2 The Calculus of Variations

2.1 Preliminary motivational examples

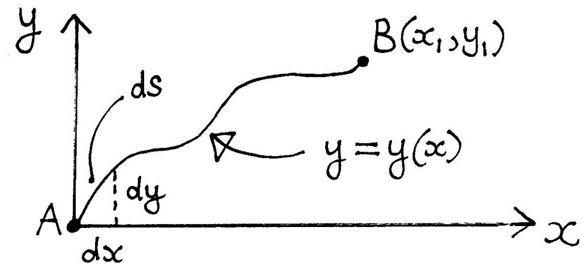

Figure 1: The figure for Example 1.

Example 1. *Shortest path between 2 points*
Suppose we have two points $A(0,0)$ and $B(x_1, y_1)$. The length l of a curve $y(x)$ joining the two points is (see figure 1):

The shortest path can be found by finding the $y(x)$ which minimizes this integral. Intuition suggests that it is a straight line. We will return to this problem later.

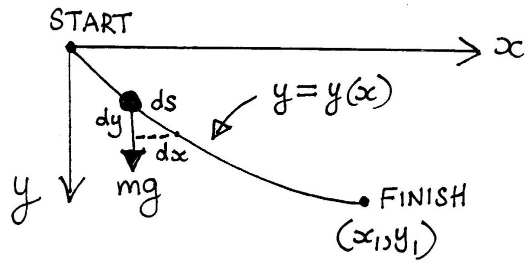

Figure 2: The brachistochrone problem

Example 2. *Curve of quickest descent ('brachistochrone')*

A slightly less trivial example is the following. A particle starts from rest at the origin and travels under gravity along a smooth curve until it reaches the point (x_1, y_1). What shape of curve should it travel along in order that the time of descent is a minimum?

If s is distance along the curve then as in the first example

$$ds = \left(1 + (dy/dx)^2\right)^{1/2} dx,$$

where $y(x)$ is the path. As the particle travels, it converts potential energy into kinetic energy while respecting the overall conservation of energy principle:

where y is measured vertically downwards from the origin, $v(x)$ is the velocity at location $(x, y(x))$ and m is the mass of the particle. Therefore we have

Rearranging:

Thus, the time τ taken to travel to x_1 along $y(x)$ is

The curve of quickest descent is found by minimizing this integral. This time the answer is far from obvious.

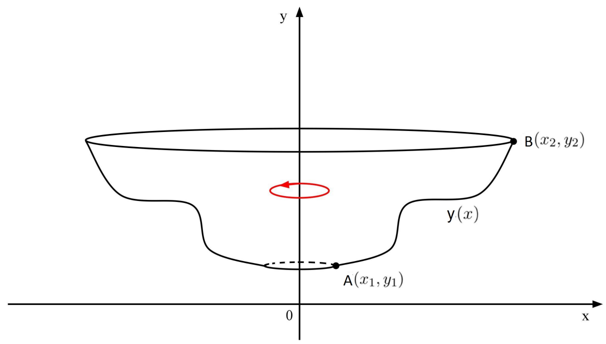

Figure 3: Surface of revolution

Example 3. *Minimal surface of revolution*
Consider a curve $y = y(x)$ joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. We now consider the surface formed by rotating this curve about the $y-$axis. The surface area is given by

Using the expression for arclength as in the first two examples, this can be rewritten as

It is of interest to find the curve $y(x)$ which minimizes $\mathcal{A}$. Again the answer is not obvious.

2.2 'The Vanishing Lemma'

Before we proceed with the general theory we need the following result. If g is a continuous function such that

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx = 0$$

for all smooth functions $\eta(x)$, with $\eta(x_1) = \eta(x_2) = 0$, then $g(x) \equiv 0$.

Proof

Assume for a contradiction that there is a point $x_0 \in [x_1, x_2]$ for which $g(x_0) \neq 0$. Let's assume without loss of generality that $g(x_0) > 0$. Since g is continuous there is a neighbourhood of x_0 in which g remains positive. Denote this neighbourhood by NH.

If x_0 is not equal to x_1 or x_2 then we can take $NH = (x_0 - \epsilon, x_0 + \epsilon)$ with $\epsilon > 0$. If $x_0 = x_1$ then $NH = [x_1, x_1 + \epsilon)$ and if $x_0 = x_2$ then $NH = (x_2 - \epsilon, x_2]$. In each case $g(x) > c > 0$ for all $x \in NH$.

Consider now a smooth function $h(x)$ on $[x_1, x_2]$ with the following properties†

(i) $h(x) = 0$ for all x outside the neighbourhood;

(ii) $\int_{x_1}^{x_2} h(x)\,dx = \int_{NH} h(x)\,dx > 0$.

It follows then that

and hence leads to a contradiction.

†For an example of such a function $h(x)$ see problem sheet 5.

2.3 General theory for 1D integrals

The examples mentioned above are special cases of the integral

$$I = \int_{x_1}^{x_2} L(x, y, y')\, dx$$

where $y' = dy/dx$. In example 1, $L = (1 + (y')^2)^{1/2}$. L is known as a *functional.*

Suppose $y = y(x)$ passes through $A(x_1, y_1)$ and $B(x_2, y_2)$. What is the particular $y(x)$ which minimizes/maximizes (extremizes) the integral I? If $y = Y(x)$ is the extremal curve, how do we find it?

Consider the family of curves

where ε is any real number and η is a smooth curve with $\eta(x_1) = \eta(x_2) = 0$. Each member of the family passes through A and B. It follows that

The integral I takes on its extreme value when $\varepsilon = 0$ (since then $y = Y$). Therefore we must have

Now

When $\varepsilon = 0$ we have $y = Y$ and $y' = Y'$, and so

We now integrate by parts to get

The integrated term vanishes since $\eta(x_1) = \eta(x_2) = 0$ and we are left with

Since $\eta(x)$ is an arbitrary smooth curve we can use the Vanishing Lemma above to deduce that Y satisfies

$$\frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) = 0 \tag{1}$$

which is known as the **Euler-Lagrange equation** in one dimension.

2.3.1 Remarks

(i) In order to integrate by parts we have assumed that the curve $Y(x)$ is of the class C^2 (i.e. the derivatives Y' and Y'' exist and are continuous).
(ii) $Y(x)$ renders I stationary, not necessarily a maximum or minimum, so the Euler-Lagrange equation is a necessary but not sufficient condition for $Y(x)$ to minimize I. In order to prove it definitely gives a (local) minimum we have to show that $I''(0) > 0$ (which is complicated to establish except for very simple examples).
(iii) We usually refer to $Y(x)$ as an *extremal curve* of I.
(iv) The Euler-Lagrange equation is an equation to determine $Y(x)$; the functional L is known for a given problem and is referred to as the *Lagrangian*.
(v) From now on we will replace Y by y, i.e. we will denote the extremal curve by $y(x)$.

2.3.2 Short forms of the 1D Euler-Lagrange equation

The equation simplifies if the functional L is independent of one or more of the variables x, y, y'.
Case 1. L is explicitly independent of y.
Here $L = L(x, y')$ and so $\partial L/\partial y = 0$. Thus the E-L equation reduces to

$$-\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

and hence

Case 2. $L = L(x, y)$ so that $\partial L/\partial y' = 0$. In this case the E-L equation reduces to

Case 3. $L = L(y, y')$ so that $\partial L/\partial x = 0$, but $dL/dx \neq 0$. Using the chain rule

$$\begin{aligned}\frac{dL}{dx} &= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{dy'}{dx} \\ &= y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'}.\end{aligned}$$

Using the E-L equation, the RHS can be rewritten as

Therefore we see that

$$\frac{dL}{dx} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

and hence the E-L equation reduces in this case to

$$L - y'\frac{\partial L}{\partial y'} = \text{ constant}.$$

It's useful to remember the short forms, but the most important equation to remember is the original Euler-Lagrange equation (1). Now that we have this we can revisit our motivational examples.

2.4 Revisiting our examples

Example 1 revisited: *shortest path between 2 points.*
Here the integral to minimize is

$$I = \int_0^{x_1} \left(1 + (y')^2\right)^{1/2} dx.$$

and hence $L = \left(1 + (y')^2\right)^{1/2}$, explicitly independent of x and y. Therefore the E-L equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

reduces to

Substituting for L we find:

This implies

and hence

Therefore the extremal curve is of the form

with m, C found from the conditions that y passes through $(0, 0)$ and (x_1, y_1). In this case:

Thus the answer is a straight line as expected. In this case we can check explicitly that $I''(0) > 0$ and hence demonstrate rigorously that this is a minimum rather than a maximum (although here of course it is obvious there is no maximal curve).

Example 2 revisited: brachistochrone

Here the integral to minimize is

$$\tau = \frac{1}{(2g)^{1/2}} \int_0^{x_1} y^{-1/2} \left(1 + (y')^2\right)^{1/2} dx$$

and so we can take

$$L = y^{-1/2}(1 + (y')^2)^{1/2}.$$

Since this is independent of x we can use the appropriate short form (case 3) of the E-L equation, namely:

$$L - y' \frac{\partial L}{\partial y'} = \text{ constant.}$$

Substituting for L :

Putting over a common denominator:

where α is an arbitrary constant. We now separate the variables and integrate, setting $y = 0$ when $x = 0$ as this is the initial location of the particle. This gives

To solve the integral we make the substitution $y = \alpha^2 \sin^2 \theta, dy = 2\alpha^2 \sin \theta \cos \theta$. Thus:

We take the positive sign so that x increases as θ increases (i.e. the parameter θ increases as the particle moves along the curve from left to right). Thus the parametric form of the mimimizing curve is:

$$x = \alpha^2(\theta - \frac{1}{2}\sin 2\theta), \;\; y = \frac{1}{2}\alpha^2(1 - \cos 2\theta), \;\; (0 \le \theta \le \theta_1),$$

where α and θ_1 can be expressed in terms of x_1 and y_1 from the condition that $x = x_1, y = y_1$ when $\theta = \theta_1$. The solution is the arc of a cycloid. A sketch is shown in figure 4. Recall that y is measured downwards. The resulting shape is a compromise between travelling the shortest distance (a straight line) and achieving the highest speed (moving vertically downwards and then horizontally).

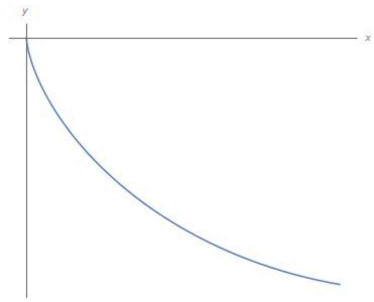

Figure 4: The curve of quickest descent under gravity

Example 3 revisited: *minimal surface of revolution*
Here we want to minimize the area

$$\mathcal{A} = 2\pi \int_{x_1}^{x_2} x\left(1+(y')^2\right)^{1/2} dx.$$

We take $L = x\left(1+(y')^2\right)^{1/2}$, which is explicitly independent of y (case 1). Hence the E-L equation is $\partial L/\partial y' =$ constant, i.e.

This can be rearranged into the form

which can be integrated to give

$$y = \pm\beta \cosh^{-1}(x/\beta) + \gamma.$$

When written in the form $x = x(y)$ this curve is known as a **catenary**. The curve has the shape shown on the left in figure 5. On the right we show a sample surface of revolution linking two circles of different radii - the surface is known as a **catenoid**.

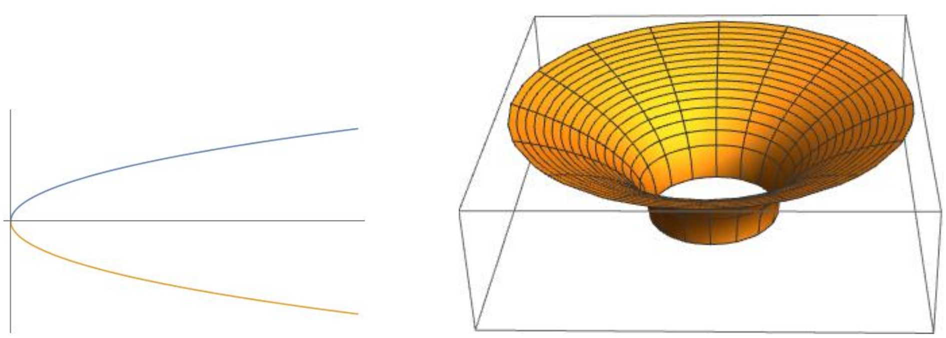

Figure 5: Left: the catenary curve $x = \cosh y$. Right: a surface of revolution formed from a section of a catenary.

Recall that the boundary conditions are such that $y(x_1) = y_1, y(x_2) = y_2$ and we can take $y_1 = 0$ without loss of generality so that one of our rings lies in the plane $y = 0$. We therefore need to choose β and γ such that

$$x_1 = \beta \cosh\left(\frac{\gamma}{\beta}\right), \quad x_2 = \beta \cosh\left(\frac{y_2 - \gamma}{\beta}\right).$$

However for some boundary conditions this is not possible: in particular if x_1 and x_2 are small, but y_2 is large. This means that there is no continuous minimal surface between small rings a large distance apart. This has applications to soap films among other things and there are some interesting videos you can find online.

2.5 Extension of the Euler-Lagrange equation to more variables

Suppose we now have an integral of the form

$$I = \int_{t_1}^{t_2} L(t, x_1(t), x_2(t), \ldots, x_n(t), x_1'(t), x_2'(t), \ldots x_n'(t))\, dt$$

so that L is a scalar function of $(2n+1)$ variables. For simplicity let's write

$$\mathbf{x} = (x_1(t), x_2(t), \ldots, x_n(t)), \;\; \mathbf{x}' = (x_1'(t), x_2'(t), \ldots x_n'(t))$$

If we suppose that the extremal solution is

$$\mathbf{X} = (X_1(t), X_2(t), \ldots, X_n(t)),$$

then in a similar way to our earlier proof we can consider a perturbation to this solution of the form

$$\mathbf{x}(t, \varepsilon) = \mathbf{X}(t) + \varepsilon \boldsymbol{\eta}(t)$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots \eta_n)$ is a smooth $n-$dimensional vector function of t, with $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0$. We then seek a solution for which

$$dI/d\varepsilon = 0 \text{ when } \varepsilon = 0.$$

Thus

using the chain rule. We can integrate by parts to get

Since $\eta_i(t_1) = \eta_i(t_2) = 0$ for all i, this reduces to

Since the η_i are arbitrary smooth functions, the Vanishing Lemma implies that

$$\frac{\partial L}{\partial X_i} - \frac{d}{dt}\frac{\partial L}{\partial X_i'} = 0 \qquad (2)$$

for all $i = 1, 2, \ldots, n$. Thus rather than having one E-L equation we now have a set of n simultaneous E-L equations to solve for the function $\mathbf{X} = (X_1, X_2, \ldots, X_n)$.

Example 4. A trivial example of this is to consider the area $\mathcal{A}$ enclosed by a simple closed curve in the $x-y$ plane. In Part 1 on Green's theorem we showed that if the boundary is denoted by C, then

$$\mathcal{A} = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Writing this in parametric form:

So here we have $\mathbf{x} = (x, y)$ and we can apply the theory above to find the closed curve which extremizes the area. We therefore need to solve the simultaneous E-L equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'} = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'} = 0,$$

where

$$L(t, x, y, x', y') = \frac{1}{2}xy' - \frac{1}{2}yx'.$$

Substituting for L the equations become

In this case we can see that the only solution is that x and y are both constant. in other words the E-L equation has led us to the minimum area of zero which is obtained by shrinking the curve C to a point. This of course is self-evident but the problem becomes more interesting if we restrict our attention to closed curves that have a fixed length l say. This is equivalent to imposing the arclength constraint

We would then hope to obtain a non-trivial answer to our problem of maximising/minimizing $\mathcal{A}$. We will return to this problem later. This example motivates our study of finding extremal solutions subject to constraints in the next section.

2.6 Variational problems involving constraints

We will start with the 1D case again as it is easier to visualize before generalizing to vector functions. Suppose we wish to find the curve $y(x)$ with $y(x_1) = y_1, y(x_2) = y_2$ such that

$$I = \int_{x_1}^{x_2} L(x, y, y')\, dx$$

is stationary, and

$$J = \int_{x_1}^{x_2} g(x, y, y')\, dx$$

is a fixed constant, J_0 say. As usual, L and g are known functionals. As before we consider a family of functions

where $Y(x)$ is the desired solution to the problem and η is a smooth function which satisfies $\eta(x_1) = \eta(x_2) = 0$ so that each member of the family passes through the end points. We therefore have

and

We want I to be stationary and so

J is a constant and so in particular

Calculating $I'(0)$ and $J'(0)$ by the same method as in the unconstrained case we arrive at the following conclusion:

$$\int_{x_1}^{x_2} \eta(x) \left\{ \frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) \right\} dx = 0$$

for all smooth functions $\eta(x)$ vanishing at the end points which satisfy

$$\int_{x_1}^{x_2} \eta(x) \left\{ \frac{\partial g}{\partial Y} - \frac{d}{dx}\left(\frac{\partial g}{\partial Y'}\right) \right\} dx = 0.$$

If follows (see problem sheet 5) that there exists a scalar λ (a **Lagrange multiplier**) such that and hence we have

$$\frac{\partial}{\partial Y}(L+\lambda g) - \frac{d}{dx}\left(\frac{\partial}{\partial Y'}(L+\lambda g)\right) = 0. \tag{3}$$

We therefore retain the familiar Euler-Lagrange equation but with L simply replaced by $L+\lambda g$. As before we will now use y rather than Y to denote the (constrained) extremal curve.

The solution procedure is as follows: if we solve equation (3) we obtain $y = y(x, \lambda, C_1, C_2)$ where C_1, C_2 are constants of integration. Then applying the boundary conditions we can reduce this to $y = y(x, \lambda)$. Finally, substituting into the integral constraint will give us the value of λ.

Example 5
Find the form of $y(x)$ which extremizes the integral

$$I = \int_0^{\pi/2} (y')^2 - y^2 + 2xy\, dx$$

subject to $y(0) = y(\pi/2) = 0$ and the constraint $\int_0^{\pi/2} y\, dx = \pi^2/8$.

2.7 Extension of the constrained case to more variables

As in the unconstrained case the method can easily be extended to problems in which we want to find the extremal solution $\mathbf{x}(t)$ (where $\mathbf{x}$ is an $n-$dimensional vector) of an integral

$$I = \int_{t_1}^{t_2} L(t, \mathbf{x}(t), \mathbf{x}'(t))\, dt$$

subject to the constraint

$$J = \int_{t_1}^{t_2} g(t, \mathbf{x}(t), \mathbf{x}'(t))\, dt = J_0.$$

As before we need to solve n simultaneous E-L equations, but now they are for the functional $L + \lambda g$, i.e.

$$\frac{\partial}{\partial X_i}(L + \lambda g) - \frac{d}{dt}\frac{\partial}{\partial X_i'}(L + \lambda g) = 0$$

for $i = 1, \ldots, n$.

Example 4 revisited.
Let's return to example 4 where we computed the area enclosed by a simple closed curve but now let us impose the constraint that the length of the curve is fixed. Our problem is to find a relation between $x(t), y(t)$ such that the area

$$\mathcal{A} = \frac{1}{2}\int_{t_1}^{t_2} \left(x(t)y'(t) - y(t)x'(t)\right)\, dt$$

is rendered stationary, subject to

$$\int_{t_1}^{t_2} (x'(t)^2 + y'(t)^2)^{1/2} dt = l,$$

where l is a constant representing the length of the closed curve. For this problem the minimum area of zero is clearly achieved if the curve collapses to a straight line. We might hope that a variational approach to the constrained problem leads to the determination of the curve that encloses the *maximum* area. We apply the Euler-Lagrange equations

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0, \quad \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0$$

to the functional $f = L + \lambda g$ where

The equations become

Integrating we obtain

where a and b are constants. Squaring and adding we find that

and so the extremal curve is a circle of radius λ. Since the perimeter is fixed equal to l then we must have $\lambda = l/2\pi$ and therefore $\mathcal{A} = l^2/4\pi$. From what we have said earlier we expect this curve maximizes (rather than minimizes) the area enclosed and this is indeed the case: the circle gives the largest area for a fixed perimeter l. Thus for any simple closed curve we have the **isoperimetric inequality**

$$4\pi\mathcal{A} \leq l^2,$$

where equality holds only when the curve is a circle.

2.8 The Euler-Lagrange equation for higher-dimensional integrals

In the final part of Chapter 1 we showed that the area of surface of a function $z = f(x, y)$ is given by the integral

$$I = \int_\Sigma (1 + |\boldsymbol{\nabla} f|^2)^{1/2}\, dx\, dy$$

where Σ is the projection of the surface onto the $x - y$ plane. Suppose that the surface is bounded by a closed curve γ lying in 3D space. If a wire loop is bent into this shape and dipped into a soap solution, a film will form. It turns out that the soap film will assume a shape which has the least surface area, at least locally, compared to all other surfaces that span the wire loop. If we want to find this shape we need to find the function f which minimizes I. Since I is a surface integral, if we want to use a variational approach we need to extend our Euler-Lagrange formulation. We will return to this example once we have derived the general theory.

2.8.1 Euler-Lagrange theory for surface integrals

We consider integrals of the form

$$I = \int_R L(\mathbf{r}, f(\mathbf{r}), \boldsymbol{\nabla} f(\mathbf{r}))\, dx\, dy$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector in $\mathbb{R}^2$. Let C denote the boundary of R and suppose f is prescribed on C. Suppose $F(\mathbf{r})$ is the extremal function we are trying to find. Consider a family of functions

$$f(\mathbf{r}) = F(\mathbf{r}) + \varepsilon\eta(\mathbf{r}),$$

where η is a smooth function which vanishes on C so that all members of the family take on the same prescribed values on the boundary. We write

$$I(\varepsilon) = \int_R L(\mathbf{r}, F + \varepsilon\eta, \boldsymbol{\nabla} F + \varepsilon \boldsymbol{\nabla}\eta)\, dx\, dy.$$

Since we require I to be stationary when $\varepsilon = 0$ we have

$$I'(0) = 0$$

as in our earlier formulations. Using the chain rule:

$$\frac{dI}{d\varepsilon} = \int_R \left(\eta \frac{\partial L}{\partial f} + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}_{\boldsymbol{\nabla} f} L \right) dx\, dy. \tag{4}$$

Here we adopt the notation

$$\boldsymbol{\nabla}_{\mathbf{p}} \equiv \mathbf{i}\frac{\partial}{\partial p_1} + \mathbf{j}\frac{\partial}{\partial p_2}$$

for any vector $\mathbf{p}$ in $\mathbb{R}^2$ and we have used the result from early in the course (Sheet 1 Q3) that

$$\frac{d}{d\varepsilon} f(\mathbf{g}(\varepsilon)) = \mathbf{g}'(\varepsilon) \cdot \boldsymbol{\nabla}_{\mathbf{g}} f.$$

Setting $\varepsilon = 0$ in (4) we therefore have

$$0 = \int_R \left(\eta \frac{\partial L}{\partial F} + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}_{\boldsymbol{\nabla}F} L \right) dx\, dy. \tag{5}$$

Now since η vanishes on the boundary C of R, the divergence theorem tells us that

$$\int_R \boldsymbol{\nabla}\eta \cdot \mathbf{A}\, dx\, dy = -\int_R \eta\, \mathrm{div}\mathbf{A}\, dxdy$$

for any vector field $\mathbf{A}$ (see Problem Sheet 3, Q1). Thus choosing

$$\mathbf{A} = \boldsymbol{\nabla}_{\boldsymbol{\nabla}F} L,$$

(5) can be rewritten in the form

$$\int_R \eta \left(\frac{\partial L}{\partial F} - \mathrm{div}(\boldsymbol{\nabla}_{\boldsymbol{\nabla}F} L) \right) dxdy = 0.$$

Since η is arbitrary, and using an appropriate extension of the Vanishing Lemma to higher dimensions, we conclude that

$$\frac{\partial L}{\partial F} - \mathrm{div}(\boldsymbol{\nabla}_{\boldsymbol{\nabla}F} L) = 0, \tag{6}$$

which is the generalization of the Euler-Lagrange equation we derived for 1D integrals. Again, henceforth we use f rather than F to denote the extremal function.

2.8.2 Remarks

(i) The equation holds for volume integrals and in fact also for n-dimensional integrals.
(ii) Constraints can be accommodated in a similar way to before.

Example 6

We conclude by revisiting the minimal surface area (soap film) example. Here we wish to minimize the integral

$$I = \int_{\Sigma} (1 + |\boldsymbol{\nabla} f|^2)^{1/2} \, dx \, dy$$

and so

$$L = (1 + |\boldsymbol{\nabla} f|^2)^{1/2},$$

which is explicitly independent of position $\mathbf{r}$ and the function f. The E-L equation (6) therefore becomes

Writing $\boldsymbol{\nabla} f = (f_x, f_y)$ we have

and so the minimal surface equation is

After some algebra (problem sheet 5) the equation can be written as the following non-linear second order partial differential equation:

$$(1 + f_y^2) f_{xx} + (1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} = 0.$$

Some solutions to this equation are investigated on sheet 5.