Imperial College London

MATH70046

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series Analysis

Date: 10 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer Questions 1-2, and Questions 3-5 in Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary, unless stated otherwise. All processes are real-valued, unless stated otherwise. The sample interval is unity, unless stated otherwise. $\Delta = 1 - B$ denotes the difference operator, where B denotes the backward shift operator.

1. (a) Define what it means for a stochastic process to be stationary. (3 marks)

   (b) Fully justifying your answer, determine whether each of the following statements are *True* or *False*.

   (i) The ARMA(2,1) process $\{X_t\}$ defined through the model

   $$X_t + \frac{1}{2}X_{t-1} - \frac{15}{16}X_{t-2} = \epsilon_t - \frac{1}{2}\epsilon_{t-1}$$

   is both stationary and invertible. (3 marks)

   (ii) The process $\{X_t\}$, defined as a sequence of independent random variables with $X_t \sim N(1,1)$ when t is even and $X_t \sim$ Poisson(1) when t is odd, is second order stationary but not strictly stationary. (Hint: if $Y \sim$ Poisson(μ), then $E\{Y\} = \mathrm{Var}\{Y\} = \mu$).
   (3 marks)

   (c) Let $\{Y_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{Y,\tau}\}$. For real constant $\gamma \neq 0$, define

   $$X_t = (1 - \gamma B)Y_t,$$
   $$W_t = (1 - \gamma^{-1} B)Y_t.$$

   (i) Express the autocovariance sequences of $\{X_t\}$ and $\{W_t\}$ in terms of $\{s_{Y,\tau}\}$. (3 marks)

   (ii) Show that $\{X_t\}$ and $\{W_t\}$ have the same autocorrelation sequence. (4 marks)

   (iii) Using the autocorrelation sequence derived in (ii), determine the maximum value ρ_1 can take for an MA(1) process. Here ρ_1 is the $\tau = 1$ term of its autocorrelation sequence.
   (4 marks)

(Total: 20 marks)

2. (a) Consider the system $Y_t = L\{X_t\}$, where $\{X_t\}$ is a zero mean stationary process and $L\{\cdot\}$ is a linear time invariant (LTI) filter. It can be shown that $S_Y(f) = |G(f)|^2 S_X(f)$ where $S_X(f)$ and $S_Y(f)$ are the spectral density functions of $\{X_t\}$ and $\{Y_t\}$, respectively, and $G(f)$ is the frequency response function of $L\{\cdot\}$.

(i) State the three defining conditions of a LTI filter. (3 marks)

(ii) Consider the LTI filter

$$L\{X_t\} = X_t - 2\alpha X_{t-1} + X_{t-2}.$$

What value of α will completely suppress oscillations at frequency $f = 1/6$. (Recall: $\cos(\pi/3) = -\cos(2\pi/3) = 1/2$) (6 marks)

(b) (i) Show that the spectral density function of an ARMA(p, q) process, in terms of its parameters $\phi_{1,p}, ..., \phi_{p,p}$, $\theta_{1,q}, ..., \theta_{q,q}$ and σ_ϵ^2, is given as

$$S(f) = \sigma_\epsilon^2 \frac{|1 - \theta_{1,q}\mathrm{e}^{-\mathrm{i}2\pi f} - ... - \theta_{q,q}\mathrm{e}^{-\mathrm{i}2\pi f q}|^2}{|1 - \phi_{1,p}\mathrm{e}^{-\mathrm{i}2\pi f} - ... - \phi_{p,p}\mathrm{e}^{-\mathrm{i}2\pi f p}|^2}.$$

(3 marks)

(ii) Consider the zero mean stationary ARMA(1,1) process $X_t - \alpha X_{t-1} = \epsilon_t + 2\alpha\epsilon_{t-1}$, where $\sigma_\epsilon^2 = 1$ and $0 < |\alpha| < 1$. The spectral density function of $\{X_t\}$ is

$$S(f) = \frac{8 + 8\cos(2\pi f)}{5 - 4\cos(2\pi f)}.$$

Determine the value of α. (4 marks)

(c) Let $\{Y_t\}$ be a process defined as $Y_t - \beta Y_{t-1} = X_t + \eta_t$, where $0 < |\beta| < 1$, $\{X_t\}$ is a zero mean stationary process with spectral density $S_X(f)$, and $\{\eta_t\}$ is a white noise process with variance $\sigma_\eta^2 = 1$, uncorrelated with $\{X_t\}$. Show

$$S_Y(f) = \frac{S_X(f) + 1}{1 + \beta^2 - 2\beta\cos(2\pi f)}.$$

(4 marks)

(Total: 20 marks)

3. Let $X_1, ..., X_N$ be a portion of a stationary process $\{X_t\}$ with autocovariance sequence $\{s_\tau\}$ and spectral density function $S(f)$. The spectral representation of a process with mean μ is

$$X_t - \mu = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ft} \mathrm{d}Z(f),$$

where $\{Z(f)\}$ is a zero mean orthogonal increment process, and $S(f)\mathrm{d}f \equiv E\{|\mathrm{d}Z(f)|^2\}$.

(a) Assume the mean of the process is known to be μ. Consider the following estimator of its autocovariance sequence,

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \mu)(X_{t+|\tau|} - \mu), \quad \tau = 0, \pm 1, ..., \pm(N-1).$$

(i) By considering $E\{\hat{s}_\tau^{(p)}\}$, show $|\mathrm{Bias}\{\hat{s}_\tau^{(p)}\}| \leq |\tau| s_0 / N$. (4 marks)

(ii) Let $\{X_t\}$ be a white noise process with known mean. Show that $\hat{s}_\tau^{(p)}$ is an unbiased estimator of s_τ, for all $\tau = 0, \pm 1, \pm 2, ..., \pm N - 1$. (3 marks)

(iii) Consequently, for a white noise process of known mean, show $\mathrm{MSE}\{\hat{s}_\tau^{(p)}\} \leq \mathrm{MSE}\{\hat{s}_\tau^{(u)}\}$, where

$$\hat{s}_\tau^{(u)} = \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \mu)(X_{t+|\tau|} - \mu), \quad \tau = 0, \pm 1, ..., \pm(N-1).$$

(4 marks)

(b) (i) The periodogram $\hat{S}^{(p)}(f)$ of a process that is known to be zero mean is given as

$$\hat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

Show the periodogram can also be expressed as

$$\hat{S}^{(p)}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} X_t \mathrm{e}^{-\mathrm{i}2\pi ft} \right|^2. \tag{$\dagger$}$$

(3 marks)

(ii) Suppose one makes an incorrect assumption that $\{X_t\}$ is zero mean, when in fact it has a non-zero mean of μ. If one continues to use the estimator in ($\dagger$) without subtracting μ, show

$$E\{\hat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} S(f')\mathcal{F}(f - f')\mathrm{d}f' + \mu^2 \mathcal{F}(f),$$

where $\mathcal{F}(f) = \frac{1}{N} |\sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi ft}|^2$. (6 marks)

(Total: 20 marks)

4. (a) (i) Define what it means for two processes $\{X_t\}$ and $\{Y_t\}$ to be jointly stationary. (3 marks)

(ii) Let $\{W_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{W,\tau}\}$ and spectral density function $S_W(f)$. Define the processes

$$X_t = AW_t + \epsilon_t,$$
$$Y_t = BW_t + \eta_t,$$

where $\{\epsilon_t\}$ and $\{\eta_t\}$ are individually unit variance zero mean white noise processes uncorrelated with $\{W_t\}$ and each other. Random variables A and B are both independent of $\{W_t\}$, $\{\epsilon_t\}$ and $\{\eta_t\}$, and the vector $(A, B)^T$ is zero mean with covariance matrix

$$\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix},$$

where $|\alpha| < 1$. Show $\{X_t\}$ and $\{Y_t\}$ are jointly stationary, deriving the autocovariance sequences $\{s_{X,\tau}\}$ and $\{s_{Y,\tau}\}$, and cross-covariance sequence $\{s_{XY,\tau}\}$. (4 marks)

(b) Let $\{V_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{V,\tau}\}$ and spectral density function $S_V(f)$. Consider the bivariate system

$$X_t = \sum_{k=-\infty}^{\infty} g_k V_{t-k} + \epsilon_t,$$
$$Y_t = \sum_{\ell=-\infty}^{\infty} h_\ell V_{t-\ell} + \xi_t.$$

where $\{g_k\}$ and $\{h_\ell\}$ are deterministic sequences and $\{\epsilon_t\}$ and $\{\xi_t\}$ are white noise processes, uncorrelated with each other and $\{V_t\}$.

(i) Derive the autocovariance sequences of $\{X_t\}$ and $\{Y_t\}$ and the cross-covariance sequence of $\{X_t\}$ and $\{Y_t\}$. (4 marks)

(ii) Show the cross spectrum $S_{XY}(f)$ of $\{X_t\}$ and $\{Y_t\}$ is given as

$$S_{XY}(f) = G(f)H^*(f)S_V(f),$$

where $G(f)$ is the Fourier transform of $\{g_k\}$ and $H(f)$ is the Fourier transform of $\{h_\ell\}$. (4 marks)

(iii) Show the magnitude squared coherence, defined as

$$\gamma^2_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)},$$

is given as

$$\gamma^2_{XY}(f) = \left[1 + \frac{S_V(f)\{\sigma^2_\xi|G(f)|^2 + \sigma^2_\epsilon|H(f)|^2\} + \sigma^2_\epsilon\sigma^2_\eta}{|G(f)|^2|H(f)|^2S^2_V(f)}\right]^{-1}.$$

Comment on the result when the noise processes $\{\epsilon_t\}$ and $\{\xi_t\}$ are removed from the system. (5 marks)

(Total: 20 marks)

5. (a) Let $X_1, X_2, ..., X_N$ be a portion of a zero mean stationary process $\{X_t\}$ with spectral density function $S(f)$. Consider the multitaper estimator

$$\hat{S}^{(MT)}(f) = \frac{1}{K}\sum_{k=0}^{K-1} \hat{S}_k^{(MT)}(f) \qquad \text{with} \qquad \hat{S}_k^{(MT)}(f) = \left|\sum_{t=1}^{N} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2,$$

where $\{h_{k,t}\}$ is the data taper for the kth direct spectral estimator $\hat{S}_k^{(MT)}(f)$. We assume $\sum_{t=1}^{N} h_{k,t}^2 = 1$ for all $k = 0, ..., K-1$.

(i) Show

$$\mathrm{Var}\{\hat{S}^{(MT)}(f)\} = \frac{1}{K^2}\sum_{k=0}^{K-1} \mathrm{Var}\{\hat{S}_k^{(MT)}(f)\} + \frac{2}{K^2}\sum_{j<k} \mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\}.$$

(3 marks)

(ii) It can be shown that

$$\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\} = |E\{J_j(f)J_k^*(f)\}|^2 + |E\{J_j(f)J_k(f)\}|^2,$$

where $J_k(f) = \sum_{t=1}^{N} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}$. When $\{h_{j,t}\}$ and $\{h_{k,t}\}$ are orthonormal tapers and $\{X_t\}$ is a white noise process with variance σ_X^2, show $E\{J_j(f)J_k^*(f)\} = 0$ and hence

$$\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\} = \sigma_X^4 \left|\sum_{t=1}^{N} h_{j,t} h_{k,t} \mathrm{e}^{-\mathrm{i}4\pi f t}\right|^2.$$

(6 marks)

(iii) Under certain symmetry conditions on the tapers, you can take it as given that

$$\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\} = \sigma_X^4 \left|\sum_{t=1}^{N} h_{j,t} h_{k,t}\right|^2.$$

Comment on the advantages of the multitaper spectral estimator versus using a direct (single taper) spectral estimator, particularly when the K tapers are orthonormal.

(3 marks)

QUESTION 5 CONTINUES ON THE NEXT PAGE

(b) Let $\{-a, 0, a, 0\}$ be a realisation of the portion X_1, X_2, X_3, X_4 of a zero mean stationary process $\{X_t\}$. Consider the following three tapers $\{h_{0,t}\}$, $\{h_{1,t}\}$ and $\{h_{2,t}\}$, each of length $N = 4$,

$$\{h_{0,t}\} = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \qquad \{h_{1,t}\} = \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \qquad \{h_{2,t}\} = \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}.$$

(i) Show $\{h_{0,t}\}$, $\{h_{1,t}\}$ and $\{h_{2,t}\}$ are an orthonormal set of tapers. (2 marks)

(ii) Assuming the distributional result

$$\hat{S}^{(MT)}(f) \stackrel{\mathrm{d}}{=} \begin{cases} S(f)\chi^2_{2K}/2K, & 0 < |f| < 1/2; \\ S(f)\chi^2_{K}/K, & |f| = 0 \text{ or } 1/2, \end{cases}$$

which of the following is the 95% confidence interval for $S(1/4)$, where $Q_\nu(p)$ is the $p \times 100$ percentile of the χ^2_ν distribution? Fully justify your answer.

$$A. \quad \left[\frac{3a^2}{3Q_6(0.975)}, \frac{3a^2}{3Q_6(0.025)}\right] \qquad B. \quad \left[\frac{6a^2}{Q_3(0.975)}, \frac{6a^2}{Q_3(0.025)}\right]$$

$$C. \quad \left[\frac{4a^2}{Q_6(0.975)}, \frac{4a^2}{Q_6(0.025)}\right] \qquad D. \quad \left[\frac{4a^2}{Q_6(0.95)}, \frac{4a^2}{Q_6(0.05)}\right]$$

Hint: Use the relationship $P[Q_\nu(p) \le \chi^2_\nu \le Q_\nu(1-p)] = 1 - 2p$. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH60046/70046

Time Series Analysis (Solutions)

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1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. $\boxed{\text{seen} \Downarrow}$ $\boxed{3, \text{A}}$

(b) (i) This ARMA(2,1) process can be written as $\boxed{\text{sim. seen} \Downarrow}$

$$(1 + \frac{1}{2}B - \frac{15}{16}B^2)X_t = (1 - \frac{1}{2}B)\epsilon_t,$$

giving characteristic polynomials $\Phi(z) = 1 + \frac{1}{2}z - \frac{15}{16}z^2$ and $\Theta(z) = 1 - \frac{1}{2}z$. The root of $\Theta(z)$ is $z = 2$, which clearly lies outside of the unit circle, hence the process is invertible. To check for stationarity, we find the roots of $\Phi(z)$. We can write $\Phi(z) = (1 - \frac{3}{4}z)(1 + \frac{5}{4}z)$, therefore the roots of $\Phi(z)$ are $z = \frac{4}{3}$ and $z = -\frac{4}{5}$. One of these is inside the unit circle and therefore this is not a stationary process, so the statement is false. Full marks awarded for just showing it is not stationary. $\boxed{3, \text{A}}$

(ii) The process is second order stationary, because $E(X_t) = 1$ and $\text{var}(X_t) = 1$ for all t (odd and even), and $\text{cov}(X_t, X_{t+\tau}) = 0$ for all $\tau \neq 0$, due to independence. It is not strictly stationary because, for example, the distribution of X_1 is not equal to the distribution of X_2. Therefore, this is true. $\boxed{3, \text{A}}$

(c) (i) First consider $X_t = (1 - \gamma B)Y_t = Y_t - \gamma Y_{t-1}$. $\{X_t\}$ will be a zero-mean process, therefore

$$\begin{aligned}
\text{cov}(X_t, X_{t+\tau}) &= E(X_t X_{t+\tau}) \\
&= E\{(Y_t - \gamma Y_{t-1})(Y_{t+\tau} - \gamma Y_{t-1+\tau})\} \\
&= E(Y_t Y_{t+\tau}) - \gamma E(Y_{t-1} Y_{t+\tau}) - \gamma E(Y_t Y_{t-1+\tau}) + \gamma^2 E(Y_{t-1} Y_{t-1+\tau}) \\
&= s_{Y,\tau} - \gamma s_{Y,\tau+1} - \gamma s_{Y,\tau-1} + \gamma^2 s_{Y,\tau} \\
&= (1 + \gamma^2) s_{Y,\tau} - \gamma s_{Y,\tau+1} - \gamma s_{Y,\tau-1}.
\end{aligned}$$

The result for $\{W_t\}$ follows by just substituting in γ^{-1} instead, i.e.

$$\text{cov}(W_t, W_{t+\tau}) = (1 + \gamma^{-2}) s_{Y,\tau} - \gamma^{-1} s_{Y,\tau+1} - \gamma^{-1} s_{Y,\tau-1}.$$

$\boxed{3, \text{B}}$

(ii) We have

$$
\begin{aligned}
\rho_{X,\tau} &= s_{X,\tau}/s_{X,0} \\
&= \frac{(1+\gamma^2)s_{Y,\tau} - \gamma s_{Y,\tau+1} - \gamma s_{Y,\tau-1}}{(1+\gamma^2)s_{Y,0} - 2\gamma s_{Y,1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{W,\tau} &= s_{W,\tau}/s_{W,0} \\
&= \frac{(1+\gamma^{-2})s_{Y,\tau} - \gamma^{-1} s_{Y,\tau+1} - \gamma^{-1} s_{Y,\tau-1}}{(1+\gamma^{-2})s_{Y,0} - 2\gamma^{-1} s_{Y,1}} \\
&= \frac{\gamma^2}{\gamma^2} \cdot \frac{(1+\gamma^{-2})s_{Y,\tau} - \gamma^{-1} s_{Y,\tau+1} - \gamma^{-1} s_{Y,\tau-1}}{(1+\gamma^{-2})s_{Y,0} - 2\gamma^{-1} s_{Y,1}} \\
&= \frac{(1+\gamma^2)s_{Y,\tau} - \gamma s_{Y,\tau+1} - \gamma s_{Y,\tau-1}}{(1+\gamma^2)s_{Y,0} - 2\gamma s_{Y,1}} = \rho_{X,\tau}.
\end{aligned}
$$

$\boxed{4, \text{B}}$

(iii) If $\{Y_t\}$ is a white noise process, then $\{X_t\}$ is an MA(1) process with parameter $\theta_{1,1} = \gamma$, and with $s_{Y,\tau} = 0$ for all $|\tau| \neq 0$, we have $\boxed{\text{unseen} \Downarrow}$

$$
\rho_{X,1} = \frac{(1+\gamma^2)s_{Y,1} - \gamma s_{Y,2} - \gamma s_{Y,0}}{(1+\gamma^2)s_{Y,0} - 2\gamma s_{Y,1}} = \frac{-\gamma}{(1+\gamma^2)}.
$$

Finding the zero derivatives;

$$
\frac{\mathrm{d}\rho_{X,1}}{\mathrm{d}\gamma} = \frac{2\gamma^2}{(1+\gamma^2)^2} - \frac{1}{(1+\gamma^2)} = \frac{(\gamma^2-1)}{(1+\gamma^2)^2}
$$

implies a stationary point at $\gamma = \pm 1$. Therefore, there is clearly a maximum at $\gamma = -1$, with $\rho_{X,1} = 1/2$. $\boxed{4, \text{D}}$

2. (a) (i) · Scale preservation:

$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$

· Superposition:

$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$

· Time invariance: if $L\{x_t\} = y_t$ then $L\{x_{t+\tau}\} = y_{t+\tau}$. $\boxed{3, \text{A}}$

(ii) The frequency response function is found by considering

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft} - 2\alpha e^{i2\pi f(t-1)} + e^{i2\pi f(t-2)} = e^{i2\pi ft}(1 - 2\alpha e^{-i2\pi f} + e^{-i4\pi f}).$$

Therefore $G(f) = 1 - 2\alpha e^{-i2\pi f} + e^{-i4\pi f}$. Alternatively, this could be found by taking the Fourier transform of the impulse response sequence $g_0 = 1, g_1 = -2\alpha, g_2 = 1$ and $g_k = 0$ for all $k \neq 0, 1, 2$. Therefore

$$\begin{aligned}|G(f)|^2 &= (1 - 2\alpha e^{-i2\pi f} + e^{-i4\pi f})(1 - 2\alpha e^{i2\pi f} + e^{i4\pi f})\\ &= 2 + 4\alpha^2 - 8\alpha\cos(2\pi f) + 2\cos(4\pi f).\end{aligned}$$

For oscillations at frequency $f = 1/6$ to be suppressed by the linear filter, we require $|G(1/6)|^2 = 0$. Therefore, we require

$$\begin{aligned}2 + 4\alpha^2 - 8\alpha\cos(2\pi f) + 2\cos(4\pi f) &= 2 + 4\alpha^2 - 8\alpha\cos(\pi/3) + 2\cos(2\pi/3)\\ &= 4\alpha^2 - 4\alpha + 1\\ &= (2\alpha - 1)^2 = 0.\end{aligned}$$

This gives $\alpha = 1/2$. $\boxed{6, \text{D}}$

(b) (i) We can express an ARMA(p, q) process as $L_\phi\{X_t\} = L_\theta\{\epsilon_t\}$ where $L_\phi\{X_t\} = X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p}$ and $L_\theta\{\epsilon_t\} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q}$. Letting $Y_t = L_\phi\{X_t\}$, we have

$$S_Y(f) = |G_\phi(f)|^2 S_X(f) = |1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2 S_X(f).$$

Letting $Y_t = L_\theta\{\epsilon_t\}$, we have

$$S_Y(f) = |1 - \theta_{1,q}e^{-i2\pi f} - \ldots - \theta_{q,q}e^{-i2\pi fq}|^2 S_\epsilon(f).$$

Recognising that $S_\epsilon(f) = \sigma_\epsilon^2$ and setting the two expressions equal to one another, we get

$$S_X(f) = \sigma_\epsilon^2 \frac{|1 - \theta_{1,q}e^{-i2\pi f} - \ldots - \theta_{q,q}e^{-i2\pi fq}|^2}{|1 - \phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

$\boxed{3, \text{A}}$

(ii) The spectra density function for the stated ARMA(1,1) model is sim. seen ⇓

$$S(f) = \frac{|1 + 2\alpha e^{-i2\pi f}|^2}{|1 - \alpha e^{-i2\pi f}|^2} = \frac{1 + 4\alpha^2 + 4\alpha\cos(2\pi f)}{1 + \alpha^2 - 2\alpha\cos(2\pi f)}.$$

We therefore have

$$\frac{8 + 8\cos(2\pi f)}{5 - 4\cos(2\pi f)} = \frac{1 + 4\alpha^2 + 4\alpha\cos(2\pi f)}{1 + \alpha^2 - 2\alpha\cos(2\pi f)},$$

and

$$\begin{aligned} 8 + 8\cos(2\pi f) &= C \cdot \{1 + 4\alpha^2 + 4\alpha\cos(2\pi f)\} \\ 5 - 4\cos(2\pi f) &= C \cdot \{1 + \alpha^2 - 2\alpha\cos(2\pi f)\}. \end{aligned}$$

This gives $C(1 + 4\alpha^2) = 8$ and $C(1 + \alpha^2) = 5$. Dividing one by the other gives $5 + 20\alpha^2 = 8 + 8\alpha^2$, implying $\alpha = \pm 1/2$. To check the sign, we have $C = 4$ and equating cosine components shows we are dealing with the positive solution, i.e. $\alpha = 1/2$. 4, B

(c) Using LTI filters, we can write this as $L_\beta\{Y_t\} = X_t + \eta_t$, where $L_\beta\{Y_t\} = 1 - \beta Y_{t-1}$. Therefore $|G_\beta(f)|^2 S_Y(f) = S_{X+\eta}(f)$, where $S_{X+\eta}(f)$ is the spectral density function for $\{X_t + \eta_t\}$. With $\{X_t\}$ and $\{\eta_t\}$ uncorrelated, we have $S_{X+\eta}(f) = S_X(f) + S_\eta(f) = S_X(f) + 1$. Furthermore, $|G_\beta(f)|^2 = |1 - \beta e^{-i2\pi f}|^2 = 1 + \beta^2 - 2\beta\cos(2\pi f)$, and therefore unseen ⇓

$$S_Y(f) = \frac{S_X(f) + 1}{1 + \beta^2 - 2\beta\cos(2\pi f)}.$$

4, C

3. (a) (i) Taking expectations, we have

seen ⇓

$$E(\widehat{s}_\tau^{(p)}) = \frac{1}{N}\sum_{t=1}^{N-|\tau|} E\left\{(X_t-\mu)(X_{t+|\tau|}-\mu)\right\} = \frac{1}{N}(N-|\tau|)s_\tau = \left(1-\frac{|\tau|}{N}\right)s_\tau.$$

unseen ⇓

It therefore follows that $\mathrm{Bias}(\widehat{s}_\tau^{(p)}) \equiv E(\widehat{s}_\tau^{(p)}) - s_\tau = -|\tau|s_\tau/N$. We further know that $|s_\tau| \le s_0$, combining gives $|\mathrm{Bias}(\widehat{s}_\tau^{(p)})| \le |\tau|s_0/N$.

4, A

(ii) From Part (i), we have that $\mathrm{Bias}(\widehat{s}_\tau^{(p)}) = -|\tau|s_\tau/N$. When $\tau = 0$, $\mathrm{Bias}(\widehat{s}_\tau^{(p)})$ clearly equals 0, and when $|\tau| \ne 0$, $s_\tau = 0$ because $\{X_t\}$ is a white noise process. Therefore, $\widehat{s}_\tau^{(p)}$ is unbiased for all τ.

3, A

sim. seen ⇓

(iii) We have,

$$\begin{aligned}
\mathrm{MSE}\{\widehat{s}_\tau^{(p)}\} &= \mathrm{Bias}^2(\widehat{s}_\tau^{(p)}) + \mathrm{var}\{\widehat{s}_\tau^{(p)}\}\\
&= \mathrm{var}\{\widehat{s}_\tau^{(p)}\}\\
&= \frac{1}{N^2}\,\mathrm{var}\left\{\sum_{t=1}^{N-|\tau|}(X_t-\mu)(X_{t+|\tau|}-\mu)\right\}\\
&\le \frac{1}{(N-|\tau|)^2}\,\mathrm{var}\left\{\sum_{t=1}^{N-|\tau|}(X_t-\mu)(X_{t+|\tau|}-\mu)\right\}\\
&= \mathrm{var}\{\widehat{s}_\tau^{(u)}\} = \mathrm{MSE}\{\widehat{s}_\tau^{(u)}\},
\end{aligned}$$

as $\widehat{s}_\tau^{(u)}$ is also unbiased.

4, C

seen ⇓

(b) (i) It follows that

$$\begin{aligned}
\widehat{S}^{(p)}(f) &= \sum_{\tau=-(N-1)}^{N-1} \widehat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}\mathrm{e}^{-\mathrm{i}2\pi f\tau}\\
&= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N} X_j X_k \mathrm{e}^{\mathrm{i}2\pi f(k-j)},
\end{aligned}$$

by swapping diagonal summation for row summation. This is equal to

$$\widehat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2.$$

3, A

(ii) Defining $J(f) = N^{-1/2}\sum_{t=1}^{N} X_t e^{-i2\pi ft}$, we have $\widehat{S}^{(p)}(f) = |J(f)|^2 = J(f)J^*(f)$. Using the spectral representation of X_t, we have $\boxed{\text{unseen} \Downarrow}$

$$J(f) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N}\left(\mu + \int_{-1/2}^{1/2} e^{i2\pi f' t} dZ(f')\right) e^{-i2\pi ft}$$
$$= \frac{\mu}{\sqrt{N}}\sum_{t=1}^{N} e^{-i2\pi ft} + \int_{-1/2}^{1/2}\sum_{t=1}^{N}\frac{1}{\sqrt{N}} e^{-i2\pi(f-f')t} dZ(f').$$

Therefore

$$E\{\widehat{S}^{(p)}(f)\} = E\left[\left\{\frac{\mu}{\sqrt{N}}\sum_{t=1}^{N} e^{-i2\pi ft} + \int_{-1/2}^{1/2}\sum_{t=1}^{N}\frac{1}{\sqrt{N}} e^{-i2\pi(f-f')t} dZ(f')\right] \times \right.$$
$$\left.\left[\frac{\mu}{\sqrt{N}}\sum_{t=1}^{N} e^{i2\pi ft} + \int_{-1/2}^{1/2}\sum_{t=1}^{N}\frac{1}{\sqrt{N}} e^{i2\pi(f-f'')t} dZ^*(f'')\right]\right\}$$
$$= \frac{\mu^2}{N}\left|\sum_{t=1}^{N} e^{-i2\pi ft}\right|^2 + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{1}{N}\sum_{t=1}^{N} e^{-i2\pi(f-f')t}\sum_{s=1}^{N} e^{i2\pi(f-f'')s} E\left\{dZ(f')dZ(f'')\right\},$$

with other terms being zero because $E\{dZ(f)\} = E\{dZ^*(f)\} = 0$. Using

$$E\{dZ(f')dZ^*(f'')\} = \begin{cases} S(f')df', & f' = f'' \\ 0, & f' \neq f'', \end{cases}$$

and $\mathcal{F}(f) = N^{-1}\left|\sum_{t=1}^{N} e^{-i2\pi ft}\right|^2$, we have

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} S(f')\mathcal{F}(f-f')df' + \mu^2\mathcal{F}(f).$$

$\boxed{6, \text{D}}$

4. (a) (i) Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary if they are each, separately, stationary processes, and $\text{cov}\{X_t, Y_{t+\tau}\}$ is a function of τ only. seen $\Downarrow$

3, A

sim. seen $\Downarrow$

(ii) First show each process is individually stationary. Considering $\{X_t\}$, we have $E\{X_t\} = 0$, and therefore

$$\begin{aligned}\text{cov}\{X_t, X_{t+\tau}\} &= E\{(AW_t + \epsilon_t)(AW_{t+\tau} + \epsilon_{t+\tau})\} \\ &= E\{A^2\}E\{W_t W_{t+\tau}\} + E\{A\}E\{W_t \epsilon_{t+\tau}\} + E\{A\}E\{W_{t+\tau}\epsilon_t\} + E\{\epsilon_t \epsilon_{t+\tau}\} \\ &= s_{W,\tau} + \delta_{0,\tau}.\end{aligned}$$

An identical argument follows for $\{Y_t\}$, with $\text{cov}\{Y_t, Y_{t+\tau}\} = s_{W,\tau} + \delta_{0,\tau}$. Therefore each process is individually stationary. Considering the cross-covariance sequence, we have

$$\begin{aligned}\text{cov}\{X_t, Y_{t+\tau}\} &= E\{(AW_t + \epsilon_t)(BW_{t+\tau} + \eta_{t+\tau})\} \\ &= E\{AB\}E\{W_t W_{t+\tau}\} + E\{A\}E\{W_t \epsilon_{t+\tau}\} + E\{B\}E\{W_{t+\tau}\epsilon_t\} + E\{\epsilon_t \eta_{t+\tau}\} \\ &= \alpha s_{W,\tau}.\end{aligned}$$

Therefore, the processes are jointly stationary with cross-covariance sequence as given.

4, A

(b) (i) First consider $\{X_t\}$. We have $E\{X_t\} = 0$, and therefore

$$\begin{aligned}\text{cov}\{X_t, X_{t+\tau}\} &= E\{X_t X_{t+\tau}\} \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_k g_m E\{V_{t-k}V_{t+\tau-m}\} + E\{\epsilon_t \epsilon_{t+\tau}\}.\end{aligned}$$

All other terms are zero due to $\{\epsilon_t\}$ being uncorrelated with $\{V_t\}$. Therefore

$$s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_k g_m s_{V,\tau-m+k} + \sigma_\epsilon^2 \delta_{0,\tau}.$$

Likewise,

$$s_{Y,\tau} = \text{cov}\{Y_t, Y_{t+\tau}\} = \sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_l h_n s_{V,\tau-n+l} + \sigma_\xi^2 \delta_{0,\tau}.$$

For the cross-covariance sequence, we have

$$\begin{aligned}\text{cov}\{X_t, Y_{t+\tau}\} = E\{X_t Y_{t+\tau}\} &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g_k h_l E\{V_{t-k}V_{t+\tau-l}\} \\ &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g_k h_l s_{V,\tau-l+k}.\end{aligned}$$

4, B

(ii) The cross-spectrum is the Fourier transform of the cross-covariance sequence, namely

$$\begin{aligned}
S_{XY}(f) &= \sum_{\tau=-\infty}^{\infty} s_{XY,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \sum_{\tau=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g_k h_l s_{V,\tau-l+k} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \sum_{\tau'=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g_k h_l s_{V,\tau'} \mathrm{e}^{-\mathrm{i}2\pi f(\tau'+l-k)} \qquad (\tau' = \tau - l + k) \\
&= \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{-\mathrm{i}2\pi fk} \sum_{l=-\infty}^{\infty} h_l \mathrm{e}^{\mathrm{i}2\pi fl} \sum_{\tau'=-\infty}^{\infty} s_{V,\tau'} \mathrm{e}^{-\mathrm{i}2\pi f\tau'} \\
&= G(f)H^*(f)S_V(f).
\end{aligned}$$

$\boxed{4, \mathrm{C}}$

(iii) Following a similar argument to (b)(ii), we have

$$S_X(f) = \sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{\mathrm{i}2\pi f\tau} = |G(f)|^2 S_V(f) + \sigma_\epsilon^2$$

$$S_Y(f) = \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} \mathrm{e}^{\mathrm{i}2\pi f\tau} = |H(f)|^2 S_V(f) + \sigma_\xi^2.$$

Therefore,

$$\begin{aligned}
\gamma_{XY}^2(f) &= \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)} \\
&= \frac{|G(f)|^2|H(f)|^2 S_V^2(f)}{[|G(f)|^2 S_V(f) + \sigma_\epsilon^2][|H(f)|^2 S_V(f) + \sigma_\xi^2]} \\
&= \left[\frac{\{|G(f)|^2 S_V(f) + \sigma_\epsilon^2\}\{|H(f)|^2 S_V(f) + \sigma_\xi^2\}}{|G(f)|^2|H(f)|^2 S_V^2(f)}\right]^{-1} \\
&= \left[1 + \frac{S_V(f)\{\sigma_\xi^2|G(f)|^2 + \sigma_\epsilon^2|H(f)|^2\} + \sigma_\epsilon^2\sigma_\eta^2}{|G(f)|^2|H(f)|^2 S_V^2(f)}\right]^{-1}.
\end{aligned}$$

When there is no noise in the system this is equivalent to setting $\sigma_\epsilon^2 = \sigma_\xi^2 = 0$, the second term becomes zero and $\gamma_{XY}^2(f) = 1$ for all frequencies. $\boxed{5, \mathrm{B}}$

5. (a) (i) Using the bilinear form of the covariance operator, we have

$$
\begin{aligned}
\operatorname{var}\{\widehat{S}^{MT}(f)\} &= \operatorname{cov}\{\widehat{S}^{MT}(f), \widehat{S}^{MT}(f)\} \\
&= \operatorname{cov}\left\{\frac{1}{K}\sum_{j=0}^{K-1}\widehat{S}_j^{(MT)}(f), \frac{1}{K}\sum_{k=0}^{K-1}\widehat{S}_k^{(MT)}(f)\right\} \\
&= \frac{1}{K^2}\sum_{j=0}^{K-1}\sum_{k=0}^{K-1}\operatorname{cov}\{S_j^{(MT)}(f), S_k^{(MT)}(f)\} \\
&= \frac{1}{K^2}\sum_{k=0}^{K-1}\operatorname{var}\{\widehat{S}_k^{(MT)}(f)\} + \frac{2}{K^2}\sum_{j<k}\operatorname{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\},
\end{aligned}
$$

because $\operatorname{cov}\{S_k^{(MT)}(f), S_k^{(MT)}(f)\} = \operatorname{var}\{S_k^{(MT)}(f)\}$, and $\operatorname{cov}\{S_j^{(MT)}(f), S_k^{(MT)}(f)\} = \operatorname{cov}\{S_k^{(MT)}(f), S_j^{(MT)}(f)\}$. $\boxed{3, \mathrm{M}}$

(ii) First consider $E\{J_j(f)J_k^*(f)\}$. We have

$$
\begin{aligned}
E\{J_j(f)J_k^*(f)\} &= E\left\{\sum_{t=1}^{N}\sum_{s=1}^{N} h_{j,t}h_{k,s}X_tX_s\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{e}^{\mathrm{i}2\pi fs}\right\} \\
&= \sum_{t=1}^{N}\sum_{s=1}^{N} h_{j,t}h_{k,s}E\{X_tX_s\}\mathrm{e}^{-\mathrm{i}2\pi f(t-s)} \\
&= \sigma_X^2\sum_{t=1}^{N} h_{j,t}h_{k,t} = 0
\end{aligned}
$$

from orthogonality of the tapers and the fact $E\{X_tX_s\} = 0$ except when $t = s$. Next we consider $E\{J_j(f)J_k(f)\}$. We have

$$
\begin{aligned}
E\{J_j(f)J_k(f)\} &= E\left\{\sum_{t=1}^{N}\sum_{s=1}^{N} h_{j,t}h_{k,s}X_tX_s\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{e}^{-\mathrm{i}2\pi fs}\right\} \\
&= \sum_{t=1}^{N}\sum_{s=1}^{N} h_{j,t}h_{k,s}E\{X_tX_s\}\mathrm{e}^{-\mathrm{i}2\pi f(t+s)} \\
&= \sigma_X^2\sum_{t=1}^{N} h_{j,t}h_{k,t}\mathrm{e}^{-\mathrm{i}4\pi ft}
\end{aligned}
$$

because $E\{X_tX_s\} = 0$ except when $t = s$.The result therefore follows $\boxed{6, \mathrm{M}}$

(iii) When the tapers are orthogonal, the covariance terms go to zero and we have

$$
\operatorname{var}\{\widehat{S}^{MT}(f)\} = \frac{1}{K^2}\sum_{k=0}^{K-1}\operatorname{var}\{\widehat{S}_k^{(MT)}(f)\}.
$$

Given $\operatorname{var}\{\widehat{S}_k^{(MT)}(f)\}$ are approximately the same and approximately equal to the variance from using just a single taper, there will be a reduction in the variance of the estimator by a factor of K. $\boxed{3, \mathrm{M}}$

(b) (i) We need to verify that $\sum_{t=1}^{K} h_{k,t}^2 = 1$ for all $k = 0, 1, 2$, and that $\sum_{t=1}^{K} h_{k,t} h_{j,t} = 0$ for $j \neq k$. This is clearly true for the stated tapers. $\boxed{2, \mathrm{M}}$

(ii) Let us first compute $\widehat{S}^{(MT)}(1/4)$ using $\widehat{S}_k^{(MT)}(f) = \left|\sum_{t=1}^{4} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2$.

$$
\begin{aligned}
\widehat{S}_0^{(MT)}(1/4) &= \left|-\frac{a}{2}\mathrm{e}^{-\mathrm{i}\pi/2} + \frac{a}{2}\mathrm{e}^{-\mathrm{i}3\pi/2}\right|^2 \\
&= \frac{a^2}{2} - \frac{a^2}{4}\mathrm{e}^{\mathrm{i}\pi} - \frac{a^2}{4}\mathrm{e}^{-\mathrm{i}\pi} = \frac{a^2}{2} - \frac{a^2}{2}\cos(\pi) = a^2. \\
\widehat{S}_1^{(MT)}(1/4) &= \left|\frac{a}{2}\mathrm{e}^{-\mathrm{i}\pi/2} + \frac{a}{2}\mathrm{e}^{-\mathrm{i}3\pi/2}\right|^2 \\
&= \frac{a^2}{2} + \frac{a^2}{4}\mathrm{e}^{\mathrm{i}\pi} + \frac{a^2}{4}\mathrm{e}^{-\mathrm{i}\pi} = \frac{a^2}{2} + \frac{a^2}{2}\cos(\pi) = 0. \\
\widehat{S}_2^{(MT)}(1/4) &= \left|\frac{a}{2}\mathrm{e}^{-\mathrm{i}\pi/2} - \frac{a}{2}\mathrm{e}^{-\mathrm{i}3\pi/2}\right|^2 \\
&= \frac{a^2}{2} - \frac{a^2}{4}\mathrm{e}^{\mathrm{i}\pi} - \frac{a^2}{4}\mathrm{e}^{-\mathrm{i}\pi} = \frac{a^2}{2} - \frac{a^2}{2}\cos(\pi) = a^2.
\end{aligned}
$$

Therefore $\widehat{S}^{(MT)}(1/4) = 1/3\sum_{k=0}^{2} \widehat{S}_k^{(MT)}(1/4) = 2a^2/3$. To get the confidence interval, we have

$$
P\left[Q_6(0.025) \leq \frac{6\widehat{S}^{(MT)}(f)}{S(f)} \leq Q_6(0.975)\right] = 0.95,
$$

implying

$$
P\left[\frac{6\widehat{S}^{(MT)}(f)}{Q_6(0.975)} \leq S(f) \leq \frac{6\widehat{S}^{(MT)}(f)}{Q_6(0.025)}\right] = 0.95.
$$

Hence

$$
\left[\frac{4a^2}{Q_6(0.975)}, \frac{4a^2}{Q_6(0.025)}\right]
$$

is the 95% confidence interval for $S(f)$. Following correct computation of $\widehat{S}^{(MT)}(1/4)$, the correct CI could also be obtained by inspection for full marks.

$\boxed{6, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60046/70046	1	This question was, on the whole, answered well. b(ii) caused a few problems, rigorous arguments for why it is not completely stationary were often lacking, and showing it was second order stationary was often made overly complicated. c(iii), a common mistake was to make Y_t a MA(1) process, you need to make it a white noise process and then X_t is MA(1).
MATH60046/70046	2	This questions was, on the whole, answered well. Marks were often dropped on a(ii) for not explaining why G(1/6) is of interest to look at. c(ii) one cannot simply equate the numerators (or denominators). Simultaneous equations must be constructed with a scaling factor.
MATH60046/70046	3	No Comments Received
MATH60046/70046	4	No Comments Received
MATH70046	5	No Comments Received