

ESTIMATING THE MEAN OF A SAMPLE

random variables : X_1, X_2, \dots, X_n

→ assume all follow distribution F_X

→ all independent (i.i.d.)

→ Let mean of F_X be $\theta \in \mathbb{R}$
where θ is unknown

Q: How to estimate θ ?

Prop 1.2.6 $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \Rightarrow E[\bar{X}] = \theta$

Now suppose we observe $x_1, x_2, \dots, x_n \in \mathbb{R}$

→ observations of X_1, X_2, \dots, X_n , respectively

To estimate θ , we compute

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

estimate of θ

$$\hat{\theta} = x_1 + 2x_2 + x_3^5$$

Point estimators ✓

$$\hat{\theta} = \hat{\theta}(x_1, x_2, \dots, x_n) = r(x_1, \dots, x_n)$$

example $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$

Definition 1.5.18

The ESTIMATION ERROR of the estimator $\hat{\theta}$ of a parameter θ is defined as $\hat{\theta} - \theta$

Definition 1.5.19 The BIAS of the estimator $\hat{\theta}$ of a parameter θ is denoted $b_{\theta}(\hat{\theta})$ and is defined as $b_{\theta}(\hat{\theta}) = E[\hat{\theta}] - \theta$

$$E[\hat{\theta} - \theta] \stackrel{\text{lin. of } E}{=} E[\hat{\theta}] - \theta = b_{\theta}(\hat{\theta})$$

Definition : If $E[\hat{\theta}] = \theta$
the $\hat{\theta}$ is UNBIASED

because $b_{\theta}[\hat{\theta}] = E[\hat{\theta}] - \theta = 0$

$\hat{\theta} = \bar{X}$ as an estimator of
mean θ of F_X (in Prop 1.2.4)

$$E[\hat{\theta}] = E[\bar{X}] = \theta$$

\Rightarrow unbiased

$\hat{\sigma}^2 = s^2$ estimator of variance σ^2
of F_X

$$E[\hat{\sigma}^2] = E[s^2] = \sigma^2 \quad (\text{in 1.2.6})$$

\Rightarrow unbiased

Definition 1.5.23

The MEAN SQUARED ERROR of
the estimator $\hat{\theta}$ of parameter θ
is defined as $E[(\hat{\theta} - \theta)^2]$

expected squared deviation of $\hat{\theta}$ from θ

Theorem 1.5.24

The mean squared error of an estimator $\hat{\theta}$ of a parameter θ can be expressed in terms of its bias and variance

$$E[(\hat{\theta} - \theta)^2] = [b_{\theta}(\hat{\theta})]^2 + \text{Var}[\hat{\theta}]$$

Proof:

$$E[X^2] = (E[X])^2 + \text{Var}[X]$$

$$E[(\hat{\theta} - \theta)^2] = (E[\hat{\theta} - \theta])^2 + \text{Var}[\hat{\theta} - \theta] \quad (\text{Exercise 1.1.5})$$

$$\text{bias: } b_{\theta}[\hat{\theta}] = E[\hat{\theta}] - \theta = E[\hat{\theta} - \theta]$$

$$\text{Var}[\hat{\theta} - \theta] = \text{Var}[\hat{\theta}]$$

$$\Rightarrow E[(\hat{\theta} - \theta)^2] = (b_{\theta}[\hat{\theta}])^2 + \text{Var}[\hat{\theta}]$$