

Topic: Probability and conditional probability

In today's problem class we will be reviewing the probability axioms and we will study problems involving conditional probabilities.

1. Given events $E, F, G \subseteq \Omega$, prove that

$$(a) \quad P(E^c \cap F) = P(F) - P(E \cap F)$$

$$(b) \quad P(E \cup F) \leq P(E) + P(F)$$

$$(c) \quad E \subseteq F, F \subseteq G \implies P(E) \leq P(G)$$

$$(d) \quad P(E \cap F) \geq P(E) + P(F) - 1$$

[(d) is known as *Bonferroni's Inequality*.]

2. Suppose that E and F are events such that $P(E) = x$, $P(F) = y$ and $P(E \cap F) = z$. Express the following terms in terms of x , y and z :

$$(a) \quad P(E^c \cup F^c)$$

$$(b) \quad P(E^c \cap F)$$

$$(c) \quad P(E^c \cup F)$$

$$(d) \quad P(E^c \cap F^c)$$

3. A crime has been committed and a suspect is being held by police. He is either guilty, G , or not, G^c , and the probability of his being guilty on the basis of current evidence is $P(G) = p$, say. Forensic evidence is now produced which shows that the criminal must have a property, A , which occurs in a proportion, π , of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population, so that $P(A|G^c) = \pi$.

The suspect is now interrogated and found to have property A. Show that the odds on his guilt have now risen from $\frac{P(G)}{P(G^c)} = p/(1-p)$ to $\frac{P(G|A)}{P(G^c|A)} = \frac{P(G)}{\pi P(G^c)}$.

Note: The odds on an event E are defined to be the ratio $P(E)/P(E^c)$, the odds-against E are $P(E^c)/P(E)$.

4. A shop sells fuses produced by three manufacturers; each manufacturer supplies a deluxe and a standard type of fuse. A mixed batch of 500 fuses is sold, and the number of faulty fuses of each type and for each manufacturer is recorded. By considering the following events; $M_i \equiv$ “fuse produced by manufacturer i ” for $i = 1, 2, 3$, $D \equiv$ “Deluxe type of fuse” and $F \equiv$ “Fuse Faulty”, a summary of the data can be presented as a 3-way table

	M_1		M_2		M_3	
	D	D^c	D	D^c	D	D^c
F	20	16	30	20	15	10
F^c	100	64	120	30	60	15

so that, for example, the number of deluxe fuses from manufacturer 1 that are faulty is 20, whereas the number of standard fuses from manufacturer 1 that are faulty is 16, etc.

- (a) A fuse is selected with equal probability from the 500. What is the probability that
 - i. it is faulty?
 - ii. it was produced by manufacturer 1?
- (b) Given that the selected fuse is faulty, what is the conditional probability that
 - i. it is a deluxe fuse?
 - ii. it is a fuse produced by manufacturer 1?
 - iii. it is a deluxe fuse produced by manufacturer 1?
- (c) Describe, evaluate, and comment on the following conditional probabilities:
 - i. $P(F|M_1)$, $P(F|M_2)$, $P(F|M_3)$
 - ii. $P(F|D)$, $P(F|D^c)$
 - iii. $P(F|M_1 \cap D)$, $P(F|M_2 \cap D)$, $P(F|M_3 \cap D)$.
 - iv. $P(F|M_1 \cap D^c)$, $P(F|M_2 \cap D^c)$, $P(F|M_3 \cap D^c)$.