Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2013

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Applied Probability

Date: Wednesday, 29 May 2013. Time: 10.00am. Time allowed: 2 hours.
This paper has FOUR questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the main book is full.
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (a) Let $X = \{X_n\}_{n\in\mathbb{N}_0}$ denote a discrete–time stochastic process taking values in a state space $E \subseteq \mathbb{Z}$. State the Markov condition which ensures that X is a Markov chain.

(b) Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a homogeneous Markov chain on a space $E \subseteq \mathbb{Z}$ with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$. Show that for any $x_0, x_1, \dots, x_N \in E$, $N \in \mathbb{N}$, we have

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots, X_N = x_N) = \mathbb{P}(X_0 = x_0) p_{x_0x_1} p_{x_1x_2} \cdots p_{x_{N-1}x_N}.$$

(c) Consider a homogeneous Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also, let $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = 2) = \mathbb{P}(X_0 = 3) = \mathbb{P}(X_0 = 4) = 1/4$.

(i) Find $\mathbb{P}(X_0 = 1, X_1 = 2, X_3 = 3, X_4 = 2)$.

(ii) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent.

(iii) Is the Markov chain irreducible?

(iv) Find a stationary distribution for the Markov chain.

(v) Decide whether or not the stationary distribution is unique. If the stationary distribution is not unique, find all possible stationary distributions.

Please note that you need to justify your answers in (ii)-(v).

2. Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a Markov chain on a state-space $E \subseteq Z$.

(a) Define what a recurrent and a transient state is.

(b) Let $i, j \in E$. Under which condition is j accessible from i? When do i and j communicate?

(c) Suppose $i, j \in E$ and i and j communicate. Prove that i is recurrent if and only if j is recurrent.

(d) Suppose that the state space E is finite. Show that at least one state is recurrent.

(e) You always have two books on your bookshelf. The books are either probability books or statistics books. Each day, a book on your bookshelf is randomly chosen and replaced by a new book. The probability of replacing a book by a book of the same topic is 0.7. The probability of replacing a book by a book of the opposite topic is 0.3. Initially, you have two probability books. Suppose you can model the number of probability books on your bookshelf by a Markov chain. More precisely, for $n \in \mathbb{N}$ let X_n denote the number of probability books after the nth selection and subsequent replacement on the nth day. Also, X_0 denotes the number of probability books, which are initially on your bookshelf. Specify the dynamics of the corresponding Markov chain.

3. (a) Suppose Z is a continuous random variable with values in $[0, \infty)$.

   (i) State the lack of memory property.

   (ii) Suppose Z satisfies the lack of memory property. State (without proof) which distribution Z has.

(b) Let $N = \{N_t\}_{t \geq 0}$ be a homogeneous Poisson process of rate $\lambda > 0$. Let $0 < s < t$.

   (i) For $k \in \mathbb{N}_0$, show that

$$\mathbb{P}(N_s = k | N_t) = \begin{cases} \binom{N_t}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{N_t - k}, & \text{if } k \leq N_t, \\ 0, & \text{if } k > N_t. \end{cases}$$

   (ii) Derive $\mathbb{E}(N_s | N_t)$.

(c) Let $N = \{N_t\}_{t \geq 0}$ be a homogeneous Poisson process of rate $\lambda > 0$. Let T denote an exponentially distributed random variable with mean $1/\alpha$ for $\alpha > 0$, which is independent of N. Find the probability mass function of N_T.

4. (a) Define the generator of a continuous-time Markov chain.

(b) Suppose the generator of a continuous-time Markov chain with state space $E = \{1, 2, 3, 4\}$ is given by

$$\mathbf{G} = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & -5 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

What are the transition probabilities of the corresponding jump-chain?

(c) Consider a birth-death process with birth rates $\lambda_n = \lambda(n+1)$ and death rates $\mu_n = \mu n^2$ for $n \in \mathbb{N}_0$ and $0 < \lambda < \mu$.

   (i) Write down the generator.

   (ii) Find the stationary distribution.

(d) Consider a birth process with birth rates $\lambda_n = \lambda(n+1)$ for $n \in \mathbb{N}_0$ and $\lambda > 0$. Does this process explode? Justify your answer.

1. (a) The Markov condition is given by seen ⇓

$$\mathbb{P}(X_n = s|X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = s|X_{n-1} = x_{n-1})$$

for all integers $n \geq 1$ and for all $s, x_0, \ldots, x_{n-1} \in E$. 3

(b) seen ⇓

$$\begin{aligned}
&\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_N = x_N) \\
&= \mathbb{P}(X_N = x_N|X_0 = x_0, X_1 = x_1, \ldots, X_{N-1} = x_{N-1}) \\
&\quad \cdot \mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_{N-1} = x_{N-1}) \\
&= \mathbb{P}(X_N = x_N|X_{N-1} = x_{N-1})\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_{N-1} = x_{N-1}),
\end{aligned}$$

where we used the Markov property. By iterating the argument, we get

$$\begin{aligned}
&\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_N = x_N) \\
= \quad &\mathbb{P}(X_N = x_N|X_{N-1} = x_{N-1}) \cdots \mathbb{P}(X_1 = x_1|X_0 = x_0)\mathbb{P}(X_0 = x_0).
\end{aligned}$$

Using the notation from the lecture notes, this can be written as

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_N = x_N) = \mathbb{P}(X_0 = x_0)p_{x_0x_1}p_{x_1x_2} \cdots p_{x_{N-1}x_N}.$$

4

(c) (i) Using the formula from part (b) and the law of total probability, we get sim. seen ⇓

$$\begin{aligned}
\mathbb{P}(X_0 = 1, X_1 = 2, X_3 = 3, X_4 = 2) &= \mathbb{P}(X_0 = 1)p_{12}p_{22}p_{23}p_{32} \\
&+ \mathbb{P}(X_0 = 1)p_{12}p_{23}p_{33}p_{32} + 0 + 0 = 2\frac{1}{4}\frac{9}{10}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{9}{160}.
\end{aligned}$$

1

(ii) We have a finite state space which can be divided into three communicating classes: The class $T = \{1\}$ is not closed and hence transient. The classes $C_1 = \{2, 3\}$ and $C_2 = \{4\}$ are closed and hence positive recurrent. 4

(iii) The Markov chain is not irreducible since it has more than one communicating class. 1

(iv) Choose $\boldsymbol{\pi} := (0, 0.5, 0.5, 0)$. Clearly $\boldsymbol{\pi}$ is a distribution since it has non-negative entries, which sum up to one. Also, it is stationary, since $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. 2

(v) The stationary distribution is not unique since we have two closed (essential) communicating classes.
We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \qquad \mathbf{P}(C_2) := 1.$$

Let $\boldsymbol{\pi}(C_1)$ denote a 2–dimensional row vector. We need to solve $\boldsymbol{\pi}(C_1)\mathbf{P}(C_1) = \boldsymbol{\pi}(C_1)$. Also, let $\boldsymbol{\pi}(C_2)$ denote a scalar and solve $\boldsymbol{\pi}(C_2)\mathbf{P}(C_2) = \boldsymbol{\pi}(C_2)$. Then $\boldsymbol{\pi}(C_1) = (0.5a, 0.5a)$ and $\boldsymbol{\pi}(C_2) = b$ for constants $a, b \in \mathbb{R}$. Now we define $\boldsymbol{\pi} := (0, 0.5a, 0.5a, b)$ for constants $a, b \geq 0$ such that $a + b = 1$. Then $\boldsymbol{\pi}$ is a stationary distribution, since $\pi_i \geq 0$ for $i = 1, \ldots, 5$ and $\sum_{i=1}^{5} \pi_i = 1$. Also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. 5

2. (a) A state $i \in E$ is recurrent if seen ⇓

$$\mathbb{P}(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$$

that is, the probability of returning to i, starting from i is 1. If the probability is less than 1, the state i is transient. 2

(b) We say that state j is accessible from state i, written $i \to j$, if the chain may ever visit state j, with positive probability, starting from i. In other words, $i \to j$ if there exist $m \geq 0$ such that $p_{ij}(m) > 0$, where $p_{ij}(m)$ denotes the m-step transition probability of going from state i to state j in m steps. Also, i and j communicate if $i \to j$ and $j \to i$, written $i \leftrightarrow j$. seen ⇓

2

(c) Let $i \leftrightarrow j$ and let i be recurrent. Then there exist integers $n, m \geq 0$ such that $p_{ij}(n) > 0$ and $p_{ji}(m) > 0$. For any integer $l \geq 0$, we have (using the Chapman Kolmogorov equations) seen ⇓

$$p_{jj}(m + l + n) \geq \underbrace{p_{ji}(m)}_{>0} p_{ii}(l) \underbrace{p_{ij}(n)}_{>0}.$$

Then

$$\sum_{l=1}^{\infty} p_{jj}(l) \geq \sum_{l=1}^{\infty} p_{jj}(m + l + n) \geq p_{ji}(m) p_{ij}(n) \sum_{l=1}^{\infty} p_{ii}(l) = \infty.$$

From the lectures we know that the fact that $\sum_{l=1}^{\infty} p_{jj}(l) = \infty$ is equivalent to j being a recurrent state, which completes the proof. 5

(d) We prove this by contradiction. Suppose that all states are transient; then according to a Corollary proved in the lectures we know that $\lim_{n\to+\infty} p_{ij}(n) = 0$ for all $i, j \in E$. Since the transition matrix is stochastic, we have $\sum_{j \in E} p_{ij}(n) = 1$. Then take the limit through the summation sign to obtain $\lim_{n\to+\infty} \sum_j p_{ij}(n) = 0$. However, this is a contradiction as the L.H.S must be 1. Hence there is at least one recurrent state. seen ⇓

5

(e) The dynamics of the Markov chain are determined by the initial distribution and the transition matrix. The state space, corresponding to the number of probability books is given by $E = \{0, 1, 2\}$. The initial distribution is given by $\boldsymbol{\nu}^{(0)} = (0, 0, 1)$. The transition matrix is given by unseen ⇓

$$\mathbb{P} = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.15 & 0.7 & 0.15 \\ 0 & 0.3 & 0.7 \end{pmatrix}.$$

The first and third row follow immediately. For the second row, we argue as follows: In order to go from 1 to 0, the selected book must be the probability book (which happens with probability 0.5), which is then replaced by a statistics book (which occurs with probability 0.3). Hence $p_{10} = 0.5 \cdot 0.3 = 0.15$. Similarly, we get that $p_{12} = 0.15$. Then $p_{11} = 1 - 0.15 - 0.15 = 0.7$.

6

3. (a) (i) The random variable Z satisfies the lack of memory property if for any $x, y > 0$, we have

$$\mathbb{P}(Z > x + y | Z > x) = \mathbb{P}(Z > y).$$

seen ⇓

2

(ii) A non-negative continuous random variable satisfying the lack of memory property has exponential distribution.

1

(b) (i) Using the definition of conditional probability, we get for $n \in \mathbb{N}_0$

unseen ⇓

$$\mathbb{P}(N_s = k | N_t = n) = \frac{\mathbb{P}(N_s = k, N_t = n)}{\mathbb{P}(N_t = n)} = \frac{\mathbb{P}(N_s = k, N_t - N_s = n - k)}{\mathbb{P}(N_t = n)}.$$

Note that a Poisson process is a counting process starting from 0. For $n < k$, we would have $N_t - N_s = n - k < 0$, which happens with probability zero. Hence, $\mathbb{P}(N_s = k | N_t = n) = 0$ for $n < k$. In the following, suppose that $n \geq k$. Then

$$\begin{aligned}
\frac{\mathbb{P}(N_s = k, N_t - N_s = n - k)}{\mathbb{P}(N_t = n)} &= \frac{\mathbb{P}(N_s = k)\mathbb{P}(N_t - N_s = n - k)}{\mathbb{P}(N_t = n)} \\
&= \frac{(\lambda s)^k}{k!} e^{-\lambda s} \frac{(\lambda(t-s))^{n-k}}{(n-k)!} e^{-\lambda(t-s)} \left(\frac{(\lambda t)^n}{n!} e^{-\lambda t} \right)^{-1} \\
&= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k},
\end{aligned}$$

since the increments of a Poisson process over disjoint intervals are independent and $N_s \sim Poi(\lambda s)$ and $N_t - N_s \sim Poi(\lambda(t-s))$. Hence,

$$\mathbb{P}(N_s = k | N_t = n) = \begin{cases} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

which implies the result.

7

(ii)

$$\begin{aligned}
\mathbb{E}(N_s | N_t = n) &= \sum_{k=0}^{\infty} k \mathbb{P}(N_s = k | N_t = n) = \sum_{k=1}^{\infty} k \mathbb{P}(N_s = k | N_t = n) \\
&= \sum_{k=1}^{n} k \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}.
\end{aligned}$$

Note that

$$k \binom{n}{k} = k \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{(n-k)!(k-1)!} = n \binom{n-1}{k-1}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n} k \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} &= n \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{s}{t}\right)^{k+1} \left(1 - \frac{s}{t}\right)^{n-1-k} \\
&= n \frac{s}{t} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-1-k} = n \frac{s}{t},
\end{aligned}$$

where we used the Binomial theorem. Altogether, we have $\mathbb{E}(N_s | N_t) = \frac{s}{t} N_t$.

Note that the following alternative solution will be given full marks, too: From (b)(i) we find that $N_s|N_t \sim Binomial(N_t, s/t)$, which implies immediately that $\mathbb{E}(N_s|N_t) = \frac{s}{t}N_t$.

5

(c) Let $n \in \mathbb{N}_0$. We need to find $\mathbb{P}(N_T = n)$. Note that $T \sim \text{Exp}(\alpha)$. From the law of total probability, we get

unseen ⇓

$$\mathbb{P}(N_T = n) = \int_0^\infty \mathbb{P}(N_T = n|T = x) f_T(x)dx = \int_0^\infty \mathbb{P}(N_x = n|T = x) f_T(x)dx.$$

Since N and T are independent, we get $\mathbb{P}(N_x = n|T = x) = \mathbb{P}(N_x = n)$. Also, $N_x \sim \text{Poi}(\lambda x)$ and $f_T(x) = \alpha \exp(-\alpha x)$ for $x \geq 0$. Altogether, we get

$$\begin{aligned}\mathbb{P}(N_T = n) &= \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x} \alpha e^{-\alpha x} dx = \alpha \frac{\lambda^n}{n!} \int_0^\infty x^n \exp^{-(\lambda+\alpha)x} dx \\ &= \alpha \frac{\lambda^n}{n!} (\lambda + \alpha)^{-n-1} \int_0^\infty z^n e^{-z} dz = \frac{\alpha \lambda^n \Gamma(n+1)}{n!(\lambda+\alpha)^{n+1}} = \frac{\alpha \lambda^n}{(\lambda+\alpha)^{n+1}},\end{aligned}$$

where we changed variables by setting $z = (\lambda + \alpha)x$ and used the fact that $\Gamma(n+1) = n!$.

5

4. (a) Let $\{\mathbf{P}_t, t \geq 0\}$ denote the stochastic semigroup, assumed to be standard, associated with the Markov chain. The generator is defined as seen ⇓

$$\mathbf{G} := \lim_{\delta \downarrow 0} \frac{1}{\delta}[\mathbf{P}_\delta - \mathbf{I}].$$

3

(b) From the lectures, we know that the transition probabilities, denoted by p_{ij} for $i, j \in E$, of the corresponding jump chain are given by $p_{ij} = g_{ij}/(-g_{ii})$ for $i \neq j$ if $-g_{ii} > 0$. If $-g_{ii} = 0$, then the state i is absorbing. Noting that the row elements in the transition matrix have to sum up to one, we get unseen ⇓

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ \frac{3}{5} & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

5

(c) (i) The generator is sim. seen ⇓

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(2\lambda + \mu) & 2\lambda & 0 & 0 & \cdots \\ 0 & 4\mu & -(3\lambda + 4\mu) & 3\lambda & 0 & \cdots \\ 0 & 0 & 9\mu & -(4\lambda + 9\mu) & 4\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

4

(ii) As in the lectures, we use the result that $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$; then

$$\begin{aligned} -\lambda_0\pi_0 + \mu_1\pi_1 &= 0 \\ \lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\pi_n + \mu_{n+1}\pi_{n+1} &= 0 \quad n \geq 1. \end{aligned}$$

This leads to

$$\pi_n = \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\pi_0$$

for any $n \in \mathbb{N}$. (Formally, this can be shown by induction, but the proof is not required here.) Such a vector $\boldsymbol{\pi}$ is a stationary distribution if and only if $\sum_n \pi_n = 1$; that is

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} < +\infty$$

with the first term ($n = 0$) defined to be 1, i.e. $\lambda_0\lambda_{-1}/\mu_1\mu_0 := 1$. Given this condition, it follows that

$$\pi_0 = \left(\sum_{n=0}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\right)^{-1}.$$

Here, we get

$$\pi_0 = \left(\sum_{n=0}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n n!}{\mu^n (n!)^2}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^n n!}\right)^{-1} = e^{-\lambda/\mu},$$

and for $n \in \mathbb{N}$, we have

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} e^{-\lambda/\mu}.$$

5

(d) No, this process does not explode, since for any initial value $n_0 \in \mathbb{N}_0$ we have $\sum_{i=n_0}^{\infty} \frac{1}{\lambda_n} = \frac{1}{\lambda}\sum_{i=n_0}^{\infty} \frac{1}{n+1} \leq \frac{1}{\lambda}\sum_{i=1}^{\infty} \frac{1}{n} = \infty$, which implies that there is no explosion. unseen ⇓

3