MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Problem Sheet 8

Exercise 36.

(i) This linear differential equation $\dot{x} = Ax$ is already given in (real) Jordan normal form, and it follows that the eigenvalues are given by $-1 \pm i$. Hence, we have

$$\varphi(t, x) = e^{At}x \to 0 \quad \text{as } t \to \infty \quad \text{for all } x \in \mathbb{R}^2$$

and

$$\|\varphi(t, x)\| = \|e^{At}x\| \to \infty \quad \text{as } t \to -\infty \quad \text{for all } x \neq 0\,.$$

This implies that $\omega(x) = \{(0, 0)\}$ for all $x \in \mathbb{R}^2$, and

$$\alpha(x) = \begin{cases} \{(0,0)\} & : \quad x = (0,0)\,, \\ \emptyset & : \quad x \neq (0,0)\,. \end{cases}$$

(ii) This linear differential equation $\dot{x} = Ax$ has two real eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the trivial equilibrium is a saddle, and while in the eigenspace $\operatorname{span}(v_1)$ of the negative eigenvalue λ_1, solutions converge to 0 forward in time, the norm of all other solutions converges to ∞. Hence,

$$\omega(x) = \begin{cases} \{(0,0)\} & : \quad x \in \operatorname{span}(v_1)\,, \\ \emptyset & : \quad x \notin \operatorname{span}(v_1)\,. \end{cases}$$

Note that it is immediately clear that $\omega(x)$ for the differential equation $\dot{x} = Ax$ is the same as $\alpha(x)$ for the differential equation $\dot{x} = -Ax$ (since analysing the flow $e^{At}x$ for $t \to \infty$ is the same as analysing $e^{-At}x$ for $t \to -\infty$). Note that the eigenvectors v_1, v_2 are the same for $\dot{x} = -Ax$, but the eigenvalues switch to $\tilde{\lambda}_1 = -\lambda_1 = 2$ and $\tilde{\lambda}_2 = -\lambda_2 = -4$. So $\dot{x} = -Ax$ is a saddle system as well, and the same arguments as above imply for $\alpha(x)$ of $\dot{x} = Ax$ that

$$\alpha(x) = \begin{cases} \{(0,0)\} & : \quad x \in \operatorname{span}(v_2)\,, \\ \emptyset & : \quad x \notin \operatorname{span}(v_2)\,. \end{cases}$$

(iii) To understand eigenvalues of this linear differential equation $\dot{x} = Ax$, we use the trace-determinate rule from Repetition Material 5, and get that $\operatorname{tr} A = 6$ and $\det A = 33$. This implies that both eigenvalues have positive real part. Hence,

$$\omega(x) = \begin{cases} \{(0,0)\} & : \quad x = (0,0)\,, \\ \emptyset & : \quad x \neq (0,0)\,. \end{cases}$$

Using the arguments in (ii), and using the fact that the eigenvalues of $\dot{x} = -Ax$ have negative real part, we obtain $\alpha(x) = \{(0, 0)\}$ for all $x \in \mathbb{R}^2$.

(iv) All solutions are monotone for this autonomous one-dimensional differential equation (Exercise 16 (i)) and if they converge, the limit must be an equilibrium (Exercise 16 (ii)). This differential equation has the two equilibria -1 and 0, and the right hand side is positive for $x < -1$ and $x > 0$, and it is negative for $x \in (-1, 0)$. Using Exercise 16, we get

$$\omega(x) = \begin{cases} \{-1\} & : \quad x \in (-\infty, 0)\,, \\ \{0\} & : \quad x = 0\,, \\ \emptyset & : \quad x \in (0, \infty)\,, \end{cases}$$

and

$$\alpha(x) = \begin{cases} \emptyset & : \quad x \in (-\infty, -1)\,, \\ \{-1\} & : \quad x = -1\,, \\ \{0\} & : \quad x \in (-1, \infty)\,. \end{cases}$$

Exercise 37.

We first look at the differential equation

$$\begin{aligned} \dot{x} &= -y + x^3\,, \\ \dot{y} &= x + y^3\,. \end{aligned}$$

The linearisation in the equilibrium $(0,0)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\pm i$ and lie on the imaginary axis, so we cannot conclude stability for the nonlinear system from its linearisation. Consider the function $V(x,y) := x^2 + y^2$. Its orbital derivative is given by

$$\dot{V}(x,y) = (2x, 2y)\begin{pmatrix} -y + x^3 \\ x + y^3 \end{pmatrix} = 2x(-y + x^3) + 2y(x + y^3) = 2x^4 + 2y^4\,,$$

which is positive except when $(x,y) = (0,0)$. This means that the flow φ increases its distance from the equilibrium forward in time. We now show that this implies instability. Choose $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then

$$m := \min\left\{\dot{V}(x,y) = 2(x^4 + y^4) : x^2 + y^2 = x_0^2 + y_0^2\right\} > 0$$

has a positive minimum on the compact circle with radius $\sqrt{x_0^2 + y_0^2}$, and we even get that

$$m = \min\left\{\dot{V}(x,y) = 2(x^4 + y^4) : x^2 + y^2 \geq x_0^2 + y_0^2\right\}.$$

It follows that

$$V(\varphi(t, x_0, y_0)) - V(x_0, y_0) = \int_0^t \dot{V}(\varphi(s, x_0, y_0))\,\mathrm{d}s \geq \int_0^t m\,\mathrm{d}s \to \infty \text{ as } t \to \infty\,.$$

This implies that $\|\varphi(t, x_0, y_0)\| \to \infty$ as $t \to \sup J_{max}(x_0, y_0)$. Hence, the equilibrium $(0,0)$ is repelling and thus unstable.

We now consider the differential equation

$$\begin{aligned} \dot{x} &= 2xy + x^3\,, \\ \dot{y} &= -x^2 + y^5\,. \end{aligned}$$

We compute $f'(0,0) = 0 \in \mathbb{R}^{2\times 2}$, so both eigenvalues are 0, and stability cannot be analysed using the linearisation. We show that $(0,0)$ is unstable similarly to the above example. We aim at finding a quadratic function of the form $V(x,y) = ax^2 + bxy + cy^2$ whose orbital derivative is positive outside of the trivial equilibrium. We calculate

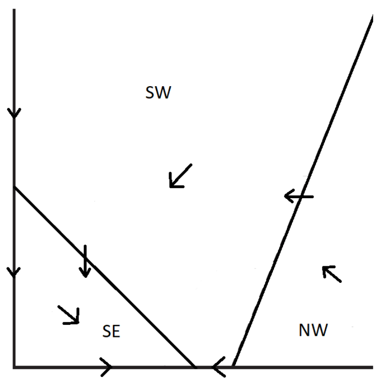
$$\dot{V}(x,y) = 2ax(2xy + x^3) + b(2xy + x^3)y + bx(-x^2 + y^5) + 2cy(-x^2 + y^5)$$

If we set $a = 1$, $c = 2$ and $b = 0$, then we ensures that $\dot{V}(x,y) = 2x^4 + 4y^6 > 0$ whenever $(x,y) \neq (0,0)$. Similarly to the first example, one establishes that the trivial equilibrium is repelling and thus unstable.

Exercise 38.

We restrict the analysis to $x, y \geq 0$.

(i) The nullcline $\dot{x} = 0$ is given by $x = 0$ and $y = 1 - x$. The nullcline $\dot{y} = 0$ is given by $y = 0$ and $y = x - 2$, and illustrated in the figure below. We also draw the arrows on the nullclines to understand



how the flow moves at the nullclines (note that $x = 0$ and $y = 0$ are invariant sets, which is special for nullclines, but also occurred in Exercise 31). We see that we have the three regions SW (on which $\dot{x} < 0$ and $\dot{y} < 0$), SE (on which $\dot{x} > 0$ and $\dot{y} < 0$) and NW (on which $\dot{x} < 0$ and $\dot{y} > 0$).

(ii) The equilibria in the relevant quadrant are $(0,0)$, $(1,0)$ (note that both $(0,-2)$ and $(\frac{3}{2}, -\frac{1}{2})$ are outside of this quadrant). If f denotes the right hand side of this differential equation, then we get

$$f'(x,y) = \begin{pmatrix} 1 - y - 2x & -x \\ y & x - 2 - 2y \end{pmatrix} .$$

The linearisation at $(0,0)$ is thus

$$f'(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} ,$$

so this is a saddle and thus unstable. The linearisation at $(1,0)$ is given by

$$f'(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} ,$$

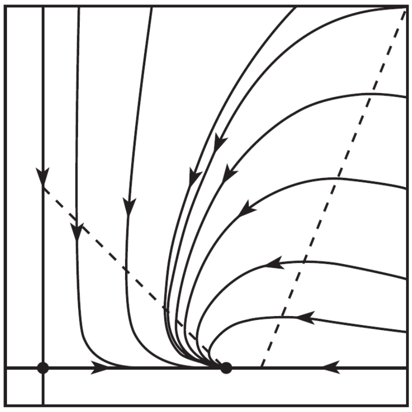
and we see that both eigenvalues are -1, so the equilibrium is exponentially stable using the linearised stability result.

(iii) It follows from the arrows that the region SE is positively invariant, and since SE is bounded, the flow in SE is bounded forward in time, so it has to converge to an equilibrium due to Exercise 16 (ii). Due to moving to the right, it can only converge to the exponentially stable equilibrium $(1,0)$, since the other equilibrium $(0,0)$ is excluded for this reason.

If the flow starts in SW, then it either stays in SW or leaves to SE. If it stays in SW, then it must be bounded forward in time (which is clear from the picture), so it needs to converge to an equilibrium, and the only equilibrium close to SW is $(1,0)$, so it must converge to this equilibrium . If it does not stay in SW, it must move to SE, and then, as we have seen above, it will converge to $(1,0)$. (As a side remark, note that it is possible to show that the orbit cannot stay in SW forward in time, which follows from the linearisation of $(1,0)$, which is approached via one tangent; but this insight is not necessary for the solution to the exercise).

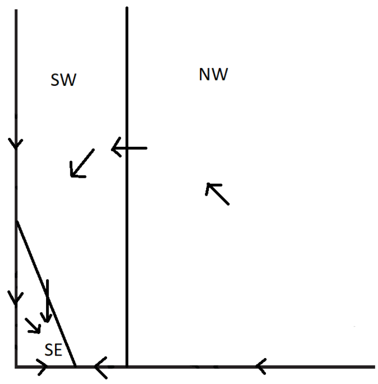
If the flow starts in NW, then it either stays in NW or leaves to SW. If it stays in SW, then it must be bounded forward in time (which is clear from the picture), so it needs to converge to an equilibrium, but there is no equilibrium close to NW, so it must go to SW forward in time, and the above reasoning shows that it will then converge to $(1, 0)$.

The phase portrait of this differential equation in the first quadrant is given as follows.



Exercise 39.

(i) $\dot{x} = 0 = x(-y + 1 - 2x)$ if and only if $x = 0$ or $y = 1 - 2x$. $\dot{y} = 0 = y(x - 1)$ if and only if $y = 0$ or $x = 1$. The nullclines bound three regions NW, SW and SE. Setting both $\dot{x} = 0$ and $\dot{y} = 0$ gives two equilibria in the first quadrant: $(0, 0)$ and $(\frac{1}{2}, 0)$.



(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote the right hand side of this differential equation. Then the derivative of f is given by

$$f'(x, y) = \begin{pmatrix} -y + 1 - 4x & -x \\ y & x - 1 \end{pmatrix},$$

and we get for the equilibrium $(0, 0)$ that

$$f'(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so $(0, 0)$ is hyperbolic and a saddle. For the equilibrium $(\frac{1}{2}, 0)$, we get

$$f'(\tfrac{1}{2}, 0) = \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix},$$

so this equilibrium is hyperbolic and attractive.

(iii) The region M_R is bounded by four line segments, and we study the vector field along these line segments to conclude M_R is positively invariant for $R > 2$.

(a) The line segment $x = 0$, $y > 0$ is invariant and thus the flow cannot leave via this line segment.

(b) The line segment $y = 0$, $x > 0$ is invariant and thus the flow cannot leave via this line segment.

(c) On the line segment $y = R - 1$, $0 \le x \le 1$, we have $\dot{y} < 0$, so the vector field points into the region M_R and thus the flow cannot leave via this line segment.

(d) We finally consider the line segment $x + y = R$, where $1 \le x \le R$ and conclude that on this line segment we have

$$\dot{x} + \dot{y} = x(-y + 1 - 2x) + y(x - 1) = x(-R + 1 - x) + y(x - 1) < 0\,,$$

since $R > 2$, so the flow cannot leave this line segment as well forward in time.

We thus have established that the region M_R is positively invariant whenever $R > 2$.

(iv) Let $(x, y) \in (0, \infty)^2$. Then there exists an $R > 0$ such that $(x, y) \in M_R$. Hence $\varphi(t, x, y) \in M_R$ for all $t \ge 0$, and since M_R is compact, the solution cannot escape to ∞ in finite time. The theorem on maximal solutions then implies that $J_{max}((x, y)) = \infty$.

(v) Let $(x, y) \in (0, \infty)^2$. If $x \le 1$, nothing needs to be shown (note that on the line $x = 1$, we have $\dot{x} < 0$. If $x > 1$, then the flow starts in NW. We know that the flow is bounded forward in time from (iv), and if we assume that the flow stays in NW, then it needs to converge due to monotonicity in both components. However, there are no equilibria in (or close to) NW, so the flow has to leave NW. Leaving NW means that the first component needs to cross the nullcline $x = 1$, which we needed to show.

(vi) A periodic orbit needs to intersect all four regions NW, SW, SE, and NE. Since there is no region NE in the first quadrant, we cannot have a periodic orbit in the first quadrant.

(vii) We have already established in (v) that if we start the flow in NW, then we need to move to SW. If the flow is in SW and stays there, then it must converge to an equilibrium (with the same reasoning as in (v)). There is only the equilibrium $(\frac{1}{2}, 0)$ close to SW, so we either converge to this equilibrium (which would finish the proof), or we need to end up in SE, if we start somewhere in $(0, \infty)^2$. SE is invariant (which follows from the arrows), so the flow has to converge to an equilibrium (with the same reasoning as in (v)). There are two equilibria close to SE: $(0, 0)$ and $(\frac{1}{2}, 0)$. We cannot converge to the saddle equilibrium $(0, 0)$, since its stable manifold is one-dimensional and given by the y-axis. This means that we must converge to $(\frac{1}{2}, 0)$ forward in time, which finishes the proof.

Exercise 40.

(i) The linearisation of this differential equation in the equilibrium $(0, 0)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\pm i$ and lie on the imaginary axis, so we cannot conclude stability for the nonlinear system from its linearisation.

(ii) We make a quadratic polynomial ansatz for the Lyapunov function $V(x, y) = ax^2 + bxy + cy^2$, and we get for its orbital derivative

$$\begin{aligned} \dot{V}(x, y) &= (2ax + by, bx + 2cy) \begin{pmatrix} -y - x^3 \\ x - y^3 \end{pmatrix} \\ &= -2axy - by^2 + bx^2 + 2cyx - 2ax^4 - bx^3y - bxy^3 - 2cy^4\,, \end{aligned}$$

and setting $a = c = 1$ and $b = 0$ gives

$$\dot{V}(x, y) = -2x^4 - 2y^4\,.$$

We have $V(0,0) = 0$, $V(x,y) > 0$ for all $(x,y) \neq (0,0)$, $\dot{V}(0,0) = 0$ and $\dot{V}(0,0) < 0$ for all $(x,y) \neq (0,0)$. This means that Lyapunov's direct method for asymptotic stability is applicable, which implies asymptotic stability of $(0,0)$. In addition, Corollary 4.30 says that all sublevel sets

$$S_c := \left\{ x \in \mathbb{R}^2 : V(x) \leq c \right\},$$

where $c > 0$, are subsets of the domain of attraction if they are compact. It is clear that S_c is compact for the above Lyapunov function, and moreover, we have

$$\bigcup_{c>0} S_c = \mathbb{R}^2,$$

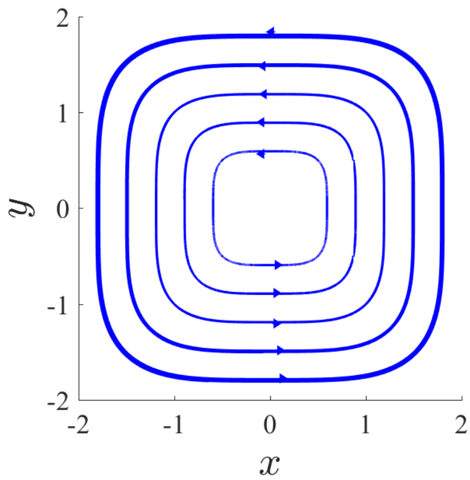
so the equilibrium is globally attracting, i.e. $W^s(0,0) = \mathbb{R}^2$.

Exercise 41.

We consider the function $V(x,y) := x^4 + y^4$. V is a Lyapunov function, since we have

$$\dot{V}(x,y) = (4x^3, 4y^3) \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix} = -4x^3y^3 + 4y^3x^3 = 0\,.$$

Since $V(0,0) = 0$ and $V(x,y) > 0$, Lyapunov's direct method for stability implies that $(0,0)$ is stable. The equilibrium is not asymptotically stable, however, since the V is constant along all orbits, so if $(x,y) \neq (0,0)$, then we have $V(x,y) > 0$ and $V(\varphi(t,x,y)) = V(x,y) > 0$ for all $t \geq 0$. Then we cannot have $\lim_{t\to\infty} \varphi(t,x,y) = (0,0)$, since this would imply that $\lim_{t\to\infty} V(\varphi(t,x,y)) = V(0,0) = 0$, which contradicts the previous statement. The level sets of V form closed curves around $(0,0)$. The phase portrait has a unique equilibrium at $(0,0)$ and closed curves with periodic solutions around it, turning anti-clockwise, indicative of the flow on the level sets.



Exercise 42.

(i) Consider the function $V(x,y) = x^2 + y^2$, and we compute the orbital derivative

$$\begin{aligned} \dot{V}(x,y) &= (2x, 2y) \begin{pmatrix} y \\ -x - (2x^2 + 3y^2 - 1)y \end{pmatrix} = 2xy + 2y(-x - (2x^2 + 3y^2 - 1)y) \\ &= 2y^2(1 - 2x^2 - 2y^2 - y^2)\,. \end{aligned}$$

For $x^2 + y^2 < \frac{1}{3}$, we have

$$\dot{V}(x,y) > 2y^2(1 - \tfrac{2}{3} - y^2) > 0\,,$$

and for $x^2+y^2>\frac{1}{2}$, we have
$$\dot V(x,y) < 2y^2(1-1-y^2) < 0\,.$$
Note that $\dot V>0$ along the inner boundary of the annulus means that orbits starting on the inner boundary flow inside the annulus, and similarly $\dot V<0$ along the outer boundary means that orbits starting on the outer boundary flow also inwards the annulus. Hence, the annulus is positively invariant.

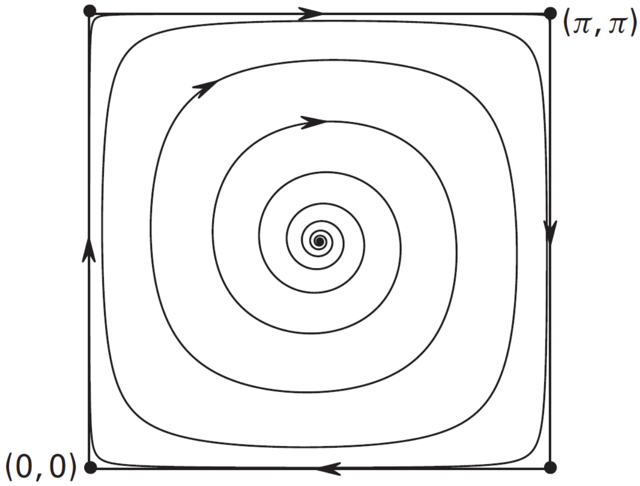
(ii) We compute the equilibria of this differential equation. $\dot x=0$ implies that $y=0$. Then the $\dot y=0$ reads as $0=-x$, so the trivial equilibrium is the only equilibrium. In particular, the positively invariant annulus does not contain an equilibrium, so the corollary of the Poincaré–Bendixson theorem implies the omega limit set of all orbits starting in the annulus is a periodic orbit. Hence, there exists a periodic orbit in the annulus.

Exercise 43.

(i) We first compute the equilibria. Suppose that $\dot x=0$ and $\dot y=0$. The first equation the implies that either $\sin(x)=0$ or $-\frac{1}{10}\cos(x)-\cos(y)=0$. If $\sin(x)=0$ (meaning that $x\in\{0,\pi\}$), then the second equation implies that $\sin(y)=0$, since $\cos(x)-\frac{1}{10}\cos(y)\neq 0$ for all $y\in[0,\pi]$ (note that $x\in\{0,\pi\}$). This means that $(0,0),(0,\pi),(\pi,0)$ and (π,π) are equilibria. Similarly, $\sin(y)=0$ from the second equation implies that $\sin(x)=0$, and this does not lead to additional equilibria. Additional equilibria may be found when $-\frac{1}{10}\cos(x)-\cos(y)=0$ and $\cos(x)-\frac{1}{10}\cos(y)=0$. This corresponds to $\cos(x)=\cos(y)=0$, giving a fifth equilibrium $(\frac{\pi}{2},\frac{\pi}{2})$.
One verifies that the linearisation at $(0,0)$ is
$$\begin{pmatrix}-1.1 & 0\\ 0 & 0.9\end{pmatrix},$$
so $(0,0)$ is a saddle. It is also straightforward to see that $(0,\pi),(\pi,0)$ and (π,π) are saddles. In addition, the linearisation at $(x,y)=(\frac{\pi}{2},\frac{\pi}{2})$ is given by
$$\begin{pmatrix}0.1 & 1\\ -1 & 0.1\end{pmatrix},$$
So the eigenvalues there are $0.1\pm i$, and $(\frac{\pi}{2},\frac{\pi}{2})$ is an unstable focus.

(ii) To determine the global dynamics, we consider the function $V(x,y):=\sin(x)\sin(y)$ for all $0\le x,y\le\pi$. Then V is positive in the interior of $[0,\pi]^2$, so on $(0,\pi)^2$, and it is zero on the boundary. Note that
$$\begin{aligned}\dot V(x,y) &= \cos(x)\sin(y)\big(\sin(x)(-0.1\cos(x)-\cos(y))\big)+\sin(x)\cos(y)\big(\sin(y)(\cos(x)-0.1\cos(y))\big)\\ &= -0.1\sin(x)\cos^2(x)\sin(y)-0.1\sin(x)\sin(y)\cos^2(y)\,.\end{aligned}$$
Hence $\dot V(x,y)<0$ if and only if $(x,y)\in(0,\pi)^2\setminus\{\frac{\pi}{2},\frac{\pi}{2}\}$. La Salle's principle implies that
$$\omega(x,y)\subset\big\{(\bar x,\bar y)\in[0,\pi]^2:\dot V(\bar x,\bar y)=0\big\}=\partial[0,\pi]^2\cup\{\tfrac{\pi}{2},\tfrac{\pi}{2}\}\quad\text{for all }\ (x,y)\in(0,\pi)^2\setminus\{\tfrac{\pi}{2},\tfrac{\pi}{2}\}\,.$$
It is clear that $\{\frac{\pi}{2},\frac{\pi}{2}\}$ cannot be in the omega limit set, since this equilibrium is repelling. Since the four equilibrium points in the boundary are saddles, it follows that the omega limit set is given by the whole boundary of the square $[0,\pi]^2$. More precisely, if the omega limit set is a proper subset of the boundary of the square, then having saddle equilibria implies that the flow will leave any neighbourhood of the omega limit set infinitely often, which would imply that the omega limit set is bigger (since outside of this neighbourhood, the flow will accumulate somewhere due to compactness of the space).

(iii) The phase portrait is given as follows: (please see next page)



Exercise 44.

Assume that $\omega(x)$ is disconnected. Then there exist two disjoint nonempty and closed sets $M_1 \subset \mathbb{R}^d$ and $M_2 \subset \mathbb{R}^d$ such that

$$\omega_1 := M_1 \cap \omega(x) \neq \emptyset\,, \quad \omega_2 := M_2 \cap \omega(x) \neq \emptyset\,, \quad \text{and} \quad \omega(x) \subset M_1 \cup M_2\,.$$

We have $\omega(x) = \omega_1 \cup \omega_2$, and the compactness of $\omega(x)$, which we get from Proposition 4.21, implies that ω_1 and ω_2 are compact. We show that there exists an $\varepsilon > 0$ such that $B_\varepsilon(\omega_1) \cap \omega_2 = \emptyset$. Assume that that this is not true, which implies that

$$B_{\frac{1}{n}}(\omega_1) \cap \omega_2 \neq \emptyset.$$

Hence there exists a sequence in $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n \in B_{\frac{1}{n}}(\omega_1) \cap \omega_2$ for all $n \in \mathbb{N}$, and due to compactness of ω_2, we get a convergence subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ that converges to an element in $\omega_1 \cap \omega_2 = \emptyset$, so we have a contradiction. Hence, there exists an $\varepsilon > 0$ such that $B_\varepsilon(\omega_1) \cap \omega_2 = \emptyset$, and in particular

$$B_{\frac{\varepsilon}{2}}(\omega_1) \cap B_{\frac{\varepsilon}{2}}(\omega_2) = \emptyset \quad \text{for all } n \in \mathbb{N}\,. \tag{D}$$

Since $\varphi(t_n, x) \in B_{\frac{\varepsilon}{2}}(\omega_1)$ for a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$, and $\varphi(s_n, x) \in B_{\frac{\varepsilon}{2}}(\omega_2)$ for a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to \infty$ as $n \to \infty$, and (D), due to continuity of φ, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\varphi(u_n, x) \notin B_{\frac{\varepsilon}{2}}(\omega_1) \cup B_{\frac{\varepsilon}{2}}(\omega_2)$. The sequence

$$\gamma_n := \varphi(u_n, x) \quad \text{for all } n \in \mathbb{N}$$

is bounded and has a convergent subsequence $\{\gamma_{n_\ell}\}_{\ell\in\mathbb{N}}$, where $\gamma_{n_\ell} \to \Gamma$ as $\ell \to \infty$. Clearly, $\Gamma \in \omega(x)$, but $\Gamma \notin \omega_1 \cup \omega_2$, which is a contradiction.