Imperial College London

MATH97044 MATH70056

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebraic Geometry

Date: 11 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In this exam, all algebraic sets (=varieties) are defined over an algebraically closed field k. It may have any characteristic, unless otherwise specified. You can use results from lectures, the lecture notes, problem sheets, and coursework. You may use the statements of earlier parts of a question without proof in solutions of later parts of that question. *Budget your time—attempt all questions!* They do not always get harder.

1. **The Zariski topology of $\mathbb{A}^n$.** In this question we work in $\mathbb{A}^n = k^n$. Ideals are in $R = k[x_1, \dots, x_n]$. An affine algebraic set (or variety) is a closed subset of $\mathbb{A}^n$ for some n.

   (a) Let $W \subseteq \mathbb{A}^n$ be an affine algebraic set. Prove that every open subset of W is a union of principal affine open subsets $D(f) := W \setminus \mathbb{V}(f)$. (2 marks)

   (b) Continuing with the notation of (a), prove that each principal affine open subset $D(f) \subseteq W$ is homeomorphic to an affine algebraic set in $\mathbb{A}^{n+1}$. (3 marks)

   (c) For each of the following sets, say whether, in the Zariski topology in $\mathbb{A}^n$, they are: (A) closed, (B) locally closed but not closed, (C) constructible but not locally closed, or (D) not constructible. Recall that "locally closed" means an open subset of a closed set, and "constructible" means a finite union of locally closed sets. *Justify your answers.*

      (i) The locus $\{(x^2 + x)(y^2 + z^2) = 0\}$ in $\mathbb{A}^3$.

      (ii) The locus $\{z \neq 0\} \cup \{(0, 0, 0)\}$ in $\mathbb{A}^3$.

      (iii) The locus $\{z \neq 0\} \cap \{x^2 + y^2 = z\}$ in $\mathbb{A}^3$.

      (iv) The locus $\{(\frac{1}{n}, n) \mid n \geq 1\} \subseteq \mathbb{A}^2$, in the case that $\operatorname{char} k = 0$. (6 marks)

   (d) Now, for each example (i)–(iv) from Part (c), say whether it is, in the Zariski topology, (A) irreducible; (B) connected but reducible; or (C) disconnected.

   (6 marks)

   (e) Prove that a nonconstant map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is continuous in the Zariski topology if and only if it has finite fibres. (3 marks)

(Total: 20 marks)

2. **Rational and regular maps.**

(a) Compute the ideal $\mathbb{I}(V)$ in terms of generators, and compute the ring of regular functions $k[V]$, on each of the following:

(i) $\{y = 0\} \cup \{y = x^2\}$ inside $\mathbb{A}^2$ with coordinates x, y. (3 marks)

(ii) $\{(t^2, t^3, t^4) \mid t \in k\} \cup \{x\text{-axis}\}$ inside $\mathbb{A}^3$ with coordinates x, y, z. (3 marks)

(b) Let $f : V \to W \subseteq \mathbb{A}^n$ be a regular map of affine varieties. Prove that there exist $m \geq n$, an affine algebraic set $V' \subseteq \mathbb{A}^m$, and an isomorphism $\iota_V : V \to V'$, such that f can be expressed as

$$f = \pi \circ \iota_V,$$

for $\pi : \mathbb{A}^m \to \mathbb{A}^n, \pi(a_1, \ldots a_m) = (a_1, \ldots, a_n)$ the projection onto the first n coordinates. (4 marks)

(c) Let $f : \mathbb{A}^3 \to \mathbb{A}^3$ be given by $f(x, y, z) = (x, xy, xyz)$.

(i) Prove that f is birational and find a rational inverse to f. (2 marks)

(ii) Find the regular locus (or "domain") of f^{-1}, with proof. (3 marks)

(d) Let $V = \{xy = 0\} \subseteq \mathbb{A}^2$. Find all regular isomorphisms $V \to V$. (5 marks)

(Total: 20 marks)

3. **Quasi-projective varieties.** In this question V and W are quasi-projective varieties (=algebraic sets). Recall the following useful facts from problem sheets: *Every rational map* $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ *is regular; every regular map* $f : \mathbb{P}^m \to \mathbb{P}^n$ *is globally defined by polynomials,* $f([x_0 : \cdots : x_m]) = [f_0 : \cdots : f_n]$.

(a) Let $C = \{(t^2, t^3, t^4) \mid t \in k\} \subseteq \mathbb{A}^3_0 \subseteq \mathbb{P}^3$. Find $\overline{C} \subseteq \mathbb{P}^3$, and find a homogeneous ideal I such that $\overline{C} = \mathbb{V}(I)$, explicitly in terms of generators. (4 marks)

(b) (i) Let $\operatorname{char} k \neq 3$, and $E = \mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$. Let $\pi : \mathbb{P}^2 \setminus \{[1:0:0]\} \to \mathbb{P}^1$ be the projection from the origin $[1:0:0]$ to the line $\mathbb{V}(x_0)$ at infinity. Show that π restricts to a regular map $\pi|_E : E \to \mathbb{P}^1$ whose fibres are all of size three except over three points, where the fibres have size one.

(4 marks)

(ii) Now let $\operatorname{char} k$ be arbitrary. Show that a regular map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined globally by polynomials of degree n which has the property that $f^{-1}([0:1]) = \{[0:1]\}$ and $f^{-1}([1:0]) = \{[1:0]\}$ must have the form $f([x:y]) = [cx^n : y^n]$ for some n and $c \in k^\times$. Show that all other fibres have the same size, equal to the number of n-th roots of unity. (3 marks)

(iii) Deduce from the preceding parts that $E = \mathbb{V}(x_0^2 + x_1^3 + x_2^3)$ is isomorphic to $\mathbb{P}^1$ if and only if $\operatorname{char} k = 3$. (4 marks)

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(c) (i) Prove that two nonempty open subsets $U, V \subseteq \mathbb{P}^1$ are isomorphic quasi-projective varieties if and only if there is an automorphism of $\mathbb{P}^1$ sending U^c to V^c. (2 marks)

(ii) Show that if $U, V \subseteq \mathbb{P}^1$ are open subsets such that $|\mathbb{P}^1 \setminus U| = |\mathbb{P}^1 \setminus V| \leq 3$, then $U \cong V$. (3 marks)

(Total: 20 marks)

4. **Dimension theory.**

(a) Show that, if $W = \mathbb{V}(f_1, \ldots, f_m) \subseteq \mathbb{P}^m$ for $f_1, \ldots, f_m$ homogeneous of positive degree, then every irreducible component of W has dimension $\geq n - m$. (2 marks)

(b) Let V, W be irreducible and quasi-projective, and let $\varphi : V \to W$ be a regular map. Show that there is an open dense subset $U \subseteq \varphi(V)$ which is quasi-projective, such that $\varphi^{-1}(U) \to U$ has constant fibre dimension d and $\dim V = \dim U + d$. You may use the fibre dimension theorem and the fact (from Chevalley's theorem) that $\varphi(V)$ contains an open dense quasi-projective set.

(2 marks)

(c) Let $V \subseteq \mathbb{P}^n$ be closed and irreducible of dimension m, and let $p \in V$. Let $\pi_p : \mathbb{P}^n \backslash \{p\} \to \mathbb{P}^{n-1}$ be the projection, and let $W := \pi_p(V \setminus \{p\})$.

(i) Show that W is irreducible and of dimension m or $m - 1$.
Hint: apply part (b).

(3 marks)

(ii) Show that $\dim W = m - 1$ if and only if $V = \pi_p^{-1}(W) \cup \{p\}$ (a cone with vertex p).
(4 marks)

(d) Let $C_{n,m} \subseteq (\mathbb{P}^n)^m = \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ be the set of m-tuples of points in $\mathbb{P}^n$ which are all collinear. Let $\Delta_m := \{(p, \ldots, p) \mid p \in \mathbb{P}^n\} \subseteq (\mathbb{P}^n)^m$, called the diagonal.

(i) Let $\pi_{i,j} : C_{n,m} \subseteq \mathbb{P}^n \times \cdots \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the projection of $C_{n,m}$ to the i-th and j-th factors: $\pi_{i,j}(p_1, \ldots, p_m) = (p_i, p_j)$. Prove that the fibres of $\pi_{i,j}$ over $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta_2$ are isomorphic to $(\mathbb{P}^1)^{m-2}$. (2 marks)

(ii) Using the previous part and part (b), show that $\pi_{i,j}^{-1}((\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta_2)$ has dimension $2n + (m - 2)$. (5 marks)

(iii) Conclude that $\dim C_{n,m} = 2n + (m - 2)$. (2 marks)

(Total: 20 marks)

5. **Mastery: MaxSpec and Spec.**

(a) (i) Define a Jacobson ring. (1 mark)

(ii) Prove that for such a ring, the inclusion $\operatorname{MaxSpec} R \subseteq \operatorname{Spec} R$ induces a bijection on closed subsets. (4 marks)

(iii) Prove that it also induces a bijection on irreducible closed subsets. (2 marks)

(iv) Given the Nullstellensatz for $\operatorname{Spec} R$, with R Jacobson, how do we conclude the Nullstellensatz for $\operatorname{MaxSpec} R$? (3 marks)

(b) Consider $V := \mathbb{V}(x^3 - 1) \subseteq \operatorname{MaxSpec} \mathbb{Z}[x]$. (You may use that $\mathbb{Z}[x]$ is Jacobson.)

(i) Find the irreducible components of V. (3 marks)

(ii) For every prime integer $p \geq 2$, find the intersection $V \cap \mathbb{V}(p)$ (break into natural cases). Compute also the ring $\mathbb{Z}[x]/(x^3 - 1, p)$, and compare the two. Facts you may use: Modulo p (in $\mathbb{F}_p$), there is a primitive n-th root of unity if and only if $n \mid p - 1$. Also, if $f \in F[x]$ is an irreducible polynomial over a field, then (f) is maximal. (3 marks)

(c) Consider the local ring $R := k[x, y]_{(x)}$.

(i) Compute $\operatorname{Spec} R$ and $\operatorname{MaxSpec} R$. (2 marks)

(ii) Compute the topological dimension of $\operatorname{Spec} R$ and $\operatorname{MaxSpec} R$. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH70056/97044

Algebraic Geometry (Solutions)

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1. **The Zariski topology of $\mathbb{A}^n$.** *In this question we work in $\mathbb{A}^n = k^n$. Ideals are in $R = k[x_1, \ldots, x_n]$. An affine algebraic set (or variety) is a closed subset of $\mathbb{A}^n$ for some n.*

sim. bookwork ⇓

(a) *Let $W \subseteq \mathbb{A}^n$ be an affine algebraic set. Prove that every open subset of W is a union of principal affine open subsets $D(f) := W \setminus \mathbb{V}(f)$.*
If $W = \mathbb{V}(f_1, \ldots, f_m)^c$, then $W = \bigcup_{i=1}^m D(f_i)$. 2, A

sim. bookwork ⇓

(b) *Continuing with the notation of (a), prove that each principal affine open subset $D(f) \subseteq W$ is homeomorphic to an affine algebraic set in $\mathbb{A}^{n+1}$.*
Consider the subset $W' := \mathbb{V}(1 - x_{n+1} f(x_1, \ldots, x_n)) \subseteq \mathbb{A}^{n+1}$. The regular projection $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ to the first n factors maps W' bijectively to $D(f) \subseteq W$. The inverse is given by the rational map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n)^{-1})$ which is regular on $D(f)$. Thus the sets are isomorphic, and in particular homeomorphic. [Note: in lectures and supplementary material, we first explained this only as a homeomorphism; see CW1, #4 from 2021.] 3, A

(c) *For each of the following sets, say whether, in the Zariski topology in $\mathbb{A}^n$, they are: (A) closed, (B) locally closed but not closed, (C) constructible but not locally closed, or (D) not constructible. Recall that "locally closed" means an open subset of a closed set, and "constructible" means a finite union of locally closed sets.* Justify your answers.

sim. bookwork ⇓

(i) *The locus $\{(x^2 + x)(y^2 + z^2) = 0\}$ in $\mathbb{A}^3$.*
(A): this is by definition $\mathbb{V}((x^2 + x)(y^2 + z^2))$, which is closed. 1, A

(ii) *The locus $\{z \neq 0\} \cup \{(0, 0, 0)\}$ in $\mathbb{A}^3$.* sim. seen ⇓
This is a union of two locally closed sets $\{z \neq 0\}$ (open) and $\{(0, 0, 0)\}$ (closed), so constructible. It is not (A) because the closure is everything (an open subset of $\mathbb{A}^3$ is dense as $\mathbb{A}^3$ is irreducible) and not (B) because it would then have to be open in its closure, hence open, but $\mathbb{V}(z) \setminus \{(0, 0, 0)\}$ is not closed (its closure is $\mathbb{V}(z)$). So the answer is (C). 2, B

(iii) *The locus $\{z \neq 0\} \cap \{x^2 + y^2 = z\}$ in $\mathbb{A}^3$.* sim. bookwork ⇓
This is the intersection of an open and closed set, so by definition locally closed. 1, A

(iv) *The locus $\{(\frac{1}{n}, n) \mid n \geq 1\} \subseteq \mathbb{A}^2$, in the case that* $\operatorname{char} k = 0$. sim. seen ⇓
The closure of this set is $\mathbb{V}(xy - 1)$, since $\mathbb{V}(xy - 1) \cong \mathbb{A}^1 \setminus \{0\}$ is closed, has the cofinite topology, and the set is an infinite subset thereof. An infinite subset of $\mathbb{V}(xy - 1)$ is not constructible, since the proper locally closed subsets are all finite, and an infinite set is not a finite union of finite sets. So the answer is (D). 2, B

(d) Now, for each example (i)–(iv) from Part (c), say whether it is, in the Zariski topology, (A) irreducible; (B) connected but reducible; or (C) disconnected.

sim. seen ⇓

(i) This set is reducible, as it is the union of $\mathbb{V}(x^2 + x)$ and $\mathbb{V}(y^2 + z^2)$. These sets intersect at $\mathbb{V}(x^2 + x, y^2 + z^2)$ which is nonempty, so the set is (B) connected but reducible. 1, A

(ii) This set is dense since it contains the open dense subset $\{z \neq 0\}$ of $\mathbb{A}^3$ (which is irreducible). So it is (A) irreducible. sim. bookwork $\Downarrow$

1, A

(iii) This set is dense in the irreducible subset $\mathbb{V}(x^2 + y^2 - z) \cong \mathbb{A}^2$, so it is (A) irreducible. sim. seen $\Downarrow$

2, A

(iv) The closure of this set is $\mathbb{V}(xy-1)$ which is irreducible, so it is (A) irreducible. sim. seen $\Downarrow$

2, A

(e) *Prove that a nonconstant map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is continuous in the Zariski topology if and only if it has finite fibres.* sim. seen $\Downarrow$

To be continuous, we require that the preimage of closed sets is closed. As $\mathbb{A}^1$ has the cofinite topology, this means that the preimage of finite sets is finite or all of $\mathbb{A}^1$. The latter can only happen if the map is constant. The former happens if and only if each fibre is finite. 3, A

2. **Rational and regular maps.**

(a) *Compute the ideal $\mathbb{I}(V)$ in terms of generators, and compute the ring of regular functions $k[V]$, on each of the following:* sim. seen $\Downarrow$

(i) $\{y = 0\} \cup \{y = x^2\}$ *inside* $\mathbb{A}^2$ *with coordinates* x, y.
Let V be the locus in question. It is cut out by the intersection $I := (y) \cap (y - x^2) = (y(y - x^2))$ of ideals of vanishing of the two components. By the Nullstellensatz, provided I is radical, we obtain that $I = \mathbb{I}(V)$ and that $k[V] \cong k[x, y]/I$ is the ring of regular functions. To see that I is radical, note that $f^m \in (y(y - x^2))$ if and only if $y \mid f^m$ and $(y - x^2) \mid f^m$, by unique factorisation. Again applying unique factorisation, this is true if and only if $y \mid f$ and $(y - x^2) \mid f$, so $f \in (y(y - x^2))$, as desired. 3, A

(ii) $\{(t^2, t^3, t^4) \mid t \in k\}$ *inside* $\mathbb{A}^3$ *with coordinates* x, y, z. sim. seen $\Downarrow$
This is again similar. Let V be the locus in question. We claim that the locus $\{(t^2, t^3, t^4)\}$ is cut out by $x^3 - y^2$ and $x^2 - z$: if $x^3 = y^2$ and t is any solution to the equation $t^2 = x$, then the solutions to the equation $y^2 = x^3 = t^6$ are of the form $y = \pm t^3$. Then up to replacing t with its negative, we see that $x = t^2, y = t^3$. Also $z = x^2 = t^4$. So $(x, y, z) = (t^2, t^3, t^4)$ for some t. Conversely it is clear that $(x, y, z) = (t^2, t^3, t^4)$ satisfies the equations $x^3 = y^2, x^2 = z$. Let $I := (x^3 - y^2, x^2 - z)$. Then $\mathbb{V}(I) = \{(t^2, t^3, t^4)\}$. It remains to show that I is radical; then $k[V] \cong k[x, y, z]/I$ follows. To see this, note that $k[x, y, z]/I \cong k[x, y]/(x^3 - y^2)$ (by the map sending z to x^2). In turn, the latter is isomorphic to $k[t^2, t^3]$ by the map $x \mapsto t^2, y \mapsto t^3$. Since $k[t^2, t^3] \subseteq k[t]$, it is an integral domain, so in particular it has no nilpotent elements (in fact, I is prime). 3, C

(b) *Let $f : V \to W \subseteq \mathbb{A}^n$ be a regular map of affine varieties. Prove that there exist $m \geq n$, an affine algebraic set $V' \subseteq \mathbb{A}^m$, and an isomorphism $\iota_V : V \to V'$, such that f can be expressed as*

$$f = \pi \circ \iota_V,$$

for $\pi : \mathbb{A}^m \to \mathbb{A}^n, \pi(a_1, \dots a_m) = (a_1, \dots, a_n)$ the projection onto the first n coordinates. sim. bookwork $\Downarrow$

First of all assume that $V \subseteq \mathbb{A}^\ell$ and $W \subseteq \mathbb{A}^n$. Set $m := \ell + n$. Then we can let ι_V be the graph mapping $V \to \Gamma'_f := \{(f(v), v) \in \mathbb{A}^n \times V\}$, given by $v \mapsto (f(v), v)$. This is an isomorphism onto the permuted graph of V as shown in lectures (the inverse is the projection map to the factor of V, i.e., to the last ℓ coordinates). Then let $\pi : \mathbb{A}^{n+\ell} \to \mathbb{A}^n$ be the projection to the first n coordinates. We then have that $\pi \circ \iota_V : V \to \mathbb{A}^n$ is just the map f with image W. 4, A

(c) *Let $f : \mathbb{A}^3 \to \mathbb{A}^3$ be given by $f(x, y, z) = (x, xy, xyz)$.* sim. seen $\Downarrow$

(i) *Prove that f is birational and find a rational inverse to f.*
The rational inverse is $(u, v, w) \mapsto (u, v/u, w/v)$. It is clear that the composition in either direction is the identity. So f is birational. 2, A

(ii) *Find the regular locus (or "domain") of f^{-1}, with proof.* sim. seen $\Downarrow$
The regular locus clearly contains the locus $\{u \neq 0, v \neq 0\}$. To show the opposite containment, suppose that (f_1, f_2, f_3) is an alternative expression for $(u, v/u, w/v)$. Then $f_2 = \frac{p_2}{q_2}$ for polynomials p_2, q_2 such that $p_2 u = q_2 v$. But $k[u, v, w]$ is a UFD, so this implies that $u \mid q_2$. Similarly, if $f_3 = p_3 q_3$, then $v \mid q_3$. 3, A

(d) *Let $V = \{xy = 0\} \subseteq \mathbb{A}^2$. Find all regular isomorphisms $V \to V$.* unseen $\Downarrow$

A regular isomorphism $V \to V$ in particular gives a regular isomorphism of irreducible components. The irreducible components are $V_1 = \{x = 0\}$ and $V_2 = \{y = 0\}$. So one either permutes these components or preserves them. Note that there is an automorphism which swaps the components: $x \leftrightarrow y$. So up to composing with this automorphism, we can restrict our attention to automorphisms preserving V_1 and V_2. These components are each isomorphic to $\mathbb{A}^1$. Let us look at automorphisms of $\mathbb{A}^1$. These correspond to k-linear automorphisms of the algebra of regular functions $k[\mathbb{A}^1] = k[x]$, in turn this is given by $x \mapsto f(x)$ which is invertible, i.e., f must be linear. We claim that in this case also $f(0) = 0$. This is true because 0 is the intersection of the two irreducible components. So our automorphism preserving V_1 and V_2 must have the form $(x, y) \mapsto (ax, by)$ for $a \in k^\times$. Conversely these are clearly automorphisms of V. So all automorphisms of V have the form $(x, y) \mapsto (ax, by)$ or $(x, y) \mapsto (by, ax)$, for $a, b \in k^\times$. 5, D

3. **Quasi-projective varieties.** *In this question V and W are quasi-projective varieties (=algebraic sets). Recall the following useful facts from problem sheets:* Every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular; every regular map $f : \mathbb{P}^m \to \mathbb{P}^n$ is globally defined by polynomials, $f([x_0 : \cdots : x_m]) = [f_0 : \cdots : f_n]$.

sim. seen ⇓

(a) *Let $C = \{(t^2, t^3, t^4) \mid t \in k\} \subseteq \mathbb{A}^3_0 \subseteq \mathbb{P}^3$. Find $\overline{C} \subseteq \mathbb{P}^3$, and find a homogeneous ideal I such that $\overline{C} = \mathbb{V}(I)$, explicitly in terms of generators.*

Suppose that $f(x_0, x_1, x_2, x_3)$ is a homogeneous polynomial vanishing on C. Then $x_3/x_0 = (x_1/x_0)^2$ on $\mathbb{A}^3_0$, hence $x_3x_0 = x_1^2$ on a dense subset, hence on all of $\overline{C}$. Plugging in $x_0 = 0$ we get $x_1 = 0$ in this case. Similarly, $(x_1/x_0)(x_3/x_0) = (x_2/x_0)^2$ on $\mathbb{A}^3_0$, hence $x_1x_3 = x_2^2$ on a dense subset, hence on all of $\overline{C}$. Now when $x_1 = 0$ we get also $x_2 = 0$. Thus the intersection of C with $\mathbb{V}(x_0)$ can only include the point $[0 : 0 : 0 : 1]$. On the other hand, this intersection cannot be zero, as we know from lectures (the intersection of a positive-dimensional closed subset of $\mathbb{P}^n$ with a hyperplane is nonempty; this curve is infinite hence of positive dimension). So $\overline{C} = C \cup \{[0 : 0 : 0 : 1]\}$.

In the preceding paragraph we found that the homogeneous ideal contains the elements $x_3x_0 - x_1^2$ and $x_1x_3 - x_2^2$. These elements are enough to guarantee that the intersection with $\mathbb{V}(x_0)$ is just a point. Also, on $\mathbb{A}^3_0$, these equations yields $z = x^2, xz = y^2$, which are enough to guarantee that $(x, y, z) \in C$ (substituting, the second equation implies $x^3 = y^2$, now see the solution to 2(a)(ii)). So setting $I := (x_3x_0 - x_1^2, x_1x_3 - x_2^2)$, we get $\overline{C} = \mathbb{V}(I)$.

(Remark: One can actually see that this homogeneous ideal is radical: if $f^m \in I$, then when we set $x_0 = 1$, the inhomogenisation of f is in the inhomogenisation of I. Hence $x_0^m f \in I$ for some I. Thus it suffices to show that if $x_0 f \in I$, then $f \in I$. Now if $x_0 f = (x_3x_0 - x_1^2)g + (x_1x_3 - x_2^2)h$, then $x_0 \mid -x_1^2 g + (x_1x_3 - x_2^2)h$, so this equation also holds for the parts g_0, h_0 of g, h which are constant in x_0. As $x_1^2, x_1x_3 - x_2^2$ are relatively prime, this can only happen if $g_0 = \overline{g}(x_1x_3 - x_2^2), h_0 = \overline{h}x_1^2$ for some $\overline{g}, \overline{h}$. Then we get $(x_3x_0 - x_1^2)(x_1x_3 - x_2^2) + x_1^2(x_1x_3 - x_2^2) = x_3x_0(x_1x_3 - x_2^2)$ and this is indeed in $x_0 I$. Thus also $x_0 f \in x_0 I$, so $f \in I$.) 4, C

(b) (i) *Let $\operatorname{char} k \neq 3$, and $E = \mathbb{V}(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$. Let $\pi : \mathbb{P}^2 \setminus \{[1 : 0 : 0]\} \to \mathbb{P}^1$ be the projection from the origin $[1 : 0 : 0]$ to the line $\mathbb{V}(x_0)$ at infinity. Show that π restricts to a regular map $\pi|_E : E \to \mathbb{P}^1$ whose fibres are all of size three except over three points, where the fibres have size one.* unseen ⇓

We saw in lectures that π is regular (on the complement of $\{[1 : 0 : 0]\}$). Since $[1 : 0 : 0] \notin E$, it follows that $\pi|_E$ is regular. The fibres are the intersections of lines through $[1 : 0 : 0]$ with E. Given such a line $\ell_{[a:b]} := [s : at : bt]$ (for fixed $[a : b] \in \mathbb{P}^1$), the intersection $\ell_{[a:b]} \cap E$ is isomorphic to $\mathbb{V}(s^3 + (a^3 + b^3)t^3) \subseteq \mathbb{P}^1 = \{[s : t]\}$. This is a cubic polynomial which has three distinct roots, $[1 : \zeta]$ for $\zeta^3 = a^3 + b^3$, unless $a^3 + b^3 = 0$, in which case there is only a unique root. The latter situation only happens for $b \in \{-a, -\omega a, -\omega^2 a\}$, where ω is a primitive cube root of unity. Note also that $\ell_{[a:b]} \cap \mathbb{V}(x_0) = [a : b] \in \mathbb{P}^1$. So the fibres all have size three except for the fibres over $p \in \{[1 : -1], [1 : -\omega]$, and $[1 : -\omega^2]\}$, where the fibre is just the single point $[0 : p]$, i.e., $[0 : 1 : -1], [0 : 1 : -\omega]$, and $[0 : 1 : -\omega^2]$, respectively. 4, D

unseen ⇓

(ii) *Now let* $\operatorname{char} k$ *be arbitrary. Show that a regular map* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *defined globally by polynomials of degree* n *which has the property that* $f^{-1}([0:1]) = \{[0:1]\}$ *and* $f^{-1}([1:0]) = \{[1:0]\}$ *must have the form* $f([x:y]) = [cx^n : y^n]$ *for some* n *and* $c \in k^\times$. *Show that all other fibres have the same size, equal to the number of* n*-th roots of unity.*

We have $f([x:y]) = [f:g]$ for $f, g \in k[x_0, x_1]$ of degree n with no common roots (note that f and g factor linearly, as k is algebraically closed). The assumption guarantees that $f = ax_0^n$ and $g = bx_1^n$ for some nonzero $a, b \in k$. Up to rescaling we get the form guaranteed in the question. Finally, all other fibres are solutions of the equation $ax_0^n + bx_1^n = 0$ for some $a, b \in k^\times$. This equation has at least one root, and the number of roots equals the number of n-th roots of unity, since the quotient of two roots is such a root of unity. $\boxed{3, \text{B}}$

(iii) *Deduce from the preceding parts that* $E = \mathbb{V}(x_0^3 + x_1^3 + x_2^3)$ *is isomorphic to* $\mathbb{P}^1$ *if and only if* $\operatorname{char} k = 3$. $\boxed{\text{unseen} \Downarrow}$

First, if $\operatorname{char} k = 3$, then $x_0^3+x_1^3+x_2^3 = (x_0+x_1+x_2)^3$, so $E = \mathbb{V}(x_0+x_1+x_2)$, the equation of a line, so $E \cong \mathbb{P}^1$. Otherwise, by (i), we have a regular map $\varphi : E \to \mathbb{P}^1$ with all fibres of size three except for exactly three points. Now suppose that there exists an isomorphism $\psi : E \to \mathbb{P}^1$. By composition we have a regular map $\varphi \circ \psi^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ with all fibres of size three except for exactly three fibres of size one. Note also that there are automorphisms of $\mathbb{P}^1$ sending any pair of points to $[1:0], [0:1]$: this is just a linear transformation of k^2 which sends a given basis to the standard one. Precomposing and postcomposing by such automorphisms, we can assume the fibres over $[0:1]$ and over $[1:0]$ are the same points. Now, the fact mentioned at the start of the problem says that $\varphi \circ \psi^{-1}$ must be defined by global polynomials. Applying (ii), we then have that all fibres over points other than $[1:0]$ and $[0:1]$ have the same size, which is a contradiction. $\boxed{4, \text{D}}$

.

$\boxed{\text{sim. seen} \Downarrow}$

(c) (i) *Prove that two nonempty open subsets* $U, V \subseteq \mathbb{P}^1$ *are isomorphic quasi-projective varieties if and only if there is an automorphism of* $\mathbb{P}^1$ *sending* U^c *to* V^c.

Any such isomorphism would define a birational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. Such a map is regular by problem sheets. So would be its inverse, so that the isomorphism $U \to V$ extends to an automorphism of $\mathbb{P}^1$. $\boxed{2, \text{A}}$

(ii) *Show that if* $U, V \subseteq \mathbb{P}^1$ *are open subsets such that* $|\mathbb{P}^1 \setminus U| = |\mathbb{P}^1 \setminus V| \leq 3$, *then* $U \cong V$. $\boxed{\text{unseen} \Downarrow}$

The automorphisms of $\mathbb{P}^1$ are all linear, of the form $[x:y] \mapsto [ax+by : cx+dy]$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible (this follows since they are globally defined). So by the previous part, it suffices to show that every pair of sets of equal size at most three can be related by an automorphism. It suffices to show this for sets of size exactly three. Furthermore, it is enough by composing automorphisms to prove that, if $V \subseteq \mathbb{P}^1$ with $|V| = 3$, then there exists an automorphism φ of $\mathbb{P}^1$ such that $\varphi(V) = \{0, 1, \infty\}$. Let $V = \{[a:b], [c:d], [e:f]\}$. By a linear transformation sending $[a:b]$ to $[1:0]$ and $[c:d]$ to $[0:1]$

(this is possible since $[a : b] \neq [c : d]$), we can assume $[a : b] = [1 : 0]$ and $[c : d] = [0 : 1]$. Automorphisms fixing these two points have the form $[x : y] \mapsto [x : \lambda y]$ for $\lambda \in k^\times$. It is clear that these automorphisms act transitively on $\mathbb{P}^1 \setminus \{[0 : 1], [1 : 0]\}$, and more precisely, to take $[e : f]$ to $[1 : 1]$ we simply apply the automorphism $[x : y] \mapsto [x : \frac{e}{f} y]$. 3, D

4. **Dimension theory.**

sim. seen ⇓

(a) *Show that, if $W = \mathbb{V}(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ for $f_1, \ldots, f_m$ homogeneous of positive degree, then every irreducible component of W has dimension $\geq n - m$.*

We prove by induction on m. This is clear for $m = 0$. By the theorem on intersections with hypersurfaces, for each irreducible component $W' \subseteq W = \mathbb{V}(f_1, \ldots, f_m)$, the intersection $W' \cap \mathbb{V}(f_{m+1})$ has components of dimension either $\dim W'$ or $\dim W' - 1$. This completes the induction. 2, A

sim. seen ⇓

(b) *Let V, W be irreducible and quasi-projective, and let $\varphi : V \to W$ be a regular map. Show that there is an open dense subset $U \subseteq \varphi(V)$ which is quasi-projective, such that $\varphi^{-1}(U) \to U$ has constant fibre dimension d and $\dim V = \dim U + d$. You may use the fibre dimension theorem and the fact (from Chevalley's theorem) that $\varphi(V)$ contains an open dense quasi-projective set.*

Let $U_1 \subseteq \varphi(V)$ be the open dense quasi-projective subset guaranteed by Chevalley's theorem. Since V is irreducible, so is $\varphi(V)$, hence also U_1. Now $\varphi^{-1}(U_1) \subseteq V$ is open and hence dense by irreducibility of V. Then $\varphi^{-1}(U_1) \to U_1$ is a surjective regular map of irreducible quasi-projective sets. The fibre dimension theorem yields a dense open subset $U \subseteq U_1$ over which the fibres have constant dimension d and $\dim \varphi^{-1}(U_1) = \dim U_1 + d$. The result follows because the dimension of an open dense subset equals the dimension of the entire set. 2, B

(c) *Let $V \subseteq \mathbb{P}^n$ be closed and irreducible of dimension m, and let $p \in V$. Let $\pi_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ be the projection, and let $W := \pi_p(V \setminus \{p\})$.*

sim. seen ⇓

(i) *Show that W is irreducible and of dimension m or $m - 1$.*
Hint: apply part (b).
First, W is irreducible since it is the continuous image of an irreducible set. Next, by part (b), there is an open dense subset $U \subseteq W$ which is irreducible and quasi-projective, with $\pi_p^{-1}(U) \to U$ having constant fibre dimension d, and $\dim(V \setminus \{p\}) = \dim \pi_p(V \setminus \{p\}) + d$. However, the fibres of π_p have dimension at most one, as it is a projection map. This gives the desired statement. 3, B

(ii) *Show that $\dim W = m - 1$ if and only if $V = \pi_p^{-1}(W) \cup \{p\}$ (a cone with vertex p).* sim. seen ⇓

By the reasoning in the previous part, $\dim W = m - 1$ if and only if, over an open dense subset U of W, the fibres have dimension $\dim V' - \dim U = 1$. But this is also the minimum fibre dimension, so all fibres have dimension one. Since the fibres are closed in $\mathbb{P}^n \setminus \{p\}$, these fibres must be the entire lines through p excepting the point p itself. But this means that for every $q \in V \setminus \{p\}$, the line L_{pq} is contained in V. So $V = \pi_p^{-1}(W) \cup \{p\}$ is indeed a cone with vertex p. 4, B

(d) *Let $C_{n,m} \subseteq (\mathbb{P}^n)^m = \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ be the set of m-tuples of points in $\mathbb{P}^n$ which are all collinear. Let $\Delta_m := \{(p, \ldots, p) \mid p \in \mathbb{P}^n\} \subseteq (\mathbb{P}^n)^m$, called the diagonal.*

unseen ⇓

(i) *Let $\pi_{i,j} : C_{n,m} \subseteq \mathbb{P}^n \times \cdots \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the projection of $C_{n,m}$ to the i-th and j-th factors: $\pi_{i,j}(p_1, \ldots, p_m) = (p_i, p_j)$. Prove that the fibres of $\pi_{i,j}$ over $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta_2$ are isomorphic to $(\mathbb{P}^1)^{m-2}$.*

This is because a point is collinear with two distinct points if and only if it is on the line they form. Thus, for $p_i \neq p_j$, the fibre $\pi_{i,j}^{-1}(p_i, p_j)$ consists of tuples $(p_1, \ldots, p_m)$ with $p_\ell \in L_{p_i,p_j} \cong \mathbb{P}^1$, the line through p_i and p_j, for all ℓ.

2, B

(ii) *Using the previous part and part (b), show that $\pi_{i,j}^{-1}((\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta_2)$ has dimension $2n + (m-2)$.*

unseen ⇓

For each irreducible component $V \subseteq \pi_{i,j}^{-1}(\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta_2)$, part (i) and part (b) show that $\dim V \leq \dim(\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta_2) + (m-2)$. For this we used that $\dim((\mathbb{P}^1)^m) = m$, which is a special case of a result from lecture, but can also be seen because an open dense subset is $(\mathbb{A}^1)^m = \mathbb{A}^m$. Now $\dim(\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta_2) = \dim \mathbb{P}^n \times \mathbb{P}^n = 2n$. Applying part (b) gives that $\dim V \leq 2n + (m-2)$. We get equality if $\pi_{i,j}(V)$ is dense in $\mathbb{P}^n \times \mathbb{P}^n$ and the generic fibre dimension is $m-2$. But, since every fibre has dimension $m-2$, one of the irreducible components of $\pi_{i,j}^{-1}(\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta_2)$ must have generic fibre dimension $m-2$ as well (since the dimension of a union is the maximum of dimensions). So for some irreducible component we get the equality of dimensions desired.

5, C

(iii) *Conclude that $\dim C_{n,m} = 2n + (m-2)$.*

unseen ⇓

The subsets given in (ii) are all open in $C_{n,m}$. Their union is not necessarily all of $C_{n,m}$, but the complement lies in the diagonal $\Delta_m \subseteq (\mathbb{P}^n)^m$. As this diagonal has dimension $n < 2n + (m-2)$, the result follows from the fact that the dimension of a union is the maximum of dimensions.

2, B

5. **Mastery: MaxSpec and Spec.**

sim. bookwork ⇓

(a) (i) *Define a Jacobson ring.*

A Jacobson ring is one for which every prime ideal is an intersection of maximal ideals. 1, M

(ii) *Prove that for such a ring, the inclusion* $\operatorname{MaxSpec} R \subseteq \operatorname{Spec} R$ *induces a bijection on closed subsets.* sim. seen ⇓

This is equivalent to the assertion that, if $X \subseteq \operatorname{Spec} R$ is closed, then $X = \overline{\operatorname{MaxSpec} R \cap X}$, since the Zariski topology on $\operatorname{MaxSpec} R$ coincides with the subspace topology from $\operatorname{Spec} R$. Suppose that $X = \mathbb{V}(I)$. Then we need to show that, for every ideal J which is contained in every maximal ideal contining I, we also have that J contains X, i.e., every prime ideal containing I. For each $\mathfrak{p} \supseteq I$, we have that J is contained in every maximal ideal containing $\mathfrak{p}$. The intersection of these maximal ideals is $\mathfrak{p}$, by the Jacobson property. So $J \subseteq \mathfrak{p}$, as desired. 4, M

(iii) *Prove that it also induces a bijection on irreducible closed subsets.* sim. seen ⇓

To see this, recall that a closed set is irreducible if and only if it is not a union of proper closed subsets. The bijection from (ii) preserves the operation of taking unions. Therefore the image of a closed set X is irreducible if and only if X is. 2, M

(iv) *Given the Nullstellensatz for* $\operatorname{Spec} R$*, with* R *Jacobson, how do we conclude the Nullstellensatz for* $\operatorname{MaxSpec} R$*?* sim. bookwork ⇓

Given a closed subset $X \subseteq \operatorname{MaxSpec}(R)$, we have $X = \overline{X} \cap \operatorname{MaxSpec}(R)$. By definition of the map $\mathbb{I}$, we have $\mathbb{I}_{\operatorname{MaxSpec}(R)}(X) = \mathbb{I}_{\operatorname{Spec}(R)}(X)$. Then by definition of the Zariski topology, $\mathbb{I}(\overline{X}) = \mathbb{I}(X)$. So the map $\mathbb{I}$ respects the bijection in part (ii).

On the other hand, given an ideal $I \subseteq R$, $\mathbb{V}_{\operatorname{MaxSpec}(R)}(I) = \mathbb{V}_{\operatorname{Spec}(R)}(I) \cap \operatorname{MaxSpec}(R)(I)$, by definition. So the map $\mathbb{V}$ respects the bijection in part (ii) as well.

Therefore, if $\mathbb{I}$ and $\mathbb{V}$ give inverse bijections between closed subsets of $\operatorname{Spec}(R)$ and radical ideals, the same is true for closed subsets of $\operatorname{MaxSpec}(R)$. 3, M

(b) *Consider* $V := \mathbb{V}(x^3 - 1) \subseteq \operatorname{MaxSpec} \mathbb{Z}[x]$*. (You may use that* $\mathbb{Z}[x]$ *is Jacobson.)* sim. seen ⇓

(i) *Find the irreducible components of* V*.*

We have $V = \mathbb{V}(x-1) \cap \mathbb{V}(x^2+x+1)$, factoring x^3-1. We claim that these sets are irreducible. Since $\mathbb{Z}[x]$ is Jacobson, hence the Nullstellensatz for MaxSpec holds, it suffices to note that the quotients $\mathbb{Z}[x]/(x-1) \cong \mathbb{Z}$ and $\mathbb{Z}[x]/(x^2+x+1) \cong \mathbb{Z}[\omega] \subseteq \mathbb{C}$ are both integral domains (here ω is a primitive complex cube root of unity). 3, M

(ii) *For every prime integer* $p \geq 2$*, find explicitly the points of the intersection* $V \cap \mathbb{V}(p)$ *(break into natural cases). Facts you may use: Modulo* p *(in* $\mathbb{F}_p$*), there is a primitive* n*-th root of unity if and only if* $n \mid p-1$*. Also, if* $f \in F[x]$ *is an irreducible polynomial over a field, then* (f) *is maximal.* sim. seen ⇓

The intersection is the collection of maximal ideals containing p and x^3-1. If $p = 3$ then $x^3 - 1 \equiv (x-1)^3$ modulo p, so that the unique maximal ideal containing (p, x^3-1) is its radical, $(p, x-1)$. If $p \neq 3$ then x^3-1 has distinct roots over an extension field; there are then two possibilities: either

$x^2 + x + 1$ is irreducible over $\mathbb{F}_p$, or it splits into two factors modulo p. This is equivalent to asking for there to be a primitive cube root of unity modulo p. In turn this happens if and only if $3 \mid (p-1)$. So for $p \equiv 1 \pmod 3$, this intersection consists of three points, the three cube roots of unity. For $p \equiv 2 \pmod 3$, the intersection consists of two points: the maximal ideals $(p, x-1)$ and (p, x^2+x+1). These are indeed maximal since any ideal containing them reduces modulo p to an ideal containing $(x-1)$ or (x^2+x+1), but these ideals are maximal in $\mathbb{F}_p[x]$ as the polynomials are irreducible. $\boxed{3, \mathrm{M}}$

(c) *Consider the local ring* $R := k[x,y]_{(x)}$. $\boxed{\text{sim. seen} \Downarrow}$

(i) *Compute* $\operatorname{Spec} R$ *and* $\operatorname{MaxSpec} R$.
Since the ring is local, $\operatorname{MaxSpec} R = \{(x)\}$ consists of just the maximal ideal. On the other hand, $\operatorname{Spec}(R)$ consists of all prime ideals of $k[x,y]$ contained in (x). These are just (x) itself and (0). $\boxed{2, \mathrm{M}}$

(ii) *Compute the topological dimension of* $\operatorname{Spec} R$ *and* $\operatorname{MaxSpec} R$. $\boxed{\text{sim. seen} \Downarrow}$
Since $\operatorname{MaxSpec} R$ is a point, it has topological dimension zero: the only irreducible subset is the entire set. On the other hand, in $\operatorname{Spec} R$ the closed subsets are $\operatorname{Spec} R$ itself and $\{(x)\}$: this is because $(x) \supseteq (0)$, so any ideal containing (0) also contains (x). As a result, $\operatorname{Spec} R$ is irreducible. Therefore the chain $\operatorname{Spec} R \supsetneq \{(x)\}$ exhibits the topological dimension of $\operatorname{Spec} R$ as one (as the size of $\operatorname{Spec} R$ is two, the topological dimension cannot be greater). $\boxed{2, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 100 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH70056/97044	1	This problem was one of the easier ones, but still many students had difficulties with part (b) (although this is a construction we used multiple times in lectures), or with some of the parts of (c) and (d). Parts (a) and (e) seemed doable.
	2	2.(ii) was unintentionally too hard: it is doable if we only require finding **some** ideal whose vanishing locus is as required (without requiring to show that it is radical). For the marking, I did not require showing that the ideal found is radical. Alternatively, without the {x-axis} part finding the radical ideal is doable. The other parts were doable, even the last part (d), although that was definitely harder.
	3	This problem was difficult, especially (3b), one of the hardest parts of the exam. Also 3c(ii) was hardly done by any of the students, perhaps because of lack of time. I think that time pressure was unfortunately very acute throughout this exam.
	4	This problem was also difficult, perhaps a bit less than (3), although it required the student to understand the concepts of dimension theory. Particularly (4c).(ii) and (4c).(iii) had few complete solutions. For (4c).(iii) the intention had been that with parts (i) and especially (ii) assumed, there is not much left, but this eluded most students.
	5	The mastery was approachable this year. Many students got many of these parts correct, a testament to their preparation. Some got a bit confused in (5c), perhaps due to time pressure.