Imperial College London

MATH97029 MATH97106

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Asymptotic Analysis

Date: Monday, 10 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider the small positive root x^* of the equation

$$x \ln \frac{1}{x} = \epsilon \quad \text{as} \quad \epsilon \searrow 0. \tag{1}$$

(i) By taking the logarithm of both sides of (1), show that $\ln x^* \sim \ln \epsilon$. Can a leading-order approximation for x^* be deduced from this result? Explain.

(3 marks)

(ii) Find a two-term asymptotic expansion for x^*. *Clue:* make the substitution $x = \epsilon X$.

(7 marks)

(b) Find leading-order asymptotic approximations for the solutions of

$$\frac{d^2y}{dx^2} - \frac{1}{x}y = 0 \quad \text{as} \quad x \to +\infty.$$

(10 marks)

(Total: 20 marks)

2. Consider the integral

$$I(x) = \int_0^1 (t - t^2) e^{ix(t-t^2)}\, dt \quad \text{as} \quad x \to +\infty.$$

(a) Derive a leading-order approximation using the method of stationary phase.

(6 marks)

(b) Derive a two-term asymptotic expansion using the method of steepest descent.

Include a schematic showing the constant-phase paths through the boundary and saddle points and indicate your choice for the integration contour.

(10 marks)

(c) Is the next term also a saddle contribution? Estimate its asymptotic order.

(4 marks)

Useful integrals:

$$\int_{-\infty}^{\infty} e^{-ixp^2}\, dp = e^{-i\pi/4}\sqrt{\frac{\pi}{x}},$$

$$\int_{-\infty}^{\infty} e^{-xp^2}\, dp = \sqrt{\frac{\pi}{x}}, \qquad \int_{-\infty}^{\infty} p^2 e^{-xp^2}\, dp = \frac{\sqrt{\pi}}{2x^{3/2}}.$$

(Total: 20 marks)

3. Consider the boundary-value problem

$$\epsilon\frac{d^2y}{dx^2} + (2-x)\frac{dy}{dx} = -1, \quad y(0) = 0, \quad y(1) = 0. \tag{2}$$

Your goal is to find asymptotic approximations to the boundary derivative

$$q = \left.\frac{dy}{dx}\right|_{x=0} \quad \text{as} \quad \epsilon \searrow 0.$$

(a) Show that an outer approximation cannot be made to vanish at both boundary points.

(3 marks)

(b) Determine the location and scaling of the boundary layer.

(4 marks)

(c) Give a scaling argument for the asymptotic order of q.

(3 marks)

(d) Obtain an approximation for q to ord(1).

(10 marks)

Note that one particular solution of

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} = 2Xe^{-2X}$$

is

$$\int_0^X s^2 e^{-2s}\,dt,$$

which equals $1/4$ for $X = \infty$.

(Total: 20 marks)

4. Consider the nonlinear oscillator described by the ordinary differential equations

$$\frac{d^2x}{dt^2} + 2\epsilon y\frac{dx}{dt} + x = 0, \quad \frac{dy}{dt} = \epsilon x^2.$$

Use the method of multiple scales to obtain a leading-order approximation in the form

$$x(t;\epsilon) \sim a(\epsilon t)\cos[t + \phi(\epsilon t)], \quad y(t;\epsilon) \sim b(\epsilon t),$$

which is valid for $t = O(1/\epsilon)$ as $\epsilon \searrow 0$. Specifically,

(a) Derive a system of differential equations governing the slow-time dynamics of the amplitude a, phase ϕ and damping coefficient b. You do not need to find the general solution.

(15 marks)

(b) Qualitatively describe the dynamics of the amplitude a depending on the *initial* sign of the damping coefficient b. It may be useful to consider the quantity

$$E = \frac{1}{2}a^2 + b^2.$$

.

(5 marks)

(Total: 20 marks)

5. Consider the boundary-value problem

$$\frac{d^2y}{dx^2} - \epsilon^2 y = \frac{1}{x^2} + \frac{1}{x^3}, \quad y(1) = 0, \quad \lim_{x\to+\infty} y = 0. \tag{3}$$

Derive the expansion

$$\left.\frac{dy}{dx}\right|_{x=1} \sim \alpha + \epsilon \ln \frac{1}{\epsilon} + \epsilon\beta \quad \text{as} \quad \epsilon \searrow 0,$$

where α and β are constants that you need to identify.

You may assume that the equation

$$\frac{d^2Y}{dX^2} - Y = \frac{1}{X^2}$$

possesses a particular solution $G(X)$ that decays as $X \to +\infty$ and satisfies

$$G(X) = -\ln X - \gamma + O(X^2 \ln X) \quad \text{as} \quad X \searrow 0,$$

where $\gamma \doteq 0.5722\ldots$ is Euler's constant.

(20 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH96020/MATH97029/MATH97106

Asymptotic analysis (Solutions)

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1. (a) (i) Taking the logarithm gives $\boxed{\text{sim. seen} \Downarrow}$

$$\ln x + \ln \ln \frac{1}{x} = \ln \epsilon. \tag{1}$$ $\boxed{3, \text{A}}$

On the left-hand side, the $\ln x$ term is dominant for small x. The problem is that the dominant balance $\ln x \sim \ln \epsilon$ does not uniquely determine x to leading asymptotic order. In particular, it does not imply that $x \sim \epsilon$.

$\boxed{\text{meth seen} \Downarrow}$

(ii) Making the substitution $x = \epsilon X$, the original equation becomes $\boxed{7, \text{B}}$

$$X \left(\ln \frac{1}{\epsilon} + \ln \frac{1}{X} \right) = 1. \tag{2}$$

It is easy to see that the only consistent dominant balance is

$$X \sim \frac{1}{\ln \frac{1}{\epsilon}}. \tag{3}$$

It follows that

$$X = \frac{1}{\ln \frac{1}{\epsilon}} + \bar{X}, \quad \text{where} \quad \bar{X} \ll \frac{1}{\ln \frac{1}{\epsilon}}. \tag{4}$$

Substituting into (2) gives

$$\left(1 + \bar{X} \ln \frac{1}{\epsilon}\right) \ln \left(\frac{1}{\ln \frac{1}{\epsilon}} + \bar{X} \right) = \bar{X} \ln^2 \frac{1}{\epsilon}. \tag{5}$$

Given the constraint on $\bar{X}$, the dominant balance is

$$\bar{X} \sim -\frac{\ln \ln \frac{1}{\epsilon}}{\ln^2 \frac{1}{\epsilon}}. \tag{6}$$

We therefore find the two-term expansion

$$x \sim \frac{\epsilon}{\ln \frac{1}{\epsilon}} - \epsilon \frac{\ln \ln \frac{1}{\epsilon}}{\ln^2 \frac{1}{\epsilon}}. \tag{7}$$

$\boxed{\text{meth seen} \Downarrow}$

(b) We look for solutions in the form $y = e^s$ and define $p = ds/dx$. A leading-order approximation for y requires s to ord(1). $\boxed{10, \text{A}}$

The ODE gives the following equation for p:

$$p^2 + \frac{dp}{dx} = \frac{1}{x}. \tag{8}$$

The only consistent dominant balance is $p^2 \sim 1/x$, hence we have two solutions:

$$p_\pm = \pm \frac{1}{x^{1/2}} + \bar{p}_\pm, \quad \text{where} \quad \bar{p}_\pm \ll \frac{1}{x^{1/2}}. \tag{9}$$

From (8), the correction satisfies

$$\pm \frac{2\bar{p}_\pm}{x^{1/2}} + \bar{p}_\pm^2 + \frac{d\bar{p}_\pm}{dx} = \pm \frac{1}{2x^{3/2}}. \tag{10}$$

On the left-hand side, the first term dominates the second term. Furthermore, balances involving the derivative give a contradiction. Thus,

$$\bar{p}_\pm \sim \frac{1}{4x}. \tag{11}$$

We have shown that

$$p_{\pm} \sim \pm\frac{1}{x^{1/2}} + \frac{1}{4x} \tag{12}$$

and it is readily seen that the remainder is algebraically smaller than $1/x$. The expansions can therefore be integrated to give

$$s_{\pm} \sim \pm 2x^{1/2} + \frac{1}{4}\ln x + c_{\pm}, \tag{13}$$

where $c_{\pm}$ are arbitrary constants.

We conclude that there are solutions

$$y_{\pm} \sim a_{\pm}x^{1/4}e^{\pm 2x^{1/2}}, \tag{14}$$

where $a_{\pm}$ are arbitrary constants.

2. (a) The phase $\psi = t - t^2$ is stationary at the interior point $t = 1/2$. This is a simple stationary point and

meth seen $\Downarrow$

6, A

$$\psi \sim \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \quad \text{as} \quad t \to \frac{1}{2}. \tag{1}$$

The method of stationary gives the leading-order approximation

$$I(x) \sim \frac{1}{4} e^{ix/4} \int_{-\infty}^{\infty} e^{-ixp^2}\, dp = \sqrt{\frac{\pi}{16x}} e^{ix/4 - i\pi/4} \quad \text{as} \quad x \to +\infty. \tag{2}$$

(b & c) Define $h(t) = t - t^2$ and $\rho(t) = i(t - t^2)$. With $t = u + iv$ and $\rho = \phi + i\psi$, we have

meth seen $\Downarrow$

6, A

4, B

2, B

2, D

$$\phi = v(2u - 1), \quad \psi = u - u^2 + v^2. \tag{3}$$

There is a simple saddle at $t = 1/2$. Note that

$$h = \rho/i = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2, \tag{4}$$

thus the constant phase paths through the saddle are, exactly, the straight lines $v = u - 1/2$ and $v = -u + 1/2$, the steepest-descent path from the saddle being the latter.

The constant-phase paths through the boundary points, where $\psi = 0$, are

$$v = \pm(u^2 - u)^{1/2}, \tag{5}$$

where we note that the $+$ sign gives the steepest descent path from $t = 0$ and the $-$ sign gives the steepest descent path from $t = 1$.

We deform the original contour of integration into a new contour

$$\mathcal{C}' = \mathcal{C}_0 + \mathcal{C}_s + \mathcal{C}_1 \tag{6}$$

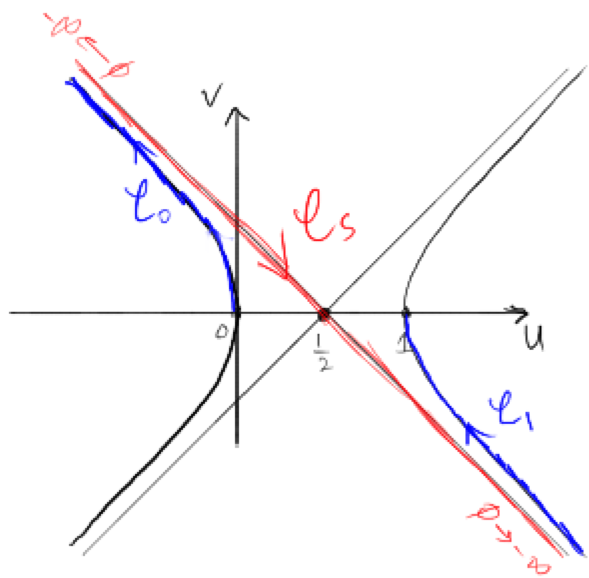
as shown in the sketch. Note that the level function ϕ vanishes at $t = 0, 1$ and $1/2$. The leading contribution from the saddle is ord$(1/x^{1/2})$. The leading contribution from the boundary points is ord$(1/x^2)$, rather than ord$(1/x)$, since $h(t)$ happens to vanish linearly there.

The saddle contribution can be evaluated exactly using the change of variables $t = 1/2 + pe^{-i\pi/4}$:

$$I_{\mathcal{C}_s} = e^{ix/4 - i\pi/4} \int_{-\infty}^{\infty} \left(\frac{1}{4} + ip^2\right) e^{-xp^2}\, dp = \sqrt{\frac{\pi}{16x}} e^{ix/4 - i\pi/4} \left(1 + \frac{2i}{x}\right). \tag{7}$$

Since the boundary contributions are ord$(1/x^2)$, we have

$$I(x) = \sqrt{\frac{\pi}{16x}} e^{ix/4 - i\pi/4} \left(1 + \frac{2i}{x}\right) + \text{ord}(1/x^2) \quad \text{as} \quad x \to +\infty. \tag{8}$$



$-\infty \leftarrow \phi$
v
ℓ_0
ℓ_s
u
0
$\frac{1}{2}$
1
ℓ_1
$\phi \to -\infty$

3. (a) The outer balance $(2-x)y' \sim -1$ gives $y \sim \ln(2-x)+a$, where a is a constant. Clearly, no choice of a makes this approximation vanish at both boundaries. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{3, \text{A}}$

(b) We look for an inner, boundary-layer, balance involving the second derivative. Let the thickness of the boundary layer be order δ and note that the outer approximation and boundary conditions suggest $y = \text{ord}(1)$ there. The ODE then implies $\delta = \epsilon$ and that the boundary-layer balance is $\epsilon y'' \sim -cy'$, where $c = 2 - \bar{x}$ and $\bar{x}$ is the boundary-layer position. Since $0 \le x \le 1$, we have the derivative attenuating exponentially to the right, which means that a boundary layer is only possible at $\bar{x} = 0$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{4, \text{B}}$

(c) The above considerations imply that the solution in the boundary layer is order unity and varies on a short, order ϵ, scale. We therefore anticipate that $q = \text{ord}(1/\epsilon)$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{3, \text{B}}$

(d) Consider first the outer region, where we anticipate an expansion $\boxed{\text{meth seen} \Downarrow}$

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x) \quad \text{as} \quad \epsilon \searrow 0, \tag{1}$$

with $0 < x \le 1$ fixed. From (a) and applying the boundary condition $y_0(1) = 0$, we have

$$y_0(x) = \ln(2-x). \tag{2}$$

At the next order we have

$$(2-x)\frac{dy_1}{dx} = -\frac{d^2y_0}{dx^2} \quad \Rightarrow \quad \frac{dy_1}{dx} = \frac{1}{(2-x)^3}. \tag{3}$$

Solving together with the boundary condition $y_1(1) = 0$, we find

$$y_1(x) = \frac{1}{2}\left\{\frac{1}{(2-x)^2} - 1\right\}. \tag{4}$$

$\boxed{5, \text{C}}$

Now consider the inner region. We write $y(x;\epsilon) = Y(X;\epsilon)$ and find the inner problem

$$\frac{d^2Y}{dX^2} + (2-\epsilon X)\frac{dY}{dX} = -\epsilon, \quad Y(0;\epsilon) = 0. \tag{5}$$

We anticipate an expansion

$$Y(X;\epsilon) \sim Y_0(X) + \epsilon Y_1(X) \quad \text{as} \quad \epsilon \searrow 0, \tag{6}$$

with $X \ge 0$ fixed. At leading order, we have

$$\frac{d^2Y_0}{dX^2} + 2\frac{dY_0}{dX} = 0, \quad Y_0(0) = 0, \tag{7}$$

with solution

$$Y_0(X) = A(1 - e^{-2X}), \tag{8}$$

wherein A is a constant. At the next order,

$$\frac{d^2Y_1}{dX^2} + 2\frac{dY_1}{dX} = X\frac{dY_0}{dX} - 1, \quad Y_1(0) = 0, \tag{9}$$

with solution

$$Y_1 = B(1 - e^{-2X}) + A\int_0^X s^2 e^{-2s}\, ds - \frac{X}{2}. \tag{10}$$

Matching the inner and outer expansions (e.g., using van Dyke's principle at orders $\mu_i = \mu_o = \epsilon$), we find[1]

$$A = \ln 2, \quad B = -\frac{3}{8} - \ln 2 \int_0^\infty s^2 e^{-2s}\, ds = -\frac{3}{8} - \frac{1}{4}\ln 2. \tag{11}$$

An expansion to order unity of the boundary derivative q follows as

$$q \sim 2A\frac{1}{\epsilon} + \left(2B - \frac{1}{2}\right) \quad \text{as} \quad \epsilon \searrow 0, \tag{12}$$

or

$$q \sim 2\ln 2\frac{1}{\epsilon} - \left(\frac{5}{4} + \frac{1}{2}\ln 2\right) \quad \text{as} \quad \epsilon \searrow 0. \tag{13}$$

(For $\epsilon = 0.01$, we have $q \approx 137.033$ from the two-term expansion and $q \approx 137.048$ from a numerical solution.) $\boxed{5, \mathrm{D}}$

[1] The matching here is obvious, and a formal procedure can be supressed. Matching to leading order is just $\lim_{X\to\infty} Y_0 = \lim_{x\to 0} y_0$, as these limits exist; this gives A. Next, the linear term in (10) matches identically with the linear perturbation of y_0 at $x = 0$; it remains to compare $\lim_{x\to 0} y_1$, which again exists, with the limit of $Y_1 + X/2$ as $X \to \infty$; this gives B.

4. (a) We know from the lectures that the weak damping term has an appreciable effect on x on a time scale $1/\epsilon$, which is long compared to the 2π period of the unperturbed oscillator. Here, the damping coefficient y is not constant; rather, it grows monotonically at a rate ϵx^2 — also on a time scale $1/\epsilon$. Intuitively, it is clear that if y is initially positive then we will have decaying oscillations; if y is initially negative, the oscillations will first grow and then decay. Furthermore, averaged over the fast time scale, x^2 attenuates exponentially, eventually; we therefore expect y to approach some limiting value. unseen $\Downarrow$

This description can be verified using the method of multiple scales. We extend $x(t;\epsilon)$ and $y(t;\epsilon)$ into two-variable functions $X(\tau,T;\epsilon)$ and $Y(\tau,T;\epsilon)$, respectively, such that $X(\tau,T;\epsilon)=x(t;\epsilon)$ and $Y(\tau,T;\epsilon)=y(t;\epsilon)$ on the physical diagonal

$$\tau = t, \quad T = \epsilon t, \tag{1}$$

and X and Y satisfy the PDE set

$$X_{\tau\tau} + X + \epsilon\left(2X_{\tau T} + 2YX_\tau\right) + \epsilon^2\left(X_{TT} + 2YX_T\right) = 0, \tag{2}$$
$$Y_\tau + \epsilon Y_T = \epsilon X^2, \tag{3}$$

which reduces to the original ODE set when restricted to the physical diagonal. The additional freedom associated with this extension will be used to satisfy necessary conditions for the expansions

$$X(\tau,T;\epsilon) \sim X_0(\tau,T) + \epsilon X_1(\tau,T), \quad Y(\tau,T;\epsilon) \sim Y_0(\tau,T) + \epsilon Y_1(\tau,T), \tag{4}$$

to be uniformly valid as $\epsilon \searrow 0$. 4, A

At leading order we have

$$X_{0\tau\tau} + X_0 = 0, \quad Y_{0\tau} = 0. \tag{5}$$

We write the solutions

$$X_0(\tau,T) = a(T)\cos(\tau + \phi(T)), \quad Y_0(\tau,T) = b(T), \tag{6}$$

where $a(T) > 0$ without loss of generality. At the next order,

$$X_{1\tau\tau} + X_1 = -2X_{0\tau T} - 2Y_0X_{0\tau}, \tag{7}$$
$$Y_{1\tau} = -Y_{0T} + X_0^2. \tag{8}$$

The right-hand sides are 2π-period in τ and can therefore be written as Fourier series. In the first equation, first harmonics would generate secular terms in X_1. In the second equation, the zeroth harmonic will result in secular terms in Y_1. Such harmonics on the right-hand side must be eliminated if the expansions (4) are to be uniform. These solvability conditions yield the slow system 5, C

$$\frac{da}{dT} = -ba, \tag{9}$$
$$\frac{db}{dT} = \frac{a^2}{2}, \tag{10}$$

together with $a\,d\phi/dT = 0$, which simply shows that that the phase is constant, say $\phi = \phi_0$.[2] 6, D

(b) The fixed points $(a, b) = (0, b)$ are linearly unstable for $b < 0$ and linearly stable for $b > 0$. Furthermore, it is easy to see that unseen ⇓

$$E = \frac{1}{2}a^2 + b^2 = \text{const.} \tag{11}$$

This defines an elliptic orbit in phase space with the direction of propagation $db > 0$ following from (10). For any initial conditions, the amplitude a tends to zero and the damping coefficient b tends to $\sqrt{E}$. Furthermore, if b is initially positive then a decays monotonically; if b is initially negative then a grows and then decays. This confirms the intuitive picture laid out at the beginning of the solution. 2, C

3, D

Comment. The slow system can be integrated in closed form to give

$$a(T) = \frac{\sqrt{2E}}{\cosh\{\sqrt{E}(T - \chi)\}}, \quad b(T) = \sqrt{E}\tanh\{\sqrt{E}(T - \chi)\}, \tag{12}$$

where E and χ, together with the constant phase ϕ_0, are determined from the initial conditions. Below we compare the multiple-scale approximation with numerics, for $\epsilon = 0.2$ and initial conditions corresponding to $E = 1$, $\chi = -1$ and $\phi_0 = 0$. In that scenario b is initially negative hence the oscillations grow before attenuating. The agreement is fantastic even for this not so small ϵ.

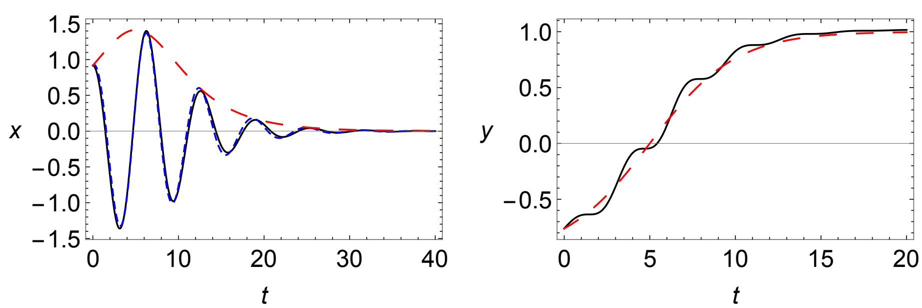


Figure 1: Comparison between asymptotic solution [cf. (6) and (12)] with $\phi = 0$ and initial conditions $a(0)$ and $b(0)$ corresponding to parameters $E = 1$ and $\chi = -1$, with numerical solution for $x(0) = a(0)$, $\dot{x}(0) = 0$ and $y(0) = b(0)$, for $\epsilon = 0.2$. Left: Solid is numerical, dashed blue is asymptotic and dashed red is the envelope a. Right: Solid is numerical and dashed red is b.

[2] As for the linear damped oscillator, going one higher order will reveal that the leading-order phase in fact varies on a super-slow timescale $1/\epsilon^2$.

unseen ⇓

20, M

5. The form of the problem suggests an outer expansion starting at order unity,

$$y(x;\epsilon) \sim y_0(x) \quad \text{as} \quad \epsilon \to 0, \tag{1}$$

with $x \geq 1$ fixed. Then $y_0(x)$ solves

$$\frac{d^2 y_0}{dx^2} = \frac{1}{x^2} + \frac{1}{x^3}. \tag{2}$$

We write the general solution as

$$y_0(x) = ax + b - \ln x + \frac{1}{2x}. \tag{3}$$

We see that this 'outer' approximation cannot satisfy attenuation at infinity. This suggests a 'boundary' layer at infinity and a dominant balance of the ODE readily shows that the thickness of such an 'inner' region would be $x = \text{ord}(1/\epsilon)$.

The outer approximation can, however, satisfy the boundary condition at $x = 1$, which at this order is $y_0(1) = 0$. This gives

$$b = -a - \frac{1}{2}. \tag{4}$$

To analyse the boundary layer at infinity we define $y(x;\epsilon) = Y(X;\epsilon)$, where $X = \epsilon x$. The inner problem then consists of the equation

$$\frac{d^2 Y}{dX^2} - Y = \frac{1}{X^2} + \frac{\epsilon}{X^3}, \tag{5}$$

the attenuation condition

$$Y \to 0 \quad \text{as} \quad X \to +\infty, \tag{6}$$

and matching with the outer region.

Equation (5) possesses the decaying homogeneous solution $\exp(-X)$, and we are told that the leading-order balance of (5) possesses a particular decaying solution which is logarithmically singular as $X \to 0$. Thus, the inner and outer approximations are of the same algebraic order. It follows that $a = 0$ and hence the leading-order outer approximation becomes The leading-order outer approximation as

$$y_0 = -\ln x + \frac{1}{2}\left(\frac{1}{x} - 1\right). \tag{7}$$

The above considerations imply that the inner approximation $Y(X;\epsilon)$ includes terms of order $\ln \epsilon$ and 1. It will be convenient to group together terms of the same algebraic order. Thus, we anticipate the expansion

$$Y(X;\epsilon) \sim Y_0(X;\epsilon) \quad \text{as} \quad \epsilon \to 0, \tag{8}$$

with $X > 0$ fixed, where Y_0 depends logarithmically upon ϵ and the error is algebraically small.

The function $Y_0(X;\epsilon)$ satisfies

$$\frac{d^2 Y_0}{dX^2} - Y_0 = \frac{1}{X^2}, \quad \lim_{X\to\infty} Y_0 = 0, \tag{9}$$

together with matching conditions. Using the information provided in the question, the general solution is written

$$Y_0(X;\epsilon) = A(\epsilon)e^{-X} + G(X), \tag{10}$$

where $A(\epsilon)$ depends logarithmically upon ϵ and

$$G(X) \sim -\ln X - \gamma + O(X^2 \ln X) \quad \text{as} \quad X \searrow 0. \tag{11}$$

Applying Van Dyke's matching principle between leading algebraic orders gives

$$A(\epsilon) = -\ln\frac{1}{\epsilon} - \frac{1}{2} + \gamma. \tag{12}$$

The logarithmic dependence of A together with the behaviour $\exp(-X) \sim 1 - X$ as $X \to 0$ implies terms in the outer of order $\epsilon \ln \epsilon$ followed by ϵ. We therefore extend the outer expansion as

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x;\epsilon) \quad \text{as} \quad \epsilon \searrow 0, \tag{13}$$

with $x \geq 1$ fixed, wherein $y_1(x;\epsilon)$ depends logarithmically upon ϵ and the error is algebraically smaller than ϵ.

The function $y_1(x;\epsilon)$ satisfies the homogeneous equation $d^2y_1/dx^2 = 0$ and homogeneous boundary condition $y_1(1;\epsilon) = 0$. Thus,

$$y_1(x;\epsilon) = a_1(\epsilon)x - a_1(\epsilon), \tag{14}$$

where $a_1(\epsilon)$ is allowed to depend logarithmically upon ϵ. At this algebraic order the outer approximation is forced solely by matching with the inner region.

Thus, van Dyke's matching principle between the outer approximation to order ϵ and the inner approximation to order unity, gives $a_1 = -A$. The outer expansion therefore implies

$$\left.\frac{dy}{dx}\right|_{x=1} \sim \alpha + \epsilon \ln\frac{1}{\epsilon} + \epsilon\beta \quad \text{as} \quad \epsilon \searrow 0, \tag{15}$$

where

$$\alpha = -\frac{3}{2}, \quad \beta = \frac{1}{2} - \gamma. \tag{16}$$

We know that the error in (15) is algebraically smaller than ϵ and it is easy to see that it is order $\epsilon^2 \ln \epsilon$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96020 MATH97029 MATH97106	1	a. (i) Most did not give a convincing explenation. Some were confused regarding the growth rate of ln(lnx). (ii) b. No one managed to obtain the second term correctly. b. Most did this reasonably well, though often with calculation errors.
MATH96020 MATH97029 MATH97106	2	a. Most did this well, sometimes with calculation errors. b. Few noticed that the contribution from the steepest-descent contour through the saddle point terminates after two terms. c. Contribution from boundary points overlooked by most.
MATH96020 MATH97029 MATH97106	3	Some incorrectly kept the "-1" on the RHS of the ODE beyond l.o.. This results in outer terms that cannot be matched.
MATH96020 MATH97029 MATH97106	4	A few did this problem well for the most part. Some obtained the slow-time equation for x but not for y.
MATH96020 MATH97029 MATH97106	5	Most identified that there is a boundary layer at infinity. Few made substantial progress in the calculation.