

Theorem 3.19: Algebra of limits

If $a_n \rightarrow a$ and $b_n \rightarrow b$ then:

1. $a_n + b_n \rightarrow a + b$,
2. $a_n b_n \rightarrow ab$,
3. $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$ if $b \neq 0$.

Proof of 1. Fix any $\epsilon > 0$. Then

Set $N = \max\{N_a, N_b\}$, so $\forall n \geq N$,

Rough working for 2: First a bit of a trick,

We can easily make $|a_n - a||b| < \frac{\epsilon}{2}$ if we take $|a_n - a| < \frac{\epsilon}{2|b|}$.

But we **cannot** deduce $|b_n - b| < \frac{\epsilon}{2|a_n|}$ from $b_n \rightarrow b$ because in the definition, ϵ has to be independent of n .

Instead we bound $|a_n| < A$ by Proposition 3.16; then we can take $|b_n - b| < \frac{\epsilon}{2A}$.

Proof of 2. $a_n \rightarrow a \implies \exists A > 0$ such that $|a_n| < A \forall n \in \mathbb{N}_{>0}$ by Proposition 3.16.

Fix $\epsilon > 0$. Then

(We added 1 to $2|b|$ to handle the case $|b| = 0$.)

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$,

Alternative trick-less proof of 2: Write $a_n = a + e_n$ and $b_n = b + f_n$ so that (easy exercise!) $e_n, f_n \rightarrow 0$. Then

Now the idea is that if we make $|e_n|, |f_n| < \epsilon$, the last term is $< \epsilon^2$ which *should be* even smaller. In fact this only works if $\epsilon \leq 1$ so we need to ensure this.

So now fix $\epsilon > 0$ and set
 $\forall n \geq N$,

Then $\exists N \in \mathbb{N}$ such that

Since $\epsilon' \leq 1$ we know $(\epsilon')^2 \leq \epsilon'$ so we get $|a_n b_n - ab| < \epsilon'(|a| + |b| + 1) \leq \epsilon$, so $a_n b_n \rightarrow ab$.

I deliberately missed out the rough working of how to choose ϵ' . Tonight **close your notes and write out your own proof of this result**. Do the rough working first, then write a concise, precise, logical proof. Don't be afraid to have several goes until the end result is undeniably a correct proof.

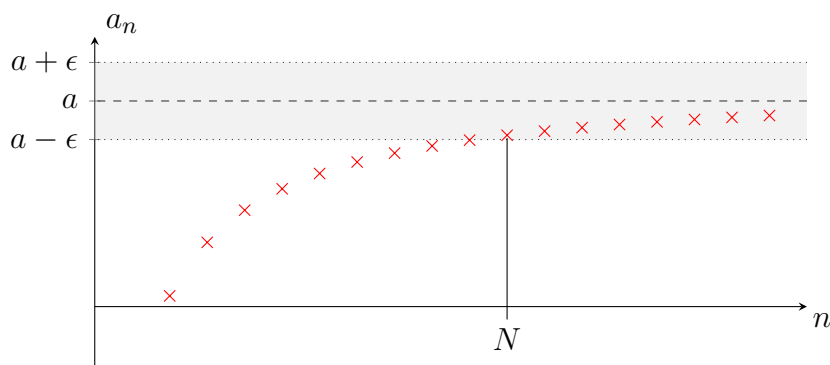
See exercise sheet for proof of (3). □

Remark 3.20. Now it's easier to handle things like $a_n = \frac{n^2 + 5}{n^3 - n + 6}$.

Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i : i \in \mathbb{N}_{>0}\}$. We write $a_n \uparrow a$.

Idea: Eventually we get in the ϵ -corridor around a (the shaded area) because $a - \epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}_{>0}\}$. We stay in there because a_n is monotonic and bounded above by a .



Proof.

Example 3.22. Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \leq b_n \forall n$ and $a_n \rightarrow a$, $b_n \rightarrow b$. Prove that $a \leq b$.

Example 3.23. Prove that if

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L < 1$$

then $a_n \rightarrow 0$.

Idea: $a_n \approx c \cdot L^n$ for $n \gg 0$, $L < 1 \implies a_n \rightarrow 0$.

Since $|a_{n+1}/a_n|$ is not exactly L , to turn this in to a proof, we must instead estimate/bound it by $|a_{n+1}/a_n| < L'$ for some $L' < 1$. Though we cannot take $L' = L$ we can take $L' = L + \epsilon$ (because $|a_{n+1}/a_n| \rightarrow L$). So we need $L + \epsilon < 1$, so let's take $\epsilon = \frac{1-L}{2}$.

Proof.

[Exercise sheet: $\alpha^k \rightarrow 0$ as $k \rightarrow \infty$ if $|\alpha| < 1$.]

We apply this to $\alpha = L' < 1$. Fixing a new $\epsilon > 0$, $\exists M > 0$ such that $\forall k \geq M$,

$$(L')^k < \frac{\epsilon}{1 + |a_N|}. \quad (**)$$

(We wanted to write $\frac{\epsilon}{|a_N|}$ but we have to beware the case $|a_N| = 0$.)

So by (*) and (**) we have

$$|a_{N+k}| < \frac{\epsilon}{1 + |a_N|} |a_N| < \epsilon \quad \forall k \geq M.$$

Rewriting this:

$$\forall n \geq N + M, \quad |a_n| < \epsilon.$$

□

3.2 Cauchy Sequences

We're now world experts at proving a_n converges if we know what the limit is. Cauchy sequences gives us a way to prove convergence *without* knowing the limit.

Definition. $(a_n)_{n \geq 1}$ is called a *Cauchy* sequence if and only if

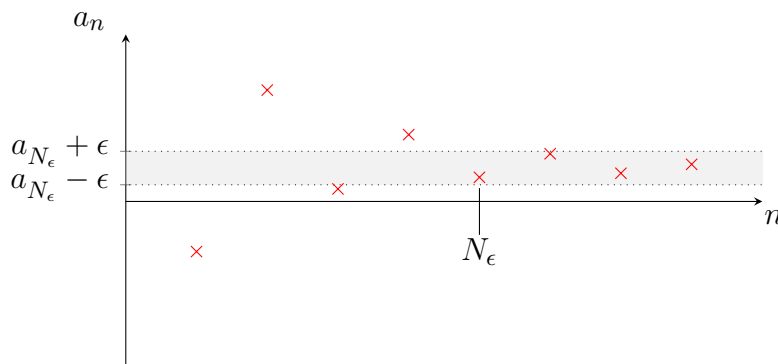
Remark 3.24. $m, n \geq N$ are arbitrary. It is not enough to say that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See exercise sheet.

Proposition 3.25. If $a_n \rightarrow a$ then (a_n) is *Cauchy*.

Proof.

Next we want to prove the converse: Cauchy \implies convergence.

We need a candidate for the limit a .



We will produce an auxiliary sequence which is *monotonic* (and bounded) \implies convergent. Let $b_n := \sup\{a_i : i \geq n\}$. Then picture shows that $b_{N_\epsilon} \in (a_{N_\epsilon} - \epsilon, a_{N_\epsilon} + \epsilon]$ and b_n s are monotonically *decreasing* because $\{a_i : i \geq n+1\} \subseteq \{a_i : i \geq n\}$ so $b_{n+1} = \sup \leq \sup = b_n$.

So b_n s converge to $\inf\{b_n : n \in \mathbb{N}\}$. We will show that a_n s converge to same number, a , using the Cauchy condition.

Lemma 3.26. (a_n) is Cauchy $\implies (a_n)$ is bounded.

Proof.

Theorem 3.27

If (a_n) is a Cauchy sequence of real numbers then a_n converges.

Corollary 3.28. (a_n) Cauchy $\iff (a_n)$ convergent.

Exercise 3.29. Show this is not true in \mathbb{Q} : there exist Cauchy sequences (a_n) with $a_n \in \mathbb{Q}$ with no limit in \mathbb{Q} .

Proof. Since (a_n) is Cauchy, it is bounded by Lemma 3.26: $|a_n| \leq A$. So we can define

Then $b_n \geq a_i \forall i \geq n$ so $b_n \geq a_i \forall i \geq n+1$ is an upper bound for $\{a_i : i \geq n+1\}$, so $b_n \geq \sup\{a_i : i \geq n+1\} = b_{n+1}$. So the sequence (b_n) is monotonically decreasing. And $b_n \geq a_n \geq -A$ shows it is also bounded below.

So we can define $a := \inf\{b_n : n \in \mathbb{N}\}$ and $b_n \downarrow a$. We claim that $a_n \rightarrow a$.

□

In the proof we twice used:

Exercise 3.30. If $S \subseteq \mathbb{R}$ satisfies $x < M \quad \forall x \in S$ then $\sup S \leq M$.

Example 3.31 (Decimals). Suppose we didn't use decimals to construct \mathbb{R} (e.g. if we used Dedekind cuts, or we just used the axioms without worrying about constructing the set).

Then using Cauchy sequences we can now make sense of the decimal $a_0.a_1a_2a_3\dots$ as follows. (Here we fix $a_0 \in \mathbb{Z}$ and $a_1, a_2, a_3, \dots \in \{0, 1, \dots, 9\}$.)

Let $(A_n)_{n \geq 1}$ be the sequence of rational numbers defined by

$$A_n := a_0 + \frac{a_1}{10} + \dots + \frac{a_n}{10^n}.$$

(A_n is the approximation to our decimal given by truncating at the n th place.)

Exercise: for all $n, m \geq N$ we have $|A_n - A_m| < 10^{-N}$.

Thus (A_n) is a Cauchy sequence: $\forall \epsilon > 0$ we can take $N > \epsilon^{-1}$ so that $10^{-N} > \epsilon^{-1}$ so that $10^{-N} < \epsilon$.

Thus it converges to a limit in \mathbb{R} . We call this limit $a_0.a_1a_2a_3\dots$.