

Sheet 2, question 6 (replacement)

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\exp(-x \tanh x)}{(1+x^2)} dx$$

and the trapezium rule approximation by I_h and $I_h^{(N)}$.

- (a) Estimate the discretisation error (do not attempt to bound the integrand in the analytic strip).
- (b) Estimate the truncation error.
- (c) Determine the optimal meshwidth h .

Solution

a) Consider $f(z) = \frac{\exp(-z \tanh z)}{1+z^2}$ for $z \in \mathbb{C}$.

This has singularities at $z = \pm i$, but is analytic in the strip $\{|Im\{z\}| < 1\}$. So by Theorem 2.12 of the lecture notes¹ (see * below for more detail) with $\alpha < 1$, we have, for the discretization error,

$$|I - I_h| \leq \frac{2M}{e^{2\pi\alpha/h} - 1}$$

$$= O(e^{-2\pi\alpha/h})$$

$e^{2\pi\alpha/h} \gg 1$
 \hookrightarrow as $h \rightarrow 0$, so

$$\frac{2M}{e^{2\pi\alpha/h} - 1} \approx \frac{2M}{e^{2\pi\alpha/h}}$$

as $h \rightarrow 0$.

* More detail:

Check the necessary conditions for Thm 2.12 as follows.

Note that for all $a' \in (-a, a)$,

$$|f(x + ia')| = \left| \frac{\exp(-(x + ia') \tanh(x + ia'))}{1 + (x + ia')^2} \right|$$

And

$$\tanh z = \frac{e^{2z} - 1}{e^{2z} + 1}$$

$$= \frac{(e^{2z} - 1)(e^{2\bar{z}} + 1)}{(e^{2z} + 1)(e^{2\bar{z}} + 1)} \quad \left\{ \begin{array}{l} \text{with} \\ z = x + iy \end{array} \right.$$

$$= \frac{e^{4x} - 1 + e^{2x}(e^{2iy} - e^{-2iy})}{e^{4x} + 1 + e^{2x}(e^{2iy} + e^{-2iy})}$$

$$= \frac{e^{4x} - 1 + 2ie^{2x} \sin y}{e^{4x} + 1 + 2e^{2x} \cos y}$$

$$\Rightarrow \operatorname{Re} \{ z \tanh z \} =$$

$$= \frac{x(e^{4x} - 1) - 2e^{2x}y \sin y}{e^{4x} + 1 + 2e^{2x} \cos y}$$

$$\approx \pm x \quad \text{as} \quad x \rightarrow \pm \infty \quad \text{for finite } y$$

So

$$|\exp(-(x+ia') \tanh(x+ia'))|$$

$$\approx \exp(\mp x) \quad \text{as} \quad x \rightarrow \pm \infty$$

$$\Rightarrow |f(x+ia')| \approx \frac{\exp(\mp x)}{x^2} \quad \text{as} \quad x \rightarrow \pm \infty$$

$$< \frac{1}{x^2} \quad \text{as} \quad x \rightarrow \pm \infty$$

$$\Rightarrow |f(x+ia')| \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm \infty$$

and furthermore $\int_{-\infty}^{\infty} |f(x+ia')| dx$ is bounded

(Since the contributions to this integral due the sections of the range of integration local to $\pm\infty$ are zero, since

$$\int \frac{1}{x^2} dx = \frac{1}{x} \rightarrow 0 \text{ as } x \rightarrow \pm\infty$$

b) The truncation error is defined to be

$$I_h - I_h^{[N]} = h \left(\sum_{n=-\infty}^{-N-1} + \sum_{n=N+1}^{\infty} f(x_n) \right)$$

$x_n = nh$

$$= 2h \sum_{n=N+1}^{\infty} f(nh)$$

Since $f(x)$ is even.

As shown in part (a),

$$|f(x)| \sim \frac{e^{-x}}{x^2} \text{ as } x \rightarrow +\infty$$

And, it follows Lemma 2.13 of the lecture notes with $g(x) \equiv x$, that

$$h \sum_{n=N+1}^{\infty} e^{-nh} = O(e^{-Nh}) \quad \text{as } N \rightarrow \infty$$

(assuming that $h \rightarrow 0$ with $h \gg 1/N$). Then, since

$$|f(x_n)| = |f(nh)| < e^{-nh} \quad \text{as } n \rightarrow \infty,$$

we can deduce from the above that

$$h \sum_{n=N+1}^{\infty} |f(x_n)| = O(e^{-Nh}) \quad \text{as } N \rightarrow \infty.$$

Thus, we have

$$|I_h - I_h^{[N]}| = O(e^{-Nh}).$$

c) To determine the optimal mesh width h
we equate the discretisation and truncation errors:

$$-\frac{2\pi a}{h} = -Nh$$

$$\Rightarrow h^2 = \frac{2\pi a}{N}$$

$$\Rightarrow h = \left(\frac{2\pi a}{N}\right)^{1/2} \quad \text{for any } a < 1$$

$$\Rightarrow \text{take } h = \left(\frac{2\pi}{N}\right)^{1/2}.$$