IMPERIAL

MATH70135

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Analytic Methods in Partial Differential Equations

Date: Wednesday, May 29, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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1. Throughout the following problem, take $\Omega \subset \mathbb{R}^n$ to be an open, bounded set with smooth boundary.

(a) Let $U \in C^\infty(\Omega)$ such that $0 < c < U(x) < C < \infty \ \forall x \in \Omega$, and let $f \in L^2(\Omega)$. Give a weak formulation for the Dirichlet problem

$$\begin{cases} \text{div}\left(\frac{\nabla u}{U}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Use the Riesz representation theorem to show that the problem above admits a unique weak solution if $f \in L^2(\Omega)$. (8 marks)

(b) Show that the Dirichlet problem

$$\begin{cases} \text{div}\left(\frac{\nabla u}{\sqrt{1+u^2}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique classical solution if $f \in C^\infty(\Omega)$. (4 marks)

(c) Assume that $f \in C^\infty(\mathcal{O})$ for open $\mathcal{O}$ which compactly contains Ω. Show that the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

admits a unique classical solution.
(Hint: convert the given problem to a Dirichlet problem for the Poisson equation.)

(8 marks)

(Total: 20 marks)

2. Let $\Omega = B(0,1) \subset \mathbb{R}^n$ be the open unit ball in $\mathbb{R}^n$. Throughout the following problem, you may apply Stokes' theorem without justification.

(a) For $u_0 \in C_c^\infty(\Omega)$, assume that u is a classical solution, with $u \in L^\infty([0,T]; L^2(\Omega))$, for the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u, \\ u|_{t=0} = u_0, \\ u|_{[0,T]\times\partial\Omega} = 0. \end{cases} \qquad (*)$$

(i) State and prove the weak maximum principle for $(*)$. (6 marks)

(ii) Show that if $u_0(x) \geq 0 \ \forall x \in \Omega$, then $u(t,x) \geq 0 \ \forall t \geq 0, x \in \Omega$. Deduce that

$$\|u(t,\cdot)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}.$$

(3 marks)

(iii) Show that $\|u(t,\cdot)\|_{L^2(\Omega)}$ decays exponentially as $t \longrightarrow \infty$. (3 marks)

(b) Show that the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta^2 u, \\ u|_{t=0} = u_0, \\ u|_{[0,T]\times\partial\Omega} = 0, \\ \frac{\partial}{\partial n_{\partial\Omega}} u|_{[0,T]\times\partial\Omega} = 0 \end{cases} \qquad (**)$$

is ill-posed for initial data in $L_0^2(\Omega)$.
(Hint: one way to do this is to use the basis of L_0^2 consisting of eigenfunctions of Δ^2 in $H_0^2(\Omega)$ to argue by contradiction that the map $S : L^2(\Omega) \longmapsto L^2(\Omega), u_0 \longmapsto u(t,\cdot)$ is discontinuous. You may assume that there exists a countably infinite basis of $L_0^2(\Omega)$ consisting of smooth eigenfunctions to Δ^2 with an unbounded set of positive eigenvalues).

(8 marks)

(Total: 20 marks)

3. (a) (i) State Holmgren's global uniqueness theorem. (6 marks)

(ii) Consider a classical solution u to the one-dimensional heat equation

$$\partial_t u = \partial_x^2 u$$

defined on the half space $\mathbb{R}^2_{t\geq 0}$. Assume that u arises from Schwarz data u_0 at $t = 0$, and suppose $u = u_x = 0$ holds on the segment $\{x = 0\} \times (1, 2)$. On what region does u necessarily vanish? (4 marks)

(b) (i) Show that all surfaces are nowhere characteristic for the Laplace equation $\Delta u = 0$. (4 marks)

(ii) State the unique continuation property for linear elliptic equations with analytic coefficients. (3 marks)

(iii) Prove that if a harmonic function u in $\mathbb{R}^n$ vanishes on $\{x_1 = 0, x_2^2 + \cdots + x_n^2 \leq 1\}$ then it must vanish everywhere on $\mathbb{R}^n$. (3 marks)

(Total: 20 marks)

4. (a) Let Σ be a surface in $\mathbb{R}^n$ given by the zero locus of a smooth function $\psi : \mathbb{R}^n \mapsto \mathbb{R}$. Let u_0 be a classical solution to the m^{th} order PDE

$$F(x, u, \partial^\beta u; |\beta| \leq m) = 0.$$

For $\bar{x} \in \Sigma$, define what it means for Σ to be non-characteristic at $\bar{x}$ on u_0.

(4 marks)

(b) (i) For $\Omega \subset \mathbb{R}^n$ open, let $V : \Omega \longmapsto \mathbb{R}^n$ be a smooth vector field on Ω. For $x \in \Omega$, Σ a smooth surface in $\mathbb{R}^n$ containing x, and $f : \Sigma \mapsto \mathbb{R}$ smooth, give sufficient criteria on Σ and V for a solution to the problem

$$\begin{aligned} V^i(x)\, \partial_i \phi &= 0, \\ \phi|_\Sigma &= f, \end{aligned}$$

to be unique in some neighbourhood of x. (4 marks)

(ii) Let $f \in C^\infty(\mathbb{R}^n)$. Assume that ϕ_1, ϕ_2 solve

$$\begin{aligned} |\nabla \phi|^2 &= 1, \\ \phi|_\Sigma &= f|_\Sigma \end{aligned}$$

in a neighbourhood of $x \in \Sigma$, and assume that Σ is non-characteristic at x on both ϕ_1 and ϕ_2. Show that if $\nabla(\phi_1 - \phi_2)|_\Sigma = 0$, then $\phi_1 = \phi_2$. (4 marks)

(iii) Let $f \in C^\infty(\mathbb{R}^n)$. Assume that ϕ_1, ϕ_2, ϕ_3 solve the problem

$$\begin{aligned} |\nabla \phi|^2 &= 1, \\ \phi|_\Sigma &= f|_\Sigma, \end{aligned}$$

in a neighbourhood of $x \in \Sigma$, and that Σ is non-characteristic at x on any of ϕ_1, ϕ_2, ϕ_3. Conclude that at most two of $\{\phi_1, \phi_2, \phi_3\}$ are distinct. (3 marks)

(c) Find all solutions to the problem

$$\begin{aligned} &|\partial_r \phi|^2 + \frac{1}{r^2}\left[|\partial_\theta \phi|^2 + \frac{1}{\sin^2\theta}|\partial_\varphi \phi|^2\right] = 1, \\ &\phi|_\Sigma = C, \end{aligned}$$

where C is a real number and Σ is the sphere of radius $R > 0$. (5 marks)

(Total: 20 marks)

5. Let $a^{ij} \in C^\infty(\mathbb{R}^n); x \in \mathbb{R}^n \longmapsto a^{ij}(x), i, j = 1, \dots, n$, and assume that $a^{ij}(x) \le A\delta^{ij}$ for some $A > 0$, that $|\nabla a| + |\nabla^2 a| \le B$ for $B > 0$, and that the operator

$$P = \partial_i(a^{ij}\partial_j)$$

is uniformly elliptic. Consider the wave equation

$$(-\partial_t^2 + \sum_{i,j=1}^{n} \partial_i(a^{ij}\partial_j))\phi = 0. \tag{††}$$

(a) Let $V = (v, V^1, \dots, V^n)$ be a non-spacelike vector at the point $(t, x) \in \mathbb{R}^{1+n}$. Show that there exists a constant $\theta > 0$ such that $-v^2 + \theta|V|^2 < 0$. (3 marks)

(b) Assume that $\phi : \mathbb{R}^{1+n}_{t\ge 0} \longmapsto \mathbb{R}$ be a classical solution to (††) arising from data $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ at $t = 0$.

(i) Show the identity

$$\sum_{i,j=1}^{n} 4\frac{x^j}{r}\partial_t\phi a^{ij}\partial_i\phi = \sum_{i,j=1}^{n} a^{ij}\left[\partial_i\phi\partial_j\phi - X^iX^j + 4\frac{x^ix^j}{r^2}(\partial_t\phi)^2\right],$$

where $X^i = \partial_i\phi - 2\frac{x^i}{r}\partial_t\phi$. (1 mark)

(ii) Show that for any fixed $T \ge 0$, $\phi|_{t=T}$ is compactly supported. (9 marks)
(Hint: Consider cones of the form $K_{R,T}\{(t, x); |x| = R - \lambda t, t \le T\}$ for suitable λ).

(iii) Show that there exists a constant

$$C = C(\phi_0, \nabla\phi_0, \nabla^2\phi_0, \phi_1, \nabla\phi_1, a, \nabla a)$$

such that $\|\nabla^2\phi(t, \cdot)\|_{L^2(\mathbb{R}^n)} < C$.

(You may assume the result of Part(b)(ii)). (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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XXX (Solutions)

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1. (a) We say $u \in H_0^1(\Omega)$ is a weak solution to the equation if for any $v \in C_c^\infty(\Omega)$ we have $\boxed{\text{meth seen} \Downarrow}$ $\boxed{8, \text{B}}$

$$\int_\Omega dx \, \frac{1}{U} \nabla u \cdot \nabla v = \int_\Omega dx \, fv.$$ (1 mark)

To apply Riesz's representation theorem, we must that

$$\langle v_1, v_2 \rangle_H := \int_\Omega dx \, \frac{1}{U} \nabla v_1 \cdot \nabla v_2$$

is a well-defined inner product on H_0^1 and that $(H_0^1(\Omega), \langle \, , \, \rangle_H)$ is a Hilbert space. It is clear that $\langle \, , \, \rangle_H$ is linear and symmetric. To show that it is positive definite, we note that if $\langle u, v \rangle_H = 0$ for any $v \in C_c^\infty(\Omega)$ then $\nabla u = 0$. Since $u \in H_0^1(\Omega)$ (i.e. it has a vanishing trace on $\partial\Omega$), A Poincaré inequality then gives that $\|u\|_{L^2(\Omega)} = 0$, thus $u = 0$. (2 marks)

Let $\| \, \|_H$ be the norm induced by $\langle u, v \rangle_H$. Since U is bounded from below away from 0, we have

$$\|u\|_H^2 \leq \frac{1}{\min_\Omega U} \|u\|_{H^1(\Omega)}^2.$$ (1 mark)

To show that $\| \, \|_H$ is equivalent to $\| \, \|_{H^1}$ on $H_0^1(\Omega)$, we apply Poincaré's inequality to get

$$\begin{aligned} \|u\|_{H^1(\Omega)}^2 &\leq (1 + C(\Omega)) \|\nabla u\|_{L^2(\Omega)}^2 && \text{by Poincaré's inequality,} \\ &\leq (1 + C(\Omega))(\max_\Omega |U|) \|u\|_H^2 && \text{using that } U \text{ is bounded.} \end{aligned}$$ (2 marks)

Note that the functional $v \longmapsto \int_\Omega dx \; fv$ is a continuous linear map from $L_0^2(\Omega)$ to $(H_0^1(\Omega), \langle \, , \, \rangle_H)$ since

$$\begin{aligned} \int_\Omega dx \; fv &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} && \text{by Cauchy–Schwartz.} \\ &\leq C(\Omega, \max_\Omega U) \|f\|_{L^2(\Omega)} \|u\|_H. \end{aligned}$$ (1 mark)

Therefore, the Riesz representation theorem applies, and there exists a unique $u \in H_0^1(\Omega)$ which is a weak solution. (1 mark)

(b) We know that the Dirichlet problem $\boxed{\text{unseen} \Downarrow}$ $\boxed{4, \text{B}}$

$$\begin{cases} \Delta g = f & \text{in } \Omega, \\ g = 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique weak solution (by Part (a) above with $U = 1$, or otherwise citing standard elliptic theory). (2 marks)

Take $u = \sinh g$. u is well-defined and has vanishing trace on $\partial\Omega$. Then the chain rule shows that u solves the given problem. (2 marks)

(c) Since $f \in C^\infty(\overline{\Omega})$ and Ω has compact closure, we have that $D^\alpha f \in L^2(\Omega)$ for any multiindex α. (1 mark)

$\boxed{\text{unseen} \Downarrow}$

$\boxed{4, \text{B}}$

Assume that u solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{H}$$

Then $v := u - f$ is a classical solution to the Dirichlet problem

$$\begin{cases} \Delta v = -\Delta f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{I}$$

Conversely, if v solves (I) then $v + f$ solves (H). (2 marks)

We know (by Part (a), or otherwise from standard elliptic theory) that there is unique solution v to (I). Thus there exists a unique u satisfying (H). (1 mark)

2. (a) (i) Statement of the weak maximum principle: For Ω open and bounded, and for $T > 0$, define $\mathcal{U} := (0,T) \times \Omega$. Let $u \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ be such that $(\partial_t - \Delta)u \leq 0$. Then

$\boxed{\text{seen} \Downarrow}$ $\boxed{6, \text{A}}$

$$\max_{\overline{\mathcal{U}}} u = \max_{\partial'\mathcal{U}} u,$$

where $\partial'\mathcal{U} := \{t = 0\} \times \Omega \sqcup (0,T) \times \partial\Omega$. (3 marks)

Proof: Let $\mathcal{U}_\epsilon := (0, T-\epsilon) \times \Omega$. Assume $(\partial_t - \Delta)u < 0$. If the maximum of u occurs in $\mathcal{U}_\epsilon$, then since u is C^2 we have $\partial_t u = 0$, $\Delta u \leq 0$. Similarly, if the maximum of u occurs at $t = T - \epsilon$, we have $\partial_t u \geq 0$, $\Delta u \leq 0$. Thus the maximum of u must occur in $\partial\mathcal{U}_\epsilon$. Since u is continuous on $\overline{\mathcal{U}}$ the conclusion follows in the case $(\partial_t - \Delta)u < 0$ on $\mathcal{U}$. (2 marks)

Now for u such that $(\partial_t - \Delta)u \leq 0$, note $v = u - \delta t$ satisfies $(\partial_t - \Delta)v < 0$ for $\delta > 0$. Thus we have

$$\max_{\overline{\mathcal{U}}} u \leq \delta T + \max_{\overline{\mathcal{U}}} v \leq \delta T + \max_{\partial'\mathcal{U}} v \leq \delta T + \max_{\partial'\mathcal{U}} u.$$

Taking the limit $\delta \to 0$ achieves the result. (1 mark)

$\boxed{\text{unseen} \Downarrow}$ $\boxed{4, \text{B}}$

(ii) Apply the weak maximum principle to $-u$. Since $-u_0(x) \leq 0 \ \forall x \in \Omega$ and $u|_{(0,T)\times\partial\Omega} = 0$, we must have $-u < 0$ on $\mathcal{U}$. (1 mark)

Thus $\int_\Omega dx \ |u(t,\cdot)| = \int_\Omega dx \ u(t,\cdot)$. Integrating the heat equation over $\mathcal{U}$, Stokes' theorem gives that

$$\int_{\mathcal{U}} dtdx \ \Delta u = \int_0^T dt \int_\Omega dx \ \Delta u = \int_0^T dt \int_{\partial\Omega} dx \ \frac{\partial u}{\partial n}.$$

Note that $\frac{\partial u}{\partial n} \leq 0$ since $u|_{\partial'\mathcal{U}} = 0$ and $u \geq 0$ on $\mathcal{U}$. (2 marks)

Therefore,

$$\int_\Omega dx \ u(t,\cdot) \leq \int_\Omega dx \ u(0,\cdot).$$ (1 mark)

$\boxed{\text{seen} \Downarrow}$ $\boxed{3, \text{A}}$

(iii) We multiply the heat equation with u, integrate by parts over Ω. Stokes' theorem gives

$$\int_\Omega dx \ u\Delta u|_{(t,x)} = \int_{\partial\Omega} dx \ u\frac{\partial u}{\partial n}|_{(t,x)} - \int_\Omega |\nabla u(t,x)|^2 = \int_\Omega |\nabla u(t,x)|^2$$

Since $u|_{\partial'\Omega} = 0$. Meanwhile, since $u(t,\cdot) \in C^1(\Omega)$ we have

$$\int_\Omega dx \ 2u\partial_t u|_{t,x} = \partial_t \int_\Omega dx \ |u(t,x)|^2.$$

Therefore, we have

$$\partial_t \|u(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} = 0.$$ (1 mark)

Since $u(t,\cdot) \in H_0^1(\Omega)$, we may apply Poincaré's inequality to get

$$\int_0^T \|u(t,\cdot)\|^2_{L^2(\Omega)} \leq C(\Omega) \int_0^T \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} \qquad \text{(1 mark)}$$

for a constant $C(\Omega) > 0$ that depends on Ω only. Thus

$$\partial_t \|u(t,\cdot)\|^2_{L^2(\Omega)} + 2C(\Omega)\|u(t,\cdot)\|^2_{L^2(\Omega)} \leq 0.$$

Grönwall's inequality (in differential form) then gives

$$\|u(t,\cdot)\|_{L^2(\Omega)} \leq e^{-C(\Omega)t}\|u(0,\cdot)\|_{L^2(\Omega)}. \qquad \text{(1 mark)}$$

unseen ⇓

(b) Assume that there exists a constant C that depends on t and Ω such that

8, D

$$\|u(t,\cdot)\|_{L^2(\Omega)} \leq C(t,\Omega)\|u(0,\cdot)\|_{L^2(\Omega)}. \qquad \text{(assumption)} \quad \text{(1 mark)}$$

Let $\{v_i\}_{i=0}^\infty$ be the eigenfunctions of Δ in $H_0^1(\Omega)$, i.e. $\Delta v_i = -\lambda_i v_i$ for $\lambda_i \geq 0$. Note that by the Fredholm alternative, the set of eigenvalues is unbounded. (1 marks)
For $N \in \mathbb{N}$, take

$$u_0 = \sum_{i=0}^N a_i v_i. \qquad \text{(data)}$$

for a_i real constants. We may find a solution to the given heat equation

$$u(t,x) = \sum_{i=0}^N c_i(t) v_i(x)$$

using the orthonormality of $\{v_i\}_{i=0}^\infty$ to derive ODEs

$$\partial_t c_i = \lambda_i^2 c_i,$$

which gives

$$u(t,x) = \sum_{i=0}^N a_i \, e^{\lambda_i^2 t} v_i(x). \qquad \text{(2 marks)}$$

Note that this solution is the unique solution that attains the data given in (data), since if v is another solution then $v - u|_{t=0}$=0, which implies by (assumption) that $u = v$. (2 marks)
We now find

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} = \sum_{i=0}^N a_i^2 e^{2\lambda_i^2 t}.$$

Note that $\lambda_i > 0$ since a nontrivial $v_i \in H_0^1(\Omega)$ must have $\nabla v_i \neq 0$

Let $\{v^{(n)}\}$ be solutions to the given heat equation arising from data $v_0^{(n)}$, with $v_0^{(n)}$ given by (data) with $a_i = \delta_{in}$. Then $\|v_0^{(n)}\|_{L^2(\Omega)} = 1$ for all $n \geq 0$, while $\|v^{(n)}(t,\cdot)\|_{L^2(\Omega)} = e^{2\lambda_n^2 t}$. Thus $\{v^{(n)}\}$ is an unbounded sequence in $L^2(\Omega)$, contradicting (assumption). (2 marks)

3. (a) (i) Let Ω be an open region of $\mathbb{R}^d$. Let P be a linear partial differential operator of order m with analytic coefficients and $\Sigma \subset \Omega$ an embedded non-characteristic hypersurface for P. seen $\Downarrow$ 6, A

Suppose that $\mathcal{U}$ is an open bounded set of $\mathbb{R}^{d-1}$, and let $\sigma : [0,1] \times \mathcal{U} \longmapsto \Omega$ is continuous, with $\sigma_\lambda = \sigma|_{\{\lambda\}\times\mathcal{U}}$ is a C^m embedding of of a non-characteristic hypersurface Σ_λ, with $\Sigma_0 \subset \Sigma$ and $\sigma([0,1] \times \partial\mathcal{U}) \subset \Sigma$.

If $u \in C^m(\Omega)$ satisfies $Pu = 0$ in Ω and $D^\alpha u = 0$ on Σ for all $|\alpha| \le m-1$. Then, we have $D^\alpha u = 0$ for $|\alpha| \le m-1$ on all of $\sigma([0,1] \times \overline{\mathcal{U}})$.

unseen $\Downarrow$ 6, C

(ii) To start with, the solution must vanish at any $(t_0, x_0) \in (1,2) \times \mathbb{R}$. To see this, consider the family of parabolae given by

$$\Sigma_\lambda = \{(t,x); x = \lambda c(1-t)(t-2), t \in [1,2]\},$$

where $c := -\frac{x_0}{(t_0-1)(t_0-2)}$ and $\lambda \in [0,1]$. Note that Σ_λ is a non-characteristic surface, since

$$(-\partial_t + \partial_x^2)((x + \lambda c(t-1)(t-2))^2)|_{\Sigma_\lambda} = 2 \neq 0.$$

Thus Holmgren's global uniqueness theorem (Part (i) above) gives $u = 0$ at (t_0, x_0), and the argument above applies for any $(t_0, x_0) \in (1,2) \times \mathbb{R}$. (2 marks)

Since we assume that $u(t,\cdot) \in \mathcal{S}(\mathbb{R})$, we know that $\|u(t,\cdot)\|_{L^2(\mathbb{R})}$
$\le \|u(1,\cdot)\|_{L^2(\mathbb{R})}$. Thus $u(t,x) = 0$ if $t \ge 1$. (2 mark)

In fact, we may apply a Fourier transform in x since $u(t,\cdot) \in \mathcal{S}(\mathbb{R})$. The resulting ODE for the Fourier transform of u, $\hat{u}$, reads

$$\partial_t \hat{u} = -\xi^2 \hat{u}.$$

Thus $\hat{u} = 0$ on $t \in [0,1]$ for any ξ. Thus u vanishes identically. (2 mark)

seen $\Downarrow$ 2, A

(b) (i) Let Σ be a smooth surface given by $\Sigma = \{\psi = 0\}$ for a smooth function $\psi : \mathbb{R}^n \mapsto \mathbb{R}$. We say that Σ is characteristic at point x if $\nabla\psi|_x \neq 0$ and

$$\Delta(\psi^2) = 0. \quad \text{(1 marks)}$$

However, a computation gives $\Delta(\psi^2) = 2|\nabla\psi|^2 > 0$ if $\nabla\psi|_x \neq 0$. Therefore, Σ cannot be characteristic at any point. (1 mark)

seen $\Downarrow$ 6, A

(ii) Statement: Let $P(x,\partial)$ be a linear, m^{th} order, elliptic partial differential operator with analytic coefficients in an open connected set $\Omega \in \mathbb{R}^n$ and let Σ be a piece of C^m hypersurface in Ω. Then, the following statement holds: If

1. $u \in C^m(\Omega)$ satisfies $P(x,\partial)u = 0$ in Ω, and
2. $D^\alpha u = 0$ on Σ for $|\alpha| \le m-1$,

then $u = 0$ on all of Ω. (3 marks)

Proof: Assume $x \in \Sigma$ and $\mathcal{N}$ is a neighborhood of x in Σ such that $D^\alpha u = 0$ on $\mathcal{N}$ for $|\alpha| \le m-1$. (3 marks) meth seen $\Downarrow$

3, C

(iii) We know that $u = 0$ in the set $x_1^2 + \cdots + x_n^2 \leq 1$: given $x \in B(0,1)$, take S_λ to be the surface $\{(x_1, \ldots, x_n); x_1 = \lambda\sqrt{1 - x_2^2 - \cdots - x_n^2}\}$. Then Holmgren's global uniqueness theorem applies to give $u(x) = 0$ as S_λ is nowhere characteristic for any $\lambda \in [0,1]$. (1 marks) Now for any $y \in \mathbb{R}^n$, let P be the plane orthogonal to the vector y, given by $\{x\mathbb{R}^n; x \cdot y = 0\}$, and let S_λ be the sphere with center λy and which contain $P \cap S(0,1)$. Since S_λ is nowhere characteristic, we may Holmgren's global uniqueness theorem again to conclude that $u(y) = 0$.. (2 marks)

4. (a) A surface Σ is non-characteristic on u at $\bar{x}$ if Σ is non-characteristic at $\bar{x}$ for the linearisation P of F on u. That is, for Σ given by $\{x; \psi(x) = 0\}$ with $\nabla\psi|_{\bar{x}} \neq 0$, and P the linear partial differential operator given by

seen $\Downarrow$

4, A

$$P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha, \qquad a_\alpha(x) := \frac{\partial F}{\partial(\partial^\alpha u)}\Big|_{x,\partial^\beta u(x);|\beta|\leq m},$$

then Σ must satisfy

$$P(\psi^m)|_{\bar{x}} \neq 0.$$ (4 marks)

unseen $\Downarrow$

4, B

(b) (i) If Σ is non-characteristic at x with respect to the linear operator $\sum_{i=1}^n V^i(x)\partial_i$, then the method of characteristics implies the existence of a unique solution in a neighbourhood of x. (2 marks)

Let Ψ be such that $y \in \Sigma$ iff $\psi(y) = 0$, and assume $\nabla\psi|_x \neq 0$. Then Σ is non-characteristic at x if and only if

$$\sum_{i=1}^n V^i(x)\partial_i\psi \neq 0,$$ (2 marks)

so we need that Σ is not tangent to V at x.

unseen $\Downarrow$

4, D

(ii) We have that

$$0 = |\nabla\phi_1|^2 - |\nabla\phi_2|^2 = \nabla(\phi_1 + \phi_2)\nabla(\phi_1 - \phi_2). \qquad \text{(diff)}$$

If $\nabla(\phi_1 + \phi_2)|_\Sigma$ is orthogonal to n_Σ, then on Σ we have

$$n_\Sigma \cdot \nabla\phi_1 = \frac{1}{2}n_\Sigma \cdot \nabla(\phi_1 + \phi_2) + \frac{1}{2}n_\Sigma \cdot \nabla(\phi_1 - \phi_2) = 0,$$

since $\nabla(\phi_1 - \phi_2)|_\Sigma = 0$. This implies that Σ is characteristic on ϕ_1 at x. Therefore, since we assume Σ is non-characteristic on ϕ_1 at x, we have that n_Σ is not orthogonal to $\nabla(\phi_1 + \phi_2)$ on Σ. (2 marks)

Letting $V^i(x) = \partial_i(\phi_1 + \phi_2)(x)$, we now know by Part(b)(i) that the equation

$$\sum_{i=1}^n V^i(x)\partial_i\phi = 0$$

only admits the zero solution for $\phi|_\Sigma = 0$. Since $\phi_1 - \phi_2$ satisfies (diff) and has vanishing data on Σ, we conclude that $\phi_1 = \phi_2$. (2 marks)

unseen $\Downarrow$

3, D

(iii) We have

$$\nabla(\phi_1 + \phi_2)\nabla(\phi_1 - \phi_2) = 0,$$
$$\nabla(\phi_2 + \phi_3)\nabla(\phi_2 - \phi_3) = 0,$$
$$\nabla(\phi_3 + \phi_1)\nabla(\phi_3 - \phi_1) = 0.$$

Assume that ϕ_1, ϕ_2, ϕ_3 are distinct. Then since the above implies $\nabla(\phi_1+\phi_2)$, $\nabla(\phi_2+\phi_3)$, $\nabla(\phi_3+\phi_1)$ are all tangent to Σ. Therefore,

$$\nabla(\phi_1+\phi_2) = \nabla(\phi_2+\phi_3) = \nabla(\phi_3+\phi_1). \qquad \text{(1 mark)}$$

This implies in particular that

$$\nabla(\phi_1-\phi_2) = \nabla(\phi_2-\phi_3) = \nabla(\phi_3-\phi_1) = 0. \qquad \text{(1 mark)}$$

Therefore, we must have at least two of ϕ_1, ϕ_2, ϕ_3 that have identical gradients at Σ. Say $\nabla(\phi_1-\phi_2)|_\Sigma = 0$. Then by Part(b)(ii) we must have that $\phi_1 = \phi_2$.

(1 mark)

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5, D

(c) Since ϕ is constant on $S(0,R)$, we necessarily have that $\nabla\phi$ is orthogonal to Σ. Therefore, $S(0,R)$ is nowhere characteristic for any solution. So we have at most two solutions by Part(b). (3 marks)

Since the data is spherically symmetric, we may find 2 distinct solutions by setting $\partial_\theta\phi = \partial_\varphi\phi = 0$ and solve

$$\partial_r\phi = \pm 1$$

to get

$$\phi = C \pm r. \qquad \text{(2 marks)}$$

5. (a) Since $(v, V^1, \ldots, V^n)$ is spacelike we have

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3, M

$$-v^2 + \sum_{i,j=1}^{n} a^{ij} V^i V^j \leq 0. \qquad \text{(1 mark)}$$

Since $(a^{ij})_{i,j=1}^n$ defines a uniformly elliptic linear operator there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij} V^i V^j \geq \theta |V|^2, \qquad \text{(1 mark)}$$

where $|V|^2 = \sum_{i=1}^n (V^i)^2$. Therefore, we have

$$-v^2 + \frac{\theta}{2}|V|^2 < 0. \qquad \text{(1 mark)}$$

(b) (i) Note that

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1, M

$$\begin{aligned} &a^{ij}\left(\partial_i\phi - 2\frac{x^i}{r}\partial_t\phi\right)\left(\partial_j\phi - 2\frac{x^j}{r}\partial_t\phi\right) \\ &= a^{ij}\partial_i\phi\partial_j\phi + 4\frac{x^i x^j}{r^2}(\partial_t\phi)^2 - 4\frac{x^j}{r}\partial_t\phi a^{ij}\partial_i\phi. \end{aligned} \qquad \text{(1 marks)}$$

(ii) (It is permitted to use the Einstein summation convention, where summation is implied over repeated indices)

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9, M

The equation implies

$$\partial_t\left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right] - 2\partial_i(a^{ij}\partial_j\phi\partial_t\phi) = 0. \qquad (\|) \quad \text{(2 marks)}$$

Let X be the vector field with components

$$\begin{aligned} X^0 &= (\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi, \\ X^i &= -2a^{ij}\partial_j\phi\partial_t\phi. \end{aligned}$$

We integrate the identity $\mathrm{div}X = 0$ over the cone

$$K_{R,T}(x_0) = \{(t,x); |x - x_0| = R - \lambda t, t \leq T\}$$

for

$$\lambda = 4A. \qquad \text{(slope)}$$

and $T \leq \lambda^{-1}R$. On integrating $\mathrm{div}X = 0$ over $K_{R,T}(0)$, we find

$$\begin{aligned} &\int_{B(0,R-\lambda T)} dx\, \lambda\left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=T} - \int_{B(0,R)} dx\, \lambda\left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=0} \\ &= -\int_0^T dt \int_{S(0,R-\lambda t)} dx\, \left[\lambda(\partial_t\phi)^2 + \lambda a^{ij}\partial_i\phi\partial_j\phi - 2a^{ij}\frac{x_i}{r}\partial_j\phi\partial_t\phi\right] \end{aligned} \qquad \text{(2 marks)}$$

We want to show that the integrand on the right hand side above is non-negative. The identity of Part(b)(i) gives

$$\lambda(\partial_t\phi)^2 + \lambda a^{ij}\partial_i\phi\partial_j\phi - 2a^{ij}\frac{x_i}{r}\partial_j\phi\partial_t\phi = \left(\lambda - 2a^{ij}\frac{x^ix^j}{r^2}\right)(\partial_t\phi)^2 + \frac{\lambda}{2}a^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2\lambda}a^{ij}\left(\partial_i\phi - 2\frac{x^i}{r}\partial_t\phi\right)\left(\partial_j\phi - 2\frac{x^j}{r}\partial_t\phi\right).$$

Since $\lambda = 4A$, we have $\lambda - 2a^{ij}\frac{x^ix^j}{r^2} \geq 2A$. Moreover,

$$a^{ij}\left(\lambda\partial_i\phi\partial_j\phi\right) + \frac{1}{\lambda}\left(\partial_i\phi - 2\frac{x^i}{r}\partial_t\phi\right)\left(\partial_j\phi - 2\frac{x^j}{r}\partial_t\phi\right) \geq \theta(\lambda|\nabla\phi|^2 + \lambda^{-1}|V|^2) \geq 0, \quad \text{(2 marks)}$$

where $V^i = \partial_i\phi - 2\frac{x^i}{r}\partial_t\phi$. We deduce

$$\int_{B(0,R-\lambda T)} dx\ \left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=T} \leq \int_{B(0,R)} dx\ \left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=0}. \quad (\ddagger)$$

(1 mark)

Now assume that ϕ arises from $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ at $t = 0$. Assume that the support of (ϕ_0, ϕ_1) is contained in $B(0, R)$. For any $T > 0$, we have that $(T, x) \in K_{a,T}(x)$ for $|x| > R + \lambda T$ and $a = \lambda T$. Note that $(\phi_0, \phi_1) = 0$ on $B(x, \lambda T)$, since $y \in B(x, \lambda T)$ implies $|y| > R + \lambda T - \lambda T = R$. We conclude by $\ddagger$ that $\phi = 0$ at (T, x). (2 marks)

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7, M

(iii) By Part(b)(ii), $\phi(t,\cdot)$ and $\partial_t\phi(t,\cdot)$ have compact support. We may then integrate ($\|$) in the region $[0,T] \times \mathbb{R}^3$ to get

$$\int_{\mathbb{R}^3} dx\ \left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=T} = \int_{\mathbb{R}^3} dx\ \left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=0}. \quad \text{(1 mark)}$$

The uniform ellipticity of a implies there exists $\theta > 0$ such that

$$\int_{\mathbb{R}^3} dx\ \left[(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]|_{t=T} \geq \int_{\mathbb{R}^3} dx\ \left[(\partial_t\phi)^2 + \theta|\nabla\phi|^2\right]|_{t=T}.$$

Thus we have

$$\|\phi(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\partial_t\phi(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} \leq \theta^{-1}\int_{\mathbb{R}^3} dx\ \left[(\phi')^2 + a^{ij}\partial_i\phi\partial_j\phi\right]. \quad \text{(1 mark)}$$

Commuting the equation by ∂_t, we may apply the above argument again to $\partial_t\phi$. Noting that the equation implies

$$\partial_t^2\phi|_{t=0} = \partial_i(a^{ij}\partial_j\phi),$$

we get

$$\|\partial_t\phi(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\partial_i(a^{ij}\partial_j\phi)(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} \leq \theta^{-1}\int_{\mathbb{R}^3} dx\ \left[(\partial_i(a^{ij}\partial_j\phi))^2 + a^{ij}\partial_i\phi'\partial_j\phi'\right].$$

(2 marks)

We now apply Stokes' theorem to compute $\|\partial_i(a^{ij}\partial_j\phi)(t,\cdot)\|^2_{L^2(\mathbb{R}^3)}$ and find

$$\int_{\mathbb{R}^3} dx\ \partial_i(a^{ij}\partial_j\phi)\partial_k(a^{k\ell}\partial_\ell\phi) = -\int_{\mathbb{R}^3} dx\ a^{ij}\partial_j\phi\partial_k\partial_i a^{k\ell}\partial_\ell\phi + \int_{\mathbb{R}^3} dx\ \partial_k a^{ij}\partial_j\phi a^{k\ell}\partial_i\partial_\ell\phi \\ + \int_{\mathbb{R}^3} dx\ a^{ij}\partial_k\partial_j\phi a^{k\ell}\partial_i\partial_\ell\phi. \qquad \text{(1 mark)}$$

The last term on the right hand side satisfies

$$\int_{\mathbb{R}^3} dx\ a^{ij}\partial_k\partial_j\phi a^{k\ell}\partial_i\partial_\ell\phi \geq \theta^2 \int_{\mathbb{R}^3} dx\ |\nabla^2\phi|^2,$$

whereas we have by Cauchy–Schwartz

$$\int_{\mathbb{R}^3} dx\ a^{ij}\partial_j\phi\partial_k\partial_i a^{k\ell}\partial_\ell\phi \leq AB\|\nabla\phi\|_{L^2(\mathbb{R}^3)},$$

and Young's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^3} dx\ \partial_k a^{ij}\partial_j\phi a^{k\ell}\partial_i\partial_\ell\phi &\leq \frac{B}{4\epsilon}\|\nabla\phi\|_{L^2(\mathbb{R}^3)} + \epsilon A\int_{\mathbb{R}^3} dx\ |\nabla^2\phi|^2 \\ &\leq \frac{B}{4\epsilon}\theta^{-1}\int_{\mathbb{R}^3} dx\ \left[(\Phi')^2 + a^{ij}\partial_i\Phi\partial_j\Phi\right] + \epsilon A\int_{\mathbb{R}^3} dx\ |\nabla^2\phi|^2\end{aligned}$$

for any $\epsilon > 0$. Choosing $\epsilon = \frac{1}{2A\theta^2}$ (or some other suitable choice), we get

$$\int_{\mathbb{R}^3} dx\ |\nabla^2\phi|^2 \leq C(\theta, A, B)\left[\int_{\mathbb{R}^3} dx\ \left[(\partial_i(a^{ij}\partial_j\Phi))^2 + a^{ij}\partial_i\Phi'\partial_j\Phi' + (\Phi')^2 + a^{ij}\partial_i\Phi\partial_j\Phi\right]\right].$$

(2 marks)

Review of mark distribution:
Total A marks: 27 of 27 marks
Total B marks: 24 of 24 marks
Total C marks: 9 of 9 marks
Total D marks: 20 of 20 marks
Total marks: 80 of 80 marks
Total Mastery marks: 20 of 20 marks