Introduction to Quantum Mechanics – Problem sheet 4

1. Commutator as Poisson bracket

Check that commutator of two operators $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ fulfils all the properties to be a Poisson bracket (see week 1).

2. Commutator practice

Using the fundamental commutator $[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{I}$, calculate the following commutators.

(a) $[\hat{q}, \hat{p}^2]$

(b) $[\hat{q}, \hat{p}^n]$

(c) $\left[\hat{q}, e^{\hat{p}}\right]$

3. The principles of quantum mechanics - measurements

Consider a system with a three-dimensional complex Hilbert space and two observables $\hat{A}$ and $\hat{B}$ given by

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}. \tag{1}$$

(a) Calculate the eigenvalues and eigenvectors of $\hat{A}$ and $\hat{B}$.

(b) Assume that at a given time t_1 the system is in the state

$$\psi(t_1) = \frac{1}{\sqrt{35}} \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \tag{2}$$

(i) Calculate the expectation values of A and B in the state (2).

(ii) Assume that a measurement of the observable A is performed at time t_1, followed instantly by a measurement of B. Find the probability of obtaining 1 for A and 2 for B, and the probability of obtaining 2 for A and 1 for B.

(iii) The system is then reset to the original state (2) and the measurement is now performed in reverse. First the measurement of B is performed followed immediately by the measurement of A. Find the probability of obtaining 2 for B and 1 for A, and the probability of obtaining 1 for B and 2 for A.

4. Time evolution operator

The time-evolution operator for a system with time-independent Hamiltonian is given by $\hat{U}(t) = \exp\left[-i\hat{H}t/\hbar\right]$, where the operator exponential is defined via the Taylor expansion. That is,

$$\hat{U}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \hat{H}^n t^n. \tag{3}$$

(a) What is $\hat{U}(0)$?

(b) Show that if $|\psi\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E it holds that $\hat{U}(t)|\psi\rangle = e^{-iEt/\hbar}|\psi\rangle$.