1. Four tourists arrive in London and, independently of each other, embark on sightseeing tours of some of the major sights.

   All four tourists start their tours at the same time, and each of them takes an exponentially distributed amount of time with mean 2 hours to complete their tour.

   What is the expected number of hours after which all four tourists will have completed their tours?

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We denote by J_i, the number of hours it takes tourist $i \in \{1, 2, 3, 4\}$ to complete their tour. Using order statistics, we write $J_{(1)} < J_{(2)} < J_{(3)} < J_{(4)}$.

We define $T_1 = J_{(1)}$ to be the number of hours it takes the first tourist to complete the tour, $T_2 = J_{(2)} - J_{(1)}$ is the additional time spent by the second tourist to complete the tour, $T_3 = J_{(3)} - J_{(2)}$ the additional time for the third tourist to complete the tour and $T_4 = J_{(4)} - J_{(3)}$ the additional time it takes the fourth tourist to complete the tour. By the lack of memory property of the exponential distribution and a result from lectures, we conclude that $T_1 \sim \mathrm{Exp}(4\lambda)$, $T_2 \sim \mathrm{Exp}(3\lambda)$, $T_3 \sim \mathrm{Exp}(2\lambda)$ and $T_4 \sim \mathrm{Exp}(\lambda)$, where $\lambda = 1/2$. Let $T = J_{(4)}$ denote the time until all tourists will have completed their tours. Then

$$\mathrm{E}(T) = \sum_{i=1}^{4} \mathrm{E}(T_i) = \frac{1}{4\lambda} + \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{1}{2} + \frac{2}{3} + 1 + 2 \approx 4.167.$$

2. You go bird watching at the London Wetland Centre. You are told that visitors who spend five hours at the centre have a probability of 0.8 of seeing at list one skylark. You may assume that the arrival of skylarks at the centre can be modelled by a homogeneous Poisson process.

   You are only able to spend two hours at the centre. What is the probability that you will see at list one skylark during your visit?

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We denote by $N = (N_t)_{t\geq 0}$ the homogeneous PP of rate $\lambda > 0$, which models the arrival of the skylarks at the London Wetland Centre. We denote by t the time in hours. We note that

$$\mathrm{P}(N_5 \geq 1) = \frac{80}{100} = \frac{4}{5}.$$

Hence, $\mathrm{P}(N_5 = 0) = 1/5$ and

$$\mathrm{P}(N_5 = 0) = \exp(-5\lambda) = 1/5 \Leftrightarrow \lambda = \frac{1}{5}\log(5) \approx 0.322.$$

We compute

$$\mathrm{P}(N_2 = 0) = \exp(-2\lambda) = \exp\left(-\frac{2}{5}\log(5)\right) \approx 0.525.$$

Hence

$$\mathrm{P}(N_2 \geq 1) = 1 - \exp(-2\lambda) \approx 0.475.$$

3. Multiple answer:

   Please select all correct statements.

   a) A Poisson process is a Markov chain in continuous time.
   b) Let $N = (N_t)_{t \geq 0}$ denote a Poisson process with rate $\lambda > 0$. Then $\mathrm{E}(N_t) = \lambda$ for $t \geq 0$.
   c) A Poisson process has independent increments.
   d) The sample paths of a Poisson process are continuous.
   e) A Poisson process is continuous in probability.

Solution: a), c), e).

4. Let X denote an exponential random variable with mean 0.5. Find

$$\mathrm{P}(X > 5|X > 2).$$

Please record your answer in decimals rounding to three decimal places if needed.

Solution: Let $X \sim \mathrm{Exp}(\lambda)$. Then $\mathrm{E}(X) = 1/\lambda = 1/2$. Hence $\lambda = 2$. By the lack of memory property we know that

$$\mathrm{P}(X > 5|X > 2) = \mathrm{P}(X > 3) = \exp(-3\lambda) = \exp(-6) \approx 0.002.$$

5. Consider a non-homogeneous Poisson process with intensity function $\lambda(t) = t + \exp(2t)$ for $t \geq 0$. Find $\mathrm{P}(N_4 = 2.5)$.
   Please record your answer in decimals rounding to three decimal places if needed.

   Solution: 0. We note that a non-homogeneous PP process takes values in $\{0, 1, 2, \ldots\}$.

6. Consider a non-homogeneous Poisson process with intensity function $\lambda(t) = t + \exp(-2t)$ for $t \geq 0$. Find $\mathrm{E}(N_5 - N_3)$.

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We define

$$m(t) := \int_0^t \lambda(s)ds = 0.5(t^2 - \exp(-2t) + 1).$$

Then

$$m(5) - m(3) = 0.5(25 - \exp(-10) - 9 + \exp(-6)) \approx 8.001.$$

We note that $N_5 - N_3 \sim \mathrm{Poi}(m(5) - m(3))$. Hence

$$\mathrm{E}(N_5 - N_3 = 1) = (m(5) - m(3)) \approx 8.001.$$

7. Consider a non-homogeneous Poisson process with intensity function $\lambda(t) = t + \exp(-2t)$ for $t \geq 0$. Find $\mathrm{P}(N_1 = 0)$.
   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We define

$$m(t) := \int_0^t \lambda(s)ds = 0.5(t^2 - \exp(-2t) + 1).$$

We note that $N_1 \sim \mathrm{Poi}(m(1))$, where $m(1) \approx 0.932$. Hence

$$\mathrm{P}(N_1 = 0) = \exp(-m(1)) = \exp\left(-0.5(2 - \exp(-2))\right) \approx 0.394.$$

8. Consider a continuous-time homogeneous Markov chain with generator given by

$$\mathbf{G} = \begin{pmatrix} -2 & 1/2 & 1/4 & 5/4 \\ 1/3 & -1 & 1/3 & 1/3 \\ 1/2 & 1/4 & -2 & 5/4 \\ 1/3 & 1/3 & 1/3 & -1 \end{pmatrix}.$$

Find the stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ and report the product of its components, i.e. find $\prod_{i=1}^4 \pi_i$.

Please record your answer in decimals rounding to three decimal places if needed.

Solution: From lectures, we know that we can find the stationary distribution by solving $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$. Equivalently, we can solve $\mathbf{G}^\top \boldsymbol{\pi}^\top = \mathbf{0}^\top$, including the additional condition that the elements of $\boldsymbol{\pi}$ need to be non-negative and sum up to 1. This leads to $\boldsymbol{\pi} = (20/131, 273/1048, 18/131, 471/1048)$. Hence

$$\prod_{i=1}^4 \pi_i = \frac{20}{131} \cdot \frac{273}{1048} \cdot \frac{18}{131} \cdot \frac{471}{1048} \approx 0.002.$$

9. Consider a continuous-time homogeneous Markov chain with generator given by

$$\mathbf{G} = \begin{pmatrix} -2 & 1/2 & 1/4 & 5/4 \\ 1/3 & -1 & 1/3 & 1/3 \\ 1/2 & 1/4 & -2 & 5/4 \\ 1/3 & 1/3 & 1/3 & -1 \end{pmatrix}.$$

Let Z denote the corresponding jump chain. Find the stationary distribution $\boldsymbol{\pi}^Z = (\pi_1^Z, \pi_2^Z, \pi_3^Z, \pi_4^Z)$ of the jump chain and report the product of its components, i.e. find $\prod_{i=1}^4 \pi_i^Z$.

Please record your answer in decimals rounding to three decimal places if needed.

Solution: The transition matrix of the jump chain Z is given by

$$\mathbf{P}_Z = \begin{pmatrix} 0 & 1/4 & 1/8 & 5/8 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/4 & 1/8 & 0 & 5/8 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}.$$

We need to solve $\boldsymbol{\pi}^Z \mathbf{P}_Z = \boldsymbol{\pi}^Z$, where the elements of $\boldsymbol{\pi}^Z$ need to be non-negative and sum up to 1. We find that $\boldsymbol{\pi}^Z = (40/169, 21/104, 36/169, 471/1352)$. Hence

$$\prod_{i=1}^4 \pi_i^Z = \frac{40}{169} \cdot \frac{21}{104} \cdot \frac{36}{169} \cdot \frac{471}{1352} \approx 0.004.$$

10. True/False question: Consider a birth process $N = (N_t)_{t \geq 0}$ starting at 1, i.e. $N_0 = 1$, with birth rates given by $\lambda_n = \log(n)$ for $n \in \mathbb{N}$. Is the following statement true or false?

    For this birth process, explosion occurs with probability 1.

    Solution: In order to answer this question, we need to check the convergence of the infinite series

    $$S := \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{\log(n)}.$$

    This series is divergent since $\log(n) < n$, hence $1/n < 1/\log(n)$ and the harmonic series diverges and, hence, S diverges as well. This implies, that explosion occurs with probability 0. Hence, the answer is FALSE.