Network Science
Autumn 2022
Problem class 4

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1. How many distinct matchings are generated by the configuration model for a degree sequence with total degree, K?
Solution: $(K-1)(K-3)(K-5)$.... Stub 1 will connect to one of $K-1$ other stubs. For each of these $K-1$ pairings, choose one of the remaining stubs. It will connect to one of $K-3$ stubs. And so on.

2. Consider the set of graphs with N nodes generated by the configuration model with a specified degree sequence. Assess the claim that p_{ij}, the probability that nodes i and j are linked, is at most $k_i k_j/(K-1)$. Provide explanations for the specific cases where $k_i = 1$ and $k_i = 2$, and then provide a more general argument for cases with $k_i > 2$ (e.g. using Markov's inequality).
Solution: First, $k_i = 1$: $p_{ij} = k_j/(K-1)$.
Now: $k_i = 2$: Label the stubs on node i as s_1, s_2. Let $s_1 \sim s_2$ indicate that stubs s_1 and s_2 on node i are linked and $s_1 \sim n_j$ indicate that stub s_1 is linked to node j. We will consider the 2 stubs on node i in order. Stub 1: $P(s_1 \sim n_j) = k_j/(K-1)$. Now consider:
$$P(s_1 \nsim n_j, s_1 \nsim s_2, s_2 \sim n_j).$$
Note that $s_2 \sim n_j$ implies that $s_1 \nsim s_2$, and I have included that latter expression as a "reminder" about where s_1 has linked. It is helpful to rewrite the joint probability in terms of a conditional probability,
$$P(s_1 \nsim n_j, s_1 \nsim s_2, s_2 \sim n_j) = P(s_2 \sim n_j | s_1 \nsim s_2, s_1 \nsim n_j) P(s_1 \nsim s_2, s_1 \nsim n_j)$$
which can be further modified to,
$$P(s_1 \nsim n_j, s_1 \nsim s_2, s_2 \sim n_j) = [k_j/(K-3)] \left[(K-(2+k_j))/(K-1)\right].$$
We know that $(K-(2+k_j))/(K-3) \leq 1$, so it follows that if $k_i = 2$, $p_{ij} \leq k_j/(K-1) + k_j/(K-1) = 2k_j/(K-1)$.
The general case: here we avoid computing increasingly complicated probabilities, and instead use Markov's inequality. Let l_{ij} be the expected number of links between nodes i and j. We know that $\langle l_{ij} \rangle = k_i k_j/(K-1)$, and Markov's inequality tells us that, $P(l_{ij} \geq 1) \leq k_i k_j/(K-1)$ and $P(l_{ij} \geq 1)$ is the probability that nodes i and j are connected.

3. In problem sheet 6, you were asked to consider the following modification to the simple Barabasi-Albert model ($N_0 = 2, q = 1$): a new node connects to any node in the graph with equal probability.
   (a) For this model,
   $$(3+t)p_1(t+1) = (2+t)p_1(t) - p_1(t) + 1.$$
   Assume that p_1 becomes stationary, when $t \to \infty$. What will p_1 be in this limit?
   Solution: If $p_1(t+1) = p_1(t) = p_{1,\infty}$, our result from PS6, question 2 becomes $p_{1,\infty} = 1/2$

(b) For this model when $k > 1$,

$$\langle N_k(t+1) \rangle = \langle N_k(t) \rangle + \left(\langle N_{k-1}(t) - N_k(t) \rangle \right) / (2+t).$$

Compare $p_{k,\infty}$ for the simple B-A model and this modified model when $k = 50$
Solution: For the B-A model, we have $p_{k,\infty} = 4/(k+2)(k+1)(k) \approx 3e-5$, and for the modified model, $p_{k,\infty} = 1/2p_{k-1,\infty} = 2^{-50} \approx 9e-16$. With preferential attachment, large-degree nodes are much more likely to be found.