Imperial College London

MATH97020 MATH97098

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Stochastic Differential Equations

Date: Monday, 24 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1.

(a) Let $\mathbf{W}_t := \mathbf{W}(t,\omega)$ denote a $d-$dimensional Brownian motion. Use Kolmogorov's continuity theorem to prove that for almost all ω and any $T > 0$, the sample path $t \mapsto \mathbf{W}(t,\omega)$ is uniformly Hölder continuous on $[0,T]$ for every $\gamma \in (0, \frac{1}{2})$.

(4 marks)

(b) Let $\mathbf{W}_t$ be a standard $d-$dimensional Brownian motion and let Q be an orthogonal $d \times d$ real matrix. Show that $\mathbf{B}_t = Q\mathbf{W}_t$ is also a Brownian motion.

(4 marks)

(c) Show that the one-dimensional standard Brownian motion is a Markov process and give the formula of the corresponding Markov semigroup.

(4 marks)

(d) Let W_t be a standard $1-$dimensional Brownian motion. Show that for every $f : \mathbb{R}^2 \mapsto \mathbb{R}$,

$$\mathbb{E}(f(W_{t_1}, W_{t_2})) = \int_{\mathbb{R}^2} f(x_1,x_2)\gamma(x_1,t_1|0)\gamma(x_2,t_2-t_1|x_1)\,dx_1dx_2,$$

where $\gamma(x,t|y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$.

(8 marks)

(Total: 20 marks)

2. Let W_t be a standard one dimensional Brownian motion.

(a) Let $\lambda > 0$ arbitrary. Show that $M_t^\lambda := e^{\lambda W_t - \frac{\lambda^2}{2}t}$ is a martingale with respect to the natural filtration of Brownian motion.

(4 marks)

(b) Use Itô's formula to show that

$$dW_t^2 = dt + 2W_t\,dW_t.$$

Use this to show that for $n = 1, 2, \dots, N$,

$$\int_{n\Delta t}^{(n+1)\Delta t} \int_{n\Delta t}^{s} dW_\ell dW_s = \frac{1}{2}\Big(\Delta W_n^2 - \Delta t\Big),$$

where $\Delta W_n = W_{(n+1)\Delta t} - W_{n\Delta t}$.

(8 marks)

(c) Let $N\Delta t = T$, $t_n = n\Delta t$, $n = 0, \dots N$, $\lambda \in [0,1]$ and $\tau_n^\lambda = (1-\lambda)t_n + \lambda t_{n+1}$. Define

$$I_T^\lambda := \int_0^T W_s \circ^\lambda dW_s = \lim_{N\to+\infty} \sum_{n=0}^{N-1} W(\tau_n^\lambda)\Delta W_n, \quad \text{in } L^2(\Omega),$$

where $\Delta W_k = W_{t_{n+1}} - W_{t_n}$. Calculate I_T^λ and its mean and variance. (8 marks)

(Total: 20 marks)

3.

(a) Solve the Itô SDE

$$dX_t = -\frac{1}{2}e^{-2X_t}\,dt + e^{-X_t}\,dW_t, \quad X_0 = x > 0.$$

(5 marks)

(b) Solve the Itô SDE

$$\mathrm{d}X_t = (1 - \ln X_t)X_t \mathrm{d}t + X_t\,\mathrm{d}W_t, \qquad X_0 = x > 0.$$

(5 marks)

(c) Consider the Itô SDE

$$dX_t = (-X_t^3 + X_t)\,dt + \sqrt{2}\,dW_t, \quad X_0 = x, \tag{1}$$

where $x \sim \rho_0(x)$ with finite moments of all orders.

1. Write down the generator and the backward and forward Kolmogorov equations for the process X_t. (2 marks)
2. Use either Itô's formula or the forward Kolmogorov equation to obtain a system of equations for the moments of X_t, $m_n(t) := \mathbb{E}X_t^n$, $n = 0, 1, 2 \dots$. (4 marks)
3. Show that X_t defined in (1) is an ergodic Markov process and obtain a formula for the invariant measure. (4 marks)

(Total: 20 marks)

4. Let $\pi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$ be a probability density function and suppose that we want to calculate the expectation

$$\mathbb{E}_\pi f = \int_{\mathbb{R}^d} f(\mathbf{x})\pi(\mathbf{x})\,d\mathbf{x}, \tag{2}$$

where $f(\mathbf{x})$ is an arbitrary function such that $\mathbb{E}_\pi f < +\infty$.

(a) Explain how you can use a diffusion process of the form

$$d\mathbf{X}_t = -\nabla V(\mathbf{X}_t)\,dt + \sqrt{2}\,d\mathbf{W}_t, \tag{3}$$

where $\mathbf{W}_t$ denotes standard Brownian motion on $\mathbb{R}^d$ in order to calculate $\mathbb{E}_\pi f$. (4 marks)

(b) Let $S \in \mathbb{R}^{d\times d}$ be a symmetric positive definite matrix. Explain how you can use Part (a) to calculate the matrix inverse S^{-1}. (8 marks)

(c) Consider perturbations of the diffusion process (3):

$$d\mathbf{X}_t^b = (-\nabla V(\mathbf{X}_t^b) + \mathbf{b}(\mathbf{X}_t^b))\,dt + \sqrt{2}\,d\mathbf{W}_t.$$

Find conditions on the vector field $\mathbf{b}(\mathbf{x})$ so that $\mathbf{X}_t^b$ can also be used in order to calculate $\mathbb{E}_\pi f$. Does the diffusion process $\mathbf{X}_t^b$ satisfy the detailed balance condition?

(8 marks)

(Total: 20 marks)

5. Consider the second order Itô stochastic differential equation, written formally as

$$\ddot{q}(t) = -V'(q(t)) - \gamma(q(t))\dot{q}(t) + \sqrt{2\gamma(q(t))}\,\dot{W}_t, \tag{4}$$

where V, γ are smooth functions with $\gamma(q) > 0$, the prime denotes differentiation with respect to q, V is a confining potential and $q(0) = q_0$, $\dot{q}(0) = p_0$ are deterministic.

(a) Write (4) as a first order system of SDEs for $q(t)$ and $\dot{q}(t) = p(t)$. Write down the generator and the backward and forward Kolmogorov equations.

(3 marks)

(b) Show that the probability density

$$\rho_\infty(q,p) = \frac{1}{Z} e^{-\left(V(q)+\frac{1}{2}p^2\right)} \tag{5}$$

is a stationary density and obtain the normalization constant. Does the formula for the stationary density depend on the interpretation of the stochastic integral in (4), i.e. Itô or Stratonovich?

(5 marks)

(c) Let $H = L^2(\mathbb{R}^2\ \rho_\infty(q,p)dqdp)$ denote the Hilbert space of square integrable functions with respect to the invariant distribution (5) and let $\gamma(q) = \gamma$, constant. Show that the generator $\mathcal{L}$ of the process can be written in the form

$$\mathcal{L} = \mathcal{A} + \gamma \mathcal{S}, \tag{6}$$

where $\mathcal{A}$ and $\mathcal{S}$ are antisymmetric and symmetric operators, respectively, in H.

(6 marks)

(d) Assume that $\gamma(q) = \gamma > 0$, constant. Let $\rho(q,p,t)$ denote the solution of the forward Kolmogorov equation and define $h(q,p,t) = \rho(q,p,t)\rho_\infty^{-1}(q,p)$. Obtain an equation for h.

(6 marks)

(Total: 20 marks)

1. (a) We calculate, with $r = t - s > 0$,

$$\begin{aligned}
\mathbb{E}(|W_t - W_s|^{2m}) &= \frac{1}{(2\pi r)^{d/2}} \int_{\mathbb{R}^d} |x|^{2m} e^{-\frac{|x|^2}{2r}}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} r^m \int_{\mathbb{R}^m} |y|^{2m} e^{-\frac{|y|^2}{2}}\, dy \quad \left(y = \frac{x}{\sqrt{r}}\right) \\
&= Cr^m = C|t-s|^m,
\end{aligned}$$

for some (explicitly computable) constant C. we apply now Kolmogorov's theorem (in the form $\mathbb{E}|X_t - X_s|^\beta \leq C|t-s|^{1+\alpha}$) with $\beta = 2m$ and $\alpha = m-1$. We conclude that the multidimensional Brownian motion is Hölder continuous with exponent

$$0 < \gamma < \frac{\alpha}{\beta} = \frac{1}{2} - \frac{1}{2m}.$$

[4] MARKS –A

(b) (There are several different proofs of this result) We check that $\mathbf{B}_t$ and $\mathbf{W}_t$ have the same finite dimensional distributions:

$$\begin{aligned}
&\mathbb{P}[B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k] = \mathbb{P}[W_{t_1} \in Q^T F_1, \ldots, W_{t_k} \in Q^T F_k] \\
&= \int_{Q^T F_1 \cdots \times Q^T F_k} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \ldots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \ldots dx_k \\
= &\int_{F_1 \cdots \times F_k} p(t_1, 0, y_1) p(t_2 - t_1, y_1, y_2) \ldots p(t_k - t_{k-1}, y_{k-1}, y_k)\, dy_1 \ldots dy_k \\
= &\ \mathbb{P}[W_{t_1} \in F_1, \ldots, W_{t_k} \in F_k].
\end{aligned}$$

In the above we have made the change of variables $y_j = Qx_j$ and we have used the fact that, since Q is an orthogonal transformation, $|Qx_j - Qx_{j-1}|^2 = |x_j - x_{j-1}|^2$.

[4] MARKS –A

(c) First we give the definition of a Markov process (students are not asked to do this). Let $\{\mathcal{F}_t^X\}_{t \in [0,+\infty)}$ denote the filtration generated by $\{X_t\}_{t \in [0,+\infty)}$. The process is Markov if, for all bounded Borel functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $t,\, h \geq$, we have

$$\mathbb{E} f((X_{t+h}) | \mathcal{F}_h^X) = \mathbb{E}(f(X_{t+h}) | X_h).$$

The Markov property of Brownian motion follows from the fact that Brownian motion has independent increments:

$$\mathbb{E}(f(W_{t+h}) | \mathcal{F}_h) = \mathbb{E}(f(W_{t+h} - W_h + W_h) | \mathcal{F}_h) = \mathbb{E}(f(W_{t+h}) | W_s).$$

The Markov semigroup, applied to a measurable function f is given by

$$(P_t f)(x) = \mathbb{E}(f(W_t) | W_0 = x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\, dy$$

with $P_0 = I$.

[4] MARKS –A

(d) We write $Y_i = W(t_i)$, $Y_i = X_i - X_{i-1}$, $i = 1, 2$. We also define

$$h(y_1, y_2) := f(y_1, y_1 + y_2).$$

Then

$$\begin{aligned}
\mathbb{E} f(W_{t_1} W_{t_2}) &= \mathbb{E} h(Y_1, Y_2) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} h(y_1, y_2) \gamma(y_1, t_1|0) \gamma(y_2, t_2 - t_1|0) dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2) \gamma(x_1, t_1|0) \gamma(y_2, t_2 - t_1|x_1) dx_1 dx_2.
\end{aligned}$$

In the above we have used the facts that the random variables Y_1 and Y_2 are independent, that $Y_1 \sim \mathcal{N}(0, t_1)$, $Y_1 \sim \mathcal{N}(0, t_2 - t_1)$. We also made the change of variables $y_1 = x_1$, $y_2 = x_2 - x_1$ and $x_0 = 0$. The Jacobian of the transformation is 1.

[8] –C

2. First we immediately see that M_t^λ is $\mathcal{F}_t-$measurable, where $\mathcal{F}_t$ is the natural filtration generated by Brownian motion. Furthermore, we have that $\mathbb{E}|M_t| < +\infty$ and therefore $M_t^\lambda \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Now we calculate, using the fact that Brownian motion has independent increments with $W_t - W_s \sim \mathcal{N}(0, t - s)$ and the formula for the characteristic function of a Gaussian random variable:

$$\begin{aligned}
\mathbb{E}\big(M_t^\lambda \,|\, \mathcal{F}_t\big) &= \mathbb{E}\big(e^{\lambda W_t - \frac{\lambda^2}{2} t} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda W_t - \frac{\lambda^2}{2} t - \lambda W_s + \lambda W_s} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda(W_t - W_s) - \frac{\lambda^2}{2}(t-s)} e^{\lambda W_s - \frac{1}{2}\lambda^2 s} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda(W_t - W_s)} \,|\, \mathcal{F}_t\big) e^{\lambda W_s - \frac{1}{2}\lambda^2 s} e^{-\frac{1}{2}\lambda^2 (t-s)} \\
&= e^{\frac{1}{2}\lambda^2(t-s)} M_s^\lambda e^{-\frac{1}{2}\lambda^2(t-s)} \\
&= M_s^\lambda.
\end{aligned}$$

[4] MARKS –A

(a) The generator of Brownian motion is $\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2}$.From Itô's formula we have

$$df(W_t) = (\mathcal{L} f)(W_t)\, dt + f'(W_t)\, dW_t.$$

We apply this to $f(x) = x^2$ to deduce

$$dW_t^2 = dt + 2W_t\, dW_t. \tag{1}$$

Furthermore,

$$\int_{n\Delta t}^{(n+1)\Delta t} \int_{n\Delta t}^{s} dW_\ell dW_s = \int_{n\Delta t}^{(n+1)\Delta t} W_s\, dW_s - W_{n\Delta t}\Big(W_{(n+1)\Delta t} - W_{n\Delta t}\Big).$$

From (1) we deduce that

$$\int_{n\Delta t}^{(n+1)\Delta t} W_s\, dW_s = \frac{1}{2}\Big(W^2_{(n+1)\Delta t} - W^2_{n\Delta t}\Big) - \frac{1}{2}\Delta t.$$

We combine these two equations to obtain

$$\begin{aligned}
\int_{n\Delta t}^{(n+1)\Delta t}\int_{n\Delta t}^{s} dW_\ell dW_s &= \frac{1}{2}\Big(W^2_{(n+1)\Delta t}-W^2_{n\Delta t}\Big)-\frac{1}{2}\Delta t - W_{n\Delta t}\Big(W_{(n+1)\Delta t}-W_{n\Delta t}\Big)\\
&= \frac{1}{2}W^2_{(n+1)\Delta t}+\frac{1}{2}W^2_{n\Delta t}-\frac{1}{2}\Delta t - W_{n\Delta t}\,W_{(n+1)\Delta t}\\
&= \frac{1}{2}\big(\Delta W_n^2-\Delta t\big).
\end{aligned}$$

[8] MARKS –B

(c) We want to prove that

$$\begin{aligned}
&\lim_{n\to+\infty}\sum_k W\left((1-\lambda)t_k+\lambda t_{k+1}\right)\left(W_{k+1}-W_k\right)\\
&=\frac{W_T^2}{2}+\left(\lambda-\frac{1}{2}\right)T \text{ in } L^2(\Omega)
\end{aligned}$$

We need to show that

$$R_n^\lambda=\frac{W_T^2}{2}-\frac{1}{2}\sum_k(\Delta W_k)^2+\sum_k\Big(W(\tau_k^\lambda)-W_k\Big)^2+\sum_k\Big(W_{k+1}-W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda)-W_k\Big) \tag{2}$$

The second and third term of (2) converges to these limits:

$$\frac{1}{2}\sum_k(\Delta W_k)^2\to\frac{T}{2}\text{ in } L^2(\Omega)\quad\text{(quadratic variation of Brownian motion)}$$

$$\begin{aligned}
\sum_k\Big(W(\tau_k^\lambda)-W_k\Big)^2 &= \sum_k\Big(W(\tau_k^\lambda)-W_k\Big)^2\\
&\approx\sum_k\Big(\tau_k^\lambda-t_k\Big)\\
&=\lambda\sum_k(t_{k+1}-t_k)\\
&\to\lambda T\text{ in } L^2(\Omega)
\end{aligned}$$

For the last term of (2),

$$\mathbb{E}\left[\sum_k\Big(W(\tau_k^\lambda)-W_k\Big)^2+\sum_k\Big(W_{k+1}-W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda)-W_k\Big)\right]^2 \tag{3}$$

$$=\sum_k\mathbb{E}\Big(W_{k+1}-W(\tau_k^\lambda)\Big)^2\,\mathbb{E}\Big(W(\tau_k^\lambda)-W_k\Big)^2 \tag{4}$$

$$=\sum_k(1-\lambda)(t_{k+1}-t_k)\lambda(t_{k+1}-t_k) \tag{5}$$

$$\le\lambda(1-\lambda)T\,|P^n|\to 0\text{ in } L^2(\Omega) \tag{6}$$

$$\implies\mathbb{E}\left[\sum_k\Big(W(\tau_k^\lambda)-W_k\Big)^2+\sum_k\Big(W_{k+1}-W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda)-W_k\Big)\right]^2\to 0\text{ in } L^2(\Omega) \tag{7}$$

where (4) follows from the fact the Brownian motion has independent increments. Hence,

$$\begin{aligned} R_n^\lambda &\to \frac{W_T^2}{2} - \frac{T}{2} + \lambda T \\ &= \frac{W_T^2}{2} + \left(\lambda - \frac{1}{2}\right) T \text{ in } L^2(\Omega) \end{aligned}$$

To show (2), we need to show that

$$R_n^\lambda = \sum_k W(\tau_k^\lambda)(W_{k+1} - W_k)$$

$$\begin{aligned} R_n^\lambda &= \frac{W_T^2}{2} - \frac{1}{2}\sum_k (\Delta W_k)^2 + \sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 + \sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 \\ &\quad + \sum_k \left(W_{k+1} - W(\tau_k^\lambda)\right)\left(W(\tau_k^\lambda) - W_k\right) \\ &= \sum_k \left[\frac{W_{k+1}^2}{2} - \frac{W_k^2}{2} - \frac{1}{2}W_{k+1}^2 + W_k W_{k+1} - \frac{1}{2}W_k^2\right] \\ &\quad + \sum_k \left(W_{k+1} - W(\tau_k^\lambda)\right)\left(W(\tau_k^\lambda) - W_k\right) \\ &= \sum_k \left[-W(\tau_k^\lambda)W_k + W_{k+1}W(\tau_k^\lambda)\right] \\ &= \sum_k W(\tau_k^\lambda)(W_{k+1} - W_k) \end{aligned}$$

The mean of I_T^λ is

$$\mathbb{E}I_T^\lambda = \frac{T}{2} + \left(\lambda - \frac{1}{2}\right) T = \frac{\lambda}{T}.$$

The variance is

$$\mathbb{E}(I_T^\lambda - \mathbb{E}I_T^\lambda)^2 = \frac{\mathbb{E}W_T^4}{4} + \frac{T^2}{4} - \frac{T\mathbb{E}W_T^2}{2} = \frac{T^2}{2}.$$

[8] MARKS –C

3. (a) The generator of the process is

$$\mathcal{L} = -\frac{1}{2}e^{-2x}\partial_x + \frac{1}{2}e^{-2x}\partial_x^2.$$

We apply Itô's formula to the function $f(x) = e^x$, $Y_t = f(X_t)$:

$$dY_t = (\mathcal{L}f)(X_t)\,dt + (\partial_x f)(X_t)e^{-X_t}\,dW_t = dW_t.$$

Therefore:

$$X_t = \ln\left(W_t + e^{X_0}\right).$$

[5] MARKS –A

(b) Note that $Y(0) = \ln(x)$. We compute by Itô's formula that $\mathrm{d}Y = \frac{1}{X}\mathrm{d}X - \frac{1}{2}\mathrm{d}t = (\frac{1}{2} - Y)\mathrm{d}t + \mathrm{d}W$. Therefore

$$\mathrm{d}(\mathrm{e}^t Y(t)) = \mathrm{e}^t Y(t) + \mathrm{e}^t \mathrm{d}Y(t) = \frac{1}{2}\mathrm{e}^t + \mathrm{e}^t W(t).$$

Thus

$$\mathrm{e}^t Y(t) = \ln(x) + \frac{1}{2}(\mathrm{e}^t - 1) + \int_0^t \mathrm{e}^s W(s)\mathrm{d}s,$$

so that

$$Y(t) = \mathrm{e}^{-t}\ln(x) + \frac{1}{2}(1 - \mathrm{e}^{-t}) + \int_0^t \mathrm{e}^{s-t} W(s)\mathrm{d}s,$$

and

$$X(t) = \exp\left[\mathrm{e}^{-t}\ln(x) + \frac{1}{2}(1 - \mathrm{e}^{-t}) + \int_0^t \mathrm{e}^{s-t} W(s)\mathrm{d}s\right].$$

[5] MARKS –A

(c) (i) The generator is

$$\mathcal{L} = (-x^3 + x)\partial_x + \partial_x^2.$$

The Fokker-Planck operator is the $L^2(\mathbb{R})-$adjoint of the generator:

$$\mathcal{L}^*\rho = \partial_x\left((x^3 - x)\rho + \partial_x\rho\right).$$

The backward and forward Kolmogorov equations are

$$\partial_t u = \mathcal{L}u, \quad u(x,t) = \mathbb{E}(f(X_t)|X_0 = x)$$

and

$$\partial_t\rho = \mathcal{L}^*\rho, \quad \rho(x,0) = \rho_0(x).$$

[2] MARKS–A

(ii) The nth moment M_n is defined as

$$m_n(t) = \int x^n\rho(x,t)\,dx.$$

We multiply the Fokker-Planck equation by x^n, integrate over $\mathbb{R}$ and integrate by parts on the right hand side of the equation to obtain

$$\begin{aligned}\dot{m}_n(t) &= -n\int(-x^3 + x)x^{n-1}\rho\,dx + n(n-1)\int x^{n-2}\rho\,dx \\ &= -nm_{n+2}(t) + nm_{n-1}(t) + n(n-1)m_{n-2}(t).\end{aligned}$$

[4] MARKS – A

(iii) We note that the generator of the process is of the form

$$\mathcal{L} = -V' \partial_x + \partial_x^2 \quad \text{with} V(x) = \frac{x^4}{4} - \frac{x^2}{2}.$$

We check that

$$Z := \int_{\mathbb{R}} e^{-V(x)} \, dx < +\infty.$$

The theory that we presented in the lectures on reversible diffusions applies and we deduce that X_t is ergodic with respect to the invariant measure

$$\rho_\infty(x) \, dx = \frac{1}{Z} e^{-V(x)} \, dx.$$

Alternatively, we can check that ρ_∞ is the stationary solution of the Fokker-Planck equation. We then write $\rho = h\rho_\infty$ to rewrite the stationary Fokker-Planck equation as

$$\mathcal{L}h = 0.$$

We multiply this equation by $h\rho_\infty$, integrate over $\mathbb{R}$ and then integrate by parts to obtain

$$\int_{\mathbb{R}} |h'|^2 \rho_\infty \, dx = 0,$$

from which we deduce that $h = \text{const}$. We conclude that ρ_∞ is the unique stationary solution of the stationary Fokker-Planck equation from which the ergodicity of the process follows.
[4] MARKS – B

4. (a) We want to choose $V(x)$ in such a way that X_t is an ergodic diffusion process with $\pi(x)$ as its invariant distribution. We need

$$e^{-V} = \pi \quad \Rightarrow \quad V(x) = -\log(\pi(x)).$$

The SDE for X_t becomes

$$dX_t = \nabla \log(\pi(X_t))\, dt + \sqrt{2}\, dW_t.$$

We have that

$$\lim_{t \to +\infty} \mathbb{E} f(X_t) = \mathbb{E}_\pi f,$$

where $\mathbb{E}$ denotes expectation with respect to the Brownian motion W_t and the initial conditions (if we take X_0 to be random). Using the ergodicity of the process X_t we can write

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(X_s)\, ds = \mathbb{E}_\pi f.$$

[4] MARKS – C

(b) Consider the previous problem with

$$\pi(x) = \frac{1}{Z} e^{-\frac{1}{2} x^T S x},$$

where Z denotes the normalization constant and S is the symmetric positive definite matrix whose inverse we want to calculate. We have that $\nabla \log(\pi(x)) = -Sx$. The corresponding SDE is

$$dX_t = -S X_t\, dt + \sqrt{2} dW_t. \tag{8}$$

Since $\pi(x)$ is a Gaussian distribution with mean 0 and covariance matrix S^{-1}, we have that

$$S^{-1}_{ij} = \int_{\mathbb{R}^d} x_i x_j \pi(x)\, dx, \quad i, j = 1, \dots d.$$

We use Part (i) to conclude that we can calculate the inverse of the matrix S by running the SDE and then calculating the expectation:

$$S^{-1}_{ij} = \lim_{T \to +\infty} \mathbb{E}(X_i(s) X_j(s)) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T X_i(s) X_j(s)\, ds, \quad i, j = 1, \dots d.$$

[8] MARKS – D

(c) We want X_t^b to have the same invariant distribution as X_t. The stationary Fokker-Planck equation reads

$$\nabla \cdot ((\nabla V - b) e^{-V} + \nabla e^{-V}) = 0,$$

from which we conclude that

$$\nabla \cdot (b(x) e^{-V}) = 0,$$

or, equivalently,

$$\nabla \cdot b - \nabla V \cdot b = 0.$$

In order for the diffusion process to satisfy detailed balance, the probability flux has to vanish at equilibrium. In our case we have

$$J_s(x) = (\nabla V - b) e^{-V} + \nabla e^{-V} = -b e^{-V} \neq 0.$$

Hence, X_t^b does not satisfy detailed balance.

[8] MARKS – D

5. (a) Introducing the momentum $p_t = \dot{q}_t$ (we write $q_t = q(t)$, $p_t = p(t)$) we can write the Langevin equation as a system of first order stochastic differential equations in phase space $(q,\, p) \in \mathbb{R}^{2d}$

$$
\begin{aligned}
dq_t &= p_t\, dt, && (9a)\\
dp_t &= f(q_t)\, dt - \gamma(q_t) p_t\, dt + \sqrt{2\gamma(q_t)}\, dW_t. && (9b)
\end{aligned}
$$

The position and momentum $\{q_t,\, p_t\}$ define a Markov process with generator

$$
\mathcal{L} = p\partial_q + f(q)\partial_p + \gamma(q)(-p\partial_p + \partial_p^2). \tag{10}
$$

The backward Kolmogorov equation for $u(q,p,t) = \mathbb{E}(g(q_t, p_t)|q_0 = q, p_0 = p)$ is

$$
\frac{\partial u}{\partial t} = \mathcal{L}u, \quad u(q,p,0) = g(q,p).
$$

Assume that initially the position and momentum are distributed according to a distribution $\rho_0(q,p)$ (When q_0, p_0 are deterministic and the initial distribution is $\rho_0(q,p) = \delta(q-q_0)\delta(p-p_0)$). The evolution of the probability distribution function of the Markov process $\{q_t,\, p_t\}$ is governed by the Fokker-Planck/forward Kolmogorov equation

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -p\partial_q \rho - f(q)\partial_p \rho + \gamma(q)\left(\partial_p(p\rho) + \partial_p^2\rho\right), && (11a)\\
\rho(q,p,0) &= \rho_0(q,p). && (11b)
\end{aligned}
$$

[3] UNSEEN

(b) Let $f = -V'(q)$ be a smooth confining potential. We need to check that $\rho_\infty(q,p)$ is a solution of the stationary Fokker-Planck equation:

$$
\begin{aligned}
\mathcal{L}^* e^{-\left(V(q)+\frac{1}{2}p^2\right)} &= -p\partial_q e^{-\left(V(q)+\frac{1}{2}p^2\right)} + V'(q)\partial_p e^{-\left(V(q)+\frac{1}{2}p^2\right)}\\
&\quad + \gamma(q)\partial_p\left((pe^{-\frac{1}{2}p^2}) + \partial_p e^{-\frac{1}{2}p^2}\right) e^{-V(q)}\\
&= 0 + \gamma(q) \times 0 = 0.
\end{aligned}
$$

By assumption, $V(q)$ is a confining potential. Furthermore, the density ρ_∞ is the product of $e^{-V(q)}$ and of the Gaussian $e^{-\frac{1}{2}p^2}$. Therefore, $\rho_\infty \in L^1(dqdp)$. The normalization constant Z is

$$
Z = \int_{\mathbb{R}^2} e^{-\left(V(q)+\frac{1}{2}p^2\right)}\, dpdq = (2\pi)^{1/2} \int_{\mathbb{R}} e^{-V(q)}\, dq.
$$

The Itô and Stratonovich interpretation of the stochastic integrals in (9) are the same since $q \in C^{1+\delta}$ for $\delta \in (0, 1/2)$.

[5] UNSEEN

(c) We note again that the invariant density can be written in the product form

$$
\rho_\infty = \left(\frac{1}{\int_{\mathbb{R}} e^{-V(q)}\, dq} e^{-V(q)}\right) \times \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}p^2}\right).
$$

The $p-$dependent part of ρ_{infty} is the invariant density of the Ornstein-Uhlenbeck process with generator $\mathcal{S} := -p\partial_p + \partial_p^2$. We know from the lecture notes that $\mathcal{S}$ is symmetric in $L^2(\mathbb{R}; \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}p^2})$

and consequently symmetric in the space H. We know check the operator $\mathcal{A} = p\partial_q - \partial_q V \partial_p$. Let $f,\ h$ be $C^1(\mathbb{R}^2)$ functions that vanish at infinity. We calculate:

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathcal{A} f h \rho_\infty \, dqdp &= \int_{\mathbb{R}^2} \big(p\partial_q f - \partial_q V \partial_p f\big) h \rho_\infty \, dqdp \\
&= -\int_{\mathbb{R}^2} f \big(p\partial_q h - \partial_q V \partial_p h\big) \rho_\infty \, dqdp \\
&= -\int_{\mathbb{R}^2} f \mathcal{A} h \rho_\infty + 0 \\
&= -\langle f, \mathcal{A} h \rangle_H .
\end{aligned}$$

Therefore, $\mathcal{A}$ is antisymmetric in H and we conclude that the generator can be written in the form

$$\mathcal{L} = \mathcal{A} + \gamma \mathcal{S}, \quad \mathcal{A}^* = -\mathcal{A}, \ \ \mathcal{S}^* = \mathcal{S},$$

with $\mathcal{A} = p\partial_q - \partial_q V \partial_p$ and $\mathcal{S} = -p\partial_p + \partial_p^2$.
[6] UNSEEN

(d) Let $\rho(q, p, t)$ denote the solution of the Fokker-Planck equation and write

$$\rho(q, p, t) = h(q, p, t) \rho_\infty(q, p). \tag{12}$$

From the previous calculations and the properties of the generator of the Ornstein-Uhlenbeck processes from the notes we have that

$$\partial_p(p\rho) + \partial_p^2 \rho = (-p\partial_p h + \partial_p^2 h)\rho_\infty = \mathcal{S} h \rho_\infty .$$

Furthermore, the calculation that we did in the integration by parts above yields:

$$p\partial_q \rho - \partial_q V \partial_p \rho = (p\partial_q h - \partial_q V \partial_p h)\rho_\infty .$$

We conclude that

$$\mathcal{L}^* \rho = \big(-\mathcal{A} h + \gamma \mathcal{S} h\big) \rho_\infty .$$

We conclude that the function $h(q, p, t)$ is the solution to the PDE

$$\frac{\partial h}{\partial t} = \big(-p\nabla_q + \nabla_q V \nabla_p \big) h + \gamma \left(-p\nabla_p + \beta^{-1}\partial_p^2 \right) h =: -\mathcal{A} h + \gamma \mathcal{S} h.$$

with $h(q, p, 0) = \rho_\infty^{-1} \rho_0(q, p)$
[6] MARKS UNSEEN

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97020MATH97098	1	Most students answered this question correctly. I was pleased to see that most students have familrized themselves with the basic properties of Brownian motion such as the Markov property, regularity of paths etc.
MATH97020MATH97098	2	overall, most students answered this question correctly. There were some issues with the calculation of the double integral in part (b) of the question and also with the calculation of the mean and variance of the integral in part c.
MATH97020MATH97098	3	most students had no problems with the application of Ito's formula and Lamperti's transformation. Most students were also able to obtain the equations for the moments in the third part of the question. Some students had difficulties in showing that the SDE in part c is ergodic. this follows directly from the material in the lecture notes.
MATH97020MATH97098	4	several students could answer parts a and c of the exam. Very few students recognized the connection between the Ornstein-Uhlenbeck process and the calculation of the inverse of a matrix.
MATH97020MATH97098	5	many students had difficulties with this question. I was a little surprised by this, since this question was similar to questions from previous exam papers and also from the problem sheets.