Applied Complex Analysis - Lecture Fifteen

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February 2025

Trapezium rule(s) for unbounded contours

- Consider
$$I = \int_{-\infty}^{\infty} f(x)dx,$$
for some f analytic on $\mathbb{R}$, with appropriate decay of f such that $I < \infty$.
- We've seen techniques for evaluating these by hand - not always possible.
- For $h > 0$ we define the *unbounded* Trapezium rule $I_h \approx I$ as
$$I_h := h \sum_{j=-\infty}^{\infty} f(x_j),$$
where $x_j = jh$.
- We define the *truncated* Trapezium rule $I_h^{(N)} \approx I$ as
$$I_h^{(N)} := h \sum_{j=-N}^{N} f(x_j), \quad \text{for } N \in \mathbb{N}_0.$$

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Convergence theorem

Suppose $f(z)$ is analytic in the complex strip $|\mathrm{Im}(z)| < a$ for some $a > 0$. Suppose further that $f(z) \to 0$ uniformly as $|z| \to 0$ in the strip and

$$\int_{-\infty}^{\infty} |f(t + ia')| dt \leq M,$$

for all $a' \in (-a, a)$. Then I_h satisfies

$$|I - I_h| \leq \frac{2M}{e^{2\pi a/h} - 1}.$$

- This result is about the *unbounded* trapezium rule.
- This is called the *discretisation* error.

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The truncation error

Defined as, for $x_n = hn$

$$|I_h - I_h^{(N)}| = \left| h \sum_{n=-\infty}^{\infty} f(x_n) - h \sum_{n=-N}^{N} f(x_n) \right|$$

$$= \left| h \sum_{n=-\infty}^{-(N+1)} f(x_n) - h \sum_{n=N+1}^{\infty} f(x_n) \right|$$

- Practically, we care about

$$|I - I_h^{(N)}| \leq |I - I_h| + |I_h - I_h^{(N)}|$$

- Often, it is enough to bound by a constant multiplied by the first term.

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Bound on (half of the) truncation error

- Suppose that, for some $\alpha > 0$ independent of $y_0 > 0$, the function g satisfies the mild growth condition

$$g(y+\delta) - g(y) \geq \alpha\delta, \tag{1}$$

for all $\delta > 0$ and $y > y_0$.

- Furthermore, suppose either that (i) the meshwidth h is independent of N, or (ii) the meshwidth $h \to 0$ as $N \to \infty$, but with a rate $1/N \ll h$.
- Then the positive terms in the truncation error satisfies:

$$h \sum_{n=N+1}^{\infty} e^{-g(hn)} = O(e^{-g(h(N+1))}), \quad N \to \infty.$$

- Proof

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Examples

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- $$\mathrm{erfc}(z) = \frac{2e^{-z^2}}{\pi} \int_0^{\infty} \frac{e^{-z^2t^2}}{t^2+1} dt, \quad z \in \mathbb{R}.$$

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Resdiue correction to trapezium rule

Recall the *exact* representation of the error:

$$I_h - I = -\sum_{\pm} \int_{-\infty \pm ia'}^{\infty \pm ia'} \frac{f(z)}{1 - e^{\mp 2\pi i x/h}} dz,$$

Returning to the following example:

$$I = \int_{-\infty}^{\infty} \frac{e^{-z^2}\sqrt{z-2i}}{z-i} dz.$$

- Recap
- Optimal h before and after the residue correction.
- Generalisations of this idea

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An Introduction to steepest descent

Oscillatory integrals

Unbounded integrals in which the integrand is rapidly oscillating:

$$\int_{-\infty}^{\infty} f(x)e^{i\omega g(x)}dx, \quad \text{for big } \omega,$$

occur in many applications:

- Gravitational waves
- Quantum mechanics
- Rainbows
- Twinkling stars

Approximating directly via standard quadrature rules (e.g. Gauss, trapezium) typically carries a computational cost $N = O(\omega)$.

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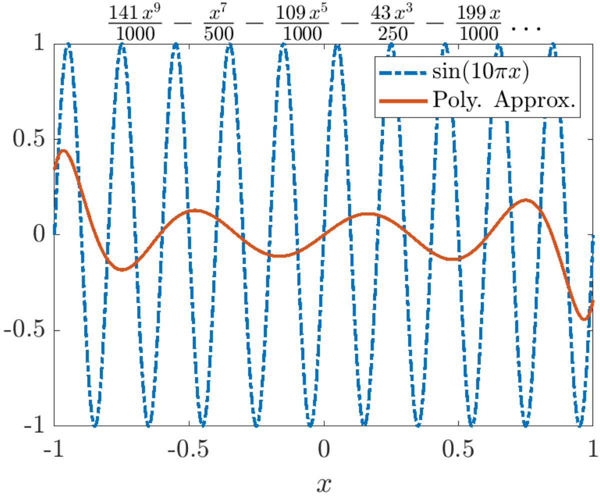
Approximating a wave with polynomials, $\omega = 10$



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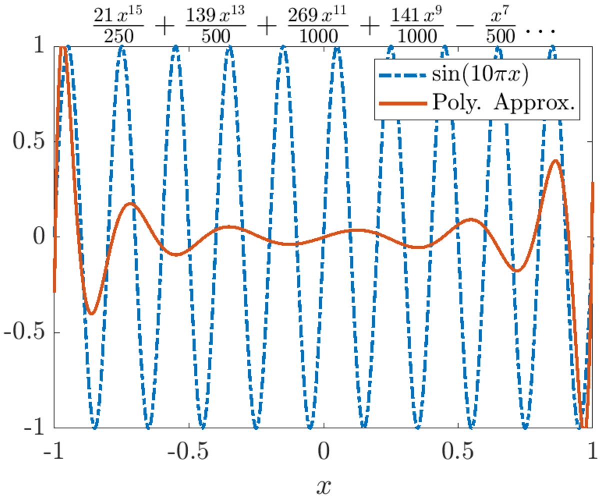

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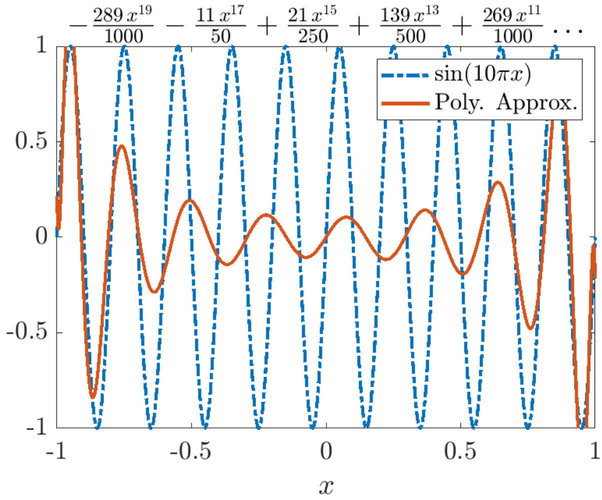

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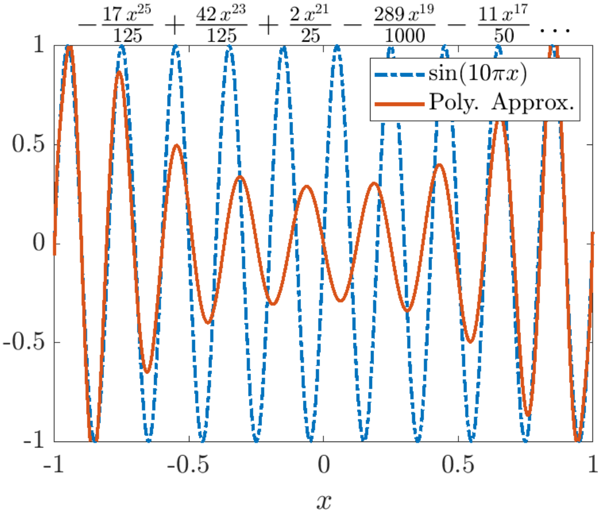

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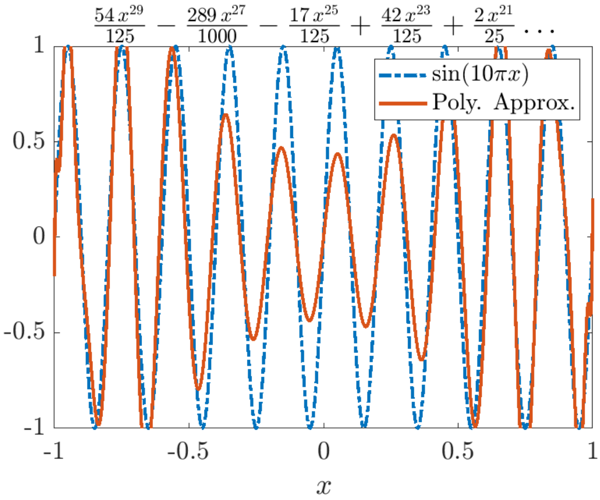

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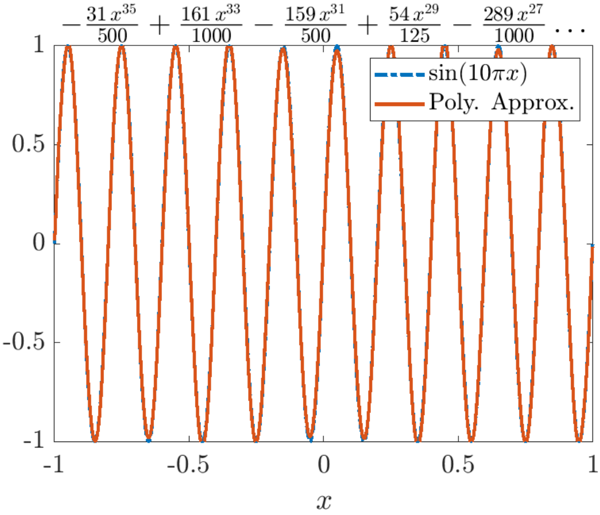

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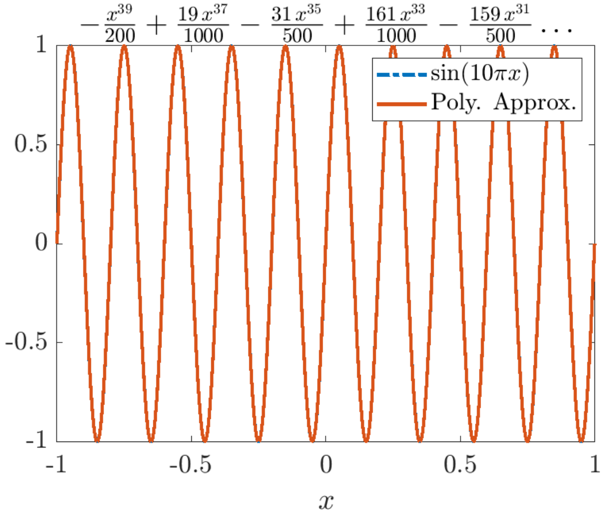

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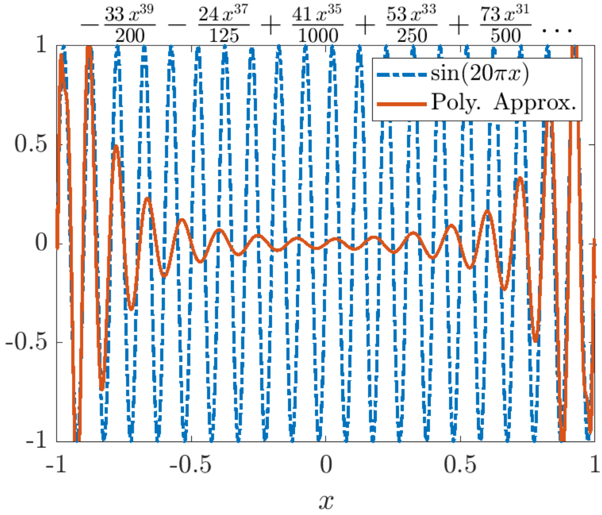

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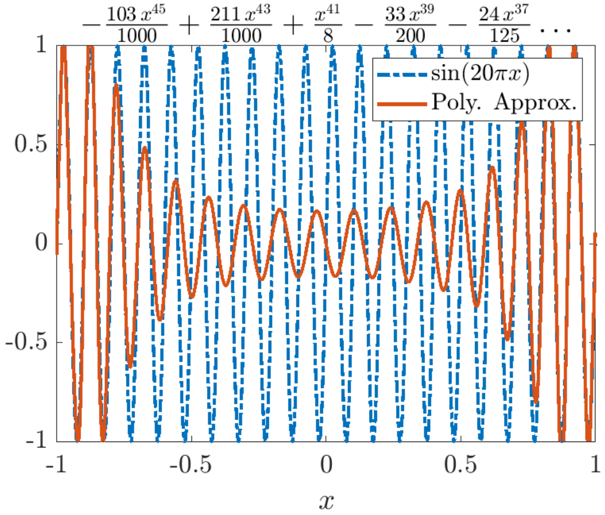

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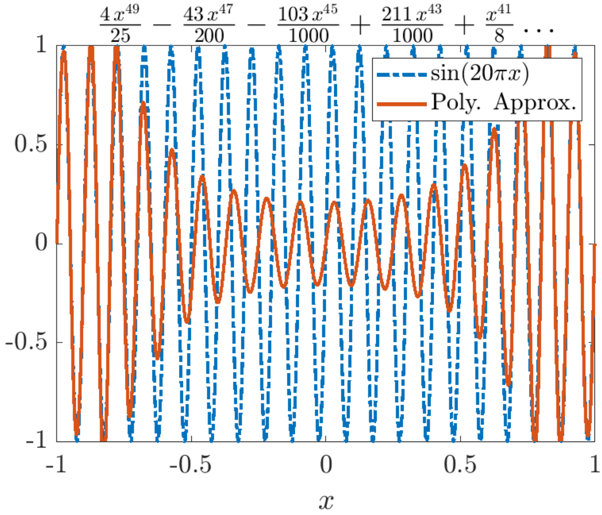

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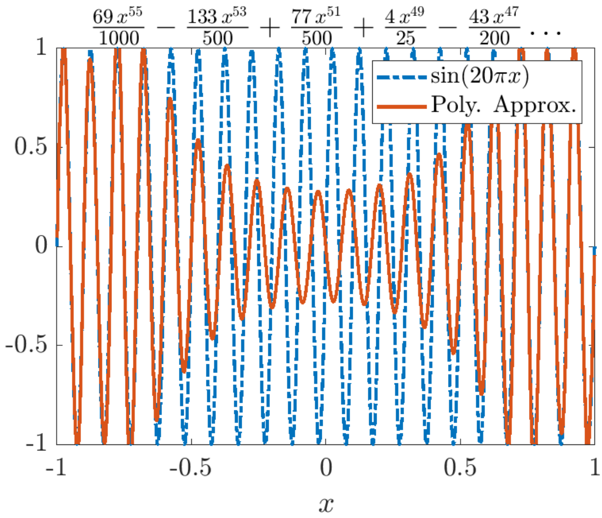

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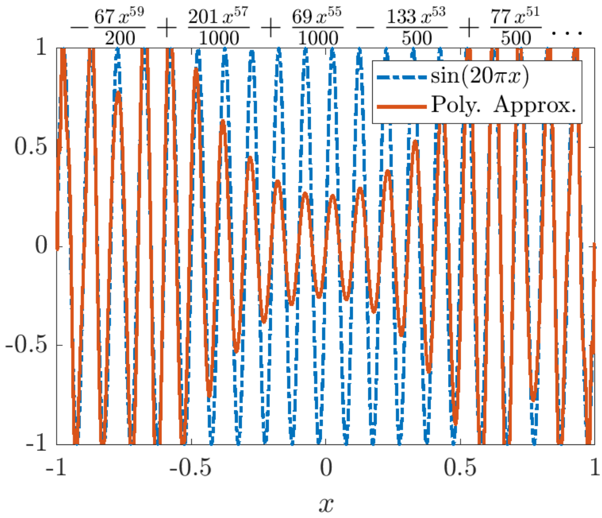

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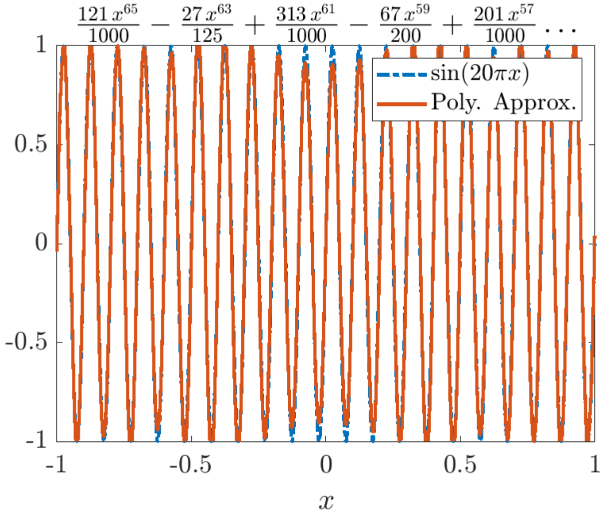

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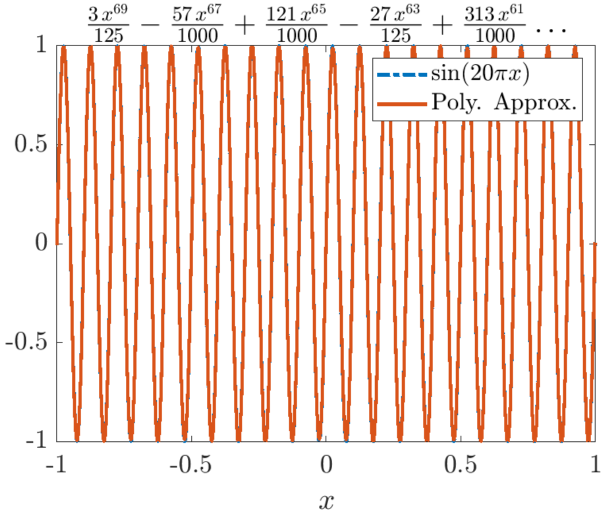

The idea behind steepest descent

The idea behind steepest descent is to deform the integration contour into the complex plane, **swapping oscillations for exponential decay**.

- Exponential decay is easier to work with, in either asymptotic or numerical approximation.
- Asymptotic methods: Watson's Lemma, Laplace's method, etc.
- Numerical methods: Gauss Quadrature rules can be constructed for *positive* weight functions.
- Those things haven't been covered in this course - so we will consider an example using the trapezium rule.

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Steepest descent feat. Trapezium rule

Consider

$$I = \int_{-\infty}^{\infty} f(x) e^{i\omega x^2} = e^{i\pi/4} \int_{-\infty}^{\infty} f(re^{i\pi/4}) e^{-\omega r^2}$$

for $f(x)$ bounded as $|x| \to \infty$, and suitable assumptions about analyticity of f.

- Cauchy's theorem tells us that deforming integral does not change value - but deforming an approximation *can* change its value! So we want to choose the deformation which is optimal for approximation.
- Not necessarily absolutely convergent!
- Converting to steepest descent path
- Choosing optimal parameters with trapezium rule
- Numerical experiments for multiple f, and h

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