Imperial College London

Lecture 08: Confidence Intervals

Statistical Modelling I

Dr. Riccardo Passeggeri

Outline

Introduction

Motivation

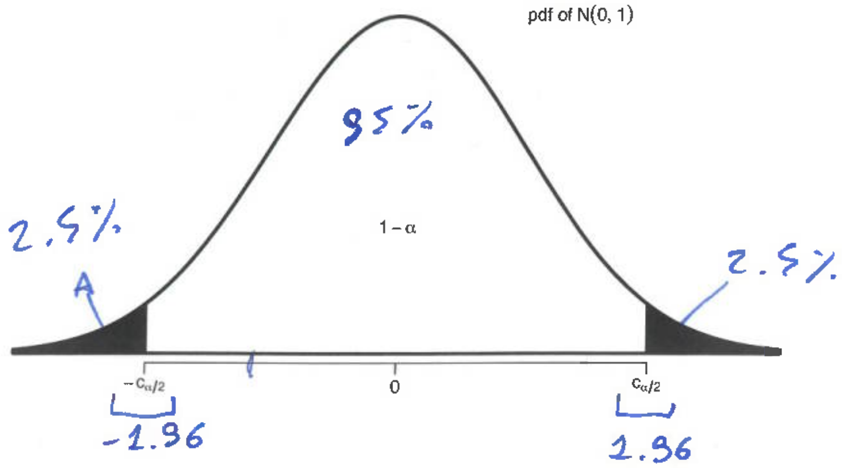
- **(Point) estimator**: one number only (does not reflect uncertainty)
- **Confidence interval**: random interval that contains the true parameter with a certain probability

Example: $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma_0^2)$, σ_0^2 known

Want: random interval that contains μ with probability $1 - \alpha$ for some $\alpha > 0$, e.g. $\alpha = 0.05$

- $\bar{Y} = \frac{1}{n} \sum Y_i \sim N(\mu, \sigma_0^2/n)$
- $\frac{\bar{Y} - \mu}{\sigma_0/\sqrt{n}} \sim N(0,1) \quad \forall \mu \in \mathbb{R}$
- $\Phi(c_{\alpha/2}) = 1 - \alpha/2$

$\Phi(x) := \mathbb{P}(X \le x)$, WHERE $X \sim N(0,1)$



$$95\% = 1 - \alpha = P\left(-c_{\alpha/2} < \overbrace{\frac{\bar{Y} - \mu}{\sigma_0/\sqrt{n}}}^{\sim N(0,1)} < c_{\alpha/2}\right)$$

$$= P(\underbrace{\bar{Y} + c_{\alpha/2}\sigma_0/\sqrt{n}}_{\text{random}} > \underbrace{\mu}_{\text{non-random}} > \underbrace{\bar{Y} - c_{\alpha/2}\sigma_0/\sqrt{n}}_{\text{random}}) = P(\mu \in I)$$

$$I = (\bar{Y} - c_{\alpha/2}\sigma_0/\sqrt{n}, \ \bar{Y} + c_{\alpha/2}\sigma_0/\sqrt{n})$$

Example: $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma_0^2)$, σ_0^2 known

The interval $(\bar{Y} - c_{\alpha/2}\sigma_0/\sqrt{n}, \bar{Y} + c_{\alpha/2}\sigma_0/\sqrt{n})$ is a random interval which contains the true μ with probability $1 - \alpha$.

The observed value of the random interval is $(\bar{y} - c_{\alpha/2}\sigma_0/\sqrt{n}, \bar{y} + c_{\alpha/2}\sigma_0/\sqrt{n})$.

This is called a $1 - \alpha$ **confidence interval** for μ.

Remarks

- α is usually small, often $\alpha = 0.05$.
- When speaking of a confidence interval we can either mean the realisation of the random interval or the random interval itself (this should hopefully be clear from the context).
- Could use asymmetrical values, but symmetrical values ($\pm c_{\alpha/2}$) give the shortest interval in this case.
- The value $\sigma_0/\sqrt{n}$ is exactly the **standard error** of $\bar{Y}$.

Example

In an industrial process, past experience shows it gives components whose strengths are $N(40, 1.21^2)$. The process is modified but s.d.(=1.21) remains the same.

After modification, $n = 12$ components give an average of 41.125.

New strength $\sim N(\mu, 1.21^2)$.
$n = 12$, $\sigma_0 = 1.21$, $\bar{y} = 41.125$, $\alpha = 0.05$, $c_{\alpha/2} \approx 1.96$.
$\rightarrow$ a 95% CI for μ is $(40.44, 41.81)$.

Note that our CI does not include 40 - an indication that the modification seems to have increased strength ($\rightarrow$ hypothesis testing)

This does **not** mean that we are 95% confident that the true μ lies in $(40.44, 41.81)$.
It means that if we were to take an infinite number of (indep) samples then in 95% of cases the calculated CI would contain the true value.

$1-\alpha$ Confidence interval

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A $1-\alpha$ confidence interval for θ is a random interval I that contains the 'true' parameter with probability $\geq 1-\alpha$, i.e.

$$P_\theta(\theta \in I) \geq 1-\alpha \quad \boxed{\forall \theta \in \Theta}$$

In the above, I can be any type of interval. For example, if L and U are random variables with $L \leq U$ then I could be the open interval (L, U), the closed interval $[L, U]$, the unbounded interval $[L, \infty)$, ...

Example: $X \sim \text{Bernoulli}(\theta)$ $\theta \in [0, 1]$ unknown

Want: $1 - \alpha$ CI for θ (suppose $0 < \alpha < 1/2$).

EX: IS $[L,U]$ A CI FOR θ WHEN $\alpha > \frac{1}{2}$?

Let

$$[L, U] = \begin{cases} [0, 1-\alpha], & \text{for } X = 0 \\ [\alpha, 1], & \text{for } X = 1 \end{cases}$$

This is indeed a $1 - \alpha$ CI, since

$$P_\theta(\theta \in [L, U]) = \begin{cases} P_\theta(X = 0) = 1 - \theta \geq 1 - \alpha & \text{for } \theta < \alpha, \\ 1 & \text{for } \alpha \leq \theta \leq 1 - \alpha, \\ P_\theta(X = 1) = \theta \geq 1 - \alpha & \text{for } \theta > 1 - \alpha. \end{cases}$$

$$P_\theta(\theta \in [L,U]) = P_\theta(\theta \in [L,U], X=0) + P_\theta(\theta \in [L,U], X=1) =$$

$$= P_\theta(\theta \in [0, 1-\alpha])P(X=0) + P_\theta(\theta \in [\alpha, 1])P(X=1) \geq 1-\alpha$$

SO $[L,U]$ IS A CI FOR θ

P_θ

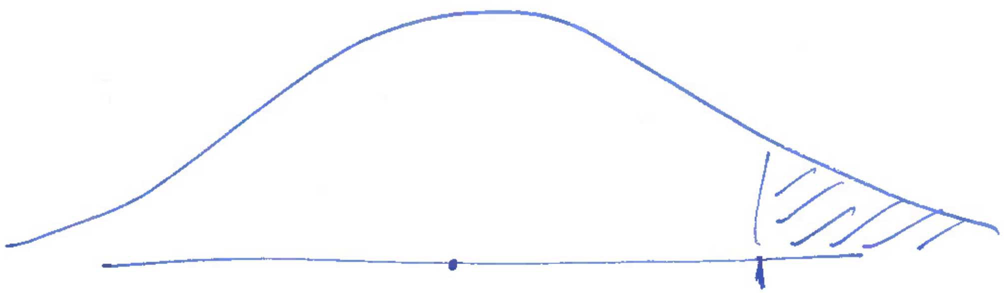
Example: One-sided confidence intervals

Suppose $Y_1, \ldots, Y_n$ are independent measurements of a pollutant θ, where higher values indicate worse pollution.

We want a $1-\alpha$ CI for θ of the form $(-\infty, h(y)]$.

For this h needs to be a function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$P_\theta(\theta \leq h(Y)) = 1 - \alpha \quad \forall \theta.$$



Constructing confidence intervals

Definition: Pivotal quantity (pivotal statistic)

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A **pivotal quantity** for θ is a function $t(Y, \theta)$ of the data and θ (and **not** any further nuisance parameters) s.t. the distribution of $t(Y, \theta)$ is known, i.e. does **not** depend on **any** unknown parameters.

This mirrors features of $\frac{\bar{Y} - \mu}{\sigma_0/\sqrt{n}}$ in the first example (where σ_0 is known)

$$t(Y, \mu) = \frac{\bar{Y} - \mu}{\sigma_0/\sqrt{n}}$$

Constructing confidence intervals via pivotal quantities

Suppose $t(Y, \theta)$ is a pivotal quantity for θ. Then we can find constants a_1, a_2 s.t.

$$P(a_1 \leq t(Y, \theta) \leq a_2) \geq 1 - \alpha$$

$a_1 = -c_{\alpha/2}$

$a_2 = c_{\alpha/2}$

because we know the distribution of $t(Y, \theta)$.

In many cases (as above) we can rearrange terms to give

$$P(h_1(Y) \leq \theta \leq h_2(Y)) \geq 1 - \alpha$$

$[h_1(Y), h_2(Y)]$ is a random interval. The observed interval

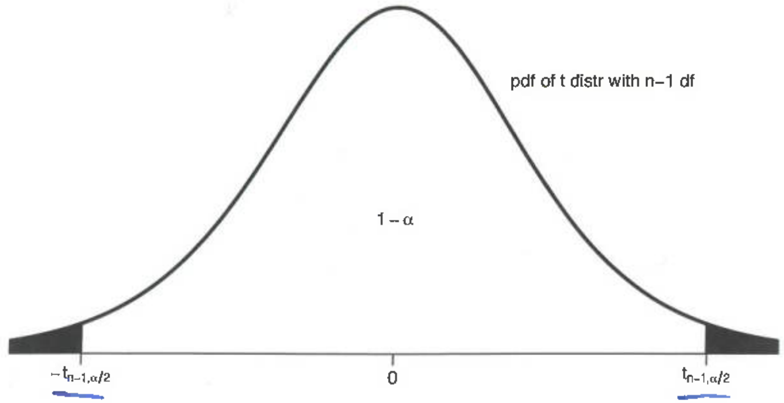
$$[\underbrace{h_1(y)}_{\text{lower confidence limit}}, \underbrace{h_2(y)}_{\text{upper confidence limit}}]$$

is a $1 - \alpha$ confidence interval for θ.

Example: $Y_1, \ldots, Y_n$ i.i.d $N(\mu, \sigma^2)$, μ, σ^2 both unknown

Want: confidence interval for μ, but σ is unknown $\Longrightarrow$ can't use $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ as a pivotal quantity!

- $S^2 = \frac{1}{n-1}\sum(Y_i - \bar{Y})^2$
- $T = \frac{\sqrt{n}}{S}(\bar{Y} - \mu)$.
- T follows a Student-t distribution with $n-1$ degrees of freedom



$t_{n-1,\alpha/2}$: $P(X \le t_{n-1,\alpha/2}) = 1 - \frac{\alpha}{2}$

WHERE $X \sim t_{n-1}$

$$1 - \alpha \text{ CI is } \left(\bar{y} - \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}, \bar{y} + \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}\right)$$

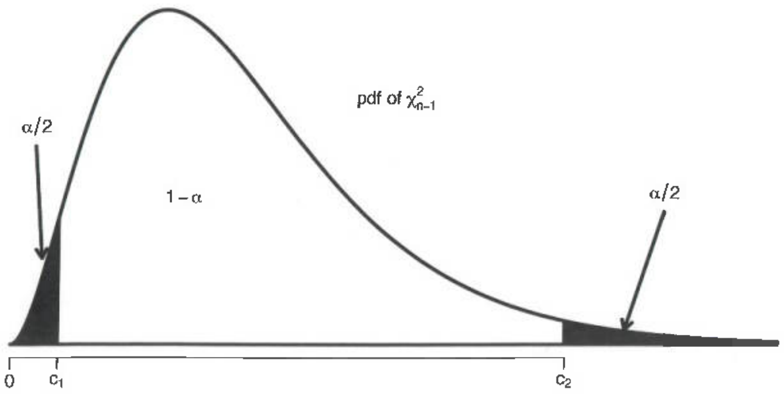
Example: $Y_1, \ldots, Y_n$ i.i.d $N(\mu, \sigma^2)$, μ, σ^2 both unknown

Want: confidence interval for σ (or σ^2)

$X \sim \chi^2_{n-1} \iff X = \sum_{i=1}^{n-1} V_i^2$ WHERE $V_i \overset{iid}{\sim} N(0,1)$

- $S^2 = \frac{1}{n-1}\sum(Y_i - \bar{Y})^2$
- $\frac{\sum(Y_i-\bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1}$ → CHI 2 - DISTRIBUTION
- c_1 and c_2 such that

$$P\left(c_1 \le \frac{\sum(Y_i-\bar{Y})^2}{\sigma^2} \le c_2\right) = 1-\alpha$$



$$1-\alpha \text{ CI for } \sigma \text{ is } \left(\sqrt{\frac{\sum(y_i-\bar{y})^2}{c_2}}, \sqrt{\frac{\sum(y_i-\bar{y})^2}{c_1}}\right)$$

Asymptotic confidence intervals

Definition: Asymptotic $1-\alpha$ confidence interval

[redacted]

A sequence of random intervals I_n is called an asymptotic $1-\alpha$ CI for θ if

$$\lim_{n\to\infty} P_\theta(\theta \in I_n) \geq 1-\alpha \quad \forall \theta \in \Theta.$$

If $\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \xrightarrow{d} N(0,1)$, then (approximately)

$$\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \stackrel{\cdot}{\sim} N(0,1)$$

and we can use the LHS as a pivotal quantity.