Math40002 Analysis 1 Problem Sheet 2

1. Fix $S \subset \mathbb{R}$ with an upper bound, and suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Give proofs or counterexamples to each of the following statements.

   (a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.

   (b) If $S \subset \mathbb{R}\backslash\mathbb{Q}$ then $\sup S \in \mathbb{R}\backslash\mathbb{Q}$.

   (c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.

   (d) $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}_{>0},\ m \leq 10^{100}\right\}$ has a minimum if it is nonempty.

   (e) There exists a $\max S$ if and only if $\sup S \in S$.

   (f) $\sup S = \inf(\mathbb{R}\backslash S)$.

   (g) $\sup S = \inf(\mathbb{R}\backslash S) \iff S$ is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

(a) False, eg $S = \{x \in \mathbb{Q} \colon x < \sqrt{2}\}$ with $\sup S = \sqrt{2}$.

(b) False, eg $S = \{x \in \mathbb{R}\backslash\mathbb{Q} \colon x < 0\}$ with $\sup S = 0$.

(c) True. Pick $s_0 \in S$ and an integer N larger than a given upper bound for S. Then $[s_0, N] \cap \mathbb{Z}$ is a finite set, so $[s_0, N] \cap S$ is also finite and nonempty set of integers so has a maximum $m \in \mathbb{Z}$. By (e) this is also $\sup S$.

(d) True. Since it is $\neq \emptyset$ pick an element s_0. Then $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}_{>0},\ m \leq 10^{100}\right\} \cap [0, s_0]$ is nonempty and *finite*! (Because m runs through the finite set $\{1, 2, \ldots, 10^{100}\}$ and n runs through the finite set $\{1, 2, \ldots, \lfloor s10^{100} \rfloor\}$.) It therefore has a minimum, which is clearly the minimum of $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}_{>0},\ m \leq 10^{100}\right\}$ because it is in the set and is $\leq$ all elements s of the set that satisfy $s \leq s_0$. It is therefore $\leq$ elements s of the set that satisfy $s > s_0$ too!

(e) True. If $\sup S \in S$ then it is the maximum element because it is an upper bound, so $\sup S \geq s$ for all $s \in S$.

Conversely if there exists $m = \max S$ then m is an upper bound, and given any other upper bound M, $M \geq m$ by definition of upper bound because $m \in S$. Therefore m is the least upper bound, i.e. $\sup S = m \in S$.

(f) False. E.g. $S = \{0\}$ has $\sup S = 0$ but $\mathbb{R}\backslash S$ is not even bounded below so has no infimum.

(g) True. If $S = (-\infty, a)$ then $\sup S = a$ while $\mathbb{R}\backslash S = [a, \infty)$ so $\inf S^c = a$ also.

If $S = (-\infty, a]$ then $\sup S = a$ while $\mathbb{R}\backslash S = (a, \infty)$ so $\inf S^c = a$ also.

Conversely, if $\sup S = \inf S^c$ then call this number a. Any $x < a$ must be in S: if not then $x \in S^c$ but $x < \inf S^c$, a contradiction. Similarly any $x > a$ must be in S^c: if not then $x \in S$ but $x > \sup S$, a contradiction.

Therefore $(-\infty, a) \subseteq S$ and $(a, \infty) \subseteq S^c$. Finally either $a \in S$ or $a \in S^c$, making S equal to $(-\infty, a]$ or $(-\infty, a)$ respectively.

2. Fix nonempty sets $S_n \subset \mathbb{R},\ n = 1, 2, 3, \ldots$. Prove that

$$\sup\{\sup S_1, \sup S_2, \sup S_3, \ldots\} \ = \ \sup\left(\bigcup_{n=1}^{\infty} S_n\right),$$

   in the sense that if either exists then so does the other, and they are equal.

Upper bound is $M_n \in \mathbb{R}$ such that $M_n \geq s\ \forall s \in S_n$.

$\sup S_n \in \mathbb{R}$ is an upper bound for S_n such that $\sup S_n \leq M_n$ for all upper bounds M_n. It exists if and only if S_n is nonempty and has an upper bound.

Suppose the left hand side exists, call is M. Then $M \geq \sup S_i \geq s$ for each i and for each $s \in S_i$, so M is an upper bound for $\bigcup_{n=1}^{\infty} S_n$, so the right hand side N exists and $M \geq N$.

Similarly if the right hand side N exists then it is an upper bound for each S_i, so each $\sup S_i$ exists and is $\leq N$. Therefore the set on the left hand side has an upper bound N, therefore its supremum M exists and $M \leq N$.

Therefore if either exists so does the other. In this case we have shown that $M \geq N$ and $M \leq N$.

Therefore $M = N$.

3. Take bounded, nonempty $S,\, T \subset \mathbb{R}$. Define $S+T := \{s+t\colon s \in S,\, t \in T\}$. Prove

$$\sup(S+T) \;=\; \sup S + \sup T.$$

Any element $s+t$ of $S+T$ is $\leq \sup S + \sup T$, so $S+T$ is nonempty and has an upper bound $\sup S + \sup T$, so it has a supremum $\leq \sup S + \sup T$.

Given $\epsilon > 0$ we know $\sup S - \epsilon$ is not an upper bound for S because it is smaller than the *least* upper bound, so $\exists s \in S$ such that $s > \sup S - \epsilon$. Similarly $\exists t \in T$ such that $t > \sup T - \epsilon$.

Therefore $\exists\, s+t \in S+T$ such that $s+t > \sup S + \sup T - 2\epsilon$, so we have proved that

$$\sup S + \sup T - 2\epsilon \;<\; \sup(S+T) \;\leq\; \sup S + \sup T \quad \forall \epsilon > 0,$$

which implies $\sup(S+T) = \sup S + \sup T$.

4.$*$ Fix $a \in (0,\infty)$ and $n \in \mathbb{N}_{>0}$ with $n > 1$. We will prove $\exists\, x \in \mathbb{R}$ such that $x^n = a$. Set

$$S_a \;:=\; \{s \in [0,\infty) \,:\, s^n < a\}$$

and show S is nonempty and bounded above, so we may define $x := \sup S_a$.

For $\epsilon \in (0,1)$ show $(x+\epsilon)^n \leq x^n + \epsilon[(x+1)^n - x^n]$. (Hint: multiply out.)
Hence show that if $x^n < a$ then $\exists\, \epsilon \in (0,1)$ such that $(x+\epsilon)^n < a$. $(*)$

If $x^n > a$ deduce from $(*)$ that $\exists\, \epsilon \in (0,1)$ such that $(\frac{1}{x}+\epsilon)^n < \frac{1}{a}$. $(**)$

Deduce contradictions from $(*)$ and $(**)$ to show that $x^n = a$.

By the binomial theorem,

$$\begin{aligned}(x+\epsilon)^n &= x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon^2 x^{n-2} + \ldots + \epsilon^n \\ &< x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon x^{n-2} + \ldots + \epsilon \\ &= x^n + \epsilon((x+1)^n - x^n),\end{aligned}$$

for $\epsilon \in (0,1)$ (so that $\epsilon^k < \epsilon$) and $x > 0$.

Therefore, if $x^n < a$ we can set $\epsilon := \min\left(\frac{1}{2}, \frac{a-x^n}{(x+1)^n - x^n}\right) \in (0,1)$ so that

$$(x+\epsilon)^n \;<\; x^n + \epsilon((x+1)^n - x^n) \;\leq\; x^n + (a - x^n) \;=\; a,$$

as required. So $x+\epsilon$ is both in S_a and $> \sup S_a$ – a contradiction.

If $x^n > a$ then $\left(\frac{1}{x}\right)^n < \frac{1}{a}$ so by $(*)$ applied to $\frac{1}{x}$ and $\frac{1}{a}$ we find $\exists\, \epsilon \in (0,1)$ such that $\left(\frac{1}{x}+\epsilon\right)^n < \frac{1}{a}$.

Thus $\left(\frac{x}{1+\epsilon x}\right)^n > a \;\Rightarrow\; y^n > a$ for all $y \geq \frac{x}{1+\epsilon x}$. That is, $\frac{x}{1+\epsilon x}$ is an upper bound for S_a, but is $< x = \sup S_a$ – a contradiction.

5. Suppose $0 < q \in \mathbb{Q}$ and $a \in (0, \infty)$. Write $q = \frac{m}{n}$ with $m, n \in \mathbb{N}_{>0}$ and define

$$a^q \; := \; x^m,$$

where $x =: a^{1/n}$ is defined in the last question. Show this is well defined, and make a definition of a^{-q}.

Show that $(ab)^q = a^q b^q$ and $(a^{q_1})^{q_2} = a^{q_1 q_2}$ for any $a, b \in (0, \infty)$ and $q, q_1, q_2 \in \mathbb{Q}$.

To show this is well defined we have to show that if we replace $\frac{m}{n}$ by $\frac{pm}{pn}$ for some $p \in \mathbb{N}_{>0}$ then we get the same answer. (Why is this enough?) That is, we must show

$$\left(a^{\frac{1}{n}}\right)^m \; = \; \left(a^{\frac{1}{pn}}\right)^{pm}.$$

It would be sufficient to prove

$$a^{\frac{1}{n}} \; = \; \left(a^{\frac{1}{pn}}\right)^p.$$

The LHS is the unique (why?) $x \in (0, \infty)$ such that $x^n = a$. So it is enough to show the RHS has this property, i.e. that

$$\left(\left(a^{\frac{1}{pn}}\right)^p\right)^n \; = \; a.$$

But the LHS is

$$\left(a^{\frac{1}{pn}}\right)^{np},$$

which is indeed a by the definition of $a^{\frac{1}{pn}}$.

Next, we'll prove that, for any $a, b \in (0, \infty)$ and $0 < q \in \mathbb{Q}$, we have the identity $(ab)^q = a^q b^q$. Writing $q = \frac{m}{n}$ with $m, n \in \mathbb{N}$, what we must show is that $((ab)^{1/n})^m = (a^{1/n})^m (b^{1/n})^m = (a^{1/n} b^{1/n})^m$ - in the last inequality here we are allowed to play with the exponent m like this since $m \in \mathbb{N}$. Again, since $m \in \mathbb{N}$, it suffices to show that

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Now, $(ab)^{1/n}$ is the unique positive real number that satisfies $((ab)^{1/n})^n = a$, so we just need to show that $a^{1/n} b^{1/n}$ satisfies this same property (since it is is also a positive real number). We see that this is true since $(a^{1/n} b^{1/n})^n = (a^{1/n})^n (b^{1/n})^n = ab$ where in the first equality we are using the fact that we can distribute the exponent n since $n \in \mathbb{N}$.

For showing $(a^{q_1})^{q_2} = a^{q_1 q_2}$ we write $q_1 = m_1/n_1$ and $q_2 = m_2/n_2$ so the desired identity is

$$(((a^{1/n_1})^{m_1})^{1/n_2})^{m_2} = (a^{1/(n_1 n_2)})^{m_1 m_2}$$

Notice that we know the righthand side above is equal to $((a^{1/(n_1 n_2)})^{m_1})^{m_2}$ (natural number exponents behave nicely) and so it suffices to show

$$((a^{1/n_1})^{m_1})^{1/n_2} = (a^{1/(n_1 n_2)})^{m_1}$$

Now, just like last time, by the uniqueness of $1/n_2$ roots, to show the above identity it suffices to show that

$$((a^{1/(n_1 n_2)})^{m_1})^{n_2} = (a^{1/n_1})^{m_1}$$

Now, again we know that $((a^{1/(n_1 n_2)})^{m_1})^{n_2} = (a^{1/(n_1 n_2)})^{m_1 n_2} = ((a^{1/(n_1 n_2)})^{n_2})^{m_1}$. Again, we "cancel" the m_1's so it suffices to show

$$(a^{1/(n_1 n_2)})^{n_2} = a^{1/n_1}$$

Now are are done since we know $((a^{1/(n_1 n_2)})^{n_2})^{n_1} = a$ (skipping a few steps here).

For a^{-q}, you should define $a^{-q} := 1/a^q$.

6. For real numbers x, y, z, consider the following inequalities.

(a)	$\lvert x+y\rvert \le \lvert x\rvert + \lvert y\rvert$	(e)	$\lvert x\rvert \le \lvert y\rvert + \lvert x-y\rvert$
(b)	$\lvert x+y\rvert \ge \lvert x\rvert - \lvert y\rvert$	(f)	$\lvert x\rvert \ge \lvert y\rvert - \lvert x-y\rvert$
(c)	$\lvert x+y\rvert \ge \lvert y\rvert - \lvert x\rvert$	(g)	$\lvert x-y\rvert \le \lvert x-z\rvert + \lvert y-z\rvert$
(d)	$\lvert x-y\rvert \ge \Big\lvert \lvert x\rvert - \lvert y\rvert \Big\rvert$		

Prove (a) from first principles. Why is it called the "triangle inequality"?

Deduce (b,c,d,e,f,g) from (a).

First prove that $x \le |x|$ by considering the two cases $x \le 0$, $x > 0$.

Then it follows that $x + y \le |x| + |y|$ and $-(x+y) \le |-x| + |-y| = |x| + |y|$. Combining these two gives (a).

Called triangle rule because it says that if you make a triangle with vertices at $0, x, x+y$ then the length of the side $[0, x+y]$ is $\le$ the sum of the lengths of the other two sides $[0, x]$ and $[x, x+y]$. It's a bit more convincing when $0, x, x+y$ are all vectors in higher dimensions, eg. R^2, where the same result holds.

For (b) replace x, y in (a) by $x+y, -y$ and rearrange. Similarly for all the others. For (g) write $x - y = (x-z) - (y-z)$.

*Starred questions * are good to prepare to discuss with your Personal Tutor.*