IMPERIAL

MATH50007

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Network Science

Date: Wednesday, May 15, 2024

Time: 10:00 – 12:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Consider a simple connected graph G of 5 nodes. Number the nodes from 1 to 5, and assume node 1 is connected to every other node in the graph by one link, and no other links exists.

(i) What is the adjacency matrix for G? (3 marks)

(ii) What is the Laplacian matrix $\mathbf{L}$ for G? (3 marks)

(b) Consider a simple connected graph, G_N with N nodes, and adjacency matrix A. Number the nodes from 1 to N, where a link is placed between node 1 and every other node in the graph, but no other links exists.

(i) Find the eigenvalue of A whose corresponding eigenvector contains the eigenvector centrality of nodes 1 to N. (4 marks)

(ii) What is the cosine similarity matrix $\boldsymbol{\sigma}$ for G_N? (3 marks)

(c) Consider a path graph with N nodes, numbered from 1 to N, where the links are $1-2, 2-3, 3-4, ..., (N-1)-N$.

(i) Suppose the path graph is partitioned into two sets, S_1 and S_2, using a cut size of 1. Show that the modularity M of this partition is given by

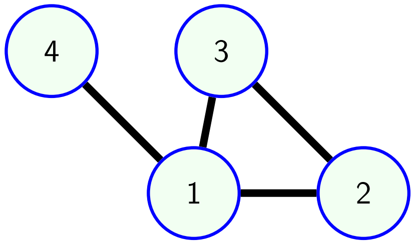
$$M = \frac{3 - 4N + 4rN - 4r^2}{2(N-1)^2},$$

where r is the number of nodes in S_1 after the partition. (4 marks)

(ii) Show that the maximum modularity for dividing the path graph into two, with a cut size of 1, is produced by a division that splits the network exactly down the middle. (3 marks)

(Total: 20 marks)

2. (a) Consider the graph G drawn below, generated by a configuration model. What is the corresponding degree sequence?



(2 marks)

(b) Suppose a simple connected graph with $N \gg 3$ nodes has been generated by the G_{Np} model with probability $p = N^{-z}$, where $z \geq 0$. Define a *two-link set of three nodes* to be formed when three nodes are connected by two edges only. Thus, one node has degree two, and the other two nodes have degree one. Let z_c be the critical value of z, such that when $z > z_c$ there exists no *two-link sets of three nodes* with high probability, as $N \to \infty$. Determine z_c.

(6 marks)

(c) Consider the following modification to the Barabasi-Albert model. Suppose we start with three connected nodes, with two links in total. At each iteration a pair of nodes are added, where the two nodes are connected by one link. They are connected to a node in the existing graph by adding one link, where the node in the existing graph is selected using preferential attachment. Therefore, at each iteration, two nodes and two links are added to the graph.

(i) What is the number of nodes $N(t)$ and the number of links $L(t)$ after t iterations?

(2 marks)

(ii) Show that the degree distribution for $k = 2$ at time $t + 1$, defined as $p_2(t + 1)$, can be written in terms of the degree distribution at time t for $k = 1, 2$, defined as $p_{1,2}(t)$, as

$$5p_2(t+1) - 3p_2(t) + p_2(t)\frac{3+2t}{2(1+t)} = -2t(p_2(t+1) - p_2(t)) + p_1(t)\frac{3+2t}{4(1+t)} + 1.$$

(6 marks)

(iii) Assume a stationary probability distribution exists. Determine the stationary probability distribution as $t \to \infty$ for $k = 2$, by imposing the condition $t(p_2(t+1) - p_2(t)) \to 0$ as $t \to \infty$. Write your solution in terms of a recurrence relation for $p_{1\infty}$ and $p_{2\infty}$, the probability stationary distributions as $t \to \infty$ for $k = 1$ and $k = 2$, respectively.

(4 marks)

(Total: 20 marks)

3. Consider an undirected, connected, N-node weighted graph G, with weight matrix $\mathbf{W}$, and diagonal degree matrix $\hat{\mathbf{D}}$, where $\hat{\mathrm{D}}_{ii} = \sum_{j=1}^{N} \mathrm{W}_{ij} = k_i$, and k_i is the degree of node i. Assume no self-loops or multiedges exist in G. Consider the following system of N differential equations

$$\frac{dx_i}{dt} = -x_i^2(x_i - \alpha) + \sum_{j=1}^{N} \sqrt{k_j}\hat{\mathrm{L}}_{ij} x_j, \tag{1}$$

where $\hat{\mathrm{L}}_{\mathrm{ij}}$ corresponds to the weighted normalised Laplacian of the graph G, and $\alpha > 0$ a parameter of the system. You should assume that $\hat{\mathbf{L}}$ is symmetric and a complete set of mutually orthogonal eigenvectors for $\hat{\mathbf{L}}$ exists, along with their corresponding eigenvalues.

(a) Show that the weighted normalised Laplacian $\hat{\mathbf{L}}$ has a zero eigenvalue, and the corresponding N-element eigenvector $\boldsymbol{v}$ has elements $v_j = \sqrt{k_j}$. You may assume the result that the weighted Laplacian, $\mathbf{L} = \hat{\mathbf{D}} - \mathbf{W}$, satisfies $\mathbf{L}\boldsymbol{z} = 0$, when $\boldsymbol{z}$ is a column vector of N ones.

(2 marks)

(b) Show that $x_i = \alpha$ for all i is an equilibrium solution of (1). (4 marks)

(c) Consider Equation (1), and $x_i(t) = \alpha + \epsilon y_i(t)$ for all i, where $\epsilon \ll 1$. Show that in the limit $\epsilon \to 0$ the N-element vector $\boldsymbol{y}$ satisfies the following system of ODEs, written in matrix form as

$$\frac{d\boldsymbol{y}}{dt} = -\alpha^2\boldsymbol{y} + \hat{\mathbf{L}}\hat{\mathbf{D}}^{\frac{1}{2}}\boldsymbol{y}. \tag{2}$$

(5 marks)

(d) Now, consider a graph where $\sqrt{k_j} = \beta$ for all j. Explain how to construct a matrix $\mathbf{B}$ so that the transformation $\boldsymbol{w} = \mathbf{B}\boldsymbol{y}$ allows the system of coupled ODEs given by Equation (2), to be written as

$$\frac{d\boldsymbol{w}}{dt} = \mathbf{M}\boldsymbol{w},$$

a decoupled system of ODEs, where $\boldsymbol{w}$ is a $N \times 1$ column vector and $\mathbf{M}$ is a diagonal $N \times N$ matrix that should be specified. (5 marks)

(e) Suppose that the initial condition is a known constant, such that $w_i(0) = w_{i0}$ is a known constant for $i = 1, \ldots, N$, where $w_{i0} > 0$. What condition on α will result in solutions to Equation (2) decaying as $t \to \infty$? Write this condition in terms of β and the spectral radius of $\hat{\mathbf{L}}$, defined as $\rho(\hat{\mathbf{L}})$. (4 marks)

(Total: 20 marks)

4. Consider a modification to the network-SI model, where upon infection, a person can recover or die. Consider the spread of a disease on a N node simple connected graph G, with adjacency matrix A and four state variables $s_i(t)$, $x_i(t)$, $r_i(t)$ and $d_i(t)$, where $i = 1, 2, ..., N$, and each variable can be 0 or 1. A node is either susceptible $s_i = 1$, infected $x_i = 1$, recovered (and not susceptible) $r_i = 1$, or dead $d_i = 1$. The variables are governed by the coupled ODEs,

$$\frac{d\langle s_i\rangle}{dt} = -\beta \sum_{j=1}^{N} A_{ij}\langle s_i x_j\rangle, \tag{3}$$

$$\frac{d\langle x_i\rangle}{dt} = \beta \sum_{j=1}^{N} A_{ij}\langle s_i x_j\rangle - \mu\langle x_i\rangle - \gamma\langle x_i\rangle, \tag{4}$$

$$\frac{d\langle r_i\rangle}{dt} = \gamma\langle x_i\rangle, \tag{5}$$

$$\frac{d\langle d_i\rangle}{dt} = \mu\langle x_i\rangle, \tag{6}$$

where the probability that a node which is susceptible at time t becomes infectious at time $t + \Delta t$ via a link to an infectious node is $\beta\Delta t$, the probability that a node which is infectious at time t becomes recovered at time $t + \Delta t$ is $\gamma\Delta t$ and the probability that a node which is infectious at time t dies at time $t + \Delta t$ is $\mu\Delta t$. Note that $s_i(t) + x_i(t) + r_i(t) + d_i(t) = 1$.

(a) The master equation for $x_i(t)$ takes the following form

$$P(x_i(t + \Delta t) = 1) = T_1 + \beta\Delta t \sum_{j=1}^{N} A_{ij} P(x_i(t) = 0, x_j(t) = 1) + O(\Delta t^2), \quad i = 1, 2, ..., N.$$

Determine the term T_1 .

(4 marks)

(b) Use a degree based approximation to show that the non-linear ODE for ϕ_k, the probability that a node with degree k is infectious for this model, is given by

$$\frac{d\phi_k}{dt} = -(\gamma + \mu)\phi_k + k\beta(1 - \phi_k) \sum_{k'=1}^{k'_{max}} \theta(k, k')\phi_{k'-1}, \tag{7}$$

where the function $\theta(k, k')$ gives the probability of a link on a node with degree k being connected to a node with degree k'. You should assume that θ and ϕ_k are independent for any k. You may use the result

$$P(x_j(t) = 1 | x_i(t) = 0, k_i = k) = \sum_{k'=1}^{k'_{max}} \theta(k, k')\phi_{k'-1}$$

without discussion.

(6 marks)

(c) Suppose that $\theta(k, k') = k' p_{k'}/K$, where $K = \sum_{i=1}^{N} k_i$, p_k is the fraction of nodes with degree k, and k_i is the degree of node i. Consider $\phi_k = \epsilon\tilde{\phi}_k$, where $\epsilon \ll 1$. Show that in the limit $\epsilon \to 0$, Equation (7) reduces to the linearised equation

$$\frac{d\tilde{\phi}_k}{dt} = -(\gamma + \mu)\tilde{\phi}_k + \frac{\beta k}{K} \sum_{k'=1}^{k'_{max}} k' p_{k'} \tilde{\phi}_{k'-1}. \tag{8}$$

(2 marks)

(d) Suppose at time $t = 0$, $\tilde{\phi}_k(0) = \tilde{\phi}_k^0$. Find the solution to Equation (8). Then determine how it behaves as t grows, where you should comment on this in terms of $K, \bar{k}, \bar{k^2}, \beta, \gamma$ and μ. You may find it useful to use $\bar{k} = \sum_{k=1}^{k_{max}} p_k k$, $\bar{k^2} = \sum_{k=1}^{k_{max}} p_k k^2$. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

MATH5007

Network Science (Solutions)

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1. (a) (i) sim. seen ⇓

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3, A

(ii) The Laplacian matrix L is given by $\mathrm{L} = \mathrm{D} - \mathrm{A}$. D is a diagonal matrix with $D_{ii} = k_i$, where the degree of node i is k_i. Now,

$$\mathrm{D} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore, subtracting A, found in (i), gives

$$\mathrm{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3, A

unseen ⇓

(b) (i) We want to find the largest most positive eigenvalue of A, and solve $\mathrm{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, where x_i is the i^{th} component of the eigenvector $\boldsymbol{x}$. Define x_1 to correspond to the central node (node 1), then by symmetry, every other component can be called x_s for $i = 2 \ldots N$. We are therefore solving the system

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_s \\ x_s \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_s \\ x_s \\ \vdots \end{bmatrix}.$$

It follows that $(N-1)x_s = \lambda x_1$ and $x_1 = \lambda x_s$, to give $\lambda = \pm\sqrt{N-1}$. To see that $\lambda = \sqrt{N-1}$ is the largest most positive eigenvalue, notice that when $\lambda = \sqrt{N-1}$, x_1 and x_s have the same sign. Since every node is reachable from every other node, and it is undirected, then A is irreducible. Therefore, by Perron-Frobenius theorem (version 2), there exists only one eigenvector where all elements have the same sign, and this corresponds to the largest most positive eigenvalue. Thus, $|\lambda| = \sqrt{N-1}$ and the largest most positive eigenvalue, is $\lambda = \sqrt{N-1}$. 3, B 1, C

unseen ⇓

(ii) We use the formula $n_{ij} = \sum_{l=1}^{N} \mathrm{A}_{il}\mathrm{A}_{lj}$. Evident from the structure of the graph, or the summation, node 1 has no common neighbours with any other nodes, and therefore $n_{1j} = 0$ for $j > 1$, but $n_{11} = N-1$, and $k_i = k_j = N-1$, thus $\sigma_{11} = 1$. Now, $n_{ij} = 1$ when $i, j = 2, ..., N$. Since, $k_i = 1$ for $i > 1$, it follows that $\sigma_{ij} = 1$ for $i = 2, ..., N$ and $j = 2, ..., N$. Since it is a simple graph, we have a symmetric matrix of the form

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3, B

meth seen ⇓

(c) (i) If we divide the network into two sets S_1 and S_2 with a cut size of 1, such that S_1 has r nodes and S_2 has $N - r$ nodes. We define the end nodes to be node 1 and r in S_1, and $r + 1$ and N in S_2. Total number of links in S_1 is $r - 1$, and total number of links in S_2 is $N - r - 1$. Note $L = N - 1$. Then the total number of stubs attached to nodes in S_1 is $K_1 = 2r - 1$ and in S_2, $K_2 = 2(N - r) - 1$. Therefore the modularity for S_1 is given by

$$M_1 = \frac{1}{2L}\left(- \frac{K_1^2}{2L} + \sum_{i \in S_1} \sum_{j \in S_1} \left(\mathrm{A}_{ij} \right) \right) = \frac{4(N-1)(r-1) - (2r-1)^2}{4(N-1)^2}$$

and the modularity for S_2 is

$$M_2 = \frac{1}{2L}\left(- \frac{K_2^2}{2L} + \sum_{i \in S_2} \sum_{j \in S_2} \left(\mathrm{A}_{ij} \right) \right) = \frac{4(N-1)(N-r-1) - (2N-2r-1)^2}{4(N-1)^2}.$$

Therefore

$$M = \frac{4(N-1)(r-1) - (2r-1)^2 + 4(N-1)(N-r-1) - (2N-2r-1)^2}{4(N-1)^2},$$

which reduces to

$$M = \frac{3 - 4N + 4rN - 4r^2}{2(N-1)^2}.$$

4, D

sim. seen ⇓

(ii) The maximum of M w.r.t. to r is found by calculating

$$dM/dr = \frac{2(N-2r)}{(N-1)^2} = 0.$$

Thus when $r = N/2$, M is maximised, since $d^2M/dr^2 < 0$. 3, B

2. (a) Node 1 has three stubs, node 2 and 3 have two stubs, and node 4 has 1 stub. This gives the degree sequence $d = [3, 2, 2, 1]$.

sim. seen ⇓

2, A

sim. seen ⇓

(b) Let $X_{ijk} = 1$ if nodes (i, j, k) form a *two-link set of three nodes* and $X_{ijk} = 0$ otherwise. The probability of (i, j, k) forming a *two-link set of three nodes* depends on the results of $3 + 3(N - 3)$ Bernoulli trials of which 2 must be successful and the others unsuccessful. Then $\langle X_{ijk}\rangle = P(X_{ijk} = 1) = p^2(1-p)^{3(N-3)+1}$. Let X be the total number of distinct *two-link set of three nodes* in a graph then

$$X = \sum_{distinct(i,j,k)} X_{ijk}.$$

Using linearity of expectation

$$\langle X\rangle = \sum_{distinct(i,j,k)} \langle X_{ijk}\rangle.$$

There are $\binom{N}{3}$ distinct triples of nodes, so

$$\langle X\rangle = \binom{N}{3} p^2(1-p)^{3(N-3)+1}$$

$$\langle X\rangle = \frac{N(N-1)(N-2)}{6} p^2(1-p)^{3(N-3)+1}$$

Then substitute $p = N^{-z}$ to get,

$$\langle X\rangle = \frac{N(N-1)(N-2)}{6} N^{-2z}(1-N^{-z})^{3(N-3)+1}.$$

For $N \to \infty$

$$\langle X\rangle \approx \frac{N^{3-2z}}{6}.$$

4, A

unseen ⇓

So for $z > 3/2$, $\langle X\rangle \to 0$ as $N \to \infty$. Hence, $P(X \geq 1) \leq \langle X\rangle \to 0$ by Markov's inequality. It follows that $z_c = 3/2$

2, B

(c)

sim. seen ⇓

(i) $N(t) = 3 + 2t, \quad L(t) = 2 + 2t$

2, A

sim. seen ⇓

(ii) First, note that at each time step, one node with degree two and one node with degree one will be added. Define the number of nodes with degree 2 as N_2. We will consider three cases. Case A: a node in the existing graph with degree 1 receives a link, in which case N_2 increases by two. Case B: a node in the existing graph with degree 2 receives a link and therefore N_2 stays the same. Case C: neither A or B occur and N_2 increases by 1. Using the formulation from the lecture,

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[P(A)(N_2(G_m(t)) + 2) + P(B)(N_2(G_m(t)))$$

$$+(1-P(A)-P(B))(N_2(G_m(t))+1)\Big].$$

This simplifies to

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[(N_2(G_m(t))+P(A)-P(B)+1\Big].$$

Then the linear preferential attachment models tells us $P(A)=\frac{N_1}{2L}$, $P(B)=\frac{2N_2}{2L}$, and therefore

$$\langle N_2(t+1)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)\Big[(N_2(G_m(t))+\frac{N_1(G_m(t))}{2L}-\frac{2N_2(G_m(t))}{2L}+1\Big]$$

Substituting

$$\langle N_k(t)\rangle = \sum_{m=1}^{N_G(t)} P(G_m)N_k(G_m(t))$$

gives

$$\langle N_2(t+1)\rangle = \langle N_2(t)\rangle\left(1-\frac{1}{(2+2t)}\right)+\frac{\langle N_1(t)\rangle}{2(2+2t)}+1,$$

and substituting $\langle N_k(t)\rangle = p_k N(t)$ gives

$$N(t+1)p_2(t+1) = N(t)p_2(t)\left(1-\frac{1}{2+2t}\right)+p_1(t)\frac{N(t)}{4(1+1t)}+1.$$

Since $N(t)=3+2t$ we get

$$(2t+5)p_2(t+1) = p_2(t)\left(3+2t-\frac{3+2t}{2+2t}\right)+p_1(t)\frac{3+2t}{4(1+1t)}+1.$$

Re-arranging gives

$$5p_2(t+1)-3p_2(t)+p_2(t)\frac{3+2t}{2+2t} = -2t(p_2(t+1)-p_2(t))+p_1(t)\frac{3+2t}{4(1+t)}+1$$

$\boxed{6, \text{C}}$

$\boxed{\text{sim. seen} \Downarrow}$

(iii) Taking the limit as $t\to\infty$ and imposing the condition gives

$$2p_{2\infty}+p_{2\infty} = \frac{1}{2}p_{1\infty}+1$$

and therefore

$$p_{2\infty} = \frac{1}{6}p_{1\infty}+\frac{1}{3}.$$

$\boxed{4, \text{A}}$

3. (a) The normalised Laplacian is $\hat{\mathrm{L}} = \hat{\mathrm{D}}^{-\frac{1}{2}}(\hat{\mathrm{D}} - \mathrm{W})\hat{\mathrm{D}}^{-\frac{1}{2}}$. Then since a column vector of N ones, $\boldsymbol{z}$, satisfies $\mathrm{L}\boldsymbol{z} = \boldsymbol{0}$, where $\mathrm{L} = \hat{\mathrm{D}} - \mathrm{W}$, if $\hat{\mathrm{D}}^{\frac{-1}{2}}\boldsymbol{v} = \boldsymbol{z}$ then $\hat{\mathrm{L}}\boldsymbol{v} = 0$, and the normalised Laplacian has a zero eigenvalue. This is satisfied with $v_j = \sqrt{k_j}$, the corresponding eigenvector, as requested. seen ⇓

2, A

sim. seen ⇓

(b) When $x_i = \alpha$ the first term is zero by inspection. Substituting $x_i = \alpha$ into the summation term results in $\sum_{j=1}^{N}\sqrt{k_j}\hat{\mathrm{L}}_{ij}x_j = \sum_{j=1}^{N}\alpha\hat{\mathrm{L}}_{ij}\sqrt{k_j}$. Now, we have shown that if $\boldsymbol{x}$ has elements $x_j = \sqrt{k_j}$ then $\hat{\mathrm{L}}\boldsymbol{x} = 0$, and therefore $\sum_{j=1}^{N}\hat{\mathrm{L}}_{ij}x_j = 0$, if $x_j = \sqrt{k_j}$. It follows that $\alpha\hat{\mathrm{L}}\boldsymbol{x} = 0$ and $\alpha\sum_{j=1}^{N}\hat{\mathrm{L}}_{ij}x_j = 0$ if $x_j = \sqrt{k_j}$. Thus $\sum_{j=1}^{N}\alpha\hat{\mathrm{L}}_{ij}\sqrt{k_j} = 0$ and we have an equilibrium solution.

4, A

sim. seen ⇓

(c) Substituting $x_i(t) = \alpha + \epsilon y_i(t)$ in Equation (1) gives

$$\epsilon\frac{dy_i}{dt} = -\epsilon y_i(\alpha + \epsilon y_i)(\alpha + \epsilon y_i) + \sum_{j=1}^{N}\sqrt{k_j}\hat{\mathrm{L}}_{ij}(\alpha + \epsilon y_j) \tag{1}$$

We know from (b) that $\sum_{j=1}^{N}\sqrt{k_j}\hat{\mathrm{L}}_{ij}(\alpha) = 0$. Then diving both sides by ϵ gives

$$\frac{dy_i}{dt} = -y_i(\alpha^2 + 2\epsilon\alpha y_i + \epsilon^2 y_i) + \sum_{j=1}^{N}\sqrt{k_j}\hat{\mathrm{L}}_{ij}y_j \tag{2}$$

and taking the limit as $\epsilon \to 0$ gives

$$\frac{dy_i}{dt} = -\alpha^2 y_i + \sum_{j=1}^{N}\sqrt{k_j}\hat{\mathrm{L}}_{ij}y_j. \tag{3}$$

Write this as

$$\frac{dy_i}{dt} = -\alpha^2 y_i + \sum_{j=1}^{N}\hat{\mathrm{L}}_{ij}\sqrt{k_i}\delta_{ij}y_j. \tag{4}$$

In matrix form this becomes

$$\frac{d\boldsymbol{y}}{dt} = -\alpha^2\boldsymbol{y} + \hat{\mathrm{L}}\hat{\mathrm{D}}^{\frac{1}{2}}\boldsymbol{y}. \tag{5}$$

5, B

sim. seen ⇓

(d) We can orthogonally diagonalize the normalised Laplacian as $\hat{\mathrm{L}} = \hat{\mathrm{V}}\Lambda\hat{\mathrm{V}}^T$, where the columns of $\hat{\mathrm{V}}$ contain the eigenvectors of $\hat{\mathrm{L}}$, normalised to have length one, and Λ is a diagonal matrix where Λ_{ii} is the eigenvalue corresponding to the eigenvector stored in the i^{th} column of $\hat{\mathrm{V}}$. Equation (**??**) can be written as

$$\frac{d\boldsymbol{y}}{dt} = -\alpha^2\boldsymbol{y} + \beta\hat{\mathrm{V}}\Lambda\hat{\mathrm{V}}^T\boldsymbol{y} \tag{6}$$

and writing $\alpha^2\boldsymbol{y} = \alpha^2\hat{\mathrm{V}}\mathrm{I}\hat{\mathrm{V}}^T\boldsymbol{y}$ we get

$$\frac{d\boldsymbol{y}}{dt} = \Big(-\alpha^2\hat{\mathrm{V}}\mathrm{I}\hat{\mathrm{V}}^T + \beta\hat{\mathrm{V}}\Lambda\hat{\mathrm{V}}^T\Big)\boldsymbol{y}. \tag{7}$$

Define a new matrix $\mathrm{M} = -\alpha^2 \mathrm{I} + \beta\Lambda$ to give

$$\frac{d\boldsymbol{y}}{dt} = \hat{\mathrm{V}}\ \mathrm{M}\hat{\mathrm{V}}^T \boldsymbol{y}. \tag{8}$$

Then define $\boldsymbol{w} = \hat{\mathrm{V}}^T \boldsymbol{y}$ results in

$$\hat{\mathrm{V}} \frac{d\boldsymbol{w}}{dt} = \hat{\mathrm{V}}\mathrm{M}\boldsymbol{w}, \tag{9}$$

since $\boldsymbol{y} = \mathrm{V}\boldsymbol{w}$. Multiplying by $\hat{\mathrm{V}}^T$ results in

$$\frac{dw_i}{dt} = M_{ii} w_i, \quad i = 1, 2, ..., N, \tag{10}$$

where $M_{ii} = \beta\lambda_i - \alpha^2$ and λ_i is the i^{th} eigenvalue of $\hat{\mathrm{L}}$ and $k_j = \beta^2$. 5, C

sim. seen ⇓

(e) Equation (10) has the general solution $w_i = w_{i0} e^{M_{ii} t}$, where $M_{ii} = \beta\lambda_i - \alpha^2$. Now, $\boldsymbol{y} = \mathrm{V}\boldsymbol{w}$, so

$$\boldsymbol{y} = w_{10} e^{(\beta\lambda_1 - \alpha^2)t} \boldsymbol{v_1} + w_{20} e^{(\beta\lambda_2 - \alpha^2)t} \boldsymbol{v_2} + ... w_{N0} e^{(\beta\lambda_N - \alpha^2)t} \boldsymbol{v_N}$$

unseen ⇓

Since the graph in question is a simple connected graph, all eigenvalues are positive or zero, and there is a maximum $\rho(\hat{\mathrm{L}})$ that the eigenvalues are less than or equal too. Consequently, all solutions will decay if $\beta\rho < \alpha^2$ 4, D

4. (a) The master equation for $x_i(t)$ is given by

sim. seen ⇓

$$P(x_i(t+\Delta t)=1)=P(x_i(t)=1,x_i\nrightarrow 0)+\beta\Delta t\sum_{j=1}^{N}A_{ij}P(x_i(t)=0,x_j(t)=1)+O(\Delta t^2).$$

The first term accounts for an infected node not recovering or dying between t and Δt. Write this as a conditional probability,

$$P(x_i(t)=1,x_i\nrightarrow 0)=P(x_i\nrightarrow 0|x_i(t)=1)P(x_i(t)=1).$$

Then since the probability of an infected node recovering or dieing in the time step Δt, is $\gamma\Delta t$ or $\mu\Delta t$, respectively, it follows that the probability of these events not happening is $1-\gamma\Delta t-\mu\Delta t$, and therefore

$$P(x_i\nrightarrow 0|x_i(t)=1)=1-\gamma\Delta t-\mu\Delta t.$$

Substituting gives

$$P(x_i(t+\Delta t)=1)=(1-(\gamma+\mu)\Delta t)P(x_i(t)=1)+\beta\Delta t\sum_{j=1}^{N}A_{ij}P(x_i(t)=0,x_j(t)=1)+O(\Delta t^2),$$

and therefore $T_1=(1-(\gamma+\mu)\Delta t)P(x_i(t)=1)$.

4, A

(b) The master degree equation for this model was derived in (a),

sim. seen ⇓

$$P(x_i(t+\Delta t)=1)=(1-(\gamma+\mu)\Delta t)P(x_i(t)=1)+\beta\Delta t\sum_{j=1}^{N}A_{ij}P(x_i(t)=0,x_j(t)=1)+O(\Delta t^2),$$

where now all probabilities are conditional on node i having the degree $k_i=k$. The summation can be re-written as summing over the neighbours of i, N_i, as

$$\sum_{j=1}^{N}A_{ij}P(x_i(t)=0,x_j(t)=1,k_i=k)=\sum_{j\in N_i}P(x_i(t)=0,x_j(t)=1,k_i=k).$$

Writing the joint probability as a conditional probability gives

2, A

$$P(x_i(t)=0,x_j(t)=1,k_i=k)=P(x_j(t)=1|x_i(t)=0,k_i=k)P(x_i(t)=0,k_i=k),$$

such that

$$P(x_i(t+\Delta t)=1)=(1-(\gamma+\mu)\Delta t)P(x_i(t)=1)+$$
$$\beta\Delta tP(x_i(t)=0,k_i=k)\sum_{j\in N_i}P(x_j(t)=1|x_i(t)=0,k_i=k)+O(\Delta t^2).$$

Using the result given in the question and the definition $P(x_i(t)=1|k_i=k)=\phi_k$ gives

$$\phi_k(t+\Delta t)=(1-(\gamma+\mu)\Delta t)\phi_k(t)+\beta\Delta t(1-\phi_k)\sum_{j\in N_i}\sum_{k'=1}^{k'_{max}}\theta(k,k')\phi_{k'-1}+O(\Delta t^2),$$

which simplifies to

$$\phi_k(t+\Delta t)=(1-(\gamma+\mu)\Delta t)\phi_k(t)+k\beta\Delta t(1-\phi_k)\sum_{k'=1}^{k'_{max}}\theta(k,k')\phi_{k'-1}+O(\Delta t^2).$$

Dividing both sides by Δt and letting $\Delta t\to 0$ gives

$$\frac{d\phi_k}{dt}=-(\gamma+\mu)\phi_k+k\beta(1-\phi_k)\sum_{k'=1}^{k'_{max}}\theta(k,k')\phi_{k'-1}$$

4, B

(c) Substituting $\phi_k = \epsilon\tilde{\phi}_k$ gives

sim. seen ⇓

$$\epsilon\frac{d\tilde{\phi}_k}{dt} = -\epsilon(\gamma+\mu)\tilde{\phi}_k + \epsilon k\beta(1-\epsilon\tilde{\phi}_k)\sum_{k'=1}^{k'_{max}}\theta(k,k')\tilde{\phi}_{k'-1} + O(\epsilon^2).$$

Divide through by ϵ and take the limit as $\epsilon \to 0$.

$$\frac{d\tilde{\phi}_k}{dt} = -(\gamma+\mu)\tilde{\phi}_k + k\beta\sum_{k'=1}^{k'_{max}}\theta(k,k')\tilde{\phi}_{k'-1}.$$

Substitute $\theta(k,k')$ to give the required result.

2, A

unseen ⇓

(d) Re-write the summation in the linearised equation as

$$\frac{d\tilde{\phi}_k}{dt} = -(\gamma+\mu)\tilde{\phi}_k + \frac{\beta k}{K}\sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\tilde{\phi}_{k'}, \tag{11}$$

Multiply both sides of Equation (**??**) by $e^{(\gamma+\mu)t}$ to re-write the equation as

$$\frac{d}{dt}\left(e^{(\gamma+\mu)t}\tilde{\phi}_k\right) = e^{(\gamma+\mu)t}\frac{\beta k}{K}\sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\tilde{\phi}_{k'}.$$

sim. seen ⇓

Define

$$\psi(t) = \sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\tilde{\phi}_{k'}e^{(\gamma+\mu)t}, \tag{12}$$

such that

$$\frac{d}{dt}\left(e^{(\gamma+\mu)t}\tilde{\phi}_k\right) = \frac{\beta k}{K}\psi.$$

Differentiate (**??**) w.r.t. t to give

$$\frac{d\psi}{dt} = \sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\frac{d}{dt}\left(e^{(\gamma+\mu)t}\tilde{\phi}_{k'}\right).$$

Therefore,

$$\frac{d\psi}{dt} = \frac{\beta\psi}{K}\sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}k',$$

Now,

$$\sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}k' = \sum_{k'=1}^{k'_{max}}k'p_{k'}(k'-1) = \bar{k^2} - \bar{k}$$

by the definition for $p_{k'}$, and therefore

$$\frac{d\psi}{dt} = \frac{\beta(\bar{k^2}-\bar{k})}{K}\psi.$$

Therefore,

$$\psi = \psi_0 e^{Ct}$$

where $\psi_0 = \sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\tilde{\phi}^0_{k'}$ and $C = \beta(\bar{k^2} - \bar{k})/K$. Since

$$\frac{d}{dt}\left(e^{(\gamma+\mu)t}\tilde{\phi}_k\right) = \frac{\beta k}{K}\psi_0 e^{Ct},$$

it follows that

$$\tilde{\phi}_k = \frac{k}{\bar{k^2} - \bar{k}}\left(e^{Ct}\sum_{k'=0}^{k'_{max}-1}(k'+1)p_{k'+1}\tilde{\phi}^0_{k'} - 1\right)e^{(-\gamma-\mu)t} + \tilde{\phi}^0_k.$$

Consequently, solutions decay when $\gamma + \mu > \beta(\bar{k^2} - \bar{k})/K$. $\boxed{8, \mathrm{D}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks

MATH50007 Network Science

Question	Marker's comment
1	Students did very well in (a). In (b)(i), students did well, but only a few students commented on why the positive root should be taken, and rarely was this linked with Perron-Frobenius theorem. Often, the diagonal elements of the similarity matrix in (b)(ii) contained errors, or students did not give sufficient reasoning behind their answers. In part (c) students often lost marks calculating the total number of stubs in each set incorrectly. Please note that in part (b)(ii) the word vector, should be replaced by matrix and this was announced in the exam. nbsp;
2	Part (a) was done very well. Students did well in (b) and used the correct method to solve the problem, but most students failed to notice that thenbsp;set of three nodesnbsp;were isolated from the rest of the graph, resulting in the wrong number of Bernoulli trials. (c)(i) was done very well.nbsp;In (c)(ii), rarely did students consider the sum over all realisations of the graph at timenbsp;t,nbsp;and only a few correctly used the linear preferential attachment model. (c)(iii) was done well mostly, but some students struggled to take the limits of the fractions involvingnbsp;t correctly.nbsp;
3	For part (a), some students struggled to see how to use the zero eigenvector of the Laplacian to obtain the needed result for the normalised Laplacian. Students did well on part (b). For part (c), many did not explain how to go from the equation for the ith element in y to the required equation in matrix form but otherwise did well. Part (d) was a struggle for many students who did not seem to understand how/when to use orthogonal diagonalisation. For (e), many were on the right track with their final answer, however most did not explain how the behavior of w was related to y, and few explicitly stated that the relevant eigenvalues were real which was an important step needed to justify using the spectral radius.