Imperial College London

MATH70132

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Logic

Date: 12 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Work in ZFC throughout, unless indicated otherwise. The notation is as used in lectures. In particular: both $\mathbb{N}$ and ω denote the set of natural numbers; L is the formal system for propositional logic; if $\mathcal{L}$ is a first-order language, then $K_{\mathcal{L}}$ is the associated formal system for first-order logic.

Unless indicated otherwise, you may use results from the notes if these do not depend on what is being asked in the question. Results quoted from the notes should be stated clearly.

1. (a) (i) Find a formula in disjunctive normal form which is logically equivalent to $(\neg\theta)$, where θ is the formula:

$$((p_1 \to (p_2 \to (\neg p_3))) \wedge (p_2 \to (p_3 \to p_1))).$$

(3 marks)

(ii) For $n \geq 2$, find a formula involving only the connectives $\neg$, $\to$ which is logically equivalent to $p_1 \wedge p_2 \wedge \cdots \wedge p_n$. Justify your answer. (4 marks)

(b) How many truth functions of two variables can be constructed using the set of connectives $\{\vee, \wedge\}$? Explain your answer. (4 marks)

(c) Suppose $\Sigma \cup \{\psi\}$ is a set of L-formulas. Define what is meant by a *deduction of* ψ from Σ and explain the notation $\Sigma \vdash_L \psi$. (3 marks)

In each of the following cases, decide whether the given statement is true for all sets of L-formulas $\Gamma \cup \{\phi, \psi\}$ and give a proof or counterexample in each case.

(i) If $\Gamma \vdash_L (\phi \to \psi)$, then $\Gamma \cup \{\phi\} \vdash_L \psi$. (2 marks)

(ii) If $\Gamma \vdash_L ((\neg\phi) \to \psi)$ and $\Gamma \vdash_L ((\neg\phi) \to (\neg\psi))$, then $\Gamma \vdash_L \phi$. (4 marks)

(Total: 20 marks)

2. (a) For an L-formula ϕ, let $V[\phi]$ denote the set of propositional valuations v with $v(\phi) = T$. Prove that, for all L-formulas ϕ, ψ, the following statements are equivalent:

(1) $V[\phi] \subseteq V[\psi]$;

(2) $(\phi \to \psi)$ is a tautology;

(3) $(\phi \to \psi)$ is a theorem of L.

Results from the notes may be used if clearly stated. (6 marks)

(b) Formulate and state a result about the formal system $K_{\mathcal{L}}$ for first-order logic which is similar to the result in part (a). How does the proof in part (a) need modifying? Does this sort of result about $K_{\mathcal{L}}$ require the formulas to be closed? (4 marks)

(c) Suppose that $\mathcal{L}$ is a first-order language, Δ is a consistent set of closed $\mathcal{L}$-formulas and θ, χ are closed $\mathcal{L}$-formulas. Suppose that $\Delta \cup \{(\theta \to \chi)\}$ is inconsistent. Prove that

$$\Delta \vdash_{K_{\mathcal{L}}} \theta \text{ and } \Delta \vdash_{K_{\mathcal{L}}} (\neg\chi).$$

You should use syntactic arguments here; do not use the Completeness Theorem or the Model Existence Theorem for $K_{\mathcal{L}}$, or consequences of these. You may quote other standard theorems or results of $K_{\mathcal{L}}$. (10 marks)

(Total: 20 marks)

3. In this question, suppose that $\mathcal{L}^{=}$ is the first-order language with equality which has no constant or function symbols and which has as relation symbols the equality symbol and two further 2-ary relation symbols R and S.

(a) (i) Describe the terms and atomic formulas of the language $\mathcal{L}^{=}$. (3 marks)

(ii) Give an example of an $\mathcal{L}^{=}$-formula $\theta(x_1)$ and a normal $\mathcal{L}^{=}$-structure $\mathcal{A}$ such that $((\forall x_1)\theta(x_1) \to (\forall x_2)\theta(x_2))$ is not true in $\mathcal{A}$. (3 marks)

(b) Consider the following three normal $\mathcal{L}^{=}$-structures $\mathcal{A}_i$ for $i = 1, 2, 3$. Each $\mathcal{A}_i$ has domain $\mathbb{N}$ and in $\mathcal{A}_i$, the relation symbol R is interpreted as the 2-ary relation $\{(m, m+1) : m \in \mathbb{N}\} \subseteq \mathbb{N}^2$ and S is interpreted as the relation $\{(m, m+i+1) : m \in \mathbb{N}\} \subseteq \mathbb{N}^2$, for $i = 1, 2, 3$.

For each $1 \leq i \leq 3$, find a closed $\mathcal{L}^{=}$-formula η_i such that $\mathcal{A}_j \models \eta_i$ iff $i = j$ (for $1 \leq j \leq 3$). (3 marks)

(c) For each natural number $n > 0$, write down a closed $\mathcal{L}^{=}$-formula σ_n such that for every normal $\mathcal{L}^{=}$-structure $\mathcal{M}$ (with domain M), $\mathcal{M} \models \sigma_n$ if and only if $|M| \geq n$. Your formula should not involve the relation symbols R, S. (2 marks)

(d) Let Δ_1 be a set of closed $\mathcal{L}^{=}$-formulas in which the relation symbol S does not appear. Similarly let Δ_2 be a set of closed $\mathcal{L}^{=}$-formulas in which the relation symbol R does not appear. Suppose that each of Δ_1 and Δ_2 has a normal model.

$(*)$ Assume further that each of Δ_1 and Δ_2 contains all of the formulas σ_n from part (c).

(i) Explain why each of Δ_1, Δ_2 has a countably infinite normal model. (2 marks)

(ii) Prove that $\Delta_1 \cup \Delta_2$ has a normal model. (4 marks)

(iii) Without the additional assumption $(*)$, is it necessarily true that $\Delta_1 \cup \Delta_2$ has a normal model? Justify your answer. (3 marks)

(Total: 20 marks)

4. In part (a) you should work in ZF. In the rest of the question, work in ZFC.

(a) If A, B are sets, let B^A denote the set of functions from A to B.

Suppose that A is an infinite set and $|A| = |A \times A|$. Prove that $|2^A| = |A^A|$. Comment on what this implies if we work in ZFC. (7 marks)

(b) (i) Define what it means for a linear ordering $(A; \leq)$ to be a *well ordered set*. (1 mark)

(ii) Prove that a linear ordering $(A; \leq)$ is a well ordered set if and only if there is no function $f : \omega \to A$ with $f(n+1) < f(n)$ for all $n \in \omega$. (5 marks)

(iii) Decide whether the following statement is true (in ZFC). Either give a proof or a counterexample:

A linear ordering $(A; \leq)$ is a well ordered set if and only if every countable, non-empty subset of A has a least element. (3 marks)

(c) Let $\kappa < \lambda$ be cardinals with λ infinite. Suppose that X is a set of cardinality λ and, for each $i < \kappa$, that Y_i is a subset of X whose complement $X \setminus Y_i$ has cardinality at most κ. Prove that $\bigcap_{i<\kappa} Y_i$ has cardinality λ. (3 marks)

Give an example to show that this is not necessarily true if $\kappa = \lambda$. (1 mark)

[You may assume a suitable version of De Morgan's Laws.]

(Total: 20 marks)

5. (a) Suppose that α is either ω or a natural number. Let $\mathcal{L}_\alpha^=$ denote the first-order language with equality which has a binary relation symbol $\leq$ and constant symbols c_k for each $k < \alpha$ (there are no other relation, function or constant symbols, apart from these and the relation symbol for equality).

Let T_α be the $\mathcal{L}_\alpha^=$-theory consisting of the axioms for a dense linear order without endpoints and the formulas $((c_i \leq c_{i+1}) \wedge (c_i \neq c_{i+1}))$, for $i < \alpha$.

(i) Suppose that α is a natural number. Prove that T_α is a complete $\mathcal{L}_\alpha^=$-theory, stating what this means. (6 marks)

(ii) Prove that T_ω is a complete $\mathcal{L}_\omega^=$-theory. (2 marks)

(iii) Find three countable models $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ of T_ω such that no two of these are isomorphic. Explain your answer briefly. (3 marks)

(b) Let $\mathcal{L}^=$ be the usual language for groups, having a binary function symbol (for the group operation), a unary function symbol (for the inverse operation), a constant symbol (for the identity element) and the equality symbol. Suppose that for each $n < \omega$, the normal $\mathcal{L}^=$-structure $\mathcal{M}_n$ is a group. Let $\mathcal{F}$ be a non-principal ultrafilter on ω and let $\mathcal{M}$ be the ultraproduct $(\Pi_{n<\omega}\mathcal{M}_n)/\mathcal{F}$.

With these assumptions, decide in each of the following cases whether the given statement is always true. Justify your answer in each case. Standard results about ultraproducts may be used if clearly stated.

(i) If each $\mathcal{M}_n$ is an abelian group, then so is $\mathcal{M}$.

(ii) If each $\mathcal{M}_n$ is non-abelian, then so is $\mathcal{M}$.

(iii) If each $\mathcal{M}_n$ has a non-identity element of finite order, then the same is true of $\mathcal{M}$.

(iv) If $\mathcal{M}$ has an element of order 3, then $\mathcal{M}_n$ has an element of order 3, for each $n < \omega$.

(9 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH6/70132

Mathematical Logic (Solutions)

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1. **Comments:** (a)(i) Standard question; (a)(ii) unseen; (b) seen similar; (c) Essentially bookwork, though (ii) is unseen in this form.

(a(i)) For a valuation v, we have $v(\theta) = F$ iff $v(p_1 \to (p_2 \to (\neg p_3))) = F$ or $v(p_2 \to (p_3 \to p_1)) = F$. The first case holds iff $v(p_1, p_2, p_3) = (T, T, T)$. The second case holds iff $v(p_1, p_2, p_3) = (F, T, T)$. Thus the d.n.f. of $(\neg\theta)$ is

$$(p_1 \wedge p_2 \wedge p_3) \vee ((\neg p_1) \wedge p_2 \wedge p_3).$$

$\boxed{3, \text{A}}$

(a(ii)) For $n \geq 2$, define the formula ϕ_n inductively by: ϕ_2 is $(p_2 \to (\neg p_1))$ and ϕ_{k+1} is $(p_{k+1} \to \phi_k)$. A simple induction then shows that a valuation v has $v(\phi_n) = F$ iff $v(p_i) = T$ for all $i \leq n$. Thus the formula $(\neg\phi_n)$ is logically equivalent to the given formula and involves only the connectives $\neg, \to$. $\boxed{4, \text{C}}$

(b) There are 4 such truth functions. Suppose that ϕ is a formula constructed using variables p_1, p_2 and connectives $\vee, \wedge$. Then its truth function F_ϕ satisfies $F_\phi(T, T) = T$ and $F_\phi(F, F) = F$ (by a simple induction on the length of ϕ). So at most 4 truth functions of 2 variables can arise as such F_ϕ. By considering ϕ to be the formulas $p_1, p_2, p_1 \vee p_2, p_1 \wedge p_2$, we see that all 4 of these do arise. $\boxed{4, \text{B}}$

(c) A deduction of ψ from Σ is a finite sequence of L-formulas such that: ϕ is the final formula in the sequence, and each formula in the sequence is an axiom of L, or a formula in Σ, or follows from earlier formulas in the sequence using the deduction rule Modus Ponens. The notation $\Sigma \vdash_L \psi$ means that there is a deduction of ψ from Σ. $\boxed{3, \text{A}}$

(i) This is TRUE. Take a deduction of $(\phi \to \psi)$ from Γ. We obtain a deduction of ψ from $\Gamma \cup \{\phi\}$ be adding two extra formulas at the end of the sequence: ϕ (a formula in $\Gamma \cup \{\phi\}$), ψ (from ϕ, $(\phi \to \psi)$ and MP). $\boxed{2, \text{A}}$

(ii) This is TRUE. Using the second of the statements, Axiom A3 of L and MP, we obtain $\Gamma \vdash (\psi \to \phi)$. Combining this with the first statement we obtain $\Gamma \cup \{(\neg\phi)\} \vdash \phi$. Thus, by the Deduction Theorem $\Gamma \vdash ((\neg\phi) \to \phi)$. From the notes, $(((\neg\phi) \to \phi) \to \phi)$ is a theorem of L. Thus, using MP, we obtain $\Gamma \vdash \phi$, as required. [A semantic argument would also be accepted here.] $\boxed{4, \text{D}}$

2. **Comments:** (a) and (b) test some standard results in a slightly unusual form; (c) is done for propositional logic L in the notes (and that was question 1(b) in the 2018/19 paper). It's the same proof, but it requires a good understanding of the material.

(a) $(1) \Rightarrow (2)$: Suppose (1) holds and v is any valuation. If $v(\phi \to \psi) = F$ then $v(\phi) = T$ and $v(\psi) = F$. This contradicts $V[\phi] \subseteq V[\psi]$. Thus $v(\phi \to \psi) = T$ for all valuations v.

$(2) \Rightarrow (3)$: This follows from the Completeness Theorem for L.

$(3) \Rightarrow (1)$: Suppose (3) holds. By the Soundness Theorem for L, if v is any valuation, then $v(\phi \to \psi) = T$. Thus, if $v(\phi) = T$, then $v(\psi) = T$. $\boxed{6, \text{A}}$

(b) Suppose that $\mathcal{L}$ is a first-order language and ϕ, ψ are $\mathcal{L}$-formulas. The following are equivalent:

(1) Whenever $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$ with $v[\phi] = T$, then $v[\psi] = T$;

(2) The $\mathcal{L}$-formula $(\phi \to \psi)$ is logically valid;

(3) The $\mathcal{L}$-formula $(\phi \to \psi)$ is a theorem of $K_{\mathcal{L}}$. $\boxed{2, \text{A}}$

The proof is essentially the same in $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$. In $(2) \Rightarrow (3)$ we use the Completeness Theorem for $K_{\mathcal{L}}$. When proving this we stated the result for closed formulas; however, we observed afterwards that it remains valid for arbitrary formulas. So we do not require ϕ, ψ in the above to be closed. $\boxed{2, \text{B}}$

(c) We use the result from the lectures that, with the notation as in the question, if η is a closed $\mathcal{L}$-formula and $\Delta \nvdash_{K_{\mathcal{L}}} \eta$, then $\Delta \cup \{\neg\eta\}$ is consistent. $\boxed{2, \text{B}}$

Suppose for a contradiction that $\Delta \nvdash \theta$. Then $\Delta' = \Delta \cup \{\neg\theta\}$ is consistent. It follows that any set of consequences of Δ' is also consistent. Using Axiom A1, $\vdash (\neg\theta \to (\neg\chi \to \neg\theta))$, so by MP and Axiom A3, $\Delta' \vdash (\theta \to \chi)$. In particular, $\Delta \cup \{(\theta \to \chi)\}$ is consistent, a contradiction. $\boxed{4, \text{D}}$

Similarly, suppose that $\Delta \nvdash \neg\chi$. Then $\Delta \cup \{\neg\neg\chi\}$ is consistent. As $(\neg\neg\chi \to \chi)$ is a theorem of $K_{\mathcal{L}}$ (from notes), we obtain that $\Delta \cup \{\chi\}$ is consistent. Using the A1 axiom $(\chi \to (\theta \to \chi))$, we obtain that $\Delta \cup \{(\theta \to \chi)\}$ is consistent - a contradiction. $\boxed{4, \text{C}}$

3. **Comments:** (a) seen similar, (ii) is similar to questions on previous papers; (b) unseen example; (c) seen; (d) unseen.

(a) (i) The language has no function or constant symbols, so the terms are just the variables. The atomic formulas are of the form $x_i = x_j$, $R(x_i, x_j)$ and $S(x_i, x_j)$ for variables x_i, x_j. $\boxed{3, \text{A}}$

(ii) Take $\theta(x_1)$ to be the formula $((\forall x_2)R(x_1, x_2) \to S(x_1, x_1))$. Let $\mathcal{A}$ be the normal $\mathcal{L}^=$-structure with domain $\mathbb{N}$, $R(x_1, x_2)$ interpreted as '$x_1 \leq x_2$' and $S(x_1, x_2)$ interpreted as '$x_1 = 0$'. Then $\mathcal{A} \models (\forall x_1)\theta(x_1)$, but $\mathcal{A} \not\models (\forall x_2)\theta(x_2)$. $\boxed{3, \text{B}}$

(b) Let η_1 be the formula

$$(\forall x)(\forall y)(S(x, y) \leftrightarrow (\exists z)(R(x, z) \wedge R(z, y))).$$

Similarly let η_2 be

$$(\forall x)(\forall y)(S(x, y) \leftrightarrow (\exists z_1)(\exists z_2)(R(x, z_1) \wedge R(z_1, z_2) \wedge R(z_2, y))).$$

and let η_3 be

$$(\forall x)(\forall y)(S(x, y) \leftrightarrow (\exists z_1)(\exists z_2)(\exists z_3)(R(x, z_1) \wedge R(z_1, z_2) \wedge R(z_2, z_3) \wedge R(z_3, y))).$$

$\boxed{3, \text{B}}$

(c) Let σ_n be the formula $(\exists x_1) \ldots (\exists x_n) \bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j)$. $\boxed{2, \text{A}}$

(d) (i) The language is countable, so by the Model Existence Theorem (or by the downward Löwenheim - Skolem theorem) each Δ_i has a countable normal model $\mathcal{A}_i$. As $\sigma_n \in \Delta_i$ for all n, each $\mathcal{A}_i$ is infinite. $\boxed{2, \text{C}}$

(ii) With the same notation as in (i), note that we may assume that the domains of $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same set A (because both are countably infinite sets, and so there is a bijection between them). We define the normal $\mathcal{L}^=$-structure $\mathcal{A}$ on domain A so that the interpretation of R is as in $\mathcal{A}_1$ and the interpretation of S is as in $\mathcal{A}_2$. If δ is a closed $\mathcal{L}^=$-formula not involving the symbol S, then $\mathcal{A}_1 \models \delta$ iff $\mathcal{A} \models \delta$ (this is a straightforward induction involving valuations to prove a more general statement for formulas which are not closed, but we do not require a proof - it is like the proof that isomorphism preserves the validity of formulas). In particular, $\mathcal{A}$ is a normal model of Δ_1. A similar argument shows that $\mathcal{A}$ is a normal model of Δ_2. $\boxed{4, \text{D}}$

(iii) This is not necessarily true without the additional assumption (*). For example, let Δ_1 contain $\sigma_2 \wedge (\neg\sigma_3)$ and Δ_2 contain σ_4 (and R, S can both be equality). Then a normal model of Δ_1 has size ≤ 2, whereas a normal model of Δ_2 has at least 4 elements. $\boxed{3, \text{A}}$

4. **Comments:** (a) Special case of argument seen on a problem sheet; (b) unseen; (c) unseen application of standard results.

(a) Note that each function $f : A \to A$ is a subset of $A \times A$, thus $|A^A| \leq |\mathcal{P}(A \times A)|$. By assumption, there is a bijection between A and $A \times A$, and this induces a bijection between their power sets. Thus $|A^A| \leq |\mathcal{P}(A)|$. Using characteristic functions, we have a bijection between $\mathcal{P}(A)$ and 2^A (where $2 = \{0, 1\}$, as usual). Thus we have $|A^A| \leq |2^A|$. $\boxed{3, \text{B}}$

As A is an infinite set, we can take distinct elements a_0, a_1 in A and obtain an injective function $2^A \to A^A$. Thus $|2^A| \leq |A^A|$. By Cantor - Schröder - Bernstein (which does not require AC), we therefore obtain $|2^A| = |A^A|$. $\boxed{2, \text{C}}$

If we work in ZFC, then we have the Fundamental Theorem of Cardinal Arithmetic, which states that $|A| = |A \times A|$ for any infinite set A. So, assuming ZFC, the above result holds in general for any infinite set A. $\boxed{2, \text{A}}$

(b) (i) This means that every non-empty subset of A has a least element (with respect to the ordering $\leq$) . $\boxed{1, \text{A}}$

(ii) Suppose there is such a function f and consider the subset $\{f(n) : n \in \omega\}$ of A. This has no least element (as $f(n+1) < f(n)$ for all $n \in \omega$), so $(A; \leq)$ is not well-ordered. $\boxed{2, \text{A}}$

Conversely suppose that $(A; \leq)$ is not a well-ordered set and let $\emptyset \neq X \subseteq A$ be a subset with no least element. Let F be a choice function on the non-empty subsets of X. We define recursively a function $f : \omega \to A$ by setting $f(0) = F(X)$ and $f(n+1) = F(\{x \in X : x < f(n)\})$. Note that this is defined for all n (otherwise X has a minimal element) and $f(n+1) < f(n)$ for all $n \in \omega$. $\boxed{3, \text{B}}$

(iii) The statement is TRUE. Certainly $\Rightarrow$ holds, just by definition. For the converse, note that (ii) shows that if $(A; \leq)$ is not a well-ordered set, then some countable subset $X = \{f(n) : n \in \omega\}$ of A has no least element. $\boxed{3, \text{A}}$

(c) By de Morgan's Law:

$$X \setminus \bigcap_{i<\kappa} Y_i = \bigcup_{i<\kappa} (X \setminus Y_i).$$

This is a union of κ-many sets, each of cardinality $\leq \kappa$. Thus it has cardinality at most $|\kappa \times \kappa|$. If κ is infinite, this cardinality is κ (by FTCA). If κ is finite then it is also finite, so less than λ. In either case, we obtain $|X \setminus \bigcap_{i<\kappa} Y_i| < \lambda$. Now, $\lambda = |X| = |\bigcap_{i<\kappa} Y_i| + |X \setminus \bigcap_{i<\kappa} Y_i|$. It follows that, as the second term is $< \lambda$, the first term is λ (addition of two infinite cardinals just takes the maximum), as required. $\boxed{3, \text{D}}$

If $\kappa = \lambda = X$, let $Y_i = X \setminus \{i\}$ for $i < \lambda$. Then $\bigcap_{i<\lambda} Y_i = \emptyset$. $\boxed{1, \text{D}}$

5. **Comments:** (a) Unseen application of standard result (DLOWE was done in the notes); (b) Seen similar examples.

(a) (i) We need to show that for every closed $\mathcal{L}^{=}_{\alpha}$-formula ϕ we have either $T_\alpha \vdash \phi$ or $T_\alpha \vdash \neg\phi$. $\boxed{1, \mathrm{M}}$

To do this, we show that any two countable normal models $\mathcal{A}_1, \mathcal{A}_2$ of T_α are isomorphic and then use the Łos - Vaught test (also called Vaught's Theorem in Cori - Lascar).

Write $\mathcal{A}_i = \langle A_i :\leq_i, d^i_0, \ldots, d^i_{\alpha-1}\rangle$ (where d^i_j is the interpretation of the constant symbol c_j in $\mathcal{A}_i$). So $d^1_0 <_1 \ldots <_1 d^1_{\alpha-1}$ and $d^2_0 <_2 \ldots <_2 d^2_{\alpha-1}$. Note that the $\mathcal{A}_i$ are countable dense linear orders without endpoints. By the back-and-forth proof of Cantor's theorem given in the directed reading, there is an order-preserving bijection $f : A_1 \to A_2$ with $f(d^1_j) = d^2_j$ for $j < \alpha$. So $\mathcal{A}_1, \mathcal{A}_2$ are isomorphic. (Alternatively, using Cantor's Theorem, we can identify A_i and their orderings with the rational numbers and the usual ordering. Then we just have to observe that given two finite sequences of rationals $d_0 < \ldots < d_{\alpha-1}$ and $e_0 < \ldots < e_{\alpha-1}$, there is an order-preserving bijection $f : \mathbb{Q} \to \mathbb{Q}$ with $f(d_j) = e_j$ for $j < \alpha$.) $\boxed{5, \mathrm{M}}$

(ii) Suppose that ϕ is a closed $\mathcal{L}^{=}_{\omega}$-formula. As ϕ is a finite sequence of symbols, there is a finite α such that ϕ is an $\mathcal{L}^{=}_{\alpha}$-formula. By (i), either $T_\alpha \vdash \phi$ or $T_\alpha \vdash \neg\phi$. As $T_\alpha \subseteq T_\omega$, we therefore obtain (respectively) $T_\omega \vdash \phi$ or $T_\omega \vdash \neg\phi$. $\boxed{2, \mathrm{M}}$

(iii) We take the domain and ordering for each of the $\mathcal{M}_i$ to be the rationals with their usual ordering. We describe the sequence of interpretations of the constant symbols in each of the $\mathcal{M}_i$: call the three sequences $(a_j : j < \omega)$, $(b_j : j < \omega)$, $(d_j : j < \omega)$ (in $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ respectively). Each of these is an increasing sequence of rational numbers.

For (a_j) we take an unbounded sequence (for example $a_j = j$ for $j < \omega$).

For (b_j) we take a bounded sequence which has a least upper bound in $\mathbb{Q}$.

For (c_j) we take a bounded sequence which has no least upper bound in $\mathbb{Q}$ (for example, c_j is the truncation to the j-place of the decimal expansion of $\sqrt{2}$).

It is easy to see that any order-preserving bijection of the rationals maps a bounded sequence to a bounded sequence, and preserves the set of upper bounds. It follows that no two of the $\mathcal{M}_i$ are isomorphic. $\boxed{3, \mathrm{M}}$

(b) We use the Łos ultraproduct theorem: with the notation in the question, suppose that θ is a closed $\mathcal{L}^{=}$-formula. Then $\mathcal{M} \models \theta$ iff $\{n : \mathcal{M}_n \models \theta\} \in \mathcal{F}$. $\boxed{1, \mathrm{M}}$

Denote the group operation by $\cdot$ and the identity element by e.

(i) TRUE: $\omega \in \mathcal{F}$, so apply the Łos theorem to the formula $(\forall x)(\forall y)(x \cdot y = y \cdot x)$. $\boxed{2, \mathrm{M}}$

(ii) TRUE: Similarly, apply the Łos theorem to the formula $(\exists x)(\exists y)(x \cdot y \neq y \cdot x)$. $\boxed{2, \mathrm{M}}$

(iii) FALSE: Let $\mathcal{M}_n$ be the cyclic group of order p_n, the n-th prime number (in increasing order). Suppose that $\mathcal{M}$ has a non-identity element of finite order k. This is expressible by an $\mathcal{L}^{=}$-formula, so the set S of n such that $\mathcal{M}_n$ has an element of order k is in $\mathcal{F}$. But this set is finite (with at most one element). A non-principal ultrafilter cannot contain a finite set, so this is impossible and therefore $\mathcal{M}$ has no non-trivial element of finite order. $\boxed{2, \mathrm{M}}$

(iv) FALSE: Let $\mathcal{M}_0$ be the trivial group and for $n > 0$, let $\mathcal{M}_n$ be the cyclic group of order 3. As $\{n \in \omega : n > 0\}$ is in $\mathcal{F}$ (as in (iii)), $\mathcal{M}$ has an element of order 3, by the Łos theorem. $\boxed{2, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total Mastery marks: 20 of 20 marks
Total marks: 100 of 100 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60132/70132	1	This was reasonably well done by most students.In 1(a)(i), I accepted the answer p_2 \wedge p_3, even though it's not 'officially' in d.n.f. In 1(c) I accepted semantic arguments, even though formal proofs/ deductions would have been better.
MATH60132/70132	2	2(a), (b) were reasonably well done. Note that in (b) we do not require the formulas to be closed.A few people gave semantic arguments (involving valuations) for (c) despite being explicitly told not to in the question.
MATH60132/70132	3	3(a)(i) A number of people wrote out the general definitions. This was not what the question asked for - I wanted to see what the definitions mean in the given language.Lots of people came up with suitable formulas for (b) (some simpler than my specimen solution). 3(d)(ii) Taking the 'union' of a model of \Delta_1 and a model of \Delta_2 does not work. Likewise, answers which just tried to say 'by compactness ....' did not receive marks.
MATH60132/70132	4	Many people had not engaged with this part of the module and produced some rather strange answers. People who had been following the material did reasonably well.Note that in (a) we do not assume AC, and so cannot use general results of cardinal arithmetic from the end of the module. In (b) there was a mark available for saying that AC is used in the construction of the function f.In (c) some students produced simpler counterexamples than mine.
MATH70132	5	(a) was done well by people who had engaged with the material, although no-one managed to do (iii) correctly (and a number of people just write down three non-isomorphic countable linear orders, which was not the point).(b) was also reasonably well done - I wanted to see that you know Los' theorem and most of you wrote it down explicitly.