MATH50004/MATH50015/MATH50019 Differential Equations
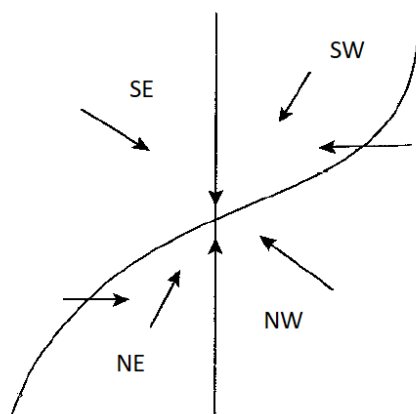## Spring Term 2023/24
Solutions to Problem Sheet 7

Exercise 31.

(i) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote the right hand side. All equilibria (x, y) are zeros of f and thus need to satisfy $-x = 0$ and $-2y + 2x^3 = 0$. It follows directly that $(x, y) = (0, 0)$, so this is the only equilibrium.

(ii) We calculate

$$f'(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

which is already given in Jordan normal form, and we see that -1 and -2 are the eigenvalues. The theorem of linearised stability implies that the equilibrium is exponentially stable and thus attractive.

(iii) The nullcline $\dot{x} = 0$ is given by $x = 0$ and the nullcline $\dot{y} = 0$ is given by $y = x^3$, and illustrated in the following figure (apologies to the $y = x^3$ function that is not plotted properly, but that does not matter for our analysis).

The two nullclines separate each other into four parts:
(N1): $x = 0$ and $y > 0$ (on which $\dot{x} = 0$ and $\dot{y} < 0$),
(N2): $x = 0$ and $y < 0$ (on which $\dot{x} = 0$ and $\dot{y} > 0$),
(N3): $y = x^3$ and $x > 0$ (on which $\dot{y} = 0$ and $\dot{x} < 0$),
(N4): $y = x^3$ and $x < 0$ (on which $\dot{y} = 0$ and $\dot{x} > 0$).

They also divide the plane into four regions:
(SW): $x > 0$ and $y > x^3$ (on which $\dot{x} < 0$ and $\dot{y} < 0$),
(SE): $x < 0$ and $y > x^3$ (on which $\dot{x} > 0$ and $\dot{y} < 0$),
(NE): $x < 0$ and $y < x^3$ (on which $\dot{x} > 0$ and $\dot{y} > 0$),
(NW): $x > 0$ and $y < x^3$ (on which $\dot{x} < 0$ and $\dot{y} > 0$).

(iv) As can be see clearly from the arrows in the regions and on the nullclines, region SW is positively invariant, but not negatively invariant (and thus also not invariant). Region SE is negatively invariant, but not positively invariant (and thus also not invariant). Region NE is positively invariant, but not negatively invariant (and thus not invariant). Region NW is negatively invariant, but not positively invariant (and thus not invariant). This analysis also implies that forward in time,

- we cannot leave SW (for all points),

- we can move from SE to NE (for certain points, not necessarily all points),

- we cannot leave NE (for all points),

- we can move from NW to SW (for certain points, not necessarily all points).

It is possible to show that a periodic orbit needs to intersect all four regions (by means of index theory, one can show that the vector field along a periodic orbit rotates exactly once within a period), but we do not show this here. In our situation, this is not possible, so no periodic orbit exists (note that it will follow also from (v)).

(v) Note first that Exercise 16 (ii) implies that if a solution stays in one of the four regions forward in time and is bounded forward in time, then it must converge to an equilibrium. This follows, since in all regions, each component of the solutions are monotone (e.g. in SW, first and second component decreasing), and if a solution is bounded forward in time, then it must converge, since both components must converge. Then Exercise 16 (ii) yields that it converges to an equilibrium.

If the flow starts in either SW or NE, then we stay in these areas, and it is also clear from the picture and the arrows that the flow is bounded forward in time in both regions (no further proof required, because the picture is accurate). So we converge using Exercise 16 (ii) to the only equilibrium $(0,0)$, since we stay in the region. If we start in region NW, then we either stay in this region (and in this case we also converge to $(0,0)$ with the same reasoning as above), or we can move to SW, in which we will stay, since this region is positively invariant, and, as shown above, we move to $(0,0)$. A similar argument shows that when starting in SE, we also have to converge to $(0,0)$ (either by staying in the region, or by moving into the positively invariant region NE).

Exercise 32.

We first show that hint and suppose that $\lambda : I \to \mathbb{R}^d$ is a solution of $\dot{x} = f(x)$. Define $\mu(t) := T^{-1}\lambda(t)$ for all $t \in I$. Then

$$\dot{\mu}(t) = T^{-1}\dot{\lambda}(t) = T^{-1}f(\lambda(t)) = T^{-1}f(T\mu(t)) \quad \text{for all } t \in I \,,$$

which shows that μ is a solution of $\dot{y} = T^{-1}f(Ty)$. The other direction of the hint is shown by replacing T^{-1} by T and repeating these arguments. Denote the flow of $\dot{x} = f(x)$ by φ and the flow of $\dot{y} = T^{-1}f(Ty)$ by $\bar{\varphi}$, and the hint implies that $\bar{\varphi}(t,y) = T^{-1}\varphi(t,Ty)$.

Let x^* be a stable equilibrium of $\dot{x} = f(x)$. This implies that $f(x^*) = 0$. Thus, $g(y^*) = g(T^{-1}x^*) = T^{-1}f(x^*) = 0$, so y^* is an equilibrium of $\dot{y} = T^{-1}f(Ty)$.

To prove that y^* is stable for $\dot{y} = T^{-1}f(Ty)$, let $\varepsilon > 0$. Since x^* is stable, there exists $\bar{\delta} > 0$ such that $x \in B_{\bar{\delta}}(x^*)$ implies

$$\|\varphi(t,x) - x^*\| < \frac{\varepsilon}{\|T^{-1}\|} \quad \text{for all } t \geq 0 \,. \tag{A}$$

Define $\delta := \frac{\bar{\delta}}{\|T\|}$. Choose $y \in B_\delta(y^*)$. We get $\|Ty - x^*\| = \|T(y - y^*)\| \leq \|T\|\|y - y^*\| < \|T\|\delta = \bar{\delta}$. Then we get from (A) that

$$\|\varphi(t,Ty) - x^*\| < \frac{\varepsilon}{\|T^{-1}\|} \quad \text{for all } t \geq 0 \,,$$

and it follows that

$$\|\bar{\varphi}(t,y) - y^*\| = \|T^{-1}\varphi(t,Ty) - T^{-1}x^*\| = \|T^{-1}(\varphi(t,Ty) - x^*)\| \leq \|T^{-1}\|\|\varphi(t,Ty) - x^*\| < \varepsilon$$

for all $t \geq 0$, which proves that y^* is stable.

Exercise 33.

x^* is attractive, and this means that there exists a $\delta > 0$ such that

$$\lim_{t \to \infty} \varphi(t, x) = x^* \quad \text{for all } x \in B_\delta(x^*). \tag{B}$$

To prove that $W^s(x^*)$ is open, let $z \in W^s(x^*)$. Since $\lim_{t \to \infty} \varphi(t, z) = x^*$, there exists a $\tau > 0$ such that $\varphi(\tau, z) \in B_\delta(x^*)$. Since $B_\delta(x^*)$ is open, there exists a $\varepsilon > 0$ such that $B_\varepsilon(\varphi(\tau, z)) \subset B_\delta(x^*)$. Since φ is a continuous function, the function $\varphi(\tau, \cdot)$ is continuous, and hence, there exists a $\gamma > 0$ such that

$$\varphi(\tau, B_\gamma(z)) \subset B_\varepsilon(\varphi(\tau, z)) \subset B_\delta(x^*).$$

So for any $y \in B_\gamma(z)$, we get $\varphi(\tau, y) \in B_\delta(x^*)$, and thus (C) implies that

$$\lim_{t \to \infty} \underbrace{\varphi(t, \varphi(\tau, y))}_{=\varphi(t+\tau, y)} = x^*,$$

where we have used the group property of φ. This means that $y \in W^s(x^*)$ and $W^s(x^*)$ is open for this reason.

Exercise 34.

Firstly note that ∂M can be empty, and nothing needs to be shown in this case. It is important that we consider the metric space D (as a subset of $\mathbb{R}^d$, inheriting the metric from $\mathbb{R}^d$) in this question. Thus, the boundary ∂M of M needs to be taken in the topology of the open set D, and we have $\partial M = \partial_{\mathbb{R}^d} M \cap D$, where $\partial_{\mathbb{R}^d} M$ is the boundary of M in $\mathbb{R}^d$.

Let $x \in \partial M$ and assume, there exists a $\tau \in J_{max}(x)$ such that $\varphi(\tau, x) \notin \partial M$. Since ∂M is closed (in the open set D), there exists an $\varepsilon > 0$ such that either

$$B_\varepsilon(\varphi(\tau, x)) \subset \text{int } M \quad \text{or} \quad B_\varepsilon(\varphi(\tau, x)) \subset D \setminus \overline{M}, \tag{B}$$

where int denotes the interior of M (i.e. the set of inner points of M). Continuity of $\varphi(\tau, \cdot)$ then implies that there exists a $\delta > 0$ such that

$$\varphi(\tau, B_\delta(x)) \subset B_\varepsilon(\varphi(\tau, x)). \tag{C}$$

We consider the two cases from (B).

Case 1. $B_\varepsilon(\varphi(\tau, x)) \subset \text{int } M$.
(C) implies $\varphi(-\tau, \varphi(\tau, B_\delta(x))) \subset \varphi(-\tau, B_\varepsilon(\varphi(\tau, x)))$, which gives

$$B_\delta(x) \subset \varphi(-\tau, B_\varepsilon(\varphi(\tau, x))) \subset M,$$

since M is invariant. This contradicts $x \in \partial M$.

Case 2. $B_\varepsilon(\varphi(\tau, x)) \subset D \setminus \overline{M}$.
Since $x \in \partial M$, there exists a $y \in B_\delta(x) \cap M$. (C) implies that $\varphi(t, y) \in B_\varepsilon(\varphi(\tau, x)) \subset D \setminus \overline{M}$. This contradicts the invariance of M and finishes the proof.

Exercise 35.

Denote by $\lambda_{max} : I_{max}(t_0, x_0) \to \mathbb{R}^d$ the maximal solution to the given initial value problem. We only show that $I_+(t_0, x_0) = \sup J$ and note that proving $I_-(t_0, x_0) = \inf J$ is analogous. Due to Proposition 2.1, we have

$$\lambda_{max}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_{max}(s)) \, ds \quad \text{for all } t \in I_{max}(t_0, x_0).$$

The triangle inequality and Lemma 2.9 imply

$$
\begin{aligned}
\|\lambda_{max}(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|f(s, \lambda_{max}(s))\| \, \mathrm{d}s \\
&\leq \|x_0\| + \int_{t_0}^{t} \big(\rho(s)\|\lambda_{max}(s)\| + \sigma(s)\big) \, \mathrm{d}s \\
&\leq \underbrace{\|x_0\| + \int_{t_0}^{t} \sigma(s) \, \mathrm{d}s}_{=:\alpha(t)} + \int_{t_0}^{t} \underbrace{\rho(s)}_{=:\beta(s)} \|\lambda_{max}(s)\| \, \mathrm{d}s \, .
\end{aligned}
$$

Assume now that $I_+(t_0, x_0) < \sup J$. Then the continuous functions α and β have a maximum on the compact set $[t_0, I_+(t_0, x_0)]$, given by c and d, respectively, and we get

$$
\|\lambda_{max}(t)\| \leq c + d \int_{t_0}^{t} \|\lambda_{max}(s)\| \, \mathrm{d}s \quad \text{for all } t \in [t_0, I_+(t_0, x_0)) \, .
$$

To this inequality, the Gronwall lemma can be applied with $u(t) := \|\lambda_{max}(t)\|$, and we get

$$
\|\lambda_{max}(t)\| \leq c e^{d(t-t_0)} \quad \text{for all } t \in [t_0, I_+(t_0, x_0)) \, . \tag{D}
$$

Then the maximal solution λ_{max} stays bounded on $[t_0, I_+(t_0, x_0))$, since the right hand side of (D) takes a maximum on the compact interval $[t_0, I_+(t_0, x_0)]$ (note that we assumed $I_+(t_0, x_0) < \sup J$). Then Theorem 2.17 (i) implies that λ_{max} needs to converge to the boundary of $D = J \times \mathbb{R}^d$, but this is not possible due to

$$
\big([t_0, \infty) \times \mathbb{R}^d\big) \cap \partial D = \begin{cases} \{\sup J\} \times \mathbb{R}^d & : \quad \sup J < \infty \, , \\ \emptyset & : \quad \sup J = \infty \, . \end{cases}
$$

This contradiction shows $I_+(t_0, x_0) = \sup J$ and finishes the proof.