IMPERIAL

MATH50004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Multi-variable Calculus and Differential Equations

Date: Monday, May 20, 2024

Time: 10:00 – 13:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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1. (a) (i) Show that

$$\mathbf{A} \cdot \operatorname{curl} \mathbf{B} = -\varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i},$$

with summation over i, j, k implied. (2 marks)

(ii) Hence or otherwise, simplify the expression

$$\operatorname{div}(\mathbf{A} \times \mathbf{B}) + \mathbf{A} \cdot \operatorname{curl} \mathbf{B}.$$ (4 marks)

(iii) Suppose a closed surface S with unit outward normal $\hat{\mathbf{n}}$ encloses a volume V. Using your result from (ii), and the divergence theorem, establish the result

$$\int_V \mathbf{A} \cdot \operatorname{curl} \mathbf{B} \, dV = -\oint_S (\mathbf{A} \times \mathbf{B}) \cdot \hat{\mathbf{n}} \, dS + \int_V \mathbf{B} \cdot \operatorname{curl} \mathbf{A} \, dV.$$ (2 marks)

(b) Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the Cartesian unit vectors. Consider the vector field

$$\mathbf{F} = \mathbf{j} \sin x,$$

and let Q be the quadrilateral in the $x - y$ plane with vertices $(\pm a, 0)$ and $(0, \pm b)$.

(i) Calculate $\operatorname{curl} \mathbf{F}$. (1 mark)

(ii) Write down the equations representing the sides of Q. (3 marks)

(iii) Green's theorem can be written in the form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \; dx \, dy.$$

What do the quantities C, $d\mathbf{r}$ and R represent? (2 marks)

(iv) Verify Green's theorem for $\mathbf{F}$ given above, and C the boundary of Q. (6 marks)

(Total: 20 marks)

2. Consider the vector field

$$\mathbf{A} = (x+z)\,\mathbf{i} + y\,\mathbf{j},$$

where $\mathbf{i}, \mathbf{j}$ are the usual Cartesian unit vectors.

(a) Calculate the divergence, curl and Laplacian of $\mathbf{A}$. (4 marks)

Consider the closed surface S whose side is the cylinder $x^2 + y^2 = 4,\ z > 0$. The plane $z = 3 + x$ provides a top cap while the bottom is closed with a circular disc at $z = 0$.

(b) Show that the outward unit normal to the curved part of S is

$$\widehat{\mathbf{n}} = \frac{1}{2}(x\,\mathbf{i} + y\,\mathbf{j}),$$

and find the corresponding outward unit normals to the other two faces. (4 marks)

(c) By evaluating the three surface integrals individually, determine

$$\oint_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS. \qquad \text{(8 marks)}$$

(d) Verify your answer by using the divergence theorem and evaluating the corresponding volume integral. (4 marks)

(Total: 20 marks)

3. A curve $y = y(x)$ joins the points $(\pm 1/2, 1)$ in the $x - y$ plane. Consider the integrals

$$I = \int_{-1/2}^{1/2} y \left(1 + (dy/dx)^2\right)^{1/2} dx, \quad J = \int_{-1/2}^{1/2} \left(1 + (dy/dx)^2\right)^{1/2} dx.$$

(a) Interpret I and J geometrically. (3 marks)

(b) Show, using the Euler-Lagrange equation in the form $L - y' \partial L / \partial y' =$ constant, that the extremal curve of I is given by

$$y = a \cosh\left(\frac{x}{a}\right),$$

and write down the equation satisfied by the constant a. (5 marks)

(c) Show that the corresponding stationary value of I is given by

$$I = \frac{a}{2} + \sqrt{1 - a^2}.$$ (4 marks)

The constraint $J = 2$ is now added to the problem.

(d) Use the Euler-Lagrange equation to show that the constrained stationary curve of I is

$$y = b \cosh\left(\frac{x}{b}\right) - \lambda,$$

where λ and b are constants. (4 marks)

(e) Write down the relation linking λ and b, and show that

$$\sinh\left(\frac{1}{2b}\right) = \frac{1}{b}.$$ (4 marks)

(Total: 20 marks)

4. (a) Let $n \in \mathbb{N}$, and consider the initial value problem

$$\dot{x} = \frac{t^n}{x}, \qquad x(0) = 1,$$

where the right hand side of this differential equation is defined for $(t, x) \in \mathbb{R} \times (0, \infty)$.

(i) Demonstrate that this differential equation is not globally Lipschitz continuous, as required in the global version of the Picard–Lindelöf theorem. (3 marks)

(ii) Explain why, nevertheless, there exists a unique local solution to this initial value problem. (2 marks)

(iii) For all $n \in \mathbb{N}$, compute the maximal solution to this initial value problem using separation of variables, and determine the maximal existence interval. Check that the maximal solution you found is indeed maximal and cannot be continued. (6 marks)

(b) Decide for each of the following three statements whether it is true or false. Justify your answer briefly.

(i) The function $t \mapsto |t|$, where $t \in \mathbb{R}$, solves an ordinary differential equation $\dot{x} = f(t, x)$ with a continuously differentiable right hand side $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. (3 marks)

(ii) The function $t \mapsto \sin(t)$, where $t \in \mathbb{R}$, solves an autonomous ordinary differential equation with a continuously differentiable right hand side $f : \mathbb{R} \to \mathbb{R}$. (3 marks)

(iii) There exists a strictly monotonically increasing function $\mu : \mathbb{R} \to \mathbb{R}$ that solves the one-dimensional differential equation

$$\dot{x} = x^2 - 2\mu(t)x + \mu(t)^2 .$$

(3 marks)

(Total: 20 marks)

5. (a) Consider the linear differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

You may use without proof that the coefficient matrix A has the two eigenvectors

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

where the eigenvalue corresponding to v is given by -2, and the eigenvalue corresponding to w is given by 1. The flow of this differential equation is given by $\varphi(t,x,y) = e^{At}\binom{x}{y}$.

(i) Clarify whether the equilibrium $(0,0)$ is attractive. (3 marks)

(ii) Does this differential equation have a periodic orbit? Justify your answer briefly. (3 marks)

(iii) Find a vector $(x,y) \in \mathbb{R}^2$ such that

$$\lim_{t\to\infty} \|\varphi(t,x,y)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \|\varphi(t,x,y)\| = \infty\,?$$

Justify your answer. (4 marks)

(iv) What are the Lyapunov exponents of the two solutions

$$t \mapsto \varphi(t,-1,1) \quad \text{and} \quad t \mapsto \varphi(t,1,1)?$$

You do not need to justify your answer. (4 marks)

(b) Let $A \in \mathbb{R}^{d\times d}$, and assume that all eigenvalues of A have zero real part. Show that

$$\limsup_{t\to\infty} \frac{\|e^{At}x\|}{t^{d-1}} < \infty \quad \text{for all } x \in \mathbb{R}^d.$$

<u>Hint.</u> The answer does not need to be long and fully rigorous, but should consist of a short explanation of all crucial components that are necessary to establish this statement. (6 marks)

(Total: 20 marks)

6. (a) Consider the two-dimensional differential equation

$$\dot{x} = y - x^2\,,$$
$$\dot{y} = -x - y^2\,,$$

which has the trivial equilibrium $(0,0)$.

(i) Say what it means for a continuously differentiable function $V : \mathbb{R}^2 \to \mathbb{R}$ to be a Lyapunov function. (2 marks)

(ii) Clarify whether the asymptotic stability of the equilibrium $(0,0)$ can be concluded from the linearisation in $(0,0)$. (2 marks)

(iii) According to Lyapunov's direct method for asymptotic stability, what conditions need to be satisfied for a Lyapunov function so that asymptotic stability of the trivial equilibrium can be concluded. (3 marks)

(iv) Find a Lyapunov function and prove that the trivial equilibrium $(0,0)$ is stable. (5 marks)

(b) Consider the two-dimensional differential equation

$$\dot{x} = x(1 - y - \tfrac{1}{4}x)\,,$$
$$\dot{y} = y(-1 + x)\,,$$

which we consider in the (open) first quadrant, i.e. for $x, y > 0$. Show that there does not exist a periodic orbit (in the first quadrant). (8 marks)

Hint. Analyse the scalar-valued function $V(x,y) = x - \ln(x) + y - \frac{3}{4}\ln(y)$ along orbits of the above differential equation.

(Total: 20 marks)

Imperial College London
MATH 50004/50015 Multivariable Calculus and Differential Equations
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SOLUTIONS

Question One Solution

(a)(i)

$$\mathbf{A}\cdot\operatorname{curl}\mathbf{B} = A_j[\operatorname{curl}\mathbf{B}]_j = A_j\varepsilon_{jik}\frac{\partial}{\partial x_i}B_k = -\varepsilon_{ijk}A_j\frac{\partial B_k}{\partial x_i}. \;[\mathbf{2\ marks,\ Cat\ A}]$$

(a)(ii)

$$\begin{aligned}\operatorname{div}(\mathbf{A}\times\mathbf{B}) + \mathbf{A}\cdot\operatorname{curl}\mathbf{B} &= \frac{\partial}{\partial x_i}[\mathbf{A}\times\mathbf{B}]_i + \mathbf{A}\cdot\operatorname{curl}\mathbf{B} = (\text{using (i)}) = \frac{\partial}{\partial x_i}[\mathbf{A}\times\mathbf{B}]_i - \varepsilon_{ijk}A_j\frac{\partial B_k}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}(\varepsilon_{ijk}A_jB_k) - \varepsilon_{ijk}A_j\frac{\partial B_k}{\partial x_i} = \varepsilon_{ijk}\left(B_k\frac{\partial A_j}{\partial x_i} + A_j\frac{\partial B_k}{\partial x_i} - A_j\frac{\partial B_k}{\partial x_i}\right)\\ &= \varepsilon_{ijk}B_k\frac{\partial A_j}{\partial x_i} = B_k\varepsilon_{kij}\frac{\partial A_j}{\partial x_i}\\ &= B_k[\operatorname{curl}\mathbf{A}]_k\\ &= \mathbf{B}\cdot\operatorname{curl}\mathbf{A}. \;[\mathbf{4\ marks,\ Cat\ B}]\end{aligned}$$

(a)(iii) Applying the divergence theorem to $\mathbf{A}\times\mathbf{B}$:

$$\begin{aligned}\oint_S(\mathbf{A}\times\mathbf{B})\cdot\widehat{\mathbf{n}}\,dS &= \int_V \operatorname{div}(\mathbf{A}\times\mathbf{B})\,dV \;[\mathbf{\ 1\ mark,\ Cat\ A}]\\ &= (\text{using (ii)}) = \int_V(\mathbf{B}\cdot\operatorname{curl}\mathbf{A} - \mathbf{A}\cdot\operatorname{curl}\mathbf{B})\,dV.\end{aligned}$$

Rearranging:

$$\int_V \mathbf{A}\cdot\operatorname{curl}\mathbf{B}\,dV = -\oint_S(\mathbf{A}\times\mathbf{B})\cdot\widehat{\mathbf{n}}\,dS + \int_V \mathbf{B}\cdot\operatorname{curl}\mathbf{A}\,dV, \;[\mathbf{1\ mark,\ Cat\ D}]$$

as required.

(b)(i)

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & \sin x & 0\end{vmatrix} = (\cos x)\,\mathbf{k}. \;[\mathbf{1\ mark,\ Cat\ A}]$$

(b)(ii) Starting with the first quadrant, the sides are

$$\begin{aligned}y &= b(1-x/a),\; 0\le x\le a,\\ y &= b(1+x/a),\; -a\le x\le 0,\\ y &= -b(1+x/a),\; -a\le x\le 0,\\ y &= -b(1-x/a),\; 0\le x\le a. \;[\mathbf{3\ marks,\ Cat\ A}]\end{aligned}$$

(b)(iii) C is a closed curve with R its interior. [**1 mark, Cat A**]
$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy$ and is a tangent vector to C. [**1 mark, Cat B**]

(b)(iv) Starting with the path integral, and traversing the sides of Q in an anti-clockwise fashion [**1 mark, Cat C**], we have $C = C_1 \cup C_2 \cup C_3 \cup C_4$ with

$$\begin{aligned}
\text{On } C_1 &: \; y = b(1 - x/a), \;\; dy = -(b/a)dx \text{ with } x \text{ from } a \text{ to } 0; \\
\text{On } C_2 &: \; y = b(1 + x/a), \;\; dy = +(b/a)dx \text{ with } x \text{ from } 0 \text{ to } -a; \\
\text{On } C_3 &: \; y = -b(1 + x/a), \;\; dy = -(b/a)dx \text{ with } x \text{ from } -a \text{ to } 0; \\
\text{On } C_4 &: \; y = -b(1 - x/a), \;\; dy = +(b/a)dx \text{ with } x \text{ from } 0 \text{ to } a.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \sum_{i=1}^{4} \int_{C_i} \sin x\,dy \\
&= \int_a^0 -(b/a)\sin x\,dx + \int_0^{-a} (b/a)\sin x\,dx + \int_{-a}^0 -(b/a)\sin x\,dx + \int_0^a (b/a)\sin x\,dx \\
&= (b/a)(1 - \cos a + 1 - \cos(-a) + 1 - \cos(-a) + 1 - \cos a) \\
&= (4b/a)(1 - \cos a). \;\; [\textbf{2 marks, Cat C}]
\end{aligned}$$

Turning our attention to the double integral, we have

$$\begin{aligned}
\int_R (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\;dx\,dy &= \int_R \cos x\,dx\,dy \\
&= \int_{-a}^0 \int_{-b(1+x/a)}^{+b(1+x/a)} \cos x\,dy\,dx + \int_0^a \int_{-b(1-x/a)}^{+b(1-x/a)} \cos x\,dy\,dx \\
&= \int_{-a}^0 2b(1 + x/a)\cos x\,dx + \int_0^a 2b(1 - x/a)\cos x\,dx \\
&= \int_0^a 4b(1 - x/a)\cos x\,dx \\
&= 4b\left([(1 - x/a)\sin x]_0^a - \int_0^a (-1/a)\sin x\,dx \right) \\
&= (4b/a)(1 - \cos a), \;\; [\textbf{3 marks, Cat D}]
\end{aligned}$$

so that Green's theorem is verified.

Question Two Solution

(a)

$$\mathrm{div}\mathbf{A} = \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y) = 2, \;\; \mathrm{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+z & y & 0 \end{vmatrix} = \mathbf{j}, \;\; \nabla^2\mathbf{A} = \frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2} = 0.$$

[4 marks, Cat A]

(b) On the curved part of S we have that $\phi = x^2 + y^2$ is constant, and so the normal is in the direction $\pm\nabla\phi = \pm(2x\mathbf{i} + 2y\mathbf{j})$. For the outward normal we take the + sign. The unit normal is therefore

$$\widehat{\mathbf{n}} = (x\mathbf{i} + y\mathbf{j})/\sqrt{x^2+y^2} = (x\mathbf{i} + y\mathbf{j})/2 = \widehat{\mathbf{n}}_c, \text{say.}$$

The unit outward normal to the bottom face is simply $\widehat{\mathbf{n}}_{base} = -\mathbf{k}$, while on the top surface we have that $z - x$ is a constant and therefore the normal is in the direction $\pm\nabla(z-x) = \pm(\mathbf{k} - \mathbf{i})$. Taking the + sign for the outward normal we have

$$\widehat{\mathbf{n}}_{top} = (\mathbf{k} - \mathbf{i})/\sqrt{2}.$$

[4 marks, Cat A]

(c) Let the three surfaces be S_c, S_{base} and S_{top}. For the contribution over S_c we have

$$\int_{S_c} \mathbf{A}\cdot\widehat{\mathbf{n}}_c\,dS = \frac{1}{2}\int_{S_c} x^2 + xz + y^2\,dS = \frac{1}{2}\int_{S_c} 4 + xz\,dS.$$

Since the surface is cylindrical with radius 2 we have $dS = 2\,dz\,d\theta$ in cylindrical polar coordinates, with $x = 2\cos\theta$. Therefore the integral becomes

$$\begin{aligned}
\int_0^{2\pi}\int_0^{3+2\cos\theta}(4 + 2z\cos\theta)\,dz\,d\theta &= \int_0^{2\pi}\left[4z + z^2\cos\theta\right]_0^{3+2\cos\theta}d\theta \\
&= \int_0^{2\pi} 12 + 8\cos\theta + (9 + 12\cos\theta + 4\cos^2\theta)\cos\theta\,d\theta \\
&= \int_0^{2\pi} 12 + 12\cos^2\theta\,d\theta \\
&= 24\pi + 12\pi = 36\pi.\;\; \textbf{[3 marks, Cat B]}
\end{aligned}$$

Here we see that the remaining contributions to the integral are zero since $\cos\theta$ and $\cos^3\theta$ are both odd about $\theta = \pi$.

Alternatively, show explicitly that $\int_0^{2\pi}\cos^3\theta\,d\theta = \int_0^{2\pi}\cos\theta - \sin^2\theta\cos\theta\,d\theta = -[\sin^3\theta/3]_0^{2\pi} = 0$.

The contribution from the base:

$$\int_{S_{base}} \mathbf{A}\cdot\widehat{\mathbf{n}}_{base}\,dS = \int_{S_{base}} -((\mathbf{x}+\mathbf{z})\mathbf{i} + \mathbf{y}\mathbf{j})\cdot\mathbf{k}\;dS = 0,\;\; \textbf{[1 mark, Cat B]}$$

while the contribution from the top is

$$\int_{S_{top}} \mathbf{A}\cdot\widehat{\mathbf{n}}_{top}\,dS = -\frac{1}{\sqrt{2}}\int_{S_{top}}(x+z)\,dS = -\frac{1}{\sqrt{2}}\int_{S_{top}}(2x+3)\,dS,\;\; \textbf{[1 mark, Cat B]}$$

after substituting $z = 3 + x$ on S_{top}. To evaluate this integral we can project onto $z = 0$ where the projection is a circular disc of radius 2. (Alternatively, a Jacobian can be used).

We then obtain

$$-\frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^2 (2r\cos\theta+3)\frac{rdrd\theta}{|\widehat{\mathbf{n}}_{top}\cdot\mathbf{k}|} = -\int_0^{2\pi}\int_0^2 (2r^2\cos\theta+3r)\,dr\,d\theta = -2\pi\int_0^2 3rdr = -12\pi.$$

[3 marks, Cat C].

Adding together the three contributions we have

$$\begin{aligned}\oint_S \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS &= \int_{S_c}\mathbf{A}\cdot\widehat{\mathbf{n}}_c\,dS + \int_{S_{base}}\mathbf{A}\cdot\widehat{\mathbf{n}}_{base}\,dS + \int_{S_{top}}\mathbf{A}\cdot\widehat{\mathbf{n}}_{top}\,dS \\ &= 36\pi - 12\pi = 24\pi.\end{aligned}$$

(d) The equivalent volume integral is

$$\begin{aligned}\int_V \mathrm{div}\mathbf{A}\,dV &= 2\int_V dV = 2\int_0^{2\pi}\int_0^2\int_0^{3+r\cos\theta} rdzdrd\theta \\ &= 2\int_0^{2\pi}\int_0^2 (3r+r^2\cos\theta)\,drd\theta \\ &= 4\pi\int_0^2 3r\,dr \\ &= 24\pi,\ \ [\mathbf{4\ marks,\ Cat\,D}]\end{aligned}$$

in agreement with part (c).

Question Three Solution

(a) $I/2\pi$ represents the area of surface of revolution generated by rotating $y = y(x)$ around the x-axis, while J represents the total length of the curve between the end points at $(\pm 1/2, 1)$.
[**3 marks, Cat A**]

(b) Let $L = y(1 + (y')^2)^{1/2}$. L is explicitly independent of x so we can use the Euler-Lagrange short form $L - y'\partial L/\partial y' =$ constant, as given in the question. This becomes

$$\begin{gathered} y(1+(y')^2)^{1/2} - y'yy'(1+(y')^2)^{-1/2} = \text{constant} = a, \text{ say} \\ \Rightarrow [y(1+(y')^2) - y(y')^2] = a(1+(y')^2)^{1/2} \\ \Rightarrow y^2 = a^2(1+(y')^2) \ \ [\textbf{2 marks, Cat A}] \end{gathered}$$

which leads to the ODE:

$$\frac{a\,dy}{(y^2-a^2)^{1/2}} = \pm dx.$$

Letting $y = a\cosh t$:

$$\begin{gathered} \int \frac{a\sinh t}{\sinh t}\,dt = \pm(x+\beta) \\ \Rightarrow at = a\cosh^{-1}(y/a) = \pm(x+\beta) \\ \Rightarrow y = a\cosh\left(\frac{x+\beta}{a}\right). \end{gathered}$$

Applying the boundary conditions $y = 1$ when $x = \pm 1/2$ shows that $\beta = 0$ and that

$$1 = a\cosh(1/2a). \ \ [\textbf{3 marks, Cat C}]$$

(c) From earlier in part (b) we have that

$$(y')^2 = y^2/a^2 - 1 = \cosh^2(x/a) - 1 = \sinh^2(x/a).$$

Thus

$$\begin{aligned} I_{stat} &= \int_{-1/2}^{1/2} a\cosh^2(x/a)\,dx \\ &= \frac{a}{2}\int_{-1/2}^{1/2} \left(\cosh(2x/a) + 1\right)\,dx \\ &= \frac{a}{2}\left[\frac{a}{2}\sinh(2x/a) + x\right]_{-1/2}^{1/2} \\ &= \frac{a^2}{2}\sinh(1/a) + \frac{a}{2}. \ \ [\textbf{2 marks, Cat A}\ ] \end{aligned}$$

But:

$$\begin{aligned} \sinh(1/a) &= 2\sinh(1/2a)\cosh(1/2a) = 2\sqrt{\cosh^2(1/2a) - 1}\,\cosh(1/2a) \\ &= \text{(using (b))} = \frac{2}{a}\sqrt{\frac{1}{a^2} - 1} = \frac{2}{a^2}\sqrt{1-a^2}. \ \ [\textbf{2 marks, Cat D}] \end{aligned}$$

Thus $I_{stat} = \sqrt{1-a^2} + a/2$, as required.

(d) For the constrained problem we apply the E-L equation to a linear combination of the integrands of I and J, i.e. we take $L = (y+\lambda)(1+(y')^2)^{1/2}$, which is again explicitly independent of x, so that the same short form of the E-L equation is relevant. Substituting in:

$$\begin{aligned}(y+\lambda)(1+(y')^2)^{1/2} - y'(y+\lambda)y'(1+(y')^2)^{-1/2} &= \text{constant} = b, \text{ say}\\ \Rightarrow [(y+\lambda)(1+(y')^2) - (y+\lambda)(y')^2] &= b(1+(y')^2)^{1/2}\\ \Rightarrow (y+\lambda)^2 &= b^2(1+(y')^2), \ [\ \textbf{2 marks, Cat B}]\end{aligned}$$

which is almost identical to the previous algebra. Rearranging as before, we have

$$y' = \pm\left\{\left(\frac{y+\lambda}{b}\right)^2 - 1\right\}^{1/2}.$$

Proceeding as in part (b):

$$y+\lambda = b\cosh\left(\frac{x+\beta}{b}\right).$$

Applying the end conditions $y = 1$ at $x = \pm 1/2$ again implies that $\beta = 0$ and so we have

$$y = b\cosh\left(\frac{x}{b}\right) - \lambda, \quad [\textbf{2 marks, Cat B}]$$

as required.

(e) From the end condition:

$$1 = b\cosh\left(\frac{1}{2b}\right) - \lambda \ [\textbf{1 mark, Cat A}]$$

is the desired relation between b and λ. To obtain the relation for b we apply the integral constraint [**1 mark, Cat B**], using the fact that, from (d): $(1+(y')^2)^{1/2} = \cosh(x/b)$. We therefore have

$$J = 2 = \int_{-1/2}^{1/2} \cosh(x/b)\, dx = 2b\sinh\left(\frac{1}{2b}\right).$$

It follows that

$$b\sinh\left(\frac{1}{2b}\right) = 1, \ [\textbf{2 marks, Cat D}]$$

as required.

4. (a) (i) If $f(t,x) = \frac{t^n}{x}$ denotes the right hand side, then the partial derivative

seen $\Downarrow$

3, A

$$\frac{\partial f}{\partial x}(t,x) = -\frac{t^n}{x^2}$$

becomes unbounded for $x \to 0$ (for any fixed t), and thus, it can be concluded from the mean value theorem that, under the assumption that a global Lipschitz condition holds, then the assumed Lipschitz constant is too small for sufficiently small $x > 0$.

sim. seen $\Downarrow$

2, A

(ii) The derivative of f is given by

$$f'(t,x) = \left(\frac{nt^{n-1}}{x}, -\frac{t^n}{x^2}\right),$$

which is obviously a continuous function, and thus, the right hand side is continuous and locally Lipschitz continuous with respect to x. This implies applicability of the local version of the Picard–Lindelöf theorem.

meth seen $\Downarrow$

3, A

2, B

1, C

(iii) We get

$$\int_1^x y\,\mathrm{d}y = \int_0^t s^n \mathrm{d}s \quad \Rightarrow \quad \frac{1}{2}x^2 - \frac{1}{2} = \frac{t^{n+1}}{n+1}$$
$$\overset{x>0}{\Rightarrow} \quad x = \sqrt{\frac{2t^{n+1}}{n+1} + 1}\,,$$

so the function $\lambda(t) = \sqrt{\frac{2t^{n+1}}{n+1} + 1}$ solves the initial value problem and is defined for $t \in \mathbb{R}$ (whenever n is odd) and

$$t \in \Big(\underbrace{-\sqrt[n+1]{\frac{1}{(n+1)2}}}_{:=t_-}, \infty\Big) \qquad \text{(whenever } n \text{ is even).}$$

The solution is obviously maximal for n even. For n odd, due to $\lim_{t\to t_-}(t, \lambda(t)) = (t_-, 0) \in \partial D$, where $D = \mathbb{R} \times (0, \infty)$, the solution cannot be extended and is maximal.

unseen $\Downarrow$

3, A

(b) (i) This statement is false. The given function is not even differentiable at $t = 0$.

unseen $\Downarrow$

3, B

(ii) This statement is false. As established on the problem sheets, solutions to one-dimensional differential equations are monotone, but the given function is not monotone.

unseen $\Downarrow$

3, B

(iii) This statement is false. Such a function satisfies $\dot{\mu}(t) = \mu(t)^2 - 2\mu(t)\mu(t) + \mu(t)^2 = 0$ for all $t \in \mathbb{R}$, from which it follows that μ is constant, so not strictly monotonically increasing.

5. (a) (i) The equilibrium $(0,0)$ is not attractive. Due to the eigenvalue corresponding to the eigenvector w being positive, solutions in the corresponding eigenspace are exponentially increasing forward in time, which implies

meth seen ⇓

3, A

$$\lim_{t\to\infty} \|e^{At}\delta w\| = \delta \lim_{t\to\infty} \|e^{At}w\| = \infty \quad \text{for all } \delta > 0\,,$$

from which it follows that $(0,0)$ is not attractive.

seen ⇓

(ii) This differential equation does not have a periodic orbit, since, as discussed in the course, any linear differential equation with a periodic orbit needs to have a non-zero eigenvalue on the imaginary axis, which is not the case here.

3, A

unseen ⇓

(iii) Define $(x,y) := v + w$. Then

4, C

$$\varphi(t,x,y) = e^{At}v + e^{At}w\,,$$

and the result follows readily from $e^{At}v \to 0$ as $t\to\infty$, $\|e^{At}v\|\to\infty$ as $t\to-\infty$, $e^{At}w\to 0$ as $t\to-\infty$, and $\|e^{At}w\|\to\infty$ as $t\to\infty$. Note that all these limits follow from the exponential rates -2 and 1 in the eigenspaces spanned by v and w.

sim. seen ⇓

(iv) Both Lyapunov exponents are given by 1.

2, A

(b) Transform A into Jordan normal form J, and we get $e^{At} = Te^{Jt}T^{-1}$ for some invertible matrix T. According to a result presented in the lectures, each entry of the matrix e^{Jt} can be written as the product of a bounded function $g(t)$, a polynomial of in t (the degree of which is at most $d-1$) and an exponential term e^{at}, where a is a real part eigenvalue. The latter term is equal to 1 due to $a=0$ and does not need to be considered. Due to the above relation between e^{At} and e^{Jt}, every entry of e^{At} is a finite sum of the product of a bounded function with a polynomial (degree at most $d-1$). This implies the result if the matrix norm is the maximum norm, but since all norms are equivalent, this also holds for the operator norm.

2, C

unseen ⇓

2, C

4, D

6. (a) (i) V is called a Lyapunov function if $\dot{V}(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}^2$, where $\dot{V}$ is the orbital derivative, defined by $\dot{V}(x,y) = V'(x,y) \cdot f(x,y)$ (f denoting the right hand side). seen ⇓ 2, A

(ii) The linearisation in the trivial equilibrium is given by sim. seen ⇓ 2, B

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which has the eigenvalues $\pm i$, and thus is non-hyperbolic. Conclusions about asymptotic stability of the trivial equilibrium cannot be made using linearisation.

(iii) To use Lyapunov's direct method for concluding that the trivial equilibrium is asymptotically stable, we require (1) $\dot{V}(x,y) < 0$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, (2) $V(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, (3) $V(0,0) = 0$, and (4) $\dot{V}(0,0) = 0$ (although the latter is automatically satisfied). seen ⇓ 3, A

(iv) Corrected solution after the examination took place: the answer to this question is much more difficult than anticipated (or potentially even impossible) since the nonlinear terms in the differential equation have an even degree (it would be straightforward to show this if the nonlinear terms were of odd degree, with the negative sign). In fact, it is possible to show that any quadratic function $V(x,y) = ax^2 + by^2 + cxy$, where $a, b > 0$ and $c \in \mathbb{R}$ are suitably chosen such that $V(x,y) > 0$ for all $(x,y) \in \mathbb{R}^d \setminus \{(0,0)\}$, is not a Lyapunov function for this differential equation. sim. seen ⇓ 5, B

(b) Denote by φ the flow of this differential equation, and compute unseen ⇓ 8, D

$$\begin{aligned}\dot{V}(x,y) &= (1 - \tfrac{1}{x})x(1 - y - \tfrac{1}{4}x) + (1 - \tfrac{3}{4y})y(-1 + x) \\ &= (x-1)(1 - y - \tfrac{1}{4}x) + (y - \tfrac{3}{4})(x-1) = -\tfrac{3}{4}(x-1)^2 \leq 0\,.\end{aligned}$$

The function $t \mapsto V(\varphi(t,x,y))$ thus is monotonically decreasing for all $(x,y) \in (0,\infty) \times (0,\infty)$, and assume there exists a periodic orbit $O(x_0,y_0)$, then $\dot{V}(x,y) = 0$ for all $(x,y) \in O(x_0,y_0)$. This implies, using the above calculation for $\dot{V}$, that the periodic orbit lies in the one-dimensional set $\{1\} \times (0,\infty)$. This implies, using the $\dot{x}$-part of the differential equation, that $y = \frac{3}{4}$, which means that the periodic orbit lies in the zero-dimensional set $\{(1, \frac{3}{4})\}$, but any periodic orbit has more than one point.

Review of mark distribution:
Total A marks: 24 of 24 marks
Total B marks: 15 of 15 marks
Total C marks: 9 of 9 marks
Total D marks: 12 of 12 marks
Total marks: 60 of 60 marks

MATH50004 Multi-variable Calculus and Differential Equations

Question	Marker's comment
1	This question was done well by the majority of students. The most common mistake (which I found very surprising and couldn't have anticipated) was mistaking the quadrilateral with vertices (+/-a,0) and (0,+/-b) for a rectangle with sides x=+/-a,y=+/-b. This then makes the (incorrect) calculations trivial.
2	This question was not done particularly well, especially in view of the fact that it was rather similar to a quiz question given during the module.In part (a) many students did not understand the meaning of Laplacian of a vector.Part (b) (determining the normals) was done well.Part (c): this was a struggle for many. The integral over the curved part should be parameterized and a Jacobian found: projecting leads to a very long-winded procedure even if one exploits symmetry.(d): There were many ingenious ways of working out the volume. As long as they were explained clearly, full marks were given.
4	This questions consisted of two parts. The first part (a) was dealing with existence,nbsp; uniqueness and maximal solutions for a specific one-dimensional nonautonomous differential equations. Overall, this was done well by most students, in particular, the more computational parts in (iii). I was, however, expecting slightly better results in the seemingly more routine parts (i) and (ii) (which were asked similarly in some of the years before). The second part (b) was dealing with three multiple choice questions that required justification. Overall, the results were mixed, and the required justifications were often not to the point.
5	The first half of this question (a(i)-a(iii)) was generally done well, but the second half (a(iv), b) not as well.nbsp;(a)(i), (ii): Most answered well. To show there wasn't a periodic orbit, some didn't notice that the equation was linear and tried (some successfully) using Poincare-Bendixson.(a)(iii): Part was done well, and some also arguing from the phase-portrait. The most common error was not realising that the question was asking for one single vector satisfying both conditions, and instead stated two different vectors (e.g. the two eigenvectors).(a)(iv): A minority were able to write down the Lyapunov exponents without calculation. Most who tried to calculate them knew the definition, but some forgot about the natural log, hence finding infinity or zero. Frequently, a mistake was made evaluating the limit t -gt; infinity.nbsp;(b) Not answered well in general. Students either knew the intended argument well or not at all. Common mistakes were to only consider d=2, semi-simple eigenvalues, or using an exponential bound (based on the max real part eigenvalue, which is zero here) and not accounting for the t^(d-1) factor.nbsp;

MATH50015 Multivariable Calculus for JMC

Question	Marker's comment
1	This question was done well by the majority of students. The most common mistake (which I found very surprising and couldn't have anticipated) was mistaking the quadrilateral withnbsp;verticesnbsp;(+/-a,0) and (0,+/-b) for a rectangle withnbsp;sidesnbsp;x=+/-a,y=+/-b. This then makes the (incorrect) calculations trivial.
2	This question was not done particularly well, especially in view of the fact that it was rather similar to a quiz question given during the module.In part (a) many students did not understand the meaning of Laplacian of a vector.Part (b) (determining the normals) was done well.Part (c): this was a struggle for many. The integral over the curved part should be parameterized and a Jacobian found: projecting leads to a very long-winded procedure even if one exploits symmetry.(d): There were many ingenious ways of working out the volume. As long as they were explained clearly, full marks were given.

MATH50019 Differential Equations for JMC

Question	Marker's comment
1	This questions consisted of two parts. The first part (a) was dealing with existence,nbsp; uniqueness and maximal solutions for a specific one-dimensional nonautonomous differential equations. Overall, this was done well by most students, in particular, the more computational parts in (iii). I was, however, expecting slightly better results in the seemingly more routine parts (i) and (ii) (which were asked similarly in some of the years before). The second part (b) was dealing with three multiple choice questions that required justification. Overall, the results were mixed, and the required justifications were often not to the point.
2	The first half of this question (a(i)-a(iii)) was generally done well, but the second half (a(iv), b) not as well.nbsp;(a)(i), (ii): Most answered well. To show there wasn't a periodic orbit, some didn't notice that the equation was linear and tried (some successfully) using Poincare-Bendixson.(a)(iii): Part was done well, and some also arguing from the phase-portrait. The most common error was not realising that the question was asking for one single vector satisfying both conditions, and instead stated two different vectors (e.g. the two eigenvectors).(a)(iv): A minority were able to write down the Lyapunov exponents without calculation. Most who tried to calculate them knew the definition, but some forgot about the natural log, hence finding infinity or zero. Frequently, a mistake was made evaluating the limit t -gt; infinity.nbsp;(b) Not answered well in general. Students either knew the intended argument well or not at all. Common mistakes were to only consider d=2, semi-simple eigenvalues, or using an exponential bound (based on the max real part eigenvalue, which is zero here) and not accounting for the t^(d-1) factor.nbsp;