

Problem Sheet 4

1. Consider the most general linear second-order partial differential equation in two independent variables (x, y)

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = R(x, y)$$

If a change of variable $\xi \equiv \xi(x, y)$, $\eta \equiv \eta(x, y)$ is made so that this equation now reads

$$A' \frac{\partial^2 u}{\partial \xi^2} + B' \frac{\partial^2 u}{\partial \xi \partial \eta} + C' \frac{\partial^2 u}{\partial \eta^2} + D' \frac{\partial u}{\partial \xi} + E' \frac{\partial u}{\partial \eta} + F' u = R(\xi, \eta)$$

Show that

$$B'^2 - 4A'C' = (B^2 - 4AC) \left[\frac{\partial(\xi, \eta)}{\partial(x, y)} \right]^2$$

Conclude that a new choice of independent variables cannot change the type of a linear second-order PDE.

2. Use the method of separation of variables to find the solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

with the perfectly-insulated boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$$

and the initial condition $u(x, 0) = f(x)$, $0 < x < L$.

Find the particular solution for the case where:

$$(a) \quad f(x) = x^2 \quad \text{and} \quad (b) \quad f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

3. Show that the solution to the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

with the initial temperature distribution

$$u(x, 0) = e^{-|x|}$$

is given by

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\omega x)}{1 + \omega^2} e^{-\omega^2 \kappa t} d\omega$$

4. At the surface of the earth, daily and annual variations of temperature can be represented by sinusoidal oscillations with equal amplitudes and periods of 1 day and 365 days, respectively. Here, we assume that for the angular frequency ω the temperature at depth x in the earth is given by $u(x, t) = A \sin(\omega t + \mu x) \exp(-\lambda x)$, where λ and μ are constants.

- (a) Use the diffusion equation to determine the constants λ and μ .
- (b) Find the ratio of the depths below the surface at which the two amplitudes have dropped to $1/20$ of their surface values.
- (c) At what time of the year is the soil coldest at the greater of these depths, assuming that the smoothed annual variation in temperature at the surface has a minimum on February 1st?
5. Consider a cube made of a material whose thermal conductivity is given by k (not to be confused with thermal diffusivity κ !). We will assume that the cube has, as its six faces, the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$ and that it does not contain any internal heat sources. Verify that the temperature distribution
- $$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp \left(-\frac{2\kappa\pi^2 t}{a^2} \right)$$
- obeys the appropriate diffusion equation. If you define the heat flux as $-k\nabla u$, across which faces is the heat flux not zero? What is the direction and rate of the heat flux at the point $(3a/4, a/4, a)$ at time $t = a^2/(\kappa\pi^2)$.
6. So far in lectures and problem sheets, we have only used the method of separation of variables for problems with homogeneous boundary conditions. Here, we will see how one can extend this method in problems which involve a nonhomogeneous boundary condition. In a biology lab, you have placed a slab of biological material of thickness L on a glass plate (as shown on Fig. 1). At $t = 0$, you cover the slab with a solution containing a chemical species at a concentration C_0 (for instance, fluorescent or radioactive molecules). The chemicals of interest will slowly diffuse in the biological material (which originally did not contain any) with diffusivity D . You wish to find the profile of concentration in chemicals in the slab over time.

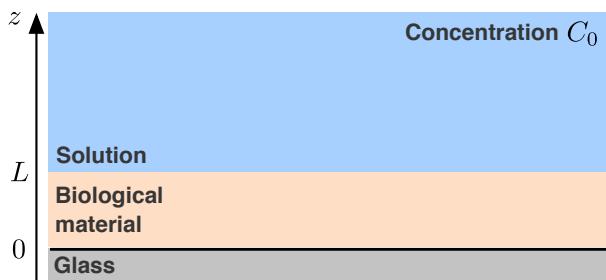


Figure 1: Schematic of the experiment described in Q6.

- (a) Formulate a 1D diffusion problem satisfied by the concentration in chemicals in the slab $u(z, t)$.
- (b) What is the steady-state solution $U(z)$ to this problem?
- (c) Show that the concentration $u(z, t)$ is given by

$$u(z, t) = C_0 + \sum_{n=1}^{\infty} A_n \cos \left[\frac{(2n-1)\pi z}{2L} \right] \exp \left[-\frac{(2n-1)^2\pi^2}{4L^2} Dt \right]$$

[Hint: first define an auxiliary variable $v(z, t) = u(z, t) - U(z)$ and use the subtraction principle to get back to a problem with homogeneous boundary conditions.]

- (d) What are the values of the coefficients A_n ?
- (e) In your experiment, it is important to make sure that you are close enough to the steady-state concentration profile. So you need to estimate how fast your solution will converge

to its steady-state value. To do so, we introduce the so-called relaxation time τ which is defined as

$$\frac{1}{\tau} = - \lim_{t \rightarrow \infty} \frac{1}{t} \ln |u(z, t) - U(z)|$$

Show that

$$\frac{1}{\tau} = \frac{\pi^2 D}{4L^2}$$

[Hint: think about the asymptotic behavior of your solution.]

Discuss briefly the significance of this result.

7.  Here, we are interested in the so-called *cable equation* (or telegrapher's equation). Note that it may require of you to read up on some concepts like Ohm's law and Kirchhoff's law, we will obviously not assume that you would know about this in an exam setting but this is an interesting application of the diffusion equation.

In the middle of the twentieth century, it was shown that the cable equation may be a good model for **signal transmission in cells and in neurons in particular**. But the history of the cable equation finds its **roots in the nineteenth century and the success of the telegraph**. In 1850, underwater cable lines were laid down between France and Britain to facilitate communication. However, it was quickly realized that this resulted in very disappointing transmission rates! The British scientist William Thomson set out to understand what was causing this problem.

The cable underwater is modeled as a system of outer and inner conductors separated by an insulating layer as shown in Fig. 2. The problem is intrinsically axisymmetric so we will consider it as one-dimensional. We will denote x the position along the cable. Fig. 2 shows a small segment of the cable. Both conductors are characterized by their linear resistivity r_i and r_o , while the insulator is modeled as a resistor r_s in parallel with a capacitor C_s .

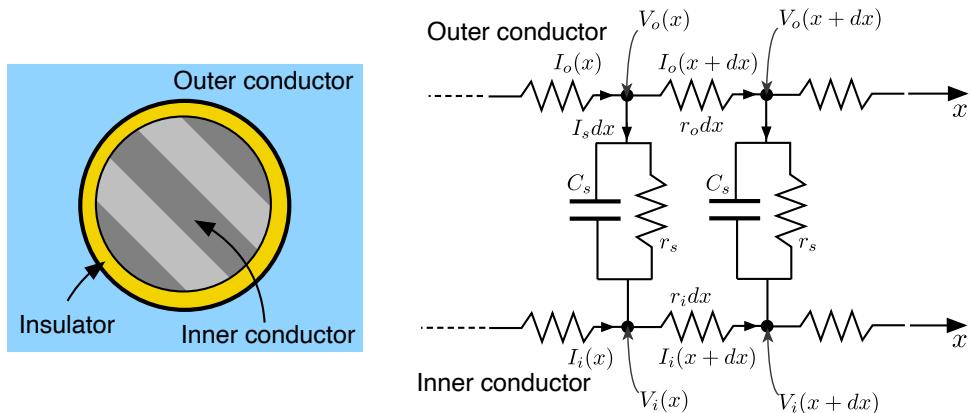


Figure 2: Schematic of the cable system

N.B.: Note that in the case of signal transmission in neurons, the insulating layer represents the membrane of the cell and the current consists of ions, mainly in the form of potassium, calcium and sodium ions.

In this derivation, we consider a small section of the cable found between x and $x + dx$.

- (a) Using [Ohm's law](#), show that in the limit $dx \rightarrow 0$ voltage and current in the conductors are related by

$$\frac{\partial V_i}{\partial x} = -r_i I_i(x) \quad \text{and} \quad \frac{\partial V_o}{\partial x} = -r_o I_o(x)$$

[Hint: first, think about how you would write Ohm's law for the small section of outer and inner conductors between x and $x + dx$. Remember that r_i and r_o are linear resistivities.]

- (b) Using [Kirchoff's law](#) (i.e. the current conservation equation), show that in the limit $dx \rightarrow 0$, we have

$$I_s = \frac{\partial I_i}{\partial x} = -\frac{\partial I_o}{\partial x}$$

- (c) We define the transinsulator potential as $V = V_i - V_o$. As the insulator is modeled as a resistor and capacitor in parallel, the current in the insulator reads

$$I_s = -\frac{1}{r_s}V - C_s \frac{\partial V}{\partial t}$$

Show that V is governed by the following equation

$$\frac{\partial V}{\partial t} = D \frac{\partial^2 V}{\partial x^2} - \beta V$$

with D and β constant parameters which you will define as a function of r_o , r_i , r_s and C_s . This is the so-called cable equation.

- (d) Discuss qualitatively why the capacitance of the insulating layer may be the source of the transmission problem.

[Hint: at this point, it is useful to analyze the terms in the cable equation.]

- (e) To understand the problem of long-range transmission, we consider that an emitter is located in $x = 0$ and that a receiver is located far enough that we can consider the case of a semi-infinite interval; this leads to the following boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t} &= D \frac{\partial^2 V}{\partial x^2} - \beta V & 0 < x < \infty \\ V(0, t) &= A \cos(\omega t) \\ V(x, t) &\rightarrow 0, x \rightarrow \infty \end{aligned}$$

Here, we seek solutions which will have propagation and oscillation properties but also decay with distance to the source. Therefore, we seek solutions of the form

$$V(x, t) = Av(x) \cos(\omega t - kx)$$

Find the functional form for $v(x)$ and show that the frequency ω , wave number k and the coefficient D must satisfy the following dispersion relation

$$\omega = 2Dk \sqrt{k^2 + \frac{\beta}{D}}$$

- (f) In the limit where the transinsulator resistivity is large, show that the source of the transmission problem is indeed the capacitance of the insulating layer.

8.  **[Harder]** At the end of Chapter 2, we concluded that a more realistic model of traffic flow could include viscous effects; under this assumption, we showed that the car density $\rho(x, t)$ was governed by the following second-order nonlinear equation

$$\frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}$$

where the wave speed given by Greenshields law was written as follows

$$c(\rho) = v_m \left(1 - 2 \frac{\rho}{\rho_m} \right)$$

In this problem, we will go through the whole procedure of using the Cole-Hopf transformation to solve an initial value problem for the viscous Burgers equation.

- (a) Using dimensional analysis, show that this equation can be written in dimensionless form

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + (1 - 2\tilde{\rho}) \frac{\partial \tilde{\rho}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2}$$

where $\tilde{\rho}$, \tilde{x} and \tilde{t} are dimensionless variables.

- (b) In its canonical form, the viscous Burgers equation reads

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2}$$

with $u(x, t)$ a scalar field. Using the Cole-Hopf transformation, show that solving the viscous Burgers equation for $u(x, t)$ can be reduced to solving a linear diffusion equation for the auxiliary function $\phi(x, t)$.

- (c) If you consider a general initial condition $u(x, 0) = u_0(x)$, what initial condition for the diffusion equation does this prescribe? Find the general solution to the viscous Burgers equation, i.e. show that the solution to the canonical Burgers equation can be written

$$u(x, t) = \frac{\int_{-\infty}^{\infty} e^{-F(x,y,t)/2\kappa} (x-y)/t dy}{\int_{-\infty}^{\infty} e^{-F(x,y,t)/2\kappa} dy}$$

with

$$F(x, y, t) = \int_0^y u_0(y') dy' + \frac{(x-y)^2}{2t}$$

- (d) Consider now that the initial conditions are given by

$$\rho(x, 0) = \begin{cases} \rho_m, & \text{for } x \leq 0 \\ 0, & \text{for } x > 0 \end{cases}$$

What is the solution to our traffic flow problem in this case?

[Hint: you will have to express the solution in terms of complementary error functions $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = (2/\sqrt{\pi}) \int_x^{\infty} e^{-u^2} du$.]

9.  **[Not examinable]** In the lecture notes, Section 3.3.7 establishes a link between the 1D diffusion equation and a symmetric random walks in 1D. In particular, we showed that if you consider a symmetric random walker starting from $x = 0$, the probability of finding this particle at position x at time t obeys the linear diffusion equation in the continuous limit. What about a biased random walker? As in the notes, consider the case where the particle makes step of size h in a time interval of size τ but now assume that the particle makes a step to the right with probability p_0 and to the left with probability $1 - p_0$ (where $p_0 \neq 1/2$). What equation for $p(x, t)$ do you obtain? Discuss the physical meaning of each of the terms in this equation.