MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Hints for Problem Sheet 8

Exercise 36

For the linear systems in (i)–(iii), consider the asymptotic behaviour of the solutions to the linear equations in and outside of the eigenspaces. For (iv), involve Exercise 16 (i) and (ii).

Exercise 37

Find a scalar-valued function V whose orbital derivative is positive except outside of the trivial equilibrium. Use the fact that the scalar-valued function increases along solutions to prove instability.

Exercise 38

This is in parts similar to Exercise 31, and Exercise 16 turns out to be crucial for the an analysis of the limiting behaviour. The four regions help to get an understanding of movements of the flow.

Exercise 39

This exercise is similar to Exercise 38, but involves also to check the positive invariance of a compact set. This is done by looking at the dynamical behaviour at the boundary of the set. In particular, the flow needs to be directed inside the set, which will prove positive invariance.

Exercise 40

This is an application of Lyapunov's direct method and involves finding a Lyapunov function. For the domain of attraction, sublevel sets have be considered (see also Corollary 4.33).

Exercise 41

This is similar to Exercise 40, and here, a (nontrivial) Lyapunov function can be found where the orbital derivative is equal to 0. This helps to plot the phase portrait.

Exercise 42

The solution of this exercise is similar to Example 4.36.

Exercise 43

(i) is standard material, and consider the Lyapunov function $V(x, y) = \sin(x)\sin(y)$ for (ii).

Exercise 44

Argue indirectly and assume that the omega limit set is disconnected. According to the definition given, this implies the existence of two disjoint closed sets. Now consider the fact that the flow needs to spend a lot of time in both sets (in the limit $t \to \infty$), but also needs to be outside of this set on a subsequence of times converging to ∞. This establishes another omega limit point outside the omega limit set, which is a contradiction.