Imperial College London

MATH97026 MATH70021

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Advanced Topics in Partial Differential Equations

Date: 23 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let $\alpha \in \mathbb{R}$. Consider the function

$$f(x,y) = \left(\sqrt{x^2+y^2}\right)^{\alpha}, \quad (x,y) \in D,$$

where $D = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, (x^2+y^2) < 1\}$.

(a) Let us recall that the distributional derivatives of f are

$$\partial_x f(x,y) = \alpha x \left(x^2+y^2\right)^{\frac{\alpha-2}{2}}, \quad \partial_y f(x,y) = \alpha y \left(x^2+y^2\right)^{\frac{\alpha-2}{2}}.$$

Determine all $\alpha \in \mathbb{R}$ for which the function f belongs to $H^1(D)$. (8 marks)

(b) Determine all $\alpha \in \mathbb{R}$ for which the function f belongs to $L^2(\partial D)$. (7 marks)

(c) Use the Sobolev embedding theorem to determine all $\alpha \in \mathbb{R}$ for which the following inequality

$$\left| \int_D f(x,y) v(x,y)\, dx\, dy \right| \leq C_f \|v\|_{H^1(D)}, \quad \forall\, v \in H^1(D),$$

holds for some constant C_f depending on f. Then, deduce that, for such values of α, the function f belongs to $(H^1(D))'$, the dual space of $H^1(D)$. (5 marks)

(Total: 20 marks)

2. Let Ω be a bounded and Lipschitz domain in $\mathbb{R}^n$.

(a) Show that there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C \left(\|\nabla u\|_{L^2(\Omega)} + \|\gamma(u)\|_{L^2(\partial\Omega)} \right), \quad \forall\, u \in H^1(\Omega),$$

where γ is the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. (7 marks)

Hint: For $u \in \mathcal{D}(\overline{\Omega})$, apply the divergence theorem to the function $\frac{x}{n}u^2(x)$. Recall the Young inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, for any $a, b, \varepsilon > 0$, and that the outward normal vector ν on $\partial\Omega$ exists almost everywhere.

(b) Let $f \in L^2(\Omega)$ and $\kappa > 0$ (κ is constant). Consider the variational formulation

$$\text{find } u \in H^1(\Omega): \quad a(u,v) = L(v), \quad \forall\, v \in H^1(\Omega), \tag{1}$$

where

$$a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \kappa \int_{\partial\Omega} \gamma(u)\gamma(v)\, d\sigma,$$

and

$$L(\cdot) : H^1(\Omega) \to \mathbb{R}, \quad L(v) = \int_\Omega f\, v\, dx.$$

(i) Prove that the bilinear form a is continuous. (4 marks)

(ii) Prove that the bilinear form a is coercive. (5 marks)

(iii) Show that there exists a unique solution to (1). (4 marks)

(Total: 20 marks)

3. (a) Let $\beta \in (0,1)$. Consider the following inequality

$$\|u\|_{L^3(\mathbb{R}^3)} \leq C\|u\|_{L^2(\mathbb{R}^3)}^{\beta}\|\nabla u\|_{L^2(\mathbb{R}^3)}^{1-\beta}, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^3). \tag{2}$$

Use a scaling argument, i.e. for $\lambda > 0$ and $u \in \mathcal{D}(\mathbb{R}^3)$, define $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}^3$, to determine the only value of β such that (2) holds. (8 marks)

(b) Let Ω be an open set in $\mathbb{R}^n$. Show that

$$\|u\|_{L^3(\Omega)} \leq \|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{L^6(\Omega)}^{\frac{1}{2}}, \quad \forall\, u \in L^6(\Omega).$$

Hint: Write $u^3 = u^{3s}u^{3(1-s)}$ for $s \in (0,1)$ and apply the Hölder inequality. (7 marks)

(c) Let Ω be a bounded and Lipschitz domain in $\mathbb{R}^3$. Prove that

$$\|u\|_{L^3(\Omega)} \leq C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall\, u \in H^1(\Omega).$$

(5 marks)

(Total: 20 marks)

4. Let Ω be a bounded domain of class $\mathcal{C}^2$ in $\mathbb{R}^2$ and $T > 0$. Consider the Cauchy-Dirichlet parabolic problem

$$\begin{cases} \partial_t u - \Delta u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{3}$$

(a) State the variational formulation associated to problem (3). (4 marks)

(b) Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be the sequence of eigenvalues associated with $-\Delta$ with Dirichlet boundary conditions and let $\{v_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega) \cap H^2(\Omega)$ be the corresponding sequence of eigenfunctions. Recall that $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$. For any $m \in \mathbb{N}$, we define $V_m = \text{Span}\{v_1, \dots, v_m\}$.

(i) For any $m \in \mathbb{N}$, introduce the form of the Galerkin approximation u_m and write the approximated problem corresponding to (3). (4 marks)

(ii) Find the vector-valued function $\mathbf{C} : [0,T] \to \mathbb{R}^m$, the matrix $A \in \mathbb{R}^{m\times m}$ and the vector $\mathbf{G} \in \mathbb{R}^m$ such that the approximated problem introduced in part (b)(i) (problem 4) is equivalent to a system of linear ODEs of the form

$$\dot{\mathbf{C}}(t) + A\mathbf{C}(t) = \mathbf{0}, \quad \mathbf{C}(0) = \mathbf{G}. \tag{4}$$

Then, discuss the solution of (4) and the existence of u_m. (4 marks)

(iii) Show that $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. (8 marks)

(Total: 20 marks)

5. *Mastery question concerning the application of fixed point theorems to PDEs*

Let Ω be a bounded and Lipschitz domain in $\mathbb{R}^n$. Consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u + u = \frac{u}{1+u^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

(a) (i) Given $u \in L^2(\Omega)$, show that

$$\left\| \frac{u}{1+u^2} \right\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}$$

(4 marks)

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(s) = \frac{s}{1+s^2}$. Show that $\|f'\|_{L^\infty(\mathbb{R})} \leq 1$. (2 marks)

(b) Consider the map $T : L^2(\Omega) \to L^2(\Omega)$ defined as $T(w) = u$, where u is the weak solution to the linear problem

$$\begin{cases} -\Delta u + u = \frac{w}{1+w^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

(i) Show that T is well-defined. (3 marks)

(ii) Show that $T(L^2(\Omega))$ is a bounded subset of $H_0^1(\Omega)$. (3 marks)

(iii) Show that T is continuous. (4 marks)

(iv) Prove the existence of a weak solution to (5). (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH60021/70021/97026

Advanced Topics in Partial Differential Equations (Solutions)

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1. (a) First, we check that $f \in L^2(D)$. By exploiting the polar coordinates, we have seen ⇓ 8, A

$$\|f\|^2_{L^2(D)} = \int_D (x^2+y^2)^\alpha \, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^1 r^{2\alpha+1} \, dr\, d\theta = \frac{\pi}{2} \int_0^1 r^{2\alpha+1} \, dr.$$

The latter integral converges if and only if $2\alpha+1>-1$. Thus, $f \in L^2(D)$ if and only if $\alpha > -1$.

Next, we determine whether $f \in H^1(D)$. We need to show that $\partial_x f$ and $\partial_y f$ belong to $L^2(D)$. Notice that, if $\alpha = 0$, then $f \in H^1(D)$. For $\alpha \neq 0$, using the polar coordinates, we compute

$$\begin{aligned}\|\partial_x f\|^2_{L^2(D)} &= \alpha^2 \int_D x^2 \left(x^2+y^2\right)^{\alpha-2} \, dx\, dy = \alpha^2 \int_0^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)^2 r^{2\alpha-4+1} \, dr\, d\theta \\ &\leq \frac{\alpha^2 \pi}{2} \int_0^1 r^{2\alpha-1} \, dr.\end{aligned}$$

The latter integral converges if and only if $2\alpha-1>-1$, namely for $\alpha>0$. Thus, $\partial_x f \in L^2(D)$ if and only if $\alpha \geq 0$. The same argument also applies for $\partial_y f$. Therefore, $f \in H^1(D)$ if and only if $\alpha \geq 0$.

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(b) We notice that f has a singularity in the origin $(x,y)=(0,0)$ for $\alpha<0$, and that f is constant on the arc

$$\{(x,y):\ x>0,\ y>0,\ x^2+y^2=1\} \subset \partial D.$$

In order to determine if $\|f\|_{L^2(\partial D)} < \infty$, it is sufficient by symmetry to study $\|f\|_{L^2(S)}$ where

$$S = \{(x,y):\ 0<x<1,\ y=0\}.$$

We have

$$\|f\|^2_{L^2(S)} = \int_0^1 |f(x,0)|^2 \, dx = \int_0^1 x^{2\alpha} \, dx,$$

which converges if and only if $2\alpha > -1$, namely $\alpha > -\frac{1}{2}$. Thus, $f \in L^2(\partial D)$ if and only if $\alpha > -\frac{1}{2}$.

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(c) By the Sobolev embedding theorem in two dimensions, for any $1 \leq p < \infty$, there exists $C_p > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C_p \|v\|_{H^1(\Omega)}, \quad \forall\, v \in H^1(D). \tag{1}$$

By the Hölder inequality and (1), for $q>1$, we have

$$\left|\int_D f(x,y) v(x,y) \, dx\, dy\right| \leq \|f\|_{L^q(D)} \|v\|_{L^{\frac{q}{q-1}}(D)} \leq C_{\frac{q}{q-1}} \|f\|_{L^q(D)} \|v\|_{H^1(D)}.$$

Therefore, the inequality

$$\left|\int_D f(x,y)\, v(x,y) \, dx\, dy\right| \leq C_f \|v\|_{H^1(D)}, \quad \forall\, v \in H^1(D),$$

holds provided that $f \in L^q(D)$ for some $q>1$. Now, we compute

$$\|f\|^q_{L^q(D)} = \int_D \left(x^2+y^2\right)^{\frac{q\alpha}{2}} \, dx\, dy = \frac{\pi}{2} \int_0^1 r^{q\alpha+1} \, dr,$$

which converges if and only if $q\alpha + 1 > -1$. This condition is verified for some $q > 1$ if $\alpha > -2$.
As a consequence, for $\alpha > -2$, we deduce that

$$\|f\|_{(H^1(D))'} = \sup_{v \in H^1(\Omega): \|v\|_{H^1(D)} \leq 1} \left| \int_D f(x,y)\, v(x,y)\, dx\, dy \right| \leq C_f,$$

which implies that $f \in (H^1(D))'$.

2. (a) For any $u \in \mathcal{D}(\overline{\Omega})$, we have

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$$\int_\Omega \operatorname{div}\left(\frac{x}{n}u(x)^2\right)\,dx = \int_\Omega u(x)^2\,dx + \int_\Omega \left(\frac{2x}{n}\cdot\nabla u(x)\right)u(x)\,dx.$$

On the other hand, by the divergence theorem, we find

$$\int_\Omega \operatorname{div}\left(\frac{x}{n}u(x)^2\right)\,dx = \int_{\partial\Omega}\left(\frac{\sigma}{n}\cdot\nu(\sigma)\right)u(\sigma)^2\,d\sigma,$$

where ν is the outward normal vector on $\partial\Omega$. Then, we obtain

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega u(x)^2\,dx \le \left|\int_\Omega \left(\frac{2x}{n}\cdot\nabla u(x)\right)u(x)\,dx\right| + \left|\int_{\partial\Omega}\left(\frac{\sigma}{n}\cdot\nu(\sigma)\right)u(\sigma)^2\,d\sigma\right|.$$

Since Ω is bounded and Lipschitz, we notice that

$$\operatorname*{ess\,sup}_{x\in\Omega}\left|\frac{2x}{n}\right| \le M_\Omega, \quad \operatorname*{ess\,sup}_{\sigma\in\partial\Omega}\left|\frac{\sigma}{n}\cdot\nu\right| \le M'_\Omega.$$

By the Cauchy-Schwarz inequality and by Young's inequality, we deduce that

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &\le M_\Omega\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + M'_\Omega\|u\|^2_{L^2(\partial\Omega)}\\ &\le \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{M_\Omega^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + M'_\Omega\|u\|^2_{L^2(\partial\Omega)},\end{aligned}$$

which implies that

$$\|u\|_{L^2(\Omega)} \le \sqrt{\max\{M_\Omega^2, 2M'_\Omega\}}\left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\right), \quad \forall\, u\in\overline{\mathcal{D}}.$$

Finally, since $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$ (with respect to the $\|\cdot\|_{H^1(\Omega)}$) and $\gamma : H^1(\Omega)\to L^2(\partial\Omega)$ is a linear and continuous operator, we extend by density argument the above inequality, so that

$$\|u\|_{L^2(\Omega)} \le C_P\left(\|\nabla u\|_{L^2(\Omega)} + \|\gamma(u)\|_{L^2(\partial\Omega)}\right), \quad \forall\, u\in H^1(\Omega), \tag{2}$$

where $C_P = \sqrt{\max\{M_\Omega^2, 2M'_\Omega\}}$.

(b) (i) We recall the trace inequality

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$$\|\gamma(u)\|_{L^2(\partial\Omega)} \le C_\gamma\|u\|_{H^1(\Omega)}, \quad \forall\, u\in H^1(\Omega). \tag{3}$$

By the Cauchy-Schwarz inequality and (3), we find

$$\begin{aligned}|a(u,v)| &\le \left|\int_\Omega \nabla u\cdot\nabla v\,dx\right| + \kappa\left|\int_{\partial\Omega}\gamma(u)\gamma(v)\,d\sigma\right|\\ &\le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \kappa\|\gamma(u)\|_{L^2(\partial\Omega)}\|\gamma(v)\|_{L^2(\partial\Omega)}\\ &\le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \kappa C_\gamma^2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\\ &\le \left(1+\kappa C_\gamma^2\right)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

Thus, a is continuous in $H^1(\Omega)\times H^1(\Omega)$.

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(ii) By exploiting (2) and the basic inequality $(a+b)^2 \le 2a^2 + 2b^2$, we compute

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\begin{aligned}
a(u,u) &= \|\nabla u\|^2_{L^2(\Omega)} + \kappa \|\gamma(u)\|^2_{L^2(\partial\Omega)} \\
&= \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \kappa \|\gamma(u)\|^2_{L^2(\partial\Omega)} \\
&\ge \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \min\left\{\frac{1}{2}, \kappa\right\} \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\gamma(u)\|^2_{L^2(\partial\Omega)}\right) \\
&\ge \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \min\left\{\frac{1}{4}, \frac{\kappa}{2}\right\} \left(2\|\nabla u\|^2_{L^2(\Omega)} + 2\|\gamma(u)\|^2_{L^2(\partial\Omega)}\right) \\
&\ge \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \min\left\{\frac{1}{4}, \frac{\kappa}{2}\right\} \left(\|\nabla u\|_{L^2(\Omega)} + \|\gamma(u)\|_{L^2(\partial\Omega)}\right)^2 \\
&\ge \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{C_P^2}\min\left\{\frac{1}{4}, \frac{\kappa}{2}\right\} \|u\|^2_{L^2(\Omega)} \\
&\ge \widetilde{\alpha}\|u\|^2_{H^1(\Omega)},
\end{aligned}
$$

where

$$
\widetilde{\alpha} = \min\left\{\frac{1}{2}, C_P^2 \min\left\{\frac{1}{4}, \frac{\kappa}{2}\right\}\right\}.
$$

This implies that a is coercive in $H^1(\Omega)$.

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(iii) Since $f \in L^2(\Omega)$, we observe that

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|L(v)| = \left|\int_\Omega f\, v\, dx\right| \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}.
$$

This gives that $L \in (H^1(\Omega))'$. Thus, thanks to the Lax-Milgram theorem, there exists a unique function $u \in H^1(\Omega)$ such that

$$
a(u,v) = L(v), \quad \forall\, v \in H^1(\Omega).
$$

3. (a) For any $u \in \mathcal{D}(\mathbb{R}^3)$ and $\lambda > 0$, let us define $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}^3$. By assumption, we have

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$$\|u_\lambda\|_{L^3(\mathbb{R}^3)} \le C\|u_\lambda\|^\beta_{L^2(\mathbb{R}^3)}\|\nabla u_\lambda\|^{1-\beta}_{L^2(\mathbb{R}^3)}, \quad \forall\, \lambda > 0. \tag{4}$$

We now compute

$$\begin{aligned}\|u_\lambda\|_{L^3(\mathbb{R}^3)} &= \left(\int_{\mathbb{R}^3} |u(\lambda x)|^3\, dx\right)^{\frac{1}{3}} \\ &= \left(\frac{1}{\lambda^3}\int_{\mathbb{R}^3} |u(z)|^3\, dz\right)^{\frac{1}{3}} \\ &= \frac{1}{\lambda}\|u\|_{L^3(\mathbb{R}^3)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|u_\lambda\|_{L^2(\mathbb{R}^3)} &= \left(\int_{\mathbb{R}^3} |u(\lambda x)|^2\, dx\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{\lambda^3}\int_{\mathbb{R}^3} |u(z)|^2\, dz\right)^{\frac{1}{2}} \\ &= \frac{1}{\lambda^{\frac{3}{2}}}\|u\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

On the other hand, we find

$$\begin{aligned}\|\nabla u_\lambda\|_{L^2(\mathbb{R}^3)} &= \left(\int_{\mathbb{R}^3} |\nabla\left(u(\lambda x)\right)|^2\, dx\right)^{\frac{1}{2}} \\ &= \left(\lambda^2\int_{\mathbb{R}^3} |(\nabla u)(\lambda x)|^2\, dx\right)^{\frac{1}{2}} \\ &= \left(\frac{\lambda^2}{\lambda^3}\int_{\mathbb{R}^3} |(\nabla u)(z)|^2\, dz\right)^{\frac{1}{2}} \\ &= \lambda^{-\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Collecting the above equality in (4), we find

$$\|u\|_{L^3(\mathbb{R}^3)} \le C\lambda^{1-\frac{3\beta}{2}-\frac{1-\beta}{2}}\|u\|^\beta_{L^2(\mathbb{R}^3)}\|\nabla u\|^{1-\beta}_{L^2(\mathbb{R}^3)}, \quad \forall\, \lambda > 0.$$

If the exponent $1 - \frac{3\beta}{2} - \frac{1-\beta}{2} > 0$ (resp. < 0), then, letting $\lambda \to 0$ (resp. $\lambda \to \infty$), we will have $u = 0$. In order to avoid this contradiction, we need to impose the condition

$$1 - \frac{3\beta}{2} - \frac{1-\beta}{2} = 0,$$

which gives $\beta = \frac{1}{2}$.

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(b) By the Hölder inequality with $p > 1$, we have

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$$\begin{aligned}\|u\|^3_{L^3(\Omega)} &= \int_\Omega u^3\, dx = \int_\Omega u^{3s}u^{3(1-s)}\, dx \\ &\le \left(\int_\Omega u^{3sp}\, dx\right)^{\frac{1}{p}}\left(\int_\Omega u^{3(1-s)\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}.\end{aligned}$$

By solving the system

$$\begin{cases} 3sp = 2 \\ 3(1-s)\frac{p}{p-1} = 6 \end{cases} \quad \Rightarrow \begin{cases} p = \frac{4}{3} \\ s = \frac{1}{2}. \end{cases}$$

Thus, we obtain

$$\|u\|^3_{L^3(\Omega)} \leq \left(\int_\Omega u^2\,dx\right)^{\frac{3}{4}} \left(\int_\Omega u^6\,dx\right)^{\frac{1}{4}} = \|u\|^{\frac{3}{2}}_{L^2(\Omega)} \|u\|^{\frac{3}{2}}_{L^6(\Omega)},$$

which implies the desired conclusion.

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(c) Since Ω is bounded and Lipschitz in $\mathbb{R}^3$, the Sobolev embedding theorem implies that there exists $C_\Omega > 0$ such that

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$$\|u\|_{L^6(\Omega)} \leq C_\Omega \|u\|_{H^1(\Omega)}, \quad \forall\, u \in H^1(\Omega). \tag{5}$$

Therefore, exploiting (5) in the inequality proved in part (b), we obtain

$$\|u\|_{L^3(\Omega)} \leq C_\Omega^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{H^1(\Omega)}, \quad \forall\, u \in H^1(\Omega).$$

4. (a) The variational formulation of the Cauchy-Dirichlet parabolic problem is: find $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ such that

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   1. the function u satisfies

$$\langle \partial_t u(t), v\rangle_* + (\nabla u(t), \nabla v) + \int_\Omega (\partial_x u(t) + \partial_y u(t))\, v\, dx = 0, \quad \forall\, v \in H_0^1(\Omega),$$

   for almost every $t \in (0,T)$;

   2. $u(0) = u_0$.

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(b) (i) For any $m \in \mathbb{N}$, we define the Galerkin approximation as

$$u_m(x,t) = \sum_{k=1}^{m} c_k^m(t) v_k(x),$$

where $c_1^m(t), \dots, c_m^m(t)$ are functions of time and $v_1, \dots, v_m$ are the first m eigenfunctions associated with $-\Delta$ subject to Dirichlet boundary conditions. The approximated problem is: find $u_m \in \mathcal{C}^1([0,T]; V_m)$ such that

   1. the function u_m satisfies

$$(\partial_t u_m(t), v_s) + (\nabla u_m(t), \nabla v_s) + \int_\Omega (\partial_x u_m(t) + \partial_y u_m(t))\, v_s\, dx = 0, \quad (6)$$

   for all $s = 1, \dots, m$, for every $t \in (0,T)$;

   2. $u_m(0) = \sum_{k=1}^m (u_0, v_k) v_k$.

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(ii) Let us define

$$\mathbf{C}(t) = \left(c_1^m(t), \dots, c_m^m(t)\right)^T, \quad \mathbf{G} = \left((u_0, v_1), \dots, (u_0, v_m)\right).$$

We now recall that $\{v_n\}_{n\in\mathbb{N}}$ is an orthonormal base of $L^2(\Omega)$ and an orthogonal base of $H^1(\Omega)$. Then, we notice that

$$(\partial_t u_m(t), v_s) = \sum_{k=1}^m \dot{c}_k^m(t)\,(v_k, v_s) = \dot{c}_s^m(t), \quad s = 1, \dots, m. \quad (7)$$

Similarly, we have

$$(\nabla u_m(t), \nabla v_s) = \sum_{k=1}^m c_k^m(t)\,(\nabla v_k, \nabla v_s) = \lambda_s\, c_s^m(t). \quad (8)$$

Moreover, we find

$$\int_\Omega (\partial_x u_m(t) + \partial_y u_m(t))\, v_s\, dx = \sum_{k=1}^m c_k^m(t) \int_\Omega (\partial_x v_k + \partial_y v_k)\, v_s\, dx. \quad (9)$$

Clearly, since $\{v_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$, we have that $\int_\Omega (\partial_x v_k + \partial_y v_k)\, v_s\, dx < \infty$ for any $k \in \mathbb{N}, s \in \mathbb{N}$. Collecting (7), (8) and (9) together, we find

$$\dot{c}_s^m(t) + \lambda_s c_s^m(t) + \sum_{k=1}^m c_k^m(t) \left(\int_\Omega (\partial_x v_k + \partial_y v_k)\, v_s\, dx \right) = 0, \quad (10)$$

for any $s = 1, \dots, m$. Besides, $u_m(0) = \sum_{k=1}^m (u_0, v_k) v_k$ is equivalent to

$$c_s^m(0) = (u_0, v_s), \quad s = 1, \dots, m. \quad (11)$$

We now define $A \in \mathbb{R}^{m\times m}$ as follows

$$A_{ij} = \lambda_i \delta_{ij} + \int_\Omega (\partial_x v_j + \partial_y v_j)\, v_i\, dx. \tag{12}$$

Therefore, in light of (10), (11) and (12), we deduce that the problem (6) is equivalent to the linear systems of ODEs

$$\dot{\mathbf{C}}(t) + A\mathbf{C}(t) = \mathbf{0}, \quad \mathbf{C}(0) = \mathbf{G}. \tag{13}$$

Since the matrix A is constant in time, it follows from the global version of the Cauchy-Lipschitz theorem that there exists a unique solution $\mathbf{C} : [0,T] \to \mathbb{R}^m$ that solves (13). As a consequence, there exists a unique $u_m \in \mathcal{C}^1([0,T]; V_m)$ solving the approximated problem (6) and the initial condition $u_m(0) = \sum_{k=1}^m (u_0, v_k) v_k$.

$\boxed{\text{sim. seen} \Downarrow}$

$\boxed{8, \text{B}}$

(iii) Multiplying (6) by $c_s^m(t)$ and summing over s, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} + \int_\Omega (\partial_x u_m(t) + \partial_y u_m(t))\, u_m(t)\, dx = 0.$$

We present two possible solutions.
Version 1. Since $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(\overline{\Omega})$, it follows that $\gamma(u_m(t)) = u_m(t)$ on $\partial\Omega$. Setting $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we infer from the Green's formula that

$$\begin{aligned}
&\int_\Omega (\partial_x u_m(t) + \partial_y u_m(t))\, u_m(t)\, dx \\
&\quad = \int_\Omega \mathbf{b} \cdot \nabla \left(\frac{1}{2} u_m(t)^2 \right) dx \\
&\quad = -\int_\Omega \underbrace{(\mathrm{div}\mathbf{b})}_{=0} \frac{1}{2} u_m(t)^2\, dx + \int_{\partial\Omega} \frac{1}{2} \underbrace{u_m(t)^2}_{=0} \mathbf{b}\cdot\nu\, d\sigma = 0.
\end{aligned}$$

Thus, we end up with

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)} = 0.$$

Integrating in time from 0 to $t \in [0,T]$, we find

$$\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)}\, ds = \frac{1}{2}\|u_m(0)\|^2_{L^2(\Omega)}.$$

Since

$$\|u_m(0)\|^2_{L^2(\Omega)} = \sum_{k=1}^m \|(u_0, v_k)|^2 \le \sum_{k=1}^\infty |(u_0, v_k)|^2 = \|u_0\|^2_{L^2(\Omega)}, \tag{14}$$

we obtain

$$\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)}\, ds \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

This gives

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \le \|u_0\|_{L^2(\Omega)}, \quad \|u_m\|_{L^2(0,T;H^1_0(\Omega))} \le C\|u_0\|_{L^2(\Omega)}, \tag{15}$$

where C depends on the Poincaré constant.

Version 2. By Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}\int_\Omega (\partial_x u_m(t) + \partial_y u_m(t))\, u_m(t)\, dx &\le \|\nabla u_m(t)\|_{L^2(\Omega)} \|u_m(t)\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\nabla u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_m(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)}.$$

Integrating in time from 0 to $t \in [0,T]$ and exploiting (14), we find

$$\begin{aligned}&\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)}\, ds \\ &\qquad \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \frac{1}{2}\|u_m(s)\|^2_{L^2(\Omega)}\, ds.\end{aligned} \tag{16}$$

By the Gronwall lemma, we deduce that

$$\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} e^t, \quad \forall\, t \in [0,T]. \tag{17}$$

By using (17) in (16), we reach

$$\int_0^t \|\nabla u_m(s)\|^2_{L^2(\Omega)}\, ds \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} e^t, \quad \forall\, t \in [0,T]. \tag{18}$$

Finally, there exists $\widetilde{C}$ depending on T and $\|u_0\|_{L^2(\Omega)}$ such that

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} \le \widetilde{C}, \quad \|u_m\|_{L^2(0,T;H^1_0(\Omega))} \le \widetilde{C}. \tag{19}$$

5. (a) (i) Since $1+s^2 \geq 1$ for any $s \in \mathbb{R}$, we have $\boxed{4, \mathrm{M}}$

$$\left\| \frac{u}{1+u^2} \right\|_{L^2(\Omega)}^2 = \int_\Omega \frac{u^2}{(1+u^2)^2}\, dx \leq \int_\Omega \frac{u^2}{1+u^2}\, dx \leq \int_\Omega 1\, dx = |\Omega|.$$

Thus, $\frac{u}{1+u^2} \in L^2(\Omega)$ and $\left\| \frac{u}{1+u^2} \right\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}$.

$\boxed{2, \mathrm{M}}$

(ii) Observing that $f \in \mathcal{C}^1(\mathbb{R})$, we compute

$$f'(s) = \frac{1-s^2}{(1+s^2)^2}$$

and we deduce that

$$\|f'\|_{L^\infty(\mathbb{R})} \leq 1.$$

(b) For any $w \in L^2(\Omega)$, we consider the linear problem

$$\begin{cases} -\Delta u + u = \frac{w}{1+w^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

We define the corresponding weak formulation: find $u \in H_0^1(\Omega)$ such that

$$(u, v)_{H_0^1(\Omega)} = \left(\frac{w}{1+w^2}, v \right), \quad \forall\, w \in H_0^1(\Omega). \tag{21}$$

$\boxed{3, \mathrm{M}}$

(i) Since $\frac{w}{1+w^2} \in L^2(\Omega)$, the unique solution $u \in H_0^1(\Omega)$ solving (21) follows from the Riesz representation theorem (or alternatively from the Lax-Milgram theorem). This implies that the map $T : L^2(\Omega) \to L^2(\Omega)$ where $T(w) = u$ is well-defined. $\boxed{3, \mathrm{M}}$

(ii) For any $w \in L^2(\Omega)$, taking $v = T(w)$ in (21), we have

$$\|T(w)\|_{H_0^1(\Omega)}^2 = \left(\frac{w}{1+w^2}, T(w) \right) \leq \left\| \frac{w}{1+w^2} \right\|_{L^2(\Omega)} \|T(w)\|_{L^2(\Omega)},$$

which implies that

$$\|T(w)\|_{H_0^1(\Omega)} \leq \sqrt{|\Omega|}.$$

$\boxed{4, \mathrm{M}}$

(iii) Let $w_n \to w$ in $L^2(\Omega)$. Notice that

$$(T(w) - T(w_n), v)_{H_0^1(\Omega)} = \left(\frac{w}{1+w^2} - \frac{w_n}{1+w_n^2}, v \right), \quad \forall\, v \in H_0^1(\Omega). \tag{22}$$

In light of part (a) (ii), we observe that

$$\begin{aligned} \left\| \frac{w}{1+w^2} - \frac{w_n}{1+w_n^2} \right\|_{L^2(\Omega)}^2 &= \|f(w) - f(w_n)\|_{L^2(\Omega)}^2 \\ &\leq \int_\Omega \left(\sup_{s \in \mathbb{R}} |f'(s)|^2 \right) |w - w_n|^2\, dx \\ &\leq \|w - w_n\|_{L^2(\Omega)}^2. \end{aligned}$$

Taking $v = T(w) - T(w_n)$ in (22), it is easily seen that

$$\|T(w_n) - T(w)\|_{H_0^1(\Omega)} \leq \|w - w_n\|_{L^2(\Omega)},$$

which entails that $T(w_n) \to T(w)$ in $H_0^1(\Omega)$.

4, M

(iv) Let $X = L^2(\Omega)$ and $K = B_{H_0^1(\Omega)}(\sqrt{|\Omega|})$, where B denotes the closed ball in $H_0^1(\Omega)$ of radius $\sqrt{|\Omega|}$ centered at the origin. By the Rellich theorem, K is compact in X. Also, K is convex. Thanks to part (ii) and (iii), it follows that $T : K \to K$ is continuous. Therefore, by the Schauder theorem, we conclude that there exists $u \in K$ such that $u = T(u)$, which implies that $u \in H_0^1(\Omega)$ solves

$$(u, v)_{H_0^1(\Omega)} = \left(\frac{u}{1+u^2}, v \right), \quad \forall\, w \in H_0^1(\Omega). \tag{23}$$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

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ExamModuleCode	QuestionNumber	Comments for Students
Advanced Topics in Partial Differential Equations_MATH60021 MATH97026 MATH70021	1	In part a) a common mistake is to not consider the case \alpha=0. In part b) the most common mistake is to use the trace theorem, instead of computing the norm in L^2 on the boundary. In fact, the range of admissible values of \alpha is different with these two approaches. In part c) the Sobolev embedding theorem has not been used in the optimal way, thereby the admissible value of \alpha was not correct.
Advanced Topics in Partial Differential Equations_MATH60021 MATH97026 MATH70021	2	In part a) the density argument to show the validity of the generalized Poincaré inequality for any function in H^1 is sometimes missing. In part b)-(ii) an occasional mistake is to use the classical Poincaré inequality.
Advanced Topics in Partial Differential Equations_MATH60021 MATH97026 MATH70021	3	In part a) the common mistake is the computation of the norm in L^2 of the gradient of u_\lambda.
Advanced Topics in Partial Differential Equations_MATH60021 MATH97026 MATH70021	4	In part a) a frequent mistake is to write the term involving the time derivative of the solution as an integral. This should be a duality pairing. In part b)-(iii), the integral of (1 1)\cdot \nabla u_m * u_m=0.
Advanced Topics in Partial Differential Equations_MATH60021 MATH97026 MATH70021	5	No specific comments to make