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Computational PDEs MATH60025/70025: Problem sheet 1

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1. The equation $u_t = u_{xx}$ may be modelled by a two-step method which is centered in both space and time, namely

$$\frac{U_n^{j+1} - U_n^{j-1}}{2k} = \frac{U_{n+1}^j - 2U_n^j + U_{n-1}^j}{h^2} \tag{2.1}$$

   What is the order of the truncation error of this scheme?

   Use the Fourier method to investigate the stability of Eqn. 2.1 for all positive values of $r = k/h^2$.

2. The stability behaviour of the method (Eqn. 2.1) can be improved by replacing the term $2U_n^j$ by the average $(U_n^{j+1} + U_n^{j-1})$:

$$U_n^{j+1} - U_n^{j-1} = 2r\left(U_{n+1}^j - U_n^{j+1} - U_n^{j-1} + U_{n-1}^j\right) .$$

   Calculate the truncation error of this scheme up to and including terms of $O(k^2)$ and $O(h^2)$.

   Under what circumstances *i.e.* as $h \to 0$ and $k \to 0$, does the truncation error of the finite difference approximation tend to zero?

3. The three-dimensional diffusion equation

$$u_t = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

   is to be solved on a rectangular grid with step-lengths (h_x, h_y, h_z) using the θ-method

$$U_{lmn}^{j+1} - U_{lmn}^j = k\left[\frac{1}{h_x^2}\delta_x^2 + \frac{1}{h_y^2}\delta_y^2 + \frac{1}{h_z^2}\delta_z^2\right]\left(\theta U_{lmn}^{j+1} + (1-\theta)U_{lmn}^j\right) ,$$

   in the usual notation. Use the Fourier method to show that this scheme is unconditionally stable for $1 \geqslant \theta \geqslant 1/2$ while if $0 \leqslant \theta < 1/2$ it is stable provided

$$k\left\{\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right\} \leqslant \frac{1}{2(1-2\theta)}.$$

4. Given a function $u(x)$ and writing $u_n = u(nh)$, show that

$$\frac{12u_{1/3} - 3u_{2/3} - 9u_0}{2h} = u'(0) + O(h^2) .$$

5. The steady equations of motion of inviscid, isentropic compressible fluid flow are

$$\begin{aligned} \rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p &= 0 \\ \mathbf{u}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{u} &= 0 \end{aligned} \tag{2.2}$$

   where $\mathbf{u}$ is the velocity vector and the pressure, p, is a function of the density, ρ, only: $p = p(\rho)$. Defining the local sound speed c by $c^2 = \frac{dp}{d\rho}$, show that these equations imply that [skip this bit if you have no knowledge of fluid dynamics]

$$\mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}) = c^2\nabla\cdot\mathbf{u}\ .$$

   If the flow is both two-dimensional and irrotational ($\nabla\wedge\mathbf{u}=0$), then **u** can be written in terms of a potential, $\phi(x,y)$, so that $\mathbf{u}=(\phi_x,\ \phi_y,\ 0)$. Obtain an equation for ϕ, and determine the conditions for it to be hyperbolic, elliptic or parabolic, commenting on the physical meaning of the results.

6. **Hard:** Use the Matrix Method to examine the stability of the scheme

$$(1+\theta)\left[\frac{U_n^{j+1}-U_n^j}{k}\right]-\theta\left[\frac{U_n^j-U_n^{j-1}}{k}\right]=\frac{U_{n+1}^j-2U_n^j+U_{n-1}^j}{h^2}$$

   where $0\le\theta\le 1$. (Note $\theta=-\frac{1}{2}$ corresponds to (3).)

7. Pollutant is introduced at a rate $Q(x,y)$ into a river flowing with velocity $u(x)$ in the y-direction. Equilibrium between downstream advection and cross-channel diffusion is reached when the pollutant concentration $C(x,\ y)$ satisfies

$$uC_y = DC_{xx}+Q\ , \qquad \text{where } D \text{ is constant.}$$

   The river occupies $-d < x < d$, and a parabolic velocity profile $u = u_0(d^2-x^2)$ is assumed. It is desired to solve this equation subject to boundary conditions on $x=\pm d$ and an initial condition at $y=0$ by an explicit method. If the step length used across the river is 1/20 of its width, what is the maximum step length along the river which will guarantee stability of the algorithm?

   Should the direction of solution be upstream or downstream? Is the method a good one?