Answers to Test 2

1. (a)

$$L = \frac{1}{2}\dot{q}^2 + a\dot{q}\sin q \sin t + b\cos q.$$

$H = p\dot{q} - L(q, \dot{q}, t)$, where $\dot{q}$ is eliminated through

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} + a\sin q \sin t$$

or $\dot{q} = p - a\sin q \sin t$.

$$H = p(p - a\sin q \sin t) - \frac{1}{2}(p - a\sin q\sin t)^2 - a(p - a\sin q \sin t)\sin q \sin t - b\cos q$$

$$= \frac{1}{2}\left(p - a\sin q \sin t\right)^2 - b\cos q.$$

H is not a constant of the motion as $\dot{H} = \partial H/\partial t$ is non-zero. (7 marks)

(b) $Q = q$, $P = p - a\sin q \sin t$ or $p = P + a\sin q\sin t = \partial F_2/\partial q$, $Q = q = \partial F_2/\partial P$. Integrating gives

$$F_2 = qP - a\cos q \sin t + f(P), \quad F_2 = qP + g(q),$$

so that $F_2 = qP - a\cos q\sin t$ is a type-2 generating function.

(5 marks)

(c)

$$K = H + \partial F_2/\partial t = \frac{1}{2}P^2 - b\cos q - a\cos q\cos t = \frac{1}{2}P^2 - b\cos Q - a\cos Q\cos t.$$

(4 marks)

(d) Hamilton's equations in the new variables:

$$\dot{Q} = \frac{\partial K}{\partial P} = P, \quad \dot{P} = -\frac{\partial K}{\partial Q} = -\sin Q(b + a\cos t),$$

so that

$$\ddot{Q} = \dot{P} = -(a\cos t + b)\sin Q.$$

(4 marks)

(e)

$$H = \frac{1}{2}(p_1 - a\sin q_1 \sin q_2)^2 + p_2 - b\cos q_1.$$

Hamilton's equations:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 - a\sin q_1\sin q_2, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1} = a\cos q_1 \sin q_2(p_1 - a\sin q_1\sin q_2) - b\sin q_1,$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = 1, \qquad \dot{p}_2 = -\frac{\partial H}{\partial q_2} = a \sin q_1 \cos q_2 (p_1 - a \sin q_1 \sin q_2).$$

The first two equations agree with Hamilton's equations deriving from the Hamiltonian in part (a) if q_1, p_1 and q_2 are identified with q, p, t, respectively. Identifying q_2 with time is consistent with the third equation $\dot{q}_2 = 1$.

(5 marks)

(Total: 25 marks)

Remarks

(i) The Lagrangian describes the motion of a simple pendulum with a vertically oscillating support. Can you demonstrate this?

(ii) The $N = 1$ Hamiltonian considered in part (a) has explicit time-dependence. In part (e) the same dynamics is obtained from a time-independent $N = 2$ Hamiltonian. Using this 'trick' any time-dependent Hamiltonian can be replaced with a time-independent Hamiltonian at the expense of increasing N by one.