Problem Sheet 4, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed plane curve parametrised by arc length, say $\gamma(t) = (x(t), z(t))$ with $x(t) > 0$ for all t, and let $S \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating γ around the z -axis.

(a) Prove that S has area $2\pi \int_a^b x(t)dt$.

(b) Compute $\int_S K dA$, where K is Gaussian curvature.

Solution: We parametrise all of S except for a pair of regular curves by

$$\phi : (a, b) \times (0, 2\pi) \to \mathbb{R}^3, \quad \phi(u, v) = (x(u)\cos(v), x(u)\sin(v), z(u)).$$

Then we compute

$$\phi_u = \big(x'(u)\cos(v), x'(u)\sin(v), z'(u)\big), \quad \phi_v = (-x(u)\sin(v), x(u)\cos(v), 0).$$

so

$$|\phi_u \times \phi_v| = \big|(-xz'\cos(v), -xz'\sin(v), xx')\big| = x\sqrt{(z')^2 + (x')^2} = x.$$

We now have

$$\text{area}(S) = \int_0^{2\pi}\int_a^b |\phi_u \times \phi_v|\, dudv = \int_0^{2\pi}\int_a^b x(u)dudv = 2\pi \int_a^b x(u)du.$$

In Problem sheet 3 we saw that $K(\phi(u, v)) = -\frac{x''(u)}{x(u)}$, and hence

$$\begin{aligned}
\int_S K dA &= \int_0^{2\pi}\int_a^b K(\phi(u, v))\, |\phi_u \times \phi_v|\, dudv \\
&= \int_0^{2\pi}\int_a^b \left(-\frac{x''(u)}{x(u)}\right) x(u)dudv \\
&= -2\pi \int_a^b x''(u)du = -2\pi \left[x'(u)\right]_{u=a}^{u=b} = 0
\end{aligned}$$

since the fact that γ is closed implies $\gamma'(a) = \gamma'(b)$ and hence $x'(a) = x'(b)$.

Problem 2. Let $S \subset \mathbb{R}^3$ be a compact, connected, nonempty surface whose curvature K is everywhere positive. Prove that $\int_S K dA \geq 4\pi$. (Hint: use the Gauss map $N : S \to \mathbb{S}^2$ to compare this to an integral over a sphere.)

Solution: At each point $p \in S$, the Gauss map $N : S \to \mathbb{S}^2$ has invertible derivative

$$dN_p : T_pS \to T_{N(p)}\mathbb{S}^2 \cong T_pS,$$

because its determinant is the curvature $K(p) > 0$. Thus, by a result we prove in the lectures (Inverse Function Theorem for surfaces), N is a local diffeomorphism. Thus, the image of N is open, since some open neighbourhood of each $p \in S$ is mapped diffeomorphically to an open neighbourhood of $N(p)$. On the other hand, the image is also closed since it is the image of the compact set S. Because $\mathbb{S}^2$ is connected, the only set which is both open and closed is all of the set or the empty set. These imply that the image is all of $\mathbb{S}^2$. In other words, N is surjective. (This is also established in one of the problem sheets.)

Let $\phi : U \to S$ be a chart at p, and shrink U, if necessary, so that N restricts to a diffeomorphism of $\phi(U) \subset S$ onto its image. Then $\psi = N \circ \phi : U \to \mathbb{S}^2$ is a chart for $\mathbb{S}^2$ at $N(p)$, and we recognize that $K(\phi(u, v)) = \det dN_{\phi(u,v)}$, so from $\psi = N \circ \phi$ we get

$$\int_{\phi(C)} K\, dA = \int_C \det\left(dN_\phi\right) |\phi_u \times \phi_v|\, dudv = \int_C |\psi_u \times \psi_v|\, dudv = \text{area}(\psi(C))$$

for any compact set $C \subset U$. If we cover S by such compact sets $\phi(C)$, then some of them may overlap, but their images under N cover all of $\mathbb{S}^2$ and so we conclude that

$$\int_S K\, dA \geq \text{area}\left(\mathbb{S}^2\right) = 4\pi.$$

Problem 3. Let $\gamma : (a, b) \to \mathbb{R}^3$ be a regular curve which has no self intersections, and let S be the surface parametrised by

$$\phi(u, v) = \gamma(u) + v b(u), \quad a < u < b, -\epsilon < v < \epsilon,$$

where $b(u)$ is the binormal vector to γ at time u. Show that there is $\epsilon > 0$ so that S is a regular surface. Prove that γ is a geodesic in S.

Solution: The first part of the problem was discussed in the problem class, so refer to the recorded videos. We discuss the second part below.

Assume without loss of generality that γ is parametrised by arc length, and let T, n, b denote the Frenet frame at $\gamma(t)$. Since $\phi_u(t, 0) = \gamma'(t) = T(t)$ and $\phi_v(t, 0) = b(t)$, the vector

$$\left(\phi_u \times \phi_v\right)(t, 0) = T(t) \times b(t) = -n(t)$$

is normal to S at $\gamma(t) = \phi(t, 0)$, and hence $N(\phi(t, 0)) = \pm n(t)$. The geodesic curvature of γ at $\gamma(t) = \phi(t, 0)$ is then given by

$$k_g(t) = \left\langle \gamma''(t), \left(N(\phi(t, 0)) \times \gamma'(t)\right)\right\rangle = \langle k(t)n(t), \pm n(t) \times T(t)\rangle = 0,$$

since $n \times T = -b$ is orthogonal to n.