Analysis II, Complex Analysis

Solutions, CW1

Q1 (5p)

Note that

$$f(z) = u + iv = \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

Therefore

a) (1p) Let $\Omega = \{z : |z| < 1, -\pi/2 < \operatorname{Arg} z < \pi/2\}$. Then

$$f(\Omega) = \{w = u + iv : u < 0, |v| < \pi/2\}.$$

b) (1p) Let $\Omega = \{z : |z| > 1, 0 \le \operatorname{Arg} z \le \pi\}$. Then

$$f(\Omega) = \{w = u + iv : u > 0, 0 \le v \le \pi\}.$$

c) (3p) Note that

$$\operatorname{Log}\left(\frac{z-i}{z+i}\right) = \operatorname{Log}\left(1 - \frac{2i}{z+i}\right) = \operatorname{Log}(w),$$

where $w = 1 - \frac{2i}{z+i}$. The principle value of Log is defined by the cut $w \in (-\infty, 0]$. Therefore the branch cut for $\operatorname{Log}\left(\frac{z-i}{z+i}\right)$ is defined by points z:

$$z = \frac{i(w+1)}{1-w}, \quad \text{where} \quad w \in (-\infty, 0].$$

Answer: The branch cut is the closed interval defined by

$$\{z = x + iy \in \mathbb{C} : z = iy, y \in [-1, 1]\}.$$

Q2a) (2p) Clearly the inequality $|\sin z| \le 1$ is equivalent to $|\sin z|^2 \le 1$, or to

$$|\sin z|^2 = \frac{1}{4}\left|e^{ix-y} - e^{-ix+y}\right|^2 = \frac{1}{4}\left|e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right|^2$$

$$= \frac{1}{4}\left((e^{-y} - e^{y})^2\cos^2 x + (e^{-y} + e^{y})^2 \sin^2 x\right)$$

$$= \frac{1}{4}\left((e^{-2y} + e^{2y} - 2)\cos^2 x + (e^{-2y} + 2 + e^{2y})\sin^2 x\right)$$

$$= \frac{1}{4}\left((e^{-2y} + e^{2y}) - 2(\cos^2 x - \sin^2 x)\right)$$

$$= \frac{1}{4}\left((e^{-2y} + e^{2y}) - 2\cos 2x\right) \le 1.$$

Therefore $|\sin z| \leq 1$ is equivalent to

$$\cosh 2y - \cos 2x \leq 2.$$

Q2 b) (3p) Using partial fraction decomposition we find

$$f(z) = \frac{1}{(z+i)(z-2)} = \frac{1}{2+i}\left(\frac{1}{z-2} - \frac{1}{z+i}\right) =$$

We have

$$\frac{1}{z-2} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n$$

and

$$\frac{1}{z+i} = \frac{1}{1+i+(z-1)} = \frac{1}{1+i}\,\frac{1}{1+\frac{z-1}{1+i}} = \frac{1}{1+i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^n}\,(z-1)^n.$$

Therefore we finally obtain

$$f(z) = \frac{1}{(z+i)(z-2)} = \frac{1}{2+i}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}}{(1+i)^{n+1}} - 1\right)(z-1)^n.$$

Note

$$\frac{(-1)^{n+1}}{(1+i)^{n+1}} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the series is convergent for all z such that $|z-1| < 1$.

Q3.a) (1p)

$$\oint_{\{z:\,|z|=5\}}\left(\frac{3}{z+2} - \frac{1}{z-2i}\right)\,dz = 2\pi i\,(3-1) = 4i\pi.$$

Q3.b) (1p)

$$\oint_{\{z:\,|z-2i|=1/2\}}\left(\frac{3}{z+2} - \frac{1}{z-2i}\right)\,dz = -2\,i\,\pi.$$

Q3.c) (3p) Using the deformation theorem we obtain

$$\begin{aligned}
\oint_{\gamma}\frac{8z-3}{z^2-z}\,dz &= \oint_{\gamma}\frac{8z-3}{z(z-1)}\,dz \\
&= \oint_{\{z:\,|z|=1/2\}}\frac{8z-3}{z(z-1)}\,dz - \oint_{\{z:\,|z-1|=1/2\}}\frac{8z-3}{z(z-1)}\,dz \\
&= 2\,i\,\pi\left(\frac{-3}{-1} - \frac{8-3}{1}\right) = -4\,i\,\pi.
\end{aligned}$$

Q4 (5p) Define the entire function (why is it entire?)

$$g(z) = \frac{f(z) - f(0) - f'(0)\,z}{z^2}.$$

The condition $\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^2} = 0$ implies that $g(z)$ is bounded and thus by Liouville's theorem g is a constant. Since

$$\lim_{|z|\to\infty} g(z) = \lim_{|z|\to\infty} \left|\frac{f(z) - f(0) - f'(0)\,z}{z^2}\right| = \lim_{|z|\to\infty} \left|\frac{f(z)}{z^2}\right| = 0,$$

we have $g(z) = 0$ and hence

$$f(z) - f(0) - f'(0)\,z = 0.$$