Problem Sheet 3

1. A region V is enclosed by a surface S. ϕ and $\mathbf{A}$ are scalar and vector fields with ϕ vanishing on S. Show, by applying the divergence theorem to $\phi\mathbf{A}$, that

$$\int_V \phi \operatorname{div} \mathbf{A} \, dV = -\int_V \mathbf{A} \cdot \nabla\phi \, dV.$$

Deduce that if $\mathbf{A}$ is solenoidal throughout V then

$$\int_V \mathbf{A} \cdot \nabla\phi \, dV = 0.$$

Show that in two dimensions the corresponding result is

$$\int_R \phi \operatorname{div} \mathbf{A} \, dx \, dy = -\int_R \mathbf{A} \cdot \nabla\phi \, dx \, dy,$$

where ϕ vanishes on the closed curve C which bounds the region R.

2. Evaluate

$$\int_S \mathbf{r} \cdot \widehat{\mathbf{n}} \, dS$$

where S is any closed surface enclosing a volume V, and $\mathbf{r}$ is the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

3. Show that

$$\int_S \frac{\mathbf{r} \cdot \widehat{\mathbf{n}}}{r^2} \, dS = \int_V \frac{dV}{r^2},$$

where S is a closed surface enclosing the volume V, and $\mathbf{r}$ is as above, with $r = |\mathbf{r}|$.

4. Use the divergence theorem to prove the following results, where S is a closed surface with unit outward normal $\widehat{\mathbf{n}}$ enclosing the volume τ, $\phi(x, y, z)$ is a scalar and $\mathbf{A}(x, y, z)$ a vector function of position.

$$\text{(i)} \int_S \widehat{\mathbf{n}} \, \phi \, dS = \int_\tau \nabla\phi \, d\tau. \quad \text{(ii)} \int_S \widehat{\mathbf{n}} \times \mathbf{A} \, dS = \int_\tau \operatorname{curl} \mathbf{A} \, d\tau.$$

5. Verify the divergence theorem for the case when

$$\mathbf{A} = x \, \mathbf{i}$$

and V is the cube $|x| \leq a$, $|y| \leq a$, $|z| \leq a$.

6. Let S be the piecewise smooth closed surface consisting of the surface of the cone $z = (x^2 + y^2)^{1/2}$ for $x^2 + y^2 \leq 1$, together with the flat cap consisting of the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$. Verify the divergence theorem

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}} \, dS = \int_V \operatorname{div} \mathbf{A} \, dV$$

for this surface, when $\mathbf{A} = (x + y)\mathbf{i} + (y - x - z)\mathbf{j} + (z - y)\mathbf{k}$. You may assume $dx \, dy = r \, dr \, d\theta$ in plane polar coordinates.

7. By converting into an appropriate line integral, use Stokes theorem to evaluate

$$\int_S (\nabla \times \mathbf{A}) \cdot \widehat{\mathbf{n}} \, dS$$

where $\mathbf{A} = (y - z, z - x, x - y)$. Here S is the upper half of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ with $z \geq 0$, and $\widehat{\mathbf{n}}$ is the unit normal to S with $\widehat{\mathbf{n}} \cdot \mathbf{k} > 0$.

8. Verify Stokes theorem for the vector field

$$\mathbf{A} = (3x - y, -yz^2/2, -y^2z/2)$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = a^2$, so that the closed curve C is a circle in the $x - y$ plane.

Hint: to evaluate the surface integral use spherical polar coordinates $x = a\sin\theta\cos\phi, y = a\sin\theta\sin\phi, z = a\cos\theta$, with $dS = a^2\sin\theta\, d\theta\, d\phi$, and $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$.

9. Let S consist of the part of the cone $z = (x^2 + y^2)^{1/2}$ for $x^2 + y^2 \le 9$ and suppose

$$\mathbf{A} = (-y, x, -xyz).$$

Verify that Stokes theorem is satisfied for this choice of $\mathbf{A}$ and S.

10. Verify Stokes theorem for the same field $\mathbf{A}$, but with S now being the section of the cone $z = (x^2 + y^2)^{1/2}$ with $4 \le x^2 + y^2 \le 9$.

Sheet 3 Answers

2. $3V$.
5. LHS = RHS = $8a^3$.
6. LHS = RHS = π.
7. $-2\pi ab$.
8. LHS = RHS = πa^2.
9. LHS = RHS = 18π.
10. LHS = RHS = 10π.