Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

M3S1/M4S1 Statistical Theory I

Date: Thursday, 27 May 2010 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (i) Find the total efficient score $U_\bullet(\theta)$ and total Fisher information $I_\bullet(\theta)$ for independent random variables $X_1,\, X_2,\, \ldots\,,\, X_n$ from $N\,(\theta,\, \theta^2)$.

Obtain (minimal) sufficient statistics for θ.

(ii) What is an *ancillary statistic*?

If X and Y are independent random variables from $N\,(\mu,\, \sigma^2)$, show that $T = X - Y$ is ancillary for unknown μ if σ^2 is known.

(iii) Let $x_1,\, x_2,\, \ldots\,,\, x_n$ be a random sample from $N\,(\theta,\, 1)$.

(a) Show that $\overline{x}$ is sufficient for θ.

(b) Explain without proof why $\overline{x}$ is complete.

(c) Show that $Z = e^{\overline{X} - 1/(2n)}$ is unbiased for e^θ.

(d) Explain without proof why Z is also a uniformly minimum variance unbiased estimator of e^θ.

[You may use without proof that, if Y is $N\,(\mu,\, \sigma^2)$, then $E(e^{\beta Y}) = \exp\,(\mu\beta + \frac{1}{2}\sigma^2\beta^2)$.]

(iv) Let $\boldsymbol{x} = \{x_1,\, x_2,\, \ldots\,,\, x_n\}$ be a random sample from *Uniform* $(-\theta,\, \theta)$, where $\theta > 1$, and the prior probability density is $\pi(\theta) = 1/\theta^2$ on $\theta > 1$.

(a) Obtain the posterior probability density function $\pi(\theta \,|\, \boldsymbol{x})$.

(b) Find the Bayes decision for estimating θ under squared error loss $L\,(\theta,\, d(\boldsymbol{x})) \;=\; (d(\boldsymbol{x}) - \theta)^2$.

2. A genetics experiment has four possible outcomes with probabilities $\frac{1}{4}(2+\theta)$, $\frac{1}{4}(1-\theta)$, $\frac{1}{4}(1-\theta)$, and $\frac{1}{4}\,\theta$. From a random sample of size n, the numbers that fell independently into the four categories were x_1, x_2, x_3 and x_4 respectively.

(a) Show that there can be no unbiased estimator for θ having a variance at the Cramér-Rao lower bound.

(b) Find the Cramér-Rao lower bound.

(c) Explain without proof why $4x_4/n$ is an unbiased estimate of θ. Find its efficiency.

(d) Evaluate the efficiency when $\theta = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}$ and 1.

[You may leave your results as fractions.]

3. (i) Explain what is meant by a *uniformly most powerful test of size* α of a simple hypothesis against a composite hypothesis.

(ii) Let $x_1, x_2, \ldots, x_n$ be a random sample from the family of distributions having the probability density function

$$f(x \mid \theta) = \frac{\theta e^{\theta x}}{e^{\theta} - 1} \quad (0 \le x \le 1).$$

(a) Show that, at $\theta = 0$, this becomes a *Uniform* $(0, 1)$ distribution.

(b) Show that there exists a uniformly most powerful test of size α of the null hypothesis $H_0 : \theta = 0$ against the alternative hypothesis $H_1 : \theta > 0$.

(c) Find the exact form of the critical region of the UMP test of size α $(0 < \alpha < 1)$ when $n = 2$.

(d) Find an asymptotic approximation to the form of the critical region of the UMP test when n is large.

[Note: If X is *Uniform* $(0, 1)$ then $\mathrm{var}(X) = \frac{1}{12}$.]

4. (i) Explain how applying the Central Limit Theorem to the Delta Method can give the asymptotic distributions of estimators.

(ii) Let $X_1, X_2, \ldots, X_n$ be independently *Poisson* (λ) distributed. We seek to estimate $\theta = e^{-\lambda} = P(X_i = 0)$.

Consider the two estimators $\widehat{\theta}_n = e^{-\widehat{\lambda}}$ and $\widetilde{\theta}_n =$ the proportion of zeros, where the sample mean, $\widehat{\lambda} = \overline{X}_n$, is the maximum likelihood estimator of λ.

(a) Show that $\sqrt{n}(\widehat{\theta}_n - \theta) \longrightarrow N(0, \lambda e^{-2\lambda})$.

(b) Show that $\sqrt{n}(\widetilde{\theta}_n - \theta) \longrightarrow N(0, e^{-\lambda}(1 - e^{-\lambda}))$.

(c) Write down the asymptotic relative efficiency ARE$(\widehat{\theta}_n , \widetilde{\theta}_n)$, and show that it is approximately 1 when λ is small.