DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2023–2024

Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Final Exam

1. **Total: 20 Marks**

(a) Let

$$X_{(a,b,r)} = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r^2\}$$

and

$$I = \{(a,b,r) \in \mathbb{R}^3 : 0 \leq a \leq 2, 0 \leq b \leq 2, r = 2\}.$$

i. For $a, b \in \mathbb{R}$ show that $X_{(a,b,r)} \subseteq X_{(a,b,s)}$ if and only if $r \leq s$. 4 Marks

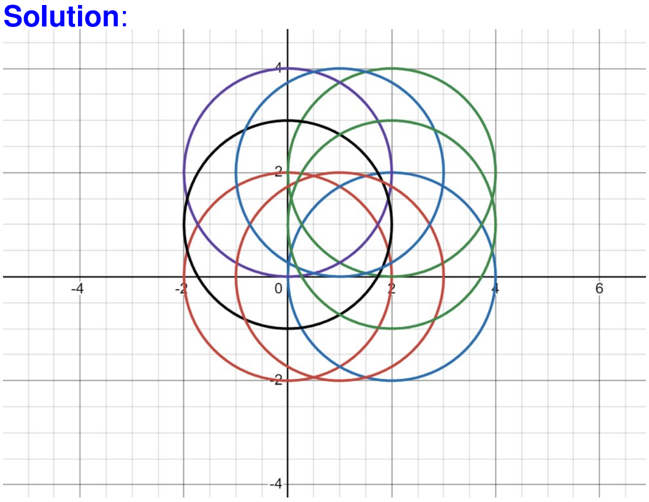
Solution:
$(\Rightarrow)$: Prove the contrapositive. Assume that $r > s$ then show that $X_{(a,b,r)}$ is not a subset of $X_{(a,b,s)}$. To show this it is sufficient to find an element $(x,y) \in X_{(a,b,r)} \setminus X_{(a,b,s)}$. Consider $(a+t, b)$: check:

$$((a+t)-a)^2 + (b-b)^2 = r^2 > s^2$$

So $(a+t, b) \in X_{(a,b,r)} \setminus X_{(a,b,s)}$.

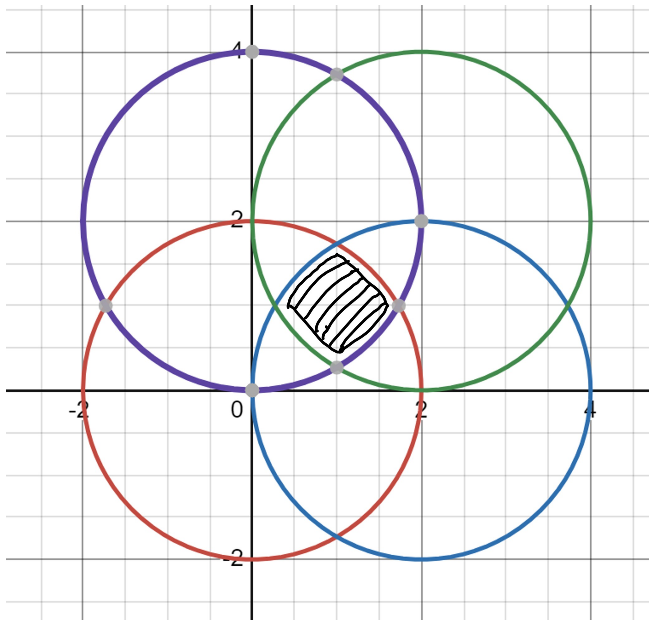
$(\Leftarrow)$: Assume $r \leq s$, suppose $(x,y) \in X_{(a,b,r)}$ then $(x-a)^2 + (y-b)^2 \leq r^2 \leq s^2$ thus $(x,y) \in X_{(a,b,s)}$.

ii. Describe $\bigcup_{i \in I} X_i$ as a union of a finite number of subsets of $\mathbb{R}^2$. 2 Marks
Solution:



$\bigcup_{i \in I} X_i = X_{(0,0,2)} \cup X_{(0,2,2)} \cup X_{(2,0,2)} \cup X_{(2,2,2)} \cup Y_1 \cup Y_2$ where
$Y_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, -2 \leq y \leq 4\}$
$Y_2 = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 4, 0 \leq y \leq 2\}$

iii. Describe $\bigcap_{i \in I} X_i$ as an intersection of a finite number of subsets of $\mathbb{R}^2$. 2 Marks
Solution:



$\bigcap_{i\in I} X_i = X_{(0,0,2)} \cap X_{(0,2,2)} \cap X_{(2,0,2)} \cap X_{(2,2,2)} \cap Y_1 \cup Y_2$ where

iv. A family of sets $\{X_j\}_{j\in J}$ is called *pairwise disjoint* if any two sets in the family, X_i and X_j, their intersection is empty, i.e. $X_i \cap X_j = \emptyset$.

Find an infinite set $J \subseteq \mathbb{R}^3$ such that $\{X_j\}_{j\in J}$ is pairwise disjoint. 3 Marks

Solution:

There are lots of solutions to this, but the basic idea is to make sure the centres of the circles in the family are further apart than twice the radii of the circles. So for example:

$$J = \mathbb{N} \times \mathbb{N} \times \{\frac{1}{4}\}$$

(b) Let X, Y be sets, $f : X \to Y$ be a function define a relation, R_f on X by:

$$R_f(x_1, x_2) \text{ if and only if } f(x_1) = f(x_2)$$

i. Prove carefully that R_f is an equivalence relation (you may not assume the result from lectures). 6 Marks

Solution:

- ☐ Reflexive: Let $x \in X$ as f is a (well-defined) function we have $f(x) = f(x)$ and thus $R_f(x, x)$. So R_f is reflexive.
- ☐ Symmetric: Let $x_1, x_2 \in X$ and suppose $R_f(x_1, x_2)$ then $f(x_1) = f(x_2)$ so $f(x_2) = f(x_1)$ so by definition $R_f(x_2, x_1)$. So R_f is symmetric.
- ☐ Transitive: Let $x_1, x_2, x_3 \in X$ and suppose $R_f(x_1, x_2)$ and $R_f(x_2, x_3)$ then $f(x_1) = f(x_2)$ and $f(x_2) = f(x_3)$so $f(x_1) = f(x_3)$ so by definition $R_f(x_1, x_3)$. So R_f is transitive.

ii. Suppose in addition that $|X| = 3$ and $|Y| = 2$ (i.e. X contains 3 element and Y contains 2 elements). For every equivalence relation E on X is there a corresponding f such that $E = R_f$?

Recall that two binary relations R and S on X are equal if $\forall x_1, x_2 \in X$, $R(x_1, x_2) \Leftrightarrow S(x_1, x_2)$. 3 Marks **Solution**:

As Y has only two elements it is not possible to define an f such that R_f has 3 equivalence classes. In particular let E be the equivalence class given by equality.

Now as $|X| = 3$ and $|Y| = 2$ we must have that f sends at least two distinct element $x_1, x_2 \in X$ to the same element $y \in Y$, in which case $R_f(x_1, x_2)$ with $x_1 \neq x_2$ thus $R_f \neq E$.

2. $\boxed{\textbf{Total: 20 Marks}}$

(a) For $x, y \in \mathbb{N}$, we define $x < y$ to mean $x \leq y$ and $x \neq y$. For this question you can use everything you know about the natural numbers.

i. Show that every nonempty subset of $\mathbb{N}$ has a unique least element. $\boxed{\text{2 Marks}}$

Solution:
Let S be a nonempty subset of $\mathbb{N}$. Since it is nonempty, by the well-ordering principle, S has a least element. We just need to show that it is unique. Assume it is not. This means that we have two distinct least elements x_1 and x_2 such that $x_1 \leq y$ and $x_2 \leq y$ for all y in $\mathbb{N}$. But then $x_1 \leq x_2$ (since y is arbitrary) and similarly $x_2 \leq x_1$. By antisymmetry, $x_1 = x_2$, which is a contradiction.

ii. Show that $x < y$ if and only if $S(x) \leq y$, where $S : \mathbb{N} \to \mathbb{N}$ is the successor function. $\boxed{\text{2 Marks}}$

Solution:
Suppose that $x < y$. This means that $y = x + b$ for some $b \in \mathbb{N}$ and $x \neq y$. This means that $b \neq 0$, since otherwise $y = x + 0 = x$. Now $b = S(a)$ for some a. Then $y = x + S(a) = S(x + a) = S(a + x) = a + S(x) = S(x) + a$. This means that $S(x) \leq y$.

(b) Let z be an integer and $+_{\mathbb{Z}}$ the addition we defined on the integers. Prove the following results using only the definition of $\mathbb{Z}$ and the addition $+_{\mathbb{Z}}$. You can use everything you know about the addition on $\mathbb{N}$.

i. $(z, 0) +_{\mathbb{Z}} (0, z) = 0$ $\boxed{\text{3 Marks}}$

Solution:
By definition of the integers

$$a - b := cl((a, b)) = \{(x, y) \in \mathbb{N} \times \mathbb{N} | a + y = x + b\}$$

Now by definition of addition on the integers $(z, 0) +_{\mathbb{Z}} (0, z) = (z, z)$. But clearly $z + 0 = 0 + z$, by commutativity of addition, hence (z, z) is in the same equivalence class as $(0, 0)$, which proves the result.

ii. $-z$ is the unique integer such that $z +_{\mathbb{Z}} (-z) = 0$ $\boxed{\text{3 Marks}}$

Solution:
Assume that (a, b) is another inverse of addition. Then $(z, 0) +_{\mathbb{Z}} (a, b) = (z + a, b)$ is in the equivalence class of $(0, 0)$. So $z + a + 0 = b + 0$, and hence $z + a = b$. We then have that $a + z = 0 + b$, so that the equivalence class of (a, b) is the same as that of $(0, z)$, as desired.

(c) Let $P(z) = a_0 + a_1 z + \ldots a_n z^n$ be a polynomial of degree $n \geq 1$ over $\mathbb{Z}_p$ with p a prime number. Show that P has at most n roots modulo p. $\boxed{\text{4 Marks}}$

Solution:
We do this by induction on the degree of P. If $n = 1$, then $P(z) = a_0 + a_1 z$, then $z = -\frac{a_0}{a_1}$ is the unique root. Assume now that any polynomial of degree n has at most n roots modulo P. Let Q be a polynomial of degree n+1. If Q has no roots we are done. Assume it has at least one root z_0. Now $Q(z) \equiv 0 \mod p$ means that $Q(z) = (z - z_0)P(z) \equiv 0 \mod p$, with P a polynomial of degree n. In other words $p | (z - z_0)P(z)$. Since p is prime either $p | (z - z_0)$ or $p | P(z)$. In the first case we have the root z_0, in the second case we have $P(z) \equiv 0 \mod p$. But by the induction hypothesis, P has at most n roots, hence Q has at most n+1 roots.

(d) Let $\mathbb{F}$ be a field. Prove the following statements. Use only the axioms of a field and the results proved in lecture following from those.

i. Prove that $-0 = 0$. $\boxed{\text{3 Marks}}$
Solution:
We need to check that $0 + 0 = 0$ but this is true by the additive identity.

ii. Prove that $-a = (-1) \times a$. $\boxed{\text{3 Marks}}$
Solution:
we just have to check that $(-1) \times a + a = 0$. But $a = 1 \times a$ (multiplicative identity) and so $(-1) \times a + a = (-1) \times a + 1 \times a = (-1 + 1) \times a$ (by distributivity) which equals $0 \times a$ by additive inverse, which is 0.

3. **Total: 20 Marks**

(a) Consider the minimal field $\mathbb{F}_2$ with elements $0, 1$, and multiplication defined in the usual way, and an "addition" $\oplus$: $\mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$ defined by the rules

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1 = 1 \oplus 0, \quad 1 \oplus 1 = 0.$$

Consider the vector space $V = \mathbb{F}_2^2$ over this field, with vector addition and scalar multiplication defined element-wise as

$$\mathbf{u} \oplus \mathbf{v} := (u_1 \oplus v_1, u_2 \oplus v_2) \quad \text{and} \quad \alpha\mathbf{u} := (\alpha u_1, \alpha u_2), \quad \forall \mathbf{u}, \mathbf{v} \in V, \alpha \in \mathbb{F}_2.$$

i. List all vectors in V. What is the dimension of V?

The four elements of V are $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. $\boxed{\text{1 Mark}}$
The dimension of the vector space V is two. $\boxed{\text{1 Mark}}$

ii. Is the map $V \times V \to \mathbb{F}$ defined as $\langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 \oplus u_2 v_2$ an inner product on V? Give a reason for your answer.

There are various reasons why this vector space doesn't have any inner product (there isn't even an order on the field $\mathbb{F}_2$, so positivity doesn't make sense) answers along this direction should get full marks. An alternative answer would show that $\langle (1, 1), (1, 1) \rangle = 1 \oplus 1 = 0$ even though $(1, 1) \neq (0, 0)$ and thus the map is not an inner product. $\boxed{\text{3 Marks}}$

(b) Consider the three vectors $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 3, 1)$ and $\mathbf{w} = (3, 1, 2)$ in $\mathbb{R}^3$.

i. Calculate the volume of the parallelepiped spanned by the three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

This can be calculated as

$$\text{Volume} = |\det(\mathbf{u}, \mathbf{v}, \mathbf{w})| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Either of these expressions would do. $\boxed{\text{2 Marks}}$
With the particular values we find

$$\left| \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \right| = |18 - 36| = 18.$$

$\boxed{\text{1 Mark}}$

ii. Are the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ linearly independent? Provide a reasoning for your answer.

They are linearly independent, which can be seen from the fact that the volume in part (i) is non-zero. $\boxed{\text{2 Marks}}$

iii. Find parametric equations for the line $\mathcal{L}_1$ going through the points U (with position vector $\mathbf{u}$) and V (with position vector $\mathbf{v}$), and the line $\mathcal{L}_2$ going through the points U and W (with position vector $\mathbf{w}$).

The line $\mathcal{L}_1$ is going through the vector $\mathbf{u}$ and directed by the vector $\mathbf{u}-\mathbf{v}$, i.e. it can be parameterised as

$$\mathcal{L}_1: \ \mathbf{r} = \mathbf{u} + \lambda(\mathbf{u}-\mathbf{v}), \quad \lambda \in \mathbb{R}.$$

And similarly

$$\mathcal{L}_2: \ \mathbf{r} = \mathbf{u} + \lambda(\mathbf{u}-\mathbf{w}), \quad \lambda \in \mathbb{R}.$$

Or any other correct parameterisation. $\boxed{\text{2 Marks}}$
With the actual numbers inserted we have

$$\mathcal{L}_1: \ \mathbf{r} = (1,2,3) + \lambda(-1,-1,2), \quad \lambda \in \mathbb{R}.$$

and

$$\mathcal{L}_2: \ \mathbf{r} = (1,2,3) + \lambda(-2,1,1), \quad \lambda \in \mathbb{R}.$$

or again any other correct parameterisation $\boxed{\text{2 Marks}}$

iv. Find a vector that is orthogonal to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

$\mathcal{L}_1$ is parallel to the vector $\mathbf{u}-\mathbf{v} = (-1,-1,2)$ and $\mathcal{L}_2$ is parallel to the vector $\mathbf{u}-\mathbf{w} = (-2,1,1)$. Thus, a vector $\mathbf{a}$ that is orthogonal to both lines can be found as

$$\mathbf{a} = (\mathbf{u}-\mathbf{v}) \times (\mathbf{u}-\mathbf{w}) = -(3,3,3).$$

Two marks for the idea, one for the numbers. $\boxed{\text{3 Marks}}$

(c) Consider the vector space $\mathbb{R}^5$, with the orthonormal basis $\{\hat{\mathbf{e}}_j\}$. Is the map $T: \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}^5$ defined as

$$T(\mathbf{u}) = \mathbf{u} + 5\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_4, \quad \forall\, \mathbf{u} \in \mathbb{R}^5$$

a linear map? Is it invertible? Provide reasonings for your answers.

The map is not linear. There are many ways to see this, one is to notice that $T(\mathbf{0}) \neq 0$ or $T(\alpha\mathbf{u}) \neq \alpha\mathbf{u}$ for $\alpha \neq 1$. $\boxed{\text{2 Marks}}$
The map is invertible. The inverse map R is given by $R(\mathbf{u}) := \mathbf{u} - 5\hat{\mathbf{e}}_1 - 3\hat{\mathbf{e}}_4$. $\boxed{\text{1 Mark}}$