IMPERIAL

MATH70062

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Lie Algebras

Date: Thursday, May 15, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

Every Lie algebra and every module over a Lie algebra are finite dimensional over $\mathbb{C}$. You can use, without proof, any results from the course provided you state them correctly and clearly.

1. (a) Give the definition of the *lower central series* of a Lie algebra L. Give the definition of a *nilpotent* Lie algebra. (3 marks)

   (b) Describe explicitly the lower central series of the Lie algebra L when

   (i) $L = \mathfrak{gl}(n)$, (2 marks)

   (ii) $L = \mathfrak{u}(n)$. (2 marks)

   You do not need to justify your answer.

   (c) Let L be a solvable Lie algebra with $\dim(L) = n$. Show that for every integer $0 \leq k \leq n$ there is a subalgebra $K \subseteq L$ of dimension k. (6 marks)

   (d) Let L be a Lie algebra, and let $A, B \subseteq L$ be two subalgebras of dimension 2.

   (i) Show that A and B are isomorphic when L is nilpotent. (3 marks)

   (ii) Given an example of a solvable Lie algebra L and A, B as above such that A and B are not isomorphic. Justify your answer. (2 marks)

   (iii) Given an example of a non-solvable Lie algebra L and A, B as above such that A and B are not isomorphic. Justify your answer. (2 marks)

(Total: 20 marks)

2. (a) Let A, B be two solvable subalgebras of $\mathfrak{sl}(n)$.

   (i) Show that $\dim(A) \leq \frac{n(n+1)}{2} - 1$.

   (3 marks)

   (ii) Assume that $\dim(A) = \dim(B) = \frac{n(n+1)}{2} - 1$. Show that there is an isomorphism $\phi : \mathfrak{sl}(n) \to \mathfrak{sl}(n)$ of Lie algebras such that $\phi(A) = B$.

   (7 marks)

   (b) Give the definition of the *rank* of the root system of a semi-simple Lie algebra L.

   (1 mark)

   (c) What is the rank of the root system of $\mathfrak{sl}(n)$? Justify your answer.

   (2 marks)

   (d) Let $\mathfrak{g} \subseteq \mathfrak{gl}(n)$ be a semi-simple Lie subalgebra which contains a solvable subalgebra of dimension $\frac{n(n+1)}{2} - 1$. Show that $\mathfrak{g} = \mathfrak{sl}(n)$.

   (7 marks)

(Total: 20 marks)

3. (a) Assume that a solvable Lie algebra L has an irreducible and faithful representation. Show that L is abelian. (2 marks)

(b) Is it possible that the adjoint representation of a solvable Lie algebra is faithful? Justify your answer. (2 marks)

For every representation $\rho : L \to \mathfrak{gl}(n)$ of a Lie algebra L we define the pairing:

$$K_\rho(x, y) = \mathrm{Tr}(\rho(x)\rho(y)) \quad (\forall x, y \in L).$$

(c) Show that K_ρ is a symmetric bilinear pairing. (2 marks)

(d) Let ρ, σ be two isomorphic representations of a Lie algebra L. Prove that the pairings K_ρ and K_σ are equal. (3 marks)

(e) Show that for every representation $\rho : L \to \mathfrak{gl}(n)$ the centre $Z(L)$ is in the orthogonal complement of L' with respect to $K_\rho(\cdot, \cdot)$. (4 marks)

(f) Let $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(n+1)$ be an irreducible representation with $n \geq 1$. Prove that the symmetric bilinear pairing K_ρ is non-degenerate. (7 marks)

(Total: 20 marks)

4. (a) Let L be a semisimple Lie algebra.

(i) What is a root space decomposition of L? (3 marks)

(ii) Give an example of a solvable subalgebra B of L such that $2\dim(B) > \dim(L)$ in terms of a root space decomposition of L. You do not need to justify your answer. (5 marks)

(b) Give an example of a semisimple Lie algebra of dimension 11. Justify your answer. (2 marks)

(c) Show that there is no simple Lie algebra of dimension 11. (5 marks)

We say that a subset S generates a Lie algebra L if the smallest subalgebra of L containing S is L itself.

(d) Show that a Lie algebra L is solvable if and only if every subalgebra of L generated by two elements is solvable. (5 marks)

(Total: 20 marks)

5. Let $R \subset \mathbb{R}^n$ be the set:

$$R = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid 0 < \sum_{i=1}^{n} a_i^2 \leq 2\}.$$

(a) Show that R is a root system with respect to the dot product on $\mathbb{R}^n$. (You can use any other results from the course provided you state them correctly and clearly.) (10 marks)

Let $R \subset \mathbb{R}^n$ be an irreducible root system with respect to the dot product on $\mathbb{R}^n$. Let $\Lambda(R)$ denote the abelian subgroup of $\mathbb{R}^n$ generated by R.

(b) Show that $\Lambda(R)$ is isomorphic to $\mathbb{Z}^n$. (3 marks)

(c) Let $(\cdot, \cdot)$ denote the dot product on $\mathbb{R}^n$. Show that there is a positive real number λ such that $\lambda(x, y) \in \mathbb{Z}$ for every $x, y \in \Lambda(R)$. (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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MATH70062

Lie Algebras (Solutions)

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1. (a) The lower central series of L is a descending series of ideals seen ⇓

$$L = L^0 \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(i)} \supseteq \cdots$$

such that $L^{(i)} = [L, L^{i-1}]$ for $i \geq 1$.

A Lie algebra L is nilpotent if $L^{(i)} = 0$ for some i. 3, A

seen ⇓

(b) (i) We have $L^{(0)} = \mathfrak{gl}(n)$ and $L^{(i)} = \mathfrak{sl}(n)$ when $i \geq 1$. 2, A

seen ⇓

(ii) We have

$$L^{(i)} = \{(a_{jk}) \in \mathfrak{gl}(n) \mid a_{jk} = 0 \text{ if } k \leq j + i\}.$$

2, A

unseen ⇓

(c) By the Jordan–Hölder theorem for Lie algebras we have a series of subalgebras:

$$\{0\} = L^0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_i \subsetneq L_k = L$$

such that L_{i-1} is a maximal proper ideal in L_i for every $i > 0$. Since subalgebras and quotients of solvable Lie algebras are solvable, we get that L_i/L_{i-1} is a solvable Lie algebra with only the zero ideal as a proper ideal. Therefore $L_i/L_{i-1} \cong \mathbb{C}$, and hence $\dim(L_i) = i$ for each i. 6, B

meth seen ⇓

(d) (i) Since A, B are subalgebras of a nilpotent Lie algebra, they are nilpotent. Since a nilpotent Lie algebra of dimension 2 is abelian, both A and B are abelian. However abelian Lie algebras of the same dimension are isomorphic. 3, B

meth seen ⇓

(ii) Let $L = \mathfrak{t}(2)$ and set:

$$A = \{(a_{ij}) \in \mathfrak{t}(2) \mid a_{12} = 0\}, \quad B\{(a_{ij}) \in \mathfrak{t}(2) \mid a_{22} = 0\}.$$

Since A is abelian while B is not, they are not isomorphic. 2, A

meth seen ⇓

(iii) Set $L = \mathfrak{gl}(2)$. It contains the simple Lie algebra $\mathfrak{sl}(2)$, so it is not solvable. It contains A, B in part (ii) above which are not isomorphic. 2, A

2. (a) (i) By Lie's theorem there is a basis of $\mathbb{C}^n$ such that with respect to this basis we have $A \subseteq \mathfrak{t}(n)$. Let $\mathfrak{st}(n) = \mathfrak{sl}(n) \cap \mathfrak{t}(n)$ be the kernel of $\mathrm{Tr}|_{\mathfrak{t}(n)}$. Clearly $A \subseteq \mathfrak{st}(n)$. Since $\dim(\mathfrak{st}(n)) = \frac{n(n+1)}{2} - 1$, the claim follows. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{3, \mathrm{A}}$

$\boxed{\text{unseen} \Downarrow}$

(ii) As we saw above there are two bases $\mathbf{e} = e_1, \ldots, e_n$ and $\mathbf{f} = f_1, \ldots, f_n$ of $\mathbb{C}^n$ such that A, B is the subalgebra of upper triangular matrices with trace 0 in the basis $\mathbf{e}, \mathbf{f}$, respectively. Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be the unique linear map which maps $\mathbf{e}$ to $\mathbf{f}$ (respecting the order), and let $\phi : \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ be the map given by the rule $X \mapsto \Phi^{-1} X \Phi$. Since $\phi(XY) = \phi(X)\phi(Y)$, it induces an automorphism of the Lie algebra $\mathfrak{gl}(n)$. Clearly ϕ maps A to B, and it maps $\mathfrak{sl}(n)$ bijectively onto $\mathfrak{sl}(n)$, since conjugation does not change the trace. $\boxed{7, \mathrm{C}}$

(b) The rank of the root system of a semi-simple Lie algebra L is the dimension of any of its Cartan subalgebras. $\boxed{\text{seen} \Downarrow}$ $\boxed{1, \mathrm{A}}$

$\boxed{\text{seen} \Downarrow}$

(c) The subalgebra $\mathfrak{sd}(n)$ of diagonal matrices of trace zero is a Cartan sublagebra of $\mathfrak{sl}(n)$, so the rank of the root system of $\mathfrak{sl}(n)$ is $n - 1$. $\boxed{2, \mathrm{A}}$

$\boxed{\text{meth seen} \Downarrow}$

(d) Since $\mathrm{Tr} : \mathfrak{gl}(n) \to \mathfrak{gl}(1)$ is a Lie algebra homomorphism, we have $\mathfrak{g} \subseteq \mathfrak{sl}(n)$. By part (a) we may assume that $\mathfrak{g}$ contains $\mathfrak{st}(n)$. So it contains $\mathfrak{sd}(n)$, too. Every element of $\mathfrak{sd}(n)$ is semi-simple in $\mathfrak{gl}(n)$, so it is semi-simple in $\mathfrak{g}$, too. Since $\mathfrak{sd}(n)$ is abelian, and maximal with respect to inclusion in $\mathfrak{sl}(n)$ among subalgebras with these two properties, it is also maximal in $\mathfrak{g}$, i.e. a Cartan subalgebra. If $i < j$ then $E_{ij} \in \mathfrak{st}(n) \subseteq \mathfrak{g}$, which spans the root space for a certain root $\lambda : \mathfrak{sd}(n) \to \mathbb{C}$. Since $\mathfrak{sd}(n)$ is a Cartan subalgebra in the semi-simple Lie algebra $\mathfrak{g}$, we get that the root space for $-\lambda$ in $\mathfrak{g}$ is non-empty. The root space for $-\lambda$ in $\mathfrak{sl}(n)$ contains the former, and it is a one-dimensional space spanned by E_{ji}. Therefore $\mathfrak{g}$ contains E_{ji}. The matrices $\{E_{ij}, E_{ji} \mid i < j\}$ and $\mathfrak{sd}(n)$ span $\mathfrak{sl}(n)$. $\boxed{7, \mathrm{D}}$

3. (a) By Lie's theorem every irreducible representation of a solvable Lie algebra L is one-dimensional. Since L has a one-dimensional faithful representation, it is isomorphic to a subalgebra of $\mathbb{C}$, so it is abelian. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{2, \text{A}}$

(b) The adjoint representation of a Lie algebra L is faithful if its kernel, the centre of L, is trivial. One such solvable Lie algebra is the unique non-abelian 2-dimensional solvable Lie algebra. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{2, \text{A}}$

(c) Since K_ρ is the composition of the linear map ρ, the bilinear matrix product and the linear trace map, it is a bilinear pairing. Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it is also symmetric. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{2, \text{A}}$

(d) Let $\phi : V \to W$ be the isomorphism from $\rho : L \to V$ to $\sigma : L \to W$. Then $\rho = \phi^{-1} \circ \sigma \circ \phi$, and hence $\boxed{\text{meth seen} \Downarrow}$

$$K_\rho(x, y) = \mathrm{Tr}(\rho(x)\rho(y)) = \mathrm{Tr}((\phi^{-1}\sigma(x)\phi)(\phi^{-1}\sigma(y)\phi)) =$$

$$\mathrm{Tr}(\phi^{-1}\sigma(x)\sigma(y)\phi) = \mathrm{Tr}(\sigma(x)\sigma(y)) = K_\sigma(x, y),$$

since conjugations do not change the trace. $\boxed{3, \text{A}}$

(e) Since ρ is a homomorphism, we have $0 = \rho([z, y]) = \rho(z)\rho(y) - \rho(y)\rho(z)$ for every $y \in L$ and $z \in Z(L)$. Therefore $\boxed{\text{meth seen} \Downarrow}$

$$K_\rho(z, [xy]) = \mathrm{Tr}(\rho(z)\rho(x)\rho(y) - \rho(z)\rho(y)\rho(x)) =$$

$$\mathrm{Tr}((\rho(z)\rho(x))\,\rho(y) - \rho(y)\,(\rho(z)\rho(x))) = 0$$

for every $x, y \in L$ and $z \in Z(L)$. Since these elements span L' and K_ρ is bilinear, the claim follows. $\boxed{4, \text{B}}$

(f) Recall that ρ is isomorphic to the representation ϕ with underlying vector space V_n with basis $b_0, \ldots, b_n$ such that $\phi(e)(b_i) = (i+1)b_{i+1}$, $\phi(h)b_i = (n-2i)b_i$, and $\phi(f)(b_i) = (n+1-i)b_{i-1}$ (where $b_{-1} = 0 = b_{n+1}$ by convention). By part (d) it is enough to show that K_ϕ is non-degenerate. In the basis e, h, f of $\mathfrak{sl}(2)$ the latter has the matrix: $\boxed{\text{meth seen} \Downarrow}$

$$\begin{pmatrix} 0 & 0 & \sum_{i=1}^n i(n+1-i) \\ 0 & \sum_{i=0}^n (n-2i)^2 & 0 \\ \sum_{i=1}^n i(n+1-i) & 0 & 0 \end{pmatrix}$$

whose determinant is $-(\sum_{i=1}^n i(n+1-i))^2(\sum_{i=0}^n (n-2i)^2) < 0$, so K_ϕ is non-degenerate. $\boxed{7, \text{B}}$

4. (a) (i) For a Cartan subalgebra H of L we have: seen $\Downarrow$

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha,$$

where

$$L_\alpha = \{x \in L \mid [hx] = \alpha(h)x \text{ for all } h \in H\}$$

and $\Phi \subset H^* - \{0\}$ is the set of roots of L. 3, A

unseen $\Downarrow$

(ii) Let B be a base of Φ and let P be the positive roots of Φ with respect to B. Then

$$B = H \oplus \bigoplus_{\alpha\in P} L_\alpha$$

is a solvable subalgebra of L such that $2\dim(B) > \dim(L)$. 5, D

meth seen $\Downarrow$

(b) Since $\mathfrak{sl}(2) \oplus \mathfrak{sl}(3)$ is the direct sum of simple Lie algebras, it is semisimple. Its dimension is $(2^2 - 1) + (3^2 - 1) = 11$. 2, A

meth seen $\Downarrow$

(c) Assume that there is a such an L. Let C be a Cartan subalgebra of L, and let Φ be the roots of C. Then $\dim(L) = r + |\Phi|$, where $\dim(C) = r$. Since $2r \leq |\Phi|$, and $r = 1$ implies $|\Phi| = 2$, we get that only $r = 3$ and $|\Phi| = 8$ is possible. Since L is simple, its root system is irreducible. However the only irreducible root systems of rank 3 are A_3, B_3, C_3, which has cardinality $12, 18, 18$, respectively, a contradiction.

5, C

unseen $\Downarrow$

(d) Every subalgebra of a solvable Lie algebra is solvable, so the only if conclusion is true. If L is not solvable and R is its radical, then L/R is semisimple, so it contains a subalgebra $S \subseteq L/R$ isomorphic to $\mathfrak{sl}(2)$. Let x, y be two elements of L mapping to the generators e, f of $S \cong \mathfrak{sl}(2)$ under the quotient map $L \to L/R$. The subalgebra $K \subseteq L$ generated by x, y maps to S surjectively under the quotient map $L \to L/R$, so it is not solvable. 5, D

5. (a) Note that $R = R_1 \cup R_2$, where $R_i = \{\underline{a} \in R \mid (\underline{a}, \underline{a}) = i\}$. Clearly R does not contain zero and since R_1 contains a basis, the set R spans $\mathbb{R}^n$. Since elements of R_1 and R_2 have length 1 and $\sqrt{2}$, respectively, if $\alpha, \beta \in R$ are proportional then $\alpha = \pm\beta$ or $\alpha = \pm\sqrt{2}\beta$, after reordering α, β. The latter is not possible, since all elements of R have integral co-ordinates. The abelian subgroup $L \subset \mathbb{R}^n$ generated by R is $\mathbb{Z}^n$, and hence $(\alpha, x) \in \mathbb{Z}$ for every $x \in L$ and $\alpha \in R$. Therefore $2(\alpha, x)/(\alpha, \alpha) \in \mathbb{Z}$ for every x and α as above. We get that reflexions $x \mapsto x - 2(\alpha, x)/(\alpha, \alpha)\alpha$ with respect to the elements $\alpha \in R$ preserve L, since the latter is closed under addition. Since R_1 and R_2 are exactly the elements of L of square length 1 and 2, respectively, these reflexions, which are orthogonal transformations, preserve R, too. meth seen $\Downarrow$ 10, M

(b) Let B be a base of R. The abelian subgroup $L \subset \mathbb{R}^n$ generated by B is isomorphic to $\mathbb{Z}^n$, as B is linearly independent. Since $B \subset R$, we have $L \subseteq \Lambda(R)$. But every element of R is an integral linear combination of the elements of B, so $R \subset L$, and hence $\Lambda(R) \subseteq L$. unseen $\Downarrow$ 3, M

(c) Let B be as above, and pick an $e \in B$. After rescaling $(\cdot, \cdot)$ we may assume that $(e, e) \in \mathbb{Q}$. We will show that $(f, g) \in \mathbb{Q}$ for every $f, g \in B$. This is enough; then there is a positive integer n such that $n(f, g) \in \mathbb{Z}$ for every $f, g \in B$, since B is finite. Since every element of L is an integer linear combination of elements of B, we get that $n(f, g) \in \mathbb{Z}$ for every $f, g \in L$ by bilinearity. Since $(f, g)/(g, g) \in \mathbb{Q}$, it is enough to show that $(g, g) \in \mathbb{Q}$ for each $g \in B$. Clearly $(e, g) \in \mathbb{Q}$ as $(e, g)/(e, e) \in \mathbb{Q}$, and $(e, g)/(g, g) \in \mathbb{Q}$, so $(g, g) \in \mathbb{Q}$ holds if $(e, g) \neq 0$. Because R is irreducible, its Dynkin diagram Γ is connected, so we get the general case by induction on the distance of g from e in Γ. unseen $\Downarrow$ 7, M

Review of mark distribution:
Total A marks: 31 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 17 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70062 Lie Algebras Markers Comments

Question 1	This is was a routine problem for the most part, and therefore not surprisingly the students did well.
Question 2	Not surprisingly part (a) (ii) and (d) turned out to be more challenging than the rest of the question, although many students managed to do (a) (ii) well. Many students missed the simple answer in part (b).
Question 3	Most of this question were pretty straightforward, but part (f) turned out to be only did well by a relatively few students. The Lie algebra part of this problem was very modest, so probably the reason why students found this difficult is due to the fact that they are not very used with quadratic forms, which is a basic material from linear algebra.
Question 4	This problem had 3 parts which were more challenging, (a) (ii), (c) and (d). The first two were done by a surprisingly large number of students, or they at least saw in (c) that the claim boils down to the classification of Dynkin diagrams. Part (d) turned out to be too hard for most students.
Question 5	Part (a) was rather routine, so many students did it, or at least made large strides. However many students misinterpreted parts (b) and (c), probably because they were not reading the instructions carefully.