Classical Dynamics **Autumn Term 2020**

Answers to Test 2

1. (i)

$$L(y, \dot{y}, t) = e^{\gamma t}\left(\frac{1}{2}\dot{y}^2 - gy\right),$$

so that $p = p_y = \partial L/\partial \dot{y} = e^{\gamma t}\dot{y}$ and $H = p\dot{y} - L(y, \dot{y}, t)$ with $\dot{y} = e^{-\gamma t}p$ giving

$$H = p \cdot e^{-\gamma t}p - e^{\gamma t}\left(\frac{1}{2}e^{-2\gamma t}p^2 - gy\right) = \frac{1}{2}e^{-\gamma t}p^2 + ge^{\gamma t}y.$$

(5 marks)

(ii)

$$L = \frac{1}{2}\dot{q}^2\cosh^2 q - g\cosh q,$$

where g is a positive constant.

(a) $p = \partial L/\partial \dot{q} = \dot{q}\cosh^2 q$ and $H = p\dot{q} - L(q, \dot{q})$ with $\dot{q} = p/\cosh^2 q$ giving

$$H = p \cdot \frac{p}{\cosh^2 q} - \left(\frac{1}{2}\frac{p^2}{\cosh^4 q}\cosh^2 q - g\cosh q\right) = \frac{p^2}{2\cosh^2 q} + g\cosh q,$$

as required.

Hamilton's equations:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\cosh^2 q}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = \frac{p^2\sinh q}{\cosh^3 q} - g\sinh q.$$

(7 marks)

(b) As the canonical transformation is time-independent $K = H$ so that

$$K = \frac{p^2}{2\cosh^2 q} + g\cosh q = \frac{P^2}{2} + g\sqrt{1 + \sinh^2 q} = \frac{P^2}{2} + g\sqrt{1 + Q^2}.$$

(4 marks)

(c) type 2: $p = \partial F_2/\partial q, \quad Q = \partial F_2/\partial P$

Here $Q = \sinh q$. Integrating the second equation gives $F_2 = P\sinh q$. Inserting this into the first equation $p = \partial F_2/\partial q = P\cosh q$ as required.

type 3: $q = -\partial F_3/\partial p, \quad P = -\partial F_3/\partial Q$

$q = \sinh^{-1} Q$. Integrating the first equation gives $F_3 = -p\sinh^{-1} Q$. This is consistent with the second equation since $-\partial F_3/\partial Q = p/\sqrt{1 + Q^2} = p/\cosh q$ as required.

type 1 : The idea is to assume a type 1 generating function exists and show that $\{Q, P\} = 1$ cannot be true. If a type 1 generating function exists then $P = -\partial F/\partial Q$ can be expressed as a function of q and Q, i.e. $P = f(q, Q)$. This is impossible since

$$\{Q, P\} = \{Q, f(q, Q)\} = 0,$$

using $\partial Q/\partial p = 0$.

Remark : This is a special case of the result that type 1 generating functions do not exist for *point transformations.*

Bonus question: Is there a type 4 generating function for the given canonical transformation?

(9 marks)

(Total: 25 marks)