Imperial College
London

Department of Mathematics
Imperial College London

Computational PDEs MATH60025/70025, 2023-2024
Released : 25 February 2025
Upload Deadline : 1.00 pm, 12 March 2025
The project mark, will be weighted to comprise 40% of the overall Module.

You are required to investigate Questions below and summarise your findings in form of a well written project report – on which you will be assessed.

Please name your files in following way:

- Technical report : **CPDES_Q2_yourCID.pdf** (limit your report to 20 pages or less (including plots). **Anything beyond the 20 page limit will NOT be marked!**
- All your code(s), label as follows :
  CPDES_Q2_of__X_yourCID.m (Matlab scripts example) or
  CPDES_Q2_of__X_yourCID.py (Python scripts).
  Zip all program files and call your zipped folder: **CPDES_Q2_programs_yourCID.zip**

Where in the above **CID** will be your College ID number.

Notes: *Important*

1. Marking will consider both the correctness of your code as well as the soundness of your analysis and clarity and legibility of the technical report.
2. Exam mark will primarily be based on contents of your written technical report. You are warned that if you ONLY submit the codes for the work with **NO technical report**, you can **NOT** expect a pass mark.
3. Do **NOT** include source code listing in your technical report.
4. All figures created by your code should be well-made and properly labelled in the technical pdf report.
5. The codes **must** be submitted.

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CPDEs CW2 : Fluid Problem

Part A: *15 marks*

Laplace's equation for $\psi(x,\, y)$ in a rectangle is defined as follows :

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \; \in [\, 0 < x < 1, \;\; 0 < y < 1\,], \tag{1.1}$$

with the boundary conditions

$$\psi(1, y) = 1 \;\; \text{along} \;\;\; 1/2 - \delta \le y \le 1/2 + \delta \tag{1.2}$$

with $\delta = 1/4$. On the rest of the boundary the following is satisfied

$$\psi(0, y) = \psi(x, 0) = \psi(x, 1) = 0. \tag{1.3}$$

Using the very coarse square mesh of $h = 1/4$ $(N = 4)$ in both (x, y) directions, discretise Eqn. 1.1, for the 9 interior grid values.

1. Set up a matrix $A\hat{\psi} = \hat{r}$ and use a direct method to solve for the 9 interior unknowns. List the numerical values you attain from your solution.
2. Next solve the equations using the Jacobi, Gauss-Seidel and SOR methods.
3. Describe criteria used to decide that the iterations have converged; make comparison of your results with solutions obtained with those in Part (1).
4. Make appropriate plots showing the convergence properties of the three iterative methods. Show in a reasonably concise form, by way of plots your findings.
5. Determine an optimal relaxation parameter for the SOR technique – make an appropriate plot to illustrate how you did this.
6. Discuss your numerical findings with theoretical expectations.
7. Next for the SOR approach, increase N successively, and make plots of the how the optimal relaxation factor ω varies. How do you ascertain that you have a grid independent result?
8. How many grid points are required for a reasonably resolved result? How do you prescribe, or assume that you have a good solution to the problem ?
9. Can you deduce an exact solution, using separation of variables ? Deduce the number of Fourier terms required to obtain a reasonable match with the most highly resolved finite-differenced result you obtain in part 7. In your report mention the order of accuracy that you hope to achieve and discuss your findings. (No marks for this part, but perhaps a bonus if you can do this).

Part B: *15 marks*
Problem description :

Fluid flows into an irregular region Γ, as shown in Fig. 1 through the gap $a-b$, and possibly out through the gap $c-d$. The problem is modelled by the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0, \tag{1.4}$$

at every point $\Gamma(x,y)$ of the section shown in Fig. 1, and is subject to the following boundary conditions

1. $\psi=0$ on $a-o-h-g-f-e-d$
2. $\partial\psi/\partial y=0$ on $c-d$
3. $\psi=1$ on $b-c$
4. $\psi=x-0.5$ on $a-b$

The velocity fields $(u,\ v)$ are given by the derivatives of the stream-function $\psi(x,y)$, namely

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x}.$$

Curves defined by $\psi=constant$ are called streamlines, and they represent flow lines of the fluid particles.

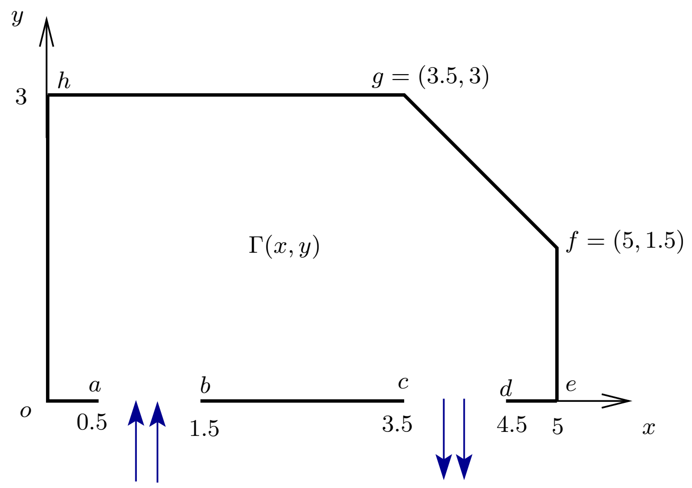

Figure 1: fluid flow problem.

Using the “adaptive grid”, or otherwise,

1. Develop a code to solve Eqn. 1.4; allow for an arbitrary number of grid points in both x and y, taking particular care with how you treat the $f-g$ boundary. Investigate and

identify a reasonable number of points which gives you a grid-independent solution using both the SOR and Gauss-Seidel methods.

2. Discuss in your report how you mesh and discretise grid points around the g-f region.
3. Compute a solution for the Gauss-Seidel (G-S) approach.
4. Construct plots comparing SOR and G-S methods, their convergence history as the number of grid points varies.
5. How many grid points are required for a reasonably resolved result? How do you prescribe, or assume that you have a good solution to the problem ?
6. Make contour plots of the $\psi, u{-}v$ fields suitably contoured, only for your grid-independent set of data points. For the ψ field draw contour levels at $\psi = (0.1, 0.2, \ldots, 1)$.
7. Evaluate the $(u,\ v)$ velocity field distributions and make appropriate plots showing any interesting features. Which parts of the region experience the largest flow gradients? Investigate whether these values are grid-independent. Make a plot of the $(u,\ v$ velocity fields along $c - d$ (*i.e.* $3.5 \leq x \leq 4.5$).
8. Make grid-resolved line plots of the ψ (u, v)-fields along x at $y = 3$ and $y = 0.5$. Similarly make plots along y at $x = 1$ and $x = 4$.

Part C: *10 marks*
Next, use a coordinate map to transform the problem into a uniform square based upon ideas discussed in Lecture.

1. Transform Eqn. 1.4 into the mapped space, discretise the equations and compute a numerical solution with the Gauss-Seidel approach.
2. In your report, describe the mapping used and the transformed PDE you solve.
3. Make appropriate plots to show correctness or grid independence of your numerical solution, with those generated in Part B.
4. Make Contour plots of the ψ-field in the region $\Gamma(x, y)$, contrasting with solutions obtained in Part B. For the ψ field draw contour levels at $\psi = (0.1, 0.2, \ldots, 1)$.