Math40003 Linear Algebra and Groups **Problem Sheet 3**
Seen B Solutions

B.1. For which a, b, c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

Since $\{(1,3,1),(2,1,1)\}$ is linearly independent, we're interested in when there are λ and μ such that

$$\lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We can eliminate the λ and μ via some clever algebra, but there's a nice geometric way that I want to show you. The vectors being linearly dependent is the same as the vector lying on the plain that contains $(1,3,1)$, $(2,1,1)$, and the origin. We take the vector product of $(1,3,1)$ and $(2,1,1)$ to get $(2,1,-5)$. Therefore (a,b,c) lies on the plain if and only if $2a+b-5c=0$.

B.2. Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

(a) If $\{v_1,\dots,v_n\}$ is a basis, for V, and $\{x_1,\dots,x_r\}$ is a linearly independent subset of V with $r<n$, and if $v_i \notin Span\{x_1,\dots,x_r\}$ for all $i=1,\dots,n$, then $\{x_1,\dots,x_r,v_{r+1},\dots,v_n\}$ is a basis for V.

(b) If U is a subspace of V, then $U+U=U$.

(c) If U and W are subspaces of V, and $\dim U + \dim W = \dim V$, then $U \cap W = \{0_V\}$.

(d) If $\dim V = n$ and $v_1 \in V$, then there exist vectors $v_2,\dots,v_n$ in V such that $\{v_1,\dots,v_n\}$ spans V.

(e) If W is a subspace of V , then $\dim W \le \dim V$ and $\dim W = \dim V$ if and only if $W=V$.

(a) False: take $\{v_1,v_2,v_3\}$ to be the standard basis of $\mathbb{R}^3$, with $s=1$ and $x_1=(0,1,1)$.

(b) True: since U is closed under addition, we have $U+U \subseteq U$; but since $0_V \in U$ we have $u = u + 0_V \in U+U$ for all $u \in U$, and so $U \subseteq U+U$.

(c) False: take $V=\mathbb{R}^2$, and $U=W=Span\{(1,0)\}$.

(d) False: but only because v_1 might be 0_V. For any other v_1 it is true.

(e) True: Let B be a basis for W. Now consider B as a subset of V. If $Span(B)=V$ then B is also a basis for V, so $\dim W = \dim V$. Otherwise, we have $v \in V \setminus Span(B)$, now $B' = \{v\} \cup B$ is LI. If this does not span V we continue to add vectors until we get a spanning set B^*. Now B^* will be basis for V and $B \subseteq B^*$, so $|B| \le |B^*|$.
It remains to show that if $\dim W = \dim V$ then $V=W$. Suppose $v \in V \setminus W$ then $v \notin Span(B)$, so $\{v\} \cup B$ is an LI subset of V, so $dim(V) \ge dim(W)+1$ giving us a contradiction.

B.3. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$).

(a) Show that the functions

$$f_1(x)=1, \quad f_2(x)=1+x+x^2, \quad f_3(x)=\sin x, \quad f_4(x)=\cos x$$

are linearly independent.

(b) Which of the following functions lie in $Span(f_1,f_2,f_3,f_4)$?

$$5-3x-3x^2, \quad \tan x, \quad 10-x-x^2+\sin(x+\pi/3).$$

(a) If $\exists \lambda_i \in \mathbb{R}$ such that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 = \mathbf{0}$ (the zero function), then

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x) = 0 \text{ for all } x \in \mathbb{R}.$$

Putting $x=0,\pi,2\pi$ gives the equations $\lambda_1+\lambda_2+\lambda_4=0$, $\lambda_1+(1+\pi+\pi^2)\lambda_2-\lambda_4=0$, $\lambda_1+(1+2\pi+4\pi^2)\lambda_2+\lambda_4=0$, from which it easily follows that $\lambda_1=\lambda_2=\lambda_4=0$. Now put $x=\pi/2$ to get $\lambda_3=0$ also. So f_1,f_2,f_3,f_4 are linearly independent.

(b) The function $\tan x$ is not in $Span(f_1, \ldots, f_4)$ — otherwise $\exists \lambda_i \in \mathbb{R}$ such that $\tan x = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x)$ for all $x \in \mathbb{R}$, which we check to be impossible by putting $x = 0, \pi, 2\pi$ etc. as above.

The other two functions are in $Span(f_1, \ldots, f_4)$: use the addition formula to show that for the last one.

B.4. (a) Describe an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

(b) Find a basis for $W = Span(x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1) \leq \mathbb{R}[x]$.

(a) If you want to describe any infinite list in a finite amount of time, you're going to need parameters. Here's one possible answer:

$$\left\{ \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix} \right\} : t \in \mathbb{R} \right\}$$

(b) Call these polynomials $p_1, \ldots, p_5$. Since $p_1 \neq 0$ we keep it. Since p_2 is not a scalar multiple of p_1, we keep it. Next, $p_3 = 2(p_2 - p_1) \in Span(p_1, p_2)$, so we throw it away. Now p_4 has an x term, so cannot be in $\langle p_1, p_2 \rangle$, so keep it. Finally, $p_5 = p_4 + \frac{1}{2} p_3 \in Span(p_1, p_2, p_4)$ so discard it. So a basis is $\{p_1, p_2, p_4\}$.

B.5. Let V be the vector space of all 3×3 matrices over $\mathbb{R}$.

(a) Find a basis of V consisting of invertible matrices.

(b) Let $W = \{A \in V \colon A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

(c) Let $W \subset V$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.

(a) E.g. $I, I - 2E_{33}, I - 2E_{22}$ together with $I + E_{ij}$ for all $i \neq j$, where E_{ij} has a 1 in the ijth position only.

(b) The general symmetric matrix

$$\begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$$

is $aE_{11} + bE_{22} + cE_{33} + x(E_{12} + E_{21}) + y(E_{13} + E_{31}) + z(E_{23} + E_{32})$. So a basis for this subspace is $E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}$, and its dimension is 6.

(c) First show the middle entry must be 0 — e.g. add each diagonal and the middle row to get that $0 =$ the sum of the 1st and third columns plus twice the middle entry.

So the magic matrices have the form $\begin{pmatrix} a & b & -a-b \\ c & 0 & -c \\ -a-c & -b & -a \end{pmatrix}$, with $2a + b + c = 0$ to get the last row to add to 0.

Get a basis by taking first $a = 1, b = 0$ then $a = 0, b = 1$. So dimension $= 2$.