MVC: Hints, tips, answers Sheet 4

1/ Consider Jacobian $\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$
& show that $\det(J) \neq 0$ when $(u,v)=(1,0)$.

To find u_x, v_x diff. implicitly wrt x and then rewrite as

$$\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

Then invert to get $\begin{pmatrix} u_x \\ v_x \end{pmatrix}$ after substituting $(u,v)=(1,0)$.

To find u_y, v_y diff expressions for x & y wrt y and proceed as above.

2/ (i) Diff implicitly wrt x. Same method as Q1 leads to

$$\begin{pmatrix} 2u & 4v \\ vy & uy-xy \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 2x \\ vy \end{pmatrix}$$

Soln exists provided det of 2×2 matrix $\neq 0$ when evaluated at (x_0, y_0, u_0, v_0).

Diff wrt y to get similar matrix equation for $\begin{pmatrix} u_y \\ v_y \end{pmatrix}$.

Then show that $u^2+2v^2+y^2-x^2=6$ & $uvy-vxy=1$ are satisfied for both given choices of (x_0, y_0, u_0, v_0).

For $(1,1,2,1)$ the determinant is $\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix} = 0$

For $(1+\sqrt{2}, -1-\sqrt{2}, 2, 1)$ we have $\begin{vmatrix} 4 & 4 \\ -1-\sqrt{2} & 1 \end{vmatrix} \neq 0$

(ii) $\begin{pmatrix} \partial F_1/\partial u & \partial F_1/\partial v \\ \partial F_2/\partial u & \partial F_2/\partial v \end{pmatrix} = \begin{pmatrix} 2u & 4v \\ vy & uy-xy \end{pmatrix}$ and so is the same as matrix in (i)

(iii) Think about whether map is 1–1.

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3/ Find $\frac{\partial \underline{r}}{\partial \xi}$, $\frac{\partial \underline{r}}{\partial \eta}$ & show dot product is zero.

Then $h_1 = \left| \frac{\partial \underline{r}}{\partial \xi} \right| = \cdots = \frac{c}{\cosh\xi - \cos\eta}$

$h_2 = h_1$

$h_3 = 1$

4/ Can proceed as in Q3 to find $h_1 = h_2 = (u^2+v^2)^{1/2}$, $h_3 = 1$.

Then $\hat{e}_1 = \frac{\partial \underline{r}/\partial u}{|\partial \underline{r}/\partial u|} = \frac{u\hat{\imath} + v\hat{\jmath}}{(u^2+v^2)^{1/2}}$

Similarly $\hat{e}_2 = (-v\hat{\imath} + u\hat{\jmath})/(\quad)^{1/2}$

& $\hat{e}_3 = \hat{k}$.

5/ (i), (ii) use expressions for div, curl in curvilinear coordinates

(iii) use unit vectors from Q4

In Cartesians, $\text{div}\,\underline{F} = 8x$, $\text{Curl}\,\underline{F} = -8y\underline{\hat{k}}$.

6/ $\hat{r} = (\cos\varphi)\hat{\imath} + (\sin\varphi)\hat{\jmath}$

$\hat{\varphi} = -\hat{\imath}\sin\varphi + \hat{\jmath}\cos\varphi$, $\hat{\underline{z}} = \underline{\hat{k}}$

& then $\hat{\imath} = \hat{r}\cos\varphi - \hat{\varphi}\sin\varphi$

$\hat{\jmath} = \hat{r}\sin\varphi + \hat{\varphi}\cos\varphi$

& $F_r = r\sin\varphi\cos\varphi + z\sin\varphi$, $F_\varphi = -r\sin^2\varphi + z\cos\varphi$, $F_z = r\cos\varphi$

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7. Find $|\det J| = 4(x^2+y^2)$. Integral $\to \frac{1}{4}\int_{v=2}^{v=4}\int_{u=-1}^{1}(u^2+v^2)\,du\,dv$.

$= \dots = 29/3$.

8. $\det J = r$. Integral $\to \int_0^{2\pi}\int_0^1 (r^4\cos^4\theta + r^4\sin^4\theta)\,r\,dr\,d\theta$

Then write in terms of $\cos 4\theta$.

9. $\det J = 2$. Integral $\to \int_{v=0}^{1}\int_{u=-v}^{v} \frac{v^2}{2}\cos(uv)\,du\,dv$

$= \dots = \frac{1}{2}(1-\cos(1))$

10. Calculate $\underline{J} = \frac{\partial \underline{r}}{\partial \lambda} \times \frac{\partial \underline{r}}{\partial s} = (\sin s, -\cos s, \lambda)$

Integral $\to \int_{s=0}^{2\pi}\int_{\lambda=0}^{1} (1+\lambda^2)^{1/2}\,d\lambda\,ds$ Solve using $\lambda = \sinh t$

11. $|\underline{J}| = \left|\frac{\partial \underline{r}}{\partial t} \times \frac{\partial \underline{r}}{\partial \theta}\right| = \dots = b(a+b\cos t)$

Then $\int_S z^2\,dS = \int_{\theta=0}^{2\pi}\int_{t=0}^{2\pi} (a+b\cos t)\,b^3\sin^2 t\,dt\,d\theta$.