Applied Complex Analysis - Lecture Eleven

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The trapezium rule

Trapezium rule on the unit circle

$$I = f(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} dz$$

- Changing variables $z = e^{i\theta}$, for $\theta \in [0, 2\pi]$

$$I = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

- Trapezium rule approx:

$$I \approx I_N = \frac{1}{N} \sum_{j=1}^{N} f(z_j), \quad \text{where } z_j = e^{2\pi i j/N}, \quad \text{for } j = 1, \ldots, N$$

- **Thm**: Suppose f is analytic and satisfies $|f(z)| < M$ inside the complex disk $|z| < r$ for some $r > 1$. Then

$$|I - I_N| \leq \frac{M}{r^N - 1} = O(r^{-N}) \quad \text{as } N \to \infty.$$

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A useless example

$$\cos(0) = 1$$

- *Complex averaging* interpretation
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Application: Rounding errors

Double precision floating point numbers are stored as

$$(-1)^S \times 2^{E-1023} \times (1+M)$$

where one bit is allocated for the sign S, 11 bits are allocated for the exponent E, and 52 bits are allocated for the Manissa M, this gives us 16 digits of precision.

- Taking differences of small numbers is OK
- Taking differences of larger numbers can be an issue
- For example:

$$1.00000012345678 - 1.00000000000000 = 0.00000012345678$$

- Sometimes, lots of digits are needed!
- Errors are exaggerated when dividing by differences of large numbers
- Example - removable singularity:

$$f(z) = \frac{e^z - 1 - z}{z^2}, \quad f(10^{-8}) = 0.500000001666667$$

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Approximating derivatives

- A popular (but dodgy) way to approximate derivatives is to use a 'finite difference approach', for EG:

$$f'(z) = \frac{f(z+\epsilon) - f(z-\epsilon)}{2\epsilon}, \quad \text{for small } \epsilon.$$

- Choosing the right ϵ is difficult.
- Too big $\implies$ inaccurate
- Too small $\implies$ rounding errors (also inaccurate!)
- for high order derivatives this can get very messy.
- Thanks to Cauchy's integral formula, we can easily approximate derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^n} d\xi \approx \frac{n!}{N}\sum_{j=1}^{N} \frac{f(re^{2\pi i j/N})}{r^j e^{2\pi i \theta j(n-1)/N}}$$

- But, our current estimates cannot be applied here...

Trapezium rule for periodic analytic functions

Trapezium rule for periodic functions

Thm: For

$$I = \int_0^{2\pi} f(\theta)d\theta \approx I_N = \frac{1}{N}\sum_{n=1}^{N} f(\theta_n), \quad \theta_n = 2\pi n/N,$$

if f is 2π-periodic and analytic in the strip $S_a := \{\theta : -a < \mathrm{Im}\{\theta\} < a\}$ for $a > 0$, then

$$|I - I_N| \leq \frac{4\pi M}{e^{aN} - 1},$$

where $M := \sup_{\theta \in S_a} |f(\theta)|$.

- Convergence of real integral depends on complex behavior
- More general class: may be applied to non-circlular contour integrals $\oint_\gamma f$, where f is analytic in an annulus containing γ
- Requires the parametrisation of γ to be analytic
- Proof

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Exactness of trapezium rule for periodic functions

Thm: If a function f has a Laurent expansion of the form

$$f(z) = \sum_{j=-N}^{N-2} a_j (z - z_0)^j,$$

for z in some annulus D, then an N-point trapezium rule I_N can exactly approximate

$$I = \oint_\gamma f(z)dz,$$

where γ is a closed anti-clockwise-oriented contour in D.

- Analagous to N-point Gauss quadrature integrating degree $(2N-1)$-degree integrals exactly
- Not particularly useful - but a helpful way to check your code!
- Proof
- Example
- Numerical examples

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The argument principle

For f meromorphic and g analytic in Ω,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}g(z)dz = \sum_{a\in\{\text{zeros of } f\}} g(a)m_a - \sum_{b\in\{\text{poles of } f\}} g(b)m_b.$$

where m_a and m_b represent the order of the zeros and poles respectively, γ is a closed contour in Ω with no loops, such that $f(z) \neq 0$ for $z \in \gamma$.

- Proof
- Consequences and applications
- Approximation trick when $g = 1$

Application: Root-finding