Imperial College London

MATH40004 - Calculus and Applications - Term 2

Problem Sheet 6 with solutions

You should prepare starred question, marked by * to discuss with your personal tutor.

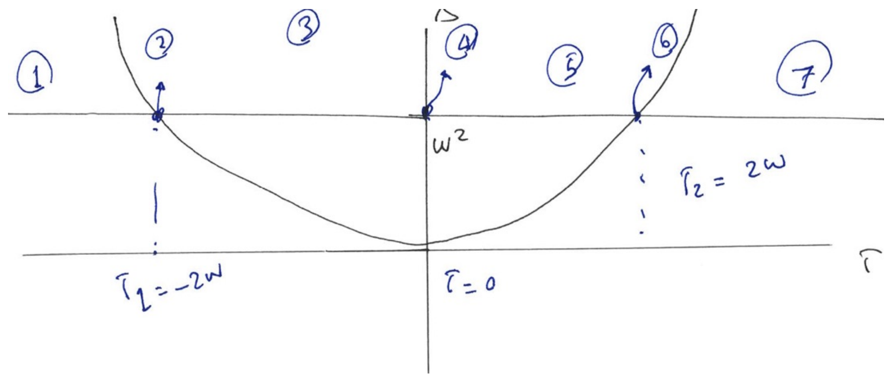
1. Consider again the second order differential equation of the damped harmonic oscillator we saw in the previous problem sheet:
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0,$$
where k and ω are positive constants representing the damping of the medium and the intrinsic frequency of the system, respectively.

   We will now examine the system when the parameter k is tuned experimentally.

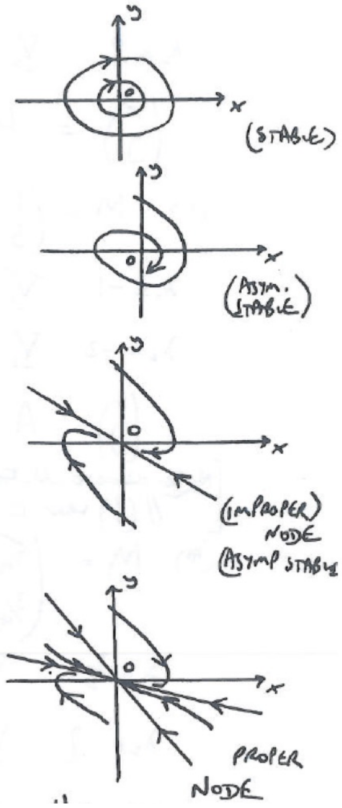
   (a) Sketch all the qualitatively different phase portraits that can appear as k is varied, and describe the associated asymptotic behaviours.

   (b) Establish all the values of k at which the system changes qualitative behaviour.

   We have seen this example in the lectures. Varying k only changes the trace of the corresponding matrix k. We have
$$\tau = -2k; \quad \Delta = \omega^2$$
   So, as we vary k there are 7 different qualitatitive behaviour, as seen in the picture below. At $\tau = k = 0$, we have a bifurcation as we go from region 3 that is assymptotically stable to the point 4 that is Lyapanov stable and then region 5 that is unstable. As negative k is not physical, we explore positive k (negative τ) regions (1-4) below.



   As negative k is not physical, we explore positive k (negative τ) regions (1-4) below. Point 4 is a circle, region 3 is stable spiral, point 3, is a stable degenerate case and region 1 is stable node. See the phase portraits on next page.

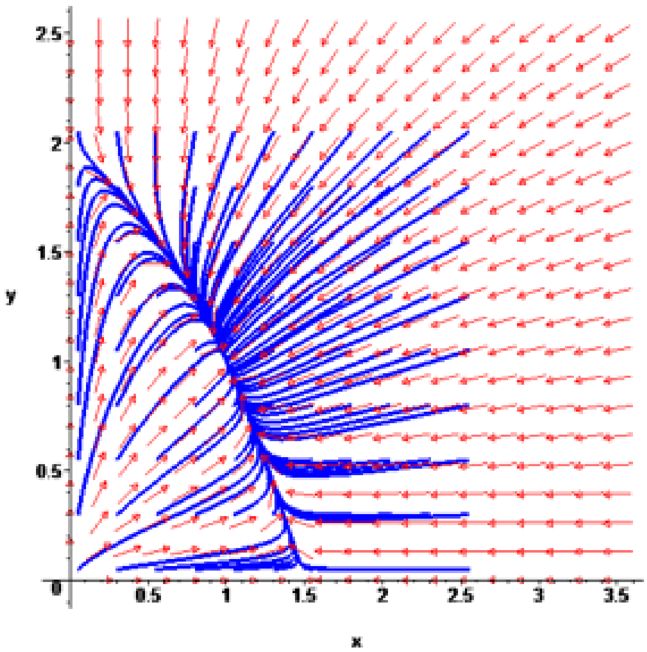


2. * You are given a system of two coupled nonlinear ODEs

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix},$$

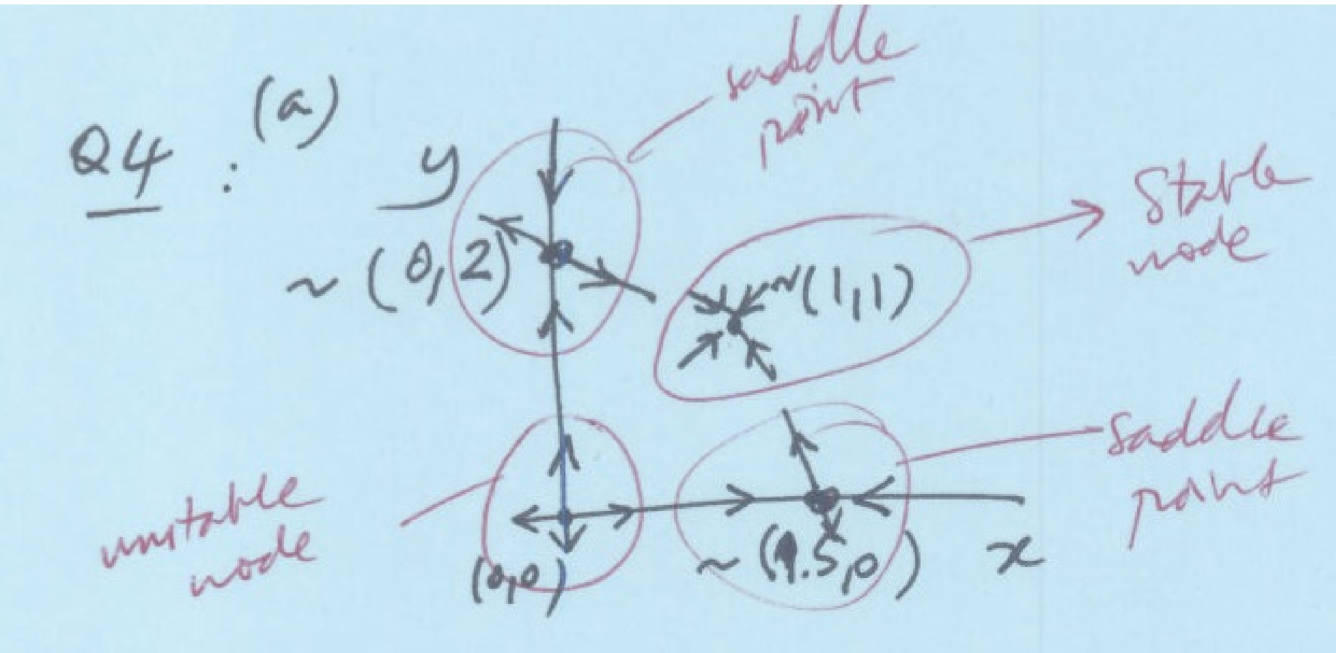
which contain some parameters. The system describes two populations x and y which compete for resources.

(a) If treated correctly, MATLAB or Maple can be your friends (really!). The following figure of the vector field and some sample trajectories can be obtained computationally. Note that x and y are populations, so the phase portrait is only of interest for $x, y > 0$.

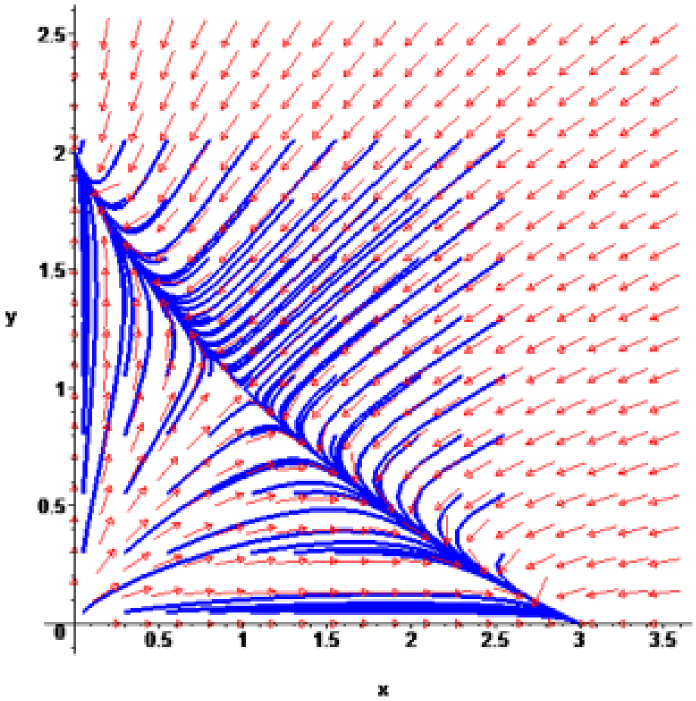


Using the flows depicted in the figure, identify and classify all the fixed points of the system. What will be the long term behaviour of the system?

Asymtotically the system goes to $(1, 1)$ in long term.

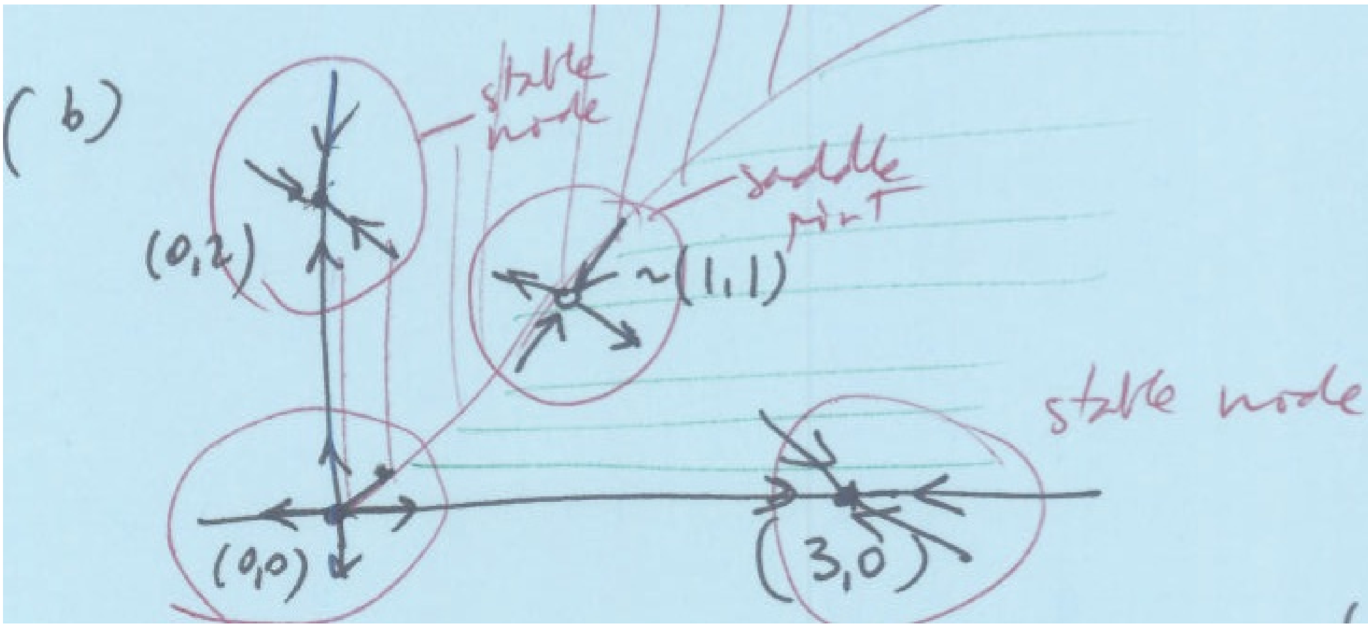


(b) As some of the parameters in the system change, we recompute the vector field and phase portrait and we obtain the following figure:



Again, use the flows depicted in the figure to identify and classify the fixed points of the system. What will be the long term behaviour of the system?

Depending on initial condition the system asymtotically goes to $(0, 2)$ or $(3, 0)$.



3. Consider the following three first order *nonlinear* ODEs:

$$\text{(a)}\ \frac{dx}{dt} = \frac{x}{\alpha + x} - x \qquad \text{(b)}\ \frac{dx}{dt} = \frac{1}{\alpha + x} - x \qquad \text{(c)}\ \frac{dx}{dt} = \frac{x}{\alpha + x^2} - x$$

where $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ is a parameter.

For each case:

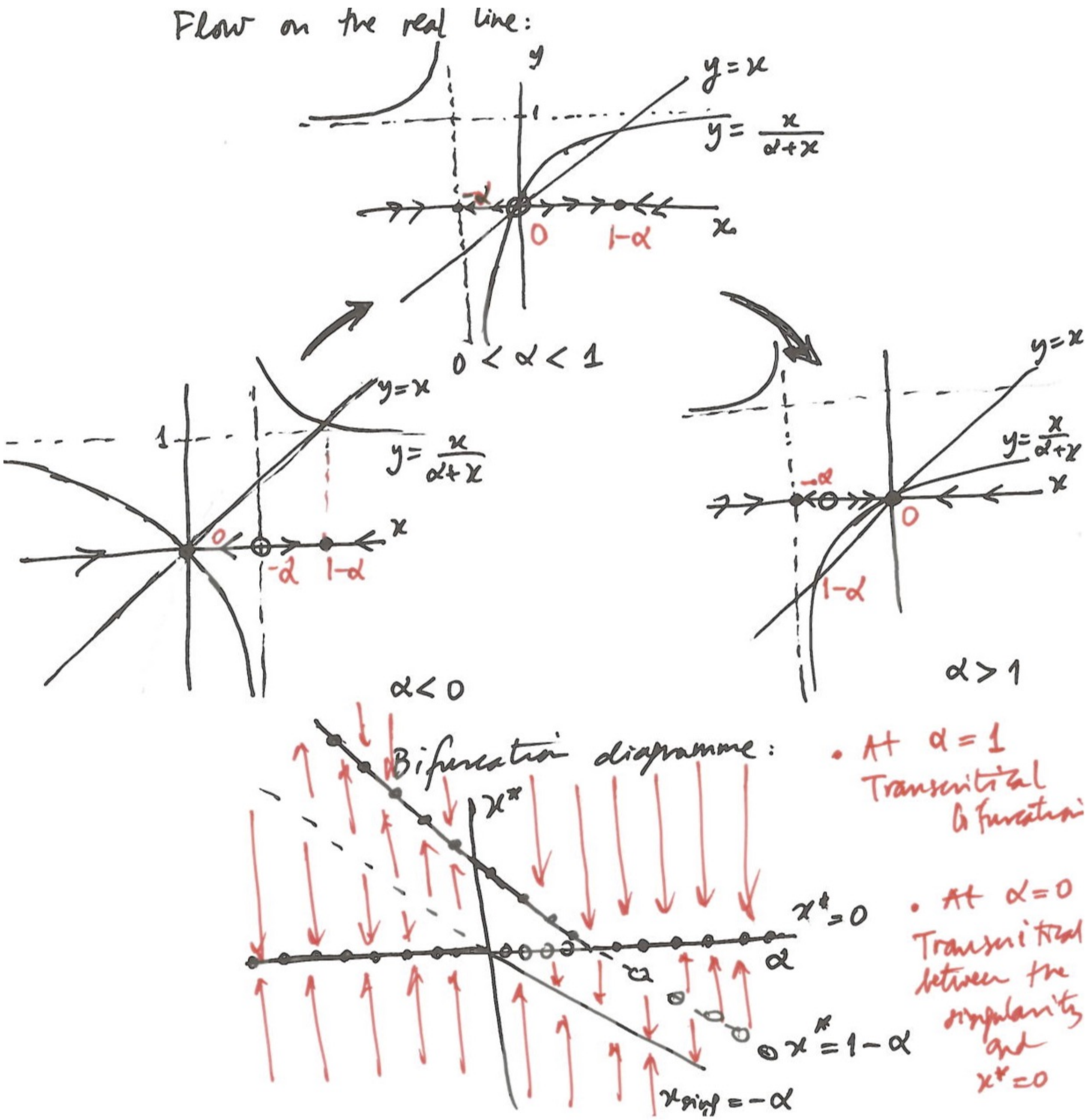
(i) Perform the global stability analysis of the ODE, i.e., find the fixed (stationary) points x^* and their stability as a function of α.

(ii) Draw the bifurcation diagram x^* vs. α and classify any bifurcations observed.

(a)

$$\frac{dx}{dt} = \frac{x}{\alpha + x} - x \qquad \Rightarrow \quad \text{Fixed points:} \quad \frac{x}{\alpha + x} - x = 0$$

We obtain $x^* = 0$ and $x^* = 1 - \alpha$. Also, we have $x_{sing} = -\alpha$. Now as shown in the figure below we sketch the flow on the real line for different value of α. And based on this we draw the bifurcation diagram. At $\alpha = 1$ there is a transcritical bifurcation and at $\alpha = 0$ there is another transcrictical bifurcation between a singularity and the fixed point at $x^* = 0$.



(b)

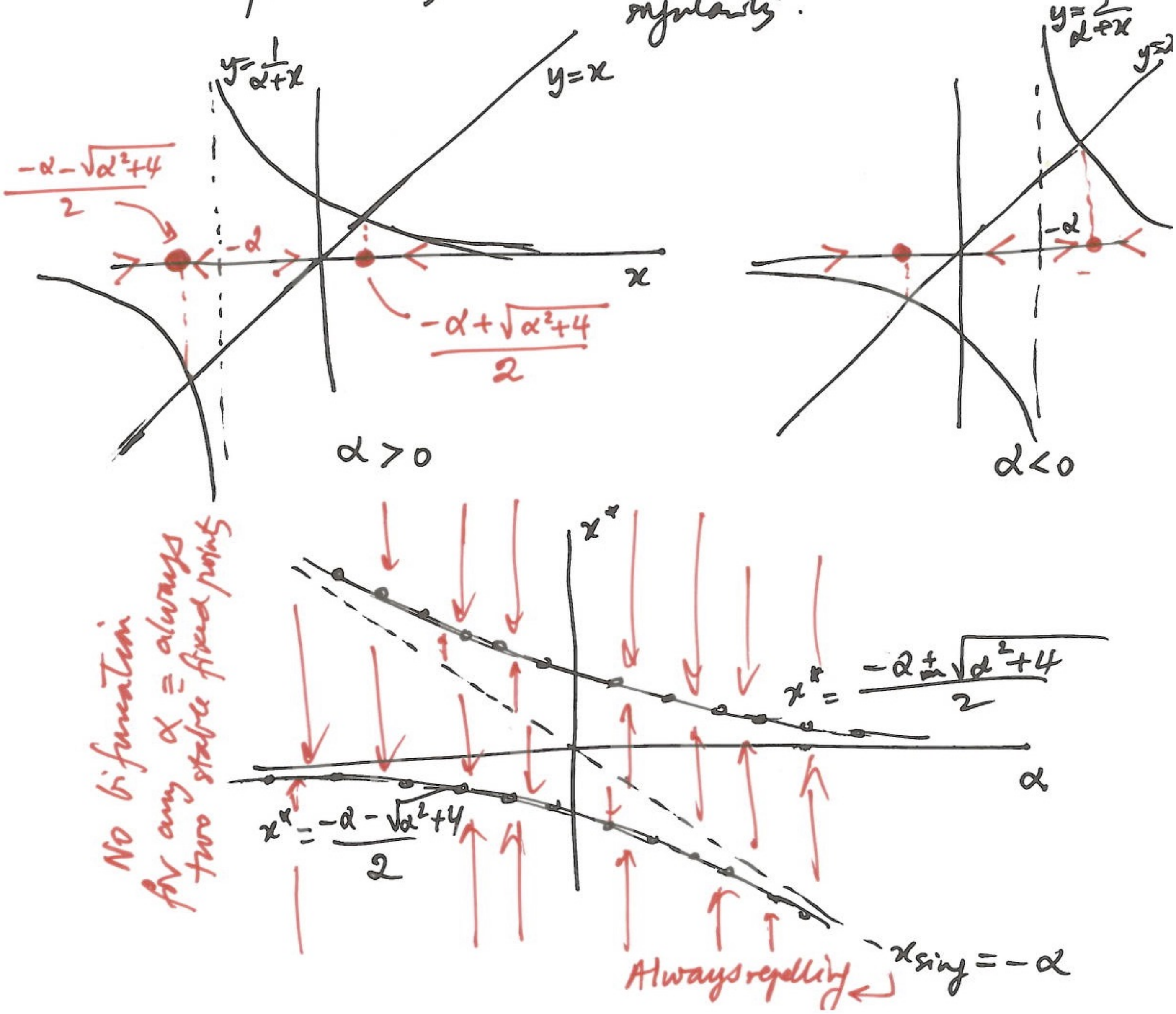
$$\frac{dx}{dt} = \frac{1}{\alpha + x} - x \qquad \Rightarrow \quad \text{Fixed points:} \quad \frac{1}{\alpha + x} - x = 0$$

We obtain a quadratic and two fixed points and also, we have $x_{sing} = -\alpha$.

$$x^* = \frac{-\alpha \pm \sqrt{\alpha^2 + 4}}{2}.$$

Now as shown in the figure below we sketch the flow on the real line for different value of α, which shows nothing changes qualitatively. And based on this we draw the bifurcation diagram, which shows no bifurcation as for all values of *alpha* there are two fixed points and a singularity.

As α changes, nothing changes qualitatively: two fixed points separated by singularity.



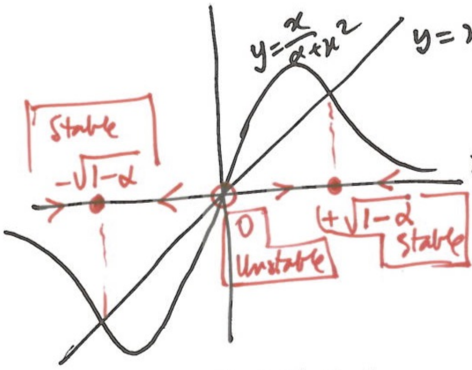
(c)

$$\frac{dx}{dt} = \frac{x}{\alpha + x^2} - x \qquad \Rightarrow \quad \text{Fixed points:} \quad \frac{x}{\alpha + x^2} - x = 0$$

Now we have upto 3 fixed points and a upto two singular points at $x_{sing} = \pm\sqrt{\alpha}$.

$$x^* = 0, \quad x^* = \pm\sqrt{1-\alpha}.$$

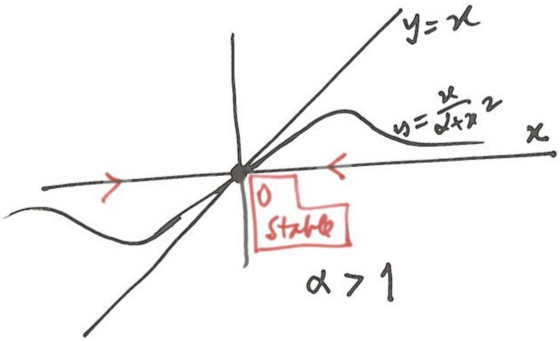
Now as shown in the figure below we sketch the flow on the real line for different value of α. And based on this we draw the bifurcation diagram, which shows there are pitchfork bifurcations at $\alpha = 1$ and $\alpha = 0$.



Slope of $\left.\frac{x}{\alpha+x^2}\right|_{x=0} = \frac{1}{\alpha}$

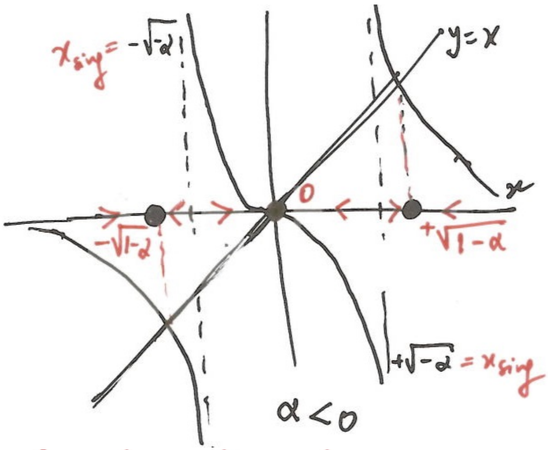
$0 < \alpha < 1$

3 fixed points

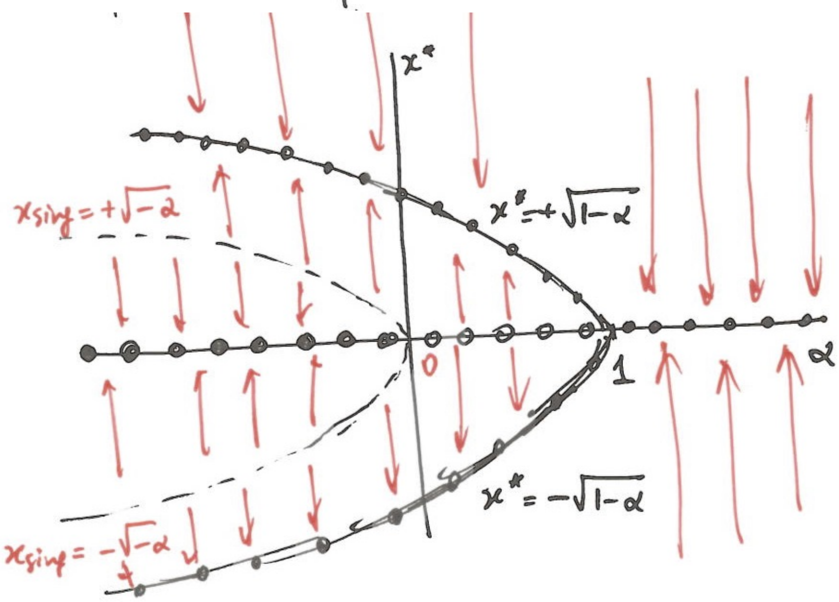


One fixed point.

$\alpha > 1$



$\alpha < 0$



4. * The following first order *nonlinear* ODE

$$\frac{dx}{dt} = x(1-x) - h$$

describes a simple model of a fishery, where $x \in \mathbb{R}^+$ is the population of fish and $h \in \mathbb{R}^+$ is a parameter.

(a) Explain the meaning of each term in the equation.

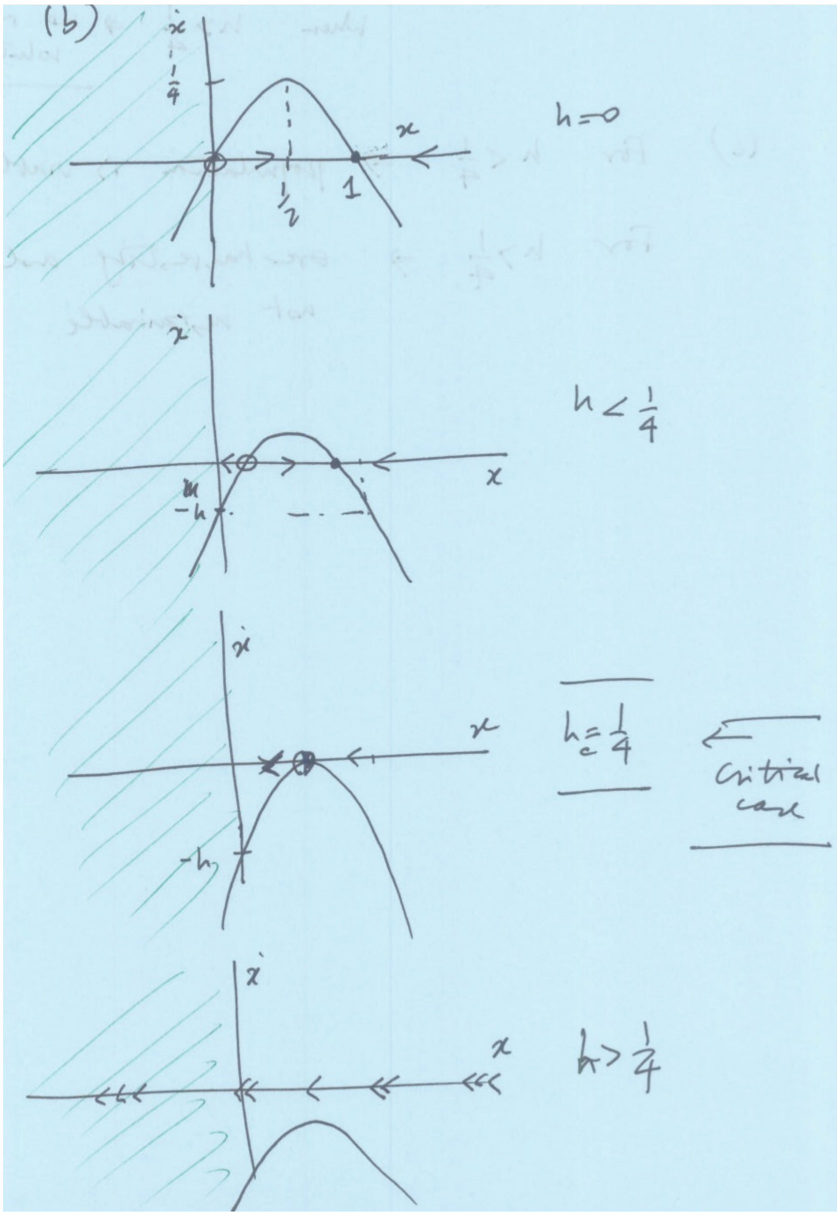
The first term is logistic growth and the second term is depression caused by fishing.

(b) Plot the vector field (on the real line) for different values of h. Show that a bifurcation occurs at a certain value $h = h_c$ and classify this bifurcation.

We have upto two fixed points:

$$x(1-x) - h = 0 \quad \Rightarrow \quad x^* = \frac{1 \pm \sqrt{1-4h}}{2}$$

As shown by the graphs below there is saddle node bifurcation at $h_c = 1/4$.



(c) Discuss the long term behaviour of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

For $h < 1/4$ the fish population is viable. But for $h > 1/4$ there is over harvesting and not sustainable and x goes to zero.