Introduction to Quantum Mechanics – Solutions for Problem sheet 4

1. Commutator as Poisson bracket

(i) The first property is trivially fulfilled: $[A, A] = A^2 - A^2 = 0 \quad \square$

(ii) Linearity in first element

$$\begin{aligned}
[c_1 A + c_2 B, C] &= (c_1 A + c_2 B)C - C(c_1 A + c_2 B) \\
&= c_1 AC - c_1 CA + c_2 BC - c_2 CB \\
&= c_1 (AC - CA) + c_2 (BC - CB) \\
&= c_1 [A, C] + c_2 [B, C] \qquad \square
\end{aligned}$$

(iii) Anti-symmetry with respect to interchange of elements

$$[B, A] = BA - AB = -(AB - BA) = -[A, B] \qquad \square$$

(iv) Poisson bracket with constant is zero

$$[c, A] = cA - Ac = cA - cA = 0 \qquad \square$$

(v) Leinbniz rule

$$\begin{aligned}
[AB, C] &= ABC - CAB \\
&= ABC - ACB + ACB - CAB \\
&= A(BC - CB) + (AC - CA)B \\
&= A[B, C] + [A, C]B \qquad \square
\end{aligned}$$

(vi) Jacobi identity

$$\begin{aligned}
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= A[B, C] - [B, C]A + B[C, A] - [C, A]B \\
&\quad + C[A, B] - [A, B]C \\
&= ABC - ACB - BCA + CBA \\
&\quad + BCA - BAC - CAB + ACB \\
&\quad + CAB - CBA - ABC + BAC \\
&= 0 \qquad \square
\end{aligned}$$

2. Commutator practice

(a) We have

$$\begin{aligned}
[\hat{q}, \hat{p}^2] &= \hat{p}[\hat{q}, \hat{p}] + [\hat{q}, \hat{p}]\hat{p} \\
&= i\hbar\hat{p} + i\hbar\hat{p} \\
&= 2i\hbar\hat{p}
\end{aligned}$$

(b) Let us calculate $[\hat{q}, \hat{p}^n]$ from $[\hat{q}, \hat{p}^{n-1}]$. We have

$$\begin{aligned}
[\hat{q}, \hat{p}^n] &= \hat{p}[\hat{q}, \hat{p}^{n-1}] + [\hat{q}, \hat{p}]\hat{p}^{n-1} \\
&= \hat{p}[\hat{q}, \hat{p}^{n-1}] + i\hbar\hat{p}^{n-1}
\end{aligned}$$

Starting from $[\hat{q}, \hat{p}^2] = \mathrm{i}\hbar\hat{p}$, we thus deduce by induction that

$$[\hat{q}, \hat{p}^n] = n\mathrm{i}\hbar\hat{p}^{n-1}.$$

(c) We calculate

$$\begin{aligned}
[\hat{q}, e^{\hat{p}}] &= \left[\hat{q}, \sum_n \frac{\hat{p}^n}{n!}\right] \\
&= \sum_n \left[\hat{q}, \frac{\hat{p}^n}{n!}\right] \\
&= \sum_n i\hbar n \frac{\hat{p}^{n-1}}{n!} \\
&= \sum_n i\hbar \frac{\hat{p}^{n-1}}{(n-1)!} \\
&= i\hbar e^{\hat{p}}
\end{aligned}$$

3. The principles of quantum mechanics - measurements

(a) Starting with A, We notice that A is already partially diagonalised. Thus instead of solving the full characteristic equation

$$\det(A - \lambda\mathbb{I}) = 0$$

we can immediately see that $\lambda = 1$ is an eigenvalue and that $\phi^A_{\lambda=1} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ is the corresponding normalised eigenvector (up to a phase). This allows us to simplify the problem by only considering the lower diagonal block for the remaining solutions. We have

$$\begin{aligned}\begin{vmatrix}1-\lambda & 1 \\ 1 & 1-\lambda\end{vmatrix} &= (1-\lambda)^2 - 1 \\ &= \lambda(\lambda - 2) \\ &= 0\end{aligned}$$

from which we can see that the eigenvalues of A are given by $\{1, 0, 2\}$. For the corresponding eigenvectors we solve $\phi^A_{\lambda=0}$:

$$\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

giving (up to a phase) $\phi^A_{\lambda=0} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & -1\end{pmatrix}^T$. For $\phi^A_{\lambda=2}$ we find

$$\begin{pmatrix}-1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

giving (up to a phase) $\phi^A_{\lambda=2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 1\end{pmatrix}^T$

The situation is similar for B in that we can immediately see that $\lambda_B = 2$ is an eigenvalue of $\hat{B}$ and that $\phi^B_{\lambda=2} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ is the corresponding normalised eigenvector. We consider the lower diagonal block for the remaining solutions. We have

$$\begin{aligned}\begin{vmatrix}-\lambda & \mathrm{i} \\ -\mathrm{i} & -\lambda\end{vmatrix} &= \lambda^2 - 1 \\ &= 0\end{aligned}$$

We therefore find that the eigenvalues of $\hat{B}$ are given by $\{2, 1, -1\}$. For the corresponding eigenvectors we solve for $\phi^B_{\lambda=1}$:

$$\begin{aligned}\begin{pmatrix}0 & \mathrm{i} \\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} &= \begin{pmatrix}x \\ y\end{pmatrix} \\ \implies x &= \mathrm{i}y,\end{aligned}$$

giving (up to a phase) $\phi^B_{\lambda=1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & \mathrm{i} & 1\end{pmatrix}^T$.

And for $\phi^B_{\lambda=-1}$:

$$\begin{aligned}\begin{pmatrix}0 & \mathrm{i} \\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} &= -\begin{pmatrix}x \\ y\end{pmatrix} \\ \implies x &= -\mathrm{i}y,\end{aligned}$$

giving (up to a phase) $\phi^B_{\lambda=-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\mathrm{i} & 1\end{pmatrix}^T$

(b) (i) For $\langle\psi|\hat{A}|\psi\rangle$ we have

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= \frac{1}{35}\begin{pmatrix}5 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\ 1\\ 3\end{pmatrix}\\ &= \frac{1}{35}\begin{pmatrix}5 & 1 & 3\end{pmatrix}\begin{pmatrix}5\\ 4\\ 4\end{pmatrix}\\ &= \frac{41}{35}.\end{aligned}$$

We can use the fact that $\langle\psi|A|\psi\rangle$ must be between λ^A_{min} and λ^A_{max} as a useful check and we indeed find $0 \leq \frac{41}{35} \leq 2$

For $\langle\psi|B|\psi\rangle$ we have

$$\begin{aligned}\langle\psi|\hat{B}|\psi\rangle &= \frac{1}{35}\begin{pmatrix}5 & 1 & 3\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & \mathrm{i}\\ 0 & -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}5\\ 1\\ 3\end{pmatrix}\\ &= \frac{1}{35}\begin{pmatrix}5 & 1 & 3\end{pmatrix}\begin{pmatrix}10\\ 3\mathrm{i}\\ -\mathrm{i}\end{pmatrix}\\ &= \frac{10}{7}.\end{aligned}$$

This is between 2 and -1 as required.

(ii) If we measure A first the outcome will determine the state directly after the measurement. We measure $A = 1$ with probability

$$\begin{aligned}P(A=1) &= |\left\langle\phi^A_{\lambda=1}\middle|\psi\right\rangle|^2\\ &= \frac{1}{35}\left|\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\cdot\begin{pmatrix}5\\ 1\\ 3\end{pmatrix}\right|^2\\ &= \frac{5}{7}\end{aligned}$$

If we measured $A = 1$ we know that the system must now be in state $\left|\phi^A_{\lambda=1}\right\rangle$, which we use to calculate the probability of measuring $B = 2$ as

$$\begin{aligned}P(B=2) &= |\left\langle\phi^B_{\lambda=2}\middle|\phi^A_{\lambda=1}\right\rangle|^2\\ &= \left|\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right|^2\\ &= 1\end{aligned}$$

We have

$$P(A=1\&B=2) = P(A=1)P(B=2) = \frac{5}{7}$$

On the other hand, the initial probability of measuring $A = 2$ is given by

$$\begin{aligned}P(A=2) &= |\left\langle\phi^A_{\lambda=2}\middle|\psi\right\rangle|^2\\ &= \frac{1}{70}\left|\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}\cdot\begin{pmatrix}5\\ 1\\ 3\end{pmatrix}\right|^2\\ &= \frac{8}{35}.\end{aligned}$$

If a measurement yields this outcome, we know that the system must now be in state $\left|\phi^A_{\lambda=2}\right\rangle$, which we use to calculate the probability of measuring $B = 1$ as

$$\begin{aligned} P(B=1) &= |\left\langle\phi^B_{\lambda=1}\middle|\phi^A_{\lambda=2}\right\rangle|^2 \\ &= \frac{1}{4}\left|\begin{pmatrix}0\\-\mathrm{i}\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\right|^2 \\ &= \frac{1}{2} \end{aligned}$$

That is the probability of measuring $A = 2$ and then $B = 1$ is given by

$$P(A = 1\&B = 2) = P(A = 2)P(B = 1) = \frac{4}{35}$$

(iii) We measure $B = 2$ with probability

$$\begin{aligned} P(B=2) &= |\left\langle\phi^B_{\lambda=2}\middle|\psi\right\rangle|^2 \\ &= \frac{1}{35}\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix}\right|^2 \\ &= \frac{5}{7} \end{aligned}$$

If we measured $B = 2$ we know that the system must now be in state $\left|\phi^B_{\lambda=2}\right\rangle$, which we use to calculate the probability of measuring

$$\begin{aligned} P(A=1) &= |\left\langle\phi^A_{\lambda=1}\middle|\phi^B_{\lambda=2}\right\rangle|^2 \\ &= \left|\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 \\ &= 1 \end{aligned}$$

We thus have

$$P(B = 2\&A = 1) = P(B = 2)P(A = 1) = \frac{5}{7}.$$

On the other hand we have for the initial B mewasurement

$$\begin{aligned} P(B=1) &= |\left\langle\phi^B_{\lambda=1}\middle|\psi\right\rangle|^2 \\ &= \frac{1}{70}\left|\begin{pmatrix}0\\-\mathrm{i}\\1\end{pmatrix}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix}\right|^2 \\ &= \frac{1}{7} \end{aligned}$$

we know that the system must now be in state $\left|\phi^B_{\lambda=1}\right\rangle$, which we use to calculate the probability of measuring

$$\begin{aligned} P(A=2) &= |\left\langle\phi^A_{\lambda=2}\middle|\phi^B_{\lambda=1}\right\rangle|^2 \\ &= \frac{1}{4}\left|\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\-\mathrm{i}\\1\end{pmatrix}\right|^2 \\ &= \frac{1}{2} \end{aligned}$$

Thus we have

$$P(B = 1\&A = 2) = P(B = 1)P(A = 2) = \frac{1}{14}$$

Note that this value is different from $P(A = 2\&B = 1)$ from the previous part as measurements on A and B do not commute.

4. Time evolution operator

$$\begin{aligned}\hat{U}(t) &= e^{-i\hat{H}t/h} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar})^n \hat{H}^n t^n\end{aligned}$$

(a)

$$\hat{U}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar})^n \hat{H}^n 0^n = \hat{I}$$

(b) We have

$$\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle,$$

From which it follows that

$$\hat{H}^n\left|\psi\right\rangle = E^n\left|\psi\right\rangle.$$

Thus we find

$$\hat{U}(t)\left|\psi\right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar}\hat{E}t)^n\left|\psi\right\rangle = e^{-iEt/\hbar}\left|\psi\right\rangle. \quad \Box$$