Imperial College London

Problem Sheet 7 Solutions

MATH50011
Statistical Modelling 1

Week 9

Lecture 15: Multivariate Normal Distributions

1. Let X and B be independent random variables such that $X \sim N(0,1)$ and $B \in \{-1,1\}$ with $P(B = 1) = P(B = -1) = \frac{1}{2}$. Let $Z = XB$.

   (a) Find $Cov(X, Z)$.

   (b) Show that $Z \sim N(0,1)$.

   (c) Are X and Z independent?

Solution. $E(XZ) = E(X^2)E(B) = 0$, $E(X)E(Z) = 0$.
Hence, $Cov(X, Z) = E(XZ) - E(X)E(Z) = 0$.

$Z \sim N(0,1)$. Indeed,

$$\begin{aligned} P(Z \le t) &= P(X \le t|B = 1)P(B = 1) + P(-X \le t|B = -1)P(B = -1) \\ &= P(X \le t)\frac{1}{2} + P(-X \le t)\frac{1}{2} = \frac{1}{2}(\Phi(t) + 1 - \Phi(-t)) = \Phi(t), \end{aligned}$$

where Φ is the cdf of a standard normal r.v.

Applying continuous functions to independent random variables preserves independence. If X and Z were independent then so would $|X|$ and $|Z|$.

However, $|X| = |Z|$ and they are not constant - showing that $|X|$ and $|Z|$ are not independent.

You can also see this by showing that X and Z are not jointly Normal. In particular, consider the vector $(1,1)$ then $(1,1)(X,Z)^T = X + Z$. Notice that when $B = -1$ we have $X + Z = 0$ and when $B = 1$ $X + Z = 2X$. Thus, $X + Z$ is Normal only when $B = 1$ and it is equal to 0 when $B = -1$, thus $X + Z$ is not Normal and by Definition 23 in the notes we conclude that (X, Z) is not a multivariate Normal distribution.

2. Suppose $X \sim N\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

   (a) What is the distribution of $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} X + \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$?

(b) Are any of the components of Z independent?

(c) Let $Y \sim N\left(\begin{pmatrix}2\\3\\2\end{pmatrix}, \begin{pmatrix}2 & 1 & 0\\1 & 2 & 0\\0 & 0 & 9\end{pmatrix}\right)$. What components of Y are independent?

Solution. $E(Z) = \begin{pmatrix}5\\3\\2\end{pmatrix} + \begin{pmatrix}-1\\-3\\2\end{pmatrix} = \begin{pmatrix}4\\0\\4\end{pmatrix}$ $Cov(Z) = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1 & 1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}$

Thus,

$$N\left(\begin{pmatrix}4\\0\\4\end{pmatrix}, \begin{pmatrix}2 & 1 & 1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}\right)$$

Z_2 and Z_3 are independent, as

$$\begin{pmatrix}Z_2\\Z_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\end{pmatrix} Z \sim N\left(\begin{pmatrix}0\\4\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)$$

(Y_1, Y_2) and Y_3 are independent.

3. Let

$$\begin{pmatrix}Y\\X\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_Y\\\mu_X\end{pmatrix}, \begin{pmatrix}\sigma_Y^2 & \rho\sigma_Y\sigma_X\\\rho\sigma_Y\sigma_X & \sigma_X^2\end{pmatrix}\right).$$

(a) Find the conditional distribution of $Y|X = x$ (it will be a univariate normal distribution).

(b) Express the conditional mean $E(Y|X = x)$ as a linear function $\beta_0 + \beta_1 x$. What are β_0 and β_1 in terms of the parameters of the bivariate normal distribution?

Solution.

(a) Using the formula $f_{y|x}(y|x) = f_{x,y}(x, y)/f_x(x)$, we find that

$$Y|X = x \sim N(\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_x), \sigma_Y^2(1 - \rho^2))$$

(b) The conditional expectation is

$$E(Y|X = x) = \mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_x) = \beta_0 + \beta_1 x$$

for $\beta_1 = \rho(\sigma_Y/\sigma_X)$ and $\beta_0 = \mu_Y - \beta_1\mu_X$. Hence, the conditional distributions for a bivariate normal distribution induce a linear model for $E(Y|X = x)$.

Lecture 16: Distributions and Independence Results

4. In the lecture we had the following definition:

Let $Z \sim N(\mu, I_n)$, where $\mu \in \mathbb{R}^n$. $U = Z^T Z$ is said to have a *non-central* χ^2*-distribution* with n degrees of freedom (d.f.) and non-centrality parameter $\delta = \sqrt{\mu^T\mu}$. Notation: $U \sim \chi_n^2(\delta)$.

(a) Show that the $\chi^2_n(\delta)$-distribution depends on μ only through δ.

(b) Show that $E(U) = n + \delta^2$ and $Var(U) = 2n + 4\delta^2$.

(c) Show that if $U_i \sim \chi^2_{n_i}(\delta_i)$, $i = 1, \dots, k$, and $U_1, \dots, U_k$ are independent then $\sum_{i=1}^k U_i \sim \chi^2_{\sum n_i}(\sqrt{\sum \delta_i^2})$.

Hint: Use moment-generating functions.

Solution.

(a) Will show that the mgf of U equals

$$M_U(t) = \frac{1}{(1-2t)^{n/2}} \exp\left(\frac{t\delta^2}{1-2t}\right)$$

Indeed, $M_U(t) = E(e^{t\sum_i Z_i^2}) = \prod_i E(e^{tZ_i^2})$ (independence)
Furthermore,

$$\begin{aligned} E(e^{tZ_i^2}) &= \int e^{tz^2} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-\mu_i)^2}{2}\right) dz \\ &= \int \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Big(\underbrace{(z-\mu_i)^2 - 2tz^2}_{\substack{=(1-2t)(z^2-2\frac{\mu_i}{1-2t}z+\frac{\mu_i^2}{1-2t}) \\ =(1-2t)(z-\frac{\mu_i}{1-2t})^2-\frac{\mu_i^2}{1-2t}+\mu_i^2 \\ =(1-2t)(z-\frac{\mu_i}{1-2t})^2-2\frac{t\mu_i^2}{1-2t}}} \Big)\right) dz \qquad \text{(compl. the square)} \\ &= \exp\Big(\frac{\mu_i^2 t}{1-2t}\Big) \frac{1}{\sqrt{1-2t}} \underbrace{\int (\text{normal pdf}) dz}_{=1} \end{aligned}$$

(b) Directly using rules for E, Var or quicker from the MGF.

(c) Is immediate by considering the MGFs of the U_i (which we have computed in the first part of this question).

5. In the lectures, we showed that for a sequence $T_n \sim t_n(0)$, $T \to_d N(0,1)$. Similar results can be derived for the χ^2_n and $F_{m,n}$ distributions.

(a) Let $Z_1, \dots, Z_n$ be iid $N(0,1)$ and define $U_n = \sum_i Z_i^2$. Use large sample properties of U_n to derive a normal approximation to the χ^2_n distribution.

(b) For m fixed and $n \to \infty$, show that $F_n \sim F_{m,n}$ converges in distribution to a χ^2_m random variable.

Solution.

(a) We see that $\bar{Z}^2 = n^{-1}U_n$ so that, by the central limit theorem

$$\sqrt{n}(\bar{Z}^2 - E(Z_1^2)) \to_d N(0, Var(Z_1^2)).$$

Since $Z_1^2 \sim \chi^2_1$, we have $E(Z_1^2) = 1$ and $Var(Z_1^2) = 2$ (see question 1(b)). Putting this together, we have

$$\sqrt{n}(\bar{Z}^2 - 1) \to_d N(0, 2)$$

and, approximately,

$$\bar{Z}^2 = n^{-1}U_n \sim N(1, 2/n).$$

Using linearity properties of the normal distribution, we arrive at the approximation $U_n \sim N(n, 2n)$. Since $U_n \sim \chi^2_n$, we observe that the approximation is exactly

$$U_n \sim N(E(U_n), Var(U_n)).$$

(b) The result as stated holds for $m = 1$. Let $U_m \sim \chi^2_m$ be independent of $V_n \sim \chi^2_n$. Then

$$F_n = \frac{U_m/m}{V_n/n} \sim F_{m,n}.$$

However, V_n/n has the same distribution as $V_n/n = n^{-1}\sum_{i=1}^n W_i$ where the W_is are iid χ^2_1. By the weak law of large numbers, $V_n/n \to_p 1$. Hence, by Slutsky's lemma

$$F_n = \frac{U_m/m}{V_n/n} = \frac{1}{V_n/n}\frac{U_m/m}{1} \to_p U_m/m$$

which is proportional to a χ^2_m random variable.

6. Revise the proofs of Lemmas 16-20 and the Fisher-Cochran theorem.

Solution. See notes.