

## Unseen problem sheet 2

### Analysis 1 - Spring 2023

Choose ONE of the following problems for your CW2 on 3 Mar 2023 by 1PM.

1. This question is shamelessly stolen from Example 6.2.9(b) of “Introduction to Real Analysis” by R. Bartle and D. Sherbert.

(a) Prove that if  $c \in (100, 105)$  then  $10 < \sqrt{c} < 11$ .

(b) Use the Mean Value Theorem to show that

$$\frac{5}{22} < \sqrt{105} - 10 < \frac{1}{4}$$

(c) Can you improve this estimate?

2. Examine the following proofs that use l’Hôpital’s rule, and explain why they’re invalid. Can you prove the statement anyway?

**Theorem 1.**  $\lim_{x \rightarrow 0} \frac{1}{\tan(x)} = 0$ .

*Proof.* Let  $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be given by  $f(x) = \cos(x)$ , and let  $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be given by  $g(x) = \sin(x)$ . Then  $f'(x) = -\sin(x)$  and  $g'(x) = \cos(x)$ . By l’Hôpital’s rule:

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \tan(0) = 0. \quad \square$$

**Theorem 2.**  $\lim_{x \rightarrow 0} \frac{\sin^2(x)}{1 - \cos(x)} = 2$ .

*Proof.* Let  $f(x) = \sin^2(x)$  and let  $g(x) = 1 - \cos(x)$ . Then  $f$  and  $g$  are differentiable on the interval  $(-\pi, \pi)$ , and  $\lim_{x \rightarrow 0} f(x) = 0$  and  $\lim_{x \rightarrow 0} g(x) = 0$ .

We calculate that  $f'(x) = 2 \sin(x) \cos(x)$  and  $g'(x) = \sin(x)$ , so

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{2 \sin(x) \cos(x)}{\sin(x)} = 2$$

By l’Hôpital’s rule, we get that

$$\lim_{x \rightarrow 0} \frac{\sin^2(x)}{1 - \cos(x)} = 2. \quad \square$$

**Theorem 3.**  $\lim_{x \rightarrow 0} \frac{1}{\sin(\frac{1}{x})} = 0$ .

*Proof.* Let  $f(x) = x^3$  and let  $g(x) = x^3 \sin(\frac{1}{x})$ . By choosing an  $\epsilon$  to be sufficiently small, we find that  $f$  and  $g$  are differentiable on the interval  $(-\epsilon, \epsilon) \setminus \{0\}$ , and that  $g'(x) \neq 0$  on  $(-\epsilon, \epsilon) \setminus \{0\}$ .

We calculate that  $\lim_{x \rightarrow 0} f(x) = 0$  and  $\lim_{x \rightarrow 0} g(x) = 0$ , and we also calculate that

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{3x^2}{3x^2 \sin(\frac{1}{x}) - x \cos(\frac{1}{x})} = 0.$$

By l'Hôpital's rule, we get that

$$\lim_{x \rightarrow 0} \frac{1}{\sin(\frac{1}{x})} = 0. \quad \square$$

3. Prove the following generalisation of the mean value theorem known as Taylor's theorem. Let  $f : I \rightarrow \mathbb{R}$  be twice differentiable on  $I$  and  $a \in I$ . Then, for each  $x \in I \setminus \{a\}$ , there exists  $\theta$  between  $a$  and  $x$  such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(\theta)}{2}(x - a)^2.$$

*Hint:* prove that you can apply the extended mean value theorem to obtain  $\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(\theta)}{G'(\theta)}$ , where  $F(t) = f(t) + f'(t)(x - t)$  and  $G(t) = (x - t)^2$ .

4. Newton's Method is a way of estimating the root of a function that you may already be familiar with. You can find an animation of Newton's Method at work at <https://www.intmath.com/applications-differentiation/newtons-method-interactive.php> (Thanks to Murray Bourne for allowing classroom use of his website, please see the 'About' page of the website for the terms of use.) We'll be proving the validity of the method rigorously here.

**Theorem 4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be twice differentiable on  $[a, b]$ . Suppose that  $f(a) < 0 < f(b)$  and that there are constants  $m$  and  $M$  such that  $0 < m \leq |f'(x)|$  and  $|f''(x)| \leq M$  for all  $x \in [a, b]$ .

Then there are  $a', b' \in [a, b]$  such that

- there is an  $r \in [a', b']$  such that  $f(r) = 0$ , and
- the sequence  $(x_n)$ , starting at some  $x_0 \in [a', b']$ , defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

is contained in  $[a', b']$  and  $\lim_{x \rightarrow \infty} x_n = r$ .

- (a) Prove that there is a unique  $r \in [a, b]$  such that  $f(r) = 0$ .  
 (b) Prove that for all  $x_0 \in [a, b]$  there is a  $c$  such that

$$-f(x_0) = f'(x_0)(r - x_0) + \frac{1}{2}f''(c)(r - x_0)^2.$$

Hint: use the Taylor's theorem from a previous item of this homework.

(c) Using the definition of  $x_1$  from the theorem, prove that

$$|x_1 - r| \leq \frac{M}{2m} |x_0 - r|^2.$$

(d) Prove that there is a  $\delta$  such that if  $|x_0 - r| < \delta$  then  $|x_n - r| < \delta$  for all  $n$ .

(e) Prove that  $\lim_{n \rightarrow \infty} x_n = r$ .

This definition, which we will study in detail on the lectures, is used in the following exercises:

**Definition.** A function  $f : I \rightarrow \mathbb{R}$  is called *convex* if it satisfies  $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$  for all  $a, b \in I$  and  $t \in [0, 1]$ .

5. (a) Prove Jensen's inequality: Let  $f : I \rightarrow \mathbb{R}$  be a convex function on an interval  $I$ . Let  $x_1, \dots, x_k \in I$  and let  $a_1, \dots, a_k > 0$ . Then

$$f\left(\frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i}\right) \leq \frac{\sum_{i=1}^k a_i f(x_i)}{\sum_{i=1}^k a_i}.$$

When is this an equality?

- (b) Prove the inequality of arithmetic and geometric means (AM-GM inequality): Let  $x_1, \dots, x_n \geq 0$ , then

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

6. Let  $I \subseteq \mathbb{R}$  be some interval.  $f : I \rightarrow \mathbb{R}$  is *halving convex* if for all  $x_1, x_2 \in I$ :

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

- (a) Prove that if  $f : I \rightarrow \mathbb{R}$  is halving convex, then for every  $k, n \in \mathbb{N}$  such that  $t = \frac{k}{2^n} \in [0, 1]$ :

$$\forall x_1, x_2 \in I : f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

- (b) Prove that if  $f : I \rightarrow \mathbb{R}$  is halving convex and continuous, then it is convex.

7. Prove that a convex function on an open interval is continuous. Is it true for a closed interval?

8. Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$ . Prove that the following are equivalent:

- (a)  $f$  is convex.
- (b) For all  $a, b \in I$  such that  $a < b$ : for all  $x \in [a, b]$ :  

$$f(x) \leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$
- (c) For all  $a, b \in I$  such that  $a < b$ : for all  $x \in I \setminus [a, b]$ :  

$$f(x) \geq f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$