

MATH50001 - Problems Sheet 6

1. Compute the integral

$$\int_0^{2\pi} \frac{\sin^2 \theta}{2 + \cos \theta} d\theta.$$

2. Let $\text{Log}(z)$ be the principle value of $\log(z)$ and let

$$f(z) = -\text{Log}(z) + \int_1^z \frac{e^\eta}{\eta} d\eta.$$

a) Prove that $f(z)$ has holomorphic continuation to the whole complex plane \mathbb{C} .

b) Find Taylor series for $f(z)$ at $z_0 = 0$.

3. Let f be continuous on $\gamma = \{z \in \mathbb{C} : |z| = 1\}$. Prove

$$\overline{\oint_\gamma f(z) dz} = - \oint_\gamma \frac{\overline{f(z)}}{z^2} dz.$$

4. Compute

$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z(z-w)}$$

for all $w : |w| \neq 1$, where

$$\gamma = \{z \in \mathbb{C} : |z| = 1\}.$$

5. Let $A = \{z : r \leq |z| \leq R\}$, where $0 < r < R < \infty$. Show that there is a positive number ε such that for an arbitrary polynomial p

$$\max_{z \in A} |p(z) - z^{-1}| > \varepsilon.$$

6. How many roots has the polynomial

$$w(z) = z^3 + 5z + 1 \quad \text{if } |z| > 1?$$

7. Show that the polynomial $z^5 + 15z + 1$ has precisely four zeros in the annular region $\{z : 3/2 < |z| < 2\}$.

8. Let $w(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$. How many zeros (counting multiplicities) does w has in the unit disc.

9. How many zeros has the complex polynomial $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ in the annulus $\{z : 1 < |z| < 2\}$?

10. Let $a > e$. Show that the equation $e^z = az^n$ has n roots inside the circle $|z| < 1$.