

Exercise 7.1. Let (X, d) be a metric space, and V be a subset of X . Show that the set V is closed if and only if $\overline{V} = V$.

Hint: use the definition of closed sets, and the definition of the closure of a set.

Exercise 7.2. Let V and W be subsets of a metric space (X, d) . The following properties hold:

- (i) if $V \subset W$, then $V^\circ \subset W^\circ$,
- (ii) if $V \subset W$, then $\overline{V} \subset \overline{W}$,

Exercise 7.3. Let V and W be subsets of a metric space (X, d) . Prove that

$$\overline{V \cup W} = \overline{V} \cup \overline{W}.$$

Give an example of (X, d) , V and W such that

$$(V \cup W)^\circ \neq V^\circ \cup W^\circ.$$

Exercise 7.4. Let (A_1, d_1) and (A_2, d_2) be metric spaces. A map $f : A_1 \rightarrow A_2$ is continuous if and only if the pre-image of any closed set in A_2 is a closed set in A_1 .

Exercise 7.5. Recall that the set of all continuous functions from $[0, 1]$ to \mathbb{R} is denoted by $C([0, 1])$. We also defined the metrics d_1 , d_2 and d_∞ on $C([0, 1])$. Consider the map

$$\Phi : C([0, 1]) \rightarrow \mathbb{R},$$

defined as

$$\Phi(f) = f(1/2).$$

- (i) Is the map Φ from the metric space $(C([0, 1]), d_\infty)$ to (\mathbb{R}, d_1) continuous? Justify your answer.
- (ii) Is the map Φ from the metric space $(C([0, 1]), d_1)$ to (\mathbb{R}, d_1) continuous? Justify your answer.
- (iii) Is the map Φ from the metric space $(C([0, 1]), d_2)$ to (\mathbb{R}, d_1) continuous? Justify your answer.

Hint: draw the graphs of few functions, and think about what it means for two functions in $C([0, 1])$ to be close together in each of those metrics.

Exercise 7.6. Consider the metric spaces $X = (\mathbb{R}, d_1)$ and $Y = (\mathbb{R}, d_{\text{disc}})$. Show that the map $f(x) = x$ from X to Y is not continuous. Show that the map $g(x) = x$ from Y to X is continuous.

Exercise 7.7. Consider the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$, for $n \geq 1$, defined as

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : [0, 1] \rightarrow \mathbb{R}$ be the constant map $f \equiv 0$.

- (i) Show that the sequence $(f_n)_{n \geq 1}$ in $C([0, 1])$ converges to f in the metric space $(C([0, 1]), d_1)$.
- (ii) Show that the sequence $(f_n)_{n \geq 1}$ in $C([0, 1])$ does not converge to f in the metric space $(C([0, 1]), d_\infty)$.
- (iii) Conclude that the identity map

$$\text{id} : (C([0, 1]), d_1) \rightarrow (C([0, 1]), d_\infty)$$

is not continuous.

Exercise 7.8. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \rightarrow Y$ be a surjective map. Show that if f is bi-Lipschitz, then it is a homeomorphism.