Imperial College London

MATH97064 MATH70010

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometric Mechanics

Date: 17 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. **Noether's Theorem for geodesic motion on a Riemannian manifold**

Consider a geodesic flow on a Riemannian manifold Q with metric $ds^2 = dq^i g_{ij}(q) dq^i$ for $q \in Q$ in which $i, j = 1, 2, \ldots, n$ for $\dim Q = n$.

The objective of this problem is to prove that when Hamilton's principle with Lagrangian

$$L_{geo}(q, \dot{q}) = \frac{1}{2} \dot{q}^i g_{ij}(q) \dot{q}^j$$

admits the isometry group $G : G \times Q \to Q$ of the Riemannian metric, then the Poisson brackets among the conservation laws of the corresponding geodesic Euler-Lagrange (EL) flow with Lagrangian $L_{geo}(q, \dot{q})$ are isomorphic to the Lie algebra $\mathfrak{g} \simeq T_e G$ of the isometry group G.

Proceed as follows:

(a) Explain in a very brief statement how the isometry group G is related to the invariance group for the Lagrangian $L_{geo}(q, \dot{q}) = \frac{1}{2} \dot{q}^i g_{ij}(q) \dot{q}^j$. (2 marks)

(b) (i) Compute the Noether quantity. (This is the endpoint term in Hamilton's principle for $0 = \delta S = \delta \int_a^b L_{geo}(q, \dot{q}) dt$ when $\delta q = \mathcal{L}_\xi q$.) (2 marks)

(ii) Show that the Noether quantity yields a momentum map $J(q, p)$ associated to the canonical G-action on T^*Q with Hamiltonian function of the form $J_\xi(q, p) = \langle J(q, p), \xi \rangle$, where the symbol $\langle \cdot \,, \cdot \rangle$ denotes pairing of the isometry Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. (4 marks)

(iii) Explicitly write the momentum map $J(q, p)$ of the isometry group G of $L_{geo}(q, \dot{q})$. (2 marks)

(c) Compute the canonical Poisson brackets $\{J_\xi(q, p), J_\eta(q, p)\}_{can}$ among the Noether Hamiltonians for the symmetries of the Lagrangian $L_{geo}(q, \dot{q})$ and show they are isomorphic to the Lie algebra $\mathfrak{g}$ of the isometry group G. (10 marks)

(Total: 20 marks)

2. **Euler-Lagrange equations for geodesics on $SO(3)$**

A three-dimensional spatial rotation is described by multiplication of a spatial vector in $\mathbb{R}^3$ by a 3×3 special orthogonal matrix, denoted $O \in SO(3)$,

$$O^T \mathbb{I} O = \mathbb{I}\,, \quad \text{so that} \quad O^{-1} = O^T \quad \text{and} \quad \det O = 1\,,$$

where $\mathbb{I}$ is the 3×3 identity matrix.

Geodesic motion on the space of rotations in three dimensions may be represented as a curve $O(t) \in SO(3)$ depending on time t. Its angular velocity is defined as the 3×3 matrix $\widehat{\Omega}$,

$$\widehat{\Omega}(t) = O^{-1}(t)\dot{O}(t) \in \mathfrak{so}(3)\,.$$

(a) Show that $\widehat{\Omega}(t) \in \mathfrak{so}(3)$ is skew symmetric. (3 marks)

(b) Show that the variational derivative $\delta\widehat{\Omega} = \frac{d}{ds}\Big|_{s=0}\widehat{\Omega}(t,s)$ of angular velocity $\widehat{\Omega} = O^{-1}\frac{d}{dt}O \in \mathfrak{so}(3)$ satisfies

$$\delta\widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,,$$

in which $\widehat{\Xi}(t) = O^{-1}(t)\delta O(t) = O^{-1}(t)\frac{d}{ds}\Big|_{s=0}O(t,s) \in \mathfrak{so}(3)$. (3 marks)

(c) Compute the Euler-Lagrange equations for Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int L(\widehat{\Omega})\,dt\,,$$

for the quadratic Lagrangian $L : TSO(3) \to \mathbb{R}$,

$$L(\widehat{\Omega}) = -\frac{1}{2}\mathrm{tr}(\widehat{\Omega}\mathbb{A}\widehat{\Omega})\,,$$

in which $\mathbb{A}$ is a symmetric, positive-definite, invertible 3×3 matrix. (8 marks)

(d) The solution of the resulting Euler-Lagrange equations represents a geodesic flow on the Lie group manifold $SO(3)$ with constant Riemannian metric $\mathbb{A} = \mathbb{A}^T$.

(i) What is the isometry group G of this Riemannian metric? (2 marks)

(ii) What conservation law is implied by the action of the isometry group on the 3×3 symmetric matrix $\mathbb{A}$? (4 marks)

(Total: 20 marks)

3. **Dynamics for a composition of two maps**

The Lagrangian for the dynamics of a flywheel mounted on the 2-axis of a rigid body is defined on $\mathfrak{so}(3) \times TS^1 \simeq \mathbb{R}^3 \times TS^1$. In coordinates, this Lagrangian is denoted as $L(\mathbf{\Omega}, \Omega_2 + \dot{\phi})$, with $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T \in \mathbb{R}^3$ and $\dot{\phi} := d\phi/dt \in TS^1$.

(a) Apply Hamilton's principle to derive the combined Euler-Poincaré equations and Euler-Lagrange equations for this Lagrangian.

(5 marks)

(b) Legendre transform the Lagrangian to the Hamiltonian formulation for this system and write its combined equations of motion in Poisson matrix form. (5 marks)

(c) Derive the Poisson matrix form of this system and identify a symplectic 2-cocycle.

(10 marks)

(Total: 20 marks)

4. Calculus on differential forms

The objective of this question is to verify fundamental formulas for operations of vector fields (v) on differential forms by using the equivalence of the dynamical form and the geometric (or, Cartan) form of the operation of the Lie derivative $(\mathcal{L}_v)$ on differential k-forms in terms of insertion of vector fields (i_v) and differential operator (d),

$$\mathcal{L}_v = \frac{d}{dt}\phi_t^*\Big|_{t=0} = (i_v)d + d(i_v) = v \,\lrcorner\, d + d(v \,\lrcorner\,)\,,$$

as well as, for example, $d^2 = 0$, $i_v i_v = 0 = v \,\lrcorner\, (v \,\lrcorner\,)$ and the product rule for the pullback ϕ_t^*.

(a) Verify by a direct calculation that the formula $\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0}\phi_t^* Y = [X, Y]$, where $[X, Y] = XY - YX$ is the commutator of the vector fields X and Y. (2 marks)

(b) Verify the formula $i_{[X\,,Y]} = \Big[\mathcal{L}_X\,,\, i_Y\Big]$. (3 marks)

(c) Use (b) to verify $\mathcal{L}_{[X\,,Y]} = \Big[\mathcal{L}_X\,,\, \mathcal{L}_Y\Big]$. (3 marks)

(d) Use the pull-back relation $\phi_t^*\Big[X,\, Y\Big] = \Big[\phi_t^* X,\, \phi_t^* Y\Big]$ and the product rule to verify the Jacobi identity $[Z, [X, Y]] + cyclic\ permutations\ = 0$ for $\frac{d}{dt}\phi_t^*\Big|_{t=0} = \mathcal{L}_Z$. (No need to compute the Jacobiator explicitly!) (2 marks)

(e) Use Cartan's form of the Lie derivative operator $\mathcal{L}_v$ to verify that

$$\phi_t^* = e^{t\mathcal{L}_v} = \sum_0^\infty \frac{t^n}{n!}\Big(d\, i_v\Big)^n + \sum_0^\infty \frac{t^n}{n!}\Big(i_v d\Big)^n = e^{t\,(d\,i_v)} + e^{t\,(i_v\,d)}$$

for

$$\left[\frac{d}{dt}\phi_t^*\right]_{t=0} =: \mathcal{L}_v \quad \text{with} \quad v = \left[\phi_t^{-1}\frac{d\phi_t}{dt}\right]_{t=0}\,,$$

in the sense of Taylor expansions of exponentials of operators acting on differential k-forms. (10 marks)

(Total: 20 marks)

5. **Elastic spherical pendulum – swinging spring – in body variables**

In body variables, the Lagrangian for the dynamics of a swinging spring under gravity g is given by

$$L_b(\mathbf{\Omega}, \mathbf{\Gamma}, R, \dot{R}) = \frac{m}{2} R^2(t)\, |\mathbf{\Omega} \times \mathbf{r}_0|^2 + \frac{m}{2} \dot{R}^2(t) |\mathbf{r}_0|^2 - mg\, \mathbf{\Gamma} \cdot R(t) \mathbf{r}_0 - \frac{k}{2} \Big(R(t) - 1\Big)^2 |\mathbf{r}_0|^2 \,.$$

Here, $\mathbf{r}_0$ is the vector from the point of the pendulum's support to the bob of mass, m, at the initial unstretched length, $|\mathbf{r}_0|$. The scalar function $R(t)$ measures the stretch or contraction of the pendulum. (The pendulum does *not* bend as it swings.)

The time-dependent solution path $\mathbf{x}(t) \in \mathbb{R}^3$ for the Euler–Lagrange equations as seen from the body frame has been lifted into the Lie group $SO(3) \times \mathbb{R}_+$ of rotating and scaling vectors in $\mathbb{R}^3$ by specifying its direct-product action on an initial position $\mathbf{x}_0 \in \mathbb{R}^3$ as the composition of these two maps:

$$\mathbf{x}(t) = R(t) \mathbf{r}(t) \,, \quad \mathbf{r}(t) = O(t) \mathbf{r}_0 \,, \quad |\mathbf{r}(t)|^2 = |\mathbf{r}_0|^2 \,, \quad \text{for} \quad \Big(R(t), O(t)\Big) \in \mathbb{R}_+ \times SO(3) \,,$$

with body angular velocity $O^{-1}\dot{O}(t) = \mathbf{\Omega}\times$ and body orientation of the vertical $\mathbf{\Gamma}(t) := O^{-1}\hat{\boldsymbol{e}}_3 \in \mathbb{R}^3$. For fixed $R(t) = 1$ and $\dot{R} = 0$ the length is fixed at $|\mathbf{r}_0|$, and the problem reduces to dynamics of the standard spherical pendulum.

(a) Take variations of the Lagrangian and derive the Euler-Poincaré equation for $\mathbf{\Pi}(t)$ and the evolution equation for $\mathbf{\Gamma}(t)$ that arises from its definition. In this derivation, note that for a point-mass pendulum bob, the body angular momentum $\mathbf{\Pi} = \partial L_b / \partial \mathbf{\Omega}$ and body angular velocity $\mathbf{\Omega}$ will be co-linear, so the Lagrangian is hyperregular. (10 marks)

(b) Legendre transform the Lagrangian $L_b(\mathbf{\Omega}, \mathbf{\Gamma}, R, \dot{R})$ written above to obtain the corresponding Hamiltonian in the body variables $H_b(\mathbf{\Pi}, \mathbf{\Gamma}, R, P)$. (2 marks)

(c) Write the corresponding Hamiltonian system in Poisson matrix form and identify a symplectic 2-cocycle.

(8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH60010/70010/97064

Geometric Mechanics (Solutions)

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Holm	Alexander	Wu

1. (a) For symmetry generators that have no explicit time dependence, the Lie symmetry group of the Lagrangian $L_{geo}(q,\dot q)$ is also the isometry group G of the metric. seen ⇓

2, A

(b) (i) A standard calculation of the EL equations from Hamilton's principle with Lagrangian $L(q,\dot q)$ yields the Euler-Lagrange equations, plus an endpoint term obtained from integration by parts in time defined by the pairing seen ⇓

$$\Big\langle \frac{\partial L}{\partial \dot q}\,,\,\delta q\Big\rangle_{TQ}\Big|_a^b =: \langle p\,,\,\delta q\rangle_{TQ}\Big|_a^b .$$

2, A

sim. seen ⇓

(ii) The *Noether quantity* is defined to be $\langle p\,,\,\delta q\rangle_{TQ}$ for $\delta q = \mathcal{L}_\xi q$. Hence, the momentum map is obtained in the required form as

$$\langle p\,,\,\delta q\rangle_{TQ} = \langle p\,,\,\mathcal{L}_\xi q\rangle_{TQ} =: \big\langle -\,p\diamond q\,,\,\xi\big\rangle_{\mathfrak{g}} = \langle J(q,p),\xi\rangle = J_\xi(q,p)\,.$$

4, D

seen ⇓

(iii) The momentum map $J(q,p)$ of the isometry group G of $L_{geo}(q,\dot q)$ is the quantity $J(q,p) = -\,p\diamond q$. This form holds in general for geodesic problems.

2, A

sim. seen ⇓

(c) The canonical Poisson brackets $\{J_\xi(q,p),J_\eta(q,p)\}_{can}$ among the Noether Hamiltonians for the symmetries of the Lagrangian $L_{geo}(q,\dot q)$ are shown to be isomorphic to the Lie algebra $\mathfrak{g}$ of the isometry group G, as follows.

$$\begin{aligned}
\{J_\xi(q,p),J_\eta(q,p)\}_{can} &= \{J_\xi(q,p),\langle p\diamond q\,,\,\eta\rangle_{\mathfrak{g}}\}_{can}\\
&= \langle \{J_\xi,p\}_{can}\diamond q + p\diamond\{J_\xi,q\}_{can}\,,\,\eta\rangle_{\mathfrak{g}}\\
&= \langle (-\mathcal{L}_\xi^T p)\diamond q + p\diamond(\mathcal{L}_\xi q)\,,\,\eta\rangle_{\mathfrak{g}}\\
&= \langle \mathcal{L}_\xi(p\diamond q)\,,\,\eta\rangle_{\mathfrak{g}} = \langle \mathrm{ad}^*_\xi(p\diamond q)\,,\,\eta\rangle_{\mathfrak{g}}\\
&= \langle p\diamond q\,,\,\mathrm{ad}_\xi\eta\rangle_{\mathfrak{g}} = -\langle p\,,\,\mathcal{L}_{[\xi,\eta]}q\rangle_{TQ} = J_{[\eta,\xi]}(q,p)
\end{aligned}$$

10, D

2. (a) The problem states, $\boxed{\text{meth seen} \Downarrow}$

$$O^T(t)\mathbb{I}O(t) = \mathbb{I}\,, \quad \text{or, equivalently,} \quad \mathbb{I}^{-1}O^T(t)\mathbb{I} = O^{-1}(t)\,,$$

for the 3×3 symmetric matrix $\mathbb{I} = \mathbb{I}^T$.
The matrices $O(t)$ are *not* orthogonal, unless $\mathbb{I} = \mathrm{Id}$. $\boxed{3,\ \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(b) The required variational formula

$$\delta\widehat{\Omega} = \widehat{\Xi}^{\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,,$$

in which $\widehat{\Xi} = O^{-1}\delta O$ follows by subtracting the time derivative $\widehat{\Xi}^{\cdot} = (O^{-1}\delta O)^{\cdot}$ from the variational derivative $\delta\widehat{\Omega} = \delta(O^{-1}\dot{O})$ in the relations

$$\begin{aligned}
\delta\widehat{\Omega} &= \delta(O^{-1}\dot{O}) = -\,(O^{-1}\delta O)(O^{-1}\dot{O}) + \delta\dot{O} = -\,\widehat{\Xi}\widehat{\Omega} + \delta\dot{O}\,,\\
\widehat{\Xi}^{\cdot} &= (O^{-1}\delta O)^{\cdot} = -\,(O^{-1}\dot{O})(O^{-1}\delta O) + (\delta O)^{\cdot} = -\,\widehat{\Omega}\widehat{\Xi} + (\delta O)^{\cdot}\,,
\end{aligned}$$

and using equality of cross derivatives $\delta\dot{O} = (\delta O)^{\cdot}$. $\boxed{4,\ \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(c) Taking matrix variations in this Hamilton's principle yields

$$\begin{aligned}
\delta S =: \int_a^b \left\langle \frac{\delta L}{\delta\widehat{\Omega}}\,,\,\delta\widehat{\Omega}\right\rangle dt &= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\frac{\delta L}{\delta\widehat{\Omega}}\right) dt\\
= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\mathbb{A}\widehat{\Omega} + \delta\widehat{\Omega}\widehat{\Omega}\mathbb{A}\right) dt &= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,(\mathbb{A}\widehat{\Omega} + \widehat{\Omega}\mathbb{A})\right) dt\\
= -\frac{1}{2}\int_a^b \mathrm{tr}\left(\delta\widehat{\Omega}\,\widehat{\Pi}\right) dt &= \int_a^b \left\langle \widehat{\Pi}\,,\,\delta\widehat{\Omega}\right\rangle dt\,.
\end{aligned}$$

$\boxed{8,\ \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(d) (i) The invariance group of the Lagrangian

$$L(\widehat{\Omega}) = \frac{1}{2}\mathrm{tr}(\widehat{\Omega}^T\mathbb{A}\widehat{\Omega})$$

is the Adjoint action $\mathrm{Ad}_{SO(3)} : SO(3)\times\mathfrak{so}(3)\to\mathfrak{so}(3)$. $\boxed{2,\ \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(ii) This Adjoint action preserves the eigenvalues of the 3×3 symmetric matrix $\mathbb{A}$. The corresponding conserved quantities are the three components of the spatial angular momentum.

$$\frac{d}{dt}\Big(\mathrm{Ad}^*_{O^{-1}(t)}\widehat{\Pi}(t)\Big) = \mathrm{Ad}^*_{O^{-1}(t)}\Big(\frac{d}{dt} + \mathrm{ad}^*_{\widehat{\Omega}}\widehat{\Pi}(t)\Big) = 0\,.$$

$\boxed{3,\ \mathrm{C}}$

3. (a) Hamilton's principle for this system is given by $\boxed{\text{seen} \Downarrow}$

$$0 = \delta S = \delta \int_a^b L(\boldsymbol{\Omega}, \Xi_2)\, dt = \int_a^b \frac{\partial L}{\partial \boldsymbol{\Omega}} \cdot \delta \boldsymbol{\Omega} + \frac{\partial L}{\partial \Xi_2}(\delta \Omega_2 + \delta \dot{\phi})\, dt\,,$$

with $\Xi_2 := \Omega_2 + \dot{\phi}$.

The equations of motion for

$$\boldsymbol{\Pi} = \frac{\partial L}{\partial \boldsymbol{\Omega}}\Big|_{Tot} = \Big(\frac{\partial L}{\partial \Omega_1}, \frac{\partial L}{\partial \Omega_2} + \frac{\partial L}{\partial \Xi_2}, \frac{\partial L}{\partial \Omega_3}\Big)^T = \frac{\partial L}{\partial \boldsymbol{\Omega}} + \frac{\partial L}{\partial \Xi_2}\hat{\boldsymbol{e}}_2 = \frac{\partial L}{\partial \boldsymbol{\Omega}} + N\hat{\boldsymbol{e}}_2$$

and

$$N = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \Xi_2}$$

are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\Omega}}\Big|_{Tot} = -\boldsymbol{\Omega} \times \frac{\partial L}{\partial \boldsymbol{\Omega}}\Big|_{Tot} \quad \text{and} \quad \frac{dN}{dt} = \frac{\partial L}{\partial \phi} = 0\,.$$

$\boxed{5, \text{B}}$

$\boxed{\text{seen} \Downarrow}$

(b) The Legendre transformation is

$$H(\boldsymbol{\Pi}, N, \phi) = \boldsymbol{\Pi} \cdot \boldsymbol{\Omega} + N\dot{\phi} - L(\boldsymbol{\Omega}, \Xi_2)\,.$$

Consequently,

$$dH(\boldsymbol{\Pi}, N, \phi) = \boldsymbol{\Omega} \cdot d\boldsymbol{\Pi} + \Big(\boldsymbol{\Pi} - \frac{\partial L}{\partial \boldsymbol{\Omega}} - \frac{\partial L}{\partial \Xi_2}\hat{\boldsymbol{e}}_2\Big) \cdot d\boldsymbol{\Omega} + \dot{\phi}dN + \Big(N - \frac{\partial L}{\partial \Xi_2}\Big)d\dot{\phi}\,.$$

$\boxed{5, \text{A}}$

$\boxed{\text{seen/sim.seen} \Downarrow}$

(c) The Hamiltonian equations are written in Poisson matrix form as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\Pi} \\ N \\ \phi \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}\times & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial H/\partial \boldsymbol{\Pi} = \boldsymbol{\Omega} \\ \partial H/\partial N = \dot{\phi} \\ \partial H/\partial \phi = 0 \end{bmatrix}$$

This Poisson matrix has a symplectic 2-cocycle between N and ϕ. Being symplectic, it satisfies the definition of a 2-cocycle. $\boxed{10, \text{C}}$

4. (a) Denote the vector fields in components as

seen $\Downarrow$

$$X = X^i(q)\frac{\partial}{\partial q^i} = \frac{d}{dt}\Big|_{t=0}\phi_t^* \quad \text{and} \quad Y = Y^j(q)\frac{\partial}{\partial q^j}\,.$$

Then, by the pull-back relation a direct computation yields, on using the matrix identity $dM^{-1} = -\,M^{-1}dM M^{-1}$,

$$\begin{aligned}
\mathcal{L}_X Y &= \frac{d}{dt}\Big|_{t=0}\phi_t^* Y = \frac{d}{dt}\Big|_{t=0}\left(Y^j(\phi_t q)\frac{\partial}{\partial(\phi_t q)^j}\right) \\
&= \frac{d}{dt}\Big|_{t=0}\left(Y^j(\phi_t q)\left[\frac{\partial(\phi_t q)}{\partial q}^{-1}\right]^k_j \frac{\partial}{\partial q^k}\right) \\
&= \left(X^j\frac{\partial Y^k}{\partial q^j} - Y^j\frac{\partial X^k}{\partial q^j}\right)\frac{\partial}{\partial q^k} \\
&= [X,\, Y]\,.
\end{aligned}$$

$\boxed{2,\ \mathrm{A}}$

seen $\Downarrow$

(b) Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y \,\lrcorner\, \alpha) = \phi_t^* Y \,\lrcorner\, \phi_t^*\alpha\,.$$

A direct computation using the dynamical definition of the Lie derivative

$$\mathcal{L}_Y \alpha = \frac{d}{dt}|_{t=0}(\phi_t^*\alpha)\,,$$

then yields

$$\frac{d}{dt}\Big|_{t=0}\phi_t^*\big(Y \,\lrcorner\, \alpha\big) \;=\; \left(\frac{d}{dt}\Big|_{t=0}\phi_t^* Y\right)\lrcorner\, \alpha + \; Y \,\lrcorner \left(\frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha\right).$$

Hence, the desired formula will appear as a rearrangement of the product rule for Lie derivatives,

$$\mathcal{L}_X(Y \,\lrcorner\, \alpha) = (\mathcal{L}_X Y)\,\lrcorner\, \alpha + Y \,\lrcorner\, (\mathcal{L}_X \alpha)\,.$$

Substituting the relation $\mathcal{L}_X Y = [X,\, Y]$ into the product rule above in part (b) and rearranging yields

$$[X,\, Y]\,\lrcorner\, \alpha = \mathcal{L}_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\mathcal{L}_X \alpha)\,,$$

as required, for an arbitrary k-form α.

Hence, for an arbitrary differential form α one finds $i_{(\mathcal{L}_X Y)} = i_{[X\,,Y]} = \big[\mathcal{L}_X\,,\, i_Y\big]$.

$\boxed{3,\ \mathrm{B}}$

(c) From part (b) we have $\boxed{\text{seen} \Downarrow}$

$$[X\,,\,Y] \lrcorner\, \alpha = \mathcal{L}_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\mathcal{L}_X \alpha).$$

Now use Cartan's formula

$$\mathcal{L}_X \alpha = \frac{d}{dt}\bigg|_{t=0} (\phi_t^* \alpha) = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha),$$

to compute the required result,

$$\mathcal{L}_{[X\,,\,Y]} = \big[\mathcal{L}_X\,,\,\mathcal{L}_Y\big],$$

as

$$\begin{aligned}
\mathcal{L}_{[X\,,\,Y]}\alpha &= d([X\,,\,Y] \lrcorner\, \alpha) + [X\,,\,Y] \lrcorner\, d\alpha \\
&= d\big(\,\mathcal{L}_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\mathcal{L}_X \alpha)\,\big) \\
&\quad + \mathcal{L}_X(Y \lrcorner\, d\alpha) - Y \lrcorner\, (\mathcal{L}_X d\alpha) \\
&= \mathcal{L}_X d(Y \lrcorner\, \alpha) - d(Y \lrcorner\, (\mathcal{L}_X \alpha) \\
&\quad + \mathcal{L}_X(Y \lrcorner\, d\alpha) - Y \lrcorner\, d(\mathcal{L}_X \alpha) \\
&= \mathcal{L}_X(\mathcal{L}_Y \alpha) - \mathcal{L}_Y(\mathcal{L}_X \alpha)\,.
\end{aligned}$$

$\boxed{3,\ \mathrm{B}}$

(d) The Jacobi identity $[Z,[X,Y]] + cp = 0$ follows directly by applying $\frac{d}{dt}\big|_{t=0}$ to the pull-back relation $\boxed{\text{seen} \Downarrow}$

$$\phi_t^*\big[X,\,Y\big] = \big[\phi_t^* X,\,\phi_t^* Y\big],$$

since

$$\frac{d}{dt}\phi_t^*[X,Y]\big|_{t=0} = \mathcal{L}_Z[X,Y] = [Z,[X,Y]].$$

$\boxed{2,\ \mathrm{A}}$

$\boxed{\text{unseen} \Downarrow}$

(e) In the sense of Taylor expansions of exponentials of operators on differential forms, one has from Cartan's formula and the product rule for the pullback, that

$$\frac{d}{dt}\phi_t^*\bigg|_{t=0} = e^{t\mathcal{L}_v} = \sum_0^\infty \frac{t^n}{n!}\big(\mathcal{L}_v\big)^n = \sum_0^\infty \frac{t^n}{n!}\big(i_v d + d i_v\big)^n = \sum_0^\infty \frac{t^n}{n!}\Big(\big(i_v d\big)^n + \big(d i_v\big)^n\Big),$$

after the cross terms in $\big(i_v d + d i_v\big)^n$ have vanished by $d^2 = 0$ and $i_v i_v = 0$, for

$$\left[\frac{d}{dt}\phi_t^*\right]_{t=0} = \left[\frac{d}{dt}e^{t(di_v + i_v d)}\right]_{t=0} = \mathcal{L}_v\,, \quad \text{with} \quad v = \left[\phi_t^{-1}\frac{d\phi_t}{dt}\right]_{t=0}.$$

$\boxed{10,\ \mathrm{B}}$

5. (a) The variations of $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ arising from variations of $O(t) \in SO(3)$ in $\mathbb{R}^3$ gives $\boxed{\text{seen/sim.seen} \Downarrow}$

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Xi}} + \mathbf{\Omega} \times \mathbf{\Xi} \quad \text{and} \quad \delta\mathbf{\Gamma} = -\,\mathbf{\Xi} \times \mathbf{\Gamma} \quad \text{with} \quad \mathbf{\Xi}\times = O^{-1}\delta O\,.$$

Likewise, the body orientation of the vertical $\mathbf{\Gamma}(t) := O^{-1}\hat{\boldsymbol{e}}_3 \in \mathbb{R}^3$ evolves as

$$\frac{d\mathbf{\Gamma}}{dt} = -\,\mathbf{\Omega} \times \mathbf{\Gamma}\,.$$

Variations of the Lagrangian L_b yield the following equations,

$$\begin{aligned}
\delta\dot{R}: &\quad P := \frac{\partial L_b}{\partial \dot{R}} = m\dot{R}|\mathbf{r}_0|^2\,,\\
\delta R: &\quad \dot{P} = \frac{\partial L_b}{\partial R} = mR\,|\mathbf{\Omega}\times\mathbf{r}_0|^2 - mg\,\mathbf{\Gamma}\cdot\mathbf{r}_0 - k|\mathbf{r}_0|^2(R-1)\,,\\
\delta\mathbf{\Gamma}: &\quad -\mathbf{\Xi}\times\mathbf{\Gamma}\cdot(-mgR\,\mathbf{r}_0) = mgR\,\mathbf{\Gamma}\times\mathbf{r}_0\cdot\mathbf{\Xi}\,,\\
\delta\mathbf{\Omega}: &\quad \mathbf{\Pi} = \frac{\partial L_b}{\partial \mathbf{\Omega}} = mR^2(\mathbf{\Omega}\times\mathbf{r}_0)\times\mathbf{r}_0 = mR^2(t)|\mathbf{r}_0|^2\mathbf{\Omega}\,.
\end{aligned}$$

With these variational derivatives, the corresponding Euler-Poincaré equation for $\mathbf{\Pi}$ is found to be

$$\left(\frac{d}{dt} + \mathbf{\Omega}\times\right)\mathbf{\Pi} = mgR(t)\,\mathbf{\Gamma}\times\mathbf{r}_0 \quad \text{with} \quad \mathbf{\Omega}\times\mathbf{\Pi} = 0\,.$$

$\boxed{10,\ \mathrm{M}}$

(b) The Legendre transform yields the following Hamiltonian in the body variables $(\mathbf{\Pi}, \mathbf{\Gamma})$, $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}
H_b(\mathbf{\Pi}, \mathbf{\Gamma}, R, P) &= \mathbf{\Pi}\cdot\mathbf{\Omega} + P\dot{R} - L_b(\mathbf{\Omega}, R, \dot{R}, \mathbf{\Gamma})\\
&= \frac{|\mathbf{\Pi}|^2}{2mR^2|\mathbf{r}_0|^2} + \frac{P^2}{2m|\mathbf{r}_0|^2} + mg\,\mathbf{\Gamma}\cdot R(t)\mathbf{r}_0 + \frac{k}{2}\big(R(t)-1\big)^2|\mathbf{r}_0|^2\,.
\end{aligned}$$

$\boxed{2,\ \mathrm{M}}$

(c) After making the appropriate standard substitutions, the resulting Poisson matrix form of the Hamiltonian dynamics turns out to possess a symplectic 2-cocycle, as follows, $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{d}{dt}\begin{bmatrix}\mathbf{\Pi}\\ \mathbf{\Gamma}\\ P\\ R\end{bmatrix} = \begin{bmatrix}\mathbf{\Pi}\times & \mathbf{\Gamma}\times & 0 & 0\\ \mathbf{\Gamma}\times & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\partial H_b/\partial\boldsymbol{\pi} = \mathbf{\Pi}/(mR(t)^2|\mathbf{r}_0|^2) = \mathbf{\Omega}\\ \partial H_b/\partial\mathbf{\Gamma} = mgR(t)\,\mathbf{r}_0\\ \partial H_b/\partial P = P/(m|\mathbf{r}_0|^2)\\ \partial H_b/\partial R = -|\mathbf{\Pi}|^2/(mR^3|\mathbf{r}_0|^2) + mg\,\mathbf{\Gamma}\cdot\mathbf{r}_0 + k\big(R(t)-1\big)|\mathbf{r}_0|^2\end{bmatrix}.$$

$\boxed{8,\ \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 21 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 14 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Geometric Mechanics MATH97064 MATH70010	1	Q1(c) was a bit abstract compared to other more explicit questions on which people seemed to score well. (Q1(c): Show that the canonical Poisson brackets among the Noether conservation laws for a geodesic flow are isomorphic to the Lie algebra of its isometry group.)
Geometric Mechanics MATH97064 MATH70010	2	questions were closely aligned to the main stream of the lectures and most people scored well on them
Geometric Mechanics MATH97064 MATH70010	3	questions were closely aligned to the main stream of the lectures and most people scored well on them
Geometric Mechanics MATH97064 MATH70010	4	The Lie derivative operator identity $\exp{tL_v}=\exp{ t i_v d}+\exp{t d i_v}$ in Q4(e) had not been seen in class. Although it can be proved directly by expanding the exponentials, many people didn't notice that approach and, hence, scored less than I expected.