Applied complex analysis: notes for Spring 2025 exam

(Last updated 7/4/2025)

With regard to the questions on singular integral equations, special functions and the Wiener-Hopf method in this year's exam, please refer to the solutions for questions on these topics in the past papers written by Dr. Brzezicki for an indication of the level of detail required in your answers. I will mark the questions on these topics in this year's exam similarly.

1 Sections of the lecture notes that you can ignore

Below is a list of some sections of Dr. Brzezicki's typed lecture notes on singular integral equations, special functions and the Wiener-Hopf method that we followed in class in the second half of the course, that are **not** going to be examined in your forthcoming exam.

- On singular integral equations:
  - Section 2.8, Ideal flow past a flat plate.
  - Section 2.9, Electrostatic potential of a point charge near a flat plate.
- On special functions:
  - Section 3.3, Hypergeometric series.
  - Section 3.4, Euler integral representation.
  - Section 3.5, Gauss summation theorem.
  - Section 3.6, The hypergeometric equation.
- On the Wiener-Hopf method:
  - Section 5.1, Riemann–Hilbert problems.
  - Section 5.5, The Wiener-Hopf product and sum decompositions.

2 Some things that you will not need to memorise

You will **not** be expected to remember any of the following formulae/identities/definitions; if any are required, they will be given in the question. Please note that this list is not necessarily exhaustive.

- The Hilbert inversion formula for the solution of a singular integral equation (equation (34) of the typed notes on singular integral equations).
- The gamma function $\Gamma(z)$ is defined for $\mathrm{Re}\{z\} > 0$ by $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$.

- The identity $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

- The beta function $B(z, w)$ is defined for $\mathrm{Re}\{z\}, \mathrm{Re}\{w\} > 0$ by $B(z, w) = \displaystyle\int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t$.

- The identity $B(z, w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.

- The definition of a right-sided transform, i.e., for a function $f(x)$, $F_+(s) \equiv \displaystyle\int_0^\infty f(x)e^{\mathrm{i}sx}\mathrm{d}x$ for $s \in \mathbb{C}$.

- The (ordinary) Fourier transform, $\hat{K}(s)$, of a function $k(x) = ce^{-\gamma|x|}$, for constant c and $\gamma > 0$: $\hat{K}(s) = \displaystyle\int_\infty^\infty k(x)e^{\mathrm{i}sx}\mathrm{d}x = \dfrac{2c\gamma}{s^2+\gamma^2}$ for s with $|\mathrm{Im}\{s\}| < \gamma$.

3 Some things that you can quote without proof

- If $|f(x)| < Ae^{\alpha x}$ as $x \to +\infty$ for some $A, \alpha \in \mathbb{R}$, then the right-sided transform $F_+(s) \equiv \displaystyle\int_0^\infty f(x)e^{\mathrm{i}sx}\mathrm{d}x$ of $f(x)$ exists and is analytic for $\mathrm{Im}\{s\} > \alpha$.

- If $|g(x)| < Be^{\beta x}$ as $x \to -\infty$ for some $B, \beta \in \mathbb{R}$, then the left-sided transform $G_-(s) \equiv \displaystyle\int_{-\infty}^0 g(x)e^{\mathrm{i}sx}\mathrm{d}x$ of $g(x)$ exists and is analytic for $\mathrm{Im}\{s\} < \beta$.

- The expansion as $|s| \to \infty$ of the right-sided transform $F_+(s)$ of a function $f(x)$, i.e., $F_+(s) = \dfrac{\mathrm{i}f(0)}{s} - \dfrac{f'(0)}{s^2} + \mathcal{O}\left(\dfrac{1}{s^3}\right)$.

- The right-sided transform $\displaystyle\int_0^\infty f'(x)e^{\mathrm{i}sx}\mathrm{d}x$ of the derivative $f'(x)$ of a function $f(x)$ is $-f(0) - \mathrm{i}sF_+(s)$, where $F_+(s)$ the right-sided transform of $f(x)$.

- The inversion formula for a right-sided transform, i.e., for $x \geq 0$, $f(x) = \dfrac{1}{2\pi}\displaystyle\int_P F_+(s)e^{-\mathrm{i}sx}\mathrm{d}s$ where P is a horizontal line extending to infinity in both directions in the region where $F_+(s)$ exists.

- In the final stage of the Wiener-Hopf method, when we invert the right-sided transform to get the final solution for $x \geq 0$, we close the infinite horizontal line along which we must integrate (see the previous item in this list) with a semi-circle beneath it. **I am happy for you just to state that the integral along this semi-circle is zero**. This is the case for all the integral equations that you will ever have to solve in this course using the Wiener-Hopf method.

- Differentiating (with respect to x) both sides of the equation

$$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|\mathrm{d}t = g(x), \quad x \in (-1,1),$$

  results in the equation

$$\frac{1}{\pi}⨍_{-1}^{1} \frac{f(t)}{t-x}\mathrm{d}t = -g'(x), \quad x \in (-1,1),$$

- The general binomial expansion for $\alpha \in \mathbb{R}$ and $|z| < 1$:

$$(1+z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \dots$$

4 Preparation for the questions set by Dr. Gibbs

To prepare for the exam questions set Dr. Gibbs, I would recommend that you work through the Problem Sheet 2 that he set you; his exam questions will be of a similar style. In terms of the lecture material, I would recommend that you prioritise studying the worked examples. You will **not** be expected to memorise any of the following theorems/results or their proofs:

- Theoreom 2.2 (Quadratic convergence of the trapezium rule).
- Theorem 2.3 (Exponential convergence of the trapezium rule for functions analytic inside a circle).
- Theorem 2.4 (Exponential convergence of the trapezium rule for periodic analytic functions).
- Theorem 2.7 (Exactness of the trapezium rule).
- Theorem 2.9 (The argument principle).
- Theorem 2.12.
- Lemma 2.13.

If any of these results are required, they will be given in the question. That said, however, you should of course be comfortable with the application of these results (as is required for the worked examples in the lecture notes and the problems on sheet 2).