Imperial College London

MATH97047 MATH70062

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Lie Algebras

Date: 30 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

On intermediate steps you can use results from the course provided you state them clearly.

1. (a) Classify up to isomorphism the complex Lie algebras $\mathfrak{g}$ whose centres are 1-dimensional and the corresponding quotients Lie algebras are abelian. (6 marks)

   (b) Let $\mathfrak{u}(n)$ be the subalgebra of $\mathfrak{gl}(n)$ formed by the upper triangular matrices with zeros on main diagonal. Prove from first principles that every element of $\mathfrak{u}(n)$ is nilpotent. Prove that the whole $\mathfrak{u}(n)$ is nilpotent. (6 marks)

   (c) Calculate with justification the nilpotency class of $\mathfrak{u}(n)$ and determine the upper central series. (3 marks)

   (d) What is the dimension of the largest subalgebra in $\mathfrak{gl}(n)$ whose centre is the 1-dimensional subalgebra spanned by the matrix with the only non-zero entry 1 in the right top corner and such that the corresponding factor algebra is abelian. (5 marks)

(Total: 20 marks)

2. (a) Let M_1 and M_2 be two modules of a Lie algebra $\mathfrak{g}$. Prove that the tensor product $M_1 \otimes M_2$ is a module subject to the following rule:

$$g \cdot (m_1 \otimes m_2) = (g \cdot m_1) \otimes m_2 + m_1 \otimes (g \cdot m_2)$$

   for all $g \in \mathfrak{g}$. (3 marks)

   (b) Let V_2 be the standard irreducible 2-dimensional module of the complex Lie algebra $\mathfrak{sl}(2)$. Decompose the tensor product $V_2 \otimes V_2$ into direct sum $\mathfrak{sl}(2)$-irreducibles computing the bases of the two representations in terms of the standard e, f, h-basis of $\mathfrak{sl}(2)$. (5 marks)

   (c) Let W be the tensor product of n copies of V_2. What are the dimensions of the weight spaces

$$W_m = \{w \in W \mid h \cdot w = mw\}$$

   for all integer numbers m? (5 marks)

   (d) Determine the irreducible components of W with their multiplicities (you can use results on the representation theory of $\mathfrak{sl}(2)$ provided you give a correct statement with a reference). (7 marks)

(Total: 20 marks)

3. Let $\mathfrak{g}$ be the simple Lie algebra of type G_2.

(a) What is the Cartan matrix of $\mathfrak{g}$? (3 marks)

(b) Compute the roots of the root system R of type G_2 and calculate the dimension of $\mathfrak{g}$. (5 marks)

(c) For every root in R exhibit a non-zero vector in the corresponding root space as a successive commutator of the Chevalley generators. (7 marks)

(d) Describe the action on $\mathfrak{g}$ of the first Chevalley generator e_α in terms of a basis of $\mathfrak{g}$. You may use the root vectors from (c) as part of this basis. (5 marks)

(Total: 20 marks)

4. Let $\mathfrak{g}$ be the simple Lie algebra of type A_3 and let e_i, f_i, h_i be the Chevalley generators, where $1 \leq i \leq 3$ and 2 is the central node of the corresponding Dynkin diagram.

(a) Write down the product rule in terms of the Chevalley basis. (5 marks)

(b) Let τ be the permutation of the Chevalley generators induced by the permutation $(1\,3)$ of the indexes. Prove that τ extends to an involutory (order 2) automorphism of $\mathfrak{g}$. (3 marks)

(c) Let $\mathfrak{h}$ be the set of elements of $\mathfrak{g}$ fixed by τ. Prove that $\mathfrak{h}$ is a subalgebra and that $\mathfrak{h}$ is a simple Lie algebra. (6 marks)

(d) Prove that $\mathfrak{h}$ is isomorphic to the simple Lie algebra of type B_2. (6 marks)

(Total: 20 marks)

5. Let A be a real algebra with a basis x, y, z, and u, with skew-symmetric product $[\cdot,\cdot] : A \times A \to A$ such that $[x,y] = 0 = [y,u] = [z,u]$, $[x,z] = \alpha x + \beta y$, $[y,z] = \gamma x + \delta y$, and $[x,u] = \epsilon x + \sigma y$.

(a) Let φ_z be the adjoint action of z on $B = \langle x, y \rangle$, that is, $\varphi_z : B \to B$ be defined by $\varphi_z(b) = [z,b]$ for all $b \in B$. Similarly, let ϕ_u be the adjoint action of u on B. Show that A is a Lie algebra if and only if φ_z and φ_u commute. (6 marks)

In particular, if A is a Lie algebra, show that, if $\gamma \neq 0$ then $\epsilon = 0 = \sigma$. (5 marks)

(b) Assuming that A is a Lie algebra, show that A is soluble. Explain why A is not necessarily nilpotent. (5 marks)

(c) Suppose that $\beta = 0 = \gamma = \delta = \epsilon = \sigma$. Show that the algebras A corresponding to different non-zero values of α are isomorphic. (4 marks)

(Total: 20 marks)

1.

(a) Let $\mathfrak{g}$ be an algebra under consideration, and let z be a non-zero vector in the centre. Define the mapping φ of $(\mathfrak{g}/\mathbb{C}z)^2$ to $\mathbb{C}$ by the following rule:

$$\varphi(x + \mathbb{C}z, y + \mathbb{C}z) = \alpha,$$

whenever

$$[x, y] = \alpha z.$$

By the conditions (1) φ is well defined, (2) φ determines the product rule on the whole algebra, (3) it is an alternating form on the quotient algebra, (4) the form φ is non-singular. Depending on the background one can use the classification of alternating forms to complete the classification, so that the isomorphism type is determined by the dimension which has to be an odd number. **(6 marks similar seen)**

(b) Let $b_1, b_2, ..., b_n$ be the basis of the underlying vector space. Then every element of $\mathfrak{u}(n)$ sends b_i in a vector contained in the subspace spanned by b_j for $j > i$. From this it is clear that the $(n-1)$-th power of that element sends every vector to the zero vector, hence nilpotent. **(6 marks unseen)**

(c) By (b) the nilpotency class is at most $n-1$ and since it is easy to find an element whose $(n-2)$-th power is non-zero, it is exactly $n-1$. The subalgebra spanned by the elements of $\mathfrak{u}(n)$, all whose non-zero entries have difference of row and column number less than i is a member of the central series. **(3 marks seen)**

(d) The largest subalgebra is the one having zero in all entries except the first row and the last column. Its dimension is $2n-1$. The non-zero commutator relations are $[e_{1i}, e_{i1}] = e_{1,n} =: z$. This follows from the commutator relations between the elementary matrices. **(5 marks unseen)**

2.

(a) This is by checking the definition of a module. **(3 marks seen)**

(b) Let (u, d) be the basis of V_2 so that the actions of the standard generators e, f, h are the following:

$$e : d \mapsto u \mapsto 0;$$

$$f : u \mapsto d \mapsto 0;$$

$$h : u \mapsto u, d \mapsto -d.$$

Notice that u and d are eigenvectors of h with eigenvalues 1 and -1, respectively.

In the tensor square of V_2 we choose the basis $u \otimes u$, $d \otimes d$, $u \otimes d$, $d \otimes u$. These are eigenvectors of h with eigenvalues 2, -2, 0 and 0 respectively. Further, the vector $u \otimes d - d \otimes u$ is annihilated by both e and f thus supporting the trivial representation. On the other hand, the subspace spanned by the former two basis vectors and the sum of the latter two supports the 3-dimensional irreducible representation isomorphic to the adjoint representation. **(5 marks unseen)**

(c) Choose the basis in W consisting of tensor products of copies of u and copies of d as in the tensor square example. The basis is of size 2^n, which is the dimension of W. Every such basis vector is an eigenvector of h with eigenvalue equal to the number of u's minus the number of d's. If there are k entries of d then the eigenvalue is $(n-k)-k = n-2k$ and its multiplicity is $\binom{n}{k}$. In particular, the tensor power is a direct sum of h-eigenspaces spanned by basis vectors. **(5 marks unseen)**

(d) The unique irreducible representation with largest h-eigenvalue m has dimension $2m+1$ and involves (up to scalar multiple) one i-eigenvector for every number i with $-m < i < m$. This rule enables to distribute the h-eigenvectors among irreducible components. To start the largest h-eigenvector is n with multiplicity 1 and the corresponding irreducible has dimension $2n + 1$ and its involves one eigenvector with eigenvalue $n - 1$. So, each of the remaining $n - 1$ eigenvectors gives rise to a next irreducible of dimension $2n - 1$ and so on. **(7 marks similar seen)**

3.

(a) The Cartan matrix is

$$\binom{2 \quad -3}{-1 \quad 2}$$

(3 marks seen)

(b) The roots are α, β, $\alpha + \beta$, $(2\alpha + \beta)$, $(3\alpha + \beta)$, $(3\alpha + 2\beta)$ and their negatives. The dimension of the algebra is 2 (for the dimension of the Cartan subalgebra)+12 (the number or roots), which is 14. **(5 marks seen)**

(c) We start with e_α, e_β, $e_{-\alpha}$, $e_{-\beta}$ then $e_{\alpha+\beta} := [e_\alpha, e_\beta]$, $e_{2\alpha+\beta} := [e_{\alpha+\beta}, e_\alpha]$ and so for all the roots. **(7 marks part seen)**

(d) from the expressions in (c) we write down the action of a Chevalley generator simplifying it making use of the Jacobi relation. **(5 marks seen)**

4.

(a) It might be convenient to write down the commutator relation using the realization by 4×4 matrices. **(5 marks seen)**

(b) It is manifestly from the commutators relations in (a) that τ induces an automorphism of the algebra. **(3 marks similar seen)**

(c) The fixed vectors in the Cartan subalgebra generate the Cartan subalgebra of the centralizer and the fixed vectors in the root space span the root space of the centralizer. From this it is easy to deduce that the algebra is simple of dimension 10. **(6 marks unseen)**

(d) Calculating the Cartan matrix of the centralizer subalgebra with identifying it with the B_2 subalgebra. **(6 marks part seen)**

5. (a) The result comes from the direct check of the Lie algebra axioms. **(11 marks similar seen)**

(b) The subspace X spanned by x, y is an abelian ideal while the subspace Y spanned by z and u is an abelian subalgebra and $A = X \oplus Y$ as a vector space. Hence A is solvable. On the other hand if $\alpha = 1$, $\beta = 0$ then the subalgebra spanned by z and x is 2-dimensional solvable non-nilpotent, hence the whole A is not always nilpotent (although it is abelian if all the six parameters are zeros). **(5 marks similar seen)**

(c) The isomorphism can be achieved by rescaling z. **(4 marks seen)**