Question 1

(a) Prove that for any two random variables X and Y,

$$\mathrm{Cov}\left(X,Y\right)=\mathrm{E}\left(XY\right)-\mathrm{E}\left(X\right)\mathrm{E}\left(Y\right)$$

(b) Prove that for random variables $X_1,X_2,\ldots,X_n$ and $Y_1,Y_2,\ldots,Y_m$ that

$$\mathrm{Cov}\left(\sum_{i=1}^{n}a_iX_i,\sum_{j=1}^{m}b_jY_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j\mathrm{Cov}\left(X_i,Y_j\right)$$

Question 2

Young's inequality states that if a and b are non-negative real numbers, and p and q are any positive numbers such that

$$\frac{1}{p}+\frac{1}{q}=1,$$

then

$$\frac{1}{p}a^p+\frac{1}{q}b^q\geq ab.$$

Use Young's inequality to prove Hölder's Inequality: Let X and Y be random variables and let p and q be two positive numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\left|\mathrm{E}\left(XY\right)\right|\leq\left(\mathrm{E}\left(|X|^p\right)\right)^{1/p}\left(\mathrm{E}\left(|Y|^q\right)\right)^{1/q}.$$

(a) If Z is a non-negative random variable, i.e. $Z\geq 0$, prove that $\mathrm{E}\left(Z\right)\geq 0$.

(b) Prove that $\mathrm{E}\left(|XY|\right)\geq 0$.

(c) Prove that $\left|\mathrm{E}\left(XY\right)\right|\leq\mathrm{E}\left(|XY|\right)$.

(d) Use Young's inequality to prove $\mathrm{E}\left(|XY|\right)\leq\left(\mathrm{E}\left(|X|^p\right)\right)^{1/p}\left(\mathrm{E}\left(|Y|^q\right)\right)^{1/q}$.

(e) Conclude that Hölder's Inequality is true.

(f) Use Hölder's Inequality to prove the Cauchy-Schwarz Inequality: $\left|\mathrm{E}\left(XY\right)\right|\leq(\mathrm{E}(|X|^2))^{1/2}(\mathrm{E}(|Y|^2))^{1/2}$.

(g) Use the Cauchy-Schwarz inequality to prove Theorem 4.1.6: $\left|\mathrm{Cov}\left(X,Y\right)\right|\leq\sigma_X\sigma_Y$, where σ_X^2 and σ_Y^2 are the variances of X and Y, respectively.

R question

There is no R question this week, please work on your coursework which is due on 10 March.