IMPERIAL

MATH70142

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematics of Business and Economics

Date: Friday, May 23, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Let $D \subseteq \mathbb{R}^n$ (for integer $n \geq 1$) be convex. The function $f : D \to \mathbb{R}$ is said to be quasi-concave if

$$f(\lambda \underline{x} + (1-\lambda)\underline{x}') \geq \min\{f(\underline{x}), f(\underline{x}')\}$$

for all $\underline{x}, \underline{x}' \in D$ and all $\lambda \in [0,1]$.

(i) Suppose that D is an interval of the real line and that f is monotonically increasing on D. Show that f is quasi-concave. (2 marks)

(ii) For general $D \subseteq \mathbb{R}^n$, show that if f is quasi-concave then for all $y \in \mathbb{R}$ the set $\{\underline{x} | f(\underline{x}) \geq y\}$ is convex. (2 marks)

(iii) For general $D \subseteq \mathbb{R}^n$ with $n \geq 2$, suppose that f is monotonically increasing on D. Is f necessarily quasi-concave? Justify your answer. (2 marks)

(b) A firm's production process requires two inputs and produces a single output. For $i = 1, 2$, let $x_i \geq 0$ denote the quantity of the i-th input that the firm uses, and $w_i > 0$ denote the cost to the firm of each unit of this input. The firm's production function is given by

$$f(x_1, x_2) = \min\{\sqrt{x_1 x_2}, x_2\}.$$

(i) For $y > 0$, sketch the input requirement set $\{(x_1, x_2) | f(x_1, x_2) \geq y\}$. (2 marks)

(ii) State the equation of the isoquant $f(x_1, x_2) = y$ in the form $x_2 = g(x_1, y)$ for some function $g(x_1, y)$ that you are to define. (3 marks)

(iii) Suppose that $w_1 \geq w_2 > 0$. Determine the input bundle that produces y units of output at the minimum cost to the firm. (*Note that you do not have to check the second-order conditions for any optimisation that you perform.*) (5 marks)

(iv) Suppose that the firm sells each unit of its output at a price $p > 0$. Still assuming that $w_1 \geq w_2 > 0$, write down a necessary condition on p in terms of w_1 and w_2 if the firm is to be able to make a positive profit from producing $y > 0$ units of output at the minimum cost. (2 marks)

(v) Suppose now that $w_2 > w_1 > 0$. What now is the input bundle that produces y units of output at the minimum cost to the firm? (2 marks)

(Total: 20 marks)

2. (a) (i) Let $\pi^*(\underline{p}, \underline{w}) : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ denote a firm's maximised profit function, where $\underline{p}$ and $\underline{w}$ are the vectors of prices at which it sells a unit of each of its m outputs and buys a unit of each of its n inputs, respectively. Show that $\pi^*(\underline{p}, \underline{w})$ is convex in $(\underline{p}, \underline{w})$, i.e., that

$$\pi^*\left(\lambda \underline{p} + (1-\lambda)\underline{p}', \lambda \underline{w} + (1-\lambda)\underline{w}'\right) \leq \lambda \pi^*(\underline{p}, \underline{w}) + (1-\lambda)\pi^*(\underline{p}', \underline{w}')$$

for all $\underline{p}$, $\underline{p}' \in \mathbb{R}^m_{\geq 0}$, all $\underline{w}$, $\underline{w}' \in \mathbb{R}^n_{\geq 0}$, and all $\lambda \in [0, 1]$. (5 marks)

(ii) Suppose that a firm's production process requires two inputs and produces a single output. Suppose also that the price at which the firm sells a unit of its output is fixed all year round, but that for three quarters of the year $\underline{w} = (5, 7)$ while for the remaining quarter $\underline{w} = (1, 3)$. Would the firm make a larger maximum profit if instead $\underline{w} = (4, 6)$ all year round? (3 marks)

(b) (i) At time t, a firm inputs into its production process quantities x_1 and x_2 of inputs 1 and 2 which it buys at prices w_1 and w_2 per unit, respectively, and produces a quantity y of its single output which it sells at a price p per unit. The following table gives the values of these quantities and prices observed at two different times, $t = 1, 2$:

t	p	w_1	w_2	x_1	x_2	y
1	5	2	3	4	5	5
2	3	1	2	5	8	9

Use this data to show whether or not the firm's actions obey the weak axiom of profit maximisation, and whether or not they obey the weak axiom of cost minimisation.

(6 marks)

(ii) Suppose that for another firm, these values are instead:

t	p	x_1	x_2	y
1	4	3	6	6
2	3	4	6	8

but that the prices w_1 and w_2 that this firm pays for each unit of inputs 1 and 2, respectively, are not observed. Assuming this firm is acting to maximise its profits, what can we deduce about the price w_1 that the firm pays for a unit of input 1 at time $t = 1$ compared to at $t = 2$? (6 marks)

(Total: 20 marks)

3. (a) (i) Briefly explain the difference between Marshallian demand and Hicksian demand. (2 marks)

(ii) Briefly describe what are meant by the "substitution effect" and the "income effect" on a consumer's demand for a good when the price of the good changes. (2 marks)

(b) A consumer is faced with choosing quantities x_1 and x_2 of goods 1 and 2, respectively. Suppose that the consumer has a budget of $m > 0$, and that for $i = 1, 2$, the price per unit of the i-th good is $p_i > 0$. The consumer's preferences can be represented by the utility function

$$u(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$$

Derive expressions for the following:

(i) the consumer's Marshallian demand functions, $x_1^*(\underline{p}, m)$ and $x_2^*(\underline{p}, m)$, where $\underline{p} = (p_1, p_2)$ (*note that you do not have to check the second-order conditions for any optimisation that you perform*); (4 marks)

(ii) the consumer's indirect utility function, $v(\underline{p}, m)$; (2 marks)

(iii) the consumer's expenditure function, $e(\underline{p}, u)$; (2 marks)

(iv) the consumer's Hicksian demand functions, $x_{H,1}^*(\underline{p}, u)$ and $x_{H,2}^*(\underline{p}, u)$. (2 marks)

Suppose now that in fact $m = 30$ and that initially $\underline{p} = \underline{p}^{(0)} = (2, 2)$, but that later this changes to $\underline{p} = \underline{p}^{(1)} = (2, 4)$, i.e., the price p_2 per unit of good 2 increases from 2 to 4, while the price of good 1 remains unchanged.

(v) What is the change in the consumer's Marshallian demand for each of goods 1 and 2 due to this price change? (2 marks)

(vi) How much of the change in the consumer's Marshallian demand for good 2 that results from this price change can be attributed to the substitution effect? (4 marks)

(Total: 20 marks)

4. Suppose that the market demand for a particular good is given by $X^*(p) = 4/p^2$, where p is the price per unit of the good.

(a) (i) Suppose that in the short-run, only a single firm produces the good, and that the cost function of this firm is given by

$$c^*(y) = \frac{1}{3} + y^2,$$

where y is the quantity of the good that the firm produces. Determine the profit-maximising output for the firm as a function of p.

(4 marks)

(ii) Determine both the price per unit of the good and the quantity of it traded when the market is at equilibrium. (2 marks)

(b) (i) Suppose now that more firms producing the good enter the market, all with the same cost function as in part (a) and all acting to maximise their profit. What is the maximum number $J_{\max}$ of firms that can operate in this market in the long-run? (6 marks)

(ii) What are the price per unit of the good and the quantity of it traded when the market is at equilibrium in the long-run, i.e., when $J_{\max}$ firms are operating in the market.

(2 marks)

(iii) Compute the long-run producers' surplus, consumers' surplus and community surplus.

(3 marks)

(iv) Compute the maximum profit that each firm makes if it sells the good at the long-run market equilibrium price (per unit) and the maximum profit that it makes if it sells the good for nothing. Compare the difference of these last two values with the portion of the producers' surplus that is attributable to each firm (i.e., the producers' surplus divided by the number of firms, $J_{\max}$, in the market). (3 marks)

(Total: 20 marks)

5. A firm's production process requires n inputs and produces a single output. For $i = 1, \ldots, n$, let $x_i \geq 0$ denote the quantity of the i-th input that the firm uses, and $w_i > 0$ denote the cost to the firm of each unit of this input. Let y denote the quantity of output that the firm produces, and $p > 0$ denote the price at which it sells each unit of this output. Furthermore, let $f(\underline{x})$ denote the production function for this process and assume this to be a continuously differentiable function.

(a) The profit made by the firm in the long-run is given by

$$\pi(\underline{x}, p, \underline{w}) = pf(\underline{x}) - \underline{w}\,\underline{x}^T.$$

Suppose that $\pi(\underline{x}, p, \underline{w})$ is maximised with $\underline{x} = \underline{x}^*(p, \underline{w})$.

(i) State the necessary first order conditions on $\underline{x}^*(p, \underline{w})$. (2 marks)

(ii) State necessary and sufficient second order conditions on $\underline{x}^*(p, \underline{w})$. (2 marks)

Define the maximum profit function by $\pi^*(p, \underline{w}) = \pi(\underline{x}^*(p, \underline{w}), p, \underline{w})$ and the corresponding output by $y^*(p, \underline{w}) = f(\underline{x}^*(p, \underline{w}))$.

(iii) Derive the relation

$$\frac{\partial}{\partial p}\pi^*(p, \underline{w}) = y^*(p, \underline{w}).$$

(4 marks)

Now suppose that in the short-run, the firm is required to fix the quantities of some of its inputs. Suppose that the quantities of these fixed inputs are given by the vector $\underline{x}_F$. Denote the short-run maximised profit function by $\pi_S^*(p, \underline{w}, \underline{x}_F)$ and the corresponding output by $y_S^*(p, \underline{w}, \underline{x}_F)$. Suppose that the short-run maximised profit function coincides with the long-run maximised profit function if $\underline{x}_F = \underline{x}_F^*(p, \underline{w})$, i.e., $\pi_S^*(p, \underline{w}, \underline{x}_F^*(p, \underline{w})) = \pi^*(p, \underline{w})$.

(iv) By considering

$$\Delta(p) = \pi^*(p, \underline{w}) - \pi_S^*(p, \underline{w}, \underline{x}_F^*(p', \underline{w})),$$

where $p' > 0$ is fixed, show that

$$\frac{\partial y^*(p, \underline{w})}{\partial p} \geq \frac{\partial y_S^*(p, \underline{w}, \underline{x}_F)}{\partial p}$$

when $p = p'$ and $\underline{x}_F = \underline{x}_F^*(p', \underline{w})$. (6 marks)

(v) Briefly explain why this last result makes intuitive sense. (1 mark)

(b) Suppose now that, rather than maximise its profit, the firm's intention is to instead maximise its output, but subject to a fixed budget for its inputs, equal to $m > 0$, say. Assuming that the firm spends all of its budget, it therefore wishes to determine the input bundle $\underline{x}$ that maximises $f(\underline{x})$ subject to the constraint $\underline{w}\,\underline{x}^T = m$. In the case where $n = 2$, if $\underline{x} = \hat{\underline{x}}^*(p, \underline{w})$ is such an optimising input bundle, derive a necessary second order condition in terms of the determinant of the Hessian matrix of the function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\,(w_1 x_1 + w_2 x_2 - m)$$

where $\lambda \in \mathbb{R}$ and we now write $f(\underline{x}) = f(x_1, x_2)$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60142/MATH70142

The Mathematics of Business and Economics (Solutions)

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1. (a) (i) For all x, $x' \in D$ and $\lambda \in [0,1]$, we have seen $\Downarrow$

$$\lambda x + (1-\lambda)x' \geq \min\{x, x'\}, \tag{1}$$

and so, since f is an increasing function,

$$f(\lambda x + (1-\lambda)x') \geq f(\min\{x, x'\}) = \min\{f(x), f(x')\}. \tag{2}$$

Thus f is quasi-concave. 2, A

seen $\Downarrow$

(ii) Suppose $f(\underline{x}) \geq y$ and $f(\underline{x}') \geq y$. Let $\underline{x}'' = \lambda \underline{x} + (1-\lambda)\underline{x}'$, where $\lambda \in [0,1]$. Then, since f is quasi-concave,

$$f(\lambda \underline{x} + (1-\lambda)\underline{x}') \geq \min\{f(\underline{x}), f(\underline{x}')\} \geq y. \tag{3}$$

Hence the set $\{\underline{x} | f(\underline{x}) \geq y\}$ is convex. 2, A

seen $\Downarrow$

(iii) No. A counterexample is $f(\underline{x}) = x_1^2 + x_2^2$ defined on $\mathbb{R}^2$. For this, for any $y > 0$, the set $\{\underline{x} | f(\underline{x}) \geq y\}$ is the exterior of a circle (centred on the origin of radius $\sqrt{y}$), which is obviously not a convex region. Hence by the result of part (ii), f cannot be quasi-concave. 2, A

sim. seen $\Downarrow$

(b) (i) See figure 1. The bounding isoquant consists of two sections: one is a section of the line $x_2 = y^2/x_1$ and the other is a section of the line $x_2 = y$. These sections meet at the point (y, y). 2, A

sim. seen $\Downarrow$

(ii) The equation of the isoquant $f(x_1, x_2) = y$ can be expressed as $x_2 = g(x_1, y)$ where we define

$$g(x_1, y) = \begin{cases} \dfrac{y^2}{x_1} & \text{for } 0 < x_1 \leq y \\ y & \text{for } x_1 > y. \end{cases} \tag{4}$$

3, B

(iii) The cost-minimising bundle that produces an output y corresponds to the point on the isoquant $f(x_1, x_2) = y$ that coincides with a point on the isocost $w_1 x_1 + w_2 x_2 = c$ for the smallest value of c. meth seen $\Downarrow$

1, C

For $0 < x_1 < y$, the gradient of this isoquant is $-(y/x_1)^2$, which decreases from minus infinity to -1 as x_1 varies over this range. For $x_1 > y$, the gradient of this isoquant is 0. The isocost lines are parallel lines of constant gradient $-w_1/w_2$. Evidently, with $w_1 \geq w_2$, we have $-w_1/w_2 \leq -1$. Hence, in this case, the cost-minimising bundle will be a point on the section of the aforementioned isoquant along which $x_2 = y^2/x_1$. 2, D

More specifically, it occurs where $-(y/x_1)^2 = -w_1/w_2$; one may deduce that it is given by:

$$(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)) = \left(\sqrt{\frac{w_2}{w_1}}y, \sqrt{\frac{w_1}{w_2}}y\right). \tag{5}$$

2, B

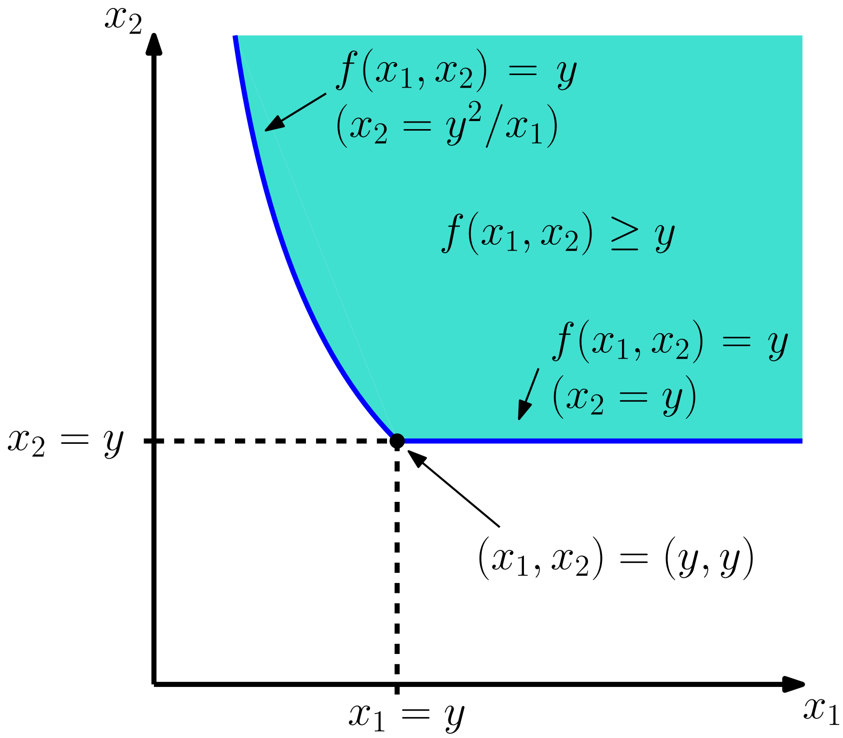

Figure 1: Sketch of the input requirement set $\{(x_1, x_2)|f(x_1, x_2) \geq y\}$ (shaded turquoise) and its bounding isoquant $f(x_1, x_2) = y$ (blue) for question 1(b)(ii).

(iv) In this case, with the input bundle as given by (5), the minimum cost to the firm is

sim. seen ⇓

$$c^*(w_1, w_2, y) = w_1 x_1^* + w_2 x_2^* = 2\sqrt{w_1, w_2}y. \tag{6}$$

Therefore the profit made by the firm is

$$\pi(w_1, w_2, y) = py - c^*(w_1, w_2, y) = (p - 2\sqrt{w_1, w_2})y. \tag{7}$$

Therefore, in order for the firm to be able to make a positive profit from producing $y > 0$ units of output, it is necessary that

$$p > 2\sqrt{w_1 w_2}. \tag{8}$$

2, C

(v) With $w_2 > w_1 > 0$, the cost-minimising input bundle that produces an output y is (y, y) - the vertex of the input requirement set.

sim. seen ⇓

2, D

2. (a) (i) Let $\underline{p}'' = \lambda \underline{p} + (1-\lambda)\underline{p}'$ and $\underline{w}'' = \lambda \underline{w} + (1-\lambda)\underline{w}'$. Also let $\pi\left(\underline{x}, \underline{p}, \underline{w}\right)$ denote the profit made with an input bundle $\underline{x}$ for the price vectors $\underline{p}$ and $\underline{w}$, and let $\underline{y}(\underline{x})$ denote the corresponding vector of output quantities. Then

$$\pi\left(\underline{x}, \underline{p}, \underline{w}\right) = \underline{p}\left(\underline{y}(\underline{x})\right)^T - \underline{w}\left(\underline{x}\right)^T . \tag{9}$$

Also let $\underline{x}^*(\underline{p}, \underline{w})$ denote the profit-maximising input bundle and $\underline{y}^*(\underline{p}, \underline{w})$ denote the corresponding vector of output quantities. So $\pi^*(\underline{p}, \underline{w}) = \pi\left(\underline{x}^*(\underline{p}, \underline{w}), \underline{p}, \underline{w}\right)$. Then we have

$$\begin{aligned} \pi^*\left(\underline{p}'', \underline{w}''\right) &= \underline{p}''\left(y^*\left(\underline{p}'', \underline{w}''\right)\right)^T - \underline{w}''\left(x^*\left(\underline{p}'', \underline{w}''\right)\right)^T \\ &= \lambda\pi\left(x^*\left(\underline{p}'', \underline{w}''\right), \underline{p}, \underline{w}\right) + (1-\lambda)\pi\left(x^*\left(\underline{p}'', \underline{w}''\right), \underline{p}', \underline{w}'\right) \quad (10) \\ &\leq \lambda\pi^*\left(\underline{p}, \underline{w}\right) + (1-\lambda)\pi^*\left(\underline{p}', \underline{w}'\right), \end{aligned}$$

as required, where we have used the facts that $\pi\left(x^*\left(\underline{p}'', \underline{w}''\right), \underline{p}, \underline{w}\right) \leq \pi^*\left(\underline{p}, \underline{w}\right)$ and $\pi\left(x^*\left(\underline{p}'', \underline{w}''\right), \underline{p}', \underline{w}'\right) \leq \pi^*\left(\underline{p}', \underline{w}'\right)$.

seen ⇓ | 5, B

(ii) Notice that $(0.75 \times (5, 7)) + (0.25 \times (1, 3)) = (4, 6)$. It then follows from the result of part (i) that

$$\pi^*\left(p, (4, 6)\right) \leq \left(0.75 \times \pi^*(p, (5, 7))\right) + \left(0.25 \times \pi^*(p, (1, 3))\right). \tag{11}$$

So the firm would not make a larger maximum profit if $\underline{w} = (4, 6)$ all year round.

unseen ⇓ | 3, C

(b) Note that in the following, we use a superscript (in parentheses) to indicate the corresponding time point, e.g., $p^{(2)}$ denotes the price p at time $t = 2$. Also, we define $\underline{r}^{(t)} = (p^{(t)}, w_1^{(t)}, w_2^{(t)})$ and $\underline{z}^{(t)} = (y^{(t)}, -x_1^{(t)}, -x_2^{(t)})$.

sim. seen ⇓

(i) The profit the firm makes at time $t = 1$ is

$$p^{(1)}y^{(1)} - \left(w_1^{(1)}x_1^{(1)} + w_2^{(1)}x_2^{(1)}\right) = 25 - (8 + 15) = 2. \tag{12}$$

However,

$$p^{(1)}y^{(2)} - \left(w_1^{(1)}x_1^{(2)} + w_2^{(1)}x_2^{(2)}\right) = 45 - (10 + 24) = 11. \tag{13}$$

Hence the firm's actions do not obey the weak axiom of profit maximisation - the firm would have made a larger profit at time $t = 1$, when prices are given by $\underline{r}^{(1)}$, if it had operated with output/inputs as given by $\underline{z}^{(2)}$ rather than those given by $\underline{z}^{(1)}$ that it actually operated with.

3, A

On the other hand, $y^{(1)} < y^{(2)}$ and the total cost to the firm at $t = 1$ is

$$w_1^{(1)}x_1^{(1)} + w_2^{(1)}x_2^{(1)} = 8 + 15 = 23 \tag{14}$$

whereas

$$w_1^{(1)}x_1^{(2)} + w_2^{(1)}x_2^{(2)} = 10 + 24 = 34 \tag{15}$$

Hence the firm's actions do obey the weak axiom of cost minimisation - the firm produces less output at $t = 1$ than at $t = 2$, but its costs at time $t = 1$ are at least less than those at $t = 2$.

3, A

(ii) By the axiom of profit maximisation, we should have $\boxed{\text{unseen} \Downarrow}$

$$\underline{r}^{(t)} \left(\underline{z}^{(t)}\right)^T \geq \underline{r}^{(t)} \left(\underline{z}^{(s)}\right)^T \quad \text{for all } s, t = 1, 2. \tag{16}$$

Thus (with $t = 1$ and $s = 2$),

$$p^{(1)}y^{(1)} - \left(w_1^{(1)}x_1^{(1)} + w_2^{(1)}x_2^{(1)}\right) \geq p^{(1)}y^{(2)} - \left(w_1^{(1)}x_1^{(2)} + w_2^{(1)}x_2^{(2)}\right) \tag{17}$$

and (with $t = 2$ and $s = 1$)

$$p^{(2)}y^{(2)} - \left(w_1^{(2)}x_1^{(2)} + w_2^{(2)}x_2^{(2)}\right) \geq p^{(2)}y^{(1)} - \left(w_1^{(2)}x_1^{(1)} + w_2^{(2)}x_2^{(1)}\right). \tag{18}$$

$\boxed{2, \text{A}}$

These last two relations can be rearranged into the equivalent forms

$$p^{(1)} \left(y^{(1)} - y^{(2)}\right) - \sum_{i=1}^{2} w_i^{(1)} \left((x_i^{(1)} - x_i^{(2)}\right) \geq 0 \tag{19}$$

and

$$p^{(2)} \left(y^{(2)} - y^{(1)}\right) - \sum_{i=1}^{2} w_i^{(2)} \left((x_i^{(2)} - x_i^{(1)}\right) \geq 0, \tag{20}$$

respectively. These then sum to give

$$\left(p^{(2)} - p^{(1)}\right) \left(y^{(2)} - y^{(1)}\right) - \sum_{i=1}^{2} \left(w_i^{(2)} - w_i^{(1)}\right) \left((x_i^{(2)} - x_i^{(1)}\right) \geq 0. \tag{21}$$

Note that $x_2^{(1)} = x_2^{(2)}$, so (21) reduces to

$$\left(p^{(2)} - p^{(1)}\right) \left(y^{(2)} - y^{(1)}\right) - \left(w_1^{(2)} - w_1^{(1)}\right) \left((x_1^{(2)} - x_1^{(1)}\right) \geq 0. \tag{22}$$

Now substituting in with the remaining given data, (22) reduces to

$$\begin{aligned} (3-4) \cdot (8-6) - &\left(w_1^{(2)} - w_1^{(1)}\right) \cdot (4-3) \geq 0 \\ \implies w_1^{(1)} &\geq w_1^{(2)} + 2. \end{aligned} \tag{23}$$

$\boxed{4, \text{D}}$

3. (a) (i) Marshallian demand is the consumer's demand for a good based on their budget, while Hicksian demand is their demand for a good based on the level of utility they seek.

seen ⇓

2, A

(ii) The substitution effect is the change in the consumer's (Marshallian) demand for the good as they seek to buy more (or less) of "substitute" goods and less (or more) of the good whose price has changed, in an attempt to maintain the level of utility of their consumption bundle given their fixed income. On the other hand, the income effect is the change in their demand resulting from the effective change in their purchasing power; they can buy less (more) of the good if its price increases (decreases), respectively.

seen ⇓

2, A

(b) (i) We seek (x_1, x_2) to maximise $u(x_1, x_2)$ subject to the constraint that $p_1x_1 + p_2x_2 = m$. Introduce the Lagrangian

sim. seen ⇓

$$L(x_1, x_2, \lambda) = x_1^{1/3}x_2^{2/3} - \lambda(p_1x_1 + p_2x_2 - m). \tag{24}$$

Then the first order conditions are

1, B

$$\frac{\partial L}{\partial x_1} = 0 \implies \frac{1}{3}\left(\frac{x_2^*}{x_1^*}\right)^{2/3} = \lambda p_1 \tag{25}$$

$$\frac{\partial L}{\partial x_2} = 0 \implies \frac{2}{3}\left(\frac{x_1^*}{x_2^*}\right)^{1/3} = \lambda p_2 \tag{26}$$

and

$$\frac{\partial L}{\partial \lambda} = 0 \implies p_1x_1^* + p_2x_2^* = m \tag{27}$$

Dividing (25) by (26) gives

$$\frac{1}{2}\frac{x_2^*}{x_1^*} = \frac{p_1}{p_2} \implies x_2^* = \frac{2p_1}{p_2}x_1^*. \tag{28}$$

Then (27) gives $3p_1x_1^* = m$, from which it follows that

$$x_1^* = x_1^*(\underline{p}, m) = \frac{m}{3p_1}. \tag{29}$$

Hence it follows from (28) that

$$x_2^* = x_2^*(\underline{p}, m) = \frac{2m}{3p_2}. \tag{30}$$

3, C

(ii) We have

sim. seen ⇓

$$\begin{aligned} v(\underline{p}, m) &= u(x_1^*(\underline{p}, m), x_2^*(\underline{p}, m)) \\ &= \left(\frac{m}{3p_1}\right)^{1/3}\left(\frac{2m}{3p_2}\right)^{2/3} \\ &= \frac{4^{1/3}m}{3(p_1p_2^2)^{1/3}}. \end{aligned} \tag{31}$$

2, A

(iii) Inverting (31) gives

$$m = \frac{3(p_1 p_2^2)^{1/3} v(\underline{p}, m)}{4^{1/3}}, \tag{32}$$

sim. seen ⇓

from which we identify

$$e(\underline{p}, u) = \frac{3(p_1 p_2^2)^{1/3} u}{4^{1/3}}. \tag{33}$$

2, B

sim. seen ⇓

(iv) We have

$$\begin{aligned} x^*_{H,1}(\underline{p}, u) &= x^*_1(\underline{p}, e(\underline{p}, u)) \\ &= \frac{3(p_1 p_2^2)^{1/3} u}{4^{1/3}} \cdot \frac{1}{3p_1} \\ &= \left(\frac{p_2}{2p_1}\right)^{2/3} u. \end{aligned} \tag{34}$$

where the second equality follows from (29) and (33). Alternatively, the student might use Shephard's Lemma, i.e.,

$$x^*_{H,1}(\underline{p}, u) = \frac{\partial e(\underline{p}, u)}{\partial p_1}. \tag{35}$$

Similarly,

$$\begin{aligned} x^*_{H,2}(\underline{p}, u) &= x^*_2(\underline{p}, e(\underline{p}, u)) \\ &= \frac{3(p_1 p_2^2)^{1/3} u}{4^{1/3}} \cdot \frac{2}{3p_2} \\ &= \left(\frac{2p_1}{p_2}\right)^{1/3} u. \end{aligned} \tag{36}$$

2, B

(v) Recalling (29) and (30), we have:

sim. seen ⇓

$$\underline{x}^*(\underline{p}^{(0)}, m) = \left(\frac{30}{3 \cdot 2}, \frac{2 \cdot 30}{3 \cdot 2}\right) = (5, 10) \tag{37}$$

and

$$\underline{x}^*(\underline{p}^{(1)}, m) = \left(\frac{30}{3 \cdot 2}, \frac{2 \cdot 30}{3 \cdot 4}\right) = (5, 5)\,. \tag{38}$$

Hence there is no change in Marshallian demand for good 1, but a decrease in Marshallian demand of 5 units for good 2.

2, A

(vi) The change in the consumer's Marshallian demand for good 2 due to the substitution effect is given by

sim. seen ⇓

$$x^*_2(\underline{p}^{(0)}, m) - x_{H,2}(\underline{p}^{(1)}, v(\underline{p}^{(0)}, m)) \tag{39}$$

Recalling (31),

$$v(\underline{p}^{(0)}, m) = \frac{4^{1/3} \cdot 30}{3 \cdot (2 \cdot 2^2)^{1/3}} = 4^{1/3} \cdot 5. \tag{40}$$

Then, recalling (36),

$$x_{H,2}(\underline{p}^{(1)}, v(\underline{p}^{(0)}, m)) = 4^{1/3} \cdot 5. \tag{41}$$

Hence

$$x_2^*(\underline{p}^{(0)}, m) - x_{H,2}(\underline{p}^{(1)}, v(\underline{p}^{(0)}, m)) = 10 - 4^{1/3} \cdot 5 = 10(1 - 2^{-1/3}). \tag{42}$$

(Note that this is less than 5 (it is equal to $5(2 - 2^{2/3})$), which is the *total* decrease in the consumer's Marshallian demand for good 2.) $\boxed{4, \mathrm{D}}$

4. (a) (i) The firm's profit function is

sim. seen ⇓

$$\pi(p, y) = py - c^*(y) = py - \frac{1}{3} - y^2. \tag{43}$$

Solving $\partial\pi(p, y)/\partial p = 0$ for y gives

1, A

$$y = y^*(p) = \frac{p}{2}. \tag{44}$$

1, A

This corresponds to a maximum (rather than a minimum) of $\pi(p, y)$ since the latter is a quadratic function in y with a negative coefficient of y^2. This is therefore the profit-maximising output. Note that it is never more profitable for the firm to shut down - i.e., produce nothing - since $p/2 > 0$ and so is always the largest value of $\pi(p, y)$ for all $y \geq 0$.

1, B

1, B

(ii) In this case of a single producer, market supply, $Y^*(p)$, is given by $y^*(p)$, i.e., $Y^*(p) = p/2$. At equilibrium, market demand and market supply are equal, i.e.,

sim. seen ⇓

$$\begin{aligned} X^*(p) &= Y^*(p), \\ \implies \frac{4}{p^2} &= \frac{p}{2} \\ \implies p &= 2 \end{aligned} \tag{45}$$

is the market equilibrium price per unit of the good, and

$$X^*(2) = Y^*(2) = 1 \tag{46}$$

is the market equilibrium quantity of it traded.

2, A

unseen ⇓

(b) (i) Since each firm operates with the same cost function (and acts to maximise profits), they all produce the same amount as the original single firm, as found in part (a). Thus, if there are J firms in total, the market supply is now

$$Y^*(p, J) = \frac{Jp}{2}. \tag{47}$$

Now, at equilibrium,

1, B

$$\begin{aligned} X^*(p) &= Y^*(p, J) \\ \implies \frac{4}{p^2} &= \frac{Jp}{2} \\ \implies p &= \frac{2}{J^{1/3}} = \hat{p}(J), \text{ say.} \end{aligned} \tag{48}$$

1, A

Then, recalling (43) and (44), the profit made by each firm is

$$\begin{aligned} \pi^*(\hat{p}, y^*(\hat{p}(J))) &= \hat{p}(J)y^*(\hat{p}(J)) - \frac{1}{3} - (y^*(\hat{p}(J)))^2 \\ &= \frac{(\hat{p}(J))^2}{4} - \frac{1}{3} \\ &= \frac{1}{J^{2/3}} - \frac{1}{3} \end{aligned} \tag{49}$$

The largest number of firms that can operate in the long-run is the largest integer J such that $\pi^*(\hat{p}, y^*(\hat{p}(J))) > 0$, i.e., such that $J^{2/3} < 3$, or $J < \sqrt{27}$. Thus $J_{\max} = 5$. $\boxed{4, \text{D}}$

$\boxed{\text{sim. seen} \Downarrow}$

(ii) Substituting $J = 5$ into (48) gives the long-run market equilibrium price per unit of the good as

$$\hat{p}(5) = \frac{2}{5^{1/3}}. \tag{50}$$

And the market equilibrium quantity traded is

$$X^*(\hat{p}) = Y^*(\hat{p}(5), 5) = 5^{2/3}. \tag{51}$$

$\boxed{2, \text{A}}$

(iii) The long-run producers' surplus is $\boxed{\text{sim. seen} \Downarrow}$

$$PS = \int_0^{\hat{p}(5)} Y^*(p, 5)\mathrm{d}p = \int_0^{\hat{p}(5)} \frac{5p}{2}\mathrm{d}p = \frac{5}{4}(\hat{p}(5))^2 = 5^{1/3}. \tag{52}$$

$\boxed{1, \text{B}}$

The long-run consumers' surplus is

$$CS = \int_{\hat{p}(5)}^{\infty} X^*(p)\mathrm{d}p = \int_{\hat{p}(5)}^{\infty} \frac{4}{p^2}\mathrm{d}p = \frac{4}{\hat{p}(5)} = 2 \cdot 5^{1/3}. \tag{53}$$

$\boxed{1, \text{B}}$

The long-run community surplus is $PS + CS = 3 \cdot 5^{1/3}$. $\boxed{1, \text{A}}$

$\boxed{\text{seen} \Downarrow}$

(iv) Recalling (43), the maximum profit made by each producer if they sell the good for nothing is $-1/3$ (in this case they should produce $y = 0$ units of output). On the other hand, recalling (49) now with $J = 5$, the maximum profit made by each producer if they sell the good for the long-run market equilibrium price is $5^{-2/3} - (1/3)$. The difference between these values is thus $5^{-2/3}$. This equals the producers' surplus as given by (52), divided by the number of firms, 5. $\boxed{3, \text{C}}$

5. (a) (i) The necessary first order conditions are $\boxed{\text{seen} \Downarrow}$

$$\frac{\partial}{\partial x_i}\pi(\underline{x}, p, \underline{w}) = 0 \quad \text{for } i = 1, \ldots, n, \tag{54}$$

which are equivalent to

$$p\frac{\partial f(\underline{x})}{\partial x_i} = w_i \quad \text{for } i = 1, \ldots, n. \tag{55}$$

$\boxed{2, \text{M}}$

(ii) A necessary second order condition is that the Hessian matrix of $f(\underline{x})$ at $\underline{x} = \underline{x}^*(\underline{p}, \underline{w})$ should be negative semi-definite. A sufficient second order condition is that this Hessian matrix should be negative definite. $\boxed{\text{seen} \Downarrow}$ $\boxed{2, \text{M}}$

(iii) We have $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}
\frac{\partial}{\partial p}\pi^*(p, \underline{w}) &= \frac{\partial}{\partial p}\left(pf(\underline{x}^*(p, \underline{w})) - \underline{w}(\underline{x}^*(p, \underline{w}))^T\right) \\
&= f(\underline{x}^*(p, \underline{w})) + \sum_{i=1}^{n} p\frac{\partial f(\underline{x}^*(p, \underline{w}))}{\partial x_i^*(p, \underline{w})}\frac{\partial}{\partial p}x_i^*(p, \underline{w}) - \sum_{i=1}^{n} w_i\frac{\partial}{\partial p}x_i^*(p, \underline{w}) \\
&= f(\underline{x}^*(p, \underline{w})) + \sum_{i=1}^{n}\left(p\frac{\partial f(\underline{x}^*(p, \underline{w}))}{\partial x_i^*(p, \underline{w})} - w_i\right)\frac{\partial}{\partial p}x_i^*(p, \underline{w}) \\
&= y^*(p, \underline{w}),
\end{aligned} \tag{56}$$

where the final line follows from (55) and the fact that $\underline{x} = \underline{x}^*(p, \underline{w})$ maximises $\pi(\underline{x}, p, \underline{w})$, along with the definition of $y^*(p, \underline{w})$. $\boxed{4, \text{M}}$

(iv) (Le Chatelier's principle.) Evidently, $\Delta(p') = 0$. Meanwhile, for all values of p not equal to p', $\Delta(p) \geq 0$. The latter is because, for fixed p and $\underline{w}$, $\pi_S^*(p, \underline{w}, \underline{x}_F^*(p', \underline{w}))$ is the maximum profit found as $\underline{x}$ varies over a subset of $\mathbb{R}^n_{\geq 0}$, whereas $\pi^*(p, \underline{w})$ is the maximum profit found as $\underline{x}$ varies over the *whole* of $\mathbb{R}^n_{\geq 0}$. Thus $\Delta(p)$ is minimised at $p = p'$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{3, \text{M}}$
Hence, at $p = p'$,

$$\begin{aligned}
&\frac{\mathrm{d}^2\Delta}{\mathrm{d}p^2} \geq 0 \\
\implies &\frac{\partial^2}{\partial p^2}\pi^*(p, \underline{w}) - \frac{\partial^2}{\partial p^2}\pi_S^*(p, \underline{w}, \underline{x}_F^*(p', \underline{w})) \geq 0 \\
\implies &\frac{\partial}{\partial p}y^*(p, \underline{w}) - \frac{\partial}{\partial p}y_S^*(p, \underline{w}, \underline{x}_F^*(p', \underline{w})) \geq 0,
\end{aligned} \tag{57}$$

which gives the required result. Here the final line follows from the result of part (b), and the analogous result $\partial\pi_S^*(p, \underline{w}, \underline{x}_F)/\partial p = y_S^*(p, \underline{w}, \underline{x}_F)$ which one can deduce in a similar manner. $\boxed{3, \text{M}}$

(v) The result shows that the firm can change its output at a faster rate in response to changes in p if none of its inputs are restricted to be fixed. This makes intuitive sense, as such restrictions would be expected to limit the firm's production capabilities. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{1, \text{M}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) The students have seen this mathematical problem and the derivation given here (of the analogous second order condition) in the context of a consumer maximising their utility function subject to a fixed budget, rather than in the context of a firm maximising its output subject to a fixed budget.

At the point $(\hat{x}_1^*, \hat{x}_2^*)$ in the (x_1, x_2)-plane, the isocost $w_1 x_1 + w_2 x_2 = m$ is tangential to an isoquant of f. Parametering this isocost by $t = x_1$, say, a necessary condition for the maximum of $f(x_1, x_2)$ at points along this isocost to occur at $\underline{x} = \underline{\hat{x}}^*(p, \underline{w})$, is that

$$\frac{\mathrm{d}^2 f(x_1, x_2)}{\mathrm{d}t^2} \leq 0 \quad \text{at } \underline{x} = \underline{\hat{x}}^*(p, \underline{w}). \tag{58}$$

But

$$\begin{aligned} \frac{\mathrm{d}f(x_1, x_2)}{\mathrm{d}t} &= \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} \\ &= \left(\frac{\partial}{\partial x_1} - \frac{w_1}{w_2}\frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial x_1}, \end{aligned} \tag{59}$$

where the second equality follows from the fact that $x_2 = (m - w_1 x_1)/w_2$ at points on this isocost. Hence,

$$\begin{aligned} \frac{\mathrm{d}^2 f(x_1, x_2)}{\mathrm{d}t^2} &= \left(\frac{\partial}{\partial x_1} - \frac{w_1}{w_2}\frac{\partial}{\partial x_2}\right)^2 \frac{\partial}{\partial x_1}, \\ &= \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} + \frac{w_1^2}{w_2^2}\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - 2\frac{w_1}{w_2}\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}. \end{aligned} \tag{60}$$

Furthermore, the Hessian matrix of the given Lagrangian function L is given by

$$H_L(x_1, x_2, w_1, w_2) = \begin{pmatrix} 0 & -w_1 & -w_2 \\ -w_1 & \dfrac{\partial^2 f(x_1, x_2)}{\partial x_1^2} & \dfrac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \\ -w_2 & \dfrac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f(x_1, x_2)}{\partial x_2^2} \end{pmatrix},$$

and the determinant of this is

$$|H_L(x_1, x_2, w_1, w_2)| = -w_2^2\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} + \frac{w_1^2}{w_2^2}\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - 2\frac{w_1}{w_2}\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}\right). \tag{61}$$

Recalling (60), it follows that

$$|H_L(x_1, x_2, w_1, w_2)| = -w_2^2 \frac{\mathrm{d}^2 f(x_1, x_2)}{\mathrm{d}t^2} \tag{62}$$

at points along the isocost. It then follows from (58), that a necessary condition for the maximum of $f(x_1, x_2)$ at points along the isocost $w_1 x_1 + w_2 x_2 = m$ to occur at $\underline{x} = \underline{\hat{x}}^*(p, \underline{w})$, is that

$$|H_L(x_1, x_2, w_1, w_2)| \geq 0 \quad \text{at } \underline{x} = \underline{\hat{x}}^*(p, \underline{w}). \tag{63}$$

$\boxed{5, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70142 Mathematics of Business and Economics Markers Comments

Question 1

Many students struggled to sketch the input requirement set for part (bi), but were still able to answer subsequent parts of the question. Many students left their answer for $g(x_1, y)$ for part b(ii) (their expressions for this function and/or their expressions for the relevant defining ranges of x_1) in an un-simplified form. However, I did not deduct marks for such answers. For part (biii), many students appeared not to realise that one could deduce the relevant form for the production function in this case (namely, $\sqrt{x_1 x_2}$ over the range $0\ x_1\ y$) from the fact that its gradient should be less than or equal to -1 at the point corresponding to the cost-minimising input bundle (equal to the gradient $-w_1/w_2$ of the corresponding isocost, to which it should be tangential at this point). Instead they considered both possible forms for the production (i.e., $f(x_1,x_2)=\sqrt{x_1 x_2}$ and $f(x_1,x_2)=x_2$) and attempted to solve the relevant minimisation problem (using a Lagrangian) for both of them, before eventually arriving at the correct conclusion.

Question 2

A lot of people did not correctly verify the convexity condition correction in the first part of the question, many simply wrote down the inequality without justification as to why it should hold. Otherwise performance was good in the sections which follows, I was lenient on the bounds which were derived for w1 at time 1 and 2 in b ii), anything reasonable was accepted so long as both time points were considered; a small penalty was applied if the student used strict inequalities.

Question 3

For parts (biii) and (biv), a number of students chose to solve an additional optimisation problem (using a Lagrangian or equivalent, in addition to the maximisation problem already solved for the Marshallian demand functions for part (bi)), rather than simply invert the expression for the indirect utility function (obtained for part (bii)) to obtain the expenditure function and hence the Hicksian demand functions. Many students left their answers for parts (bi)--(biv) in an un-simplified form, for example, leaving $(3^{1/3})(3^{2/3})$ as such rather than writing it simply as 3. However, I did not deduct marks for such answers. I also allowed for errors carried forward, deducting only mark(s) for the initial mistake. For part (bvi), many students attempted to apply Slutsky's equation or answered by saying something about substituting good 1 for good 2, rather than recalling that the substitution effect corresponds to the demand for a bundle of the previous level of utility and hence relates to the Hicksian demand for a bundle of that level.

Question 4	Many students did not check the boundary conditions for maximisation in the first part of the question, a small penalty was applied in this case. Otherwise performance was overall good, calculation mistakes were penalised only once and then carried through to the other parts of the question that were linked.
Question 5	Many students struggled with this question as a whole. Parts (aiv) and (b) were the most challenging (especially the latter), but many students also dropped marks for part (aiii).