Assessed Coursework 2

You may discuss these problems with other students, but you must write up your own solutions.

Problem 1. Let $C : (-\infty, +\infty) \to \mathbb{R}^2$ be a regular curve with no self-intersections, and consider the surface

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in C\left((-\infty, +\infty)\right)\right\}.$$

Show that there is an isometry from the xy-plane onto S, and determine the Gaussian curvature of S at each point.

Solution: If the map C is defined on a finite interval $[a, b]$ we know from the lectures that it is possible to re-parametrise the curve by arc-length. We cannot directly apply that result here since the domain is not bounded. However, a similar argument can be repeated here, when the domain is not bounded. Define the function h on $(-\infty, +\infty) = \operatorname{Dom} C$ as

$$h(t) = \int_0^t C'(s)\, ds.$$

This is a well-define function for all real values of t. We also have $|h'(t)| = |C(t)| \neq 0$ for all $t \in \mathbb{R}$. Thus, $h'(t)$ is either always positive, or always negative. This implies that h is either strictly increasing or strictly decreasing. It follows that the image of h is equal to an open set, which might be equal to $\mathbb{R}$, or it may be equal to a set of the form $(a, +\infty)$ or of the form $(-\infty, b)$. In either way, let U be the image of h. We can consider the inverse of h, the map $f : U \to \mathbb{R}$. As in the lectures, it follows that $C \circ f : U \to \mathbb{R}^2$ is a regular curve parametrised by arc-length.

Let us assume that $U = \mathbb{R}$, and let us write $C(t) = (x(t), y(t))$, for $t \in \mathbb{R}$.

Consider the set

$$P = \{(u, v, 0) \mid u, v \in \mathbb{R}\} \subset \mathbb{R}^3.$$

We also consider the map $F : P \to S$, defined as

$$F(u, v, 0) = (x(u), y(u), v).$$

Then P has a chart $\phi(u, v) = (u, v, 0)$, and this gives a chart $\psi(u, v) = F(\phi(u, v)) = (x(u), y(u), v)$ on S as well. We have

$$\frac{\partial \psi}{\partial u} = dF_p\left(\frac{\partial \phi}{\partial u}\right), \qquad \frac{\partial \psi}{\partial v} = dF_p\left(\frac{\partial \phi}{\partial v}\right).$$

We compute the first fundamental form in each chart as

$$\frac{\partial \phi}{\partial u} = (1, 0, 0), \quad \frac{\partial \phi}{\partial v} = (0, 1, 0) \quad \Longrightarrow \quad g_\phi = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_u, \phi_u \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whereas

$$\frac{\partial \psi}{\partial u} = (x_u, y_u, 0)\,, \quad \frac{\partial \psi}{\partial v} = (0, 0, 1) \quad \Longrightarrow \quad g_\psi = \begin{pmatrix} \langle \psi_u, \psi_u \rangle & \langle \psi_u, \psi_v \rangle \\ \langle \psi_v, \psi_u \rangle & \langle \psi_u, \psi_u \rangle \end{pmatrix} = \begin{pmatrix} x_u^2 + y_u^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $(x(t), y(t))$ is a parametrisation by arc length, we have $x_u^2 + y_u^2 = 1$ and so $g_\psi = I$. We conclude that dF_p preserves the first fundamental forms, so F is a local isometry, and hence an isometry since it is a bijection. The plane has Gaussian curvature $K = 0$, therefore, by the Theorema Egregium we conclude that $K = 0$ at every point in S.

In general, if $U \neq \mathbb{R}$ (this can happen if the length of the curve C is finite), then the statement of the problem is not correct as it is written. In that case, the curve S is isometric to an open set in $\mathbb{R}^2$. For that case, one can repeat the above argument, only replacing P by $P_U = \{(u, v, 0) \mid u \in U, v \in \mathbb{R}\}$. (Only one student (Zhenkai Pan) observed this point.)

Problem 2. Let $S \subset \mathbb{R}^3$ be a regular surface, and $\phi : U \to S$ be a chart. Assume that there is a smooth function $\lambda : U \to \mathbb{R}$ such that the first fundamental form of S at each point $\phi(u, v)$ is

$$\begin{pmatrix} e^{\lambda(u,v)} & 0 \\ 0 & e^{\lambda(u,v)} \end{pmatrix}.$$

(Such coordinates are called isothermal.)

(a) Show that the Christoffel symbols satisfy

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \lambda_u/2, \quad \Gamma_{22}^1 = -\lambda_u/2, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = \lambda_v/2, \quad \Gamma_{11}^2 = -\lambda_v/2.$$

(b) Show that the Gaussian curvature K on $\phi(U)$ satisfies

$$\Delta\lambda + 2Ke^\lambda = 0,$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ is the Laplacian.

Solution: (a) We differentiate $\phi_u \cdot \phi_u = e^\lambda$ with respect to u to get

$$e^\lambda \lambda_u = 2\phi_u \cdot \phi_{uu} = 2\phi_u \cdot \left(\Gamma_{11}^1 \phi_u + \Gamma_{11}^2 \phi_v + A_{11} N\right) = 2\Gamma_{11}^1 (\phi_u \cdot \phi_u) = 2\Gamma_{11}^1 e^\lambda$$

since ϕ_u is orthogonal to ϕ_v by assumption and also to N. Similarly, differentiating the same equation with respect to v gives

$$e^\lambda \lambda_v = 2\phi_u \cdot \phi_{uv} = 2\phi_u \cdot \left(\Gamma_{12}^1 \phi_u + \Gamma_{12}^2 \phi_v + A_{12} N\right) = 2\Gamma_{12}^1 (\phi_u \cdot \phi_u) = 2\Gamma_{12}^1 e^\lambda$$

and differentiating $\phi_v \cdot \phi_v = e^\lambda$ with respect to v gives

$$e^\lambda \lambda_v = 2\phi_v \cdot \phi_{vv} = 2\phi_v \cdot \left(\Gamma_{22}^1 \phi_u + \Gamma_{22}^2 \phi_v + A_{22} N\right) = 2\Gamma_{22}^2 (\phi_v \cdot \phi_v) = 2\Gamma_{22}^2 e^\lambda$$

while differentiating with respect to u instead gives

$$e^\lambda \lambda_u = 2\phi_v \cdot \phi_{uv} = 2\phi_v \cdot \left(\Gamma_{12}^1 \phi_u + \Gamma_{12}^2 \phi_v + A_{12} N\right) = 2\Gamma_{12}^2 (\phi_v \cdot \phi_v) = 2\Gamma_{12}^2 e^\lambda$$

Thus $\lambda_u = 2\Gamma_{11}^1 = 2\Gamma_{12}^2$ and $\lambda_v = 2\Gamma_{12}^1 = 2\Gamma_{22}^2$
Similarly, we differentiate $\phi_u \cdot \phi_v = 0$ with respect to u to get $\phi_{uu} \cdot \phi_v + \phi_u \cdot \phi_{uv} = 0$, or $\Gamma_{11}^2 e^\lambda + \Gamma_{12}^1 e^\lambda = 0$, from which $\Gamma_{11}^2 = -\Gamma_{12}^1 = -\frac{1}{2}\lambda_v$. If we differentiate it with respect to v instead, we get $\phi_u \cdot \phi_{vv} + \phi_{uv} \cdot \phi_v = 0$, or $\Gamma_{22}^1 e^\lambda + \Gamma_{12}^2 e^\lambda = 0$, and so $\Gamma_{22}^1 = -\Gamma_{12}^2 = -\frac{1}{2}\lambda_u$

(b) We use the Gauss equation

$$Kg_{11} = \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + \left(\Gamma^2_{11}\right)_v - \left(\Gamma^2_{12}\right)_u$$

noting that $g_{11} = e^\lambda$ and plugging in the values for each Christoffel symbol from above:

$$Ke^\lambda = \left(\frac{1}{4}\lambda_u^2\right) + \left(-\frac{1}{4}\lambda_v^2\right) - \left(-\frac{1}{4}\lambda_v^2\right) - \left(\frac{1}{4}\lambda_u^2\right) + \left(-\frac{1}{2}\lambda_v\right)_v - \left(\frac{1}{2}\lambda_u\right)_u = -\frac{1}{2}\Delta\lambda$$

which after some slight rearranging becomes $\Delta\lambda + 2Ke^\lambda = 0$.

Problem 3. Let S be the unit sphere in $\mathbb{R}^3$. Using the map

$$\phi(u,v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)),$$

compute the Christoffel symbols $\Gamma^k_{i,j}$, for $i,j,k = 1,2$, at each point in $S \setminus \{\pm(0,0,1)\}$.

Solution: We compute

$$\phi_u = (-\sin(u)\cos(v), \cos(u)\cos(v), 0), \quad \phi_v = (-\cos(u)\sin(v), -\sin(u)\sin(v), \cos(v))$$

from which $\phi_u \cdot \phi_u = \cos^2(v), \phi_u \cdot \phi_v = 0$, and $\phi_v \cdot \phi_v = 1$. Since ϕ_u, ϕ_v, and N are all mutually orthogonal, we can take the dot product of both sides of $\frac{\partial^2\phi}{\partial x_i \partial x_j} = \Gamma^1_{ij}\frac{\partial\phi}{\partial x_1} + \Gamma^2_{ij}\frac{\partial\phi}{\partial x_2} + A_{ij}N$ with $\frac{\partial\phi}{\partial x_k}$ to obtain

$$\frac{\partial^2\phi}{\partial x_i \partial x_j} \cdot \frac{\partial\phi}{\partial x_k} = \Gamma^k_{ij}\left(\frac{\partial\phi}{\partial x_k} \cdot \frac{\partial\phi}{\partial x_k}\right) \quad \Longrightarrow \quad \Gamma^k_{ij} = \frac{\frac{\partial^2\phi}{\partial x_i \partial x_j} \cdot \frac{\partial\phi}{\partial x_k}}{\left(\frac{\partial\phi}{\partial x_k} \cdot \frac{\partial\phi}{\partial x_k}\right)}$$

We compute the terms in the numerator using

$$\frac{\partial^2\phi}{\partial x_1 \partial x_1} = (-\cos(u)\cos(v), -\sin(u)\cos(v), 0)$$
$$\frac{\partial^2\phi}{\partial x_1 \partial x_2} = (\sin(u)\sin(v), -\cos(u)\sin(v), 0)$$
$$\frac{\partial^2\phi}{\partial x_2 \partial x_2} = (-\cos(u)\cos(v), -\sin(u)\cos(v), -\sin(v))$$

from which $\Gamma^1_{11} = 0, \Gamma^2_{11} = \sin(v)\cos(v), \Gamma^1_{12} = \frac{-\cos(v)\sin(v)}{\cos^2(v)} = -\tan(v)$, and $\Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0$
Remark: Since the surface is the unit sphere, we know that the outward unit normal is $N(\phi(u,v)) = \phi(u,v)$, and since $\phi_{vv} = -\phi$ it follows that ϕ_{vv} is orthogonal to the tangent vectors ϕ_u and ϕ_v. This tells us that $\Gamma^1_{22} = \Gamma^2_{22} = 0$ without any further computation.

Problem 4. Let S_1, S_2 be regular surfaces in $\mathbb{R}^3$, and assume that the maps

$$\phi(u,v) = (u\cos(v), u\sin(v), \log u), \quad \psi(u,v) = (u\cos(v), u\sin(v), v)$$

are charts for S_1 and S_2, respectively, for (u,v) in some open set with $u > 0$.

(a) Show that the Gaussian curvature of S_1 at $\phi(u,v)$ is equal to the Gaussian curvature of S_2 at $\psi(u,v)$.

(b) Show that the map $F : S_1 \to S_2$ defined as $F(\phi(u,v)) = \psi(u,v)$, that is, $F = \psi \circ \phi^{-1}$, is not a local isometry.

Solution: For S_1, we compute that

$$\phi_u = \left(\cos(v), \sin(v), \frac{1}{u}\right), \quad \phi_v = (-u\sin(v), u\cos(v), 0)$$

so the first fundamental form and normal vector are given by

$$g_\phi = \begin{pmatrix} 1+1/u^2 & 0 \\ 0 & u^2 \end{pmatrix}, \quad N_\phi = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-\cos(v), -\sin(v), u)}{(u^2+1)^{1/2}}$$

and the second fundamental form is

$$A_\phi = \begin{pmatrix} N_\phi \cdot \phi_{uu} & N_\phi \cdot \phi_{uv} \\ N_\phi \cdot \phi_{vu} & N_\phi \cdot \phi_{vv} \end{pmatrix} = \frac{1}{(u^2+1)^{1/2}} \begin{pmatrix} -1/u & 0 \\ 0 & u \end{pmatrix}$$

so the curvature is

$$K_\phi = \frac{\det(A_\phi)}{\det(g_\phi)} = \frac{-1/\left(u^2+1\right)}{(u^2+1)} = \frac{-1}{(u^2+1)^2}$$

We repeat these calculations for S_2 to see that

$$\psi_u = (\cos(v), \sin(v), 0), \quad \psi_v = (-u\sin(v), u\cos(v), 1)$$

from which we determine

$$g_\psi = \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix}, \quad N_\psi = \frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|} = \frac{(\sin(v), -\cos(v), u)}{(u^2+1)^{1/2}}$$

and

$$A_\psi = \begin{pmatrix} N_\psi \cdot \psi_{uu} & N_\psi \cdot \psi_{uv} \\ N_\psi \cdot \psi_{vu} & N_\psi \cdot \psi_{vv} \end{pmatrix} = \frac{1}{(u^2+1)^{1/2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

so that S_2 has Gaussian curvature

$$K_\psi = \frac{\det(A_\psi)}{\det(g_\psi)} = \frac{-1/\left(u^2+1\right)}{u^2+1} = \frac{-1}{(u^2+1)^2}$$

Although we have shown that $K_\phi(u,v) = K_\psi(u,v)$ for all u, v, the first fundamental forms of ϕ and ψ are distinct: for example, we have $\langle \phi_u, \phi_u \rangle = 1 + u^{-2}$ whereas

$$\left\langle dF_{\phi(u,v)}(\phi_u), dF_{\phi(u,v)}(\phi_u) \right\rangle = \langle \psi_u, \psi_u \rangle = 1$$

and so $dF_{\phi(u,v)}$ does not preserve lengths. Thus F is not a local isometry.