Applied Probability Tips for Exams

When writing solutions for exams, write some justifications between equations, e.g.
- By Law of total probability
- by independence of X, Y
- By time-homogeneity
- By linearity of expectation

In person exams will focus more on the proofs from lecture notes. Unseen proofs (like the ones in Grimmett&Stirzaker) will not be assessed

Basics
N does not include 0, N_0 does

Remember where sum/definition begins!
- accessible states i → j : exists m ≥ 0 s.t. $p_{ij}(m)>0$
- T_j first hitting time: starts with time n = 1
- N_j time spent on state j: start with time n = 0
- $N_i(j)$ number of visits to i before reaching j: from time n = 1
- generating function: x = 0
- lemma 3.4.7: from l = 1 (value at l = 0 is 0, ignored)

conditional probability cannot be 0

when writing pdf, remember to include the case "0"

Remember to square constant when taken out of Var() !!!!

If E implies F, then P(E)≤ P(F)

For any positive random variable X, use rigorous definition of X = ∞ below to prove properties:

$$P(X = \infty) = \lim_{N \to \infty} P(X \geq N)$$

P(X = ∞) = 1 ⇔ E(exp(-X)) = 0

if a random variable/event consists of several independent random

variables/events
- conditioning on variable(s)/event(s) helps

try use MGF E(e^(tX)) when stuck on finding moments(e.g. mean, variance)
try use PGF when the distribution of random variable is difficult to find
- e.g. when you are trying to verify whether a process is Poisson process

NOT confuse $E(X^2)$ and Var(X)
- in particular, formulae for var of compound Poisson uses $E(X^2)$

You can use pdf of any continuous distribution to evaluate integrals, and similarly PMF of any discrete distribution to evaluate summations
- do NOT use Poisson pmf for integral calculation, because it is discrete. When you see some integrand looking like Poisson, try using Gamma function or Gamma distribution

Discrete-time Markov Chain

When calculating probabilities on Markov chain, you may have to change its form i.e. P(E) = P(F)
- Carefully check if events E, F implies each other!!
- for Poisson process, usually all events are transformed to $N_t - N_s = k$ where $0 \le s < t$
  - Common mistakes: $P(J_1 < t) \neq P(N_t = 1)$

Two conditions for P to be stochastic matrix:
- Row sum is 1 (NOT column)
- **All elements ≥ 0**

When calculating mean recurrence time, don't forget the case Ti = ∞

Recurrent state may also have infinite mean recurrence time ($\mu_i = \infty$),
- but only when card(E) = ∞.

period = n does NOT mean you are always able to return in n steps.
- But you can only return at time kn where k is some integer.

before proving properties of a new chain given by some transition probabilities, first verify it is a Markov chain!
- Time-homogenous
- Markov property

- probabilities sum to 1

closed communicating class may be transient (when |C| is infinite)

Independence of DTMC:
- If (X_n) is DTMC, $\{X_n\}$ can be mutually independent. This happens when all marginal distributions are the same (i.e. marginal distribution is the stationary distribution)
- If $\{X_n\}$ is mutually independent, then it is trivially a Markov chain

Poisson Process

In this course, using Poisson distribution to estimate binomial only requires qualitative argument on n being enormous, p being sufficiently small
- no need for $np < 5$

sum of two dependent Poisson process may not be Poisson process

the only way to show a defined Stochastic process is Poisson process is to use the definitions (verify the 4 conditions) NO SHORTCUT

When calculating probabilities for Poisson process, it is always safer to write any event using N_t, or increments $N_{s+t} - N_s$ (e.g. one event in [0, 3], two events in (3, 5])
if you are studying Z_t = something^(N_t), find Z_{t+s} / Z_t to use independence of increment

For increments like $N_{s+t} - N_s$, it is very difficult to study its properties alone, so usually the value of N_s is conditioned.

Dealing with joint distributions:
- conditioning always helps
- Transformations may be used to make the variables independent. e.g. J_i, J_{i+1} can be made independent by $U_1 = J_i$, $U_2 = J_{i+1} - J_i$, which are inter-arrival times, independent.

Non-homogeneous Poisson have independent increments, but NOT stationary increments!
- Many other properties do not hold. e.g. not memoryless (increment interval no longer exponential)

Intensity function of non-homogeneous Poisson must be **CONTINUOUS**

Compound Poisson process of N_t, Y_t may not have the same jump times as the original process N_t:
- if Yi may take value 0. e.g. Yi are Bernoulli

Continuous-time Markov Chain(CTMC)

for matrix P^Z, $[p^Z]_{ii} = 0$ unless i is absorbing state

Two ways of describing CTMC:
- jump chain (Z_n) i.e. the transition matrix P^Z & rates of exponential distribution for holding times q_i
- q_{ij} transition rates (of exponential alarm clocks)

Generator:
- g_{ii} will be non-positive! Because $p_{ii}(0) = 1$
- row sum is 0 NOT 1
  - so generator G is NOT stochastic matrix

arrows going out from states on transition diagram need not have sum 1. Because numbers on the arrows are transition rates q_{ij}, not probabilities.

for finite state Markov process, solutions to Kolmogorov forward & backward equations are both given by $P(t) = \exp(tG)$
- usually in practice, forward equation is used

useful transition from continuous to discrete:
Given CTMC X_t, you can build discrete Markov chain $Y_n := X_{t\delta}$ for some $\delta > 0$.
- Y_n is guaranteed to be aperiodic
- For two rational δ_1, δ_2, corresponding $\{n\delta_1\}$, $\{n\delta_2\}$ will intersect at infinitely many points.
- This may help to prove uniqueness. For the gaps between rationals, continuity can be used. (treat every real number as limit of rationals)

Jump chain and CTMC
jump chain is unique for given CTMC But when given a discrete MC, there are multiple ways to construct CTMC from it,

For $i \in E$, let g_i denote non-negative constants. Define

$$g_{ij} = \begin{cases} g_i p^Z_{ij}, & \text{if } i \neq j, \\ -g_i & \text{if } i = j. \end{cases}$$

Class structure, recurrence, transience of CTMC are determined by the jump chain.

Some properties of CTMC are quite extreme:

- either $p_{ij}(t) = 0$ for all t, or $p_{ij}(t) > 0$ for all t (in which case we write i→j)
- $\{X_t = i\}$ is unbounded with probability either 1 or 0.
- Explosion probability $P(J_\infty < \infty)$ is either 1 or 0.

all because holding time $H_{|i}$ can take any non-negative real value

Don't forget **initial/boundary condition** for forward, backward equations!!

Rate of birth process only depends on the current state, not on time. So birth process is time-homogeneous.