Mathematical Logic (MATH6/70132;P65)
Problem Sheet 4

[1] In each of the following cases a first-order language $\mathcal{L}_i$ and two $\mathcal{L}_i$-structures $\mathcal{A}_i$, $\mathcal{B}_i$ are given. In each case, write down a sentence of $\mathcal{L}_i$ which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$. Explain your answers briefly (your argument need not involve valuations).
(a) $\mathcal{L}_1$ has a single binary relation symbol R. The domain of $\mathcal{A}_1$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$. The domain of $\mathcal{B}_1$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$.
(b) $\mathcal{L}_2$ has a single binary relation symbol R. The domain of $\mathcal{A}_2$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$. The domain of $\mathcal{B}_2$ is $\mathbb{Q}$ (the set of rational numbers) and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$.
(c) $\mathcal{L}_3$ has a single unary function symbol f and a single binary relation symbol E. The domain of $\mathcal{A}_3$ is $\mathbb{N}$ and f is interpreted as the function $x_1 \mapsto x_1 + 1$. The domain of $\mathcal{B}_3$ is $\mathbb{Z}$ and f is interpreted as the function $x_1 \mapsto x_1 + 2$. In both structures E is interpreted as equality.
(d) $\mathcal{L}_4$ has a single binary relation symbol R. The domain of $\mathcal{A}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 3'. The domain of $\mathcal{B}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 5'.

Solution (a) $\langle \mathbb{N}; \leq \rangle$ has a least element, whereas $\langle \mathbb{Z}; \leq \rangle$ does not. So the sentence

$$(\exists x_1)(\forall x_2)R(x_1, x_2)$$

is true in $\mathcal{A}_1$ but not in $\mathcal{B}_1$.

(b) Between any two distinct rational numbers we can find a third: this is not true of natural numbers. So

$$(\forall x_1)(\forall x_2)(\exists x_3)(R(x_1, x_2) \to (R(x_1, x_3) \wedge R(x_3, x_2))$$

is true in $\mathcal{B}_2$ but not in $\mathcal{A}_2$. Thus its negation is true in $\mathcal{A}_2$ but not in $\mathcal{B}_2$.

(c) In $\mathcal{A}_3$ the function is not onto, whereas in $\mathcal{B}_3$ the function is onto. We can express this by saying that the formula

$$(\exists x_1)(\forall x_2)\neg(E(f(x_2), x_1))$$

is true in $\mathcal{A}_3$ but not in $\mathcal{B}_3$.

(d) In $\mathcal{A}_4$ there are 3 equivalence classes (- any natural number is congruent modulo 3 to $0, 1$ or 2), whereas in $\mathcal{B}_4$ there are 5. In particular the formula

$$(\exists x_1)(\exists x_2)(\exists x_3)(\forall x_4)(R(x_1, x_4) \vee R(x_2, x_4) \vee R(x_3, x_4))$$

is true in $\mathcal{A}_4$, but not in $\mathcal{B}_4$.

[2] (a) Show (by giving an argument involving valuations) that for any formula ϕ the following formula is logically valid:

$$((\exists x_1)(\forall x_2)\phi \to (\forall x_2)(\exists x_1)\phi).$$

(b) Give an example of a formula ϕ and an interpretation where the following is false:

$$((\forall x_1)(\exists x_2)\phi \to (\exists x_2)(\forall x_1)\phi).$$

Solution: (a) Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$ which satisfies $(\exists x_1)(\forall x_2)\phi$. We must show v satisfies $(\forall x_2)(\exists x_1)\phi$. So suppose v' is x_2-equivalent to v. We need to show v' satisfies $(\exists x_1)\phi$. Now, there is a valuation v'' which is x_1-equivalent to v and which satisfies $(\forall x_2)\phi$ (2.2.10 in notes). Let w be the valuation given by

$$w(x_i) = \begin{cases} v''(x_1) & \text{if } i = 1 \\ v'(x_2) & \text{if } i = 2 \\ v(x_i) & \text{if } i > 2 \end{cases}.$$

Then w is x_2-equivalent to v'' and so satisfies ϕ. But w is also x_1-equivalent to v', so (by 2.2.10) v' satisfies $(\exists x_1)\phi$, as required.

(b) Keep things very simple: take a language with a single binary relation symbol R and the interpretation with domain $\mathbb{N}$ and R interpreted as equality. Let ϕ be the formula $R(x_1, x_2)$. Then $(\forall x_1)(\exists x_2)\phi$ says 'for every x_1, there is some x_2 with $x_1 = x_2$, ' which is obviously true in this interpretation; but $(\exists x_2)(\forall x_1)\phi)$ says 'there is some value of x_2 with $x_1 = x_2$ for all x_1,' which is clearly false in this interpretation. Thus $((\forall x_1)(\exists x_2)\phi \to (\exists x_2)(\forall x_1)\phi)$ is false in this interpretation.

[3] Let $\mathcal{L}$ be a first-order language with a binary relation symbol R. A *strict partial order* is an $\mathcal{L}$-structure which is a model of the closed formula ϕ:

$$(\forall x_1)(\forall x_2)(\forall x_3)((\neg R(x_1, x_1)) \wedge ((R(x_1, x_2) \wedge R(x_2, x_3)) \to R(x_1, x_3))).$$

(So in a model of this formula, the interpretation of R behaves like $<$.)

(a) Show that in any model of ϕ the formula ψ given by:

$$(\forall x_1)(\forall x_2)(R(x_1, x_2) \to (\neg R(x_2, x_1)))$$

is true.

(b) Write down an $\mathcal{L}$-formula χ which has a model and is such that any $\mathcal{L}$-structure which is a model of χ is infinite.

Solution: (a) Suppose $\mathcal{A} \models \phi$ and $a, b \in A$ are such that $\bar{R}(a, b)$. We need to show that $\bar{R}(b, a)$ does not hold in $\mathcal{A}$. Well, suppose it did. Then we have both $\bar{R}(a, b)$ and $\bar{R}(b, a)$, so by the second clause in ϕ we get that $\bar{R}(a, a)$ holds in $\mathcal{A}$. But this contradicts the first clause in ϕ. So indeed $\mathcal{A} \models \psi$.

(b) Consider the formula χ given by:

$$\phi \wedge (\forall x_1)(\exists x_2)R(x_2, x_1).$$

Then any model of χ is a strict partial order in which for every element, there is an element less than it. This certainly has a model (for example $\langle \mathbb{Z}; < \rangle$) and has no finite model (in a finite strict partial order, there is always a least element).

[4] Suppose $\mathcal{L}$ is a first-order language and $\phi(x_1)$ is an $\mathcal{L}$-formula with a free variable x_1 and possibly other free variables. Under what circumstances is the formula

$$((\forall x_1)\phi(x_1) \to (\forall x_2)\phi(x_2))$$

logically valid? Justify your answer.

Solution: Have a look at the notes on the proof of 2.5.3 (made available in week 7): in particular 2.5.8 (2). This tells you that if x_2 is free for x_1 in ϕ and does not occur free in ϕ, then the given formula is a theorem of $K_{\mathcal{L}}$ and therefore logically valid. You can also do this 'semantically' using 2.3.7. and 2.3.3.

You should be able to construct examples where the formula is not logically valid in the cases where x_2 is not free for x_1 in ϕ (see question 3 on the 2017-18 exam paper) and where x_2 is free in ϕ.