Imperial College
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MATH97044 MATH97153

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Algebraic Geometry

Date: Friday, 21 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In this exam, all algebraic sets (=varieties) are defined over an algebraically closed field k. You can use results from lectures, the lecture notes, and problem sheets. *Budget your time—attempt all questions!* They do not always get harder.

1. **The Zariski topology of $\mathbb{A}^n$.** In this question we work in $\mathbb{A}^n = k^n$. Ideals are in $R = k[x_1, \ldots, x_n]$. An affine algebraic set (or variety) is a closed subset of $\mathbb{A}^n$ for some n.

(a) Recall the definition of the radical, $\operatorname{rad} I$, of an ideal I. Prove that $\mathbb{V}(\operatorname{rad} I) = \mathbb{V}(I)$. (3 marks)

(b) For each of the following sets, say whether, in the Zariski topology in $\mathbb{A}^n$, they are: (A) closed, (B) locally closed but not closed, (C) constructible but not locally closed, or (D) not constructible. Recall that "locally closed" means an open subset of a closed set, and "constructible" means a finite union of locally closed sets. *Justify your answers.*

1. The union $\{x = 0\} \cup \{y = 0\}$ in $\mathbb{A}^2$.
2. The locus $\{y^3 = xy\} \setminus \{(0, 0)\}$ in $\mathbb{A}^2$.
3. The image $\operatorname{im}(\varphi)$, for the map $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ defined by $(x, y) \mapsto (x^2 y, y)$.
4. The locus $\mathbb{Z} \subseteq \mathbb{A}^1$, *in the case that* $\operatorname{char} k = 0$. (6 marks)

(c) Now do the same thing as in the previous example, but for each say whether it is, in the Zariski topology, (A) irreducible; (B) connected but reducible; or (C) disconnected. (6 marks)

(d) Let V_1, V_2 be affine algebraic sets. Are the open subsets of $V_1 \times V_2$ precisely the unions of products of open sets $U_1 \times U_2$? Give a proof or counterexample. (2 marks)

(e) Let $V \subseteq \mathbb{A}^n$ be any subset, equipped with the subspace topology. Show that V is quasi-compact, i.e., every open covering of V has a finite subcovering. (3 marks)

(Total: 20 marks)

2. **Rational and regular maps.**

(a) Compute the ideal $\mathbb{I}(V)$ in terms of generators, and compute the ring of regular functions $k[V]$, on each of the following:

(i) $V = \{x\text{-axis}\} \cup \{x = y\} \cup \{y\text{-axis}\}$ inside $\mathbb{A}^2$ with coordinates x, y. (2 marks)

(ii) $V = \{x\text{-axis}\} \cup \{y\text{-axis}\} \cup \{z\text{-axis}\}$ inside $\mathbb{A}^3$ with coordinates x, y, z. (2 marks)

(b) (i) Let V by an affine algebraic set. Given a union $V = V_1 \cup \cdots \cup V_n$ with V_i closed subsets of V, define the pullbacks $k[V] \to k[V_i]$ of the inclusions $V_i \to V$. Use this to construct an embedding of $k[V]$ as a subalgebra of $k[V_1] \times \cdots \times k[V_n]$. (2 marks)

(ii) Let $V = V_1 \cup V_2 \cup V_3$ with $V_1 = \{y = 0\}$, $V_2 = \{x = 0\}$, and $V_3 = \{x = y\}$. Prove that, under the embedding of part (b)(i) above,

1. $k[V_i \cup V_j] = \{(f_i, f_j) \in k[V_i] \times k[V_j] \mid f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}\}$ for all i, j, but
2. $k[V] \subsetneq \{(f_1, f_2, f_3) \in k[V_1] \times k[V_2] \times k[V_3] \mid f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}, \forall i, j\}$. (4 marks)

(c) Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $f(x, y) = (x, xy)$.

(i) Prove that f is birational and find a rational inverse to f. (2 marks)

(ii) Find the regular locus (or "domain") of f^{-1}, with proof. (3 marks)

(d) Let $V = \{xy = 1\} \subseteq \mathbb{A}^2$ (the "hyperbola"). Find all regular isomorphisms $V \to V$. (5 marks)

(Total: 20 marks)

3. **Quasi-projective varieties.** In this question V and W are quasi-projective varieties (=algebraic sets). Recall the following useful fact from problem sheets: *Every rational map* $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ *is regular.*

(a) Let $f \in k[x_1, \ldots, x_n]$ and $V := \mathbb{V}(f)$.

(i) Explain how to embed $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$. (2 marks)

(ii) Define a homogeneous polynomial $\widetilde{f} \in k[x_0, \ldots, x_n]$ such that $\mathbb{V}(\widetilde{f}) = \overline{V} \subseteq \mathbb{P}^n$. You do not have to verify that this holds. (2 marks)

(b) Now let $f \in k[x_1, x_2]$ be an irreducible quadratic polynomial and $C := \mathbb{V}_{\mathbb{A}^2}(f) \subseteq \mathbb{A}^2$ be a quadric curve. We consider as before $\overline{C} = \mathbb{V}(\widetilde{f}) \subseteq \mathbb{P}^2$ for $\widetilde{f} \in k[x_0, x_1, x_2]$ the homogenisation.

(i) For any $p \in \overline{C}$, consider the projection in $\mathbb{P}^2$ from p to the line: $\{[a : b : 0] \mid a, b \in k\} \subseteq \mathbb{P}^2$. Show that this is a rational map which is regular on $\overline{C} \setminus \{p\}$. (2 marks)

(ii) Construct a rational inverse $\pi^{-1} : \{[a : b : 0] \mid a, b \in k\} \dashrightarrow \overline{C}$. (Hint: change coordinates so that $p = [0 : 0 : 1]$, and write $\widetilde{f}$ in the form $\sum_i g_i(x_0, x_1)x_2^i$). (3 marks)

(iii) Show that π^{-1} is regular, $\pi^{-1} : \{[a : b : 0] \mid a, b \in k\} \cong \mathbb{P}^1 \to \overline{C}$. (2 marks)

(iv) Prove that π is also regular and hence that $\overline{C} \cong \mathbb{P}^1$ (Hint: don't forget that the point $p \in \overline{C}$ was arbitrary). (4 marks)

(c) Prove that every birational map $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is actually regular and of the form $\varphi([x : y]) = [ax + by : cx + dy]$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible and uniquely defined up to scaling. (5 marks)

(Total: 20 marks)

4. **Dimension theory.**

(a) Let $\varphi : \mathbb{P}^m \to \mathbb{P}^n$ be a regular map and let $\Gamma_\varphi = \{(p, \varphi(p)) \mid p \in \mathbb{P}^m\} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ be the graph.

(i) Show that the map $\mathbb{P}^m \to \Gamma_\varphi$, $p \mapsto (p, \varphi(p))$ is an isomorphism. (2 marks)

(ii) Using the preceding, show that Γ_φ is closed, irreducible, and of dimension m. (2 marks)

(b) Prove that every regular map $\varphi : \mathbb{P}^m \to \mathbb{P}^n$ is constant if $m > n$. (Hint: You may use the fact from the first problem of Problems Sheet 6–7, showing this is defined by a single tuple of polynomials.) (4 marks)

(c) Let $\Sigma \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the subset of triples of collinear points (meaning they all lie on the same line).

(i) Let $T := \{\mathbb{V}_{\mathbb{P}^2}(ax_0 + bx_1 + cx_2) \mid (a, b, c) \in k^3 \setminus \{(0, 0, 0)\}\}$ be the set of all lines in $\mathbb{P}^2$. Give a bijection between T and $\mathbb{P}^2$. (1 mark)

(ii) Define the variety

$$\widetilde{\Sigma} := \{(p, q, r, \ell) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times T \mid \ p,\ q,\ r \text{ lie on the line } \ell\},$$

viewing T as $\mathbb{P}^2$. Show that $\widetilde{\Sigma}$ is closed. (2 marks)

(iii) Show that $\widetilde{\Sigma}$ is irreducible of dimension five (Hint: look at the projection to the last factor T). (3 marks)

(iv) Using the preceding parts, show that Σ is closed and irreducible of dimension five. (4 marks)

(v) Find an explicit trilinear equation which defines Σ. (2 marks)

(Total: 20 marks)

5. **Mastery: MaxSpec and Spec.**

(a) Consider $V := \mathbb{V}(3x^2 + 3y^2) \subseteq \operatorname{MaxSpec} \mathbb{Z}[x, y]$.

(i) Find the irreducible components of V. (3 marks)

(ii) Now consider $V_p := V \cap \mathbb{V}(p)$ for p prime. Show how to realise V_p as a subset of $\mathbb{A}^2_{\mathbb{F}_p} := \operatorname{MaxSpec} \mathbb{F}_p[x, y]$, compatibly with the Zariski topology. (2 marks)

(iii) Continuing as in the preceding part, show the following:

1. If $p = 2$, then V_p is a line in $\mathbb{A}^2_{\mathbb{F}_2}$, i.e., the solution of a linear equation.
2. If $p = 3$, then $V_p = \mathbb{A}^2_{\mathbb{F}_3}$.
3. If $p > 3$ is congruent to 1 modulo 4, then V_p is a union of two lines.
4. If $p > 3$ is congruent to 3 modulo 4, then V_p is the vanishing locus of an irreducible quadratic polynomial. (4 marks)

(b) State the Nullstellensatz for $\operatorname{Spec} R$. Give an example of a commutative ring R where the analogous statement does not hold for $\operatorname{MaxSpec} R$ and explain how it fails. (3 marks)

(c) Let $R := k[x]_{(x)}$, the local ring (recall it is given from $k[x]$ by inverting all elements which are not multiples of x). Recall that the prime ideals are (0) and (x). Recall also that given a topological space X, the topological dimension is given as the maximum length n of a chain $X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n$, where X_i are all closed irreducible subsets of X.

(i) Find all closed subsets of $\operatorname{Spec} R = \{0, (x)\}$. (2 marks)

(ii) Find all irreducible subsets of $\operatorname{Spec} R$. (2 marks)

(iii) Compute the topological dimension of $\operatorname{Spec} R$. (2 marks)

(iv) Does $\operatorname{Spec} R$ satisfy the property that, if $X, Y \subseteq \operatorname{Spec} R$ are arbitrary subsets, then $\dim(X \cup Y) = \max(\dim X, \dim Y)$? Why or why not? (2 marks)

(Total: 20 marks)

Solutions: Algebraic Geometry, 2021 exam

Question 1. ***The Zariski topology of*** $\mathbb{A}^n$**.** *In this question we work in* $\mathbb{A}^n = k^n$*. Ideals are in* $R = k[x_1, \ldots, x_n]$*. An affine algebraic set (or variety) is a closed subset of* $\mathbb{A}^n$ *for some* n*.*

Part 1(a). *Recall the definition of the radical,* $\operatorname{rad} I$*, of an ideal* I*. Prove that* $\mathbb{V}(\operatorname{rad} I) = \mathbb{V}(I)$*.*

3, A

Define $\operatorname{rad} I = \{f \in R \mid f^m \in I \text{ for some } m\}$. Then $x \in \mathbb{V}(\operatorname{rad} I)$ if and only if $f(x) = 0$ for all $f \in \operatorname{rad} I$, which is true if and only if $f(x)^n = 0$ for each $n \geq 1$, so if and only if $g(x) = 0$ for all $g \in I$.

Part 1(b). *For each of the following sets, say whether, in the Zariski topology in* $\mathbb{A}^n$*, they are: (A) closed, (B) locally closed but not closed, (C) constructible but not locally closed, or (D) not constructible. Recall that "locally closed" means an open subset of a closed set, and "constructible" means a finite union of locally closed sets.* Justify your answers.

(1) The union $\{x = 0\} \cup \{y = 0\}$ *in* $\mathbb{A}^2$*.*
(2) The locus $\{y^3 = xy\} \setminus \{(0,0)\}$ *in* $\mathbb{A}^2$*.*
(3) The image $\operatorname{im}(\varphi)$*, for the map* $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ *defined by* $(x, y) \mapsto (x^2y, y)$*.*
(4) The locus $\mathbb{Z} \subseteq \mathbb{A}^1$*,* in the case that $\operatorname{char} k = 0$.

6, A

(1) (A). This is the locus $\mathbb{V}(xy)$, hence closed.

(2) (B). This is locally closed since $\mathbb{V}(y^3 - xy)$ is closed as is $\{0, 0\}$, but it is not closed: Every function vanishing on this locus, written as a polynomial of x with coefficients in a polynomial in y, specialises at $y = 0$ to a polynomial in x which vanishes everywhere except at $x = 0$, and only the zero polynomial has this property. Thus every function vanishing on this locus is a multiple of y and hence vanishes at the origin.

(3) (C). This locus is constructible: it is a union of $\mathbb{A}^2 \setminus \{y = 0\}$ with the origin, and these are open and closed respectively. But it is not locally closed, since its closure is $\mathbb{A}^2$ itself, and the difference is $\{y = 0\} \setminus \{(0, 0)\}$ which we showed was not locally closed in (2).

(4) (D). This is not constructible: the Zariski topology is the cofinite one, so that locally closed subsets are all either finite (hence closed) or cofinite (hence open). Finite unions of these sets are the same, so in a cofinite topological space, locally closed and constructible coincide. But $\mathbb{Z}$ is neither finite nor cofinite.

Part 1(c). *Now do the same thing as in the previous example, but for each say whether it is, in the Zariski topology, (A) irreducible; (B) connected but reducible; or (C) disconnected.*

6, B

(1) (B). This is reducible, with connected components the two coordinate axes. But it is connected: since if we write it as a disjoint union of two open subsets, one of them must contain the origin, and any open set containing the origin has intersection with each line $x = 0$ and $y = 0$ which is open and dense. Then since the lines have the cofinite topology, the other set must be finite, but this cannot be open as then its restriction on at least one line would be cofininte, a contradiction.

(2) (C). This is disconnected, as it is the union of the two disjoint open subsets $\{y = 0\} \setminus \{0, 0\}$ and $\{y^2 = x\} \setminus \{0, 0\}$. These sets are open since they are the complements of the closed sets $\{y^2 = x\}$ and $\{y = 0\}$.

(3) (A). This set is irreducible, since its closure is all of $\mathbb{A}^2$, which is irreducible.

(4) (A). This set is also irreducible for the same reason as in (3): its closure is all of $\mathbb{A}^1$, which is irreducible.

Part 1(d). *Let V_1, V_2 be affine algebraic sets. Are the open subsets of $V_1 \times V_2$ precisely the unions of products of open sets $U_1 \times U_2$? Give a proof or counterexample.*

2, B

No, they are not. For example, let $V_1 = V_2 = \mathbb{A}^1$ and consider the complement of the diagonal. The products of open subsets all contain one of the form $U \times U$ for some cofinite set U, but this is not contained in the complement of the diagonal.

Part 1(e). *Let $V \subseteq \mathbb{A}^n$ be any subset, equipped with the subspace topology. Show that V is quasi-compact, i.e., every open covering of V has a finite subcovering.*

3, B

Let's pick a sequence of open subsets $U_{i_1}, U_{i_2}, \ldots$ of V such that, at each stage, either $U_{i_1} \cup \cdots \cup U_{i_n} = V$ in which case we are done, or else $U_{i_1} \cup \cdots \cup U_{i_{n-1}} \subsetneq U_{i_1} \cup \cdots \cup U_{i_n}$. This is possible since the entire union is V. Taking complements we have a strictly decreasing chain of closed subsets of V. These are the intersections of closed subsets $Z_1, Z_2, \ldots$ of $\mathbb{A}^n$ with V, by definition of the subspace topology on V. We can replace Z_i with $Z_1 \cap \cdots \cap Z_i$ so that this chain is also strictly decreasing: $Z_1 \supsetneq Z_2 \supsetneq \cdots$. By Hilbert's basis theorem this must terminate, so for some n we arrive at $U_{i_1} \cup \cdots \cup U_{i_n} = V$.

Question 2. ***Rational and regular maps.***

Part 2(a). *Compute the ideal $\mathbb{I}(V)$ in terms of generators, and compute the ring of regular functions $k[V]$, on each of the following:*

Subpart 2(a)(i). $V = \{x - axis\} \cup \{x = y\} \cup \{y - axis\}$ *inside* $\mathbb{A}^2$ *with coordinates* x, y.

2, A

This is cut out by the equation $xy(x-y)$, so $\mathbb{I}(V)$ is the radical of $(xy(x-y))$, but this is radical since it has no irreducible factors of multiplicity greater than one and $k[x,y]$ is a UFD. Thus $k[V] = k[x,y]/\mathbb{I}(V) = k[x,y]/(xy(x-y))$.

Alternative proof: if a polynomial $f(x,y)$ has $f(t,0) = 0 = f(0,t) = f(t,t)$ for all t, then $f(t,0), f(0,t), f(t,t) \in k[t]$ are all identically zero since k is infinite, and then we get that it is a multiple of xy with the property that the sum of the coefficients in each degree is zero, i.e., a multiple of $xy(x-y)$).

Note: we may embed $k[V]$ into $k[x] \oplus k[x] \oplus k[y]$ via the map $f \mapsto (f|_{y=0}, f|_{y=x}, f|_{x=0})$, and the image is the subalgebra generated by $(x, x, 0)$ and $(0, x, y)$.

Subpart 2(a)(ii). $V = \{x - axis\} \cup \{y - axis\} \cup \{z - axis\}$ *inside* $\mathbb{A}^3$ *with coordinates* x, y, z.

2, B

This is cut out by xy, xz, and yz, and again (xy, xz, yz) is radical as it is the intersection of radical ideals $(xy), (xz), (yz)$, radical since they are elements with no irreducible factors of multiplicity greater than one, and $k[x,y,z]$ is a UFD. Thus $k[V] = k[x,y,z]/(xy, xz, yz)$.

Alternative proof: An element which is not in the ideal must be a polynomial with some term which is a monomial in only one of the variables, but then the same is true for any power of this element.

Note: $k[V] = k[x,y,z]/(xy, xz, yz) \cong \{(f, g, h) \in k[x] \times k[y] \times k[z] \mid f(0) = g(0) = h(0)\}$, or concretely $k \oplus xk[x] \oplus yk[y] \oplus zk[z]$ with $xy = yz = xz = 0$ but with $c \in k$ acting by usual scalar multiplication.

Part 2(b).

Subpart 2(b)(i). *Let V by an affine algebraic set. Given a union $V = V_1 \cup \cdots \cup V_n$ with V_i closed subsets of V, define the pullbacks $k[V] \to k[V_i]$ of the inclusions $V_i \to V$. Use this to construct an embedding of $k[V]$ as a subalgebra of $k[V_1] \times \cdots \times k[V_n]$.*

2, A

The maps are given by the restriction, $k[V] \to k[V_i]$, $f \mapsto f|_{V_i}$. The resulting map $f \mapsto (f|_{V_1}, \ldots, f|_{V_n})$ is injective because if $f|_{V_i} = 0$ for all i then $f = 0$. It is clearly an algebra morphism.

Subpart 2(b)(ii). *Let $V = V_1 \cup V_2 \cup V_3$ with $V_1 = \{y = 0\}$, $V_2 = \{x = 0\}$, and $V_3 = \{x = y\}$. Prove that, under the embedding of part (b)(i) above,*

(1) $k[V_i \cup V_j] = \{(f_i, f_j) \in k[V_i] \times k[V_j] \mid f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}\}$ for all i, j, but
(2) $k[V] \subsetneq \{(f_1, f_2, f_3) \in k[V_1] \times k[V_2] \times k[V_3] \mid f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}, \forall i, j\}$.

4, C

For the first part, take $i = 1, j = 2$. Then we are saying that $k[x, y]/(xy) \to k[x, y]/(x) \times k[x, y]/(y)$ has image $k(1, 1) + (yk[y] \times xk[x])$. Since the source is spanned by k, $xk[x]$ and $yk[y]$, we can check this on each of these, where the images are $k(1, 1)$, $\{0\} \times xk[x]$, and $yk[y] \times \{0\}$, respectively.

For the second part, note that the map preserves polynomial degree: if $f(x, y)$ is homogeneous of degree d, then $f(x, 0)$, $f(0, y)$, and $f(x, x)$ are all also homogeneous of degrees d. Looking at the subspaces of degree 1, the source $k[V]_1$ has dimension ≤ 2 (since it is spanned by x and y), but the target has dimension three (with basis $(x, 0, 0), (0, y, 0)$, and $(0, 0, x)$). So the embedding from (i) cannot be surjective.

Part 2(c). *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $f(x, y) = (x, xy)$.*

Subpart 2(c)(i). *Prove that f is birational and find a rational inverse to f.*

2, A

A rational inverse is given by $(u, v) \mapsto (u, v/u)$.

Subpart 2(c)(ii). *Find the regular locus (or "domain") of f^{-1}, with proof.*

3, A

The regular locus is $\mathbb{A}^2 \setminus \{(0, v) \mid v \in k\}$. This is because u is regular, but if f/g is any other fraction which is an expression for v/u, then in lowest terms it must be v/u as $k[u, v]$ is a UFD. Hence $u \mid g$.

Part 2(d). *Let $V = \{xy = 1\} \subseteq \mathbb{A}^2$ (the "hyperbola"). Find all regular isomorphisms $V \to V$.*

5, D

This is equivalent to automorphisms of the ring of regular functions $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$. Note that the only invertible elements of $k[x, x^{-1}]$ are constant multiples of powers of x: indeed, $\frac{f(x)}{x^m} \frac{g(x)}{x^n} = 1$ implies that $f(x)g(x) = x^{m+n}$, and by unique factorisation, each of $f(x)$ and $g(x)$ is a constant multiple of a power of x. Now an automorphism must send x to an invertible element, hence to cx^m for some $c \in k^\times$. Since then x^{-1} has to map to $c^{-1}x^{-m}$, the choice of c and m uniquely determines a homomorphism $k[x, x^{-1}] \to k[x, x^{-1}]$, whose image consists of polynomials in x^m. To be surjective we need $m = \pm 1$. Then we see that the result is invertible, with inverse $x \mapsto (c^{-1}x)^m$. Thus these all define automorphisms. Translating back to geometry the resulting automorphisms are: $(x, y) \mapsto (cx, c^{-1}y)$ and $(x, y) \mapsto (cy, c^{-1}x)$, for $c \in k^\times$.

Question 3. *Quasi-projective varieties.* *In this question V and W are quasi-projective varieties (=algebraic sets). Recall the following useful fact from problem sheets:* Every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.

Part 3(a). *Let $f \in k[x_1, \ldots, x_n]$ and $V := \mathbb{V}(f)$.*

Subpart 3(a)(i). *Explain how to embed $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$.*

2, A

We can identify $\mathbb{A}^n$ with $\{[x_0 : \cdots : x_n] \mid x_i \neq 0\} \subseteq \mathbb{P}^n$ via the map $(x_1, \ldots, x_n) \mapsto [x_1 : x_2 : \ldots : x_i : 1 : x_{i+1} : \cdots : x_n]$. Below we set $i = 0$.

Subpart 3(a)(ii). *Define a homogeneous polynomial $\widetilde{f} \in k[x_0, \ldots, x_n]$ such that $\mathbb{V}(\widetilde{f}) = \overline{V} \subseteq \mathbb{P}^n$. You do not have to verify that this holds.*

2, A

This is the homogenisation: if $f = \sum_{j=0}^{d} f_j$ where f_j is homogeneous of degree j and $d = \deg f$, then we set $\widetilde{f} = \sum_{j=0}^{d} x_0^{d-j} f_j$.

Why this holds (*not required in solution*): Like this we have $\mathbb{V}(\widetilde{f}) \cap \mathbb{A}_0^n = \mathbb{V}(f)$ since $\widetilde{f}|_{x_0=1} = f$. Therefore, since $\mathbb{V}(\widetilde{f})$ is closed, we have $\mathbb{V}(\widetilde{f}) \supseteq \overline{V}$. Conversely, suppose that $g \in \mathbb{I}(\overline{V})$. Then $g|_{x_0=1}$ vanishes on V. By the Nullstellensatz, $(g|_{x_0=1})^m = g^m|_{x_0=1}$ is a multiple of f for some $m \geq 1$. Note that every homogeneous polynomial whose restriction to $x_0 = 1$ is h is a multiple of $\widetilde{h}$. Thus g^m is a multiple of $\widetilde{f}$. We get therefore that $\mathbb{I}(\overline{V}) \subseteq \operatorname{rad}(\widetilde{f})$. So $\overline{V} \supseteq \mathbb{V}(\widetilde{f})$. We thus have $\overline{V} = \mathbb{V}(\widetilde{f})$.

Part 3(b). *Now let $f \in k[x_1, x_2]$ be an irreducible quadratic polynomial and $C := \mathbb{V}_{\mathbb{A}^2}(f) \subseteq \mathbb{A}^2$ be a quadric curve. We consider as before $\overline{C} = \mathbb{V}(\widetilde{f}) \subseteq \mathbb{P}^2$ for $\widetilde{f} \in k[x_0, x_1, x_2]$ the homogenisation.*

Subpart 3(b)(i). *For any $p \in \overline{C}$, consider the projection in $\mathbb{P}^2$ from p to the line: $\{[a : b : 0] \mid a, b \in k\} \subseteq \mathbb{P}^2$. Show that this is a rational map which is regular on $\overline{C} \setminus \{p\}$.*

2, A

Note: there is a minor error in the statement: one should require that p not be in the line, otherwise the projection is not defined (or is degenerate).

This was done in lectures, but we give the argument. It is enough to show that the map is regular on $\overline{C} \setminus \{p\}$ since then it is rational by definition. On $\mathbb{P}^2$ the projection is defined in homogeneous coordinates by the linear projection $k^3 \to k^2 = \{(a, b, 0)\}$ with kernel the line corresponding to p. This is clearly regular everywhere except at p.

Subpart 3(b)(ii). *Construct a rational inverse $\pi^{-1} : \{[a : b : 0] \mid a, b \in k\} \dashrightarrow \overline{C}$. (Hint: change coordinates so that $p = [0 : 0 : 1]$, and write $\widetilde{f}$ in the form $\sum_i g_i(x_0, x_1) x_2^i$).*

3, B

Up to change of coordinates we can assume that $p = [0 : 0 : 1]$ (and we project onto $\{[a : b : 0]\}$). Then the projection is $[a : b : c] \mapsto [a : b : 0]$. Since $\widetilde{f}$ vanishes at p it can be written as $\widetilde{f} = g_0(x_0, x_1) + g_1(x_0, x_1)x_2$ where g_0 is homogeneous quadratic and g_1 is linear. Since it is quadratic and irreducible, and the field is algebraically closed, we must have both g_0 and g_1 nonzero, and g_0 cannot be a multiple of g_1. Given $\lambda = [a : b] \in \mathbb{P}^1$, the line through λ and p is $[at : bt : s]$, and its intersection with $\overline{C} \setminus \{p\}$ is at $[a : b : t]$ where $g_0(a, b) + g_1(a, b)t = 0$. Thus we get a rational map $\mathbb{P}^1 \to \overline{C}$ given by $[a : b] \mapsto [ag_1(a, b) : bg_1(a, b) : -g_0(a, b)]$. By construction it is inverse to the projection.

Subpart 3(b)(iii). *Show that π^{-1} is regular, $\pi^{-1} : \{[a : b : 0] \mid a, b \in k\} \cong \mathbb{P}^1 \to \overline{C}$.*

2, A

This follows from the useful fact mentioned, or we can just see from the proof that this inverse is regular: there we see that it would only be irregular at the common zero locus of $g_0, g_1 \in k[x_0, x_1]$, but these are homogeneous of degrees two and one, so this can only happen if $g_1 \mid g_0$. In this case though $\widetilde{f}$ would be reducible, which is a contradiction.

Subpart 3(b)(iv). *Prove that π is also regular and hence that $\overline{C} \cong \mathbb{P}^1$ (Hint: don't forget that the point $p \in \overline{C}$ was arbitrary).*

4, D

From the preceding we get that $\overline{C}\setminus\{p\}$ is isomorphic to its image in $\mathbb{P}^1$. As the point p was arbitrary, we can also take another point $q \in \overline{C}$ and get $\pi_q : \overline{C} \setminus \{q\} \cong U \subseteq \mathbb{P}^1$. We can also restrict π to a rational map $\pi : \overline{C} \setminus \{q\} \dashrightarrow \mathbb{P}^1$. By the preceding isomorphism π_q, we can view π as a rational map $\mathbb{P}^1 \supseteq \overline{C} \setminus \{q\} \dashrightarrow \mathbb{P}^1$, so it extends to a regular map over $\mathbb{P}^1$, and in particular over $\overline{C} \setminus \{q\} \subseteq \mathbb{P}^1$. But then π is regular at p, hence regular (since we already knew it was regular elsewhere). Now π and its inverse are regular maps which are inverse as rational maps, hence their compositions are the identity and they are isomorphisms.

Part 3(c). *Prove that every birational map $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is actually regular and of the form $\varphi([x : y]) = [ax + by : cx + dy]$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible and uniquely defined up to scaling.*

5, C

By the fact at the beginning of the problem, the first statement is clear. The rational inverse must then also be regular. As the compositions are then regular maps which are the identity over a dense open subset, they are the identity, and the map φ is a regular automorphism.

First note that the transformations given, $\varphi([x : y]) = [ax+by : cx+dy]$ for invertible matrices, are automorphisms since they are projective changes of coordinates (the inverse is given by the inverse matrix). We have to prove the converse.

Let φ be a regular automorphism of $\mathbb{P}^1$. If $\varphi(\infty) = \infty := [0 : 1]$, we obtain an automorphism of $\mathbb{A}^1_0$. If not, say $\varphi(\infty) = \lambda$, we can compose by $x \mapsto (x - \lambda)^{-1}$, $[x_0 : x_1] \mapsto [x_1 - \lambda x_0 : x_0]$, so that the result is an automorphism fixing ∞. Since the desired transformations are stable under composition (multiplication of linear maps), it suffices to therefore assume $\varphi(\infty) = \infty$. Now we are reduced to the statement that an automorphism of $\mathbb{A}^1$ is an invertible linear transformation, done on the Problem Sheets. Here is a quick proof: if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is invertible for $a_n \neq 0$, then plugging in its inverse we have to get $f(g(x)) = x$. But the degree of $f(g(x))$ is the product of the degrees, and for 1 to be a multiple of n, we need $n = 1$.

Question 4. ***Dimension theory.***

Part 4(a). *Let $\varphi : \mathbb{P}^m \to \mathbb{P}^n$ is a regular map and let $\Gamma_\varphi = \{(p, \varphi(p)) \mid p \in \mathbb{P}^m\} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ be the graph.*

Subpart 4(a)(i). *Show that the map $\mathbb{P}^m \to \Gamma_\varphi$, $p \mapsto (p, \varphi(p))$ is an isomorphism.*

2, A

This map is regular since it is a product of regular maps and, composing with the Segre embedding, one gets a regular map to $\mathbb{P}^{(m+1)(n+1)-1}$. It is inverted by the first projection map (which is obviously regular).

Subpart 4(a)(ii). *Using the preceding, show that Γ_φ is closed, irreducible, and of dimension m.*

2, A

It follows from the preceding part that Γ_φ is irreducible of dimension m since $\mathbb{P}^m$ is. Now we explained in lectures (end of Lecture 21) that this was closed: it is the preimage under $\varphi \times \mathrm{Id} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ of the diagonal, and the latter is defined by equations $x_i y_j = x_j y_i$ and hence closed. Alternatively, Γ_φ is the image of a regular map from $\mathbb{P}^m$ which is complete (by a theorem from lectures), hence it is complete and therefore closed.

Part 4(b). *Prove that every regular map $\varphi : \mathbb{P}^m \to \mathbb{P}^n$ is constant if $m > n$. (Hint: You may use the fact from the first problem of Problems Sheet 6–7, showing this is defined by a single tuple of polynomials.)*

4, B

Thanks to the fact mentioned in the hint, we can assume that $\varphi(p) = [f_0(p) : \cdots : f_n(p)]$ for all $p \in \mathbb{P}^m$. Now, the regular locus is the complement of $\mathbb{V}(f_0, \ldots, f_n)$. If $m \geq n+1$, then by a theorem from lectures, this is nonempty of dimension at least $m - n - 1$.

Part 4(c). *Let $\Sigma \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the subset of triples of collinear points (meaning they all lie on the same line).*

Subpart 4(c)(i). *Let $T := \{\mathbb{V}_{\mathbb{P}^2}(ax_0 + bx_1 + cx_2) \mid (a, b, c) \in k^3 \setminus \{(0, 0, 0)\}\}$ be the set of all lines in $\mathbb{P}^2$. Give a bijection between T and $\mathbb{P}^2$.*

1, A

The bijection is $[a : b : c] \mapsto \mathbb{V}(ax_0 + bx_2 + cx_2)$. This map is well-defined because $\mathbb{V}(f)$ does not depend on a nonzero scaling of f.

Subpart 4(c)(ii). *Define the variety*

$$\widetilde{\Sigma} := \{(p, q, r, \ell) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times T \mid \ p,\ q,\ r \text{ lie on the line } \ell,$$

viewing T as $\mathbb{P}^2$. Show that $\widetilde{\Sigma}$ is closed.

2, A

The condition $p \in \ell$ is itself closed: it is defined by a bilinear equation in p and ℓ, constant in q and r. So we intersect these conditions for p, q, and r.

Subpart 4(c)(iii). *Show that $\widetilde{\Sigma}$ is irreducible of dimension five (Hint: look at the projection to the last factor T).*

3, C

The fourth projection $\pi_4 : \widetilde{\Sigma} \to \mathbb{P}^2$ is surjective. The fibre over ℓ is ℓ^3, which is irreducible and of dimension three. Thus by a theorem from lectures on fibre dimension, $\widetilde{\Sigma}$ is irreducible of dimension $3 + 2 = 5$.

Subpart 4(c)(iv). *Using the preceding parts, show that Σ is closed and irreducible of dimension five.*

4, D

We can project $\widetilde{\Sigma}$ to the first three factors, $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. This is *not* surjective since not all triples of points are collinear. But when three points are collinear, if they are not all equal (which is an open condition), there is a unique line through them. So the generic fibre over the image of this projection is a point, which is zero-dimensional. By the other fibre dimension theorem, the image of $\widetilde{\Sigma}$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is $5 - 0 = 5$-dimensional. Since $\widetilde{\Sigma}$ is irreducible, so is this image. But this image is just Σ itself. Since it is the image of a projective, hence complete, variety (by a theorem from lectures), it is itself complete, and hence closed.

Subpart 4(c)(v). *Find an explicit trilinear equation which defines Σ.*

2, D

This is the determinant: writing the three vectors into a three-by-three matrix, the determinant is the condition that the three are linearly independent, which is equivalent to the condition that the three vectors span at most a two-dimensional subspace, i.e., the three points lie on a line.

Question 5. ***Mastery: MaxSpec and Spec.***

Part 5(a). *Consider* $V := \mathbb{V}(3x^2 + 3y^2) \subseteq \operatorname{MaxSpec} \mathbb{Z}[x, y]$.

Subpart 5(a)(i). *Find the irreducible components of* V.

3, M

We can factor $3x^2 + 3y^2 = 3(x^2 + y^2)$. Now $x^2 + y^2$ is irreducible, so we get $V = \mathbb{V}(3) \cup \mathbb{V}(x^2 + y^2)$. Moreover since $\mathbb{Z}[x, y]$ is a UFD, the ideals (3) and $(x^2 + y^2)$ are prime, hence these are the irreducible components.

Subpart 5(a)(ii). *Now consider* $V_p := V \cap \mathbb{V}(p)$ *for* p *prime. Show how to realise* V_p *as a subset of* $\mathbb{A}^2_{\mathbb{F}_p} := \operatorname{MaxSpec} \mathbb{F}_p[x, y]$, *compatibly with the Zariski topology.*

2, M

Note that $V_p = \{\mathfrak{m} \subseteq \mathbb{Z}[x, y] \text{ maximal}, \mathfrak{m} \supseteq (3x^2+3y^2, p)$. By the third isomorphism theorem we can identify this with $\{\mathfrak{m} \subseteq \mathbb{Z}[x, y]/(p) \mid \mathfrak{m} \supseteq (3x^2 + 3y^2)\} \subseteq \mathbb{A}^2_{\mathbb{F}_p}$.

Subpart 5(a)(iii). *Continuing as in the preceding part, show the following:*

(1) If $p = 2$, *then* V_p *is a line in* $\mathbb{A}^2_{\mathbb{F}_2}$, *i.e., the solution of a linear equation.*
(2) If $p = 3$, *then* $V_p = \mathbb{A}^2_{\mathbb{F}_3}$.
(3) If $p > 3$ *is congruent to* 1 *modulo* 4, *then* V_p *is a union of two lines.*
(4) If $p > 3$ *is congruent to* 3 *modulo* 4, *then* V_p *is the vanishing locus of an irreducible quadratic polynomial.*

4, M

(1) When $p = 2$ we get $V_2 = \mathbb{V}_{\mathbb{A}^2_{\mathbb{F}_2}}(x^2 + y^2)$, but $x^2 + y^2 = (x + y)^2$ modulo two, which gives the statement.

(2) When $p = 3$ we get $V_3 = \mathbb{V}_{\mathbb{A}^2_{\mathbb{F}_3}}(0) = \mathbb{A}^2_{\mathbb{F}_3}$.

(3) When $p > 3$ we get $V_3 = \mathbb{V}_{\mathbb{A}^2_{\mathbb{F}_p}}(x^2 + y^2)$. This factors into linear factors, $(x + \zeta y)(x - \zeta y)$ where $\zeta^2 = -1$ (such a solution exists in $\mathbb{F}_p$ since $\mathbb{F}_p^\times$ is cyclic of order a multiple of four).

(4) Now $x^2 + y^2$ is irreducible (it factors over the algebraic closure, but this requires -1 to be a square, which it is not in $\mathbb{F}_p$). So this follows as in (iii).

Part 5(b). *State the Nullstellensatz for* $\operatorname{Spec} R$. *Give an example of a commutative ring* R *where the analogous statement does not hold for* $\operatorname{MaxSpec} R$ *and explain how it fails.*

3, M

The Nullstellensatz says that $\mathbb{V}$ and $\mathbb{I}$ give inverse bijections between radical ideals and closed subsets of $\operatorname{Spec} R$; or sometimes it is stated as the weaker property that $\mathbb{I}(\mathbb{V}(J)) = \operatorname{rad} J$. If $R = \mathbb{Z}[1/p]_{p \text{ prime}, p>3}$, then note that $6 \in R$ vanishes at both maximal ideals (2) and (3), and it is in fact the intersection of these maximal ideals. Hence $\mathbb{I}(\mathbb{V}(0)) = (6)$. (This happens because the prime ideal (0) is not an intersection of maximal ideals.)

Part 5(c). *Let* $R := k[x]_{(x)}$, *the local ring (recall it is given from* $k[x]$ *by inverting all elements which are not multiples of* x*). Recall that the prime ideals are* (0) *and* (x). *Recall also that given a topological space* X, *the topological dimension is given as the maximum length* n *of a chain* $X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n$, *where* X_i *are all closed irreducible subsets of* X.

Subpart 5(c)(i). *Find all closed subsets of* $\operatorname{Spec} R = \{0, (x)\}$.

2, M

The closed subsets are $\mathbb{V}(0) = \operatorname{Spec} R$, $\mathbb{V}(x^m) = \{(x)\}$, and $\emptyset$. (So everything except $\{0\}$, as 0 is not a closed point.)

Subpart 5(c)(ii). *Find all irreducible subsets of* $\operatorname{Spec} R$.

2, M

All non-empty subsets are irreducible. This is obvious for the single element sets. To see that $\operatorname{Spec} R$ itself is irreducible, note that there are only two closed subsets and one is all of $\operatorname{Spec} R$, so any union of closed sets which contains $\operatorname{Spec} R$ must have one of the sets itself containing $\operatorname{Spec} R$.

Subpart 5(c)(iii). *Compute the topological dimension of* $\operatorname{Spec} R$.

2, M

We may take the chain $\operatorname{Spec} R = \operatorname{Spec} R \supsetneq \{(0)\}$, which shows that $\dim R = 1$ since $\operatorname{Spec} R$ is itself irreducible, and there could be no longer chain as $|\operatorname{Spec} R| = 2$.

Subpart 5(c)(iv). *Does* $\operatorname{Spec} R$ *satisfy the property that, if* $X, Y \subseteq \operatorname{Spec} R$ *are arbitrary subsets, then* $\dim(X \cup Y) = max(\dim X, \dim Y)$*? Why or why not?*

2, M

No: $\operatorname{Spec} R = \{(0)\} \cup \{(x)\}$ even though the dimension of the LHS is two, and the dimension of a set of just one element is always one (it is always closed and irreducible in itself).

Total A marks: 33 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 15 of 16 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97044MATH97153	1	This was one of the easier questions. The students did not completely follow the instructions on problem 1b, sometimes just showing "locally closed" for example instead of "(B) locally closed but not closed". There was some confusion in 1(c) in realising that Z is irreducible---although in the complex topology it is discrete, it is not in the Zariski topology which is the cofinite one. The students could have supplied more details in 1(d) as to why the diagonal is not closed in the product topology.
MATH97044MATH97153	2	This was a bit more difficult. Many students incorrectly wrote V(xyz) for the union of the coordinate axes (a.(ii)), which is not correct (this is the union of coordinate hyperplanes). 2(d) was one of the hardest parts of the exam---few students found both the inverse dilations of x and y **and** swapping x and y, and still fewer proved this was everything.
MATH97044MATH97153	3	3(b) was quite difficult, particularly finding the inverse map in 3(b)(ii). Part 3(b)(iv) was one of the hardest parts of the exam, with few students seeing how to make a rigorous argument using the idea of changing the point p of the projection (the idea of the solution is only to use this to show that an open neighbourhood of the point p embeds inside the projective line). Also, few students gave a complete solution to 3(c), neglecting for example to explain why the degree must be one to get an isomorphism.

MATH97044MATH97153	4	Surprisingly many students did quite well on this question, even if they had trouble on Q2 or Q3. I think they prepared well. Some students neglected to show that the graph was closed in 4a.(ii) (this depends on the ambient space so can't be concluded from the isomorphism alone). 4(c) was more difficult, and fewer students were able to argue 4c(iii) and (iv) correctly. Not very many found the determinant in 4c(v).
MATH97044MATH97153	5	This was surprisingly manageable. Many students left some time for this question, even if they skipped other parts. Still, it suffered from being the last question. I think that students were on the whole able to show that they had some grasp of what happens in algebraic geometry outside of the algebraically closed field situation, which is great. Few students were able to completely correctly work out the example in 5c, although this is surely due partly to time constraints or fatigue.