MATH40005 Probability and Statistics

Midterm [20 points] **23 November 2021**

Please remember to define your notation and justify all your answers.

Question 1: Three fair six-sided dice are rolled together once.

(a) (1 point) Find the sample space for this experiment.

(b) (2 points) What is the probability that one die shows 4, another 3, and another 1?

Solution:

(a) The sample space is given by

$$\begin{aligned}\Omega &= \{1,2,3,4,5,6\}^3 \\ &= \{\omega = (\omega_1, \omega_2, \omega_3) : \omega_i \in \{1,2,3,4,5,6\}, \text{for } i = 1,2,3\}.\end{aligned}$$

[**1 point**]

(b) From lectures (by the multiplication principle) we know that $\text{card}(\Omega) = 6^3 = 216$. For the three possible outcomes involving the numbers 4,3,1 there are 3! possible arrangements since the order does not matter. Hence, using the naive definition of probability, we get for the event A :="one die shows 4, another 3, and another 1" that

$$\mathrm{P}(A) = \frac{\text{card}(A)}{\text{card}(\Omega)} = \frac{3!}{6^3} = \frac{6}{216} = \frac{1}{36}.$$

[**1 point for the justification and 1 point for the correct probability**]

Question 2: (4 points) Suppose there are three identical urns. Each urn contains 6 identical balls labelled with the numbers $\{1,2,3,4,5,6\}$. Suppose you take one ball from each urn. What is the probability that one of the numbers on the three labels equals the sum of the other two numbers?

Solution: The sample space is given by $\Omega = \{1,2,3,4,5,6\}^3$ with $\text{card}(\Omega) = 6^3 = 216$. Let A denote the event that one of the numbers on the three labels equals the other two numbers.

We can list the possible outcomes and the number of possible arrangements for each number combination, where we need to adjust for over-counting if a number appears twice.

- Numbers: 1,1,2, Possible cases: 3!/2!=3
- Numbers: 1,2,3, Possible cases: 3!
- Numbers: 1,3,4, Possible cases: 3!
- Numbers: 1,4,5, Possible cases: 3!
- Numbers: 1,5,6, Possible cases: 3!
- Numbers: 2,2,4 Possible cases: 3!/2!=3

- Numbers: 2,3,5 Possible cases: 3!
- Numbers: 2,4,6 Possible cases: 3!
- Numbers: 3,3,6 Possible cases: 3!/2!=3

Using the naive probability, we get that

$$P(A) = \frac{\text{card}(A)}{\text{card}(\Omega)} = \frac{3 \cdot 3 + 6 \cdot 3!}{216} = \frac{45}{216} = \frac{5}{24}.$$

[**2 points for the justification and 2 points for the correct probability**]

Question 3: If the letters H, I, P, P, O, P, O, T, A, M, U, S are arranged at random, what is the probability that

(a) (2 points) the arrangements spell the word HIPPOPOTAMUS?

(b) (2 points) the arrangements have three adjacent P's?

Solution:

(a) Let Ω denote the sample space consisting of all possible arrangements of the letters H, I, P, P, O, P, O, T, A, M, U, S. Using the multiplication principles, there are 12! possibilities of arranging the 12 letters. Note that P appears three times, we need to adjust for overcounting, these letters can be arranged in 3! possible ways; also, O appears twice, we need to adjust for overcounting, these letters can be arranged in 2! possible orders. Altogether we have

$$\text{card}(\Omega) = \frac{12!}{3!2!} = 11! \left[= \frac{479001600}{12} = 39916800\right].$$

[**The factorials do not need to be expanded for full marks.**]

We define the event E= the arrangements spell the word HIPPOPOTAMUS. Since only one arrangement spells the word HIPPOPOTAMUS, the corresponding probability is given by

$$P(E) = \frac{\text{card}(E)}{\text{card}(\Omega)} = \frac{1}{11!}.$$

[**1 point for the justification and 1 point for the correct probability**]

(b) Consider removing the P's: _ H _ I _ O _ O_ T_ A_ M_ U_ S_. We can arrange the remaining letters in 9!/2! possible ways, where we adjust for the fact that the letter O appears twice. In order to ensure that we have three adjacent P's, we must choose one of the blank spaces _ to put the three P's. We can do this in $\binom{10}{1} = 10$ possible ways. We define the event E= the arrangements have three adjacent P's. Then

$$\text{card}(E) = \frac{9!}{2!} \cdot 10 = 9! \cdot 5[= 1814400].$$

Hence,

$$P(E) = \frac{\text{card}(E)}{\text{card}(\Omega)} = \frac{9! \cdot 5}{11!} = \frac{5}{110}[= 0.04545455].$$

[**1 point for the justification and 1 point for the correct probability**]

Question 4: (2 points) Consider a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $A, B \in \mathcal{F}$. Suppose that $\mathrm{P}(A) = 0$. Can we conclude that $\mathrm{P}(A \cap B) = 0$? If so, prove the result, otherwise give a counterexample.

Solution: Yes, the stated implication holds. We prove this as follows. We note that $A \cap B \subseteq A$. Hence, from lectures (Thm 4.2.3), we deduce that $\mathrm{P}(A \cap B) \leq \mathrm{P}(A)$. Since the probability measure takes values between 0 and 1, we deduce that $0 \leq \mathrm{P}(A \cap B) \leq \mathrm{P}(A) = 0$ (by assumption). This implies indeed that $\mathrm{P}(A \cap B) = 0$.

[**2 points for the justification** Please note that $\mathrm{P}(A) = 0$ does not in general imply that $A = \emptyset$. So any justification using such an argument is false.]

Question 5: Suppose there are only two types of books on your bookshelf, they either cover probability or analysis (but never both topics). 70% of your books are analysis books. 65% of the probability books contain graphics, whereas only 30% of the analysis books contain graphics, the other books only contain text and formulas.

(a) (3 points) Suppose you randomly take one book from your bookshelf. What is the probability that it does not contain any graphics?

(b) (1 point) Suppose that one of your books does not contain any graphics. What is the probability that this book is a probability book?

Solution: We define the following events P:=book is a probability book, A:=book is an analysis book, G:= book has graphics. We read off the following probabilities:

$$\mathrm{P}(P) = 30/100 = 3/10, \quad \mathrm{P}(A) = 70/100 = 7/10, \quad \mathrm{P}(G|P) = 65/100 = 13/20, \quad \mathrm{P}(G|A) = 30/100 = 3/10.$$

[1 marks for defining the events and stating the (conditional) probabilities.]

(a) We apply the law of total probability and find that

$$\begin{aligned} \mathrm{P}(G) &= \mathrm{P}(G|P)\mathrm{P}(P) + \mathrm{P}(G|A)\mathrm{P}(A) \\ &= \frac{13}{20}\frac{3}{10} + \frac{3}{10}\frac{7}{10} = \frac{81}{200}[= 0.405]. \end{aligned}$$

Hence,

$$\mathrm{P}(G^c) = 1 - \mathrm{P}(G) = \frac{119}{200}[= 0.595].$$

[1 mark for the computation, 1 mark for stating the law of total probability.]

(b) We note that $\mathrm{P}(G^c|P) = 1 - \mathrm{P}(G|P) = 7/20$. Using Bayes' theorem, we deduce that

$$\mathrm{P}(P|G^c) = \frac{\mathrm{P}(G^c|P)\mathrm{P}(P)}{\mathrm{P}(G^c)} = \frac{7}{20}\frac{3}{10}\frac{200}{119} = \frac{21}{119}[= 0.1764706].$$

[1 mark for the computation and justification (stating Bayes' theorem)]

Question 6: Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space and consider events $A, B \in \mathcal{F}$. Suppose that $\mathrm{P}(A) = 1/3, \mathrm{P}(B) = 2 \cdot \mathrm{P}(A), \mathrm{P}(A \cap B) = \mathrm{P}(A)/2$.

(a) (2 points) Find $\mathrm{P}(A) + \mathrm{P}(B)$. Can you conclude that $A \cup B = \Omega$?

(b) (1 point) Find $\mathrm{P}(B^c|A)$.

Solution:

(a) We note that $\mathrm{P}(A) = 1/3, \mathrm{P}(B) = 2 \cdot \mathrm{P}(A) = 2/3, \mathrm{P}(A \cap B) = \mathrm{P}(A)/2 = 1/6$. Hence, $\mathrm{P}(A) + \mathrm{P}(B) = 1$. [**1 mark**]

However, we cannot conclude that $\Omega = A \cup B$, since (by a result from lectures)

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6} < 1 = \mathrm{P}(\Omega).$$

[**1 mark for the justification**]

(b) Using the definition of conditional probability, we need to compute

$$\mathrm{P}(B^c|A) = \frac{\mathrm{P}(B^c \cap A)}{\mathrm{P}(A)}.$$

By the law of total probability, we have

$$\frac{1}{3} = \mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^c) = \frac{1}{6} + \mathrm{P}(A \cap B^c),$$

hence $\mathrm{P}(A \cap B^c) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$.

Altogether, we get

$$\mathrm{P}(B^c|A) = \frac{\mathrm{P}(B^c \cap A)}{\mathrm{P}(A)} = \frac{1/6}{1/3} = \frac{1}{2}.$$

[**1 mark**]