IMPERIAL

MATH70046

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series Analysis

Date: Tuesday, May 7, 2024
Time: 10:00 – 12:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer Each Question in a Separate Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary, unless stated otherwise. All processes are real-valued, unless stated otherwise. The sample interval is unity, unless stated otherwise. $\Delta = 1 - B$ denotes the difference operator, where B denotes the backward shift operator.

1. (a) Define what it means for a stochastic process to be second-order stationary. (3 marks)

   (b) For the following processes, determine whether they are second-order stationary. For those that you determine are second-order stationary, find their autocovariance sequence.

   (i) $X_t = (-1)^t + \epsilon_t - \frac{1}{2}\epsilon_{t-1}$. (2 marks)

   (ii) $X_t = 2X_{t-1} - \frac{15}{16}X_{t-2} + \epsilon_t$ (3 marks)

   (iii) $X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k}$, where $g_k = (k+1)^{-1}$. (3 marks)

   (iv) $X_t = A_t \cos(\omega t) + B_t \sin(\omega t) + \epsilon_t$, where ω is a constant. Random processes $\{A_t\}$ and $\{B_t\}$ are second-order stationary, uncorrelated with $\{\epsilon_t\}$, and have zero means and autocovariance sequences $s_{A,\tau} = s_{B,\tau} = \alpha^{|\tau|}$, where $\alpha \neq 0$ is a constant. Furthermore, $\mathrm{Cov}(A_s, B_t) = 0$, for all s and t.

   (4 marks)

   (c) Consider the second-order stationary autoregressive process $\{X_t\}$ with mean μ, where

$$X_t - \mu = \phi_{1,p}(X_{t-1} - \mu) + ... + \phi_{p,p}(X_{t-p} - \mu) + \epsilon_t.$$

   Show that $\sigma_X^2 \equiv \mathrm{Var}\{X_t\}$ satisfies the relationship

$$\sigma_\epsilon^2 = (1 - \phi_{1,p}\rho_1 - ... - \phi_{p,p}\rho_p)\sigma_X^2,$$

   where $\{\rho_\tau\}$ is the autocorrelation sequence of $\{X_t\}$. (5 marks)

(Total: 20 marks)

2. (a) Consider the AR(2) process

$$X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + \epsilon_t.$$

(i) Express $\{X_t\}$ in terms of the backward shift operator. (2 marks)

(ii) Show that $\{X_t\}$ is a second-order stationary process. (2 marks)

(iii) Show $\{X_t\}$ takes the general linear process form

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k},$$

where $g_k = \frac{1}{3^{k+1}}\left(2^{k+1} + (-1)^k\right)$. (4 marks)

(b) Consider a stationary process $\{X_t\}$ that can be written as a general linear process,

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t.$$

We wish to construct the l-step ahead forecast of the form

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}.$$

(i) Show that the l-step prediction variance $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

(4 marks)

(ii) Show the l-step ahead forecast can be written in the form

$$X_t(l) = \Psi^{(l)}(B)\Psi^{-1}(B)X_t,$$

where $\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k$. (2 marks)

(iii) Consider the AR(2) process of Part (a). Show that the forecast $X_N(l)$ for X_{N+l}, given the entire sequence $\{X_t : t \leq N\}$, can be written in the form $a_l X_N + b_l X_{N-1}$ for sequences of constants $\{a_l\}$ and $\{b_l\}$. In doing so, find a_l and b_l. (6 marks)

(Total: 20 marks)

3. (a) Let $\{X_t\}$ be a stationary process with variance σ_X^2 and let $\bar{X} = \frac{1}{N}\sum_{t=1}^{N} X_t$. Show that we can write

$$\mathrm{Var}\{\bar{X}\} = \frac{\sigma_X^2}{N}\left(1 + \frac{2}{N}\sum_{i=1}^{N-1}\sum_{j>i}^{N}\rho_{j-i}\right),$$

where $\{\rho_\tau\}$ is the autocorrelation sequence of $\{X_t\}$.

(6 marks)

(b) Consider the stationary AR(1) process $\{X_t\}$, defined as

$$X_t = \phi X_{t-1} + \epsilon_t.$$

Show that its autocovariance sequence, $\{s_\tau\}$, is given as $s_\tau = \sigma_X^2 \phi^{|\tau|}$, where $\sigma_X^2 = \sigma_\epsilon^2/(1-\phi^2)$.

(4 marks)

(c) For the AR(1) process in (b), show

$$\sum_{j>i}^{N}\rho_{j-i} = \frac{\phi(1-\phi^{N-i})}{1-\phi}.$$

Hence show that for the AR(1) process,

$$\mathrm{Var}\{\bar{X}\} = \frac{\sigma_\epsilon^2}{N(1-\phi^2)}\left(1 + \frac{2\phi}{N(1-\phi)}\left[N - \frac{(1-\phi^N)}{(1-\phi)}\right]\right). \qquad (\dagger)$$

(5 marks)

(d) Using linear filters, derive the spectral density function $S(f)$ of an AR(1) process. Verify using $(\dagger)$ that, if we ignore terms of order $1/N^2$, we obtain the approximation

$$\mathrm{Var}\{\bar{X}\} \approx \frac{S(0)}{N}.$$

Hint: you may take as given that if $L\{\cdot\}$ is a linear time-invariant filter with frequency response function $G(f)$, and $Y_t = L\{X_t\}$, then $S_Y(f) = |G(f)|^2 S_X(f)$.

(5 marks)

(Total: 20 marks)

4. (a) Consider the process $Y_t = \nu_t + X_t$, where $\{X_t\}$ is a zero mean stationary process, and $\{\nu_t\}$ is a deterministic periodic sequence with period $s \in \mathbb{Z}^+$.

(i) Show the operator $1 - B^s$ when applied to $\{Y_t\}$ removes the seasonal component.

(3 marks)

(ii) Considering $1 - B^s$ as a linear time-invariant filter, derive its frequency response function $G(f)$. (3 marks)

(iii) By considering the (squared) gain of the linear filter, describe the effect it is having from a frequency domain point of view. (3 marks)

(b) Let $\{X_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{X,\tau}\}$ and spectral density function $S_X(f)$. Define random processes

$$\begin{aligned} W_t &= X_t + X_{t-2} + \xi_t \\ V_t &= X_{t-1} + X_{t-3} + \eta_t, \end{aligned}$$

where $\{\xi_t\}$ and $\{\eta_t\}$ are both zero mean and unit variance white noise processes, uncorrelated with each other and with $\{X_t\}$.

(i) State the definition of joint stationarity, and show $\{W_t\}$ and $\{V_t\}$ are jointly stationary.

(4 marks)

(ii) Compute the cross-spectrum, $S_{WV}(f)$, of $\{W_t\}$ and $\{V_t\}$ in terms of $S_X(f)$. Hint: you may find it useful to define the process $Y_t = X_t + X_{t-2}$.

(5 marks)

(iii) The group delay between two jointly stationary processes is defined as

$$-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\theta(f),$$

where $\theta(f)$ is the cross-phase spectrum. Compute the group delay of $\{W_t\}$ and $\{V_t\}$. Interpret the result.

(2 marks)

(Total: 20 marks)

5. PRELIMINARY INFORMATION

- In this question, you may assume the following results.

$$\sum_{t=0}^{N-1} \cos^2(2\pi f_j t) = \sum_{t=0}^{N-1} \sin^2(2\pi f_j t) = \frac{N}{2}$$
$$\sum_{t=0}^{N-1} \cos(2\pi f_j t)\sin(2\pi f_j t) = \sum_{t=0}^{N-1} \cos(2\pi f_j t)\sin(2\pi f_k t) = 0$$
$$\sum_{t=0}^{N-1} \cos(2\pi f_j t)\cos(2\pi f_k t) = \sum_{t=0}^{N-1} \sin(2\pi f_j t)\sin(2\pi f_k t) = 0,$$

  where $f_j = j/N$ and $f_k = k/N$ with j and k both integers such that $j \neq k$ and $1 \leq j, k < N/2$.

- If $Y_1, Y_2, ..., Y_\nu$ are independent zero mean, unit variance Gaussian random variables, then $\chi^2_\nu \equiv Y_1^2 + Y_2^2 + ... + Y_\nu^2$ has a chi-square distribution with ν degrees of freedom.

- Let χ^2_ν denote a chi-square distributed random variable with ν degrees of freedom, then $E\{\chi^2_\nu\} = \nu$ and $\mathrm{Var}\{\chi^2_\nu\} = 2\nu$.

QUESTION BEGINS ON NEXT PAGE

(a) Let $X_0, ..., X_{N-1}$ be a portion of a zero mean Gaussian white noise process $\{X_t\}$ with variance σ^2.

For $f_k \neq 0$ or $1/2$, show that the periodogram is distributed

$$\hat{S}_X^{(p)}(f_k) \overset{\mathrm{d}}{=} S(f_k)\frac{\chi_2^2}{2},$$

where $\overset{\mathrm{d}}{=}$ denotes *equal in distribution* and χ_ν^2 denotes a chi-square distributed random variable with ν degrees of freedom.

Hint: Consider representing the periodogram in terms of $A(f) = N^{-1/2}\sum_{t=0}^{N-1} X_t \cos(2\pi f t)$ and $B(f) = N^{-1/2}\sum_{t=0}^{N-1} X_t \sin(2\pi f t)$. (8 marks)

(b) Let $X_0, ..., X_{N-1}$ be a portion of a Gaussian zero mean stationary process $\{X_t\}$ with spectral density function $S(f)$. Consider the weighted multitaper estimator

$$\hat{S}^{(WMT)}(f) = \sum_{k=0}^{K-1} d_k \hat{S}_k^{(MT)}(f) \qquad \text{with} \qquad \hat{S}_k^{(MT)}(f) = \left|\sum_{t=0}^{N-1} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2,$$

where $\{h_{k,t}\}$ is the data taper for the kth direct spectral estimator $\hat{S}_k^{(MT)}(f)$ and $d_0, .., d_{K-1}$ are weights with $\sum_{k=0}^{K-1} d_k = 1$. We assume $\sum_{t=0}^{N-1} h_{k,t}^2 = 1$ for all $k = 0, ..., K-1$.

(i) Show that if $\hat{S}_k^{(MT)}(f)$ is an unbiased estimator of $S(f)$ for all $k = 0, ..., K-1$, then $\hat{S}^{(WMT)}(f)$ is also an unbiased estimator of $S(f)$. (2 marks)

(ii) It can be shown that $\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\} \approx S^2(f)\left|\sum_{t=0}^{N-1} h_{j,t} h_{k,t}\right|^2$, for $0 < f < 1/2$. Let $\{h_{0,t}\}, ..., \{h_{K-1,t}\}$ be an orthonormal set of tapers. Show that

$$\mathrm{Var}\{\hat{S}^{(WMT)}(f)\} \approx S^2(f)\sum_{k=0}^{K-1} d_k^2.$$

(5 marks)

(iii) We wish to find an approximate distribution for $\hat{S}^{(WMT)}(f)$, assuming that

$$\hat{S}^{(WMT)}(f) \overset{\mathrm{d}}{=} a\chi_\nu^2.$$

Match this with the expectation and variance of $\hat{S}^{(WMT)}(f)$ to find scaling factor a and effective degrees of freedom ν (which can be non-integer valued).

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60046/70046

Time Series Analysis (Solutions)

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1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓

3, A

(b) (i) Check the mean of this process. $E\{X_t\} = (-1)^t$, which depends on t and is not constant. Therefore the process is non-stationary.

2, A

(ii) This AR(2) process can be written as $(1 - 2B - \frac{15}{16}B^2)X_t = \epsilon_t$. To check for stationarity we need to check whether the roots of the characteristic polynomial $\Phi(z) = 1 - 2z - \frac{15}{16}z^2$ lie outside of the unit circle. We can write $\Phi(z) = (1 - \frac{5}{4}z)(1 - \frac{3}{4}z)$, for which the roots are $4/5$ and $4/3$. As one of these roots lies inside the unit circle, this process is non-stationary.

3, A

(iii) The condition for stationarity with a general linear process is that $\sum_{k=0}^{\infty} g_k^2 < \infty$, which is true here because $\sum_{k=0}^{\infty} g_k^2 = \sum_{k=1}^{\infty} 1/k^2 < \infty$.
For the acvs, we have

$$\begin{aligned} s_\tau &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k g_{k+\tau} \\ &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right)\left(\frac{1}{k+\tau+1}\right). \end{aligned}$$

3, A

(iv) The process $\{X_t\}$ is mean zero, therefore

sim. seen ⇓

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t+\tau}) &= E(X_t X_{t+\tau}) \\ &= E(A_t A_{t+\tau})\cos(\omega t)\cos(\omega(t+\tau)) + \\ &\quad E(B_t B_{t+\tau})\sin(\omega t)\sin(\omega(t+\tau)) + E(\epsilon_t \epsilon_{t+\tau}), \end{aligned}$$

with all other terms equal to zero due to being uncorrelated. Therefore

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t+\tau}) &= \alpha^{|\tau|}[\cos(\omega t)\cos(\omega(t+\tau)) + \sin(\omega t)\sin(\omega(t+\tau))] + \sigma_\epsilon^2 \delta_{0,\tau} \\ &= \alpha^{|\tau|}[\cos(\omega(t+\tau) - \omega t)] + \sigma_\epsilon^2 \delta_{0,\tau} \\ &= \alpha^{|\tau|}\cos(\omega\tau) + \sigma_\epsilon^2 \delta_{0,\tau}, \end{aligned}$$

which does not depend on t. Therefore $\{X_t\}$ is stationary with acvs as given.

4, B

(c) Let $Y_t = X_t - \mu$, giving

sim. seen ⇓

$$Y_t = \phi_{1,p} Y_{t-1} + ... + \phi_{p,p} Y_{t-p} + \epsilon_t,$$

the defining equation for a zero-mean AR(p) process $\{Y_t\}$. We note $\mathrm{Cov}\{X_t, X_{t+\tau}\} = \mathrm{Cov}\{Y_t, Y_{t+\tau}\} = E\{Y_t Y_{t+\tau}\}$. Multiplying both sides by Y_t and take expectations gives

$$E\{Y_t Y_t\} = E\{Y_t(\phi_{1,p} Y_{t-1} + ... + \phi_{p,p} Y_{t-p} + \epsilon_t)\}$$
$$s_0 = \phi_{1,p} s_1 + ... + \phi_{p,p} s_p + E\{Y_t \epsilon_t\}.$$

Now consider the term $E\{Y_t \epsilon_t\}$, which we can also express as

$$E\{(\phi_{1,p} Y_{t-1} + ... + \phi_{p,p} Y_{t-p} + \epsilon_t)\epsilon_t\}.$$

We note that Y_{t-k} is uncorrelated with ϵ_t for all $k \geq 1$, and therefore $E\{Y_t \epsilon_t\} = E\{\epsilon_t^2\} = \sigma_\epsilon^2$. Therefore

$$s_0 = \phi_{1,p} s_1 + ... + \phi_{p,p} s_p + \sigma_\epsilon^2.$$

Dividing through by $s_0 = \sigma_X^2$ gives the desired result. $\boxed{5, \mathrm{C}}$

2. (a) (i) This can be written as $\boxed{\text{sim. seen} \Downarrow}$

$$X_t - \frac{1}{3}X_{t-1} - \frac{2}{9}X_{t-2} = \epsilon_t$$
$$(1 - \frac{1}{3}B - \frac{2}{9}B^2)X_t = \epsilon_t.$$

$\boxed{2, \text{A}}$

(ii) To show it is second order stationary, we are required to show the roots of the characteristic polynomial $\Phi(z) = 1 - \frac{1}{3}z - \frac{2}{9}z^2$ lie outside of the unit circle. We can write $\Phi(z) = (1 + \frac{1}{3}z)(1 - \frac{2}{3}z)$. This has roots -3 and $3/2$, which both lie outside of the unit circle. Hence, the process is stationary. $\boxed{2, \text{A}}$

(iii)

$$\begin{aligned} X_t &= \frac{1}{(1+\frac{1}{3}B)(1-\frac{2}{3}B)}\epsilon_t \\ &= \left(\frac{1}{3}\cdot\frac{1}{1+\frac{1}{3}B} + \frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}B}\right)\epsilon_t \\ &= \left(\frac{1}{3}\cdot\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k B^k + \frac{2}{3}\cdot\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k B^k\right)\epsilon_t \\ &= \sum_{k=0}^{\infty} g_k B^k \epsilon_t, \end{aligned}$$

where $g_k = \frac{1}{3^{k+1}}(2^{k+1} + (-1)^k)$. $\boxed{4, \text{B}}$

(b) (i) Using the GLP representation, we have $X_{t+l} = \sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k}$. We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} E\{(X_{t+l} - X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k} - \sum_{k=0}^{\infty}\delta_k\epsilon_{t-k}\right)^2\right\} \\ &= E\left\{\left(\sum_{k=0}^{l-1}\psi_k\epsilon_{t+l-k} + \sum_{k=0}^{\infty}[\psi_{k+l} - \delta_k]\epsilon_{t-k}\right)^2\right\} \\ &= \sigma_\epsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right) + \sum_{k=0}^{\infty}(\psi_{k+l} - \delta_k)^2\right\}. \end{aligned}$$

The first term is independent of the choice of $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. $\boxed{4, \text{A}}$

(ii) Part (i) means the l-step ahead forecast can be written $X_t(l) = \Psi^{(l)}(B)\epsilon_t$, where $\Psi^{(l)}(z)$ is as stated in the question. Given $X_t = \Psi(B)\epsilon_t$, we have $\epsilon_t = \Psi^{-1}(B)X_t$, giving $X_t(l) = \Psi^{(l)}(B)\Psi^{-1}(B)X_t$. $\boxed{2, \text{A}}$

unseen $\Downarrow$

(iii) From (b)(ii)

$$X_t(l) = \sum_{k=0}^{\infty} g_{k+l}\epsilon_{t-k} = \sum_{k=0}^{\infty}\left(\left(\frac{2}{3}\right)^{l+k+1} - \left(-\frac{1}{3}\right)^{l+k+1}\right)\epsilon_k,$$

i.e.

$$\Psi^{(l)}(B) = \sum_{k=0}^{\infty}\left(\left(\frac{2}{3}\right)^{l+k+1} - \left(-\frac{1}{3}\right)^{l+k+1}\right)B^k.$$

We know that $\Psi^{-1}(B) = (1-\frac{2}{3}B)(1+\frac{1}{3}B)$, therefore,

$$\begin{aligned}
X_N(l) &= \left[\sum_{k=0}^{\infty}\left\{\left(\frac{2}{3}\right)^{l+k+1} - \left(-\frac{1}{3}\right)^{l+k+1}\right\}B^k\right](1-\frac{2}{3}B)(1+\frac{1}{3}B)X_N \\
&= \left\{\left(\frac{2}{3}\right)^{l+1}\frac{1}{1-\frac{2}{3}B} - \left(-\frac{1}{3}\right)^{l+1}\frac{1}{1+\frac{1}{3}B}\right\}(1-\frac{2}{3}B)(1+\frac{1}{3}B)X_N \\
&= \left\{\left(\frac{2}{3}\right)^{l+1}(1+\frac{1}{3}B) - \left(-\frac{1}{3}\right)^{l+1}(1-\frac{2}{3}B)\right\}X_N \\
&= \left\{\left(\frac{2}{3}\right)^{l+1} - \left(-\frac{1}{3}\right)^{l+1}\right\}X_N + \left\{\frac{1}{3}\left(\frac{2}{3}\right)^{l+1} - \frac{2}{3}\left(-\frac{1}{3}\right)^{l+1}\right\}X_{N-1},
\end{aligned}$$

as required, where

$$a_l = \left(\frac{2}{3}\right)^{l+1} - \left(-\frac{1}{3}\right)^{l+1}, \qquad b_l = \frac{1}{3}\left(\frac{2}{3}\right)^{l+1} - \frac{2}{3}\left(-\frac{1}{3}\right)^{l+1}.$$

6, D

3. (a) We have $E\{\bar{X}\} = \mu$, where $\mu \equiv E\{X_t\}$. Therefore,

sim. seen ⇓

$$\begin{aligned}\mathrm{Var}\{\bar{X}\} &= E\left\{(\bar{X}-\mu)^2\right\} \\ &= E\left\{\left(\frac{1}{N}\sum_{t=1}^{N}(X_t-\mu)\right)^2\right\} \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\{(X_i-\mu)(X_j-\mu)\} \\ &= \frac{\sigma_X^2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{j-i}.\end{aligned}$$

Using the fact that $\rho_{i-j} = \rho_{j-i}$, and there's N lots of $\rho_0 = 1$, we can rewrite the double sum to give

$$\mathrm{Var}\{\bar{X}\} = \frac{\sigma_X^2}{N^2}\left(N + 2\sum_{i=1}^{N-1}\sum_{j>i}^{N}\rho_{j-i}\right),$$

and hence the desired result.

6, B

(b) For $\tau > 0$,

seen ⇓

$$\begin{aligned}X_tX_{t-\tau} &= \phi X_{t-1}X_{t-\tau} + X_{t-\tau}\epsilon_t \\ E\{X_tX_{t-\tau}\} &= \phi E\{X_{t-1}X_{t-\tau}\} + E\{X_{t-\tau}\epsilon_t\} \\ s_\tau &= \phi s_{\tau-1} + 0\end{aligned}$$

because $X_{t-\tau}$ and ϵ_t are uncorrelated. Iterating back gives $s_\tau = \phi^\tau s_0$, and symmetry of the acvs gives $s_\tau = \phi^{|\tau|}s_0$, for all τ. We're just left with finding $s_0 = \sigma_X^2$. We know it is stationary, and therefore $\mathrm{Var}\{X_t\} = \mathrm{Var}\{X_{t-1}\} \equiv \sigma_X^2$. Taking the variance of both sides of the defining equation,

$$\begin{aligned}\mathrm{Var}\{X_t\} &= \phi^2\,\mathrm{Var}\{X_{t-1}\} + \mathrm{Var}\{\epsilon_t\} + \mathrm{Cov}\{X_{t-1},\epsilon_t\} \\ \sigma_X^2 &= \phi^2\sigma_X^2 + \sigma_\epsilon^2 + 0,\end{aligned}$$

giving $\sigma_X^2 = \sigma_\epsilon^2/(1-\phi^2)$.

4, A

(c) We begin with

unseen ⇓

$$\begin{aligned}\sum_{j>i}^{N}\rho_{j-i} &= \sum_{k=1}^{N-i}\rho_k \\ &= \sum_{k=1}^{N-i}\phi^k = \frac{\phi(1-\phi^{N-i})}{1-\phi}.\end{aligned}$$

Therefore, using part (a), we have

$$\begin{aligned}
\mathrm{Var}\{\bar{X}\} &= \frac{\sigma_X^2}{N}\left(1 + \frac{2}{N}\sum_{i=1}^{N-1}\sum_{j>i}^{N}\rho_{j-i}\right)\\
&= \frac{\sigma_X^2}{N}\left(1 + \frac{2}{N}\sum_{i=1}^{N-1}\frac{\phi(1-\phi^{N-i})}{1-\phi}\right)\\
&= \frac{\sigma_X^2}{N}\left(1 + \frac{2\phi}{N(1-\phi)}\sum_{i=1}^{N-1}(1-\phi^{N-i})\right)\\
&= \frac{\sigma_X^2}{N}\left(1 + \frac{2\phi}{N(1-\phi)}\left(N-1-\sum_{i=1}^{N-1}\phi^i\right)\right)\\
&= \frac{\sigma_X^2}{N}\left(1 + \frac{2\phi}{N(1-\phi)}\left(N-1-\left(\frac{1-\phi^N}{1-\phi}-1\right)\right)\right)\\
&= \frac{\sigma_\epsilon^2}{N(1-\phi^2)}\left(1 + \frac{2\phi}{N(1-\phi)}\left[N - \frac{(1-\phi^N)}{(1-\phi)}\right]\right).
\end{aligned}$$

$\boxed{5, \mathrm{C}}$

(d) To get the spectral density function of an AR(1) process, we consider the linear filter representation writing it as $\epsilon_t = L\{X_t\} = X_t - \phi X_{t-1}$. The frequency response function of this linear filter is $G(f) = 1 + \phi \mathrm{e}^{-\mathrm{i}2\pi f}$. The spectral density function of $\{\epsilon_t\}$ is σ_ϵ^2, and therefore

$$S(f) = \frac{\sigma_\epsilon^2}{|1-\phi\ \mathrm{e}^{-\mathrm{i}2\pi f}|^2}.$$

Ignoring $1/N^2$ terms gives

$$\mathrm{Var}\{\bar{X}\} \approx \frac{\sigma_\epsilon^2}{N(1-\phi^2)}\left(1+\frac{2\phi}{1-\phi}\right) = \frac{\sigma_\epsilon^2}{N(1-\phi^2)}\left(\frac{1+\phi}{1-\phi}\right) = \frac{\sigma_\epsilon^2}{N(1-\phi)^2}.$$

With the derived form for $S(f)$, we see that

$$S(0) = \frac{\sigma_\epsilon^2}{(1-\phi)^2},$$

and the result follows. $\boxed{5, \mathrm{D}}$

4. (a) (i)

$$
\begin{aligned}
(1-B^s)Y_t &= (1-B^s)(v_t + X_t) \\
&= \nu_t + X_t - \nu_{t-s} - X_{t-s} \\
&= \nu_t - \nu_{t-s} + X_t - X_{t-s} \\
&= X_t - X_{t-s},
\end{aligned}
$$

because $\nu_t = \nu_{t-s}$ due to the periodicity of ν_t. $\boxed{3, \mathrm{A}}$

(ii) As a LTI filter, we have $L\{Y_t\} = Y_t - Y_{t-s}$. The impulse response sequence for this filter is $g_0 = 1, g_s = -1$ and $g_k = 0$ for all other k. Therefore the frequency response function, taking the Fourier transform of the impulse response sequence (the method that looks at the response on $\mathrm{e}^{\mathrm{i}2\pi ft}$ is also valid), is given as

$$G(f) = \sum g_k \mathrm{e}^{-\mathrm{i}2\pi ft} = 1 - \mathrm{e}^{-\mathrm{i}2\pi fs}.$$

$\boxed{3, \mathrm{A}}$

(iii) The squared gain is

$$|G(f)|^2 = (1-\mathrm{e}^{-\mathrm{i}2\pi fs})(1-\mathrm{e}^{\mathrm{i}2\pi fs}) = 2 - 2\cos(2\pi fs).$$

$\boxed{1, \mathrm{A}}$

This function takes a value of zero at $f = 1/s$, thus it removes oscillations that have a period of s, i.e. the seasonal component. It also removes all integer multiples of this frequency, i.e. $f = k/s$, $k \in \mathbb{Z}$. $\boxed{2, \mathrm{B}}$

(b) (i) Processes $\{W_t\}$ and $\{V_t\}$ are said to be jointly stationary if they are both individually stationary and the cross-covariance sequence $\mathrm{Cov}\{W_t, V_{t+\tau}\}$ depends only on τ.

They are clearly both stationary as they are the linear combination of zero mean stationary processes. To check joint stationarity, consider

$$
\begin{aligned}
\mathrm{Cov}\{W_t, V_{t+\tau}\} &= E\{W_t V_{t+\tau}\} \\
&= E\left\{(X_t + X_{t-2} + \xi_t)(X_{t-1+\tau} + X_{t-3+\tau} + \eta_{t+\tau})\right\} \\
&= E\{X_t X_{t-1+\tau}\} + E\{X_t X_{t-3+\tau}\} + E\{X_{t-2} X_{t-1+\tau}\} + E\{X_{t-2} X_{t-3+\tau}\} \\
&= s_{X,\tau-1} + s_{X,\tau-3} + s_{X,\tau+1} + s_{X,\tau-1}.
\end{aligned}
$$

All other terms are zero due to uncorrelated terms. This clearly depends only of τ, and hence the processes are jointly stationary. $\boxed{4, \mathrm{B}}$

(ii) Letting $Y_t = X_t + X_{t-2}$, we have

$$W_t = Y_t + \xi_t$$
$$V_t = Y_{t-1} + \eta_t.$$

We now have $s_{WV,\tau} = E\{Y_t Y_{t+\tau-1}\} = s_{Y,\tau-1}$. Therefore

$$\begin{aligned} S_{WV}(f) &= \sum_{\tau=-\infty}^{\infty} s_{WV,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau-1} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \mathrm{e}^{-\mathrm{i}2\pi f} \sum_{\tau=-\infty}^{\infty} s_{Y,\tau-1} \mathrm{e}^{-\mathrm{i}2\pi f(\tau-1)} = \mathrm{e}^{-\mathrm{i}2\pi f} S_Y(f). \end{aligned}$$

We can express $S_Y(f) = |G(f)|^2 S_X(f)$, where $G(f)$ is the frequency response function for the LTI filter $L\{X_t\} = X_t + X_{t-2}$. For this filter, $G(f) = 1+\mathrm{e}^{-\mathrm{i}4\pi f}$. Therefore $S_Y(f) = [2+2\cos(4\pi f)]S_X(f)$, and $S_{VW}(f) = \mathrm{e}^{-\mathrm{i}2\pi f}\,[2 + 2\cos(4\pi f)]\, S_X(f)$.

$\boxed{5, \mathrm{D}}$

(iii) The cross spectrum, in the form $|S_{VW}(f)|\,\mathrm{e}^{\mathrm{i}\theta(f)}$, has $\theta(f) = -2\pi f$. The group delay, as defined, is therefore given as

$$-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\theta(f) = 1.$$

It is clear to see, particularly using the transformation $Y_t = X_t + X_{t-2}$, that there is a lag/delay between $\{W_t\}$ and $\{V_t\}$ of 1, which exactly matches with the group delay computed.

$\boxed{2, \mathrm{C}}$

5. (a) The periodogram is defined as $\boxed{\text{seen} \Downarrow}$

$$\widehat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2.$$

This can be written as

$$\widehat{S}^{(p)}(f) = |A(f) + \mathrm{i}B(f)|^2 = |A(f)|^2 + |B(f)|^2,$$

where

$$A(f) = N^{-1/2}\sum_{t=0}^{N-1} X_t \cos(2\pi f t) \quad \text{and} \quad B(f) = N^{-1/2}\sum_{t=0}^{N-1} X_t \sin(2\pi f t).$$

Both $A(f)$ and $B(f)$ are a finite linear combination of independent Gaussian distributed random variables, and are hence themselves Gaussian.
It is immediate that $E\{A(f)\} = E\{B(f)\} = 0$, and therefore

$$\mathrm{Var}\{A(f_k)\} = E\{A^2(f_k)\} = N^{-1}\sum_{t=0}^{N-1}\sum_{t'=0}^{N-1} E\{X_t X_{t'}\}\cos(2\pi f_k t)\cos(2\pi f_k t').$$

With $\{X_t\}$ a white noise process, it follows that

$$\mathrm{Var}\{A(f_k)\} = \frac{\sigma^2}{N}\sum_{t=0}^{N-1}\cos^2(2\pi f_k t) = \frac{\sigma^2}{N}\cdot\frac{N}{2} = \frac{\sigma^2}{2}$$

using the given identities. The result for $\mathrm{Var}\{B(f_k)\}$ follows in an identical way. Furthermore

$$\begin{aligned}
\mathrm{Cov}\{A(f_k), B(f_k)\} &= E\{A(f_k)B(f_k)\} \\
&= \sum_{t=0}^{N-1}\sum_{t'=0}^{N-1} h_t h_{t'} E\{X_t X_{t'}\}\cos(2\pi f_k t)\sin(2\pi f_k t') \\
&= \frac{\sigma^2}{N}\sum_{t=0}^{N-1}\cos(2\pi f_k t)\sin(2\pi f_k t) = 0 \text{ for all } f_j \text{ and } f_k.
\end{aligned}$$

Both $\sqrt{(2/\sigma^2)}A(f_k)$ and $\sqrt{(2/\sigma^2)}B(f_k)$ are unit variance zero mean Gaussian random variables, and furthermore are independent by the above (uncorrelated Gaussian rvs $\Longrightarrow$ independence). Therefore, $(2/\sigma^2)(A^2(f_k) + B^2(f_k)) = (2/\sigma^2)S^{(p)}(f_k) \stackrel{\mathrm{d}}{=} \chi_2^2$, which gives $S^{(p)}(f_k) \stackrel{\mathrm{d}}{=} (\sigma^2/2)\chi_2^2$. $\boxed{8, \mathrm{M}}$

(b) (i) We have

$$E\left\{\widehat{S}^{(WMT)}(f)\right\} = \sum_{k=0}^{K-1} d_k E\left\{\widehat{S}_k^{(MT)}(f)\right\} = S(f)\sum_{k=0}^{K-1} d_k = S(f),$$

hence, it is unbiased. $\boxed{2, \text{M}}$

(ii)

$$\begin{aligned}
\mathrm{Var}\{\widehat{S}^{(WMT)}(f)\} &= \mathrm{Cov}\{\widehat{S}^{(WMT)}(f), \widehat{S}^{(WMT)}(f)\} \\
&= \mathrm{Cov}\left\{\sum_{k=0}^{K-1} d_k \widehat{S}_k^{(MT)}(f), \sum_{k=0}^{K-1} d_k \widehat{S}_k^{(MT)}(f)\right\} \\
&= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} d_j d_k \,\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\}.
\end{aligned}$$

When $j = k$ we have $\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} = \mathrm{Var}\{\widehat{S}_k^{(MT)}(f)\}$. We also have $\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} = \mathrm{Cov}\{\widehat{S}_k^{(MT)}(f), \widehat{S}_j^{(MT)}(f)\}$. Therefore,

$$\sum_{j=0}^{J-1}\sum_{k=0}^{K-1} d_j d_k \,\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} =$$

$$\sum_{k=0}^{K-1} d_k^2 \;\mathrm{Var}\{\widehat{S}_k^{(MT)}(f)\} + 2\sum_{j<k} d_j d_k \;\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\}.$$

With the orthonormality of the tapers, the approximation stated in the question reduces to $\mathrm{Cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} \approx 0$ and $\mathrm{Var}\{\widehat{S}_k^{(MT)}(f)\} \approx S^2(f)$. The result follows. $\boxed{5, \text{M}}$

(iii) Using the provided result, we have $E\{a\chi_\nu^2\} = a\nu$ and $\mathrm{Var}\{a\chi_\nu^2\} = 2a^2\nu$. $\boxed{\text{unseen} \Downarrow}$ Matching these with the derived results, we have

$$a\nu = S(f) \quad \text{and} \quad 2a^2\nu = S^2(f)\sum_{k=0}^{K-1} d_k^2.$$

Solving for a and ν, we have

$$2aS(f) = S^2(f)\sum_{k=0}^{K-1} d_k^2 \implies a = \frac{S(f)}{2}\sum_{k=0}^{K-1} d_k^2$$

and

$$\nu = \frac{2}{\sum_{k=0}^{K-1} d_k^2}.$$

$\boxed{5, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks