IMPERIAL

MATH70009

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Bifurcation Theory

Date: Wednesday, May 8, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the map

$$\begin{cases} \bar{x} = y, \\ \bar{y} = a - b^2 x - y^2. \end{cases}$$

with two parameters a and b.

(a) How many fixed points does this map have at most? (4 marks)

(b) Find the bifurcation sets in the (a, b)-plane corresponding to the following bifurcations of the fixed points.

(i) Saddle-node bifurcation. (5 marks)

(ii) Period-doubling bifurcation. (5 marks)

(iii) Neimark-Sacker bifurcation. (6 marks)

(Total: 20 marks)

2. Consider the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + (1-a)\varepsilon y + bxy + (\varepsilon a - 1)y^2. \end{cases}$$

(a) Show that the first Lyapunov coefficient for the equilibrium $(x, y) = (0, 0)$ at $\varepsilon = 0$ is given by $L = -\dfrac{b}{8}$. (6 marks)

(b) Suppose $a > 3, b \neq 0$.

(i) What is the stability of the equilibrium $(0, 0)$ at $\varepsilon = 0$? (2 marks)

(ii) Determine the values of b for which there exists a periodic orbit when $\varepsilon > 0$? (4 marks)

(c) Find out how many periodic orbits can exist for small $\varepsilon > 0$ in the following two cases: (Please justify your answer.)

(i) $a \neq 1, b = 0$. (4 marks)

(ii) $a = 1, b = 0$. (4 marks)

(Total: 20 marks)

3. Consider the system

$$\begin{cases} \frac{dx}{dt} = -\varepsilon^2 x^2 + \varepsilon xy + xz + x^3, \\ \frac{dy}{dt} = -\frac{1}{2}y + \varepsilon x^2 + y^2 z^2, \\ \frac{dz}{dt} = z + 2x^2 + \varepsilon x^4, \end{cases}$$

where ε is a small parameter.

(a) Find the expression of the defining function $(y, z) = \phi(x, \varepsilon)$ of the center manifold up to order 5. (Hint: treat ε as a new variable.) (5 marks)

(b) Suppose that the system restricted to the center manifold assumes the form

$$\frac{dx}{dt} = f(x, \varepsilon) = -\varepsilon^2 x^2 + (2\varepsilon^2 - 3)x^3 + o(x^3).$$

(i) Write the normal form for the bifurcation of the equilibrium $x = 0$, and specify the control parameters as functions of ε. (6 marks)

(ii) Sketch the bifurcation diagram in the space of the control parameters and explain the dynamics. (5 marks)

(iii) Describe the dynamics near $x = 0$ for all small ε. (4 marks)

(Total: 20 marks)

4. Consider the map

$$\bar{x} = f_a(x) = a(x - x^8),$$

which is defined on $[0, 1]$ with a parameter a.

(a) (i) Determine the stability of the fixed point $x = \sqrt[7]{\frac{2}{9}}$ at $a = \frac{9}{7}$. (8 marks)

(ii) Determine the stability of the fixed point $x = 0$ at $a = 1$ and $a = -1$, respectively. (6 marks)

(b) Find the parameter values for which the map has a stable fixed point. (6 marks)

(Total: 20 marks)

5. Consider a smooth family $\bar{x} = f(x, \varepsilon)$ of two-dimensional maps (i.e., $x \in \mathbb{R}^2$). Suppose that at $\varepsilon = 0$ the map $f(x, 0)$ has a fixed point $x = x_0$ with multipliers $e^{\pm i\omega_0}$ for some $\omega_0 \in (0, \pi)$.

(a) (i) The fixed points are given by solutions of $x - f(x, \varepsilon) = 0$. Show that for all small ε the map has a fixed point near $x = x_0$. (3 marks)

(ii) Suppose $\varepsilon = (\varepsilon_1, \varepsilon_2) := (\mu, \omega - \omega_0)$ is such that the fixed point found in (a) has multipliers $(1 + \mu)e^{\pm i\omega}$. By counting resonant terms, show that if $\omega_0 = 2\pi/5$, then, for all small ε (i.e., μ close to 0 and ω close to $2\pi/5$), the normal form of this family is given by

$$\bar{z} = (1 + \mu)e^{i\omega} \left(z(1 + (L + i\Omega)|z|^2) + A(z^*)^4 + O(|z|^5) \right),$$

where z^* is the complex conjugate of z and L, Ω, A are coefficients depending on parameters. (5 marks)

(b) Assume $L = -1$ for all small parameters so that in the polar coordinates $z = re^{i\varphi}$ the normal form is given by

$$\bar{r} = (1 + \mu)r(1 - r^2 + g_1(r, \varphi, \mu, \omega)), \qquad \bar{\varphi} = \omega + \varphi + g_2(r, \varphi, \mu, \omega),$$

where $g_1 = O(r^3)$ and $g_2 = O(r^2)$ are smooth functions. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$. The invariant curve attracts all orbits from a small neighbourhood of the fixed point, independent of parameters. It can be shown that the curve has an equation $r = h(\varphi)$ where h is a smooth, positive, periodic function of φ. Show that

$$h(\varphi) = \sqrt{\mu} + O(\mu).$$

(8 marks)

(c) Let $\omega_0 = 2\pi/5$. With the result in (b), show that in the (μ, ω)-plane there exist parameter values corresponding to the existence of periodic orbit of period 5. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60009/MATH70009

Bifurcation Theory (Solutions)

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1. (a) Fixed points of the map are solutions of $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{cases} x = y, \\ y = a - b^2x - y^2. \end{cases}$$

Substituting the first equation into the second one, yields

$$y^2 + (1 + b^2)y - a = 0. \tag{1}$$

Thus, there can be at most two fixed points. $\boxed{4, \text{A}}$

(b) (i) The Jacobian matrix of the system is $\boxed{\text{sim. seen} \Downarrow}$

$$J = \begin{pmatrix} 0 & 1 \\ -b^2 & -2y \end{pmatrix}.$$

A saddle-node bifurcation corresponds to a $+1$ multiplier, which is equivalent to $\det(J - I) = 0$ where I is the 2 by 2 identity matrix. Solving the equation gives

$$1 + b^2 + 2y = 0. \tag{2}$$

Thus, fixed points having a $+1$ multiplier correspond to the solutions of the system consisting of (1) and (2). Eliminating y in these equations gives the sought bifurcation curve

$$s_1 = \{(a, b) \mid a = -\frac{(1 + b^2)^2}{4}\}.$$

$\boxed{5, \text{A}}$

(ii) Similarly, having -1 as a multiplier means $\det(J + I) = 0$, or,

$$1 + b^2 - 2y = 0.$$

Combining this equation with (1) gives the bifurcation curve for the period-doubling bifurcation:

$$s_2 = \{(a, b) \mid a = \frac{3(1 + b^2)^2}{4}\}.$$

$\boxed{5, \text{A}}$

(iii) At the critical moment, the multipliers are $\lambda_{1,2} = e^{\pm i\theta}$ with $\theta \in (0, \pi)$. The necessary condition is

$$\det J = \lambda_1 \lambda_2 = 1,$$

which gives

$$1 - b^2 = 0. \tag{3}$$

$\boxed{2, \text{C}}$

Next, one needs to exclude the case of $\lambda_1 = \lambda_2^{-1}$ being real. That is, the characteristic equation of J: $\lambda^2 + 2y\lambda + b^2 = 0$ has no real roots, which by (3) amounts to $4y^2 - 4 < 0$ or $|y| < 1$. We find y from (1) as

$$y = -1 \pm \sqrt{1 + a}.$$

Then, $|y| < 1$ is equivalent to

$$-1 < -1 - \sqrt{1+a} < 1 \quad \text{or} \quad -1 < -1 + \sqrt{1+a} < 1.$$

The first inequality cannot be satisfied since $\sqrt{1+a} \geqslant 0$ (y must be real). The second inequality implies $-1 < a < 3$, which together with (3) gives the bifurcation curve

$$s_3 = \{(a, b) \mid -1 < a < 3, b = \pm 1\}.$$

$\boxed{4, \text{C}}$

2. (a) We diagonalise the linearisation matrix by taking $z = x - iy$ (so that $x = (z+z^*)/2$ and $y = (z^* - z)/2i$). The system at $\varepsilon = 0$ then reads

sim. seen ⇓

$$\frac{dz}{dt} = iz + \frac{b}{4}(z+z^*)(z-z^*) - \frac{i}{4}(z-z^*)^2 = iz + \frac{b-i}{4}z^2 + \frac{i}{2}zz^* - \frac{b+i}{4}(z^*)^2.$$

By the formula for the first Lyapunov coefficient, we obtain

3, A

$$L = -\frac{1}{1}\mathrm{Im}(\frac{b-i}{4} \cdot \frac{i}{2}) = -\frac{b}{8}.$$

3, A

meth seen ⇓

(b) (i) The equilibrium is stable when $L < 0$, or $b > 0$; it is unstable $L > 0$, or $b < 0$.

2, A

(ii) Periodic orbits are born when the equilibrium changes its stability.
The stability depends on the real part of the eigenvalues of its linearisation matrix.
If we denote the matrix by A, then the real part is given by $\frac{1}{2}tr(A) = \frac{1}{2}(1-a)\varepsilon < -\varepsilon$ by assumption.

2, A

So, for $\varepsilon > 0$ the equilibrium is stable. There is a stability change if the equilibrium is unstable at $\varepsilon = 0$, that is, $b > 0$.

2, A

meth seen ⇓

(c) (i) When $b = 0$, the system at $\varepsilon = 0$ is reversible by $(x = -x, t = -t)$. Thus, there are infinitely many periodic orbits around the equilibrium at $\varepsilon = 0$.

2, D

Since by (b.ii) the real part of the eigenvalues is given by $\frac{1}{2}(1-a)\varepsilon$, we have

$$\frac{d\mathrm{Re}}{d\varepsilon} = -\frac{a}{2} \neq 0.$$

Therefore, we can apply the Hopf theorem which implies that no periodic orbits can exist for $\varepsilon > 0$.

2, D

(ii) In this case one cannot apply the Hopf theorem since the real part does not change when ε changes. In fact, the system is always reversible when $a = 1$, and hence there are infinitely many periodic orbits for $\varepsilon > 0$.

4, D

3. (a) Adding $\frac{d\varepsilon}{dt} = 0$ to the system, we find that $(x, \varepsilon), y, z$ are central, stable and unstable variables, respectively. To find the desired formula for the center manifold, we kill powers of central variables in the equations of $\frac{dy}{dt}, \frac{dz}{dt}$ up to order 5, by the coordinate transformation $\boxed{\text{sim. seen} \Downarrow}$

$$y^{new} = y - 2\varepsilon x^2, \qquad z^{new} = z + 2x^2 + \varepsilon x^4.$$

In the new coordinates, the center manifold is given by $(y^{new}, z^{new}) = O(|x, \varepsilon|^6)$, and hence in the original coordinates, the center manifold is given by $\boxed{2, \text{B}}$

$$y = 2\varepsilon x^2 + O(|x, \varepsilon|^6), \qquad z = -2x^2 - \varepsilon x^4 + O(|x, \varepsilon|^6).$$

$\boxed{3, \text{B}}$

$\boxed{\text{unseen} \Downarrow}$

(b) (i) The system at $\varepsilon = 0$ is $\frac{dx}{dt} = -3x^3 + o(x^3)$. So, the first non-zero Lyapunov coefficient is $\ell_3 = -3$. To obtain the normal form of the bifurcation:

$$\frac{du}{dt} = \mu_0(\varepsilon) + \mu_1(\varepsilon)u + \ell_3 u^3 + o(u^3),$$

we need to move origin to the the solution of $\boxed{2, \text{C}}$

$$0 = \frac{\partial^2 f(x, \varepsilon)}{\partial x^2} = -2\varepsilon^2 + 6(2\varepsilon^2 - 3),$$

which is given by

$$x = x^*(\varepsilon) = \frac{\varepsilon^2}{3(2\varepsilon^2 - 3)}.$$

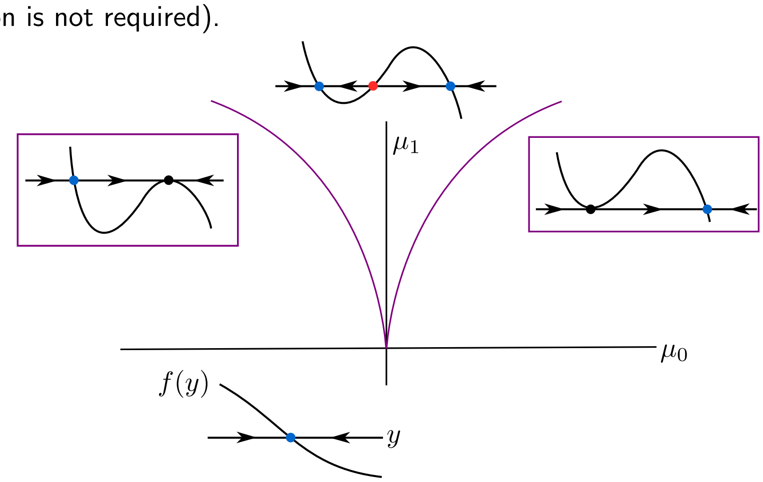
The control parameters are $\boxed{2, \text{C}}$

$$\mu_0 = f(x^*, \varepsilon) = -\frac{2\varepsilon^6}{27(2\varepsilon^2 - 3)^2}, \qquad \mu_1 = \frac{\partial f(x^*, \varepsilon)}{\partial x} = -\frac{\varepsilon^4}{3(2\varepsilon^2 - 3)^2}$$

$\boxed{2, \text{C}}$

$\boxed{\text{unseen} \Downarrow}$

(ii) Only the bifurcation curve, the cusp, is required for marks (the graph of the function is not required).



3, B

On the bifurcation curve, the equilibrium $x = 0$ decomposes into two equilibria, one stable and another one semi-stable; and after crossing the curve, the semi-stable equilibrium can further decompose into two equilibria, where one is stable and the other one is unstable. 2, B

sim. seen $\Downarrow$

(iii) Since μ_0 and μ_1 are negative for all small ε, only the dynamics in the third quadrant can be realized, that is, we always have only one stable equilibrium near $x = 0$. 4, B

4. (a) (i) The multiplier is given $\lambda = a(1-8x^7)$. $\boxed{\text{sim. seen} \Downarrow}$

At $a = \frac{9}{7}$ the fixed point $x = \sqrt[7]{\frac{2}{9}}$ has multiplier -1. The sign of the first Lyapunov coefficient equal to the Schwarzian derivative

$$S = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

We have

$$S = 336ax^3 - \frac{3}{2}(56ax^4)^2 = ax^5(336 - 3\cdot 28\cdot 56ax^7) = ax^5(336 - 3\cdot 28\cdot 56\cdot\frac{2}{7}) < 0.$$

So, the fixed point is stable. $\boxed{4, \mathrm{D}}$

(ii) At $a = 1$ the fixed point $x = 0$ has multiplier $+1$. The stability is determined by the first non-zero Lyapunov coefficient. It is the first non-zero Lyapunov coefficient is equal to the coefficient of the lowest order nonlinear term, which is $\ell_8 = -1$ is this case. $\boxed{3, \mathrm{B}}$

So, the fixed point is semi-stable at $a = 1$, unstable from the left and stable from the right. $\boxed{3, \mathrm{B}}$

At $a = -1$ the fixed point $x = 0$ has multiplier -1. the Schwarzian derivative vanishes. So, we determine the stability by considering the second iteration of the map:

$$\bar{\bar{x}} = -(\bar{x} - \bar{x}^8) = x - x^8 + (x - x^8)^8 = x - 8x^{15} + o(x^{15}),$$

so

$$\bar{\bar{x}} = |x|(1 - 8x^{14} + o(x^{14})) < |x|,$$

for all small $x \neq 0$. So, $x = 0$ is stable. $\boxed{4, \mathrm{D}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) Solve the equation $x = a(x - x^8)$ with $x \neq 0$ gives the fixed point $O_2 : x = (1 - \frac{1}{a})^{\frac{1}{7}}$ for $a \neq 1$. The multiplier is $\lambda = a(1-8x^7)$. $\boxed{2, \mathrm{A}}$

First we know that a fixed point is stable when $|\lambda| < 1$ and unstable when $|\lambda| > 1$. For O_1, we have $|\lambda| < 1$ if $a \in (-1, 1)$. With the stability at $|\lambda| = 1$ found in a(ii), we conclude that O_1 is stable if $a \in [-1, 1)$. $\boxed{2, \mathrm{A}}$

For O_2, we have $|\lambda| < 1$ if $a \in (1, \frac{9}{7})$. Since O_2 does not exist at $a = 1$, it cannot have $\lambda = 1$. With the stability at $\lambda = -1$ found in a(i), we conclude that O_2 is stable if $a \in (1, \frac{9}{7}]$. Thus, the map has a stable fixed point when $a \in [-1, 1) \cup (1, \frac{9}{7}]$. $\boxed{2, \mathrm{A}}$

5. (a) (i) Since $\omega_0 \in (0, \pi)$, the linearization matrix $\frac{\partial f(x,\varepsilon)}{\partial x}|_{x=x_0}$ is invertible. Since $x = x_0$ is a special solution to the equation $x - f(x, \varepsilon)$, it follows from the Implicit Function Theorem that one can find a solution $x = x^*(\epsilon)$ which is defined near $x = x_0$ and satisfies $x^*(\varepsilon) - f(x^*(\varepsilon), \varepsilon) = 0$. This solution is the desired fixed point.

unseen ⇓

3, M

sim. seen ⇓

(ii) A monomial $z^m(z^*)^n$ is resonant if

$$e^{i\frac{2\pi}{5}} = e^{i\frac{2m\pi}{5}} e^{i\frac{-2n\pi}{5}},$$

or

$$1 = m - n + 5k, \qquad k \in \mathbb{Z}.$$

Thus, resonant monomials of order larger than 1 and smaller than 5 satisfy

2, M

$k = 0:$ $m = 2$ and $n = 1$,

$k = 1:$ $m = 0$ and $n = 4$,

and hence are $z^2 z^*$ and $(z^*)^4$.

3, M

sim. seen ⇓

(b) It suffices to show that there exists some constant $K > 0$ such that the annulus

$$|r - \sqrt{\mu}| \le K\mu \tag{4}$$

is forward invariant, since the invariant curve attracts all nearby orbits in a small neighborhood of the fixed point 0.

3, M

By assumption, there exists $C > 0$ such that $|g_1| < Cr^3$ and $|g_2| < Cr^2$. Hence,

$$(1 + \mu)r(1 - r^2 - Cr^3) < \bar{r} < (1 + \mu)r(1 - r^2 + Cr^3). \tag{5}$$

It suffices to show that for all small μ, $\bar{r} > r$ if the point is on the inner circle of the annulus, that is, $r = \sqrt{\mu} - K\mu$; and $\bar{r} < r$ if the point is on the outer circle of the annulus, that is, $r = \sqrt{\mu} + K\mu$.

3, M

By (5), to have $\bar{r} > r$ on the inner circle, we need

$$\bar{r} > (1+\mu)r(1-(\sqrt{\mu}-K\mu)^2-C(\sqrt{\mu}-K\mu)^3) = r(1+(2K-C)\mu\sqrt{\mu}+O(\mu^2)) > r,$$

which leads to $K > C/2$. Similarly, to have $\bar{r} < r$ on the outer circle, we need

$$\bar{r} < (1+\mu)r(1-(\sqrt{\mu}-K\mu)^2+C(\sqrt{\mu}-K\mu)^3) = r(1-(2K-C)\mu\sqrt{\mu}+O(\mu^2)) < r,$$

which also leads to $K > C/2$. Thus, for any $K > C/2$, the annulus defined by (4) is forward invariant.

2, M

sim. seen ⇓

(c) The map on the invariant circle found in (b) is given by

$$\bar{\varphi} = \omega + \varphi + O(\mu).$$

At $\mu = 0$, the rotation number of this map is $\omega/2\pi$, so, if ω varies near $\omega_0 = 2\pi/5$, the rotation number changes from smaller than $1/5$ to larger than $1/5$. By continuity, this observation also holds for all small μ. Thus, there are parameter values in $\{\mu > 0\}$ (the region where the invariant curve exists) for which the rotation number is exactly $1/5$, which gives rise to periodic orbits of period 5.

4, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH60009 Bifurcation Theory

Question	Marker's comment
1	Most students did well in finding the critical parameter values responsiblenbsp;for different bifurcations.
2	Most students did well. Some students could not explain well why the system is Hamiltonian/reversible.nbsp;
3	This seems to be the hardest question, where bothnbsp; computational errors and misunderstanding of concepts occurred.Computation for center manifolds seems harder than expected. Just need to directly use the standard formula for Normal form transformation, or find the transformation by comparing terms of the same order.The control parameters are functions of the given parameters in a system. Need to bring the system to the normal form to find the expressions of thesenbsp;control parameters.
4	Most of the students know how to determine the stability of the fixed points, but the number of arithmetic errors was unexpected.

MATH70009 Bifurcation Theory

Question	Marker's comment
1	Most students did well in finding the critical parameter values responsiblenbsp;for different bifurcations. Some students have difficulty with the Neimark-Sacker bifurcation.
2	Most students did well. Some students could not explain well why the system is Hamiltonian/reversible.nbsp;
3	This seems to be the hardest question, where both computational errors and misunderstanding of concepts occurred.Computation for center manifolds seems harder than expected. Just need to directly use the standard formula for Normal form transformation, or find the transformation by comparing terms of the same order.The control parameters are functions of the given parameters in a system. Need to bring the system to the normal form to find the expressions of these control parameters.
4	Most of the students know how to determine the stability of the fixed points, but the number of arithmetic errors was unexpected.
5	This question contains a lot of information but it is not difficult. The big variation of performance is not expected.Most students failed because not have a clear idea about the research procedure in the Neimark-Sacker bifurcation, including normal form transformation and how to find the invariant curve.