IMPERIAL

MATH70060

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Complex Manifolds

Date: Wednesday, May 29, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) For $n \geq 1$, define the complex manifold $\mathbb{P}^n$. (4 marks)

(b) Let Z be the closed subset of $\mathbb{P}^3$ given by

$$Z = \left\{ [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

(i) Prove that Z is a complex submanifold of $\mathbb{P}^3$. (4 marks)

(ii) Find the dimension of Z. (2 marks)

(c) Construct an immersion of $\mathbb{P}^2$ into $\mathbb{C}^3$, or prove that it does not exist. (5 marks)

(d) Construct an immersion of $\mathbb{P}^2$ into $\mathbb{P}^3$, or prove that it does not exist. (5 marks)

(Total: 20 marks)

2. (a) Define the notion of an *almost complex structure* on a real manifold X.

(3 marks)

(b) Which of the following real manifolds admit an almost complex structure and which do not? Explain your answer.

(i) S^2 (two-dimensional sphere)

(3 marks)

(ii) S^3 (three-dimensional sphere)

(4 marks)

(iii) $\mathbb{R}^4$

(2 marks)

(c) Let X be the real manifold $\mathbb{R}^2$, and $T_{X,\mathbb{R}}$ be its real tangent bundle. Let J be the bundle morphism from $T_{X,\mathbb{R}}$ to itself given by

$$J\left(\frac{\partial}{\partial x}\right) = e^y \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$J\left(\frac{\partial}{\partial y}\right) = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y},$$

where x and y are the standard coordinates on $\mathbb{R}^2$ and f, g are unspecified functions.

(i) Find an example of a pair of functions f, g that make J an almost complex structure.

(4 marks)

(ii) Is this almost complex structure J integrable? Explain why. (4 marks)

(Total: 20 marks)

3. (a) For a complex manifold X and a holomorphic vector bundle E over X, define the *Dolbeault cohomology* $H^q(X, E)$, $q \geq 0$. (4 marks)

(b) Let $X = \mathbb{P}^1$, and T^*_X be the holomorphic cotangent bundle of X. Calculate the dimension of $H^0(X, T^*_X)$. (4 marks)

(c) State the *Serre duality* theorem. (4 marks)

(d) Calculate the dimension of $H^1(\mathbb{P}^1, T^*_{\mathbb{P}^1})$ (4 marks)

(e) Calculate the dimension of $H^2(\mathbb{P}^1, T^*_{\mathbb{P}^1})$. (4 marks)

(Total: 20 marks)

4. (a) Let E be a complex vector bundle over a real manifold X.
Define the notion of a *connection* ∇ on E. (4 marks)

(b) Let $\nabla = d + A$ in a local trivialization of E, where A is an $r \times r$ matrix of complexified 1-forms on X. Define the 2-form

$$\omega = \mathrm{Tr}(dA + A \wedge A),$$

where Tr denotes the trace.

(i) Prove that ω is independent of the chosen local trivialization. (4 marks)

(ii) Prove that $d\omega = 0$. (4 marks)

(c) Let X be a complex manifold and E be a holomorphic vector bundle over X. Let $\langle\ ,\ \rangle$ be an Hermitian metric on E. Define the notion of the *Chern connection* of $(E, \langle\ ,\ \rangle)$.
(4 marks)

(d) Let X be an open subset of $\mathbb{C}$, and E be the trivial holomorphic line bundle over X. Let the Hermitian metric on E be given by

$$\langle s, t\rangle = (1 + |z|^2)^2\ s\ \bar{t},$$

where z is the standard coordinate on $\mathbb{C}$, and $s, t \in C^\infty(X, E)$.
Calculate the Chern connection $\nabla = d + A$ of $(E, \langle\ ,\ \rangle)$. (4 marks)

(Total: 20 marks)

5. (a) Define the notion of a *Kähler form* on a complex manifold X.

(4 marks)

(b) Let $X = \mathbb{C}^2$ with the standard complex coordinates z_1, z_2. For each of the following complexified 2-forms on X, determine if it is Kähler or not, and briefly explain why.

(i) $\omega = \frac{i}{2} dz_1 \wedge dz_2 + \frac{i}{2} d\overline{z}_1 \wedge d\overline{z}_2$. (2 marks)

(ii) $\omega = \frac{i}{2} dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$. (2 marks)

(iii) $\omega = \frac{i}{2}(1 - |z_1|^2) dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$. (2 marks)

(iv) $\omega = \frac{i}{2}(1 + |z_2|^2) dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$. (2 marks)

(v) $\omega = \frac{i}{2} dz_1 \wedge d\overline{z}_2 + \frac{i}{2} dz_2 \wedge d\overline{z}_1$. (2 marks)

(c) State the *Hodge Decomposition Theorem*. (3 marks)

(d) Draw the Hodge diamond for $\mathbb{P}^3$. You do not need to include the proof. Explain what the numbers in the diamond mean. (3 marks)

(Total: 20 marks)

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MATH70060

Complex Manifolds (Solutions)

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1. (a) For $n \geq 1$, define the complex manifold $\mathbb{P}^n$. seen ⇓

As a topological space, $\mathbb{P}^n$ is defined as $(\mathbb{C}^{n+1} \setminus \{(0, ..., 0)\})/\sim$, where $u \sim v$ if $\exists \lambda \in \mathbb{C}^*$ such that $v = \lambda u$. Denoting the equivalence class of $(x_0, ..., x_n)$ by $[x_0 : ... : x_n]$, we construct the following $n+1$ open sets 4, A

$$U_i = \Big\{[x_0 : ... : x_n] \in \mathbb{P}^n : x_i \neq 0\Big\}, \quad i = 0, ..., n.$$

and the chart maps $\phi_i : U_i \to \mathbb{C}^n$, $i = 0, ..., n$,

$$\phi_i\Big([x_0 : ... : x_n]\Big) = \left(\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right).$$

(b) Let Z be the closed subset of $\mathbb{P}^3$ given by

$$Z = \Big\{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\Big\}.$$

sim. seen ⇓

(i) Prove that Z is a complex submanifold of $\mathbb{P}^3$.

It is enough to check that $U_i \cap Z$ is a submanifold of U_i for each $i = 0, 1, 2, 3$. Consider the case $i = 0$ (the remaining cases are analogous). Consider the coordinates $y_j = \frac{x_j}{x_0}$, $j = 1, 2, 3$, on U_0. The subset $U_0 \cap Z$ of U_0 is given by the equation $F = 0$, where 4, A

$$F(y_1, y_2, y_3) = 1 + y_1^3 + y_2^3 + y_3^3.$$

Let us prove that the holomorphic map $F : U_i \to \mathbb{C}$ is submersion in a neighborhood of Z. We have

$$F'_{y_1} = 3y_1^2, \quad F'_{y_2} = 3y_2^2, \quad F'_{y_3} = 3y_3^2.$$

At least one of the derivatives is non-zero, unless $(y_1, y_2, y_3) = (0, 0, 0)$. Note that the point $(0, 0, 0)$ does not belong to $U_0 \cap Z$. Therefore, F is a submersion in a neighborhood of Z. By the Implicit Function Theorem, $U_0 \cap Z$ is a submanifold of U_0.

sim. seen ⇓

(ii) Find the dimension of Z.

The set Z is cut out by one equation in a three dimensional manifold. Therefore the dimension of Z equals $3 - 1 = 2$. 2, A

sim. seen ⇓

(c) Construct an immersion of $\mathbb{P}^2$ into $\mathbb{C}^3$, or prove that it does not exist.

A holomorphic map $\mathbb{P}^2 \to \mathbb{C}^3$ consists of three holomorphic functions on $\mathbb{P}^2$. Since $\mathbb{P}^2$ is compact, each holomorphic function on it has to be constant. Therefore, each holomorphic map $\mathbb{P}^2 \to \mathbb{C}^3$ is constant, hence cannot be an immersion. 5, B

sim. seen ⇓

(d) Construct an immersion of $\mathbb{P}^2$ into $\mathbb{P}^3$, or prove that it does not exist.

Let $F : \mathbb{P}^2 \to \mathbb{P}^3$ be given by 5, A

$$F(x_0 : x_1 : x_2) = [x_0 : x_1 : x_2 : 0].$$

Let us show that F is an immersion. It is enough to show that for each standard open set $U_i \subset \mathbb{P}^2$, $i = 0, 1, 2$, the restriction $F : U_i \to \mathbb{P}^3$ is an immersion. Without loss of generality, let $i = 0$. Note that $F(U_0)$ lies completely in the standard open set $V_0 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : x_0 \neq 0\}$. In the standard coordinates on U_0 and V_0, the map F has the expression:

$$F(y_1, y_2) = (y_1, y_2, 0).$$

The derivative matrix of F is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It has rank 2 for every point (y_1, y_2). Therefore, F is an immersion on U_0, and on the whole of $\mathbb{P}^2$.

2. (a) Define the notion of an almost complex structure on a real manifold X. seen ⇓

An almost complex structure on X is a vector bundle morphism $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\mathrm{Id}$, where $T_{X,\mathbb{R}}$ is the real tangent bundle of X. 3, A

(b) Which of the following real manifolds admit an almost complex structure and which do not? Explain your answer. meth seen ⇓

(i) S^2 (two-dimensional sphere) 3, B

The real manifold S^2 admits a structure of a complex manifolds, namely $\mathbb{P}^1$. Therefore, it also admits an almost complex structure. meth seen ⇓

(ii) S^3 (three-dimensional sphere) 4, C

We are going to prove that no three-dimensional manifold admits an almost complex structure. It is enough to show that there is no $J : \mathbb{R}^3 \to \mathbb{R}^3$ such that $J^2 = -\mathrm{Id}$. Suppose such J exists. Consider its complexification $J_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$. Let $V^{1,0}$ be the i-eigenvalue of $J_{\mathbb{C}}$, and $V^{0,1}$ be the $-i$-eigenvalue of $J_{\mathbb{C}}$. Since $J^2 = -\mathrm{Id}$, we have $\mathbb{C}^3 = V^{1,0} \oplus V^{0,1}$. At the same time, we have $\overline{V^{1,0}} = V^{0,1}$. Therefore, $V^{1,0}$ and $V^{0,1}$ have the same dimension d such that $d + d = 3$, a contradiction. seen ⇓

(iii) $\mathbb{R}^4$ 2, A

Let x_1, y_1, x_2, y_2 be the standard coordinates on $\mathbb{R}^4$. We can define the almost complex structure

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad i = 1, 2.$$

(c) Let X be the real manifold $\mathbb{R}^2$, and $T_{X,\mathbb{R}}$ be its real tangent bundle. Let J be the bundle morphism from $T_{X,\mathbb{R}}$ to itself given by

$$J\left(\frac{\partial}{\partial x}\right) = e^y \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$J\left(\frac{\partial}{\partial y}\right) = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y},$$

where x and y are the standard coordinates on $\mathbb{R}^2$ and f, g are unspecified functions. sim. seen ⇓

(i) Find an example of a pair of functions f, g that make J an almost complex structure. 4, B

We have

$$J^2\left(\frac{\partial}{\partial x}\right) = J\left(e^y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = (e^{2y} - f)\frac{\partial}{\partial x} + (e^y + g)\frac{\partial}{\partial y},$$

$$J^2\left(\frac{\partial}{\partial y}\right) = J\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) = (e^y + g)f\frac{\partial}{\partial x} + (f + g^2)\frac{\partial}{\partial y}.$$

We see that $J^2 = -\mathrm{Id}$ if and olny if $g = -e^y$, $f = -e^{2y} - 1$. sim. seen ⇓

(ii) Is this almost complex structure J integrable? Explain why. 4, D

We calculate that $T^{0,1}$, the $-i$-eigenbundle of J, is spanned by

$$v = (e^y - i)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

For any two sections $h_1 v$, $h_2 v$ of $T^{0,1}$, we have

$$[h_1 v, h_2 v] = \big(h_1 v(h_2) - h_2 v(h_1)\big)v \in C^\infty(\mathbb{R}^2, T^{0,1}).$$

Therefore J is integrable.

3. (a) For a complex manifold X and a holomorphic vector bundle E over X, define the Dolbeault cohomology $H^q(X,E)$, $q \geq 0$.

seen ⇓

4, A

Let $\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q}(E)) \to C^\infty(X, \Omega_X^{0,q+1}(E))$ be the delbar operator of E. We define

$$H^q(X,E) := \frac{\mathrm{Ker}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q}(E)) \to C^\infty(X, \Omega_X^{0,q+1}(E))\right)}{\mathrm{Im}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q-1}(E)) \to C^\infty(X, \Omega_X^{0,q}(E))\right)}$$

(b) Let $X = \mathbb{P}^1$, and T_X^* be the holomorphic cotangent bundle of X. Calculate the dimension of $H^0(X, T_X^*)$.

sim. seen ⇓

4, D

The space $H^0(X, T_X^*)$ consists of the holomorphic sections of T_X^*. Let U_0 and U_1 be the standard open sets covering $\mathbb{P}^1$, s and t be the standard coordinates on U_0 and U_1, respectively. We have $s = \frac{1}{t}$ on $U_0 \cap U_1$.

Let $\alpha = f(s)ds$ be a holomorphic section of T_X^* over U_0. Since $f(s)$ is holomorphic, we have $f(s) = \sum_{n=0}^{\infty} a_n s^n$, $s \in \mathbb{C}$. Over U_1, we have

$$\alpha = f\left(\frac{1}{t}\right) d\left(\frac{1}{t}\right) = -\frac{1}{t^2}\left(\sum_{n=0}^{\infty} a_n \frac{1}{t^n}\right) dt.$$

We see that α has is holomorphic at $t = 0$ if and only if $a_n = 0$ for all $n \geq 0$. Therefore, the only holomorphic section of T_X^* is the zero section. Hence the dimension of $H^0(X, T_X^*)$ is zero.

seen ⇓

4, A

(c) State the Serre duality theorem.

Let X be a compact complex manifold of dimension n. Let E be a holomorphic vector bundle over X. Then for all $0 \leq q \leq n$ we have

$$H^q(X,E)^* = H^{n-q}(X, E^* \otimes K_X),$$

where $K_X = \Omega_X^n$ is the canonical line bundle of X.

seen ⇓

4, C

(d) Calculate the dimension of $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*)$.

Note that since $\dim(\mathbb{P}^1) = 1$, we have $K_{\mathbb{P}^1} = T_{\mathbb{P}^1}^*$. By Serre duality, we obtain

$$H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*)^* = H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes T_{\mathbb{P}^1}^*) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}.$$

Therefore, the dimension of $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}^*)$ is one.

meth seen ⇓

4, B

(e) Calculate the dimension of $H^2(\mathbb{P}^1, T_{\mathbb{P}^1}^*)$.

Since the dimension of $\mathbb{P}^1$ is one, the bundle $\Omega_{\mathbb{P}^1}^{0,2}$ is the zero bundle. Hence, we obtain $H^2(\mathbb{P}^1, T_{\mathbb{P}^1}^*) = 0$.

4. (a) Let E be a complex vector bundle over a real manifold X. $\boxed{\text{seen} \Downarrow}$

Define the notion of a connection ∇ on E. $\boxed{4, \text{A}}$

A connection on E is a map

$$\nabla\colon C^\infty(X,E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

such that

$$\nabla(f\sigma) = f\nabla\sigma + df \otimes \sigma \qquad \text{for any } f \in C^\infty(X), \sigma \in C^\infty(X,E).$$

(b) Let $\nabla = d + A$ in a local trivialization of E, where A is an $r \times r$ matrix of complexified 1-forms on X. Define the 2-form

$$\omega = \mathrm{Tr}(dA + A \wedge A),$$

where Tr denotes the trace.

$\boxed{\text{seen} \Downarrow}$

(i) Prove that ω is independent of the chosen local trivialization. $\boxed{4, \text{C}}$

In a different trivialization, the connection will be $\nabla = d + B$, with $B = g^{-1}dg + g^{-1}Ag$, for some invertible matrix of functions g. In lecture, we have proved that

$$dB + B \wedge B = g^{-1}(dA + A \wedge A)g.$$

Therefore, $\mathrm{Tr}(dB + B \wedge B) = \mathrm{Tr}(dA + A \wedge A)$.

$\boxed{\text{seen} \Downarrow}$

(ii) Prove that $d\omega = 0$. $\boxed{4, \text{D}}$

Since A is a matrix of 1-forms, we have $\mathrm{Tr}(A \wedge A) = 0$. Therefore, $\omega = \mathrm{Tr}(dA)$. Hence, we have

$$d\omega = d(\mathrm{Tr}(dA)) = \mathrm{Tr}(d^2A) = 0.$$

$\boxed{\text{seen} \Downarrow}$

(c) Let X be a complex manifold and E be a holomorphic vector bundle over X. Let $\langle\ ,\ \rangle$ be an Hermitian metric on E. Define the notion of the Chern connection of $(E, \langle\ ,\ \rangle)$. $\boxed{4, \text{B}}$

Chern connection is a unique connection

$$\nabla\colon C^\infty(X,E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

such that

$$\nabla^{0,1} = \overline{\partial}_E$$

and such that ∇ is compatible with the metric $\langle\ ,\ \rangle$, in the sense that

$$d\langle \sigma, \tau\rangle = \langle \nabla\sigma, \tau\rangle + \langle \sigma, \nabla\tau\rangle, \quad \sigma, \tau \in C^\infty(X,E).$$

$\boxed{\text{meth seen} \Downarrow}$

(d) Let X be an open subset of $\mathbb{C}$, and E be the trivial holomorphic line bundle over X. Let the Hermitian metric on E be given by $\boxed{4, \text{D}}$

$$\langle s, t\rangle = (1 + |z|^2)^2\ s\,\bar{t},$$

where z is the standard coordinate on $\mathbb{C}$, and $s, t \in C^\infty(X,E)$.

Calculate the Chern connection $\nabla = d + A$ of $(E, \langle\ ,\ \rangle)$.

Let $h(z) = (1 + |z|^2)^2$. Then the only entry of A is

$$a = \frac{\partial\overline{h}}{\overline{h}} = \frac{\partial h}{h} = 2\frac{\partial(1 + |z|^2)}{1 + |z|^2} = \frac{4\overline{z}\ dz}{1 + |z|^2}.$$

5. (a) Define the notion of a Kähler form on a complex manifold X. seen ⇓

A Kähler form is a real positive $(1,1)$-form ω such that $d\omega = 0$. 4, M

(b) Let $X = \mathbb{C}^2$ with the standard complex coordinates z_1, z_2. For each of the following complexified 2-forms on X, determine if it is Kähler or not, and briefly explain why. unseen ⇓

(i) $\omega = \frac{i}{2} dz_1 \wedge dz_2 + \frac{i}{2} d\overline{z}_1 \wedge d\overline{z}_2$.
It is not Kähler, since it is not a $(1,1)$-form. 2, M

seen ⇓

(ii) $\omega = \frac{i}{2} dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$.
It is the standard Kähler form on $\mathbb{C}^2$. 2, M

unseen ⇓

(iii) $\omega = \frac{i}{2}(1 - |z_1|^2) dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$.
It is not Kähler, since for $|z_1| \geq 1$ it fails to be positive. 2, M

unseen ⇓

(iv) $\omega = \frac{i}{2}(1 + |z_2|^2) dz_1 \wedge d\overline{z}_1 + \frac{i}{2} dz_2 \wedge d\overline{z}_2$.
It is not Kähler, since it is not closed. 2, M

unseen ⇓

(v) $\omega = \frac{i}{2} dz_1 \wedge d\overline{z}_2 + \frac{i}{2} dz_2 \wedge d\overline{z}_1$.
It is not Kähler, since it is not positive. Indeed, the matrix 2, M

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not positive definite, because

$$\begin{pmatrix} 1 & 0 \end{pmatrix} H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

(c) State the Hodge Decomposition Theorem. seen ⇓

Let X be a compact Kähler manifold. 3, M

Then

$$H^k(X, \mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X) \quad \text{and} \quad H^{p,q}(X) = \overline{H^{q,p}(X)},$$

where $H^k(X,\mathbb{C})$ denotes the de Rham cohomology of X and $H^{p,q}(X)$ denotes the Dolbeault cohomology of X.

(d) Draw the Hodge diamond for $\mathbb{P}^3$. You do not need to include the proof. Explain what the numbers in the diamond mean. seen ⇓ 3, M

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Here $h^{p,q} = \dim H^{p,q}(\mathbb{P}^3)$.

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks