Imperial College London

DEPARTMENT OF MATHEMATICS
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**Introduction to University Mathematics** **MATH40001/MATH40009**

## Part I – Solutions to Problem Sheet 2: Functions

1. First consider the sets $X = \{1\}$, $Y = \{2,3\}$ and $Z = \{4\}$, and consider $f : X \to Y$ defined by $f(1) = 2$ and $g : Y \to Z$ defined by $g(2) = g(3) = 4$ (in fact this is the only function from $Y$ to $Z$). Then $g \circ f$ is the bijection from $X$ to $Z$ sending 1 to 4, so $g \circ f$ is both injective and surjective. Because $f$ isn't surjective and $g$ isn't injective, this gives counterexamples to (b)and (c), so if you thought you had proved these two then I would recommend comparing your proof with this counterexample and trying to figure out what went wrong.

   However (a) and (d) are true, and here are the proofs.

   (a) Say $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. We want to prove $x_1 = x_2$. We know $f(x_1) = f(x_2)$ so by applying $g$ we deduce $g(f(x_1)) = g(f(x_2))$ or, in other words, $(g \circ f)(x_1) = (g \circ f)(x_2)$. By injectivity of $g \circ f$ we deduce that $x_1 = x_2$, which is what we wanted to prove.

   (d) We want to prove that for all $z \in Z$ there exists $y \in Y$ with $g(y) = z$, so let $z \in Z$ be arbitrary. By surjectivity of $g \circ f$, there exists $x \in X$ with $(g \circ f)(x) = z$, or equivalently $g(f(x)) = z$. Setting $y = f(x)$ this implies $g(y) = z$, which is what we had to show.

2. (a) This function is bijective. Perhaps the easiest way of checking this is to observe that for all $x$, $f(f(x)) = x$ (or in other words, $f \circ f$ is the identity function). Indeed, if $x \neq 0$ then $1/x \neq 0$ and so $f(f(x)) = 1/(1/x)) = x$, and if $x = 0$ then $f(f(0)) = f(0) = 0$. This means that $f$ is its own two-sided inverse, and any function with a two-sided inverse is a bijection.

   (b) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{y-1}{2}$. Then for all $x \in \mathbb{R}$ we have $g(f(x) = \frac{(2x+1)-1}{2} = x$, and for all $y \in \mathbb{R}$ we have $f(g(y)) = 2(\frac{y-1}{2}) + 1 = y$. This means that $g$ is a two-sided inverse for $f$ and hence $f$ is a bijection. Note that we really do have to do both of these calulations, they feel similar but they're different – remember that $\sin^{-1}$ is only a one-sided inverse for $\sin : \mathbb{R} \to [-1,1]$.

   (c) The function from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x+1$ is injective, because if $2x_1+1 = 2x_2+1$ then subtracting one and dividing by $2$ we deduce $x_1 = x_2$. However it is not surjective, as if $n$ is an integer then $f(n) = 2n+1$ is always odd, so if $y$ is, for example, the even integer $2$ then there is no solution to $f(x) = y$ with $x$ in our codomain.

   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3-x$ if the Riemann hypothesis is true, and $f(x) = 2+x$ if not. This function is bijective, and the two-sided inverse function is $g$ defined by $g(y) = 3-y$ if the Riemann hypothesis is true, and $g(y) = y-2$ if it is false. A case by case check shows that whether or not the Riemann Hypothesis is true, $f \circ g$ and $g \circ f$ are both the identity function.

   (e) The function $f(n) = n^3-2n^2+2n-1$ is injective but not surjective. Indeed, $f(n+1)-f(n) = 3n^2+3n+1-4n-2+2 = 3n^2-n+1 = 3(n-\frac{1}{6})^2+\frac{11}{12} > 0$ for every integer $n$, and hence $f(n+1) > f(n)$ for every integer $n$. Now an easy induction on $m$ implies that $f(n+m) > f(n)$ for every positive integer $m$. As a consequence we deduce that if $p, q$ are integers with $p < q$ then, setting $n = p$ and $m = q-p > 0$, we deduce $f(p) < f(q)$. In partiular if $a \neq b$ then we can prove $f(a) \neq f(b)$, because without loss of generality $a < b$ and then writing $a = n$ and $b = n+m$ with $m > 0$ we deduce $f(a) < f(b)$. This does injectivity.

   Surjectivity is not true. One checks easily that $f(1) = 0$ and $f(2) = 3$, and now I claim that there cannot exist any integer $n$ with $f(n) = 2$. Indeed, we've just seen that $n = 1$ and $n = 2$ don't work, and if $n > 2$ then $f(n) > f(2) = 3 > 2$, and similarly if $n < 1$ then $f(n) < f(1) = 0 < 2$, so no cases can work.

3. (a) $f$ is not defined at zero, as $1/0$ is not a real number.
   (b) The question specifies that the domain and codomain are both $\mathbb{R}$, but if the input $x$ is negative then $\sqrt{x}$ is not a real number.
   (c) $f(0) = 1/2$ which is not in the codomain.
   (d) We don't say which solution, and there are sometimes more than one (for example $y^3 - y = 0$ has three solutions for $y$). If we were to give a careful recipe saying exactly which one we always choose when there is a choice, then we could make this into a function, but as it stands there's not enough information.
   (e) You might think $1 + x + x^2 + x^3 + \cdots = 1/(1-x)$, but this is only true for $|x| < 1$. If $x = 10$, for example, then this function is not well-defined, because the sum diverges to infinity (which is not a real number).

4. The meaning of "$g$ is a two-sided inverse for $f$" is $(\forall x \in X, g(f(x)) = x) \wedge (\forall y \in Y, f(g(y)) = y)$. The meaning of "$f$ is a two-sided inverse for $g$" is $(\forall y \in Y, f(g(y)) = y) \wedge (\forall x \in X, g(f(x)) = x)$, so the result follows because $P \wedge Q \iff Q \wedge P$ ;-)

   From the lecture notes we know that a function is a bijection if and only if it has a two-sided inverse. Hence if $f$ is a bijection, and $g$ is a two-sided inverse, then $f$ is a two-sided inverse for $g$, and hence $g$ is a bijection.

5. This question is an $\iff$ question so we have to prove both implications. Note: this question is very long.

   Let's first prove that if $f$ is friends with $g$ then the ranges $\mathrm{range}(f)$ and $\mathrm{range}(g)$ of $f$ and $g$ are equal. Recall that $\mathrm{range}(f)$ is defined to be $\{z \in Z \mid \exists x \in X, f(x) = z\}$, and that some people call this the image of $f$. To prove that $\mathrm{range}(f) = \mathrm{range}(g)$ we need to show $\forall z \in Z$, $z \in \mathrm{range}(f) \iff t \in \mathrm{range}(g)$, which is again an $\iff$ question, so we again have two jobs to do. We are assuming that $f$ is friends with $g$, so let's choose a bijection $h : X \to Y$ such that $f = g \circ h$.

   Let $z \in Z$ be arbitrary. First we prove that $z \in \mathrm{range}(f) \implies z \in \mathrm{range}(g)$ (assuming that $f$ is friends with $g$). Well, $z \in \mathrm{range}(f)$ implies that there exists $x \in X$ with $f(x) = z$. Define $y = h(x)$. It's now a straightforward calculation to see that $z = g(y)$, because $g(y) = g(h(x)) = (g \circ h)(x) = f(x) = z$. In particular, this means $z \in \mathrm{range}(g)$, which was what we wanted.

   The other way is a bit trickier. We know that $h$ is a bijection, so it has a two-sided inverse function $h^{-1} : Y \to X$. We have $z \in Z$ arbitrary, and we want to prove $z \in \mathrm{range}(g) \implies z \in \mathrm{range}(f)$. So let's assume $z \in \mathrm{range}(g)$, and let's choose $y \in Y$ with $g(y) = z$. Now let's set $x = h^{-1}(y)$. Then $h(x) = h(h^{-1}(y)) = y$ because of the definition of two-sided inverse, and applying $g$ we deduce that $g(h(x)) = g(y) = z$. Because $f = g \circ h$ we deduce $f(x) = z$, and hence $z$ is in the image of $f$, which is what we wanted.

   We are now half way through the question. We now need to go the other way, and prove that if $\mathrm{range}(f) = \mathrm{range}(g)$ then $f$ is friends with $g$. To do this we are going to have to do more than prove something, we are going to have to construct a function $h : X \to Y$ and then prove that it is a bijection and that $f = g \circ h$. So let's start by constructing this function from $X$ to $Y$. Let $x \in X$ be arbitrary. We need to construct some element $h(x) \in Y$. First we observe that $f(x)$ is in the image of $f$, and hence, by assumption, in the image of $g$. This means that there exists some $y \in Y$ such that $g(y) = f(x)$. Furthermore, such a $y$ is unique, because $g$ is injective; if $g(y_1) = g(y_2) = f(x)$ then $y_1 = y_2$. So we can define $h(x) = y$.

   We now need to prove that $h$ is a bijection, and that $f = g \circ h$. Let's first prove that $f = g \circ h$. Say $x \in X$ is arbitrary. By definition of $h(x)$, we know that $g(h(x)) = f(x)$. But this just says that $(g \circ h)(x) = f(x)$. Because $x$ was arbitrary, we have proved that $g \circ h = f$.

   Finally we need to prove that $h$ is a bijection. Injectivity is not so hard – if $x_1, x_2 \in X$ and $h(x_1) = h(x_2)$, then applying $g$ we deduce that $g(h(x_1)) = g(h(x_2))$ and hence $f(x_1) = f(x_2)$. But $f$ is assumed injective, and hence $x_1 = x_2$.

The only thing left is surjectivity of $h$. So say $y \in Y$ is arbitrary. Then $g(y)$ is in the image of $g$, which by assumption is the image of $f$. Hence there exists some $x \in X$ with $f(x) = g(y)$. I claim that $h(x) = y$. How do we know this? Well, $g$ is injective, so it suffices to prove that $g(h(x)) = g(y)$, but $g(h(x)) = f(x)$, and we know that $f(x) = g(y)$ is true, so we are done.

6. One way of thinking about it: imagine building a subset of $T$. for each of the $n$ elements of $T$ we need to make an independent yes/no decision about whether it's in the subset or not. That is $n$ independent choices of two things, so there are $2^n$ ways of making those choices.

   Another way of thinking about it: a subset of $T$ is the same thing as a predicate on $T$ which is a map from $T$ to $\{\texttt{true}, \texttt{false}\}$. And the number of maps from a set of size $n$ to a set of size 2 is $2^n$ by a result from the course.

7. (a) Assume for a contradiction that $X = f(t)$ for some $t \in \alpha$. The key insight (which is hard to spot!) is that now $t \in f(t)$ iff $t \in X$ (by defininition of $t$) iff $t \notin f(t)$ (by definition of $X$). However $t \in f(t)$ and $t \notin f(t)$ are opposites of other so they can never be logically equivalent, a contradiction. This proves that our assumption that $X$ is not in the range of $f$.
   (b) If $f : \alpha \to \mathcal{P}(\alpha)$ was a surjection then the $X$ defined in the previous part would have to be in the range of $f$, because the range of $f$ is all of $\mathcal{P}(\alpha)$ for a surjection. This contradicts the previous part.
   (c) If $2^n \leq n$ for some natural number $n$ then letting $\alpha$ be a set of size $n$ (for example $\{1, 2, 3, \ldots, n\}$), the size of $\mathcal{P}(\alpha)$ would be $2^n$ (by the previous question), and so if $2^n \leq n$ we would be able to write down a surjection from $\alpha$ to $\mathcal{P}(\alpha)$, and this contradicts the previous part of this question.

8. (a) $g_*(\{1\})$ is the $b \in \{8, 9, 10\}$ for which there exists some $a \in \{1\}$ with $g(a) = b$. The only possible value of $a$ is $a = 1$, and $g(1) = 8$, so $f_*(\{1\}) = \{8\}$. Similarly for $g_*(\{1, 2\})$ the only possible values of $a$ ar $a = 1$ and $a = 2$, and $g(1) = g(2) = 8$, so again $g_*(\{1, 2\}) = \{8\}$.
   (b) $g_*$ is not injective because we just wrote down two distinct subsets of $\mathcal{P}(\alpha)$ which get sent to the same element of $\mathcal{P}(\beta)$ by $g_*$.
   (c) $g_*$ is not surjective either. One way to show this would be by counting: the domain of $g_*$ has size $2^2 = 4$ and the codomain has size $2^3 = 8$ which is larger than 4 so there are no surjective functions from $\mathcal{P}(\alpha)$ to $\mathcal{P}(\beta)$.
   (d) $g^*(\{8\})$ is the elements $a$ of $\alpha$ such that $g(a) \in \{8\}$ or, in other words, such that $g(a) = 8$. Trying all the elements of $\alpha$ we see that both $a = 1$ and $a = 2$ works. So $g^*(\{8\}) = \{1, 2\}$. Similarly $g^*(8, 9)$ is the elements $a$ such that $g(a) = 8$ or $g(a) = 9$, and again both $a = 1$ and $a = 2$ work. So $g^*(\{8, 9\}) = \{1, 2\}$ as well.
   (e) $g^*$ is not injective because we just saw two distinct elements of $\mathcal{P}(\beta)$ which got mapped to the same element of $\mathcal{P}(\alpha)$.
   (f) $g^*$ is not surjective either, although this is a little trickier. I claim that there can be no subset $Y$ of $\beta$ such that $g^*(Y) = \{1\}$. Let's prove it by contradiction. Assume that such a subset $Y$ existed. Then $1 \in g^*(Y)$ so by definition of $g^*$ we know that $g(1) \in Y$. Hence $8 \in Y$ so $g(2) \in Y$, and so again by definition of $g^*$ we have $2 \in g^*(Y)$. But this is a contradiction because $g^*(Y) = \{1\}$.
   (g) No they certainly are not, because two-sided inverses of functions are bijections and neither $g_*$ nor $g^*$ are bijections.

9. We have $b \in f_*(X) \iff \exists a \in X, f(a) = b$, so $f_*(X) = \{f(a) \,|\, a \in X\}$. The assertion $f_*(X) \subseteq Y$ thus means $\forall a \in X, f(a) \in Y$. Similarly $X \subseteq f^*(Y)$ means $\forall a \in X, a \in f^*(Y)$, and $a \in f^*(Y)$ means $f(a) \in Y$ by definition of $f^*$. Hence the two statements are both equivalent to $\forall a \in X, f(a) \in Y$ and are thus logically equivalent to each other.

10. By the previous part, $X \subseteq f^*(f_*(X)) \iff f_*(X) \subseteq f_*(X)$, and the latter statement is obviously true (because $P \implies P$ for all propositions $P$). Similarly $f_*(f^*(Y)) \subseteq Y \iff f^*(Y) \subseteq f^*(Y)$ which again is obviously true.