MATH50004/MATH50015/MATH50019 Differential Equations

Spring Term 2023/24

Solutions to Quiz 6

Question 1. Correct answer: (b).

Consider the matrices

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{and note that } A + B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$A + B$ only has eigenvalue 0, while A has eigenvalues -2 and 0, and B has eigenvalue 1. Thus, are Lyapounov exponents $\sigma_A = -2$ for A and $\sigma_B = 1$ for B, but $\sigma_A + \sigma_B = -1 \neq 0$, which is the only Lyapunov exponent for $A + B$.

Question 2. Correct answer: (a).

Stability follows looking at the sign of the right hand side, which is negative for small $x > 0$ and positive for negative values of x close to 0. A rigorous argument requires the use of Exercise 16 (see also Exercise 17). The result also follows from Exercise 27 (ii).

Question 3. Correct answer: (a).

Due to monotonicity for solutions of one-dimensional differential equations, one can show that attractivity of the equilibrium x^* for $\dot{x} = f(x)$ implies that there exists a $\delta > 0$ such that $f(x) < 0$ for all $x \in (x^*, x^* + \delta)$ and $f(x) > 0$ for all $x \in (x^* - \delta, x^*)$. For the function $g(x) = -f(-x)$, for all $x \in (-x^* - \delta, -x^*)$, we have

$$g(x) = -\underbrace{f(\underbrace{-x}_{\in(x^*,x^*+\delta)})}_{<0} > 0$$

and for all $x \in (-x^*, -x^* + \delta)$, we have

$$g(x) = -\underbrace{f(\underbrace{-x}_{\in(x^*-\delta,x^*)})}_{>0} < 0$$

Since $-g(-x) = f(x)$, this implies that if and only if statement.

Question 4. Correct answer: (a).

As in Question 3, attractivity of the equilibrium x^* for $\dot{x} = f(x)$ implies that there exists a $\delta > 0$ such that $f(x) < 0$ for all $x \in (x^*, x^* + \delta)$ and $f(x) > 0$ for all $x \in (x^* - \delta, x^*)$. Clearly, this equilibrium cannot be repulsive, since in this case the signs need to be exactly the opposite in this one-dimensional case.

Question 5. Correct answer: (a).

We do not provide a completely rigorous proof here. Without loss of generality, let x_1^* be the equilibrium such that $\lim_{t\to-\infty} \varphi(t, x) = x_1^*$. Choose an $\varepsilon > 0$ small enough such that the heteroclinic orbit is not completely contained in $B_\varepsilon(x_1^*)$. Then it is straightforward to see that in each δ-neighbourhood of x_1^*, there are points that forward in time leave the ε-neighbourhood. So x_1^* is unstable.