MATH50010: Probability for Statistics
Problem Sheet 7

Questions marked with (†) build on material from Tuesdays lecture.

1. Suppose $(\Omega, \mathcal{F}\,\Pr)$ is a probability space and $A, B, C \in \mathcal{F}$. Show that

$$\Pr(A \cap B|C) = \Pr(A|B \cap C)\Pr(B|C).$$

2. (a) (†) Let $\mathbf{P}$ be the transition matrix of a discrete Markov chain $(X_n)_{n\geq 0}$. Show by induction that the n-step transition matrix satisfies $\mathbf{P}_n = \mathbf{P}^n$.

   (b) (†) Show that a *stochastic matrix* has at least one eigenvalue equal to 1. Hence show that if $\mathbf{P}$ is a stochastic matrix, then so is $\mathbf{P}^n$ for all $n \in \mathbb{N}$.

3. For each matrix, decide whether it is stochastic. If it is, draw the corresponding transition diagram (assuming the state space is given by $E = \{1, 2, 3\}$).

$$\text{(a)} \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & -1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0.6 & 0.35 & 0.05 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.05 & 0.85 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Suppose we use a Markov chain to model the autumn weather in Bretagne (France), from day to day, as follows. If it is raining today, there is a probability 0.3 that it will rain again tomorrow. Similarly, if it is raining today, the probability of cloudy weather (with no rain) the next day is 0.5, and the probability of sunshine the next day is 0.2. Taking the states in order as rainy, cloudy, sunny, the full transition matrix for the Markov chain is:

$$\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

   Suppose today is a Friday in Bretagne, in mid-October, and it is raining.

   (a) Calculate the probability that both the next two days, Saturday and Sunday, will be sunny.

   (b) Calculate the probability of rain on Sunday.

   (c) Suggest aspects of this model that are unrealistic. How might the model be improved?

5. Let X_n be the maximum reading obtained in the first n rolls of a fair die. Show that $\{X_n\}$ is a Markov chain, and give the transition probabilities.

6. (†) Let (S_n) be a symmetrical random walk on the state space $\mathbb{Z}$, so that if $S_{n-1} = k$, then $S_n = k - 1$ with probability $\frac{1}{2}$ and $S_n = k + 1$ with probability $\frac{1}{2}$.

   Determine whether each of these processes below is a time homogeneous Markov chain and, if so, find the transition matrix.

   (a) $A = (S_n)_{n\geq 0}$,

   (b) $B = (S_n + n)_{n\geq 0}$,

   (c) $C = (S_n + n^2)_{n\geq 0}$,

   (d) $D = (S_n + (-1)^n)_{n\geq 0}$,

(e) $E = (|S_n|)_{n\geq 0}$,

(f) $F = (S_n^2 - n)_{n\geq 0}$,

(g) $G = (S_{2n})_{n\geq 0}$.

For discussion

7. (†) Define the sequence $Y_1, Y_3, Y_5, \ldots$ of independent and identically distributed random variables by

$$\Pr(Y_{2k+1} = -1) = \Pr(Y_{2k+1} = 1) = \frac{1}{2}, \qquad k \geq 0.$$

Further, define $Y_{2k} = Y_{2k-1}Y_{2k+1}$

(a) Show that $(Y_{2k})_{k\geq 0}$ is a sequence of independent, identically distributed random variables, with the same distribution as the odd Ys.

(b) Show that $Y_1, Y_2, \ldots$ is a sequence of pairwise independent random variables.

(c) Show further that $p_{ij}(n) = \Pr(Y_n = j|Y_0 = i)$ satisfies the Chapman-Kolmogorov equations.

(d) Explain why (Y_k) is not a Markov chain (hence the C-K equations are necessary but not sufficient for a stochastic process to be Markov).

(e) Show that $Z_n = (Y_n, Y_{n+1})$ is a (non-homogeneous) Markov chain.