Example 4.1.5. *Question:* Find the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

Proposition 4.1.6. Let V and W be vector spaces over F. Let $\{v_1, ..., v_n\}$ be a basis for V. Let $w_1, ..., w_n$ be any n vectors from W (these don't need to be distinct). Then there is a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all i.

Remark 4.1.7. This shows that once we know what a linear transformation does to a basis we know what the transformation is.

Example 4.1.8. Let V be the space of all polynomials in x over $\mathbb{R}$ with degree less than or equal to 2. A basis for this is $\{1, x, x^2\}$. We can pick any three arbitary vectors in V for example:

$$
\begin{aligned}
w_1 &= 1 + x \\
w_2 &= x - x^2 \\
w_3 &= 1 + x^2
\end{aligned}
$$

By Proposition 4.1.6 there is a linear transformation $T : V \to V$ such that $T(1) = w_1$, $T(x) = w_2$, $T(x^2) = w_3$.

We can work out what T does to a general element of V. A general element is of the form $v = a1 + bx + cx^2$, so

$$
\begin{aligned}
T(v) &= \\
&= \\
&=
\end{aligned}
$$

4.2 Image and Kernel

Definition 4.2.1. Let $T : V \to W$ be a linear transformation:

- The *Image of* T is the set $Im\, T = \{T(v) \in W : v \in V\} \subseteq W$.
- The *Kernel of* T is the set $Ker\, T = \{v \in V : T(v) = 0_W\} \subseteq V$.

Example 4.2.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix}$$

- The image of T is the set of all vectors in $\mathbb{R}^2$ of the form $\begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix}$ for $x_1, x_2, x_3 \in \mathbb{R}$. This is the space:

- The kernel of T is the set of vectors in $\mathbb{R}^3$ such that $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_W$ that is so say such that:

Proposition 4.2.3. Let $T : V \to W$ be a linear transformation. Then:

1. $Im\, T$ is a subspace of W.
2. $Ker\, T$ is a subspace of V.

Note: In general we write $U \leq V$ to mean U is a subspace of V, so with this notation we are saying $Im\, T \leq W$ and $Ker\, T \leq V$.

Example 4.2.4. Let V_n be the vector space of polynomials in x over $\mathbb{R}$ of degree $\leq n$. We have $V_0 \leq V_1 \leq V_2 \ldots$. Define:

$$T : V_n \to V_{n-1},$$
$$T(f(x)) = f'(x).$$

Note: T is linear.

$$\begin{aligned} Ker\,T &= \\ &= \\ &= \end{aligned}$$

Suppose $g(x)$ has degree $\leq n-1$. Then by integrating $g(x)$ we can find $f(x)$ such that $f'(x) = g(x)$ and $deg(f(x)) = 1 + deg(g(x))$, so $deg(f(x)) \leq n$. Hence $ImT =$.

Of course the $f(x)$ such that $f'(x) = g(x)$ is not unique - if c is a constant then $f(x) + c$ also has this property. In fact we get the set $\{h(x) : h'(x) = g(x)\}$ consists of polynomials $f(x) + k(x)$ where $k(x) \in Ker\,T$.

Proposition 4.2.5. Let $T : V \to W$ be a linear tranformation and let $v_1, v_2 \in V$. Then

$$T(v_1) = T(v_2) \text{ iff } v_1 - v_2 \in Ker\, T.$$

Proposition 4.2.6. Let $T : V \to W$ be a linear transformation. Suppose that $\{v_1, ..., v_n\}$ is a basis for V. Then $Im\, T = Span\{T(v_1), .., T(v_n)\}$.

Proposition 4.2.7. Let A be an $m \times n$ matrix. Let $T : F^n \to F^m$ be given by $T(v) = Av$. Then:

1. $Ker\, T$ is the solution space to $Av = 0$.
2. $Im\, T$ is the column space of A.
3. $dim(Im\, T) = rankA$.

Theorem 4.2.8. ***The rank nulity theorem:*** *We've seen that when $Tv = Av$, $rank(A) = dim(Im\,T)$. An old fashioned name for $dim(Ker\,T)$ is the nulity of A*

Let $T : V \to W$ be a linear transformation. Then

$$dim(Im\,T) + dim(Ker\,T) = dim(V)$$

Example 4.2.9.

Let $a, b, c \in \mathbb{R}$, define $U = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$. U is a subspace of $\mathbb{R}^3$.

We can find dimension of U by defining:

$$T : \mathbb{R}^3 \to \mathbb{R}$$
$$T(x, y, z) = (a, b, c)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Now $U = \ker T$, and clearly $ImT = \mathbb{R}$ (as not all $a, b, c = 0$), thus $dim(ImT) = 1$. So

$$\begin{aligned} dimU &= dim(\ker T) \\ &= \\ &= \end{aligned}$$

Corollary 4.2.10. A system of linear equations in n unknowns with co-efficients in F:

$$\begin{array}{llll} a_{11}x_1 & +a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

is called *homogeneous* if $b_1 = b_2 = ... = b_m = 0$.

We know in this case that we will always get at least a trivial solution to the system - and we saw in the test that the set of solutions forms a subspace of F^n, but what dimension will this subspace have?

We can use the rank-nulity theorem to work this out:

We know that if we let $A = (a_{ij})$, then this system of linear equations can be represented as $Ax = 0$. We also know that A can be seen as a linear transformation $A : F^n \mapsto F^m$.

By Proposition 4.2.7 the set of solutions in this case is $ker(A)$, and by the rank nulity we get

$dim(ker(A)) = dim(F^n) - dim(Im(A))$

Now the $dim(Im(A)) = rank(A)$ thus the we can work out how many solutions we have to a set of homogeneous equations with n unknowns:

- If $rank(A) \geq n$ we get one solution (the trivial one i.e. 0_V)
- If $rank(A) < n$ we get infinitely many solutions (assuming F is infinite)

Exercise 4.2.11. In this case the rank of the augmented matrix $(\mathsf{A}|0)$ is the same as that of A.

How does this work for a non homogeneous system of linear equations?

Essentially almost the same except - but we are taking a coset of the system of equations and we have to account for the case were $rank(A) < rank(A|b)$