

LECTURE 1. THE SUBJECT OF THE BIFURCATION THEORY

Bifurcation theory = part of Dynamical Systems (DS) theory

To be more precise, it is one of the methods of Dynamical Systems theory. What is the Dynamical Systems theory? There are different point of views, however one may argue that Dynamical Systems theory is, essentially, or at least when we talk about Dynamical Systems with *continuous time*, a theory of Differential Equations.

By differential equation we will always mean the following equation:

$$\frac{dx}{dt} = f(x), \quad (1.1)$$

where $x = x(t) \in \mathbb{R}^n$ is an *unknown* vector-function depending on time t , $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *given* smooth vector-function, i.e., we have a *system* of n equations with n variables:

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \dots, x_n), \\ \frac{dx_2}{dt} = f_2(x_1, \dots, x_n), \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, \dots, x_n). \end{cases} \quad (1.2)$$

In this course, we will consider the *autonomous* differential equations only, so we assume from now on that $f(x)$ doesn't depend explicitly on t .

It is well-known that a system of differential equations can be explicitly solved (e.g. in elementary functions) in very exceptional cases only, so in the DS theory, instead of solving differential equations explicitly, we investigate them qualitatively. "To investigate" means to establish certain general/specific statements about the behaviour of solutions.

One of such statements which all of us are supposed to know, is the existence and uniqueness theorem. Indeed, if we assume that the function f is smooth (which we will always do in this course) and suppose for simplicity that all partial derivatives $\frac{\partial f_i}{\partial x_j}$ are globally bounded in \mathbb{R}^n , then, for a given initial data $x_0 \in \mathbb{R}^n$, system (0.1) possesses a unique solution $x(t, x_0)$ such that $x(0, x_0) = x_0$ and this solution is defined for all time $t \in \mathbb{R}$. Moreover, the solution $x(t, x_0)$ depends smoothly on time $t \in \mathbb{R}$ and on the initial data $x_0 \in \mathbb{R}^n$, so the solution map $(t, x_0) \rightarrow x(t, x_0)$ is well-defined and smooth, see the course of Ordinary Differential Equations.

Recall that this may be not true without the global boundedness assumption on the derivatives of f . For instance, in the simplest example of a scalar

equation

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0,$$

the general solutions have a form $x(t) = \frac{x_0}{1-x_0 t}$ and we see that all solutions except of $x(t) \equiv 0$ blow up in finite time $t = \frac{1}{x_0}$. So, one should be careful here and always remember that without the boundedness assumption, defining $x(t, x_0)$ for all times t can be impossible.

There are two points of view on the solution of (0.1). First, we can fix initial condition x_0 and change time t . Then we get what is called an orbit of x_0 :

$$\text{Orb}(x_0) := \{x(t, x_0), t \in \mathbb{R}\}.$$

Alternatively, we can fix time t and consider the solution map $x_0 \rightarrow x(t, x_0)$ for different x_0 , then for each $t \in \mathbb{R}$, we have a smooth map $S(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is generated by our system of differential equations:

$$S(t)(x_0) := x(t, x_0).$$

This family of maps $S(t)$, $t \in \mathbb{R}$, is often called a DS generated by systems of differential equations (0.1). (By the above remark, S is not necessarily defined for all t and x_0 , so we consider this map as for such t and x_0 for which it is defined.)

Another type of DS are DS with discrete time:

$$x_{k+1} = g(x_k), \quad k = 0, 1, \dots, \tag{1.3}$$

where $x_k \in \mathbb{R}^n$, $k = 0, 1, \dots$, a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given smooth vector-function. Given x_0 , its orbit

$$\text{Orb}(x_0) = \{x_k\}_{k=0}^{\infty}$$

is uniquely defined and since g is smooth $x_k = x_k(x_0)$ is a smooth function of x_0 . Thus, if fix k , then for each x_0 , we have a smooth map $S(k) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $S(k)(x_0) := x_k(x_0)$. Thus, we have a family of smooth maps $S(k)$ parameterized by the discrete parameter k which now plays the role of time.

Dynamical systems with discrete time and continuous time are closely related with each other. But essentially the theory of dynamical systems was invented in order to understand differential equations, so in this course we will mostly talk about differential equations and DS with continuous time.

We think about our DS $S(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as describing some natural process evolving in time, i.e., we have some system (hopefully existing in nature) and the state of this system is determined by some set of variables $x_0 \in \mathbb{R}^n$, and the maps $S(t)$ give the evolution of the state variables in time. So, if we know the initial state x_0 for time $t = 0$, by the existence and uniqueness theorem, the state $x(t) = S(t)(x_0)$ is predetermined for all times.

The problem here is that typically we cannot solve the corresponding differential equations explicitly and we do not know the solution $x(t)$, but want to

know at least something about this solution. In the DS theory we are mainly interested in the behaviour of solutions for large times, i.e., what happens with the solution $x(t)$ as time tends to infinity or, in other words, what can be the limit of an orbit of a given system as time (discrete or continuous) tends to infinity. An orbit is a set, for systems with continuous time this is a curve, for systems with discrete time this is just a sequence of points. The limit points of an orbit $\text{Orb}(x_0)$ as time tends to infinity belong to the closure $\overline{\text{Orb}(x_0)}$, so we want to understand how the closure of a given orbit looks like.

We also assume that the orbits $\text{Orb}(x_0)$ are bounded sets. It is a standard assumptions in the theory of DS since it is somehow difficult to work with trajectories tending to infinity. In this case the closure $\overline{\text{Orb}(x_0)}$ is bounded and closed and therefore is a *compact* set in \mathbb{R}^n . Moreover, it is also *invariant* with respect to the considered DS. We recall that, by definition, a set $A \subset \mathbb{R}^n$ is invariant with respect to DS $S(t)$ if $x_0 \in A$ implies that $S(t)(x_0) \in A$ for all times t .

Then, since an invariant set consists of orbits, it is almost a tautology to say that $\overline{\text{Orb}(x_0)}$ is an invariant set. It is also not difficult to prove that the closure of an invariant set is an invariant set again. So, the closure $\overline{\text{Orb}(x_0)}$ is a *compact invariant set*.

Thus, we come to the ultimate goal (impossible to achieve, but still) of the DS theory:

for a given DS (a system of differential equations) describe all its compact invariant sets.

If we describe the structure of these invariant sets, then we understand behaviour of a system for all possible initial conditions x_0 when time goes to infinity without solving the corresponding system of differential equations explicitly. The problem is that this goal is too complicated, and in the most of examples cannot be reached (in some cases we even can prove that it is impossible to do in some logic-theoretical sense). Nevertheless, we want to do something in this direction, e.g., to get some partial description or description of some specific compact invariant sets. And at this moment we may start to speak about bifurcation theory as one of the approaches to this impossible goal.

Below we will formulate the approach of a bifurcation theory in a bit abstract and schematic way. It will be clarified later in more concrete examples.

Suppose we *know* a compact invariant set C . Then we try to build upon this knowledge. Take a sufficiently small neighbourhood U of C . Let N be a set of all orbits which lie entirely in U , see Figure 1 below. The set N is not empty: at least orbits from the set C never leave U , since C is invariant. *Suppose we have all possible information about the set C , then bifurcation theory provides you tools how to describe N using the information available about C .* This is useful, because C can be very simple and easy to find, while N can be very nontrivial. This is the first step, and the next step is to consider systems which



FIGURE 1

depend on parameters (for example, we may have a function f in the right-hand side of (0.1) which is polynomial, this polynomial has coefficients, which can be considered as the parameters in our system). Then we ask a question:

“What will happen to our system if we change these parameters?”

More precisely, what will happen with the set N when parameters change? N can be trivial ($N = C$) for some values of parameters, but can become non-trivial (we say it undergoes a bifurcation) when we change parameters. Thus, the bifurcation theory provides us regular methods how to get an information about much richer and interesting set N starting from the basic, simple set C .

This was a very general introduction and now let us consider some examples. We start with the trivial example where nothing interesting happens, but some tools and computations which we present in this example will be used throughout the course.

Example 1.1. Let us consider the following system:

$$\frac{dx}{dt} = f(x, \varepsilon), \quad (1.4)$$

where $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, \dots, f_n)$, and $\varepsilon \in \mathbb{R}^k$ is a parameter. We assume that ε is close to zero, and start with considering system (0.4) with $\varepsilon = 0$. Suppose that there is a point x^* , such that $f(x^*, 0) = 0$. Then $x(t) = x^*$, for any $t \in \mathbb{R}$, is a solution of (0.4), which is called a *stationary solution* or an *equilibrium*. Let us assume that the equilibrium x^* is *exponentially stable*, i.e. *all eigenvalues of the Jacobi matrix*

$$\frac{\partial f}{\partial x} \Big|_{x=x^*} = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \Big|_{x=x^*} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

have negative real parts.

The equilibrium $x = x^*$ is a compact invariant set; in accordance with our general scheme we denote it by C . We take a small neighbourhood U of C and ask a question:

“What will be the set of all orbits which never leave the neighbourhood U neither in forward time nor in backward time?”

Our claim is that this set consists just from one point $x^*(\varepsilon)$, i.e.,

$$N = C = \{x = x^*(\varepsilon)\}$$

and all other orbits which start in U tends to x^* when time tends to plus infinity and live the neighbourhood U when time tends to minus infinity.

We start with the following simple, but important statement.

Theorem 1.2. *If all eigenvalues of the matrix $\frac{\partial f}{\partial x}|_{x=x^*, \varepsilon=0}$ are non-zero, then for all small ε , system (0.4) has an equilibrium $x^*(\varepsilon)$, which smoothly depends on a parameter ε , such that $x^*(0) = x^*$.*

Proof. Apply the implicit function theorem (IFT) to equation $f(x^*(\varepsilon), \varepsilon) = 0$. \square

Let us remind the statement of the IFT which is one of the key theorems of Analysis and which any intelligent person must know. Unfortunately, generations of students attending this course claim to have never heard about it..... - so let us remind IFT. Let us start with the scalar case. Suppose that we are given a smooth function of two scalar variables $f(x, \varepsilon)$ such that $f(0, 0) = 0$ and we want to solve the equation

$$f(x, \varepsilon) = 0$$

in a small neighbourhood of $(0, 0)$, i.e., to find a function $x = x(\varepsilon)$ such that $f(x(\varepsilon), \varepsilon) \equiv 0$. When is it possible? If we linearize f near $(0, 0)$, we formally get

$$\frac{\partial f}{\partial x}(0, 0)x(\varepsilon) + \frac{\partial f}{\partial \varepsilon}(0, 0)\varepsilon \approx 0$$

and this is solvable if $\frac{\partial f}{\partial x}(0, 0) \neq 0$. Then the IFT tells us that under this condition, the above formal calculation can be justified and the desired function $x(\varepsilon)$ indeed exists and smooth.

Similar arguments work in the vector case as well, but we now need the Jacobi matrix

$$\frac{\partial f}{\partial x}(0, 0) := \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}(0, 0)$$

to be invertible. In other words, the determinant of $\frac{\partial f}{\partial x}$ should be non-zero at $(0, 0)$ and this is *exactly* the condition that all eigenvalues of this matrix are not zeros.

We now state the main result about exponential stability.

Theorem 1.3. *If all eigenvalues of the matrix $\frac{\partial f}{\partial x}|_{x=x^*, \varepsilon=0}$ have negative real parts, then all orbits in the neighbourhood U of the equilibrium state $x^*(\varepsilon)$ exponentially tend to $x^*(\varepsilon)$:*

$$\|x(t, x_0) - x^*(\varepsilon)\| \leq \|x_0 - x^*(\varepsilon)\|e^{-\alpha t}, \quad \alpha > 0, \quad (1.5)$$

for all $t \geq 0$ and $x_0 \in U$.

Remark 1.4. The inequality (0.5) depends on the choice of a norm. Of course, all norms in a finite-dimensional vector space are equivalent, but this means that if we choose an equivalent norm, then the inequality (0.5) will transform to

$$\|x(t, x_0) - x^*(\varepsilon)\| \leq C \|x_0 - x^*(\varepsilon)\| e^{-\alpha t}, \quad \alpha > 0,$$

for some positive C . In other words, an orbit $x(t, x_0)$ will tend to $x^*(\varepsilon)$ eventually, but we cannot guarantee that the distance to the equilibrium x^* is a decreasing function of time. Thus, we should note that the inequality (0.5) is satisfied only for the specific choice of a norm and the precise formulation of the theorem is that *there exists* a norm on \mathbb{R}^n for which inequality (0.5) holds.

Proof. Let us denote $A(\varepsilon) := \frac{\partial f}{\partial x}\Big|_{x=x^*(\varepsilon)}$. For $\varepsilon = 0$, we know that, all eigenvalues of A lie to the left of imaginary axis.

Exercise 1.5. Prove that the same is true for all small ε .

The first step is a coordinate transformation, where we put the equilibrium at the origin:

$$x = y + x^*(\varepsilon).$$

Remark 1.6. Coordinate transformations is one of the main methods of the dynamical systems theory and differential equations. In general, it is almost always a good idea that instead of trying to solve a differential equation, one does coordinate transformations, which simplify the equation as much as possible, ideally - until there remains nothing to solve.

Rewriting equation (0.4) in new coordinates and Taylor expanding function f near $x^*(\varepsilon)$, we obtain

$$\frac{dy}{dt} = f(y + x^*(\varepsilon)) = f(x^*(\varepsilon)) + \frac{\partial f}{\partial x}\Big|_{x=x^*(\varepsilon)} y + o(y) = A(\varepsilon)y + o(y). \quad (1.6)$$

Our next coordinate transformation is

$$y = Qz, \quad \text{where } Q \text{ is some constant non-degenerate } n \times n \text{ matrix.}$$

Then, in the new coordinates, equation (0.6) attains the form

$$\frac{dz}{dt} = Q^{-1}AQz + o(z) = \tilde{A}z + o(z).$$

Since the matrix Q is non-degenerate, the matrix \tilde{A} is similar to A , so it has the same eigenvalues $\{\lambda_j\}_{j=1}^n$. If all λ_j are different, then Q can be chosen in such a way that \tilde{A} is diagonal. In general, Q can be chosen such that \tilde{A} has a *special Jordan form*.

Let us remind that the standard Jordan form is

$$A = \begin{pmatrix} \lambda_1 & \delta_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_{n-1} & \delta_{n-1} \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\delta_j = 0$ or 1, if $\delta_j = 1$, then $\lambda_j = \lambda_{j+1}$.

Special Jordan form is exactly the same, but you can choose $\delta_j = 0$ or ν , where ν is any given constant which is not equal to zero.

As we know from Linear Algebra, any matrix can be brought to the standard Jordan form, so let us show that it can be brought to the special Jordan form. Indeed, in the standard basis

$$A\vec{e}_j = \lambda_j \vec{e}_j + \delta_{j-1} \vec{e}_{j-1},$$

so, if we chose the special basis $\vec{e}_j^{new} = \nu^j \vec{e}_j^{old}$, $j = 1, \dots, n$, then

$$A\vec{e}_j^{new} = \lambda_j \vec{e}_j^{new} + \nu \delta_{j-1} \vec{e}_{j-1}^{new},$$

which was required to prove.

Now our original system (0.4) can be rewritten in the new basis, which corresponds to the special Jordan form of the matrix A , with $\nu > 0$ sufficiently small,

$$\frac{dx}{dt} = Ax + o(x). \quad (1.7)$$

Since eigenvalues of the new matrix A are the same as for the old one, we still have the following picture

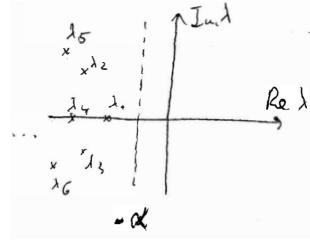


FIGURE 2

where all eigenvalues $\{\lambda_j\}_{j=1}^n$, at the picture $n = 6$, lie to the left of line $\operatorname{Re} \lambda = -\alpha$, for some positive α , i.e., $\operatorname{Re} \lambda_j < -\alpha$, for $j = 1, \dots, n$.

The desired norm in which we will obtain the inequality (0.5) is $\|x\|^2 = \sum_{j=1}^n |x_j|^2$, where x is taken in the new basis. Of course, in this basis x may become complex, and scalar product (dot product) should be defined as $x \cdot y = \sum_{j=1}^n \bar{x}_j y_j$, where \bar{x} means complex conjugate of x .

Having in hand our equation written in the form (0.7), let us estimate the norm of x . Let x be an arbitrary solution of (0.7), then

$$\begin{aligned} \frac{d}{dt}\|x\|^2 &= \sum_{j=1}^n (\bar{x}_j \frac{dx_j}{dt} + \frac{d\bar{x}_j}{dt} x_j) = 2 \operatorname{Re} \sum_{j=1}^n \bar{x}_j (Ax)_j + o(\|x\|^2) = \\ &= 2 \operatorname{Re} \sum_{j=1}^n (\bar{x}_j \lambda_j x_j + \delta_j \bar{x}_j x_{j-1}) + o(\|x\|^2) \leq \\ &\leq 2 \sum_{j=1}^n \left(\operatorname{Re} \lambda_j |x_j|^2 + \nu \frac{|x_j|^2 + |x_{j-1}|^2}{2} \right) + o(\|x\|^2) \leq \\ &\leq 2 \left(\max_{j=1,\dots,n} \{\operatorname{Re} \lambda_j\} + \nu \right) \sum_{j=1}^n |x_j|^2 + o(\|x\|^2) \leq -2\alpha \|x\|^2. \end{aligned}$$

Here we used that x is close to zero (therefore $o(\|x\|^2)$ is negligible) and that $\nu > 0$ can be taken arbitrary small.

Thus, we obtained that $\frac{d}{dt}\|x\|^2 \leq 0$ as far as $\|x(t)\|$ is small, hence $\|x(t)\|$ does not increase with time, as long as it remains small. So, if $\|x_0\|$ is small enough, then $\|x(t)\| \leq \|x_0\|$ for all $t \geq 0$.

Now it only remains to solve the equation

$$\frac{d}{dt}\|x\|^2 \leq -2\alpha \|x\|^2, \text{ for all } t \geq 0.$$

Direct calculations give

$$\frac{1}{\|x\|} \frac{d\|x\|}{dt} = \frac{d}{dt} \log \|x(t)\| \leq -\alpha, \text{ for all } t \geq 0, \text{ so}$$

$$\log \|x(t)\| - \log \|x_0\| \leq -\alpha t \Rightarrow \|x(t)\| \leq e^{-\alpha t} \|x_0\|,$$

and the theorem is proved. \square

We got that if an equilibrium is exponentially stable, then any solution, which starts in a sufficiently small neighbourhood U of it, will tend exponentially to this equilibrium when time goes to plus infinity, which means that when time tends to minus infinity these solutions will increase exponentially and leave the neighbourhood U . Thus, indeed, the only orbit which never leaves the neighbourhood U is the equilibrium.

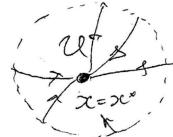


FIGURE 3

In the next lectures, there will be more examples, which are more interesting. Throughout the course we will study the simplest cases: C consists of one (equilibrium, periodic orbit) or two (equilibrium with homoclinic) orbits, but this is a typical situation in the bifurcation theory, it is quite rare when people consider more orbits. The whole idea is to take a very small set C , look inside its small neighbourhood, and then obtain many interesting and unexpected effects.

LECTURES 2 AND 3. CENTER MANIFOLD THEOREM

In the previous lecture we discussed the behaviour of systems of differential equations near a stable equilibrium state. Recall that the equilibrium state is asymptotically exponentially stable when all eigenvalues of the linearization matrix have negative real parts and the result we obtained is this: all trajectories from a small neighbourhood of such equilibrium state tend to the equilibrium state as time goes to plus infinity, and as time goes to minus infinity they just go away from the neighbourhood.

The main question for this lecture is:

What can we say in a general situation when not all of the eigenvalues have negative real parts?

As in the previous lecture, we consider a general system of differential equations:

$$\frac{dx}{dt} = f(x), \quad (2.1)$$

where $x = x(t) \in \mathbb{R}^n$ is an unknown vector-function depending on time t , $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given smooth vector-function. We also assume that we have an equilibrium state $x(t) \equiv x_0$, i.e., $f(x_0) = 0$. We take a small neighbourhood of this equilibrium state and want to understand the behaviour in this neighbourhood.

The main tool for this problem is a wonderful result which we discuss in this lecture. It is the *Center Manifold Theorem*. The proof of this theorem will not be given in the course, instead we spend this lecture on formulating this theorem and extracting important consequences of it.

The main consequence of this theorem is that the problem of studying the behaviour in a small neighbourhood of an equilibrium state allows for a tremendous *reduction of dimension*. Indeed, we start with a system which may have arbitrary dimension (x belongs to \mathbb{R}^n and n can be anything; similar results hold also for partial differential equations which are infinite-dimensional dynamical systems) and in a small neighbourhood of an equilibrium we end up with some very low-dimensional problem generated by a system of differential equations similar to (2.1), but the number of equations is reduced from n to

$$p = \text{the number of eigenvalues with zero real part.}$$

Since a generic matrix $A = f'(x_0)$ do not have eigenvalues with zero real part (and for a generic $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ all of the equilibria $f(x_0) = 0$ have eigenvalues with non-zero real parts only), typically $p = 0$ and this reduced system is zero-dimensional, but interesting cases start from $p = 1$ (which is typical for one-parametrical families $f = f(x, \varepsilon)$, $\varepsilon \in \mathbb{R}$). So, in the case $p = 1$ we start with a system of n equations and, by the center manifold theorem,

we reduce our problem to a *scalar* first-order differential equation, which is of course much easier to study. In more general situations the reduced system can become two-dimensional, sometimes three-dimensional, in any case the dimension of the reduced system is very-low.

We now turn to precise formulation of the center manifold theorem and its meaning. We are given an equilibrium $x_0 \in \mathbb{R}^n$, and making the coordinate transform

$$y = x - x_0,$$

we move this equilibrium to the origin $y = 0$, then we take function f and Taylor-expand it:

$$\frac{dy}{dt} = Ay + o(y). \quad (2.2)$$

Since we are interested in the behaviour of the system in a small neighbourhood of the equilibrium $y = 0$ - which corresponds to small y , the leading order approximation is described by the linear term Ay . So, we first look at the linear part Ay (although the non-linear terms are also important and will be treated later).

We split the eigenvalues of the matrix A into three groups. The first one consists of eigenvalues with real parts less than zero, we denote them by

$$\lambda_1, \dots, \lambda_k, \quad \operatorname{Re} \lambda_j < 0.$$

These eigenvalues are called *stable* because would it be no others eigenvalues, the behaviour of the system (2.2) will be very simple, everything will go to zero exponentially, as time goes to plus infinity, as it is proven in Lecture 1 (without nonlinear terms this fact is self-evident - any solution of the linear problem is just a combination of exponents $e^{\lambda_j t}$, maybe with some polynomial factors, and when the real part of λ is negative, $e^{\lambda t}$ goes to zero, as t goes to plus infinity).

The second group consists of eigenvalues with real parts greater than zero, we denote them by

$$\gamma_1, \dots, \gamma_m, \quad \operatorname{Re} \gamma_j > 0.$$

These eigenvalues are called *unstable* by similar reasons.

The remaining third group consists of eigenvalues whose real parts are equal to zero. We denote them by

$$\eta_1, \dots, \eta_p, \quad \operatorname{Re} \eta_j = 0.$$

These eigenvalues are called *neutral* or *central*.

Recall that, since A is a real matrix, complex eigenvalues go in pairs, i.e., if a complex number is an eigenvalue, then its complex conjugate is also an eigenvalue. So, we have real eigenvalues and pairs of complex conjugate ones.

At the next step, we move to a Jordan base of the matrix A . To be more precise, we do a coordinate transformation $y = Q\tilde{y}$ (where Q is a transition matrix generated by the corresponding eigenvectors) such that the corresponding

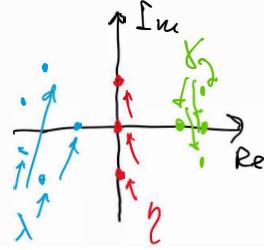


FIGURE 1

transformed matrix $Q^{-1}AQ$ has the following form:

$$A \rightarrow Q^{-1}AQ = \begin{pmatrix} \mu_1 & \delta_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \mu_{n-1} & \delta_{n-1} \\ 0 & \cdots & \cdots & 0 & \mu_n \end{pmatrix}. \quad (2.3)$$

We have the eigenvalues μ_j on the diagonal and above the diagonal $\delta_k = 0$ or $\delta_k = \nu$, where ν can be made arbitrarily small, see Lecture 1 and the course of Linear Algebra.

Since the above δ_k s may be not equal to zero only if the eigenvalues are equal, we may order the eigenvalues in such a way that the first ones will be λ s (negative real parts), the second - eigenvalues γ (positive real parts), and the third are η s (zero real parts), and in these new coordinates the matrix A will have the following *block-diagonal* form:

$$Q^{-1}AQ = \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix}, \quad (2.4)$$

where

$$C = \begin{pmatrix} \gamma_1 & \delta_{2,1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \gamma_{m-1} & \delta_{2,m-1} \\ 0 & \cdots & \cdots & 0 & \gamma_m \end{pmatrix}, \quad D = \begin{pmatrix} \eta_1 & \delta_{3,1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \eta_{p-1} & \delta_{3,p-1} \\ 0 & \cdots & \cdots & 0 & \eta_p \end{pmatrix},$$

and the matrix B also has the similar form.

Actually, it is not so important for us that the matrices B, C, D are in Jordan normal form, only the block-diagonal structure (2.4) is crucial for what follows. We have used the Jordan base only in order to verify that such a block-diagonal reduction always exists. Thus, we have verified that the matrix A acquires a block diagonal form (2.4) where the $k \times k$ block B corresponds to stable eigenvalues, then the $m \times m$ block C corresponds to unstable eigenvalues, and the last $p \times p$ block D corresponds to neutral eigenvalues.

In the constructed new coordinates \tilde{y} (which we will from now on denote also by y for simplicity) the vector $y \in \mathbb{R}^n$ can be split on 3 components:

$$y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \text{where} \quad \begin{aligned} u &\in \mathbb{R}^k \\ v &\in \mathbb{R}^m \\ w &\in \mathbb{R}^p \end{aligned} \quad \text{and} \quad n = k + m + p.$$

which correspond to blocks B , C and D . Then our equation (2.2) reads

$$\begin{cases} \frac{du}{dt} = Bu + f_1(u, v, w) & \text{Spectrum}(B) \subset \{Re < 0\}, \\ \frac{dv}{dt} = Cv + f_2(u, v, w) & , \quad \text{Spectrum}(C) \subset \{Re > 0\}, \\ \frac{dw}{dt} = Dw + f_3(u, v, w) & \text{Spectrum}(D) \subset \{Re = 0\}, \end{cases} \quad (2.5)$$

where f_j , with $j = 1, 2, 3$, are given non-linearities satisfying

$$f_j(y)|_{y=0} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}|_{y=0} = \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}|_{(u,v,w)=0} = 0,$$

so, if we do the Taylor expansion of a function f near the zero equilibrium, we should start with terms of order two.

If we formally throw away nonlinear terms, then the system (2.5) becomes decoupled: stable (u), unstable (v), and central (w) variables will change independently. The behavior of the linear system is described as follows.

Since $\text{Spectrum}(B)$ (the spectrum of a matrix is the collection of all its eigenvalues) lies to the left of the imaginary axis, then as we proved on the previous lecture any solution of the equation $\frac{du}{dt} = Bu$ goes to zero exponentially, when time goes to plus infinity, so the u variables independently of v and w go to zero.

The behaviour of the v variables becomes stable, if we change the direction of time. Indeed, if instead of time t we consider time $(-t)$:

$$\frac{dv}{dt} = Cv \quad \xrightarrow{t \mapsto -t} \quad -\frac{dv}{dt} = Cv \quad \Rightarrow \quad \frac{dv}{dt} = -Cv,$$

then we note that since the matrix C has eigenvalues with positive real parts, the matrix $-C$ will have eigenvalues with negative real parts, which means that v goes to zero backward in time.

For the w variables, we do not know what will happen, but since the real parts of η eigenvalues are equal to zero, the corresponding solution cannot grow or decay exponentially (in the case where all eigenvalues are simple, the solution $w(t)$ will not leave the small neighbourhood of zero, but it may grow polynomially if there are nontrivial Jordan cells). Thus, the u variables give stability, the v variables gives instability, and only the w variable may give something more interesting.

The above conclusion is true only for the linear system, when we throw away nonlinear terms. In this case, when we go to a Jordan form, matrix A acquires block-diagonal form with blocks B, C and D , and therefore linear system decouples. But we have nonlinear terms, and these nonlinear terms make a coupling between the u, v and w variables. For example, we know that

the v variable decay in a backward time which means that in a forward time it grows. Thus, since f_1 in (2.5) contains variable v , it may also grow. As a result, for the equation on the u variables, we have competing factors: there is a term Bu which pushes u to decay, and the term $f_1(u, v, w)$, which may push u to grow. Summing-up, we have a non-trivial question:

“How adding the nonlinear terms f_1, f_2, f_3 affects the behaviour of stable, unstable and central variables?”

The center manifold theorem solves this problem.

The center manifold theorem gives an analogue of the decoupling in the presence of nonlinear terms too. It says that there is a nonlinear coordinate transformation, such that unstable variables v grow with time, stable variables u grow in backward time, and *only central variables w can demonstrate the nontrivial behaviour*.

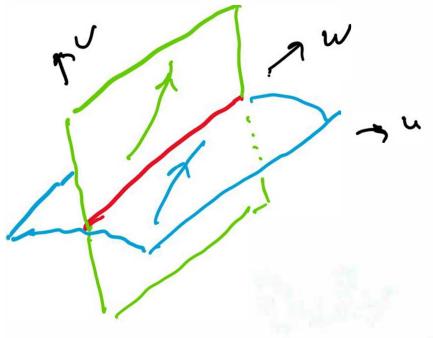


FIGURE 2

We now turn to the precise statements starting from the so-called Central-Stable Manifold Theorem.

Theorem 2.1 (Central-Stable Manifold Theorem). *There exists a smooth function $\phi_{cs} : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ such that its graph*

$$W^{cs} := \{(u, v, w) \in \mathbb{R}^n : (u, w) \in \mathbb{R}^k \times \mathbb{R}^p, v = \phi_{cs}(u, w)\}$$

is a locally invariant manifold. W^{cs} contains zero and is tangent at zero to $v = 0$, i.e., $\phi_{cs}(0) = 0$ and $\phi'_{cs}(0) = \frac{\partial \phi_{cs}}{\partial (u, w)}|_{(u, w)=0} = 0$.

The theorem gives us only the *existence* of such a function which may be not unique. We just know that at least one such a surface exists and the main characterization properties of it are: smooth, locally invariant, it goes through zero, and is tangent at zero to $v = 0$.

The theorem says that function ϕ_{cs} is smooth, but how smooth? The precise answer is given below:

- (1) if $f \in C^r$ (f has r continuous derivatives with respect to all variables), then $\phi_{cs} \in C^r$;
- (2) if $f \in C^\infty$ (f has infinitely many derivatives with respect to all variables), then we can only guarantee finitely many derivatives for ϕ_{cs} ,

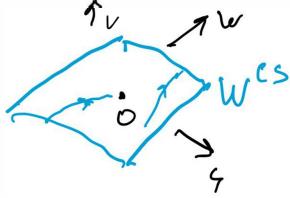


FIGURE 3

$\phi_{cs} \in C^r$, and the number r depends on the size of the neighbourhood. Since this is a local manifold, we have a small neighbourhood of zero, where the manifold W^{cs} is actually defined and invariant. If we want to make r larger, then we should take smaller neighbourhood. In other words, for an arbitrarily large (but fixed) r , we may find a sufficiently small neighbourhood where the function ϕ_{cs} will be C^r -smooth. At the point O , we have infinitely many derivatives, but, in general, in a small neighbourhood of O we only have finitely many derivatives.

In particular, it is important, that the center-stable manifold W^{cs} is very rarely analytic, i.e., function f can be analytic, but function ϕ_{cs} typically not, so it cannot be written in the form of convergent Taylor series.

Thus, W^{cs} is a graph of a smooth function ϕ^{cs} , and we call the surface W^{cs} center-stable manifold, because coordinates on W^{cs} are stable variables u and central variables w . Here the word “manifold” means nothing more than a *surface* presented as a graph of a function. Recall also that any smooth submanifold of \mathbb{R}^n can be *locally* presented as a graph of a function over the tangent space and our manifold W^{cs} is actually defined only locally (in a small neighbourhood of O).

Up to the moment, we postulated that there exists a smooth surface, which goes through zero, and is tangent at $(u, w) = 0$ to $v = 0$. Of course, there are a lot of such surfaces, what is important is that this surface not just smooth, but it is locally invariant, i.e., the orbit of any point in W^{cs} stays in W^{cs} all the time until it remains in U (a small neighbourhood of O). In other words,

$$v(0) = \phi_{cs}(u(0), w(0)), \text{ then } v(t) = \phi_{cs}(u(t), w(t)) \text{ for all } t \in I,$$

where I is an interval around $t = 0$, such that $\|(u(t), v(t), w(t))\| < \delta$ for any $t \in I$. Here δ is a sufficiently small positive number which determines the neighbourhood U of the origin where our manifold is invariant.

Note that existence of a locally invariant manifold without requiring smoothness is also trivial. Just take any point, take its orbit until it stays in a small neighbourhood of O , take another point and its orbit, then take a union of these points, you will have a locally invariant set which however will not necessarily be smooth. The whole trick is to guarantee that this locally invariant set is a smooth surface as stated in the center-stable manifold theorem.

The proof of the theorem is not trivial. It is based on the contraction mapping principle and the implicit function theorem in the Banach space, but

in this course we omit the proof and concentrate on the formulation of the theorem, its explanation and applications.

We first state the analogue of the above theorem for the unstable directions.

Theorem 2.2 (Central-Unstable Manifold Theorem). *There exists a smooth function ϕ_{cu} such that the graph $u = \phi_{cu}(v, w)$ is a locally invariant manifold W^{cu} containing O and tangent at O to $u = 0$.*

Proof. Let us rewrite system (2.5) in backward time:

$$\begin{cases} \frac{du}{dt} = -Bu - f_1(u, v, w), \\ \frac{dv}{dt} = -Cv - f_2(u, v, w), \\ \frac{dw}{dt} = -Dw - f_3(u, v, w). \end{cases}, \quad (2.6)$$

The stable directions become unstable, and vice versa.

What is the spectrum of matrix $-B$? Spectrum of B used to be on the left of the imaginary axis, but after reversion of time (changing B to $-B$), it moves to the right. Spectrum of $-D$ remains on the imaginary axis, and spectrum of $-C$ now is on the left of the imaginary axis. This means, that u variables become unstable and v variables become stable.

We see that the reversion of time and changing u to v and v to u gives a system of the same form. We note that local invariance is the same in forward and backward time, the only change is that u variables become v variables, and vice versa. Thus, applying Theorem 2.1 to the system (2.6), we finish the proof of the theorem. \square

We are now ready to state the promised center manifold theorem.

Theorem 2.3 (Central Manifold Theorem). *There exists a locally invariant p -dimensional smooth manifold W^c (center manifold) which passes through O and is tangent at this point to the plane $u = v = 0$. This manifold can be presented as a graph of a smooth function $\phi_c : \mathbb{R}^p \rightarrow \mathbb{R}^k \times \mathbb{R}^m$:*

$$(u, v) = \phi_c(w), \text{ where } \phi_c(0) = 0, \quad \phi'_c(0) = 0. \quad (2.7)$$

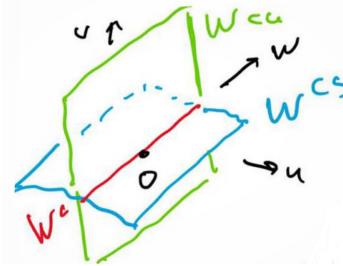


FIGURE 4

Proof. We define $W^c = W^{cs} \cap W^{cu}$. Then local invariance of W^c is a straightforward corollary of Theorems 2.1 and 2.2. Indeed, if an orbit starts in the center manifold, then, since the center manifold belongs to the center-stable

manifold, i.e., $W^c \subset W^{cs}$, and W^{cs} is locally invariant, it should remain in W^{cs} , but on the other hand, $W^c \subset W^{cu}$, and by invariance the orbit should remain in the center-unstable manifold as well, thus, it should stay in their intersection W^c .

Thus, we only need to prove that W^c is indeed given by equation (2.7) with a smooth ϕ_c , i.e., that for each small w the system

$$\begin{cases} u = \phi_{cu}(v, w) \\ v = \phi_{cs}(u, w) \end{cases}$$

has a uniquely defined small solution (u, v) which depends smoothly on w .

But this is an obvious corollary of the implicit function theorem. Indeed, the matrix

$$\frac{\partial(u - \phi_{cu}(v, w), v - \phi_{cs}(u, w))}{\partial(u, v)} \Big|_{(u,v,w)=0} = \text{Id}$$

is invertible, since $\phi'_{cs}(0) = 0$ and $\phi'_{cu}(0) = 0$. The theorem is proved. \square

Exercise 2.4. Try to recall or find in the literature what is the Implicit Function Theorem. How is it proved? Some textbooks offer a proof by induction in dimension, do not use them! Prove it in a Banach space. How is it related to a contraction mapping principle?

Exercise 2.5. How does the Implicit Function Theorem imply $\phi_c(0) = 0$ and $\phi'_c(0) = 0$?

The next corollary of the center manifold theorem is crucial for the understanding of the local behaviour of trajectories near the equilibrium O .

Theorem 2.6. *If a forward orbit of a point $M \in U$ (where U is a small neighbourhood of zero), never leaves U , then $M \in W^{cs}$. If a backward orbit of a point $M \in U$ never leaves U , then $M \in W^{cu}$.*

From this theorem we infer that:

If the whole orbit of $M \in U$ stays in U entirely, then $M \in W^c$.

So, if we are interested in local dynamics, i.e., in the structure of the set N of all orbits that never leave U , we only need to look at initial conditions on the center manifold. Recall that this is a locally invariant set, which means that if the initial condition belongs to W^c , then the whole orbit belongs to W^c . Therefore, the set N lies entirely in W^c .

We have that dynamics in U are described completely by the dynamics of the w -variables only: the values of (u, v) are uniquely defined (by the formula $(u, v) = \phi_c(w)$) by the w -variables at every moment of time for every orbit in the center manifold - hence for every orbit from the set N . Thus, in the small neighbourhood U of the point O system (2.5) is reduced to

$$\frac{dw}{dt} = Dw + f_3(\phi_c(w), w), \quad w \in \mathbb{R}^p. \quad (2.8)$$

This system is called the reduction of the original system to invariant manifold. This is a p -dimensional system: our original equation may have as large dimension as we want, dimensions of u and v may be anything, but the dynamics of our system is completely and accurately described by the reduced system (2.8).

We repeat that the Center Manifold Theorem is truly remarkable because it gives a dramatic reduction of the dimension of the problem. Indeed, instead of original $(m+k+p)$ -dimensional system, we can consider only a p -dimensional one and p is the number of eigenvalues on the imaginary axis which is usually very small!

We now turn to the proof of Theorem 2.6.

Proof. We will only prove the second statement of the theorem, that

(*) *if the backward orbit of $M \in U$ stays in U for all $t \leq 0$, then $M \in W^{cu}$.*

Then, by the symmetry of the problem, reversing time t to $-t$, we will get the similar statement for the center-stable manifold W^{cs} .

Proving (*) is the same as to show that forward orbits that stay in U are attracted to W^{cu} . Namely, we will show that there exists $\alpha > 0$ such that if $M(t) \in U$ for $t \in [t_1, t_2]$, then

$$\text{dist}(M(t_2), W^{cu}) \leq e^{-\alpha(t_2-t_1)} \text{dist}(M(t_1), W^{cu}). \quad (2.9)$$

After that we take $t_2 = 0$. If the backward orbit of M stays in U for all $t \leq 0$, then, in the limit where t_1 tends to minus infinity, $\text{dist}(M(t_1), W^{cu})$ remains bounded and the inequality (2.9) gives $M = M(0) \in W^{cu}$. So, we only need to prove inequality (2.9).

By Theorem 2.2, we know that the system

$$\begin{cases} \frac{du}{dt} = Bu + f_1(u, v, w) = F_1 & \text{Spectrum}(B) \subset \{\text{Re } < 0\}, \\ \frac{dv}{dt} = Cv + f_2(u, v, w) = F_2 & \text{Spectrum}(C) \subset \{\text{Re } > 0\}, \\ \frac{dw}{dt} = Dw + f_3(u, v, w) = F_3 & \text{Spectrum}(D) \subset \{\text{Re } = 0\}, \end{cases} \quad (2.10)$$

has a smooth locally invariant manifold $u = \phi_{cu}(v, w)$, where $\phi_{cu}(0) = 0$ and $\phi'_{cu}(0) = 0$. We will show that this fact together with the fact that all eigenvalues of B have negative real parts imply that the manifold W^{cu} is exponentially attracting (i.e., inequality (2.9) holds indeed).

To this end, we use a very useful trick called the *straightening of the invariant manifold*. We will straighten W^{cu} - recall that this is the graph of $u = \phi_{cu}(v, w)$. In general, whenever we have an invariant surface, it often helps to try to replace it by an invariant plane making the initial surface straight. In our case, this is done by performing the following coordinate transformation:

$$u^{new} = u - \phi_{cu}(v, w). \quad (2.11)$$

To implement this coordinate transformation, we need to answer the question “*What is the form of the system (2.10) in the new coordinates?*”

One thing which we already know is that in the new coordinates W^{cu} is given by $u^{new} = 0$. It is a locally invariant manifold, i.e., if $u^{new}(0) = 0$ then

$u^{new}(t) \equiv 0$ as long as you are in a small neighbourhood of O . Therefore,

$$\frac{du^{new}}{dt} = 0 \quad \text{whenever} \quad u^{new} = 0, \quad (2.12)$$

no matter what v and w are.

Now let us actually do the coordinate transformation (2.11) and look how the system (2.10) changes in the new coordinates.

$$\begin{aligned} \frac{du^{new}}{dt} &= \frac{du}{dt} - \frac{\partial \phi_{cu}}{\partial v} \frac{dv}{dt} - \frac{\partial \phi_{cu}}{\partial w} \frac{dw}{dt} = F_1(u^{new} + \phi_{cu}(v, w), v, w) - \\ &\quad \frac{\partial \phi_{cu}}{\partial v} F_2(u^{new} + \phi_{cu}(v, w), v, w) - \frac{\partial \phi_{cu}}{\partial w} F_3(u^{new} + \phi_{cu}(v, w), v, w). \end{aligned} \quad (2.13)$$

From (2.13) and (2.12), it follows that

$$F_1(\phi_{cu}(v, w), v, w) = \frac{\partial \phi_{cu}}{\partial v} F_2(\phi_{cu}(v, w), v, w) + \frac{\partial \phi_{cu}}{\partial w} F_3(\phi_{cu}(v, w), v, w). \quad (2.14)$$

Consequently equation (2.13) transforms to

$$\begin{aligned} \frac{du^{new}}{dt} &= [F_1(u^{new} + \phi_{cu}(v, w), v, w) - F_1(\phi_{cu}(v, w), v, w)] - \\ &\quad \frac{\partial \phi_{cu}}{\partial v} [F_2(u^{new} + \phi_{cu}(v, w), v, w) - F_2(\phi_{cu}(v, w), v, w)] - \\ &\quad \frac{\partial \phi_{cu}}{\partial w} [F_3(u^{new} + \phi_{cu}(v, w), v, w) - F_3(\phi_{cu}(v, w), v, w)]. \end{aligned} \quad (2.15)$$

Taking into account the form of functions F_j , $j = 1, 2, 3$, we rewrite (2.15) as

$$\frac{du^{new}}{dt} = Bu^{new} + \Phi(u^{new} + \phi_{cu}(v, w), v, w) - \Phi(\phi_{cu}(v, w), v, w), \quad (2.16)$$

where $\Phi = f_1 - \frac{\partial \phi_{cu}}{\partial v} f_2 - \frac{\partial \phi_{cu}}{\partial w} f_3$, so

$$\Phi(0) = 0, \quad \frac{\partial \Phi}{\partial u}(0) = 0.$$

Indeed, since $\phi_{cu}(v, w)$ does not depend on u , only the functions f_1, f_2, f_3 need to be differentiated with respect to u , but all derivatives of f_j , $j = 1, 2, 3$, at the origin are zeroes and, therefore, $\frac{\partial \Phi}{\partial u}(0)$ is also equal to zero. Also, we note that by the definition of Φ , we lose one derivative with respect to v and w , but do not lose any derivatives with respect to u . Thus, function Φ may be only continuous with respect to v and w , but it is at least C^1 -smooth with respect to u .

To proceed further, let us show that

$$\|\Phi(u^{new} + \phi_{cu}(v, w), v, w) - \Phi(\phi_{cu}(v, w), v, w)\| \leq \beta \|u^{new}\| \quad (2.17)$$

for some small $\beta > 0$, provided that (u, v, w) belongs to a small neighbourhood of O . Let us for a moment assume that u is a scalar function and Φ doesn't depend on v and w , then the right-hand side of (2.16) has a particular form,

i.e., it contains a difference of values of a function Φ at points ϕ_{cu} and $u^{new} + \phi_{cu}$. Thus, by the Lagrange formula,

$$\Phi(a) - \Phi(b) = \Phi'(\xi)(a - b), \text{ where } \xi \in (a, b), \quad (2.18)$$

where in our case $a = u^{new} + \phi_{cu}$ and $b = \phi_{cu}$. Since both a and b are close to zero, ξ is also close to zero, and from the condition $\Phi'(0) = 0$ and the continuity of Φ' we conclude that $|\Phi'(\xi)| \leq \beta$ for some small β and the desired estimate holds.

It is well-known that the Lagrange formula fails in the vector case, but a slightly modified version of it (which is often referred as integral mean value theorem) remains true in a vector case. Namely, we want to rewrite (2.16) in the form

$$\frac{du^{new}}{dt} = [B + \Psi(u^{new}, v, w)]u^{new}, \quad \Psi(0) = 0. \quad (2.19)$$

What is Ψ here? It has to satisfy the following identity:

$$\Psi(u^{new}, v, w)u^{new} = \Phi(u^{new} + \phi_{cu}(v, w), v, w) - \Phi(\phi_{cu}(v, w), v, w).$$

Define $\xi(s) = \Phi(su^{new} + \phi_{cu}(v, w), v, w)$, then

$$\begin{aligned} \Phi(u^{new} + \phi_{cu}(v, w), v, w) - \Phi(\phi_{cu}(v, w), v, w) &= \xi(1) - \xi(0) = \\ &= \int_0^1 \frac{d\xi}{ds} ds = \int_0^1 \frac{\partial \Phi}{\partial u}(su^{new} + \phi_{cu}(v, w), v, w) dsu^{new}. \end{aligned}$$

Thus,

$$\Psi(u^{new}, v, w) = \int_0^1 \frac{\partial \Phi}{\partial u}(su^{new} + \phi_{cu}(v, w), v, w) ds$$

and, indeed, Ψ is a continuous function and it is small in a small neighbourhood of O . Therefore, we have estimate (2.17) in a general case as well.

Thus, in (2.19) we have a matrix B whose eigenvalues have negative real parts and a small correction Ψ . In Lecture 1, when we studied an exponentially stable equilibrium, we showed in a very similar situation, that

$$\frac{d}{dt} \|u^{new}(t)\| \leq -\alpha \|u^{new}(t)\|, \quad (2.20)$$

where the norm is computed in a special Jordan base and $\alpha > 0$ is some constant which is less than minus $\min_{j=1,\dots,k} \operatorname{Re} \lambda_j$. Integrating (2.20) we get the exponential convergence:

$$\|u^{new}(t_2)\| \leq e^{-\alpha(t_2-t_1)} \|u^{new}(t_1)\|, \quad (2.21)$$

where t_1 and t_2 are such that a point $M(t) = (u(t), v(t), w(t))$ lies in a small neighbourhood of O for all $t \in [t_1, t_2]$. Recall that $u^{new} := u - \phi_{cu}(v, w)$, so $\|u^{new}(t)\|$ is naturally interpreted as a distance to a center-unstable manifold and (2.21) gives the desired estimate (2.9) and finishes the proof of the theorem. \square

The next natural question to ask is:

“What happens when the system depends on parameters?”

Recall that the bifurcation theory is not just about the dynamics in a small neighbourhood of something, it is also about what happens with the dynamics when we change parameter.

So, let us consider the following system:

$$\frac{dy}{dt} = Ay + f(y, \varepsilon), \quad f(0, 0) = 0, \quad \frac{\partial f}{\partial y}(0, 0) = 0, \quad (2.22)$$

where $y = y(t) \in \mathbb{R}^n$ is an unknown vector-function depending on time t , A is $n \times n$ matrix, $\varepsilon \in \mathbb{R}^s$ is a parameter, $f : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^n$ is a given smooth vector-function. At $(0, 0)$ system (2.22) has equilibrium and $f(y, \varepsilon)$ at $\varepsilon = 0$ is a nonlinearity (with respect to y only) which starts from quadratic terms. When we change ε , the function f changes, for example, some linear terms may appear, etc. The equilibrium that used to be at zero for $\varepsilon = 0$ can also move or even disappear, eigenvalues at the equilibrium can change - in particular, they can move away from the imaginary axis, etc.

Could we use the center manifold theorem in order to understand the behaviour of trajectories in this case? In particular, is the dimension reduction described above still possible? The answer to both these questions is positive. We can still use the central manifold theorem with the help of the following trick: we add the equation $\frac{d\varepsilon}{dt} = 0$ to the system (2.22). We can do this since ε is a parameter which does not change in time. Then the pair $(y, \varepsilon) = (0, 0)$ is an equilibrium of the extended $(n+s)$ -dimensional system

$$\begin{cases} \frac{dy}{dt} = Ay + f(y, \varepsilon) \\ \frac{d\varepsilon}{dt} = 0. \end{cases} \quad (2.23)$$

The eigenvalues of the linear part of this system consist of the eigenvalues which come from the first equation of (2.23) and the eigenvalues, which come from the second equation $\frac{d\varepsilon}{dt} = 0$, i.e., from the 0-matrix - and all the eigenvalues of 0-matrix are zero. So, by adding the equation $\frac{d\varepsilon}{dt} = 0$, we may treat the parameters as *extra central variables*. Thus, we now have $p+s$ center variables (w, ε) and the conditions of the center manifold theorem are satisfied for these new extended system and we have the following result.

Theorem 2.7. *There exists a smooth locally invariant center manifold W_c of equation (2.23) which is the graph of a function $\phi_c : \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^k \times \mathbb{R}^m$:*

$$W^c := \left\{ (u, v, w, \varepsilon) \in \mathbb{R}^{n+s} : (w, \varepsilon) \in \mathbb{R}^{p+s}, (u, v) = \phi_c(w, \varepsilon) \right\},$$

where $\phi_c(0, 0) = 0$ and $\phi'_c(0, 0) = 0$ such that the all orbits of the extended system (2.23) which never leave a small neighbourhood U of $y = \varepsilon = 0$ (neither in forward, nor in backward time) must belong to the manifold W^c .

Thus, we have formally obtained a $(p+s)$ -dimensional reduced system on a center manifold W_c . The equation $\frac{d\varepsilon}{dt} = 0$ is not modified by this reduction, so the reduced system has the form

$$\frac{dw}{dt} = Dw + f_3(\phi_c(w, \varepsilon), \varepsilon), \quad \frac{d\varepsilon}{dt} = 0 \quad (2.24)$$

Thus, for any fixed small $\varepsilon = \varepsilon_0 \in \mathbb{R}^s$, the restriction $W_c(\varepsilon_0) := W_c|_{\varepsilon=\varepsilon_0}$ is a locally invariant p -dimensional smooth manifold of the initial system (2.22) with $\varepsilon = \varepsilon_0$, and the dynamics on it are determined by the first equation of (2.24) (with $\varepsilon = \varepsilon_0$). So, the local behaviour of trajectories of (2.22) for a fixed small ε is completely determined by the reduced p -dimensional system (2.24).

We repeat once more that

$$\dim W^c(\varepsilon) = p = \text{"the number of central eigenvalues at } \varepsilon = 0\text{"}$$

which is usually very small, no matter how large n is. As we already mentioned, typically, if we do not have parameters, the matrix $A = f'(x_0)$ does not have eigenvalues on the imaginary axis, so $\dim W^c = 0$, i.e., W^c is the equilibrium $x = x_0$. When we perturb the system slightly, the center manifold remains zero-dimensional, hence nothing interesting happens (the equilibrium persists, and all other orbits leave the neighbourhood).

In contrast to this, in one-parameter families of matrices ($\varepsilon \in \mathbb{R}^1$), we typically may have values of parameters for which there is either a single 0 eigenvalue (then $\dim W^c = 1$), or a pair of eigenvalues $\pm i\omega$ (then $\dim W^c = 2$) on the imaginary axis.

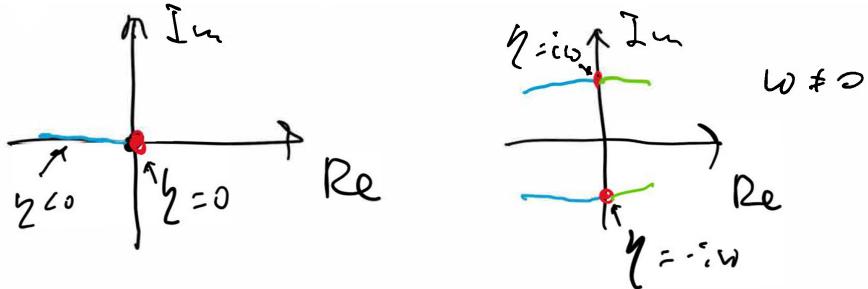


FIGURE 5

In principle, it may happen that two complex conjugate eigenvalues and one real eigenvalue (or two pairs of complex conjugate eigenvalues) cross the imaginary axis at the same value of the parameter. Then the reduced dimension will be $p = 3$ (or $p = 4$), but this is not typical (different eigenvalues “do not see” each other) and may simultaneously come to the imaginary axis in some exceptional cases only. Typically, we need one more parameter in order to “synchronize” their crossing of the imaginary axis. Thus, if we want to have more eigenvalues on the imaginary axis, we need more parameters (typically).

In this course, we will consider only two cases

Case I: $\eta_1 = 0 \Rightarrow \dim W^c = 1$;

Case II: $\eta_{1,2} = \pm i\omega \Rightarrow \dim W^c = 2$.

We will also build a similar theory for periodic orbits.

LECTURE 4. SINGLE ZERO EIGENVALUE

We have learned that the behaviour near equilibrium is determined by the number of eigenvalues on the imaginary axis. Typically the number of such eigenvalues (i.e., with zero real part) is very low. For example, if we have a one-parameter family $f(x, \varepsilon)$, $\varepsilon \in \mathbb{R}^1$, then typically we have only the equilibria with simple eigenvalue $\lambda = 0$ or a pair of complex conjugate eigenvalues on the imaginary axis.

In this lecture, we consider the case where we have exactly one simple eigenvalue $\lambda = 0$ and no more eigenvalues at the imaginary axis. In this case there is a one-dimensional center manifold, and all orbits which never leave a small neighbourhood of our equilibrium (for the original system as well as for the perturbed system, when we change parameters slightly) must belong to this one-dimensional invariant manifold (this manifold is just a curve). So, let us put the equilibrium at zero and consider the restriction of the original system to the center manifold

$$\frac{dw}{dt} = g(w, \varepsilon), \quad w \in \mathbb{R}^1, \quad (4.1)$$

where $g(0, 0) = 0$, and $\frac{\partial g(w, \varepsilon)}{\partial w}|_{(w, \varepsilon)=(0,0)} = 0$, so $\varepsilon = 0$ is exactly the moment of the bifurcation when our system has one zero eigenvalue. The question which we ask is:

“What happens with the solutions of the equation (4.1) in a small neighbourhood of the equilibrium state, i.e., for small values of w and ε ?”

Note that the reduced equation (4.1) is *scalar*, i.e., $w \in R^1$ is not a vector. Such equations are very simple and can be solved by the separation of variables:

$$\frac{dw}{dt} = g(w) \Rightarrow \int \frac{dw}{g(w)} = t + C.$$

You have studied this at school or at the first year in the University, but the problem is that it could be difficult to take this integral explicitly, and also one has to divide by g and g may be zero, etc. So, this is a nice formula which is not particularly useful for our purposes. What we usually do is: we do not solve the equation (4.1) explicitly, instead we just investigate it qualitatively. And the investigation is based on the following observation:

$$\begin{aligned} g(w) > 0 &\Rightarrow \frac{dw}{dt} > 0 \Rightarrow w(t) \text{ grows;} \\ g(w) < 0 &\Rightarrow \frac{dw}{dt} < 0 \Rightarrow w(t) \text{ decreases.} \end{aligned}$$

Thus, we draw a graph of the function g and see when $g(w) = 0$ which corresponds to equilibria of the system (4.1). If on the left of an equilibrium the function g is positive and on the right it is negative, then the equilibrium is stable, and vice versa: if g changes sign from negative to positive, then the equilibrium is unstable.

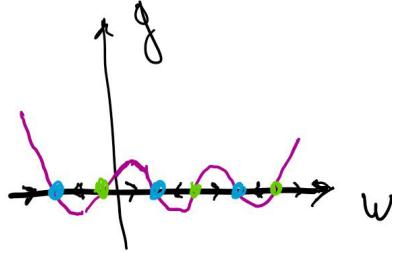


FIGURE 1

In the figure, stable and unstable equilibria are indicated by blue and green colors respectively. This picture is based on a general theory describing the qualitative behaviour of solutions for equations on a straight line which we formulate as an exercise below.

Exercise 4.1. a) Prove that every orbit of

$$\frac{dw}{dt} = g(w), \quad w \in I := [a, b] \subset R^1,$$

either leaves I as time grows, or tends to an equilibrium state.

b) Prove the same as t decreases.

c) Prove that an equilibrium state w_0 (where $g(w_0) = 0$) is stable if $g'(w_0) < 0$ and unstable if $g'(w_0) > 0$.

Some comments to the exercise. If the orbit starts at equilibrium state, it remains at equilibrium state. Suppose the orbit does not start at equilibrium state, but starts somewhere in between two equilibria. It cannot cross the equilibrium state, because the orbits of smooth differential equations cannot have common points. But in-between the equilibria the function $g(w)$ preserves sign, so the orbit $w(t)$ is a monotone bounded function, therefore, it must have a limit: $\lim_{t \rightarrow \infty} w(t) = w_0^*$. Then, from the equation we conclude that the derivative $w'(t) = \frac{dw}{dt}(t)$ also has a limit which is equal to $g(w_0^*)$. We claim that $g(w_0^*) = 0$ and, therefore, w_0^* is an equilibrium. Indeed, due to the Lagrange formula (mean value theorem), we have

$$w(n+1) - w(n) = w'(\xi_n), \quad \xi_n \in (n, n+1),$$

hence $0 = \lim_{n \rightarrow \infty} w(n+1) - \lim_{n \rightarrow \infty} w(n) = \lim_{n \rightarrow \infty} w'(\xi_n) = \lim_{t \rightarrow \infty} w'(t) = g(w_0^*)$.

Thus, the theory says that the full behaviour of the system is determined by the number of equilibria and their stability. Moreover, in a typical situation when we have finitely many equilibria and all of them are simple (i.e., $g'(w_0^*) \neq 0$), their stability is alternating: if the first one is stable, the second one is unstable, the third one is stable again, etc. By this reason, we are mainly interested in the stability of utmost left equilibria, e.g., if we know the utmost left equilibrium is stable, then the next one will be unstable and so on. Or we can start with the utmost right equilibrium: if for example it is unstable, then the next one will be stable and so on.

Thus, when we consider bifurcations of an equilibrium with one zero eigenvalue, the main question is how many equilibria the system on the center manifold may have and what are their stabilities.

Let us study this question. At $\varepsilon = 0$ we have an equilibrium which we put this equilibrium at the origin. Let us Taylor expand the function g at zero. We note that the linear term is equal to zero, because we assume the zero eigenvalue. So, we start with quadratic terms:

$$\frac{dw}{dt} = g(w) = l_2 w^2 + l_3 w^3 + \dots$$

The coefficients of this expansion are called Lyapunov coefficients. Lyapunov lived from the middle of nineteenth to twentieth centuries. He made an enormous contribution to the dynamical systems theory, so many things in dynamical systems are called by his name. There exist Lyapunov exponents, Lyapunov metric, Lyapunov function (actually, when we did computations in a special Jordan base when proving the exponential stability of an equilibrium state with negative real parts of the eigenvalues = negative Lyapunov exponents, we constructed a Lyapunov metric and showed that its square is a Lyapunov function). We will mostly deal with Lyapunov coefficients.

The next theorem determines the number of equilibria in the perturbed system.

Theorem 4.2. *If the first non-zero Lyapunov coefficient is l_k , then no more than k equilibria can be born under a small smooth perturbation.*

This means that if we allow g to be a function of a parameter ε , no more than k equilibria can exist for small ε in a small neighbourhood of zero.

Proof. At $\varepsilon = 0$ our system has a form $\frac{dw}{dt} = g(w) = l_k w^k + o(w^k)$, so

$$\frac{\partial^k g}{\partial w^k}(0) = k! l_k \neq 0.$$

Therefore, $\frac{\partial^k g}{\partial w^k} \neq 0$ for all small w and ε . This already implies the desired property that $g(\cdot, \varepsilon)$ has no more than k zeros in a small neighbourhood of

zero for every fixed small ε . Indeed, due to Lagrange formula, between any two zeroes of g there is a zero of g' , so if g has more than k zeroes, then g' has more than $k - 1$ zeroes. Similarly g'' has more than $k - 2$ zeroes, \dots , $g^{(k)}$ has more than $k - k = 0$ zeroes, which is a contradiction. \square

This estimate is sharp: if $l_2 = \dots = l_{k-1} = 0$, $l_k \neq 0$, then up to k equilibria can be born at arbitrary small ε . Indeed, at $\varepsilon = 0$ the system reads

$$\frac{dw}{dt} = l_k w^k + o(w^k) = w^k \psi(w),$$

where ψ is equal to $l_k \neq 0$ plus something small, so it is non-zero. Then we replace w^k by $(w - \varepsilon)(w - 2\varepsilon)\dots(w - k\varepsilon)$:

$$\frac{dw}{dt} = (w - \varepsilon)(w - 2\varepsilon)\dots(w - k\varepsilon)\psi(w). \quad (4.2)$$

The right-hand side smoothly depends on a parameter ε and, when $\varepsilon = 0$, we get the original system, which means that this is indeed a small perturbation of the original system. The right-hand side of (4.2) has k different real roots: $w = \varepsilon, \dots, w = k\varepsilon$, i.e. the system (4.2) has exactly k equilibria.

The next question is the stability of the newly born equilibria. As before, our equation is $\frac{dw}{dt} = g(w, \varepsilon)$, $w \in I \subset \mathbb{R}^1$. The stability of any equilibrium

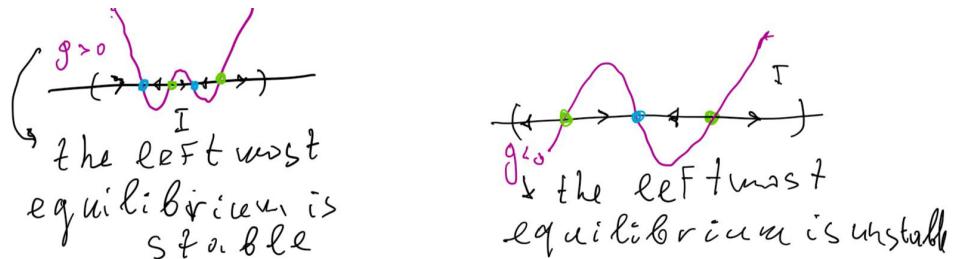


FIGURE 2

depends on whether g is negative or positive to the left (or to the right) of the equilibrium, and since the stability type alternates, it is enough to know the stability type of the utmost left or utmost right equilibrium.

Obviously, the stability type of the utmost left equilibrium after the bifurcation coincides, for all small ε , with the stability from the left of the zero equilibrium at the moment of bifurcation. Indeed, the stability of the original equilibrium at the critical moment $\varepsilon = 0$, when the system is given by $\frac{dw}{dt} = g(w, 0)$, is determined as follows:

$$\begin{aligned} g > 0 \text{ at } w < 0 &\Rightarrow \text{the stability from the left} \\ g < 0 \text{ at } w < 0 &\Rightarrow \text{the instability from the left.} \end{aligned}$$

If we take any negative point such that $g \neq 0$, at this point, the sign of g cannot change at a sufficiently small change of ε . Therefore, if the equilibrium were

stable from the left at the critical moment, then for all small ε we will have $g(w, \varepsilon) > 0$ at negative w which are bounded away from $w = 0$, so the utmost left equilibrium will always be stable from the left in this case. Similarly, if the original equilibrium were unstable from the left at the critical moment, then we will have $g < 0$ at small negative w which stay at a sufficient distance from zero, meaning the instability from the left of the utmost left equilibrium.

It remains to figure out the stability type of the origin at $\varepsilon = 0$. Recall the equation:

$$\frac{dw}{dt} = g(w) = l_k w^k + o(w^k).$$

There are two different cases: when k is even, and when it is odd.

Suppose $k = 2p$, i.e. it is even. Then, let us draw a graph of the function w^{2p} . It is always positive (except for $w = 0$) and everything is determined by the sign of l_{2p} (since $o(w^{2p})$ is too small to change the sign of g). Thus, if l_{2p} is positive, the function g will be always positive and $\frac{dw}{dt}$ also will be positive, i.e., w will grow, and if l_{2p} is negative, then w will decay (except for $w = 0$ where we have the equilibrium).

Thus, if l_{2p} is positive, the equilibrium is stable from the left and unstable from the right. We will have the similar situation if l_{2p} is negative: then our equilibrium state will be unstable from the left and stable from the right. This kind of equilibria is called semi-stable.

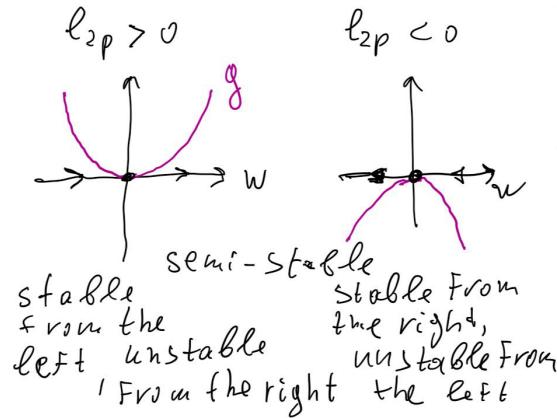


FIGURE 3

We have completely different picture if $k = 2p + 1$, i.e., if it is odd. Then the sign of w^{2p+1} is the same as the sign of w . If l_{2p+1} is negative, then the sign of $\frac{dw}{dt}$ will be opposite to the sign of w which gives us stability. Indeed, if we start from the left, we have to grow because g is positive, and if we start from the right, we have to decay because g is negative.

If l_{2p+1} is positive, then the sign of $\frac{dw}{dt}$ is the same as the sign of w and thus, the equilibrium is unstable (see Figure 4).

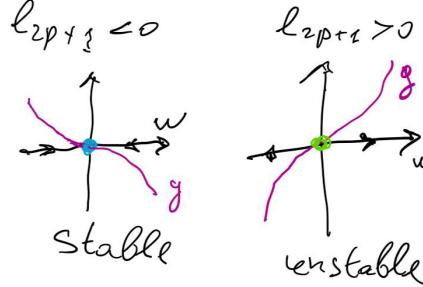


FIGURE 4

This is the full description at $\varepsilon = 0$. The results for $\varepsilon \neq 0$ follow immediately, as was explained before.

Now let us discuss some philosophy behind the stability theory, which works not only in one-dimensional case. As we have already seen, in one-dimensional case the stability/instability of the left/right most equilibrium depends on the stability of the equilibrium at the bifurcation moment (at $\varepsilon = 0$). The similar statement is true in general, when we consider arbitrary bifurcations of an equilibrium. Indeed, suppose we have an equilibrium and it is asymptotically stable. It means that there exists a small neighbourhood of the equilibrium such that any orbit which starts inside of this neighbourhood or on the boundary of this neighbourhood tends to the equilibrium as time goes to plus infinity. In particular, any orbit which starts at the boundary goes inside, and no orbit which starts inside can get out. Now we add a perturbation which may lead to a very complicated bifurcation, i.e., instead of one equilibria, a very complicated set can be created, but the fact that if you start on the boundary of that fixed neighbourhood, then all orbits must go inside, persists. So, we do not know in general what happens after the bifurcation, but if we know that

the equilibrium is asymptotically stable at $\varepsilon = 0 \Rightarrow$ the bifurcation is safe,

i.e. no orbit will go away and if we start in a small neighbourhood of the equilibrium we will remain in it.

An opposite happens in the case when

the equilibrium is unstable at $\varepsilon = 0 \Rightarrow$ the bifurcation is dangerous.

Why is it dangerous? Since the instability at $\varepsilon = 0$ means there exists a neighbourhood such that you can find initial condition which is arbitrarily close to your equilibrium but the corresponding orbit goes out of this neighbourhood. Now you add a perturbation, but you reach this fixed boundary

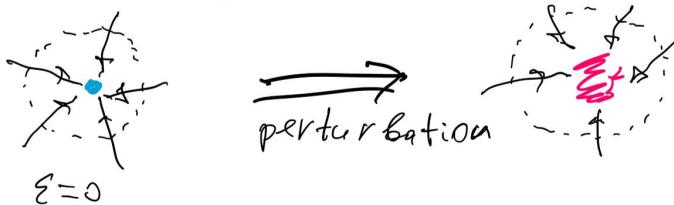


FIGURE 5

in a finite-time, so if you perturb slightly your system, you will still be able to reach this boundary starting from a small neighbourhood of the original equilibrium (whose size depends on a size of the perturbation). So, maybe after the perturbation the equilibrium will even become stable, but sufficiently close to it there will be orbits which drive your system far away. This is dangerous. Imagine you start with a stable equilibrium (in Figure 6, the right picture) which is born from the bifurcation of an unstable one, and everything sufficiently close is perfectly stable. Then a small disturbance happens and the system goes far away from where it was supposed to be - a catastrophe happens.

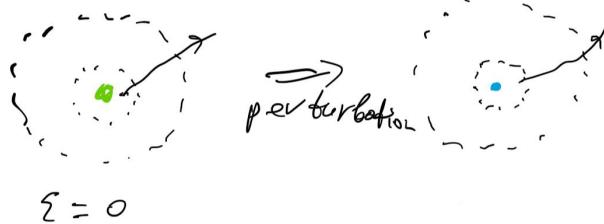


FIGURE 6

The next topic which we will consider is the stability of the whole system. Everything we have considered before was the stability on the central manifold, but for our equilibrium state, except for one zero eigenvalue, we may have others, some of them with positive real parts and some of them with negative ones. If there exist unstable eigenvalues ($\text{Re } \gamma > 0$), then the equilibrium is unstable no matter what is the stability on the central manifold W^c .

Indeed, we have proved that the distance to a center-unstable manifold goes to zero exponentially in forward time, and that the distance to a center-stable manifold goes to zero exponentially in backward time. So, if we are not on a center-stable manifold, then in forward time the distance to it grows. This exactly means instability. Thus, in the case when we have unstable eigenvalues, what happens on the center manifold is not very important from the point of view of the stability theory.

If there are no unstable eigenvalues, then the stability is the same as on the center manifold W^c . Indeed, in this case the whole space is the center-stable

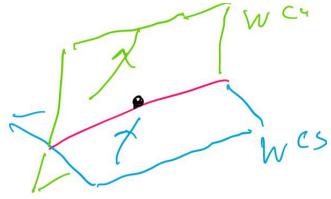


FIGURE 7

manifold, so we have some behaviour on the center manifold and transverse to it we have contraction. This is not a proof, in order to prove this statement you need the so-called invariant foliation theorem, and also to work more on the invariant manifold theorem which we did not do, but this is a general principle in which I hope you believe. So, we have some slow dynamics on a center manifold and behaviour transverse to it is just an exponential contraction.

For example, if we have a semi-stable equilibrium on the center manifold, i.e., $\frac{dw}{dt} = l_{2p}w^{2p} + o(w^{2p})$, $l_{2p} \neq 0$, then transverse to W^c we have contraction and, as a result, in the whole space we have the so-called saddle-node (see Figure 8). There is a certain boundary, which is called the strong stable manifold. To the right of this boundary you have a saddle-like behaviour, i.e. you converge in u direction, then follow the center manifold and go away. And on the left of the boundary you converge both in u and w directions. This is the so-called node behaviour.

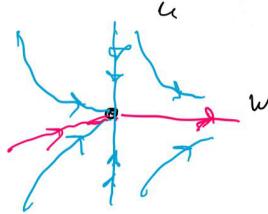


FIGURE 8

In the case of a stable equilibrium on the center manifold, i.e., $\frac{dw}{dt} = l_{2p+1}w^{2p+1} + o(w^{2p+1})$, $l_{2p+1} < 0$, the equilibrium for the whole system will be also stable (recall that we assumed there are no unstable eigenvalues).

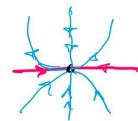


FIGURE 9

If the equilibrium is unstable on the center manifold, i.e., $\frac{dw}{dt} = l_{2p+1}w^{2p+1} + o(w^{2p+1})$, $l_{2p+1} > 0$, then in terms of the whole system we have saddle behaviour.

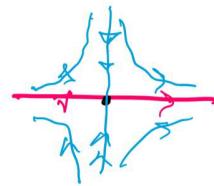


FIGURE 10

Conclusion: Bifurcations of equilibria with one zero eigenvalue are determined by the sign and the number k of the first non-zero Lyapunov coefficient l_k at $\varepsilon = 0$. To be more precise:

1. Up to k equilibria, of alternating stability can be born.
2. If k is even, then the equilibrium is semi-stable on W^c . The bifurcation is dangerous, and the equilibrium can disappear at the bifurcation.

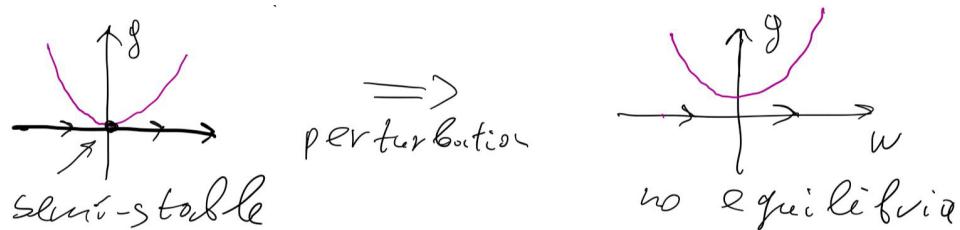


FIGURE 11

3. If k is odd, then the equilibrium is *stable* on W^c if $l_k < 0$, and whatever perturbation you do at least one equilibrium will exist. This is a *safe* bifurcation. After the bifurcation you may have up to k equilibria, but the most left and the most right ones will be stable. Inside you may have instability, but this instability does not lead to a catastrophe.

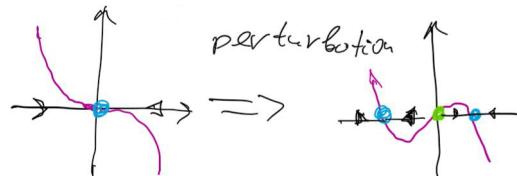


FIGURE 12

If $l_k > 0$, then again after the perturbation an equilibrium will not disappear and up to k equilibria may be born, but this a *dangerous* bifurcation, since at $\varepsilon = 0$ the equilibrium is *unstable* and therefore after the perturbation the most left and the most right equilibria will be unstable. Inside you may have a stable equilibrium, but its domain of attraction is always bounded by the nearby unstable equilibria, so it is very small.

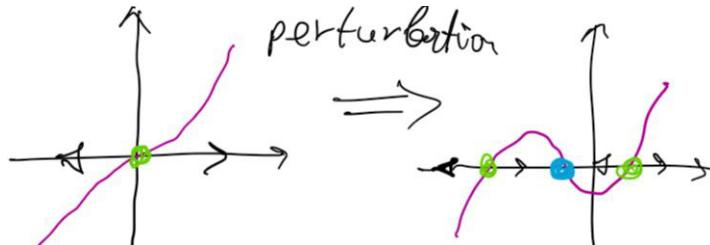


FIGURE 13

LECTURE 5. SINGLE ZERO EIGENVALUE ($l_2 \neq 0$)

We continue studying bifurcations of an equilibrium with a single zero eigenvalue, i.e., $\dim W^c = 1$. The restriction of our system to the center manifold W^c reads:

$$\frac{dw}{dt} = g(w, \varepsilon), \quad w \in \mathbb{R}^1,$$

where g is a smooth function of w and ε and we look at the behaviour in a small neighbourhood of the equilibrium. We also assume that our system depends on parameters ε and when the parameters change bifurcations may happen (we fix $\varepsilon = 0$ as a bifurcation point). Therefore, at $\varepsilon = 0$ there is an equilibrium with a simple zero eigenvalue. Without loss of generality we may put this equilibrium to zero, i.e. $g(0, 0) = 0$, and since we have a zero eigenvalue, $\frac{\partial g}{\partial w}(0, 0) = 0$ (this is a corollary of conditions which we pose on the system saying that $\varepsilon = 0$ is a bifurcation point). We do not want to impose more conditions, so we assume that the second derivative of g at zero is not zero, i.e.,

$$\frac{\partial^2 g}{\partial w^2}(0, 0) = 2l_2 \neq 0.$$

It is what typically happens. “Typically” is a crucial word here, because there may be untypical situations, when $l_2 = 0$ or some more coefficients in the Taylor expansion of g are equal to zero, but typically it does not happen. As usual, the question is:

“What happens when we change ε ? ”

We start with one equilibrium but, when ε changes, the number of equilibria may change, and since $l_2 \neq 0$, no more than two equilibria can emerge. Another question is the stability of these equilibria. As we discussed, the stability is alternating, for example, the left equilibrium is stable and the right one is unstable, or right is stable, left is unstable. We also established that when $l_2 \neq 0$ the equilibrium is semi-stable. So, practically we know everything, but we want to investigate this problem a little bit more - in order to show how the bifurcation theory works. This is the simplest example of a bifurcation, with minimal assumptions.

We recall that, due to the above assumptions, the nonlinearity near $w = 0$ has a form

$$g(w, \varepsilon) = l_2 w^2 + a_0(\varepsilon) + a_1(\varepsilon)w + a_2(\varepsilon)w^2 + \dots,$$

where $a_i(\varepsilon)$ tend to zero as $\varepsilon \rightarrow 0$. The first thing we want to do is to get rid of the term $a_1(\varepsilon)w$ by doing a simple coordinate transformation. To this end, we note that, since $\frac{\partial g}{\partial w}(0, 0) = 0$ and $\frac{\partial^2 g}{\partial w^2}(0, 0) \neq 0$, by the implicit function

theorem, for all small ε there exists a unique $w^*(\varepsilon)$ such that $\frac{\partial g}{\partial w}(w^*(\varepsilon), \varepsilon) \equiv 0$. Clearly $w = w^*(\varepsilon)$ is a critical point of $g(w, \varepsilon)$ (minimum or maximum) for every fixed small ε .



FIGURE 1

Doing the Taylor expansion of the function g at the critical point w^* we get:

$$\begin{aligned} \frac{dw}{dt} = g(w^*(\varepsilon), \varepsilon) + \frac{\partial g}{\partial w}(w^*(\varepsilon), \varepsilon)(w - w^*(\varepsilon)) + \\ \frac{\partial^2 g}{\partial w^2}(w^*(\varepsilon), \varepsilon)(w - w^*(\varepsilon))^2 + o((w - w^*(\varepsilon))^2). \end{aligned}$$

We denote the first term by $\mu(\varepsilon) := g(w^*(\varepsilon), \varepsilon)$. The second term is zero, since $\frac{\partial g}{\partial w}(w^*(\varepsilon), \varepsilon) = 0$. The third term is non-zero – by continuity since $\frac{\partial^2 g}{\partial w^2}(0, 0) \neq 0$.

Now we perform the promised coordinate transformation. Note again, that coordinate transformations are one of the main tools in the theory of dynamical systems, as well as in the theory of differential equations: again and again, whenever we have a differential equation, we actively avoid solving it, and instead we do coordinate transformations which simplify the equation. And if we are lucky enough, after several transformations our system becomes so simple that there is nothing to study. This is the idea, and in our case the desired transformation is

$$w = w^* + y.$$

Basically, what we do is we put the origin not at the equilibrium state, but at the critical point w^* . Then we get:

$$\frac{dy}{dt} = \mu(\varepsilon) + l_2(\varepsilon)y^2 + o(y^2). \quad (5.1)$$

The term $\mu(\varepsilon)$ is very important and it is called *the control parameter*.

In bifurcation theory people often talk about control parameters. What is a control parameter? It is some function of the parameters of the system, such that the value of this function determines what happens to the system, e.g. changes of the value of this function lead to changes in system's behaviour.

Let us first consider the case when l_2 is positive. Then at the bifurcation moment $\mu = 0$, l_2 is positive, y^2 is positive (everywhere except for $y = 0$), and $o(y^2)$ is too small and can not change the sign of l_2y^2 . Thus, we have the following picture: 0 is an equilibrium state, if we start on the left of 0 we

move to the right, and if we start on the right we also move to the right. So the equilibrium is stable from the left and unstable from the right, so it is a semi-stable point.

Now we perturb the parameters. Suppose μ is positive. If $\mu > 0$, then our “parabola” $l_2 y^2 + o(y^2)$ moves up, which means that the right-hand side of (5.1) is always positive (since μ is the value of the right-hand side at the minimal point). So, at every point, $\frac{dy}{dt}$ is positive, i.e., there are no equilibrium states any more, all orbits come from the left and go out to the right.

If μ becomes negative, then the “parabola” $l_2 y^2 + o(y^2)$ moves down and there appear some values of y for which the right-hand side is negative. There also remain values where it is positive, so there must be values where it is equal to zero. The right-hand side of (5.1) is positive on the left and on the right and it is negative in the middle. We have also proved that if $l_2 \neq 0$, there could be no more than two equilibria, so it must be exactly one root on the left of the critical point and one root on the right. We have stability from the left, so the left equilibrium is stable, and instability from the right, so the right one is unstable.

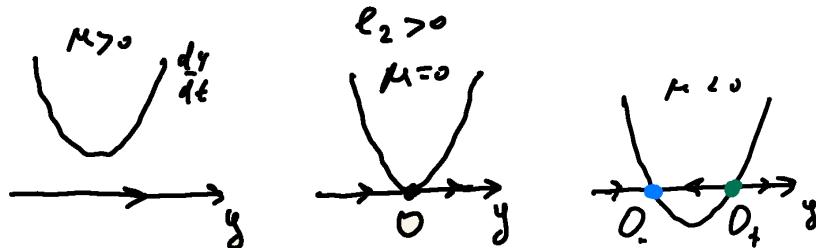


FIGURE 2

The similar picture we have for $l_2 < 0$. At the moment of bifurcation ($\mu = 0$), the right-hand side $l_2 y^2 + o(y^2)$ is negative everywhere, except for $y = 0$, where it is equal to zero. So, $\frac{dy}{dt}$ is negative, and, thus, $y(t)$ is a decreasing function. Everything moves to the left, except of the point O ($y = 0$). In other words, at zero we have a semi-stable equilibrium, which is stable from the right and unstable from the left.

When we make μ negative, at the point of maximum the right-hand side is negative ($= \mu$), so it is negative everywhere. So, the orbit just comes from the right and goes to the left, i.e., there are no equilibrium states.

When μ is positive, we again have two equilibrium states: the left one is unstable, because the equilibrium was unstable from the left at the moment of bifurcation, and the right one is stable, since at $\mu = 0$ the equilibrium was stable from the right.

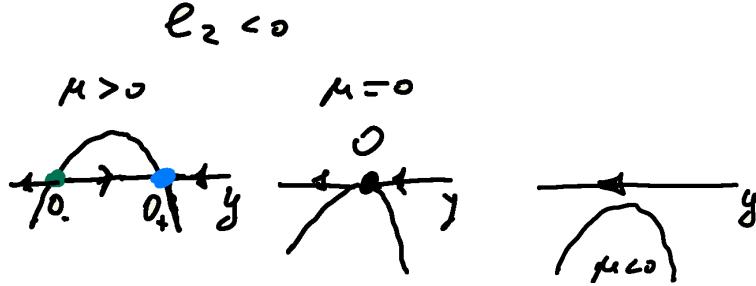


FIGURE 3

We summarize the obtained results in the following theorem.

Theorem 5.1. *When $\mu = 0$, there is only one equilibrium O , semi-stable. When $\mu \cdot l_2 > 0$, the equilibrium disappears. When $\mu \cdot l_2 < 0$, two equilibria are born,*

$$O_{\pm} : y = \pm \sqrt{-\frac{\mu}{l_2}} + o(\sqrt{|\mu|}),$$

one of them is stable and another one is unstable (on W^c).

Proof. Actually, everything is already proven, except for the explicit formula for O_{\pm} . So, we take the equation

$$\mu + l_2 y^2 + o(y^2) = 0 \quad (5.2)$$

for $\mu \cdot l_2 < 0$ and try to solve it. Without the term $o(y^2)$ the solution of (5.2) is exactly $y = \pm \sqrt{-\frac{\mu}{l_2}}$. Thus, what we need to prove is that adding the term $o(y^2)$ will only lead to an $o(\sqrt{|\mu|})$ -correction to the solution of (5.2). How do we do this? We do a coordinate transformation (again!). This time it is the scaling transformation:

$$y = Y \sqrt{-\frac{\mu}{l_2}} \quad (5.3)$$

It is a usual trick in asymptotic analysis: whenever you have a small parameter (in our case μ), it is useful to scale your equation, such that the equation becomes regularized.

After the scaling (5.3), our problem (5.2) transforms to

$$\mu - l_2 \frac{\mu}{l_2} Y^2 + o(\mu) = 0.$$

Then we divide by μ , since we assume that $\mu \neq 0$, so we can always do this:

$$1 - Y^2 + o(1)_{\mu \rightarrow 0} = 0, \quad (5.4)$$

where $o(1)_{\mu \rightarrow 0}$ means that this term goes to zero, when μ goes to zero. In order to solve (5.4), i.e., find Y as a function of μ , we use the implicit function theorem (IFT).

If $\mu = 0$ the equation (5.4) has two solutions, $Y = \pm 1$. If we differentiate (5.4) with respect to Y at $\mu = 0$, the derivative will be $-2Y$, which is not equal to zero, since $Y \neq 0$. Thus, by the IFT, we have two roots: $Y = \pm 1 + o(1)_{\mu \rightarrow 0}$. The IFT gives us a unique solution, so how it can be two solutions? Actually, there is no contradiction here because the IFT gives us a unique solution of the perturbed equation only in a *small* neighbourhood of the non-perturbed solution. In the non-perturbed case (at $\mu = 0$) we have two non-degenerate solutions $Y = 1$ and $Y = -1$, so when we add a perturbation, by the IFT, we will have exactly one perturbed solution in each of the small neighbourhoods of these two unperturbed solutions. By the IFT these solutions also depend smoothly on the parameters.

Returning to the original variables, we get

$$y = \pm \sqrt{-\frac{\mu}{l_2}} + o(\sqrt{|\mu|})$$

and the theorem is proven. \square

Let us have a small discussion. Why do we need to do scaling in (5.3)? Why we can not directly apply the IFT to equation (5.2)? Because at $\mu = 0$, the solution of (5.2) is $y = 0$, and it is degenerate. Namely, when we differentiate (5.2) with respect to y we will get $2l_2y$ which is equal to zero, when y is equal to zero, which means that the IFT is not applicable.

Now let us consider the multi-dimensional case, where at $\mu = 0$ we have a saddle-node. In this case, on the center manifold W^c we have a semi-stable point, whereas in the directions transverse to W^c we have a contraction.

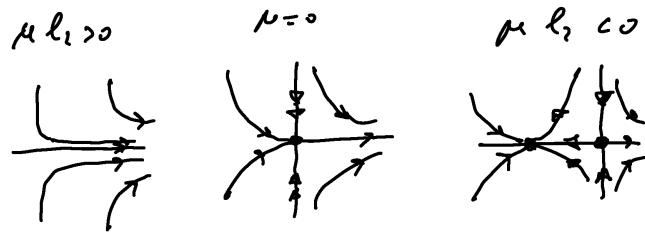


FIGURE 4

So, in the Figure 4 (in the middle), if we start somewhere on the left, i.e., in the region of stability, we will first contract very close to the center manifold and then just follow it and tend to the equilibrium point. Thus, on the left we have a node behaviour. And on the right, on the center manifold we have instability. So, as before we will first contract to the center manifold, then follow it, until we leave the small neighbourhood of the equilibrium. Thus, on the right we have a saddle behavior.

When equilibrium disappears, i.e., $\mu \cdot l_2 > 0$, then everything goes away. We contract close to the center manifold, then follow the center manifold and go away (see Figure 4 (in the left)).

When $\mu \cdot l_2 < 0$, then the original equilibrium decomposes into two. On the center manifold one point is stable and another one is unstable. Then for the whole system the stable point remains stable, but the unstable point on the center manifold will become saddle for the whole system (see Figure 4 (in the right)).

So, in the multi-dimensional case the saddle-node bifurcation is not a decomposition to stable and unstable points, it is a decomposition to stable point and saddle. You can think like that: first you had one node point, then a saddle point comes from outside (the right picture in Figure 4), then it comes closer and closer and eventually collides creating a saddle-node (the middle picture in Figure 4), after further change of parameters saddle-node also disappears (the left picture in Figure 4). Or vice versa, there was nothing, then out of blue sky a saddle-node is appeared, then after another small change of parameters it decomposes into two points (node and saddle).

If we also have an unstable direction, then for $\mu = 0$ we have what is called saddle-saddle equilibrium state, see the middle picture of Figure 5.

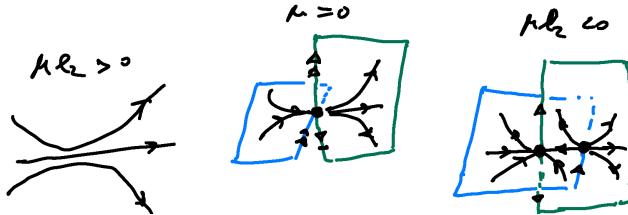


FIGURE 5

We have as before a one-dimensional center manifold. Assume for simplicity that we have a three-dimensional system with one positive and one negative eigenvalue as plotted in Figure 5. Then both center-stable and center-unstable manifolds are 2D surfaces (close to planes), which intersect over the one-dimensional center manifold. The left half part of the center-stable manifold is shown by blue in Figure 5) and on this semi-plane we have the contraction. So, the left part of this manifold is a half plane on which everything is converging to the equilibrium point. On the right half part of this manifold (which is not shown at the picture) we will have a weak expansion along the direction of the center manifold and the contraction in the transversal direction.

The right half part of the center-unstable manifold is shown by green. On this half plane all orbits diverge from the equilibrium state (i.e., converge to it when time tends to minus infinity). On the non-plotted left part of this

manifold we have a saddle behavior with slow convergence along the center manifold and divergence in the transversal direction. All other orbits leave the neighbourhood of the equilibrium in both backward and forward time.

When we perturb the parameters, this saddle-saddle point may disappear (if $\mu \cdot l_2 > 0$), so all orbits go away, see the left picture of Figure 5. Or, when $\mu \cdot l_2$ becomes negative, we will have stable point and unstable point on the center manifold, but transverse to it we always have contraction and expansion, so both of these points will become saddles, see the right picture of Figure 5. Let us look at what is the difference between these points. Since we assume that there is exactly one unstable and one stable direction in the non-perturbed problem, then for the equilibrium which is stable on the center manifold its unstable manifold (in the full system) is one-dimensional, but for the point, which is unstable on the center manifold, its unstable manifold (again in the full system) is the whole plane, i.e. it is two-dimensional. We have the similar picture for stable manifolds.

So, if we go from right to left in Figure 5, first we have two equilibrium states, both of them are saddles but with different dimensions of stable and unstable manifolds, then they collide, form a saddle-saddle and eventually disappear. Or, if we go from left to right, first we have nothing, then from nothing creates saddle-saddle, and then it decomposes into two different saddles.

Let us conclude the lecture by discussing the nature of *control parameters* in the bifurcation theory. Recall that the initial system of differential equations (which usually describes the behaviour of real systems arising in physics, biology, chemistry, etc.) is given to us and the parameters ε also have physical meaning and also are given to us. But as we have seen in the previous example, only a certain function of them $\mu = \mu(\varepsilon)$ is really important for understanding what happens under the bifurcation and the perturbations of ε which do not change the value of the control parameter μ change nothing and are not important for us. In particular, most important for us are the values of ε such that $\mu(\varepsilon) = 0$. So, we need to understand how the surface $\mu(\varepsilon) = 0$ looks like in the space of physical parameters ε .

In a typical situation, we may do this using the IFT. To this end we need the non-degeneracy condition at $\varepsilon = 0$ ($\mu'(0) \neq 0$) to be satisfied. In some situations dependence of μ on parameters ε is degenerate, i.e., $\frac{\partial \mu}{\partial \varepsilon} = 0$ at zero, but typically it is not the case, i.e. $\frac{\partial \mu}{\partial \varepsilon} \neq 0$. What does it mean? The parameter μ in our case is a scalar function, but ε can be a vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ and the non-degeneracy condition

$$\frac{\partial \mu}{\partial \varepsilon} = \left(\frac{\partial \mu}{\partial \varepsilon_1}, \frac{\partial \mu}{\partial \varepsilon_2}, \dots, \frac{\partial \mu}{\partial \varepsilon_k} \right) \neq 0$$

means that at least one component of this vector is not equal to zero. In the simplest case, when ε is scalar, if $\frac{d\mu}{d\varepsilon} \neq 0$ then μ is a monotone function of ε . As a result, μ can be taken as a new parameter of our system instead of ε .

If ε is multi-dimensional, the non-degeneracy condition means that for some j we have $\frac{\partial\mu}{\partial\varepsilon_j} \neq 0$ at $\varepsilon = 0$, so by the IFT, the equation

$$\mu(\varepsilon_1, \dots, \varepsilon_j, \dots, \varepsilon_k) = 0 \quad (5.5)$$

defines a smooth codimension-1 surface in the ε -space

$$\varepsilon_j = \phi(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_{j+1}, \dots, \varepsilon_k). \quad (5.6)$$

If we look at the equation (5.5), we see that at $\mu = 0$ bifurcation happens, i.e., when μ is positive we have one behaviour and when μ is negative we have another behaviour, and the boundary is exactly $\mu = 0$, thus the surface (5.6) divides a neighbourhood near 0 of the space of parameters into two regions, in one region μ is positive and in another one it is negative. So, above this surface we have no equilibrium states, and below it we have two equilibrium states. When ε is a scalar, (5.6) is not a surface, it is just a point, where μ is zero, and by this point the line of parameters is divided into two regions, one where μ is positive, and another one where it is negative.

The surface (5.6) is of codimension-1, meaning that its dimension is the dimension of the space of parameters minus one, i.e., $k - 1$.

In general, bifurcation theory studies much more complicated situations, so people tend to classify bifurcations by codimension, where the *codimension of bifurcation* is the number of control parameters we need to somehow describe what is going on. We deliberately make it sound vague, because the same bifurcation can be treated as a bifurcation of codimension 1, codimension 2, and so on. It depends exactly on which questions we ask about the system. In the situation we consider, we want to describe everything and we end up with a very good description. Here one parameter is definitely enough, so, this is a bifurcation of codimension 1. But in more complicated situations we actually know (and may even prove this rigorously), that we can never give a complete description of what is going on. Whatever description you give, it will be incomplete (only partial) and it will just answer certain particular questions. We choose the number of control parameters in dependence on these particular questions.

But in the cases we study in this course, we typically know how many control parameters we need and only at the end of the course we will see the situation when the number of these parameters becomes an issue.

Control parameters are somehow intrinsic to dynamics, but usually the dependence on parameters has nothing to do with dynamics, so the relation between true parameters and control parameters is important and usually people

require non-degeneracy. If you have a bifurcation of codimension d , then the non-degeneracy means that the Jacobi matrix

$$\frac{\partial \mu}{\partial \varepsilon} = \frac{\partial(\mu_1, \dots, \mu_d)}{\partial(\varepsilon_1, \dots, \varepsilon_n)} = \begin{pmatrix} \frac{\partial \mu_1}{\partial \varepsilon_1} & \frac{\partial \mu_1}{\partial \varepsilon_2} & \dots & \frac{\partial \mu_1}{\partial \varepsilon_k} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \mu_d}{\partial \varepsilon_1} & \frac{\partial \mu_d}{\partial \varepsilon_2} & \dots & \frac{\partial \mu_d}{\partial \varepsilon_k} \end{pmatrix}$$

has a full rank. In the non-degenerate case bifurcations of codimension d correspond to smooth surfaces of codimension d in the parameter space. For example, if we have d parameters, then a bifurcation of codimension d corresponds to a point, if we have $d + 1$ parameters, then a bifurcation of codimension d corresponds to a curve, if we have $d + k$ parameters, then a bifurcation of codimension d corresponds to a k -dimensional surface, and so on.

LECTURE 6. SINGLE ZERO EIGENVALUE ($l_3 \neq 0$)

We continue studying bifurcations of an equilibrium with a single zero eigenvalue which means that we can restrict our system to a one dimensional central manifold, $\dim W^c = 1$. The reduced system on this manifold reads

$$\frac{dw}{dt} = g(w, \varepsilon), \quad w \in \mathbb{R}^1,$$

where g is a smooth function of w and ε is a bifurcation parameter(s). As usual, we look at the behaviour in the small neighbourhood of the equilibrium state $w = 0$ and small ε ($\varepsilon = 0$ is a bifurcation point). But now we consider the case of a codimension-two bifurcation, the so-called *cusp bifurcation* (we will see why it is called like this).

What does it mean “codimension 2”? It means that we impose more conditions on our system. To be more precise, we assume that at $\varepsilon = 0$

$$g(0, 0) = 0, \quad \frac{\partial g}{\partial w}(0, 0) = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial w^2}(0, 0) = 0.$$

At the previous lecture we required only the first two conditions to be satisfied, where the first one is not actually a condition, it just means that we put the origin at the equilibrium. But now we add the condition that the second derivative of g with respect to w at zero is equal to zero. We also assume that the third Lyapunov exponent l_3 is not zero (which is typically true):

$$\frac{\partial^3 g}{\partial w^3}(0, 0) = 6l_3 \neq 0.$$

Thus, we only have two true conditions ($\frac{\partial g}{\partial w}(0, 0) = 0$ and $\frac{\partial^2 g}{\partial w^2}(0, 0) = 0$) and therefore this is a bifurcation of codimension 2.

One should be careful, there is a relation between a number of conditions we impose on our system and the number of control parameters we need. Usually each parameter is responsible for controlling one condition, but for some more complicated bifurcations the number of conditions may be less than the number of control parameters. But this does not happen in the considered case of a cusp bifurcation: as we will see below, two control parameters is enough to understand this bifurcation completely. This, in this case the number of conditions imposed equals to the number of control parameters.

Similarly to the previous situation, we get rid of one term in the Taylor expansion of $g(w, \varepsilon)$ by a coordinate transformation. Now it will be not linear, but the quadratic term. Indeed, since $\frac{\partial^2 g}{\partial w^2}(0, 0) = 0$ and $\frac{\partial^3 g}{\partial w^3}(0, 0) = 6l_3 \neq 0$,

by the implicit function theorem (IFT), for all sufficiently small ε , there exists a unique $w^*(\varepsilon)$ such that $\frac{\partial^2 g}{\partial w^2}(w^*(\varepsilon), \varepsilon) \equiv 0$.

Now we do the coordinate transformation

$$y = w - w^*,$$

where we put the origin into the point where the second derivative of g with respect to w vanishes. Then we do expansion at the point w^*

$$\frac{dy}{dt} = g(w^*(\varepsilon), \varepsilon) + \frac{\partial g}{\partial w}(w^*(\varepsilon), \varepsilon)y + \frac{\partial^3 g}{\partial w^3}(w^*(\varepsilon), \varepsilon)y^3 + o(y^3), \quad (6.1)$$

denote

$$\mu_0 := g(w^*(\varepsilon), \varepsilon), \quad \mu_1 := \frac{\partial g}{\partial w}(w^*(\varepsilon), \varepsilon), \quad \text{and} \quad l_3(\varepsilon) := \frac{1}{6} \frac{\partial^3 g}{\partial w^3}(w^*(\varepsilon), \varepsilon)$$

and rewrite (6.1) as

$$\frac{dy}{dt} = f(y, \varepsilon) = \mu_0 + \mu_1 y + l_3(\varepsilon)y^3 + o(y^3), \quad y \in \mathbb{R}^1. \quad (6.2)$$

As we will see, μ_0 and μ_1 are the desired control parameters.

This is an equation on a straight line, so the problem reduces to finding the number of equilibria, i.e., roots of $f(y, \varepsilon)$ for every small ε . The other question is their stability, but the stability types of equilibria just alternate (when all equilibria correspond to simple roots) and the stability on the most left one is determined by the stability at the point of bifurcation. As we already discussed, if l_3 is negative, then this equilibrium is stable, and if l_3 is positive, then this equilibrium is unstable. Thus, after the bifurcation we may have up to three equilibria and if l_3 is negative, then the most left and most right ones will be stable and the middle one will be unstable. If $l_3 > 0$ then the most left and the most right equilibria will be unstable, but the middle one will be stable. So, the stability is not an issue, the issue is only how many equilibrium states we have.

The idea is (instead of trying to solve the equation $g(w, \varepsilon) = 0$) to look for which ε the number of equilibria may change. If for some ε the function f has a simple root (such that $\frac{\partial f}{\partial y} \neq 0$), then by the IFT this root does not disappear, moreover it depends smoothly on ε . The only possibility for the number of roots to change is when we have a non-simple root (this is a root at which not just the function is equal to zero but also its derivative). For example, in Figure 1 (right picture), if we change f a little bit (like going up), then the right root will disappear. This is exactly a saddle-node bifurcation, which we have studied before, i.e. the bifurcation of a semi-stable equilibrium. In contrast to this, the simple root in Figure 1 (left picture) cannot disappear:

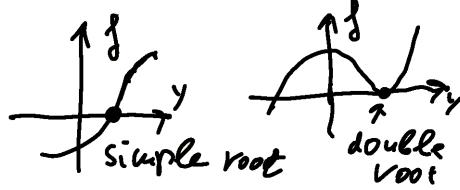


FIGURE 1

The equation for non-simple roots

$$\begin{cases} f(y, \varepsilon) = 0 \\ \frac{\partial f}{\partial y}(y, \varepsilon) = 0 \end{cases} \quad (6.3)$$

defines a *bifurcation set* in the parameter space. Indeed, (6.3) is a system of two equations on many variables, y is one-dimensional, whereas ε may be multidimensional. Suppose that, from one of the equations (6.3), we can express y as a function of ε . Then we substitute it to the remaining equation and get the equation on ε which already does not contain the parameter y and therefore defines a certain subset of the parameter space, which is exactly a bifurcation set.

Usually by a bifurcation set we refer to any set in the space of parameters where any bifurcation happens. But in this particular setting, when we study systems on \mathbb{R}^1 , bifurcations correspond only to changing of the number of roots, so (6.3) defines exactly the bifurcation set for our problem.

We assume non-degeneracy, i.e., that the matrix

$$\frac{\partial \mu}{\partial \varepsilon} = \begin{pmatrix} \frac{\partial \mu_0}{\partial \varepsilon_1} & \frac{\partial \mu_0}{\partial \varepsilon_2} & \dots & \frac{\partial \mu_0}{\partial \varepsilon_k} \\ \frac{\partial \mu_1}{\partial \varepsilon_1} & \frac{\partial \mu_1}{\partial \varepsilon_2} & \dots & \frac{\partial \mu_1}{\partial \varepsilon_k} \end{pmatrix}$$

has a full rank. Moreover, in order to be able to realize any values of μ_0 and μ_1 close to zero, the vector ε must be at least two-dimensional, i.e., $k \geq 2$. Otherwise (for $k = 1$), we will typically have only a curve in the control parameter space defined implicitly by $\mu_0 = \mu_0(\varepsilon)$ and $\mu_1 = \mu_1(\varepsilon)$. For simplicity, we will consider below only the case where the number of parameters is exactly two, then $\frac{\partial \mu}{\partial \varepsilon}$ is an invertible matrix at zero. Therefore, by the IFT, we can choose μ_0 and μ_1 as new parameters and ε will be a smooth function of $\mu = (\mu_0, \mu_1)$.

Now, let us substitute the explicit formula for the function f into the system (6.3):

$$\begin{cases} f(y, \mu) = \mu_0 + \mu_1 y + l_3(\mu) y^3 + o(y^3) = 0, \\ \frac{\partial f}{\partial y}(y, \mu) = \mu_1 + 3l_3(\mu) y^2 + o(y^2) = 0. \end{cases} \quad (6.4)$$

We see that we have two equations on three variables (y, μ_0 and μ_1). We want to express μ_0 and μ_1 as functions of y , thus by the IFT we should check

that

$$\det \frac{\partial(f, \frac{\partial f}{\partial y})}{\partial(\mu_0, \mu_1)} \Big|_{(y, \mu)=(0,0)} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0. \quad (6.5)$$

This gives us that the functions $\mu_0 = \mu_0(y)$ and $\mu_1 = \mu_1(y)$ solving (6.4) are uniquely defined and smooth for small y and we may expand them in Taylor series. How do we do this? We note that when μ is small $l_3(\mu)$ is close to $l_3(0)$, thus in equations (6.4) we replace $l_3(\mu)$ by $l_3(0)$ (in the leading order) and get from the second equation that

$$\mu_1 = -3l_3(0)y^2 + o(y^2).$$

Substituting this expression to the first equation, we get the desired presentation of the bifurcation set (the bifurcation curve in our case) in the parametric form:

$$\begin{cases} \mu_1 = -3l_3(0)y^2 + o(y^2), \\ \mu_0 = 2l_3(0)y^3 + o(y^3). \end{cases} \quad (6.6)$$

Finally, excluding the parameter y from this equation, we will get this curve as a graph $\mu_1 = \mu_1(\mu_0)$. Indeed, dropping out the non-essential terms $o(y^2)$ and $o(y^3)$, we get the expression

$$\mu_1 = \sqrt[3]{\frac{27l_3}{4}\mu_0^2}.$$

It can be shown that in a general situation, the terms which we have dropped out will just correct this formula by non-essential terms:

$$\mu_1 = \sqrt[3]{\frac{27l_3}{4}\mu_0^2} \left(1 + o(1)_{\mu_0 \rightarrow 0} \right).$$

Plotting the graph, we see that the bifurcation curve has a characteristic cusp shape (see Figure 2 red curve). Recall that by the construction, this is the curve which corresponds to double roots.

First, let us assume that $l_3 < 0$. Then the bifurcation diagram is depicted in Figure 2. Indeed, the bifurcation curve divides the plane of parameters into two regions and the number of roots can change only when we cross this red curve. In other words, the number of roots is the same for all points belonging to the same region. By this reason, it is enough to compute the number of root at one specific point of every region. For instance, for the region below the curve, we may take a point $\mu_0 = 0, \mu_1 < 0$. Then we have

$$\frac{dy}{dt} = g = \mu_1 y + l_3(\mu)y^3 + o(y^3), \quad \mu_1 < 0, \quad l_3 < 0.$$

Since μ_1 and l_3 are both negative, the function g is monotonically decreasing in a neighbourhood of $y = 0$ and, therefore $y = 0$ is the only root. Thus, we have only one equilibrium in this region which is stable (recall that the stability is

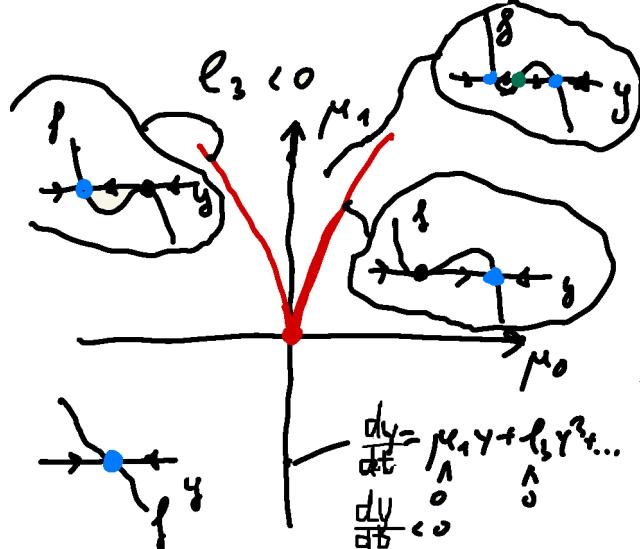


FIGURE 2

determined by the stability of the non-perturbed equilibrium). Analogously, we may take $\mu_1 > 0$ and see that in this case we have three equilibria, the left one is stable, the middle one is unstable and the right one is again stable.

Finally, we may take a point on the left branch of the bifurcation curve and check that the left root will be simple and stable whereas the right double one will be semi-stable. On the right branch the situation is opposite, the left one is double semi-stable and the right one is simple and stable. At the origin all three roots collide.

To illustrate how the behaviour of our system changes, let us imagine that we start from the region below the red bifurcation curve, then change the parameters in such a way that at some moment we cross this curve, go inside the region above the curve, then cross it again and return back to the region below the curve (assume also that we do not cross this curve through the cusp point at the origin). What happens then with our system?

We have initially started with a stable single equilibrium. Then at the moment when we cross the red curve first time, an extra equilibrium appears from nothing (which is degenerate and semi-stable). When we move further into the region above the red curve this semi-stable equilibrium decomposes into stable and unstable equilibria (we already studied this, this is exactly a saddle node bifurcation from the previous lecture). After that, when we are crossing the red curve again, the unstable equilibrium comes closer and closer to another stable equilibrium and collides with it when we are on the red curve, again generating a semi-stable equilibrium. After the further small change of

parameters, this semi-stable equilibrium also disappears and we again have only one stable equilibrium.

We have a very similar picture when $l_3 > 0$. All reasoning is the same, but since l_3 is now positive, the bifurcation curve is in the lower half-plane of a parameter plane. Then we compute the value of a function $f = \mu_0 + \mu_1 y + l_3(\mu)y^3 + o(y^3)$ at one point, for example, at $\mu_0 = 0$ and $\mu_1 > 0$, and see that there is exactly one equilibrium in the region above the curve, which is unstable; when we cross the bifurcation curve and get to the region below, we get two more equilibria (stable and unstable).

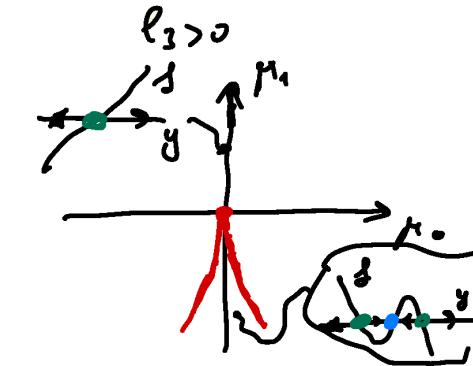


FIGURE 3

To summarize, we emphasize a very important principle:

bifurcations happen only when the bifurcation set is crossed.

Mention also that Figures 2 and 3 are called *bifurcation diagrams*, i.e., bifurcation diagram is the decomposition of the space of parameters into regions of different behaviour. Once bifurcation diagram is built and bifurcations are analyzed, the behaviour for every parameter value from a given region can be reconstructed from the behaviour at one parameter value from this region.

LECTURE 7. ANDRONOV-HOPF BIFURCATION

We have, essentially, finished studying of bifurcations of equilibria with a single zero eigenvalue and now turn to the study of the case of two complex conjugate eigenvalues on the imaginary axis. In this case we have $\lambda_{1,2} = \pm i\omega$ with some $\omega > 0$ and the corresponding center manifold is two-dimensional: $\dim W^c = 2$. Thus, the reduced system is also two-dimensional, so we need to consider systems on a plane.

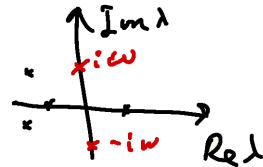


FIGURE 1

Recall that our original system is real, so all non-real eigenvalues always go in complex conjugate pairs. In particular, the non-real eigenvalues crossing the imaginary axis also form pairs of complex conjugate, so their number cannot be less than two and the corresponding center manifold cannot be one dimensional.

The case of a single pair of non-real eigenvalues on the imaginary axis is again a codimension-1 phenomenon. Indeed, if you have a one-parameter family of systems such that, for some parameter values, a pair of eigenvalues is on the left of the imaginary axis and, for some other parameter values, this pair is on the right of it, then, by the continuity, there should be some parameter value, when this pair of complex conjugate eigenvalues lies on the imaginary axis. Any small change in the family will not break this fact, meaning that this bifurcation occurs in one-parameter families in a robust way.

Now, suppose that for some parameter value all eigenvalues are on the left of the imaginary axis. Then we change parameters and pair of eigenvalues crosses the imaginary axis and goes to the right, so equilibrium state becomes unstable (note that the equilibrium does not disappear as it does not have 0 as an eigenvalue). How does this change of stability happen? The answer is given by the theorem which was proven by Andronov and Leontovich (for two-dimensional systems) in 1937. In 1942 it was also proven by Hopf (for multidimensional systems) from a little bit different perspective. At that time

there was no center manifold theorem, but now we know that the center manifold theorem allows one to reduce everything to dimension two. So, you may take results for the two-dimensional case and they are automatically applicable for the higher-dimensional case. In the works of Andronov, Leontovich and Hopf it was proven that the stability loss when eigenvalues go from the left to the right is accompanied with a birth of *periodic orbits*.

Thus, the Andronov-Hopf bifurcation = a birth of periodic orbits from an equilibrium state with a pair of purely imaginary eigenvalues.

The reduced system on the center manifold is

$$\frac{dx}{dt} = Ax + o(x), \quad x \in \mathbb{R}^2, \quad (7.1)$$

where

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}. \quad (7.2)$$

As usual, we put the origin at the equilibrium state. We write the matrix A in this way because it is a well-known fact from Linear Algebra that any 2×2 matrix with eigenvalues $\pm i\omega$ can be brought to the specific form (7.2). Note, in particular, that $\text{Tr } A = 0$, as the sum of the eigenvalues $\lambda_1 + \lambda_2 = i\omega - i\omega = 0$, and $\det A = \omega^2$ since it is the product of the eigenvalues ($\lambda_1 \cdot \lambda_2 = \omega^2 > 0$).

It is important that the determinant of the matrix A is non-zero. This implies, again by the implicit function theorem (IFT), when we change parameters, the equilibrium does not disappear and it is a smooth function of parameters. For every small ε , we make a coordinate transformation which puts the origin at this equilibrium. So, without loss of generality, our system can, for all small ε , be brought to the form

$$\frac{dx}{dt} = A(\varepsilon)x + o(x), \quad x \in \mathbb{R}^2, \quad (7.3)$$

where

$$A(\varepsilon) = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix} \quad (7.4)$$

and μ and ω depend smoothly on the parameters ε . The eigenvalues of the matrix $A(\varepsilon)$ are clearly $\lambda_{1,2} = \mu \pm i\omega$. This is just the general form of the pair of complex-conjugate eigenvalues and, any matrix with these eigenvalues can be brought to the form given by (7.4) (check that the linear transformation that brings the matrix to such form smoothly depends on parameters ε !). Note also that $\text{Tr } A(\varepsilon) = 2\mu$ which is exactly $\lambda_1 + \lambda_2$, and $\det A(\varepsilon) = \mu^2 + \omega^2$ which is, of course, $\lambda_1 \cdot \lambda_2$.

We have denoted the real part of the eigenvalues by μ since this is exactly a control parameter:

$$\begin{aligned}\mu > 0 &\Rightarrow \text{the equilibrium is unstable,} \\ \mu < 0 &\Rightarrow \text{the equilibrium is stable.}\end{aligned}$$

The question we want to answer is:

“What happens when the control parameter μ goes from negative values to positive ones?”

We already know that equilibrium changes stability, but the next theorem by Andronov, Leontovich, and Hopf adds more information.

Theorem 7.1. *The change of stability of the equilibrium at the origin is accompanied by the birth of a periodic orbit (maybe, several periodic orbits).*

Let us show how this works. To this end, we rewrite a system (7.4) in a coordinate-wise form. We obtain:

$$\begin{cases} \frac{dx_1}{dt} = \mu x_1 + \omega x_2 + o(x), \\ \frac{dx_2}{dt} = -\omega x_1 + \mu x_2 + o(x). \end{cases} \quad (7.5)$$

At the next step, we write our system in polar coordinates: $x_1 = r \cos \phi$ and $x_2 = r \sin \phi$:

$$\begin{cases} \frac{dr}{dt} \cos \phi - r \sin \phi \frac{d\phi}{dt} = \mu r \cos \phi + \omega r \sin \phi + o(r), \\ \frac{dr}{dt} \sin \phi + r \cos \phi \frac{d\phi}{dt} = -\omega r \cos \phi + \mu r \sin \phi + o(r). \end{cases} \quad (7.6)$$

If we multiply the first equation of (7.6) by $\cos \phi$ and the second one by $\sin \phi$ and sum them up, we get the equation for r ; and if we multiply the first equation by $\sin \phi$ and the second one by $\cos \phi$ and subtract them, we get the equation for ϕ :

$$\begin{cases} \frac{dr}{dt} = \mu r + o(r), \\ \frac{d\phi}{dt} = -\omega + o(1)_{r \rightarrow 0} \neq 0. \end{cases} \quad (7.7)$$

When we look at these equations, we see that r is a slow variable, since μ is small, and ϕ is a fast variable. So, our trajectories rotate clockwise in a plane around the origin, while the amplitude r may slowly increase or decrease. If, after one full round, the value of r decreases or increases, the orbit has a spiral shape. However, it may happen that r returns exactly to the same value when ϕ changes to 2π . This gives us a periodic orbit.

Now, we may have different cases.

First, we assume that $\mu < 0$. We start very close to zero, then $o(r)$ is much smaller than μr , so everything is determined by the term μr which is negative, therefore $\frac{dr}{dt} < 0$ and r decreases. Thus, taking into account

the second equation of (7.7), we obtain a spiral which goes closer and closer to equilibrium, so the equilibrium is stable. Analogously, when $\mu > 0$, the equilibrium state is unstable, and we have a spiral that goes away from the equilibrium.

One logical possibility is this: if we had, at $\mu = 0$, an asymptotically stable equilibrium, then when we change parameters, we should still continue to come close to this equilibrium, but the equilibrium becomes unstable, so there is something which separates orbits that converge close to the equilibrium and those diverging from it. This is a system on a plane, hence, by the Poincare-Bendixson theorem, this “separating something” should be either equilibria or periodic orbits. Since for such kind of problems we may only have one equilibrium (new equilibria might be born only if we had zero eigenvalues), this dividing object may only be a periodic orbit and it inherits the stability of the original equilibrium, so it must be stable.

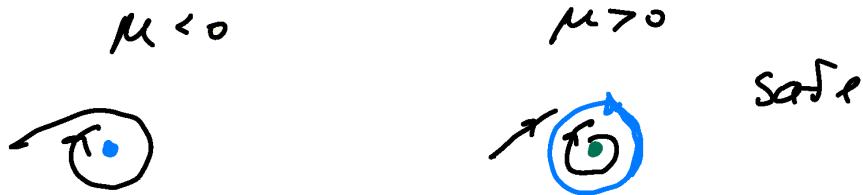


FIGURE 2

There may be other possibilities, for example, at $\mu = 0$ the equilibrium is unstable and all orbits spiral out of it. When we change parameters and make μ negative, the equilibrium becomes stable, but very close to it there must exist a periodic orbit, which must be unstable, so if we start inside this periodic orbit we converge to the equilibrium, and if we start outside we just go away.



FIGURE 3

The theorem, how it is formulated above, does not tell us which situation actually happens. The first case is when we have a stable equilibrium state, perturb it (change the value of μ from negative to positive) and still have some kind of stability. The equilibrium becomes unstable, but orbits do not go away from a small neighbourhood due to the presence of the stable periodic orbit. This is a safe bifurcation (Figure 2).

An example of a dangerous bifurcation is when we have a stable equilibrium and we do not know that very close to it there is an unstable periodic orbit. Then, because of some noise or a slight change of the parameter we can leave the safety zone and go far away (Figure 3).

Also, we are not obliged to have only one periodic orbit, there may be two, three, or even infinitely many of them in some situations. So, a lot may happen, but the first important thing is to know whether the bifurcation is safe or dangerous. How many periodic orbits are born is also interesting, but less important, because all of them are more or less the same, you just rotate around the origin. What is really important is their stability. The analysis here is not as straightforward as it was before, it requires the so-called normal forms theory, which we will discuss in the next lecture.

What distinguishes this case from the previous one (where we had a single zero eigenvalue) is that previously we started with an equilibrium, perturbed it, and only the number of equilibria changed. Here the situation is completely different: we start with an equilibrium state, perturb it, and have a new object born out of the equilibrium which is not an equilibrium, but a periodic orbit.

The previously studied case is the subject of the singularity theory which is the science about how the structure of the set of roots of a function changes when parameters change. The singularity theory is somehow regular: you have a certain object, a point which is a root of a function, then you change parameters and a new object which appears is still of the same nature, namely, it is still a point. Here we start with a point, change parameters and get a completely new object, which is a closed curve. This is the main feature of the bifurcation theory. In the bifurcation theory we always (except for our first example) see this phenomenon – objects of an arbitrary nature are born from the objects of a completely different nature when we perturb parameters of the system. Because of this, there is no pre-described set of methods which suits the bifurcation theory: we can never *a priori* predict the nature of objects which will emerge at bifurcations.

LECTURE 8. NORMAL FORM THEORY

We continue studying Andronov-Hopf bifurcation. Recall that in this case we have an equilibrium with a pair of complex conjugate eigenvalues (of the corresponding linearization matrix) which moves from the left half-plane to the right half-plane crossing the imaginary axis, and this is accompanied by the birth of periodic orbits. To investigate what happens here we need the *normal form* theory. This theory goes back to Poincare and this lecture is devoted to formulating and proving the main theorem of this theory.

Consider a system of differential equations:

$$\frac{dx}{dt} = Ax + o(x), \quad (8.1)$$

where $x = x(t) \in \mathbb{R}^n$ is an unknown vector-function depending on time t , i.e., $x = (x_1, x_2, \dots, x_n)$, and A is a real $n \times n$ matrix with constant coefficients; let its eigenvalues be $\lambda_1, \dots, \lambda_n$. We want to do coordinate transformations which make the system simpler. For instance, we may do *linear* transformations which bring the matrix A to a Jordan form. This simplifies the linear part of our equation. But we may also do *non-linear* transformations, and the general normal form theory is exactly about how to simplify the system near the equilibrium state by doing the non-linear transformations. Our goal is to make the system near an equilibrium as simple as possible, and the theory of normal forms gives us a way of doing this.

First, we bring the matrix A to a Jordan form. We assume, that the matrix A is diagonalizable (for example, all eigenvalues are simple). The same results hold true for a general Jordan form, but in order to simplify computations we consider only diagonalizable matrices. Thus, we do a coordinate transformation after which the matrix A becomes diagonal:

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

and our system reads:

$$\frac{dx_i}{dt} = \lambda_i x_i + \sum_{|m| \geq 2} a_m^{(i)} x_1^{m_1} \cdots x_n^{m_n}, \quad (8.2)$$

where $i = 1, \dots, n$, $m = (m_1, \dots, m_n)$, m_i are non-negative integers, and we denote $|m| = m_1 + \cdots + m_n$. So, instead of $o(x)$ we explicitly write all possible monomials of all degrees greater or equal to two with some coefficients $a_m^{(i)}$.

In other words, we expand the nonlinearity in Taylor series near $x = 0$. We mention from the very beginning that the analyticity of the non-linearity is not necessary for this theory, so the series may be formal (not necessarily convergent), moreover, we will actually use only finitely many terms in these expansions.

We start with a real matrix A , but it may have pairs of complex conjugate eigenvalues. Thus, after we diagonalize A and write the system (8.1) in the form (8.2), the vector x becomes, in general, complex. But it has a special form, i.e., when λ_i is complex conjugate to λ_{i+1} , then x_i is also complex conjugate to x_{i+1} . Moreover, all coordinate transformations, which we will do further, respect this relation. Thus, the transformed equations will be also split on pairs of complex conjugate ones, so there is not a problem at all to return back to real-valued coordinates.

We already brought our system to the normal form of order one (i.e. Jordan normal form) which greatly simplifies the linear part of the system (matrix A). Now we want to do a coordinate transformation which will also simplify the nonlinear part of the system.

Before formulating the theorem which allows us to do this, we need the following key definition.

Definition 8.1. The monomial $a_m^{(i)} x_1^{m_1} \cdots x_n^{m_n}$ in the i th equation of (8.2) is called *non-resonant* if $\lambda_i \neq m_1\lambda_1 + \dots + m_n\lambda_n$.

Further we will also use notation $(m, \lambda) := m_1\lambda_1 + \dots + m_n\lambda_n$.

Theorem 8.2. For any $N \geq 2$, there exists a local coordinate transformation $y = x + P(x)$, where the polynomial $P(x)$ does not contain zero and first order terms, such that all non-resonant monomials up to degree N get killed, i.e., in the new local coordinates y equations (8.2) read:

$$\frac{dy_i}{dt} = \lambda_i y_i + \sum_{|m| \geq 2, \lambda_i = (m, \lambda)}^N b_m^{(i)} y_1^{m_1} \cdots y_n^{m_n} + O(|y|^{N+1}). \quad (8.3)$$

where $b_m^{(i)} \in \mathbb{C}$ are some new coefficients.

Here $P = (P_1, \dots, P_n)$ is a vector polynomial, since x and y are vectors. In order to be able to say that $y = x + P(x)$ is a local coordinate transformation, we must be sure that it is invertible near zero. This local invertibility follows from the inverse (or implicit) function theorem and the condition that $P(x)$ starts from the quadratic terms. Indeed, due to this condition, we know that $P(0) = P'(0) = 0$, hence

$$\frac{dy}{dx}|_{x=0} = \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}|_{x=0} = \text{Id} + P'(0) = \text{Id}.$$

Since the derivative matrix of the transformation (the identity matrix) is invertible, the transformation itself is also locally invertible. Note that it is not necessarily globally invertible (for all $y \in \mathbb{R}^n$), so the transformation defined in this theorem works only in a small neighbourhood of zero, where system (8.1) is actually considered.

“Killing a monomial” is a standard terminology in this theory. This means that you make a coefficient in front of this monomial equal to zero, so all non-resonant monomials in the right-hand side of (8.3) are indeed “killed”.

The power of this theorem is that resonant monomials are rare and most of monomials are usually non-resonant, so removing the non-resonant monomials makes a great simplification of (8.1). We will see this in the example of Andronov-Hopf bifurcation in next lectures.

Proof. We will prove the theorem by induction by the order $M \leq N$ of the monomials. So, first we will kill all non-resonant monomials of order two, then of order three, and so on. Assume that all non-resonant monomials of order $|m| \leq M - 1 < N$ have already been killed and we need now to kill the monomial $x_1^{m'_1} \cdots x_n^{m'_n}$ of order $|m'| = m'_1 + \cdots + m'_n = M \geq 2$. We want to do this in the i th equation of (8.2) ($i = 1, 2, \dots, n$) by the following *explicitly given* transformation:

$$\begin{cases} y_i = x_i + cx_1^{m'_1} \cdots x_n^{m'_n} \\ y_j = x_j, \text{ for } j \neq i, \end{cases} \quad (8.4)$$

where $c \in \mathbb{C}$ is a number which will be determined later.

Since $|m'| \geq 2$, this transformation is locally invertible and the inverse is smooth and even analytic, by the implicit function theorem. It is important for us that the Taylor expansion for the inverse transform has the form:

$$x_i = y_i - cy_1^{m'_1} \cdots y_n^{m'_n} + O(y^{|m'|+1}). \quad (8.5)$$

Indeed, this can be verified by solving the equation $y = x + P(x)$ by successive iterations. Namely,

$$x^0 := y, \quad x^{(n+1)} = y - P(x^{(n)}), \quad x = \lim_{n \rightarrow \infty} x^{(n)}.$$

The existence of the limit is guaranteed by the Banach contraction theorem (and the iteration scheme is exactly what we usually do in order to prove the inverse/implicit function theorem). Then, one can see formula (8.5) already at step one and as it is also immediate to see that further iterations will not affect these lower order terms.

The statement of the theorem will follow from the following two elementary facts.

1. The terms of order up to $|m'| = M$ in the equations for $\frac{dy_j}{dt} = \frac{dx_j}{dt}, j \neq i$, are not affected by the coordinate transformation (8.4). Indeed, the original equations for $\frac{dx_j}{dt}$ are

$$\frac{dx_j}{dt} = \lambda_j x_j + \sum_{|m| \geq 2} a_m^{(j)} x_1^{m_1} \cdots x_n^{m_n}, j \neq i.$$

For monomials $x_1^{m_1} \cdots x_n^{m_n}$ which do not contain x_i the statement is obvious, since we may directly substitute $y_j = x_j$. So, we only need to consider monomials which contain x_i . The general form for such monomials is

$$x_i Q(x), \quad (8.6)$$

where $Q(x)$ is a monomial of any order greater or equal to one (whether or not $Q(x)$ contains x_i is not important). We substitute (8.5) into (8.6) and get

$$\begin{aligned} y_i Q(x) + (-cy_1^{m'_1} \cdots y_n^{m'_n} + O(y^{|m'|+1}))Q(x) = \\ y_i Q(x) + O(y^{|m'|})Q(x) = y_i Q(y) + O(y^{|m'|+1}), \end{aligned} \quad (8.7)$$

where we have also implicitly used that $x \sim y$ in the leading order.

2. In the equation for $\frac{dy_i}{dt}$ the only monomial of order less or equal to $|m'|$ affected by the transformation is exactly $a_{m'}^{(i)} x_1^{m'_1} \cdots x_n^{m'_n}$.

Indeed, using (8.2) we compute

$$\frac{d}{dt}(x_j^{m'_j}) = m'_j x_j^{m'_j-1} \frac{dx_j}{dt} = m'_j \lambda_j x_j^{m'_j} + h.o.t., \quad (8.8)$$

where $j = 1, \dots, n$, and *h.o.t.* are terms of order greater than m'_j . Now, with (8.8) in hands, we express $\frac{dy_i}{dt}$ in terms of x :

$$\frac{dy_i}{dt} = \frac{dx_i}{dt} + c(m'_1 \lambda_1 + m'_2 \lambda_2 + \cdots + m'_n \lambda_n) x_1^{m'_1} \cdots x_n^{m'_n} + O(x^{|m'|+1}),$$

then

$$\frac{dy_i}{dt} - \lambda_i y_i = \frac{dx_i}{dt} - \lambda_i x_i + c((m', \lambda) - \lambda_i) x_1^{m'_1} \cdots x_n^{m'_n} + O(x^{|m'|+1}),$$

and, finally, using the equation for $\frac{dx_i}{dt}$ we get

$$\begin{aligned} \frac{dy_i}{dt} = \lambda_i y_i + \sum_{|m| \geq 2, m' \neq m}^M a_m^{(i)} x_1^{m_1} \cdots x_n^{m_n} + \\ [a_{m'}^{(i)} + c((m', \lambda) - \lambda_i)] x_1^{m'_1} \cdots x_n^{m'_n} + O(x^{|m'|+1}). \end{aligned} \quad (8.9)$$

As we have already seen, if we substitute y instead of x into our monomials, all corrections will be of order greater or equal to $|m'| + 1$. So, we can rewrite

(8.9) as

$$\frac{dy_i}{dt} = \lambda_i y_i + \sum_{\substack{|m|=2, m' \neq m \\ |m'| \geq 2}}^M a_m^{(i)} y_1^{m_1} \cdots y_n^{m_n} + [a_{m'}^{(i)} + c((m', \lambda) - \lambda_i)] y_1^{m'_1} \cdots y_n^{m'_n} + O(y^{|m'|+1}).$$

We see that indeed the only monomial of order less or equal to $|m'|$ which may be affected is $y_1^{m'_1} \cdots y_n^{m'_n}$ in the i -th equation. If this term is resonant, i.e., $(m', \lambda) - \lambda_i = 0$, then actually nothing changes and we can not kill this monomial. But if this term is non-resonant, then choosing

$$c = \frac{a_{m'}^{(i)}}{\lambda_i - (m', \lambda)},$$

we get rid of the monomial $y_1^{m'_1} \cdots y_n^{m'_n}$. Thus, all of the non-resonant monomials of order $|m| \leq N$ can be killed by the above described procedure and the theorem is proved. \square

Let us summarize. We take an equation (8.2) and start to modify monomials one by one. First we take monomials of order two, if the non-resonant condition for them are satisfied, then we can kill them, if not, then we leave them as they are and go to monomials of order three and so on. If we want to kill the monomial $a_{m'}^{(i)} x_1^{m'_1} \cdots x_n^{m'_n}$ then the desired change of variables is

$$y_i = x_i + \frac{a_{m'}^{(i)}}{\lambda_i - (m', \lambda)} x_1^{m'_1} \cdots x_n^{m'_n}, \quad y_j = x_j, \quad i \neq j,$$

and the final transformation which kills all of the non-resonant monomials up to order N will be a finite composition of these elementary transformations. Note that in general we cannot pass to the limit $N \rightarrow \infty$ and kill the non-resonant terms of all orders simultaneously, since this sequence of transformations is usually *divergent* as $N \rightarrow \infty$ even in the case when all of the non-linearities are real analytic, so you can find such a transformation only in the form of a *formal* series which has zero radius of convergence.

LECTURE 9. STABILITY OF THE A-H BIFURCATION

We return to the study of the Andronov-Hopf bifurcation. Recall that after the reduction to the center manifold the original system transforms to

$$\frac{dx}{dt} = Ax + o(x), \quad x \in \mathbb{R}^2, \quad (9.1)$$

and

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix},$$

where we put the origin to the equilibrium state (as we always do) and assume that at the moment of bifurcation there are two complex conjugate eigenvalues with zero real parts, i.e., $\lambda_{1,2} = \pm i\omega$, $\omega > 0$. Coordinate-wise, we can rewrite (9.1) as follows:

$$\begin{cases} \frac{dx_1}{dt} = \omega x_2 + o(x), \\ \frac{dx_2}{dt} = -\omega x_1 + o(x). \end{cases} \quad (9.2)$$

Our first question to answer is:

“What is the stability of the equilibrium state at the moment of bifurcation?”

This question is very important since we know that the stability of the equilibrium point at the critical moment determines the stability for all small values of parameters. In particular, it determines whether the bifurcation is safe or dangerous.

We will use the theory of normal forms in order to simplify the nonlinearity in the reduced system (9.2) on the center manifold by doing the appropriate coordinate transformations. To this end, the first thing we need to do is to bring the linear part of system (9.1) to diagonal form. To do this we introduce the following change of variables:

$$z = x_1 - ix_2, \quad z^* = x_1 + ix_2,$$

where the star denotes the complex conjugation. The inverse transform is also easy to write:

$$x_1 = \operatorname{Re} z = \frac{z + z^*}{2}, \quad x_2 = -\operatorname{Im} z = \frac{z^* - z}{2i}.$$

Then, in these new coordinates system (9.2) has a form

$$\begin{cases} \frac{dz}{dt} = i\omega z + \sum_{m+n \geq 2, m,n \geq 0} a_{mn} z^m (z^*)^n, \\ \frac{dz^*}{dt} = -i\omega z^* + \sum_{m+n \geq 2, m,n \geq 0} a_{mn}^* (z^*)^m z^n. \end{cases} \quad (9.3)$$

Note that the coefficients a_{mn} are now complex, but the equation for $\frac{dz^*}{dt}$ is complex conjugate to the equation for $\frac{dz}{dt}$. This reflects the fact that our original system was real.

Then, by the normal form theory, we can kill all non-resonant terms in the equation for $\frac{dz}{dt}$ (this transformation will automatically also kill the non-resonant terms in the equation for $\frac{dz^*}{dt}$ since it will remain complex conjugate). Thus, all terms $a_{mn}z^m(z^*)^n$ such that

$$\lambda_1 \neq m\lambda_1 + n\lambda_2,$$

where $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ will be killed. Since $\omega \neq 0$, the resonance condition is written as

$$i\omega \neq mi\omega - ni\omega \Rightarrow m \neq n + 1,$$

which means that only the terms where $m = n + 1$ survive. In particular, all terms of even order can be killed. Thus, the normal form for the Andronov-Hopf bifurcation reads:

$$\frac{dz}{dt} = i\omega z + \sum_{n=1}^M (L_n + i\Omega_n)z^{n+1}(z^*)^n + o(|z|^{2M+1}), \quad (9.4)$$

where M is an arbitrary finite integer. Real parts of the coefficients of the resonant terms L_n are called Lyapunov coefficients (as usual), and the imaginary parts of the coefficients of the resonant terms Ω_n are called Birkhoff coefficients. In our theory Birkhoff coefficients are not important, but they become important in the theory of Hamiltonian systems.

The next trick is to go to polar coordinates:

$$z = re^{i\phi} \quad \text{and} \quad z^* = re^{-i\phi}$$

and rewrite the equation (9.4) as

$$\frac{dr}{dt}e^{i\phi} + ire^{i\phi}\frac{d\phi}{dt} = re^{i\phi}[i\omega + \sum_{n=1}^M (L_n + i\Omega_n)r^{2n}] + o(r^{2M+1}).$$

Then, we multiply this equation by $e^{-i\phi}$ and write separately equations for real and imaginary parts:

$$\begin{cases} \frac{dr}{dt} = r[L_1r^2 + L_2r^4 + \cdots + L_Mr^{2M}] + o(r^{2M+1}) \\ \frac{d\phi}{dt} = \omega + \Omega_1r^2 + \Omega_2r^4 + \cdots + \Omega_Mr^{2M} + o(r^{2M}). \end{cases} \quad (9.5)$$

As we see, up to some very small terms of order $o(r^{2M+1})$ and $o(r^{2M})$ in the first and second equations of (9.5) respectively, the right-hand sides do not depend on ϕ . Since $\omega \neq 0$ and positive and r is sufficiently small, the behaviour of ϕ is just a counter-clockwise rotation. The stability or instability depends on the behaviour of r , and the behaviour of r is determined by the Lyapunov coefficients. We look for the first non-zero Lyapunov coefficient. Let it be L_k

(typically this is L_1 , but may be L_2, L_3 and so on). Then we fix M to be equal to k and write (9.5) as

$$\begin{cases} \frac{dr}{dt} = L_k r^{2k+1} + o(r^{2k+1}) = r^{2k+1}(L_k + o(1)_{r \rightarrow 0}) \\ \frac{d\phi}{dt} = \omega + \Omega_1 r^2 + \dots + \Omega_k r^{2k} + o(r^{2k}) > 0. \end{cases} \quad (9.6)$$

Thus,

$$\begin{aligned} \frac{dr}{dt} &> 0 \text{ for all small } r, & \text{if } L_k > 0 \\ \frac{dr}{dt} &< 0 \text{ for all small } r, & \text{if } L_k < 0, \end{aligned}$$

which means that when L_k is positive, our equilibrium state is unstable, and when L_k is negative, the equilibrium state is stable.

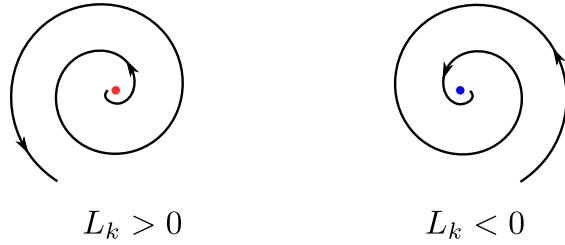


FIGURE 1

To summarize, we start with the original equation, restrict it to the center manifold, and assume that there are only two eigenvalues on the imaginary axis while the rest of them is to the left of it (since if we have some eigenvalues to the right of the imaginary axis, the equilibrium state will be unstable no matter what is going on on the center manifold). We put the origin into the equilibrium and diagonalize the linear part of the system. Then we do the normal form transformation, i.e., we kill all non-resonant terms up to order $2k+1$, where k is chosen as follows. First, we kill quadratic terms, after that kill all non-resonant cubic terms, then we go to polar coordinates and look at the coefficient L_1 ; if $L_1 \neq 0$, we stop. If $L_1 = 0$, we kill all terms of order four and all non-resonant terms of order five, go to polar coordinates again, see whether L_2 is equal to zero and so on. Finally we find k such that $L_k \neq 0$ and look whether it is negative or positive.

If L_k is negative, we have stability at the critical moment.. Indeed, since $\frac{d\phi}{dt} > 0$, we have oscillations in ϕ variable, but the amplitude of these oscillations will decay, since $\frac{dr}{dt} < 0$ and r will decrease. We explained in previous lectures that the stability at the critical moment implies that this bifurcation is safe. If L_k is positive, the situation is opposite: we start nearby the equilibrium but since $\frac{dr}{dt} > 0$, the amplitude of oscillations will grow which means that the equilibrium state is unstable at the critical moment and the bifurcation is dangerous.

I emphasize once more that the stability at the critical moment is very important. For example, when we fly at an aircraft: whenever turbulence happens, it means that we are somewhere in the situations shown in the figure (since we used to move stationary but now we start to have oscillations). Then everything depends on the sign of the first non-zero Lyapunov coefficient. If it is negative then no problem, we will just have some oscillations, but survive, whereas if the Lyapunov coefficient is positive, then for this particular aircraft this means there is a chance it will never reach the destination.

It is interesting that this important “life or death” parameter of the system, the first non-zero Lyapunov coefficient, cannot be immediately felt, or sensed, or grasped by some physical intuition - you cannot simply write the Taylor expansion of the system on the center manifold and just see the Lyapunov coefficients. One needs, first, to perform the computations prescribed by the normal form theory, and this is a non-intuitive procedure. It follows from the normal form theorem that the Lyapunov coefficients L_k are some algebraic functions of all coefficients of the original system up to order $2k + 1$. There is a sufficiently short formula for the most important Lyapunov coefficient L_1 which we will obtain at the next lecture. The formulas for other coefficients also can be written explicitly, but they are not that nice as the formula for L_1 .

LECTURE 10. A-H BIFURCATION. COMPUTATION OF THE FIRST LYAPUNOV COEFFICIENT

As we learned at the previous lecture, the stability of the equilibrium state at the critical moment is completely determined by the sign of the first non-zero Lyapunov coefficient. Let us derive the formula for L_1 . As we have seen, we cannot just see the value of L_1 directly from the coefficients of the system. First, we should bring our system to normal form and this requires some amount of computations. So, we will do this computations in the general case and obtain a formula for L_1 .

As we have already shown at the previous lecture, the reduced system at the center manifold can be brought to the diagonal form

$$\begin{cases} \frac{dz}{dt} = i\omega z + az^2 + bzz^* + c(z^*)^2 + dz^2z^* + \text{the rest}, \\ \frac{dz^*}{dt} = -i\omega z^* + a^*(z^*)^2 + b^*zz^* + c^*z^2 + d^*(z^*)^2z + \text{the rest}, \end{cases} \quad (10.1)$$

where $\omega > 0$, z is a complex variable, z^* is complex conjugate to z , by “the rest” we mean all non-resonant cubic terms and all higher order terms (as we will see below the non-resonant cubic terms cannot affect the Lyapunov coefficient, so we do not write them explicitly). We also recall that the equation for $\frac{dz^*}{dt}$ is complex conjugate to the equation for $\frac{dz}{dt}$. By this reason, we will consider below the equation for $\frac{dz}{dt}$ only.

In the previous lecture, we learned that all quadratic terms can be killed by the normal form transformations, but this will lead to the change in the higher order terms. In particular, the coefficient in front of the resonant term z^2z^* will be also changed, and our goal is to express the real part of this transformed coefficient (which is exactly L_1) in terms of the coefficients a, b, c , and d .

In the lecture on normal forms, we obtained a general formula that allows us to kill the non-resonant monomial of the form $\alpha x_1^{m_1} \cdots x_n^{m_n}$, namely,

$$y_i = x_i + \frac{\alpha}{\lambda_i - (m, \lambda)} x_1^{m_1} \cdots x_n^{m_n}.$$

In our case x_1 is z , $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$ and the value of the vector m depends on which monomial we want to kill, i.e., for az^2 - we have $m = (2, 0)$, for bzz^* - we have $m = (1, 1)$, and for $c(z^*)^2$ - we have $m = (0, 2)$. The desired coordinate transformation is

$$u = z + \frac{a}{-i\omega}z^2 + \frac{b}{i\omega}zz^* + \frac{c}{3i\omega}(z^*)^2. \quad (10.2)$$

2 A-H BIFURCATION. COMPUTATION OF THE FIRST LYAPUNOV COEFFICIENT

This coordinate transformation is identity plus quadratic terms. The quadratic terms do not give contributions to linear terms, therefore the linear part of the equation will be the same. In order to figure out the nonlinear part of the equation, we compute as follows:

$$\begin{aligned} \frac{du}{dt} - i\omega u &= \frac{dz}{dt} - \frac{2a}{i\omega} z \frac{dz}{dt} + \frac{b}{i\omega} z \frac{dz^*}{dt} + \frac{b}{i\omega} z^* \frac{dz}{dt} + \frac{2c}{3i\omega} z^* \frac{dz^*}{dt} - i\omega z + az^2 - \\ bzz^* - \frac{c}{3}(z^*)^2 &= -\frac{2a}{i\omega} z \frac{dz}{dt} + \frac{b}{i\omega} z \frac{dz^*}{dt} + \frac{b}{i\omega} z^* \frac{dz}{dt} + \frac{2c}{3i\omega} z^* \frac{dz^*}{dt} + \\ az^2 - bzz^* - \frac{c}{3}(z^*)^2 + az^2 + bzz^* + c(z^*)^2 + dz^2 z^* + \text{the rest}. \end{aligned}$$

Now we collect the terms of equal powers and, using (10.1), obtain

$$\begin{aligned} \frac{du}{dt} - i\omega u &= -\frac{2a}{i\omega} z \frac{dz}{dt} + \frac{b}{i\omega} z \frac{dz^*}{dt} + \frac{b}{i\omega} z^* \frac{dz}{dt} + \frac{2c}{3i\omega} z^* \frac{dz^*}{dt} + 2az^2 + \frac{2}{3}c(z^*)^2 + \\ dz^2 z^* + \text{the rest} &= -2az^2 - \frac{2ab}{i\omega} z^2 z^* - bzz^* + \frac{bb^*}{i\omega} z^2 z^* + bzz^* + \frac{ab}{i\omega} z^2 z^* - \\ \frac{2c}{3}(z^*)^2 + \frac{2cc^*}{3i\omega} z^2 z^* + 2az^2 + \frac{2}{3}c(z^*)^2 + dz^2 z^* + \text{the rest} = \\ \left(-\frac{ab}{i\omega} + \frac{bb^*}{i\omega} + \frac{2cc^*}{3i\omega} + d \right) z^2 z^* + \text{the rest}. \quad (10.3) \end{aligned}$$

We need to express z as a function of u in the right-hand side. But in (10.3) we are interested only in cubic terms, and if in cubic terms we replace z by u plus small correction (i.e. by the inverse of (10.2) which is $z = u + O(u^2)$, see lecture 9 for details), the contribution of this correction will be only in higher order terms, which means that we can simply replace z by u :

$$\frac{du}{dt} - i\omega u = \left(i \frac{ab}{\omega} - i \frac{bb^*}{\omega} - i \frac{2cc^*}{3\omega} + d \right) u^2 u^* + \text{the rest}.$$

All cubic terms in “the rest” are non-resonant, so we can do the coordinate transformation which kills all of them without affecting the resonant cubic term. So, we may assume that these transformations are already done and all non-resonant cubic terms are killed, i.e., the rest will be of order at least four. Thus, finally we have the desired normal form:

$$\frac{du}{dt} = i\omega u + \left(i \frac{ab}{\omega} - i \frac{|b|^2}{\omega} - i \frac{2|c|^2}{3\omega} + d \right) u^2 u^* + O(|u|^4),$$

and the first Lyapunov coefficient is given by

$$L_1 = \operatorname{Re} d - \frac{1}{\omega} \operatorname{Im}(ab).$$

LECTURE 11. ANDRONOV-HOPF BIFURCATION. NUMBER OF PERIODIC ORBITS

Recall that at the Andronov-Hopf bifurcation we have an equilibrium with two complex-conjugate eigenvalues which cross the imaginary axis when parameters ε change: at the critical moment $\varepsilon = 0$ we have exactly one pair of complex conjugate eigenvalues $\lambda_{1,2} = \pm i\omega$ with $\omega \neq 0$. The claim is that the change in ε leads to the birth of periodic orbits and the question is:

“What type of periodic orbits and how many of them?”

At the previous lecture we showed that the stability at the moment of bifurcation is determined by the sign of the first non-zero Lyapunov coefficient. Now we study bifurcations, i.e., what happens when eigenvalues move across the imaginary axis (for small $\varepsilon \neq 0$). We always assume that equilibrium is at zero. Indeed, we do not have a zero eigenvalue, therefore, the equilibrium does not disappear, and no new equilibria are born when we change parameters. Moreover the equilibrium depends smoothly on parameters, by the implicit function theorem, so for any value of parameters we may put the origin at the equilibrium state.

Then we linearize and look at the eigenvalues

$$\lambda_{1,2} = \mu_0(\varepsilon) \pm i\omega(\varepsilon).$$

Here ε is some set of parameters, it may be one scalar parameter or a vector parameter. Since the critical moment is $\varepsilon = 0$, we have $\mu_0(0) = 0$ and $\omega(0) = \omega$. When parameters change, the real part of the eigenvalues may also change: when it is negative, the equilibrium is stable, and when it is positive, the equilibrium state is unstable. In what follows, for shortness, we will not explicitly state the dependence on ε .

Now we diagonalize our system

$$\frac{dz}{dt} = (\mu_0 + i\omega)z + \sum_{m+n \geq 2, m, n \geq 0} a_{m,n} z^m (z^*)^n, \quad (11.1)$$

and bring it to a normal form, i.e., we kill all non-resonant terms up to some order k , where k is an arbitrary fixed finite number. We recall that the monomial $a_{m,n} z^m (z^*)^n$ is non-resonant if

$$\lambda_1 \neq m\lambda_1 + n\lambda_2, \quad (11.2)$$

where $\lambda_1 = \mu_0(\varepsilon) + i\omega(\varepsilon)$ and $\lambda_2 = \mu_0(\varepsilon) - i\omega(\varepsilon)$.

Suppose we have the relation (11.2) for $\varepsilon = 0$, then by continuity we have the same relation for all small ε , i.e., if some term is non-resonant at $\varepsilon = 0$, then it is non-resonant for all small ε . The same is true for any finite number of non-resonant terms. Then for any given k the system (11.1) can be brought to the normal form

$$\frac{dz}{dt} = (\mu_0 + i\omega)z + (L_1 + i\Omega_1)z^2 z^* + \cdots + (L_k + i\Omega_k)z^{k+1}(z^*)^k + o(|z|^{2k+1}) \quad (11.3)$$

for all small ε . The normal form theorem gives explicit formulas for the normal form transformation, and one can see from these formulas that the transformation is smooth in ε . Note that resonant terms may become non-resonant when ε changes, but in (11.3) we keep all terms which are resonant at $\varepsilon = 0$ (this is necessary if we want the normal form transformation to be defined for all small ε and be smooth in ε). As we mentioned, we can do it up to any finite order k : if we want to take a larger order k , then we need to take smaller ε (such that inequality (11.2) remains fulfilled).

System (11.3) has the same form as for the case $\varepsilon = 0$, but now all the coefficients are functions of ε , and we have a new coefficient μ_0 (a real part of $\lambda_{1,2}$) which was equal to zero before; now it is some function of ε , which does not stay zero for $\varepsilon \neq 0$, in general.

It makes sense to stop the expansion once we found the first non-zero Lyapunov coefficient L_m at $\varepsilon = 0$. As we have already told, typically m is one, but it may be larger. To stress that some Lyapunov coefficients are zero at $\varepsilon = 0$, we denote by μ_j the coefficients which vanish at $\varepsilon = 0$ and rewrite (11.3) as follows:

$$\frac{dz}{dt} = z[\mu_0 + \cdots + \mu_{k-1}|z|^{2k-2} + L_k|z|^{2k} + i(\omega + \Omega_1|z|^2 + \cdots + \Omega_k|z|^{2k})] + o(|z|^{2k+1}),$$

where $\mu_j = 0$ at $\varepsilon = 0$, $j = 0, \dots, k-1$, and $\omega \neq 0$, $L_k \neq 0$ for all small ε . As we learned, the stability at the critical moment is determined by the sign of L_k , i.e., if $L_k > 0$ the equilibrium state is unstable, if $L_k < 0$ the equilibrium state is stable.

Now let us go to polar coordinates $z = re^{i\phi}$. Arguing exactly as we did at the previous lecture, we obtain

$$\begin{cases} \frac{dr}{dt} = r[\mu_0 + \cdots + \mu_{k-1}r^{2(k-1)} + L_kr^{2k}] + o(r^{2k+1}), \\ \frac{d\phi}{dt} = \omega + \Omega_1r^2 + \cdots + \Omega_kr^{2k} + o(r^{2k}). \end{cases} \quad (11.4)$$

The terms $o(r^{2k+1})$ and $o(r^{2k})$ in the first and second equations of (11.4) may depend on r and ϕ , but what is important is that the other terms in the right-hand side of (11.4) do not depend on the polar angle ϕ .

We now state the main theorem of this lecture.

Theorem 11.1. *If $L_k \neq 0$, then up to k (and not more) periodic orbits can be born when parameters change.*

The picture is like that

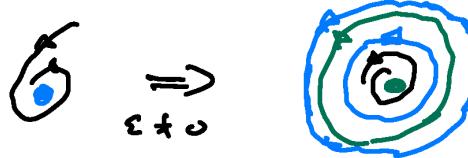


FIGURE 1

At the critical moment, we have an equilibrium state, which is stable (for example). Then the orbits starting at the boundary of a certain neighbourhood U of the equilibrium go inside this neighbourhood, and this remains true when we perturb our system, i.e., add terms μ_j , $j = 0, \dots, k-1$. The equilibrium state may become unstable, but then there should be at least one stable periodic orbit which separates orbits winding out of the equilibrium state and the ones going inside U . We may have other periodic orbits too, and some can be unstable, but since at the critical moment the equilibrium state was stable, then from the outside we must have stability, which means that we should have an outer periodic orbit stable. This is a general picture: we have a finite nested set of periodic orbits with cardinality maximum equal to k and their stability alternates: the outer most periodic orbit inherits the stability of the equilibrium state at the critical moment.

We now start the proof of the theorem. Recall that the behaviour in ϕ is always rotation with a non-zero speed. We take a positive real axis, $\phi = 0$ or, which is the same $\phi = 2\pi$, take any initial condition r_0 on this line ($\phi = 0$), and issue an orbit from r_0 . This orbit goes around the equilibrium state and then returns to some point on the real axis ($\phi = 2\pi$). We denote this point by $T(r_0)$, so the map $r_0 \rightarrow T(r_0)$ is well defined. Obviously, if we have a periodic orbit, then $T(r_0)$ must be equal r_0 , i.e., r_0 must be a fixed point of the map T .

So, the local dynamics of our system is completely determined by the map T . In particular, to say that up to k periodic orbits can be born is the same as to say that there can be up to k fixed points of the map T . The map T is called the *Poincare map*.

Thus, we need to find a usable formula for the Poincare map near $r_0 = 0$. Let us denote

$$G(r) := \omega + \Omega_1 r^2 + \dots + \Omega_k r^{2k} \quad (11.5)$$

and

$$F(r) := \mu_0 + \dots + \mu_{k-1} r^{2(k-1)} + L_k r^{2k}. \quad (11.6)$$

The following lemma is the main technical tool for the proof of the theorem.

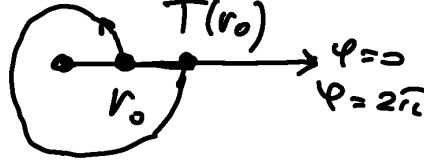


FIGURE 2

Lemma 11.2. *Under the above assumptions, the Poincare map T satisfies:*

$$T(r_0) = r_0 + \frac{2\pi r_0}{G(r_0)} [F(r_0) + o(r^{2k})] (1 + o(1)_{r_0 \rightarrow 0, \varepsilon \rightarrow 0}). \quad (11.7)$$

Assume that this lemma is proved. Thus, in order to find periodic orbits we should solve equation $T(r_0) = r_0$, or alternatively, the equation

$$F(r_0) + o(r^{2k}) = 0. \quad (11.8)$$

The function $F(r_0)$ is a polynomial of order $2k$; if we differentiate it $2k$ times, the $2k$ -th derivative will be non-zero, since $L_k \neq 0$. As we have proved in previous lectures, this means that equation (11.8) can have no more than $2k$ solutions. Note that half of the solutions will be positive and half will be negative, because each solution is a point of intersection of a periodic orbit with the x -axis and there are two such points (positive and negative) for each periodic orbit. Thus, we indeed have no more than k periodic orbits. This finishes the proof of the theorem, modulo the lemma.

Proof of the lemma. In order to compute the Poincare map, we need to solve the equation (11.4) with the initial condition r_0 , i.e., to find formulas for $r(t, r_0)$ and $\phi(t, r_0)$. Then we need to solve the equation $\phi(t, r_0) = 2\pi$ and thus find the time moment $t = t(r_0)$ when the orbit crosses the line $\phi = 2\pi$. Finally, we substitute this value of t into $r(t, r_0)$ and this gives us exactly the value $T(r_0)$ of the Poincare map at r_0 .

This scheme can be simplified by introducing a new time ϕ , i.e., instead of time t we use $\phi(t)$ as a new time variable (we can do this since $G(r) \neq 0$ for small r). Then, due to the chain rule,

$$\frac{dr}{d\phi} = \frac{\frac{dr}{dt}}{\frac{dt}{d\phi}},$$

i.e., we obtain the equation for $\frac{dr}{d\phi}$ by dividing the first equation of (11.4) to the second one:

$$\frac{dr}{d\phi} = r \frac{F(r) + o(r^{2k})}{G(r) + o(r^{2k})}. \quad (11.9)$$

In these new variables, the problem of finding the time of intersection becomes very simple: this time is exactly $\phi = 2\pi$.

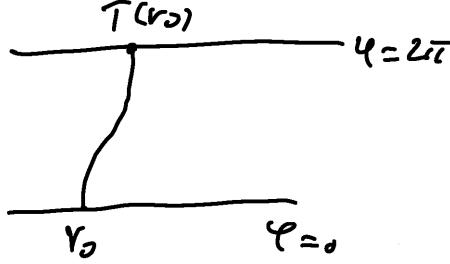


FIGURE 3

The next trick we do is the scaling $r = Rr_0$. Then (11.9) transforms to

$$\frac{dR}{d\phi} = R \frac{F(Rr_0) + o(r_0^{2k})}{G(Rr_0) + o(r_0^{2k})}, \quad R|_{\phi=0} = 1. \quad (11.10)$$

In this new system, r_0 is just a small parameter which is no more involved into the initial conditions. This is a regular differential equation and it can be further simplified: namely, since $G(r_0 R)$ is non-zero for all small r_0 and for all not extremely large R , we have

$$\frac{dR}{d\phi} = R \frac{F(Rr_0)}{G(Rr_0)} + o(r_0^{2k}), \quad R|_{\phi=0} = 1. \quad (11.11)$$

We know that if the difference of the vector fields of two differential equations is bounded by some constant C , then, for any fixed time T , their solutions satisfy $\|x(T, x_0) - x'(T, x_0)\| = O(C)$. Thus the difference between the solution of the shortened equation

$$\frac{dR}{d\phi} = R \frac{F(Rr_0)}{G(Rr_0)}, \quad R|_{\phi=0} = 1 \quad (11.12)$$

and the solution of our original equation (11.11), for time interval $[0, 2\pi]$, is of order $o(r_0^{2k})$. If we return back to original coordinate r , then the difference will be of order $o(r_0^{2k+1})$ (since $r = Rr_0$). Thus, the map $T(r_0)$ is $o(r_0^{2k+1})$ close to the 2π -map of the system

$$\frac{dr}{d\phi} = r \frac{F(r)}{G(r)}. \quad (11.13)$$

The advantage is that the original system (11.9) had a dependence on ϕ , whereas the right-hand side of (11.13) is independent of ϕ , so it even can be solved by separation of variables. However, the analysis of the explicit formula for the solution is still rather complicated, so we prefer not to go this way. Instead, we do the second scaling:

$$r = r_0(1 + uF(r_0)). \quad (11.14)$$

We can do this scaling if $F(r_0) \neq 0$. If $F(r_0) = 0$, then r_0 will be the equilibrium of (11.13) and the solution will be $r_0 = \text{const}$, hence $r(2\pi, r_0) = r_0$ (with

$r(\phi, r_0)$ being the solution of (11.13)). As a result, $T(r_0) = r(2\pi, r_0) + o(r_0^{2k+1})$ automatically satisfies the formula in the statement of the lemma.

Substituting (11.14) into (11.13), we obtain

$$\frac{du}{d\phi} r_0 F(r_0) = r_0 (1 + uF(r_0)) \frac{F(r_0(1 + uF(r_0)))}{G(r_0(1 + uF(r_0)))}.$$

Dividing both sides of this equation to $r_0 F(r_0)$, we obtain

$$\frac{du}{d\phi} = \frac{1 + uF(r_0)}{G(r_0(1 + uF(r_0)))} \frac{F(r_0(1 + uF(r_0)))}{F(r_0)}.$$

The first fraction $\frac{1 + uF(r_0)}{G(r_0(1 + uF(r_0)))}$ is regular, whereas the second fraction $\frac{F(r_0(1 + uF(r_0)))}{F(r_0)}$ looks irregular, because we have some small number divided by another small number. To verify that it is actually also regular, we use the Lagrange formula:

$$F(r_0(1 + uF(r_0))) = F(r_0 + ur_0 F(r_0)) = F(r_0) + F'(\tilde{r}_0)ur_0 F(r_0),$$

where $\tilde{r}_0 \in (r_0, r_0(1 + uF(r_0)))$. Then

$$\frac{F(r_0(1 + uF(r_0)))}{F(r_0)} = 1 + F'(\tilde{r}_0)ur_0.$$

So, the second fraction equals to 1 up to a small corrector (it is also not difficult to verify that this corrector is smooth and its derivatives are also small).

Finally, (11.13) can be rewritten as follows:

$$\frac{du}{d\phi} = \frac{1 + \text{small}}{\omega + \text{small}} (1 + \text{small}), \quad (11.15)$$

where ‘‘small’’ denotes terms that go to zero as r_0 and μ_j go to zero. The solution of (11.15) is, obviously,

$$u(\phi) = u(0) + \frac{\phi}{\omega} + \text{small}.$$

Indeed, this follows from the smooth dependence on parameters, i.e., if we add a small perturbation to a right-hand side, this leads to a small perturbation of the solution.

We also note that $u(0) = 0$ (due to the way we scaled the variable r). Hence,

$$u(2\pi) = \frac{2\pi}{\omega} (1 + \text{small}),$$

and

$$r = r_0 \left(1 + \frac{2\pi}{\omega} F(r_0) (1 + \text{small})\right)$$

is the Poincare map for the system (11.13). To find the Poincare map for the full system we need to add the term $o(|r|^{2k+1})$:

$$\begin{aligned} T(r_0) &= r_0 \left(1 + \frac{2\pi}{\omega} F(r_0)(1 + \text{small}) \right) + o(|r_0|^{2k+1}) = \\ &= r_0 \left(1 + \frac{2\pi}{\omega} (F(r_0) + o(|r_0|^{2k}))(1 + \text{small}) \right). \end{aligned}$$

Since $G(r_0) = \omega + \text{"small"}$, the obtained formula coincides with (11.7). Thus, the lemma is proven and the theorem is also proven. \square

Let us emphasize again that periodic orbits correspond to positive roots of $F(r_0) + o(r_0^{2k}) = 0$, or more explicitly to positive roots of

$$\mu_0 + \mu_1 \rho + \cdots + \mu_{k-1} \rho^{k-1} + L_k \rho^k + o(\rho^k) = 0, \quad (11.16)$$

where $\rho := r_0^2$. Since the k^{th} derivative of (11.16) is equal to $L_k + o(1) \neq 0$ then it may have no more than k roots. Thus, we have no more than k periodic orbits.

A brief summary: the sign of the first non-zero Lyapunov coefficient gives us stability/instability at the critical moment, and the order number of this Lyapunov coefficient (for example, if L_k is the first non-zero Lyapunov coefficient, then this number is k) tells us how many periodic orbits can be born at bifurcation of our system. Stability of these periodic orbits alternates in typical situations, i.e., when no periodic orbits correspond to non-simple roots of (11.16).

In the next lecture, we will study how exactly periodic orbits depend on parameters μ_j , $j = 0, \dots, k-1$; what really happens when we change parameters; how many periodic orbits we have for different values of parameters; and how to build the bifurcation diagram.

LECTURE 12. ANDRONOV-HOPF BIFURCATION. THE CASES $L_1 \neq 0$, AND $L_1 = 0$ BUT $L_2 \neq 0$

We continue looking at the Andronov-Hopf bifurcation and now start with the most basic case, where the first Lyapunov coefficient is non-zero: $L_1 \neq 0$. Let me recall that we restrict our system on the center manifold, so the system is two-dimensional, i.e., $\dim W^c = 2$. The eigenvalues at the equilibrium state are $\lambda_{1,2} = \mu(\varepsilon) \pm i\omega(\varepsilon)$, where ε are parameters and $\mu(0) = 0$, i.e., at the critical moment we have the Andronov-Hopf bifurcation; for small perturbations, the value of $\mu(\varepsilon)$ is also small. We ask what happens when the real part of these eigenvalues goes from positive to negative and the equilibrium state changes stability. We are interested in periodic orbits, so we use the Poincare map defined as follows:

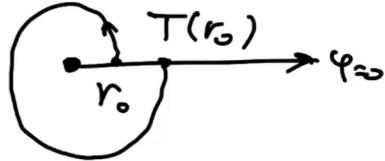


FIGURE 1

We take the line $\phi = 0$, which is a positive semiaxis at the (r, ϕ) -plane (in polar coordinates). We know that the ϕ variable rotates with the velocity close to ω . Therefore, if we start with any point r_0 then, after time close to $\frac{2\pi}{\omega}$, the trajectory returns to the positive semiaxis $\phi = 0$ (or $\phi = 2\pi$) at the point $T(r_0)$ which defines the desired Poincare map. The formula for $T(r_0)$ is derived at the previous lecture:

$$T(r_0) = r_0 + r_0 \frac{2\pi}{\omega} (\mu + L_1 r_0^2 + o(r_0^2)) (1 + o(1)_{r_0 \rightarrow 0, \varepsilon \rightarrow 0}).$$

The periodic orbits correspond to fixed points of the map T or, in other words, to positive roots of the equation

$$\mu + L_1 r_0^2 + o(r_0^2) = 0. \quad (12.1)$$

So, the question about bifurcations of the equilibrium state is reduced to the study of positive solutions of (12.1). Denoting $\rho = r_0^2$ we rewrite (12.1) as

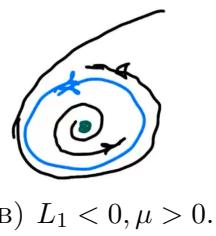
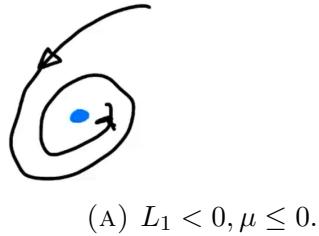
$$\mu + L_1 \rho + o(\rho) = 0. \quad (12.2)$$

Assume that $L_1 \neq 0$. Then the solution of (12.2) is

$$\rho = -\frac{\mu}{L_1} + o(\mu).$$

We solved (12.2) by the Implicit Function Theorem. Indeed, since $L_1 \neq 0$ the derivative of (12.2) with respect to ρ is non-zero. Thus, ρ is uniquely expressed as a function of μ and we just expand the solution in powers of μ . Of course, ρ must be positive, which means that $\mu \cdot L_1 < 0$ (the $o(\mu)$ -term is too small to change the sign of ρ).

There are two different cases depending on the sign of the Lyapunov coefficient. We start with $L_1 < 0$:



We know that at the critical moment the equilibrium state is stable and all trajectories converge to this equilibrium state. We have the same picture for $\mu < 0$ and $\mu = 0$. Indeed, since μ is a real part of the eigenvalue, it determines the stability of the equilibrium state: when μ is negative, the equilibrium state is stable, moreover, there can be no periodic orbits in this case, since $\mu \cdot L_1 > 0$ which means that the condition on the existence of periodic orbits is violated ($\rho = -\frac{\mu}{L_1} < 0$).

When $\mu > 0$ the situation is different, because we know that when $\mu > 0$ we have a single periodic orbit, since $\mu \cdot L_1 < 0$. It must be stable, because it inherits stability from the equilibrium at the critical moment. Inside this periodic orbit we still have an equilibrium, but it becomes unstable, since $\mu > 0$.

Let us now look to the case $L_1 > 0$:

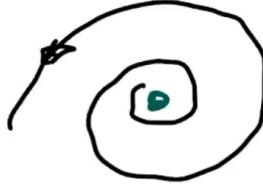


FIGURE 3

The picture is very different, because at the critical moment $\mu = 0$ we have instability, i.e., all trajectories which start at a small neighbourhood of zero go away. We have the same picture when $\mu > 0$, because at positive μ the equilibrium state is unstable and no periodic orbits can be born ($\mu \cdot L_1 > 0$).

Before the bifurcation, when $\mu < 0$, we have a stable equilibrium state and a single unstable periodic orbit, (the stability alternates). If we start inside this periodic orbit, we go to zero, and if we start outside, we go away.

We emphasize once more that there is a huge difference between cases $L_1 > 0$ and $L_1 < 0$. In Figure 3b, even when the equilibrium is stable, the region of stability is very small: the smaller μ gets – the smaller the region of stability becomes. Thus, this situation is dangerous. Imagine that we stay in an equilibrium state, then we give some random perturbation to our system, and if this perturbation kicks us beyond the periodic orbit, the system goes far away from the equilibrium. So, this unstable periodic orbit plays a role of the boundary of the stability region of the equilibrium state before the bifurcation happens. In contrast to this, the bifurcation is safe if $L_1 < 0$.

We finished discussing the picture on the center manifold. Now, imagine we are not on a center manifold. Suppose our system is not two dimensional, but has other variables along which, for example, we have contraction. Then, for $L_1 < 0$ the picture is like that:

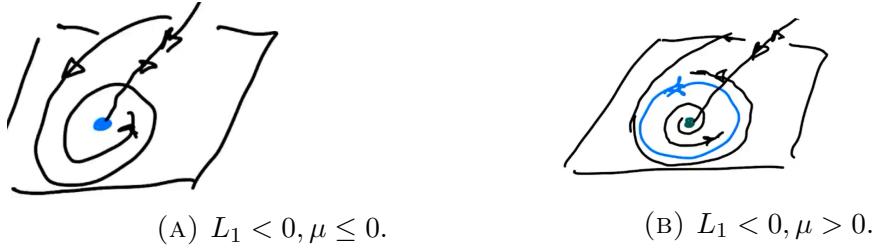


FIGURE 4

Let us look at the case $L_1 < 0$ and $\mu > 0$. What happens to the periodic orbit? Actually, nothing special, it was stable on the center manifold, transverse to it we have stability, so it remains stable. But if we look at the equilibrium, we see that it is not fully unstable, now it has certain directions of contraction. So, at the Andronov-Hopf bifurcation in the higher-dimensional case, we have not only the stability inherited by the periodic orbit, but also the initial equilibrium becomes a saddle-focus. Indeed, it has one contraction direction and a two-dimensional unstable manifold. This unstable manifold is a small disc bounded by the stable periodic orbit, see Figure 4b.

This is the picture for small values of the parameter μ only. In this case, all trajectories which start inside the disk tend to the periodic orbit, so the whole unstable manifold of the equilibrium belongs to the basin of attraction of the periodic orbit. When parameter μ grows, the periodic orbit goes further away from the equilibrium state and it may eventually happen that some trajectories miss this periodic orbit and go away, so the unstable manifold may leave the basin of attraction of the periodic orbit. It may also happen that for some large

parameter values the unstable manifold of the equilibrium intersects with its stable manifold, then some trajectory from the unstable manifold returns back to the equilibrium as $t \rightarrow \infty$, forming the so-called Shilnikov homoclinic loop (a trajectory that tends to the saddle-focus equilibrium in both forward and backward times).



FIGURE 5. Shilnikov loop.

This behaviour often leads to chaos under certain additional conditions. So, the Andronov-Hopf bifurcation in higher-dimensional cases creates one of the main objects responsible for chaotic behaviour, namely, a saddle-focus with one-dimensional stable manifold and two dimensional unstable manifold. When parameter μ grows sufficiently, the saddle-focus equilibrium can acquire the Shilnikov loop, which will lead to chaos.

Let us now consider the case $L_1 > 0$ and add again a stable transversal direction:

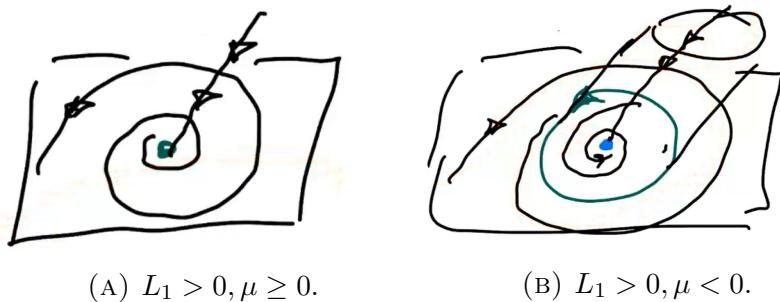


FIGURE 6

In this case we again notice that for $\mu > 0$ the equilibrium which is unstable on the center manifold becomes a saddle-focus. In the case $\mu < 0$, the equilibrium is stable, but the periodic orbit is not fully unstable, it is saddle, since it has a two-dimensional stable manifold corresponding to the contraction in the transverse direction (this manifold is a cylinder) and a two-dimensional unstable manifold which is a part of the center manifold. If you start on the stable manifold of the periodic orbit, then you will rotate along the cylinder and converge to the unstable periodic orbit. In this situation we again may have

what is called homoclinic orbits, i.e., orbits which start on the unstable manifold, then go away, and return along the stable manifold, but now we speak about a homoclinic orbit not to the equilibrium, but to the periodic orbit. The existence of such homoclinic orbits again leads to chaotic behaviour.

This is an important difference between the cases of positive and negative Lyapunov coefficients: when we have positive Lyapunov coefficient (with the extra stable directions), it might happen that the Andronov-Hopf bifurcation is accompanied by the birth of chaotic behaviour, due to the global effects, e.g. due to the formation of orbits which go away from the small neighbourhood and then return (a similar picture for $L_1 < 0$ is possible only if μ is sufficiently large).

Now we consider the degenerate case $L_1 = 0$ and $L_2 \neq 0$. The defining condition for periodic orbits is given by the following equation:

$$\mu_0 + \mu_1\rho + L_2\rho^2 + o(\rho^2) = 0, \quad (12.3)$$

where $\rho = r_0^2$. Thus, at the moment of bifurcation, we have $L_1 = 0$, so we need two control parameter. The first one is μ_0 which governs the stability of the equilibrium state, i.e., μ_0 is the real part of the eigenvalue at the equilibrium state, and the second one is μ_1 which is the first Lyapunov coefficient. At the moment of bifurcation both μ_0 and μ_1 are zeroes, but when we change the parameters they may become non-zero. We need to analyse bifurcations in equation (12.3), which is almost the same as we did when we study saddle-node bifurcations – the only difference is that we are looking not for any root of (12.3), but only for positive roots. Bifurcations may happen, for example, when equation (12.3) has a non-simple root, which corresponds to the derivative of (12.3) equals to zero, i.e.,

$$\mu_1 + 2L_2\rho + o(\rho) = 0. \quad (12.4)$$

Thus, (12.3) and (12.4) together give one bifurcation condition.

There is also another possibility of bifurcations, when a root of (12.3) changes from negative to positive, and vice versa, that is, a periodic orbit is born or disappears. The critical moment is when $\rho = 0$ is a root. Substituting $\rho = 0$ into (12.3), we get the second bifurcation condition

$$\mu_0 = 0. \quad (12.5)$$

From the first condition, solving (12.4) with respect to ρ (we can do this due to the implicit function theorem), we obtain

$$\rho = -\frac{\mu_1}{2L_2} + o(\mu).$$

Substituting this value of ρ into (12.3) and noting that we are only interested in positive values of ρ , we get the relation between μ_0 and μ_1 :

$$\begin{cases} \mu_0 = \frac{1}{4} \frac{\mu_1^2}{L_2} + o(\mu_1), \\ \mu_1 L_2 < 0. \end{cases} \quad (12.6)$$

Thus, our bifurcation curves are given by (12.5) and (12.6), and all possible bifurcations may happen only on these bifurcation curves. The curve $\mu_0 = 0$ corresponds to the birth of periodic orbit out of the equilibrium state (the Andronov-Hopf bifurcation), the curve (12.6) corresponds to the collision of periodic orbits and their disappearance after that.

Let us first consider the case $L_2 < 0$. First, we draw the bifurcation curves and then study separately each region.

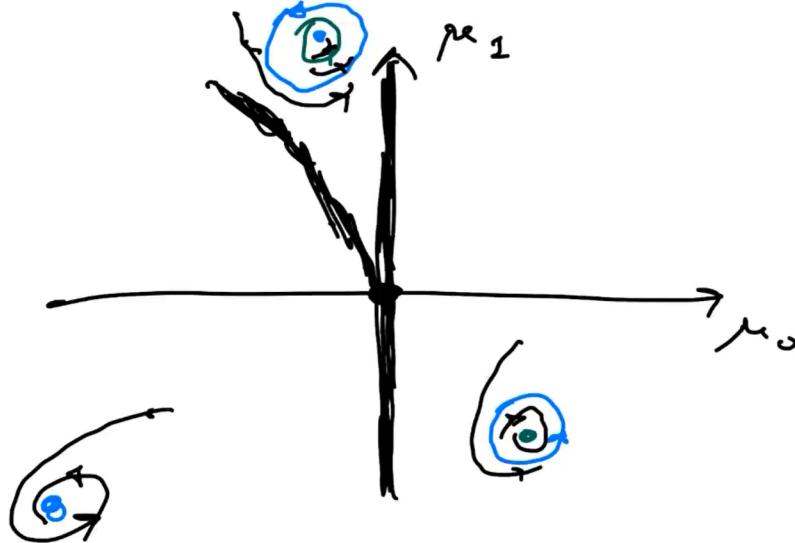


FIGURE 7. $L_2 < 0$.

Let me recall that we study the following system

$$\begin{cases} \frac{dr}{dt} = r(\mu_0 + \mu_1 r^2 + L_2 r^4 + o(r^4)), \\ \frac{d\phi}{dt} = \omega + \dots \end{cases}$$

So, for the ϕ variable we just have rotation, and the most interesting behaviour is in r . Let us start with the region where μ_0 is negative, and chose $\mu_1 = 0$, then μ_0 and L_2 are both negative, which means that $\frac{dr}{dt}$ is negative and r is decaying, i.e., the equilibrium state is stable and no periodic orbits can be born. The same behaviour will be everywhere in this region.

Now, imagine that we start somewhere in the previous region and let us change parameters and cross the line $\mu_0 = 0$. At this moment μ_0 becomes

positive and we have the Andronov-Hopf bifurcation, i.e., the periodic orbit is born out of the equilibrium state. This periodic orbit is stable, since the first Lyapunov coefficient μ_1 at the moment of the bifurcation is negative, and the equilibrium state becomes unstable. This is the behaviour very close to the line $\mu_0 = 0$, but since there are no bifurcations everywhere in the right half-plane, this will be the behaviour everywhere in this region, just the size of the periodic orbit may change.

Then, going counter-clockwise we come to some point $\mu_0 = 0$, $\mu_1 > 0$, where we again have the Andronov-Hopf bifurcation, but now, since μ_1 is positive, we will have unstable periodic orbit born out of the equilibrium state and the equilibrium state will become stable again. Note that nothing will happen to the outermost stable periodic orbit.

On the line (12.6), the stable periodic orbit and the unstable one collide and disappear and we again have one stable equilibrium point.

This is the full description for the case $L_2 < 0$. We note that because the equilibrium state was stable at the moment of bifurcation, this bifurcation is safe (the outermost periodic orbit is stable).

To finish our today's lecture, let us discuss the case $L_2 > 0$. Again, first we draw the bifurcation curves and then analyse the behaviour of our system in each region.

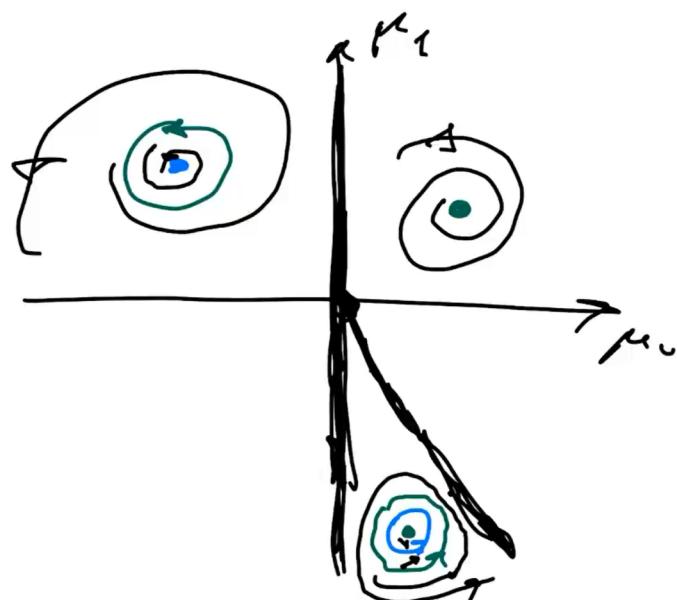


FIGURE 8. $L_2 > 0$.

We start with the region with the simplest behaviour, i.e., the one which contains points $\mu_0 > 0$ and $\mu_1 = 0$. Then, $\frac{dr}{dt}$ is positive for all points except for $r = 0$, since $\mu_0 > 0$ and $L_2 > 0$. Then r is growing if it is not equal to zero initially, and the equilibrium state is unstable.

Next, we cross the line $\mu_0 = 0$ where we have the Andronov-Hopf bifurcation with the positive first Lyapunov coefficient μ_1 . This means that when the equilibrium state becomes stable ($\mu_0 < 0$) the unstable periodic orbit is born.

Continuing to go counter-clockwise, we come to the point $\mu_0 = 0$ and $\mu_1 < 0$; we again have the Andronov-Hopf bifurcation, but since μ_1 is negative, the stable periodic orbit is born out of the equilibrium state, whereas the equilibrium state itself becomes unstable. The outermost periodic orbit does not change, it remains unstable.

When we cross the line (12.6), the stable and unstable periodic orbits collide and disappear, and as expected only the unstable equilibrium point remains.

This was the full picture for the case $L_2 \neq 0$. One can go further and consider the case where $L_2 = 0$ and $L_3 \neq 0$. Then you will need three parameters and will get a three-parameter bifurcation diagram. But we do not do this. The next lecture will be still about the Andronov-Hopf bifurcation.

LECTURE 13. ANDRONOV-HOPF BIFURCATION. THE HOPF THEOREM

We almost finished studying the Andronov-Hopf bifurcation in the previous lectures. Today we look at it from a bit different angle. We consider just a one-parameter family of systems and assume that at $\varepsilon = 0$ we have an equilibrium state at zero. Eigenvalues are functions of ε , and when ε changes, they cross imaginary axis. So, at $\varepsilon = 0$ the system has an equilibrium state with eigenvalues $\lambda_1, \dots, \lambda_n$, and $\lambda_{1,2} = \pm i\omega$, $\operatorname{Re} \lambda_j \neq 0$ if $j > 2$. We also assume that

$$\frac{d \operatorname{Re} \lambda_{1,2}}{d \varepsilon} \neq 0,$$

which means that when we change ε , we cross the imaginary axis with a non-zero speed. Then the following theorem is valid.

Theorem 13.1. *There exists a smooth function $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}$, which vanishes along its first derivatives at $(0,0)$, such that for any point $(x,y) \in W^c$ in a small neighbourhood of the equilibrium $(0,0)$, it lies on a periodic orbit at some value of ε if and only if*

$$\varepsilon = \xi(x, y).$$

In other words, periodic orbits near the equilibrium are given by the level set of ξ .

In other words, all of the periodic orbits of the considered system in a small neighbourhood of the origin at small values of ε are given by the level sets of the function ξ .

To understand what is going on, let us look at the picture.



FIGURE 1

Usually we draw a picture in the phase space (plane (x, y)), but now our picture is three-dimensional, since we have an additional ε -direction. The theorem says that the surface on Figure 1 is smooth and filled by periodic orbits, i.e., whenever we are on the surface, we belong to a periodic orbit

of the system at a given value of ε , and if we are not on the surface, then the corresponding orbit is not periodic. In addition, we have $\xi(0,0) = 0$ and $\frac{\partial \xi}{\partial(x,y)} = 0$, so this surface is tangent to $\varepsilon = 0$. When $(x,y) = (0,0)$, of course, we have an equilibrium state, but except for the point $(0,0)$ this surface is filled by periodic orbits. We know that, depending on the Lyapunov coefficient, these periodic orbits can exist for ε positive or ε negative. Also we have seen pictures when for some parameter values there are two periodic orbits, or there can be even more periodic orbits, it does not matter. If we consider the extended phase space (center manifold (x,y) and the parameter ε), all these periodic orbit lie on a nice smooth surface.

We will prove this theorem in a moment, but let us look at some clarifying examples before. It is easier to understand what is going on if we forget about the angle variable ϕ and draw only the amplitude of our periodic orbit r (computed, say, at $\phi = 0$). We also recall that our standing assumption is $\frac{d \operatorname{Re} \lambda_{1,2}}{d\varepsilon} \neq 0$ – and $\operatorname{Re} \lambda_{1,2}$ is exactly a parameter μ in our notations. So we may write

$$\frac{d\mu}{d\varepsilon} \neq 0,$$

which means that we can assume that $\varepsilon = \mu$. Since μ is a monotone function of ε , this is just a smooth change of parameters.

As we already seen, if the Lyapunov coefficient $L_1 < 0$, we have a stable periodic orbit that exists for all small positive μ .

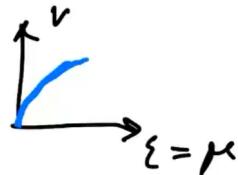


FIGURE 2

The Hopf theorem says that μ is a smooth function of r and ϕ . To depict dependence on ϕ we can take the curve in Figure 2 and “rotate” it around the ε -axis (to be more precise, we need to solve our ODE $\frac{dr}{d\phi} = \dots$ with the initial condition $(r_0, \phi_0) = (r(\mu), 0)$; we also know that the dependence on ϕ in this equation appears in higher order terms only, so at the leading order approximation this is just a rotation). Since each point in this curve corresponds to a periodic orbit, we obtain the following smooth surface:

To be rigorous, we still need to prove that the smooth curve $\mu = \mu(r)$ gives a smooth surface after such a “rotation”, but we will not discuss this technical exercise here.

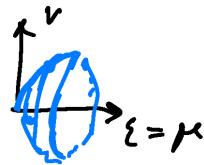


FIGURE 3

For each positive value of μ , we have a unique value of r , for which we have a periodic orbit through the point r . Since μ is a smooth function of r and the derivative of μ with respect to r is zero (by the Hopf theorem), this means that the derivative $\frac{dr}{d\mu}$ is infinite. So, we have a much better picture if we look at μ as a function of r , then by the Hopf theorem for each value r there is a unique value of μ , for which we have a periodic orbit on the plane.

Now let $L_1 > 0$, then we have an unstable periodic orbit which exists at negative μ (see Figure 4).



FIGURE 4

If L_1 is close to zero and positive, but $L_2 < 0$, then we will have the following picture: first for small and negative values of μ and for small r we have an unstable periodic orbit, then as r grows we collide with a stable periodic orbit, and eventually for larger values of r and positive μ we have only a stable periodic orbit. So, for some values of μ we have one periodic orbit, for some we have two: stable and unstable, and for some we have nothing; but anywhere μ is a nice smooth function of r (see Figure 5).

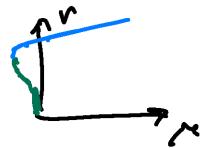


FIGURE 5

We draw the picture for the specific choice of L_1 and L_2 , but no matter what is L_1 and L_2 , the Hopf theorem gives that μ is a smooth function of r , whereas how r depends on μ may be complicated.

Now let us prove the Hopf theorem.

Proof. Let us recall that periodic orbits are defined by the following formula:

$$\mu + L_1 r^2 + L_2 r^4 + \dots = 0, \quad (13.1)$$

where r is an “amplitude” of the periodic orbit and the coefficients μ, L_1, L_2, \dots depend on a parameter ε . Since μ smoothly depends on ε , we can say that all coefficients L_j depend on μ . We should prove that for any value of μ the solutions of the equation (13.1) all lie on the surface $\mu = \xi(r)$ for some smooth function ξ . This statement follows from the implicit function theorem, indeed, the derivative of (13.1) with respect to μ at $(r, \mu) = (0, 0)$ is not zero (equal to one), so μ can be expressed as a smooth function of r :

$$\mu = -L_1(0)r^2 - L_2(0)r^4 + \dots$$

Then, by the “rotation” we got a smooth surface $\mu = \xi(x, y)$ that is tangent at zero to zero (since its Taylor expansion starts from the quadratic terms).



FIGURE 6

□

Now we will look at examples where this theorem helps.

Example 13.2. Let us consider the following system:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \mu y - x + x^2. \end{cases} \quad (13.2)$$

This system has an equilibrium at $(x, y) = (0, 0)$. The linearization matrix at the equilibrium state is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

and $\det A = 1$ and $\text{Tr } A = \mu$. So, at $\mu = 0$ the eigenvalues are $\lambda_{1,2} = \pm i$ and this is the Andronov-Hopf bifurcation. We also note the real part of eigenvalues changes with non-zero velocity when μ changes, this is true since $\text{Tr } A = \lambda_1 + \lambda_2 = \mu$ and $\text{Re } \lambda_{1,2} = \frac{\mu}{2}$ and $\frac{d \text{Re } \lambda_{1,2}}{d\mu} = \frac{1}{2} \neq 0$.

Thus, for this system we know from the Hopf theorem that for any values of x and y there is a unique value of μ for which there is a periodic orbit going through this point (x, y) for this value of μ .

Let us look at the system (13.2) with $\mu = 0$.

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + x^2. \end{cases}$$

We should compute the first Lyapunov coefficient, to do this we should bring our system to diagonal form, then do normal form transformations, after that go to polar coordinates and analyze the Poincare map. But when we look at this system, we immediately see that we do not need to compute Lyapunov coefficients, because all of them are zero, since the system is Hamiltonian. Indeed, this is a system of classical mechanics: suppose we have a particle on a line, x is a coordinate of this particle, then y is a velocity and $\frac{dy}{dt}$ is acceleration, and by the Newton law acceleration is equal to force. So, this is a Newtonian system and we know that such systems conserve energy. For this system the energy is

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{3},$$

where $\frac{y^2}{2}$ is the kinetic energy, and $\frac{x^2}{2} - \frac{x^3}{3}$ is the potential energy (which is minus integral of the force). From the course of mechanics you know that kinetic plus potential energy is conserved with time, i.e.,

$$\frac{d}{dt}H(x(t), y(t)) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} = -\frac{dy}{dt} \frac{dx}{dt} + \frac{dx}{dt} \frac{dy}{dt} = 0.$$

So, at $\mu = 0$, any solution of this system is a level set of $H(x, y)$, i.e., is a curve along which $H(x, y)$ is a constant. Without term $-\frac{x^3}{3}$ these curves would be just circles $\frac{y^2}{2} + \frac{x^2}{2} = c$; but since we are in a small neighbourhood of zero, adding the term $-\frac{x^3}{3}$ leads only to a slight distortion of the shape of these circles. This means that at $\mu = 0$ a small neighbourhood of zero is filled by periodic orbits, i.e., every point (x, y) near the origin belongs to a periodic orbit at $\mu = 0$.

Thus, in the space (μ, x, y) we have a surface $\mu = 0$, such that every point is periodic. By the Hopf theorem, all periodic orbits which exist for any μ belong to a single smooth surface, but we already found this surface, this is $\mu = 0$.

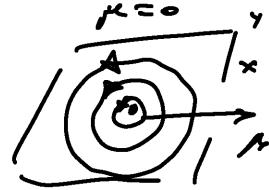


FIGURE 7

Thus, since we proved that at $\mu = 0$ every orbit is periodic, there are no periodic orbits at $\mu \neq 0$ by the Hopf theorem. In particular, all Lyapunov

coefficients are zeros, because if L_n is the first non-zero Lyapunov coefficient, then the equilibrium state is asymptotically stable if it is negative and asymptotically unstable if it is positive. But in our case we see a neutral behaviour, i.e., we start near the equilibrium state and rotate around it all the time. This is called Lyapunov stability, basically this is something in between stability and instability. This can happen only if all Lyapunov coefficients are zero. Of course, it would be very laborious task to compute all of them, because there are infinitely many of them. This trick with finding the Hamiltonian function solves this problem.

The picture is like that:



FIGURE 8

For $\mu = 0$ we have a center equilibrium state (Lyapunov stable), for $\mu > 0$ there are no periodic orbits, the equilibrium state is unstable, and for $\mu < 0$ the equilibrium is stable and any orbit converges to it, also no periodic orbits.

At the previous lecture we discussed that the Andronov-Hopf bifurcation is always accompanied by a birth of periodic orbits, when the orbit loses stability. In this particular case we have continuum of periodic orbits and all of them exist exactly at the moment when the equilibrium state loses stability. What does it mean? When $\mu < 0$ there are spirals that go to zero, but then the distance between two consecutive curls of the spiral becomes smaller and smaller, and at $\mu = 0$ all of them become circles. When $\mu > 0$, all periodic orbits disappear and all orbits start slowly wing away. Of course, this is a dangerous bifurcation. We have stability at the moment of bifurcation, but this stability is too weak to guarantee all orbits not go away when we just a little bit change a parameter.

Let us consider a different example.

Example 13.3.

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \mu y - x + y^2. \end{cases}$$

We have the same linear part and the same bifurcation, going from stability to instability. We again claim that all Lyapunov coefficients are zero.

Indeed, at $\mu = 0$ the system is

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + y^2, \end{cases} \quad (13.3)$$

and this system is reversible.

Definition 13.4. The system is time-reversible if there exists a change of coordinates such that in the new coordinates the system coincides with the original one up to the reversion of time.

So, this is the trick: we change the time t by minus t , which is equivalent to multiplying the right-hand side of the system by minus one. We also change the variable y to minus y , and indeed see that the system (13.3) does not change, i.e., it is time reversible. Note that instead of y^2 in the right-hand side of the second equation, we can write any function of x and y^2 , i.e., $f(x, y^2)$.

What does reversibility mean? Let us draw a line $y = 0$, take any point in this line, and consider an orbit around zero, which starts at our chosen point. We know that all orbits are spirals around zero; and when they spiral they come again to the line $y = 0$. What is so special about the line $y = 0$? This line is a line consisting of fixed points of the transformation $y \rightarrow -y$. The invariance with respect to the transformations

$$\begin{aligned} t &\rightarrow -t \\ y &\rightarrow -y \end{aligned}$$

means that if we take a piece of orbit, take a symmetric image of this piece of orbit under the transformation $y \rightarrow -y$ and change the direction of time it will be again an orbit of our system. Thus, if we start at some point with $y = 0$ and issue orbits in forward and backward time these two orbits will be symmetric with respect to the line $y = 0$. The symmetric image of the forward orbit is a backward orbit, and this backward orbit due to the symmetry comes exactly to the same point on $y = 0$ as a forward orbit, which means that the forward orbit is a closed curve, i.e., a periodic orbit. We can take a different point and repeat the same argument, and the reversibility implies that every orbit around zero must be periodic (see Figure 9). Note that we have reversibility only at $\mu = 0$. By the Hopf theorem, if $\mu \neq 0$, there are no periodic orbits and we have the same picture as in the conservative case (Example 1). If $\mu < 0$ there is a stable equilibrium state and everything converges to it; if $\mu > 0$ there is an unstable equilibrium state and everything diverges from it.

Concluding remarks: The Hopf theorem says that a parameter μ is a smooth function of coordinates of periodic orbits x and y , and all periodic orbits lie on this nice smooth surface $\mu = \xi(x, y)$. We had also an interesting observation that if at the moment of bifurcation every orbit around the equilibrium state is a periodic orbit (such equilibrium state is called center), then when we change



FIGURE 9

parameters, we do not have periodic orbits. This center case usually does not happen, because this means infinitely many conditions on the system, but this may happen when the system at $\mu = 0$ is Newtonian or in general has a conserved quantity, or is reversible.

LECTURE 14. FINDING CENTER MANIFOLD

We have finished studying local bifurcations of equilibrium states. The next step will be studying local bifurcations of periodic orbits. Before we do it, we return to the question of how to find the center manifold. We know from the center manifold theorem that the problem of bifurcations in a small neighbourhood of the equilibrium state is reduced to studying bifurcations on the center manifold. This is a great advantage because the center manifold is often low-dimensional, like dimension 1 (for the saddle-node bifurcation) or dimension 2 (for the Andronov-Hopf bifurcation). Now, the question is:

“How to write the restriction of the original system to the center manifold?”

To do this, we should take our system and substitute the equation of the center manifold into this system. So the problem is to find this equation. We use normal forms for that. Actually, you can do it in many ways, but quite effective method is to bring a system to a normal form, such that the center manifold in these new coordinates becomes zero plus some very small terms. Then you can return to original coordinates, i.e., do the inverse normal form transformation, and obtain the desired center manifold. Let me explain how to do it.

Recall that when we have an equilibrium state, we put the origin to the equilibrium state, and then we can write our equation like that

$$\frac{dx}{dt} = Ax + O(x^2), \quad (14.1)$$

where Ax is the linear part and $O(x^2)$ is the nonlinearity. Then we look at the spectrum of A and group eigenvalues into three groups:

$$\begin{aligned} & \lambda_1, \dots, \lambda_m, \text{ where } \operatorname{Re} \lambda_j < 0, \\ & \gamma_1, \dots, \gamma_n, \text{ where } \operatorname{Re} \gamma_j > 0, \\ & \eta_1, \dots, \eta_k, \text{ where } \operatorname{Re} \eta_j = 0. \end{aligned}$$

Note that η_j are eigenvalues which correspond to the center manifold. Next we go to the Jordan base and introduce variables

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \text{ where } \begin{array}{ll} u \in \mathbb{R}^m \\ v \in \mathbb{R}^n \\ w \in \mathbb{R}^k \end{array}$$

We know that this system has a pair of manifolds: a center-stable manifold W^{cs} which is parametrised by the variables u and w , and a center-unstable manifold W^{cu} which is parametrised by the variables v and w . The center manifold

W^c is the intersection of W^{cs} and W^{cu} . On the center-unstable manifold, orbits go away from center manifold in forward time. On the center-stable manifold, orbits tend to the center manifold in forward time and go away in backward time. The only orbits that may never leave a small neighbourhood of the equilibrium state are orbits that belong to the center manifold. The center manifold is tangent to w -axis, and on the center manifold, v and u are functions of w , i.e., $W^c : (u, v) = f(w) = O(w^2)$.

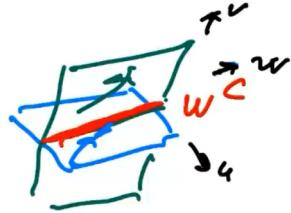


FIGURE 1

Then we restrict our equation on the center manifold, i.e., we do not write equations for u and v variables, but take the equation for $\frac{dw}{dt}$ which depends on u, v, w and, instead of (u, v) , substitute $f(w)$. To do this we need to know a formula for the function f .

So, let us write the equation in a Jordan form, which means that instead of x we should substitute u, v and w into (14.1) and get

$$\begin{cases} \frac{du}{dt} = Bu + O(u^2 + v^2 + w^2), \\ \frac{dv}{dt} = Cv + O(u^2 + v^2 + w^2), \\ \frac{dw}{dt} = Dw + O(u^2 + v^2 + w^2), \end{cases} \quad (14.2)$$

where the matrix B has eigenvalues $\lambda_1, \dots, \lambda_m$; the eigenvalues of C are $\gamma_1, \dots, \gamma_n$, and the eigenvalues of D are η_1, \dots, η_k . If instead of u and v we substitute $f(w)$ into the equation for $\frac{dw}{dt}$, we will get terms of at least third order (if there are terms uw or vw). So, once we brought our system to the block-diagonal form (14.2), we do not need to know the equation for the center manifold in order to find the quadratic terms in the system for $\frac{dw}{dt}$ – we just substitute zero instead of u and v and get the quadratic terms.

But to find cubic terms, we really need to know the equation for the center manifold up to quadratic terms. Why do we need cubic terms, in general? When we studied the saddle-node bifurcation, we noticed that usually everything is determined by the first Lyapunov coefficient (a coefficient in front of the quadratic term) if it is non-zero. So, in this case we do not need to know the equation for the center manifold. However, if we have the Andronov-Hopf bifurcation, everything again is determined by the sign of the first Lyapunov coefficient, but it depends on cubic terms, so we need to know the equation for the center manifold (function f) at least up to quadratic terms, otherwise

we cannot compute the first Lyapunov coefficient and determine whether this bifurcation is safe or dangerous.

This was a motivation, and now a small observation. If $\frac{du}{dt} = 0$ at $u = 0$ (this means that there are no terms with w^2 and v^2 in the first equation of (14.2)) and $\frac{dv}{dt} = 0$ at $v = 0$, then $\{u = 0\}$ and $\{v = 0\}$ are invariant manifolds. Moreover, the manifold $\{u = 0\}$ is, of course, tangent to $u = 0$ and thus we can say that it is parameterized by variables v and w , so this is the center-unstable manifold, i.e., $W^{cu} = \{u = 0\}$. Analogously, $W^{cs} = \{v = 0\}$ is the center-stable manifold. Since the center manifold $W^c = W^{cs} \cap W^{cu}$, so $W^c = \{u = 0\} \cap \{v = 0\}$ and $f(w) = 0$.

In fact, if we want to have $f(w) = 0$, it is enough to have more relaxed conditions, namely,

$$\begin{aligned} \frac{du}{dt} &= 0 \text{ at } (u, v) = 0 \\ \frac{dv}{dt} &= 0 \text{ at } (u, v) = 0 \end{aligned} \Rightarrow f(w) = 0. \quad (14.3)$$

In this case, the center-stable and center-unstable manifolds may be a little bit curved but $(u, v) = 0$ still will be an invariant manifold and it is tangent to w axis, so indeed, this is the center manifold.

The conditions (14.3) mean that there are no terms in the first and second (14.2) equations that depend only on w . So, the idea of our method is to kill the terms depending only on w in the first and second (14.2) equations up to some power k , using normal form transformations.

Let us bring the system (14.2) to a normal form. For simplicity, we assume that all eigenvalues are different and linearization matrix is diagonal, but the same result holds true if we have Jordan cells. First, we write equations for the u variables.

$$\frac{du_j}{dt} = \lambda_j u_j + \sum_{|p|+|q|+|s|\geq 2} a_{(p,q,s)} u^p v^q w^s, \quad (14.4)$$

where $j = 1, \dots, n$; and p, q, s are multi-indices, i.e., $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_m)$, $s = (s_1, \dots, s_k)$ and

$$u^p v^q w^s = u_1^{p_1} \dots u_n^{p_n} v_1^{q_1} \dots v_m^{q_m} w_1^{s_1} \dots w_k^{s_k}.$$

The normal form theorem says that the monomial $a_{(p,q,s)} u^p v^q w^s$ in the equation (14.4) can be killed if it is non-resonant. Non-resonance means that

$$\lambda_j \neq (p, \lambda) + (q, \gamma) + (s, \eta).$$

Recall that we should only kill the terms which do not contain u and v , which means that $p = q = 0$. Thus, we only need to kill $w_1^{s_1} \dots w_k^{s_k}$ terms – and all of them are non-resonant. Indeed,

$$\lambda_j \neq s_1 \eta_1 + \dots + s_k \eta_k,$$

since $\operatorname{Re} \lambda_j < 0$, but $\operatorname{Re}(s_1\eta_1 + \dots + s_k\eta_k) = 0$. Thus, in any equation for the u variables, up to any order, we can kill all terms that depend only on w . By the same logic, if you have equations for v

$$\frac{dv_j}{dt} = \gamma_j v_j + \sum_{|p|+|q|+|s|\geq 2} a_{(p,q,s)} u^p v^q w^s, \quad (14.5)$$

the non-resonance condition will be

$$\gamma_j \neq (p, \lambda) + (q, \gamma) + (s, \eta)$$

and, again, if $p, q = 0$, then the real part of the right-hand side is zero and the real part of the left-hand side is positive; so we can kill all monomials, up to any order, that depend only on w .

To summarize: for any N we can do normal form transformations and bring our system to the form

$$\begin{aligned} \frac{du}{dt} &= Bu + O(u^2 + v^2) + O(\|u\| + \|v\|)w + O(\|w\|^N), \\ \frac{dv}{dt} &= Cv + O(u^2 + v^2) + O(\|u\| + \|v\|)w + O(\|w\|^N), \\ \frac{dw}{dt} &= Dw + O(u^2 + v^2 + w^2). \end{aligned}$$

So, we have terms of order N and higher which we do not know. They can be anything, but terms of order smaller than N vanish identically when u and v are equal to zero. Then, if we go through the proof of the center manifold theorem (which we actually did not prove) we see that the center manifold is found as a fixed point of some contraction operator in a function space via the Banach contraction mapping principle. Fixed points of contracting operators depend nicely on perturbations, so if we find a fixed point for the system

$$\begin{aligned} \frac{du}{dt} &= Bu + O(u^2 + v^2) + O(\|u\| + \|v\|)w, \\ \frac{dv}{dt} &= Cv + O(u^2 + v^2) + O(\|u\| + \|v\|)w, \\ \frac{dw}{dt} &= Dw + O(u^2 + v^2 + w^2), \end{aligned}$$

and then add to the system a small correction $O(\|w\|^N)$, then a correction which is added to a fixed point will be also of order $O(\|w\|^N)$. This is a general idea: whenever you use the Banach contraction mapping principle and you perturb your operator, then corrections to fixed points are proportional to the corrections to the operator. In our case, corrections to the system are of order N and because our system is the initial data for the contraction operator it follows that corrections to the fixed point are also of order N .

This means that once we kill all terms up to the order $(N - 1)$ in the equations for $\frac{du}{dt}$ and $\frac{dv}{dt}$, then the center manifold is

$$W^c : (u, v) = O(\|w\|^N).$$

A brief summary: we can kill all non-resonant monomials, in particular we can kill monomials that do not depend on u and v in the equations for $\frac{du}{dt}$ and $\frac{dv}{dt}$. Once we do this, we know that in the new variables the equation for the

center manifold is zero up to some small correction – returning to the original variables we get the result.

We illustrate this method with the following example. In general, computations may be sufficiently long, but the system we consider is as easy as possible, just to show how it works.

Example 14.1. Let us consider the following system

$$\begin{cases} \frac{du}{dt} = -2u + u^2 + uw + w^2, \\ \frac{dw}{dt} = u^2 - uw, \end{cases}$$

and find the equation of the center manifold up to cubic terms.

We should kill the term w^2 in the first equation and we do this via the normal forms transformation:

$$u^{new} = u - \frac{1}{2}w^2.$$

Let us check that this formula gives the correct transformation:

$$\begin{aligned} \frac{du^{new}}{dt} + 2u^{new} &= \frac{du}{dt} - w\frac{dw}{dt} + 2u - w^2 = u^2 + uw + w^2 - \\ &\quad wu^2 + w^2u - w^2 = u^2 + uw - wu^2 + w^2u. \end{aligned}$$

Thus, indeed all terms (in particular, all quadratic terms) are proportional to u , which means that we have, in the new coordinates, $W^c : u^{new} = O(w^3)$, and in original coordinates

$$W^c : u = \frac{w^2}{2} + O(w^3).$$

If we want to find the equation of the system on the center manifold, we substitute this u into the equation for $\frac{dw}{dt}$ and get

$$\frac{dw}{dt} = -\frac{w^3}{2} + O(w^4).$$

This means that we have a safe cusp bifurcation ($\ell_2 = 0$ and $\ell_3 = -\frac{1}{2} < 0$). Because the sign of ℓ_3 is negative, it follows that on the center manifold at the critical moment the equilibrium state is stable, transverse to the center manifold it is also stable, because we have contraction in the u variable. So, at the critical moment the picture is like that:



FIGURE 2

When we perturb the system, since the first non-zero term is cubic, the equilibrium can be decomposed to three stationary points: two stable points and one saddle point in between.

LECTURE 15. LOCAL BIFURCATIONS OF PERIODIC ORBITS

Our next topic will be bifurcations of periodic orbits. There are many similarities between bifurcations of periodic orbits and bifurcations of equilibrium states, but there are also differences. Things in common are that there is also the center manifold theorem and that the method of normal forms is also applicable and useful.

The subject is this: suppose we have a system of differential equations and suppose this system has a periodic orbit. Then we ask the following question:

What happens in a small neighbourhood of the periodic orbit?



FIGURE 1

Let us start by constructing the Poincare section. The problem of studying the behaviour near a periodic orbit in a system of differential equations by this approach is reduced to the study of behaviour near the fixed point of some map. We take a periodic orbit L and an arbitrary point O on it. Then we take a surface S transverse to this periodic orbit at the point O . If we are on a plane, then S is just a curve. If we are in a three dimensional space, then



FIGURE 2

the cross-section S is a two-dimensional surface. In higher dimensional spaces this will be a hypersurface of co-dimension one. Now let us take any point x

on the surface S in a small neighbourhood of the point O and issue an orbit through x . If x coincides with O , then after a finite time (the period of L) we will return to the surface S exactly at the point O . If we start near the point O at some point x , the solution will go close to the curve L (since we have a continuous dependence of solutions on the initial data) and eventually cross the surface S somewhere close to the point O . Indeed, we have a system

$$\frac{dx}{dt} = f(x),$$

where $\frac{dx}{dt}$ is the velocity with which the point moves along the solution of the differential equation. On the periodic orbit, $f(x)$ is nowhere zero, because otherwise we would have an equilibrium state, not a periodic orbit. Thus, when we come close to the point O , we have $f(x) \neq 0$, i.e., we move with non-zero velocity and cross the cross-section S at some point \bar{x} , see Figure 3.

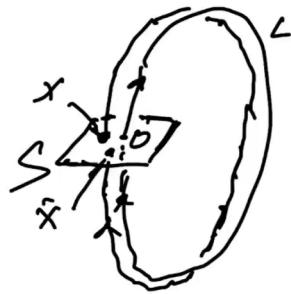


FIGURE 3

Thus, the map $\bar{x} = T(x)$ is defined, which is called the Poincaré map. Let us start at any point x which is close to O and lies on the cross-section S . After finite time it returns to S , then we continue and the orbit comes again to the cross-section S and so on – as long as an orbit stays in a small neighbourhood of the point O it will return to S . So, studying dynamics near a periodic orbit L is the same as studying dynamics of the map T near the fixed point O . This principle was introduced by Poincaré. He figured out that it is a good idea to represent solutions of differential equations by orbits in a phase space and, when we consider periodic orbits, to take a cross-section and to consider the corresponding map. This is indeed a good idea, because if a system of differential equations is defined in an $(n+1)$ -dimensional space, then the corresponding Poincaré's map is defined in n -dimensional space. Moreover, instead of studying the dynamics around curves (periodic orbits), we need to study the dynamics around a fixed point, which should be a much simpler task.

What can we say about the map T ? Since $f(x)$ is a smooth function and solutions of differential equations smoothly depend on initial conditions, we

expect that T is also a smooth function and smoothly depends on initial conditions. Indeed, let us take a cross-section S , a point x_0 on it, and issue a trajectory $x(t, x_0)$ from this point (see Figure 4). The function $x(t, x_0)$ is a smooth function of both arguments, since the considered system of differential equations is smooth. Then the Poincare map is $T(x_0) = x(\tau(x_0), x_0)$, where $\tau(x_0)$ is the return time to the cross-section. We know that $x(t, x_0)$ is a smooth function of time t and the initial condition x_0 , but a priori it is not immediate that the return time is a smooth function of the initial condition. So, we will prove that the Poincare map is smooth if we prove that $\tau(x_0)$ is a smooth function of x_0 .

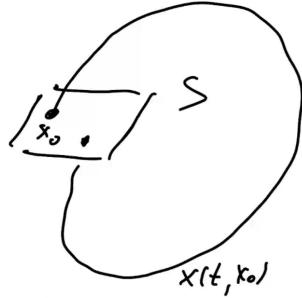


FIGURE 4

Time τ can be found from its definition

$$x(\tau, x_0) \in S. \quad (15.1)$$

To do this we need to write (15.1) in the form of equation, then we will use the Implicit Function Theorem to express τ as a function of x_0 .

First we should write the expression for S . On the cross-section S we have a point O which corresponds to the periodic orbit, at O we have a non-zero vector $\dot{x} = f(x)$ which is tangent to the periodic orbit. Now, let us choose a coordinate system. We put the origin to the point O and choose coordinates $(x_1, \dots, x_n, x_{n+1})$ in such a way that $f(0)$ is tangent to the x_{n+1} axis, i.e., $f(0) = (0, \dots, 0, 1)$, and the equation for S will be $x_{n+1} = 0$ (we can do this since we assume that the orbit L is transversal to the surface S). Thus, (15.1) can be rewritten as

$$x_{n+1}(\tau, x_0) = 0.$$

We also know that $x_{n+1}(\tau_0, 0) = 0$, where $\tau_0 > 0$ is the return time (period) for the orbit L . By the Implicit Function Theorem we can express τ as a smooth function of x_0 near $(\tau_0, 0)$ if the derivative of x_{n+1} with respect to τ is non-zero at $(\tau_0, 0)$. The derivative of x_{n+1} with respect to time is

$$\frac{\partial x_{n+1}}{\partial \tau}|_{(\tau_0, 0)} = f_{n+1}(0) = 1 \neq 0.$$

Thus, we can express τ as a smooth function of x_0 and therefore the Poincare map $T(x_0) = x(\tau(x_0), x_0)$ is smooth. In fact we have more. Indeed, it is obvious that if we start at the point $T(x_0)$ and go backward in time, then we get the inverse of the Poincare map. Thus, the inverse of the Poincare map T^{-1} exists and it is the Poincare map for the same cross-section but for the flow obtained by the inversion of time in the original system, i.e., for the system

$$\dot{x} = -f(x).$$

This is a smooth system of differential equations, hence T^{-1} is also smooth. So, not just the Poincare map is smooth, its inverse exists and is also smooth, which means that the Poincare map is diffeomorphism.

We have just shown that the problem of analysing the local behaviour near a periodic orbit is reduced to analysing the local behaviour of iterations of a diffeomorphism (the Poincare map) near a fixed point. In general we do not need to think about a differential equation at all, instead, from the very beginning we can assume that we have a diffeomorphism $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(0) = 0$.

There is another situation where a similar construction appear. Suppose we have a map $x \rightarrow g(x)$, where $g(x)$ is smooth, and this map has a periodic point O of period k , i.e., $g^k(O) = O$. Then, the map $T = g^k$ is the Poincare map,

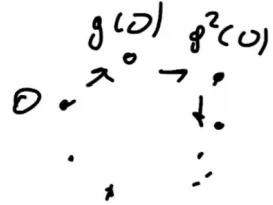


FIGURE 5

because the point O is a fixed point of the map g^k . We only need to look at the behaviour of the map T in a small neighbourhood of the fixed point O .

Now, given a map $T : x \mapsto \bar{x}$ with a fixed point at zero, write the Taylor expansion:

$$\bar{x} = T(x) = T(0) + T'(0)x + O(x^2),$$

where $T(0) = 0$ and $T'(0)$ is some matrix. Denoting it by A , we get

$$\bar{x} = Ax + o(x), \quad x \in \mathbb{R}^n. \tag{15.2}$$

The question is to study iterations of such maps. Let $\lambda_1, \dots, \lambda_n$ be the spectrum of A . Then we can always bring A to the special Jordan form as we

did it in the first lecture of this course:

$$A = \begin{pmatrix} \lambda_1 & \delta_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_{n-1} & \delta_{n-1} \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\delta_j = 0$ or $|\delta_j| < \varepsilon$ for some sufficiently small number ε ; if $\delta_j \neq 0$, then $\lambda_j = \lambda_{j+1}$. Define the norm of A as follows: first take a sum of absolute values of all elements in the row and then take maximum over all rows. Then we obtain

$$\|A\| \leq \max_{j=1,\dots,n} |\lambda_j| + \varepsilon.$$

This matrix norm corresponds to the norm $\|x\| = \max_{i=1,\dots,n} |x_i|$ in \mathbb{R}^n , so the key inequality $\|Ax\| \leq \|A\| \|x\|$ is valid. Using (15.2), we compute the norm of \bar{x} in a small neighbourhood of the fixed point O (i.e. $\|x\|$ is small):

$$\|\bar{x}\| \leq \|x\|(\|A\| + o(1)),$$

where $o(1)$ can be made as small as we want by decreasing the size of the neighbourhood of O , in particular it can be made smaller than ε , which gives us

$$\|\bar{x}\| \leq \|x\|(\max_{j=1,\dots,n} |\lambda_j| + 2\varepsilon).$$

This is a very simple estimate, much simpler than we did in differential equations, but the idea is similar. If all λ_j , $j = 1, \dots, n$, are smaller than 1 in the absolute value, then after we add 2ε (where ε is small enough) the obtained number will be still smaller than 1 in the absolute value. Then, if we take some point and start to iterate, the norm of this point will become smaller and smaller: since at each iteration we will multiply the norm to some factor smaller than 1, we will exponentially converge to zero. Moreover, since the norm of x after iterations can become only smaller, we can never leave the neighbourhood where we started and, thus, the norm of the $o(1)$ -term cannot become greater than ε . So, we proved the following theorem.

Theorem 15.1. *If all multipliers $\lambda_1, \dots, \lambda_n$ are smaller than 1 in the absolute value, then O is exponentially stable.*

If all multipliers are larger than 1 in the absolute value, then we consider the inverse map. For the inverse map, the derivative will be the inverse matrix, so we should deal with the matrix A^{-1} and all its eigenvalues $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ will be smaller than 1 in the absolute value, which means that we will have stability in backward time.

So, it is natural to expect that if we want to study bifurcations we should look on the eigenvalues on the unit circle, i.e., eigenvalues whose absolute values are equal to 1.

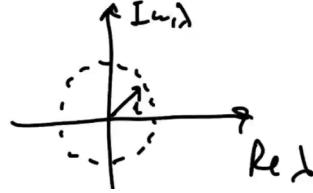


FIGURE 6

Now we do exactly what we did in the case of equilibrium states. Write x as

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \begin{array}{l} u \in \mathbb{R}^m \\ v \in \mathbb{R}^p, \quad m + p + q = n. \\ w \in \mathbb{R}^q \end{array}$$

Let the variables u correspond to eigenvalues inside the unit circle, v correspond to eigenvalues outside of the unit circle, and w correspond to critical eigenvalues (critical multipliers, neutral multipliers) which lie on the unit circle. From now on, by λ_j we will denote only the eigenvalues which are smaller than 1 in the absolute value. The spectrum of A can be grouped as $\{\lambda_1, \dots, \lambda_m, \gamma_1, \dots, \gamma_p, \eta_1, \dots, \eta_q\}$, where $|\lambda_j| < 1$, $|\gamma_j| > 1$ and $|\eta_j| = 1$, and after we go to the special Jordan base, we see that the matrix A becomes block-diagonal:

$$A = \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix},$$

where B corresponds to the eigenvalues λ_j , C corresponds to γ_j and D corresponds to η_j . So, we can write our system (15.2) exactly in the same form as we did for the case of differential equations:

$$\begin{aligned} \bar{u} &= Bu + h_1(u, v, w), \\ \bar{v} &= Cv + h_2(u, v, w), \\ \bar{w} &= Dw + h_3(u, v, w), \end{aligned}$$

where $h_{1,2,3} = o(u, v, w)$ are non-linearities. Analogously to the case of differential equations, we have the following theorem on the existence of a *center unstable* (W^{cu}) and a *center stable* (W^{cs}) invariant manifolds whose intersection gives us a smooth *center manifold* $W^c = W^{cu} \cap W^{cs}$ (red coloured in the figure).

Theorem 15.2. *There exist two locally invariant manifolds $W^{cu} : u = \psi^{cu}(v, w)$ and $W^{cs} : v = \psi^{cs}(u, w)$ such that $\psi(0) = 0$ and $\psi'(0) = 0$.*

The picture is the same as we had for differential equations: The term “local invariance” in the theorem means that the orbit lies on the manifold as long as it stays in a small neighbourhood of the origin. In exactly the same way as we did for the system of differential equations, it follows from the previous theorem and from the fact that the variables u are contracting and the variables

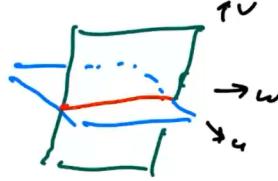


FIGURE 7

v are expanding, that if a *forward* orbit stays near O forever, then it belongs to W^{cs} , and if a *backward* orbit stays near O forever, then it belongs to W^{us} . So, the following theorem holds true.

Theorem 15.3. *If an orbit does not leave a small neighbourhood of zero both forwards and backwards, then it lies in $W^c = W^{cs} \cap W^{cu} : (u, v) = \psi^c(w)$.*

Thus, for the orbits that never leave a small neighbourhood of O , the variables u and v are given functions of w , so

$$\bar{w} = Dw + h_3(\psi^c(w), w),$$

which means that we get a huge reduction of the dimension. This is the same principle as in the case of local bifurcations of equilibria for differential equations.

If there is no eigenvalue on the unit circle, then W^c is just a point. By the preceding theorem, this means that the only point which never leaves a small neighbourhood of zero is zero itself.

The center manifold W^c is smooth, and it smoothly depends on parameters. So, let us, at a certain moment, have eigenvalues on the unit circle. When we change parameters, eigenvalues can move, but the center manifold will persist and we can always restrict our system to the center manifold, even when parameters change. Smoothness of the center manifold means that if a system is C^r -smooth for some finite r , then the center manifold will be also C^r -smooth. If the system was C^∞ -smooth, we are still able to guarantee only finite smoothness, which depends on the neighbourhood where we look. If we take a smaller neighbourhood, the smoothness can get higher, but in principle we can never guarantee that a center manifold is C^∞ . At the point O itself we can have infinitely many derivatives (if the right-hand side is infinitely differentiable). These derivatives can be found via the Taylor expansion, which is usually not convergent, since in general ψ^c is not an analytic function. This Taylor expansion can be found, for example, using the method of normal forms for maps which is analogous to what we studied in previous lectures for the case of differential equations.

LECTURE 16. LOCAL BIFURCATIONS OF PERIODIC ORBITS. CASE 1: $k = 1, \eta = 1$

As we discussed in the previous lecture, the problem of studying local bifurcations of periodic orbits can be reduced to the analysis of the dynamics in a small neighbourhood of a fixed point of the Poincare map. The map near the fixed point is written as

$$\bar{x} = Ax + o(x).$$

Like in the case of equilibrium states, there is a center manifold and all interesting dynamics are on the center manifold (the orbits which are not in the center manifold leave a neighborhood of zero at forward or backward iterations or the map). In the typical situation, the center manifold is just the zero point itself – this happens when the matrix A has no eigenvalues on the unit circle. If we have some eigenvalues on the unit circle, then we have a non-trivial center manifold, and the restriction of our system to the center manifold is:

$$\bar{w} = Dw + o(w).$$

The spectrum of D is $\{\eta_1, \dots, \eta_k\}$, where η_j all lie on a unit circle, i.e., $|\eta_j| = 1$, $j = 1, \dots, k$. There are two possibilities: the eigenvalues can form complex conjugate pairs and they can be real, i.e., equal to either plus one or minus one. A general system does not have multipliers on the unit circle at all, because by a small perturbation we can always move them out of the circle. But, if we have a one-parameter family of maps, it might happen that for some value of a parameter we will have one eigenvalue equal to $+1$ or -1 , or a pair of complex conjugate eigenvalues on the unit circle. This is the case of co-dimension 1, the case where we need just one parameter. If we have two parameters, then we may have two pairs of complex conjugate eigenvalues or one pair of complex conjugate ones and $+1$ or -1 , or a double ± 1 , and so on.

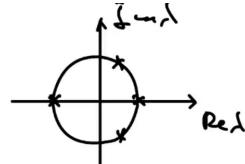


FIGURE 1

In this course, we will consider only the most basic cases:

1. The dimension of the center manifold $\dim W^c = 1$ and $\eta = 1$;
2. $\dim W^c = 1$ and $\eta = -1$;
3. $\dim W^c = 2$ and $\eta = e^{\pm i\omega}, 0 < \omega < \pi$.

The first case is very simple and it is indistinguishable from the case of one zero eigenvalue for an equilibrium state. The second and third cases are different but both of them have something in common with the Andronov-Hopf bifurcation.

Let us start with the first case, where $\eta = 1$, and thus, matrix D is not a matrix but a scalar $D = 1$. Then, equation on the center manifold has a form:

$$\bar{w} = w + h(w, \varepsilon). \quad (16.1)$$

At $\varepsilon = 0$, we have $h(0, 0) = 0$ and $\frac{\partial h}{\partial w}(0, 0) = 0$. It is important that

$$\frac{d\bar{w}}{dw} = 1 + \text{small terms} > 0,$$

which means that the map $w \rightarrow \bar{w}$ is monotonically increasing.

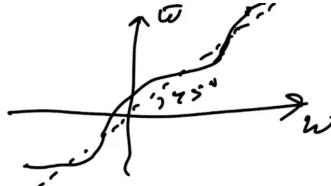


FIGURE 2

The intersection of the graph with the 45° -line, corresponds to the fixed points of the map, i.e., $\bar{w} = w$. The following statement holds true: every orbit either converges to a fixed point or goes away. Indeed, consider an orbit $\{w_n\}$:

$$w_{n+1} = f(w_n) := w_n + h(w_n, \varepsilon).$$

Let us assume that $w_{n+1} < w_n$, then since f is monotonically increasing, we obtain that $w_{n+2} = f(w_{n+1}) < f(w_n) = w_{n+1}$ and, consequently, the sequence w_n is monotonically decreasing. Analogously, if $w_{n+1} > w_n$, then the sequence w_n is monotonically increasing. In any case w_n is a monotone sequence and by this reason it should have a limit, if it is bounded (if not, then it should go away). Thus, if the limit exists, using continuity of f we get

$$\lim_{n \rightarrow \infty} w_{n+1} = \lim_{n \rightarrow \infty} f(w_n) = f(\lim_{n \rightarrow \infty} w_n) = f(\lim_{n \rightarrow \infty} w_{n+1}),$$

i.e., in the limit we have a fixed point, as claimed.

In principle, there may be no fixed points, but if there are fixed points, then they have alternating stability. A fixed point is stable if the derivative at it is smaller than one; and if the derivative is larger than one, then this fixed point is unstable. So, when we study the map (16.1), the only things that we should study are the fixed points of this map and their stability. Obviously, to find the fixed points we should solve the equation

$$h(w, \varepsilon) = 0,$$

and to determine stability we should look at the derivative of $h(w, \varepsilon)$:

$$\begin{aligned}\frac{\partial h}{\partial w}(\text{fixed point}) &< 0 \Rightarrow \text{stability}, \\ \frac{\partial h}{\partial w}(\text{fixed point}) &> 0 \Rightarrow \text{instability}.\end{aligned}$$

Thus, the situation for maps ($\bar{w} = w + h(w, \varepsilon)$) is the same as we had for equilibria of differential equations ($\dot{w} = h(w, \varepsilon)$). The only difference is that, in the case of differential equations, on the interval between fixed points there is only one trajectory which connects them, whereas for maps there is a continuum set of trajectories, which are discrete sets of points. But the rest is the same.

Let us summarize: we consider a map

$$\bar{w} = w + h(w, \varepsilon).$$

At $\varepsilon = 0$, we have $h(w) = l_2 w^2 + l_3 w^3 + \dots$. As in the case of differential equations, everything is determined by the number and sign of the first non-zero Lyapunov coefficient: let $l_k \neq 0$ and $l_j = 0$ for $j = 2, \dots, k-1$, then

if k is even, the fixed point is semi-stable;

if k is odd, the fixed point is stable if $l_k < 0$ and unstable if $l_k > 0$.

Now we will consider examples. Let

$$\bar{w} = w + l_k w^k + \dots, \quad (16.2)$$

where k is even and l_k is positive. Then we have instability to the right of zero, while from the left we have stability (see Figure 3).

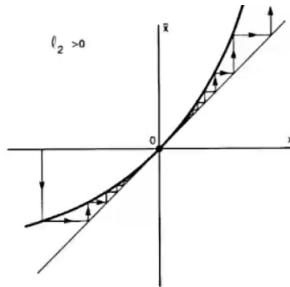


FIGURE 3. k is even, $l_k > 0$

If k is even, but l_k is negative, then we have stability from the right and instability from the left.

The picture is completely different if k is odd. First we assume that $l_k < 0$ in (16.2). Then, when w is positive, $l_k w^k$ is negative and vice versa, so zero is a stable point. If k is odd, but l_k is positive, then zero is an unstable fixed point, see Figure 4.

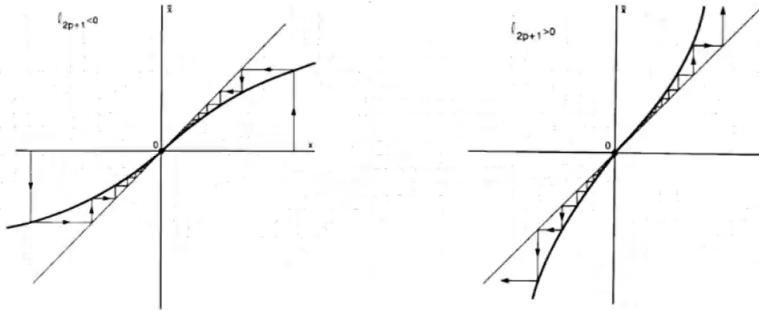


FIGURE 4. k is odd

When we perturb our system ($\varepsilon \neq 0$), if l_k is the first non-zero Lyapunov coefficient, then *up to k fixed points can be born*. These fixed points will have alternating stability and the most left and the most right one will have the same stability type as the fixed point has at the critical moment.

Main case: $l_2 \neq 0$. Then

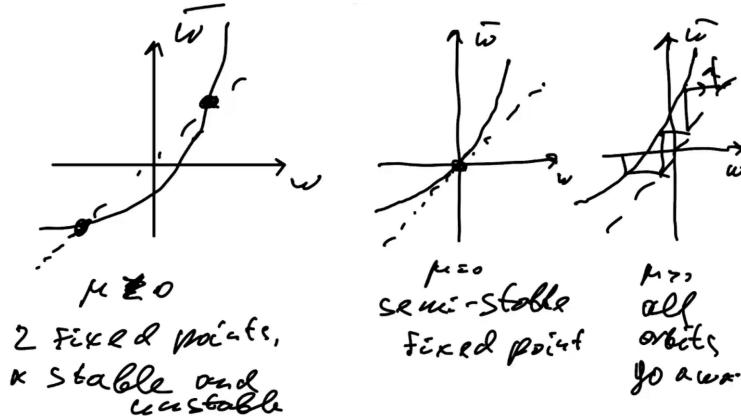
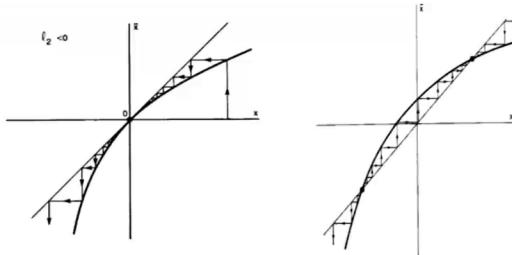
$$\bar{w} = w + \mu(\varepsilon) + l_2(\varepsilon)w^2 + o(w^2). \quad (16.3)$$

In principle there can be a linear term in w , but as we showed for the case of an equilibrium with one zero eigenvalue, this linear term can be killed by the shift of coordinates, whereas a constant term μ cannot be killed, in general. As we remember, the behaviour of the system is determined by the sign of $\mu \cdot l_2$.

Let us consider the case $l_2 > 0$ (see Figure 5).

For $\mu = 0$, we already know that zero is a semi-stable fixed point. When μ is negative, we simply move the graph of the map down and see that there are two fixed points, the left one is stable and the right one is unstable (because the stability is inherited). The proof is exactly the same as for the case of the equilibrium with one zero eigenvalue, you just need to equate the right side of (16.3) to w and analyze this equation. When $\mu > 0$ we move the graph up, and see that there is no intersections with the line $\bar{w} = w$ anymore, since zero was a tangency point; so, all orbits go away after $\frac{1}{\sqrt{\mu}}$ iterations.

When $l_2 < 0$ the picture is opposite. For $\mu = 0$ there is a semi-stable fixed point which is stable on the right and unstable on the left, for $\mu < 0$ there are no fixed points, and for $\mu > 0$ there are two fixed points, see Figure 6 for $\mu = 0$ and $\mu > 0$.

FIGURE 5. $l_2 > 0$ FIGURE 6. $l_2 < 0$

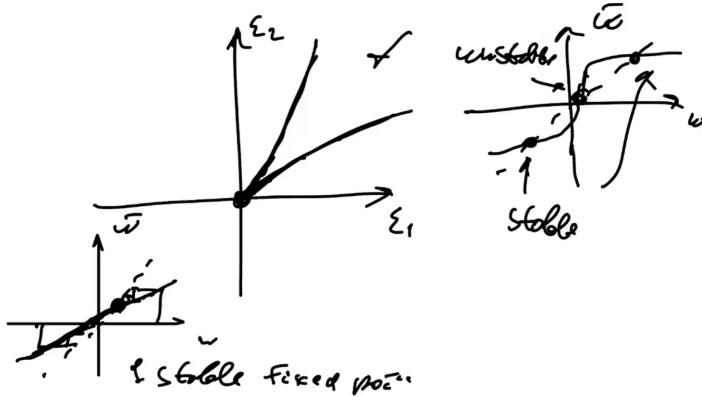
Next, consider the cusp bifurcation:

$$\bar{w} = w + \mu_0 + \mu_1 w + l_3 w^3 + o(w^3). \quad (16.4)$$

We suppose that $l_2 = 0$ and start with $l_3 \neq 0$. When we perturb our system, then all coefficients start to depend on a parameter ε and, in addition, we will have terms $\mu_0(\varepsilon)$ and $\mu_1(\varepsilon)w$ (we may also have quadratic terms, but they can be killed by a change of variables).

The picture is similar to the case of the equilibrium with one zero eigenvalue (see Figure 7).

We assume that l_3 is negative. Parameters ε_1 and ε_2 are related with μ_0 and μ_1 by a smooth transformation. Boundaries of the cusp correspond to the saddle-node bifurcation, which we just studied, and divide the plane of parameters into two regions: outside of the cusp there is only one fixed point (not necessarily at zero); this point is stable, so everything converges to it. For parameters inside the cusp, there are three fixed points, the middle one is unstable and two others are stable, since the most left and the most right points inherit the stability of the fixed point at the critical moment. On the boundaries, the unstable point moves to one of the stable ones and disappear,

FIGURE 7. $l_3 < 0$

thus we indeed have one stable point outside of the cusp region. So, for $l_3 < 0$ the bifurcation is safe.

If l_3 is positive, then we should inverse the picture 7, i.e., everywhere outside of the cusp region we will have one unstable fixed point, and inside of it we will again have three fixed points, but the middle one will be stable and the boundary points will be unstable. Thus, for $l_3 > 0$ the bifurcation is dangerous.

As we see, all the theory which we have for the bifurcations of the equilibrium with one zero eigenvalue is transferred word by word to the theory of bifurcations of fixed points with one multiplier equal to +1.

Now let us look what is going on in terms of periodic orbits. Consider the case $l_2 \neq 0$ and draw a picture on the plane, see Figure 8. For $\mu = 0$ we have a semi-stable periodic orbit, when $\mu \cdot l_2 < 0$. This periodic orbit must decompose into two periodic orbits, one stable and one unstable. When $\mu \cdot l_2 > 0$ periodic orbits disappear; the time you spend on the place where you used to have a semi-stable periodic orbit is proportional to $\frac{1}{\sqrt{\mu}}$.

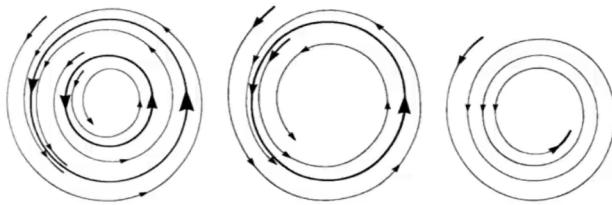


FIGURE 8

This was a picture on the center manifold. It is one-dimensional for maps and two-dimensional for systems of differential equations.

If we add contraction transverse to the center manifold (for maps) then stable point will remain stable, whereas unstable point will become a saddle (Figure 9, left picture). Then we change parameters and at a certain moment these two points collide, forming a saddle-node (Figure 9, middle picture). On the right we have a saddle behavior and on the left we have contraction, like in the case of equilibrium states, the only difference is that orbits are just sequences of points now, not curves. When we further change parameters, the fixed point disappears and all orbits go away (Figure 9, right picture).

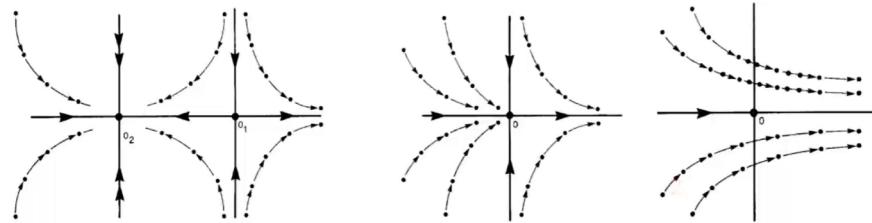


FIGURE 9

If Figure 9 is a picture on a cross-section, then for the system of differential equations we just need to add rotation transverse to the cross-section. Then we will have the following picture, see Figure 10.

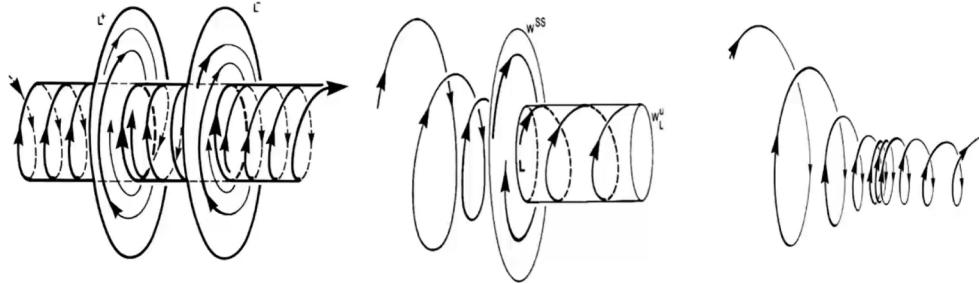


FIGURE 10

On Figure 10, left, we have two periodic orbits, the left one is stable and the right one is saddle. Then at certain values of parameters they collide (Figure 10, middle picture) and form a saddle-node, so from the left we converge to the periodic orbit and from the right we have an unstable manifold which has a form of cylinder and all orbits converge to this cylinder, rotate and go away. When we further change parameters, a periodic orbit disappears and all orbits rotate and eventually go away (Figure 10, right picture).

Now let us assume that $l_2 = 0$ but $l_3 < 0$ and, first, add a contraction direction, then a stable point on the center manifold will remain stable. But if we add an expanding direction, then a stable point will become saddle, see Figure 11.

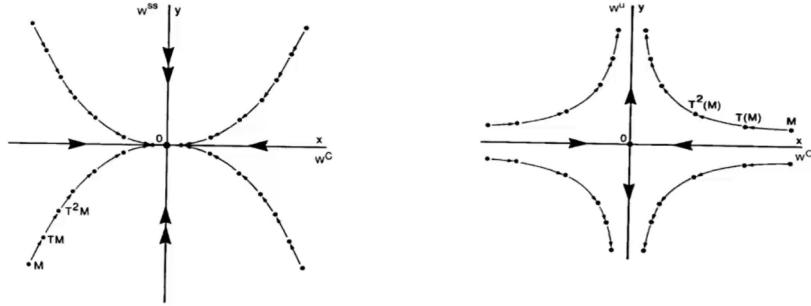


FIGURE 11

The main message of this lecture: bifurcations of periodic orbits with one multiplier equal to +1 follow exactly the same rules as bifurcations of equilibria with one eigenvalue equal to zero. The main bifurcation is a saddle-node bifurcation ($l_2 \neq 0$), when two periodic orbits come close together, collide and disappear. The only thing to remember is that we are not talking about equilibria – we are talking about fixed points of the Poincaré map which correspond to closed phase curves of systems of differential equations.

LECTURE 17. LOCAL BIFURCATIONS OF PERIODIC ORBITS. CASE 2: $k = 1$, $\eta = -1$ (PERIOD DOUBLING)

In this lecture we start to study a new case: period-doubling bifurcation that corresponds to one multiplier equal to minus one ($\eta = -1$). There can be other eigenvalues inside or outside a unit circle, but after we restrict our system to the center manifold this is not important. If there is only one eigenvalue on a unit circle, the center manifold is one-dimensional. So, the map is one-dimensional (a map of a line $x \in \mathbb{R}$ into itself). Since the multiplier is equal to -1 , with each iteration we will jump from positive x to negative x and vice versa. As we will see, everything again depends on the Lyapunov coefficients. As we will also see, they are not just coefficients of the Taylor expansions, but some functions of the coefficients of the Taylor expansions obtained by the method of normal forms. If the first non-zero Lyapunov coefficient is negative, then the fixed point at a critical moment is stable (see Figure 1).

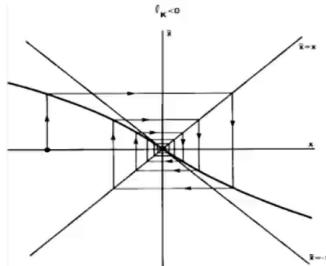


FIGURE 1

If the first non-zero Lyapunov coefficient is positive, then the fixed point at a critical moment is unstable (see Figure 2).

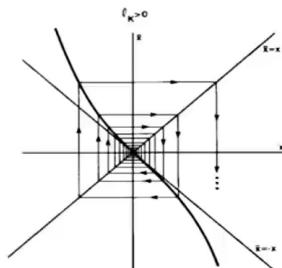


FIGURE 2

If we add contraction transverse to the center manifold (we also assume that the first Lyapunov coefficient is negative), then we will have stability, but

again jump from positive x to negative x forward and backward (see Figure 3, left picture). After the bifurcation, our stable point gives birth to a new stable periodic orbit of period two; whereas a fixed point itself loses stability and becomes saddle (see Figure 3, right picture). This is why this bifurcation is called “period doubling”.

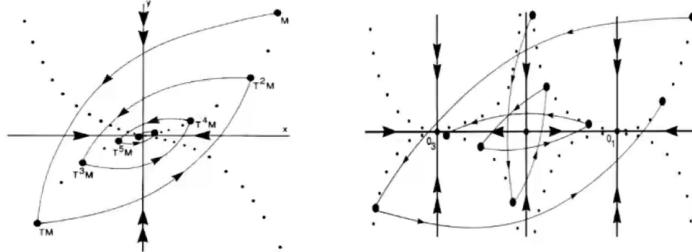


FIGURE 3

If we add a contraction direction transverse to the center manifold where we have an unstable fixed point and bifurcate, then an unstable point will become saddle and the newly born orbit of period 2 will also be saddle (see Figure 4). What is important is that after the bifurcation we do not obtain two new fixed points of the Poincare map, but end up with just one orbit of period two.

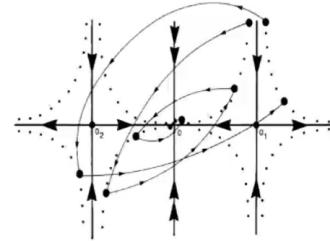


FIGURE 4

The questions which we will study are:

How to compute Lyapunov coefficients? How to prove that if the first Lyapunov coefficient is negative, we have stability and if positive - instability? How to prove that indeed orbits of period two are born? How many orbits of period two are born?

All these questions will be discussed in this and next lectures.

Before that, it is instructive to look at the pictures for the system of differential equations (what we discussed before is a Poincare map). First, let us consider what is happening on the center manifold. After the bifurcation, a periodic orbit does not disappear. If we start on the center manifold, after one round near a periodic orbit the Poincare map must change the sign of x .

Thus, we see that the surface formed by trajectories is actually a Möbius strip (it is not a cylinder as it was for the previous case where the multiplier was equal to plus one). When this new orbit of period two is born on the center manifold, the original periodic orbit becomes unstable and the boundary of its unstable manifold is a new stable periodic orbit for the system of differential equations, which corresponds to the period-two orbits for the Poincaré map on the cross-section, and its period is close to twice the period of the original orbit (see Figure 5, left picture).

Next, let us look what will happen if we add a contraction transverse to the Möbius strip. Figure 5, right picture, is a three-dimensional picture. L is a periodic orbit, the unstable manifold is a Möbius strip. It also has a stable manifold, but if you want to close it, you will see that it is again a Möbius strip. In spite of this strange topological structure – that stable and unstable manifolds to periodic orbits become Möbius strips, this bifurcation happens extremely often, people find it in many systems of differential equations of vastly different nature.

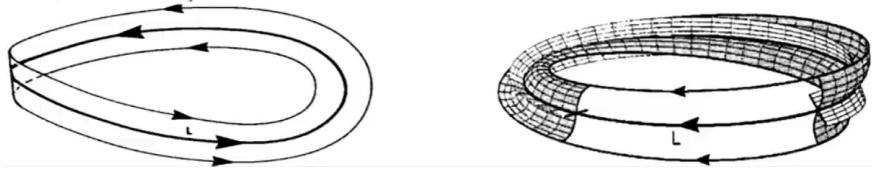


FIGURE 5

This was an overview of what we are going to do and now we start to answer questions which we asked. We write our map on the center manifold

$$\bar{w} = -w + h(w, \varepsilon), \quad w \in \mathbb{R}^1.$$

At $\varepsilon = 0$, we assume that

$$h(0, 0) = 0 \text{ and } \frac{\partial h}{\partial w}(0, 0) = 0. \quad (17.1)$$

The map $\bar{w}(w)$ is monotonically decreasing. Indeed,

$$\frac{\partial \bar{w}}{\partial w} = -1 + \frac{\partial h}{\partial w} < 0,$$

since h is a smooth function and its derivative is small at small w by (17.1). The graph of $\bar{w}(w)$ at $\varepsilon \neq 0$ is schematically plotted in Figure 6. At $\varepsilon = 0$ the graph should go through zero. The intersection of this graph with line $\bar{w} = w$ by definition is a fixed point.

Obviously, for any small ε , there is always only one fixed point. Indeed, the equation for fixed points is:

$$w = -w + h(w, \varepsilon),$$

or

$$w - \frac{1}{2}h(w, \varepsilon) = 0.$$

By the Implicit Function Theorem there is only one fixed point $w = w^*(\varepsilon)$ which smoothly depends on ε .



FIGURE 6

Let us introduce a new variable $w^{new} = w - w^*(\varepsilon)$. From now on instead of w^{new} we will write w . Then

$$\bar{w} = -w + h(w, \varepsilon) \quad \text{and} \quad h(0, \varepsilon) \equiv 0 \text{ for all } \varepsilon,$$

so, zero is a unique fixed point of our map, for all small ε . The only thing which may happen to this fixed point is that it may change stability. For this bifurcation, as we told, when the fixed point changes stability, there appears an orbit of period two. Let us show that the map cannot have orbits of higher period. This is true since the second iteration $\bar{\bar{w}}$ is a monotonically increasing function of w . Indeed,

$$\frac{d\bar{w}}{dw} = \frac{d\bar{w}}{d\bar{w}} \frac{d\bar{w}}{dw} > 0,$$

since the map itself is monotonically decreasing and thus $\frac{dw}{d\bar{w}} < 0$ and $\frac{d\bar{w}}{dw} < 0$. As we discussed, for monotonically increasing maps every orbit either goes away or tends to a fixed point. So, for the original map every orbit tends to an orbit of period two or to the fixed point (or goes away). Of course, stability of points of period two is alternating and the outermost one inherits stability of the fixed point at the critical moment. So, once we know the stability of the fixed point at the critical moment, then we know the stability of all orbits of period two. Actually, the only problem is to figure out how many orbits of period two we have.

To simplify further computations, let us first do a small detour to a normal forms theory for maps. We will consider a general map:

$$\bar{x} = Ax + f(x), \quad x \in \mathbb{R}^n, \tag{17.2}$$

where A is a matrix and $f(x) = o(x)$ is a non-linearity. We want to do a coordinate transformation to make the map (17.2) simpler. First thing to do

is to bring a matrix A to a Jordan form. For simplicity we will consider only the case when A is diagonalizable:

$$\bar{x}_j = \lambda_j + \sum_{|m| \geq 2, m=(m_1, \dots, m_n)} a_{j,m} x_1^{m_1} \cdots x_n^{m_n}.$$

Theorem 17.1. *All non-resonant monomials can be killed up to any given order.*

This means that after we do coordinate transformations, we can make coefficients $a_{j,m}$ of all non-resonant monomials equal to zero up to any order (the order is $|m| = m_1 + \dots + m_n$). It is in exact parallel to what we did for equilibrium states. The only things to explain are what is a non-resonant monomial and what are the coordinate transformation that kill them. To kill the monomial $a_{j,m} x_1^{m_1} \cdots x_n^{m_n}$ the desired coordinate transformation is

$$x_j^{\text{new}} = x_j + \frac{a_{j,m}}{\lambda_j - \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n}} x_1^{m_1} \cdots x_n^{m_n}.$$

We can kill monomials one by one in the increase of order: first we kill non-resonant monomials of order two – this affects coefficients of monomials of order three and higher, then we kill monomials of order three – this affects monomials of order four and higher but do not affect monomials of order two and other monomials of order three, and so on.

Definition 17.2. A monomial $a_{j,m} x_1^{m_1} \cdots x_n^{m_n}$ is called resonant if

$$\lambda_j = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n}.$$

Many of the resonant monomials cannot be killed at all, some of them can be killed by some other coordinate transformations, but not all of them.

Suppose we perturb the system a little bit, then the eigenvalues $\lambda_1, \dots, \lambda_n$ will change only a little bit, which means that if for the unperturbed system a certain monomial was non-resonant, it remains non-resonant for the perturbed system. So, when we decide which monomials are resonant, we decide this only at the critical moment

This was the general theory, and now we return to our particular map and consider it at the moment of bifurcation:

$$\bar{w} = -w + \sum_{|m| \geq 2} a_m w^m.$$

The resonance condition in our case is

$$-1 = (-1)^m,$$

and we see that *all monomials with odd m are resonant, and all monomials with even m are non-resonant*.

So, we can bring the map to the form

$$\bar{w} = -w(1 + L_1 w^2 + \cdots + L_M w^{2M}) + o(|w|^{2M+1}) \quad \text{at } \varepsilon = 0.$$

Let $L_k \neq 0$ be the first non-zero Lyapunov coefficient. Then we take $M = k$ and write

$$\bar{w} = -w(1 + L_k w^{2k} + o(|w|^{2k})).$$

Let us compute the absolute value of \bar{w} .

$$|\bar{w}| = |w|(1 + (L_k + \text{small})w^{2k}).$$

Thus,

$$\begin{aligned} |\bar{w}| &< |w| & \text{if } L_k < 0 & \Rightarrow \text{stability}, \\ |\bar{w}| &> |w| & \text{if } L_k > 0 & \Rightarrow \text{instability}. \end{aligned}$$

This means that whenever we iterate our map, the value of w will decrease if L_k is negative, unless $w = 0$; and it will increase if L_k is positive, see Figures 1 and 2.

In the next lecture, we will start with discussing how many orbits of period two can be born at the bifurcation.

LECTURE 18. PERIOD DOUBLING BIFURCATION. SCHWARZ DERIVATIVE.

Today we will finish discussion of the period doubling bifurcation and, in particular, will derive a formula for the first Lyapunov coefficient. The map we consider is

$$\bar{w} = -w + aw^2 + bw^3 + \dots.$$

So, we have the fixed point at zero, the multiplier equals to minus one, a and b are arbitrary coefficients of the Taylor expansion. We stopped on a third order term because we know that the first Lyapunov coefficient is determined by the terms up to the third order. Let us do coordinate transformation which kills quadratic terms:

$$w^{new} = w + cw^2.$$

We know that if we want to kill a non-resonant term, we should perform a coordinate transformation w plus the term we want to kill but with a different coefficient. Let us do this transformation and see what is this coefficient.

$$\begin{aligned} \bar{w}^{new} + w^{new} &= \bar{w} + cw^2 + w + cw^2 = -w + aw^2 + bw^3 + cw^2(1 - aw + \dots)^2 + \\ &\quad w + cw^2 + o(w^3) = (a + 2c)w^2 + (b - 2ac)w^3 + o(w^3). \end{aligned}$$

To kill the quadratic term, we choose $c = -\frac{1}{2}a$, then

$$w^{new} = -w^{new} + (b + a^2)(w^{new})^3 + o((w^{new})^3),$$

where we substituted w^3 by $(w^{new})^3$, because the difference between w^{new} and w is a higher order term. Thus,

$$L_1 = -a^2 - b.$$

Let us denote $f(w) = -w + aw^2 + bw^3 + o(w^3)$, then

$$a = \frac{1}{2}f''(0), \quad b = \frac{1}{6}f'''(0) \quad \text{and } f'(0) = -1.$$

So, in terms of the function f

$$L_1 = \frac{1}{6} \frac{f'''(0)}{f'(0)} - \frac{1}{4} \left(\frac{f''(0)}{f'(0)} \right)^2 = \frac{1}{6} \left[\frac{f'''(0)}{f'(0)} - \frac{3}{2} \left(\frac{f''(0)}{f'(0)} \right)^2 \right] = \frac{1}{6} S_{[f]}(0),$$

where $S_{[f]}$ is the Schwarz derivative (or Schwarzian derivative) of function f . The Schwarz derivative was invented more than a hundred years before the bifurcation theory was invented – because of needs of the cartography and it also pops up in different branches of mathematics. It is a small miracle that this object also appeared in the bifurcation theory. Many interesting properties of one-dimensional maps are related to its Schwarz derivative; in particular, if you know that the Schwarz derivative has a certain sign (negative

or positive), you do not need to compute the first Lyapunov coefficient at period-doubling bifurcations (recall that the only important information about Lyapunov coefficient is its sign).

The important formula for the Schwarz derivative of the composition of two maps is

$$S_{[f \circ g]} = S_{[f]}(g(x))(g'(x))^2 + S_{[g]}(x). \quad (18.1)$$

Let me explain why this formula is important. Suppose functions f and g both have negative Schwartz derivatives, then it follows from the formula (18.1) that the Schwarz derivative of the composition of f and g is also negative. Suppose we have a map f and compose it to itself, then we will have second iteration of the map. So, if the map f has negative Schwarz derivative, then its second iteration also has negative Schwarz derivative. Continuing this process, we see that any iteration of the map will have negative Schwarz derivative.

Thus, let us consider a one-dimensional map and take a periodic orbit of any period, say period n , and suppose that it undergoes a period-doubling bifurcation. We know that the first Lyapunov coefficient is a Schwarz derivative of a periodic orbit of period n , which is the Schwarz derivative of the n -th iteration of f at the point of period n . If we know that the Schwarz derivative of f is negative, then we do not need to compute the n -th iteration of the map because we automatically know that the Schwarz derivative of f^n is also negative. Hence, for such maps the first Lyapunov coefficient for any period doubling bifurcation of any orbit is always negative.

Let us now demonstrate formula (18.1). First, we compute the derivatives of $h(x) = f(g(x))$:

$$\begin{aligned} h'(x) &= f'(g(x))g'(x), & h''(x) &= f''(g(x))g'(x)^2 + f'(g(x))g''(x), \\ h'''(x) &= f'''(g(x))g'(x)^3 + 3f''(g(x))g''(x)g'(x) + f'(g(x))g'''(x). \end{aligned}$$

Now we can compute the Schwarz derivative for $h(x)$:

$$\begin{aligned} S_{[h]}(x) &= \frac{h'''(x)}{h'(x)} - \frac{3}{2} \left(\frac{h''(x)}{h'(x)} \right)^2 = \frac{f'''(g(x))}{f'(g(x))} g'(x)^2 + 3 \frac{f''(g(x))}{f'(g(x))} g''(x) + \frac{g'''(x)}{g'(x)} - \\ &\quad \frac{3}{2} \left[\frac{f''(g(x))}{f'(g(x))} g'(x) + \frac{g''(x)}{g'(x)} \right]^2 = S_{[f]}(g(x))g'(x)^2 + S_{[g]}(x), \end{aligned}$$

and this is the required formula (18.1).

Let us conclude: *if the map $x \rightarrow f(x)$ has negative Schwarz derivative for all x , then any its iteration also has negative Schwarz derivative. In particular, all period-doubling bifurcations are soft (not dangerous).*

Example 18.1. Let us consider a map

$$\bar{x} = M - x^2,$$

where M is some parameter, not necessarily small.

Then,

$$f'(x) = -2x, \quad f''(x) = -2, \quad f'''(x) = 0$$

and the Schwarz derivative is

$$S_{[f]}(x) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 < 0.$$

Suppose that M is sufficiently small, then the graph will be a parabola, which is below the line $\bar{x} = x$ (see Figure 1). So, we do not have any fixed points or periodic orbits and every orbit goes to minus infinity.

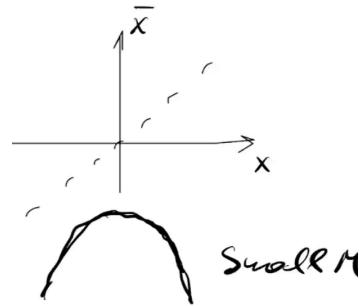


FIGURE 1

When M is large, we have two unstable fixed points. Indeed, the derivative is equal to $-2x$, and when $|x|$ is large (larger than $1/2$) the absolute value of it is larger than 1, which means that the fixed points are unstable. Let us denote the left fixed point by $-a$ and consider the interval $I_0 := [-a, a]$. It is obvious that any bounded orbit of f must be in I_0 , see Figure 2.

Let us consider the preimage $I_{-1} = f^{-1}([-a, a])$ of the interval I_0 for sufficiently large M . Namely, we want the preimages $\pm b$, $b > 0$ of the point a to exist (an easy calculation shows that the equation $f(b) = a$ is solvable if $M > 2$). In this case, we see that

$$I_{-1} = [-a, -b] \cup [b, a]$$

is a disjoint union of two intervals and the restriction of f to any of these intervals is a smooth one-to-one function. Moreover, $f([-a, -b]) = f([b, a]) = [a, b]$. We can iterate this procedure. For instance, the inverse image of both $[-a, -b]$ and $[b, a]$ will consist again of two disjoint intervals, so we have

$$I^{-2} := f^{-2}([-a, a]) = [-a, -b_1] \cup [-b_2, -b] \cup [b, b_2] \cup [b_1, a],$$

see Figure 3, and so on.

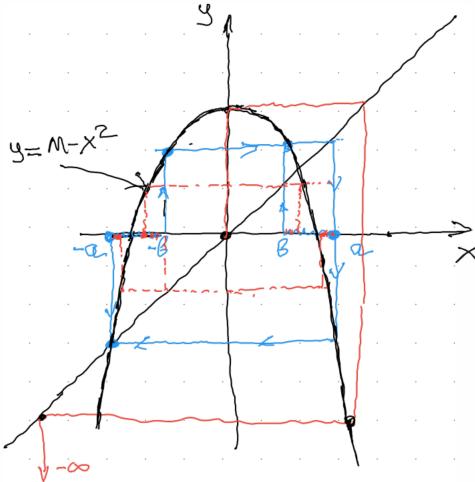


FIGURE 2

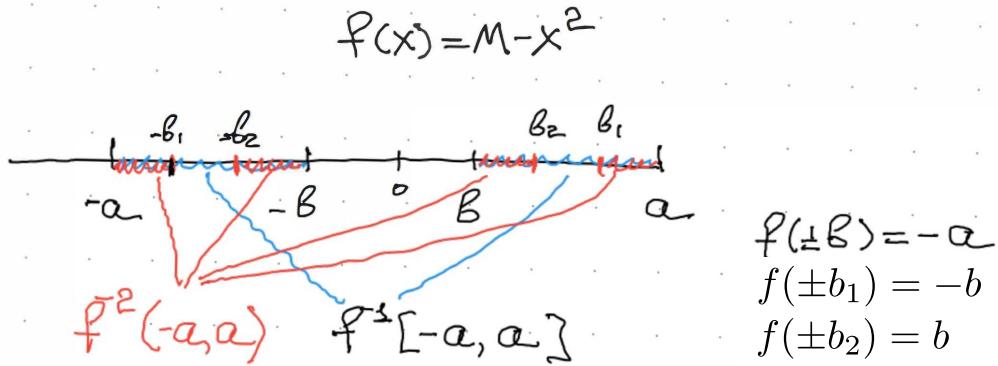


FIGURE 3

Thus, $I^{-n}([-a, a])$ consists of 2^n disjoint closed intervals. Finally, if we take a (non-empty) intersection

$$K = \cap_{n \in \mathbb{N}} I^{-n}$$

of the nested closed sets I^{-n} , we obtain a Cantor-type set K of points whose orbits remain bounded for all n . Moreover, as follows from the construction, for any coding $\omega = \{\omega_n\}_{n=1}^\infty$ (which is a sequence of zeros and ones: $\omega_n = \{0, 1\}$), we can find $x_0 \in K$ such that $f^n(x_0) > 0$ if $\omega_n = 1$ and $f^n(x_0) < 0$ if $\omega_n = 0$. Actually, the correspondence between coding sequences ω and points $x_0 \in K$ is one-to-one, but this requires a proof (for instance if M is large enough the restriction of f to I_{-1} will be expanding $|f'(x)| > 1$, so the inverse will be contracting, and we may use the contraction mapping arguments).

This means a chaotic behaviour. The dynamics on K is indistinguishable from what I get if I flip a coin. Indeed, if I am flipping a coin, every time I

get a head or a tail (0 or 1,) so I obtain a symbol $\omega_n \in \{0, 1\}$. The result of infinitely many such flips is exactly a (random) coding sequence ω but every such random sequence is represented by a unique orbit of my map. This gives us a random, i.e., chaotic, behavior of iterations of this map. The construction is strongly based on the possibility to split the inverse image of some interval into smaller disjoint intervals in such a way that the map is one-to-one at every such subinterval and the image of every of such smaller interval covers the initial big interval.

As we see at small M we have a very simple behaviour (everything goes to infinity), and at large M we have chaotic behaviour, very developed chaos, where for every sequence of 0s and 1s we have the orbit with the corresponding code. Moreover, we can prove that if the code is periodic, then the corresponding orbit is also periodic; which means that we have infinitely many periodic orbits of all codes and we can also prove that all these periodic orbits are unstable. So, there is a question: “What happens when the parameter M grows?” At the very beginning, when M is small, we have nothing, but when M is large, we have infinitely many periodic orbits. Where did all these periodic orbits come from? They came from bifurcations. We know two possible bifurcations: a saddle-node bifurcation, when a pair of fixed points is born, after which one point (which is on the left) is unstable but another one (which is on the right) is stable; when M grows the stable point also becomes unstable, and the only way for it to become unstable without disappearing is a period-doubling bifurcation. Since the Schwartz derivative is negative, we know that only stable periodic orbits can undergo a period-doubling bifurcation. So, after the first bifurcation, the stable fixed point becomes unstable and a stable orbit of period two is born. When we further change parameters this orbit of period two also becomes unstable, giving a stable periodic orbit of period 4. Then we change parameters again and so on. Thus, before we get chaotic behaviour, we have an infinite sequence of period-doubling bifurcations, and only after that we have chaos.

Infinite cascade of period doubling bifurcations exist also in the case of higher-dimensional maps but there is no nice formula for the first Lyapunov coefficient of such maps. So, in the higher-dimensional case we really need to compute it separately each time a period-doubling happens, or prove that it is always negative by different methods. But for the one-dimensional case this for free due to the formula:

$$L_1 = \frac{1}{6} S_{[f]}(0).$$

LECTURE 17. PERIOD-DOUBLING BIFURCATIONS CONTINUED

We continue analyzing period-doubling bifurcations. In the previous lecture, using the method of normal forms, we found that on the center manifold our map can be reduced to

$$\bar{w} = -w - L_k w^{2k+1} + o(w^{2k+1}), \quad (17.1)$$

where the even-order terms were killed by normalizing transformation and $L_k w^{2k+1}$ is the first non-zero nonlinear term of an odd order. We checked that at the critical moment the fixed point is stable if L_k is negative and unstable if L_k is positive. When we perturb our map, we know that the fixed point does not disappear. So, we put the origin into the fixed point. Thus, the fixed point will be always at zero. We also know that when we perturb our system, some number of orbits of period 2 can be born at this bifurcation.

Theorem 17.1. *If L_k is the first non-zero Lyapunov coefficient, then up to k (and no more) orbits of period 2 can be born at the bifurcation.*

Proof. Let us write the second iteration of the map (17.1):

$$\begin{aligned} \bar{\bar{w}} &= -\bar{w} - L_k \bar{w}^{2k+1} + o(\bar{w}^{2k+1}) = w + L_k w^{2k+1} - \\ &L_k(-w^{2k+1})(1 + \text{ small }) + o(w^{2k+1}) = w + 2L_k w^{2k+1} + o(w^{2k+1}), \end{aligned}$$

or

$$\bar{\bar{w}} - w = 2L_k w^{2k+1} + o(w^{2k+1}). \quad (17.2)$$

The derivative of order $2k + 1$ from the right hand side of (17.2) is non-zero, since $L_k \neq 0$. Thus, when we perturb our map (17.1), this will result in a small perturbation of the second iteration of the map, which means that again the derivative of order $2k + 1$ of $(\bar{\bar{w}} - w)$ will be non-zero.

Therefore, after a perturbation, equation (17.2) can have no more than $2k+1$ zeros, which correspond to $2k + 1$ fixed points of the second iteration. One of these fixed points is zero, it is a fixed point of the original map and it is also a fixed point of the second iteration. What remains are $2k$ fixed points which correspond to k orbits of period two of the original map (we recall that we have exactly one fixed point in a small neighborhood of the origin, so the remaining roots of $\bar{\bar{w}} - w$ correspond to points of period two of the original map). \square

Next, for small k (since for large k the situation is very degenerate) we want to figure out what are the control parameters and how the number of orbits of period 2 changes when we change control parameters.

We start with the general case and write the perturbed version of the map (17.1):

$$\bar{w} = -w(1 + \mu_0 + \sum_{j \leq k-1} \mu_j w^{2j} + L_k w^{2k} + o(w^{2k})),$$

where the coefficients μ_0, μ_j, L_k and the term $o(w^{2k})$ all depend on small parameters ε . At the critical moment all μ_0 and μ_j are equal to zero. When the bifurcation happens, the μ 's may become non-zero, in particular:

- $\mu_0 < 0 \Rightarrow \quad w = 0$ is stable (see Figure 1, left picture),
- $\mu_0 > 0 \Rightarrow \quad w = 0$ is unstable (see Figure 1, right picture).

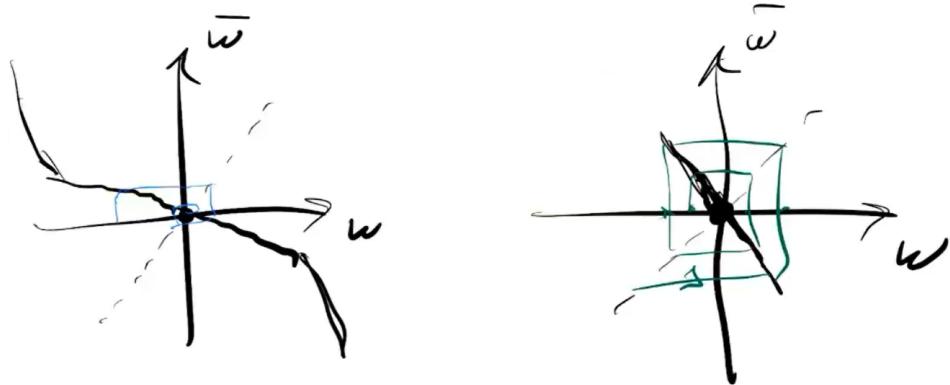


FIGURE 1

Note that trajectories are not the spirals in this figure, but the intersections of these spirals with the line $\bar{w} = 0$.

Let us look at the case $k = 1$, so there is no term $\sum_{j \leq k-1} \mu_j w^{2j}$ and we have only one control parameter μ_0 . Assume first that $L_1 < 0$. We know that in this case the fixed point is stable at the critical moment. We know that, when we perturb our system, we may have up to one orbit of period 2, and the stability of this orbit inherits the stability of the fixed point at the critical moment. Moreover its stability must be different from the stability of the fixed point. So, this unique orbit of period 2 must be stable, which means that it exists if and only if the fixed point becomes unstable. This happens when μ_0 is positive, see Figures 2 and 3.

We have the similar result when $k = 1$ and L_1 is positive. For $\mu_0 \geq 0$ we only have one fixed point, and this fixed point is unstable (see Figure 4); for $\mu_0 < 0$

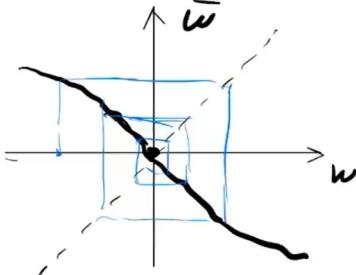


FIGURE 2. $L_1 < 0$,
 $\mu_0 \leq 0$

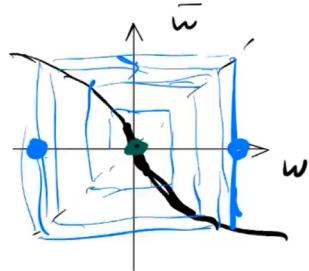


FIGURE 3. $L_1 < 0$,
 $\mu_0 > 0$

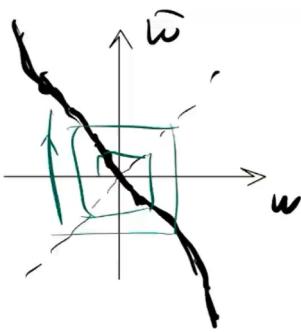


FIGURE 4. $L_1 > 0$,
 $\mu_0 \geq 0$

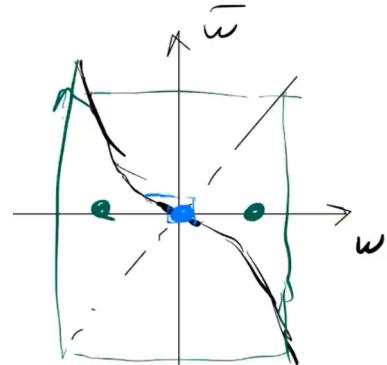


FIGURE 5. $L_1 > 0$,
 $\mu_0 < 0$

the fixed point becomes stable and an unstable orbit of period 2 appears (see Figure 5).

Let us look at the case $k = 2$. Then our map has a form

$$\bar{w} = -w(1 + \mu_0 + \mu_1 w^2 + L_2 w^4 + o(w^4)), \quad (17.3)$$

and we know that up to two orbits of period 2 can be born. These orbits are comprised by fixed points of the second iteration of the map, excluding the fixed point 0 for the original map. One can show by direct calculations that the equation for these fixed points is

$$\bar{\mu}_0 + \bar{\mu}_1 w^2 + L_2 w^4 + o(w^4) = 0, \quad (17.4)$$

where $(\bar{\mu}_0, \bar{\mu}_1)$ are smooth functions of (μ_0, μ_1) which are close to identity near $(0, 0)$. For simplicity, we will denote them by (μ_0, μ_1) again.

Solutions of the equation (17.4) go in pairs, one corresponds to positive value of w and another one corresponds to negative value of w . Denoting $\rho = w^2$, we get

$$\mu_0 + \mu_1 \rho + L_2 \rho^2 + o(\rho^2) = 0. \quad (17.5)$$

This equation is exactly the same equation as we have for the Andronov-Hopf bifurcation. So, whatever bifurcation diagram we had for the Andronov-Hopf bifurcation, we have here as well. We look for the bifurcations of positive roots of the equation (17.5). When the number of roots may change? When two roots collide and disappear and when one of the roots becomes negative. The first case corresponds to derivative with respect to ρ equal to zero at the root and the second case corresponds to $\rho = 0$.

Thus, bifurcation equations are

$$\mu_0 = 0 \quad (\rho = 0)$$

and

$$\begin{cases} \mu_0 + \mu_1\rho + L_2\rho^2 + o(\rho^2) = 0, \\ \mu_1 + 2L_2\rho + o(\rho) = 0, \quad \rho > 0. \end{cases} \quad (17.6)$$

Thus, from the last equation we see that $\mu_1 L_2 < 0$ and by the Implicit Function Theorem

$$\rho = -\frac{\mu_1}{2L_2} + o(|\mu|),$$

where $\mu = (\mu_0, \mu_1)$. Substituting this value of ρ into the first equation of (17.6), we get

$$\mu_0 = \frac{\mu_1^2}{4L_2} + o(\mu_1^2).$$

Now we know the bifurcation set on the plane of parameters and, thus, can recover the behaviour.

Let us consider the case $L_2 < 0$ and draw the bifurcation diagram. We start with the region where $\mu_0 < 0$ and choose $\mu_1 = 0$ – then we see that $\mu_0 + \mu_1\rho + L_2\rho^2 + o(\rho^2) = 0$ cannot have solutions and by this reason there is only a stable fixed point for all μ from this region. When we cross the line $\mu_0 = 0$ at some negative μ_1 , by the previous theory (for $L_1 \neq 0$), we have one stable orbit of period 2 and one unstable fixed point. When we go further counterclockwise and again cross $\mu_0 = 0$, but already at some positive μ_1 , we see that the first Lyapunov coefficient is positive and so we get an additional unstable orbit of period 2, see Figure 6.

As an exercise, you can draw a similar picture for the case $L_2 > 0$.

In the next lecture, we will discuss the theory of the period-doubling bifurcation in more details for the case of one-dimensional maps. There are some interesting phenomena there. But today we want to discuss what is going on in higher dimensions. Suppose we have a Poincare map, or in other words we have a fixed point and let us assume that it is stable (see Figure 7, left picture). Now we start to change parameters and period-doubling bifurcation may occur, i.e. the fixed point becomes a saddle (we assume that we are in

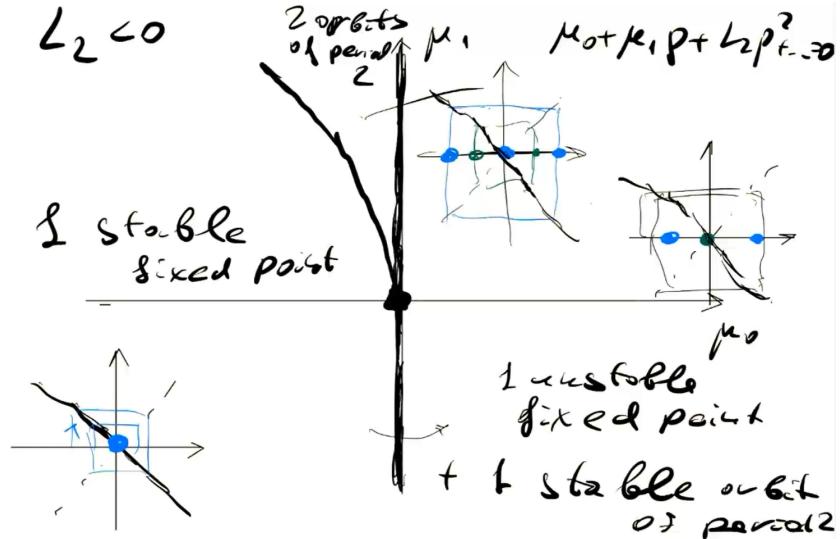


FIGURE 6

higher dimensions and have an additional contraction direction) and there appears a stable orbit of period 2 (see Figure 7, right picture). After we further change parameters, another period-doubling bifurcation may happen – now with the orbit of period 2. So, from the orbit of period 2 there can emerge an orbit of period 4 which inherits the stability, while the period-2 orbit becomes a saddle. So, now we have three orbits: one saddle fixed point, one saddle orbit of period 2 and one stable orbit of period 4 (see Figure 7, middle-down picture).

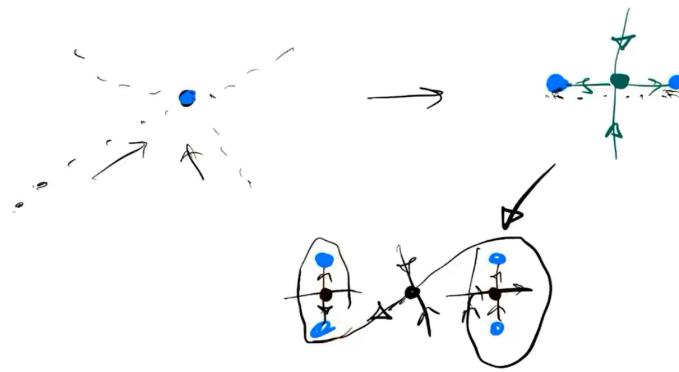


FIGURE 7

Then we further change parameters and obtain a stable orbit of period 8 while all the previous orbits become saddles, and so on. What people noticed is that, for a *finite* change in the parameter values, we may have *infinitely many bifurcations of period doubling*. After each bifurcation, one more periodic orbit becomes saddle and you have one more stable orbit of a doubled period. At the end you have infinitely many periodic orbits all of which are unstable on

the center manifold. After that you have what is called chaotic dynamics. Actually, this cascade of period doublings is the most popular route to chaos.

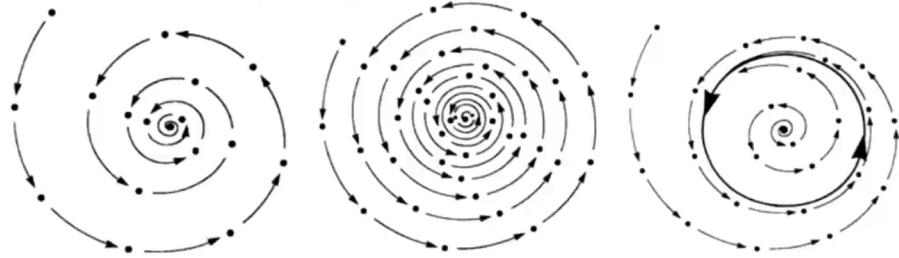
The theory of these infinite cascades exists for the one-dimensional case from the end of the 70s. For some classes of two-dimensional maps, the mathematical theory was built only 15 years ago. For three-dimensional maps, such rigorous theory still does not exist, although people see it in numerous experiments and everyone believes in it.

LECTURE 20. BIRTH OF INVARIANT TORUS.

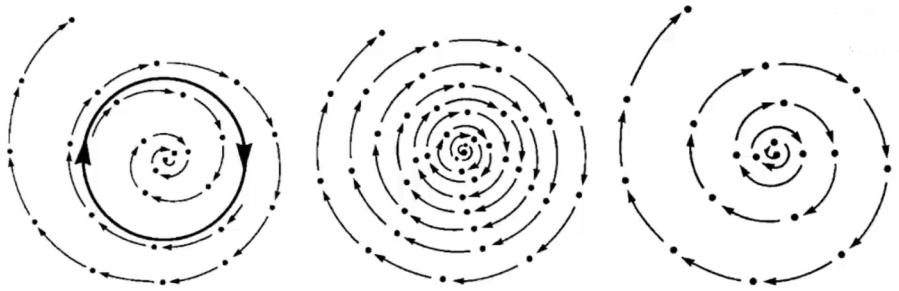
The next bifurcation leads to the creation of an invariant torus. Let me recall that we start with a periodic orbit, take a cross-section and consider the Poincare map on the cross-section. The periodic orbit becomes a fixed point of the Poincare map. Suppose it is originally stable. Stability means that eigenvalues of a linearization of the Poincare map at the fixed point are inside the unit circle. These eigenvalues are called multipliers. Then, when we start to change the parameters, at some parameters value, which we always take equal to zero, the fixed point may lose stability, which means that the multipliers come to the unit circle. What happens after they cross the unit circle? We have already considered the cases, when a multiplier is equal to one or minus one, and now we consider the bifurcation where there are multipliers are equal to $e^{\pm i\omega}$, $0 < \omega < \pi$.

As multipliers go from the inside of the unit circle to the outside, we assume that they are different from 1, so the fixed point does not disappear, but it changes stability: it used to be stable and now it becomes unstable. We restrict our considerations to the center manifold; there are two multipliers on the unit circle, so the center manifold is two-dimensional. Thus, we should consider two-dimensional maps near a fixed point. To understand what happens when we change parameters, we should compute Lyapunov coefficients. As in the previously considered cases, the answer depends on the sign of the first non-zero Lyapunov coefficient. When it is negative, the fixed point is stable at the moment of bifurcation (center picture in Figure 1). After the bifurcation, it loses stability, but a closed invariant curve emerges nearby, diffeomorphic to a circle, such that all orbits tend to it, and if an orbit starts on this curve, then it remains on the curve (right picture in Figure 1). So, this is a safe bifurcation. This is very similar to the Andronov-Hopf bifurcation, where first we have an equilibrium state and, after the bifurcation, a periodic orbit arises. An important difference is that, in the Andronov-Hopf bifurcation, the circle is just one periodic orbit. Here it is not, here we have many orbits and they all lie on the invariant closed curve.

When the first Lyapunov coefficient is positive (Figure 2), at the moment of bifurcation the fixed point is unstable, so this bifurcation is dangerous. Before the bifurcation the fixed point is stable, but around it there is an unstable invariant curve, i.e., if we start on this curve, we remain on it, and if we start inside it, we converge to the fixed point, but if we start outside, we diverge. When we change parameters, the basin of attraction of the fixed point becomes

FIGURE 1. $L < 0$

smaller and smaller, and at the moment of bifurcation any small perturbation takes the initial orbit far away.

FIGURE 2. $L > 0$

There are several differences between the Andronov-Hopf bifurcation and the birth of invariant torus:

- (1) The first difference is that the result is not true when $\omega = \pi/2$ and $\omega = 2\pi/3$. These are the so-called cases of *strong resonances* which have to be excluded. Some of the people who first studied this bifurcation did not notice this restriction and did mistakes. Do not follow their example.
- (2) A new object that is born from a fixed point is an invariant curve, but it is not one trajectory, it is a collection of trajectories that somehow decided to group into a smooth invariant curve. Motion along this curve is not periodic, but could be more complicated and requires analysis.

It makes sense to look at the pictures for the flow. Let us have a three-dimensional system of differential equations, so the pictures that we had before (Figures 1 and 2) are pictures on the cross-section. The fixed point of the Poincare map on this cross-section corresponds to a periodic orbit. In Figure 3, left picture, the periodic orbit is stable, which means that if we start nearby, any orbit will converge to it, and since the eigenvalues are complex, we will go around it in a spiral. After the bifurcation, we have a two-dimensional invariant torus in three-dimensional space, and the periodic orbit, which used to

be stable, now becomes unstable. So, if we start at any point on the torus, we move along the torus and never leave it. On the torus we have two frequencies of the motion: one frequency corresponds to motion transverse to the periodic orbit, and the other corresponds to motion along the periodic orbit. To summarize: we start with a periodic orbit (one-frequency regime) which after the bifurcation loses stability and gives rise to a torus (a collection of two-frequency regimes). Of course, different points on the torus correspond to different trajectories, i.e., this is a collection of orbits, not just one orbit.

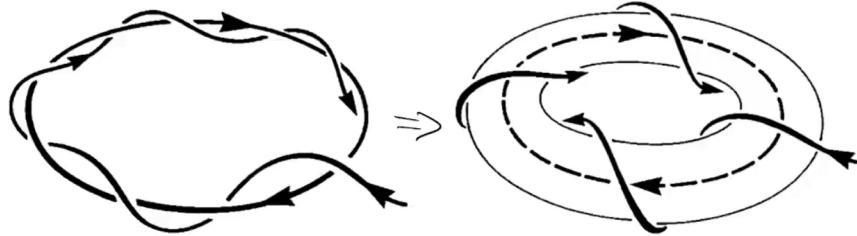


FIGURE 3

- (3) We may also consider the case where the first Lyapunov coefficient is equal to zero. This will be the codimension-two bifurcation, like for the Andronov-Hopf bifurcation. One of the parameters (μ_0) corresponds to the change of stability and the birth of the invariant curve, and the other one (μ_1) is responsible for the sign of the first Lyapunov coefficient. So, like in the Andronov-Hopf bifurcation, there will be a region where we have just one stable fixed point, a region where we have one invariant curve which is stable and an unstable fixed point, and then there will be a region where we have two invariant curves, stable and unstable, and a stable fixed point, see Figure 4.

It is tempting to think that when we go from a region with two invariant curves to a region without invariant curves, these curves will simply collide and disappear, as in the Andronov-Hopf bifurcation, but, strictly speaking, they cannot collide without additional assumptions, since they consist of many different orbits. There is no single moment of collision of these two invariant curves; when they collide, a certain chaotic dynamics arises. This is a very complicated process that has not been completely understood by now. There is even a theorem that it can never be fully understood. So, in contrast to the Andronov-Hopf bifurcation, it is very difficult and, up to a certain degree, impossible to go beyond the case where the first Lyapunov coefficient is nonzero. This is because a torus is not a single orbit, but a collection of orbits.

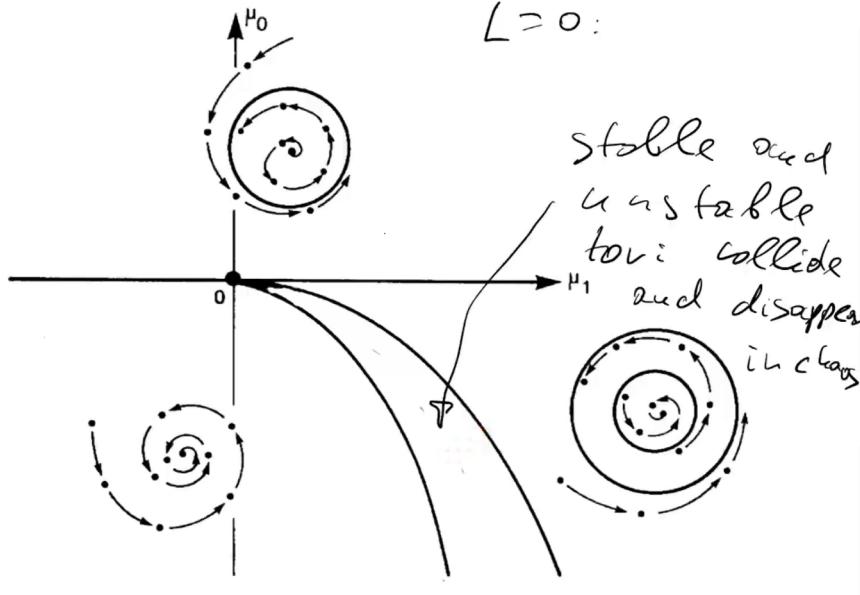


FIGURE 4

That was the introduction, and now we move to the formal exposition. We consider the following map on a two-dimensional center manifold:

$$\bar{x} = Ax + o(x), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad (20.1)$$

where

$$A = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix},$$

and its eigenvalues are $\lambda_{1,2} = e^{\pm i\omega}$. We assume that $0 < \omega < \pi$, so eigenvalues are complex conjugate (not real) and lie on the unit circle. Next, as usual, we perform a normal form transformation to see which nonlinear terms can be killed. First we introduce a new variable

$$z = x_1 + ix_2.$$

Then the system (20.1) becomes diagonal

$$\bar{z} = e^{i\omega} z + \sum_{m+n \geq 2} a_{mn} z^m (z^*)^n,$$

and we also have the complex conjugate equation for z^* . Now we can compute which terms are resonant. If the following condition

$$e^{i\omega} = (e^{i\omega})^m (e^{-i\omega})^n \quad (20.2)$$

is satisfied, then the term $a_{mn} z^m (z^*)^n$ is resonant and it is hard to kill it by a smooth coordinate transformation, and we do not do it. On the other hand,

if the resonance condition (20.2) is not satisfied, then the corresponding term can be killed.

Let us write (20.2) as

$$e^{i\omega} = e^{i\omega(m-n)}$$

and take the logarithm to obtain

$$i\omega = i\omega(m-n) + 2\pi ik, \text{ where } k \in \mathbb{Z}. \quad (20.3)$$

The part $i\omega = i\omega(m-n)$ is exactly the same as in the Andronov-Hopf bifurcation, but since the logarithm is a multivalued function, there is an additional term $2\pi ik$. Since $\omega \neq 0$ we can divide both parts of (20.3) by $i\omega$ and get

$$1 = m - n + \frac{2\pi}{\omega}k. \quad (20.4)$$

There are two cases here:

- (1) $\frac{2\pi}{\omega}$ is irrational. Then the only resonant terms are those for which $m = n+1$ (at $k=0$). This is exactly the same condition which we had for the Andronov-Hopf bifurcation. So, resonant terms are $z(zz^*)^n$.
- (2) $\frac{2\pi}{\omega}$ is rational, i.e., $\frac{2\pi}{\omega} = \frac{q}{p}$, where p and q are positive and coprime.

If $k=0$, then terms that satisfy condition

$$m = n + 1 \quad (20.5)$$

are again resonant. But now we have more possibilities for resonant terms, namely $k = ps$. Then the resonance condition (20.4) reads

$$m = n + 1 - qs, \text{ where } s \in \mathbb{Z}, q \in \mathbb{N} \text{ and coprime with } p. \quad (20.6)$$

Alternatively, for $s < 0$ we can rewrite (20.6) as

$$m = n + 1 + q|s| \quad (20.7)$$

and for $s > 0$ as

$$n = m - 1 + qs. \quad (20.8)$$

Let us write out the first nonresonant terms in these series of resonances. First we note that since $0 < \omega < \pi$, then $\frac{2\pi}{\omega} > 2$ and consequently $q \geq 3$.

We see that if in (20.7) we take $n = 0$ and $s = -1$, we will get $m = q+1$, which is equal to at least four. So, the term of the lowest order in the series of resonances (20.7) is of order four, which is higher than the term of the lowest order in the series (20.5), i.e., higher than three.

In the series (20.8) the terms of the lowest order correspond to

$$m = 0, \text{ and } n = q - 1.$$

Thus, if $q = 3$ or $q = 4$ the term $(z^*)^{q-1}$ is comparable with $z^2 z^*$ which comes from the series (20.5) and should be included in the normal form.

Let us summarize. If $\frac{\omega}{2\pi}$ is irrational, then the corresponding normal form is

$$\bar{z} = e^{i\omega}(z + (L_1 + i\Omega_1)z|z|^2 + \dots).$$

If $\frac{\omega}{2\pi} = \frac{p}{q}$, i.e., it is rational, then the normal form reads

$$\bar{z} = e^{i\omega}(z + (L_1 + i\Omega_1)z|z|^2 + A(z^{*})^{q-1} + \dots).$$

If $q \neq 3$ and $q \neq 4$, then the term $A(z^{*})^{q-1}$ is of order $o(|z|^3)$, which means that it is not important and we do not need to write it. But if $q = 3$, then we get an additional quadratic term which should be included; if $q = 4$, then $A(z^{*})^{q-1}$ will be of the same order as $(L_1 + i\Omega_1)z|z|^2$ and by this reason we cannot neglect it.

To have $0 < \omega < \pi$, when $q = 3$ we have that p must be equal to one so $\omega = \frac{2\pi}{3}$ in this case; when $q = 4$, we again have p equal to one, and thus, $\omega = \frac{\pi}{2}$.

Let me remind you that at the very beginning of the lecture we already mentioned that the cases of $\omega = \frac{2\pi}{3}$ and $\omega = \frac{\pi}{2}$ should be excluded and now we see that indeed these values of ω correspond to the so-called strong resonances, and that the behaviour in these cases can differ from the general one.

Thus, if $q \neq 3$ or $q \neq 4$, then the normal form reads

$$\bar{z} = e^{i\omega}(z + (L_1 + i\Omega_1)z|z|^2 + O(|z|^4)), \quad (20.9)$$

where L_1 is the first Lyapunov coefficient. The following theorems hold.

Theorem 20.1. *The fixed point is stable at the moment of bifurcation if $L_1 < 0$ and is unstable if $L_1 > 0$.*

This can be verified by passing to polar coordinates in the normal form equation derived above and will be done in the next lecture. We also state the main theorem which gives the existence of an invariant curve which will be proved in the next two lectures.

Theorem 20.2. *Suppose $L_1 \neq 0$, $\omega \neq \frac{2\pi}{3}$, and $\omega \neq \frac{\pi}{2}$. When the map (20.9) is perturbed and the fixed point O changes stability, a closed invariant curve is born. It is stable if $L_1 < 0$ and unstable if $L_1 > 0$. It attracts all forward orbits if $L_1 < 0$ and all backward orbits if $L_1 > 0$ from a small neighbourhood of O (except for O itself).*

LECTURE 21. BIRTH OF INVARIANT TORUS. CONTINUED.

Let me recall that we consider the bifurcation corresponding to a pair of multipliers lying on the unit circle, $\lambda_{1,2} = e^{\pm i\omega}$, where $0 < \omega < \pi$ and $\omega \neq \frac{\pi}{2}, \frac{2\pi}{3}$. After we brought a system to the normal form and killed all nonresonant terms, we obtain the following map:

$$\bar{z} = e^{i\omega}(z + (L_1 + i\Omega_1)z|z|^2 + O(|z|^4)),$$

where L_1 is the first Lyapunov coefficient; we assume that it is not equal to zero. Note that before defining L_1 we took $e^{i\omega}$ out of the brackets.

We want to prove that if $L_1 < 0$, then the fixed point is stable, and the orbits converge to it. When we perturb our system, the fixed point may change stability, and then a closed invariant curve surrounding this fixed point will be born. This curve attracts all the orbits from a small neighbourhood except for the fixed point itself.

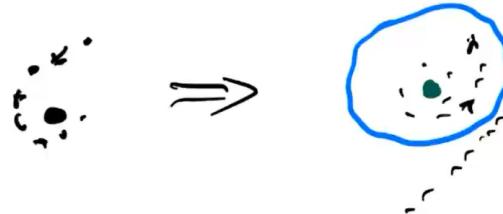


FIGURE 1. $L_1 < 0$

When $L_1 > 0$ we have the similar picture but with the opposite direction of iterations.

Proof of the stability theorem. The first thing to notice is that when we perturb the system, the fixed point does not disappear, since there is no multiplier equal to one. So we can always put the fixed point at zero. The eigenvalues of the linearization matrix can move and consequently their absolute value may change. Thus, the normal form for the perturbed system is

$$\bar{z} = e^{i\omega}(z(1 + \mu) + (L_1 + i\Omega_1)z|z|^2 + O(|z|^4)), \quad (21.1)$$

where all coefficients depend on the parameter ε . When μ is positive, the fixed point is unstable, and it is stable when μ is negative.

The first claim to show is: if $L_1 < 0$ and $\mu \leq 0$, then $z = 0$ is a stable fixed point and it attracts all the orbits from a small neighbourhood. Let us rewrite the normal form (21.1) in polar coordinates $z = re^{i\phi}$:

$$\bar{r}e^{i\bar{\phi}} = e^{i(\omega+\phi)}(r(1 + \mu) + (L_1 + i\Omega_1)r^3 + O(r^4)), \quad (21.2)$$

and then separate equations for r and ϕ . To obtain the equation for r , let us take the absolute value of both parts of (21.2):

$$\begin{aligned} \bar{r} &= r|1 + \mu + (L_1 + i\Omega_1)r^2 + O(r^3)| = r\sqrt{(1 + \mu)^2 + 2L_1(1 + \mu)r^2 + O(r^3)} = \\ &= r(1 + \mu)\sqrt{1 + \frac{2L_1}{1 + \mu}r^2 + O(r^3)} = r(1 + \mu)\left(1 + \frac{L_1}{1 + \mu}r^2 + O(r^3)\right). \end{aligned}$$

Thus,

$$\bar{r} = r(1 + \mu + L_1r^2 + O(r^3)). \quad (21.3)$$

To obtain the equation for $\bar{\phi}$ we divide (21.2) by (21.3) and obtain:

$$e^{i\bar{\phi}} = e^{i(\omega+\phi)} \left(1 + i\frac{\Omega_1r^2}{1 + \mu} + O(r^3)\right).$$

By taking the logarithm, we finally get

$$\bar{\phi} = \phi + \omega + O(r^2).$$

As we see, the ϕ behaviour is very simple: at each step a certain angle is added to it, i.e., we are close to the rotation on angle ω .

So, most of the interesting things happen with r . If $\mu \leq 0$ and $L_1 < 0$, then

$$\bar{r} \leq r(1 + (L_1 + O(r))r^2).$$

Since $L_1 + O(r) < 0$ for small r , we have $1 + (L_1 + O(r))r^2 < 1$, so r is monotonically decreasing and, thus, it must have a limit and this limit is zero.

The opposite fact is also true: if $\mu \geq 0$ and $L_1 > 0$, then $1 + \mu + L_1r^2 + O(r^3) > 1$, which means that the forward iterations lead the orbits away, whereas the backward iterations converge to zero. Thus, in this case the fixed point is unstable. \square

The next question is: “What happens when μ and L_1 have opposite signs?”

Theorem 21.1. *If $\mu > 0$ and $L_1 < 0$, then all forward orbits (except for the fixed point at zero) enter the annulus $A = \{|r - \sqrt{-\frac{\mu}{L}}| \leq C\mu\}$ with some constant $C > 0$ and never leave it.*



Proof. We prove that

- (1) if $r \leq \sqrt{-\frac{\mu}{L}} - C\mu$, i.e., we start inside the inner disk, then $\bar{r} > r$, i.e., the value of r grows monotonically.
- (2) if $r \geq \sqrt{-\frac{\mu}{L}} + C\mu$, i.e., we start outside the outer boundary of the annulus, then $\bar{r} < r$, i.e., the value of r decreases monotonically.

We start with the first statement and want to prove that $1 + \mu + L_1 r^2 + O(r^3) > 1$. Indeed,

$$\begin{aligned} 1 + \mu - |L_1|r^2 + O(r^3) &> 1 + \mu - |L_1| \left(\sqrt{-\frac{\mu}{L}} - C\mu \right)^2 - K\mu^{3/2} = \\ &= 1 + \mu - \mu + 2C|L_1|\mu\sqrt{-\frac{\mu}{L}} - C^2\mu^2|L_1| - K\mu^{3/2} > 1. \end{aligned}$$

Here K is some constant which is independent of C and we replaced r^2 in the term $-|L_1|r^2$ by its worst (largest) value $\sqrt{-\frac{\mu}{L}} - C\mu$.

The term $C^2\mu^2|L_1|$ is not important since it is of a second order, and the constant C may be chosen sufficiently large such that $2C|L_1|\mu\sqrt{-\frac{\mu}{L}} > K\mu^{3/2}$. So, if we start inside the inner disk after some sufficiently large number of iterations we must get inside the annulus.

By analogous computations we obtain (2). Thus, if C is sufficiently large, then when we start inside the inner disk or outside the outer boundary of the annulus we must get inside the annulus.

We proved that when we start inside the inner disk, or on its boundary, \bar{r} is always larger than r . This means that after finitely many iterations we will get outside the inner disk. We also proved that when we start outside of the outer boundary or exactly on it, $\bar{r} < r$. This means that this annulus is forward invariant, i.e., if we start inside the outer boundary we remain inside the outer boundary and if we start outside the inner boundary we remain outside it. So, if we start between these two boundaries, we remain between them. This proves the theorem. \square

The found forward-invariant annulus is very small. Indeed, the center line is at a distance of the order $\sqrt{\mu}$ from zero, and the width of the annulus is of order μ (see Figure 2).

The next step would be to prove that once we get inside the annulus, we converge to the invariant curve which lies inside it. This will be the topic of the next lecture. We will take the inner boundary and look at its image, we know that it will lie strictly inside the annulus. The inner boundary is a perfect circle, we will show that image of it is almost a perfect circle, and image of this



FIGURE 2

image is again almost a perfect circle and so on. The sequence of these images converges to uniquely defined curve which is also almost a perfect circle, and because it is a limit of iterations of a curve, then, of course, it is an invariant curve. We will also show that we can start with any perfect circle inside the annulus and its image will again converge to the same invariant curve. To do this, we will need to recall the Banach contraction mapping principle and check that conditions of this principle are fulfilled.

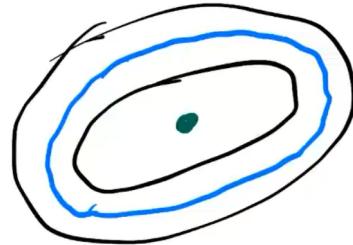


FIGURE 3

LECTURE 22. PROOF OF THE INVARIANT CURVE THEOREM.

We continue the proof of the invariant curve theorem. Recall that we have a fixed point which undergoes a bifurcation, where a pair of multipliers go through the unit circle, i.e., the multipliers have a form $\lambda_{1,2} = e^{\pm i\omega}(1 + \mu)$. When $\mu = 0$, we have a pair of multipliers lying on the unit circle; when $\mu < 0$, the multipliers go inside the unit circle; and when $\mu > 0$ they go outside the unit circle. Thus, a change in the sign of μ corresponds to a change in the stability of the fixed point. We assume that $\omega \in (0, \pi)$ and $\omega \neq \frac{\pi}{2}, \frac{2\pi}{3}$, so there are no strong resonances. Then, in the previous lecture, we went to normal form on the center manifold and assumed that the first Lyapunov coefficient $L \neq 0$, in particular, we considered the case where $L < 0$. We showed that the Poincare map near zero fixed point in polar coordinates can be written in the form:

$$F : \begin{cases} \bar{r} = r(1 + \mu + Lr^2 + O(r^3)), \\ \bar{\phi} = \phi + \omega + \Omega r^2 + O(r^3). \end{cases}$$

We also proved that, when L is negative, the fixed point is stable at the critical moment $\mu = 0$. Then, at positive μ , it loses stability and every orbit goes inside the annulus

$$\mathcal{A} : \left| r - \sqrt{-\frac{\mu}{L}} \right| \leq C\mu,$$

which is invariant, i.e., $F(\mathcal{A}) \subset \mathcal{A}$. The width of \mathcal{A} is of the order of μ . The middle line of the annulus is a perfect circle $r = \sqrt{-\frac{\mu}{L}}$.



FIGURE 1

If we look at the iterations of the ϕ variable, we see that the behaviour of ϕ is more or less uniform rotation.

The question is: “What is the behaviour of the r variable?” The claim is that inside this annulus there is an invariant closed curve which attracts all orbits.

To prove this claim, we will start with any circle (or smooth closed curve) inside this annulus and iterate it to show that images of this circle converge to some curve. Because images of every curve converge to this particular curve, images of every point inside the annulus also converge to the same invariant curve.

We will draw the annulus like a strip and assume periodicity, i.e., we do not think of ϕ as an angle, but assume it is defined on a real line. We will take any curve $r = r(\phi)$ which lies in the annulus (and $r(\phi)$ is a smooth 2π -periodic function) and apply the map

$$F : \{r = r(\phi)\} \rightarrow \{\bar{r} = \tilde{r}(\bar{\phi})\}$$

to it. How do we build the image \tilde{r} ? We take any point on the curve $r(\phi)$ and consider its image, then take another point and consider its image, and then the union of all images will be the image of the curve. It is important that our curve is a graph of r as a function of ϕ , which means that only one point on the curve corresponds to each value of ϕ ; but it may happen that this condition is true for the original curve, whereas for its image, for some values of ϕ , there may be several corresponding values of r . To exclude this possibility, we will consider only smooth and almost straight curves, i.e., such that $|r'(\phi)| \leq \kappa|\mu|$ for some coefficient κ . Then the following lemma holds.

Lemma 22.1. *There exists $\kappa > 0$ such that if you take any periodic curve inside the annulus \mathcal{A} of the form $r = r(\phi)$ such that $|r'(\phi)| \leq \kappa|\mu|$, then its image under the map F is again a periodic curve of the same form $\bar{r} = \tilde{r}(\bar{\phi})$ and it satisfies the condition*

$$|\tilde{r}'(\bar{\phi})| \leq \kappa|\mu| \tag{22.1}$$

with the same constant κ .

Proof. It is enough to prove (22.1), then automatically \bar{r} will be a function of $\bar{\phi}$. Indeed, to have several values of \bar{r} for a given value of $\bar{\phi}$, we need the derivative to be infinite at some values of $\bar{\phi}$, but exactly the condition (22.1) forbids this. Thus, we only need to estimate $\frac{d\bar{r}}{d\bar{\phi}}$. Let us write formulas for these differentials:

$$\begin{cases} d\bar{r} = (1 + \mu + 3Lr^2 + O(r^3))dr + O(r^4)d\phi, \\ d\bar{\phi} = (1 + O(r^3))d\phi + O(r)dr, \end{cases}$$

and use that $(r, \phi) \in \mathcal{A}$, i.e., $r = \sqrt{-\frac{\mu}{L}} + O(\mu)$. Then

$$\begin{cases} d\bar{r} = (1 - 2\mu + O(\mu^{3/2}))dr + O(\mu^2)d\phi, \\ d\bar{\phi} = (1 + O(\mu^{3/2}))d\phi + O(\mu^{1/2})dr, \end{cases} \tag{22.2}$$

and, consequently,

$$\frac{d\bar{r}}{d\bar{\phi}} = \frac{(1 - 2\mu + O(\mu^{3/2})) \frac{dr}{d\phi} + O(\mu^2)}{1 + O(\mu^{3/2}) + O(\mu^{1/2}) \frac{dr}{d\phi}}. \quad (22.3)$$

Thus, we expressed the derivative of the image as a function of the derivative of the preimage. Since by our assumptions $\left| \frac{dr}{d\phi} \right| \leq \kappa|\mu|$, the denominator of (22.3) can be estimated as

$$1 + O(\mu^{3/2}) + O(\mu^{1/2}) \frac{dr}{d\phi} = 1 + O(\mu^{3/2}),$$

which means that

$$\frac{d\bar{r}}{d\bar{\phi}} = (1 - 2\mu + O(\mu^{3/2})) \frac{dr}{d\phi} + O(\mu^2)$$

and

$$\begin{aligned} \left| \frac{d\bar{r}}{d\bar{\phi}} \right| &= \left| (1 - 2\mu + O(\mu^{3/2})) \right| \left| \frac{dr}{d\phi} \right| + O(\mu^2) \leq (1 - 2\mu + O(\mu^{3/2})) \kappa|\mu| + O(\mu^2) \\ &\leq \kappa|\mu| - 2\kappa\mu^2(1 + O(\sqrt{\mu})) + O(\mu^2) \leq \kappa|\mu|, \end{aligned}$$

if κ is large enough and μ is small enough. We also used here that the term $O(\mu^2)$ does not depend on κ . The lemma is proved. \square

Thus, the map F induces a map \mathcal{F} in the space X of 2π -periodic functions with values in the annulus, which are Lipschitz continuous (with the Lipschitz constant $\kappa\mu$), i.e.:

$$|\Delta r| \leq \kappa|\mu||\Delta\phi|. \quad (22.4)$$

The space X becomes a metric space if we endow it with the standard sup-metric (the metric of uniform convergence):

$$\text{dist}(r_1, r_2) = \sup_{\phi} |r_1(\phi) - r_2(\phi)|.$$

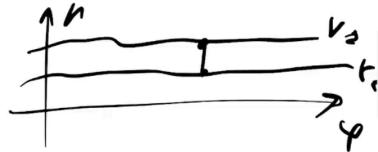


FIGURE 2

The metric space X is also complete, so any Cauchy sequence in this space is convergent. This follows from the fact that the annulus is closed and the space of continuous functions of real numbers with the sup-metric is a complete metric space. Lipschitz property is also preserved under the uniform limit. Indeed, if you have a sequence of functions that are Lipschitz with the same Lipschitz constant, then its limit is also a Lipschitz function with the same

Lipschitz constant. Thus, we have a map on a complete metric space, and for such maps the Banach contraction mapping principle holds.

Theorem 22.2 (Banach contraction mapping principle). *For any complete metric space X , if a map $\mathcal{F} : X \rightarrow X$ is contracting, then it has a fixed point $\rho^* = \mathcal{F}\rho^*$ and $\rho^* = \lim_{n \rightarrow +\infty} \mathcal{F}^n \rho_0$ for every $\rho_0 \in X$.*

Contraction means that for any two functions ρ_1 and ρ_2 the following condition is satisfied:

$$\text{dist}(\mathcal{F}(\rho_1), \mathcal{F}(\rho_2)) \leq q \text{dist}(\rho_1, \rho_2)$$

with some constant $q < 1$.

The theorem tells us that we can take any initial guess, start to iterate, and the sequence of iterations will converge. Indeed, our map is contracting, and thus the distances between points form a geometric progression and the sum of geometric progression converges. The limit is a fixed point, and this fixed point is unique by contraction. Indeed, if you have two fixed points their images must be closer to each other than the fixed points themselves, but this is only possible if this is the same fixed point.

If we prove that, for any Lipschitz curve from X , taking its image by the map F on the annulus is a contraction in X , then a sequence of images will converge to a certain Lipschitz invariant curve. So, it remains to check that our map \mathcal{F} is a contraction.

We have two functions $r = \rho_1(\phi)$ and $r = \rho_2(\phi)$, take images of them by the map \mathcal{F} and obtain two different functions $r = \tilde{\rho}_1(\phi)$ and $r = \tilde{\rho}_2(\phi)$. We need to prove that the distance between $\tilde{\rho}_1(\phi)$ and $\tilde{\rho}_2(\phi)$ for any ϕ is smaller than the distance between ρ_1 and ρ_2 , and there is a contraction factor which is smaller than one, namely, $|\tilde{\rho}_1(\phi) - \tilde{\rho}_2(\phi)| < q \text{dist}(\rho_1, \rho_2)$.

To do this, we take a preimage of $\tilde{\rho}_1(\phi)$, this will be a point (ϕ_1, r_1) , and a preimage of $\tilde{\rho}_2(\phi)$, which is a point (ϕ_2, r_2) , see Figure 3.

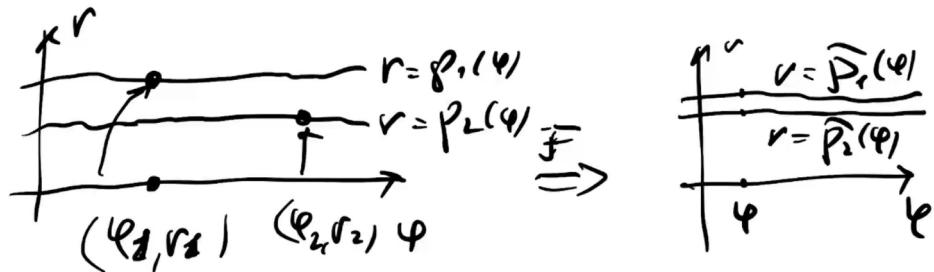


FIGURE 3

Replacing dr by $\Delta r = r_1 - r_2$ and $d\phi$ by $\Delta\phi = \phi_1 - \phi_2$ in (22.2) and using the mean value theorem, we obtain

$$\begin{cases} \tilde{\rho}_1(\phi) - \tilde{\rho}_2(\phi) = (1 - 2\mu + O(\mu^{3/2})) (r_1 - r_2) + O(\mu^2)(\phi_1 - \phi_2), \\ 0 = (1 + O(\mu^{3/2})) (\phi_1 - \phi_2) + O(\mu^{1/2})(r_1 - r_2). \end{cases} \quad (22.5)$$

We see that there is a certain relation between $r_1 - r_2$ and $\phi_1 - \phi_2$; in particular, from the second equation we have

$$\phi_1 - \phi_2 = O(\mu^{1/2})(r_1 - r_2),$$

which means that $O(\mu^2)(\phi_1 - \phi_2) = O(\mu^{5/2})(r_1 - r_2)$, i.e., this term is negligible, and consequently,

$$|\tilde{\rho}_1(\phi) - \tilde{\rho}_2(\phi)| = |1 - 2\mu + O(\mu^{3/2})| |r_1 - r_2|.$$

The problem is that $|r_1 - r_2|$ is not a distance between curves, since r_1 is a value at ϕ_1 and r_2 is a value at ϕ_2 . To overcome this difficulty we take an additional point with a coordinate ϕ_1 but at the curve ρ_2 , then by the triangle inequality:

$$|r_1 - r_2| \leq \text{dist}(\rho_1, \rho_2) + \kappa\mu|\phi_1 - \phi_2| \leq \text{dist}(\rho_1, \rho_2)(1 + O(\mu^{3/2})),$$

where we have used (22.4) and the second equation of (22.5). So, up to a small correction, $|r_1 - r_2|$ is, indeed, a distance between curves ρ_1 and ρ_2 , which means that

$$|\tilde{\rho}_1(\phi) - \tilde{\rho}_2(\phi)| = |1 - 2\mu + O(\mu^{3/2})| \text{dist}(\rho_1, \rho_2). \quad (22.6)$$

The right-hand side of (22.6) does not depend on ϕ , so we can take the supremum on both sides and obtain

$$\text{dist}(\tilde{\rho}_1, \tilde{\rho}_2) = \sup_{\phi} |\tilde{\rho}_1(\phi) - \tilde{\rho}_2(\phi)| = |1 - 2\mu + O(\mu^{3/2})| \text{dist}(\rho_1, \rho_2).$$

For small μ , the factor $1 - 2\mu + O(\mu^{3/2})$ is smaller than one. Therefore, our map is contraction and, thus, there exists an invariant curve that lies on the annulus, i.e., $r = \rho^*(\phi) = \sqrt{-\frac{\mu}{L}} + O(\mu)$.

This proves the existence of a Lipschitz invariant curve, because we worked at the space of Lipschitz functions, but we did not prove the existence of the smooth invariant curve. Of course, the map F is smooth and if we start with a smooth curve $r(\phi)$ the image of it will be smooth, but we did not prove contraction in the space of smooth curves (did not prove contraction for derivatives).

In fact, the invariant curve is smooth (as smooth as we want). Similarly to center manifolds, it cannot be C^∞ -smooth, and for any finite μ we have some finite smoothness, which can decrease when μ grows, but for sufficiently small μ we have sufficiently many derivatives. The proof of this statement is technical and by this reason is omitted.