Imperial College London
Department of Mathematics

Unseen 4

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

In this exercise, we generalize the definition of the inner product you saw in the lectures. To avoid confusion we refer to this new definition as a "general inner product"

Definition 1. Let V be an $\mathbb{R}$-vector space. A *general inner product* on V is a binary function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that the following properties hold for all $v, u, w \in V$, $\alpha \in \mathbb{R}$:

(i) $\langle v, u \rangle = \langle u, v \rangle$.

(ii) $\langle \alpha v + u, w \rangle = \alpha \langle v, w \rangle + \langle u, w \rangle$.

(iii) $\langle v, v \rangle \geq 0$.

(iv) if $\langle v, v \rangle = 0$, then $v = 0$.

Similarly, we define a general norm on V: $\|v\| := \sqrt{\langle v, v \rangle}$.

1. For each of the following, determine whether $\langle \cdot, \cdot \rangle$ is a general inner product:
   (a) Let $V = M_{n \times m}(\mathbb{R})$ and let $\langle A, B \rangle := \operatorname{trace}(AB^\top)$.
   (b) Let V be the space of all continuous functions on the interval $[a, b]$, with point-wise addition and scalar multiplication. Let $\langle g, f \rangle := \int_a^b f(x)g(x)dx$.
   (c) Let V be the set of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\langle X, Y \rangle := \mathbb{E}[X \cdot Y]$.
2. Prove that every n-dimensional inner product space is isomorphic to $\mathbb{R}^n$ with the inner product from class. Namely: Let $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle_s := \sum_{i=1}^n a_i b_i$ (we call this the *standard inner product*, a.k.a the dot product). Let V be an $\mathbb{R}$-vector space with general inner product $\langle \cdot, \cdot \rangle$. Prove that there is an invertible linear transformation $T : \mathbb{R}^n \to B$ such that $\forall v, u \in \mathbb{R}^n : \langle v, u \rangle_s = \langle T(v), T(u) \rangle$.
3. Prove the following for general inner products: Let V be an $\mathbb{R}$-vector space and let $\langle \cdot, \cdot \rangle$ be a general inner product on V. Prove:
   (a) Cauchy-Schwarz inequality:
      i. $\forall v, u \in V : \langle v, u \rangle \leq \|v\| \cdot \|u\|$
      ii. $\forall v, u \in V : \langle v, u \rangle = \|v\| \cdot \|u\| \iff \{v, u\}$ are linearly dependent.
   (b) The triangle inequality: $\forall v, u \in V : \|v + w\| \leq \|v\| + \|w\|$.
   (c) The Pythagorean theorem:
   $\forall v, u \in V : \langle v, u \rangle = 0 \iff \|v\|^2 + \|u\|^2 = \|u + v\|^2$.
   (d) The Parallelogram law: $\forall v, u \in V : \|v + u\|^2 + \|v - u\|^2 = 2\|v\|^2 + 2\|u\|^2$.