MATH50001 - Problems Sheet 8

1. a) Compute formally

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

where $\Delta\varphi(x,y) = \varphi''_{xx}(x,y) + \varphi''_{yy}(x,y)$.

b) Show that if f is holomorphic, then

$$\Delta|f(z)|^2 = 4|f'_z(z)|^2.$$

c) Prove that if $f = u + iv$ is holomorphic then

$$|f'(z)|^2 = \det\begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix}.$$

2. Harmonic conjugates:
Show that the following functions u are harmonic and find their corresponding harmonic conjugate v and holomorphic $f = u + iv$:

a) $u(x,y) = x^3 - 3xy^2 - 2y$.

b) $u(x,y) = x - xy$.

c) Let $u(x,y) = xe^x\cos y - ye^x\sin y$. Find the holomorphic function $f(z)$ (as functions of z) with the real part $u = xe^x\cos y - ye^x\sin y$ and such that $f(i\pi) = 0$.

3.* Let f be holomorphic in an open connected set Ω. Consider

$$g(x,y) = |f(x+iy)|^2, \quad x+iy \in \Omega.$$

Show that if g is harmonic in Ω then f is a constant function.

4.* Show that if $u(x,y)$ is a harmonic real valued function, then

$$\Delta(u^2) \geq 0 \quad \text{and} \quad \Delta^2(u^2) = \Delta\left(\Delta(u^2)\right) \geq 0.$$

5. Show that if $\varphi(x,y)$ and $\psi(x,y)$ are harmonic, then u and v defined by

$$u(x,y) = \varphi'_x(x,y)\,\varphi'_y(x,y) + \psi'_x(x,y)\,\psi'_y(x,y)$$

and

$$v(x,y) = \frac{1}{2}\left((\varphi'_x(x,y))^2 + (\psi'_x(x,y))^2 - \left(\varphi'_y(x,y)\right)^2 - \left(\psi'_y(x,y)\right)^2\right)$$

satisfy the Cauchy-Riemann equations.

6. Find a Möbius transformation that takes the points $z_1 = 2$, $z_2 = i$ and $z_3 = -1$ onto the given points $w_1 = 2i$, $w_2 = -2$, and $w_3 = -2i$, respectively.

6′. Find a Möbius transformation that takes the points $z_1 = 2$, $z_2 = 1+i$ and $z_3 = 0$ onto the given points $w_1 = 1$, $w_2 = i$, and $w_3 = -i$, respectively.

7. Let $f : \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \to \Omega$, such that

$$f(z) = \frac{z - i}{z + i}.$$

Describe Ω.

8. Find a Möbius transformation $w = f(z)$ such that the points

$$f(-2i) = 0, \qquad f(-2) = i, \qquad f(0) = 1.$$

Show that

$$D_1 = \{z : |z + 1 + i| < \sqrt{2}\}$$

maps onto

$$D_2 = \left\{z : \left|z - \frac{1}{2} - \frac{i}{2}\right| < \frac{1}{\sqrt{2}}\right\}.$$

9. Find the Möbits transformation $w = f(z)$ that maps the points $z_1 = -2$, $z_2 = -1 - i$ and $z_3 = 0$ onto the points $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$ respectively. Show that this transformation maps the disk $|z + 1| < 1$ onto the upper half plane.

10. Let $\alpha \in (0, \pi)$. Find a transformation conformal in

$$\{r\, e^{i\theta} : r > 0,\ -\pi < \theta < \pi\}$$

that maps the sector $\{r\, e^{i\theta} : r > 0,\ 0 < \theta < \alpha\}$ onto the half-plane

$$\{w : \operatorname{Im} w > 0\}.$$

11. Find a conformal mapping that transforms the sector $\{z : 0 < \arg z < \pi/4\}$ onto the disc $\{w : |w - 1| < 2\}$.