IMPERIAL

MATH70060

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Complex Manifolds

Date: Wednesday, May 7, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Let X be a complex manifold. Define what it means for a real $(1,1)$ form $\omega \in C^\infty(X, \Omega_X^{1,1})$ to be *positive*. (4 marks)

(b) Let X_1 and X_2 be Kähler manifolds. Prove that the manifold $X_1 \times X_2$ admits a positive $(1,1)$-form. (5 marks)

(c) Let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$. Show that the torus $\mathbb{C}^n/\Lambda$ is Kähler. (3 marks)

(d) Let X be a Kähler manifold of complex dimension n with Kähler form ω. Show that

$$\star\omega = \frac{1}{(n-1)!}\omega^{n-1}.$$

Here, $\omega^{n-1} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-1 \text{ times}}$. Deduce that ω is harmonic. (8 marks)

(Total: 20 marks)

2. (a) Give the definition of the tautological line bundle $\pi : \mathcal{O}(-1) \to \mathbb{CP}^1$ on $\mathbb{CP}^1$. Let $U_i = \{[x_0 : x_1] \in \mathbb{CP}^1 : x_i \neq 0\}$ for $i \in \{0,1\}$ be the standard affine cover of $\mathbb{CP}^1$. Define local trivialisations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ for $i \in \{0,1\}$ so that the transition map $g_{0,1} = \psi_0 \circ \psi_1^{-1} : (U_0 \cap U_1) \times \mathbb{C} \to (U_0 \cap U_1) \times \mathbb{C}$ satisfies

$$g_{0,1}([x_0 : x_1], v) = \left([x_0 : x_1], \frac{x_0}{x_1}v\right).$$

(6 marks)

(b) Choose local trivialisations over U_1 and U_2 for the bundle $\mathcal{O}(2)$ and compute the induced transition map.

You can use the trivialisations ψ_i for $i = \{0,1\}$ from the previous part to construct trivialisations for $\mathcal{O}(2)$.

(4 marks)

(c) Prove that the only holomorphic section of $\mathcal{O}(-1)$ is the zero section.

(4 marks)

(d) Let L be a holomorphic line bundle over a compact complex manifold X. Prove that L is isomorphic to the trivial line bundle if and only if the bundles L and L^* have a global holomorphic section that is not constant 0.

(6 marks)

(Total: 20 marks)

3. Let X be a real manifold with almost complex structure J.

(a) Show that the real dimension of X is an even number.

(4 marks)

(b) Define the holomorphic and anti-holomorphic tangent space of X.

(4 marks)

(c) Define the lattice $\Lambda_1 = \langle 1, i \rangle \subset \mathbb{C}$ and the two-dimensional real manifold $E_1 = \mathbb{C}/\Lambda_1$. For each $x \in E_1$ we have a canonical identification $T_x E_1 \cong \mathbb{R}^2$. Define the almost complex structure $J_1 : TE_1 \to TE_1$ via

$$(J_1)_x : T_x E_1 \cong \mathbb{R}^2 \to T_x E_1 \cong \mathbb{R}^2$$
$$(v, w) \mapsto (-v - w, 2v + w)\,.$$

Check that J_1 is integrable. (6 marks)

(d) Let $\Lambda_2 = \langle 1, 1+i \rangle$. Then $E_2 = \mathbb{C}/\Lambda_2$ is a complex manifold and its complex structure induces an almost complex structure J_2. Find a diffeomorphism $F : E_1 \to E_2$ such that

$$J_2(\mathrm{d}\, F(v)) = \mathrm{d}\, F(J_1(v)) \quad \text{for all } v \in TE_1.$$

(You must prove that your map F is well defined and bijective.) (6 marks)

(Total: 20 marks)

4. In this question we denote the Fubini-Study form on $\mathbb{CP}^n$ by $\omega_{FS,n}$ to highlight dependence on n.

(a) Give the definition of $\omega_{FS,n}$. (4 marks)

(b) For $n \geq 1$, define the embedding

$$F : \mathbb{CP}^{n-1} \to \mathbb{CP}^n$$
$$[x_0 : \cdots : x_{n-1}] \mapsto [x_0 : \cdots : x_{n-1} : 0].$$

Prove that for $n \geq 1$ we have $F^*(\omega_{FS,n}) = \omega_{FS,n-1}$. (6 marks)

(c) Prove that

$$\int_{\mathbb{CP}^1} \omega_{FS,1} = 1.$$

(6 marks)

(d) Let $x_1, y_1, \ldots, x_n, y_n$ be coordinates on $\mathbb{C}^n$. Let $\omega = \mathrm{d}\, x_1 \wedge \mathrm{d}\, y_1 + \cdots + \mathrm{d}\, x_n \wedge \mathrm{d}\, y_n$. Find a function $f \in C^\infty(\mathbb{C}^n, \mathbb{R})$ such that

$$\omega = \frac{i}{2}\partial\overline{\partial} f,$$

and give a proof that your choice of f satisfies this equation. (4 marks)

(Total: 20 marks)

5. (a) Define the first Chern class of a complex vector bundle. (4 marks)

(b) Let X be a real manifold and let E_1 and E_2 be complex vector bundles over X. Let ∇^i be a connection on E_i for $i \in \{1, 2\}$. Show that $\nabla^{\otimes}$ defined by

$$\begin{aligned} \nabla^{\otimes} : C^{\infty}(X, E_1 \otimes E_2) &\to C^{\infty}(X, \Omega^1_{X,\mathbb{C}} \otimes E_1 \otimes E_2) \\ v \otimes w &\mapsto (\nabla^1 v) \otimes w + v \otimes (\nabla^2 w) \end{aligned}$$

is a connection on $E_1 \otimes E_2$. (4 marks)

(c) Find a formula for the curvature of the connection $\nabla^{\otimes}$ in terms of the curvatures of the connections ∇^1 and ∇^2.

Use this to find a formula for $c_1(E_1 \otimes E_2)$ in terms of $c_1(E_1)$ and $c_1(E_2)$. (10 marks)

(d) If E_1 and E_2 are line bundles, then the formula

$$c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$$

holds. Use this to show the following: for any line bundle L on a real manifold X, we have that $c_1(L^*) = -c_1(L)$. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

EXAM MONTH 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70060

Complex Manifolds (Solutions)

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1. (a) Let X be a complex manifold. Define what it means for a real $(1,1)$ form $\omega \in C^\infty(X, \Omega_X^{1,1})$ to be *positive*. RELATED ⇓

Let $n = \dim(X)$. The form ω is called *positive* if for all $x \in X$ there exists a neighbourhood $U \subset X$ of x and a chart $\phi : U \to V \subset \mathbb{C}^n$ such that $(\phi^{-1})^*\omega$ is positive, i.e.

$$(\phi^{-1})^*\omega = \frac{i}{2} \sum_{1 \le j,k \le n} h_{jk} dz_j \wedge d\overline{z_k}$$

with the matrix $(h_{jk}) : V \to \mathbb{R}^{n\times n}$ being positive definite in every point. 4, A

(b) Let X_1 and X_2 be Kähler manifolds. Prove that the manifold $X_1 \times X_2$ admits a positive $(1,1)$-form. SIMILAR ⇓

Let X_1, X_2 be two Kähler manifolds of dimension n_1, n_2 with Kähler forms ω_1, ω_2. Denote by $p_s : X_1 \times X_2 \to X_s$ the projection onto the s-th component for $s \in \{1,2\}$. Define $\omega = p_1^*\omega_1 + p_2^*\omega_2$.

Claim: the form ω is a Kähler form on $X_1 \times X_2$.

Proof of claim: let $(x_1, x_2) \in X_1 \times X_2$. For $s \in \{1,2\}$ let ϕ_s be a chart around x_s such that

$$(\phi_s^{-1})^*\omega = \frac{i}{2} \sum_{1 \le j,k \le n_s} h_{jk}^{(s)} dz_j^{(s)} \wedge d\overline{z_k}^{(s)}$$

with the matrix $(h_{jk}^{(s)})$ being positive definite. Then $\phi = (\phi_1, \phi_2)$ is a chart around (x_1, x_2). 1, A

We have

$$(\phi^{-1})^*\omega = \frac{i}{2} \sum_{1 \le j,k \le n_1+n_2} H_{jk} dz_j \wedge d\overline{z_k}$$

for

$$H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix},$$

which shows that ω is a $(1,1)$-form. The matrix H is positive definite because $h^{(s)}$ are positive definite by assumption, which proves the claim. 4, C

(c) Let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$. Show that the torus $\mathbb{C}^n/\Lambda$ is Kähler. RELATED ⇓

Denote by $\omega \in \Omega^2(\mathbb{C}^n)$ the standard Kähler form on $\mathbb{C}^n$.

We must prove that the form ω descends to the quotient T^{2n}. 1, A

To this end, let $\lambda \in \Lambda$ and write

$$f_\lambda : \mathbb{C}^n \to \mathbb{C}^n$$
$$x \mapsto x + \lambda.$$

Then $f_\lambda^*(\omega) = \omega$, so ω is well-defined as a form on T^{2n}. 2, B

(d) Let X be a Kähler manifold of complex dimension n with Kähler form ω. Show that UNSEEN ⇓

$$\star\omega = \frac{1}{(n-1)!}\omega^{n-1}.$$

Deduce that ω is harmonic. Here, $\omega^{n-1} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-1 \text{ times}}$.

Let $x \in X$. There exist coordinates (z_j) around x such that, at the point x:

$$\omega_x = \sum_{j=1}^n dz_j \wedge d\overline{z_j}.$$

Then

$$\star\omega_x = \sum_{j=1}^{n} dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge \widehat{dz_j} \wedge \widehat{d\overline{z_j}} \wedge \ldots dz_n \wedge d\overline{z_n},$$

where $\widehat{\bullet}$ denotes omitting a factor. $\boxed{4, \text{C}}$

Also,

$$\begin{aligned}(\omega_x)^{n-1} &= (dz_1 \wedge d\overline{z_1} + \ldots dz_n \wedge d\overline{z_n})^{n-1} \\ &= (n-1)! \sum_{j=1}^{n} dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge \widehat{dz_j} \wedge \widehat{d\overline{z_j}} \wedge \ldots dz_n \wedge d\overline{z_n}.\end{aligned}$$

This proves the claim.

To see that ω is harmonic, note that $d\omega = 0$ by the definition of Kähler form. Furthermore,

$$d(\star\omega) = \frac{1}{(n-1)!} d\left(\omega^{n-1}\right) = \frac{1}{(n-1)!} \sum_{j=1}^{n-1} \omega \wedge \cdots \wedge \underbrace{d\omega}_{j\text{-th factor}} \wedge \cdots \wedge \omega = 0,$$

so $d^*\omega = 0$ and therefore $\Delta\omega = 0$. $\boxed{4, \text{D}}$

2. (a) Give the definition of the tautological line bundle $\pi : \mathcal{O}(-1) \to \mathbb{CP}^1$ on $\mathbb{CP}^1$. Let $U_i = \{[x_0 : x_1] \in \mathbb{CP}^1 : x_i \neq 0\}$ for $i \in \{0,1\}$ be the standard affine cover of $\mathbb{CP}^1$. Define local trivialisations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ for $i \in \{0,1\}$ so that the transition map $g_{0,1} = \psi_0 \circ \psi_1^{-1} : (U_0 \cap U_1) \times \mathbb{C} \to (U_0 \cap U_1) \times \mathbb{C}$ satisfies

RELATED ⇓

$$g_{0,1}([x_0 : x_1], v) = \left([x_0 : x_1], \frac{x_0}{x_1} v\right).$$

The bundle $\mathcal{O}(-1)$ is defined to be

$$\mathcal{O}(-1) = \{(x, v) \in \mathbb{CP}^1 \times \mathbb{C}^2 : v \in x\}$$

with projection map

$$\begin{aligned} \pi : \mathcal{O}(-1) &\to \mathbb{CP}^1 \\ (x, v) &\mapsto x. \end{aligned}$$

2, A

Define

$$\begin{aligned} \psi_0([1 : x_1], \lambda \cdot (1, x_1)) &= ([1 : x_1], \lambda), \\ \psi_1([x_0 : 1], \lambda \cdot (x_0, 1)) &= ([x_0 : 1], \lambda). \end{aligned}$$

Then

$$\begin{aligned} g_{0,1}([x_0 : x_1], v) &= \psi_0 \psi_1^{-1}\left(\left[\frac{x_0}{x_1} : 1\right], v\right) \\ &= \psi_0\left(\left[1 : \frac{x_1}{x_0}\right], v \cdot \left(\frac{x_0}{x_1}, 1\right)\right) \\ &= \psi_0\left(\left[1 : \frac{x_1}{x_0}\right], v \cdot \frac{x_0}{x_1}\left(1, \frac{x_1}{x_0}\right)\right) \\ &= \left([x_0 : x_1], v \cdot \frac{x_0}{x_1}\right). \end{aligned}$$

4, B

(b) Choose local trivialisations over U_1 and U_2 for the bundle $\mathcal{O}(2)$ and compute the induced transition map.

UNSEEN ⇓

You can use the trivialisations ψ_i for $i = \{0,1\}$ from the previous part to construct trivialisations for $\mathcal{O}(2)$.

By definition, $\mathcal{O}(2) = \mathcal{O}(1) \otimes \mathcal{O}(1)$, so we can define for $i \in \{0,1\}$ the following (inverses of) trivialisations of $\mathcal{O}(2)$:

$$\begin{aligned} \phi_i^{-1} : U_i \times \mathbb{C} &\to \pi^{-1}(U_i) \otimes \pi^{-1}(U_i) \\ (x, v) &\mapsto \psi_i^{-1}(x, v) \otimes \psi_i^{-1}(x, 1). \end{aligned}$$

Then, for $(x, v) \in (U_0 \cap U_1) \times \mathbb{C}$ with $x = [x_0 : x_1]$:

$$\begin{aligned} \phi_0 \phi_1^{-1}(x, v) &= \phi_0(\psi_1^{-1}(x, v) \otimes \psi_1^{-1}(x, 1)) \\ &= \phi_0(\psi_0^{-1}\psi_0\psi_1^{-1}(x, v) \otimes \psi_0^{-1}\psi_0\psi_1^{-1}(x, 1)) \\ &= \phi_0\left(\psi_0^{-1}\left(x, \frac{x_0}{x_1} v\right) \otimes \psi_0^{-1}\left(x, \frac{x_0}{x_1} 1\right)\right) \\ &= \left(\frac{x_0}{x_1}\right)^2 \cdot \phi_0(\psi_0^{-1}(x, v) \otimes \psi_0^{-1}(x, v)) \\ &= \left(\frac{x_0}{x_1}\right)^2 (x, v). \end{aligned}$$

4, D

(c) Prove that the only holomorphic section of $\mathcal{O}(-1)$ is the zero section. UNSEEN ⇓

Let $s \in H^0(\mathbb{CP}^1, \mathcal{O}(-1))$ be a global section. By definition of $\mathcal{O}(-1)$, we have that $s = (s_1, s)2)$, where $s_1 : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the identity and $s_2 : \mathbb{CP}^1 \to \mathbb{C}^2$ is a holomorphic map. The manifold $\mathbb{CP}^1$ is compact, so s_2 is a constant map. From the condition that $s(x) \in \mathcal{O}(-1)_x$ for every $x \in \mathbb{CP}^1$ we have that $s_2(x) = 0$ for all $x \in \mathbb{CP}^1$, which proves the claim. 4, C

(d) Let L be a holomorphic line bundle over a compact complex manifold X. Prove that L is isomorphic to the trivial line bundle if the bundles L and L^* have a global holomorphic each section that is not constant 0. SIMILAR ⇓

Let $s_1 \in H^0(X, L)$ and $s_2 \in H^0(X, L^*)$. Then $s_1 \otimes s_2 \in H^0(X, L \otimes L^*)$. We have that $L \otimes L^* = \underline{\mathbb{C}}$ is the trivial bundle. 2, A

Thus, we can view $s_1 \otimes s_2$ as a holomorphic function from X to $\mathbb{C}$. Because X is compact, $s_1 \otimes s_2$ must be constant. By assumption, s_1 and s_2 are not constant zero, so $s_1 \otimes s_2$ must be constant non-zero. Thus s_1 and s_2 are nowhere zero. Therefore s_1 defines an isomorphism between L and the trivial bundle $\underline{\mathbb{C}}$. 4, B

3. Let X be a real manifold with almost complex structure J. RELATED $\Downarrow$

(a) Show that the real dimension of X is an even number.

Let $\dim(X) = n$. Then

$$\det(J)^2 = \det(J^2) = \det(-\mathrm{Id}) = (-1)^n.$$

But $\det(J)$ is a real number, and this equation can only have a solution for even n. 4, A

(b) Define the holomorphic and anti-holomorphic tangent space of X. RELATED $\Downarrow$

Let $x \in X$. Extend J complex linearly to $J : (T_xX)_{\mathbb{C}} \to (T_xX)_{\mathbb{C}}$. Then

$T_x^{1,0} = \{v \in (T_xX)_{\mathbb{C}} : J(v) = iv\}$ is the holomorphic tangent space, and
$T_x^{0,1} = \{v \in (T_xX)_{\mathbb{C}} : J(v) = -iv\}$ is the anti-holomorphic tangent space.

4, A

(c) Define the lattice $\Lambda_1 = \langle 1, i\rangle \subset \mathbb{C}$ and the two-dimensional real manifold $E_1 = \mathbb{C}/\Lambda_1$. For each $x \in E_1$ we have a canonical identification $T_xE_1 \cong \mathbb{R}^2$. Define the almost complex structure $J_1 : TE_1 \to TE_1$ via SIMILAR $\Downarrow$

$$\begin{aligned}(J_1)_x : T_xE_1 \cong \mathbb{R}^2 &\to T_xE_1 \cong \mathbb{R}^2\\ (v, w) &\mapsto (-v - w, 2v + w)\,.\end{aligned}$$

Check that J_1 is integrable.

Under the identification $T_xE_1 \cong \mathbb{R}^2$ we have for $x \in E_1$:

$$T_x^{0,1} = \left\{ \begin{pmatrix} v \\ (-1-i)v \end{pmatrix} \in \mathbb{R}^2 : v \in \mathbb{R} \right\}.$$

Thus, we can write vector fields $X, Y \in C^\infty(T^{0,1})$ as

$$\begin{aligned} X &= f_X \cdot \begin{pmatrix} 1 \\ -1-i \end{pmatrix},\\ Y &= f_Y \cdot \begin{pmatrix} 1 \\ -1-i \end{pmatrix} \end{aligned}$$

for smooth functions f_X, f_Y. Then

$$\begin{aligned}[X, Y] &= X(Y) - Y(X)\\ &= f_X \begin{pmatrix} 1 \\ -1-i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1-i \end{pmatrix}(f_Y) - f_Y \begin{pmatrix} 1 \\ -1-i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1-i \end{pmatrix}(f_X),\end{aligned}$$

where in the first step we used the formula for the Lie bracket on $\mathbb{R}^n$, and in the last line wrote $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}(f_Y)$ for the directional derivative of f_Y in the direction of $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$. The last expression is again a section of $T^{0,1}$, which proves the claim. 6, B

(d) Let $\Lambda_2 = \langle 1, 1+i \rangle$. Then $E_2 = \mathbb{C}/\Lambda_2$ is a complex manifold and its complex structure induces an almost complex structure J_2. Find a diffeomorphism $F : E_1 \to E_2$ such that

$$J_2(\mathrm{d}\, F(v)) = \mathrm{d}\, F(J_1(v)) \quad \text{for all } v \in TE_1.$$

(You must prove that your map F is well defined and bijective.)

UNSEEN ⇓

Define

$$\begin{aligned} F' : \mathbb{C} \cong \mathbb{R}^2 &\to \mathbb{C} \cong \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x \\ x+y \end{pmatrix}. \end{aligned}$$

By definition, $F'(z+\lambda) = F'(z) + F'(\lambda)$ for $\lambda \in \Lambda_1$. One checks that $F'(\lambda) \in \Lambda_2$, so F' descends to a map $F : E_1 \to E_2$.

Because F' is bijective and $F'(\Lambda_1) = \Lambda_2$ we have that F is bijective. Smoothness of F follows from smoothness of F'.

Then for $x \in E_1$ and $\begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^2 \cong T_x E_1$:

$$\begin{aligned} J_2\left(dF\begin{pmatrix} v \\ w \end{pmatrix}\right) &= J_2\begin{pmatrix} v \\ v+w \end{pmatrix} = \begin{pmatrix} -v-w \\ v \end{pmatrix} \\ dF\left(J_1\begin{pmatrix} v \\ w \end{pmatrix}\right) &= dF\begin{pmatrix} -v-w \\ 2v+w \end{pmatrix} = \begin{pmatrix} -v-w \\ -v-w+2v+w \end{pmatrix} = \begin{pmatrix} -v-w \\ v \end{pmatrix}, \end{aligned}$$

which proves the claim.

6, D

4. In this question we denote the Fubini-Study form on $\mathbb{CP}^n$ by $\omega_{FS,n}$ to highlight dependence on n. RELATED $\Downarrow$

(a) Give the definition of $\omega_{FS,n}$.

Let $\mathbb{CP}^n = \bigcup_{j=1}^n U_j$ be the standard open covering of projective space. On U_j define

$$\omega_{j,n} = \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_j}\right|^2\right) \in \Omega^2(U_j).$$

These forms satisfy $\omega_{j,n}|_{U_j\cap U_k} = \omega_{k,n}|_{U_j\cap U_k}$, so they define a global form $\omega_{FS,n} \in \Omega^2(\mathbb{CP}^n)$. 4, A

(b) For $n \geq 1$, define the embedding SIMILAR $\Downarrow$

$$\begin{aligned} F : \mathbb{CP}^{n-1} &\to \mathbb{CP}^n \\ [x_0 : \cdots : x_{n-1}] &\mapsto [x_0 : \cdots : x_{n-1} : 0]. \end{aligned}$$

Prove that for $n \geq 1$ we have $F^*(\omega_{FS,n}) = \omega_{FS,n-1}$.

Restricting F to the affine patch U_j we find:

$$\begin{aligned} F^*(\omega_{j,n}) &= \frac{i}{2\pi}\partial\overline{\partial}\log F^*\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_j}\right|^2\right) \\ &= \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{l=0}^{n-1}\left|\frac{z_l}{z_j}\right|^2\right) \\ &= \omega_{j,n-1}, \end{aligned}$$

where we used in the first step that the pullback of forms commutes with taking the exterior derivative, and we used that F is holomorphic, so it commutes with ∂ and $\overline{\partial}$. 6, A

(c) Prove that

$$\int_{\mathbb{CP}^1} \omega_{FS,1} = 1.$$

$\boxed{\text{UNSEEN} \Downarrow}$

On the affine patch U_0 we have that

$$\begin{aligned}\omega_{FS,1} &= \frac{i}{2\pi}\frac{1}{(1+|z_1|^2)^2} dz_1 \wedge d\overline{z}_1 \\ &= \frac{1}{\pi} \cdot \frac{1}{(1+|z+1|^2)^2}\, dx \wedge dy,\end{aligned}$$

so

$$\begin{aligned}\int_{\mathbb{CP}^1} \omega_{FS,1} &= \int_{U_0} \omega_{FS,1} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{(1+|z+1|^2)^2}\, dx \wedge dy \\ &= \frac{1}{\pi}\int_{\theta\in[0,2\pi)}\int_{r\in[0,\infty)} \frac{1}{(1+r^2)^2} \cdot r\, dr \\ &= \left[-\frac{1}{1+r^2}\right]_0^\infty \\ &= 1,\end{aligned}$$

where in the first step we used that $\mathbb{CP}^1 \setminus U_0$ is a set of measure zero, and in the third step we used the transformation formula for the integral after changing to polar coordinates. $\boxed{4, \text{B}}$

In the fourth step we guessed the antiderivative of $\frac{1}{(1+r^2)^2} \cdot r$. $\boxed{2, \text{D}}$

(d) Let $x_1, y_1, \ldots, x_n, y_n$ be coordinates on $\mathbb{C}^n$. Let $\omega = \mathrm{d}\, x_1 \wedge \mathrm{d}\, y_1 + \cdots + \mathrm{d}\, x_n \wedge \mathrm{d}\, y_n$. Find a function $f \in C^\infty(\mathbb{C}^n, \mathbb{R})$ such that $\boxed{\text{RELATED} \Downarrow}$

$$\omega = \frac{i}{2}\partial\overline{\partial} f,$$

and give a proof that your choice of f satisfies this equation.

Let

$$f = \sum_{k=1}^n |z_k|^2,$$

where $z_k = x_k + iy_k$. We have

$$\partial\overline{\partial}|z_k|^2 = \partial\overline{\partial} z_k\overline{z_k} = 1 \cdot dz_k \wedge d\overline{z_k},$$

so

$$\frac{i}{2}\partial\overline{\partial} f = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\overline{z_k} = \sum_{k=1}^n dx_i \wedge dy_i.$$

$\boxed{4, \text{A}}$

5. (a) Define the first Chern class of a complex vector bundle. RELATED $\Downarrow$

Let ∇ be any connection on E and θ_∇ its curvature. Then

$$c_1(E) := \left[\frac{i}{2\pi}\mathsf{Tr}\,\theta_\nabla\right] \in H^2_{dR}(X,\mathbb{C})$$

is the first Chern class of E. 4, M

(b) Let X be a real manifold and let E_1 and E_2 be complex vector bundles over X. Let ∇^i be a connection on E_i for $i \in \{1,2\}$. Show that $\nabla^\otimes$ defined by SIMILAR $\Downarrow$

$$\begin{aligned}\nabla^\otimes : C^\infty(X, E_1 \otimes E_2) &\to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E_1 \otimes E_2)\\ v \otimes w &\mapsto (\nabla^1 v) \otimes w + v \otimes (\nabla^2 w)\end{aligned}$$

is a connection on $E_1 \otimes E_2$.

The $\mathbb{C}$-linearity is clear, and it remains to check the Leibniz rule. To this end, let $v \otimes w \in H^0(X, E_1 \otimes E_2)$ and $f \in C^\infty(X)$ and V a vector field on X. We then have:

$$\begin{aligned}\nabla^\otimes_V(f \cdot v \otimes w) &= \nabla^\otimes_V((f\cdot v)\otimes w)\\ &= \nabla^1_V(fv)\otimes w + (fv)\otimes \nabla^2_V w\\ &= V(f)\cdot v\otimes w + f \cdot \nabla^\otimes_V(v\otimes w),\end{aligned}$$

where in the last step we used the Leibniz rule for the connection ∇^1. 4, M

(c) Find a formula for the curvature of the connection $\nabla^\otimes$ in terms of the curvatures of the connections ∇^1 and ∇^2. UNSEEN $\Downarrow$

Use this to find a formula for $c_1(E_1 \otimes E_2)$ in terms of $c_1(E_1)$ and $c_1(E_2)$.

Let $v \otimes w \in H^0(X, E_1 \otimes E_2)$ and A, B be vector fields. Then

$$\begin{aligned}&\theta_{\nabla^\otimes}(A,B)(v\otimes w)\\ &= \nabla^\otimes_A\nabla^\otimes_B v\otimes w - \nabla^\otimes_B\nabla^\otimes_A v\otimes w - \nabla^\otimes_{[A,B]}v\otimes w\\ &= ((\nabla^1_A\nabla^1_B - \nabla^1_B\nabla^1_A - \nabla^1_{[A,B]})v)\otimes w + v\otimes((\nabla^2_A\nabla^2_B - \nabla^2_B\nabla^2_A - \nabla^2_{[A,B]})w)\\ &= (\theta_{\nabla^1}(A,B)v)\otimes w + v\otimes(\theta_{\nabla^2}(A,B)w).\end{aligned}$$

8, M

Thus,

$$\mathsf{Tr}\,(\theta_{\nabla^\otimes}) = \mathsf{rk}(E_2)\cdot \mathsf{Tr}\,(\theta_{\nabla^1}) + \mathsf{rk}(E_1)\cdot\mathsf{Tr}\,(\theta_{\nabla^2})$$

and therefore

$$c_1(E_1\otimes E_2) = \mathsf{rk}(E_2)\cdot c_1(E_1) + \mathsf{rk}(E_1)\cdot c_1(E_2).$$

2, M

(d) If E_1 and E_2 are line bundles, then the formula UNSEEN $\Downarrow$

$$c_1(E_1\otimes E_2) = c_1(E_1) + c_1(E_2)$$

holds. Use this to show the following: for any line bundle L on a real manifold X, we have that $c_1(L^*) = -c_1(L)$.

We have

$$0 = c_1(\underline{\mathbb{C}}) = c_1(L^*\otimes L) = c_1(L^*) + c_1(L),$$

where we used the formula for the Chern class of tensor products in the last step. Rearranging yields $c_1(L^*) = -c_1(L)$.

2, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70060 Complex Manifolds Markers Comments

Question 1

Parts a and b were done well, overall. Part c (showing that the torus is Kaehler) caused some trouble. One must show that the Kaehler structure from C^n is invariant under the lattice. Many people put this in words, but were imprecise in doing this and lost marks. A quick way to write this down is shown in the example solution. Part d caused big problems. Many people checked $\omega \wedge * \omega=\omega \wedge \omega^{n-1}$, up to a constant. One cannot deduce $*\omega=\omega^{n-1}$ from this. To solve this problem, one must exactly compute $* \omega$.

Question 2

Part a caused no problems. In part b I meant to ask for O(-2) rather than O(2). As written, the question is still solvable but a bit harder than what I intended. The example solution provided by me is for O(-2) rather than O(2), so it is not correct. Of course, I marked the question as it was written on the exam paper, not according to the example solution.

Almost no one solved part c in the intended way. Many people checked it in local trivialisations, but it also follows immediately from the fact that a holomorphic map into C^2 from the compact CP^1 is constant. A section of the tautological bundle can be viewed as a map into C^2.

In part d there is another minor discrepancy between question paper and solution paper. The question asks for "if and only if", the solution only proves "if". This is the hard direction, the other direction is almost tautological.

Question 3

Overall no problems in a and b. The worked solution for part c gives an explicit formula for the anti-holomorphic tangent space, and then checks integrability on it. Many students found shortcuts without computing this explicitly. At the end of the day, ever almost complex structure on a complex one-dimensional manifold are integrable, so it is possible to avoid the exact formulae. That may not be the easiest thing to do, though. The worked solution contains a typo: the second entry of the vector in the formula for the anti-holomorphic tangent space should be $(-1+i)v$ rather than $(-1-i)v$.

Part d was rarely solved correctly. I would have expected to see formulae for J_1, J_2, and based on that find a formula for dF. There's then a linear map F with the same equation which turns out to be well-defined on the tori. Instead, most people guessed a map and then went to work, often realising very late that it doesn't intertwine J_1 and J_2 as demanded.

Question 4

Unfortunately many students got the definition of the Fubini-Study form in part a wrong, which then made part b and c difficult. Here, I didn't mind the sign (plus or minus) of the given form. In the lecture I had used a wrong sign. Plus or minus were both marked as correct. Many students explained a solution to part b in words, but were imprecise. No matter the definition of the Fubini-Study form, one can write this down in formulae. This is easiest if one uses the local formulae for the Fubini-Study form. If one uses the global description, one has to prove that F pulls back the Chern connection to the Chern connection. That's not trivial.
Part d went smoothly.

Question 5

Almost no one got the definition of first Chern class right. It's (up to a factor) the de Rham cohomology class of the curvature of ANY connection, not just the Chern connection. The Chern connection only exists on a holomorphic bundle with Hermitian metric, but not on every complex bundle.
Part b posed no problems.
Part c was often solved well, employing every possible definition of curvature. Even faster than the example solution is to use $F=\nabla \nabla$, but it is hard to see how some things cancel out in this case.
Part d was rarely solved as intended. It's just that $L \otimes L^*$=trivial bundle, so $0=c1(\text{trivial bundle})=c1(L \otimes L^*)=c1(L)+c1(L^*)$. That was the intended solution.