MATH60005/70005: Optimisation (Autumn 24-25)

Chapter 7: exercises

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1. Solve the problem

$$\begin{array}{ll} \min & x_1^2 + 2x_2^2 + 4x_1x_2 \\ \text{s.t.} & \mathbf{x} \in \Delta_2 \,. \end{array}$$

2. **Orthogonal regression**. Suppose we have $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. For a given $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$, we define the hyperplane:

$$H_{\mathbf{x},y} := \left\{ \mathbf{a} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{a} = y \right\}$$

In the orthogonal regression problem, we seek to find a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$ such that the sum of squared Euclidean distances between the points $\mathbf{a}_1, \ldots, \mathbf{a}_m$ to $H_{\mathbf{x},y}$ is minimal:

$$\min_{\mathbf{x},y} \left\{ \sum_{i=1}^{m} d\left(\mathbf{a}_i, H_{\mathbf{x},y}\right)^2 : \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R} \right\}$$

Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}$$

Show the optimal solution of the orthogonal regression problem is given by $\mathbf{x}$ which is an eigenvector of the matrix $\mathbf{A}^\top(\mathbb{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^\top)\mathbf{A}$ associated with the minimum eigenvalue and $y = \frac{1}{m}\sum_{i=1}^{m} \mathbf{a}_i^\top \mathbf{x}$.

3. Consider the problem

$$\begin{array}{ll} \min & x_1^2 - x_2 \\ \text{s.t.} & x_2 = 0 \,, \end{array}$$

and its equivalent formulation

$$
\begin{array}{ll}
\min & x_1^2 - x_2 \\
\text{s.t.} & x_2^2 \leq 0\,.
\end{array}
$$

Determine KKT conditions for both problems, are they equivalent and solvable?