L9. $\mathcal{L}$: 1st order language.

(2.2.3) Def. Define the $\mathcal{L}$-formulas inductively:

1. An atomic formula of $\mathcal{L}$ is of the form $R(t_1, \dots, t_n)$ where R is an n-ary relation symbol (of $\mathcal{L}$) and $t_1, \dots, t_n$ are terms.

2. i) Any atomic formula is an $\mathcal{L}$-formula;
   ii) If ϕ, ψ are $\mathcal{L}$-formulas then
   $$(\neg \phi)$$
   $$(\phi \to \psi)$$
   $$(\forall x)\phi$$
   are $\mathcal{L}$-fmlas. (where x is any variable).
   iii) Any $\mathcal{L}$-fmla arises in this way.

Example Suppose $\mathcal{L}$ has

2-ary fn. symbol f
1-ary fn. symbol g
2-ary rel. symbol R
constant symbol c_1

Some terms:

$x_1, x_2, c_1, \quad f(g(x_2), c_1), \dots$

Atomic formula

$R(f(g(x_2), c_1), x_1) \quad R(x_1, x_2)$

$\mathcal{L}$-formulas

$$(\forall x_1)\left(R(f(g(x_2), c_1), x_1) \to R(x_1, x_2)\right)$$

Ex: Take the signature for groups (in 2.1.3): R, m, i, e & write some terms + formulas. How can you express the group axioms as formulas? (Assuming R is equality.)

Eg. $(\forall x_1) R(m(i(x_1), x_1), e)$

etc.

(2.2.4) Def. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure. A valuation in $\mathcal{A}$ is a function v from the set of terms of $\mathcal{L}$ to A (the domain of $\mathcal{A}$) satisfying:

a) $v(c_k) = \bar{c}_k$

b) If $t_1, \ldots, t_m$ are terms and f is an m-ary function symbol then

$$v(f(t_1, \ldots, t_m)) = \bar{f}(v(t_1), \ldots, v(t_m)).$$

(2.2.5) Lemma Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and $a_0, a_1, \ldots \in A$. Then there is a unique valuation v (in $\mathcal{A}$) with

$$v(x_\ell) = a_\ell \quad (\text{for all } \ell \in \mathbb{N}).$$

Pf: (Sketch) By induction of length of term t to define $v(t)$: let

i) $v(x_i) = a_i \quad (\forall i \in \mathbb{N})$

ii) $v(c_k) = c_k \quad (k \in K)$

iii) $v(f(t_1, \ldots, t_m)) = \bar{f}(v(t_1), \ldots, v(t_m))$.

Show this is a well-defined function. #

Example Groups

Signature: R, m, i, e

$\mathcal{G} = \langle \mathbb{Z} ; =, +, -, 0 \rangle$

Suppose $v(x_0) = 3$ & $v(x_1) = -4$

For v a valuation v:

$v(m(x_0, x_1)) = \bar{m}(v(x_0), v(x_1))$
$= 3 + (-4) = -1$.

$v(m(e, i(m(x_0, x_1)))) = 1$

(2.2.6) Def. Suppose v, w are valuations in an $\mathcal{L}$-str. $\mathcal{A}$. and x_ℓ is a variable. Say that v, w are <u>x_ℓ-equivalent</u> if $v(x_m) = w(x_m)$ when $m \neq \ell$.

[equal 'up to x_ℓ'].

(2.2.7) <u>Def.</u> (1) Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$. Define inductively, for an $\mathcal{L}$-formula ϕ what is meant by

v satisfies ϕ (in $\mathcal{A}$)

(abbreviated as $v[\phi] = T$)

[Negation: v does not satisfy ϕ (in $\mathcal{A}$)
(dns) $v[\phi] = F$]

(i) <u>Atomic formula</u>:
Suppose R is an n-ary rel. symbol and $t_1, \ldots, t_n$ are terms then v satisfies the atomic fmla

$$R(t_1, \ldots, t_n) \quad \text{(in } \mathcal{A}\text{)}$$

iff $\bar{R}(v(t_1), \ldots, v(t_n))$ holds in $\mathcal{A}$.

(ii) Suppose ϕ, ψ are $\mathcal{L}$-fmlas (& we ~~ka~~ know about valuations satisfying these)

(a) $v[(\neg\phi)] = T$ iff $v[\phi] = F$.

(b) Say $v[(\phi \to \psi)] = F$ iff $v[\phi] = T$ and $v[\psi] = F$.

(c) Say v satisfies $(\forall x_\ell)\phi$ (in $\mathcal{A}$) iff for every valuation w (in $\mathcal{A}$) which is x_ℓ-equivalent to v, we have $w[\phi] = T$.

(2) Suppose ϕ is an $\mathcal{L}$-fmla and $\mathcal{A}$ is an $\mathcal{L}$-str.. If every valuation in $\mathcal{A}$ satisfies ϕ then say that ϕ is <u>true in $\mathcal{A}$</u> (or $\mathcal{A}$ is a model of ϕ) and write $\mathcal{A} \models \phi$

(\models)

If $\mathcal{A} \models \phi$ for every $\mathcal{L}$-structure $\mathcal{A}$, say that ϕ is <u>logically valid</u> & write

$\models \phi$.

(Analogues of the propositional tautologies.)

Examples:

1) Suppose $\mathcal{L}$ has a 2-ary rel. symbol R. The $\mathcal{L}$ fmla: x_1

$$\phi : \left(R(x_1, x_2) \to \left(R(x_2, x_3) \to R(x_1, x_3)\right)\right)$$

is true in

$\mathcal{A} = \langle \mathbb{N} ; < \rangle$ $\quad \mathcal{A} \models \phi$.

R interpreted as $<$.

Also $\mathcal{A} \models (\forall x_1)(\forall x_2)(\forall x_3)\phi$.

2) $\mathcal{B} = \langle \mathbb{N} ; \neq \rangle$

(R read as $x_1 \neq x_2$)

then $\mathcal{B} \not\models \phi$

($\mathcal{B}$ is not a model of ϕ).

Eg take val. v with

$v(x_1) = v(x_3) = 1$ & $v(x_2) = 2$.