Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2023-2024

Partial Differential Equations in Action **MATH50008**

Solutions to Midterm Exam

1. **Total: 10 Marks** Consider the following first-order PDE

$$2\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = e^{x+5y} - u \quad \text{with} \quad x, y \in \mathbb{R}$$

(a) First, we use the method of characteristics which we can write for this problem

$$\frac{du}{dy} = -\frac{1}{2}\left(e^{x+5y} - u\right) \quad \text{on} \quad \frac{dx}{dy} = -1$$

By integration, we obtain

$$x = -y + c_1$$

where c_1 is an integration constant to be determined. Further, we rewrite the second ODE as

$$\frac{du}{dy} = -\frac{1}{2}\left(e^{x+5y} - u\right) \Rightarrow \frac{du}{dy} - \frac{u}{2} = -\frac{1}{2}e^{4y+c_1}$$

where we have used the fact that we are working on the characteristic with equation $x+y = c_1$. For this, we can use an integrating factor and write

$$\begin{aligned} u &= e^{y/2}\left(\int \left(-\frac{1}{2}e^{4y+c_1}\right) e^{-y/2} dy\right) + c_2 e^{y/2} \\ &= -\frac{1}{2}e^{y/2+c_1}\int e^{7y/2}\, dy + c_2 e^{y/2} \\ &= -\frac{1}{7}e^{4y+c_1} + c_2 e^{y/2} \end{aligned}$$

so relating our integration constant c_1 and c_2 such that $c_2 = f(c_1)$ and using the fact that $c_1 = x + y$, we finally obtain the desired general solution

$$u(x, y) = f(x + y)e^{y/2} - \frac{1}{7}e^{x+5y}$$

with f an arbitrary function to be determined.

6 Marks

(b) To find the particular solution to this PDE, we use the fact that $u(x, 0) = 0$ for all $x \in \mathbb{R}$. This translates to

$$u(x, 0) = 0 \Rightarrow f(x) - \frac{1}{7}e^{x} = 0 \Rightarrow f(x) = \frac{1}{7}e^{x}$$

and we conclude that the particular solution is finally given by

$$u(x, y) = \frac{1}{7}e^{x+3y/2} - \frac{1}{7}e^{x+5y} = \frac{1}{7}e^{x+3y/2}\left(1 - e^{7y/2}\right)$$

4 Marks

2. **Total: 20 Marks** Here, we consider Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

where we have assumed that the initial conditions were given by

$$u(x,0) = \begin{cases} 3, & x < 0 \\ 3(1 - x/6), & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

(a) The initial conditions are given on Fig. 1.

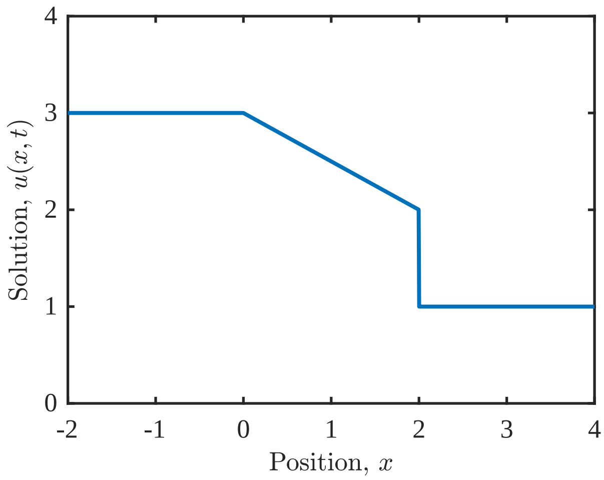

Figure 1: Sketch of the initial conditions for Q2.

2 Marks

(b) The method of characteristics gives us here that

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u, \quad x(0) = \xi$$

which means that

$$u = u(\xi, 0) \quad \text{on} \quad x = u(\xi, 0)t + \xi$$

So based on the initial conditions, we obtain the following equation for the characteristics

$$\begin{cases} \text{I} — \xi < 0: & x = 3t + \xi \\ \text{II} — 0 < \xi < 2: & x = 3(1 - \xi/6)t + \xi \\ \text{III} — \xi > 2: & x = t + \xi \end{cases}$$

2 Marks

The diagram of characteristics is given on Fig. 2.

2 Marks

To show that a shock is present for all $t > 0$, simply notice that the characteristics from region II are crossing the characteristics from region III even for $t \to 0^+$. This is simply due to the fact that the slope of the characteristics from region III is 1, while the slope of say the right most characteristic in region II is $1/2$.

1 Mark

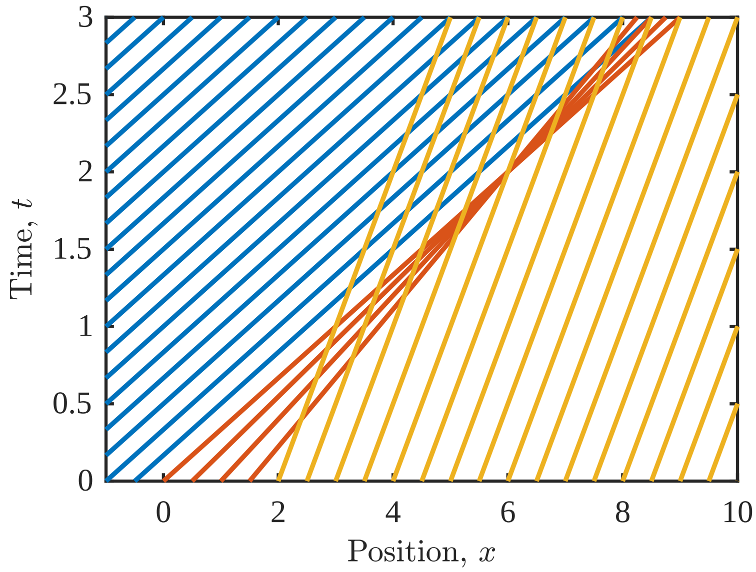

Figure 2: Diagram of characteristics with characteristics from region I (blue), region II (orange), region III (yellow). This diagram does not show any rarefaction fan. Characteristics from the region II and III cross for all times $t > 0$, leading to shock formation.

(c) To find an explicit solution $u(x,t)$, we first need to deal with the shock! To do so, we want to apply the RH jump condition which requires the knowledge of the solution on either side of the shock. We easily have the solution in region III as along the yellow characteristics, $u(x,t) = 1$. If we forget that the shock exists, we can write the explicit solution in region II as follows. The method of characteristics in region II gave us

$$u = 3(1 - \xi/6) \quad \text{on} \quad x = 3(1 - \xi/6)t + \xi$$

In particular, we conclude that $\xi = (x - 3t)/(1 - t/2)$, which means that

$$u(x,t) = 3\left(1 - \frac{1}{6}\frac{2x - 6t}{2 - t}\right) = \frac{6 - x}{2 - t}$$

If we denote $s(t)$ the position of the shock, we thus have

After the shock: $\quad u_+ = 1 \quad$ (which comes from region III)

Before the shock: $\quad u_- = \dfrac{6 - s(t)}{2 - t} \quad$ (which comes from region II)

The Rankine-Hugoniot jump condition reads

$$\frac{ds}{dt} = \frac{[u^2/2]}{[u]} = \frac{1}{2}(u_+ + u_-) = \frac{1}{2}\left[1 + \frac{6 - s(t)}{2 - t}\right]$$

this finally leads to the ODE

$$\frac{ds}{dt} + \frac{s}{4 - 2t} = \frac{1}{2} + \frac{3}{2 - t}$$

subject to the initial condition $s(0) = 2$.

3 Marks

We can integrate this equation with the integration factor

$$\int \frac{dt}{4 - 2t} = -\frac{1}{2}\ln(4 - 2t) = -\ln\sqrt{4 - 2t}$$

leading to

$$
\begin{aligned}
s(t) &= \exp[\ln\sqrt{4-2t}]\left[\int\left(\frac{1}{2}+\frac{3}{2-t}\right)\exp[-\ln\sqrt{4-2t}]dt\right] + C\exp[\ln\sqrt{4-2t}] \\
&= \sqrt{4-2t}\left[\frac{1}{2}\int\frac{dt}{\sqrt{4-2t}} + 3\int\frac{dt}{(2-t)\sqrt{4-2t}}\right] + C\sqrt{4-2t} \\
&= \sqrt{4-2t}\left[\frac{1}{2}\int\frac{dt}{\sqrt{4-2t}} + \frac{3}{\sqrt{2}}\int\frac{dt}{(2-t)^{3/2}}\right] + C\sqrt{4-2t} \\
&= \sqrt{4-2t}\left[-\frac{1}{2}\sqrt{4-2t} + \frac{3}{\sqrt{2}}\frac{2}{\sqrt{2-t}}\right] + C\sqrt{4-2t} \\
&= -\frac{4-2t}{2} + 6 + C\sqrt{4-2t} \\
&= 4 + t + C\sqrt{4-2t}
\end{aligned}
$$

where C is an integration constant to be determined. We use the initial condition above to write $2 = 4 + C\sqrt{2} \Rightarrow C = -\sqrt{2}$ and so conclude that

$$s(t) = 4 + t - \sqrt{4-2t}$$

2 Marks

The explicit solution can thus be written as follows

$$
u(x,t) = \begin{cases} 3, & x < 3t \\ (6-x)/(2-t), & 3t \le x < 4 + t - \sqrt{4-2t} \\ 1, & x \ge 4 + t - \sqrt{4-2t} \end{cases}
$$

2 Marks

This solution is valid until the moment where the shock path encounters the characteristics from region I; this condition can be written as $s(t) = 3t$, i.e.

$$4 + t - \sqrt{4-2t} = 3t \Rightarrow t = 3/2 \quad \text{or} \quad t = 2$$

So our solution is only valid until $t_s = 3/2$ at which point the RH condition changes.

1 Mark

(d) For $t > t_s$, we need to change the RH condition as the solution on the left of the shock is now given by the solution in region I. We thus have

After the shock: $u_+ = 1$ (which comes from region III)
Before the shock: $u_- = 3$ (which comes from region I)

The Rankine-Hugoniot jump condition reads

$$\frac{ds}{dt} = \frac{1}{2}(u_+ + u_-) = 2 \Rightarrow s(t) = 2t + C$$

where C is an integration constant which we can determine with $s(3/2) = 4 + 3/2 - 4\sqrt{4 - 2(3/2)} = 9/2$. We conclude that

$$s(t) = \frac{3}{2} + 2t$$

2 Marks

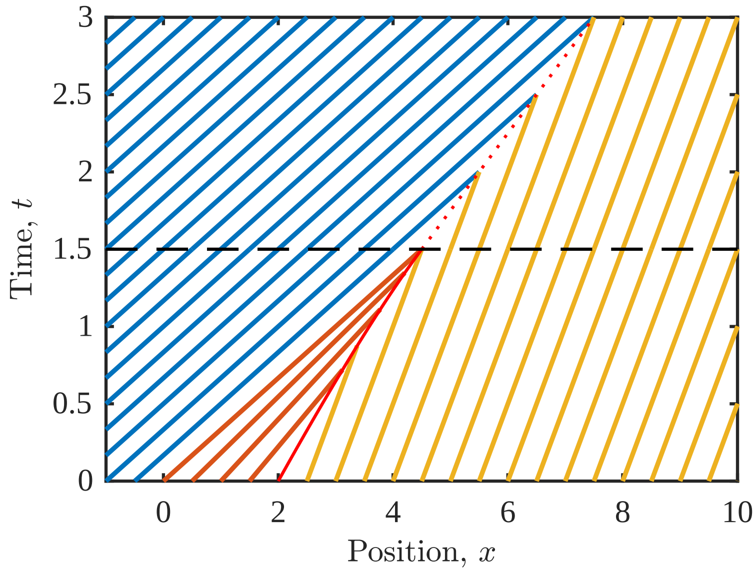

Figure 3: Amended diagram of characteristics showing the shock path obtained first for $t \leq t_s$(solid red line) and then for $t > t_s$ (dotted red line).

The explicit solution is thus given by

$$u(x,t) = \begin{cases} 3, & x < 3/2 + 2t \\ 1, & x > 3/2 + 2t \end{cases}$$

1 Mark

The amended diagram of characteristics is given in Fig.3.

2 Marks

As can be seen from the diagram of characteristics, the shock will propagate forever in a straight line as solution before and after the shock remain constant in time. Our solution is thus valid for all $t > t_s$.