Mock Exam 1: Mastery Question (Q5)

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Consider the problem

$$\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & \tfrac{1}{2}(x(T))^2 \\
\text{subject to} \quad & \dot{x}(t) = u(t) \\
& x(0) = x_0 \text{ given} \\
& u(t) \in [-1, 1], \text{ for all } t \in \mathbb{R}
\end{aligned}$$

i Using the PMP, find an expression for the optimal control as a feedback law.

ii Find an explicit expression for the optimal value function of the problem.

Solutions:

i The Hamiltonian is given by

$$H(t, x, u, \lambda) = \lambda u$$

Pointwise minimization yields

$$\mu(t, x) = \arg\min_{|u| \le 1} \lambda u = \begin{cases} 1, & \lambda < 0 \\ -1, & \lambda > 0 \\ \tilde{u}, & \lambda = 0 \end{cases} = -\operatorname{sign}(\lambda)$$

where $\tilde{u} \in [-1, 1]$ is arbitrary. The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x, u, \lambda) = 0, \quad \lambda(T) = \frac{\partial \phi}{\partial x}(x(T)) = x(T)$$

which has the solution $\lambda(t) = x(T)$. We now have to cases:

- $x(T) \neq 0$: In this case $\lambda(t) \neq 0$ for all t and we can write

$$\mu(t, x) = -\operatorname{sign}(\lambda) = -\operatorname{sign} x(T) = -\operatorname{sign} x$$

The last equality holds since x will have the same sign as $x(T)$ during the whole state trajectory.

- $x(T) = 0$: In this case $\lambda = 0$ for all t and we may use any control signal $\tilde{u} \in [-1, 1]$, which obeys the constraint $x(T) = 0$. One such control signal is

$$\mu(t, x) = -\operatorname{sign} x$$

since this will drive x to zero and stay there.

Consequently, one optimal control is

$$\mu^*(t, x) = -\operatorname{sign}(x)$$

ii Since $J^*(t, x) = \frac{1}{2}(x^*(T))^2$, we need to find $x^*(T)$. It holds that

$$x(T) - x(t) = \int_t^T \dot{x}(\tau)d\tau = \int_t^T u(\tau)d\tau$$

which can be written as

$$x(T) = x(t) - \int_t^T \operatorname{sign}\{x(\tau)\}d\tau \qquad \text{(SE)}$$

There are two cases:

– $x(t) > 0$: In this case the controller will decrease $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$x(T) = \max\{0, \overbrace{x(t) - (T - t)}^{\text{from (SE)}}\}$$
$$= \max\{0, |x(t)| - (T - t)\}$$

– $x(t) < 0$: In this case the controller will increase $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$x(T) = \min\{0, \overbrace{x(t) + (T - t)}^{\text{from (SE)}}\} = -\max\{0, -x(t) - (T - t)\}$$
$$= -\max\{0, |x(t)| - (T - t)\}$$

Thus, the only difference between the two cases are the sign in front of the max and the optimal value function becomes

$$V(t, x) = J^*(t, x) = \frac{1}{2}(x^*(T))^2 = \frac{1}{2}(\max\{0, |x| - (T - t)\})^2.$$