Math40002 Analysis 1 — Unseen 4

1. Let $s_n = \sum_{k=1}^{n} \frac{1}{n+k}$. Prove s_n converges.

2. Define a sequence by $a_1 = 1$ and $a_{n+1} = (a_n + 1)^{1/2}$. Prove that $a_n \to (1+\sqrt{5})/2$.

3. In Unseen 2, for a sequence $(a_n)_{n=1}^{\infty}$, we defined $\limsup(a_n)_{n=1}^{\infty}$ to be $\inf_{m\geq 1} \left\{ \sup_{n\geq m} \{ a_n \} \right\}$. Prove:
$$\limsup(a_n)_{n=1}^{\infty} = \lim_{m\to\infty} \left(\sup_{n\geq m} \{ a_n \} \right)$$
in the sense that if one side of the equation exists, then so does the other and then they are equal.

   In the same fashion, give two definitions for lim inf and show that they are equivalent in the same sense as above.

4. Let (a_n) be a sequence. Prove that $a_n \to a$ if and only if $a_{2n} \to a$ and $a_{2n+1} \to a$.

   Try to generalize.

5. The sequence b_n has b_1 and b_2 positive, and $b_{n+2} = b_n + b_{n+1}$ (note that then $b_n > 0$ for all n). Define $a_n = b_{n+1}/b_n$. Prove that (a_n) converges, and find the limit.