IMPERIAL

MATH70059

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Differential Topology

Date: Thursday, May 29, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Show that there exists a closed 2-form on $\mathbb{R}^3 - \{0\}$ that is not exact. (4 marks)

(b) If M is a compact orientable manifold with nonempty boundary, show that there is no smooth map $f : M \to \partial M$ with $f|_{\partial M} = Id_{\partial M}$. (4 marks)

(c) Let M be a smooth oriented manifold of dimension n, ω a closed k-form on M with compact support. Let $S \subset M$ be a compact oriented k-dimensional submanifold without boundary. Suppose $\int_S \omega \neq 0$. Show the following:

(i) ω is not exact on M and $\omega|_S$ is not exact on S. (3 marks)

(ii) S does not bound a compact oriented $(k+1)$-dimensional submanifold $N \subset M$. (3 marks)

(d) Let $M \hookrightarrow \mathbb{R}^{n+1}$ be an orientable closed hypersurface, so $\dim M = n$. Let $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the negation map, $\iota(x) = -x$. Assume that $M = \iota(M)$. Prove the formula:

$$\iota^*|_{H^n(M)} = (-1)^{n+1} I_{H^n(M)}.$$

(6 marks)

(Total: 20 marks)

2. (a) Consider the product of spheres $M := \mathbb{S}^2 \times \mathbb{S}^1$.

(i) Compute the de Rham cohomology of M using the Mayer–Vietoris sequence. (5 marks)

(ii) Is M homotopy equivalent to $\mathbb{S}^3$? Justify your answer. (1 mark)

(iii) Let N be a manifold of Euler characteristic equal to m. Compute the Euler characteristic of $\chi(M \times \mathbb{S}^n)$ for all n. (3 marks)

(b) In this part consider the real projective space $\mathbb{RP}^n = \mathbb{S}^n / \sim$ where $\sim$ is the equivalence relation that identifies each point $x \in \mathbb{S}^n$ with its antipodal point $-x$. Let $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ be the quotient map.

(i) Show that, if $[\omega] \in H^n(\mathbb{RP}^n)$, then $\pi^*[\omega]$ is invariant under ι^*. (2 marks)

(ii) Deduce that $\mathbb{RP}^n$ is nonorientable if n is even. (4 marks)

(iii) Conversely, if n is odd, prove that $\mathbb{RP}^n$ is orientable by constructing a nonvanishing volume form. (5 marks)

(Total: 20 marks)

3. Let $\mathbb{T}^2 := (\mathbb{S}^1)^2$, and consider the map $f_{m,n} : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$f_{m,n}(z, w) = (z^m, w^n),$$

where $z, w \in \mathbb{S}^1$, $m, n \in \mathbb{Z}$, and $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

(a) Define the *degree* of a smooth map of closed oriented n-manifolds. (2 marks)

(b) Fix an orientation on $\mathbb{T}^2$. Compute the degree of the map $f_{m,n}$. (2 marks)

(c) Compute the map $f_{m,n}^*$ on cohomology $H^k(\mathbb{T}^2)$, in all degrees k. (3 marks)

(d) Give an example, with proof, of two maps of closed oriented manifolds which have the same degree but are not homotopic. (4 marks)

(e) Let $f : M \to N$ and $g : N \to P$ be smooth maps of closed oriented n-manifolds. Show that $\deg(g \circ f) = \deg(g) \deg(f)$. (3 marks)

(f) Now suppose that M is a closed oriented manifold and that $f : M \to M$ is a smooth map such that f^m is the identity for $m \geq 1$, with m the minimal possible positive integer. What are the possible degrees of f? For each $m \geq 1$ and each possible degree, find an M and an f achieving this degree. (6 marks)

(Total: 20 marks)

4. Let $M = (\mathbb{S}^1)^3 = \mathbb{T}^3$ be the three-torus and let $f : M \to \mathbb{R}$ be the function

$$f(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) = \sin\theta_1 + \sin\theta_2 + \sin\theta_3.$$

Let g be the metric on M obtained from the product of the standard metrics on $\mathbb{S}^1$, i.e., via the embedding $M \subseteq \mathbb{C}^3 \cong (\mathbb{R}^2)^3 \cong \mathbb{R}^6$.

(a) Compute the critical points of f. (6 marks)

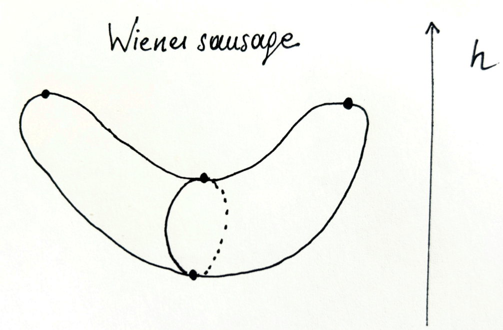
(b) Show that f is a Morse function, and compute the indices of its critical points. (4 marks)
[Hint: You may use the fact that $e^{i\theta} \to \sin\theta$ is a Morse function on the circle.]

(c) Describe the stable and unstable manifolds of all the critical points of f. (5 marks)
[Hint: You may shorten your answer on this part and the next by using some available symmetry.]

(d) Show that the pair (f, g) is Morse-Smale. (5 marks)

(Total: 20 marks)

5. Let M be the Wiener sausage 2-dimensional surface depicted below, equipped with the metric and outward orientation from its embedding into $\mathbb{R}^3$:



For the purposes of this problem, it is fine to locate critical points and draw flow lines using the picture.

(a) Let h be the height function, which you may assume is Morse–Smale.

(i) Write down the underlying vector spaces of the Morse complex. (3 marks)

(ii) Compute the differential of the Morse complex. (6 marks)

(iii) Compute the Morse homology of the Wiener sausage. (4 marks)

(b) Let N be a closed manifold with a Morse–Smale function h whose number of critical points equals the total dimension of the de Rham cohomology.

(i) Prove that the differentials of the associated Morse complex are zero (over $\mathbb{Z}$). (4 marks)

(ii) In the preceding situation, show that the Morse homology groups (over $\mathbb{Z}$) are also free $\mathbb{Z}$-modules. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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MATH70059

Differential Topology (Solutions)

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1. (a) $\mathbb{R}^3 - \{0\}$ is homotopy equivalent to $\mathbb{S}^2$, hence they have isomorphic de Rham cohomology. Since $H^2(\mathbb{S}^2) \neq 0$, this shows the claim. sim. seen ⇓

4, A

(b) Suppose there exists such an f. Let ω be a volume form on ∂M (this is possible since M and hence ∂M are orientable). Then $\int_{\partial M} \omega \neq 0$. Since f is the identity on ∂M we have $f^*\omega|_{\partial M} = \omega$. Hence $\int_{\partial M} f^*\omega \neq 0$. But using the fact that ω is closed, since it is a volume form, we have unseen ⇓

$$0 \neq \int_{\partial M} f^*\omega = \int_M d(f^*\omega) = \int_M f^* d\omega = 0.$$

This contradicts Stokes' Theorem. 3, C

(c) (i) Assume $\omega|_S$ is exact. Then there exists a $k-1$-form η on S such that $d\eta = \omega|_S$. Hence unseen ⇓

$$0 \neq \int_S \omega = \int_S d\eta = \int_{\partial S} \eta = 0,$$

since S has no boundary, which is a contradiction. Hence $\omega|_S$ is inexact on S, and consequently ω is inexact on M. 3, A

(ii) Assume that there exists a $k+1$-dimensional submanifold $N \subset M$ such that $\partial N = S$. Then since ω is closed unseen ⇓

$$0 = \int_N d\omega = \int_{\partial N} \omega = \int_S \omega,$$

which is a contradiction. 4, B

(d) Let ω be the Euclidean volume form on M. The antipodal map is a Euclidean symmetry, so it preserves unsigned volume. As a result, $\iota^*\omega = \pm\omega$. So we just have to find the sign. (Alternatively, if one completes problem 3 first, one can argue that the degree of *any* diffeomorphism, or even homotopy equivalence, is necessarily ± 1 since the product of this degree and the degree of the inverse is one, see 3e and its solution). The sign is determined by whether or not the orientation is preserved. unseen ⇓

Notice that the map ι on $\mathbb{R}^{n+1}$ has the property that $d\iota = -I$, identifying all tangent spaces with $\mathbb{R}^{n+1}$ in the usual way. As the determinant is $(-1)^{n+1}$, it preserves or reverses orientation depending on whether $n+1$ is even or odd, respectively.

Next, let us consider the outward orientation on M, meaning that a frame on M is positively oriented if, after adjoining an outward-pointing vector from the hypersurface, one obtains the positive orientation on $\mathbb{R}$. This yields a consistent orientation on M. Note that $d\iota$ sends an outward-pointing vector to another outward pointing vector, since the outside and the inside of M are also preserved by ι, noting that the outside is unbounded whereas the inside bounded. Applying $d\iota$ to a positively oriented frame on M, we get a frame that, together with an outward pointing vector, is the orientation of the preceding paragraph, i.e., positive or negative depending on whether $n+1$ is even or odd. Thus, overall, orientation on M is preserved if and only if $n+1$ is even.

Finally we obtain that $\iota^*_{H^n(M)} = (-1)^{n+1} I_{H^n(M)}$, as desired. 6, D

2. (a) (i) We use the covering $U := \mathbb{S}^2 \times S^1 \setminus \{p\}$ and $V := \mathbb{S}^2 \times S^1 \setminus \{q\}$ for two distinct points $p, q \in S^1$. Then $U, V \cong \mathbb{S}^2 \times (0,1)$ which is homotopic to $\mathbb{S}^2$, and $U \cap V \cong \mathbb{S}^2 \times ((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))$ which is homotopic to the disjoint union of two 2-spheres. We get the sequence: meth seen ⇓

$$0 \to H^0(\mathbb{S}^2 \times S^1) = \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1(\mathbb{S}^2 \times S^1) \to 0$$
$$\to 0 \to H^2(\mathbb{S}^2 \times S^1) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^3(\mathbb{S}^2 \times S^1) \to 0.$$

The Euler characteristic of the first part between zeros is zero, so we conclude that $\dim H^1(\mathbb{S}^2 \times S^1) = 1$. From the next part between zeros we get that H^2 and H^3 of $\mathbb{S}^2 \times S^1$ have the same dimension. By Poincaré duality, since S^1 and S^2 and hence the product is orientable and connected, we get that $H^3(\mathbb{S}^2 \times S^1) \cong \mathbb{R}$, so the same is true for H^2. We get that all cohomology groups (from degree zero to three) of $S^2 \times S^1$ are one-dimensional. 5, B

sim. seen ⇓

(ii) No, because $H^1(M)$ and $H^2(M)$ are nonzero, unlike the case of $\mathbb{S}^3$. 2, A

(iii) By the Künneth theorem, $H^k(N \times \mathbb{S}^n) \cong \bigoplus_{i+j=k} H^i(N) \otimes H^j(\mathbb{S}^n)$. Applying the cohomology of the sphere, this implies that $\chi(N \times \mathbb{S}^m) = \sum_{i=0}^{\dim N} (-1)^i \dim H^i(N) + (-1)^{n-i} \dim H^i(N)$. Rewriting this, we get $\chi(N \times \mathbb{S}^m) = (1 + (-1)^n)m$. unseen ⇓ 3, B

unseen ⇓

(b) (i) This is because $\pi \circ \iota = \pi$, so that $\pi^*[\omega] = (\pi \circ \iota)^*[\omega] = \iota^* \pi^*[\omega]$. 2, A

(ii) We have, if n is even, then for every class $[\omega] \in H^n(\mathbb{RP}^n)$, then $\pi^*[\omega] = -\pi^*[\omega]$, so that $\pi^*[\omega] = 0$. If $\mathbb{RP}^n$ is orientable then there is a nonvanishing volume form ω, but then $\pi^*\omega$ is also a nonvanishing volume form, which is a contradiction since its cohomology class is zero. 3, C

(iii) The antipodal map is the restriction of the negation map on $\mathbb{R}^{n+1}$. The standard volume form on $\mathbb{S}^n$, for n odd, is invariant under this because this map preserves Euclidean volume, but also preserves orientation because the degree of the antipodal map is positive. Alternatively, since the antipodal map commutes with Euclidean symmetries, ι^* of the standard (symmetric) volume form must be another symmetric volume form, but this form is unique and is the standard Euclidean volume.

At every point $p \in \mathbb{S}^n$, the differential dp gives an isomorphism $T_p\mathbb{S}^n \to T_{\pi(p)}\mathbb{RP}^n$. Using the dual of this isomorphism we can define at every point of $\mathbb{RP}^n$ a differential n-form from each point of its preimage. However there are two points in each preimage. The form we get from $T_{\iota(p)}\mathbb{S}^n \to T_{\pi(p)}\mathbb{RP}^n$ is the same as the one we get from the composition $\pi^* \circ \iota^* : T_p\mathbb{S}^n \to T_{\pi(p)}\mathbb{RP}^n$. But as ι^* preserves the volume form the two images are the same. So the two possible forms we can define at each $p \in \mathbb{RP}^n$ are the same, and we get a well-defined n-form on $\mathbb{RP}^n$. It is nonvanishing everywhere by construction. Taking the resulting consistent orientation, we conclude that $\mathbb{RP}^n$ is orientable.

5, D

3. (a) For a map $f : X \to Y$, this is the number m such that, if ω_Y is a volume form ($\int_Y \omega_Y = 1$), then $m = \int_X f^*\omega_Y$. In other words, $f^*[\omega_Y] = m[\omega_X]$ for $[\omega_X]$ a volume form on X. seen $\Downarrow$

2, A

(b) We have $f^*_{m,n}(d\theta_1 \wedge d\theta_2) = mn\, d\theta_1 \wedge \theta_2$, so the degree is mn. sim. seen $\Downarrow$

[Note in this part and next that students may assume that one has the basis we use of cohomology of $\mathbb{T}^2$, but alternatively this can be deduced from the cohomology of $\mathbb{S}^1$ (assumed) together with the Künneth theorem.] 2, A

unseen $\Downarrow$

(c) A basis for cohomology in degrees less than two is given by $[1], [d\theta_1], [d\theta_2]$. The same argument shows that the map $f^*_{m,n}$ has the form: $f^*_{m,n}d\theta_1 = m\, d\theta_1$ and $f^*_{m,n}d\theta_2 = n\, d\theta_2$. Of course, $f^*_{m,n}$ is the identity in degree zero. 3, B

(d) We saw in (c) that $f^*_{m,n}$ changes when we alter values of m, n, so $f_{m,n}$ and $f_{m',n'}$ cannot be homotopic unless $(m, n) = (m', n')$. On the other hand, by (b), to have the same degree we only need $mn = m'n'$. So an example is furnished by $f_{1,4}$ and $f_{2,2}$ (or more simply, $f_{0,0}$ and $f_{0,1}$). unseen $\Downarrow$

4, A

(e) This is by definition of degree in (a): we just need to see what multiple of a volume form we get. But given ω_P, a volume form on P, we have $f^*g^*[\omega_P] = \deg(g)f^*[\omega_N] = \deg f \deg g\omega_M$, for ω_M, ω_N volume forms on M and N, respectively. sim. seen $\Downarrow$

3, A

(f) We have that $\deg(f)^m = 1$ by (e). So if m is odd, then $\deg(f) = 1$. An example is furnished by a rotation of $M = \mathbb{S}^2$ about some axis by $2\pi/m$. Next, if m is even, then we obtain $\deg(f) = \pm 1$. To obtain degree 1, we can use the same example as before. To obtain degree -1, we can compose the rotation with the antipodal map. Since the antipodal map commutes with every linear transformation of $\mathbb{R}^3 \supseteq \mathbb{S}^2$, it commutes with the rotation, so the order of f is then still $m = \mathrm{lcm}(m, 2)$. unseen $\Downarrow$

6, C

4. (a) The function f is given by: meth seen $\Downarrow$

$$f(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) = \sin\theta_1 + \sin\theta_2 + \sin\theta_3$$

We compute the gradient of f by differentiating each factor with respect to the corresponding θ_i:

$$\nabla f = \left(\frac{\partial f}{\partial\theta_1}, \frac{\partial f}{\partial\theta_2}, \frac{\partial f}{\partial\theta_3}\right) = (\cos\theta_1, \cos\theta_2, \cos\theta_3)$$

Critical points occur where the gradient is zero, i.e., when $\cos\theta_i = 0$ for all i. This happens when:

$$\theta_i = \pm\frac{\pi}{2} \quad \text{for each} \quad i = 1, 2, 3$$

Thus, there are $2^3 = 8$ critical points corresponding to all possible sign combinations of $\pm\frac{\pi}{2}$. These critical points are, in terms of the complex coordinates on the torus:

$$(\pm i, \pm i, \pm i).$$

6, A

(b) Since $\sin\theta$ itself is a Morse function on the circle, by the Morse lemma in a neighborhood of a critical point we can choose coordinates so that each $\cos\theta_i$ becomes $\pm 1 \mp \theta_i^2$. Therefore each critical point is nondegenerate and the index is the sum of the indices of $\sin\theta_i$ for each factor. So there is one critical point of index $0 + 0 + 0 = 0$, one of index $1 + 1 + 1 = 3$, and three each of index $1+0+0 = 0+1+0+0+0+1 = 1$ and $1+1+0 = 1+0+1 = 0+1+1 = 2$. meth seen $\Downarrow$

4, A

(c) The gradient flow downward on a single factor $\mathbb{S}^1$, equipped with the Morse function $\sin\theta$, takes every point except for the critical point i in the limit $t \to \infty$ to the critical point $-i$. So, on $\mathbb{T}^3$, the limit of the gradient flow beginning at a point (a, b, c) is the critical point $(\pm i, \pm i, \pm i)$ with all signs negative except for those where the coordinate was initially i, in which case that coordinate remains constant. As a result, the stable manifold of the critical point $(-i, -i, -i)$ is $\mathbb{T}^3$ take away the three coordinate tori $\mathbb{T}^2$ obtained by setting one of the coordinates to i. Similarly, the stable manifold of the critical point $(-i, -i, i)$ is then $\mathbb{T}^2 \times \{i\}$ take away the two circles $\mathbb{S}^1 \times \{i\} \times \{i\}$ and $\{i\} \times \mathbb{S}^1 \times \{i\}$. The stable manifold of the critical point $(-i, i, i)$ is then $\mathbb{S}^1 \times \{i\} \times \{i\} \setminus \{(i, i, i)\}$. Finally, the stable manifold of the critical point (i, i, i) is itself. The other stable manifolds can be obtained from these by permuting coordinates. meth seen $\Downarrow$

The unstable manifolds can be obtained from the stable manifolds by negating all of the θ_i, by symmetry. For instance, the unstable manifold of (i, i, i) is $\mathbb{T}^3$ take away the three coordinate tori $\mathbb{T}^2$ obtained by setting one of the coordinates to $-i$.

5, D

unseen $\Downarrow$

(d) Finally, to show that (f, g) is Morse-Smale, we need to show that a stable and an unstable manifold intersect transversely. The intersection of the unstable manifold $M^u(\varepsilon_1 i, \varepsilon_2 i, \varepsilon_3 i)$ with the stable manifold $M^s(\delta_1 i, \delta_2 i, \delta_3 i)$ is nonempty if and only if $\delta_i \leq \varepsilon_i$ for all i and, provided it is nonempty, it has dimension $\frac{1}{2}\sum_i(\varepsilon_i - \delta_i)$. This is just $\dim M^u + \dim M^s - 3$, which can be seen because it is true if $\varepsilon_i = \delta_i = 1$

for all i (where $\dim M^u = 3$ and $\dim M^s = 0$), and it remains true if we modify just one of the δ_i or ε_i preserving the inequalities $\delta_i \leq \varepsilon_i$ for all i. So when stable and unstable manifolds intersect, they intersect transversely. 5, B

meth seen ⇓

5. (a) (i) There are four critical points, of indices $2, 2, 1,$ and 0, as we easily see from the picture. So the Morse complex is $\mathbb{Z}^2 \to \mathbb{Z} \to \mathbb{Z}$. 3, M

meth seen ⇓

(ii) The differentials can be computed as follows: We have to find the flow lines passing through two of the critical points. There is a unique flow line from each index two point to the index one point. To orient these for the purposes of the Morse complex, pick an orientation of M, say outward-normal, and pick a co-orientation of the index one point, say into the page of the picture. Then the orientation of the flow line is given by the right hand rule of the flow direction and the co-orientation for the index one point, compared against the outward normal direction. Clearly the two flow lines have opposite orientations. So the differential $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is given by $(a, b) \mapsto a - b$ (or the negative of this, depending on choices). 6, M

meth seen ⇓

(iii) Next, there are two flow lines from the index-one critical point to the index-zero one, forming a circle together with the two critical points. The orientations on these are opposite: they are given by comparing against the co-orientation we pick at the index-zero point. So the complex, up to rescaling individual indices, becomes

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

The homology here is then $\mathbb{Z}$ in degrees zero and two. 4, M

(b) (i) In this case the Morse complex over $\mathbb{R}$ has total dimension, before taking cohomology, equal to the total dimension of the de Rham cohomology. Since the Morse homology must actually be isomorphic to the de Rham cohomology, this implies the differentials are zero, otherwise we would be taking a quotient of a proper vector subspace, which then has smaller dimension; the dimension is also finite since the manifold N is closed. As a result the sums of orientations defining the differential are zero, and the differentials are all zero over any ring, including over $\mathbb{Z}$. 4, M

unseen ⇓

unseen ⇓

(ii) Now, if we take homology over $\mathbb{Z}$ we find again that the Morse homology is a free $\mathbb{Z}$-module—it is the same as the original complex. 3, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70059 Differential Topology Markers Comments

Question 1	Q1 was well answered.
Question 2	Mostly fine. 2a)iii) seemed to be hard: almost no one thought of using the sum formula for Euler characteristic and if they did, they didn't get the correct answer. Several mistakes in b)ii) and iii). For ii) some students didn't understand how to use the previous part. For iii) Some students used the differential of the projection but didn't account for the fact that there are two preimages which need to be compatible (they are as the antipodal map preserves orientation when n odd).
Question 3	Mostly well answered. Finding examples in e) seemed hard.
Question 4	This was probably the hardest question. a) and b) were well answered, but almost no one was able to figure out the stable and unstable manifolds, partly because students didn't really think to look at a gradient flow on one factor S1…This of course made it impossible to do d) entirely.
Question 5	Answers to this question were mixed. Some students got a)ii) wrong because they were not consistent with the orientations. b)ii) Clearly some students forgot what a free Z-module is. But otherwise it was well answered.