

# MATH50001 Problems Sheet 3

## Solutions

1). Because  $1/(z^2 - 4)$  is holomorphic at all points

$$\{z : |z| \leq 4\} \setminus \left\{ \{z : |z + 2| \leq 1\} \cup \{z : |z - 2| \leq 1\} \right\}.$$

we obtain

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = \oint_{\gamma_1} \frac{1}{(z + 2)(z - 2)} dz + \oint_{\gamma_2} \frac{1}{(z + 2)(z - 2)} dz,$$

where  $\gamma_1 : |z + 2| = 1$  and  $\gamma_2 : |z - 2| = 1$ .

Note that

$$\frac{1}{(z + 2)(z - 2)} = -\frac{1}{4} \frac{1}{z + 2} + \frac{1}{4} \frac{1}{z - 2}.$$

It follows that

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = \frac{1}{4} \oint_{\gamma_1} \left( \frac{1}{z - 2} - \frac{1}{z + 2} \right) dz + \frac{1}{4} \oint_{\gamma_2} \left( \frac{1}{z - 2} - \frac{1}{z + 2} \right) dz.$$

Because  $1/(z - 2)$  is holomorphic inside and on  $\gamma_1$  its integral around  $\gamma_1$  is zero (by the Cauchy-Goursat theorem). Similarly, the contour integral of  $1/(z + 2)$  around  $\gamma_2$  vanishes. Besides,

$$\oint_{\gamma_1} \frac{1}{z + 2} dz = 2\pi i \quad \text{and} \quad \oint_{\gamma_2} \frac{1}{z - 2} dz = 2\pi i.$$

Consequently

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = -\frac{1}{4} 2\pi i + \frac{1}{4} 2\pi i = 0.$$

2).

$$\oint_{\gamma} \frac{1}{z^3 - 1} dz = \oint_{\gamma} \frac{1}{(z - 1)(z - e^{i2\pi/3})(z - e^{i4\pi/3})} dz.$$

Note that

$$\begin{aligned} & \frac{1}{(z - 1)(z - e^{i2\pi/3})(z - e^{i4\pi/3})} \\ &= \frac{1}{3} \frac{1}{z - 1} - \frac{e^{-i\pi/3}}{3} \frac{1}{z - e^{i2\pi/3}} + \frac{e^{i\pi/3}}{3} \frac{1}{z - e^{i4\pi/3}} \end{aligned}$$

Both functions  $1/(z - 1)$  and  $1/(z - e^{i4\pi/3})$  are holomorphic inside and on  $\gamma = \{z : |z - i| = 1\}$ . Therefore

$$\oint_{\gamma} \frac{1}{z - 1} dz = \oint_{\gamma_1} \frac{1}{z - e^{i4\pi/3}} dz = 0.$$

Since  $e^{i2\pi/3}$  is inside the domain bounded by  $\gamma$  we have

$$\oint_{\gamma} \frac{1}{z - e^{i2\pi/3}} dz = 2\pi i.$$

Thus

$$\oint_{\gamma} \frac{1}{z^3 - 1} dz = -\frac{e^{-i\pi/3}}{3} 2\pi i = -\frac{1 - i\sqrt{3}}{6} 2\pi i = -\frac{\pi(\sqrt{3} + i)}{3}.$$

3). a) No, b) Yes, c) No.

4).

$$\oint_{\gamma} \frac{e^z \sin z}{z - 5} dz = 2\pi i e^z \sin z \Big|_{z=5} = 2\pi i e^5 \sin 5.$$

5). Let  $\operatorname{Re} z > 0$ , choose any  $\beta > |z|$ . We consider the interval  $\gamma_1 = [i\beta, -i\beta]$ , half circle

$$\gamma_2 = \{z : z = \beta e^{i\theta}, -\pi/2 < \theta < \pi/2\}$$

and denote  $\gamma = \gamma_1 \cup \gamma_2$ . Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta - z} d\eta = \frac{1}{2\pi i} \int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z} d\eta + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\eta)}{\eta - z} d\eta.$$

Let  $r = \min_{\eta \in \gamma_2} |\eta - z|$ . Using our assumptions we find that if  $\eta \in \gamma_2$

$$\left| \frac{f(\eta)}{\eta - z} \right| = \left| \frac{f(\eta)\eta^k}{\eta^k(\eta - z)} \right| \leq \frac{M}{|\eta|^k} \frac{1}{|\eta - z|} \leq \frac{M}{\beta^k r}.$$

According to the ML-inequality

$$\left| \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\eta)}{\eta - z} d\eta \right| \leq \frac{M}{2\pi\beta^k r} \cdot \pi\beta = \frac{M}{2\beta^{k-1} r} \rightarrow 0,$$

as  $\beta \rightarrow \infty$ . Thus

$$f(z) = \frac{1}{2\pi i} \lim_{\beta \rightarrow \infty} \int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z} d\eta = -\frac{1}{2\pi i} \lim_{\beta \rightarrow \infty} \int_{-i\beta}^{i\beta} \frac{f(\eta)}{\eta - z} d\eta.$$

6). We use Cauchy's theorem

$$\begin{aligned}
 f(z_0) &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - z_0} dz \\
 &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - z_0} dz - \frac{1}{2i\pi} \oint_{|z|=1} \frac{f(z)}{z - \bar{z}_0^{-1}} dz \\
 &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{-\bar{z}_0^{-1} + z_0}{(z - z_0)(z - \bar{z}_0^{-1})} f(z) dz \\
 &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{-\bar{z}_0^{-1}(1 - |z_0|^2)}{(z - z_0)(z - \bar{z}_0^{-1})} f(z) dz \\
 &= \frac{1}{2i\pi} \oint_{|z|=1} \frac{(1 - |z_0|^2)}{(z - z_0)(1 - z\bar{z}_0)} f(z) dz.
 \end{aligned}$$

7)

Let

$$\gamma_1 = \{z : z = 2e^{i\varphi}, \varphi \in [0, \pi]\}, \quad \gamma_2 = \{z : z = x, -2 < x < -1\},$$

$$\gamma_3 = \{z : z = e^{i\varphi}, \varphi \in [\pi, 0]\}, \quad \gamma_4 = \{z : z = x, 1 < x < 2\}.$$

Then

$$\int_{\gamma_1} \frac{z}{\bar{z}} dz = \int_0^\pi \frac{2e^{i\varphi}}{2e^{-i\varphi}} 2i e^{i\varphi} d\varphi = 2i \int_0^\pi e^{3i\varphi} d\varphi = \frac{2}{3} (e^{3i\pi} - 1) = -\frac{4}{3},$$

$$\int_{\gamma_2} \frac{z}{\bar{z}} dz = \int_{-2}^{-1} dx = x \Big|_{-2}^{-1} = -1 - (-2) = 1,$$

$$\int_{\gamma_3} \frac{z}{\bar{z}} dz = \int_\pi^0 \frac{e^{i\varphi}}{e^{-i\varphi}} i e^{i\varphi} d\varphi = i \int_\pi^0 e^{3i\varphi} d\varphi = \frac{1}{3} (1 - e^{3i\pi}) = \frac{2}{3},$$

$$\int_{\gamma_4} \frac{z}{\bar{z}} dz = \int_1^2 dx = x \Big|_1^2 = 1.$$

Finally we have

$$\int_\gamma \frac{z}{\bar{z}} dz = \sum_{j=1}^4 \int_{\gamma_j} \frac{z}{\bar{z}} dz = -\frac{4}{3} + 1 + \frac{2}{3} + 1 = \frac{4}{3}.$$

**8a)**

Let us introduce curves  $\gamma_n = \{z : |z - n| = 1/2\}$ ,  $n = 0, 1, \dots, k$ . Then by using the Deformation Theorem we have

$$I_k = \oint_{\gamma} \frac{dz}{z(z-1)\dots(z-k)} dz = \sum_{n=0}^k \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz.$$

Note that for any  $n : 0 < n < k$  the Cauchy's integral formula implies

$$\oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz = 2\pi i \frac{1}{n} \dots \frac{1}{1} \cdot \frac{1}{-1} \dots \frac{1}{n-k} = 2\pi i \frac{(-1)^{k-n}}{n!(k-n)!},$$

where the cases  $n = 0$  and  $n = k$  are treated similarly. Thus

$$\sum_{n=0}^k \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)} dz = 2\pi i \sum_{n=0}^k \frac{(-1)^{k-n}}{n!(k-n)!}.$$

**8b)**

Using Cauchy's integral formula we obtain (for  $k \geq 1$ )

$$J_k = \oint_{\gamma} \frac{(z-1)(z-2)\dots(z-k)}{z} dz = 2\pi i (-1)^k k!.$$

**9. Step 1.** We first prove that if  $\gamma \subset \mathbb{C}$  is a simple closed piecewise-smooth curve bounding  $\Omega$  and  $f$  and  $f'_z$  are continuous inside and on  $\gamma$  then

$$\oint_{\gamma} f(z) dz = 2i \iint_{\Omega} \frac{\partial f(z)}{\partial \bar{z}} dx dy.$$

Indeed, this is a direct corollary of Green's theorem

$$\begin{aligned} \oint_{\gamma} f(z) dz &= \oint_{\gamma} (u + iv) d(x + iy) = \oint_{\gamma} u dx - v dy + i \oint_{\gamma} v dx + u dy \\ &= \iint_{\Omega} (-v'_x - u'_y + i u'_x - i v'_y) dx dy \\ &= 2i \iint_{\Omega} \frac{1}{2} \left( u'_x - \frac{1}{i} u'_y \right) + \frac{i}{2} \left( v'_x - \frac{1}{i} v'_y \right) dx dy \\ &= 2i \iint_{\Omega} \frac{d}{d\bar{z}} (u + iv) dx dy = 2i \iint_{\Omega} \frac{d}{d\bar{z}} f(z) dx dy. \end{aligned}$$

**Step 2.** Let  $r > 0$  is small enough, so that  $B_r(z_0) = \{z : |z - z_0| \leq r\} \subset D$ . Using the same argument as in the proof of the Deformation theorem we

find

$$\oint_{|z-z_0|=1} \frac{f(z)}{z-z_0} dz - \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz$$

$$= 2i \iint_{D \setminus B_r(z_0)} \frac{d}{d\bar{z}} \left( \frac{f(z)}{z-z_0} \right) dx dy = 2i \iint_{D \setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0} dx dy,$$

where we have used the fact that  $1/(z-z_0)$  is holomorphic in  $D \setminus B_r(z_0)$  and therefore  $d(z-z_0)^{-1}/d\bar{z} = 0$ . It remains to note that

$$\lim_{r \rightarrow 0} \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz = \lim_{r \rightarrow 0} \oint_{|z-z_0|=r} \frac{f(z_0) + f(z) - f(z_0)}{z-z_0} dz = 2\pi i f(z_0)$$

and that

$$\lim_{r \rightarrow 0} \iint_{D \setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0} dx dy = \iint_D \frac{df(z)/d\bar{z}}{z-z_0} dx dy.$$