ESTIMATING THE MEAN OF A SAMPLE

random variables : $X_1, X_2, ---, X_n$

→ assume all follow distribution F_X

→ all independent (i.i.d.)

→ Let mean of F_X be $\theta \in \mathbb{R}$

where θ is unknown

Q: How to estimate θ?

Prop 1.2.6 $\quad \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \Rightarrow E[\overline{X}] = \theta$

Now suppose we observe $x_1, x_2, --, x_n \in \mathbb{R}$

→ observations of $X_1, X_2, --, X_n$, respectively

To estimate θ, we compute

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x} \qquad \hat{\theta} = x_1 + 2x_2 + x_3^5$$

estimate of θ

Point estimators ✓

$$\hat{\Theta} = \hat{\Theta}(x_1, x_2, \ldots, x_n) = r(x_1, \ldots, x_n)$$

example $\hat{\Theta} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Definition 1.5.18

The ESTIMATION ERROR of the estimator $\hat{\Theta}$ of a parameter θ is defined as $\hat{\Theta} - \theta$

Definition 1.5.19 The BIAS of the estimator $\hat{\Theta}$ of a parameter θ is denoted $b_\theta(\hat{\Theta})$ and is defined as $b_\theta(\hat{\Theta}) = E[\hat{\Theta}] - \theta$

$$E[\hat{\Theta} - \theta] \overset{\text{lin. of } E}{=} E[\hat{\Theta}] - \theta = b_\theta(\hat{\Theta})$$

Definition: If $E[\hat{\Theta}] = \theta$ the $\hat{\Theta}$ is UNBIASED

because $b_\theta[\hat{\Theta}] = E[\hat{\Theta}] - \theta = 0$

$\hat{\Theta} = \overline{X}$ as an estimator of mean θ of F_X (in Prop 1.2.6)

$E[\hat{\Theta}] = E[\overline{X}] = \theta$

$\Rightarrow$ unbiased

$\hat{\Gamma} = S^2$ estimator of variance σ^2 of F_X

$E[\hat{\Gamma}] = E[S^2] = \sigma^2$ (in 1.2.6)

$\Rightarrow$ unbiased

Definition 1.5.23

The MEAN SQUARED ERROR of the estimator $\hat{\Theta}$ of parameter θ is defined as $E[(\hat{\Theta} - \theta)^2]$

expected squared deviation of $\hat{\Theta}$ from θ

Theorem 1.5.24

The mean squared error of an estimator $\hat{\Theta}$ of a parameter θ can be expressed in terms of its bias and variance

$$E\left[(\hat{\Theta}-\theta)^2\right] = \left[b_\theta(\hat{\Theta})\right]^2 + Var[\hat{\Theta}]$$

Proof:

X

$$E[X^2] = (E[X])^2 + Var[X]$$

(Exercise 1.1.5)

$$E\Big[\overbrace{(\hat{\Theta}-\theta)}^{X^2}{}^2\Big] = (E[\hat{\Theta}-\theta])^2 + Var[\hat{\Theta}-\theta]$$

bias: $b_\theta[\hat{\Theta}] = E[\hat{\Theta}]-\theta = E[\hat{\Theta}-\theta]$

$$Var[\hat{\Theta}-\theta] = Var[\hat{\Theta}]$$

$$\Rightarrow E\left[(\hat{\Theta}-\theta)^2\right] = (b_\theta[\hat{\Theta}])^2 + Var[\hat{\Theta}]$$