(1) $\mathcal{L}^{=}$ $\leq$

$\mathcal{Q} = \langle \mathbb{Q} ; \leq \rangle$

$\mathcal{R} = \langle \mathbb{R} ; \leq \rangle$

$\Sigma = \Delta \cup \Sigma_{=}$

<u>2.7.2</u> If ϕ is a closed $\mathcal{L}^{=}$-fmla then

$$\left.\begin{array}{l} \mathcal{Q} \models \phi \\ \iff \Sigma \vdash \phi \\ \iff \mathcal{R} \models \phi . \end{array}\right\}$$

<u>Pf:</u> $\Leftarrow$: As $\mathcal{Q} \models \Sigma$ if $\Sigma \vdash \phi$, then $\mathcal{Q} \models \phi$ by Generalised Soundness 2.4.7.

<u>$\Rightarrow$</u>: If $\Sigma \nvdash \phi$

By 2.7.5 $\Sigma \vdash (\neg \phi)$

So $\mathcal{Q} \models (\neg \phi)$, therefore $\mathcal{Q} \not\models \phi$.

Similarly $\mathcal{R} \models \phi \iff \Sigma \vdash \phi$.

#

Pf. shows that if $\mathcal{A}$ is a normal model of Δ then $Th(\mathcal{A}) = \{$closed ϕ : $\Sigma_{=} \cup \Delta \vdash \phi \}$.

Δ <u>axiomatises</u> $Th(\mathcal{A})$

(2.7.6) Theorem There is an algorithm which decides; given a closed $\mathcal{L}^=$-fmla ϕ, whether $\langle \mathbb{R}; \leq \rangle \models \phi$ or $\langle \mathbb{R}; \leq \rangle \not\models \phi$.

(equivalent to $\langle \mathbb{R}; \leq \rangle \models (\neg\phi)$.

Pf: $\Sigma = \Delta \cup \Sigma_=$ is a recusively enumerable set of fmlas. ~~As~~ The set of axioms of $K_{\mathcal{L}^=}$ is also recursively enumerable.

"So" the set of consequences of Σ is also recursively enumerable.

Method. Run the method which generates all consequences of Σ. As Σ is complete it will eventually give ϕ or $(\neg\phi)$. Then stop. #.

① Depends on

- Recursively enumerable set of axioms Σ (which ~~[illegible]~~ are complete)
-
- Completeness Thm.

(3)

(2) For $\langle \mathbb{Q}; \leq \rangle$ more practical methods exists.

(3) Works for some other structures.

(4) No such algorithm for $\langle \mathbb{N}; +, \cdot, 0, 1 \rangle$.

Gödel's Incompleteness Thm.

Mathematical Logic (MATH6/70132;P65)
Problem Class, week 8

[1] Let $\mathcal{L}^=$ be the usual language for rings, with binary function symbols $+, \cdot, -$ and constant symbols $0, 1$. Let Φ consist of the usual axioms for fields. So a field is a normal model of Φ.

Recall that for each prime number p there is a field $\mathbb{F}_p$ with p elements (take the integers modulo p).

Using the compactness theorem for normal models, prove the following:

Suppose ϕ is a closed $\mathcal{L}^=$-formula with the property that for infinitely many primes p, we have $\mathbb{F}_p \models \phi$. Then there is an infinite field F with $F \models \phi$.

If you know what the characteristic of a field is, show that we can also take F to be of characteristic 0.

[2] Suppose $\mathcal{L}^=$ is a first order language with equality ($=$) and a single binary relation symbol R.

(i) Write down a set Σ of closed $\mathcal{L}^=$-formulas such that the normal models of Σ are the normal $\mathcal{L}^=$-structures in which R is interpreted as an equivalence relation in which there are infinitely many equivalence classes and all equivalence classes are infinite.
(ii) Explain why any two countable normal models of Σ are isomorphic.

(iii) Find two non-isomorphic normal models of Σ with the same domain.

(iv) Prove that if $\mathcal{A}_1, \mathcal{A}_2$ are two normal models of Σ and ϕ is a closed $\mathcal{L}^=$-formula, then $\mathcal{A}_1 \models \phi \Leftrightarrow \mathcal{A}_2 \models \phi$.

[3] Suppose $\mathcal{L}^=$ is a language with equality and a single 2-ary relation symbol R. A graph $\mathcal{A} = \langle A; \bar{R}\rangle$ is a normal model of

$$(\forall x_1)(\forall x_2)(\neg R(x_1, x_1) \wedge (R(x_1, x_2) \to R(x_2, x_1))).$$

So $\bar{R}$ is symmetric and irreflexive. The elements of A are usually called *vertices.*

A clique in a graph is a set C of vertices such that any two distinct vertices in C are related by $\bar{R}$; a co-clique is a set K of vertices such that no pair of vertices in K is related by $\bar{R}$.

(i) For $n \in \mathbb{N}$, express the properties 'there is a clique of size n' and 'there is a co-clique of size n' by closed formulas μ_n and λ_n.

(ii) The infinite version of Ramsey's Theorem says that an infinite graph has an infinite clique or an infinite co-clique. Using this and the Compactness Theorem deduce the finite version of the theorem:

For every $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if $\mathcal{A}$ is a graph with at least N vertices, then $\mathcal{A}$ has a clique of size n or a co-clique of size n.

Problem class.

1/ For $n \in \mathbb{N}$ let σ_n be

$$(\exists x_1) \cdots (\exists x_n) \bigwedge_{1 \le i < j \le n} (x_i \ne x_j).$$

Consider

$$\Sigma = \{\phi\} \cup \{\sigma_n : n \in \mathbb{N}\} \cup \Phi.$$

~~Every finite~~ Suppose $\Sigma_0 \subseteq \Sigma$ is finite.

Can assume $\Sigma_0 = \{\phi, \sigma_1, \dots, \sigma_n\} \cup \Phi$

given in question!

Let $p \geqslant n$ be such that $\mathbb{F}_p \models \phi$ (there is such).

Then $\mathbb{F}_p \models \Sigma_0$.

By compactness thm. for normal $\mathcal{L}^{=}$-strs.

Σ has a normal model. $\cancel{\ }$.

For "char $= 0$" add into Σ_0: τ_n $\underbrace{1 + \dots + 1}_{n} \ne 0$.

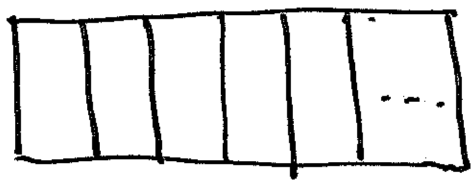
(all $n \in \mathbb{N}$)

(2)

2/ Σ axioms for eq. rel. $\quad \mathcal{L}^{=} \quad R$

$$\eta_n : (\exists x_1) \ldots (\exists x_n) \bigwedge_{1 \le i < j \le n} \neg E(x_i, x_j).$$

$$K_n \quad (\forall x)(\exists x_1) \cdots (\exists x_n) \bigwedge_{1 \le i < j \le n} R(x, x_i) \wedge (x_i \ne x_j)$$



a_{ij} $\quad i, j \in \mathbb{N}$

$$a_{ij} \, E \, a_{i'j'} \iff i = i'.$$

3

3/. $\mu_n : (\exists x_1)\cdots(\exists x_n) \bigwedge_{1\le i<j\le n} R(x_i, x_j)$

$\lambda_n : (\exists x_1)\cdots(\exists x_n) \bigwedge_{1\le i<j\le n} \left(\neg R(x_i, x_j) \wedge (x_i \neq x_j)\right)$.

(ii) Assume for a contradiction there is $n \in \mathbb{N}$ such that for every m, there is a graph Γ_m with $\geq m$ vertices and $\Gamma_m \models (\neg\mu_n) \wedge (\neg\lambda_n)$.

Consider $\Sigma \cup \{\gamma\} \cup \{\sigma_m : m \in \mathbb{N}\} \cup \{(\neg\mu_n) \wedge (\neg\lambda_n)\}$

graph axiom

Every finite subset of this has a normal model (by assumption).
So by CT, Σ has a normal model. This contradicts the infinite Ramsey thm. #.