Applied Complex Analysis - Lecture Nine

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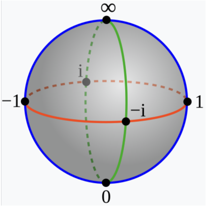
Behaviour at complex ∞



- Approach is to analyse $f(1/z)$ for small z
- $f(z) = z$ has a simple pole at ∞
- Argument of $f(z)$ as $z \to \infty$ depends on direction of approach (like singularities at zero)
- $f(z)$ analytic at $z = 0$ if and only if $f(1/z)$ analytic at ∞
- Detecting branch points: analyse $f(1/z)$ and take a little circle around $z = 0$, look for a jump, as before
- This is conceptually challenging - equivalent to taking a little circle around complex ∞.

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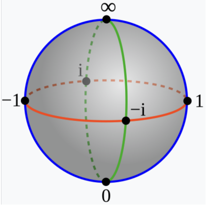

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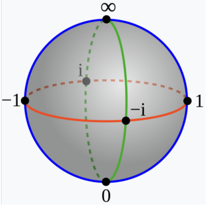

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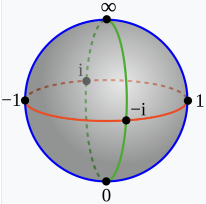

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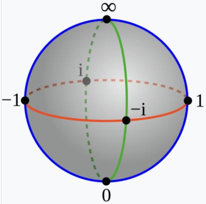

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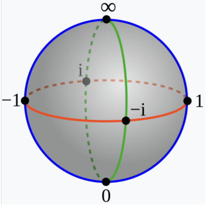

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Examples

For $w = 1/z$:

- $\log(w)$
- $\sqrt{w}$
- $\sqrt{(w-1)(w+1)}$

Branch points, branch cuts, and branches

- **Branch points** are a type of singular points - a rotation will give us a jump (in multi-valued case)
- To make things easy on ourselves, we choose single-valued versions of multi-valued functions. But we have to choose the jump somewhere. This jump is called a **branch cut**.
- The description of the branch cut (e.g. $\arg z \in [-\pi, \pi)$) determines a single **branch** of the multi-valued function.
- Branch cuts may be interpreted as stitching together single-valued branches to reconstruct the multi-valued function
- The number of branches is equal to the number of values of the multi-valued function (away from the branch points).
- We often say (perhaps ambiguously) the square root function has two branches to mean it has two branches *given a branch cut*
- But there are infinitely many choices for branch cuts!
- Examples.

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Example problems

- $$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \alpha \in (0,1)$$

- $$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

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Principal value integrals

- We say integral is *weakly singular* if $f(z) = O(|z|^\alpha)$ for some $\alpha \in (-1, 0)$ as $z \to 0$. The integrand diverges at zero, but the integral is absolutely convergent.
- Such integrals are absolutely convergent, similar in many ways to integrals of smooth functions.
- When $\alpha = -1$ (or worse), integrals are not absolutely convergent. But they may converge in a different sense.

$$\fint_a^b f(x)dx = \lim_{\epsilon \to 0+} \left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right).$$

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Examples

- $$\int_{-1}^{1} \frac{1}{x} dx$$

- $$\int_{-1}^{1} \frac{x^{\alpha-1}}{1-x}, \quad \alpha \in (0,1)$$

Next week

- We will move on to a numerical section of the course
- There will be live coding demos, in Julia notebooks
- These are available online if you want to follow along
- Alternatively, you can install Julia, Anaconda (python distribution), and IJulia (Julia package)