Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 10
Algebra of limits,
Monotone and bounded sequences

Ajay Chandra

Theorem 3.19 - Algebra of limits

If $a_n \to a$ and $b_n \to b$ then:

1. $a_n + b_n \to a + b$,
2. $a_n b_n \to ab$,
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$. ← Problem Sheet

Proof

① Let $\varepsilon > 0$, There exists $N_1 \in \mathbb{N}_0$ st $\forall n \geq N_1, |a_n - a| < \varepsilon/2$
$N_2 \in \mathbb{N}_0$ $\forall n \geq N_2, |b_n - b| < \varepsilon/2$

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$

$|a_n + b_n - (a+b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon \implies a_n + b_n \longrightarrow a + b$ ∎

② Want $a_nb_n \to ab$

Rough work: $|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab|$

$|a_n-a|, |b_n-b|$ are small

$\le |a_nb_n - a_nb| + |a_nb - ab|$ Now factor

$\le \underbrace{|a_n|\cdot|b_n-b|}_{(i)} + \underbrace{|b|\cdot|a_n-a|}_{(ii)}$

Wanted (i) $< \varepsilon/2$

(ii) $< \varepsilon/2$

$\le (|a_n|+|b|)\varepsilon$ Not enough, bound should be independent of n

$\le (M+|b|)\varepsilon$

$|a_n|$ is bounded by some $M>0$

Proof: Let $\varepsilon>0$, Then $\exists M_1\in\mathbb{N}$, $\exists M_2\in\mathbb{N}$ s.t. $|a_n-a|<\frac{\varepsilon}{2(|b|+1)}$ $\forall n\ge M_1$

$|b_n-b| < \frac{\varepsilon}{2M}$ $\forall n\ge M_2$

where $M>0$ is an upper bound for $|a_n|$ (a_n is bounded since it converges)

Then for $n\ge \max(M_1,M_2)$

$$|a_nb_n - ab| \le |a_nb_n - a_nb| + |a_nb-ab| \le |a_n|\cdot\overset{n\ge M_2}{|b_n-b|} + |b|\cdot\overset{n\ge M_1}{|a_n-a|}$$

$$< M\frac{\varepsilon}{2M} + |b|\cdot\frac{\varepsilon}{2(|b|+1)} \le \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \frac{|b|}{|b|+1}<1$$

So $a_nb_n \to ab$ ■

Remark 3.20

Now it's easier to handle things like $a_n = \dfrac{n^2 + 5}{n^3 - n + 6}$.

$a_n \to 0$ Rough work

Using algebra of limits

$$a_n = \frac{\overbrace{1/n + 5/n^3}^{b_n}}{\underbrace{1 - 1/n^2 + 6/n^3}_{c_n}}$$

dividing n^3 on top and bottom

We know $\frac{1}{n} \to 0$ + Algebra of limits

$$\lim_{n\to\infty} \frac{1}{n} + \frac{5}{n^3} = \lim_{n\to\infty} \frac{1}{n} + 5 \lim_{n\to\infty} \left(\frac{1}{n}\right)^3 = 0$$

Can also show $c_n \to 1$

Can show $a_n \to 0$

Warning

Not true if $a_n = n$, $b_n = -n$

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Only makes sense if you ~~kn~~ know a_n, b_n converge

Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i : i \in \mathbb{N}_{>0}\}$. We write $a_n \uparrow a$.

$j > i \Rightarrow a_j \geq a_i$

Since a is the supremum,

$\forall \varepsilon > 0$, $N \in \mathbb{N}_{>0}$, with $a_N > a - \varepsilon$

$\forall n \geq N \quad a - \varepsilon < a_N \leq a_n \leq a$

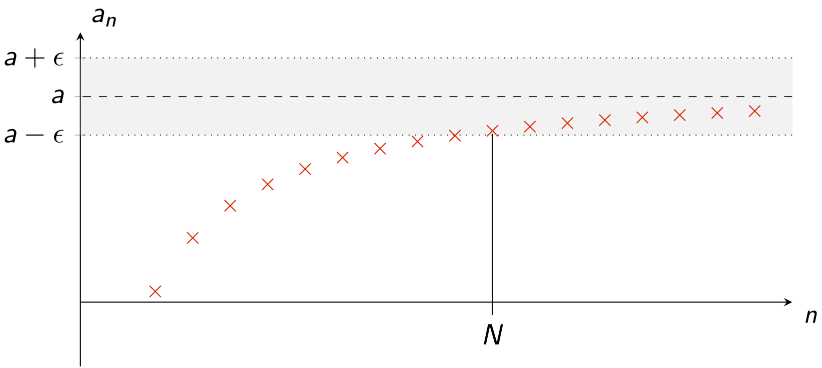
So $\forall n \geq N \quad a_n \in (a-\varepsilon, a+\varepsilon) \Leftrightarrow |a_n - a| < \varepsilon$

So $a_n \to a$

$(a_n \uparrow a)$

Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i : i \in \mathbb{N}_{>0}\}$. We write $a_n \uparrow a$.



Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i \colon i \in \mathbb{N}_{>0}\}$. We write $a_n \uparrow a$.