

Mathematics Pre-arrival course

Solutions to Problem Sheet 1 – Language of Mathematics and Real Functions

The starred questions on this problem sheet are for you to think about — we will not be giving solutions to them in the pre-arrival course. Instead these will form the basis of discussion in your first *MATH40001/MATH40009 - Introduction to University Mathematics* session once you arrive at Imperial.

Language of Mathematics

1. Rewrite the following statements formally with quantifiers (don't forget to refer to the mathematical notation handout).
 - (a) If x and y are real numbers and y is strictly positive, $x + y$ is always bigger than x .
 $\forall x \forall y (((x \in \mathbb{R} \wedge y \in \mathbb{R}) \wedge y > 0) \Rightarrow x + y > x)$
 - (b) Every real number has a complex square root.
 $\forall x (x \in \mathbb{R} \Rightarrow \exists y (y \in \mathbb{C} \wedge y^2 = x))$
 - (c) The average of two positive integers is positive.
 $\forall x \forall y (((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \wedge (x > 0 \wedge y > 0)) \Rightarrow (x + y)/2 > 0)$
 - (d) The difference of two negative integers is not necessarily negative.
 $\exists x \exists y \neg ((x < 0 \wedge y < 0) \Rightarrow x - y < 0)$
2. Let S be the set of all people living in London, and A a function that associates to every member of S their age.
 - (a) Write the function formally.

$$A : S \rightarrow \mathbb{N} \cup \{0\}$$

$$s \mapsto A(s) = \text{age of } s$$

- (b) Rewrite with quantifiers the following statement: in London, everybody is older than somebody.

$$\forall s \in S, \exists t \in S : A(t) < A(s)$$

3. ★ Let n be an integer. Prove (carefully) that

- (a) if 2 divides n , then 2 divides n^2 .

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

- (b) if 2 divides n^2 , then 2 divides n .

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

4. Show that $\sqrt{2} \notin \mathbb{Q}$.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then $\exists m, n \in \mathbb{Z}, n \neq 0 : \sqrt{2} = \frac{m}{n}$.

Wlog, assume that all common factors have been cancelled, so that m, n have no common factors other than 1 or -1 .

Then

$$2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2 \quad (1)$$

So $2|m^2$ (this notation means that 2 divides m^2 , said differently, $\exists k \in \mathbb{Z} : m^2 = 2k$)

Using the result from the previous question, we conclude that then $2|m$ (*this really needs a proof...have a think about how you would prove it — see previous question!*). This means that $\exists r \in \mathbb{Z} : m = 2r$. Substituting this into Equation 1 gives

$$(2r)^2 = 4r^2 = 2n^2 \quad \text{so} \quad n^2 = 2r^2$$

This means that $2|n^2$, so again, $2|n$.

Drawing together our results, we have shown that $2|m$ and $2|n$. But this contradicts our assumption that m, n have no common factors other than 1 and -1 .

Hence, our original assumption, that $\sqrt{2} \in \mathbb{Q}$, must be false. So $\sqrt{2} \notin \mathbb{Q}$ ■

5. Prove by induction that 3 divides $n^3 - n$ for all integers $n \geq 0$.

To prove a proposition by induction, we need to proceed in two steps:

- **Base case:** for $n = 1$, $n^3 - n = 0$, so we have checked that 3 trivially divides $n^3 - n$ for $n = 1$.
- **Induction step:** assume that for some $m \in \mathbb{N}$, $m > 1$, we have $3|m^3 - m$, we want to show that $3|(m+1)^3 - (m+1)$. But we know that

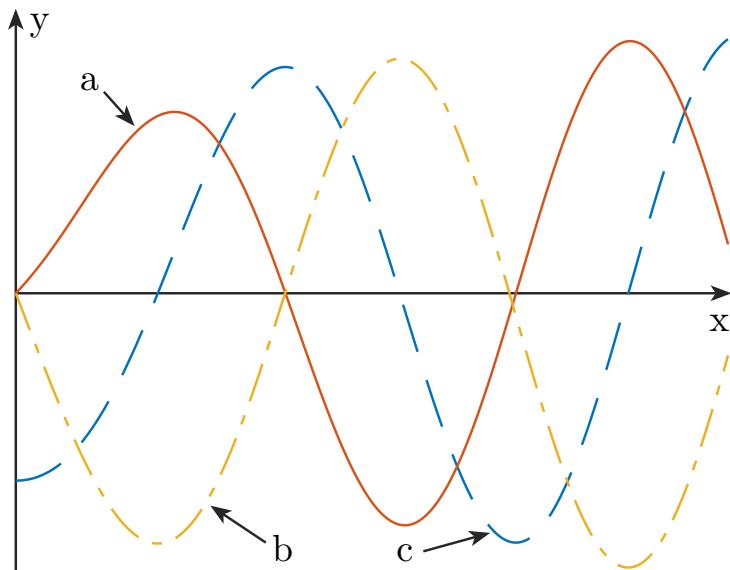
$$(m+1)^3 - (m+1) = m^3 + 3m^2 + 3m + 1 - m - 1 = m^3 - m + 3(m^2 + m)$$

By assumption, we know that $\exists k \in \mathbb{Z} : m^3 - m = 3k$, so we can write $(m+1)^3 - (m+1) = 3(k + m^2 + m)$, defining $k' = k + m^2 + m$, we have thus found $k' \in \mathbb{Z}$ such that $(m+1)^3 - (m+1) = 3k'$ and thus $3|(m+1)^3 - (m+1)$. By induction, we conclude that

$$\forall n \in \mathbb{N}, 3|n^3 - n$$

Real Functions

6. The following figure shows the graph of a function $f(x)$, its derivative $f'(x)$ and the definite integral $F(x) = \int_0^x f(t)dt$. Can you identify each graph? Explain your reasoning.



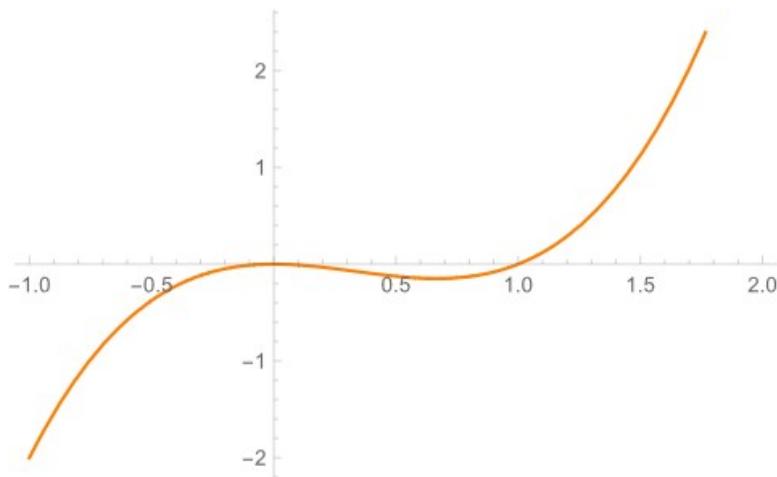
At $x = 0$, the slope of a is positive, which means neither b nor c can be its derivative. Therefore, $a = f'(x)$.

At $x = 0$, the slope of b is negative, but a is positive, so c is our only candidate for $f(x)$, and thus $b = \int_0^x f(t)dt$.

7. Try to sketch by hand the following functions:

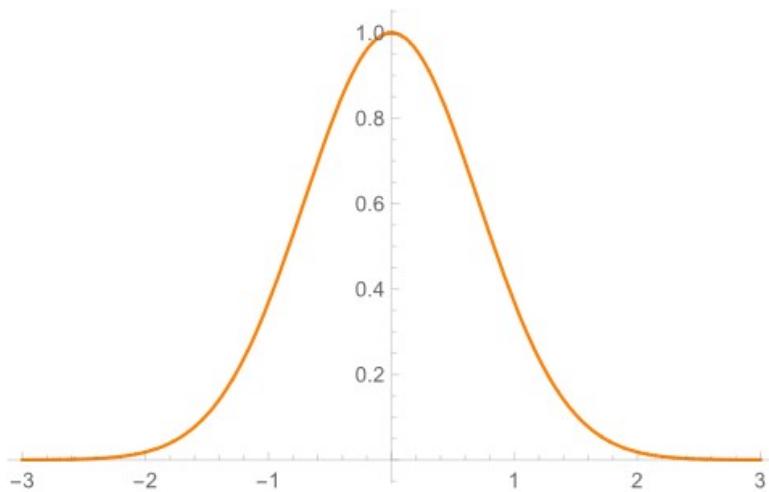
$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto x^3 - x^2$$



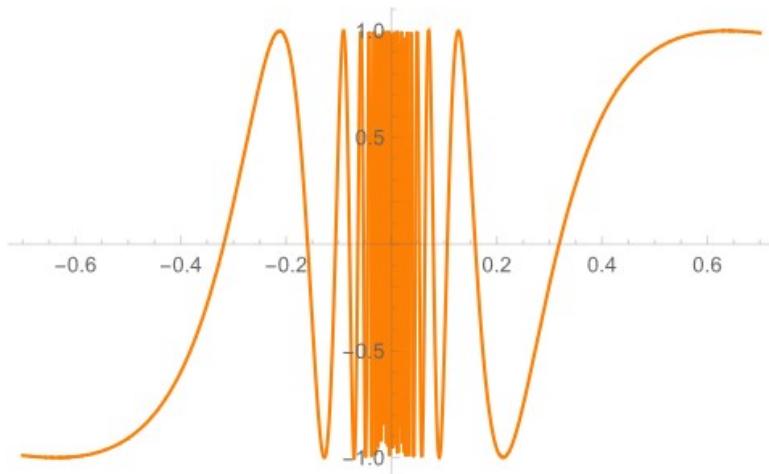
$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto e^{-x^2}$$



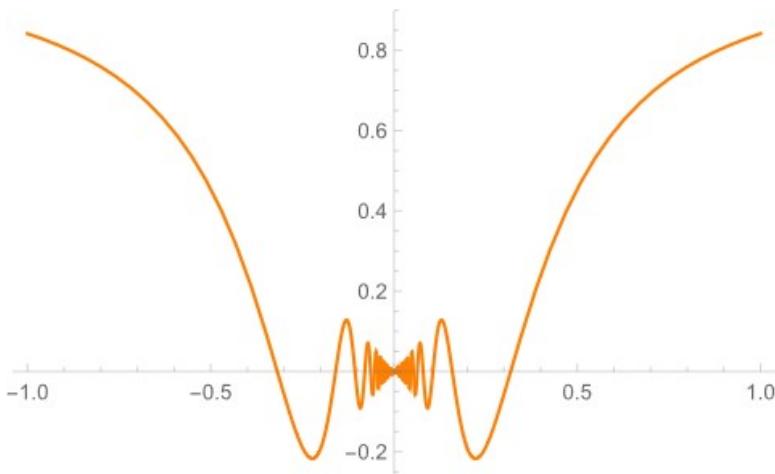
$$f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$$

$$x \mapsto \sin\left(\frac{1}{x}\right)$$



$$f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$$

$$x \mapsto x \sin\left(\frac{1}{x}\right)$$



8. Can you think of a **continuous function which is not differentiable?**

An example is given by

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad \text{with} \quad f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{otherwise} \end{cases}$$

9. ★ Can you think of a function which is **discontinuous everywhere?**

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

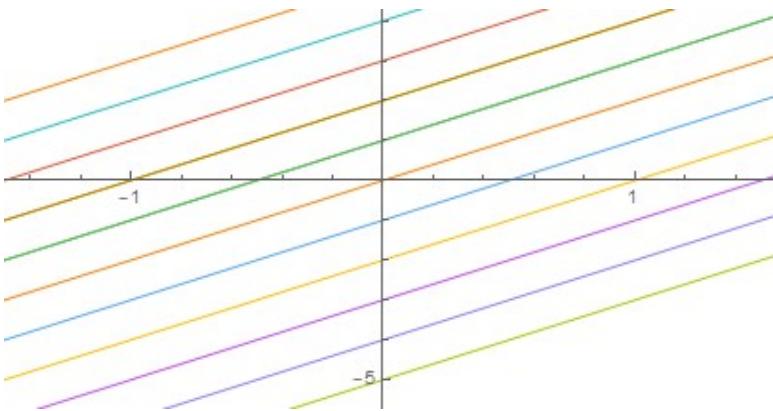
10. ★ What about one which is **differentiable everywhere, but with a discontinuous derivative?**

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

Differential Equations

11. Find the family of solutions to $\frac{dy}{dx} = 2$, and draw them on the plane.

The family of solutions is given by $y = 2x + c$, for $c \in \mathbb{R}$



12. Find the solution to the initial value problem:

$$\frac{dy}{dx} = y, \quad y(0) = 2$$

At A-levels, you have seen that first-order ordinary differential equations with constant coefficients of this kind have solutions of the form

$$y(x) = A \exp(x)$$

This can easily be checked by substituting in the equation for instance. Further, $y(0) = 2 \Rightarrow A \exp(0) = 2 \Rightarrow A = 2$. So we conclude that the solution is given by

$$y(x) = 2 \exp(x)$$

If you do not remember the form of the solutions, these can be rederived easily by separation of variables and direct integration

$$\frac{dy}{dx} = y \Rightarrow \frac{dx}{dy} = \frac{1}{y}$$

Thus

$$x = \int \frac{1}{y} dy = \log(y) + c$$

where $\log(x)$ represents the natural logarithm, also denoted $\ln(x)$. So $e^x = e^{(\log y + c)} = ye^c$ and plugging in the initial value, we get $e^0 = 2e^c$ so $e^c = \frac{1}{2}$, so we get:

$$y = 2e^x$$

13. Check that $y(x) = a \sin(x + \lambda) + b \cos(x + \mu)$ is a solution to:

$$\frac{d^2y}{dx^2} = -y$$

We can sub in the equation to find:

$$\begin{aligned} \frac{dy}{dx} &= a \cos(x + \lambda) - b \sin(x + \mu) \\ \frac{d^2y}{dx^2} &= -a \sin(x + \lambda) - b \cos(x + \mu) = -y \end{aligned}$$

14. ★ Can an initial value problem have more than one solution?

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

15. ★ Consider a box containing radioactive atoms, at any time t , we denote $x(t)$ the number of radioactive atoms remaining in the box. With time, these atoms decay; as each atom has the same chance to decay, the rate of change of atom number is proportional to the number of atoms remaining. Can you write an ordinary differential equation governing the number of atoms in the box? Find a solution to this problem, knowing that there was initially x_0 atoms in the box.

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.