L24

(3.4.7) Corollary.

(i) If X is a set of ordinals then $\bigcup X$ is an ordinal.

(ii) ω is an ordinal.

Pf: (i) By 3.4.4 $\bigcup X$ is a set of ordinals & by 3.4.6 it is well ordered by $\in$.

Check: $\bigcup X$ is a transitive set.

(ii) ω is a set of ordinals (3.4.3)

By (i) it's enough to show

$\bigcup \omega = \omega$.

$\subseteq$: If $m \in \bigcup\omega$ then $m \in n \in \omega$. Then $m \in \omega$.

$\supseteq$: If $n \in \omega$ then $n \in n^+ \in \omega$

so $n \in \bigcup \omega$. #

Can now form other 'infinite' ordinals:

$\omega^+ = \{0, 1, 2, \ldots, \omega\}$

$(\omega^+)^+$ etc. //.

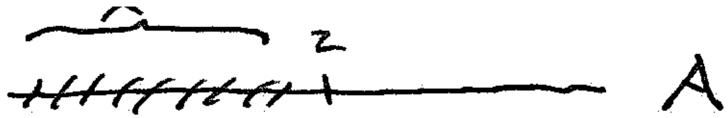
(3.4.8) Theorem. If $(A; \leq)$ is any well ordered set, then there is a unique ordinal which is similar to $(A; \leq)$

[called the order-type of $(A; \leq)$.]

Eg. there is an ordinal similar to $\mathbb{N} \times \mathbb{N}$.

(3.4.9) Definition Suppose $(A; \leq)$ is a w.o. set.

① Say that $X \subseteq A$ is an initial segment of A if whenever $x \in X$ and $y \in A$ with $y \leq x$, then $y \in X$.



Eg If α, β are ordinals and $\alpha < \beta$ then α is an initial segment of β:

$$\alpha = \{ \delta \in \beta : \delta < \alpha \}$$

The initial segment X of A is proper if $X \neq A$.

(2) If $z \in A$ write

$$A[z] = \{ a \in A : a < z \}$$

this is a proper initial segment of A.

(3.4.10) Lemma. Suppose $(A; \leq)$ is a w.o. set and $X \subseteq A$ is a proper initial segment. Then there is $z \in A$ with $X = A[z]$.

Pf: Take z the least elt. of $A \setminus X$ and show that it works. #

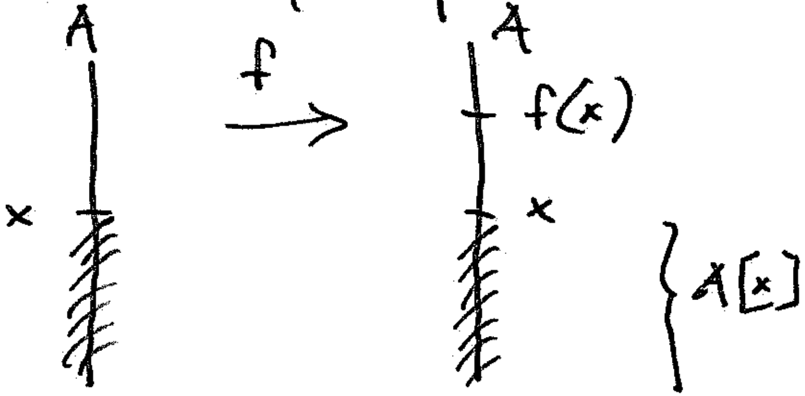
(3.4.11) Proposition Suppose $(A; \leq)$ is a w.o. set and $f : A \to A$ is order preserving [ie f is injective and if $a_1, a_2 \in A$

$$a_1 \leq a_2 \Rightarrow f(a_1) \leq f(a_2)]$$

Suppose $f(A)$ is an initial segment of A. Then $f(x) = x \quad \forall x \in A$.

Pf Suppose not. Let x be the least element of $\{ y \in A : f(y) \neq y \}$. So if $z < x$ then $f(z) = z$.

ie f restricted to $A[x]$ is the identity map.



As $f(x) \neq x$ and f is injective we have $f(x) > x$.

But then if $y > x$ we have $f(y) > f(x) > x$, so there is no y with $f(y) = x$.

This contradicts that $f(A)$ is an initial segment and $f(x) > x$. $\lightning$.

So $f(x) = x \quad \forall x \in A$ #

(3.4.12) <u>Cor</u>. If $\alpha \neq \beta$ are ordinals then α is not similar to β (ie. $\alpha \not\simeq \beta$).

<u>Pf</u>: By 3.4.5 can assume $\beta < \alpha$. Then $\beta \subset \alpha$ and β is a proper initial segment of α.

By 3.4.11 it follows that β is not similar to α. #

Proof of 3.4.8.

Given: w.o. set $(A; \leq)$

Find a unique ordinal similar to $(A; ~~\leq~~ \leq)$

Uniqueness: By 3.4.12.

Existence: Consider

$$X = \{x \in A : A[x] \text{ is similar to an ordinal}\}$$

By uniqueness, if $x \in X$ there is a unique ordinal α_x similar to $A[x]$.

Let $S = \{\alpha_x : x \in X\}$

... (*)

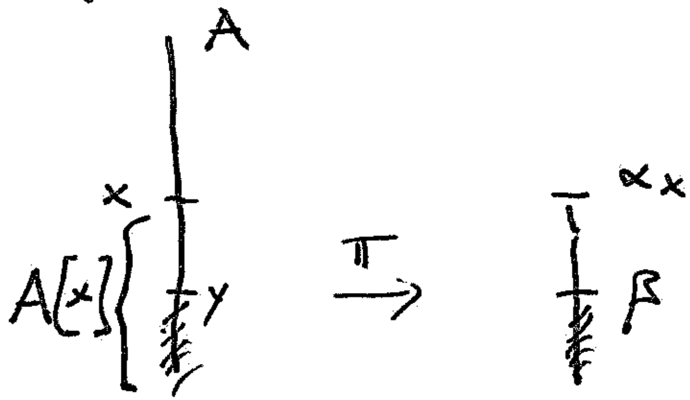
Claim: S is an ordinal

(1) S is a set of ordinals so (by 3.4.6) $\in$ is a strict well ordering on S.

(2) S is a transitive set

i.e. if $\beta \in \alpha_x$ then $\beta \in S$:

$x \in X$



Let $\pi : A[x] \to \alpha_x$ be a similarity.

Let $\pi^{-1}(\beta) = y$.

Then π restricted to $A[y]$ gives a similarity $A[y] \to$
$\{\delta \in \alpha_x : \delta < \beta\} = \beta$

thus $y \in X$, and

$\alpha_y = \beta \in S$. //.

Denote S by α.

AIM $X = A$:

once we have this the map

$$A \longrightarrow \alpha$$
$$x \longmapsto \alpha_x$$

is a similarity.

Know: X is an initial segment of A.

if $X \neq A$ there is $z \in A \setminus X$ with $X = A[z]$ (by 3.3.6)

We know that $x \mapsto \alpha_x$ is a similarity from $X = A[z]$ to $\alpha = S$.

This gives $z \in X$. Contradiction

#.

Problem: Why is S a set (in [illegible])

- Needs ZF 8

Axiom of Replacement.