Imperial College London

MATH97028 MATH97105

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Complex Analysis

Date: Wednesday, 26 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Use the Plemelj theorem to calculate

$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(t+2)(t-z)} \mathrm{d}t, \qquad z \notin [-1,1].$$

Demonstrate that the proposed solution satisfies all four requirements of the Plemelj theorem.

(7 marks)

(b) Find a closed form expression for all solutions $u(x)$ to the equation

$$\frac{1}{\pi} \fint \frac{u(t)}{x-t} \mathrm{d}t = x, \qquad x \in [-1,1],$$

using the Hilbert inversion formula:

$$u(x) = -\frac{1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{x-t} \mathrm{d}t + \frac{C}{\sqrt{1-x^2}},$$

where f is the Hilbert transform of u and C is an arbitrary constant.

Hint: you may use

$$\sqrt{1 \pm \frac{1}{z}} = 1 \pm \frac{1}{2z} - \frac{1}{8z^2} + \mathcal{O}\left(z^{-3}\right), \qquad z \to \infty.$$

(7 marks)

(c) The Cauchy transform of $f(x)$ on the real line is defined by

$$\mathcal{C}[f](z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \mathrm{d}t.$$

Let

$$\phi(z) = \frac{\mathrm{e}^{-z^2}}{2\pi \mathrm{i}} \left[\log\left(-\frac{1}{z}\right) + \log z - \pi \operatorname{erfi}(z)\right],$$

where

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{t^2} \mathrm{d}t.$$

You are given that

$$\lim_{z\to\infty} \phi(z) = 0.$$

Show that

$$\mathcal{C}\left[\mathrm{e}^{-x^2}\right](z) = \phi(z),$$

by verifying that $\phi(z)$ satisfies the conditions of the Plemelj theorem.

(6 marks)

(Total: 20 marks)

2. Consider the solution of the following Laplace's equation, using $z = x + \mathrm{i}y$:

1. $v_{xx} + v_{yy} = 0$ for $z \notin [-1, 1] \cup \{\pm\mathrm{i}\}$,
2. $v(x, y) = \frac{1}{2}\log|z \pm \mathrm{i}| + O(1)$ as $z \to \pm\mathrm{i}$,
3. $v(x, y) = \log|z| + o(1)$ as $z \to \infty$, and
4. $v(x, 0) = \kappa$, for $-1 < x < 1$ where κ is an unknown constant.

This equation models the potential field of two positive charges of strength $1/2$ at $\pm\mathrm{i}$ with a metal sheet that has no net charge placed on $[-1, 1]$.

(a) By writing

$$v(x, y) = \int_{-1}^{1} u(t)\log|z - t|\mathrm{d}t + \frac{1}{2}\log|z - \mathrm{i}| + \frac{1}{2}\log|z + \mathrm{i}|,$$

show that the problem of finding $v(x, y)$ can be reformulated as finding $u(x)$ such that

$$\int_{-1}^{1} u(t)\log|x - t|\mathrm{d}t = g(x),$$

where

$$\int_{-1}^{1} u(x)\mathrm{d}x = 0.$$

What is $g(x)$ in this equation? Explain why $v(x, y)$ will thereby satisfy the required four conditions. (5 marks)

(b) Find $u(x)$. Hint: reduce the problem to one of inverting the Hilbert transform. You may use

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\left(\frac{\sqrt{\mathrm{i} - 1}\sqrt{\mathrm{i} + 1}}{x - \mathrm{i}} + \frac{\sqrt{-\mathrm{i} - 1}\sqrt{-\mathrm{i} + 1}}{x + \mathrm{i}}\right)\mathrm{d}x = -2\pi, \qquad \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\mathrm{d}x = \pi.$$

(7 marks)

(c) Find an explicit formula for κ in terms of an integral. (3 marks)

(d) Express $v(x, y)$ in terms of the Cauchy transform of $\int_x^1 u(t)\mathrm{d}t$. Note: knowing $u(x)$ explicitly is not required to complete this part. (5 marks)

(Total: 20 marks)

3. The Hermite polynomials

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2} = 2^n x^n + \mathcal{O}(x^{n-1}), \qquad n \geq 0,$$

are orthogonal on $(-\infty, \infty)$ with respect to the weight $w(x) = \mathrm{e}^{-x^2}$ and they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x} [w(x) H_n(x)] = -w(x) H_{n+1}(x), \qquad n \geq 0,$$

and

$$x H_n(x) = n H_{n-1}(x) + \frac{H_{n+1}(x)}{2}, \qquad n \geq 1.$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = 2n H_{n-1}(x).$$

(4 marks)

(b) Let

$$\boldsymbol{H} = \begin{pmatrix} H_0(x) \\ H_1(x) \\ \vdots \end{pmatrix}.$$

Give the operators J and D such that

$$x\boldsymbol{H} = J\boldsymbol{H} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{H} = D\boldsymbol{H}.$$

(2 marks)

(c) Suppose $u(x), x \in \mathbb{R}$ has a weighted Hermite expansion:

$$u(x) = \sum_{k=0}^{\infty} u_k \mathrm{e}^{-x^2/2} H_k(x) = \mathrm{e}^{-x^2/2} \begin{pmatrix} H_0(x) & H_1(x) & \cdots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \boldsymbol{u}.$$

Use the operators J and D to represent the ordinary differential operator

$$u''(x) - x^4 u(x), \qquad x \in (-\infty, \infty),$$

as an operator acting on $\boldsymbol{u} = \begin{pmatrix} u_0 & u_1 & \cdots \end{pmatrix}^\intercal$. (5 marks)

(d) The log transform on the real line is defined by

$$M[f](z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \log(z - x) \mathrm{d}x.$$

Express the log transform of weighted Hermite polynomials, $M[wH_k](z)$, in terms of a Cauchy transform for $k \geq 1$. (3 marks)

(e) Give the recurrence relationship for the Cauchy transform of weighted Hermite polynomials

$$\mathcal{C}\left[wH_k\right](z), \qquad k = 0, 1, \ldots.$$

Use the fact that $\int_{-\infty}^{\infty} w(x) \mathrm{d}x = \sqrt{\pi}$ and the function $\phi(z)$ in question 1 (c). (3 marks)

(f) Use the function $\phi(z)$ in question 1 (c) to find the Hilbert transform of $H_0 w$, i.e., $\mathcal{H}[H_0 w](x)$. (3 marks)

(Total: 20 marks)

4. Let $u(x)$ solve the integral equation

$$\frac{2}{3}u(x) + \int_0^\infty K(t-x)u(t)\mathrm{d}t = f(x) \qquad \text{for} \qquad x \geq 0,$$

where

$$K(x) = \mathrm{e}^{-|x|} \qquad \text{and} \qquad f(x) = 1 + \mathrm{e}^{-x}.$$

We will use the notations

$$g_{\mathrm{L}}(x) := \begin{cases} g(x) & x < 0 \\ 0 & x \geq 0 \end{cases}, \qquad g_{\mathrm{R}}(x) := \begin{cases} 0 & x < 0 \\ g(x) & x \geq 0 \end{cases},$$

and, for the Fourier transform,

$$\hat{f}(s) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t.$$

(a) What are the regions of analyticity of $\widehat{K}(s)$, and $\widehat{f_{\mathrm{R}}}(s)$? Assuming that $|u(x)|$ is bounded, what is the region of analyticity of $\widehat{u_{\mathrm{R}}}(s)$? Justify your answers without explicit calculation. (3 marks)

(b) Show that the Fourier transforms are given by

$$\widehat{K}(s) = \frac{2}{1+s^2} \qquad \text{and} \qquad \widehat{f_{\mathrm{R}}}(s) = \frac{1+2\mathrm{i}s}{\mathrm{i}s - s^2}.$$

(3 marks)

(c) For the integral equation above, set up a Riemann–Hilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s) \qquad \text{for} \qquad s \in (-\infty, \infty) + \mathrm{i}\delta,$$

where $\Phi_+(s)$ is analytic above $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_-(s)$ is analytic below $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_\pm(s)$ decay at infinity, and

$$g(s) = \frac{\frac{2}{3}(s^2+4)}{1+s^2}.$$

Explain the choice of δ and the definition of $\Phi_\pm(s)$, $g(s)$ and $h(s)$ in terms of the Fourier transforms of u, f, and K. (3 marks)

(d) Show that the winding number of $g(s)$ about the origin is zero for $s \in (-\infty,\infty)+\mathrm{i}\delta$. (3 marks)

(e) Find a solution to the homogeneous Riemann–Hilbert problem

$$\kappa_+(s) = g(s)\kappa_-(s) \qquad \text{for} \qquad s \in (-\infty,\infty)+\mathrm{i}\delta$$

such that $\kappa_+(s) = \frac{2}{3} + \mathcal{O}(s^{-1})$ and $\kappa_-(s) = 1 + \mathcal{O}(s^{-1})$ as $s \to \infty$, where δ is the same constant as in (c). (3 marks)

(f) Determine $u(x)$. (5 marks)

(Total: 20 marks)

5. (a) The poles of the function

$$f(x) = \frac{1}{2 - \cos \pi x}, \qquad x \in [-1, 1],$$

that are closest to $[-1, 1]$ are at $x = \pm \mathrm{i} \log(2 + \sqrt{3})/\pi$. The function has a Fourier expansion

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}\pi k x}$$

and its Fourier coefficients are bounded by

$$\left| \hat{f}_k \right| \leq M R^{-|k|}, \qquad k \in \mathbb{Z},$$

where $1 < R < 2 + \sqrt{3}$. The maximum modulus of f attained on the strip $|\mathrm{Im}\, x| \leq \log R$ is

$$M = \frac{2}{4 - R^{-1} + R}.$$

(i) Let $F_n(x)$ denote the truncated Fourier expansion

$$F_n(x) = \sum_{k=-n}^{n} \hat{f}_k \mathrm{e}^{\mathrm{i}\pi k x}.$$

Find a bound on the error $|f(x) - F_n(x)|$ for $x \in [-1, 1]$. (3 marks)

(ii) Denote a truncated Chebyshev expansion of f by

$$f_n(x) = \sum_{k=0}^{n-1} f_k T_k(x).$$

Find the largest Bernstein ellipse within which $f_n(x)$ converges to f. Call this ellipse E_{ρ_*}. (3 marks)

(iii) Find a bound on the maximum modulus of f on a Bernstein ellipse E_ρ with $\rho < \rho_*$. Call this bound M.

(2 marks)

(iv) Find a bound on the error $|f(x) - f_n(x)|$ for $x \in [-1, 1]$. You may use the bound on the Chebyshev coefficients:

$$|f_k| \leq 2M\rho^{-k}, \qquad k \geq 1.$$

(2 marks)

(v) Does an interpolant in n equally spaced points in $[-1, 1]$ converge to $f(x)$ on $[-1, 1]$ as $n \to \infty$? Explain. (3 marks)

(b) Let $x_1, \ldots, x_n \in [-1, 1]$ be distinct interpolation nodes and suppose we let the node polynomial have a root of multiplicity two at x_n:

$$\ell(x) = (x - x_1) \cdots (x - x_{n-1})(x - x_n)^2.$$

Let Γ be a positively orientated, simple, closed contour that encloses $x_1, \ldots, x_n$ and x. Suppose f is analytic on and inside Γ. Use the residue theorem to evaluate

$$F(x) := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\ell(x)}{\ell(t)} \frac{f(t)}{t - x} \mathrm{d}t.$$

Then evaluate $F(x_k)$, $k = 1, \ldots, n$. (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

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MATH96019/MATH97028/MATH97105

Applied Complex Analysis (Solutions)

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1. (a) According to the Plemelj theorem, the Cauchy transform of f seen $\Downarrow$

$$\mathcal{C}f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(t)}{t-z} \mathrm{d}t$$

is given by $\phi(z)$ if (i)

$$\phi_+(x) - \phi_-(x) = f(x), \qquad -1 < x < 1,$$

where

$$\phi_\pm(x) = \lim_{\epsilon \to 0^+} \phi(x \pm \mathrm{i}\epsilon);$$

(ii)

$$\lim_{z\to\infty} \phi(z) = 0;$$

(iii) $\phi(z)$ is analytic on $\bar{\mathbb{C}} \setminus [-1,1]$; and (iv) the singularities of $\phi(z)$ are weaker than poles. For

$$f(x) = \frac{\sqrt{1-x^2}}{x+2},$$

the function

$$\psi(z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(z+2)},$$

has the correct jump,

$$\psi_+(x) - \psi_-(x) = \frac{\mathrm{i}\sqrt{1-x}\sqrt{x+1}}{2\mathrm{i}(x+2)} - \frac{-\mathrm{i}\sqrt{1-x}\sqrt{x+1}}{2\mathrm{i}(x+2)} = f(x), \qquad -1 < x < 1.$$

To satisfy properties (ii)–(iv), we need to subtract off $\psi(\infty) = \frac{1}{2\mathrm{i}}$ and the pole at $z=-2$, hence

$$\mathcal{C}\left[\frac{\sqrt{1-\diamond^2}}{\diamond+2}\right](z) = \phi(z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(z+2)} - \frac{1}{2\mathrm{i}} + \frac{\sqrt{3}}{2\mathrm{i}(z+2)}.$$

7, A

(b) We have $f(x) = x$ and thus the solution is seen $\Downarrow$

$$u(x) = -\frac{1}{\sqrt{1-x^2}}\mathcal{H}\left[\diamond\sqrt{1-\diamond^2}\right](x) + \frac{C}{\sqrt{1-x^2}},$$

where C is an arbitrary constant. We calculate the Hilbert transform $\mathcal{H}$ via the Cauchy transform (since the Hilbert and Cauchy transforms are related according to $(\mathcal{C}^+ + \mathcal{C}^-)f(x) = \mathrm{i}\mathcal{H}f(x)$). The function

$$\psi(z) = \frac{1}{2\mathrm{i}} z\sqrt{z-1}\sqrt{z+1},$$

has the correct jump

$$\psi_+(x) - \psi_-(x) = \frac{1}{2}x\sqrt{1-x}\sqrt{1+x} - \left(-\frac{1}{2}x\sqrt{1-x}\sqrt{1+x}\right) = x\sqrt{1-x^2}.$$

Using the hint, the large-z behaviour of ψ is

$$\begin{aligned}\psi &= \frac{z^2}{2\mathrm{i}}\sqrt{1-\frac{1}{z}}\sqrt{1+\frac{1}{z}} \\ &= \frac{z^2}{2\mathrm{i}}\left(1 - \frac{1}{2z^2} + \mathcal{O}(z^{-4})\right) = \frac{1}{2\mathrm{i}}\left(z^2 - \frac{1}{2}\right) + \mathcal{O}(z^{-2}), \qquad z \to \infty.\end{aligned}$$

By the Plemelj theorem, we therefore have

$$\phi(z) = \mathcal{C}\left[\diamond\sqrt{1-\diamond^2}\right](z) = \frac{1}{2\mathrm{i}}z\sqrt{z-1}\sqrt{z+1} - \frac{1}{2\mathrm{i}}\left(z^2 - \frac{1}{2}\right),$$

thus

$$\mathcal{H}\left[\diamond\sqrt{1-\diamond^2}\right](x) = -\mathrm{i}\left(\phi_+(x) + \phi_-(x)\right) = x^2 - \frac{1}{2},$$

and

$$u(x) = -\frac{x^2 - \frac{1}{2}}{\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}.$$

7, A

(c) First note that in the upper half-plane

unseen ⇓

$$\log\left(-\frac{1}{z}\right) + \log z = \mathrm{i}\pi,$$

and in the lower half-plane

$$\log\left(-\frac{1}{z}\right) + \log z = -\mathrm{i}\pi.$$

Hence,

$$\phi(z) = \begin{cases} \mathrm{e}^{-z^2}\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\mathrm{erfi}(z)\right) & \text{if } \mathrm{Im}\, z > 0, \\ \mathrm{e}^{-z^2}\left(-\frac{1}{2} + \frac{\mathrm{i}}{2}\mathrm{erfi}(z)\right) & \text{if } \mathrm{Im}\, z < 0, \end{cases}$$

so

$$\phi_+(x) - \phi_-(x) = \mathrm{e}^{-x^2}, \qquad x \in \mathbb{R}.$$

The imaginary error function is an entire function because

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\mathrm{e}^{z^2}.$$

The function $\phi(z)$ satisfies all the conditions of the Plemelj theorem: it has the right jump, $\phi(\infty) = 0$, it is analytic off the real axis and its singularities are weaker than poles.

6, D

2. (a) Writing $\boxed{\text{seen} \Downarrow}$

$$v(x,y) = \int_{-1}^{1} u(t)\log|z-t|\mathrm{d}t + \frac{1}{2}\log|z-\mathrm{i}| + \frac{1}{2}\log|z+\mathrm{i}|.$$

This solves Laplace's equation (1) away from $[-1,1]$ and $\pm\mathrm{i}$ as it is the real part of an analytic function. It also solves (2) since as $z \to \pm\mathrm{i}$ the logarithmic terms dominate. For (3), we have

$$\frac{1}{2}\log|z-\mathrm{i}| + \frac{1}{2}\log|z+\mathrm{i}| = \log|z| + \frac{1}{2}\log|1-\mathrm{i}/z| + \frac{1}{2}\log|1+\mathrm{i}/z| = \log|z| + o(1)$$

The condition $\int_{-1}^{1} u(t)\mathrm{d}t = 0$ ensures that (3) is satisfied since

$$\int_{-1}^{1} u(t)\log|z-t|\mathrm{d}t = \int_{-1}^{1} u(t)\mathrm{d}t\log|z| + \int_{-1}^{1} u(t)\log|1-t/z|\mathrm{d}t = \int_{-1}^{1} u(t)\log|1-t/z|\mathrm{d}t$$

and $\log|1-t/z| \to 0$. The condition (4) that $v(x,0) = \kappa$ then reduces to the integral equation, with

$$g(x) = \kappa - \frac{1}{2}\log(x^2+1)$$

$\boxed{5, \text{A}}$

(b) Differentiating and multiplying by $1/\pi$ we get $\boxed{\text{sim. seen} \Downarrow}$

$$\mathcal{H}u(x) = \frac{1}{\pi}⨍ \frac{u(t)}{x-t}\mathrm{d}t = -\frac{x}{\pi(x^2+1)} := f(x).$$

From the inverse Hilbert transform formula we know

$$u(x) = -\frac{1}{\sqrt{1-x^2}}\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{x-t}\mathrm{d}t + \frac{C}{\sqrt{1-x^2}}$$

To find the Hilbert transform of $h(x) := -f(x)\sqrt{1-x^2}$, we first find the Cauchy transform

$$\phi(z) = \mathcal{C}h(z) = \frac{z\sqrt{z-1}\sqrt{z+1}}{2\pi\mathrm{i}(z^2+1)} - \frac{1}{2\pi\mathrm{i}} + \frac{\mathrm{i}\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4\pi(z-\mathrm{i})} + \frac{\mathrm{i}\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4\pi(z+\mathrm{i})},$$

then

$$\mathcal{H}h(x) = -\mathrm{i}\left(\phi_+(x)+\phi_-(x)\right) = -\frac{\mathrm{i}}{\pi}\left(-\frac{1}{\mathrm{i}} + \frac{\mathrm{i}\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{2(x-\mathrm{i})} + \frac{\mathrm{i}\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{2(x+\mathrm{i})}\right)$$

and the solution is (after relabelling the constant)

$$u(x) = \frac{\mathcal{H}h(x)+D}{\sqrt{1-x^2}} = \frac{1}{2\pi\sqrt{1-x^2}}\left(\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{x-\mathrm{i}} + \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{x+\mathrm{i}}\right) + \frac{C}{\sqrt{1-x^2}}.$$

Using the hint,

$$\int_{-1}^{1} u(x)\mathrm{d}x = -1 + C\pi = 0 \quad \Rightarrow \quad C = \frac{1}{\pi}.$$

$\boxed{7, \text{C}}$

(c) Setting $z=0$ in the representation for v and using $v(0,0)=\kappa$, we have that $\boxed{\text{seen} \Downarrow}$

$$\kappa = \int_{-1}^{1} u(t)\log|t|\mathrm{d}t.$$

$\boxed{3, \text{B}}$

(d) We can write

sim. seen $\Downarrow$

$$v(x,y)-\frac{1}{2}\log|z-\mathrm{i}|-\frac{1}{2}\log|z+\mathrm{i}| = \mathrm{Re}\int_{-1}^{1} u(t)\log(z-t)\mathrm{d}t = \mathrm{Re}\,2\pi\mathrm{i}\int_{z}^{\infty}\mathcal{C}u(z)\mathrm{d}z,$$

where $\mathcal{C}u(z) = \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} u(t)/(t-z)\mathrm{d}t$ is the Cauchy transform. Define

$$\phi(z) = \int_{z}^{\infty}\mathcal{C}u(z)\mathrm{d}z.$$

Inspecting the jump of $\phi(z)$ on $[-1,1]$ we find that

$$\phi_{+}(x)-\phi_{-}(x) = \int_{x}^{1}(\mathcal{C}^{+}-\mathcal{C}^{-})u(x)\mathrm{d}x = \int_{x}^{1} u(x)\mathrm{d}x$$

For $x<-1$ we have

$$\phi_{+}(x)-\phi_{-}(x) = \int_{-1}^{1} u(x)\mathrm{d}x = 0$$

and for $x>1$ we also have $\phi_{+}(x)=\phi_{-}(x)$ by analyticity, as we avoid any branch cuts. Hence by Plemelj

$$\phi(z) = \mathcal{C}[\int_{x}^{1} u(t)\mathrm{d}t](z)$$

and therefore

$$v(x,y) = \mathrm{Re}\,2\pi\mathrm{i}\mathcal{C}[\int_{x}^{1} u(t)\mathrm{d}t](z) + \frac{1}{2}\log|z-\mathrm{i}| + \frac{1}{2}\log|z+\mathrm{i}|$$

5, B

3. (a) First we check that H_n' is orthogonal to polynomials of degree $< n-1$ with respect to the weight $w = \mathrm{e}^{-x^2}$. Let p_m be a polynomial of degree m, then

$$\boxed{\text{sim. seen} \Downarrow}$$

$$\begin{aligned}
\langle H_n', p_m \rangle_w &= \int_{-\infty}^{\infty} H_n'(x) p_m(x) \mathrm{e}^{-x^2} \mathrm{d}x \\
&= H_n(x) p_m(x) \mathrm{e}^{-x^2} \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_n(x) \left[p_m'(x) - 2x p_m(x) \right] \mathrm{e}^{-x^2} \mathrm{d}x \\
&= -\langle H_n, p_m'(x) - 2x p_m(x) \rangle_w
\end{aligned}$$

This inner product is zero provided $m+1 < n$. This shows that $H_n' = c_n H_{n-1}$, where c_n is a constant. To determine c_n, note that

$$H_n' = n 2^n x^{n-1} + \mathcal{O}(x^{n-2}) = 2n k_{n-1} x^{n-1} + \mathcal{O}(x^{n-2}).$$

This implies that $c_n = 2n$.

$$\boxed{4, \text{B}}$$

(b) The result in (a), the three-term recurrence and the fact that $H_0 = 1$ imply that the differential and Jacobi operators are

$$\boxed{\text{sim. seen} \Downarrow}$$

$$D = \begin{pmatrix} 0 & & & \\ 2 & & & \\ & 4 & & \\ & & \ddots & \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & \frac{1}{2} & & & \\ 1 & 0 & \frac{1}{2} & & \\ & 2 & 0 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

$$\boxed{2, \text{A}}$$

(c) We have

$$\boxed{\text{meth seen} \Downarrow}$$

$$u'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \boldsymbol{u} = \mathrm{e}^{-x^2/2} \left(-x + \frac{\mathrm{d}}{\mathrm{d}x} \right) \boldsymbol{H}^\intercal \boldsymbol{u} = \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \left(-J^\intercal + D^\intercal \right) \boldsymbol{u},$$

thus

$$\begin{aligned}
u''(x) &= \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \left(-J^\intercal + D^\intercal \right)^2 \boldsymbol{u}, \\
x^4 u(x) &= x^4 \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \boldsymbol{u} = \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \left(J^\intercal \right)^4 \boldsymbol{u},
\end{aligned}$$

and

$$u''(x) - x^4 u(x) = \mathrm{e}^{-x^2/2} \boldsymbol{H}^\intercal \left[\left(-J^\intercal + D^\intercal \right)^2 - \left(J^\intercal \right)^4 \right] \boldsymbol{u}.$$

In coefficient space the differential operator is represented by

$$\left(-J^\intercal + D^\intercal \right)^2 - \left(J^\intercal \right)^4.$$

$$\boxed{5, \text{C}}$$

(d) Integrating by parts and using $[-w(x) H_{k-1}(x)]' = w(x) H_k$, it follows that

$$\boxed{\text{unseen} \Downarrow}$$

$$\begin{aligned}
M[w H_k](z) &= \frac{1}{\pi} \int_{-\infty}^{\infty} H_k(x) w(x) \log(z - x) \mathrm{d}x \\
&= \frac{1}{\pi} \left(\left[-\mathrm{e}^{-x^2} H_{k-1}(x) \log(z - x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x^2} H_{k-1}(x)}{z - x} \mathrm{d}x \right) \\
&= 2\mathrm{i} \mathcal{C} \left[w H_{k-1} \right](z), \qquad k \geq 1.
\end{aligned}$$

$$\boxed{3, \text{D}}$$

(e) Let

seen ⇓

$$C_k(z) = \mathcal{C}\left[H_k w\right](z), \qquad k \geq 0,$$

then

$$C_0(z) = \phi(z),$$

$$zC_0 = -\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} w(x)\mathrm{d}x + \frac{1}{2}C_1(z) = \frac{\mathrm{i}}{2\sqrt{\pi}} + \frac{1}{2}C_1(z),$$

and for $k \geq 1$, the $C_k(z)$ satisfy the same recurrence as the Hermite polynomials:

$$zC_k(z) = kC_{k-1}(z) + \frac{1}{2}C_{k+1}(z), \qquad k \geq 1.$$

3, B

(f) Using the expression derived in question 1 (c),

unseen ⇓

$$\begin{aligned}\mathcal{H}\left[H_0 w\right](x) = \mathcal{H}\left[w\right](x) &= -\mathrm{i}\left(\phi^+(x) + \phi^-(x)\right) \\ &= -\mathrm{i}\mathrm{e}^{-x^2}\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\mathrm{erfi}(x) - \frac{1}{2} + \frac{\mathrm{i}}{2}\mathrm{erfi}(x)\right) \\ &= \mathrm{e}^{-x^2}\mathrm{erfi}(x)\end{aligned}$$

3, D

4. (a) Since $f(x)\mathrm{e}^{\gamma x}$ is absolutely integrable for any $\gamma < 0$ we know that $\widehat{f_{\mathrm{R}}}(s)$ is analytic for $\operatorname{Im} s < 0$. For the same reason $\widehat{u_{\mathrm{R}}}(s)$ is analytic for $\operatorname{Im} s < 0$. On the other hand, $K(x)\mathrm{e}^{\gamma x}$ is absolutely integrable for $-1 < \gamma < 1$ hence $\hat{K}(s)$ is analytic in the strip $-1 < \operatorname{Im} s < 1$. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{3, \mathrm{B}}$

(b) These are straightforward integration by parts calculations: $\boxed{\text{sim. seen} \Downarrow}$

$$\hat{K}(s) = \int_{-\infty}^{\infty} K(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t = \int_{-\infty}^{0} \mathrm{e}^{(1-\mathrm{i}s)t}\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{(-1-\mathrm{i}s)t} = \frac{1}{1-\mathrm{i}s} - \frac{1}{-1-\mathrm{i}s} = \frac{2}{1+s^2}$$

$$\widehat{f_{\mathrm{R}}}(s) = \int_{0}^{\infty} \left(1+\mathrm{e}^{-t}\right)\mathrm{e}^{-\mathrm{i}ts}\mathrm{d}t = \frac{1}{\mathrm{i}s} + \frac{1}{1+\mathrm{i}s} = \frac{1}{\mathrm{i}s} + \frac{\mathrm{i}}{\mathrm{i}-s} = \frac{1+2\mathrm{i}s}{\mathrm{i}s-s^2}$$

$\boxed{3, \mathrm{A}}$

(c) We can write the integral equation as $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{2}{3}u_{\mathrm{R}}(x) + \int_{-\infty}^{\infty} K(t-x)u_{\mathrm{R}}(t)\mathrm{d}t = f_{\mathrm{R}}(x) + p_{\mathrm{L}}(x)$$

where

$$p(x) := \int_{-\infty}^{\infty} K(t-x)u_{\mathrm{R}}(t)\mathrm{d}t.$$

Transforming into Fourier space we have

$$\widehat{p_{\mathrm{L}}}(s) - \left(\frac{2}{3} + \hat{K}(s)\right)\widehat{u_{\mathrm{R}}}(s) = -\widehat{f_{\mathrm{R}}}(s)$$

for $s \in (-\infty,\infty)+\mathrm{i}\delta$ for any $-1 < \delta < 0$. As discussed in lectures $\widehat{p_{\mathrm{L}}}(s)$ is analytic for $\operatorname{Im} s \geq \delta$ while $\widehat{u_{\mathrm{R}}}(s)$ is analytic for $\operatorname{Im} s \leq \delta$, thus define $\Phi_+(s) = \widehat{p_{\mathrm{L}}}(s)$, $\Phi_-(s) = \widehat{u_{\mathrm{R}}}(s)$,

$$g(s) = \frac{2}{3} + \hat{K}(s) = \frac{\frac{2}{3}(s^2+4)}{1+s^2}$$

and $h(s) = -\widehat{f_{\mathrm{R}}}(s) = \frac{1+2\mathrm{i}s}{s^2-\mathrm{i}s}$. $\boxed{3, \mathrm{A}}$

(d) Let $s = t + \mathrm{i}\delta$, $t \in \mathbb{R}$, $-1 < \delta < 0$ then $\boxed{\text{meth seen} \Downarrow}$

$$\operatorname{Re} g(s) = \frac{2}{3}\frac{(t^2-\delta^2+1)(t^2-\delta^2+4)+4t^2\delta^2}{(t^2-\delta^2+4)^2+4t^2\delta^2} > 0, \quad t \in \mathbb{R},$$

and this implies that the winding number of g about the origin is zero since the image of g is in the right half-plane for $t \in \mathbb{R}$. $\boxed{3, \mathrm{D}}$

(e) Factorising $\boxed{\text{sim. seen} \Downarrow}$

$$g(s) = \frac{2}{3}\frac{(s+2\mathrm{i})(s-2\mathrm{i})}{(s+\mathrm{i})(s-\mathrm{i})} = \kappa_+(s)\kappa_-(s)^{-1}$$

by location of the poles we see that $\kappa_+(s) = \frac{2}{3}\frac{(s+2\mathrm{i})}{(s+\mathrm{i})}$ and $\kappa_-(s) = \frac{(s-\mathrm{i})}{(s-2\mathrm{i})}$ satisfy the required properties. $\boxed{3, \mathrm{A}}$

(f) We write $\boxed{\text{sim. seen} \Downarrow}$

$$\Phi_\pm(s) = \kappa_\pm(s) Y_\pm(s)$$

so that

$$\Phi_+(s) - g(s)\Phi_-(s) = \kappa_+(s)(Y_+(s) - Y_-(s))$$

Thus we want to solve, using partial fraction expansion to expand, $\boxed{2, \text{A}}$

$$Y_+(s) - Y_-(s) = \frac{h(s)}{\kappa_+(s)} = \frac{1+2\mathrm{i}s}{s^2 - \mathrm{i}s}\frac{3}{2}\frac{(s+\mathrm{i})}{(s+2\mathrm{i})} = \frac{3\mathrm{i}}{4s} + \frac{\mathrm{i}}{s-\mathrm{i}} + \frac{5\mathrm{i}}{4(s+2\,\mathrm{i})}.$$

By splitting the poles we get

$$Y_+(s) = \frac{5\mathrm{i}}{4(s+2\,\mathrm{i})}$$
$$Y_-(s) = -\frac{3\mathrm{i}}{4s} - \frac{\mathrm{i}}{s-\mathrm{i}}$$

This ensures that

$$\Phi_+(s) = \frac{5\mathrm{i}}{4(s+2\,\mathrm{i})}\frac{2}{3}\frac{(s+2\mathrm{i})}{(s+\mathrm{i})} = \frac{5\mathrm{i}}{6(s+\mathrm{i})}$$
$$\Phi_-(s) = \left(-\frac{3\mathrm{i}}{4s} - \frac{\mathrm{i}}{s-\mathrm{i}}\right)\frac{s-\mathrm{i}}{s-2\mathrm{i}} = \left(-\frac{3(s-\mathrm{i})}{4s} - 1\right)\frac{\mathrm{i}}{s-2\mathrm{i}}$$

We then recover the solution from the inverse Fourier transform $\boxed{2, \text{B}}$

$$u(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\delta}^{\infty+\mathrm{i}\delta} \Phi_-(s)\mathrm{e}^{\mathrm{i}sx}\mathrm{d}s = \mathrm{i}\left(\operatorname*{Res}_{s=0} + \operatorname*{Res}_{s=2\mathrm{i}}\right)\Phi_-(s)\mathrm{e}^{\mathrm{i}sx}$$
$$= \mathrm{i}\left(-\frac{3\mathrm{i}}{8} - \frac{11\mathrm{i}}{8}\mathrm{e}^{-2x}\right) = \frac{3}{8} + \frac{11}{8}\mathrm{e}^{-2x}$$

$\boxed{1, \text{D}}$

5. (a) (i) We have

$$|f(x) - F_n(x)| = \left| \left(\sum_{k=-\infty}^{-n-1} + \sum_{k=n+1}^{\infty} \right) \hat{f}_k \mathrm{e}^{\mathrm{i}\pi k x} \right| \leq 2 \sum_{k=n+1}^{\infty} |\hat{f}_k| = 2M \frac{R^{-n}}{R-1}.$$

$\boxed{3,\ \mathrm{M}}$

(ii) A Bernstein ellipse E_ρ is defined by

$$E_\rho = \left\{ x \in \mathbb{C} \,:\, \left(\frac{\operatorname{Re} x}{\alpha} \right)^2 + \left(\frac{\operatorname{Im} x}{\beta} \right)^2 = 1,\ \alpha = \frac{1}{2}\left(\rho + \rho^{-1}\right), \beta = \frac{1}{2}\left(\rho - \rho^{-1}\right), \rho > 1 \right\}.$$

The largest Bernstein ellipse within which f_n converges to $f(x)$ intersects the imaginary axis at $\pm i\beta_*$, where

$$\beta_* = \frac{\log(2+\sqrt{3})}{\pi} \approx 0.419$$

and

$$\rho_* = \beta_* + \sqrt{\beta_*^2 + 1} \approx 1.503511210.$$

$\boxed{3,\ \mathrm{M}}$

(iii) The maximum modulus of $f = 1/(2 - \cos \pi x)$ on a Bernstein ellipse E_ρ with $\beta < \log(2+\sqrt{3})/\pi$ is attained where the ellipse intersects the imaginary axis, hence

$$M = \frac{2}{4 - \exp(-\pi\beta) + \exp(\pi\beta)}.$$

$\boxed{2,\ \mathrm{M}}$

(iv) The error on $[-1, 1]$ is bounded by

$$|f(x) - f_n(x)| = \left| \sum_{k=n}^{\infty} f_k T_k(x) \right| \leq \sum_{k=n}^{\infty} |f_k| = 2M \frac{\rho^{-n}}{1-\rho^{-1}}$$

$\boxed{2,\ \mathrm{M}}$

(v) As discussed in the Mastery Material, for an interpolant in equally spaced points to converge to a function f on $[-1, 1]$, f needs to be analytic in an eye-shaped region that encloses the interval $[-1, 1]$ and which intersects the imaginary axis at $\approx \pm 0.52\mathrm{i}$. Since f has poles at $\pm 0.419\mathrm{i}$, f is not analytic in the eye-shaped region and therefore an equispaced interpolant will not converge to f on $[-1, 1]$. $\boxed{3,\ \mathrm{M}}$

(b) First calculate the residue of the double pole at $t = x_n$:

$$\begin{aligned}
\operatorname*{Res}_{t=x_n}\left[\frac{\ell(x)}{\ell(t)}\frac{f(t)}{t-x}\right] &= \lim_{t\to x_n}\frac{\mathrm{d}}{\mathrm{d}t}(t-x_n)^2\left[\frac{\ell(x)}{\ell(t)}\frac{f(t)}{t-x}\right] \\
&= \ell(x)\lim_{t\to x_n}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f(t)}{(t-x_1)\cdots(t-x_{n-1})(t-x)}\right] \\
&= \ell(x)\left[\frac{f'(x_n)}{(x_n-x_1)\cdots(x_n-x_{n-1})(x_n-x)}\right. \\
&\quad -\frac{f(x_n)}{(x_n-x_1)^2(x_n-x_2)\cdots(x_n-x_{n-1})(x_n-x)} \\
&\quad \left. -\cdots-\frac{f(x_n)}{(x_n-x_1)\cdots(x_n-x_{n-1})(x_n-x)^2}\right]
\end{aligned}$$

$$\begin{aligned}
F(x) &= \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\ell(x)}{\ell(t)}\frac{f(t)}{t-x}\mathrm{d}t = \left(\operatorname*{Res}_{t=x} + \sum_{j=1}^{n-1}\operatorname*{Res}_{t=x_j} + \operatorname*{Res}_{t=x_n}\right)\frac{\ell(x)}{\ell(t)}\frac{f(t)}{t-x} \\
&= f(x) - \sum_{j=1}^{n-1}\frac{\ell(x)}{\ell'(x_j)}\frac{f(x_j)}{x-x_j} \\
&\quad + \ell(x)\left[\frac{f'(x_n)}{(x_n-x_1)\cdots(x_n-x_{n-1})(x_n-x)} - \frac{f(x_n)}{(x_n-x_1)^2(x_n-x_2)\cdots(x_n-x_{n-1})(x_n-x)}\right. \\
&\quad \left. -\cdots-\frac{f(x_n)}{(x_n-x_1)\cdots(x_n-x_{n-1})(x_n-x)^2}\right]
\end{aligned}$$

It follows that

$$F(x_k) = f(x_k) - f(x_k) = 0, \qquad k = 1,\ldots,n.$$

$\boxed{7,\ \mathrm{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96019 MATH97028 MATH97015	1	Questions (a) and (b) were answered very well but only a few managed to score full marks on the final question, which was the most challenging.
MATH96019 MATH97028 MATH97015	2	For the most part, students performed very well on this question but many in question (a) didn't simplify the logarithms correctly to find the function g(x).
MATH96019 MATH97028 MATH97015	3	Most students did well on questions (a), (b) and (c) but less so on the remaining questions. This is probably because questions (d), c) and e) involved Cauchy, Hilbert and Log transforms on the line whereas in most of the lecture notes, we considered these transforms on a compact interval.
MATH96019 MATH97028 MATH97015	4	Overall this question was answered well, however many students factored the function g as g = \kappa_+ * \kappa_- instead of g = \kappa_+ / \kappa_-. Only a few students managed to find the correct partial fraction factorisation and decomposition into Y_+ and Y_-.
MATH96019 MATH97028 MATH97015	5	Most students fared poorly in this question on the mastery material. Some students didn't have enough time for the question. It worth noting that the mastery material is new (different from last year's) and there was no recorded lecture on the material. There's a typo in question 5 (a): M should be 2/(4 - R^(-1) - R). In the solution to question (a) (iii), M should be 2/(4 - exp(-\pi \beta) - exp(\pi \beta).