Geometry Course Work 1

Q1. The function $x \mapsto x^2$ is monotonically increasing for $x \geq 0$ and $|\phi(t)| \geq 0$ so if $|\phi(t)|$ is maximised at t_0, $|\phi(t)|^2$ is also maximised at t_0

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=t_0} |\phi(t)|^2 = 0 \;, \quad \left.\frac{d^2}{dt^2}\right|_{t=t_0} |\phi(t)|^2 \leq 0$$

$$\frac{d}{dt}|\phi(t)|^2 = \frac{d}{dt}\langle \phi(t), \phi(t)\rangle = \langle \phi'(t), \phi(t)\rangle + \langle \phi(t), \phi'(t)\rangle$$
$$= 2\langle \phi(t), \phi'(t)\rangle$$

$$\frac{d^2}{dt^2}|\phi(t)|^2 = \frac{d}{dt} 2\langle \phi(t), \phi'(t)\rangle = 2\langle \phi'(t), \phi'(t)\rangle$$
$$+ 2\langle \phi(t), \phi''(t)\rangle$$

so we have $2\langle \phi(t_0), \phi'(t_0)\rangle = 0$

$$2|\phi'(t_0)|^2 + 2\langle \phi(t_0), \phi''(t_0)\rangle \leq 0$$
$$\Rightarrow \langle \phi(t_0), \phi''(t_0)\rangle \leq -|\phi'(t_0)|^2 \leq 0$$

If ϕ is parametrisation by arc length,

$$|\langle \phi(t_0), \phi''(t_0)\rangle| = -\langle \phi(t_0), \phi''(t_0)\rangle \geq |\phi'(t_0)|^2 = 1$$

~~as curve is parametrised by arc length.~~

By Cauchy-Schwartz inequality

$$|\phi(t_0)| \, k(t_0) = |\phi(t_0)| \, |\phi''(t_0)| \geq |\langle \phi(t_0), \phi''(t_0)\rangle| \geq 1$$

$$\Rightarrow k(t_0) \geq \frac{1}{|\phi(t_0)|} \qquad \square$$

~~So $k(t_0) = |k(t_0)| \geq \frac{1}{|\phi(t_0)|}$~~

~~since by definition $k(t) \geq$~~

If ϕ is not parametrised by arc length, By lemma 1.2, one can always find smooth function $f: [c,d] \to [a,b]$ s.t. $\psi := \phi \circ f$ is parametrisation by arc-length.

$$\max_{s\in[c,d]} |\psi(s)| = \max_{s\in[c,d]} |\phi(f(s))| = \max_{t\in[a,b]} |\phi(t)|$$

use change of variable $t = f(s)$

so $|\psi(s_0)| = |\phi(t_0)|$ where s_0 is the value where $|\psi(s)|$ reaches maximum

Since curvature does not depend on parametrisation,

the inequality $k(t_0) \geq \frac{1}{|\phi(t_0)|}$ still holds.

Q2. By lectures, $T'(t) = k(t)N(t)$

$B'(t) = -\tau(t)N(t)$

Since $\tau(t) = ck(t)$,

$B'(t) = -Ck(t)N(t) = -CT'(t)$

$\Rightarrow \frac{d}{dt}(B(t) + CT(t)) \equiv 0 \quad \forall t.$

$\Rightarrow B(t) + CT(t) = \text{constant}.$

let $v := B(t) + CT(t)$

$\langle T(t), v\rangle = \langle T(t), B(t)\rangle + C\langle T(t), T(t)\rangle$

By definition, $B(t) = T(t) \times N(t)$ so

$\langle T(t), B(t)\rangle = 0$

ϕ is parametrisation by arc length so

$\langle T(t), T(t)\rangle = |T(t)|^2 = |\phi'(t)|^2 = 1$

$\Rightarrow \langle T(t), v\rangle = C$

If θ is angle between $T(t)$, v,

$\cos\theta = \frac{\langle T(t), v\rangle}{|T(t)||v|} \Rightarrow \theta = \cos^{-1}\left(\frac{C}{|v|}\right)$

which is constant.

$\square$

Q3. $\gamma'(t) = (-a\sin(t), b\cos(t))$

$\gamma''(t) = (-a\cos(t), -b\sin(t))$

$n(t) = \frac{(-b\cos(t), -a\sin(t))}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}}$ $a, b \neq 0$.

$\cancel{\langle\gamma''}$ $k(t) = \dfrac{\langle \gamma''(t), n(t)\rangle}{|\phi'(t)|^2}$

$$= \frac{1}{(a^2\sin^2(t)+b^2\cos^2(t))^{\frac{3}{2}}}\left(ab\cos^2(t)+ab\sin^2(t)\right)$$

$$= \frac{ab}{(a^2\sin^2(t)+b^2\cos^2(t))^{\frac{3}{2}}}$$

This is the curvature at $(a\cos(t), b\sin(t))$, $\forall t\in\mathbb{R}$.

Q4. As in proof of proposition 5.6,

if $\phi: U\to S$ is chart for the surface S, where $U\subset\mathbb{R}^2$ is open set, write,

$$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

Given $p\in S$, let $q=\phi^{-1}(p)$,

$$d\phi_q = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$ has rank 2 so WLOG

say $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is invertible, extend the map ϕ to

$$\hat{\phi}: U\times\mathbb{R}\to\mathbb{R}^3 \quad (U\times\mathbb{R}\subset\mathbb{R}^3)$$

by $\hat{\phi}(u,v,w) = (x(u,v), y(u,v), z(u,v)+w)$

note $\hat{\phi}|_{\{(u,v,w):w=0\}} = \phi$ $\quad$ $\hat{\phi}$ is smooth as it is $\phi + r$ where $r(u,v,w) = (0,0,w)$ is smooth and ϕ is smooth by def.

$$d\hat{\phi}_q = \begin{pmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ z_u & z_v & 1 \end{pmatrix}$$ ~~first two~~ ~~rows~~ columns are ~~LI~~ by ~~assumption, and third row~~ e_3 ~~is LI to first two columns~~

page 3 Jacobian $= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ by assumption.

so $d\hat{\phi}_q$ is invertible.

By inverse function theorem, there is neighborhood $\hat{U}$ ~~of~~ in $U \times \mathbb{R}$ containing q s.t. $\hat{\phi}: \hat{U} \to \hat{\phi}(\hat{U})$ is smooth ~~id~~ diffeomorphism. let $V := \hat{\phi}(\hat{U})$

Define $F: V \to \mathbb{R}$ by $F = p_z \circ \hat{\phi}^{-1}$

where $p_z(u,v,w) := w$. is map from $\mathbb{R}^3$ to $\mathbb{R}$.

$F^{-1}(0) = \{(x,y,z) : p_z(\hat{\phi}^{-1}(x,y,z)) = 0\}$

$= \{(x,y,z) : \hat{\phi}^{-1}(x,y,z) \cdot e_3 = 0\} \quad e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \{(x,y,z) : \hat{\phi}^{-1}(x,y,z) \in U\} = \{\phi(u,v) : (u,v) \in U\} = S$

because $\hat{\phi}|_{\{(u,v,w): w=0\}} = \phi$ by definition

F is smooth as p_z, $\hat{\phi}$ are smooth by chain rule.

$\nabla F(q) = (d\hat{\phi}_p^{-1})^T \nabla p_z$ where $\nabla p_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$=$ last row of $d\hat{\phi}_p^{-1}$.

$d\hat{\phi}_q$ is invertible so $d\hat{\phi}_p^{-1}$ is also invertible.

$\Rightarrow \nabla F(p) \neq 0$ if $p \in V$. $\square$

Take surface $\{0 < x^2+y^2 < 1, z = 0\} =: S$

if $\exists$ smooth F s.t. $S = F^{-1}(c)$, as otherwise, $(0,0,0) \in S = F^{-1}(c)$

$F(0,0,0) \neq c$ but $F(\varepsilon, 0, 0) = c$

$\forall \varepsilon \in (0,1)$ so $\lim_{\varepsilon \to 0} F(\varepsilon, 0, 0) = c \neq F(0,0,0)$

F is not ~~smooth~~ continuous $\divideontimes$.

so S is not regular level set of any smooth function F

Q5. (i) let $F(x,y,z) = \left(e^z x, e^z y, \frac{1}{e^{2z}}\right)$

~~if~~ $F: C \to \mathbb{R}^3$

Note if $x^2+y^2=1$, ~~$F(x,y,z)=$~~

$$\frac{1}{e^{2z}} = \frac{1}{((e^z x)^2+(e^z y)^2)} = \frac{1}{e^{2z}(x^2+y^2)}$$

so $F(x,y,z) \in D$

i.e. F is a function from C to D.

$z \mapsto e^z$, $z \mapsto e^{-2z}$ are smooth,

so F is smooth.

~~And~~ $F^{-1}(u,v,w) = \left(\sqrt{w}\,u, \sqrt{w}\,v, -\frac{1}{2}\ln(w)\right)$

~~is~~ with $F^{-1}: D \to C$ is smooth

as $w > 0$ ~~on D~~ for $(u,v,w) \in D$

and $w \mapsto \sqrt{w}$, $w \mapsto \ln(w)$ are smooth on $(0,\infty)$

so C, D are diffeomorphic

(ii). let $F: S_1 \to S_2$ be

$F(x,y,z) = (x, y, \sqrt{x^2+y^2})$

x^2+y^2 ~~$\neq 0$~~ > 0 on S_1 so F is smooth.

$F^{-1}(u,v,w) = (u,v,0)$ $\quad F^{-1}: S_2 \to S_1$

because $(F^{-1} \circ F)(x,y,z) = F^{-1}(x,y,\sqrt{x^2+y^2}) = (x,y,0)$

$z=0$ on S_1.

and $F \circ F^{-1}(u,v,w) = F(u,v,0) = (u,v,\sqrt{u^2+v^2})$

Indeed if $(u,v,w) \in S_2$, $w = \sqrt{u^2+v^2}$.

so F is bijective and F^{-1} is also smooth.

So. S_1, S_2 are diffeomorphic.