Mathematics Year 1, Calculus and Applications I

Midterm Exam, November 26 2020

You have 40 minutes to complete the paper

The question is worth 50 marks.

This question is concerned with the function

$$f(x) = [x] + (x - [x])^{1/2} \qquad x \geq 0. \tag{1}$$

It is understood that $[x]$ denotes the integer part of a real number x.

1. Is $f(x)$ continuous for all $x > 0$? Justify your answer - there is no need to provide $\varepsilon - \delta$ proofs. **5 marks**

2. Is $f(x)$ differentiable for all $x > 0$? Justify your answer with a proof using the definition of the derivative as a limit. **5 marks**

3. Sketch $f(x)$ for $x \geq 0$ and $g(x) = f'(x)$ for $x > 0$. **6 marks**

4. Calculate $\int_0^x f(s)ds$ for arbitrary x. [Give a formula involving x alone. Hint: Think of the area under the curve.] **8 marks**

5. A bug starts at the origin $(0, 0)$ and moves in the positive $x-$direction by walking on the curve defined by $f(x)$. Its speed is initially V, but every time it completes a climb of unit vertical distance, its speed decreases by a factor of 2 due to exhaustion. How long will it take it to get to the point (N, N) where N is a positive integer. **8 marks**

6. The region between the function $y = f(x)$ and the $x-$axis is revolved about the $x-$axis to produce a solid of revolution $\mathcal{D}_1$, and also revolved about the $y-$axis to produce a solid of revolution $\mathcal{D}_2$. The corresponding surfaces of revolution are denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$.

   (a) Consider the interval $(k-1) \leq x \leq k$ where $k \geq 1$ is a positive integer, write down integral expressions for $\mathcal{S}_1$, $\mathcal{S}_2$ and show that they can be expressed as

   $$\mathcal{S}_1 = \int_0^1 2\pi \left[(k-1) + s^{1/2}\right] \left[1 + \frac{1}{4s}\right]^{1/2} ds$$
   $$\mathcal{S}_2 = \int_0^1 2\pi[(k-1) + s] \left[1 + \frac{1}{4s}\right]^{1/2} ds$$

   Without calculating the integrals explicitly, prove that

   $$\mathcal{S}_1 > \mathcal{S}_2.$$

   [Hint: Consider the quantity $\mathcal{S}_1 - \mathcal{S}_2$ and prove that it is positive.] **8 marks**

   (b) Explain this inequality intuitively using geometry? **2 marks**

(c) Now calculate the integrals in 6(a) above explicitly and hence give formulas for $\mathcal{S}_1$ and $\mathcal{S}_2$ for the interval $0 \leq x \leq M$ where M is a positive integer. Confirm that $\mathcal{S}_1 > \mathcal{S}_2$.

[You may leave part of the answer for $\mathcal{S}_2$ in terms of $\mathcal{I} = \int_0^1 (1 + 4t^2)^{3/2} dt$.] **8 marks**

Solutions

1. The only points where things could go wrong are $x = 1, 2, \ldots$. Suppose $x = n + \varepsilon$ where $n \geq 1$ is an integer. If $\varepsilon > 0$ we have $f(x) = n + (n+\varepsilon-n)^{1/2} = n + \varepsilon^{1/2} \to n$ as $\varepsilon \to 0+$. If $\varepsilon < 0$, then $f(x) = (n-1) + (n + \varepsilon - (n-1))^{1/2} \to n$ as $\varepsilon \to 0-$. Hence the function is continuous. **5 marks**

2. The function is not differentiable. Can see this for the point $x = 1$, for example. Consider
$$f'(1) := \lim_{h\to 0} \frac{f(1+h) - f(1)}{h},$$
and calculate
$$\lim_{h\to 0, h>0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0,h>0} \frac{1 + (1+h-1)^{1/2} - 1}{h} = \lim_{h\to 0,h>0} \frac{1}{h^{1/2}} = \infty,$$
and
$$\lim_{h\to 0,h<0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0,h<0} \frac{(1+h)^{1/2} - 1}{h} = \lim_{h\to 0,h<0} \frac{[(1+h)^{1/2}-1][(1+h)^{1/2}+1]}{h[(1+h)^{1/2}+1]} = \frac{1}{2}$$
5 marks

3. The function is a step function and is defined in a piecewise manner for $k-1 \leq x \leq k$, $k = 1, 2, \ldots$, i.e.
$$f(x) = (k-1) + [x - (k-1)]^{1/2}, \qquad k-1 \leq x \leq k,$$
and
$$g(x) := f'(x) = \frac{1}{2}\frac{1}{[x-(k-1)]^{1/2}}, \qquad k-1 < x \leq k.$$
Clearly $g \to \infty$ as $x \to (k-1)$ from above. The derivative as the point $x = k$ is approached from below is defined and equal to $1/2$. A plot is given in figure 1.
6 marks - 3 for each plot

4. Think of the area under the curve as suggested. Given x, define $n = [x]$ so that $n \geq 0$. Then
$$\begin{aligned}
\int_0^x f(s)ds &= \int_0^n f(s)ds + \int_n^x f(s)ds \\
&= n\int_0^1 x^{1/2}dx + (1 + 2 + \ldots + (n-1)) + \int_n^x (n + (s-n)^{1/2})ds \\
&= \frac{2}{3}n + \sum_{i=1}^{n-1} i + \left[ns + \frac{2}{3}(s-n)^{3/2}\right]_n^x \\
&= \frac{2}{3}n + \frac{1}{2}n(n-1) + n(x-n) + \frac{2}{3}(x-n)^{3/2} \\
&= \frac{2}{3}[x] + \frac{1}{2}[x]([x]-1) + [x](x-[x]) + \frac{2}{3}(x-[x])^{3/2}
\end{aligned}$$
8 marks

5. What we really need is the length of the curve between any interval $[k-1,k]$ as above. The length is the same so enough to calculate it once for $0 \le x \le 1$. From the formula we have

$$L = \int_0^1 \sqrt{1+\frac{1}{4x}}dx = \int_0^1 \frac{1}{2}x^{-1/2}(1+4x)^{1/2}dx$$

Substitute $x^{1/2} = t$, i.e. $\frac{1}{2}x^{-1/2}dx = dt$, i.e.

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+4t^2}dt \overset{2t=\sinh\theta}{=} \int_0^{\sinh^{-1}(1/2)} \sqrt{1+\sinh^2\theta}\frac{1}{2}\cosh\theta\, d\theta \\
&= \int_0^{\sinh^{-1}(1/2)} \frac{1}{2}\cosh^2\theta d\theta = \int_0^{\sinh^{-1}(1/2)} \frac{1}{4}(1+\cosh 2\theta)d\theta \\
&= \frac{1}{4}\left(\theta_0 + \frac{1}{2}\sinh(2\theta_0)\right) \qquad \text{where} \quad \theta_0 = \sinh^{-1}(1/2).
\end{aligned}$$

Fine to leave it as this - it can in fact be written in terms of logs but there is no need.

Having found the length and since we know the speed, we can write down the time taken

$$T = \frac{L}{V} + \frac{2L}{V} + \frac{2^2L}{V} + \ldots + \frac{2^{N-1}L}{V} = \frac{L}{V}\left(1+2+2^2+\ldots+2^{N-1}\right).$$

[Can sum the geometric series to find $T = \frac{L}{V}(2^N - 1)$ but that is not necessary.]

8 marks

6. (a) We can use the formulas we derived in class to find

$$\begin{aligned}
\mathcal{S}_1 &= \int_{k-1}^{k} 2\pi\left[(k-1)+(x-(k-1))^{1/2}\right]\left[1+\frac{1}{4(x-(k-1))}\right]^{1/2} dx \\
\mathcal{S}_2 &= \int_{k-1}^{k} 2\pi x\left[1+\frac{1}{4(x-(k-1))}\right]^{1/2} dx
\end{aligned}$$

Can transform these by letting $x = (k-1)+s$ so that the integrals become:

$$\begin{aligned}
\mathcal{S}_1 &= \int_0^1 2\pi\left[(k-1)+s^{1/2}\right]\left[1+\frac{1}{4s}\right]^{1/2} ds \\
\mathcal{S}_2 &= \int_0^1 2\pi[(k-1)+s]\left[1+\frac{1}{4s}\right]^{1/2} ds
\end{aligned}$$

Now

$$\mathcal{S}_1 - \mathcal{S}_2 = 2\pi\int_0^1 (s^{1/2}-s)\left[1+\frac{1}{4s}\right]^{1/2} ds$$

The function $h(s) = s^{1/2} - s$ satisfies $h(s) \ge 0$ for $0 \le s \le 1$ since $h(s) = s^{1/2}(1-s^{1/2})$ and $0 \le s^{1/2} \le 1$ on $[0,1]$. Hence the integrand is positive hence $\mathcal{S}_1 > \mathcal{S}_2$. **8 marks**

(b) The inequality follows from geometrical intuition as follows. For $\mathcal{S}_1$ the radial distance from the $x-$axis to a given point on the curve, is larger than the corresponding distance to the same point from the $y-$axis on $\mathcal{S}_2$. As a result, the area swept out in making $\mathcal{S}_1$ is larger than that for $\mathcal{S}_2$.

2 marks

(c) Explicit calculations of the integrals. Recall the $L = \int_0^1 \sqrt{1+4t^2}dt$ that was found in 5 above. This appears again and will be used as L. Start with $\mathcal{S}_1$,

$$\begin{aligned}
\mathcal{S}_1 &= \int_0^1 2\pi \frac{\left[(k-1)+s^{1/2}\right]}{2s^{1/2}}[1+4s]^{1/2}ds \overset{s^{1/2}=t}{=} 2\pi \int_0^1 \left[(k-1)+t\right]\sqrt{1+4t^2}dt \\
&= 2\pi(k-1)L + 2\pi\left[(1+4t^2)^{3/2}\frac{2}{3}\cdot\frac{1}{8}\right]_0^1 \\
&= 2\pi(k-1)L + \frac{\pi}{6}(5^{3/2}-1)
\end{aligned}$$

and using the same ideas and substitution $s^{1/2} = t$ we find

$$\begin{aligned}
\mathcal{S}_2 &= 2\pi \int_0^1 \left[(k-1)+t^2\right]\sqrt{1+4t^2}\,dt \\
&= 2\pi(k-1)L + 2\pi\int_0^1 t\cdot t(1+4t^2)^{1/2}dt \\
&= 2\pi(k-1)L \overset{\text{by parts}}{+} 2\pi\left\{\left[t\cdot(1+4t^2)^{3/2}\frac{2}{3}\cdot\frac{1}{8}\right]_0^1 - \frac{1}{12}\int_0^1 (1+4t^2)^{3/2}dt\right\} \\
&= 2\pi(k-1)L + \frac{\pi}{6}\left(5^{3/2} - \int_0^1 (1+4t^2)^{3/2}dt\right) \\
&= 2\pi(k-1)L + \frac{\pi}{6}\left(5^{3/2} - \mathcal{I}\right)
\end{aligned}$$

[Of course $\mathcal{I}$ can be found as $\int_0^{\theta_0} \frac{1}{2}\cosh^4\theta d\theta$ which can be found using elementary methods but not needed here.]

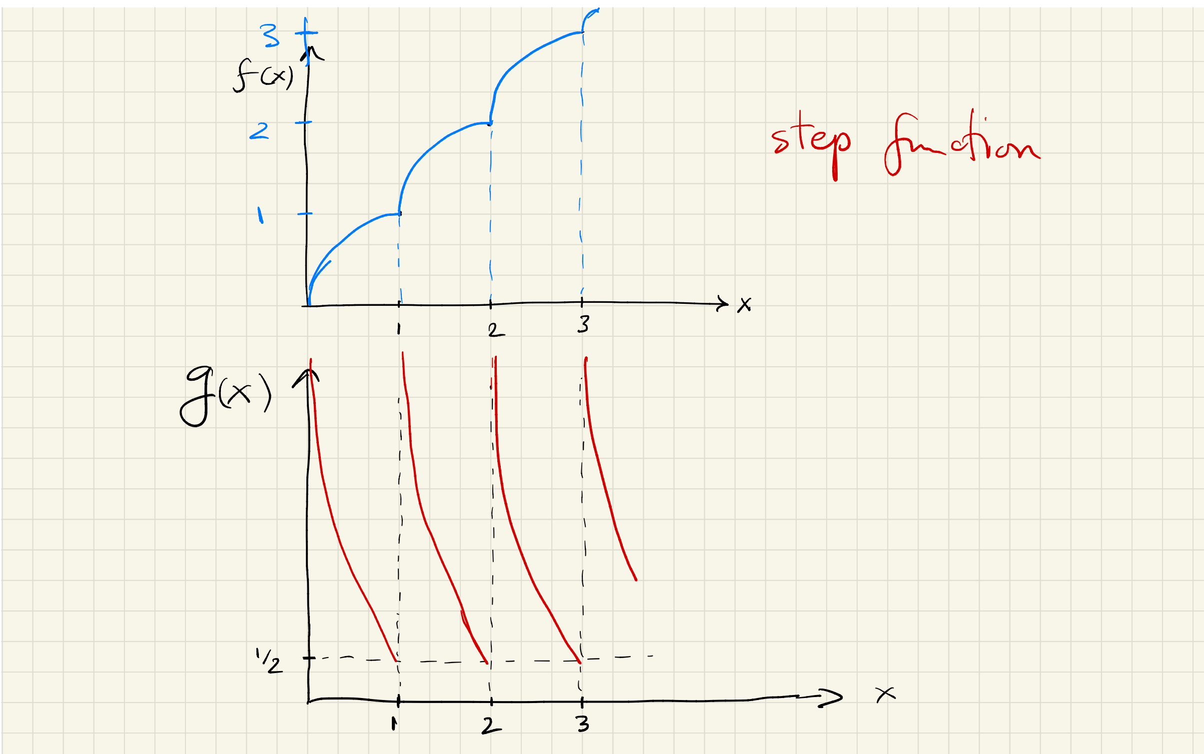
Clearly $\mathcal{I} > 1$ and hence $\mathcal{S}_1 > \mathcal{S}_2$ as proved earlier.

Finally, we need to sum all areas from $x = 0$ to $x = M$, or equivalently from $k = 1$ to $k = M$, to obtain

$$\mathcal{S}_1^{total} = 2\pi L\sum_{k=1}^{M}(k-1) + \frac{\pi}{6}(5^{3/2}-1)M = \pi LM(M-1) + \frac{\pi}{6}(5^{3/2}-1)M,$$

$$\mathcal{S}_1^{total} = 2\pi L\sum_{k=1}^{M}(k-1) + \frac{\pi}{6}(5^{3/2}-\mathcal{I})M = \pi LM(M-1) + \frac{\pi}{6}(5^{3/2}-\mathcal{I})M.$$

8 marks



f(x)
3
2
1
x
1
2
3
step function
g(x)
1/2
x
1
2
3