Classical Dynamics **Autumn Term 2024**

Answers to Problem Sheet 1

1.
$$F_x = -\frac{\partial V}{\partial x} = axy, \qquad F_y = -\frac{\partial V}{\partial y} = bx^2.$$

Integrating the first equation gives $V = -\frac{1}{2}ax^2y + f(y)$. Inserting into the second equation gives $\frac{1}{2}ax^2 - f'(y) = bx^2$. The force is conservative for $b = \frac{1}{2}a$.

2.
$$\mathbf{F} = -\lambda(x\mathbf{i} + y\mathbf{j})(x^2 + y^2 - 1),$$

$$V = \frac{\lambda}{4}(x^2 + y^2 - 1)^2.$$

The minimum of the potential is the circle $x^2 + y^2 = 1$. For large λ the motion is confined to the unit circle. This is a model of a constraint force.

3.
$$\mathbf{F} = y\mathbf{i} - x\mathbf{j}$$

is not conservative as $\nabla \times \mathbf{F} = (\partial_x F_y - \partial_y F_x)\mathbf{k} = -2\mathbf{k} \neq 0$.

4. (i) A central force has the form

$$\mathbf{F}(\mathbf{r}, t) = F(r, t)\frac{\mathbf{r}}{\mathbf{r}},$$

which derives from the potential energy function

$$V(r, t) = -\int F(r, t)dr.$$

A central force is conservative.

(ii) The applied torque is

$$\mathbf{K} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F(r, t)\frac{\mathbf{r}}{r} = 0.$$

Hence angular momentum is a constant of the motion.

5. The work done by the force

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}}\, dt = T(t_2) - T(t_1)$$

is zero as $\mathbf{F} \cdot \dot{\mathbf{r}} = q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \dot{\mathbf{r}} = 0$. Hence the kinetic energy is a constant of the motion.

6. Here the force is

$$F = -\frac{dV}{dx} = \frac{2}{x^3} - 2x,$$

which is zero at $x = 1$. N2 is $\ddot{x} = 2x^{-3} - 2x$. Taylor expand F about $x = 1$; $F = F(1) + F'(1)(x-1) + \ldots$ Now $F'(x) = -6x^{-4} - 2$ so that $F'(1) = -8$, that is $\ddot{x} = -8(x-1) + \ldots$ or $\ddot{z} = -8z + \ldots$ where $z = x - 1$. For small z the system is a simple harmonic oscillator with angular frequency $\sqrt{8}$.

7. Integrating $T - V$ over a period $\mathcal{T} = 2\pi/\omega$

$$\int_0^{\mathcal{T}} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right) dt$$

$$= \frac{m}{2}\dot{x}x\Big|_0^{\mathcal{T}} - \int_0^{\mathcal{T}} \frac{m}{2}\ddot{x}x\, dt - \int^{\mathcal{T}} \frac{1}{2} m\omega^2 x^2 dt,$$

using integration by parts on the first term ($u = \dot{x}$ and $\dot{v} = \dot{x}$). The boundary term vanishes by periodicity and the two integrals cancel using N2 $\ddot{x} = -\omega^2 x$. Hence the integral of $T - V$ over a full period is zero (or the average kinetic energy equals the average potential energy).

8. For the Kepler problem the force has the form $\mathbf{F} = -k\mathbf{r}/r^3$, where k is a positive constant. As this is a central force the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a constant of the motion.

   (i) using $\dot{\mathbf{L}} = 0$

$$\dot{\mathbf{A}} = \dot{\mathbf{p}} \times \mathbf{L} - km\frac{d}{dt}\frac{\mathbf{r}}{r} = -k\frac{\mathbf{r}}{r^3} \times (\mathbf{r} \times \mathbf{p}) - km\frac{r^2\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}}{r^3}$$

$$= -\frac{k}{r^3}\left[(\mathbf{r} \cdot \mathbf{p})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{p}\right] - km\frac{r^2\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}}{r^3} = 0.$$

   Note that $\dot{r} = d(x^2+y^2+z^2)^{1/2}/dt = (x^2+y^2+z^2)^{-1/2}(x\dot{x} + y\dot{y} + z\dot{z}) = (\mathbf{r} \cdot \dot{\mathbf{r}})/r$.

   (ii) $\mathbf{A} \cdot \mathbf{L} = 0$ since $\mathbf{p} \times \mathbf{L}$ is perpendicular to $\mathbf{L}$ and $\mathbf{r}$ is perpendicular to $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

$$A^2 = |\mathbf{p} \times \mathbf{L}|^2 + m^2k^2 - \frac{2km}{r}\mathbf{r} \cdot \mathbf{p} \times \mathbf{L}.$$

Exchanging a dot and cross product yields

$$\mathbf{r} \cdot \mathbf{p} \times \mathbf{L} = \mathbf{r} \times \mathbf{p} \cdot \mathbf{L} = L^2.$$

Accordingly,

$$A^2 = p^2L^2 + m^2k^2 - \frac{2km}{r}L^2 = 2mL^2\left(\frac{p^2}{2m} - \frac{k}{r}\right) + m^2k^2 = 2mEL^2 + m^2k^2.$$

9. A particle of unit mass is subject to the central force

$$\mathbf{F} = -\mu\mathbf{r},$$

where μ is a constant.

As this is a central force the motion is planar. Without loss of generality consider motion in the $z = 0$ plane. The equations of motion for x and y are

$$m\ddot{x} = -\mu x, \quad m\ddot{y} = -\mu y.$$

Surprisingly, it is not easier to analyse the motion in polar coordinates!

For $\mu > 0$ these are simple harmonic oscillators with solutions

$$x = A\cos(\omega t + \alpha), \quad y = B\cos(\omega t + \beta),$$

where $\omega = \sqrt{\mu/m}$. If A and B are non-zero and $\alpha \neq \beta$ the orbits are ellipses in the $z = 0$ plane. Why is this?

For $\mu < 0$ the solutions are

$$x = A\cosh(\sqrt{-\mu/m}t + \alpha), \quad y = B\cosh(\sqrt{-\mu/m}t + \beta),$$

The particle moves on a branch of a hyperbola.

Remark: The motion resembles the elliptical orbits in the Kepler problem. There is a difference. In the Kepler problem the origin coincides with a focus. In the present problem the origin coincides with the centre of the ellipse.