IMPERIAL

MATH70143

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamics, Symmetry and Integrability

Date: Wednesday, May 1, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. **Foucault pendulum: Oscillation dynamics in a rotating frame**

Hamilton's action principle for the Foucault pendulum is given in planar polar coordinates $(r, \theta, \dot{r}, \dot{\theta}) \in T\mathbb{R}^2$ by

$$0 = \delta S = \delta \int_0^T L\, dt = \delta \int_0^T \frac{m}{2}\Big(\dot{r}^2 + r^2(\dot{\theta} + \Omega)^2 - \omega^2 r^2\Big) dt\,. \tag{1}$$

This system has two frequencies: Ω about the vertical axis (e.g., Earth's rotation) and oscillation frequency $\omega = \sqrt{g/l_0}$ with gravitational acceleration g and pendulum length l_0.

(a) What is the symmetry group G of the Lagrangian L in the equation above?

(2 marks)

(b) Write the Euler-Lagrange equations for this Lagrangian in terms of the canonical variables.

(2 marks)

(c) Legendre transform the Lagrangian and write the Hamiltonian.

(2 marks)

(d) Derive Hamilton's canonical equations for this system.

(2 marks)

(e) Explain why the radial motion for this system is periodic.

(4 marks)

(f) Although the radial motion for this system is periodic,
show that the angular motion is not periodic.

(8 marks)

(Total: 20 marks)

2. **Adjoint and co-Adjoint motion**

(a) Verify that

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\mathrm{g}_t^{-1}}\zeta = -\,\mathrm{ad}_\xi \zeta \,.$$

for any *fixed* $\zeta \in \mathfrak{g}$ and $\xi = \dot{g}_t g_t^{-1}\Big|_{t=0}$.

(4 marks)

(b) Prove the following identities with $\xi_t = g_t^{-1}\dot{g}_t$ for *fixed* $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$:

$$\left\langle \frac{d}{dt}\Big(\mathrm{Ad}_{\mathrm{g}_t}\eta\Big)\,,\,\mu \right\rangle = \Big\langle \mathrm{Ad}_{\mathrm{g}_t}\Big(\mathrm{ad}_{\mathrm{g}_t^{-1}\dot{\mathrm{g}}_t}\eta\Big)\,,\,\mu\Big\rangle = \Big\langle \mathrm{Ad}_{\mathrm{g}_t}\Big(\mathrm{ad}_{\xi_t}\eta\Big)\,,\,\mu\Big\rangle \;,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes a nondegenerate real-valued pairing which defines dual quantities.

(4 marks)

(c) Prove the following dynamical relation:

$$\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}\right) = \mathrm{Ad}^*_{g^{-1}(t)}\left(\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}\right) = 0\,.$$

Note that the last equation is the Euler-Poincaré equation.

(8 marks)

(d) Prove that the Euler-Poincaré equation arises from the constrained variational principle,

$$0 = \delta S = \delta \int_0^T \ell(\xi) + \Big\langle \mu\,,\, g^{-1}\dot{g} - \xi \Big\rangle\, dt\,,$$

where δg vanishes at the endpoints in time. Also calculate the Noether quantity.

Hint: A side calculation of $\delta\xi - \dot{\eta} = \mathrm{ad}_\xi \eta$ with $\xi = g^{-1}\dot{g}$ and $\eta := g^{-1}\delta g$ may be useful.

(4 marks)

(Total: 20 marks)

3. **Properties of the diamond operator**

From its definition,

$$\langle p\,,\,\delta q\rangle = \langle p\,,\,-\pounds_\xi q\rangle_{TQ} =: \langle p\diamond q\,,\,\xi\rangle_{\mathfrak{g}} =: J^\xi(q,p)\,,$$

the properties of diamond operator ($\diamond$) are inherited from the properties of the Lie derivative.

(a) Let the Lagrangian $L : TQ \to \mathbb{R}$ in Hamilton's principle $\delta S = 0$ with constrained action integral $S = \int_0^T L(q,v) + \left\langle p\,,\,\frac{dq}{dt} - v\right\rangle dt$ be invariant under a Lie-group action $g_\epsilon q(t) = q(t,\epsilon) : G\times Q \to Q$ where the curve $g_\epsilon \in G$ is the flow of G parameterised by ϵ which becomes the identity transformation at $\epsilon = 0$.

Prove Noether's theorem for this situation in terms of the corresponding diamond operator.

(6 marks)

(b) Show that under the canonical Poisson bracket, the quantity $J^\xi(q,p)$ generates δq and δp.

(6 marks)

(c) Calculate Poisson bracket $\{J^\xi(p,q), H(p,q)\}$ for $J^\xi(p,q) = \langle p\diamond q\,,\,\xi\rangle$ and Hamiltonian $H(p,q)$.

(6 marks)

(d) State and prove Noether's theorem in phase space.

(2 marks)

(Total: 20 marks)

4. **Hamilton-Pontryagin principle for heavy top dynamics**

Consider the following ***Hamilton-Pontryagin principle***

$$0 = \delta S = \delta \int_a^b \ell(\Omega, g^{-1}(t)e_3) + \Big\langle \Pi\,,\, g^{-1}\dot{g}(t) - \Omega \Big\rangle \, dt\,.$$

Physically, when variable $(\Omega, \Pi, \Gamma \in \mathbb{R}^3$ with constant $e_3 \in \mathbb{R}^3$ and the Lie group is $g = SO(3)$, then the resulting dynamics would describe ***heavy top dynamics***.

(a) Calculate the equations of motion for this Hamilton-Pontryagin principle. That is, show that

$$\frac{d\Pi}{dt} = \mathrm{ad}^*_\Omega \Pi + \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma \quad \text{and} \quad \frac{\mathrm{d}\Gamma}{\mathrm{dt}} = \Omega \,.\, \Gamma\,,$$

where $\Omega := g^{-1}\dot{g}(t)\,, \quad \Gamma := g^{-1}(t)e_3, \quad \Omega \,.\, \Gamma$ denotes matrix Lie algebra action of $\Omega \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$. (The matrix Lie algebra action $\Omega \,.\, \Gamma$ in the heavy top case is $\Omega \,.\, \Gamma = -\Omega \times \Gamma$ for $\Omega \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$.)

$$\Pi := \frac{\partial \ell}{\partial \Omega} \quad \text{and} \quad \Big\langle \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma \,,\, \Xi \Big\rangle = \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, -\Xi \,.\, \Gamma \Big\rangle, \quad \Xi = g^{-1}\delta g\,, \quad (\Omega, \Xi) \in \mathfrak{g} \times V\,,$$

and $\Xi \,.\, \Gamma$ represents the Lie algebra action of $\Xi \in \mathfrak{so}(3)$ on $\Gamma \in \mathbb{R}^3$..

(10 marks)

(b) Use the reduced Legendre transform to determine the corresponding Hamiltonian and express its partial derivatives with respect to Π and Γ.

(2 marks)

(c) Calculate the Hamiltonian equations in (Π, Γ) and write them in Lie-Poisson matrix form.

(4 marks)

(d) Calculate the corresponding Lie-Poisson bracket, $\{f, h\}(\Pi, \Gamma)$.

(4 marks)

(Total: 20 marks)

5. Euler-Poincaré Hamilton's principle for ideal fluids

(a) Consider the Euler-Poincaré Hamilton's principle for ideal fluid dynamics with advected quantities, whose reduced action integral S_{red} with Lagrangian $\ell : u \in \mathfrak{X}(\mathcal{D}) \times a \in V(\mathcal{D}) \to \mathbb{R}$ is given by

$$S_{red} = \int_0^T \ell(u, a)\, dt$$

Compute the corresponding Euler-Poincaré equations.

(6 marks)

(b) Use the reduced Legendre transform to express the corresponding Hamiltonian,

(2 marks)

(c) Determine the Hamiltonian's functional derivatives with respect to μ and a

(2 marks)

(d) Write the Hamiltonian formulation of fluid dynamics with advected quantities in Lie-Poisson matrix form.

(4 marks)

(e) Show that the Lie-Poisson bracket is dual to a semidirect-product Lie algebra and identify this Lie algebra.

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60143/70143

Symmetry, Dynamics & Integrability (Solutions)

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1. **Foucault pendulum: Oscillation dynamics in a rotating frame**

seen ⇓

(a) The symmetry group of this Lagrangian is S^1 (angular translations in θ).

xx

2, A

(b) The Noether endpoint terms define linear momentum & angular momentum as

seen ⇓

$$p_r := \frac{\partial L}{\partial \dot r} = m\dot r \quad \text{and} \quad p_\theta := \frac{\partial L}{\partial \dot\theta} = mr^2(\dot\theta + \Omega)\,,$$

which satisfy Euler-Lagrange equations

$$\frac{dp_r}{dt} = mr\big((\dot\theta+\Omega)^2 - \omega^2\big) = \frac{p_\theta^2}{mr^3} - m\omega^2 r \quad \text{and} \quad \frac{dp_\theta}{dt} = 0\,.$$

xx

2, A

(c) The Legendre transform to the Hamiltonian $H(r, p_r, p_\theta)$ is given by

seen ⇓

$$\begin{aligned} H(r,p_r,p_\theta) &:= \langle p_r\,,\, \dot r\rangle + \left\langle p_\theta\,,\, \dot\theta\right\rangle - L(r,\dot r,\dot\theta) \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \Omega p_\theta + \frac{m}{2}\omega^2 r^2\,. \end{aligned}$$

xx

2, A

(d) Hamilton's canonical equations for the Foucault pendulum are given by

meth seen ⇓

$$\dot r = \frac{\partial H}{\partial p_r} = p_r/m\,, \quad \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} - \Omega\,,$$

$$\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - m\omega^2 r\,, \quad \dot p_\theta = -\frac{\partial H}{\partial \theta} = 0\,.$$

xx

2, B

(e) The radial phase portrait shows that the radial motion for this system is periodic, because the radial potential is convex upward.

meth seen ⇓

xx

4, B

seen ⇓

(f) Since p_θ is constant, the solution for $\theta(t)$ is given by direct integration as

$$\int_0^T \dot\theta(t)dt = \theta(T) - \theta(0) = \int_0^T \frac{p_\theta}{mr^2(t)}\,dt - \Omega T\,.$$

Consequently, this solution is *not* periodic. Instead, after each closed orbit $r(t)$ with period T, the angle $\theta(t)$ has precessed by the angle $\Delta\theta = \Omega T$.

xx

8, B

2. **Adjoint and co-Adjoint motions** $\boxed{\text{meth seen} \Downarrow}$

(a)

$$
\begin{aligned}
\frac{d}{dt}\mathrm{Ad}_{g_t^{-1}}\zeta = \frac{d}{dt}(g_t^{-1}\zeta g_t) &= -(g_t^{-1}\dot{g}_t)g_t^{-1}\zeta g_t + g_t^{-1}\zeta\dot{g}_t \\
&= -\,\xi_t(\mathrm{Ad}_{g_t^{-1}}\zeta) + (g_t^{-1}\zeta g_t)(g_t^{-1}\dot{g}_t) = -\,\xi_t(\mathrm{Ad}_{g_t^{-1}}\zeta) + (\mathrm{Ad}_{g_t^{-1}}\zeta)\xi_t \\
&= -\,\mathrm{ad}_{\xi_t}(\mathrm{Ad}_{g_t^{-1}}\zeta)
\end{aligned}
$$

Then evaluating $\mathrm{Ad}_{g_t^{-1}}\big|_{t=0} = Id$ yields the result.

xx $\boxed{4, \text{A}}$

(b) $\boxed{\text{meth seen} \Downarrow}$

$$
\begin{aligned}
\left\langle \frac{d}{dt}\big(\mathrm{Ad}_{g_t}\eta\big)\,,\,\mu \right\rangle = \left\langle \frac{d}{dt}\big(g_t\eta g_t^{-1}\big)\,,\,\mu \right\rangle &= \left\langle \dot{g}_t\eta g_t^{-1} - g_t\eta g_t^{-1}\dot{g}_t g_t^{-1}\,,\,\mu \right\rangle \\
&= \left\langle \mathrm{Ad}_{g_t}\big((g_t^{-1}\dot{g}_t)\eta\big) - \mathrm{Ad}_{g_t}\big(\eta(g_t^{-1}\dot{g}_t)\big)\,,\,\mu \right\rangle \\
&= \left\langle \mathrm{Ad}_{g_t}\big(\mathrm{ad}_{g_t^{-1}\dot{g}_t}\eta\big)\,,\,\mu \right\rangle = \left\langle \mathrm{Ad}_{g_t}\big(\mathrm{ad}_{\xi_t}\eta\big)\,,\,\mu \right\rangle\,.
\end{aligned}
$$

xx $\boxed{4, \text{D}}$

(c) For a fixed $\zeta \in \mathfrak{g}$, calculate the time derivative of the following pairing $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
\frac{d}{dt}\left\langle \mathrm{Ad}^*_{g^{-1}}\frac{\partial \ell}{\partial \xi}, \zeta \right\rangle &= \frac{d}{dt}\left\langle \frac{\partial \ell}{\partial \xi}(t), \mathrm{Ad}_{g^{-1}(t)}\zeta \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \mathrm{Ad}_{g^{-1}(t)}\zeta \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}, \frac{d}{dt}(g^{-1}(t)\zeta g(t)) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \mathrm{Ad}_{g^{-1}(t)}\zeta \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}, -g^{-1}\dot{g}g^{-1}\zeta g + g^{-1}\zeta\dot{g} \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \mathrm{Ad}_{g^{-1}(t)}\zeta \right\rangle + \left\langle \frac{\partial \ell}{\partial \xi}, -\xi\mathrm{Ad}_{g^{-1}}\zeta + (g^{-1}\zeta g)(g^{-1}\dot{g}) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi}, \mathrm{Ad}_{g^{-1}(t)}\zeta \right\rangle - \left\langle \frac{\partial \ell}{\partial \xi}, \mathrm{ad}_\xi(\mathrm{Ad}_{g^{-1}}\zeta) \right\rangle \\
&= \left\langle \frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}, \mathrm{Ad}_{g^{-1}}\zeta \right\rangle = \left\langle \mathrm{Ad}^*_{g^{-1}}\left(\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}\right), \zeta \right\rangle = 0
\end{aligned}
$$

Since the fixed Lie-algebra element ζ is arbitrary, we have

$$
\frac{d}{dt}\left(\mathrm{Ad}^*_{g^{-1}(t)}\frac{\partial \ell}{\partial \xi}\right) = \mathrm{Ad}^*_{g^{-1}(t)}\left(\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\frac{\partial \ell}{\partial \xi}\right) = 0\,.
$$

xx $\boxed{8, \text{D}}$

(d) The Euler-Poincaré equation arises from the variational principle as follows $\boxed{\text{meth seen} \Downarrow}$

$$
\begin{aligned}
0 = \delta S = \delta\int_0^T \ell(\xi) &+ \left\langle \mu\,,\,g^{-1}\dot{g} - \xi \right\rangle dt \\
= \int_0^T &\left\langle \frac{\partial \ell}{\partial \xi} - \mu\,,\,\delta\xi \right\rangle + \left\langle \mu\,,\,\frac{d\eta}{dt} + \mathsf{ad}_\xi\eta \right\rangle + \left\langle \delta\mu\,,\,g^{-1}\dot{g} - \xi \right\rangle dt \\
= \int_0^T &\left\langle \frac{\partial \ell}{\partial \xi} - \mu\,,\,\delta\xi \right\rangle - \left\langle \frac{d\mu}{dt} - \mathsf{ad}^*_\xi\mu\,,\,\eta \right\rangle + \left\langle \delta\mu\,,\,g^{-1}\dot{g} - \xi \right\rangle dt + \left\langle \mu\,,\,\eta \right\rangle\Big|_0^T\,.
\end{aligned}
$$

where $\xi = g^{-1}\dot{g}$ and the quantity $\eta := g^{-1}\delta g$ vanishes at the endpoints in time. The side calculation of $(\delta\xi - \dot{\eta}) = \mathsf{ad}_\xi\eta$ is useful in the second line.

xx $\boxed{4, \text{D}}$

3. **Properties of the diamond operator** seen $\Downarrow$

(a) The corresponding infinitesimal Lie G-symmetry $\delta q = -\,\pounds_\xi q$ where $\xi \in \mathfrak{g} \simeq T_eG$ leaves the Lagrangian L invariant and implies conservation of the endpoint term,

$$\left\langle \frac{\partial L}{\partial \dot q}\,,\,\delta q \right\rangle_{TQ} =: \langle p\,,\,\delta q\rangle_{TQ} = \langle p\,,-\pounds_\xi q\rangle_{TQ} =: \langle p \diamond q\,,\,\xi\rangle_{\mathfrak{g}} =: \langle J(q,p)\,,\xi\rangle_{\mathfrak{g}} =: J^\xi(q,p)\,.$$

xx 6, A

(b) Under the canonical Poisson bracket, $\{\,q\,,\,p\,\}_{can} = Id$, one calculates easily that meth seen $\Downarrow$

$$\delta q = \big\{q, J^\xi(q,p)\big\}_{can} = -\,\pounds_\xi q \quad \text{and} \quad \delta p = \big\{p, J^\xi(q,p)\big\}_{can} = -\,\pounds_\xi^T p\,.$$

xx 6, B

(c) meth seen $\Downarrow$

$$\begin{aligned}-\frac{d}{dt}J^\xi(p,q) = \{H(p,q), J^\xi(p,q)\}_{can} &= \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p \\ &= -\,\frac{\partial H}{\partial q}\pounds_\xi q - \frac{\partial H}{\partial p}\pounds_\xi^T p = -\,\pounds_\xi H(p,q)\,.\end{aligned}$$

xx 6, C

(d) Invariance of the Hamiltonian $H(p,q)$ under the Lie transformation of phase space generated by the infinitesimal canonical transformation meth seen $\Downarrow$

$$\{\,H(p,q)\,,\,J^\xi(p,q)\}_{can} = -\,\pounds_\xi H(p,q) = 0$$

implies Invariance of the phase space generator $J^\xi(p,q)$ under the dynamics generated by $H(p,q)$

$$\{\,J^\xi(p,q\,,\,H(p,q)\}_{can} = \frac{d}{dt}J^\xi(p,q) = 0\,.$$

Hence, we have the desired statement of Noether's theorem in phase space:
Lie invariance of the Hamiltonian implies conservation of the infinitesimal generator of the Lie symmetry.

xx 2, A

4. **Hamilton-Pontryagin principle for the heavy top** $\boxed{\text{sim. seen} \Downarrow}$

(a) First calculate

$$\delta\Omega = \delta(g^{-1}\dot{g}) = \frac{d\Xi}{dt} + \big[\Omega, \Xi\big] \quad \text{and} \quad \delta g^{-1}(t)e_3 = -\,\Xi\,.\,\Gamma \quad \text{with} \quad \Gamma := g^{-1}(t)e_3$$

Then insert the results into the variational principle to find

$$\begin{aligned}
0 = \delta S = \delta \int_a^b \ell(\Omega, g^{-1}(t)e_3) + \big\langle \Pi\,,\, g^{-1}\dot{g}(t) - \Omega \big\rangle\, dt \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle \Pi\,,\, \frac{d\Xi}{dt} + \big[\Omega, \Xi\big] \Big\rangle + \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, -\,\Xi\,.\,\Gamma \Big\rangle \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle \Pi\,,\, \frac{d\Xi}{dt} + \mathrm{ad}_\Omega \Xi \Big\rangle + \Big\langle \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma\,,\, \Xi \Big\rangle \\
= \int_a^b \Big\langle \frac{\partial \ell}{\partial \Omega} - \Pi\,,\, \delta\Omega \Big\rangle + \Big\langle -\,\frac{d\Pi}{dt} + \mathrm{ad}^*_\Omega \Pi + \frac{\partial \ell}{\partial \Gamma} \diamond \Gamma\,,\, \Xi \Big\rangle\,.
\end{aligned}$$

xx $\boxed{10, \text{A}}$

(b) The Hamiltonian is found from the reduced Legendre transform, $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned}
h(\Pi, \Gamma) &= \big\langle \Pi\,,\, \Omega \big\rangle - \ell(\Omega, \Gamma) \\
dh &= \big\langle d\Pi\,,\, \Omega \big\rangle + \Big\langle \Pi - \frac{\partial \ell}{\partial \Omega}\,,\, \delta\Omega \Big\rangle - \Big\langle \frac{\partial \ell}{\partial \Gamma}\,,\, \delta\Gamma \Big\rangle \\
\frac{\delta h}{\delta \Pi} &= \Omega \quad \text{and} \quad \frac{\partial h}{\partial \Gamma} = -\,\frac{\partial \ell}{\partial \Gamma}\,.
\end{aligned}$$

xx $\boxed{2, \text{A}}$

(c) Hamiltonian equations in Lie-Poisson matrix form are found as $\boxed{\text{unseen} \Downarrow}$

$$\frac{d}{dt}\begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} \mathrm{ad}^*_\square \Pi & \square \diamond \Gamma \\ \square\,.\,\Gamma & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial \Pi = \Omega \\ \partial h/\partial \Gamma = -\,\partial \ell/\partial \Gamma \end{bmatrix}$$

xx $\boxed{4, \text{C}}$

(d) The corresponding Lie-Poisson bracket is given by $\boxed{\text{unseen} \Downarrow}$

$$\frac{df(\Pi, \Gamma)}{dt} = \Bigg\langle \begin{bmatrix} \partial f/\partial \Pi \\ \partial f/\partial \Gamma \end{bmatrix}, \begin{bmatrix} \mathrm{ad}^*_\square \Pi & \square \diamond \Gamma \\ \square\,.\,\Gamma & 0 \end{bmatrix} \begin{bmatrix} \partial h/\partial \Pi = \Omega \\ \partial h/\partial \Gamma = -\,\partial \ell/\partial \Gamma \end{bmatrix} \Bigg\rangle =: \{f, h\}$$

xx $\boxed{4, \text{A}}$

5. **Euler-Poincaré compressible adiabatic fluid dynamics**

seen ⇓

(a) The symmetry reduced Hamilton's principle with a Lagrangian functional for ideal fluid dynamics in our standard form with Euler-Poincaré variations is given by

$$
\begin{aligned}
0 = \delta S_{red} = \delta \int_0^T \ell(u,a)\,dt &= \int_0^T \Big\langle \frac{\delta \ell}{\delta u}\,,\,\delta u \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a}\,,\,\delta a \Big\rangle\,dt \\
&= \int_0^T \Big\langle \frac{\delta \ell}{\delta u}\,,\,\partial_t \xi - \mathrm{ad}_{\mathrm{u}}\xi \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a}\,,\,-\pounds_\xi a \Big\rangle\,dt \\
&= \int_0^T \Big\langle -(\partial_t + \mathrm{ad}^*_{\mathrm{u}})\frac{\delta \ell}{\delta \mathrm{u}} + \frac{\delta \ell}{\delta \mathrm{a}} \diamond \mathrm{a}\,,\,\xi \Big\rangle\,dt\,.
\end{aligned}
$$

The resulting Euler-Poincaré equations are

$$
\big(\partial_t + \mathrm{ad}^*_{\mathrm{u}}\big)\frac{\delta \ell}{\delta \mathrm{u}} = \frac{\delta \ell}{\delta \mathrm{a}} \diamond \mathrm{a} \quad \text{with advection relation} \quad \big(\partial_{\mathrm{t}} + \pounds_{\mathrm{u}}\big)\mathrm{a} = 0\,.
$$

xx 6, M

(b) The reduced Legendre transformation is

meth seen ⇓

$$
h(\mu,a) := \langle \mu\,,\,u\rangle - \ell(u,a)\,,
$$

xx 2, M

(c) The functional derivatives

meth seen ⇓

$$
\begin{aligned}
\delta h(\mu,a) &= \langle \delta\mu\,,\,u\rangle + \Big\langle \mu - \frac{\delta \ell}{\delta u}\,,\,\delta u \Big\rangle - \Big\langle \frac{\delta \ell}{\delta a}\,,\,\delta a \Big\rangle \\
&= \Big\langle \frac{\delta h}{\delta \mu}\,,\,\delta\mu \Big\rangle + \Big\langle \frac{\delta h}{\delta a}\,,\,\delta a \Big\rangle\,,
\end{aligned}
$$

yield the required variational relations

$$
\delta\mu : \frac{\delta h}{\delta \mu} = u, \qquad \delta u : \mu = \frac{\delta \ell}{\delta u}, \qquad \delta a : \frac{\delta h}{\delta a} = -\frac{\delta \ell}{\delta a}\,.
$$

xx 2, M

(d) The resulting Lie-Poisson equations may then be written as

sim. seen ⇓

$$
\big(\partial_t + \mathrm{ad}^*_{\delta \mathrm{h}/\delta\mu}\big)\mu = -\frac{\delta \mathrm{h}}{\delta \mathrm{a}} \diamond \mathrm{a} \quad \text{with advection relation} \quad \big(\partial_{\mathrm{t}} + \pounds_{\delta \mathrm{h}/\delta\mu}\big)\mathrm{a} = 0\,.
$$

These equations are expressed in Lie-Poisson matrix form, as follows

$$
\partial_t \begin{bmatrix} \mu \\ a \end{bmatrix} = - \begin{bmatrix} \mathrm{ad}^*_\square \mu & \square \diamond a \\ \pounds_\square a & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta\mu \\ \delta h/\delta a \end{bmatrix}
$$

xx 4, M

(e) The Lie-Poisson bracket is seen to be dual to a semidirect-product Lie algebra by isolating multiples of the fluid variables, as follows. $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
\frac{d}{dt}f(\mu,a) &= -\Big\langle \Big(\frac{\delta f}{\delta \mu}\,,\, \frac{\delta f}{\delta a} \Big)\,,\, \Big(\mathrm{ad}^*_{\delta \mathrm{h}/\delta\mu}\mu + \frac{\delta \mathrm{h}}{\delta \mathrm{a}} \diamond \mathrm{a}\,,\, \pounds_{\delta \mathrm{h}/\delta\mu}\mathrm{a} \Big) \Big\rangle \\
&=: -\Big\langle \frac{\delta f}{\delta \mu}\,,\, \mathrm{ad}^*_{\delta \mathrm{h}/\delta\mu}\mu + \frac{\delta \mathrm{h}}{\delta \mathrm{a}} \diamond \mathrm{a} \Big\rangle - \Big\langle \frac{\delta f}{\delta a}\,,\, \pounds_{\delta h/\delta\mu} a \Big\rangle \\
&= -\Big\langle \frac{\delta f}{\delta \mu}\,,\, \mathrm{ad}^*_{\delta \mathrm{h}/\delta\mu}\mu \Big\rangle - \Big\langle \frac{\delta f}{\delta a}\,,\, \pounds_{\delta h/\delta\mu} a \Big\rangle - \Big\langle \frac{\delta f}{\delta \mu}\,,\, \frac{\delta h}{\delta a} \diamond a \Big\rangle \\
&= -\Big\langle \mathrm{ad}_{\delta \mathrm{h}/\delta\mu} \frac{\delta \mathrm{f}}{\delta \mu}\,,\, \mu \Big\rangle - \Big\langle \pounds^T_{\delta h/\delta\mu} \frac{\delta f}{\delta a} - \pounds^T_{\delta f/\delta\mu} \frac{\delta h}{\delta a}\,,\, a \Big\rangle \\
&= -\Big\langle (\mu\,,\,a)\,,\, \Big(\pounds_{\delta h/\delta\mu} \frac{\delta f}{\delta \mu}\,,\, \pounds^T_{\delta h/\delta\mu} \frac{\delta f}{\delta a} - \pounds^T_{\delta f/\delta\mu} \frac{\delta h}{\delta a} \Big) \Big\rangle =: \big\{ f, h \big\}(\mu, a)\,.
\end{aligned}
$$

The last line pairs $(\mu, a) \in \mathfrak{X}^* \times V$ with the semidirect-product Lie algebra action of vector fields $(X, \bar{X}) \in \mathfrak{X}$ acting on a vector space V^* dual to V with elements $(a^*, \bar{a}^*) \in V^*$. Formally, the action is

$$
\big[(X, a^*), (\bar{X}, \bar{a}^*)\big] = \big([X, \bar{X}]\,,\, X\bar{a}^* - \bar{X}a^* \big)\,.
$$

Thus, the matrix form of the Hamiltonian equations defines the Lie-Poisson bracket in as the pairing of the semidirect-product Lie algebra $\mathfrak{X} \circledS V^*$ with its coordinates $\mu \in \mathfrak{X}^*$ dual to $\delta h/\delta\mu \in \mathfrak{X}$ and $a \in V$ dual to $\delta h/\delta a \in V^*$.

xx $\boxed{6, \mathrm{M}}$

Review of mark distribution:

Total A marks: 34 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 10 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks