Mathematics Year 1, Calculus and Applications I

Midterm Exam, November 22, 2022

1. (a) Consider the function $f(x) = x^x$, $x \geq 0$.

   (i) Find $\lim_{x \to 0+} f(x)$. **3 marks**

   (ii) Calculate $f'(x)$ and find $\lim_{x \to 0+} f'(x)$. **3 marks**

   (iii) Find the minimum of $f(x)$ in the interval $0 \leq x < \infty$. **3 marks**

   (iv) Sketch the graph $y = f(x)$. **3 marks**

   (b) Now consider the function $g(x) = x^x \sin x$, $x \geq 0$.

   (i) Find $\lim_{x \to 0+} g(x)$ and $\lim_{x \to 0+} g'(x)$. **2 marks**

   (iii) Prove that there exists a critical point $x = \xi$, say, of $g(x)$ in the interval $(0, \pi)$ (state any theorems you use but do not prove them). Is the critical point at $x = \xi$ a maximum or a minimum? **3 marks**

   (iv) Prove that $\frac{\pi}{2} < \xi < \pi$ (once again state any theorems you use but do not prove them). **3 marks**

2. Consider the function $f(x) = \frac{x(\alpha^2 - x^2)}{\sqrt{x^2 - 1}}$ where x is real and $\alpha \neq 1$ is a non-negative constant.

   (a) Find the domain of $f(x)$. Is the function even, odd or neither?

   2 marks

   (b) (i) If $\alpha = 2$ show that $f(x)$ decreases monotonically in the interval $1 < x < \infty$.

   4 marks

   (ii) If $\alpha = 1/2$ then there exists a unique critical point of $f(x)$ in the interval $1 < x < \infty$. Find and classify it (there is no need to find the second derivative). **4 marks**

   (iii) Sketch $f(x)$ for all $x > 1$ in the cases $\alpha = 2$ and $\alpha = 1/2$.

   2 marks

   (c) If $\alpha > 1$, use integration by parts to calculate the improper integral $\int_1^\alpha f(x) dx$.

   4 marks

   (d) Show that in the case $\alpha > 1$ there exists at least one point $1 < x < \infty$ where $f''(x) = 0$. Do this without explicitly calculating $f''(x)$ for general x. [Hint: Consider the behaviour of $f(x)$ near $x = 1$ and for x very large.]
   4 marks