MATH50004 Differential Equations
Spring Term 2022/23
Solutions to the mid-term exam

Question 1

(i) λ solves this initial value problem if $\dot{\lambda}(t) = t\cos(\lambda(t))$ for all $t \in I$, and we have $\lambda(0) = 0$.

[1 point for solution identity; 1 point for initial value property]

(ii) Let $f(t,x) = t\cos(x)$. We get for all $t \in \mathbb{R}$ that

$$\begin{aligned}
\lambda_0(t) &= 0\,,\\
\lambda_1(t) &= 0 + \int_0^t f(s,\lambda_0(s))\,\mathrm{d}s = \int_0^t s\,\mathrm{d}s = \frac{1}{2}t^2\,,\\
\lambda_2(t) &= 0 + \int_0^t f(s,\lambda_1(s))\,\mathrm{d}s = \int_0^t s\cos(\tfrac{1}{2}s^2)\,\mathrm{d}s = \big[\sin(\tfrac{1}{2}s^2)\big]_{s=0}^{s=t} = \sin(\tfrac{1}{2}t^2)\,.
\end{aligned}$$

[7 points = 1 point for λ_0; 2 points for λ_1; 4 points for λ_2]

Question 2

(i) Consider a continuous function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ that satisfies

$$\|f(t,x) - f(t,y)\| \le K\|x-y\| \quad \text{for all } t \in \mathbb{R} \text{ and } x,y \in \mathbb{R}^d\,,$$

where $K > 0$ is a constant. Define $h := \frac{1}{2K}$. Then every initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, admits a unique solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$.

[2 points; it is okay to not specify h precisely]

(ii) Assume there exists a $\tilde{T} > t_0 + h$ such that all solutions to the initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, do not exist until time $\tilde{T}$. Define

$$T := \inf\big\{\tilde{T} > t_0 + h : \text{ all solutions satisfying } x(t_0) = x_0 \text{ do not exist until time } \tilde{T}\big\}\,.$$

This implies that there exists a solution $\lambda : [t_0 - h, T - \frac{h}{2}] \to \mathbb{R}^d$ of the initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$. Consider the initial condition $x(T - \frac{h}{2}) = \lambda(T - \frac{h}{2})$. The Picard–Lindelöf theorem implies that there exists a solution $\mu : [T - \frac{3h}{2}, T + \frac{h}{2}] \to \mathbb{R}^d$ of this initial value problem. Since $\lambda(T - \frac{h}{2}) = \mu(T - \frac{h}{2})$, both solutions coincide on the intersection of their domains, and λ can be extended up to time $t = T + \frac{h}{2}$ using the solution μ. This contradicts the definition of T. Similarly, one shows existence of solutions on intervals unbounded below.

[6 points]

(iii) The derivative of the right hand side is clearly unbounded, as a polynomial of degree 7, and using the mean value theorem, this implies that there cannot be a global Lipschitz condition.

[3 points]

(iv) The solution to the initial value problem cannot cross the constant solutions, given by 0 and 1, both forward and backward in time. Hence, the solution cannot be unbounded on an interval of finite length, and since the differential equation is globally defined, the theorem on maximal solutions implies that the maximal solution exists globally.

[4 points]

Question 3

(i) Consider $\dot{x} = x$ with $x = 0$. Clearly $\varphi(t, 0) = 0$ and both sides are given by $\{0\}$.

(ii) Consider $\dot{x} = x$ with $x = 1$. Clearly $\varphi(t, 1) = e^t$ and $\cap_{t>0} O^+(\varphi(t, 1)) = \cap_{t>0}(e^t, \infty) = \emptyset$.

[for each (i) and (ii): 2 points for the example with point x, and 1 point for justification]