Imperial College London
Department of Mathematics

Solutions to Question Sheet 2 - Probl. Class week 4

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

This is the problem sheet for the problem classes in week 4. All questions can be attempted with the material in lectures 1–5. Solutions will be released after the classes on Monday of week 4.

Question 1 Let $n \in \mathbb{N}$, $n \geq 2$. Suppose $D : M_n(\mathbb{R}) \to \mathbb{R}$ is a function on which elementary row operations have the same effect as they do for det (for example, if B is obtained from $A \in M_n(\mathbb{R})$ by interchanging two rows, then $D(B) = -D(A)$, etc.). Suppose also that $D(I_n) = 1$. Prove that $D(C) = \det(C)$ for all $C \in M_n(\mathbb{R})$.
Harder: What if we replace $\mathbb{R}$ by an arbitrary field F?

Solution: Suppose first that C is row equivalent to the identity matrix I_n. So there is a sequence of elementary row operations which turns I_n into C. Then, by induction on the number of operations used, $\det(C) = D(C)$ (the base step, where there are no operations used and $C = I_n$ is the assumption $D(I_n) = 1$).

If C is not row equivalent to I_n, then there is a sequence of elementary row operations which turns C into a matrix with a row of zeros. Multiplying this row by a non-zero constant $\alpha \neq 1$ we get $(\alpha - 1)D(C) = 0$.

For an arbitrary field, the same argument will work unless $F = \{0, 1\}$: then we cannot find a non-zero constant $\alpha \neq 1$. So the argument will fail in $\mathbb{F}_2$, the field of integers modulo 2. The result also breaks down in this case: for example in $M_2(\mathbb{F}_2)$ we can let $D(C) = 1$ for all rank 1 matrices C and $D(C) = \det(C)$ in other cases, and the required properties hold.

Question 2 For each of the following linear maps $T : V \to V$, choose a basis B for V and compute $[T]_B$. Hence, or otherwise, compute $\det(T)$.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) V is the vector space of all 2×2 matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(iii) V is the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

Solution: (i) The matrix of T with respect to the standard basis is $\begin{pmatrix} -1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$.

So the determinant of T is 2.

(ii) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$.

So the determinant is $\det(M)^2 = 36$.

(iii) T sends $1 \mapsto 0$, $x \mapsto 3x$, $x^2 \mapsto x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. Thus $\det(T) = 0$ (which we can also see from the fact that T is singular).

Question 3 Suppose $n \geq 2$ and $A \in M_n(F)$. The adjugate matrix $adj(A)$ is the transpose of the matrix of cofactors of A and we showed that $adj(A)A = \det(A)I_n$. Give an expression for $adj(adj(A))$ in the case where A is invertible.

Solution: Using the given equation, we have $adj(adj(A))adj(A) = \det(adj(A))I_n$ and (taking determinants) $\det(adj(A))\det(A) = \det(A)^n$. So if $\det(A) \neq 0$ we obtain that $adj(A)$ is invertible and

$$adj(adj(A)) = \det(A)^{n-1}(adj(A))^{-1} = \det(A)^{n-2}A.$$

Question 4 Suppose F is a field. Let $n \in \mathbb{N}$ and $a_0, ..., a_{n-1} \in F$, not all zero. Using the Vandermonde determinant, prove that the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

has at most $n - 1$ distinct roots in F, i.e. there are at most $n - 1$ distinct $\alpha \in F$ such that $f(\alpha) = 0$.

Solution: Suppose $x_1, ..., x_n \in F$ are roots, so $f(x_i) = a_0 + a_1 x_i + \cdots + a_{n-1}x_i^{n-1} = 0$, for $i = 1, ..., n$. Then

$$a_0 \begin{pmatrix} 1 \\ 1 \\ \ddots \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ \ddots \\ x_n \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} x_1^{n-1} \\ x_2^{n-1} \\ \ddots \\ x_n^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ddots \\ 0 \end{pmatrix}.$$

Hence the columns of the Vandermonde determinant are linearly dependent so the determinant is 0. Hence $x_i = x_j$ for some $i \neq j$.

Question 5 Suppose U, V, W are vector spaces over a field F and $T : U \to V$ and $S : V \to W$ are linear transformations. Show that the composition $S \circ T : U \to W$ is a linear transformation. If U, V, W are finite dimensional with bases B, C, D, prove that

$${}_D[S \circ T]_B = {}_D[S]_C\, {}_C[T]_B.$$

Solution: To check that $S \circ T$ is a linear transformation, just use the definition.

For the next part, take a deep breath and repeatedly use a result 4.3.4(?) from last term:

Let $u \in U$. Then:

$${}_D[S \circ T]_B[u]_B = [S(T(u))]_D = {}_D[S]_C[T(u)]_C = {}_D[S]_C\, {}_C[T]_B[u]_B.$$

As the vector u is arbitrary, $[u]_B$ takes all possible values in F^n ($n = \dim U$). The required matrix equality now follows from Lemma 4.3.4 in the second term notes.

Question 6 Let V be a vector space over a field F and $T : V \to V$ be a linear transformation. Suppose that $\lambda \in F$ is an eigenvalue of T. Let $m \geq 1$ be an integer and denote by T^m the composition $T \circ \ldots \circ T$ (m times). Note that this is a linear transformation $V \to V$.

i) Show that λ^m is an eigenvalue of T^m.

ii) If $a_0, \ldots, a_m \in F$ are such that $a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m = 0$, show that λ is a root of the polynomial $p(x) = a_0 + a_1 x + \ldots + a_m x^m$.

Solution:

i) Suppose $0 \neq v \in V$ is an eigenvector with eigenvalue λ. Then $T(v) = \lambda(v)$. Furthermore (see by induction or otherwise) $T^m(v) = \lambda^m(v)$. So v is an eigenvector of T^m with eigenvalue λ^m.

ii) Again, let v be an eigenvector of T with eigenvalue λ. Then

$$0 = 0v = (a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m)v = a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \ldots + a_m \lambda^m v = p(\lambda)v.$$

As $v \neq 0$ this implies that $p(\lambda) = 0$.

Question 7 Suppose that $T : V \to V$ is a linear map with the property that $T(T(v)) = T(v)$ for all $v \in V$.

(i) Show that

$$V = \ker(T) + \operatorname{im}(T) \text{ and } \ker(T) \cap \operatorname{im}(T) = \{0\}.$$

Hint: Note that if $v \in V$ then $v = (v - T(v)) + T(v)$.

(ii) Deduce that if V is of dimension n, then there is a basis B of V such that

$$[T]_B = \begin{pmatrix} I_s & 0_{s\times(n-s)} \\ 0_{(n-s)\times s} & 0_{(n-s)\times(n-s)} \end{pmatrix},$$

where $s = \dim(\operatorname{im}(T))$.

Solution: (i) We need to show that $\ker(T) \cap \operatorname{im}(T) = \{0\}$ and that $V = \ker(T) + \operatorname{im}(T)$. Suppose that $v \in \ker(T) \cap \operatorname{im}(T)$. Then there exists $w \in V$ such that $v = T(w)$. Then

$$0 = T(v) = (T(T(w))) = T(w) = v$$

so that $\ker(T) \cap \operatorname{im}(T) = \{0\}$. Now suppose $v \in V$, so that $v = (v - T(v)) + T(v)$. Note that $T(v) \in \operatorname{im}(T)$. Also $T(v - T(v)) = T(v) - T^2(v) = 0$ so $v - T(v) \in \ker(T)$. So $v \in \ker(T) + \operatorname{im}(T)$.

(ii) Note that all (non-zero) vectors in $\ker(T)$ are eigenvectors with eigenvalue 0. Moreover, if $v \in \operatorname{im}(T)$ there is $w \in V$ with $T(w) = v$, so $T(v) = T(T(w)) = T(w) = v$. So all non-zero vectors in $\operatorname{im}(T)$ are eigenvectors with eigenvalue 1. Let $v_1, \ldots, v_t$ be a basis for $ker(T)$ and $w_1, \ldots, w_s$ a basis for $\operatorname{im}(T)$. By rank + nullity $\dim(V) = t + s$ and by (i), $w_1, \ldots, w_s, v_1, \ldots, v_t$ span V. So these $s + t$ vectors form a basis B for V. The matrix $[T]_B$ is of the required form.
[Remark: Such a T is called a *projection*: can you see why?]