# Imperial College London

**MATH40004 - Calculus and Applications - Term 2**

## Problem Sheet 1 with solutions

*Questions marked by $^*$ are good candidates for discussion at the tutorials.*

1. Find the Fourier transforms of the following functions (with $a > 0$). Also, obtain the Fourier sine transform for the function in (ii) and Fourier cosine transform for the function in (iv).

(i) $f(x) = \begin{cases} e^{-ax}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

$\mathcal{F}\{f(x)\} = \int_0^\infty \exp(-ax)e^{-i\omega x}\, dx$

$= \int_0^\infty e^{-(a+i\omega)x}\, dx = \frac{1}{a+i\omega}$

(ii) $f(x) = \operatorname{sgn}(x)\exp(-a\,|x|)$; $[\operatorname{sgn}(x) = 1$ if $x > 0$ and $-1$ if $x < 0]$.

$\mathcal{F}\{\operatorname{sgn}(x)\exp(-a\,|x|)\} = \int_{-\infty}^0 (-1)e^{ax}e^{-i\omega x}\, dx + \int_0^\infty e^{-ax}e^{-i\omega x}\, dx = -\frac{1}{a-i\omega} + \frac{1}{a+i\omega} = -\frac{2i\omega}{a^2+\omega^2}$.

From the lectures we have $\hat{f}(\omega) = -2i\hat{f}_s(\omega)$, therefor $\hat{f}_s(\omega) = \frac{\omega}{a^2+\omega^2}$.

(iii) $f(x) = 2a/(a^2 + x^2)$;

We know from lecture that if $f(x) = \exp(-a\,|x|)$, then $\widehat{f}(\omega) = 2a/(a^2 + \omega^2)$

$\Rightarrow \widehat{f}(x) = 2a/(a^2 + x^2)$.

By the symmetry formula $\mathcal{F}\{\widehat{f}(x)\} = 2\pi f(-\omega) = 2\pi\exp(-a\,|\omega|)$.

(iv) $f(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise;

$f(x) = 1 - x^2$ for $|x| \leq 1 \Rightarrow \widehat{f}(\omega) = \int_{-1}^1 (1 - x^2)e^{-i\omega x}dx$

$= \int_{-1}^1 (1 - x^2)\cos\omega x\, dx - i\int_{-1}^1 (1 - x^2)\sin\omega x\, dx$.

The second integral is zero since we are integrating an odd function.

The first integral is an even function so can be written as twice the integral over $[0, 1]$.

Thus $\widehat{f}(\omega) = 2\int_0^1 (1 - x^2)\cos\omega x\, dx = \cdots$(by parts twice)$\cdots = -(4/\omega^2)\cos\omega + (4/\omega^3)\sin\omega$.

From the lectures we have $\hat{f}(\omega) = 2\hat{f}_c(\omega)$, therefor $\hat{f}_c(\omega) = -(2/\omega^2)\cos\omega + (2/\omega^3)\sin\omega$.

(v) $f(x) = \sin(ax)/(\pi x)$; (Hint: use the transform of a rectangular pulse from the lectures and the symmetry formula).

From your result in part (v), deduce that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

From lectures, if $h(x) = 1$ for $|x| \leq a$ and zero otherwise, then $\widehat{h}(\omega) = (2/\omega)\sin(a\omega)$.

Then by the symmetry formula, $\mathcal{F}\{(2/x)\sin(ax)\} = 2\pi h(-\omega) = 2\pi h(\omega)$ since $h$ is even.

Thus, $\mathcal{F}\{\sin(ax)/\pi x\} = h(\omega)$. We therefore have that $\int_{-\infty}^\infty (\sin(ax)/\pi x)\, e^{-i\omega x}dx = h(\omega)$.

Setting $\omega = 0$ and $a = 1$ : $\int_{-\infty}^\infty (\sin x)/x\, dx = \pi h(0) = \pi$.

The integrand is even about $x = 0$, and so $\int_0^\infty (\sin x)/x\, dx = \pi/2$ as required.

2. If a function has Fourier transform $\widehat{f}(\omega)$, find the Fourier transform of $f(x)\sin(ax)$ in terms of $\widehat{f}$.

$\mathcal{F}\{f(x)\sin ax\} = \int_{-\infty}^{\infty} f(x)\sin ax \, e^{-i\omega x} \, dx = \frac{1}{2i} \int_{-\infty}^{\infty} f(x)(e^{iax} - e^{-iax})e^{-i\omega x} \, dx$

$= \frac{1}{2i} \int_{-\infty}^{\infty} f(x)e^{-i(\omega-a)x} \, dx - \frac{1}{2i} \int_{-\infty}^{\infty} f(x) \, e^{-i(\omega+a)x} \, dx = \underline{\frac{1}{2i}\widehat{f}(\omega-a) - \frac{1}{2i}\widehat{f}(\omega+a)}.$

3. By applying the inversion formula to the transforms obtained in quiz 1 and problem 1(iv), establish the following results:

$$\text{(i)} \int_0^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = \frac{\pi e^{-a}}{2a} \text{ if } a > 0;$$

From the quiz in the lectures $\mathcal{F}\{\exp(-a\,|x|)\} = 2a/(a^2 + \omega^2)$.

Therefore using the inversion formula:

$\exp(-a\,|x|) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2a/(a^2 + \omega^2)\right) e^{i\omega x} \, d\omega = (a/\pi)\left(\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{a^2+\omega^2} \, d\omega + i \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{a^2+\omega^2} \, d\omega\right)$

The second integral is zero since the integrand is odd in $\omega$,

while the first integral has an even integrand and so doubles up over $[0, \infty]$.

Thus $\exp(-a\,|x|) = (2a/\pi) \int_0^{\infty} \cos(x\omega)/(a^2 + \omega^2) \, d\omega$.

This expression is true for any $x$. Setting $x = 1$ :

$$\underline{\frac{\pi e^{-a}}{2a} = \int_0^{\infty} \frac{\cos \omega}{a^2 + \omega^2} \, d\omega}$$

as required.

$$\text{(ii)} \int_{-\infty}^{\infty} \frac{\sin x - x\cos x}{x^3} \, dx = \frac{\pi}{2}.$$

From 1(iv) if we define $g(x) = 1 - x^2$ for $|x| \leq 1$ and zero otherwise, then by inversion:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega\right) e^{i\omega x} \, d\omega.$$

Set $x = 0$ and rearrange to obtain desired result.

4.* Sketch the function given by

$$f(x) = \begin{cases} 2d - |x| & \text{for } |x| \leq 2d, \\ 0 & \text{otherwise.} \end{cases},$$

and show that $\widehat{f}(\omega) = (2/\omega)^2 \sin^2(\omega d)$.

Use the energy theorem to demonstrate that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 \, dx = \frac{2\pi}{3}.$$

The function $f(x)$ is sketched in Figure 1.

$\widehat{f}(\omega) = \int_{-2d}^{2d}(2d - |x|)e^{-i\omega x} dx = \int_{-2d}^{2d}(2d - |x|)\cos\omega x \, dx - i\int_{-2d}^{2d}(2d - |x|)\sin\omega x \, dx.$

The second integral is zero since the integrand is odd in $x$.
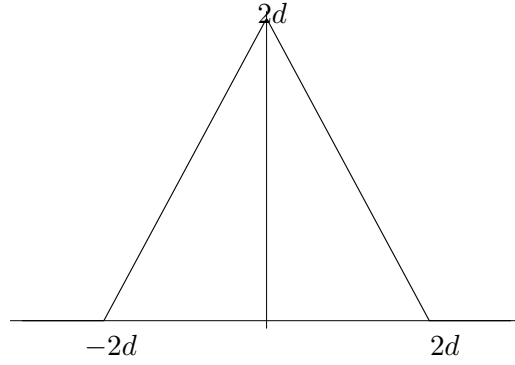
Figure 1: The function $f(x)$ in Question 4

The first integral has an even integrand and so doubles up over $[0, 2d]$.

Thus $\widehat{f}(\omega) = 2 \int_0^{2d} (2d - x) \cos \omega x \, dx = \cdots \text{(by parts)}$

$\cdots = (2/\omega^2)(1 - \cos(2\omega d)) = (4/\omega^2) \sin^2(\omega d)$

Therefore $\left| \widehat{f}(\omega) \right|^2 = \underline{(16/\omega^4) \sin^4(\omega d)}$.

Now $\int_{-\infty}^{\infty} (f(u))^2 \, du = \int_{-2d}^{2d} (2d - |u|)^2 du = 2 \int_0^{2d} (2d - u)^2 du$

$= \cdots = \underline{(16/3)d^3}$.

By the energy theorem we have $32\pi d^3/3 = 16 \int_{-\infty}^{\infty} \sin^4(\omega d)/\omega^4 d\omega$.

Setting $d = 1$ we get

$$\underline{\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4} \, dx = \frac{2\pi}{3}},$$

as required.

5. Show that the Fourier transform of $\exp(-cx)H(x)$, where $H$ is the Heaviside function and $c$ is a positive constant, is given by $1/(c + i\omega)$. Hence use the convolution theorem to find the inverse Fourier transform of

$$\frac{1}{(a + i\omega)(b + i\omega)},$$

where $a > b > 0$.

If $f(x) = \exp(-cx)H(x)$ then $\widehat{f}(\omega) = \int_{-\infty}^{\infty} (e^{-cx}H(x)) \, e^{-i\omega x} \, dx = \int_0^{\infty} e^{-(c+i\omega)x} dx = \underline{1/(c + i\omega)}$.

Convolution $\Rightarrow (\mathcal{F})^{-1}(\widehat{g}(\omega)\widehat{h}(\omega)) = g(x) * f(x)$.

Let $\widehat{g}(\omega) = 1/(a + i\omega) \Rightarrow g(x) = \exp(-ax)H(x)$.

Let $\widehat{h}(\omega) = 1/(b + i\omega) \Rightarrow h(x) = \exp(-bx)H(x)$.

$\Rightarrow (\mathcal{F})^{-1}((a + i\omega)^{-1}(b + i\omega)^{-1}) = (\exp(-ax)H(x)) * (\exp(-bx)H(x))$.

RHS $= \int_{-\infty}^{\infty} \exp(-a(x - u))H(x - u)\exp(-bu)H(u) \, du$

$= \int_0^{\infty} \exp(-a(x - u))H(x - u)\exp(-bu) \, du$

The function $H(x - u)$ is non-zero (and equal to 1) only if $0 < u < x$.

Therefore RHS $= \int_0^x \exp(-ax)\exp((a - b)u) \, du = \cdots = \underline{(\exp(-bx) - \exp(-ax))/(a - b) \ \ (x > 0)}$.

$\underline{\text{RHS} = 0 \text{ if } x < 0}$.

6. Use the symmetry rule to show that

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}(\widehat{f}(\omega) * \widehat{g}(\omega)).$$

Convolution $\Rightarrow \mathcal{F}\{\widehat{f}(x) * \widehat{g}(x)\} = \mathcal{F}\{\widehat{f}(x)\}\mathcal{F}\{\widehat{g}(x)\}$

$= \text{(symmetry formula)} = 4\pi^2 f(-\omega)g(-\omega).$

Take RHS, change $\omega$ to $x$ and take Fourier transform again from both sides:

$\mathcal{F}\{4\pi^2 f(-x)g(-x)\} = 2\pi(\widehat{f}(-\omega) * \widehat{g}(-\omega))$ using the symmetry rule again.

Thus: $\underline{\mathcal{F}\{f(x)g(x)\} = (\widehat{f}(\omega) * \widehat{g}(\omega))/(2\pi)}$, as required.


7. Suppose that $f(x)$ is a function such that $\widehat{f}(\omega) = 0$ for all $\omega$ with $|\omega| > M$, where $M$ is a positive constant. Let $g(x) = \sin(ax)/(\pi x)$. Show that if the constant $a > M$ :

$$f(x) * g(x) = f(x).$$

Hint: Use the transform of $g(x)$ from Q1(v).

From 1(v) we have that $\widehat{g}(\omega) = 1$ if $|\omega| \leq a$ and zero otherwise.

By convolution: $f(x) * g(x) = (\mathcal{F})^{-1}(\widehat{f}(\omega)\widehat{g}(\omega)).$

Inversion formula $\Rightarrow$ RHS $= (1/2\pi) \int_{-\infty}^{\infty} \widehat{f}(\omega)\widehat{g}(\omega)e^{i\omega x}d\omega = (1/2\pi) \int_{-a}^{a} \widehat{f}(\omega)e^{i\omega x}\,d\omega$

$= (1/2\pi) \int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega$ (since $a > M$).

Thus: $f(x) * g(x) = (\mathcal{F})^{-1}(\widehat{f}(\omega)) \underline{= f(x)}$, as required.


8.* By considering suitable integration formulae, establish the following results involving the Dirac delta function:

(i) $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$; (ii) $x\delta'(x) = -\delta(x)$; (iii) $\delta(-x) = \delta(x)$.

Here $f(x)$ is continuous. [In each case multiply by an arbitrary continuous test function $\phi(x)$ and integrate from $-\infty$ to $\infty$].

(i) $\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\phi(x)\,dx = f(x_0)\phi(x_0) = f(x_0) \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\,dx$

$= \int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\phi(x)\,dx$

$\Rightarrow \int_{-\infty}^{\infty} [f(x)\delta(x - x_0) - f(x_0)\delta(x - x_0)]\phi(x)\,dx = 0$ for arbitrary $\phi$.

$\Rightarrow \underline{f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)}.$

(ii) $\int_{-\infty}^{\infty} x\phi(x)\delta'(x)\,dx = \text{(by parts)} = [\delta(x)x\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)\,(\phi(x) + x\phi'(x))\,dx.$

Term in square brackets is zero.

Integral reduces to $-\phi(0)$ which can also be written as $- \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx.$

Thus $\underline{x\delta'(x) = -\delta(x)}.$

(iii) $\int_{-\infty}^{\infty} \delta(-x)\phi(x)\,dx = \text{(subst } x = -s) = \int_{-\infty}^{\infty} \delta(s)\phi(-s)\,ds = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx$

$\Rightarrow \underline{\delta(-x) = \delta(x)}.$