Imperial College London

PDEs in Action

An Introduction to Modelling in Applied Mathematics

MATH50008

Dr. T. Bertrand

Department of Mathematics
Imperial College London

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Foreword

The present set of notes represents an introduction to partial differential equations and modelling in applied mathematics. They are intended as a companion to the lectures of **MATH50008 - PDEs in Action** a second-year undergraduate elective module delivered in the Spring term at Imperial College London. Note that some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working.

Inevitably, typos will have found their way through these few chapters; the author would be grateful if such typos can be reported via the Ed discussion forum associated to the module or directly by email to t.bertrand@imperial.ac.uk.

There are many good books dealing with partial differential equations at an undergraduate level. Below, I provide a short list of reference texts that I find informative and that curious students should consult to delve in deeper in particular topics or as a complement to these lectures notes:

- “An introduction to Partial Differential Equations”, Y. Pinchover and J. Rubinstein (Cambridge University Press)
- “Introduction to PDEs”, P. J. Olver (Springer)
- “Partial Differential Equations: An introduction”, W. Strauss (Wiley)
- “Applied Partial Differential Equations”, Ockendon, Howison, Lacey and Movchan (Oxford University Press)
- “Applied Partial Differential Equations”, J.D. Logan (Springer)
- “Wave Motion”, J. Billingham and A.C. King (Cambridge).

We recommend that the students taking this module make sure that they are in command of the material presented in the first year module **MATH40004 - Calculus and Applications** and the first term of the second year module **MATH50004 - Multi-variable Calculus and Differential Equations**.

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