Quantum Mechanics II **Spring 2025**

First Coursework

Due Date: Wednesday, 19 Feb 2025 to be submitted electronically on Blackboard.

Please make the solutions you hand in as neat and concise as possible. Show your work. As some problems are shorter than others, the mark scheme will not allocate equal marks to each problem. You are encouraged to perform any necessary integrals, but looking them up or using software to compute them is also acceptable. As always, cite any sources used.

1. Shorter problems on the first chapter.
   (a) Suppose $\phi(x) = Axe^{-x^2/a^2}$ where a is a real constant describes a quantum mechanical state in the position basis. Determine A.
   (b) Determine $\tilde{\phi}(p)$ of the state from (a).
   (c) Suppose that the wavefunction in the position basis is even, $\phi(x) = \phi(-x)$. Show this state expressed in the momentum basis is also even (in p).
   (d) Suppose that the wavefunction in the position basis satisfies $\phi(x) = [\phi(-x)]^*$. What does this tell us about the state in the momentum basis?

2. Consider the Hamiltonian
$$\hat{\mathcal{H}} = -w\left[|1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1| + \text{h.c.}\right].$$
   Here, w is a positive real parameter, $\{|1\rangle, |2\rangle, |3\rangle\}$ is an orthonormal basis describing the Hilbert space of interest, and "h.c." denotes the Hermitian conjugate of the three terms preceding it (making the full Hamiltonian Hermitian, as it must be). It may be helpful to think of this Hamiltonian as describing a particle that can jump between vertices of an equilateral triangle.
   (a) Take a rotation operator in this context to be given by $\hat{R} = |3\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3|$ and a reflection operator to be given by $\hat{\sigma} = |1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 3|$. Show that these operators are unitary and commute with the Hamiltonian. Can you think of any other distinct unitary symmetry operators for this Hamiltonian?
   (b) Find a common eigenbasis (and corresponding eigenvalues) for $\hat{R}$ and $\hat{\mathcal{H}}$. Next, find a common eigenbasis of $\hat{\sigma}$ and $\hat{\mathcal{H}}$.
   (c) Work out an explicit expression for $[\hat{R}, \hat{\sigma}]$. Let $|\phi\rangle$ be the non-degenerate eigenstate that you found already in (b). Determine $[\hat{R}, \hat{\sigma}]\,|\phi\rangle$. Is this consistent with our considerations from Section 3.2 of the notes?

3. Consider a particle governed by the 1d harmonic oscillator Hamiltonian. Suppose the particle is in the initial state $|\psi(0)\rangle = (|\phi_0\rangle + |\phi_1\rangle)/\sqrt{2}$ where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the ground and first excited states of the Hamiltonian. Find the expectation value of the position operator as a function of time.

Suggested approach: Use ladder operators and the Heisenberg equations of motion.

Something more challenging to think about (but not assessed/marked): Determine the time evolution of the position and momentum variance and discuss within the context of the Heisenberg uncertainty relation.