1 (a) There are two ways to do this. The easiest way is simply to go around each node and add up the currents out of each node. For example, for node a, the net current *out* is

$$-1 + 1 = 0. \quad (1)$$

Going around all 6 nodes in this way confirms that KCL is satisfied at all of them.

Another way is to use the fact that the net current out of the set of nodes is given by

$$-\mathbf{A}^T\mathbf{w} \quad (2)$$

and it can be checked that this equals the zero vector. Of course, for this, you must write down the incidence matrix (details omitted here). If $-\mathbf{A}^T\mathbf{w} = \mathbf{0}$ this current satisfies the Kirchhoff current law at all nodes.

(b) The two loops in this circuit – forming a basis of the left nullspace of $\mathbf{A}$ — are

$$\mathbf{w}_1 = (+1, 0, -1, +1, 0, -1, 0)^T, \qquad \mathbf{w}_2 = (0, +1, 0, -1, +1, 0, -1)^T. \quad (3)$$

(c) The solution $\mathbf{w}$ in part (a) must be a linear combination of these two left nullspace vectors:

$$\mathbf{w} = c_1\mathbf{w_1} + c_2\mathbf{w_2}. \quad (4)$$

It is easily checked that $c_1 = -1, c_2 = -3$.

(d) We know that there does not exist a potential vector $\mathbf{x}$ for which

$$\mathbf{w} = -\mathbf{CAx} = -\mathbf{Ax}. \quad (5)$$

One way to see this directly is to ground node a, say, and then try to construct the other potentials by adding up the potential differences given by the w_i: if $x_a = 0$ then it is clear that we must have $x_b = +1$ and $x_d = -1$ in order to get the currents in those edges. However, then, starting from node d we find $x_e = -2$ but starting at node b we find $x_e = -1$. This is a contradiction: such a vector $\mathbf{x}$ does not exist.

(e) It is easy to compute the associated vector $\mathbf{f}$ – i.e. the vector of net currents **out** of each node:

$$\mathbf{f} = (+4, -1, -2, -6, +5, 0)^T. \quad (6)$$

This is a non-zero vector so the current does not satisfy Kirchhoff's current law at all nodes (although it does satisfy KCL at node 6).

Note: it can be checked that this satisfies $\mathbf{x}_0^T\mathbf{f} = 0$ where $\mathbf{x}_0 = [1,\ 1,\ 1,\ 1,\ 1,\ 1]^T$.

(f) The vector

$$\mathbf{e} = (+1, +1, +3, -1, -1, -3, +1)^T \quad (7)$$

comes from the vector of potentials

$$\mathbf{e} = -\mathbf{A}\mathbf{x}, \tag{8}$$

where

$$\mathbf{x} = (0, -1, -2, -3, 0, -1)^T. \tag{9}$$

(g) By direct calculation,

$$\begin{aligned} \mathbf{e}^T\mathbf{w} &= (+1, +1, +3, -1, -1, -3, +1)^T(-1, -3, +1, +2, -3, +1, +3) \\ &= -1 - 3 + 3 - 2 + 3 - 3 + 3 \\ &= 0. \end{aligned} \tag{10}$$

(h) Since we know that **e** comes from a vector of potentials, i.e.,

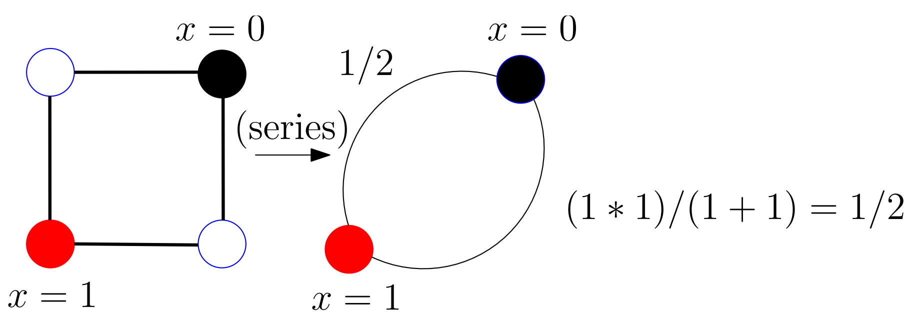
$$\mathbf{e} = -\mathbf{A}\mathbf{x} \tag{11}$$

then

$$\mathbf{e}^T\mathbf{w} = -\mathbf{x}^T\mathbf{A}^T\mathbf{w} \tag{12}$$

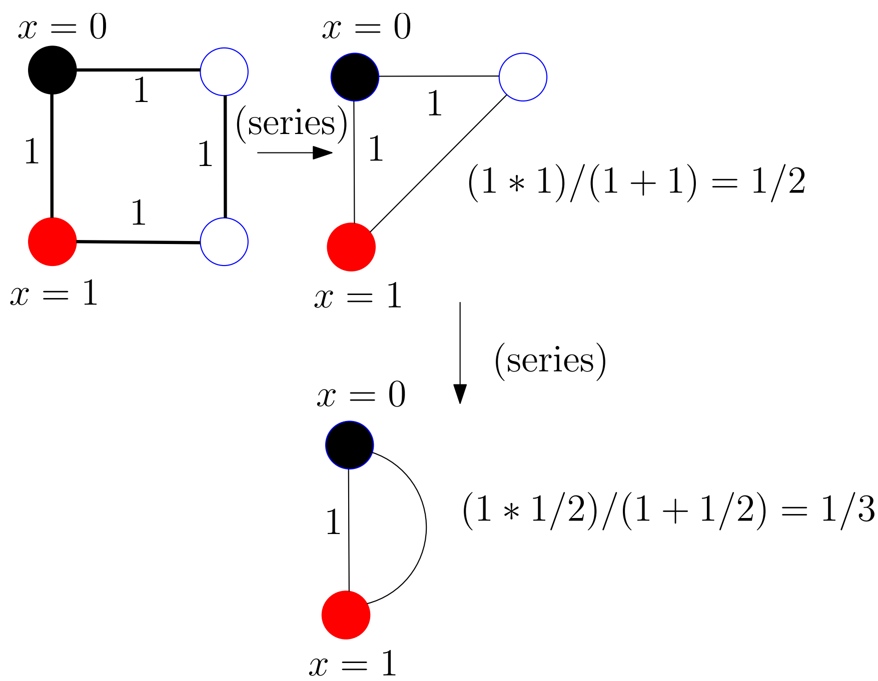
but this equals zero since $-\mathbf{A}^T\mathbf{w} = 0$.

2 (a) The following diagrams show a succession of equivalent circuits, using the rule for resistors in series, that reduce each given circuit to two resistors in parallel:



Hence the effective resistance of this circuit is (two resistors in parallel)

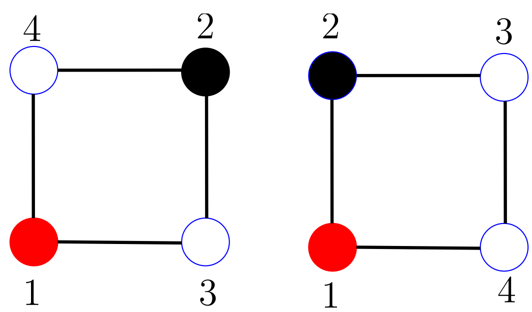
$$\frac{1}{2} + \frac{1}{2} = 1. \tag{13}$$



Hence the effective resistance of this circuit is (two resistors in parallel)

$$1+\frac{1}{3}=\frac{4}{3}. \tag{14}$$

(b) To carry out the linear algebra exercise we label the nodes as in this figure:



For the first circuit the Laplacian matrix is

$$\mathbf{K}=\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}=\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \tag{15}$$

where

$$\mathbf{A}=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}. \tag{16}$$

The connection between the node voltages and net currents out is

$$\mathbf{K}\begin{bmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f \\ -f \\ 0 \\ 0 \end{bmatrix}. \tag{17}$$

Setting

$$\mathbf{e}_1 = [1\ 0]^T, \qquad \mathbf{f}_1 = [f\ \ -f]^T, \qquad \hat{\mathbf{x}} = [x_3\ x_4]^T \tag{18}$$

the system becomes

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}\hat{\mathbf{x}} = \mathbf{f}_1, \qquad \mathbf{B}\mathbf{e}_1 + \mathbf{A}\hat{\mathbf{x}} = 0. \tag{19}$$

Hence

$$\hat{\mathbf{x}} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{e}_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \tag{20}$$

Therefore

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}\hat{\mathbf{x}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} f \\ -f \end{pmatrix}. \tag{21}$$

We read off the effective conductance $f = 1$, in agreement with part (a).

For the second circuit the Laplacian matrix is

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{23}$$

The connection between the node voltages and net currents out is

$$\mathbf{K}\begin{bmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f \\ -f \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

Setting

$$\mathbf{e}_1 = [1\ 0]^T, \qquad \mathbf{f}_1 = [f\ \ -f]^T, \qquad \hat{\mathbf{x}} = [x_3\ x_4]^T \tag{25}$$

the system becomes

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}\hat{\mathbf{x}} = \mathbf{f}_1, \qquad \mathbf{B}\mathbf{e}_1 + \mathbf{A}\hat{\mathbf{x}} = 0. \tag{26}$$

Hence

$$\hat{\mathbf{x}} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{e}_1 = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}. \tag{27}$$

Therefore,

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}\hat{\mathbf{x}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -4/3 \end{pmatrix} = \begin{pmatrix} f \\ -f \end{pmatrix}. \tag{28}$$

We read off the effective conductance $f = 4/3$, in agreement with part (a).

3 (a) The Laplacian matrix is

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \tag{29}$$

(b) The sub-blocks are

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{30}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \tag{31}$$

(c) We have

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \tag{32}$$

where

$$\mathbf{x} = \begin{pmatrix} \mathbf{e}_1 \\ \hat{\mathbf{x}} \end{pmatrix}, \qquad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f \\ -f \\ \mathbf{0} \end{pmatrix}. \tag{33}$$

Hence

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \hat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} f \\ -f \\ \mathbf{0} \end{pmatrix}. \tag{34}$$

This implies

$$\mathbf{B}\mathbf{e}_1 + \mathbf{C}\hat{\mathbf{x}} = 0, \qquad \text{or} \qquad \hat{\mathbf{x}} = -\mathbf{C}^{-1}\mathbf{B}\mathbf{e}_1. \tag{35}$$

This means that

$$\begin{pmatrix} f \\ -f \end{pmatrix} = \mathbf{A}\mathbf{e}_1 - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B}\mathbf{e}_1. \tag{36}$$

The effective conductance, which is defined to be the net current out of node 1, is therefore given by the analytical expression

$$f = C_{\text{eff}} = \mathbf{e}_1^T[\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B}]\mathbf{e}_1. \tag{37}$$

Note: the matrix

$$\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B} \tag{38}$$

is the Schur complement of **C** in **K**.

(d) Using `MATLAB`, for example, it can be determined that the quantity (37) is

$$f = C_{\text{eff}} = \frac{4}{5}. \tag{39}$$

4. We must solve

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \tag{40}$$

where it can be found that the weighted Laplacian is

$$\mathbf{K} = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 3 & -2 & -1 \\ -1 & -2 & 5 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{bmatrix}, \tag{41}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}. \tag{42}$$

The connection between the node voltages and net currents out is

$$\mathbf{K} \begin{bmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f \\ -f \\ 0 \\ 0 \end{bmatrix}. \tag{43}$$

Setting

$$\mathbf{e}_1 = [1\ 0]^T, \qquad \mathbf{f}_1 = [f\ \ -f]^T, \qquad \hat{\mathbf{x}} = [x_3\ x_4]^T \tag{44}$$

the system becomes

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}^T\hat{\mathbf{x}} = \mathbf{f}_1, \qquad \mathbf{B}\mathbf{e}_1 + \mathbf{C}\hat{\mathbf{x}} = 0. \tag{45}$$

Hence since

$$\mathbf{C}^{-1} = \frac{1}{16}\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \tag{46}$$

then

$$\hat{\mathbf{x}} = -\mathbf{C}^{-1}\mathbf{B}\mathbf{e}_1 = \frac{1}{16}\begin{pmatrix} 6 \\ 7 \end{pmatrix}. \tag{47}$$

Hence

$$\mathbf{A}\mathbf{e}_1 + \mathbf{B}\hat{\mathbf{x}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} -13 \\ -19 \end{pmatrix} = \begin{pmatrix} 19/16 \\ -19/16 \end{pmatrix} = \begin{pmatrix} f \\ -f \end{pmatrix}. \tag{48}$$

We read off the effective conductance to be

$$f = C_{\text{eff}} = \frac{19}{16}. \tag{49}$$

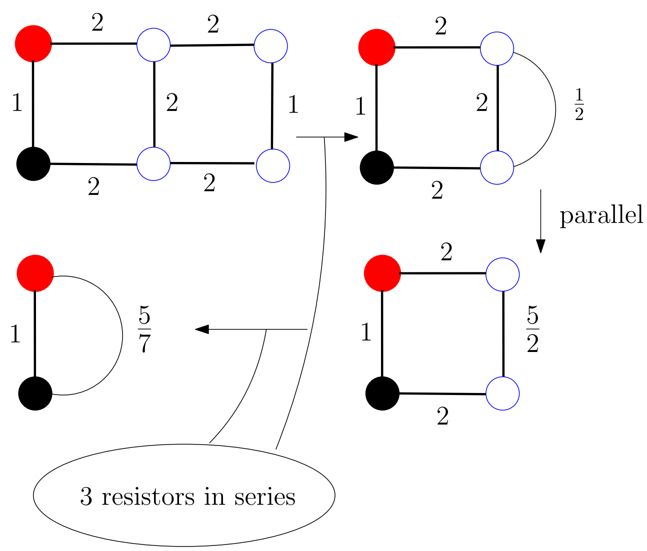
5. The rule for constructing the weighted Laplacian is as straightforward as the unweighted case. At each node, node i say, add up all the conductances c_{ij} of the edges emanating from node i and going to node j (for some set of j's); call the answer c_i, i.e.,

$$c_i = \sum_j c_{ij} \tag{50}$$

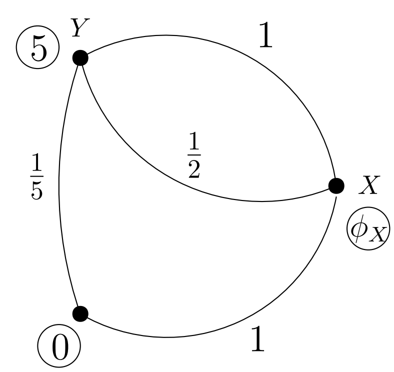
where the sum is over the nodes connected to node i. The diagonal term of the i-th row of $K = A^TCA$ corresponding to node i is c_i. All other components of the i-th row are zero except, if node i is connected to node j, then $-c_{ij}$ appears in the j-th column. Note that all elements of each row add to zero.

6. The circuit can be reduced to simpler circuits by noticing (a) 3 resistors in series; (b) 2 resistors in parallel; (c) 3 resistors in series; (d) the final 2 resistors in parallel gives the effective conductance of the circuit as

$$1 + \frac{5}{7} = \frac{12}{7}. \tag{51}$$



7. When all the edges of zero resistance are connected together (which is permissible since those nodes must all be at the same potential/voltage), then the circuit simplifies to the graph shown in the figure (where the conductances rather than the voltages are shown):



(a) This graph is so simple there is no need to solve the problem using linear algebra; it can be done directly. To find the voltage at X, call it ϕ_X, we can apply Kirchhoff's current law at node X. The net current (either in or out) of node X should be zero:

$$1 \times \phi_X + \frac{1}{2} \times (\phi_X - 5) + 1 \times (\phi_X - 5) = 0. \tag{52}$$

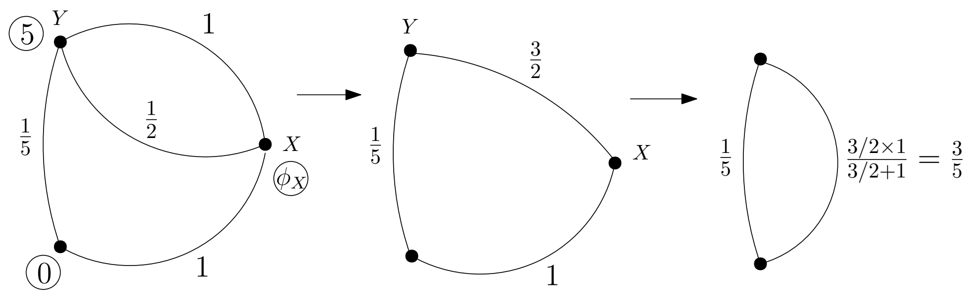
We find

$$\phi_X = 3. \tag{53}$$

(b) There are two ways to solve this part. One way is to directly solve for the current out of node Y since we know the voltages at all nodes in the circuit:

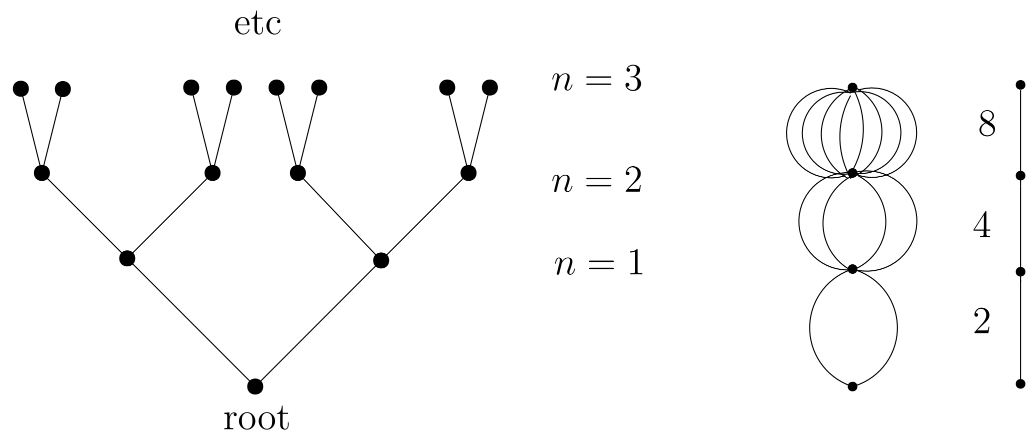
$$1 \times (5-3) + \frac{1}{2} \times (5-3) + 5 \times (5-0) = 4. \tag{54}$$

Another way is to solve for the *effective conductance* by simplifying to "equivalent circuits" using the laws for resistors in series and parallel:



The conductance of the final parallel resistors equals their sum, which is $4/5$. This is, by definition, the net current out of node Y assuming unit voltage. However the voltage there is 5, hence the current out of node Y is $5 \times 4/5 = 4$.

8. By the symmetry of the configuration, with the root at unit voltage and all generation 3 nodes grounded, all nodes at each generation will be at the same potential. We can therefore identify each generation as a single node. This leads to an equivalent circuit with 3 resistors in series with conductances $2, 4$ and 8:



Using the rule for resistors in series the conductances satisfy

$$\frac{1}{C_{\text{eff}}^{(3)}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2}\right] = \frac{1}{2}\left[\frac{1-(1/2)^3}{1-(1/2)}\right] = 1 - \left(\frac{1}{2}\right)^3, \tag{55}$$

where we have used the formula for the sum of a finite geometric series. Hence

$$C_{\text{eff}}^{(3)} = \frac{8}{7}. \tag{56}$$

(b) It is clear from the solution in part (a) that the general result for any n is

$$\frac{1}{C_{\text{eff}}^{(n)}} = 1 - \left(\frac{1}{2}\right)^n, \tag{57}$$

so that

$$C_{\text{eff}}^{(n)} = \frac{2^n}{2^n - 1}. \tag{58}$$

(c) The limiting result as $n \to \infty$ is

$$C_{\text{eff}}^{(\infty)} = 1. \tag{59}$$