Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$\langle x, y \rangle = \langle y, x \rangle, \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \langle ax, y \rangle = a \langle x, y \rangle.$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0 \tag{1}$$

(c) By thinking of (1) as a quadratic in t, and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|\langle x, y \rangle| \leq \|x\| \|y\|. \tag{2}$$

When does equality hold?

(d) Deduce the triangle inequality for the norm on $\mathbb{R}^n$.

(e) Show the reverse triangle inequality:

$$\big| \|x\| - \|y\| \big| \leq \|x - y\|$$

Exercise 1.2. Suppose $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

(i) Show that:

$$\max_{k=1,\ldots,n} \left| x^k \right| \leq \|x\|.$$

(ii) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left| x^k \right|.$$

[Hint: write out $\|x\|^2$ in coordinates and estimate]

Exercise 1.3. Suppose that $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i \to \infty} x_i = x, \qquad \lim_{i \to \infty} y_i = y.$$

(a) Show that

$$\lim_{i \to \infty} (x_i + y_i) = x + y.$$

(b) Show that

$$\lim_{i \to \infty} \langle x_i, y_i \rangle = \langle x, y \rangle,$$

and deduce that

$$\lim_{i \to \infty} \|x_i\| = \|x\|.$$

[Hint: Write $\langle x_i, y_i \rangle - \langle x, y \rangle = \langle x_i - x, y_i - y \rangle + \langle x_i - x, y \rangle + \langle x, y_i - y \rangle$ and use the Cauchy-Schwartz inequality (2)]

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence of real numbers with $a_i \to a$ as $i \to \infty$. Show that

$$\lim_{i \to \infty} (a_i x_i) = ax.$$

[Hint: Write $a_i x_i - ax = (a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)$ and use the properties of the norm.]

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional