Axiom of Replacement (ZF8)

(3.4.13) **Def.** (Operation on sets.)

Suppose $F(x, y, z_1, \ldots, z_r)$ is a formula (in $\mathcal{L}^{=}_{\in}$) with the property:

"whenever $s_1, \ldots, s_r$ are sets and a is a set there is a unique set b such that

$$F(a, b, \underbrace{s_1, \ldots, s_r}_{\text{"parameters"}}) \text{ holds.}"$$

$z_1, \ldots, z_r$ "parameter variables".

With $s_1, \ldots, s_r$ fixed $F(x, y, s_1, \ldots, s_r)$ gives a 'function'

$a \mapsto b$.

F is called an operation on sets.

Example.

1) Without parameters
$F(a, b)$ 'b is the power set of a.'

2) $F(a, b, s_1)$ (parameter: s_1)
'b is the set of functions from a to s_1'.

3) $F(a, b)$ says:
" $a = (U; \leq)$ is a w.o. set and b is an ordinal similar to a; otherwise b is $\emptyset$."

Axiom (scheme) of replacement

(ZF8)

Suppose $F(x, y, z_1, \ldots, z_r)$ is an operation on sets; $s_1, \ldots, s_r$ and A are sets. Then there is a set B with

$$B = \{ b : F(a, b, s_1, \ldots, s_r) \text{ holds for some } a \in A \}$$

B is constructed by "replacing" every elt. $a \in A$ by the corresponding set b.

// In pf. of 3.4.8 :

X $\qquad \alpha_x$ ordinal

$\{ \alpha_x : x \in X \}$

(3.5) Transfinite induction

(3.5.1) Thm. Suppose $P(x)$ is a 1st order formula, or property of sets. Assume that for all ordinals α

(**) if $P(\beta)$ holds for all $\beta < \alpha$ then $P(\alpha)$ holds.

Then $P(\alpha)$ holds for all ordinals α.

Note: (**) includes the "base case" $\alpha = 0 = \emptyset$.

Pf: Suppose for a contradiction that there is an ordinal γ where $P(\gamma)$ does not hold. Consider

$$\{ \delta : \delta \text{ is an ordinal} \leq \gamma \text{ a } P(\delta) \text{ does not hold} \}$$

$$\subseteq \gamma^+ .$$

this is a non-empty set of ordinals, so has a least elt. α. Then if $\beta < \alpha$ (& β an ordinal), $P(\beta)$ holds. So by (**) $P(\alpha)$ holds. ↯. #

(3.5.2) <u>Thm</u>. Suppose α is an infinite ordinal. Then $|\alpha \times \alpha| = |\alpha|$.

(3.5.3) <u>Cor</u>. If $(A; \leq)$ is an infinite w.o. set, then

$$|A \times A| = |A|.$$

<u>Pf</u>: By 3.4.8 then there is an ordinal α with $(A; \leq) \simeq (\alpha; \in)$

therefore $|A \times A| = |\alpha \times \alpha| = |\alpha|$
$= |A|$. #

<u>Pf of (3.5.2)</u>.

(0) Result holds if $\alpha = \omega$ (or indeed, any countably infinite set). So may assume that α is uncountable

(1) <u>Assume</u> that if $\omega \leq \beta < \alpha$ then $|\beta| = |\beta \times \beta|$

<u>Show</u> that $|\alpha| = |\alpha \times \alpha|$.

- the result follows by transfinite induction.

(2) Assuming α is uncountable, & may assume that if $\beta < \alpha$ then $|\beta| < |\alpha|$.

It then follows then $|\beta^+| < |\alpha|$ (ex. 4 on p sheet 7).

(3) Enough to show $|\alpha \times \alpha| \leq |\alpha|$ (as $|\alpha| \leq |\alpha \times \alpha|$
$\gamma \longmapsto (\gamma, 0)$).

STEP 1. Suppose we have a well-ordering $\leq$ of $A = \alpha \times \alpha$ such that for all $x \in A$

$$|\underbrace{A[x]}_{\{y \in A \,:\, y < x\}}| < |\alpha| \quad \ldots (*)$$

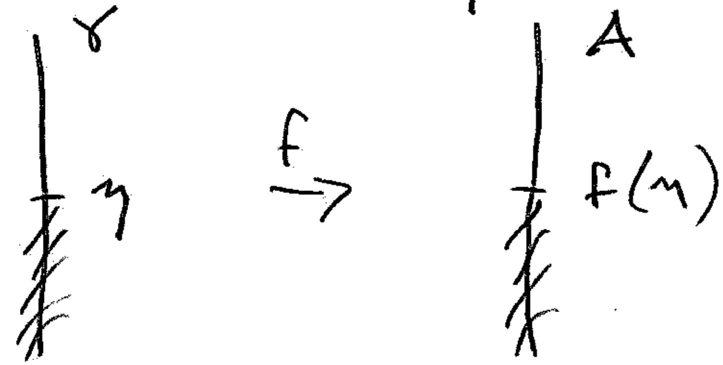
Then $|\alpha \times \alpha| \leq |\alpha|$.

Pf: By 3.4.8 there is an ordinal γ which is similar to $(A; \leq)$. Let $f: \gamma \to A$ be the similarity.

Show $\gamma \subseteq \alpha$.

[Then $|\gamma| \leq |\alpha|$, so $|\alpha \times \alpha| = |\gamma| \leq |\alpha|$.]

Let $\eta \in \gamma$. So $\eta < \gamma$



As f is a similarity it gives a bijection

$$\eta = \{\delta \in \gamma : \delta < \eta\} \to A[f(\eta)]$$

So $|\eta| = |A[f(\eta)]| < |\alpha|$ by $(*)$.

Thus $\eta < \alpha$ (otherwise $\alpha \leq \eta$, so $\alpha \subseteq \eta$, then $|\alpha| \leq |\eta|$ - contradiction).

So $\eta \in \alpha$. Thus $\gamma \subseteq \alpha$. //

Step 1.