Problem Sheet 3

1). Using the two identities

$$\Gamma(1+z) = z\Gamma(z),$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

express $|\Gamma(iy)|$ (for $y \in \mathbb{R}$) in terms of the hyperbolic function sinh. You may use the fact that $\overline{\Gamma(z)} = \Gamma(\overline{z})$ for all z.

2). Consider the integral $I = \int_0^1 \log(\Gamma(x))dx$.

(a). Show that $I = \frac{1}{2}(\log \pi - J)$, where $J = \int_0^1 \log(\sin(\pi x))dx$. (Hint: Consider I also written in terms of the variable $u = 1 - x$).

(b). Show that $J = -\log 2$ and hence that $I = \log \sqrt{2\pi}$. (Hint: Consider the variable $v = \frac{\pi}{2}x$).

3). By using techniques involving the Gamma and Beta functions, find the exact value of

$$\int_0^4 x^2\sqrt{16 - x^2}dx.$$

4). Verify the following identities. Do this using the series representation for $F(a, b; c; z)$ and comparing it with the Taylor series expansions (about $z = 0$) of the left-hand sides.

(a). $(1 - z)^{-a} = F(a, b; b, z)$,

(b). $\log(1 - z) = -zF(1, 1; 2; z)$.