Imperial College London

MATH97067_MATH97178

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Dynamics, Symmetry and Integrability

Date: Tuesday, 25 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. ***Scale-shear Lie group:*** $\mathfrak{S} \times \mathbb{R}^3 \to \mathbb{R}^3$

Consider the action of the two parameter ***scale-shear Lie group*** $\mathfrak{S}$ on $\mathbb{R}^3$, represented by multiplication of an $\mathbb{R}^3$ vector by 3×3 matrices $M(s)$ as

$$M(s)\mathbf{x} = \begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - a_1 x_3 \\ x_2 + a_1 x_3 \\ a_2 x_3 \end{bmatrix},$$

for real group parameters $a_k(s)$, for $k = 1, 2$, depending smoothly on the real parameter s. The corresponding infinitesimal transformations are given by computing the matrix tangent to the two dimensional scale-shear matrix $M(s)$ at the identity $s = 0$, given by the 3×3 matrix

$$\xi = M'(s)\Big|_{s=0} = \begin{bmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \end{bmatrix} \in \mathfrak{s}, \quad \text{with} \quad \xi_k = a_k'(0), \quad \text{for} \quad k = 1, 2.$$

Thus, a natural matrix basis for the Lie algebra $\mathfrak{s}$ is given by $\xi = \sum_{k=1}^{2} \xi_k m_k$ for the constant matrices

$$m_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The corresponding infinitesimal transformations of the Lie algebra $\mathfrak{s}$ acting on $\mathbb{R}^3$ are represented by

$$\Phi_\xi(\mathbf{x}) = \xi\mathbf{x} = \begin{bmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\xi_1 x_3 \\ \xi_1 x_3 \\ \xi_2 x_3 \end{bmatrix} \tag{1}$$

Questions.

(a) (2 marks)

Show that the 3×3 scale-shear matrices form a Lie group under matrix multiplication, by calculating the product of two such matrices and then using the result to find the inverse matrices.

(b) (2 marks)

Use the 3×3 matrix representation of the scale-shear group $\mathfrak{S}$ to show that it is a semidirect product Lie group, $\mathfrak{S} = S \circledS \Sigma$ in which S is the scale subgroup, Σ is the shear subgroup, and the latter is a normal subgroup.

(c) (1 mark)

Make a table of commutators $[m_i, m_j] = c_{ij}^k m_k$ in the basis of matrices m_k for $k = 1, 2$, defined in equation (1) and, thus, determine the structure constants c_{ij}^k of the scale-shear Lie algebra $\mathfrak{s}$ in this basis.

(d) (5 marks)

Compute the components (J_1, J_2) of the momentum map in $\mathbf{p} \cdot \Phi_\xi(\mathbf{x}) = \mathbf{J}(\mathbf{x}, \mathbf{p}) \cdot \boldsymbol{\xi}$ for the basis (m_1, m_2) by using Hamilton's canonical equations to transform the infinitesimal spatial transformations in (1) into derivatives of a Hamiltonian with respect to canonically conjugate momentum, $\mathbf{p}$, such that $\{\mathbf{x}, \mathbf{p}\} = 1$.

(e) (3 marks)

Lift the spatial infinitesimal transformations of the scale-shear Lie algebra acting on $\mathbb{R}^3$ given in (1) to the corresponding infinitesimal actions on phase space $T^*\mathbb{R}^3$ by constructing its Hamiltonian vector field using the canonical Poisson bracket.

(f) (3 marks)

Show that the momentum map $\mathbf{J} : T^*\mathbb{R}^3 \to \mathfrak{s}^*$ is Poisson, by making a table of the Poisson brackets among its components $\{J_1, J_2\}$, then comparing them with the commutator table for the structure constants c_{ij}^k determined earlier in Part (c) for the Lie algebra $\mathfrak{s}$.

(g) (1 mark)

Explain why this table of canonical Poisson brackets produces a valid Poisson bracket among the momentum map components J_1 and J_2.

(h) (1 mark)

Write the equations of motion for $\mathbf{J}(t) = (J_1, J_2) \in \mathfrak{s}^* \simeq \mathbb{R}^2$, for the Hamiltonian function $H(\mathbf{J}) = \frac{1}{2}(J_1^2 + J_2^2)$. That is, compute $\dot{J}_1 = \{J_1, H\}$ and $\dot{J}_2 = \{J_2, H\}$.

(i) (2 marks)

Describe the properties of this solution for each component of $\mathbf{J}(t) = (J_1, J_2)$.
$\boxed{\textbf{\textit{Hint:}}}$ The result will yield the equations for geodesic motion on the Lobachevsky half plane we discussed in class. For the solutions, keep the $\tanh$ function in mind.

(Total: 20 marks)

2. A *loop group* $\widetilde{G}$ is a Lie group of G-valued smooth functions $\widetilde{G} = C^\infty(S^1, G)$, with pointwise multiplication. The corresponding loop Lie algebra $\widetilde{\mathfrak{g}}$ is the Lie algebra of $\mathfrak{g}$-valued functions on the circle with pointwise commutator.

(a) (6 marks)

Compute the Euler-Poincaré equation for $S = \int_{t_1}^{t_2} \ell(\xi)dt = \int_{t_1}^{t_2} \oint_{S^1} L(\xi(x,t))dxdt$ for $\xi \in \widetilde{\mathfrak{g}}$, for a *left-invariant* reduced Lagrangian $\ell(\xi)$ given by the squared H^s Sobolev norm of $\mathfrak{g}$-valued smooth functions $\xi(x)$

$$\ell(\xi) = \frac{1}{2}\|\xi\|^2_{H^s} = \frac{1}{2}\oint_{S^1} \mathrm{tr}\Big(\xi(x)(1-\alpha^2\partial_x^2)^s\xi(x)\Big)\,dx =: \frac{1}{2}\Big\langle \xi(x)^\flat\,,\,\xi(x)\Big\rangle$$

with trace pairing and *flat map* $\flat : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}^*$, so that $(1-\alpha^2\partial_x^2)^s\xi(x) = \xi(x)^\flat \in \widetilde{\mathfrak{g}}^*$ plays the role of the moment of inertia tensor in the case of a rigid body.

(b) Legendre transform this Lagrangian and determine its associated:

(i) loop Hamiltonian, (3 marks)

(ii) and loop Lie-Poisson bracket. (4 marks)

(c) (7 marks)

Explain how the Casimirs for a loop Lie Poisson bracket differ from the finite dimensional case.

(Total: 20 marks)

3. Dynamics of N point vortices interacting in the plane.

For 2D incompressible flow, the vorticity ω satisfies a scalar vorticity transport equation,

$$\partial_t \omega = -\boldsymbol{v} \cdot \nabla \omega \,, \text{ with } \omega := \hat{\boldsymbol{z}} \cdot \mathrm{curl} \boldsymbol{v} \text{ and } \boldsymbol{v}(\boldsymbol{x}, t) = \nabla^{\perp} \psi := \hat{\boldsymbol{z}} \times \nabla \psi = (-\psi_y, \psi_x) \,,$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity and $\psi(\boldsymbol{x}, t)$ is called the *stream function* of the flow. By its definition the stream function satisfies Poisson's equation $\Delta \psi = \omega(\boldsymbol{x}, t)$ at each time, t. This means that for appropriate boundary conditions, we have $\psi(\boldsymbol{x}, t) = G * \omega$ where G is the Green's function for the Laplacian in the domain of flow, which we take to be the $\mathbb{R}^2$ plane; so $G(\boldsymbol{x}, \boldsymbol{x}') \approx \log(|\boldsymbol{x} - \boldsymbol{x}'|)$.

(a) (6 marks)

Show that in addition to conserving energy $H = \frac{1}{2} \int |\boldsymbol{v}|^2 \, dx \, dy = -\frac{1}{2} \int \psi \Delta \psi \, dx \, dy$ vorticity transport implies an infinite number of conservation laws,

$$C_{\Phi} = \int_{\mathbb{R}^2} \Phi(\omega) \, dx \, dy \quad \text{with} \quad \omega = \Delta \psi \,,$$

for an arbitrary differentiable function Φ, provided $\lim_{|\boldsymbol{x}| \to \infty} \omega = 0$.

(b) (i) (3 marks)

Show that the vorticity transport equation admits singular solutions representing a set of M interacting point vortices with

$$\omega(\boldsymbol{x}, t) = \hat{\boldsymbol{z}} \cdot \mathrm{curl} \boldsymbol{v}(\boldsymbol{x}, t) = \sum_{A=1}^{M} \Gamma_A(t) \delta(\boldsymbol{Q}_A(t) - \boldsymbol{x}) \,,$$

with strengths $\Gamma_A(t)$, located at $\boldsymbol{x} = \boldsymbol{Q}_A(t)$, respectively.

(ii) (3 marks)

Show that the Stokes version of the Kelvin-Noether theorem for the circulation dynamics of planar Euler fluid dynamics implies for the point vortex solution that the circulations Γ_A, $A = 1, 2, \ldots, M$, are all constants.

(iii) (2 marks)

Show that the constancy of the vortex strengths Γ_A implies that each point vortex moves with the local fluid flow along Lagrangian trajectories. In particular, show that the paths of the point vortices obey the Lagrangian relation, $\dot{\boldsymbol{Q}}_A(t) = \boldsymbol{v}(\boldsymbol{Q}_A(t), t)$.

(c) (6 marks)

Write the dynamical equations for the entire set of N point vortices arising from the total Hamiltonian given by

$$H(\boldsymbol{Q}_A) = -\frac{1}{2} \int \psi \Delta \psi \, dx \, dy = -\frac{1}{2} \sum_{A,B=1}^{M} \Gamma_A \Gamma_B G\Big(\boldsymbol{Q}_A(t) - G(\boldsymbol{Q}_B(t)\Big) \,,$$

which is obtained by substituting the point vortex stream function into the total energy for Euler fluid motion.

(Total: 20 marks)

4. Untangling a coupled fluid system

(a) Consider the Clebsch action integral (in standard notation for this class), describing a fluid Lagrangian with its advection constraint which are coupled to a canonical subsystem evolving with the moving fluid,

$$
\begin{aligned}
S(u,a,b,q,p) = \int_{t_1}^{t_2} \underbrace{\ell(u,a)dt}_{\text{`Fluid' Lagrangian}} \; &+ \int_{t_1}^{t_2} \underbrace{\Big\langle b\,,\,\partial_t a + \mathcal{L}_u a \Big\rangle_V dt}_{\text{Advection Constraint}} \\
&+ \int_{t_1}^{t_2} \underbrace{\Big\langle p\,,\,\partial_t q + \mathcal{L}_u q \Big\rangle_V - \mathcal{H}(q,p)\,dt}_{\text{Canonical phase-space Lagrangian}}
\end{aligned}
$$

(i) Compute the independent variational derivatives of the action integral. (5 marks)

(ii) Derive the Euler-Poincaré equations for this action integral using the Clebsch approach. (4 marks)

(iii) Write the Kelvin-Noether theorem for this system. (3 marks)

(b) The Legendre transform for this system yields following the Hamiltonian,

$$
h(m,a;p,q) = \Big\langle m\,,\,u \Big\rangle + \Big\langle p\,,\,\partial_t q \Big\rangle - \ell(u,a) - \Big\langle p\,,\,\partial_t q + \mathcal{L}_u q \Big\rangle + \mathcal{H}(q,p)
$$

(i) Compute the variational derivatives of the Hamiltonian $h(m,a;p,q)$. (4 marks)

(ii) Express the equations for this system in Lie-Poisson bracket form, including the auxiliary advection equation $\partial_t a + \mathcal{L}_u a = 0$. (4 marks)

(Total: 20 marks)

5. Gyrostat: A rigid body with flywheel attached.

Consider a rigid body rotating in $\mathbb{R}^3$ with a flywheel attached along its intermediate principle axis in the body. Just as for the isolated rigid body itself, this problem involves only kinetic energy. We couple the flywheel to the rigid body by adding the phase space Lagrangian for the dynamics of the flywheel to the reduced Lagrangian for the rigid body on $\mathfrak{so}(3)$, so the gyrostat is rotating in the frame of rotation about the intermediate principal axis in the body. This approach results in the following Lagrangian for the system, given by $L:\ \mathfrak{so}(3)\times TS^1\to\mathbb{R}$, as

$$L(\mathbf{\Omega},\,\dot{\phi};p_\phi)=\frac{1}{2}I_1\Omega_1^2+\frac{1}{2}I_2\Omega_2^2+\frac{1}{2}I_3\Omega_3^2+p_\phi(\dot{\phi}+\Omega_2)-H(p_\phi)\,.$$

Here, $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)\in\mathfrak{so}(3)$ are the components of the angular velocity vector of the rigid body; I_1, I_1, I_3 are its positive constant principal moments of inertia; $\dot{\phi}\in TS^1$ is the rotational frequency relative to Ω_2 of the flywheel about the intermediate principal axis of the rigid body; p_ϕ is angular momentum the flywheel; and $H(p_\phi)=p_\phi^2/(2J_2)$ is its kinetic energy Hamiltonian.

(a) Use Hamilton's principle to compute the equations of motion for this system. (5 marks)

(b) Legendre-transform the Lagrangian to compute:

(i) the system's Hamiltonian; (2 marks)

(ii) conservation laws; (2 marks)

(iii) Poisson bracket (direct sum) (2 marks)

(c) Does the addition of the gyrostat have a qualitative effect on the rigid body motion? Explain. (5 marks)

(d) Is this system integrable? Explain. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

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MATH97067/MATH97178

Dynamics, Symmetry and Integrability (Solutions)

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1. (a) Consider the matrices in $\mathfrak{S}$ given by unseen $\Downarrow$ 2, A

$$A = \begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_1 \\ 0 & 0 & a_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -b_1 \\ 0 & 1 & b_1 \\ 0 & 0 & b_2 \end{bmatrix}.$$

Then the matrix product gives another element of $\mathfrak{S}$

$$AB = \begin{bmatrix} 1 & 0 & -b_1 - a_1 b_2 \\ 0 & 1 & b_1 + a_1 b_2 \\ 0 & 0 & a_2 b_2 \end{bmatrix}$$

The identity element is the usual matrix identity, when $b_1 = -\,a_1 b_2$ and $a_2 b_2 = 1$. Thus, the inverse is given uniquely by

$$A^{-1} = \begin{bmatrix} 1 & 0 & a_1/a_2 \\ 0 & 1 & -a_1/a_2 \\ 0 & 0 & a_2^{-1} \end{bmatrix}.$$

Hence, we are dealing with a two-parameter matrix (Lie) group. unseen $\Downarrow$

(b) Conjugation ABA^{-1} of an element of the shear subgroup $B \in \Sigma$ by an element of the full group $A \in \mathfrak{S}$ yields 2, B

$$\begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_1/a_2 \\ 0 & 1 & -a_1/a_2 \\ 0 & 0 & a_2^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(\alpha - a_1)/a_2 \\ 0 & 1 & (\alpha - a_1)/a_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, shears form a normal subgroup, since $\mathfrak{S}\Sigma\mathfrak{S}^{-1} \subset \Sigma$ for all elements of $\mathfrak{S}$. unseen $\Downarrow$

(c) The table of commutators $[m_i, m_j] = c_{ij}^k m_k$ is easily found by direct computation as 1, B

$$[m_i,\, m_j] = c_{ij}^k m_k = \begin{array}{|c|cc|} \hline [\,\cdot\,,\,\cdot\,] & m_1 & m_2 \\ \hline m_1 & 0 & m_1 \\ m_2 & -m_1 & 0 \\ \hline \end{array}.$$

The nonzero structure constants are read off the table as $c_{12}^1 = 1 = -c_{21}^1$. unseen $\Downarrow$

(d) The infinitesimal spatial transformations in $\mathfrak{s}$ correspond to the canonical Poisson bracket 5, C

$$\delta\mathbf{x} = \frac{d\mathbf{x}}{ds}\Big|_{s=0} = \Phi_\xi(\mathbf{x}) = \{\mathbf{x}, J^\xi(\mathbf{x},\mathbf{p})\} = \frac{J^\xi(\mathbf{x},\mathbf{p})}{\partial\mathbf{p}}$$

Thus, the Hamiltonian $J^\xi(\mathbf{x},\mathbf{p})$ which generates these infinitesimal spatial transformations is given by

$$\begin{aligned} J^\xi(\mathbf{x},\mathbf{p}) &= \mathbf{p}\cdot\Phi_\xi(\mathbf{x}) = p_1(-\xi_1 x_3) + p_2(\xi_1 x_3) + p_3(\xi_2 x_3) \\ &= x_3(p_2 - p_1)\xi_1 + x_3 p_3 \xi_2 = \mathbf{J}(\mathbf{x},\mathbf{p})\cdot\boldsymbol{\xi} \end{aligned}$$

Thus, the components (J_1, J_2) of the momentum map dual to components (ξ_1, ξ_2) in the basis (m_1, m_2) of $\mathfrak{s}$ are $J_1 = x_3(p_2 - p_1)$ and $J_2 = x_3 p_3$. unseen $\Downarrow$

(e) Hamilton's canonical equations lift the infinitesimal spatial transformations $\Phi_\xi(\mathbf{x})$ to infinitesimal transformations in phase space as 3, D

$$\delta\mathbf{p} = \frac{d\mathbf{p}}{ds}\Big|_{s=0} = \{\mathbf{p}, J^\xi\} = -\frac{\partial J^\xi}{\partial\mathbf{x}} = -\,p_j\frac{\partial}{\partial\mathbf{x}}\Phi_\xi^j(\mathbf{x})\,.$$

Thus, the Hamiltonian vector field for the Hamiltonian $J^\xi(\mathbf{x},\mathbf{p})$

$$X_{J^\xi} = \{\,\cdot\,,\, J^\xi\,\} = \frac{\partial J^\xi}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{\partial J^\xi}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{p}} = \Phi_\xi(\mathbf{x})\cdot\frac{\partial}{\partial \mathbf{x}} - \left(p_j\frac{\partial}{\partial \mathbf{x}}\Phi^j_\xi(\mathbf{x})\right)\cdot\frac{\partial}{\partial \mathbf{p}}$$

In the present case, $J^\xi = p_1(-\xi_1 x_3) + p_2(\xi_1 x_3) + p_3(\xi_2 x_3)$, so its Hamiltonian vector field is

$$X_{J^\xi} = -\xi_1 x_3\frac{\partial}{\partial x_1} + \xi_1 x_3\frac{\partial}{\partial x_2} + \xi_2 x_3\frac{\partial}{\partial x_3} + 0\frac{\partial}{\partial p_1} + 0\frac{\partial}{\partial p_2} + (p_1\xi_1 - p_2\xi_1 - p_3\xi_2)\frac{\partial}{\partial p_3}\,.$$

Note that p_1 and p_2 are left invariant. unseen ⇓

(f) The Poisson brackets among the momentum map components $J_1 = x_3(p_2 - p_1)$ and $J_2 = x_3 p_3$ are, in tabular form, 3, B

$$\{J_i,\, J_j\} = c^k_{ij}J_k = \begin{array}{|c|cc|} \hline \{\,\cdot\,,\,\cdot\,\} & J_1 & J_2 \\ \hline J_1 & 0 & J_1 \\ J_2 & -J_1 & 0 \\ \hline \end{array}\,.$$

That is, $\{J_i, J_j\} = c^k_{ij}J_k$. Thus, the entries in this Poisson bracket table match the nonzero structure constants $c^1_{12} = 1 = -c^1_{21}$ in the $\mathfrak{s}$ commutator table (). For two functions $F(\mathbf{J})$ and $H(\mathbf{J})$ we have

$$\{F,H\} = \frac{\partial F}{\partial J_i}\{J_i,J_j\}\frac{\partial H}{\partial J_j} = J_k\,c^k_{ij}\frac{\partial F}{\partial J_i}\frac{\partial H}{\partial J_j} = J_2\left(\frac{\partial F}{\partial J_1}\frac{\partial H}{\partial J_2} - \frac{\partial F}{\partial J_2}\frac{\partial H}{\partial J_1}\right)$$

unseen ⇓

(g) The bracket $\{F,H\}$ is bilinear, skew, Leibnitz, and satisfies Jacobi as a result of its coefficients matching those of the commutator table for the Lie algebra $\mathfrak{s}$. It has no Casimir function. In fact, it is canonical in $q = J_1$ and $p = \log(|J_2|)$ for $J_2 \neq 0$. 1, B

unseen ⇓

(h) We have $\frac{dF}{dt} = \{F,H\} = J_2\left(\frac{\partial F}{\partial J_1}\frac{\partial H}{\partial J_2} - \frac{\partial F}{\partial J_2}\frac{\partial H}{\partial J_1}\right)$. Hence, for $F = (J_1, J_2)$ 1, C

$$\dot J_1 = J_1 J_2 \quad\text{and}\quad \dot J_2 = -\,J_1^2$$

unseen ⇓

(i) These equations describe geodesic motion on the Lobachevsky half-plane. Keeping the $\tanh$ function in mind and recalling that 2, B

$$\frac{d\tanh(ct)}{dt} = c\,\mathrm{sech}^2(ct) \qquad \frac{d\,\mathrm{sech}(ct)}{dt} = -\,c\,\mathrm{sech}(ct)\tanh(ct),$$

we find, for $J_1(0) = c$ and $J_2(0) = 0$,

$$J_2(t) = -c\tanh(ct) \quad\text{and}\quad J_1(t) = c\,\mathrm{sech}(ct),$$

and of course we check that the Hamiltonian H is conserved, by

$$2H = J_1^2 + J_2^2 = c^2(\tanh^2 + \mathrm{sech}^2) = c^2\,.$$

We have $\lim_{t\to\infty}(J_1(t), J_2(t)) = (0, -\,c)$. Consequently, the quantity $J_1(t)$ falls exponentially with time, from $J_1(0)$ toward the line of fixed points at $J_1 = 0$, while $J_2(t)$ goes to a constant equal to $-\,c = -J_2(0)$.

2. (a) The Euler-Poincaré equation for $S = \int_{t_1}^{t_2} \ell(\xi)dt = \int_{t_1}^{t_2} \oint_{S^1} L(\xi(x,t))dxdt$ is

unseen ⇓

6, A

$$
\begin{aligned}
0 = \delta S &= \int_{t_1}^{t_2} \left\langle \frac{\delta \ell}{\delta \xi} \,,\, \delta\xi \right\rangle dt = \int_{t_1}^{t_2} \left\langle \frac{\delta \ell}{\delta \xi} \,,\, \partial_t \eta + \mathrm{ad}_\xi \eta \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle -\partial_t \frac{\delta \ell}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} \,,\, \delta\xi \right\rangle dt + \left\langle \frac{\delta \ell}{\delta \xi} \,,\, \eta \right\rangle \Big|_{t_1}^{t_2}
\end{aligned}
$$

with left-invariant $\eta(x) = g^{-1}\delta g(x,t) \in \widetilde{\mathfrak{g}}$ which vanishes at the endpoints in time.

(b) Legendre transform this Lagrangian and determine its associated:

meth seen ⇓

(i) Loop Hamiltonian,

3, A

$$
\begin{aligned}
h(\xi^\flat) &= \left\langle \xi^\flat \,,\, \xi \right\rangle - \ell(\xi) = \frac{1}{2}\left\langle \xi^\flat \,,\, (\xi^\flat)^\sharp \right\rangle \\
&= \frac{1}{2} \oint_{S^1} \mathrm{tr}\Big(\xi(x)^\flat \big((1-\alpha^2\partial_x^2)^s\big)^{-1} * \xi(x)^\flat \Big)\, dx
\end{aligned}
$$

in which $*$ denotes convolution with the Green's function of $(1-\alpha^2\partial_x^2)^s$, so that

$$
\frac{\delta h}{\delta \xi^\flat} = \xi \,.
$$

meth seen ⇓

(ii) The loop Lie-Poisson bracket for left action is.

4, A

$$
\{f \,,\, h\}(\xi^\flat) = -\left\langle \xi^\flat \,,\, \left[\frac{\delta f}{\delta \xi^\flat} \,,\, \frac{\delta h}{\delta \xi^\flat} \right] \right\rangle = \left\langle \xi^\flat \,,\, \mathrm{ad}_{\frac{\delta h}{\delta \xi^\flat}} \frac{\delta f}{\delta \xi^\flat} \right\rangle
$$

Thus, one recovers the EP equation from the Lie Poisson bracket as

$$
\partial_t \xi^\flat = \{\xi^\flat \,,\, h\} = \mathrm{ad}^*_{\frac{\delta h}{\delta \xi^\flat}} \xi^\flat = \mathrm{ad}^*_\xi \xi^\flat
$$

meth seen ⇓

(c) By inspection, the null eigenvectors of the loop Lie-Poisson bracket in Part (b)(ii) are the same (pointwise) as for the finite dimensional case. Therefore the Casimirs are the same (pointwise) as for the finite dimensional case.

7, A

3. (a) meth seen ⇓

6, A

$$\frac{d}{dt}C_\Phi[\omega] = \int_S \partial_t \Phi(\omega)\,dxdy = -\int_S \boldsymbol{v}\cdot\nabla\Phi(\omega)\,dxdy$$
$$= -\int_S \operatorname{div}(\boldsymbol{v}\Phi(\omega))\,dxdy = -\oint_{\partial S} \hat{\boldsymbol{n}}\cdot\boldsymbol{v}\,\Phi(\omega)ds = 0\,.$$

3, B

(b) (i) The stream function for such a set of point vortices would be obtained from the corresponding Green's function for the Laplacian as,

$$\psi(\boldsymbol{x},t) = G*\omega(\boldsymbol{x},t) = \sum_{A=1}^{M}\Gamma_A(t)G(\boldsymbol{Q}_A(t)-(\boldsymbol{x},t))\,.$$

From this stream function one may obtain the velocity $\boldsymbol{v} = \nabla^\perp\psi$ and then the vorticity $\omega = \Delta\psi$ with no difficulty.

3, B

(ii) $\Gamma_A(t) = \Gamma_A = \text{constant}$ follows from

$$\frac{d}{dt}\oint_{\partial S(t)=c(\boldsymbol{v})}\omega(\boldsymbol{x},t)dx\,dy = \frac{d}{dt}\sum_{A=1}^{M}\Gamma_A(t) = 0.$$

2, C

(iii) Preservation of scalar vorticity inside any Kelvin loop moving with the flow implies that each point vortex moves with the local fluid flow along Lagrangian trajectories.

$$\frac{d}{dt}\int_{S(t)}\omega(\boldsymbol{x},t)\,dx\,dy = \int_{S(t)}(\partial_t\omega + bsv\cdot\nabla\omega)\,dx\,dy$$
$$= \sum_{A=1}^{M}\int_{S(t)}\Gamma_A\Big(\dot{\boldsymbol{Q}}_A(t)-\boldsymbol{v}\Big)\delta'(\boldsymbol{Q}_A(t)-\boldsymbol{x}) = 0\,.$$

In particular, the paths of the point vortices obey the Lagrangian relation,

$$\dot{\boldsymbol{Q}}_A(t) = \boldsymbol{v}(\boldsymbol{Q}_A(t),t)\,.$$

6, D

(c) For all $A = 1,...,M$, the equations of motion of each point vortex at $\boldsymbol{Q}_A(t) = (X_A(t),Y_A(t))$ can be cast into canonical Hamiltonian form, as

$$\frac{dX_A}{dt} = -\frac{\partial\psi^A}{\partial Y_A} \quad\text{and}\quad \frac{dY_A}{dt} = \frac{\partial\psi^A}{\partial X_A}\,,$$

with the individual Hamiltonian / stream function $\psi^A(\boldsymbol{Q}_A(t),t)$ satisfying

$$\psi^A(\boldsymbol{Q}_A(t),t) = G*\omega(\boldsymbol{Q}_A(t),t) = \sum_{B=1}^{M}\Gamma_B G(\boldsymbol{Q}_A(t)-\boldsymbol{Q}_B(t))\,,$$

where $G(\boldsymbol{x},\boldsymbol{y}) = \log(|\boldsymbol{x}-\boldsymbol{y}|)$ is the Green's function of the Laplacian in the plane. The canonical equations above then imply,

$$\Gamma_A\frac{dX_A}{dt} = -\frac{\partial H}{\partial Y_A} \quad\text{and}\quad \Gamma_A\frac{dY_A}{dt} = \frac{\partial H}{\partial X_A}\,,$$

4. (a) (i) Compute the elemental variational derivatives of the action integral.

seen ⇓

5, B

$$
\begin{aligned}
&\delta u: \quad \frac{\delta \ell}{\delta u} - b \diamond a - p \diamond q = 0\,,\\
&\delta b: \quad \partial_t a + \mathcal{L}_u a = 0\,, \qquad \delta a: \quad \frac{\delta \ell}{\delta a} - \partial_t b + \mathcal{L}_u^T b = 0\,,\\
&\delta p: \quad \partial_t q + \mathcal{L}_u q - \frac{\delta H}{\delta p} = 0\,, \qquad \delta q: \quad \partial_t p - \mathcal{L}_u^T p + \frac{\delta H}{\delta q} = 0\,.
\end{aligned}
$$

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(ii) Using these variational relations in the Clebsch approach leads to the following Euler-Poincaré motion equation,

4, A

$$
\partial_t \frac{\delta \ell}{\delta u} + \mathrm{ad}^*_u \frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta a} \diamond a = - \frac{\delta\,\mathcal{H}(q,p)}{\delta q} \diamond q + p \diamond \frac{\delta\,\mathcal{H}(q,p)}{\delta p}\,.
$$

(iii) The Kelvin-Noether theorem for this system is

3, A

meth seen ⇓

$$
\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\frac{\delta \ell}{\delta u} = \oint_{c(u)} \frac{1}{D}\left(\frac{\delta \ell}{\delta a} \diamond a - \frac{\delta\,\mathcal{H}(q,p)}{\delta q} \diamond q + p \diamond \frac{\delta\,\mathcal{H}(q,p)}{\delta p} \right)
$$

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(b) (i) Compute the variational derivatives of the Hamiltonian $h(m,a;p,q)$.

4, B

$$
\begin{aligned}
\delta h(m,a;p,q) &= \Big\langle \delta m\,,\, u \Big\rangle + \Big\langle m - \frac{\delta \ell}{\delta u} + p \diamond q\,,\, \delta u \Big\rangle \\
&\quad - \Big\langle \delta p\,,\, \mathcal{L}_u q - \frac{\delta \mathcal{H}}{\delta p} \Big\rangle - \Big\langle \mathcal{L}_u^T p - \frac{\delta \mathcal{H}}{\delta q}\,,\, \delta q \Big\rangle
\end{aligned}
$$

In particular, $m = \frac{\delta \ell}{\delta u} - p \diamond q$.

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(ii) The Hamiltonian equations may be written in Poisson matrix form, including the auxiliary advection equation $\partial_t a + \mathcal{L}_u a = 0$, as

4, C

$$
\frac{d}{dt}\begin{bmatrix} m \\ a \\ p \\ q \end{bmatrix} = - \begin{bmatrix} \partial m + m \partial & \Box \diamond a & 0 & 0 \\ \mathcal{L}_\Box a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta m = u \\ \delta h/\delta a = -\delta \ell/\delta a \\ \delta h/\delta p = -\mathcal{L}_u q + \delta \mathcal{H}/\delta p \\ \delta h/\delta q = -\mathcal{L}_u^T p + \delta \mathcal{H}/\delta q \end{bmatrix}.
$$

The corresponding Lie-Poisson bracket form is the following direct sum,

$$
\{f\,,\,h\} = -\int \frac{\delta f}{\delta m}(\partial m + m\partial)\frac{\delta h}{\delta m} + \frac{\delta f}{\delta a}\mathcal{L}_{\frac{\delta h}{\delta m}} a - \frac{\delta h}{\delta a}\mathcal{L}_{\frac{\delta f}{\delta m}} a + \frac{\delta f}{\delta q}\frac{\delta h}{\delta p} - \frac{\delta h}{\delta q}\frac{\delta f}{\delta p}\, dV\,.
$$

5. (a) Use Hamilton's principle to compute the equations of motion for this system. Hamilton's principle is,

meth seen ⇓

5, D

$$0 = \delta S = \delta \int_{t_1}^{t_2} L(\mathbf{\Omega}, \dot{\phi}; p_\phi)\, dt \quad \text{with} \quad \delta\mathbf{\Omega} = \dot{\mathbf{\Xi}} + \mathbf{\Omega} \times \mathbf{\Xi}$$

$$= \int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial \mathbf{\Omega}}\,,\, \delta\mathbf{\Omega} \right\rangle + \left\langle \frac{d\phi}{dt} + \Omega_2 - \frac{\partial H}{\partial p_\phi}\,,\, \delta p_\phi \right\rangle - \left\langle \frac{dp_\phi}{dt}\,,\, \delta\phi \right\rangle dt + \Big\langle p_\phi\,,\, \delta\phi \Big\rangle\Big|_{t_1}^{t_2}$$

Hence the equations of motion are

$$\mathbf{\Xi} : \frac{d}{dt}\frac{\partial L}{\partial \mathbf{\Omega}} + \mathbf{\Omega} \times \frac{\partial L}{\partial \mathbf{\Omega}} = 0\,, \quad \text{with} \quad \frac{\partial L}{\partial \mathbf{\Omega}} = (I_1\Omega_1, I_2\Omega_2 + p_\phi, I_3\Omega_3) =: \mathbf{\Pi}\,,$$

$$\delta p_\phi : \frac{d\phi}{dt} + \Omega_2 - \frac{\partial H}{\partial p_\phi} = 0\,, \quad \text{with} \quad H(p_\phi) = \frac{p_\phi^2}{2J_2}\,,$$

$$\delta\phi : \frac{dp_\phi}{dt} = 0\,.$$

meth seen ⇓

(b) Legendre-transform the Lagrangian to compute the system's

(i) Hamiltonian (show work)

2, D

$$H(\mathbf{\Pi}, p_\phi) = \mathbf{\Pi}\cdot\mathbf{\Omega} + p_\phi\dot{\phi} - \left(\frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}I_3\Omega_3^2 + p_\phi(\dot{\phi} + \Omega_2) - H(p_\phi)\right)$$

$$= \frac{\Pi_1^2}{2I_1} + \frac{(\Pi_2 - p_\phi)^2}{2I_2} + \frac{\Pi_3^2}{2I_3} + \frac{p_\phi^2}{2J_2}$$

(ii) Conversation laws (by Noether's theorem)

2, B

$H(\mathbf{\Pi}, p_\phi)$, $|\mathbf{\Pi}|^2$, p_ϕ are all three conserved.

(iii) Poisson bracket (direct sum)

2, D

The Hamiltonian equations may be written in Poisson matrix form as

$$\frac{d}{dt}\begin{bmatrix} \mathbf{\Pi} \\ \phi \\ N \end{bmatrix} = -\begin{bmatrix} \mathbf{\Pi}\times & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial H/\partial\mathbf{\Pi} = \mathbf{\Omega} := \Big(I_1{}^{-1}\Pi_1, I_2{}^{-1}(\Pi_2 - p_\phi), I_3{}^{-1}\Pi_3\Big) \\ \partial H/\partial\phi = 0 \\ \partial H/\partial p_\phi = J_2{}^{-1}p_\phi - I_2{}^{-1}\big(\Pi_2 - p_\phi\big) \end{bmatrix}.$$

The corresponding Poisson bracket form is

$$\{F\,,\,H\} = -\,\mathbf{\Pi}\cdot\left(\frac{\partial F}{\partial\mathbf{\Pi}} \times \frac{\partial H}{\partial\mathbf{\Pi}}\right) + \frac{\partial F}{\partial\phi}\frac{\partial H}{\partial p_\phi} - \frac{\partial H}{\partial\phi}\frac{\partial F}{\partial p_\phi}\,.$$

(c) Does the addition of the gyrostat have a qualitative effect on the rigid body motion?

unseen ⇓

5, D

Yes. The motion takes place in $\mathbf{\Pi} \in \mathbb{R}^3$ along the intersections of the Hamiltonian ellipsoid and the angular momentum sphere. Shifting the intermediate axis of the Hamiltonian ellipsoid destroys the heteroclinic orbits on the angular momentum sphere and creates different critical points (equilibria) with different stable and unstable orbits and new homoclinic connections. Thus, the gyrostat make a global rearrangement of the classic rigid body motion on the angular momentum sphere.

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(d) Is this system integrable? Explain. Yes, the system is still integrable, since the motion remains on the intersections of the Hamiltonian and angular momentum sphere, and the gyrostat dynamics form a separate action-angle pair.

4, C