1(a). Labelling the top wall as node 1 and going vertically down to nodes 2, 3 and 4 (node 4 being the lower wall) the weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & C+1 & -C \\ 0 & 0 & -C & C \end{bmatrix}.$$

1(b). The system to solve for the equilibrium displacements is

$$\mathbf{Kx} = \mathbf{f},$$

where

$$\mathbf{x} = \begin{bmatrix} 0 \\ \phi_2 \\ \phi_3 \\ 0 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} r_1 \\ mg \\ mg \\ r_2 \end{bmatrix}.$$

The middle two equations are easily solved by hand:

$$\phi_2 = \frac{(C+2)mg}{2C+1}$$

and

$$\phi_3 = \frac{3mg}{(2C+1)}.$$

1(c). As $C \to 0$,

$$\phi_2 = 2mg, \qquad \phi_3 = 3mg.$$

As $C \to \infty$,

$$\phi_2 = mg/2, \qquad \phi_3 = 0.$$

1(d). To compute the internal spring forces we need the incidence matrix

$$\mathbf{A} = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \end{bmatrix}$$

and spring constant matrix $\mathbf{C}$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & C \end{bmatrix}$$

leading to the internal spring forces being

$$\mathbf{CAx} = mg \begin{bmatrix} \dfrac{C+2}{2C+1} \\[2ex] \dfrac{1-C}{2C+1} \\[2ex] -\dfrac{3C}{2C+1} \end{bmatrix}.$$

As $C \to 0$, we find

$$\mathbf{CAx} \to mg \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

As $C \to \infty$, we find

$$\mathbf{CAx} \to mg \begin{bmatrix} 1/2 \\ -1/2 \\ -3/2 \end{bmatrix}.$$

2(a). The weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1 + c_2 + c_3 & -c_3 & -c_2 \\ 0 & -c_3 & c_3 & 0 \\ 0 & -c_2 & 0 & c_2 \end{bmatrix}.$$

2(b). The system to solve for the equilibrium displacements is

$$\mathbf{Kx} = \mathbf{f},$$

where

$$\mathbf{x} = \begin{bmatrix} 0 \\ \phi_1 \\ \phi_2 \\ 0 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} r_1 \\ m_1 g \\ m_2 g \\ r_2 \end{bmatrix}.$$

The middle two equations are easily solved by hand:

$$\phi_1 = \frac{(m_1 + m_2)g}{c_1 + c_2}$$

and

$$\phi_2 = \frac{m_2 g}{c_3} + \frac{(m_1 + m_2)g}{c_1 + c_2}.$$

2(c). The first and third equations give the reaction forces at the walls:

$$r_1 = -c_1\phi_2 = -c_1\frac{(m_1+m_2)g}{c_1+c_2}$$

and

$$r_2 = -c_2\phi_2 = -c_2\frac{(m_1+m_2)g}{c_1+c_2}.$$

Note that

$$r_1 + r_2 = -(m_1+m_2)g.$$

2(d). Assuming the walls at top and bottom are fixed and only the two masses can move, the free oscillations – that is, the oscillations when there are no external forces on the masses, only the internal spring forces – are the solutions of the governing equations which reduce to

$$-\hat{\mathbf{K}}\mathbf{x} = \mathbf{M}\frac{d^2\mathbf{x}}{dt^2},$$

where $\mathbf{x} = [\phi_1(t)\ \phi_2(t)]^T$ are the displacements of the two masses and

$$\hat{\mathbf{K}} = \begin{bmatrix} C & -c_3 \\ -c_3 & c_3 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

where, for convenience, we introduce the notation $C = c_1 + c_2 + c_3$. To find the natural modes of oscillation of the system we let

$$\mathbf{x} = \mathbf{\Phi}e^{\mathrm{i}\omega t}$$

for some real ω (this is the frequency of free, or natural, oscillation). Then, on substitution into the governing equation above we find

$$-\hat{\mathbf{K}}\mathbf{\Phi} = -\omega^2\mathbf{M}\mathbf{\Phi}.$$

If we let $\lambda = \omega^2$ then we need to find the eigenvalues λ satisfying

$$\begin{bmatrix} C/m_1 & -c_3/m_1 \\ -c_3/m_2 & c_3/m_2 \end{bmatrix}\mathbf{\Phi} = \lambda\mathbf{\Phi}.$$

The values of λ are the solutions of the characteristic equation

$$\det\begin{bmatrix} C/m_1-\lambda & -c_3/m_1 \\ -c_3/m_2 & c_3/m_2-\lambda \end{bmatrix} = 0.$$

It is easily found that

$$\lambda = \frac{1}{2}\left[\frac{C}{m_1}+\frac{c_3}{m_2} \pm \left[\left(\frac{C}{m_1}+\frac{c_3}{m_2}\right)^2 - 4\left(\frac{Cc_3-c_3^2}{m_1m_2}\right)\right]^{1/2}\right].$$

It follows that the natural frequencies of oscillation are given by

$$\omega = \pm \left[\frac{1}{2} \left[\frac{C}{m_1} + \frac{c_3}{m_2} \pm \left[\left(\frac{C}{m_1} + \frac{c_3}{m_2} \right)^2 - 4 \left(\frac{Cc_3 - c_3^2}{m_1 m_2} \right) \right]^{1/2} \right] \right]^{1/2}.$$

Note: the next question helps us see why these frequencies are all real.

3(a). To find the natural modes of oscillation of the system we let

$$\mathbf{x} = \mathbf{\Phi} e^{i\omega t}$$

for some real ω. Then, on substitution into the governing equation we find

$$-\hat{\mathbf{K}}\mathbf{\Phi} = -\omega^2 \mathbf{M}\mathbf{\Phi}$$

But $\mathbf{M}$ is clearly invertible with

$$\mathbf{M}^{-1} = \begin{bmatrix} 1/m_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/m_2 & 0 & \cdots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & \cdots & 0 & 1/m_N \end{bmatrix}.$$

Thus $\mathbf{\Phi}$ satisfies

$$\mathbf{M}^{-1}\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{\Phi}$$

from which we see that $\mathbf{\Phi}$ is an eigenvector of $\mathbf{M}^{-1}\hat{\mathbf{K}}$ and ω^2 is an eigenvalue.

(b). Even though $\hat{\mathbf{K}}$ is symmetric, once we multiply on the left by $\mathbf{M}^{-1}$ it is clear that the first row gets multiplied by $1/m_1$ while the second row gets multiplied by $1/m_2$. If $m_1 \neq m_2$ this clearly destroys the symmetry since the $(1,2)$ term in the matrix will now be different from the $(2,1)$ term. If $m_1 = m_2$ this argument will pertain at some later pair (i,j) if there exists some $m_i \neq m_j$, as has been assumed.

(c). Despite this lack of symmetry, we can still prove that $\mathbf{M}^{-1}\hat{\mathbf{K}}$ has N real eigenvalues and eigenvectors. To see this, notice that

$$\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{M}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{M}^{1/2}\mathbf{\Phi},$$

where $\mathbf{M}^{1/2}$ will also be diagonal with positive entries. It is clear that it is invertible with

$$\mathbf{M}^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 & 0 & \cdots & 0 \\ 0 & 1/\sqrt{m_2} & 0 & \cdots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & \cdots & 0 & 1/\sqrt{m_N} \end{bmatrix}.$$

Hence we can write

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{\Phi}$$

This can be rewritten as

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\underbrace{\mathbf{M}^{-1/2}\mathbf{M}^{+1/2}}_{\text{identity}}\mathbf{\Phi} = \omega^2 \mathbf{M}^{1/2}\mathbf{\Phi}$$

or as

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}\mathbf{\Psi} = \omega^2\mathbf{\Psi},$$

where

$$\mathbf{\Psi} = \mathbf{M}^{+1/2}\mathbf{\Phi}.$$

Now the matrix

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}$$

can be shown to be positive definite and symmetric, which means that is has N real eigenvalues and N real orthogonal eigenvectors, i.e., there are N solutions Ψ_j for $j = 1, \ldots, N$ satisfying

$$\mathbf{M}^{-1/2}\hat{\mathbf{K}}\mathbf{M}^{-1/2}\mathbf{\Psi}_j = \lambda_j \mathbf{\Psi}_j,$$

where λ_j is real and positive. The real values $\{\lambda_j | j = 1, \ldots, N\}$ are the squares of the natural frequencies ω_j^2. Note also that

$$\mathbf{\Psi}_i^T \mathbf{\Psi}_j = 0$$

if $i \neq j$. This means that

$$\mathbf{\Phi}_i^T \mathbf{M} \mathbf{\Phi}_j = 0$$

if $i \neq j$. We see that while the vectors $\mathbf{\Phi}_j$ are not orthogonal in the usual sense, they satisfy this generalized condition of "M-orthogonality".

4. On taking a derivative of the given quantity we find, using the product rule,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\hat{\mathbf{K}}\mathbf{x}\right] = \frac{1}{2}\left[\ddot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{M}\ddot{\mathbf{x}} + \dot{\mathbf{x}}^T\hat{\mathbf{K}}\mathbf{x} + \mathbf{x}^T\hat{\mathbf{K}}\dot{\mathbf{x}}\right].$$

Now we can use

$$M\ddot{\mathbf{x}} = -\hat{\mathbf{K}}\mathbf{x}, \qquad \ddot{\mathbf{x}}^T\mathbf{M} = -\mathbf{x}^T\hat{\mathbf{K}}$$

to eliminate the second derivatives and we find everything cancels. Hence the given quantity does not change in time, and is conserved by the dynamics.

5(a). The weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & C+1 & -C & 0 \\ 0 & -C & C+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

For the motion of the two masses the submatrix

$$\hat{\mathbf{K}} = \begin{bmatrix} C+1 & -C \\ -C & C+1 \end{bmatrix}$$

is relevant. According to Newton's second law the governing system of differential equations for the displacements ϕ_1 (left mass) and ϕ_2 (right mass) is

$$\hat{\mathbf{f}} - \hat{\mathbf{K}}\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2},$$

where

$$\mathbf{x} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \qquad \hat{\mathbf{f}} = \begin{bmatrix} \cos\Omega t \\ 0 \end{bmatrix}.$$

(b) To find the particular solution, let

$$\mathbf{x}^{PS} = \mathbf{\Phi}\cos\Omega t.$$

On substitution into the equations derived in part (a), and cancellation of the common factor of $\cos\Omega t$,

$$\hat{\mathbf{f}}_0 - \hat{\mathbf{K}}\mathbf{\Phi} = -\Omega^2\mathbf{\Phi},$$

where

$$\hat{\mathbf{f}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{\mathbf{K}}\mathbf{\Phi} - \Omega^2\mathbf{\Phi}.$$

We proceed by finding the eigenvalues and eigenvectors of $\hat{\mathbf{K}}$. The determinant condition

$$\det\begin{bmatrix} C+1-\lambda & -C \\ -C & C+1-\lambda \end{bmatrix} = 0$$

yields

$$\lambda_1 = 1, \qquad \lambda_2 = 1+2C$$

with corresponding eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We now write

$$\mathbf{\Phi} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$$

and, on noticing that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$$

we can write the equation for $\mathbf{\Phi}$ as

$$\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 = \lambda_1 a_1 \mathbf{e}_1 + \lambda_2 a_2 \mathbf{e}_2 - \Omega^2 a_1 \mathbf{e}_1 - \Omega^2 a_2 \mathbf{e}_2.$$

On equating coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ we find

$$a_1 = \frac{1}{2(\lambda_1 - \Omega^2)}, \qquad a_2 = \frac{1}{2(\lambda_2 - \Omega^2)}.$$

Thus the particular solution is

$$\left[\frac{1}{2(\lambda_1 - \Omega^2)}\mathbf{e}_1 + \frac{1}{2(\lambda_2 - \Omega^2)}\mathbf{e}_2\right] \cos \Omega t.$$

The general solution is obtained – using linearity – by adding a solution of the homogeneous system, i.e., a solution of

$$-\hat{\mathbf{K}}\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2}.$$

On letting

$$\mathbf{x} = \mathbf{\Psi} e^{\mathrm{i}\omega t}$$

for some vector $\mathbf{\Psi}$ we must solve the eigenvalue problem

$$\hat{\mathbf{K}}\mathbf{\Psi} = \omega^2 \mathbf{\Psi}$$

which we already know has solutions

$$c_1 \mathbf{e}_1 e^{\mathrm{i}t} + c_2 \mathbf{e}_2 e^{\mathrm{i}\sqrt{1+2C}t}, \qquad c_1, c_2 \in \mathbb{C}.$$

The general solution is therefore given by

$$\begin{aligned}\mathbf{x} = &\left[\frac{1}{2(\lambda_1 - \Omega^2)}\mathbf{e}_1 + \frac{1}{2(\lambda_2 - \Omega^2)}\mathbf{e}_2\right] \cos \Omega t \\ &+ \mathbf{e}_1[A\cos t + B\sin t] + \mathbf{e}_2[D\cos(\sqrt{1+2C}t) + E\sin(\sqrt{1+2C}t)].\end{aligned}$$

(c) It is clear that the solution of part (b) is valid provided that

$$\Omega^2 \neq \lambda_1, \lambda_2.$$

These are the "resonant" values where the forcing frequency equals one of the natural frequencies of the system.

6(a). We already computed the eigenvalues/eigenvectors so the general solution for free oscillation is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [a\cos t + b\sin t] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [d\cos(\sqrt{1+2C}t) + e\sin(\sqrt{1+2C}t)].$$

Since the initial velocities are zero we immediately deduce

$$b = e = 0.$$

Hence

$$\mathbf{x} = \begin{bmatrix} a\cos t + d\cos(\sqrt{1+2C}t) \\ a\cos t - d\cos(\sqrt{1+2C}t) \end{bmatrix}.$$

But at $t = 0$,

$$\mathbf{x} = \begin{bmatrix} 0 \\ A \end{bmatrix}$$

implying that

$$a = -d, \qquad 2a = A.$$

Hence,

$$\mathbf{x} = \frac{A}{2}\begin{bmatrix} \cos t - \cos(\sqrt{1+2C}t) \\ \cos t + \cos(\sqrt{1+2C}t) \end{bmatrix}.$$

On use of trigonometric identities we can write this as

$$\mathbf{x} = A\begin{bmatrix} \sin\Omega t \sin\epsilon t \\ \cos\Omega t \cos\epsilon t \end{bmatrix},$$

where

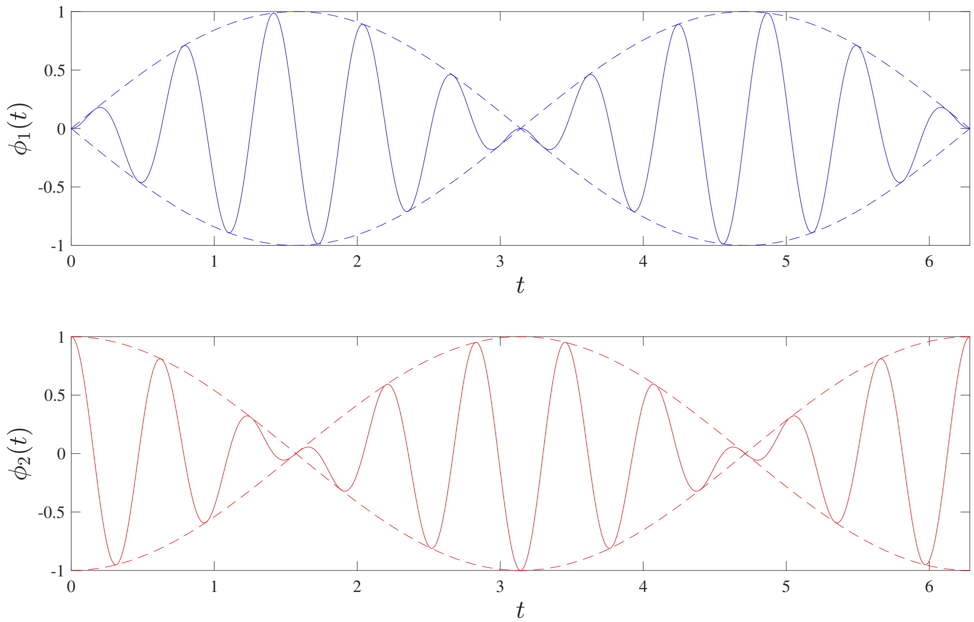
$$\Omega t - \epsilon t = t,$$
$$\Omega t + \epsilon t = \sqrt{1+2C}t.$$

It follows that

$$\Omega = \frac{\sqrt{1+2C}+1}{2}, \qquad \epsilon = \frac{\sqrt{1+2C}-1}{2}.$$

(b) It is clear that we need $C \ll 1$ if we require $\epsilon \ll \Omega$.

(c) If $\epsilon \ll \Omega$ then the displacements comprise a fast oscillation, with frequency Ω, with an amplitude that changes over a slow time scale with frequency $\epsilon \ll \Omega$. Also, for early times $t \ll 1$ the amplitude of displacement of the left mass is small, and the displacement of the right mass is large; however around $\epsilon t \approx \pi/2$ this situation reverses and it is the left mass where all the energy of the system of concentrated (with the right mass hardly moving). This exchange of energy continues. Here are some typical graphs of $\phi_1(t)$ and $\phi_2(t)$ as functions of t with the "envelopes" shown as dotted lines:


$\phi_1(t)$
t
$\phi_2(t)$
t