2D continuum limit of the applications framework

1. Consider the voltage distribution in a circular conductor of unit radius, and with unit conductivity, centred at the origin given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{m}{2\pi} \log z.$$

(a) Verify, by direct calculation of the derivatives, that $\nabla^2 \phi = 0$ everywhere inside the conductor except at $(0, 0)$ where $\phi(x, y)$ is not defined.

(b) Show that $\phi = 0$ on the boundary $|z| = 1$ of the conductor.

(c) Find an expression for $J^{(x)} - \mathrm{i}J^{(y)}$ where $(J^{(x)}, J^{(y)})$ is the current density vector.

(d) Using the result of part (c), calculate the net current leaving the conductor through its boundary $|z| = 1$ by integrating the normal component of the current density vector around the boundary.

(e) Could you have anticipated your answer to part (d)?

2. Consider the voltage distribution in a square conductor, with unit conductivity,

$$-L/2 < x < L/2, \qquad -L/2 < y < L/2$$

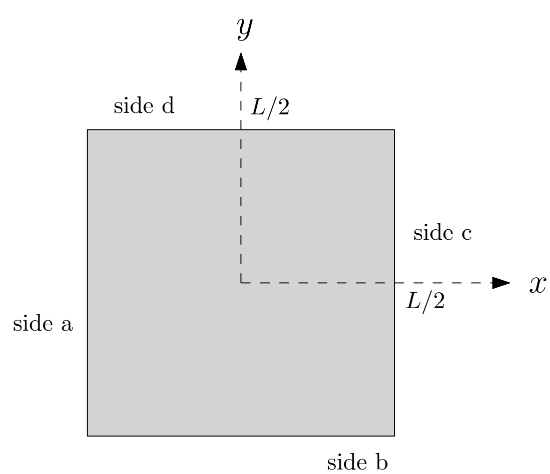
centred at the origin and with sides of length $L > 0$ given by

$$\phi(x, y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{z}{L} - \frac{m}{2\pi} \log z. \tag{1}$$

Let the 4 sides of the conductor be denoted a, b, c and d as shown in the Figure.



(a) Verify, by direct calculation of the derivatives, that $\nabla^2\phi = 0$ everywhere inside the conductor except at $(0,0)$ where $\phi(x,y)$ is not defined.

(b) Find an expression for $J^{(x)} - \mathrm{i}J^{(y)}$.

(c) Use your answer in part (b) to calculate the net current leaving the conductor through **each** of its four sides a, b, c and d.

(d) The complex potential (1) for this square conductor is made up of two contributions: one is

$$-\frac{z}{L}$$

which resembles the complex potential for a uniform current across the conductor from left to right running parallel to the x axis as considered in the lectures; there is a second contribution

$$-\frac{m}{2\pi}\log z$$

which is the complex potential associated with a *point current source* at the origin. Based on these observations can you see another way to find the four answers to part (c)?

3. In consideration of the strip conductor $-\infty < x < \infty, -\pi/2 < y < \pi/2$ in the lecture notes it was determined that the current density component $J^{(y)}$ on the top boundary $y = \pi/2$ is

$$J^{(y)} = \frac{m}{2\pi}\mathrm{sech}x.$$

Use this result to calculate the total current leaving the strip through this top boundary of the strip by integrating this normal current density function. Could you have anticipated your result in advance?

4. Consider the voltage distribution in a circular conductor of unit radius centred at the origin, and of unit conductivity $\hat{c} = 1$, given by

$$\phi(x,y) = \mathrm{Re}[h(z)], \qquad z = x + \mathrm{i}y,$$

where

$$h(z) = -\frac{m}{2\pi}\log\left(\frac{z^2 - a^2}{z^2a^2 - 1}\right), \qquad 0 < a < 1,$$

where a is a real parameter. The outer boundary of the conductor is the circle $|z| = 1$.

(a) Verify that $\nabla^2\phi = 0$ everywhere inside the conductor except at the two points $(a,0)$ and $(-a,0)$.

(b) Show that $\phi = 0$ on the boundary $|z| = 1$ of the conductor.

(c) Show that the current density in the direction normal to the conductor boundary is

$$\frac{m(a^4-1)}{\pi}\frac{1}{2a^2\cos 2\theta-(1+a^4)},$$

where the angle θ is used to parametrize a point $(\cos\theta, \sin\theta)$ on the conductor boundary $|z| = 1$.

(d) Use your answer to part (c) to calculate the net current leaving the conductor through its boundary $|z| = 1$ by integrating the normal component of the current density vector. (*Hint:* you may find the "t-substitution" from calculus helpful to carry out the integration).

(e) Could you have deduced the answer to part (d) using other arguments?