2.3 (1) Bound + free variables.

1) Eg. ψ_1

free bound free bound

$$(R_1(x_1,x_2) \to (\forall x_3)\underbrace{R_2(x_1,x_3)}_{\text{scope}})$$

(2.3.1) Def. Suppose ϕ, ψ are $\mathcal{L}$-fmlas and $(\forall x_i)\phi$ occurs as a subformula of ψ

ie. ψ is $\ldots\ldots(\forall x_i)\phi\ldots$

We say that ϕ is the scope of $(\forall x_i)$ here.

An occurrence of a variable x_j in ψ is bound if it is within the scope of a quantifier $(\forall x_j)$ in ψ (or it is this x_j).

Otherwise it is a free occurence of x_j in ψ.

Free variables in ψ: variables with a free occurrence in ψ.

An $\mathcal{L}$-formula with no free variables is called a closed formula (or a sentence).

Examples.

2) ψ_2

bound free

$$((\forall x_1)\underbrace{R_1(x_1,x_2)}_{\text{scope}} \to R_2(x_1,x_2))$$

3) Compare with ψ_3

$$(\forall x_1)\underbrace{\big(R_1(x_1,x_2) \to R_2(x_1,x_2)\big)}_{\text{scope}} \qquad \text{free variable } x_2 .$$

4) ψ_4

$$\Big(\big(\exists \overset{\text{bound}}{x_3}\big)\, \underbrace{R_1(x_1,x_2)}_{\text{scope}} \to \big(\forall \overset{\text{bound}}{x_2}\big)\, \underbrace{R_2(\overset{\text{bound}}{x_2},x_3)}_{\text{scope}}\Big)$$

(2.3.2) <u>Notation</u> If ψ is an $\mathcal{L}$-fmla. with free variables amongst $x_1,\dots,x_n$ we might sometimes write $\psi(x_1,\dots,x_n)$ to show this.

If $t_1,\dots,t_n$ are terms, then by $\psi(t_1,\dots,t_n)$ we mean the $\mathcal{L}$-fmla obtained by replacing each <u>free</u> occurrence of x_i in ψ by t_i (for $i=1,\dots,n$).

Eg. $\psi(x_1, x_2)$

$$((\forall x_1) R_1(x_1, x_2) \to (\forall x_3) R_3(x_1, x_2, x_3))$$

$t_1 \quad f_1(x_1)$

$t_2 \quad f_2(x_1, x_2)$

$\psi(t_1, t_2)$

$$((\forall x_1) R_1(x_1, f_2(x_1, x_2)) \to (\forall x_3) R_3(f_1(x_1), f_2(x_1, x_2), x_3))$$

(2.3.3) Theorem.

Suppose ϕ is a closed $\mathcal{L}$-fmla and $\mathcal{A}$ is an $\mathcal{L}$-structure.

then either $\mathcal{A} \models \phi$

or $\mathcal{A} \models (\neg \phi)$.

More generally if ϕ has free variables amongst $x_1, \ldots, x_n$ and v, w are valuations in $\mathcal{A}$ with $v(x_i) = w(x_i)$ for $i \leq n$

then $v[\phi] = T \iff w[\phi] = T$.

(allow $n = 0$ here : ie. no free variables).

Pf: The first statement follows from the second : if ϕ has no free variables any two valuations v, w in $\mathcal{A}$ agree on the free variables in ϕ

So $v[\phi] = w[\phi]$. //

Prove the generalisation by induction on the number of connectives & quantifiers in ϕ.

Base case. ϕ is atomic ie.

ϕ is $R(t_1, \ldots, t_m)$

$t_1, \ldots, t_m$ terms.

the only variables used in $t_1, \ldots, t_m$ are amongst $x_1, \ldots, x_n$.

So $v(t_j) = w(t_j)$ for $j = 1, \ldots, m$.

(see 2.2.5).

Then $v[R(t_1, \ldots, t_m)] = T$

$\iff \bar{R}(v(t_1), \ldots, v(t_m))$ holds in $\mathcal{A}$

$\iff \bar{R}(w(t_1), \ldots, w(t_m))$ holds

$\iff w[\phi] = T$. //

Inductive step. ϕ is $(\neg\psi)$, $(\psi \to \chi)$ or $(\forall x_i)\psi$

First two cases: Ex.

Suppose ϕ is $(\forall x_i)\psi$

Suppose $v[\phi] = F$

Want to show $w[\phi] = F$.

By Def 2.2.7 there is a valuation v' (in $\mathcal{A}$) with v' x_i-equivalent to v and $v'[\psi] = F$.

The free variables of ψ are amongst $x_1, \ldots, x_n, x_i$

Let w' be the valuation x_i-equivalent to w with $w'(x_i) = v'(x_i)$.

Claim: v', w' agree on the free variables in ψ.

Given this the ind. hyp. applied to ψ gives $w'[\psi] = v'[\psi] = F$.

w' is x_i-equiv. to w

so $w[(\forall x_i)\psi] = F$

ie. $w[\phi] = F$.

Pf of Claim:

Case 1 $\quad i \leq n$.

Free vars. of ϕ are amongst $x_1, \dots, x_n$.

If $j \leq n$ & $j \neq i$ then

$v'(x_j) = v(x_j) = w(x_j)$
$= w'(x_j)$.

Also $w'(x_i) = v'(x_i)$

so v', w' agree on $x_1, \dots, x_n$.

Case 2 $\quad i > n$.

Same argument works!

//.

#.