Sheet 1

1. **(Demand functions)** Students 1, 2 and 3 are interviewed about their coffee consumption. More specifically, they report their *reservation prices* for the quantity of cups of coffee they would be willing to purchase. The reservation price is the highest price per quantity such that they are willing to buy that specific quantity.

Student 1:

Quantity	1	2	3
Price in £	3.00	2.00	1.00

Student 2:

Quantity	1	2	3
Price in £	4.00	1.50	0.50

Student 3:

Quantity	1	2	3
Price in £	1	0.00	0.00

a) Calculate the respective individual demand functions D_i, $i \in \{1,2,3\}$ (as functions in the price) and the inverse demand functions P_i, $i \in \{1,2,3\}$ (as functions in the quantity demanded). If you properly consider the functions as maps $Q_i\colon A \to B$ and $P_i\colon C \to E$, what sets A, B, C, E are most appropriate? should be D_i

$$D_1(p) = \begin{cases} 3 & \text{for } 0 \leq p \leq 1 \\ 2 & \text{for } 1 < p \leq 2 \\ 1 & \text{for } 2 < p \leq 3 \\ 0 & \text{for } 3 < p \end{cases}$$

$$= 3 \cdot \mathbb{1}_{[0,1]}(p) + 2\mathbb{1}_{(1,2]}(p) + \mathbb{1}_{(2,3]}(p)$$

$$= \mathbb{1}_{[0,1]}(p) + \mathbb{1}_{[0,2]}(p) + \mathbb{1}_{[0,3]}(p)$$

Similarly,

$$D_2(p) = \begin{cases} 3 & \text{for} \quad 0 < p \leq 0.5 \\ 2 & \text{for} \quad 0.5 < p \leq 1.5 \\ 1 & \text{for} \quad 1.5 < p \leq 4 \\ 0 & \text{for} \quad 4 < p \end{cases}$$

$$= \mathbb{1}_{(0,0.5]}(p) + \mathbb{1}_{(0,1.5]}(p) + \mathbb{1}_{(0,4]}(p)$$

And

$$D_3(p) = \begin{cases} 1 & \text{for} \quad 0 < p \leq 1 \\ 0 & \text{for} \quad 1 < p \end{cases}$$

$$= \mathbb{1}_{(0,1]}(p)$$

And

(we are treating price as a continuous variable)

$$D_i(p) : \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0} \quad \text{for} \quad i = 1, 2, 3$$

Note that $D_i(p)$ is not defined for $p = 0$; it could be infinite.

Inverse demand functions. The question doesn't make it clear, but it is looking for the reservation price for a given quantity (as I mentioned in lectures). So

$P_1(1) =$ max. price per cup that Student 1 will pay if they are to buy 1 cup

$= 3$

$P_1(2) = 2$

$P_1(3) = 1$

$P_1(n) = 0$ for $n \geq 4$

Note, Student 1 will buy just 1 cup if the price per cup is $> £2$ but $\leq £3$, but obviously this price is <u>not</u> unique. But the corresponding reservation price <u>is</u> <u>unique</u>.

Similarly,

$P_2(1) = 4$

$P_2(2) = 1.5$

$P_2(3) = 0.5$

$P_2(n) = 0$ for $n \geq 4$

And

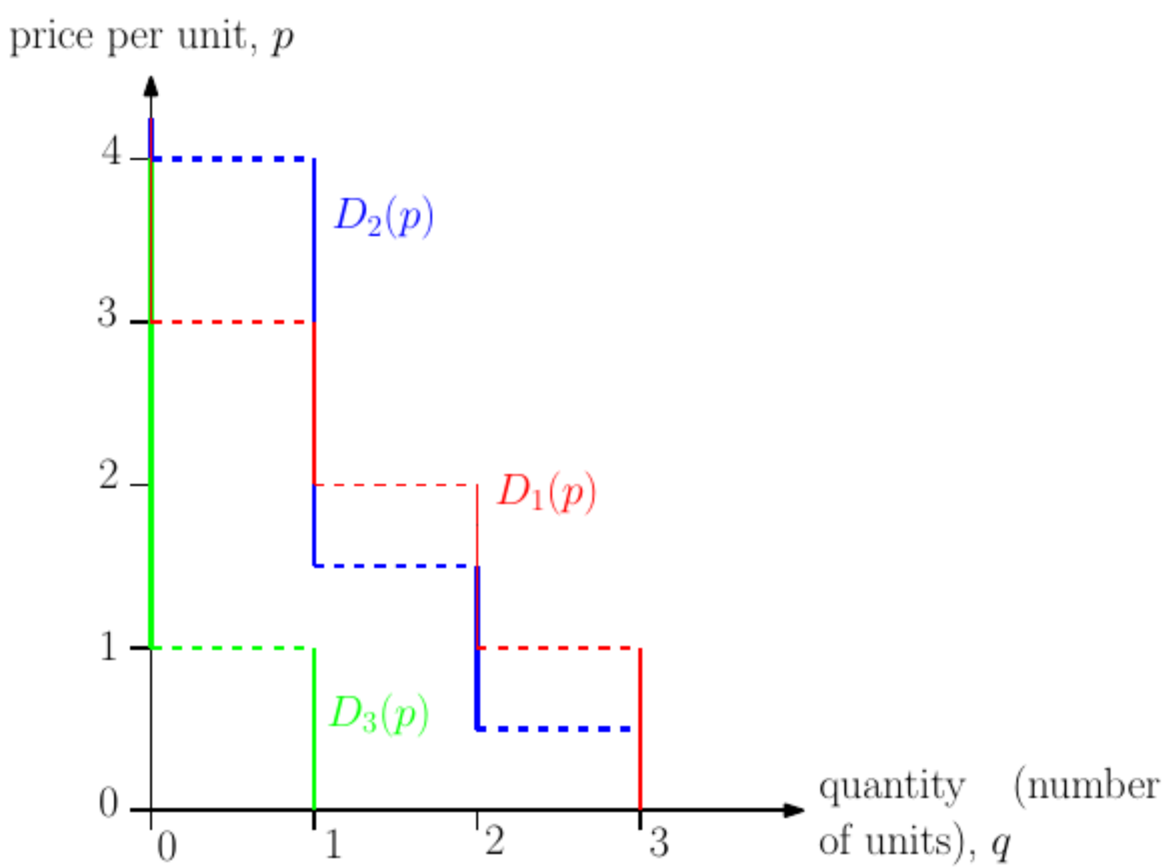
$$P_3(1) = 1$$

$$P_3(n) = 0 \quad \text{for} \quad n \geq 2$$

And

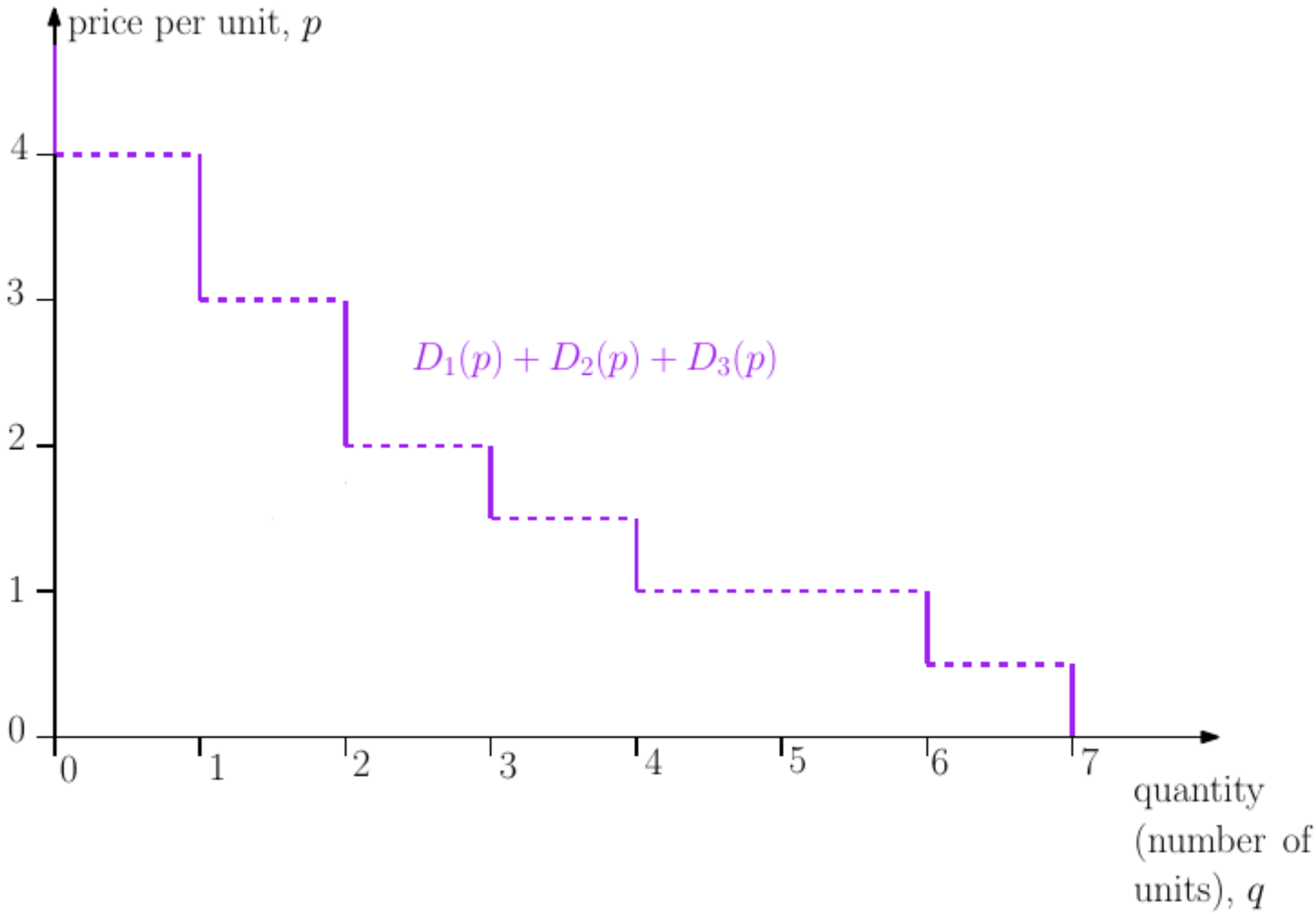
$$P_i : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0} \quad \text{for} \quad i = 1, 2, 3$$

Note that $P_i(0)$ is not uniquely defined.



b) Calculate the aggregate demand function D and the corresponding inverse P of the aggregate demand function.

$D(p) = D_1(p) + D_2(p) + D_3(p)$



Inverse of D is defined by

$P(1) =$ max price per cup the students will pay if between them they buy 1 cup

$= 4$

$P(2) = 3$

$P(3) = 2$

$P(4) = 1.5$

$P(5) =$ undefined

Solutions say $P(5) = 1$
This is correct if one
defines $P(q)$ as in solutions.

That is, $P(q) = \max\{p \geq 0 \,|\, D(p) \geq q\}$.

$P(6) = 1$

$P(7) = 0.5$

$P(n) = 0$ for $n \geq 8$

c) If the price for one cup of coffee is £ 0.75, how many cups will be sold in total?

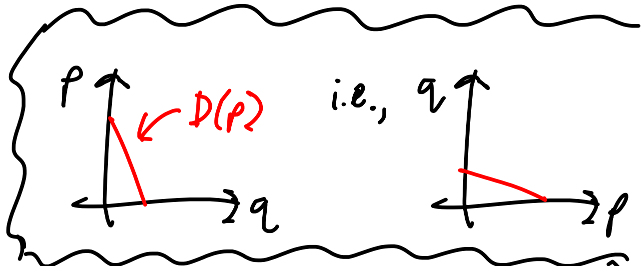
Solution: We calculate $D(0.75) = 6$. Indeed, at a price of £ 0.75, Student 1 will buy 3 cups, Student 2 will demand 2 cups and Student 3 will demand 1 cup.

$D(0.75) = 6$ – see graph above.

2. **(Price elasticity of demand and supply)**

a) Student A claims that since a linear demand function has constant slope, it also exhibits constant price elasticity. Under which conditions is Student A correct? Justify your answer.

$$\varepsilon_D = \frac{\partial D}{\partial p} \cdot \frac{p}{D}$$



$D(p)$ a linear demand function means

$$D(p) = a - bp$$

for some constants $a \geq 0, b \geq 0$, for $0 < p \leq \frac{a}{b}$

($D(p)$ should be a decreasing function of p, and ≥ 0. And it doesn't make sense to define D for $p = 0$.)

So $$\varepsilon_D = \frac{-b \cdot p}{a - bp}.$$

So $\varepsilon_D =$ constant, c, say

$$\Leftrightarrow \quad c = \frac{-bp}{a-bp}$$

$$\Rightarrow \quad a = 0 \quad \text{or} \quad b = 0$$

But $a = 0 \Rightarrow b = 0$ (as otherwise $D < 0$) $\Rightarrow D \equiv 0$

And $b = 0 \Rightarrow D(p) = a$

So ε_D is constant, equal to 0, if $D(p)$ is constant. This includes if $D(p) \equiv 0$. Note that the formula $\varepsilon_D = \frac{\partial D}{\partial p}\frac{p}{D}$ does not make sense if $D \equiv 0$, but in this case we should think of ε_D as defined as 0. Note also that we refer to the case of $\varepsilon_D = 0$ as that of perfect inelasticity. This is an exception to the Law of demand.

b) In case you agree with Student A, are there other possible demand functions that exhibit a constant price elasticity? If you disagree with Student A, are there alternatives to linear demand functions that exhibit a constant price elasticity?

Consider just $D \equiv D(p)$. Then

$$\varepsilon_D = D'(p) \cdot \frac{p}{D}$$

So

$$\varepsilon_D = c \iff \frac{D'}{D} = \frac{c}{p}$$

$$\iff \log D = c \log p + b \quad \text{for some constant } b$$

$$\iff D = a p^c$$

for some constant $a \; (= e^b)$.

Note that a should be ≥ 0 (which it is for $b \in \mathbb{R}$) so that $D \geq 0$, and c should be ≤ 0 so that

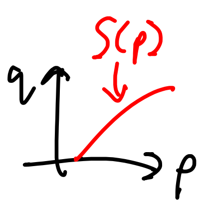
D is a decreasing function of p (in accordance with the law of demand).

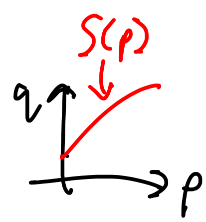
c) Now, Student B claims that since a linear supply function has constant slope, it also exhibits constant price elasticity. Under which conditions is Student B correct? Justify your answer.

Linear supply function $S(p)$ means

$$S(p) = a + bp$$

for some constants a, b with $a \in \mathbb{R}$, $b \geq 0$ and $p \geq -a/b \; (>0)$ if $a < 0$, $p \geq 0 \; (> -a/b)$ if $a > 0$, so $p \geq \max(0, -a/b)$ (whether $a<0$ or $a>0$).

(want: 

or 

)

$a < 0$,
$S(p) = 0$ for $p = -a/b > 0$

$a > 0$. $S(0) > 0$.
Seller can supply a certain amount for free. Is this realistic?

Then

$$\varepsilon_s = S'(p) \cdot \frac{p}{S} = \frac{bp}{a+bp}$$

So $\varepsilon_s = c \iff \dfrac{bp}{a+bp} = c$

$\Leftrightarrow \quad a = 0 \quad \text{or} \quad b = 0$

Note, in this case we can have $a = 0$ with $b \neq 0$.

d) In case you agree with Student B, are there other possible supply functions that exhibit a constant price elasticity? If you disagree with Student B, are there alternatives to linear supply functions that exhibit a constant price elasticity?

As for (b), get

$$S(p) = a p^c$$

where now $a \geq 0$ and $c \geq 0$ so that $S(p)$ is a non-negative and non-decreasing function of p (the latter being in accordance with the law of supply).

3. **(Quasi-Concavity)** We have seen in the lecture that one of the assumptions of a production function is quasi-concavity. If $D \subseteq \mathbb{R}^n$ is a convex subset of $\mathbb{R}^n$, we say that a function $f\colon D \to \mathbb{R}$ is quasi-concave if

$$f((1-\lambda)y + \lambda x) \geq \min\{f(x), f(y)\} \qquad \forall x, y \in D,\ \forall \lambda \in [0,1]. \tag{1}$$

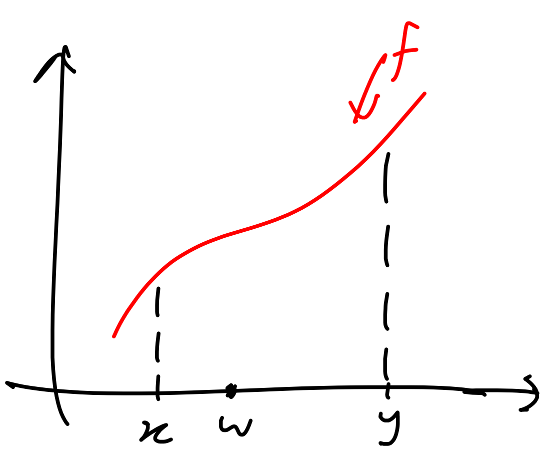
a) Show that for an interval $I \subseteq \mathbb{R}$ ~~and~~ $f\colon I \to \mathbb{R}$ is quasi-concave if and only if

(i) f is monotonically increasing; or
(ii) f is monotonically decreasing; or
(iii) f is monotonically increasing and then monotonically decreasing.

First note that any interval $I \subseteq \mathbb{R}$ is convex.
Also, (1) holds for any f for $x = y$; hence assume $x \neq y$.

Also $w = (1-\lambda)y + \lambda x = y + \lambda(x-y)$ for $\lambda \in [0,1]$ is a point on the (straight) line segment that joins x and y. Without loss of generality, assume that $x < y$.

Now, assume (i)



Then $f(w) \geq f(x) \geq \min\{f(x), f(y)\}\ \forall w \in [x,y]$.

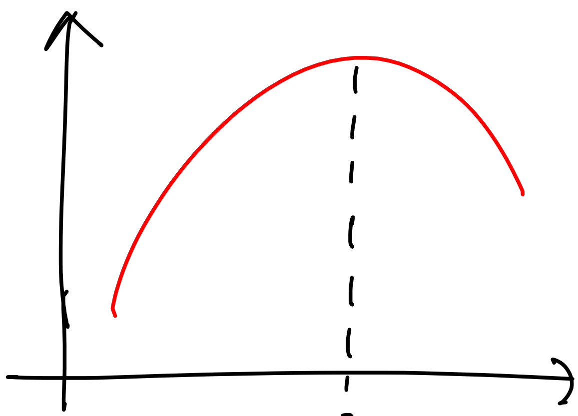
So (1) holds.

Similarly, if (ii) holds then

$$f(w) \geq f(y) \geq \min\{f(x), f(y)\} \quad \forall w \in [x,y].$$

So again (1) holds.

Now suppose (iii) holds.



Let z denote the point where f switches from being monotonic increasing to being monotonic decreasing.

If $x < y \leq z$ then $f(w) \geq f(x) \geq \min\{f(x), f(y)\}$ $\forall w \in [x,y]$ and so (1) holds.

Similarly, if $z \leq x < y$, then $f(w) \geq f(y)$ $\forall w \in [x,y]$ and so again (1) holds.

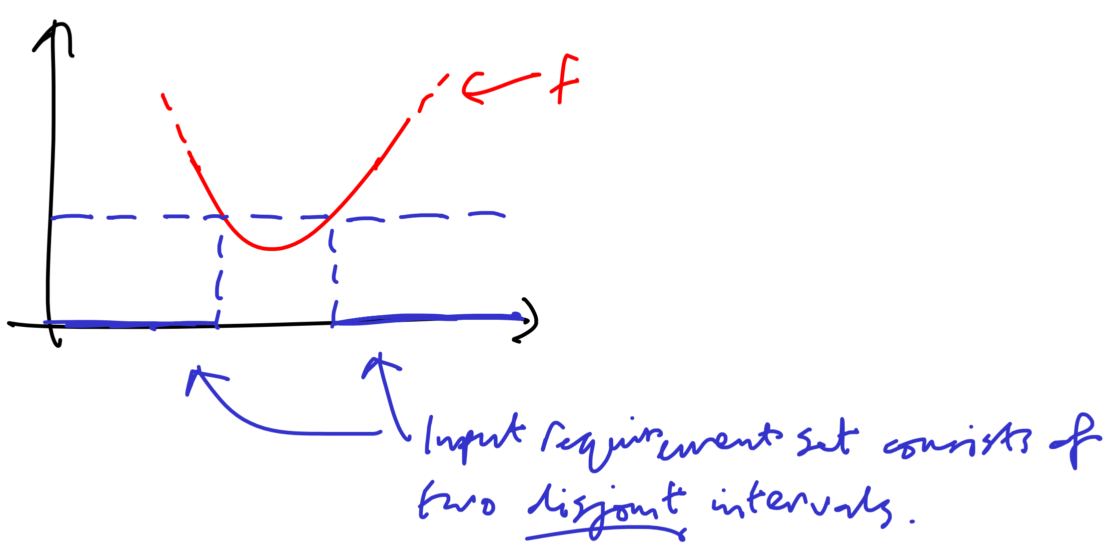
Finally, if $x \leq z \leq y$, then $f(w) \geq f(x)$ $\forall w \in [x,z]$ and $f(w) \geq f(y)$ $\forall w \in [z,y]$. Hence $f(w) \geq \min\{f(x), f(y)\}$ $\forall w \in [x,y]$ and so again (1) holds.

So in all cases, if (iii) holds then so does (i).

Now assume that f does not satisfy either (i), (ii) or (iii). Then there exist $x, y, w \in I$ with $w \in [x, y]$ such that $f(w) < f(x)$ and $f(w) < f(y)$.

But this contradicts (1). //

Note that the input requirement sets for an f such as in the last sketch are <u>not</u> convex:



Note also that it is not true that a monotonic increasing (or decreasing) function of two variables is quasi-concave. For example, consider

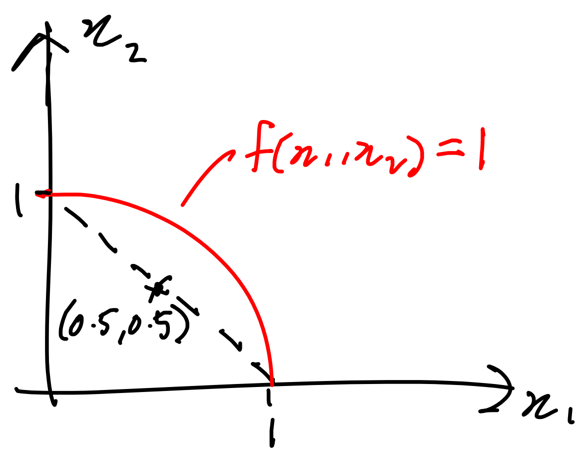
$$f(x_1, x_2) = x_1^2 + x_2^2 .$$

This is monotonic increasing: if $x_i^* \geq x_i$ for $i = 1, 2$, then $f(x_1^*, x_2^*) \geq f(x_1, x_2)$. However,

$$f(1,0) = f(0,1) = 1$$

But $f(0.5, 0.5) = \frac{1}{2} < \min\{f(1,0), f(0,1)\}$

and $(0.5, 0.5) = (1-\frac{1}{2})(1,0) + \frac{1}{2}(0,1)$, so f is not quasi-concave.



← Note: The input requirement sets for this f are not convex.

b) Let $D \subseteq \mathbb{R}^n$ be convex. Show that if a function $f : D \to \mathbb{R}$ is concave, it is also quasi-concave. Show that the reverse implication does not hold by giving a counterexample.

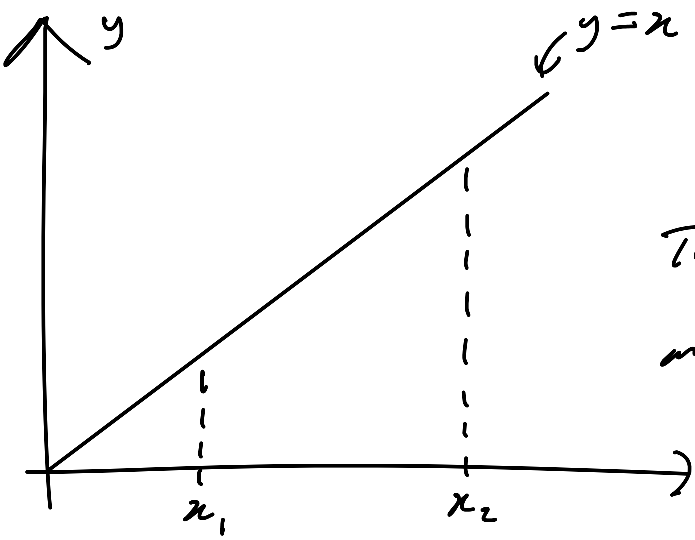
Suppose that f is concave. Then by definition,

$$f\left((1-\lambda)\underline{x} + \lambda \underline{y}\right) \geq (1-\lambda) f(\underline{x}) + \lambda f(\underline{y}) \quad \forall \underline{x}, \underline{y} \in D,$$
$$\forall \lambda \in [0,1]$$

Note that for $\lambda \in [0,1]$, $(1-\lambda) f(\underline{x}) + \lambda f(\underline{y})$ is some value in the interval between $f(\underline{x})$ and $f(\underline{y})$, so it is $\geq \min\{f(\underline{x}), f(\underline{y})\}$. It follows that f is quasi-concave.

Conversely:

First try to picture general quasi-concave and concave functions of a single variable (the latter for simplicity):



Try $x_1 = 0$, $x_2 = 1$.
with $f(0) = 0$ and $f(1) = 1$.

Then $f((1-\lambda)x_1 + \lambda x_2) = f(\lambda)$

and $(1-\lambda) f(x_1) + \lambda f(x_2) = \lambda$

But for any $\lambda \in [0,1]$ and $k > 1$, we have $\lambda^k < \lambda$

In particular,
Consider $f(x) = x^2$ on $\mathbb{R}_{\geq 0}$.

$f(x)$ is monotonically increasing on D and hence (by (a)) is quasi-concave on D. However, take $x = 0$, $y = 1$ and $\lambda = 0.5$. Then

$$f((1-\lambda)x + \lambda y) = f(0.5)$$
$$= 1/4$$

But

$$(1-\lambda)f(x) + \lambda f(y) = 0.5 f(0) + 0.5 f(1)$$
$$= 0.5$$
$$> 1/4$$

So on $\mathbb{R}_{\geq 0}$ $f(x)$ is <u>not</u> concave but <u>is</u> quasi-concave.

c) Let $D \subseteq \mathbb{R}^n$ be convex and $f\colon D \to \mathbb{R}$ a continuously differentiable function. Show that the following assertions are equivalent:

(i) f is quasi-concave.
(ii) For all $y \in \mathbb{R}$ the set $f^{-1}([y,\infty))$ is convex.
(iii) For all $x, y \in D$: If $f(y) \geq f(x)$, then $\nabla f(x)^\top (y-x) \geq 0$.

Proof that (i) $\Rightarrow$ (ii):

f is quasi-concave. Now consider $y \in \mathbb{R}$. We want to show that

$$(1-\lambda)\underline{x} + \lambda \underline{z} \in f^{-1}([y,\infty)) \quad \forall\, \underline{x}, \underline{z} \in f^{-1}([y,\infty)),$$
$$\forall\, \lambda \in [0,1].$$

i.e., that

$$f((1-\lambda)\underline{x} + \lambda \underline{z}) \geq y \quad \forall\, \underline{x}, \underline{z} \in f^{-1}([y,\infty)),$$
$$\forall\, \lambda \in [0,1].$$

But by assumption of the quasi-concavity of f on D,

$$f((1-\lambda)\underline{x} + \lambda \underline{z}) \geq \min\{f(\underline{x}), f(\underline{z})\}$$

$$\geq y$$ since $f(\underline{x}), f(\underline{z})$ are both $\geq y$.

Hence $f^{-1}([y,\infty))$ is convex for all $y \in \mathbb{R}$. //

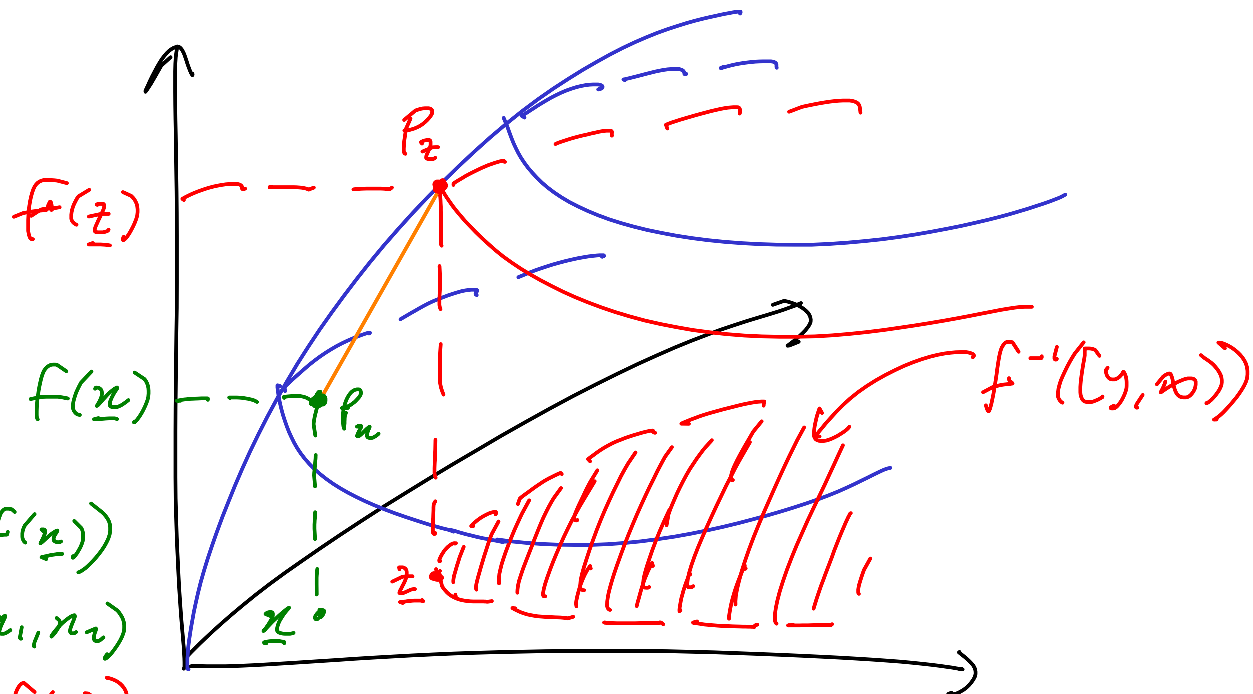
Proof that (ii) $\Rightarrow$ (iii).

Note that $f: D \to \mathbb{R}$ is continuously differentiable means that the derivative of f exists as we approach any point $\underline{x} \in D$, and is the same no matter which direction we approach $\underline{x}$ from.

Consider $\underline{x}, \underline{z} \in D$ with $f(\underline{z}) \geq f(\underline{x})$.

Note that $(\nabla f(\underline{x}))^T(\underline{z}-\underline{x})/|\underline{z}-\underline{x}|$ is the derivative of $f(\underline{x})$ at $\underline{x}$ in the direction of $(\underline{z}-\underline{x})$.

eg for $n=2$



$P_x = (x_1, x_2, f(\underline{x}))$

where $\underline{x} = (x_1, x_2)$

$P_z = (z_1, z_2, f(\underline{z}))$

Assuming (ii), we want to show that

$$(\nabla f(\underline{x}))^T(\underline{z}-\underline{x}) \geq 0$$

i.e., that if we move from $\underline{x}$ in the direction of $\underline{z}-\underline{x}$, then $f(\underline{x})$ should not decrease.

By assumption of (ii), $f^{-1}([f(\underline{x}), \infty))$ and $f^{-1}([f(\underline{z}), \infty))$ are both convex. Note that $\underline{z} \in f^{-1}([f(\underline{x}), \infty))$ since $f(\underline{z}) \geq f(\underline{x})$. Then by the convexity of $f^{-1}([f(\underline{x}), \infty))$,

$$(1-\lambda)\underline{z} + \lambda \underline{x} \in f^{-1}([f(\underline{x}), \infty)) \quad \forall \lambda \in [0,1].$$

$$\Rightarrow \quad f((1-\lambda)\underline{z} + \lambda \underline{x}) \geq f(\underline{x}) \quad \forall \lambda \in [0,1] \quad 1$$

Now note that

$$(\nabla f(\underline{x}))^T \frac{(\underline{z} - \underline{w})}{|\underline{z} - \underline{w}|} = \underbrace{\lim_{|\delta \underline{w}| \to 0} \frac{f(\underline{x} + \delta \underline{w}) - f(\underline{x})}{|\delta \underline{w}|}}$$

This limit exists and is indep. of the direction of $\delta \underline{w}$ by the assumption that f is continuously differentiable on D.

where

$$\begin{aligned} \underline{x} + \delta \underline{w} &= \underline{x} + \varepsilon(\underline{z} - \underline{x}) \\ &= (1-\varepsilon)\underline{x} + \varepsilon \underline{z} \\ &= \lambda \underline{x} + (1-\lambda)\underline{z} \end{aligned}$$

where $\lambda = 1 - \varepsilon$

and we let $\varepsilon \to 0$, or equivalently, $\lambda \to 1$.

Thus,

$$(\nabla f(\underline{x}))^T \frac{(\underline{z} - \underline{w})}{|\underline{z} - \underline{w}|} = \lim_{|\delta \underline{w}| \to 0} \frac{f(\underline{x} + \delta \underline{w}) - f(\underline{x})}{|\delta \underline{w}|}$$

$$= \lim_{\lambda \to 1} \frac{f(\lambda \underline{x} + (1-\lambda)\underline{z}) - f(\underline{x})}{|\delta \underline{v}|}$$

by (1)

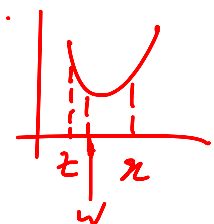
$$\geq 0 \qquad //$$

Proof that (iii) $\Rightarrow$ (i) :

Assume that (i) does not hold. Then $\exists\ \underline{x}, \underline{z} \in D$ and $\lambda_0 \in (0,1)$ such that

$$f((1-\lambda_0)\underline{z} + \lambda_0 \underline{x}) < \min\{f(\underline{z}), f(\underline{x})\}$$

eg.



Without loss of generality, we may assume that $f(\underline{x}) \leq f(\underline{z})$.

Then

$$f((1-\lambda_0)\underline{z} + \lambda_0 \underline{x}) < f(\underline{x})$$

just a rearrangement of argument of f on LHS

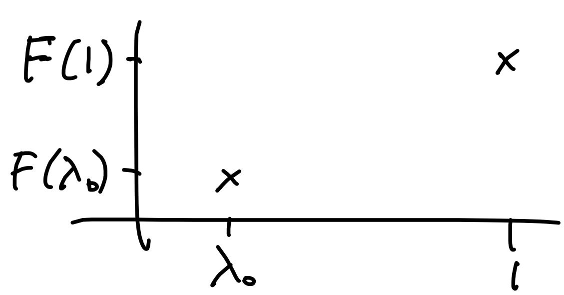
i.e., $$f(\underline{z} + \lambda_0(\underline{x} - \underline{z})) < f(\underline{x})$$

So if we define $F(\lambda) = f(\underline{z} + \lambda(\underline{x} - \underline{z}))$ then $F(\lambda_0) < F(1)$.

Then for some value $\lambda_1 \in [\lambda_0, 1]$, we must have

$$F(\lambda_1) < F(1) \text{ and } F'(\lambda_1) > 0$$

(because:



)

But,

$$F'(\lambda) = \lim_{\delta\lambda \to 0} \frac{F(\lambda+\delta\lambda) - F(\lambda)}{\delta\lambda}$$

$$= \lim_{\delta\lambda \to 0} \frac{f(\underline{z} + \lambda(\underline{x}-\underline{z}) + \delta\lambda(\underline{x}-\underline{z})) - f(\underline{z}+\lambda(\underline{x}-\underline{z}))}{\delta\lambda}$$

$$= \left(\nabla f(\underline{z}+\lambda(\underline{x}-\underline{z}))\right)^T (\underline{x}-\underline{z})$$

Thus

$$\left(\nabla f(\underline{z}+\lambda_1(\underline{x}-\underline{z}))\right)^T (\underline{x}-\underline{z}) > 0$$

& $\lambda_1 > 0$

$$\Rightarrow \left(\nabla f(\underline{z}+\lambda_1(\underline{x}-\underline{z}))\right)^T \lambda_1(\underline{x}-\underline{z}) > 0$$

$$\Rightarrow \left(\nabla f(\underline{z}+\lambda_1(\underline{x}-\underline{z}))\right)^T \left(\underline{z} - (\underline{z}+\lambda_1(\underline{x}-\underline{z}))\right) < 0$$

But $f(\underline{z}+\lambda_1(\underline{x}-\underline{z})) < f(\underline{x}) \leq f(\underline{z})$. So (iii) does not hold either.

//

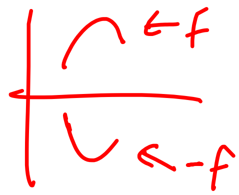
d) We say a function $g\colon D \to \mathbb{R}$ is quasi-convex if $f = -g$ is quasi-concave. State the equivalence in **c)** directly in terms of g.

$g = -f$.

Then we can rewrite statements (i) – (iii) of (c) as follows.

(i). $f((1-\lambda)\underline{x} + \lambda \underline{z}) \geq \min\{f(\underline{x}), f(\underline{z})\} \quad \forall \underline{x}, \underline{z} \in D, \ \forall \lambda \in [0,1]$

becomes $\Big\downarrow$ eg



$-f((1-\lambda)\underline{x} + \lambda \underline{z}) \leq \max\{-f(\underline{x}), -f(\underline{z})\} \quad \forall \underline{x}, \underline{z} \in D, \ \forall \lambda \in [0,1]$

i.e.,

$g((1-\lambda)\underline{x} + \lambda \underline{z}) \leq \max\{g(\underline{x}), g(\underline{z})\} \quad \forall \underline{x}, \underline{z} \in D, \ \forall \lambda \in [0,1]$

(ii) 'For all $y \in \mathbb{R}$, the set $f^{-1}([y, \infty))$ is convex.'

becomes $\Big\downarrow$

$$f^{-1}([y,\infty)) = \{\underline{x} \in D : y \leq f(\underline{x}) < \infty\}$$
$$= \{\underline{x} \in D : -\infty < g(\underline{x}) \leq -y\}$$
$$= g^{-1}((-\infty, -y])$$

'For all $y \in \mathbb{R}$, the set $g^{-1}((-\infty, y])$ is convex.'

(iii) $\forall \underline{x}, \underline{z} \in D$: If $f(\underline{z}) \geq f(\underline{x})$ then $(\nabla f(\underline{x}))^T(\underline{z} - \underline{x}) \geq 0$.

becomes

$\forall \underline{x}, \underline{z} \in D$: If $g(\underline{z}) \leq g(\underline{x})$ then $(\nabla g(\underline{x}))^T(\underline{z} - \underline{x}) \leq 0$.

e) Prove that the following production functions are increasing and quasi-concave.

- Leontief production function $f(x_1, x_2) = \min(ax_1, bx_2)$;
- linear production function $f(x_1, x_2) = ax_1 + bx_2$;
- Cobb-Douglas production function $f(x_1, x_2) = Ax_1^a x_2^b$,

where $A, a, b, x_1, x_2 \geq 0$.

$f(x_1, x_2)$ is increasing if

$x_i^* \geq x_i$ for $i = 1, 2$ $\Rightarrow$ $f(x_1^*, x_2^*) \geq f(x_1, x_2)$

It is straightforward to check that all three of these production functions are increasing. Now check that they are quasi-concave.

Leontief: $f(\underline{x}) = \min\{ax_1, bx_2\}$

Consider $\underline{x}, \underline{y} \in \mathbb{R}^2_{\geq 0}$ and $\lambda \in [0, 1]$. Then

$f((1-\lambda)\underline{x} + \lambda \underline{y}) =$

$$= \min\{a[(1-\lambda)x_1 + \lambda y_1],\ b[(1-\lambda)x_2 + \lambda y_2]\}$$

$\downarrow$ for $\lambda \in [0,1]$, $(1-\lambda)x_i + \lambda y_i \geq \min\{x_i, y_i\}$

for $i = 1, 2$

$$\geq \min\{a\min\{x_1, y_1\},\ b\min\{x_2, y_2\}\}.$$

$$= \min\{\min\{ax_1, ay_1\},\ \min\{bx_2, by_2\}\}$$

$$= \min\{\min\{ax_1, bx_2\},\ \min\{ay_1, by_2\}\}$$

$$= \min\{f(\underline{x}), f(\underline{y})\}. \quad //$$

Linear : $f(\underline{x}) = ax_1 + bx_2$

Consider $\underline{x}, \underline{y} \in \mathbb{R}^2_{\geq 0}$ and $\lambda \in [0,1]$. Then

$$f((1-\lambda)\underline{x} + \lambda \underline{y})$$

$$= a[(1-\lambda)x_1 + \lambda y_1] + b[(1-\lambda)x_2 + \lambda y_2]$$

$$= (1-\lambda)(ax_1 + bx_2) + \lambda(ay_1 + by_2)$$

$$= (1-\lambda) f(\underline{x}) + \lambda f(\underline{y})$$

$\geq \min\{f(\underline{x}), f(\underline{y})\}$ //

<u>Cobb-Douglas</u> : $f(\underline{x}) = A x_1^a x_2^b$

Assume $A \neq 0$.

If $\min\{a,b\} = 0$ then $f(\underline{x})$ is a monotonically increasing function of a single variable and so is quasi-concave by Q3.

So suppose $a, b > 0$. We will show that for all $y \in \mathbb{R}$, the set $f^{-1}([y,\infty))$ is convex. It then follows from Q2(c) that f is quasi-concave.

For $y \leq 0$, $f^{-1}([y,\infty)) = \mathbb{R}^2_{\geq 0}$, which is convex.

Consider now $y > 0$.

$$f(\underline{x}) = y \iff A x_1^a x_2^b = y$$

$$\iff x_2 = \left(\frac{y}{A x_1^a}\right)^{1/b}$$

← Note, $y/(A x_1^a)$ is > 0 for $x_1 > 0$

Take the positive real root, as $x_2 \geq 0$.

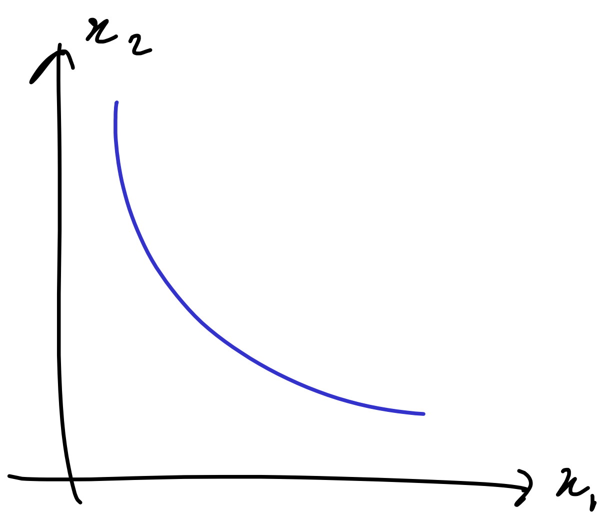
$$= \left(\frac{y}{A}\right)^{1/b} \cdot (x_1)^{-a/b} \equiv g_y(x_1), \text{ say}$$

$$\Rightarrow g_y'(x_1) = -\frac{a}{b}\left(\frac{y}{A}\right)^{1/b} \cdot x_1^{-\frac{a}{b}-1}$$

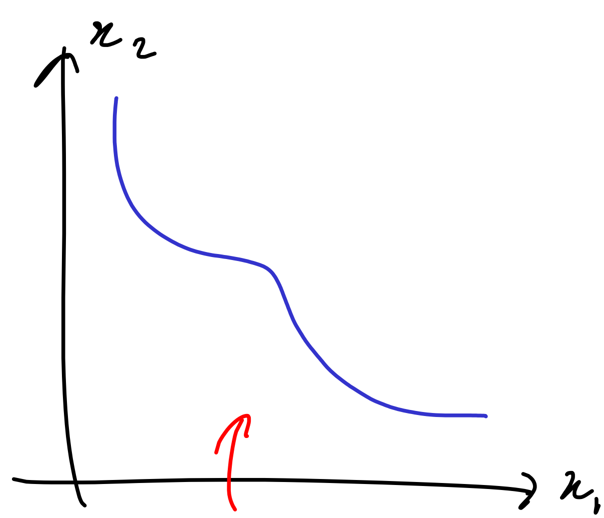
$$g_y''(x_1) = \left(\frac{a}{b}\right)\cdot\left(\frac{a}{b}+1\right)\left(\frac{y}{A}\right)^{1/b} x_1^{-\frac{a}{b}-2}$$

So, since $a, b > 0$ and we are taking positive roots, then $\forall x_1 > 0$ we have $g_y'(x_1) < 0$ and $g_y''(x_1) > 0$.

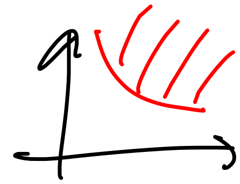
The latter means that $g_y'(x_1)$ is monotonic increasing as $x_1 > 0$ increases. So $g_y'(x_1)$ starts off at $-\infty$ for $x_1 = 0$ and then as x_1 increases, $x_2'(x_1)$ increases monotonically to 0. So $g_y(x_1)$ looks like

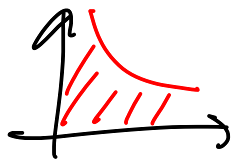


not



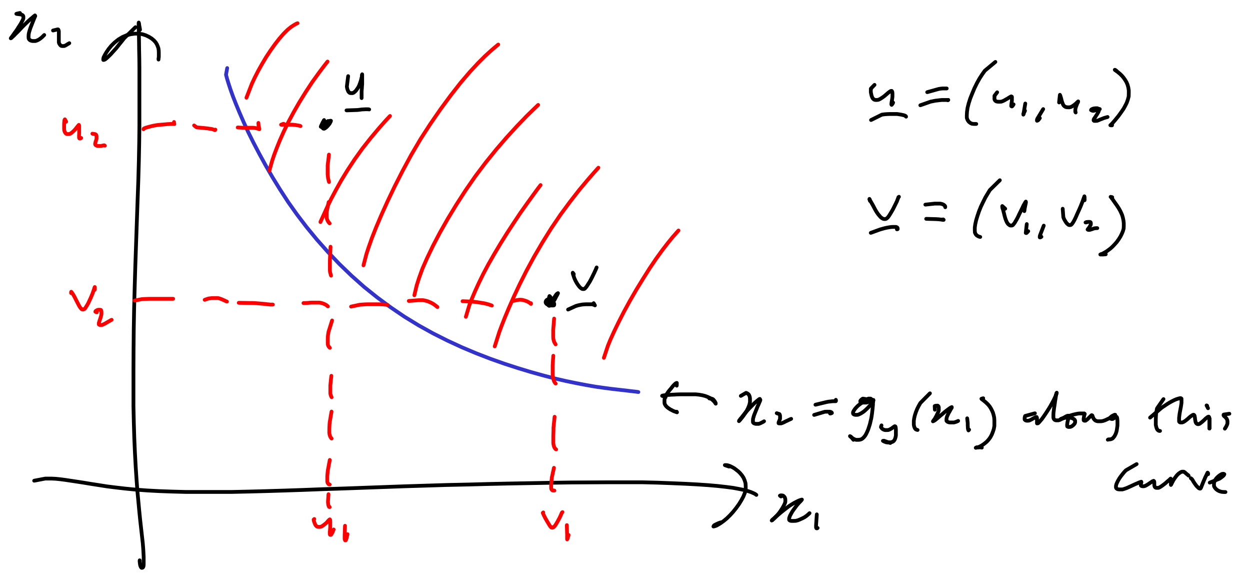
Neither region on either side of this curve is convex.

And $f^{-1}([y,\infty))$ is the region  not

 because $f(\underline{x})$ increases as either x_1 or x_2 increases. Such a region would appear to be convex. One can prove this rigorously using the fact (see the additional result proved below) that

$$g_y(\lambda x_1 + (1-\lambda)x_1^*) \leq \lambda g_y(x_1) + (1-\lambda) g_y(x_1^*) \quad (*)$$

$\forall x_1, x_1^* \in \mathbb{R}_{\geq 0}$, $\forall \lambda \in [0,1]$. Indeed, consider two points $\underline{u}, \underline{v}$ in the region $f^{-1}([y,\infty))$:



Since $\underline{u}, \underline{v} \in f^{-1}([y,\infty))$ then $u_2 \geq g_y(u_1)$ and $v_2 \geq g_y(v_1)$.

Consider now $\underline{w} = (w_1, w_2) = \lambda \underline{u} + (1-\lambda)\underline{v}$. Then

$$
\begin{aligned}
w_2 &= \lambda u_2 + (1-\lambda) v_2 \\
&> \lambda g_y(u_1) + (1-\lambda) g_y(v_1) \\
&\geq g_y(\lambda u_1 + (1-\lambda) v_1) \quad \text{by } \circledast \\
&= g_y(w_1)
\end{aligned}
$$

$\Rightarrow \quad \underline{w} \in f^{-1}([y,\infty))$.

This is true $\forall \lambda \in [0,1]$. Hence $f^{-1}([y,\infty))$ is convex.

//

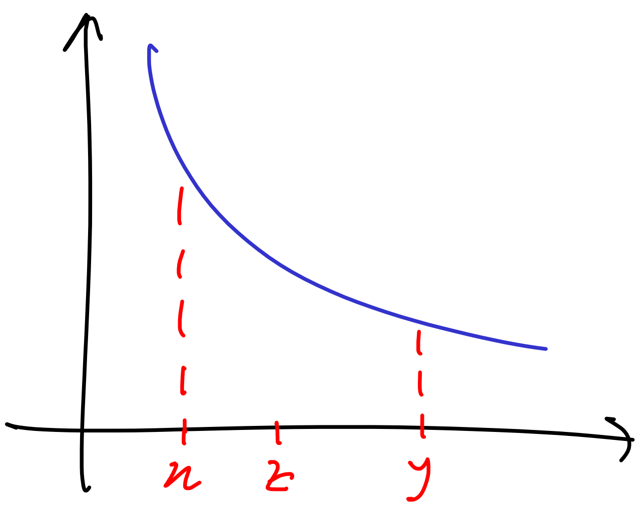
Additional result:

If $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is twice differentiable and satisfies $f''(x) \geq 0$ for all $x \in \mathbb{R}_{\geq 0}$ then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \dagger$$

$\forall\, x, y \in \mathbb{R}_{\geq 0}$, $\forall\, \lambda \in [0,1]$. ($f$ is said to be convex)

Proof of $\dagger$:



Without loss of generality assume that $x < y$.

Let $z = \lambda x + (1-\lambda) y$.

$\exists\, z_1 \in (x, z)$ such that $f'(z_1) = \dfrac{f(z) - f(x)}{z - x}$.

$\exists\, z_2 \in (z, y)$ such that $f'(z_2) = \dfrac{f(y) - f(z)}{y - z}$.

Then

$$\lambda f(x) + (1-\lambda) f(y) =$$

$$= \lambda\left(f(z) - (z-x) f'(z_1)\right)$$

$$+ (1-\lambda)\left(f(z) + (y-z) f'(z_2)\right)$$

$$= f(z) + \underbrace{\lambda(1-\lambda)}_{\substack{\geq 0 \\ \text{since } \lambda \in [0,1]}} \underbrace{(y-x)}_{\geq 0} \underbrace{\left(f'(z_2) - f'(z_1)\right)}_{\substack{\geq 0 \\ \text{since } z_2 \geq z_1 \text{ and} \\ f''(x) \geq 0 \ \forall x \geq 0.}}$$

$$\geq f(z)$$

$$= f(\lambda x + (1-\lambda) y) \quad //$$