

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Representation Theory

Date: Wednesday, May 7, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

You may use all results from the course, including lectures, lecture notes, problems sheets and courseworks without proof, unless otherwise specified. You may also use the assertions of previous parts of a question in solving later ones, without proof. All parts require full justification unless noted to the contrary.

1. Let G be a finite group, and let V be a representation of G .

- (a) Let $S : V \rightarrow V$ be a linear map. Define what is meant by S being
- (i) G -linear; (1 mark)
 - (ii) a projection operator. (1 mark)

- (b) Let $\theta : G \rightarrow \mathbf{C}^*$ be a one-dimensional representation of G . Define a linear map $S : V \rightarrow V$ by,

$$S(v) = |G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(g)(v),$$

- (i) Show that for all $v \in V$ and all $h \in G$, $S(\rho(h)(v)) = \theta(h) \cdot S(v)$. (2 marks)
- (ii) Show that for all $v \in V$ and all $h \in G$, $\rho(h)(S(v)) = \theta(h) \cdot S(v)$. (2 marks)
- (iii) Let

$$W = \{v \in V \mid \forall h \in G, \rho(h)(v) = \theta(h) \cdot v\}.$$

- Show that for all $w \in W$, $S(w) = w$. (2 marks)
- (iv) Show that S is a G -linear projection operator and $\text{im}(S) = W$. (4 marks)

- (c) Let $G = C_4 = \{1, g, g^2, g^3\}$ be the cyclic group of order 4.

Let X be the set of size-2 subsets of G (so $|X| = 6$). G acts on itself by left-multiplication, and there is an induced action of G on X , and associated representation $V = \mathbf{C}[X]$ of G .

Let $\theta : G \rightarrow \mathbf{C}^*$ be the 1-dimensional representation of G defined by $\theta(g^k) = (-1)^k$.

Give a basis for the subrepresentation W described in (b)(iii). (4 marks)

- (d) Now we return to the case of a general G , V and θ .

Suppose that V is irreducible. Show that either $S = 0$ or V is isomorphic to (\mathbf{C}, θ) .

(4 marks)

(Total: 20 marks)

2. Let G be the group of 3×3 matrices with entries drawn from ± 1 and 0, with exactly one nonzero entry in each row and in each column, and with determinant 1.

- (a) Show that $|G| = 24$. (3 marks)
- (b) Since G is defined as a subgroup of $\text{Mat}_3(\mathbf{C})$, this provides a 3-dimensional representation of G . Prove that this representation is irreducible. (4 marks)
- (c) Show that the map $\theta : G \rightarrow \mathbf{C}^*$, which sends a matrix in G to the product of its nonzero entries, is a one-dimensional representation of G . (3 marks)
- (d) Give a second irreducible 3-dimensional representation of G . Include a proof that it is not isomorphic to the 3-dimensional representation from (b). (4 marks)
- (e) Consider the action of G by left-multiplication on the set

$$X = \left\{ \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} \right\}.$$

(Thus $|X| = 3$, and each element of X is a set of size 2. G can be thought of as the group of orientation-preserving symmetries of the octahedron, and this is the induced action on the set of pairs of opposite vertices.)

Show that there is an irreducible two-dimensional subrepresentation of $\mathbf{C}[X]$. (4 marks)

- (f) Give a full list (up to isomorphism) of the irreducible representations of G . (2 marks)

(Total: 20 marks)

3. Let G be a group of size 12, with six conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ of sizes as listed below, and with two characters as follows:

set	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6
size	1	2	2	1	3	3
χ_{V_2}	1	1	1	1	-1	-1
χ_{V_3}	2	1	-1	-2	0	0

- (a) Let V_1 be the trivial irreducible representation. Compute its character. (1 mark)
- (b) Prove that V_3 is irreducible. (3 marks)
- (c) Prove the existence of a new two-dimensional irreducible representation, call it V_4 . Give its character.
(Hint: study tensor products of the known representations.) (6 marks)
- (d) Determine how many more irreducible representations there are (up to isomorphism), and find their dimensions. (2 marks)
- (e) Now we are given the further information that at least one of the remaining irreducible representations is isomorphic to its dual. Determine the characters of all the remaining irreducible representations. (8 marks)

(Total: 20 marks)

4. (a) Consider the commutative algebra $A = \mathbf{C}[x]$ of polynomials in the variable x . Define a function $\rho : A \rightarrow \mathbf{C}$ by, for a polynomial $f(x) \in A$,

$$\rho(f) = f(3).$$

Since $\text{End}(\mathbf{C})$ can be canonically identified with \mathbf{C} , the function ρ can be thought of as a function from A to $\text{End}(\mathbf{C})$.

- (i) Show that (\mathbf{C}, ρ) is a left A -module. (4 marks)
- (ii) Find (with proof) an A -submodule W of A such that $(\mathbf{C}, \rho) \cong A/W$. (5 marks)
- (b) Consider the algebra $A = \text{Mat}_2(\mathbf{C})$ of 2×2 matrices. For a nonzero vector $x \in \mathbf{C}^2$, define

$$V_x := \{M \in A \mid Mx = 0\}.$$

- (i) Prove that V_x is a proper nonzero submodule of A . (3 marks)
- (ii) Prove that all proper nonzero submodules of A are of this form. (8 marks)

(Total: 20 marks)

5. (a) Let G be a group, let H be a subgroup of G , and let K be a subgroup of H . Let (V, ρ) be a representation of K . In this problem we will construct a canonical isomorphism

$$\text{coInd}_H^G \text{coInd}_K^H V \cong \text{coInd}_K^G V.$$

- (i) Denote by \tilde{V} the vector space of functions $F : G \rightarrow \text{Fun}(H, V)$, such that
- for all $g \in G$, $h \in H$, and $k \in K$, $F(g)(kh) = \rho(k)F(g)(h)$;
 - for all $g \in G$ and for all $h, h' \in H$, $F(hg)(h') = F(g)(h'h)$.

Prove that

$$\text{coInd}_H^G \text{coInd}_K^H V = (\tilde{V}, \tilde{\rho}),$$

where $\tilde{\rho} : G \rightarrow \text{End}(\tilde{V})$ is a function satisfying, for all $g, g' \in G$,

$$\tilde{\rho}(g)(F)(g') = F(g'g).$$

(5 marks)

- (ii) Let $f : G \rightarrow V$ be an element of $\text{coInd}_K^G V$. Show that the function $F_f : G \rightarrow \text{Fun}(H, V)$ defined by,

$$F_f(g)(h) = f(hg)$$

is an element of \tilde{V} . (3 marks)

- (iii) Let $T : \text{coInd}_K^G V \rightarrow \text{coInd}_H^G \text{coInd}_K^H V$ denote the function $f \mapsto F_f$. Show that T is a homomorphism of representations. (4 marks)

- (iv) Show that T is bijective. (5 marks)

- (b) Let G be a group. Prove that for any subgroup H of G , there exists a representation V of H , such that the regular representation $\mathbf{C}[G]$ of G is isomorphic to $\text{coInd}_H^G V$.

(3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60039/MATH70039

Group Representation Theory (Solutions)

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1. (a) (i) $S : V \rightarrow V$ is *G-linear* if for all $g \in G$ and $v \in V$, $S(\rho(g)v) = \rho(g)S(v)$.

seen ↓

(ii) $S : V \rightarrow V$ is a *projection operator* if for all $v \in V$, $S(S(v)) = S(v)$.

1, A

seen ↓

1, A

(b) (i) For $v \in V$ and $h \in G$,

meth seen ↓

$$\begin{aligned} S(\rho(h)(v)) &= |G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(gh)(v) \\ &= |G|^{-1} \sum_{g \in G} \theta(gh^{-1})^{-1} \cdot \rho(g)(v) \\ &= \theta(h) \cdot \left(|G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(g)(v) \right) \\ &= \theta(h) \cdot S(v). \end{aligned}$$

2, A

(ii) For $v \in V$ and $h \in G$,

meth seen ↓

$$\begin{aligned} \rho(h)(S(v)) &= |G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(hg)(v) \\ &= |G|^{-1} \sum_{g \in G} \theta(h^{-1}g)^{-1} \cdot \rho(g)(v) \\ &= \theta(h) \cdot \left(|G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(g)(v) \right) \\ &= \theta(h) \cdot S(v). \end{aligned}$$

2, A

(iii) For $w \in W$,

meth seen ↓

$$\begin{aligned} S(w) &= |G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \rho(g)(w) \\ &= |G|^{-1} \sum_{g \in G} \theta(g)^{-1} \cdot \theta(g) \cdot w \\ &= w. \end{aligned}$$

2, A

meth seen ↓

(iv) The linearity of S is clear. The *G-linearity* of S comes from (i) and (ii):

$$S(\rho(h)(v)) = \theta(h) \cdot S(v) = \rho(h)(S(v)).$$

For any $v \in V$, by (ii) $S(v) \in W$, and so by (iii) $S(S(v)) = S(v)$. Thus S is a projection operator.

Finally, by (ii), $\text{im}(S) \subseteq W$, and by (iii), $W \subseteq \text{im}(S)$. Thus $\text{im}(S) = W$.

4, C

- (c) Take a generic vector in $\mathbf{C}[X]$:

unseen ↓

$$v = a\{1, g\} + b\{g, g^2\} + c\{g^2, g^3\} + d\{g^3, 1\} + e\{1, g^2\} + f\{g, g^3\}$$

We have that

$$\rho(g)v = a\{g, g^2\} + b\{g^2, g^3\} + c\{g^3, 1\} + d\{1, g\} + e\{g, g^3\} + f\{g^2, 1\}.$$

So for a 1-dimensional representation θ , $v \in W_\theta$ if and only if

2, A

$$\begin{aligned} & a\{g, g^2\} + b\{g^2, g^3\} + c\{g^3, 1\} + d\{1, g\} + e\{g, g^3\} + f\{g^2, 1\} \\ &= \rho(g)v \\ &= \theta(g) \cdot v \\ &= \theta(g) (a\{1, g\} + b\{g, g^2\} + c\{g^2, g^3\} + d\{g^3, 1\} + e\{1, g^2\} + f\{g, g^3\}); \end{aligned}$$

that is, if and only if

$$\begin{aligned} a &= \theta(g)b, & c &= \theta(g)c, & e &= \theta(g)f, \\ b &= \theta(g)c, & d &= \theta(g)a, & f &= \theta(g)e. \end{aligned}$$

In this case $\theta(g) = -1$, so

$$\begin{aligned} a &= -b, & c &= -c, & e &= -f, \\ b &= -c, & d &= -a, & f &= -e. \end{aligned}$$

Therefore the following two vectors form a basis for W :

1, A

$$\begin{aligned} v &= \{1, g\} - \{g, g^2\} + \{g^2, g^3\} - \{g^3, 1\} \\ w &= \{1, g^2\} - \{g, g^3\}. \end{aligned}$$

1, B

- (d) Since V is irreducible, the subrepresentation $W = \text{im}(S)$ from (b) is either (0) or all of V .

1, A

If the former, then $S = 0$. If the latter, then for all $v \in V$ and all $g \in G$,

$$\rho(g)(v) = \theta(g) \cdot v. \quad (\star)$$

Let v be a nonzero vector in V . By (\star) , Cv is a subrepresentation and is isomorphic to (\mathbf{C}, θ) . Since V is irreducible, the only nonzero subrepresentation is V , so $Cv = V$.

3, B

2. (a) An element of G is a 3×3 permutation matrix, adjusted by changing the signs of some of the nonzero entries. There are six permutations of $\{1, 2, 3\}$. Each of these yields four elements of G : if the associated permutation matrix has determinant 1 then elements of G are obtained by negating either 0 or 2 of the entries, if the associated permutation matrix has determinant -1 then elements of G are obtained by negating either 1 or 3 of the entries.

unseen ↓

- (b) Since this representation is 3-dimensional, to check irreducibility it suffices to check that there are no one-dimensional subrepresentations, that is, no nonzero common eigenvectors of all the elements of G .

3, B

meth seen ↓

1, A

Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a common eigenvector of the elements of G .

Since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in G$, either $a = 0$ or $b = c = 0$.

Since $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in G$, v is a multiple of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad \text{or } \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$$

(where ω is the third root of unity $e^{2\pi i/3}$). Thus, since at least one entry of v is zero, v is zero.

- (c) Clearly $\theta(I_3) = 1$. We must check that θ is multiplicative.

3, C

unseen ↓

Let M and N be matrices in G . Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the nonzero entries of M , and let η_1, η_2, η_3 be the nonzero entries of N . Then the nonzero entries of MN are $\epsilon_1\eta_{\sigma(1)}, \epsilon_2\eta_{\sigma(2)}, \epsilon_3\eta_{\sigma(3)}$ for some permutation $\sigma \in S_3$, so

$$\theta(MN) = (\epsilon_1\eta_{\sigma(1)}) (\epsilon_2\eta_{\sigma(2)}) (\epsilon_3\eta_{\sigma(3)}) = (\epsilon_1\epsilon_2\epsilon_3) (\eta_1\eta_2\eta_3) = \theta(M)\theta(N).$$

- (d) The tensor product of the representation ρ from (b) and the representation θ from (c) is a 3-dimensional irreducible representation of G .

3, B

sim. seen ↓

2, A

Consider the element

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

of G . We have that

$$\chi_\rho(M) = 1, \quad \theta(M) = -1, \quad \chi_{\theta\rho}(M) = -1.$$

Since ρ and $\theta\rho$ have different characters they are not isomorphic.

2, B

- (e) The following is a two-dimensional subrepresentation of $\mathbf{C}[X]$:

sim. seen ↓

$$V = \left\{ \sum_{x \in X} a_x x \mid \sum_{x \in X} x = 0 \right\}.$$

2, A

This is irreducible: it is obtained from the (irreducible) reflection representation of the permutation group S_X by pulling back under the surjective homomorphism

$$\varphi : G \rightarrow S_X$$

which comes from the group action of G on X .

For marking: student must mention the surjectivity condition but need not explicitly prove that φ is surjective.

- (f) From (c) and the trivial representation we have two 1-dimensional representations, which are nonisomorphic since there are matrices M for which $\theta(M) \neq 1$. (See the solution to (d) for an example.)

2, C

meth seen ↓

From (e) we have one 2-dimensional irreducible representation, and from (b) and (d) we have two non-isomorphic 3-dimensional irreducible representations. Since $1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24 = |G|$, there can be no more irreducible representations.

2, A

	C_1	C_2	C_3	C_4	C_5	C_6
χ_{V_1}	1	1	1	1	1	1

seen ↓

This is identical (apart from the difference in group) to, e.g., 2020 3(a), but I think this is fine for a 1-point intro question.

(b)

$$\begin{aligned}\langle \chi_{V_3}, \chi_{V_3} \rangle &= \frac{1}{12} (1 \cdot 2^2 + 2 \cdot 1^2 + 2 \cdot (-1)^2 + 1 \cdot (-2)^2 + 3 \cdot 0^2 + 3 \cdot 0^2) \\ &= 1,\end{aligned}$$

so V_3 is irreducible.(c) The character of $\chi_{V_3 \otimes V_3}$ is obtained by squaring that of V_3 :

	C_1	C_2	C_3	C_4	C_5	C_6
χ_{V_3}	4	1	1	4	0	0

3, B

unseen ↓

1, A

We have that

$$\begin{aligned}\langle \chi_{V_3 \otimes V_3}, \chi_{V_1} \rangle &= \frac{1}{12} (1 \cdot 4 \cdot 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + 1 \cdot 4 \cdot 1 + 3 \cdot 0 \cdot 1 + 3 \cdot 0 \cdot 1) \\ &= 1,\end{aligned}$$

$$\begin{aligned}\langle \chi_{V_3 \otimes V_3}, \chi_{V_2} \rangle &= \frac{1}{12} (1 \cdot 4 \cdot 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + 1 \cdot 4 \cdot 1 + 3 \cdot 0 \cdot -1 + 3 \cdot 0 \cdot -1) \\ &= 1,\end{aligned}$$

so V_1 and V_2 each appear once in the decomposition of $V_3 \otimes V_3$ into irreducible subrepresentations. Letting $V_4 \subseteq V_3 \otimes V_3$ be a complementary subrepresentation to $V_1 \oplus V_2$, we have that $\chi_{V_4} = \chi_{V_3 \otimes V_3} - \chi_{V_1} - \chi_{V_2}$, i.e.

	C_1	C_2	C_3	C_4	C_5	C_6
χ_{V_4}	2	-1	-1	2	0	0

2, B

Now observe that

$$\begin{aligned}\langle \chi_{V_4}, \chi_{V_4} \rangle &= \frac{1}{12} (1 \cdot 2^2 + 2 \cdot (-1)^2 + 2 \cdot (-1)^2 + 1 \cdot 2^2 + 3 \cdot 0^2 + 3 \cdot 0^2) \\ &= 1,\end{aligned}$$

so V_4 is irreducible.(d) We have found two 1-dimensional representations (V_1 and V_2) and two 2-dimensional representations (V_3 and V_4). The sum of the squares of the dimensions of the remaining irreducible representations is $12 - 1^2 - 1^2 - 2^2 - 2^2 = 2$, so there must be two remaining one-dimensional irreducible representations and no others.

3, D

meth seen ↓

(e) Introduce variables for the unknown entries in the character of V_5 and V_6 , so that the full full character table, according to our knowledge so far, is:

set	C_1	C_2	C_3	C_4	C_5	C_6
size	1	2	2	1	3	3
χ_{V_1}	1	1	1	1	1	1
χ_{V_2}	1	1	1	1	-1	-1
χ_{V_3}	2	1	-1	-2	0	0
χ_{V_4}	2	-1	-1	2	0	0
χ_{V_5}	1	a_5	b_5	c_5	d_5	e_5
χ_{V_6}	1	a_6	b_6	c_6	d_6	e_6

2, A

unseen ↓

Ingredients: Here are some identities (not all are needed for a solution).

1 mark in Category B for each of these observations, up to a maximum of 3.

3, B

Orthogonality with first column/Decomposition of regular representation: From the orthogonality of \mathcal{C}_1 with, respectively, $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$, we have,

$$\begin{aligned} a_5 + a_6 &= (1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 + 2 \cdot -1 + 1 \cdot a_5 + 1 \cdot a_6) - 2 \\ &= 0 - 2 = -2, \\ b_5 + b_6 &= (1 \cdot 1 + 1 \cdot 1 + 2 \cdot -1 + 2 \cdot -1 + 1 \cdot b_5 + 1 \cdot b_6) + 2 \\ &= 0 + 2 = 2, \\ c_5 + c_6 &= (1 \cdot 1 + 1 \cdot 1 + 2 \cdot -2 + 2 \cdot 2 + 1 \cdot c_5 + 1 \cdot c_6) - 2 \\ &= 0 - 2 = -2, \\ d_5 + d_6 &= 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 0 + 2 \cdot 0 + 1 \cdot d_5 + 1 \cdot d_6 \\ &= 0, \\ e_5 + e_6 &= 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 0 + 2 \cdot 0 + 1 \cdot e_5 + 1 \cdot e_6 \\ &= 0. \end{aligned}$$

Column norms: From the norm of the column \mathcal{C}_5 , we have,

$$\begin{aligned} |d_5|^2 + |d_6|^2 &= (1^2 + (-1)^2 + 0^2 + 0^2 + |d_5|^2 + |d_6|^2) - 2 \\ &= \frac{12}{3} - 2 \\ &= 2, \end{aligned}$$

and arguing similarly with \mathcal{C}_6 we get $|e_5|^2 + |e_6|^2 = 2$.

Orthogonality of \mathcal{C}_5 and \mathcal{C}_6 : This gives,

$$\begin{aligned} d_5\bar{e_5} + d_6\bar{e_6} &= (1 \cdot 1 + -1 \cdot -1 + 0 \cdot 0 + 0 \cdot 0 + d_5 \cdot \bar{e_5} + d_6 \cdot \bar{e_6}) - 2 \\ &= 0 - 2 = -2. \end{aligned}$$

Orthogonality with first row: From the orthogonality of χ_{V_1} with χ_{V_5} , we now have,

$$\begin{aligned} 0 &= 1 \cdot 1 + 2 \cdot -1 + 2 \cdot 1 + 1 \cdot -1 + 3 \cdot d_5 + 3 \cdot e_5 \\ &= 3(d_5 + e_5), \end{aligned}$$

so $e_5 = -d_5$, and similarly arguing on χ_{V_6} we see $e_6 = -d_6$.

Tensor product: V_6 is necessarily $V_5 \otimes V_2$, so

$$a_6 = a_5, \quad b_6 = b_5, \quad c_6 = c_5, \quad d_6 = -d_5, \quad e_6 = -e_5.$$

Self-duality of V_5 : This means that the character of V_5 is real:

$$\bar{a_5} = a_5, \quad \bar{b_5} = b_5, \quad \bar{c_5} = c_5, \quad \bar{d_5} = -d_5, \quad \bar{e_5} = -e_5.$$

Main argument: The identities above can be combined in various ways to get a solution. Two sample arguments are below.

5 marks in Category D for a full solution.

5, D

Method 1: From the tensor product identities, let's introduce variables a, b, c, d, e with

$$a_6 = a_5 = a, \quad b_6 = b_5 = b, \quad c_6 = c_5 = c, \quad d_6 = -d_5 = -d, \quad e_6 = -e_5 = -e.$$

From the first-column-orthogonality identities, we find

$$2a = -2, \quad 2b = 2, \quad 2c = -2,$$

so $a = c = -1$ and $b = 1$. From the column-norm identities,

$$2|d|^2 = 2|e|^2 = 2,$$

so $|d| = |e| = 1$. From the orthogonality of \mathcal{C}_5 and \mathcal{C}_6 , $2d\bar{e} = -2$, so $d\bar{e} = -1$. From the self-duality of V_5 , d and e are both real.

Thus $\{d, e\} = \{1, -1\}$.

WLOG let us say $d = 1$, $e = -1$ (the other choice just swaps V_5 and V_6).

Method 2: From the first-column-orthogonality identities, we have

$$a_5 + a_6 = -2, \quad b_5 + b_6 = 2, \quad c_5 + c_6 = -2,$$

and since all these variables are roots of unity, by the sharp version of the triangle inequality

$$a_5 = a_6 = c_5 = c_6 = -1, \quad b_5 = b_6 = 1.$$

From the first-row-orthogonality identities, let's introduce a variable x with

$$d_5 = -e_5 = x.$$

From the tensor product identities,

$$d_6 = -d_5 = -x, \quad e_6 = -e_5 = x.$$

From the column-norm identities, we have

$$2|x|^2 = |d_5|^2 + |d_6|^2 = 2,$$

so $|x| = 1$. This means $x \in \{-1, 1\}$, by the self-duality of V_5 . If $x = 1$ we get

$$d_5 = e_6 = 1, \quad d_6 = e_5 = -1,$$

and if $x = -1$ this just reverses the rows.

Final answer: Up to reversing the rows, the two missing characters are

	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6
χ_{V_5}	1	-1	1	-1	1	-1
χ_{V_6}	1	-1	1	-1	-1	1

4. (a) (i) We must show that ρ is an algebra homomorphism. (We will check this for ρ considered as a map $A \rightarrow \mathbf{C}$, but the identification between \mathbf{C} and $\text{End}(\mathbf{C})$ also identifies them as algebras, so the same will be true for ρ considered as a map $A \rightarrow \text{End}(V)$.)

unseen ↓

It is clearly \mathbf{C} -linear, and clearly $\rho(1) = 1$. Now, given polynomials $f(x)$ and $g(x)$, we have

$$\rho(fg) = (fg)(3) = f(3)g(3) = \rho(f)\rho(g).$$

4, A

- (ii) From (i), $\rho : A \rightarrow \mathbf{C}$ is an algebra homomorphism. Therefore it is a homomorphism of A -modules:

unseen ↓

$$\rho(\rho_A(f)g) = \rho(fg) = \rho(f)\rho(g).$$

Moreover ρ is surjective, since for all $c \in \mathbf{C}$, $\rho(c) = c$.

Therefore, by the First Isomorphism Theorem, $(\mathbf{C}, \rho) \cong A/\ker(\rho)$. It therefore suffices to describe $\ker(\rho)$.

In fact, $\ker(\rho)$ is the ideal $\langle x - 3 \rangle$ in A : for a polynomial $f(x) \in A$, we have that $\rho(f) = f(3)$ vanishes if and only if $f(x)$ is a multiple of $x - 3$.

- (b) (i) Given $M \in V_x$ we have $Mx = 0$, so for all $N \in A$,

5, C

unseen ↓

$$(NM)x = N(Mx) = N(0) = 0.$$

Also, for nonzero x , V_x is two-dimensional, so is a proper nonzero submodule of A .

2, A

1, A

unseen ↓

- (ii) **Method 1:** Let V be a proper nonzero submodule of A . By Theorem 4.4.1 in the lecture notes, V is isomorphic as an A -module to $(\mathbf{C}^2)^m$, for some m . In fact, since $\dim A = 4$, to have V be proper and nonzero we must have $m = 1$. Therefore there exists an injective homomorphism of A -modules,

$$T : \mathbf{C}^2 \rightarrow A,$$

with $\text{im}(T) = V$. Write $p_1, p_2 : A \rightarrow \mathbf{C}^2$ for the projections from A onto the first and second columns, respectively. Since p_1 is A -linear, the map $p_1 \circ T : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ is A -linear, so by Schur's lemma there exists $\lambda \in \mathbf{C}$ with $p_1 = \lambda \circ \text{id}_{\mathbf{C}^2}$. Similarly there exists $\mu \in \mathbf{C}$ with $p_2 = \mu \circ \text{id}_{\mathbf{C}^2}$. We then have that for all $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{C}^2$,

$$T(v) = \begin{pmatrix} \lambda a & \mu a \\ \lambda b & \mu b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \lambda & \mu \end{pmatrix} = v \begin{pmatrix} \lambda & \mu \end{pmatrix}.$$

So (by the **Lemma** at the end of the solution)

$$V = \text{im}(T) = \left\{ v \begin{pmatrix} \lambda & \mu \end{pmatrix} \mid v \in \mathbf{C}^2 \right\} = V_x,$$

$$\text{for } x = \begin{pmatrix} \mu \\ -\lambda \end{pmatrix}.$$

8, D

Method 2: Let V be a proper nonzero submodule of A . Then there exists a nonzero linear map $\psi : A \rightarrow \mathbf{C}$ such that $V \subseteq \ker(\psi)$. Let us say,

$$\psi \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = kx + ly + mz + nw,$$

for some $k, l, m, n \in \mathbf{C}$ not all zero. In particular, either $(k, l) \neq 0$ or $(m, n) \neq 0$. WLOG assume the former. We will show that $V = V_x$, for $x = \begin{pmatrix} k \\ l \end{pmatrix}$.

Indeed, let $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in V$. Then we have,

$$V \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix},$$

so

$$0 = \psi \left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right) = kx + ly.$$

Similarly arguing with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we see that $0 = kz + lw$. Therefore

$$0 = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

This shows that $V \subseteq V_x$. In fact V_x is simple, so, since V is nonzero, $V = V_x$.

To see V_x is simple, either appeal to Theorem 4.4.1 ($\text{Mat}_2(\mathbf{C})$ -modules are all isomorphic to $(\mathbf{C}^2)^{\oplus n}$ for some n , and here since $\dim V_x = 2$ we have $n = 1$)

or prove simplicity directly: Let $x = \begin{pmatrix} k \\ l \end{pmatrix}$; then

$$V_x = \left\{ v \begin{pmatrix} -l & k \end{pmatrix} \mid v \in \mathbf{C}^2 \right\}.$$

For any nonzero $M \in V_x$, we have $M = v \begin{pmatrix} -l & k \end{pmatrix}$ for some nonzero $v \in \mathbf{C}^2$, and then, for any $N = w \begin{pmatrix} -l & k \end{pmatrix} \in V_x$, by choosing $P \in A$ such that $Pv = w$ we have $PM = N$; thus $AM = V_x$.

Lemma: Let $x = \begin{pmatrix} k \\ l \end{pmatrix}$ be a nonzero vector; then

$$V_x = \left\{ v \begin{pmatrix} -l & k \end{pmatrix} \mid v \in \mathbf{C}^2 \right\}.$$

Proof: Since x is nonzero, either k or l is nonzero; WLOG let l be nonzero.

Clearly for any $v \in \mathbf{C}^2$ we have

$$v \begin{pmatrix} -l & k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = v \cdot 0 = 0,$$

so $v \begin{pmatrix} -l & k \end{pmatrix} \in V_x$. Conversely if

$$0 = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} kx + ly \\ kz + lw \end{pmatrix}$$

then $y = -k/lx$, $w = -k/lz$ so

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & -k/lx \\ z & -k/lz \end{pmatrix} = \begin{pmatrix} -x/l \\ -z/l \end{pmatrix} \begin{pmatrix} -l & k \end{pmatrix}.$$

5. (a) (i) An element of $\text{coInd}_H^G \text{coInd}_K^H V$ is a function $F : G \rightarrow \text{Fun}(H, V)$ such that
- for all $g \in G$, $F(g) \in \text{coInd}_K^H V$;
 - for all $g \in G$ and all $h \in H$, $F(hg) = \rho_{\text{coInd}_K^H V}(h)F(g)$.

unseen ↓

Let us unpack these and see that they agree with the two conditions in the definition of \tilde{V} .

First, for $g \in G$, $F(g) \in \text{coInd}_K^H V$ means that for all $h \in H$ and $k \in K$, $F(g)(kh) = \rho(k)F(g)(h)$.

Second, for $g \in G$ and $h \in H$, $F(hg) = \rho_{\text{coInd}_K^H V}(h)F(g)$ means that for all $h' \in H$,

$$F(hg)(h') = \rho_{\text{coInd}_K^H V}(h)(F(g))(h') = F(g)(h'h).$$

Let us now check that $\rho_{\text{coInd}_H^G \text{coInd}_K^H V}$ behaves as described. Indeed, for $F \in \tilde{V}$ and $g, g' \in G$,

$$\rho_{\text{coInd}_H^G \text{coInd}_K^H V}(g)(F)(g') = F(g'g) = \tilde{\rho}(g)(F)(g').$$

- (ii) For $g \in G$, $h \in H$, and $k \in K$,

$$F_f(g)(kh) = f(khg) = \rho(k)f(hg) = \rho(k)F_f(g)(h).$$

5, M

unseen ↓

For $g \in G$ and $h, h' \in H$,

$$F_f(hg)(h') = f(h'hg) = F_f(g)(h'h).$$

- (iii) The function T is clearly linear. Let us check the G -linearity.

Let $f \in \text{coInd}_K^G V$, let $g, g' \in G$, and let $h \in H$. Then

$$\begin{aligned} T(\rho_{\text{coInd}_K^G V}(g)f)(g')(h) &= F_{\rho_{\text{coInd}_K^G V}(g)f}(g')(h) \\ &= (\rho_{\text{coInd}_K^G V}(g)f)(hg') \\ &= f(hg'g) \\ &= F_f(g')(h) \\ &= (\tilde{\rho}(g)F_f)(g')(h) \\ &= (\tilde{\rho}(g)T(f))(g')(h) \end{aligned}$$

3, M

unseen ↓

- (iv) **Injectivity:** Let $f \in \text{coInd}_K^G V$ and suppose that $T(f) = 0$. Then for all $g \in G$,

$$0 = T(f)(g)(1) = F_f(g)(1) = f(1 \cdot g) = f(g).$$

4, M

unseen ↓

So $f = 0$.

Surjectivity: Given $F \in \tilde{V}$, define $f : G \rightarrow V$ by $f(g) := F(g)(1)$. We have that $f \in \text{coInd}_K^G(V)$, since for $g \in G$ and $k \in K$,

$$f(kg) = F(kg)(1) = F(g)(1 \cdot k) = F(g)(k \cdot 1) = \rho(k)F(g)(1) = \rho(k)f(g).$$

So f is in the domain of T . We then have that for all $g \in G$ and all $h \in H$,

$$\begin{aligned} F(g)(h) &= F(g)(1 \cdot h) \\ &= F(hg)(1) \\ &= f(hg) \\ &= T(f)(g)(h). \end{aligned}$$

Therefore $F = T(f)$.

2, M

3, M

(b) Let \mathbf{C} denote the trivial representation of the trivial group. By (a) we have,

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$$\mathbf{C}[G] \cong \text{coInd}_{\{1\}}^G \mathbf{C} \cong \text{coInd}_H^G \text{coInd}_{\{1\}}^H \mathbf{C}.$$

3, M

In fact,

$$\mathbf{C}[G] \cong \text{coInd}_H^G \mathbf{C}[H].$$

For marking: Student need not make this last observation.

Review of mark distribution:

Total A marks: 30 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 14 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70039 Group Representation Theory Markers Comments

- Question 1** This problem was done well. The trickier pieces were (b)(iv), (c) and (d).
In (b)(iv) and (d), most students realised the ideas which should be used, but not all could fit these ideas together correctly.
In (c), many students had trouble writing down the C_4 -action on X concretely.
- Question 2** Most students could make a serious attempt at this problem, but there were several tricky sub-parts.
In the irreducibility argument in (b) and the non-isomorphism argument in (d), many students made careless (hasty?) errors despite having the right ideas.
The multiplicativity of the map in (c) and the irreducibility argument in (e) were difficult for most students.
- Question 3** This problem was done fairly well. Since (c) was a prerequisite for (e), the large minority of students who did not solve (c) scored low overall.
On (e), most students who could attempt it had all the necessary ideas for a solution, but many seem to have worked too quickly, making careless errors or leaving logical gaps.
- Question 4** This problem was difficult, largely due to (b)(ii). Most students did well on the other parts, although minor oversights (such as the surjectivity argument in (a)(ii)) were common.
- Question 5** This problem was done either very well or very poorly, more commonly the latter: the abstraction seems to have been difficult for most students.