MATH60005/70005/97405: Optimization (Spring 21-22)

Coursework Solutions

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1. (6 marks) Consider the function

$$f(\mathbf{x}) = x_1^2 - x_1^2 x_2^2 + x_2^4 \,.$$

Find all the stationary points of f and classify them (local/global min/max, saddle point). Justify your answer.

2. Let $\mathbf{z} \in \mathbb{R}^n$ be a signal that is related to the input $\mathbf{u} \in \mathbb{R}^n$ by

$$z_k = \sum_{j=1}^{k} w_j u_{k-j+1} \,, \quad k = 1, 2, \ldots, n \,,$$

where $\mathbf{w} \in \mathbb{R}^n$ is a given signal. The goal of this problem is to find an optimal input $\mathbf{u}^*$ to achieve different objectives:

i) Tracking a reference signal $\bar{\mathbf{z}} \in \mathbb{R}^n$. The tracking error is defined as

$$f_t := \sum_{k=1}^{n} (z_k - \bar{z}_k)^2 \,.$$

ii) Minimizing the energy of the input signal, expressed as

$$f_e := \sum_{k=1}^{n} (u_k)^2 \,.$$

iii) A second-order regularizer of the input, formulated as

$$f_s := \sum_{k=2}^{n-1} (u_{k+1} - 2u_k + u_{k-1})^2 \,.$$

2.1 (5 marks) Formulate the joint minimization of f_t, f_e, and f_s as a regularized least squares problem for $\mathbf{u}$ of the form

$$\min_{\mathbf{u}\in\mathbb{R}^n} \left\{ \mathcal{J}(\mathbf{u}) := \|T\mathbf{u} - \mathbf{b}\|^2 + \delta\|\mathbf{E}\mathbf{u}\|^2 + \eta\|\mathbf{S}\mathbf{u}\|^2 \right\} ,$$

giving precise definitions for $\mathbf{T}$, $\mathbf{b}$, $\mathbf{E}$, and $\mathbf{S}$. Here $\mathbf{E}$ and $\mathbf{S}$ represent matrices related to f_e and f_s, respectively.The regularisation parameters δ and η are non-negative. Determine an explicit expression for the solution to this problem, and discuss the existence and uniqueness of a solution.

2.2 (4 marks) Taking $n = 200$ and $\mathbf{w}$ given by

$$w_k = \begin{cases} 1 & k \leq 100\,, \\ 0 & \text{otherwise} \end{cases}$$

and the reference $\bar{\mathbf{z}} \in \mathbb{R}^{200}$ given by

$$\bar{z}_k = \begin{cases} -1 & \text{for } 1 \leq k \leq 50\,, \\ 0 & \text{for } 51 \leq k \leq 90\,, \\ 1 & \text{for } 91 \leq k \leq 140\,, \\ 0 & \text{for } 141 \leq k \leq 200\,, \end{cases}$$

analyse the effect of the different regularization parameters. For this, show and discuss your observations for each of the following:

2.2.1 (2 marks) A plot of $\mathcal{J}(\mathbf{u}^*)$ versus δ from 0 to 10^5, taking $\eta = 0$. What is the asymptotic behaviour of $\mathbf{u}^*$ and $\mathcal{J}(\mathbf{u}^*)$ as δ grows?

2.2.2 (2 marks) A single plot illustrating the reference signal $\bar{\mathbf{z}}$, and reconstructed signals $\mathbf{z}^*$ for $\delta = 0$ and $\eta = 0, 1, 10, 100, 1000$. What is the effect of the regularizer? Explain why the reconstructed signal for $(\delta, \eta) = (0, 0)$ coincides exactly with the reference signal.

2.3 (5 marks) Create the noisy signal $\hat{\mathbf{y}} = \bar{\mathbf{z}} + \mathcal{N}$, where $\mathcal{N}$ is the noise sequence in the companion file `noise.mat`. Formulate the denoising problem for computing the denoised signal $\mathbf{z}$ using a term of the form f_s as a regularizing term for $\mathbf{z}$ and study the differences with the regularization given by

$$f_{tv} := \sum_{k=1}^{n-1} (z_{k+1} - z_k)^2\,.$$

Compare your solutions against the original noisy signal for both regularizers separately, when $\lambda = 1$ and $\lambda = 100$. Derive an explicit gradient iteration for solving the regularisation problem if an exact line search is used, expressing your results depending on λ and a regularization matrix $\mathbf{R}$.

Solutions

1. For $f(\mathbf{x}) = x_1^2 - x_1^2 x_2^2 + x_2^4$, we begin by finding the stationary points from $\nabla f(\mathbf{x}) = 0$, which leads to

$$
\begin{aligned}
2x_1(1 - x_2^2) &= 0\,, \\
2x_2(2x_2^2 - x_1^2) &= 0\,,
\end{aligned}
$$

which leads to the following stationary points: $(0, 0)$, $(\pm\sqrt{2}, \pm 1)$.

For $(0, 0)$ we first note that this is a local minimum using the definition. Note that $f(\mathbf{x}) = x_1^2(1 - x_2^2) + x_2^4$, so we can take a ball centred at the origin, for instance $\|\mathbf{x}\| < 1$, where $f(\mathbf{x}) \geq f(0, 0) = 0$ since $(1 - x_2^2) > 0$. We cannot claim that $(0, 0)$ is a global minimum since f is not coercive. To see this, take $x_2 = 1$, so that $f(x_1, x_2) = 1$ regardless of x_1 (we can also see this from the curves $(x_1, \sqrt{x_1})$, where $f(x_1) = 2x_1^2 - x_1^3$). For the remaining 4 stationary points we note that the Hessian matrix is given by

$$
\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2(1 - x_2^2) & -4x_1x_2 \\ -4x_1x_2 & 12x_2^2 - 2x_1^2 \end{bmatrix}.
$$

For all these stationary points we note that $\det(\nabla^2 f) = -32$, hence these are all saddle points.

Marks: 2 for stationary points, 2 for saddle points, 2 for identifying local min (f not coercive and local minimizer).

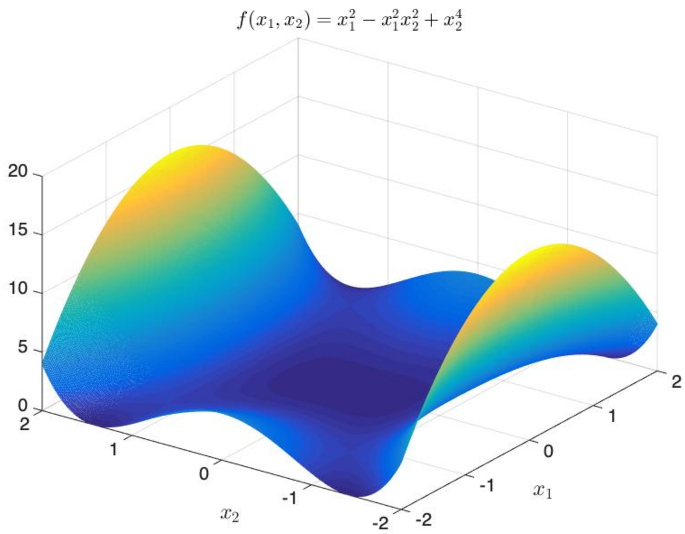


Figure 1: The graph of $f(\mathbf{x}) = x_1^2 - x_1^2 x_2^2 + x_2^4$.

2.1 The relation between $\mathbf{z}$ and $\mathbf{u}$ can be expressed as

$$\mathbf{z} = \mathbf{T}\mathbf{u}\,,$$

where the matrix $\mathbf{T} \in \mathbb{R}^{n \times n}$ is of the form

$$\mathbf{T} = \begin{bmatrix} w_1 & & & & & & & & & \\ w_2 & w_1 & & & & & & & & \\ w_3 & w_2 & w_1 & & & & & & & \\ \vdots & w_3 & w_2 & w_1 & & & & & & \\ & \vdots & w_3 & w_2 & w_1 & & & & & \\ & & \vdots & w_3 & w_2 & w_1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ h_n & h_{n-1} & h_{n-2} & \dots & & & & & & h_1 \end{bmatrix},$$

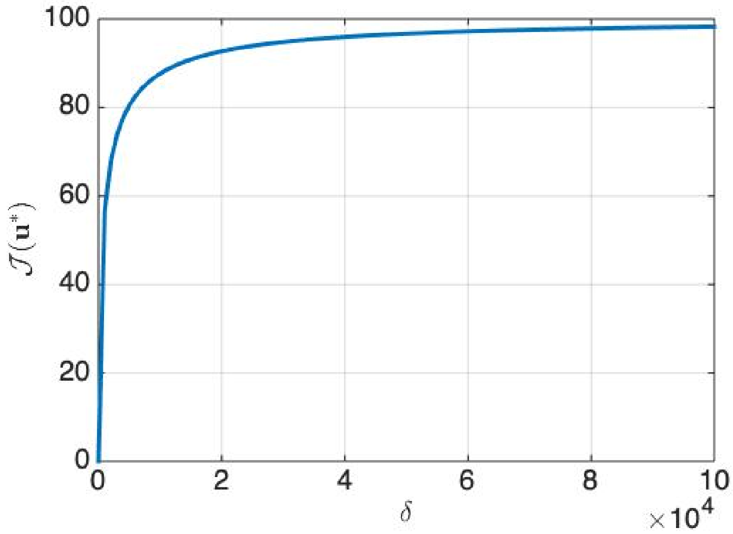
with 0's above the diagonal. The vector $\mathbf{b} \in \mathbb{R}^n$ arising from the tracking term is given by $\mathbf{b} = \bar{\mathbf{z}}$. It is easy to see that $f_e = \|\mathbf{u}\|^2$ and thus $\mathbf{E} \in \mathbb{R}^{n \times n}$ is the identity matrix $\mathbf{I}_n$. Finally, f_s is expressed as $\|\mathbf{S}\mathbf{u}\|^2$, where $\mathbf{S} \in \mathbb{R}^{n-2 \times n}$ is given by

$$\mathbf{S} = \begin{bmatrix} 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -2 & 1 \end{bmatrix},$$

with 0's above and below the diagonal. In the lectures we have seen the derivation of the regularized linear least squares solution with a single regularization term, the derivation here is analogous, computing $\nabla \mathcal{J}(\mathbf{u}) = 0$ leads to the stationary point (and global minimum)

$$\mathbf{u}^* = (\mathbf{T}^\top \mathbf{T} + \delta \mathbf{E}^\top \mathbf{E} + \eta \mathbf{S}^\top \mathbf{S})^{-1} \mathbf{T}^\top \mathbf{b}\,.$$

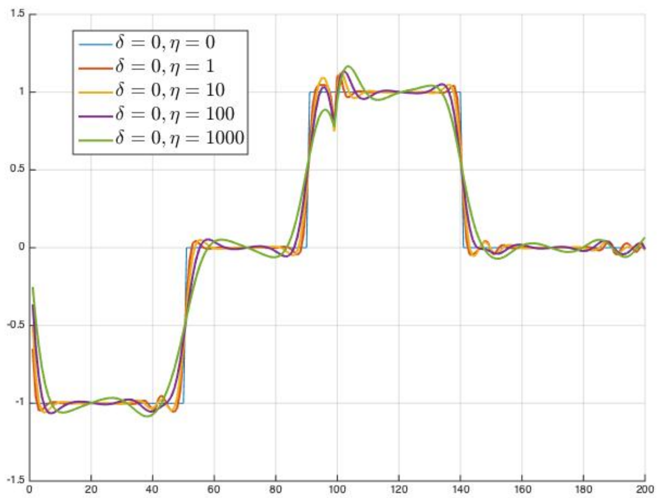
There exists a unique solution provided that $\text{null}(\mathbf{T}) \cap \text{null}(\mathbf{E}) \cap \text{null}(\mathbf{S}) = \{0\}$, which is satisfied immediately if $\delta > 0$. In the case $\delta = 0$, we note that the nullspace of $\mathbf{S}$ is non-trivial, as it includes constant and vectors of the type $v_i = i$ (which would be need to checked against $\text{null}(\mathbf{T})$). However, if $w_1 \neq 0$, then $\text{null}(\mathbf{T}) = \{0\}$. **Marks:** 2 marks for correct matrices, 1 mark for problem solution, 2 marks for existence and uniqueness discussion, which should be based around δ, η, w_1 and kernel conditions.



2.2.1

Figure 2: As δ increases, the minimization of $\delta||\mathbf{u}||^2$ takes over, which has a unique global minimum at $\mathbf{u} = \mathbf{0}$. When $||\mathbf{u}|| \longrightarrow 0$, the tracking term $f_t \longrightarrow ||\bar{\mathbf{z}}||^2$ and $\mathcal{J}(\mathbf{u}^*) \longrightarrow ||\bar{\mathbf{z}}||^2 = 100$ (direct by inspection of $\bar{\mathbf{z}}$).

Marks: 1 mark for correct plot, 1 mark for discussion, which should observe some asymptotic behaviour of $\mathcal{J}$.



2.2.2

Figure 3: As η increases, the smoothening effect of the regularizer becomes evident. The discontinuities of the reference signal are smeared out. In the absence of regularizers ($\delta = \eta = 0$), we are left with a linear least squares problem where $\mathbf{T}$ is invertible, and therefore the reference signal can be recovered exactly.

Marks: 1 mark for correct plot, 1 mark for discussion (smoothing effect and recovery of LLS solution if $\delta = \eta = 0$.)

2.3 As discussed during the linear least squares lectures, the denoising problem is written as

$$\min_{\mathbf{z}\in\mathbb{R}^n} \|\hat{\mathbf{y}} - \mathbf{z}\|^2 + \lambda\|\mathbf{S}\mathbf{z}\|^2\,, \quad \lambda > 0\,,$$

where $\mathbf{S}$ corresponds to $\mathbf{S}$ in f_s. The denoising problem for f_{tv} reads

$$\min_{\mathbf{z}\in\mathbb{R}^n} \|\hat{\mathbf{y}} - \mathbf{z}\|^2 + \lambda\|\mathbf{L}\mathbf{z}\|^2\,, \quad \lambda > 0\,,$$

where $\mathbf{L} \in \mathbb{R}^{n-1\times n}$ is the matrix

$$\mathbf{L} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

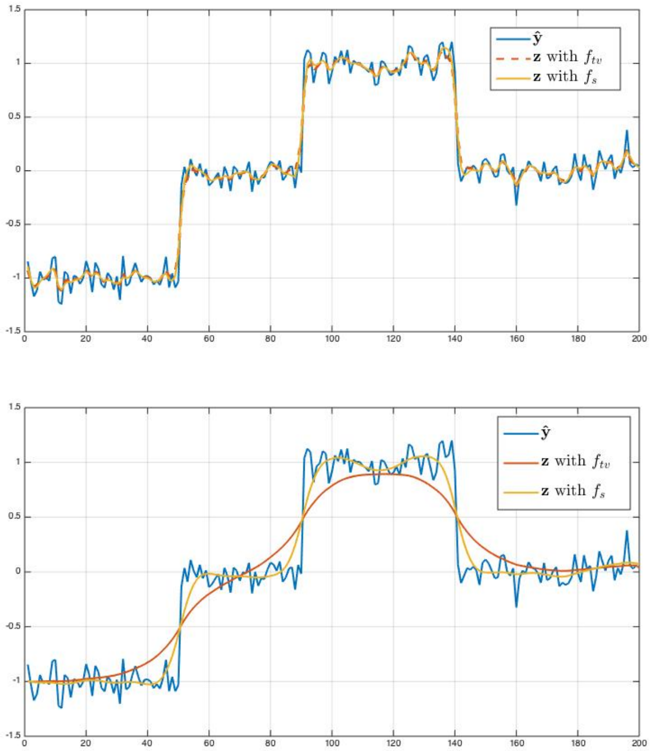


Figure 4: For $\lambda = 1$, we can see that both regularized signals sightly denoise $\hat{\mathbf{y}}$. Instead, for $\lambda = 100$ we observe that the regularization imposed by f_s is excessive, whereas f_{tv} shows a better balance between denoising and fitting with respect to $\hat{\mathbf{y}}$.

For a generic denoising problem of the form

$$\min_{\mathbf{z}\in\mathbb{R}^n} \|\hat{\mathbf{y}} - \mathbf{z}\|^2 + \lambda\|\mathbf{R}\mathbf{z}\|^2\,, \quad \lambda > 0\,,$$

it suffices to express it as a quadratic function. Defining $f(\mathbf{z}) := \|\hat{\mathbf{y}} - \mathbf{z}\|^2 + \lambda \|\mathbf{R}\mathbf{z}\|^2$, we have

$$\begin{aligned} f(\mathbf{z}) &= (\hat{\mathbf{y}} - \mathbf{z})^\top (\hat{\mathbf{y}} - \mathbf{z}) + \lambda \mathbf{z}^\top \mathbf{R}^\top \mathbf{R}\mathbf{z}\,, \\ &= \mathbf{z}^\top (\mathbf{I} + \lambda \mathbf{R}^\top \mathbf{R})\mathbf{z} - 2\hat{\mathbf{y}}\mathbf{z} + \|\mathbf{y}\|^2\,, \end{aligned}$$

and therefore, as seen in lectures, we can use exact linesearch with the stepsize given by t=

$$\frac{\|\nabla f(\mathbf{z})\|^2}{2\nabla f(\mathbf{z})^\top A \nabla f(\mathbf{z})}$$

where $A := (\mathbf{I} + \lambda \mathbf{R}^\top \mathbf{R})$ and $\nabla f(\mathbf{z}) = 2(A\mathbf{z} + \mathbf{b})$, where $\mathbf{b} = -\hat{\mathbf{y}}$.

Marks: 3 marks for the first part (statement of the denoising problem and plots) and 2 marks for the line search iteration. For the line search, it was allowed to link the exact formula form the live session and apply it to this particular problem.