MATH50010 – Autumn 2021 – Midterm

You should state carefully any results from lectures that are used, and justify briefly why they are applicable.

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \Pr)$.

(a) (1 mark) State a necessary and sufficient condition in terms of subsets of Ω of the form $\{X \leq x\}$ for the function $X : \Omega \to \mathbf{R}$ to be a random variable with respect to $\mathcal{F}$.

(b) (2 marks) Show that if F_X is the cumulative distribution function of a random variable X, then $\lim_{x\to-\infty} F_X(x) = 0$.

(c) (2 marks) Show that if X and Y are random variables with respect to $\mathcal{F}$, then so is $Z = \max\{X, Y\}$.

In the remainder of the question, let X be an absolutely continuous random variable with probability density function given by

$$f_X(x) = nx^{n-1}, \quad \text{for} \quad 0 < x < 1,$$

and zero otherwise, where $n \in \{1, 2, \ldots\}$.

(d) (1 mark) Write down the cumulative distribution function of X.

(e) (3 marks) Determine the probability density function of the random variable $Y = \frac{X}{1+X}$.

(f) (4 marks) Show that X has the same distribution as $\max\{U_1, U_2, \ldots, U_n\}$, where the random variables $U_i \sim \textsc{Uniform}(0, 1)$ are independent.

(g) (4 marks) Find the covariance between the random variables $V = X^p$ and $W = X^q$, where $p, q \geq 1$.

(h) (3 marks) Find the monotonic *decreasing* function H such that the random variable T, defined by $T = H(X)$, has a probability density function that is constant on the interval $(0, 1)$, and zero otherwise.