Classical Dynamics **Autumn Term 2024**

Problem Sheet 5

1. The following transformation is canonical

$$Q = \log p, \quad P = -qp.$$

Verify that $\{Q, P\} = 1$ and find a generating function for the canonical transformation.

2. The Hamiltonian for the Kepler problem in polar coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}.$$

(i) Determine the Poisson brackets

(a) $\{p_\theta, H\}$ (b) $\{p_r, H\}$ (c) $\{r^{-1}, H\}$ (d) $\{e^{i\theta}, H\}$.

(ii) Show that the complex quantity

$$V = e^{i\theta}\left(\frac{p_\theta^2}{r} - ip_r p_\theta - mk\right)$$

satisfies $\{V, H\} = 0$. What is $\{p_\theta, V\}$?

(iii) Show that if $p_\theta \neq 0$,

$$r = \frac{L}{1 + e\cos(\theta - \alpha)},$$

where L, e and α are constants of the motion. Comment on the result.

3. Suppose that A and B are functions of q_i, p_i and t ($i = 1, ..., N$).

(i) Show that

$$\frac{\partial}{\partial t}\{A, B\} = \Big\{\frac{\partial A}{\partial t}, B\Big\} + \Big\{A, \frac{\partial B}{\partial t}\Big\}.$$

(ii) Show that if A and B are constants of the motion then so is $\{A, B\}$.

4. [1] (i) The Hamiltonian for a system has the form

$$H(q,p) = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right).$$

Find the equation of motion for q.

(ii) Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part (i) is satisfied.

5. Find a time-independent canonical transformation mapping the Hamiltonian $H(q,p) = p^2 + e^q$ to $K(Q,P) = P^2$.

   Hint: consider a generating function of the form $F_4(p,P)$.

[1] from Goldstein, Safko and Poole (exercise 25, chapter 9).