ZF 1–9 Zermelo-Fraenkel Set Theory, ZF.

4. Axiom of Choice + Consequences

(4.1) Statement + WO principle.

(4.1.1) **Def.** Axiom of Choice (AC)

Suppose A is a set of non-empty sets. Then there is a function

$$f : A \to \bigcup A$$

such that for $a \in A$ we have

$$f(a) \in a .$$

Axioms ZF1–9 + AC : ZFC.

Example (4.1.2)

Suppose X is a (non-empty) set and $A = \mathcal{P}(X) \setminus \{\emptyset\}$ (i.e. the non-empty subsets of A).

By AC there is a function

$$f : A \to X \quad \text{such that}$$

$$f(Y) \in Y \text{ for every } \emptyset \neq Y \subseteq X .$$

Such an f is called a choice function on X.

Note: If $(X; \leq)$ is a w.o. set then we automatically have a choice function f on X: if $\emptyset \neq Y \subseteq X$ let $f(Y) = \min_{\leq}(Y)$ (least elt. of Y).

(4.1.3) Theorem (ZF) Suppose X is a non-empty set and $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ is a choice function. Then there is a well-ordering $\leq$ of X, i.e. $(X ; \leq)$ is a w.o. set.

Pf: Idea: Use transfinite recursion to construct a bijection between X and some ordinal.

$G(0) \quad G(1) \quad G(2) \quad \ldots$

$$G(\alpha) = f\Big(\underbrace{X \setminus \{G(\beta) : \beta < \alpha\}}_{\text{if this is } \neq \emptyset}\Big)$$

At stage α $G \restriction \alpha$ is an injective function $\alpha \to X$.

Why does this 'terminate'?

(4.1.4) Thm (Hartogs' Lemma, ZF)

For any set X there is an ordinal α such that there is no injective function $h : \alpha \to X$. //.

Pf of (4.1.3) (Given 4.1.4)

Let ∞ be some set with $\infty \notin X$.

Consider $\tilde{X} = X \cup \{\infty\}$

Using Transfinite Recursion, define an operation G :

For an ordinal γ define

$$G(\gamma) = \begin{cases} f(X \setminus \{G(\beta) : \beta < \gamma\}) & \text{if } X \setminus \{G(\beta) : \beta < \gamma\} \neq \emptyset \\ \infty & \end{cases}$$

<u>Note</u>: If $\infty \notin \operatorname{im}(G \restriction \gamma)$ then $G \restriction \gamma$ is an injective function $\gamma \to X$.

By Hartogs' Lemma, there is some ordinal α with $G(\alpha) = \infty$

Take the least such α. Then

$g : G \restriction \alpha : \alpha \to X$ is an injective function which is <u>surjective</u>. i.e. g is a bijection.

Define $\leq$ on X by: $x_1 \leq x_2 \iff g^{-1}(x_1) \leq g^{-1}(x_2)$

ordering on α.

#.

Pf of 4.1.4. (ZF)

X set; find an ordinal α s.t. there is no injective $h : \alpha \to X$.

Consider the set

$$\mathcal{Y} = \left\{ (Y ; \leq_Y) : \begin{array}{l} Y \subseteq X \text{ and} \\ \leq_Y \text{ is a w.o.} \\ \text{on } Y \end{array} \right\}$$

Let

$$S = \left\{ \beta : \begin{array}{l} \beta \text{ is an ordinal} \\ \text{similar to some} \\ (Y ; \leq_Y) \in \mathcal{Y} \end{array} \right\}$$

- A set, using Axiom of Replacement.

$$S = \left\{ \beta : \begin{array}{l} \beta \text{ is an ordinal} \\ \text{and there is an} \\ \text{injective function} \\ \beta \to X \end{array} \right\}.$$

Let $\sigma = \bigcup S$.

This is an ordinal (3.4.7) & $\beta \leq \sigma$ for all $\beta \in S$.

Let $\alpha = \sigma^+$. Then α is an ordinal & for $\beta \in S$ $\beta \leq \sigma < \alpha$.

So $\alpha \notin S$. //. #.

(4.1.5) Cor. (ZF)

AC is equivalent to WO (Well Ordering Principle)

If A is any set then there $\leq_A \subseteq A \times A$ such that $(A; \leq_A)$ is a w.o. set.

$[\ ZF \vdash (AC \leftrightarrow WO).\ ]$

Pf: $AC \Rightarrow WO$ AC gives a choice function on A; then use 4.1.3. //.

$WO \Rightarrow AC$. If A is any set of non-empty sets let $B = \bigcup A$. By WO there is a w.o. $\leq_B$ on B. Define $f : A \to \bigcup A$ by $f(a) = \min_{\leq_B}(a)$. #.

(4.1.6) Cor. (ZFC).

(i) If A is an set, there is an ordinal α with $|A| = |\alpha|$.

(ii) If A, B are sets, then $|A| \leq |B|$ or $|B| \leq |A|$.

(iii) (Fundamental theorem of Cardinal Arithmetic).

If A is any infinite set then $|A \times A| = |A|$.

Pf: (i) By WO there is a w.o. set $(A; \leq_A)$. This is similar to some ordinal α, then $|A| = |\alpha|$.

(ii) By (i) there are ordinals α, β with $|A| = |\alpha|$ & $|B| = |\beta|$. By 3.4.6 $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

(iii) By (i) there is an ordinal α with $|\alpha| = |A|$. Then use $|\alpha| = |\alpha \times \alpha|$ - 3.5.3 #