Answers to Problem Sheet 3

1.
$$\frac{dH}{dt} = \sum_{i=1}^{N}(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i) - \frac{dL}{dt} = \sum_{i=1}^{N}\left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) - \frac{dL}{dt},$$

using the Euler-Lagrange equations, $\dot{p}_i = \partial L/\partial q_i$, and the definition $p_i = \partial L/\partial \dot{q}_i$. This is zero as the chain rule gives

$$\frac{d}{dt}L(q,\dot{q}) = \sum_{i=1}^{N}\left(\frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right).$$

2. Here the Lagrangian is the kinetic energy T as the potential energy is a constant which can be taken to be zero. On the wire $r = 1 + \cos\theta$ which gives $\dot{r} = -\sin\theta\ \dot{\theta}$ and so

$$L = T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{m}{2}\left[\sin^2\theta\dot{\theta}^2 + (1+\cos\theta)^2\dot{\theta}^2\right]$$

$$= \frac{m}{2}\left[\sin^2\theta + \left(1 + \cos^2\theta + 2\cos\theta\right)\right]\dot{\theta}^2 = m(1+\cos\theta)\dot{\theta}^2.$$

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \dot{\theta}\cdot 2m(1+\cos\theta)\dot{\theta} - m(1+\cos\theta)\dot{\theta}^2 = T$$

is a constant of the motion. This gives

$$\frac{d\theta}{dt} = \frac{C}{\sqrt{1+\cos\theta}}.$$

Now use $1 + \cos\theta = 2\cos^2(\theta/2)$

$$\cos(\theta/2)d\theta = C'dt$$

which integrates to $2\sin(\theta/2) = C't +$ constant or

$$\theta(t) = 2\sin^{-1}(at + b)$$

where a and b are constants.

Unless $a = 0$ (which gives θ=constant) the solution is only valid for a finite time period for which $-1 \leq at + b \leq 1$. This is because the bead reaches the cusp at $\theta = \pi$.

3. (i)
$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}).$$
Expanding out the dot products the Lagrangian can be written
$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\left(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}\right).$$
Therefore
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x.$$
To obtain the Lorentz force law one must regard A_x, A_y and A_z as arbitrary functions of x, y, z. Accordingly,
$$\frac{\partial L}{\partial x} = q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right)$$
and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + q\frac{dA_x}{dt} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right).$$
This gives
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + q\left[\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\dot{y} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\dot{z}\right]$$
$$= m\ddot{x} + q[-B_z\dot{y} + B_y\dot{z}] = m\ddot{x} + q(\mathbf{B}\times\mathbf{v})_x = 0,$$
which is the x-component of the Lorentz force law. Similarly, the other two Euler-Lagrange equations yield the y and z components of the Lorentz force law.

(ii) $A_y = bx$, $A_x = A_z = 0$ gives $\mathbf{B} = \nabla\times\mathbf{A} = b\mathbf{k}$ so a suitable Lagrangian is
$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qbx\dot{y}.$$
Here y and z are cyclic. However, this answer is not unique, e.g. $A_x = -by$, $A_y = A_z = 0$ also works which leads to a Lagrangian cyclic in x and z.

4. (i) The E-L equations give (dropping R)
$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \qquad \frac{d}{dt}(\sin^2\theta\dot{\phi}) = 0.$$

ϕ=constant yields $\ddot{\theta} = 0$ or $\theta = \alpha t + \beta$.

θ =constant yields $\sin\theta\cos\theta\dot{\phi}^2 = 0$ and $\sin^2\theta\dot{\phi} = C$. This works if $C = 0$ (no motion). If $C \neq 0$ this requires that $\theta = \pi/2$ (on the equator) or $\theta = 0,\ \pi$ (no motion - fixed at the poles).

(ii) The E-L equations are now

$$R^2(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) - \alpha\sin\theta\dot{\phi} = 0, \qquad \frac{d}{dt}[R^2\sin^2\theta\dot{\phi} + \alpha(1-\cos\theta)] = 0.$$

ϕ =constant yields $\ddot{\theta} = 0$ *and* $\cos\theta$ =constant meaning that the only solutions where ϕ=constant are static in that θ is also constant.

θ =constant yields $\sin\theta\dot{\phi}(R^2\cos\theta\dot{\phi} + \alpha) = 0$. This allows the static solutions $\theta = 0,\ \pi$ (stuck at poles) and $\dot{\phi} = 0$. As in part (i) circular orbits with $\dot{\phi} \neq 0$ are possible. However, unlike in part (i) these occur for any $\theta \neq \pi/2$.

Comment: The solutions to part (i) are either great circles or a point (no motion) whereas the solutions to part (ii) are non-great circles or a single point.