Imperial College London

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
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This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 2

Date: 14th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

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Formula Sheet

Relations satisfied by the spin operators and eigenstates:

$$\hat{S}^2 \left| s, m \right\rangle = \hbar^2 s(s+1) \left| s, m \right\rangle$$
$$\hat{S}_z \left| s, m \right\rangle = \hbar m \left| s, m \right\rangle$$
$$\hat{S}_+ \left| s, m \right\rangle = \hbar \sqrt{s(s+1) - m(m+1)} \left| s, m+1 \right\rangle$$
$$\hat{S}_- \left| s, m \right\rangle = \hbar \sqrt{s(s+1) - m(m-1)} \left| s, m-1 \right\rangle .$$

Results from non-degenerate perturbation theory:

$$E_n^{(1)} = \left\langle n \right| \hat{V} \left| n \right\rangle$$
$$E_n^{(2)} = \sum_{n' \neq n} \frac{| \left\langle n' \right| \hat{V} \left| n \right\rangle |^2}{\varepsilon_n - \varepsilon_{n'}}.$$

Result from first-order degenerate perturbation theory:

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{P}(\hat{\mathcal{H}}_0 + \lambda \hat{V})\hat{P}$$

where $\hat{P}$ projects into the subspace of degenerate unperturbed states.

1. This question involves several (mostly) unrelated short problems covering the material appearing throughout the module. These problems should only require short calculations, if any.

(a) Evaluate and simplify $[\hat{a}, (\hat{a}^\dagger)^2]$ where $\hat{a}$ is a bosonic annihilation (also known as destruction) operator. (2 marks)

(b) Evaluate and simplify $[\hat{x}\hat{p}, (\hat{x}\hat{p})^\dagger]$ where $\hat{x}$ and $\hat{p}$ are the position and momentum operators. (3 marks)

(c) Simplify $\hat{c}\hat{c}^\dagger\hat{c}$ where $\hat{c}$ is a fermionic annihilation operator that satisfies $\{\hat{c}, \hat{c}^\dagger\} = 1$. (3 marks)

(d) State how the position and momentum operators transform under the operation of time reversal. For systems involving spin, what are the possible values for the square of the time-reversal operator. (3 marks)

(e) State Kramer's theorem (involving the relation between time reversal and ground state degeneracy). What does Kramer's theorem tell us about the ground state of the Harmonic oscillator Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$? (3 marks)

(f) Consider a composite particle consisting of two fundamental particles, each having spin half: $s_1 = s_2 = 1/2$. What are the possible values of the total spin s of the composite particle? Is the composite particle a fermion or boson? (3 marks)

(g) Show that the eigenvalues of the particle exchange operator can only take on the value of $+1$ or -1. (3 marks)

(Total: 20 marks)

2. In this question, we consider the harmonic oscillator Hamiltonian under a perturbation. In particular, we consider $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{V}$ where

$$\hat{\mathcal{H}}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\hat{V} = \frac{\gamma}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

In this, λ is the usual parameter used in perturbation theory and $\gamma > 0$.

(a) As discussed in lecture, by introducing the ladder operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}}\hat{p}$, the Harmonic oscillator Hamiltonian takes the form $\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. Show that $\hat{V}$ can be written in terms of ladder operators as

$$\hat{V} = i\frac{\gamma}{2}\hbar(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}).$$

(5 marks)

(b) Determine the first-order perturbative contribution (in λ) to all eigenenergies of this Hamiltonian. (4 marks)

(c) Using second-order perturbation theory determine the second-order correction (in λ) to the ground state energy of this Hamiltonian. (6 marks)

(d) Determine the exact ground state energy of the full Hamiltonian $\hat{\mathcal{H}}$ when $\omega > |\lambda|\gamma$ and show that it is consistent with the result of (c). Hint: It might be helpful to factor the Hamiltonian as

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{p} + \lambda\gamma m\hat{x})^2 + \frac{1}{2}m\Omega^2\hat{x}^2$$

where Ω is a constant to be determined and apply a suitable unitary transformation.

(5 marks)

(Total: 20 marks)

3. Consider the Hermitian operators $\hat{\gamma}$ and $\hat{\xi}$ that satisfy the following relations: $\hat{\gamma}^2 = \hat{\xi}^2 = 1$ and $\hat{\gamma}\hat{\xi} = -\hat{\xi}\hat{\gamma}$. From these we form the Hamiltonian

$$\hat{\mathcal{H}} = iw\hat{\gamma}\hat{\xi}$$

where w is a potentially complex quantity having units of energy. Such operators correspond to *Majorana Fermions* which are being pursued as potential hardware for quantum computers.

(a) What condition does the requirement of $\hat{\mathcal{H}}$ being Hermitian place on w? (4 marks)

(b) Determine the Heisenberg equations of motion for the operators $\hat{\gamma}$ and $\hat{\xi}$. Solve these equations to show that

$$\hat{\gamma}_H(t) = \cos(2wt/\hbar)\hat{\gamma} + \sin(2wt/\hbar)\hat{\xi}$$
$$\hat{\xi}_H(t) = -\sin(2wt/\hbar)\hat{\gamma} + \cos(2wt/\hbar)\hat{\xi}.$$

(6 marks)

(c) Show explicitly from the result of (b) that $[\hat{\gamma}_H(t)]^2$ is time independent. Explain how this can be deduced alternatively from the relation $\hat{\gamma}_H(t) = \hat{\mathcal{U}}^\dagger(t)\hat{\gamma}\hat{\mathcal{U}}(t)$ where $\hat{\mathcal{U}}(t)$ is the time-evolution operator.

(5 marks)

(d) Suppose that at $t = 0$ the initial state is an eigenstate of $\hat{\gamma}$ with eigenvalue one: $\hat{\gamma}\,|\psi(0)\rangle = |\psi(0)\rangle$. Determine the expectation value of $\hat{\gamma}$ at later times. At what times will $\langle\psi(t)|\,\hat{\gamma}\,|\psi(t)\rangle$ take on its maximum value?

(5 marks)

(Total: 20 marks)

4. Spin-half systems

(a) Write down the commutation relations between the spin operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. Alternatively write down a single relation involving the Levi-Civita symbol containing each of these relations as we did in lecture. Using results from the Formula Page or otherwise, for a spin-half system, show that $\hat{S}_+ \left|\downarrow\right\rangle = \hbar \left|\uparrow\right\rangle$ and $\hat{S}_- \left|\uparrow\right\rangle = \hbar \left|\downarrow\right\rangle$ where $\hat{S}_\pm = \hat{S}_x + i\hat{S}_y$ are the spin raising and lowering operators. Also recall that $\hat{S}_- \left|\downarrow\right\rangle = \hat{S}_+ \left|\uparrow\right\rangle = 0.$ (5 marks)

(b) Now let us consider the case of two spin-half particles governed by the Hamiltonian

$$\hat{\mathcal{H}}_2 = -J\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$$

where J is a positive constant. Show that $\left|\uparrow\uparrow\right\rangle$ (using the notation of lecture) is an eigenstate of this Hamiltonian. What is the corresponding eigenvalue? Hint: it is helpful to show and use the following relation $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 = \hat{S}_{1,z}\hat{S}_{2,z} + \frac{1}{2}\hat{S}_{1,+}\hat{S}_{2,-} + \frac{1}{2}\hat{S}_{1,-}\hat{S}_{2,+}$. (5 marks)

(c) Now let us generalise the above to the case of N spin-half spins governed by the Hamiltonian

$$\hat{\mathcal{H}}_N = -J\sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}$$

where we let $\hat{\mathbf{S}}_{N+1} = \hat{\mathbf{S}}_1$ (periodic boundary conditions). Show that the state with all spins pointing up $\left|\phi\right\rangle = \left|\uparrow\uparrow\ldots\uparrow\right\rangle$ is an eigenstate of this Hamiltonian, and find the corresponding eigenvalue. Hint: it is helpful to express the terms $\hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}$ in a way that involves spin raising and lowering operators.

(5 marks)

(d) Consider the collection of N states having one of the N spins down and the rest up. Denote the state with the first spin flipped as $\left|n=1\right\rangle = \left|\downarrow\uparrow\uparrow\ldots\uparrow\right\rangle$, the state with the second spin flipped as $\left|n=2\right\rangle = \left|\uparrow\downarrow\uparrow\uparrow\ldots\uparrow\right\rangle$ and so on. (This orthonormal collection of N states are eigenstates of the z-component of the total spin operator with eigenvalue $\frac{\hbar}{2}(N-2)$.)

Determine the eigenenergies and eigenstates of $\hat{\mathcal{H}}_N$ within this subspace of N states.

(5 marks)

(Total: 20 marks)

5. Multiple-Particle Systems

(a) (i) Suppose that a single particle governed by the Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$ has eigenstates and eigenenergies given as $\hat{\mathcal{H}}\,|\phi_n\rangle = E_n\,|\phi_n\rangle$. Now consider two particles in this potential, which are described by the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2m}\hat{p}_2^2 + V(\hat{x}_1) + V(\hat{x}_2).$$

Write down expressions for the two-particle eigenstates and eigenenergies of this Hamiltonian. (4 marks)

(ii) Show that the two-particle Hamiltonian of (a)(i) commutes with the particle exchange operator $\hat{\mathscr{P}}$. Find the simultaneous eigenstates of $\hat{\mathcal{H}}$ and $\hat{\mathscr{P}}$ for the two-particle system. (4 marks)

(b) (i) Now consider the Hamiltonian

$$\hat{\mathcal{H}}_0 = \frac{1}{2}U\hat{\rho}_1(\hat{\rho}_1 - 1) + \frac{1}{2}U\hat{\rho}_2(\hat{\rho}_2 - 1) + \frac{1}{2}U\hat{\rho}_3(\hat{\rho}_3 - 1)$$

where U is a positive parameter and $\hat{\rho}_n = \hat{a}_n^\dagger \hat{a}_n$ $(n = 1, 2, 3)$. The creation and annihilation operators ($\hat{a}_n^\dagger$ and $\hat{a}_n$) are taken to be bosonic. Determine three degenerate ground states of this Hamiltonian for $N = 2$ particles and the associated eigenenergy. For the case of $N = 3$ particles determine the eigenenergies and associated degeneracies (no need to write out all of the eigenstates). (5 marks)

(ii) Consider the Hamiltonian of (b)(i) under the perturbation

$$\hat{V} = -w\sum_{n=1}^{2}(\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n)$$

where w is a positive parameter. For the case of $N = 2$ particles, use degenerate perturbation theory to determine the ground state of $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\hat{V}$ to **first** order in λ. (7 marks)

(Total: 20 marks)

Solutions for Quantum Mechanics II Exam, 2020

1. This question involves several (mostly) unrelated short problems covering the material appearing throughout the module. These problems should only require short calculations, if any.

   (a) Evaluate and simplify $[\hat{a}, (\hat{a}^\dagger)^2]$ where $\hat{a}$ is a bosonic annihilation (also known as destruction) operator.

   Seen.

   The result follows from the application of the identity $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$ and the bosonic commutation relations stated in the problem. Using this, one finds $[\hat{a}, (\hat{a}^\dagger)^2] = 2\hat{a}^\dagger$.

   (b) Evaluate and simplify $[\hat{x}\hat{p}, (\hat{x}\hat{p})^\dagger]$ where $\hat{x}$ and $\hat{p}$ are the position and momentum operators.

   Seen.

   First note that $(\hat{x}\hat{p})^\dagger = \hat{p}^\dagger\hat{x}^\dagger = \hat{p}\hat{x} = [\hat{p}, \hat{x}] + \hat{x}\hat{p} = -i\hbar + \hat{x}\hat{p}$. Putting this into the commutator, we find $[\hat{x}\hat{p}, (\hat{x}\hat{p})^\dagger] = [\hat{x}\hat{p}, -i\hbar + \hat{x}\hat{p}] = 0 + 0 = 0$.

   (c) Simplify $\hat{c}\hat{c}^\dagger\hat{c}$ where $\hat{c}$ is a fermionic annihilation operator.

   Unseen. A calculation gives: $\hat{c}\hat{c}^\dagger\hat{c} = (\{\hat{c}, \hat{c}^\dagger\} - \hat{c}^\dagger\hat{c})\hat{c} = \hat{c}$.

   (d) State how the position and momentum operators transform under the operation of time reversal. For systems involving spin, what are the possible values for the square of the time-reversal operator.

   Seen.

   Under time reversal the position operator is unchanged while the momentum operator changes sign. The possible values for $\hat{T}^2$ are ± 1.

   (e) State Kramer's theorem (involving the relation between time reversal and ground state degeneracy). What does Kramer's theorem tell us about the ground state of the Harmonic oscillator Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$.

   Seen/Unseen

   Kramer's theorem says that if a Hamiltonian is time reversal invariant and the time reversal operator satisfies $\hat{T}^2 = -1$ then all eigenstates of the Hamiltonian are at least doubly degenerate. The harmonic oscillator hamiltonian is TRI but the time-reversal operator satisfies $\hat{T}^2 = 1$. Therefore Kramer's theorem tells us nothing – indeed all of the eigenstates of the HO are non-degenerate.

   (f) Consider a composite particle consisting of two fundamental particles, each having spin half: $s_1 = s_2 = 1/2$. What are the possible values of the total spin s of the composite particle? Is the composite particle a fermion or boson?

   Seen

   The possible values are $s = 1$ or $s = 0$. The composite particle is a boson (according to the spin-statistics theorem).

(g) Show that the eigenvalues of the particle exchange operator can only take on the value of $+1$ or -1.

Seen

The key ingredient is that $\hat{\mathscr{P}}^2 = 1$ as discussed in lecture. Suppose $\hat{\mathscr{P}} |\phi\rangle = \lambda |\phi\rangle$. Applying $\hat{\mathscr{P}}$ on the left gives $|\phi\rangle = \lambda \hat{\mathscr{P}} |\phi\rangle = \lambda^2 |\phi\rangle$. Therefore $\lambda^2 = 1$ and so the result follows.

2. In this question, we consider the harmonic oscillator Hamiltonian under a perturbation: $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{V}$ where

$$\hat{\mathcal{H}}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$
$$\hat{V} = \frac{\gamma}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

In this, λ is the usual parameter used in perturbation theory and $\gamma > 0$.

(a) As discussed in lecture, by introducing the ladder operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}}\hat{p}$, the Harmonic oscillator Hamiltonian takes the form $\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. Show that $\hat{V}$ can be written in terms of ladder operators as

$$\hat{V} = i\frac{\gamma}{2}\hbar(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}).$$

Unseen

First one needs to invert the expression for $\hat{a}$ and $\hat{a}^\dagger$ to find expressions for the position and momentum operators: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$, $\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$. Substituting this into $\hat{V}$ a calculation gives the result.

(b) Determine the first-order perturbative contribution (in λ) to all eigenenergies of this Hamiltonian.

Unseen

The relevant expression from first-order perturbation theory is $E_n^{(1)} = \langle n| \hat{V} |n\rangle$. One can compute this directly. OR, one can notice from the revealing form of $\hat{V}$ from the previous part that the expectation values $\langle n| \hat{V} |n\rangle$ always vanish due to the properties of the Harmonic oscillator eigenstates. $E_n^{(1)} = 0$.

(c) Using second-order perturbation theory determine the second-order correction (in λ) to the ground state energy of this Hamiltonian.

Unseen

For this one applies a central result from second-order non-degenerate perturbation theory

$$E_n^{(2)} = \sum_{n' \neq n} \frac{|\langle n'| \hat{V} |n\rangle|^2}{\varepsilon_n - \varepsilon_{n'}}.$$

The key is using the result of (a) and the properties of the harmonic oscillator eigenstates to determine which term contributes. The result is $E_0^{(2)} = -\gamma^2\hbar/(4\omega)$.

(d) Determine the exact ground state energy of the full Hamiltonian $\hat{\mathcal{H}}$ when $\omega > |\lambda|\gamma$ and show that it is consistent with the result of (c). Hint: It might be helpful to factor the Hamiltonian as

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{p} + \lambda\gamma m\hat{x})^2 + \frac{1}{2}m\Omega^2\hat{x}^2$$

where Ω is a constant to be determined.

Unseen

Factoring $\hat{\mathcal{H}}$ in the suggested way gives $\Omega = \sqrt{\omega^2 - \lambda^2\gamma^2}$. Next with the unitary operator $\hat{U} = e^{i\frac{\lambda}{2}\gamma m\hat{x}^2/\hbar}$, one can work out that $\hat{U}\hat{p}\hat{U}^\dagger = \hat{p} - \lambda\gamma m\hat{x}$. Applying this to the Hamiltonian then gives

$$\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\Omega^2\hat{x}^2.$$

This transformed Hamiltonian will have the same spectrum as the original Hamiltonian. With what we know about the harmonic oscillator, the eigenenergies are $E_n = \hbar\Omega(n + 1/2)$ and the exact ground state is $E_{\mathrm{gs}} = \hbar\Omega/2$. To second order in λ, $\Omega = \omega - \lambda^2\gamma^2/(2\omega)$. Substituting this into E_{gs} recovers the results from perturbation theory.

3. Consider the Hermitian operators $\hat{\gamma}$ and $\hat{\xi}$ that satisfy the following relations: $\hat{\gamma}^2 = \hat{\xi}^2 = 1$ and $\hat{\gamma}\hat{\xi} = -\hat{\xi}\hat{\gamma}$. From these we form the Hamiltonian

$$\hat{\mathcal{H}} = iw\hat{\gamma}\hat{\xi}$$

where w is a potentially complex quantity having units of energy. Such operators correspond to *Majorana Fermions* which are being pursued as potential hardware for quantum computers.

(a) What condition does the requirement of $\hat{\mathcal{H}}$ being Hermitian place on w?

Seen

Taking the adjoint of the Hamiltonian gives $\hat{\mathcal{H}}^\dagger = -iw^*\hat{\xi}^\dagger\hat{\gamma}^\dagger$. Using the given information that $\hat{\xi}$ and $\hat{\gamma}$ are Hermitian and that they anti-commute gives $\hat{\mathcal{H}}^\dagger = -iw^*\hat{\gamma}\hat{\xi}$. Comparing this with the $\hat{\mathcal{H}}$ then requires $w = w^*$ and so w is real.

(b) Determine the Heisenberg equations of motion for the operators $\hat{\gamma}$ and $\hat{\xi}$. Solve these equations to determine $\hat{\gamma}_H(t)$ and $\hat{\xi}_H(t)$.

Seen

The equations of motion are

$$\hbar\frac{d}{dt}\hat{\gamma}_H = 2w\hat{\xi}_H$$
$$\hbar\frac{d}{dt}\hat{\xi}_H = -2w\hat{\gamma}_H.$$

Solving these gives

$$\begin{aligned}\hat{\gamma}_H &= \cos(2wt/\hbar)\hat{\gamma} + \sin(2wt/\hbar)\hat{\xi} \\ \hat{\xi}_H &= -\sin(2wt/\hbar)\hat{\gamma} + \cos(2wt/\hbar)\hat{\xi}.\end{aligned}$$

(c) Show explicitly from the result of (b) that $[\hat{\gamma}_H(t)]^2$ is time independent. Explain how this can be deduced alternatively from the relation $\hat{\gamma}_H(t) = \hat{\mathcal{U}}^\dagger(t)\hat{\gamma}\hat{\mathcal{U}}(t)$ where $\hat{\mathcal{U}}(t)$ is the time-evolution operator.

Unseen

Squaring the expression from the previous part gives

$$[\hat{\gamma}_H(t)]^2 = \hat{\gamma}^2\cos^2(2wt/\hbar) + \hat{\xi}^2\sin^2(2wt/\hbar) + \cos(2wt)\sin(2wt)(\hat{\gamma}\hat{\xi} + \hat{\xi}\hat{\gamma}).$$

Using the relations satisfied by the operators given in the problem it then follows that $[\hat{\gamma}_H(t)]^2 = 1$ and hence it is time independent.

This can be deduced in a simpler way as

$$[\hat{\gamma}_H(t)]^2 = \hat{\mathcal{U}}^\dagger(t)\hat{\gamma}\hat{\mathcal{U}}(t)\hat{\mathcal{U}}^\dagger(t)\hat{\gamma}\hat{\mathcal{U}}(t) = \hat{\mathcal{U}}^\dagger(t)\hat{\gamma}^2\hat{\mathcal{U}}(t) = 1.$$

(d) Suppose that at $t = 0$ the initial state is an eigenstate of $\hat{\gamma}$ with eigenvalue one: $\hat{\gamma}|\psi(0)\rangle = |\psi(0)\rangle$. Determine the expectation value of $\hat{\gamma}$ at later times. At what times will $\langle\psi(t)|\hat{\gamma}|\psi(t)\rangle$ take on its maximum value?

Similar Seen

A key point to realise is that $\langle\psi(t)|\hat{\gamma}|\psi(t)\rangle = \langle\psi(0)|\hat{\gamma}_H|\psi(0)\rangle$. Therefore,

$$\langle\gamma\rangle(t) = \cos(2wt/\hbar) + \sin(2wt)\langle\psi(0)|\hat{\xi}|\psi(0)\rangle.$$

What is $\langle\psi(0)|\hat{\xi}|\psi(0)\rangle$? It can be seen to vanish as follows. Noting that we are working with an eigenstate of $\hat{\gamma}$,

$$\langle\psi(0)|\hat{\xi}|\psi(0)\rangle = \langle\psi(0)|\gamma^2\hat{\xi}|\psi(0)\rangle = \langle\psi(0)|\gamma\hat{\xi}|\psi(0)\rangle.$$

The far LHS and far RHS can be seen to be pure real and pure imaginary, respectively. So both vanish. So we have

$$\langle\gamma\rangle(t) = \cos(2wt/\hbar)$$

which takes on its maximum value for times when $2wt/\hbar = 2\pi n$ for integer n.

4. Spin-half systems

(a) Write down the commutation relations between the spin operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. Alternatively write down a single relation involving the Levi-Civita symbol containing each of these relations as we did in lecture. Using results from the Formula Page or otherwise, for a spin-half system, show that $\hat{S}_+ |\downarrow\rangle = \hbar |\uparrow\rangle$ and $\hat{S}_- |\uparrow\rangle = \hbar |\downarrow\rangle$ where $\hat{S}_\pm = \hat{S}_x + i\hat{S}_y$ are the spin raising and lowering operators. Also recall that $\hat{S}_- |\downarrow\rangle = \hat{S}_+ |\uparrow\rangle = 0$.

Seen

The commutation relations are $[\hat{S}_a, \hat{S}_b] = i\varepsilon_{abc}\hat{S}_c$. The other parts follow from expressions from the formula sheet. One needs to recognise that $s = 1/2$ and $m = 1$ corresponds to the up spin while $m = -1$ corresponds to the down spin.

(b) Now let us consider the case of two spin-half particles governed by the Hamiltonian

$$\hat{\mathcal{H}}_2 = -J\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$$

where J is a positive constant. Show that $|\uparrow\uparrow\rangle$ (using the notation of lecture) is an eigenstate of this Hamiltonian. What is the corresponding eigenvalue? Hint: it is helpful to show and use the following relation $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 = \hat{S}_{1,z}\hat{S}_{2,z} + \frac{1}{2}\hat{S}_{1,+}\hat{S}_{2,-} + \frac{1}{2}\hat{S}_{1,-}\hat{S}_{2,+}$.

Similar Seen

It is first useful to write the dot product between the spins in a way that the raising and lowering operators appear explicitly as suggested. The relation from the hint can be arrived at by expressing $\hat{S}_x$ and $\hat{S}_y$ in terms of the raising and lowering operators and substituting this into the expression involving the spin dot product. So the Hamiltonian expressed this way is

$$\hat{\mathcal{H}}_2 = -J(\hat{S}_{1,z}\hat{S}_{2,z} + \frac{1}{2}\hat{S}_{1,+}\hat{S}_{2,-} + \frac{1}{2}\hat{S}_{1,-}\hat{S}_{2,+}).$$

Next we note that the last two terms in this expression involve raising operators which will give zero when acting on an up spin. Therefore,

$$\hat{\mathcal{H}}_2 |\uparrow\uparrow\rangle = -J\hat{S}_{1z}\hat{S}_{2z} |\uparrow\uparrow\rangle = -J\frac{\hbar^2}{4} |\uparrow\uparrow\rangle .$$

So the state under consideration is an eigenstate with eigenvalue can be read off of the above expression.

(c) Now let us generalise the above to the case of N spin-half spins governed by the Hamiltonain

$$\hat{\mathcal{H}}_N = -J\sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}$$

where we let $\hat{\mathbf{S}}_{N+1} = \hat{\mathbf{S}}_1$ (periodic boundary conditions). Show that the state with all spins pointing up $|\phi\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$ is an eigenstate of this Hamiltonian, and find the corresponding eigenvalue. Hint: it is helpful to express the terms $\hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}$ in a way that involves spin raising and lowering operators.

Unseen

Here it is useful to note that $\hat{S}_{n,x}\hat{S}_{n+1,x} + \hat{S}_{n,y}\hat{S}_{n+1,y} = \frac{1}{2}(\hat{S}_{n,+}\hat{S}_{n+1,-} + \hat{S}_{n,-}\hat{S}_{n+1,+}$. As both of these terms involve spin raising operators, $\hat{S}_{n,x}\hat{S}_{n+1,x} + \hat{S}_{n,y}\hat{S}_{n+1,y}$ will anihiliate $|\phi\rangle$. Therefore the full Hamiltonian acts on our state as

$$\hat{\mathcal{H}}\,|\phi\rangle = -J\sum_n \hat{S}_{n,z}\hat{S}_{n+1,z}\,|\phi\rangle = -J\sum_n \left(\frac{\hbar}{2}\right)^2 |\phi\rangle = -JN\frac{\hbar^2}{4}\,|\phi\rangle\,.$$

Therefore $|\phi\rangle$ is an eigenstate and the eigenvalue can be read off of the above equation.

(d) Consider the collection of N states having one of the N spins down and the rest up. Denote the state with the first spin flipped as $|n = 1\rangle = |{\downarrow\uparrow\uparrow}\ldots\uparrow\rangle$, the state with the second spin flipped as $|n = 2\rangle = |{\uparrow\downarrow\uparrow\uparrow}\ldots\uparrow\rangle$ and so on. (This orthonormal collection of N states are eigenstates of the z-component of the total spin operator with eigenvalue $\frac{\hbar}{2}(N-2)$.)

Determine the eigenenergies and eigenstates of $\hat{\mathcal{H}}_N$ within this subspace of N states.

Unseen

First let's evaluate how the Hamiltonian acts on these states. It is most convenient to express $\hat{\mathcal{H}}$ in terms of the raising and lowering operators as was done in (c). Those terms will move the flipped spin to the right and left. More specifically,

$$\hat{\mathcal{H}}\,|n\rangle = \bar{E}\,|n\rangle - J\frac{\hbar^2}{2}\,|n+1\rangle - J\frac{\hbar^2}{2}\,|n-1\rangle$$

where $\bar{E} = -J\frac{\hbar^2}{2}(N-2)$. From this, we can see that the Fourier transform of these states $|k\rangle = \frac{1}{\sqrt{N}}\sum_k e^{ikn}\,|n\rangle$ (where k takes on values $\frac{2\pi}{N}, \frac{2\pi}{N}2, \ldots, 2\pi$) are eigenstates of the Hamiltonian with eigenvalues

$$E_k = \bar{E} - J\hbar^2\cos(k).$$

5. Multiple-Particle Systems

(a) i. Suppose that a single particle governed by the Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$ eigenstates and eigenenergies given as $\hat{\mathcal{H}}\,|\phi_n\rangle = E_n\,|\phi_n\rangle$. Now consider two particles in this potential, which are described by the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2m}\hat{p}_2^2 + V(\hat{x}_1) + V(\hat{x}_2).$$

Seen

The important thing to notice is that there is no interaction between these particles. Therefore the eigenstates of this two-particle Hamiltonian will simply be tensor products of the single-particle eigenstates. In particular, the eigenstates are $|\phi_n\phi_m\rangle$ which have the corresponding eigenenergies $E_n + E_m$.

ii. Show that the two-particle Hamiltonian of (a) commutes with the particle exchange operator $\hat{\mathscr{P}}$. Find the simultaneous eigenstates of $\hat{\mathcal{H}}$ and $\hat{\mathscr{P}}$ for the two-particle system.

Seen

The exchange operator $\hat{\mathscr{P}}$ operates on the position operators as $\hat{\mathscr{P}}\hat{x}_1\hat{\mathscr{P}} = \hat{x}_2$ and $\hat{\mathscr{P}}\hat{x}_2\hat{\mathscr{P}} = \hat{x}_1$ with similar relations for the momentum operators and squares to one. Notice that the Hamiltonian is symmetric under the exchanges $\hat{x}_1 \leftrightarrow \hat{x}_2$ and $\hat{p}_1 \leftrightarrow \hat{p}_2$. Therefore $[\hat{\mathscr{P}}, \hat{\mathcal{H}}] = (\hat{\mathscr{P}}\hat{\mathcal{H}}\hat{\mathscr{P}} - \hat{\mathcal{H}})\hat{\mathscr{P}} = 0$.

The eigenstates with $+1$ eigenvalue of exchange are

$$\frac{1}{\sqrt{2}}(|\phi_n\phi_m\rangle + |\phi_m\phi_n\rangle)$$

while those with -1 eigenvalue of exchange are

$$\frac{1}{\sqrt{2}}(|\phi_n\phi_m\rangle - |\phi_m\phi_n\rangle).$$

(b) i. Now consider the Hamiltonian

$$\hat{\mathcal{H}}_0 = \frac{1}{2}U\hat{\rho}_1(\hat{\rho}_1 - 1) + \frac{1}{2}U\hat{\rho}_2(\hat{\rho}_2 - 1) + \frac{1}{2}U\hat{\rho}_3(\hat{\rho}_3 - 1)$$

where U is a positive parameter and $\hat{\rho}_n = \hat{a}_n^\dagger \hat{a}_n$ $(n = 1, 2, 3)$. The creation and annihilation operators ($\hat{a}_n^\dagger$ and $\hat{a}_n$) are taken to be bosonic. Determine three degenerate ground states of this Hamiltonian for $N = 2$ particles and the associated eigenenergies. For the case of $N = 3$ particles determine the eigenenergies and associated degeneracies (no need to write down all eigenstates).

Unseen

The eigenstates with the lowest energy 0 are $\hat{a}_2^\dagger\hat{a}_3^\dagger|0\rangle$, $\hat{a}_1^\dagger\hat{a}_3^\dagger|0\rangle$, and $\hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle$. (There are another three eigenstates with energy U that have double occupancy).

For three particles, there are $\frac{4!}{2!2!} = 10$ eigenstates. In particular, there is one with $E = 0$ that has only single occpupancy. There are six with $E = U$ that have one state with single occupancy and one with double occupancy. Finally there are three with $E = 3U$ that have one state with triple occupancy.

ii. Consider the Hamiltonian of (b)(i) under the perturbation

$$\hat{V} = -w \sum_{n=1}^{2} (\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n)$$

where w is a positive parameter. For the case of $N = 2$ particles, use degenerate perturbation theory to determine the ground state of $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{V}$ to **first** order in λ.

Unseen

This problem involves degenerate first order perturbation theory. It is sufficient to diagonalise $\hat{V}$ within the subspace of degenerate ground states. The corresponding matrix is

$$V = -w \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The eigenvalues are $\pm w\sqrt{2}$ and 0. Therefore the ground state energy to first order in the perturbation is $E_{\mathrm{gs}} = -w\lambda\sqrt{2}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97021 MATH97099	1	This question was composed of several short sub-problems that focused on material throughout the module. This was intended to be the most straightforward question on the exam and most did well on this question.	
MATH97021 MATH97099	2	The goal here was to apply perturbation theory (which was covered in detail during the module) to an unfamiliar example. Marking a through c it became clear that most are very competant with the method. Part (d) was exceptionally challenging -- but there were some good solutions here.	
MATH97021 MATH97099	3	This question was a variation of an un-assessed coursework problem. Nearly everyone received close to full marks on the bit (a,b) that was close to the coursework. Parts c and d were a little tricky but many did well here too.	
MATH97021 MATH97099	4	This question focused on spin systems. Most did well on a and b. Part c and d should have been fairly unfamiliar and challenging, but some did well here too.	
MATH97021 MATH97099	5	The mastery question focused on many particle systems. The marks on the question were a bit lower than the rest (nobody received full marks on it). Parts (a) was bookwork and most received high marks there. Part (b) was more unfamiliar but by reading some solutions, I felt that many of you have grasped second quantisation well and have an intuitive understanding of it.	