Asymptotic Methods

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Part I

Basics of asymptotic approximations

1 Introduction

Some things in maths are easy to understand: x^a, e^x, $\ln x$, and sums of products of these.

Most things in maths are hard to understand: Complicated combinations of the above, integrals, equations, differential equations.

Asymptotic methods allow us to turn hard things into easy things, when the value of a parameter is **very small or very large**. For example, Taylor series are an example of an asymptotic expansion, and tell you what the function looks like near the expansion point (when the distance to the point is very small). However, asymptotic expansions are much more general than that, and in this introduction we'll see some brief examples of what is to come in this module.

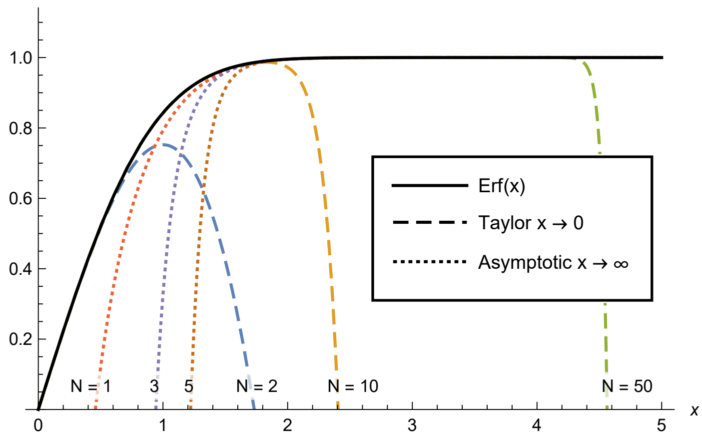
Note that the focus of this module is on methods and ideas for tackling problems rather than rigorous proofs. Some of the problems will be motivated by physical applications, but only the methods themselves will be examinable.

Example 1.1. Consider the error function $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$, one of the simplest integrals that can't be evaluated in terms of elementary functions.

Using the Taylor series of e^{-t^2} about $t = 0$, we can integrate each term to obtain a Taylor series

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}. \tag{1.1}$$

This Taylor series actually converges for all x, but it does so very slowly when x is large. Also, the function is nearly constant for $|x| > 2$, but this is not at all obvious from the Taylor series.



Let's try to find an approximation of $\mathrm{erf}(x)$ for large x instead. What's the simplest approximation we can do? The function looks nearly constant, so let's just take the limit

$$\mathrm{erf}(x) \approx \mathrm{erf}(+\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = 1. \tag{1.2}$$

Can we improve on this trivial approximation? Let's write down its error

$$\mathrm{erf}(x) - 1 = \frac{2}{\sqrt{\pi}}\left[\int_0^x - \int_0^{\infty}\right]\mathrm{e}^{-t^2}\,\mathrm{d}t = -\frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-t^2}\,\mathrm{d}t, \tag{1.3}$$

and try to approximate it for large x. The trick in this case is to integrate by parts by inserting an extra factor t to make $t\mathrm{e}^{-t^2}$,

$$\operatorname{erf}(x) - 1 = -\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{1}{t}\,t\mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\left\{\left[\frac{1}{2t}\mathrm{e}^{-t^2}\right]_x^\infty + \int_x^\infty \frac{1}{2t^2}\mathrm{e}^{-t^2}\,\mathrm{d}t\right\} \approx \frac{2}{\sqrt{\pi}}\left\{-\frac{1}{2x}\mathrm{e}^{-x^2}\right\}. \tag{1.4}$$

Was the last step, discarding the next integral, a good approximation? It certainly looks so in the figure, but how do we justify it without plotting it on a computer? Let's look at the discarded integral, and try to make $t\mathrm{e}^{-t^2}$ again, but this time doing some simplifying approximations using $t \geq x$ before evaluating the integral:

$$\left|\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{1}{2t^2}\mathrm{e}^{-t^2}\,\mathrm{d}t\right| = \left|\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{1}{2t^3}t\mathrm{e}^{-t^2}\,\mathrm{d}t\right| \leq \left|\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{1}{2x^3}t\mathrm{e}^{-t^2}\,\mathrm{d}t\right| = \frac{2}{\sqrt{\pi}}\frac{1}{4x^3}\mathrm{e}^{-x^2}. \tag{1.5}$$

We summarise our result as

$$\operatorname{erf}(x) = 1 + \frac{2}{\sqrt{\pi}}\mathrm{e}^{-x^2}\left\{-\frac{1}{2x} + O(x^{-3})\right\}, \tag{1.6}$$

where the O stands for "a term that's bounded by some constant times this thing". When x is large, the $O(x^{-3})$ error is small relative to the term $-1/2x$, so we have a good approximation.

We could continue to integrate by parts to obtain arbitrarily many terms, making the remainder smaller each time,

$$\operatorname{erf}(x) = 1 + \frac{2}{\sqrt{\pi}}\mathrm{e}^{-x^2}\left\{-\frac{1}{2x} + \frac{3}{4x^3} - \frac{3\times 5}{8x^5} + \cdots + (-1)^N\frac{3\times 5\times\cdots\times(2N-1)}{2^N x^{2N-1}} + O(x^{-(2N+1)})\right\}. \tag{1.7}$$

We say that we have a full asymptotic expansion for erf, and write

$$\operatorname{erf}(x) \sim 1 + \frac{2}{\sqrt{\pi}}\mathrm{e}^{-x^2}\sum_{n=1}^\infty (-1)^n\frac{3\times 5\times\cdots\times(2n-1)}{2^n x^{2n-1}}, \tag{1.8}$$

but note that the thing on the right-hand side is just a formal series and cannot be evaluated, because it diverges for any fixed value of x. The problem is that although each term is smaller than the previous one for sufficiently large x, the value of x required for this to hold increases with n, so at each fixed value of x the terms eventually start to grow in size instead of shrink.

Example 1.2. Here is an integral example with a very different flavour. Let's estimate the factorial $n!$ for large n.

The trick (or rather, one possible trick) is to use the Gamma function integral expression and make a change variables,

$$n! = \Gamma(n+1) = \int_0^\infty t^n\mathrm{e}^{-t}\,\mathrm{d}t \overset{t=ns}{=} n^{n+1}\int_0^\infty s^n\mathrm{e}^{-ns}\,\mathrm{d}s = n^{n+1}\int_0^\infty \mathrm{e}^{n(\ln s - s)}\,\mathrm{d}s. \tag{1.9}$$

According to the Laplace method, which we will learn about in this module, this integral will be dominated by the contribution from near the maximum of $f(s) = \ln s - s$, which occurs at $s = 1$, and the dominant contribution comes from the Taylor expansion of f up to the first non-constant term, so we obtain the estimate

$$n! \approx n^{n+1}\int_{-\infty}^{+\infty} \mathrm{e}^{n(-1-(s-1)^2/2)}\,\mathrm{d}s = n^{n+1}\mathrm{e}^{-n}\sqrt{\frac{2\pi}{n}} = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{1.10}$$

This result is called Stirling's approximation, and by keeping more terms in the Taylor expansion of f it's possible to obtain more terms,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots\right]. \tag{1.11}$$

Again, this asymptotic expansion is not a convergent series. We note that the first correction, $1/12n$ is surprisingly small even for the not-very-large number $n = 1$, and in fact the leading-order approximation is accurate to within 8% for $n \geq 1$, and within 1% for $n \geq 10$.

Example 1.3. Let's move on to solving equations. Consider the equation $x\mathrm{e}^x = y$ for large $y \to +\infty$. (The exact solution is known as the Lambert W-function, $x = W(y)$.)

It is clear that, in order for y to be large, either x or e^x (or both) must be large. Since e^x is much larger than x, we might expect to obtain a reasonable approximation by approximating

$$\mathrm{e}^x \approx y \quad \Rightarrow \quad x \approx \ln y. \tag{1.12}$$

Based on this dominant balance, we can set up an iteration

$$x = \ln y - \ln x, \tag{1.13}$$

in which we repeatedly substitute our estimate for x into the right-hand side to obtain a more accurate estimate. Starting from $x \approx \ln y$, we find

$$x = \ln y - \ln x \qquad = \ln y + O(\ln\ln y) \tag{1.14}$$

$$\Rightarrow \quad x = \ln y - \ln(\ln y + O(\ln\ln y)) \qquad = \ln y - \ln\ln y + O\left(\frac{\ln\ln y}{\ln y}\right) \tag{1.15}$$

$$\Rightarrow \quad x = \ln y - \ln\left(\ln y - \ln\ln y + O\left(\frac{\ln\ln y}{\ln y}\right)\right) = \ln y - \ln\ln y + \frac{\ln\ln y}{\ln y} + O\left(\frac{(\ln\ln y)^2}{(\ln y)^2}\right). \tag{1.16}$$

This is an example of an asymptotic expansion whose form would be very difficult to guess beforehand!

Example 1.4. We finish with an example of a differential equation, solved using the method of matched asymptotic expansions. Consider $\varepsilon y'' + y' - y = 0$ with boundary conditions $y(0) = 0$, $y(1) = 1$, for small $\varepsilon > 0$.

We first set $\varepsilon = 0$ and try to solve the resulting equation,

$$y' - y \approx 0 \quad \Rightarrow \quad y \approx C\mathrm{e}^x = \mathrm{e}^{x-1}, \tag{1.17}$$

where we have chosen C to satisfy the boundary condition $y(1) = 1$. As setting $\varepsilon = 0$ has reduced the order of the differential equation, this solution cannot be made to satisfy the other boundary condition too.

In order to satisfy the boundary condition at $x = 0$, we need to make the second derivative term dominant again, which we achieve by assuming that the solution varies on a much smaller length scale, comparable to ε, near the boundary $x = 0$. We represent this rapid variation using a change of variables $x = \varepsilon X$, noting that $\mathrm{d}/\mathrm{d}x = \varepsilon^{-1}\mathrm{d}/\mathrm{d}X$. Writing $y(x) = Y(X)$ for the solution in this region, we obtain

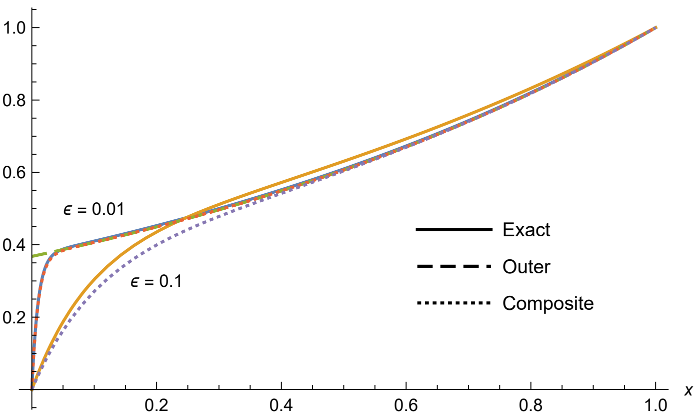
$$\frac{1}{\varepsilon}Y''(X) + \frac{1}{\varepsilon}Y'(X) - Y(X) = 0 \quad \Rightarrow \quad Y'' + Y' \approx 0 \quad \Rightarrow \quad Y = A + B\mathrm{e}^{-X} = A(1 - \mathrm{e}^{-X}), \tag{1.18}$$

where we have chosen B to satisfy the boundary condition $Y(0) = 0$. The second constant A is clearly somehow determined by the other condition $y(1) = 1$, but how?

The answer is to "match" the "outer" solution $y(x) = \mathrm{e}^{x-1}$ and the "inner" solution $Y(X) = A(1-\mathrm{e}^{-X})$ in an intermediate "overlap region". At leading order, we can simply observe that for small $x = \varepsilon X$, the outer solution tends to the value e^{-1}, while for large $X = x/\varepsilon$, the inner solution tends to the value $Y = A$. Thus, the only consistent choice is $A = \mathrm{e}^{-1}$. We can then form a "composite" solution by adding the inner and outer solutions and subtracting the common overlap

$$y(x) + Y(X) - \mathrm{e}^{-1} = \mathrm{e}^{x-1} + \mathrm{e}^{-1}\left(1 - \mathrm{e}^{-x/\varepsilon}\right) - \mathrm{e}^{-1} = \mathrm{e}^{-1}\left(\mathrm{e}^x - \mathrm{e}^{-x/\varepsilon}\right). \tag{1.19}$$

As seen in the figure below, the asymptotic result agrees well with the exact solution for small ε, and the agreement improves as $\varepsilon \to 0$. We could in principle calculate the higher-order corrections by expanding $y = y_0 + \varepsilon y_1 + \cdots$ and $Y = Y_0 + \varepsilon Y_1 + \cdots$ and then solving and matching order by order in ε. However, already at leading order we have obtained a good approximation as well as information about the spatial structure of the solution and insight into which terms form the dominant balance in each region.


1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
x
ϵ = 0.01
ϵ = 0.1
Exact
Outer
Composite

2 Definitions

The goal of the various methods taught in this module is to obtain approximate explicit expressions for quantities that it may be difficult or impossible to obtain exact explicit expressions for. Thus, we need to start by defining what it means for something to be a good approximation. (This section contains a lot of mathematical rigour, but don't worry – we'll ignore most of it later!)

2.1 Order notation

Definition 2.1. Let z_0 be a complex number. We say that

- $f(z) = O(\phi(z))$ ["$f(z)$ is big-oh of $\phi(z)$"] as $z \to z_0$ if

$$\begin{gathered} f/\phi \text{ is bounded as } z \to z_0, \\ \text{i.e. there exist } M > 0 \text{ and } \delta > 0 \text{ such that } |f| \leq M|\phi| \text{ for } 0 < |z - z_0| < \delta. \end{gathered} \tag{2.1}$$

- $\begin{cases} f(z) = o(\phi(z)) \text{ [“}f(z)\text{ is little-oh of }\phi(z)\text{”]} \\ \text{or } f(z) \ll \phi(z) \text{ [“}f(z)\text{ is much smaller than }\phi(z)\text{”]} \\ \text{or } \phi(z) \gg f(z) \text{ [“}\phi(z)\text{ is much larger than }f(z)\text{”]} \end{cases}$ as $z \to z_0$ if

$$\begin{gathered} f/\phi \to 0 \text{ as } z \to z_0, \\ \text{i.e. for all } M > 0 \text{ there exists } \delta > 0 \text{ such that } |f| \leq M|\phi| \text{ for } 0 < |z - z_0| < \delta. \end{gathered} \tag{2.2}$$

- $f(z) = \mathrm{ord}(\phi)$ ["$f(z)$ is order $\phi(z)$"] as $z \to z_0$ if

$$\begin{gathered} f/\phi \text{ is bounded and bounded away from zero as } z \to z_0, \\ \text{i.e. there exist } m > 0,\ M > 0 \text{ and } \delta > 0 \text{ such that } m|\phi| \leq |f| \leq M|\phi| \text{ for } 0 < |z - z_0| < \delta. \end{gathered} \tag{2.3}$$

Remark(s) 2.1. • Analogous definitions apply for $z_0 = \pm\infty$, replacing the conditions $0 < |z - z_0| < \delta$ with $z > 1/\delta$ or $z < -1/\delta$.

- The O, o and ord are like asymptotic analogues of $\leq$, $<$ and $=$ that ignore $O(1)$ prefactors.

Example 2.1. • If $A, B \neq 0$ and $\alpha < \beta$ then $Ax^\alpha \ll Bx^\beta$ as $x \to \infty$, and $Ax^\alpha \gg Bx^\beta$ as $x \to 0$.

- $2x^5 - 4x^4 + 2x^3$ is $\mathrm{ord}(x^5)$ as $x \to \infty$, $\mathrm{ord}(x^3)$ as $x \to 0$, and $\mathrm{ord}((x-1)^2)$ as $x \to 1$.
- Exponentials beat powers: If $A > 0$ and $\alpha > 0$ then $\exp(Ax^\alpha) \gg Bx^\beta$ as $x \to \infty$ and $\exp(Ax^{-\alpha}) \gg Bx^\beta$ as $x \to 0$, for all B and β.
- Powers beat logarithms: If $A \neq 0$ and $\alpha > 0$ then $x^\alpha \gg B(\ln x)^\beta$ as $x \to \infty$ and $x^{-\alpha} \gg B(\ln x)^\beta$ as $x \to 0$, for all B and β.
- If $f = o(\phi)$ then $f = O(\phi)$. If $f = \mathrm{ord}(\phi)$ then $f = O(\phi)$. If $f = o(\phi)$ then $f \neq \mathrm{ord}(\phi)$.
- But there exist $f = O(\phi)$ that are neither $o(\phi)$ nor $\mathrm{ord}(\phi)$.

Remark(s) 2.2. • For real z and z_0, the limit can be specified as being from one side only, using "$z \searrow z_0$" or "$z \to z_0^+$" for approaching from the right, and "$z \nearrow z_0$" or "$z \to z_0^-$" for approaching from the left. For complex z and z_0, the limit is sometimes specified as being in a certain sector, i.e. for a specific range of values of $\arg(z - z_0)$.

- For finite z_0, to save writing $(z - z_0)$ repeatedly we will almost always shift to expanding about zero, e.g. by defining $\varepsilon = z - z_0 \to 0$.
- We're too lazy to write out "as $z \to z_0$" every time, so will drop it when obvious.
- We will rarely do manipulations with ord in this module, as we simply want to bound errors from above using O and o. When bounding errors I will often accidentally say "order" when I mean "big-oh"...
- If ϕ has infinitely many zeroes as $z \to z_0$ (e.g. $\sin z$ as $z \to +\infty$), then the definitions using f/ϕ don't make sense, but those with $|\phi|$ multiplied out still do. (However, in practice ϕ usually does not do this.)

- Use of O, o and ord in equalities is an abuse of notation, as the symbols are treated differently on the left- and right-hand sides of an equation. For example when we write $(x + O(x^3))^3 = x^3 + o(x^4)$, we mean that for every possible $O(x^3)$ function on the left-hand side, the equality holds for some $o(x^4)$ function on the right-hand side. Thus, for example, $o(\phi) = O(\phi)$ but $O(\phi) \neq o(\phi)$.
- Various things you can do with O, o:
  - Add O and o together (keeping only the "largest"), e.g. $O(x^2) + o(x^2) + o(x)$ is $O(x^2)$ as $x \to \infty$ and $o(x)$ as $x \to 0$.
  - Multiply by a function or multiply together, e.g. $o(f)o(g) = o(fg)$ and $f\,O(g) = O(fg)$.
  - Integrate the arguments (if not changing sign), e.g. $\int_0^x O(f(t))\,\mathrm{d}t = O(\int_0^x f(t)\,\mathrm{d}t)$ provided $f(t) \geq 0$.
- Various things you cannot do with O, o:
  - Integrate the arguments if changing sign (due to risk of cancellation).
  - Differentiate (due to risk of oscillation).
- Manipulating ord is more difficult due to the risk of cancellation, e.g.

$$x = \mathrm{ord}(x) \quad \text{and} \quad -x = \mathrm{ord}(x) \quad \text{but} \quad x + (-x) = 0 \neq \mathrm{ord}(x), \tag{2.4}$$

  so we avoid it and settle for converting ord to O before manipulation.

2.2 Asymptotic expansions

Definition 2.2. We say that $f(z) \sim g(z)$ ["$f(z)$ is asymptotic to $g(z)$" or "$f(z)$ twiddles $g(z)$"] as $z \to z_0$ if

$$f/g \to 1 \text{ as } z \to z_0, \qquad \text{i.e. } f - g = o(g) \tag{2.5}$$

Remark(s) 2.3. • The $\sim$ is stricter than ord as it does distinguish $O(1)$ prefactors.

- If $f \sim \phi$ then $f = \mathrm{ord}(\phi)$.

Definition 2.3. We say that $f(z)$ is an asymptotic approximation of $g(z)$ to order $\delta(z)$ as $z \to z_0$ if

$$f - g = o(\delta). \tag{2.6}$$

Definition 2.4. Given a sequence $\delta_0(z) \gg \delta_1(z) \gg \delta_2(z) \gg \cdots$, we say that $f(z)$ has the (finite) asymptotic expansion

$$f(z) \sim \sum_{n=0}^{N} a_n \delta_n(z) \qquad \text{if} \qquad f(z) = \sum_{n=0}^{N} a_n \delta_n(z) + o(\delta_N(z)), \tag{2.7}$$

i.e. the error is smaller than the last term in the expansion. We say that $f(z)$ has the (infinite/full) asymptotic expansion

$$f(z) \sim \sum_{n=0}^{\infty} a_n \delta_n(z) \qquad \text{if} \qquad f(z) = \sum_{n=0}^{N} a_n \delta_n(z) + o(\delta_N(z)) \text{ for all } N, \tag{2.8}$$

i.e. for any partial sum the error is smaller than the last term kept.

Remark(s) 2.4. • We have now redefined the "$\sim$" symbol to have a stricter meaning than before (the difference must be small relative to the last term kept in the sum on the right-hand side, rather than just small relative to the whole left- or right-hand side).

- A common mistake is to treat the infinite asymptotic expansion as a function itself, or try to sum up the tail of the series, but this is not correct, as it can be divergent.
- Asymptotic expansions can be added and multiplied (provided the terms are appropriately reordered and/or deleted in the result).

- Like o and O, asymptotic expansions can be integrated with respect to the parameter (provided the δ_n do not change sign), but not differentiated (although we'll try anyway).

Definition 2.5. The sequence $\delta_n(z)$ is called the asymptotic scale for the expansion.

Remark(s) 2.5. • A Taylor series is an asymptotic expansion whose scale functions are natural powers.

- Often asymptotic expansions are of the form $f(x) \sim \mathrm{e}^{Ax^\alpha} x^\beta \sum_{n=0}^{\infty} a_n (x^\gamma)^n$, i.e. have terms that reduce by a factor x^γ each step, with possibly a pre-multiplying power and/or exponential.
- Sometimes there are logarithms inserted, e.g.

$$f(x) \sim (\cdots)\left[a_0 + a_{1,1} x \ln x + a_{1,0} x + a_{2,2} x^2 (\ln x)^2 + a_{2,1} x^2 (\ln x) + a_{2,0} x^2 + \cdots\right] \tag{2.9}$$

- For a given function f and asymptotic scale $\{\delta_n(z)\}$, the coefficients a_N can be determined by

$$a_N = \lim_{z \to z_0} \frac{f(z) - \sum_{n=0}^{N-1} a_n \delta_n(z)}{\delta_N(z)}, \tag{2.10}$$

  so the expansion is unique. But a given function f can have different expansions for different scales (see the tan example below).
- There is usually only one reasonable/simplest choice of asymptotic scale for a function, and it comes out naturally from the calculation of the expansion.
- Different functions can have the same asymptotic expansion, if they differ by a quantity smaller than all the scale functions:

$$\text{If} \quad f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n} \text{ as } x \nearrow \infty \quad \text{then} \quad f(x) + \mathrm{e}^{-x} \sim \sum_{n=0}^{\infty} a_n x^{-n} \text{ too.} \tag{2.11}$$

The simplest approximation task we might think of is converting an exact (but complicated) explicit expression into an approximate (but simple) explicit expression, like determining the Taylor expansion of a function, but more general.

Example 2.2. Well-known Taylor expansions for small x:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \cdots, \qquad \sin(x) = x - \frac{x^3}{6} + \cdots, \qquad \cos(x) = 1 - \frac{x^2}{2} + \cdots \tag{2.12}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \qquad (1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + \frac{p(p-1)(p-2)}{6} x^3 + \cdots, \tag{2.13}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots. \tag{2.14}$$

These can be combined to deduce other Taylor expansions, such as

$$\tan x = \frac{\sin x}{\cos x} = \left[x - \frac{x^3}{6} + O(x^5)\right]\left[1 - \frac{x^2}{2} + O(x^4)\right]^{-1} = x + \frac{x^3}{3} + O(x^5), \tag{2.15}$$

as well as non-Taylor expansions, such as

$$\tan x = \sin x\,(1 - (\sin x)^2)^{-1/2} = (\sin x)\left[1 + \frac{1}{2}(\sin x)^2 + O((\sin x)^4)\right]^{-1/2} = \tag{2.16}$$

$$= \sin x + \frac{(\sin x)^3}{2} + O((\sin x)^5), \tag{2.17}$$

$$\sqrt{\frac{\cos x}{x}} = \frac{1}{x^{1/2}}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right]^{1/2} = \tag{2.18}$$

$$= \frac{1}{x^{1/2}}\left[1 + \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right) - \frac{1}{8}\left(-\frac{x^2}{2} + O(x^4)\right)^2 + O(x^6)\right] \tag{2.19}$$

$$= \frac{1}{x^{1/2}}\left[1 - \frac{x^2}{4} - \frac{x^4}{96} + O(x^6)\right] = x^{-1/2} - \frac{x^{3/2}}{4} - \frac{x^{7/2}}{96} + O(x^{11/2}). \tag{2.20}$$

Remark(s) 2.6. When substituting expansions into other functions and re-expanding, it is important to pay attention to what point is being expanded about. If x is the leading-order term and $y \ll x$ represents all the corrections, then we need to expand a function of $(x+y)$ about x. For powers and logarithms, we can use

$$(x+y)^p = x^p(1+(y/x))^p = x^p(1+p(y/x)+p(p-1)(y/x)^2/2+\cdots), \tag{2.21}$$

$$\ln(x+y) = \ln x + \ln(1+(y/x)) = \ln x + (y/x) - (y/x)^2 + \cdots\,, \tag{2.22}$$

and in particular at leading order we obtain just x^p and $\ln(x)$.

However, for exponentials we have $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$, which does not reduce to e^x in general. (The same applies to $\cos(x+y)$ and $\sin(x+y)$, but with the trigonometric addition formulae.) We need to additionally have $y \ll 1$ in order to deduce

$$\mathrm{e}^{x+y} = \mathrm{e}^x(1+y+y^2/2+\cdots) \sim \mathrm{e}^x. \tag{2.23}$$

Example 2.3. As $x \searrow 0$, we have

$$(\sin x)^{-2} = \left[x\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)\right]^{-2} = \tag{2.24}$$

$$= \frac{1}{x^2}\left[1-2\left(-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+3\left(\frac{x^2}{6}+O(x^4)\right)^2+O(x^6)\right] = \tag{2.25}$$

$$= \frac{1}{x^2}\left[1+\frac{x^2}{3}+\frac{x^4}{15}+O(x^6)\right] = \frac{1}{x^2}-\frac{1}{3}+\frac{x^2}{15}+O(x^4) \tag{2.26}$$

$$\Rightarrow \quad \mathrm{e}^{(\sin x)^{-2}} = \exp\left[\frac{1}{x^2}-\frac{1}{3}+\frac{x^2}{15}+O(x^4)\right] = \exp\left[\frac{1}{x^2}-\frac{1}{3}\right]\left(1+\frac{x^2}{15}+O(x^4)\right). \tag{2.27}$$

3 Equations with a small/large parameter

In this section, we seek to obtain explicit approximations to the solutions of algebraic (i.e. non-differential) equations.

3.1 Polynomial equations

We start with polynomial equations, i.e. equations of the form

$$a_n(\varepsilon)x^n + \cdots + a_1(\varepsilon)x + a_0(\varepsilon) = 0, \tag{3.1}$$

where the goal is to obtain an expansion for the solution $x = x(\varepsilon)$ as $\varepsilon \searrow 0$.

Example 3.1. Let's solve the equation $\varepsilon x^2 + (\cos\varepsilon)x - \mathrm{e}^{\varepsilon} = 0$ as $\varepsilon \searrow 0$.

Let's forget for a moment that we can solve general quadratic equations exactly, and instead attempt to solve this equation using asymptotic methods. We note that $\cos\varepsilon$ and e^{ε} are easily expanded as $\varepsilon \searrow 0$ so won't pose a big problem.

What is a good leading-order approximation x_0 to the solution when ε is very small? Let's just set $\varepsilon = 0$:

$$x_0 - 1 = 0 \quad \Rightarrow \quad x_0 = 1. \tag{3.2}$$

How do we improve on this estimate? The most naive **power-series method** is to assume that the solution takes the form of a power series $x \sim \sum_{n=0}^{\infty} x_n\varepsilon^n$. In order to find the first four terms, we then substitute

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + O(\varepsilon^4) \tag{3.3}$$

into the equation and collect terms of the same order:

$$\begin{aligned}
0 &= \varepsilon\left[x_0^2 + \varepsilon(2x_0x_1) + \varepsilon^2(2x_0x_2 + x_1^2) + O(\varepsilon^3)\right] && (3.4)\\
&\quad + \left[1 - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4)\right]\left[x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + O(\varepsilon^4)\right] && (3.5)\\
&\quad - \left[1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{6}\varepsilon^3 + O(\varepsilon^4)\right] = && (3.6)\\
&= x_0 - 1 + \varepsilon\left[x_0^2 + x_1 - 1\right] + \varepsilon^2\left[2x_0x_1 - \tfrac{1}{2}x_0 + x_2 - \tfrac{1}{2}\right] && (3.7)\\
&\quad + \varepsilon^3\left[2x_0x_2 + x_1^2 - \tfrac{1}{2}x_1 + x_3 - \tfrac{1}{6}\right] + O(\varepsilon^4). && (3.8)
\end{aligned}$$

(By thinking ahead we knew we could stop the expansion in the first term at the $O(\varepsilon^3)$ remainder.) Now it's simply a question of solving the equation order by order to determine the coefficients x_n:

$$\begin{aligned}
O(\varepsilon^0):&\quad 0 = x_0 - 1 && \Rightarrow \quad x_0 = 1 && (3.9)\\
O(\varepsilon^1):&\quad 0 = x_0^2 + x_1 - 1 = x_1 && \Rightarrow \quad x_1 = 0 && (3.10)\\
O(\varepsilon^2):&\quad 0 = 2x_0x_1 - \tfrac{1}{2}x_0 + x_2 - \tfrac{1}{2} = x_2 - 1 && \Rightarrow \quad x_2 = 1 && (3.11)\\
O(\varepsilon^3):&\quad 0 = 2x_0x_2 + x_1^2 - \tfrac{1}{2}x_1 + x_3 - \tfrac{1}{6} = x_3 + \tfrac{11}{6} && \Rightarrow \quad x_3 = -\tfrac{11}{6}. && (3.12)
\end{aligned}$$

We conclude that one solution is

$$x = 1 + \varepsilon^2 - \frac{11}{6}\varepsilon^3 + O(\varepsilon^4), \tag{3.13}$$

and we note the expansion can be continued to arbitrary order, as the $O(\varepsilon^n)$ equation will be of the form $0 = \cdots + x_n + \cdots$, which always has a single solution x_n in terms of the previous coefficients. This is an example of a **regular** solution.

A more general method of finding the corrections to the leading-order result is the **iteration method**. The idea is to emulate the solution steps for the leading-order result while treating all other occurrences of x as constant, in order to rewrite the equation to the form $x = f(x)$, where f does not depend on x at leading order. We can then iteratively apply f to our approximation and obtain successively better approximations to the solution.

In our example, we can rewrite the equation as follows:

$$\varepsilon x^2 + (\cos\varepsilon)x - \mathrm{e}^{\varepsilon} = 0 \quad \Rightarrow \quad x = \frac{\mathrm{e}^{\varepsilon} - \varepsilon x^2}{\cos\varepsilon}. \tag{3.14}$$

We can then iteratively produce, starting from the leading-order scaling $x = O(1)$,

$$x = \frac{\mathrm{e}^{\varepsilon} - \varepsilon O(1)}{\cos\varepsilon} = 1 + O(\varepsilon) \tag{3.15}$$

$$\Rightarrow \quad x = \frac{\mathrm{e}^{\varepsilon} - \varepsilon(1 + O(\varepsilon))}{\cos\varepsilon} = \frac{1 + \varepsilon - \varepsilon + O(\varepsilon^2)}{1 + O(\varepsilon^2)} = 1 + O(\varepsilon^2) \tag{3.16}$$

$$\Rightarrow \quad x = \frac{\mathrm{e}^{\varepsilon} - \varepsilon(1 + O(\varepsilon^2))}{\cos\varepsilon} = \frac{1 + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)}{1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)} = 1 + \varepsilon^2 + O(\varepsilon^3) \quad \text{etc.} \tag{3.17}$$

Why did we gain one more power of ε of accuracy in each iteration? If x denotes the exact solution then the nth error E_n satisfies

$$x + E_{n+1} = f(x + E_n) \quad \Rightarrow \quad E_{n+1} = f(x + E_n) - x \approx f(x) + f'(x)E_n - x = f'(x)E_n, \tag{3.18}$$

and since we ensured that f only depends on x in the small corrections, in this case $O(\varepsilon)$ terms, we have $f'(x) = O(\varepsilon)$.

We note that the number of solutions to the equation changed when we set $\varepsilon = 0$, because the highest-degree term in the equation is multiplied by a quantity that vanishes in the limit $\varepsilon \searrow 0$. Where is the second solution? Clearly, we need the quadratic term to become important, so we might expect x to be larger than $O(1)$ and diverge as $\varepsilon \searrow 0$. This is called a **singular** solution.

Let's use the **method of dominant balance** to look for a rescaling of x that yields the singular solution. We define

$$x = \delta(\varepsilon)X, \tag{3.19}$$

where δ is meant to capture the approximate size of x (e.g. a power of ε) while $X = \mathrm{ord}(1)$ captures its specific value. This yields the equation

$$\begin{array}{ccccccc} \varepsilon\delta^2X^2 & + & (\cos\varepsilon)\delta X & - & \mathrm{e}^{\varepsilon} & = 0. \\ \mathrm{ord}(\varepsilon\delta^2) & & \mathrm{ord}(\delta) & & \mathrm{ord}(1) & \end{array} \tag{3.20}$$

If δ is such that only one of these three terms is the largest while the other two are smaller, then we do not obtain a valid solution X as the only leading-order solution is $X = 0 \neq \mathrm{ord}(1)$. Hence, we need at least two of these terms to be the largest at the same time, while the remaining term may be smaller.

We've already seen that balancing the second and third terms yields the scaling for the regular solution,

$$\mathrm{ord}(\delta) = \mathrm{ord}(1) \quad \Rightarrow \quad \delta = 1. \tag{3.21}$$

(Strictly speaking, we could have δ be any multiple of 1, and have smaller corrections, but we take the simplest possible expression.) The first term, $\mathrm{ord}(\varepsilon)$, is then smaller than the two terms we balanced, which are $\mathrm{ord}(1)$, so we have a consistent dominant balance. This recovers the first solution with $x = X$.

Let's try balancing the first and the third term,

$$\mathrm{ord}(\varepsilon\delta^2) = \mathrm{ord}(1) \quad \Rightarrow \quad \delta = \varepsilon^{-1/2}. \tag{3.22}$$

This yields the neglected term as $\mathrm{ord}(\varepsilon^{-1/2})$ which is larger than the supposedly dominant terms which are $\mathrm{ord}(1)$, so this is not a consistent balance and does not yield a solution.

Finally, we balance the first and the second term,

$$\mathrm{ord}(\varepsilon\delta^2) = \mathrm{ord}(\delta) \quad \Rightarrow \quad \delta = \varepsilon^{-1}. \tag{3.23}$$

This yields the two balanced terms as $\mathrm{ord}(\varepsilon^{-1})$, while the neglected third term is $\mathrm{ord}(1)$, so we again have a consistent dominant balance. We obtain the rescaled equation

$$\frac{1}{\varepsilon}X^2 + \frac{\cos\varepsilon}{\varepsilon}X - \mathrm{e}^{\varepsilon} = 0 \quad \Rightarrow \quad X^2 + (\cos\varepsilon)X - \varepsilon\mathrm{e}^{\varepsilon} = 0. \tag{3.24}$$

Taking the limit $\varepsilon \searrow 0$ now yields the dominant balance

$$X^2 + X \approx 0 \quad \Rightarrow \quad X \approx -1, \tag{3.25}$$

(The discarded solution $X \approx 0$ represents the regular solution with $x = \text{ord}(1)$ and $X = \text{ord}(\varepsilon)$.) We can proceed to find the corrections to this result using $X \sim \sum_{n=0}^{\infty} X_n \varepsilon^n$ or using an iteration with

$$X = -\cos\varepsilon + \varepsilon \frac{e^{\varepsilon}}{X}, \tag{3.26}$$

which yields

$$X = -1 + O(\varepsilon) \tag{3.27}$$

$$\Rightarrow \quad X = -1 + O(\varepsilon^2) + \varepsilon \frac{1 + O(\varepsilon)}{-1 + O(\varepsilon)} = -1 - \varepsilon + O(\varepsilon^2) \tag{3.28}$$

$$\Rightarrow \quad X = -1 + \tfrac{1}{2}\varepsilon^2 + \varepsilon \frac{1 + \varepsilon + O(\varepsilon^2)}{-1 - \varepsilon + O(\varepsilon^2)} = -1 + \tfrac{1}{2}\varepsilon^2 - \varepsilon(1 + O(\varepsilon^2)) = -1 - \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3), \tag{3.29}$$

and hence

$$x = \varepsilon^{-1} X = -\varepsilon^{-1} - 1 + \tfrac{1}{2}\varepsilon + O(\varepsilon^2). \tag{3.30}$$

We can also find the corrections to a leading-order result using a version of the method of dominant balance. Let's consider the regular solution $x \approx 1$ again, and write

$$x = 1 + \delta_1 x_1 + o(\delta_1), \quad \text{where} \quad \delta_1 \ll 1, \quad x_1 = \text{ord}(1). \tag{3.31}$$

We substitute this into the equation to find

$$\varepsilon(1 + \delta_1 x_1 + o(\delta_1))^2 + (\cos\varepsilon)(1 + \delta_1 x_1 + o(\delta_1)) - e^{\varepsilon} = 0. \tag{3.32}$$

We now drop as many terms as possible, in particular anything smaller than $o(\delta_1)$ from the middle term, and expand in ε and drop higher-order terms,

$$0 = \varepsilon + (1 - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4))(1 + \delta_1 x_1 + o(\delta_1)) - (1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)) \tag{3.33}$$

$$= -\tfrac{1}{2}\varepsilon^2 + \delta_1 x_1 + o(\delta_1) - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3) = \delta_1 x_1 - \varepsilon^2 + o(\delta_1) + O(\varepsilon^3). \tag{3.34}$$

In this case, the $\text{ord}(\varepsilon)$ terms surprisingly cancelled, so we had to make sure to expand to $\text{ord}(\varepsilon^2)$. We conclude that the only (simplest) consistent choice is

$$\delta_1 = \varepsilon^2, \qquad x_1 = 1. \tag{3.35}$$

We could then continue to find the next correction with $x = 1 + \varepsilon^2 + \delta_2 x_2 + o(\delta_2)$ and $\delta_2 \ll \varepsilon^2$, $x_2 = \text{ord}(1)$...

Remark(s) 3.1. • The difficult bit is often finding each leading-order balance, which can involve rescaling and nonlinearity (and the leading-order problem might require numerical solution using a computer). After that, the corrections are usually easier to find, as their equations are linear, or they just involve expanding the iterated function. This principle is often true for more general asymptotics problems too.

- There are two competing definitions for **regular** and **singular**. Some people say that a solution is regular if it has a power-series expansion, and singular if not. Some people say that a solution is regular if it converges and singular if not. Thus, everyone agrees that a power-series expansion is regular, and that a divergent expansion is singular, but a convergent non-power-series expansion (e.g. $x \sim \varepsilon^{1/2}$) is disputed. An equation, or a more general asymptotic problem, is classified as regular if all its solutions/results are regular, and singular otherwise. Singular problems are the most exciting, as the basic characteristics of the problem (e.g. number of solutions, convergence of an integral, order of a differential equation) are different at small $\varepsilon > 0$ compared with at exactly $\varepsilon = 0$.
- Consider a general nth degree polynomial equation where the coefficients have been rescaled so that all the coefficients converge as $\varepsilon \to 0$ and at least one is $\text{ord}(1)$.

  - If some of the coefficients don't have power-series expansions as $\varepsilon \to 0$, but contain more exotic terms (logarithms, fractional powers, or exponentials), then we can't expect the solution to have a power-series expansion either.
  - If any of the leading-order solutions is a multiple root (e.g. $x^2 - \varepsilon = 0$), then there is a risk that a fractional power of ε appears in the expansion
  - If the highest-degree term is multiplied by a small factor (e.g. $\varepsilon x - 1 = 0$), then at least one solution is divergent so the problem is singular.

  If none of these exceptions hold, then we expect all solutions to have power-series expansions.

- In the method of dominant balance, we've split up the unknown x as a scale $\delta(\varepsilon)$ times an ord(1) quantity X. In practice, it's often fine to just work with directly with x in the scaling argument if you're lazy, but it is strongly recommended to rescale to $X = \operatorname{ord}(1)$ before trying to solve the equation in detail.
- Sometimes, more than two terms can form part of the dominant balance. (You will find that when trying to balance two terms, another term becomes the same order at the same time.)
- When there are many terms to balance, it is better to work through the possible balances systematically rather than trying all pairs, as shown in the next example.

Example 3.2. Let's find the correct scalings for the solutions of the equation

$$\tan(\varepsilon^2)x^4 + \varepsilon x^3 + (\cos\varepsilon)x^2 + \varepsilon^2 x - \varepsilon = 0 \tag{3.36}$$

as $\varepsilon \searrow 0$.

This time, let's work with x directly and estimate the terms as

$$\operatorname{ord}(\varepsilon^2 x^4), \quad \operatorname{ord}(\varepsilon x^3), \quad \operatorname{ord}(x^2), \quad \operatorname{ord}(\varepsilon^2 x), \quad \operatorname{ord}(\varepsilon). \tag{3.37}$$

For very very small x, the constant term $\operatorname{ord}(\varepsilon)$ alone is dominant. As x increases, the other terms increase in magnitude until one of them becomes equal to $\operatorname{ord}(\varepsilon)$. Which one? Let's try the lowest degree, x^1, first. If $\operatorname{ord}(\varepsilon^2 x) = \operatorname{ord}(\varepsilon)$ then $x = \operatorname{ord}(\varepsilon^{-1})$. This is not consistent since the x^2 term becomes $\operatorname{ord}(x^2) = \operatorname{ord}(\varepsilon^{-2}) \gg \operatorname{ord}(\varepsilon)$. Since the x^2 term posed a problem by being too large, we try to use it next. If $\operatorname{ord}(x^2) = \operatorname{ord}(\varepsilon)$ then $x = \operatorname{ord}(\varepsilon^{1/2})$. This is a consistent balance since the other terms become $\operatorname{ord}(\varepsilon^4)$, $\operatorname{ord}(\varepsilon^{5/2})$ and $\operatorname{ord}(\varepsilon^{5/2})$ which are all smaller. We conclude that the x^2 and ε terms have a dominant balance at $x = \operatorname{ord}(\varepsilon^{1/2})$.

As x increases past $\operatorname{ord}(\varepsilon^{1/2})$, the x^2 term becomes dominant alone. The lower-degree terms that are negligible will not become dominant again, as they grow slower with x than the x^2 term. Which is the next term to become part of a dominant balance together with x^2? Let's try the next term up, $\operatorname{ord}(\varepsilon x^3) = \operatorname{ord}(x^2)$. This yields $x = \operatorname{ord}(\varepsilon^{-1})$, so these two terms become $\operatorname{ord}(\varepsilon^{-2})$. We find that the last remaining term, $\operatorname{ord}(\varepsilon^2 x^4)$ is also $\operatorname{ord}(\varepsilon^{-2})$, so this is a balance between three terms.

As x increases past $\operatorname{ord}(\varepsilon^{-1})$, the highest-degree term becomes dominant alone, and all other terms become small compared with it.

The dominant balances can be represented graphically (figure 3.1).

3.2 Transcendental equations

Moving on to non-polynomial equations, the principle for how to find a dominant balance between terms is the same, but one difficulty is how to obtain an initial guess for the leading-order value of the root. For this, we'll need to use some intuition, or maybe draw a sketch.

Example 3.3. We first consider a simple but important example, namely inverting a function using its expansion near a known point. For example, let's find an expansion for the inverse $x = f^{-1}(y)$ of the function $f(x) = x^3 \mathrm{e}^x$ near $x = 0$.

Near $x = 0$, we simply expand the function

$$f(x) = x^2 \mathrm{e}^x = x^3 + x^4 + \frac{x^5}{2} + O(x^6), \tag{3.38}$$

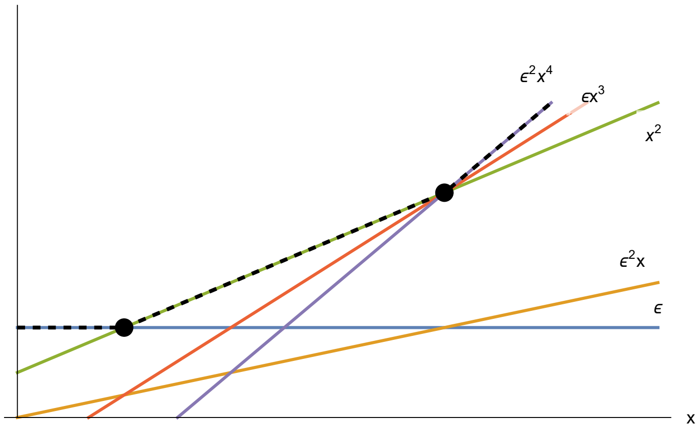


Figure 3.1: Schematic log-log plot of the magnitude of the terms in the equation (3.36). The black circles indicate a dominant balance, while dotted lines indicate a lone dominant term.

and use the leading-order term to obtain the leading-order approximation,

$$y \approx x^3 \quad \Rightarrow \quad x \approx \pm y^{1/3}. \tag{3.39}$$

The corrections are easily obtained using the iteration method, by moving all remaining terms to the other side:

$$x^3 = y - x^4 - \frac{x^5}{2} + O(x^6) \tag{3.40}$$

$$\Rightarrow \quad x = \left(y - x^4 - \frac{x^5}{2} + O(x^6)\right)^{1/3} = y^{1/3}\left[1 - \frac{x^4}{y} - \frac{x^5}{2y} + O\left(\frac{x^6}{7}\right)\right]^{1/3}. \tag{3.41}$$

Starting from $x \approx y^{1/3}$ we then obtain

$$x = y^{1/3}\left[1 + O(y^{1/3})\right] \tag{3.42}$$

$$\Rightarrow \quad x = y^{1/3}\left[1 - y^{1/3} + O(y^{2/3})\right]^{1/3} = y^{1/3}\left[1 - \frac{1}{3}y^{1/3} + O(y^{2/3})\right] \tag{3.43}$$

$$\Rightarrow \quad x = y^{1/3}\left[1 - y^{1/3}\left(1 - \frac{4}{3}y^{1/3}\right) - \frac{1}{2}y^{2/3} + O(y)\right]^{1/3} \tag{3.44}$$

$$= y^{1/3}\left[1 + \frac{1}{3}\left(-y^{1/3} + \frac{5}{6}y^{2/3}\right) - \frac{1}{9}\left(-y^{1/3}\right)^2 + O(y)\right] \tag{3.45}$$

$$= y^{1/3}\left[1 - \frac{1}{3}y^{1/3} + \frac{1}{6}y^{2/3} + O(y)\right] = y^{1/3} - \frac{1}{3}y^{2/3} + \frac{1}{6}y + O(y^{4/3}). \tag{3.46}$$

Example 3.4. Let's find the positive solutions to the equation $2\cosh x + x^{-1} = y$ as $y \nearrow \infty$.

Sketching the function $\mathrm{e}^x + \mathrm{e}^{-x} + x^{-1}$ reveals that there are three solutions for large positive y: a small positive one and two large positive and negative ones.

If x is small, then x^{-1} dominates so we expect $x \approx y^{-1}$. We use the iteration

$$x = \frac{1}{y - 2\cosh x} \tag{3.47}$$

to obtain

$$x = \frac{1}{y + O(1)} = \frac{1}{y}\left(1 + O(y^{-1})\right) = y^{-1} + O(y^{-2}) \tag{3.48}$$

$$\Rightarrow \quad x = \frac{1}{y - 2\cosh(y^{-1} + O(y^{-2}))} = \frac{1}{y - 2 - y^{-2} + O(y^{-3})} = \tag{3.49}$$

$$= \frac{1}{y(1 - 2y^{-1} + y^{-3} + O(y^{-4}))} = \frac{1}{y}\left(1 + 2y^{-1} + 4y^{-2} + 7y^{-3} + O(y^{-4})\right) \tag{3.50}$$

$$= y^{-1} + 2y^{-2} + 4y^{-3} + 7y^{-4} + O(y^{-5}). \tag{3.51}$$

If x is large and positive, then e^x dominates so we expect $x \approx \ln y$. We use the iteration

$$x = \ln(y - x^{-1} - \mathrm{e}^{-x}) \tag{3.52}$$

to obtain

$$x = \ln\left[y - O((\ln y)^{-1})\right] = \ln y + \ln\left[1 - O(y^{-1}(\ln y)^{-1}\right] = \ln y + O(y^{-1}(\ln y)^{-1}). \tag{3.53}$$

$$\Rightarrow \quad x = \ln\left[y - \frac{1}{\ln y + O(y^{-1}(\ln y)^{-1})} - y^{-1} + O(y^{-2}(\ln y)^{-2})\right] = \tag{3.54}$$

$$= \ln\left[y - (\ln y)^{-1} - y^{-1} + O(y^{-1}(\ln y)^{-2})\right] \tag{3.55}$$

$$= \ln y - y^{-1}(\ln y)^{-1} - y^{-2} + O(y^{-2}(\ln y)^{-2}). \tag{3.56}$$

Example 3.5. Let's find approximations to the real solutions of $(\sin x)^2 = \varepsilon x$, as $\varepsilon \searrow 0$.

Sketching the functions $(\sin x)^2$ and εx reveals that the solutions are near the solutions $\sin x = 0$, i.e. $n\pi$ for integer $n \geq 0$. Additionally, there are two such solutions for each n. We write

$$x = n\pi + x_1, \qquad x_1 \ll 1 \tag{3.57}$$

and expand

$$(\sin x)^2 = (\sin x_1)^2 = \left(x_1 - \frac{x_1^3}{6} + O(x_1^5)\right)^2 = x_1^2 - \frac{x_1^4}{3} + O(x_1^6), \tag{3.58}$$

resulting in the equation

$$\varepsilon n\pi + \varepsilon x_1 = x_1^2 - \frac{x_1^4}{3} + O(x_1^6). \tag{3.59}$$

Let's first consider the case $n > 0$. As $\varepsilon \searrow 0$ with $x_1 \ll 1$, we find that the main balance must be

$$\varepsilon n\pi \sim x_1^2 \quad \Rightarrow \quad x_1 \approx \pm\sqrt{\varepsilon n\pi}. \tag{3.60}$$

We can improve on this estimate using the iteration

$$x_1 = \pm\sqrt{\varepsilon n\pi + \varepsilon x_1 + \frac{x_1^4}{3} + O(x_1^6)}. \tag{3.61}$$

For the case $n = 0$, one root is exactly $x = 0$ and the other satisfies

$$\varepsilon = x - \frac{x^3}{3} + O(x^5) \quad \Rightarrow \quad x \approx \varepsilon, \quad x = \varepsilon + \frac{x^3}{3} + O(x^5) = \varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5). \tag{3.62}$$

Part II
Integrals

4 Term-by-term integration

In simple cases, an integral might be calculated by asymptotically expanding the integrand and then integrating the result term by term. Note that the integration can be either with respect to the small/large parameter of the integrand expansion or with respect to another quantity.

Remark(s) 4.1. When integrating with respect to the parameter of the expansion, we use the fact that, provided $f(x)$ does not change sign,

$$\int_{x_0}^{x} O\left(f(t)\right)\,\mathrm{d}t = O\left(\int_{x_0}^{x} f(t)\,\mathrm{d}t\right), \tag{4.1}$$

where the O are valid for the limit $t \to x_0$ and $x \to x_0$. The same relation holds for o, and also for infinite x_0.

Make sure to check that the limit of the integral is the point that you are expanding about, and that the integrand cannot be expanded further first (for example, when expanding $\int_0^x t^{1/2}\mathrm{e}^{-t}\,\mathrm{d}t$ as $x \searrow 0$, we can expand the exponential about $t = 0$).

Example 4.1. Let's obtain an expansion for $I(x) = \displaystyle\int_x^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t$ as $x \searrow 0$.

We first note that there is no singularity at $t = 0$, as the integrand tends to 1 there. We can then shift to an integral from 0 to x by "splitting" the integral into two parts,

$$I(x) = \int_0^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t - \int_0^x \frac{\ln(1+t)}{t}\,\mathrm{d}t = A - J(x). \tag{4.2}$$

The first integral looks very difficult to evaluate, but it's equal to some constant A independent of x, so from an asymptotic standpoint we don't need to do anything further with it. (According to Mathematica, the result is $A = \pi^2/12$.) In the second integral, we can now use the small-t expansion of the integrand,

$$J(x) = \int_0^x \left(\sum_{n=1}^{N}(-1)^{n+1}\frac{t^{n-1}}{n}\right) + o(t^{N-1})\,\mathrm{d}t = \sum_{n=1}^{N}(-1)^{n+1}\frac{x^n}{n^2} + o(x^N). \tag{4.3}$$

We conclude that

$$I(x) \sim A + \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^2} \qquad \text{as} \qquad x \searrow 0. \tag{4.4}$$

Example 4.2. Let's obtain an expansion for $I(x) = \displaystyle\int_x^1 \frac{\ln(1+t)}{t^3}\,\mathrm{d}t$ as $x \searrow 0$.

This integrand is singular at $t = 0$, specifically

$$\frac{\ln(1+t)}{t^3} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{n-3}}{n} = \frac{1}{t^2} - \frac{1}{2t} + O(1) \qquad \text{as} \qquad t \searrow 0, \tag{4.5}$$

so we need to subtract the (non-integrably) singular terms before changing the limits of the integral:

$$I(x) = \int_x^1 \frac{1}{t^2} - \frac{1}{2t}\,\mathrm{d}t + \int_x^1 \frac{\ln(1+t)}{t^3} - \frac{1}{t^2} + \frac{1}{2t}\,\mathrm{d}t = \tag{4.6}$$

$$= \frac{1}{x} - 1 + \frac{1}{2}\ln x + \left(\int_0^1 - \int_0^x\right)\frac{\ln(1+t)}{t^3} - \frac{1}{t^2} + \frac{1}{2t}\,\mathrm{d}t \tag{4.7}$$

$$\sim \frac{1}{x} + \frac{1}{2}\ln x + A + \sum_{n=3}^{\infty}(-1)^n\frac{x^{n-2}}{n(n-2)}, \qquad A = -1 + \int_0^1 \frac{\ln(1+t)}{t^3} - \frac{1}{t^2} + \frac{1}{2t}\,\mathrm{d}t. \tag{4.8}$$

(According to Mathematica, $A = -3/4$.)

Remark(s) 4.2. Sometimes we end up needing to integrate a term that has not just a power of x but also a multiplying exponential and/or a logarithm. In these cases, the integral might not have a simple form, but we can instead obtain an asymptotic expansion of it, using **integration by parts**:

Example 4.3. Let's find an expansion for the exponential integral $E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t$, as $x \nearrow \infty$.

We seek to integrate by parts, and since x and t are large we want to have higher and higher powers of t in the denominator. So let's repeatedly integrate the exponential and differentiate the power:

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = \left[\mathrm{e}^{-t}\left(-\frac{1}{t}\right)\right]_x^\infty - \int_x^\infty \frac{\mathrm{e}^{-t}}{t^2}\,\mathrm{d}t = \left[\mathrm{e}^{-t}\left(-\frac{1}{t}+\frac{1}{t^2}\right)\right] + 2\int_x^\infty \frac{\mathrm{e}^{-t}}{t^3}\,\mathrm{d}t \tag{4.9}$$

$$= \left[\mathrm{e}^{-t}\left(-\frac{1}{t}+\frac{1}{t^2}-\frac{2}{t^3}\right)\right] - 6\int_x^\infty \frac{\mathrm{e}^{-t}}{t^4}\,\mathrm{d}t \quad = \cdots \tag{4.10}$$

$$= \left[\mathrm{e}^{-t}\left(-\frac{1}{t}+\frac{1}{t^2}-\frac{2}{t^3}+\frac{6}{t^4}-\cdots+\frac{(-1)^N(N-1)!}{t^N}\right)\right]_x^\infty + (-1)^N N!\int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t \tag{4.11}$$

$$= \mathrm{e}^{-x}\left(\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\cdots+\frac{(-1)^{N-1}(N-1)!}{x^N}\right) + R_N(x), \tag{4.12}$$

where we can bound the remainder using

$$|R_N(x)| = N!\int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t \le N!\int_x^\infty \frac{\mathrm{e}^{-t}}{x^{N+1}}\,\mathrm{d}t = \frac{N!\,\mathrm{e}^{-x}}{x^{N+1}}. \tag{4.13}$$

Since this is smaller than the last kept term (in fact it is equal to the next term!), we conclude that

$$E_1(x) \sim \mathrm{e}^{-x}\sum_{n=1}^\infty \frac{(-1)^{n-1}(n-1)!}{x^n}. \tag{4.14}$$

Note here that when estimating the remainder, we replaced the power with a constant and integrated the exponential. It's also possible to replace the exponential with a constant and integrate the power. This makes your error bound be $O(\mathrm{e}^{-x}/x^N)$, i.e. the same as the last term kept, which isn't a problem since this means that the remainder from truncating one term earlier, after the e^{-x}/x^{N-1} term, is of the correct size $O(\mathrm{e}^{-x}/x^N)$.

Remark(s) 4.3. When integrating an asymptotic expansion with respect to another quantity (i.e. not the small/large parameter of the expansion), we need to ensure that the expansion is uniformly valid throughout the range of integration. This just means that in the formula

$$f(t;\varepsilon) = \sum_{n=0}^N a_n(t)\delta_n(\varepsilon) + R_N(t;\varepsilon), \qquad \frac{R_N(t;\varepsilon)}{\delta_N(\varepsilon)} \to 0 \tag{4.15}$$

the last limit is uniform in t, i.e. for every $M > 0$ there exists $\varepsilon_* > 0$ **independent of** t such that $|R_N/\delta_N| < M$ for all $|\varepsilon| < \varepsilon_*$. Assuming all functions are integrable with respect to t, we then have

$$\int_{t_1}^{t_2} f(t;\varepsilon)\,\mathrm{d}t = \sum_{n=0}^N \left(\int_{t_1}^{t_2} a_n(t)\,\mathrm{d}t\right)\delta_n(\varepsilon) + o(\delta_N(\varepsilon)). \tag{4.16}$$

Example 4.4. Consider the integral $I(\varepsilon) = \int_0^1 \frac{\sin(\varepsilon t)}{t}\,\mathrm{d}t$ as $\varepsilon \searrow 0$.

The integrand has the expansion

$$\frac{\sin(\varepsilon t)}{t} = \sum_{n=0}^N (-1)^n \frac{\varepsilon^{2n+1}t^{2n}}{(2n+1)!} + o(\varepsilon^{2N+1}), \tag{4.17}$$

which is uniformly valid for $0 \le t \le 1$, so we can integrate to obtain

$$I(\varepsilon) = \sum_{n=0}^N (-1)^n \frac{\varepsilon^{2n+1}}{(2n+1)(2n+1)!} + o(\varepsilon^{2N+1}). \tag{4.18}$$

5 Splitting the range of integration

Sometimes the integrand cannot be expanded in way that is valid throughout the entire range of integration, but does have different expansions that are valid in different regions. We can then split the range of integration into separate parts, which can be calculated using different expansions of the integrand, and the final result is the sum of the separate results. We first try to solve an example without splitting:

Example 5.1. Consider $I(\varepsilon) = \int_0^1 (\varepsilon + x)^{-1/2}\,\mathrm{d}x$.

If we naively expand the integrand for small ε then we obtain

$$I \approx \int_0^1 x^{-1/2}\left[1 - \frac{\varepsilon}{2x} + \cdots\right]\mathrm{d}x = \int_0^1 x^{-1/2}\,\mathrm{d}x - \frac{\varepsilon}{2}\int_0^1 x^{-3/2}\,\mathrm{d}x + \cdots . \tag{5.1}$$

The first term yields $I \approx 2$, but the second term is divergent! The problem is that the expansion is not uniformly valid, as it requires ε/x to be small and hence breaks down for $x = \mathrm{ord}(\varepsilon)$. We can estimate the **local** contribution to the integral from $x = \mathrm{ord}(\varepsilon)$ as being the integrand scale, $\mathrm{ord}(\varepsilon^{-1/2})$, times the interval length, $\mathrm{ord}(\varepsilon)$, which yields $\mathrm{ord}(\varepsilon^{1/2})$ – larger than the next term in the naive expansion. For comparison, the **global** contribution is an ord(1) integrand over a ord(1) interval, resulting in an ord(1) contribution (namely $I \approx 2$).

We can calculate the first correction (local contribution) by subtracting the leading-order result and rescaling x to be small,

$$I - 2 = \int_0^1 (\varepsilon + x)^{-1/2} - x^{-1/2}\,\mathrm{d}x \overset{x=\varepsilon X}{=} \varepsilon^{1/2}\int_0^{1/\varepsilon} (1+X)^{-1/2} - X^{-1/2}\,\mathrm{d}X \approx \tag{5.2}$$

$$\approx \varepsilon^{1/2}\int_0^{\infty} (1+X)^{-1/2} - X^{-1/2}\,\mathrm{d}X = 2\varepsilon^{1/2}\left[(1+X)^{1/2} - X^{1/2}\right]_0^{\infty} = -2\varepsilon^{1/2}, \tag{5.3}$$

where we have made use of the large-X expansion

$$(1+X)^{1/2} - X^{1/2} = X^{1/2}\left[1 + O(X^{-1})\right] - X^{1/2} = O(X^{-1/2}) \to 0. \tag{5.4}$$

Note here how the subtracted global term helped "regularise" the local integral which would otherwise diverge as $X \nearrow \infty$. Similarly, in other cases we could find that the global integral is divergent at e.g. $x = 0$ which indicates that the local contribution is dominant, and subtracting the divergent term(s) which are associated with the dominant local contribution yields the correction from the global contribution.

We conclude that $I \sim 2 - 2\varepsilon^{1/2}$, but it's difficult to continue. Instead, we can use the more systematic method of **splitting the range of integration**, but we will show this method using a different example.

Example 5.2. Consider $I(\varepsilon, \alpha) = \int_0^\infty \frac{\mathrm{d}x}{(1+x)(\varepsilon+x)^\alpha}$ for fixed $\alpha > 0$ as $\varepsilon \searrow 0$.

We observe two key scales for x in the integrand, and we estimate the order of magnitude of the contribution to the integral from each:

$$x = \mathrm{ord}(1): \quad \text{Integrand: } \sim \frac{1}{(1+x)x^\alpha} = \mathrm{ord}(1), \quad \text{Contribution to integral: } \mathrm{ord}(1), \tag{5.5}$$

$$x = \mathrm{ord}(\varepsilon): \quad \text{Integrand: } \sim \frac{1}{(\varepsilon+x)^\alpha} = \mathrm{ord}(\varepsilon^{-\alpha}), \quad \text{Contribution to integral: } \mathrm{ord}(\varepsilon^{1-\alpha}). \tag{5.6}$$

Thus, if $0 < \alpha < 1$ then we expect that the **global** contribution from $x = \mathrm{ord}(1)$ dominates and the integral is ord(1), while for $\alpha > 1$ the **local** contribution from $x = \mathrm{ord}(\varepsilon)$ dominates and the integral is $\mathrm{ord}(\varepsilon^{-(\alpha-1)}) \gg 1$. We can in fact calculate the leading-order result in each case directly, by approximating

$$0 < \alpha < 1, \quad x = \mathrm{ord}(1): \quad I \sim \int_0^\infty \frac{1}{(1+x)x^\alpha} = \frac{\pi}{\sin(\pi\alpha)}, \tag{5.7}$$

$$1 < \alpha, \quad x = \mathrm{ord}(\varepsilon): \quad I \overset{x=\varepsilon X}{\sim} \int_0^\infty \frac{\varepsilon\,\mathrm{d}X}{\varepsilon^\alpha(1+X)^\alpha} = \frac{\varepsilon^{-(\alpha-1)}}{\alpha-1}. \tag{5.8}$$

(The first integral is calculated using complex methods, by considering a keyhole contour integral and picking up a residue at -1, which is outside the scope of this module.) As expected, the global integral would be divergent at 0 for $\alpha \geq 1$ and the local integral would be divergent at ∞ for $\alpha \leq 1$.

What are the corrections to the leading-order results? In order to calculate these systematically, we split the integration range at an intermediate scale δ with $\varepsilon \ll \delta \ll 1$, i.e. $\delta = C\varepsilon^\beta$ with $0 < \beta < 1$. We then obtain

$$I = I_1 + I_2, \quad I_1 = \int_0^\delta \frac{\mathrm{d}x}{(1+x)(\varepsilon+x)^\alpha}, \quad I_2 = \int_\delta^\infty \frac{\mathrm{d}x}{(1+x)(\varepsilon+x)^\alpha}. \tag{5.9}$$

Let's consider the specific value $\alpha = 1/2$.

In the first integral, I_1, which represents the local contribution, we change to the local scale $x = \varepsilon X$. Note that although we think of the local contribution as coming from $x = \mathrm{ord}(\varepsilon)$ and hence $X = \mathrm{ord}(1)$, the integration range in fact stretches from $X = 0$ to $X = \delta/\varepsilon \gg 1$. Nevertheless, we only need the fact that $\varepsilon X = O(\varepsilon\delta) \ll 1$ in order to expand the $1/(1+\varepsilon X)$ in the integrand,

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon^{1/2}\,\mathrm{d}X}{(1+\varepsilon X)(1+X)^{1/2}} = \varepsilon^{1/2}\int_0^{\delta/\varepsilon} \frac{1}{(1+X)^{1/2}}\left[1 - \varepsilon X + O(\varepsilon^2X^2)\right]\,\mathrm{d}X. \tag{5.10}$$

$$= \left[\varepsilon^{1/2}\,2(1+X)^{1/2} + \varepsilon^{3/2}\left(-\frac{2}{3}(1+X)^{3/2} + 2(1+X)^{1/2}\right) + O(\varepsilon^{5/2}X^{5/2})\right]_0^{\delta/\varepsilon}. \tag{5.11}$$

In the second integral, I_2, which represents the global contribution, x ranges from $x = \delta \ll 1$ to ∞, but we only need the fact that $\varepsilon/x = O(\varepsilon/\delta) \ll 1$ in order to expand the $1/(\varepsilon+x)^{1/2}$ in the integrand,

$$I_2 = \int_\delta^\infty \frac{1}{(1+x)x^{1/2}}\left[1 - \varepsilon\frac{1}{2x} + O(\varepsilon^2/x^2)\right]\,\mathrm{d}x \overset{x=y^2}{=} 2\int_{\delta^{1/2}} \frac{1}{1+y^2}\left[1 - \varepsilon\frac{1}{2y^2} + O(\varepsilon^2/y^4)\right]\,\mathrm{d}y \tag{5.12}$$

$$= \left[2\arctan(y) + \varepsilon\left(\frac{1}{y} + \arctan(y)\right) + O(\varepsilon^2/y^3)\right]_{\delta^{1/2}}^\infty. \tag{5.13}$$

In principle, adding together the two results gives us the answer, with errors $O(\delta^{5/2})$ and $O(\varepsilon^2/\delta^{3/2})$. Since δ is still unspecified, we can choose it to give us the smallest possible error, so we equate $\delta^{5/2} = \varepsilon^2/\delta^{3/2}$ and find $\delta = \varepsilon^{1/2}$ with optimal error $O(\varepsilon^{5/4})$. Before we substitute this specific value, we expand both integrals in large δ/ε and small δ, respectively, keeping sufficiently many terms that the resulting error is at most $O(\varepsilon^{5/4})$ when $\delta = \varepsilon^{1/2}$:

$$I_1 = \varepsilon^{1/2}\left[2\left(1+\frac{\delta}{\varepsilon}\right)^{1/2} - 2\right] + \varepsilon^{3/2}\left[-\frac{2}{3}\left(1+\frac{\delta}{\varepsilon}\right)^{3/2} + 2\left(1+\frac{\delta}{\varepsilon}\right)^{1/2} + \frac{4}{3}\right] + O(\delta^{5/2}) \tag{5.14}$$

$$= \left[2\delta^{1/2} + \frac{\varepsilon}{\delta^{1/2}} - 2\varepsilon^{1/2} + O(\varepsilon^2/\delta^{3/2})\right] + \left[-\frac{2}{3}\delta^{3/2} + O(\varepsilon\delta^{1/2})\right] + O(\delta^{5/2}), \tag{5.15}$$

$$I_2 = \left[\pi - 2\arctan\delta^{1/2}\right] + \varepsilon\left[-\frac{1}{\delta^{1/2}} + \frac{\pi}{2} - \arctan\delta^{1/2}\right] + O(\varepsilon^2/\delta^{3/2}) \tag{5.16}$$

$$= \left[\pi - 2\delta^{1/2} + 2\frac{\delta^{3/2}}{3} + O(\delta^{5/2})\right] + \varepsilon\left[-\frac{1}{\delta^{1/2}} + \frac{\pi}{2} + O(\delta^{1/2})\right] + O(\varepsilon^2/\delta^{3/2}). \tag{5.17}$$

$$\Rightarrow \quad I = \pi - 2\varepsilon^{1/2} + \frac{\pi}{2}\varepsilon + O(\delta^{5/2}, \varepsilon\delta^{1/2}, \varepsilon^2/\delta^{3/2}) \overset{\delta=\varepsilon^{1/2}}{=} \pi - 2\varepsilon^{1/2} + \frac{\pi}{2}\varepsilon + O(\varepsilon^{5/4}). \tag{5.18}$$

We note that all terms depending on δ have cancelled out, as the result should not depend on where exactly we split the integrals, so the terms in the result all come from the local $X = 0$ endpoint and global $x = 1$ endpoint. The fact that the δ-dependent terms are the same (but with opposite sign) between the two integrals also reflects the fact that the integrand can be doubly expanded on the intermediate scale

$\varepsilon \ll x \ll 1$:

$$\int x^{-1/2}(1+x)^{-1}(1+\varepsilon/x)^{-1/2}\,\mathrm{d}x = \int x^{-1/2}\left(1-x+x^2-\cdots\right)\left(1-\frac{\varepsilon}{2x}+\frac{3\varepsilon^2}{8x^2}+\cdots\right)\mathrm{d}x = \tag{5.19}$$

$$= \int x^{-1/2}\begin{pmatrix} 1 & -x & +x^2 & \cdots \\ -\dfrac{\varepsilon}{2x} & +\dfrac{\varepsilon}{2} & -\dfrac{\varepsilon}{2}x & \cdots \\ +\dfrac{3\varepsilon^2}{8x^2} & -\dfrac{3\varepsilon^2}{8x} & +\dfrac{3\varepsilon^2}{8} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\mathrm{d}x = \begin{pmatrix} 2x^{1/2} & -\dfrac{2}{3}x^{3/2} & +\dfrac{2}{5}x^{5/2} & \cdots \\ \dfrac{\varepsilon}{x^{1/2}} & +\varepsilon x^{1/2} & -\dfrac{\varepsilon}{3}x^{3/2} & \cdots \\ -\dfrac{\varepsilon^2}{4x^{3/2}} & \dfrac{3\varepsilon^2}{4x^{1/2}} & +\dfrac{3\varepsilon^2}{4}x^{1/2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + C. \tag{5.20}$$

Let's also consider the special case $\alpha = 1$, which results in subtleties with logarithms. In this case,

$$I_1 = \int_0^{\delta/\varepsilon} \frac{1}{1+X}\left[1-\varepsilon X + O(\varepsilon^2 X^2)\right]\,\mathrm{d}X = \left[\ln(1+X) + \varepsilon(-X+\ln(1+X)) + O(\varepsilon^2 X^2)\right]_0^{\delta/\varepsilon} \tag{5.21}$$

$$I_2 = \int_\delta^\infty \frac{1}{(1+x)x}\left[1-\frac{\varepsilon}{x}+O(\varepsilon^2/x^2)\right]\,\mathrm{d}x = \left[\ln\frac{x}{1+x} + \varepsilon\left(\frac{1}{x}+\ln\frac{x}{1+x}\right) + O(\varepsilon^2/x^2)\right]_\delta^\infty. \tag{5.22}$$

We equate the errors $O(\delta^2) = O(\varepsilon^2/\delta^2)$ to find $\delta = \varepsilon^{1/2}$ and the overall error $O(\varepsilon)$. Then

$$I_1 = \left[\ln\frac{\delta}{\varepsilon} + \frac{\varepsilon}{\delta} + O(\varepsilon^2/\delta^2)\right] + \left[-\delta + \varepsilon\ln\frac{\delta}{\varepsilon} + O(\varepsilon^2/\delta)\right] + O(\delta^2) \tag{5.23}$$

$$I_2 = \left[-\ln\delta + \delta + O(\delta^2)\right] + \left[-\frac{\varepsilon}{\delta} - \varepsilon\ln\delta + O(\varepsilon\delta)\right] + O(\varepsilon^2/\delta^2) \tag{5.24}$$

$$\Rightarrow \quad I = \ln\frac{1}{\varepsilon} - \varepsilon\ln\frac{1}{\varepsilon} + O(\delta^2, \varepsilon\delta, \varepsilon^2/\delta^2) = \ln\frac{1}{\varepsilon} + \varepsilon\ln\frac{1}{\varepsilon} + O(\varepsilon). \tag{5.25}$$

Note that our naive estimates of the integral contribution as being the integrand scale times the size of the interval fails, as the logarithm is special. The dominant contribution comes from the intermediate region $\varepsilon \ll x \ll 1$ and is larger, $\mathrm{ord}(\ln\varepsilon)$, than the estimates $\mathrm{ord}(1)$ for either side, but not by very much. Although strictly speaking the leading-order $\mathrm{ord}(\ln\varepsilon)$ result is independent of the local and global details, in practice their $\mathrm{ord}(1)$ contributions (which turned out to vanish in this particular example) should be calculated if possible.

6 Integrals with a large exponent

Integrals of the form

$$I(x) = \int_a^b \mathrm{e}^{xh(t)} f(t)\,\mathrm{d}t \tag{6.1}$$

often occur as the result of Fourier transforms, Laplace transforms, or other methods for solving differential equations. We can exploit the singular behaviour of the exponential function for large arguments to obtain an asymptotic expansion for $I(x)$ as $x \nearrow \infty$.

Remark(s) 6.1. We will make frequent use of the following facts about the Gamma function:

- The integral formula $\Gamma(x) = \displaystyle\int_0^\infty \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t$.
- The recurrence relation $\Gamma(x+1) = x\Gamma(x)$ and relation to the factorial $n! = \Gamma(n+1)$.
- The half-integer values are given by

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u = \sqrt{\pi}, \quad \Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \tag{6.2}$$

  where the double factorial is defined by $(2n-1)!! = (2n-1)\times(2n-3)\times\cdots\times 3\times 1$, and $(-1)!! = 1$.
- The related integral

$$\int_0^\infty \mathrm{e}^{-u^2} u^n\,\mathrm{d}u = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t} t^{(n-1)/2}\,\mathrm{d}t = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right), \tag{6.3}$$

  where the Γ function can be further simplified depending on whether n is odd or even.

6.1 Exponential integrals

In this section we consider

$$I(x) = \int_a^b \mathrm{e}^{xh(t)} f(t)\,\mathrm{d}t, \quad x \nearrow \infty, \tag{6.4}$$

where the parameter x, the integration variable t and the function $h(t)$ in the exponent are all real. (The limits a and b can also be infinite, provided that the integral converges.)

Example 6.1. Consider the integral

$$I(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t}\,\mathrm{d}t, \qquad x \nearrow \infty. \tag{6.5}$$

The integrand is ord(1) for $t = O(1/x)$, and exponentially small for $t \gg 1/x$, so we would expect to obtain all algebraic contributions from the local region $t = \mathrm{ord}(1/x)$. Hence, we might expect to be able to use the small-t expansion of $1/(1+t)$ to say that

$$I(x) \sim \sum_{n=0}^\infty \int_0^\infty \mathrm{e}^{-xt}(-t)^n\,\mathrm{d}t \overset{t=s/x}{=} \sum_{n=0}^\infty \frac{(-1)^n}{x^{n+1}}\int_0^\infty \mathrm{e}^{-s} s^n\,\mathrm{d}s = \sum_{n=0}^\infty (-1)^n \frac{n!}{x^{n+1}}. \tag{6.6}$$

Note that the result is not a convergent series, so there is no way this could be a valid manipulation. However, according to Watson's lemma below, this result is correct. (Also, note that the substitution $t = (s/x) - 1$ can be used to relate this integral to the exponential integral $E_1(x)$ which we have expanded before, and the expansions agree.)

6.1.1 Watson's lemma

Theorem 6.1 (Watson's lemma)**.** If $f(t)$ is continuous and has the asymptotic power-law expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha_n} \text{ as } t \searrow 0, \qquad \text{where } -1 < \alpha_0 < \alpha_1 < \cdots, \tag{6.7}$$

then the following integral, where $0 < T \leq \infty$, has an asymptotic expansion given by

$$\int_0^T e^{-xt} f(t)\,dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n+1)}{x^{\alpha_n+1}} \text{ as } x \nearrow \infty, \tag{6.8}$$

provided that $\int_0^T |f(t)|\,dt$ exists, or more generally that $\int_0^T e^{-Xt}|f(t)|\,dt$ exists for some X.

Remark(s) 6.2. • The result should not be a surprise – the theorem simply states that, although not allowed, substituting the expansion and integrating term-by-term yields the correct answer:

$$\int_0^T e^{-xt} f(t)\,dt \sim \sum_{n=0}^{\infty} \int_0^T e^{-xt} a_n t^{\alpha_n}\,dt \overset{s=xt}{\sim} \sum_{n=0}^{\infty} \frac{a_n}{x^{\alpha_n+1}} \int_0^{\infty} e^{-s} s^{\alpha_n}\,ds \tag{6.9}$$

- The result is valid also for a finite number of terms in the expansion of f.
- The various conditions ensure that the integral exists: Continuity of $f(t)$ ensures that the integrand is continuous and hence (locally) integrable. The limitation $-1 < \alpha_0$ ensures that the integral doesn't diverge at $t = 0$, and the limitation that $e^{-Xt}|f(t)|$ is integrable ensures that, for sufficiently large x, the integral exists (and in particular doesn't diverge at infinity).
- The key point is that any contribution to the integral from $t \geq \delta$ has an $O(e^{-\delta x})$ suppression factor, which makes it exponentially small compared with the terms $O(1/x^{\alpha_n+1})$ even if δ is very small (as long as it remains fixed, independent of x). Thus, we were able to use the asymptotic expansion for $f(t)$, which is only known to be valid for small t, to obtain a full asymptotic expansion for the whole integral.

Proof. Here is a sketch of a proof of Watson's lemma, the details of which are not important.

We denote the integral by $I(x)$, fix $N \geq 0$, and fix a small value of $\delta > 0$ such that

$$f(t) = \sum_{n=0}^{N} a_n t^{\alpha_n} + R_N(t), \tag{6.10}$$

where the remainder is uniformly $O(t^{\alpha_{N+1}})$ in the range $0 \leq t \leq \delta$, i.e. bounded by $Mt^{\alpha_{N+1}}$ for some M independent of t. When we split the integral at δ,

$$I(x) = \underbrace{\int_0^{\delta} e^{-xt} f(t)\,dt}_{I_1} + \underbrace{\int_{\delta}^{T} e^{-xt} f(t)\,dt}_{I_2}, \tag{6.11}$$

we can then neglect the contribution I_2 from $t \geq \delta$ as being exponentially smaller, which we justify by splitting the exponent into two halves and using the bound

$$|I_2| \leq e^{-x\delta/2} \int_{\delta}^{T} e^{-xt/2}|f(t)|\,dt \leq e^{-\delta x/2} \int_0^T e^{-Xt}|f(t)|\,dt = O(e^{-x\delta/2}). \tag{6.12}$$

We then expand the lower part to obtain

$$I_1(x) = \left(\sum_{n=0}^{N} a_n \int_0^{\delta} e^{-xt} t^{\alpha_n}\,dt \right) + \int_0^{\delta} e^{-xt} R_N(t)\,dt. \tag{6.13}$$

In each of the main terms, we use

$$\int_0^{\delta} e^{-xt} t^{\alpha_n}\,dt = \int_0^{\infty} e^{-xt} t^{\alpha_n}\,dt - \int_{\delta}^{\infty} e^{-xt} t^{\alpha_n}\,dt = \frac{\Gamma(\alpha_n+1)}{x^{\alpha_n+1}} + O(e^{-x\delta/2}), \tag{6.14}$$

where the $t > \delta$ integral is exponentially small for the same reason as I_2. Similarly, the integral of the remainder term is bounded by

$$\left|\int_0^\delta e^{-xt} R_N(t)\,dt\right| \le \left|\int_0^\delta e^{-xt} M t^{\alpha_N+1}\,dt\right| \le M\int_0^\infty e^{-xt} t^{\alpha_N+1}\,dt = O(x^{-(\alpha_N+2)}). \tag{6.15}$$

Collecting these together, we find that $I(x)$ is the required partial sum plus a sufficiently small error. □

Example 6.2. The confluent hypergeometric function ${}_1F_1(a;b;x)$ is proportional to the integral

$$I(x) = \int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1}\,dt \quad \text{for} \quad \operatorname{Re} b > \operatorname{Re} a > 0. \tag{6.16}$$

(You can check that this satisfies Kummer's equation $xy''(x) + (b-x)y'(x) - ay = 0$.)

What is the behaviour of $I(x)$ as $x \searrow -\infty$? We note that the integral is of the required form for Watson's lemma, with the large parameter being $(-x) \nearrow +\infty$ and

$$f(t) = t^{a-1}(1-t)^{b-a-1} \sim \sum_{n=0}^\infty (-1)^n \frac{\Gamma(b-a)}{\Gamma(b-a-n)n!} t^{a-1+n} \quad \text{as} \quad t \searrow 0. \tag{6.17}$$

We thus obtain

$$I(x) = \int_0^1 e^{-(-x)t} f(t)\,dt \sim \sum_{n=0}^\infty (-1)^n \frac{\Gamma(b-a)\Gamma(a+n)}{\Gamma(b-a-n)n!}\frac{1}{(-x)^{a+n}}. \tag{6.18}$$

What is the behaviour of $I(x)$ as $x \nearrow \infty$? Now the integrand is exponentially largest near $t = 1$ instead of $t = 0$, so we use the transformation $t = 1 - s$ to rewrite the integral as

$$I(x) = \int_0^1 e^{x(1-s)}(1-s)^{a-1}s^{b-a-1}\,ds = e^x \int_0^1 e^{-xs}(1-s)^{a-1}s^{b-a-1}\,ds, \tag{6.19}$$

and can then expand $f(s)$ and apply Watson's lemma as before.

Remark(s) 6.3. Watson's lemma can also be used when the exponent has more complicated form, provided that it is x times a function that decreases with t, as we can then rewrite the integral using a change of variables $h(t) = h(a) - s$ to

$$\int_a^b e^{xh(t)} f(t)\,dt = e^{xh(a)} \int_0^{h(a)-h(b)} e^{-xs} \underbrace{\frac{f(h^{-1}(h(a)-s))}{-h'(h^{-1}(h(a)-s))}}_{\text{new } f(s)}\,ds, \tag{6.20}$$

but this can be very tedious due to the need to expand the new $f(s)$.

6.1.2 Laplace's method

When the function $h(t)$ in the exponent is complicated, we can instead use Laplace's method, in which we expand $h(t)$ in place in the exponent. Note that when using Laplace's method we typically do not justify all the approximations made in detail, but the principle is the same as for Watson's lemma: The dominant contribution to the integral comes from a small region near the maximum of h, and any contribution from outside this region is exponentially smaller, compared with at the maximum. This allows us to expand the integrand near the maximum and evaluate the integral. Often, a function h attains its maximum on an interval of integration either at one of its endpoints, or at a stationary point with $h' = 0$ and $h'' < 0$, so let's consider those cases first.

Example 6.3. Let's find a few terms in an asymptotic expansion for

$$I(x) = \int_0^1 \mathrm{e}^{-x\mathrm{e}^t} \cos\sqrt{t}\,\mathrm{d}t \quad \text{as} \quad x \nearrow \infty. \tag{6.21}$$

We identify $h(t) = -\mathrm{e}^t$ and $f(t) = \cos\sqrt{t}$. For $0 \le t \le 1$, h attains its maximum value $h = -1$ at $t = 0$, so the (exponentially) dominant contribution comes from near $t = 0$, and we only need to consider this local contribution, in which we can expand h (and later f):

$$I(x) = \int_0^\delta \mathrm{e}^{x(-1-t-t^2/2+O(t^3))} \cos\sqrt{t}\,\mathrm{d}t + \mathrm{EST} = \mathrm{e}^{-x}\int_0^\delta \mathrm{e}^{-xt}\mathrm{e}^{-xt^2/2+O(xt^3)} \cos\sqrt{t}\,\mathrm{d}t + \mathrm{EST}, \tag{6.22}$$

where "EST" stands for "exponentially **smaller** terms". In the expansion for $h(t)$ in the exponent, the constant term simply yields an exponential prefactor. The first varying term, which in this case is linear, is responsible for the exponential suppression of the integrand away from the maximum so that only the region $t = O(1/x)$ contributes. We thus rescale $t = u/x$ to make that term ord(1) and clarify the order of the contributions from the other terms. The higher-order terms, which have higher powers of t than the ord(1) term, become small corrections in the exponent, which allows their exponential to be expanded:

$$\exp\left[-\frac{xt^2}{2} + O(xt^3)\right] = \exp\left[-\frac{u^2}{2x} + O(x^{-2})\right] = 1 - \frac{u^2}{2x} + O(x^{-2}). \tag{6.23}$$

Similarly, the higher-order terms in the expansion of f also become small corrections:

$$\cos\sqrt{t} = 1 - \frac{t}{2} + O(t^2) = 1 - \frac{u}{2x} + O(x^{-2}). \tag{6.24}$$

Hence,

$$I(x) = \mathrm{e}^{-x}\int_0^{x\delta} \mathrm{e}^{-u}\left[1 + \frac{-u^2}{2x} + O(x^{-2})\right]\left[1 - \frac{u}{2x} + O(x^{-2})\right]\frac{\mathrm{d}u}{x} + \mathrm{EST} \tag{6.25}$$

We assume that although δ is small, $x\delta$ is large, so that we can replace the integration limit by ∞ and only pick up another "EST" error. Then

$$I(x) = \frac{\mathrm{e}^{-x}}{x}\int_0^\infty \mathrm{e}^{-u}\left[1 + \frac{-u^2}{2x} - \frac{u}{2x} + O(x^{-2})\right]\mathrm{d}u + \mathrm{EST} = \frac{\mathrm{e}^{-x}}{x}\left[1 - \frac{3}{2x} + O(x^{-2})\right]. \tag{6.26}$$

(Strictly speaking, the big-O bounds in the intermediate steps are wrong because u can be as large as $\delta x \gg 1$, so we should be keeping track of the powers of u, but since the exponential at the end suppresses any contribution from $u \gg 1$, this is not a problem and we're too lazy to do it.)

Example 6.4. We shall calculate the first few terms in an asymptotic expansion of the integral

$$I(x) = \int_0^\infty \mathrm{e}^{x(\ln t - t)}\,\mathrm{d}t, \quad x \nearrow \infty. \tag{6.27}$$

We first check where $h(t) = \ln t - t$ attains its maximal value on the integration interval $0 \le t \le \infty$: The function decays to $-\infty$ at both endpoints, and it has a single stationary point, $0 = h'(t) = 1/t - 1 \quad \Rightarrow t = 1$. We expand $h(t)$ near this point

$$h(1+s) = \ln(1+s) - (1+s) = -1 - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + O(s^5), \tag{6.28}$$

and verify that it's a maximum since the first varying term is negative.

The integral is exponentially dominated by the contribution from near $t = 1$, so

$$I(x) = \int_{-\delta}^{\delta} \exp\left[x\left(-1 - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + O(s^5)\right)\right] \mathrm{d}t + \text{EST}. \tag{6.29}$$

In order to make the first varying term be ord(1), we set $s = (2/x)^{1/2}u$. (The choice of whether to include the numerical factor 2 in the rescaling is up to you.) Thus,

$$I(x) = \sqrt{\frac{2}{x}}\mathrm{e}^{-x} \int_{-\sqrt{x/2}\,\delta}^{\sqrt{x/2}\,\delta} \mathrm{e}^{-u^2} \exp\left(\frac{2^{3/2}u^3}{3x^{1/2}} - \frac{u^4}{x} + O(x^{-3/2})\right) \mathrm{d}u + \text{EST} \tag{6.30}$$

$$= \sqrt{\frac{2}{x}}\mathrm{e}^{-x} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2} \left[1 + \frac{2^{3/2}u^3}{3x^{1/2}} + \left(\frac{4u^6}{9x} - \frac{u^4}{x}\right) + O(x^{-3/2})\right] \mathrm{d}u + \text{EST} \tag{6.31}$$

$$= \sqrt{\frac{2\pi}{x}}\mathrm{e}^{-x} \left[1 + \frac{0}{x^{1/2}} + \frac{1}{x} \underbrace{\left(\frac{4}{9} \times \frac{15}{8} - \frac{3}{4}\right)}_{1/12} + O(x^{-3/2})\right]. \tag{6.32}$$

The first correction term in the integrand was an odd function, so its integral from $-\infty$ to $+\infty$ cancelled out. In fact, this occurs at every other order, resulting in the corrections being integer powers of x (relative to the leading order).

Remark(s) 6.4. • The main steps of the Laplace method are:

- Identify the maximum of $h(t)$ in the integration interval. Claim that the integral is exponentially dominated by the sum of the contributions from these regions.
- In each region $|t - t_*| \leq \delta$ (one-sided if on the boundary), expand h and f for small $s = t - t_*$.
- Rescale s to make the first varying term in h be ord(1). Keep this term in the exponent.
- Expand the exponential of any higher-order terms in h and multiply with the expansion for f.
- Replace the integration limit(s) with ∞ and evaluate the integral (using the Γ function).

• The general leading-order contribution for a linear endpoint maximum at $t = a$ or $t = b$ is given by

$$\int_a^b \mathrm{e}^{xh(t)} g(t)\,\mathrm{d}t \sim \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{xh(a) - x(-h'(a))s} g(a)\,\mathrm{d}t = \frac{\mathrm{e}^{xh(a)}g(a)}{x(-h'(a))} & \text{or} \\ \displaystyle\int_0^\infty \mathrm{e}^{xh(b) - xh'(b)s} g(b)\,\mathrm{d}t = \frac{\mathrm{e}^{xh(b)}g(b)}{xh'(b)}, & \end{cases} \tag{6.33}$$

and for a quadratic interior maximum at $t = c$,

$$\int_a^b \mathrm{e}^{xh(t)} g(t)\,\mathrm{d}t \sim \int_{-\infty}^{\infty} \mathrm{e}^{xh(c) - x(-h''(c)/2)s^2} g(c)\,\mathrm{d}t = \frac{\sqrt{2\pi}\mathrm{e}^{xh(c)}g(c)}{\sqrt{(-h''(c))x}}. \tag{6.34}$$

If the quadratic maximum is at either endpoint, then one of the integral limits is replaced with 0, resulting in the contribution being halved.

• What terms to expect:

- In general, for an nth order maximum, i.e. $h(t_* + s) = h(t_*) + As^n + o(s^n)$ with $A \neq 0$, the dominant contribution comes from a region of size $s = \text{ord}(x^{-1/n})$, and hence the integral is $\text{ord}(\mathrm{e}^{xh(t_*)}/x^{1/n})$.
- If additionally the multiplying function f vanishes or diverges like s^α (with $\alpha > -1$ so that the integral still exists), then Laplace's method can proceed as usual with f expanded in (possibly non-integer) powers of s starting from s^α, and the integral will be $\text{ord}(\mathrm{e}^{xh(t_*)}/x^{(1+\alpha)/n})$.
- If the corrections come from integer powers of s, then they become powers of $\text{ord}(x^{-1/n})$.
- At an interior maximum, for integer powers of s, every other term vanishes due to being an integral of an odd function.

- If instead the pre-multiplying function f vanishes or diverges exponentially at $t = t_*$, then that's an indication that we should reconsider the integrand as

$$\mathrm{e}^{xh(t)} f(t) = \mathrm{e}^{xh(t)+\ln f(t)} \tag{6.35}$$

  and try to regroup the two terms in the exponent to obtain a different $h(t)$ which achieves its maximum without $f(t)$ vanishing or diverging exponentially, as seen in the next example.

Example 6.5. Let's calculate the large-n behaviour of $n!$, using the integral expression for the Gamma function,

$$n! = \Gamma(n+1) = \int_0^\infty \mathrm{e}^{-t} t^n \,\mathrm{d}t. \tag{6.36}$$

Here, the large parameter n is in the exponent once we rewrite $t^n = \mathrm{e}^{n \ln t}$. However, $h(t) = \ln t$ grows indefinitely as $t \to \infty$, instead of attaining a maximum. The integral doesn't diverge, though, because of the exponential decay of the e^{-t} term. Combining the two terms and seeking a maximum yields

$$\mathrm{e}^{-t} t^n = \mathrm{e}^{-t+n\ln t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(-t + n\ln t) = -1 + \frac{n}{t} \quad \Rightarrow \quad t_* = n. \tag{6.37}$$

This is a **moving maximum**, and hints that we should rescale the integration variable as $t = ns$, to obtain an exponent which attains its maximum at a fixed location $s = 1$,

$$n! = \Gamma(n+1) = n\int_0^\infty \mathrm{e}^{-ns+n\ln n+n\ln s}\,\mathrm{d}t = n^{n+1}\int_0^\infty \mathrm{e}^{n(-s+\ln s)}\,\mathrm{d}s. \tag{6.38}$$

We calculated this integral in the previous example, so using that result we obtain

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left[1 + \frac{1}{12n} + O(n^{-2})\right]. \tag{6.39}$$

The moving maximum could also be handled without rescaling: When we expand in small $s = t - n$ the exponent becomes

$$-n - s + n\ln(n+s) = -n + n\ln n - s + n\ln\left(1 + \frac{s}{n}\right) = -n + n\ln n - \frac{s^2}{2n} + \frac{s^3}{3n^2} + O\left(\frac{s^4}{n^3}\right). \tag{6.40}$$

Although this is not of the standard form $nh(t)+\ln f(t)$ (plus some n-dependent "constant") for Laplace's method, we can still proceed in the same way. Looking at the first varying term in the expansion, $-s^2/2n$, and insisting that it becomes ord(1) yields the rescaling $s^2/2n = u^2$, after which the calculation becomes identical to the previous one.

6.2 Oscillatory integrals

In this section we consider the case of a purely imaginary exponent, i.e.

$$I(x) = \int_a^b \mathrm{e}^{\mathrm{i}x\psi(t)} f(t)\,\mathrm{d}t, \quad x \nearrow \infty, \tag{6.41}$$

where the parameter x, the variable t and the phase function $\psi(t)$ are all real. Note that by taking the real or imaginary part we can obtain results for integrals with $\cos(x\psi(t))$ or $\sin(x\psi(t))$. (Again, we allow a and/or b to be infinite provided that the integral converges.)

Example 6.6. Consider the simple case

$$I(x) = \int_a^b \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t, \quad x \nearrow \infty. \tag{6.42}$$

If f is (continuously) differentiable, then we can integrate by parts,

$$I(x) = -\frac{\mathrm{i}}{x}\left[\mathrm{e}^{\mathrm{i}xt} f(t)\right]_a^b + \frac{\mathrm{i}}{x}\int_a^b \mathrm{e}^{\mathrm{i}xt} f'(t)\,\mathrm{d}t. \tag{6.43}$$

If f is N times (continuously) differentiable, then we can repeat this N times to obtain,

$$I(x) = -\left[\mathrm{e}^{\mathrm{i}xt}\left(\frac{\mathrm{i}}{x}f(t) + \frac{\mathrm{i}^2}{x^2}f'(t) + \frac{\mathrm{i}^3}{x^3}f''(t) + \cdots + \frac{\mathrm{i}^N}{x^N}f^{(N-1)}(t)\right)\right]_a^b + \frac{\mathrm{i}^N}{x^N}\int_a^b \mathrm{e}^{\mathrm{i}xt} f^{(N)}(t)\,\mathrm{d}t. \tag{6.44}$$

Do the first $N-1$ terms form an asymptotic expansion for I? It turns out that the answer is yes, because Riemann-Lebesgue lemma below proves that the final integral is $o(1)$ as $x \nearrow \infty$.

Note that the terms are endpoint contributions, and the leading-order term is $O(1/x)$. (If a or b is infinite then, provided f and its derivatives decay, there is no endpoint contribution from that end.)

Theorem 6.2 (Riemann–Lebesgue lemma)**.** For $-\infty \le a < b \le \infty$, if $\int_a^b |f(t)|\,\mathrm{d}t$ exists, then

$$\int_a^b \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t \to 0 \quad \text{as } x \nearrow \infty. \tag{6.45}$$

Why should we expect this to hold? The idea is that for large x the integrand oscillates rapidly (on the scale $\mathrm{ord}(1/x)$) while the amplitude of oscillations changes on a much longer scale ($\mathrm{ord}(1)$), so each oscillation approximately cancels itself out (figure 6.1(a)). We can justify this for sufficiently well-behaved functions, by integrating by parts once,

$$\int_a^b \mathrm{e}^{\mathrm{i}xt} f(t)\,\mathrm{d}t = -\frac{\mathrm{i}}{x}\underbrace{\left[\mathrm{e}^{\mathrm{i}xt} f(t)\right]_a^b}_{=O(1)} + \frac{\mathrm{i}}{x}\underbrace{\int_a^b \mathrm{e}^{\mathrm{i}xt} f'(t)\,\mathrm{d}t}_{\text{bounded by } \int_a^b |f'(t)|\,\mathrm{d}t} = O(1/x) \to 0, \tag{6.46}$$

but a proof for more general $f(t)$ is outside the scope of this module.

Remark(s) 6.5. What is the implication for more general oscillatory integrals, with a general function $\psi(t)$ in the exponent? If ψ is monotonic and ψ' bounded away from 0, then we could just use the change of variables $s = \psi(t)$ to change the integral into the previous form

$$I(x) = \int_{\psi(a)}^{\psi(b)} \mathrm{e}^{\mathrm{i}xs}\frac{f(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))}\,\mathrm{d}s \tag{6.47}$$

and obtain $O(1/x)$ terms from the endpoints. However, if $\psi' = 0$ somewhere then the transformation cannot be used, as the integrand would become singular. This indicates that the dominant contribution will come from near the point of stationary phase. Intuitively, the rapid oscillations (due to the rapid change of the phase $x\psi(t)$ with t) which cause cancellation slow down near the point of stationary phase, resulting in less cancellation there (figure 6.1(b)). We use the **method of stationary phase** to obtain the leading-order contribution:

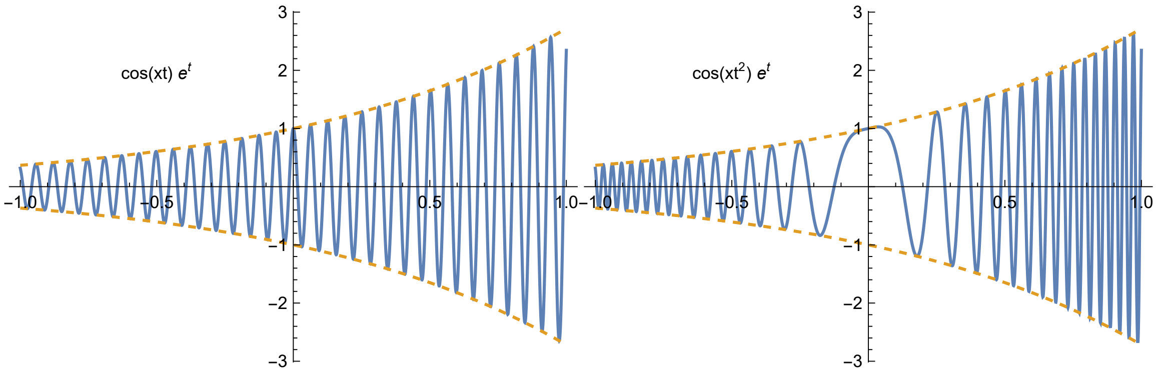


Figure 6.1: Oscillations with amplitude $f(t) = \mathrm{e}^t$ and phase function (a) $\psi = t$, (b) $\psi = t^2$ for $x = 100$.

Example 6.7. Let's find the leading-order behaviour of the zeroth-order Bessel function of the first kind $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x \sin t)\,\mathrm{d}t$ as $x \nearrow \infty$.

We write $J_0(x) = (1/\pi)\,\mathrm{Re}\, I(x)$, where $I(x) = \displaystyle\int_0^\pi \mathrm{e}^{\mathrm{i}x\sin t}\,\mathrm{d}t$. The phase function $\psi(t) = \sin t$ is stationary at $t = \pi/2$, so that's where we expect the dominant contribution to come from, while the contributions away from the stationary point are $O(1/x)$ as they are integrals over an interval where ψ' is bounded away from zero.

Since, unlike in Laplace's method, there is no exponential decay away from the stationary point, we can only work at leading order and will not keep track of any errors. We thus expand ψ near the stationary point to only the first varying term, $\sin(\pi/2 + s) = \cos(s) \approx 1 - s^2/2$, and similarly if we had an $f(t)$ then we'd just take the first term in its expansion, which typically means evaluating it at the stationary point. Thus, we obtain

$$I(x) \sim \int_{-\delta}^{\delta} \mathrm{e}^{\mathrm{i}x(1-s^2/2)}\,\mathrm{d}s \overset{s=(2/x)^{1/2}u}{\sim} \mathrm{e}^{\mathrm{i}x}\sqrt{\frac{2}{x}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}u^2}\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}x}\sqrt{\frac{2\pi}{x}}\mathrm{e}^{-\mathrm{i}\pi/4}. \tag{6.48}$$

In the last step, we had to evaluate the Fresnel integral $\int_0^\infty \mathrm{e}^{\mathrm{i}u^2}\,\mathrm{d}u$, which we explain how to do below. It's possible to cheat by using the formula $\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-au^2}\,\mathrm{d}u = \sqrt{\frac{\pi}{a}}$ with $a = \mathrm{i}$, but you have to make sure you choose the correct branch of the square root, which is that the argument of a should be the principal one, $\pi/2$.

We conclude that $J_0(x) \sim \sqrt{2/\pi x}\cos(x - \pi/4)$, although strictly speaking we shouldn't use the $\sim$ symbol as the right-hand side has zeroes in not quite the same locations as the left-hand side. What we really mean is that the error is $o(x^{-1/2})$, without including the $\cos(x - \pi/4)$ inside the $o(\cdot)$.

Example 6.8. In general, the method of stationary phase ends with needing to evaluate an integral of the form

$$\int_0^\infty \mathrm{e}^{\mathrm{i}u^p}\,\mathrm{d}u \overset{u=v^{1/p}}{=} \frac{1}{p}\int_0^\infty \mathrm{e}^{\mathrm{i}v}v^{(1/p)-1}\,\mathrm{d}v, \tag{6.49}$$

where $p > 1$. We would like to convert the exponential to e^{-w} to make a Gamma function, so we rotate the contour from the positive real axis to the positive imaginary axis, and obtain, with $v = \mathrm{i}w = \mathrm{e}^{\mathrm{i}\pi/2}w$,

$$\frac{1}{p}\mathrm{e}^{\mathrm{i}\pi/2p}\int_0^\infty \mathrm{e}^{-w}w^{(1/p)-1}\,\mathrm{d}w = \frac{\mathrm{e}^{\mathrm{i}\pi/2p}\Gamma(1/p)}{p}. \tag{6.50}$$

(If you want to be extra careful, you check that the contour rotation is allowed since, firstly, the contribution from a quarter circle at infinity vanishes, which follows from Jordan's lemma since $|v^{(1/p)-1}| \searrow 0$ as $|v| \nearrow \infty$ for $p > 1$, and, secondly, the contribution from a quarter circle avoiding the branch point at the origin vanishes, which follows from estimating the modulus of the integrand as $|v|^{(1/p)-1}$ and multiplying by the arclength $|v|\pi/2$.)

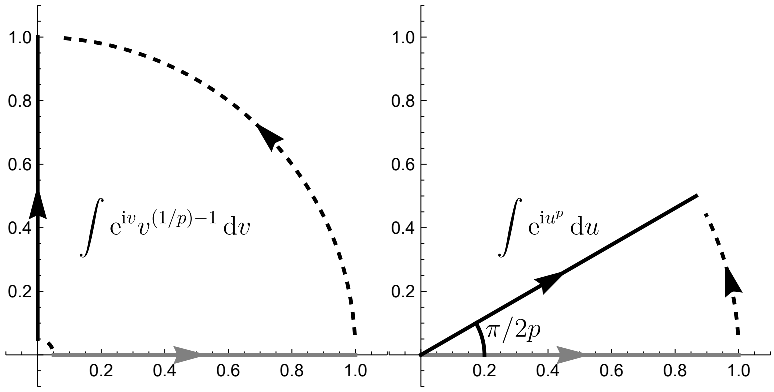


Figure 6.2: Contour deformation for calculating the integral $\displaystyle\int_0^\infty e^{iu^p}\, du$.

It is also possible to do the contour rotation first, by an angle $\pi/2p$, to obtain

$$\int_0^\infty e^{iu^p}\, du \overset{u=e^{i\pi/2p}v}{=} e^{i\pi/2p} \int_0^\infty e^{-v^p}\, dv \overset{v=w^{1/p}}{=} \frac{e^{i\pi/2p}}{p} \int_0^\infty e^{-w} w^{(1/p)-1}\, dw. \tag{6.51}$$

(But in order to prove that the contribution from infinity vanishes, you'd still have to do the other variable change and apply Jordan's lemma.)

If the exponent has the opposite sign (i.e. if the pth derivative of ψ is negative), then we get the complex conjugate of the result. If the integral is double-ended, then we split it into two parts, and flip the sign of s in the lower half, so that for even p we get twice this result, while for odd p we get twice the real part of this result.

Remark(s) 6.6. The method of stationary phase has several drawbacks due to the lack of exponential decay away from the stationary point

- We can only obtain the leading-order result – trying to put in any corrections results in a divergent integral of the form $\displaystyle\int e^{iu^2} u^n\, du$ with $n \geq 1$.
- It is important that the multiplying function $f(t)$ is bounded over the whole range of integration including the endpoints, as the estimate $O(1/x)$ for the global contribution relies on an integration by parts.
- Also, if $f(t)$ vanishes at the point of stationary phase then the local contribution from there is smaller, which is problematic if the end result becomes no larger than the $O(1/x)$ endpoint contributions.

We can avoid these issues by using the method of steepest descent instead, which is the next topic.

6.3 Method of steepest descent

We finally consider the case of

$$I(x) = \int_C \mathrm{e}^{xh(t)} f(t)\,\mathrm{d}t, \qquad x \to +\infty, \tag{6.52}$$

where $h(t)$ can be complex, and the curve C is taken in the complex plane with finite or infinite endpoints. We will start with an example application of the **method of steepest descent** before discussing the finer details of the method.

Example 6.9. Let revisit the stationary-phase problem $I(x) = \displaystyle\int_0^\pi \mathrm{e}^{\mathrm{i}x \sin t}\,\mathrm{d}t$.

The main idea is to deform the contour in the complex t-plane so as to make the exponent have constant imaginary part, leaving just its real part varying with t. The imaginary part becomes a constant multiplicative factor outside the integral, and we are left with an integral that can be estimated using Laplace's method.

We say that a contour with constant imaginary part is a steepest-descent (SD) contour, for reasons to be explained later. What do the SD contours look like? We split $t = p + \mathrm{i}q$ into real and imaginary parts, and find

$$h(t) = \mathrm{i}\sin(p + \mathrm{i}q) = \mathrm{i}(\sin p \underbrace{\cos \mathrm{i}q}_{\cosh q} + \cos p \underbrace{\sin \mathrm{i}q}_{\mathrm{i}\sinh q}) = \underbrace{-\cos p \sinh q}_{h_\mathrm{r}} + \mathrm{i}\underbrace{\sin p \cosh q}_{h_\mathrm{i}}, \tag{6.53}$$

and we seek the solutions with constant h_i. For $h_\mathrm{i} = 0$ the solutions are vertical lines, $\sin p = n\pi$ with any q. For $h_\mathrm{i} \neq 0$, we obtain

$$q = \pm \operatorname{acosh}(h_\mathrm{i}/\sin p), \tag{6.54}$$

and as $p \to n\pi$ we have $q \to \pm\infty$, so the contours are asymptotic to the vertical lines $h_\mathrm{i} = 0$. Near the points of stationary phase $t = (n + \frac{1}{2})\pi$, we can expand in small $(\Delta p, \Delta q) = (p - (n + \frac{1}{2}\pi), q)$ and obtain $h_\mathrm{i} \approx (-1)^n[1 + \frac{1}{2}(\Delta_q^2 - \Delta_p^2)]$, so the contours form a pair of crossing 45-degree lines $\Delta_q = \pm\Delta_p$ with hyperbolae around them. (In general it is not possible to solve for one coordinate in terms of the other, in which case we don't get an explicit expression for the contour path.)

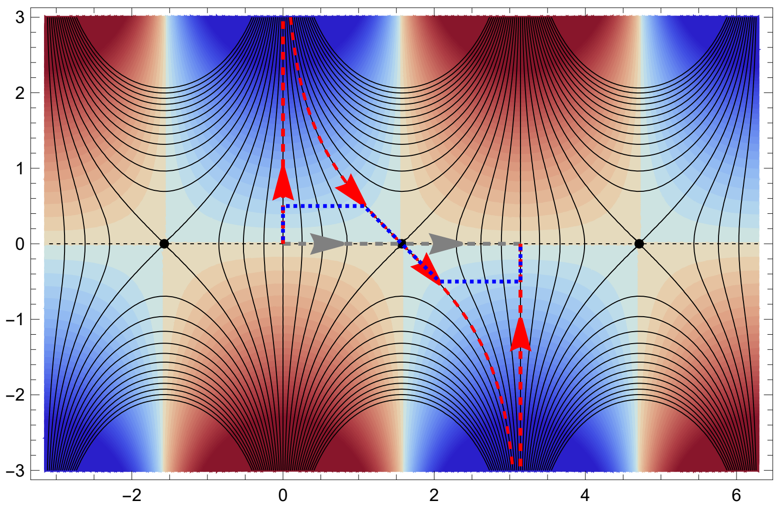


Figure 6.3: Steepest-descent contours (i.e. curves with h_i = constant) for $h(t) = \mathrm{i}\sin t$. Colours indicate the value of h_r (positive red, negative blue). The horizontal dashed grey contour is the original contour of integration. The dashed red contour is an exact steepest-descent contour. The dotted blue contour is an alternative piecewise linear contour that can also be used.

Now the idea is to deform the integration contour to follow the SD contours, and in particular $h_\mathrm{r} = -\cos p \sinh q$ should decrease as we move away from the real line. This means that we should go into the upper-half plane $q > 0$ for $0 \le p \le \pi/2$ and into the lower-half plane $q < 0$ for $\pi/2 \le p \le \pi$. A suitable contour is shown in figure 6.3. Note that although we have $\operatorname{Im} h(t)$ constant on each individual SD contour, the value of $\operatorname{Im} h(t)$ can change as we jump from one contour to a neighbouring one at infinity (or, in rare cases, at a finite pole or singularity). (Strictly speaking, we can't deform onto the infinite contours directly, but instead we connect the separate contours with bypass contours at say $q = \pm q_{max}$, and verify that the contribution from the bypass contours vanishes as $q_{max} \to \infty$, which is typically due to the integrand decaying exponentially.)

We can now calculate the contributions from each part separately: On the first vertical contour, we have $t = \mathrm{i}q$ for q increasing from 0 to ∞,

$$I_1 = \int_0^\infty \mathrm{e}^{x\mathrm{i}\sin(\mathrm{i}q)}\mathrm{i}\,\mathrm{d}q = \mathrm{i}\int_0^\infty \mathrm{e}^{-x\sinh q}\,\mathrm{d}q \sim \mathrm{i}\int_0^\infty \mathrm{e}^{-xq}\,\mathrm{d}q = \frac{\mathrm{i}}{x}. \tag{6.55}$$

On the last vertical contour, we have $t = \pi + \mathrm{i}q$ for q increasing from $-\infty$ to 0,

$$I_3 = \mathrm{i}\int_{-\infty}^0 \mathrm{e}^{x\mathrm{i}\sin(\pi+\mathrm{i}q)}\,\mathrm{d}q \overset{q=-u}{=} \mathrm{i}\int_0^\infty \mathrm{e}^{-x\sinh u}\,\mathrm{d}u \sim \frac{\mathrm{i}}{x}. \tag{6.56}$$

On the curved middle contour, we first try to do the calculation directly, which is a bit tedious: We have $h_\mathrm{i} = 1$ (by evaluating at $t = \pi/2$) and

$$h_r = -\cos p \sinh q = -\cos p\left(\pm\sqrt{\cosh^2 q - 1}\right) = -\cos p\left(\pm\sqrt{\frac{1}{\sin^2 p} - 1}\right) = -\frac{\cos^2 p}{\sin p}, \tag{6.57}$$

where we use the fact that q, and hence $\sinh q$, should have the same sign as $\cos p$. Note that this expression is ≤ 0, as expected if the maximum is going to be at $p = \pi/2$. For convenience, we define the shifted variable $s = p - \pi/2$, and obtain

$$h = -\frac{\sin^2 s}{\cos s} + \mathrm{i}, \quad t = \left(s + \frac{\pi}{2}\right) + \mathrm{i}\left(-\operatorname{sign}(s)\operatorname{acosh}\frac{1}{\cos s}\right), \tag{6.58}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1 + \mathrm{i}\left(-\operatorname{sign}(s)\frac{1}{\sqrt{(1/\cos^2 s) - 1}}\frac{\sin s}{\cos^2 s}\right) = 1 - \mathrm{i}\frac{1}{\cos s}. \tag{6.59}$$

Hence, the contribution from this contour is

$$I_2 = \int_{-\pi/2}^{\pi/2} \exp\left[x\left(-\frac{\sin^2 s}{\cos s} + \mathrm{i}\right)\right]\left(1 - \mathrm{i}\frac{1}{\cos s}\right)\mathrm{d}s \sim \mathrm{e}^{\mathrm{i}x}\int_{-\infty}^\infty \mathrm{e}^{-xs^2}(1-\mathrm{i})\,\mathrm{d}s = \tag{6.60}$$

$$= \sqrt{\frac{\pi}{x}}\mathrm{e}^{\mathrm{i}x}(1-\mathrm{i}) = \sqrt{\frac{2\pi}{x}}\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{-\mathrm{i}\pi/4}. \tag{6.61}$$

Since the stationary point is an internal maximum, we know that the next correction is not $\operatorname{ord}(\mathrm{e}^{\mathrm{i}x}/x)$, but rather $O(\mathrm{e}^{\mathrm{i}x}/x^{3/2})$, so we conclude that

$$I(x) = \sqrt{\frac{2\pi}{x}}\mathrm{e}^{\mathrm{i}(x-\pi/4)} + \frac{2\mathrm{i}}{x} + O(x^{-3/2}). \tag{6.62}$$

Compared with the method of stationary phase, the method of steepest descent has allowed us to calculate the next corrections (and more terms could be included), and has shown that the dominant contributions come from the neighbourhoods of the stationary point and the endpoints, while any other contributions are exponentially small. In effect, near each dominant point we have rotated the contour to make the oscillatory integral become exponentially decaying, in an analogous way to how we integrated $\mathrm{e}^{\mathrm{i}u^p}$, which then allows us to expand near each point to arbitrary order.

How do we calcualte the middle integral more easily? The point is that most of the effort spent on calculating the full SD contour is wasted since we then discard the contributions from the exponentially smaller parts of the contour, i.e. anything away from the dominant point(s). Hence, in practice we can simply use the tangent lines to the full contour near the dominant point(s), joined up in arbitrary way away from the points where the integrand is exponentially smaller (figure 6.3). For I_2, we would then use $t = \pi/2 + \mathrm{e}^{-\mathrm{i}\pi/4}s$ and write down

$$h = \underbrace{h(\pi/2)}_{\mathrm{i}} + \underbrace{h'(\pi/2)}_{0}(t-\pi/2) + \frac{1}{2}\underbrace{h''(\pi/2)}_{-\mathrm{i}}(t-\pi/2)^2 + O((t-\pi/2)^3) = \mathrm{i} - \frac{1}{2}s^2 + O(s^3), \tag{6.63}$$

$$\Rightarrow \quad I_2(x) \sim \int_{-\delta}^{\delta} \mathrm{e}^{x(\mathrm{i}-s^2/2)}\mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{d}s \sim \sqrt{\frac{2\pi}{x}}\mathrm{e}^{\mathrm{i}(x-\pi/4)} \tag{6.64}$$

Note that the correct choice of tangent angle $-\pi/4$ results in the first varying term becoming purely negative – if it doesn't, then there's a mistake somewhere. In this case, the integrals I_1 and I_3 already use straight lines so are unchanged, except that we would immediately write down the limits as being $\pm\delta$, rather than starting with $\pm\infty$.

(The full SD contour is not completely useless though – when evaluating the integral numerically without asymptotic approximation, using the SD contour avoids any issues with oscillation and cancellations.)

Remark(s) 6.7. • Since we're using contour deformation, we need all parts of the integrand to be complex analytic in the region we're deforming over. We can pass through poles and pick up residues, but must avoid crossing branch cuts. (Since the specific path taken by the contour doesn't matter, we use $\int_a^b$ to mean the integral over any contour from a to b, but need to specify which way the contour goes around any singularities. We also allow a and b to be infinite with a direction, e.g. $\mathrm{e}^{\mathrm{i}\pi/4}\infty$.)

• Since h is complex analytic, the contours of its real and imaginary parts are perpendicular. We can see this by expanding, with $A = h'(t_*)$,

$$h(t_* + s) - h(t_*) \approx As = |A|\,|s|\,\exp\left[\mathrm{i}(\arg A + \arg s)\right]. \tag{6.65}$$

The directions in which the real part is does not vary with $|s|$, i.e. $\exp = \pm\mathrm{i}$, are perpendicular to the directions in which the imaginary part does not vary with $|s|$, i.e. $\exp = \pm 1$. The latter is also the direction in which the real part decreases (or increases) the fastest, which is why the constant-h_{i} curves are called curves of steepest descent (or ascent, depending on which direction you walk in).

• The above expansion fails when $h' = 0$, and we instead obtain, if the first non-zero derivative is the nth one ($n \geq 2$), with $A = h^{(n)}(t_*)/n!$,

$$h(t_* + s) - h(t_*) \approx As^n = |A|\,|s|^n\,\exp\left[\mathrm{i}\,(\arg A + n\arg s)\right]. \tag{6.66}$$

This is a saddle point of order n, with n equally spaced increasing directions and n equally spaced decreasing directions, as shown in figure 6.4. Note that the real and imaginary parts do not have maximum or minimum points – only saddles. The directions of steepest descent from the saddle are found by setting $\exp = -1$,

$$\arg A + n\arg s = (2k+1)\pi \quad \Rightarrow \quad \arg s = \frac{\pi - \arg A}{n} + \frac{2\pi k}{n}, \quad k = 0, 1, \cdots, n-1. \tag{6.67}$$

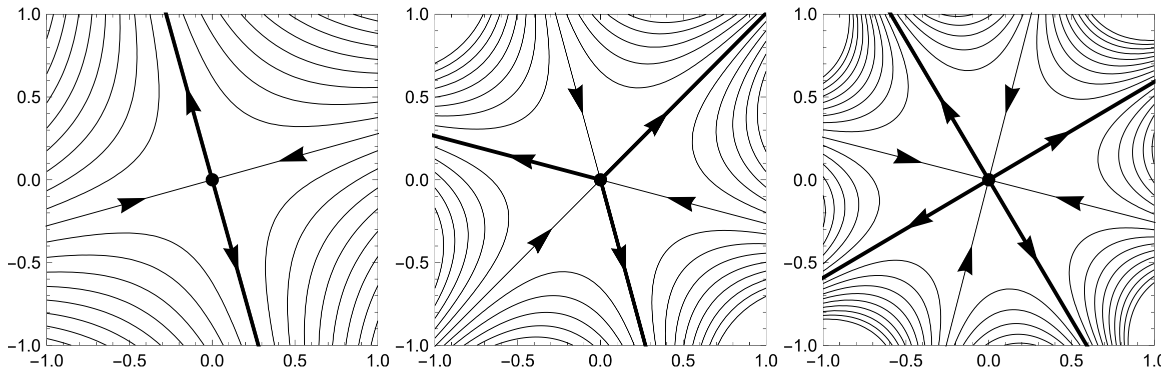


Figure 6.4: Shape of the contours of $\mathrm{Im}(h)$ near a saddle point of order 2, 3 or 4. The arrows indicate directions of descent, and the thick lines are the directions of steepest descent from the saddle.

- The near-saddle expansion also describes the behaviour at infinity, i.e. for large $|t|$, if $h(t) \sim At^n$ as $|t| \nearrow \infty$.
- How do we sketch the steepest-descent contours without the aid of a computer? We identify the saddles by solving $h'(t) = 0$ and sketch the local behaviour near the saddles as above, and try to solve explicitly for the contours crossing at the saddles. We also try to sketch the large-$|t|$ behaviour. Then, we simply join up the contours in a reasonable way.
- How do we know which integration contour to choose? To answer this, we must first understand why contour deformation is needed in general. Consider the exponential $\mathrm{e}^{xh(t)}$ in the integrand. We can estimate its magnitude using

$$|\mathrm{e}^{xh(t)}| = |\mathrm{e}^{x \operatorname{Re} h(t)}|\,|\mathrm{e}^{x\mathrm{i} \operatorname{Im} h(t)}| = |\mathrm{e}^{x \operatorname{Re} h(t)}|, \tag{6.68}$$

so naively we would expect the integral over a general contour to be dominated simply by the point t_* on the contour at which $\operatorname{Re} h(t)$ attains its maximum. After all, away from this maximum the integrand is exponentially smaller than $\mathrm{e}^{x \operatorname{Re} h(t_*)}$, so should be negligible in comparison.

However, if we try to proceed as in the Laplace method, and calculate the contribution from a neighbourhood of the maximum t_*, in which we have the expansion, say,

$$h(t_* + s) \approx h(t_*) + \mathrm{i}As - Bs^2, \tag{6.69}$$

where the ord(s) term is purely imaginary due to the real part having a maximum, then we obtain the local contribution

$$\int_{-\delta}^{\delta} \mathrm{e}^{x(h(t_*) - Bs^2 + \mathrm{i}As)}\,\mathrm{d}t = \mathrm{e}^{xh(t_*)}\sqrt{\frac{\pi}{Bx}}\mathrm{e}^{-A^2x/4B}, \tag{6.70}$$

so, if $A \neq 0$, then the local contribution is also exponentially smaller than the estimate $\mathrm{e}^{x \operatorname{Re} h(t_*)}$, and it is not clear whether the local or the global contribution is dominant.

The key problem is that the oscillations, due to the real part varying with t, make the integral smaller than the estimate we obtain from just looking at the magnitude of the integrand. The point of the method of steepest descent is to deform the contour of integration so that **the maximum value of $\operatorname{Re} h(t)$ on the contour is obtained without oscillation, i.e. with $\operatorname{Im} h(t)$ approximately constant near the maximum/maxima**. The lack of oscillation will ensure that the local contribution to the integral is as estimated (just divided by some fractional power of x), and hence indeed exponentially larger than the contributions from elsewhere.

The condition that $\operatorname{Im} h(t)$ is approximately constant means that we must follow the direction of steepest descent/ascent of $\operatorname{Re} h(t)$, near where the maximum $\operatorname{Re} h(t)$ is attained. If the maximum is not at an endpoint, then this means that $\operatorname{Re} h(t)$ attains a local internal maximum, near which it varies very little, despite moving in the direction of its fastest variation. This can only happen if it is stationary in all directions at the maximum, i.e. the maximum is attained at a saddle. (An alternative argument is that we need $A = 0$ in the calculation above, which means that $h' = 0$.) This shows that for any suitable contour, **the maximum of $\operatorname{Re} h(t)$ on the contour must be attained at an endpoint or a saddle point, while moving in the SD direction**.

This principle guides our choice of contour, and there are two possible strategies: One strategy is to write down all finite endpoints and saddle points, noting down their values of $\operatorname{Re} h(t)$, and try to make each dominant in turn and seeing if it's possible to deform the contour to make this happen. Another strategy is to start by following the exact SD contour from any finite endpoint downhill, continuing downhill until a point at infinity is reached (where $\operatorname{Re} h \searrow -\infty$), and then try to join up the points at infinity with an exact SD contour that passes through a saddle point.

We illustrate this with a few more examples.

Example 6.10. Consider the integral

$$I(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} \cos(4xt) f(t)\,\mathrm{d}t = \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{e}^{x(-t^2+4\mathrm{i}t)} f(t)\,\mathrm{d}t, \tag{6.71}$$

where f is a suitably well-behaved function (e.g. a polynomial).

On the integration contour, i.e. the real line, the exponential attains its maximum at $t = 0$, but it does so with oscillations which make the integral exponentially smaller (figure 6.5(a)). To find the exact SD contours, we could solve for constant $\mathrm{Im}\, h(p + \mathrm{i}q)$, but in this case it's easier to complete the square and find

$$h(t) = -t^2 + 4\mathrm{i}t = -(t - 2\mathrm{i})^2 - 4, \tag{6.72}$$

so the contours look like a quadratic saddle centred at $t_* = 2\mathrm{i}$, descending horizontally and ascending vertically. It is clear that we should simply translate the contour up to $\mathrm{Im}\, t = 2$ to obtain the appropriate full SD contour. (Indeed, there are no finite endpoints and only one saddle, so the saddle must be the dominant point.)

Using the parametrisation $t = 2\mathrm{i} + s$ with s real going from $-\infty$ to $+\infty$, we obtain the result

$$\int_{-\infty}^{\infty} \mathrm{e}^{x(-s^2-4)} f(2\mathrm{i} + s)\,\mathrm{d}s \overset{s=u/\sqrt{x}}{=} \frac{\mathrm{e}^{-4x}}{\sqrt{x}} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2} \left[f(2\mathrm{i}) + f'(2\mathrm{i}) \frac{u}{\sqrt{x}} + O(x^{-1}) \right] \mathrm{d}u \tag{6.73}$$

$$= \mathrm{e}^{-4x} \sqrt{\frac{\pi}{x}} \left[f(2\mathrm{i}) + O(x^{-1}) \right]. \tag{6.74}$$

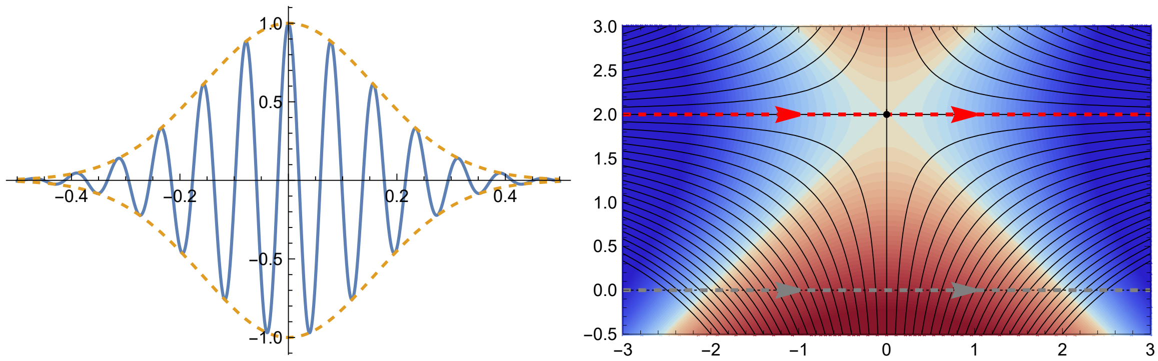


Figure 6.5: (a) Plot of $\mathrm{e}^{-xt^2} \cos(4xt)$ for $x = 20$. (b) Steepest-descent contours for $h(t) = -t^2 + 4\mathrm{i}t$. Colours indicate the value of h_r (positive red, negative blue). The dashed grey contour is the original contour. The dashed red contour is the exact SD contour.

Example 6.11. The Airy function $\mathrm{Ai}(z)$ is a solution of the equation $y''(z) - zy(z) = 0$ that decays as $z \nearrow \infty$, and has the integral expression

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(zs+s^3/3)}\,\mathrm{d}s. \tag{6.75}$$

How exactly does it decay as $z \nearrow \infty$? (This will become important for a later part of this module.)

We note that the original form of the integral is not very suitable since the supposedly large term zs becomes dominated by $s^3/3$ for large s, so we rewrite it using a change of variables that balances zs with $s^3/3$, i.e. $s = z^{1/2}t$,

$$\mathrm{Ai}(z) = \frac{z^{1/2}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{z^{3/2}\mathrm{i}(t+t^3/3)}\,\mathrm{d}t, \qquad h(t) = \mathrm{i}\left(t + \frac{1}{3}t^3\right). \tag{6.76}$$

Where will the dominant contribution come from? We note that the contour has no finite endpoints, so it must be one or more of the saddle points. We calculate

$$h'(t) = \mathrm{i}\left(1 + t^2\right) \quad \Rightarrow \quad t_* = \pm\mathrm{i}, \quad h''(t_*) = 2\mathrm{i}t_* = \mp 2. \tag{6.77}$$

The expansions tell us which the steepest-descent directions are

$$t_* = +\mathrm{i}: \quad h(t_*+s) \approx -\frac{2}{3} - s^2 \quad \Rightarrow \quad \text{S.D. } \arg s = 0,\, \pi \tag{6.78}$$

$$t_* = -\mathrm{i}: \quad h(t_*+s) \approx +\frac{2}{3} + s^2 \quad \Rightarrow \quad \text{S.D. } \arg s = \pi/2,\, -\pi/2. \tag{6.79}$$

To complete our picture of the $h(t)$ landscape, we note that for large $|t|$ we have

$$h(t) \sim \mathrm{i}\frac{t^3}{3} = \frac{|t|^3}{3}\mathrm{e}^{\mathrm{i}(\pi/2+3\arg t)} \quad \Rightarrow \quad \text{decay for} \begin{cases} 0 < \arg t < \pi/3, \\ 2\pi/3 < \arg t < \pi, \\ 4\pi/3 < \arg t < 5\pi/3. \end{cases} \tag{6.80}$$

Now we sketch the landscape and attempt to find an appropriate contour of integration (figure 6.6(a)). It is clear that the infinite endpoints can be bent upward to ensure exponential decay, and that the saddle at $+\mathrm{i}$ is suitable to pass through. (The saddle at $-\mathrm{i}$ has a larger value so would be exponentially dominant if we could pass through it, but any contour between the two endpoints that passes through that saddle will pass through a higher ridge somewhere else, which violates the condition that the maximum on the contour is attained at the saddle.)

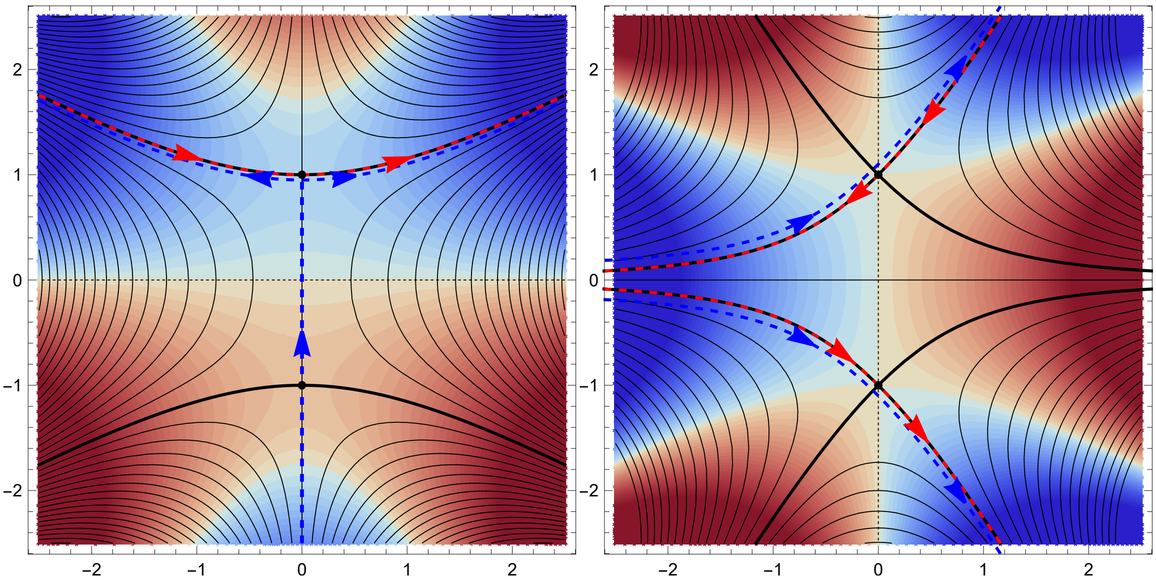


Figure 6.6: Steepest-descent contours for (a) $h(t) = \mathrm{i}(t + t^3/3)$ and (b) $h(t) = t + t^3/3$. Colours indicate the value of h_r (positive red, negative blue). The dashed red contour is the exact SD contour for $\mathrm{Ai}(z)$. The dashed blue contours are the exact SD contours for $\mathrm{Bi}(z)$.

We can then choose any contour that passes through the saddle horizontally, such as a local straight line $t = \mathrm{i} + s$ with s real, which yields

$$\mathrm{Ai}(z) \sim \frac{z^{1/2}}{2\pi}\int_{-\delta}^{\delta} \mathrm{e}^{z^{3/2}(-\frac{2}{3}-s^2)}\,\mathrm{d}s \sim \frac{z^{1/2}}{2\pi}\mathrm{e}^{-\frac{2}{3}z^{3/2}}\sqrt{\frac{\pi}{z^{3/2}}} = \frac{\mathrm{e}^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}, \qquad \text{as } z \nearrow \infty. \tag{6.81}$$

(In fact, since the local expansion just has one neglected term, namely the cubic term of $h(\mathrm{i}+s) = -\frac{2}{3} - s^2 + \mathrm{i}\frac{1}{3}s^3$ in the exponent, we can obtain a full asymptotic expansion by simply Taylor expanding $\exp(\mathrm{i}\frac{1}{3}s^3)$ and integrating.)

A second independent solution of the Airy equation is $\mathrm{Bi}(z)$, sometimes called the "Bairy" function, which can be defined as the same integral over two different contours, added together,

$$\mathrm{Bi}(z) = \frac{1}{2\pi\mathrm{i}}\left[\int_{-\mathrm{i}\infty}^{-\infty} + \int_{-\mathrm{i}\infty}^{\infty}\right]\mathrm{e}^{\mathrm{i}(zs+\frac{1}{3}s^3)}\,\mathrm{d}s = \frac{z^{1/2}}{2\pi\mathrm{i}}\left[\int_{-\mathrm{i}\infty}^{-\infty} + \int_{-\mathrm{i}\infty}^{\infty}\right]\mathrm{e}^{\mathrm{i}(zs+\frac{1}{3}s^3)}\,\mathrm{d}s. \tag{6.82}$$

We deform the contours to pass over the saddle at $-\mathrm{i}$, which is dominant, and then continue to the saddle at $+\mathrm{i}$ where they turn left or right and continue downhill to $\pm\infty$ (figure 6.6(a)). However, the contours don't have to follow this path exactly, as long as they pass over the saddle at $-\mathrm{i}$ in the correct direction, and do not visit anywhere with $\operatorname{Re} h > \operatorname{Re}(-\mathrm{i}) = 2/3$. Since the dominant part of the two integrals is the same, the result is, using $t = -\mathrm{i} + \mathrm{i}s$ with s real,

$$\operatorname{Bi}(z) \sim \frac{z^{1/2}}{2\pi \mathrm{i}} 2 \int_{-\delta}^{\delta} \mathrm{e}^{z^{3/2}(\frac{2}{3}+(\mathrm{i}s)^2)} \mathrm{i}\,\mathrm{d}s \sim \frac{z^{1/2}}{\pi} \mathrm{e}^{\frac{2}{3}z^{3/2}} \sqrt{\frac{\pi}{z^{3/2}}} = \frac{\mathrm{e}^{\frac{2}{3}z^{3/2}}}{\sqrt{\pi} z^{1/4}} \qquad \text{as } z \nearrow \infty. \tag{6.83}$$

What about $z \searrow -\infty$? Let's write $z = -r$, where $r \nearrow \infty$. Choosing the branch $z^{1/2} = \mathrm{i}r^{1/2}$ to use in the variable transformation $s = z^{1/2} t = \mathrm{i} r^{1/2} t$, we obtain

$$\operatorname{Ai}(z) = \frac{\mathrm{i}r^{1/2}}{2\pi} \int_{\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{e}^{r^{3/2}(t+t^3/3)}\,\mathrm{d}t, \quad h(t) = t + \frac{t^3}{3}, \quad h'(t) = 1 + t^2, \quad h''(t) = 2t. \tag{6.84}$$

We again find saddles at $t = \pm \mathrm{i}$, but their properties are different:

$$t_* = +\mathrm{i}: \quad h(t_* + s) \approx \frac{2}{3}\mathrm{i} + \mathrm{i}s^2 \qquad \Rightarrow \quad \text{S.D. } \arg s = \pi/4,\ -3\pi/4 \tag{6.85}$$

$$t_* = -\mathrm{i}: \quad h(t_* + s) \approx -\frac{2}{3}\mathrm{i} - \mathrm{i}s^2 \qquad \Rightarrow \quad \text{S.D. } \arg s = -\pi/4,\ 3\pi/4. \tag{6.86}$$

Also, the large-$|t|$ behaviour has rotated so that the positive real axis is growing while the negative real axis is decaying, and in fact the real axis is an SD contour. This is sufficient information to crudely sketch the SD contours (figure 6.6(b)) and determine that the appropriate full SD contour for Ai must come down from $\mathrm{e}^{\mathrm{i}\pi/3}\infty$ and cross i in the $-3\pi/4$ direction before heading to $-\infty$, then come back and cross $-\mathrm{i}$ in the $-\pi/4$ direction before heading to $\mathrm{e}^{-\mathrm{i}\pi/3}\infty$.

We conclude that

$$\operatorname{Ai}(z) \sim \frac{\mathrm{i}r^{1/2}}{2\pi} \int_{-\delta}^{\delta} \mathrm{e}^{r^{3/2}(\frac{2}{3}\mathrm{i}+\mathrm{i}(\mathrm{e}^{-\mathrm{i}3\pi/4}s)^2)} \mathrm{e}^{-\mathrm{i}3\pi/4} + \mathrm{e}^{r^{3/2}(-\frac{2}{3}\mathrm{i}-\mathrm{i}(\mathrm{e}^{-\mathrm{i}\pi/4}s)^2)} \mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{d}s \tag{6.87}$$

$$\sim \frac{\mathrm{i}}{2\sqrt{\pi} r^{1/4}} \left[-\mathrm{e}^{\frac{2}{3}\mathrm{i}r^{3/2}+\mathrm{i}\pi/4} + \mathrm{e}^{-\frac{2}{3}\mathrm{i}r^{3/2}-\mathrm{i}\pi/4} \right] = \frac{\sin\left(\frac{2}{3}r^{3/2} + \pi/4\right)}{\sqrt{\pi} r^{1/4}} \qquad \text{as } z = -r \searrow -\infty. \tag{6.88}$$

Similarly, for $\operatorname{Bi}(z)$ we obtain the sum of two contours that start at $-\infty$ and go to $\pm \mathrm{i}\infty$, so it is clear that each crosses one saddle $\pm \mathrm{i}$ in the direction $\pm\pi/4$ and continues towards $\pm \mathrm{e}^{\mathrm{i}\pi/3}\infty$ (figure 6.6(b)). Thus, the contribution from $-\mathrm{i}$ is the same as for Ai, while the contribution from $+\mathrm{i}$ is the opposite, due to the contour going in the opposite direction. We conclude that

$$\operatorname{Bi}(z) \sim \frac{1}{2\sqrt{\pi} r^{1/4}} \left[\mathrm{e}^{\frac{2}{3}\mathrm{i}r^{3/2}+\mathrm{i}\pi/4} + \mathrm{e}^{-\frac{2}{3}\mathrm{i}r^{3/2}-\mathrm{i}\pi/4} \right] = \frac{\cos\left(\frac{2}{3}r^{3/2} + \pi/4\right)}{\sqrt{\pi} r^{1/4}} \qquad \text{as } z = -r \searrow -\infty. \tag{6.89}$$

To summarise, we have

$$\operatorname{Ai}(z) \sim \frac{\mathrm{e}^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi} z^{1/4}}, \qquad \operatorname{Bi}(z) \sim \frac{\mathrm{e}^{\frac{2}{3}z^{3/2}}}{\sqrt{\pi} z^{1/4}} \qquad \text{as } z \nearrow \infty, \tag{6.90}$$

$$\operatorname{Ai}(z) \sim \frac{\sin\left(\frac{2}{3}(-z)^{3/2} + \pi/4\right)}{\sqrt{\pi}(-z)^{1/4}}, \quad \operatorname{Bi}(z) \sim \frac{\cos\left(\frac{2}{3}(-z)^{3/2} + \pi/4\right)}{\sqrt{\pi}(-z)^{1/4}} \qquad \text{as } z \searrow -\infty. \tag{6.91}$$

These are plotted together with the exact Ai and Bi in figure 6.7.

Remark(s) 6.8. Another method to obtain the full SD contour is to do the transformation $h(t) = h(t_*) - s$ where s is real and increases from zero, or $h(t) = h(t_*) - s^2$ where s is real and increases past zero, and solve for h. This is guaranteed to be a steepest-descent contour, but you still have to put in the effort of working out which point t_* is dominant, and which of the descending directions from t_* to use!

Example 6.12. Let's revisit an earlier integral but insert a pole,

$$I(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} \cos(4xt) \frac{1}{1+t^2}\,\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{e}^{x(-t^2+4\mathrm{i}t)} \frac{1}{1+t^2}\,\mathrm{d}t. \tag{6.92}$$

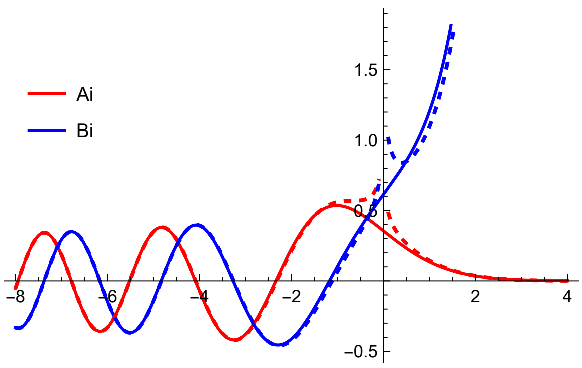


Figure 6.7: Plots of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ (solid curves) together with their asymptotic approximations as $z \to \pm\infty$ (dashed curves).

We still have a saddle at $t = 2\mathrm{i}$ with a horizontal full SD contour, but deforming up to that contour picks up the residue from the pole at $t = \mathrm{i}$,

$$2\pi\mathrm{i}\, \frac{\mathrm{e}^{x(-t^2+4\mathrm{i}t)}}{(t+\mathrm{i})}\bigg|_{t=\mathrm{i}} = \pi\mathrm{e}^{-3x}. \tag{6.93}$$

Here, we've use the residue formula: If $f(t)$ is analytic near t_* then the integral of $f(t)/(t-t_*)$ anti-clockwise around t_* is

$$\oint \frac{f(t)}{t-t_*}\,\mathrm{d}t \overset{t=t_*+\varepsilon\mathrm{e}^{\mathrm{i}\theta}}{=} \lim_{\varepsilon\searrow 0}\int_{\theta=0}^{2\pi} \frac{f(t_*+\varepsilon\mathrm{e}^{\mathrm{i}\theta})}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,\mathrm{d}\theta = \int_{\theta=0}^{2\pi} f(t_*)\mathrm{i}\,\mathrm{d}\theta = 2\pi\mathrm{i}f(t_*). \tag{6.94}$$

We conclude that

$$I(x) = \pi\mathrm{e}^{-3x} + \mathrm{e}^{-4x}\sqrt{\frac{\pi}{x}}\left[-\frac{1}{3} + O(x^{-1})\right]. \tag{6.95}$$

Note that the contribution from the pole is exponentially dominant, but it is exact (i.e. there is no asymptotic approximation/expansion made when deforming the contour and picking up the residue) so there are no $O(\mathrm{e}^{-3x}/x^{\alpha})$ errors that would hide the $\mathrm{ord}(\mathrm{e}^{-4x}/x^{1/2})$ contribution from the saddle point. Also note that deforming to any suitable SD contour (not just the full one) will require crossing the pole and picking up that residue.

On the other hand, the pole at $t_* = -\mathrm{i}$ does not contribute with its $\mathrm{ord}(\mathrm{e}^{+5x})$ residue, as we did not pass it while deforming, and indeed there is no suitable SD contour that would involve passing it during deformation, as any such contour will have its maximum $\operatorname{Re} h$ attained somewhere other than the saddle.

If the pole were instead located say at $t = 2+\mathrm{i}$, where $h(t) = -7+6\mathrm{i}$ has smaller real part than at the saddle, $h(2\mathrm{i}) = -4$, then deforming onto the full SD contour crosses the pole and picks up the residue while there are alternative choices of contour that do not. However, this makes no difference since the residue is exponentially smaller than the saddle contribution, so is beyond all orders in the asymptotic expansion.

Part III
Ordinary differential equations

7 Local analysis of ordinary differential equations

We seek to determine the local behaviour of the solutions to differential equations, either near a finite value of x which we can shift to be $x = 0$, or as $x \to \pm\infty$.

7.1 Linear equations

Example 7.1. Let's find expansions as $x \nearrow \infty$ for the solutions $y(x)$ of the equation $y'' - x^{k-2}y = 0$, for various values of k.

Suppose the leading-order behaviour of y is a power law,

$$y \sim y_0, \qquad y_0 = Cx^\alpha, \quad y_0' = C\alpha x^{\alpha-1}, \quad y_0'' = C\alpha(\alpha-1)x^{\alpha-2}, \qquad \text{where} \quad C \neq 0. \tag{7.1}$$

Substitution into the equation then yields

$$C\alpha(\alpha-1)x^{\alpha-2} - x^{k-2}Cx^\alpha \approx 0 \quad \Rightarrow \quad \alpha(\alpha-1) - x^k \approx 0. \tag{7.2}$$

We find a dominant balance between the two terms for the specific value $k = 0$. In this case, we obtain the equation

$$\alpha(\alpha-1) = 1 \quad \Rightarrow \quad \alpha = \frac{1 \pm \sqrt{5}}{2}. \tag{7.3}$$

For this simple example, these happen to be exact solutions of the equation, so we're done.

For $k = -m < 0$, the first term, representing y'', is always dominant. We thus obtain the equation

$$\alpha(\alpha-1) = 0 \quad \Rightarrow \quad \alpha = 0 \text{ or } \alpha = 1, \tag{7.4}$$

which is essentially the statement that if $y'' \approx 0$ then $y \approx Cx$ or $y \approx C$. Let's calculate the corrections to the smaller solution, $y \sim C$. Writing $y = C + y_1$ with $y_1 \ll 1$ yields

$$y_1'' - x^{-m-2}(C + y_1) = 0 \quad \Rightarrow \quad y_1'' \approx Cx^{-m-2} \quad \Rightarrow \quad y_1 \approx \frac{Cx^{-m}}{m(m+1)}. \tag{7.5}$$

(Note here that y_1 cannot have a constant or linear term since we assume that it is $\ll 1$.) We can continue to find the corrections y_2, y_3, etc. by repeatedly solving $y_{n+1}'' = x^{-m-2}y_n$, and find that

$$y \sim C\left[1 + \frac{x^{-m}}{m(m+1)} + \frac{x^{-2m}}{m(m+1)\,2m(2m+1)} + \frac{x^{-3m}}{m(m+1)\,2m(2m+1)\,3m(3m+1)} + \cdots\right]. \tag{7.6}$$

For the larger solution, $y \sim Dx$, we similarly obtain

$$y \sim Dx\left[1 + \frac{x^{-m}}{(m-1)m} + \frac{x^{-2m}}{(m-1)m\,(2m-1)2m} + \frac{x^{-3m}}{(m-1)m\,(2m-1)2m\,(3m-1)3m} + \cdots\right], \tag{7.7}$$

plus an arbitrary multiple of the smaller solution. (The latter expansion is different if $1/m$ is an integer, as eventually the solution will involve integrating $1/x$ and result in logarithmic terms.)

As a side note, this method for $k > 0$, in which we treat the derivative y'' as dominant so that the leading-order "balance" is $y'' = 0$, was essentially iterating the equation

$$y(x) = C + Dx + \int_{x_1=0}^{x}\int_{x_2=0}^{x_1} x_2^{k-2}y(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1. \tag{7.8}$$

(But beware – the iteration method does not work in general on differential equations, due to the possibility of homogeneous solutions.)

Finally, we consider $k > 0$. In this case, the $x^{k-2}y$ term becomes the sole dominant term, and we find that

$$x^{k-2}y \approx 0 \quad \Rightarrow \quad y \approx 0. \tag{7.9}$$

This is not very useful – we need the derivative term to remain in the dominant balance to get a non-trivial result. Let's take inspiration from the case $k = 2$:

$$y'' - y = 0 \quad \Rightarrow \quad y = C_\pm \mathrm{e}^{\pm x}. \tag{7.10}$$

Whereas differentiating a power law is like multiplying by $1/x$, differentiating an exponential results in multiplying by the derivative of the exponent. Hence, we might expect to be able to use exponentials to make the derivative term "larger than expected" and achieve a balance with the other term. The following method is known as one version of the **WKB (Wentzel–Kramers–Brillouin) method**, or as the **Liouville–Green (LG) method**.

We make a change of variables

$$y(x) = \mathrm{e}^{S(x)} \quad \Rightarrow \quad y' = S'\mathrm{e}^S, \quad y'' = (S'^2 + S'')\mathrm{e}^S \tag{7.11}$$

$$\Rightarrow \quad (S'^2 + S'')\mathrm{e}^S - x^{k-2}\mathrm{e}^S = 0 \quad \Rightarrow \quad S'^2 + S'' = x^{k-2}, \tag{7.12}$$

which converts the second-order homogeneous linear equation for y into a first-order inhomogeneous nonlinear equation for S'. The key point was for the derivative to yield a larger result than dividing by x, so we assume that $S' \gg 1/x$. Assuming S has a power-law behaviour so that $S'' = O(S'/x)$, it then follows that $S'^2 \gg S''$. Thus, the leading-order balance is

$$S \sim S_0, \quad S_0'^2 = x^{k-2} \quad \Rightarrow \quad S_0' = \pm x^{(k-2)/2}, \quad S_0 = \pm\frac{2}{k}x^{k/2}. \tag{7.13}$$

(Throughout this we ignore the constant of integration for S as it corresponds to a constant prefactor in y.) So is the leading-order solution given by $y \sim \exp(S_0)$? Only if the correction to S is $\ll 1$. Let's calculate it:

$$S' = S_0' + S_1' + \cdots \quad \Rightarrow \quad 2S_0'S_1' + S_1'^2 + S_0'' + S_1'' + \cdots = 0 \quad \Rightarrow \quad 2S_0'S_1' + S_0'' = 0 \tag{7.14}$$

$$\Rightarrow \quad S_1' = -\frac{S_0''}{2S_0'} = -\frac{k-2}{4x} \quad \Rightarrow \quad S_1 = -\frac{k-2}{4}\ln x. \tag{7.15}$$

This is $\gg 1$, so must be included in the leading-order approximation for y. The next correction, S_2, will have $S_2' \ll 1/x$ and hence $S_2 \ll 1$, so is not needed. We conclude that two (independent) solutions have the asymptotic behaviour

$$y \sim \exp\left(\pm\frac{2}{k}x^{k/2}\right) x^{-(k-2)/4}\left[1 + o(1)\right]. \tag{7.16}$$

A general solution would be a linear combination of these two, but since the + solution is exponentially dominant, any solution that doesn't decay will have an expansion given only by the + solution.

Note that $k = 3$ yields the Airy equation $y'' - xy = 0$, and we recover the behaviours

$$\mathrm{Ai}(x) \sim A\frac{\mathrm{e}^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}, \quad \mathrm{Bi}(z) \sim B\frac{\mathrm{e}^{+\frac{2}{3}x^{3/2}}}{x^{1/4}} \quad \text{as } x \nearrow \infty, \tag{7.17}$$

but with the constants A and B undetermined. A similar analysis yields the general form

$$C\frac{\mathrm{e}^{\frac{2}{3}\mathrm{i}(-x)^{3/2}}}{(-x)^{1/4}} + D\frac{\mathrm{e}^{-\frac{2}{3}\mathrm{i}(-x)^{3/2}}}{(-x)^{1/4}} \quad \text{as } x \searrow -\infty, \tag{7.18}$$

and suitable values of C and D yield the trigonometric results for $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ derived from their integral expressions.

Remark(s) 7.1. • Obtaining expansions near $x = 0$ (or other finite x) proceeds in the same way, except that the corrections are higher powers of x, and the WKB solution exponent S will start with a negative power of x.

- For equations with more complicated coefficients, we expand the coefficients.
- Overall, the rule of thumb is that once the equation is written in "equidimensional" form,

$$p_n(x)x^n y^{(n)} + p_{n-1}(x)x^{n-1}y^{(n-1)} + \cdots + p_0(x)y = 0, \tag{7.19}$$

the term(s) whose coefficient $p_k(x)$ is dominant in the relevant limit will have the dominant scale when $y \propto x^\alpha$. If it's a lone term, then we effectively have the dominant balance $y^{(k)} \approx 0$ and obtain k independent solutions starting from x^0, x^1, $\cdots$, x^{k-1}. If it's multiple terms, with the highest being of order k, then attempting x^α yields a kth order polynomial equation for α and hence k independent solutions, each starting from a different root.

If $k < n$, then the remaining $n - k$ solutions are found using the WKB method, by substituting $y = \mathrm{e}^S$ with $S \gg 1$, which yields $y^{(k)} \sim S'^k \mathrm{e}^S$ and hence effectively a polynomial equation for S' at leading order, which we solve by seeking dominant balances, at least one of which will involve S'^n. We always need to calculate the first correction, or more, to ensure that we have the correct leading-order behaviour of e^S.

- Various things can go wrong on the way, typically requiring the use of logarithms – we won't explore the possibilities here. (There is some theory, the Frobenius method, for the case when all coefficients p_k can be expanded using integer powers of x.)
- For inhomogeneous equations,

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = f(x), \tag{7.20}$$

we know that the general solution is given by homogeneous solutions, which we solve as above, plus a particular solution, which we try to find by similarly seeking a dominant balance that involves some terms on the left-hand side and $f(x)$ on the right-hand side. For example, if $f(x)$ expands to a power of x, then we guess that y is a power too and identify the dominant term on the left-hand side as before. If $f(x)$ has some exponential dependence on x, then we guess that y is some power times the same exponential.

7.2 Nonlinear equations

For nonlinear equations we simply try to guess the leading-order balance. Perhaps the dominant balance involves linear terms only, so we can apply the same methods as before. This includes the case where one lone term dominates, so we can write $y^{(k)} \approx 0$ and start y with some integer power less than k. Or we might find a dominant nonlinear balance between multiple terms for some power-law behaviour $y = Ax^\alpha$.

Once we've found a leading-order solution $y \approx y_0$, writing $y = y_0 + y_1$ and substituting into the equations, we should find that any nonlinear terms in y_1 are negligible, so the next dominant balance will be linear in y_1, and we can follow the strategies outlined above.

Typically, the number of undetermined constants should match the order of the differential equation.

Example 7.2. A small droplet spreading on a horizontal surface has an advancing contact line, near which its shape $y(x)$ is governed by the equation $(y^2 + y)y''' = -1$ where $x, y(x) \geq 0$, with $y(0) = 0$ and $y(+\infty) = +\infty$. What is the local behaviour of y near $x = +\infty$?

Since $y \gg 1$ we can neglect $y \ll y^2$ and obtain $y^2 y''' \approx -1$. Using intuition, we can see that $y''' \approx -1/y^2$ is $\ll 1$, so we might expect $y \sim Ax^2$. To check, we try a power law $y = Ax^\alpha$ and obtain

$$-1 \approx y^2 y''' = A^2 x^{2\alpha}\, A\alpha(\alpha-1)(\alpha-2)x^{\alpha-3} = A^3\alpha(\alpha-1)(\alpha-2)x^{3\alpha-3}. \tag{7.21}$$

For $\alpha > 1$, the right-hand side is dominant so we must have $\alpha(\alpha-1)(\alpha-2) = 0$, i.e. $\alpha = 2$. Writing $y = Ax^2 + y_1$ with $y_1 \ll x^2$, we find $y^2 + y \sim y^2 \sim A^2x^4$ so

$$y_1''' = -\frac{1}{(y^2 + y)} = -\frac{1}{A^2x^4}(1 + o(1)) \quad \Rightarrow \quad y_1 \sim Bx + C + \frac{1}{6A^2x} + o(x^{-1}). \tag{7.22}$$

This solution has 3 independent parameters, A (which must be positive to get $y \geq 0$), B and C, which might be expected from a third-order differential equation.

Let's now consider solutions that grow slower, in essence excluding the quadratic solution by enforcing $y'' \to 0$ as $x \nearrow \infty$. We return to the balance $-1 \approx A^3\alpha(\alpha-1)(\alpha-2)x^{3\alpha-3}$ where we previously considered a dominant right-hand side, and instead seek a dominant balance between the two sides. Equating the exponents x^0 and $x^{3\alpha-3}$ yields $\alpha = 1$, but we already know that the right-hand side vanishes for $\alpha = 1$, so it doesn't work. Another way to obtain the same contradiction is to insist on writing $y = Bx + y_1$ with $y_1 \ll x$ and find

$$y_1''' \sim \frac{1}{B^2x^2} \quad \Rightarrow \quad y_1'' \sim -\frac{1}{B^2x} \quad \Rightarrow \quad y_1' \sim -\frac{1}{B^2}\ln x \quad \Rightarrow \quad y_1 \sim -\frac{1}{B^2}x(\ln x - 1), \tag{7.23}$$

which contradicts $y_1 \ll x$.

The rule of thumb is that when a balance determines the exponent in $y = Bx^\alpha$ but the equation for the coefficient B ends up with B dropping out leaving a contradiction (e.g. $1 = B^3 \times 0$ above) is to try the same exponent but multiply by a logarithm to some power. Thus, we try

$$y \sim Bx^1(\ln x)^\beta \ \Rightarrow \ y' \sim B(\ln x)^\beta \ \Rightarrow \ y'' \sim B\beta(\ln x)^{\beta-1}/x \ \Rightarrow \ y''' \sim -B\beta(\ln x)^{\beta-1}/x^2. \tag{7.24}$$

Note here that we have only kept the largest term, and discarded terms that are smaller by only a logarithm (e.g. $(x^\alpha \ln x)' = \alpha x^{\alpha-1}\ln x + x^\alpha x^{-1} \sim \alpha x^{\alpha-1}\ln x$). This means that in every step we differentiate the power rather than the logarithm, except when the power is x^0. We then balance

$$-B^3\beta(\ln x)^{3\beta-1} = -1 \quad \Rightarrow \quad \beta = \frac{1}{3}, \quad B = \beta^{-1/3} = 3^{1/3}. \tag{7.25}$$

Thus, we have leading-order behaviour $y \sim x(3\ln x)^{1/3}$, but from the approximations we've made we can expect to get corrections that are only logarithmically ($1/\ln x$, rather than some $1/x^\alpha$) smaller.

It turns out that the next few terms are given by

$$y \sim x(3\ln x)^{1/3}\left[1 + \frac{C}{\ln x} - \frac{\frac{10}{27} + C^2}{(\ln x)^2} + O((\ln x)^{-3})\right], \tag{7.26}$$

but calculating them is not very instructive. This result turns out to be important for determining the evolution of the radius of the droplet using matched asymptotic expansions, and the leading-order result is fully determined, but to obtain the first corrections (which are only $O(1/\ln x)$ smaller) we need C which can only be determined numerically.

8 Differential equations with a small parameter

We now turn to solving differential equations with a small parameter ε (or a large parameter $1/\varepsilon$). One consequence of this is that we are no longer dealing with functions $f(\varepsilon)$ of the small parameter only, but instead functions $f(x, \varepsilon)$ depending on another variable x too. Asymptotic expansions of such functions are sometimes called **parameter expansions**.

Definition 8.1. The obvious generalisation of the definition of an asymptotic expansion to have a dependence on x is to allow the coefficients to depend on x,

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) \quad \Leftrightarrow \quad f(x,\varepsilon) - \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) = o(\delta_N(\varepsilon)) \tag{8.1}$$

as $\epsilon \searrow 0$. This is called a **Poincaré expansion** or a **classical expansion**. If the little-o is uniform for x in the relevant domain, i.e. for all $M > 0$ there exists $\varepsilon_* > 0$ independent of x such that

$$\left| f(x,\varepsilon) - \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) \right| \le M\delta_N(\varepsilon) \quad \text{for } 0 < \varepsilon < \varepsilon_*, \tag{8.2}$$

then we say that the expansion is **uniformly valid**. (For this module, the main point of this definition won't be to prove when things are uniformly valid, but just to point out when they are not.) If we additionally allow the coefficients to also depend on ε, then we have a **generalised asymptotic expansion**

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x,\varepsilon)\delta_n(\varepsilon) = \sum_{n=0}^{N} f_n(x,\varepsilon) \quad \Leftrightarrow \quad f(x,\varepsilon) - \sum_{n=0}^{N} f_n(x,\varepsilon) = o(f_N(x,\varepsilon)). \tag{8.3}$$

Remark(s) 8.1. For a differential equation, the parameter ε can appear in the coefficients of the equation, any forcing terms, and the boundary conditions. If the perturbation is regular in the sense that the solution has a power-series expansion, then we can write

$$y(x) \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \tag{8.4}$$

and substitute into the equation, then solve for $y_n(x)$ at each order. Since it is not at all obvious when this will be the case, a good start is to try it out and see if we can spot any problems.

Example 8.1. Consider the equation $y'' + \varepsilon y = 0$, $y(0) = 0$, $y(1) = \mathrm{e}^{\varepsilon}$.

We make the ansatz

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots \tag{8.5}$$

and substitute into the governing equations to find

$$(y_0'' + \varepsilon y_1'' + \cdots) + \varepsilon(y_0 + \varepsilon y_1 + \cdots) = 0, \tag{8.6}$$

$$y_0(0) + \varepsilon y_1(0) + \cdots = 0, \quad y_0(1) + \varepsilon y_1(1) + \cdots = 1 + \varepsilon + \cdots \tag{8.7}$$

Solving order by order then yields

$$y_0'' = 0, \quad y_0(0) = 0, \quad y_0(1) = 1 \quad \Rightarrow \quad y_0 = ax + b = x \tag{8.8}$$

$$y_1'' = -y_0 = -x, \quad y_1(0) = 0, \quad y_1(1) = 1 \quad \Rightarrow \quad y_1 = -\frac{x^3}{6} + cx + d = -\frac{x^3}{6} + \frac{7x}{6}. \tag{8.9}$$

etc.

So we claim that

$$y \sim x + \varepsilon\left[-\frac{x^3}{6} + \frac{7x}{6}\right], \tag{8.10}$$

and it doesn't look like there will be any problems, so this is probably a uniformly valid Poincaré expansion.

In the next few sections, we will consider different singular perturbation problems and the methods for solving them.

9 Matched asymptotic expansions

Example 9.1. Consider $\varepsilon y'' + y' + y = 0$ with boundary conditions $y(0) = -1$, $y(1) = \mathrm{e}^{-1}$, for $\varepsilon \searrow 0$.

The equations and boundary conditions just contain integer powers of ε, so let's try a classical power-series expansion

$$y \sim y_0 + \varepsilon y_1 + \cdots . \tag{9.1}$$

Solving order by order we obtain

$$O(\varepsilon^0): \quad y_0' + y_0 = 0 \qquad \Rightarrow \quad y_0 = C_0 \mathrm{e}^{-x}, \tag{9.2}$$

$$O(\varepsilon^1): \quad y_1' + y_1 = -y_0'' = -C_0 \mathrm{e}^{-x} \quad \Rightarrow \quad y_1 = -C_0 x \mathrm{e}^{-x} + C_1 \mathrm{e}^{-x}. \tag{9.3}$$

We expect the constants C_0 and C_1 to be determined by the boundary conditions, but there are two conditions at each order and they yield conflicting values. For example, at $\mathrm{ord}(\varepsilon^0)$ we have

$$-1 = y_0(0) = C_0 \Rightarrow C_0 = -1, \qquad \text{but} \qquad \mathrm{e}^{-1} = y_0(1) = C_0 \mathrm{e}^{-1} \qquad \Rightarrow C_0 = +1. \tag{9.4}$$

$$0 = y_1(0) = 0 \quad \Rightarrow C_1 = 0, \qquad \text{but} \qquad 0 = y_1(1) = -\mathrm{e}^{-1}(-C_0 + C_1) \Rightarrow C_1 = C_0. \tag{9.5}$$

The problem is that the small parameter multiplies the highest-derivative term in the equation, so the leading-order problem (and indeed the problem at any order) has lower derivatives than the original – this is a singular perturbation! As a result we can only satisfy one of the two boundary conditions.

It turns out that we should choose to satisfy the condition at $x = 1$, i.e. $C_0 = 1$ and $C_1 = 1$. (We'll discuss why below.) The resulting expansion

$$y(x) = \mathrm{e}^{-x} + \varepsilon(1 - x)\mathrm{e}^{-x} + O(\varepsilon^2) \tag{9.6}$$

will be valid for the **outer region** $x = \mathrm{ord}(1)$, or more specifically for any fixed x in the range $0 < x \leq 1$ as $\varepsilon \searrow 0$. We call this the **outer expansion**.

To satisfy the boundary condition at $x = 0$, we introduce a **boundary layer** near $x = 0$, i.e. a small region in which y varies rapidly so that $\varepsilon y''$ becomes significant. What scaling should it have? Let's set

$$y(x) = Y(X), \quad x = \varepsilon^\alpha X, \qquad \text{where } \alpha > 0 \text{ and } X = \mathrm{ord}(1). \tag{9.7}$$

and try to determine α. The derivatives then become larger,

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}X}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}X} = \frac{1}{\varepsilon^\alpha}\frac{\mathrm{d}}{\mathrm{d}X} \qquad \Rightarrow \qquad \varepsilon^{1-2\alpha} Y'' + \varepsilon^{-\alpha} Y' + Y = 0. \tag{9.8}$$

We now seek a leading-order balance between these terms (i.e. equate the exponents of ε), similar to how we solved polynomial equations. Balancing the Y' and Y terms leads to $-\alpha = 0$ and thus the outer scaling $\alpha = 0$. Balancing the Y'' and Y terms leads to $1 - 2\alpha = 0$ and hence $\alpha = 1/2$ which is inconsistent since the Y' term at $\mathrm{ord}(\varepsilon^{-1/2})$ will be larger. Balancing the Y'' and Y' terms yields the correct balance, $1 - 2\alpha = -\alpha$ i.e. $\alpha = 1$.

We have thus identified the scaling for the **inner region**, $x = \varepsilon X$. The governing equations become

$$\varepsilon^{-1} Y'' + \varepsilon^{-1} Y' + Y = 0 \quad \Rightarrow \quad Y'' + Y' = -\varepsilon Y, \qquad \text{with } Y(0) = -1. \tag{9.9}$$

We obviously do not try to impose the boundary condition at $x = 1$, which would be at $X = 1/\varepsilon$. We again try a classical power-series expansion, which we call the **inner expansion**, $Y = Y_0 + \varepsilon Y_1 + \cdots$, and obtain, to leading order,

$$Y_0'' + Y_0' = 0, \quad Y_0(0) = -1 \quad \Rightarrow \quad Y_0 = A_0 \mathrm{e}^{-X} + B_0 = A_0 \mathrm{e}^{-X} - A_0 - 1. \tag{9.10}$$

The unknown constant is determined by **matching** the inner and outer expansions. In this simple case, we can simply equate the limiting values

$$\lim_{X \nearrow \infty} Y_0(X) = \lim_{x \searrow 0} y_0(x) \quad \Rightarrow \quad -A_0 - 1 = 1 \quad \Rightarrow \quad A_0 = -2, \quad Y_0 = 1 - 2\mathrm{e}^{-X}. \tag{9.11}$$

Let's now continue to next order in the inner expansion,

$$Y_1'' + Y_1' = -Y_0 = -1 + 2\mathrm{e}^{-X}, \quad Y_1(0) = 0 \tag{9.12}$$

$$\Rightarrow \quad Y_1 = -X - 2X\mathrm{e}^{-X} + A_1\mathrm{e}^{-X} + B_1 = -X - 2X\mathrm{e}^{-X} + A_1(\mathrm{e}^{-X} - 1). \tag{9.13}$$

To determine A_1 by matching, we need to do something more sophisticated.

There are two methods used for asymptotic matching. The straightforward method is to use **Van Dyke's matching principle**, which states that taking the outer expansion to $\mathrm{ord}(\varepsilon^P)$, substituting in the inner variable and re-expanding to $\mathrm{ord}(\varepsilon^Q)$ should yield the same result as taking the inner expansion to $\mathrm{ord}(\varepsilon^Q)$, substituting in the outer variable and re-expanding to $\mathrm{ord}(\varepsilon^P)$. Taking $P = Q = 1$, and introducing the non-standard notation O_x and O_X (indicating which variable is held fixed in the limit $\varepsilon \searrow 0$), we obtain

$$\begin{aligned}
y(\varepsilon X) &= \left[\mathrm{e}^{-\varepsilon X}\right] + \varepsilon\left[(1-\varepsilon X)\mathrm{e}^{-\varepsilon X}\right] + O_x(\varepsilon^2) && (9.14)\\
&= \left[1 - \varepsilon X + O_X(\varepsilon^2)\right] + \varepsilon\left[1 + O_X(\varepsilon)\right] + O_x(\varepsilon^2) && (9.15)\\
&= 1 - \varepsilon X + \varepsilon + O_X(\varepsilon^2) + O_x(\varepsilon^2) && (9.16)\\
Y(x/\varepsilon) &= \left[1 - 2\mathrm{e}^{-x/\varepsilon}\right] + \varepsilon\left[-(x/\varepsilon) - 2(x/\varepsilon)\mathrm{e}^{-x/\varepsilon} + A_1(\mathrm{e}^{-x/\varepsilon} - 1)\right] + O_X(\varepsilon^2). && (9.17)\\
&= \left[1 + O_x(\varepsilon^2)\right] + \varepsilon\left[-x/\varepsilon + O_x(\varepsilon) - A_1 + O_x(\varepsilon)\right] + O_X(\varepsilon^2) && (9.18)\\
&= 1 - x - \varepsilon A_1 + O_x(\varepsilon^2) + O_X(\varepsilon^2). && (9.19)
\end{aligned}$$

(In fact, in the inner expansion the errors are exponentially smaller, but $O_x(\varepsilon^2)$ is sufficient.) Comparing these results, we find that they agree provided that $A_1 = -1$.

We have now determined both the outer and inner expansion to $\mathrm{ord}(\varepsilon)$,

$$y(x) = \mathrm{e}^{-x} + \varepsilon(1-x)\mathrm{e}^{-x} + O_x(\varepsilon^2), \tag{9.20}$$

$$Y(X) = \left[1 - 2\mathrm{e}^{-X}\right] + \varepsilon\left[1 - X - (1+2X)\mathrm{e}^{-X}\right] + O_X(\varepsilon^2), \tag{9.21}$$

which together describe the behaviour of $y(x)$ as $\varepsilon \searrow 0$.

We can form an **(additive) composite approximation** by adding the inner and outer approximations (expressed in terms of x) and subtracting their common overlap behaviour found above,

$$\begin{aligned}
y(x) &\approx \left\{\mathrm{e}^{-x} + \varepsilon(1-x)\mathrm{e}^{-x}\right\} + \left\{1 - 2\mathrm{e}^{-x/\varepsilon} - x + \varepsilon - (2x+\varepsilon)\mathrm{e}^{-x/\varepsilon}\right\} - \{1 - x + \varepsilon\} && (9.22)\\
&= \mathrm{e}^{-x}\left[1 + \varepsilon(1-x)\right] + \mathrm{e}^{-x/\varepsilon}\left[-2 - (2x+\varepsilon)\right]. && (9.23)
\end{aligned}$$

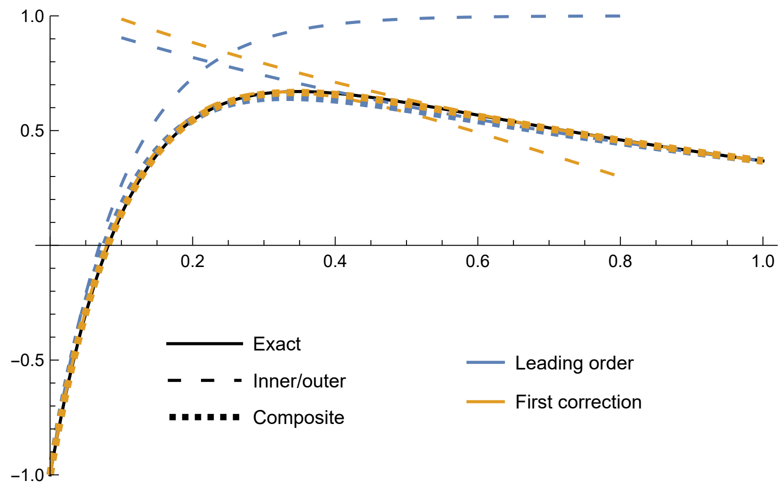


Figure 9.1: Solution of $\varepsilon y'' + y' + y = 0$, $y(0) = -1$, $y(1) = \mathrm{e}^{-1}$ for $\varepsilon = 0.1$.

Remark(s) 9.1. • Why did the boundary layer have to be at $x = 0$? If we try a boundary layer at $x = 1$, then we'll find $x = 1+\varepsilon X$, but the inner expansions will contain e^{-X} which grow exponentially in the matching limit $X \searrow -\infty$ and hence cannot match to the outer. In general, there can be one or more boundary layers, located at the endpoints of the interval as well as in the interior (also known as **interior layers**), and nested within each other.

- In this particular example, the outer solution was fully determined by the boundary conditions, and we only needed matching to determine the unknown constants of the inner solution. In general, it can also be the other way around, or both regions can have unknown constants.

- The Van Dyke matching principle can also be applied to more exotic scale functions, but it does not always work. A more robust method for matching is to use an **intermediate variable** $\xi = x/\varepsilon^\beta = \varepsilon^{1-\beta}X$ for some fixed β in the range $0 < \beta < 1$. We substitute this into the expansions and re-expand each term:

$$y(\mathrm{e}^\beta\xi) = \left[\mathrm{e}^{-\varepsilon^\beta\xi}\right] + \varepsilon\left[(1-\varepsilon^\beta\xi)\mathrm{e}^{-\varepsilon^\beta\xi}\right] + O_x(\varepsilon^2) = \tag{9.24}$$

$$= \left[1 - \varepsilon^\beta\xi + \varepsilon^{2\beta}\frac{\xi^2}{2} + O_\xi(\varepsilon^{3\beta})\right] + \left[\varepsilon + O_\xi(\varepsilon^{1+\beta})\right] + O_x(\varepsilon^2), \tag{9.25}$$

$$Y(\xi/\varepsilon^{1-\beta}) = \left[1 - 2\mathrm{e}^{-\xi/\varepsilon^{1-\beta}}\right] + \varepsilon\left[-\frac{\xi}{\varepsilon^{1-\beta}} - 2\frac{\xi}{\varepsilon^{1-\beta}}\mathrm{e}^{-x/\varepsilon^{1-\beta}} + A_1(\mathrm{e}^{-x/\varepsilon^{1-\beta}} - 1)\right] + O_X(\varepsilon^2) = \tag{9.26}$$

$$= [1 + \mathrm{EST}] + \left[-\varepsilon^\beta\xi + \mathrm{EST} - \varepsilon A_1 + \mathrm{EST}\right] + O_X(\varepsilon^2). \tag{9.27}$$

The two results agree provided that $A_1 = -1$ and that the $\varepsilon^{2\beta}$ term in the outer is small compared with the $\mathrm{ord}(\varepsilon)$ terms we kept, so we need to choose $1/2 < \beta < 1$. (This term will match a term coming from the next term in the inner expansion.)

- The outer scaling $x = \mathrm{ord}(1)$ and inner scaling $x = \mathrm{ord}(\varepsilon)$ are **distinguished limits**, in which at least two terms of the governing equation balance. It's possible to solve the governing equations on any scale $x = \mathrm{ord}(\varepsilon^\alpha)$, but the result will be the same as taking the limit of either the outer or the inner expansion (or both).

- The key principle for matching is that the outer and inner expansions represent the same function, and the assumption is that both expansions are valid, in an **overlap region** on an intermediate scale. If the regions of validity do not overlap, this is likely due to an overlooked distinguished limit on an intermediate scale, which then matches correctly to both the smaller and the larger scales.

Example 9.2. Consider

$$(x+\varepsilon)y''+y'=0, \quad y(0)=1, \quad y(1)=2, \qquad \varepsilon \searrow 0. \tag{9.28}$$

(This is easily solved exactly, and in fact the inner problem turns out to be equivalent to the full problem, but the asymptotic calculation is instructive.)

In the outer region, we expand $y = y_0 + \varepsilon y_1 + \cdots$ and obtain

$$xy_0''+y_0'=0 \quad \Rightarrow \quad xy_0' = A_0 \quad \Rightarrow \quad y_0 = A_0 \ln x + B_0. \tag{9.29}$$

This time we have two unknown constants, but the solution has a singularity at $x=0$ so imposing $y(0)=0$ would require $A_0 = B_0 = 0$, which is incompatible with the other condition $y(1) = B_0 = 2$. Again it turns out that the boundary layer must be at $x=0$, so we impose the condition at $x=1$ and obtain $y_0 = A_0 \ln x + 2$.

The boundary layer again has the scaling $x = \varepsilon X$, so

$$(X+1)Y''+Y'=0, \quad Y(0)=1. \tag{9.30}$$

Expanding $Y = Y_0 + \varepsilon Y_1 + \cdots$ yields

$$(X+1)Y_0''+Y_0'=0, \quad Y_0(0)=1 \quad \Rightarrow \quad (X+1)Y_0' = C_0 \tag{9.31}$$

$$\Rightarrow \quad Y_0 = C_0 \ln(1+X) + D_0 = C_0\ln(1+X)+1. \tag{9.32}$$

We now try to match the outer and inner solutions using Van Dyke's matching principle:

$$y(\varepsilon X) = A_0 \ln(\varepsilon X) + 2 + O_x(\varepsilon) = A_0 \ln\varepsilon + A_0 \ln X + 2 + O_x(\varepsilon), \tag{9.33}$$

$$= A_0 \ln x + 2 + O_x(\varepsilon), \tag{9.34}$$

$$Y(x/\varepsilon) = C_0 \ln(1+x/\varepsilon) + 1 + O_X(\varepsilon) = 1 + C_0\left[\ln\frac{x}{\varepsilon} + \ln\left(\frac{\varepsilon}{x}+1\right)\right] + O_X(\varepsilon) \tag{9.35}$$

$$= 1 + C_0 \ln x - C_0 \ln\varepsilon + O_x(\varepsilon) + O_X(\varepsilon). \tag{9.36}$$

Here, we have rewritten the first result in terms of x again to facilitate the comparison of the two results. In order for the $\ln x$ terms to match, we need $A_0 = C_0$, but we are left trying to match 2 with $1 - C_0 \ln\varepsilon$.

So we wish to effectively have $-C_0 \ln\varepsilon = 1$, i.e. $A_0 = C_0 = -1/\ln(\varepsilon)$, which is strictly speaking achieved by setting $C_0 = 0$ but introducing new terms in the inner and outer that are $\operatorname{ord}(1/\ln\varepsilon)$. Thus, we obtain the results

$$y(x) = 2 - \frac{1}{\ln\varepsilon}\ln x + O_x(\varepsilon), \qquad Y(X) = 1 - \frac{1}{\ln\varepsilon}\ln(1+X) + O_X(\varepsilon). \tag{9.37}$$

Remark(s) 9.2. • Logarithmic terms are special because they unexpectedly jump order when the variables are rescaled, e.g.

$$\ln(x) = \operatorname{ord}(1) \quad \text{but} \quad \ln(\varepsilon X) = \ln\varepsilon + \ln X = \operatorname{ord}(\ln\varepsilon). \tag{9.38}$$

This has two important consequences:

- The matching process may requires us to introduce terms at an unexpected order, in the sense that the governing equations have no forcing terms at that order. This phenomenon is sometimes called **switchbacking**. In the example above, the governing equations imply corrections at $\operatorname{ord}(\varepsilon)$, but we had to introduce $\operatorname{ord}(1/\ln\varepsilon)$ corrections (which are thus necessarily homogeneous solutions of the equations).
- When matching expansions involving logarithms, **you must not cut between logarithmic orders**. In this example, we had both outer and inner terms of $\operatorname{ord}(1)$ and $\operatorname{ord}(1/\ln\varepsilon)$. If we try to keep only the $\operatorname{ord}(1)$ terms in the matching and discard the $\operatorname{ord}(1/\ln\varepsilon)$ terms then we obtain

$$y(x) = 2 + O_x(1/\ln\varepsilon), \qquad Y(X) = 1 + O_X(1/\ln\varepsilon), \tag{9.39}$$

and after trivially substituting $x = \varepsilon X$ or $X = x/\varepsilon$ and re-expanding we are still left with the constants 2 and 1 that don't match – we need the $\ln x/\ln\varepsilon$ term to jump order and contribute at $\operatorname{ord}(1)$. Thus, when matching for example

$$[\operatorname{ord}(\ln\varepsilon) + \operatorname{ord}(1)] + \varepsilon\left[\operatorname{ord}((\ln\varepsilon)^2) + \operatorname{ord}(\ln\varepsilon) + \operatorname{ord}(1)\right] + \varepsilon^2[\cdots] + \cdots, \tag{9.40}$$

you should only cut between the algebraic orders and thus either keep or neglect all terms within the same square bracket.

Example 9.3. Consider

$$\varepsilon e^{x-1}y'' + (x-1)y' - y = 0, \quad y(0) = 2, \quad y(2) = 1, \qquad \varepsilon \searrow 0. \tag{9.41}$$

The leading-order outer equation yields

$$(x-1)y_0' - y_0 = 0 \quad \Rightarrow \quad y_0 = C(x-1), \tag{9.42}$$

$$y_0(0) = 2 \ \Rightarrow\ C = -2, \quad \text{but} \quad y_0(2) = 1 \quad \Rightarrow \quad C = 1. \tag{9.43}$$

It seems we need a boundary layer somewhere, but where?

In a boundary layer at any x_0, we would expect $y' \gg y$, so might expect a balance between the y'' and y' terms, with $x - x_0 = \varepsilon X$. The boundary-layer equations would then be

$$e^{x_0-1+\varepsilon X}Y'' + (x_0 - 1 + \varepsilon X)Y' - \varepsilon Y = 0 \tag{9.44}$$

$$\Rightarrow \quad e^{x_0-1}Y_0'' + (x_0-1)Y_0' = 0 \quad \Rightarrow \quad Y_0 = A_0 + B_0 \exp\left[\frac{1-x_0}{e^{x_0-1}}X\right]. \tag{9.45}$$

Thus, for $x_0 < 1$ the exponential term blows up as $X \to \infty$ and cannot match to an outer in $x > x_0$, and for $x_0 > 1$ it blows up as $X \to -\infty$ and cannot match to an outer in $x < x_0$. (So the former could work if the right-hand boundary of the domain were at $x_0 < 1$ instead, and the latter could work if the left-hand boundary were at $x_0 > 1$.)

The only remaining possibility is $x_0 = 1$, so the boundary layer is more accurately called an **interior layer**. At $x = 1$ the coefficient of y' vanishes, so the scaling will be different. Hence we try $x - 1 = \varepsilon^\alpha X$ and find

$$\varepsilon^{1-2\alpha}e^{\varepsilon^\alpha X}Y'' + XY' - Y = 0, \tag{9.46}$$

from which we deduce (since the exponential is ord(1)) that $\alpha = 1/2$ and we have a full balance between the three terms,

$$(1 + \varepsilon^{1/2}X + \varepsilon X^2/2 + \cdots)Y'' + XY' - Y = 0, \tag{9.47}$$

but the inner problem is still simpler than the full problem due to the e^{x-1} being Taylor expanded.

Since we don't have a boundary condition to impose on Y, it is fully determined by matching to the outer solutions, so let's look at their leading-order matching behaviour. In this case, the outer solutions both vanish linearly as $x \to 1$, so that

$$X < 0: \quad y_0(1 + \varepsilon^{1/2}X) \sim -2\varepsilon^{1/2}X, \qquad X > 0: \quad y_0(1 + \varepsilon^{1/2}X) \sim \varepsilon^{1/2}X. \tag{9.48}$$

According to Van Dyke matching, these should be the leading-order behaviours of $Y(X)$ as $X \to \pm\infty$. This suggests that $Y = \text{ord}(\varepsilon^{1/2})$, and so we write $Y \sim \varepsilon^{1/2}Y_{1/2}(X)$ with

$$Y_{1/2}'' + XY_{1/2}' - Y_{1/2} = 0, \qquad Y_{1/2} \sim -2X \text{ as } X \searrow -\infty, \quad Y_{1/2} \sim X \text{ as } X \nearrow \infty. \tag{9.49}$$

This leading-order inner problem has a unique solution, which is plotted in figure 9.2(a).

We have already performed the matching to obtain the two far-field conditions on $Y_{1/2}$ as $X \to \pm\infty$, but we could double-check using Van Dyke matching that, for example, matching to the left side, the two results agree at leading order:

$$y(x) = -2(x-1) + o_x(1) = -2\varepsilon^{1/2}X + o_x(1) = -2\varepsilon^{1/2}X + o_X(\varepsilon^{1/2}) + o_x(1), \tag{9.50}$$

$$Y(X) = \varepsilon^{1/2}Y_{1/2}(X) + o_X(\varepsilon^{1/2}) = \varepsilon^{1/2}(-2(x-1)/\varepsilon^{1/2} + o_(\varepsilon^{-1/2})) + o_X(\varepsilon^{1/2}) = \tag{9.51}$$

$$= -2(x-1) + o_x(1) + o_X(\varepsilon^{1/2}). \tag{9.52}$$

Since the leading-order outer solution y_0 is equal to the far-field expansion of the inner solution and hence equal to the overlap behaviour, the additive composite is simply the inner solution.

We note that even if we need to solve the inner equation numerically it's not a complete failure, as we still end up with a universal numerical solution that applies to all small ε, as opposed to an exact numerical

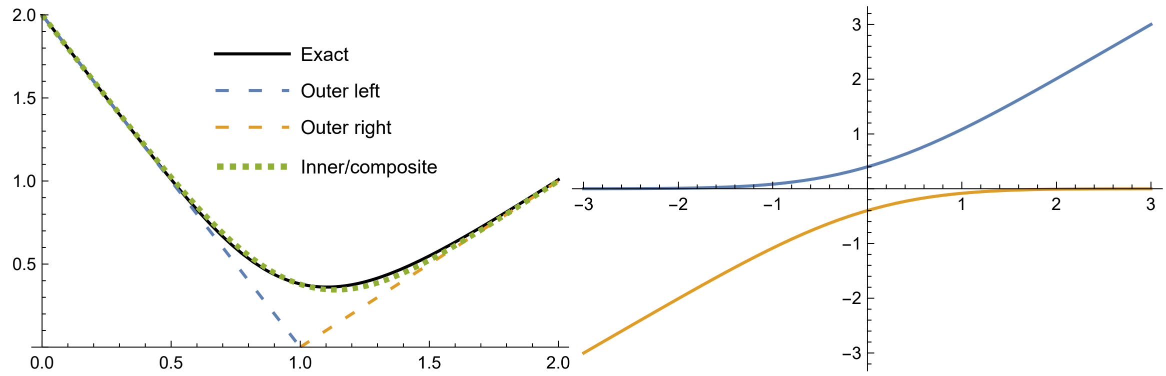


Figure 9.2: (a) Solution of $\varepsilon e^{x-1}y'' + (x-1)y' - y = 0$, $y(0) = 2$, $y(2) = 1$ for $\varepsilon = 0.1$. (b) Two independent solutions of the leading-order inner equation.

solution that would have to be recomputed for every value of ε, and we have also gained insight into the boundary-layer structure of the problem.

It turns out that the inner problem in this case happens to have an analytical solution, which can be found by observing that $Y_{1/2} = X$ is a solution and then solving for $Z(X) = Y_{1/2}/X$:

$$Y_{1/2} = X\,Z(X) \quad\Rightarrow\quad (XZ'' + 2Z') + X(XZ' + Z) - XZ = 0 \quad\Rightarrow\quad XZ'' + (X^2+2)Z' = 0 \tag{9.53}$$

$$\Rightarrow\quad Z' = \exp\left(-\int \frac{X^2+2}{X}\,\mathrm{d}X\right) = \exp\left(-X^2/2 - 2\ln X + C\right) = AX^{-2}e^{-X^2/2} \tag{9.54}$$

$$\Rightarrow\quad Y_{1/2} = XZ = X\int AX^{-2}e^{-X^2/2}\,\mathrm{d}X = AX\left[-X^{-1}e^{-X^2/2} - \int e^{-X^2/2}\,\mathrm{d}X\right] = \tag{9.55}$$

$$= -A\left[e^{-X^2/2} + X\sqrt{\frac{\pi}{2}}\,\mathrm{erf}(X/\sqrt{2})\right] + BX. \tag{9.56}$$

The result can be written in the form

$$Y_{1/2} = \alpha_+\left[X\frac{1+\mathrm{erf}(X/\sqrt{2})}{2} + \frac{e^{-X^2/2}}{\sqrt{2\pi}}\right] + \alpha_-\left[X\frac{1-\mathrm{erf}(X/\sqrt{2})}{2} - \frac{e^{-X^2/2}}{\sqrt{2\pi}}\right], \tag{9.57}$$

with $\alpha_+ = \lim_{X\nearrow\infty} Y'_{1/2} = 1$, $\alpha_- = \lim_{X\searrow-\infty} Y'_{1/2} = -2$. Each square bracket is a solution that has zero slope at one end $X \to \pm\infty$ and unit slope at the other end $X \to \mp\infty$, as plotted in figure 9.2(b).

Remark(s) 9.3. In this example the outer solution turned out to be continuous across the boundary layer while its derivative was discontinuous and hence changed rapidly in the boundary layer, which is consequently also called a **derivative layer**. In particular, this meant that at leading order the inner solution was simply equal to the value of the outer solution at the boundary layer (in this case zero), while the change in derivative occurs at higher order, $\mathrm{ord}(\varepsilon^{1/2})$. When boundary conditions are imposed on the derivative of the function, e.g. $y'(0) = 1$, then the resulting boundary/derivative layer similarly only has a higher-order inner correction.

Example 9.4. We finish this section with a famous nonlinear example, the Van der Pol oscillator. The full analysis is somewhat tedious, so we will only cover some highlights here, and even less in the lecture. (See e.g. Hinch for more details.) Consider the following equation for $x(t)$:

$$x'' = -\mu(x^2-1)x' - x, \qquad \mu > 0. \tag{9.58}$$

The idea is that we have a standard harmonic oscillator $x'' = -x$, such as a particle attached to a spring that provides a restoring force towards equilibrium, but with a nonlinear friction term. For $|x| > 1$, the friction term has the opposite sign to x' so provides a force that slows down the particle as usual, but for $|x| < 1$ the friction term has the same sign as x' so provides a force in the same direction as the particle is already moving. This means in particular that the trivial solution $x = 0$ is unstable, and instead the particle settles into a unique periodic oscillation (a "limit cycle") for each value of $\mu > 0$, that crosses between $|x| < 1$ where the friction is putting energy into the system and $|x| > 1$ where the friction is drawing energy out of the system.

We consider the limit $\mu = \varepsilon^{-1} \nearrow \infty$, and start by rescaling $t \to t/\varepsilon$ to obtain an ord(1) balance between the friction and the restoring force on the right-hand side,

$$\varepsilon^2 x'' = -(x^2-1)x' - x. \tag{9.59}$$

A numerical solution is plotted in figure 9.3(a), and we can see that the limit cycle consists of alternating slow "relaxation" phases (outer solutions) where x decreases from ± 2 to ± 1 on an ord(1) timescale and fast phases (interior layers) where x almost instantaneously jumps from ± 1 to ∓ 2. Let's try to describe these oscillations using the method of matched asymptotic expansions.

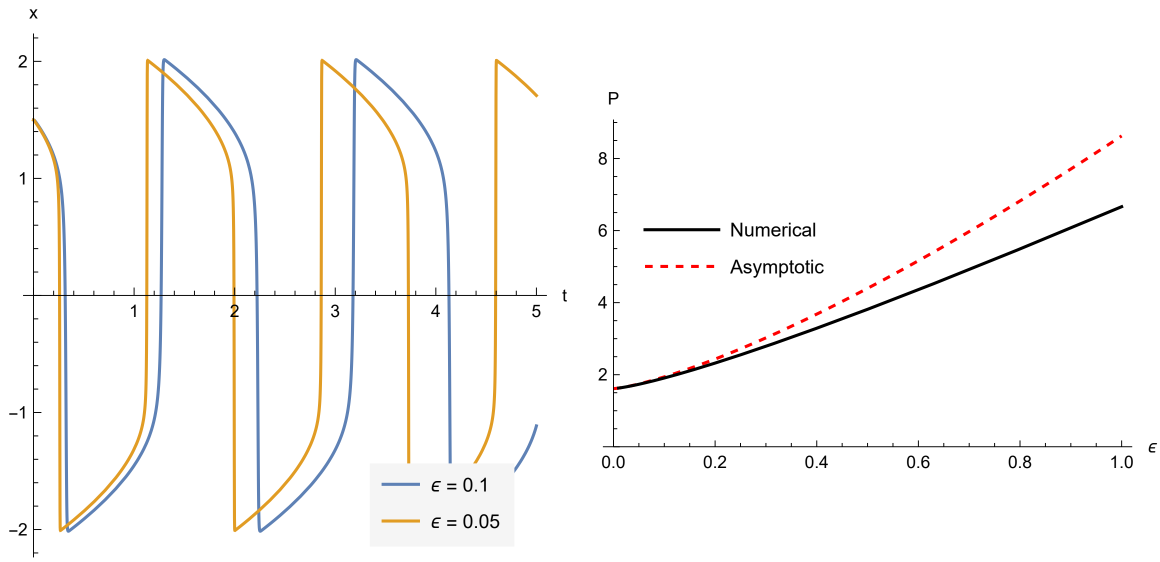


Figure 9.3: Numerical results for the Van der Pol oscillator. (a) Evolution of $x(t)$. (b) Dependence of (rescaled) period on ε.

Let's start with a positive branch of the outer solution. The leading-order outer equation can be solved using separation of variables,

$$(x_0^2-1)x_0' + x_0 = 0 \quad \Rightarrow \quad (x_0 - x_0^{-1})x_0' = -1 \quad \Rightarrow \quad \frac{x_0^2}{2} - \ln x_0 = t_0 - t, \tag{9.60}$$

where the constant of integration t_0 is irrelevant. It's possible to express x_0 explicitly in terms of t using the Lambert W-function, but we won't need it.

Where is the boundary layer? The coefficient of x_0' vanishes at $x_0 = 1$ so that's a good guess. Looking more closely, the left-hand side $\frac{1}{2}x_0^2 - \ln x_0$ is minimal for $x_0 = 1$, taking the value $1/2$, so the outer solution ceases to exist as $t \nearrow t_0 - 1/2$ and $x_0 \nearrow 1$, which is thus where we expect to find a boundary layer. We choose $t_0 = 1/2$ for convenience and further expand near this point to find

$$\tfrac{1}{2} + (x_0-1)^2 + O((x_0-1)^3) = \tfrac{1}{2} - t \quad \Rightarrow \quad x_0 = 1 + (-t)^{1/2} + O(t) \text{ as } t \nearrow 0. \tag{9.61}$$

What is the scaling for the boundary layer? Since the problem is nonlinear the scaling for x matters as well as the scaling for t. Additionally, what matters is the changes in each variable, and not their absolute values. The key is to **use the local expansion of the outer solution** to link the scales for Δx and Δt, so that we still only have one unknown scale to solve for. In our case, the outer solution comes in with a square-root behaviour, so $\Delta x = \mathrm{ord}(\Delta t^{1/2})$ and we write

$$t = \varepsilon^\alpha T, \quad x = 1 + \varepsilon^{\alpha/2} X \quad \Rightarrow \quad x^2 - 1 = (x-1)(x+1) = \varepsilon^{\alpha/2} X(2 + \varepsilon^{\alpha/2} X) \tag{9.62}$$

$$\Rightarrow \quad \varepsilon^{2+\alpha/2-2\alpha} X'' = -\varepsilon^{\alpha/2+\alpha/2-\alpha} X(2 + \varepsilon^{\alpha/2} X) X' - 1 - \varepsilon^{\alpha/2} X. \tag{9.63}$$

We see that the square-root coupling between the scales of x and t from the outer solution has resulted in the two right-hand side terms remaining ord(1), and so the inner distinguished scaling is when all three terms are in balance, i.e. $2 + \alpha/2 - 2\alpha = 0$. We thus obtain

$$\alpha = 4/3, \quad t = \varepsilon^{4/3} T, \quad x = 1 + \varepsilon^{2/3} X, \quad X'' = -(2 + \varepsilon^{2/3} X) X X' - 1 - \varepsilon^{2/3} X. \tag{9.64}$$

(An alternative way of finding the scaling is to simply ask when the neglected term $\varepsilon^2 x_0'' = \mathrm{ord}(\varepsilon^2 (-t)^{-3/2})$ becomes comparable to the kept terms $x_0 = \mathrm{ord}(1)$ and $(x_0^2 - 1)x_0' = \mathrm{ord}((-t)^{1/2}(-t)^{-1/2}) = \mathrm{ord}(1)$, which yields $(-t) = \mathrm{ord}(\varepsilon^{4/3})$.)

The leading-order boundary layer equations are then

$$X_0'' = -2X_0 X_0' - 1 \quad \Rightarrow \quad X_0' = -X_0^2 - T + T_0, \tag{9.65}$$

where T_0 is a constant of integration. This equation can be transformed into the Airy equation using $X_0(T) = Z'(T)/Z(T)$, but we shall simply proceed by local analysis. As $T \searrow -\infty$ it turns out that the dominant balance is $-X_0^2 - T \approx 0 \Rightarrow X_0 \sim (-T)^{1/2}$ which matches the outer solution. As T increases, the right-hand side must become negative eventually, and then X_0 becomes increasingly negative and decreases increasingly faster until we obtain a dominant balance

$$X_0' \approx -X_0^2 \quad \Rightarrow \quad \frac{X_0'}{X_0^2} \sim -1 \quad \Rightarrow \quad -\frac{1}{X} \sim T_* - T \quad \Rightarrow \quad X \sim -\frac{1}{T_* - T}. \tag{9.66}$$

Hence, the solution blows up negatively at a finite T, indicating that there is a further nested boundary layer, and this $\varepsilon^{4/3}$ layer was just an intermediate transition layer.

We again use the limiting behaviour of the current layer, $X = \mathrm{ord}((T_* - T)^{-1})$, to inform the scaling of the next layer,

$$t = \varepsilon^{4/3} T = \varepsilon^{4/3}(T_* + \varepsilon^\beta)\tau, \quad x = 1 + \varepsilon^{2/3} X = 1 + \varepsilon^{2/3-\beta}\xi \tag{9.67}$$

$$\Rightarrow \quad \varepsilon^{-3\beta}\xi'' = -\varepsilon^{-3\beta}(2 + \varepsilon^{2/3-\beta}\xi)\xi\xi' - 1 - \varepsilon^{2/3-\beta}\xi \tag{9.68}$$

We see that the ξ'' and ξ' terms remain in balance, provided $2/3 - \beta \geq 0$, so seeking a distinguished limit we choose to increase $\beta \geq 0$ until more terms come into balance, which occurs at $\beta = 2/3$, when the rescalings from x to X and from X to ξ cancel out so that $x = 1 + \xi$. Choosing to define ξ without the shift, we obtain

$$t = \varepsilon^{4/3} T = \varepsilon^{4/3} T_* + \varepsilon^2 \tau, \quad x = \xi, \quad \xi'' = -(\xi^2 - 1)\xi' - \varepsilon^2 \xi. \tag{9.69}$$

We recognise this as the fast scale that one might have obtained from naively seeking a boundary-layer scaling in the original governing equations by balancing $\varepsilon^2 x''$ with x', which would have missed out the intermediate transition layer caused by the coefficient of x' vanishing at $x = 1$.

The leading-order equation can be integrated once, with the constant determined by the matching condition $\xi - 1 \sim \tau^{-1}$ from the intermediate layer, and then integrated again using separation of variables and partial fractions,

$$\xi_0' = -\tfrac{1}{3}\xi_0^3 + \xi_0 - \tfrac{2}{3} = -\tfrac{1}{3}(\xi_0 - 1)^2(\xi_0 + 2) \tag{9.70}$$

$$\Rightarrow \quad \tau - \tau_0 = \int \frac{-3\,\mathrm{d}\xi_0}{(\xi_0 - 1)^2(\xi_0 + 2)} = \int \frac{-1}{(\xi_0 - 1)^2} + \frac{1/3}{\xi_0 - 1} - \frac{1/3}{\xi_0 + 2}\,\mathrm{d}\xi_0 = \frac{1}{\xi_0 - 1} + \frac{1}{3}\ln\frac{\xi_0 - 1}{\xi_0 + 2}. \tag{9.71}$$

We see how $\xi_0 \nearrow 1$ yields $\tau \sim (\xi_0 - 1)^{-1}$ which matches the intermediate layer, while

$$\xi_0 \searrow -2 \quad \text{yields} \quad \tau \sim \tfrac{1}{3}\ln(\xi_0 + 2) \quad \Rightarrow \quad \xi_0 = -2 + \mathrm{EST}. \tag{9.72}$$

We conclude that the next outer solution must start at $x = -2$, and then go through the process with the opposite sign, returning to the $x > 0$ outer solution at $x = 2$.

We can now approximate the period of the oscillation to leading order, by just taking the time spent in the outer regions,

$$P = 2\left[-\frac{x_0^2}{2} + \ln x_0\right]_{x_0=2}^{1} = 2\left(\frac{3}{2} - \ln 2\right) \approx 1.614. \tag{9.73}$$

To calculate the correction to this result, we need to track the corrections to the leading-order solutions more carefully. If we assume that we start with the smallest possible correction, $\mathrm{ord}(\varepsilon^2)$, in the outer, then it is possible to calculate the $\mathrm{ord}(\varepsilon^{4/3})$ additional time spent in the transition region from slow to fast. It is tempting to conclude that this yields the correction to P, but there is another effect: The local expansion in the transition region generates an $\mathrm{ord}(\varepsilon^{4/3})$ correction in the fast region, which propagates into an $\mathrm{ord}(\varepsilon^{4/3})$ correction in the next slow region. This correction turns out to be of the form $C\varepsilon^{4/3}x_0'(t)$, which represents a shift in time that we can undo,

$$x_0(t) + C\varepsilon^{4/3}x_0'(t) + \mathrm{ord}(\varepsilon^2) = x_0(t + C\varepsilon^{4/3}) + \mathrm{ord}(\varepsilon^2), \tag{9.74}$$

in order to truly return to the (mirror of the) original slow solution with $\mathrm{ord}(\varepsilon^2)$ corrections that we started with. This time shift must be added to the estimate of the period, and hence produces another $\mathrm{ord}(\varepsilon^{4/3})$ correction.

In the end, the result (which is plotted in figure 9.3(b)) turns out to be

$$P = 2\left(\frac{3}{2} - \ln 2\right) + \varepsilon^{4/3}3a + \mathrm{ord}(\varepsilon^2 \ln \varepsilon) \approx 1.614 + 7.014\varepsilon^{4/3}, \tag{9.75}$$

where $-a = 2.338$ is the smallest zero of the Airy function $\mathrm{Ai}(z)$.

10 WKB method

We now encounter the WKB (Wentzel–Kramers–Brillouin) method again, and like before it is used in linear differential equations to balance a higher-order derivative term that appears to be small with a lower-order term that appears to be large. The difference is that instead of looking at local expansions of $y(x)$ as x tends to a limit, we're looking at parameter expansions of $y(x, \varepsilon)$ as ε tends to a limit.

10.1 WKB solution away from turning points

Example 10.1. The most classical WKB example is the equation

$$\varepsilon^2 y'' - q(x)y = 0, \qquad \varepsilon \searrow 0, \tag{10.1}$$

for $y(x)$, where $q(x)$ is a given real function of x.

If we wanted to balance the two terms in the equation, then we could rescale $x = \varepsilon X$, and let $y(x) = Y(X)$, to obtain

$$Y'' - q(\varepsilon X)Y = 0, \tag{10.2}$$

which is a differential equation whose coefficient $q(\varepsilon X)$ varies slowly, relative to the natural length scale $X = \mathrm{ord}(1)$ of the equation. Thus, we might expect to be able to naively approximate q as being a constant, resulting in two independent solutions of the form

$$Y = A_\pm(\varepsilon X)\mathrm{e}^{\pm\sqrt{q(\varepsilon X)}X} = A_\pm(x)\mathrm{e}^{\pm\varepsilon^{-1}\sqrt{q(x)}\,x}, \tag{10.3}$$

where the coefficients $A_\pm(\varepsilon X)$ vary slowly on the X-scale. However, as we shall see, this would not remain valid throughout a long interval up to $X = \mathrm{ord}(1/\varepsilon)$, corresponding to $x = \mathrm{ord}(1)$, due to errors accumulating in the exponent. Instead, we need the exponent to still be $\mathrm{ord}(X) = \mathrm{ord}(x/\varepsilon)$ but of a more general form.

This is the motivation for the WKB method, which starts with the (exact) change of variables

$$y = \mathrm{e}^{S(x)/\delta} \quad \Rightarrow \quad y' = \mathrm{e}^{S/\delta}\frac{S'}{\delta}, \quad y'' = \mathrm{e}^{S/\delta}\left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right), \tag{10.4}$$

where the scaling for $\delta \ll 1$ in terms of ε needs to be determined (although in this example we can already suspect it's going to be $\delta = \varepsilon$). Substitution into the governing equation yields

$$\varepsilon^2\mathrm{e}^{S/\delta}\left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right) - q(x)\mathrm{e}^{S/\delta} = 0 \quad \Rightarrow \quad \varepsilon^2\left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right) - q(x) = 0. \tag{10.5}$$

We must first determine the correct scaling of δ, by identifying the dominant balance in this equation. Since $1/\delta^2 \gg 1/\delta$, the only non-trivial balance is between the first and last terms, so we choose $\delta = \varepsilon$ and obtain

$$y = \mathrm{e}^{S(x)/\varepsilon}, \qquad S'^2 + \varepsilon S'' = q(x). \tag{10.6}$$

We have thus exactly converted the singularly perturbed linear homogeneous second-order differential equation for $y(x)$ into a regularly perturbed non-linear inhomogeneous first-order differential equation for $S'(x)$.

It is now clear that we should expand

$$S = S_0 + \varepsilon S_1 + O(\varepsilon^2) \quad \Rightarrow \quad S_0'^2 + \varepsilon 2S_0'S_1' + \varepsilon S_0'' + O(\varepsilon^2) = 0 \tag{10.7}$$

$$\Rightarrow \quad S_0'^2 = q, \quad 2S_0'S_1' + S_0'' = 0. \tag{10.8}$$

$$\Rightarrow \quad S_0' = \pm q^{1/2}, \quad S_1' = -\frac{S_0''}{2S_0'} = -\left(\frac{1}{2}\ln S_0'\right)' = -\left(\frac{1}{4}\ln q\right)'. \tag{10.9}$$

We see that there are two different solutions, depending on the sign choice of S_0', which is good since the original equation for y is of second order so should have two independent solutions. (Coincidentally, the value of S_1' is the same for both choices.) We conclude that the original solutions are

$$y_\pm = \exp\left[\pm\frac{1}{\varepsilon}\int_{x_0}^{x} q^{1/2}\,\mathrm{d}x + \frac{1}{4}\ln q + O(\varepsilon)\right] = \frac{1}{q^{1/4}}\exp\left[\pm\frac{1}{\varepsilon}\int_{x_0}^{x} q^{1/2}\,\mathrm{d}x\right][1 + O(\varepsilon)]\,. \tag{10.10}$$

The general leading-order solution would be a linear combination of these, i.e.

$$y \sim A_+ y_+ + A_- y_-. \tag{10.11}$$

As before, adding a constant of integration is equivalent to changing $A_\pm$, so any lower limit x_0 can be used.

For $q(x) < 0$ the result is valid using any branch for the roots, but it is nicer to rewrite the result as

$$y_\pm = \exp\left[\pm\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}(-q)^{1/2}\,\mathrm{d}x + \frac{1}{4}\ln(-q) + O(\varepsilon)\right] = \frac{1}{(-q)^{1/4}}\exp\left[\pm\frac{\mathrm{i}}{\varepsilon}\int_{x_0}^{x}(-q)^{1/2}\,\mathrm{d}x\right][1+O(\varepsilon)], \tag{10.12}$$

and the general solution can be written as either

$$y \sim B_+ y_+ + B_- y_- \text{ or } y \sim \frac{C}{(-q)^{1/4}}\cos\left[\frac{1}{\varepsilon}\int_{x_0}^{x}(-q)^{1/2}\,\mathrm{d}x\right] + \frac{D}{(-q)^{1/4}}\sin\left[\frac{1}{\varepsilon}\int_{x_0}^{x}(-q)^{1/2}\,\mathrm{d}x\right]. \tag{10.13}$$

Remark(s) 10.1.
- The WKB method requires a linear equation (as we need to be able to cancel the $\mathrm{e}^{S/\delta}$ to cancel) and, roughly speaking, either ord(1) terms with coefficients that vary slowly or coefficients varying at ord(1) but a small parameter multiplying a higher-order derivative.
- The key results are that the phase, i.e the imaginary part of the exponent, varies like $\varepsilon^{-1}\int(-q)^{1/2}\,\mathrm{d}x$ rather than just $\varepsilon^{-1}(-q)^{1/2}\,x$, and that the amplitude has a $|q|^{-1/4}$ variation.
- Sometimes the alternative ansatz $y(x) = A(x)\mathrm{e}^{\psi(x)/\varepsilon}$ is used (possibly with a factor of i in the exponent), which is equivalent to our $\mathrm{e}^{S_0(x)/\varepsilon + S_1(x)} = \mathrm{e}^{S_1(x)}\mathrm{e}^{S_0(x)/\varepsilon}$.

Example 10.2. Consider the one-dimensional wave equation in a medium for $u(X,t)$ where the wave speed $c(\varepsilon X)$ varies slowly with position (i.e. on a length scale that is long compared with the wavelength),

$$\partial_t^2 u - c(\varepsilon X)^2 \partial_X^2 u = 0. \tag{10.14}$$

Assuming a time-harmonic solution (or equivalently taking a Fourier transform in time), and rescaling $X = x/\varepsilon$ for convenience, yields the 1D Helmholtz equation:

$$u(X,t) = \mathrm{e}^{-\mathrm{i}\omega t}y(X,t) \quad\Rightarrow\quad \mathrm{e}^{-\mathrm{i}\omega t}\left[-\omega^2 y - \varepsilon^{-2}c(x)^2\partial_x^2 y\right] = 0 \quad\Rightarrow\quad \varepsilon^2\partial_x^2 y + k(x)^2 y = 0, \tag{10.15}$$

where $k(x) = \omega/c(x) > 0$ is the wavenumber.

When $k(x)$ is constant, the solutions are given by

$$y = A_+\mathrm{e}^{+\mathrm{i}\varepsilon^{-1}kx} + A_-\mathrm{e}^{-\mathrm{i}\varepsilon^{-1}kx} \quad\Rightarrow\quad u = A_+\mathrm{e}^{\mathrm{i}k(\varepsilon^{-1}x - ct)} + A_-\mathrm{e}^{-\mathrm{i}k(\varepsilon^{-1}x + ct)}, \tag{10.16}$$

which we identify as rightward and leftward propagating waves with speed $\mathrm{d}x/\mathrm{d}t = \pm\varepsilon c$ (i.e. $\mathrm{d}X/\mathrm{d}t = \pm c$), respectively. (For example, the first term is constant for $x - \varepsilon ct = \text{const}$, i.e. $x = \text{const} + \varepsilon ct$.)

For varying $k(x)$, we have the standard WKB equation with $q = -k^2 < 0$ and can thus write down the WKB solutions,

$$y \sim \frac{A_+}{k(x)^{1/2}}\mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int k(x)\,\mathrm{d}x} + \frac{A_-}{k(x)^{1/2}}\mathrm{e}^{-\mathrm{i}\varepsilon^{-1}\int k(x)\,\mathrm{d}x}, \tag{10.17}$$

and the coefficients A_+ and A_- would be determined by boundary conditions. Again, the two terms represent rightward and leftward propagating waves, as the imaginary part of the exponents are increasing and decreasing, respectively. For example, for the first term, if we follow a trajectory $x = x(t)$, then the imaginary part of the overall exponent (i.e. including the $-\mathrm{i}\omega t$) is constant if

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\varepsilon^{-1}\int^{x(t)} k(x)\,\mathrm{d}x - \omega t\right] = \varepsilon^{-1}x'k(x) - \omega = \varepsilon^{-1}k(x)\left[x' - \varepsilon c(x)\right], \tag{10.18}$$

i.e. if we are travelling at the local wave speed $\mathrm{d}x/\mathrm{d}t = \varepsilon^{-1}c(x)$.

As a specific problem, we consider the example of a wave with frequency $\omega = 1$ travelling in a mostly uniform medium, say $c = 1$ (i.e. $k = 1$), except that there is a finite region in which the speed varies, i.e. c and k not necessarily equal to 1. Assuming the wave is incident from the left, the far-field boundary conditions are

$$y \sim A_i \mathrm{e}^{+\mathrm{i}\varepsilon^{-1}x} + A_r \mathrm{e}^{-\mathrm{i}\varepsilon^{-1}x} \text{ as } x \searrow -\infty, \quad y \sim A_t \mathrm{e}^{+\mathrm{i}\varepsilon^{-1}x} + 0\,\mathrm{e}^{-\mathrm{i}\varepsilon^{-1}x} \text{ as } x \nearrow \infty. \tag{10.19}$$

Here, the complex amplitudes are the incident amplitude A_i which is known, and the reflected amplitude A_r and the transmitted amplitude A_t which are unknown. Note that the incidence condition imposes two boundary conditions, namely the incident amplitude A_i from the left, and the incident amplitude 0 from the right.

Arbitrarily taking the phase integrals to have lower limit $x = 0$, we write down the general solution

$$y \sim \frac{A_+}{k^{1/2}} \mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^x k\,\mathrm{d}x} + \frac{A_-}{k^{1/2}} \mathrm{e}^{-\mathrm{i}\varepsilon^{-1}\int_0^x k\,\mathrm{d}x}. \tag{10.20}$$

Comparing with the conditions at $x \nearrow \infty$, recalling that $k \to 1$, we immediately obtain

$$A_- = 0, \quad A_+ = \lim_{x\nearrow\infty} \frac{A_t \mathrm{e}^{\mathrm{i}\varepsilon^{-1}x}}{\mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^x k\,\mathrm{d}x}} = \lim_{x\nearrow\infty} A_t \mathrm{e}^{\mathrm{i}\varepsilon^{-1}(x - \int_0^x k\,\mathrm{d}x)} = A_t \mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^\infty 1-k\,\mathrm{d}x}. \tag{10.21}$$

Then, comparing with the conditions at $x \searrow -\infty$, again with $k \to 1$, we obtain

$$A_r = 0, \quad A_i = \lim_{x\searrow-\infty} \frac{A_+ \mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^x k\,\mathrm{d}x}}{\mathrm{e}^{\mathrm{i}\varepsilon^{-1}x}} = \lim_{x\searrow-\infty} A_+ \mathrm{e}^{\mathrm{i}\varepsilon^{-1}(\int_0^x k\,\mathrm{d}x - x)} = A_+ \mathrm{e}^{\mathrm{i}\varepsilon^{-1}(\int_{-\infty}^0 1-k\,\mathrm{d}x)}. \tag{10.22}$$

Recalling that this is a leading-order calculation, we thus conclude that actually

$$A_r = o(1), \quad A_t \sim A_i \mathrm{e}^{\mathrm{i}\varepsilon^{-1}(-\int_0^\infty 1-k\,\mathrm{d}x - \int_{-\infty}^0 1-k\,\mathrm{d}x)} = A_i \mathrm{e}^{\mathrm{i}\varepsilon^{-1}(\int_{-\infty}^\infty k-1\,\mathrm{d}x)}. \tag{10.23}$$

This shows that, to leading order, the wave passes through the varying region without reflection, and although its amplitude varies inside that region, in the end it returns to its original amplitude ($|A_t| \sim |A_i|$) but picks up a phase shift $\varepsilon^{-1}\int_{-\infty}^{\infty} k(x) - 1\,\mathrm{d}x$.

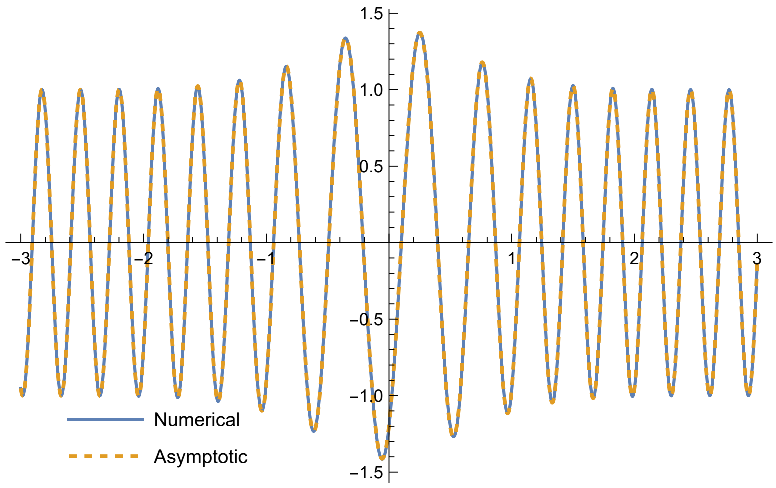


Figure 10.1: Real part of solution $y(x)$ for a wave incident from the left ($A_i = 1$) in a medium with $k = 1 - \frac{1}{2}\mathrm{e}^{-x^2}$, for $\varepsilon = 0.05$.

10.2 WKB solution near turning points

Example 10.3. A famous application of the WKB method is to quantum mechanics. The steady-state wavefunction $y(x)$ for a particle with energy E in a given potential $V(x)$ satisfies the time-independent Schrödinger equation, which in rescaled form is

$$-\varepsilon^2 y'' + V(x)y = Ey \quad \Rightarrow \quad \varepsilon^2 y'' - q(x)y = 0, \quad q(x) = V(x) - E. \tag{10.24}$$

(The first term in the original equation represents kinetic energy, because velocity is like $-\varepsilon \mathrm{i}\partial_x$ so velocity squared is like $-\varepsilon^2\partial_x^2$, and the other two terms represent potential energy and total energy.) The WKB approximation applies in the limit $\varepsilon \searrow 0$, when $q(x)$ varies on a much larger scale than the natural length scale of the system.

In regions where $E > V(x)$ (which correspond to the regions where a classical, i.e. non-quantum, particle can exist), we have $q < 0$ so the solutions are oscillatory. Specifically, the solution with $+\mathrm{i}$ in the exponent corresponds to a rightward propagating wave, and the $-\mathrm{i}$ solution corresponds to a leftward one. In regions where $E < V(x)$ (where the classical particle cannot exist), we have $q > 0$ so the solutions are exponential.

Let's first consider a particle reflecting off a potential barrier at $x = 0$, i.e. suppose that $q < 0$ (oscillatory solutions) for $x < 0$ and $q > 0$ (exponential solutions) for $x > 0$. We thus have the leading-order WKB solutions, valid in the two separate regions,

$$x < 0: \quad y \sim \frac{A_-}{(-q)^{1/4}} \mathrm{e}^{-\mathrm{i}\int_x^0 (-q)^{1/2}\,\mathrm{d}x/\varepsilon} + \frac{A_+}{(-q)^{1/4}} \mathrm{e}^{\mathrm{i}\int_x^0 (-q)^{1/2}\,\mathrm{d}x/\varepsilon} \tag{10.25}$$

$$x > 0: \quad y \sim \frac{B_-}{q^{1/4}} \mathrm{e}^{-\int_0^x q^{1/2}\,\mathrm{d}x/\varepsilon} + \frac{B_+}{q^{1/4}} \mathrm{e}^{+\int_0^x q^{1/2}\,\mathrm{d}x/\varepsilon}. \tag{10.26}$$

(Note here that we have chosen to place the integral limits at the turning point $x = 0$, which will make the asymptotic matching easier, and flipped the signs of the $x < 0$ integrals so that the integrals themselves are positive, which is just a convention to make it easy to translate the method to other cases.)

The conditions describing a particle (wave) incident from the left and bouncing against the potential barrier is that the incident coefficient A_- (which corresponds to the exponent whose imaginary part increases with x) is known as $x \searrow -\infty$, and that $y \to 0$ as $x \nearrow +\infty$ so that $B_+ = 0$. The reflection coefficient A_+ is to be determined, as well as the coefficient B_- of the exponential decay inside the barrier.

The point $x = 0$ where $q = 0$ is called a **turning point**, and the WKB approximations blow up there as the denominators $|q|^{1/4}$ vanish, so we need to obtain an inner asymptotic expansion of y near $x = 0$ to obtain the correct relationship between $A_\pm$ and $B_\pm$. We assume that q can be Taylor expanded with a non-zero, and hence positive, derivative $m = q'(0)$, so that

$$q(x) = mx + O(x^2) \approx mx \quad \text{as } x \to 0. \tag{10.27}$$

We will only do a leading-order approximation, so neglect the higher-order term here. Substituting this into the original equation yields

$$\varepsilon^2 y'' - mxy \approx 0. \tag{10.28}$$

We seek a rescaling of x to balance the two terms in the equation, and it is convenient to include the factor m in the rescaling as well, so we define

$$x = \varepsilon^{2/3} m^{-1/3} X, \quad y(x) = Y(X) \quad \Rightarrow \quad Y'' - XY = 0 \tag{10.29}$$

We recognise this as the Airy equation, with the general solution given by

$$Y \sim C\,\mathrm{Ai}(X) + D\,\mathrm{Bi}(X). \tag{10.30}$$

Since the outer solution on the right decays exponentially with x, we expect it to successfully match to $\mathrm{Ai}(X)$ which also decays exponentially as $X \nearrow \infty$, but not $\mathrm{Bi}(X)$ which grows exponentially. Hence, we set $D = 0$ and expand the Ai to find

$$X \nearrow \infty: \quad Y \sim C\frac{\mathrm{e}^{-\frac{2}{3}X^{3/2}}}{2\sqrt{\pi}\,X^{1/4}}, \tag{10.31}$$

$$X \searrow -\infty: \quad Y \sim C\frac{\sin(\frac{2}{3}(-X)^{3/2}+\frac{\pi}{4})}{\sqrt{\pi}\,(-X)^{1/4}} = C\frac{\mathrm{e}^{\mathrm{i}\frac{2}{3}(-X)^{3/2}-\mathrm{i}\pi/4}+\mathrm{e}^{-\mathrm{i}\frac{2}{3}(-X)^{3/2}+\mathrm{i}\pi/4}}{2\sqrt{\pi}\,(-X)^{1/4}}. \tag{10.32}$$

(Although this result is tricky to memorize, a trick for determining the relative values of the coefficients is to remember that one side is proportional to the exponentially decaying WKB solution, while the other side consists of two terms that are just the analytical continuation of the exponentially decaying term to the negative real axis counterclockwise and clockwise, respectively, in the complex plane.)

We now match with the inner limit of the outer WKB solutions. We expand, for $x > 0$ and $x < 0$ respectively, using $q \approx mx$ as $x \to 0$,

$$\int_0^x q^{1/2}\,\mathrm{d}x \approx \int_0^x (mx)^{1/2}\,\mathrm{d}x = \tfrac{2}{3}m^{1/2}x^{3/2} = \varepsilon\tfrac{2}{3}X^{3/2}, \tag{10.33}$$

$$q^{1/4} \approx m^{1/4}x^{1/4} = \varepsilon^{1/6}m^{1/6}X^{1/4}, \tag{10.34}$$

$$\int_x^0 (-q)^{1/2}\,\mathrm{d}x \approx \int_x^0 (-mx)^{1/2}\,\mathrm{d}x = \left[-\tfrac{2}{3}m^{1/2}(-x)^{3/2}\right]_x^0 = \tfrac{2}{3}m^{1/2}(-x)^{3/2} = \varepsilon\tfrac{2}{3}(-X)^{3/2}, \tag{10.35}$$

$$(-q)^{1/4} \approx (-mx)^{1/4} = \varepsilon^{1/6}m^{1/6}(-X)^{1/4}. \tag{10.36}$$

Thus, we have

$$x \searrow 0: \quad y \sim \frac{B_-}{\varepsilon^{1/6}m^{1/6}X^{1/4}}\mathrm{e}^{-\frac{2}{3}X^{3/2}}, \qquad x \nearrow 0: \quad y \sim \frac{A_-\mathrm{e}^{-\mathrm{i}\frac{2}{3}(-X)^{3/2}} + A_+\mathrm{e}^{+\mathrm{i}\frac{2}{3}(-X)^{3/2}}}{\varepsilon^{1/6}m^{1/6}(-X)^{1/4}}. \tag{10.37}$$

Comparison of the expansions of the inner and outer solutions then yields

$$B_- = \frac{\varepsilon^{1/6}m^{1/6}C}{2\sqrt{\pi}}, \quad A_- = \frac{\varepsilon^{1/6}m^{1/6}C\mathrm{e}^{-\mathrm{i}\pi/4}}{2\sqrt{\pi}}, \quad A_+ = \frac{\varepsilon^{1/6}m^{1/6}C\mathrm{e}^{\mathrm{i}\pi/4}}{2\sqrt{\pi}} \tag{10.38}$$

$$\Rightarrow \quad A_- = B_-\mathrm{e}^{\mathrm{i}\pi/4}, \quad A_+ = B_-\mathrm{e}^{-\mathrm{i}\pi/4}. \tag{10.39}$$

This final result simply relates the outer coefficients on either side of the inner region, and is called a **connection formula**.

We conclude that $A_+ = A_-\mathrm{e}^{-\mathrm{i}\pi/2}$, i.e. the reflected coefficient is equal to the incident coefficient in amplitude but has a phase change.

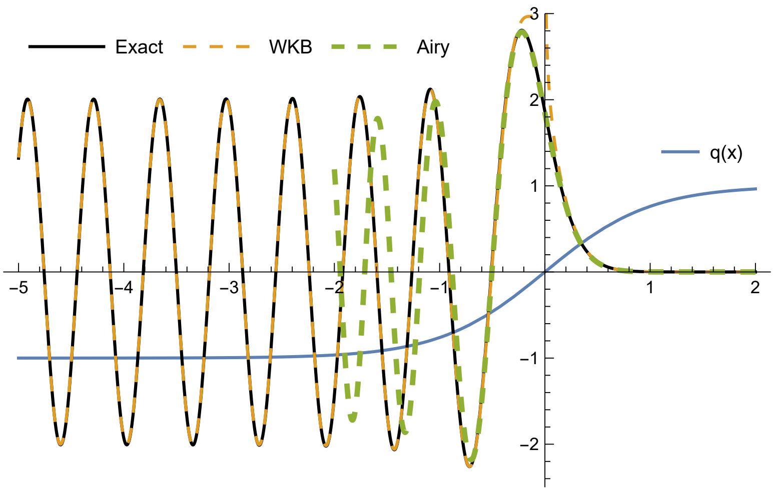


Figure 10.2: Reflection against a potential barrier with $q(x) = \tanh(x)$, $\varepsilon = 0.1$.

Remark(s) 10.2. We can restate the connection formula in terms of the outer WKB solutions, for a turning point at some $x = a$ with $q'(a) > 0$, as

$$y \sim \frac{A}{q^{1/4}} \mathrm{e}^{-\int_a^x q^{1/2}\,\mathrm{d}x/\varepsilon} \quad \text{for } x > a \tag{10.40}$$

$$\Rightarrow \quad y \sim \frac{A}{(-q)^{1/4}} \left[\mathrm{e}^{\mathrm{i}\int_x^a (-q)^{1/2}\,\mathrm{d}x/\varepsilon - \mathrm{i}\pi/4} + \mathrm{e}^{-\mathrm{i}\int_x^a (-q)^{1/2}\,\mathrm{d}x/\varepsilon + \mathrm{i}\pi/4} \right] = \tag{10.41}$$

$$= \frac{A}{(-q)^{1/4}} 2\cos\left[\frac{1}{\varepsilon}\int_x^a (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] \quad \text{for } x < a. \tag{10.42}$$

For the left–right mirrored case, where $q'(a) < 0$ so that the exponential side is $x < a$ and the oscillatory side is $x > a$, we simply need to reverse the inequality signs and flip the integral limits $\int_a^x \leftrightarrow \int_x^a$ so that again each integral is positive.

Example 10.4. We again consider the Schrödinger equation

$$-\varepsilon^2 y'' + V(x)y = Ey \quad \Rightarrow \quad \varepsilon^2 y'' - q(x)y = 0, \quad q(x) = V(x) - E, \tag{10.43}$$

but suppose that $V(x)$ has a typical potential well shape, similar to a parabola. Then we expect to have some interval $a < x < b$ in which $E > V(x)$ and the solution is oscillatory, while the outer regions $x < a$ and $x > b$ have $E < V(x)$ and exponential solutions. We seek a bound-state solution, i.e. one in which $y \to 0$ as $x \to \pm\infty$. This is an eigenvalue problem for E, meaning that a nontrivial ($y \not\equiv 0$) solution only exists for specific values of E, and we would like to determine these values of E.

We already have all the building blocks needed to solve this problem. Considering the decay in $x > b$ matching via a turning point to the region $a < x < b$, we find that, for some constant A_R,

$$y \sim \frac{A_R}{q^{1/4}} \mathrm{e}^{-\int_b^x q^{1/2}\,\mathrm{d}x/\varepsilon} \text{ for } x > b \quad \Rightarrow \quad y \sim \frac{A_R}{(-q)^{1/4}} 2\cos\left[\frac{1}{\varepsilon}\int_x^b (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] \text{ for } a < x < b. \tag{10.44}$$

Similarly, the decay in $x < a$ matching via a turning point to $a < x < b$ yields

$$y \sim \frac{A_L}{q^{1/4}} \mathrm{e}^{-\int_x^a q^{1/2}\,\mathrm{d}x/\varepsilon} \text{ for } x < a \tag{10.45}$$

$$\Rightarrow \quad y \sim \frac{A_L}{(-q)^{1/4}} 2\cos\left[\frac{1}{\varepsilon}\int_a^x (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] \text{ for } a < x < b. \tag{10.46}$$

We now have two different expressions for the leading-order behaviour in the middle range $a < x < b$, so we require that they be equal,

$$A_R \cos\left[\frac{1}{\varepsilon}\int_x^b (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] = A_L \cos\left[\frac{1}{\varepsilon}\int_a^x (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] \quad \text{for } a < x < b. \tag{10.47}$$

The arguments of the cosines can't be equal (up to a constant phase shift), because one increases and the other decreases with x. Instead, they must be opposite, up to a phase shift of $n\pi$, and the amplitudes must be equal too, up to a sign difference of $(-1)^n$. Hence,

$$\left[\frac{1}{\varepsilon}\int_x^b (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] = -\left[\frac{1}{\varepsilon}\int_a^x (-q)^{1/2}\,\mathrm{d}x - \frac{\pi}{4} \right] + n\pi, \quad A_L = A_R(-1)^n \tag{10.48}$$

$$\Rightarrow \quad \frac{1}{\varepsilon}\int_a^b (-q)^{1/2}\,\mathrm{d}x = \left(n + \frac{1}{2} \right)\pi, \qquad n = 0, 1, 2, \cdots. \tag{10.49}$$

This quantisation condition is a key result in quantum mechanics, and (in principle) allows us to calculate the allowed values of E by substituting in the expression for q and integrating. Note that the $1/\varepsilon$ on the left-hand side implies that the approximation is only valid (i.e. $\varepsilon \ll 1$) for $n \gg 1$.

As a concrete example, the harmonic oscillator with $V(x) = x^2$ yields

$$q = V(x) - E = x^2 - E \quad \Rightarrow \quad a = -\sqrt{E},\ b = \sqrt{E}, \tag{10.50}$$

$$\varepsilon(n + \tfrac{1}{2})\pi = \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E - x^2}\,\mathrm{d}x \overset{x=\sqrt{E}u}{=} E\int_{-1}^{1} \sqrt{1 - u^2}\,\mathrm{d}u = E\pi/2, \tag{10.51}$$

using the area of a semicircle or a substitution $u = \sin v$ and the double angle formula. We conclude that $E = \varepsilon(2n+1)$ for $\varepsilon \ll 1$ and $n \gg 1$. Famously, for the harmonic oscillator this approximation happens to be exact for all ε and n, and the exact solution is given by $y(x) \propto H_n(x/\sqrt{\varepsilon})\mathrm{e}^{-x^2/2\varepsilon}$ where H_n is the Hermite polynomial of degree n.

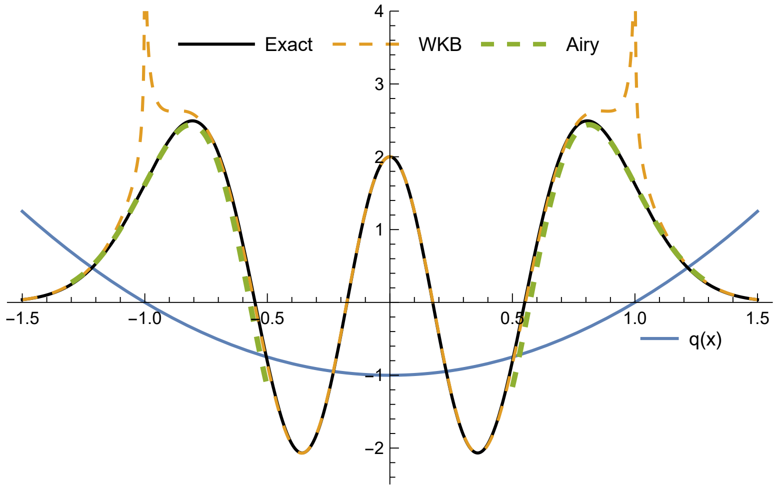


Figure 10.3: Solution for a potential well $V = x^2$ with $n = 4$, $\varepsilon = 1/9$, $E = \varepsilon(2n+1) = 1$, $q = E - V = 1 - x^2$.

11 Perturbed harmonic oscillators

In this section we study ordinary differential equations that are small perturbations to the harmonic oscillator, which after a suitable rescaling takes the form

$$y''(t) + y(t) = 0. \tag{11.1}$$

As we shall see, the small perturbations can accumulate over long time scales to affect the leading-order oscillating solution $y = A\cos t + B\sin t$.

11.1 Poincaré–Lindstedt method / method of strained coordinates

Example 11.1. The equation for the angle $\theta(\tau)$ of a pendulum can be written as

$$m\ell^2\theta''(\tau) + mg\ell\sin\theta(\tau) = 0. \tag{11.2}$$

We consider small-amplitude oscillations of the pendulum, $\theta \ll 1$, for which we can Taylor expand the sin. For convenience, we write $\theta(\tau) = (6\varepsilon)^{1/2}y(t)$, where $t = (\ell/g)^{1/2}\tau$, $\varepsilon \ll 1$ and $y = \mathrm{ord}(1)$. The governing equation then simplifies to

$$0 = y'' + (6\varepsilon)^{-1/2}\sin\left[(6\varepsilon)^{1/2}y\right] = y'' + y - \varepsilon y^3 + \varepsilon^2\frac{3y^5}{10} + O(\varepsilon^3). \tag{11.3}$$

We consider just the terms down to $O(\varepsilon)$,

$$y'' + y - \varepsilon y^3 = 0. \tag{11.4}$$

This equation is a form of Duffing's oscillator with no damping, no forcing, and a restoring force that has a weak nonlinearity. How does the small nonlinear term change the oscillation?

We first attempt a naive expansion

$$y(t) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2), \tag{11.5}$$

and find the leading-order solution

$$y_0'' + y_0 = 0 \qquad \Rightarrow \qquad y_0 = A_0\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}. \tag{11.6}$$

Here, in order to simplify later calculations, we have chosen to use a complex representation with a complex constant of integration A_0, and " + cc" denotes adding the complex conjugate, resulting in the real expression

$$A_0\mathrm{e}^{\mathrm{i}t} + A_0^*\mathrm{e}^{-\mathrm{i}t} = 2\,\mathrm{Re}\left[A_0\mathrm{e}^{\mathrm{i}t}\right] = 2\left[(\mathrm{Re}\,A_0)\cos t - (\mathrm{Im}\,A_0)\sin t\right]. \tag{11.7}$$

Note that since $\mathrm{Re}\,A_0$ and $\mathrm{Im}\,A_0$ represent two independent real constants, this is indeed a most general form of a solution y_0 if we assume that it must be real, which is the only physically relevant case.

Carefully taking the cube of the complex expression,

$$(A_0\mathrm{e}^{\mathrm{i}t} + \mathrm{cc})^3 = (A_0^3\mathrm{e}^{\mathrm{i}t})^3 + 3(A_0\mathrm{e}^{\mathrm{i}t})^2(A_0^*\mathrm{e}^{-\mathrm{i}t}) + 3(A_0\mathrm{e}^{\mathrm{i}t})(A_0^*\mathrm{e}^{-\mathrm{i}t})^3 + (A_0^*\mathrm{e}^{-\mathrm{i}t})^3, \tag{11.8}$$

then yields the forcing for the $O(\varepsilon)$ corrections,

$$y_1'' + y_1 = -y_0^3 = A_0^3\mathrm{e}^{3\mathrm{i}t} + 3A_0|A_0|^2\mathrm{e}^{\mathrm{i}t} + \mathrm{cc} \quad \Rightarrow \quad y_1 = -\frac{A_0^3}{8}\mathrm{e}^{3\mathrm{i}t} - \frac{3A_0|A_0|^2\mathrm{i}}{2}t\,\mathrm{e}^{\mathrm{i}t} + A_1\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}. \tag{11.9}$$

We thus obtain the result

$$y(t) = A_0\mathrm{e}^{\mathrm{i}t} + \varepsilon\left[-\frac{A_0^3}{8}\mathrm{e}^{3\mathrm{i}t} - \underbrace{\frac{3A_0|A_0|^2\mathrm{i}}{2}t\,\mathrm{e}^{\mathrm{i}t}}_{\text{secular}} + A_1\mathrm{e}^{\mathrm{i}t}\right] + O(\varepsilon^2) + \mathrm{cc}. \tag{11.10}$$

Here, the $t\,\mathrm{e}^{\mathrm{i}t}$ term yields oscillations that grow in amplitude, and we refer to it as a "secular" term. We also refer to the original forcing term, that was proportional to $\mathrm{e}^{\mathrm{i}t}$, as "secular". Thus, although the expansion is valid for $t = O(1)$, it becomes invalid for large $t = \mathrm{ord}(1/\varepsilon)$, due to the secular term no longer being a small correction to the leading-order term.

However, we expect the exact solutions to the original equation to oscillate periodically and remain bounded, because we can multiply the equation by y' and integrate to obtain an energy conservation equation,

$$0 = y'\left(y'' + y - \varepsilon y^3\right) = \left(\frac{y'^2}{2} + \frac{y^2}{2} - \varepsilon\frac{y^4}{4}\right)' \quad \Rightarrow \quad \frac{y'^2}{2} + \frac{y^2}{2} - \varepsilon\frac{y^4}{4} = \text{const.} \tag{11.11}$$

The problem with our naive expansion comes from the fact that its oscillation frequency is wrong. To illustrate how an error in frequency generates a secular term, we consider naively expanding an oscillation with angular frequency $(1+\varepsilon)$:

$$\cos((1+\varepsilon)t) = \cos(t+\varepsilon t) = \cos(t) - \varepsilon t\,\sin(t) + O(\varepsilon^2). \tag{11.12}$$

Again, we see a secular term appearing, which invalidates the expansion when $t = \mathrm{ord}(1/\varepsilon)$. In essence, although in $\cos(t+\varepsilon t)$ the correction εt is always small relative to t, since cos is an exponential-type function the expansion is only valid if εt is small relative to 1.

With the insight that the oscillation frequency needs adjustment, we arrive at the **Poincaré–Lindstedt method**. The idea is to use the exact, but unknown, angular frequency ω of the oscillator (with $\omega = 1 + o(1)$) to introduce a “strained” time coordinate

$$s = \omega t, \quad \frac{\mathrm{d}}{\mathrm{d}t} = \omega\frac{\mathrm{d}}{\mathrm{d}s}, \qquad \omega = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + O(\varepsilon^3). \tag{11.13}$$

We thus obtain the equation (where primes now denote differentiation with respect to s) and condition

$$\omega^2 y'' + y - \varepsilon y^3 = 0, \qquad y \text{ is } 2\pi\text{-periodic in } s. \tag{11.14}$$

The expansion $y(s) = y_0(s) + \varepsilon y_1(s) + \cdots$ then yields

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0(s) = A_0\mathrm{e}^{\mathrm{i}s} + \mathrm{cc}, \tag{11.15}$$

$$y_1'' + y_1 = y_0^3 - 2\omega_1 y_0'' = A_0^3\mathrm{e}^{3\mathrm{i}s} + 3A_0|A_0^2|\mathrm{e}^{\mathrm{i}s} + 2\omega_1 A_0\mathrm{e}^{\mathrm{i}s} + \mathrm{cc}. \tag{11.16}$$

The condition that y_1 is 2π-periodic in s requires that the secular terms (i.e. the ones proportional to $\mathrm{e}^{\mathrm{i}s}$) cancel out, which determines the frequency correction ω_1 and yields the general solution for y_1:

$$3A_0|A_0|^2 + 2\omega_1 A_0 = 0 \quad \Rightarrow \quad \omega_1 = -\frac{3}{2}|A_0|^2, \qquad y_1 = -\frac{A_0^3}{8}\mathrm{e}^{3\mathrm{i}s} + A_1\mathrm{e}^{\mathrm{i}s} + \mathrm{cc}. \tag{11.17}$$

We conclude that the answer is

$$y = [A_0 + \varepsilon A_1]\,\mathrm{e}^{\mathrm{i}s} - \varepsilon\frac{3|A_0|^2}{8}\mathrm{e}^{3\mathrm{i}s} + O(\varepsilon^2) + \mathrm{cc}, \qquad s = \left(1 - \tfrac{3}{2}\varepsilon|A_0|^2 + O(\varepsilon^2)\right)t, \tag{11.18}$$

where the constants of integration $A_0,\ A_1,\ \cdots$ are determined by, for example, initial conditions. Dropping the $O(\varepsilon^2)$ terms yields a result that is valid up to $t = O(1/\varepsilon)$, but again fails at $t = \mathrm{ord}(1/\varepsilon^2)$ due to the $\mathrm{ord}(\varepsilon^2)$ correction to the frequency multiplying with the $\mathrm{ord}(1/\varepsilon^2)$ time resulting in a $\mathrm{ord}(1)$ change in phase of the oscillation. Note that expanding the result naively for $t = \mathrm{ord}(1)$ recovers the secular term from before, since

$$A_0\mathrm{e}^{\mathrm{i}(t - \frac{3}{2}\varepsilon|A_0|^2 t + O(\varepsilon^2))} = A_0\mathrm{e}^{\mathrm{i}t} - \varepsilon\frac{3\mathrm{i}|A_0|^2 A_0 t}{2}\mathrm{e}^{\mathrm{i}t} + O(\varepsilon^2) \text{ for } t = \mathrm{ord}(1). \tag{11.19}$$

As seen in figure 11.1 which shows a range of times $t = O(1/\varepsilon)$, the leading-order result $y_0 = \cos(t)$ has a slightly wrong period. The naive correction looks like it has a corrected period to begin with, but grows without bound due to the secular term. The Poincaré–Lindstedt result agrees excellently with the exact result, both in period and amplitude, over the range shown.

Remark(s) 11.1. • It is also possible to do the calculation in real form, using trigonometric functions. To keep the number of terms small we rewrite the classical general solution using an amplitude and a phase instead, $y_0 = a_0\cos s + b_0\sin s = a_0\cos(s+\phi_0)$, and we further write $\hat{s} = s + \phi_0$ for simplicity.

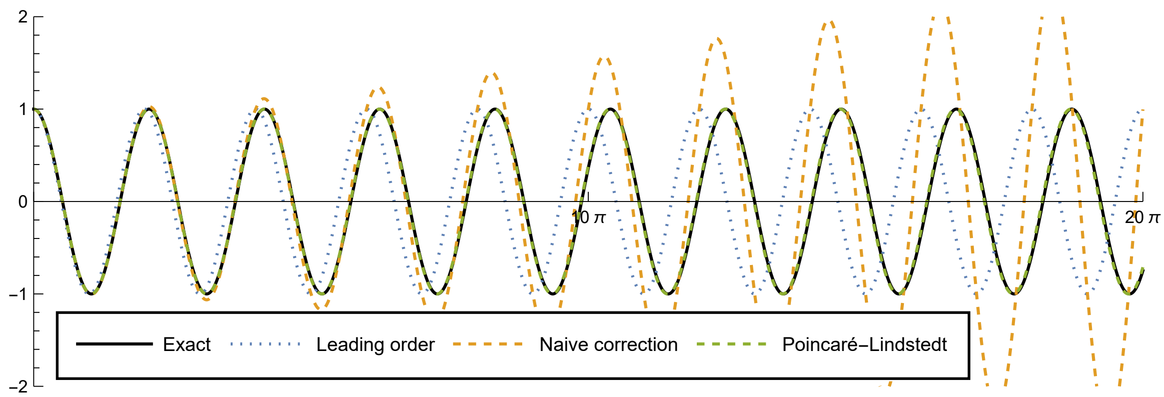


Figure 11.1: Solution of the duffing oscillator $y'' + y - \varepsilon y^3 = 0$, $y(0) = 1$, $y'(0) = 0$, with $\varepsilon = 0.1$.

We must then use a trigonometric multiple-angle formula to convert the nonlinear term into a sum of linear ones,

$$y_1'' + y_1 = y_0^3 - 2\omega_1 y_0'' = a_0^3(\cos\hat{s})^3 + 2\omega_1 a_0 \cos\hat{s} = a_0^3 \frac{\cos 3\hat{s} + 3\cos\hat{s}}{4} + 2\omega_1 a_0 \cos\hat{s} \tag{11.20}$$

$$\Rightarrow \quad 2\omega_1 a_0 + \frac{3a_0^3}{4} = 0 \quad \Rightarrow \quad \omega_1 = -\frac{3a_0^2}{8}. \tag{11.21}$$

- When comparing complex and real forms, don't forget the factor two in $A_0 \mathrm{e}^{\mathrm{i}t} + \mathrm{cc} = 2|A_0|\cos(t + \arg A_0)$, i.e. $|A_0| = a_0/2$.
- We see that nonlinearities in general generate "harmonics", i.e. oscillations at integer multiples of the original frequency.
- In general, we end up with $y_n'' + y_n = f_n(t)$ (but with $t = s$) where the forcing $f_n(t)$ is 2π-periodic, and it may be difficult to identify the secular terms in $f_n(t)$ if it has a complicated expression. If we are able to determine the Fourier series of f_n in any of the equivalent forms

$$f_n(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mt + b_m \sin mt) = \frac{a_0}{2} + \sum_{m=1}^{\infty} c_m \cos(mt + \phi_m) = \sum_{m=0}^{\infty} A_m \mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \tag{11.22}$$

then the coefficients corresponding to secular terms are a_1, b_1, c_1 and A_1, respectively. Equivalently, using the integral expression for the Fourier coefficients, the secularity condition on f_n is that

$$\int_0^{2\pi} \cos t\, f_n(t)\,\mathrm{d}t = \int_0^{2\pi} \sin t\, f_n(t)\,\mathrm{d}t = 0, \quad \text{or} \quad \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t} f_n(t)\,\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{+\mathrm{i}t} f_n(t)\,\mathrm{d}t = 0. \tag{11.23}$$

These relationships can also be obtained from the governing equation by integrating by parts, e.g.

$$\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t} f_n(t)\,\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t}(y_n'' + y_n)\,\mathrm{d}t = \left[\mathrm{e}^{-\mathrm{i}t}(y_n' + \mathrm{i}y_n)\right]_0^{2\pi} = 0, \tag{11.24}$$

since y_n is required to be periodic. (In fact, for more general linear equations $\mathcal{L}y_n = f_n$, you may be able to find an "adjoint" function $y^\dagger$ such that the equation has appropriate solutions only if $\int y^\dagger f_n\,\mathrm{d}t = 0$, where $y^\dagger$ satisfies an "adjoint equation" $\mathcal{L}^\dagger y^\dagger = 0$, which is different unless $\mathcal{L}$ is "self-adjoint", i.e. $\mathcal{L} = \mathcal{L}^\dagger$.)

- When defining a strained time, instead of the angular frequency ω it is also possible to use the "angular period" $\chi = \omega^{-1}$, to equivalently obtain

$$t = \chi s, \quad \frac{\mathrm{d}}{\mathrm{d}t} = \chi^{-1}\frac{\mathrm{d}}{\mathrm{d}s}, \quad \chi = 1 + \varepsilon\chi_1 + \varepsilon^2\chi_2 + O(\varepsilon^3), \tag{11.25}$$

and proceed as usual.

- The **method of strained coordinates** sometimes refers to using the more general transformation

$$t = s + \varepsilon t_1(s) + \varepsilon^2 t_2(s) + O(\varepsilon^3), \tag{11.26}$$

where $t_1(s)$, $t_2(s)$, etc. are general functions of s which are determined (although not uniquely) during the calculation, and the method can be applied to non-periodic solutions as well.

11.2 Method of multiple scales

In the previous subsection we saw how to use the Poincaré–Lindstedt method of strained coordinates to calculate the solutions of weakly perturbed oscillators that remain periodic. We now turn to more general weak perturbations of oscillators, which result in non-periodic motion. In fact, the motion turns out to be quasi-periodic, in the sense that there is a fast oscillation on an ord(1) time scale, whose amplitude and phase varies slowly with time.

Example 11.2. Let's revisit the Van der Pol oscillator, but now consider the limit of small friction,

$$y''(t) + \varepsilon(y(t)^2 - 1)y'(t) + y(t) = 0. \tag{11.27}$$

As before, a naive expansion $y(t) = y_0(t) + \varepsilon y_1(t) + \cdots$ results in an oscillatory leading-order term

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0 \mathrm{e}^{\mathrm{i}t} + \mathrm{cc}. \tag{11.28}$$

Then correction

$$\begin{aligned}
y_1'' + y_1 &= (1 - y_0^2)y_0' = \left[1 - 2|A_0|^2 - A_0^2\mathrm{e}^{2\mathrm{i}t} - (A_0^*)^2\mathrm{e}^{-2\mathrm{i}t}\right]\left[\mathrm{i}A_0\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_0^*\mathrm{e}^{-\mathrm{i}t}\right] = &(11.29)\\
&= (1 - 2|A|^2)\mathrm{i}A_0\mathrm{e}^{\mathrm{i}t} + A_0^2\mathrm{e}^{2\mathrm{i}t}\mathrm{i}A_0^*\mathrm{e}^{-\mathrm{i}t} - A_0^2\mathrm{e}^{2\mathrm{i}t}\mathrm{i}A_0\mathrm{e}^{\mathrm{i}t} + \mathrm{cc} &(11.30)\\
&= \underbrace{\mathrm{i}(1 - |A_0|^2)A_0\mathrm{e}^{\mathrm{i}t}}_{\text{secular}} - \mathrm{i}A_0^3\mathrm{e}^{3\mathrm{i}t} + \mathrm{cc}. &(11.31)
\end{aligned}$$

has a secular term that yields an oscillation with amplitude growing like t, resulting in the expansion breaking down at $t = \mathrm{ord}(1/\varepsilon)$.

The **method of multiple scales** proceeds as follows: We introduce a "**slow time**" $T = \varepsilon t$ and treat it as a separate independent variable to the "**fast time**" t, with $y = y(t, T)$. The derivatives of y become

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}, \tag{11.32}$$

which converts the ordinary differential equation into a partial differential equation, which we solve while ensuring that no secular terms arise, and at the end we set $T = \varepsilon t$ to obtain the final result.

The introduction of slow time $T = \varepsilon t$ converts the ODE into the PDE

$$\partial_t^2 y + 2\varepsilon\partial_t\partial_T y + \varepsilon^2\partial_T^2 y + \varepsilon(y^2 - 1)(\partial_t y + \varepsilon\partial_T y) + y = 0. \tag{11.33}$$

We expand $y(t, T) = y_0(t, T) + \varepsilon y_1(t, T) + \cdots$ and recover the same leading-order behaviour as before

$$\partial_t^2 y_0 + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0(T)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \tag{11.34}$$

with the difference that the amplitude $A_0(T)$ is now a function of the slow time T. The point is that the oscillator oscillates on the fast time scale $t = \mathrm{ord}(1)$, with its (complex) amplitude evolving on the slower time scale $t = \mathrm{ord}(1/\varepsilon)$ represented by T.

The slow derivative of A_0 comes in at next order, resulting in the same equation as before but with an extra derivative term

$$\partial_t^2 y_1 + y_1 = -2\partial_t\partial_T y_0 + (1 - y_0^2)\partial_t y_0 = \mathrm{i}\left[-2\partial_T A_0 + (1 - |A_0|^2)A_0\right]\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A_0^3\mathrm{e}^{3\mathrm{i}t} + \mathrm{cc}. \tag{11.35}$$

The condition that there is no secular term, so that the expansion remains valid for $t = \mathrm{ord}(1/\varepsilon)$, yields a condition on the time derivative, i.e. an evolution equation for $A_0(T)$,

$$\partial_T A_0 = \frac{1 - |A_0|^2}{2}A_0. \tag{11.36}$$

To solve this, we write $A_0(T) = r(T)\mathrm{e}^{\mathrm{i}\phi(T)}$ where the magnitude $r = |A_0| > 0$ and argument $\phi = \arg A_0$ are real, resulting in

$$A_0' = r'\mathrm{e}^{\mathrm{i}\phi} + r\,\mathrm{i}\phi'\mathrm{e}^{\mathrm{i}\phi} = (r' + \mathrm{i}r\phi')\mathrm{e}^{\mathrm{i}\phi} \quad \Rightarrow \quad r' + r\,\mathrm{i}\phi' = \frac{1 - r^2}{2}r \quad \Rightarrow \quad r' = \frac{1 - r^2}{2}r, \quad \phi' = 0. \tag{11.37}$$

Hence the argument ϕ is constant while the magnitude equation can be integrated using separation of variables and partial fractions,

$$T = \int \frac{2\,\mathrm{d}r}{(1-r^2)r} = \int \frac{2}{r} + \frac{1}{1-r} - \frac{1}{1+r}\,\mathrm{d}r = \ln\frac{r^2}{1-r^2} + \text{const} \tag{11.38}$$

$$\Rightarrow \quad C\mathrm{e}^T = \frac{r^2}{1-r^2} = \frac{1}{r^{-2}-1} \quad \Rightarrow \quad r = \frac{1}{\sqrt{1+C^{-1}\mathrm{e}^{-T}}} = \frac{1}{\sqrt{1+D\mathrm{e}^{-T}}}. \tag{11.39}$$

We conclude that the leading-order behaviour, valid up to $t = O(1/\varepsilon)$, is

$$y(t) = \frac{\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{1+D\mathrm{e}^{-\varepsilon t}}}\mathrm{e}^{\mathrm{i}t} + \mathrm{cc} + O(\varepsilon), \tag{11.40}$$

where the constants ϕ and D are determined by initial conditions. We could further calculate the $\operatorname{ord}(\varepsilon)$ correction,

$$\partial_t^2 y_1 + y_1 = -\mathrm{i}A_0^3\mathrm{e}^{3\mathrm{i}t} + \mathrm{cc} \quad \Rightarrow \quad y_1 = \mathrm{i}\frac{A_0^3}{8}\mathrm{e}^{3\mathrm{i}t} + A_1(T)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \tag{11.41}$$

but the evolution of $A_1(T)$ is only determined by considering the secularity condition at $\operatorname{ord}(\varepsilon^2)$. This turns out to yield a T-secular term in A_1 resulting in εA_1 no longer being small relative to A_0 at $T = \operatorname{ord}(1/\varepsilon)$, i.e. $t = \operatorname{ord}(1/\varepsilon^2)$. We would then need to introduce a slow-slow time $T_2 = \varepsilon^2 t$, and seek an expansion for $y(t, T, T_2)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T} + \varepsilon^2\frac{\partial}{\partial T_2}. \tag{11.42}$$

Figure 11.2 shows that $A_0(T)$ correctly captures the slow variation of the amplitude of the oscillator. (There is no change in phase/frequency at this order, since ϕ is constant.)

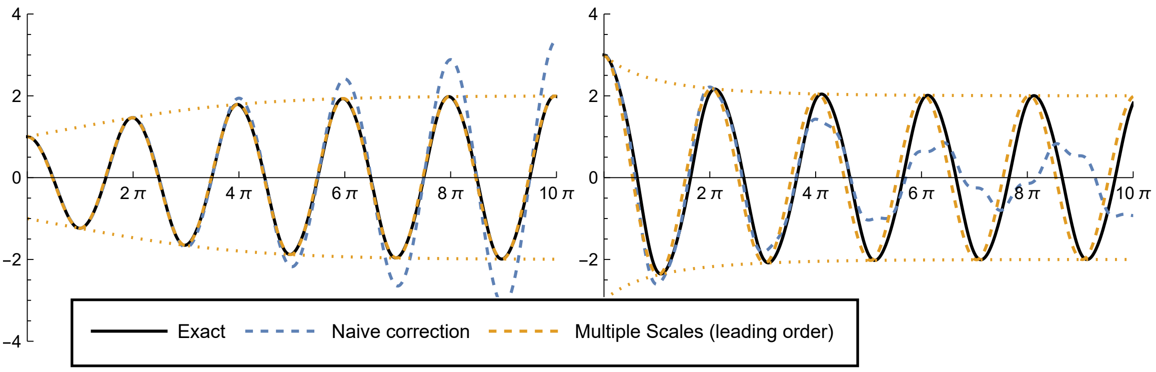


Figure 11.2: Solution of the Van der Pol oscillator $y'' + \varepsilon(y^2-1)y' + y = 0$, (a) $y(0) = 1$ or (b) $y(0) = 3$, and $y'(0) = 0$, with $\varepsilon = 0.2$.

Remark(s) 11.2. • The evolution equation for $A_0(T)$ is typically of the form $A_0'(T) = f(|A_0|^2)A_0$, where f is a polynomial, because if the terms in the equation are of the form

$$\alpha\left[A_0\mathrm{e}^{\mathrm{i}t}\right]^p\left[A_0^*\mathrm{e}^{-\mathrm{i}t}\right]^q + \mathrm{cc} = \alpha A_0^p(A_0^*)^q\mathrm{e}^{\mathrm{i}(p-q)t} + \mathrm{cc} \tag{11.43}$$

then the secular terms with $\mathrm{e}^{+\mathrm{i}t}$ have $(p-q) = 1$ and hence are of the form $\alpha|A_0|^{2q}A_0\mathrm{e}^{\mathrm{i}t}$.

- The evolution equation for $A_0(T)$ is sometimes simplified by splitting A_0 into magnitude and argument. In particular, for the above form,

$$A_0'(T) = f(|A_0|^2)A_0, \quad A_0(T) = r(T)\mathrm{e}^{\mathrm{i}\phi(T)} \quad \Rightarrow \quad r' + \mathrm{i}r\phi' = f(r^2)r. \tag{11.44}$$

If f is purely real then the argument is constant and the magnitude evolves. If f is purely imaginary then the magnitude is constant while the argument evolves (grows linearly with time).

- For more general f, the evolution equation for $A_0(T)$ might not have an explicit solution. In that case, we'd have to solve the equation for $A_0(T)$ numerically, but it's still an improvement to only have to resolve the evolution of the amplitude and phase on the slow time scale, corresponding to $t = \mathrm{ord}(1/\varepsilon)$, compared with solving the original equation where we would have to resolve the oscillations on the $t = \mathrm{ord}(1)$ timescale.

- Instead of the complex amplitude $A_0(T)$ it is also possible to use the real form

$$y_0 = a_0(T)\cos(t + \phi(T)) \quad \Rightarrow \quad \partial_T y_0 = a_0' \cos\hat{t} - a_0\phi' \sin\hat{t}, \quad \text{where } \hat{t} = t + \phi(T). \tag{11.45}$$

  This would yield, with $\hat{t} = t + \phi(T)$,

$$\begin{aligned}
\partial_t^2 y_1 + y_1 &= -2\partial_t\partial_T y_0 + (1 - y_0^2)\partial_t y_0 = 2\left(a_0' \sin\hat{t} + a_0\phi' \cos\hat{t}\right) - a_0 \sin\hat{t} + a_0^3 \underbrace{(\cos\hat{t})^2 \sin\hat{t}}_{=(\sin 3\hat{t} + \sin\hat{t})/4} & (11.46)\\
&= [2a_0\phi'] \cos\hat{t} + \left[2a_0' - a_0 + \tfrac{1}{4}a_0^3\right] \sin\hat{t} + \left[\tfrac{1}{4}a_0^3\right] \sin 3\hat{t} & (11.47)\\
&\Rightarrow \quad \phi' = 0, \quad a_0' = \frac{1 - \frac{1}{4}a_0^2}{2} a_0. & (11.48)
\end{aligned}$$

  These agree with our previous equations for $\arg A_0 = \phi$ and $|A_0| = a_0/2$.

11.3 Resonance

Resonance occurs when a time-dependent term in the equation excites oscillations at the natural frequency of the harmonic oscillator. We study this phenomenon using a harmonic oscillator with weak damping to consume any energy generated by the resonance, so as to avoid the amplitude diverging to infinity,

$$y''(t) + 2\varepsilon y'(t) + y(t) = 0, \tag{11.49}$$

which we will perturb in various ways.

Example 11.3. We illustrate the basic principle of resonance by considering a forcing term with some constant frequency ω and amplitude $2F > 0$:

$$y''(t) + 2\varepsilon y'(t) + y(t) = 2F\cos(\omega t) = F\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc}. \tag{11.50}$$

This equation is simple enough that we can solve it explicitly, by seeking a solution oscillating with the same frequency as the forcing,

$$y(t) = A\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} \quad \Rightarrow \quad (-\omega^2 + 2\mathrm{i}\omega\varepsilon + 1)A = F \quad \Rightarrow \quad A = \frac{F}{1 - \omega^2 + 2\varepsilon\mathrm{i}\omega}. \tag{11.51}$$

(We ignore the homogeneous solution $C\mathrm{e}^{-\varepsilon t + \mathrm{i}\sqrt{1-\varepsilon^2}t} + \mathrm{cc}$, which is simply damped oscillations that decay as $t \nearrow \infty$.) We note that the amplitude of the resulting oscillations is

$$|A| = \left|\frac{F}{1 - \omega^2 + 2\varepsilon\mathrm{i}\omega}\right| = \frac{|F|}{\sqrt{(1-\omega^2)^2 + 4\varepsilon^2\omega^2}}, \tag{11.52}$$

and thus identify two different regimes of behaviour.

In the "off-resonance" regime, where $\omega - 1 \gg \varepsilon$ (which is equivalent to $\omega^2 - 1 \gg \varepsilon$), the amplitude of the response is approximately $A \sim F/(1-\omega^2)$. This corresponds to a leading-order balance between the oscillator terms and the forcing, with damping being negligible:

$$y_0'' + y_0 = F\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} \quad \Rightarrow \quad y_0 = \frac{F}{1-\omega^2}\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc}. \tag{11.53}$$

However, in the "on-resonance" regime, where $\omega - 1 = O(\varepsilon)$, the amplitude becomes much larger, $A = \mathrm{ord}(1/\varepsilon)$. Importantly, if we look back at the terms in the governing equation, the terms that dominate are y'' and y which are both $\mathrm{ord}(1/\varepsilon)$, so the resonant solution satisfies the homogeneous equation to leading order while the forcing and damping terms that drive the leading-order oscillation only come in at next order.

Example 11.4. Let's now consider a resonance example that we can't solve exactly, by adding in a nonlinear term. For convenience, instead of having an ord(1) forcing resulting in an $\mathrm{ord}(1/\varepsilon)$ solution, we make the forcing be $\mathrm{ord}(\varepsilon)$ and expect to obtain an ord(1) solution. We thus consider the equation

$$y'' + 2\varepsilon y' + y + k\varepsilon y^3 = \varepsilon F\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc}, \tag{11.54}$$

where k is a given real ord(1) constant. (This is the Duffing equation again, now including damping and forcing.) We have chosen the coefficient of the nonlinear term to be $\mathrm{ord}(\varepsilon)$ so that it comes in at that order together with the forcing and damping terms. Further, we represent the closeness of the forcing frequency ω to the resonant frequency 1 by substituting

$$\omega = 1 + \varepsilon\omega_1, \tag{11.55}$$

and using the **detuning parameter** $\omega_1 = \mathrm{ord}(1)$ as the parameter instead of ω. (Note that this is an exact relation between the parameter ω and ω_1, and not an expansion of some solution in powers of ε.)

We now use the method of multiple scales,

$$T = \varepsilon t, \quad \frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T} \tag{11.56}$$

$$\Rightarrow \quad \partial_t^2 y + 2\varepsilon\partial_t\partial_T y + \varepsilon^2\partial_T^2 y + 2\varepsilon(\partial_t y + \varepsilon\partial_T y) + y + k\varepsilon y^3 = \varepsilon F\mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}\omega_1 T} + \mathrm{cc}. \tag{11.57}$$

Here, we have made the key substitution $\omega t = (1 + \varepsilon\omega_1)t = t + \omega_1 T$, in order to capture the long-term effect of the small frequency difference $\omega - 1 = \varepsilon\omega_1$ at leading order.

We then expand $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ and obtain $y_0 = A_0 \mathrm{e}^{\mathrm{i}t} + \mathrm{cc}$ followed by

$$\partial_t^2 y_1 + y_1 = \left[F\mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}\omega_1 T} + \mathrm{cc}\right] - 2\partial_t\partial_T y_0 - 2\partial_t y_0 - k y_0^3 = \tag{11.58}$$

$$= F\mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}\omega_1 T} - 2\mathrm{i}\partial_T A_0 \mathrm{e}^{\mathrm{i}t} - 2\mathrm{i}A_0\mathrm{e}^{\mathrm{i}t} - 3k|A_0|^2 A_0 \mathrm{e}^{\mathrm{i}t} - kA_0^3\mathrm{e}^{3\mathrm{i}t} + \mathrm{cc}. \tag{11.59}$$

We thus identify the secularity condition

$$2\mathrm{i}A_0' + 2\mathrm{i}A_0 + 3k|A_0|^2 A_0 = F\mathrm{e}^{\mathrm{i}\omega_1 T} \quad \Rightarrow \quad A_0' + (1 - \tfrac{3}{2}\mathrm{i}k|A_0|^2)A_0 = -\tfrac{1}{2}\mathrm{i}F\mathrm{e}^{\mathrm{i}\omega_1 T}. \tag{11.60}$$

The forcing term comes in with T-dependence $\mathrm{e}^{\mathrm{i}\omega_1 T}$, due to the forcing frequency $\omega = 1 + \varepsilon\omega_1$ being slightly different from the natural frequency 1, so we change variables to remove this,

$$A_0 = \bar{A}_0\mathrm{e}^{\mathrm{i}\omega_1 T} \quad \Rightarrow \quad \bar{A}_0' = (\bar{A}_0' + \mathrm{i}\omega_1\bar{A}_0)\mathrm{e}^{\mathrm{i}\omega_1 T} \quad \Rightarrow \quad \bar{A}_0' + (1 + \mathrm{i}\omega_1 - \tfrac{3}{2}k\mathrm{i}|\bar{A}_0|^2)\bar{A}_0 = -\tfrac{1}{2}\mathrm{i}F. \tag{11.61}$$

We can again decompose $\bar{A}_0(T) = r(T)\mathrm{e}^{\mathrm{i}\phi(T)}$ into magnitude and argument, but this time we are unable to solve for the evolution r and ϕ explicitly,

$$\hat{A}_0' = (r' + \mathrm{i}r\phi')\mathrm{e}^{\mathrm{i}\phi} \quad \Rightarrow \quad r' + \mathrm{i}r\phi' + (1 + \mathrm{i}\omega_1 - \tfrac{3}{2}\mathrm{i}kr^2)r = -\tfrac{1}{2}\mathrm{i}F\mathrm{e}^{-\mathrm{i}\phi} = \tfrac{1}{2}F(-\sin\phi - \mathrm{i}\cos\phi) \tag{11.62}$$

$$\Rightarrow \quad r' + r = \tfrac{1}{2}F\sin\phi, \quad r\phi' + (\omega_1 - \tfrac{3}{2}kr^2)r = -\tfrac{1}{2}F\cos\phi. \tag{11.63}$$

We instead seek just the steady-state solutions $r' = \phi' = 0$ (corresponding to steady periodic oscillations at this order), which yields

$$r = \tfrac{1}{2}F\sin\phi, \quad (\omega_1 - \tfrac{3}{2}kr^2)r = -\tfrac{1}{2}F\cos\phi \tag{11.64}$$

$$\Rightarrow \quad r^2\left[1 + (\omega_1 - \tfrac{3}{2}kr^2)^2\right] = \tfrac{1}{4}F^2, \quad \cot\phi = \tfrac{3}{2}kr^2 - \omega_1. \tag{11.65}$$

This is the amplitude–frequency response equation. The dependence of r on ω_1 can be plotted parametrically by solving for $\omega_1 = \frac{3}{2}kr^2 \pm \sqrt{F^2/(4r^2) - 1}$ as a function of r instead (figure 11.3). For sufficiently large nonlinearity $|k|$, there is a range of frequencies ω_1 for which three steady amplitudes r are possible, and it turns out that the top and bottom branches are stable while the middle branch is unstable. With two stable branches coexisting, the value selected by the system depends on the initial conditions, and the system can exhibit hysteresis (i.e. the state of the system is dependent not only on the parameter values but also on its history): If the frequency ω_1 is slowly increased, the amplitude r will follow one solution branch until it ceases to exist, at which point r will jump onto the other solution branch. If the frequency is then slowly decreased, the amplitude r stays on the other branch and does not jump back until the other branch ceases to exist, resulting in a "hysteresis loop".

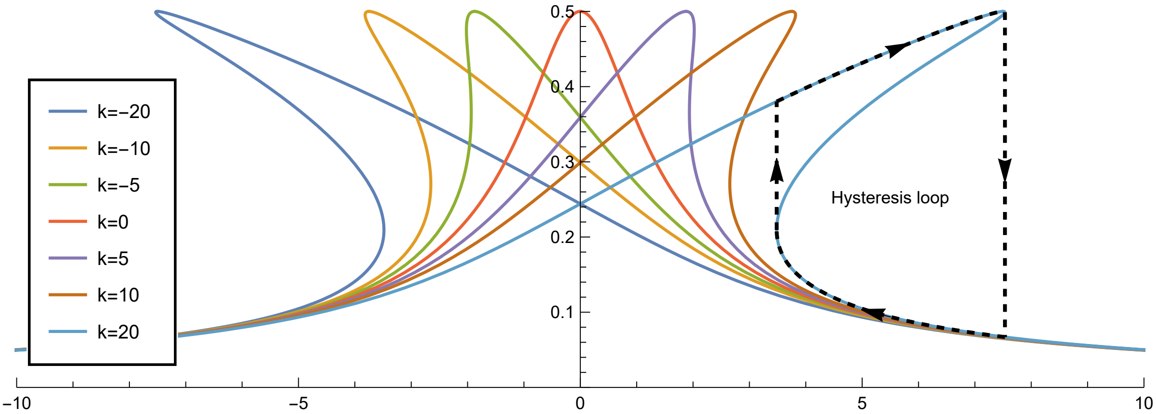


Figure 11.3: Dependence of steady oscillation amplitude $r = |A_0|$ on detuning frequency $\omega_1 = (\omega - 1)/\varepsilon$ for the Duffing equation $\varepsilon y'' + 2\varepsilon y' + y + k\varepsilon y^3 = \varepsilon\, 2F\cos(\omega t)$, with $F = 1$ and various values of k.

Remark(s) 11.3. We considered the example when the forcing resonates with the fundamental mode, i.e. the forcing frequency ω_f equals the frequency ω_0 of the oscillator (which we have taken to be $\omega_0 = 1$), but with nonlinearities we can also have resonance due to the forcing nonlinearly generating a harmonic with frequency $n\omega_f$ that equals ω_0, and, perhaps more surprisingly, a forcing of frequency $\omega_f = n\omega_0$ interacting nonlinearly with a natural ω_0 oscillation to provide the driving force that sustains it.

Example 11.5. We finally consider an example of parametric resonance.

The angle $\theta(\tau)$ of a pendulum whose pivot point is oscillated up and down with amplitude $2a$ and frequency Ω satisfies the equation

$$m\ell^2\theta''(\tau) + m(g + 2a\Omega^2\cos(\Omega\tau))\ell\sin\theta(\tau) = 0. \tag{11.66}$$

Rescaling time by $\tau = (\ell/g)^{1/2}t$, assuming a small angle $\theta = \delta y \ll 1$ (this time ignoring the nonlinearity completely), and introducing a damping term, yields the Mathieu equation

$$y'' + (1 + \varepsilon\, 2\cos(\omega t))y = 0, \quad \varepsilon = a\Omega^2/g, \quad \omega = (\ell/g)^{1/2}\Omega, \tag{11.67}$$

and we consider the limit $\varepsilon \searrow 0$ where the oscillations are small. We note that the oscillation in the equation causes the instantaneous frequency $\sqrt{1 + \varepsilon\, 2\cos(\omega t)}$ to oscillate. Unlike the case when there is an oscillating forcing term, this equation is linear and homogeneous, so does have a trivial solution $y = 0$, but there are also nontrivial solutions which may be amplified due to the resonance.

Let's start with a naive expansion $y(t) = y_0(t) + \varepsilon y_1(t) + \cdots$, which yields

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \tag{11.68}$$

$$y_1'' + y_1 = -2\cos(\omega t)y_0 = -\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc}\right)\left(A_0\mathrm{e}^{\mathrm{i}t} + cc\right) = A_0\mathrm{e}^{\mathrm{i}(\omega+1)t} + A_0^*\mathrm{e}^{\mathrm{i}(\omega-1)t} + \mathrm{cc}. \tag{11.69}$$

We obtain a secular term if $\omega + 1 = \pm 1$ or $\omega - 1 = \pm 1$, for which the only positive solution is $\omega = 2$. (The case $\omega = 0$ also yields a secular term, but looking back at the original equation we recognise that this just changes the oscillator frequency to a new constant $\sqrt{1 + 2\varepsilon}$.)

We conclude that this parametric resonance occurs when the driving frequency is (nearly) double the oscillator frequency. To investigate this phenomenon in more detail, we use the method of multiple scales and introduce a slow time $T = \varepsilon t$, and further we allow for driving frequency to be slightly different from the resonant value 2,

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}, \quad \omega = 2 + \varepsilon\omega_1 \quad \Rightarrow \quad \partial_t^2 y + \varepsilon 2\partial_t\partial_T y + \varepsilon^2\partial_T^2 y + \left[1 + \varepsilon(\mathrm{e}^{2\mathrm{i}t}\mathrm{e}^{\mathrm{i}\omega_1 T} + \mathrm{cc})\right]y = 0. \tag{11.70}$$

We expand $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ and obtain $y_0 = A_0(T)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}$ followed by

$$y_1'' + y_1 = -2\partial_t\partial_T y_0 - (\mathrm{e}^{2\mathrm{i}t}\mathrm{e}^{\mathrm{i}\omega_1 T} + \mathrm{cc})y_0 = \mathrm{e}^{\mathrm{i}t}\left[-2\mathrm{i}A_0'(T) - A_0^*\mathrm{e}^{\mathrm{i}\omega_1 T}\right] - \mathrm{e}^{3\mathrm{i}t}A_0\mathrm{e}^{\mathrm{i}\omega_1 T} + \mathrm{cc}. \tag{11.71}$$

The secularity condition then yields

$$2\mathrm{i}A_0' + A_0^*\mathrm{e}^{\mathrm{i}\omega_1 T} = 0 \quad \Rightarrow \quad A_0' = \tfrac{\mathrm{i}}{2}A_0^*\mathrm{e}^{\mathrm{i}\omega_1 T}. \tag{11.72}$$

This equation can be solved as follows. We eliminate the T-variation by defining

$$A_0 = \bar{A}_0\mathrm{e}^{\mathrm{i}\omega_1 T/2} \quad \Rightarrow \quad A_0' = (\bar{A}_0' + \tfrac{\mathrm{i}}{2}\omega_1\bar{A}_0)\mathrm{e}^{\mathrm{i}\omega_1 T/2} \quad \Rightarrow \quad \bar{A}_0' + \tfrac{\mathrm{i}}{2}\omega_1\bar{A}_0 = \tfrac{\mathrm{i}}{2}\bar{A}_0^*. \tag{11.73}$$

Splitting $\bar{A}_0 = \bar{A}_\mathrm{r} + \mathrm{i}\bar{A}_\mathrm{i}$ into real and imaginary parts then yields

$$\bar{A}_\mathrm{r}' + \mathrm{i}\bar{A}_\mathrm{i}' = -\frac{\mathrm{i}}{2}\omega_1(\bar{A}_\mathrm{r} + \mathrm{i}\bar{A}_\mathrm{i}) + \frac{\mathrm{i}}{2}(\bar{A}_\mathrm{r} - \mathrm{i}\bar{A}_\mathrm{i}) = \frac{\omega+1}{2}\bar{A}_\mathrm{i} + \mathrm{i}\frac{-\omega_1+1}{2}\bar{A}_\mathrm{r} \tag{11.74}$$

$$\Rightarrow \quad \bar{A}_\mathrm{r}'' = \frac{\omega+1}{2}\,\frac{-\omega_1+1}{2}\bar{A}_\mathrm{r} = \frac{1-\omega_1^2}{4}\bar{A}_\mathrm{r} \quad \Rightarrow \quad \bar{A}_\mathrm{r} \propto \mathrm{e}^{\pm\sqrt{(1-\omega_1^2)/4}T}. \tag{11.75}$$

We see that if the frequency is sufficiently close to the resonant one, $|\omega_1| < 1$, then there is an exponentially growing solution, while if the frequency is farther away, $|\omega_1| > 1$, then the solutions oscillate.

Since the original equation was linear, the amplitude of the solution is arbitrary, and it is not surprising that the solution that grows does so exponentially without bound. Including a linear damping term $+2\mu\varepsilon y'$ would simply modify the growth criterion and rate, but a suitable nonlinear term could cause the amplitude to saturate at a finite value instead of growing indefinitely.