Imperial College London

MATH97066 MATH97177

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Bifurcation Theory

Date: Tuesday, 11 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider a system of differential equations in $\mathbb{R}^4$. How many stable equilibria and periodic orbits can be born at the following bifurcations:

(a) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{3} \pm i$, $\pm i$, and the Lyapunov coefficients $L_1 = L_2 = 0,\ L_3 > 0$? (5 marks)

(b) A periodic orbit with the multipliers $-\frac{1}{3} \pm i, -1$ and the first Lyapunov coefficient negative? (5 marks)

(c) A periodic orbit with the multipliers $(-1, -1, 1)$? (5 marks)

(d) A homoclinic loop to an equilibrium with the eigenvalues of the linearisation matrix equal to $-2 \pm i, -2, 1$? (5 marks)

(Total: 20 marks)

2. (a) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following system of differential equation to the center manifold:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + y + x^2 + z^2, \\ \dot{z} = x^2 + xz - yz + z^2. \end{cases}$$

(12 marks)

(b) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following map to the center manifold:

$$\begin{cases} \bar{x} = -x + x^3 - xy, \\ \bar{y} = x + 2y + 4x^2. \end{cases}$$

(8 marks)

(Total: 20 marks)

3. Consider the following map on the interval $[0,1]$:

$$\bar{x} = f(x) = a(\sqrt{x} - x)$$

with a positive parameter a.

(a) For which values of the parameter a does the map have a fixed point with the multiplier $+1$ or -1? (12 marks)

(b) For which values of the parameter a does the map have a stable fixed point? (8 marks)

(Total: 20 marks)

4. Consider the system

$$\begin{cases} \dot{x} = y + x^2, \\ \dot{y} = -x + \varepsilon y + axy - y^2, \end{cases}$$

that depends on parameters ε and a.

(a) Write the normal form, up to the terms of the third order, for this system near the equilibrium $(x,y) = (0,0)$ at $\varepsilon = 0$. (12 marks)

(b) Find the first Lyapunov coefficient. Determine for which values of ε and a the equilibrium state at $(x,y) = (0,0)$ is asymptotically stable. How many periodic orbits can exist near $(x,y) = (0,0)$ for small $\varepsilon > 0$ at $a = 2$? At $a = -2$? At $a = 0$? (8 marks)

(Total: 20 marks)

5. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω near $2\pi/5$.

(a) By counting resonant terms, show that the normal form for such map is given by

$$\bar{z} = (1+\mu)e^{\pm i\omega} z[1 + (L + i\Omega)|z|^2 + A(z^*)^4 + O(|z|^5)],$$

where z is a complex variable, z^* is complex-conjugate to z, and $A = ae^{i\psi}$, L, and Ω are constants. (6 marks)

(b) Assume that the first Lyapunov coefficient satisfies $L < 0$. By scaling z we can always make $L = -1$ in this case. In the polar coordinates $z = re^{i\varphi}$ the normal form recasts as

$$\bar{r} = (1+\mu)\ r\ (1 - r^2 + ar^3 \cos(5\varphi - \psi) + O(r^4)), \quad \bar{\varphi} = \varphi + \frac{2\pi}{5} + \delta + \Omega r^2 - ar^3 \sin(5\varphi - \psi) + O(r^4),$$

where $\delta = \omega - 2\pi/5$ is a small parameter; you do not need to verify this formula. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$. The invariant curve attracts all orbits from a small neighbourhood of the fixed point, independent of μ and δ. It can be shown that the curve has an equation $r = f(\varphi)$ where f is a smooth, positive, periodic function of φ. Show that

$$|f(\varphi) - \sqrt{\mu}| \leq (a+1)\mu.$$

(8 marks)

(c) Show that in the (μ, δ)-plane near the origin there exist parameter values corresponding to the existence of orbits of period 5. You may use the fact shown in part (b), that the invariant curve satisfies $r = f(\varphi) = O(\sqrt{\mu})$. Show that for all small positive μ the set of values of δ for which the map has orbits of period 5 has diameter $O(\mu^{3/2})$. (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96041/MATH97066/MATH97177/MATH97242

Bifurcation Theory (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. Consider a system of differential equations in $\mathbb{R}^4$. How many stable equilibria and periodic orbits can be born at the following bifurcations: $\boxed{\text{seen/sim.seen} \Downarrow}$

(a) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{3} \pm i$, $\pm i$, and the Lyapunov coefficients $L_1 = L_2 = 0$, $L_3 > 0$?

The equilibrium can become stable when the eigenvalues move to the left of the imaginary axis. No other equilibria can be born. Up to 3 periodic orbits can be born - only one of them can be stable. $\boxed{5, \text{A}}$

(b) A periodic orbit with the multipliers $-\frac{1}{3} \pm i, -1$ and the first Lyapunov coefficient negative?

No equilibria can be born out of a periodic orbit. The periodic orbit has multipliers outside of the unit circle, so no stable periodic orbits can be born. $\boxed{5, \text{A}}$

(c) A periodic orbit with the multipliers $(-1, -1, 1)$?

Two multipliers can be moved to the unit circle and one strictly inside of the unit circle. After that a stable invariant curve can be born, on which there can be infinitely many resonant stable periodic orbits. $\boxed{5, \text{A}}$

(d) A homoclinic loop to an equilibrium with the eigenvalues of the linearisation matrix equal to $-2 \pm i, -2, 1$?

The nearest to the imaginary axis eigenvalue is real, so only 1 stable periodic orbit can be born when the homoclinic loop splits. $\boxed{5, \text{A}}$

2. (a) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following system of differential equation to the center manifold: unseen $\Downarrow$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + y + x^2 + z^2, \\ \dot{z} = x^2 + xz - y^2 + z^2. \end{cases}$$

The linearisation matrix is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and the eigenvalues are $\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ and $\lambda_3 = 0$. The center manifold is one-dimensional and has an equation of the type

$$(x, y) = f(z).$$

After the coordinate transformation $x^{new} = x - z^2$ the system takes the form

$$\begin{cases} \dot{x}^{new} = y + O_3, \\ \dot{y} = -x^{new} + y + (x^{new})^2 + O_3, \\ \dot{z} = (x^{new})^2 + x^{new}z - y^2 + z^2 + 2x^{new}z^2 + z^3 + O_4, \end{cases}$$

where O_3 stands for cubic and higher order terms O_4 stands for quartic and higher order terms. Since the equations for $\dot{x}^{new}$ and $\dot{y}$ have no quadratic terms depending on z, the equation for the center manifold is

$$x^{new} = 0 + O_3, \quad y = 0 + O_3.$$

Substituting this into the equation for $\dot{z}$, we obtain the system on the center manifold: $\dot{z} = z^2 + z^3 + O_4.$ 12, D

(b) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following map to the center manifold: seen/sim.seen $\Downarrow$

$$\begin{cases} \bar{x} = -x + x^3 - xy, \\ \bar{y} = x + 2y + 4x^2. \end{cases}$$

Do the coordinate transformation $y^{new} = y + ax + bxy$ with indeterminate coefficients a and b. We obtain

$$\bar{y}^{new} = x + 2y + 4x^2 + a(-x + x^3 - xy) + b(-x + x^3 - xy)(x + 2y + 4x^2) = x(1-a) + 2y + (4-b)x^2 - (a+2b)xy + O_3.$$

If we take $a = 1/3$, we get

$$\bar{y}^{new} = 2y^{new} + (4-b)x^2 - (\frac{1}{3} + 4b)xy + O_3 = y^{new} + [(4-b) + \frac{1}{3}(\frac{1}{3} + 4b)]x^2 - (\frac{1}{3} + 4b)xy^{new} + O_3.$$

By taking $b = -\frac{37}{3}$, we make the coefficient of x^2 vanish, so the equation of the center manifold becomes $y^{new} = O_3$. Returning to the old coordinate y, we obtain

$$y(1 - \frac{37}{3}x) + \frac{1}{3}x = O_3 \Longrightarrow y = -\frac{1}{3}x - 4\frac{1}{9}x^2 + O_3.$$

Substituting this into the equation for $\bar{x}$, we obtain

$$\bar{x} = -x + \frac{1}{3}x^2 + 5\frac{1}{9}x^3 + O_4.$$

8, C

3. Consider the following map on the interval $[0,1]$: $\boxed{\text{seen/sim.seen} \Downarrow}$

$$\bar{x} = f(x) = a(\sqrt{x} - x)$$

with a positive parameter a.

(a) For which values of the parameter a does the map have a fixed point with the multiplier $+1$ or -1?

The fixed points are $x = 0$ and $x_a = \left(\frac{a}{a+1}\right)^2$. The derivative at $x = 0$ is $+\infty$, so this fixed point is always unstable. The derivative at $x = x_a$ is

$$a(\frac{1}{2\sqrt{x_a}} - 1) = \frac{1-a}{2}.$$

It cannot be equal to 1 at positive a. It equals to -1 at $a = 3$. $\boxed{12, \text{A}}$

(b) For which values of the parameter a does the map have a stable fixed point?

We have $|f'(x_a)| < 1$ for $a < 3$ and $|f'(x_a)| > 1$ for $a > 3$. So the fixed point $x = x_a$ is stable for $a < 3$ and unstable for $a > 3$. To determine the stability at $a = 3$, compute the Schwarz derivative at $a = 3$ at the point x_a:

$$f'(x_a) = -1, \quad f''(x_a) = -\frac{a}{4}x_a^{-3/2} = -\frac{16}{9}, \quad f'''(x_a) = \frac{3a}{8}x_a^{-5/2} = \frac{128}{27}.$$

The Schwarz derivative $S = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ is negative, so the fixed point is also stable at $a = 3$. $\boxed{4, \text{B}}$

$\boxed{4, \text{C}}$

seen/sim.seen ⇓

4. Consider the system

$$\begin{cases} \dot{x} = y + x^2, \\ \dot{y} = -x + \varepsilon y + axy - y^2, \end{cases}$$

that depends on parameters ε and a.

(a) Write the normal form, up to the terms of the third order, for this system near the equilibrium $(x, y) = (0, 0)$ at $\varepsilon = 0$.

Let $z = x - iy$ (so $x = (z + z^*)/2$ and $y = (z^* - z)/2i$). The system at $\varepsilon = 0$ takes the form

$$\dot{z} = iz + \frac{1}{4}(z+z^*)^2 + \frac{a}{4}(z^2 - (z^*)^2) - \frac{i}{4}(z - z^*)^2 = iz + \frac{1+a-i}{4}z^2 + \frac{1+i}{2}zz^* + \frac{1-a-i}{4}(z^*)^2.$$

Introduce new variable

$$w = z + i\frac{1+a-i}{4}z^2 - i\frac{1+i}{2}zz^* - i\frac{1-a-i}{12}(z^*)^2.$$

We get

$$\dot{w} - iw = \dot{z} + i\frac{1+a-i}{2}z\dot{z} - i\frac{1+i}{2}(\dot{z}z^* + z\dot{z}^*) - i\frac{1-a-i}{6}z^*\dot{z}^* - iz + \frac{1+a-i}{4}z^2 - \frac{1+i}{2}zz^* - \frac{1-a-i}{12}(z^*)^2 =$$

$$= -(\frac{a}{8} + \frac{8 - 5a + a^2}{24})z^2 z^* + \dots,$$

where the dots stand for non-resonant cubic terms and terms of higher order. By killing the non-resonant cubic terms, we obtain the sought normal form:

$$\dot{w} = iw - (\frac{a}{8} + \frac{8 - 5a + a^2}{24})w^2 w^* + O_4.$$

12, B

(b) Find the first Lyapunov coefficient. Determine for which values of ε and a the equilibrium state at $(x, y) = (0, 0)$ is asymptotically stable. How many periodic orbits can exist near $(x, y) = (0, 0)$ for small $\varepsilon > 0$ at $a = 2$? At $a = -2$? At $a = 0$?

The first Lyapunov coefficient is

$$L_1 = -a/8.$$

The fixed point is asymptotically stable at $\varepsilon < 0$ (as the trace of the linearization matrix is negative in this case) and unstable at $\varepsilon > 0$. The stability at $\varepsilon = 0$ is determined by the sign of L_1: the equilibrium at zero for $\varepsilon = 0$ is asymptotically stable when $a > 0$ and unstable when $a < 0$. The stability (but not asymptotic stable) at $a = 0$, $\varepsilon = 0$ follows from the reversibility of the system (the symmetry with respect to $t \to -t$, $(x, y) \to (y, x)$).
At $a = 2$ the Andronov-Hopf bifurcation is supercritical ($L_1 < 0$), so one periodic orbit is born at $\varepsilon > 0$. It is subcritical for $a = -2$ ($L_1 > 0$), so no periodic orbit is born at $\varepsilon > 0$. By the reversibility, a neighbourhood of the equilibrium is filled by periodic orbits at $a = 0, \varepsilon = 0$, so no periodic orbits are born at $\varepsilon > 0$ by Hopf theorem.

4, B

4, D

5. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω near $2\pi/5$.

(a) By counting resonant terms, show that the normal form for such map is given by

$$\bar{z} = (1+\mu)e^{\pm i\omega} z[1 + (L+i\Omega)|z|^2 + A(z^*)^4 + O(|z|^5)],$$

where z is a complex variable, z^* is complex-conjugate to z, and $A = ae^{i\psi}$, L, and Ω are constants.

It is enough to show it for $\mu = 0$, $\omega = 2\pi/5$. The term $z^m(z^*)^n$ is resonant (hence is present in the normal form) if

$$e^{\frac{2\pi i}{5}} = e^{\frac{2\pi i}{5}(m-n)}.$$

Taking logarithm, we obtain

$$\frac{2\pi i}{5} = \frac{2\pi i}{5}(m-n) + 2\pi i k \Longrightarrow m = n+1-5k$$

for an integer k. Only $(m,n) = (2,1)$ and $(m,n) = (0,4)$ give terms of order less than 5. $\boxed{6,\ \mathrm{M}}$

(b) Assume that the first Lyapunov coefficient satisfies $L < 0$. By scaling z we can always make $L = -1$ in this case. In the polar coordinates $z = re^{i\varphi}$ the normal form recasts as

$$\bar{r} = (1+\mu)\, r\, (1-r^2+ar^3\cos(5\varphi-\psi)+O(r^4)), \qquad \bar{\varphi} = \varphi+\frac{2\pi}{5}+\delta+\Omega r^2-ar^3\sin(5\varphi-\psi)+O(r^4),$$

where $\delta = \omega - 2\pi/5$ is a small parameter; you do not need to verify this formula. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$. The invariant curve attracts all orbits from a small neighbourhood of the fixed point, independent of μ and δ. It can be shown that the curve has an equation $r = f(\varphi)$ where f is a smooth, positive, periodic function of φ. Show that

$$|f(\varphi) - \sqrt{\mu}| \le (a+1)\mu.$$

It is enough to show that the annulus $|r - \sqrt{\mu}| \le (a+1)\mu$ is taken inside itself by the map. We have

$$\bar{r} \le (1+\mu)\, r\, (1 - r^2 + ar^3 + O(r^4)),$$

so if $r = \sqrt{\mu} + (a+1)\mu$, then

$$\frac{\bar{r}}{r} \le (1+\mu)\, (1-\mu-2(a+1)\mu^{3/2}+a\mu^{3/2}+O(\mu)) = 1-(a+2)\mu^{3/2}+O(\mu) < 1.$$

On the other hand,

$$\bar{r} \ge (1+\mu)\, r\, (1 - r^2 - ar^3 + O(r^4)),$$

so if $r = \sqrt{\mu} - (a+1)\mu$, then

$$\frac{\bar{r}}{r} \ge (1+\mu)\, (1-\mu+2(a+1)\mu^{3/2}-a\mu^{3/2}+O(\mu)) = 1+(a+2)\mu^{3/2}+O(\mu) > 1.$$

This proves the claim. $\boxed{8,\ \mathrm{M}}$

(c) Show that in the (μ, δ)-plane near the origin there exist parameter values corresponding to the existence of orbits of period 5. You may use the fact shown in part (b), that the invariant curve satisfies $r = f(\varphi) = O(\sqrt{\mu})$. Show that for all small positive μ the set of values of δ for which the map has orbits of period 5 has diameter $O(\mu^{3/2})$.

The map restricted to the invariant curve $r = \sqrt{\mu} + O(\mu)$ is

$$\bar{\varphi} = \varphi + \frac{2\pi}{5} + \delta + \Omega\mu + O(\mu^{3/2}).$$

When $\delta + \Omega\mu \ll -\mu^{3/2}$, the rotation number is less than $1/5$, while it is larger than $1/5$ when $\delta + \Omega\mu \gg \mu^{3/2}$. By continuity, it follows that for every small δ there is an interval of μ (of length no more than $O(\mu^{3/2})$ where the rotation number equals $1/5$, which corresponds to the existence of points of period 5. $\boxed{6, \mathrm{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96041 MATH97066 MATH97177	1	When all multipliers are on the unit circle, one can perturb the system such that two complex-conjugate multipliers remain on the unit circle, while others get inside. This corresonds to a stable torus bifurcation, leading to an unlimited number of stable resonant periodic orbits
MATH96041 MATH97066 MATH97177	2	Normalising transformations are easy and work well when the linear part of the system is brought to diagonal (or Jordan) form first.
MATH96041 MATH97066 MATH97177	3	Most students did it well
MATH96041 MATH97066 MATH97177	4	Most students did it well
MATH96041 MATH97066 MATH97177	5	Most students did it well