BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Optimisation Mock Exam

Date: Wednesday, 11th May 2021

Time: 09:00-11:00

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Consider the function

$$f(\mathbf{x}) = f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2 , \quad \mathbf{x} \in \mathbb{R}^2 ,$$

with

$$f_1(\mathbf{x}) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2 ,$$
$$f_2(\mathbf{x}) = -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2 .$$

Knowing that there exists an $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = 0$, find a minimizer for this function, and discuss whether it is local or global. (10 marks)

(b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

(i) $f(\mathbf{x}) = \sum_{i=1}^n x_i \ln(x_i) - (\sum_{i=1}^n x_i) \ln(\sum_{i=1}^n x_i)$ over $\mathbb{R}^n_{++}$ (5 marks)

(ii) $f(\mathbf{x}) = \sqrt{\mathbf{x}^T Q \mathbf{x} + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix. (5 marks)

(Total: 20 marks)

2. (a) Find the optimal solution of the problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^3} \quad & 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - 3x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(10 marks)

(b) Consider the problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n , \ \mathbf{e}^\top \mathbf{x} = 1\} ,$$

where f is a continuously differentiable function. Show that $\mathbf{x}^*$ is a stationary point of this problem if and only if

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \ldots = \frac{\partial f}{\partial x_n}(\mathbf{x}^*) .$$

(10 marks)

(Total: 20 marks)

3. Consider the problem

$$\begin{aligned} \min \quad & -x_1 x_2 x_3 \\ \text{s.t.} \quad & x_1 + 3x_2 + 6x_3 \leq 48 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(i) Write down the KKT conditions. (10 marks)

(ii) Find the optimal solution of the problem using the KKT system. (10 marks)

(Total: 20 marks)

4. Consider the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{m} \|\mathbf{A}_i\mathbf{x}+\mathbf{b}_i\|$$

where $\mathbf{A}_i \in \mathbb{R}^{k_i\times n}, \mathbf{b}_i \in \mathbb{R}^{k_i}, i = 1, 2, \ldots, m.$

(i) Using auxiliary variables $\mathbf{y}_i = \mathbf{A}_i\mathbf{x}+\mathbf{b}_i$, write an equivalent constrained optimization problem, derive its Lagrangian, and the dual objective function. (10 marks)

(ii) Derive an explicit formulation of the dual problem. (10 marks)

(Total: 20 marks)

5. **Mastery question.** We consider a set of N robots with state (position,velocity) $(x_i(t), v_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ interacting under second-order dynamics of the form

$$\frac{dx_i}{dt} = v_i \,, \tag{1}$$
$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) + u_i(t)\,, \tag{2}$$
$$x_i(0) = x_0\,, \quad v_i(0) = v_0\,, \qquad i = 1, \dots, N\,, \tag{3}$$

where $u_i(t) \in \{u : \mathbb{R}_+ \to \mathbb{R}^2\}$ correspond to control signals for each robot, and $a(r)$ is a communication kernel of the type

$$a(r) = \frac{1}{(1 + r^2)}\,.$$

Our goal is to drive the system to consensus, that is, to converge towards a configuration in which

$$v_i = \bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j \quad \text{for all } i.$$

For this, we write a finite horizon control problem of the form

$$\min_{\mathbf{u}(\cdot)} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \left(\|\bar{v} - v_j\|^2 + \gamma\|u_j\|^2\right) dt\,,$$

with $\gamma > 0$, subject to the dynamics (1)-(3).

(i) Write the necessary optimality conditions for this problem, giving an explicit expression of the optimal control as a function of the adjoint variable. (10 marks)

(ii) The result in (a) gives the optimal control for a given initial condition. If an optimal feedback was sought instead, explain what is the practical difficulty associated to its synthesis, and how could we circumvent it. (10 marks)

(Total: 20 marks)