MATH50001 Problems Sheet 5
Solutions

1)
a)

$$\frac{1}{2}\cdot\frac{1}{z-2i}+\sum_{n=0}^{\infty}\frac{i^{n-1}}{2^{2n+3}}(z-2i)^n, \qquad 0<|z-2i|<4.$$

b)

$$\sum_{n=-1}^{\infty}\frac{1}{e(n+1)!}(z+1)^n, \qquad |z+1|>0.$$

2)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n(z-1)^n}-\sum_{n=0}^{\infty}(z-1)^n.$$

3) Obviously

$$\frac{9}{(z-4)(z+5)}=\frac{1}{z-4}-\frac{1}{z+5}.$$

3a) If $|z|<4$ we have

$$\frac{1}{z-4}=-\frac{1}{4-z}=-\frac{1}{4}\frac{1}{1-z/4}=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{4^n}z^n$$

and

$$-\frac{1}{z+5}=-\frac{1}{5}\frac{1}{1-(-z/5)}=-\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}z^n.$$

Therefore

$$\frac{9}{(z-4)(z+5)}=-\sum_{n=0}^{\infty}\left(\frac{1}{4^{n+1}}+\frac{(-1)^n}{5^{n+1}}\right)z^n.$$

3b) If $2<|z|<5$, then

$$\frac{1}{z-4}=\frac{1}{z}\frac{1}{1-4/z}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{4^n}{z^n}$$

and thus

$$\frac{9}{(z-4)(z+5)} = \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} z^n$$
$$= \sum_{n=-\infty}^{-1} 4^{-n-1} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} z^n.$$

3c) If $5 < |z|$, then

$$-\frac{1}{z+5} = -\frac{1}{z} \frac{1}{1-(-5/z)} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{(-5)}{z}\right)^n$$

which implies

$$\frac{9}{(z-4)(z+5)} = \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{(-5)}{z}\right)^n$$
$$= \sum_{n=-\infty}^{-1} \left(4^{-n-1} - (-5)^{-n-1}\right) z^n.$$

4) Let $f(z) = z\, e^z$ and $z_0 = 2$. Then

$$f(z) = (z-2)\, e^{z-2} e^2 + 2e^{z-2} e^2 = \sum_{n=0}^{\infty} \frac{e^2}{n!} (z-2)^{n+1} + \sum_{n=0}^{\infty} \frac{2e^2}{n!} (z-2)^n$$
$$= 2e^2 + \sum_{n=1}^{\infty} \left(\frac{e^2}{(n-1)!} + \frac{2e^2}{n!} \right) (z-2)^n.$$

5)
a) If f is holomorphic at z_0 and has a zero of order m at z_0, then there is $g(z)$ holomorphic at z_0, $g(z_0) \neq 0$ such that $f(z) = (z-z_0)^m\, g(z)$. Therefore

$$\frac{1}{f(z)} = \frac{1}{(z-z_0)^m} \cdot \frac{1}{g(z)}.$$

Thus $1/f$ has a pole of order m at z_0.

b)

$$\left(2\cos z - 2 - z^2\right)^2 = \left(2\left(1 - \frac{1}{2} z^2 + \frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \dots\right) - 2 + z^2\right)^2$$
$$= 4\left(\frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \dots\right)^2 = z^8 \cdot g(z),$$

where $g(z)$ is holomorphic at 0 and $g(0) \neq 0$. Therefore $\frac{1}{(2\cos z - 2 + z^2)^2}$ has a pole at 0 of order 8.

6)

a) $z = 0$, essential singularity;

b) $z = 0$, pole of order 4;

c) $z = n\pi$, $(6n \pm 1)\pi/3$, poles of order 1.

7) The function $f(z) = \frac{e^z}{z(z-2)^3}$ has two poles inside $\gamma = \{|z| = 3\}$. One of them is at $z_1 = 0$ of order one and the other one is at $z_2 = 2$ of order three. Therefore

$$\begin{aligned}\oint_\gamma \frac{e^z}{z(z-2)^3}\,dz &= 2\pi i\,(\operatorname{Res}[f, z_1] + \operatorname{Res}[f, z_2]) \\ &= 2\pi i \left(\frac{e^0}{(-2)^3} + \lim_{z\to 2} \frac{1}{2} \frac{d^2}{dz^2} \frac{(z-2)^3 e^z}{z(z-2)^3} \right) \\ &= -\frac{\pi i}{4} + \pi i \lim_{z\to 2} \left(\frac{e^z}{z} - 2\frac{e^z}{z^2} + 2\frac{e^z}{z^3} \right) \\ &= -\frac{\pi i}{4} + \pi i\, e^2 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4} \right) = \frac{\pi(e^2-1)i}{4}.\end{aligned}$$

8) Substituting $z = e^{i\theta}$ gives

$$\begin{aligned} I := \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} &= \frac{1}{i} \oint_{|z|=1} \frac{1}{-az^2 + (1+a^2)z - a}\,dz \\ &= \frac{1}{i} \oint_{|z|=1} \frac{1}{(z-a)(1-az)}\,dz. \end{aligned}$$

If $|a| < 1$ there is a pole inside $|z| = 1$ at a with the residue $1/(1-a^2)$. Therefore $I = 2\pi/(1-a^2)$.
If $|a| > 1$ the pole inside $|z| = 1$ is at $1/a$ and the residue is

$$\lim_{z\to 1/a} \frac{z - 1/a}{(z-a)(az-1)} = \frac{1}{a^2-1}.$$

Hence $I = 2\pi/(a^2-1)$.

9)

$$\oint_\gamma \frac{e^z-1}{z^2(z-1)} = 2\pi\,i\,2\,\mathrm{Res}\left[\frac{e^z-1}{z^2(z-1)},1\right] - 2\pi\,i\,2\,\mathrm{Res}\left[\frac{e^z-1}{z^2(z-1)},0\right]$$

$$= 4\pi i\left\{\frac{e^z-1}{z^2}\Big|_{z=1} - \frac{z}{dz}\,\frac{e^z-1}{z-1}\Big|_{z=0}\right\}$$

$$= 4\pi i\left\{e-1-\frac{e^z(z-1)-(e^z-1)}{(z-1)^2}\Big|_{z=0}\right\}$$

$$= 4\pi i\left\{e-1+1\Big|_{z=0}\right\} = 4\pi i e.$$

10) Indeed,

$$\frac{1}{2\pi i}\oint_{|z|=r} z^{n-1}\,|f(z)|^2\,dz = \frac{1}{2\pi i}\int_0^{2\pi} r^{n-1}\,e^{i(n-1)\theta}\,|f(re^{i\theta})|^2\,i\,e^{i\theta}\,r\,d\theta$$

$$= \frac{r^n}{2\pi}\int_0^{2\pi} e^{i(n-1)\theta}\left(\sum_{k=0}^{n} a_k r^k e^{ik\theta}\right)\overline{\left(\sum_{m=0}^{n} a_m r^m e^{im\theta}\right)}\,e^{i\theta}\,d\theta.$$

The only term that survive if $n-1+k-m+1=0$. The only possibility for that is $k=0$ and $m=n$ and thus

$$\frac{1}{2\pi i}\oint_{|z|=r} z^{n-1}\,|f(z)|^2\,dz = a_0\,\bar{a}_n\,r^{2n}.$$