MVC: Sheet 3 Hints, tips, answers

1/ For Q1 see page 3.

2/ This is equal to $\int_V \underline{\nabla} \cdot \underline{r}\, dV = 3V$.

3/ $\oint_S \frac{\underline{r} \cdot \underline{\hat{n}}}{r^2}\, dS = \int_V \underline{\nabla} \cdot \left(\frac{\underline{r}}{r^2}\right) dV = \dots = \int_V \frac{1}{r^2}\, dV$

4/ (i) First let $\underline{A} = \varphi(x,y,z)\hat{\imath}$; $\hat{n} = (l, m, n)$

then div thm $\Rightarrow \oint_S l\varphi\, dS = \int_\tau \frac{\partial \varphi}{\partial x}\, d\tau$

Then consider $\varphi\hat{\jmath}$ and $\varphi\hat{k}$ in turn to get similar results

Combine all 3 to get desired result.

(ii) Expand out $\underline{\hat{n}} \times \underline{A}$; surface integral is

$$\oint_S \hat{\imath}(mA_3 - nA_2) - \hat{\jmath}(lA_3 - nA_1) + \hat{k}(lA_2 - mA_1)\, dS$$

Then use results obtained in part (i) to turn this into $\int_\tau \text{Curl}\,\underline{A}\, d\tau$

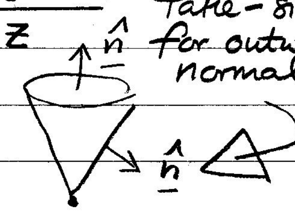
5/ We have $\underline{\nabla} \cdot \underline{A} = 1$ & so $\int_V \underline{\nabla} \cdot \underline{A}\, dV = \dots = 8a^3$.

The 4 faces of cube with normals in $\pm\hat{\jmath}, \pm\hat{k}$ directions have $\underline{A} \cdot \underline{\hat{n}} = 0$ and so give no contribution. On the faces with normals in the $\pm\hat{\imath}$ direction we have $\underline{A} = \pm a\hat{\imath}$. Adding these 2 contributions gives $4a^3 + 4a^3 = 8a^3$.

6/ $\text{div}\,\underline{A} = \dots = 3$ & so $\int_V \text{div}\,\underline{A}\, dV = 3 \times \text{volume of cone} = \dots = \pi$

On surface of cone, unit normal is $\pm \frac{(-x\hat{\imath} - y\hat{\jmath} + z\hat{k})}{\sqrt{2}\, z}$ take - sign for outward normal

Then we find $\underline{A} \cdot \underline{\hat{n}} = 0$ on cone surface.



Then we need contribution from flat cap.

Here $\underline{\hat{n}} = \underline{\hat{k}}$ & $\underline{A} \cdot \underline{\hat{n}} = 1 - y$

Switch to polars to do integral over disk radius 1.

Find that contribution is equal to π.

MVC: Sheet 3 Hints, tips, answers (ctd)

7/ In this case the closed curve (γ) for Stokes thm is the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $z=0$.

Now $\underline{A}\cdot d\underline{r}$ on $z=0$ is equal to $y\,dx - x\,dy$.

On the ellipse $x = a\cos\theta$, $y = b\sin\theta$ & so $\oint_\gamma \underline{A}\cdot d\underline{r} = \ldots = -2\pi ab$

(note that since $\hat{\underline{n}}$ is outward to S we should travel anti-clockwise around γ).

8/ $\text{Curl}\,\underline{A} = \ldots = \hat{k}$. Unit normal $\hat{\underline{n}} = \ldots = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})/a$.

Thus $\text{curl}\,\underline{A}\cdot\hat{\underline{n}} = \ldots = \cos\theta$.

Using the dS given in the question $\int_S \text{curl}\,\underline{A}\cdot\hat{n}\,dS = \ldots = \pi a^2$.

γ is a circle of radius a in the plane $z=0$ and

$\underline{A}\cdot d\underline{r} = (3x-y)\,dx$. Then use plane polars to get

$\oint_\gamma \underline{A}\cdot d\underline{r} = \ldots = \pi a^2$.

9/ In this case γ is circle around the top of cone i.e. $x^2+y^2=9$ in the plane $z=3$.

Then $\oint_\gamma \underline{A}\cdot d\underline{r} = \oint_\gamma -y\,dx + x\,dy \underset{\begin{pmatrix}\text{use polars}\\ x=3\cos\theta\\ y=3\sin\theta\end{pmatrix}}{=} \ldots = 18\pi$. (traversing γ anti-clockwise)

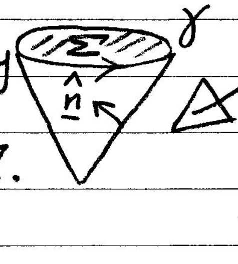
For the surface integral, $\text{curl}\,\underline{A} = -xz\hat{\imath} + yz\hat{\jmath} + 2\hat{k}$.

& $\hat{\underline{n}} = (-x\hat{\imath} - y\hat{\jmath} + z\hat{k})/(\sqrt{2}\,z)$ (think about sign!)

so that on S, $\text{curl}\,\underline{A}\cdot\hat{n} = \dfrac{1}{\sqrt{2}}(x^2 - y^2 + 2)$

To do integral, project onto $z=3$ where projected shape Σ is the circle $x^2+y^2=9$

Then $\ldots \int_S (\text{Curl}\,\underline{A}\cdot\hat{\underline{n}})\,dS = \ldots = \int_\Sigma (x^2-y^2+2)\,dx\,dy$



Then use plane polars $x = r\cos\theta$, $y = r\sin\theta$ to cover Σ.

$\ldots = 18\pi$.

MVC : Sheet 3 Hints, tips, answers (ctd)

1/ Applying div thm $\Rightarrow \int_V \underline{\nabla}\cdot(\varphi\underline{A})\,dV = 0$
then use $\underline{\nabla}\cdot(\varphi\underline{A}) \equiv \underline{A}\cdot\underline{\nabla}\varphi + \varphi \operatorname{div}\underline{A} \leftarrow$ (=0 if $\underline{A}$ Solenoidal)

In 2D $\int_R \underline{\nabla}\cdot(\varphi\underline{A})\,dx\,dy = \oint_C \varphi\underline{A}\cdot\hat{\underline{n}}\,ds$ (= 0 since $\varphi = 0$ on C)

Then proceed as in 3D

10/ Proceed as in Q9. This time the projection onto $z=0$ is an annulus, For the path integral there are two boundaries to consider.