Imperial College
London

M4A34/M5A34

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Dynamics, Symmetry & Integrability

Date: **Wednesday 11th May 2016**

Time: **09.30 – 12.00**

Time Allowed: **2 Hours 30 Mins**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. **Constrained Hamilton's principle on (semisimple) Lie algebras**

Consider Hamilton's principle $\delta S = 0$ with constrained action

$$S(\xi, q, p) = \int l(\xi, q)dt + \int \langle p, \dot{q} + \mathrm{ad}_\xi q \rangle dt\,, \tag{1}$$

where $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of the Lie algebra $\mathfrak{g}$ on itself. The canonical variables $(q, p) \in T^*_e G \simeq \mathfrak{g} \times \mathfrak{g}^*$, are $q \in \mathfrak{g}$ and $p \in \mathfrak{g}^*$. Here p is a Lagrange multiplier which enforces the dynamical evolution of q given by the Lie algebra action of $\xi \in \mathfrak{g}$, as $\dot{q} + \mathrm{ad}_\xi q = 0$, via the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$. The problem simplifies considerably for $\mathfrak{g} = so(3) \simeq \mathbb{R}^3$, since in that case we may replace $\mathrm{ad}_\xi q \to \xi \times q$ and the pairing becomes $\langle p, \mathrm{ad}_\xi q \rangle \to p \cdot \xi \times q$ in $\mathbb{R}^3$ vector notation, as done in class.

(a) By taking variations of S with respect to ξ, q and p, with $\delta q = 0$ at the endpoints in time, derive a set of three equations for the three variables, q, p and $\mu := \partial l / \partial \xi$.

(b) Pair the variational equation for $\mu := \partial l / \partial \xi$ with a fixed Lie algebra vector η and show that the three variational equations from the previous part imply the Euler-Poincaré equation,

$$\frac{d}{dt}\frac{\partial l(\xi, q)}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial l(\xi, q)}{\partial \xi} = \mathrm{ad}^*_q \frac{\partial l}{\partial q}\,. \tag{2}$$

(c) Write Euler-Poincaré equation (2) in vector notation via the hat map for $\mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3$.

2. **Hamiltonian formulation on (semisimple) Lie algebras**

(a) Assume that the Lagrangian $l(\xi, q)$ in the previous question is hyperregular, so one may solve uniquely for $\xi = \Xi(\mu, q)$. Calculate the Legendre transformation to the Hamiltonian formulation, via the formula

$$h(\mu, q) = \langle \mu \,,\, \xi \rangle - l(\xi, q)\,, \tag{3}$$

and compute the partial derivatives of the Hamiltonian in terms of partial derivatives of the Lagrangian.

(b) Write the Euler-Poincaré equation and the auxiliary constraint equation $\dot{q} + \mathrm{ad}_\xi q = 0$ as Hamiltonian equations of motion in Lie-Poisson bracket form.

(c) For the case when ξ, q and p are all vectors in $\mathbb{R}^3$, write the Hamiltonian equations of motion.

(d) Find two conservation laws in this vector notation which are conserved for any Hamiltonian. (These are the Casimirs for the Lie-Poisson bracket.)

(e) Write the equations of motion for the heavy top, whose Hamiltonian in this vector notation is given by

$$h(\mu, q) = \frac{1}{2}\mu \cdot I^{-1}\mu + gq \cdot \chi$$

where I is the diagonal moment of inertia tensor in the rigid body of unit mass, χ is a vector fom the point of support to the centre of mass and g is the constant acceleration of gravity.

3. $SO(n)$ **rigid-body dynamics**

$SO(n)$ rigid-body dynamics for the angular momentum matrix $M(t)$ is given by

$$\frac{dM}{dt} = [M\,,\,\Omega] = M\Omega - \Omega M \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}\,, \tag{4}$$

for evolution of the $n \times n$ skew-symmetric matrices M, Ω, with $n \times n$ constant symmetric $\mathbb{A}$.

(a) Show that the solution of $SO(n)$ rigid-body dynamics (4) is given by a similarity transformation (also known as co-Adjoint motion),

$$M(t) = O(t)^{-1}M(0)O(t) =: \mathrm{Ad}^*_{O(t)}M(0)\,,$$

with $O(t) \in SO(n)$ and $\Omega := O^{-1}\dot{O}(t)$.

(b) Prove that the initial eigenvalues of the matrix $M(0)$ are preserved by $SO(n)$ rigid-body dynamics; that is, $d\lambda/dt = 0$ in

$$M(t)\psi(t) = \lambda\psi(t)\,,$$

provided its eigenvectors $\psi \in \mathbb{R}^n$ of the angular momentum matrix $M(t)$ evolve according to

$$\psi(t) = O(t)^{-1}\psi(0)\,.$$

That is, prove that the evolution of $M(t)$ is **isospectral.**

(c) Prove that the matrix invariants of $M(t)$ under $SO(n)$ rigid-body dynamics are preserved:

$$\frac{d}{dt}\mathrm{tr}(M - \lambda\mathrm{Id})^K = 0\,,$$

for every integer power $K > 0$.

(d) Show that isospectrality $d\lambda/dt = 0$ allows the quadratic rigid-body dynamics (4) on $SO(n)$ to be rephrased as a system of two coupled linear equations: the eigenvalue problem for M and an evolution equation for its eigenvectors ψ, as follows:

$$M\psi = \lambda\psi \quad \text{and} \quad \frac{d\psi}{dt} = -\,\Omega\psi\,, \quad \text{with} \quad \Omega = O^{-1}\dot{O}(t)\,.$$

4. EPDiff equation

The EPDiff(H^1) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the square of the H^1 norm on the real line of the right-invariant vector field of velocity $u = \dot{g}g^{-1} \in \mathfrak{X}(\mathbb{R})$, $g \in \mathrm{Diff}(\mathbb{R})$. Namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int_{-\infty}^{\infty} u^2 + u_x^2 \, dx \,. \tag{5}$$

(Assume u vanishes as $|x| \to \infty$ for $x \in \mathbb{R}$.)

(a) For variations $\delta u = \partial_t \xi - \mathrm{ad}_u \xi$ (note that the sign of $\mathrm{ad}_u \xi$ is correct for right-invariant vector fields, u and ξ, for which $\mathrm{ad}_u \xi = -[u, \xi] = u_x \xi - \xi_x u$) use Hamilton's principle $\delta S = 0$ with $S = \int l(u)\, dt$ to derive the EPDiff(H^1) equation,

$$\partial_t m = -\mathrm{ad}_u^* m = -(\partial_x m + m\partial_x)u \,, \tag{6}$$

on the real line in terms of its velocity u and its momentum

$$m = \frac{\delta l}{\delta u} = u - u_{xx} = (1 - \partial_x^2)u \,,$$

in one spatial dimension. The inverse of the latter relation may be written in terms of the Green's function for the Helmholtz operator $(1 - \partial_x^2)$ as

$$u = K * m = \int_{-\infty}^{\infty} K(x,y)m(y)\, dy = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} m(y)\, dy \,. \tag{7}$$

(b) Legendre transform the Lagrangian in (5) to find the Hamiltonian $h(m)$ and write the EPDiff equation (6) in terms of a Lie-Poisson bracket for EPDiff(H^1), given by

$$\partial_t m = -B_2 \frac{\delta h}{\delta m} = \{\, m \,,\, h \,\}_2 \quad \text{with} \quad \{\, f \,,\, h \,\}_2 \,. \tag{8}$$

Identify the operator B_2 in (8) and show that it defines a Lie-Poisson bracket.

(c) Write the definition of a Casimir and determine the Casimir for the Lie-Poisson bracket with Hamiltonian operator B_2 in (8).

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(d) Show that equation (6) conserves the following integral quantities

$$
\begin{aligned}
C_{-1} &= \int_{-\infty}^{\infty} \sqrt{m}\, dx \quad \text{with} \quad \frac{\delta C_{-1}}{\delta m} = \frac{1}{2\sqrt{m}}\,, \\
C_0 &= \int_{-\infty}^{\infty} m\, dx \quad \text{with} \quad \frac{\delta C_0}{\delta m} = 1\,, \\
C_1 &= \frac{1}{2}\int_{-\infty}^{\infty} mu\, dx \quad \text{with} \quad \frac{\delta C_1}{\delta m} = u\,, \\
C_2 &= \frac{1}{2}\int_{-\infty}^{\infty} u(u^2 + u_x^2)\, dx \quad \text{with} \quad \frac{\delta C_2}{\delta m} = \frac{1}{2} K * (3u^2 - u_x^2 - 2uu_{xx})\,.
\end{aligned}
\tag{9}
$$

(e) Verify that the relations

$$
\partial_{\tau_{n+1}} m = B_1 \frac{\delta C_n}{\delta m} = B_2 \frac{\delta C_{n-1}}{\delta m} \quad \text{for} \quad n+1 = 0, 1, 2, 3,
$$

are satisfied by the four conservation laws in (9).

5. **Constrained Hamilton's principle for fluids**

Consider a constrained Hamilton's principle of the following form appropriate for fluid dynamics,

$$S(\mathbf{u}, D, \theta; \phi, \beta) = \int_a^b \Big(l(\mathbf{u}, D) + \big\langle \phi\,,\, D_t + \mathrm{div}(D\mathbf{u}) \big\rangle - \big\langle \beta\,,\, \theta_t + \mathbf{u}\cdot\nabla\theta \big\rangle \Big) dt\,, \qquad (10)$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the L^2 pairing, integrated over the fixed spatial domain of flow, $\mathcal{D}$. For example, $\langle\, f\,,\, h\,\rangle = \int_{\mathcal{D}} f g\, d^n x$, for L^2 functions f and g defined over $\mathcal{D} \subset \mathbb{R}^n$ with appropriate boundary conditions on $\partial\mathcal{D}$.

The physical variables $(\mathbf{u}, D, \theta)$, are, respectively, the fluid velocity vector, $\mathbf{u}$, mass density, D, and scalar entropy per unit mass (or potential temperature), θ. The other variables ϕ and β are Lagrange multipliers which enforce the advection relations for density D and scalar θ, respectively. Note: the Lagrangian $l(\mathbf{u}, D)$ does not depend on the variable θ.

(a) Compute the variational equations for Hamilton's principle $\delta S = 0$, for variations of (10) in the set of variables $\{\mathbf{u}, D, \theta, \phi, \beta\}$.

(b) Show that these variational equations imply the momentum map relation

$$\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} = D d\phi + \beta d\theta\,. \qquad (11)$$

(c) Using the result (11), show that, together, the other variational equations imply the motion equation

$$(\partial_t + \pounds_u)\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x}\Big) = d\frac{\delta l}{\delta D}\,. \qquad (12)$$

(d) Show that this motion equation implies conservation of the following circulation integral around any loop $c(u)$ moving with the fluid,

$$\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} = \oint_{c(u)} (\partial_t + \pounds_u)\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x}\right) = \oint_{c(u)} d\frac{\delta l}{\delta D} = 0\,. \qquad (13)$$

(e) In the two-dimensional case ($n = 2$), use the Stokes theorem to show that the 2-form given by $d(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x}) = \omega d^2 x$ satisfies a continuity equation,

$$(\partial_t + \pounds_u)(\omega d^2 x) = \big(\omega_t + \mathrm{div}(\omega \mathbf{u})\big) d^2 x = 0\,. \qquad (14)$$

(f) Again for $n = 2$, show that the continuity equation for vorticity (14) implies an infinite number of conservation laws for the integrals $C_\Phi = \int_{\mathcal{D}} D\Phi(\omega/D) d^2 x$, for any differentiable choice of the function Φ, provided the normal component of the velocity $\hat{\mathbf{n}}\cdot\mathbf{u}$ vanishes on the boundary of the fixed two-dimensional flow domain.