BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH40004

Calculus and Applications (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

4 . (a) To check linear independence we calculate the Wronskian which is in this case: sim. seen ⇓

$$W = \det \begin{bmatrix} x & x+2 & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} = -4 \neq 0.$$

So, these functions are really independent. 3, A

(b (i)) We try $y = e^{\lambda x}$ and we obtain

$$\lambda^2 - 6\lambda + 9 = 0$$

which has a repeated root of $\lambda = 3$, so as the basis for the solution space we can use $\{e^{\lambda t}, te^{\lambda t}\}$ (which are linearly independent). So, we have

$$y_{CF} = c_1 e^{3x} + c_2 x e^{3x}.$$

3, A

(b (ii)) We try the ansatz sim. seen ⇓

$$y_{PI} = Ae^{-3x} + Bx^2 e^{3x}.$$

By substituting into the ODE, We obtain $A = \frac{1}{6}$ and $B = 3$. So we have:

$$y_{GS} = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{6} e^{-3x} + 3x^2 e^{3x}.$$

Using the initial conditions at $x = 0$, we obtain: 5, B

$$c_1 + \frac{1}{6} = 0,$$

$$3c_1 + c_2 - \frac{1}{2} = 0.$$

Which gives $c_1 = -\frac{1}{6}$ and $c_2 = 1$. So the solution is:

$$y = -\frac{1}{6} e^{3x} + x e^{3x} + \frac{1}{6} e^{-3x} + 3x^2 e^{3x}.$$

2, A

(c) Using the inverse Fourier transform we have: unseen ⇓

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\delta(a\omega + b) + \delta(a\omega - b)]\, e^{i\omega x} d\omega.$$

3, C

We evaluate each part of this integral by a change of variable $\theta = a\omega \pm b$ and considering the $a > 0$ and $a < 0$ case separately to obtain:

$$f(x) = \frac{1}{2\pi |a|} e^{-\frac{ixb}{a}} + \frac{1}{2\pi |a|} e^{\frac{ixb}{a}} = \frac{1}{\pi |a|} \cos \frac{xb}{a}.$$

4, D

5 (a (i)) By setting $u(x, y) = 0$ we have: meth seen ⇓

$$x = 1, \quad y = 1, \quad \text{and} \quad x + y = 4.$$

2, A

(a (ii)) We take partial derivatives of u with respect to x and y and set them to zero to obtain the stationary points. meth seen ⇓

$$\frac{\partial u}{\partial x} = 2xy + y^2 - 6y - 2x + 5 = 0;$$

$$\frac{\partial u}{\partial y} = 2xy + x^2 - 6x - 2y + 5 = 0;$$

By subtracting the two equations we obtain $x = y$ or $x + y = 4$ and by substituting these into the first or second equation we obtain the following 4 stationary points.

$$P_1 = (1, 1), \quad P_2 = (1, 3), \quad P_3 = (3, 1), \quad P_4 = (\frac{5}{3}, \frac{5}{3}).$$

We use the trace and determinant of the Hessian Matrix to classify the stationary point. We have:

$$\frac{\partial^2 u}{\partial x^2} = 2y - 2, \quad \frac{\partial^2 u}{\partial y^2} = 2x - 2, \quad \frac{\partial^2 u}{\partial x \partial y} = 2x + 2y - 6.$$

So we have:

$$H(P_1) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \quad \Rightarrow \quad \tau = 0, \, \Delta = -4 < 0 \quad \Rightarrow \quad P_1 \text{ is saddle point.}$$

$$H(P_2) = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \quad \Rightarrow \quad \tau = 4 > 0, \, \Delta = -4 < 0 \quad \Rightarrow \quad P_1 \text{ is saddle point.}$$

$$H(P_3) = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \quad \Rightarrow \quad \tau = 4 > 0, \, \Delta = -4 < 0 \quad \Rightarrow \quad P_1 \text{ is saddle point.}$$

$$H(P_4) = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \quad \Rightarrow \quad \tau = \frac{8}{3} > 0, \, \Delta = \frac{4}{3} > 0 \quad \Rightarrow \quad P_1 \text{ is minimum.}$$

6, A

(a (iii)) Sketch is presented in Figure 1. sim. seen ⇓

3, C

(b (i)) We set the RHS of the ODE to zero and we obtain the following two roots for $r > 2$ (there is one repeated root $y^* = 1$ at $r = 2$). meth seen ⇓

$$y^* = 1 \pm \sqrt{r - 2}.$$

Using plots of the $\frac{dy}{dt}$ against y we obtain the stability of the fixed points as shown in Figure 2.

3, B

(b (ii)) The bifurcation diagram is sketched in Figure 3. At $r = 2$ there is saddle-node bifurcation. unseen ⇓

4, D

(b (ii)) From the bifurcation diagram it is evident that for $r < 2$, where there are no fixed points, $y(t)$ diverge to $+\infty$ as $t \to \infty$ regardless of initial values of $y(0)$. unseen ⇓

2, B

6 . (a) $\boxed{\text{meth seen} \Downarrow}$

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -3 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = -\frac{5}{2}, \quad \lambda_2 = -1.$$

We can obtain the corresponding eigenvectors.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So the solution in the vector form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{5}{2}t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}.$$

$\boxed{4, \text{A}}$

There is a stable fixed points at $(0, 0)$ as both eigenvalues are negative. $\boxed{2, \text{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(a (ii)) The phase portrait can be seen in Figure 4.

$\boxed{3, \text{B}}$

(a (iii)) We have $\tau = -\frac{7}{2}$ and $\Delta = \frac{5}{2} + \epsilon$. So, we see in the (τ, Δ) plane that as we vary ϵ at $\epsilon = -\frac{5}{2}$, we have $\Delta = 0$ and the fixed point changes stability from stable to unstable. $\boxed{3, \text{C}}$

(b (i)) Using the definition of the Jacobian of a transformation we have: $\boxed{\text{meth seen} \Downarrow}$

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}.$$

$\boxed{2, \text{A}}$

The infinitesimal element of area in the (u, v) coordinate system is:

$$dA = |det J|\, du\, dv = 4(u^2 + v^2)\, du\, dv.$$

$\boxed{2, \text{B}}$

$\boxed{\text{unseen} \Downarrow}$

(b (ii)) The partial derivatives requested are the elements of the first row of the matrix J^{-1}. So, by inverting J we have:

$$J^{-1} = \frac{1}{4(u^2 + v^2)} \begin{bmatrix} 2u & 2v \\ -2v & 2u \end{bmatrix}.$$

So we have:

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{2u}{4(u^2 + v^2)} \quad \text{and} \quad \left(\frac{\partial u}{\partial y}\right)_x = \frac{2v}{4(u^2 + v^2)}.$$

$\boxed{4, \text{D}}$

Review of mark distribution:

Total A marks: 24 of 32 marks

Total B marks: 15 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks

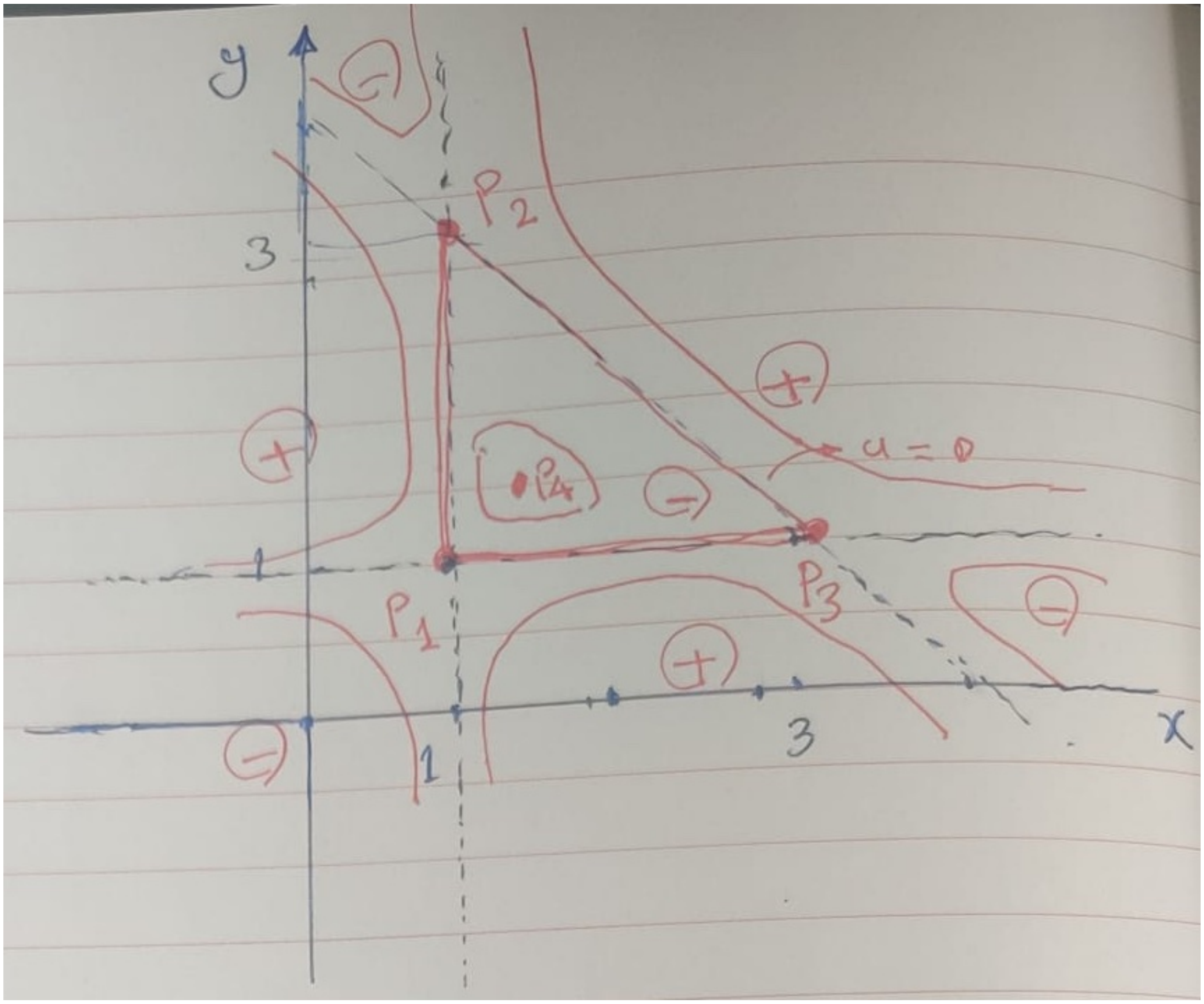

Figure 1: A contour plot for 5a(iii).

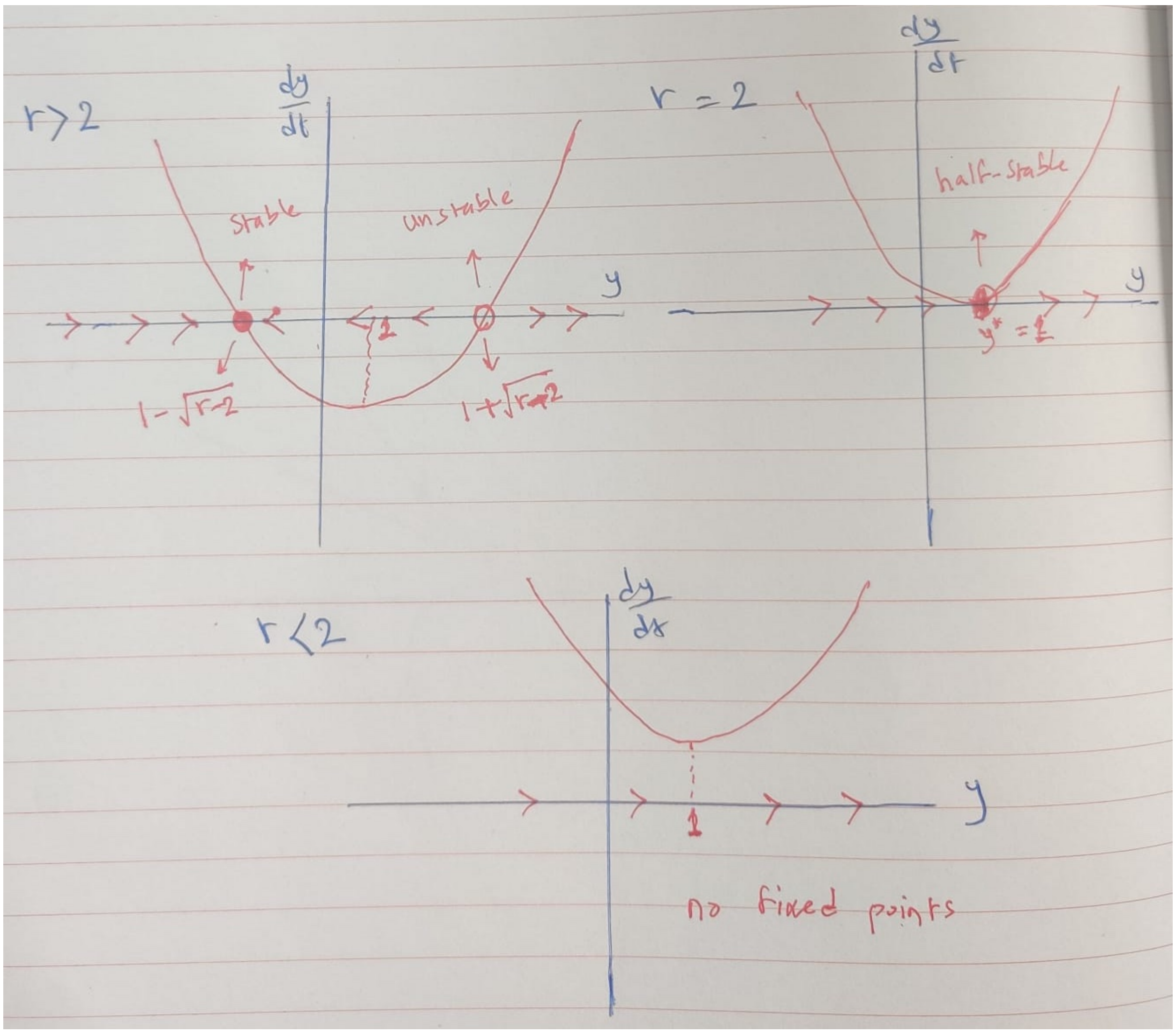

Figure 2: Stability diagrams for question 5b(i).

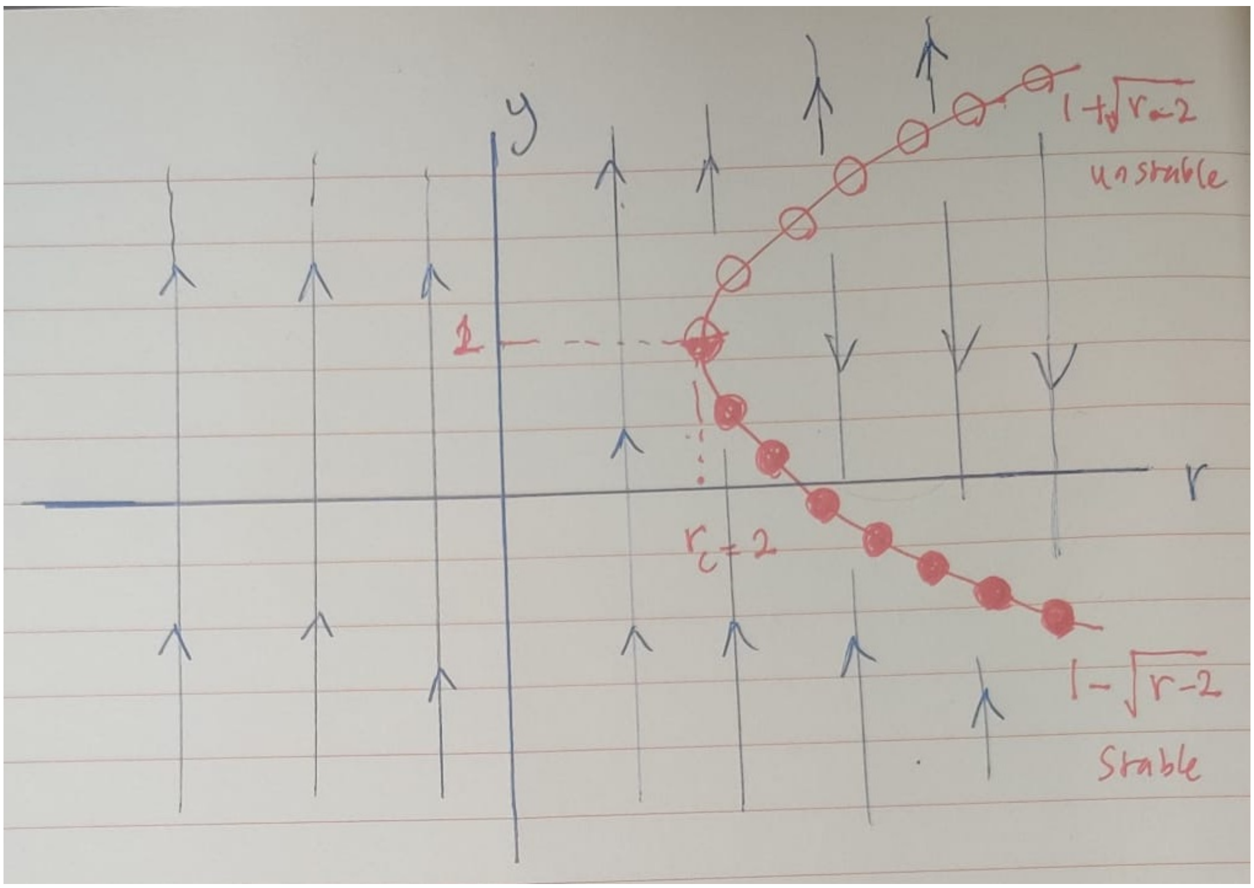

Figure 3: Bifurcation diagram for question 5b(ii).

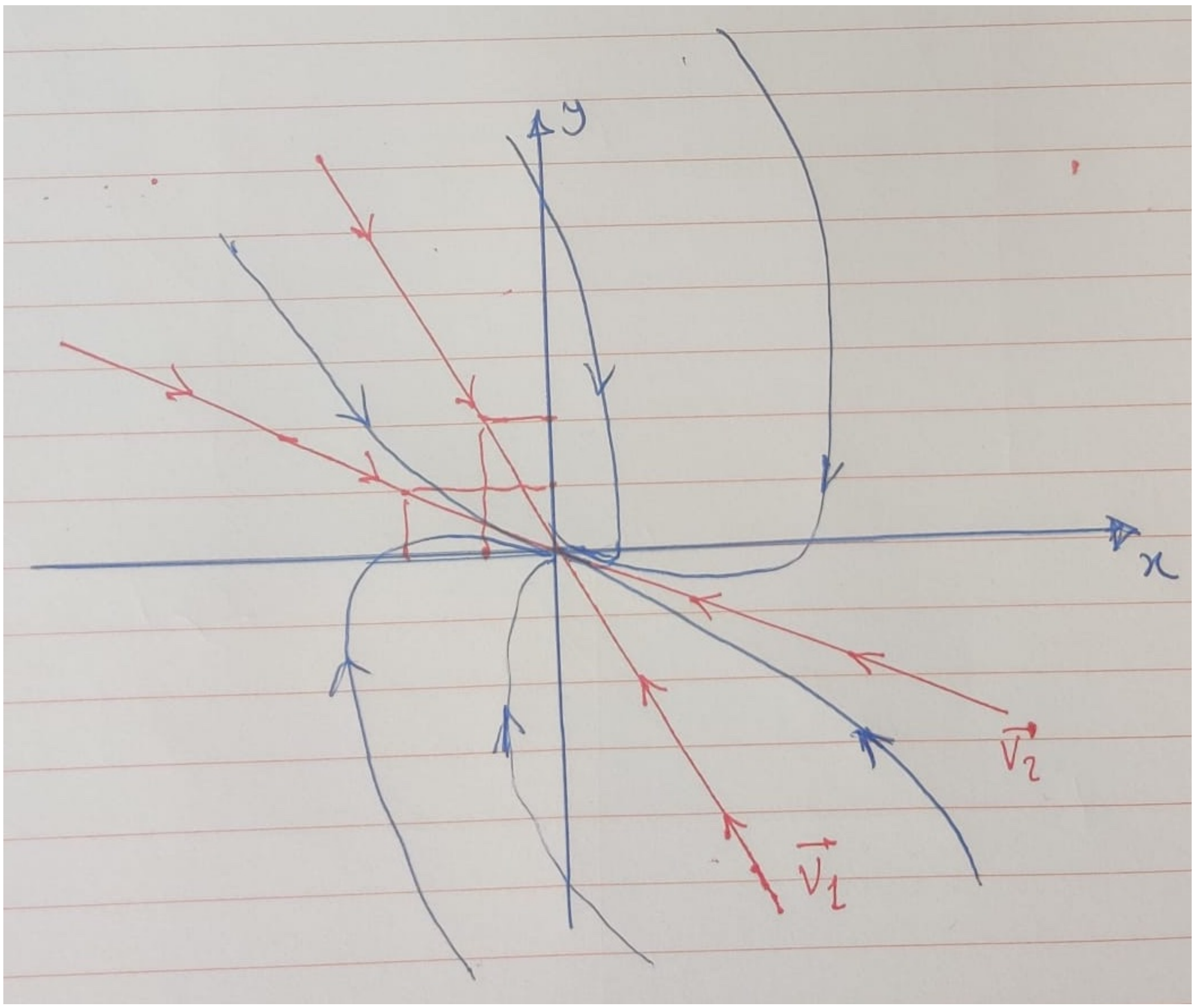

Figure 4: Phase portrait for question 6a(ii).