Problem sheet 1

Analysis 1 - Spring 2023

The problems on this sheet cover the topics from the lectures from 13 Jan until 30 Jan (Lectures 1 to 6). Attempt as many as you can before the problem sessions on 24 Jan and 31 Jan, so that you can use the time of the problem session to ask questions and show us your solutions. Solutions will be released on 24 Jan after the problem session.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(\mathbb{R}) \subset \mathbb{Q}$. Prove that f is constant.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Is this still true if we only assume that $f(x) = g(x)$ for $x \in \mathbb{Z}$?

3. Prove the following:

   (a) $|\sin(x)| \le |x|$, for all $x \in \mathbb{R}$.

   (b) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is uniformly continuous. (Hint: first prove the identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$.)

4. Consider the function $f : [1,2] \cap \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = \left|x - \sqrt{2}\right|$. Prove that f does *not* have a minimum value. Why doesn't the extreme value theorem apply?

5. (*) Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n. \end{cases}$$

   Here all rational numbers $x = \frac{m}{n}$ are written in lowest terms, with $n > 0$.

   (a) Prove that if x is rational, then f is not continuous at x.

   (b) Prove that if x is irrational, then f is continuous at x.

6. Let $f : [a,b] \to \mathbb{R}$ be continuous, and suppose that $f(a) \le y \le f(b)$.

   (a) Let $(a_0, b_0) = (a, b)$, and for all $n \ge 0$, define $m_n = \frac{a_n+b_n}{2}$ and

$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, m_n), & f(m_n) > y \\ (m_n, b_n), & f(m_n) \le y. \end{cases}$$

   Prove that the sequences $\big(a_n\big)$ and $\big(b_n\big)$ converge to the same limit $L \in [a,b]$.

   (b) Prove that $f(L) = y$, concluding a new proof of the intermediate value theorem.

7. For any nonempty set $S \subset \mathbb{R}$, define $d_S : \mathbb{R} \to \mathbb{R}$ by $d_S(x) = \inf\limits_{s \in S} |x - s|$.

   (a) Describe or draw graphs of d_S when S is each of $\{0\}$, $\{-1, 3\}$, $\mathbb{Z}$, $\mathbb{Q}$.

(b) Prove that $|d_S(y) - d_S(x)| \leq |y - x|$ for all $x, y \in \mathbb{R}$, and conclude that d_S is continuous.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, not necessarily continuous. Define $S(x) = \sup_{y<x} f(y)$ and $I(x) = \inf_{y>x} f(y)$.

   (a) Prove for all $x \in \mathbb{R}$ that $S(x) \leq f(x) \leq I(x)$.

   (b) Prove for all $x \in \mathbb{R}$ that $S(x) = I(x)$ if and only if f is continuous at x.

   (c) Find an injective mapping

   $$\{x \in \mathbb{R} \mid f \text{ is not continuous at } x\} \to \mathbb{Q}.$$

   Conclude that f is continuous at all but at most countably many real numbers.

9. Remember that a set U is open if for all $x \in U$, there exist $\delta = \delta(x) > 0$, such that $(x - \delta, x + \delta) \subset U$. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage

   $$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$

   is open.

10. Give an example of a bounded open set $S \subset \mathbb{R}$ and a continuous function $f : S \to \mathbb{R}$ which does *not* satisfy the intermediate value theorem: in other words, there are points $a < b$ in S and some x between $f(a)$ and $f(b)$ such that $f(c) \neq x$ for all $c \in S$.

11. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is closed.

12. Let $\big(S_n\big)_{n\in\mathbb{N}}$ denote a decreasing sequence of nonempty subsets of $\mathbb{R}$, meaning that

    $$S_1 \supset S_2 \supset S_3 \supset \dots.$$

    Let $S = \bigcap_{n=1}^{\infty} S_n$ be their intersection.

    (a) Give an example where all of the S_n are open and S is empty.

    (b) Prove that if all of the $S_n = [a_n, b_n]$ are closed intervals, then S is nonempty. (Hint: consider the sequence a_n.)

13. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}$ is a closed and bounded interval, then the image $f(S)$ is also a closed bounded interval. Is the same true for open intervals?

14. (a) Show that $f(x) = x^{1/2}$ is differentiable on $(0, \infty)$, and compute its derivative.

    (b) Do the same for $f(x) = x^{1/n}$, where n is any positive integer.

    (c) Now do the same for $f(x) = x^{m/n}$, where m and n are positive integers.

15. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Hölder continuous* with exponent $\alpha > 0$ if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbb{R}$. Show that if $\alpha > 1$ then f is differentiable, and $f'(x) = 0$.

Remark: We will see in lecture soon that if $f' \equiv 0$ then f must be constant.

16. Find all $x \in \mathbb{R}$ where $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ x^2, & x \in \mathbb{Q} \end{cases}$ is differentiable and compute its derivative.

17. ($*$) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. We will prove that $f'(x)$ has the *intermediate value property* even though it may not be continuous. In both parts we will suppose that $f'(a) < f'(b)$ and fix some t such that $f'(a) < t < f'(b)$.

    (a) Let $g(x) = f(x) - tx$. Prove that there is some $c \in (a, b)$ such that $g(c) < g(a)$. (Hint: what is $g'(a)$?) Similarly, prove that there is some $d \in (a, b)$ such that $g(d) < g(b)$. In other words, $g(x)$ is not minimized at $x = a$ or at $x = b$.

    (b) Show that $g'(y) = 0$ for some $y \in (a, b)$, and deduce that $f'(y) = t$.