MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 5

Exercise 21.

For (i), show that the statement holds for finite approximations of e^C and e^B, and then consider the limit. For (ii), use Step 1 of the proof of Theorem 3.3. For (iii), show first that the function $g(t) := e^{(B+C)t}e^{-Ct}e^{-Bt}$ is constant. For (iv), understand how powers of block diagonal matrices look like.

Exercise 22.

Show that every trajectory of $\dot{x} = Jx$ is mapped onto a trajectory of $\dot{x} = Ax$ via the invertible linear mapping T. More precisely, show that $TO_J(x) = T\{e^{Jt}x : t \in \mathbb{R}\} = O_A(Tx)$ for all $x \in \mathbb{R}^d$, where O_J and O_A denote the orbits of the differential equations $\dot{x} = Jx$ and $\dot{x} = Ax$, respectively.

Exercise 23.

For (i), note first that it is clear how to decompose the two given matrices $A = D + P$ the in the required form, and after showing that $PD = DP$, Exercise 21 (iii) can be applied. The main task remains to find a representation of e^{Pt}. Note that similar calculations have been made in the proof of Proposition 3.8. For (ii), the matrix exponential function e^{At} can be found using the Jordan normal form of the matrix A, and to get the phase portrait, consider Exercise 22.

Exercise 24.

Note that this matrix is given in Jordan normal form, and Proposition 3.8 can be used to obtain the flow. Then analyse the flow $\varphi(t, x)$ (which is equivalent to analysing all solutions).

Exercise 25.

To get a continuous-time model in form of a differential equation, discretise into small time steps and then let the length of these time steps converge to 0. It is easier to first understand how to model the changes in salt concentration using a simpler situation with only one tank K with 1000 litres, and assume that at time $t_0 = 0$, per minute b litres of saline solution flow out and b litres of pure water flow in. Fix a time $t > 0$, and find, using a discretisation, the salt concentration $s(t)$, given an initial salt concentration $s(0)$ at time $t_0 = 0$. Note that the model you get for the described situation is a linear differential equation, and (i) can be answered using the matrix exponential function. Note that the situation in (ii) can be modeled by an inhomogeneous linear differential equation, where the coefficient matrix is given by matrix obtained in (i).