IMPERIAL

MATH70039

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Representation Theory

Date: Monday, May 13, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

You may use all results from the course, including lectures, lecture notes, problems sheets and courseworks without proof, unless otherwise specified. You may also use the assertions of previous parts of a question in solving later ones, without proof. All parts require full justification unless noted to the contrary.

1. (a) (i) Define a homomorphism of representations of a group G (or G-linear map). (2 marks)

   (ii) Prove that if (V, ρ_V) and (W, ρ_W) are representations of a group G and W is irreducible, then any nonzero homomorphism $V \to W$ is surjective and any nonzero homomorphism $W \to V$ is injective. (3 marks)

   (b) Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternionic group of order eight and let (V, ρ_V) be an irreducible representation of G of dimension greater than one.

   (i) Show that for every one-dimensional representation L of G, we have $L \otimes V \cong V$. Use this to prove that, furthermore, $\dim \mathrm{Hom}_G(L, V \otimes V) = 1$. (4 marks)

   (ii) Decompose $V \otimes V$ into irreducible representations of G, up to isomorphism. (3 marks)

   (iii) Now do the same as in (ii) for $V^{\otimes 3} := V \otimes V \otimes V$. (2 marks)

   (iv) Now generalise to $V^{\otimes n}$ for all $n \geq 1$. (3 marks)

   (v) Is the answer to parts (i)–(iv) any different if we replace G by the dihedral group D_4 of order eight? Why or why not (briefly)? (3 marks)

(Total: 20 marks)

2. (a) Let G be the group of affine-linear transformations of $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$:

$$G = \{T_{a,b} : a \in \mathbb{Z}_m^\times, b \in \mathbb{Z}_m\} \subseteq \mathrm{Perm}(\mathbb{Z}_m), \quad T_{a,b}(x) = ax + b.$$

   (i) Show that for every one-dimensional representation $\chi : \mathbb{Z}_m^\times \to \mathbb{C}^\times$ of $\mathbb{Z}_m^\times$, there is an irreducible representation of G, $\widetilde{\chi} : G \to \mathbb{C}^\times, \widetilde{\chi}(T_{a,b}) = \chi(a)$. (2 marks)

   (ii) For every $\zeta \in \mathbb{C}^\times$ with $\zeta^m = 1$, show that there is a representation $(\mathbb{C}[\mathbb{Z}_m^\times], \rho_\zeta)$ of G given as follows: $\rho_\zeta(T_{a,b})(a') = \zeta^{(aa')^{-1}b} aa'$. (3 marks)

   (iii) Assume that $m = p$ is a prime and $\zeta = e^{2\pi i/p}$. Prove that ρ_ζ is irreducible.
   [Hint: Compute and use the character of ρ_ζ.] (4 marks)

   (iv) For $m = p$ prime, classify all irreducible representations of G. (2 marks)

   (b) (i) For a vector space $V \neq 0$, show that there is a nonzero linear map $\iota : V \otimes V^* \to \mathrm{End}(V)$ satisfying $\iota(v \otimes f)(w) = f(w)v$, and that ι is surjective if $\dim V < \infty$. (2 marks)

   (ii) Suppose that V is a representation of G. Show that $\mathrm{End}(V)$ is a representation of $G \times G$ in a way that the map in (i) is $G \times G$-linear, using the external product action on $V \otimes V^*$, $\rho(g,h)(v \otimes f) = \rho_V(g)(v) \otimes \rho_{V^*}(h)(f)$. (2 marks)

   (iii) Prove that for V a representation of G, $\mathrm{End}(V)$ is an irreducible representation of $G \times G$ if and only if V is an irreducible finite-dimensional representation of G.
   [Hint: You may use the facts: 1. The map ι is not surjective if V is infinite-dimensional; 2. If V and W are irreducible finite-dimensional representations of G and H, then $V \otimes W$ is an irreducible $G \times H$-representation.] (5 marks)

(Total: 20 marks)

3. Let G be a group of size 24 and seven conjugacy classes with one character as follows, for $\omega = e^{2\pi i/3}$:

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
1	1	ω^2	ω	1	ω^2	ω

(a) Find two more characters of one-dimensional representations of G. (2 marks)

(b) Find the dimensions of all irreducible representations of G. (2 marks)

(c) Let V, V' be two-dimensional irreducible representations of G. Show that $V' \cong L \otimes V$ for some representation L. (Do not use information given in (e) below). (5 marks)

[Hint: Show that, if not, the character values for $\mathcal{C}_i, i \in \{3, 4, 6, 7\}$ are zero for all irreducible representations of dimension bigger than one, and apply column orthogonality.]

(d) Now we are given another irreducible character:

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
2	-2	-1	-1	0	1	1

Using this, write the complete character table of G. (4 marks)

(e) Find the conjugacy classes in the centre of G. (2 marks)

(f) Find all normal subgroups of G, as unions of conjugacy classes. (3 marks)

(g) Find the commutator subgroup $[G, G]$, as a union of conjugacy classes. (2 marks)

(Total: 20 marks)

4. (a) Define the algebra $A_n := \mathbb{C}[x]/(x^n) = \{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{C}, x^n = 0\}$.

(i) Prove that the submodules of A_n are all of the form $x^m A_n$ for $0 \le m \le n$. (3 marks)

(ii) Deduce that there is a unique simple A_n-module up to isomorphism. (2 marks)

(iii) Now let (V, ρ_V) be a finite-dimensional A_n-module. Using the Jordan decomposition of the element $\rho_V(x)$, prove that V is isomorphic to a direct sum of modules $A_n/x^m A_n$ for $0 < m \le n$, and show that these are indecomposable. (3 marks)

(b) Suppose that A is an algebra with exactly two simple modules up to isomorphism, of dimensions two and three.

(i) Prove that $\dim A \ge 13$, with equality if and only if A is semisimple. (3 marks)

(ii) Prove that $\dim A \ne 14$. (2 marks)

(iii) Prove that every simple A-bimodule, or equivalently $A \otimes A^{\mathrm{op}}$-module, has dimension at least four. (5 marks)

(iv) Prove that $\dim A \notin \{15, 16\}$. (2 marks)

[Hint: A two-sided ideal of A is an A-bimodule.]

(Total: 20 marks)

5. (a) (i) Let $H \leq G$ be groups, (V, ρ_V) a representation of G, and (W, ρ_W) a representation of H. State the *Frobenius reciprocity theorem* for coinduced representations, giving explicit maps between the relevant Hom spaces. (2 marks)

(ii) Prove the theorem in (i), using that the explicit maps in (i) are well-defined. (4 marks)

(b) Let G be any group (not necessarily finite).

(i) Show that $\mathrm{Fun}(G, \mathbb{C})$ admits an action of $G \times G$ via $(g_1, g_2)(\varphi)(h) = \varphi(g_1^{-1} h g_2)$. (2 marks)

(ii) Show that there is a $G \times G$-linear isomorphism $F : \mathrm{Fun}(G, \mathbb{C}) \to \mathrm{coInd}_{G_\Delta}^{G \times G}(\mathbb{C})$, for $G_\Delta := \{(g, g) : g \in G\} \leq G \times G$, given by $F(\varphi)(g, h) = \varphi(g^{-1} h)$. (5 marks)

(iii) For every finite-dimensional representation (V, ρ_V) of G, use Frobenius reciprocity and the preceding parts to construct a nonzero $G \times G$-linear map $\mathrm{End}(V) \to \mathrm{Fun}(G, \mathbb{C})$.
Note that the action of $G \times G$ on $\mathrm{End}(V)$ is defined in Q2. (2 marks)

(iv) In the case of G a finite group, using the preceding parts, conclude that $\mathrm{Fun}(G, \mathbb{C}) \cong \bigoplus_V \mathrm{End}(V)$ as $G \times G$-representations, where the sum is over all irreducible representations V of G up to isomorphism.
[Hint: see the hint to Q2.(b).(iii).] (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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XXX (Solutions)

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1. (a) (i) This is a linear map $T : V \to W$ such that $T \circ \rho_V(g) = \rho_W(g) \circ T$ for all $g \in G$. seen $\Downarrow$ 2, A

(ii) Given that $T : V \to W$ is G-linear and nonzero, its image is nonzero, so if W is irreducible then the image must be all of W as it is a subrepresentation. Similarly, if $T : W \to V$ is nonzero and G-linear, its kernel is a proper subrepresentation of W, hence zero. 3, A

(b) (i) Since G has a unique two-dimensional irrep (as we saw, or because $|G| = 8$ and G has the trivial one-dimensional representation as well), and $L \otimes V$ is still a two-dimensional irrep (irreducible since $L^* \otimes L \otimes V \cong V$ is), it is isomorphic to V. Then $\mathrm{Hom}_G(L, V \otimes V) \cong \mathrm{Hom}_G(V^*, L^* \otimes V)$. Now V^* is still irreducible as V is irreducible and finite-dimensional; also as we showed $L^* \otimes V \cong V$. Put together, $\dim \mathrm{Hom}_G(L, V \otimes V) = 1$. sim. seen $\Downarrow$ 4, B

(ii) There are four one-dimensional representations of G up to isomorphism, and they all appear in $V \otimes V$ by the preceding. So $V \otimes V \cong L_1 \oplus L_2 \oplus L_3 \oplus L_4$ for L_i the one-dimensional representations up to isomorphism. 3, A

(iii) Now we just tensor $V \otimes L_i \cong V$ by (i), to get that $V^{\otimes 3} \cong V^4$. 2, A

(iv) We have by induction that $V^{\otimes n} \cong \begin{cases} L_1^{2^{n-2}} \oplus \cdots \oplus L_4^{2^{n-2}}, & n \text{ even}, \\ V^{2^{n-1}}, & n \text{ odd}. \end{cases}$ 3, C

(v) No, because the dihedral group of order 8, and the quaternionic group have the same character table. 3, B

2. (a) (i) This is one-dimensional, so automatically irreducible. We just need to observe that $\widetilde{\chi}$ defines a homomorphism. This is true because $T_{a,b}T_{c,d} = T_{ac,b+ad}$. unseen ⇓ 2, A

(ii) This is an explicit check: 3, B

$$\rho_\zeta(T_{a,b}T_{c,d})(a') = \rho_\zeta(T_{ac,b+ad})(a') = \zeta^{(aca')^{-1}(b+ad)}aca'$$
$$= \zeta^{(aca')^{-1}b+(ca')^{-1}d}aca' = \rho_\zeta(T_{a,b})\rho_\zeta(T_{c,d})(a').$$

(iii) The character of ρ_ζ is nonzero only on $T_{a,b}$ such that $a = 1$. There $\chi_\zeta(T_{1,b}) = \begin{cases} -1, & b \neq 0 \\ p-1, & b = 0. \end{cases}$
Now let's take the inner product of this character with itself: $\langle \chi_\zeta, \chi_\zeta \rangle = |G|^{-1}((p-1)^2 + (p-1)) = 1$. So it is indeed irreducible, by orthonormality of irreducible characters. 4, C

(iv) We have found $p-1$ distinct one-dimensional matrix representations, together with one irreducible $p-1$ dimensional representation. By the sum of squares formula, these must be all of the irreducible representations of G. sim. seen ⇓ 2, A

(b) (i) This map is evidently nonzero since $f(w)v$ can be nonzero, as $V \neq 0$. For $v_1, \ldots, v_n$ a basis of V and $v_1^\vee, \ldots, v_n^\vee$ a dual basis of V^*, we have $T = \sum_{i=1}^n \iota(T(v_i) \otimes v_i^\vee)$, as can be seen by evaluating on any basis vector. seen ⇓ 2, A

(ii) Simply let $\rho(g,h)(T) = \rho_g(T) \circ T \circ \rho_h(T)^{-1}$. Then $\rho(g,h)\iota(v \otimes f) = \iota(\rho_V(g)(v) \otimes \rho_{V^*}(h)(f))$, as desired. unseen ⇓ 2, C

(iii) We have by Hint (1) that $V \otimes V^* \to \mathrm{End}(V)$ has image a subrepresentation which is proper and nonzero if V is infinite-dimensional. So $V \otimes V^*$ can only be irreducible if V is finite-dimensional; it is also clear that $\mathrm{End}(V)$ can only be irreducible if V is irreducible, as otherwise we can tensor a proper nonzero subrepresentation with V^* to obtain a $G \times G$-subrepresentation of $V \otimes V^*$. Conversely for V finite-dimensional and irreducible, we showed in lectures that V^* also has these properties, and by Hint (2), the external product $V \boxtimes V^*$ must be irreducible too. Now the surjective nonzero map $V \boxtimes V^* \to \mathrm{End}(V)$ must be an isomorphism, so $\mathrm{End}(V)$ is also irreducible. unseen ⇓ 5, D

3. (a) Simply take the trivial character $(1,1,1,1,1,1,1)$ and the character of the dual, $(1,1,\omega,\omega^2,1,\omega,\omega^2)$ (which is also the tensor square of the given one). sim. seen ⇓ 2, A

(b) Recall that the one-dimensional representations form an abelian group, whose order divides that of G. Here the given representation generates a subgroup of this group of size 3. So the only possibility if we have more one-dimensional representations is to have $2 \cdot 3 = 6$ of them, but as the group has size 24 and $24-6=18$ is not a square, this is impossible. We then have $1^2+1^2+1^2+2^2+2^2+2^2+3^2 = 24$ as the only possibility, any other possibility will be too large except $1^2+1^2+1^2+2^2+2^2+2^2+2^2 < 24$. meth seen ⇓ 2, A

(c) Let L be the one-dimensionl representation with character given at the beginning of the statement. If V is a two-dimensional irrep, then either $V, V\otimes L, V\otimes L\otimes L$ are all nonisomorphic, or else they are all isomorphic, since $L^{\otimes 3}$ is trivial (and hence $(L^{\otimes 2})^{\otimes 2} \cong L$). For a contradiction, suppose $V \cong V \otimes L$. Then $\chi_V(\mathcal{C}_i) = 0$ for $i \in \{3,4,6,7\}$. Also the three-dimensional irrep must have value 0 for these conjugacy classes, since it also is isomorphic to its tensor product by L. Then all irreducible characters of $\mathcal{C}_i$ for $i \in \{3,4,6,7\}$ are zero except for the three one-dimensional representations. By column orthogonality, these four conjugacy classes all have size $24/3 = 8$, which is impossible. unseen ⇓ 5, D

(d) By (c) we can tensor this by the one-dimensional irreps (the given one and its conjugate) to get all the two-dimensional irreps (this is also clear without using (c), since tensoring is guaranteed to produce irreps). This gives six irreducible characters, call them $\chi_1,\ldots,\chi_6$, all but one. To get the last one, of dimension three, we can apply the regular character, so this one is given as $\chi_7 = \frac{1}{3}(\chi_{\mathbb{C}[G]} - \sum_{i=1}^6 \chi_i)$. Put together we get the table: sim. seen ⇓

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
1	1	1	1	1	1	1
1	1	ω^2	ω	1	ω^2	ω
1	1	ω	ω^2	1	ω	ω^2
2	-2	-1	-1	0	1	1
2	-2	$-\omega^2$	$-\omega$	0	ω^2	ω
2	-2	$-\omega$	$-\omega^2$	0	ω	ω^2
3	3	0	0	-1	0	0

4, B

seen ⇓

(e) These are the ones whose columns have all entries of absolute value equal to the dimension, i.e., $\mathcal{C}_1$ and $\mathcal{C}_2$, so that $Z(G) \cong C_2$. 2, A

(f) For an irrep we look at the conjugacy classes whose character value equals the dimension. The kernel of the two nontrivial one-dimensional irreps is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_5$. The three two-dimensional irreps are all faithful. The three-dimensional irrep has kernel $\mathcal{C}_1 \cup \mathcal{C}_2$. These subgroups along with the trivial subgroup and the entire group fit into a single chain, $\{e\} = \mathcal{C}_1 \subseteq \mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_5 \subseteq G$. Intersections of these subgroups therefore yield exactly the same ones listed, which are all the normal subgroups. seen ⇓ 3, A

(g) This is the intersection of the kernels of the one-dimensional representations; by the preceding part we see that it is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_5$. (Note that $|[G,G]| = 1+1+6 = 8$, so $|G_{\mathsf{ab}}| = 24/8 = 3$, as it should be since there are three one-dimensional representations.) seen ⇓ 2, A

4. (a) (i) We claim that the submodules of A_n are precisely $x^m A_n$ for $0 \le m \le n$. These are clearly submodules since A_n is commutative. Conversely, if $V \subseteq A_n$ is a nonzero submodule and $0 \le m < n$ is minimal such that there is an element $v \in V$ in $x^m A_n$ but not in $x^{m+1}A_n$, that is, $v = ax^m +$ (higher degree terms) for nonzero a (note that such an element must exist because every nonzero element has this form for some $0 \le m < n$), then up to nonzero scaling, we can assume $v = x^m +$ (higher degree terms). Then $v, xv, \ldots, x^{n-m-1}v$ are a strictly upper-triangular change of basis (i.e, by adding multiples of later basis elements to each basis element) from $x^m, x^{m+1}, \ldots, x^{n-1}v$ for the span, $x^m A_n$, hence $V = x^m A_n$. The zero module equals $x^n A_n$. sim. seen ⇓ 3, C

(ii) Recall that for every simple A_n-module V, there is a surjection $A_n \to V$, of the form $a \mapsto av$ for fixed nonzero $v \in V$. By the first isomorphism theorem, $V \cong A_n/x^m A_n$ for some m, and to be simple we require $m = 1$. So we get $V \cong A_n/xA_n$, the unique simple A_n-module up to isomorphism. 2, A

(iii) The operator $T = \rho_V(x)$ is nilpotent: $T^n = 0$. So the Jordan normal form of T on V is a direct sum of Jordan blocks with 0's on the diagonal, of sizes $\le n$. So V decomposes as a direct sum of such cases where there is a single Jordan block. For the case of a single Jordan block of size m, this module is spanned by $v, Tv, \ldots, T^{m-1}v$ for some nonzero v, with $T^m = 0$. This means that the map $A_n/x^m A_n \to V, f \mapsto fv$ is an isomorphism. These modules are indecomposable because, by (i), every proper submodule is a submodule of $xA_n/x^m A_n$. 3, B

(b) (i) The inequality is by the Jacobson Density theorem: $\Phi : A \to \mathrm{End}(V) \oplus \mathrm{End}(W)$ is surjective for V, W the simples of dimension $2, 3$ respectively, and the RHS has dimension $4+9 = 13$. Then, if $\dim A = 13$, Φ is an isomorphism, and then Artin–Wedderburn implies that A is semisimple. meth seen ⇓ 3, B

(ii) The kernel of Φ cannot be one-dimensional since there is no one-dimensional A-module, hence if A is not semisimple its dimension must strictly exceed $13 + 1 = 14$. sim. seen ⇓ 2, A

(iii) To see this, if we have an A-bimodule E which had dimension only 2 or 3, then as an A-module it would be isomorphic to V or W, hence simple (otherwise, if 3-dimensional and reducible, it would have V as a submodule and the quotient would be one-dimensional, a contradiction). Then $\mathrm{End}_A E \cong \mathbb{C}$ by Schur's Lemma. But the right action would produce a homomorphism $A^{\mathsf{op}} \to \mathrm{End}_A E \cong \mathbb{C}$, or equivalently since $\mathbb{C}$ is commutative, a homomorphism $A \to \mathbb{C}$. This is impossible since A has no one-dimensional module. unseen ⇓ 5, D

(Remark: one can go further and argue that all simple finite-dimensional $A \otimes B$-modules are tensor products of simple A- and B-modules, respectively; this means that the only simple A-bimodules in the example have dimension $4, 6$, and 9.)

(iv) As the hint points out, a two-sided ideal of A is an A-bimodule. So the kernel of the map $A \to \mathrm{End}(V) \oplus \mathrm{End}(W)$, which is a bimodule, if nonzero, has dimension at least four. We conclude that, if A is not semisimple, $\dim A \ge 13 + 4 = 17$. unseen ⇓ 2, B

5. (a) (i) The theorem states that $\Phi : \mathrm{Hom}_G(V, \mathrm{coInd}_H^G W) \leftrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G V, W) : \Psi$ with maps $\boxed{\text{seen} \Downarrow}$

$\boxed{2, \mathrm{M}}$

$$\Phi(\phi)(v) = \phi(v)(1), \quad \Psi(\phi')(v)(g) = \phi'(\rho_V(g)(v)).$$

(ii) To see they are inverse, note that $\phi(v)(g) = \rho_{\mathrm{coInd}_H^G W}(g)\phi(v)(1) = \phi(\rho_V(g)v)$. Thus $\Phi \circ \Psi(\phi')(v) = \phi'(v)$ and $\boxed{4, \mathrm{M}}$

$$\Psi \circ \Phi(\phi)(v)(g) = \Phi(\phi)(\rho_V(g)v) = \phi(\rho_V(g)v)(1) = \phi(v)(g).$$

(Not required: to check that the maps Φ, Ψ are well-defined, note that $\Phi(\phi)(\rho_V(h)v) = \phi(\rho_V(h)v)(1) = \phi(v)(h) = \rho_V(h)\phi(v)(1) = \rho_V(h)\Phi(\phi)(v)$, and $\Psi(\phi')(\rho_V(g)v)(g') = \phi'(\rho_V(g'g)v) = \Psi(\phi')(v)(g'g) = \rho_{\mathrm{coInd}_H^G W}(g)\Psi(\phi')(v)(g')$.) $\boxed{\text{unseen} \Downarrow}$

(b) (i) To check that the given formula is an action, observe that $(g_1, g_2)(g_3, g_4)(\varphi)(h) = \varphi(g_1^{-1}g_3^{-1}hg_3g_1) = (g_1g_3, g_2g_4)(\varphi)(h)$. $\boxed{2, \mathrm{M}}$

(ii) Let us check that this is well-defined: we have $F(\varphi)(gg_1, gg_2) = \varphi(g_1^{-1}g^{-1}gg_2) = \varphi(g_1^{-1}g_2) = F(\varphi(g_1, g_2)$. (This is also because the map is given by Frobenius reciprocity from the G-linear map $\mathrm{Fun}(G, \mathbb{C}) \to \mathbb{C}$, $f \mapsto f(1)$.) Let us check that it is $G \times G$ linear: we have

$$\begin{aligned}(g,h)(F(\varphi))(g',h') &= F(\varphi)(g'g, h'h) = \varphi(g^{-1}(g')^{-1}h'h) \\ &= ((g,h)\varphi)(g'^{-1}h') = F((g,h)\varphi)(g',h').\end{aligned}$$

To check that it is an isomorphism, we can take the inverse map F' given by $F'(\psi)(g) = \psi(1, g)$. Note that $F'F(\varphi)(g) = F\varphi(1, g) = \varphi(g)$. Then $FF'(\psi)(g, h) = F'(\psi)(g^{-1}h) = \psi(1, g^{-1}h) = \psi(g, h)$ where we use the G_Δ invariance at the end. $\boxed{5, \mathrm{M}}$

(iii) Thanks to (ii) we just need to construct a nonzero G_Δ-linear map $\mathrm{End}(V) \to \mathbb{C}$, i.e., a G-linear map for the usual action; this is given by the trace map, since V is finite-dimensional. $\boxed{2, \mathrm{M}}$

(iv) We obtain a $G \times G$-linear map $\Phi : \bigoplus_V \mathrm{End}(V) \to \mathrm{Fun}(G, \mathbb{C})$, with each summand mapping in a nonzero way to the target. Now for V irreducible, and finite-dimensional since G is finite, we have $\mathrm{End}(V) \cong V \boxtimes V^*$ as representations of $G \times G$, which is irreducible. For V, V' nonisomorphic, we have $\mathrm{End}(V) \ncong \mathrm{End}(V')$, as this is already true as a representation of just $\{1\} \times G$ by Schur's Lemma (where $\mathrm{End}(V) \cong V^{\dim V}$). So all submodules of the sum $\bigoplus_V \mathrm{End}(V)$ are just given by a subset of the V (by a result from the course). Since Φ is injective on each summand, the kernel of Φ must be zero. Since the dimensions of the source and target are equal, it is an isomorphism.

$\boxed{5, \mathrm{M}}$

Review of mark distribution:

Total A marks: 31 of 32 marks

Total B marks: 22 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 15 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH60039 Group Representation Theory

Question	Marker's comment
1	This question was, along with Q3, the easiest on this challenging paper. nbsp;The main difficulty was in 1b, where students had to realise the existence of a unique two-dimensional irreducible representation, and use this to deduce part (i). Most students were able to apply (i) to parts (ii) and (iii), but obtaining the general formula in (iv) was not easy. nbsp;Many students also overlooked in (b)(v) the fact that the properties of the group used in (b)(i)--(b)(iv) also apply to the dihedral group of order eight.
2	This was, along with Q4, one of the most difficult questions. nbsp;For part (a), many students realised that in (i) and (ii) one only needs to verify that the given map is a homomorphism, and this was readily done in (a)(i); part (iii) posed the real difficulty, with only a few students successfully computing the character, and even fewer verifying its inner product with itself as one. nbsp;Part (b) was a bit easier, as many students understood how to obtain the required properties from the natural isomorphisms when V is finite-dimensional.
3	Students were well-prepared for this, with part (c) posing the greatest challenge; a few students seemed to forget the process for deducing parts (e),(f), and (g) from the character table.
4	This was a difficult question, with (a)(i) and (b)(ii) (and consequently (b)(iv)) posing the most difficulty. nbsp;Students also were often confused in (a)(ii) on the difference between simple A-modules and simple A-submodules of A. nbsp;Many students succeeded in applying the statement of (a)(i) to nbsp;(a)(ii) and even (a)(iii).nbsp;

MATH70039 Group Representation Theory

Question	Marker's comment
1	This question was, along with Q3, the easiest on this challenging paper. nbsp;The main difficulty was in 1b, where students had to realise the existence of a unique two-dimensional irreducible representation, and use this to deduce part (i). Most students were able to apply (i) to parts (ii) and (iii), but obtaining the general formula in (iv) was not easy. nbsp;Many students also overlooked in (b)(v) the fact that the properties of the group used in (b)(i)--(b)(iv) also apply to the dihedral group of order eight.
2	This was, along with Q4, one of the most difficult questions. nbsp;For part (a), many students realised that in (i) and (ii) one only needs to verify that the given map is a homomorphism, and this was readily done in (a)(i); part (iii) posed the real difficulty, with only a few students successfully computing the character, and even fewer verifying its inner product with itself as one. nbsp;Part (b) was a bit easier, as many students understood how to obtain the required properties from the natural isomorphisms when V is finite-dimensional.
3	Students were well-prepared for this, with part (c) posing the greatest challenge; a few students seemed to forget the process for deducing parts (e),(f), and (g) from the character table.
4	This was a difficult question, with (a)(i) and (b)(ii) (and consequently (b)(iv)) posing the most difficulty. nbsp;Students also were often confused in (a)(ii) on the difference between simple A-modules and simple A-submodules of A. nbsp;Many students succeeded in applying the statement of (a)(i) to nbsp;(a)(ii) and even (a)(iii).nbsp;
5	nbsp;Many students appeared to be pressed for time before reaching this question. nbsp;Those who managed a serious attempt performed admirably on (a)(i) and (b)(i), and sometimes on (a)(ii). However, due to time constraints, or potentially unfamiliarity with the situation of G x G representations, (b)(ii)--(b)(iv) posed a challenge.nbsp;