Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2023-2024

Partial Differential Equations in Action **MATH50008**

Solutions to Problem Sheet 4

1. In this problem the algebra is a bit lengthy but rather straightforward. Considering a change of variables $\xi \equiv \xi(x,t)$ and $\eta \equiv \eta(x,t)$, the differential operators with respect to x and y take the following form (by the chain rule)

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}$$

where we have used the shorthand notation $\xi_x = \frac{\partial \xi}{\partial x}$, $\eta_x = \frac{\partial \eta}{\partial x}$ etc. for simplicity. Then, if one write $u(x,t) = v(\xi,\eta)$, the second derivative of u with respect to x becomes

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left(\xi_x \frac{\partial v}{\partial \xi} + \eta_x \frac{\partial v}{\partial \eta}\right) \\ &= \xi_x^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_x\eta_x \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2 v}{\partial \eta^2} + \xi_x \frac{\partial \xi_x}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_x \frac{\partial \xi_x}{\partial \eta}\frac{\partial v}{\partial \xi} + \xi_x \frac{\partial \eta_x}{\partial \xi}\frac{\partial v}{\partial \eta} + \eta_x \frac{\partial \eta_x}{\partial \eta}\frac{\partial v}{\partial \eta}\end{aligned}$$

We obtain similar expressions for the other two second-order derivatives:

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \xi_x\xi_y \frac{\partial^2 v}{\partial \xi^2} + (\xi_x\eta_y + \xi_y\eta_x)\frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x\eta_y \frac{\partial^2 v}{\partial \eta^2} + \xi_x \frac{\partial \xi_y}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_x \frac{\partial \xi_y}{\partial \eta}\frac{\partial v}{\partial \eta} + \xi_x \frac{\partial \eta_y}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_x \frac{\partial \eta_y}{\partial \eta}\frac{\partial v}{\partial \eta} \\ \frac{\partial^2 u}{\partial y^2} &= \xi_y^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_y\eta_y \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_y^2 \frac{\partial^2 v}{\partial \eta^2} + \xi_y \frac{\partial \xi_y}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_y \frac{\partial \xi_y}{\partial \eta}\frac{\partial v}{\partial \xi} + \xi_y \frac{\partial \eta_y}{\partial \xi}\frac{\partial v}{\partial \eta} + \eta_y \frac{\partial \eta_y}{\partial \eta}\frac{\partial v}{\partial \eta}\end{aligned}$$

As the nature of a linear second-order PDE is determined purely by the sign of $B^2 - 4AC$, we only need to look at the second-order derivatives terms and none of the first-order derivatives (i.e. none of the terms appearing in the first-order derivatives of u either!). The terms we are interested in are then only:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &: \quad \xi_x^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_x\eta_x \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2 v}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial x \partial y} &: \quad \xi_x\xi_y \frac{\partial^2 v}{\partial \xi^2} + (\xi_x\eta_y + \xi_y\eta_x)\frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x\eta_y \frac{\partial^2 v}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial y^2} &: \quad \xi_y^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_y\eta_y \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_y^2 \frac{\partial^2 v}{\partial \eta^2}\end{aligned}$$

The coefficient A', B' and C' of the transformed equations are therefore given by

$$\begin{aligned}A' &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B' &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C' &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\end{aligned}$$

The final step is to compute the quantity $D' = B'^2 - 4A'C'$; the algebra is a little messy but straightforward

$$
\begin{aligned}
D' &= (2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y)^2 - 4\left(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\right)\left(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right) \\
&= 4A^2\xi_x^2\eta_x^2 + 4C^2\xi_y^2\eta_y^2 + B^2(\xi_x\eta_y + \xi_y\eta_x)^2 \\
&\quad + 4AB\xi_x\eta_x(\xi_x\eta_y + \xi_y\eta_x) + 4BC\xi_y\eta_y(\xi_x\eta_y + \xi_y\eta_x) + 8AC\xi_x\xi_y\eta_x\eta_y \\
&\quad - 4A^2\xi_x^2\eta_x^2 - 4AB\xi_x^2\eta_x\eta_y - 4AC\xi_x^2\eta_y^2 \\
&\quad - 4AB\xi_x\xi_y\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y - 4BC\xi_x\xi_y\eta_y^2 \\
&\quad - 4AC\xi_y^2\eta_x^2 - 4BC\xi_y^2\eta_x\eta_y - 4C^2\xi_y^2\eta_y^2 \\
&= B^2\left[(\xi_x\eta_y + \xi_y\eta_x)^2 - 4\xi_x\xi_y\eta_x\eta_y\right] - 4AC\left[\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y\right] \\
&\quad + AB\left[4\xi_x\eta_x(\xi_x\eta_y + \xi_y\eta_x) - 4\xi_x^2\eta_x\eta_y - 4\xi_x\xi_y\eta_x^2\right] \\
&\quad + BC\left[4\xi_y\eta_y(\xi_x\eta_y + \xi_y\eta_x) - 4\xi_x\xi_y\eta_y^2 - 4\xi_y^2\eta_x\eta_y\right] \\
&\quad + A^2\left[4\xi_x^2\eta_x^2 - 4\xi_x^2\eta_x^2\right] + C^2\left[4\xi_y^2\eta_y^2 - 4\xi_y^2\eta_y^2\right] \\
&= \left(B^2 - 4AC\right)\left[\xi_x\eta_y - \xi_y\eta_x\right]^2
\end{aligned}
$$

where we recognize $\xi_x\eta_y - \xi_y\eta_x$ to be the jacobian of the transformation. We conclude that

$$
B'^2 - 4A'C' = \left(B^2 - 4AC\right)\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2
$$

Since the square of the jacobian is positive, D' has the same sign as $B^2 - 4AC$, showing that the nature of the PDE is not affected by the change of independent variables.

2. We use the method of separation of variables. Let $u(x,t) = X(x)T(t)$. Substituting in the heat equation, we obtain

$$
XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda^2
$$

as we want a solution periodic in x. We can integrate these ODEs to conclude that

$$
\begin{aligned}
X(x) &= A_1\sin\lambda x + A_2\cos\lambda x \\
T(t) &= B_1\exp\left(-\lambda^2 t\right)
\end{aligned}
$$

Using the boundary conditions will give us conditions on the integration constants and the separation constant λ. Here, we find that

$$
\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow X'(0) = X'(L) = 0 \Rightarrow A_1 = 0 \text{ and } \sin(\lambda L) = 0 \Rightarrow A_1 = 0 \text{ and } \lambda = \frac{n\pi}{L}
$$

So the general solution to this problem is of the form

$$
u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\exp\left(\frac{n^2\pi^2}{L^2}t\right)
$$

The coefficients a_i are set by the initial conditions; indeed, the half-range Fourier cosine series gives us

$$
f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)
$$

with

$$
a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \cdots
$$

(a) Here, we have

$$
f(x) = x^2
$$

which leads to

$$a_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = (-1)^n \left(\frac{2L}{n\pi}\right)^2, \quad n \geq 1$$
$$a_0 = \frac{2L^2}{3}$$

where have used two integration by parts. Finally, we write the solution to this problem

$$u(x,t) = \frac{L^2}{3} + \left(\frac{2L}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2}t\right)$$

(b) Here, we have

$$f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

which leads to

$$a_n = \frac{2}{L}\int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right) dx = \left(\frac{2}{n\pi}\right) \sin\left(\frac{n\pi}{2}\right), \quad n \geq 1$$
$$a_0 = 1$$

Finally, we write the solution to this problem

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2}{L^2}t\right)$$

Solution profiles are shown in Fig. 1.

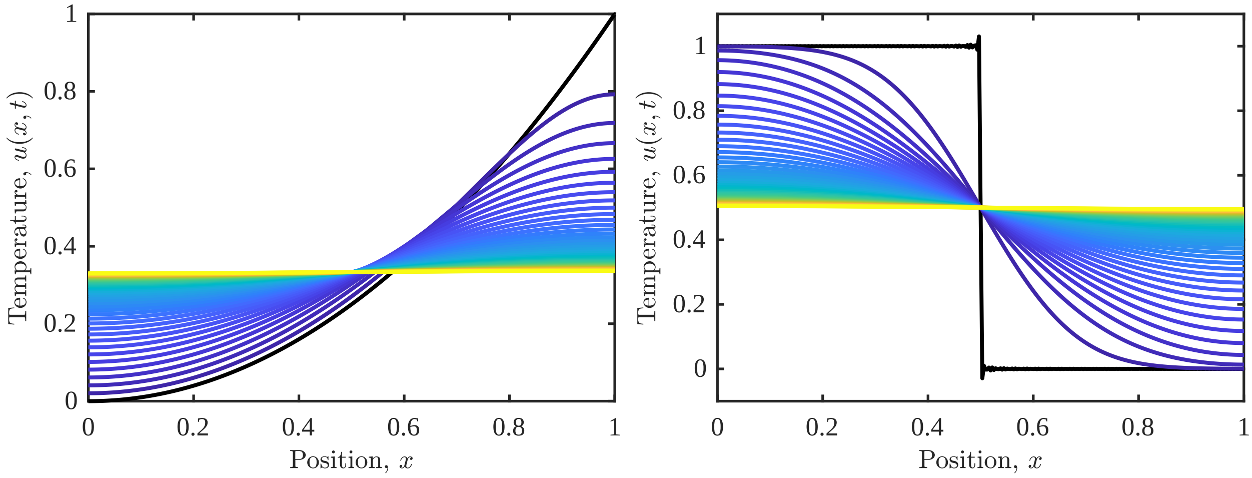

Figure 1: Temperature distributions as a function of time for (a) $f(x) = x^2$ (left panel) and (b) $f(x) = \mathbf{1}_{(0,L/2)}$ (right). The profiles were obtained by summing over the first 1000 Fourier modes.

3. In this problem, we want to solve the heat equation on an infinite domain subject to the initial temperature profile

$$u(x,0) = e^{-|x|}$$

As we are on an infinite domain, we will use integral transform methods (here, a Fourier transform method). Taking a Fourier transform in x, one gets

$$\frac{\partial \widehat{u}}{\partial t} = -\omega^2 \kappa \widehat{u}$$

which we can integrate in time to obtain

$$\widehat{u} = A(\omega)e^{-\omega^2 \kappa t}$$

The initial conditions will help us determine $A(\omega)$. For $t = 0$, we have

$$\widehat{u} = A(\omega) = \mathcal{F}\left\{e^{-|x|}\right\}$$

We can evaluate this Fourier transform as follows

$$\begin{aligned}
\mathcal{F}\left\{e^{-|x|}\right\} &= \int_{-\infty}^{+\infty} \exp(-|x|)e^{-i\omega x}dx \\
&= \int_{-\infty}^{+\infty} \exp(-|x| - i\omega x)dx \\
&= \int_{-\infty}^{0} \exp((1 - i\omega)x)dx + \int_{0}^{+\infty} \exp(-(1 + i\omega)x)dx \\
&= \frac{1}{1 - i\omega} + \frac{1}{1 + i\omega} = \frac{2}{1 + \omega^2}
\end{aligned}$$

Finally, to find the solution in real space, we need to apply the inversion formula and we obtain

$$\begin{aligned}
u(x,t) = \mathcal{F}^{-1}\left\{\widehat{u}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2}{1+\omega^2}e^{-\omega^2\kappa t}e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} \frac{2}{1+\omega^2}e^{-\omega^2\kappa t}e^{i\omega x}d\omega + \int_{0}^{+\infty} \frac{2}{1+\omega^2}e^{-\omega^2\kappa t}e^{i\omega x}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{+\infty} \frac{2}{1+\omega^2}e^{-\omega^2\kappa t}(e^{i\omega x} + e^{-i\omega x})d\omega \\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos(\omega x)}{1+\omega^2}e^{-\omega^2\kappa t}d\omega
\end{aligned}$$

4. In this question, we are interested in the diffusion of surface temperature fluctuations inside the ground.

   (a) We can substitute the solution we are given $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$ back into the one-dimensional diffusion equation,

   $$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$$

   This gives

   $$\frac{\partial^2 u}{\partial x^2} = -A\mu^2\sin(\omega t + \mu x)e^{-\lambda x} - 2A\lambda\mu\cos(\omega t + \mu x)e^{-\lambda x} + A\lambda^2\sin(\omega t + \mu x)e^{-\lambda x}$$

   and

   $$\frac{\partial u}{\partial t} = A\omega\cos(\omega t + \mu x)e^{-\lambda x}$$

   By comparing the coefficients in front of the cosine and sine functions, we can conclude that we necessarily have

   $$\lambda^2 = \mu^2 \quad \text{and} \quad -2\kappa\mu\lambda = \omega$$

   Now, ω being a frequency, we know that $\omega \geq 0$. Similarly, $\kappa \geq 0$ as it is the thermal diffusivity of soil. Finally, we expect the surface temperature fluctuations to be attenuated as they travel deeper in the ground and not diverge, so we have $\lambda \geq 0$. From all this, we can conclude that

   $$\lambda = -\mu = \sqrt{\frac{\omega}{2\kappa}}$$

(b) Without loss of generality, we consider that the surface is at $x = 0$ and that x increases as we go deeper in the ground. For the two sinusoids to be attenuated by a factor of $1/20$, we necessarily have

$$\exp(-\lambda_d x_d) = \frac{1}{20} = \exp(-\lambda_y x_y)$$

where the subscript d signifies the daily fluctuations and the subscript y the yearly ones. We can conclude that the ratio of the depths is given by

$$\frac{x_d}{x_y} = \frac{\lambda_y}{\lambda_d} = \sqrt{\frac{\omega_y}{\omega_d}} = \sqrt{\frac{1}{365}}$$

So the yearly surface temperature fluctuations are travelling deeper than the daily fluctuations.

(c) As we have shown that $x_y > x_d$. At the greater depth x_y, only the yearly variations are important. The question here asks at what time of the year is the soil coldest at x_y if the coldest day on the surface was found to be February 1st. This is thus a question of finding the difference in phase between surface temperature and the temperature at depth x_y. This is given by $\mu_y x_y$; but we know that $\mu_y x_y = -\lambda_y x_y$. So in particular, considering the definition of x_y, we have

$$e^{-\lambda_y x_y} = \frac{1}{20} \Rightarrow \lambda_y x_y = \ln 20$$

This means that the temperature at the depth x_y is $\ln(20)/(2\pi)$ of a year behind the surface temperature. If the coldest day on the surface is measured on February 1, then we conclude that the coldest day at depth x_y will be

$$1 \text{ February} + \left(\frac{\ln 20}{2\pi} \times 365\right) \text{ days} \approx 23 \text{ July}$$

5. The equation governing the temperature distribution in the cube is the three-dimensional diffusion equation, that is

$$\frac{\partial u}{\partial t} - \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

Substituting in this equation the solution that is provided

$$u(x, y, z, t) = A\cos\frac{\pi x}{a}\sin\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

we find that

$$-\frac{2\kappa\pi^2}{a^2}u - \kappa\left(-\frac{\pi^2}{a^2} + 0 - \frac{\pi^2}{a^2}\right)u = 0$$

which means that the diffusion equation is satisfied and we have indeed been presented with a solution.

Heat will flow across a face if at any point on it the normal temperature gradient $\partial u/\partial n$ is not equal to zero; where we define $\mathbf{n}$ to be the local outward normal vector to the face.

- ☐ For the faces $x = \pm a$, we write the normal temperature gradient as

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial x} = \pm\left(-\frac{\pi}{a}\right)A\sin\frac{\pi x}{a}\sin\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

So we find that while there may be some heat flow in the x-direction inside the cube, at the surface of the faces, we have $x = \pm a$ and

$$\frac{\partial u}{\partial n} = 0$$

☐ For the faces $y = \pm a$, we write the normal temperature gradient as

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial y} = 0, \quad \text{for all } x \text{ and } z$$

So we conclude that there is no heat flow in the y-direction, not inside the cube or on any of the two faces.

☐ For the faces $z = \pm a$, we write the normal temperature gradient as

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial z} = \pm\left(\frac{\pi}{a}\right) A\cos\frac{\pi x}{a}\cos\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right) \neq 0 \quad \text{for general } x.$$

So we conclude that heat will flow across the faces $z = \pm a$.
The point $(x, y, z) = (3a/4, a/4, a)$ lies on the face $z = a$. Now, at time $t = a^2/(\kappa\pi^2)$, we find

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \left(\frac{\pi}{a}\right) A\cos\frac{3\pi}{4}\cos\pi\exp\left(-\frac{2\kappa\pi^2}{a^2}\frac{a^2}{\kappa\pi^2}\right)$$

which leads to the following heat flux across the surface

$$-k\frac{\partial u}{\partial n} = -\frac{kA\pi e^{-2}}{\sqrt{2}a}$$

and we conclude that the heat flux into the cube is given by $kA\pi e^{-2}/(\sqrt{2}a)$.
As a side note: the thermal conductivity k and the thermal diffusivity κ are related by $\kappa = k/(\rho c)$ where ρ is the density of the material and c is the specific heat capacity.

6. In this question, we are interested in studying the diffusion of a chemical substance inside a biological tissue.

   (a) *Formulating a 1D diffusion problem satisfied by the concentration in chemicals in the slab* $u(z, t)$ — In this problem, we have placed a slab of biological material of thickness L on top of a glass plate and covered it by a solution containing a chemical specie of interest at concentration C_0. Over time, the chemical specie will diffuse inside the biological material. At the interface between the slab and the solution, the concentration will be maintained constant and equal to C_0 (Dirichlet boundary condition), this is valid as long as the volume of solution we place above the biological material is large enough. Glass being impermeable, we need to impose a no flux boundary condition at the bottom of the slab (Neumann boundary condition). As the biological material does not contain any of the chemical at first, the concentration throughout at $t = 0$ will be equal to zero. The PDE problem we are trying to solve is then:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial z^2}, \quad 0 < z < L,\ t > 0\\
&\frac{\partial u}{\partial z}(0, t) = 0, \quad t > 0\\
&u(L, t) = C_0, \quad t > 0\\
&u(z, 0) = 0, \quad 0 \leq z < L
\end{aligned}$$

   (b) *Steady-state solution* $U(z)$ *to this problem* — In steady-state, the time derivative disappears and $U(z)$ is then solution to

$$\frac{d^2U}{dz^2} = 0$$

   whose general solution is $U(z) = Az + B$. Boundary conditions impose

$$U'(0) = 0 \Rightarrow A = 0$$
$$U(L) = C_0 \Rightarrow B = C_0$$

   So the steady-state solution is $U(z) = C_0$.

(c) *General solution for the concentration* $u(z,t)$ — The boundary conditions of the original problem are not homogeneous, so technically, we do not know how to solve this problem easily. However, if we define $v(z,t) = u(z,t) - U(z)$, using the subtraction principle, we easy realize that $v(z,t)$ is a solution to the following problem

$$\begin{aligned}
&\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial z^2}, \quad 0 < z < L,\ t > 0 \\
&\frac{\partial v}{\partial z}(0,t) = 0 \\
&v(L,t) = 0 \\
&v(z,0) = -C_0
\end{aligned}$$

which we do know how to solve!

In particular, we can use the method of separation of variables and seek separated solutions of the form

$$v(z,t) = Z(z)T(t)$$

Substituting this in the diffusion equation, we find

$$ZT' = DZ''T \Rightarrow \frac{Z''}{Z} = \frac{T'}{DT} = K$$

where $K < 0$ is the only possibility as we need our solution v to decay over time (so that our solution u converges to the steady-state solution eventually). One could also check all the cases and show that $K < 0$ is the only case leading to non-trivial solutions. We denote $K = -\lambda,\ \lambda > 0$. Physically, this condition is also equivalent to stating that the concentration in the biological material cannot explode! First, we obtain the following general solution

$$Z(z) = \alpha\cos\left(\sqrt{\lambda}z\right) + \beta\sin\left(\sqrt{\lambda}z\right)$$

Boundary conditions impose that

$$Z'(0) = 0 \Rightarrow \beta = 0$$

and

$$Z(L) = 0 \Rightarrow \sqrt{\lambda}L = \left(n - \frac{1}{2}\right)\pi$$

which leads to a solution of the form

$$Z_n(z) = \alpha_n\cos(\mu_n z)$$

where $\mu_n = (2n-1)\pi/(2L)$. Further, going back to the ODE for T, we obtain the following solution

$$T_n(t) = \beta_n\exp\left(-\mu_n^2 Dt\right)$$

A general solution to our reduced PDE problem is then given by

$$v(z,t) = \sum_{n=1}^{\infty} A_n\cos(\mu_n z)\exp\left(-\mu_n^2 Dt\right)$$

where $\mu_n = (2n-1)\pi/(2L)$. And we finally conclude that the general solution for the concentration $u(z,t)$ is given by

$$u(z,t) = C_0 + \sum_{n=1}^{\infty} A_n\cos\left[\frac{(2n-1)\pi z}{2L}\right]\exp\left[-\frac{(2n-1)^2\pi^2}{4L^2}Dt\right]$$

(d) *Calculating the coefficients* A_n — To compute the coefficients A_n, we use the initial conditions. The initial conditions for the reduced problem are given by $v(z,0) = -C_0$. At this point, it is useful to recall the orthogonality of the cosine functions, i.e.

$$\int_0^L \cos\left(\frac{(2n-1)\pi z}{2L}\right)\cos\left(\frac{(2m-1)\pi z}{2L}\right)dz = \begin{cases} L/2, & n = m \\ 0, & n \neq m \end{cases}$$

So we easily obtain the coefficients as

$$\begin{aligned} A_n &= \frac{\int_0^L v(z,0)\cos(\mu_n z)dz}{\int_0^L \cos^2(\mu_n z)dz} = -\frac{2C_0}{L}\int_0^L \cos(\mu_n z)dz \\ &= \frac{4(-1)^n C_0}{(2n-1)\pi} \end{aligned}$$

which leads to the following solution

$$u(z,t) = C_0\left\{1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left[\frac{(2n-1)\pi z}{2L}\right]\exp\left[-\frac{(2n-1)^2\pi^2}{4L^2}Dt\right]\right\}$$

Figure 2 shows the solution to this diffusion problem over time.

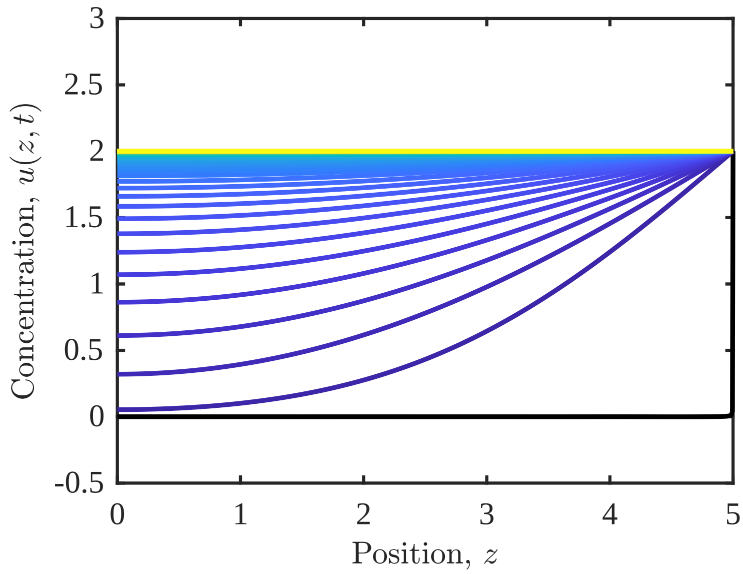

Figure 2: Concentration of chemicals in the slab over time with $D = 1$, $L = 5$ time increasing from blue to yellow starting from initial conditions $u(z,0) = 0$, $0 \leq z < L$.

(e) *Relaxation time* — In your experiment, it is important to make sure that you are close enough to the steady-state concentration profile. So you need to estimate how fast your solution will converge to its steady-state value. To do so, we can introduce the so-called relaxation time τ which is defined as

$$\frac{1}{\tau} = -\lim_{t\to\infty}\frac{1}{t}\ln|u(z,t) - U(z)|$$

Here, we need to think about the asymptotic behavior of our solution. We know that our solution will converge over time to the steady-state solution $U(z)$ but the question is how fast. For this, consider the solution we obtained above

$$u(z,t) = C_0 + \sum_{n=1}^{\infty} A_n \cos\left[\frac{(2n-1)\pi z}{2L}\right]\exp\left[-\frac{(2n-1)^2\pi^2 D}{4L^2}t\right]$$

where $A_n = 4C_0(-1)^n/(2n-1)\pi$ and the steady state solution is given by $U(z) = C_0$. So one can write

$$u(z,t) - U(z) = \sum_{n=1}^{\infty} A_n \cos\left[\frac{(2n-1)\pi z}{2L}\right] \exp\left[-\frac{(2n-1)^2\pi^2 D}{4L^2}t\right]$$

i.e.

$$u(z,t) - U(z) = A_1 \cos\left[\frac{\pi z}{2L}\right] \exp\left[-\frac{\pi^2 D}{4L^2}t\right] + \sum_{n=2}^{\infty} A_n \cos\left[\frac{(2n-1)\pi z}{2L}\right] \exp\left[-\frac{(2n-1)^2\pi^2 D}{4L^2}t\right]$$

As we are interested in the $t \to \infty$ limit, we are only interested in the asymptotic behavior of these terms. In particular, we can write that

$$\begin{aligned} u(z,t) - U(z) &= A_1 \cos\left[\frac{\pi z}{2L}\right] \exp\left[-\frac{\pi^2 D}{4L^2}t\right] + \mathcal{O}\left(e^{-4\pi^2 Dt/4L^2}\right) \\ &= A_1 \cos\left[\frac{\pi z}{2L}\right] \exp\left[-\frac{\pi^2 D}{4L^2}t\right] \left\{1 + \mathcal{O}\left(e^{-3\pi^2 Dt/4L^2}\right)\right\} \end{aligned}$$

So taking the logarithm, we conclude that

$$\ln|u(z,t) - U(z)| = \ln|A_1 \cos(\pi z/2L)| - \frac{\pi^2 D}{4L^2}t + \ln\left[1 + \mathcal{O}\left(e^{-3\pi^2 Dt/4L^2}\right)\right]$$

The last term tends to zero when $t \to \infty$; the first term is independent of t and so will vanish when dividing by t and taking the limit, so we are left with

$$\frac{1}{\tau} = -\lim_{t\to\infty} \frac{1}{t} \ln|u(z,t) - U(z)| = \frac{\pi^2 D}{4L^2}$$

How should you understand this relaxation time? — Well, you can see that the time-dependent solution converges to the steady-state solution $U(z)$. We have represented the solution as an infinite sum over modes which are products of decaying exponentials with a sinusoidal function; the convergence of the transient solution to the steady-state solution is thus controlled by the decay rates in the exponentials. The relaxation time, i.e. the time which controls how fast your solution converges to the steady-state solution, is given by the slowest of the decaying rates of these exponentials. The slowest mode to decay is your limiting factor for the convergence to the steady-state solution and this is given by the decay rate in the exponential associated to the first mode. Basically, this tells you that diffusion smoothes short wavelength fluctuations faster than the long wavelength ones.

7. In this problem, we are interested in the so-called *cable equation*.

   (a) *Ohm's law links voltage and current in the conductors* — In the outer conductor, Ohm's Law relates the difference of potential between the nodes at potential $V_o(x)$ and $V_o(x+dx)$, the resistance of this small slice of conductor and the current $I_o(x)$ crossing it; it is given by

$$V_o(x+dx) - V_o(x) = -I_o(x+dx)r_o dx$$

Similarly, for the inner conductor, we can write

$$V_i(x+dx) - V_i(x) = -I_i(x+dx)r_i dx$$

In the limit $dx \to 0$, we can then write

$$\boxed{\frac{\partial Vo}{\partial x} = -r_o I_o(x) \quad \text{and} \quad \frac{\partial Vi}{\partial x} = -r_i I_i(x)}$$

(b) *Kirchoff's law (aka conservation of the currents)* — We can know take care of expressing the current in the system using the idea of current conservation (i.e. Kirchhoff's law). Kirchhoff's law states that at any node the currents entering equal the current leaving the node. Here, we then write

$$I_i(x+dx) = I_i(x) + I_s dx \quad \text{and} \quad I_o(x+dx) = I_o(x) - I_s dx$$

As before, in the limit $dx \to 0$, we obtain

$$\boxed{I_s = \frac{\partial I_i}{\partial x} = -\frac{\partial I_o}{\partial x}}$$

(c) *Deriving the cable equation* — We define the transinsulator potential as $V = V_i - V_o$. As the insulator is modeled as a resistor and capacitor in parallel, the current in the insulator reads

$$I_s = -\frac{1}{r_s}V - C_s\frac{\partial V}{\partial t}$$

So by taking a derivative with respect to x, we obtain

$$I_s = -\frac{1}{r_i + r_o}\frac{\partial^2 V}{\partial x^2}$$

Using the fact that

$$I_s = -\frac{1}{r_s}V - C_s\frac{\partial V}{\partial t}$$

We obtain

$$\frac{1}{r_s}V + C_s\frac{\partial V}{\partial t} = \frac{1}{r_i + r_o}\frac{\partial^2 V}{\partial x^2}$$

and we conclude that

$$\boxed{\frac{\partial V}{\partial t} = D\frac{\partial^2 V}{\partial x^2} - \beta V}$$

where we have defined

$$D = \frac{1}{C_s(r_i + r_o)} \quad \text{and} \quad \beta = \frac{1}{r_s C_s}$$

This is the so-called cable equation.

(d) *What can be the source of the transmission problem?* — In deriving the equation governing the transinsulator potential, we can see that the capacitance of the insulating layer is the term which leads to a diffusion equation. A diffusion process smoothes fluctuations and so in particular, any oscillatory signal imposed in the inner conductor will be dampened because of this diffusion process. The capacitance is thus the source of the transmission problem.

(e) *Understanding the problem of long-range transmission* — To understand the problem of long-range transmission, we consider that an emitter is located in $x = 0$ and that a receiver is located far enough that we can consider the case of a semi-infinite interval; this leads to the following boundary value problem

$$\begin{aligned}
&\frac{\partial V}{\partial t} = D\frac{\partial^2 V}{\partial x^2} - \beta V \quad 0 < x < \infty \\
&V(0,t) = A\cos(\omega t) \\
&V(x,t) \to 0,\ x \to \infty
\end{aligned}$$

Here, we seek solutions which will have propagation and oscillation properties but also decay with distance to the source. Therefore, we seek solutions of the form

$$V(x,t) = Av(x)\cos(\omega t - kx)$$

i.e. a propagating wave with amplitude varying with x.

To save some space, let us define $\phi = \omega t - kx$ before we substitute this solution in the diffusion equation to get

$$-\omega v \sin\phi = D\left(\frac{\partial^2 v}{\partial x^2}\cos\phi + 2k\frac{\partial v}{\partial x}\sin\phi - k^2 v\cos\phi\right) - \beta v\cos\phi$$

Equating the coefficients of the $\cos\phi$ term, we find

$$\frac{\partial^2 v}{\partial x^2} - \left(k^2 + \frac{\beta}{D}\right)v = 0$$

The boundary conditions given above imply that $v(0) = 1$ and $v(\infty) = 0$. We conclude that

$$\boxed{v(x) = \exp\left[-\sqrt{k^2 + \frac{\beta}{D}}x\right]}$$

If now we equate the coefficients of the $\sin\phi$ term, we find

$$-\omega \exp\left[-\sqrt{k^2 + \frac{\beta}{D}}x\right] = -2Dk\sqrt{k^2 + \frac{\beta}{D}}\exp\left[-\sqrt{k^2 + \frac{\beta}{D}}x\right]$$

which implies that $v(x)$ is indeed a solution is k, ω and D are related by the following relation

$$\boxed{\omega = 2Dk\sqrt{k^2 + \frac{\beta}{D}}}$$

this is an example of what is called a *dispersion relation* (i.e. a relation between wave number k and frequency ω).

(f) *Solving the transmission problem in the large transinsulator resistivity limit* — In the limit where the transinsulator is large, $\beta \ll 1$. Let's take the limit where $\beta = 0$. In this case, we find that the transinsulator voltage and dispersion relation become

$$V(x,t) = Ae^{-kx}\cos(\omega t - kx) \quad \text{and} \quad \omega = 2Dk^2$$

The cable transmission rate is the quantity of signal (or information) transmitted per unit time. It has for dimensions the inverse of a time, we can then interpret the quantity ω as the transmission rate. Further, $1/k$ has the dimensions of a length. Writing that $1/k = L$, we find that the transmission rate is

$$\omega = 2DL^{-2} \tag{1}$$

Note that the transmission decays with the distance between emitter and receiver as expected. By definition, we have

$$D = \frac{1}{C_s(r_i + r_o)}$$

We note that there is only so much that can be done to reduce the resistance of the cable (which are copper for most of them). So we conclude that the important factor here the capacitance of the insulating layer and that if C_s is too low, the transmission rate will be low as well. A low capacitance insulating layer is a common problem in low quality cables. Thomson's analysis was instrumental and he was later asked to consult when it was decided to lay underwater cable from Britain to the USA in 1865. This was made possible by drastic improvements in the quality of the cable production. For these contributions, Thomson was made Lord Kelvin in 1866.

8. In this problem, we consider the following traffic flow model

$$\frac{\partial \rho}{\partial t} + v_m \left(1 - 2\frac{\rho}{\rho_m}\right)\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}$$

(a) First, we will use dimensional analysis to nondimensionalize this equation. Let us write the following change of variable

$$x = \tilde{x} x_c \quad t = \tilde{t} t_c \quad \rho = \tilde{\rho}\rho_m$$

where x_c and t_c are the characteristic length and time. Note that we have already taken that the characteristic density here is ρ_m. Substituting for these in the model above, we obtain

$$\frac{\rho_m}{t_c}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\rho_m}{x_c} v_m (1 - 2\tilde{\rho}) \frac{\partial \tilde{\rho}}{\partial \tilde{x}} = \nu \frac{\rho_m}{x_c^2}\frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2} \Rightarrow \frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \Pi_1 (1 - 2\tilde{\rho}) \frac{\partial \tilde{\rho}}{\partial \tilde{x}} = \Pi_2 \frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2}$$

where we have introduced two dimensionless groups $\Pi_1 = v_m t_c / x_c$ and $\Pi_2 = \nu t_c / x_c^2$. As we have two characteristic quantities to determine, we set

$$\Pi_1 = 1 \Rightarrow t_c = \frac{x_c}{v_m}$$
$$\Pi_2 = 1 \Rightarrow t_c = \frac{x_c^2}{\nu}$$

which means that $x_c = \nu / v_m$ and $t_c = \nu / v_m^2$. Under this transformation, our traffic flow model takes the form

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + (1 - 2\tilde{\rho}) \frac{\partial \tilde{\rho}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2}$$

(b) Here, we consider the canonical viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2}$$

Here, we use the Cole-Hopf transformation to reduce this problem to a linear diffusion equation as is done in the lecture notes. According to the Cole-Hopf transformation, we introduce the auxiliary function $\phi(x,t)$ such that

$$u = -\frac{2\kappa}{\phi}\frac{\partial \phi}{\partial x}$$

Let us define ψ such that

$$u = \frac{\partial \psi}{\partial x}, \quad \psi = -2\kappa \ln \phi$$

Substituting in the Burgers equation, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)^2 = \kappa \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

By integration with respect to x, we obtain

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \kappa \frac{\partial^2 \psi}{\partial x^2} + f(t)$$

where $f(t)$ is a time-dependent source term. Now as we consider here a system with no entrances or no exits on the road, there are no sources or sinks and so $f(t) = 0$. So we conclude that

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \kappa \frac{\partial^2 \psi}{\partial x^2}$$

Substituting for the definition of $\psi = -2\nu \ln \phi$, we write

$$\frac{-2\kappa}{\phi}\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\frac{-2\kappa}{\phi}\frac{\partial \phi}{\partial x}\right]^2 = \kappa \frac{\partial}{\partial x}\left[-2D\frac{1}{\phi}\frac{\partial \phi}{\partial x}\right]$$
$$= \frac{-2\kappa^2}{\phi}\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2}\left[\frac{2\kappa}{\phi}\frac{\partial \phi}{\partial x}\right]^2$$

which simplifies to

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2}$$

We conclude that the auxiliary function ϕ is solution to the linear diffusion equation.

(c) Let us consider a general initial condition $u(x,0) = u_0(x)$, then the initial conditions for the auxiliary function are given by

$$\phi_0(x) = \exp\left[-\frac{1}{2\kappa}\int_0^x u_0(y)dy\right]$$

by definition of the auxiliary function. As ϕ is solution of the diffusion equation on the real line, we know that we can use the heat kernel

$$K(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-x^2/(4\kappa t)}$$

to write that the solution to this problem is given by the following convolution

$$\phi(x,t) = \frac{1}{\sqrt{4\pi D t}}\int_{-\infty}^{+\infty} \phi_0(y) e^{-(x-y)^2/(4\kappa t)} dy$$

By differentiating with respect to x, we obtain

$$\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{4\pi D t}}\int_{-\infty}^{+\infty} \phi_0(y)\frac{(x-y)}{2\kappa t} e^{-(x-y)^2/(4\kappa t)} dy$$

By definition of the auxiliary function, we have

$$u(x,t) = -\frac{2\kappa}{\phi}\frac{\partial \phi}{\partial x} = \frac{\int_{-\infty}^{+\infty} dy\ \phi_0(y) e^{-(x-y)^2/4\kappa t}(x-y)/t}{\int_{-\infty}^{+\infty} dy\ \phi_0(y) e^{-(x-y)^2/4\kappa t}}$$

which can be rewritten as

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} dy\ e^{-F(x,y,t)/2\kappa}(x-y)/t}{\int_{-\infty}^{+\infty} dy\ e^{-F(x,y,t)/2\kappa}}$$

where we have defined

$$F(x,y,t) = \int_0^y u_0(y')dy' + \frac{(x-y)^2}{2t}$$

(d) Finally, we go back to our original traffic flow problem and consider that the initial conditions are given by

$$\rho(x,0) = \begin{cases} \rho_m, & \text{for } x \le 0 \\ 0, & \text{for } x > 0 \end{cases}$$

This translate to initial conditions

$$\tilde{\rho}(\tilde{x},0) = \begin{cases} 1, & \text{for } \tilde{x} \le 0 \\ 0, & \text{for } \tilde{x} > 0 \end{cases}$$

for the nondimensional problem. Using the transformation $u = 1 - 2\tilde{\rho}$, we can show that the equation governing the density of cars is equivalent to the canonical Burgers equation with $\kappa = 1$. Using this transformation, we can write that the initial conditions for $u(x,t)$ are given by

$$u_0(x) = \begin{cases} -1, & \text{for } x \leq 0 \\ 1, & \text{for } x > 0 \end{cases}$$

where we have dropped the tilde for simplicity. Thus, we get that

$$F(x,y,t) = |y| + \frac{(x-y)^2}{2t} \tag{2}$$

and

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} dy\, e^{-F(x,y,t)/2}(x-y)/t}{\int_{-\infty}^{+\infty} dy\, e^{-F(x,y,t)/2}}$$

If we denote

$$G(x,t) = \int_{-\infty}^{+\infty} \exp\left[-F(x,y,t)/2\right] dy$$

Then, we can rewrite the solution

$$u(x,t) = -\frac{2}{G}\frac{\partial G}{\partial x}$$

Let's calculate $G(x,t)$, we have

$$\begin{aligned} G(x,t) &= \int_{-\infty}^{+\infty} \exp\left[-\frac{|y|}{2} - \frac{(x-y)^2}{4t}\right] dy \\ &= \int_{-\infty}^{0} \exp\left[\frac{y}{2} - \frac{(x-y)^2}{4t}\right] dy + \int_{0}^{+\infty} \exp\left[-\frac{y}{2} - \frac{(x-y)^2}{4t}\right] dy \\ &= \int_{0}^{+\infty} \exp\left[-\frac{y}{2} - \frac{(x+y)^2}{4t}\right] dy + \int_{0}^{+\infty} \exp\left[-\frac{y}{2} - \frac{(x-y)^2}{4t}\right] dy \end{aligned}$$

With the change of variable $w = (x+y+t)/2\sqrt{t}$, we obtain that $w^2 = (t/4 + x/2) + y/2 + (x+y)^2/4t$ and $dy = 2\sqrt{t}dw$, which transform the first integral in

$$\int_{0}^{+\infty} \exp\left[-\frac{y}{2} - \frac{(x+y)^2}{4t}\right] dy = 2\sqrt{t}e^{t/4+x/2}\int_{\frac{t+x}{2\sqrt{t}}}^{\infty} e^{-w^2}dw = \sqrt{\pi t}e^{t/4+x/2}\,\mathrm{erfc}\left(\frac{t+x}{2\sqrt{t}}\right)$$

with $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = (2/\sqrt{\pi})\int_x^{\infty} e^{-t^2}dt$ the complementary error function. Similarly, we can use the change of variable $w = (-x+y+t)/2\sqrt{t}$ to obtain

$$\int_{0}^{+\infty} \exp\left[-\frac{y}{2} - \frac{(x-y)^2}{4t}\right] dy = 2\sqrt{t}e^{t/4-x/2}\int_{\frac{t-x}{2\sqrt{t}}}^{\infty} e^{-w^2}dw = \sqrt{\pi t}e^{t/4-x/2}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{t}}\right)$$

So we conclude that

$$G(x,t) = \sqrt{\pi t}e^{t/4}\left[e^{-x/2}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{t}}\right) + e^{x/2}\,\mathrm{erfc}\left(\frac{t+x}{2\sqrt{t}}\right)\right]$$

After some straightforward algebra, we obtain the full solution

$$u(x,t) = \frac{2\,\mathrm{erfc}\left((t-x)/2\sqrt{t}\right)}{\mathrm{erfc}\left((t-x)/2\sqrt{t}\right) + e^{x}\,\mathrm{erfc}\left((t+x)/2\sqrt{t}\right)} - 1$$

Remember that we are still dealing in dimensionless quantities here! To avoid confusion, let us reintroduce the tilde notation

$$\tilde{\rho}(\tilde{x},\tilde{t}) = 1 - \frac{\operatorname{erfc}\left((\tilde{t}-\tilde{x})/2\sqrt{\tilde{t}}\right)}{\operatorname{erfc}\left((\tilde{t}-\tilde{x})/2\sqrt{\tilde{t}}\right) + e^{\tilde{x}}\operatorname{erfc}\left((\tilde{t}+\tilde{x})/2\sqrt{\tilde{t}}\right)}$$

In particular, we have

$$\tilde{\rho} = \frac{\rho}{\rho_m}, \quad \tilde{x} = \frac{x v_m}{\nu}, \quad \tilde{t} = \frac{t v_m^2}{\nu}$$

which means that

$$\frac{\tilde{t}\pm\tilde{x}}{2\sqrt{\tilde{t}}} = \frac{\frac{tv_m^2}{\nu} \pm \frac{xv_m}{\nu}}{2\sqrt{tv_m^2/\nu}} = \frac{v_m}{\nu}\frac{\sqrt{\nu}}{v_m}\frac{v_m t \pm x}{2\sqrt{t}} = \frac{v_m t \pm x}{\sqrt{4\nu t}}$$

So after all this effort, we finally conclude that the solution to our problem is given by

$$\rho(x,t) = \rho_m\left[1 - \frac{\operatorname{erfc}\left((v_m t - x)/\sqrt{4\nu t}\right)}{\operatorname{erfc}\left((v_m t - x)/\sqrt{4\nu t}\right) + e^{xv_m/\nu}\operatorname{erfc}\left((v_m t + x)/\sqrt{4\nu t}\right)}\right]$$

The solution is shown on Fig. 3.

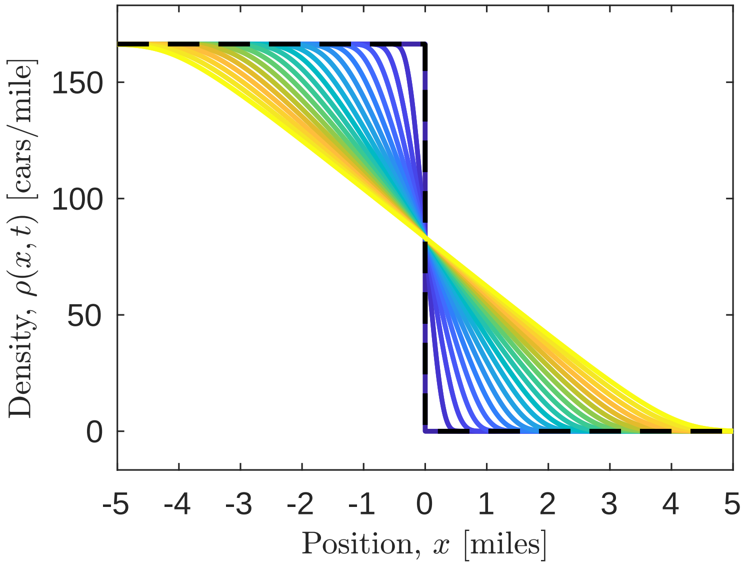

Figure 3: Solutions to the traffic flow problem with $\rho_m = 166.4$ cars/mile, $v_m = 36.8$ mph and $\nu = 1$, initial conditions are shown as black dashed line and time increases from blue to yellow.

9. Here, we consider a random walker which makes steps of size $h > 0$ every time interval of size $\tau > 0$. We assume that it starts from $x = 0$ and that the particle makes a step to the right with probability p_0 and to the left with probability $q_0 = 1 - p_0$ (where $p_0 \neq 1/2$), independently of the previous step.

   We call this a biased random walk as the symmetry of the walk is broken by the fact that $p_0 \neq 1/2$. Thus, this random walk models the tendency of the particle to move to the right (respectively, to the left) if $p_0 > 1/2$ (respectively, $p_0 < 1/2$). We denote $p(x,t)$ the probability of finding a particle at position $x = mh$ at time $t = \tau N$. If a particle is in position x at time $t+\tau$, then it was in position $x-h$ with probability p_0 and in position $x+h$ with probability q_0 at time t. So we write

$$p(x,t+\tau) = p_0 p(x-h,t) + q_0 p(x+h,t) \quad (\star)$$

as in the notes, the initial conditions are given by

$$p(0,0) = 1 \quad \text{and} \quad p(x,0) = 0, \quad \text{if } x \neq 0$$

Keeping x and t fixed, we want to explore the limit $h, \tau \to 0$. We Taylor expand as follows

$$\begin{aligned} p(x, t+\tau) &= p(x,t) + \frac{\partial p}{\partial t}(x,t)\tau + o(\tau) \\ p(x \pm h, t) &= p(x,t) \pm \frac{\partial p}{\partial x}(x,t)h + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)h + o(h^2) \end{aligned}$$

We can substitute this in $(\star)$ to get

$$\frac{\partial p}{\partial t}\tau + o(\tau) = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}h^2 + (q_0 - p_0)h\frac{\partial p}{\partial x} + o(h^2)$$

Dividing by τ, we obtain

$$\frac{\partial p}{\partial t} + o(1) = \frac{h^2}{2\tau}\frac{\partial^2 p}{\partial x^2} + (q_0 - p_0)\frac{h}{\tau}\frac{\partial p}{\partial x} + o(h^2/\tau)$$

We can see that a new term appeared: $(q_0 - p_0)\frac{h}{\tau}\frac{\partial p}{\partial x}$. Here, we proceed as in the notes and require that

$$\frac{h^2}{\tau} = 2D$$

to obtain a non trivial result. But what we saw in the notes is that if we keep p_0 and q_0 constant then

$$(q_0 - p_0)\frac{h}{\tau} \to \infty$$

which is in contradiction with the equation we just obtained! Let us write

$$(q_0 - p_0)\frac{h}{\tau} = \frac{(q_0 - p_0)}{h}\frac{h^2}{\tau}$$

so if we also require here that

$$\frac{(q_0 - p_0)}{h} \to \beta$$

with β finite, then we have

$$\frac{(q_0 - p_0)}{h}\frac{h^2}{\tau} \to 2D\beta \equiv v$$

and in the limit of $h, \tau \to 0$, we obtain

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + v\frac{\partial p}{\partial x}$$

How can we interpret this equation? We already know that the term Dp_{xx} models diffusion. Now looking at the dimensions of v, we realize that v is a velocity and so this is an advection term which quantifies the tendency of the limiting continuous motion to move in a preferred direction. If $v < 0$, then the particle will move preferentially to the right and if $v > 0$, the particle will move preferentially to the left. This is called a diffusion process with drift and the equation we obtained is an advection-diffusion equation.