BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Optimisation Mock Exam

Date: Wednesday, 11th May 2021
Time: 09:00-11:00
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Given the function

$$f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$$

(i) Determine its stationary points. (5 marks)

(ii) Classify the stationary points found in i). (5 marks)

(b) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{g} \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which define the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top H\mathbf{x} + \mathbf{g}^\top\mathbf{x}$. Does there exists a vector $\mathbf{u} \in \mathbb{R}^3$ such that $f(t\mathbf{u}) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct $\mathbf{u}$. (10 marks)

(Total: 20 marks)

2. (a) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

(i) $f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$ (5 marks)

(ii) $h(\mathbf{x}) = (\|\mathbf{x}\|^2 + 1)^2$, $\mathbf{x}$ in $\mathbb{R}^n$. (5 marks)

(b) Consider the problem

$$\begin{aligned} \text{(P)} \quad & \min \quad f(\mathbf{x}) \\ & \text{s.t.} \quad g(\mathbf{x}) \leq 0 \\ & \qquad\quad \mathbf{x} \in X \end{aligned}$$

where f, g are convex an $X \subseteq \mathbb{R}^n$ is convex. Suppose $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{aligned} & \min \quad f(\mathbf{x}) \\ & \text{s.t.} \quad \mathbf{x} \in X \end{aligned}$$

(10 marks)

(Total: 20 marks)

3. Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(i) Is the problem convex?. (6 marks)

(ii) Find all KKT points of the problem. (8 marks)

(iii) Find the optimal solution of the problem. (6 marks)

(Total: 20 marks)

4. Consider the minimization

$$\begin{aligned} \min \quad & x_1 - 4x_2 + x_3^4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3^2 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

(i) Formulate the dual problem. (10 marks)

(ii) Solve the dual problem. (10 marks)

(Total: 20 marks)

5. **Mastery question.** A community living around a lake wants to maximize the yield of fish taken out of the lake. The amount of fish at a certain time is denoted x. The growth rate of the fish is kx and fish is captured with a rate ux where u is the control variable, which is assumed to satisfy $0 \le u \le u_{\max}$. The dynamics of the fish population is then given by

$$\dot{x} = (k - u)x, \quad x(0) = x_o$$

Here $k > 0$ and $x_o > 0$. The total amount of fish obtained during a time period T is

$$J = \int_0^T uxdt$$

(i) Derive the necessary conditions given by the PMP for the problem of maximizing J. (8 marks)

(ii) Show that the necessary conditions are satisfied by a bang-bang control, that is, it only takes boundary values of the constraint set. How many switching times are there? (6 marks)

(iii) Determine an equation for calculating the switching time(s). (6 marks)

(Total: 20 marks)