Mathematical Logic (MATH6/70132;P65)
Solutions to Sheet 5

[1] Prove (or at least, sketch a proof of) the following version of the Lindenbaum Lemma (2.5.2) which was used in the proof of 2.5.3:

Suppose $\mathcal{L}$ is a countable first-order language and Σ is a consistent set of closed $\mathcal{L}$-formulas. Then there is a consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-formulas such that, for every closed $\mathcal{L}$-formula ϕ either $\Sigma^* \vdash \phi$ or $\Sigma^* \vdash (\neg\phi)$.

Solution: We use the fact (proved exactly as in 1.3.7) that, for every closed formula ϕ, if $\Sigma \not\vdash_{K_\mathcal{L}} \phi$, then $\Sigma \cup \{\neg\phi\}$ is consistent.

Then enumerate the *closed* $\mathcal{L}$-formulas as $\phi_0, \phi_1, \phi_2, \ldots$ and define sets $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$ inductively. We take Σ_{n+1} to equal Σ_n if $\Sigma \vdash \phi_n$ and $\Sigma_n \cup \{\neg\phi_n\}$ otherwise. By the previous paragraph each Σ_n is consistent, so their union Σ^* is consistent and has the required property.

[2] Describe a language with equality $\mathcal{L}^=$ which is appropriate for groups (see the example after 2.2.7).

(a) Write down a closed $\mathcal{L}^=$-formula γ whose normal models are precisely the groups. (You can use traditional mathematical notation if you wish.)

Solution: Use a language with equality having a binary function symbol (for multiplication); a 1-ary function symbol (for inversion) and a constant symbol (for the identity element). Let γ consist of the usual group axioms in this language (joined together using $\wedge$).

(b) Write down a set Σ of closed $\mathcal{L}^=$-formulas whose normal models are precisely the infinite groups.

Solution For $n \in \mathbb{N}$, let σ_n be the usual formula

$$(\exists x_1)\ldots(\exists x_n) \bigwedge_{1\le i<j\le n} (x_i \neq x_j).$$

Then take Σ to be:

$$\{\gamma, \sigma_1, \sigma_2, \ldots\}.$$

(c) Suppose that ϕ is a closed $\mathcal{L}^=$-formula such that for every $n \in \mathbb{N}$ there is a group with at least n elements which is a model of ϕ. Show that there is an infinite group which is a model of ϕ.

Solution: Consider the set of formulas $\Sigma_1 = \Sigma \cup \{\phi\}$. Every finite subset of this has a normal model (by the assumption). So by the compactness theorem for normal models, Σ_1 has a normal model. So this is an infinite group which is a model of ϕ.

(d) Show that there is no set Δ of closed $\mathcal{L}^=$-formulas whose normal models are precisely the finite groups.

Solution: Suppose there is such a set Δ. Consider $\Delta_1 = \Delta \cup \{\Sigma_1, \sigma_2, \ldots\}$. Then every finite subset of Δ_1 has a normal model (take a finite group of sufficiently large size, for example, a sufficiently large finite cyclic group). So Δ_1 has a normal model. This is a contradiction: it must be infinite (because of the formulas σ_n) and a finite group (because $\Delta \subseteq \Delta_1$).

(e) Show that there is no closed $\mathcal{L}^=$ formula whose normal models are precisely the infinite groups.

Solution: If there were such a formula ψ, then the normal models of $((\neg\psi) \wedge \gamma)$ are precisely the finite groups. This contradicts (d).

(f) (Harder) Is there a closed $\mathcal{L}^=$-formula σ which has a normal model and is such that any normal model of σ is an infinite group?

Solution: You may have come up with a different solution here. We require a property of groups (expressible by an $\mathcal{L}^=$ - formula) which is true in some group, but not in any finite group. Consider the property which says that we have a group with at least 3 elements in which any two non-identity elements are conjugate.

This can be expressed by an $\mathcal{L}^{=}$-formula. Moreover, there are no finite groups with this property. (The order of every non-identity element would have to be a prime p; then the order of the group would have to be a power of p and so the group would have a non-trivial centre. So as all non-identity elements are conjugate, the group is abelian and its size is therefore at most 2.) It is much harder to see that there is some (infinite) group with this property, but if you know about HNN-extensions, you can construct one.

[3] Suppose $\mathcal{L}^{=}$ is a first-order language with equality and a single binary relation symbol R. A graph is a normal model of the closed formula γ:

$$(\forall x_1)(\forall x_2)((\neg R(x_1, x_1)) \wedge (R(x_1, x_2) \to R(x_2, x_1))).$$

(i) Find a closed formula τ with the property that there is a finite normal model of $\gamma \wedge \tau$ whose domain has n elements iff n is divisible by 3.

Solution: Consider the formula τ which says that 'every element is related to exacly two other elements and these are also related'. So a normal model of this looks like a collection of triangles. You can take τ to be:

$$(\forall x_1)(\exists x_2)(\exists x_3)(R(x_1, x_2) \wedge R(x_1, x_3) \wedge R(x_2, x_3) \wedge (\forall x_4)(R(x_1, x_4) \to (x_4 = x_2) \vee (x_4 = x_3))).$$

(ii) (Hard) Can you find a closed $\mathcal{L}^{=}$-formula which has no finite models and has some infinite graph as a normal model?

Solution: It helps to know some theorems in graph theory to do this. The following is given in Peter Cameron's book mentioned on the reading list.

The Friendship Theorem of Erdös, Rényi and Sós says the following. Suppose we have a finite group of people with the property that for every two people in the group, they have exactly one friend in common in the group. The conclusion is that there is someone in the group who is everyone's friend. (Here we assume that friendship is symmetric and we don't consider someone to be their own friend.) Clearly this can be expressed as a theorem in graph theory where friendship is the graph relation. As an $\mathcal{L}^{=}$-formula it is of the form $(\alpha \to \beta)$ where β says $(\exists x_1)(\forall x_2)((x_1 = x_2) \vee R(x_1, x_2))$ and α expresses the hypothesis (Exercise - write it down). It is easy to construct an infinite graph satisfying the hypothesis and where the conclusion does not hold. So we can take the formula to be

$$\gamma \wedge \alpha \wedge (\neg \beta).$$

[4] Suppose $\mathcal{L}^{=}$ is a first-order language with equality ($=$) and a single binary relation symbol $\leq$. A linear order is a normal $\mathcal{L}^{=}$-structure $\langle A; \leq_A \rangle$ such that the relation $\leq_A$ is reflexive, transitive and such that for distinct $a, b \in A$ exactly one of $a \leq_A b$, $b \leq_A a$ holds.

Let Σ be the set of all closed $\mathcal{L}^{=}$-formulas which are true in all *finite* linear orders.
(i) Prove that any normal model of Σ is a linear order with a least element and a greatest element.
(ii) Prove that any normal model of Σ (with at least 2 elements) is not dense.
(iii) Prove that Σ has an infinite normal model.
(iv) Find a closed $\mathcal{L}^{=}$-formula ϕ such that neither ϕ nor $(\neg \phi)$ is a consequence of Σ.

Solution (i) First, we can write down closed $\mathcal{L}^{=}$-formulas (as in the notes) such that the normal models of these formulas are precisely the linear orders. So these formulas are in Σ, and any normal model of Σ is a linear order. Any finite linear order has a least element (otherwise we find an infinite descending sequence of element $a_0 > a_1 > a_2 \ldots$, contradicting the finiteness). We can express the property of having a least element by a closed $\mathcal{L}^{=}$-formula

$$(\exists x_1)(\forall x_2)(x_1 \leq x_2).$$

Thus this formula is in Σ, and so any normal model of Σ has a least element. We can clearly do something similar for greatest elements.

(ii) Density of a linear order is expressed by a closed formula ψ (as in the notes). Now, no finite linear order with at least two elements is dense (for example, such an order has a least element a and a next-least element b and there is no c with $a < c < b$). Thus the closed formula:

$$((\exists x_1)(\exists x_2)(x_1 \neq x_2) \to (\neg\psi))$$

is in Σ. So any normal model of Σ with at least 2 elements is not dense, as required.

(iii) Let σ_n be the closed formula from the notes expressing (in a normal structure) 'there are at least n elements.' Then every finite subset of

$$\Gamma = \Sigma \cup \{\sigma_n : n \in \mathbb{N}\}$$

has a normal model (because there is a linear order with n elements for each natural number n). So by the compactness theorem, Γ has a normal model, and this is an infinite linear order in which all formulas in Σ are true.

(iv) Take any σ_n. There is a finite linear order in which σ_n is true, and one in which it is false. So we have a normal model of Σ in which σ_n is true and (at least) one in which it is false. So neither $\neg\sigma_n$ nor σ_n is a consequence of Σ.

[5] Suppose $\mathcal{L}^=$ is a first order language with equality ($=$) and a single binary relation symbol R. Write down what it means for two normal $\mathcal{L}^=$-structures to be isomorphic (see the problem class in week 6)?
(i) Write down a set Σ of closed $\mathcal{L}^=$-formulas such that the normal models of Σ are the normal $\mathcal{L}^=$-structures in which R is interpreted as an equivalence relation in which all equivalence classes have size 2 or 3 and there are infinitely many equivalence classes of size 2 and infinitely many of size 3.
(ii) Explain why any two countable normal models of Σ are isomorphic.

Solution: We say that two $\mathcal{L}^=$-structures $\mathcal{A} = (A; \bar{R}_A)$ and $\mathcal{B} = (B; \bar{R}_B)$ are isomorphic if there is a bijection $\alpha : A \to B$ such that for $a, a' \in A$, we have $\bar{R}_A(a, a')$ holds in $\mathcal{A}$ iff $\bar{R}_B(\alpha(a), \alpha(a'))$ holds in $\mathcal{B}$.

(i) Let η be the formula whose normal models are sets with an equivalence relation. Let $\tau(x_1)$ be the formula
$(\exists x_2)((x_2 \neq x_1) \wedge R(x_1, x_2) \wedge (\forall x_3)(R(x_1, x_3) \to (x_3 = x_1) \vee (x_3 = x_2)))$
So $\tau(x_1)$ says that 'x_1 lies in an equivalence class of size 2'. Similarly let $\rho(x_1)$ be the formula
$(\exists x_2)(\exists x_3)((x_2 \neq x_1) \wedge (x_2 \neq x_3) \wedge (x_1 \neq x_3) \wedge R(x_1, x_2) \wedge R(x_1, x_3) \wedge$
$\wedge (\forall x_4)(R(x_1, x_4) \to (x_4 = x_1) \vee (x_4 = x_2) \vee (x_4 = x_3))$
Thus $\rho(x_1)$ says 'x_1 lies in an equivalence class of size 3.'

So the formula χ given by $(\forall x_1)(\tau(x_1) \vee \rho(x_1))$ says that all equivalence classes have 2 or 3 elements.

For $n \in \mathbb{N}$, let θ_n be the formula
$(\exists x_1)\ldots(\exists x_n)(\exists x_{n+1})\ldots(\exists x_{2n})(\bigwedge_{1 \leq i < j \leq n}(\neg R(x_i, x_j)) \wedge \bigwedge_{n+1 \leq i < j \leq 2n}(\neg R(x_i, x_j)) \wedge$
$\wedge \bigwedge_{1 \leq i \leq n} \tau(x_i) \wedge \bigwedge_{n+1 \leq j \leq 2n} \rho(x_i))$.

Then θ_n says that there are at least n equivalence classes with 2 elements and at least n equivalence classes with 3 elements.

So let $\Sigma = \{\eta, \chi, \theta_1, \theta_2, \theta_3, \ldots\}$. Then in any normal of Σ, the relation symbol R is interpreted as an equivalence relation with all equivalence classes having 2 or 3 elements, and there are infinitely many equivalence classes of each size.

(ii) Suppose $\mathcal{A} = \langle A; \bar{R}_A \rangle$ is a countable normal model of Σ. So there are countably many equivalence classes $A_0, A_1, A_2, \ldots$ with two elements and countably many $A'_0, A'_1, \ldots$ with 3 elements. Write $A_i = \{a_{i0}, a_{i1}\}$, and $A'_j = \{a'_{j0}, a'_{j1}, a'_{j2}\}$. Now suppose $\mathcal{B} = \langle B; \bar{R}_B \rangle$ is another countable normal model. We can list the classes of sizes 2 and 3 as B_i, B'_j $(i, j \in \mathbb{N})$ as for $\mathcal{A}$ and label the elements of the classes b_{ik} and $b'_{j\ell}$ (for $k = 0, 1$ and $\ell = 0, 1, 2$). Then the map α from A to B given by $\alpha(a_{ik}) = b_{ik}$ and $\alpha(a_{j\ell}) = b'_{j\ell}$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

[6] Suppose $\mathcal{L}^=$ is a first-order language with equality having just a single 1-ary function symbol f (and no other relation, function or constant symbols apart from $=$).

(i) What does it mean to say that two normal $\mathcal{L}^{=}$-structures $\mathcal{A}$, $\mathcal{B}$ are isomorphic?
(ii) Write down a set Σ of closed $\mathcal{L}^{=}$-formulas such that $\langle A; \bar{f}\rangle$ is a normal model of Σ if and only if: $\bar{f} : A \to A$ is a bijection and for every $n \in \mathbb{N}$, the function $\bar{f}^n : A \to A$ (obtained by applying $\bar{f}$ n times) has no fixed points.

(iii) Find countable normal models $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of Σ such that no two of these models are isomorphic and any countable model of Σ is isomorphic to one of these.

Solution: (i) $\mathcal{A} = (A; \bar{f}_A)$ and $\mathcal{B} = (B; \bar{f}_B)$ are isomorphic if there is a bijection $\alpha : A \to B$ such that for $a \in A$, we have $\bar{f}_B(\alpha(a)) = \alpha(\bar{f}_A(a))$.

(ii) To say that $\bar{f}$ is a bijection use the formula β:
$(\forall x_2)(\exists x_1)(f(x_1) = x_2) \wedge (\forall x_1)(\forall x_2)((f(x_1) = f(x_2)) \to (x_1 = x_2))$.

To say that $\bar{f}^n$ has no fixed points (for $n \geq 1$) write $f^n(x_1)$ as shorthand for the term $f(f(\ldots(f(x_1))\ldots))$ (with n f's) and use the formula γ_n:
$(\forall x_1)(f^n(x_1) \neq x_1)$.

So let $\Sigma = \{\beta, \gamma_1, \gamma_2, \gamma_3, \ldots\}$.

(iii) If $n \geq 1$, let $\mathcal{A}_n = \langle \mathbb{Z}; \bar{f}_n\rangle$ with $\bar{f}_n : \mathbb{Z} \to \mathbb{Z}$ given by $\bar{f}_n(x) = x + n$. Then each $\mathcal{A}_n$ is a (normal) model of Σ and $\bar{f}_n$ moves the integers in n infinite 'orbits' (i.e. subsets obtained by applying $\bar{f}_n$ or $\bar{f}_n^{-1}$ repeatedly to a particular integer). We also have a countable model $\mathcal{A}_0$ with countably many orbits. No two of the $\mathcal{A}_i$ are isomorphic as they have different numbers of orbits. On the other hand if $\mathcal{B}$ is a countable model of Σ the function permutes the elements of $\mathcal{B}$ in infinite orbits. If there are n of these where n is finite then $\mathcal{B}$ is isomorphic to $\mathcal{A}_n$. If there are infinitely many, there are countably many and $\mathcal{B}$ is isomorphic to $\mathcal{A}_0$.