**Linear Algebra** **Unseen Question 8**

In this project we will prove Theorem 16.2 of the lecture notes (an interesting result which I did not prove). In case you haven't looked at this week's notes yet, we need a couple of definitions to get started. If $V$ is a vector space over a field $F$, we say a map $(\ ,\ ) : V \times V \mapsto F$ is a *bilinear form* if is linear on both sides, ie. left-linear: $(\alpha v_1 + \beta v_2, w) = \alpha(v_1, w) + \beta(v_2, w)$, and similarly right-linear. (Eg. an inner product on $\mathbb{R}^n$ is bilinear, but not one on $\mathbb{C}^n$.)

In the lectures we study two main types of bilinear form:

*symmetric*: $(v, u) = (u, v)$ for all $u, v \in V$; and

*skew-symmetric*: $(v, u) = -(u, v)$ for all $u, v \in V$.

This is "justified" by the theorem in question, 16.2. This result concerns the orthogonality condition

$$(v, w) = 0 \quad \Leftrightarrow \quad (w, v) = 0 \quad \text{for all } v, w \in V. \qquad (*)$$

**Theorem 16.1** *A bilinear form satisfies condition* $(*)$ *iff it is symmetric or skew-symmetric.*

The right-to-left implication in the theorem is obvious: symmetric and skew-symmetric bilinear forms satisfy (*). For the converse, here are the steps in the proof for you to carry out.

(1) Show that if $(v, v) = 0$ for all $v \in V$ then $(\ ,\ )$ is skew-symmetric. (Hint: consider $(u + v, u + v)$.)

**From now on, suppose** $(\ ,\ )$ **satisfies (*).**

(2) Let $u, v, w \in V$ and $x = (w, u)v - (v, u)w$. Show $(x, u) = 0$, and deduce using (*) that

$$(w, u)(u, v) = (u, w)(v, u).$$

Putting $w = u$, deduce that

$$(u, v) \neq (v, u) \Rightarrow (u, u) = (v, v) = 0.$$

(3) Now suppose that $(\ ,)$ is *not symmetric*. So $\exists u_0, v_0$ such that $(u_0, v_0) \neq (v_0, u_0)$, hence by (2), also $(u_0, u_0) = (v_0, v_0) = 0$.

We aim to prove that $(w, w) = 0$ for all $w \in V$ (which implies $(\ ,\ )$ skew-symmetric by (1)). So assume for a contradiction that there exists $w$ such that $(w, w) \neq 0$.

(4) Using (2), show $(u_0, w) = (w, u_0)$ and $(v_0, w) = (w, v_0)$.

(5) In (2), let $u = u_0, v = v_0$ and deduce that $(u_0, w)\left((u_0, v_0) - (v_0, u_0)\right) = 0$. Hence show

$$(u_0, w) = (w, u_0) = (v_0, w) = (w, v_0) = 0.$$

(6) Now apply the implication in (2) with $u = v_0 + w, v = u_0$ to show that $(v_0 + w, v_0 + w) = 0$.

(7) Deduce finally that $(w, w) = 0$, as required!!