Example 4.1.5. *Question:* Find the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

Answer: Note that $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ form a basis for $\mathbb{R}^2$, a general vector of $\mathbb{R}^2$ is $\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So we must have:

$$
\begin{aligned}
T \begin{pmatrix} a \\ b \end{pmatrix} &= T \left(a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= aT \begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= a \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} a \\ -a + b \\ 2a + 3b \end{pmatrix}
\end{aligned}
$$

This map is linear as $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$, so a matrix transformation.

Proposition 4.1.6. Let V and W be vector spaces over F. Let $\{v_1, ..., v_n\}$ be a basis for V. Let $w_1, ..., w_n$ be any n vectors from W (these don't need to be distinct). Then there is a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all i.

Proof: Suppose that $v \in V$, then there exist $\lambda_1, ..., \lambda_n$ such that $v = \lambda_1 v_1 + ... + \lambda_n v_n$. Define the following map:

$$
\begin{aligned}
&T : V \to W \\
&T(v) = \lambda_1 w_1 + ... + \lambda_n w_n
\end{aligned}
$$

Claim: T is a linear transformation.

- *T preserves addition:*. Suppose $v, u \in V$, so we have $v = \lambda_1 v_1 + ... + \lambda_n v_n$ and $u = \mu_1 v_1 + ... + \mu_n v_n$. So:

$$
\begin{aligned}
T(v + u) &= T(\lambda_1 v_1 + ... + \lambda_n v_n + \mu_1 v_1 + ... + \mu_n v_n) \\
&= T((\lambda_1 + \mu_1)v_1 + ... + (\lambda_n + \mu_n)v_n) \\
&= (\lambda_1 + \mu_1)w_1 + ... + (\lambda_n + \mu_n)w_n \\
&= \lambda_1 w_1 + ... + \lambda_n w_n + \mu_1 w_1 + ... + \mu_n w_n \\
&= T(v) + T(u)
\end{aligned}
$$

- *T preserves scalar multiplication:* Suppose $v \in V$ and $\alpha \in F$, we have $v = \lambda_1 v_1 + ... + \lambda_n v_n$.

So

$$
\begin{aligned}
T(\alpha v) &= T(\alpha(\lambda_1 v_1 + ... + \lambda_n v_n)) \\
&= T(\alpha \lambda_1 v_1 + ... + \alpha \lambda_n v_n) \\
&= \alpha \lambda_1 w_1 + ... + \alpha \lambda_n w_n \\
&= \alpha(\lambda_1 w_1 + ... + \lambda_n w_n) \\
&= \alpha T(v)
\end{aligned}
$$

So it remains to check uniqueness. Suppose that we have a linear transformation S such that $S(v_i) = w_i$ for all i. Then we have:

$$
\begin{aligned}
S(\lambda_1 v_1 + ... + \lambda_n v_n) &= \lambda_1 S(v_1) + ... + \lambda_n S(v_n) \\
&= \lambda_1 w_1 + ... + \lambda_n w_n
\end{aligned}
$$

So $T = S$ proving uniqueness.

Remark 4.1.7. This shows that once we know what a linear transformation does to a basis we know what the transformation is.

Example 4.1.8. Let V be the space of all polynomials in x over $\mathbb{R}$ with degree less than or equal to 2. A basis for this is $\{1, x, x^2\}$. We can pick any three arbitary vectors in V for example:

$$
\begin{aligned}
w_1 &= 1 + x \\
w_2 &= x - x^2 \\
w_3 &= 1 + x^2
\end{aligned}
$$

By Proposition 4.1.6 there is a linear transformation $T : V \to V$ such that $T(1) = w_1$, $T(x) = w_2$, $T(x^2) = w_3$.

We can work out what T does to a general element of V. A general element is of the form $v = a1 + bx + cx^2$, so

$$
\begin{aligned}
T(v) &= T(a1 + bx + cx^2) \\
&= a(1 + x) + b(x - x^2) + c(1 + x^2) \\
&= (a + c) + (a + b)x + (-b + c)x^2
\end{aligned}
$$

4.2 Image and Kernel

Definition 4.2.1. Let $T : V \to W$ be a linear transformation:

- The *Image of T* is the set $Im\,T = \{T(v) \in W : v \in V\} \subseteq W$.

- The *Kernel of T* is the set $Ker\,T = \{v \in V : T(v) = 0_W\} \subseteq V$.

Example 4.2.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix}$$

- The image of T is the set of all vectors in $\mathbb{R}^2$ of the form $\begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix}$ for $x_1, x_2, x_3 \in \mathbb{R}$. This is the space:

$$\left\{ x_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\} = CSp(\begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}) = \mathbb{R}^2$$

- The kernel of T is the set of vectors in $\mathbb{R}^3$ such that $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_W$ that is so say such that:

$$\begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Alternatively this is the solution space of $Ax = 0$. In this case the kernel is $Sp \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$.

Proposition 4.2.3. Let $T : V \to W$ be a linear transformation. Then:

1. $Im\,T$ is a subspace of W.

2. $Ker\,T$ is a subspace of V.

Note: In general we write $U \leq V$ to mean U is a subspace of V, so with this notation we are saying $Im\,T \leq W$ and $Ker\,T \leq V$.

Proof: For both we need to check the vector space criterion.

1.
   - Certainly $Im\,T \neq \emptyset$, since $T(0) \in Im\,T$.
   - Suppose $w_1, w_2 \in Im\,T$ then there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Now,
   $$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$$
   So $w_1 + w_2 \in Im\,T$.

- Suppose $w \in Im\,T$ and let $\lambda \in F$. We have $w = T(v)$ for some $v \in V$, now $T(\lambda v) = \lambda T(v) = \lambda w$. So $\lambda w \in Im\,T$

So $Im\,T \leq W$.

Example 4.2.4. Let V_n be the vector space of polynomials in x over $\mathbb{R}$ of degree $\leq n$. We have $V_0 \leq V_1 \leq V_2....$ Define:

$$T : V_n \to V_{n-1},$$
$$T(f(x)) = f'(x).$$

Note: T is linear.

$$
\begin{aligned}
Ker\,T &= \{f(x) : f'(x) = 0\} \\
&= \{\text{constant polys}\} \\
&= V_0
\end{aligned}
$$

Suppose $g(x)$ has degree $\leq n-1$. Then by integrating $g(x)$ we can find $f(x)$ such that $f'(x) = g(x)$ and $deg(f(x)) = 1 + deg(g(x))$, so $deg(f(x)) \leq n$. Hence $ImT = V_{n-1}$.

Of course the $f(x)$ such that $f'(x) = g(x)$ is not unique - if c is a constant then $f(x) + c$ also has this property. In fact we get the set $\{h(x) : h'(x) = g(x)\}$ consists of polynomials $f(x) + k(x)$ where $k(x) \in Ker\,T$.

Proposition 4.2.5. Let $T : V \to W$ be a linear tranformation and let $v_1, v_2 \in V$. Then

$$T(v_1) = T(v_2) \text{ iff } v_1 - v_2 \in Ker\,T.$$

Proof:

$$
\begin{array}{rll}
T(v_1) = T(v_2) & \text{iff} & T(v_1) - T(v_2) = 0 \\
& \text{iff} & T(v_1 - v_2) = 0 \\
& \text{iff} & v_1 - v_2 \in Ker\,T
\end{array}
$$

Proposition 4.2.6. Let $T : V \to W$ be a linear transformation. Suppose that $\{v_1, ..., v_n\}$ is a basis for V. Then $Im\,T = Span\{T(v_1), .., T(v_n)\}$.

Proof: Clearly $Span\{T(v_1), .., T(v_n)\} \subseteq Im\,T$. Conversely, let $w \in Im\,T$. Then $w = Tv$ for some $v \in V$. Since $\{v_1, ..., v_n\}$ is a basis for V we can find scalars λ_i such that

$$
\begin{array}{rll}
v & = & \lambda_1 v_1 + ...\lambda_n v_n \\
w & = & T(v) \\
& = & T(\lambda_1 v_1 + ...\lambda_n v_n) \\
& = & \lambda_1 T(v_1) + ...\lambda_n T(v_n) \in Span\{T(v_1), ..., T(v_n)\}
\end{array}
$$

Proposition 4.2.7. Let A be an $m \times n$ matrix. Let $T : F^n \to F^m$ be given by $T(v) = Av$. Then:

1. $Ker\,T$ is the solution space to $Av = 0$.

2. $Im\,T$ is the column space of A.

3. $dim(Im\,T) = rank A$.

Proof:

1. Immediate from definitions

2. Take the "standard" basis for F^n that is:

$$
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad
e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
$$

   By proposition 4.2.6 we have $Im\,T = Span\{T(e_1), ..., T(e_n)\}$. Now $T(e_i) = Ae_i = c_i$ where c_i is the ith column of A. SO $Im\,T = Span\{c_1, ..., c_n\} = CSp(A)$.

3. By (ii) $dim(Im\,T) = dim(CSp(A)) = $ column rank of $A = rk(A)$

Theorem 4.2.8. *The rank nulity theorem: We've seen that when $Tv = Av$, $rank(A) = dim(Im\, T)$. An old fashioned name for $dim(Ker\, T)$ is the nulity of A*

Let $T : V \to W$ be a linear transformation. Then

$$dim(Im\, T) + dim(Ker\, T) = dim(V)$$

Proof: Let $\{u_1, ...u_s\}$ be a basis for $\ker T$, and let $\{w_1, ..., w_r\}$ be a basis for $Im\, T$. For each $w_i \in Im\, T$, and so $\exists v_i \in V$ with $Tv_i = w_i$. We claim that $B = \{u_1, ..., u_s\} \cup \{v_1, ...v_r\}$ is a basis for V.

- *Spanning set:* Let $v \in V$ since $Tv \in Im\, T$ we can write $Tv = \lambda_1 w_1 + ...\lambda_r w_r$ for scalars λ_i. So

$$
\begin{aligned}
Tv &= \lambda_1 w_1 + ...\lambda_r w_r \\
&= T(\lambda_1 v_1 + ...\lambda_r v_r)
\end{aligned}
$$

Now by proposition 4.2.5 $v - \lambda_1 v_1 + ...\lambda_r v_r \in \ker T$ so $v - \lambda_1 v_1 + ...\lambda_r v_r = \mu_1 u_1 + ... + \mu_s u_s$. Thus

$$v = \mu_1 u_1 + ... + \mu_s u_s + \lambda_1 v_1 + ...\lambda_r v_r \in span(B)$$

- *Linear independence* Suppose:

$$\lambda_1 v_1 + ...\lambda_r v_r + \mu_1 u_1 + ... + \mu_s u_s = 0$$

By applying T we get:

$$
\begin{aligned}
0 &= T(\lambda_1 v_1 + ...\lambda_r v_r + \mu_1 u_1 + ... + \mu_s u_s) \\
&= \lambda_1 T(v_1) + ...\lambda_r T(v_r) + \mu_1 T(u_1) + ... + \mu_s T(u_s) \\
&= \lambda_1 w_1 + ...\lambda_r w_r
\end{aligned}
$$

Thus $\lambda_1 = ... = \lambda_r = 0$, so we get that $\mu_1 u_1 + ... + \mu_s u_s = 0$, so $\mu_1 = ... = \mu_s = 0$.

Example 4.2.9.

Let $a, b, c \in \mathbb{R}$, define $U = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$. U is a subspace of $\mathbb{R}^3$.

We can find dimension of U by defining:

$$
T : \mathbb{R}^3 \to \mathbb{R}
$$
$$
T(x, y, z) = (a, b, c) \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

Now $U = \ker T$, and clearly $Im T = \mathbb{R}$ (as not all $a, b, c = 0$), thus $dim(Im T) = 1$. So

$$
\begin{aligned}
dim U &= dim(\ker T) \\
&= dim(\mathbb{R}^3) - dim(Im T) \\
&= 3 - 1 = 2
\end{aligned}
$$

Corollary 4.2.10. A system of linear equations in n unknowns with co-efficients in F:

$$
\begin{array}{llll}
a_{11}x_1 & +a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 & +a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & = & b_2 \\
\vdots & \vdots & & \vdots \quad \vdots \\
a_{m1}x_1 & +a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n & = & b_m
\end{array}
$$

is called *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$.

We know in this case that we will always get at least a trivial solution to the system - and we saw in the test that the set of solutions forms a subspace of F^n, but what dimension will this subspace have?

We can use the rank-nulity theorem to work this out:

We know that if we let $A = (a_{ij})$, then this system of linear equations can be represented as $Ax = 0$. We also know that A can be seen as a linear transformation $A : F^n \mapsto F^m$.

By Proposition 4.2.7 the set of solutions in this case is $ker(A)$, and by the rank nulity we get

$dim(ker(A)) = dim(F^n) - dim(Im(A))$

Now the $dim(Im(A)) = rank(A)$ thus the we can work out how many solutions we have to a set of homogeneous equations with n unknowns:

- If $rank(A) \geq n$ we get one solution (the trivial one i.e. 0_V)

- If $rank(A) < n$ we get infinitely many solutions (assuming F is infinite)

Exercise 4.2.11. In this case the rank of the augmented matrix $(A|0)$ is the same as that of A.

How does this work for a non homogeneous system of linear equations?

Essentially almost the same except - but we are taking a coset of the system of equations and we have to account for the case were $rank(A) < rank(A|b)$

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