L2

$$\Big(\underbrace{\big((p \to q) \wedge (q \to (\neg p))\big)}_{\psi} \to \underbrace{(\neg p)}_{\chi}\Big) \quad : \phi$$

p	q		ψ	χ	ϕ
T	T		F	F	T
T	F		F	F	T
F	T		T	T	T
F	F		T	T	T

(1.1.4) Def. ① A formula ϕ is a tautology if its truth function F_ϕ always has value T.

② Say that formulas ψ, χ are logically equivalent (l.e.) if and only if $(\psi \leftrightarrow \chi)$ is a tautology.

(1.1.5) Remark: ① If ψ, χ have variables amongst $p_1, \ldots, p_n$ they are l.e. if and only if F_ψ, F_χ (fns. of n variables) are the same.

Eg $((p \to q) \wedge (q \to (\neg p)))$ is l.e to $(\neg p)$.

② Suppose ϕ is a fmla with variables $p_1, \ldots, p_n$ and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots, q_m$. For each $i \leq n$ substitute ϕ_i for p_i in ϕ. Then

(i) the result is a formula θ

(ii) if ϕ is a tautology, then so is θ.

(1.1.6) Examples ① Check

$$(((\neg p_2) \to (\neg p_1)) \to (p_1 \to p_2))$$

is a tautology. So if ϕ_1, ϕ_2 are formulas, then

$$(((\neg \phi_2) \to (\neg \phi_1)) \to (\phi_1 \to \phi_2))$$

(2) Warning:

$$(p_1 \to (\neg p_1))$$

is not a tautology, But we can find a formula ϕ_1 st. $(\phi_1 \to (\neg \phi_1))$ is a tautology.

(3) Examples of l.e. formulas:

1) $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. to $((p_1 \wedge p_2) \wedge p_3)$ [so usually omit brackets here]

2) Same, with $\vee$

3) $(p_1 \vee (p_2 \wedge p_3))$ is l.e. to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

3)' Same with $\vee, \wedge$ interchanged

4) $(\neg(\neg p_1))$ is l.e. to p_1.

5) $(\neg(p_1 \wedge p_2))$ is l.e. to $((\neg p_1) \vee (\neg p_2))$ (de Morgan)

5') Same with $\wedge, \vee$ interchanged

By 1.1.5, obtain, eg. for formulas ϕ_1, ϕ_2, ϕ_3 $(\phi_1 \wedge (\phi_2 \wedge \phi_3))$ is l.e. to $((\phi_1 \wedge \phi_2) \wedge \phi_3)$ etc.

(So omit brackets).

(1.1.7) Lemma. There are 2^{2^n} truth functions of n variables.

Pf: A truth fn. is a fn.

$$G : \{T,F\}^n \longrightarrow \{T,F\}$$

size 2^n size 2

So 2^{2^n} possibilities. #

(1.1.8) Def. Say that a set of connectives is adequate if for every $n \geq 1$ every truth function of n variables can be expressed as the truth fn. of a formula which involves on connectives from the set (and $p_1, \dots, p_n$).

(1.1.9) Thm The set $\{\neg, \vee, \wedge\}$ is adequate.

[Uses: disjunctive normal form.]

Proof: Let $G : \{T,F\}^n \to \{T,F\}$ be given.

Case 1 Suppose $G(\bar{v}) = F$ for all $\bar{v} \in \{T,F\}^n$.

Take ϕ to be $((p_1) \wedge (\neg p_1))$.

Then $F_\phi = G$. //

Case 2 Not Case 1.

List the $\bar{v}$ with $G(\bar{v}) = T$ as $\bar{v}_1, \dots, \bar{v}_r$.

Write $\bar{v}_i = (v_{i1}, \dots, v_{in})$ (each $v_{ij} \in \{T,F\}$).

Define

$$q_{ij} = \begin{cases} p_j & \text{if } v_{ij} = T \\ (\neg p_j) & \text{if } v_{ij} = F \end{cases}$$

Let ψ_i be

$$(q_{i1} \wedge q_{i2} \wedge \dots \wedge q_{in})$$

Then

$F_{\psi_i}(\bar{v}) = T \iff$

each q_{ij} has value T

$\iff$ each p_j has value v_{ij}

$\iff \bar{v} = \bar{v}_i$.

Example: $n = 3$ $\quad \bar{v}_1 = (T, F, T)$

ψ_1 is $(p_1 \wedge (\neg p_2) \wedge p_3)$

Now let

ϕ be $\psi_1 \vee \psi_2 \vee \dots \vee \psi_r$

Then $F_\phi(\bar{v}) = T$

$\iff F_{\psi_i}(\bar{v}) = T$ for some $i \leq r$.

$\iff \bar{v}$ is one of $\bar{v}_1, \dots, \bar{v}_r$.

$\iff G(\bar{v}) = T$.

So $F_\phi = G$, as required.

$\#$

A formula ϕ as in Case 2 is said to be in <u>disjunctive normal form</u>.

Cor (1.1.10) Suppose χ is a formula whose truth fn. is not always F. Then χ is l.e. to a formula in d.n.f.

[Apply Case 2 to $G = F_\chi$.]

Eg. χ: $((p_1 \to p_2) \to (\neg p_2))$

$n=2$ $\quad F_\chi(\bar{v}) = T$

$\iff \bar{v} = (T,F)$ or (F,F)

d.n.f.

$(p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2))$

(1.1.11) Cor. The following sets of connectives are adequate: 5

1) $\{\neg, \vee\}$

2) $\{\neg, \wedge\}$

3) $\{\neg, \to\}$

Pf: 1) By (1.1.9) ETS that we can express $\wedge$ in terms of $\neg, \vee$:

$(p_1 \wedge p_2)$ is l.e. to

$(\neg((\neg p_1) \vee (\neg p_2)))$.

2) Similar.

3) Express $\vee$ using $\neg, \to$:

$(p_1 \vee p_2)$ is l.e. to $((\neg p_1) \to p_2)$