Imperial College
London

M3P5

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: **Thursday 26th May 2016**

Time: **14.00 – 16.00**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and $S = \{(x, y, f(x, y)) | (x, y) \in \mathbb{R}^2\}$ be a surface in $\mathbb{R}^3$.

(a) Show that S is diffeomorphic to $\mathbb{R}^2$.

(b) Find an expression for the length of the curve $C = \{(x, y, f(x, y)) | x^2 + y^2 = 1\} \subset S$.

(c) Classify all regular closed curves on S up to a regular homotopy.

2. Let S_1 and S_2 be two surfaces and $f : S_1 \to S_2$ be a smooth map, which is onto.

(a) Show that there is a point $p \in S_1$, so that the differential df_p is invertible.

(b) Suppose $\langle v, w \rangle = \langle df_p(v), df_p(w) \rangle$ for every point $p \in S_1$ and every two vectors v and w in the tangent plane T_pS_1. Express the Gaussian curvature of S_1 at p in terms of the Gaussian curvature of S_2 at $f(p)$.

(c) Assuming the same condition as in part (b) holds, can you conclude that S_1 and S_2 are isometric? Give a proof or a counterexample.

4. (a) S_1, S_2 and S_3 are compact connected surfaces without boundary. You are given the following 3 statements:

(STATEMENT 1) The connected sum of S_1 and S_2 has Euler characteristic 0, while the connected sum of S_2 and S_3 has Euler characteristic 1.

(STATEMENT 2) S_1 is orientable and locally isometric to the plane.

(STATEMENT 3) S_2 is a surface in $\mathbb{R}^3$. Its mean curvature $H(p)$ and Gaussian curvature $K(p)$ satisfy $H^2 = K$ for every point $p \in S_2$.

Find S_1, S_2 and S_3 (up to diffeomorphism).

(b) Let T be a surface in $\mathbb{R}^3$ diffeomorphic to a torus. Prove that T has points of positive, negative and zero Gaussian curvature.

(c) *Turning a sphere inside out in* $\mathbb{R}^4$. Let $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ denote the standard sphere. Construct a homotopy $f_t : S^2 \to \mathbb{R}^4$, $t \in [0, 1]$, such that for each t the image of f_t is isometric to S^2, $f_0(x, y, z) = (x, y, z, 0)$ and $f_1(x, y, z) = (x, y, -z, 0)$. (Hint: think of an analogous problem for a circle in $\mathbb{R}^3$).

3. (a) Let γ be a curve on a surface S in $\mathbb{R}^3$ and N be a unit normal vector field on S. Show that $\langle \gamma''(0), N(\gamma(0)) \rangle = -\langle \gamma'(0), dN_{\gamma(0)}(\gamma'(0)) \rangle$.

(b) Recall that the second fundamental form $\sigma_p(X, Y)$ is defined as $\sigma_p(X, Y) = -\langle X, dN_p(Y) \rangle$. Let X be a unit vector in T_pS. Use part (a) to give a geometric interpretation of $\sigma_p(X, X)$. Let P denote the plane intersecting S at p and spanned by vectors N and X. What can be said about intersection $P \cap S$?

(c) Let $\lambda_2(p) \geq \lambda_1(p) \geq 0$ denote the principal curvatures of S at the point p. Let γ be a geodesic in S with $\gamma(0) = p$. Show that $\lambda_1 \leq |\gamma''(0)| \leq \lambda_2$.