MATH50001/18, Complex Analysis, 2024

MTExam

Solutions

1. [5p]

Find the set $\Omega \subset \mathbb{C}$, where the function

$$\mathrm{Log}\left(\frac{z}{2z+i}\right)$$

is holomorphic. Here $\mathrm{Log}(w) = \ln|w| + i\,\mathrm{Arg}\,w$ and $\mathrm{Arg}\,w$ is the principal value of the argument.

Solution: By definition the principal value of $\mathrm{Log}(w)$ is holomorphic for $w \in \mathbb{C} \setminus \{(-\infty, 0]\}$. Note that

$$w = \frac{z}{2z+i} \quad \Longrightarrow \quad z = \frac{iw}{1-2w}.$$

The inclusion $w \in (-\infty, 0]$ implies $z \in \gamma = \{z : \, z = iy, \, y \in [-1/2, 0]\}$. Thus we conclude that $\mathrm{Log}\left(\frac{z}{z+1}\right)$ is holomorphic in $\Omega = \mathbb{C} \setminus \gamma$.

2. [5p]

Compute the integral

$$\oint_{|z-i|=2} \frac{1}{(z+2)(z-2i)^2}\, dz.$$

Solution: We find

$$\oint_{|z-i|=2} \frac{1}{(z+2)(z-2i)^2}\, dz = \oint_{|z-i|=2} \frac{f(z)}{(z-2i)^2}\, dz,$$

where $f(z) = \frac{1}{z+2}$.

Note that $f(z)$ is holomorphic for $\{z : \, |z-i| \le 2$. Therefore by the Generalised Cauchy's integral formula we obtain

$$\oint_{|z-i|=2} \frac{1}{(z+2)(z-2i)^2}\, dz = 2\pi i f'(z)\Big|_{z=2i} = -\frac{2\pi i}{(2+2i)^2}.$$

3. [5p]

Find Taylor series for

$$f(z) = \frac{1}{z^2+1}$$

about $z_0 = 1$. What is its radius of convergence?

Solution:

Note that

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

We find

$$\frac{1}{z-i} = \frac{1}{1-i+(z-1)} = \frac{1}{1-i}\,\frac{1}{1+\frac{z-1}{1-i}}$$
$$= \frac{1}{1-i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-i)^n}(z-1)^n.$$

Similarly

$$\frac{1}{z+i} = \frac{1}{1+i+(z-1)} = \frac{1}{1+i}\,\frac{1}{1+\frac{z-1}{1+i}}$$
$$= \frac{1}{1+i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^n}(z-1)^n.$$

Finally we have

$$f(z) = \frac{1}{z^2+1} = \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(1-i)^{n+1}} - \frac{(-1)^n}{(1+i)^{n+1}}\right)(z-1)^n. \quad (\mathbf{4p})$$

The series converges if $|z-1| < \sqrt{2}$. **(1p)**

4. [5p]

Let f and g be entire. Assume that $\operatorname{Im} f \le \operatorname{Im} g$. Show that $f(z) = g(z) + \mathrm{const}$.

Solution: Let $h(z) = f(z) - g(z)$ and consider $e^{-ih(z)}$. Since functions f and g are entire, then both functions h and e^{-ih} are also entire. The inequality $\operatorname{Im} f \le \operatorname{Im} g$ implies

$$\left|e^{-ih(z)}\right| = \left|e^{\operatorname{Im}(f(z)-g(z)) - i\operatorname{Re}(f(z)-g(z))}\right| = e^{\operatorname{Im}(f(z)-g(z))} \le 1.$$

By using Liouville's theorem we find that $e^{-ih(z)}$ is a constant. Thus $f-g$ must be a constant too.