Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Final Exam

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You may assume, without proof, any results from the lectures, lecture notes and videos, unless you are explicitly asked to prove them.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

1. **Total: 20 Marks**

   (a) Give the definitions of what it means for a binary relation R on a set X to be

   i. reflexive; 1 Mark

   ii. symmetric; 1 Mark

   iii. antisymmetric. 1 Mark

   (b) Let X be a set and let R be a binary relation on X. Let's define S to be the "opposite" binary relation on X, by which I mean: if $x, y \in X$ then $S(x,y) = \neg R(x,y)$. In other words, $S(x,y)$ is true if and only if $R(x,y)$ is false.

   For example, if X is the real numbers and R is the binary relation $\leq$ then S would be the binary relation $>$.

   i. Assume that X is nonempty, and R is reflexive. Prove that S is not reflexive. 1 Mark

   ii. Give a proof or counterexample: If R is symmetric, then S is symmetric. 2 Marks

   iii. Give a proof or counterexample: if R is antisymmetric, then S is antisymmetric. 2 Marks

   (c) Now let X be a set and let R_1 and R_2 be two binary relations on X. Let's define a new binary relation S on X as the "union" of R_1 and R_2, by which I mean: $S(x,y) = R_1(x,y) \vee R_2(x,y)$. In other words, $S(x,y)$ is true if and only if at least one of $R_1(x,y)$ and $R_2(x,y)$ is true.

   i. Say R_1 and R_2 are reflexive. Prove that S is reflexive. 2 Marks

   ii. Give a proof or counterexample: if R_1 and R_2 are symmetric, then S is symmetric. 2 Marks

   iii. Give a proof or counterexample: if R_1 and R_2 are antisymmetric, then so is S. 2 Marks

   (d) Finally, let A and B be sets, and let $f : A \to B$ be a function. Say R is a binary relation on B, and define a binary relation S on A by, for $a_1, a_2 \in A$, defining $S(a_1, a_2) = R(f(a_1), f(a_2))$. In other words, to see if two elements of A are related by S, apply f to them both and see if the resulting elements of B are related by R.

   i. Say R is reflexive. Prove that S is reflexive. 2 Marks

   ii. Give a proof or counterexample: if R is symmetric, then S is symmetric. 2 Marks

   iii. Give a proof or counterexample: if R is antisymmetric, then S is antisymmetric. 2 Marks

2. **Total: 20 Marks**

(a) i. Write the definition of the addition on the natural numbers. 2 Marks

ii. Let $X \subseteq \mathbb{N}$ be a set which is closed under addition: if $x, y \in X$, then $x + y \in X$. Prove using (P5) that, for every natural number $n \geq 1$ and every $x \in X$, we have $n \cdot x \in X$. 4 Marks

(b) i. Write the principle of strong induction. 2 Marks

ii. Show that if $P_0,\ P_1,\ P_2 \ldots P_n$ are $n+1$ propositions, then

$$\neg(P_0 \vee P_1 \vee \cdots \vee P_n) \iff \neg P_0 \wedge \neg P_1 \wedge \cdots \wedge \neg P_n.$$

3 Marks

(c) In the following, use only the axioms of the reals.

i. Let x and y be real numbers. Show that if $x + y = x$, then $y = 0$. 3 Marks

ii. Show that 0 is the unique neutral element for addition. 2 Marks

(d) Let b be a natural number with $b \geq 2$. Let $b_1 := b$ and let $\{b_n\}$ be a sequence such that $b_{n+1} = b \cdot b_n$ for all $n \geq 1$. Let $x \in \mathbb{R}$ be a real number. Prove that there is a unique sequence of integers a_n for $n \geq 1$ such that $\frac{a_n}{b_n} \leq x < \frac{a_n+1}{b_n}$ for all n. 4 Marks

3. **Total: 20 Marks**

(a) i. Find a parametric equation for the line $\mathcal{L}$ passing through the points $A(1,2,3)$ and $B(2,3,2)$ in $\mathbb{R}^3$. 2 Marks

ii. Does the point $C(-1,0,1)$ lie on $\mathcal{L}$? 1 Mark

iii. Compute the distance between $\mathcal{L}$ and the point $D(1,-2,2)$. 3 Marks

iv. Does $\mathcal{L}$ intersect the plane with cartesian equation $x + y - z - 9 = 0$? If yes, where? If no, why not? 2 Marks

(b) Consider the curve

$$\mathbf{r}(t) = (t\cos(2t) - \sin(2t))\mathbf{i} + t^2\mathbf{j} + (t\sin(2t) + \cos(2t))\mathbf{k},$$

parameterised by $t \geq 0$ in $\mathbb{R}^3$.

i. Find the length of the curve on the interval $0 \leq t \leq 3$. 3 Marks

ii. Find the unit tangent vector $\mathbf{T}(t)$ to the curve. 1 Mark

iii. Find the unit normal vector $\mathbf{N}(t)$ to the curve. 2 Marks

(c) Initially, a particle is stationary at the origin. Its acceleration at time t is given by

$$\mathbf{a}(t) = \frac{1}{t+1}\mathbf{i} + e^{-t}\mathbf{j} + \sin(t)\mathbf{k}.$$

Find an expression for the trajectory of the particle. 6 Marks

Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Final Exam

1. $\boxed{\textbf{Total: 20 Marks}}$

   (a) i. It means $\forall a \in X, R(a,a)$.

   ii. It means $\forall a, b \in X, R(a,b) \implies R(b,a)$.

   iii. It means $\forall a, b \in X, R(a,b) \wedge R(b,a) \implies a = b$.

   (b) i. X is nonempty, so choose $t \in X$. Now R is reflexive, so $R(t,t)$ is true, so $S(t,t)$ is false. This means that $\forall a \in X, S(a,a)$ is false, because $a = t$ is a counterexample. So S is not reflexive.

   ii. This is true. Say $a, b \in X$ are arbitrary and $S(a,b)$ is true; we need to show $S(b,a)$ is true. Let's do it by contradiction. Assume $S(b,a)$ is false. Then by definition of S, we know $R(b,a)$ is true. But R is symmetric, so $R(a,b)$ must then be true, meaning that $S(a,b)$ is false, contradicting our assumptions.

   iii. This is not true in general. For example let X be the set $\{1,2\}$ (or any set with two elements) and say $R(1,1)$ and $R(2,2)$ are true, but $R(1,2)$ and $R(2,1)$ are false. Then R is antisymmetric, because if $R(a,b)$ and $R(b,a)$ are true then the only possibilities are $a = b = 1$ or $a = b = 2$ by definition of $R(a,b)$. However the "opposite" relation S has $S(1,2)$ and $S(2,1)$ both true, but $1 \neq 2$, so S is not antisymmetric.

   (c) i. Say $a \in X$ is arbitrary; we need to prove $S(a,a)$ is true. Then we know $R_1(a,a)$ is true and $R_2(a,a)$ is true, because both R_1 and R_2 are reflexive. Hence $R_1(a,a) \vee R_2(a,a)$ is true, so by definition $S(a,a)$ is true.

   ii. This is true. Suppose $a, b \in X$ are arbitrary and $S(a,b)$ is true; our goal is to prove $S(b,a)$ is true. By definition of $S(a,b)$ we know that either $R_1(a,b)$ or $R_2(a,b)$ is true. If $R_1(a,b)$ is true then symmetry of R_1 tells us that $R_1(b,a)$ is true. Conversely if $R_2(a,b)$ is true then by symmetry of R_2 we deduce $R_2(b,a)$ is true. Hence at least one of $R_1(b,a)$ and $R_2(b,a)$ are true, so $S(b,a)$ is true, which is what we wanted to prove.

   iii. This is not true. Suppose $X = \{3,4\}$ (or any set with two elements) and define R_1 by $R_1(3,4)$ is false and all other possibilities are true, and let's define R_2 by $R_2(4,3)$ is false and all other possibilities are true. I claim that R_1 and R_2 are antisymmetric. Indeed if $R_1(a,b)$ and $R_1(b,a)$ are true then a and b can't be different, because the only way they can be different is that one is 3 and one is 4, and $R_1(3,4)$ is false. Similarly R_2 is antisymmetric. However S is not antisymmetric, because $S(3,4)$ is true (as $R_2(3,4)$ is true) and $S(4,3)$ is true (as $R_1(4,3)$ is true), but $3 \neq 4$.

   (d) i. Let $a \in A$ be arbitrary; we want to prove that $S(a,a)$ is true. By definition $S(a,a) = R(f(a), f(a))$. But R is reflexive, so $R(f(a), f(a))$ is true, and we're done.

   ii. This is true. Assume R is symmetric. Say $a_1, a_2 \in A$ are arbitrary and $S(a_1, a_2)$ is true. We want to prove $S(a_2, a_1)$ is true. By definition of S we know $R(f(a_1), f(a_2))$ is true. By symmetry of R we deduce $R(f(a_2), f(a_1))$ is true. By definition of S we deduce $S(a_2, a_1)$ is true, and this was what we wanted.

   iii. This is not true. Here is a counterexample. Let $A = \{5,6\}$ (or any set with two elements), let $B = \{7\}$ (or any set with one element), and define $f : A \to B$ by $f(5) = f(6) = 7$. Let R be the binary relation on B defined by $R(7,7)$ is true. Then R is antisymmetric, because if $x, y \in B$ and $R(x,y)$ and $R(y,x)$ are true then x must equal y, as $x, y \in B$ so they're both equal to 7 and hence equal to each other. However

S is not antisymmetric, because $S(5,6)$ is true (by definition it equals $R(7,7)$, which is true) and $S(6,5)$ is also true (by definition it also equals $R(7,7)$), but $5 \neq 6$.

2. **Total: 20 Marks**

(a) i. Write the definition of the addition on the natural numbers. 2 Marks

Proof. The addition is the unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that

$$\forall x \in \mathbb{N}, x + 0 = x. \tag{1}$$

$$\forall x, y \in \mathbb{N},\ x + \nu(y) = \nu(x + y). \tag{2}$$

$\square$

ii. Let $X \subseteq \mathbb{N}$ be a set which is closed under addition: if $x, y \in X$, then $x + y \in X$. Prove using (P5) that, for every natural number $n \geq 1$ and every $x \in X$, we have $n \cdot x \in X$. 4 Marks

Proof. This is proved by induction. Fix $x \in X$. Let $A := \{a \in \mathbb{N} \mid a \cdot x \in X\} \cup \{0\}$. We claim that $A = \mathbb{N}$. 0 is trivially in the set. By P5 it is enough to prove that $a \in A$ implies $\nu(a) \in A$. Suppose that $a \cdot x \in X$. Then $\nu(a) \cdot x = (a+1) \cdot x = a \cdot x + a \in X$. Finally, we need to show that $\nu(0) \in A$. But $\nu(0) = 1$ and $1 \cdot x = x \in X$ by a result from lecture. So $\nu(0) \in A$. $\square$

(b) i. Write the principle of strong induction 2 Marks

Proof. For each $n \in \mathbb{N}$, let $P(n)$ be a proposition depending on the number n. If

1) $P(n_0)$ is true for some $n_0 \in \mathbb{N}$,

2) for all $n \in \mathbb{N}$, $P(k)$ is true for all $n_0 \leq k \leq n$ implies that $P(n+1)$ is also true,

then $P(n)$ is true for all $n \geq n_0$. $\square$

ii. Show that if $P_0,\ P_1,\ P_2 \ldots P_n$ are $n+1$ propositions, then

$$\neg(P_0 \vee P_1 \vee \cdots \vee P_n) = \neg P_0 \wedge \neg P_1 \wedge \cdots \wedge \neg P_n.$$

4 Marks

Proof. We prove this by induction. For $n = 0$ the statement is trivially true. $n = 1$ is De Morgan's law seen in Part I. Let $k \geq 1$ and let $P(k)$ be the statement we're proving for $n = k$. Assume $P(k)$ is true, hence the induction hypothesis is $\neg(P_0 \vee \cdots \vee P_k) \iff \neg P_0 \wedge \cdots \wedge \neg P_k$. We want to show that $P(k+1)$ is true. We have by De Morgan's law (or using strong induction), and then using the induction hypothesis

$$\begin{aligned} \neg(P_0 \vee \cdots \vee P_{k+1}) \quad &= \quad \neg((P_0 \vee \cdots \vee P_k) \vee P_{k+1}) \\ &\iff \quad \neg(P_0 \vee \cdots \vee P_k) \wedge \neg P_{k+1} \\ &\iff \quad (\neg P_0 \wedge \cdots \wedge \neg P_k) \wedge \neg P_{k+1} = \neg P_0 \wedge \cdots \wedge \neg P_{k+1}, \end{aligned}$$

which proves the result. $\square$

(c) In the following, use only the axioms of the reals.

i. Let x and y be real numbers. Show that if $x + y = x$, then $y = 0$. 3 Marks

Proof. We use the existence of the additive inverse $-x$ and get $-x + (x + y) = -x + x$. Then using associativity of the addition, this is equivalent to $(-x + x) + y = -x + x$. Now using the property of the additive inverse we get $0 + y = 0$ and finally $y = 0$ by the property of the neutral element of addition. $\square$

ii. Show that 0 is the unique neutral element for addition. 2 Marks

Proof. Assume there is another neutral element we call $0'$, such that $x + 0' = 0' + x = x$, for all x. Then by the previous part $0' = 0$. $\square$

(d) Let b be a natural number with $b \geq 2$. Let $b_1 := b$ and let $\{b_n\}$ be a sequence such that $b_{n+1} = b \cdot b_n$ for all $n \geq 1$.
Let $x \in \mathbb{R}$ be a real number. Prove that there is a unique sequence of integers a_n for $n \geq 1$ such that $\frac{a_n}{b_n} \leq x < \frac{a_n+1}{b_n}$ for all n. $\boxed{\text{4 Marks}}$

Proof. For each $n \geq 1$, we claim that there is a unique integer a_n such that $a_n \leq x \cdot b_n < a_n + 1$. Let $y := x \cdot b_n$. If $y > 0$, this is a result from lecture. If y is an integer, then $a_n := y$ satisfies $a_n \leq y < a_n + 1$. Conversely if $a_n \leq y < a_n + 1$ and a_n is an integer, if $a_n \neq y$, then $y - a_n \geq 1$, so that $a_n + 1 \leq y$, a contradiction. Finally, if y is negative and not an integer, then $a_n \leq y < a_n + 1$ for a_n an integer if and only if $a_n < y < a_n + 1$. This is true if and only if $-a_n > -y > -a_n - 1$, i.e., for $a'_n := -a_n - 1$, $a'_n < -y < a'_n + 1$. As $-y > 0$, such a'_n exists and is unique as a result of lectures (as $-y$ is not an integer).
Dividing by b_n, we get $\frac{a_n}{b_n} \leq x < \frac{a_n+1}{b_n}$. Conversely, if this inequality is true then by multiplying by b_n we get that a_n is the unique sequence mentioned previously. □

3. **Total: 20 Marks**

(a) i. *Find a parametric equation for the line $\mathcal{L}$ passing through the points $A(1,2,3)$ and $B(2,3,2)$ in $\mathbb{R}^3$.* $\boxed{\text{2 Marks}}$

A vector directed along the line is given by $\overrightarrow{AB} = (2,3,2) - (1,2,3) = (1,1,-1)$ and we know a point on the line, for example $A(1,2,3)$. Hence a vector/parametric equation of the line may be written as

$$\mathbf{r} = (x,y,z) = (1,2,3) + \lambda(1,1,-1) = (1+\lambda, 2+\lambda, 3-\lambda),$$

for $\lambda \in \mathbb{R}$.

ii. *Does the point $C(-1,0,1)$ lie on $\mathcal{L}$?* $\boxed{\text{1 Mark}}$
If C lies on the line then simultaneously we must have

$$\begin{aligned} 1+\lambda &= -1, \\ 2+\lambda &= 0, \\ 3-\lambda &= 1. \end{aligned}$$

The first and second of these give $\lambda = -2$ but the third gives $\lambda = 2$ hence the point C does not lie on the line.

iii. *Compute the distance between $\mathcal{L}$ and the point $D(1,-2,2)$.* $\boxed{\text{3 Marks}}$

The distance between the line and point D can be calculated as

$$d(\mathcal{L}, D) = \frac{|\overrightarrow{AD} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|}.$$

Now $\overrightarrow{AD} = (1,-2,2) - (1,2,3) = (0,-4,-1)$, so then

$$\begin{aligned} \overrightarrow{AD} \times \overrightarrow{AB} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & -1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \mathbf{i}(4-(-1)) - \mathbf{j}(0-(-1)) + \mathbf{k}(0-(-4)) \\ &= 5\mathbf{i} - \mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Now finding the moduli

$$\begin{aligned} |\overrightarrow{AD} \times \overrightarrow{AB}| &= \sqrt{5^2 + (-1)^2 + 4^2} = \sqrt{42}, \\ |\overrightarrow{AB}| &= \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}, \end{aligned}$$

which leads to the distance between the point D and the line being

$$d(\mathcal{L}, D) = \frac{|\overrightarrow{AD} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|} = \sqrt{14}.$$

iv. *Does $\mathcal{L}$ intersect the plane with cartesian equation $x + y - z - 9 = 0$? If yes, where? If no, why not?* 2 Marks

We know on the line that

$$\begin{aligned} x &= 1 + \lambda, \\ y &= 2 + \lambda, \\ z &= 3 - \lambda. \end{aligned}$$

So, if the line intersects the plane then we must have that

$$(1 + \lambda) + (2 + \lambda) - (3 - \lambda) - 9 = 0,$$

for some value of λ. Indeed, solving this we find $\lambda = 3$ and so the line intersects the plane at the point $(4, 5, 0)$.

(b) *Consider the curve*

$$\mathbf{r}(t) = (t\cos(2t) - \sin(2t))\mathbf{i} + t^2\mathbf{j} + (t\sin(2t) + \cos(2t))\mathbf{k},$$

parameterised by $t \geq 0$ in $\mathbb{R}^3$.

i. *Find the length of the curve on the interval $0 \leq t \leq 3$.* 3 Marks

Differentiating $\mathbf{r}(t)$ gives

$$\mathbf{r}'(t) = -2t\sin(2t)\mathbf{i} + 2t\mathbf{j} + 2t\cos(2t)\mathbf{k}.$$

This has modulus given by

$$|\mathbf{r}'(t)| = \sqrt{(-2t\sin(2t))^2 + (2t)^2 + (2t\cos(2t))^2} = 2\sqrt{2}t,$$

taking the positive square root since $t \geq 0$. Finally computing the length of the curve gives

$$L = 2\sqrt{2}\int_0^3 t\,dt = \sqrt{2}[t^2]_0^3 = 9\sqrt{2}.$$

ii. *Find the unit tangent vector $\mathbf{T}(t)$ to the curve.* 1 Mark

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\ &= \frac{-2t\sin(2t)\mathbf{i} + 2t\mathbf{j} + 2t\cos(2t)\mathbf{k}}{2\sqrt{2}t} \\ &= -\frac{\sin(2t)}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{\cos(2t)}{\sqrt{2}}\mathbf{k}. \end{aligned}$$

iii. *Find the unit normal vector $\mathbf{N}(t)$ to the curve.* 2 Marks

Let's compute the derivative of $\mathbf{T}(t)$:

$$\mathbf{T}'(t) = -\sqrt{2}(\cos(2t)\mathbf{i} + \sin(2t)\mathbf{k}).$$

Then the normal can be calculated as

$$\begin{aligned}
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \\
&= \frac{-\sqrt{2}(\cos(2t)\mathbf{i} + \sin(2t)\mathbf{k})}{\sqrt{(-\sqrt{2}\cos(2t))^2 + 0^2 + (-\sqrt{2}\sin(2t))^2}} \\
&= \frac{-\sqrt{2}(\cos(2t)\mathbf{i} + \sin(2t)\mathbf{k})}{\sqrt{2}} \\
&= -(\cos(2t)\mathbf{i} + \sin(2t)\mathbf{k}).
\end{aligned}$$

(c) *Initially, a particle is stationary at the origin. Its acceleration at time t is given by*

$$\mathbf{a}(t) = \frac{1}{t+1}\mathbf{i} + e^{-t}\mathbf{j} + \sin(t)\mathbf{k}.$$

Find an expression for the trajectory of the particle. $\boxed{\textit{6 Marks}}$

Let's integrate to find an expression for the velocity:

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s)ds \\
&= 0 + \int_0^t \frac{1}{s+1}ds\mathbf{i} + \int_0^t e^{-s}ds\mathbf{j} + \int_0^t \sin(s)ds\mathbf{k} \\
&= [\log(s+1)]_0^t\,\mathbf{i} - \left[e^{-s}\right]_0^t\mathbf{j} - [\cos(s)]_0^t\,\mathbf{k} \\
&= \log(t+1)\mathbf{i} + (1-e^{-t})\mathbf{j} + (1-\cos(t))\mathbf{k}.
\end{aligned}$$

Now let's integrate again to find an expression for the position of the particle:

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(s)ds \\
&= 0 + \int_0^t \log(s+1)ds\mathbf{i} + \int_0^t (1-e^{-s})ds\mathbf{j} + \int_0^t (1-\cos(s))ds\mathbf{k} \\
&= \left([s\log(s+1)]_0^t - \int_0^t \frac{s}{s+1}ds\right)\mathbf{i} + \left[s+e^{-s}\right]_0^t\mathbf{j} + [s-\sin(s)]_0^t\,\mathbf{k} \\
&= \left(t\log(t+1) + [\log(s+1)-s]_0^t\right)\mathbf{i} + (t+e^{-t}-1)\mathbf{j} + (t-\sin(t))\mathbf{k} \\
&= ((t+1)\log(t+1)-t)\mathbf{i} + (t+e^{-t}-1)\mathbf{j} + (t-\sin(t))\mathbf{k},
\end{aligned}$$

where to reach the third line we have integrated $\log(s+1)$ by parts and to reach the fourth line we have written $s/(s+1)$ in the form $1-\frac{1}{s+1}$ to perform the integration.