Chapter 2: Singular Integral Equations

In this chapter we will learn how to solve equations of the form

$$\frac{1}{\pi}\int_a^b k(t-x)f(t)dt = g(x), \tag{14}$$

where $a < x < b$, a, b, finite, and $g(x)$ is given, for the unknown function $f(x)$. There is no explicit solution known for equations of this form for a general kernel function k, however, notable exceptions are when k is of the form

$$k(t-x) = \frac{1}{t-x}, \quad \text{or} \quad k(t-x) = \log(t-x).$$

Notice in these cases the associated integral equation (14) has a singularity along the integration path (when $t = x$), hence the name **singular integral equations**.

2.1 The Cauchy and Hilbert Transforms

In order to solve equations of the form (14) we will make use of the **Cauchy Transform** (of a function $f(\xi)$ over a path γ) (sometimes simply referred to as a Cauchy-type integral) defined by

$$C(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi, \tag{15}$$

where γ is a smooth contour and $f(\xi)$ is continuous for $\xi \in \gamma$. In general this transform is defined for complex valued functions $f(\xi)$, and γ can be any contour in the complex plane, but for simplicity in this course we will only consider the case when γ is an interval of the real line (in particular we will set $\gamma = [-1, 1]$). Since the transform depends on $f(\xi)$ and γ it is usually denoted by $C_{[f,\gamma]}(z)$ (or some other way illustrating the explicit f and γ dependence), but within our course it should be sufficiently clear what we mean so we drop this notation and simply use $C(z)$.

Proposition 2.37. *The Cauchy transform is* ***analytic*** *for all z, including as $z \to \infty$ (in fact $C(z) \sim O(\frac{1}{z})$ as $z \to \infty$),* ***except*** *for $z \in \gamma$.*

Proof. The analyticity extends from Cauchy's Integral Theorem. The proof of the decay can be found in 'Trogdon and Olver: Riemann-Hilbert Problems, Their Numerical Solution, and the Computation of Nonlinear Special Functions' on pages 34-35. □

For $z \in \gamma$, we define the **Hilbert Transform** (of a function $f(\xi)$ over a path γ) to be

$$H(z) = \frac{1}{2\pi i}\fint_\gamma \frac{f(\xi)}{\xi - z}d\xi, \tag{16}$$

where γ is a smooth contour and $f(\xi)$ is continuous for $\xi \in \gamma$. As before this transform is defined for arbitrary smooth curves $\gamma \in \mathbb{C}$, but we will restrict our attention to the case when γ is an interval of the real line. Similarly, the transform is usually denoted by $H_{[f,\gamma]}(z)$ (or some equivalent way), but for our purposes the notation $H(z)$ should be clear.

Remark: Note that for $z \in \gamma$, $H(z)$ has a singularity along the integration path, hence the use of the principal value integral sign.

Note: In other sources/previous versions of this course, the definition of $H(z)$ may have a different factor outside of the integral, like $-\frac{1}{\pi}$ for instance.

2.2 The Plemelj Formulae

Consider again the Cauchy transform (of a function $f(\xi)$ over a path γ) given by

$$C(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

Let's investigate what happens as $z \to z_0$, where z_0 is **any** point on γ (except for the end points of γ). To do this, we introduce the notation where, looking in the direction of integration along γ, we denote:

$C_+(z_0) =$ the limiting value of $C(z)$ as $z \to z_0$ from the **left** of γ.
$C_-(z_0) =$ the limiting value of $C(z)$ as $z \to z_0$ from the **right** of γ.

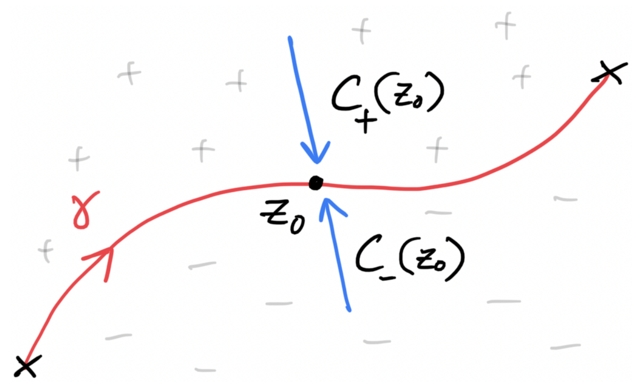

Figure 32

To examine $C_+(z_0)$, we consider the integral along the deformed contour $\gamma_\varepsilon + c_\varepsilon^+$ as shown in figure 33 and take the limit as $\varepsilon \to 0$.

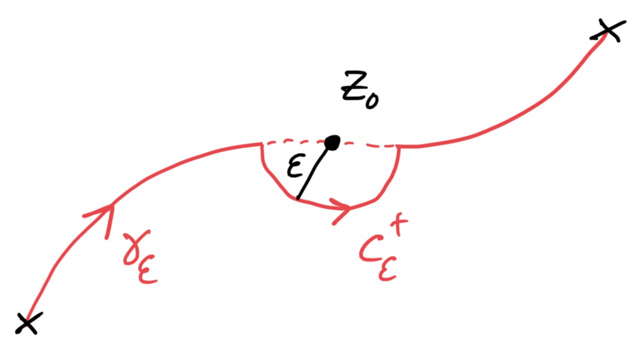

Figure 33

Here $\gamma_\varepsilon = \gamma$ with a section of length 2ε about z_0 removed and $c_\varepsilon^+ =$ semi-circle, centre z_0, radius ε. So, we have

$$C_+(z_0) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon + c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi.$$

On c_ε^+ we have $z = z_0 + \varepsilon e^{i\theta}$, where $\theta_0 \le \theta \le \theta_0 + \pi$ (for some angle θ_0). Hence

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi &= \lim_{\varepsilon\to 0} \int_{\theta_0}^{\theta_0+\pi} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta \\ &= i\pi f(z_0),\end{aligned}$$

where we have used the Taylor expansion of f about z_0 given by $f(z_0 + \varepsilon e^{i\theta}) = f(z_0) + O(\varepsilon)$ and then taken the limit $\varepsilon \to 0$ to reach the result. Hence we get

$$2\pi i C_+(z_0) = \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z_0} d\xi + \lim_{\varepsilon\to 0} \int_{c_\varepsilon^+} \frac{f(\xi)}{\xi - z_0} d\xi,$$

giving

$$C_+(z_0) = H(z_0) + \frac{1}{2} f(z_0), \tag{17}$$

where

$$H(z_0) = \frac{1}{2\pi i} ⨍_\gamma \frac{f(\xi)}{\xi - z_0} d\xi,$$

is the previously defined Hilbert transform. Now consider $C_-(z_0)$.

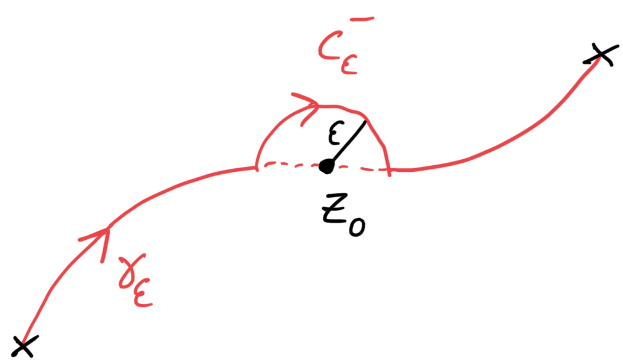

Figure 34

Here γ_ε is as before and c_ε^- is as c_ε^+ but the other half of the circle. Using similar analysis to before, but now noting $\theta_0 \ge \theta \ge \theta_0 - \pi$ (note this is $-\pi$ not $+\pi$ as we have gone around z_0 clockwise this time), we find (exercise):

$$\begin{aligned}C_-(z_0) &= \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\gamma_\varepsilon + c_\varepsilon^-} \frac{f(\xi)}{\xi - z_0} d\xi \\ &= H(z_0) - \frac{1}{2} f(z_0).\end{aligned} \tag{18}$$

Finally considering (17) + (18) and (17) - (18) we obtain:

$$C_+(z_0) + C_-(z_0) = 2H(z_0) \tag{19}$$

$$C_+(z_0) - C_-(z_0) = f(z_0). \tag{20}$$

These two equations are together known as the **Plemelj formulae**.

Example (to illustrate the Plemelj formulae in action):
Let's show that the Plemelj formulae hold for an example case. Let's take $f(x) = 1$ and take γ to be the segment of the real axis between -1 and 1: $-1 < \xi < 1$.

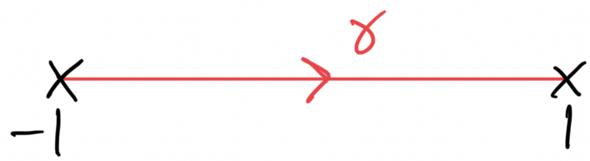

Figure 35: The path γ between $[-1, 1]$.

Thus, the **Cauchy transform** gives:

$$C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{1}{\xi - z} d\xi = \frac{1}{2\pi i}\log\left(\frac{z-1}{z+1}\right).$$

Next we want to consider $C_\pm(x)$, for $-1 < x < 1$. Note that $\log(\frac{z-1}{z+1})$ is a multi-valued function with branch points at ± 1. Taking a branch cut along γ gives a branch of this function analytic everywhere except for a jump discontinuity across γ. Introducing local coordinates:

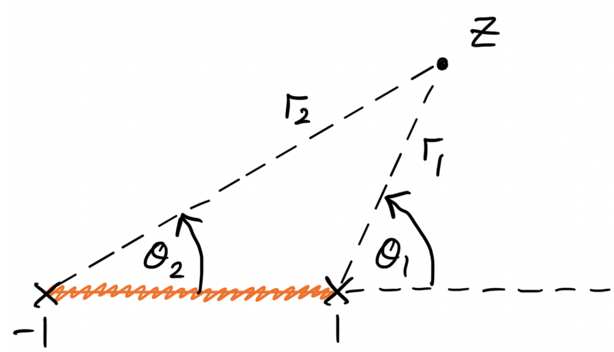

Figure 36

Here $r_1 = |z-1|$, $\theta_1 = \arg\{z-1\}$, $r_2 = |z+1|$, $\theta_2 = \arg\{z+1\}$ and let's take $-\pi < \theta_1, \theta_2 < \pi$. Then

$$C(z) = \frac{1}{2\pi i}\left(\log\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2)\right). \tag{21}$$

Now let $x^{\pm} = x \pm i\delta$ for $-1 < x < 1$ and δ small. At x^{+}: $\theta_1 = \pi$, $\theta_2 = 0$, so we get $\theta_1 - \theta_2 = \pi$. At x^{-}: $\theta_1 = -\pi$, $\theta_2 = 0$, so we get $\theta_1 - \theta_2 = -\pi$. At both $x^{\pm}$ we have $r_1 = 1 - x$ and $r_2 = 1 + x$. Then, using (21):

$$C_+(z) = \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right) + \frac{1}{2},$$
$$C_-(z) = \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right) - \frac{1}{2}.$$

Finally let's work out $H(x)$:

$$\begin{aligned} H(x) &= \frac{1}{2\pi i}\fint_{-1}^{1}\frac{1}{\xi - x}d\xi \\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{1}{\xi - x}d\xi + \int_{x+\varepsilon}^{1}\frac{1}{\xi - x}d\xi\right] \\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\log|-\varepsilon| - \log|-1-x| + \log|1-x| - \log|\varepsilon|\right) \\ &= \frac{1}{2\pi i}\log\left(\frac{1-x}{1+x}\right). \end{aligned}$$

Then, for $x \in \gamma$, we have shown:

$$\begin{aligned} C_+(x) + C_-(x) = \frac{1}{\pi i}\log\left(\frac{1-x}{1+x}\right) &= 2H(x), \\ C_+(x) - C_-(x) = 1 &= f(x), \end{aligned}$$

both expected by Plemelj.

2.3 The Converse Problem

Our main goal of this chapter will be to learn how to solve equations of the form of (14). In order to motivate the solution scheme for this, let us for a moment consider the converse to this problem. That is, suppose, conversely, we are given a function $f(z)$ which is continuous along some smooth path γ, and our goal is to find a function, $G(z)$ say, which is analytic for **all** z **except** on γ, where for $z_0 \in \gamma$ it has the jump discontinuity: $G_+(z_0) - G_-(z_0) = f(z_0)$, and furthermore which vanishes at infinity.
The solution to this problem is **unique** and is given by

$$G(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi - z}d\xi, \tag{22}$$

i.e the Cauchy transform of f on γ!

Why? Let's check this:

1). Analytic at all z except on γ:
 $G(z)$ is the Cauchy transform of $f(z)$ which has this analyticity property.

2). $G_+(z_0) - G_-(z_0) = f(z_0)$, $z_0 \in \gamma$:
 This is true by the Plemelj formulae.

3). $G(z)$ vanishes at infinity:
This is true since another property of the Cauchy transform is that $G(z) \sim O(1/z)$ as $z \to \infty$. So we have decay at infinity.

4). Uniqueness of solution:
Let's verify this. Suppose G_1 and G_2 are two solutions. Then the difference $G_1 - G_2$ is analytic for **all** z including on γ (since G_1 and G_2 have the same jump discontinuity across γ) and $G_1 - G_2$ vanishes as $z \to \infty$. Hence, by Liouville's theorem: $G_1 - G_2 = \text{constant} = 0$, by noting the behaviour at infinity. Thus $G_1 = G_2$, i.e we have uniqueness.

Remark: We required $G(z) \to 0$ as $z \to \infty$, but we can weaken this to allow for algebraic growth, i.e $G(z) \sim O(|z|^n)$ as $z \to \infty$, for some $n \in \mathbb{N}$.
In this case, we can check (exercise) using the extension of Liouville's theorem that $G(z)$ is determined up to a polynomial of degree n:

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi + p_n(z),$$

where $p_n(z)$ is an arbitrary polynomial of degree n (this may be determined by additional conditions if we have them).

2.4 The Inversion Problem: Cauchy Kernel

Let us return to our main focus of interest, namely problem (14). That is to find a function $f(x)$ satisfying

$$\frac{1}{\pi} \fint_a^b k(t-x) f(t) dt = g(x),$$

for $a < x < b$, where $k(t-x)$ and $g(x)$ are given functions. As mentioned earlier, for the kernel functions $k(t-x) = 1/(t-x)$ and $k(t-x) = \log(t-x)$ there are known techniques to solve the equation. Let's start with the so called **Cauchy kernel** $k(t-x) = 1/(t-x)$. We will also restrict ourselves to the case where $a = -1$, $b = 1$ throughout this course for simplicity (though in general this is not necessary). So, our aim is to find a function $f(x)$ satisfying

$$\frac{1}{\pi} \fint_{-1}^1 \frac{f(t)}{t-x} dt = g(x), \quad -1 < x < 1. \tag{23}$$

In terms of the Hilbert transform, we can rephrase this problem as

$$2iH(x) = g(x). \tag{24}$$

Let's solve for $f(x)$ as follows: first introduce the Cauchy transform:

$$C(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{f(t)}{t-z} dt.$$

This satisfies the Plemelj formulae

$$C_+(x) + C_-(x) = 2H(x), \tag{25}$$
$$C_+(x) - C_-(x) = f(x), \tag{26}$$

where

$$H(z) = \frac{1}{2\pi i}\fint_{-1}^{1} \frac{f(t)}{t-z}dt,$$

is the Hilbert transform of $f(x)$ on $[-1,1]$. Now using (24), the first Plemelj equation (25) gives

$$C_+(x) + C_-(x) = -ig(x). \tag{27}$$

Now recall the 'converse problem' from section 2.3. We can solve problems of the form

$$G_+(x) - G_-(x) = \text{known function}. \tag{28}$$

So this motivates a new goal; can we convert (27) to an equation of the form of (28).
Suppose now we have some function $\phi(z)$, which is analytic everywhere except for a jump discontinuity across γ of the form

$$\phi_+(x) = -\phi_-(x), \tag{29}$$

for $-1 < x < 1$. Then consider $w(z) = \phi(z)C(z)$. For $-1 < x < 1$, we have

$$\begin{aligned} w_+(x) - w_-(x) &= \phi_+(x)C_+(x) - \phi_-(x)C_-(x) \\ &= \phi_+(x)(C_+(x) + C_-(x)) \\ &= -i\phi_+(x)g(x), \end{aligned} \tag{30}$$

where we have used (29) to reach the second line and (27) to reach the final line. So, provided we can find such a function $\phi(z)$, then (30) is of the form of the 'converse problem', whose solution is known.

Finding $\phi(z)$

For $\phi(z)$ we need a function analytic everywhere except for the jump discontinuity $\phi_+(x) = -\phi_-(x)$ across $-1 < x < 1$. We have already encountered a function satisfying this in the course already!
We take $\phi(z) = \sqrt{z^2-1}$, with a branch cut along $-1 < x < 1$.
Let's double check this gives the required jump in $\phi(z)$ across $[-1,1]$. As usual, introduce local coordinates r_1, r_2, θ_1, θ_2 as shown.

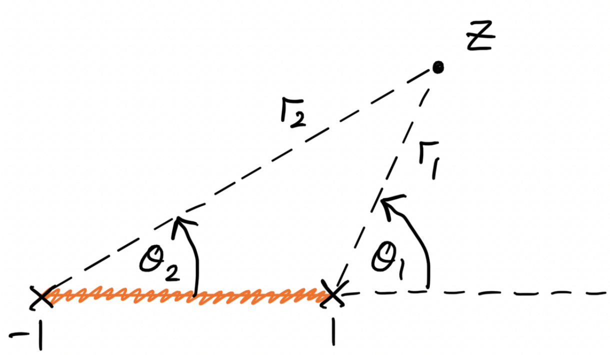

Figure 37: Usual local coordinates.

Here $r_1 = |z-1|$, $\theta_1 = \arg\{z-1\}$, $r_2 = |z+1|$, $\theta_2 = \arg\{z+1\}$ and let's take $-\pi \leq \theta_1, \theta_2 \leq \pi$ (recall that this choice corresponds to the branch that looks like $+z$ as $z \to \infty$).
Now $\phi(z) = \sqrt{z^2-1} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{\Theta}{2}}$, where $\Theta = \theta_1 + \theta_2$.
On the upperside of the branch cut, for $-1 < x < 1$, $y = \delta$ ($\delta > 0$ small), we have: $\theta_1 = \pi$, $\theta_2 = 0$, giving $\Theta = \pi$, resulting in $\phi(x) = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i\sqrt{1-x^2}$ (where we have used the usual facts that $r_1 = |z-1| = -(x-1) = 1-x$ and $r_2 = |z+1| = x+1$).
On the lowerside of the branch cut, for $-1 < x < 1$, $y = -\delta$, we have: $\theta_1 = -\pi$, $\theta_2 = 0$, giving $\Theta = -\pi$, resulting in $\phi(x) = (r_1 r_2)^{\frac{1}{2}} e^{-i\frac{\pi}{2}} = -i\sqrt{1-x^2}$.
Hence $\phi_+(x) = -\phi_-(x)$ for $-1 < x < 1$ as we required.

Returning to where we were

Let's return back to our solution scheme; we now have

$$w_\pm(x) = \phi_\pm(x) C_\pm(x) = \pm i\sqrt{1-x^2} C_\pm(x). \tag{31}$$

This means that (30) gives $w_+(x) - w_-(x) = -i\phi_+(x) g(x) = \sqrt{1-x^2} g(x)$. We know the solution to this problem as it is of the form of the 'converse problem'. The general solution is

$$w(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2} g(t)}{t-z} dt + p_n(z), \tag{32}$$

where $p_n(z)$ is a polynomial. We can determine $p_n(z)$ as follows; as $z \to \infty$, $\phi(z) \sim O(z)$, $C(z) \sim O(1/z)$, so since $w(z) = \phi(z) C(z)$, then $w(z) \sim O(1)$. But in (32), the Cauchy-type integral $\sim O(1/z)$, hence it must be the case that we have $p_n(z) = \text{constant} = A_1$, say. So,

$$w(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\sqrt{1-t^2} g(t)}{t-z} dt + A_1.$$

Now recall the other Plemelj equation (26) we haven't used yet

$$\begin{aligned} f(x) &= C_+(x) - C_-(x) \\ &= \frac{1}{i\sqrt{1-x^2}} [w_+(x) + w_-(x)], \end{aligned} \tag{33}$$

using (31). Now, using the Plemelj formulae for the function $w(z)$, we deduce that, for $-1 < x < 1$:

$$w_+(x) + w_-(x) = 2H_w(x),$$

where

$$H_w(x) = \frac{1}{2\pi i} \fint_{-1}^{1} \frac{\sqrt{1-t^2} g(t)}{t-x} dt + A_2,$$

for some constant A_2. Here I have used the subscript w to denote that this is the Hilbert transform corresponding to function $w(z)$ not $C(z)$. Hence, plugging this into (33) gives

$$f(x) = \frac{2}{i\sqrt{1-x^2}} H_x(x),$$

or

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} \fint_{-1}^{1} \frac{\sqrt{1-t^2} g(t)}{t-x} dt + \frac{A}{\sqrt{1-x^2}}, \tag{34}$$

for a constant $A \in \mathbb{R}$. (34) is known as the **Hilbert Inversion Formula**.

2.5 Example Problem

Let's do an example to illustrate how we use the inversion formula and solve for $f(x)$.
Example: Find a function $f(x)$ satisfying

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x}dt = 1, \quad -1 < x < 1.$$

i.e. the function $g(x) = 1$ here.
Solution: Applying the **Hilbert inversion formula** (34), we have

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} I(x) + \frac{A}{\sqrt{1-x^2}}, \tag{35}$$

where

$$I(x) = ⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}dt.$$

All that's left to do is to determine $I(x)$. Let's consider two different methods:

Method 1 (Plemelj formulae):

Introduce the Cauchy transform of $\sqrt{1-z^2}$ on $[-1, 1]$:

$$C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-z}dt.$$

Then, applying the **Plemelj formulae** (and noting that $I(x) = 2\pi i H(x)$ for $\sqrt{1-x^2}$ on $[-1, 1]$):

$$I(x) = \pi i(C_+(x) + C_-(x)), \tag{36}$$

for $-1 < x < 1$. Now let's find $C_\pm(x)$ as follows: take $z = x \in \mathbb{R} > 1$. Let $t = \cos\theta$. Then

$$\begin{aligned}
C(x) &= \frac{1}{2\pi i}\int_{\pi}^{0} \frac{-\sin^2\theta}{\cos\theta - x}d\theta \\
&= \frac{1}{2\pi i}\int_{0}^{\pi} \frac{1-\cos^2\theta}{\cos\theta - x}d\theta \\
&= \frac{-1}{2\pi i}\int_{0}^{\pi} \frac{(x+\cos\theta)(x-\cos\theta) - (x^2-1)}{(x-\cos\theta)}d\theta \\
&= \frac{1}{2\pi i}\Bigg[\underbrace{-\int_{0}^{\pi}(x+\cos\theta)d\theta}_{=-\pi x} + (x^2-1)\underbrace{\int_{0}^{\pi}\frac{d\theta}{x-\cos\theta}}_{=\frac{\pi}{\sqrt{x^2-1}}}\Bigg],
\end{aligned}$$

where the second integral is left as an exercise in complex analysis and residue theory (see similar examples on problem sheet 1: consider $1/2\int_{-\pi}^{\pi}$ and convert to an integral around the unit circle). Hence we find

$$C(x) = \frac{1}{2i}(-x + \sqrt{x^2-1}),$$

for $x > 1$. Then, by analytic continuation, we must have

$$C(z) = \frac{1}{2i}(-z + \sqrt{z^2 - 1}),$$

for complex z. Then, recall that for $-1 < x < 1$, we know $\sqrt{z^2-1}|_{z=x\pm i\delta(\delta<<1)} = \pm i\sqrt{1-x^2}$. Hence for $-1 < x < 1$ we have

$$C_{\pm}(x) = \frac{1}{2i}(-x \pm i\sqrt{1-x^2}).$$

Thus (36) gives

$$\begin{aligned} I(x) &= \pi i(\frac{1}{2i}(-2x)) \\ &= -\pi x, \end{aligned} \tag{37}$$

which, on substitution into (35) gives

$$f(x) = \frac{(x + A)}{\sqrt{1-x^2}}.$$

To determine the constant A we need extra information given in the problem (for example the value of $f(x)$ at some point x_0.

Method 2 (Contour integration):

Consider the analytic continuation of the integrand of $I(x)$ into the complex z-plane given by

$$\frac{\sqrt{z^2-1}}{z-x}.$$

Note the difference inside the square root. Let's expand this function as $z \to \infty$:

$$\begin{aligned} \frac{\sqrt{z^2-1}}{z-x} &= \frac{\sqrt{1-\frac{1}{z^2}}}{1-\frac{x}{z}} \\ &= \left(1 - \frac{1}{2z^2} + O(\frac{1}{z^4})\right)\left(1 + \frac{x}{z} + O(\frac{1}{z^2})\right) \\ &= 1 + \frac{x}{z} + O(\frac{1}{z^2}), \end{aligned} \tag{38}$$

where we have used the expansions $(1-X)^{-1} = 1 + X + X^2 + \cdots$ and $(1+X)^n = 1 + nX + \frac{n(n-1)}{2!}X^2 + \cdots$ in the second line. Owing to the result (38) we consider the contour integral

$$\oint_\gamma \left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz,$$

where γ is the contour $\gamma = \gamma_1 + \gamma_2 + \gamma_+ + \gamma_- + \gamma_R + \gamma_{\varepsilon_1} + \gamma_{\varepsilon_2} + \gamma_{\varepsilon_+} + \gamma_{\varepsilon_-}$, as shown in figure 38.

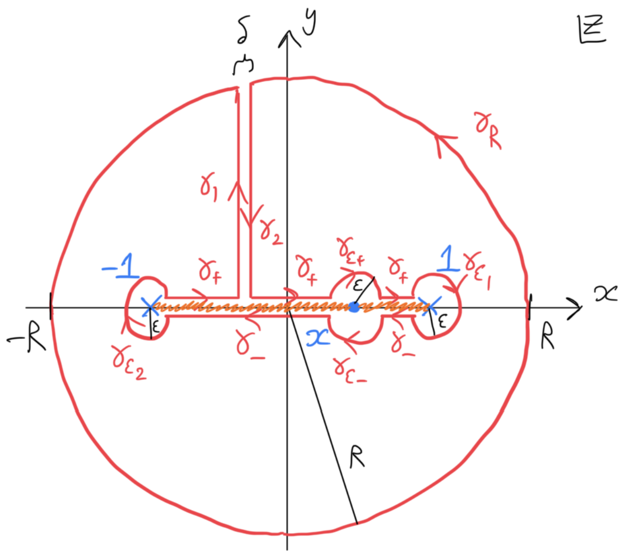

Figure 38: Contour path.

Here γ_R is a circle, centre 0, radius R. $\gamma_{\varepsilon_{1,2}}$ are circles, radius ε with centres on $1, -1$. $\gamma_{\varepsilon_\pm}$ are semi-circles, centre x, radius ε. $\gamma_\pm$ are straight lines that hug the top and bottom sides respectively of the branch cut for the multivalued function $\sqrt{z^2-1}$ taken along the real axis between $[-1, 1]$ (we choose the branch that $\sim +z$ as $z \to \infty$). Now consider the limit as $\varepsilon \to 0$, $\delta \to 0$ and $R \to \infty$.
There are no singularities inside the contour, hence by Cauchy's theorem

$$\oint_\gamma \left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz = 0.$$

Now let's consider the integrals along the separate segments of γ. First note that

$$\int_{\gamma_1+\gamma_2} = 0,$$

since the integral is continuous between γ_1 and γ_2 and we integrate along them in opposite directions. Secondly, we can check that (exercise)

$$\lim_{\varepsilon\to 0}\int_{\varepsilon_{1,2}} = 0.$$

Now let's see what's happening on γ_R. Here $z = Re^{i\theta}$, where θ takes any range of 2π. We find

$$\begin{aligned}
\lim_{R\to\infty}\int_{\gamma_R}\left[\frac{\sqrt{z^2-1}}{z-x} - 1\right] dz &= \lim_{R\to\infty}\int_0^{2\pi}\left[\left(1 + \frac{x}{Re^{i\theta}} + O(\frac{1}{R^2})\right) - 1\right] iRe^{i\theta} d\theta \\
&= \lim_{R\to\infty}\int_0^{2\pi}\left(ix + O(\frac{1}{R})\right) d\theta \\
&= 2\pi i x,
\end{aligned}$$

where we have used the fact that we know the integrand behaves as found in (38) as $z \to \infty$. Now recall that on the top and bottom sides of the branch cut we have $\sqrt{z^2-1}|_{\gamma_\pm,\gamma_{\varepsilon_\pm}} = \pm i\sqrt{1-x^2}$. Also note that on γ_{ε_+} we have $z = x + \varepsilon e^{i\theta}$, $0 \le \theta \le \pi$. Thus

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\gamma_+ + \gamma_{\varepsilon_+}} &= \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon}\frac{i\sqrt{1-t^2}}{t-x}dt + \int_{x+\varepsilon}^{1}\frac{i\sqrt{1-t^2}}{t-x}dt + \int_{\pi}^{0}\frac{i\sqrt{1-x^2}}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta - \int_{-1}^{1}1dt\right]\\ &= i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt + \pi\sqrt{1-x^2} - 2.\end{aligned}$$

Similarly, on γ_-, γ_{ε_-} (recalling $\sqrt{z^2-1} = -i\sqrt{1-x^2}$ and with $z = x + \varepsilon e^{i\theta}$ where $-\pi \le \theta \le 0$ on γ_{ε_-}), it can be shown that (exercise):

$$\lim_{\varepsilon\to 0}\int_{\gamma_- + \gamma_{\varepsilon_-}} = i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt - \pi\sqrt{1-x^2} + 2.$$

Thus, summing up all the contributions and equating this to the result from Cauchy's theorem

$$2i⨍_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}dt + 2\pi i x = 0,$$

or in other words

$$I(x) = -\pi x, \tag{39}$$

as found via method 1. As before, substituting for $I(x)$ in (35) gives the unknown function $f(x) = \frac{x+A}{\sqrt{1-x^2}}$.

2.6 The Inversion Problem: Logarithmic Kernel

Once again let us return to problem (14), but now consider the **logarithmic kernel** given by $k(t-x) = \log(t-x)$ and once again consider the case where $a=-1$, $b=1$ for simplicity. This results in us trying to find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = g(x), \quad -1 < x < 1. \tag{40}$$

Assuming $f(x)$ is 'nice' (e.g analytic), then in fact such an integral is non-singular even though the integrand itself is (roughly speaking this is because it is possible to show that the integral goes like $\varepsilon\log\varepsilon \to 0$ as $\varepsilon \to 0$). Such equations are sometimes referred to as **weakly** singular integral equations (this is why we don't use the $⨍$ symbol here).

Let's solve for $f(x)$ as follows

$$\begin{aligned}\int_{-1}^{1} f(t)\log|t-x|dt &= \int_{-1}^{x} f(t)\log(x-t)dt + \int_{x}^{1} f(t)\log(t-x)dt\\ &= \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon} f(t)\log(x-t)dt + \int_{x+\varepsilon}^{1} f(t)\log(t-x)dt\right],\end{aligned}$$

because the integral along the section of length 2ε about x tends to zero as $\varepsilon \to 0$ (again glossing over some details this is because $\varepsilon\log\varepsilon \to 0$ as $\varepsilon \to 0$). Now we will differentiate with respect to x under the integral sign. Note here however, that the variable x is also present in the limits of the integral, hence denoting $h(x,t) = f(t)\log(x-t)$, we find

$$\begin{aligned}\frac{d}{dx}\left\{\int_{-1}^{x-\varepsilon} f(t)\log(x-t)dt\right\} &= h(x,x-\varepsilon)\cdot\frac{d}{dx}(x-\varepsilon) - h(x,-1)\cdot\frac{d}{dx}(-1) + \int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt \\ &= \int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt + f(x-\varepsilon)\log\varepsilon,\end{aligned}$$

where in the first line the full formula for differentiating under the integral sign when limits are non-constant was used. Similarly, we can show

$$\frac{d}{dx}\left\{\int_{x+\varepsilon}^{1} f(t)\log(t-x)dt\right\} = \int_{x+\varepsilon}^{1}\frac{-f(t)}{t-x}dt - f(x+\varepsilon)\log\varepsilon.$$

Hence, we have

$$\begin{aligned}\frac{d}{dx}\left\{\int_{-1}^{1} f(t)\log|t-x|dt\right\} &= \lim_{\varepsilon\to 0}\left\{\int_{-1}^{x-\varepsilon}\frac{f(t)}{x-t}dt + \int_{x+\varepsilon}^{1}\frac{f(t)}{x-t}dt + (f(x-\varepsilon)-f(x+\varepsilon))\log\varepsilon\right\} \\ &= -⨍_{-1}^{1}\frac{f(t)}{t-x}dt,\end{aligned}$$

where in the first line the term $(f(x-\varepsilon)-f(x+\varepsilon))\log\varepsilon \to 0$ as $\varepsilon \to 0$ (again provided $f(x)$ is 'nice'). Thus we have a strategy to solve (40). Namely differentiating (40) with respect to x gives

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{f(t)}{t-x}dt = -g'(x), \quad -1 < x < 1. \tag{41}$$

This is now an equation of the form of (23) for which we can apply the Hilbert inversion formula to solve for $f(x)$.

2.7 Example Problems

Let's look at a couple of examples where the logarithmic kernel is used.

Example 1:

Find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = 3, \quad -1 < x < 1.$$

Solution: Differentiating wrt x we find

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{f(t)}{t-x}dt = 0, \quad -1 < x < 1.$$

Then, applying the Hilbert inversion formula (34) we get

$$f(x) = \frac{A}{\sqrt{1-x^2}},$$

where A is a constant. With logarithmic singular integral equations we can determine the value of A by substituting our form of $f(x)$ back into the original equation. Here we get

$$\frac{1}{\pi}\int_{-1}^{1}\frac{A\log|t-x|}{\sqrt{1-t^2}}dt = 3, \quad -1 < x < 1.$$

This equation holds for all x between -1 and 1. So, in particular, it holds for $x = 0$, which means

$$\frac{A}{\pi}\int_{-1}^{1}\frac{\log|t|}{\sqrt{1-t^2}}dt = 3,$$

or $A = \frac{3\pi}{I}$, where

$$I = \int_{-1}^{1}\frac{\log|t|}{\sqrt{1-t^2}}dt.$$

Now,

$$I = 2\underbrace{\int_{0}^{1}\frac{\log t}{\sqrt{1-t^2}}dt}_{=I_0} = 2I_0.$$

Let's compute I_0. To do this let $t = \sin\theta$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}\log(\sin\theta)d\theta \\ &= \frac{1}{2}\int_0^{\pi}\log(\sin\theta)d\theta, \end{aligned}$$

using symmetry arguments. Now put $\alpha = \theta/2$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}\log(\sin 2\alpha)d\alpha \\ &= \int_0^{\pi/2}\log(2\sin\alpha\cos\alpha)d\alpha \\ &= \int_0^{\pi/2}(\log 2 + \log(\sin\alpha) + \log(\cos\alpha))d\alpha. \end{aligned}$$

Now put $\beta = \pi/2 - \alpha$, then

$$\begin{aligned} I_0 &= \int_0^{\pi/2}(\log 2 + 2\log(\sin\beta))d\beta \\ &= \frac{\pi}{2}\log 2 + 2I_0. \end{aligned}$$

Giving

$$I_0 = -\frac{\pi}{2}\log 2.$$

Thus we have

$$\begin{aligned} A &= \frac{3\pi}{2I_0} \\ &= -\frac{3}{\log 2}. \end{aligned}$$

Hence the solution to the original problem is given by

$$f(x) = \frac{-3}{\log 2}\cdot\frac{1}{\sqrt{1-x^2}}.$$

Example 2:

Find a function $f(x)$ satisfying

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = x, \quad -1 < x < 1.$$

Solution: Differentiating wrt x we find

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x}dt = -1, \quad -1 < x < 1.$$

Hence, applying the Hilbert inversion formula (34) we get

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

where A is a constant. But recall from earlier, in section 2.5 equations (37) and (39), we showed that

$$I(x) = ⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}dt = -\pi x.$$

Thus, we have

$$f(x) = \frac{A-x}{\sqrt{1-x^2}}.$$

Again, to determine A, substitute for $f(x)$ into the original equation

$$\frac{1}{\pi}\int_{-1}^{1} \left[\frac{-t}{\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}}\right]\log|t-x|dt = x, \quad -1 < x < 1.$$

This time, checking what happens for $x = 0$, we find

$$\frac{1}{\pi}\int_{-1}^{1} \left[\frac{-t}{\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}}\right]\log|t|dt = 0.$$

Now the first term here is an **odd** function, so it integrates to 0 over $[-1, 1]$, leaving us with

$$\frac{A}{\pi}\int_{-1}^{1} \frac{\log|t|}{\sqrt{1-t^2}}dt = 0,$$

and one can check that the integrand of this is < 0 for all values of t, hence this integral is **non-zero**. Thus we conclude

$$A = 0,$$

giving

$$f(x) = \frac{-x}{\sqrt{1-x^2}}.$$

2.8 Ideal Fluid Flow past a Flat Plate

Understanding branch cuts and Cauchy transforms allows us to solve problems which can be reduced to singular integral equations. A classic example of this is the Laplace equation for ideal fluid flow. We consider the case of uniform flow with angle α around an infinitesimally thin plate on $[-1, 1]$. This can be modelled as

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= 0, \text{ everywhere off } [-1,1], \\ \psi(x, 0) &= 0, \text{ for } -1 < x < 1, \\ \psi(x, y) &\sim y \cos\alpha - x \sin\alpha, \text{ as } x^2 + y^2 \to \infty, \end{aligned} \tag{42}$$

where $\psi(x, y)$ is some real-valued function (called the streamfunction) which corresponds in some way to the fluid trajectories (for those taking fluids courses, or those simply interested, we have what are called streamlines, lines which the fluid follows over time, when $\psi(x, y) = \text{constant}$). Using the techniques we have developed over the last lectures, let's see how we can obtain a nice, closed-form expression for the solution $\psi(x, y)$ as the imaginary part of an analytic function.

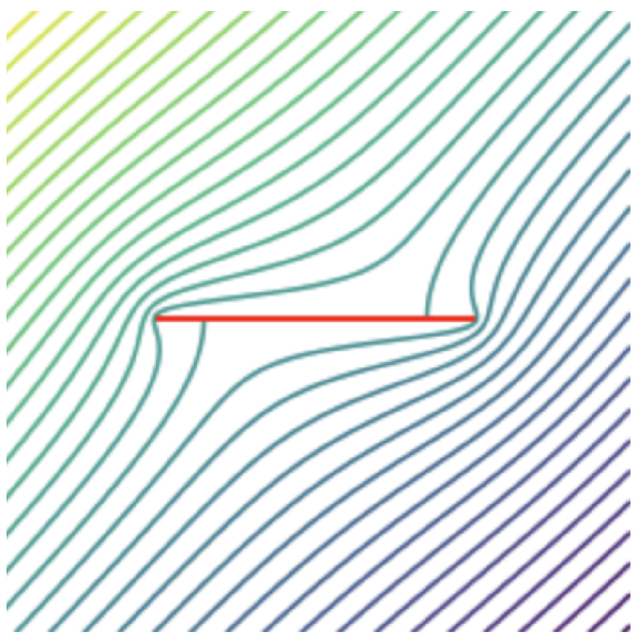

Figure 39: Ideal fluid flow at an angle of $\pi/4$ past a flat plate along $[-1, 1]$.

The figure above shows a plot of the solution we will find. The lines are called streamlines and represent how the fluid flows around the flat plate on $[-1, 1]$. In the plot $\alpha = \pi/4$ radians. We will solve this problem in stages:

1). Rephrasing the problem in $\psi(x, y)$ to a complex analytical problem in $w(z)$.

2). Reduction to a singular integral equation.

3). Calculating the inverse Hilbert transform using the inversion formula.

4). Calculating the remaining Cauchy transform.

Steps 3 and 4 involve the usual solution methods we have employed in our example problems so far.

Step 1: The real and imaginary parts of an analytic function satisfy Laplace's equation: that is if $w(z) = \phi(x, y) + i\psi(x, y)$, where ϕ and ψ are the real and imaginary parts, then

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

We proved this result in chapter 1. Therefore we can write the ideal fluid flow problem as a problem of calculating $w(z) = \phi(x, y) + i\psi(x, y)$ whose imaginary part, $\psi(x, y)$, is the solution to the ideal fluid flow pde (we are not interested in finding $\phi(x, y)$ here, although this can readily be found from $w(z)$ and does have meaning in a fluid mechanics context; it is known as the velocity potential and its x and y partial derivates give the horizontal and vertical velocity components of the fluid).

So, rephrased, we want to find an **analytic** function $w(z)$ satisfying

$$\begin{aligned} \mathrm{Im}\{w(x)\} &= 0, \quad \text{for } -1 < x < 1, \\ w(z) &\sim e^{-i\alpha} z, \quad \text{as } z \to \infty. \end{aligned}$$

(Exercise: check that $e^{-i\alpha}z$ gives the correct behaviour corresponding to (42)).

Step 2: Now $w(z)$ must be a function analytic everywhere off the plate between $[-1, 1]$ with the correct far field and plate boundary conditions. Let's think about what the Cauchy transform does here... well, the Cauchy transform of some function, say $f(z)$, on $[-1, 1]$ generates an analytic function everywhere off $[-1, 1]$ that decays as $z \to \infty$. This is exactly what we need when paired with the correct far-field behaviour! So, let's make the ansatz

$$w(z) = \underbrace{e^{-i\alpha} z + c}_{\text{the behaviour at } \infty} + \underbrace{\frac{1}{2\pi i} \int_{-1}^{1} \frac{f(t)}{t - z} dt}_{= C(z)}, \tag{43}$$

where $c \in \mathbb{C}$ is a constant. Here $C(z)$ is as mentioned the Cauchy transform of some unknown function $f(z)$, which gives us a function analytic everywhere off the plate, which decays at ∞. It remains to satisfy the plate condition $\mathrm{Im}\{w(x)\} = 0$ on $-1 < x < 1$. Before we check this note that

$$\overline{C(z)} = \frac{-1}{2\pi i} \int_{-1}^{1} \frac{f(t)}{t - \overline{z}} dt = -C(\overline{z}),$$

so long as $f(x)$ is a real-valued function (for all z off $[-1, 1]$). It then follows using this fact that, if we take $z = x + i\delta$, where $\delta << 1$ is real, and let $\delta \to 0$

$$\overline{C_+(x)} = \overline{C}_+(x) = -C_-(x). \tag{44}$$

Thus, we can find the imaginary part of the Cauchy transform, using the Plemelj formulae, by

$$\begin{aligned} 2H(x) = C_+(x) + C_-(x) &= C_+(x) - \overline{C_+(x)} = 2i\mathrm{Im}\{C_+(x)\} \\ &= C_-(x) - \overline{C_-(x)} = 2i\mathrm{Im}\{C_-(x)\}, \end{aligned}$$

where we have used (44) in each line to arrive at the two different forms for $2H(x)$. This means that we must have $\mathrm{Im}\{C_+(x)\} = \mathrm{Im}\{C_-(x)\} = \mathrm{Im}\{C(x)\}$, giving

$$H(x) = i\mathrm{Im}\{C(x)\}. \tag{45}$$

Now applying the plate condition on $z = x$ where $-1 < x < 1$:

$$\begin{aligned} 0 = \mathrm{Im}\{w(x)\} &= \mathrm{Im}\{(\cos\alpha - i\sin\alpha)(x)\} + \mathrm{Im}\{c\} + \mathrm{Im}\{C(x)\} \\ &= -x\sin\alpha + \mathrm{Im}\{c\} - iH(x), \end{aligned}$$

upon using result (45). Hence we arrive at the **singular integral equation**

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{f(t)}{t-x}dt = -2x\sin\alpha + 2\mathrm{Im}\{c\}, \quad -1 < x < 1,$$

which is an equation of the form (23) which we have learnt how to solve.

Step 3: For simplicity I'll suppose now $c = 0$, so $\mathrm{Im}\{c\} = 0$, but in principle you could solve this equation with the methods we have learned with c arbitrary and be given extra information later on to determine it. Then we have

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{f(t)}{t-x}dt = -2x\sin\alpha, \quad -1 < x < 1.$$

Applying the Hilbert inversion formula, we find

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\fint_{-1}^{1}\frac{\sqrt{1-t^2}(-2t\sin\alpha)}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

or

$$f(x) = \frac{2\sin\alpha}{\pi\sqrt{1-x^2}}I(x) + \frac{A}{\sqrt{1-x^2}},$$

where

$$I(x) = \fint_{-1}^{1}\frac{t\sqrt{1-t^2}}{t-x}dt.$$

Exercise (Problem Sheet 2): show that $I(x) = \frac{\pi}{2} - \pi x^2$. Then

$$f(x) = \frac{\sin\alpha - 2\sin\alpha x^2 + A}{\sqrt{1-x^2}}.$$

Now from physical principles the solution should not blow-up. If $f(x)$ blows-up, then so does its Cauchy transform, and hence our solution. This means at the end points where $x = \pm 1$ we need to ensure $f(x)$ stays regular. Therefore, we choose $A = \sin\alpha$, so that

$$f(x) = \frac{2\sin\alpha(1-x^2)}{\sqrt{1-x^2}} = 2\sin\alpha\sqrt{1-x^2}, \tag{46}$$

which is regular at $x = \pm 1$.

Step 4: Now substituting back for $f(x)$ from (46) into our expression (43) for $w(z)$ gives

$$w(z) = e^{-i\alpha}z + \frac{1}{2\pi i}\int_{-1}^{1} \frac{2\sin\alpha\sqrt{1-t^2}}{t-z}dt.$$

But now recall from section 2.5 in Method 1, we showed

$$C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-z}dt = \frac{1}{2i}(-z+\sqrt{z^2-1}).$$

Hence

$$w(z) = e^{-i\alpha}z - i\sin\alpha(-z+\sqrt{z^2-1}),$$

is the complex potential function we are looking for whose imaginary part gives the function $\psi(x,y)$ desired:

$$\psi(x,y) = \operatorname{Im}\{e^{-i\alpha}z - i\sin\alpha(-z+\sqrt{z^2-1})\}. \tag{47}$$

This is the solution for the streamfunction $\psi(x,y)$ for uniform flow at an angle α past a flat plate along the real axis between $[-1,1]$. Plotting $\psi(x,y) =$ constant, for a range of different constants gives a plot like the one shown in figure 39.

2.9 Electrostatic Potential of a point charge near a flat plate

Another application of Laplace's equation is that of electrostatics. Suppose we are interested in determining the electric field surrounding a plate where a nearby point source is located. Let's suppose the point source is at $x = 2$, say, and the plate is located on $[-1,1]$ and is being held at some constant potential, k, which we don't know. This problem can be modelled as

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \text{ for } z \text{ off } [-1,1] \text{ and } x = 2,\\
V(z) &\sim \log|z-2| + O(1), \text{ for } z \to 2,\\
V(z) &\sim \log|z| + o(1), \text{ for } z \to \infty,\\
V(x) &= k, \text{ for } -1 < x < 1.
\end{aligned}$$

Here $V(z)$ is the electrostatic potential which we want to solve for. It turns out (we omit the details here) that we can represent the solution in the form

$$V(z) = \frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-z|dt + \log|z-2|. \tag{48}$$

On $-1 < x < 1$, this satisfies

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|dt = k - \log(2-x).$$

Differentiating wrt x

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{f(t)}{t-x}dt = -\frac{1}{2-x}.$$

Now applying the Hilbert inversion formula we find

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}} \underbrace{⨍_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-2)(t-x)}dt}_{=I(x)} + \frac{A}{\sqrt{1-x^2}}.$$

Exercise (Problem Sheet 2): Show that $I(x) = \frac{-\pi\sqrt{3}}{x-2}$. Thus

$$f(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{A}{\sqrt{1-x^2}}.$$

To determine the constant A we can look at the far-field behaviour of our solution. We know $V(z) \sim \log|z|$ as $z \to \infty$. It is also possible to show that (we omit the details here)

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-z|dt \sim \frac{1}{\pi}\int_{-1}^{1} f(t)dt \log|z|, \quad \text{as } z \to \infty.$$

Thus, since $\log|z-2| \sim \log|z|$ as $z \to \infty$, we must set

$$\frac{1}{\pi}\int_{-1}^{1} f(t)dt = 0, \tag{49}$$

since then in (48), $V(z) \sim 0 \times \log|z| + \log|z-2| \sim \log|z|$ as $z \to \infty$ as required.
Exercise (Problem Sheet 2): (49) gives

$$\frac{1}{\pi}\int_{-1}^{1} \left(\frac{\sqrt{3}}{(t-2)\sqrt{1-t^2}} + \frac{A}{\sqrt{1-t^2}} \right) dt = 0,$$

which leads to $A = 1$. Hence

$$f(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}},$$

meaning that upon back substitution into (48) we have

$$V(z) = \frac{1}{\pi}\int_{-1}^{1} \left[\frac{\sqrt{3}}{(t-2)\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} \right] \log|t-z|dt + \log|z-2|.$$

For any point z in the plane this integral can be calculated numerically and so we can plot the solution over a range of z values.