frac{\left(1 + (\frac{dy}{dx})^2\right)^{3/2}}{\frac{d^2y}{dx^2}}.$$

Intro to ODEs - Second Order ODEs

$$\text{let } X = \frac{u}{\sqrt{1+u^2}} \quad \implies \quad X = \frac{x}{R} + c_1 \quad \implies \quad dX = \frac{1}{R}dx$$

Intro to ODEs - Linear ODEs

The general form of linear ODEs of order k

$$\alpha_k(x)\frac{d^k y}{dx^k} + \alpha_{k-1}(x)\frac{d^{k-1} y}{dx^{k-1}} + ... + \alpha_1(x)\frac{dy}{dx} + \alpha_0(x)y = f(x),$$

where $\alpha_k(x)$,.., $\alpha_0(x)$ and $f(x)$ are functions of only the independent variable x. The ODE is called *homogeneous* if $f(x) = 0$ and *inhomogeneous* otherwise.

Examples:

- First order ODE
- Bessel's equation
- Legendre's equation

Intro to ODEs - Linear ODEs

Linear Operators

We define the differential operator: $\mathcal{D}[f] \equiv \frac{d}{dx}[f]$.
Differential operator is a linear operator:

$$\mathcal{D}[\lambda_1 f_1 + \lambda_2 f_2] = \lambda_1 \mathcal{D}[f_1] + \lambda_2 \mathcal{D}[f_2]$$

Linear ODEs are associated with a linear operator:
$\mathcal{L}[y] \equiv \sum_{i=0}^{k} \alpha_i(x) \mathcal{D}^i[y]$.

$$\mathcal{L}[\lambda_1 f_1 + \lambda_2 f_2] = \lambda_1 \mathcal{L}[f_1] + \lambda_2 \mathcal{L}[f_2]$$

Intro to ODEs - Linear ODEs

Linear independence: A set $\{f_i(x)\}_{i=1}^k$ is said to be linearly independent if f_i's satisfy the following condition:

Statement

The solutions of the homogeneous linear ODE $\mathcal{L}[y] = 0$ form a vector space (see MATH40003: Linear Algebra and Groups) of dimension k, where k is the order of the ODE. Therefore, the general solution of a linear homogeneous ODE can be written as

$$y_{GS}^H(x; c_1, ..., c_k) = c_1 y_1 + c_2 y_2 + ... + c_k y_k,$$

where $B = \{y_i(x)\}_{i=1}^k$ is a set of linearly independent solutions forming a basis for the linear homogeneous ODE's solution vector space.

Intro to ODEs - Linear ODEs

Statement

To test the linear independence of a set of functions $\{y_i(x)\}_{i=1}^k$, we calculate the *Wronskian*, which is the determinant of the Wronskian matrix ($\mathbb{W}_{k\times k}$):

$$W[\{y_i(x)\}_{i=1}^k] = \det\mathbb{W} = \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_k(x) \\ \frac{dy_1}{dx}(x) & \frac{dy_2}{dx}(x) & \cdots & \frac{dy_k}{dx}(x) \\ \vdots & \vdots & & \vdots \\ \frac{d^{k-1}y_1}{dx^{k-1}}(x) & \frac{d^{k-1}y_2}{dx^{k-1}}(x) & \cdots & \frac{d^{k-1}y_k}{dx^{k-1}}(x) \end{bmatrix}$$

The set $\{y_i(x)\}_{i=1}^k$ is linearly independent if

Example

Show that $\sin(x)$ and $\cos(x)$ are linearly independent.

Intro to ODEs - Linear ODEs

Quiz

Condsier the functions $y_1(x) = x^2$ and $y_2(x) = |x|x$. Evaluate the Wronskian of these two functions. Are these functions linearly dependent?

Intro to ODEs - Linear ODEs

General solution of the non-homogeneous linear ODE

$$\mathcal{L}[y] = f(x)$$

We split the problem into two simpler steps.

1. We consider the corresponding homogeneous linear ODE $\mathcal{L}[y] = 0$. We obtain the general solution, which is also known as *complementary function* (y_{CF}):

$$y_{CF} = y_{GS}^{H}(x; c_1, \cdots, c_k) = \sum_{i=1}^{k} c_i y_i(x),$$

   where, $\{y_i\}_{i=1}^{k}$ are the basis of solution vector space (a set of linearly independent solutions of the homogeneous linear ODE).

Intro to ODEs - Linear ODEs

General solution of the non-homogeneous linear ODE

2. We obtain any/one solution of the full non-homogeneous ODE, which is also known as *particular integral* (y_{PI}):

$$\mathcal{L}[y_{PI}] = f(x)$$

Then for the solution to the full problem by combining the results above and due to linearity, we have:

.

Intro to ODEs - Linear ODEs

Linear ODEs with constant coefficients

The general linear ODE is not always analytically solvable. Next year, you will see approximative and numerical methods to solve this kind of ODEs. In the following, we will focus on the case of linear ODEs with constant coefficients (α_is not depending on independent variable x):

$$\mathcal{L}[y] = \sum_{i=0}^{k} \alpha_i \mathcal{D}^i[y] = f(x)$$

First order linear ODEs with constant coefficients

As seen before the general case can be solved using the integrating factor.

Intro to ODEs - Linear ODEs

$$\frac{dy}{dx} + \frac{\alpha_0}{\alpha_1} y = \frac{f(x)}{\alpha_1}$$

Example 1: $f(x) = x$

Intro to ODEs - Linear ODEs

Alternative method:

1. Solve the corresponding homogeneous ODE:

$$\mathcal{L}[y_{CF}] = \alpha_1 \frac{dy}{dx} + \alpha_0 y = 0$$

.

2. Find a particular integral for the full ODE: $\mathcal{L}[y_{PI}] = f(x) = x$. This is done by *Ansatz*, which is an educated guess using the *method of undetermined coefficients*:

Intro to ODEs - Linear ODEs

$$\mathcal{L}[y_{PI}] = \alpha_1(2Ax + B) + \alpha_0(Ax^2 + Bx + C) = x$$

Intro to ODEs - Linear ODEs

Example 2: $f(x) = e^{bx}; b \neq -\frac{\alpha_0}{\alpha_1}$

$$\mathcal{L}[y] = \alpha_1 \frac{dy}{dx} + \alpha_0 y = e^{bx}$$

Intro to ODEs - Linear ODEs

What if $b = -\frac{\alpha_0}{\alpha_1}$?

Naive ansatz $y_{PI} = Ae^{bx}$ does not work, since $\mathcal{L}[y_{PI}] = 0$. A not-so-naive ansatz is:

$$y_{PI} = A(x)e^{bx}$$

Here we are looking for an unknown function $A(x)$, so we will obtain an ODE. This is called the *method of variation of parameters*, developed by Euler and Lagrange.

Intro to ODEs - Linear ODEs

$$\alpha_1 \frac{dA}{dx} = 1$$

Quiz

Show that if we have obtained two different particular integrals for the non-homogeneous ODE $\mathcal{L}[y] = f(x)$, the resulting general solutions are equivalent (i.e. specify the same family of solutions).

Intro to ODEs - Linear ODEs

Second order linear ODEs with constant coefficients

$$\mathcal{L}[y] = \alpha_2 \frac{d^2y}{dx^2} + \alpha_1 \frac{dy}{dx} + \alpha_0 y = f(x)$$

$B = \{y_1(x), y_2(x)\}$ is a basis for the solution vector space of the homogeneous ODE: $\mathcal{L}[y^H] = 0$.

Intro to ODEs - Linear ODEs

Solving the homogeneous second order linear ODE

$$\mathcal{L}[y] = \alpha_2 \frac{d^2 y}{dx^2} + \alpha_1 \frac{dy}{dx} + \alpha_0 y = 0$$

Ansatz: $y^H = e^{\lambda x}$

Intro to ODEs - Linear ODEs

Testing the linear independence of the solutions

We should evaluate the Wronskian:

$$W(x) = \det \begin{bmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{bmatrix} = e^{(\lambda_1+\lambda_2)x}(\lambda_2 - \lambda_1)$$

For the case of $\lambda_1 = \lambda_2 = -\frac{\alpha_1}{2\alpha_2}$, we have $y_1 = e^{\lambda_1 x}$, what about the second solution y_2?

This is similar to the method of variation of parameters. In the context of 2nd order linear ODEs, when we have one of the solutions and looking for the second solution, this method is called *reduction of order*.

Intro to ODEs - Linear ODEs

$$\alpha_0 \left[A y_1 \right] + \alpha_1 \left[\frac{dA}{dx} y_1 + A \frac{dy_1}{dx} \right] + \alpha_2 \left[\frac{d^2 A}{dx^2} y_1 + 2 \frac{dA}{dx} \frac{dy_1}{dx} + A \frac{d^2 y_1}{dx^2} \right] = 0$$

Intro to ODEs - Linear ODEs

Testing the linear independence of the solutions

We should evaluate the Wronskian:

$$W(x) = \det \begin{bmatrix} e^{\lambda_1 x} & x e^{\lambda_1 x} \\ \lambda_1 e^{\lambda_1 x} & e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x} \end{bmatrix}$$

Intro to ODEs - Linear ODEs

Possible behaviours of the 2nd order linear homogeneous ODE

If $\lambda_1 \neq \lambda_2$ then $y_{CF} = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

$$\lambda_{1,2} = -\frac{\alpha_1}{2\alpha_2} \pm \sqrt{\frac{\alpha_1^2 - 4\alpha_0\alpha_2}{4\alpha_2^2}}$$

1. $\alpha_1^2 - 4\alpha_0\alpha_2 > 0 \quad \Longrightarrow \quad \lambda_{1,2} \in \mathbb{R}$

$\lambda_{1,2}$ can be both positive, both negative or one positive/one negative.

Intro to ODEs - Linear ODEs

2. $\alpha_1^2 - 4\alpha_0\alpha_2 < 0 \quad \Longrightarrow \quad \lambda_{1,2} \in \mathbb{C}$

$$\left|\frac{\alpha_1^2 - 4\alpha_0\alpha_2}{4\alpha_2^2}\right| = \omega^2 \quad \Longrightarrow \quad \lambda_{1,2} = -\frac{\alpha_1}{2\alpha_2} \pm i\omega$$

Intro to ODEs - Linear ODEs

i. If $\frac{\alpha_1}{2\alpha_2} < 0$

ii. If $\frac{\alpha_1}{2\alpha_2} > 0$

iii. If $\frac{\alpha_1}{2\alpha_2} = 0 \quad \Longrightarrow \quad y_{CF} = A\cos(\omega x - \phi)$

Intro to ODEs - Linear ODEs

Quiz

Solve the following ODE:

$$\mathcal{L}[y] = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x + 3$$

Intro to ODEs - Linear ODEs

- Second step: Finding a particular integral $\mathcal{L}[y_{PI}] = f(x)$.
Ansatz: $y_{PI} = bx + c$; using the method of undetermined coefficients to find b and c.

Intro to ODEs - Linear ODEs

Example 1

$$\mathcal{L}[y] = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$$

- First step: $y_{CF} = c_1 e^{2x} + c_2 e^{x}$.
- Second step: Ansatz $y_{PI} = Ae^{2x}$?

We turn to the method of variation of parameter.

Intro to ODEs - Linear ODEs

$$\frac{d^2A}{dx^2} + \frac{dA}{dx} = 1$$

Let $u = \frac{dA}{dx} \quad \Longrightarrow \quad \frac{du}{dx} + u = 1$

Intro to ODEs - Linear ODEs

Example 2

$$\mathcal{L}[y] = \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

- First step: Solving the Homogeneous problem $\mathcal{L}[y^H] = 0$

Intro to ODEs - Linear ODEs

- Second step: Finding a particular integral $\mathcal{L}[y_{PI}] = f(x)$.

1st naive Ansatz: $y_{PI} = Ae^{-2x}$

2nd naive Ansatz: $y_{PI} = Axe^{-2x}$

Good Ansatz: $y_{PI} = A(x)e^{-2x}$

Intro to ODEs - Linear ODEs

Quiz

$$\mathcal{L}[y] = \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cosh(2x)$$

Intro to ODEs - Linear ODEs

- Second step: Finding a particular integral $\mathcal{L}[y_{PI}] = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$.

Intro to ODEs - Linear ODEs

kth order Linear ODEs with constant coefficients

$$\mathcal{L}[y] = \sum_{i=0}^{k} \alpha_i \mathcal{D}^i[y] = f(x); \quad \alpha_i \in \mathbb{R}$$

$$y_{GS}(x; c_1, \cdots, c_k) = y_{CF} + y_{PI} = y^H_{GS}(x; c_1, \cdots, c_k) + y_{PI}(x)$$

- First step: Solving the Homogeneous problem $\mathcal{L}[y^H] = 0$

Intro to ODEs - Linear ODEs

- Case 1: k roots of the characteristic polynomial are distinct:

$$\mathbb{W}(x) = \begin{bmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_k x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_k e^{\lambda_k x} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} e^{\lambda_1 x} & \lambda_2^{k-1} e^{\lambda_2 x} & \cdots & \lambda_k^{k-1} e^{\lambda_k x} \end{bmatrix}$$

$$W(x) = \det \mathbb{W}(x) = e^{\sum_{i=1}^{k} \lambda_i x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix} =$$

$$e^{\sum_{i=1}^{k} \lambda_i x} \prod_{1 \le i < j \le k} (\lambda_i - \lambda_j) \neq 0; \quad \text{(Vandermonde determinant)}$$

Intro to ODEs - Linear ODEs

- Case 2: Not all of the k roots are distinct:

- Second step: Finding a particular integral $\mathcal{L}[y_{PI}] = e^{bx}$

Intro to ODEs - Linear ODEs

Euler-Cauchy equation

An example of linear ODE with non-constant coefficients:

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \beta_{k-1} x^{k-1} \frac{d^{k-1} y}{dx^{k-1}} + \cdots + \beta_1 x \frac{dy}{dx} + \beta_0 y = f(x)$$

Using the change of variable $x = e^z$, the Euler-Cauchy equation can be transformed into a linear ODE with constant coefficients.

Example

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = x^3$$

Intro to ODEs - Linear ODEs

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} =$$

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = e^{3z}$$

Intro to ODEs - Linear ODEs

Quiz

Solve the following 3rd order linear ODE:

$$\mathcal{L}[y] = \frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = \sin x.$$

Intro to ODEs - using FT

Using Fourier Transforms to solve linear ODEs

As Fourier transform is a linear operation and given the properties we had for Fourier transforms of derivatives of a function:

One can use Fourier transforms to solve linear ODEs or find particular integrals. This is particularly relevant for solving partial differential equations as discussed in the second year. Here we discuss an example.

Example:

Find a solution for the following ODE, known as the *Airy* equation or the *Stokes* equation, which arises in different areas of physics. Assume $\lim_{x\to\pm\infty} y(x) = 0$.

$$\frac{d^2y}{dx^2} - xy = 0.$$

Intro to ODEs - using FT

This is a linear 2nd order ODE with non-constant coefficients and so far we have not seen a method of solving it. Let's take Fourier transform from this ODE and see if we can solve for the Fourier transform.

Intro to ODEs - Systems of ODEs

Definition. Systems of Ordinary Differential Equations

$$G_1(x, y_1, y_2, \cdots, y_n, \frac{dy_1}{dx}, \cdots, \frac{dy_n}{dx}, \cdots, \frac{d^{k_1}y_1}{dx^{k_1}}, \cdots, \frac{d^{k_n}y_n}{dx^{k_n}}) = 0$$

$$\vdots$$

$$G_n(x, y_1, y_2, \cdots, y_n, \frac{dy_1}{dx}, \cdots, \frac{dy_n}{dx}, \cdots, \frac{d^{k_1}y_1}{dx^{k_1}}, \cdots, \frac{d^{k_n}y_n}{dx^{k_n}}) = 0$$

The system is *ordinary* as we still have one independnet variable x, but now in contrast to single ODEs, we have n functions of independent variable $y_1(x), y_2(x), \cdots, y_n(x)$ to be found. This is the implicit form but the systems of ODEs can be written in explicit from as well.

Intro to ODEs - Definitions and Notations

Examples: System of ODEs

1. Predator-prey system *e.g.* rabbits and foxes (Lotka-Volterra model, 1925/26).

2. Chemistry and biochemistry

Intro to ODEs - Systems of ODEs

Examples: System of ODEs

3. Coupled spring-mass systems

Intro to ODEs - Systems of ODEs

Examples: System of ODEs

4. SIR model of desease propagation in a pandemic

Intro to ODEs - Systems of ODEs

Systems of ODEs of general order can be rewritten in terms of systems of first order ODEs, so the following system written in explicit form is more general than it seems.

$$\frac{dy_1}{dx} = F_1(x, y_1, y_2, \cdots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = F_n(x, y_1, y_2, \cdots, y_n)$$

Intro to ODEs - Systems of ODEs

Turning a higher order ODE into systems of 1st order ODEs

Example: Damped Harmonic spring

$$m\frac{d^2x}{dt^2} + \eta\frac{dx}{dt} + kx = F(t)$$

Intro to ODEs - Systems of ODEs

Quiz

How many first order ODEs are required to represent the following higher order system of two ODEs:

$$G_1(t, x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}) = 0,$$

$$G_2(t, x, y, \frac{dx}{dt}, \frac{d^2y}{dt^2}, \frac{d^3y}{dt^3}) = 0.$$

Intro to ODEs - Systems of ODEs

The general vector notation for systems of 1st order ODEs

$$\frac{d\vec{y}_{n\times 1}}{dt} = \vec{F}_{n\times 1}(t, \vec{y}_{n\times 1})$$

Here n is the number of equations, t is the independent variable and $\vec{y}$ is the function we are looking for. Next, we discuss an important subclass of these systems

Systems of linear 1st order ODEs with constant coefficients

Intro to ODEs - Systems of ODEs

We can write the system of linear ODEs in matrix form:

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

Since both martix A and derivative $\frac{d}{dt}$ are linear operators, the operator associated to systems of linear ODEs $\mathcal{L}$ is also a linear operator. By linearity we have:

Therefore, the solutions of the homogenous systems of linear ODEs $\mathcal{L}[\vec{y}_H] = 0$ forms a vector space of dimension n. So a set of linearly independent solutions $B = \{\vec{y}_i\}_{i=1}^{n}$ form a basis for this space. Therefore, similar to linear ODEs, the general solution can be written as

Intro to ODEs - Systems of ODEs

The general solution of non-homogenous systems of 1st order linear ODEs

Similar to the case of linear ODEs, here also we find the general solution in two steps

1. Obtain complimentary function $\vec{y}_{CF}$ by solving the corresponding homogenous systems of ODEs.
2. Find a particular integral $\vec{y}_{PI}$ that satisfies the full non-homogenous systems of ODEs.

Then, for the general solution $\vec{y}_{GS}$ we have:

Intro to ODEs - Systems of ODEs

Solving the homogenous problem

$$\mathcal{L}[\vec{y}_H] = 0 \quad \Longrightarrow \quad \frac{d\vec{y}_H}{dt} = A\vec{y}_H$$

First, we consider the case where matrix A is diagonalizable.

Intro to ODEs - Systems of ODEs

We use the eigenvectors matrix V to obtain the solution of the homogenous system.

$$\frac{d\vec{y}}{dt} = A\vec{y} \quad \Longrightarrow \quad V^{-1}\frac{d\vec{y}}{dt} = V^{-1}AVV^{-1}\vec{y}$$

Intro to ODEs - Systems of ODEs

$$\vec{Z} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \implies \vec{y}_H = V\vec{Z} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

Finding particular integral

$$\mathcal{L}[\vec{y}_{PI}] = \vec{g}(t)$$

We will use Ansatz and use the methods of undetermined coefficients and variation of parameters as done for the linear ODEs.

Intro to ODEs - Systems of ODEs

Example 1: Solve the system of ODEs for $\{x(t), y(t)\}$

$$\frac{dx}{dt} = -4x - 3y - 5$$
$$\frac{dy}{dt} = 2x + 3y - 2$$

Intro to ODEs - Systems of ODEs

We obtain the eigenvalues λ_1 and λ_2 and eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

$$\lambda^2 + \lambda - 6 = 0 \quad \Longrightarrow \quad \lambda_1 = 2,\ \lambda_2 = -3$$

Intro to ODEs - Systems of ODEs

$$\vec{y}_{CF} = \vec{y}_H^{GS}(t; c_1, c_2) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

2nd step: Find any particular integral $\vec{y}_{PI}(t)$ (Quiz)

Intro to ODEs - Systems of ODEs

$$\vec{y}_{PI} = \begin{bmatrix} a \\ b \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} 3 & 3 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -7/2 \\ 3 \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

When A has repeated eigenvalues

Case 1: A is still diagonalizable (it has n linearly independent eigenvectors). Then we can still use the method described. For example for $n = 2$ we have:

Intro to ODEs - Systems of ODEs

When A has repeated eigenvalues

Case 2: A is not diagonalizable (it has less than n linearly indepenent eigenvectors). Then we will use the *Jordan normal form*. We first discuss an example and then see the general case.

Example:

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

Eigenvectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad \Longrightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix} = 2 \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

We look for similarity transformation to a Jordan normal form. We look for a matrix of the form:

$$W = \begin{bmatrix} 1 & \alpha \\ -1 & \beta \end{bmatrix}$$

So that:

$$W^{-1}AW = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

$$\begin{aligned} \alpha - \beta &= 1 + 2\alpha \\ \alpha + 3\beta &= -1 + 2\beta \end{aligned} \implies \alpha + \beta = -1 \implies \vec{w}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

The Jordan normal form allows us to solve the non-diagonalizable systems of ODEs:

$$\frac{d\vec{y}}{dt} = A\vec{y} \quad \implies \quad W^{-1}\frac{d\vec{y}}{dt} = \left[W^{-1}AW\right] W^{-1}\vec{y}$$

Intro to ODEs - Systems of ODEs

$$\begin{aligned}\frac{dz_1}{dt} &= 2z_1 + z_2 \\ \frac{dz_2}{dt} &= 2z_2\end{aligned}$$

Intro to ODEs - Systems of ODEs

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

Intro to ODEs - Systems of ODEs

Now consider the case of a non-diagonalizable $A_{n \times n}$ with one repeated eigenvalue λ. Assume λ is associated with only a single eigenvector. We can use the Jordan normal form to obtain a solution to the associated systmes of linear ODEs.

Intro to ODEs - Systems of ODEs

$$
\begin{aligned}
\frac{dz_n}{dt} &= \lambda z_n \\
\frac{dz_{n-1}}{dt} &= \lambda z_{n-1} + z_n \\
\frac{dz_{n-2}}{dt} &= \lambda z_{n-2} + z_{n-1} \\
&\vdots \\
\frac{dz_1}{dt} &= \lambda z_1 + z_2
\end{aligned}
$$

Intro to ODEs - Systems of ODEs

Quiz: Find the general solution for this system of ODEs

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Intro to ODEs - Qualitative analysis of linear ODEs

Qualitative analysis of solutions of (systems of) linear ODEs

So far we have focused to obtain analytical solution to ODEs, but this is not always possible. Even, when it is possible, it is not always very insightful. In this section, we will focus on qualitative analysis of ODEs and as before, we mostly focus on linear ODEs. We'll discuss assymptotics behavior, fixed points (and their stability) and phase plane analysis.

Assymptotic behaviour

Assymptotic behaviour of $y(t)$ as $t \to \infty$.

Example Population growth

Intro to ODEs - Qualitative analysis of linear ODEs

Fixed points of systems of ODEs

$\vec{y}^*$ is a *fixed point* or an *equilibrium point* of a system of first order ODEs, if once $\vec{y}(t_0) = \vec{y}^*$ at some time t_0 then for all future times $t > t_0$, state vector $\vec{y}$ remains equal to $\vec{y}^*$. Thus, at fixed point we have:

$$\left[\frac{d\vec{y}}{dt}\right]_{\vec{y}=\vec{y}^*} = 0$$

Example 1 Logistic growth

Example 2 Systems of Linear homogeneous ODEs

Intro to ODEs - Qualitative analysis of linear ODEs

Informally, a fixed point is stable if whenever the initial state is near that point, the state remains near it, perhaps even tending toward the equilibrium point as time increases. Formally, we have two types of stability as described below.

Lyapunov and Asymptotic stability

- A fixed point $\vec{y}^*$ is said to be *Lyapunov stable*, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, then for $\forall t \geq 0$, we have $\|\vec{y}(t) - \vec{y}^*\| < \epsilon$.
  It means no blow up but also not approaching to the fixed point.

- A fixed point $\vec{y}^*$ is said to be *Asymptotically stable*, if it is Lyapunov stable and there exists a $\delta > 0$ such that, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, then we have $\lim_{t \to \infty} \|\vec{y}(t) - \vec{y}^*\| = 0$.

Intro to ODEs - Qualitative analysis of linear ODEs

Phase plane analysis

The general solution of systems of ODEs is given by the family of parameteric curves, specified by the initial condition:

$$\vec{y}(t; c_1, \cdots, c_n) \in \mathbb{R}^n$$

The phase plane for the 2 dimensional systmes:

Intro to ODEs - Qualitative analysis of linear ODEs

Interpretation of the phase plane in terms of dynamics

The general solution $\vec{y}(t)$ corresponds to a trajectory of a point moving on the phase plane with velocity $\frac{d\vec{y}}{dt}(t)$.

For a system of first order ODEs of the form: $\frac{d\vec{y}}{dt} = F(\vec{y})$
Where, there is no explicit dependence on the independent variable (time) on the right hand side, the velocity is a vector defined at every point of phase plane and is tangent to the trajectory. This is called the *vector field*.

Intro to ODEs - Qualitative analysis of linear ODEs

Uniqueness of solutions of ODEs

Solutions of ODEs are uniquely defined by initial conditions except at special points in the phase plane (no proof now, you will see rigorous proof next year). Trajectories in the phase plane cannot cross (except at some special points) as this would be equivalent of non-uniquenss of solutions. The special points are fixed points or singular points where trajectories start or end.

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz

Illustrate some trajectories in the phase plane near an unstable, aymptotically stable or Lyapanov stable fixed point.

Intro to ODEs - Qualitative analysis of linear ODEs

Phase plane analysis for the linear systems of first order ODEs

For linear systems, the vector field has some very nice properties.

$$\frac{d\vec{y}}{dt} = A\vec{y}$$

We have eigenvectors defining very special directions in the phase plane $A\vec{v}_1 = \lambda_1\vec{v}_1$. The line defined by $\vec{v}_1$ in the phase plane is an *invariant*, meaning that if we start on $\vec{v}_1$, we will remain on it.

Intro to ODEs - Qualitative analysis of linear ODEs

To check this explicitly, we go back to the general solution of systems of ODE (2 dimensional case):

$$\vec{y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Intro to ODEs - Qualitative analysis of linear ODEs

First example of qualitative and phase plane analysis of linear systems of ODE

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix}$$

Intro to ODEs - Qualitative analysis of linear ODEs

- Compute the vector field:

- Asymptotically, solutions blow up parallel to $\vec{v}_1$.

- Asymtotically, if we start on $\vec{v}_2$, we approach $\vec{y}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Phase portrait

Aim of phase plane analysis is to obtain the *phase portrait* of the system, which is a summary of all distinct solutions, with qualitatively different trajectories in the phase plane.

Intro to ODEs - Qualitative analysis of linear ODEs

Obtaining the family of solutions explicitly

$$\begin{aligned}\frac{dx}{dt} &= -4x - 3y \\ \frac{dy}{dt} &= 2x + 3y\end{aligned} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2x + 3y}{-4x - 3y}$$

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz: Alternatively, obtain the trajectories in the phase portrait by using the general soltuion

$$\vec{y}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Intro to ODEs - Qualitative analysis of linear ODEs

$$\frac{(v_{1y}x - y)^{\lambda_1}}{(v_{2y}x - y)^{\lambda_2}} = \frac{[c_2(v_{1y} - v_{2y})]^{\lambda_1}}{[c_1(v_{2y} - v_{1y})]^{\lambda_2}}$$

Intro to ODEs - Qualitative analysis of linear ODEs

Qualitative analysis of the general system of linear ODEs in 2 dimension

Consider the general 2 dimensional system ($n = 2$):

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Intro to ODEs - Qualitative analysis of linear ODEs

Catalogue of qualitative behaviours in terms of τ and Δ

$$\tau = \mathrm{trace}(A); \quad \Delta = \mathrm{Det}(A); \quad \lambda_1, \lambda_2 = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Try this online applet:
http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/

Intro to ODEs - Qualitative analysis of linear ODEs

- 1: Saddle-point or Hyperbolic profile. $\Delta < 0$ (Lower half of (τ, Δ) plane)

$$\Delta < 0 \quad \Longrightarrow \quad \tau^2 - 4\Delta > \tau^2 > 0$$

Intro to ODEs - Qualitative analysis of linear ODEs

- 2.1.1: Repelling or unstable node ($0 < \Delta < \frac{\tau^2}{4}$; $\tau > 0$)

Starting on $\vec{v}_2$ blow-up along the direction of $\vec{v}_2$. Otherwise, blow up in the direction of $\vec{v}_1$.

Intro to ODEs - Qualitative analysis of linear ODEs

- 2.1.2: Attracting or stable node ($0 < \Delta < \frac{\tau^2}{4}$; $\tau < 0$)

Starting on $\vec{v}_2$ decays to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ along the direction of $\vec{v}_2$.

Otherwise, decays along the direction of $\vec{v}_1$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Intro to ODEs - Qualitative analysis of linear ODEs

- 2.2.1: Centre or elliptic profile ($\Delta > \frac{\tau^2}{4}$; $\tau = 0$)

Periodic behaviour corresponds to closed curves in the phase plane. For linear systems the closed curves are ellipses.

Intro to ODEs - Qualitative analysis of linear ODEs

To know the direction of motion we evaluate the vector field:

Intro to ODEs - Qualitative analysis of linear ODEs

- 2.2.2: Repelling or unstable spiral ($\Delta > \frac{\tau^2}{4}$; $\tau > 0$)

$$\vec{y} = e^{\frac{\tau}{2}t}\left[c_1 e^{i\omega t}\vec{v}_1 + c_2 e^{-i\omega t}\vec{v}_2\right]$$

Intro to ODEs - Qualitative analysis of linear ODEs

- 2.2.3: Attracting or stable spiral ($\Delta > \frac{\tau^2}{4}$; $\tau < 0$)

$$\vec{y} = e^{\frac{\tau}{2}t}\left[c_1 e^{i\omega t}\vec{v}_1 + c_2 e^{-i\omega t}\vec{v}_2\right]$$

Intro to ODEs - Qualitative analysis of linear ODEs

- 3.1: line of repelling or unstable fixed points

$$\Delta = 0; \quad \tau > 0; \quad \lambda_1 = \tau; \quad \lambda_2 = 0$$

$$\vec{y} = c_1 e^{\tau t} \vec{v}_1 + c_2 \vec{v}_2$$

Intro to ODEs - Qualitative analysis of linear ODEs

- 3.2: line of attracting or stable fixed points

$$\Delta = 0; \quad \tau < 0; \quad \lambda_1 = \tau; \quad \lambda_2 = 0$$

$$\vec{y} = c_1 e^{\tau t} \vec{v}_1 + c_2 \vec{v}_2$$

Intro to ODEs - Qualitative analysis of linear ODEs

- 4.1: Repelling and attracting star node

$$\tau^2 - 4\Delta = 0; \quad \lambda = \lambda_1 = \lambda_2 = \frac{\tau}{2}$$

A is diagonalizable.

The trajectory is always defined by $\vec{y}(0)$.

Intro to ODEs - Qualitative analysis of linear ODEs

- 4.2.1: Unstable improper or degenerate node
$$\tau^2 - 4\Delta = 0; \quad \lambda = \lambda_1 = \lambda_2 = \frac{\tau}{2}$$
A is non-diagonalizable and $\tau > 0$.

We estimate vector field at specific points to helps draw the phase portrait:

Intro to ODEs - Qualitative analysis of linear ODEs

- 4.2.2: stable improper or degenerate node
$$\tau^2 - 4\Delta = 0; \quad \lambda = \lambda_1 = \lambda_2 = \frac{\tau}{2}$$
A is non-diagonalizable and $\tau < 0$. We have a similar phase portrait to the previous case.

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz: Draw the phase portrait for the following system of linear ODEs
$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{bmatrix} 1 & \frac{1}{2} \\ -2 & -1 \end{bmatrix}.$$

The catalogue of phase portraits for the 2 dimensional linear systems of ODEs

2D nonlinear systems of ODE

Phase plane analysis still very useful tool. One obtains the special points (fixed points) of the system and could linearise the equations in the vicinity of these points to get insight from the linear system. Vector field and asymptotic behaviour is useful to identify the trajectories in the phase plane.

Example 1: Synthetic Biology: Genetic Toggle Switch

Example 2: Lotka-Voltera Model

Intro to ODEs - Qualitative analysis of linear ODEs

Extension of phase plane analysis to higher dimensional systems

- $\vec{y}(t)$ can be considered a trajectory in $\mathbb{R}^n$, where n is the dimensionality of the system.
- From each initial condition there is a unique trajectory and trajectories do not cross except at some special points.
- We can consider asymptotic behavior to draw the trajectories.
- We can compute vector field:
- Fixed points ($\vec{y}^*$) are obtained by:

Intro to ODEs - Qualitative analysis of linear ODEs

The approach directly generalises for the linear systems

Solution is given by the eigenvectors and eigenvalues of matrix A:

Stability of linear n dimensional systems

For the 2 dimensional case we had stability where $\tau \leq 0$ and $\Delta \geq 0$. In terms of eigenvalue characterization this means:

Similarly, for the general n dimensional case we have:

Intro to ODEs - Qualitative analysis of linear ODEs

Lorenz system (1933, Edward Lorenz)

This is a system of 3 nonlinear equations that is a model of atmospheric convection. This rather simple model for certain values of parameters ($\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$), exhibits a complex non-periodic dynamics that is an example of chaos. This dynamical behavior is signfied by the divergence of trajectories in the phase plane starting from near identical initial conditions.

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z.$$

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz: Summerise the stability of 2 dimensional systems of linear ODEs in the (τ, Δ) plane

Intro to ODEs - Bifurcations

Bifurcations in a dynamical system (System of ODEs)

Bifurcations describe the qualitative change in behavior under a variation or change of some parameters of the system. Parameters are constants that are tunable.

Linear Systems

Example: Taking k to be the tuning parameter in the damped Harmonic Oscilator system:

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$$

Intro to ODEs - Bifurcations

How does the qualitative behavior of $\vec{y}(t;k)$ change when k is varied?

Intro to ODEs - Bifurcations

Bifurcations in Linear systems

In linear systmes the bifurcations are related to changes in the stability of the system.

Bifurcations in non-Linear systems

In non-linear systmes there is a whole zoo of bifurcations and we will not cover these here but we will consider 1 dimensional systems.

Intro to ODEs - Bifurcations

Qualitative behavior of non-linear 1 D systems

$$\frac{dy}{dt} = f(y); \quad y \in \mathbb{R}^1$$

Phase plane for 1D systems:

- $y(t)$ are trajectories on the real line.
- vector field describing how we move is the velocity and is scalar in 1D case. : $f(y) \in \mathbb{R}$
- special points:
  - fixed points:
  - singularities:

Intro to ODEs - Bifurcations

First (trivial) example of the phase plane analysis in 1D

$$\frac{dy}{dt} = ky$$

Intro to ODEs - Bifurcations

Bifurcation diagram

A bifurcation diagram summarises all possible behaviours of the system as a parameter is varied. It represents all fixed points of the system and their stability as a function of the varing parameter.

Intro to ODEs - Bifurcations

There are only 3 kinds of nonlinear 1D systems in terms of their bifurcation

1. Saddle-node bifurcation

This is a basic mechanism for creation and destroying fixed points. The prototypical example of saddle-node bifurcation is given by:

$$\frac{dy}{dt} = r + y^2$$

Intro to ODEs - Bifurcations

For $r < 0$ we have 2 fixed points (one stable, one unstable), at $r = 0$ we have one half-stable fixed point and for $r > 0$ we have no fixed point. There is a Saddle-node bifurcation at $r = 0$. These changes in the number and stability of the fixed points can be summerised using the bifurcation diagram, which is a plot of fixed points vs parameter r.

Intro to ODEs - Bifurcations

2. Transcritical bifurcation

In certain sytems a fixed point must exist for all values of a parameter. The prototypical example of this form of bifurcation is given by:

$$\frac{dy}{dt} = ry - y^2$$

Intro to ODEs - Bifurcations

For $r < 0$ we have 2 fixed points (one stable, one unstable), at $r = 0$ we have one half-stable fixed point and for $r > 0$ go back to two fixed points. At the bifurcation point $r = 0$ an exchange of stabilities takes place between the two fixed points.

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz: Consider the following ODE with r a real constant and find the fixed points and draw vector fields as a function of r.

$$\frac{dy}{dt} = r - \cos x.$$

Intro to ODEs - Qualitative analysis of linear ODEs

Quiz: Sketch the bifurcation diagram and find values of r at which a bifurcation occurs.

Intro to ODEs - Bifurcations

3. Pitchfork bifurcation

This kind of bifurcation is common in physical systems that have a symmetry (The buckling example). There are two subtypes of pitchfork bifurcation.

3.1 Supercritical pitchfork bifurcation

The prototypical example of supercritical pitchfork bifurcation is given by:

$$\frac{dy}{dt} = ry - y^3$$

Note that the equation is invariant under the change of variable $y \to -y$.

Intro to ODEs - Bifurcations

For $r < 0$ we have 1 stable fixed point, at $r = 0$ we have still one stable fixed point and for $r > 0$ we have three fixed points (two stable and a middle one that is unstable). At the bifurcation point $r = 0$ an exchange of stabilities takes place between the two fixed points.

Intro to ODEs - Bifurcations

Supercritical Pitchfork Bifurcation diagram

Intro to ODEs - Bifurcations

3.2 Subcritical pitchfork bifurcation

When in contrast to the supercritical case the cubic term is destablizing:

$$\frac{dy}{dt} = ry + y^3$$

Intro to ODEs - Bifurcations

Subcritical Pitchfork Bifurcation diagram

Intro to ODEs - Bifurcations

Singular points

As discussed the special points can be the fixed points y^* or the singularities:

Example:

$$\frac{dy}{dt} = \frac{K}{y}$$

Intro to ODEs - Bifurcations

Explicit solution and the bifurcation diagram

$$\frac{dy}{dt} = \frac{K}{y} \quad \Longrightarrow \quad \int y dy = \int K dt$$

Intro to ODEs - Bifurcations

Impossibility of oscillations for one dimensional systems

Fixed points dominate the dynamics of first-order systems. The phase point never reverses direction and the approach to equilibrium is always monotonic, hence there is no over-shoot, damped oscillations or periodic solutions. This is a topological constraint, if you flow monotonically on a line, you'll never come back to your starting point. Of course, if we were moving on a circle rahter than a line, we could eventually return to starting point.

Example:

Intro to ODEs - Bifurcations

Linear stability analysis

So far we have relied on graphical methods to determine stability of fixed points. A quantitative measure can be obtained by linearizing about the fixed point. Let $\eta = y - y^*$ be a small perturbation away from y^*. We have:

Intro to ODEs - Bifurcations

Quiz: Sketch the bifurcation diagram for the following ODE

$$\frac{dy}{dt} = ry + y^3 - y^5$$

Intro to Multivariate Calculus - Definitions

Definitions

So far we have considered functions of single independent variables (ordinary functions):

We now consider functions of several variables or *Multivariable or Multivariate functions*.

For every n-tuple of $\{x_i\}_{i=1}^n$, where $x_i \in \mathbb{R}$, there exists an image in $\mathbb{R}$.

An important example is functions of two variables:

Intro to Multivariate Calculus - Definitions

Representation

1. 3D representation where $f(x, y)$ is the hight:

2. Level curves $\vec{x}_C = (x, y)_C$, where $f(\vec{x}_C) = C$. For each C there will be a set of points that fulfill this condition.

Intro to Multivariate Calculus - Definitions

Notions for calculus

1. Limit: $\lim_{\vec{x} \to \vec{x}^*} f(\vec{x}) = C$

2. Continuity: $\lim_{\vec{x} \to \vec{x}^*} f(\vec{x}) = f(\vec{x}^*)$

Intro to Multivariate Calculus - Differentiation

Derivatives of functions of several variabiles: Partial differentiation

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}$$

Higher order partial derivatives

Considring a function of two variable $f(x, y)$, we denote the first partial derivatives as:

$$g_1(x, y) = \left(\frac{\partial f}{\partial x}\right)_y \quad ; \quad g_2(x, y) = \left(\frac{\partial f}{\partial y}\right)_x$$

We have:

Symmetry of mixed derivatives or equality of mixed derivatives:

If the second partial derivatives are continuous, we have:

Intro to Multivariate Calculus - Differentiation

Operationally, calculations are simple. Partial derivatives are obtained by keeping the other variables constant.

Example:

$$u(x, y) = x^2 \sin y + y^3$$

Intro to Multivariate Calculus - Differentiation

Total differentiation of a function of several variables

Total derivative evaluates the infinitesimal change of $f(\vec{x})$ when all the variables are allowed to change infinitesimally.

Intro to Multivariate Calculus - Differentiation

$$df = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Chain rule for functions of several variables

Reminder: Chain rule for ordinary functions.

What is the equivalent for multivariable functions?

$$u = u(x, y) = u(\vec{x}); \quad \vec{x} \in \mathbb{R}^2$$

But we have $x = x(t)$ and $y = y(t)$ with $t \in \mathbb{R}$. What is $\dfrac{du}{dt}$?

This generalizes to functions of n variables:

Example:

Consider a cylinder that its radius and height are expanding with time:

$$r(t) = 2t; \qquad h(t) = 1 + t^2$$

Evaluate the rate of change in volume $\frac{dV}{dt}$.

Intro to Multivariate Calculus - Differentiation

Another example of chain rule: taking care of all dependencies

$$u = u(x, y); \quad \text{with} \quad y = y(t, x)$$

Intro to Multivariate Calculus - Differentiation

Dependencies on another set of coordinates

Let $\quad h = h(x, y); \quad$ with $\quad x = x(u, v) \quad$ and $\quad y = y(u, v)$

Intro to Multivariate Calculus - Differentiation

Quiz: For the function given below obtain $\left(\frac{\partial u}{\partial x}\right)_t$ and $\left(\frac{\partial u}{\partial t}\right)_x$.

$$\text{Let} \quad u(x,y) = xy + y^2; \quad \text{with} \quad y = x + t$$

Intro to Multivariate Calculus - Differentiation

Implicit functions

Explicit form (reminder): $y = f(x); \quad x \in \mathbb{R}$
Implicit form: $F(x, y) = 0$
Trivially, if we have the explicit form we also have an implicit form:

For functions of two variables, we also have explicit form:

$$z = z(x, y)$$

Implicit form:

$$F(x, y, z) = 0$$

Intro to Multivariate Calculus - Differentiation

Differentiation using the Implicit form

Taking total differential from the implicit form $F(x, y, z) = 0$, we obtain:

$$dF = \left(\frac{\partial F}{\partial x}\right)_{y,z} dx + \left(\frac{\partial F}{\partial y}\right)_{x,z} dy + \left(\frac{\partial F}{\partial z}\right)_{x,y} dz = 0$$

Taking total differential from the explicit form $z = z(x, y)$, we obtain:

$$dz = \left(\frac{\partial z}{\partial x}\right)_{y} dx + \left(\frac{\partial z}{\partial y}\right)_{x} dy$$

We thus have the following relationship between derivatives of the implicit and explicit form:

Intro to Multivariate Calculus - Differentiation

Example:

$$\text{Let} \quad z(x, y) = x^2 + y^2 - 5$$

Intro to Multivariate Calculus - Differentiation

Taylor expansion for functions of two variables

Taylor expansion for functions of one variable (reminder):

$$\text{Let} \quad f(x) : \mathbb{R} \to \mathbb{R} \quad \text{and consider} \quad x_0 \in \mathbb{R}$$

$$f(x_0+\Delta x) = f(x_0)+\left(\frac{df}{dx}\right)_{x_0}\Delta x+\frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0}(\Delta x)^2+\frac{1}{3!}\left(\frac{d^3 f}{dx^3}\right)_{x_0}(\Delta x)^3+\cdots$$

Let us consider $f(\vec{x})$, $\vec{x} \in \mathbb{R}^2$, we assume suitable conditions of differentiability.

Intro to Multivariate Calculus - Differentiation

Upto to 3rd order we have: $f(\vec{x_0}+\Delta\vec{x}) = f(x_0+\Delta x, y_0+\Delta y) =$

$$f(x_0, y_0+\Delta y) + \left(\frac{\partial f}{\partial x}\right)_{x_0, y_0+\Delta y} \Delta x + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{x_0, y_0+\Delta y} (\Delta x)^2 + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_{x_0, y_0+\Delta y} (\Delta x)^3 + \cdots =$$

$$f(x_0, y_0) + \left(\frac{\partial f}{\partial y}\right)_{\vec{x_0}} \Delta y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{\vec{x_0}} (\Delta y)^2 + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial y^3}\right)_{\vec{x_0}} (\Delta y)^3 + \cdots$$

$$+\Delta x \left[\left(\frac{\partial f}{\partial x}\right)_{\vec{x_0}} + \left(\frac{\partial^2 f}{\partial y \partial x}\right)_{\vec{x_0}} \Delta y + \frac{1}{2}\left(\frac{\partial^3 f}{\partial y^2 \partial x}\right)_{\vec{x_0}} (\Delta y)^2 + \cdots\right]$$

$$+\frac{1}{2}(\Delta x)^2 \left[\left(\frac{\partial^2 f}{\partial x^2}\right)_{\vec{x_0}} + \left(\frac{\partial^3 f}{\partial y \partial x^2}\right)_{\vec{x_0}} \Delta y + \cdots\right] + \frac{1}{3!}(\Delta x)^3 \left[\left(\frac{\partial^3 f}{\partial x^3}\right)_{\vec{x_0}} + \cdots\right] =$$

Intro to Multivariate Calculus - Differentiation

We can write the Taylor expansion up to the second order in a vector-matrix form:

We define *Gradient* of the function f evaluated at point $\vec{x}_0$ as:

$$\vec{\nabla} f_{\vec{x}_0} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix}_{\vec{x}_0}$$

Hessian Matrix associated with the function f evaluated at the point $\vec{x}_0$ is defined as:

$$H_{ij}(\vec{x}_0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{\vec{x}_0}$$

Intro to Multivariate Calculus - Differentiation

We can write the Taylor expansion up to the second order in terms of the Gradient and Hessian:

$$f(\vec{x}_0 + \Delta\vec{x}) = f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0)^T . \Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^T H(\vec{x}_0)\Delta\vec{x} + \cdots$$

This generalizes to n dimensions:

Example (approximation):

$$\text{Let} \quad A(x, y) = xy$$

Expand A around $\vec{x}_0 = (x_0, y_0)$

Using Taylor Expansion for error analysis

Example: What is the error in h given errors in x and θ:

$$h(x,\theta) = x\tan\theta$$

Intro to Multivariate Calculus - Differentiation

Quiz:

$$\text{Let} \quad u(x, y) = e^{2x-y}$$

Expand u around $\vec{x}_0 = (0, 0)$

Intro to Multivariate Calculus - Applications

Change of Coordinates

Nonlinear transformations representing the change of coordinates from polar ($\vec{x} = (r, \theta)$) to cartesian ($\vec{x} = (x, y)$) and vice versa.

Intro to Multivariate Calculus - Applications

Cartesian to Polar

Let $x = x(r, \theta); \quad y = y(r, \theta)$, we have:

$$dx = \left(\frac{\partial x}{\partial r}\right)_\theta dr + \left(\frac{\partial x}{\partial \theta}\right)_r d\theta$$
$$dy = \left(\frac{\partial y}{\partial r}\right)_\theta dr + \left(\frac{\partial y}{\partial \theta}\right)_r d\theta$$

Intro to Multivariate Calculus - Applications

Polar to Cartesian

Let $r = r(x, y)$; $\quad \theta = \theta(x, y)$, we have:

$$dr = \left(\frac{\partial r}{\partial x}\right)_y dx + \left(\frac{\partial r}{\partial y}\right)_x dy$$
$$d\theta = \left(\frac{\partial \theta}{\partial x}\right)_y dx + \left(\frac{\partial \theta}{\partial y}\right)_x dy$$

Intro to Multivariate Calculus - Applications

Application 1

$$ds^2 = dx^2 + dy^2$$

Intro to Multivariate Calculus - Applications

Application 2: Infinitesimal element of area in polar coordinate

$$dA = dx dy$$

Intro to Multivariate Calculus - Applications

How to generalise area element for a general transformation

Consider a general transformation in two-dimensions:

$$x = x(u, v) \quad \text{and} \quad y = y(u, v)$$

We have:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = J \begin{bmatrix} du \\ dv \end{bmatrix}$$

where J is the Jacobian of the transformation.

Intro to Multivariate Calculus - Applications

For the polar coordinate we obtain:

$$dxdy = |\mathrm{det}J|\, drd\theta$$

Generalisation to higher dimensions: volume element

Intro to Multivariate Calculus - Applications

Quiz: Obtain the area of a circle of radius R by using element of area in polar coordinates.

Intro to Multivariate Calculus - Applications

Partial Differential Equations (PDEs)

Find $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ satisfying:

$$f(x_1, \cdots, x_n, f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \frac{\partial^2 f}{\partial x_i x_j}, \cdots) = 0$$

Examples:

- Laplace Equation

- Wave Equation

Intro to Multivariate Calculus - Applications

Transforming a PDE under a change of coordinates

$$u(x, y) \quad \longleftrightarrow \quad u(r, \theta)$$

Intro to Multivariate Calculus - Applications

Example: Laplace Equation in polar coordinates

Laplace equation in Cartesian coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In polar coordinates we have:

$$\frac{\partial}{\partial x}[u] = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Intro to Multivariate Calculus - Applications

$$\frac{\partial^2}{\partial x^2}[u] = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial y^2}[u] = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Find function $u(r)$ fulfilling the Laplace equation.

Quiz: D'Alembert solution of the Wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Show $u(x, t)$ having the D'Alembert form:

$$u(\xi); \quad \xi = x - ct$$

is a general solution.

For example a particular case is:

$$u(x,t) = \sin(kx - wt) = \sin(k\xi)$$

In general the form of the function $u(x) = u(x, t = 0)$ is fixed by the initial condition. This means that under the wave equation u(x) travels at constant speed c without distortion.

Going back to first order nonlinear ODEs: Exact ODEs

Given first order ODE:

$$\frac{dy}{dx} = \frac{-F(x,y)}{G(x,y)}$$

If we have a solution of the ODE in implicit form $u(x,y) = 0$, Assuming u is continuous with continuous derivatives. For total derivative of u we have:

Intro to Multivariate Calculus - Applications

Example for exact ODEs

Is the following ODE exact?

$$\frac{dy}{dx} = \frac{-2xy - \cos x \cos y}{x^2 - \sin x \sin y}$$

Intro to Multivariate Calculus - Applications

Since the ODE is exact, we can look for a solution in implicit form $u(x, y) = 0$ such that:

$$F(x, y) = \frac{\partial u}{\partial x} \quad \text{and} \quad G(x, y) = \frac{\partial u}{\partial y}$$

Intro to Multivariate Calculus - Applications

When the ODE is not exact, sometimes we can find a function (an integrating factor) that will make the equation exact. Given:

$$F(x, y)dx + G(x, y)dy = 0$$

is not exact, we look for a function $\lambda(x)$ or $\lambda(y)$ such that:

$$\lambda F(x, y)dx + \lambda G(x, y)dy = 0$$

is exact. Note that an integrating factor can in general be a function of both x and y, but in this case we cannot find an explicit solution for λ, and it is for this reason we can not solve very many ODEs.

Example

$$\frac{dy}{dx} = \frac{xy - 1}{x(y - x)}$$

Intro to Multivariate Calculus - Applications

We will try to find a $\lambda(x)$ (or $\lambda(y)$ that will make the ODE exact).
We need to find $\lambda(x)$ such that:

$$\frac{\partial[\lambda(x)(xy-1)]}{\partial y} = \frac{\partial[\lambda(x)(x^2-xy)]}{\partial x}$$

Intro to Multivariate Calculus - Applications

Quiz: Solve the exact ODE we have obtained:

$$(y - \frac{1}{x})dx + (x - y)dy = 0$$

Intro to Multivariate Calculus - Applications

Sketching functions of two variables

Similar to the sketching of functions of one variable, we will use the following steps:

- Check continuity and find singularities.
- Find asymptotic
- Obtain some level curves, *e.g.* $f(\vec{x}) = 0$.
- Find stationary points: minimum, maximum, saddle nodes

Intro to Multivariate Calculus - Applications

Stationary points for functions of two variables

Reminder for functions of one variable:

For functions of two variables:

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The type of stationary points can be determined by the Hessian matrix

$$f(\vec{x}^* + \Delta\vec{x}) = f(\vec{x}^*) + \begin{bmatrix} \frac{\partial f}{\partial x}(\vec{x}^*) & \frac{\partial f}{\partial y}(\vec{x}^*) \end{bmatrix} .\Delta\vec{x}$$

$$\frac{1}{2}\Delta\vec{x}^T . \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x}(\vec{x}^*) & \frac{\partial^2 f}{\partial x \partial y}(\vec{x}^*) \\ \frac{\partial^2 f}{\partial y \partial x}(\vec{x}^*) & \frac{\partial^2 f}{\partial^2 y}(\vec{x}^*) \end{bmatrix} .\Delta\vec{x}$$

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$$f(\vec{x}^* + \Delta\vec{x}) - f(\vec{x}^*) = \frac{1}{2}\Delta\vec{x}^T \left[V((\vec{x}^*)\Lambda(\vec{x}^*)V^T(\vec{x}^*) \right] \Delta\vec{x}$$

$$f(\vec{x}^* + \Delta\vec{x}) - f(\vec{x}^*) = \frac{1}{2}\Delta\vec{z}^T\Lambda(\vec{x}^*)\Delta\vec{z} = \frac{1}{2}\left[(\Delta z_1)^2\lambda_1 + (\Delta z_2)^2\lambda_2\right]$$

Intro to Multivariate Calculus - Applications

Example for sketching functions of two variables

$$u(x,y) = (x-y)(x^2+y^2-1)$$

Intro to Multivariate Calculus - Applications

We next find the stationary points.

$$\frac{\partial u}{\partial x}(\vec{x}^*) = \frac{\partial u}{\partial y}(\vec{x}^*) = 0$$

Intro to Multivariate Calculus - Applications

We classify the stationary points.

$$H(\vec{x}) = \begin{bmatrix} 6x - 2y & 2y - 2x \\ 2y - 2x & 2x - 6y \end{bmatrix}$$

Intro to Multivariate Calculus - Applications

Given the location and stability of the stationary points, we complete our sketch of

$$u(x, y) = (x - y)(x^2 + y^2 - 1)$$