Mathematical Logic (MATH6/70132;P65)
Problem Sheet 5

[1] Prove (or at least, sketch a proof of) the following version of the Lindenbaum Lemma (2.5.2) which was used in the proof of 2.5.3:

Suppose $\mathcal{L}$ is a countable first-order language and Σ is a consistent set of closed $\mathcal{L}$-formulas. Then there is a consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-formulas such that, for every closed $\mathcal{L}$-formula ϕ either $\Sigma^* \vdash \phi$ or $\Sigma^* \vdash (\neg\phi)$.

[2] Describe a language with equality $\mathcal{L}^=$ which is appropriate for groups (see the example after 2.2.5).

(a) Write down a closed $\mathcal{L}^=$-formula γ whose normal models are precisely the groups. (You can use traditional mathematical notation if you wish.)

(b) Write down a set Σ of closed $\mathcal{L}^=$-formulas whose normal models are precisely the infinite groups. [Hint: you may use the formulas σ_n mentioned in Week 6 Problem class/ Lecture 17.]

(c) Suppose that ϕ is a closed $\mathcal{L}^=$-formula such that for every $n \in \mathbb{N}$ there is a group with at least n elements which is a model of ϕ. Show that there is an infinite group which is a model of ϕ.

(d) Show that there is no set Δ of closed $\mathcal{L}^=$-formulas whose normal models are precisely the finite groups.

(e) Show that there is no closed $\mathcal{L}^=$ formula whose normal models are precisely the infinite groups.

(f) (Much harder; really only do-able if you have done a course on infinite groups) Is there a closed $\mathcal{L}^=$-formula σ which has a normal model and is such that any normal model of σ is an infinite group?

[3] Suppose $\mathcal{L}^=$ is a first-order language with equality and a single binary relation symbol R. A graph is a normal model of the closed formula γ:

$$(\forall x_1)(\forall x_2)((\neg R(x_1, x_1)) \wedge (R(x_1, x_2) \to R(x_2, x_1))).$$

(i) Find a closed formula τ with the property that there is a finite normal model of $\gamma \wedge \tau$ whose domain has n elements iff n is divisible by 3.

(ii) (Much Harder) Can you find a closed $\mathcal{L}^=$-formula which has no finite models and has some infinite graph as a normal model?

[4] Suppose $\mathcal{L}^=$ is a first-order language with equality ($=$) and a single binary relation symbol $\leq$. A linear order is a normal $\mathcal{L}^=$-structure $\langle A; \leq_A \rangle$ such that the relation $\leq_A$ is reflexive, transitive and such that for distinct $a, b \in A$ exactly one of $a \leq_A b$, $b \leq_A a$ holds.

Let Σ be the set of all closed $\mathcal{L}^=$-formulas which are true in all *finite* linear orders.
(i) Prove that any normal model of Σ is a linear order with a least element and a greatest element.
(ii) Prove that any normal model of Σ (with at least 2 elements) is not dense.
(iii) Prove that Σ has an infinite normal model.
(iv) Find a closed $\mathcal{L}^=$-formula ϕ such that neither ϕ nor $(\neg\phi)$ is a consequence of Σ.

[5] Suppose $\mathcal{L}^=$ is a first order language with equality ($=$) and a single binary relation symbol R. Write down what it means for two normal $\mathcal{L}^=$-structures to be isomorphic (see the problem class in week 6)?
(i) Write down a set Σ of closed $\mathcal{L}^=$-formulas such that the normal models of Σ are the normal $\mathcal{L}^=$-structures in which R is interpreted as an equivalence relation in which all equivalence classes have size 2 or 3 and there are infinitely many equivalence classes of size 2 and infinitely many of size 3.
(ii) Explain why any two countable normal models of Σ are isomorphic.

[6] Suppose $\mathcal{L}^=$ is a first-order language with equality having just a single 1-ary function symbol f (and no other relation, function or constant symbols apart from $=$).

(i) What does it mean to say that two normal $\mathcal{L}^=$-structures $\mathcal{A}$, $\mathcal{B}$ are isomorphic?
(ii) Write down a set Σ of closed $\mathcal{L}^=$-formulas such that $\langle A; \bar{f} \rangle$ is a normal model of Σ if and only if: $\bar{f} : A \to A$ is a bijection and for every $n \in \mathbb{N}$, the function $\bar{f}^n : A \to A$ (obtained by applying $\bar{f}$ n times) has no fixed points.

(iii) Find countable normal models $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of Σ such that no two of these models are isomorphic and any countable model of Σ is isomorphic to one of these.

David Evans, February 2025.