

ME4 Q4

$$\min x_1 - 4x_2 + x_3^4$$

$$\text{s.t. } x_1 + x_2 + x_3^2 \leq 2$$

$$x_1 \geq 0 \checkmark \quad x_2 \geq 0 \checkmark$$

i) Cost is convex $\nabla^2 f = \begin{bmatrix} \text{sketch of a circle} & 12x_3^2 \end{bmatrix} \succeq 0$

$$\begin{array}{l} x_3^* = 0 \quad x_2^* = 2 \\ x_1^* = 0 \end{array} \quad \Bigg|$$

Constraints are convex $\begin{bmatrix} \text{sketch of an ellipse} & 2 \end{bmatrix} \succeq 0$
(Do KKT).

ii) Duality: $X = \mathbb{R}^3$

$$\begin{aligned} L(\underline{x}, d) = & x_1 - 4x_2 + x_3^4 + d_1(x_1 + x_2 + x_3^2 - 2) \\ & + d_2(-x_1) \\ & + d_3(-x_2) \end{aligned}$$

$$\begin{aligned} \min_{x \in \mathbb{R}^3} L(\underline{x}, d) = & x_1(1 + d_1 - d_2) \\ & + x_2(-4 + d_1 - d_3) \\ & + x_3^4 + d_1 x_3^2 - 2d_1 \end{aligned}$$

$$\Rightarrow \min L = \begin{cases} \min_{\underline{x} \in \mathbb{R}^3} x_3^4 + d_1 x_3^2 - 2d_1 & \begin{array}{l} 1 + d_1 - d_2 = 0 \text{ and} \\ -4 + d_1 - d_3 = 0 \end{array} \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_3} x_3^4 + d_1 x_3^2 - 2d_1$$

$$f'_3 = 0 \Rightarrow 4x_3^3 + 2d_1 x_3 = 0$$

$$2x_3(2x_3^2 + d_1) = 0$$

$$\underbrace{2x_3}_{\boxed{x_3=0}} \underbrace{(2x_3^2 + d_1)}_{=0}, \text{ not possible because } d_1 \geq 0$$

$$\min_{x \in \mathbb{R}^2} L = \begin{cases} -2d_1 & 1 + d_1 - d_2 = 0 \\ & -4 + d_1 - d_3 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual:

$$\begin{array}{l} \max -2d_1 \\ \text{s.t. } d_1 \geq 0 \\ 1 + d_1 - d_2 = 0 \\ -4 + d_1 - d_3 = 0 \end{array} \quad \begin{array}{l} d_1 = 4 \\ d_2 = 5 \\ d_3 = 0 \\ \boxed{x_3 = 0} \end{array}$$

$$\underline{\hspace{10em}} = 0$$

$$d_1 (x_1 + x_2 + x_3^2 - 2) = 0$$

$$d_2 (-x_1) = 0$$

$$d_3 (-x_2) = 0$$

$$d_1 = 4 \Rightarrow x_1 + x_2 - 2 = 0$$

$$d_2 = 5 \Rightarrow \boxed{x_1 = 0}$$

$$\Rightarrow \boxed{x_2 = 2}$$

Extra: Using KKT for the primal.

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = x_1 - 4x_2 + x_3^4 + \lambda_1(x_1 + x_2 + x_3^2 - 2) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

$$\nabla_{\underline{x}} \mathcal{L} = 0 \Leftrightarrow 1 + \lambda_1 - \lambda_2 = 0 \quad (1)$$

$$-4 + \lambda_1 - \lambda_3 = 0 \quad (2)$$

$$4x_3^3 + 2\lambda_1 x_3 = 0 \quad (3)$$

$$\lambda_1(x_1 + x_2 + x_3^2 - 2) = 0 \quad (4)$$

$$\lambda_2(-x_1) = 0 \quad (5)$$

$$\lambda_3(-x_2) = 0 \quad (6)$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Case $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$

$$\lambda_1 \neq 0 \Rightarrow x_1 + x_2 + x_3^2 - 2 = 0$$

$$\lambda_2 \neq 0 \Rightarrow x_1 = 0$$

$$\Rightarrow x_2 + x_3^2 - 2 = 0$$

$$\text{Also, (3)} \Leftrightarrow 2x_3(2x_3^2 + \lambda_1) = 0$$

$$\hookrightarrow \lambda_1 \neq 0 \Rightarrow x_3 = 0$$

Therefore $\underline{x}^* = (0, 2, 0)$ solves KKT + convexity $x_2 = 2$
+ Slater ($\underline{x} = (0, 0, 0); 0 + 0 + 0^2 < 2$)
 \Rightarrow KKT are nec. and sufficient $\Rightarrow \underline{x}^*$ is optimal.