Linear Algebra **Unseen Question 6**

This project is devoted to calculating the number of conjugacy classes of the finite general linear groups $GL(n, \mathbb{F}_p)$. We know by the RCF Theorem that this is equal to the number of different RCFs of invertible $n \times n$ matrices over $\mathbb{F}_p$. We have seen a few examples of this calculation in the lecture notes and on Sheet 7, Q6.

(A) First consider $GL(2, \mathbb{F}_p)$.

 (1) For each $\alpha \in \mathbb{F}_p \setminus 0$, calculate the number of RCFs with characteristic polynomial $(x-\alpha)^2$.

 (2) For any other monic quadtratic polynomial $p(x) \in \mathbb{F}_p[x]$ with $p(0) \neq 0$, show that there is exactly one RCF with characteristic polynomial $p(x)$.

 (3) Using (1) and (2) compute the total number of conjugacy classes of $GL(2, \mathbb{F}_p)$.

(B) Now do the same for $GL(3, \mathbb{F}_p)$.

 (1) For each $\alpha \in \mathbb{F}_p \setminus 0$, calculate the number of RCFs with characteristic polynomial $(x-\alpha)^3$.

 (2) For distinct $\alpha, \beta \in \mathbb{F}_p \setminus 0$, calculate the number of RCFs with characteristic polynomial $(x-\alpha)^2(x-\beta)$.

 (3) For any other monic cubic polynomial $p(x) \in \mathbb{F}_p[x]$ with $p(0) \neq 0$, show that there is exactly one RCF with characteristic polynomial $p(x)$.

 (4) Hence compute the total number of conjugacy classes of $GL(3, \mathbb{F}_p)$.

(C) Try to repeat for $GL(4, \mathbb{F}_p)$,... and beyond, if you feel like it!