Imperial College
London

Autumn Term, 2022

Analysis 1A

Lecture 3 - Countability

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Definition

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with no repeats (all s_i distinct). Here $s_n := f(n)$.

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Intuition: Given sets A and B, the existence of a bijection $f : A \to B$ means A and B have the same "number of elements".

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More precisely, we define $f : \mathbb{N}_{>0} \to S$ inductively as follows:

- $f(1) = \min S$,

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More precisely, we define $f : \mathbb{N}_{>0} \to S$ inductively as follows:

- $f(1) = \min S$,
- Assume $f(1), \ldots, f(n-1)$ are defined already. Since S is infinite the set $S \backslash \{f(1), \ldots, f(n-1)\}$ is nonempty and so we may define

$$f(n) \; := \; \min \Big(S \backslash \{f(1), \ldots, f(n-1)\} \Big).$$

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- To show f is surjective, we argue by contradiction. If f isn't surjective, then $\exists$ smallest $s \in S \backslash \mathrm{im}(f)$. Since $s \neq \min S$ (because $f(1) = \min S$) we know $\exists s' \in S$ such that $s' < s$ – picking the largest such s', then $s' = f(n)$ for some $n \in \mathbb{N}_{>0}$ and by our rule we must have $s = f(n+1)$.

$\square$

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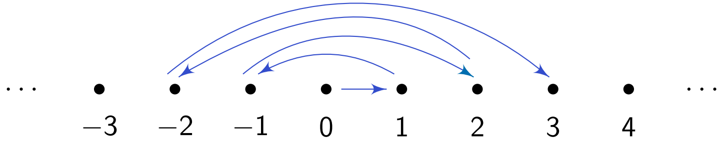

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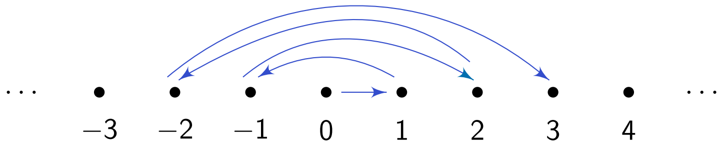

Formally, define a bijection $f : \mathbb{N}_{>0} \to \mathbb{Z}$ by declaring, for $k \geq 1$,

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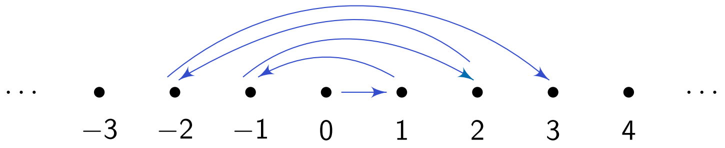

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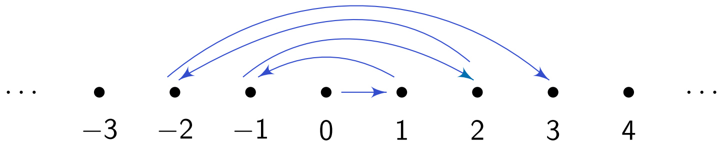

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Show similarly than A, B countable $\Rightarrow A \cup B$ countable.

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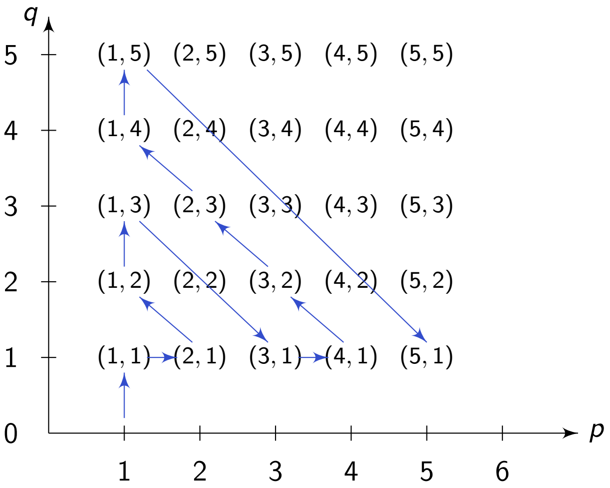

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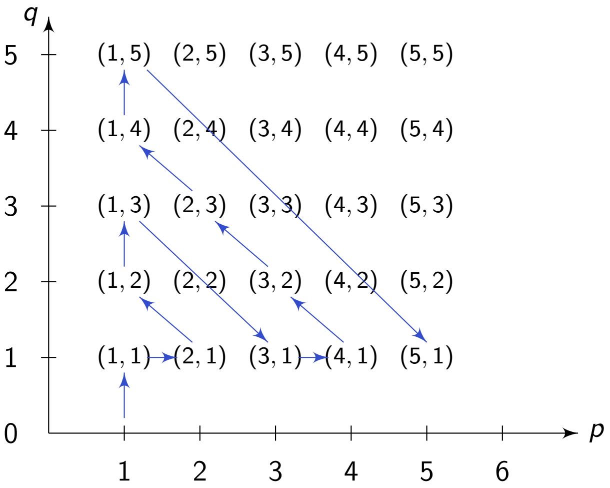

Now list the pairs according to the path shown, *missing out pairs which aren't in lowest terms*. It is not clear how to write an explicit formula for this list....

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Now we show the countability of $\mathbb{Q}$ - define a bijection $g : \mathbb{N}_{>0} \to \mathbb{Q}$ we then set

$$g(1) := 0 \quad \text{and} \quad \begin{cases} g(2k) & := F(k), \\ g(2k+1) & := -F(k). \end{cases}$$

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$$g(1) \; := \; 0 \quad \text{and} \quad \begin{cases} g(2k) & := \; F(k), \\ g(2k+1) & := \; -F(k). \end{cases}$$

That is, if $q_1, q_2, \ldots$ $(q_i := F(i))$ is our list of elements of $\mathbb{Q}_{>0}$ then our new list is $0, q_1, -q_1, q_2, -q_2, \ldots$. $\square$

Theorem 2.21

$\mathbb{R}$ is uncountable.

Proof: (Cantor's Diagonal Argument) We argue by contradiction, suppose that you can "list" all the real numbers.

We write this list as follows, using decimal expansions with no $\overline{9}$s:

$$
\begin{aligned}
x_1 &= a_1.\, a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots \\
x_2 &= a_2.\, a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots \\
x_3 &= a_3.\, a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots \\
x_4 &= a_4.\, a_{41}\, a_{42}\, a_{43}\, a_{44} \ldots \\
&\vdots \\
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As usual $a_1, a_2, a_3, \dots \in \mathbb{Z}$ and $a_{11}, a_{12}, \dots \in \{0, 1, 2, \dots, 9\}$.

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- Then $\forall i \geq 1$, we see $x \neq x_i$ because it differs in the ith decimal place $a_{i\,i}$.

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Therefore we have found an $x \in \mathbb{R}$ not on the list. $\square$

- There's a set $\mathbb{A}$ with $\mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$ called the set of *algebraic numbers*: $\mathbb{A}$ is the collection of $x \in \mathbb{R}$ which satisfy a polynomial equation $p(x) = 0$, where p has integer coefficients.

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- Any rational number p/q satisfies an equation $p(x) := qx - p = 0$, so we indeed have $\mathbb{Q} \subset \mathbb{A}$

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- Examples of transcendental numbers are e and π (proving e or π is transcendental is harder than proving $\sqrt{3}$ is irrational....).