Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 17

More series tests, and rearrangements

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We call a sequence a_n *alternating* if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ $\forall n$ (or the opposite).

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Theorem 4.20 - Alternating Series Test

Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

$|a_n|$ monotonically decrease to 0

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Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. Consider the partial sums $s_n = \sum_{i=1}^{n} (-1)^i b_i$.

$\sum \frac{(-1)^n}{n}$ $\quad b_n = \frac{1}{n}$

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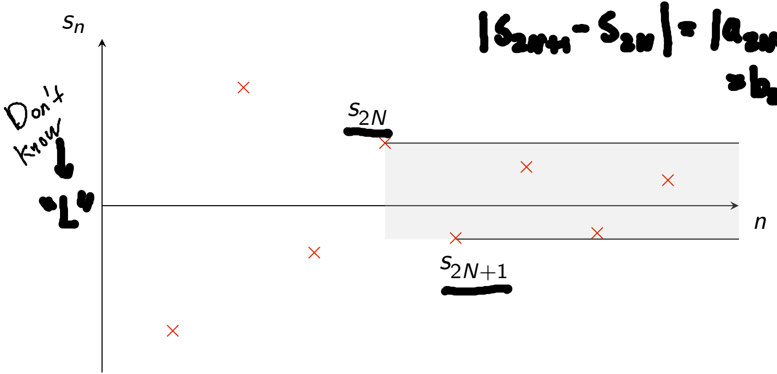
$|s_{2N+1} - s_{2N}| = |a_{2N+1}|$
$= b_{2N+1}$



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Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$.

Fix n WTS that for $k \geq 2n+1$, $S_{2n+1} \leq S_k \leq S_{2n}$

Showing $S_k \leq S_{2n}$: Suppose $k = 2j$ is even

$$S_{2j} = S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \underbrace{(-b_{2n+3} + b_{2n+4})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0}$$

So $S_{2j} \leq S_{2n}$. For $k = 2j+1$, $S_{2j+1} = S_{2j} - b_{2j+1} \leq S_{2n}$

Showing $S_{2n+1} \leq S_K$ Exercise

Now, we claim S_j is Cauchy. Let $\varepsilon > 0$. $\exists N$ s.t $|b_{2N+1}| < \varepsilon$

But then $\forall\, n, m \geq 2N+1$

$$S_{2N+1} \leq S_n, S_m \leq S_{2N}, \text{ so } |S_n - S_m| \leq |S_{2N+1} - S_{2N}| = b_{2N+1} < \varepsilon.$$

∎

Theorem 4.20 - Alternating Series Test

Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. We claim

(1) $s_i \leq s_{2n} \ \forall i \geq 2n$,

(2) $s_i \geq s_{2n+1} \ \forall i \geq 2n+1$.

Exercise 4.21

What do you think about the infinite sum

$$\left(1-\frac{1}{2}\right)-\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{5}\right)-\frac{1}{6}+\left(\frac{1}{7}-\frac{1}{8}\right)-\frac{1}{9}+\left(\frac{1}{10}-\ldots\right)\ldots?$$

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$
5. Other

$\sum(\;)$ convergent series

$-\frac{1}{3}\sum\frac{1}{n}$

Exercise 4.22

The alternating sequence $a_n = \begin{cases} \frac{1}{n^2} + \frac{1}{n} & n \text{ even}, \\ -\frac{1}{n^2} & n \text{ odd}, \end{cases}$

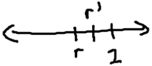
has sum $\sum a_n$ which is

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$ ✓
4. Divergent to $-\infty$
5. Other

Theorem 4.23 - Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Proof



Let $\varepsilon = \frac{1-r}{2}$, then $\exists N$ s.t $\forall n \geq N$,

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| < \varepsilon \implies \left| \frac{a_{n+1}}{a_n} \right| < r + \varepsilon = \frac{1+r}{2} = r'$$

Then for $n \geq N$, $|a_n| \leq (r')^{n-N} |a_N|$

Then compare $\sum_{n \geq N} |a_n|$ with $\sum_{n \geq N} (r')^{n-N} |a_N|$

So $\sum_{n \geq N} |a_n|$ convergent

$$= (r')^{-N} |a_N| \underbrace{\sum_{n \geq N} (r')^n}_{\text{convergent geometric series}}$$

So $\sum a_n$ is absolutely convergent.

Theorem 4.23 - Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Example 4.25

Let

$$a_n = \frac{100^n(\cos n\theta + i \sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge?

Theorem 4.23 - Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

$\to 1$

Inconclusive

Example 4.25

Let

$$a_n = \frac{100^n(\cos n\theta + i \sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge?

Remark 4.24

The ratio test only applies when a_n decays at least exponentially in n. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.