Classical Dynamics **Autumn Term 2024**

Problem Sheet 7

1. The Hamiltonian for a simple pendulum is

$$H(\theta, p) = \frac{p^2}{2} - \cos\theta.$$

Here the mass, length and the acceleration due to gravity have been set to unity.

(i) Using elementary considerations (conservation of energy) show that the period of oscillations is given by

$$T = 2\sqrt{2}\int_0^{\cos^{-1}(-E)} \frac{d\theta}{\sqrt{E+\cos\theta}} \qquad (-1 \le E < 1).$$

(ii) Compute the action variable

$$J = \oint p \; d\theta,$$

and use the result to rederive the result quoted in part (i).

Hint: express J as an integral - do not attempt to evaluate the integral.

(iii) What happens if $E \ge 1$?

2. The Hamiltonian for the Kepler problem (with $m = 1$) is

$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \frac{k}{r}.$$

Compute the action variable J_r as a function of $J_\theta = 2\pi p_\theta$ and $\alpha_1 = E$ and use the result to write H as a function of J_r and J_θ.

Hint: $$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{x}\, dx = \frac{\pi}{2}\left(a + b - 2\sqrt{ab}\right) \qquad (b > a > 0).$$

3. [1] Consider the Hamiltonian

$$H = \frac{p^2}{2} + \lambda |x|.$$

(i) Sketch the trajectories in the xp plane (take $\lambda = 1$).

(ii) Treating λ as a positive constant, write H as a function of λ and the action variable

$$J = \oint p \, dx.$$

(iii) Suppose that λ is slowly (adiabatically) increased until it is twice its initial value. What is the effect of this change on the frequency of oscillation?

4. The motion of two particles in one dimension is governed by the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U(x_1 - x_2).$$

Here U is an arbitrary function, m_1 and m_2 are the masses of the particles.

(i) Find two constants of the motion and identify the associated Noether symmetries.

(ii) Is the Hamiltonian Liouville-integrable?

5. (i) Obtain a type 2 generating function for the identity transformation

$$Q_i = q_i, \quad P_i = p_i.$$

(ii) Consider the canonical transformation

$$Q_i = q_i + s\{q_i, \alpha\}, \qquad P_i = p_i + s\{p_i, \alpha\}, \tag{1}$$

where s is a small deformation parameter and $\alpha = \alpha(q_1, ..., q_N, p_1, ..., p_N, t)$ is a function of the phase space variables and time. Find a type 2 generating function for this transformation (neglect terms of order s^2).

(iii) Noether's theorem states that if $K(Q_i, P_i, t) = H(Q_i, P_i, t)$ then α is a constant of the motion. Prove this (neglect terms of order s^2).

[1] This is Q5 of the 2020 examination.