IMPERIAL

MATH70058

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Manifolds

Date: Tuesday, April 29, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the set $X = \{x^2 + y^2 - z^2 = 0\} \subset \mathbb{R}^3$ with the subspace topology. First show any point p in $X - (0,0,0)$ admits a neighborhood in X which is homeomorphic to an open 2-disk. Is X a topological manifold? (6 marks)

(b) Let S^n be the standard n-sphere.

(i) Construct a submersion from $\mathbb{R}$ to S^2, or show it is impossible. (3 marks)

(ii) Construct a submersion from $\mathbb{R}^2$ to S^2, or show it is impossible. (4 marks)

(iii) Recall that S^2 is diffeomorphic to the complex projective plane. Construct a submersion from S^3 to S^2. Could it be non-surjective?. (4 marks)

(c) Give one example of an immersion which is not an embedding. (3 marks)

(Total: 20 marks)

2. (a) Show $F : (\mathbb{R}^2, x, y) \to (\mathbb{R}^2, u, v)$ given by

$$F(x,y) = (xe^y + y, xe^y - y).$$

is a diffeomorphism.

(4 marks)

(b) Compute the following for the above F.

(i) $F^*(3du - dv)$. (3 marks)

(ii) $F_*(y\dfrac{\partial}{\partial x})$. (4 marks)

(c) Consider the 3-sphere $S^3 := \{x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$.

(i) Consider the vector field

$$V := -3y\frac{\partial}{\partial x} + 3x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}$$

on $\mathbb{R}^4$. Show it restricts to a nowhere vanishing vector field on S^3. (4 marks)

(ii) A point $p \in S^3$ is called a periodic point of V, if there exists an integral curve $\gamma : \mathbb{R} \to S^3$ with $\gamma(0) = p$ and $\gamma(t) = p$ for some $t > 0$. Find all periodic points of V . (5 marks)

(Total: 20 marks)

3. (a) Let $F : M \to N$ be a smooth map between two smooth manifolds.

   (i) Show the set of critical points of F is closed. (4 marks)

   (ii) If M is compact, show the regular values of F form an open subset of N. (3 marks)

(b) Let $F : S^n \to S^n$ be a smooth map between two standard spheres, and let p be a regular value of F.

   (i) Show $F^{-1}(p)$ contains a finite number of points. (4 marks)

   (ii) Give one example, for some n and a regular value p, such that $F^{-1}(p)$ contains three points. (3 marks)

(c) Recall any continuous map from S^2 to N admits a continuous lift from S^2 to the universal cover of N. Show any smooth map $F : S^2 \to S^1 \times S^1$ has a critical point. (6 marks)

(Total: 20 marks)

4. For a connected manifold M, let $H^p_{dR}(M)$ be the de Rham cohomology group of M at degree p.

(a) Let ω, η be two closed differential forms on M.

   (i) Show that the de Rham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η. (4 marks)

   (ii) Show the above wedge product gives the de Rham cohomology $R(M) := \oplus_p H^p_{dR}(M)$ a ring structure. Is it commutative? What is the unit of this ring $R(M)$? (4 marks)

(b) Let $F : M \to N$ be a smooth map. Show the pull back operation $F^* : \omega \mapsto F^*\omega$ induces a well-defined ring homomorphism from $R(N)$ to $R(M)$. (3 marks)

(c) Recall that the de Rham cohomology of $\mathbb{R}^n$ is

$$H^0_{dR}(\mathbb{R}^n) \cong \mathbb{R}, \quad H^p_{dR}(\mathbb{R}^n) \cong \{0\}, \forall p \neq 0.$$

   (i) For any smooth map $F : M \to \mathbb{R}$, show F^* is injective at degree zero. (3 marks)

   (ii) Let M be simply connected, and let $F : M \to T^n$ be a smooth map from M to the n-dimensional torus T^n. Determine F^*. (6 marks)

(Total: 20 marks)

5. (a) Let $\pi : E \to M$ be a vector bundle over a smooth manifold M. Define the dual bundle $E^* \to M$ whose total space is

$$E^* := \sqcup_{p \in M} E_p^*,$$

where E_p^* is the dual space of E_p, with the obvious projection map. Show E^* is a vector bundle over M. Given local transition matrices of E, find the local transition matrices of E^*. (6 marks)

(b) A vector bundle $E \to M$ is called orientable if it admits a collection of local trivializations whose transition matrices have positive determinant everywhere.

(i) Show M is orientable if and only if its tangent bundle TM is orientable. (4 marks)

(ii) Show a rank one vector bundle is orientable if and only if it is trivial. (6 marks)

(c) Let M be a regular level set of a function F on $\mathbb{R}^n$. Show TM is orientable. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70058

Manifolds (Solutions)

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1. (a) For any point p in X which is not the origin, it is a regular point of the function $x^2+y^2-z^2$. Hence it admits a neighborhood in X which is homeomorphic to an open 2-disk. On the other hand, $X-(0,0,0)$ has two components $X \cap \{z>0\}$ and $X \cap \{z<0\}$. Therefore the origin cannot admit a neighborhood in X which is homeomorphic to an open 2-disk. This shows that X is not a topological manifold. seen ⇓

6, B

(b) (i) In order to have a submersion, the dimension of the domain should be at least the dimension of the target. Hence there is no such submersion. 3, A

(ii) The stereographic graphic projection gives a diffeomorphism from S^2 minus the north pole to $\mathbb{R}^2$. Hence its inverse gives a submersion from $\mathbb{R}^2$ to S^2. Explicitly, such a map is

$$\sigma(x,y) = \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1}.$$

4, A

(iii) Consider the projection map unseen ⇓

$$\pi : \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1, \quad (z_1, z_2) \mapsto [z_1, z_2].$$

It induces a well-defined smooth map from S^3 to $\mathbb{C}P^1$. At a point $p=(z_1,z_2)$, the kernel of $d\pi$ is spanned, over $\mathbb{R}$, by two vectors (z_1,z_2) and $i(z_1,z_2)$. When restricted to $S^3 \subset \mathbb{C}^2$, the kernel of $d\pi$ is spanned by $i(z_1,z_2)$. Hence $d\pi \mid_{T_pS^3}$ has a one dimensional kernel and $\pi \mid_{S^3} : S^3 \to \mathbb{C}P^1$ is a submersion.
Let $F : S^3 \to S^2$ be a submersion. By the rank theorem, it is an open map. On the other hand, $F(S^3)$ is a compact subset in S^2 since S^3 is compact and F is continuous. Therefore F is surjective. 4, D

(c) It suffices to give an immersion which is not injective. For example, seen ⇓

$$F : \mathbb{R} \to S^1 = \{x^2+y^2=1\} \subset \mathbb{R}^2, \quad F(t) = (\cos(t), \sin(t)).$$

3, A

2. (a) The simplest way is to compute the inverse of F. Explicitly, it is $\boxed{\text{seen} \Downarrow}$

$$F^{-1}(u,v) = \left(\frac{u+v}{2}e^{\frac{v-u}{2}}, \frac{u-v}{2}\right)$$

Then we can see both F and F^{-1} are smooth, since their components are smooth functions. $\boxed{4,\ \mathrm{A}}$

(b) (i)

$$F^*(3du - dv) = 2e^y dx + (2xe^y + 4)dy.$$

$\boxed{3,\ \mathrm{A}}$

(ii) Recall that for a general vector field V, we have

$$(F_*V)_{(u,v)} = dF_{F^{-1}(u,v)}(V_{F^{-1}(u,v)}).$$

Hence we can compute

$$F_*\left(y\frac{\partial}{\partial x}\right)_{(u,v)} = \frac{u-v}{2}e^{\frac{u-v}{2}}\frac{\partial}{\partial u} + \frac{u-v}{2}e^{\frac{u-v}{2}}\frac{\partial}{\partial v}.$$

$\boxed{4,\ \mathrm{A}}$

(c) (i) The 3-sphere is a regular level set of the function $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2$. Note that $\boxed{\text{meth seen} \Downarrow}$

$$df = 2xdx + 2ydy + 2zdz + 2wdw.$$

By the characterization of the tangent space of a regular level set, a tangent vector of $\mathbb{R}^4$ is tangent to S^3 if and only if it is in the kernel of df. Then we can verify that

$$V = -3y\frac{\partial}{\partial x} + 3x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}$$

is indeed a tangent vector to S^3 at every point in S^3. It is nowhere zero since at least one of the coordinate is non-zero. $\boxed{4,\ \mathrm{B}}$

(ii) Let $p = (a,b,c,d)$ be a point in S^3. Consider a curve

$$\gamma : \mathbb{R} \to \mathbb{R}^4, \gamma(t) = (x(t), y(t), z(t), w(t))$$

satisfying the differential equations

$$x'(t) = -3y(t), y'(t) = 3x(t), z'(t) = -w(t), w'(t) = z(t); \quad \gamma(0) = (a,b,c,d).$$

The first two equations are independent of the last two equations. Hence we solve them separately to get

$$x(t) = a\cos(3t) - b\sin(3t), y(t) = a\sin(3t) + b\cos(3t),$$
$$z(t) = c\cos(t) - d\sin(t), w(t) = c\sin(t) + d\cos(t).$$

Then we can verify γ is a smooth map to S^3 and is an integral curve of V. Moreover, we have $\gamma(2\pi) = (a,b,c,d)$. Hence every point in S^3 is a periodic point of V. $\boxed{5,\ \mathrm{C}}$

3. (a) (i) Let $p \in M$ be a regular point of F. It means the map $dF_p : T_pM \to T_{F(p)}N$ is surjective. Recall the space of full rank matrices is an open subset of the space of all matrices, and the map dF_p depends on p continuously. We get the set of regular points of F is open. Hence the set of critical points is a closed subset of M. meth seen ⇓ 4, C

(ii) By (i), we know the set $C(F)$ of critical points is a closed subset of M. If M is compact, $C(F)$ is also compact. Since F is continuous, $F(C(F))$ is a compact subset of N. But N is a Hausdorff space, which implies that $F(C(F))$ is closed. Hence the regular values form an open subset of N. 3, C

(b) (i) By the regular value theorem, the set $F^{-1}(p)$ is a properly embedded submanifold of S^n of dimension zero. Hence it is a collection of points with discrete topology. Since S^n is compact, there are only finitely many of them. 4, A

(ii) Identify $\mathbb{R}^2$ with $\mathbb{C}$. So the unit circle is identified with $S^1 = \{|z|^2 = 1\}$. The map $z \mapsto z^3$ induces a smooth map from S^1 to S^1. For example, picking a chart $S^1 - \{1\}$ we can use the angle coordinate $\theta \in (0, 2\pi)$. Locally the map in this coordinate is $\theta \mapsto 3\theta$. It is a local diffeomorphism hence every point is a regular point. For any point in the circle it has three preimages. 3, B

(c) The universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$. Identifying S^1 with the unit circle in $\mathbb{C}$, the covering map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ can be explicitly realized as $(x, y) \mapsto (e^{ix}, e^{iy})$. A smooth map $F : S^2 \to S^1 \times S^1$ admits a continuous lift $G : S^2 \to \mathbb{R}^2$, meaning that $\pi \circ G = F$. Note that π is not only continuous, but also a local diffeomorphism. This implies that G is also smooth. Suppose that F does not have any critical point. By the chain rule of the differential unseen ⇓

$$dF = d(\pi \circ G) = dG \circ d\pi,$$

we know G has no critical point. This is impossible since there is no submersion from S^2 to $\mathbb{R}^2$. 6, D

4. (a) (i) First note that

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (d\eta) = 0 + 0 = 0.$$

seen ⇓

Hence the form $\omega \wedge \eta$ is closed and represents some class in de Rham cohomology. Next consider any other representatives $\omega + d\alpha, \eta + d\beta$ of the classes $[\omega], [\eta]$. We have

$$(\omega + d\alpha) \wedge (\eta + d\beta) = \omega \wedge \eta \pm d(\alpha \wedge \eta) \pm d(\omega \wedge \beta) \pm d(\alpha \wedge d\beta).$$

The latter three signs depend on the degrees of ω, η which are not important here. Hence $(\omega + d\alpha) \wedge (\eta + d\beta)$ represents the same class as $\omega \wedge \eta$.

4, B

(ii) Since the wedge product is distributive with addition

$$\omega \wedge (\eta + \sigma) = \omega \wedge \eta + \omega \wedge \sigma,$$

it induces a ring structure on $R(M)$. It is not commutative since $\omega \wedge \eta \neq \eta \wedge \omega$ in general. Let f be the constant function on M with value one. It is a closed zero-form. For any closed form ω, we have $f \wedge \omega = f \cdot \omega = \omega$. Hence the class of f is the unit of this ring.

4, A

(b) Recall that the exterior derivative commutes with pull back. Hence closed forms are pulled back to closed forms, and exact forms are pulled back to exact forms. The pull back map induces a well-defined map on the de Rham cohomology. Moreover, it is linear and $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. The pull back map is a ring homomorphism.

meth seen ⇓

3, A

(c) (i) We have showed in class that $H^0_{dR}(M) \cong \mathbb{R}$ if M is connected, and $H^0_{dR}(M)$ is identified with the space of constant functions. Note that the pull back of a constant function is again a constant function. Therefore the map F^* in degree zero is an isomorphism.

3, B

(ii) Suppose there is a smooth map $F : M \to T^n$. Since M is simply connected, we have a smooth lift $G : M \to \mathbb{R}^n$ satisfying $\pi \circ G = F$ where $\pi : \mathbb{R}^n \to T^n$ is the exponential covering map. See $3(c)$. Recall the composition law of pull backs

$$F^* = (\pi \circ G)^* = G^* \circ \pi^*.$$

unseen ⇓

The map G^* is zero at any degree $p \neq 0$, since the de Rham cohomology groups of $\mathbb{R}^n$ vanish at these degrees. Therefore F^* is also zero when $p \neq 0$. At degree zero, F^* is given by pulling back constant functions.

6, D

5. (a) We use the vector bundle chart lemma to show E^* is a vector bundle over M. Let $\{U_\alpha\}$ be an open cover of M such that for any α we have a local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. Now let $\pi' : E^* \to M$ be the projection map. We define a local trivialization $\Psi_\alpha : (\pi')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ in the following way. Pick $f \in E_p^*$ for $p \in U_\alpha$. Define meth seen ⇓

$$\Psi_\alpha(p, f) := (p, (f(\Phi_\alpha^{-1}(p, e_1)), \cdots, f(\Phi_\alpha^{-1}(p, e_k))),$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^k$. Then for each fix p, $\Psi_\alpha(p, \cdot)$ is a vector space isomorphism. Suppose $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$, we have

$$\begin{aligned} \Psi_\beta(p, f) :=& (p, (f(\Phi_\beta^{-1}(p, e_1)), \cdots, f(\Phi_\beta^{-1}(p, e_k))) \\ =& (p, (f(\Phi_\alpha^{-1}(p, \tau_{\alpha\beta}(p)e_1)), \cdots, f(\Phi_\alpha^{-1}(p, \tau_{\alpha\beta}(p)e_k))). \end{aligned}$$

Therefore $\Psi_\alpha \circ \Psi_\beta^{-1}(p, v) = (p, (\tau_{\alpha\beta}^{-1})^T(p)v)$, where T stands for the transpose of a matrix. Since $\tau_{\alpha\beta}(p)$ depends on p smoothly, $(\tau_{\alpha\beta}^{-1})^T(p)$ also depends on p smoothly. All the conditions in the vector bundle chart lemma are satisfied and $E^* \to M$ is equipped with a vector bundle structure. 6, M

(b) (i) Let $\{U_\alpha, \phi_\alpha\}$ be charts on M. The transition matrices of TM are the Jacobian matrices of the coordinate transition maps. Then TM is oriented if and only if their determinants are positive everywhere, which is exactly the definition of M being oriented. 4, M

(ii) First since a trivial rank one bundle admits a global trivialization, it is orientable. Next suppose a rank one bundle $\pi : E \to M$ is orientable. There exist local trivializations $\{U_\alpha, \phi_\alpha\}$ of E whose transition functions are positive everywhere. Pick a partition of unity $\{\rho_\alpha\}$ subcoordinate to $\{U_\alpha\}$. Define a map $s : M \to E$ as unseen ⇓

$$s(p) := \sum_\alpha \rho_\alpha(p)\phi_\alpha^{-1}(p, 1).$$

Since the transition functions are positive, the definition of s gives a nowhere vanishing section of E. Hence E is trivial. 6, M

(c) Consider the gradient vector field meth seen ⇓

$$\nabla F = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots \frac{\partial f}{\partial x_n}\frac{\partial}{\partial x_n}.$$

It is a smooth vector field on $\mathbb{R}^n$ which is nowhere tangent to M because M is a regular level set of F. Define a differential $(n-1)$ form

$$\eta(\cdot, \ldots, \cdot) := (dx_1 \wedge \cdots \wedge dx_n)(\nabla F, \cdot, \ldots, \cdot).$$

For any $p \in M$, there exist $v_1, \cdots, v_{n-1} \in T_pM$ such that $\{\nabla F, v_1, \cdots, v_{n-1}\}$ form a basis of $T_p\mathbb{R}^n$, since ∇F is nowhere tangent to M. This implies that η restricts to a nowhere zero $(n-1)$ form on M. We have shown in class that M is orientable if and only if it admits such a form. By the above $b(i)$, TM is orientable.

4, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70058 Manifolds Markers Comments

Question 1	This question tests basic notions about manifolds and smooth maps. People are doing fine. It is important to keep in mind some concrete examples, besides the abstract theory.
Question 2	This question asks for explicit computations about push forward, pull back and integral curves. Some students are not familiar with the formula.
Question 3	This question tests properties of smooth maps. Certain topology background is required. Some notions are not in the prerequisite of the module, due to setter's overlook.
Question 4	This is about properties of differential forms. Probably since it is covered in the last several lectures, some students are not familiar with such notions. The last part (c)(ii) assumes that people know what the universal cover of a torus is. It is not a prerequisite for this module. This makes the question largely unattended.
Question 5	This question asks for generalization of what was covered in class. Like the relations between tangent and cotangent bundles, orientation on manifold. Part (c) is an analogue of previous year's question, replacing the two sphere by a regular level set. The proof is more or less identical. However, few people noticed this.