MVC : Sheet 5 Hints, tips, answers

1/ Need to show $0 = \lim_{x \to 0+} q = \lim_{x \to 0+} q'$ etc.

Can use that product of 2 smooth fns is smooth
and the product of 2 exponential fns is always +ve
To change interval consider mapping between $[x_1, x_2]$ & $[0,1]$.

2/ We find $I'(0) = \int_0^1 12x \sin 2\pi x + 12\pi x^2 \cos 2\pi x \, dx \underset{\text{(by parts)}}{=} 0$
E-L equation leads to $y'' = 6x \Rightarrow y = x^3$
Then stat. val. of I is $\int_0^1 12x^4 + 9x^4 \, dx = 21/5$.

3/ E-L eqn leads to $y'' + y = 0$ Applying b.c.'s $\Rightarrow y = \sin x$

4/ We have $x_1 = \beta \cosh(\gamma/\beta)$ Then x_1 small $\Rightarrow \beta$ small $\Rightarrow \gamma$ small
Similarly from $x_2 = \beta \cosh((y_2 - \gamma)/\beta)$ we need $y_2 - \gamma$ small
– but y_2 is large so not possible

5/ E-L eqn leads to $\theta = \int c_1 / (r(r^6 - c_1^2)^{1/2}) \, dr + \text{const.}$
Try subst. $r^3 = c_1 \sec u$. Find $\theta = u/3 + \text{const.}$

6/ Recall that in spherical polars $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta \, (d\varphi)^2$
On sphere of radius 1 we have $r = 1$ & $dr = 0$.
E-L eqn is $\partial f / \partial \varphi' = \text{const.}$
$\Rightarrow \varphi' = K \operatorname{cosec}\theta / (\sin^2\theta - K^2)^{1/2}$
Use subst. $u = \cot\theta$, $du = -\operatorname{cosec}^2\theta \, d\theta$

7/ Let $v(x) = f(x) - \lambda g(x)$
Show that $\int_{x_1}^{x_2} g(x) v(x) \, dx = 0$ and $\therefore \int_{x_1}^{x_2} f(x) v(x) \, dx = 0$

However $\int_{x_1}^{x_2} f(x) v(x) \, dx = \cdots = \int_{x_1}^{x_2} (v(x))^2 \, dx$
$\Rightarrow v(x) \equiv 0$.

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8/ Integrand indept of x Apply E-L to $y'^2+\lambda y$
$\Rightarrow \lambda y - y'^2 = \text{const}$. Integrate to get $x+C = \left(\frac{2}{\lambda}\right)(\lambda y+k)^{1/2}$
b.c.'s $\Rightarrow y = (\lambda x/4)(x+1)$
Then find $\lambda = 24/5$ from integral constraint
Then $I = \int_0^1 y'^2\,dx = \left(\frac{36}{25}\right)\int_0^1 (2x+1)^2\,dx = 156/25$

9/ E-L leads to $x^2y''+2xy'-2y=0$ Cauchy-Euler type ODE
$\Rightarrow \ldots \Rightarrow y = Bx + A/x^2$.
End condns $\Rightarrow A = -4/7,\ B = 4/7$.
With constraint ODE is $x^2y''+2xy'-2y = \lambda/2x$
Soln $y = Bx + A/x^2 - \lambda/4x$
End condns $\Rightarrow A = -4/7 + 3\lambda/14,\ B = 4/7 + \lambda/28$
Subst into constraint to get $\lambda = 12$ ($\Rightarrow A=2, B=1$)

10/ E-L leads to $y''+m^2y = -(\lambda/2)\cos nx$
If $m \neq n$ then PS is $-(\lambda/2)\cos nx/(m^2-n^2)$
Proceed by applying end condns & constraint
If $m=n$ then PS is modified to $-(\lambda/4m)\,x\sin mx$
Then apply end condns & integral constraint
Eventually find $\lambda = -4m^2$.

11/ Write out equation in Cartesians. Helpful to let g (say) be equal to $1+f_x^2+f_y^2$.
If $f = ax+by+c$ eqn is satisfied trivially
For the Scherk surface $f_x = -\tan x$, $f_y = \tan y$, $f_{xy}=0$
& $f_{xx} = -\sec^2 x$, $f_{yy} = \sec^2 y$