MATH50010 – Autumn 2021 – Midterm

You should state carefully any results from lectures that are used, and justify briefly why they are applicable.

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \Pr)$.

(a) (1 mark) State a necessary and sufficient condition in terms of subsets of Ω of the form $\{X \leq x\}$ for the function $X : \Omega \to \mathbf{R}$ to be a random variable with respect to $\mathcal{F}$.

X is a a random variable if and only if $\{X \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbf{R}$.

(b) (2 marks) Show that if F_X is the cumulative distribution function of a random variable X, then $\lim_{x\to-\infty} F_X(x) = 0$.

Take any sequence $(x_n)_{n\geq 1}$ such that $x_n \downarrow -\infty$. Define the decreasing sequence of events $A_n = \{X \leq x_n\}$.

Then, by the continuity property of $\Pr$ on decreasing sequences of events,

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \Pr(A_n) = \Pr\left(\lim_{n\to\infty} \bigcap_{i=1}^{n} A_i\right) = \Pr(\emptyset) = 0.$$

(c) (2 marks) Show that if X and Y are random variables with respect to $\mathcal{F}$, then so is $Z = \max\{X, Y\}$.

By part (a), it is enough to show that $\{\omega \in \Omega : Z(\omega) \leq z\} \in \mathcal{F}$ for all $z \in \mathbf{R}$.

Note that $\max(X, Y) \leq z$ if and only if both $X \leq x$ and $Y \leq y$. Hence we have the equality of events

$$\{\omega \in \Omega : Z(\omega) \leq z\} = \{\omega \in \Omega : X(\omega) \leq z\} \cap \{\omega \in \Omega : Y(\omega) \leq z\}.$$

As X and Y are random variables, we see that $\{\omega \in \Omega : X(\omega) \leq z\} \in \mathcal{F}$ and $\{\omega \in \Omega : Y(\omega) \leq z\} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under intersections, we have shown that W is a random variable.

In the remainder of the question, let X be an absolutely continuous random variable with probability density function given by

$$f_X(x) = nx^{n-1}, \quad \text{for} \quad 0 < x < 1,$$

and zero otherwise, where $n \in \{1, 2, \ldots\}$.

(d) (1 mark) Write down the cumulative distribution function of X.

For $x \in (0, 1)$,

$$F_X(x) = \int_0^x nt^{n-1}\,dt = x^n,$$

so that

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^n & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

(e) (3 marks) Determine the probability density function of the random variable $Y = \frac{X}{1+X}$.

$Y = g(X) = \frac{X}{1+X}$, so that the inverse function is given by $X = g^{-1}(Y) = \frac{Y}{1-Y}$.

Note that g is a monotone, and therefore one-one function,. The derivative of g^{-1} is given by

$$\frac{dg^{-1}}{dy} = \frac{d}{dy}\left(-1 + \frac{1}{1-y}\right) = \frac{1}{(1-y)^2} > 0,$$

which is continuous on the domain of g^{-1}, which is $(0, \frac{1}{2})$.

The theorem for univariate transformations then gives

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dg}{dy}\right| = n\left(\frac{y}{1-y}\right)^{n-1} \frac{1}{(1-y)^2} = \frac{ny^{n-1}}{(1-y)^{n+1}}, \qquad y \in \left(0, \frac{1}{2}\right).$$

(f) (4 marks) Show that X has the same distribution as $\max\{U_1, U_2, \ldots, U_n\}$, where the random variables $U_i \sim \textsc{Uniform}(0,1)$ are independent.

The CDF of any of the uniform random variables is $F_U(u) = u$ for $u \in (0,1)$. As in (c), for $x \in (0,1)$ we have the equality of events

$$\{\max\{U_1, U_2, \ldots, U_n\} \leq x\} = \bigcap_{i=1}^{n} \{U_i \leq x\}.$$

Using the independence of the U_i, we see that

$$\Pr\left(\bigcap_{i=1}^{n} \{U_i \leq x\}\right) = \prod_{i=1}^{n} \Pr\left(U_i \leq x\right) = x^n.$$

As $\max\{U_1, U_2, \ldots, U_n\}$ has the same cumulative distribution function as X, we conclude that the random variables are identically distributed.

(g) (4 marks) Find the covariance between the random variables $V = X^p$ and $W = X^q$, where $p, q \geq 1$.

$$\mathrm{Cov}(X^p, X^q) = \mathrm{E}(X^{p+q}) - \mathrm{E}(X^p)\mathrm{E}(X^q).$$

Hence need to calculate

$$E(X^m) = \int_{-\infty}^{\infty} x^m f_X(x)\, dx = \int_0^1 x^m n x^{n-1}\, dx = \left[\frac{nx^{m+n}}{m+n}\right]_0^1 = \frac{n}{m+n}.$$

Then

$$\begin{aligned}\mathrm{Cov}(X^p, X^q) &= \frac{n}{p+q+n} - \frac{n}{(p+n)}\frac{n}{(q+n)} = \frac{n\left[(p+n)(q+n) - n(p+q+n)\right]}{(p+q+n)(p+n)(q+n)} \\ &= \frac{npq}{(p+q+n)(p+n)(q+n)}.\end{aligned}$$

(h) (3 marks) Find the monotonic *decreasing* function H such that the random variable T, defined by $T = H(X)$, has a probability density function that is constant on the interval $(0,1)$, and zero otherwise.

To find the decreasing function H on $(0,1)$; need $F_T(t) = t,\ 0 < t < 1$, that is, need

$$\begin{aligned}\Pr(T \le t) &= \Pr[\ H(X) \le t\ ] = t \Longrightarrow \Pr[\ X \ge H^{-1}(t)\ ] = t \Longrightarrow 1 - \Pr[\ X < H^{-1}(t)\ ] = t \\ &\Longrightarrow \left\{H^{-1}(t)\right\}^n = 1 - t \textit{ and hence } H(x) = 1 - x^n.\end{aligned}$$