Math40003 Linear Algebra and Groups **Problem Sheet 4**

Seen A

A.1. Exercise 3.6.2: Verify the Steinitz Exchange Lemma where:

- $V = \mathbb{R}^3$
- $X = \{e_1, e_2\}$
- $u = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

You may find Lemma 3.6.1. useful here.

We're asked to verify the Steinitz Exchange Lemma for these objects, so the first thing we should do is check that these objects satisfies all the conditions of the SEL. If they don't then the SEL is trivially true, and we can stop worrying. This V is clearly a vector space, and the field is implicitly $\mathbb{R}$. It's certainly true that $X \subseteq V$, and since $u = 2e_1 + 3e_2$ we also have that $u \in \text{Span}(X)$, and $u \notin \text{Span}(X \setminus \{e_1\})$, and $u \notin \text{Span}(X \setminus \{e_2\})$. Since we can use both e_1 and e_2 as the 'v' in SEL, we need to verify both.

That's not hard to do, we just need to check that $e_1 \in \{u, e_2\}$ and $e_2 \in \{e_1, u\}$.

$$e_1 = \frac{1}{2}(u - 3e_2) \quad e_2 = \frac{1}{3}(u - 2e_1)$$

A.2. Prove Lemma 3.6.8: Suppose that $\dim(V) = n$. Then the following statements are true:

(a) Any spanning set of size n is a basis.
(b) Any linearly independent set of size n is a basis.
(c) S is a spanning set if and only if it contains a basis (as a subset).
(d) S is linearly independent if and only if it is contained in a basis (i.e. it's a subset of a basis).
(e) Any subset of V of size $> n$ is linearly dependent.

The definitions of span, linear independence, basis, and dimension are crucial here. You might also want to use Corollary 3.6.4..

(a) Let $S \subseteq V$ be a spanning set with n elements. If S is not linearly independent, i.e. there is an $s \in S$ such that $s \in \text{Span}(S \setminus \{s\})$, then $\text{Span}(S) = \text{Span}(S \setminus \{s\})$. We keep removing elements of S in this way, until we get a linearly independent $S' \subseteq S$ such that S' is also spanning. But then S' is a basis, and by Corollary 3.6.4. we have that $|S'| = n$. Therefore $S' = S$, and so S must have been linearly independent in the first place.

(b) Let S be a linearly independent set with n elements. We can do something very similar to what we did in 1., but instead of taking things out, we add them. If S is not spanning then there is $s \in V \setminus \text{Span}(S)$. Then $S \cup \{s\}$ is still linearly independent. We repeat this until we get a basis S', with $S \subseteq S'$ and $|S'| = n$. Therefore $S = S'$, and S is a basis.

(c) In 1., we found a basis which is a subset of an arbitrary spanning set.

(d) In 2., we extended an arbitrary linearly independent set to a basis.

(e) Let $S \subseteq V$. Then $\text{Span}(S)$ is a subspace of V, and hence a vector space in its own right. $\dim(\text{Span}(S)) \leq n$. By 3., S contains a basis of $\text{Span}(S)$, which we can call S'. We know that $|S'| \leq n < |S|$. Therefore S is not a basis, and hence is not linearly independent.

A.3. Exercise 3.7.4: Let U and W be subspaces of V, a vector space over F. Then $U + W$ and $U \cap W$ are subspaces of V.

Use Definition 3.7.1. for the definition of $+$ and $\cap$ of subspaces, and see how those definitions interact with what it means to be a subspace.

(a) $U+W$ is a subspace: Clearly $U+W \subset V$, so we can apply the subspace test:

- $0 \in U$ and $0 \in W$ so $0+0=0 \in U+W$.
- Suppose $v_1, v_2 \in U+W$ then $v_1 = u_1+w_1$ and $v_2 = u_2+w_2$ for some $u_i \in U$ and $w_i \in W$. Consider

$$\begin{array}{rcll} v_1+v_2 & = & (u_1+w_1) + (u_2+w_2) & \\ & = & \underbrace{(u_1+u_2)}_{\in U} + \underbrace{(w_1+w_2)}_{\in W} & \begin{array}{l} + \text{ in } V \text{ is commutative and associative} \\ U, W \text{ closed under } + \end{array} \end{array}$$

  So $v_1+v_2 \in U+W$
- Let $\lambda \in \mathbb{R}$ and $v \in U+W$ then $v = u+w$ for some $u \in U$ and $w \in W$. Consider

$$\begin{array}{rcll} \lambda v & = & \lambda(u + w) & \\ & = & \underbrace{\lambda u}_{\in U} + \underbrace{\lambda w}_{\in W} & \begin{array}{l} \text{by distributivity in } V \\ U, W \text{ closed under scalar} \times \end{array} \end{array}$$

  So $\lambda v \in U+W$

(b) $U \cap W$ is a subspace: Clearly $U \cap W \subseteq V$, so we can appy the subspace test:

- $0 \in U$ and $0 \in W$ so $0 \in U \cap W$.
- Suppose $v_1, v_2 \in U \cap W$. Then $v_1, v_2 \in U$, and U is a subspace, so $v_1+v_2 \in U$. Similarly $v_1+v_2 \in W$. Therefore $v_1+v_2 \in U \cap W$.
- Suppose $v \in U \cap W$ and $\lambda \in F$. Then $v_1 \in U$, and U is a subspace, so $\lambda v \in U$. Similarly $\lambda v \in W$. Therefore $\lambda v \in U \cap W$.

A.4. Let V be an n-dimensional vector space. Prove that for all $i \leq n$ there is a subspace U of V such that U has dimension i.

Whenever you see the words "an n-dimensional vector space", you should, as a reflex, think "Let B be a basis of V. Then $|B| = n$."

Let $\{b_1, \ldots, b_n\}$ be a basis of V. Let $B_i = \{b_1, \ldots, b_i\}$. Then $\operatorname{Span}(B_i)$ is a subspace of V with dimension i.