Problem Sheet 3, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let $S \subset \mathbb{R}$ be a *helicoid*, that is, the surface parametrised by

$$\phi(u, v) = (u\sin(v), -u\cos(v), v).$$

Compute the Gaussian and mean curvatures of S at each point $\phi(u, v)$.

Solution: We aim to use Prop 12.1 to calculate the Gaussian and mean curvatures.

The map ϕ is injective and smooth, with partial derivatives

$$\phi_u(u, v) = (\sin(v), -\cos(v), 0), \quad \phi_v(u, v) = (u\cos(v), u\sin(v), 1)$$

which are linearly independent at each (u, v). Thus, ϕ is a (global) chart for S. The unit normal vector to S is given by

$$N(u, v) = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}(u, v) = \frac{(-\cos(v), -\sin(v), u)}{(1 + u^2)^{1/2}}.$$

We also have

$$\phi_{uu}(u, v) = (0, 0, 0), \quad \phi_{uv}(u, v) = (\cos(v), \sin(v), 0), \quad \phi_{vv}(u, v) = (-u\sin(v), u\cos(v), 0).$$

Then, the first fundamental form at $\phi(u, v)$ is

$$g = \begin{pmatrix} \langle \phi_u(u, v), \phi_u(u, v) \rangle & \langle \phi_u(u, v), \phi_v(u, v) \rangle \\ \langle \phi_v(u, v), \phi_u(u, v) \rangle & \langle \phi_v(u, v), \phi_v(u, v) \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + u^2 \end{pmatrix}$$

and the second fundamental form at $\phi(u, v)$ is

$$A = \begin{pmatrix} \langle N(u, v), \phi_{uu}(u, v) \rangle & \langle N(u, v), \phi_{uv}(u, v) \rangle \\ \langle N(u, v), \phi_{uv}(u, v) \rangle & \langle N(u, v), \phi_{vv}(u, v) \rangle \end{pmatrix} = \frac{1}{(1 + u^2)^{1/2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

from which we compute

$$\sigma = g^{-1}A = \frac{1}{(1 + u^2)^{3/2}} \begin{pmatrix} 1 + u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \frac{-1}{(1 + u^2)^{3/2}} \begin{pmatrix} 0 & 1 + u^2 \\ 1 & 0 \end{pmatrix}$$

Then, by Prop 12.1,

$$K = \det(\sigma) = \frac{1}{(1 + u^2)^2}, \qquad H = \frac{1}{2}\operatorname{tr}(\sigma) = 0.$$

Problem 2. Let $S \subset \mathbb{R}^3$ be the graph of $z = f(x, y)$, where f is a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. Compute the Gaussian curvature of S at each point $(x, y, f(x, y))$.

Solution: We aim to use prop 12.1. We use a chart $\phi : \mathbb{R}^2 \to S$ of the form $\phi(u, v) = (u, v, f(u, v))$. Then,

$$\phi_u = (1, 0, f_u), \qquad \phi_v = (0, 1, f_v),$$

and hence the normal vector at $\phi(u, v)$ is

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-f_u, -f_v, 1)}{(1 + f_u^2 + f_v^2)^{1/2}}.$$

We also have

$$\phi_{uu} = (0, 0, f_{uu}), \quad \phi_{vv} = (0, 0, f_{vv}), \quad \phi_{uv} = \phi_{vu} = (0, 0, f_{uv}).$$

Then, the first fundamental form is

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}$$

and the second fundamental form is

$$A = \begin{pmatrix} \langle \phi_{uu}, N(\phi) \rangle & \langle \phi_{vu}, N(\phi) \rangle \\ \langle \phi_{uv}, N(\phi) \rangle & \langle \phi_{vv}, N(\phi) \rangle \end{pmatrix} = \frac{1}{(1 + f_u^2 + f_v^2)^{1/2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}.$$

Thus, at $\phi(x, y) = (x, y, f(x, y))$, we have

$$G = \frac{\det(A)}{\det(g)} = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2}.$$

Problem 3. Let $\gamma(t) = (x(t), z(t))$ be a plane curve (in the xz-plane) parametrised by arc length, with $x(t) > 0$ for all $t \in \mathbb{R}$. Let $S \subset \mathbb{R}^3$ denote the surface of revolution formed by rotating $\gamma(\mathbb{R})$ about the z-axis.

(a) Using the parametrisation

$$\phi(u, v) = (x(u) \cos(v), x(u) \sin(v), z(u))$$

of S, prove that the Gaussian curvature of S at a point $\phi(u, v)$ is given by

$$K(\phi(u, v)) = \frac{-x''(u)}{x(u)}.$$

(b) Characterise the planar points of S in terms of z and its derivatives.

Hint for part (a): at some point you may wish to differentiate the equation $(x')^2 + (z')^2 = 1$ to help simplify things, and prepare to do considerable calculations!

Solution: We compute the partial derivatives

$$\phi_u(u,v) = (x_u(u)\cos(v), x_u(u)\sin(v), z_u(u)), \quad \phi_v(u,v) = (-x(u)\sin(v), x(u)\cos(v), 0)$$

and hence the normal vector

$$\begin{aligned} N(\phi(u,v)) = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}(u,v) &= \frac{(-x(u)z_u(u)\cos(v), -x(u)z_u(u)\sin(v), x(u)x_u(u))}{\left((x(u)z_u(u))^2 + (x(u)x_u(u))^2\right)^{1/2}} \\ &= \frac{(-z_u(u)\cos(v), -z_u(u)\sin(v), x_u(u))}{(x_u^2(u) + z_u^2(u))^{1/2}}. \end{aligned}$$

We also have

$$\begin{aligned} \phi_{uu}(u,v) &= (x_{uu}(u)\cos(v), x_{uu}(u)\sin(v), z_{uu}(u)), \\ \phi_{uv}(u,v) &= (-x_u(u)\sin(v), x_u(u)\cos(v), 0), \\ \phi_{vv}(u,v) &= (-x(u)\cos(v), -x(u)\sin(v), 0). \end{aligned}$$

The first fundamental form is

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} = \begin{pmatrix} x_u^2 + z_u^2 & 0 \\ 0 & x^2 \end{pmatrix}$$

and the second fundamental form is

$$A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{uv} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix} = \frac{1}{(x_u^2 + z_u^2)^{1/2}} \begin{pmatrix} x_u z_{uu} - x_{uu} z_u & 0 \\ 0 & x z_u \end{pmatrix}.$$

Therefore,

$$\sigma = g^{-1} A = \frac{1}{(x_u^2 + z_u^2)^{1/2}} \begin{pmatrix} \frac{x_u z_{uu} - x_{uu} z_u}{x_u^2 + z_u^2} & 0 \\ 0 & \frac{z_u}{x} \end{pmatrix} = \begin{pmatrix} x_u z_{uu} - x_{uu} z_u & 0 \\ 0 & \frac{z_u}{x} \end{pmatrix}.$$

where in the above equation we have used $x_u^2 + z_u^2 = 1$. Thus, we get

$$K = \det(\sigma) = \frac{x_u z_u z_{uu} - x_{uu} z_u^2}{x}$$

and from differentiating $x_u^2 + z_u^2 = 1$ we get $x_u x_{uu} + z_u z_{uu} = 0$, so that

$$K = \frac{x_u(-x_u x_{uu}) - x_{uu} z_u^2}{x} = \frac{-x_{uu}(x_u^2 + z_u^2)}{x} = -\frac{x_{uu}}{x}.$$

For part (b), we recall that the planar points are the points where the principal curvatures are both zero. Thus, σ must be the zero matrix. In this case, we have $\frac{z_u}{x} = 0$, so $z_u = 0$, and then $0 = x_u z_{uu} - x_{uu} z_u = x_u z_{uu}$. If $z_u = 0$ then x_u is nonzero, since $x_u^2 + z_u^2 = 1$, so the last equation gives $z_{uu} = 0$. The conditions $z_u = 0$ and $z_{uu} = 0$ certainly suffice to give $\sigma = 0$, so we conclude that a point $p = \phi(u,v)$ is planar if and only if $z'(u) = z''(u) = 0$.

Problem 4. Let S be a regular surface and $C \subset S$ an *asymptotic line*, meaning that C is a regular curve whose normal curvature is zero.

(a) Prove that $K(p) \leq 0$ at all points $p \in C$,

(b) If the curvature of C is non-zero everywhere, then its torsion satisfies $|\tau(p)| = \sqrt{-K(p)}$.

Hint for part (a): parametrise the curve C by arc length, and use the definition of normal curvature in terms of the inner product with N, and show that $A(C', C') = 0$.

Hint for part (b): use the relation in part (a) to show that the normal to the curve n_C is orthogonal to the normal to the surface N. Then, think about the Frenet frames, and use the Frenet equations. This might be a difficult problem!

Solution: Part (a): Fix an arbitrary $p \in C$, and consider an arc-length parametrisation $\alpha : [a, b] \to C$ near p. The normal curvature at each point is given by $k_n = \langle kn, N \rangle$, where n is the unit normal vector to C and N is the unit normal vector to S at each point $\alpha(t)$. Since $k_n = 0$ and $\alpha''(t) = kn$, we have

$$\langle \alpha''(t), N(\alpha(t)) \rangle = 0, \quad \forall t \in [a, b]. \tag{1}$$

We now differentiate the equation $\langle \alpha'(t), N(\alpha(t)) \rangle = 0$ to get

$$\langle \alpha''(t), N(\alpha(t)) \rangle + \langle \alpha'(t), dN_{\alpha(t)} \left(\alpha'(t) \right) \rangle = 0.$$

The first term on the left hand side of the above equation is 0 by hypothesis, and the second term is $-A(\alpha'(t), \alpha'(t))$, so we must have $A(\alpha'(t), \alpha'(t)) = 0$ at each point $\alpha(t)$. If the principal curvatures at $\alpha(t)$ are $\lambda_1 \leq \lambda_2$, then since $|\alpha'(t)| = 1$ we have

$$\lambda_1 \leq A\left(\alpha'(t), \alpha'(t)\right) \leq \lambda_2$$

since $\lambda_1 = \min A(v, v)$ and $\lambda_2 = \max A(v, v)$ among all unit vectors v. Then, $\lambda_1 \leq 0$ and $\lambda_2 \geq 0$, and so $K = \lambda_1 \lambda_2 \leq 0$ at all points $\alpha(t)$ along C.

Part (b): Now if $k_\alpha(t) \neq 0$ for all t, we can divide the Equation (1) by $k_\alpha(t)$ to get

$$\langle n(t), N(\alpha(t)) \rangle = 0.$$

The unit tangent vector $T(t) = \alpha'(t)$ and normal vector $n(t)$ to C are both orthogonal to each other and to $N(\alpha(t))$, which is itself a unit vector, so we must have

$$N(\alpha(t)) = \pm(T(t) \times n(t)) \quad \Longrightarrow \quad B(t) = \pm N(\alpha(t)).$$

We differentiate this last equation and use the Frenet equation $B' = -\tau n$ to get

$$-\tau(t) n(t) = B'(t) = \pm dN_{\alpha(t)} \left(\alpha'(t) \right).$$

Taking inner products with $-n(t)$ gives $\tau(t) = \pm A(n(t), T(t))$. In the orthonormal basis $(T(t), n(t))$ of T_pS we can write $K = \det\left(g^{-1}A\right)$, where

$$g = \begin{pmatrix} \langle T, T \rangle & \langle T, n \rangle \\ \langle n, T \rangle & \langle n, n \rangle \end{pmatrix} = I, \quad A = \begin{pmatrix} A(T, T) & A(n, T) \\ A(T, n) & A(n, n) \end{pmatrix} = \begin{pmatrix} 0 & \pm\tau \\ \pm\tau & A(n, n) \end{pmatrix}$$

and so $K = -\tau^2$. This implies that $|\tau| = \sqrt{-K}$ as claimed.

Problem 5. Let S be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, with the chart $\phi : \mathbb{R}^2 \to S$ given by

$$\phi(u, v) = (u, v, f(u, v)).$$

Compute the Christoffel symbols Γ_{ij}^k.

Solution: We have $\phi_u = (1, 0, f_u)$ and $\phi_v = (0, 1, f_v)$, so the normal vector is

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-f_u, -f_v, 1)}{(1 + f_u^2 + f_v^2)^{1/2}} = \frac{(-f_u, -f_v, 1)}{(1 + |\nabla f|^2)^{1/2}}.$$

Let $x_1 = u$ and $x_2 = v$, and observe that

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \left(0, 0, \frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

Then, the entries of the matrix of the second fundamental form are

$$A_{ij} = \left\langle \frac{\partial^2 \phi}{\partial x_i \partial x_j}, N \right\rangle = \frac{\frac{\partial^2 f}{\partial x_i \partial x_j}}{(1 + |\nabla f|^2)^{1/2}}.$$

Then, the Christoffel symbols satisfy

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x_i \partial x_j} &= \Gamma_{ij}^1 \left(1, 0, \frac{\partial f}{\partial x_1}\right) + \Gamma_{ij}^2 \left(0, 1, \frac{\partial f}{\partial x_2}\right) + A_{ij} N \\ &= \left(\Gamma_{ij}^1 - \frac{\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial x_1}}{1 + |\nabla f|^2},\ \Gamma_{ij}^2 - \frac{\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial x_2}}{1 + |\nabla f|^2},\ \Gamma_{ij}^1 \frac{\partial f}{\partial x_1} + \Gamma_{ij}^2 \frac{\partial f}{\partial x_2} + \frac{\frac{\partial^2 f}{\partial x_i \partial x_j}}{1 + |\nabla f|^2}\right). \end{aligned}$$

The first two coordinates of $\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ are zero, which implies that

$$\Gamma_{ij}^k = \frac{\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial x_k}}{1 + |\nabla f|^2}.$$