MATH50010: Probability for Statistics
Problem Sheet 5

1. In question 6 of Problem Sheet 4, you derived the cdfs of a number of random variables involving the minimum or maximum of a random sample. In this problem we will derive the limiting distribution (i.e. the distribution to which it converges in distribution), if it exists, of these same random variables.

   Suppose $(X_1, \ldots, X_n)$ is a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with pmf/pdf f_X and cdf F_X, let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from $X_1, \ldots, X_n$.

   (a) Suppose $X_1, \ldots, X_n \sim \text{Unif}(0, 1)$, that is
   $$F_X(x) = x, \text{ for } 0 \leq x \leq 1.$$
   Find the limiting distributions of Y_n and Z_n as $n \longrightarrow \infty$.

   (b) Suppose $X_1, \ldots, X_n$ have cdf
   $$F_X(x) = 1 - x^{-1}, \text{ for } x \geq 1.$$
   Find the limiting distributions of Z_n and $U_n = Z_n^n$ as $n \longrightarrow \infty$.

   (c) Suppose $X_1, \ldots, X_n$ have cdf
   $$F_X(x) = \frac{1}{1 + e^{-x}}, \text{ for } x \in \mathbb{R}.$$
   Find the limiting distributions of Y_n and $U_n = Y_n - \log n$, as $n \longrightarrow \infty$.

   (d) Suppose $X_1, \ldots, X_n$ have cdf
   $$F_X(x) = 1 - \frac{1}{1 + \lambda x}, \text{ for } x > 0.$$
   Let $U_n = Y_n/n$ and $V_n = nZ_n$. Find the limiting distributions of Y_n, Z_n, U_n, and V_n as $n \longrightarrow \infty$.

2. Show that if $X_1, \ldots, X_n$ are a sequence of random variables such that $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$ for X with cdf continuous on $\mathbb{R}$, then for $a, b > 0$, $aX_n + b \xrightarrow{\mathcal{D}} aX + b$.

3. (a) Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Show that $X_n + Y_n \xrightarrow{P} X + Y$. Does a similar result hold for convergence in distribution?

   (b) Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Show that $X_n Y_n \xrightarrow{P} XY$. Does a similar result hold for convergence in distribution?

4. (a) Show that if $X_1, X_2, \ldots$ are a sequence of random variable such that $X_n \xrightarrow{P} X$ as $n \to \infty$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous deterministic function, then $g(X_n) \xrightarrow{P} g(X)$.

   (b) Assume the conditions of part (a), does it also hold that $g(X_n) \xrightarrow{\mathcal{D}} g(X)$?

5. Show that
$$\mathrm{E}\left[\frac{|X_n|}{1 + |X_n|}\right] \to 0 \quad \text{as } n \to \infty \qquad \text{if and only if} \qquad X_n \xrightarrow{P} 0.$$

For discussion

6. **Slutsky's Theorems**

   (a) Suppose that $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} c$ where c is a constant. Show that $X_n Y_n \xrightarrow{\mathcal{D}} cX$ and that $\frac{X_n}{Y_n} \xrightarrow{\mathcal{D}} \frac{X}{c}$ if $c \neq 0$.
   [Hint: consider the case where $|Y_n - c| > \delta$ and $|Y_n - c| \leq \delta$ for some appropriately chosen $\delta > 0$]

   (b) Suppose that $X_n \xrightarrow{\mathcal{D}} 0$ and $Y_n \xrightarrow{P} Y$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x and $g(x, y)$ is continuous at $x = 0$ for all y. Show that $g(X_n, Y_n) \xrightarrow{P} g(0, Y)$.

7. Let $X_1, X_2, \ldots$ be i.i.d. $\mathrm{N}(0, 1)$ random variables.

   (a) Show that for any $x > 0$,

   $$(x^{-1} - x^{-3})e^{-x^2/2} \leq \int_x^{\infty} e^{-y^2/2}\, dy \leq x^{-1} e^{-x^2/2}$$

   (b) Show that with probability 1,

   $$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1$$

   (c) Show that,

   $$\Pr(X_n > a_n\, i.o.) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \Pr(X_1 > a_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \Pr(X_1 > a_n) = \infty \end{cases}$$