

This week's problem is about polynomial equations over matrices. These are equations of the form

$$X^k + A_{k-1}X^{k-1} + \cdots + A_1X + A_0 = 0,$$

where the variable X and the coefficients A_i are in $M_n(\mathbb{C})$, the ring of $n \times n$ matrices over \mathbb{C} . The theory of such equations is very different from the usual theory of polys, mainly because (a) unlike \mathbb{C} , $M_n(\mathbb{C})$ is not commutative, and (b) unlike \mathbb{C} , $M_n(\mathbb{C})$ has many nonzero elements that do not have inverses.

The problems we will look at just consider quadratic equations over $M_2(\mathbb{C})$, the 2×2 matrices, but this is interesting enough. Write such an equation as

$$X^2 + AX + B = 0 \quad (A, B, X \in M_2(\mathbb{C})). \quad (1)$$

1. Unlike quadratics over \mathbb{C} , not every pair of matrices are roots of a quadratic (1): show that there is no quadratic with roots $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2. The number of solutions of a quadratic (1) can vary a lot:

(a) How many solutions are there of the quadratic $X^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$?

(b) How many solutions are there of the quadratic $X^2 - I = 0$? (Hint: it's more than 2!)

3. Some theory - diagonalisable solutions: suppose X is a solution of (1), and λ an evalue of X with a corresponding evector v .

(a) Show that $\det(\lambda^2 I + \lambda A + B) = 0$ and $v \in \text{Ker}(\lambda^2 I + \lambda A + B)$.

(b) By solving the quartic equation for λ in (a) and finding corresponding vectors v , we can find the diagonalisable solutions X of (1). Do this for the equations

$$(i) \quad X^2 + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

$$(ii) \quad X^2 + X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.$$

4. More theory - non-diagonalisable solutions: suppose X is a non-diagonalisable solution of (1). Then X has a repeated evalue λ (a root of the quartic in 3(a)), and $Y = X - \lambda I$ has repeated evalue 0. Adjusting the quadratic (1), we can replace X by Y . We look for such X . As X is triangularisable, $X^2 = 0$ and there is a basis v_1, v_2 such that $Xv_1 = 0, Xv_2 = v_1$. This gives $Bv_1 = 0, Av_1 + Bv_2 = 0$. If we find all such v_1, v_2 we can find all the non-diagonalisable solutions of (1).

(i) Carry this out for the equations in 3(b).

(ii) As a final exercise, find all solutions (diagonalisable or otherwise) of the equation

$$X^2 + \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.$$

Note: goodness, I got a bit carried away. It's a nice fact (that can be proved using the theory sketched above) that the possible numbers of solutions of the quadratic (1) are 0,1,2,3,4,5,6 or ∞ . If you want to read more about this, there is a nice article by R L Wilson, "Polynomial equations over matrices", that can be found online.