

Mathematics Year 1, Calculus and Applications I

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Problem Sheet 6

1. Let $\{r_n\}$ denote the rational numbers in the interval $(0, 1)$ arranged in the sequence whose first few terms are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots$. Prove that the series $\sum_1^\infty r_n$ diverges.
2. Determine the convergence or divergence of the following infinite series:

$$\begin{aligned} (a) \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad (b) \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n \quad (c) \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} \quad (d) \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} 4^n \\ (e) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \quad (f) \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \\ (g) \sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}, \quad (h) \sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}, \quad (i) \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}} \right) \end{aligned}$$

3. (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots = 1.$$

Use the result to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and obtain upper and lower bounds for this sum.

- (b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.
- (c) Find the sum $\sum_{n=1}^{\infty} \frac{1+n}{2^n}$. [Hint: Differentiate a certain power series, justifying any operations.]
4. Suppose that $\{a_n\}$ is a decreasing sequence of positive terms such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \rightarrow 0$ as $n \rightarrow \infty$. [Hint - consider the sum $a_{n+1} + a_{n+2} + \dots + a_{2n}$.]
5. (a) For what values of α do the following series converge or diverge

$$(i) \sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} \quad (ii) \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^\alpha}$$

- (b) Show that the following series converges

$$\sum_{n=2}^{\infty} \frac{\log(n+1) - \log n}{(\log n)^2}.$$

6. For what values $p > 0$ does the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n^p}\right)^n$ converge.
7. This problem follows closely the derivation in class for the power series expansion for $\log(1+x)$.

- (a) Write down the sum of the geometric series $\sum_{k=0}^n r^k$.
 (b) Use (a) to show that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}.$$

- (c) Use (b) to show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n, \quad (1)$$

where R_n is the remainder which you should express as an integral involving x .

- (d) Show that the power series for $\tan^{-1} x$ converges absolutely for x in the closed interval $[-1, 1]$.
 (e) Use the power series to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. How many terms do we have to keep in this series in order to estimate π with accuracy to 10 decimal places, i.e. with error less than 10^{-10} ?

8. Following up from the calculation of π above, here is a much more efficient way.

- (a) Starting from the addition formula for the tangent

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

introduce the inverse functions $x = \tan^{-1} u$ and $y = \tan^{-1} v$ to show that

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1} \left(\frac{u+v}{1-uv} \right). \quad (2)$$

- (b) Show that choosing $(u+v)/(1-uv) = 1$ in expression (2), we have the following formula for π ,

$$\frac{\pi}{4} = \tan^{-1} u + \tan^{-1} v, \quad (3)$$

and that restricting u and v to be in the interval $(0, 1)$ we can express them as the one-parameter family

$$u = \frac{1-p}{1+p}, \quad v = p, \quad 0 < p < 1, \quad (4)$$

or equivalently

$$u = \frac{n-m}{n+m}, \quad v = \frac{m}{n}, \quad 0 < m < n, \quad (5)$$

where we picked p to be the rational number $p = m/n$.

Use your earlier findings regarding the power series for $\tan^{-1} x$ (equation (1)) to explain why the choices (4)-(5) are useful.

- (c) Hence show that (first derived and used by Euler)

$$\frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}. \quad (6)$$

Noting that $\frac{\frac{1}{3} + \frac{1}{7}}{1 - \frac{1}{21}} = \frac{1}{2}$, show that $\tan^{-1} \frac{1}{2} = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$, which when combined with (6) gives the formula (used by Jurij Vega, 1754-1802, a Slovenian mathematician who got 140 digits accuracy to π using this formula)

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}, \quad (7)$$

and on use of $\frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{40}} = \frac{1}{3}$ and previous results we also have

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{8}. \quad (8)$$

- (d) If we use the expressions (6), (7) and (8), respectively, how many terms in the expansion (1) do we need to compute π to 10 decimals accuracy? Compare with your answer to question 8(e).
9. (a) *Binomial Theorem.* Let $f(x) = (1+x)^s$ where s is a real number. Use induction arguments to show that $f^{(n)}(x) = s(s-1)\dots(s-n+1)(1+x)^{s-n}$ and hence write down the Taylor series for $f(x)$ including the remainder term. Hence show that $(1+x)^s$ converges uniformly (i.e. it is analytic) for $|x| < 1$.
- (b) Use the Binomial Theorem to compute $(126)^{1/3}$ and $\sqrt{96}$ to 4 decimals.
- (c) Write out the Maclaurin series for $1/\sqrt{1+x^2}$ using the binomial series. What is $\left. \frac{d^{20}}{dx^{20}} \left(\frac{1}{\sqrt{1+x^2}} \right) \right|_{x=0}$?
- (d) Find the Maclaurin series for $g(x) = \sqrt{1+x} + \sqrt{1-x}$, and hence calculate $g^{(20)}(0)$ and $g^{(2001)}(0)$.
10. Find the radius of convergence of the following series:

$$\begin{aligned} (1) \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n & \quad (2) \sum_{n=1}^{\infty} \frac{n^n}{(n!)} x^n & (3) \sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n & (4) \sum_{n=1}^{\infty} \frac{n^{5n}}{(2n)! n^{3n}} x^n \\ (5) \sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^2} x^n & (6) \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{2^n} x^n & (7) \sum_{n=1}^{\infty} \frac{\log n}{2^n} x^n & (8) \sum_{n=1}^{\infty} \frac{1 + \cos 2\pi n}{3^n} x^n \\ (9) \sum_{n=1}^{\infty} n x^n & (10) \sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{n!} x^n & (11) \sum_{n=1}^{\infty} n^2 x^n & (12) \sum_{n=1}^{\infty} \frac{\cos n^2}{n^n} x^n \\ (13) \sum_{n=1}^{\infty} \frac{n}{\log n} x^n & (14) \sum_{n=1}^{\infty} \frac{(-1)^n}{n! - 1} x^n & (15) \sum_{n=1}^{\infty} \frac{n!}{n^n} x^n & (16) \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n!} x^n \end{aligned}$$

You may use Stirling's formula

$$n! = (2\pi n)^{1/2} n^n e^{-n} e^{\theta/12n}, \quad 0 \leq \theta \leq 1,$$

in its appropriate form for large n .

[**Answers:** (1) $1/4$, (2) $1/e$, (3) 27 , (4) $4/e^2$, (5) 0 , (6) 2 , (7) 2 , (8) 3 , (9) 1 , (10) ∞ , (11) 1 , (12) ∞ , (13) 1 , (14) ∞ , (15) e , (16) ∞ .]

11. Find the Taylor series of the function $f(x) = \int_1^x \log t \, dt$ for x near 1 . Do the same for the function $x \log x$ and compare the two. What do you conclude?

12. Find the first four non-vanishing terms of the Maclaurin series for the following functions:

$$\begin{array}{lll} (a) \ x \cot x & (b) \ e^{\sin x}, & (c) \ \frac{\sqrt{\sin x}}{\sqrt{x}} \\ (d) \ e^{e^x}, & (e) \ \sec x, & (f) \ \log \sin x - \log x \end{array}$$

13. Consider the function $h(x)$ defined on the interval $[-\pi, \pi]$ and given by

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2 \sin(x/2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Use a Maclaurin expansion to show that $h(x)$ is continuous and has a continuous first derivative at $x = 0$.

14. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = f(x)/(1-x)$.

- (a) By multiplying the power series of $f(x)$ and $1/(1-x)$, show that $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $b_n = a_0 + \dots + a_n$ is the n th partial sum of the series $\sum_{n=0}^{\infty} a_n$.
- (b) Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{n \rightarrow \infty} b_n$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?
- (c) Let $\frac{e^x}{1-x} = \sum_{n=0}^{\infty} b_n x^n$. What is $\lim_{n \rightarrow \infty} b_n$?

15. (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1} x$. For what values of x do they converge?
- (b) Find the terms up to and including x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ into the series for $\sin^{-1} x$.
- (c) Use the substitution method from part (b) to obtain the first five terms of the series for $\sin^{-1} x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 to a_5 .
- (d) Find the terms up to and including x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. [Hint: Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g .]