IMPERIAL

MATH70132

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Mathematical Logic

Date: Thursday, May 1, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Work in ZFC throughout, unless indicated otherwise. The notation is as used in the lecture notes. In particular: both $\mathbb{N}$ and ω denote the set of natural numbers; L is the formal system for propositional logic; if $\mathcal{L}$ is a first-order language, then $K_{\mathcal{L}}$ is the associated formal system for first-order logic. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the set of integers, the set of rational numbers and the set of real numbers.

Unless indicated otherwise, you may use results from the lecture notes if these do not depend on what is being asked in the question. Results quoted from the lecture notes should be stated clearly.

1. (a) State the *Generalised Soundness Theorem* for the formal system L. (3 marks)

   (b) For ϕ a propositional formula of 3 variables p_1, p_2, p_3 let $W[\phi]$ denote the set of truth values for (p_1, p_2, p_3) for which ϕ has truth value F.

   (i) Write down examples of propositional formulas ϕ_1, ϕ_2, ϕ_3 where
   - $W[\phi_1] = \{(T,T,T),(T,F,T)\}$;
   - $W[\phi_2] = \{(T,F,T),(F,F,T)\}$;
   - $W[\phi_3] = \{(F,T,T),(T,F,T)\}$. (4 marks)

   (ii) Prove that none of the formulas ϕ_1, ϕ_2, ϕ_3 is a consequence of the other two formulas. (3 marks)

   (iii) Is there a formula ϕ_4 which is a consequence of $\Gamma = \{\phi_1, \phi_2, \phi_3\}$, but which is not a consequence of any proper subset of Γ? Explain your answer. (3 marks)

   (c) In this part of the question, you should give syntactic arguments, not involving truth tables, valuations, or use of the Completeness Theorem for L. You may use results from the notes and problem sheets about theorems of L. Suppose ϕ, ψ are L-formulas.

   (i) Show that $\{((\neg\phi) \to \psi), ((\neg\phi) \to (\neg\psi))\} \vdash_L \phi$.

   (ii) Show that $\vdash_L ((\phi \to \psi) \to (((\neg\phi) \to \psi) \to \psi))$.

   (7 marks)

(Total: 20 marks)

2. (a) Suppose $\mathcal{L}^=$ is a 1st-order language with equality having 2-ary relation symbols $=, R, E$, a constant symbol c and a 1-ary function symbol f.

(i) Write down:

- three terms which are not variables;
- a closed atomic formula;
- a formula with one free variable which is not atomic.

(3 marks)

(ii) Consider the $\mathcal{L}^=$-formula ϕ:

$$(\forall x_1)(R(x_1, x_2) \to ((\neg E(x_2, x_3)) \to (\forall x_2)E(x_1, x_2))).$$

- Identify the bound and free occurrences of the variables x_1, x_2, x_3 in ϕ.
- Give an example of a term which is not free for x_2 in ϕ. Are there only finitely many such terms?
- Find a normal model of $(\forall x_1)(\forall x_2)(\forall x_3)\phi$.

(9 marks)

(b) In this part, $\mathcal{L}^=$ is a 1st-order language with equality having 2-ary relation symbols $=$, R, E and no other relation, function and constant symbols. Consider the following normal $\mathcal{L}^=$-structures where in each case, R is interpreted as the reverse lexicographic ordering on the domain (with $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ having their usual orderings $\leq$), E is interpreted as the equivalence relation of having the same second coordinate and:

$\mathcal{A}_1$ has domain $\mathbb{Q} \times \mathbb{Q}$;
$\mathcal{A}_2$ has domain $\mathbb{Q} \times \mathbb{Z}$;
$\mathcal{A}_3$ has domain $\mathbb{Q} \times \mathbb{N}$.

(i) For $1 = 1, 2, 3$ find a closed $\mathcal{L}^=$-formula θ_i such that for all $j \leq 3$ we have $\mathcal{A}_j \models \theta_i$ if and only if $i = j$. Explain your answer briefly. (4 marks)

(ii) Suppose that $\mathcal{A}_4$ is as for $\mathcal{A}_1$ except that E is interpreted as the equivalence relation of having the same *first* coordinate. Are $\mathcal{A}_1$ and $\mathcal{A}_4$ isomorphic $\mathcal{L}^=$-structures? Justify your answer. (4 marks)

(Total: 20 marks)

3. In this question, $\mathcal{L}^=$ is a 1st-order language with equality having 2-ary relation symbols $=$ and R (and no other relation, function and constant symbols).

(a) (i) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula σ_n with the property that a normal $\mathcal{L}^=$-structure $\mathcal{A}$ is a model of σ_n if and only if the domain of $\mathcal{A}$ has at least n elements. (2 marks)

(ii) State the *Compactness Theorem* for normal $\mathcal{L}^=$-structures. Suppose that Σ is a set of closed $\mathcal{L}^=$-formulas such that every normal model of Σ is finite. Prove that there is $m \in \mathbb{N}$ such that every normal model of Σ has size at most m. (4 marks)

(iii) Suppose that $\mathcal{A} = \langle A; \bar{R}_A \rangle$ and $\mathcal{B} = \langle B; \bar{R}_B \rangle$ are normal $\mathcal{L}^=$-structures and $\mathcal{A}$ is finite. Suppose further that for every closed $\mathcal{L}^=$-formula ϕ we have that

$$\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi.$$

Prove that $\mathcal{B}$ is isomorphic to $\mathcal{A}$. (3 marks)

(b) By definition, a *graph* is a normal $\mathcal{L}^=$-structure $\mathcal{A} = \langle A; \bar{R} \rangle$ which is a model of the following $\mathcal{L}^=$ formula γ:

$$(\forall x_1)(\forall x_2)(R(x_1, x_2) \to (R(x_2, x_1) \wedge (x_1 \neq x_2))).$$

A graph $\mathcal{A} = \langle A; \bar{R} \rangle$ is called *rich* if for all finite subsets $C, D \subseteq A$ with $C \cap D = \emptyset$ there is $a \in A$ such that for all $c \in C$ and $d \in D$ we have $(a, c) \in \bar{R}$ and $(a, d) \notin \bar{R}$. In the following, you may assume without proof that: there is a rich graph; any rich graph is infinite; and any two countable rich graphs are isomorphic.

(i) Write down a set Δ of closed $\mathcal{L}^=$-formulas, including γ, with the property that a graph $\mathcal{A}$ is rich if and only if $\mathcal{A} \models \Delta$. (4 marks)

(ii) Write down the set $\Sigma_=$ consisting of the axioms for equality of the language $\mathcal{L}^=$. (3 marks)

(iii) Prove that the set $\Delta' = \Delta \cup \Sigma_=$ is complete: meaning that if ϕ is a closed $\mathcal{L}^=$-formula, then either ϕ or $(\neg\phi)$ is a consequence of Δ'. (4 marks)

(Total: 20 marks)

4. (a) Consider the following sets:

$S_1 = \mathbb{R}$, the set of real numbers;

$S_2 = \mathbb{C}$, the set of complex numbers;

$S_3 = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$;

S_4 is a basis for $\mathbb{C}$ considered as a vector space over $\mathbb{R}$;

S_5 is a basis for $\mathbb{C}$ considered as a vector space over $\mathbb{Q}$;

S_6 is $\mathbb{R}^{\mathbb{R}}$, the set of functions from $\mathbb{R}$ to $\mathbb{R}$;

S_7 is $\mathcal{P}(\mathcal{P}(\mathbb{N}))$.

In parts (i) and (ii) below, explanations are required (and any general results about cardinal arithmetic may be assumed, if quoted accurately); part (iii) does not require any explanation.

(i) Compare the cardinalities of S_1 and S_2. (2 marks)

(ii) Compare the cardinalities of S_1, S_6 and S_7. (4 marks)

(iii) State the relationship ($<$ or $=$) between the cardinalities $|S_i|$ of the sets S_i (for $i \leq 7$). (3 marks)

(b) Define what it means for a set α to be an *ordinal*. Prove that if α is an ordinal, then so is its successor $\alpha^\dagger = \alpha \cup \{\alpha\}$. (4 marks)

(c) Suppose γ, δ are non-zero ordinals and $f : \gamma \to \delta$ is a function with the property that for all $y \in \delta$ there is some $x \in \gamma$ such that $y \leq f(x)$ (where $\leq$ is the usual ordering on an ordinal).

(i) Prove that if γ is finite, then there exists an ordinal α with $\delta = \alpha^\dagger$. (3 marks)

(ii) Prove that if δ is the least uncountable ordinal, then γ is uncountable. (2 marks)

(iii) Give an example where δ is an uncountable ordinal and $\gamma = \omega$. Justify your answer. (2 marks)

(Total: 20 marks)

5. In this question, $\mathcal{L}^=$ denotes a first-order language with equality. All $\mathcal{L}^=$-structures considered are assumed to be normal $\mathcal{L}^=$-structures. In (a), when you are asked to provide an example you may choose the language $\mathcal{L}^=$.

(a) Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}^=$-structures with domains M and N (respectively).

(i) Define what it means to say that $\mathcal{M}$ is *elementarily equivalent* to $\mathcal{N}$ (written $\mathcal{M} \equiv \mathcal{N}$) and for $\mathcal{N}$ to be an *elementary extension* of $\mathcal{M}$ (written $\mathcal{M} \preceq \mathcal{N}$). (3 marks)

(ii) Give an example of isomorphic $\mathcal{L}^=$-structures $\mathcal{M}$, $\mathcal{N}$ where $M \subseteq N$ and $\mathcal{N}$ is not an elementary extension of $\mathcal{M}$. Justify your answer. (3 marks)

(iii) Suppose that $\mathcal{M}$ is infinite. Explain why there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ with $N \neq M$. (3 marks)

(b) Let $\mathcal{L}^=$ be the language (with equality) having 2-ary function symbols $+, \cdot$; constant symbols $0, 1, c$; 2-ary relation symbols $=, <$. Let $(a_n : n < \omega)$ be a sequence of rational numbers. For each $n < \omega$ let $\mathcal{A}_n$ be the normal $\mathcal{L}^=$-structure with domain $\mathbb{Q}$:

$$\mathcal{A}_n = \langle \mathbb{Q}; +, \cdot, 0, 1, a_n, < \rangle,$$

where $+, \cdot, 0, 1$ and $<$ have their usual meaning on $\mathbb{Q}$ and the extra constant symbol c is interpreted as a_n in $\mathcal{A}_n$.

Let $\mathcal{F}$ be a non-principal ultrafilter on ω and let

$$\mathcal{M} = \langle M; +, \cdot, 0, 1, b, < \rangle$$

be the ultraproduct $(\Pi_{n<\omega}\mathcal{A}_n)/\mathcal{F}$. So $b \in M$ is the interpretation of c in $\mathcal{M}$. In the following you should justify your answers, but standard results about ultraproducts may be used if clearly stated.

(i) Define the domain M of $\mathcal{M}$ and state how $\mathbb{Q}$ may be regarded as a subset (in fact, a subfield) of M. (3 marks)

(ii) Suppose that $a_i \neq a_j$ for all $0 \leq i < j < \omega$. Show that if $p(x)$ is a non-zero polynomial with coefficients in $\mathbb{Q} \subseteq M$, then $p(b) \neq 0$. (4 marks)

(iii) Suppose that $r \in \mathbb{R}$. Show that for some choice of the sequence $(a_i : i < \omega)$ we have that for all $q \in \mathbb{Q} \subseteq M$:

$$q < b \text{ (in } \mathcal{M}) \Leftrightarrow q < r \text{ (in } \mathbb{R}).$$

(4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH70132

Mathematical Logic (Solutions)

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1. **Comments: (a) bookwork; (b) standard use of d.n.f. but unseen in this form (the notation $W[\phi]$ is unseen); (c) seen similar, but not these examples.**

(a) Suppose $\Sigma \cup \{\phi\}$ is a set of L-formulas and v is a propositional valuation. If $\Sigma \vdash_L \phi$ and $v(\Sigma) = T$, then $v(\phi) = T$. $\boxed{3, \text{A}}$

(b) (i) Let ψ_1, ψ_2, ψ_3 be, respectively, the formulas (in disjunctive normal form): $\boxed{4, \text{A}}$
$(p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge (\neg p_2) \wedge p_3)$;
$(p_1 \wedge (\neg p_2) \wedge p_3) \vee ((\neg p_1) \wedge (\neg p_2) \wedge p_3)$;
$((\neg p_1) \wedge p_2 \wedge p_3) \vee (p_1 \wedge (\neg p_2) \wedge p_3)$.
Let ϕ_i be $(\neg \psi_i)$ (for $i = 1, 2, 3$).

(ii) For $i = 1, 2, 3$ we can find a valuation v_i such that $v_i(\phi_i) = F$ and v_i of the other two formulas is true. The result then follows from (a). For example, we can take $v_1((p_1, p_2, p_3)) = (T, T, T)$, $v_2((p_1, p_2, p_3)) = (F, F, T)$ and $v_3((p_1, p_2, p_3)) = (F, T, T)$. $\boxed{3, \text{B}}$

(iii) Yes: take ϕ_4 to be $\phi_1 \wedge \phi_2 \wedge \phi_3$. Certainly we have that ϕ_4 is a consequence of Γ. Also, note that each ϕ_i is a consequence of ϕ_4, so the required property follows directly from (b)(ii). $\boxed{3, \text{B}}$

(c) (i) Let $\Gamma = \{((\neg \phi) \to \psi), ((\neg \phi) \to (\neg \psi))\}$. Two applications of Modus Ponens give that both ψ and $(\neg \psi)$ are consequences of $\Gamma \cup \{(\neg \phi)\}$. It follows (using a result in the notes) that any formula is a consequence of $\Gamma \cup \{(\neg \phi)\}$. Let α be an axiom. Then $\Gamma \cup \{(\neg \phi)\} \vdash_L (\neg \alpha)$. Using the Deduction Theorem, we obtain $\Gamma \vdash ((\neg \phi) \to (\neg \alpha))$. Using MP and an A3 axiom then gives $\Gamma \vdash (\alpha \to \phi)$. As α is an axiom, we have $\Gamma \vdash \alpha$, so MP then gives $\Gamma \vdash \phi$, as required. (Other arguments in L are possible.) $\boxed{4, \text{D}}$

(ii) Let $\Delta = \{(\phi \to \psi), ((\neg \phi) \to \psi)\}$. Using $\vdash ((\phi \to \psi) \to (\neg \psi \to \neg \phi))$ (from notes) and MP we obtain $\Delta \vdash (\neg \psi \to \neg \phi)$. By Hypothetical Syllogism we then obtain $\Delta \vdash (\neg \psi \to \psi)$. The notes show that $\vdash ((\neg \psi \to \psi) \to \psi)$, so MP gives $\Delta \vdash \psi$. Two applications of DT then give the result. $\boxed{3, \text{C}}$

2. **Comments: (a) most of this is just testing standard definitions, the last part in (ii) is harder as the formula is complicated; (b) unseen example, but seen similar, reverse lexicographic ordering was covered in lectures and problem sheets and was on the paper last year.**

(a) (i) There are many possible answers here, for example: $\boxed{3, \mathrm{A}}$
c, $f(x_1)$, $f(c)$ are 3 terms which are not variables;
$R(c, f(c))$ is a closed atomic formula;
$(\forall x_1)R(x_1, x_2)$ is a formula with one free variable which is not atomic.

(ii) • In the formula below a hat over the variable indicates that it is a bound occurrence; all other occurrences are free: $\boxed{3, \mathrm{A}}$

$$(\forall \hat{x}_1)(R(\hat{x}_1, x_2) \to ((\neg E(x_2, x_3)) \to (\forall \hat{x}_2)E(\hat{x}_1, \hat{x}_2))).$$

• The term x_1 is not free for x_2 in ϕ. There are infinitely many terms not free for x_2 in ϕ: $f(x_1)$, $f(f(x_1))$, $f(f(f(x_1)))$, ... $\boxed{3, \mathrm{A}}$

• For the normal model $\langle A; \bar{R}, \bar{E}, \bar{c}, \bar{f}\rangle$ we keep things as simple as possible. Note that if $\bar{R}$ is the empty-set (so no elements are related by $\bar{R}$), then the formula is true. So take this to hold where A is any non-empty set, and, for example, take $\bar{E}$ to be equality, $\bar{f}$ the identity function and $\bar{c}$ any element of A. $\boxed{3, \mathrm{C}}$

(b) (i) In each case, denote by $\bar{E}((a, b))$ the equivalence class containing the pair (a, b) (in the domain A_i). $\mathcal{A}_1$ has the property that if $b' < b$ there is b'' with $b' < b'' < b$, and this is not the case in $\mathcal{A}_2$ or $\mathcal{A}_3$. Thus we can take as θ_1 the formula: $\boxed{4, \mathrm{B}}$

$$(\forall x)(\forall y)(\exists z)((\neg E(x, y) \wedge (x < y)) \to ((x < z) \wedge (z < y) \wedge (\neg E(x, z)) \wedge (\neg E(y, z)))).$$

$\mathcal{A}_2$ has the property that for all b there is b' with $b' < b$ and this does not hold in $\mathcal{A}_3$. Thus as θ_2 we may take the conjunction of $\neg\theta_1$ and

$$(\forall x)(\exists y)(\neg E(x, y) \wedge (y < x)).$$

Finally, for θ_3 we can take $(\neg\theta_1) \wedge (\neg\theta_2)$.
Note that in the above, we are using $x < y$ as an abbreviation for $(R(x, y) \wedge (x \neq y))$. We are also missing some brackets out to aid readability.

(ii) $\mathcal{A}_1$ and $\mathcal{A}_4$ are not isomorphic. In $\mathcal{A}_1$ the $\bar{E}$-equivalence classes are convex (if $(a, b) < (a', b') < (a'', b)$, then $b' = b$). But the classes are not convex in $\mathcal{A}_4$: for example $(0, 0) < (1, 1) < (0, 2)$. So $\mathcal{A}_1$ is a model of the following formula, but $\mathcal{A}_4$ is not: $\boxed{4, \mathrm{D}}$

$$(\forall x)(\forall y)(\forall z)((E(x, y) \wedge (x < z) \wedge (z < y)) \to E(x, z)).$$

3. **Comments: (a) (i), (ii) bookwork and standard argument; (a)(iii) unseen and harder; (b) unseen example, (i), (ii) bookwork and standard, (iii) seen this argument in a different context.**

(a) (i) Take σ_n to be $(\exists x_1)\dots(\exists x_n)\bigwedge_{1\le i<j\le n}(x_i \ne x_j)$. $\boxed{2, \mathrm{A}}$

(ii) The Compactness Theorem states that if Δ is a set of closed $\mathcal{L}^=$-formulas such that every finite subset of Δ has a normal model, then Δ has a normal model. For the main part, suppose not.Then for every $n \in \mathbb{N}$ such that Σ has a normal model with at least n elements. It follows that every finite subset of $\Gamma = \Sigma \cup \{\sigma_n : n \in \mathbb{N}\}$ has a normal model. By the Compactness Theorem, it then follows that Γ has a normal model $\mathcal{A}$. But then $\mathcal{A}$ is an infinite normal model of Σ, which contradicts the given property of Σ. $\boxed{4, \mathrm{B}}$

(iii) Suppose A has n elements and $A = \{a_1, \dots, a_n\}$. Let S be the set of pairs (i, j) such that $\bar{R}_A(a_i, a_j)$ holds and S' the set of pairs (i, j) such that $\bar{R}_A(a_i, a_j)$ does not hold. Consider the formula θ which is the conjunction of $\neg\sigma_{n+1}$ and $\boxed{3, \mathrm{C}}$

$$(\exists x_1)\dots(\exists x_n)((\bigwedge_{1\le i<j\le n}(x_i \ne x_j))\wedge(\bigwedge_{(i,j)\in S} R(x_i, x_j))\wedge(\bigwedge_{(i,j)\in S'} \neg R(x_i, x_j))).$$

This is a closed $\mathcal{L}^=$ formula which is true in $\mathcal{A}$. So $\mathcal{B}$ is a normal model of θ. It follows that B has exactly n elements which we may write as $b_1, \dots, b_n$ so that the map $a_i \mapsto b_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

(b) (i) Let δ_n be the formula: $\boxed{4, \mathrm{A}}$

$$(\forall x_1)\dots(\forall x_n)(\forall y_1)\dots(\forall y_n)(\exists z)(\bigwedge_{1\le i,j\le n}(x_i \ne y_j) \to \bigwedge_{1\le i\le n}(R(z, x_i)\wedge(\neg R(z, y_i))).$$

Let $\Delta = \{\gamma, \delta_n : n \in \mathbb{N}\}$.

(ii) The axioms state that $=$ is an equivalence relation and: $\boxed{2, \mathrm{A}}$

$$(\forall x_1)(\forall x_2)(\forall y_1)(\forall y_2)((x_1 = y_1) \wedge (x_2 = y_2) \wedge (R(x_1, x_2) \to R(y_1, y_2))).$$ $\boxed{1, \mathrm{B}}$

(iii) Note that any rich graph is a model of Δ', so Δ' is consistent. Suppose for a contradiction that neither formula is a consequence of Δ'. Then by a result from the notes, both $\Delta' \cup \{\neg\phi\}$ and $\Delta' \cup \{\neg\neg\phi\}$ are consistent. By the Model Existence Theorem (and the fact that these sets contain the axioms for equality), there are countable normal models $\mathcal{A}_1$ and $\mathcal{A}_2$ of the sets. Now, $\mathcal{A}_1$ and $\mathcal{A}_2$ are countable normal models of Δ, so are countable rich graphs, hence they are isomorphic (by the given facts). But also $\mathcal{A}_1 \models \neg\phi$ and $\mathcal{A}_2 \models \neg\neg\phi$, which is a contradiction. $\boxed{4, \mathrm{D}}$

4. **Comments: (a) seen similar examples in notes and on past papers; (b) bookwork; (c) unseen (cofinality was not covered in the module).**

(a) (i) The map $(a, b) \mapsto a+ib$ is a bijection between $\mathbb{R}^2$ and $\mathbb{C}$. By the Fundamental Theorem of Cardinal Arithmetic $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Thus $|S_1| = |S_2|$. $\boxed{2,\ \mathrm{A}}$

(ii) As a consequence of FTCA, if λ, κ are cardinals, κ is infinite and $2 \leq \lambda \leq \kappa$, then $\lambda^\kappa = \kappa^\kappa$. It follows that $|S_6| = 2^\kappa$, where $\kappa = |\mathbb{R}| = |S_1|$. We know (from notes) that $\kappa = 2^\omega$ and that for any set X we have $|X| < |\mathcal{P}(X)| = 2^{|X|}$ (Cantor's Theorem). It follows that $|S_1| < |S_6| < |S_7|$. $\boxed{4,\ \mathrm{B}}$

(iii) We have $\boxed{2,\ \mathrm{A}}$

$$2 = |S_4| < |S_1| = |S_2| = |S_3| = |S_5| < |S_6| < |S_7|.$$ $\boxed{1,\ \mathrm{B}}$

(b) A set α is an ordinal if it is transitive (meaning: if $x \in y \in \alpha$, then $x \in \alpha$) and the membership relation $\in$ restricted to α is a strict well-ordering on the elements of α. $\boxed{4,\ \mathrm{A}}$

The membership relation on $\alpha^\dagger$ is that of α with an extra element, greater than all elements of α. This is still a strict well ordering. For transitvity, if $x \in y \in \alpha^\dagger$, then either: $y \in \alpha$, in which case $x \in \alpha \subseteq \alpha^\dagger$; or $y = \alpha$, in which case we still have $x \in \alpha$.

(c) (i) Suppose γ is finite and (as given) γ is non-empty. Then γ has a greatest element, say z. Let $f(z) = \alpha \in \delta$. By the given property of f, we have $x \leq \alpha$ for all $x \in \delta$. Moreover $\delta = \{x \in \delta : x < \alpha\} \cup \{\alpha\} = \alpha \cup \{\alpha\} = \alpha^\dagger$, as required. $\boxed{3,\ \mathrm{C}}$

(ii) Suppose not and that γ is countable. So there is a surjection $\theta : \omega \to \gamma$. For $n \in \omega$, let $\beta_n = \{x \in \delta : x \leq f(\theta(n))\} = f(\theta(n))^\dagger$. Thus β_n is a countable ordinal (as $f(\theta(n)) \in \delta$). By the assumption on f, we have $\delta \subseteq \bigcup\{\beta_n : n \in \omega\}$. But this implies that δ is countable: a contradiction. $\boxed{2,\ \mathrm{D}}$

(iii) Define a sequence of cardinals $(\kappa_n : n \in \omega)$ by $\kappa_0 = \omega$ and $\kappa_{n+1} = 2^{\kappa_n}$. By Cantor's Theorem, $\kappa_n < \kappa_{n+1}$, so in particular κ_n is uncountable for $n \geq 1$. Let $\delta = \bigcup_{n\in\omega} \kappa_n$. This is an ordinal and $\kappa_n < \delta$ for all $n \in \omega$, so δ is certainly uncountable. Define the map $f : \omega \to \delta$ by $f(n) = \kappa_n$. If $x \in \delta$ then $x \in \kappa_n$ for some $n \in \omega$, so $x < f(n)$. $\boxed{2,\ \mathrm{D}}$

5. **Comments: (a) (i), (ii), (iii) standard bookwork; (b) (i) bookwork; (ii), (iii) seen similar, but not this example.**

(a) (i) $\mathcal{M} \equiv \mathcal{N}$ means that for every closed $\mathcal{L}^=$-formula ϕ we have $\mathcal{M} \models \phi$ iff $\mathcal{N} \models \phi$. $\mathcal{M} \preceq \mathcal{N}$ means that $M \subseteq N$ and for any $\mathcal{L}^=$-formula $\psi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$ we have $\mathcal{M} \models \psi(a_1, \ldots, a_n)$ iff $\mathcal{N} \models \psi(a_1, \ldots, a_n)$. $\boxed{3, \mathrm{M}}$

(ii) Consider the group $\mathcal{N} = \langle \mathbb{Z}; +, 0 \rangle$ (as a structure in the obvious language) and the subgroup $\mathcal{M} = \langle 2\mathbb{Z}; +, 0 \rangle$. Clearly $\mathcal{M}$ is isomorphic to $\mathcal{N}$. Let $\psi(x)$ be the formula $(\exists y)(y + y = x)$ and consider the element $2 \in M$. Then $\mathcal{N} \models \psi(2)$, but $\mathcal{M} \not\models \psi(2)$. $\boxed{3, \mathrm{M}}$

(iii) By the upward Loewenheim - Skolem Theorem there is an elementary extension $\mathcal{M} \preceq \mathcal{N}$ with $|N|$ of arbitrarily large cardinality. Taking this cardinality to be greater than $|M|$ gives what we want. $\boxed{3, \mathrm{M}}$

(b) (i) For sequences $(p_i), (q_i) \in \mathbb{Q}^\omega$ write $(p_i) \sim (q_i)$ iff $\{i \in \omega : p_i = q_i\} \in \mathcal{F}$. This is an equivalence relation on $\mathbb{Q}^\omega$ and M is the set of equivalence classes. Denote the $\sim$-class of the sequence (p_i) by $[(p_i)]$.We identify $q \in \mathbb{Q}$ with the $\sim$-class of the constant sequence $(q : n < \omega)$ in the ultraproduct. $\boxed{3, \mathrm{M}}$

(ii) Note that $b = [(a_n : n < \omega)]$. Suppose that $p(b) = 0$. We may assume that the coefficients of $p(x)$ are integers. By rearranging the equation $p(b) = 0$ into an equation $p_1(b) = p_2(b)$ where p_1, p_2 are polynomials with only non-negative integer coefficients, we see that there is an $\mathcal{L}^=$-formula $\phi_p(x)$ such that being a root of $p(x)$ is the same as satisfying $\phi_p(x)$. As $\mathcal{M} \models \phi_p(b)$ it follows by the Łos Theorem (or even from the definition of $\mathcal{M}$) that $\{n < \omega : \mathcal{A}_n \models \phi_p(a_n)\} \in \mathcal{F}$. As $\mathcal{F}$ is non-principal, all the sets in it are infinite. In particular, there are infinitely many $n < \omega$ with $p(a_n) = 0$. As p is a non-zero polynomial and the a_n are distict, this is a contradiction. [I would not expect so much detail here.] $\boxed{4, \mathrm{M}}$

(iii) Let $(a_n : n < \omega)$ be an increasing sequence of rationals converging to r. We claim that this works (again, recall that $b = [(a_n : n < \omega)]$). Suppose $q < r$. There is some $k < \omega$ such that if $n > k$, then $q < a_n$. It follows (by definition $<$ on $\mathcal{M}$ and that the cofinite set $\{n < \omega : n > k\}$ is in $\mathcal{F}$) that $q < b$. Conversely, suppose $q = [(q : n < \omega)] < b$ (in $\mathcal{M}$). So by definition of $<$ on $\mathcal{M}$, the set $\{n < \omega : q < a_n\} \in \mathcal{F}$ and so is non-empty. As $a_n < r$ it follows that $q < r$. $\boxed{4, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total Mastery marks: 20 of 20 marks
Total marks: 100 of 100 marks

MATH70132 Mathematical Logic Markers Comments

Question 1	Generally, this was done well. In (b)(ii) several people misinterpreted the question to mean (for example) that \phi_1 is not a consequence of \phi_2 and not a consequence of \phi_3, rather than \phi_1 is not a consequence of the set \{ \phi_2,\phi_3\}. But I allowed this.
Question 2	There was a small typo at the start of (b)(i) (it should say 'For i = 1,2,3 ....') but only one person commented on this. Most of the standard parts were done well, though some people had not learned the definitions correctly. (b)(ii) was challenging, but there were some correct solutions.
Question 3	There were a few incorrect statements of the Compactness Theorem and some poor attempts at using it, despite this type of question appearing on every past paper. (a)(iii) was challenging (it is not enough to say that there is a bijection from A to B). In (b)(iii) some people used Vaught's Test and \omega-categoricity from the Mastery material, which was fine.
Question 4	There were some good answers to this question, although (c)(ii), (iii) did not receive many serious attempts (in fact, (c)(iii) was the only question on the paper that no-one solved correctly).
Question 5	A few good answers here from people who had studied the Mastery material well. In (b)(ii) very few people explained why 'p(x) = 0' could be regarded as a formula in the given language, but otherwise the argument was correct.