Imperial College London
Department of Mathematics

Solutions to Question Sheet 3 - Probl. Class week 5

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

This is the problem sheet for the problem class on Monday of week 5. Solutions will be released after the last problem class on Monday of week 5.

Question 1 For each of the following matrices $A \in M_3(\mathbb{R})$, find the eigenvalues and eigenvectors. Then diagonalise A, or prove it cannot be diagonalised.

(i) $\begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$.

Solution: (i) Eigenvalues $1, 3$ with eigenvectors $a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $b\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ respectively, for any non-zero real numbers a, b. So if we set (for instance) $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(ii) Eigenvalues $1, 2, 3$ with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. So setting $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ gives $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(iii) Characteristic polynomial is $(x-1)(x-3)^2$, so eigenvalues are $1, 3$. For $\lambda = 1$, eigenvectors are scalar multiples of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. For $\lambda = 3$ eigenvectors are $\begin{pmatrix} a+b \\ a \\ b \end{pmatrix}$ for any a, b (not both zero). So taking $P = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ for instance gives $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(iv) Eigenvalues $2, -1$. For $\lambda = 2$ eigenvectors are multiples of $(1, 0, 1)^T$; for $\lambda = -1$ eigenvectors are multiples of $(1, -3, 4)^T$. So there is no basis of $\mathbb{R}^3$ consisting of eigenvectors and therefore the matrix is not diagonalisable.

Question 2 For which values of c is the matrix $\begin{pmatrix} 1-2c & 4c & -c \\ -c & 2c+1 & -c \\ 0 & 0 & -1 \end{pmatrix} \in M_3(\mathbb{R})$ diagonalisable?

Solution: The characteristic polynomial is $(1-x)^2(1+x)$, so eigenvalues are $1, -1$ with 1 repeated. For $c \neq 0$, the eigenvectors for $\lambda = 1$ are scalar multiples of $(2, 1, 0)^T$, so we cannot form a 3×3 matrix P with linearly independent eigenvectors as its columns. But for $c = 0$ the eigenvectors for $\lambda = 1$ are $(a, b, 0)^T$, so we can find an invertible P such as $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ such that $P^{-1}AP$ is $\mathrm{diag}(-1, 1, 1)$. So the matrix is diagonalisable if and only if $c = 0$.

Question 3 Let $A = \begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix} \in M_2(\mathbb{R})$.

(a) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.
(b) Find A^n, where n is an arbitrary positive integer.
(c) Find a matrix $B \in M_2(\mathbb{R})$ such that $B^3 = A$.
(d) Find a matrix $C \in M_2(\mathbb{C})$ such that $C^2 = A$.
(e) Prove that there is no $C \in M_2(\mathbb{R})$ such that $C^2 = A$.

Solution: (a) Solving for the eigenvalues $-1, 8$, let P be the matrix with columns the eigenvectors, i.e. $P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Then $P^{-1}AP = D = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$.

(b) As seen in lectures, $(P^{-1}AP)^n = P^{-1}A^nP$, hence $P^{-1}A^nP = D^n$, giving $A^n = PD^nP^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 8^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. This works out as $\begin{pmatrix} 2 \cdot (-1)^n - 8^n & 2 \cdot (-1)^n - 2 \cdot 8^n \\ (-1)^{n+1} + 8^n & (-1)^{n+1} + 2 \cdot 8^n \end{pmatrix}$.

(c) If $E = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ then $E^3 = D$, so $(PEP^{-1})^3 = PE^3P^{-1} = PDP^{-1} = A$. So take $B = PEP^{-1} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$.

(d) If $F = \begin{pmatrix} i & 0 \\ 0 & \sqrt{8} \end{pmatrix}$ then $F^2 = D$, so $C = PFP^{-1} = \begin{pmatrix} -\sqrt{8} + 2i & -2\sqrt{8} + 2i \\ \sqrt{8} - i & 2\sqrt{8} - i \end{pmatrix}$ satisfies $C^2 = A$ just as in (c).

(e) Suppose $C^2 = A$ with all entries of C real. Then $\det(C)^2 = \det(C^2) = \det(A) = -8$. This is impossible as $\det(C)$ is real.

Question 4 Suppose V is a vector space over a field F and $T : V \to V$ is linear. If $\lambda \in F$, let $E_\lambda = \{v \in V : T(v) = \lambda v\}$. Prove that this is a subspace of V and λ is an eigenvalue of T if and only if $E_\lambda \neq \{0\}$.

Solution: Either use the test for a subspace, or note that E_λ is the kernel of the linear map $(T - \lambda Id) : V \to V$, and is therefore a subspace.

Question 5 For each of the linear maps θ_i below, write down the matrix representing θ_i with respect to the standard basis. Hence find the eigenvalues of θ_i and for each eigenvalue λ, find the eigenspace E_λ. Determine whether θ_i is diagonalizable.

i) $\theta_1 : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\theta_1 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} c - b \\ a - c \\ c \end{pmatrix}.$$

ii) $\theta_2 : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\theta_2 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} c-b \\ a-c \\ c \end{pmatrix}.$$

Solution:

i) The characteristic polynomial is

$$\det \begin{pmatrix} x & 1 & -1 \\ -1 & x & 1 \\ 0 & 0 & x-1 \end{pmatrix} = \ldots = (x-1)(x^2+1)$$

So the only eigenvalue is $\lambda = 1$.

$$E_1 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : \begin{pmatrix} c-b \\ a-c \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : b = 0, a = c \right\}.$$

So this is a 1-dimensional space, with basis $\{(1,0,1)^T\}$. The only eigenvectors of θ are multiples of $(1,0,1)^T$. The map is not diagonalizable.

ii) The characteristic polynomial is again $x(x^2+1)$, but this time the roots are $1, \pm i$. Since there are 3 distinct eigenvectors, the map is diagonalizable. The eigenspaces are given by

$$E_1 = \left\{ \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} : a \in \mathbb{C} \right\}, \quad E_i = \left\{ \begin{pmatrix} ib \\ b \\ 0 \end{pmatrix} : b \in \mathbb{C} \right\}, \quad E_{-i} = \left\{ \begin{pmatrix} a \\ ia \\ 0 \end{pmatrix} : a \in \mathbb{C} \right\}.$$

A basis of eigenvectors would therefore be $(1,0,1)^T, (i,1,0)^T, (1,i,0)^T\}$.

Question 6 For each of the linear maps T in Question 2 of Sheet 2, compute the eigenvalues and eigenvectors of T and determine whether or not T is diagonalisable.

Solution: (i) The matrix of T with respect to the standard basis is $\begin{pmatrix} -1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$. So the char poly is $(x+1)^2(x-2)$. The eigenvalues are $-1, 2$. The eigenspace E_{-1} is spanned by $(-1,2,1)$; the eigenspace E_2 is spanned by $(0,1,1)$. There is no basis of eigenvectors, so T is not diagonalisable.

(ii) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$. So the char poly is $(x-2)^2(x-3)^2$, so the eignvalues are $2, 3$. The eigenspace E_2 of A is spanned by $(2,-1,0,0)^T$ and $(0,0,2,-1)^T$. So the eigenspace E_2 of T is spanned by $\begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$. Likewise the eigenspace E_3 of T is spanned by $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. So T is diagonalisable.

(iii) T sends $1 \mapsto 0$, $x \mapsto 3x$, $x^2 \mapsto x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. The eigenvalues are $0, 3, 6$. Corresponding eigenvectors are 1 (the constant function), x and $x + 3x^2$ (and their non-zero scalar multiples). T is diagonalisable.

Question 7 As in Question 9 of Sheet 1, let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the a_i are in the field F. Let $e_1, \ldots, e_n$ be the standard basis of F^n.

i) Prove that F^n is spanned by the vectors $e_1, Ae_1, \ldots, A^{n-1}e_1$. What is $A^n e_1$ as a linear combination of these?

ii) Show that for every $v \in F^n$ there is a polynomial $q(x)$ (over F) of degree at most $n - 1$ such that $v = q(A)e_1$ (where $q(A)$ is the result of substituting A for x into the polynomial q).

iii) Deduce that $\chi_A(A)$ is the zero matrix (this is a special case of the Cayley - Hamilton Theorem).

Solution: (i) Note that as the columns of A are the images of the standard basis vectors, $Ae_i = e_{i+1}$ for $1 \leq i < n$. Thus $A^i e_1 = e_{i+1}$ for $1 \leq i < n$. We also have

$$A^n e_1 = A(A^{n-1}e_1) = Ae_n = -a_0 e_1 - a_1 e_2 - \ldots - a_{n-1} e_n = -a_0 e_1 - a_1 Ae_1 - \ldots - a_{n-1} A^{n-1} e_1.$$

(ii) By (i) each e_i is in (the span of) $e_1, Ae_1, \ldots, A^{n-1}e_1$, so the span of these is the whole of F^n. So if $v \in F^n$ there are $b_0, \ldots, b_{n-1} \in F$ with

$$v = b_0 e_1 + b_1 Ae_1 + \ldots + b_{n-1} A^{n-1} e_1 = (b_0 I_n + b_1 A + \ldots + b_{n-1} A^{n-1}) e_1.$$

The result follows.

(iii) By (i) we have $p(A)e_1 = 0$, where $p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n$. Let v, q be as in (ii). Then $p(A)v = p(A)q(A)e_1 = q(A)p(A)e_1 = 0$ (as A commutes with powers of itself). So $p(A) = 0$. But by Qu 9, Sheet 1, $\chi_A(x) = p(x)$.

Question 8 In this question you can use Q7. Unless stated otherwise, you can choose which field to use.

(a) Find a 3×3 matrix which has characteristic polynomial $x^3 - 7x^2 + 2x - 3$.
(b) Find a 3×3 matrix A such that $A^3 - 2A^2 = I_3$.
(c) Find a 4×4 invertible matrix B such that $B^{-1} = B^3 + I_4$.
(d) Find a 5×5 invertible matrix B such that $B^{-1} = B^3 + I_5$.
(e) Find a real 4×4 matrix C such that $C^2 + C + I_4 = 0$.
(f) For each $n \geq 2$ find an $n \times n$ matrix C such that $C^n = I_n$ but $C \neq I_n$.

Solution: (a) $\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 7 \end{pmatrix}$ works (by Q9, Sh 1)

(b) Take $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. This works, by Qu7.

(c) Multiplying through by B, the equation is $B^4 + B - I = 0$. So we can use Qu7 to find such a matrix 4×4 matrix B. Note that as the constant term of the char poly is non-zero, B is indeed invertible.

(d) Take B_0 as in (c), and let $B = \begin{pmatrix} B_0 & 0 \\ 0 & \lambda \end{pmatrix}$, where λ is a complex root of $x^4 + x - 1$.

(e) By Qu7 the 2×2 matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ satisfies $A^2 + A + I = 0$. So take $C = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

(f) Use Qu7 to get a non-identity $n \times n$ matrix with char poly $x^n - 1$.