Introduction to Quantum Mechanics – Problem sheet 3

1. The Dirac notation

(a) For each of the following expressions decide whether it is a scalar, a ket, a bra, or an operator and give the adjoint expression

(i) $\langle\phi|\chi\rangle\langle\phi|$

(ii) $\langle\phi|\hat{A}|\chi\rangle\langle\phi|\hat{A}$

(iii) $c\langle\chi|\phi\rangle$

(iv) $c|\phi\rangle\langle\chi|$

(v) $\langle\chi|\hat{A}^{\dagger}|\phi\rangle$

where c is a scalar, and $\hat{A}$ is an operator.

(b) Consider a three-dimensional complex Hilbert space with an orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$. Consider the states $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{i}|\phi_1\rangle + |\phi_3\rangle\right)$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + \mathrm{i}|\phi_3\rangle\right)$.

(i) Verify that the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other and normalised. Find a third state $|\psi_3\rangle$ that is orthogonal to both $|\psi_1\rangle$ and $|\psi_2\rangle$ and normalised.

(ii) Verify that the three states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ are a complete set by verifying that they form a resolution of the identity, both in Dirac and traditional vector representation in $\mathbb{C}^3$.

(iii) Expand the vector $|\chi\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}|\phi_1\rangle + |\phi_2\rangle)$ in the basis $\{|\psi_{1,2,3}\rangle\}$.

2. Basis invariance of the trace

The trace of a matrix operator $\hat{O}$ on $\mathbb{C}^N$ is given by $Tr(\hat{O}) = \sum_{j=1}^{N}\langle b_j|\hat{O}|b_j\rangle$, where $\{|b_j\rangle\}$ is an orthonormal basis.

(a) Show that the trace is basis independent, i.e. that $Tr(\hat{O}) = \sum_j \langle d_j|\hat{O}|d_j\rangle$, if $\{|d_j\rangle\} \neq \{|b_j\rangle\}$ is another orthonormal basis.

(b) Show that the trace is cyclic, that is, $Tr(\hat{A}\hat{B}\hat{C}) = Tr(\hat{C}\hat{A}\hat{B}) = Tr(\hat{B}\hat{C}\hat{A})$.

(c) Use the Trace to show that there are no matrices fulfilling $[\hat{A}, \hat{B}] = c\hat{I}$, where $\hat{I}$ denotes the identity and c is a non-zero scalar.

3. The spectral theorem

In the matrix representation of an operator $\hat{A}$, the matrix elements of the matrix are given by $A_{jk} = \langle j|\hat{A}|k\rangle$, where $\{|j\rangle\}$ are an orthonormal basis.

(a) Show that every finite-dimensional operator can be expressed as $\hat{A} = \sum_{jk} A_{jk}|j\rangle\langle k|$.

(b) Show that a finite dimensional Hermitian operator can be expressed as $\hat{A} = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|$ where $\hat{A}|\phi_n\rangle = \lambda_n|\phi_n\rangle$.

4. Hermiticity in L^2

Consider the Hilbert space of square integrable functions, L^2, equipped with the standard inner product of $(\phi(x), \chi(x)) = \int_{-\infty}^{+\infty}\phi^*(x)\chi(x)\mathrm{d}x$. Verify that the operator $\hat{K} : \psi(x) \mapsto -\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}$ is Hermitian on the space of L^2 functions that are twice differentiable and whose derivatives vanish at infinity faster than any polynomial.