MATH60005/70005: Optimization (Autumn 22-24)

Week 3 solutions

Dr Dante Kalise
Department of Mathematics
Imperial College London, United Kingdom
dkaliseb@imperial.ac.uk

Polynomial fit and denoising

We will replicate the linear regression and regularized linear least squares examples from this week, with a different model. First, we will generate a noisy dataset of 200 samples coming from

$$v_i = u_i^2 + \mathcal{N}(0, 0.04)\,, \quad i = 1, \dots, 200\,,$$

where the u_i's are uniformly sampled in $[-1, 1]$, and $\mathcal{N}(0, 0.04)$ means adding Gaussian noise of mean 0 and variance 0.04 for **each** sample.

1. Generate the pairs (u_i, v_i) using suitable random generators. Make a plot illustrating (u_i, v_i). What is the model you can identify to express v as a function of u?

```
Ns=200; %number of samples
u=2*rand(Ns,1)-1;% uniform sampling in [-1;1];
u=sort(u); %sorting the samples
v=u.^2+0.2*randn(Ns,1); %sampling with Gaussian noise of mean ...
    0 and variance 0.04
figure(1) %plotting the samples
plot(u,v,'LineStyle','none','Marker','*')
```

2. Write a linear regression problem for finding the optimal parameters in a model

$$v(u) = au^2 + bu + c$$

For a linear regression with N_s samples for a model $v(u) = au^2 + bu + c$, we assemble the matrices $\mathbf{A} \in \mathbb{R}^{N_s \times 3}$ and $\mathbf{b} \in \mathbb{R}^{N_s}$

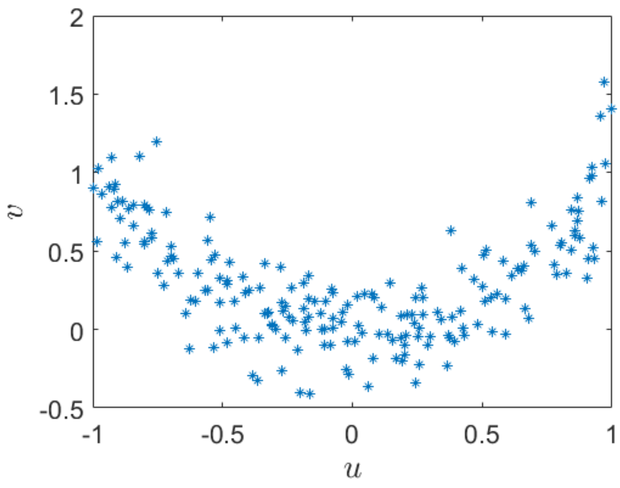

Figure 1: The sampled points suggest a quadratic dependence for $v(u)$.

```
A=[u.^2 u ones(Ns,1)]; %Ns by 3 matrix
b=v;
```

We can solve the normal equations

```
xls=(A'*A)\ (A'*b); % normal equations with backslash for the ...
    inverse
```

or, you can use backslash directly!

```
xls=A\ b;
```

MATLAB will recognize that $\mathbf{A}$ is a rectangular matrix and that you're trying to solving LLS, so instead of computing the inverse of $\mathbf{A}$, it will compute the solution to the normal equations.

3. Compute the least squares solution and compute the total least squares error in the ℓ_2 norm.

   For the error we need to compute $\|\mathbf{v} - \mathbf{A}\mathbf{x}_{LS}\|_2$,

```
 l2error=norm(v-A*xls,2);
```

4. Now, instead of solving a regression problem, use the v_i values to recover a denoised signal using regularized least squares using the same total variation regularization described the lecture notes.

   We assemble the regularization matrix $\mathbf{L}$ as in the notes

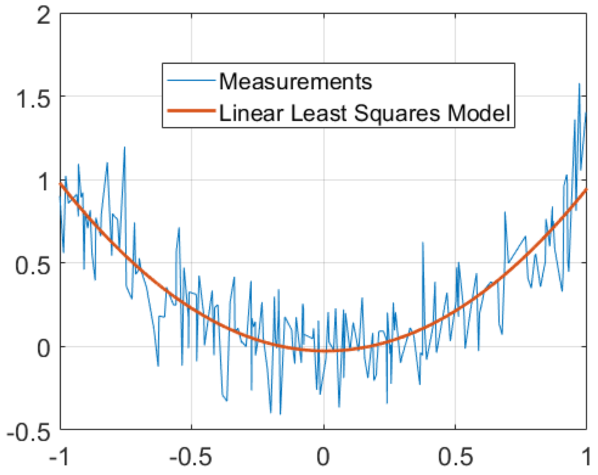

Figure 2: The measurements against linear least squares regression for $v(u) = au^2 + bu + c$.

```
L=zeros(Ns-1,Ns);
for i=1:Ns-1
L(i,i)=1;
L(i,i+1)=-1;
end
```

We set the regularization parameter $\lambda = 100$ (just to try, you need to see the effect of different values) and solve the regularized least squares solution

```
lambda=100;
xrls=(eye(Ns)+lambda*L'*L)\b;
```

and we plot the regularized signal against the noisy measurements

```
figure(3)
plot(u,v)
hold on
plot(u,xrls)
```

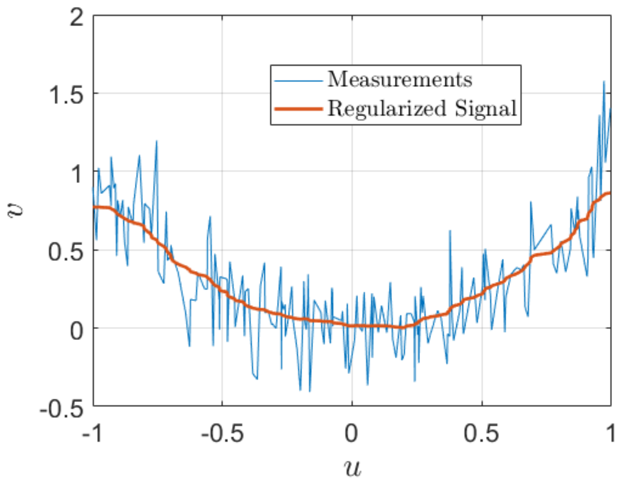

Figure 3: The regularized signal for $\lambda = 100$.