Imperial College London

MATH97017 MATH97095

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Finite Elements: Numerical Analysis and Implementation

Date: Wednesday, 5 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider the finite element $(K, P, \mathcal{N})$ given by

   1. K is the 1×1 square, with bottom-left corner at $(0, 0)$.
   2. P is the polynomial space spanned by $\{1, x, y, xy\}$.
   3. $\mathcal{N} = (N_1, N_2, N_3, N_4)$ where $N_i(p) = p(z_i)$ and (z_1, z_2, z_3, z_4) are the four corners of the square.

   Find the nodal basis for this finite element. (You may use a computational linear algebra package such as Numpy or Matlab to invert matrices but you must write the matrices that you are computing with in your solution.) (8 marks)

   (b) Consider a finite element $(K, P, \mathcal{N})$ with

   1. K is the triangle with vertices at $z_1 = (0, 0)$, $z_2 = (1, 0)$, and $z_3 = (0, 1)$.
   2. P is the polynomial space spanned by $\{1, x, y, xy(1 - x - y)\}$,

   and nodal basis

$$\begin{aligned} \psi_1 = x - 9xy(1 - x - y),\ \psi_2 = y - 9xy(1 - x - y), \\ \psi_3 = 1 - x - y - 9xy(1 - x - y),\ \psi_4 = 27xy(1 - x - y). \end{aligned} \tag{1}$$

   (i) Find a set of nodal variables $\mathcal{N}$ that corresponds to this nodal basis, justifying your answer. (6 marks)

   (ii) Provide a C^0 geometric decomposition for this element, explaining why it has the specified continuity. (6 marks)

(Total: 20 marks)

2. In this question we consider the Poisson equation

$$-\nabla^2 u = f, \tag{2}$$

on a convex domain Ω with $u = 0$ on $\partial\Omega$. We assume that f is such that $u \in H^2(\Omega)$ but $u \notin H^3(\Omega)$.

(a) Consider Theorem 5.30 of the notes. Explain why this theorem is not sufficient for estimating the convergence rate for finite element discretisation for this problem when $k = 2$.

(6 marks)

(b) For $i = 1$, propose and prove a modification of Lemma 5.28 for this case.

(7 marks)

(c) For $i = 1$, propose and justify (referring to existing proofs in the notes) a modification of Theorem 5.30 for this case. Comment on the difference in the estimate compared to the case $u \in H^3(\Omega)$. (7 marks)

(Total: 20 marks)

3. (a) Write a finite element variational problem for the following equation,

$$-\nabla^2 u = 0, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \tag{3}$$

describing the types of finite element spaces that should be used to ensure a unique solution.

(6 marks)

(b) Consider the finite element discretisation in the case where Ω is a 1×1 square and $g = \exp(\cos(x)\cos(y+x))$. Explain why your variational problem is not possible to implement exactly on a computer when using numerical quadrature, and propose a modification that is.

(6 marks)

(c) Adjust the statement and proof of Céa's Lemma to accommodate your modification from (b), and comment on the conditions for convergence of the finite element solution to the exact solution as the mesh is refined. (Hints: start from the triangle inequality for $\|u - v + v - u_h\|_{H^1(\Omega)}$ for $v \in V_h$ and then work with the term $\|v - u_h\|_{H^1(\Omega)}$.)

(8 marks)

(Total: 20 marks)

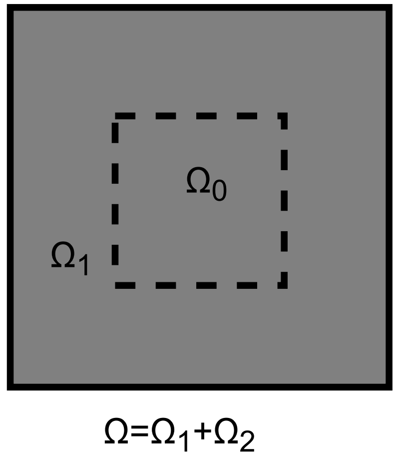

Figure 1: Domain for Question 4.

4. In this question we consider the domain displayed in Figure 1. The domain Ω is the entire square area shaded grey, with outer boundary $\partial\Omega$ shown as a continuous black line. Inside the domain is a smaller square, denoted Ω_0, with boundary Γ, shown as a dashed black line. We define Ω_1 to be the complement of Ω_0 in Ω.

We consider the following problem: find u such that

$$-\nabla^2 u = 0, \text{ in } \Omega_0 \text{ and } \Omega_1, \tag{4}$$

$$u = 0, \text{ on } \partial\Omega, \tag{5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_0} + \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_1 \cap \Gamma} = 2, \tag{6}$$

where for a domain Ω_i, $\frac{\partial u}{\partial n}|_{\Omega_i}$ is the value of the normal component of the derivative restricted to Ω_i, using the outward pointing normal to $\partial\Omega_i$. Note that since the outward pointing normals to $\partial\Omega_0$ and $\partial\Omega_1 \cap \Gamma$ are equal and opposite, this condition on Γ indicates a discontinuity in the normal derivative of u.

(a) Using the continuous Lagrange finite element space of degree k, formulate a finite element discretisation for this problem. (6 marks)

(b) Show that the finite element discretisation has a unique solution, and provide a constant γ such that the finite element solution satisfies

$$\|u_h\|_{H^1(\Omega)} \leq \gamma, \tag{7}$$

independently of h. You may quote results from the course notes without proof. (8 marks)

(c) Let u_h be the numerical solution on a mesh consisting of squares subdivided into right angled triangles, with square edge length h, and let u be the exact solution of the problem. Discuss the applicability of the bound on $\|u_h - u\|_{H^1(\Omega)}$ that we studied in the course. (6 marks)

(Total: 20 marks)

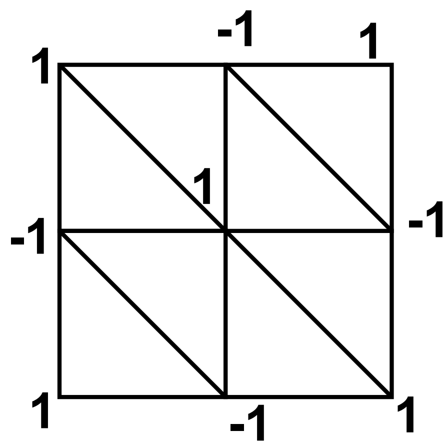

Figure 2: Example function values for Question 5.

5. (a) Let $V = (H^1(\Omega))^2$ and $Q = \mathring{L}^2(\Omega)$, where Ω is a convex polygonal domain. Let $b : V \times Q \to \mathbb{R}$ be the bilinear form

$$b(v, q) = \int_\Omega q \nabla \cdot v \, \mathrm{d}\, x. \tag{8}$$

Assuming the result that the divergence is surjective from V to Q, show that b satisfies the inf-sup condition

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta, \tag{9}$$

for some constant β.

(5 marks)

(b) Define the operator $\delta : Q \to V$ by

$$\int_\Omega w \cdot \delta q \, \mathrm{d}\, x = b(w, q), \quad \forall w \in V. \tag{10}$$

Show that the kernel of δ, $\mathrm{Ker}(\delta)$, is empty. (Hint: find a suitable choice of test function.)

(5 marks)

(c) Let $V_h \subset V$ and $Q_h \subset Q$ be finite element spaces chosen for the discretisation of Stokes' equation. Let $\Pi_h : V \to V_h$ satisfy condition 1 of Fortin's trick, i.e.

$$b(v - \Pi_h v, q) = 0, \quad \forall v \in V, \, q \in Q_h. \tag{11}$$

Define the discrete operator $\delta_h : Q_h \to V_h$ by

$$\int_\Omega w \cdot \delta_h q \, \mathrm{d}\, x = b(w, q), \quad \forall w \in V_h. \tag{12}$$

Show that $\mathrm{Ker}(\delta_h) \subseteq \mathrm{Ker}(\delta)$.

(5 marks)

(Question continues on the following page.)

(d) Consider a mesh consisting of squares subdividing into right angle triangles by joining the top left vertex and the bottom right vertex of each square, and consider the discretisation for Stokes with continuous linear Lagrange elements for each component of the velocity and continuous linear Lagrange elements for the pressure.

By considering the function $p \in Q_h$ taking vertex values in an alternating pattern as indicated in Figure 2, show that $\mathrm{Ker}(\delta_h) \not\subseteq \mathrm{Ker}(\delta)$ in this case.
Do V_h and Q_j satisfy condition 1 of Fortin's trick? (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

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XXX (Solutions)

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1. (a) The Vandermonde matrix and its inverse are

sim. seen ⇓

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

which can be computed by hand by e.g. doing back-substitution on the columns of the identity matrix. The basis is then

$$\psi_1(x,y) = 1 - x - y + xy = (1-x)(1-y), \tag{2}$$
$$\psi_2(x,y) = x - xy = x(1-y), \tag{3}$$
$$\psi_3(x,y) = y - xy = y(1-x), \tag{4}$$
$$\psi_4(x,y) = xy. \tag{5}$$

8, A

(b) (i) Suitable nodal variables are $N_i(p) = p(z_i)$ where $z_1 = (1,0)$, $z_2 = (0,1)$, $z_3 = (0,0)$, $z_4(1/3,1/3)$.

unseen ⇓

We can check that the basis is a nodal one for these nodes by noticing that the spanning set for P is the linear functions plus a cubic "bubble" function that vanishes on the triangle vertices (and the edges). Thus to make a nodal basis for our nodal variables, the basis functions 1 to 3 can just be the usual linear basis functions (that are equal to 1 on the corresponding vertex and 0 at all the others) plus a scalar multiple of the bubble function so that they vanish at z_4. The bubble vanishes at all the vertices so it just needs to be scaled appropriately to take the value 1 at z_4 as required.

6, A

(ii) We assign N_i to vertex z_i for $i = 1, 2, 3$, and the bubble function the entire cell. This is a C0 geometric decomposition because: (1) the value at each vertex can be obtained from the nodal variable assigned to that vertex (since it is just point evaluation at the vertex), (2) the value at each edge can be obtained from the nodal variables assigned to the closure of the edge, which is just vertex values at each end in this case, and the function is linear when restricted to an edge.

sim. seen ⇓

6, B

2. (a) The theorem is insufficient because $|u|_{H^3(K_1)}$ is unbounded, so it doesn't provide any bound on the error. unseen $\Downarrow$

6, B

(b) The appropriate statement is (under the same conditions as 5.28 but with $u \in H^2(K_1)$, $k = 3$), sim. seen $\Downarrow$

$$|\mathcal{I}_{K_1} u - u|_{H^1(K_1)} \leq C_1 |u|_{H^2(K_1)}. \tag{6}$$

To prove it,

$$\begin{aligned} |\mathcal{I}_{K_1} u - u|_{H^1(K_1)} &\leq \|Q_{3,B} u - u\|^2_{H^1(K_1)} + \|\mathcal{I}_{K_1}(u - Q_{3,B} u)\|^2_{H^1(K_1)} & (7)\\ &\leq (1 + C^2)|u|^2_{H^2(K_1)}, & (8) \end{aligned}$$

where $Q_{3,B}$ is the degree k averaged Taylor polynomial over a ball B inside K_1 but as large as possible, and where we used Lemmas 3.22 and Corollary 3.16. 7, B

(c) The appropriate statement is (under the same conditions as 5.30 but with $u \in H^2(\Omega)$)

$$|\mathcal{I}_K u - u|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}. \tag{9}$$

To show this, note that we can obtain the local estimate

$$|\mathcal{I}_K u - u|_{H^1(K)} \leq C_K d|u|_{H^{k+1}(K)}, \tag{10}$$

by following the steps in the proof but with k replaced by 1. Then the same technique of summing over all the cells gives the global result. 7, C

3. (a) To derive the variational form, we multiply by a test function v and integrate by parts as usual to get sim. seen ⇓

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\, x - \int_{\partial\Omega} v \underbrace{\frac{\partial u}{\partial n}}_{=g} \,\mathrm{d}\, S = 0, \tag{11}$$

so a suitable variational form is to find $v \in \bar{V}_h$ such that

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\, x = \int_{\partial\Omega} vg \,\mathrm{d}\, S, \quad \forall v \in \bar{V}_h, \tag{12}$$

where $\bar{V}_h$ is the subspace of V_h of functions that integrate to zero, and V_h is some choice of C^0 finite element space. 6, A

(b) The issue is that the integrals are not tractable in general, so we can't evaluate the RHS of the problem. A possible modification is to interpolate g to V_h in the boundary resulting in g_h, and solve the perturbed problem unseen ⇓

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\, x = \int_{\partial\Omega} vg_h \,\mathrm{d}\, S, \quad \forall v \in \bar{V}_h. \tag{13}$$

6, D

(c) The modification to Céa's Lemma is unseen ⇓

$$\|u - u_h\|_{H^1(\Omega)} \leq (1 + M/\gamma) \sup_{v \in V_h} \|u - v\|_{H^1(\Omega)} + \frac{C}{\gamma}\|g - g_h\|_{L^2(\partial\Omega)}, \tag{14}$$

so there are now two terms, a best approximation term of u in V_h, and an approximation error term for g_h.

To prove it, following the steps of Céa's Lemma, we take a test function $v \in V_h$ in both the exact and approximate equation, and compute the difference, to yield

$$a(u - u_h, v) = \int_{\partial\Omega} v(g - g_h) \,\mathrm{d}\, S, \forall v \in V_h. \tag{15}$$

Then we use coercivity to write (for any $v \in V_h$)

$$\begin{aligned}
\gamma\|u_h - v\|_{H^1(\Omega)} &\leq a(u_h - v, u_h - v), && (16)\\
&= a(u_h - u, u_h - v) + a(u - v, u_h - v), && (17)\\
&= \int_{\partial\Omega} (u_h - v)(g_h - g) \,\mathrm{d}\, S + a(u - v, u_h - v), && (18)\\
&\leq C\|u_h - v\|_{H^1(\Omega)}\|g_h - g\|_{L^2(\partial\Omega)} + M\|u - v\|_{H^1(\Omega)}\|u_h - v\|_{H^1(\Omega)}, && (19)
\end{aligned}$$

where C is the constant in the trace inequality and M is the continuity constant of the bilinear form $a(u, v)$. Then, dividing by $\|u_h - v\|_{H^1(\Omega)}$ gives

$$\gamma\|u_h - v\|^2_{H^1(\Omega)} \leq C\|g_h - g\|_{L^2(\partial\Omega)} + M\|u - v\|_{H^1(\Omega)}. \tag{20}$$

Then, combining with the triangle inequality, we get

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \|u - v\|_{H^1(\Omega)} + \|u_h - v\|_{H^1(\Omega)}, && (21)\\
&\leq (1 + M/\gamma)\|u - v\|_{H^1(\Omega)} + \frac{C}{\gamma}\|g - g_h\|_{L^2(\partial\Omega)}, && (22)
\end{aligned}$$

and minimisation over v gives the result. 8, D

4. (a) Multiplying by a test function v that vanishes on the exterior boundary and integrating by parts separately in Ω_1 and Ω_2 gives

unseen $\Downarrow$

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\, x - \int_\Gamma v \left(\frac{\partial u}{\partial n}|_{\partial\Omega_1} + \frac{\partial u}{\partial n}|_{\partial\Omega_2 \cap \Gamma} \right) \mathrm{d}\, S = 0, \tag{23}$$

and substitution of the boundary condition gives the variational problem: find $u_h \in V_h$ such that

$$\int_\Omega \nabla v \cdot \nabla u_h \,\mathrm{d}\, x = 2 \int_\Gamma v \,\mathrm{d}\, S, \quad \forall v \in V_h. \tag{24}$$

6, A

(b) We are in the case of Theorem 4.38, so we just need to check continuity of the linear form according to

sim. seen $\Downarrow$

$$F[v] = 2 \int_\Gamma v \,\mathrm{d}\, S \leq \|v\|_{L^2(\Gamma)} 2|\Gamma| \leq \|v\|_{H^1(\Omega_0)} 2|\Gamma| \leq \|v\|_{H^1(\Omega)} 2|\Gamma|. \tag{25}$$

where we have used the trace theorem for continuous finite elements (Theorem 4.4), and $|\Gamma| = \int_\Gamma \mathrm{d}\, S$. Hence F is continuous.

8, A

(c) The bound studied in the course is

unseen $\Downarrow$

$$\|u_h - u\|_{H^1(\Omega)} \leq h|u|_{H^2(\Omega)}, \tag{26}$$

but the solution has a jump in the first derivative across Γ, so $|u|_{H^2(\Omega)}$ is not finite, so the bound does not imply convergence of the numerical solution as $h \to 0$.

6, C

5. (a) The map is surjective, so there exists $v \in V$ such that $q = \nabla \cdot v$ for all $q \in Q$. Hence, using the Riesz Representation Theorem, for all $F \in Q'$, there exists q_F such that unseen $\Downarrow$

$$F[p] = \int_\Omega p q_F \,\mathrm{d}\,x, \quad \forall p \in Q. \tag{27}$$

So, for all $F \in Q'$, there exists $v \in V$ such that

$$b(v, p) = \int_\Omega \nabla \cdot v p \,\mathrm{d}\,x = F[p], \quad \forall p \in Q. \tag{28}$$

In other words, for all $F \in Q'$ there exists v such that $Bv = F$, which means that B is surjective. Then, from the notes, this implies the inf-sup condition. 5, M

(b) Let $q \in \mathrm{Ker}(\delta)$, i.e. $\delta q = 0$. Taking w such that $\nabla \cdot w = q$, we have unseen $\Downarrow$

$$0 = \int_\Omega \nabla \cdot w q \,\mathrm{d}\,x = \int_\Omega q^2 \,\mathrm{d}\,x \implies q = 0. \tag{29}$$

5, M

(c) Let $q \in \mathrm{Ker}(\delta_h)$. Then for $w \in V$, unseen $\Downarrow$

$$\int_\Omega w \cdot \delta q \,\mathrm{d}\,x = b(w, q) = b(\Pi_h w, q), \tag{30}$$

$$= \int_\Omega \Pi_h w \cdot \delta_h q \,\mathrm{d}\,x = 0, \tag{31}$$

so $q \in \mathrm{Ker}(\delta)$ as required. 5, M

(d) Considering $p \in Q_h$ having a pattern of the type given in Figure 2, it suffices to consider $b(w, p)$ for w being basis functions associated with vertices in the interior of the mesh, which span V_h (because of the zero boundary condition). The support of w consists of 6 triangles with symmetry about a diagonal line from top-left to bottom-right. The divergence of w is antisymmetric about that line, whilst p is symmetric, so the integral $b(w, p)$ vanishes. Hence, $p \in \ker(\delta_h)$. The previous result says that the Fortin Trick assumptions imply that the $\ker(\delta_h)$ is empty, V_h, Q_h must fail to satisfy the Fortin Trick assumptions. unseen $\Downarrow$ 5, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH96063 MATH97017 MATH97095	1	All students did well on this question. A common error in b(i) was trying to use the Vandermonde matrix to find the nodal variables (it can't be done).
MATH96063 MATH97017 MATH97095	2	All students could do 2(a). Some students missed the point in (b) of not jumping straight to the polynomial degree for the choice of norm in lemma 5.28.
MATH96063 MATH97017 MATH97095	3	Students did well on 3a. Many students were able to suggest an alternative using interpolation in 3b. 3c was not attempted by most students.
MATH96063 MATH97017 MATH97095	4	Students did well on 4a. In 4b, several students missed that the trace theorem needs to be used for the continuity of F, and some did not link with the correct coercivity result from the notes. Some but not all answered 4c correctly.
MATH96063 MATH97017 MATH97095	5	Generally students made a good attempt at this question. In 5a, there was some confusion between the divergence operator V->Q and the B operator. Parts b and c were frequently well answered but no candidates attempted 5d.