Applied Complex Analysis - Lecture Fourteen

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February 2025

Trapezium rule(s) for unbounded contours

- Consider
$$I = \int_{-\infty}^{\infty} f(x)dx,$$
for some f analytic on $\mathbb{R}$, with appropriate decay of f such that $I < \infty$.
- We've seen techniques for evaluating these by hand - not always possible.
- For $h > 0$ we define the *unbounded* Trapezium rule $I_h \approx I$ as
$$I_h := h \sum_{j=-\infty}^{\infty} f(x_j),$$
where $x_j = jh$.
- We define the *truncated* Trapezium rule $I_h^{(N)} \approx I$ as
$$I_h^{(N)} := h \sum_{j=-N}^{N} f(x_j), \quad \text{for } N \in \mathbb{N}_0.$$

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Convergence theorem

Suppose $f(z)$ is analytic in the complex strip $|\mathrm{Im}(z)| < a$ for some $a > 0$. Suppose further that $f(z) \to 0$ uniformly as $|z| \to 0$ in the strip and

$$\int_{-\infty}^{\infty} |f(t + ia')| dt \leq M,$$

for all $a' \in (-a, a)$. Then I_h satisfies

$$|I - I_h| \leq \frac{2M}{e^{2\pi a/h} - 1}.$$

- This result is about the *unbounded* trapezium rule.
- This is called the *discretisation* error.

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The truncation error

Defined as, for $x_n = hn$

$$|I_h - I_h^{(N)}| = \left| h \sum_{n=-\infty}^{\infty} f(x_n) - h \sum_{n=-N}^{N} f(x_n) \right|$$

$$= \left| h \sum_{n=-\infty}^{-(N+1)} f(x_n) - h \sum_{n=N+1}^{\infty} f(x_n) \right|$$

- Practically, we care about

$$|I - I_h^{(N)}| \leq |I - I_h| + |I_h - I_h^{(N)}|$$

- Often, it is enough to bound by a constant multiplied by the first term.

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Bound on (half of the) truncation error

- Suppose that, for some $\alpha > 0$ independent of $y_0 > 0$, the function g satisfies the mild growth condition

$$g(y+\delta) - g(y) \geq \alpha\delta, \tag{1}$$

  for all $\delta > 0$ and $y > y_0$.
- Furthermore, suppose either that (i) the meshwidth h is independent of N, or (ii) the meshwidth $h \to 0$ as $N \to \infty$, but with a rate $1/N \ll h$.
- Then the positive terms in the truncation error satisfies:

$$h \sum_{n=N+1}^{\infty} e^{-g(hn)} = O(e^{-g(h(N+1))}), \quad N \to \infty.$$

- Proof

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Examples

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- $$\mathrm{erfc}(z) = \frac{2e^{-z^2}}{\pi} \int_0^{\infty} \frac{e^{-z^2 t^2}}{t^2+1} dt, \quad z \in \mathbb{R}.$$
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- But it's an important step in the analysis of the method.
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Be careful what you wish for!

Resdiue correction to trapezium rule

Recall the *exact* representation of the error:

$$I_h - I = -\sum_{\pm} \int_{-\infty \pm ia'}^{\infty \pm ia'} \frac{f(z)}{1 - e^{\mp 2\pi i x/h}} dz,$$

Now consider the following example:

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