Imperial College London

MATH70060

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Complex Manifolds

Date: 23 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let U be an open subset of $\mathbb{C}$. Define the notion of a holomorphic function $f : U \to \mathbb{C}$. (4 marks)

   (b) Let U be an open subset of $\mathbb{C}^n$, $n \geq 2$. Define the notion of a holomorphic function $f : U \to \mathbb{C}$. (4 marks)

   (c) State the Hartog's theorem for a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, $n \geq 2$. (4 marks)

   (d) Provide an example of a holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that does not have an extension to a holomorphic function $\mathbb{C} \to \mathbb{C}$. (4 marks)

   (e) Provide an example of a C^∞-function $f : \mathbb{R}^4 \setminus \{(0,0,0,0)\} \to \mathbb{R}^2$ that does not have an extension to a C^∞-function $\mathbb{R}^4 \to \mathbb{R}^2$. (4 marks)

(Total: 20 marks)

2. Let $n \geq 2$, and $\sigma : \mathrm{Bl}_p\mathbb{C}^n \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ at the origin $p = (0, ..., 0)$.

(a) (i) Describe complex charts (U_i, ϕ_i) that make $\mathrm{Bl}_p\mathbb{C}^n$ a complex manifold. (4 marks)

(ii) Let $E = \sigma^{-1}(p)$. For each chart (U_i, ϕ_i), construct a holomorphic submersion $f_i : U_i \to \mathbb{C}$ such that $f_i^{-1}(0) = E \cap U_i$. (4 marks)

(iii) Prove that E is a complex submanifold of $\mathrm{Bl}_p\mathbb{C}^n$, and calculate its dimension. (2 marks)

(b) Prove or construct a counterexample to the following statement: "Let f be a holomorphic function $\mathrm{Bl}_p\mathbb{C}^n \to \mathbb{C}^k$, for some $k \geq 1$. There exists a holomorphic function $g : \mathbb{C}^n \to \mathbb{C}^k$ such that f equals the compostion $g \circ \sigma$". (4 marks)

(c) Let us now concentrate on the case $n = 2$. Let x_0, x_1 be the standard coordinates on $\mathbb{C}^2$. Consider the holomorphic vector field

$$u = (1 + x_0)\frac{\partial}{\partial x_1}$$

on $\mathbb{C}^2$. This induces a vector field v on $\mathrm{Bl}_p\mathbb{C}^2 \setminus E$, via the biholomorphism $\sigma|_{\mathrm{Bl}_p\mathbb{C}^2 \setminus E} : \mathrm{Bl}_p\mathbb{C}^2 \setminus E \to \mathbb{C}^2 \setminus \{p\}$. Does v extend to a holomorphic vector field on $\mathrm{Bl}_p\mathbb{C}^2$? Explain your answer. (6 marks)

(Total: 20 marks)

3. Consider the quotient $X = (\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{Z}$, where the group $\mathbb{Z}$ acts on $\mathbb{C}^2 \setminus \{(0,0)\}$ via $m \cdot z = 3^m z$, $m \in \mathbb{Z}$, $z \in \mathbb{C}^2 \setminus \{(0,0)\}$.

(a) Describe holomorphic charts on X that make it a complex manifolds. Explain why the transition functions for your charts are biholomorphic. (4 marks)

(b) Find the complex dimension of X. (2 marks)

(c) Let α be a complexified C^∞ differential k-form on a complex manifold Y, that is, $\alpha \in C^\infty(Y, \Omega^k_{Y,\mathbb{C}})$.

(i) What does it mean for α to be d-closed? (2 marks)

(ii) What does it mean for α to be d-exact? (2 marks)

(d) Construct a complexified 1-form on X that is d-closed, but not d-exact. Explain why the form you constructed satisfies the required conditions. (5 marks)

(e) Does there exists a holomorphic 1-form on X that is not identically zero? Justify your answer. (5 marks)

(Total: 20 marks)

4. (a) Define the holomorphic line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$, $n \geq 1$, $k \in \mathbb{Z}$. (4 marks)

(b) Given a holomorphic vector bundle E over a complex manifold X, define the Dolbeault cohomology $H^q(X, E)$, $q \geq 0$. (4 marks)

(c) Calculate $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. (5 marks)

(d) Calculate $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. (4 marks)

(e) Calculate $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$. (3 marks)

(Total: 20 marks)

5. (a) State the Hodge decomposition theorem for Kähler manifolds. (4 marks)

(b) Let X be a compact connected complex surface that is Kähler. Suppose that we know that $h^{2,0}(X) = 1$, $h^{1,2}(X) = 2$, and that the topological Euler characteristic of X is zero. Find all the Hodge numbers of X and draw its Hodge diamond. (4 marks)

(c) State the $\partial\bar{\partial}$-lemma for Kähler manifolds. (4 marks)

(d) Let Y be a compact Kähler manifold. Let ω be a holomorphic p-form on Y, for some $p \geq 0$.

(i) Prove that ω cannot be d-exact, unless $\omega = 0$. (4 marks)

(ii) Does ω have to be d-closed? Prove it or give a counterexample. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

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Complex Manifolds Exam, May 2023

Solutions

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1. (a) Let U be an open subset of $\mathbb{C}$. Define the notion of a holomorphic function $f : U \to \mathbb{C}$. $\boxed{4, \mathrm{A}}$

The function f is **holomorphic** on U if for all $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

(b) Let U be an open subset of $\mathbb{C}^n$, $n \geq 2$. Define the notion of a holomorphic function $f : U \to \mathbb{C}$. $\boxed{4, \mathrm{A}}$

The function f is **holomorphic** if f is continuous and for each $z = (z_1, \ldots, z_n) \in U$ such that $D(z, \epsilon) \subset U$ for some polyradius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, we have that the function in one variable

$$f(z_1, \ldots, z_{i-1}, \cdot, z_{i+1}, \ldots, z_n) : D(z_i, \epsilon_i) \to \mathbb{C}$$

is holomorphic.

(c) State the Hartog's theorem for a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, $n \geq 2$. $\boxed{4, \mathrm{A}}$

Let $R = (R_1, \ldots, R_n)$ and $r = (r_1, \ldots, r_n)$ such that $R_i > r_i > 0$ for $i = 1, \ldots, n$. Let

$$U = D(0, R) \setminus D(0, r) \subset \mathbb{C}^n.$$

Then any holomorphic function on U extends to a holomorphic function on $D(0, R)$.

(d) Provide an example of a holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that does not have an extension to a holomorphic function $\mathbb{C} \to \mathbb{C}$. $\boxed{4, \mathrm{A}}$

The function $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but does not extend holomorphically to $\mathbb{C}$, because it has a pole at $z = 0$.

(e) Provide an example of a C^∞-function $f : \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\} \to \mathbb{R}^2$ that does not have an extension to a C^∞-function $\mathbb{R}^4 \to \mathbb{R}^2$. $\boxed{4, \mathrm{A}}$

The function

$$f(x_1, x_2, x_3, x_4) = \frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \, (1, 1)$$

is C^∞, because it is a rational function. It does not extend to a C^∞ function $\mathbb{R}^4 \to \mathbb{R}^2$, because $\lim_{(x_1, x_2, x_3, x_4) \to (0,0,0,0)} f(x_1, x_2, x_3, x_4)$ is infinite.

2. Let $n \geq 2$, and $\sigma : \mathrm{Bl}_p\mathbb{C}^n \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ at the origin $p = (0, ..., 0)$.

(a) (i) Describe complex charts (U_i, ϕ_i) that make $\mathrm{Bl}_p\mathbb{C}^n$ a complex manifold. 4, B

Let $x_0, ..., x_{n-1}$ be the coordinates on $\mathbb{C}^n$. Recall that $\mathrm{Bl}_p(X)$ can be described as the set of pairs (ℓ, q), where ℓ is a straight line in $\mathbb{C}^n$ passing through the origin and $q \in \ell$. To define the structure of complex manifold on $\mathrm{Bl}_p(X)$, we consider n charts:

$U_i = \{(\ell, q) : \ell$ is not contained in the hyperplane $x_i = 0\}$, $i = 0, ..., n-1$.

Define $\phi_i : U_i \to \mathbb{C}$ via evaluating the coordinates x_i and s_{ij}, $j \neq i$, where $q = (x_0, ..., x_{n-1})$ and ℓ is given by equations $x'_j = s_{ij}x'_i$, $j \neq i$.

The transition function between U_i and U_j is given by

$$x_j = s_{ij}x_i, \quad s_{ji} = \frac{1}{s_{ij}}, \quad s_{jk} = \frac{s_{ik}}{s_{ij}}, \ k \neq i, j.$$

(ii) Let $E = \sigma^{-1}(p)$. For each chart (U_i, ϕ_i), construct a holomorphic submersion $f_i : U_i \to \mathbb{C}$ such that $f_i^{-1}(0) = E \cap U_i$. 4, B

Define $f_i(\ell, q) = x_i$. The holomorphic Jacobian of this function in the coordinates defined above is $(1, 0, ..., 0)$. Therefore, it is a submersion. The set $f_i^{-1}(0)$ consists precisely of the pair $(\ell, q) \in \mathrm{Bl}_p\mathbb{C}^n$ such that $q = (0, ..., 0)$. Therefore, $f_i^{-1}(0) = E$.

(iii) Prove that E is a complex submanifold of $\mathrm{Bl}_p\mathbb{C}^n$, and calculate its dimension. 2, A

Since we realized E as the level set of a submersion $\mathbb{C}^n \to \mathbb{C}$ everywhere locally, the set E is a submanifold of $\mathrm{Bl}_p\mathbb{C}^n$ of dimension $n-1$.

(b) Prove or construct a counterexample to the following statement: "Let f be a holomorphic function $\mathrm{Bl}_p\mathbb{C}^n \to \mathbb{C}^k$, for some $k \geq 1$. There exists a holomorphic function $g : \mathbb{C}^n \to \mathbb{C}^k$ such that f equals the compostion $g \circ \sigma$". 4, C

The statement is true. Let us prove it.

Define $g : \mathbb{C}^n \setminus \{p\} \to \mathbb{C}^k$ via $g = f \circ \sigma^{-1}$. This is well-defined and holomorphic, since σ is a biholomorphism between $\mathrm{Bl}_p\mathbb{C}^n \setminus E$ and $\mathbb{C}^n \setminus \{p\}$. By Hartog's theorem, the holomorphic function g extends to a holomorphic function $g : \mathbb{C}^n \to \mathbb{C}^k$. The equality $f = g \circ \sigma$ is true over $\mathrm{Bl}_p\mathbb{C}^n \setminus E$ by construction of g. Therefore, by continuity, we have $f = g \circ \sigma$ over the whole $\mathrm{Bl}_p\mathbb{C}^n$.

(c) Let us now concentrate on the case $n = 2$. Let x_0, x_1 be the standard coordinates on $\mathbb{C}^2$. Consider the holomorphic vector field

$$u = (1 + x_0)\frac{\partial}{\partial x_1}$$

on $\mathbb{C}^2$. This induces a vector field v on $\mathrm{Bl}_p\mathbb{C}^2 \setminus E$, via the biholomorphism $\sigma|_{\mathrm{Bl}_p\mathbb{C}^2 \setminus E} : \mathrm{Bl}_p\mathbb{C}^2 \setminus E \to \mathbb{C}^2 \setminus \{p\}$. Does v extend to a holomorphic vector field on $\mathrm{Bl}_p\mathbb{C}^2$? Explain your answer. 6, D

The vector field v does not extend to a holomorphic vector field on $\mathrm{Bl}_p\mathbb{C}^2$. Let us prove it.

Consider the chart U_0 with coordinates x_0 and $s_{01} = \frac{x_1}{x_0}$. We have

$$\frac{\partial}{\partial x_1} = \frac{1}{x_0} \frac{\partial}{\partial s_{01}}.$$

Therefore,

$$v = \frac{1 + x_0}{x_0} \frac{\partial}{\partial s_{01}}.$$

We see that the vector field v has a pole along the exceptional divisor $x_0 = 0$.

3. Consider the quotient $X = (\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{Z}$, where the group $\mathbb{Z}$ acts on $\mathbb{C}^2 \setminus \{(0,0)\}$ via $m \cdot z = 3^m z$, $m \in \mathbb{Z}$, $z \in \mathbb{C}^2 \setminus \{(0,0)\}$.

(a) Describe holomorphic charts on X that make it a complex manifolds. Explain why the transition functions for your charts are biholomorphic. $\boxed{4, \mathrm{B}}$

Let π be the natural projection $\mathbb{C}^2 \setminus \{(0,0)\} \to X$. The map π is a local homeomorphism, that is, for each $x \in X$, there is an open set $U_x \subset X$ containing x and an open set $V_x \subset \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\pi|_{V_x} : V_x \to U_x$ is a homeomorphism. Define the set of charts U_x, $x \in X$, with the chart maps $\phi_x : U_x \to V_x \subset \mathbb{C}^2$ given by $\phi_x = \left(\pi|_{V_x}\right)^{-1}$. Since X is second countable, we can choose countably many $x_1, x_2, ... \in X$, such that $\{U_{x_i}\}_{i \geq 1}$ is a countable cover of X. (In fact X is compact, so one can even choose a finite subcover.)

Let us check that the transition functions are biholomorphic. Let U_{x_i}, U_{x_j} be two of the charts above. We have

$$g_{i,j} = \phi_{x_i} \circ \phi_{x_j}^{-1} : V_{x_j} \to V_{x_i},$$
$$g_{ij}(z) = 3^m z, \text{ for some } m \in \mathbb{Z}.$$

The number m may change from one connected component of V_{x_j} to another. However, in each connected component it is clear from this formula that g_{ij} is a holomorphic function.

(b) Find the complex dimension of X. $\boxed{2, \mathrm{A}}$

Each chart in the construction above has a codomain that is an open subset of $\mathbb{C}^2$. Therefore, the manifold X has dimension 2.

(c) Let α be a complexified C^∞ differential k-form on a complex manifold Y, that is, $\alpha \in C^\infty(Y, \Omega^k_{Y,\mathbb{C}})$.

(i) What does it mean for α to be d-closed? $\boxed{2, \mathrm{A}}$

It means that $d\alpha = 0$.

(ii) What does it mean for α to be d-exact? $\boxed{2, \mathrm{A}}$

It means that $\alpha = d\eta$, for some complexified C^∞ differential $(k-1)$-form η on Y.

(d) Construct a complexified 1-form on X that is d-closed, but not d-exact. Explain why the form you constructed satisfies the required conditions. $\boxed{5, \mathrm{C}}$

Let β be the 1-form on $\mathbb{C}^2 \setminus \{(0,0)\}$ whose expression in the standard coordinates z_1, z_2 on $\mathbb{C}^2$ is

$$\beta = \frac{d(|z_1|^2 + |z_2|^2)}{|z_1|^2 + |z_2|^2}$$

The form β is invariant under the action of $\mathbb{Z}$; therefore, it induces a 1-form α on X. To check that the 1-form α is closed, it is enough to check that β is closed. The latter follows from the formula

$$\beta = d \log(|z_1|^2 + |z_2|^2).$$

To prove that α is not d-exact, by Stokes theorem, it is enough to present a C^∞ map $\gamma : S^1 \to X$ such that $\int_\gamma \alpha \neq 0$. (Indeed, if $\alpha = df$ for a function f, then

$\int_\gamma \alpha = \int_\emptyset f = 0$.) Let $S^1 = [0,1]/0 \sim 1$, and $\gamma(t) = (2t+1, 2t+1)$; note that $\gamma(0) = \gamma(1)$ in X, so this defines a map $S^1 \to X$. We have

$$\int_\gamma \alpha = \int_0^1 \gamma^*(\alpha) = \int_0^1 d\log(|2t+1|^2 + |2t+1|^2) = \log(3^2+3^2) - \log(1^2+1^2) = \log(18) - \log(2) \neq 0.$$

(e) Does there exists a holomorphic 1-form on X that is not identically zero? $\boxed{5, \mathrm{D}}$

The answer is no.

Let ω be a holomorphic 1-form on X. Let us consider the pullback 1-form $\sigma = \pi^*(\omega)$ on $\mathbb{C}^2 \setminus \{(0,0)\}$. Let

$$\sigma = f_1(z_1, z_2)dz_1 + f_2(z_1, z_2)dz_2,$$

where f_1,f_2 are holomorphic functions on $\mathbb{C}^2 \setminus \{(0,0)\}$. By Hartog's theorem, both the functions f_1, f_2 extend to holomorphic functions on $\mathbb{C}^2$. Therefore, σ also extends to a holomorphic 1-form on $\mathbb{C}^2$.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$, $\phi(z_1, z_2) = (3z_1, 3z_2)$. Since σ is a pullback under the quotient map π, and $\pi \circ \phi = \pi$ over $\mathbb{C}^2 \setminus \{(0,0)\}$, we have

$$\phi^*(\sigma) = \phi^*\pi^*(\omega) = \pi^*(\omega) = \sigma.$$

Let $f_1(z_1, z_2) = \sum_{i,j\geq 0} a_{ij} z_1^i z_2^j$ and $f_2(z_1, z_2) = \sum_{i,j\geq 0} b_{ij} z_1^i z_2^j$ be the power series expansions about the origin. We have

$$\sigma = \sum_{i,j\geq 0} a_{ij} z_1^i z_2^j dz_1 + \sum_{i,j\geq 0} b_{ij} z_1^i z_2^j dz_1,$$

$$\phi^*(\sigma) = \sum_{i,j\geq 0} a_{ij} 3^{i+j+1} z_1^i z_2^j dz_1 + \sum_{i,j\geq 0} b_{ij} 3^{i+j+1} z_1^i z_2^j dz_2.$$

By examining the terms of these two power series, we see that the equality $\phi^*(\sigma) = \sigma$ implies $\sigma = 0$. Therefore, we have $\omega = 0$, as claimed.

4. (a) Define the holomorphic line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$, $n \geq 1$, $k \in \mathbb{Z}$. $\boxed{4, \mathrm{B}}$

The line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ is defined as

$$\mathrm{Bl}_{\vec{0}}(\mathbb{C}^{n+1}) = \{(\ell, q) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid q \in \ell\}.$$

The line bundle projection $\pi \colon \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is the projection to the first factor. The trivialization of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over each open set in the standard cover $\{U_i\}$ of $\mathbb{P}^n$ is given by

$$\psi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{C},$$

$$\psi_i([x_0 \colon \ldots \colon x_n], (q_0, \ldots, q_n)) = ([x_0 \colon \ldots \colon x_n], q_i).$$

The transition maps $g_{i,j} = \psi_i \circ \psi_j^{-1}$ are given by

$$g_{i,j}(q_j) = \frac{x_i}{x_j} q_j.$$

The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is defined as the dual $\mathcal{O}_{\mathbb{P}^n}(-1)^*$ of $\mathcal{O}_{\mathbb{P}^n}(-1)$. Furthermore, for any $k > 0$, we have

$$\mathcal{O}(k) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes k},$$

$$\mathcal{O}_{\mathbb{P}^n}(-k) := \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}.$$

The line bundle $\mathcal{O}_{\mathbb{P}^n}(0)$ is defined to be the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$.

(b) Given a holomorphic vector bundle E over a complex manifold X, define the Dolbeault cohomology $H^q(X, E)$, $q \geq 0$. $\boxed{4, \mathrm{A}}$

Let

$$\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q}(E)) \to C^\infty(X, \Omega_X^{0,q+1}(E))$$

be the $\overline{\partial}$-operator of the holomorphic vector bundle E. Then

$$H^q(X, E) := \frac{\operatorname{Ker}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q}(E)) \to C^\infty(X, \Omega_X^{0,q+1}(E))\right)}{\operatorname{Im}\left(\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q-1}(E)) \to C^\infty(X, \Omega_X^{0,q}(E))\right)}$$

(c) Calculate $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. $\boxed{5, \mathrm{D}}$

Recall that $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ is the space of holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(3)$.

Let $U_i = \{[x_0 : x_1] \in \mathbb{P}^1 : x_i \neq 0\}$, $i = 0, 1$. Let $s = \frac{x_1}{x_0}$ be the coordinate on U_0, and $t = \frac{x_0}{x_1}$ be the coordinate on U_1. Let z be the fiberwise coordinate on $\mathcal{O}_{\mathbb{P}^1}(3)|_{U_0} \cong U_0 \times \mathbb{C}$, and w be the fiberwise coordinate on $\mathcal{O}_{\mathbb{P}^1}(3)|_{U_1} \cong U_1 \times \mathbb{C}$. The transition function between the coordinates s, z on $\mathcal{O}_{\mathbb{P}^1}(3)|_{U_0}$ and t, w on $\mathcal{O}_{\mathbb{P}^1}(3)|_{U_1}$ is given by

$$t = \frac{1}{s},$$

$$w = \frac{1}{s^3} z$$

Let $w = f(t) = \sum_{k=0}^\infty a_k t^k$ be a holomorphic section of $\mathcal{O}_{\mathbb{P}^1}(3)|_{U_1}$. In the other trivialization, this section will have the expression

$$\frac{1}{s^3} z = f\left(\frac{1}{s}\right)$$

$$z = s^3 f\left(\frac{1}{s}\right) = \sum_{k=0} a_k s^{3-k}.$$

Such a section is holomorphic over the whole $\mathbb{P}^1$ iff it extends over $s = 0$ iff $a_k = 0$, $k \geq 4$.

Therefore, the space of holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(3)$ is 4-dimensional, and is parametrized by a_0, a_1, a_3, a_4.

Answer: $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) = \mathbb{C}^4$.

(d) Calculate $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. $\boxed{4, \mathrm{B}}$

Recall that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

For the first summand, we have $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$, since every holomorphic function on $\mathbb{P}^1$ is constant.

We claim that the second summand is zero. To this end, we need to show that every holomoephic section $\psi : \mathbb{P}^1 \to \mathcal{O}_{\mathbb{P}^1}(-1)$ is the zero section. Let us consider the composition $\sigma \circ \psi : \mathbb{P}^1 \to \mathbb{C}^2$, where $\sigma : \mathcal{O}_{\mathbb{P}^1}(-1) = \mathrm{Bl}_{(0,0)}\mathbb{C}^2 \to \mathbb{C}^2$ is the blow-up map. Since $\mathbb{P}^1$ is compact, we conclude that $\sigma \circ \psi$ is constant. Moreover, σ is biholomorphism away from the exceptional divisor $E \subset \mathrm{Bl}_{(0,0)}\mathbb{C}^2$, and ψ is injective. This implies that $\sigma \circ \psi$ maps the whole $\mathbb{P}^1$ to $(0,0)$. Therefore $\psi(\mathbb{P}^1) = E$, which implies that ψ is the zero section.

Summing it up,

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathbb{C} \oplus 0 = \mathbb{C}.$$

(e) Calculate $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$. $\boxed{3, \mathrm{C}}$

Let us apply the Serre duality, having in mind that the canonical line bundle of $\mathbb{P}^1$ is $K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$:

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)^* \otimes K_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}.$$

5. (a) State the Hodge decomposition theorem for Kähler manifolds. $\boxed{4, \mathrm{M}}$

Let X be a compact Kähler manifold. Then its de Rham cohomology groups $H^k(X, \mathbb{C})$ and its Dolbeault cohomology groups $H^{p,q}(X)$ are all finite-dimensional, and

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover, $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

(b) Let X be a compact connected complex surface that is Kähler. Suppose that we know that $h^{2,0}(X) = 1$, $h^{1,2}(X) = 2$, and that the topological Euler characteristic of X is zero. Find all the Hodge numbers of X and draw its Hodge diamond. $\boxed{4, \mathrm{M}}$

Using complex conjugation, we obtain $h^{0,2} = h^{2,0} = 1$, $h^{2,1} = h^{1,2} = 2$. Since X is compact connected, we have $h^{0,0} = 1$. By Serre duality, we also have $h^{2,2} = h^{0,0} = 1$, $h^{0,1} = h^{2,1} = 1$, $h^{1,0} = h^{1,2} = 1$. Combining all this information, we get the following Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & ? & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

To determine remaining Hodge number $h^{1,1}$, we use that the Euler characteristic is zero:

$$b_0 - b_1 + b_2 - b_3 + b_4 = 0,$$

where $b_k = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ are the Betti numbers of X. The Hodge decomposition implies

$$b_0 = b_4 = 1,$$

$$b_1 = b_3 = 4,$$

$$b_2 = 2 + h^{1,1}.$$

These equalities together imply

$$1 - 4 + h^{1,1} + 2 - 4 + 1 = 0 \implies h^{1,1} = 4.$$

Therefore, the full Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

(c) State the $\partial\overline{\partial}$-lemma for Kähler manifolds. $\boxed{4, \mathrm{M}}$

Let X be a compact Kähler manifold. Let α be a d-closed (p, q)-form on X. Then α is d-exact if and only if it is $\partial\overline{\partial}$-exact.

(d) Let Y be a compact Kähler manifold. Let ω be a holomorphic p-form on Y, for some $p \geq 0$.

(i) Prove that ω cannot be d-exact, unless $\omega = 0$. $\boxed{4, \text{M}}$

The form ω is of type $(p, 0)$. If it is d-exact, then $\omega = \partial\overline{\partial}\beta$ for some form β. However, the type of β would need to be $(p-1, -1)$, which implies that $\beta = 0$. This, in turn, implies $\omega = 0$.

(ii) Does ω have to be d-closed? Prove it or give a counterexample. $\boxed{4, \text{M}}$

Yes, it does.

Let $\tau = d\omega$. Note that $\overline{\partial}\omega = 0$, therefore $\tau = \partial\omega$ is of type $(p+1, 0)$. By the $\partial\overline{\partial}$-lemma, we have $\tau = \partial\overline{\partial}\sigma$. The form σ needs to have type $(p, -1)$, which implies that $\sigma = 0$. Therefore, $\tau = 0$ as well.

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70060	1	No Comments Received
MATH70060	2	No Comments Received
MATH70060	3	No Comments Received
MATH70060	4	No Comments Received
MATH70060	5	No Comments Received