Imperial College London

M3S1

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Statistical Theory 1

Date: **Tuesday 10th May 2016**

Time: **09.30 – 11.30**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables are provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the Neyman Factorization Criterion and prove it for the case of discrete distributions.

   (b) Suppose that $X_1, ..., X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$, where $0 < \theta < 1$.

      (i) Show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ. Is T complete? Why? Also, argue whether or not T is minimal sufficient for θ.

      (ii) Find the UMVUE of $\theta(1-\theta)$.
      [Hint: $I(X_1 = 0, X_2 = 1)$ is an unbiased estimator of $\theta(1-\theta)$.]

      (iii) Compute the Cramér-Rao lower bound for the variance of unbiased estimators of $\theta(1-\theta)$. Does the UMVUE of $\theta(1-\theta)$ obtained in (ii) attain this lower bound? Why or why not?

2. Let $(X_1, Y_1),...,(X_n, Y_n)$ be independent pairs of Normal random variables where X_i and Y_i are independent $N(\mu_i, \sigma^2)$ random variables.

   (a) Find the MLEs of $\mu_1,...,\mu_n$ and σ^2.

   (b) Now, suppose we observe only $Z_1,...,Z_n$ where $Z_i = X_i - Y_i$.

      (i) Find the MLE of σ^2 based on $Z_1,...,Z_n$ and discuss whether or not it is consistent.

      (ii) Obtain a method of moments (MM) estimator of σ^2 based on $Z_1,...,Z_n$.

      (iii) Consider testing $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$. Find the UMP test at level α based on $Z_1,...,Z_n$.

      (iv) Is the UMP level α test obtained in (iii) unbiased? Justify your answer.

3. Let $X_1,\ldots,X_n$ be i.i.d. random variables from the delayed exponential distribution having the probability density function

$$f_\theta(x) = \theta e^{-\theta(x-2)}, \qquad x > 2,$$

where θ is unknown. Suppose that the prior distribution for θ is $\mathrm{Exponential}(\lambda)$ where λ is a known positive constant.

(a) Obtain the posterior distribution of θ.

(b) Is the prior here a conjugate prior? Justify your answer.

(c) Find the Bayesian point estimator of θ under the squared error loss function.

(d) Verify whether or not the Bayes estimator obtained in (c) is admissible.

4. Suppose that $X_1,\ldots,X_m \overset{\text{i.i.d.}}{\sim} \mathrm{Exponential}(\theta_1)$ and $Y_1,\ldots,Y_n \overset{\text{i.i.d.}}{\sim} \mathrm{Exponential}(\theta_2)$, and assume the X_i and the Y_i are independent. Consider testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

(a) Show that the likelihood ratio test statistic is as follows

$$\lambda(x,y) = \left(\frac{m}{m+n} + \frac{n}{m+n}\frac{\bar{Y}}{\bar{X}}\right)^{-m}\left(\frac{n}{m+n} + \frac{m}{m+n}\frac{\bar{X}}{\bar{Y}}\right)^{-n}.$$

(b) Obtain a test at level α using the test statistic $\frac{\bar{X}}{\bar{Y}}$.
[Hint: Use the fact that if χ_1^2 and χ_2^2 are two independent chi-squared random variables with degrees of freedom v_1 and v_2 respectively, then $\frac{\chi_1^2/v_1}{\chi_2^2/v_2} \sim F(v_1,v_2)$.]

(c) Obtain the likelihood ratio test using the asymptotic distribution of $-2\log(\lambda(x,y))$ under H_0.

(d) Construct a confidence interval for $\frac{\theta_1}{\theta_2}$ with confidence coefficient $1-\alpha$.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\frac{1}{\theta}$	$\frac{(1-\theta)}{\theta^2}$	$\frac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\frac{n}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\frac{n(1-\theta)}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		$0 \quad (\text{if } \nu>1)$	$\dfrac{\nu}{\nu-2} \quad (\text{if } \nu>2)$	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Imperial College London

M4S1

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Statistical Theory 1

Date: **Tuesday 10th May 2016**
Time: **09.30 – 12.00**
Time Allowed: **2 Hours 30 Mins**
This paper has Five Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
Statistical tables are provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) State the Neyman Factorization Criterion and prove it for the case of discrete distributions.

   (b) Suppose that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$, where $0 < \theta < 1$.

      (i) Show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ. Is T complete? Why? Also, argue whether or not T is minimal sufficient for θ.

      (ii) Find the UMVUE of $\theta(1-\theta)$.
      [Hint: $I(X_1 = 0, X_2 = 1)$ is an unbiased estimator of $\theta(1-\theta)$.]

      (iii) Compute the Cramér-Rao lower bound for the variance of unbiased estimators of $\theta(1-\theta)$. Does the UMVUE of $\theta(1-\theta)$ obtained in (ii) attain this lower bound? Why or why not?

2. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent pairs of Normal random variables where X_i and Y_i are independent $N(\mu_i, \sigma^2)$ random variables.

   (a) Find the MLEs of $\mu_1, \ldots, \mu_n$ and σ^2.

   (b) Now, suppose we observe only $Z_1, \ldots, Z_n$ where $Z_i = X_i - Y_i$.

      (i) Find the MLE of σ^2 based on $Z_1, \ldots, Z_n$ and discuss whether or not it is consistent.

      (ii) Obtain a method of moments (MM) estimator of σ^2 based on $Z_1, \ldots, Z_n$.

      (iii) Consider testing $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$. Find the UMP test at level α based on $Z_1, \ldots, Z_n$.

      (iv) Is the UMP level α test obtained in (iii) unbiased? Justify your answer.

3. Let $X_1,...,X_n$ be i.i.d. random variables from the delayed exponential distribution having the probability density function

$$f_\theta(x) = \theta e^{-\theta(x-2)}, \qquad x > 2,$$

where θ is unknown. Suppose that the prior distribution for θ is $\mathrm{Exponential}(\lambda)$ where λ is a known positive constant.

(a) Obtain the posterior distribution of θ.

(b) Is the prior here a conjugate prior? Justify your answer.

(c) Find the Bayesian point estimator of θ under the squared error loss function.

(d) Verify whether or not the Bayes estimator obtained in (c) is admissible.

4. Suppose that $X_1, ..., X_m \overset{\text{i.i.d.}}{\sim} \mathrm{Exponential}(\theta_1)$ and $Y_1, ..., Y_n \overset{\text{i.i.d.}}{\sim} \mathrm{Exponential}(\theta_2)$, and assume the X_i and the Y_i are independent. Consider testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

(a) Show that the likelihood ratio test statistic is as follows

$$\lambda(x,y) = \left(\frac{m}{m+n} + \frac{n}{m+n}\frac{\bar{Y}}{\bar{X}}\right)^{-m} \left(\frac{n}{m+n} + \frac{m}{m+n}\frac{\bar{X}}{\bar{Y}}\right)^{-n}.$$

(b) Obtain a test at level α using the test statistic $\frac{\bar{X}}{\bar{Y}}$.
[Hint: Use the fact that if χ_1^2 and χ_2^2 are two independent chi-squared random variables with degrees of freedom v_1 and v_2 respectively, then $\frac{\chi_1^2/v_1}{\chi_2^2/v_2} \sim F(v_1, v_2)$.]

(c) Obtain the likelihood ratio test using the asymptotic distribution of $-2\log(\lambda(x,y))$ under H_0.

(d) Construct a confidence interval for $\frac{\theta_1}{\theta_2}$ with confidence coefficient $1-\alpha$.

Mastery Question:

5. Let $X_1,\ldots,X_n$ be i.i.d. Cauchy random variables with density function

$$f_\theta(x) = \frac{1}{\pi\left(1+(x-\theta)^2\right)} \qquad x \in R,$$

and suppose outcomes $x_1,\ldots,x_n$ are observed.

(a) Write down the likelihood equation for estimating θ and discuss whether it has a unique solution for the given sample $x_1,\ldots,x_n$.

(b) Given an estimate $\hat{\theta}^{(k)}$ for θ at iteration k, obtain a new estimate $\hat{\theta}^{(k+1)}$ using the Newton-Raphson method.

(c) Show that a new estimate $\hat{\theta}^{(k+1)}$ using the Fisher scoring algorithm is

$$\hat{\theta}^{(k+1)} = \hat{\theta}^{(k)} + \frac{4}{n}\sum_{i=1}^{n}\frac{x_i-\hat{\theta}^{(k)}}{1+\left(x_i-\hat{\theta}^{(k)}\right)^2}.$$

[Hint: $\int_0^\infty \frac{1-t^2}{(1+t^2)^3}dt = \frac{\pi}{8}$.*]*

(d) Is the sample mean $\bar{x}$ an appropriate initial value for the Newton-Raphson and the Fisher scoring methods here? If not, suggest a good starting point. Briefly explain your thinking.

(e) Which method has a faster convergence: the Newton-Raphson method or the Fisher scoring algorithm? Why?

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$E_{f_X}[X]$	$\text{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad E_{f_Y}[Y] = \mu + \sigma E_{f_X}[X] \qquad \text{Var}_{f_Y}[Y] = \sigma^2\text{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	