Imperial College London

MATH70015

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 1

Date: 18 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. A quantum wave function

Consider a particle that can move on the positive half line with position coordinate $x \in \mathbb{R}^+$. Assume that the system is described by the wave function

$$\psi(x) = Ax\mathrm{e}^{-\lambda x}\mathrm{e}^{-\mathrm{i}\omega t},$$

where A, λ, ω are real positive constants.

(a) Sketch the absolute value of the wave function as a function of x. (2 marks)

(b) Find A such that the wave function is normalised to one. (6 marks)

(c) Write an expression for the probability to find the particle in the interval $[0, a]$, where a is a positive constant, in terms of the wave function $\psi(x)$. You do not need to calculate the probability. (2 marks)

(d) In the neighbourhood of which position is the particle most likely to be found? (4 marks)

(e) What is the expectation value of the position x? (6 marks)

Hint: You may use $x\mathrm{e}^{-yx} = -\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^{-yx}$ to calculate the integrals.

(Total: 20 marks)

2. **The principles of QM - a three level system**

Consider a system on the Hilbert space $\mathbb{C}^3$ spanned by the orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$. The Hamiltonian $\hat{H}$ and another operator $\hat{A}$ are given as

$$\hat{H} = E_0|\phi_1\rangle\langle\phi_1| + \frac{\mathrm{i}E_1}{2}\left(|\phi_2\rangle\langle\phi_3| - |\phi_3\rangle\langle\phi_2|\right),$$

and

$$\hat{A} = a\left(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\right),$$

where E_0, E_1 and a are real constants.

(a) Verify that the vectors

$$|\chi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\phi_2\rangle \pm \mathrm{i}|\phi_3\rangle\right)$$

are eigenvectors of the Hamiltonian $\hat{H}$ and deduce the corresponding eigenvalues.

(5 marks)

(b) Assume that at some specific time t_0 the system is in the state

$$|\psi\rangle = |\chi_+\rangle.$$

(i) With what probability does a measurement of the energy $\hat{H}$ at time t_0 return which result and what is the expectation value of the energy?

(3 marks)

(ii) With what probability does a measurement of the observable $\hat{A}$ at time t_0 return which result and what is the expectation value of $\hat{A}$?

(5 marks)

(iii) Assume now that a measurement of the observable $\hat{A}$ at time t_0 yields the result a. What is the probability that a subsequent measurement of the energy at a later time $t > t_0$ yields the result E_0?

(7 marks)

(Total: 20 marks)

3. **Energy quantisation in a non-symmetric infinite well**

Consider a potential of the form

$$V(x) = \begin{cases} \infty, & x \le -L \\ 0, & -L < x \le 0 \\ V_0, & 0 < x \le L \\ \infty, & L < x, \end{cases}$$

where V_0 and L are real and positive constants.

(a) What is the range of energies for possible bound states? (2 marks)

(b) What boundary conditions does an energy eigenfunction have to fulfill at $x = \pm L$ and $x = 0$? (4 marks)

(c) Write down an ansatz for possible bound states for the energy region $E < V_0$ using the known form of solutions of the time-independent Schrödinger equation in the different spatial regions. (4 marks)

(d) (i) Use the boundary conditions between the different regions to derive a quantisation condition for the energies of bound states with $E < V_0$ of the form

$$\sqrt{\frac{c^2}{z^2} - 1} = f(z),$$

in terms of the variable $z = \frac{L}{\hbar}\sqrt{2mE}$, where c is a constant depending on V_0, m, L, and $\hbar$.

(7 marks)

(ii) Is there a non-zero minimum value of V_0 such that an energy eigenstate with $E < V_0$ exists? Give a reason for your answer. (3 marks)

(Total: 20 marks)

4. The quantum harmonic oscillator

Consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right),$$

with the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, fulfilling $[\hat{a}, \hat{a}^\dagger] = 1$, and the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. The number operator and the ladder operators fulfil the commutation relations

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad \text{and} \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

(a) Let $|\nu\rangle$ denote an eigenvector of $\hat{N}$ with corresponding eigenvalue ν.

(i) Show that the eigenvalues of $\hat{N}$ are non-negative, and that $\hat{a}|\nu\rangle$ is the zero vector if and only if $\nu = 0$. Do this by considering the norm of the vector $\hat{a}|\nu\rangle$. (3 marks)

(ii) Show that $\hat{a}|\nu\rangle$ is either the zero vector or an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu - 1$. (3 marks)

(iii) Show that $\hat{a}^\dagger|\nu\rangle$ is either the zero vector or an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu + 1$. (2 marks)

From the above we can conclude that the eigenstates of the harmonic oscillator Hamiltonian are a set of orthonormal states $|n\rangle$ with non-negative integers n, belonging to the energy eigenvalues $\hbar\omega(n + \frac{1}{2})$.

(b) There is a non-degenerate ground state $|0\rangle$ with $\hat{N}|0\rangle = 0$. Show that from the non-degeneray of the ground state it follows that all states $|n\rangle$ are non-degenerate. (4 marks)

(c) Assume that the system at a given time t_0 is prepared in the state $|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}(|2\rangle - \mathrm{i}|3\rangle)$. What is the expectation value of the rescaled position operator $\hat{Q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ at a later time $t > t_0$? (8 marks)

Hint: In parts (b) and (c) you may use that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$.

(Total: 20 marks)

5. **Mastery: Entanglement**

(a) (i) What are the names conventionally given to the two observers A and B in quantum information protocols? (2 marks)

(ii) What is the difference between the CHSH (Clauser-Horne-Shimony-Holt) setup as compared to Bohm's version of the EPR paradox? (2 marks)

(iii) What does the no-cloning theorem state? (2 marks)

(b) Consider the two-particle spin state

$$|\psi\rangle = N\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big),$$

where the single-particle states $|\uparrow\rangle$ and $|\downarrow\rangle$ are the normalised eigenstates of the single-particle $\hat{S}_z$ operator

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues $\pm\frac{\hbar}{2}$. That is, $\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$ and $\hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$.

(i) For which value of N is the state $|\psi\rangle$ normalised to one? (2 marks)

(ii) Assume A measures the z-component of the spin of the first particle and obtains the outcome $\frac{\hbar}{2}$. What is the state after this measurement? With what probability does B obtain which outcome in a measurement of the z-component of the spin of the second particle? (3 marks)

(c) The x-component of the single particle spin is described by the operator

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Express the eigenvectors of the x-component of the single particle spin operator, $|\rightarrow\rangle$ and $|\leftarrow\rangle$, with $\hat{S}_x|\rightarrow\rangle = \frac{\hbar}{2}|\rightarrow\rangle$ and $\hat{S}_x|\leftarrow\rangle = -\frac{\hbar}{2}|\leftarrow\rangle$, in terms of the eigenstates of the z-component. (2 marks)

(ii) Express the state

$$|\psi\rangle = N\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big),$$

In terms of $|\rightarrow\rangle$ and $|\leftarrow\rangle$. (3 marks)

(iii) Assume that the initial state is again given by

$$|\psi\rangle = N\Big(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\Big),$$

and A measures the x-component of the first particle, obtaining the outcome $\frac{\hbar}{2}$. With what probability does B obtain which outcome in a measurement of the z-component of the spin of the second particle? (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60015/MATH70015

Quantum Mechanics 1 (Solutions)

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1. **A quantum wave function**

Consider a particle that can move on the positive half line with position coordinate $x \in \mathbb{R}^+$. Assume that the system is described by the wave function

$$\psi(x) = Ax\mathrm{e}^{-\lambda x}\mathrm{e}^{-\mathrm{i}\omega t},$$

where A, λ, ω are real positive constants. seen/sim.seen ⇓

(a) Sketch the absolute value of the wave function as a function of x.

Solution: Full marks for sketch depicting a linear increase from zero at x=0, a peak and asymptotic decay to zero at infinity. 2, A

(b) Find A such that the wave function is normalised to one. meth seen ⇓

Solution: We want $\int_0^\infty |\psi(x)|^2 \mathrm{d}x = 1$. We calculate

$$\int_0^\infty |\psi(x)|^2 \mathrm{d}x = A^2 \int_0^\infty x^2 \mathrm{e}^{-2\lambda x} \mathrm{d}x.$$

Using the hint we calculate 3, A

$$\begin{aligned}
\int_0^\infty x^2 \mathrm{e}^{-2\lambda x} \mathrm{d}x &= \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\int_0^\infty \mathrm{e}^{-2\lambda x}\mathrm{d}x\right) \\
&= \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[-\frac{1}{2\lambda}\mathrm{e}^{-2\lambda x}\right]_0^\infty \\
&= \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{1}{2\lambda}\right) \\
&= \frac{1}{4\lambda^3}
\end{aligned}$$

And thus we conclude

$$A = 2\lambda^{3/2}.$$

3, B

(c) What is the expression for the probability in terms of the wave function $\psi(x)$ to find the particle in an interval $[0, a]$, where a is a positive constant? You do not need to calculate the probability. seen ⇓

Solution: The probability is given by $P(x \in [0, a]) = \int_0^a |\psi(x)|^2 \mathrm{d}x$, assuming that $\psi(x)$ is normalised to one. 2, A

(d) In the neighbourhood of which position is the particle most likely to be found? unseen ⇓

Solution: We need to calculate the location of the maximum of the probability density $\rho(x) = A^2 x^2 \mathrm{e}^{-2\lambda x}$. 2, D

Taking the derivative we find for the locations of the possible extrema

$$2A^2(x - \lambda x^2)\mathrm{e}^{-2\lambda x} = 0,$$

that is

$$x(1 - \lambda x) = 0.$$

We have a minimum at $x = 0$ and a maximum at

$$x_{max} = \frac{1}{\lambda}.$$

2, C

$\boxed{\text{meth seen} \Downarrow}$

(e) What is the expectation value of the position x?

Solution: We need to calculate

$$\langle \hat{x} \rangle = \int_0^\infty x|\psi(x)|^2 \mathrm{d}x = A^2 \int_0^\infty x^3 \mathrm{e}^{-2\lambda x} \mathrm{d}x$$

$\boxed{3, \text{A}}$

Using the hint we find

$$\begin{aligned} \langle \hat{x} \rangle = \int_0^\infty x|\psi(x)|^2 \mathrm{d}x &= A^2 \int_0^\infty x^3 \mathrm{e}^{-2\lambda x} \mathrm{d}x \\ &= -\frac{1}{8} A^2 \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3} \left(\frac{1}{2\lambda} \right) \\ &= \frac{3}{8} \frac{A^2}{\lambda^4} = \frac{3}{2\lambda}. \end{aligned}$$

$\boxed{3, \text{B}}$

2. **The principles of QM - a three level system**

Consider a system on the Hilbert space $\mathbb{C}^3$ spanned by the orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$. The Hamiltonian $\hat{H}$ and another operator $\hat{A}$ are given as

$$\hat{H} = E_0|\phi_1\rangle\langle\phi_1| + \frac{\mathrm{i}E_1}{2}\left(|\phi_2\rangle\langle\phi_3| - |\phi_3\rangle\langle\phi_2|\right),$$

and

$$\hat{A} = a\left(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\right),$$

where E_0, E_1 and a are real constants. sim. seen ⇓

(a) Verify that the vectors

$$|\chi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\phi_2\rangle \pm \mathrm{i}|\phi_3\rangle\right)$$

are eigenvectors of the Hamiltonian $\hat{H}$ and deduce the corresponding eigenvalues.

Solution: To verify this, we use that $\langle\phi_j|\phi_k\rangle = \delta_{jk}$ and calculate

$$\begin{aligned}\hat{H}|\chi_+\rangle &= \mathrm{i}\frac{E_1}{2}\frac{1}{\sqrt{2}}\left(\mathrm{i}|\phi_2\rangle - |\phi_3\rangle\right)\\ &= -\frac{E_1}{2}\frac{1}{\sqrt{2}}\left(|\phi_2\rangle + \mathrm{i}|\phi_3\rangle\right) = -\frac{E_1}{2}|\chi_+\rangle,\end{aligned}$$

and

$$\begin{aligned}\hat{H}|\chi_-\rangle &= \mathrm{i}\frac{E_1}{2}\frac{1}{\sqrt{2}}\left(-\mathrm{i}|\phi_2\rangle - |\phi_3\rangle\right)\\ &= -\frac{E_1}{2}\frac{1}{\sqrt{2}}\left(|\phi_2\rangle - \mathrm{i}|\phi_3\rangle\right) = \frac{E_1}{2}|\chi_-\rangle.\end{aligned}$$

That is, $|\chi_+\rangle$ is an eigenvector of $\hat{H}$ belonging to the eigenvalue $-\frac{E_1}{2}$, and $|\chi_-\rangle$ is an eigenvector of $\hat{H}$ belonging to the eigenvalue $+\frac{E_1}{2}$. 5, B

(b) Assume that at some specific time t_0 the system is in the state

$$|\psi\rangle = |\chi_+\rangle.$$

sim. seen ⇓

(i) With what probability does a measurement of the energy $\hat{H}$ return which result and what is the expectation value of the energy?

Solution: Since $|\chi_+\rangle$ is an eigenstate of the Hamiltonian, an energy measurement yields the corresponding eigenvalue with probability one. That is, the energy is measured to be $-\frac{E_1}{2}$ with probability one and the expectation value is also given by $-\frac{E_1}{2}$. 3, A

unseen ⇓

(ii) With what probability does a measurement of the observable $\hat{A}$ return which result and what is the expectation value of $\hat{A}$?

Solution: The possible measurement outcomes are a and 0. The probability of measuring 0 is given by

$$P(0) = |\langle\phi_3|\psi\rangle|^2 = \frac{1}{2}.$$

2, D

The probability of obtaining the outcome a is given by

$$P(a) = |\langle\phi_1|\psi\rangle|^2 + |\langle\phi_2|\psi\rangle|^2 = \frac{1}{2}.$$

$\boxed{2, \mathrm{C}}$

The expectation value of $\hat{A}$ is thus found as

$$\langle\hat{A}\rangle = a \times P(a) + 0 \times P(0) = \frac{a}{2}.$$

$\boxed{1, \mathrm{A}}$

$\boxed{\text{unseen} \Downarrow}$

(iii) Assume now that a measurement of the observable $\hat{A}$ at the specific time t_0 yields the result a. What is the probability that a subsequent measurement of the energy at a later time $t > t_0$ yields the result E_0?

Solution: If the measurement of $\hat{A}$ yields the outcome a, this collapses the state to the subspace spanned by $|\phi_1\rangle$ and $|\phi_2\rangle$. Since the initial state had no contribution in $|\phi_1\rangle$ direction, the state directly after the measurement is given by

$$|\psi(t_0^+)\rangle = |\phi_2\rangle.$$

$\boxed{3, \mathrm{D}}$

Using the method of stationary states writing $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|\chi_+\rangle + |\chi_-\rangle)$, the time evolved state is then found as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\frac{E_1}{2}\frac{t-t_0}{\hbar}}|\chi_+\rangle + \mathrm{e}^{-\mathrm{i}\frac{E_1}{2}\frac{t-t_0}{\hbar}}|\chi_-\rangle).$$

Since there is no contribution of $|\phi_1\rangle$ the probability to measure E_0 is zero for all times. $\boxed{4, \mathrm{B}}$

3. **Energy quantisation in a non-symmetric infinite well**

Consider a potential of the form

$$V(x) = \begin{cases} \infty, & x \leq -L \\ 0, & -L < x \leq 0 \\ V_0, & 0 < x \leq L \\ \infty, & L < x, \end{cases}$$

sim. seen $\Downarrow$

where V_0 and L are real and positive constants.

(a) What is the range of energies for possible bound states?

Solution: The energy has to be above the minimum of the potential, i.e. $E > 0$. 2, A

(b) What boundary conditions does an energy eigenfunction have to fulfill at $x = \pm L$ and $x = 0$?

Solution: $\phi(x)$ is continuous everywhere, we have $\phi(\pm L) = 0$ and the first derivative is continuous at $\phi(x = 0)$. 4, A

(c) Write down an ansatz for possible bound states for the energy region $E < V_0$ using the known form of solutions of the time-independent Schrödinger equation in the different spatial regions.

Solution: We have

$$\phi_E(x) = \begin{cases} 0, & x \leq -L \\ A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx}, & -L < x \leq 0 \\ C\mathrm{e}^{\kappa x} + D\mathrm{e}^{-\kappa x}, & 0 \leq x < L \\ 0, & L \leq x, \end{cases}$$

with $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$.

4, A

(d) (i) Use the boundary conditions between the different regions to derive a quantisation condition for the energies of bound states with $E < V_0$ of the form

$$\sqrt{\frac{c^2}{z} - 1} = f(z),$$

in terms of the variable $z = \frac{L}{\hbar}\sqrt{2mE}$, where c is a constant depending on V_0, m, L, and $\hbar$.

Solution: We have

$$\phi_E(-L) = A\mathrm{e}^{-\mathrm{i}kL} + B\mathrm{e}^{\mathrm{i}kL} = 0,$$

that is

$$B = -A\mathrm{e}^{-2\mathrm{i}kL},$$

and

$$\phi(L) = C\mathrm{e}^{\kappa L} + D\mathrm{e}^{-\kappa L} = 0,$$

that is

$$D = -C\mathrm{e}^{2\kappa L}.$$

At $x = 0$ we have

$$\phi(0) = A + B = C + D,$$

and

$$\phi'(0) = \mathrm{i}k(A - B) = \kappa(C - D).$$

Inserting B in terms of A, and D in terms of C into the boundary conditions at $x = 0$ we find $\boxed{3,\ \mathrm{B}}$

$$\begin{aligned} A(1 - \mathrm{e}^{-2\mathrm{i}kL}) &= C(1 - \mathrm{e}^{2\kappa L}) \\ \mathrm{i}kA(1 + \mathrm{e}^{-2\mathrm{i}kL}) &= \kappa C(1 + \mathrm{e}^{2\kappa L}). \end{aligned}$$

Dividing the second equation by the first, this can be rewritten as

$$k\cot(kL) = -\kappa\coth(\kappa L).$$

Now we rewrite in terms of $z = Lk$ using

$$\kappa = \sqrt{\frac{2mV_0}{\hbar^2} - k^2} = \frac{1}{L}\sqrt{\frac{2mV_0L^2}{\hbar^2} - z^2}$$

as

$$\sqrt{\frac{c^2}{z^2} - 1} = -\frac{\cot(z)}{\coth(\sqrt{c^2 - z^2})},$$

with

$$c = \frac{\sqrt{2mV_0}\,L}{\hbar}$$

$\boxed{4,\ \mathrm{C}}$

(ii) Is there a non-zero minimum value of V_0 such that an energy eigenstate with $E < V_0$ exists? $\boxed{\text{unseen} \Downarrow}$

Solution: Yes, there is a minimum value of V_0 below which no such state exists. This can be seen by a graphic analysis of the quantisation condition. The right hand side is negative for $z < \frac{\pi}{2}$, while the left hand side is positive and vanishes at $z = c$. Thus, only for $c > \frac{\pi}{2}$ is there a bound state with $0 < E < V_0$. $\boxed{3,\ \mathrm{D}}$

4. **The quantum harmonic oscillator**

Consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right),$$

with the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, fulfilling $[\hat{a}, \hat{a}^\dagger] = 1$, and the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. The number operator and the ladder operators fulfil the commutation relations

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad \text{and} \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

(a) Let $|\nu\rangle$ denote an eigenvector of $\hat{N}$ with corresponding eigenvalue ν.

(i) Prove that the eigenvalues of $\hat{N}$ are non-negative, and that $\hat{a}|\nu\rangle$ is the zero vector if and only if $\nu = 0$. Do this by considering the norm of the vector $\hat{a}|\nu\rangle$.
Solution: We have

$$||\hat{a}|\nu\rangle||^2 = \langle\nu|\hat{a}^\dagger\hat{a}|\nu\rangle = \nu\langle\nu|\nu\rangle \geq 0.$$

Since the state $|\nu\rangle$ is not the zero vector (as it is an eigenvector of $\hat{N}$) we have $\nu \geq 0$. Further, for $\nu = 0$ we have that $||\hat{a}|\nu\rangle||^2 = 0$, which means that $\hat{a}|0\rangle$ is the zero vector. Also note that $\hat{a}|\nu\rangle$ can be the zero vector for no other value of ν, since it has non-zero norm. $\boxed{3, \text{A}}$

(ii) Prove that $\hat{a}|\nu\rangle$ is either the zero vector or an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu - 1$.
Solution: We consider $\hat{N}\hat{a}|\nu\rangle$:

$$\begin{aligned}
\hat{N}\hat{a}|\nu\rangle &= \hat{N}\hat{a}|\nu\rangle - \hat{a}\hat{N}|\nu\rangle + \hat{a}\hat{N}|\nu\rangle \\
&= [\hat{N}, \hat{a}]|\nu\rangle + \hat{a}\hat{N}|\nu\rangle \\
&= -\hat{a}|\nu\rangle + \nu\hat{a}|\nu\rangle \\
&= (\nu - 1)\hat{a}|\nu\rangle
\end{aligned}$$

That is either $\hat{a}|\nu\rangle$ is the zero vector or it is an eigenvector of $\hat{N}$ with eigenvalue $\nu - 1$. $\boxed{3, \text{A}}$

(iii) Prove that $\hat{a}^\dagger|\nu\rangle$ is an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu + 1$.
Solution: We consider $\hat{N}\hat{a}^\dagger|\nu\rangle$:

$$\begin{aligned}
\hat{N}\hat{a}^\dagger|\nu\rangle &= [\hat{N}, \hat{a}^\dagger]|\nu\rangle + \hat{a}^\dagger\hat{N}|\nu\rangle \\
&= \hat{a}^\dagger|\nu\rangle + \nu\hat{a}^\dagger|\nu\rangle \\
&= (\nu + 1)\hat{a}^\dagger|\nu\rangle
\end{aligned}$$

That is either $\hat{a}^\dagger|\nu\rangle$ is the zero vector or it is an eigenvector of $\hat{N}$ with eigenvalue $\nu + 1$. $\boxed{2, \text{A}}$

From the above we can conclude that the eigenstates of the harmonic oscillator Hamiltonian are a set of orthonormal states $|n\rangle$ with non-negative integers n, belonging to the energy eigenvalues $\hbar\omega(n + \frac{1}{2})$.

(b) There is a non-degenerate ground state $|0\rangle$ with $\hat{N}|0\rangle = 0$. Show that from the non-degeneray of the ground state it follows that all states $|n\rangle$ are non-degenerate.

Solution: Assume that the eigenvalue n is non-degenerate, but there are two linearly independent eigenstates $|\phi_{n+1}\rangle$ and $|\chi_{n+1}\rangle$ belonging to the eigenvalue $n+1$. Applying the lowering operator to these two states we can produce the states $|\phi_n\rangle = \hat{a}|\phi_{n+1}\rangle$ and $|\chi_n\rangle = \hat{a}|\chi_{n+1}\rangle$. Now, since the eigenvalue n is non-degenerate these two vectors have to be multiples of each other, i.e. there exists a $\lambda \in \mathbb{C}$ such that

$$|\phi_n\rangle = \lambda|\chi_n\rangle. \tag{1}$$

Now acting on this relation with the raising operator $\hat{a}^\dagger$ yields:

$$\hat{a}^\dagger|\phi_n\rangle = \hat{a}^\dagger\hat{a}|\phi_{n+1}\rangle = \hat{N}|\phi_{n+1}\rangle = \lambda\hat{a}^\dagger|\chi_n\rangle = \lambda\hat{a}^\dagger\hat{a}|\chi_{n+1}\rangle = \lambda\hat{N}|\chi_{n+1}\rangle, \tag{2}$$

that is $\hat{N}|\phi_{n+1}\rangle = \lambda\hat{N}|\chi_{n+1}\rangle$. But since $|\phi_{n+1}\rangle$ and $|\chi_{n+1}\rangle$ are eigenstates of $\hat{N}$ with eigenvalue $(n+1)$, we have

$$|\phi_{n+1}\rangle = \lambda|\chi_{n+1}\rangle, \tag{3}$$

which means the two assumed to be linearly independent eigenvectors are linearly dependent after all, and the eigenvalue $n+1$ is also non-degenerate. Thus, starting from $n=0$ we have shown that the eigenvalues are non-degenerate. $\square$

$\boxed{4, \mathrm{B}}$

(c) Assume that the system at a given time t_0 is prepared in the state $|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle - \mathrm{i}|3\rangle\right)$. What is the expectation value of the rescaled position operator $\hat{Q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ at a later time $t > t_0$?

Solution: The time-evolved state is given by

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\frac{5}{2}\omega(t-t_0)}|2\rangle - \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{7}{2}\omega(t-t_0)}|3\rangle) \\ &= \frac{\mathrm{e}^{-\mathrm{i}\frac{5}{2}\omega(t-t_0)}}{\sqrt{2}}(|2\rangle - \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}|3\rangle)\end{aligned}$$

We thus have $\boxed{2, \mathrm{C}}$

$$\begin{aligned}\langle\psi(t)|\hat{Q}|\psi(t)\rangle &= \frac{1}{2\sqrt{2}}\left(\langle 2| + \mathrm{i}\mathrm{e}^{\mathrm{i}\omega(t-t_0)}\langle 3|\right)(\hat{a} + \hat{a}^\dagger)\left(|2\rangle - \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}|3\rangle\right) \\ &= \frac{1}{2\sqrt{2}}\left(\langle 2| + \mathrm{i}\mathrm{e}^{\mathrm{i}\omega(t-t_0)}\langle 3|\right)\left(\sqrt{2}|1\rangle + \sqrt{3}|3\rangle - \sqrt{3}\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}|2\rangle - \sqrt{4}\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}|4\rangle\right) \\ &= \frac{1}{2\sqrt{2}}\left(-\sqrt{3}\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)} + \sqrt{3}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega(t-t_0)}\right) \\ &= \sqrt{\frac{3}{2}}\sin(\omega(t-t_0)).\end{aligned}$$

$\boxed{6, \mathrm{D}}$

5. **Mastery: Entanglement**

(a) (i) What are the names conventionally given to the two observers A and B in quantum information protocols?
Solution: Alice and Bob $\boxed{2, \text{M}}$

(ii) What is the difference between the CHSH (Clauser-Horne-Shimony-Holt) setup as compared to Bohm's version of the EPR paradox?
Solution: In the Bohm version of the EPR paradox both observers measure the spin in one specific direction, while in the CHSH setup both observers can choose between two different directions for the spin measurement. $\boxed{2, \text{M}}$

(iii) What does the no-cloning theorem state?
Solution: The no-cloning theorem states that there is no unitary operator that can create a copy of an arbitrary state while maintaining the original. $\boxed{2, \text{M}}$

(b) Consider the two-particle spin state

$$|\psi\rangle = N(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$

where the single-particle states $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the single-particle $\hat{S}_z$ operator

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with eigenvalues $\pm\frac{\hbar}{2}$. That is,

$$\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle, \quad \text{and} \quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle.$$

(i) For which value of N is the state $|\psi\rangle$ normalised to one?
Solution: The state is normalised for $N = \frac{1}{\sqrt{2}}$. $\boxed{2, \text{M}}$

(ii) Assume A measures the z-component of the first particle and obtains the outcome $\frac{\hbar}{2}$. What is the state after this measurement? With what probability does B obtain which outcome in a measurement of the z-component of the spin of the second particle?
Solution: The measurement of the z-component of the spin of the first particle with outcome $\frac{\hbar}{2}$ collapses the state to

$$|\uparrow\rangle|\downarrow\rangle.$$

Thus, B will obtain the outcome $-\frac{\hbar}{2}$ with probability one. $\boxed{3, \text{M}}$

(c) The x-component of the single particle spin is described by the operator

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Express the eigenvectors of the x-component of the single particle spin operator, $|\rightarrow\rangle$ and $|\leftarrow\rangle$, with $\hat{S}|\rightarrow\rangle = \frac{\hbar}{2}|\rightarrow\rangle$ and $\hat{S}_x|\leftarrow\rangle = -\frac{\hbar}{2}|\leftarrow\rangle$, in terms of the eigenstates of the z component.

Solution: We can write the eigenstates of the $\hat{S}_x$ operator as a superposition of the eigenstates of the $\hat{S}_z$ as

$$|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)$$

and

$$|{\leftarrow}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle),$$

with

$$\hat{S}_x|{\rightarrow}\rangle = \frac{\hbar}{2}|{\rightarrow}\rangle, \quad \text{and} \quad \hat{S}_x|{\leftarrow}\rangle = -\frac{\hbar}{2}|{\leftarrow}\rangle.$$

$\boxed{2, \text{M}}$

(ii) Express the state

$$|\psi\rangle = N(|{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle),$$

In terms of $|{\rightarrow}\rangle$ and $|{\leftarrow}\rangle$.
Solution: We have

$$|{\uparrow}\rangle = \frac{1}{\sqrt{2}}(|{\rightarrow}\rangle + |{\leftarrow}\rangle),$$

and

$$|{\downarrow}\rangle = \frac{1}{\sqrt{2}}(|{\rightarrow}\rangle - |{\leftarrow}\rangle),$$

and thus

$$|{\uparrow}\rangle|{\downarrow}\rangle = \frac{1}{2}(|{\rightarrow}\rangle|{\rightarrow}\rangle - |{\rightarrow}\rangle|{\leftarrow}\rangle + |{\leftarrow}\rangle|{\rightarrow}\rangle - |{\leftarrow}\rangle|{\leftarrow}\rangle)$$

and

$$|{\downarrow}\rangle|{\uparrow}\rangle = \frac{1}{2}(|{\rightarrow}\rangle|{\rightarrow}\rangle + |{\rightarrow}\rangle|{\leftarrow}\rangle - |{\leftarrow}\rangle|{\rightarrow}\rangle - |{\leftarrow}\rangle|{\leftarrow}\rangle)$$

That is,

$$|\psi\rangle = N(|{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle) = N(|{\leftarrow}\rangle|{\rightarrow}\rangle - |{\rightarrow}\rangle|{\leftarrow}\rangle).$$

$\boxed{3, \text{M}}$

(iii) Assume that the initial state is again given by

$$|\psi\rangle = N(|{\uparrow}\rangle|{\downarrow}\rangle - |{\downarrow}\rangle|{\uparrow}\rangle),$$

and A measures the x-component of the first particle, obtaining the outcome $\frac{\hbar}{2}$. With what probability does B obtain which outcome in a measurement of the z-component of the spin of the second particle?
Solution: Using the result from part (ii), if the outcome of A's measurement is $\frac{\hbar}{2}$ the state collapses to

$$|{\rightarrow}\rangle|{\leftarrow}\rangle.$$

Thus, the state of the second particle is

$$|{\leftarrow}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle),$$

which means that in a measurement of the z-component of the second particle the possible outcomes are $\pm\frac{\hbar}{2}$, both are obtained with probability $\frac{1}{2}$. $\boxed{4, \text{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 22 of 20 marks
Total C marks: 10 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60015/70015	1	No Comments Received
MATH60015/70015	2	No Comments Received
MATH60015/70015	3	No Comments Received
MATH60015/70015	4	No Comments Received
MATH70015	5	No Comments Received