M3S1/M4S1
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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

M3S1/M4S1 Statistical Theory I

1. (a) (i) What is the *sufficiency principle*?

(ii) What is its importance?

(iii) Explain what is meant by a *minimal sufficient statistic* for a family of distributions parameterised by an unknown parameter θ.

(b) For cases (i) and (ii) below find, giving your reasoning, a minimal sufficient statistic for positive θ.

(i) From a random sample $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ having the probability density function

$$f(x|\theta) \;=\; \begin{cases} \theta e^{\theta(\theta - x)} & (x > \theta), \\ 0 & (x \leq \theta). \end{cases}$$

(ii) In bio-assay, P(positive response at dosage z) $= P(X = 1 \,|\, z, \theta) = \dfrac{e^{\theta z}}{1 + e^{\theta z}}$

Here $X_1, X_2, \ldots, X_n$ are independent *Bernoulli* random variables with

$$P(X_k = x_k \,|\, z_k, \theta) \;=\; \frac{e^{\theta z_k x_k}}{1 + e^{\theta z_k x_k}} \quad (x_k \in \{0, 1\})$$

where $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ are known constants.

2. (a) What is the *monotone likelihood ratio criterion*?
What is its importance?

(b) (i) From the combined independent random samples $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ from *Poisson*(θ) and $\boldsymbol{y} = \{y_1, y_2, \ldots, y_n\}$ from *Poisson*($c\theta$), where $c > 0$ is a known constant, show that $t(\boldsymbol{x}, \boldsymbol{y}) = \sum_1^n (x_i + y_i)$ is sufficient for θ.

(ii) Find the most powerful size α test of $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$, where α is small. *Give your reasoning.*

(iii) Consider the size α test of $H_0^* : \theta = 1$ against $H_1 : \theta > 1$.
Obtain a normal approximation for the distribution of $T = t(\boldsymbol{X}, \boldsymbol{Y})$ for given θ and large n.
If ξ is such that $\alpha = P(T > \xi | \theta = 1)$, find an approximation for ξ.
Obtain an approximation to the power function $\beta(\theta)$.

3. (a) What is meant by a *pivotal quantity*?

(b) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from a distribution having probability density function

$$f(x|\theta) = \begin{cases} \dfrac{x}{\theta} \exp\left(-\dfrac{1}{2}\dfrac{x^2}{\theta}\right) & (x > 0), \\ 0 & (x \leq 0), \end{cases}$$

where θ is an unknown positive parameter.

(i) Obtain the efficient total score $U_\bullet(\theta)$.

(ii) From the form of $U_\bullet(\theta)$ write down

- the total Fisher information $I_\bullet(\theta)$,
- the maximum likelihood estimator $\widehat{\theta}$ of θ,
- $\operatorname{var}(\widehat{\theta})$.

(iii) Which theorem guarantees that $\operatorname{var}(\widehat{\theta})$ minimises the variance over all unbiased estimators of θ?

(iv) Show that $Z = \widehat{\theta}/\theta$ is a pivotal quantity having a *Gamma*$(n, 1)$ distribution.

(v) From (iv) construct a $100(1-\alpha)\%$ confidence interval for θ having equal tail probabilities for small α.

4. (a) State the Lehmann-Scheffé Theorem for finding a minimum variance unbiased estimator (MVUE).

(b) For a random sample $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ from the Delayed Exponential distribution having probability density function

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & (x > \theta), \\ 0 & (x \leq \theta), \end{cases}$$

by considering the pivotal quantity $Z = X - \theta$, or otherwise, find the distribution of $X_{\min}$.

Find an unbiased estimator of θ that is a function of $X_{\min}$ alone.

(c) (i) Find a non-zero function $h(t)$, for which $E\{h(T)\} = 0$, to show that the *Uniform*$(-\theta, \theta)$ family of distributions, where $\theta > 0$, is not complete.

(ii) Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a random sample from *Uniform*$(-\theta, \theta)$ $(\theta > 0)$, and let the prior probability for θ be *Pareto* with probability density function

$$\pi(\theta\,|\alpha, \beta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}\, H(\theta > \alpha)$$

where α and β are known positive constants, and $H(A) = 1$ if A is true, and 0 if A is false.

Show that the posterior distribution for θ is *Pareto*(α^*, β^*), where α^* and β^* are to be determined.

	M3S1/M4S1 EXAMINATION SOLUTIONS May-June 2014	M3S1 M4S1
Question 1		Marks & seen/unseen

Parts

a) i) The sufficiency principle is that: if statistic t is sufficient for the family of distributions parameterised by θ, then analysis of the data should be only through t. — Bookwork — 2

ii) Possible responses regarding its importance are:
- the reduction of a set of data to t,
- a simplified analysis for hypothesis testing etc ~~eg~~ criteria for most powerful tests of composite hypotheses, ~~using~~
- Identifying MVUE by Lehmann-Scheffé

2

iii) A minimal sufficient statistic is a function of every sufficient statistic, so any further reduction would not yield a sufficient statistic.
It is essentially unique through the equivalence relation $t(\underset{\sim}{x}) = t(\underset{\sim}{y})$ iff likelihood ratio $\ell(\theta;\underset{\sim}{x})/\ell(\theta;\underset{\sim}{y})$ does not depend on θ.

2

b) i) Unseen

$$\ell(\theta;\underset{\sim}{x}) = (\theta e^{\theta^2})^n e^{-n\theta\bar{x}} H(x_{min}>\theta)$$
$$= g(\theta,\underset{\sim}{x}) = g(\theta,\bar{x},x_{min})$$

so $\bar{x}$, x_{min} are jointly sufficient for θ by Neyman factorisation.

$$\frac{\ell(\theta;\underset{\sim}{x})}{\ell(\theta;\underset{\sim}{y})} = e^{-n\theta(\bar{x}-\bar{y})}\frac{H(x_{min}>\theta)}{H(y_{min}>\theta)}$$

This does not depend on θ when $\bar{x}=\bar{y}$ & $x_{min}=x_{max}$

7

ii)
$$\ell(\theta;\underset{\sim}{x},\underset{\sim}{z}) = e^{\theta\sum z_k x_k}/\prod(1+e^{\theta z_k})$$

so $t(\underset{\sim}{x}) = \sum z_k x_k$ is sufficient for θ.
Minimal sufficient because dimension 1 cannot be reduced further.

~~7~~

<u>Alt</u> loglik $L(\theta;\underset{\sim}{x},\underset{\sim}{z}) = \theta\sum z_k x_k - \sum \ln(1+e^{\theta z_k})$

$L(\theta;\underset{\sim}{x},\underset{\sim}{z}) - L(\theta;\underset{\sim}{y},\underset{\sim}{z}) = \theta\{t(\underset{\sim}{x})-t(\underset{\sim}{y})\}$ does not depend on θ if $t(\underset{\sim}{x})=t(\underset{\sim}{y})$.

7

Setter's initials	Checker's initials	Page number
RC		1

	M3S1/M4S1 EXAMINATION SOLUTIONS May-June 2014	M3S1 M4S1
Question 2		Marks & seen/unseen

Parts

a) The monotone likelihood ratio criterion holds if the likelihood ratio $\lambda(\underset{\sim}{x})$ is a non-increasing (or non-decreasing) function of $t(\underset{\sim}{x})$, a sufficient statistic for θ.
Its importance is that if the criterion is satisfied, the test is UMP. — Bookwork; 3

b) i) — Unseen

$$f_{\underset{\sim}{X},\underset{\sim}{Y}}(\underset{\sim}{x},\underset{\sim}{y}|\theta) = \ell(\theta;\underset{\sim}{x},\underset{\sim}{y},c) = \prod_i \left\{ \frac{\theta^{x_i} e^{-\theta}}{x_i!} \cdot \frac{(c\theta)^{y_i} e^{-c\theta}}{y_i!} \right\}$$
$$= \frac{c^{\Sigma y_i}}{\prod\{x_i! y_i!\}} \theta^{\Sigma(x_i+y_i)} e^{-(c+1)n\theta}$$

Let $t(\underset{\sim}{x},\underset{\sim}{y}) = \Sigma(x_i+y_i)$ & $w = (c+1)n$
then t is sufficient for θ by Neyman factorisation. — 3

ii) Loglikelihood $L(\theta;\underset{\sim}{x},\underset{\sim}{y}) = t \ln\theta - w\theta$

$$\frac{\partial L}{\partial \theta} = \frac{t}{\theta} - w, \quad \frac{\partial^2 L}{\partial \theta^2} = -\frac{t}{\theta^2} < 0 \quad \text{so } L\uparrow \text{ as } \theta\uparrow$$

So criterion is satisfied. (Note: mle $\hat{\theta} = \frac{t}{w}$) — 5

Note: $H_0: \theta=\theta_0<1$ v. $H_1: \theta=\theta_1>1$ is MP by Neyman-Pearson
Holds for all θ_0, θ_1,
so the MP test is to reject $H_0: \theta \le 1$ if t is too large. — 2

iii) Under $H_0^*: \theta=1$ v. $H_1: \theta>1$, $T = \Sigma(X_i+Y_i)$

$E(T) = w\theta$, $var(T) = w\theta$ (by independence)

$$Z = \frac{T-E(T)}{\sqrt{var(T)}} = \frac{T-w\theta}{\sqrt{w\theta}} \sim N(0,1) \text{ (for large } n)$$

$$P(T>\xi|\theta) = P\left(Z > \frac{\xi - w\theta}{\sqrt{w\theta}}\right) \sim 1-\Phi\left(\frac{\xi-w\theta}{\sqrt{w\theta}}\right)$$

Although ~~Since~~ T is integer valued, for large n we can approximate ξ.

If $\theta=1$, for given size α, $\alpha \approx 1-\Phi\left(\frac{\xi-w}{\sqrt{w}}\right)$

i.e. $\xi \approx w + \sqrt{w}\,\Phi^{-1}(1-\alpha)$

so $\beta(\theta) \approx 1-\Phi\left(\frac{\xi-w\theta}{\sqrt{w\theta}}\right)$. — 7

Setter's initials	Checker's initials	Page number
AC		2

Parts

a) A pivotal quantity $z(\underset{\sim}{x}, \theta)$ has a known sampling distribution that does not depend on θ — seen 2

b) i) unseen

$$\ln f = \ln x - \ln \theta - \frac{1}{2}\frac{x^2}{\theta}$$

$$\frac{\partial \ln f}{\partial \theta} = -\frac{1}{\theta} + \frac{1}{2}x^2\left(\frac{1}{-\theta^2}\right) \quad \text{so } U(\theta) = \frac{1}{\theta^2}\left(\frac{1}{2}X^2 - \theta\right)$$

$$U_\circ(\theta) = \frac{n}{\theta^2}\left(\frac{1}{2}\overline{X^2} - \theta\right)$$ 4

ii) $I_\circ(\theta) = \frac{n}{\theta^2}$ 2

MLE $\hat{\theta} = \frac{1}{2}\overline{x^2} = \frac{1}{2n}\Sigma x_i^2$ 2

$\text{var}(\hat{\theta}) = 1/I_\circ(\theta) = \frac{\theta^2}{n}$ 2

iii) Cramér-Rao Theorem 2

iv) Let $y = \frac{x^2}{2\theta}$ $\quad dy = \frac{x}{\theta}dx$

$$\int_0^x f(x_0|\theta)dx_0 = \int_0^x e^{-\frac{1}{2\theta}x_0^2} \cdot \frac{1}{\theta}x_0 dx_0$$

$$= \int_0^y e^{-y_0} dy_0 = 1 - e^{-y}$$

so $Y = \frac{X^2}{2\theta}$ is Exponential(1)

$Z = \sum_1^n Y_i^2 = \frac{1}{2\theta}\Sigma X_i^2 = \frac{\hat{\theta}}{\theta}$ is Gamma$(n, 1)$ 3

v) $\frac{1}{2}\alpha = P\left(\frac{\hat{\theta}}{\theta} > c_u\right) = P\left(\theta < \frac{\hat{\theta}}{c_u}\right)$

$\frac{1}{2}\alpha = P\left(\frac{\hat{\theta}}{\theta} < c_L\right) = P\left(\theta > \frac{\hat{\theta}}{c_L}\right)$

so the $100(1-\alpha)\%$ CI for θ is $\left(\frac{\hat{\theta}}{c_u}, \frac{\hat{\theta}}{c_L}\right)$ 3

Question 4		Marks & seen/unseen
Parts a)	If S is a complete sufficient statistic, then any function of S is a MVUE of its expectation.	bookwork 2
b)	Pivot $Z = X - \theta$ is Exponential(1) $P(Z_{min} > z) = P(\text{each } Z_i > z) = (e^{-z})^n = e^{-nz}$ $(z>0)$ so Z_{min} is Exponential(n) $E(Z_{min}) = \frac{1}{n}$ so $E\big(n(X_{min}-\theta)\big) = 1$ i.e. $E\left(X_{min} - \frac{1}{n}\right) = \theta$ so $X_{min} - \frac{1}{n}$ is an unbiased estimator of θ and is a function of n alone.	unseen 6
c) i)	$E(X) = 0$, $E(\bar{X}) = 0$ for example.	3
ii)	$\pi(\theta\|\alpha,\beta,\underset{\sim}{x}) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}} H(\theta>\alpha) \cdot \frac{1}{(2\theta)^n} H(-\theta < \text{each } x_i < \theta)$ $\propto \frac{1}{\theta^{\beta+n+1}} H(\theta>\alpha) H(\theta>\|x_{min}\|) H(\theta > \|x_{max}\|)$ $\propto \frac{\beta^* \alpha^{*\beta^*}}{\theta^{\beta^*}} H(\theta > \alpha^*)$ where $\alpha^* = \max\{\alpha, \|x_{min}\|, \|x_{max}\|\}$, $\beta^* = \beta + n$.	9