Linear Algebra Unseen Question 5

In this project you will prove a version of the JCF theorem for *real* matrices (even when they don't have real eigenvalues). Here is what this theorem says. For real numbers a, b such that $b \neq 0$, define the real matrix $M_{ab} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and define $J_n(a,b)$ to be the real $2n \times 2n$ matrix

$$\begin{pmatrix} M_{ab} & I & 0 & 0 & \cdots & 0 \\ 0 & M_{ab} & I & 0 & \cdots & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & M_{ab} & I \\ 0 & 0 & 0 & 0 & \cdots & M_{ab} \end{pmatrix}$$

(like a Jordan block $J_n(\lambda)$, with M_{ab} instead of λ and $I = I_2$ instead of 1).

Real JCF Theorem *Every real square matrix A is similar over $\mathbb{R}$ to a matrix of the form*

$$J = J_{m_1}(c_1) \oplus \cdots \oplus J_{m_k}(c_k) \oplus J_{n_1}(a_1, b_1) \oplus \cdots \oplus J_{n_l}(a_l, b_l),$$

where all $a_i, b_i, c_i \in \mathbb{R}$ and all $b_i \neq 0$ (and "similar over $\mathbb{R}$" means that $\exists$ a real matrix P such that $P^{-1}AP = J$).

Deduce this from the JCF theorem over $\mathbb{C}$ (Thm. 11.3 in the notes) in the following steps.

(1) 2×2 case: let $A \in M_2(\mathbb{R})$. Show that the eigenvalues of A are either real, or they are $\lambda, \bar{\lambda}$ for some complex number $\lambda = a + bi$ where $b \neq 0$.

If the evalues are real numbers c_1, c_2 then by the JCF theorem 11.3 applied with $F = \mathbb{R}$, A is similar over $\mathbb{R}$ to either $J_1(c_1) \oplus J_1(c_2)$ or $J_2(c_1)$ (with $c_1 = c_2$ in the latter case). (This is an observation, there is nothing to prove in this case.)

Show that if the evalues are $\lambda, \bar{\lambda}$ (with $\lambda = a + bi$ as above), then A is similar over $\mathbb{R}$ to M_{ab}. (Hint to get started: there is an evector $w \in \mathbb{C}^2$ with $Aw = \lambda w$. Write $w = x + yi$ with $x, y \in \mathbb{R}^2$ and show x, y is a basis of $\mathbb{R}^2$.)

(2) 4×4 case: suppose $A \in M_4(\mathbb{R})$ and A has complex evalues $\lambda, \lambda, \bar{\lambda}, \bar{\lambda}$ where $\lambda = a + bi$ and $b \neq 0$.

(a) Show that the complex JCF of A is either $J_2(\lambda) \oplus J_2(\bar{\lambda})$ or $J_1(\lambda)^2 \oplus J_1(\bar{\lambda})^2$.

(b) Deduce that A is similar over $\mathbb{R}$ to either $J_2(a,b)$ or $M_{ab} \oplus M_{ab}$ $(= J_1(a,b) \oplus J_1(a,b))$. (Hint: try to copy the argument of the last part of (1).)

(3) General case of one pair of complex evalues: prove the Real JCF theorem for real matrices A having only one pair of complex conjugate evalues $\lambda, \bar{\lambda}$ (and no real evalues).

(4) Complete the proof of the Real JCF theorem.