MATH50010 – Autumn 2023 – Midterm

You should state carefully any results from lectures that are used.

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ unless otherwise stated.

(a) (1 mark) Define mathematically what it means for $\mathcal{F}$ to be a sigma algebra.

(b) (3 marks) Let $\Omega = \{1, 2, 3, 4\}$. Give an example of a sigma algebra $\mathcal{F}$ on Ω and two functions $X_1, X_2 : \Omega \to \mathbf{R}$ such that X_1 is a random variable with respect to $\mathcal{F}$ but X_2 is not a random variable with respect to $\mathcal{F}$. Briefly justify your answers.

(c) (4 marks) Prove that for any random variable, X, and $x \in \mathbf{R}$, $Pr(X < x) = \lim_{x_n \to x} Pr(X \leq x_n)$ for any strictly increasing sequence $x_n \uparrow x$.

In the remainder of the question, let X and Y be absolutely continuous random variables with joint probability density function given by

$$f_{XY}(x, y) = \lambda^2 \exp\{-\lambda(x + y)\} \quad \text{for} \quad x > 0, y > 0 \tag{1}$$

and zero otherwise, where $\lambda > 0$ is a constant.

(d) (3 marks) Derive the marginal density functions f_X and f_Y and cumulative distribution functions F_X and F_Y of the random variables X and Y.

(e) (1 mark) Calculate $Cov(X, Y)$, the covariance between X and Y.

(f) (3 marks) Determine the joint probability density function of the random variables $S = X + Y$ and $D = X - Y$.

(g) (2 marks) Calculate $Cov(S, D)$, the covariance between S and D from part (f). Are S and D independent?

(h) (3 marks) Let $U_1 \sim \text{Uniform}[0, 1]$ and $U_2 \sim \text{Uniform}[0, 1]$ be independent uniform random variables. Explain how to use U_1 and U_2 to obtain a sample of $X + Y$.