IMPERIAL

**MATH50001**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Analysis 2**

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| Date: Thursday, May 9, 2024 |
| Time: 10:00 – 13:00 (BST) |
| Time Allowed: 3 hours |
| **This paper has 6 Questions.** |
| **Please Answer Each Question in a Separate Answer Booklet** |
| Candidates should start their solutions to each question on a new sheet of paper. |
| Supplementary books may only be used after the relevant main book(s) are full. |
| Any required additional material(s) will be provided. |
| Allow margins for marking. |
| Credit will be given for all questions attempted. |
| Each question carries equal weight. |

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$$f(x,y) = \begin{cases} (x, y^2, 0) & \text{if } y \geq 0, \\ (x, 0, -y^2) & \text{if } y < 0. \end{cases}$$

(a) Is the map $f$ continuous at $(0,0)$? Justify your answer. (6 marks)

(b) Find the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ at $(0,0)$. (7 marks)

(c) Is the map $f$ differentiable at $(0,0)$? Justify your answer. (7 marks)

(Total: 20 marks)

2. (a) Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$d(x,y) = \left| \int_x^y e^{t^2} \, dt \right|$$

is a metric on $\mathbb{R}$. (6 marks)

(b) Is the function $\|x\| = d(0,x)$ a norm function on $\mathbb{R}$? Justify your answer. (3 marks)

In the following questions, let $(X,d)$ be an arbitrary metric space, and $C \subseteq X$ be arbitrary.

(c) Define what it means for $C$ to be connected in $(X,d)$. (3 marks)

(d) Define what it means for $c \in X$ to be a limit point of $C$. (3 marks)

(e) Assume that $C$ is connected in $(X,d)$, and $C$ contains at least two elements. Prove that any $c \in C$ is a limit point of $C$. (5 marks)

(Total: 20 marks)

3. (a) Define the notions of closed set, compact set, and sequentially compact set in a metric space. (6 marks)

(b) Show that any compact set in a metric space is sequentially compact. (5 marks)

In the following questions, consider the metric space $(\mathcal{C}[0,1], d_\infty)$, where

$$\mathcal{C}[0,1] = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous on } [0,1] \}$$

$$d_\infty(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|, \qquad \text{for } f, g \in \mathcal{C}[0,1].$$

Let $g \equiv 0$ be the constant function $0$ on $[0,1]$, and define

$$D = \{ f \in \mathcal{C}[0,1] \mid d_\infty(f,g) \leq 1 \}$$

(c) Is the set $D$ closed in $(\mathcal{C}[0,1], d_\infty)$? Justify your answer. (4 marks)

(d) Is the set $D$ compact in $(\mathcal{C}[0,1], d_\infty)$? Justify your answer. (5 marks)

(Total: 20 marks)

4. (a) Let $\alpha \in \mathbb{C}$. Define $z^\alpha$, $z \in \mathbb{C}$, as a single-valued function. (2 marks)

   (b) Let $z_1, z_2, \alpha \in \mathbb{C}$. Is it true that $z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha$, where we assume that $z^\alpha$ is a single-valued function? Justify you answer. (4 marks)

   (c) Evaluate the integral

$$\oint_{|z-1-i|=2} \frac{z^3+3i}{(z^2+1)^2}\, dz.$$

(6 marks)

   (d) Find all holomorphic functions $f(z) = u(x, y) + iv(x, y)$ whose real part equals

$$u(x, y) = e^x(x\cos y - y\sin y).$$

(8 marks)

(Total: 20 marks)

5. (a) Give definitions of a removable singularity, a pole of order $m$ and an essential singularity. (3 marks)

   (b) Let

$$f(z) = \begin{cases} \frac{1}{\sin z} - \frac{1}{z}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Which singularities does $f$ have at $z = 0$? (4 marks)

   (c) Find the Laurent expansion for the function

$$f(z) = \frac{\sin z}{(z-1)^2} \quad \text{about } z_0 = 1.$$

(6 marks)

   (d) Let $f$ be holomorphic on the disc $|z| \leq R$ and assume that on the circle $|z| = R$ we have

$$|f(z)| \leq \frac{M}{|y|}, \qquad z = x + iy.$$

Prove that

$$|(z^2 - R^2)f(z)| \leq 2\,M\,R \quad \text{for all} \quad |z| \leq R.$$

(7 marks)

(Total: 20 marks)

6. (a) Formulate the Rouche's theorem (the proof is not required).

(2 marks)

(b) For each $n \geq 0$, how many zeros does the equation

$$e^z - 4z^n + 1 = 0$$

have in the disc $\{z \in \mathbb{C} : |z| < 1\}$.

(4 marks)

(c) Compute the value of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)}\, dx, \qquad \text{where} \quad 0 < a < 2.$$

*Hint:* Use integration over the boundary of the rectangle with vertices $-R, R, R+2\pi i, -R+2\pi i$.

(10 marks)

(d) Let

$$f(z) = \frac{z-i}{z+i}.$$

Describe the set $f(\Omega)$, where $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

(4 marks)

(Total: 20 marks)

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# MATH50001

# Analysis 2 / Real Analysis and Topology for JMC / Complex Analysis for JMC(Solutions)

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| ................ | ................ | ................ |

1. (a) Yes. (1 mark) $\boxed{\text{sim. seen} \Downarrow}$

Assume that $\|(x,y)\| \le 1$. For $y \ge 0$,

$$\|f(x,y)\| = \|(x,y^2,0)\| = \sqrt{x^2+y^4} \le \sqrt{x^2+y^2} = \|(x,y)\|,$$

and for $y < 0$,

$$\|f(x,y)\| = \|(x,0,-y^2)\| = \sqrt{x^2+y^4} \le \sqrt{x^2+y^2} = \|(x,y)\|.$$

( 3 mark)

Thus, for $\epsilon > 0$ we let $\delta = \min\{\epsilon, 1\}$. If $\|(x,y)\| < \delta$,

$$\|f(x,y) - f(0,0)\| = \|f(x,y)\| \le \|(x,y)\| < \delta \le \epsilon.$$

( 2 mark) $\boxed{6, \text{B}}$

(b) The partial derivatives are $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}\frac{\partial}{\partial x} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{(t,0,0)}{t} = (1,0,0).\end{aligned}$$

( 3 mark)

$$\frac{\partial}{\partial y} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}.$$

There are two possibilities for $f(0,t)$, depending on the sign of $t$.

$$\lim_{t\to 0^+} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0^+} \frac{(0,t^2,0)}{t} = (0,0,0)$$

$$\lim_{t\to 0^-} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0^+} \frac{(0,0,-t^2)}{t} = (0,0,0).$$

These imply that $\frac{\partial}{\partial y} f(0,0) = (0,0,0)$. ( 4 mark) $\boxed{7, \text{A}}$

(c) Yes ( 1 mark). $\boxed{\text{sim. seen} \Downarrow}$

We claim that the derivative of $f$ at $(0,0)$ is the linear map $\Lambda$ determined by $\Lambda(1,0) = (1,0,0)$ and $\Lambda(0,1) = (0,0,0)$. That is, $\Lambda(x,y) = (x,0,0)$. ( 2 mark)

Let $(x,y) \in \mathbb{R}^2$ be an arbitrary point. We have

$$\begin{aligned} f((0,0)+(x,y)) - f(0,0) - \Lambda[(x,y)] &= f(x,y) - (x,0,0) \\ &= \begin{cases} (0,y^2,0) & \text{if } y \ge 0 \\ (0,0,-y^2) & \text{if } y < 0 \end{cases}\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\|f((0,0)+(x,y)) - f(0,0) - \Lambda[(x,y)]\|}{\|(x,y)\|} &\le \lim_{(x,y)\to(0,0)} \frac{y^2}{\|(x,y)\|} \\ &\le \lim_{(x,y)\to(0,0)} \frac{\|(x,y)\|^2}{\|(x,y)\|} = 0.\end{aligned}$$

This shows that $f$ is differentiable at $(0,0)$. ( 4 mark) $\boxed{7, \text{C}}$

2. (a) We need to verify 3 conditions for the metric. sim. seen ⇓

Because of the modulus function, $d(x,y) \geq 0$, for all $x, y \in \mathbb{R}$. Moreover, since $e^{t^2} \geq e^0 = 1$, $d(x,y) \geq |x-y|$ which shows that $d(x,y) = 0$ iff $x = y$. ( 2 mark)

For all $x, y \in \mathbb{R}$,

$$d(x,y) = \left| \int_x^y e^{t^2}\, dt \right| = \left| - \int_y^x e^{t^2}\, dt \right| = \left| \int_y^x e^{t^2}\, dt \right| = d(y,x).$$

( 2 mark)

For all $x, y, z \in \mathbb{R}$ we have

$$\begin{aligned} d(x,z) &= \left| \int_x^z e^{t^2}\, dt \right| \\ &= \left| \int_x^y e^{t^2}\, dt + \int_y^z e^{t^2}\, dt \right| \\ &\leq \left| \int_x^y e^{t^2}\, dt \right| + \left| \int_y^z e^{t^2}\, dt \right| = d(x,y) + d(y,z). \end{aligned}$$

( 2 mark) 6, A

(b) No. ( 1 mark) unseen ⇓

For any norm function, we must have $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{R}$ and $x \in \mathbb{R}$. With $x = 1$ and $r = 2$, we have

$$\|2 \times 1\| = \int_0^2 e^{t^2}\, dt = \int_0^1 e^{t^2}\, dt + \int_1^2 e^{t^2}\, dt > \int_0^1 e^{t^2}\, dt + \int_0^1 e^{t^2}\, dt = 2\|1\|.$$

In the above equation we have used that $e^{t^2}$ is strictly increasing. ( 2 mark) 3, D

seen ⇓

(c) $C$ is called connected if there are no pairs of open sets $U$ and $V$ in $(X,d)$ such that $C \cap U \neq \emptyset$, $C \cap V \neq \emptyset$, $U \cap V = \emptyset$ and $C \subseteq U \cup V$. ( 3 mark) 3, A

seen ⇓

(d) $c$ is a limit point of $C$ if for every $\delta > 0$, $B_\delta(c) \cap C$ contains an element distinct from $c$. ( 3 mark) 3, A

unseen ⇓

(e) Let $c \in C$ be an arbitrary element. By assumption, there is $x \neq c$ in $C$. Let $r_0 = d(x,c) > 0$. We claim that for every $r \in (0, r_0)$, there is $x_r \in C$ such that $d(x_r, c) = r$. Otherwise, we consider the open sets $U = B_r(c)$ and $V = X \setminus \overline{B_r(c)}$, which are open sets, $C \subseteq U \cup V = X$, $c \in U \cap C$, and $x \in V \cap C$. This shows that $C$ is disconnected, which is a contradiction. ( 3 mark)

Fix an arbitrary $\delta > 0$. If $\delta > r_0$, then $x \in C \cap B_\delta(c)$ is distinct from $c$. If $\delta \leq r_0$, by the above paragraph, there is $c_{\delta/2} \in C$ such that $d(c, c_{\delta/2}) = \delta/2$. This shows that $c_{\delta/2} \in B_\delta(c)$ and $c_{\delta/2} \neq c$. ( 2 mark) 5, D

3. (a) $E$ is closed in $(X,d)$ if for every sequence $(x_i)_{i\geq 1}$ in $E$ which converges to some $x$ in the metric space $(X,d)$, we have $x \in E$. ( 2 mark) seen $\Downarrow$

$E$ is compact in $(X,d)$ if every open cover for $E$ has a finite subcover for $E$.

( 2 mark)

$E$ is sequentially compact, if every sequence in $E$ has a subsequence which converges to some element in $E$. ( 2 mark) 6, A

(b) Assume that $E$ is a compact set in a metric space $(X,d)$. Let $(x_i)_{i\geq 1}$ be a sequence in $E$. Assume that in the contrary this sequence has no convergent subsequence. This implies that for every $x \in E$, there is no subsequence of $(x_i)_{i\geq 1}$ which converges to $x$. Thus, for every $x \in E$ there is $r_x > 0$ such that $B_{r_x}(x)$ contains at most a finite number of elements in the sequence $(x_i)_{i\geq 1}$. The collection $\cup_{x\in E} B_{r_x}(x)$ is an open cover for $E$. Because $E$ is compact, there must be a finite subcover of this cover for $E$. This implies that there are only finitely many elements of $(x_i)_{i\geq 1}$ in $E$ which is not possible. ( 5 mark) seen $\Downarrow$ 5, C

(c) Yes. ( 1 mark) unseen $\Downarrow$

Let $(f_i)_{i\geq 1}$ be a sequence in $D$ which converges to some $f$ in $(\mathcal{C}[0,1], d_\infty)$. By assumption $d(f_i, g) \leq 1$. By the triangle inequality, for all $i \geq 1$,

$$d_\infty(f,g) \leq d_\infty(f,f_i) + d_\infty(f_i,g) \leq d_\infty(f_i,f) + 1.$$

Since $d_\infty(f_i,f) \to 0$, we conclude that $d_\infty(f,g) \leq 1$, and hence $f \in D$. This shows that $D$ is a closed set. ( 3 mark) 4, B

(d) No. ( 1 mark) unseen $\Downarrow$

If $D$ is compact, by part (b), it must be sequentially compact. However, the sequence $f_n = \sin(2\pi nx)$ belongs to $D$, but does not have any convergent subsequence. ( 2 mark for any sequence which works)

Each $f_n$ is continuous, and

$$d_\infty(f_n, g) = \sup_{t\in[0,1]} |\sin(2\pi nt) - 0| \leq 1.$$

Thus, each $f_n \in D$.

Assume that $f_n$ converges to some $h : [0,1] \to \mathbb{R}$. Since the convergence with respect to $d_\infty$ is the uniform convergence and each $f_n$ is continuous, $h$ must be continuous. Since $f_n(0) = 0$ for all $n$, we must have $h(0) = 0$. By continuity of $h$, if $k$ is a large enough integer, we must have $h(1/(4k)) < 1/2$. On the other hand, for $m \geq 1$,

$$f_{(2m+1)k}(1/(4k)) = \sin(2\pi(2m+1)k/(4k)) = \sin(\pi m + \pi/2) = \pm 1.$$

which contradicts $f_n(1/(4k)) \to h(1/(4k)) < 1/2$ as $n \to \infty$.

( 2 mark for the correct justification) 5, D

4. (a) We define $z^\alpha$ as a singe-valued function such that

seen ⇓

$$z^\alpha = e^{\alpha(\mathrm{Log} z)} = e^{\alpha(\ln|z| + i\,\mathrm{Arg}\, z)},$$

where $\mathrm{Arg}\, z$ is the principal value of the argument, namely, $-\pi < \mathrm{Arg}\, z \le \pi$.

2, A

(b) In general

sim. seen ⇓

$$z_1^\alpha z_2^\alpha \neq (z_1 z_2)^\alpha.$$

( 1 mark)

Indeed,

$$z_1^\alpha z_2^\alpha = e^{\alpha(\ln|z_1| + i\mathrm{Arg}\, z_1)} e^{\alpha(\ln|z_2| + i\mathrm{Arg}\, z_2)} = e^{\alpha(\ln|z_1 z_2| + i\mathrm{Arg}\, z_1 + i\mathrm{Arg}\, z_2)}$$

However

$$(z_1 z_2)^\alpha = e^{\alpha(\ln|z_1 z_2| + i\mathrm{Arg}\,(z_1 z_2))}.$$

Note that in general $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \neq \mathrm{Arg}\,(z_1 z_2)$.

( 3 marks)

4, A

(c) The function $f(z) = \frac{z^3+3i}{(z^2+1)^2} = \frac{z^3+3i}{(z+i)^2(z-i)^2}$ has only one pole of order two at $z = i$ in the disc $\{z : |z - 1 - i| < 2\}$. Let

sim. seen ⇓

$$g(z) = \frac{z^3 + 3i}{(z+i)^2}.$$

( 3 marks)

Then by using Cauchy's integral formula we find

$$\oint_{|z-1-i|=2} \frac{z^3+3i}{(z^2+1)^2}\, dz = \oint_{|z-1-i|=2} \frac{g(z)}{(z-i)^2}\, dz$$
$$= \frac{2\pi i}{1!}\, g'(z)\Big|_{z=i} = 2\pi i \left( \frac{3z^2}{(z+i)^2} - 2\frac{z^3+3i}{(z+i)^3} \right)_{z=i}$$
$$= 2\pi i \left( \frac{3}{4} - 2\frac{2i}{-8i} \right) = \frac{5}{2}\,\pi i.$$

( 3 marks)

6, B

(d) We first check if $u(x,y) = e^x(x\cos y - y\sin y)$ is harmonic. Indeed,

unseen ⇓

$$u'_x = e^x(x\cos y - y\sin y + \cos y), \qquad u''_{xx} = e^x(x\cos y - y\sin y + 2\cos y),$$

$$u'_y = e^x(-x\sin y - \sin y - y\cos y), \qquad u''_{yy} = e^x(-x\cos y - 2\cos y + y\sin y).$$

Thus $\Delta u = 0$.

( 2 marks)

In order to find $f(z) = u(x,y) + iv(x,y)$ we use the C-R equations

$$u'_x = v'_y, \qquad u'_y = -v'_x.$$

The first equation gives

$$v = \int e^x(x\cos y - y\sin y + \cos y)\, dy = e^x \left( x\sin y - \int y\sin y\, dy + \sin y \right)$$
$$= e^x\,(x\sin y + y\cos y - \sin y + \sin y) = e^x(x\sin y + y\cos y) + C(x).$$

( 2 marks)

To find $C(x)$ we use the second C-R equation $u'_y = -v'_x$. Then

$$-v'_x = e^x(-x\sin y - y\cos y - \sin y) + C'(x) = u'_y.$$

Therefore $C(x) = c \in \mathbb{R}$.

( 1 mark)

Finally we obtain

$$f(z) = e^x(x\cos y - y\sin y) + i\, e^x(x\sin y + y\cos y) + i\, c$$
$$= x\, e^{x+iy} + i\, y\, e^{x+iy} + i\, c = z\, e^z + i\, c.$$

( 2 marks) $\boxed{8, \mathrm{B}}$

5. (a) Definition. seen ⇓

Suppose a holomorphic function $f$ has an isolated singularity at $z_0$ and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of $f$ valid in some annulus $0 < |z - z_0| < R$. Then

* If $a_n = 0$ for all $n < 0$, $z_0$ is called a removable singularity.
* If $a_n = 0$ for $n < -m$ where $m$ a fix positive integer, but $a_{-m} \neq 0$, $z_0$ is called a pole of order $m$.
* If $a_n \neq 0$ for infinitely many negative $n$'s, $z_0$ is called an essential singularity. 3, A

(b) By using L'Hopital's rule twice we find unseen ⇓

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z - \sin z}{z \sin z} = \lim_{z\to 0} \frac{1 - \cos z}{\sin z + z\cos z}$$
$$= \lim_{z\to 0} \frac{\sin z}{\cos z + \cos z - z\sin z} = 0 = f(0).$$

( 4 marks)

Therefore $f$ is continuous at $z = 0$ and thus the function $f$ has a removable singularity at $z = 0$.

( 1 mark) 4, A

(c) We fist find sim. seen ⇓

$$f(z) = \frac{\sin z}{(z-1)^2} = \frac{1}{2i}\left(\frac{e^{iz}}{(z-1)^2} - \frac{e^{-iz}}{(z-1)^2}\right).$$

( 1 mark)

Thus

$$\frac{e^{iz}}{(z-1)^2} = e^i \frac{e^{i(z-1)}}{(z-1)^2} = \frac{e^i}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(i(z-1))^n}{n!}$$
$$= [n - 2 = k] = e^i \sum_{k=-2}^{\infty} \frac{i^{k+2}}{(k+2)!}(z-1)^k$$

and

$$\frac{e^{-iz}}{(z-1)^2} = e^{-i} \frac{e^{-i(z-1)}}{(z-1)^2} = \frac{e^{-i}}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(-i(z-1))^n}{n!}$$
$$= [n - 2 = k] = e^{-i} \sum_{k=-2}^{\infty} \frac{(-i)^{k+2}}{(k+2)!}(z-1)^k.$$

( 3 marks)

Finally we have

$$\frac{\sin z}{(z-1)^2} = \frac{1}{2i}\sum_{k=-2}^{\infty}\frac{e^i\, i^{k+2} - e^{-i}(-i)^{k+2}}{(k+2)!}(z-1)^k$$
$$= \sum_{k=-2}^{\infty}\frac{e^{i(1+\pi/2(k+2))} - e^{-i(1+\pi/2(k+2))}}{2i}\frac{1}{(k+2)!}(z-1)^k$$
$$= \sum_{k=-2}^{\infty}\frac{\sin\left[(1+\pi/2(k+2))\right]}{(k+2)!}(z-1)^k.$$

( 2 marks) 6, B

(d) Let $z = Re^{i\theta} = x + iy$. Then $\boxed{\text{unseen} \Downarrow}$

$$|z^2 - R^2| = R^2|e^{2i\theta} - 1| = 2R^2|\sin\theta| = 2R\,|y|.$$

( 2 marks)

Therefore for $z: \ |z| = R$ we have

$$|(z^2 - R^2)f(z)| = |z^2 - R^2||f(z)| \le |z^2 - R^2|\,\frac{M}{|y|} \le 2\,M\,R.$$

( 2 marks)

Finally by the maximum module principle we obtain

$$|(z^2 - R^2)f(z)| \le 2\,M\,R \quad \text{for all} \quad |z| \le R.$$

( 3 marks) $\boxed{7, \mathrm{C}}$

6. (a) Rouche's Theorem

seen ⇓

Let $f$ and $g$ be holomorphic in an open set $\Omega$ and let $\gamma \subset \Omega$ be a simple, closed, piecewise-smooth curve that contains in its interior only points of $\Omega$.

If $|g(z)| < |f(z)|$, $z \in \gamma$, then the sums of the orders of the zeros of $f+g$ and $f$ inside $\gamma$ are the same.

( 2 marks) 2, A

sim. seen ⇓

(b) Let $f(z) = 4z^n$ and $g(z) = e^z + 1$.
If $|z| = 1$, then

$$|f(z)| = 4 > \max_{|z|=1} |e^z + 1| = e + 1 \geq |g(z)|.$$

( 2 marks)

The equation $f(z) = 4z^n = 0$ has $n$ roots in the disc $D = \{z : |z| < 1\}$. Then by Rouche's theorem we find that the number of solutions of the equation $e^z - 4z^n + 1 = 0$ in $D$ equals $n$.

( 2 marks) 4, A

unseen ⇓

(c) Let us introduce

$$f(z) = \frac{e^{az}}{(e^z+1)(e^z+2)}.$$

( 1 mark)

and for $R : R > \ln 2$ consider the curve:

$$\gamma = \gamma_1 \cup \gamma_2 \cap \gamma_3 \cup \gamma_4,$$

where

$$\gamma_1 = \{z = x+iy : -R \leq x \leq R, y = 0\}, \quad \gamma_2 = \{z = x+iy : x = R, 0 \leq y \leq 2\pi\},$$

$$\gamma_3 = \{z = x+iy : R \geq x \geq -R, y = 2\pi\}, \ \gamma_4 = \{z = x+iy : x = -R, 2\pi \geq y \geq 0\}.$$

( 1 marks)

The function $f(z)$ has two simple poles within $\gamma$

$$z_1 = i\pi \quad \text{and} \quad z_2 = \ln 2 + i\pi.$$

By using Cauchy's integral formula we obtain

$$\oint_\gamma f(z)\, dz = \int_\gamma \frac{e^{az}}{(e^z+1)(e^z+2)}\, dz$$
$$= 2\pi i \left(\mathrm{Res}\,[f(z), i\pi] + \mathrm{Res}\,[f(z), \ln 2 + i\pi]\right).$$

Applying L'Hopital's rule we find

$$\mathrm{Res}\,[f(z), i\pi] = \lim_{z\to i\pi} \frac{e^{az}(z-i\pi)}{(e^z+1)(e^z+2)} = \frac{e^{ai\pi}}{e^{i\pi}+2} \lim_{z\to i\pi} \frac{1}{e^z} = \frac{e^{ai\pi}}{-1+2}(-1) = -e^{ai\pi}.$$

and

$$\begin{aligned}\text{Res}\,[f(z), \ln 2 + i\pi] &= \lim_{z\to \ln 2+i\pi} \frac{e^{az}(z - \ln 2 - i\pi)}{(e^z+1)(e^z+2)} \\ &= \frac{e^{a(\ln 2+i\pi)}}{e^{\ln 2+i\pi}+1}\,\frac{1}{e^{\ln 2+i\pi}} = \frac{2^a e^{ai\pi}}{(-2+1)}\,\frac{1}{(-2)} = 2^{a-1}e^{ai\pi}.\end{aligned}$$

Therefore

$$\oint_\gamma f(z)\,dz = 2\pi i e^{ai\pi}(-1+2^{a-1}).$$

( 3 marks)

We also find

$$\int_{\gamma_1} f(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)}\,dx, \quad \text{as} \quad R\to\infty.$$

$$\begin{aligned}\int_{\gamma_3} f(z)\,dz &\to -\int_{-\infty}^{\infty} \frac{e^{a(x+2\pi i)}}{(e^{x+2\pi i}+1)(e^{x+2\pi i}+2)}\,dx \\ &= -e^{2\pi i\,a}\int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)}\,dx, \quad \text{as} \quad R\to\infty.\end{aligned}$$

( 2 marks)

Moreover, by using the ML-inequality and since $0 < a < 2$ we obtain

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{(e^{R+iy}+1)(e^{R+iy}+2)}\,dy\right| \\ &\le \frac{e^{a\,R}}{(e^R-1)(e^R-2)}\,2\pi \to 0 \quad \text{as} \quad R\to\infty.\end{aligned}$$

Similarly

$$\left|\int_{\gamma_4} f(z)\,dz\right| \to 0 \quad \text{as} \quad R\to\infty.$$

( 2 marks)

Finally we have

$$2\pi i e^{ai\pi}(-1+2^{a-1}) = I(1-e^{2\pi i a})$$

and thus

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)}\,dx = \frac{\pi(1-2^{a-1})}{\sin(a\pi)}.$$

( 1 mark) $\boxed{10, \text{D}}$

$\boxed{\text{unseen} \Downarrow}$

(d) We first find the set of points $f(\operatorname{Im} z = 0)$. It is enough to test three points, for example, $z_1 = -1$, $z_2 = 0$ and $z_3 = 1$. We then find

$$w_1 = f(z_1) = \frac{-1-i}{-1+i} = i, \quad w_2 = f(z_2) = \frac{0-i}{0+i} = -1,$$
$$w_3 = f(z_3) = \frac{1-i}{1+i} = -i.$$

Thus the set $f(\Omega)$ is either the set of points $w \in \mathbb{C}$ such that $|w| > 1$ or $|w| < 1$.

( 3 marks)

Note that since $f(i) = 0$ we obtain $f(\Omega) = \{w \in \mathbb{C} : \; |w| < 1\}$.

( 1 mark) $\boxed{4, \text{A}}$

**Review of mark distribution:**

Total A marks: 48 of 48 marks
Total B marks: 30 of 30 marks
Total C marks: 19 of 18 marks
Total D marks: 23 of 24 marks
Total marks: 120 of 120 marks
Total Mastery marks: 0 of 20 marks

# MATH50001 Analysis 2

| Question | Marker's comment |
| --- | --- |
| 1 | This was a routine problem, but with some fine issues to analyse. Many students treated the problem as a calculus problem, which resulted in partial credit deducted. There were also near complete solutions.nbsp; |
| 2 | Many people lost a point or two on parts (a) and (b) for making claims without proof. nbsp;The most common major error was claiming falsely in parts (c) and (e) that a connected set is path-connected; others often forgot in the definition of connectedness that the sets U and V have to be open, because otherwise you could let U be a single point and V its complement and the definition would be trivially satisfied. |
| 3 | Most students did well on (a). nbsp;Part (b) was tricky; many people lost points for claiming that given a sequence (x_n) in A, the set A is covered by epsilon-neighbourhoods of the x_n; others tried to apply the Bolzano-Weierstrass or Heine-Borel theorems, which only work in R^n as opposed to an arbitrary metric space. nbsp;Similarly, some students tried to apply Arzelà-Ascoli in part (d), but many sequences in C[0,1] are not equicontinuous. nbsp;A common mistake on (c) was to find a sequence (f_n) whose pointwise limit is discontinuous: then the pointwise limit isn't in C[0,1], so the sequence (f_n) isn't actually convergent in C[0,1]. |
| 4 | Most of students were able to solve Q4 very well. However, some of them were not familiar with the notion of the principle value of the log-function. This notion was discussed at the lectures many times and I can only assume that such students simply did not come to the lectures. |
| 5 | 5a) The definition of "pole of order m" asks that the coefficients a_{-m-1}, a_{-m-2}, ... in the Laurent series are zero, and also that m is the smallest positive number satisfying this. Many students forgot the second detail.5b) This was overall done well with several different solutions used.5c) To obtain full marks, some simplification of the series was required at the end. Only one student did these simplifications.5d) Many students incorrectly used in their proof that \|y\|lt;=R implies 1/\|y\|lt;=1/R. (The second inequality is the wrong way around, which makes it incorrect.) This simplified the task so much that no points were awarded for that. However, one point was awarded if the Maximum Modulus Principle was applied, even if the rest of the solution was not correct. |

# MATH50017 Real Analysis and Topology for JMC

| Question | Marker's comment |
|---|---|
| 1 | Many students considered right and left limits in part a, which is never discussed in the lecture, and is not obviously valid in higher dimensions. As a result, little partial credit has been subtracted. In part b, many students have presented calculations without justification, despite having similar examples discussed in the lectures. In part c again, continuity of the partial derivatives have been stated without proof by many students.nbsp; |
| 2 | Many people lost a point or two on parts (a) and (b) for making claims without proof. nbsp;The most common major error was claiming falsely in parts (c) and (e) that a connected set is path-connected; others often forgot in the definition of connectedness that the sets U and V have to be open, because otherwise you could let U be a single point and V its complement and the definition would be trivially satisfied. |
| 3 | Most students did well on (a). nbsp;Part (b) was tricky; many people lost points for claiming that given a sequence (x_n) in A, the set A is covered by epsilon-neighbourhoods of the x_n; others tried to apply the Bolzano-Weierstrass or Heine-Borel theorems, which only work in R^n as opposed to an arbitrary metric space. nbsp;Similarly, some students tried to apply Arzelà-Ascoli in part (d), but many sequences in C[0,1] are not equicontinuous. nbsp;A common mistake on (c) was to find a sequence (f_n) whose pointwise limit is discontinuous: then the pointwise limit isn't in C[0,1], so the sequence (f_n) isn't actually convergent in C[0,1]. |