IMPERIAL

MATH70022

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Finite Elements: Numerical Analysis and Implementation

Date: Thursday, May 23, 2024
Time: 14:00 – 16:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer Each Question in a Separate Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. The 1D Hermite element is defined by $(K, \mathcal{P}, \mathcal{N})$, where:

   1. K is the interval $[a, b]$ (for simplicity in this question we will take $a = 0$, $b = 1$),
   2. $\mathcal{P}$ is the space of polynomials of degree 3 or less,
   3. $N_1(v) = v(a)$, $N_2(v) = v(b)$, $N_3(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(a)$, $N_4(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(b)$.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$. (5 marks)

(b) Show that the polynomial $\phi(x) = 3x^2 - 2x^3$ is one of the nodal basis functions for $\mathcal{P}$. (5 marks)

(c) Which of the following problems is this element suitable for building a Galerkin discretisation for, and why?

$$-u'' = 0, \quad u(0) = 0,\ u'(1) = 1, \tag{1}$$

$$u - u'' + u'''' = \sin(x), \quad u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2}$$

$$\begin{aligned} u - u'' + u'''' - u'''''' = \exp(x), \quad & u(0) = u'(0) = u''(0) = 0, \\ & u(1) = u'(1) = u'''(1) = 0. \end{aligned} \tag{3}$$

(5 marks)

(d) Propose a finite element that is suitable for the same problems that you indicated in Part (c), but with $\mathcal{P}$ altered to be the space of polynomials of degree 4 or less. Explain briefly why your proposed finite element is unisolvent. (5 marks)

(Total: 20 marks)

2. Consider the following linear variational problem: find $u \in H^1$ such that

$$a(u, v) = F(v), \quad \forall v \in H^1,$$

where $a(u, v)$ is a bilinear functional with continuity constant M and coercivity constant γ, and $F(v)$ is a bounded linear functional.

For a finite element space $V_h \subset H^1(\Omega)$ defined on a domain Ω, the Galerkin approximation of the linear variational problem is as follows: find $u_h \in V_h$ such that

$$a(u, v) = F(v), \quad \forall v \in V_h.$$

(a) Show that

$$a(u - u_h, v) = 0, \quad \forall v \in V_h.$$

(5 marks)

(b) Show that

$$\gamma \|u - u_h\|_{H^1} \leq M \|u - v\|_{H^1},$$

for any $v \in V_h$, where $\|f\|_{H^1}$ is the H^1 norm on Ω. (5 marks)

(c) Consider the following bilinear form on $H^1(\Omega)$,

$$a(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v + v\beta \cdot \nabla u \, \mathrm{d}\, x,$$

where β is a given vector field with $|\beta| \leq \beta_0$, $\nabla \cdot \beta = 0$ and $\beta \cdot n = 0$ on the boundary $\partial\Omega$ of Ω, where n is the unit outward pointing normal to $\partial\Omega$.

(i) Find a finite upper bound for the continuity constant of a. (3 marks)

(ii) Show that

$$a(u, u) = \int_\Omega u^2 + |\nabla u|^2 + \frac{1}{2} \nabla \cdot (\beta u^2) \, \mathrm{d}\, x.$$

(3 marks)

(iii) Hence, find a finite upper bound for the coercivity constant of a. (4 marks)

(Total: 20 marks)

3. (a) For $f \in H^1(\Omega)$, where Ω is some convex polygonal domain, the H^1 projection of f into a degree k Lagrange finite element space V is the function $u \in V$ such that

$$\int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x = \int_\Omega vf + \nabla v \cdot \nabla f \,\mathrm{d}\,x, \quad \forall v \in V.$$

Show that u exists and is unique from this definition, with

$$\|u\|_{H^1} \leq \|f\|_{H^1}.$$

(5 marks)

(b) Show that the H^1 projection is mean-preserving, i.e.

$$\int_\Omega u \,\mathrm{d}\,x = \int_\Omega f \,\mathrm{d}\,x.$$

(5 marks)

(c) Show that the H^1 projection u into V of f is the minimiser over $v \in V$ of the functional

$$J[v] = \int_\Omega (v - f)^2 + \|\nabla(v - f)\|^2 \,\mathrm{d}\,x.$$

(5 marks)

(d) Hence, show that

$$\|u - f\|_{H^1(\Omega)} < Ch^k |f|_{H^{k+1}(\Omega)},$$

where h is the maximum triangle diameter in the triangulation used to construct V. Here, you may quote results from the course without proof.

(5 marks)

(Total: 20 marks)

4. Consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0, \tag{4}$$

solved for a time-dependent function $u(x,t)$ on a closed simply connected domain Ω, with boundary conditions $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$.

(a) Given a C^0 finite element space V_h, formulate a finite element discretisation of the wave equation (4) with time dependent solution $u(x,t) \in V_h$. You may assume that the solution is twice differentiable in time, and should use a variational formulation involving integration over the spatial domain Ω only. (5 marks)

(b) Show that the discretisation can be written in the form

$$M\frac{\mathrm{d}^2}{\mathrm{d}\, t^2}\boldsymbol{u} + K\boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the vector of basis coefficients for $u \in V_h$.

(5 marks)

(c) Show that the discretisation is equivalent to the following formulation: simultaneously find $u \in V_h$ and $v \in V_h$ such that

$$\langle \phi, u_t \rangle - \langle \phi, v \rangle = 0, \quad \forall \phi \in V_h, \tag{5}$$

$$\langle \psi, v_t \rangle + \langle \nabla\psi, \nabla u \rangle = 0, \quad \forall \psi \in V_h, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product on Ω. (5 marks)

(d) Using Equations (5-6), show that the solution conserves energy,

$$E = \frac{1}{2}\int_\Omega v^2 + |\nabla u|^2 \,\mathrm{d}\, x,$$

i.e. $\dot{E} = 0$. (5 marks)

(Total: 20 marks)

5. The variational formulation of the Stokes equation seeks $(u, p) \in (V, Q)$, where $V = (\mathring{H}^1)^3$ is the subspace of $(H^1)^3$ vanishing on the boundary, and $Q = \mathring{L}^2$ is the subspace of L^2 that integrates to zero, such that

$$c((u,p),\,(v,q)) = \int_\Omega f \cdot v \,\mathrm{d}\,x, \quad \forall (v,q) \in (\mathring{H}^1(\Omega))^3 \times \mathring{L}^2(\Omega), \tag{7}$$

where

$$c((u,p),\,(v,q)) = a(u,v) + b(v,p) + b(u,q), \tag{8}$$

$$a(u,v) = 2\mu \int_\Omega \epsilon(u) : \epsilon(v) \,\mathrm{d}\,x, \quad b(u,q) = \int_\Omega q \nabla \cdot u \,\mathrm{d}\,x, \tag{9}$$

$\mu > 0$ is the viscosity (a real number), and $\epsilon(v)$ is a mapping from vector fields to tensors given by

$$\epsilon(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right). \tag{10}$$

(a) Show that the bilinear form c is not coercive. (5 marks)

(b) We define the operator $B : V \to Q'$ by

$$Bv[p] = b(v,p), \quad \forall p \in Q,$$

and the transpose operator $B^* : Q \to V'$ by

$$B^*p[v] = b(v,p), \quad \forall v \in V,$$

where V' and Q' are the dual spaces to V and Q respectively.

(i) Show that the inf-sup condition,

$$\inf_{0 \neq q \in Q} \sup_{\neq v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0,$$

is equivalent to

$$\inf_{0 \neq q \in Q} \frac{\|B^* q\|_{V'}}{\|q\|_Q} \geq \beta.$$

(5 marks)

(ii) Hence, show that the inf-sup condition is equivalent to

$$\|B^* q\|_{V'} \geq \beta \|q\|_Q, \quad \forall q \in Q.$$

(5 marks)

(iii) Hence, show that the inf-sup condition implies injectivity of B^*. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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M70022

Finite Elements (Solutions)

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1. (a) We need to show that for $v \in \mathcal{P}$ then if $N_i(v) = 0$, $i = 1, 2, 3, 4$, then $v \equiv 0$. At $x = 0$ there is a double root because $v(0) = v'(0) = 0$, and similarly at $x = 1$. Therefore v is a degree 3 polynomial with 4 roots, so $v \equiv 0$ by the fundamental theorem of algebra. seen/sim.seen $\Downarrow$ 5, A

(b) $N_1(\phi) = \phi(0) = 3 \times 0^2 - 2 \times 0^3 = 0$. $N_2(\phi) = \phi'(0) = 6 \times 0 - 6 \times 0^2 = 0$. $N_3(\phi) = \phi(1) = 3 \times 1^2 - 2 \times 1^3 = 1$. $N_4(\phi) = 6 \times 1 - 6 \times 1^2 = 0$. Hence ϕ vanishes for all nodal variables, except for N_3, so it is a nodal basis function. 5, A

(c) The finite element can be used to build a C^1 finite element space, because we have the function value and its derivative at each interval vertex, so we can enforce continuity of the function and its derivative by sharing those nodal variables between cells.

Equation (1) is a second order problem, which requires an H^1 formulation, so the finite element space must be C^0. Our finite element space is $C^1 \subset C^0$, so this is suitable.

Equation (2) is a fourth order problem, which requires an H^2 formulation, so the finite element space must be C^1. Our finite element space is C^1, so this is suitable.

Equation (3) is a sixth order problem, which requies an H^3 formulation, so the finite element space must be C^2. Our finite element space cannot be C^2, because the second derivative at a vertex cannot be computed from nodal variables stored at that vertex, so this is not suitable. 5, A

(d) We need to add an extra nodal variable to those above. For example, $N_5(v) = v((a + b)/2)$. This is unisolvent because if $N_1[v] = 0$ for all $i = 1, 2, 3, 4, 5$, v is a degree 4 polynomial with five roots, hence is the zero polynomial. This is still suitable for problems (1) and (2) because the function value and derivative at each vertex can be obtained from nodal variables associated with that vertex. 5, A

2. (a) Since $V_h \subset H^1$, we can use $v \in V_h$ in both the original variational problem and the Galerkin approximation, $\boxed{\text{seen} \Downarrow}$

$$a(u,v) = F[v],\ a(u_h, v) = F[v].$$

Subtraction and linearity in the first argument gives

$$0 = a(u,v) - a(u_h, v) = a(u - u_h, v),$$

as required. $\boxed{5, \mathrm{A}}$

(b) Continuity and coercivity mean that

$$a(u,v) \leq M\|u\|_{H^1}\|v\|_{H^1},\ \gamma\|u\|_{H^1}^2 \leq a(u,u),$$

respectively. Then, for arbitrary $v \in V_h$, we have

$$\begin{aligned}
\text{(coercivity) } \gamma\|u - u_h\|_{H^1}^2 &\leq a(u - u_h, u - u_h), \\
\text{(linearity) } &\leq a(u - u_h, u - v) + \underbrace{a(u - u_h, v - u_h)}_{=0 \text{ by part (a)}}, \\
\text{(continuity) } &\leq M\|u - u_h\|_{H^1}\|u - v\|_{H^1}.
\end{aligned}$$

Either $\|u - u_h\|_{H^1} = 0$, in which case the required result holds as required, or we can divide by $\|u - u_h\|_{H^1}$ to obtain the required result. $\boxed{5, \mathrm{B}}$

(c) (i) $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}
a(u,v) &= \langle u, v\rangle_{H^1} + \int_\Omega v\beta\cdot\nabla u \,\mathrm{d}\, x, \\
&\leq \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{L^2}\|\beta\cdot\nabla u\|_{H^1}, \\
&\leq \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{L^2}\beta_0\|\nabla u\|_{L^2}, \\
&\leq (1 + \beta_0)\|u\|_{H^1}\|v\|^{H^1}.
\end{aligned}$$

The continuity constant is $\leq (1 + \beta_0)$. $\boxed{3, \mathrm{B}}$

(ii) Substituting $v = u$ gives $\boxed{\text{unseen} \Downarrow}$

$$a(u,u) = \int_\Omega u^2 + |\nabla u|^2 + u\beta\cdot\nabla u \,\mathrm{d}\, x.$$

We have $u\nabla u = \frac{1}{2}\nabla u^2$. If $\nabla\cdot\beta = 0$ then $\beta\cdot\nabla\psi = \nabla\cdot(\beta\psi)$ for any scalar field ψ and the result follows by picking $\psi = u^2/2$. $\boxed{3, \mathrm{B}}$

(iii)

$$\begin{aligned}
a(u,u) &= \int_\Omega u^2 + |\nabla u|^2 + \frac{1}{2}\nabla\cdot(\beta u^2)\,\mathrm{d}\, x, \\
&= \int_\Omega u^2 + |\nabla u|^2 \,\mathrm{d}\, x + \frac{1}{2}\int_{\partial\Omega} \underbrace{\beta\cdot n}_{=0}\frac{1}{2}u^2 \,\mathrm{d}\, S, \\
&= \|u\|_{H^1}^2,
\end{aligned}$$

so the coercivity constant is 1. $\boxed{4, \mathrm{D}}$

3. (a) This variational problem has a bilinear form which is just the H^1 inner product. Hence it is continuous and coercive with scaling constants equal to 1. From the Lax-Milgram theorem, the solution exists and is unique. Taking $v = u$, we have $\boxed{\text{sim. seen} \Downarrow}$

$$\|u\|_{H^2}^2 = \langle u, f\rangle_{H^2} \leq \|u\|_{H^2}\|f\|_{H^2},$$

from Cauchy-Schwarz, and dividing both sides by $\|u\|_{L^2}$ gives the result. $\boxed{5, \text{B}}$

(b) Since V is a Lagrange finite element space of degree k, it contains the function $v = 1$. Taking v in the definition gives

$$\int_\Omega u \,\mathrm{d}\, x$$

on the left hand side, and

$$\int_\Omega f \,\mathrm{d}\, x$$

on the right, hence the result. $\boxed{5, \text{B}}$

(c) Method 1: solve by computing variational derivative,

$$\delta J[v; \delta v] = 2\int_\Omega \delta v(u - f) + \nabla\delta v \cdot \nabla(u - f) \,\mathrm{d}\, x = 0, \quad \forall \delta v \in V,$$

which is equivalent to our variational problem above.
Method 2: by contradiction. If u is not the minimiser, then there exists $v \in V$ with $J[v] \leq J[u]$. Then

$$\begin{aligned} J[v] &= \|v - f\|_{H^1}^2 = \|(v - u) + (u - f)\|_{H^1}^2, \\ &= \|v - u)\|_{H^1}^2 + \underbrace{\langle v - u, u - f\rangle_{H^1}}_{=0 \text{ by defn of } u} + \|u - f\|_{H^1}, \\ &= \|v - u\|_{H^1}^2 + J[u], \end{aligned}$$

and we conclude that $\|v - u\|_{H^1}^2 \leq 0$, a contradiction (since norms cannot be negative and u is assumed not equal to v). $\boxed{5, \text{C}}$

(d) Since u minimises the functional J, we have

$$\begin{aligned} \|u - f\|_{H^1(\Omega)} &= \inf_{\|v\|_{H^1(\Omega)} > 0} \|v - f\|_{H^1(\Omega)}, \\ &\leq \|I_h f - f\|_{H^1(\Omega)}, \\ &\leq Ch^{k-1}|f|_{H^2(\Omega)}, \end{aligned}$$

where I_h is the nodal interpolation operator into V, and we used the standard approximation result for I_h. $\boxed{5, \text{D}}$

4. (a) Multiplication by test function and integration by parts in the Laplacian term gives the following variational problem: find (twice time differentiable) time-dependent $u \in H^1$ such that

sim. seen ⇓

$$\langle \psi, u_{tt} \rangle + \langle \nabla \psi, \nabla u \rangle = 0, \quad \forall \psi \in H^1.$$

Our C^0 finite element space is a subset of H^1 so we may propose the following finite element discretisation, find $u_h \in V_h$ such that

$$\langle \psi, u_{h,tt} \rangle + \langle \nabla \psi, \nabla u_h \rangle = 0, \quad \forall \psi \in V_h.$$

5, A

(b) Using a basis for V_h of dimension N, we substitute basis expansions for ψ and u, leading to

$$\sum_i^N \psi_i \left(\sum_j^N \int_\Omega \phi_i \phi_j \,\mathrm{d}\, x \frac{\mathrm{d}^2}{\mathrm{d}\, t^2} u_j - \sum_j^N \nabla \phi_i \cdot \nabla \phi_j \,\mathrm{d}\, x u_j w \right) = 0.$$

Since the basis coefficients ψ_i are arbitrary, we must have

$$\sum_j^N \underbrace{\int_\Omega \phi_i \phi_j \,\mathrm{d}\, x}_{=M_{ij}} \frac{\mathrm{d}^2}{\mathrm{d}\, t^2} u_j - \sum_j^N \underbrace{\nabla \phi_i \cdot \nabla \phi_j \,\mathrm{d}\, x}_{=K_{ij}} u_j = 0, \quad i = 1, \ldots, N,$$

which is equivalent to the required form.

5, C

(c) If we introduce $v \in V_h$ such that

unseen ⇓

$$\langle \phi, u_t \rangle - \langle \phi, v \rangle = 0, \quad \forall \phi \in V_h,$$

(which is Equation (5)) then choosing $\phi = u_t - v \in V_h$ gives

$$0 = \langle u_t - v, u_t - v \rangle = \|u_t - v\|_{L^2}^2 \implies u_t = v.$$

Hence, $u_{tt} = v_t$, and we can substitute into the variational form to get Equation (6), and the two formulations are equivalent.

5, C

(d)

$$\begin{aligned} \dot{E} &= \langle v, v_t \rangle + \langle \nabla u, \nabla u_t \rangle, \\ \text{(definition of } w) &= \langle v, v_t \rangle + \langle w, u_t \rangle, \\ \text{(Equations (5-6))} &= \langle \nabla v, \nabla u \rangle - \langle w, v \rangle, \\ \text{(definition of } w) &= \langle \nabla v, \nabla u \rangle - \langle \nabla u, \nabla v \rangle = 0, \end{aligned}$$

as required.

5, D

5. (a) Taking $v = 0$, $p \neq 0$, we have

$$c((0,p),(0,p)) = a(0,0) + b(0,p) + b(0,p) = 0,$$

by bilinearity (or the definition). Hence we have a pair (v,p) with $\|v\|_V^2 + \|p\|_Q^2 > 0$, but $c((v,p),(v,p)) = 0$, i.e. c is not coercive. $\boxed{5, \mathrm{M}}$

(b) (i) We have

$$\|B^*p\|_{V'} = \sup_{0\neq v\in V} \frac{B^*p[v]}{\|v\|_V} = \sup_{0\neq v\in V} \frac{b(v,p)}{\|v\|_V},$$

and hence

$$\inf_{0\neq p\in Q} \frac{B^*p\|_{V'}}{\|p\|_Q} = \inf_{0\neq p\in Q} \frac{b(v,p)}{\|v\|_V\|p\|_Q},$$

so the two conditions are equivalent. $\boxed{5, \mathrm{M}}$

(ii) For any q,

$$\frac{\|B^*q\|_{V'}}{\|q\|_Q} \geq \inf_{0\neq p\in Q} \frac{\|B^*p\|_{V'}}{\|p\|_Q} \geq \beta,$$

by the definition of $\inf$, so

$$\|B^*q\|_{V'} \geq \beta\|q\|_Q,$$

for any q. Starting from this end, we take $q \neq 0$, divide by $\|q\|_Q$, and $\inf$ over all such q, to recover the original expression. $\boxed{5, \mathrm{M}}$

(iii) If there exists q_1, q_2 such that $B^*q_1 = B^*q_2$, and then $B^*(q_1 - q_2) = 0$ by linearity. Then Part b(ii) says that

$$0 = \|B^*(q_1 - q_2)\|_{V'} \geq \beta\|q_1 - q_2\|_Q,$$

i.e. $q_1 = q_2$, so B^* is injective. $\boxed{5, \mathrm{M}}$

Review of mark distribution:
Total A marks: 30 of 32 marks
Total B marks: 21 of 20 marks
Total C marks: 15 of 12 marks
Total D marks: 14 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH60022 Finite Elements: Numerical Analysis & Implementation

Question	Marker's comment
1	Students made a good job of parts a-b. In part c, a common problem was not considering an appropriate weak form (where derivatives can be transferred to the test function if it has sufficiently many weak derivatives), and some candidates concluded that there were too many derivatives in (2) when there were not.nbsp;Students mostly made a good effort in d, but some students added two nodal variables which was too many when only adding one basis function.
2	Students generally made a great job of this question, although there were a significant minority of students whose weakness with vector calculus was showing in confusion between gradients and divergences of scalar and vector functions respectively.
4	It's clear that many students were short on time by the point they started answering this question, which resulted many very incomplete solutions. The latter parts of this question were quite challenging for many students who did attempt them. One of the most common errors was assuming the strong form and deriving the weak form, which is not sufficient to show that two weak forms are equivalent.

MATH70022 Finite Elements: Numerical Analysis & Implementation

Question	Marker's comment
1	See comments on 60022
2	See comments on 60022
5	Students generally made an excellent job of this question testing understanding of the isomorphism between bilinear forms and operators between finite element spaces and dual finite element spaces.