MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Extra Material 2: Proof of the local version of the Picard–Lindelöf theorem

In this document, a proof of the local version of the Picard–Lindelöf theorem is provided.

Theorem 1 (Picard–Lindelöf theorem, local version)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous with respect to x. Consider for a fixed $(t_0, x_0) \in D$ the initial value problem*

$$\dot{x} = f(t, x)\,, \qquad x(t_0) = x_0\,. \tag{1}$$

Then the following two statements hold:

(i) Qualitative version. *The initial value problem* (1) *has locally a uniquely determined solution, i.e. there exists a $h = h(t_0, x_0) > 0$ such that* (1) *has exactly one solution on $[t_0 - h, t_0 + h]$.*

(ii) Quantitative version. *Consider for some $\tau, \delta > 0$ the set $W^{\tau,\delta}(t_0, x_0) := [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)}$, where $\overline{B_\delta(x_0)}$ is the closed δ-neighbourhood of x_0. We assume that $W^{\tau,\delta}(t_0, x_0) \subset D$, and we suppose that there exist $K, M > 0$ such that*

$$\|f(t, x) - f(t, y)\| \le K\|x - y\| \quad \textit{for all } (t, x), (t, y) \in W^{\tau,\delta}(t_0, x_0) \tag{2}$$

and

$$\|f(t, x)\| \le M \quad \textit{for all } (t, x) \in W^{\tau,\delta}(t_0, x_0)\,. \tag{3}$$

Then (1) *has exactly one solution on $[t_0 - h, t_0 + h]$, where $h = h(t_0, x_0) := \min\{\tau, \frac{1}{2K}, \frac{\delta}{M}\}$.*

Proof. Fix an initial pair (t_0, x_0). We first prove the quantitative version and show that the qualitative version follows from that.

Step 1. Quantitative version.
Let τ, δ be chosen such that $W^{\tau,\delta}(t_0, x_0) \subset D$, and assume that (2) and (3) hold with $K, M > 0$. We define $h := \min\{\tau, \frac{1}{2K}, \frac{\delta}{M}\}$, and as in proof of the global version of the Picard–Lindelöf theorem, we consider Picard iterates on the interval $[t_0 - h, t_0 + h]$, which are given by

$$P(u)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))\, \mathrm{d}s \quad \text{for all } t \in [t_0 - h, t_0 + h]\,.$$

We will show first that P acts on $X := C^0\big([t_0 - h, t_0 + h], \overline{B_\delta(x_0)}\big)$ (in contrast to the proof of the global version, we require that the Picard iterates map into the set $\overline{B_\delta(x_0)}$, rather than $\mathbb{R}^d$, since we have only the local Lipschitz condition (2) for x defined in this set). Note that X is not a vector space, but as a closed subset of the Banach space $C^0\big([t_0 - h, t_0 + h], \mathbb{R}^d\big)$, it is a complete metric space.

Thereto, assume that $u \in X$, and we need to show that $P(u) \in X$. For all $t \in [t_0 - h, t_0 + h]$, we have

$$\begin{aligned} \|P(u)(t) - x_0\| &= \left\| \int_{t_0}^{t} f(s, u(s))\, \mathrm{d}s \right\| \\ &\overset{\text{Lemma 2.9}}{\le} \left| \int_{t_0}^{t} \|f(s, u(s))\|\, \mathrm{d}s \right| \\ &\overset{(3)}{\le} \left| \int_{t_0}^{t} M\, \mathrm{d}s \right| \le hM \le \delta\,, \end{aligned}$$

which implies that $P(u)(t) \in \overline{B_\delta(x_0)}$ for all $t \in [t_0 - h, t_0 + h]$, and hence, $P : X \to X$ is well-defined.

Finally, we note that (2) implies that the Lipschitz condition holds on $W^{\tau,\delta}(t_0, x_0)$, and with the same analysis as in the proof of the global version, it thus follows that P is a contraction (note that $h \leq \frac{1}{2K}$ is needed here). The Banach fixed point theorem then finishes the proof of this step.

Step 2. Qualitative version.
Since f is locally Lipschitz continuous, there exists a neighbourhood $U \subset D$ of (t_0, x_0) and a constant $K > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| \quad \text{for all } (t,x),(t,y) \in U\,.$$

Let $V \subset U$ be a compact set that contains (t_0, x_0) in its interior. Since continuous functions on compact sets attain a maximum, there exists an $M > 0$ such that

$$\|f(t,x)\| \leq M \quad \text{for all } (t,x) \in V\,.$$

Finally, choose $\tau, \delta > 0$ such that

$$W^{\tau,\delta}(t_0, x_0) = [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)} \subset V \subset D\,.$$

Note that such a choice for τ and δ is possible, since V contains (t_0, x_0) in its interior. Application of the quantitative version finishes the proof of this theorem. □