# Statistical Theory - Problem Sheet 4

Spring 2023. Email corrections to Kolyan Ray: kolyan.ray@ic.ac.uk

**Instructions:** Please attempt the non-starred questions. If you have time, attempt the starred questions (they are not necessarily more difficult).

**1.** For $n \in \mathbb{N}$ fixed, suppose $X \sim \text{Binomial}(n, \theta)$, where $\theta \in \Theta = [0, 1]$.

(a) Consider a prior $\theta \sim \text{Beta}(a, b)$, $a, b > 0$. Show that the posterior distribution is $\text{Beta}(a + X, b + n - X)$ and compute the posterior mean $\bar{\theta}_n(X) = E(\theta|X)$.

(b) Show that the maximum likelihood estimator for $\theta$ is *not* identical to the posterior mean with uniform prior $\theta \sim U[0, 1]$.

(c) Assuming that $X$ is sampled from a fixed $\text{Binomial}(n, \theta_0)$ distribution, some $\theta_0 \in (0, 1)$, derive the asymptotic distribution of $\sqrt{n}(\bar{\theta}_n(X) - \theta_0)$ as $n \to \infty$.

**2.** Let $X_1, \ldots, X_n \sim^{iid} U(\theta, \theta + 1)$, where $\theta \in \mathbb{R}$, and assign to $\theta$ a discrete uniform prior on $1, \ldots, m$, i.e. $\Pi(\theta = k) = 1/m$ for $k = 1, \ldots, m$. Compute the posterior distribution for $\theta$. Find the Bayes estimator under squared error loss. Is it admissible?

**3.** Let $X$ be a real-valued random variable with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Consider the estimator $\hat{\mu}_{a,b} = aX + b$ of $\mu$. For each of the following cases, show that $\hat{\mu}_{a,b}$ is an inadmissible estimator of $\mu$ under squared error loss: (i) $a > 1$, (ii) $a < 0$, (iii) $a = 1$, $b \neq 0$.

**4.** Consider a general decision problem.

(a) Show that if a Bayes rule $\delta_\pi$ is unique, then it is admissible.

(b) Show that if a decision rule $\delta$ is admissible and has constant risk, then it is minimax.

**5*.** Consider an observation $X$ from a parametric model $\{f_\theta : \theta \in \Theta\}$ with prior $\pi$ on $\Theta \subseteq \mathbb{R}$ and a general risk function $R(\delta, \theta) = E_\theta L(\delta(X), \theta)$. Assume that there exists some decision rule $\delta_0$ such that $R(\delta_0, \theta) < \infty$ for all $\theta \in \Theta$.

(a) For the loss function $L(a, \theta) = |a - \theta|$, show that the Bayes rule associated to $\pi$ equals any median of the posterior distribution $\pi(\cdot|X)$.

(b) For weight function $w : \Theta \to [0, \infty)$ and loss function $L(a, \theta) = w(\theta)[a - \theta]^2$, show that the Bayes rule $\delta_\pi$ associated to $\pi$ is unique and equals

$$\delta_\pi(X) = \frac{E^\pi[w(\theta)\theta|X]}{E^\pi[w(\theta)|X]},$$

assuming that the expectations in the last ratio exist, are finite, and that $E^\pi[w(\theta)|X] > 0$.

**6.** Returning to the setup of Question **1.**, consider the posterior mean $\bar{\theta}_{n,a,b}(X) = \bar{\theta}_n(X)$ as a frequentist estimator.

(a) For squared error loss $L(u, v) = (u - v)^2$, compute the risk function $R(\bar{\theta}_n, \theta)$ and show that for some choice of the prior parameters, the risk is a constant function of $\theta \in [0, 1]$. Hence find the unique minimax estimator $\tilde{\theta}_n$ for $\theta$.

(b) Deduce that the maximum likelihood estimator $\hat{\theta}_n$ of $\theta$ is not minimax for fixed sample size $n \in \mathbb{N}$.

(c) By computing the respective risk functions, show that

$$\lim_{n\to\infty} \frac{\sup_\theta R(\hat{\theta}_n, \theta)}{\sup_\theta R(\tilde{\theta}_n, \theta)} = 1$$

and that

$$\lim_{n\to\infty} \frac{R(\hat{\theta}_n, \theta)}{R(\tilde{\theta}_n, \theta)} < 1 \text{ for all } \theta \in [0,1], \theta \neq \frac{1}{2}.$$

*This shows that the maximum likelihood estimator dominates $\tilde{\theta}_n$ in the large sample limit as $n \to \infty$.*

**7**. Let $X_1, \ldots, X_n \sim^{iid} f_\theta$. For each of the following parametric models of pmf/pdf's, find a complete sufficient statistic.

(a) $f_\theta(x) = \frac{2x}{\theta^2}$, $0 < x < \theta$ and $\theta > 0$.

(b) $f_\theta(x) = \frac{\theta}{(1+x)^{1+\theta}}$, $x > 0$ and $\theta > 0$.

(c) $f_\theta(x) = \frac{(\log\theta)\theta^x}{\theta - 1}$, $x \in [0,1]$ and $\theta > 1$.

(d) $f_\theta(x) = e^{-(x-\theta)}\exp(e^{-(x-\theta)})$, $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

(e) $f_\theta(x) = \binom{2}{x}\theta^x(1-\theta)^{2-x}$, $x = 0, 1, 2$ and $\theta \in [0,1]$.

*Hint: it may be useful to consider which families are exponential families.*

**8**. Let $X_1, \ldots, X_n$ be i.i.d. from a density

$$f_\theta(x) = e^{-(x-\theta)}, \quad x \in (\theta, \infty),\ \theta \in \mathbb{R}.$$

Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic for $\theta$. Show that $X_{(1)}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ are independent. Find a uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.

*Hint: you may find it helpful to consider Question 5 on Problem Sheet 3.*

**9**. Let $X_1, \ldots, X_n \sim^{iid} U(-\theta, \theta)$, where $\theta > 0$. Find, if one exists, a uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.

**10**. Let $X_1, \ldots, X_n$ be i.i.d. real-valued random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2$. Consider a *linear estimator* $\sum_{i=1}^n a_i X_i$ of $\mu$. Under what conditions on $\{a_i\}_{i=1}^n$ is the estimator unbiased? Find the unbiased estimator of this form with the minimum variance, and calculate its variance.

**11**. Let $X_1, \ldots, X_n \sim^{iid} N(\theta, 1)$, $\theta \in \mathbb{R}$. Show that the uniformly minimum variance unbiased estimator (UMVUE) of $\theta^2$ is $\bar{X}_n^2 - 1/n$. Calculate its variance and compare it to the Cramer-Rao lower bound.

*Hint: $EY^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$ for $Y \sim N(\mu, \sigma^2)$.*