Assessed Coursework 1

You may discuss these problems with other students, but you must write up your own solutions.

Problem 1. Let $\phi : [a, b] \to \mathbb{R}^n$ (n = 2 or 3) be a regular curve, and suppose that $|\phi(t)|$ realises its maximum value at some $t_0 \in (a, b)$. Prove that

$$k(t_0) \geq \frac{1}{|\phi(t_0)|}.$$

Solution: We may assume that ϕ is parametrised by arc length, since the conclusion does not depend on the parametrisation. The function $|\phi(t)|$ is maximised at $t = t_0$ if and only if the function $f(t) = |\phi(t)|^2 = \langle \phi(t), \phi(t) \rangle$ is maximised at $t = t_0$. In the latter case, we must have $f'(t_0) = 0$ and $f''(t_0) \leq 0$. We have $f'(t) = 2\langle \phi'(t), \phi(t) \rangle$ and hence

$$f''(t_0) = 2\left(\langle \phi''(t_0), \phi(t_0) \rangle + \langle \phi'(t_0), \phi'(t_0) \rangle \right) \leq 0$$

Since ϕ is parametrised by arc-length, we have $|\phi'(t_0)|^2 = 1$, so the above inequality give us

$$\langle \phi''(t_0), \phi(t_0) \rangle \leq -1.$$

Using $k(t_0) = |\phi''(t_0)|$, we get

$$k(t_0)|\phi(t_0)| = |\phi''(t_0)||\phi(t_0)| \geq \left| \langle \phi''(t_0), \phi(t_0) \rangle \right| \geq 1.$$

Problem 2. Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parametrised by arc length, and with curvature k and torsion τ. Assume that there is a constant $c \in \mathbb{R}$ such that for all $t \in [a, b]$ we have $\tau(t) = ck(t)$ and $k(t) > 0$. Prove that the tangent vectors of ϕ make a fixed angle θ with some fixed vector $v \in \mathbb{R}^3$, for all values of t.
Hint: First find a linear combination of the vectors in the Frenet frame which is independent of t.

Solution: First we look for a linear combination of the Frenet frames which is independent of t. Using the Frenet equations, for real numbers e_1, e_2, e_3, we note that

$$\begin{aligned} \frac{d}{dt}\Big(e_1 T(t) + e_2 N(t) + e_3 B(t) \Big) &= e_1(k(t)N(t)) + e_2(\tau(t)B(t) - k(t)T(t)) + e_3(-\tau(t)N(t)) \\ &= -e_2 k(t)T(t) + (e_1 k(t) - e_3\tau(t))\, N(t) + e_2\tau(t)B(t). \end{aligned}$$

The right hand side of the above equation becomes zero if we take $(e_1, e_2, e_3) = (c, 0, 1)$, where c is the constant in the hypothesis. Thus, the vector $cT(t) + B(t)$ is constant, and has length $\sqrt{c^2 + 1}$. We then see that the angle θ between T and the vector $v = cT(t) + B(t)$ satisfies

$$\cos(\theta) = \frac{\langle T(t), cT(t) + B(t) \rangle}{|T(t)||cT(t) + B(t)|} = \frac{c}{\sqrt{c^2 + 1}}$$

which is clearly independent of t.

Problem 3. Consider the ellipse

$$\gamma(t) = (a\cos(t), b\sin(t)), \quad t \in \mathbb{R},$$

where $0 < a < b$. Calculate the curvature of this curve at each point on the curve.

Solution: One may (try to) reparameterise the curve γ in terms of arc-length and then calculate the curvature using that change of coordinate. However, it is convenient to use the formula in Proposition 4.2 to calculate the signed curvature, and take take its absolute value.
We have

$$\gamma'(t) = (-a\sin t, b\cos t),$$

and the normal to the curve at the point $\gamma(t)$ is

$$n(t) = \frac{(-b\cos t, -a\sin t)}{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}.$$

Thus,

$$\kappa(t) = \frac{\langle \gamma''(t), n(t)\rangle}{|\gamma'(t)|^2} = \frac{\langle(-a\cos t, -b\sin t), (-b\cos t, -a\sin t)\rangle}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

Since the right hand side of the above equation is positive, it is the curvature of γ at $\gamma(t)$.

Problem 4. Prove that any regular surface $S \subset \mathbb{R}^3$ is locally a regular level set, that is, for every $p \in S$ there is an open set $V \subset \mathbb{R}^3$ containing p, and a smooth function $F : V \to \mathbb{R}$ such that $S \cap V = F^{-1}(0)$ and ∇F is non-zero at every point in $V \cap S$.

Is it true that every regular surface in $\mathbb{R}^3$ is the regular level set of a smooth function on $\mathbb{R}^3$?

Solution: Let S be an arbitrary regular surface, and let $p \in S$ be arbitrary. By a result in the lectures, there is a neighbourhood of p such that $V \cap S$ is the graph of a smooth function of (x, y), or (y, z), or (x, y). Without loss of generality, let us assume that it is a function of (x, y). Thus, there is an open set U in the xy plane, such that

$$V \cap S = \{(x, y, z) \in V \mid (x, y) \in U, z = f(x, y)\}.$$

Let us define the function $G : V \to \mathbb{R}$, as

$$G(x, y, z) = f(x, y) - z.$$

Clearly, this is a smooth function, since f is. Also, $G^{-1}(0) = S \cap V$. We need to show that $\nabla G(x, y, z) \neq 0$ for all $(x, y, z) \in S \cap V$. We see that

$$\nabla G(x, y, z) = (f_x(x, y), f_y(x, y), 1),$$

which is clearly non-zero.

Problem 5. Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$. We say that S_1 and S_2 are **diffeomorphic**, if there is a bijective map $f : S_1 \to S_2$ such that both $f : S_1 \to S_2$ and $f^{-1} : S_2 \to S_1$ are smooth.

i) Are the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

and the surface

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0, z = 1/(x^2 + y^2)\}$$

diffeomorphic? Justify your answer.

ii) The surfaces

$$S_1 = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 < 1, z = 0\right\},$$
$$S_2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 < 1, z = \sqrt{x^2 + y^2}\right\}$$

diffeomorphic? Justify your answer.

[You do not need to prove that these sets are regular surfaces in $\mathbb{R}^3$, although it may follow from the proofs you present for the diffeomorphic property.]

Solution: i) Let us consider the map

$$G : \mathbb{R}^3 \to \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$$

defined as

$$G(x, y, z) = (e^{-z/2}x, e^{-z/2}y, e^z).$$

It is easy to see that G is a bijection. Moreover, G is a smooth map since all its partial derivatives are defined and continuous. The Jacobian of G is

$$\begin{pmatrix} e^{-z/2} & 0 & 0 \\ 0 & e^{-z/2} & 0 \\ \frac{-x}{2}e^{-z/2} & \frac{-y}{2}e^{-z} & e^z \end{pmatrix}$$

The determinant of the above matrix is non-zero at every point in (x, y, z) in $\mathbb{R}^3$. Thus, by the inverse function theorem, the inverse of G is smooth near each point.
Now, we note that if (x, y, z) belongs to C, we have $x^2 + y^2 = 1$ and $z \in \mathbb{R}$ is arbitrary. Then, $G(x, y, z)$ belongs to D, since

$$1/(e^{-z}x^2 + e^{-z}y^2) = 1/(e^{-z}) = e^z.$$

Indeed, G is a bijection from C to D.
Let us show that $G : C \to D$ is smooth. Let $p \in C$ be an arbitrary point, and let $\phi : U \to C$ be a chart for C at p. The map $G \circ \phi : U \to \mathbb{R}^3$ is a composition of two smooth maps on open sets in Euclidean spaces. Thus, it is smooth.
Similarly, using the smoothness of G^{-1} on $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ one can see that G^{-1} is smooth at every point in D.

ii)

For this pat, we may consider the map

$$G : \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0\} \to \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0\}$$

defined as

$$G(x, y, z) = (x, y, \sqrt{x^2 + y^2} + z).$$

As in item (i), one can see that G is a smooth map with a smooth inverse. Moreover, it maps S_1 to S_2. An identical argument shows that G is a smooth map from S_1 to S_2. More details left to the reader.