ExamModuleCode	Question Number	Comments for Students
M3M6	1	All students performed well.
M3M6	2	Students did OK with this problem, though struggled with the last part.
M3M6	3	Students hadn't seemed to study this question, and performed significantly poorer than their M45M6 counterparts.
M3M6	4	I was disappointed in the results of students here, who seemed to miss the basic fact emphasised in lectures that Cauchy transforms of OPS satisfy the same recurrence. Even more surprising is that this lack of knowledge only afflicted M3M6 students, where M45M6 students did fine.
M45M6	5	Students overall did well.

Imperial College
London

M3M6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Methods of Mathematical Physics

Date: Wednesday 22 May 2019

Time: 10.00 - 12.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College
London

M4/5M6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

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Methods of Mathematical Physics

Date: Wednesday 22 May 2019
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1. (a) Use residue calculus to calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2+1}\mathrm{d}x.$$

(b) Use the Plemelj formulæ to calculate

$$\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} \frac{t}{\sqrt{1-t^2}(t-z)}\mathrm{d}t \qquad \text{for} \qquad z \notin [-1,1].$$

Demonstrate that the proposed solution satisfies all four requirements of the Plemelj formulæ.
Hint: recall that

$$\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-z)}\mathrm{d}t = \frac{\mathrm{i}}{2\sqrt{z-1}\sqrt{z+1}}.$$

(c) Find a closed form expression for all solutions $u(x)$ to the equation

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{u(t)}{t-x}\mathrm{d}t = \frac{x}{\sqrt{1-x^2}} \qquad \text{for} \qquad x \in [-1,1],$$

using the Hilbert inversion formula:

$$u(x) = -\frac{1}{\pi\sqrt{1-x^2}}⨍_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{t-x}\mathrm{d}t - \frac{C}{\sqrt{1-x^2}},$$

where f is the Hilbert transform of u.
Hint: recall that

$$\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} \frac{1}{t-z}\mathrm{d}t = \frac{\log(z-1)-\log(z+1)}{2\pi\mathrm{i}}.$$

2. Let $u(x)$ solve the integral equation

$$\int_0^\infty K(t-x)u(t)\mathrm{d}t = f(x) \qquad \text{for} \qquad x \geq 0,$$

where

$$K(x) = \mathrm{e}^{-|x|} \qquad \text{and} \qquad f(x) = 2 - \mathrm{e}^{-x}.$$

We will use the notations

$$g_{\mathrm{L}}(x) := \begin{cases} g(x) & x < 0 \\ 0 & x \geq 0 \end{cases}, \qquad g_{\mathrm{R}}(x) := \begin{cases} 0 & x < 0 \\ g(x) & x \geq 0 \end{cases},$$

and the Fourier transform

$$\hat{f}(s) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t.$$

(a) What are the regions of analyticity of $\hat{K}(s)$, and $\widehat{f_{\mathrm{R}}}(s)$? Assuming that $|u(x)|$ is bounded, what is the region of analyticity of $\widehat{u_{\mathrm{R}}}(s)$? Justify your answers without explicit calculation.

(b) Show that the Fourier transforms satisfy

$$\hat{K}(s) = \frac{2}{1+s^2} \qquad \text{and} \qquad \widehat{f_{\mathrm{R}}}(s) = \frac{2+\mathrm{i}s}{\mathrm{i}s - s^2}$$

(c) For the integral equation above, set up a Riemann–Hilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s) \qquad \text{for} \qquad s \in (-\infty,\infty) + \mathrm{i}\delta$$

where $\Phi_+(s)$ is analytic above $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_-(s)$ is analytic below $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_\pm(s)$ decay at infinity, and

$$g(s) = \frac{2}{1+s^2}.$$

Explain the choice of δ and the definition of $\Phi_\pm(s)$, $g(s)$ and $h(s)$ in terms of the Fourier transforms of u, f, and K.

(d) Is $g(s)$ degenerate? Explain why or why not.

(e) Find a solution to the homogeneous Riemann–Hilbert problem

$$\kappa_+(s) = g(s)\kappa_-(s) \qquad \text{for} \qquad s \in (-\infty,\infty) + \mathrm{i}\delta$$

such that $\kappa_+(s) = o(1)$ and $\kappa_-(s) = s + O(1)$ as $s \to \infty$, where δ is the same constant as in (c).

(f) Determine $u(x)$.

3. Consider the solution of the following Laplace's equation, using $z = x + iy$:

1. $v_{xx} + v_{yy} = 0$ for $z \notin [-1, 1] \cup \{\pm 2\}$,
2. $v(x, y) = \pm \log |z \mp 2| + O(1)$ as $z \to \pm 2$,
3. $v(x, y) = o(1)$ as $z \to \infty$, and
4. $v(x, 0) = \kappa$, for $-1 < x < 1$ where κ is an unknown constant.

This equation models the potential field of two unit charges of opposite sign at ± 2 with a metal sheet that has no net charge placed on $[-1, 1]$.

(a) By writing

$$v(x, y) = \int_{-1}^{1} u(t) \log |t - z| \mathrm{d}t + \log |z - 2| - \log |z + 2|,$$

show that the problem of finding $v(x, y)$ can be reformulated as finding $u(x)$ such that

$$\int_{-1}^{1} u(t) \log |t - x| \mathrm{d}t = f(x),$$

where

$$\int_{-1}^{1} u(x) \mathrm{d}x = 0.$$

What is $f(x)$ in this equation? Explain why $v(x, y)$ will thereby satisfy the required four conditions.

(b) Find $u(x)$. Hint: reduce the problem to one of inverting the Hilbert transform.

(c) What is the value of κ?

(d) Express $v(x, y)$ in terms of the Cauchy transform of $\int_x^1 u(t) \mathrm{d}t$. Note: knowing $u(x)$ explicitly is not required to complete this part.

4. Recall the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + O(x^{n-1})$$

for $n = 0, 1, \ldots$ and $\alpha > -1$, which are orthogonal with respect to

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x.$$

Note that Laguerre polynomials satisfy the three-term recurrence

$$x L_n^{(\alpha)}(x) = -(n+\alpha) L_{n-1}(x) + (2n + \alpha + 1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x),$$

where $L_{-1}(x) := 0$. Further, denote the weighted Cauchy transform of the Laguerre polynomials as

$$C_n^{(\alpha)}(z) := \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x) \mathrm{d}x}{x - z}.$$

Assume in this question that $-1 < \alpha < 0$.

For real a, define the incomplete Gamma function

$$\Gamma(a, z) = \int_z^\infty \zeta^{a-1} \mathrm{e}^{-\zeta} \mathrm{d}\zeta,$$

where the path of integration can be chosen to be the union of two line segments: one segment from z to 1 and another segment from 1 to $+\infty$. Further, define the Gamma function as

$$\Gamma(a) := \Gamma(a, 0).$$

(a) Express $z C_n^{(\alpha)}(z)$ in terms of $C_{n-1}^{(\alpha)}(z)$, $C_n^{(\alpha)}(z)$, and $C_{n+1}^{(\alpha)}(z)$, for $n = 0, 1, 2, \ldots$.

(b) Show that

$$\frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{x^\alpha \mathrm{e}^{-x}}{x - z} \mathrm{d}x = \frac{1}{\Gamma(-\alpha)} \frac{(-z)^\alpha \mathrm{e}^{-z} \Gamma(-\alpha, -z)}{\mathrm{e}^{-\mathrm{i}\pi\alpha} - \mathrm{e}^{\mathrm{i}\pi\alpha}}.$$

(c) Find

$$C_1^{(-1/2)}(x) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{x^{-1/2} L_1^{(-1/2)}(x) \mathrm{e}^{-x}}{x - z} \mathrm{d}x,$$

in terms of $\Gamma(1/2, -z)$.

5. Denote the Cauchy transform of f on $[0,\infty)$ by

$$\mathcal{C}f(z)=\frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{f(x)}{x-z}\mathrm{d}x.$$

Further, define the two-sheeted Riemann surface analogue of the Cauchy transform of v on $[0,\infty)$ via the following two functions:

$$\mathcal{C}^1v(z):=\mathcal{C}v(z)+\mathrm{i}\sqrt{-z}\mathcal{C}w(z) \qquad \text{and} \qquad \mathcal{C}^2v(z):=\mathcal{C}v(z)-\mathrm{i}\sqrt{-z}\mathcal{C}w(z),$$

where $w(x)=\frac{v(x)}{\sqrt{x}}$.

(a) Show that

$$\mathcal{C}f(z)=\mathcal{C}u(\sqrt{z})+\mathcal{C}u(-\sqrt{z}).$$

where $u(x)=f(x^2)$, and $f(x)$ is any smooth function that decays like $f(x)=O(x^{-2})$ as $x\to\infty$.

(b) Derive a closed form expression for

$$\int_0^\infty \frac{\mathrm{e}^{-\sqrt{x}}}{x-z}\mathrm{d}x$$

in terms of the exponential integral

$$\mathrm{Ei}(z):=\int_{-\infty}^z \frac{\mathrm{e}^\zeta}{\zeta}\mathrm{d}\zeta,$$

where the path of integration can be chosen to be the union of two line segments: one segment from $-\infty$ to -1 and another segment from -1 to z. Note: while $\mathrm{e}^{-\sqrt{x}}$ is not smooth at the origin, you can assume the formula in part (a) still applies.

(c) Express $\mathcal{C}f(z)$ in terms of $\mathcal{C}^1v(z^2)$ and $\mathcal{C}^2v(z^2)$, where $v(x)=f(\sqrt{x})$, and $f(x)$ is any smooth function that decays like $f(x)=O(x^{-2})$ as $x\to\infty$.

M345M6 2018/19 Solutions

1(a) Because it has decay in the upper-half plane, we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2+1}\,\mathrm{d}x = 2\pi\mathrm{i}\mathrm{Res}_{z=\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}z}}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{\pi}{\mathrm{e}}$$

Seen similar [6 marks]

1(b) Plemelj guarantees that $\psi(z)$ is equal to the Cauchy transform of f

$$\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{f(t)}{t-z}\,\mathrm{d}t$$

if it satisfies the following conditions:

1. $\psi(\infty)=0$,
2. $\psi(z)$ is analytic off $[-1,1]$,
3. $\psi(z)$ has weaker than pole singularities everywhere and
4. $\psi(z)$ satisfies the jump
$$\psi_+(x)-\psi_-(x)=f(x) \qquad \text{for} \qquad -1<x<1.$$

One can calculate $\psi(z)$ by taking a known solution and multiplying by a rational function so that it has the right jump on $[-1,1]$, and then subtracting off the poles to ensure that property (2) is satisfied. Therefore, take as an initial guess

$$\tilde{\phi}(z) = \frac{\mathrm{i}z}{2\sqrt{z-1}\sqrt{z+1}}$$

To satisfy (1) we need to subtract off the behaviour at infinity to get

$$\psi(z) = \frac{\mathrm{i}z}{2\sqrt{z-1}\sqrt{z+1}} - \frac{\mathrm{i}}{2}$$

(2) and (3) are clearly satisfied. Finally, it has the jump for $-1<x<1$

$$\psi_+(x)-\psi_-(x) = \frac{\mathrm{i}x}{2\mathrm{i}\sqrt{1-x}\sqrt{x+1}} - \frac{\mathrm{i}x}{-2\mathrm{i}\sqrt{1-x}\sqrt{x+1}} = \frac{x}{\sqrt{1-x^2}}$$

Seen similar [7 marks]

1(c) Here we use the Hilbert transform inversion formula:

$$\begin{aligned} u(x) &= -\frac{1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{t-x}\,\mathrm{d}t - \frac{C}{\sqrt{1-x^2}} \\ &= -\frac{1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{t}{t-x}\,\mathrm{d}t - \frac{C}{\sqrt{1-x^2}} \end{aligned}$$

Since, in this case, $f(x) = x/\sqrt{1-x^2}$.

To calculate the Hilbert transform of x, we first calculate the Cauchy transform: $\mathcal{C}[t](z)$ by taking the ansatz from the hint

$$z\frac{\log(z-1) - \log(z+1)}{2\pi \mathrm{i}}$$

We need to remove the behaviour at ∞. Using the Taylor series of $\log z$ around $z = 1$ we have

$$\log(z+c) = \log z + \log(1 + c/z) = \log z + \frac{c}{z} + O(z^{-2})$$

thus

$$\log(z-1) - \log(z+1) = \log z - 1/z - \log z - 1/z + O(z^{-2}) = -\frac{2}{z} + O(z^{-2})$$

and so we find from Plemelj that

$$\mathcal{C}[t](z) = z\frac{\log(z-1) - \log(z+1)}{2\pi \mathrm{i}} + \frac{1}{\mathrm{i}\pi}$$

we then use that the Hilbert transform satisfies

$$\mathcal{H}[t](x) = \mathrm{i}(\mathcal{C}^+ + \mathcal{C}^-)[t](x) = x\frac{\log(1-x) - \log(x+1)}{\pi} + \frac{2}{\pi}$$

Combining the $2/\pi$ constant with the constant term, we arrive at the solution

$$u(x) = -\frac{x}{\sqrt{1-x^2}}\frac{\log(1-x) - \log(x+1)}{\pi} - \frac{C}{\sqrt{1-x^2}}$$

Seen similar [7 marks]

2(a) Since $|f(x)| < C_\gamma e^{-\gamma x}$ for any $\gamma > 0$ we know from $e^{-i(p+iq)t} = e^{-ipt}e^{qt}$ that $\widehat{f_R}(s)$ is analytic for $\Im s < 0$. Similarly, $\widehat{u_R}(s)$ is analytic for $\Im s < 0$. On the other hand, $K(x)e^{\gamma x}$ decays for $-1 < \gamma < 1$ hence $\hat{K}(s)$ is analytic in the strip $-1 < \Im s < 1$.

Seen [3 marks]

2(b) These are straightforward integration by parts calculations:

$$\begin{aligned}\hat{K}(s) &= \int_{-\infty}^{\infty} K(t)e^{-ist}\,dt = \int_{-\infty}^{0} e^{t-ist}\,dt + \int_{0}^{\infty} e^{-t-ist}\,dt = \frac{1}{1-is} + \frac{1}{1+is} \\ &= \frac{2}{(1-is)(1+is)} = \frac{2}{s^2+1}\end{aligned}$$

And

$$\widehat{f_R}(s) = 2\int_0^\infty e^{-ist}\,dt - \int_0^\infty e^{-t-ist}\,dt = \frac{2}{is} - \frac{1}{1+is} = \frac{2+2is-is}{is(1+is)} = \frac{2+is}{is-s^2}$$

Seen similar [3 marks]

2(c) We can write the integral equation as

$$\int_{-\infty}^{\infty} K(t-x)u_R(t)\,dt = f_R(x) + p_L(x)$$

where

$$p(x) := \int_{-\infty}^{\infty} K(t-x)u_R(t)\,dt$$

Transforming into Fourier space we have

$$\hat{K}(s)\widehat{u_R}(s) = \widehat{f_R}(s) + \widehat{p_L(s)}$$

for $s \in (-\infty, \infty) + i\delta$ for any $-1 < \delta < 0$. As discussed in lectures $\widehat{p_L(s)}$ is analytic for $\Im s \geq \delta$ while $\widehat{u_R}(s)$ is analytic for $\Im s \leq \delta$, thus define $\Phi_+(s) = \widehat{p_L(s)}$, $\Phi_-(s) = \widehat{u_R}(s)$, and

$$g(s) = \hat{K}(s) = \frac{2}{s^2+1}$$

and $h(s) = -\widehat{f_R}(s) = \frac{2+is}{s^2-is}$.

Seen [3 marks]

2(d) It is degenerate because it tends to zero at ∞, not to 1.

Seen [3 marks]

2(e) Factorising

$$g(s) = \frac{2}{(s+i)(s-i)} = \kappa_+(s)\kappa_-(s)^{-1}$$

by location of the poles we see that $\kappa_+(s) = \frac{2}{s+i}$ and $\kappa_-(s) = s - i$ satisfies the required properties.

Seen similar [3 marks]

2(f) We write

$$\Phi_\pm(s) = \kappa_\pm(s) Y_\pm(s)$$

so that

$$\Phi_+(s) - g(s)\Phi_-(s) = \kappa_+(s)(Y_+(s) - Y_-(s))$$

Thus we want to solve, using partial fraction expansion to expand,

$$Y_+(s) - Y_-(s) = \frac{h(s)}{\kappa_+(s)} = \frac{1}{2}\frac{2+\mathrm{i}s}{s(s-\mathrm{i})}(s+\mathrm{i}) = \frac{\mathrm{i}}{2} - \frac{1}{s} + \frac{1}{s-\mathrm{i}}$$

As κ_+ itself decays we can associate the constant term with it, thus we get by splitting the poles

$$\begin{aligned} Y_+(s) &= \frac{\mathrm{i}}{2}, \\ Y_-(s) &= \frac{1}{s} - \frac{1}{s-\mathrm{i}}. \end{aligned}$$

This ensures that

$$\begin{aligned} \Phi_+(s) &= \frac{\mathrm{i}}{s+\mathrm{i}} \\ \Phi_-(s) &= \left(\frac{1}{s} - \frac{1}{s-\mathrm{i}}\right)(s-\mathrm{i}) = -\frac{\mathrm{i}}{s} \end{aligned}$$

We then recover from the inverse Fourier transform

$$u(x) = \frac{-\mathrm{i}}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \frac{\mathrm{e}^{\mathrm{i}sx}}{s}\,\mathrm{d}x = \mathrm{Res}_{z=0}\frac{\mathrm{e}^{\mathrm{i}zx}}{z} = 1$$

Seen similar [5 marks]

3(a) Writing

$$v(x,y) = \int_{-1}^{1} u(t) \log|t-z| \, \mathrm{d}t + \log|z-2| - \log|z+2|$$

This solves Laplace's equation (1) away from $[-1,1]$ and ± 2 as its the real part of an analytic function. It also solves (2) since as $z \to \pm 2$ the logarithmic terms domain. For (3), we need it to be asymptotically zero. We have

$$\log|z-2| - \log|z+2| = \log|1-2/z| - \log|1+2/z| = o(1)$$

so we need the integral term to also decay. The integral condition ensures this since

$$\int_{-1}^{1} u(t) \log|z-t| \, \mathrm{d}t = \int_{-1}^{1} u(t) \, \mathrm{d}t \log|z| + \int_{-1}^{1} u(t) \log|1-t/z| \, \mathrm{d}t = \int_{-1}^{1} u(t) \log|1-t/z| \, \mathrm{d}t$$

and $\log|1-t/z| \to 0$. The condition (4) that $v(x,0) = \kappa$ then reduces to the integral equation, with

$$f(x) = \kappa + \log(x+2) - \log(2-x).$$

Seen [5 marks]

3(b) Differentiating and multiplying by $1/\pi$ we get

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{t-x} \, \mathrm{d}t = \frac{1}{\pi} \left(\frac{1}{x-2} - \frac{1}{x+2} \right)$$

From the inverse Hilbert transform formula we know

$$u(x) = -\frac{1}{\sqrt{1-x^2}} \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{t-x} \, \mathrm{d}t + \frac{C}{\sqrt{1-x^2}}$$

for $f(x) = \frac{1}{\pi}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)$. To find the Hilbert transform of $g(x) = f(x)\sqrt{1-x^2}$, we first find the Cauchy transform

$$\mathcal{C}g(z) = f(z) \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}} - \frac{\sqrt{3}}{2\mathrm{i}\pi} \left(\frac{1}{z+2} + \frac{1}{z-2} \right)$$

where the second term is chosen to cancel the poles of $f(z)$. Thus we find

$$u(x) = -\frac{\mathrm{i}(\mathcal{C}^+ + \mathcal{C}^-)[f\sqrt{1-t^2}](x)}{\sqrt{1-x^2}} = \frac{\sqrt{3}}{\pi\sqrt{1-x^2}} \left(\frac{1}{x+2} + \frac{1}{x-2} + C \right)$$

We choose $C = 0$ to satisfy the integral condition: this follows since in that case $u(x) = -u(-x)$ and thereby $\int_{-1}^{1} u(x) \, \mathrm{d}x = 0$.

Seen similar [5 marks]

3(c) $\kappa = 0$ by the anti-symmetry of $u(x)$, that is, since $u(x) = -u(-x)$ we have:

$$\kappa = v(x,0) = \int_{-1}^{1} u(t) \log|t| \, \mathrm{d}t = 0.$$

Seen similar [5 marks]

3(d) We can write

$$v(x,y) - \log|z-2| + \log|z+2| = \Re\int_{-1}^{1} u(t)\log(z-t)\,\mathrm{d}t = \Re 2\pi\mathrm{i}\int_{z}^{\infty} \mathcal{C}u(z)\,\mathrm{d}z,$$

where $\mathcal{C}u(z) = \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} u(t)/(t-z)\,\mathrm{d}t$ is the Cauchy transform. Define

$$\phi(z) = \int_{z}^{\infty} \mathcal{C}u(z)\,\mathrm{d}z.$$

Inspecting the jump of $\phi(z)$ on $[-1,1]$ we find that

$$\phi_+(x) - \phi_-(x) = \int_{x}^{1} (\mathcal{C}^+ - \mathcal{C}^-)u(x)\,\mathrm{d}x = \int_{x}^{1} u(x)\,\mathrm{d}x$$

For $x < -1$ we have

$$\phi_+(x) - \phi_-(x) = \int_{-1}^{1} u(x)\,\mathrm{d}x = 0$$

and for $x > 1$ we also have $\phi_+(x) = \phi_-(x)$ by analyticity, as we avoid any branch cuts. hence by Plemelj

$$\phi(z) = \mathcal{C}[\int_{x}^{1} u(t)\,\mathrm{d}t](z)$$

and therefore

$$v(x,y) = -\Re 2\pi\mathrm{i}\mathcal{C}[\int_{x}^{1} u(t)\,\mathrm{d}t](z) + \log|z-2| - \log|z+2|$$

Seen similar [5 marks]

4(a) We have

$$\begin{aligned} zC_n^{(\alpha)}(z) &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{x^\alpha (z-x) L_n^{(\alpha)}(x)\mathrm{e}^{-x}\,\mathrm{d}x}{x-z} + \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{x^\alpha x L_n^{(\alpha)}(x)\mathrm{e}^{-x}\,\mathrm{d}x}{x-z} \\ &= -\frac{1}{2\pi\mathrm{i}}\int_0^\infty x^\alpha L_n^{(\alpha)}(x)\mathrm{e}^{-x}\,\mathrm{d}x - (n+\alpha)C_{n-1}^{(\alpha)}(z) + (2n+\alpha+1)C_n^{(\alpha)}(z) - (n+1)C_{n+1}^{(\alpha)}(z) \end{aligned}$$

when $n > 0$ the first integral vanishes because $L_n^{(\alpha)}(x)$ is orthogonal to all lower degree polynomials with respect to the weight $x^\alpha \mathrm{e}^{-x}$. Otherwise use

$$\int_0^\infty x^\alpha L_0^{(\alpha)}(x)\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(\alpha+1).$$

Seen similar [7 marks]

4(a) A simple integration by parts argument shows that the right-hand side decays as $z \to \infty$ hence we just need to check the jump. Note for $\phi(z) = z^\alpha \Gamma(-\alpha, z)$ we have the jump for $x < 0$

$$\phi_+(x) - \phi_-(x) = x_+^\alpha \Gamma_+(-\alpha, x) - x_-^\alpha \Gamma_-(-\alpha, x) = |x|^\alpha \int_x^\infty (\mathrm{e}^{\mathrm{i}\pi\alpha} t_+^{-\alpha-1} - \mathrm{e}^{-\mathrm{i}\pi\alpha} t_-^{-\alpha-1})\mathrm{e}^{-t}\,\mathrm{d}t.$$

Since $\mathrm{e}^{\mathrm{i}\pi\alpha} t_+^{-\alpha-1} - \mathrm{e}^{-\mathrm{i}\pi\alpha} t_-^{-\alpha-1} = 0$ for $t < 0$ this reduces to

$$(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}) \int_0^\infty t^{-\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t = (\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha})\Gamma(-\alpha).$$

We thus have for $x > 0$ that $\mathrm{e}^{-z}\phi(-z)$ has the jump

$$\begin{aligned} \mathrm{e}^{-x}\phi_-(-x) - \mathrm{e}^{-x}\phi_+(-x) &= (-x)_-^\alpha \mathrm{e}^{-x}\Gamma_-(-\alpha, -x) - (-x)_+^\alpha \mathrm{e}^{-x}\Gamma_+(-\alpha, -x) \\ &= -(\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha})\Gamma(-\alpha) x^\alpha \mathrm{e}^{-x} \end{aligned}$$

and the result follows.

Seen [7 marks]

4(c) From the recurrence in part (a) with $n = 0$ and $\alpha = -1/2$ we know

$$zC_0^{(-1/2)}(z) = -\frac{\Gamma(1/2)}{2\pi\mathrm{i}} + 1/2C_0^{(-1/2)}(z) - C_1^{(-1/2)}(z)$$

In other words,

$$C_1^{(-1/2)}(z) = -\frac{\Gamma(1/2)}{2\pi\mathrm{i}} + \frac{(1/2 - z)}{\Gamma(1/2)}\frac{(-z)^{-1/2}\mathrm{e}^{-z}\Gamma(1/2, -z)}{2\mathrm{i}}.$$

Unseen [6 marks]

5(a) Define
$$\phi(z) = \mathcal{C}u(\sqrt{z}) + \mathcal{C}u(-\sqrt{z}).$$
$\phi(z)$ is analytic off $\mathbb{R}$ and decays at ∞. For $x < 0$ we have continuity (hence analyticity) since
$$\phi_+(x) = \mathcal{C}u(\sqrt{x}_+) + \mathcal{C}u(-\sqrt{x}_+) = \mathcal{C}u(\mathrm{i}\sqrt{|x|}) + \mathcal{C}u(-\mathrm{i}\sqrt{|x|}) = \mathcal{C}u(-\sqrt{x}_-) + \mathcal{C}u(\sqrt{|x|}_-) = \phi_-(x)$$
For $x > 0$ we have
$$\phi_+(x) - \phi_-(x) = \mathcal{C}_+u(\sqrt{x}) + \mathcal{C}u(-\sqrt{x}) - \mathcal{C}_-u(\sqrt{x}) - \mathcal{C}u(-\sqrt{x}) = u(\sqrt{x}) = f(x)$$

Seen [6 marks]

5(b) Note for $x > 0$ that
$$\mathrm{Ei}_+(x) - \mathrm{Ei}_-(x) = -\oint \frac{\mathrm{e}^\zeta}{\zeta}\,\mathrm{d}\zeta = -2\pi\mathrm{i}$$
using Residue calculus. Thus we have
$$\mathcal{C}[\mathrm{e}^{-x}](z) = -\frac{\mathrm{e}^{-z}\mathrm{Ei}(z)}{2\pi\mathrm{i}}$$
this decays since, integrating by parts, we have
$$\mathrm{e}^{-z}\mathrm{Ei}(z) = \frac{1}{z} + \mathrm{e}^{-z}\int_{-\infty}^{z} \frac{\mathrm{e}^\zeta}{\zeta^2}\,\mathrm{d}\zeta = O(1/z)$$
Plugging in the formula from part (a) we get
$$\int_0^\infty \frac{\mathrm{e}^{-\sqrt{x}}}{x - z}\,\mathrm{d}x = -\mathrm{e}^{-\sqrt{z}}\mathrm{Ei}(\sqrt{z}) - \mathrm{e}^{\sqrt{z}}\mathrm{Ei}(-\sqrt{z})$$

Seen similar [7 marks]

5(c) The answer is $\mathcal{C}f(z) = \phi(z)$ for
$$\phi(z) = \begin{cases} \mathcal{C}^1 v(z^2) & \Im z > 0 \\ \mathcal{C}^2 v(z^2) & \Im z < 0. \end{cases}$$
We now verify this. Decay and analyticity off of $\Re z = 0$ follow immediately. Check the jump for $x < 0$. Note that
$$\lim_{\epsilon\to 0}\sqrt{-(x+\mathrm{i}\epsilon)^2} = \lim_{\epsilon\to 0}\sqrt{-(x^2 - \mathrm{i}\epsilon)} = \sqrt{-x^2}_+$$
hence the signs are still the same.
$$\begin{aligned}\phi_+(x) - \phi_-(x) &= \frac{\mathcal{C}_+v(x^2) + \mathrm{i}\sqrt{-x^2}_+\mathcal{C}_+w(x^2) - \mathcal{C}_-v(x^2) + \mathrm{i}\sqrt{-x^2}_-\mathcal{C}_-w(x^2)}{2} \\ &= \frac{v(x^2) - \sqrt{x^2}w(x^2)}{2} = \frac{f(x) - f(x)}{2} = 0\end{aligned}$$
On the other hand for $x > 0$ we have
$$\begin{aligned}\phi_+(x) - \phi_-(x) &= \frac{\mathcal{C}_+v(x^2) + \mathrm{i}\sqrt{-x^2}_-\mathcal{C}_+w(x^2) - \mathcal{C}_-v(x^2) + \mathrm{i}\sqrt{-x^2}_+\mathcal{C}_-w(x^2)}{2} \\ &= \frac{v(x^2) + \sqrt{x^2}w(x^2)}{2} = \frac{f(x) + f(x)}{2} = f(x)\end{aligned}$$

Unseen [7 marks]