Imperial College London
Department of Mathematics

Solutions to Blackboard Quiz 1

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

You should submit your answers on Blackboard by 10am UK time on Wednesday 25 January 2023 (week 3). The first part of the test is about the change of basis formula which was covered in Section 4 of the notes. We will be using this a lot, so you should revise this. The test is worth 1.5 percent of the marks for the module.

(A) (The following text refers to Questions 1 - 5) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x_1, x_2) = \begin{pmatrix} 3x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$. Consider the following bases of $\mathbb{R}^2$:
E : the standard basis e_1, e_2;
B : $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
C : $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Also consider the following matrices in $M_2(\mathbb{R})$:

$$U = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix};\ W = \begin{pmatrix} -6 & -5 \\ 7 & 7 \end{pmatrix};\ X = \begin{pmatrix} 5 & 1 \\ -13 & -4 \end{pmatrix};\ Y = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix};\ Z = \begin{pmatrix} 5 & 2 \\ -9 & 5 \end{pmatrix}.$$

Let Id be the identity transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Which of these (if any) is equal to:

Qu 1: $[T]_E$;

Solution: Write the images of the vectors in B to find $[T]_E = Y$.

Qu 2: $[T]_B$;

Solution: The answer is X. One can do it by writing the images of B wrt to B. Alternatively one can write

$${}_E[Id]_B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad {}_B[Id]_E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad [T]_B = {}_B[Id]_E([T]_E)\, {}_E[Id]_B = X.$$

Qu 3: $[T]_C$;

Solution: The answer is W. One can do it by writing the images of C wrt to C. Alternatively one can write

$${}_E[Id]_C = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \qquad {}_C[Id]_E = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \qquad [T]_C = {}_C[Id]_E([T]_E)\, {}_E[Id]_C = W.$$

Qu 4: ${}_C[Id]_B$;

Solution: We have already computed all the ingredients to get ${}_C[Id]_B$. Namely

$$ {}_C[Id]_B = {}_C[Id]_E\, {}_E[Id]_B = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}. $$

Thus the answer is none of the above.

Qu 5: ${}_B[Id]_C$?

Solution: Similar as above

$$ {}_B[Id]_C = {}_B[Id]_E\, {}_E[Id]_C = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} = U. $$

(B) (The following text refers to Questions 6 - 9) Decide whether each of the following statements is true for all $n \in \mathbb{N} \setminus \{0\}$ and all $A, B \in M_n(\mathbb{R})$.

Qu 6: $Au = Bu$ for all $u \in \mathbb{R}^n$ if and only if $A = B$.

Solution: True. One direction is immediate. For the other one it suffices to observe that if $Au = Bu$ for all $u \in \mathbb{R}^n$, then they coincide on the standard basis, which gives $A = B$.

Qu 7: If $\det(A) = \det(B)^2$, then $A = B^2$.

Solution: False. Take for example

$$ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = I_2. $$

Qu 8: $\det(\alpha A + B) = \alpha \det(A) + \det(B)$.

Solution: False. This is only true for $n = 1$. In all the other cases $\det(\alpha A) = \alpha^n \det(A)$. Hence if, for instance, $B = 0$, A is non-singular, and α is such that $\alpha^{n-1} \neq 1$, $\det(\alpha A) \neq \alpha \det(A)$. Thus the statement is false. The statement is also false if we restrict to $\alpha = 1$, take $A = I_2$ and $B = -I_2$. Then $\det(A) = \det(B) = 1$

$$ A + B = 0, \qquad \det(A + B) = 0 \neq \det(A) + \det(B) = 1 + 1 = 2. $$

Qu 9: If AB^t is non-singular, then B is non-singular.

Solution: True. This is because $\det(AB^t) = \det(A)\det(B^t)$ and $\det(B^t) = \det(B)$.

(C) Decide whether the following statement is true or false.

Qu 10: Let $n \in \mathbb{N} \setminus \{0\}$, and let A be a square matrix of size n with integer entries. Let p be a prime number, and assume that all the entries of A are divisible by p^m for some $m \in \mathbb{N}$. Then $\det(A)$ is divisible by p^{mn}.
(You may regard A as an element of $M_n(\mathbb{Q})$ and use Definition 5.1.3 for $\det(A)$.)

Solution: This is true, we may use the recursive formula in Definition 5.1.3 to set up an induction on the size of A.