Mathematics Year 1, Calculus and Applications I

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Problem Sheet 4

1. (a) Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. [Hint: Consider $\sum_{i=1}^{n} \left[(i+1)^3 - i^3\right]$]

   (b) Find the integral $\int_0^1 x^2 dx$ using upper Riemann sums and an equipartition of $[0,1]$.

2. In approximating the integral $\int_0^1 e^x dx$ with an upper Riemann sum, we used the result $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} e^{i/n} = e - 1$. Show this.

3. Show that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases},$$

   is not Riemann integrable. [Hint: Take *any* partition of $[0,1]$ and consider the lower and upper Riemann sums.]

4. Show that $\frac{d}{dx}(\sec x + \tan x) = \sec x(\sec x + \tan x)$. Hence show that

$$\int \sec x dx = \log(\sec x + \tan x).$$

   Note that the integral derived above makes sense only if (i) $\cos x \neq 0$, and (ii) $\sec x + \tan x > 0$. Determine an interval where such an interval can be applied.

5. Calculate

$$\int \frac{1}{(x^2+1)^3} dx, \quad \int \frac{1}{x^3-1} dx, \quad \int \frac{x^3+1}{x^3-1} dx, \quad \int x^3\sqrt{x^2+1} dx, \quad \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x + \sin^3 x}$$

6. Let $I_n = \int \frac{1}{(x^2+1)^n} dx$ where $n > 1$ is an integer (what is the integral when $n = 1$?) Starting from I_{n-1} use integration by parts to establish the recursion formula

$$2(n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}.$$

7. (a) Show that for any two integers m, n

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx dx = 0, \qquad \int_{-\pi}^{\pi} \sin mx \, \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } n = m \end{cases},$$

   and find an analogous formula for $\int_{-\pi}^{\pi} \cos mx \, \cos nx dx$.

   (b) Suppose that f is defined on $[-\pi, \pi]$ and it is a $2\pi-$*periodic function*, i.e. $f(x + 2\pi) = f(x)$. If we approximate $f(x)$ by the series $f(x) \approx a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx$, use the results above to find formulas for a_0, a_k, b_k, $k = 1, \ldots N$.

(c) Now take f to be defined on $[-\pi, \pi]$ as follows

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}.$$

Using the trigonometric approximation of part (b) above, calculate the a's and b's and confirm that $b_k = 0$ for all k. Could you have anticipated this result by considering the symmetries of f?

Congratulations! You have just computed a Fourier series.

8. Using comparison tests for improper integrals, determine convergence or divergence of the following integrals

$$\int_0^\infty e^{-x^2} dx, \qquad \int_0^\infty \frac{x^3}{(1+x^2)^2} dx, \qquad \int_0^\infty \frac{1}{\sqrt{x+x^3}} dx$$
$$\int_0^1 \frac{\sin^2 x}{1+x^2} dx, \qquad \int_0^1 \frac{1}{\log(1+x)} dx, \qquad \int_0^\infty \sin(x^2) dx.$$

9. Prove that

$$\int_0^1 \frac{x^3}{2-\sin^4 x} dx \leq \frac{1}{4} \log 2 \qquad \text{and} \qquad \left| \int_0^{\pi/2} \frac{x-\pi/2}{2+\cos x} dx \right| \leq \frac{\pi^2}{16}.$$

10. Prove the *integral mean value theorem* which generalizes a bit the theorem proved in class: Let f and g be continuous on $[a, b]$ with $g(x) \geq 0$ for $x \in [a, b]$. Then there exists a c between a and b with

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Show by finding an example that the conclusion of the theorem is wrong if the assumption $g(x) \geq 0$ is dropped.

11. Let μ be the average of the function f defined on a closed interval $[a, b]$. The average value of $(f(x) - \mu)^2$ is called the *variance* of f on $[a, b]$, and the square root of the variance is the *standard deviation* of f on $[a, b]$ and is denoted by σ. Find the average value, variance and standard deviation of each of the following functions on the specified interval:

$$x^2 \text{ on } [0,1], \qquad xe^x \text{ on } [0,1], \qquad \sin 2x \text{ on } [0, 4\pi],$$

$$f(x) = \begin{cases} 1 & \text{on } [0,1] \\ 2 & \text{on } (1,2] \end{cases}, \qquad f(x) = \begin{cases} 2 & \text{on } [0,1] \\ 3 & \text{on } (1,2] \\ 1 & \text{on } (2,3] \\ 5 & \text{on } (3,4] \end{cases}$$

12. (a) Suppose that $f(x)$ is a step function on $[a, b]$, with value k_i on the interval (x_{i-1}, x_i) belonging to the partition $(x_0, x_1, \ldots, x_n)$. Find a formula for the standard deviation of f on $[a, b]$.

    (b) Simplify your formula for the case when the partition consists of equally spaced points.

(c) Show that if the standard deviation of a step function is zero, then the function is a constant.

(d) Give a definition of the standard deviation of a list of numbers $a_1, a_2, \ldots, a_n$.

(e) What can you say about the list of numbers if its standard deviation is zero?

13. Given a positive integer n define

$$\delta_n(x) = \begin{cases} n & \text{for } |x| \le \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Consider also a continuous function $g(x)$ defined on $[a, b]$ where $a < 0$, $b > 0$.

(a) Calculate $\int_a^b g(x)\delta_n(x)dx$, and prove that $\lim_{n\to\infty} \int_a^b g(x)\delta_n(x)dx = g(0)$.

(b) Find also $\lim_{n\to\infty} \int_a^t g(x)\delta_n(x)dx$ for $t < 0$ and $t > 0$, respectively.

(c) Given arbitrary numbers t_1, t_2, define $\int_{t_1}^{t_2} \delta(x)dx$ by the $\lim_{n\to\infty} \int_{t_1}^{t_2} \delta_n(x)dx$. Use the results of part (b) above to find the former integral for $t_1 < t_2 < 0$, $0 < t_1 < t_2$ and $t_1 < 0 < t_2$.

(d) The function $\delta(x)$ is a *distribution* - it is zero everywhere except at 0 where it is infinite. What is its anti-derivative? Give an expression and sketch it.

14. Let $f(x) = [x] + 1$ and $F(x) = \int_0^x f(t)dt$ (recall that $[x]$ means the integer part of x). Find an explicit expression for $F(x)$ when $0 \le x \le 2$, and show that $F'(1) \ne f(1)$. Explain why this does not contradict the fundamental theorem of calculus.

15. By evaluating the integral $\int_1^n \log x dx$ where n is a positive integer, and comparing with the upper and lower Riemann sums associated to the partition $(1, 2, \ldots, n)$ of the interval $[1, n]$, show that

$$(n-1)! \le n^n e^{-n} e \le n!$$

Hence prove that

$$\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} = 1/e.$$

16. For any non-negative integer n, let

$$I_n = \int_0^\infty e^{-x}(\sin x)^n dx, \qquad J_n = \int_0^\infty e^{-x}(\sin x)^n \cos x dx.$$

(a) Show that $I_0 = 1$, $J_0 = 1/2$, $I_n = nJ_{n-1}$ and $J_n = nI_{n-1} - (n+1)I_{n+1}$.

(b) Using the results in part (a), show that $I_1 = 1/2$, $J_1 = 1/5$ and that for $n \ge 2$ we have the explicit recursion formulas

$$I_n = \frac{n(n-1)}{(1+n^2)} I_{n-2}, \qquad J_n = \frac{n(n-1)}{1+(n+1)^2} J_{n-2}, \quad n \ge 2.$$

(c) Find explicit expressions in the form of rational numbers for each of I_n and J_n. [Note: You need to treat n being even or odd sep separately.] Which is larger, I_n or J_n? Explain.