Imperial College London

MATH70012

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: 23 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Question 1

(Total: 20 marks)

Suppose I, a British investor, can, at any time $n = 0, 1$:

1. deposit £ in a British bank at the (domestic) interest rate $r \geq 0$
2. buy \$ at the exchange rate E_n (defined as the number of £ needed to buy one \$)
3. deposit \$ in an American bank at the (foreign) interest rate $f \geq 0$
4. buy shares in an American stock, whose price in \$ is S_n

As usual we assume I can deposit/buy a negative amount (i.e. borrow/short-sell). Denote with $g \in \mathbb{R}$ the number of \$, and with $h \in \mathbb{R}$ the number of shares, bought at time 0.

Assume that the values of (S, E) are given by the following trinomial model: on the probability space $\Omega := \{a, b, c\}$ we consider a probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$, and define $S_0 = 4, E_0 = \frac{27}{8}$,

ω	a	b	c
$S_1(\omega)$	6	2	14
$E_1(\omega)$	6	4	2

Answer the following questions and justify carefully your answers.

(a) The formula (3 marks)

$$V_1^{x,g,h} := x + (x - gE_0 - hS_0E_0)((1+r) - 1) + g((1+f)E_1 - E_0) + h(S_1E_1 - S_0E_0), \quad (1)$$

describes my total wealth $V_1 := V_1^{x,g,h}$ (in £) at time 1 if my initial capital (in £) is $V_0 = x$. At time 1, what is the value (in £) of my American investments (cash plus shares)?

Assume from now on that $r = 1, f = \frac{1}{2}$.

(b) Is the market model arbitrage-free? (4 marks)

(c) Is the market complete? (3 marks)

(d) Consider an option P on the stock, whose payoff in \$ is (5 marks)

ω	a	b	c
$P_1(\omega)$	4	8	10

What are the arbitrage-free prices of P? Give your answer both in £ and in \$.

(e) Consider the set $\mathcal{A}$ of all trading strategies which super-replicate the option P. Identify exactly the set of all $(x, g, h) \in \mathcal{A}$ with smallest initial capital, i.e. such that (5 marks)

$$x = p := \min\{x' : (x', g', h') \in \mathcal{A}\}.$$

Question 2

(Total: 20 marks)

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. Define the random variable

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	6	12

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 0$, a stock whose initial price is $S_0 = 6$, and whose final price is S_1. We also consider a put option P (with underlying S) with strike price 7. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) arbitrage-free? (2 marks)

(b) Is the market (B, S) complete? (2 marks)

(c) Is the put option P replicable (in the market (B, S))? (3 marks)

(d) What is the smallest price $u(P)$ at which an infinitely risk-averse investor would sell the put option (in the market (B, S))? (5 marks)

From now on assume that P can be liquidly traded at the initial price $P_0 = u(P)$, and consider the market (B, S, P).

(e) Is the market (B, S, P) arbitrage-free? If not, explicitly determine an arbitrage strategy. (5 marks)

(f) Is the market (B, S, P) complete? (3 marks)

Question 3

(Total: 20 marks)

Let $N \in \mathbb{N} \setminus \{0\}$. On the space $\Omega := \{H, T\}^N$, consider (as usual) the natural filtration $\mathcal{F} = \mathcal{F}^X$ of the coin tosses $(X_i)_{i=1}^N$ (i.e. the random variables $X_i(\omega) := \omega_i$, where $\omega := (\omega_i)_{i=1}^N$), i.e. $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ for $i = 1, \ldots, N$, and the probability $\mathbb{P}$ (on the σ-algebra $\mathcal{A} := \mathcal{F}_N$) for which the coin tosses are independent and

$$\mathbb{P}(X_i = H) = \frac{1}{2} = \mathbb{P}(X_i = T) \quad \text{for all } i = 1, \ldots, N.$$

On $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ consider the multi-period binomial model (B, S) with expiration N, 0 interest rate (i.e. $B_i = 1$ for all i), and stock price $S = (S_i)_{i=0}^N$ given by the asymmetric random walk

$$S_k := S_0 + \sum_{i=1}^k Y_i \quad \text{for all } k = 0, \ldots, N, \quad \text{where} \quad Y_i(\omega) := \begin{cases} 2 & \text{if } \omega_i = H \\ -1 & \text{if } \omega_i = T \end{cases} \quad i = 1, \ldots, N, \, \omega \in \Omega,$$

where $S_0 \in \mathbb{R}$ is a constant. Define the variation of S as the process

$$W_k := |S_0| + \sum_{i=1}^k |Y_i| \quad \text{for all } k = 0, \ldots, N.$$

Consider the call option with underlying W, strike price $K \in \mathbb{R}$ and expiry N, i.e. the derivative with payoff $(W_N - K)^+$ at time N; denote with V_i its arbitrage-free price at time $i = 0, \ldots, N$. Denote with $\mathbb{Q}$ the risk-neutral measure. Whenever we will say that a process is Markov, we mean that it is Markov with respect to $\mathbb{Q}$ and $\mathcal{F}$. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Are the $(Y_i)_i$ IID (i.e. independent and identically distributed) under $\mathbb{Q}$? (3 marks)

(b) Let $\mathcal{F}^Y$ be the natural filtration of Y. Does $\mathcal{F}^Y = \mathcal{F}$ hold? (1 marks)

(c) Is S a Markov process? (2 marks)

(d) Is W a Markov process? (2 marks)

(e) Is (S, W) a Markov process? (2 marks)

(f) Which of the 3 processes S, W and (S, W) are such that, for every $n = 0, \ldots, N$, V_n can be written as a function f_n of the value of such process at time n? (3 marks)

(g) For one process $A \in \{S, W, (S, W)\}$ such that V_n can be written as $V_n = f_n(A_n)$ (for some f_n) as described in item (f), write explicitly f_N and an explicit formula to express f_n in terms of f_{n+1} for $n = 0, \ldots, N-1$. (7 marks)

Question 4

(Total: 20 marks)

Consider the N-period model where the bank account has interest rate r, and the stock price process $S = (S_n)_{n=0}^N$ is given by the (arbitrage-free) binomial model with constant down and up factors d, u, defined as usual on the probability space $\Omega = \{H, T\}^N$, endowed with the natural filtration $\mathcal{F} = \mathcal{F}^S$ of S. Assume that

$$N = 3, \quad r = 0, \quad d = \frac{1}{2}, \quad u = 2, \quad S_0 = 8.$$

Define as usual $\inf \emptyset := \infty$, and consider

$$\tau := \inf\{i = 0, \ldots, N : S_i \geq 20\},$$

which is a random time (i.e. a random variable with values in $\mathbb{N} \cup \{\infty\}$). Consider the derivative D which pays S_τ at time τ if $\tau \leq N$ (where $S_\tau := S_k$ on $\{\tau = k\}$ for each $k = 0, \ldots, N$), and pays 6 at time N otherwise; in other words, D yields the payoff

$$D_\sigma := S_\tau \cdot 1_{\{\tau \leq N\}} + 6 \cdot 1_{\{\tau = \infty\}} \quad \text{at time} \quad \sigma := \tau \wedge N.$$

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Explicitly compute $\tau(\omega)$ for each $\omega \in \Omega$. (2 marks)

(b) Prove that τ is a stopping time, i.e., that $\{\tau = n\} \in \mathcal{F}_n$ for all $n = 0, \ldots, N$. (2 marks)

(c) Is σ a stopping time? (2 marks)

(d) Determine the arbitrage-free price V of D (meaning, its values V_k for $\{k \leq \sigma\}$). (7 marks)

(e) Suppose it becomes possible to trade D at price 6 at any time strictly before time σ, and at price D_σ at time σ. Identify an arbitrage in the (B, S, D) market (this market has time horizon σ), and compute its final payoff (i.e. its value at time σ). (3 marks)

(f) Consider the option C which gives you the right to choose at time 1 to immediately receive either one share of S, or one derivative D. Compute the arbitrage-free price C_0 at time 0 of such option. (4 marks)

Question 5

(Total: 20 marks)

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded), whose prices S_t^1, S_t^2 at times $t = 0, 1$ have values

$$S_0^1 = 300, \qquad S_0^2 = 200, \qquad \begin{array}{c|c|c} \omega & H & T \\ \hline S_1^1(\omega) & 360 & 270 \\ \hline S_1^2(\omega) & 200 & 260 \end{array} \tag{2}$$

Assume that the events $\{H\}$ and $\{T\}$ have probability $\frac{1}{2}$. We'll use the following standard notation: $R_1^i := \frac{S_1^i}{S_0^i} - 1$ denotes the return of asset $i = 1, 2$ between time 0 and 1, $\Sigma := \mathrm{Cov}(R_1)$ denotes the covariance matrix of the returns (i.e. $\Sigma_{i,j} := \mathbb{E}(R_1^i R_1^j) - \mathbb{E}(R_1^i)\mathbb{E}(R_1^j)$, $i = 1, 2$), a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in asset S^i. Denote with $\mu = \mu(\pi)$ the average, and with $\sigma = \sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Compute the average returns $\mu_i := \mathbb{E}[R_1^i], i = 1, 2$, and the covariance matrix Σ. Prove that the correlation ρ between R_1^1 and R_1^2 equals -1. (4 marks)

(b) Find a portfolio with average return of 8%. Does it involve any short-selling? (3 marks)

(c) Find the equation whose solution is the set S of values (σ, μ) taken by all portfolios, and draw S. (5 marks)

(d) Determine and draw the subset S_+ of S corresponding to all portfolios which do not involve short-selling. (2 marks)

(e) Find two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (3 marks)

(f) What is the value V_1^M at time 1 of the market portfolio (x_0^M, π^M) with initial value $x_0^M = 1000$? (3 marks)

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2023

MATH60012/MATH70012 Mathematical Finance: An Intro to Option Pricing

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes, though only for students who took the course last year (21-22)

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a-e) 3+4+3+5+5=20 marks; 2(a,b,c) 2+2+3=7 marks; 3(b,c) 1+2=3 marks; 4(a) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
2(d,f) 5+3=8 marks; 3(d,e) 2+2=4 marks; 4(b,c,f) 2+2+4=8 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(e) 5 marks; 4(d) 7 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
3(a,f,g) 3+3+7=13 marks; 4(e) 3 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Finance: An Intro to Option Pricing

Date: Tuesday, 23rd May 2023
Time: 10 – 12
Time Allowed: 2 Hours for MATH60 paper; 2.5 Hours for MATH70 papers
This paper has *4 Questions (MATH60 version); 5 Questions (MATH70 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

Question 1

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Suppose I, a British investor, can, at any time $n = 0, 1$:

1. deposit £ in a British bank at the (domestic) interest rate $r \geq 0$
2. buy \$ at the exchange rate E_n (defined as the number of £ needed to buy one \$)
3. deposit \$ in an American bank at the (foreign) interest rate $f \geq 0$
4. buy shares in an American stock, whose price in \$ is S_n

As usual we assume I can deposit/buy a negative amount (i.e. borrow/short-sell). Denote with $g \in \mathbb{R}$ the number of \$, and with $h \in \mathbb{R}$ the number of shares, bought at time 0.

Assume that the values of (S, E) are given by the following trinomial model: on the probability space $\Omega := \{a, b, c\}$ we consider a probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$, and define $S_0 = 4, E_0 = \frac{27}{8}$,

ω	a	b	c
$S_1(\omega)$	6	2	14
$E_1(\omega)$	6	4	2

Answer the following questions and justify carefully your answers.

(a) The formula (3 marks)

$$V_1^{x,g,h} := x + (x - gE_0 - hS_0E_0)((1 + r) - 1) + g((1 + f)E_1 - E_0) + h(S_1E_1 - S_0E_0), \qquad (1)$$

describes my total wealth $V_1 := V_1^{x,g,h}$ (in £) at time 1 if my initial capital (in £) is $V_0 = x$. At time 1, what is the value (in £) of my American investments (cash plus shares)?

Assume from now on that $r = 1, f = \frac{1}{2}$.

(b) Is the market model arbitrage-free? (4 marks)

(c) Is the market complete? (3 marks)

(d) Consider an option P on the stock, whose payoff in \$ is (5 marks)

ω	a	b	c
$P_1(\omega)$	4	8	10

What are the arbitrage-free prices of P? Give your answer both in £ and in \$.

(e) Consider the set $\mathcal{A}$ of all trading strategies which super-replicate the option P. Identify exactly the set of all $(x, g, h) \in \mathcal{A}$ with smallest initial capital, i.e. such that (5 marks)

$$x = p := \min\{x' : (x', g', h') \in \mathcal{A}\}.$$

Solution:

(a) The value in £ of my investments in \$

$$V_1^{\$} = (g(1+f) + hS_1)E_1,$$

as this describes the wealth I hold in the foreign bank account and stock. Another way to see this, is to notice that $V_1^{\$} = V_1 - V_1^{£}$, where the value in £ of my investments in £ is

$$V_1^{£} = (x - (g + hS_0)E_0)(1+r),$$

since this is the value in my British bank account.

(b,c) Since at time 1 the value (in £) of a \$ (deposited at time 0 in the foreign bank account) is $E_1(1+f)$, and of one share of the foreign stock is S_1E_1, the payoffs in £ of my investments are

event	a	b	c
foreign bank $= E_1(1+f)$	9	6	3
foreign stock $= S_1E_1$	36	8	28

and their initial values (in £) are E_0 and S_0E_0. I can of course also invest in buying £; this investment has initial value 1 and final value $1+r$.

1^{st} **solution:** Since $\mathbb{Q}$ is an EMM (equivalent martingale measure) iff $X_0(1+r) = \mathbb{E}^{\mathbb{Q}}[X_1]$ for every traded security X, $\mathbb{Q}$ is an EMM iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 1 = q_1 + q_2 + q_3 & \text{(Domestic Bank/(1+r))} \\ E_0(1+r) = 9q_1 \quad 6q_2 + 3q_3 & \text{(Foreign Bank)} \\ S_0E_0(1+r) = 36q_1 + 8q_2 + 28q_3 & \text{(Foreign Stock)} \\ q_1 > 0 \quad q_2 > 0 \quad q_3 > 0 & (\mathbb{Q} \sim \mathbb{P}) \end{cases} \tag{2}$$

The above system of 3 equalities has the only solution

$$q_1 = \frac{1}{2}, \quad q_2 = \frac{1}{4} = q_3,$$

which satisfies the inequalities $q_i > 0$, $i = 1, \ldots, 3$. Thus there exist a unique EMM, so by the fundamental theorems the model is arbitrage free and complete.

2^{nd} **solution:** By definition, (g,h) is an arbitrage iff $V_1^{0,g,h} \geq 0$ and $V_1^{0,g,h} \neq 0$. Since we can rewrite eq. (1) as

$$V_1 = V_1^{x,g,h} = x(1+r) + g((1+f)E_1 - E_0(1+r)) + h(S_1E_1 - S_0E_0(1+r)), \tag{3}$$

substituting $x = 0$ and the numerical values given in eq. (2) we find that $V_1^{0,g,h} \geq 0$ holds iff

$$\begin{cases} g(9 - \frac{27}{4}) + h(36 - 27) \geq 0 \\ g(6 - \frac{27}{4}) + h(8 - 27) \geq 0 \\ g(3 - \frac{27}{4}) + h(28 - 27) \geq 0 \end{cases} \tag{4}$$

Multiplying times 4 we get the equivalent system

$$\begin{cases} 9g & +36h & \geq 0 \\ -3g & -76h & \geq 0 \\ -15g & +4h & \geq 0 \end{cases} \tag{5}$$

Which is easily seen to have $g = 0 = h$ as only solution: one can apply the FM algorithm, or solve it geometrically, or add the 2nd eq to one third of the 1st eq to get $(-76+12)h \geq 0$, i.e. $h \leq 0$, and then the 1st eq gives $g \geq 0$, and so $-15g + 4h \leq 0$, thus the 3rd eq gives $0 \geq 15g = 4h \leq 0$ and so $g = 0 = h$. Thus $V_1^{0,g,h} \geq 0$ implies $V_1^{0,g,h} = 0$, so there is no arbitrage.

Analogously, given an arbitrary payoff X_1, we can write the replication equation $V_1^{x,g,h} = X_1$ as

$$\begin{cases} 2x & +g(9-\frac{27}{4}) & +h(36-27) & = X_1(a) \\ 2x & +g(6-\frac{27}{4}) & +h(8-27) & = X_1(b) \\ 2x & +g(3-\frac{27}{4}) & +h(28-27) & = X_1(c) \end{cases}$$

and since this is a system of 3 independent equations in the three unknowns x, g, h, it always has a solution; thus, the market is complete.

(d) The value in £ of the option at time 1 is $P_1 E_1$, i.e.

ω	a	b	c
$P_1(\omega)$	4	8	10
$P_1E_1(\omega)$	24	32	20

1^{st} **solution:** Given the EMM $\mathbb{Q}$, using the RNPF $X_0(1+r) = \mathbb{E}^{\mathbb{Q}}[X_1]$ gives that the arbitrage-free price in £ of the put option is

$$\frac{\mathbb{E}^{\mathbb{Q}}[P_1E_1]}{1+r} = \frac{24 \cdot \frac{1}{2} + 32 \cdot \frac{1}{4} + 20 \cdot \frac{1}{4}}{2} = \frac{25}{2},$$

and so its price in \$ is $P_0 = \frac{25}{2}/E_0 = \frac{100}{27}$.

2^{nd} **solution:** Since we can rewrite eq. (1) as eq. (3), we can write the replication equation $V_1^{x,g,h} = P_1E_1$ as

$$\begin{cases} 2x & +g(9-\frac{27}{4}) & +h(36-27) & = 24 \\ 2x & +g(6-\frac{27}{4}) & +h(8-27) & = 32 \\ 2x & +g(3-\frac{27}{4}) & +h(28-27) & = 20 \end{cases}$$

Subtracting $2x$ and multiplying times 4 we get the equivalent system

$$\begin{cases} g(36-27) & +4h(36-27) & = 96-8x \\ g(24-27) & +4h(8-27) & = 128-8x \\ g(12-27) & +4h(28-27) & = 80-8x \end{cases}, \quad \text{i.e.} \quad \begin{cases} 9g & +36h & = 96-8x \\ -3g & -76h & = 128-8x \\ -15g & +4h & = 80-8x \end{cases}.$$

Some algebra shows that the system has the unique solution

$$x = \frac{25}{2}, \quad g = \frac{11}{9}, \quad h = -\frac{5}{12},$$

and so the price in £is $\frac{25}{2}$, and so the price in \$ is $P_0 = \frac{25}{2}/E_0 = \frac{100}{27}$.

(e) Since p is the supremum of the set $\{\frac{25}{2}\}$ of arbitrage-free prices we have $x = p = \{\frac{25}{2}\}$. If $(x, g, h) \in \mathcal{A}$ and $x = p$ then the super-replication inequality

$$V_1^{x,g,h}(\omega) \geq P_1 E_1(\omega) \quad \forall \omega \in \Omega$$

always holds with equality: otherwise starting with zero capital, selling one put, and with the proceeds $p = x$ buying g \$ and h shares of stock would have final payoff $V_1^{x,g,h} - P_1 E_1$ and thus would be an arbitrage. Thus (x, g, h) is a replicating strategy for the put; conversely, any replicating strategy (x, g, h) satisfies $(x, g, h) \in \mathcal{A}$ and $x = p$.

To determine the set of replicating strategies we solve the replication equation $V_1^{x,g,h} = P_1 E_1$. This we have done in our 2nd solution to the previous item; notice that, if we had solved the previous item using the 1st solution method instead, we could more quickly find the replicating strategies by substituting in the replication equation the value of x, since we already know that $x = p$ and $p = \frac{25}{2}$ (because p equals the supremum of the set of arbitrage-free prices). We can then discard one of the 3 equations (as it becomes redundant), say the 2nd one, and get

$$\begin{cases} 9g & +36h & = -4 \\ -15g & +4h & = -20 \end{cases}$$

which is easier to solve. Once found the unique solution

$$g = \frac{11}{9}, \quad h = -\frac{5}{12},$$

and recalling that $x = \frac{25}{2}$, as a safety check we can then plug in these values in the discarded equation and confirm that it is indeed satisfied, as it should: indeed

$$-3g - 76h = -3 \cdot \frac{11}{9} + 76 \cdot \frac{5}{12} = \frac{84}{3} = 28 = 128 - 8x.$$

Question 2

(Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. Define the random variable

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	6	12

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 0$, a stock whose initial price is $S_0 = 6$, and whose final price is S_1. We also consider a put option P (with underlying S) with strike price 7. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) arbitrage-free? (2 marks)

(b) Is the market (B, S) complete? (2 marks)

(c) Is the put option P replicable (in the market (B, S))? (3 marks)

(d) What is the smallest price $u(P)$ at which an infinitely risk-averse investor would sell the put option (in the market (B, S))? (5 marks)

From now on assume that P can be liquidly traded at the initial price $P_0 = u(P)$, and consider the market (B, S, P).

(e) Is the market (B, S, P) arbitrage-free? If not, explicitly determine an arbitrage strategy. (5 marks)

(f) Is the market (B, S, P) complete? (3 marks)

Solution:

(a) 1^{st} **solution:** Yes, since the 3 values d, m, u of S_1/S_0 satisfy $d < 1 + r < u$, because

$$d = \frac{2}{6} = \frac{1}{3}, \quad 1 + r = 1, \quad u = \frac{12}{6} = 2.$$

2^{nd} **solution:** We will now compute the set $\mathcal{M}$ of EMMs (Equivalent Martingale Measures) and show that is not empty; from the FTAP it follows that the model is arbitrage-free. Recall that $r = 0$, so $\mathbb{Q}$ is an EMM if $S_0 = \mathbb{E}^{\mathbb{Q}}[S_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 6 = 2q_1 + 6q_2 + 12q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

Solving the system we find

$$\mathcal{M} = \left\{ q_t := \begin{pmatrix} 3t \\ 1 - 5t \\ 2t \end{pmatrix} : t \in \left(0, \frac{1}{5}\right) \right\}. \tag{6}$$

(b) 1^{st} **solution:** No, because it is a trinomial model, so the two available investments B, S span only a 2-dimensional space, and thus do not span the whole 3-dimensional space of possible payoffs.

2^{nd} **solution:** No, because the set of EMM contains more than one element

(c) 1^{st} **solution:** The put has payoff

$$P_1 = (4 - S_1)^+ = \left(\begin{pmatrix} 7-2 \\ 7-6 \\ 7-12 \end{pmatrix}\right)^+ = \left(\begin{pmatrix} 5 \\ 1 \\ -5 \end{pmatrix}\right)^+ = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

Thus, the equation $xB_1 + yS_1 = P_1$ has no solution, i.e. P is not replicable: indeed the third component of the vector equation $xB_1 + yS_1 = P_1$ is $x + 12y = 0$, which gives $x = -12y$, but

$$-12B_1 + S_1 = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix}$$

is not a multiple of P_1, so the equation $y(-12B_1 + S_1) = P_1$ has no solution.

2^{nd} **solution:** Taking $Q \in \mathcal{M}$ we compute

$$\mathbb{E}^{\mathbb{Q}}[P_1] = 5q_1 + 1q_2 + 0q_3 = 5 \cdot 3t + 1 \cdot (1 - 5t) + 0 \cdot 2t = 10t + 1$$

which spans $(1, 3)$, since $t \in \left(0, \frac{1}{5}\right)$. Thus the set of arbitrage-free prices of the put is $(1, 3)$, which is not a singleton, so the put it not replicable.

(d) 1^{st} **solution:** It is the smallest value of x such that the system of inequalities $x + h(S_1 - S_0) \geq P_1$ has a solution $h \in \mathbb{R}^3$; to solve such system, we can use the FM algorithm as follows. The system is

$$\begin{cases} x + \ \ (2-6)h \geq 5 \\ x + \ \ (6-6)h \geq 1 \\ x + (12-6)h \geq 0 \end{cases} = \begin{cases} x - 4h \geq 5 \\ x \qquad \ \geq 1 \\ x + 6h \geq 0 \end{cases} = \begin{cases} \ \ \frac{1}{4}x - \frac{5}{4} \geq h \\ -\frac{1}{6}x \qquad \leq h \\ \quad x \qquad \geq 1 \end{cases}, \tag{7}$$

so eliminating the variable h from the first two inequalities gives

$$-\frac{1}{6}x \leq \frac{1}{4}x - \frac{5}{4}, \text{ i.e. } 3 \leq x$$

and so taking into account also the third inequality $1 \leq x$ we get $3 \leq x$. Thus the solution is 3.

2^{nd} **solution:** The solution is the upper boundary of the interval of arbitrage-free prices. Since this was calculated to be $(1, 3)$, the solution is 3.

(e) Since P is not replicable, the set of its arbitrage-free prices is an open interval $(d(P), u(P))$, with $u(P) = 3$ as in the previous item. Thus, $P_0 = u(P)$ is not an arbitrage-free price, so the market (B, S, P) admits arbitrage. This happens because the price P_0 is higher than any arbitrage-free price, so to get an arbitrage we need to (short-)sell the put (because it is overpriced). Thus, to determine an arbitrage we need to find h such that $0 + h(S_1 - S_0) - (P_1 - P_0)$ is ≥ 0 but not $= 0$. To find such h we thus consider the system of inequalities in the previous item, take $x = 3$, and solve for h. Taking $x = 3$ in the last system in (7) gives

$$\begin{cases} \ \ \frac{3}{4} - \frac{5}{4} \geq h \\ -\frac{3}{6} \qquad \leq h \\ \ \ 3 \qquad \geq 1 \end{cases}.$$

The first two inequalities give $-\frac{1}{2} \leq h \leq -\frac{1}{2}$, i.e. $h = \frac{1}{2}$. For such h we have $0 + h(S_1 - S_0) - (P_1 - P_0) \geq 0$, and the third inequality is satisfied strictly and thus $0 + h(S_1 - S_0) - (P_1 - P_0)$ is not identically $= 0$. So, an arbitrage is achieved by (starting with 0 initial capital and) selling one put option, and buying $\frac{1}{2}$ shares of S.

(f) Yes, it is complete. Notice that we cannot answer using EMMs, since the market is not free of arbitrage and so the assumptions of the 2nd FTAP are not satisfied; instead, we notice that, since B_1 equals the constant 1, the vector S_1 is not a multiple of B_1, so B_1, S_1 are independent. Since P is not replicable in the market (B, S), the vector P_1 is not a linear combination of B_1 and S_1. Thus, the vectors B_1, S_1, P_1 are independent, and so they span a space of dimension 3. Since they are 3-dimensional vectors, they form a basis of $\mathbb{R}^3$, i.e. the market is complete.

Question 3 (Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

Let $N \in \mathbb{N} \setminus \{0\}$. On the space $\Omega := \{H, T\}^N$, consider (as usual) the natural filtration $\mathcal{F} = \mathcal{F}^X$ of the coin tosses $(X_i)_{i=1}^N$ (i.e. the random variables $X_i(\omega) := \omega_i$, where $\omega := (\omega_i)_{i=1}^N$), i.e. $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ for $i = 1, \dots, N$, and the probability $\mathbb{P}$ (on the σ-algebra $\mathcal{A} := \mathcal{F}_N$) for which the coin tosses are independent and

$$\mathbb{P}(X_i = H) = \frac{1}{2} = \mathbb{P}(X_i = T) \quad \text{for all } i = 1, \dots, N.$$

On $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ consider the multi-period binomial model (B, S) with expiration N, 0 interest rate (i.e. $B_i = 1$ for all i), and stock price $S = (S_i)_{i=0}^N$ given by the asymmetric random walk

$$S_k := S_0 + \sum_{i=1}^{k} Y_i \quad \text{for all } k = 0, \dots, N, \quad \text{where} \quad Y_i(\omega) := \begin{cases} 2 & \text{if } \omega_i = H \\ -1 & \text{if } \omega_i = T \end{cases} \quad i = 1, \dots, N, \, \omega \in \Omega,$$

where $S_0 \in \mathbb{R}$ is a constant. Define the variation of S as the process

$$W_k := |S_0| + \sum_{i=1}^{k} |Y_i| \quad \text{for all } k = 0, \dots, N.$$

Consider the call option with underlying W, strike price $K \in \mathbb{R}$ and expiry N, i.e. the derivative with payoff $(W_N - K)^+$ at time N; denote with V_i its arbitrage-free price at time $i = 0, \dots, N$. Denote with $\mathbb{Q}$ the risk-neutral measure. Whenever we will say that a process is Markov, we mean that it is Markov with respect to $\mathbb{Q}$ and $\mathcal{F}$. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Are the $(Y_i)_i$ IID (i.e. independent and identically distributed) under $\mathbb{Q}$? (3 marks)

(b) Let $\mathcal{F}^Y$ be the natural filtration of Y. Does $\mathcal{F}^Y = \mathcal{F}$ hold? (1 marks)

(c) Is S a Markov process? (2 marks)

(d) Is W a Markov process? (2 marks)

(e) Is (S, W) a Markov process? (2 marks)

(f) Which of the 3 processes S, W and (S, W) are such that, for every $n = 0, \dots, N$, V_n can be written as a function f_n of the value of such process at time n? (3 marks)

(g) For one process $A \in \{S, W, (S, W)\}$ such that V_n can be written as $V_n = f_n(A_n)$ (for some f_n) as described in item (f), write explicitly f_N and an explicit formula to express f_n in terms of f_{n+1} for $n = 0, \dots, N-1$. (7 marks)

Solution:

(a) Yes. Indeed, since Y_i is given by an invertible function of X_i, we have that $\sigma(Y_i) = \sigma(X_i)$, and so the $(Y_i)_i$ are independent under $\mathbb{Q}$ iff the coin tosses are independent under $\mathbb{Q}$. To show that they are, we compute

$$\tilde{p}_i := \mathbb{Q}(X_{i+1} = H | \mathcal{F}_i) = \frac{(1 + r_i) - d_i}{u_i - d_i} = \frac{1 - d_i}{u_i - d_i} = \frac{S_{i-1} - S_{i-1} d_i}{S_{i-1} u_i - S_{i-1} d_i} = \frac{-(-1)}{2 - (-1)} = \frac{1}{3};$$

this this is non-random, X_{i+1} is $\mathbb{Q}$-independent of $\mathcal{F}_i = \mathcal{F}_i^X$, and since this happens for every i the $(X_i)_i$ are $\mathbb{Q}$-independent.

Since

$$\{X_i = H\} = \{Y_1 = 2\}, \quad \{X_i = T\} = \{Y_1 = -1\},$$

we have that

$$\mathbb{Q}(Y_i = 2) = \mathbb{Q}(X_i = H) = \frac{1}{3}, \quad \mathbb{Q}(Y_i = -1) = 1 - \mathbb{Q}(Y_i = 2) = \frac{2}{3}, \quad \text{for all } i,$$

so the $(Y_i)_i$ are identically distributed (under $\mathbb{Q}$).

(b) Yes, since $\sigma(Y_i) = \sigma(X_i)$ for all i.

(c) Yes, since $S_{i+1} = S_i + Y_{i+1}$ and Y_{i+1} is independent of $\mathcal{F}_i$.

(d) Yes, since $W_{i+1} = W_i + |Y_{i+1}|$ and Y_{i+1} is independent of $\mathcal{F}_i$.

(e) 1^{st} **solution:** Since S and W are Markov, trivially (S, W) is Markov.

2^{nd} **solution:** Yes, since

$$(S_{i+1}, W_{i+1}) = (S_i, W_i) + (Y_{i+1}, |Y_{i+1}|)$$

and Y_{i+1} is independent of $\mathcal{F}_i$.

(f) Since $V_N = f_N(W_N)$ for $f_N(w) = (w - K)^+$, and W is Markov, there exist f_n such that $V_n = f_n(W_n)$ (we show this more explicitly in the next item, where we explicitly calculate f_n). In particular there exist f_n such that $V_n = f_n(S_n, W_n)$ for all n. Instead we cannot choose S, since we cannot express V_N as a function of S_N.

(g) We consider $A = W$, since it is better than considering $A = (S, W)$ (as the latter takes more values). We already stated that $V_N = f_N(W_N)$ for $f_N(w) = (w - K)^+$. Let us now compute the recursive formula for f_n. Since we can write

$$W_{i+1} = W_i + |Y_{i+1}| = h(W_i, Y_{i+1}) \quad \text{for} \quad h(w, y) := w + |y|,$$

using the independence lemma we get $\mathbb{E}^{\mathbb{Q}}[f(W_{i+1})|\mathcal{F}_i] = g(W_i)$, where

$$g(w) := \mathbb{E}^{\mathbb{Q}}[f(h(w, Y_{i+1}))] = \tilde{p}f(y+2) + (1-\tilde{p})f(y+1) \tag{8}$$

and as usual $\tilde{p} := \mathbb{Q}(X_{i+1} = H)$. Since the RNPF gives $V_n = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}_n(V_{n+1})$, assuming by backward induction that $V_{n+1} = f_{n+1}(W_{n+1})$ we get

$$V_n = \mathbb{E}^{\mathbb{Q}}_n[\frac{V_{n+1}}{1+r}] = \mathbb{E}^{\mathbb{Q}}_n[\frac{f_{n+1}(W_{n+1})}{1+r}].$$

Combining this with (8) we see that $V_n = f_n(W_n)$ for f_n given by

$$f_n(w) = \frac{1}{1+r}\left(\tilde{p}f_{n+1}(y+1) + (1-\tilde{p})f_{n+1}(y+1)\right), \quad \text{for } n = 0, \dots N-1$$

Question 4

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Consider the N-period model where the bank account has interest rate r, and the stock price process $S = (S_n)_{n=0}^N$ is given by the (arbitrage-free) binomial model with constant down and up factors d, u, defined as usual on the probability space $\Omega = \{H, T\}^N$, endowed with the natural filtration $\mathcal{F} = \mathcal{F}^S$ of S. Assume that

$$N = 3, \quad r = 0, \quad d = \frac{1}{2}, \quad u = 2, \quad S_0 = 8.$$

Define as usual $\inf \emptyset := \infty$, and consider

$$\tau := \inf\{i = 0, \dots, N : S_i \geq 20\},$$

which is a random time (i.e. a random variable with values in $\mathbb{N} \cup \{\infty\}$). Consider the derivative D which pays S_τ at time τ if $\tau \leq N$ (where $S_\tau := S_k$ on $\{\tau = k\}$ for each $k = 0, \dots, N$), and pays 6 at time N otherwise; in other words, D yields the payoff

$$D_\sigma := S_\tau \cdot 1_{\{\tau \leq N\}} + 6 \cdot 1_{\{\tau = \infty\}} \quad \text{at time} \quad \sigma := \tau \wedge N.$$

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Explicitly compute $\tau(\omega)$ for each $\omega \in \Omega$. (2 marks)

(b) Prove that τ is a stopping time, i.e., that $\{\tau = n\} \in \mathcal{F}_n$ for all $n = 0, \dots, N$. (2 marks)

(c) Is σ a stopping time? (2 marks)

(d) Determine the arbitrage-free price V of D (meaning, its values V_k for $\{k \leq \sigma\}$). (7 marks)

(e) Suppose it becomes possible to trade D at price 6 at any time strictly before time σ, and at price D_σ at time σ. Identify an arbitrage in the (B, S, D) market (this market has time horizon σ), and compute its final payoff (i.e. its value at time σ). (3 marks)

(f) Consider the option C which gives you the right to choose at time 1 to immediately receive either one share of S, or one derivative D. Compute the arbitrage-free price C_0 at time 0 of such option. (4 marks)

Solution:

(a) To identify τ, let us first draw the binomial tree of S. For later use, we also draw the known part of the binomial tree of V, i.e. we draw $V_\sigma = D_\sigma$ at time σ; after time σ (i.e. the value of V_k on $\{k > \sigma\}$) the value of V is undefined (because the value at time $n > k$, of receiving an amount of cash V_k at time k, depends on how we choose to invest at times $n \geq k$), and the value before time σ it is not known (i.e. it is not an input to the problem), though it can be calculated by replication as we do in item (d).

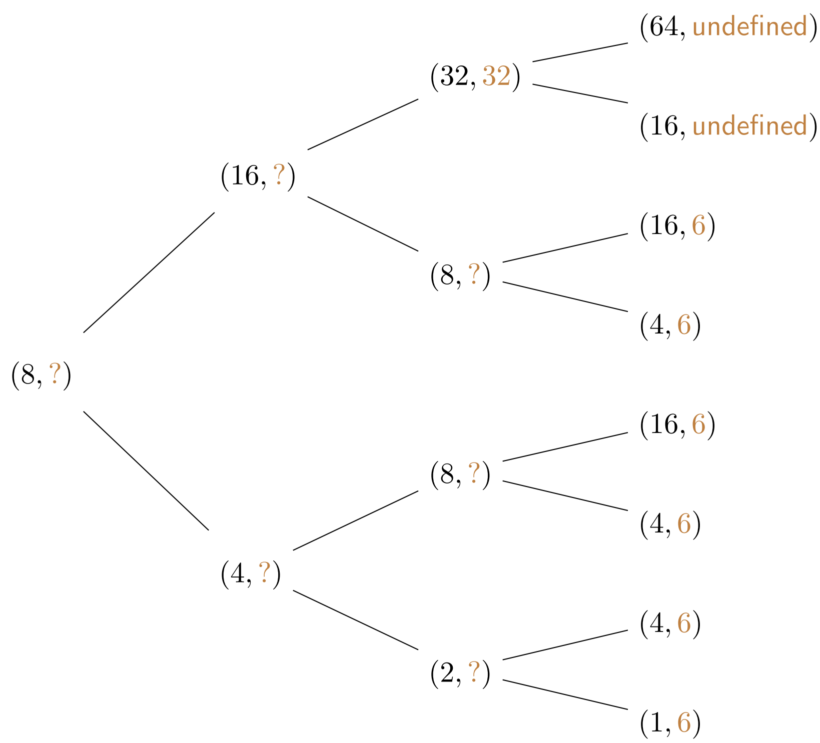

Figure 1: Tree of (S, V)

As the drawing of the binomial tree of S shows, $\tau(\omega) = 2$ if $\omega_1 = H = \omega_2$, and $\tau(\omega) = \infty$ otherwise.

(b) $\{\tau = n\} = \{S_n \geq 20\} \bigcap (\cap_{i=0}^{n-1}\{S_i < 20\}) \in \mathcal{F}_n$ for all $n \leq N$ since S is $\mathcal{F}$-adapted.

(c) Yes, since $\{\sigma = n\} = \{\tau = n\} \in \mathcal{F}_n$ for each $n = 0, \ldots, N-1$ and $\{\sigma = N\} = \Omega \setminus (\cup_{n=0}^{N-1}\{\sigma = n\}) \in \mathcal{F}_{N-1} \subseteq \mathcal{F}_N$.

(d) We know the value $V_k = D_k$ on $\{\sigma = k\}$, and we have to determine the values of V_k at previous times, i.e. on $\{k < \sigma\}$. We can do so by determining the replicating strategy H and the price V as usual, by backward induction, either by hand (i.e., by solving the replication equations), or by using the delta-hedging formula and the RNPF. We will now apply the RNPF and use that $r = 0$ and $\tilde{p} = \frac{1-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{3}$. Since $\tilde{p}6 + (1-\tilde{p})6 = 6$, the above binomial tree shows that $V_i(\omega) = 6$ if $i \in \{1, 2, 3\}$ and $\omega_1 = T$, and if $i = 2$ and $(\omega_1, \omega_2) = (H, T)$; all is left is to compute $V_1(H)$, and then V_0. The RNPF gives

$$V_1(H) = \frac{1}{3} \cdot 32 + \frac{2}{3} \cdot 6 = \frac{44}{3}, \quad V_0 = \frac{1}{3} \cdot \frac{44}{3} + \frac{2}{3} \cdot 6 = \frac{80}{9},$$

and so the tree of (S, V) is as follows

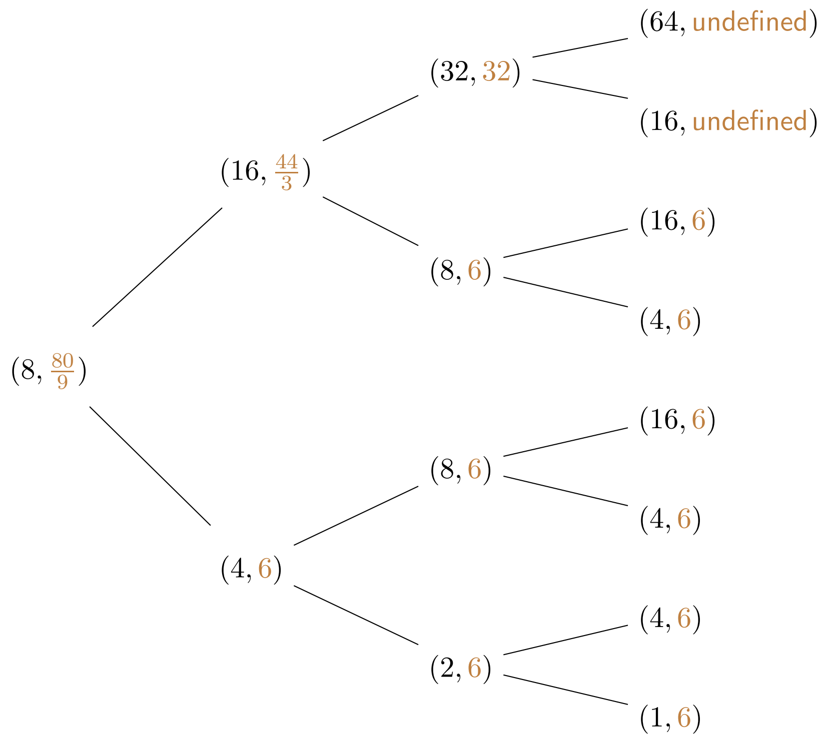

Figure 2: Tree of (S, V)

(e) We know that $V_0 = \frac{80}{9}$ is the unique value at time 0 of an arbitrage-free price process for D in the (B, S)-market. Thus, if D is traded at the price 6 at time 0, then there exists an arbitrage. Since $6 < \frac{80}{9}$, it means the derivative is being traded at a smaller value than it should be, so to make an arbitrage we can: start with zero initial capital, *buy* the derivative, hedge such a trade (i.e. replicate minus the derivative, by trading in the stock market and using the bank) until time the derivative expires at time σ, and put the remaining money in the bank. Since buying the derivative costs 6, and replicating it costs $-\frac{80}{9}$, we are left with $\frac{80}{9} - 6 = \frac{26}{9}$ at time 0, which we can invest in the bank, leading to the arbitrage profit of $\frac{26}{9} > 0$ at time σ.

(f) Since

$$S_1(H) = 16 > \frac{44}{3} = V_1(H), \quad S_1(T) = 4 < 6 = V_1(T),$$

if $\omega_1 = H$ we should choose to own S, and otherwise to own D. Doing this leads to the payoff $C_1 = S_1 \vee V_1$ given by $C_1(H) = 16, C_1(T) = 6$, which is the value at time 1 of the chooser option. Its value at time 0 is then given by the RNPF as being $C_0 = \frac{1}{3} \cdot 16 + \frac{2}{3} \cdot 6 = \frac{28}{3}$.

Question 5

(Total: 20 marks)

UNSEEN

Consider a one-period binomial market model composed only of two risky assets (in particular, no bond is traded), whose prices S_t^1, S_t^2 at times $t = 0, 1$ have values

$$S_0^1 = 300, \qquad S_0^2 = 200, \qquad \begin{array}{c|c|c} \omega & H & T \\ \hline S_1^1(\omega) & 360 & 270 \\ \hline S_1^2(\omega) & 200 & 260 \end{array} \tag{9}$$

Assume that the events $\{H\}$ and $\{T\}$ have probability $\frac{1}{2}$. We'll use the following standard notation: $R_1^i := \frac{S_1^i}{S_0^i} - 1$ denotes the return of asset $i = 1, 2$ between time 0 and 1, $\Sigma := \mathrm{Cov}(R_1)$ denotes the covariance matrix of the returns (i.e. $\Sigma_{i,j} := \mathbb{E}(R_1^i R_1^j) - \mathbb{E}(R_1^i)\mathbb{E}(R_1^j)$, $i = 1, 2$), a portfolio is denoted as (x_0, π), where x_0 is the investor's initial capital, and $\pi = (\pi_1, \pi_2)$, where π_i is the proportion of the investor's wealth that is invested in asset S^i. Denote with $\mu = \mu(\pi)$ the average, and with $\sigma = \sigma(\pi)$ the standard deviation, of the return of the portfolio (x_0, π). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Compute the average returns $\mu_i := \mathbb{E}[R_1^i], i = 1, 2$, and the covariance matrix Σ. Prove that the correlation ρ between R_1^1 and R_1^2 equals -1. (4 marks)

(b) Find a portfolio with average return of 8%. Does it involve any short-selling? (3 marks)

(c) Find the equation whose solution is the set S of values (σ, μ) taken by all portfolios, and draw S. (5 marks)

(d) Determine and draw the subset S_+ of S corresponding to all portfolios which do not involve short-selling. (2 marks)

(e) Find two portfolios π^a, π^b such that $\mu(\pi^a) > \mu(\pi^b)$ and $\sigma(\pi^a) < \sigma(\pi^b)$. Draw the two points on S which correspond to π^a, π^b. Determine which of these two portfolios is the preferable one, and explain why. (3 marks)

(f) What is the value V_1^M at time 1 of the market portfolio (x_0^M, π^M) with initial value $x_0^M = 1000$? (3 marks)

Solution:

(a) First we compute the returns

$$\begin{array}{c|c|c} \omega & H & T \\ \hline R_1^1(\omega) & \frac{1}{5} & -\frac{1}{10} \\ \hline R_1^2(\omega) & 0 & \frac{3}{10} \end{array} \tag{10}$$

and then compute their average as

$$\mu_1 = \mathbb{E}[R_1^1] = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{10}\right) = \frac{1}{20}, \qquad \mu_2 = \mathbb{E}[R_1^2] = \frac{1}{2}\left(0 - \frac{3}{10}\right) = \frac{3}{20}, \tag{11}$$

and finally we can compute Σ as follows

$$\sigma_1^2 := \Sigma_{1,1} = \mathbb{E}[(R_1^1)^2] - (\mathbb{E}[R_1^1])^2 = \frac{1}{2}\left(\frac{1}{25} + \frac{1}{100}\right) - \left(\frac{1}{20}\right)^2 = \frac{5}{200} - \frac{1}{400} = \frac{9}{400},$$
$$\sigma_2^2 := \Sigma_{2,2} = \mathbb{E}[(R_1^2)^2] - (\mathbb{E}[R_1^2])^2 = \frac{1}{2}\left(0 + \frac{9}{100}\right) - \left(\frac{3}{20}\right)^2 = \frac{9}{200} - \frac{9}{400} = \frac{9}{400},$$
$$\Sigma_{1,2} = \Sigma_{2,1} = \mathbb{E}[R_1^1 R_1^2] - \mathbb{E}[R_1^1]\mathbb{E}[R_1^2] = \frac{1}{2}\left(0 - \frac{3}{100}\right) - \frac{1}{20}\frac{3}{20} = -\frac{3}{200} - \frac{3}{400} = -\frac{9}{400},$$

and so

$$\Sigma = \frac{9}{400}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{12}$$

In particular the correlation of the returns is

$$\rho := \frac{\Sigma_{1,2}}{\sigma_1\sigma_2} = \frac{-\frac{9}{400}}{\sqrt{\frac{9}{400}}\sqrt{\frac{9}{400}}} = -1. \tag{13}$$

(b) Since the return $R(\pi)$ of a portfolio (x_0, π) depends only on $\pi = (\pi_1, \pi_2)$ and is given by $R_1(\pi) = \pi_1 R_1^1 + \pi_2 R_1^2$, using the notation $u := \pi_1$, so that $\pi_2 = 1 - u$, and using eq. (11) we compute

$$\mu(u) := \mu(\pi) = \mathbb{E}[R_1(\pi)] = u\mu_1 + (1-u)\mu_2 = \frac{u}{20} + \frac{3(1-u)}{20} = \frac{3-2u}{20}, \tag{14}$$

and so $\mathbb{E}[R_1(\pi)] = \frac{8}{100}$ becomes $\frac{3-2u}{20} = \frac{8}{100}$, and solving for u gives $3 - 2u = \frac{8}{5}$, and so $u = \frac{7}{10}$ and $\pi = (\frac{7}{10}, \frac{3}{10})$. Since $\pi_1, \pi_2 \geq 0$, this portfolio involves no short-selling.

(c) Since Σ is not invertible, S is given by the union of two half-lines. To describe S in formulas, we use eq. (14) and analogously compute the standard deviation $\sigma(u)$ of the portfolio $\pi = (u, 1-u)$ as

$$\sigma(u)^2 = \pi^T\Sigma\pi = u^2\sigma_1^2 + (1-u)^2\sigma_2^2 + 2\rho u(1-u)\sigma_1\sigma_2$$

and so we get

$$\sigma(u)^2 = \frac{9}{400}\Big(u^2 + (1-u)^2 - 2u(1-u)\Big) = \frac{9}{400}(u - (1-u))^2 = \frac{9}{400}(2u-1)^2. \tag{15}$$

Inverting eq. (14) we get

$$u = \frac{3}{2} - 10\mu \tag{16}$$

which plugged into eq. (15) gives $\sigma = \frac{3}{10}|1 - 10\mu|$, i.e.

$$S := \{(\sigma, \mu) \in \mathbb{R}_+ \times \mathbb{R} : \sigma = \frac{3}{10}|1 - 10\mu|\}, \tag{17}$$

which is a union of the half-lines starting from the point $(\sigma, \mu) = (0, \frac{1}{10})$ and with slope ± 3.

(d) By definition, the portfolio $\pi = (\pi_1, \pi_2)$ involves no short-selling if $\pi_1, \pi_2 \geq 0$ and so $\pi = (u, 1-u)$ involves no short-selling iff $u \in [0,1]$, which by eq. (16) is equivalent to $\mu \in \left[\frac{1}{20}, \frac{3}{20}\right]$, which by eq. (17) is equivalent to $\sigma \in \left[0, \frac{3}{20}\right]$. Thus

$$S_+ := \{(\sigma, \mu) \in \left[0, \frac{3}{20}\right] \times \left[\frac{1}{20}, \frac{3}{20}\right] : \sigma = \frac{3}{10}|1 - 10\mu|\}, \tag{18}$$

which is the union of the two segments starting from the point $(\sigma, \mu) = (0, \frac{1}{10})$, with endpoints $\left(\frac{3}{20}, \frac{1}{20}\right)$ and $\left(\frac{3}{20}, \frac{3}{20}\right)$.

(e) For example one can take π^a and π^b which correspond to the points $(0, \frac{1}{10})$ and $(\frac{3}{10}, 0)$ in S, since $\frac{1}{10} > 0$ and $0 < \frac{3}{10}$. To find $\pi^a = (u^a, 1-u^a), \pi^b = (u^b, 1-u^b)$ we apply eq. (16) and get

$$u^a = \frac{3}{2} - 10 \cdot \frac{1}{10} = \frac{1}{2}, \qquad u^b = \frac{3}{2} - 10 \cdot 0 = \frac{3}{2}.$$

The portfolio π^a is preferable, since the standard deviation $\sigma(\pi)$ of the returns of a portfolio π is normally used as the measure its risk, and so π^a has both a higher average return, and a lower risk, than π^b, and so any investor (which uses $\sigma(\pi)$ as the measure its risk) would prefer π^a to π^b, irrespectively of his/her attitude towards risk.

(f) By definition of market portfolio

$$\pi_1^M = \frac{S_0^1}{S_0^1 + S_0^2} = \frac{300}{300 + 200} = \frac{3}{5}, \quad \pi_2^M = \frac{S_0^2}{S_0^1 + S_0^2} = \frac{200}{300 + 200} = \frac{2}{5} = 1 - \pi_1^M$$

and plugging these values into the formula $V_1(x_0, \pi) = x_0\pi_1(1 + R_1^1) + x_0\pi_2(1 + R_1^2)$ gives

$$V_1^M := V_1(x_0^M, \pi^M) = 1000\left(\frac{3}{5}(1 + R_1^1) + \frac{2}{5}(1 + R_1^2)\right),$$

and so using the values of R_1^i from eq. (11) gives

$$V_1^M(H) = 1000\left(\frac{3}{5}\left(1 + \frac{1}{5}\right) + \frac{2}{5}(1 + 0)\right) = 1000\left(\frac{3}{5} \cdot \frac{6}{5} + \frac{2}{5}\right) = 40(18 + 10), \tag{19}$$

$$V_1^M(T) = 1000\left(\frac{3}{5} \cdot \frac{9}{10} + \frac{2}{5} \cdot \frac{13}{10}\right) = 20(27 + 26), \tag{20}$$

and so

$$V_1^M(H) = 40 \cdot 28 = 40 \cdot 30 - 40 \cdot 2 = 1200 - 80 = 1120, \tag{21}$$

$$V_1^M(T) = 20 \cdot 53 = 20 \cdot 50 + 20 \cdot 3 = 1000 + 60 = 1060. \tag{22}$$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60012/70012	1	Quite a number of students did not know at all how to deal with a market with multiple currencies (despite the fact that we had worked out several such exercises in class), and so lost many points in this (otherwise quite simple) exercise. Few students managed to solve item (e), because they did not realise how, instead of solving a system involving 3 inequalities in 3 variables, with some thinking it would enough to instead solve a system of 2 equalities in 2 variables (which is massively quicker to solve)
MATH60012/70012	2	This was the easier exercise, and most students correctly solved all parts, though a number of them failed to correctly solve part (d)
MATH60012/70012	3	This was a simple exercise on markov pricing, yet many students did not know how to formally prove that a process is markov, or how to compute conditional expectations (and thus prices) using the independence lemma. Normally students perform better on this type of exercise than they did in this exam
MATH60012/70012	4	Very few students were able to properly prove that tau and sigma were stopping times, and some did not manage to identify an arbitrage. Yet, most students were able to solve the rest of the exercise
MATH70012	5	Often students were able to solve parts (a) and (e), and sometimes part (b). Very few solved anything else, probably because I had not assigned practice exercises on the topic (as I believed this question was very simple anyway), and since the degeneracy of the covariance matrix meant that the required graphs in parts (c,d) were not of the most standard kind. So students did really poorly on this question.