Exercise 10.1. Show that any convergent sequence in a metric space, is a Cauchy sequence.

Hint: Adapt the proof of the same statement for the sequences of real numbers.

Exercise 10.2. Let (X, d) be a metric space, and assume that $(x_n)_{n\geq 1}$ is a Cauchy sequence in X. If there is a subsequence of $(x_n)_{n\geq 1}$ which converges to some $x \in X$, then the sequence $(x_n)_{n\geq 1}$ converges to x.

Hint: Adapt the proof of the same statement for the sequences of real numbers.

Exercise 10.3. Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there is $K > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous.

Hint: Show that for ϵ one can use $\delta = \epsilon/K$.

Exercise 10.4. Let $x_1 = \sqrt{2}$, and define the sequence $(x_n)_{n\geq 1}$ according to

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}.$$

Show that the sequence $(x_n)_{n\geq 1}$ converges to a root of the equation

$$x^4 - 4x^2 - x + 4 = 0$$

which lies in the interval $[\sqrt{3}, 2]$.

Hint: Work with the function $f(x) = \sqrt{2 + \sqrt{x}}$ on the interval $[\sqrt{3}, 2]$.

Exercise 10.5. Consider the map $f : (0, 1/3) \to (0, 1/3)$, defined as $f(x) = x^2$. Show that the map f is a contraction with respect to the Euclidean metric d_1. But, f has no fixed point in $(0, 1/3)$.

Hint: you may use the formula $x^2 - y^2 = (x - y)(x + y)$.

Exercise 10.6. Consider the map $f : [1, \infty) \to [1, \infty)$ defined as $f(x) = x + 1/x$. Show that $([1, +\infty), \mathrm{d}_1)$ is a complete metric space, and for all x and y in $[1, \infty)$ we have

$$\mathrm{d}_1(f(x), f(y)) \leq \mathrm{d}(x, y).$$

But, f has no fixed point.

Hint:You may use $f' < 1$ on $[1, +\infty)$.

Unseen Exercise. (unseen) Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there are $K > 0$ and $\alpha > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|^{\alpha}.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous. A function f satisfying this inequality for some K and α, is called a holder function (or an α-holder function).

Hint: Show that for ϵ one can use $\delta = (\epsilon/K)^{1/\alpha}$.

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Questions marked with $*$ are optional

Unseen Exercise. Show that the metrics d_1, d_2 and d_∞ on $C([a,b])$ satisfy the following inequalities. For all f and g in $C([a,b])$, we have

$$d_1(f,g) \leq d_2(f,g)\sqrt{b-a},$$

and

$$d_2(f,g) \leq d_\infty(f,g)(b-a)^2.$$

Conclude that if $(f_n)_{n\geq 1}$ converges to f in d_∞, it also converges in d_2. Similarly, if $(f_n)_{n\geq 1}$ converges to f in d_2, it also converges in d_1.