Imperial College London

MATH97050 MATH97161

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Riemannian Geometry

Date: Wednesday, 26 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider the topological manifold $\mathbb{R}$. Define $\psi, \varphi \colon \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = x, \qquad \varphi(x) = \begin{cases} 2x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Justify your answers to the following questions.

(i) Is $\{(\mathbb{R}, \psi)\}$ a smooth atlas for $\mathbb{R}$? (1 mark)

(ii) Is $\{(\mathbb{R}, \varphi)\}$ a smooth atlas for $\mathbb{R}$? (1 mark)

(iii) Is $\{((0,\infty), \varphi|_{(0,\infty)}), ((-\infty, 0), \varphi|_{(-\infty,0)})\}$ a smooth atlas for $\mathbb{R}$? (2 marks)

(iv) Is $\{(\mathbb{R}, \psi), (\mathbb{R}, \varphi)\}$ a smooth atlas for $\mathbb{R}$? (2 marks)

(v) Is $\{(\mathbb{R}, \psi), ((-\infty, 0), \varphi|_{(-\infty,0)})\}$ a smooth atlas for $\mathbb{R}$? (2 marks)

(b) Consider the smooth manifold $\mathbb{R}^2$, and let (x, y) denote the standard Cartesian coordinates. Let $V = x\partial_x + y\partial_y$. Compute the Lie derivatives, $\mathcal{L}_V X$, $\mathcal{L}_V Y$ and $\mathcal{L}_V T$, of the following tensor fields along V,

$$X = \partial_x + \partial_y, \qquad Y = \cos y \partial_x + \sin x \partial_y, \qquad T = y^2 dx \otimes dx + x dx \otimes dy.$$

You may use any results from the lectures, without proof, provided you state them clearly. (4 marks)

(c) Briefly justify your answers to the following questions.

(i) Is the vector field ∂_x on $(0, 1)$ complete? (2 marks)

(ii) Is every nontrivial vector field (i.e. every vector field which does not vanish identically) on $(0, 1)$ incomplete? (2 marks)

(iii) Is every vector field on $\mathbb{R}$ complete? (2 marks)

(iv) Is every vector field on S^n complete? (2 marks)

(Total: 20 marks)

2. (a) Let (x, y, z) denote the standard Cartesian coordinates on $\mathbb{R}^3$. Say whether each of the following $(0, 2)$ tensor fields on $\mathbb{R}^3$ are Riemannian metrics on $\mathbb{R}^3$. Justify your answers.

(i) $g = dx \otimes dx + dy \otimes dy + dx \otimes dy + dz \otimes dz$. (2 marks)

(ii) $g = \sin^2(xz) dx \otimes dx + \cos^2(xz) dy \otimes dy + dz \otimes dz$. (2 marks)

(ii) $g = dx \otimes dx + 2dy \otimes dy - dz \otimes dy - dy \otimes dz + dz \otimes dz$. (2 marks)

(b) Consider the metric

$$g = \frac{4}{(1 + x^2 + y^2)^2}(dx^2 + dy^2),$$

on $\mathbb{R}^2$, where (x, y) are standard Cartesian coordinates.

(i) Compute the Christoffel symbols of the Levi-Civita connection of g with respect to the frame $\{\partial_x, \partial_y\}$. (4 marks)

(ii) Show that the x axis, $\{y = 0\}$, is the image of a geodesic of g. Hint: It suffices to find a surjective function $f \colon I \to \mathbb{R}$, for some open interval $I \subset \mathbb{R}$, such that $\gamma(t) = (f(t), 0)$ satisfies the geodesic equations. You may use, without proof, the fact that $\frac{d}{dx}(\tan^{-1}(x)) = (1 + x^2)^{-1}$.

(5 marks)

(iii) Consider now the metric g on $\mathbb{R}^2 \setminus \{0\}$. Show that the map $f \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ defined by

$$f(x, y) = \frac{(x, y)}{x^2 + y^2},$$

is an isometry of $(\mathbb{R}^2 \setminus \{0\}, g)$. (5 marks)

(Total: 20 marks)

3. (a) Which one of the following three Riemannian manifolds is not geodesically complete? Very briefly justify your answer.

   * The torus

$$\{((2+\cos\theta)\cos\phi,(2+\cos\theta)\sin\phi,\sin\theta) \mid \theta,\phi\in[0,2\pi)\}\subset\mathbb{R}^3,$$

     with the induced metric from $(\mathbb{R}^3, g_{Eucl})$.

   * The annulus

$$\{(x,y)\in\mathbb{R}^2 \mid 1<x^2+y^2<2\}\subset\mathbb{R}^2,$$

     with the induced metric from $(\mathbb{R}^2, g_{Eucl})$.

   * The circle

$$\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}\subset\mathbb{R}^2,$$

     with the induced metric from $(\mathbb{R}^2, g_{Eucl})$.

(3 marks)

(b) Let $(\mathcal{M},g)$ be a Riemannian manifold. Consider normal coordinates $\{x^i\}$ centred at $p\in\mathcal{M}$. Show that the Christoffel symbols of $(\mathcal{M},g)$ with respect to the local frame $\{\partial_{x^i}\}$ vanish at p, i.e. $\Gamma^i_{jk}(p)=0$. (6 marks)

(c) Let g be a metric on $\mathbb{R}^n$ such that every geodesic of $(\mathbb{R}^n,g)$ is a constant speed straight line. Is $(\mathbb{R}^n,g)$ necessarily isometric to $(\mathbb{R}^n,g_{Eucl})$? Prove or give a counterexample. You may use any results from the lectures provided you state them clearly. (5 marks)

(d) Let $(\mathcal{M},g)$ be a Riemannian manifold and let $K\in\mathfrak{X}(\mathcal{M})$ be a Killing vector. Show that, for any geodesic $\gamma\colon(a,b)\to\mathcal{M}$, the inner product $g(K,\dot\gamma)$ is conserved along γ. (6 marks)

(Total: 20 marks)

4. (a) Define $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\} \subset \mathbb{R}^3$. Given a Riemannian metric g on B, let R denote the Riemann curvature tensor of g. Justify your answers to the following questions. You may use, without proof, any symmetries of R.

(i) Does there exist a Riemannian metric g on B such that, for all $X, Y, Z \in \mathfrak{X}(B)$,

$$R(X,Y)Z = 0?$$

(2 marks)

(ii) Does there exist a Riemannian metric g on B such that, for all $X, Y, Z \in \mathfrak{X}(B)$,

$$R(X,Y)Z = g(X,Y)Z?$$

(2 marks)

(iii) Does there exist a Riemannian metric g on B such that, for all $X, Y, Z \in \mathfrak{X}(B)$,

$$R(X,Y)Z = Z?$$

(3 marks)

(iv) Does there exist a Riemannian metric g on B such that, for all $X, Y, Z \in \mathfrak{X}(B)$,

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y?$$

(3 marks)

(b) Let $(\mathcal{M}, g)$ be a Riemannian manifold whose sectional curvatures satisfy the upper bound

$$\kappa(\Pi_p) \leq \lambda \qquad \text{for all } p \in \mathcal{M} \text{ and all 2-planes } \Pi_p \subset T_p\mathcal{M},$$

for some constant $\lambda > 0$. Let $\gamma\colon [0, \frac{\pi}{2\sqrt{\lambda}}] \to \mathcal{M}$ be a unit speed geodesic. Show that there are no points along γ conjugate to $\gamma(0)$. Hint: if c is a constant then solutions of the ordinary differential inequality $f''(s) + cf(s) \geq 0$ can be compared to solutions of the ordinary differential equation $h''(s) + ch(s) = 0$, if $f(0) = h(0) = 0$ and $f'(0) = h'(0)$, by considering $(f/h)'(s)$.

(10 marks)

(Total: 20 marks)

5. (a) Let $\mathcal{M}$ be a submanifold of $(\mathbb{R}^n, g_{Eucl})$ whose second fundamental form identically vanishes. Is $\mathcal{M}$, with the induced metric, necessarily flat? Prove or give a counterexample.

(4 marks)

(b) Consider a Riemannian manifold $(\mathcal{M}, g)$ and an oriented submanifold $\overline{\mathcal{M}}$ with $\dim\overline{\mathcal{M}} = \dim\mathcal{M} - 1$. Define the second fundamental form

$$k(X, Y) = -g(\nabla_X n, Y),$$

where n is the oriented unit normal to $\overline{\mathcal{M}}$. Show that

$$\overline{\nabla}_X k(Y, Z) - \overline{\nabla}_Y k(X, Z) = Rm(X, Y, Z, n),$$

for all $X, Y, Z \in \mathfrak{X}(\overline{\mathcal{M}})$, where $\overline{\nabla}$ is the Levi-Civita connection of the induced metric on $\overline{\mathcal{M}}$ and Rm is the Riemann curvature tensor of $(\mathcal{M}, g)$.

(6 marks)

(c) Consider the paraboloid

$$P = \{(u, v, u^2 + v^2) \mid (u, v) \in \mathbb{R}^2\},$$

as a submanifold of $(\mathbb{R}^3, g_{Eucl})$.

(i) Show that, in the ambient (x, y, z) Cartesian coordinates of $\mathbb{R}^3$, the upward unit normal to P takes the form

$$n = \frac{1}{1 + 4x^2 + 4y^2}\Big(-2x\partial_x - 2y\partial_y + \partial_z\Big).$$

Hint: first compute the vectors ∂_u and ∂_v in the ambient Cartesian coordinates.

(4 marks)

(ii) Compute the second fundamental form $k(X, Y) = -g(\nabla_X n, Y)$ of P in the (u, v) coordinate system for P. (6 marks)

(Total: 20 marks)

Riemannian geometry exam 2021 – solutions

1. (a) (i) Yes: $\mathbb{R}$ is an open cover of $\mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism. The compatibility is trivial since there is only one chart.

Seen [1 mark].

(ii) Yes: $\mathbb{R}$ is an open cover of $\mathbb{R}$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism. The compatibility is trivial since there is only one chart.

Unseen [1 mark].

(iii) No: $\{(-\infty, 0), (0, \infty)\}$ is not an open cover of $\mathbb{R}$.

Unseen [2 marks].

(iv) No: $\varphi \circ \psi^{-1}\colon \mathbb{R} \to \mathbb{R}$ is not a smooth diffeomorphism.

Unseen [2 marks].

(v) Yes: $\{(-\infty, 0), \mathbb{R}\}$ is an open cover of $\mathbb{R}$. The maps $\psi\colon \mathbb{R} \to \mathbb{R}$ and $\varphi\,|_{(-\infty,0)}\colon (-\infty, 0) \to (-\infty, 0)$ are homoemorphisms (where $(\varphi\,|_{(-\infty,0)})^{-1} = x/2$). Finally, the map

$$\psi \circ (\varphi\,|_{(-\infty,0)})^{-1}(x) = x/2,$$

is a diffeomorphism from $(-\infty, 0) \to (-\infty, 0)$ with smooth inverse $2x$.

Unseen [2 marks].

(b) Since X and Y are vector fields one can use the fact that

$$\mathcal{L}_V Z = [V, Z],$$

for any vector field Z. A simple computation gives

$$\mathcal{L}_V X = -\partial_x - \partial_y, \qquad \mathcal{L}_V Y = -(y \sin y + \cos y)\partial_x + (x \cos x - \sin x)\partial_y.$$

For $\mathcal{L}_V T$ it is easiest to use the component form of Lie derivative on $(0, 2)$ tensor fields from the lectures:

$$(\mathcal{L}_V T)_{ij} = V(T_{ij}) + \partial_i V^k\, T_{kj} + \partial_j V^k\, T_{ik}.$$

A simple computation then gives

$$(\mathcal{L}_V T)_{xx} = 4y^2, \qquad (\mathcal{L}_V T)_{xy} = 3x, \qquad (\mathcal{L}_V T)_{yx} = (\mathcal{L}_V T)_{yy} = 0,$$

and so

$$\mathcal{L}_V T = 4y^2 dx \otimes dx + 3x dx \otimes dy.$$

Unseen [4 marks].

(c) (i) No: The curve $\gamma\colon (0, 1) \to (0, 1)$ defined by $\gamma(t) = t$ is an integral curve of ∂_x which cannot be extended to an integral curve $\gamma\colon \mathbb{R} \to (0, 1)$.

Unseen [2 marks].

(ii) No: Given $\varepsilon > 0$ small, any smooth vector field X which satisfies $X(t) = 0$ for $0 < t < \varepsilon$ and $X(t) = 0$ for $1 - \varepsilon < t < 1$ is complete.

Seen as exercise [2 marks].

(iii) No: Let $X = x^2\partial_x$. The curve $\gamma\colon (-\infty, 1) \to \mathbb{R}$ defined by $\gamma(t) = \frac{1}{1-t}$ is an integral curve of X which cannot be extended to a larger interval.

Seen as exercise [2 marks].

(iv) Yes: Let X be a vector field on S^n and let $\gamma\colon (T_-, T_+) \to S^n$ be a maximal integral curve of X. If $T_+ < \infty$ then the Cauchy–Lipschitz Theorem implies that $\gamma(t)$ leaves every compact set as $t \to T_+$. Since S^n is compact this is not possible. So $T_+ = \infty$ and, by a similar argument, $T_- = -\infty$.

Unseen [2 marks].

2. (a) (i) No: g is not symmetric.

Unseen [2 marks].

(ii) No: $g|_{(0,0,0)} = dy \otimes dy + dz \otimes dz$ is clearly not positive definite, e.g. $g|_{(0,0,0)}(\partial_x, \partial_x) = 0$.

Unseen [2 marks].

(iii) Yes: g is symmetric and, for $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$,

$$g(X, X) = (X^1)^2 + (X^2)^2 + (X^2 - X^3)^2,$$

is a positive definite expression.

Unseen [2 marks].

(b) (i) A computation, using the standard formula for the Christoffel symbols, gives

$$\Gamma^x_{xx} = -\Gamma^x_{yy} = \Gamma^y_{xy} = -\frac{2x}{1 + x^2 + y^2},$$

and

$$\Gamma^x_{xy} = \Gamma^y_{yy} = -\Gamma^y_{xx} = -\frac{2y}{1 + x^2 + y^2}.$$

Unseen [4 marks].

(ii) The geodesic equations take the form

$$\ddot{\gamma}^i(t) + \dot{\gamma}^j(t)\dot{\gamma}^k(t)\Gamma^i_{jk}(\gamma(t)) = 0.$$

Look for a geodesic of the form $\gamma(t) = (f(t), 0)$, for some function f. The second geodesic equation (with $i = y$) becomes trivial since $\Gamma^y_{xx}(\gamma(t)) = 0$ for all t. The first geodesic equation, using the above expression for Γ^x_{xx}, becomes,

$$f''(t) - \frac{2f(t)}{1 + (f(t))^2}(f'(t))^2 = 0.$$

It follows that

$$(\log f'(t))' = (\log(1 + f(t)^2))',$$

and so any f satisfying

$$f'(t) = 1 + f(t)^2,$$

is a solution. By separation of variables, using the hint, one sees that $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(t) = \tan(t)$ is a solution. The image of $(-\frac{\pi}{2}, \frac{\pi}{2})$ under f is $\mathbb{R}$, and so the image of $(-\frac{\pi}{2}, \frac{\pi}{2})$ under γ is the entire x axis, $\{y = 0\}$.

Unseen [5 marks].

(iii) Note that f is smooth on $\mathbb{R}^2$ and invertible with smooth inverse $f^{-1}(x, y) = f(x, y)$. Hence f is a diffeomorphism. Now, one computes

$$f_{*(x,y)}\partial_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}\partial_x - \frac{2xy}{(x^2 + y^2)^2}\partial_y, \qquad f_{*(x,y)}\partial_y = -\frac{2xy}{(x^2 + y^2)^2}\partial_x + \frac{x^2 - y^2}{(x^2 + y^2)^2}\partial_y,$$

from which it follows that

$$f^*g_{(x,y)}(\partial_x, \partial_x) = g_{f(x,y)}(f_{*(x,y)}\partial_x, f_{*(x,y)}\partial_x) = \ldots = \frac{4}{(1+x^2+y^2)} = f^*g_{(x,y)}(\partial_y, \partial_y),$$

and

$$f^*g_{(x,y)}(\partial_x, \partial_y) = 0.$$

Hence $f^*g_{(x,y)} = g_{(x,y)}$ and f is an isometry.

Unseen [5 marks].

3. (a) The annulus is not geodesically complete. For example the curve $\gamma\colon (1,2) \to \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$ defined by $\gamma(t) = (t, 0)$ is a geodesic which cannot be extended to a larger interval. The other two manifolds are compact subsets of $\mathbb{R}^n$ and hence geodesically complete by the Hopf–Rinow Theorem. Full marks are awarded for *either* justifying that the annulus is not geodesically complete, *or* for justifying that the torus and the circle are geodesically complete.

Unseen [3 marks].

(b) Recall that, in normal coordinates centred at p, for any $v = v^i\partial_{x^i}|_p \in T_p\mathcal{M}$ the curve $\gamma(t) = (tv^1, \ldots, tv^n)$ is a geodesic. The geodesic equations,

$$\ddot{\gamma}^i(t) + \dot{\gamma}^j(t)\dot{\gamma}^k(t)\Gamma^i_{jk}(\gamma(t)) = 0,$$

at $t = 0$ then imply that

$$v^j v^k \Gamma^i_{jk}(p) = 0,$$

for all $v^1, \ldots, v^n$, from which it follows that

$$\Gamma^i_{jk}(p) = 0,$$

for all $i, j, k = 1, \ldots, n$.

Seen as exercise [6 marks].

(c) Yes. Consider Cartesian coordinates on $\mathbb{R}^n$. Given any $p \in \mathbb{R}^n$ and any $v \in \mathbb{R}^n = T_p\mathbb{R}^n$, each corresponding geodesic takes the form $\gamma_{p,v}(t) = p + tv$ since $\gamma_{p,v}$ has to be a constant speed straight line. The geodesic equations at $t = 0$ then imply that

$$\Gamma^i_{jk}(p)v^j v^k = 0,$$

for all $i = 1, \ldots, n$. Since p and v were arbitrary, it follows that $\Gamma^i_{jk}(p) = 0$ for all $i, j, k = 1, \ldots, n$, $p \in \mathcal{M}$. It follows, using the expression for the components in terms of derivatives of Christoffel symbols, that the Riemann curvature tensor vanishes identically. By a theorem from the lectures it then follows that $(\mathbb{R}^n, g)$ is isometric to $(\mathbb{R}^n, g_{Eucl})$.

Unseen [5 marks].

(d) Recall that, for any vector fields V, X, Y,

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y),$$

since ∇ is compatible with g, and the Lie derivative satisfies

$$V(g(X,Y)) = (\mathcal{L}_V g)(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}Y).$$

Hence locally (extending $\dot{\gamma}$ arbitrarily to a vector field so that the steps in the computation make sense),

$$\begin{aligned}\frac{d}{dt}g(K,\dot{\gamma}) &= g(\nabla_{\dot{\gamma}}K, \dot{\gamma}) + g(K, \nabla_{\dot{\gamma}}\dot{\gamma}) = g(\nabla_K\dot{\gamma}, \dot{\gamma}) - g([K,\dot{\gamma}], \dot{\gamma}) \\ &= \frac{1}{2}K(g(\dot{\gamma},\dot{\gamma})) - (\mathcal{L}_K g)(\dot{\gamma},\dot{\gamma}) - \frac{1}{2}K(g(\dot{\gamma},\dot{\gamma})) = 0,\end{aligned}$$

where the torsion free property of ∇ has been used, along with the fact that

$$\mathcal{L}_X Y = [X, Y],$$

for all vector fields X, Y, and the fact that

$$\mathcal{L}_K g = 0,$$

since K is a Killing vector.

Unseen [6 marks].

4. (a) (i) Yes: the Euclidean metric $g = dx^2 + dy^2 + dz^2$.

Seen [2 marks].

(ii) No: by the symmetries of R it must be the case that $R(X, Y)Z = -R(Y, X)Z$ for all X, Y, Z, but g is symmetric and so $g(X, Y)Z = g(Y, X)Z$.

Unseen [2 marks].

(iii) No: this R would satisfy

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = X + Y + Z,$$

but the first Bianchi identity implies that this would have to vanish.

Unseen [3 marks].

(iv) Yes: Consider the restriction of the round metric on $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\}$ to $\{w > 0\}$ and its pullback by the map $i(x, y, z) = (x, y, z, \sqrt{1 - x^2 - y^2 - z^2})$. We saw in the lectures that the round metric satisfies

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Seen [3 marks].

(b) Let J be a non-trivial normal Jacobi field along γ such that $J(0) = 0$. We are interested in whether there exists such a J which has another 0 along γ before time $t = \frac{\pi}{2\sqrt{\lambda}}$. Without loss of generality assume $|D_t J(0)| = 1$. One computes

$$\frac{d^2}{dt^2}(|J(t)|) = \frac{d}{dt}\Big(\frac{g(D_t J, J)}{|J|}\Big) = \frac{g(D_t^2 J, J) + |D_t J|^2}{|J|} - \frac{g(D_t J, J)}{|J|^3}.$$

By Cauchy–Schwarz

$$\frac{|D_t J|^2}{|J|} - \frac{g(D_t J, J)}{|J|^3} \geq 0,$$

and, by the Jacobi equation and the assumption on the sectional curvatures

$$\frac{g(D_t^2 J, J)}{|J|} = -\frac{Rm(\dot{\gamma}, J, J, \dot{\gamma})}{|J|} = -\kappa(\dot{\gamma}, J) \geq -\lambda.$$

Hence

$$\frac{d^2}{dt^2}(|J(t)|) + \lambda |J(t)| \geq 0.$$

Note now that $h(t) = \frac{1}{\sqrt{\lambda}} \sin(t\sqrt{\lambda})$ satisfies $h'(0) = 0$ and

$$\frac{d^2}{dt^2} h(t) + \lambda h(t) = 0.$$

This choice of h is inspired by the Jacobi fields of the round sphere, which saturates the curvature assumption. It suffices to show that $|J(t)| \geq h(t)$, since $h(t)$ has no zeros on $(0, \frac{\pi}{2\sqrt{\lambda}}]$. Let $f(t) = |J(t)|$ and note that $\lim_{t\to 0} f'(t) = 1$. Now

$$\frac{d}{dt}\Big(\frac{f}{h}\Big) = \frac{f'(t)h(t) - h'(t)f(t)}{h(t)^2}, \qquad \frac{d}{dt}(f'h - h'f) = f''(t)h(t) - h''(t)f(t) \geq 0.$$

Using the fact that $(f'h - h'f)(0) = 0$ and $\frac{f}{h}(0) = 1$ (e.g. by L'Hoptial's rule) it then follows that

$$\frac{f(t)}{h(t)} \geq 1,$$

for all t.

Unseen [10 marks].

5. (a) Yes: Recall the Gauss equation. Since the Riemann curvature tensor of $(\mathbb{R}^n, g_{Eucl})$ vanishes, it follows that

$$Rm(X,Y,Z,W) = g_{Eucl}(\Pi(X,W), \Pi(Y,Z)) - g_{Eucl}(\Pi(X,Z), \Pi(Y,W)),$$

where Rm is the Riemann curvature tensor of $\mathcal{M}$ with the induced metric and Π is the second fundamental form. If $\Pi \equiv 0$ then it follows that $Rm \equiv 0$ and so $\mathcal{M}$ with the induced metric is flat.

Unseen [4 marks].

(b) Note first that, by definition of k,

$$\nabla_X k(Y,Z) = k(\nabla_X Y, Z) + k(Y, \nabla_X Z) = k(\overline{\nabla}_X Y, Z) + k(Y, \overline{\nabla}_X Z) = \overline{\nabla}_X k(Y,Z).$$

Now

$$\begin{aligned}&\overline{\nabla}_X k(Y,Z) - \overline{\nabla}_Y k(X,Z)\\ &\qquad = X(k(Y,Z)) - k(\nabla_X Y, Z) - k(Y, \nabla_X Z) - Y(k(X,Z)) + k(\nabla_Y X, Z) + k(X, \nabla_Y Z).\end{aligned}$$

By definition of k,

$$X(k(Y,Z)) = -g(\nabla_X \nabla_Y n, Z) - g(\nabla_Y n, \nabla_X Z).$$

Similarly for $Y(k(X,Z))$ and so,

$$X(k(Y,Z)) - Y(k(X,Z)) = -Rm(X,Y,n,Z) - g(\nabla_{[X,Y]} n, Z) - g(\nabla_Y n, \nabla_X Z) + g(\nabla_X n, \nabla_Y Z).$$

Inserting into the above then gives

$$\begin{aligned}\overline{\nabla}_X k(Y,Z) - \overline{\nabla}_Y k(X,Z) &= Rm(X,Y,Z,n) - g(\nabla_{[X,Y]} n, Z) - g(\nabla_Y n, \nabla_X Z)\\ &+ g(\nabla_X n, \nabla_Y Z) - k(\nabla_X Y, Z) - k(Y, \nabla_X Z) + k(\nabla_Y X, Z) + k(X, \nabla_Y Z) = Rm(X,Y,Z,n).\end{aligned}$$

Unseen [6 marks].

(c) (i) One computes

$$\partial_u = \partial_x + 2x\partial_z, \qquad \partial_v = \partial_x + 2y\partial_z.$$

Hence, for

$$n = \frac{1}{(1 + 4x^2 + 4y^2)^{\frac{1}{2}}}\big(- 2x\partial_x - 2y\partial_y + \partial_z\big),$$

one has

$$g_{Eucl}(\partial_u, n) = g_{Eucl}(\partial_v, n) = 0.$$

Moreover,

$$g_{Eucl}(n, n) = 1.$$

Since the tangent space at each point of P is spanned by ∂_u, ∂_v, it follows that n is indeed the unit normal.

Unseen [4 marks].

(ii) Using the above expressions one computes

$$\nabla_{\partial_u} n = \ldots = \frac{-2}{(1+4x^2+4y^2)^{\frac{3}{2}}}\big[(1+4y^2)\partial_x - 4xy\partial_y + 2x\partial_z\big],$$
$$\nabla_{\partial_v} n = \ldots = \frac{-2}{(1+4x^2+4y^2)^{\frac{3}{2}}}\big[-4xy\partial_x + (1+4x^2)\partial_y + 2y\partial_z\big],$$

from which it follows that

$$g(\nabla_{\partial_u} n, \partial_u) = g(\nabla_{\partial_v} n, \partial_v) = -\frac{2}{(1+4x^2+4y^2)^{\frac{1}{2}}}, \qquad g(\nabla_{\partial_u} n, \partial_v) = 0.$$

Since $k(X,Y) = -g(\nabla_X n, Y)$, it follows that

$$k = \frac{2}{(1+4u^2+4v^2)^{\frac{1}{2}}}(du^2 + dv^2).$$

Unseen [6 marks].

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97050_MATH97161	1	The first part of this question was answered well. Many students struggled to compute the Lie derivative of T.
MATH97050_MATH97161	2	A very common mistake on this question was to fail to show that f, in part © (iii), is a diffeomorphism.
MATH97050_MATH97161	3	This question was answered well. I was impressed with the attempts at part (d)
MATH97050_MATH97161	4	Part (b) was the most difficult question on the paper and several students did not even attempt it, despite the fact that the case that all sectional curvatures are equal to a postiive constant (rather than bounded above by a positive constant) was treated explicitly in the lectures.
MATH97050_MATH97161	5	This question was answered well, but many students struggled and made mistakes with part (b).