Exercise 2.1. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^1 > 0\}$?

(d) $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^i \in [0, 1)\}$?

(e) $\mathbb{Q}^n := \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Solution: a) Open, b) Open, c) Open, d) Not open, e) Not Open.

Exercise 2.2. Let $(x_i)_{i=0}^{\infty}$ be a sequence of vectors $x_i \in \mathbb{R}^n$ with $x_i \to x$. Suppose that the x_i satisfy $\|x_i\| < r$ for all i and some $r > 0$. Show that:

$$\|x\| \leq r.$$

[Hint: work by contradiction, assume $\|x\| > r$ and show this leads to an absurdity]

Solution: Suppose in the contrary that $\|x\| = s > r$. Let $\epsilon = \frac{s-r}{2} > 0$. By the convergence of (x_i), there exists $j \in \mathbb{N}$ such that:

$$\|x_j - x\| < \epsilon.$$

By the reverse triangle inequality we have:

$$\big|\|x\| - \|x_j\|\big| \leq \|x_j - x\| < \epsilon,$$

however:

$$\big|\|x\| - \|x_j\|\big| = s - \|x_j\| \geq s - r = 2\epsilon,$$

so we conclude

$$2\epsilon < \epsilon$$

which together with the fact that $\epsilon > 0$ is a contradiction.

Exercise 2.3. (a) Show that if U_1, U_2 are open in $\mathbb{R}^n$, then so are the sets

$$i)\ \ U_1 \cup U_2 \qquad\qquad ii)\ \ U_1 \cap U_2$$

Solution: Suppose $x \in U_1 \cup U_2$. Then either $x \in U_1$ or $x \in U_2$. WLOG consider the first possibility. Then since U_1 is open, there exists $r > 0$ such that $B_r(x) \subset U_1$. But this implies $B_r(x) \subset U_1 \cup U_2$, so $U_1 \cup U_2$ is open.

Suppose $x \in U_1 \cap U_2$. Then there exist r_1, r_2 such that $B_{r_1}(x) \subset U_1$ and $B_{r_2}(x) \subset U_2$. Taking $r = \min\{r_1, r_2\}$ we have:

$$B_r(x) \subset B_{r_1}(x) \subset U_1, \qquad B_r(x) \subset B_{r_2}(x) \subset U_2,$$

so that $B_r(x) \subset U_1 \cap U_2$ and thus $U_1 \cap U_2$ is open.

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Questions marked with $*$ are optional

(b) Suppose U_α, for α in an index set I, is a collection of open sets in $\mathbb{R}^n$.

(i) Show that $\bigcup_{\alpha \in I} U_\alpha$ is open in $\mathbb{R}^n$.
[Hint: Can the proofs for part (a) adapted to this setting.]

Solution: Suppose $x \in \bigcup_{\alpha \in I} U_\alpha$. Then there exists $a \in I$ such that $x \in U_a$. Since U_a is open, there exists $r > 0$ such that $B_r(x) \subset U_a$, which implies $B_r(x) \subset \bigcup_{\alpha \in I} U_\alpha$, hence $\bigcup_{\alpha \in I} U_\alpha$ is open.

(ii) Give an example showing that $\bigcap_{\alpha \in I} U_\alpha$ need not be open.
[Hint: You may start by looking at intervals in dimension 1.]

Solution: Consider:

$$U_i = \left(-2^{-i}, 2^{-i}\right), \text{ for } i \in \mathbb{N}.$$

Then, $\bigcap_{i \in \mathbb{N}} U_i = \{0\}$, which is not open, but each set U_i is an open interval.

Exercise 2.4. Suppose $A \subset \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}^m$. Show that $\lim_{x \to p} f(x) = F$ if and only if for any sequence $(x_i)_{i=0}^\infty$ in $A \setminus \{p\}$ which converges to p we have

$$f(x_i) \to F, \qquad \text{as } i \to \infty.$$

Solution: First suppose that $\lim_{x \to p} f(x) = F$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in A$ with $0 < \|x - p\| < \delta$ we have:

$$\|f(x) - F\| < \epsilon.$$

Now let $(x_i)_{i=0}^\infty$ be any sequence with $x_i \in A, x_i \neq p$ and $x_i \to p$. Since $x_i \to p$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$0 < \|x_i - p\| < \delta,$$

so by our assumption we have

$$\|f(x_i) - F\| < \epsilon,$$

and thus $f(x_i) \to F$.

Now suppose that for any sequence $(x_i)_{i=0}^\infty$ with $x_i \in A, x_i \neq p$ and $x_i \to p$ we have:

$$f(x_i) \to F, \qquad \text{as } i \to \infty.$$

Suppose that $f(x) \not\to F$ as $x \to p$. Then there exists $\epsilon > 0$ such that for any $i \in \mathbb{N}$ we can find x_i with:

$$0 < \|x_i - p\| < 2^{-i}, \qquad \|f(x_i) - F\| \geq \epsilon.$$

Now, clearly the sequence $(x_i)_{i=0}^\infty$ converges to p, but $f(x_i) \not\to F$, so we have a contradiction.

Exercise 2.5. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \ldots, 0)$ is continuous on $\mathbb{R}$.

Solution: Suppose $p \in \mathbb{R}$. Fix $\epsilon > 0$ and suppose $x \in \mathbb{R}$ satisfies $|x - p| < \epsilon$. Then:

$$\|f(x) - f(p)\| = \|(x - p, 0, \ldots, 0)\| = |x - p| < \epsilon.$$

(b) Let $A \subset \mathbb{R}^n$ and suppose we are given a map $f : A \to \mathbb{R}^m$ where

$$f(x^1, \dots, x^n) \mapsto \left(f^1\left((x^1, \dots, x^n)\right), \dots, f^m\left((x^1, \dots, x^n)\right)\right).$$

Show that f is continuous at $p \in A$ if and only if each map $f^k : A \to \mathbb{R}$ is continuous at p, for $k = 1, \dots, m$.

Solution: First suppose that each map $f^k : \mathbb{R}^n \to \mathbb{R}$ is continuous at p, for $k = 1, \dots, m$. Fix $\epsilon > 0$. Then for each k there exists $\delta_k > 0$ such that for $x \in A$ with $\|x - p\| < \delta_k$ we have:

$$\left|f^k(x) - f^k(p)\right| < \frac{\epsilon}{\sqrt{n}}.$$

Let $\delta = \min_{k=1,\dots,m} \delta_k$. If $x \in A$, $\|x - p\| < \delta$, we have:

$$\|f(x) - f(p)\| \leq \sqrt{n} \max_{k=1,\dots,m} \left|f^k(x) - f^k(p)\right| < \sqrt{n}\frac{\epsilon}{\sqrt{n}} = \epsilon,$$

so that f is continuous at p.

Now suppose that f is continuous at p. Fix $\epsilon > 0$, then there exists $\delta > 0$ such that for all $x \in A$, $0 < \|x - p\| < \delta$ we have:

$$\|f(x) - f(p)\| < \epsilon.$$

Fix $j \in \{1, \dots, m\}$. We estimate:

$$\left|f^j(x) - f^j(p)\right| \leq \max_{k=1,\dots,m} \left|f^k(x) - f^k(p)\right| \leq \|f(x) - f(p)\| < \epsilon,$$

so that f^j is continuous at p.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f\left((x^1, x^2, \dots, x^n)\right) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$, [1].

Solution: By part a), the map from $\mathbb{R}^n$ to each coordinate is continuous, so any finite combination of sums and products of these functions is continuous.

Exercise 2.6.*

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

[Hint: Start by picking an arbitrary point w in $f^{-1}(U)$, and see how continuity of f gives you the epsilon-ball around w.]

Solution: Fix $x \in f^{-1}(U)$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. Since f is continuous, there exists $\delta > 0$ such that if $y \in \mathbb{R}^n$ with $\|y - x\| < \delta$ then $\|f(y) - f(x)\| < \epsilon$. But this implies that $f(y) \in B_\epsilon(f(x)) \subset U$, so we have that $y \in f^{-1}(U)$ provided $\|y - x\| < \delta$. Thus $B_\delta(x) \subset f^{-1}(U)$ and $f^{-1}(U)$ is indeed open.

[1] Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

Solution: Fix $x \in \mathbb{R}^n$, and let $\epsilon > 0$. Since $B_\epsilon(f(x))$ is open, we have that the set $f^{-1}(B_\epsilon(f(x)))$ is open. We note that $x \in f^{-1}(B_\epsilon(f(x)))$, thus there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Now if $y \in \mathbb{R}^n$ with $\|x - y\| < \delta$, then $y \in B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$, so that $f(y) \in B_\epsilon(f(x))$ and thus $\|f(y) - f(x)\| < \epsilon$, so that f is indeed continuous at x.

Exercise 2.7. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f(x) = x.$$

Show that f is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \mathrm{id},$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Solution: We could appeal to an example in the lecture notes (linear maps are differentiable) and note that the identity is a linear map, thus is differentiable with derivative equal to itself. Alternatively, we note that if $Df(p) = \iota$, then

$$f(p+h) - f(p) - Df(p)[h] = (p+h) - p - h = 0,$$

so we certainly have

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - Df(p)[h]\|}{\|h\|} = 0,$$

which implies f is differentiable.

Exercise 2.8. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = (2\xi \quad 2\eta)$$

Solution: Setting $h = (h_1, h_2)$, we calculate

$$\begin{aligned} f(p+h) - f(p) - Df(p)[h] &= (\xi + h_1)^2 + (\eta + h_2)^2 - \xi^2 - \eta^2 - 2\xi h_1 - 2\eta h_2 \\ &= h_1^2 + h_2^2. \end{aligned}$$

Thus we have

$$\frac{\|f(p+h) - f(p) - Df(p)[h]\|}{\|h\|} = \frac{h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} = \sqrt{h_1^2 + h_2^2},$$

so certainly

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - Df(p)[h]\|}{\|h\|} = \lim_{h\to 0} \|h\| = 0.$$

Exercise 2.9. One might hope that the differential can be calculated by finding

$$\lim_{x \to p} \frac{f(x) - f(p)}{\|x - p\|}.$$

By considering the example of Exercise 2.7 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Solution: Taking $f : \mathbb{R}^n \to \mathbb{R}^n$ to be the identity and $p = 0$, we have

$$\frac{f(x) - f(p)}{\|x - p\|} = \frac{x}{\|x\|}.$$

This function has no limit as $x \to 0$. To see this, consider $x = \lambda e_1$, then:

$$\frac{x}{\|x\|} = \frac{\lambda}{|\lambda|}.$$

The limit $\lambda \to 0$ does not exist, since $\frac{\lambda}{|\lambda|} = 1$ for $\lambda > 0$ and $\frac{\lambda}{|\lambda|} = -1$ for $\lambda < 0$.

Exercise 2.10. Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at p and

$$Dh(p) = Df(p) + Dg(p)$$

Solution: Since f and g are differentiable at p, there exist linear maps $Df(p), Dg(p) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to p} \frac{\|f(x) - f(p) - Df(p)[x - p]\|}{\|x - p\|} = 0,$$

and

$$\lim_{x \to p} \frac{\|g(x) - g(p) - Dg(p)[x - p]\|}{\|x - p\|} = 0.$$

Now we estimate by the triangle inequality

$$\begin{aligned}\frac{\|h(x) - h(p) - Dh(p)[x - p]\|}{\|x - p\|} &= \frac{\|f(x) + g(x) - f(p) - g(p) - Df(p)[x - p] - Dg(p)[x - p]\|}{\|x - p\|} \\ &\leq \frac{\|f(x) - f(p) - Df(p)[x - p]\|}{\|x - p\|} + \frac{\|g(x) - g(p) - Dg(p)[x - p]\|}{\|x - p\|},\end{aligned}$$

so that we have

$$\lim_{x \to p} \frac{\|h(x) - h(p) - Dh(p)[x - p]\|}{\|x - p\|} = 0,$$

and the conclusion follows.

Unseen Exercise. Let $\alpha \in \mathbb{R}$ be an irrational number, and for $n \in \mathbb{N}$ let

$$a_n = \frac{1}{2^n} \left(\cos(2\pi n\alpha), \sin(2\pi n\alpha)\right) \in \mathbb{R}^2.$$

(a) Show that $a_n \to (0, 0) \in \mathbb{R}^2$ as $n \to \infty$.

Solution: Let $\epsilon > 0$ be arbitrary. There is $n' \geq 1$ such that for all $n \geq n'$ we have $2^{-n} < \epsilon$. For $n \geq n'$ we have

$$\|a_n - (0,0)\| = |2^{-n}| \, \|(\cos(2\pi n\alpha), \sin(2\pi n\alpha))\| = 2^{-n} < \epsilon.$$

(b) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ according to

$$f(x) = \begin{cases} 1 & \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the map f is not continuous at $(0,0)$.

Solution: Since $a_n \neq (0,0)$ for all $n \in \mathbb{N}$, we have $f(0,0) = 0$. On the other hand $a_n \to (0,0)$ and $f(a_n) \equiv 1$ does not converge to $0 = f(0,0)$. This shows that the map f is not continuous at $(0,0)$.

(c) for every non-zero vector $u = (u^1, u^2) \in \mathbb{R}^2$, show that f is continuous in the direction of u at 0. That is, the map $t \mapsto f(tu)$ is continuous at $t = 0$.

Solution: Let us fix an arbitrary non-zero vector $u = (u^1, u^2) \in \mathbb{R}^2$. Consider the line

$$L = \{tu \mid t \in \mathbb{R}\} \subset \mathbb{R}^2.$$

We claim that there is at most one integer $n \in \mathbb{N}$ such that $a_n \in L$. Assume in the contrary that there are two such integers, say m and n with $m \neq n$. Then, there are t_n and t_m in $\mathbb{R}$ such that $a_m = t_m u$ and $a_n = t_n u$. Because a_n and a_m are non-zero, t_n and t_m must be non-zero, so we conclude that

$$u = a_m/t_m = a_n/t_n,$$

and then

$$\frac{1}{2^m t_m} (\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = \frac{1}{2^n t_n} (\cos(2\pi n\alpha), \sin(2\pi n\alpha)).$$

Since for every $\gamma \in \mathbb{R}$, $(\cos(\gamma), \sin(\gamma))$ has modulus 1, we conclude that $|2^n t_n| = |2^m t_m|$. Therefore, either

$$(\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = (\cos(2\pi n\alpha), \sin(2\pi n\alpha))$$

or

$$(\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = -(\cos(2\pi n\alpha), \sin(2\pi n\alpha)).$$

Both of these cases imply that $\cos(2\pi m\alpha) = \cos(2\pi n\alpha)$. This implies that there is $k \in \mathbb{Z}$ such that $2\pi n\alpha = 2\pi m\alpha + 2k\pi$. Therefore, $\alpha = k/(n-m)$, which contradicts α being irrational.

Let us define δ as follows. If there is no a_n in L, we define $\delta = \|u\|$. If there is $a_n \in L$, we let $\delta = \|a_n\| \,/\, \|u\|$. Since there is at most one a_n in L, this is a well-defined number.

We claim that for every $t \in \mathbb{R}$ such that $|t| < \delta$, we have $f(tu) = 0$. That is because if there is no a_n in L then $f(tu)$ is constant 0 for every t. If there is $a_n \in L$, then we have

$$\|tu\| < |t| \, \|u\| < \delta \, \| \|_u = \|a_n\|.$$

This implies that $f(tu) = 0$.

Since the map $t \mapsto f(tu)$ is constant on the interval $(-\delta, \delta)$, it is continuous at 0.

Unseen Exercise. Show that the set

$$A = \{(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n \mid x^1 > 0, x^2 > 0, \ldots, x^n > 0\}$$

is an open set in $\mathbb{R}^n$.

Solution: Let $x = (x^1, x^2, \ldots, x^n)$ be an arbitrary point in A. Define

$$r = \min\{x^1, x^2, \ldots, x^n\}.$$

Since $x \in A$, all x^i are positive, and hence $r > 0$.

Now assume that $y = (y^1, y^2, \ldots, y^n)$ is an arbitrary point in $B_r(x^1, x^2, \ldots, x^n)$. For all $i \in \{1, 2, \ldots, n\}$, we have

$$|x^i - y^i| \leq \|x - y\| < r \leq x^i.$$

This implies that

$$x^i - y^i < x^i,$$

and hence $y^i > 0$. As i was arbitrary, we conclude that $y \in A$.