Exercise 10.1. Show that any convergent sequence in a metric space, is a Cauchy sequence.

  Hint: Adapt the proof of the same statement for the sequences of real numbers.

Solution: Let $(x_n)_{n \geq 1}$ be a sequence in X which converges to $x \in X$. Fix an arbitrary $\epsilon > 0$. By the definition of convergence of sequences, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathrm{d}(x_n, x) < \epsilon/2$. Therefore, by the triangle inequality, for all $m, n \geq N$ we obtain

$$\mathrm{d}(x_n, x_m) \leq \mathrm{d}(x_n, x) + \mathrm{d}(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $(x_n)_{n \geq 1}$ is Cauchy.

Exercise 10.2. Let (X, d) be a metric space, and assume that $(x_n)_{n \geq 1}$ is a Cauchy sequence in X. If there is a subsequence of $(x_n)_{n \geq 1}$ which converges to some $x \in X$, then the sequence $(x_n)_{n \geq 1}$ converges to x.

  Hint: Adapt the proof of the same statement for the sequences of real numbers.

Solution: Let $(x_{n_k})_{k \geq 1}$ be a subsequence of $(x_n)_{n \geq 1}$ which converges to some $x \in X$. Fix $\epsilon > 0$. There is $N \in \mathbb{N}$ such that if $k \geq N$ we have $\mathrm{d}(x_{n_k}, x) < \epsilon/2$. On the other hand, there is $M \in \mathbb{N}$ such that if $m, n \geq M$, $\mathrm{d}(x_m, x_n) < \epsilon/2$. For every $n \geq M$, we may choose $k \geq N$ such that $n_k \geq M$. Then, by the triangle inequality,

$$\mathrm{d}(x_n, x) \leq \mathrm{d}(x_n, x_{n_k}) + \mathrm{d}(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $(x_n)_{n \geq 1}$ converges to x.

Exercise 10.3. Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there is $K > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous.

  Hint: Show that for ϵ one can use $\delta = \epsilon/K$.

Solution: Fix $\epsilon > 0$. Let $\delta = \epsilon/K$. For all $f \in \mathcal{C}$, and all x and y in $[a, b]$, if $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \epsilon.$$

This means that the family $\mathcal{C}$ is uniformly equi-continuous.

Exercise 10.4. Let $x_1 = \sqrt{2}$, and define the sequence $(x_n)_{n \geq 1}$ according to

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}.$$

Show that the sequence $(x_n)_{n \geq 1}$ converges to a root of the equation

$$x^4 - 4x^2 - x + 4 = 0$$

which lies in the interval $[\sqrt{3}, 2]$.

  Hint: Work with the function $f(x) = \sqrt{2 + \sqrt{x}}$ on the interval $[\sqrt{3}, 2]$.

Solution: Consider the map

$$f(x) = \sqrt{2 + \sqrt{x}}, \quad \forall x \in [\sqrt{3}, 2].$$

First we note that f maps $[\sqrt{3}, 2]$ into $[\sqrt{3}, 2]$. That is because, $f(\sqrt{3}) \geq \sqrt{3}$, $f(2) \leq 2$, and f is an increasing function. More precisely, for all $t \in [\sqrt{3}, 2]$ we have

$$\sqrt{3} \leq f(\sqrt{3}) \leq f(t) \leq f(2) \leq 2,$$

and hence $f(t) \in [\sqrt{3}, 2]$.

Now we show that f is contracting on the interval $[\sqrt{3}, 2]$. By a simple calculation we see that for all $x \in [\sqrt{3}, 2]$, we have

$$f'(x) = \frac{1}{4} \frac{1}{\sqrt{2 + \sqrt{x}}} \frac{1}{\sqrt{x}} \leq \frac{1}{4}.$$

Therefore, for all x and y in $[\sqrt{3}, 2]$, we have

$$|f(x) - f(y)| = \left| \int_x^y f'(t)\, dt \right| \leq \left| \int_x^y |f'(t)| dt \right| \leq \frac{1}{4}|x - y|.$$

This shows that f is uniformly contracting on $[\sqrt{3}, 2]$.

By definition, $x_{n+1} = f(x_n)$, the sequence $(x_n)_{n \geq 1}$ is contained in $[\sqrt{3}, 2]$. Since $(\mathbb{R}, d_1)$ is a complete metric space, and $[\sqrt{3}, 2]$ is closed in $(\mathbb{R}, d_1)$, we conclude that $[\sqrt{3}, 2]$ is a complete metric space with respect to the induced metric. By the argument in the proof of the Banach fixed point theorem, the sequence $(x_n)_{n \geq 1}$ is a Cauchy sequence. Therefore, it must converge to some limit in $[\sqrt{3}, 2]$. Moreover, the limit of the sequence, is the unique fixed point of the function f in $[\sqrt{3}, 2]$. Thus, we must have

$$x = \sqrt{2 + \sqrt{x}}$$

which implies that $x^2 = 2 + \sqrt{x}$, and hence $(x^2 - 2)^2 = x$, and hence the relation in the exercise.

Exercise 10.5. Consider the map $f : (0, 1/3) \to (0, 1/3)$, defined as $f(x) = x^2$. Show that the map f is a contraction with respect to the Euclidean metric d_1. But, f has no fixed point in $(0, 1/3)$.

Hint: you may use the formula $x^2 - y^2 = (x - y)(x + y)$.

Solution: For all x and y in $(0, 1/3)$, we have

$$|f(x) - f(y)| = |(x - y)(x + y)| < \frac{2}{3}|x - y|$$

Thus, f us contracting. However, since for all $x \in (0, 1/3)$, $f(x) < x$, f does not have a fixed point in $(0, 1/3)$. One cannot apply the Banach Fixed point theorem here since the interval $(0, 1/3)$ is not complete.

Exercise 10.6. Consider the map $f : [1, \infty) \to [1, \infty)$ defined as $f(x) = x + 1/x$. Show that $([1, +\infty), d_1)$ is a complete metric space, and for all x and y in $[1, \infty)$ we have

$$d_1(f(x), f(y)) \leq d(x, y).$$

But, f has no fixed point.

Hint: You may use $f' < 1$ on $[1, +\infty)$.

Solution: Any Cauchy sequence in $([1, +\infty), d_1)$ is a Cauchy sequence in $(\mathbb{R}, d_1)$. But the latter space is complete so that the sequence converges to some $x \in (\mathbb{R}, d_1)$. This x also belongs to $[1, +\infty)$ since this set is closed. Thus $([1, +\infty), d_1)$ is complete.

We note that for all x and y in $[1, +\infty)$, we have

$$
\begin{aligned}
|f(x) - f(y)| &= \left| x - y + \frac{1}{x} - \frac{1}{y} \right| \\
&= \left| x - y - \frac{x - y}{xy} \right| \\
&\leq |x - y| \cdot \left| 1 - \frac{1}{xy} \right| \\
&\leq |x - y|.
\end{aligned}
$$

Obviously, $f(x)$ has no fixed point, since for all x in $[1, +\infty$ we have $x \neq x + 1/x$. Note that the Banach Fixed Point Theorem cannot be applied here since there is no $K \in (0, 1)$ such that $|f(x) - f(y)| \leq K|x - y|$.

Unseen Exercise. (unseen) Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there are $K > 0$ and $\alpha > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|^\alpha.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous. A function f satisfying this inequality for some K and α, is called a holder function (or an α-holder function).

Hint: Show that for ϵ one can use $\delta = (\epsilon/K)^{1/\alpha}$.

Solution: Fix $\epsilon > 0$. Let $\delta = (\epsilon/K)^{1/\alpha}$. For all $f \in \mathcal{C}$, and all x and y in $[a, b]$, if $|x - y| < \delta$, we have
$$|f(x) - f(y)| \leq K|x - y|^\alpha < K\delta^\alpha = \epsilon.$$

This means that the family $\mathcal{C}$ is uniformly equi-continuous.

Unseen Exercise. Show that the metrics d_1, d_2 and d_∞ on $C([a, b])$ satisfy the following inequalities. For all f and g in $C([a, b])$, we have

$$d_1(f, g) \leq d_2(f, g)\sqrt{b - a},$$

and

$$d_2(f, g) \leq d_\infty(f, g)(b - a)^2.$$

Conclude that if $(f_n)_{n \geq 1}$ converges to f in d_∞, it also converges in d_2. Similarly, if $(f_n)_{n \geq 1}$ converges to f in d_2, it also converges in d_1.

Solution: We have

$$
\begin{aligned}
d_1(f, g) &= \int_a^b |f(t) - g(t)| \, dt \\
&= \int_a^b |f(t) - g(t)| \cdot 1 \, dt \\
&\leq \|f - g\|_2 \|1\|_2 \\
&= d_2(f, g)\sqrt{b - a}.
\end{aligned}
$$

In the above equation, the inequality is obtained from the Cauchy-Schwarz inequality, see Problem Sheet 6, Exercise 1-(iii). That is, for all h and k in $C([a,b])$ we have

$$\int_a^b |h(x)k(x)|\, dx \le \|h\|_2 \|k\|_2 \,.$$

(Just use $h = f - g$ and $k \equiv 1$.)

For the second inequality, we note that

$$
\begin{aligned}
d_2(f,g) &= \left(\int_a^b |f(t) - g(t)|^2\, dt \right)^2 \\
&\le \left(\int_a^b d_\infty(f,g)^2\, dt \right)^2 \\
&= d_\infty(f,g)(b-a)^2.
\end{aligned}
$$