MATH50001 Problems Sheet 4

Solutions

1)

$$\oint \frac{z^2}{(z-1)^n}\,dz = \frac{2\pi i}{(n-1)!}\,\frac{d^{n-1}}{dz^{n-1}}\,z^2 = \begin{cases} 2\pi i, & n=1, \\ 4\pi i, & n=2, \\ 2\pi i, & n=3, \\ 0, & n>3. \end{cases}\quad .$$

2 a) This is the ellipse with two focuses at 2 and -2.

2 b)

$$\oint_\gamma \frac{\sin z}{(z+2)^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\,\sin z\Big|_{z=-2} = -\pi i \sin(-2) = \pi i \sin 2.$$

3) Let p be a polynomial. Then, by Cauchy's formula

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{1-zp(z)}{z}\,dz = \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{z}\,dz = 1.$$

Therefore by using the ML-inequality we obtain

$$1 = \left|\frac{1}{2\pi i}\oint_{|z|=1} \frac{1-zp(z)}{z}\,dz\right| \le \max_{|z|=1}|1-zp(z)| = \max_{|z|=1}|z^{-1}-p(z)|.$$

4) Indeed, for any $z_0, z_1 \in \mathbb{C}$, $z_0 \neq z_1$ and R sufficiently large, we have

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{|z|=R} \frac{f(z)}{(z-z_0)(z-z_1)}\,dz \\ &\quad = \frac{1}{z_0-z_1}\left(\frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_0)}\,dz - \frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_1)}\,dz\right) \\ &\quad = \frac{1}{z_0-z_1}\left(f(z_0)-f(z_1)\right). \end{aligned}$$

Since f is bounded, there is a constant M, such that $|f(z)| \leq M$, $z \in \mathbb{C}$. Therefore using the ML-inequality we find

$$\begin{aligned}\Big|\frac{1}{2\pi i}\oint_{|z|=R}\frac{f(z)}{(z-z_0)(z-z_1)}\,dz\Big| &\leq M\,R\max_{z:|z|=R}\frac{1}{|(z-z_0)(z-z_1)|}\\ &\leq M\,R\,\frac{1}{(R-|z_0|)(R-|z_1|)}\\ &= M\,R^{-1}\,\frac{1}{(1-|z_0|/R)(1-|z_1|/R)}\to 0,\\ &\qquad\qquad \text{as}\quad R\to\infty\end{aligned}$$

This implies

$$\frac{1}{z_1-z_0}\,(f(z_0)-f(z_1))=0$$

and thus $f(z_0)=f(z_1)$. Since z_0 and z_1 are arbitrary, we finally obtain that f is a constant function.

5) Note that if $n=0$, then we simply apply Liouville's theorem.

Assume that $|f(z)| \leq C(1+|z|)^n$ with some $C>0$. Then for any $z_0\in\mathbb{C}$ we have

$$\begin{aligned}|f^{(n+1)}(z_0)| &= \left|\frac{(n+1)!}{2i\pi}\oint_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{n+2}}\,dz\right|\\ &\leq \frac{C(n+1)!}{2\pi}\max_{z:|z-z_0|=R}(1+|z|)^n\,\frac{2\pi R}{R^{n+2}}\to 0,\\ &\qquad\qquad \text{as}\quad R\to\infty.\end{aligned}$$

Therefore $f^{(n+1)}\equiv 0$ and thus $f^{(n)}$ is a constant function. We conclude that $f(z)$ is a polynomial of degree at most n by integrating $f^{(n)}(z)$ n-times.

6) Assume that $f=u+iv$ is an entire function that has a bounded real part. Then $g(z)=e^{f(z)}$ is also entire. Note that sinve u is bounded then $|g|=e^u$ is bounded. Thus g is constant. and therefore f is a constant function.

7)

a) converges,

b) converges,

$$\Big|\frac{3-(2i)^n}{\cos ni}\Big| = 2\,\Big|\frac{3-(2i)^n}{e^{-n}+e^n}\Big| = 2\,\frac{2^n}{e^n}\,\frac{|3/2^n-i^n|}{1+e^{-2n}}$$

Clearly $|3/2^n-i^n|\leq 3/2^n+1\leq 5/2$ and $1+e^{-2n}>1$. Therefore

$$\Big|\frac{3-(2i)^n}{\cos ni}\Big|\leq 5\,\frac{2^n}{e^n}.$$

Since $2 < e$ we conclude that series converges.

c) diverges, indeed:

$$\left|\frac{ni}{n+i}\right|^{n^2} = \left(\frac{n}{\sqrt{n^2+1}}\right)^{n^2} = \left(\frac{1}{\sqrt{1+1/n^2}}\right)^{n^2} = \frac{1}{(1+1/n^2)^{n^2/2}} \to e^{-1/2} \neq 0.$$

Because it is known that

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e.$$

8) $\operatorname{Re} z \leq 0$.

9) a) $|z| < 1$; b) $|z-4| < 2^{-1/4}$; c) $|z-2| < 1$.

10)

a) $\sum_{n=2}^{\infty} \frac{z^n}{2^{n-1}}, \quad |z| < 2$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} (z-i)^n, \quad |z-i| < \sqrt{2}.$

11)

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$, $|z| < \infty$.

b) $\sum_{n=0}^{\infty} \frac{e^{1+i}}{n!} (z-1-i)^n$, $|z-1-i| < \infty$.

c) $\frac{\pi i}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n i^n} (z-i)^n$, $|z-i| < 1$.

12)
Clearly
$$a_k = \frac{1}{k!} f^{(k)}(0).$$
a) Let $1 > \varepsilon > 0$. By using the generalized Couchy's formula we find
$$f^{(k)}(0) = \frac{k!}{2i\pi} \oint_{|z|=1-\varepsilon} \frac{f(z)}{z^{k+1}}\, dz.$$
Therefore by using the ML inequality and the fact that $|f(z)| < 1$ in $\mathbb{D}$ we have
$$|a_k| \leq \frac{1}{k!} \left| \frac{k!}{2i\pi} \oint_{|z|=1-\varepsilon} \frac{f(z)}{z^{k+1}}\, dz \right| \leq \frac{1}{(1-\varepsilon)^k}.$$
Letting $\varepsilon \to 0$ we obtain $|a_k| \leq 1$.

b) We now use the generalized Cauchy's formula integrating over a circle $C_r = \{z \in \mathbb{C} : |z| = r\}$.
$$f^{(k)}(0) = \frac{k!}{2i\pi} \oint_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz.$$
By applying the ML inequality we find by using the inequality $|f(z)| < (1-|z|)^{-1}$
$$|a_k| \leq \frac{r}{r^{k+1}(1-r)}$$
Note that
$$\frac{d}{dr} r^k(1-r) = kr^{k-1} - (k+1)r^k = 0$$
implies $r = k/(k+1)$ and finally we obtain
$$|a_k| \leq \frac{(k+1)^{k+1}}{k^k}.$$