Consider The problem

$$\min f(\underline{x})$$

$$\text{s.t.} \quad \underbrace{\underline{e}^T \cdot \underline{x}}_{x_1 + x_2 + \dots + x_n = 1} = 1, \quad \underline{x} \in \mathbb{R}^n$$

$$\underbrace{[1\ 1\ 1\ 1\ 1 \dots]}_{n}$$

$$\sum_{i=1}^{n} x_i = 1$$

where f is a continuously differentiable function. Show that $\underline{x}^*$ is a stationary point $\boxed{\text{if and only if}}$

$$\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$$

Answer: iif

Part (I): Stationary point $\Longrightarrow \frac{\partial}{\partial x_1} f(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$

Part (II): $\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots \frac{\partial f}{\partial x_n}(\underline{x}^*) \Rightarrow \underline{x}^*$ is stationary.

We begin showing part II:

Stationarity def.

$$\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*) \implies \boxed{\nabla f(\underline{x}^*)^T(\underline{x} - \underline{x}^*) \geq 0}$$

$\forall\ \underline{x}$ such that $\underline{e}^T \cdot \underline{x} = 1$

$$\nabla f(\underline{x}^*)^T \cdot (\underline{x} - \underline{x}^*) \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}^*)(x_i - x_i^*),$$

but recall

$$\overset{\sim}{=} \frac{\partial f}{\partial x_1}(\underline{x}^*) \sum_{i=1}^{n}(x_i - x_i^*)$$

$$= \frac{\partial f}{\partial x_1}(\underline{x}^*) \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^*\right)$$

but, $\underline{x}$ and $\underline{x}^*$ are such that $\underbrace{\underline{e}^T \cdot \underline{x}}_{x_1 + \dots + x_n}{}^{\text{or } \underline{x}^*} = 1$

$$\sum_{i=1}^{n} x_i = 1$$

$$= \frac{\partial f}{\partial x_1}(\underline{x}^*)\,(1 - 1) = 0 \geq 0 \quad \blacksquare$$

Part (I): Assume $\underline{x}^*$ such that $\nabla f(\underline{x}^*)^T(\underline{x} - \underline{x}^*) \geq 0$

$\forall\ \underline{x}$ s.t. $\underset{(n)}{\underline{e}}^T \cdot \underline{x} = 0$

$\sum_1 x_i = 0$

$$\Longrightarrow \frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$$

By Contradiction. Assume $\underline{x}^*$ stationary, but

where $\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$ does NOT hold,

or equivalently, that there exist two indexes

$i, j,\ i \neq j$ such that $\frac{\partial f}{\partial x_i}(\underline{x}^*) \neq \frac{\partial f}{\partial x_j}(\underline{x}^*)$

or $$\frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$$

We construct the following vector from the stationary point

$$x_k = \begin{cases} x_k^* & k \notin \{i, j\} \\ x_i^* - 1 & k = i \\ x_j^* + 1 & k = j \end{cases}$$

$$\underline{x}^* = [x_1^* \dots x_i^* \dots x_j^* \dots x_n^*] \qquad \frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$$

$$\underline{x} = [x_1^* \dots x_i^* - 1 \dots x_j^* + 1 \dots x_n^*]$$

Note that if $\underline{x}^*$ is such that $e^T \underline{x}^* = 1$

$\Longrightarrow$ $\underline{x}$ is such that $e^T \underline{x} = 1$

$\Rightarrow \underline{x}$ is feasible (inside the constraint set)

But then,

$$\nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) = \frac{\partial f}{\partial x_i}(\underline{x}^*)\underbrace{(x_i - x_i^*)}_{-1} \quad = x_i^* - 1 - x_i^*$$

$$+ \frac{\partial f}{\partial x_j}(\underline{x}^*)\underbrace{(x_j - x_j^*)}_{1} \quad = x_j^* + 1 - x_j^*$$

$$+ \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \frac{\partial f}{\partial x_k}(\underline{x}^*)\underbrace{(x_k - x_k^*)}_{0} \quad x_k = x_k^*$$

but recall the construction of x_k

$$\Longrightarrow \nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) = \frac{\partial f}{\partial x_i}(\underline{x}^*)(-1)$$

$$+ \frac{\partial f}{\partial x_j}(\underline{x}^*)(1)$$

$$+ \ 0$$

$$= \frac{\partial f}{\partial x_j}(\underline{x}^*) - \frac{\partial f}{\partial x_i}(\underline{x}^*) < 0$$

because we assumed $\frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$

$\Longrightarrow$ This contradicts the hypothesis that $\underline{x}^*$

is a stationary point such that

$$\nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) \geq 0 \quad \forall \underline{x}$$

such that

$$\underline{e}^T \cdot \underline{x} = 1.$$