IMPERIAL

MATH70132

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Mathematical Logic

Date: Thursday, May 2, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Work in ZFC throughout, unless indicated otherwise. The notation is as used in the lecture notes. In particular: both $\mathbb{N}$ and ω denote the set of natural numbers; L is the formal system for propositional logic; if $\mathcal{L}$ is a first-order language, then $K_{\mathcal{L}}$ is the associated formal system for first-order logic. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the set of integers, the set of rational numbers and the set of real numbers.

Unless indicated otherwise, you may use results from the lecture notes if these do not depend on what is being asked in the question. Results quoted from the lecture notes should be stated clearly.

1. (a) Suppose that n is a positive natural number and $(x_1, \ldots, x_n) \in \{T, F\}^n$. Write down a propositional formula ϕ in propositional variables $p_1, \ldots, p_n$ such that for every valuation v we have $v(\phi) = T$ if and only if $v(p_i) = x_i$ for all $i \leq n$. Explain your answer carefully. (4 marks)

   (b) Prove that the number of truth functions of n variables which can be constructed as the truth function of a formula using only the connective $\vee$ is $2^n - 1$. (4 marks)

   (c) State the Deduction Theorem for L. Using this, prove that if ϕ, θ, ψ are L-formulas, then

$$((\phi \to (\theta \to \psi)) \to (\theta \to (\phi \to \psi)))$$

   is a theorem of L. (4 marks)

   (d) In each of the following cases, decide whether the given statement is true or not. Justify your answer in each case. In parts (iii) and (iv), your justification should use syntactic arguments and not use the Completeness Theorem.

      (i) The formal system L has only finitely many axioms. (1 mark)

      (ii) If ϕ is an L-formula, then either ϕ or $(\neg\phi)$ is a theorem of L. (1 mark)

      (iii) If θ is a theorem of L and ϕ is an L-formula, then $((\neg\theta) \to \phi)$ is a theorem of L. (3 marks)

      (iv) If θ is an L-formula such that $((\neg\theta) \to \phi)$ is a theorem of L for every L-formula ϕ, then θ is a theorem of L. (3 marks)

(Total: 20 marks)

2. (a) Suppose $(X_1; \leq_1)$ and $(X_2; \leq_2)$ are linearly ordered sets. The *reverse lexicographic ordering* $\leq$ on $X_1 \times X_2$ is defined by saying that, for $x, x' \in X_1$ and $y, y' \in X_2$, we have

$$(x, y) \leq (x', y') \text{ if and only if } (y <_2 y') \text{ or } (y = y' \text{ and } x \leq_1 x').$$

Prove that $\leq$ is a linear ordering on $X_1 \times X_2$. (3 marks)

(b) Let $\mathcal{L}^=$ be a first-order language with equality having, as well as equality, a single 2-ary relation symbol R and no other relation, function or constant symbols. In each of the following cases, normal $\mathcal{L}^=$-structures $\mathcal{A}_i$ and $\mathcal{B}_i$ are given. In each case, either give a closed $\mathcal{L}^=$-formula θ_i which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$, or explain why there is no such formula.

(i) The domain of $\mathcal{A}_1$ is $\mathbb{R} \times \mathbb{R}$ and the domain of $\mathcal{B}_1$ is $\mathbb{Q} \times \mathbb{Q}$. In each case R is interpreted as the reverse lexicographic ordering coming from the usual linear orderings $\leq$ on $\mathbb{R}$ and $\mathbb{Q}$ respectively. (3 marks)

(ii) The domain of $\mathcal{A}_2$ is $\mathbb{N}$ and R is interpreted as the usual linear ordering $\leq$ on $\mathbb{N}$. The domain of $\mathcal{B}_2$ is $\mathbb{N} \times \{0, 1\}$ and R is interpreted as the reverse lexicographic ordering (coming from the usual orderings $\leq$). (3 marks)

(iii) $\mathcal{A}_3$ and $\mathcal{B}_3$ have domains $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Q})$, the sets of subsets of $\mathbb{N}$ and $\mathbb{Q}$ respectively and in each case $R(x_1, x_2)$ is interpreted as '$x_1 \subseteq x_2$'. (3 marks)

(c) Let $\mathcal{L}^=$ be the usual first-order language with equality for groups, having a 2-ary function symbol $\cdot$ for the group operation and a constant symbol e for the identity element. For each prime number p, let $\mathcal{C}_p$ denote the cyclic group of order p, considered as a normal $\mathcal{L}^=$-structure. Let Φ be the set consisting of closed $\mathcal{L}^=$-formulas ϕ having the property that:

there are only finitely many primes p with $\mathcal{C}_p \models (\neg\phi)$.

(i) Using the Compactness Theorem for normal models, prove that Φ has a normal model. (2 marks)

(ii) Prove that if $\mathcal{A}$ is a normal model of Φ, then $\mathcal{A}$ is an infinite abelian group. (3 marks)

(iii) By considering the $\mathcal{L}^=$-formula $(\forall x)(\exists y)(y \cdot y = x)$, or otherwise, show that the infinite cyclic group $(\mathbb{Z}; +)$ is not a model of Φ. (3 marks)

(Total: 20 marks)

3. (a) Suppose $\mathcal{L}$ is a first-order language and Σ is a *complete, consistent* set of closed $\mathcal{L}$-formulas.

(i) Define what is meant by *complete* and *consistent* here. (2 marks)

Assume also that the following property holds:

(*) if $\theta(x_i)$ is an $\mathcal{L}$-formula with a single free variable x_i, then there is a constant symbol c in $\mathcal{L}$ (depending on θ) such that

$$\Sigma \vdash_{K_{\mathcal{L}}} ((\neg(\forall x_i)\theta(x_i)) \to (\neg\theta(c))).$$

(ii) Describe how to construct an $\mathcal{L}$-structure $\mathcal{A}$ whose domain is the set of closed terms of $\mathcal{L}$ and which has the property that $\mathcal{A} \models \Sigma$. You do not need to give a full proof that $\mathcal{A} \models \Sigma$, but you should outline the proof and explain carefully the step where the assumption (*) is used in the proof. (8 marks)

(b) Let $\mathcal{L}^=$ be a first-order language with equality having, as well as equality, a single 2-ary relation symbol E and no other relation, function or constant symbols.

(i) Give a set Δ of closed $\mathcal{L}^=$-formulas with the property that a normal $\mathcal{L}^=$-structure $\mathcal{B} = (B; \bar{E})$ is a model of Δ if and only if $\bar{E}$ is an equivalence relation on B with infinitely many classes and all $\bar{E}$-equivalence classes are infinite. (4 marks)

(ii) For the set Δ in your answer to part (b)(i), prove that if σ is a closed $\mathcal{L}^=$-formula and $\Delta \vdash \sigma$, then there is a finite normal $\mathcal{L}^=$-structure $\mathcal{A}$ with $\mathcal{A} \models \sigma$. (4 marks)

(iii) Suppose that Δ' is a set of closed $\mathcal{L}^=$-formulas with the same models as Δ from part (b)(ii). Does the result in part (b)(ii) necessarily hold for Δ'? Explain your answer. (2 marks)

(Total: 20 marks)

4. (a) (i) State Zorn's Lemma and say how it is related to the Axiom of Choice. (4 marks)

(ii) Let $\mathcal{L}$ be a first-order language and let $\mathcal{F}$ denote the set of closed $\mathcal{L}$-formulas. Suppose $\Sigma \subseteq \mathcal{F}$ is consistent. Use Zorn's Lemma to prove that there is a subset Σ^* of $\mathcal{F}$ which contains Σ and which is consistent and complete. (4 marks)

(b) (i) Define what it means for a set to be: an *ordinal*; a *cardinal*. (2 marks)

(ii) Give an example of a set which is an ordinal but not a cardinal. (2 marks)

(iii) Explain briefly why (in ZFC), for every set A there is a unique cardinal $|A|$ which is equinumerous with A. (2 marks)

(c) If A is any set, let $S_1(A)$ denote the set of countable subsets of A. Let $S_2(A)$ denote A^ω, the set of functions from ω to A.

(i) Suppose X is an infinite set and $A = \mathcal{P}(X)$, the power set of X. Prove that $S_1(A)$ and $S_2(A)$ have the same cardinality as A. (4 marks)

(ii) Let β denote the least uncountable ordinal. How are β and the cardinality of $S_2(\beta)$ related? Explain your answer. (2 marks)

(Total: 20 marks)

5. In this question, $\mathcal{L}^=$ is a first-order language with equality and all structures considered are normal $\mathcal{L}^=$-structures.

(a) Suppose $\mathcal{L}^=$ is a countable language. For an infinite cardinal κ, an $\mathcal{L}^=$-theory T is κ-*categorical* if it has a model of cardinality κ and any two models of T of cardinality κ are isomorphic. Give brief explanations for your answers to the following.

(i) Give a language $\mathcal{L}^=$ and an example of an $\mathcal{L}^=$-theory which has an infinite model and which is not κ-categorical for any infinite cardinal κ. (2 marks)

(ii) Give a language $\mathcal{L}^=$ and an example of an $\mathcal{L}^=$-theory which is ω-categorical, but not κ-categorical for any uncountable cardinal κ. (3 marks)

(iii) Give a language $\mathcal{L}^=$ and an example of an $\mathcal{L}^=$-theory which is κ-categorical for all uncountable κ, but which is not ω-categorical. (3 marks)

(b) Let $\mathcal{L}^=$ be the language (with equality) having 2-ary function symbols $+, \cdot$; constant symbols 0,1; 2-ary relation symbols $=, \leq$; and a 1-ary function symbol F. For each $n < \omega$ let $\mathcal{A}_n$ be a normal $\mathcal{L}^=$-structure with domain $\mathbb{R}$:

$$\mathcal{A}_n = \langle \mathbb{R}; +, \cdot, 0, 1, \leq, f_n \rangle,$$

where $+, \cdot, 0, 1$ and $\leq$ have their usual meaning on $\mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$ is a given function (and F is interpreted as f_n in $\mathcal{A}_n$).

Let $\mathcal{F}$ be a non-principal ultrafilter on ω and let

$$\mathcal{M} = \langle \mathbb{R}^*; +, \cdot, 0, 1, \leq, f^* \rangle$$

be the ultraproduct $(\Pi_{n<\omega}\mathcal{A}_n)/\mathcal{F}$. In the following you should justify your answers, but standard results about ultraproducts may be used if clearly stated.

(i) Define the domain $\mathbb{R}^*$ of $\mathcal{M}$. (2 marks)

(ii) Show that if each f_n is a constant function, then f^* is also a constant function. Say what the value of f^* is in this case. (2 marks)

(iii) State the Łos Theorem for the ultraproduct $\mathcal{M}$. (1 mark)

(iv) Suppose that for each $n < \omega$ and $x \in \mathbb{R}$ we have $f_n(x) = x^n$. Under what circumstances do we have $f^*(x) \geq 0$ for all $x \in \mathbb{R}^*$? (2 marks)

(v) Show that if each function f_n is continuous and $f_n(0) \leq f_n(1)$, then for every $y \in \mathbb{R}^*$ with $f^*(0) \leq y \leq f^*(1)$ there is $x \in \mathbb{R}^*$ with $f^*(x) = y$. (2 marks)

(vi) Give an example where the functions f_n are differentiable and for all $y \in \mathbb{R}^*$ with $0 \leq y \leq 1$, the set $\{x \in \mathbb{R}^* : 0 \leq x \leq 1 \text{ and } f^*(x) = y\}$ is infinite. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

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MATH70132

Mathematical Logic (Solutions)

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1. **Comments: (a) standard question; (b) unseen; (c) seen similar; (d) (i), (ii) easy but common mistakes here; (iii), (iv) unseen in this form.**

(a) For $i \leq n$ let q_i be p_i if $x_i = T$ and $(\neg p_i)$ if $x_i = F$. Let ϕ be the propositional formula $q_1 \wedge q_2 \wedge \ldots \wedge q_n$. Then $v(\phi) = T$ iff $v(q_i) = T$ for all $i \leq n$. As $v(q_i) = T \Leftrightarrow v(p_i) = x_i$, the result follows. $\boxed{4, \mathrm{A}}$

(b) If ψ is a formula constructed from variables amongst $p_1, \ldots, p_n$ using only the connective $\vee$, then, as repetitions of a variable do not affect the truth value of such a formula and nor does interchanging the order of variables, ψ is logically equivalent to a disjunction ψ_S given by $\bigvee_{i \in S} p_i$ for some non-empty subset S of $\{1, \ldots, n\}$. Moreover, if $S \neq S'$, then $\psi_S, \psi_{S'}$ are not logically equivalent: WLOG, let $i \in S \subseteq S'$ and consider the valuation v with $v(p_i) = T$ and $v(p_j) = F$ for all $j \neq i$. Then $v(\psi_S) = T$ and $v(\psi_{S'}) = F$. Thus, the required number is equal to the number of non-empty subsets of $\{1, \ldots, n\}$, and this is $2^n - 1$. $\boxed{4, \mathrm{B}}$

(c) DT states that if $\Gamma \cup \{\phi, \psi\}$ is a set of L-formulas and $\Gamma \cup \{\phi\} \vdash_L \psi$, then $\Gamma \vdash_L (\phi \to \psi)$. $\boxed{1, \mathrm{A}}$

By two applications of Modus Ponens (MP) we have $\boxed{3, \mathrm{B}}$

$$\{(\phi \to (\theta \to \psi)), \theta, \phi\} \vdash \psi.$$

Two applications of DT then give $\{(\phi \to (\theta \to \psi))\} \vdash (\theta \to (\phi \to \psi))$. A further application of DT then gives the result (as a deduction from $\emptyset$ is the same as a proof in L).

(d) (i) FALSE: for example, for each variable p_i we have an axiom (of type A1) $(p_i \to (p_i \to p_i))$. $\boxed{1, \mathrm{A}}$

(ii) FALSE: for example, take ϕ to be the formula p_1. Every theorem is logically valid and neither p_1 nor $(\neg p_1)$ is logically valid. $\boxed{1, \mathrm{A}}$

(iii) TRUE: A result from the module is that $((\neg\theta) \to (\theta \to \phi))$ is a theorem of L. The result then follows from part (c) and Modus Ponens. $\boxed{3, \mathrm{B}}$

(iv) TRUE: Let ϕ be $(\neg\alpha)$ where α is any axiom (or other theorem of L). We have the A3 Axiom $(((\neg\theta) \to \phi) \to (\alpha \to \theta))$. Two applications of MP (using the given theorem and α) then give that θ is a theorem of L. $\boxed{3, \mathrm{C}}$

2. **Comments: (a) left as an exercise in lectures; (b)(i) unseen, (ii) seen similar, (iii) unseen; (c) unseen.**

(a) Check the axioms for a linear order. We will drop subscripts on $\leq$. $\boxed{3, \text{A}}$

It is clear that $(x,y) \leq (x,y)$.

If $(x,y) \leq (x',y')$ and $(x',y') \leq (x,y)$ then we cannot have $y < y'$ or $y' < y$, so $y = y'$. It then follows that $x = x'$, so $(x,y) = (x',y')$.

For transitivity, suppose $(x,y) \leq (x',y') \leq (x'',y'')$. Then $y \leq y' \leq y''$. If $y = y''$ then $y = y' = y''$ and therefore $x \leq x' \leq x''$, so $x \leq x''$ and $(x,y) \leq (x'',y'')$. If $y < y''$ then also $(x,y) \leq (x'',y'')$.

Finally, if $(x,y) \not\leq (x',y')$, then either $y = y'$ and $x > x'$, or $y' < y$. In either case, $(x',y') < (x,y)$.

(b) (i) There is no such formula. Both $\mathcal{A}_1$ and $\mathcal{B}_1$ are dense linear orders without endpoints and we proved in the lectures that any closed formula true in one of these is true in any other. $\boxed{3, \text{D}}$

(ii) The element $(0,1)$ of $\mathcal{B}_2$ has the property that there is something less than it, but it has no immediate predecessor. There is no such element in $\mathcal{A}_2$. The existence of such an element in $\mathcal{B}_2$ can be expressed by an $\mathcal{L}^=$-formula $(\exists x)\phi(x)$, where $\phi(x)$ is: $\boxed{3, \text{B}}$

$$(\exists y)(R_1(y,x) \wedge (\forall z)(R_1(z,x) \to R(z,y))).$$

Here we are writing $R_1(y,x)$ as shorthand for $(R(y,x) \wedge (y \neq x))$. So take the required formula to be $(\neg(\exists x)\phi(x))$.

(iii) The two structures are isomorphic, so there is no such closed formula. Note that $\mathbb{Q}$ and $\mathbb{N}$ are countably infinite, so there is a bijection between them. This bijection gives a bijection between their power sets which preserves containment and so is an isomorphism between $\mathcal{A}_3$ and $\mathcal{B}_3$. $\boxed{3, \text{C}}$

(c) (i) By the Compactness Theorem for normal models, it is enough to prove that every finite subset $\{\phi_1, \ldots, \phi_n\}$ of Φ has a normal model. For $i \leq n$, $\{p : \mathcal{C}_p \models \phi_i\}$ contains all but finitely many primes. As there are infinitely many primes, it follows that there are infinitely many p with $\mathcal{C}_p \models \phi_i$ for all $i \leq n$. Any such $\mathcal{C}_p$ is a normal model of $\{\phi_1, \ldots, \phi_n\}$. $\boxed{2, \text{A}}$

(ii) By definition, the group axioms and the formula $(\forall x)(\forall y)(x \cdot y = y \cdot x)$ are in Φ, so $\mathcal{A}$ is an abelian group. For each natural number n the formula σ_n given by $(\exists x_1) \ldots (\exists x_n) \bigwedge_{1 \leq i < j \leq n}(x_i \neq x_j)$ and expressing that 'there at least n elements' is in Φ. Thus $\mathcal{A}$ is infinite. $\boxed{3, \text{A}}$

(iii) Note that $(\mathbb{Z}; +)$ is not a model of the given formula (consider $x = 1$), so it suffices to prove that the formula is in Φ. Now, if p is an odd prime, for $x \in \mathcal{C}_p$ we can consider $y = x^{(p+1)/2} \in \mathcal{C}_p$. Then $y \cdot y = x^{p+1} = x$, as $\mathcal{C}_p$ has order p. So $\mathcal{C}_p$ is a model of the formula. $\boxed{3, \text{D}}$

3. **Comments: (a) Is testing understanding of a difficult proof (this question has not been asked before); (b)(i), (ii) are similar to a question on a problem sheet, (iii) is unseen.**

(a) (i) Σ is *complete* if for every closed $\mathcal{L}$-formula ϕ we have $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ or $\Sigma \vdash_{K_{\mathcal{L}}} (\neg\psi)$; it is *consistent* if at most one of these holds. $\boxed{2, \mathrm{A}}$

(ii) Construction of $\mathcal{A}$:

Let $A = \{\bar{t} : t$ is a closed term of $\mathcal{L}\}$. Here, the bar is a notational device to distinguish elements of A from closed terms used in formulas. Note that the closed terms are formed from constant and function symbols: there are no variables in them. We make A into an $\mathcal{L}$-structure as follows. $\boxed{3, \mathrm{B}}$

A constant symbol c in $\mathcal{L}$ is interpreted as $\bar{c} \in A$.

Suppose R is an n-ary relation symbol in $\mathcal{L}$. We interpret R as the n-ary relation $\bar{R}$ on A as follows. If $\bar{t}_1, \ldots, \bar{t}_n \in A$, then we say that $\bar{R}(\bar{t}_1, \ldots, \bar{t}_n)$ holds iff $\Sigma \vdash R(t_1, \ldots, t_n)$ (note that the latter is a closed atomic $\mathcal{L}$-formula).

Suppose f is an m-ary function symbol in $\mathcal{L}$. We interpret f as the m-ary function $\bar{f} : A^m \to A$ as follows. If $\bar{t}_1, \ldots, \bar{t}_m \in A$ let $\bar{f}(\bar{t}_1, \ldots, \bar{t}_m)$ equal $\overline{f(t_1, \ldots, t_m)}$.

This completes the definition of the $\mathcal{L}$-structure $\mathcal{A}$.

Proof that $\mathcal{A} \models \Sigma$.

We prove that if ϕ is a closed $\mathcal{L}$-formula, then:

(**) $\mathcal{A} \models \phi \Leftrightarrow \Sigma \vdash_{K_{\mathcal{L}}} \phi$.

The proof is by induction on the number of connectives and quantifiers in ϕ. The base case is when ϕ is a closed atomic formula and this follows from the definition of $\mathcal{A}$. The inductive step splits into cases depending on whether ϕ is of the form $\neg\psi$, $(\theta \to \chi)$ or $(\forall x_i)\theta(x_i)$. $\boxed{3, \mathrm{A}}$

For the first of these, we use the inductive hypothesis and that Σ is consistent and complete. The second case is routine.

In the third case , if x_i is not free in θ then the inductive step is straightforward. If it is free, then we cannot apply the induction hypothesis directly to θ as it applies to closed formulas. The assumption (*) is used in the proof of $\Rightarrow$ in (**). $\boxed{2, \mathrm{D}}$

Indeed, suppose $\mathcal{A} \models \phi$ and suppose for a contradiction that $\Sigma \not\vdash \phi$. By completeness of Σ we have $\Sigma \vdash \neg\phi$. Using $(*)$ there is a constant symbol c such that $\Sigma \vdash \neg\theta(c)$. By induction it follows that $\mathcal{A} \models \neg\theta(c)$ and this contradicts $\mathcal{A} \models (\forall x_i)\theta(x_i)$.

The other direction is more straightforward.

(b) (i) Let η be the formula expressing that $\bar{E}$ is an equivalence relation: $\boxed{4, \mathrm{A}}$

$$(\forall x_1)(\forall x_2)(\forall x_3)(E(x_1, x_1) \wedge (E(x_1, x_2) \to E(x_2, x_1)) \wedge (E(x_1, x_2) \wedge E(x_2, x_3) \to E(x_1, x_3))).$$

For $n \in \mathbb{N}$ let α_n be the formula saying 'there are at least n classes':

$$(\exists x_1) \ldots (\exists x_n) \bigwedge_{1 \leq i < j \leq n} \neg E(x_i, x_j).$$

and let β_n say that 'every class has at least $n - 1$ elements':

$$(\forall y)(\exists x_1) \ldots (\exists x_n) \bigwedge_{1 \leq i < j \leq n} \neg(x_i \neq x_j) \wedge E(y, x_i).$$

Let $\Delta = \{\eta, \alpha_n, \beta_n : n \in \mathbb{N}\}$.

(ii) If $\Delta \vdash \sigma$ then, as deductions are finite, there is some $k \in \mathbb{N}$ such that σ is a consequence of $\{\eta, \alpha_n, \beta_n : n \leq k\}$. So (by the generalised soundness result) any normal model of the latter is a model of σ. So we may take $\mathcal{A}$ to be a finite set of size k^2 partitioned by an equivalence relation into k classes of size k. $\boxed{4, \mathrm{C}}$

(iii) Yes, it does. By definition, Δ and Δ' have the same models. It follows from the Completeness Theorem that each of Δ, Δ' is a consequence of the other and they therefore have the same set of consequences. So if $\Delta' \vdash \sigma$ then $\Delta \vdash \sigma$ and we may apply (b)(ii) to find $\mathcal{A}$. $\boxed{2, \mathrm{D}}$

4. **Comments: (a)(i) bookwork, (ii) seen similar and was mentioned in lectures; (b) (i), (iii) bookwork, (ii) straightforward example; (c) unseen.**

(a) (i) ZL is the following statement. Suppose $(A; \leq)$ is a non-empty partially ordered set with the property that every chain C in $(A; \leq)$ has an upper bound in A. Then $(A; \leq)$ has a maximal element (that is, an element $b \in A$ such that $a \geq b$ implies $a = b$ for all $a \in A$). $\boxed{4, \text{A}}$

It is a consequence of the Zermelo - Fraenkel Axioms (ZF) that ZL implies AC and AC implies ZL.

(ii) Let A be the set of subsets $\Sigma_1 \subseteq \mathcal{F}$ which are consistent and which contain Σ. Then $(A; \subseteq)$ is a non-empty poset. We first show that it satisfies the hypothesis of ZL. Indeed, if $C \subseteq A$ is a chain, then we claim $\bigcup C$ is in A and so is an upper bound for C in A. If it is not consistent, then as proofs are finite there are $C_1, \ldots, C_n \in C$ such that $C_1 \cup \ldots \cup C_n$ is inconsistent. But as C is a chain we may assume that one of these sets contains the others and so is also inconsistent: a contradiction. So by ZL, there is a maximal element Σ^* of A. It remains to show that Σ^* is complete. Suppose ϕ is a closed $\mathcal{L}$-formula and $\Sigma^* \not\vdash \phi$. By a lemma in the notes, $\Sigma^* \cup \{\neg\phi\}$ is consistent. By maximality of Σ^* we therefore have $\neg\phi \in \Sigma^*$, so $\Sigma^* \vdash \neg\phi$, as required. $\boxed{4, \text{B}}$

(b) (i) A set α is an *ordinal* if every element of α is a subset of α and the membership relation $\in$ on α is a strict well ordering of α. An ordinal is a *cardinal* if it is not equinumerous with any of its elements. $\boxed{2, \text{A}}$

(ii) Consider the ordinals ω and $\omega^\dagger = \omega \cup \{\omega\}$. These are equinumerous and $\omega \in \omega^\dagger$. So $\omega^\dagger$ is not a cardinal. $\boxed{2, \text{A}}$

(iii) Using AC we can show that there is a well ordering of A, so by a result in the notes, there is a bijection between A and some ordinal α. Consider the non-empty set of ordinals $\{\beta \leq \alpha : \beta \text{ is equinumerous with } A\}$. This has a least element κ (as it is a subset of the ordinal $\alpha^\dagger$). Then κ is a cardinal which is equinumerous with A. Uniqueness is from the definition of being a cardinal. $\boxed{2, \text{C}}$

(c) (i) Let $|X| = \lambda$. Then $|A| = 2^\lambda$. We have $|S_2(A)| = (2^\lambda)^\omega = 2^{\lambda \cdot \omega} = 2^\lambda$, the last of these following from properties of cardinal arithmetic as $\omega \leq \lambda$. Moreover, there is a natural surjection $S_2(A) \to S_1(A)$ (sending a function $\omega \to A$ to its image), so $|S_1(A)| \leq |S_2(A)|$ and we have $\boxed{4, \text{D}}$

$$2^\lambda = |A| \leq |S_1(A)| \leq |S_2(A)| = 2^\lambda$$

where the first of these inequalities comes from the injective map which sends $a \in A$ to the singleton subset $\{a\}$. It follows that we have the required equalities.

(ii) Note that β is a cardinal and we have the inequalities between cardinals $\omega < \beta \leq 2^\omega$ (the second as 2^ω is uncountable). Now, $|S_2(\beta)| = \beta^\omega$ and $2^\omega \leq \beta^\omega \leq (2^\omega)^\omega = 2^\omega$. So $\beta \leq |S_2(\beta)| = 2^\omega$. We cannot say whether $\beta = 2^\omega$ or not as this statement is the Continuum Hypothesis. $\boxed{2, \text{D}}$

5. **Comments: (a) Definition and examples are mentioned in the directed reading, but categoricity has not appeared on the exam paper previously; (b) a variation on standard questions on ultraproducts.**

(a) (i) Consider $\mathcal{L}^=$ which has just equality and a 1-ary relation symbol P. Let T be empty. For every infinite κ there are (at least) two non-isomorphic models of cardinality κ: one where P is the empty set and one where it has a single element. $\boxed{2, \mathrm{M}}$

(ii) Consider the theory Δ of a equivalence relations with infinitely many classes all of which are infinite from Question 3(b). This is ω-categorical. However, if κ is an uncountable categorical we have two non-isomorphic models of cardinality κ: for example, one in which there are countably many classes all of cardinality κ and one where there are κ-many classes all of which are countable. $\boxed{3, \mathrm{M}}$

(iii) Let $\mathcal{L}^=$ be a language for $\mathbb{Q}$-vector spaces (for example, with a 2-ary function symbol for the group operation and a 1-ary function symbol f_q for scalar multiplication by q for each $q \in \mathbb{Q}$). Let T be the theory of $\mathbb{Q}$-vector spaces in this language. Finite dimensional $\mathbb{Q}$-spaces give non-isomorphic countable models of T, so T is not ω-categorical. However, if κ is an uncountable cardinal and V is a $\mathbb{Q}$-vector space of cardinality κ (this exists), then a basis of V has cardinality κ and so any two such vector spaces are isomorphic. $\boxed{3, \mathrm{M}}$

(b) (i) For sequences $(a_i), (b_i) \in \mathbb{R}^\omega$ write $(a_i) \sim (b_i)$ iff $\{i \in \omega : a_i = b_i\} \in \mathcal{F}$. This is an equivalence relation on $\mathbb{R}^\omega$ and $\mathbb{R}^*$ is the set of equivalence classes. Denote the $\sim$-class of the sequence (a_i) by $[(a_i)]$. $\boxed{2, \mathrm{M}}$

(ii) Suppose $f_n(x) = c_n$ for all $x \in \mathbb{R}$. By definition of f^* we have $f^*([(a_i)] = [(f_i(a_i)]$ for all $[(a_i)] \in \mathbb{R}^*$. So this is the constant function equal to $[(c_i)]$. $\boxed{2, \mathrm{M}}$

(iii) The Łos Theorem gives that if ϕ is a closed $\mathcal{L}^=$-formula, then $\mathcal{M} \models \phi$ iff $\{n \in \omega : \mathcal{A}_n \models \phi\} \in \mathcal{F}$. $\boxed{1, \mathrm{M}}$

(iv) By (iii) $f^*(x) \geq 0$ for all $x \in \mathbb{R}^*$ iff $\mathcal{M} \models (\forall x)(F(x) \geq 0)$ iff $\{n \in \omega : \mathcal{A}_n \models (\forall x)(F(x) \geq 0)\} \in \mathcal{F}$. By definition of the f_n here, this is the case iff $\{2n : n \in \omega\} \in \mathcal{F}$. $\boxed{2, \mathrm{M}}$

(v) Consider the closed formula ϕ: $\boxed{2, \mathrm{M}}$

$$(\forall y)(\exists x)((F(0) \leq y \leq F(1)) \to (F(x) = y)).$$

By the intermediate value theorem $\mathcal{A}_n \models \phi$. Now, $\omega \in \mathcal{F}$, so by the Łos theorem, we have $\mathcal{M} \models \phi$ and this gives the required result.

(vi) Let $f_n(x) = \sin(2\pi nx)$. Consider the closed $\mathcal{L}^=$-formula θ_n $\boxed{3, \mathrm{M}}$

$$(\forall y)((0 \leq y \leq 1) \to (\exists x_1) \ldots (\exists x_n) \bigwedge_{i<j\leq n} (0 \leq x_i \leq 1) \wedge (F(x_i) = y) \wedge (x_i \neq x_j)).$$

If m is sufficiently large (say $m > 2n$) then $\mathcal{A}_m \models \theta_n$. It follows that $\mathcal{M} \models \theta_n$ for all n, so $\mathcal{M}$ has the required property.

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total Mastery marks: 20 of 20 marks
Total marks: 100 of 100 marks

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Question	Marker's comment
1	Question 1 was reasonably well done by many people.1(b): some people missed the very simple proof here, or appeared to misunderstand the question. An inductive proof was possible, but unnecessary.1(d) (i), (ii) a surprising number of people got these wrong.(iii), (iv): As stated in the question, I was looking for syntactic proofs here, not truth tables or valuations.nbsp;
2	Most people managed to check 2(a) (though in some cases, there was too much written)(b)(i) The two structures are not isomorphic (one is uncountable and the other is countable, so there is no bijection between them). Also, you cannot immediately deduce the result from the fact that the reals and rationals (as orderings) have the same theory.
3	Only two people made an attempt at 3(a)(ii) (but both produced good answers).3(b) People who understood what was going on scored well in (i) and (ii). (iii) was a slightly strange question which few people got completely correct.
4	Students who had worked on this part of the module did reasonably well.4(a)(ii) is related to one of the courseworks, but more time spent on the exercises on Zorn's Lemma would have been beneficial. Some people used an argument based on transfinite recursion, which I allowed.4(c)(ii) Very few people noticed the connection with the continuum hypothesis and just wrote down that the same proof as in (c)(i) would work.

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Question	Marker's comment
1	See y3 paper
2	See y3 paper
3	See y3 paper
4	See y3 paper
5	Very few people had engaged with the Mastery material, though some of the solutions were very good.Part (a) was found to be harder than part (b), possibly though less familiarity from previous papers.