BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Introduction to Game Theory Mock Paper B

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH60 paper; 2.5 Hours for MATH70 paper

This paper has *4 Questions (MATH60 version); 5 Questions (MATH70 version)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Throughout this question consider a finite, two-player, simultaneous move game, G, being played between players A and B.

(a) Define what it means to have an **equilibrium** of the game. (3 marks)

(b) Define what it means for a strategy of player A to be **strictly dominated** by another strategy of player A. Define also what it means for the strategy to be **weakly dominated** by another strategy. (2 marks)

(c) Prove that if we delete a strictly dominated strategy from G then the game, say G', with this strategy removed has the same equilibria as G. (5 marks)

(d) Define what it means for G to be termed **degenerate**. (1 mark)

(e) Consider the duopoly game played between two corporate firms, firm X and firm Y, which produce quantities x and y of a particular product respectively. The profits of the two firms are given by

$$g_X(x, y) = x(16 - x - y),$$

for firm X, and

$$g_Y(x, y) = y(16 - x - y),$$

for firm Y, when the firms produce quantities x and y of the product respectively.

(i) Find all equilibria of the game when it is played over the finite strategy sets $X_S = Y_S = \{0, 2, 4, 6, 8\}$, i.e $x, y \in \{0, 2, 4, 6, 8\}$. (6 marks)

(ii) Find all equilibria of the game when it is played over the infinite strategy sets $X_S = Y_S = [0, 16]$, i.e $x, y \in [0, 16]$. (3 marks)

(Total: 20 marks)

2. Throughout this question consider a finite, two-player, zero-sum game being played between players A and B.

(a) Define what it means for a strategy of player A to be an equaliser strategy. (2 marks)

(b) Prove that if α^* is an equaliser strategy for player A and β^* is an equaliser strategy for player B then (α^*, β^*) forms an equilibrium of the game. (3 marks)

(c) Each of two players, A and B, chooses an integer from the set $\{1, 2, 3, 4\}$ independently and simultaneously. If both choose the same integer, neither gets any reward, but if they choose different integers then B must pay A the **maximum** of their two choices.

(i) Find a solution of this game. (7 marks)

Suppose now that the game remains the same except that B must now pay A the **minimum** of their two choices when they choose different integers.

(ii) By considering the 3×3 sub-game in which A never chooses 1 and B never chooses 4, find a solution to this new game. (7 marks)

(iii) Which game is more profitable to player A? (1 mark)

(Total: 20 marks)

3. Two builders, A and B, are competing for a contract to construct a new university building. Each can bid either £l million or £h million for the job, where $l < h$. The builder with the lower bid will win the contract and will be paid the value of their bid once the building is finished. If both bids are equal, a fair coin is tossed to decide who should win the contract. Each builder reckons that the real cost of completing the job is £c million, where $0 < c < 2l - h$. They decide to **collaborate**.

(a) (i) Construct a normal (strategic) form representation of the game. (1 mark)

(ii) Determine the builders' threat levels in the game. (3 marks)

(iii) Sketch the payoff set and identify the bargaining set. Indicate the pareto-optimal frontier of this set. (3 marks)

(iv) Show that each builder cannot expect to make a profit of more than £m million, where

$$m = \frac{(l-c)(3h-2l-c)}{2(h-c)}.$$

(5 marks)

(v) Write down the Nash bargaining solution for the game. (2 marks)

In the following question we consider the atomic model of flow through a congestion game.

(b) (i) Define the **price of anarchy** of a congestion game. (1 mark)

(ii) Give an example of a congestion game which has a price of anarchy equal to 1. (2 marks)

(iii) Give an example of a congestion game which has a price of anarchy greater than or equal to 2. (3 marks)

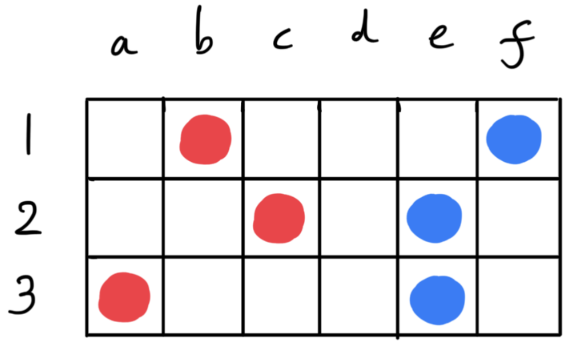
(Total: 20 marks)

4. [Throughout this question you may assume any results about impartial games and the game of Nim unless you are asked to prove them.]

(a) Define the **Nim value** of an impartial game G. (2 marks)

(b) State and prove the copycat principle for impartial games. (4 marks)

(c) Northcott's game is played on a rectangular grid of squares where a red and a blue counter are placed in each row. See the example game position below.

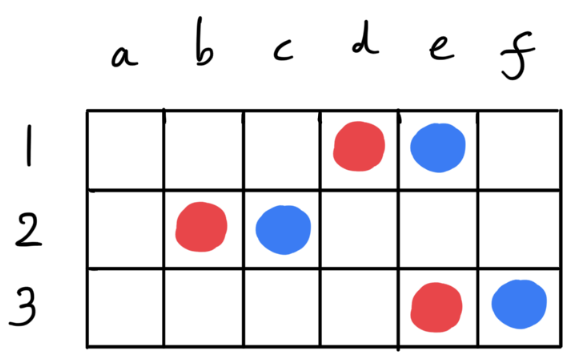


Typically, in each row, the red counters are on the left side of the blue counters, but this need not be the case for every row. Player A moves the red counters and starts the game, followed by player B who moves the blue counters. The players take turns to make a move.

In a move, a player selects a row and slides a counter of their colour to any other empty square within its row, but may not 'jump over' the other player's counter. For example, in the figure above, in row 2 player A (red) may slide their counter from square $2c$ to any of the squares $2a$, $2b$ or $2d$.

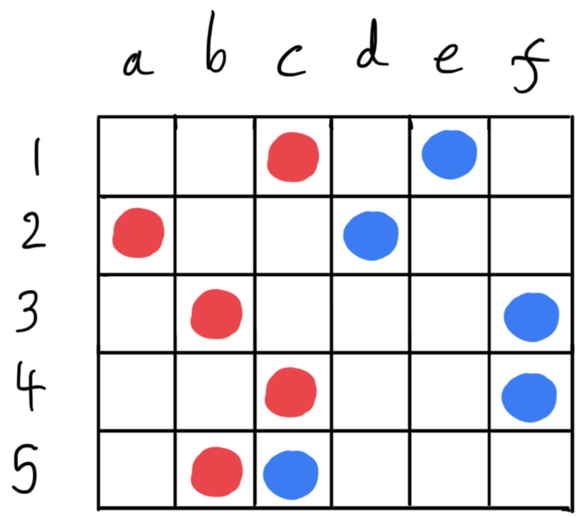
The game is played with the normal play convention, where a player who becomes unable to make a valid move on their turn loses the game.

(i) With player A (red) to move, which player will win in the following position? (2 marks)



(ii) In Northcott's game consisting of a single row, explain why the Nim value of any position is equal to $*m$, where m is the number of empty squares between the red and blue counters in the row. (3 marks)

(iii) With player A (red) to move, show that the following position is winning for player A (red). Then determine all their possible winning moves, justifying your reasoning. (6 marks)



(iv) Suppose that it were player B's (blue's) turn to move in the figure above from part (iii). Is the position still winning for player B? Do the same winning moves that were available for player A exist for player B? Justify your answers. (3 marks)

(Total: 20 marks)

5. (a) In a two-player game let α and $\hat{\alpha}$ be mixed strategies for player A and let β be a mixed strategy for player B. Suppose that (α, β) and $(\hat{\alpha}, \beta)$ are both equilibria of the game. Prove that $(k\alpha + (1-k)\hat{\alpha}, \beta)$ is also an equilibrium of the game for any $k \in [0, 1]$. (5 marks)

(b) Determine all equilibria of the following game.

		B		
		b_1	b_2	b_3
	a_1	2, 1	0, 1	1, 1
A	a_2	0, 4	3, 2	2, 1
	a_3	1, 0	2, 2	0, 3

[Hint: You might find it helpful to consider the possible sub-games where player A is restricted to playing between two of their pure strategies]. (15 marks)

(Total: 20 marks)