Imperial College
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**MATH97061 MATH97172**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Stochastic Calculus with Applications to non-Linear Filtering**

Date: Friday, 14 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

For the following questions, assume the set-up: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$ and $V$ be a standard one-dimensional $\mathcal{F}_t$-adapted Brownian motion under $\mathbb{P}$. Let $f$ and $\sigma$ be bounded Lipschitz continuous real valued functions and let $X$ be the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \int_0^t \sigma(X_s)\,dV_s. \tag{1}$$

Assume that $X_0$ has distribution $\pi_0$ at time 0, is independent of $V$ and $\mathbb{E}\left[(X_0)^2\right] < \infty$. Let $W$ be a standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of $X$, and $Y$ be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s)ds + W_t, \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. The process $Y = \{Y_t,\ t \geq 0\}$ is called the observation process. Let $\{\mathcal{Y}_t,\ t \geq 0\}$ be the filtration associated with the process $Y$, that is $\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t])$. The filtering problem consists in determining the conditional distribution $\pi_t$ of the signal $X_t$ given $\mathcal{Y}_t$. That is, $\pi_t(A) = \mathbb{E}[I_A(X_t) \mid \mathcal{Y}_t]$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-field on $\mathbb{R}$ and $I_A$ is the indicator function of the set $A$) and $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]$ for any bounded Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$.

1. Let $a, b$ be positive reals and $Z^{a,b} = \{Z_t^{a,b}, t \geq 0\}$ be the process defined by

$$Z_t^{a,b} = \exp\left(\int_0^t a h(X_s)dW_s - \frac{b^2}{2}\int_0^t h(X_s)^2\,ds\right),\ t \geq 0.$$

   (a) Deduce the evolution equation satisfied by the process $Z^{a,b}$. (5 marks)

   (b) Assuming that the process $t \to \int_0^t Z_s^{a,b} h(X_s)^2 ds$ is strictly positive $P$-almost surely, prove that $Z^{a,b}$ is an $\mathcal{F}_t$-adapted martingale if and only if $a = b$. (10 marks)

   (c) Prove that $\sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^{a,b}\right)^{-2}\right] < \infty$ for any $T > 0$. (5 marks)

(Total: 20 marks)

2. Assume that the signal process $(X)$ satisfies the following linear equation

$$X_t = X_0 + \int_0^t (aX_s + b_s)\,ds + 2\int_0^t \sigma_s dV_s,$$

   where $a < 0$ and $b_t = \sigma_t = e^{at}$. In addition, assume that $X_0 \sim N(x_0, P_0)$ and that it is independent of $V$.

   (a) Prove that $X_t = e^{at}(X_0 + t + 2V_t)$. (5 marks)

   (b) Find the prior distribution of $X_t$. (5 marks)

   (c) Find the joint distribution of $(X_1, X_2)$. (5 marks)

   (d) Assuming that $\lim_{t\to\infty} \frac{V_t}{t} = 0$, compute the limit $\lim_{t\to\infty} X_t$. (5 marks)

(Total: 20 marks)

3. Let $A$ be the second order differential operator

$$A\varphi = f\varphi' + \frac{1}{2}\sigma^2\varphi'', \quad \varphi \in C_b^2(\mathbb{R}),$$

where $C_b^2(\mathbb{R})$ is the set of all bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$ twice differentiable with bounded first and second derivatives. Next let $\mu = \{\mu_t, t \geq 0\}$ be a measure-valued process satisfying the equation

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(A\varphi)\, ds + 2\int_0^t (\mu_s(h\varphi) - \mu_s(h)\mu_s(\varphi))\, ds, \text{ for any } \varphi \in C_b^2(\mathbb{R}), \quad (3)$$

where $\mu_0$ is a probability measure. Let $\mu(\mathbf{1}) = \{\mu_t(\mathbf{1}), t \geq 0\}$ be the associated total mass process, that is $\mu_t(\mathbf{1}) := \mu_t(\mathbb{R})$ for any $t \geq 0$ and let $\nu = \{\nu_t, t \geq 0\}$ be the measure-valued process defined as

$$\nu_t(\varphi) = \mu_s(\varphi)\exp\left(2\int_0^t \mu_s(h)\, ds\right) \text{ for any } \varphi \in C_b^2(\mathbb{R}), \quad t \geq 0. \quad (4)$$

(a) Deduce the evolution equation satisfied by the mass process $\mu(\mathbf{1}) = \{\mu_t(\mathbf{1}), t \geq 0\}$. (3 marks)

(b) Prove $\mu_t$ is a probability measure for any $t \geq 0$. (5 marks)

(c) Deduce the evolution equation satisfied by the process $\nu(\varphi) = \{\nu_t(\varphi), t \geq 0\}$ for any $\varphi \in C_b^2(\mathbb{R})$. (6 marks)

(d) Assume that the measure $\nu_t$ is absolutely continuous with respect to the Lebesgue measure. Denote by $\tilde{\nu}_t$ the density of the measure $\nu_t$ with respect to the Lebesgue measure and assume that $\tilde{\nu}_t \in C_b^2(\mathbb{R})$. Deduce the partial differential equation satisfied by $\tilde{\nu} = \{\tilde{\nu}_t, t \geq 0\}$. (6 marks)

(Total: 20 marks)

4. Recall that $W$ is a standard one-dimensional Brownian motion. Let $U$ be the process defined as $U_t = \sinh(W_t)$ for any $t \geq 0$. [Recall that $\sinh x = (e^x - e^{-x})/2$.]

(a) State Itô's formula as applied to semimartingales [no proof required]. (3 marks)

(b) Deduce the evolution equation satisfied by the process $U$. (5 marks)

(c) Rewrite the evolution equation for $U$ as a stochastic differential equation. In other words find two functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$dU_t = g(U_t)\, dt + h(U)\, dW_t. \quad (5)$$

(3 marks)

(d) Does equation (5) have a unique solution? (3 marks)

(e) Compute $\mathbb{E}[U_t^2]$. (3 marks)

(f) Compute $\liminf_{t\to\infty} U_t$. (3 marks)
[You can assume, without proof, that $\limsup_{t\to\infty} W_t = -\liminf_{t\to\infty} W_t = \infty$.]

(Total: 20 marks)

5. Let $X$ be the process satisfying the SDE (1). In other words

$$X_t = X_0 + B_t + M_t,\, t \geq 0,$$

where $B = \{B_t, t > 0\}$ and $M = \{M_t, t > 0\}$ are the processes defined as

$$\begin{aligned} B_t &= \int_0^t f(X_s)\, ds,\, t \geq 0. \\ M_t &= \int_0^t \sigma(X_s)\, dV_s,\, t \geq 0. \end{aligned}$$

Let $\pi$ be a partition of the interval $[0,T]$, $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$. The mesh of the partition, denoted by $\|\pi\|$, is defined to be $\|\pi\| = \max_{1\leq i\leq n} |t_i - t_{i-1}|$.

(a) Let $f : [0,T] \to \mathbb{R}$ be a real-valued function and let $p > 0$. Define the $p$-variation of the function $f$. (2 marks)

(b) Prove that the process $M$ is a martingale. (3 marks)

(c) Compute the limit

$$\lim_{\|\pi\|\to 0} \sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}|.$$

(6 marks)

(d) Prove that the process $X$ has finite $2$-variation paths. (6 marks)

(e) Compute the 3-variation of the process $X$. (3 marks)

[You can use without proof the fact that $X$ is a continuous process and that $M$ has finite $2$-variation paths.]

(Total: 20 marks)

## Answers

**Question 1. [20 marks]**
**(a). [5 marks, seen similar]** Let $\xi = \{\xi_t, t > 0\}$ be the semimartingale defined by

$$\xi_t = \int_0^t ah(X_s)dW_s - \frac{b^2}{2}\int_0^t h(X_s)^2\, ds \qquad t \geq 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned} Z^{a,b} &= \exp(\xi_t) \\ &= \exp(\xi_0) + \int_0^t \exp(\xi_s)\, d\xi_s + \frac{1}{2}\int_0^t \exp(\xi_s)\, d\langle\xi\rangle_s \\ &= 1 + \int_0^t Z_s\left(ah(X_s)dW_s - \frac{b^2}{2}h(X_s)^2\, ds\right) + \frac{a^2}{2}\int_0^t Z_s h(X_s)^2 ds \\ &= 1 + \int_0^t aZ_s h(X_s)dW_s + \frac{a^2-b^2}{2}\int_0^t Z_s h(X_s)^2 ds \end{aligned}$$

**(b). [5 marks, not seen]** "$\Longrightarrow$". The process $Z^{a,b}$ is a semimartingale as it has a Doob-Meyer decomposition: It is a sum of a martingale part and a finite variation part. As it is a martingale, its finite variation part must be 0, by the uniqueness of the Doob-Meyer decomposition. Since the process $t \to \int_0^t Z_s h(X_s)^2 ds$ is strictly positive $P$-almost surely, we must have $a^2 - b^2 = 0$, hence $a = b$ (due to the fact that the $a, b$ are both positive reals).
**[5 marks, seen similar]** "$\Longleftarrow$" Novikov's condition states that if $u = \{u_t, t > 0\}$ is a process defined as $u_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ for $M$ a continuous local martingale, then a sufficient condition for $u$ to be a martingale is that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] < \infty, \qquad 0 \leq t < \infty.$$

In this case the process $t \to a\int_0^t h(X_s)dW_s$ is a local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by $t \to \int_0^t a^2 h(X_s)^2\, ds$ . Moreover, since $h$ is bounded, it follows that $|h(X_s)| \leq \|h\|_\infty$ hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] = \mathbb{E}\left[\exp\left(\frac{a^2}{2}\int_0^t h(X_s)^2\, ds\right)\right] \leq \exp\left(\frac{ta\,\|h\|_\infty^2}{2}\right) < \infty, \qquad 0 \leq t < \infty.$$

Hence, by Novikov's condition, the process $Z^{a,a}$ is a martingale.
**(c). [5 marks, seen similar]** Observe that

$$\frac{1}{\left(Z_t^{a,b}\right)^2} = \exp\left(\frac{1}{2}\int_0^t \left((-2a)^2 + 2b^2\right) h(X_s)^2 ds\right) \bar{z}_s \leq \exp\left(\left(2a^2+b^2\right) t \left\|h\right\|_\infty^2\right) \bar{z}_s$$

where $\bar{z} = \{\bar{z}_t, t > 0\}$ is the process defined by

$$\bar{z}_t = \exp\left(\int_0^t -2ah(X_s)dW_s - \frac{(-2a)^2}{2}\int_0^t h(X_s)^2\, ds\right),\ t \geq 0.$$

Again, by Novikov's condition, the process $\bar{z} = \{\bar{z}_t, t > 0\}$ is a martingale. Hence $\mathbb{E}[\bar{z}_s] = \mathbb{E}[\bar{z}_0] = 1$ and

$$\sup_{t\in[0,T]} \mathbb{E}\left[\frac{1}{\left(Z_t^{a,b}\right)^2}\right] \leq \sup_{t\in[0,T]} \exp\left(\left(2a^2+b^2\right) t \left\|h\right\|_\infty^2\right) \mathbb{E}[\bar{z}_t] = \exp\left(\left(2a^2+b^2\right) T \left\|h\right\|_\infty^2\right) < \infty$$

for any $T > 0$.

**Question 2. [20 marks]**
**(a). [5 marks, seen similar]** Prove that $X_t = e^{at}(X_0 + t + 2V_t)$. Let $u = \{u, t \geq 0\}$ be the (deterministic) process given by $u_t = e^{-at}$, $t \geq 0$. By integration by parts

$$X_t u_t = X_0 u_0 + \int_0^t X_s du_s + \int_0^t u_s dX_s + \langle X, u \rangle_t .$$

Since $u$ has no martingale part $\langle X, u \rangle_t = 0$, we get

$$\begin{aligned} X_t u_t &= X_0 - \int_0^t a u_s X_s ds + \int_0^t (a X_s + b_s) u_s ds + 2\int_0^t \sigma_s u_s dV_s \\ &= X_0 + \int_0^t b_s u_s ds + 2\int_0^t \sigma_s u_s dV_s \\ &= X_0 + \int_0^t 1 ds + 2\int_0^t dV_s = X_0 + t + 2V_t \end{aligned}$$

which gives us the required identity.
**(b). [5 marks, seen similar]** The random variable $X_0 + t + 2V_t$ is the sum of the two independent Gaussian random variable plus the constant $t$, hence it is itself Gaussian with mean and variance obtained by summing up the corresponding component means and variances. So

$$X_0 + t + 2V_t \sim N(x_0 + t, P_0 + 2t)$$

Since $X_t = e^{at}(X_0 + t + 2V_t)$ it follows that $X_t$ is a Gaussian random variable and

$$X_t \sim N\left(e^{at}(x_0 + t), e^{2at}(P_0 + 2t)\right).$$

**(c). [5 marks, not seen]** The pair of random variables $(X_1, X_2)$ are individually Gaussian random variables. Moreover, they are obtained via linear transformations from the independent Gaussian random variables $(X_0, V_1, V_2)$.It follows that $(X_1, X_2)$ is (jointly) a Gaussian random vector and

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = N\left(\left(\begin{array}{c} e^{a}(x_0 + 1) \\ e^{2a}(x_0 + 2) \end{array}\right), \left(\begin{array}{cc} e^{2a}(P_0 + 2) & P_{X_1 X_2} \\ P_{X_1 X_2} & e^{4a}(P_0 + 4) \end{array}\right)\right),$$

where

$$\begin{aligned} P_{X_1 X_2} &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\ &= e^{3a}\mathbb{E}[(X_0 + 1 + 2V_1)(X_0 + 2 + 2V_2)] - e^{3a}(x_0 + 1)(x_0 + 2) \\ &= e^{3a}\mathbb{E}[(X_0 + 1)(X_0 + 2)] + 4e^{3a}\mathbb{E}[V_1 V_2] - e^{3a}(x_0 + 1)(x_0 + 2) \\ &= e^{3a}\left(\mathbb{E}\left[X_0^2\right] + 3\mathbb{E}[X_0] + 2 + 4\right) - e^{3a}(x_0 + 1)(x_0 + 2) \\ &= e^{3a}\left(\left(x_0^2 + P_0 + 3x_0 + 2 + 4\right) - (x_0 + 1)(x_0 + 2)\right) \\ &= e^{3a}(P_0 + 4). \end{aligned}$$

**(d). [5 marks, not seen]** Since $\lim_{t\to\infty} e^{at}t = 0$ we deduce that.

$$\lim_{t\to\infty} X_t = \lim_{t\to\infty} e^{at}t\left(\left(\frac{X_0}{t} + 1 + 2\frac{V_t}{t}\right)\right) = 0\,(0+1+0) = 0.$$

**Question 3. [20 marks]**

**(a). [3 marks, seen similar]** Since $A\mathbf{1} = 0$, it follows from the equation that the mass process satisfies the following

$$\begin{aligned}
\mu_t(\mathbf{1}) &= \mu_0(\mathbf{1}) + \int_0^t \mu_s(A\mathbf{1})\,ds + 2\int_0^t (\mu_s(h) - \mu_s(h)\,\mu_s(1))\,ds \\
&= 1 + 2\int_0^t \mu_s(h)\,(1 - \mu_s(1))ds.
\end{aligned}$$

**(b). [5 marks, not seen]** Let $e = \{e_t, t \geq 0\}$ be the process defined as $e_t = \mu_t(\mathbf{1}) - 1$. It follows from the above equation that $e$ satisfies the linear ordinary differential equation

$$\frac{de_t}{dt} = -2\mu_s(h)\,e_t,\ \ t \geq 0, \quad e_0 = 0,$$

which has as the unique solution the trivial solution $e_t = 0$. Hence $\mu_t(\mathbf{1}) = 1$ and therefore $\mu_t$ is indeed a probability for all $t \geq 0$.

**(c). [6 marks, not seen]** Denote $m = \{m_t, t \geq 0\}$ the process defined as

$$\begin{aligned}
m_t &= \exp\left(2\int_0^t \mu_s(h)\,ds\right),\ \ t \geq 0, \\
m_t &= m_0 + 2\int_0^t \mu_s(h)\,m_s ds,\ \ t \geq 0.
\end{aligned}$$

By the product rule

$$\begin{aligned}
\nu_t(\varphi) &= \mu_t(\varphi)\,m_t \\
&= \mu_0(\varphi)\,m_0 + \int_0^t \mu_s(\varphi)\,dm_s + \int_0^t m_s d\mu_s(\varphi) \\
&= \mu_0(\varphi)\,m_0 + 2\int_0^t \mu_s(\varphi)\,\mu_s(h)\,m_s ds \\
&\quad + \int_0^t m_s\mu_s(A\varphi)\,ds + 2\int_0^t m_s(\mu_s(h\varphi) - \mu_s(h)\,\mu_s(\varphi))\,ds \\
&= \mu_0(\varphi) + \int_0^t \nu_s(A\varphi)\,ds + 2\int_0^t \nu_s(h\varphi)\,ds
\end{aligned}$$

**(d). [6 marks, seen similar].** We have that

$$\begin{aligned}
\nu_t(\varphi) &= \int \varphi(x)\tilde{\nu}_t(x)dx \\
&= \mu_0(\varphi) + \int_0^t \nu_s(A\varphi)\,ds + 2\int_0^t \nu_s(h\varphi)\,ds \\
&= \int_{\mathbb{R}} \varphi(x)\tilde{\nu}_s(x)dx + \int_0^t \int_{\mathbb{R}} (A\varphi(x) + 2h(x)\,\varphi(x))\,\tilde{\nu}_s(x)dxds
\end{aligned}$$

Let $A^* : \mathcal{C}_b^2(\mathbb{R}) \to \mathcal{C}_b(\mathbb{R})$ be the operator

$$A^*\varphi = -(f\varphi)' + \frac{1}{2}(\sigma^2\varphi)''.$$

Then, assuming that $\varphi \in \mathcal{C}_c^2(\mathbb{R})$ where the following holds

$$\int A\varphi\tilde{\nu}_s dx = \int \varphi A^*\tilde{\nu}_s dx \tag{1}$$

We can deduce that

$$\begin{aligned}\int \varphi(x)\tilde{\nu}_t(x)dx &= \int \varphi(x)\tilde{\nu}_0(x)dx + \int_0^t \int \varphi(x)A^*\tilde{\nu}_s(x)dxds \\ &= \int \varphi(x)\left[\tilde{\nu}_0(x) + \int_0^t \left(A^* + 2h\left(x\right)\right)\tilde{\nu}_s(x)ds\right]dx\end{aligned}$$

Denote by $g_t(x) = \tilde{\nu}_t(x) - \tilde{\nu}_0(x) - \displaystyle\int_0^t \left(A^* + 2h\left(x\right)\right)\tilde{\nu}_s(x)ds$, so that

$$\int \varphi(x)g_t(x)dx = 0.$$

This means that $g_t$ is orthogonal to any $\varphi \in \mathcal{C}_c^2(\mathbb{R})$. Since $\tilde{\nu}_t \in C_b^2\left(\mathbb{R}\right)$ it follows that $g_t \in C_b\left(\mathbb{R}\right)$. It follows that $g_t 1_{[-M,M]} \in L^2(\mathbb{R})$ and, since $\mathcal{C}_c^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it follows that $g_t 1_{[-M,M]} \perp g_t 1_{[-M,M]}$ for any $M > 0$. This implies that $g_t = 0$. We have deduced that

$$\frac{d\tilde{\nu}_t(x)}{dt} = \left(A^* + 2h\left(x\right)\right)\tilde{\nu}_t.$$

**Question 4. [20 marks]**

**(a). [3 marks, seen]** Let $X_t^1, \ldots, X_t^d$ be semi-martingales and $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ a function which is one time continuously differentiable with respect to $t$ and twice with respect to $x_i$, $i = 1, 2, \ldots, d$. Then

$$\begin{aligned} f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X_s)dX_s^i \\ + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)d[X^i, X^j]_s. \end{aligned}$$

**(b) [5 marks, seen similar]** Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sinh(x)$. We apply Itô's formula to the Brownian motion $W$ to get that

$$\begin{aligned} \sinh(W_t) &= \sinh(W_0) + \int_0^t \cosh(W_s)\, dW_s + \frac{1}{2}\int_0^t \sinh(W_s)\, d\langle W\rangle_s \\ &= \int_0^t \cosh(W_s)\, dW_s + \frac{1}{2}\int_0^t \sinh(W_s)\, ds \qquad (2) \end{aligned}$$

**(c). [3 marks, seen similar]**

Using the fact that $\cosh x = \sqrt{1 + (\sinh x)^2}$, we cam re-write (2) as

$$dU_t = \frac{1}{2}U dt + \sqrt{1 + (U_t)^2}dW_t. \qquad (3)$$

**(d). [3 marks, seen similar]** Both the drift and the diffusion coefficients in equation (3) are Lipschitz continuous functions (the drift is linear and the diffusion term has bounded derivative) therefore the equation has a unique solution in accordance with one of the theorems in the lectures.

**(e). [3 marks, not seen]** We have that

$$\mathbb{E}\left[U_t^2\right] = \frac{\mathbb{E}\left[e^{2W_t}\right] + \mathbb{E}\left[e^{-2W_t}\right] - 2}{2} = e^{2t} - 1.$$

**(f). [3 marks, not seen]** Observe that the hyperbolic sine function is a continuously increasing function. Therefore

$$\liminf_{t\to\infty} U_t = \sinh\left(\liminf_{t\to\infty} W_t\right) = -\infty.$$

as $\lim_{x\to-\infty} \sinh(x) = -\infty$.

## Mastery Question

**Question 5 [20 marks]**
**(a).[2 marks, seen]** The $p$-variation of the function $f$ over an arbitrary partition $\pi$ of the interval $[0,T]$ is defined to be

$$V_{T,\pi}^{(p)}(f) \triangleq \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p.$$

The total $p$-variation of $f$ over the interval $[0,T]$ is then defined as $V_T^{(p)}(f) \triangleq \lim_{\|\pi\|\to 0} V_{T,\pi}^{(p)}(f)$.
**(b).[3 marks, seen similar]** We know that if $H$ is an $\mathcal{F}_t$-previsible process such that

$$\mathbb{E}\left[\int_0^T H_t^2 dt\right] < \infty, \ \ t \geq 0.$$

then the corresponding stochastic integral $I(H) = \{I_t(H), t \geq 0\}$

$$I_t(X) = \int_0^t H_s^2 dV_s$$

is a (genuine) square integrable martingale. This is the case here as the process $\sigma(X) = \{\sigma(X_t), t \geq 0\}$ is a bounded hence square integrable and continuous, hence previsible, process.
**(c). [6 marks, not seen]** Note that the functions $s \to f(X_s)$ and $s \to |f(X_s)|$ are continuous therefore they are uniformly continuous. In other words for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|s_1 - s_2| < \delta$

$$|f(X_{s_1}) - f(X_{s_2})| < \varepsilon, \qquad ||f(X_{s_1})| - |f(X_{s_2})|| < \varepsilon.$$

Hence, if $\|\pi\| < \delta$

$$B_{t_i} - B_{t_{i-1}} = \int_{t_{i-}}^{t_i} f(X_s)\, ds = \int_{t_{i-}}^{t_i} f(X_s) - f(X_{t_i})\, ds + f(X_{t_i})(t_i - t_{i-1}).$$

Since $\left|\int_{t_{i-}}^{t_i} f(X_s) - f(X_{t_i})\, ds\right| \leq \varepsilon(t_i - t_{i-1})$ ,it follows that

$$\begin{aligned}\left|\sum_{i=1}^{n}\left(|B_{t_i} - B_{t_{i-1}}| - \left|f\left(X_{t_{i-1}}\right)\right|(t_i - t_{i-1})\right)\right| &\leq \sum_{i=1}^{n}\left|\left|B_{t_i} - B_{t_{i-1}}\right| - |f(X_s)|(t_i - t_{i-1})\right| \\ &\leq \varepsilon\sum_{i=1}^{n}(t_i - t_{i-1}) = \varepsilon T.\end{aligned}$$

Similarly, one shows that

$$\begin{aligned}\left|\int_0^T |f(X_s)|\,ds - \sum_{i=1}^n \left|f\left(X_{t_{i-1}}\right)\right|(t_i - t_{i-1})\right| &= \left|\sum_{i=1}^n \left(\int_{t_{i-1-}}^{t_i} |f(X_s)|\,ds - \left|f\left(X_{t_{i-1}}\right)\right|(t_i - t_{i-1})\right)\right| \\ &\leq \varepsilon T.\end{aligned}$$

It follows that

$$\left|\sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}| - \int_0^T |f(X_s)|\,ds\right| \leq 2\varepsilon T,$$

hence

$$V_T^{(1)}(B) = \lim_{\|\pi\|\to 0} \sum_{i=1}^n |B_{t_i} - B_{t_{i-1}}| = \int_0^T |f(X_s)|\,ds.$$

**(d). [6 marks, not seen]** Consider

$$\begin{aligned}V_{T,\pi}^{(2)}(X) &\triangleq \sum_{i=1}^n \left(X_{t_i} - X_{t_{i-1}}\right)^2 \\ &= \sum_{i=1}^n \left(B_{t_i} - B_{t_{i-1}}\right)^2 + 2\sum_{i=1}^n |\left(B_{t_i} - B_{t_{i-1}}\right)(M_{t_i} - M_{t_{i-1}}) + \sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2.\end{aligned}$$

We show that the first two terms in the above identity converge to 0. Observe that

$$\begin{aligned}\lim_{\|\pi\|\to 0} \sum_{i=1}^n \left(B_{t_i} - B_{t_{i-1}}\right)^2 &= \lim_{\|\pi\|\to 0} \left(\max_j \left|B_{t_j} - B_{t_{j-1}}\right|\right)\left(\sum_{i=1}^n \left|B_{t_i} - B_{t_{i-1}}\right|\right) \\ &= 0 \times V_T^{(1)}(B) = 0 \\ \lim_{\|\pi\|\to 0} \sum_{i=1}^n |\left(B_{t_i} - B_{t_{i-1}}\right)(M_{t_i} - M_{t_{i-1}}) &= \lim_{\|\pi\|\to 0} \left(\max_j \left|M_{t_j} - M_{t_{j-1}}\right|\right)\left(\sum_{i=1}^n \left|B_{t_i} - B_{t_{i-1}}\right|\right) \\ &= 0 \times V_T^{(1)}(B) = 0\end{aligned}$$

Moreover we know that

$$V_T^{(2)}(M) = \lim_{\|\pi\|\to 0} V_{T,\pi}^{(2)}(M) = \lim_{\|\pi\|\to 0} \sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2 < \infty.$$

(In fact we know that $V_T^{(2)}(M) = [M]_T$). From this we deduce that

$$V_T^{(2)}(X) = \lim_{\|\pi\|\to 0} V_{T,\pi}^{(2)}(X) = V_T^{(2)}(M) < \infty.$$

**(e). [3 marks, seen similar]** Observe that, since $X$ has continuous trajectories, we have that

$$\begin{aligned} V_T^{(3)}(X) &= \lim_{\|\pi\|\to 0} \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right|^3 \\ &= \lim_{\|\pi\|\to 0} \left( \max_j \left| X_{t_j} - X_{t_{j-1}} \right| \right) \left( \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right|^2 \right) = 0 \times V_T^{(2)}(X) \\ &= 0. \end{aligned}$$

**If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.**
**Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the  Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.**

| ExamModuleCode | QuestionNumber | Comments for Students |
|---|---|---|
| MATH97061MATH97172 | 1 | A question answered very well by the majority of the students.  The only if part was done completely by only a few although it was a straightfoward application of the uniqueness of the Doob-Meyer decomposition. |
| MATH97061MATH97172 | 2 | A question answered very well by the majority of the students.  There were one or two hiccups to identify the prior distribution as some students tried to use its evolution and not the explicit formula given in part (a). |
| MATH97061MATH97172 | 3 | A more challenging question. This wasn't (quite) the Zakai equation but very similar to it. Once you realized that that then the question becomes very simple as you can use the same arguments. Half the answers where along this lines. |
| MATH97061MATH97172 | 4 | The students found this question easy. Most answers were very good. |
| MATH97061MATH97172 | 5 | A true mastery question. A few excellent answers. Surprisingly most students could not do part (c) which was essentially a first year real analysis question. |