Quiz 1 – Solutions

1. *Consider the statistical model $N(\theta^2, 1)$ where θ is the unknown parameter and $\theta \in \Theta$. Is the model identifiable?*
   It depends because if $\Theta = \mathbb{R}$ then the model is not identifiable, while if $\Theta = [0, \infty)$ then it is identifiable.

2. *When estimating a parameter, is it better to use an unbiased estimator than a biased estimator?*
   As explained in class, it depends because an unbiased estimator might have a very large variance, while a biased one might have a very small MSE, which makes it more informative than the first one.

3. *Let $X_1, X_2, ...,$ be a sequence of i.i.d. random variables. Assume that the distribution of X_1 is $N(0, 1)$. We know that the sample mean converges in distribution to a random variable Y as $n \to \infty$, that is $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{d} Y$ as $n \to \infty$. What is the distribution of Y?*
   By the weak law of large numbers we know that $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mathbb{E}[X_1] = 0$ hence $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{d} 0$. So $Y = 0$ and so Y is clearly not a Normal, Poisson, Exponential, or Bernoulli random variable.

   Observe that $X_1 \sim Bernoulli(p)$ with $p = 1$ is also a possible solution, even if it is a degenerate case. Such a degenerate case is usually not considered in the definition of a Bernoulli random variable (otherwise any constant is a translated Bernoulli random variable). However, in some case they are. Further, observe that neither the Normal nor the Exponential nor the Poisson distributions can be possible solutions.

4. *Let $X = (X_1, ..., X_n)$ be a random vector with unknown parameter $\theta \in \Theta \subset \mathbb{R}$. Assume that mild regularity conditions hold and that the Cramer-Rao lower bound (CRLB) exists and is strictly positive. Is it possible to find an estimator T of θ based on X such that the variance of T is strictly lower than the CRLB?*
   Let T be a constant. Then its variance is 0 which is strictly smaller than the CRLB.

5. *Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ be two sequences of random variables with unknown parameters $\mu \in \mathbb{R}$ and $\theta \in \mathbb{R}$, respectively. Consider the sequence of unbiased estimators $T_1, T_2, ...$ for μ and the sequence of unbiased estimators $S_1, S_2, ...$ for θ. Assume that $Var(T_n - S_n) \to 0$ as $n \to \infty$. Can we conclude that $T_n - S_n$ converges in probability to $\mu - \theta$ as $n \to \infty$, for every $\mu, \theta \in \mathbb{R}$?*
   Observe that $\mathbb{E}[T_n - S_n] = \mathbb{E}[T_n] - \mathbb{E}[S_n] = \mu - \theta$ by linearity of the expectation. Thus, $T_n - S_n$ is an unbiased estimator for $\mu - \theta$ and so it is an asymptotically unbiased estimator. Since we know that $Var(T_n - S_n) \to 0$ as $n \to \infty$, by a result from the lectures we obtain that $T_n - S_n$ is a consistent estimator for $\mu - \theta$, hence $T_n - S_n \xrightarrow{p} \mu - \theta$, and this holds for every $\mu, \theta \in \mathbb{R}$.