Applied Complex Analysis - Lecture Five

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Isolated singularities

Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

Plots and some mathematical intuition

Thm: A function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

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Residue Theory

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The coefficient a_{-1} in the Laurent series expansion

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is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \text{Res}(f, z_0).$$

Why should we care?

- for $f = \frac{1}{z}$, we have $\text{Res}(f, 0) =$
- for $f = \frac{1}{z^2}$, we have $\text{Res}(f, 0) =$
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Connecting residues to closed contour integrals

Thm: Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (the radius of convergence), then

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z) dz.$$

Proof

Residue Theorem Let $f(z)$ be analytic in some $\mathcal{D} \setminus \{z_1, z_2, \ldots z_n\}$ bounded by a closed path γ, where $z_1, z_2, \ldots z_n$ are poles or essential singularities lying inside γ. Then

$$\oint_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

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Proof

Ways to compute residues

1. For

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + \cdots \,,$$

   so that $f(z)$ has a pole of order m at z_0,

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)].$$

2. For

$$f(z) = \frac{A(z)}{(z-z_0)^m},$$

   where $A(z)$ is analytic at $z = z_0$ (and that $A(z_0) \neq 0$),

$$\text{Res}(f, z_0) = \frac{A^{(m-1)}(z_0)}{(m-1)!}.$$

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Ways to compute residues (continued)

3. If $f(z)$ contains a simple pole (pole of order $m = 1$) and $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0 and B has a simple zero at z_0 ($m = 1$), with $A(z_0) \neq 0$, then

$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}.$$

Example: Residues and contour integrals of

$$f(z) = \frac{1}{1 + z^4}$$

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Using contour deformation to evaluate

$$\int_{-\infty}^{\infty} f(z)dz,$$

where f has poles.

(Some) applications

- Statistics, e.g. Cauchy-Lorentz distribution
- Fourier and (inverse) Laplace transforms
- Potential flow theory, poles represent sinks

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Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad k > 0.$$
- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1.$$

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

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Analytic Continuation

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

Connects local and global behaviour of analytic functions

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Branch points and branch cuts

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A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., analytically continuing along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

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