Topic: Discrete random variables and their distributions

In today's problem class we will be studying properties of discrete random variables.

1. Show that the function

$$p_X(x) = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x$$

   for parameter $\lambda > 0$ is a valid probability mass function for a discrete random variable X taking values on $\{0, 1, 2, ...\}$. Also, find $\mathrm{P}(X \leq x)$ for $x \in \mathbb{R}$.

2. For what values of k is the following function a valid probability mass function?

$$p_X(x) = \begin{cases} \frac{k}{x(x+1)} & \text{if } x = n, n+1, n+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   where n is a fixed positive integer.

   Hint:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

3. If $X \sim \mathrm{Poi}(\lambda)$ and we know that $\mathrm{P}(X > 0) = 1 - \mathrm{e}^{-0.5}$, determine $\mathrm{P}(X \leq 1)$.

4. If $X \sim \mathrm{Poi}(\lambda)$ find the probability that X is odd.

5. If $X \sim \mathrm{Bin}(n, \theta)$, find $g(x)$ such that

$$p_X(x+1) = g(x)p_X(x), \quad x = 0, 1, \ldots, n-1.$$

6. Let $X \sim \mathrm{Bin}(n, p)$ for $n \in \mathbb{N}, 0 < p < 1$, and let $q = 1 - p$. Show that $Y = n - X \sim \mathrm{Bin}(n, q)$.

7. Let $X \sim \mathrm{Bin}(n, p)$ for an even $n \in \mathbb{N}$, and $p = 1/2$. Show that the distribution of X is symmetric about $n/2$, i.e.

$$\mathrm{P}(X = \frac{n}{2} + j) = \mathrm{P}(X = \frac{n}{2} - j),$$

   for all nonnegative integers j.

8. A fair coin is tossed n times. Let H, T denote the discrete random variables corresponding to the number of heads and the number of tails, respectively, in n tosses of the coin. Define the discrete random variable $X = H - T$. Find the image/range and probability mass function of X.