Imperial College
London

MATH97017 MATH70022

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Finite Elements: Numerical Analysis and Implementation

Date: 01 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the following finite element

- K is a triangle,
- P is the space of polynomials of degree ≤ 2,
- N is the set of six nodal variables given by evaluation at the vertices and edge centres of K.

(a) Show that N determines P. (7 marks)

(b) Give a C^0 geometric decomposition of this finite element, showing that it is C^0. (7 marks)

(c) Show that finite element spaces built from this element are not necessarily C^1. (6 marks)

(Total: 20 marks)

2. (a) Write a C^0 finite element variational problem for the following equation,

$$\epsilon u - \nabla^2 u = \exp(xy), \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Omega = \{x, y : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$ with boundary $\partial\Omega$, and $0 < \epsilon < 1$.

(6 marks)

(b) Show that the bilinear form for the variational problem is continuous and coercive, and give bounds for the continuity and coercivity constants M and γ for this problem.
(You may make use of the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$.)

(6 marks)

(c) Assuming Ceá's Lemma and standard interpolation error estimates, derive an error bound for the H^1 error $\|u - u_h\|$ where u_h is the solution obtained by a linear Lagrange finite element approximation with maximum mesh size h, and u is the exact solution. What is happening to this error bound when ϵ is very small?

(8 marks)

(Total: 20 marks)

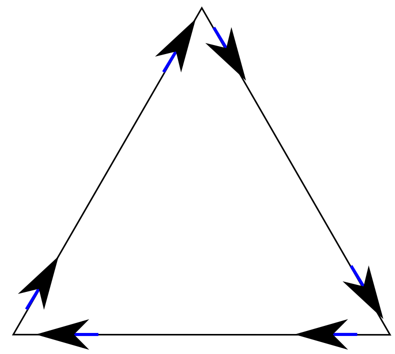

Figure 1: Nodal variables diagram for Question 4.

3. In this question we consider the following finite element.

   - K is a triangle.
   - P are vector-valued linear functions (i.e. the x- and y- components of the function are both polynomials of degree ≤ 1).
   - The six nodal variables are the components of the function tangential to the edges at the locations indicated by the arrows in Figure 1.

   (a) Describe how this element can be used to construct a finite element space V where the functions are continuous in the tangential component across each edge. Show that the finite element space does indeed have this property. (6 marks)

   (b) Consider the quadratic Lagrange finite element space P_2. Show that $\phi \in P_2 \implies \nabla\phi \in V$. (8 marks)

   (c) Provide a formula for the weak curl $\nabla^{\perp}\cdot u = -\partial u_1/\partial y + \partial u_2/\partial x$ for a function $u = (u_1, u_2) \in V$, and show that it is indeed the weak curl. (6 marks)

(Total: 20 marks)

4. Let Ω be a convex polygonal domain. Assume that you have a fast and efficient code for solving the variational problem: find $u \in V$ such that

$$\int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x = F[v], \quad \forall v \in V, \tag{2}$$

for arbitrary linear functionals $F[v]$, where V is a C^0 finite element space. However, you want to solve a different variational problem: find $u \in V$ such that

$$\int_\Omega a(x)uv + b(x)\nabla u \cdot \nabla v \,\mathrm{d}\,x = G[v], \quad \forall v \in V, \tag{3}$$

where $a(x)$ and $b(x)$ are some known functions that satisfy $0 < \alpha < a(x) < \beta < \infty$, $0 < \alpha < b(x) < \beta < \infty$, for all $x \in \Omega$. One possible approach is to apply the following iterative scheme,

$$\int_\Omega u^{k+1}v + \nabla u^{k+1} \cdot \nabla v \,\mathrm{d}\,x = F_k[v], \tag{4}$$

where

$$F_k[v] = \int_\Omega u^k v + \nabla u^k \cdot \nabla v \,\mathrm{d}\,x + \mu\left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k \cdot \nabla v \,\mathrm{d}\,x\right), \quad \forall v \in V, \tag{5}$$

where $\mu > 0$, for an iterative sequence $u^0, u^1, u^2, \ldots$ of guesses at the solution. To implement this, we choose an initial guess u^0, and then iteratively generate the sequence by solving (4) for u^{k+1} given u^k (which enables us to construct $F_k[v]$).

(a) Show that if the sequence converges to a limit $u_k \to u^*$ as $k \to \infty$, then u^* solves Equation (3). (6 marks)

(b) Defining the error $\epsilon^k = u - u^k$, where u solves (3), derive a variational problem that relates ϵ^{k+1} to ϵ^k (without explicitly involving u^{k+1} or u^k). (6 marks)

(c) Find a value of μ such that $\|\epsilon^{k+1}\|_{H^1} < \|\epsilon^k\|_{H^1}$, concluding that the iterative procedure converges. You may make use of the stability bound from Lax-Milgram, i.e. the solution u to a variational problem satisfies

$$\|u\|_{H^1} \leq \frac{1}{\gamma}\|F\|_{(H^1)^*}, \tag{6}$$

where γ is the coercivity constant of the bilinear form and F is the linear form appearing on the right hand side. (8 marks)

(Total: 20 marks)

5. (a) Let V and Q be Hilbert spaces. Let $b : V \times Q \to \mathbb{R}$ be a bilinear form. We define the operator $B : V \to Q'$ as follows. For each $v \in V$, Bv is an element of Q', defined by

$$(Bv)[p] = b(v, p), \, \forall p \in Q. \tag{7}$$

For an operator $T : X \to Y'$, we define the transpose operator $X^* : Y \to X'$ as

$$(T^* y)[x] = (Tx)[y], \quad \forall x \in X, \, y \in Y. \tag{8}$$

Use these definitions to derive a formula for B^*. (6 marks)

(b) Assuming the inf-sup condition

$$\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta, \tag{9}$$

for some $\beta > 0$, show that B^* is injective. (7 marks)

(c) Let

$$b(u, p) = \int_\Omega p \nabla \cdot u \, \mathrm{d}\, x, \tag{10}$$

for some chosen problem domain Ω such that b satisfies the inf-sup condition for some given finite element spaces V_h and Q_h. We define the "weak gradient" operator $\tilde{\nabla} : Q_h \to V_h$ such that

$$\int_\Omega w \cdot \tilde{\nabla} p \, \mathrm{d}\, x = \int_\Omega p \nabla \cdot u \, \mathrm{d}\, x. \tag{11}$$

What does the inf-sup condition imply about the operator $\tilde{\nabla}$? (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH97095

MATH97095 (Solutions)

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1. (a) Label the triangle vertices v_1, v_2, v_3, with edges Π_1 joining v_1 and v_2, Π_2 joining v_2 and v_3, and Π_3 joining v_3 and v_1. Let L_i be the non-degenerate affine function that vanishes on Π_i, for $i = 1, 2, 3$. Now let v be the quadratic polynomial that vanishes on all of the vertices and edge centres. If we can show that $v \equiv 0$, then we can conclude that $\mathcal{N}$ determines P. Consider v restricted to Π_1. Since v vanishes at 3 points on Π_1 (two vertices and an edge centre), then by the fundamental theorem of algebra, v vanishes on the entire of Π_1. Hence, $v(x) = L_1(x)Q_1(x)$ where Q_1 is a polynomial of degree at most 1. Similarly, v vanishes everywhere on Π_2, so Q_1 must vanish everywhere on Π_2, except possibly where Π_2 intersects Π_1, since $L_1(x)$ is zero there. However, Q_1 is continuous so must vanish everywhere on Π_2. Hence, $Q_1 = L_2 c$, where c is a constant, and so $v = L_1 L_2 c$. Neither L_1 nor L_2 vanish on the edge centre of Π_3, but v vanishes there, so c must be zero, and so $v \equiv 0$ as required. seen ⇓ 7, A

(b) To each vertex we associate the nodal variable corresponding to point evaluation at that vertex. To each edge we associate the nodal variable corresponding to point evaluation at the centre of that edge. sim. seen ⇓

To check it is C^0, we need to check that we can (a) recover the value of the function at each vertex using nodal variables from that vertex, and (b) recover the value of the function at each edge using nodal variables from the closure of that edge. Showing (a) is immediate. For (b), on the closure of each edge we have two vertex values and an edge value. That is enough to recover the value of the quadratic function restricted to the edge. 7, A

(c) To give a counter example, consider the unit square subdivided into two triangles by the diagonal line $x = y$. We consider the finite element function equal to $x - y$ in the top-left triangle, and equal to zero in the bottom right. This is a piecewise polynomial of degree at most 2 (even though it is only degree 1). The derivative is equal to $(1, -1)$ in the top-left and $(0, 0)$ in the bottom right, and hence it is discontinuous. Hence, the function is only in C^0 and not in C^1, as required. unseen ⇓ 6, A

2. (a) Multiplication by test function, and integration by parts gives sim. seen ⇓

$$\int_{\Omega} \epsilon uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x - \int_{\partial\Omega} v \frac{\partial u}{\partial n} \,\mathrm{d}\,S = \int_{\Omega} v \exp(xy) \,\mathrm{d}\,x. \quad (1)$$

Then, application of the boundary condition leads to the variational problem: find $u \in V$ such that

$$a(u, v) := \int_{\Omega} \epsilon uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x = F(v) := \int_{\Omega} v \exp(xy) \,\mathrm{d}\,x, \quad \forall v \in V, \quad (2)$$

where $V \subset H^1$ is a continuous finite element space. 6, A

(b) The Schwarz inequality gives sim. seen ⇓

$$\begin{aligned} a(u, v) &\leq \epsilon \|u\|_{L^2} \|v\|_{L^2} + |u|_{H^1} |v|_{H^1}, & (3) \\ &\leq \|u\|_{L^2} \|v\|_{L^2} + |u|_{H^1} |v|_{H^1}, & (4) \\ &\leq 2 \|u\|_{H^1} \|v\|_{H^1}, & (5) \end{aligned}$$

so the continuity constant is bounded above by 2 (it is actually 1 from Young's inequality), but we don't require sharper estimates here.

For coercivity,

$$u(v, v) = \epsilon \|v\|_{L^2}^2 + |v|_{H^1}^2 \geq \epsilon \|v\|_{H^1}^2, \quad (6)$$

so the continuity constant is ϵ. 6, A

(c) Combining Ceá's lemma and the interpolation error estimate, we get unseen ⇓

$$\|u - u_h\|_{H^1} \leq \frac{2}{\epsilon} h \|u\|_{H^1}. \quad (7)$$

As $\epsilon \to 0$, the error estimate becomes unbounded. 8, B

3. (a) We work with the geometric decomposition, assigning the two tangential components on each edge to that edge. Then, we require that neighbouring cells have the same nodal variable values from that edge (taking care of orientation). Since the tangential component of the function is a linear scalar function restricted to the edge, and the nodal variables give two values of that function on the edge, the tangential component is determined from those values and will be continuous across the edge.

unseen ⇓

6, C

(b) If we have $\phi \in P_2$, then ϕ is continuous across an edge. Therefore the tangential component of the derivative is the same on either side of the edge. Further, the derivative is a linear vector field, so $\nabla\phi \in V$.

unseen ⇓

8, D

(c) For $u \in V$, the weak curl is $Du \in L^2$ such that

sim. seen ⇓

$$Du|_K = \nabla^\perp \cdot u|_K, \tag{8}$$

for each triangle K in the mesh. To check that this is indeed the weak curl, take a C_0^∞ function (infinitely differentiable and all derivatives vanish on the boundary) ϕ, and compute

$$-\int_\Omega \phi Du \,\mathrm{d}\,x = -\sum_K \int_K \phi Du \,\mathrm{d}\,x, \tag{9}$$

$$= \sum_K \int_K \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \sum_K \int_{\partial K} \phi u \cdot n_K^\perp \,\mathrm{d}\,S, \tag{10}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \sum_K \int_{\partial K} \phi(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp) \,\mathrm{d}\,S, \tag{11}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \int_{\partial\Omega} \underbrace{\phi = 0}(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp) \,\mathrm{d}\,S, \tag{12}$$

$$-\int_\Gamma \phi \underbrace{(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp)}_{=0} \,\mathrm{d}\,S, \tag{13}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x, \tag{14}$$

where n_K is the outward pointing normal to each triangle K, and the facet integrals vanish due to continuity of the tangential component. This matches the definition of the weak curl.

6, C

4. (a) We have $\boxed{\text{unseen} \Downarrow}$

$$\int_\Omega u^{k+1}v + \nabla u^{k+1}\cdot\nabla v \,\mathrm{d}\,x = \int_\Omega u^k v + \nabla u^k \cdot \nabla v \,\mathrm{d}\,x$$
$$+\mu\left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V, \tag{15}$$

Assuming that the limit exists, we take the limit, and get

$$\int_\Omega u^* v + \nabla u^*\cdot\nabla v \,\mathrm{d}\,x = \int_\Omega u^* v + \nabla u^* \cdot \nabla v \,\mathrm{d}\,x$$
$$+\mu\left(G[v] - \int_\Omega a(x)u^* v + b(x)\nabla u^*\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V, \tag{16}$$

which reduces to

$$\mu\left(G[v] - \int_\Omega a(x)u^* v + b(x)\nabla u^*\cdot\nabla v\,\mathrm{d}\,x\right) = 0, \quad \forall v\in V, \tag{17}$$

which implies u^* solves our equation for $\mu > 0$. $\boxed{6,\ \mathrm{B}}$

(b) First we use that $\boxed{\text{unseen} \Downarrow}$

$$G[v] = \int_\Omega a(x)uv + b(x)\nabla u\cdot\nabla v\,\mathrm{d}\,x, \quad \forall v\in V, \tag{18}$$

so

$$\begin{aligned} &\mu\left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right)\\ =\ &\mu\left(\int_\Omega a(x)uv + b(x)\nabla u\cdot\nabla v\,\mathrm{d}\,x - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right)\\ =\ &\mu\left(\int_\Omega a(x)(u-u^k)v + b(x)\nabla(u-u^k)\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V. \end{aligned} \tag{19}$$

Then we get

$$\int_\Omega \epsilon^{k+1}v + \nabla\epsilon^{k+1}\cdot\nabla v\,\mathrm{d}\,x = \mu\int_\Omega a(x)\epsilon^k v + b(x)\nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x, \quad \forall v\in V. \tag{20}$$

$\boxed{6,\ \mathrm{B}}$

(c) Equation (20) is of the form $a(u,v) = F(v)$, with $\boxed{\text{unseen} \Downarrow}$

$$a(u,v) = \int_\Omega uv + \nabla u\cdot\nabla v\,\mathrm{d}\,x, \quad F(v) = \mu\int_\Omega a(x)\epsilon^k v + b(x)\nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x. \tag{21}$$

In this case, $a(\cdot,\cdot)$ *is* the H^1 inner product, so the continuity and coercivity constants are both 1. We have

$$|F(v)| \le \mu\beta\|\epsilon^k\|_{L^2}\|v\|_{L^2} + \mu\beta|\epsilon^k|_{H^1}|v|_{L^2} \le 2\mu\beta\|\epsilon^k\|_{H^1}\|v\|_{H^1}, \tag{22}$$

(sharper estimates are possible but the question doesn't require them). Hence, from Lax-Milgram, we have

$$\|\epsilon^{k+1}\|_{H^1} < 2\mu\beta x\|\epsilon^k\|_{H^1}, \tag{23}$$

and so the norm of the error is guaranteed to reduce if $0 < \mu < 1/(2\beta)$. $\boxed{8,\ \mathrm{D}}$

5. (a)

$$(B^*p)[v] = (Bv)[p] = b(v,p). \tag{24}$$

unseen ⇓

6, M

(b) We have

$$\|B^*p\|_{V'} = \sup_{0\neq v\in V} \frac{(B^*p)[v]}{\|v\|_V} = \sup_{0\neq v\in V} \frac{b(v,p)}{\|v\|_V} \geq \beta\|p\|_Q, \tag{25}$$

seen ⇓

by the inf-sup condition. Assume that there exists p, q such that $B^*p = B^*q$. Linearity means that $B^*(p-q) = 0$. Then,

$$0 = \|B^*(p-q)\|_{V'} \geq \beta\|p-q\|_Q \implies p = q. \tag{26}$$

Hence B^* maps each element of Q to a different element of V', i.e. it is injective.

7, M

(c) $\tilde{\nabla}$ corresponds to the operator B^*, followed by the Riesz map back into V. Since B^* is injective, and the Riesz map is invertible, therefore $\tilde{\nabla}$ is also injective.

7, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Finite Elements: Numerical Analysis and Implementation_MATH60022 MATH97017 MATH70022	1	Almost all students made a very good effort of this question, with marks just being lost for the final part, with some students not providing a proof (best way to do this is with a counterexample).
Finite Elements: Numerical Analysis and Implementation_MATH60022 MATH97017 MATH70022	2	Generally, students made a very good effort of this question. The main errors were getting the wrong sign of the inequality in the coercivity definition.
Finite Elements: Numerical Analysis and Implementation_MATH60022 MATH97017 MATH70022	3	This was the question that students found the hardest on the exam, with only about half the students getting above half marks. Most marks were lost due to mixing up normals and tangents, or by not properly treating the fact that the functions are vector valued.
Finite Elements: Numerical Analysis and Implementation_MATH60022 MATH97017 MATH70022	4	There were many completely correct answers to this question, with students just not submitted answers to later parts in cases where marks were lost.
Finite Elements: Numerical Analysis and Implementation_MATH60022 MATH97017 MATH70022	5	There was a variety of success with this question. The main challenge was formulating answers to parts b and c in logical steps.