

This week's problem is about polynomial equations over matrices. These are equations of the form

$$X^k + A_{k-1}X^{k-1} + \cdots + A_1X + A_0 = 0,$$

where the variable  $X$  and the coefficients  $A_i$  are in  $M_n(\mathbb{C})$ , the ring of  $n \times n$  matrices over  $\mathbb{C}$ . The theory of such equations is very different from the usual theory of polys, mainly because (a) unlike  $\mathbb{C}$ ,  $M_n(\mathbb{C})$  is not commutative, and (b) unlike  $\mathbb{C}$ ,  $M_n(\mathbb{C})$  has many nonzero elements that do not have inverses.

The problems we will look at just consider quadratic equations over  $M_2(\mathbb{C})$ , the  $2 \times 2$  matrices, but this is interesting enough. Write such an equation as

$$X^2 + AX + B = 0 \quad (A, B, X \in M_2(\mathbb{C})). \quad (1)$$

- 1.** Unlike quadratics over  $\mathbb{C}$ , not every pair of matrices are roots of a quadratic (1): show that there is no quadratic with roots  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

- 2.** The number of solutions of a quadratic (1) can vary a lot:

- (a) How many solutions are there of the quadratic  $X^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ ?
- (b) How many solutions are there of the quadratic  $X^2 - I = 0$ ? (Hint: it's more than 2!)

- 3.** Some theory - diagonalisable solutions: suppose  $X$  is a solution of (1), and  $\lambda$  an evalue of  $X$  with a corresponding evector  $v$ .

- (a) Show that  $\det(\lambda^2 I + \lambda A + B) = 0$  and  $v \in \text{Ker}(\lambda^2 I + \lambda A + B)$ .
- (b) By solving the quartic equation for  $\lambda$  in (a) and finding corresponding vectors  $v$ , we can find the diagonalisable solutions  $X$  of (1). Do this for the equations

$$\begin{aligned} \text{(i)} \quad & X^2 + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0, \\ \text{(ii)} \quad & X^2 + X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

- 4.** More theory - non-diagonalisable solutions: suppose  $X$  is a non-diagonalisable solution of (1). Then  $X$  has a repeated evalue  $\lambda$  (a root of the quartic in 3(a)), and  $Y = X - \lambda I$  has repeated evalue 0. Adjusting the quadratic (1), we can replace  $X$  by  $Y$ . We look for such  $X$ . As  $X$  is triangularisable,  $X^2 = 0$  and there is a basis  $v_1, v_2$  such that  $Xv_1 = 0, Xv_2 = v_1$ . This gives  $Bv_1 = 0, Av_1 + Bv_2 = 0$ . If we find all such  $v_1, v_2$  we can find all the non-diagonalisable solutions of (1).

- (i) Carry this out for the equations in 3(b).
- (ii) As a final exercise, find all solutions (diagonalisable or otherwise) of the equation

$$X^2 + \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.$$

Note: goodness, I got a bit carried away. It's a nice fact (that can be proved using the theory sketched above) that the possible numbers of solutions of the quadratic (1) are 0,1,2,3,4,5,6 or  $\infty$ . If you want to read more about this, there is a nice article by R L Wilson, "Polynomial equations over matrices", that can be found online.