

3.3 Spanning

Definition 3.3.1. Let V be an F -vector space. Let $u_1, \dots, u_m \in V$ then:

- A *Linear Combination* of u_1, \dots, u_m is a vector of the form $\alpha_1 u_1 + \dots + \alpha_m u_m$ for scalars $\alpha_1, \dots, \alpha_m \in F$. Note we can also write $\alpha_1 u_1 + \dots + \alpha_m u_m$ as $\sum_{i=1}^m \alpha_i u_i$.
- The *span* of u_1, \dots, u_m is the set of linear combinations of u_1, \dots, u_m . i.e. $\text{Span}(u_1, \dots, u_m) = \{\alpha_1 u_1 + \dots + \alpha_m u_m \in V : \alpha_1, \dots, \alpha_m \in F\}$.

NB: there are several different notations used for Span, e.g., $\text{Sp}(X)$, $\langle X \rangle$.

Lemma 3.3.2.

Let V be an F vector space, and $u_1, \dots, u_m \in V$ then $\text{Span}(u_1, \dots, u_m)$ is a subspace of V .

Proof:

Remark 3.3.3.

- By convention we take the empty sum to be 0_V , so $\text{Span} \emptyset = \{0_V\}$
- For an infinite set S we still only take finite sums i.e.

$$\text{Span}(S) = \left\{ \sum_{s_i \in S'} \alpha_i s_i : S' \subset^{finite} S, \alpha_i \in F \right\}$$

Exercise 3.3.4. Show that for an infinite subset S of an F -vector space V , $\text{Span}(S)$ is a subspace of V .

Definition 3.3.5. Let V be an F vector space, and suppose $S \subset V$ is such that $\text{Span}(S) = V$ then we say S spans V , or equivalently S is a *spanning set* for V .

Example 3.3.6.

$$\bullet \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ spans } \mathbb{R}^3.$$

$$\bullet \mathbb{R}^{\deg \leq n}[x] \text{ spanned by } \{1, x, x^2, \dots, x^n\}$$

Exercise 3.3.7. Which of the following sets span \mathbb{R}^3 :

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

In the above exercise we see that we sometimes have “redundant” vectors in a spanning set. If as well as spanning the set is linearly independent, then this won’t happen.

3.4 Linear Independence

Definition 3.4.1. Let V be an F -vector space. We say $u_1, \dots, u_m \in V$ are *linearly independent* if whenever

then it must be that

We say $\{u_1, \dots, u_m\}$ is a *linearly independent set*.

Alternatively, a set $\{u_1, \dots, u_m\}$ is *linearly dependent* if $\alpha_1 u_1 + \dots + \alpha_m u_m = 0_V$ where at least one $\alpha_i \neq 0$, and a set is linearly independent if it is not linearly dependent.

Example 3.4.2.

- The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a linearly independent subset of \mathbb{R}^3 .
- Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be functions and suppose $f(x) = x$ and $g(x) = x^2$. The set $\{f, g\}$ is a linearly independent subset of $V = \mathbb{R}^{\mathbb{R}}$.

- The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a linearly *dependent* subset of \mathbb{R}^3 .

- For V and F -vector space then $\{0_V\}$ is linearly *dependent*
- For V and F -vector space $v \in V$ then $\{v\}$ is linearly independent iff $v \neq 0_V$.

Exercise 3.4.3. Which of the following sets are linearly independent subsets of \mathbb{R}^3 :

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Lemma 3.4.4. Let v_1, \dots, v_n be linearly independent in an F -vector space V . Let v_{n+1} be such that $v_{n+1} \notin \text{Span}(v_1, \dots, v_n)$. Then v_1, \dots, v_{n+1} is linearly independent.

3.5 Bases

Definition 3.5.1.

- Let V be an F -vector space. A *basis* of V is a linearly independent spanning set of V .
- If V has a finite basis then we say V is a *finite dimensional* vector space.

Example 3.5.2.

- The set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for \mathbb{R}^3 . We have done linear independence, you must show $\text{Span}(B) = \mathbb{R}^3$.
- Let F be a field, then in F^n let e_i be the column vector with zeros everywhere except the i^{th} row. Then $\{e_1, \dots, e_n\}$ forms a basis for F^n and is known as the *standard basis*.
- $\mathbb{R}[x]$ has basis $\{1, x, x^2, \dots\}$.

Note: Not every vector space is finite dimensional. For example $\mathbb{R}[x]$ the set of real polynomials doesn't have a finite basis, but it does have infinite bases, e.g., $\{1, x, x^2, \dots\}$.

Exercise 3.5.3. Which of the following sets span \mathbb{R}^3 :

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$
4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Proposition 3.5.4. Let V be an F -vector space, let $S = \{u_1, \dots, u_m\} \subseteq V$. Then S is a basis of V if and only if every vector in V has a unique expression as a linear combination of elements of S .

Remark 3.5.5 Let $B = \{u_1, \dots, u_m\}$ be a basis for an F -vector space V . By Proposition 3.5.4 we see that we have a bijective map f from V to F^m , for $v = \alpha_1 u_1 + \dots + \alpha_m u_m$ we define $f(v) = (\alpha_1, \dots, \alpha_m)$ we call $(\alpha_1, \dots, \alpha_m)$ the co-ordinates of v

Proposition 3.5.6. Let V be a non-trivial (i.e. not $\{0\}$) F -vector space and suppose V has a finite spanning set S then S contains a linearly independent spanning set.

I.e., if V has a finite spanning set it has a basis - for cases where there is no finite spanning set we would need something called the axiom of choice to show this (see LOGIC course in year 3)