Imperial College London

MATH70029

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Functional Analysis

Date: 30 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Let $(X, \|\cdot\|)$ be a normed linear space.

(a) Which additional property is required for $(X, \|\cdot\|)$ to be a Banach space? (4 marks)

(b) Show that a Cauchy sequence in $(X, \|\cdot\|)$ is convergent if and only if it has a convergent subsequence. (4 marks)

(c) Suppose $(X, \|\cdot\|)$ has the following property: for every sequence $(x_n) \subset X$ with

$$\sum_{n=1}^{\infty} \|x_n\| < \infty,$$

the sequence (s_n) with $s_n = \sum_{k=1}^{n} x_k$ converges in $(X, \|\cdot\|)$. Show that $(X, \|\cdot\|)$ is a Banach space. *Hint:* use (a) and (b). (12 marks)

(Total: 20 marks)

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces.

(a) Let $A : D_A \subset X \to Y$ be a linear operator with graph $\Gamma_A \subset X \times Y$. Then

$$\|x\|_{\Gamma_A} \overset{\text{def.}}{=} \|x\|_X + \|Ax\|_Y$$

defined on D_A is called the graph norm. Show that if A has closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space. (8 marks)

(b) Let $(Z, \|\cdot\|_Z)$ be a third Banach space and let

$$T_1 : D_1 \subset X \to Y,$$
$$T_2 : D_2 \subset X \to Z$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that there exists $C \in (0, \infty)$ such that for all $x \in D_1$,

$$\|T_2 x\|_Z \leq C\Big(\|T_1 x\|_Y + \|x\|_X\Big).$$

Hint: choose a suitable map between the graphs. (12 marks)

(Total: 20 marks)

3. (a) Let H be an infinite-dimensional Hilbert space over $\mathbb{R}$, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let (e_n) be an orthonormal sequence in H.

(i) Show that for any $x \in H$,

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2.$$

(8 marks)

(ii) Show that (e_n) converges weakly and determine its limit. (6 marks)

(b) Let $f_n : [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Show that $f_n \xrightarrow{w} 0$ in $L^2[0, 2\pi]$ as $n \to \infty$. (6 marks)

(Total: 20 marks)

4. (a) Let X, Y, Z be Banach spaces, endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, respectively.

(i) Give the definition of a compact operator $T : X \to Y$. Be sure to specify the meaning of the word "compact." (4 marks)

(ii) Let $T : X \to Y$ be a compact operator, $S : Y \to Z$ be a bounded linear operator. Prove that the composition

$$S \circ T : X \to Z,$$

i.e. $(S \circ T)(x) = S(T(x))$ for all $x \in X$, is a compact operator. (8 marks)

(b) Consider the linear operator $T : L^2(0, 1) \to L^2(0, 1)$ defined by

$$(Tf)(x) = \sqrt{x} f(x), \text{ for all } f \in L^2(0,1) \text{ and } x \in (0,1).$$

Is T compact? Justify your answer. *Hint:* you may wish to consider the functions

$$f_n(x) = \begin{cases} n^\alpha, & \text{for } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

for suitable choice of α and $n \in \mathbb{N}$. (8 marks)

(Total: 20 marks)

5. Let $\mathbb{S}^1$ denote the unit circle and $H \subset L^2(\mathbb{S}^1)$ be given by $H = \mathrm{im}(P)$ $(= \{Pf : f \in L^2(\mathbb{S}^1)\})$, where $P : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the projection operator given by

$$(Pf)(\theta) \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}, \quad \text{for all } \theta \in [0, 2\pi].$$

with $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta$ for $n \in \mathbb{Z}$. Given any $\varphi \in C = C^0(\mathbb{S}^1)$, the space of continuous functions on $\mathbb{S}^1$, define $T_\varphi : H \to H$ by

$$T_\varphi(f) \stackrel{\text{def.}}{=} P(\varphi f), \text{ for all } f \in H.$$

(a) By explicit computation, for $\varphi(\theta) = E_k(\theta) = e^{ik\theta}$ show that $T_{E_k} T_{E_l} - T_{E_k E_l}$ is a compact operator on H for every $k, l \in \mathbb{Z}$.

(6 marks)

(b) For every $\varphi, \psi \in C$, show that $T_\varphi T_\psi - T_{\varphi\psi}$ is a compact operator on H. *Hint:* approximate φ and ψ by linear combinations of exponentials.

(8 marks)

(c) Deduce that if $\varphi \in C$ is nowhere vanishing then T_φ is a Fredholm operator. (6 marks)

(Total: 20 marks)

Module:	MATH60029/MATH70029
Setter:	PF Rodriguez
Checker:	A Chandra
Editor:	editor
External:	external
Date:	April 20, 2023
Version:	Draft version for checking

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2023

MATH60029/MATH70029 Functional Analysis

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 4 marks; 1(b) 4 marks; 2(a) 8 marks; 3(a)(ii) 6 marks; 3(b) 6 marks; 4(a)(i) 4 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(c) 12 marks; 4(a)(ii) 8 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(b) 12 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
3(a) 8 marks; 4(b) 8 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Functional Analysis

Date: ??
Time: ??
Time Allowed: 2 Hours for MATH60029 paper; 2.5 Hours for MATH70029 paper
This paper has *4 Questions (MATH60029 version); 5 Questions (MATH70029 version)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Let $(X, \|\cdot\|)$ be a normed linear space.

(a) (SEEN) Which additional property is required for $(X, \|\cdot\|)$ to be a Banach space? (4 marks)

Solution: The normed space $(X, \|\cdot\|)$ must be complete in the norm topology, i.e. every Cauchy sequence in $(X, \|\cdot\|)$ converges in X.

(b) (SEEN) Show that a Cauchy sequence in $(X, \|\cdot\|)$ is convergent if and only if it has a convergent subsequence. (4 marks)

Solution: One implication is trivial. Assume now that $(x_n) \subset X$ is Cauchy and has a convergent subsequence, say (x_{n_k}), with limit point x. We show that (x_n) converges to x. Let $\varepsilon > 0$. By the Cauchy property, one finds N such that for all $n, m \geq N$, $\|x_n - x_m\| \leq \frac{\varepsilon}{2}$. By convergence of the subsequence one finds K large enough such that $n_K \geq N$ and $\|x_{n_K} - x\| \leq \frac{\varepsilon}{2}$. This yields, for all $n \geq N$,

$$\|x_n - x\| \leq \|x_n - x_{n_K}\| + \|x_{n_K} - x\| \leq \varepsilon,$$

as desired.

(c) (SEEN SIMILAR) Suppose $(X, \|\cdot\|)$ has the following property: for every sequence $(x_n) \subset X$ with

$$\sum_{n=1}^{\infty} \|x_n\| < \infty,$$

the sequence (s_n) with $s_n = \sum_{k=1}^{n} x_k$ converges in $(X, \|\cdot\|)$. Show that $(X, \|\cdot\|)$ is a Banach space. *Hint:* use (a) and (b). (12 marks)

Solution: Suppose that $(y_n) \subset X$ is Cauchy. Then for all $k \geq 1$ there exists and integer $n_k \geq 1$ such that

$$\text{for all } n, m \geq n_k\text{: } \|y_n - y_m\| \leq 2^{-k}.$$

We may further assume that $n_{k+1} > n_k$. Let $x_k = y_{n_{k+1}} - y_{n_k}$. Then

$$\sum_{k=1}^{\infty} \|x_k\| = \sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty,$$

which by assumption implies that

$$s_k = \sum_{\ell=1}^{k} x_\ell = y_{n_{k+1}} - y_{n_1}$$

converges in X as $k \to \infty$. Hence (y_{n_k}) is a convergence subsequence of (y_n). Since (y_n) is Cauchy it follows by part (b) that (y_n) is convergent. Thus, X is complete.

(Total: 20 marks)

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces.

(a) (SEEN SIMILAR) Let $A : D_A \subset X \to Y$ be a linear operator with graph $\Gamma_A \subset X \times Y$. Then

$$\|x\|_{\Gamma_A} \stackrel{\text{def.}}{=} \|x\|_X + \|Ax\|_Y$$

defined on D_A is called the graph norm. Show that if A has closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space. (8 marks)

Solution: Let (x_n) be a Cauchy sequence in $(D_A, \|\cdot\|_{\Gamma_A})$. The definition of the graph norm implies that both (x_n) is Cauchy in $(X, \|\cdot\|_X)$ and (Ax_n) is Cauchy in $(Y, \|\cdot\|_Y)$. Since both spaces are complete by assumption, there exist $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|Ax_n - y\|_Y \to 0$ as $n \to \infty$. Since the graph of A is closed, it follows that $x \in D_A$ with $Ax = y$. Overall, this yields that $\|x_n - x\|_{\Gamma_A} \to 0$ as $n \to \infty$, which proves that $(D_A, \|\cdot\|_{\Gamma_A})$ is complete.

(b) (UNSEEN) Let $(Z, \|\cdot\|_Z)$ be a third Banach space and let

$$T_1 : D_1 \subset X \to Y,$$
$$T_2 : D_2 \subset X \to Z$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that there exists $C \in (0, \infty)$ such that for all $x \in D_1$,

$$\|T_2 x\|_Z \leq C\Big(\|T_1 x\|_Y + \|x\|_X\Big).$$

Hint: choose a suitable map between the graphs. . (12 marks)

Solution: Let Γ_1 and Γ_2 be the graphs of T_1 and T_2, respectively. Since T_1 and T_2 have closed graphs by assumption, (a) implies that $(D_1, \|\cdot\|_{\Gamma_1})$ and $(D_2, \|\cdot\|_{\Gamma_2})$ are Banach spaces. Since $D_1 \subset D_2$, we can consider the identity map

$$\text{id} : (D_1, \|\cdot\|_{\Gamma_1}) \to (D_2, \|\cdot\|_{\Gamma_2}).$$

We claim that the graph of id is closed. Indeed, assume that $(x_n) \subset D_1$ and $\|x_n - x\|_{\Gamma_1} \to 0$ as $n \to \infty$ for some $x \in D_1$ and $\|\text{id}(x_n) - y\|_{\Gamma_2} = \|x_n - y\|_{\Gamma_2} \to 0$ as $n \to \infty$ for some $y \in D_2$. The definition of the graph norm implies that both $\|x_n - x\|_X \to 0$ and $\|x_n - y\|_X \to 0$ as $n \to \infty$, which implies that $x = y$ and proves the claim. The closed graph theorem now applies and implies that id is continuous, or equivalently, bounded, which means that there exists $C \in (0, \infty)$ such that for all $x \in D_1$,

$$\|x\|_{\Gamma_2} \leq C\|x\|_{\Gamma_1}.$$

By definition of the graph norm, this implies

$$\|T_2 x\|_Z \leq C\Big(\|T_1 x\|_Y + \|x\|_X\Big) - \|x\|_X.$$

(Total: 20 marks)

3. (a) Let H be an infinite-dimensional Hilbert space over $\mathbb{R}$, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let (e_n) be an orthonormal sequence in H.

(i) (UNSEEN) Show that for any $x \in H$,

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

(8 marks)

Solution: One considers

$$x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k,$$

and notes that the left-hand side of the inequality is simply $\lim_n \|x_n\|^2$. Using orthonormality of the e_k's, one further finds that

$$\langle x - x_n, x_n \rangle = 0.$$

Thus, using Pythagoreas' theorem, one has that

$$\|x\|^2 = \|x - x_n\|^2 + \|x_n\|^2 \geq \|x_n\|^2,$$

from which the desired bound follows by letting $n \to \infty$.

(ii) (SEEN SIMILAR) Show that (e_n) converges weakly and determine its limit. (6 marks)

Solution: Fix any $x \neq 0$ in H. Then the inequality in (a) implies in particular that

$$|\langle x, e_n \rangle| \to 0 \text{ as } n \to \infty.$$

By Riesz' representation theorem, any bounded linear functional $\ell \in H^*$ is of the form $\ell(\cdot) = \langle x, \cdot \rangle$ for some $x \in H$. So the previous display precisely entails that $e_n \overset{w}{\to} 0$.

(b) (SEEN SIMILAR) Let $f_n : [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Show that $f_n \overset{w}{\to} 0$ in $L^2[0, 2\pi]$ as $n \to \infty$. (6 marks)

Solution: For any $m, n \in \mathbb{N}$, one has

$$\int_0^{2\pi} \sin(nt)\sin(mt)dt = \frac{1}{2}\int_0^{2\pi} \Big(\cos((m-n)t) - \cos((m+n)t)\Big)dt = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

Thus, $(f_n/\sqrt{\pi})$ is an orthonormal system in the Hilbert space $L^2([0, 2\pi], \mathbb{R})$ and the claim follows by applying part (a),(ii).

(Total: 20 marks)

4. (a) Let X, Y, Z be Banach spaces, endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, respectively.

(i) Give the definition of a compact operator $T : X \to Y$. Be sure to specify the meaning of the word "compact." (4 marks)

Solution: A bounded linear operator $T : X \to Y$ is compact if for any bounded set A in $(X, \|\cdot\|_X)$ we have that $\overline{T(A)}$ is compact, where $\overline{B}$ refers to the $\|\cdot\|_Y$-closure of B. Compactness refers to the sequential compactness of the space $(\overline{T(A)}, \|\cdot\|_Y)$.

(ii) Let $T : X \to Y$ be a compact operator, $S : Y \to Z$ be a bounded linear operator. Prove that the composition

$$S \circ T : X \to Z,$$

i.e. $(S \circ T)(x) = S(T(x))$ for all $x \in X$, is a compact operator. (8 marks)

Solution: As seen in the lectures, T is compact if and only if for every bounded sequence (x_n) of X, we have that (Tx_n) has a convergent subsequence in Y. With this characterization, the claim is almost immediate. Let (x_n) be as above and (Tx_{n_k}) the corresponding converging subsequence in Y. Since S is bounded, S is continuous, thus $((S \circ T)x_{n_k})$ is convergent in Z. Using the afore characterization again, it follows that $S \circ T$ is compact.

(b) Consider the linear operator $T : L^2(0,1) \to L^2(0,1)$ defined by

$$(Tf)(x) = \sqrt{x} f(x), \text{ for all } f \in L^2(0,1) \text{ and } x \in (0,1).$$

Is T compact? Justify your answer. *Hint:* you may wish to consider the functions

$$f_n(x) = \begin{cases} n^\alpha, & \text{for } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\ 0 & \text{otherwise} \end{cases}$$

for suitable choice of α and $n \in \mathbb{N}$. (8 marks)

Solution: We choose $\alpha = \frac{1}{2}$. Then a simple calculation shows that $f_n \in L^2(0,1)$ for every n with $\|f_n\|_{L^2(0,1)} = 1$. Thus $A = \{f_n : n \in \mathbb{N}\}$ is a bounded set. We show that no subsequence of (Tf_n) is Cauchy, which implies that T is not compact. Let $n \geq m$. Then

$$Tf_n - Tf_m = \begin{cases} \sqrt{x}(\sqrt{n} - \sqrt{m}), & \text{if } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}) \\ -\sqrt{xm}, & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{1}{m}) \\ 0, & \text{else.} \end{cases}$$

Hence

$$\begin{aligned} \|Tf_n - Tf_m\|_{L^2(0,1)} &= (\sqrt{n} - \sqrt{m})^2 \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} x dx + m \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} x dx \\ &\geq \frac{(\sqrt{n} - \sqrt{m})^2}{2} \left(\frac{1}{n^2} + \frac{1}{n} \right) = \left(1 - 2\sqrt{\frac{m}{n}} \right) \left(\frac{1}{2} + \frac{1}{2n} \right) \end{aligned}$$

For any given $N \in \mathbb{N}$ and subsequence (n_k) we can choose $n = n_k \geq N$ and $m = n_{k'} \geq N$ such that $n \geq m^2$. Then $\sqrt{\frac{m}{n}} \leq \frac{1}{\sqrt{m}}$ and it follows that

$$\inf_{N\to\infty} \sup_{n_k, n_{k'} \geq N} \|Tf_{n_k} - Tf_{n_{k'}}\|_{L^2(0,1)} \geq \frac{1}{2}$$

(in fact a more careful but unnecessary calculation shows that the limit exists and equals 1). This implies that (Tf_{n_k}) cannot be Cauchy.

(Total: 20 marks)

5. Let $\mathbb{S}^1$ denote the unit circle and $H \subset L^2(\mathbb{S}^1)$ be given by $H = \mathrm{im}(P)$ $(= \{Pf : f \in L^2(\mathbb{S}^1)\})$, where $P : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the projection operator given by

$$(Pf)(\theta) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}, \quad \text{for all } \theta \in [0, 2\pi].$$

with $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$ for $n \in \mathbb{Z}$. Given any $\varphi \in C = C^0(\mathbb{S}^1)$, the space of continuous functions on $\mathbb{S}^1$, define $T_\varphi : H \to H$ by

$$T_\varphi(f) \overset{\text{def.}}{=} P(\varphi f), \text{ for all } f \in H.$$

(a) By explicit computation, for $\varphi(\theta) = E_k(\theta) = e^{ik\theta}$ show that $T_{E_k}T_{E_l} - T_{E_kE_l}$ is a compact operator on H for every $k, l \in \mathbb{Z}$.

(6 marks)

Solution: By direct computation we obtain that for all $\theta \in [0, 2\pi]$,

$$(T_{E_k}T_{E_l})(f)(\theta) = \sum_{n=k}^{\infty} \hat{f}(n-(k+l))e^{in\theta},$$
$$(T_{E_kE_l})(f)(\theta) = \sum_{n=0}^{\infty} \hat{f}(n-(k+l))e^{in\theta}.$$

Hence

$$(T_{E_k}T_{E_l} - T_{E_kE_l})(f)(\theta) = \begin{cases} -\sum_{n=0}^{k-1} \hat{f}(n-(k+l))e^{in\theta}, & \text{if } k > 0 \\ 0, & \text{if } k \le 0. \end{cases}$$

Thus $T_{E_k}T_{E_l} - T_{E_kE_l}$ has finite rank, hence it is compact.

(b) For every $\varphi, \psi \in C$, show that $T_\varphi T_\psi - T_{\varphi\psi}$ is a compact operator on H. *Hint:* approximate φ and ψ by linear combinations of exponentials.

(8 marks)

Solution: By density we can find sequences (φ_j), (ψ_j) of the form

$$\varphi_j(\theta) = \sum_{k=1}^{N_j} a_k e^{in_k\theta},$$
$$\psi_j(\theta) = \sum_{k=1}^{M_j} b_k e^{im_k\theta},$$

such that $\varphi_j \to \varphi$ and $\psi_j \to \psi$ uniformly on $[0, 2\pi]$ as $j \to \infty$. Observe that

$$T_{\varphi_j}T_{\psi_j} - T_{\varphi_j\psi_j} = \sum_{k=1}^{N_j}\sum_{\ell=1}^{M_j} a_k b_\ell (T_{E_{n_k}}T_{E_{m_\ell}} - T_{E_{n_k}E_{m_\ell}}).$$

By (a) this operator is compact for every j.

Moreover,

$$
\begin{aligned}
&\|(T_{\varphi_j} - T_{\varphi_j\psi_j}) - T_\varphi T_\psi - T_{\varphi\psi}\|_{L(H)} \\
&\qquad \le \|(T_{\varphi_j}T_{\psi_j} - T_{\varphi_j})T_{\psi_j}\|_{L(H)} + \|T_\varphi(T_{\psi_j} - T_\psi)\|_{L(H)} + \|T_{\psi_j\varphi_j} - T_{\psi\varphi}\|_{L(H)} \\
&\qquad = \|T_{\varphi_j-\varphi}T_{\psi_j}\|_{L(H)} + \|T_\varphi T_{\psi_j-\psi}\|_{L(H)} + \|T_{\psi_j\varphi_j-\psi\varphi}\|_{L(H)} \\
&\qquad \le \|\varphi_j - \varphi\|_\infty \cdot \|\psi_j\|_\infty + \|\psi_j - \psi\|_\infty \cdot \|\varphi\|_\infty + \|\psi_j\varphi_j - \psi\varphi\|_\infty,
\end{aligned}
$$

which tends to 0 as $j \to \infty$. Thus, $T_\varphi T_\psi - T_{\varphi\psi}$ is compact as a limit of a sequence of compact operators.

(c) Deduce that if $\varphi \in C$ is nowhere vanishing then T_φ is a Fredholm operator. (6 marks)

Solution: Let $\psi(\theta) \overset{\text{def.}}{=} \varphi(\theta)^{-1}$ for all $\theta \in [0, 2\pi]$, which is well-defined by assumption on φ. By (b), one knows that

$$
\begin{aligned}
T_\varphi T_\psi - T_{\varphi\psi} &= T_\varphi T_\psi - \mathrm{id}_H, \\
T_\psi T_\varphi - T_{\psi\varphi} &= T_\psi T_\varphi - \mathrm{id}_H
\end{aligned}
$$

are both compact. It now follows from the characterization of Fredholm operators as "invertible modulo compact operators" (see for instance Theorem 4.3.8 in Salamon-Bühler, which was part of the reading assignment for the Mastery Question) that T_φ is Fredholm.

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60029/70029	1	1 (a) (b): solved mostly well1 (c): some students struggle to extract a subsequence satisfying the required assumption (finiteness of the sum of norms)
MATH60029/70029	2	2 (a): solved mostly well2 (b): as with numerous examples in class, this question becomes simple upon defining a suitable "identity map"
MATH60029/70029	3	Solved well but important to notice that (e_n) need not be a basis.
MATH60029/70029	4	4 (a) good4 (b) most found alpha such that boundedness of (T f_n) is guaranteed. Many struggled to formalize that no subsequence of (T f_n) can be Cauchy
MATH70029	5	No Comments Received