IMPERIAL

MATH70046

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Wednesday, May 7, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary, unless stated otherwise. All processes are real-valued, unless stated otherwise. The sample interval is unity and the index set the integers, unless stated otherwise. B denotes the backward shift operator.

1. (a) Define what it means for a stochastic process to be second-order stationary. (3 marks)

(b) Let $\{X_t\}$ be a moving average (MA) process of order q defined via

$$X_t = \mu - \theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where $\{\epsilon_t\}$ is a zero-mean white noise process with variance σ_ϵ^2.
Derive the autocovariance sequence $\{s_\tau\}$ for $\{X_t\}$. (5 marks)

(c) Let $\{X_t\}$ be a stationary MA(q_1) process, and $\{Y_t\}$ be a stationary MA(q_2) process, i.e.

$$X_t = \Theta_X(B)\epsilon_{X,t} \text{ and } Y_t = \Theta_Y(B)\epsilon_{Y,t},$$

where $\{\epsilon_{X,t}\}$ and $\{\epsilon_{Y,t}\}$ are zero-mean white noise sequences, which we assume are uncorrelated with each other at all times, and $\Theta_X(z)$ and $\Theta_Y(z)$ are characteristic polynomials of orders q_1 and q_2, respectively.

(i) Define a new process $\{W_t\}$ as $W_t = X_t + Y_t$. Derive the auto-covariance sequence $\{s_{W,\tau}\}$ of $\{W_t\}$ in terms of $\{s_{X,\tau}\}$ and $\{s_{Y,\tau}\}$, the autocovariance sequences of $\{X_t\}$ and $\{Y_t\}$, respectively. (3 marks)

(ii) Carefully considering when $\{s_{W,\tau}\}$ takes non-zero values, what about its form is consistent with $\{W_t\}$ also being a moving average process? What would be the order of this moving average process? (You do not need to formally prove that $\{W_t\}$ is a moving average process.) (3 marks)

(d) Let $\{U_t\}$ be a zero-mean stationary AR(p_1) process and $\{V_t\}$ a zero-mean stationary AR(p_2) process, i.e.

$$\Phi_U(B)U_t = \epsilon_{U,t} \text{ and } \Phi_V(B)V_t = \epsilon_{V,t},$$

where $\{\epsilon_{U,t}\}$ and $\{\epsilon_{V,t}\}$ are zero-mean white noise sequences, which we assume are uncorrelated with each other at all times, and $\Phi_U(z)$ and $\Phi_V(z)$ are characteristic polynomials of orders p_1 and p_2, respectively.

Show that $\{U_t + V_t\}$ is an autoregressive moving average (ARMA) process, stating the order of the autoregressive component.

Hint: You may use without proof that the sum of uncorrelated moving average processes is also a moving average process.

(6 marks)

(Total: 20 marks)

2. (a) Let $X_1, ..., X_N$ be a portion of a stationary process with mean μ and autocovariance sequence $\{s_\tau\}$. Consider the estimator

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+\tau} - \bar{X}) \qquad 0 \leq \tau \leq N-1,$$

where $\bar{X} = N^{-1} \sum_{t=1}^{N} X_t$.

(i) In the case where μ is known and substituted in for $\bar{X}$, show that $\hat{s}_\tau^{(p)}$ is a biased estimator of s_τ, computing its bias. Show that it is asymptotically unbiased as $N \to \infty$, for a fixed τ.

(5 marks)

(ii) The "unbiased" estimator is defined as

$$\hat{s}_\tau^{(u)} = \frac{1}{N-|\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+\tau} - \bar{X}) \qquad 0 \leq \tau \leq N-1.$$

Why is $\hat{s}_\tau^{(p)}$ often preferred over $\hat{s}_\tau^{(u)}$? (4 marks)

(b) Let $\{X_t\}$ be a zero mean stationary AR(p) process: $X_t - \phi_{1,p} X_{t-1} - \ldots - \phi_{p,p} X_{t-p} = \epsilon_t$.

(i) Derive the Yule-Walker equations $\boldsymbol{\gamma}_p = \boldsymbol{\Gamma}_p \boldsymbol{\phi}_p$ and $\sigma_\epsilon^2 = s_0 - \sum_{j=1}^{p} \phi_{j,p} s_j$, for estimation of the parameter vector $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and white noise variance σ_ϵ^2, where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$ and

$$\boldsymbol{\Gamma}_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}.$$

(6 marks)

(ii) Consider the zero-mean AR(2) process

$$X_t = \phi_{1,2} X_{t-1} + \phi_{2,2} X_{t-2} + \epsilon_t.$$

Obtain estimates of $\phi_{1,2}$, $\phi_{2,2}$ and σ_ϵ^2 when $\hat{s}_0^{(p)} = 1$, $\hat{s}_1^{(p)} = -1/5$ and $\hat{s}_2^{(p)} = 1/5$.

(5 marks)

(Total: 20 marks)

3. Consider the system $Y_t = L\{X_t\}$, where $\{X_t\}$ is a zero mean stationary process and $L\{\cdot\}$ is a linear time invariant (LTI) filter. It can be shown that $S_Y(f) = |G(f)|^2 S_X(f)$ where $S_X(f)$ and $S_Y(f)$ are the spectral density functions of $\{X_t\}$ and $\{Y_t\}$, respectively, and $G(f)$ is the frequency response function of $L\{\cdot\}$.

(a) State the three defining conditions of a LTI filter. (3 marks)

(b) The process $\{Y_t\}$ is obtained from $\{X_t\}$ by the application of a linear filter with impulse response sequence $g_0 = 1$, $g_1 = -\frac{1}{2}$, $g_2 = \frac{1}{2}$ and $g_k = 0$ otherwise, i.e.

$$Y_t = \sum_{k=-\infty}^{\infty} g_k X_{t-k}.$$

(i) Find the frequency response function for this LTI filter. (4 marks)

(ii) If the spectral density function of $\{X_t\}$ is

$$S_X(f) = \frac{5 + 3\cos(2\pi f)}{3 - 3\cos(2\pi f) + 2\cos(4\pi f)},$$

what is the spectral density function of $\{Y_t\}$? (4 marks)

(iii) Given $\{Y_t\}$ is an invertible MA process, express $\{X_t\}$ as an ARMA process, giving its orders and values of all its parameters. (6 marks)

(iv) Design a linear filter that transforms $\{Y_t\}$ into a white noise process. (3 marks)

(Total: 20 marks)

4. (a) (i) Define what it means for a pair of random processes $\{X_t\}$ and $\{Y_t\}$ to be jointly stationary, and in doing so, define the cross-covariance sequence $\{s_{XY,\tau}\}$. (2 marks)

(ii) In general it is true that $s_{XY,\tau} \neq s_{XY,-\tau}$, however show that $s_{XY,\tau} = s_{YX,-\tau}$ always. (2 marks)

(b) The jointly-stationary processes $\{X_t\}$ and $\{Y_t\}$ are related by

$$X_t = \beta Y_{t-1} + \epsilon_t$$
$$Y_t = \beta X_{t-1} + \eta_t,$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are both zero-mean white noise processes with variance σ^2, uncorrelated with each other at all times, and $|\beta| < 1$.

(i) Represent $\{X_t\}$ and $\{Y_t\}$ in AR(2) form. From this, argue that they are identical stochastic processes, and hence $s_{X,\tau} = s_{Y,\tau}$ and $s_{XY,\tau} = s_{YX,\tau}$. (6 marks)

(ii) By considering the processes $\{W_t = X_t + Y_t\}$ and $\{V_t = X_t - Y_t\}$, derive the autocovariance sequences of $\{X_t\}$ and $\{Y_t\}$, and the cross-covariance sequence of $\{X_t\}$ and $\{Y_t\}$.

Hint: you may use without proof that the autocovariance sequence for an AR(1) process $X_t = \phi X_{t-1} + \epsilon_t$ is $s_\tau = s_0 \phi^{|\tau|}$ where $s_0 = \sigma_\epsilon^2/(1-\phi^2)$. (6 marks)

(iii) The magnitude square coherence is defined as

$$\gamma_{XY}^2(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)}.$$

Show that

$$\gamma_{XY}^2(f) = \left[\frac{2\sum_{\tau=0}^{\infty} \beta^{2\tau+1}\cos[2\pi f(2\tau+1)]}{1 + 2\sum_{\tau=1}^{\infty} \beta^{2\tau}\cos(4\pi f\tau)}\right]^2.$$

(4 marks)

(Total: 20 marks)

5. PRELIMINARY INFORMATION

– Let $X_1, ..., X_N$ be a portion of a zero-mean stationary process $\{X_t\}$, and let $\{h_t\}$ be a data taper of length N such that $\sum_{t=1}^{N} h_t^2 = 1$. The direct spectral estimator is defined as

$$\hat{S}^{(d)}(f) = \left|\sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2 .$$

– In this question, you may use the following version of Isserlis' Theorem. If Z_1, Z_2, Z_3 and Z_4 are four complex valued random variables with zero means, then

$$\mathrm{Cov}\{Z_1 Z_2, Z_3 Z_4\} = \mathrm{Cov}\{Z_1, Z_3\}\mathrm{Cov}\{Z_2, Z_4\} + \mathrm{Cov}\{Z_1, Z_4\}\mathrm{Cov}\{Z_2, Z_3\}.$$

Recall: for a pair of zero-mean complex-valued random variables S and T, $\mathrm{Cov}\{S, T\} = E\{S^* T\}$.

QUESTION BEGINS ON NEXT PAGE

Let $G_1, ..., G_N$ be a portion of a zero-mean stationary Gaussian process $\{G_t\}$ with spectral density function $S_G(f)$. Define

$$J(f) \equiv \sum_{t=1}^{N} h_t G_t \mathrm{e}^{-\mathrm{i}2\pi ft}.$$

(a) Show that

$$\mathrm{Cov}\{|J(f')|^2, |J(f)|^2\} = |E\{J(f')J^*(f)\}|^2 + |E\{J(f')J(f)\}|^2.$$

(3 marks)

(b) Using the spectral representation theorem, show that

$$J(f) = \int_{-1/2}^{1/2} H(f-u)\mathrm{d}Z_G(u) \quad \text{and} \quad J(f) = -\int_{-1/2}^{1/2} H(f+u)\mathrm{d}Z_G^*(u),$$

where $\{Z_G(f)\}$ is the orthogonal increment process associated with $\{G_t\}$ and $H(\cdot)$ is the Fourier transform of $\{h_t\}$. (4 marks)

(c) Hence show that

$$\mathrm{Cov}\{\hat{S}^{(d)}(f+\eta), \hat{S}^{(d)}(f)\} = \left|\int_{-1/2}^{1/2} H^*(f+\eta-u)H(f-u)S_G(u)\mathrm{d}u\right|^2 + \left|\int_{-1/2}^{1/2} H^*(f+\eta+u)H(f-u)S_G(u)\mathrm{d}u\right|^2,$$

where $\hat{S}^{(d)}(\cdot)$ is the direct spectral estimator as defined in the Preliminary Information.

(4 marks)

(d) It can be shown for frequencies away from 0 and $1/2$ that

$$\mathrm{Cov}\{\hat{S}^{(d)}(f+\eta), \hat{S}^{(d)}(f)\} \approx S_G^2(f)\left|\int_{-1/2}^{1/2} H(u)H(\eta-u)\mathrm{d}u\right|^2.$$

Show that this can be written in terms of the data-taper as

$$\mathrm{Cov}\{\hat{S}^{(d)}(f+\eta), \hat{S}^{(d)}(f)\} \approx S_G^2(f)\left|\sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2.$$

(6 marks)

(e) Under the assumption that $S_G(f+\eta) \approx S_G(f)$, show that the correlation between $\hat{S}^{(d)}(f+\eta)$and $\hat{S}^{(d)}(f)$ is approximately

$$\left|\sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2.$$

(3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60046/70046

Time Series Analysis (Solutions)

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1. (a) Either: $\boxed{\text{seen} \Downarrow}$

$\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, $\mathrm{Var}\{X_t\}$ is a finite constant for all t, and $\mathrm{Cov}\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t.

Or, equivalently:

$\{X_t\}$ is second-order stationary if, for all $n \geq 1$, for any $t_1, t_2, ..., t_n \in T$, and for any s such that $t_1 + s, t_2 + s, ..., t_n + s \in T$, all the joint moments of order 1 and 2 of $\{X_{t_1}, X_{t_2} ..., X_{t_n}\}$ exist, are finite, and equal to the corresponding joint moments of $\{X_{t_1+s}, X_{t_2+s} ..., X_{t_n+s}\}$. $\boxed{3, \text{A}}$

(b) W.l.o.g. assume $E\{X_t\} = \mu = 0$. Then $\mathrm{Cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\}$. Since $E\{\epsilon_t \epsilon_{t+\tau}\} = 0 \ \ \forall\, \tau \neq 0$ we have for $\tau \geq 0$.

$$\begin{aligned}
\mathrm{Cov}\{X_t, X_{t+\tau}\} &= \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_{j,q} \theta_{k,q} E\{\epsilon_{t-j} \epsilon_{t+\tau-k}\} \\
&= \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q} \theta_{j+\tau,q} \qquad (k = j + \tau) \\
&\equiv s_\tau.
\end{aligned}$$

Since $s_\tau = s_{-\tau}$,

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q} \theta_{j+|\tau|,q} & |\tau| \leq q \\ 0 & |\tau| > q \end{cases}$$

$\boxed{5, \text{A}}$

(c) (i) Using the bilinear form of covariance gives $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\mathrm{Cov}\{W_t, W_{t+\tau}\} &= \mathrm{Cov}\{X_t + Y_t, X_{t+\tau} + Y_{t+\tau}\} \\
&= \mathrm{Cov}\{X_t, X_{t+\tau}\} + \mathrm{Cov}\{X_t, Y_{t+\tau}\} + \mathrm{Cov}\{Y_t, X_{t+\tau}\} + \mathrm{Cov}\{Y_t, Y_{t+\tau}\}
\end{aligned}$$

Because $\{\epsilon_{X,t}\}$ and $\{\epsilon_{Y,t}\}$ are uncorrelated at all times, if follows that $\mathrm{Cov}\{X_t, Y_{t+\tau}\} = \mathrm{Cov}\{Y_t, X_{t+\tau}\} = 0$ for all t and τ, and therefore $s_{W,\tau} = s_{X,\tau} + s_{Y,\tau}$. $\boxed{3, \text{B}}$

(ii) As $\{X_t\}$ is a MA(q_1) process, $s_{X,\tau} = 0$ for all $|\tau| > q_1$. Similarly $s_{Y,\tau} = 0$ for all $|\tau| > q_2$. Therefore, $s_{W,\tau} = 0$ for all $|\tau| > \max\{q_1, q_2\}$, which is consistent with $\{W_t\}$ being an MA(q) where $q = \max\{q_1, q_2\}$. $\boxed{3, \text{C}}$

(d) To proceed we want to get a polynomial form for $\{U_t + V_t\}$. We have Then

$$\Phi_V(B)\Phi_U(B)U_t = \Phi_V(B)\epsilon_{U,t}$$
$$\Phi_U(B)\Phi_V(B)V_t = \Phi_U(B)\epsilon_{V,t}$$

The order of the polynomials on the left of these equations may be interchanged. By adding these equations we get

$$\Phi_U(B)\Phi_V(B)[U_t + V_t] = \Phi_U(B)\epsilon_{V,t} + \Phi_V(B)\epsilon_{U,t}.$$

The product $\Phi_U(B)\Phi_V(B) = \Phi_p(B)$, a polynomial of order $p = p_1+p_2$. $\Phi_U(B)\epsilon_{V,t}$ is an MA(p_1) process and $\Phi_V(B)\epsilon_{U,t}$ is an MA(p_2) process; their driving white noises are uncorrelated with each other at all times. From the hint, we see that $\Phi_U(B)\epsilon_{V,t} + \Phi_V(B)\epsilon_{U,t}$ is an MA(q) process where $q = \max\{p_1, p_2\}$. So $\{U_t + V_t\}$ is an ARMA(p, q) process with $p = p_1 + p_2$ and $q = \max\{p_1, p_2\}$. $\boxed{6, \text{C}}$

2. (a) (i) Consider the estimator $\boxed{\text{seen} \Downarrow}$

$$\widehat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|}(X_t-\mu)(X_{t+\tau}-\mu),$$

with μ known. Taking the expected value, we get

$$\begin{aligned}
E\{\widehat{s}_\tau^{(p)}\} &= \frac{1}{N}\sum_{t=1}^{N-|\tau|}E\{(X_t-\mu)(X_{t+\tau}-\mu)\} \\
&= \frac{1}{N}\sum_{t=1}^{N-|\tau|}s_\tau \\
&= \frac{1}{N}(N-|\tau|)\,s_\tau \\
&= \left(1-\frac{|\tau|}{N}\right)s_\tau.
\end{aligned}$$

Given that bias$(\widehat{s}_\tau^{(p)}) = E\{\widehat{s}_\tau^{(p)}\} - s_\tau$, we have bias$(\widehat{s}_\tau^{(p)}) = -\frac{|\tau|}{N}s_\tau$. For fixed τ, this clearly goes to zero as $N\to\infty$. $\boxed{5, \text{A}}$

(ii)
- Even though $\widehat{s}_\tau^{(u)}$ is unbiased when μ is known, in many practical situations $\widehat{s}_\tau^{(p)}$ has superior mean square error due to its lower variance. $\boxed{2, \text{A}}$
- For processes with purely continuous spectra, it is true that $s_\tau \to 0$ as $\tau\to\infty$. Estimator $\widehat{s}_\tau^{(p)}$ mimics this property, whereas $\widehat{s}_\tau^{(u)}$ does not. $\boxed{1, \text{A}}$
- $\{\widehat{s}_\tau^{(p)}\}$ is always a positive semi-definite sequence, whereas in can be the case that $\{\widehat{s}_\tau^{(u)}\}$ is not. $\boxed{1, \text{A}}$

(b) (i) We start by multiplying the defining equation by X_{t-k}:

$$X_tX_{t-k} = \sum_{j=1}^{p}\phi_{j,p}X_{t-j}X_{t-k} + \epsilon_tX_{t-k}.$$

Taking expectations, for $k>0$:

$$s_k = \sum_{j=1}^{p}\phi_{j,p}s_{k-j}.$$

Let $k=1,2,\dots,p$ and recall that $s_{-\tau}=s_\tau$ to obtain

$$\begin{aligned}
s_1 &= \phi_{1,p}s_0 + \phi_{2,p}s_1 + \ldots + \phi_{p,p}s_{p-1} \\
s_2 &= \phi_{1,p}s_1 + \phi_{2,p}s_0 + \ldots + \phi_{p,p}s_{p-2} \\
\vdots & \quad \vdots \\
s_p &= \phi_{1,p}s_{p-1} + \phi_{2,p}s_{p-2} + \ldots + \phi_{p,p}s_0
\end{aligned}$$

or in matrix notation,

$$\boldsymbol{\gamma}_p = \Gamma_p \boldsymbol{\phi}_p,$$

where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$; $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$
and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}.$$

Furthermore multiplying the defining equation by X_t and take expectations we obtain

$$\begin{aligned} s_0 &= \sum_{j=1}^{p} \phi_{j,p} s_j + E\{\epsilon_t X_t\} \\ &= \sum_{j=1}^{p} \phi_{j,p} s_j + \sigma_\epsilon^2, \end{aligned}$$

so that

$$\sigma_\epsilon^2 = s_0 - \sum_{j=1}^{p} \phi_{j,p} s_j.$$

6, B

(ii) For the AR(2) case, we have

sim. seen ⇓

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 \\ s_1 & s_0 \end{bmatrix}, \quad \boldsymbol{\gamma}_p = [s_1, s_2]^T,$$

and $[\phi_1, \phi_2]^T = \Gamma_p^{-1}\boldsymbol{\gamma}_p$, meaning

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{s_0^2 - s_1^2} \begin{bmatrix} s_0 & -s_1 \\ -s_1 & s_0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}.$$

Plugging in our estimates for s_0, s_1 and s_2 gives

$$\begin{bmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{bmatrix} = \frac{1}{1-(1/25)} \begin{bmatrix} 1 & 1/5 \\ 1/5 & 1 \end{bmatrix} \begin{bmatrix} -1/5 \\ 1/5 \end{bmatrix} = \frac{25}{24} \begin{bmatrix} -4/25 \\ 4/25 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix}.$$

Additionally

$$\widehat{\sigma}_\epsilon^2 = \widehat{s}_0 - \sum_{j=1}^{p} \widehat{\phi}_{j,p}\widehat{s}_j = 1 - \left[\left(-\frac{1}{6}\right)\left(-\frac{1}{5}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\right] = \frac{14}{15}.$$

5, B

3. (a) A digital filter L that transforms an input sequence $\{x_t\}$ into an output sequence $\{y_t\}$ is called a linear time-invariant (LTI) digital filter if it has the following three properties: $\boxed{\text{seen} \Downarrow}$

[**1**] Scale-preservation:

$$L\{\{\alpha x_t\}\} = \alpha L\{\{x_t\}\}.$$

[**2**] Superposition:

$$L\{\{x_{t,1} + x_{t,2}\}\} = L\{\{x_{t,1}\}\} + L\{\{x_{t,2}\}\}.$$

[**3**] Time invariance:
If

$$L\{\{x_t\}\} = \{y_t\}, \quad \text{then} \quad L\{\{x_{t+\tau}\}\} = \{y_{t+\tau}\}.$$

Here τ is integer-valued, and the notation $\{x_{t+\tau}\}$ refers to the sequence whose t-th element is $x_{t+\tau}$. $\boxed{3, \text{A}}$

(b) (i) With $L\{X_t\} = X_t - \frac{1}{2}X_{t-1} + \frac{1}{2}X_{t-2}$, we can find the frequency response function of $L\{\cdot\}$ by considering $\boxed{\text{sim. seen} \Downarrow}$

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft} - \frac{1}{2}e^{i2\pi f(t-1)} + \frac{1}{2}e^{i2\pi f(t-2)} = e^{i2\pi ft}(1 - \frac{1}{2}e^{-i2\pi f} + \frac{1}{2}e^{-i4\pi f}),$$

giving $G(f) = 1 - \frac{1}{2}e^{-i2\pi f} + \frac{1}{2}e^{-i4\pi f}$.
Taking the Fourier transform of $\{g_k\}$ is also an equally valid method. $\boxed{4, \text{A}}$

(ii) If $G(f) = 1 - \frac{1}{2}e^{-i2\pi f} + \frac{1}{2}e^{-i4\pi f}$, then

$$\begin{aligned}
|G(f)|^2 &= (1 - \frac{1}{2}e^{-i2\pi f} + \frac{1}{2}e^{-i4\pi f})(1 - \frac{1}{2}e^{i2\pi f} + \frac{1}{2}e^{i4\pi f}) \\
&= \frac{3}{2} - \frac{3}{4}\left(e^{i2\pi f} + e^{-i2\pi f}\right) + \frac{1}{2}\left(e^{i4\pi f} + e^{-i4\pi f}\right) \\
&= \frac{1}{2}\left(3 - 3\cos(2\pi f) + 2\cos(4\pi f)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
S_Y(f) = |G(f)|^2 S_X(f) &= \frac{1}{2}\left(3 - 3\cos(2\pi f) + 2\cos(4\pi f)\right) \cdot \frac{5 + 3\cos(2\pi f)}{3 - 3\cos(2\pi f) + 2\cos(4\pi f)} \\
&= \frac{1}{2}(5 + 3\cos(2\pi f)).
\end{aligned}$$

$\boxed{4, \text{A}}$

(iii) Let's first identify the invertible MA process for $\{Y_t\}$. Given the spectral density function of an MA(q) process is

$\boxed{\text{unseen} \Downarrow}$

$$S(f) = \sigma_\epsilon^2 \left|1 - \theta_{1,q}\mathrm{e}^{-\mathrm{i}2\pi f} + \ldots + \theta_{q,q}\mathrm{e}^{-\mathrm{i}2\pi f q}\right|^2,$$

and the spectrum of the process in question contains only a $\cos(2\pi f)$ term, we can immediately infer it is an MA(1) process. Letting $Y_t = \epsilon_t - \theta\epsilon_{t-1}$, we have

$$S_Y(f) = \sigma_\epsilon^2(1 + \theta^2 - 2\theta\cos(2\pi f)).$$

Comparing with $S_Y(f) = \frac{1}{2}(5 + 3\cos(2\pi f))$, we have

$$\frac{5}{2} = \sigma_\epsilon^2(1+\theta^2)$$
$$\frac{3}{2} = -2\sigma_\epsilon^2\theta.$$

This gives $\sigma_\epsilon^2 = -3/4\theta$, and hence

$$\begin{aligned}\frac{5}{2} &= -\frac{3}{4\theta}\left(1+\theta^2\right)\\ \theta^2 + \frac{10}{3}\theta + 1 &= 0\\ \theta &= \frac{-\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2} = -\frac{1}{3},\ -3\end{aligned}$$

The invertible version is clearly the case of $\theta = -1/3$, giving $Y_t = \epsilon_t + \frac{1}{3}\epsilon_{t-1}$, with $\sigma_\epsilon^2 = 9/4$.

Therefore we have $Y_t = X_t - \frac{1}{2}X_{t-1} + \frac{1}{2}X_{t-2}$ and $Y_t = \epsilon_t + \frac{1}{3}\epsilon_{t-1}$, meaning $\{X_t\}$ is the ARMA$(2,1)$ process

$$X_t - \frac{1}{2}X_{t-1} + \frac{1}{2}X_{t-2} = \epsilon_t + \frac{1}{3}\epsilon_{t-1},$$

with $\sigma_\epsilon^2 = 9/4$. $\boxed{6,\ \text{D}}$

(iv) With $Y_t = \left(1 + \frac{1}{3}B\right)\epsilon_t$, we have that

$$\epsilon_t = \left(1 + \frac{1}{3}B\right)^{-1} Y_t = \sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k Y_{t-k}.$$

This defines the linear filtering on $\{Y_t\}$ that gives a white noise process. $\boxed{3,\ \text{C}}$

4. (a) (i) Processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary if they are both individually stationary and the cross-covariance sequence $s_{XY,\tau} = \mathrm{Cov}\{X_t, Y_{t+\tau}\}$ depends only on τ.

seen $\Downarrow$

2, A

(ii) By definition $s_{YX,-\tau} = \mathrm{Cov}\{Y_t, X_{t-\tau}\}$, which equals $\mathrm{Cov}\{Y_{t+\tau}, X_t\}$ by joint stationarity. This is clearly identical to $\mathrm{Cov}\{X_t, Y_{t+\tau}\} = s_{XY,\tau}$.

2, A

(b) (i) Through substitution, we have

unseen $\Downarrow$

$$\begin{aligned}
X_t &= \beta Y_{t-1} + \epsilon_t \\
&= \beta(\beta X_{t-2} + \eta_{t-1}) + \epsilon_t \\
&= \beta^2 X_{t-2} + \beta\eta_{t-1} + \epsilon_t
\end{aligned}$$

Likewise, we can show $Y_t = \beta^2 Y_{t-2} + \beta\epsilon_{t-1} + \eta_t$. Given $\{\eta_t\}$ and $\{\epsilon_t\}$ are uncorrelated with each other, $\{\beta\eta_{t-1} + \epsilon_t\}$ and $\{\beta\epsilon_{t-1} + \eta_t\}$ are both zero-mean white noise processes with variance $\sigma^2(1+\beta^2)$, and therefore $\{X_t\}$ and $\{Y_t\}$ are identical AR(2) processes, namely

$$U_t = \beta^2 U_{t-2} + \xi_t$$

where $\{\xi_t\}$ is a white noise process with variance $\sigma_\xi^2 = \sigma^2(1+\beta)$. They will therefore have the same autocovariance sequence and clearly $s_{XY,\tau} = \mathrm{Cov}(X_t, Y_{t+\tau}) = \mathrm{Cov}(Y_t, X_{t+\tau}) = s_{YX,\tau}$.

6, B

(ii) Using the defining equations

$$\begin{aligned}
W_t &= X_t + Y_t \\
&= \beta(X_{t-1} + Y_{t-1}) + \epsilon_t + \eta_t \\
&= \beta W_{t-1} + \xi_t
\end{aligned}$$

where $\{\xi_t = \epsilon_t + \eta_t\}$ is a white noise process with variance $2\sigma^2$. Therefore $\{W_t\}$ is an AR(1) process. Similarly

$$\begin{aligned}
V_t &= X_t - Y_t \\
&= \beta(Y_{t-1} - X_{t-1}) + \epsilon_t - \eta_t \\
&= -\beta V_{t-1} + \zeta_t
\end{aligned}$$

where $\{\zeta_t = \epsilon_t - \eta_t\}$ is a white noise process with variance $2\sigma^2$. Therefore $\{V_t\}$ is also an AR(1) process.
From the hint,

$$s_{W,\tau} = \frac{2\sigma^2}{1-\beta^2}\beta^{|\tau|}, \qquad s_{V,\tau} = \frac{2\sigma^2}{1-\beta^2}(-\beta)^{|\tau|}.$$

From their definitions, we have that

$$
\begin{aligned}
s_{W,\tau} = \mathrm{Cov}\{W_t, W_{t+\tau}\} &= \mathrm{Cov}\{(X_t + Y_t), (X_{t+\tau} + Y_{t+\tau})\} \\
&= s_{X,\tau} + s_{XY,\tau} + s_{YX,\tau} + s_{Y,\tau} \\
&= 2s_{X,\tau} + 2s_{XY,\tau},
\end{aligned}
$$

from Part (i). Likewise

$$
\begin{aligned}
s_{V,\tau} = \mathrm{Cov}\{V_t, V_{t+\tau}\} &= \mathrm{Cov}\{(X_t - Y_t), (X_{t+\tau} - Y_{t+\tau})\} \\
&= s_{X,\tau} - s_{XY,\tau} - s_{YX,\tau} + s_{Y,\tau} \\
&= 2s_{X,\tau} - 2s_{XY,\tau}.
\end{aligned}
$$

Adding, we get

$$
s_{W,\tau} + s_{V,\tau} = 4s_{X,\tau} = \frac{2\sigma^2}{(1-\beta^2)}(\beta^{|\tau|} + (-\beta)^{|\tau|}) \implies s_{X,\tau} = s_{Y,\tau} = \begin{cases} \frac{\sigma^2}{1-\beta^2}\beta^{|\tau|}, & \tau \text{ even} \\ 0 & \tau \text{ odd} \end{cases},
$$

and subtracting we get

$$
s_{W,\tau} - s_{V,\tau} = 4s_{XY,\tau} = \frac{4\sigma^2}{(1-\beta^2)}(\beta^{|\tau|} - (-\beta)^{|\tau|}) \implies s_{XY,\tau} = s_{YX,\tau} = \begin{cases} \frac{\sigma^2}{1-\beta^2}\beta^{|\tau|}, & \tau \text{ odd} \\ 0 & \tau \text{ even} \end{cases}.
$$

$\boxed{6, \mathrm{D}}$

(iii) We have that

$$
\begin{aligned}
S_X(f) = S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \frac{\sigma^2}{1-\beta^2} \sum_{\tau=-\infty}^{\infty} \beta^{2|\tau|} \mathrm{e}^{-\mathrm{i}4\pi f\tau} \\
&= \frac{\sigma^2}{1-\beta^2} \left(1 + 2\sum_{\tau=1}^{\infty} \beta^{2\tau} \cos(4\pi f\tau)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{XY}(f) &= \sum_{\tau=-\infty}^{\infty} s_{XY,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \frac{\sigma^2}{1-\beta^2} \sum_{\tau=-\infty}^{\infty} \beta^{2|\tau|+1} \mathrm{e}^{-\mathrm{i}2\pi f(2\tau+1)} \\
&= \frac{\sigma^2}{1-\beta^2} 2\sum_{\tau=0}^{\infty} \beta^{2\tau+1} \cos(2\pi f(2\tau+1)).
\end{aligned}
$$

Plugging these into the definition of the magnitude squared coherence gives the desired result. $\boxed{4, \mathrm{D}}$

5. (a) We know that $J(f)$ will be zero-mean complex-Gaussian because $\{G_t\}$ is a zero-mean Gaussian process. Therefore, from Isserlis' Theorem, we have $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}
\mathrm{Cov}\{|J(f')|^2, |J(f)|^2\} &= \mathrm{Cov}\{J(f')J^*(f'), J^*(f)J(f)\} \\
&= \mathrm{Cov}\{J(f'), J^*(f)\}\,\mathrm{Cov}\{J^*(f'), J(f)\} + \mathrm{Cov}\{J(f'), J(f)\}\,\mathrm{Cov}\{J^*(f'), J^*(f)\} \\
&= E\{J^*(f')J^*(f)\}E\{J(f')J(f)\} + E\{J^*(f')J(f)\}E\{J(f')J^*(f)\} \\
&= |E\{J(f')J^*(f)\}|^2 + |E\{J(f')J(f)\}|^2
\end{aligned}$$

$\boxed{3, \mathrm{M}}$

(b) The spectral representation theorem states there's an orthogonal increment process $\{Z_G(f)\}$ such that

$$G_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ft}\mathrm{d}Z_G(f).$$

Therefore

$$\begin{aligned}
J(f) \equiv \sum_{t=1}^{N} h_t G_t \mathrm{e}^{-\mathrm{i}2\pi ft} &= \sum_{t=1}^{N} h_t \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ut}\mathrm{d}Z_G(u)\mathrm{e}^{-\mathrm{i}2\pi ft} \\
&= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi t(f-u)}\mathrm{d}Z_G(u) \\
&= \int_{-1/2}^{1/2} H(f-u)\mathrm{d}Z_G(u).
\end{aligned}$$

Furthermore, we know from the properties of the orthogonal increment process that $\mathrm{d}Z_G(-u) = \mathrm{d}Z_G^*(u)$, and therefore

$$J(f) = -\int_{-1/2}^{1/2} H(f+u)\mathrm{d}Z_G(-u) = -\int_{-1/2}^{1/2} H(f+u)\mathrm{d}Z_G^*(u).$$

$\boxed{4, \mathrm{M}}$

(c) Given $\widehat{S}^{(d)}(f) = |J(f)|^2$, using Part (a), we have that

$$\begin{aligned}
\mathrm{Cov}\{\widehat{S}^{(d)}(f+\eta), \widehat{S}^{(d)}(f)\} &= \mathrm{Cov}\{|J(f+\eta)|^2, |J(f)|^2\} \\
&= |E\{J(f+\eta)J^*(f)\}|^2 + |E\{J(f+\eta)J(f)\}|^2.
\end{aligned}$$

From Part (b) we know that

$$\begin{aligned}
E\{J(f+\eta)J^*(f)\} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} H(f+\eta-u)H^*(f-u')E\{\mathrm{d}Z_G(u)\mathrm{d}Z_G^*(u')\} \\
&= \int_{-1/2}^{1/2} H(f+\eta-u)H^*(f-u)S_G(u)\mathrm{d}u,
\end{aligned}$$

and, similarly,

$$E\{J(f+\eta)J(f)\} = -\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} H(f+\eta+u)H(f+u')E\{\mathrm{d}Z_G(u)\mathrm{d}Z_G^*(u')\}$$
$$= -\int_{-1/2}^{1/2} H(f+\eta+u)H(f+u)S_G(u)\mathrm{d}u.$$

The result follows. $\boxed{4, \mathrm{M}}$

(d) Let us consider the integral $\boxed{\text{unseen} \Downarrow}$

$$\int_{-1/2}^{1/2} H(u)H(\eta-u)\mathrm{d}u = \int_{-1/2}^{1/2}\left(\sum_{s=1}^{N} h_s \mathrm{e}^{-\mathrm{i}2\pi us}\right)\left(\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi(\eta-u)t}\right)\mathrm{d}u$$
$$= \int_{-1/2}^{1/2}\sum_{s=1}^{N}\sum_{t=1}^{N} h_s h_t \mathrm{e}^{-\mathrm{i}2\pi us}\mathrm{e}^{-\mathrm{i}2\pi(\eta-u)t}\mathrm{d}u$$
$$= \sum_{s=1}^{N}\sum_{t=1}^{N} h_s h_t \mathrm{e}^{-\mathrm{i}2\pi\eta t}\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi u(t-s)}\mathrm{d}u.$$

We need to consider two cases. The first case is when $t \neq s$. In this setting

$$\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi u(t-s)}\mathrm{d}u = \frac{1}{\mathrm{i}2\pi(t-s)}\left(\mathrm{e}^{\pi(t-s)} - \mathrm{e}^{-\pi(t-s)}\right) = \frac{1}{\pi(t-s)}\sin(\pi(t-s)) = 0$$

as we have the sine of an integer multiple of π. The second case is when $t = s$. In this setting

$$\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi u(t-s)}\mathrm{d}u = \int_{-1/2}^{1/2}\mathrm{d}u = 1.$$

Therefore

$$\int_{-1/2}^{1/2} H(u)H(\eta-u)\mathrm{d}u = \sum_{s=1}^{N}\sum_{t=1}^{N} h_s h_t \mathrm{e}^{-\mathrm{i}2\pi\eta t}\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}2\pi u(t-s)}\mathrm{d}u$$
$$= \sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t},$$

and the result follows. $\boxed{6, \mathrm{M}}$

(e) Using the result from Part (d), setting $\eta = 0$, we have that $\mathrm{Var}\{\widehat{S}^{(d)}(f)\} \approx S_G^2(f)$, and given the assumption that $S_G(f) \approx S_G(f+\eta)$, we also have $\mathrm{Var}\{\widehat{S}^{(d)}(f+\eta)\} \approx S_G^2(f+\eta) \approx S_G^2(f)$. Therefore $\boxed{\text{seen} \Downarrow}$

$$\mathrm{Corr}\{\widehat{S}^{(d)}(f+\eta), \widehat{S}^{(d)}(f)\} \approx \frac{S_G^2(f)\left|\sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2}{\sqrt{S_G^2(f)S_G^2(f)}} = \left|\sum_{t=1}^{N} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t}\right|^2.$$

$\boxed{3, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70046 Time Series Markers Comments

Question 1

Answers to this question were somewhat mixed. As expected, the definition of stationarity was well answered, but deriving the acvs for an MA process caused quite a large number of issues. Often to do with the limits on the summation and handling the mean of the process (which was mu). Considering a zero-mean process w.l.o.g. was acceptable (that's what is done in the notes), but if you are going to consider the mean-mu case, it must be handled correctly. A common loss of a mark was not considering the abs(\tau)q situation. The other part which caused most difficulty was (d), which is understandable, as this was the hardest element. For those who did take the correct approach, a common mistake at the end was to say that the order of a p1-order polynomial multiplied by a p2-order polynomial was a p1 x p2 order polynomial. It's not, it's a p1 + p2 order polynomial (the highest order z term is z^p1 x z^p2 = z^{p1+p2}). Overall we get an ARMA(p1+p2 , max{p1,p2}).

Question 2

There was a reasonable amount of bookwork in this question, so on the whole it was answered quite well. To be honest, the number of people who couldn't correctly invert a 2 x 2 matrix (part b(ii)) was surprising. Also, a common loss of marks was to only compute estimates for phi_1 and phi_2. sigma_ep^2 is also a parameter of the model and needed to be computed.

Question 3

Generally done well by everyone.
(i) generally done well by everyone. (ii) Most students identified the strategy for this question. Some errors in manipulation, which were more common when people worked in trigonometric rather than exponential form.
(iii) Seemed to be the most challenging sub-question. Most people able to identify ARMA(2,q) but fewer able to get q=1 and remaining parameters. A common mistake was selecting the wrong root for theta.
(iv) Generally done well by those who completed part (iii), allowed error carried forward from selecting incorrect ARMA parameters.

Question 4

Part (a) generally done well by all students. Part (b) responses to (i) sometimes lacked specificity in the representation of $\{X_t\}$ and $\{Y_t\}$ as AR(2) processes, for example by not stating the variance of the noise term. In part (ii), several students attempted to derive the autocovariance sequences of $\{X_t\}$ and $\{Y_t\}$ (with varying degrees of success) but not by considering $\{W_t\}$ and $\{V_t\}$ as directed in the question. Part (iii), where it was attempted, often lacked sufficient description or justification of the steps taken for a "show that" question.

Question 5

Answers to this question were mixed. (a), (b) and (c) where, on the whole, answered quite well. The second part of (b) caused more issues than expected as many didn't recall the property that $dZ(f) = dZ^{ast}(-f)$.

(d) caused the most issues as this was unseen. Somewhat surprisingly, many people didn't correctly identify the multiplication of two sums as being a double sum, as opposed a sum of the squared terms (which is incorrect). Only with the integral of the complex exponential does this then reduce down to a single sum of squared terms.