IMPERIAL

MATH70014

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Biology

Date: Thursday, May 16, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

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1. (a) A biologist friend discovers a strain of bacteria whose population dynamics obeys the modified logistic equation,

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right),$$

where x is the population size and r, K, and $0 < K_0 < K$ are positive constants.

(i) Determine the fixed points and their stability. (4 marks)

(ii) Provide some interpretation of the parameter K_0. (3 marks)

(b) Your friend conducts an experiment where they add bacteria at a constant rate $H(> 0)$ to the growing population such that

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right) + H$$

(i) Sketch the bifurcation diagram showing how the fixed points and their stability vary with H, using solid lines to indicate stable branches and dashed lines to indicate unstable branches. (3 marks)

(ii) Determine, x_c, the value of the fixed point at the bifurcation, for the case where $r = 1$, $K = 2$ and $K_0 = 1$. (4 marks)

(c) Your friend also discovers that the bacteria are preyed upon by a species of nematodes (round worms) such that

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right) - y,$$
$$\frac{dy}{dt} = -y + \alpha x,$$

where y is the nematode population and α is a positive constant.

(i) Using sketches of the nullclines, show that depending on the value of α, there are either 3 fixed points, 2 fixed points, or 1 fixed point. (3 marks)

(ii) Describe what occurs to the populations at long times in the cases of 3 fixed points and 1 fixed point. (3 marks)

(Total: 20 marks)

2. (a) Consider the following system of chemical reactions

$$A \xrightarrow{1} U$$
$$2U + V \xrightarrow{1} 3U$$
$$B + U \xrightarrow{1} V$$
$$U \xrightarrow{1} 0$$

(i) Using the law of mass action, determine the system of differential equations governing the evolution of u and v, the concentrations of molecules U and V, respectively. Thus, a and b, the concentration of A and B, respectively, can be regarded as parameters in the system. (4 marks)

(ii) Determine the fixed point of the system and show that it becomes unstable when $b > 1 + a^2$. (4 marks)

(b) Consider the modified nondimensional model for glycolysis,

$$\frac{d\sigma_1}{d\tau} = \nu - \sigma_1(1+\sigma_2)^2,$$
$$\frac{d\sigma_2}{d\tau} = \eta\left(\sigma_1(1+\sigma_2)^2 - \sigma_2\right),$$

where ν and η are positive constants and where σ_1, σ_2 and τ are the dimensionless concentrations and nondimensional time, respectively.

(i) Determine the nullclines. Sketch the nullclines for a case where the system undergoes oscillations. (2 marks)

(ii) Add to your sketch the trajectory corresponding to oscillations in the limit $\eta \gg 1$. (3 marks)

(iii) Determine an expression for the approximate period of the oscillations when $\eta \gg 1$. You may leave your expression in the form of an integral, but determine the upper limit of integration. (7 marks)

(Total: 20 marks)

3. Consider the general Fisher-Kolmogorov equation,

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}$$

with $f(1) = f(0) = 0$, $f'(1) < 0$, $f'(0) > 0$, and $f(u) > 0$ for $0 < u < 1$. The boundary conditions are $u \to 1$ as $x \to -\infty$ and $u \to 0$ as $x \to \infty$. Show the following:

(a) For a travelling wave solution of the form $u(x,t) = U(\xi)$ with $\xi = x - ct$, where c is the wave speed, show that $c > 0$. (5 marks)

(b) If there is a value of c^* for which there is no travelling wave solution with $u \geq 0$, then there is no such travelling wave solution for any $c < c^*$. (5 marks)

(c) If there is a value of c^* for which there is a travelling wave solution with $u \geq 0$, then there is such a travelling wave solution for any $c > c^*$. (5 marks)

(d) Let μ be the smallest positive number for which $f(u) \leq \mu u$ for $0 \leq u \leq 1$. Show that a travelling wave solution exists for all $c \geq 2\sqrt{\mu}$. (5 marks)

(Total: 20 marks)

4. Let $X(t)$ be the random variable for the total population size of a birth-death-emmigration process and $p_i(t) = \mathrm{Prob}\{X(t) = i\}$. The infinitesimal transition probabilities are,

$$p_{i+j,i}(\Delta t) = \mathrm{Prob}\{X(t+\Delta t) = i + j | X(t) = i\}$$
$$= \begin{cases} (\mu i + \nu)\Delta t + o(\Delta t), & j = -1 \\ \lambda i^2 \Delta t + o(\Delta t), & j = 1 \\ 1 - (\lambda i^2 + \mu i + \nu)\Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1, 0, 1. \end{cases}$$

where λ, μ, and ν are positive constants. Take the initial population to be $X(0) = N$.

(a) Based on the infinitesimal transition probabilities, what is the corresponding deterministic model for the population size? (3 marks)

(b) From the transition probabilities, derive the forward Kolmogorov equation for $p_i(t)$. (4 marks)

(c) Take $\lambda = 0$ for all i, and $\nu = 0$ if $i = 0$ (i.e. there is no emmigration if the population size is zero). Show that the probability generating function, $\mathcal{P}(z,t) = \sum_{i=0}^{N} p_i(t) z^i$, satisfies

$$\frac{\partial \mathcal{P}}{\partial t} = .\mu(1 - z)\frac{\partial \mathcal{P}}{\partial z} + \nu(z^{-1} - 1)\mathcal{P}. \tag{1}$$

(9 marks)

(d) From the equation for the generating function, find the equation for the mean of the process. Recall that $m(t) = \partial \mathcal{P}/\partial z|_{z=1}$. (4 marks)

(Total: 20 marks)

5. Consider the general system of N coupled oscillators of the form

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1, j\neq i}^{N} a_{ij} \sin(\theta_j - \theta_i) \tag{2}$$

(a) What do θ_i, ω_i and a_{ij} each represent? (3 marks)

(b) Consider the case $N = 3$. Write the system of equations in terms of $\phi_1 = \theta_1 - \theta_2$, $\phi_2 = \theta_2 - \theta_3$ and $\phi_3 = \theta_3 - \theta_1$. (4 marks)

(c) Take $N = 3$, $\omega_i = \omega > 0$ and $a_{ij} = a > 0$ for all i, j. Find and discuss the steady phase differences given by the model. Consider only phase difference values that lie in the interval $[0, 2\pi)$. (5 marks)

(d) Suppose now $\omega_1 = \omega_2 = \omega$ and $\omega_3 = 0$ and, as in part (c), $a_{ij} = a > 0$ for all i, j. Find the steady phase difference with the condition that $\sum_{i=1}^{3} \sin\phi_i = 0$. Explain why this additional condition is necessary. Provide the conditions on ω and a that ensure the solution exists. (8 marks)

(Total: 20 marks)

	EXAMINATION SOLUTIONS 2023-24	Course
Question 1		Marks & seen/unseen
Parts (a) (i)	FIXED POINTS: $f(x) = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right)$ FIXED POINTS ($f(x)=0$) AT $x=0$, $x=K$, AND $x=K_0$. WE SEE THAT FOR $0<x<K_0$, $f(x)<0$. THUS, $x=0$ STABLE, $x=K_0$ UNSTABLE, $x=K$ STABLE.	UNSEEN 4 A
(ii)	K_0 IS A THRESHOLD POPULATION SIZE ABOVE WHICH THE POPULATION CAN SURVIVE. THE BACTERIA NEED OTHER BACTERIA TO SURVIVE AND REPRODUCE.	UNSEEN 3B
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	EXAMINATION SOLUTIONS 2023-24	Course
Question		Marks & seen/unseen

Parts

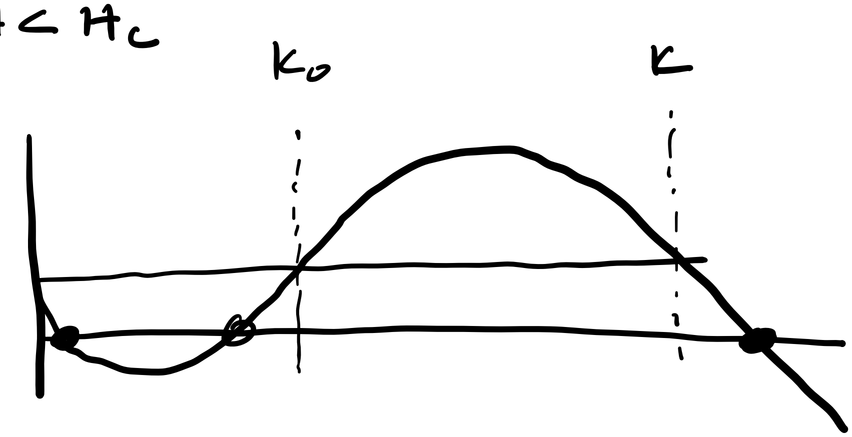
(b) (i)

UNSEEN

FIXED POINTS NOW GIVEN BY $f(x) = -H$

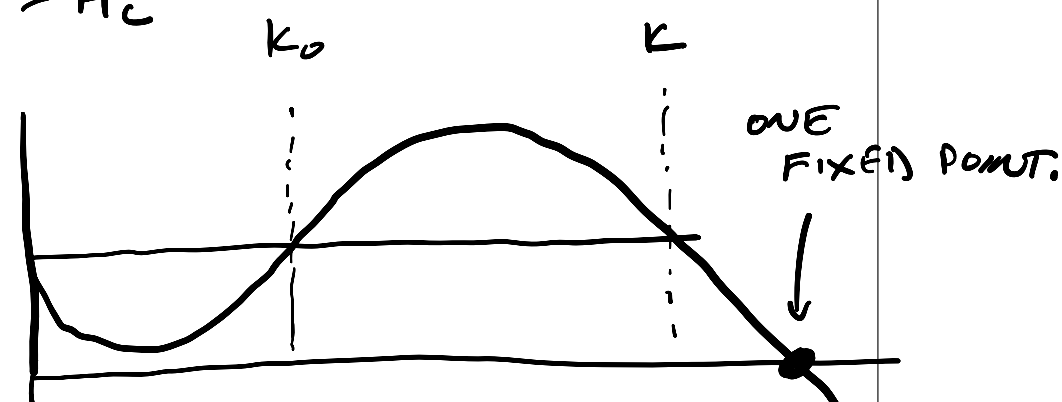
THERE IS A CRITICAL VALUE H_c SUCH THAT

$H < H_c$



THE ARE THREE FIXED POINTS.

$H > H_c$



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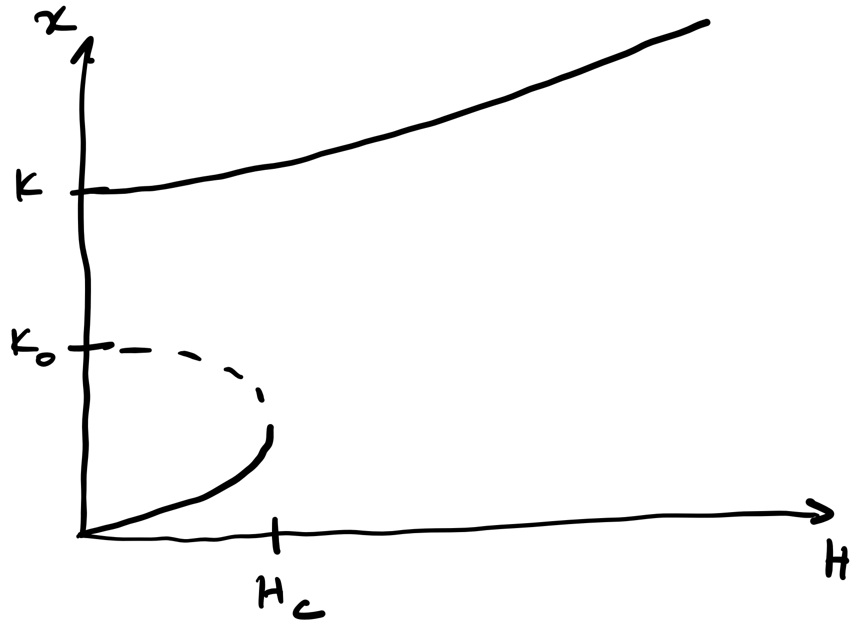
Parts

BIFURCATION DIAGRAM



3A

UNSEEN

(ii) AT THE BIFURCATION,

$$f'(x_c) = 0.$$

$$f(x) = r\left[-x + x^2\left(\frac{1}{K} + \frac{1}{K_0}\right) - x^3/KK_0\right]$$

$$f'(x) = r\left[-1 + 2x\left(\frac{1}{K} + \frac{1}{K_0}\right) - 3x^2/KK_0\right]$$

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Question		Marks & seen/unseen

Parts

USING $r=1$, $K=2$, $K_0=1$,

WE HAVE

$$f'(x) = -1 + 3x - \frac{3}{2}x^2 = 0$$

WHICH HAS SOLUTIONS

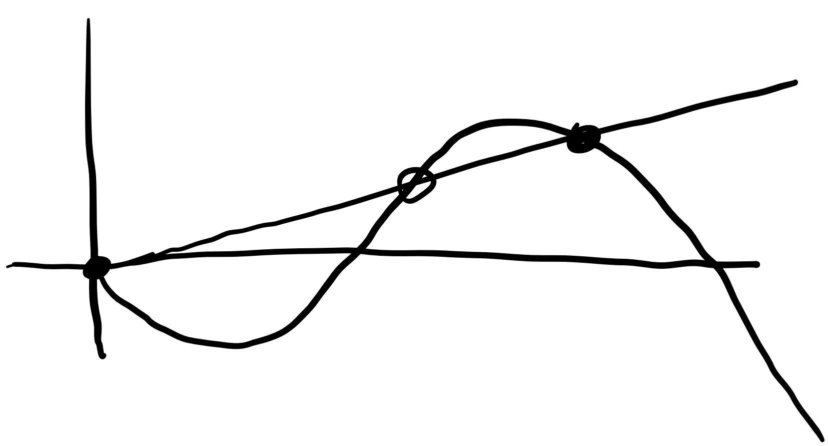
$$x = 1 \pm \frac{\sqrt{3}}{3}.$$

THUS, $x_c = 1 - \frac{\sqrt{3}}{3}$. 4B

(c)(i) THE NULLCLINES ARE UNSEEN

$$y = f(x) \text{ AND } y = \alpha x$$

3 FIXED POINTS:



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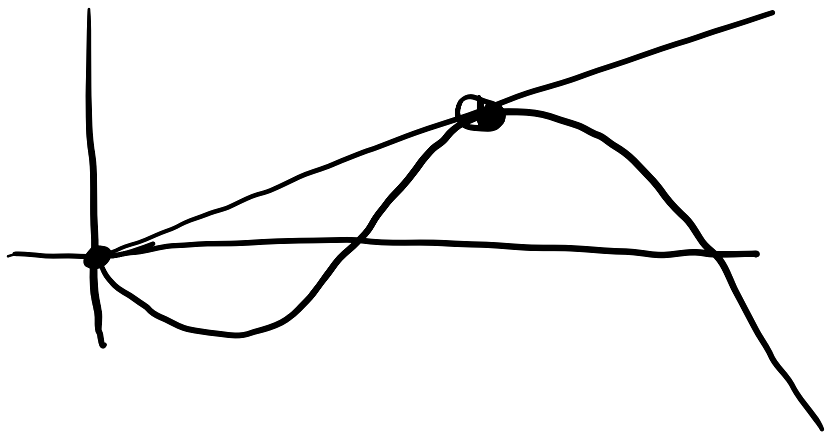
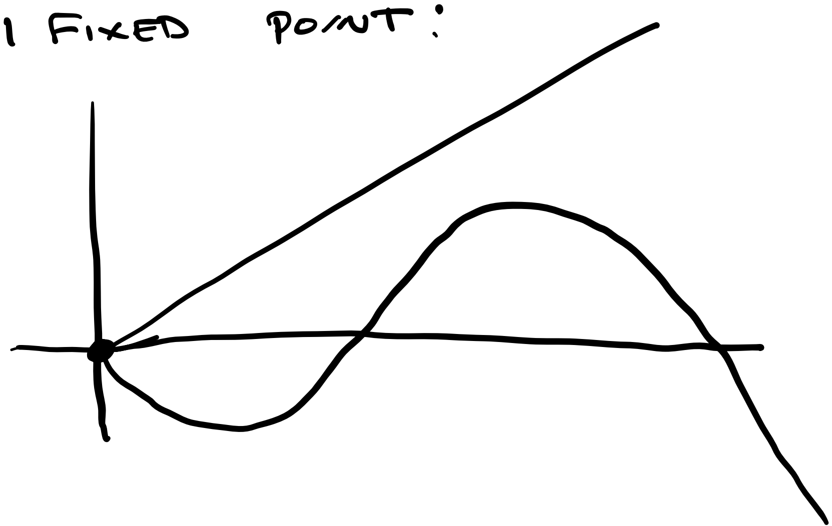
Question		Marks & seen/unseen

Parts

2 FIXED POINTS:



1 FIXED POINT:



3A

(ii) WHEN THERE ARE 3 FIXED POINTS, THERE IS BISTABILITY BETWEEN EXISTENCE AND EXTINCTION. EXISTENCE REQUIRES A SUFFICIENTLY LARGE BACTERIA POPULATION SIZE.

UNSEEN

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WITH ONE FIXED POINT THERE IS ONLY EXTINCTION. HERE, THE WORM POPULATION GROWS TOO RAPIDLY AND CANNOT BE SUSTAINED.

3C

Question		Marks & seen/unseen
2		

Parts

(i)

$$\frac{du}{dt} = a + u^2 v - (1+b)u$$

$$\frac{dv}{dt} = bu - u^2 v$$

UNSEEN

4A

SEEN

(ii) FIXED POINT

$$0 = a + u^2 v - (1+b)u \quad (1)$$

$$0 = bu - u^2 v \quad (2)$$

FROM (2), WE HAVE

$$v = b/u$$

USING THIS IN (1) GIVES

$$u = a.$$

THUS, $v = b/a.$

STABILITY:

$$J(u,v) = \begin{bmatrix} 2uv - b - 1 & u^2 \\ -2uv + b & -u^2 \end{bmatrix}$$

$$J\left(a,\frac{b}{a}\right) = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}$$

$$\text{TRACE}(J) = b - 1 - a^2$$

$$\text{DET}(J) = -a^2(b-1) + ba^2$$

$$= a^2 > 0$$

THUS, UNSTABLE IF

$$\text{TRACE}(J) > 0,$$

$$b > 1 + a^2$$

4A

(b)(i) SETTING THE RIGHTHAND SIDES EQUAL TO ZERO GIVES

SEEN

$$0 = \nu - \sigma_1(1+\sigma_2)^2$$

$$0 = \eta\left(\sigma_1(1+\sigma_2)^2 - \sigma_2\right)$$

EXAMINATION SOLUTIONS 2023-24

Course

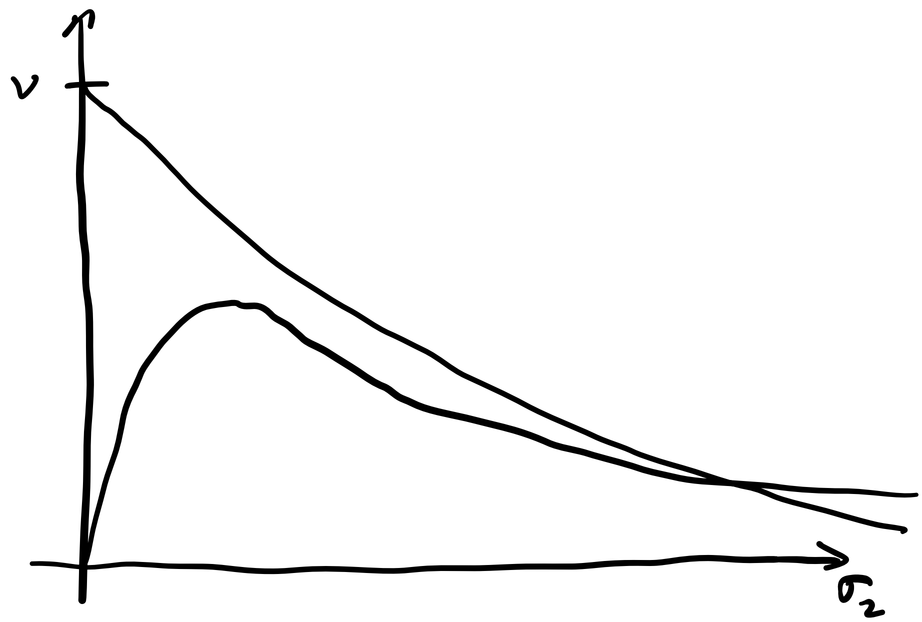
Question

Marks & seen/unseen

Parts

THUS, THE NULLCLINES ARE

$$\sigma_1 = \frac{\nu}{(1+\sigma_2)^2}, \quad \sigma_1 = \frac{\sigma_2}{(1+\sigma_2)^2}$$



* NOTICE HOW THE NULLCLINES INTERSECT AFTER THE MAXIMUM. THIS ENSURES OSCILLATIONS. 2A

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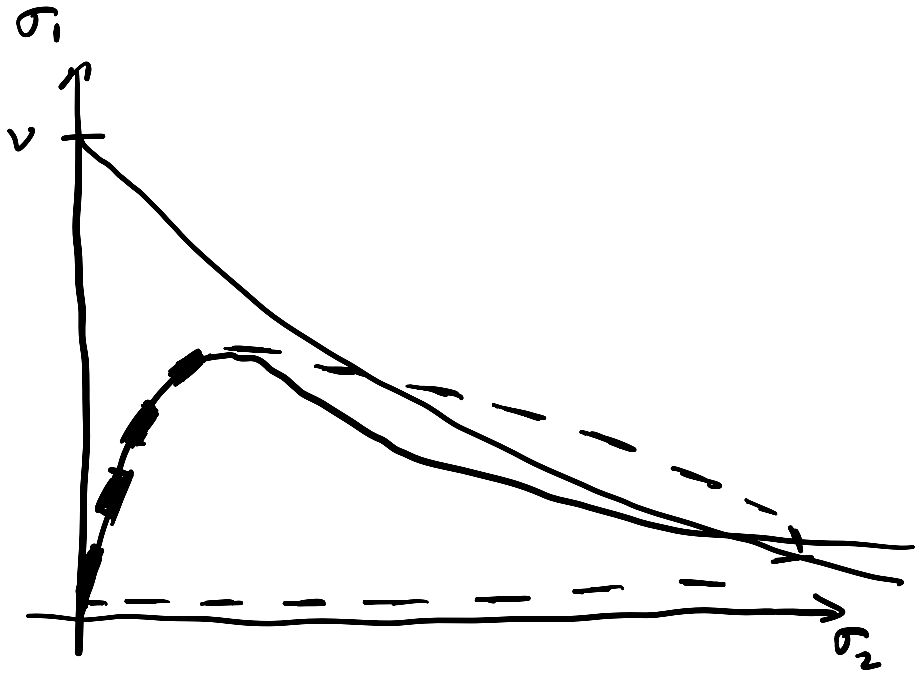
Parts (ii)	SAMPLE TRAJECTORY. (DASHED LINE)	UNSEEN



IN THIS LIMIT, THE LIMIT CYCLE FINDS ITSELF ON THE NULLCLINE BETWEEN THE ORIGIN AND THE MAXIMUM. 3B

IT IS THIS PORTION OF THE TRAJECTORY THAT WE CAN USE TO APPROXIMATE THE PERIOD.

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Parts

(iii) THUS,

$$\frac{d\sigma_1}{dt} = \nu - \sigma_1 (1+\sigma_2)^2$$

UNSEEN

INTEGRATING GIVES

$$\int_0^T dt = \int_a^b \frac{d\sigma_1}{\nu - \sigma_1 (1+\sigma_2)^2}$$

USING THE EQUATION FOR THE NULLCLINE.

$$\sigma_1 = \frac{\sigma_2}{(1+\sigma_2)^2}$$

$$\frac{d\sigma_1}{d\sigma_2} = \frac{1-\sigma_2}{(1+\sigma_2)^3}$$

WHICH GIVES:

$$T = \int_c^d \frac{1-\sigma_2}{(1+\sigma_2)^3(\nu-\sigma_2)}\, d\sigma_2$$

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Question		Marks & seen/unseen
Parts	THE UPPER LIMIT OF INTEGRATION CORRESPONDS TO THE MAXIMUM OF THE NULLCLINE. THUS, $$0 = \frac{1-\sigma_2}{(1+\sigma_2)^3} \Rightarrow \sigma_2 = 1$$ AND $\boxed{d=1}$.	7D
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Question 3 | Marks & seen/unseen

Parts

(a) SEEN

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -c\,U', \quad \frac{\partial^2 u}{\partial x^2} = U''$$

WE HAVE,

$$-c\,U' = f(U) + U''$$

MULTIPLYING BY U'

GIVES.

$$-c(U')^2 = f(U)U' + U''U'$$

IF WE NOW INTEGRATE

OVER ALL SPACE,

$$-c\int_{-\infty}^{\infty}(U')^2\,d\xi = \int_{-\infty}^{\infty} f(U)U'\,d\xi + \int_{-\infty}^{\infty} U''U'\,d\xi$$

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Question		Marks & seen/unseen
Parts		

$$\int_{-\infty}^{\infty} u'' u' \, d\xi$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\xi} \frac{(u')^2}{2} \, d\xi$$

$$= \left. \frac{(u')^2}{2} \right|_{-\infty}^{\infty} = 0 .$$

$$\int_{-\infty}^{\infty} f(u) \, u' \, d\xi = \int_{1}^{0} f(u) \, du$$

$$= -\int_{0}^{1} f(u) \, du .$$

THUS

$$c = \frac{\int_0^1 f(u) \, du}{\int_{-\infty}^{\infty} (u')^2 \, d\xi} > 0$$

SINCE $f(u) > 0$ FOR $0 < u < 1$.

5A

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	EXAMINATION SOLUTIONS 2023-24	Course
Question		Marks & seen/unseen
Parts (b)	LET $v = u'$ SO WE HAVE	UNSEEN

$$u' = v$$
$$v' = -cv - f(u)$$

THE FIXED POINTS ARE $(0,0)$ AND $(1,0)$.

$$J(u,v) = \begin{bmatrix} 0 & 1 \\ -f'(u) & -c \end{bmatrix}$$

THE EIGENVALUES ARE

$$\lambda_\pm = \frac{c \pm \sqrt{c^2 - 4f'(u)}}{2}$$

WE SEE THAT SINCE $f'(0) > 0$, WE REQUIRE

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	EXAMINATION SOLUTIONS 2023-24	Course
Question		Marks & seen/unseen
Parts	THAT $c^2 - 4f'(0) \geq 0$ FOR $u \geq 0$. THUS, IF WE HAVE $(c^*)^2 - 4f'(0) < 0$, THEN $c - 4f'(0) < 0$ IF $c^* > c$ AND WE CANNOT HAVE $u \geq 0$.	5B
(c)	SIMILARLY, IF $(c^*)^2 - 4f'(0) \geq 0$ THEN $c^2 - 4f'(0) \geq 0$ IF $c > c^*$ AND WE CAN HAVE $u \geq 0$.	UNSEEN 5B
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Parts

(d) FOR $f(u) \leq \mu u$, THIS MEANS THAT $f'(0) \leq \mu$. UNSEEN

THUS $c^2 - 4\mu \geq 0$

PROVIDES THE CONDITION FOR $u \geq 0$ AND THEREFORE

$$c \geq 2\sqrt{\mu}.$$

5C

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Question 4		Marks & seen/unseen

Parts

(a) THE DETERMINISTIC MODEL IS — UNSEEN

$$\frac{dn}{dt} = -\nu - \mu n + \lambda n^2$$

3A

(b) FROM THE INFINITESIMAL TRANSITION PROBABILITIES WE SEE THAT THE NON ZERO ENTRIES OF THE GENERATOR MATRIX ARE — UNSEEN

$$q_{ii} = -\lambda i^2 - \mu i - \nu$$

$$q_{i-1,i} = \mu i + \nu$$

$$q_{i+1,i} = \lambda i^2$$

BASED ON THIS THE FORWARD KOLMOGOROV EQN IS

$$\frac{dp_i}{dt} = [-\lambda i^2 - \mu i - \nu] p_i + \lambda (i-1)^2 p_{i-1} + [\mu (i+1) + \nu] p_{i+1}$$

4A

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Parts		
(c)	IF WE MULTIPLY THE FKE ($\lambda=0$) BY z^i AND SUM OVER i WE HAVE	UNSEEN

$$\frac{\partial}{\partial t}\sum_{i=0}^{N} p_i z^i = -\mu \sum_{i=0}^{N} i\, p_i z^i - \nu \sum_{i=0}^{N} p_i z^i + \mu \sum_{i=0}^{N} (i+1)\, p_{i+1} z^i + \nu \sum_{i=0}^{N} p_{i+1} z^i$$

FOR EACH SUM WE HAVE

$$\sum_{i=0}^{N} i\, p_i z^i = z \sum_{i=0}^{N} i\, p_i z^{i-1} = z \frac{\partial P}{\partial z}$$

$$\sum_{i=0}^{N} p_i z^i = P$$

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Question		Marks & seen/unseen
Parts	$$\sum_{i=0}^{N} (i+1) p_{i+1} z^{i} = \frac{\partial P}{\partial z}$$ $$\sum_{i=0}^{N} p_{i+1} z^{i} = z^{-1} \sum_{i=0}^{N} p_{i+1} z^{i+1}$$ SINCE $v = 0$ FOR $i = 0$, WE CAN WRITE $$v z^{-1} \sum_{i=0}^{N} p_{i+1} z^{i+1} = v z^{-1} \sum_{i=0}^{N} p_i z^i$$ $$= v z^{-1} P.$$ THUS THE EQUATION FOR P IS $$\frac{\partial P}{\partial t} = \mu (1 - z) \frac{\partial P}{\partial z} + v (z^{-1} - 1) P. \quad (3)$$	9 D
(d)	WE FIRST DIFFERENTIATE (3) WITH RESPECT TO z:	UNSEEN
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Question

Marks & seen/unseen

Parts

$$\frac{\partial}{\partial t}\left(\frac{\partial P}{\partial z}\right) = -\mu \frac{\partial P}{\partial z} + \mu(1-z)\frac{\partial^2 P}{\partial z^2} + \nu(z^{-1}-1)\frac{\partial P}{\partial z} - \nu z^{-2} P.$$

WE THEN EVALUATE THE RESULTING EXPRESSION AT $z=1$:

$$\frac{\partial}{\partial t}\left(\left.\frac{\partial P}{\partial z}\right|_{z=1}\right) = -\mu \left.\frac{\partial P}{\partial z}\right|_{z=1} + \mu \cancel{(1-z)\Big|_{z=1}}^{\,0} \left.\frac{\partial^2 P}{\partial z^2}\right|_{z=1} + \nu \cancel{(z^{-1}-1)\Big|_{z=1}}^{\,0} \left.\frac{\partial P}{\partial z}\right|_{z=1} - \nu \left. z^{-2}\right|_{z=1} \left. P\right|_{z=1}$$

THIS GIVES

$$\frac{dm}{dt} = -\mu m - \nu.$$

4 C

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	EXAMINATION SOLUTIONS 2023-24	Course
Question 5		Marks & seen/unseen

Parts

(a) θ_i IS THE PHASE OF OSCILLATOR i. ω_i IS THE FREQUENCY OF OSCILLATOR i.
a_{ij} IS THE COUPLING STRENGTH THAT QUANTIFIES HOW STRONGLY OSCILLATOR j ENTRAINS OSCILLATOR i. 3

(b) SUBTRACTING THE EQUATIONS GIVES

$$\frac{d\varphi_1}{dt} = \Omega_1 - (a_{12} + a_{21}) \sin\varphi_1 + a_{13} \sin\varphi_3 + a_{23} \sin\varphi_2$$

$$\frac{d\varphi_2}{dt} = \Omega_2 - (a_{23} + a_{32}) \sin\varphi_2 + a_{21} \sin\varphi_1 + a_{31} \sin\varphi_3$$

$$\frac{d\varphi_3}{dt} = \Omega_3 - (a_{13} + a_{31}) \sin\varphi_2 + a_{21} \sin\varphi_1 + a_{32} \sin\varphi_2$$

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Question

Marks & seen/unseen

Parts

WHERE $\Omega_1 = \omega_1 - \omega_2$, $\Omega_2 = \omega_2 - \omega_3$ AND $\Omega_3 = \omega_3 - \omega_1$.

4

(c) FOR THE GIVEN PARAMETER VALUES, THE SYSTEM OF EQUATIONS BECOMES

$$\frac{d}{dt}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix} = a\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}\sin\varphi_1\\ \sin\varphi_2\\ \sin\varphi_3\end{bmatrix}$$

THE STEADY SOLUTIONS ARE GIVEN BY

$$\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}\sin\varphi_1\\ \sin\varphi_2\\ \sin\varphi_3\end{bmatrix} = 0$$

WHICH IS SATISFIED WHEN

$$\begin{bmatrix}\sin\varphi_1\\ \sin\varphi_2\\ \sin\varphi_3\end{bmatrix} = \alpha\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

WITH $\alpha \in [-1, 1]$. THUS, WE HAVE $\varphi_i = \sin^{-1}\alpha$ for $i = 1, 2, 3$.

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WE SEE, THEREFORE THAT ANY CONSTANT PHASE DIFFERENCE IS A STEADY STATE. 5

(d) IN THIS CASE, WE HAVE THE SYSTEM

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \sin\varphi_1 \\ \sin\varphi_2 \\ \sin\varphi_3 \end{bmatrix} = \frac{\omega}{a} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

THE SOLUTION IS GIVEN BY

$$\begin{bmatrix} \sin\varphi_1 \\ \sin\varphi_2 \\ \sin\varphi_3 \end{bmatrix} = \frac{\omega}{3a} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

WHERE β IS AN ARBITRARY CONSTANT. THE CONDITION $\sum_{i=1}^{3} \sin\varphi_i = 0$ IS NEEDED TO SET β. APPLYING THIS CONDITION GIVES $\beta = 0$. THUS $\varphi_1 = 0$, $\varphi_2 = \sin^{-1}(-\omega/3a)$ AND

	EXAMINATION SOLUTIONS 2023-24	Course
Question		Marks & seen/unseen
Parts	$\varphi_3 = \sin^{-1}(w/3a)$. WE SEE THAT FOR A SOLUTION TO EXIST, WE MUST HAVE $w/3a \in [0, 1]$.	8
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	EXAMINATION SOLUTIONS 2023-24	Course
Question		Marks & seen/unseen
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Question		Marks & seen/unseen
Parts		
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MATH60014 Mathematical Biology

Question	Marker's comment
1	In general, the students handled this question very well with most receiving most marks.nbsp; If students did lose marks, it was mostly due to an incorrect sketch of the bifurcation diagram.
2	Students performed well on the first half of this question, but many lost marks on the second half when asked to sketch the nullclines, provide a sample trajectory in the relaxation oscillator limit, and, finally, establish an expression for an approximation of the period.nbsp; Deriving the expression for the period proved particularly challenging.
3	This question proved to be th most challenging for students, even though the analysis in the first part is identical to that from a problem sheet question.nbsp; I allowed marks for students that used dynamical systems arguments, though the model solution workded directly with the PDE.nbsp; Some students recovered with the remaining parts, but many did not.
4	Overall, the students did very well on this question with many receiving full marks.

MATH70014 Mathematical Biology

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4	Overall, the students did very well on this question with many receiving full marks.
5	Overall, the students handled the mastery very well.nbsp; The middle part where the phase difference was not unique (the matrix was not invertible) proved to be where most students lost marks.nbsp; Many students were able to recover and complete the remaining parts.