Are these functions convex?

(i) $f(\underline{x}) = \ln\left(\sum_{i=1}^{K} e^{\underline{a_i}^T \underline{x} + b_i}\right)$, $\underline{a_i} \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

<u>Answer</u>: We know that log-sum-exp is convex (lecture notes convex)

$$h(\underline{x}) = \ln\left(e^{x_1} + \cdots + e^{x_n}\right)$$

but we indentify

$$f(\underline{x}) = h(A\underline{x} + \underline{b}) = \ln\left(e^{\underline{a_1}^T \underline{x} + b_1} + \cdots + e^{\underline{a_m}^T \underline{x} + b_m}\right)$$

$$= \ln\left(\sum_{i=1}^{K} e^{\underline{a_i}^T \underline{x} + b_i}\right)$$

where $A = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_K^T \end{bmatrix} \in \mathbb{R}^{K \times n}$ and $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix} \in \mathbb{R}^K$

Convex $h$ composed with linear transformation

$\Rightarrow$ $f(\underline{x}) = h(A\underline{x} + \underline{b})$ is convex.

ii) $f(\underline{x}) = \|\underline{x}\|^4$

$f(\underline{x}) = h \circ g(\underline{x})$, where $h(y) = y^4$ and $g(\underline{x}) = \|\underline{x}\|$

$h$ is convex and non-decreasing in $\mathbb{R}_+$ :

$h'(y) = 4y^3 \geq 0$ for $y \in \mathbb{R}_+$

$h''(y) = 12y^2 \geq 0$

$g(\underline{x}) = \|\underline{x}\|$ is convex (convexity of norms)

and goes from $\mathbb{R}^n$ to $\mathbb{R}_+$

$\Longrightarrow$ Composition of $h \circ g(\underline{x})$ with

$h$ convex and non-decreasing on Image$(g)$

$g$ convex in $\mathbb{R}^n$ $\Longrightarrow f = h \circ g(\underline{x})$ convex

//.