Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Applied Probability

Date: Tuesday, 26 May 2015. Time: 10.00am – 12.00noon. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use TWO main answer books (A & B) for their solutions as follows: book A - solutions to questions 1, 2 & 3; book B - solution to question 4.

Supplementary books may only be used after the relevant main book(s) are full.

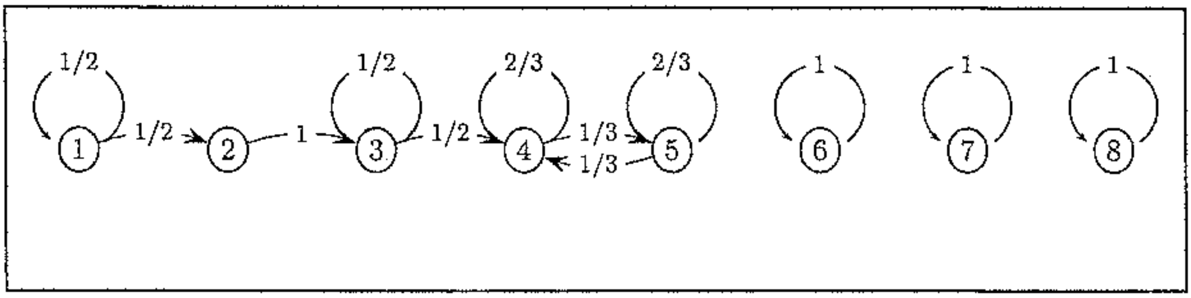
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition diagram given by



(i) Find the transition matrix.

(ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(iii) Find all possible stationary distributions.

Please note that you need to justify your answers in (ii)-(iii).

(b) Suppose we have a Markov chain with finite state space E and transition matrix $\mathbf{P}$. Suppose for some $i \in E$ that

$$p_{ij}(n) \to \pi_j \text{ as } n \to \infty \text{ for all } j \in E.$$

Show that π (which is the row vector consisting of the elements π_j for $j \in E$) is a stationary distribution. *Please make sure that you justify all steps in your proof carefully.*

2. We define an irreducible, positive recurrent Markov chain $(X_n)_{n\in\{0,1,\dots,N\}}$ for an $N \in \mathbb{N}$. We assume that π is the stationary distribution, and $\mathbf{P}$ is the transition matrix, and that for any $n \in \{0,1,\dots,N\}$ the marginal distribution of X_n is also given by π.

(a) The reversed chain is defined as

$$Y_n = X_{N-n} \quad \text{for any } n \in \{0,1,\dots,N\}.$$

Show that the sequence $Y = (Y_n)_{n\in\{0,1,\dots,N\}}$ is a Markov chain which satisfies

$$\mathbb{P}(Y_{n+1} = j | Y_n = i) = \frac{\pi_j}{\pi_i} p_{ji}.$$

(b) Give the definition for $(X_n)_{n\in\{0,1,\dots,N\}}$ to be *time-reversible.*

(c) Show that $(X_n)_{n\in\{0,1,\dots,N\}}$ is time-reversible if and only if the detailed-balance equations hold.

(d) Suppose that π is given by the hypergeometric distribution HG(d), for $d \in \mathbb{N}$, i.e.

$$\pi_i = \frac{\binom{d}{i}^2}{\binom{2d}{d}}, \text{ for } i \in \{0,1,\dots,d\}.$$

Further, suppose that

$$p_{i(i-1)} = \left(\frac{i}{d}\right)^2, \qquad p_{ii} = \frac{2i(d-i)}{d^2}, \qquad p_{i(i+1)} = \frac{(d-i)^2}{d^2},$$

and $p_{ij} = 0$ otherwise.

(i) Show that π given by the HG(d) distribution is indeed a stationary distribution for the Markov chain.

(ii) Show that the Markov chain is time-reversible.

Hint: You do NOT need to show that the HG(d) distribution is indeed a distribution.

3. Let $(N_t)_{t\geq 0}$ denote a Poisson process of rate $\lambda > 0$. Let $(X_i)_{i\in\mathbb{N}}$ denote its inter-arrival times and let $T_n = \sum_{i=1}^{n} X_i$ denote the time to the nth event for $n \in \mathbb{N}$ (also $T_0 = 0$).

(a) Show that $X_1 \sim \text{Exp}(\lambda)$.

(b) (i) Derive the Laplace transform of X_1, i.e. find $\mathbb{E}(e^{-uX_1})$ for $u > 0$.

(ii) Suppose that $Y \sim \text{Gamma}(n, \lambda)$, i.e. its probability density function is given by $f_Y(y) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}$ for $y \geq 0$ and $\lambda > 0, n \in \mathbb{N}$. Derive the Laplace transform of Y.

(c) Show that $T_n \sim \text{Gamma}(n, \lambda)$.

(d) Define $Z_t = t - T_{N_t}$ for $t \geq 0$ and show that

$$\mathbb{P}(Z_t > x) = \begin{cases} e^{-\lambda x}, & 0 \leq x < t, \\ 0, & t \leq x. \end{cases}$$

4. (a) Define a birth-death process.

(b) Consider a simple birth, simple death process with immigration denoted by $(X_t)_{t\geq 0}$. I.e. the birth rates are given by $\lambda_n = \lambda n + \alpha$ and the death rates are given by $\mu_n = \mu n$ for constants $\lambda, \alpha, \mu > 0$ and for $n \in \mathbb{N}_0$. Let $\mathbf{P}_t = (p_{ij}(t))$ denote its stochastic semigroup.

(i) Derive the forward equations for $p_{i0}(t)$ and for $p_{ij}(t)$ for $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$.

(ii) Suppose that the initial population at time 0 is of size $n_0 \in \mathbb{N}_0$. Show that

$$M(t) := \mathbb{E}(X_t) = \sum_{j=1}^{\infty} j p_{n_0 j}(t).$$

(iii) Show that $M(t)$ as defined in (ii) satisfies

$$M'(t) = \alpha + (\lambda - \mu)M(t), \quad t \geq 0.$$