Imperial College London

MATH97008 MATH70001

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fluid Dynamics 1

Date: 24 May 2022
Time: 09:00 – 11:30 (BST)
Time Allowed: 2:30 hours
Upload Time Allowed: 30 minutes
This paper has 5 Questions.
Candidates should start their solutions to each question on a new sheet of paper.
Each sheet of paper should have your CID, Question Number and Page Number on the top.
Only use 1 side of the paper.
Allow margins for marking.
Any required additional material(s) will be provided.
Credit will be given for all questions attempted.
Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Explain the concepts of *fluid particle* and *continuum hypothesis*, stating clearly the relevant length scales and limits involved. For a gas consisting of two types of molecules with masses m_1 and m_2, describe how the density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{V}(\mathbf{x}, t)$ of a fluid particle at an arbitrary point $\mathbf{x}$ and time t may be defined in terms of the mass and velocities of the molecules.

(4 marks)

(b) The trajectory of an arbitrary fluid particle in a two-dimensional flow field is described by

$$x = x_0 e^t + a\,t, \quad y = (y_0 - \tfrac{1}{2}x_0)e^{-t} + \tfrac{1}{2}x_0 e^t,$$

where (x_0, y_0) denotes the position at $t = 0$, and a is a constant.

(i) Deduce that the velocity field in terms of x, y and t is

$$(u, v) = (x - a\,t + a,\ -y + x - a\,t),$$

and verify that the flow is incompressible. (4 marks)

(ii) Suppose that dye is continuously discharged from time $t \geq t_0$ into the flow at a fixed point (x_0, y_0) in the flow field. Find the equation describing the streakline formed. Show that when $a = 0$ the streakline observed at time t is the same as the trajectory of the fluid particle starting from (x_0, y_0) at $t = t_0$. (5 marks)

(c) Follow the motion of a volume of fluid, which occupies the region $\mathcal{D}_0$ at time $t = t_0$. At time $t > t_0$, the fluid volume occupies region $\mathcal{D}$. The trajectory of each fluid particle in $\mathcal{D}_0$ defines a mapping between $\mathbf{x}_0 \equiv (x_0, y_0, z_0)$ and $\mathbf{x}(t; \mathbf{x}_0) \equiv (x, y, z)$ so that the volume integral of an arbitrary function $G(\mathbf{x}, t)$ obeys the relation

$$\iiint\limits_{\mathcal{D}} G d\tau = \iiint\limits_{\mathcal{D}_0} G J d\tau_0,$$

where J is the determinant of the Jacobian matrix $\partial(x, y, z)/\partial(x_0, y_0, z_0)$.

(i) Show that conservation of mass implies that

$$\rho J = \rho_0,$$

where ρ and ρ_0 denote the density distributions in $\mathcal{D}$ and $\mathcal{D}_0$, respectively. Hence deduce that

$$\iiint\limits_{\mathcal{D}} \rho G d\tau = \iiint\limits_{\mathcal{D}_0} G \rho_0 d\tau_0; \qquad \frac{D}{Dt}\iiint\limits_{\mathcal{D}} \rho G d\tau = \iiint\limits_{\mathcal{D}} \rho \frac{DG}{Dt} d\tau,$$

where $\frac{D}{Dt}$ denotes the material derivative. (5 marks)

(ii) Show further that

$$\frac{D}{Dt}\iiint\limits_{\mathcal{D}} G d\tau = \iiint\limits_{\mathcal{D}} \Big[\frac{DG}{Dt} + G(\nabla \cdot \mathbf{V})\Big] d\tau.$$

(2 marks)

(Total: 20 marks)

2. Consider an incompressible flow of viscous fluid that is driven by an axial pressure gradient along a straight pipe of a general cross section. The flow is steady and unidirectional, with a velocity field $\mathbf{V} = (u,\, 0,\, 0)$ in the Cartesian coordinate system (x, y, z), where the axial direction is x, and y and z are the coordinates in the cross section.

(i) Show that the pressure gradient $\partial p/\partial x$ must be a constant, and that the axial velocity u is independent of x and is governed by

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -G/\mu,$$

where $G = -\partial p/\partial x$, and μ is the viscosity coefficient. Specify the boundary conditions for u.

(4 marks)

(ii) If the cross section of the pipe is elliptical, with its boundary given by

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad (a > b),$$

show that

$$u = Ay^2 + Bz^2 + C,$$

is a solution for an appropriate choice of constants A, B and C, which you are required to determine.

Calculate the mass flux in the axial direction, where you may use the substitution: $y = a\, r\cos\theta$, $z = b\, r\sin\theta$, for which $dxdy = (ab)rdrd\theta$.

(7 marks)

(iii) Calculate the stress tensor $\mathcal{P}$, and the stress (i.e. force per unit area), $\mathbf{p}_n$, that the fluid exerts on the surface of the pipe.

(5 marks)

[*Hint*: *The unit inward normal vector* $\mathbf{n}$ *of the pipe surface is given by*

$$\mathbf{n} = (0,\, -y/a^2,\, -z/b^2)\Big/\sqrt{\frac{y^2}{a^4} + \frac{z^2}{b^4}}. \quad]$$

(iv) Consider the pressure and viscous (frictional) forces exerts on the fluid in the pipe between $x = 0$ and $x = L$. Show that in the axial direction, the total viscous force that the pipe exerts on the fluid balances the total pressure force acting on the cross sections at $x = 0$ and $x = L$.

(4 marks)

(Total: 20 marks)

3. (a) Let $\mathbf{V} = (u, v, w)$ be a three-dimensional velocity field and $\boldsymbol{\omega} = \nabla \times \mathbf{V}$ be the vorticity. Establish the identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \boldsymbol{\omega} \times \mathbf{V} + \nabla\left(\frac{V^2}{2}\right),$$

where $V^2 = u^2 + v^2 + w^2$.

Suppose that the body force $\mathbf{f}$ acting on the fluid can be written as $\mathbf{f} = -\nabla U$, where U is a scalar function. Show that when the flow is irrotational, incompressible and inviscid, the Euler equations may be reduced to the relation

$$\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}V^2 + \frac{p}{\rho} + U = \mathcal{C}(t),$$

where φ is the velocity potential, p the pressure and $\mathcal{C}(t)$ is independent of position.

(7 marks)

(b) An open tank is filled with water to a height $h = h_0$ at time $t = 0$, and under the action of the Earth's gravitional force the water flows out of the tank through a small orifice on the side wall near the bottom, as is shown in Figure 1. Let s and S denote the areas of the orifice and the tank's cross section. It is assumed that the ratio $\sigma = s/S \ll 1$.

(i) Assuming that the water flow in the tank is slow enough to be treated as quasi-steady, show that the exiting velocity V through the orifice is related to the instantaneous height h via

$$V \approx \sqrt{2gh}; \tag{1}$$

state the approximations made in your derivation.

(5 marks)

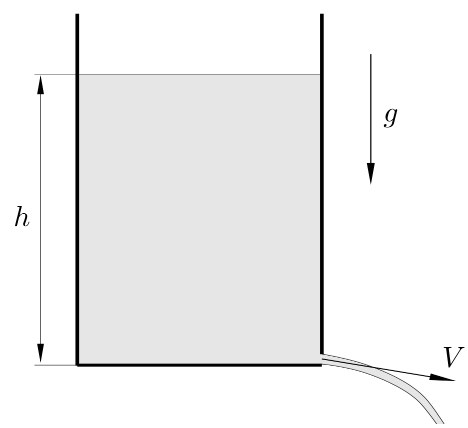

Figure 1: Water discharged through orifice from a tank.

(ii) Now investigate the problem without making the quasi-steady assumption, but assuming that the flow in the tank is one-dimensional except in a small region near the orifice, where the flow changes its direction but the change of the velocity potential is assumed to be negligible. Show that h and V satisfy the equations

$$hh''(t) + \tfrac{1}{2}h'^2 = \tfrac{1}{2}V^2, \quad h' = -\sigma V, \tag{2}$$

where a prime denotes d/dt.

(3 marks)

Question continues on the next page.

(iii) Let $E = V^2$, and show that the system (2) can be written as

$$\frac{dE}{dh} + \frac{1 - 1/\sigma^2}{h} E = -2g/\sigma^2.$$

Solve the equation for E and hence show that

$$V = \sqrt{\frac{2gh}{1 - 2\sigma^2}\Big[1 - \big(\frac{h}{h_0}\big)^{(\sigma^{-2}-2)}\Big]}.$$

Discuss the difference from, and connection with, the result (1).

(5 marks)

(Total: 20 marks)

4. Consider the incompressible inviscid potential flow past an ellipse whose major axis is aligned with the free-stream velocity vector; see Figure 2(a).

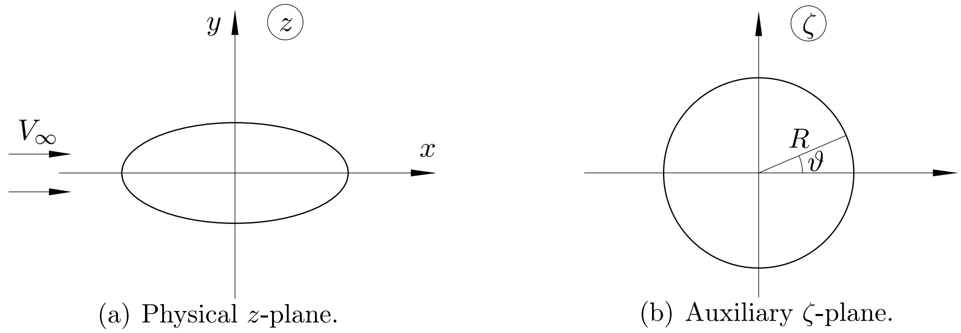

Figure 2: Flow past an ellipse.

(i) Verify that the Joukovskii transformation

$$z = \frac{1}{2}\Big(\zeta + \frac{a^2}{\zeta}\Big) \tag{3}$$

performs a conformal mapping of a circle of radius $R > a$ in the ζ-plane (Figure 2b) onto an ellipse in the z-plane (Figure 2a) with the major and minor semi-axes given by

$$\alpha = \frac{1}{2}\Big(R + \frac{a^2}{R}\Big), \qquad \beta = \frac{1}{2}\Big(R - \frac{a^2}{R}\Big).$$

(6 marks)

(ii) In the auxiliary ζ-plane, write the complex potential as

$$W(\zeta) = \widetilde{V}_\infty\Big(\zeta + \frac{R^2}{\zeta}\Big).$$

Find the free-stream velocity $\widetilde{V}_\infty$ in the auxiliary plane such that the free-stream velocity in the physical plane is V_∞.

(5 marks)

(iii) Deduce that the maximum value of the velocity on the surface of the ellipse is given by

$$V_{\max} = (1 + \beta/\alpha)V_\infty.$$

(4 marks)

(iv) The force that the fluid exerts on the ellipse may be calculated by using the Blasius-Chaplygin formula,

$$F_X - iF_Y = \tfrac{1}{2} i\rho \oint \Big(\frac{dw}{dz}\Big)^2 dz,$$

where F_X and F_Y denote the forces in the x and y directions respectively, and the integral is around the boundary of the ellipse. By converting the integral to the ζ-plane, verify that

$$F_X = 0, \qquad F_Y = 0.$$

(5 marks)

(Total: 20 marks)

5. (a) Consider the flow round the tip of a flat plate. As only the local behaviour of the flow is relevant, a semi-infinite plate, as shown in Figure 3, is considered so that the solution can be found using the conformal mapping,

$$z = \zeta^2,$$

which maps the physical flow domain to the upper half of the auxiliary ζ-plane.

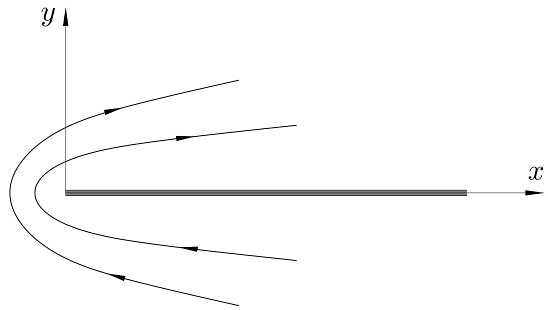

Figure 3: Flow around a plate tip.

(i) A suitable solution for the complex potential $W(\zeta)$ in the auxiliary plane is given by

$$W(\zeta) = \tilde{V}_\infty \zeta,$$

where $\tilde{V}_\infty$ is a real constant. Find the complex potential $w(z)$ in the physical plane, and show that each streamline in the physical z-plane has a parabolic shape.

(4 marks)

(ii) Choose one of the streamlines and, treating it as the surface of a solid body, calculate the integral pressure force

$$F_X \equiv 2 \int_0^\infty (p - p_\infty)\, dy,$$

which acts upon this 'body', and is parallel to the plate surface, where p_∞ denotes the pressure in the far field. Does the force F_X depend on the streamline chosen?

You may use without proof the result that $\displaystyle\int_0^\infty \frac{ds}{s^2+1} = \pi/2.$

(6 marks)

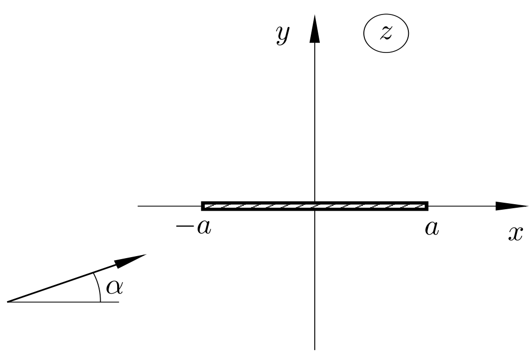

Figure 4: Flow past a flat plate at an angle of attack.

(b) Consider the flow past a flat plate at a non-zero angle of attack, as shown in Figure 4. On the one hand, according to the general theory of two-dimensional potential flows, the resultant

Question continues on the next page.

force acting on the plate is perpendicular to the free-stream velocity vector. On the other hand, "common sense" suggests that the pressure force should be directed perpendicular to the plate surface. This apparent 'contradiction' or 'paradox' can be resolved by appealing to the result obtained in Part (a). This involves analysing the behaviour near the leading edge of the complex potential for the flow, which is given in the auxiliary ζ-plane by

$$W(\zeta) = \frac{1}{2}V_\infty\Big(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}}\Big) + \frac{\Gamma}{2\pi i}\ln\zeta, \tag{4}$$

where V_∞ is the magnitude of the velocity in the far field, ζ is related to z via the Joukovskii transformation

$$\zeta = z + \sqrt{z^2 - a^2}, \tag{5}$$

and the circulation

$$\Gamma = -2\pi a V_\infty \sin\alpha\,,$$

in order to satisfy the Joukovskii-Kutta condition.

(i) Let $z = -a + z'$ with $|z'| \ll a$ near the leading edge $z = -a$. Show by Taylor expansion that the complex potential $w(z)$ in the physical plane has a similar near tip behaviour to the flow in Part (a) with an equivalent $\tilde{V}_\infty$, which you are required to find explicitly in terms of V_∞, a and α.
By using the result derived in Part (a) for F_X in terms of $\tilde{V}_\infty$, which holds for the finite-plate case, calculate the force in the x-direction, F_X, for the present flow.

(5 marks)

(ii) Given that the velocities on the upper and lower surfaces of the plate, $\overline{V}^+$ and $\overline{V}^-$, are expressed as

$$\overline{V}^\pm = V_\infty\Big(\cos\alpha \pm V_\infty \sin\alpha\sqrt{\frac{a-x}{a+x}}\Big), \tag{6}$$

calculate $p^+ - p^-$, where p^+ and p^- denote the pressure on the upper and lower surfaces, respectively. Hence, evaluate the total pressure force

$$F_Y \equiv \int_{-a}^{a}(p^- - p^+)dx,$$

which acts on, and is perpendicular to, the plate.

[You may use without proof the result that $\displaystyle\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx = a\pi$.]

(3 marks)

(iii) Show that the supposition of F_X and F_Y leads to a force perpendicular to the free-stream velocity.

(2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60001, MATH70001, MATH97008

Fluid Dynamics 1 (Solutions)

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1. (a) Part (a) A fluid particle is a collection of fluid molecules in a small volume $V \sim l^3$ with the length scale l in the range $\lambda \ll l \ll L$, where λ and L denote the mean free path λ (i.e. the mean distance molecules travel between two consecutive collisions) and the size of the fluid field respectively. On the scale L, the volume is so small that this collection of molecules behaves like a 'particle', which occupies the centre of the volume, say $\mathbf{x}$. With the condition that $l \gg \lambda$, the volume is sufficiently large that many collisions take place to ensure that statistically averaged (i.e. macroscopic) properties of all the molecules in the volume can be defined and attached to the fluid particle at $\mathbf{x}$. As this can be done for each $\mathbf{x}$, fluid particles can be considered as being continuously distributed in space, and the fluid as a continuum. seen $\Downarrow$

unseen $\Downarrow$

For a gas consisting of two type of molecules, the density of a fluid particle at a point $\mathbf{x}$ may be defined as

$$\rho(\mathbf{x}, t) = (N_1 m_1 + N_2 m_2)/V,$$

and the velocity as

$$\mathbf{V}(\mathbf{x}, t) = \frac{1}{N_1 m_1 + N_2 m_2} \sum (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2),$$

where N_1 and N_2 denote the numbers of molecules of type 1 and 2 respectively in a small volume (e.g. a sphere) centred at $\mathbf{x}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the velocities of the molecules. Here the key hypothesis made is that the averaged quantities become independent of the shape and size of the volume when $l \gg \lambda$. 4, A

(b) (i) The velocity (u, v) is given by unseen $\Downarrow$

$$u = \frac{dx}{dt} = x_0 e^t + a, \qquad v = \frac{dy}{dt} = -(y_0 - \tfrac{1}{2}x_0)e^{-t} + \tfrac{1}{2}x_0 e^t. \qquad (1)$$

This is the velocity of each particle at the current position (x, y) and time t. In order to express (u, v) in terms of (x, y), we note that

$$x_0 e^t = x - at, \qquad (y_0 - \tfrac{1}{2}x_0)e^{-t} = y - (x - at)/2,$$

which are then substituted into (1) to give

$$u = x - at + a, \qquad v = -y + x - at.$$

Note that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0,$$

and so the flow is incompressible indeed. Alternatively, consider the Jacobian matrix of the mapping $(x_0, y_0) \longrightarrow (x, y)$ as defined by the trajectory,

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} = \begin{pmatrix} e^t & 0 \\ \frac{1}{2}(e^t - e^{-t}) & e^{-t} \end{pmatrix}.$$

The determinant of the Jacobian matrix is $J = 1$, which implies incompressibility too ($\rho = \rho_0$). 4, A

(ii) In order to find the streakline, we solve the initial-value problem,

sim. seen ⇓

$$\begin{cases} \dfrac{dx}{dt} = x - at + a, \\ \dfrac{dy}{dt} = -y + x - at, \end{cases} \qquad \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Solving the equation for x, we have $x = C_1 e^t + at$. Imposing the initial condition yields the constant

$$C_1 = (x_0 - at_0)e^{-t_0},$$

and it follows that

$$x = (x_0 - at_0)e^{t-t_0} + at. \tag{2}$$

Substituting x into the equation for y, we have

$$\frac{dy}{dt} = -y + (x_0 - at_0)e^{t-t_0}.$$

The solution is found as

$$y = C_2 e^{-t} + \tfrac{1}{2}(x_0 - at_0)e^{t-t_0}.$$

Use of the initial condition determines C_2 as

$$C_2 = \left[y_0 - \tfrac{1}{2}(x_0 - at_0)\right]e^{t_0},$$

and hence

$$y = \left[y_0 - \tfrac{1}{2}(x_0 - at_0)\right]e^{-t+t_0} + \tfrac{1}{2}(x_0 - at_0)e^{t-t_0}. \tag{3}$$

The streakline formed is described by (2) and (3) with x and y being considered as functions of t_0: $0 \le t_0 \le t$ with t fixed.
When $a = 0$,

$$x = x_0 e^{t-t_0}, \qquad y = \left[y_0 - \tfrac{1}{2}x_0\right]e^{-t+t_0} + \tfrac{1}{2}x_0 e^{t-t_0}.$$

The trajectory is described by (x, y) as functions of $t \ge t_0$, while the streakline observed at time t is described by (x, y) as functions of $t_0 \le t$. Both are described by (x, y) as functions of $\tau \equiv t - t_0 \ge 0$:

$$x = x_0 e^{\tau}, \qquad y = \left[y_0 - \tfrac{1}{2}x_0\right]e^{-\tau} + \tfrac{1}{2}x_0 e^{\tau}.$$

This result is expected as the flow is steady when $a = 0$.

5, B

(c) (i) In the given relation,

sim. seen ⇓

$$\int_{\mathcal{D}} G d\tau = \int_{\mathcal{D}_0} GJ d\tau_0, \tag{4}$$

set $G = \rho$, the density of the fluid, then

$$\int_{\mathcal{D}} \rho d\tau = \int_{\mathcal{D}_0} \rho J d\tau_0.$$

On the other hand, the left-hand side, $\int_{\mathcal{D}} \rho d\tau$, stands for the mass of the fluid in the region $\mathcal{D}$, and thus remains constant during the motion due to the conservation of mass and so

$$\int_{\mathcal{D}} \rho d\tau = \int_{\mathcal{D}_0} \rho_0 d\tau_0.$$

Since $\mathcal{D}$ is arbitrary, so is $\mathcal{D}_0$, and it follows that $J\rho = \rho_0$.

With ρG taking the place of G in (4), we have

$$\int_{\mathcal{D}} \rho G d\tau = \int_{\mathcal{D}_0} J(\rho G) d\tau_0.$$

Use of the relation $\rho J = \rho_0$ gives the required identity

$$\int_{\mathcal{D}} \rho G \, d\tau = \int_{\mathcal{D}_0} \rho_0 G d\tau_0. \tag{5}$$

Let the material derivative, $\frac{D}{Dt}$, act on (5):

$$\frac{D}{Dt}\int_{\mathcal{D}} \rho G d\tau = \int_{\mathcal{D}_0} \rho_0 \frac{DG}{Dt} d\tau_0 = \int_{\mathcal{D}_0} J\big(\rho \frac{DG}{Dt}\big) d\tau_0 = \int_{\mathcal{D}} \rho \frac{DG}{Dt} d\tau,$$

where in the second last step we used the relation $\rho_0 = \rho J$ and in the last step $\rho\frac{DG}{Dt}$ takes the place of G in (4). $\boxed{5, \mathrm{C}}$

(ii) Now treating G/ρ as G in the proceeding result, we have $\boxed{\text{unseen} \Downarrow}$

$$\frac{D}{Dt}\int_{\mathcal{D}} G d\tau = \frac{D}{Dt}\int_{\mathcal{D}} \rho (G/\rho) d\tau = \int_{\mathcal{D}} \rho \frac{D}{Dt}(G/\rho) d\tau. \tag{6}$$

Note that

$$\frac{D}{Dt}(G/\rho) = \frac{1}{\rho}\frac{DG}{Dt} - \frac{G}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}\frac{DG}{Dt} - \frac{G}{\rho^2}(-\rho \nabla\cdot\mathbf{V}) = \frac{1}{\rho}\big[\frac{DG}{Dt} + G(\nabla\cdot\mathbf{V})\big],$$

where use has been made of the continuity equation. Inserting the above relation to (6), we obtain the required result

$$\frac{D}{Dt}\int_{\mathcal{D}} G d\tau = \int_{\mathcal{D}} \Big[\frac{DG}{Dt} + G(\nabla\cdot\mathbf{V})\Big] d\tau.$$

$\boxed{2, \mathrm{D}}$

2. (i) Continuity equation: $\dfrac{\partial u}{\partial x} = 0 \quad \Longrightarrow \quad u = u(y, z).$ $\boxed{\text{seen} \Downarrow}$

The momentum equations in the $y-$ and $z-$directions yield: $\dfrac{\partial p}{\partial y} = 0, \quad \dfrac{\partial p}{\partial z} = 0.$

The axial momentum equation:

$$\frac{\partial p}{\partial x} = \mu\Big(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big). \tag{7}$$

The left-hand side is a function of x whilst the right-hand side is a function of y and z. Hence both sides must be a constant, $-G$ say. There follows the required equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -G/\mu.$$

Boundary condition is specified as $u = 0$ on the surface (and u is finite within the pipe). $\boxed{4, \text{A}}$

(ii) Substituting the expression for u into equation (7) yields $\boxed{\text{meth seen} \Downarrow}$

$$2A + 2B = -G/\mu. \tag{8}$$

Applying boundary condition $u = 0$ at $y^2/a^2 + z^2/b^2 = 1$, we obtain

$$u = Ay^2 + Bb^2(1 - y^2/a^2) + C = 0,$$

which implies that

$$A - Bb^2/a^2 = 0, \quad Bb^2 + C = 0. \tag{9}$$

from (8) and (9), we find that

$$B = -\frac{Ga^2}{2\mu(a^2 + b^2)}, \quad A = -\frac{Gb^2}{2\mu(a^2 + b^2)}, \quad C = \frac{Ga^2b^2}{2\mu(a^2 + b^2)}.$$

Substitutions back to u gives

$$u = \frac{Ga^2b^2}{2\mu(a^2 + b^2)}\Big(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\Big) \equiv U_0\Big(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\Big).$$

The axial volume flux is given by the integration of u over the elliptical cross section S:

$$Q \equiv \int_S u(y, z)dydz.$$

Let $y = ar\cos\theta$ and $z = br\sin\theta$, for which $dydz = abr\,dr\,d\theta$. Hence

$$Q = U_0 \int_0^{2\pi}\int_0^1 (1 - r^2)(ab)rdr\,d\theta = \tfrac{1}{2}\pi abU_0.$$

$\boxed{7, \text{A}}$

(iii) The stress tensor $\mathcal{P}$ with the components: $\boxed{\text{unseen} \Downarrow}$

$$p_{xx} = -p, \quad p_{xy} = \mu\frac{\partial u}{\partial y} = -\frac{Gb^2}{a^2 + b^2}y, \quad p_{xz} = \mu\frac{\partial u}{\partial z} = -\frac{Ga^2}{a^2 + b^2}z,$$

$$p_{yx} = p_{xy}, \quad p_{yy} = -p, \quad p_{yz} = 0,$$

$$p_{zx} = p_{xz}, \quad p_{zy} = 0, \quad p_{zz} = -p.$$

The stress $\mathbf{p}_n$ that the fluid exerts on the pipe surface is

$$\mathbf{p}_n = \mathcal{P}\cdot\mathbf{n} = \Big(\frac{Ga^2b^2}{a^2+b^2}(y^2/a^4+z^2/b^4), py/a^2, pz/a^2\Big)/\sqrt{y^2/a^4+z^2/b^4}.$$

The first component on the right-hand side is the axial component of $\mathbf{p}_n$,

$$\mathbf{p}_n\big|_x = \frac{Ga^2b^2}{a^2+b^2}\sqrt{y^2/a^4+z^2/b^4}.$$

5, B

(iv) The pressure forces at $x = 0$ and $x = L$ are given by $p(0)(\pi ab)$ and $p(L)(\pi ab)$ respectively, and so their sum is unseen ⇓

$$F_p \equiv [p(0) - p(L)](\pi ab) = \pi abGL.$$

The force that the pipe surface exerts on the fluid has the axial component $-\mathbf{p}_n\big|_x$. Note that $-\mathbf{p}_n\big|_x$ is uniform in the axial direction but nonuniform in the circumferential direction, and so it is necessary to integrate along ellipse S

$$F_{\text{vis}} = -L\int_S \mathbf{p}_n\big|_x ds = -\frac{GLa^2b^2}{a^2+b^2}\int_S\sqrt{\frac{y^2}{a^4}+\frac{z^2}{b^4}}\,ds, \tag{10}$$

where for the ellipse, we have

$$y = a\cos\theta, \quad z = b\sin\theta,$$

and so

$$ds = \sqrt{(dy)^2+(dz)^2} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta,$$
$$\sqrt{\frac{y^2}{a^4}+\frac{z^2}{b^4}} = \frac{1}{ab}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}.$$

Thus substituting these into (10), we obtain

$$F_{\text{vis}} = -\frac{GLab}{a^2+b^2}\int_0^{2\pi}(a^2\sin^2\theta + b^2\cos^2\theta)d\theta = -(\pi ab)GL.$$

That the pressure force F_p and the viscous force F_{vis} are of equal magnitude but of opposite sign indicates that the two are in balance.

4, D

3. (a) Consider the x-component of the vector identity. The left-hand side is written as

seen $\Downarrow$

$$(\mathbf{V}\cdot\nabla)\mathbf{V}\Big|_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \tag{11}$$

Now consider the right-hand side of the identity,

$$\big[\boldsymbol{\omega}\times\mathbf{V}\big] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ u & v & w \end{vmatrix},$$

with the $x-$component being

$$\big[\boldsymbol{\omega}\times\mathbf{V}\big]\Big|_x = \omega_y w - \omega_z v. \tag{12}$$

The vorticity is calculated as

$$\boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.$$

We have

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{13}$$

Substitution of (13) into (12) gives

$$\big[\boldsymbol{\omega}\times\mathbf{V}\big]\Big|_x = w\frac{\partial u}{\partial z} - w\frac{\partial w}{\partial x} - v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y}. \tag{14}$$

The $x-$component of the second term on the right-hand side of the identify is given by

$$\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right) = \frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right) = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}. \tag{15}$$

Adding (14) and (15) together leads to

$$\big[\boldsymbol{\omega}\times\mathbf{V}\big]\Big|_x + \frac{\partial}{\partial x}\left(\frac{V^2}{2}\right) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

which coincides with (11), thus proving the vector identify holds for the x-component. This procedure can obviously be repeated for the $y-$ and $z-$components of (11).

[Full credit will be given for the proof using tensor notations.]

By using the identity established and $\mathbf{f} = -\nabla U$, the Euler equations,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \mathbf{f},$$

for an incompressible inviscid flow can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + \big[\boldsymbol{\omega}\times\mathbf{V}\big] + \nabla\Big(\frac{V^2}{2}\Big) = -\nabla\Big(\frac{p}{\rho} + U\Big).$$

When the flow is irrotational, $\boldsymbol{\omega} = 0$ and there exists a scalar φ, the velocity potential, such that $\mathbf{V} = \nabla\varphi$. Hence the Euler equations can be rewritten as

$$\nabla\Big[\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}V^2 + \frac{p}{\rho} + U\Big] = 0.$$

There follows the required relation

$$\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}V^2 + \frac{p}{\rho} + U = \mathcal{C}(t). \tag{16}$$

7, A

(b) (i) With the flow being treated as "quasi-steady", (16) simplifies to

sim. seen ⇓

$$\frac{V^2}{2} + \frac{p}{\rho} + U = \mathcal{C}. \tag{17}$$

[This equation also follows from the steady Euler equations (without making potential flow assumption), in which case this equation holds along each streamline.] The potential of the gravitational force is given by

$$U = gz + C,$$

and so (17) becomes

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = \mathcal{C}, \tag{18}$$

where the constant C has been absorbed into $\mathcal{C}$.

Now applying (18) to the water surface, where $z = h$, $p = p_a$ and $V = h'(t)$, and the orifice, where $z = 0$ and $p = p_a$, we have

$$\frac{h'^2}{2} + \frac{p_a}{\rho} + gh = \frac{V^2}{2} + \frac{p_a}{\rho}. \tag{19}$$

The conservation of mass implies that

$$Sh'(t) = -sV,$$

which is used in (19) to give

$$V = \sqrt{2gh/(1-\sigma^2)} \approx \sqrt{2gh}, \tag{20}$$

since $\sigma \ll 1$.

5, B

(ii) When the flow in the tank is treated as irrotational and one-dimensional, the velocity is

unseen ⇓

$$w = h',$$

everywhere, and so the velocity potential is

$$\varphi = h'z + \varphi_0(t).$$

Applying the relation (16) at $z = h$, where $\varphi = h'(t)h + \varphi_0(t)$, and at the orifice ($z = 0$), we have

$$h''h + \varphi_0'(t) + \frac{1}{2}h'^2 + \frac{p_a}{\rho} + gh = \frac{\partial}{\partial t}\varphi\big|_{\text{orifice}} + \frac{1}{2}V^2 + \frac{p_a}{\rho}. \tag{21}$$

We take $\varphi\big|_{\text{orifice}} = \varphi\big|_{z=0} = \varphi_0(t)$ since the variation of φ is assumed to negligible as the flow changes its direction. Again, conservation of mass means that the velocity at the orifice, V, is related to the velocity at the water surface by $sV = -Sh'$, i.e.

$$h'(t) = -\sigma V, \tag{22}$$

3, C

where the minus sign takes account of the fact that $V > 0$ while $h'(t) < 0$. $\boxed{\text{sim. seen} \Downarrow}$

(iii) Inserting (22) into (21) and rearranging, we obtain

$$h''h + \frac{1}{2}(1 - \sigma^{-2})h'^2 + gh = 0. \tag{23}$$

With V being treated as a function of h, differentiation of (22) with respect to t by using the chain rule gives

$$h''(t) = -\sigma V'(h)h'(t) = \sigma^2 V'(h)V. \tag{24}$$

Now inserting (22) and (24) into (23), we arrive at equation

$$(\sigma^2 - 1)V^2 + 2\sigma^2 hV'(h)V + 2gh = 0, \tag{25}$$

which can be written in terms of $E = V^2$ as

$$E'(h) + \frac{1 - 1/\sigma^2}{h}E = -\frac{2g}{\sigma^2}. \tag{26}$$

Equation (26) is a first-order linear ordinary differential equation for E, and so can be solved using the method of integration factor, which is found as

$$\exp\{\int \frac{\alpha}{h} dh\} = h^\alpha,$$

where we have put

$$\alpha = 1 - 1/\sigma^2.$$

After multiplying this factor to both sides of (26), we obtain

$$\frac{d}{dh}(Eh^\alpha) = -\frac{2g}{\sigma^2}h^\alpha.$$

Integrating from h_0 to h, and noting that $E = 0$ at $h = h_0$, we find that

$$Eh^\alpha = -\frac{2g}{\sigma^2(1+\alpha)}(h^{(1+\alpha)} - h_0^{(1+\alpha)}),$$

i.e.

$$E = \frac{2g}{1 - 2\sigma^2}\Big[h - h_0(\frac{h}{h_0})^{-\alpha}\Big].$$

It follows that

$$V = \sqrt{\frac{2gh}{1 - 2\sigma^2}\Big[1 - (\frac{h}{h_0})^{(\sigma^{-2}-2)}\Big]}.$$

Now with the unsteadiness included, the exiting velocity V depends not only on the instantaneous water level $h(t)$ but also on its initial level h_0. In the limit $\sigma \to 0$, $1/\sigma^2 - 2 \to \infty$ while $h/h_0 < 1$, and so $V \to \sqrt{2gh}$, recovering (20) (which is Torricelli's formula). $\boxed{5, \text{D}}$

4. (i) The equation of the circle in the ζ-plane is written as $\boxed{\text{seen} \Downarrow}$

$$\zeta = Re^{i\vartheta}. \tag{27}$$

In order to find its image in the z-plane, (27) is substituted into the Joukovskii transformation

$$z = \frac{1}{2}(\zeta + \frac{a^2}{\zeta}), \tag{28}$$

to obtain

$$x + iy = \frac{1}{2}(Re^{i\vartheta} + \frac{a^2}{R}e^{-i\vartheta}). \tag{29}$$

Separation of the real and imaginary parts of (29) yields

$$x = \frac{1}{2}(R + \frac{a^2}{R})\cos\vartheta, \qquad y = \frac{1}{2}(R - \frac{a^2}{R})\sin\vartheta,$$

or equivalently,

$$\cos\vartheta = \frac{x}{\alpha}, \qquad \sin\vartheta = \frac{y}{\beta}, \tag{30}$$

where

$$\alpha = \frac{1}{2}(R + \frac{a^2}{R}), \qquad \beta = \frac{1}{2}(R - \frac{a^2}{R}). \tag{31}$$

Parameter ϑ may be eliminated from (30) by squaring equations (30) and adding them together. This results in

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

which is the standard form of equation for an ellipse. From (31) we find

$$R = \alpha + \beta, \qquad a = \sqrt{\alpha^2 - \beta^2}. \tag{32}$$

$\boxed{6, \text{A}}$

(ii) In the auxiliary ζ-plane, the complex potential is written as $\boxed{\text{seen} \Downarrow}$

$$W(\zeta) = \tilde{V}_\infty\Big(\zeta + \frac{R^2}{\zeta}\Big).$$

The corresponding complex potential $w(z)$ in the physical plane is given by the composition of $W(\zeta)$ and $\zeta = \zeta(z)$: $w(z) = W\big(\zeta(z)\big)$. Therefore, the complex conjugate velocity in the physical z-plane is calculated using the Chain Rule as

$$\overline{V}(z) = \frac{dW}{d\zeta}\frac{1}{dz/d\zeta} = \frac{\tilde{V}_\infty\,(1 - R^2/\zeta^2)}{\frac{1}{2}\,(1 - a^2/\zeta^2)}. \tag{33}$$

In order to find $\tilde{V}_\infty$, we use the free-stream boundary condition,

$$\overline{V} \to V_\infty \quad \text{as} \quad z \to \infty.$$

Taking the limit $z \to \infty$ in (33), we find that

$$\overline{V}(z) \to 2\tilde{V}_\infty,$$

which means that

$$\tilde{V}_\infty = \tfrac{1}{2}V_\infty, \tag{34}$$

and hence

$$W(\zeta) = \tfrac{1}{2}V_\infty\Big(\zeta + \frac{R^2}{\zeta}\Big). \tag{35}$$

$\boxed{5, \text{B}}$

(iii) Substituting (34) back into (33), we have

sim. seen ⇓

$$\overline{V}(z) = V_\infty \frac{1 - R^2/\zeta^2}{1 - a^2/\zeta^2}. \tag{36}$$

A point on the ellipse corresponds to (27), which is substituted into (36), giving

$$\overline{V}(z) = V_\infty \frac{1 - e^{-2i\vartheta}}{1 - (a^2/R^2)e^{-2i\vartheta}} = V_\infty \frac{2i \sin\vartheta R^2}{(R^2 - a^2)\cos\vartheta + i(R^2 + a^2)\sin\vartheta}.$$

Hence

$$\begin{aligned} |\overline{V}| &= \frac{2V_\infty R^2 \sin\vartheta}{\left[(R^2 - a^2)^2 \cos^2\vartheta + (R^2 + a^2)^2 \sin^2\vartheta\right]^{1/2}} \\ &= \frac{2V_\infty R^2}{\left[(R^2 - a^2)^2 \cot^2\vartheta + (R^2 + a^2)^2\right]^{1/2}}, \end{aligned}$$

which indicates that $|\overline{V}|$ attains its maximum at $\vartheta = \pi/2$, which corresponds to $x = 0$ and $y = \beta$ in the z-plane, a point atop the ellipse. At this point, $\zeta = iR$, substitution of which into (36) yields

$$\overline{V}\Big|_{\zeta = iR} = \frac{2V_\infty}{1 + a^2/R^2} = (1 + \beta/\alpha)V_\infty,$$

where use has been made of (32).

4, C

unseen ⇓

(iv) Noting $dw/dz = (dW/d\zeta)/(dz/d\zeta)$ and $dz = (dz/d\zeta)d\zeta$, we rescast the Blasius-Chaplygin formula into an integral in the ζ-plane,

$$F_X - iF_Y = \tfrac{1}{2} i\rho \oint_C \Big(\frac{dW}{d\zeta}\Big)^2 \frac{1}{dz/d\zeta} d\zeta, \tag{37}$$

where C denotes the circle in the ζ-plane: $|\zeta| = R > a$. Substituting (28) and (35) into (37), we have

$$\begin{aligned} F_X - iF_Y &= \frac{1}{4} i\rho V_\infty^2 \oint_C \Big(1 - R^2/\zeta^2\Big)^2 \frac{1}{1 - a^2/\zeta^2} d\zeta \\ &= \frac{1}{4} i\rho V_\infty^2 \oint_C \Big[1 - \frac{2R^2 - a^2}{\zeta^2 - a^2} + \frac{R^4}{\zeta^2(\zeta^2 - a^2)}\Big] d\zeta. \end{aligned}$$

The integration of the first term, a constant, around a closed loop is zero. The second and third integrals can be deformed into those on a large circle: $|\zeta| = \tilde{R} \gg 1$. As the integrands are of $O(\tilde{R}^{-2})$ and $O(\tilde{R}^{-4})$ and hence the integrals are of $O(\tilde{R}^{-1})$ and $O(\tilde{R}^{-3})$, respectively. Both vanish as $\tilde{R} \to \infty$. Therefore, we conclude that

$$F_X = 0, \qquad F_Y = 0.$$

5, D

5. (a) (i) In the auxiliary ζ-plane, the given complex potential,

seen $\Downarrow$

$$W(\zeta) = \tilde{V}_\infty \zeta, \tag{38}$$

represents a uniform flow in the upper half-plane. The conformal mapping,

$$z = \zeta^2, \tag{39}$$

is inverted to give $\zeta = \sqrt{z}$. Hence, the complex potential in the physical plane is found as

$$w(z) = \tilde{V}_\infty \sqrt{z}, \tag{40}$$

featuring a 'square-root singularity', which is a generic behaviour of the flow near a tip.

In order to find the shape of the streamlines in the physical plane, we note that in the auxiliary plane all streamlines are parallel to the ξ-axis, and therefore may be represented by the equation

$$\zeta = \xi + ih \qquad (-\infty < \xi < \infty), \tag{41}$$

where h is fixed along each streamline. Substituting (41) into (39), we have

$$x + iy = (\xi + ih)^2 = \xi^2 - h^2 + 2ih\xi,$$

which implies

$$x = \xi^2 - h^2, \qquad y = 2h\xi. \tag{42}$$

Elimination of ξ from the above two equations leads to

$$x = \frac{y^2}{4h^2} - h^2, \tag{43}$$

which shows that all streamlines in the physical plane are parabolas. The upper and lower surfaces of the plate should be viewed as the singular limit of $h \to 0$.

[4 marks]

sim. seen $\Downarrow$

(ii) As suggested, we treat a streamline with a fixed h as a solid body. To find the pressure distribution along the contour of this body, we first calculate the complex conjugate velocity

$$\overline{V}(z) = \frac{dW}{d\zeta}\frac{1}{dz/d\zeta} = \frac{\tilde{V}_\infty}{2\zeta} = \frac{\tilde{V}_\infty}{2(\xi + i\eta)} = \frac{\tilde{V}_\infty}{2\sqrt{z}}, \tag{44}$$

where use is made of (38) and (39). Clearly,

$$V = \left|\overline{V}(z)\right| \to 0 \quad \text{as} \quad z \to \infty.$$

On the body contour $\eta = h$, the modulus of the velocity is

$$V = \left|\overline{V}(z)\right| = \frac{\tilde{V}_\infty}{2\sqrt{\xi^2 + h^2}} = \frac{\tilde{V}_\infty}{2\sqrt{\frac{y^2}{4h^2} + h^2}}, \tag{45}$$

where use has been made of the second of (42).

Alternatively,

$$V = \frac{\tilde{V}_\infty}{2|z|^{1/2}} = \frac{\tilde{V}_\infty}{2(x^2+y^2)^{1/4}} = \frac{\tilde{V}_\infty}{2\big[(\frac{y^2}{4h^2}-h^2)^2+y^2\big]^{1/4}} = \frac{\tilde{V}_\infty}{2\sqrt{\frac{y^2}{4h^2}+h^2}}.$$

Applying the Bernoulli equation to the streamline that lies along the body contour, we have

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho},$$

which gives

$$p - p_\infty = -\frac{\rho}{2}V^2 = -\frac{\rho}{2}\frac{\tilde{V}_\infty^2}{4\left(\frac{y^2}{4h^2}+h^2\right)},$$

where use is made of (45). Hence the x-component of the pressure force may be written as

$$F_X = 2\int_0^\infty (p-p_\infty)\,dy = -\frac{1}{4}\rho\tilde{V}_\infty^2\int_0^\infty \frac{dy}{\frac{y^2}{4h^2}+h^2}.$$

Introducing a new integration variable s such that $y = 2h^2 s$, we have

$$F_X = -\frac{\rho\tilde{V}_\infty^2}{2}\int_0^\infty \frac{ds}{(s^2+1)} = -\frac{\rho\tilde{V}_\infty^2}{2}\arctan s\Big|_0^\infty = -\frac{\pi}{4}\rho\tilde{V}_\infty^2. \qquad (46)$$

This result shows that the force acting on a body of parabola shape is indeed independent of h. It follows that the force acting on the plate, which is the limit of $h \to 0$, is also this value.

[6 marks]

unseen $\Downarrow$

(b) (i) Let $z = -a + z'$ with $|z'| \ll a$ near the leading edge $z = -a$. It follows that

$$\zeta = -a + z' + \sqrt{z'(-2a+z')} \approx -a + i\sqrt{2a}\sqrt{z'},$$

$$\frac{1}{\zeta} \approx -\frac{1}{a}(1 + i\sqrt{\frac{2}{a}}\sqrt{z'}), \quad \ln\zeta \approx \ln(-a) - i\sqrt{\frac{2}{a}}\sqrt{z'}.$$

Use of these approximations in $W(\zeta)$ leads to

$$w(z) \approx \Big[\sqrt{2a}\sin\alpha V_\infty - \frac{\Gamma}{\sqrt{2a}\,\pi}\Big]\sqrt{z'}, \qquad (47)$$

where constant terms are neglected. Inserting into (47) the value of circulation, $\Gamma = -2\pi a V_\infty \sin\alpha$ (which follows from the Joukovskii-Kutta condition), we obtain

$$w(z) \approx 2\sqrt{2a}\sin\alpha V_\infty\sqrt{z'}.$$

It transpires that the local square-root behaviour (singularity) of $w(z)$ is the same as that in the problem studied in Part (a), with the equivalent

$$\tilde{V}_\infty = 2\sqrt{2a}\sin\alpha V_\infty.$$

Use of this in (46) gives the force parallel to the plate,

$$F_X = -2\pi a\rho\sin^2\alpha V_\infty^2. \qquad (48)$$

[5 marks]

sim. seen ⇓

(iii) In order to calculate $(p^+ - p^-)$, the Bernoulli equation is applied to the upper and lower surfaces to give

$$\frac{p^+}{\rho} + \frac{(\overline{V}^+)^2}{2} = \frac{p^-}{\rho} + \frac{(\overline{V}^-)^2}{2}, \quad \text{i.e.} \quad p^+ - p^- = -\frac{1}{2}\rho\Big[(\overline{V}^+)^2 - (\overline{V}^-)^2\Big].$$

Use of the given $\overline{V}^+$ and $\overline{V}^-$ gives

$$p^+ - p^- = -\frac{1}{2}\rho V_\infty^2\left\{\left[\cos\alpha + \sin\alpha\sqrt{\frac{a-x}{a+x}}\right]^2 - \left[\cos\alpha - \sin\alpha\sqrt{\frac{a-x}{a+x}}\right]^2\right\}$$

$$= -\rho V_\infty^2(2\sin\alpha\cos\alpha)\sqrt{\frac{a-x}{a+x}}.$$

The pressure force is perpendicular to the plate is evaluated as

$$F_Y = \rho V_\infty^2(2\sin\alpha\cos\alpha)\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx = 2\pi a\rho(\sin\alpha\cos\alpha)V_\infty^2,$$

where use is made of the value of the definite integral.

[3 marks]

sim. seen ⇓

(iv) Projecting F_X and F_Y to the direction of the oncoming flow, we obtain the drag,

$$D = F_Y\sin\alpha + F_X\cos\alpha = 2\pi a\rho(\sin^2\alpha\cos\alpha)V_\infty^2 - 2\pi a\rho\sin^2\alpha\cos\alpha V_\infty^2 = 0,$$

which is the manifestation of d'Alembert's paradox. Projecting F_X and F_Y to the direction perpendicular to that of the oncoming flow, we obtain the lift,

$$L = F_Y\cos\alpha - F_X\sin\alpha = 2a\pi\rho(\sin\alpha\cos^2\alpha)V_\infty^2 + 2a\rho\pi\sin^3\alpha V_\infty^2 = 2\pi a\rho\sin\alpha V_\infty^2,$$

which is consistent with the Joukovskii-Kutta formula.

[2 marks]

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Fluid Dynamics 1_MATH60001 MATH97008 MATH70001	1	All made good or decent attempts, and got the scores accordingly. The unfamiliar or relatively hard parts were tackled rather successfully by quite a few students.
Fluid Dynamics 1_MATH60001 MATH97008 MATH70001	2	All did well or very well except two who seemed underprepared. The very last part is hard, and was tackled by just a few.
Fluid Dynamics 1_MATH60001 MATH97008 MATH70001	3	All did well or very except one who seemed underprepared. There was unfortunately a typo (a missing term) in the question. Some of you noted this and gave the corrected answer. In any case, this part was marked generously to compensate possble impact and to ensure no one is disdvantaged.
Fluid Dynamics 1_MATH60001 MATH97008 MATH70001	4	All except three did well or very well. The challenging part was tackled rather successfully by quite a few students.
Fluid Dynamics 1_MATH60001 MATH97008 MATH70001	5	The majority only managed to do the first half of the question. The second half is unfamiliar and requires a more thorough understanding of the topic. Also the paper was rather long. These will be taken into account when setting the PEM thresholds.