Imperial College London

MATH70033

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebraic Curves

Date: 19 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

If you are working on one part of a question, you may use the results of the previous parts of the same question even if you have not solved them.

1. Let C be the projective plane curve which is the zero set of $x_0^d + x_1^d + x_2^d$, for $d \geq 1$.

   (a) Prove that C is smooth. (5 marks)

   (b) Let $f : X \to Y$ be a morphism of Riemann surfaces. Define the ramification degree of f at a point $x_0 \in X$ and the degree of f. (5 marks)

   (c) Consider the projection $p : C \to \mathbb{P}^1$ given by $p([x_0, x_1, x_2]) = [x_0, x_1]$. How many ramification points does p have and what is the degree of p? (5 marks)

   (d) State the Riemann-Hurwitz formula and use it to compute the genus of C. (5 marks)

(Total: 20 marks)

2. Recall that a conic is the zero set of a degree 2 homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$ and that it is said to be degenerate if it is reducible.

   (a) Consider $\mathbb{P}^2$ and its dual, $\check{\mathbb{P}}^2$. Recall that the dual is defined to be the set of all lines in $\mathbb{P}^2$ and that the line defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ in $\mathbb{P}^2$ corresponds to the point $[a_0, a_1, a_2] \in \check{\mathbb{P}}^2$. If C is a smooth projective plane curve, then the dual, $\check{C} \subset \check{\mathbb{P}}^2$ is defined to be the set of tangent lines to C. Let $C \subset \mathbb{P}^2$ be the zero set of $x_0^2 + x_1^2 + x_2^2$. Realize the dual of C as a projective plane curve in $\check{\mathbb{P}}^2$ by giving an explicit equation. (5 marks)

   (b) Show that every conic can be given as the zero set of the equation

   $$x^T \cdot A \cdot x = 0$$

   where $x = [x_0, x_1, x_2]$ and A is a symmetric 3×3 matrix. (4 marks)

   (c) Show that a conic C is degenerate if and only if $\det(A) = 0$ where A is the matrix from part (b).
   Hint: You may want to use the classification of conics and can freely assume it. (6 marks)

   (d) Let C_1 and C_2 be two non-degenerate conics such that they are the zero sets of polynomials Q_1 and Q_2, respectively. Consider the pencil of conics given by

   $$Q_{a,b} = aQ_1 + bQ_2$$

   where $[a, b] \in \mathbb{P}^1$. Show that there are at most three singular fibers of the pencil of conics. (5 marks)

(Total: 20 marks)

3. Let C and D be two curves of degree d such that they intersect in exactly d^2 distinct points and let $S \subset C \cap D$ be a subset consisting of $d \cdot k$ points where $k < d$.

   (a) State Bezout's Theorem. (5 marks)

   (b) Let p be a point not in C or D. Construct a degree d curve that contains S and p. (5 marks)

   (c) Now suppose that all the points in S lie on an irreducible curve, F, of degree k. Prove that the remaining $d^2 - d \cdot k$ points of $C \cap D$ lie on a curve of degree $d - k$ in the case where F is a component of C. (4 marks)

   (d) Repeat part (c) but in the case where F is not a component of C or D. (6 marks)

(Total: 20 marks)

4. (a) Let C be the zero set of $x_0^3 + x_0x_1x_2$. What is the number of irreducible components of C? Find the singular points of C. (5 marks)

   (b) Consider the complete linear system of degree 2, denoted by $\mathcal{L}_2$. Recall that $\mathcal{L}_2 \simeq \mathbb{P}^N$ for some N. What is N? Briefly justify your answer. (4 marks)

   (c) What is the dimension of the linear subspace of conics that pass through all the singular points of C? Make sure to justify your answer! (6 marks)

   (d) Construct a pencil which parametrizes conics that pass through the singular points of C as well as $[1, 0, 0]$ and $[1, 1, 1]$. (5 marks)

(Total: 20 marks)

5. (a) State the Riemann-Roch theorem for a nonsingular projective plane curve. (5 marks)

(b) Let C be a nonsingular projective plane curve of genus g and let D be a divisor on C and $p \in C$. Prove that $\ell(D) \leq \ell(D + p)$. (5 marks)

(c) Prove that $0 \leq \ell(D + p) - \ell(D) \leq 1$.
Hint: Construct a function $L(D + p) \to \mathbb{C}$. (5 marks)

(d) Let κ be a canonical divisor. Prove that $\ell(\kappa - D - p) \neq \ell(\kappa - D)$ if and only if $\ell(D + p) = \ell(D)$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

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MATH60033/70033

Algebraic Curves (Solutions)

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1. (a) The vector of partial derivatives is given by $(dx_0^{d-1}, dx_1^{d-1}, dx_2^{d-1})$. This simultaneously vanishes only at $(0,0,0)$ which is not a point on the curve. $\boxed{5, \mathrm{A}}$

(b) Let $x_0 \in X$ and $y_0 = f(x_0)$. Then let (U, ψ) be a chart on Y such that $y_0 \in U$ and $\psi(y_0) = 0$. Then there is a chart (V, ϕ) for X where $x_0 \in V$ and $\phi(x_0) = 0$ such that $\varphi(V)$ is a disk with center 0 in $\mathbb{C}$ and $\psi \circ f \circ \varphi^{-1}$ is given by $z \mapsto z^m$ for some m. Then the ramification degree of f at x_0 is m and the degree of f is given by $\sum_{x \in f^{-1}}(y_0)\nu_f(x)$ for any $y_0 \in Y$. $\boxed{5, \mathrm{B}}$

(c) For a generic and fixed (x_0, x_1), there are d solutions to $x_0^d + x_1^d + x_2^d = 0$ so the degree of the morphism is d. Then the ramification points of p are points that map to $[x_0, x_1]$ such that $x_0^d + x_1^d = 0$. There are d such points and the preimage above these points all consist of a single point so there are d ramification points of index $d-1$. $\boxed{5, \mathrm{B}}$

(d) Let $f : X \to Y$ be a morphism of Riemann surfaces. The Riemann-Hurwitz formula states that

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + (\sum_{x \in X} \nu_f(x) - 1).$$

We consider the case where $X = C$ and $Y = \mathbb{P}^1$. Then since $g(\mathbb{P}^1) = -2$ and by part c the morphism is degree d with d ramification points with ramification degree $d-1$, plugging into the Riemann-Hurwitz formula we get

$$2g(C) - 2 = d(-2) + d(d-1)$$

and solving for $g(C)$ gives the desired answer. $\boxed{5, \mathrm{B}}$

2. (a) We first compute the tangent line at a point on C. Let $[p_0, p_1, p_2] \in C$. Then the tangent line is given by $2p_0x_0 + 2p_0x_1 + 2p_0x_2$ which corresponds to the point $[2p_0, 2p_1, 2p_2] \sim [p_0, p_1, p_2]$ in the dual. So we see that the dual curve is also given by the same equation. $\boxed{5, \mathrm{A}}$

(b) Every conic is given by a homogeneous quadratic equation. Suppose it is given by

$$ax_0^2 + bx_0x_1 + cx_1^2 + dx_0x_2 + ex_1x_2 + fx_2^2.$$

Then the matrix

$$A = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}$$

satisfies the desired property. $\boxed{4, \mathrm{A}}$

(c) One could do this by hand, but we can also use the fact that every smooth non-degenerate conic is projectively equivalent to $x_0^2 + x_1^2 + x_2^2$ whose associated matrix is the identity which has non-zero determinant. If a conic is given by a matrix B, then a projectively equivalent conic is given by a matrix M^tBM for some invertible matrix M, which preserves non-vanishing of determinant. In the other direction, if the conic is singular, it is either a single line or two lines. Both of whose corresponding matrices have determinant zero. $\boxed{6, \mathrm{C}}$

(d) By problem 2 part c, a conic is degenerate if and only if the associated matrix has determinant 0. Let M_1 and M_2 be the matrices associated to the conics given by Q_1 and Q_2, respectively. Then the degenerate members of the pencil are exactly when $\det(aM_1 + bM_2) = 0$. This has at most three solutions for points $[a, b]$. $\boxed{5, \mathrm{D}}$

3. (a) Let C and D be projective plane curves with no common component and let $d = \deg(C)$ and $e = \deg(D)$. Then

$$\sum_{p \in C \cap D} I(p, C, D) = de.$$

$\boxed{5, \text{A}}$

(b) Suppose C is the zero set of a polynomial P_1 and D is the zero set of a polynomial P_2. Then consider the pencil given by $aP_1 + bP_2$. Then there is a curve of degree d in this pencil that has intersection points S and point p. $\boxed{5, \text{B}}$

(c) Consider P_1. Then since F is a component of C, we can write $P_1 = QR$ where F is the zero set of Q and R is not constant and of degree $d - k$. Then the remaining $d^2 - d \cdot k$ lie on the zero set of R which is a curve of degree $d - k$. $\boxed{4, \text{A}}$

(d) If F is not on a component of C or D then there exists a point p on the component F but not on C or D. Using part a, we can find a a curve H of degree d that intersects all the points of S as well as p. Then consider $H \cap F$. By construction, this intersection has $dk + 1$ points. So by Bezout's theorem, H and F must have a common component. This means that we can write $aP_1 + bP_2 = H \cdot R$ for some polynomial R of degree $d - k$. The remaining points in questions must all lie on the zero set of R. $\boxed{6, \text{D}}$

4. (a) Note that $x_0^3 + x_0x_1x_2 = x_0(x_0^2 + x_1x_2)$. This is the union of a line and a non-degenerate conic so there are two irreducible components. Both components are smooth so the only singular points where be will the components intersect. This happens at $[0, 1, 0]$ and $[0, 0, 1]$. $\boxed{5, \mathrm{A}}$

(b) The vector space of homogeneous polynomials of degree 2 in three variables has basis given by

$$x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$$

so $N = 5$. $\boxed{4, \mathrm{A}}$

(c) The linear system of conics in part b is isomorphic to $\mathbb{P}^5$. Then imposing a one point condition drops the dimension by 1. Imposing the second point condition drops the dimension by at most 1. The key is to find a conic that goes through one singular point but not the other. For example, something like $x_0^2 + x_2^2$. $\boxed{6, \mathrm{C}}$

(d) The key is to find two conics that intersect at these 4 points and construct the pencil with equation $aQ_1 + bQ_2$ where $[a, b] \in \mathbb{P}^1$ and Q_i is the defining polynomial for the conics. Then consider $Q_1 = x_0x_1 - x_0x_2$ and $Q_2 = x_0x_1 - x_2x_2$. Both are the union of two lines, all four lines are distinct and the intersections between the two conics are the four prescribed points. One could also just do this easily from the equation of a general conic. $\boxed{5, \mathrm{D}}$

5. (a) Let D be any divisor on a nonsingular projective plane curve C of genus g and let κ be a canonical divisor on C. Then

$$\ell(D) - \ell(\kappa - D) = \deg(D) + 1 - g.$$

$\boxed{5, \mathrm{M}}$

(b) We want to show that $\ell(D) \leq \ell(D+p)$. It suffices to show that $L(D) \subset L(D+p)$. Recall that

$$L(D) := \{f \in K(C) : \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

Then since $D \leq D + p$, it is immediate that if $f \in L(D)$, then

$$\operatorname{div}(f) + D' \geq \operatorname{div}(f) + D \geq 0.$$

$\boxed{5, \mathrm{M}}$

(c) By Part b, we know that $\ell(D) \leq \ell(D+p)$. We can write $D = np + D'$ where p does not appear non-trivially in D' for some n. Then let z be a local coordinate around p and consider the function

$$\varphi : L(D+p) \to \mathbb{C}, \varphi(f) = (t^{n+1}f)p.$$

Then the kernel of φ are functions f such that $t^{n+1}f$ has a zero at p which is exactly $L(D)$. Then because the image is either $\mathbb{C}$ or $\{0\}$, we get the desired bound. $\boxed{5, \mathrm{M}}$

(d) By Riemann-Roch we have the following:

$$\ell(D) = \ell(\kappa - D) + \deg(D) + 1 - g$$

$$\begin{aligned} \ell(D+p) &= \ell(\kappa - D - p) + \deg(D+p) + 1 - g \\ &= \ell(\kappa - D - p) + \deg(D) + 2 - g \end{aligned}$$

If the two are equal then

$$\ell(\kappa - D) = \ell(\kappa - D - p) + 1$$

which shows one direction. If $\ell(D) \neq \ell(D+p)$, by the previous part, we must have that $\ell(D+p) = \ell(D) + 1$ which immediately gives the other direction. $\boxed{5, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60033/70033	1	This problem went well. There were some issues with computing the right number of ramification points which then caused some problems in part d.
MATH60033/70033	2	Parts a and b were good. Most people got the main idea of c but if you used a projective transformation to invoke the classification of conics, I wanted to see a statement about how projective transformations affect the defining matrix and determinant. Part d gave many people problems. The idea was just to immediately apply part c and consider solutions [a:b] for det(aM1 + bM2)= 0 where Mi is a symmetric matrix.
MATH60033/70033	3	Numerous papers left off assumptions in Bezout’s Theorem (like the curves being projective, or them not sharing a component). Part b was supposed to be obtained by using a linear combination of the defining equations for C and D. Most people did not get part d. This was supposed to be an application of part b and then Bezout’s Theorem to see that F is a component of the curve constructed from part b.
MATH60033/70033	4	No real issues here. The only common error I saw was in part c, where some people did not justify why the two point conditions were independent.
MATH70033	5	Parts and b went well. There were some issues about getting that upper bound on part c. The point was to construct a function to C whose kernel was exactly L(D). Part d, one direction was an immediate application of part c and the other was just using Riemann-Roch.