MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 2

Exercise 6.

For the verification that the operator norms and the max norm define norms, do verify that all three conditions of Definition 2.4 hold. To show in (ii) that the two norms are equivalent, do not aim for optimal constants K_1 and K_2 (which are difficult to get), and note that simple estimates can be done with $K_1 = 1$ and $K_2 = m\sqrt{n}$.

Exercise 7.

Make use of the mean value theorem as explained in Lecture 6 for the one-dimensional functions in (i)–(iii). For the higher-dimensional functions in (iv) and (v), the mean value inequality needs to be used, but note that the mean value inequality was formulated in terms of the operator norm. The operator norm is difficult to compute for most matrices, but to prove Lipschitz continuity, it is not necessary to find an optimal Lipschitz constant (i.e. a smallest one). By using Exercise 6 (ii), you can find bounds for the operator norm in terms of the max norm, and this is good enough here (and simplifies the task).

Exercise 8.

The intuitive idea is to glue the local solutions together for different initial value problems, and then the result seems clear, since the constant $h > 0$ in Theorem 2.11 does not depend on the pair of initial conditions (t_0, x_0), so the solution can be extended on $\mathbb{R}$ (note that some comments on this exercise along these lines were made in Lecture 8). One way to make this rigorous is to define for a given initial pair (t_0, x_0) the quantity

$$T := \inf \big\{ \tilde{T} > t_0 + h : \text{all solutions satisfying the initial condition } x(t_0) = x_0 \\ \text{do not exist at time } \tilde{T} \big\} .$$

By assuming $T < \infty$, one can extend the solution beyond T using the above intuitive idea.

Exercise 9.

For (i), all three conditions in Definition 2.4 need to be verified. (ii) is more complicated, and a hint was already given on the problem sheet. Firstly, it is necessary to verify that the given sequence of functions is a Cauchy sequence with respect to $\|\cdot\|_1$, and note that it is not a Cauchy sequence using the supremum norm $\|\cdot\|_\infty$ (why?). The pointwise limit of this sequence is obviously discontinuous, and in order to show that no continuous limit function exists (now in $\|\cdot\|_1$), you can come to a contradiction if you assume there is a continuous limit function u_∞, and then showing that $\|u_n - u_\infty\|_1 \not\to 0$ as $n \to \infty$.

Exercise 10.

Consider the sequence of functions $u_n : J \to \mathbb{R}^d$, given by $u_n(t) := f(t, \lambda_n(t))$, and show first that this sequence converges uniformly. Here you need the fact that continuous functions on compact sets are uniformly continuous, and restrict the domain of the function f appropriately to use this fact.