Introduction to Game Theory MATH60141/MATH70141

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Chapter 1: Prelude

1.1 About this Module

Welcome to Introduction to Game Theory! This course will provide you with a broad understanding of the basics of game theory and investigates several of the different types of games that we can analyse using the theory. We will endeavour to build up our knowledge in as 'hands on' a way as possible with an 'active' learning style, often by playing games in the classroom to understand the ideas and concepts. There will also be recommendations of games to play as homework as well as collaboration on group assessed tasks (more on that later). Throughout the module applications of the theory will be presented; we'll see applications in the economic behaviour of market competitors, competition between biological species, how to make informed military decisions, politics and how to run for prime minister, traffic and congestion, bargaining and coming to collaborative agreements, whether you should pay your taxes, applications in sport and also where to meet for a date (and many more applications)!

The course has been split into seven chapters, starting with chapter 1, the prelude, where we are now, which will be a short excursion exploring a few games and talking about what constitutes mathematical game theory. Chapter 2 then goes on to look at the concepts of dominance, best response and equilibria. In chapter 3 we introduce the notion of a mixed strategy and prove Nash's theorem on the existence of an equilibrium in finite games. Chapter 4 concerns zero sum games, or strictly competitive games, where we will understand the concept of max-min and min-max strategies and see Von Neumann's Minimax theorem. Chapter 5 deals with an introduction to cooperation in games, and how players may improve their success by cooperating on an agreed joint strategy. We will see the Nash bargaining solution arise from a set of axioms. In chapter 6 we will focus on congestion games, or traffic games, a type of potential game. In this context we will see how equilibria emerge and investigate whether these equilibria are always socially good by using a ratio known as the price of anarchy. Finally we arrive at chapter 7 on combinatorial games. In this chapter we will focus on the game of Nim and see how it plays a central role in the theory of these games.

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1.1.1 Accessability

This module requires few prerequisites and has been designed with one aim of trying to 'open the door' for more cross-departmental study into the mathematics department (which is typically difficult). As such, any third/fourth year Mathematics, Physics or Computer Science student will be well equipped for studying the module. For students slightly further reaching, like third/fourth year Engineers, Chemists and Biologists, it is likely that you will be able to take the module with your current mathematics knowledge but it is recommended that you speak with the lecturer about suitability so you (and me) are put at ease: there may be one or two gaps in your knowledge for which a small amount of supplementary reading can be used to bridge them.

1.1.2 Prerequisites

Though we have said the module is accessible, for those wanting a more detailed breakdown the following topics should be considered prerequisite knowledge:

- Basic Probability and Statistics;
- One-dimensional Calculus;
- Mathematical notation;
- Vectors and Matrices;
- Logic and Proof techniques; we will utilise several different methods of proof and logical arguments, this is perhaps the only area which could be lacking for students beyond Maths, Physics and Computer Science. As mentioned - if you are unsure on your current level of knowledge, please get in touch with the lecturer!

1.1.3 Study Materials and Reading Lists

There are seven problem sets to accompany these lecture notes; one for each chapter, which may build on understanding from previous chapters as we develop our knowledge further. The problems will generally start simple and progress in difficulty throughout each sheet. The majority of the problems should enrich your understanding of the concepts and prepare you well for examination (there will be nothing included that would constitute a waste of your time). Occasionally, problems with one of two symbols by the question number may appear. Firstly star problems, which are considered (very) difficult. The solution of these may include references to research papers for instance. Nevertheless they are included for interest and often take the theory in the notes a little further. Secondly diamond problems, which are not necessarily of any specific difficulty, but are open problems that have deliberately been left unsolved (no solutions will be provided for these). These are included to provide ideas for possible group coursework topics (see soon). Usually these are ideas and problems the lecturer thinks are interesting, but hasn't had a chance to investigate yet!

The course has been designed so that it is entirely self-contained; meaning that these lecture notes, along with the seven problem sets contain everything that you will need to prepare for the examination, so if you don't learn efficiently from using books and other resources don't worry, everything you need is included. Nevertheless, many students like to learn from the use of textbooks on the topic so a short reading list is included below. There are many books on game theory and really any textbook introducing the topic will contain similar information, so feel free to browse the library if you're looking for a different choice. Some useful texts include:

- Game Theory Basics (Bernhard von Stengel): For those looking for a companion book for the course this should be your number one choice. We will very closely follow parts of this book throughout several chapters of this course. In fact this book really re-shaped some parts of this module, so if you prefer to learn from textbooks then the corresponding chapters from this book are my recommended reading;
- Introduction to Game Theory (Peter Morris);
- The Theory of Games (Jianhua Wang);
- Introducing Game Theory and its Applications (Mendelson, Chapman and Hall);
- Game Theory: A Nontechnical Introduction to the Analysis of Strategy (Roger McCain);
- Winning Ways for Your Mathematical Plays (Berlekamp, Conway and Guy).

Below are some other resources that I recommend (of course I have developed these notes because I want you to use them, but hearing different explanations of the same concept can help you to master it):

- There is an excellent series of lectures on Game Theory by Ben Polak from Yale university on Youtube (YaleCourses). The lecture series is delivered primarily to Economics majors (eww), meaning the level of mathematics is generally lower than what we will employ, but nevertheless Ben does a fantastic job of explaining the concepts of the theory - for those that are interested in more detail on the social, political and economic aspects of the theory, then these lectures are highly recommended. Many of the classroom games we will play have also been taken from Ben's game theory course.
- The lecture notes on Algorithmic Game Theory by Tim Roughgarden (these are also written into a book called 'Twenty Lectures on Algorithmic Game Theory'). For our course they prove an excellent resource for Congestion Games.
- Finally, and most importantly, the lecture notes and module on the Dynamics and Learning of Iterated Games (which you should certainly consider taking if you are interested in this module) taught by Professor Sebastian van Strien here in the Mathematics department at Imperial is an excellent companion to this module. This module has been designed so that there is as minimal overlap as possible with this companion. Where we have some gaps on the theory of repeated games and dynamics within games, such as evolutionary game theory, that module delves deep into these concepts.

If you find a textbook or any resource you believe to be excellent, please let me know!

1.1.4 Assessment

The module will have an assessed **group** coursework task, worth 10%. This will take place during the Autumn term, probably released mid term and due towards the end of term - more details closer to the time! There will be a 2-hour exam with four questions worth a total of 90% in the May-June exam season. For Master's students the weightings remain the same but this exam will be 2 hours 30 minutes long and contain an extra fifth question aimed at Master's level.

1.1.5 Final Administrative Things

There will be a weekly 'office hour' (I'll be in my office and free for one hour to answer any questions you might have about anything for this module) that you are welcome to come and drop by during if you have any questions that you want to ask me. My office is room 657 in Huxley. During the first lecture we will do a vote as to when the office hour will take place.

The course has 30 hours of scheduled lectures; usually 3 hours per week for 10 weeks. We will use approximately 25 of these as lectures, where we will move through these lecture notes and cover new content. The remaining 5 hours we will dedicate to being 'problem classes'; spread approximately one every 2 weeks, where we will spend the time going through some of the problems from the problem sheets. We may also play some additional classroom games during these sessions that didn't quite fit into the main lecture notes, so don't miss out!

1.2 The Everybody Bids Game

Let's play a game. We'll involve everybody in the class (I'll play too!). Everyone in the class is going to be given a piece of paper, and on that paper you need to write down your name and a whole number between 0 and 100 inclusive. Don't let anyone see your number and don't discuss your decision with anyone! Once everyone has decided on their number, the papers will all be collected and the average (the mean) of the numbers we all chose will be calculated (any bids that are not whole numbers within the range given will be discarded and not count towards the average). Whoever has bid closest to one half of the class average will be declared the winner of the game! If there are multiple people with the same closest bid then they are all declared winners of the game.

Activity: Play the game!

1.2.1 Discussion

There are a few useful concepts to discuss here. We will generalise and formalise some of these ideas later in the course, but for now let's identify them and appreciate how they might influence dictating the play of the game.

Let's think about choosing a number in the range from 51 to 100. We can notice that this is not going to be a good choice (it's a bad choice actually, in fact a **strictly dominated strategy** - we will define what exactly a strategy is and what we mean by these terms soon), why? Well let's think, even if **everyone** in the class bid 100, then one half of the average would be $1/2 \times 100 = 50$, so 50 is definitely going to be closer to one half of the class average than 51, or 52, or 53, or anything up to and including 100, **regardless** of what everyone else is doing. So we can conclude that choosing any number from 51 to 100 is certainly not optimal, since choosing 50 would give us a bid strictly closer to one half of the average, a better choice.

Now that we know this, what can we say about the game and the players. Well it's not just us that this knowledge applies to, we are not playing against beings that can't think for themselves, it applies to **every** player in the game. So no **rational** player (or no intelligent player that wants to try and win) will have chosen a number between 51 and 100. This means that we can, in essence, re-state the game. It's the same game as before, except now we know that the players are going to choose numbers between 0 and 50 (we can 'delete' or 'remove' the numbers 51 to 100 from the game as no rational player will choose them). What does this mean about a play of the game?

Well, let's take our logic from earlier one step further, we can conclude that all the numbers from 26 to 50 are now bad choices, since even if everyone bidded 50, one half of the average would then be 25. So now no rational player will choose any number from 26 to 50. Can you see what is going to happen?

Indeed if we continue the above argument, after each iteration removing all the 'bad' choices, we will eventually reach the situation where **everyone** is choosing the number 0. There is then no better any player can do, since the average is now at 0 (if everyone is bidding 0), so bidding any other number will cause the player to be further away from one half of the average than if they were to bid 0. Each player is now **responding best** to what all the other players are doing. This situation is an important concept in game theory, known as an equilibrium, or a Nash equilibrium (named after John Nash, one of the most famous mathematicians to progress the theory). We will talk about these equilibria extensively throughout the course - in fact, so powerful is the concept of an equilibrium in a game, that one can even think of classical game theory as being the search for equilibria in games.

1.3 The Sweets Dilemma

Let's play another game. This game is played between two players and each player has the choice to play strategy 1 or strategy 2. To involve everyone in the class we are going to do the following: on your sheet of paper write your name and whether you want to play strategy 1 or strategy 2, we will then collect in all the papers and randomly assign you against each other in pairs to see what the outcomes are. If you want to discuss your ideas with those around you before you make your choice feel free to do so! Before that I need to tell you what the outcomes of the game are under each possibility.

Each player has two options available to them, strategy 1 or strategy 2. If both players choose strategy 1, then they each receive nothing. If both players choose strategy 2, then they each receive 1 sweet, but if one player chooses strategy 1 and the other player chooses strategy 2 then the player who chose strategy 1 receives their sweet and their opponent's sweet (a total of 2 sweets!) and the player who chose strategy 2 receives nothing.

Activity: Play the game!

1.3.1 Discussion

Let's discuss the game a little bit as there are a few useful things to note here. First, to try and help visualise the game, let's put the information into a table, shown in figure 1. Let's call our first player A and our second player B and denote the strategies as s_1 and s_2 (strategy 1 and strategy 2).

		B	
		s_1	s_2
A	s_1	0, 0	2, 0
	s_2	0, 2	1, 1

Figure 1: The Sweets Dilemma

We will see another example with more detail in section 1.7, but this is a common way to display certain two-player games. The rows correspond to each strategy of player A and the columns to each strategy of player B. In each cell a pair of entries are given: the first the amount of sweets player A receives if that pair of strategies is chosen, the second entry the amount of sweets player B receives. These values are widely referred to as the **payoffs** to the players in the game.

Now that the information is more clearly summarised we might notice something interesting - that playing strategy 2 feels 'a bit weaker' in some way. Indeed, notice that if our opponent plays strategy 2, then we will get more sweets (we'll get 2) by playing strategy 1 than if we played strategy 2 (where we'd get only 1 sweet), and if they play strategy 1, then we will get nothing regardless of what we choose. So it feels like strategy 1 'does better' for us than strategy 2 (indeed, as we will define more rigorously later, strategy 1 **weakly dominates** strategy 2). By this logic it feels like, since strategy 1 performs either the same or better than strategy 2, we should play strategy 1. But if both players follow this line of thought then they end up with no sweets at all! However, this **is** an (and in due course you will see how to show this is **the only**) equilibrium of this game played in this manner.

There is more to say here however. It is likely upon playing the game in class that actually some players did choose strategy 2, and moreover some of those players were paired up together and received a sweet each. Why? Strategy 1 seemed at least as good as 2 if not better, so why would some players still choose strategy 2? Analysing the game in the way we have in figure 1 is making a judgement on our players - that what they care about is getting as many sweets as possible for themselves, hence their payoffs are exactly the number of sweets received. For some of you this might be a really bad assumption on which to model your payoffs! Some of you might not like sweets at all, so might see even 2 sweets as worthless. Some players may value the thought of sharing and negatively value the thought of backstabbing their classmates, making them uneasy about the choice of strategy 1, resulting in a tendency to go for strategy 2. All of these ideas would end up changing the payoffs for a player, so it is very important to have a clear concept of what each player's payoff is in a game to be able to analyse it effectively!

Remark: In this course we will not look at the concept of a player's **utility**, but those interested in how payoffs can take account for these personal 'values' of a player might be interested in reading more about utility theory.

A final point to mention here is that of communication. In the game communication was allowed beforehand with those nearby; people you would unlikely be assigned against. Even if you could communicate with the person you were going to be playing against, would it help? Notice that even if a player says they will play strategy 2, they are still incentivised (if they care about maximising their own amount of sweets) to lie about this and switch to strategy 1 as it will give them the chance of getting more sweets than strategy 2 could. So under the premise that this game is being played **non-cooperatively**, communication doesn't help here and players are incentivised to try and convince each other to play strategy 2 so that they can profit from playing strategy 1. Ultimately this will result in both players playing strategy 1 most of the time!

However, if we could change the rules slightly, and allow the players to play the game **cooperatively** with an **enforced** or binding agreement on their strategy choices, i.e they had to agree on a pair of strategies and then were obliged to see that agreement through, then they could collectively do better, for example, by agreeing on playing strategy 2 and each receiving 1 sweet. This is much better than both players trying to backstab one another and ending up with nothing. We will see more of this idea later in chapter 5 when we discuss co-operative game theory.

1.4 Principles, Elements and Classification of Games

As we have highlighted through our discussions of the two games we've played so far, in the real world people play games with a variety of principles behind their actions and at times without thought or rationality. Although this is a very interesting philosophical and ethical problem to discuss, to build up an understanding of the core mathematical principles of games, we will make some assumptions based on how we expect our players to act which will pertain throughout the analysis we do in the course!

1.4.1 Principles of Game Theory

Some principles of mathematical game theory that we will assume throughout the course are:

- Values: In mathematical game theory we always analyse games as if every one of the players is trying to win or optimise their own score/winnings. That is, we assume the players are **rational** and **greedy**. The players don't put any value on the 'fun' of playing the game or on any exterior societal implications or morals. The players don't act in irrational ways for any reason. In any games we model we will assume that a player's payoff will already factor in their values. For example, in the case where a player would gain one sweet but their opponent nothing, a player who cares about others receiving sweets too will have a lower payoff than a player who cares only about getting as many sweets as possible for themselves. This moral value is then factored into the payoffs for the game.

- Rules: Games in this course and those which can be studied by game theory have strict rules. In the real world, sometimes thinking 'outside the box' to discover a 'new move' or option in a situation is a great skill, but in this course the games will always have specified moves which are available to the players - the skill comes in determining which are the **best** moves!
- No Cheating: Linked to the last point, in the real world cheating can happen: you can swap a card with one kept up your sleeve, you can change the state of a game while your opponent is distracted, etc. To study games analytically we will assume no cheating takes place.

1.4.2 Elements of a Game

Some games we encounter will involve different elements, possibly there are random moves incorporated or the physical location of the game matters. There are three core elements within all games we will study that dictate our analysis. These are:

- Players: the number of players in the game needs to be given.
- Moves: the list of all possible moves at each point in the game for all players in the game and the order in which these moves will take place (many games are asymmetric and different players will have different moves available to them).
- Payoffs: The list of all payoffs (outcomes at the end of the game) for all players needs to be given.

This information will usually be given at the start of a game (in everything we will look at this will be the case), though there are some interesting examples of games where this is not the case.

1.4.3 Classification of Games

There are many ways in which we can classify games, often related to the manner in which they are played. Some of the main ways are shown here:

- Number of players: games can be played with different numbers of players, even single player where the player is playing against 'nature', 'the casino' or some set 'algorithm' - things that don't have conscious choice but follow an algorithm or have chance elements.
- Whether moves are simultaneous or sequential.
- Complete or incomplete information: a game is said to have complete information if all players know the structure of the game: the amount, and order in which the players move, all possible moves in each position and the payoffs for all possible outcomes for all players (Chess and Tic-Tac-Toe for example). If any of these are not known the game has incomplete information (Bridge and many other card games for example).
- Zero-sum (or constant-sum) games: these games have the property that the sum of all the payoffs to all the players for each possible outcome of the game equals zero (or a constant).
- Communication: are the players allowed to communicate before/during the game or not at all?
- Cooperative or Non-cooperative: some games may be played cooperatively where the players try to work together to maximise their payoffs. Other games are non-cooperative where the players are playing against each other.

1.5 Strategies

Now is a good time to introduce an important definition, what we mean by the **strategy** of a player in a game. This is different from the concept of a **move** in a game and also different to an **outcome** of a game, which we will define fairly loosely here (as their meaning can become more specific depending on the type of game being played).

Definition 1.1. A **move** refers to the action a player must make on their turn to progress from one game position to the next position.

Definition 1.2. An **outcome** of a game refers to the final result (which may be win/lose/draw or a combination of payoffs for all players) of a game once the game has been played.

Definition 1.3. A **strategy** for a player involves a complete description of all the moves that will be made in any game position, including responses to any random moves (tossing coins, shuffling cards), and the opponents' moves. In other words, a strategy is a program which can be followed to play the game mechanically.

There is one last definition we make now of a **pure strategy**. These will be the type of strategies considered throughout this and the next chapter. Later, in chapter 3, we will generalise these pure strategies to include **mixed strategies** (which we will define formally later but give an example now to clarify the difference between these and pure strategies).

Definition 1.4. A **pure strategy** is a strategy that doesn't involve any self-imposed random chances of playing any moves.

This definition is based on negation, so here's a quick example to try to clarify it slightly: a fair coin is tossed and you have to guess whether it will land as heads or tails. Your pure strategies are to choose either heads or tails with absolute intent. A mixed strategy would constitute actively deciding upon a probability that you will choose heads. For example choosing heads with probability $1/2$ (and hence tails with probability $1/2$) would constitute a mixed strategy - for now don't worry about these!

Let's look at some examples to understand strategies better:

1.5.1 Examples

1). Two companies, A and B, each produce and sell a product. Each wants to increase their market share. Company A intends to do **some** of either:

- Spend 10% of profits on advertising,
- Reduce the price of the product, or
- Give a free gift on purchase of the product.

Company B intends to do **one** of either:

- Spend 5% of profits on advertising, or
- Reduce the price of the product.

The companies act **simultaneously** in this game. How many **pure strategies** does each company have?

Answer). Company A has $2^3 - 1 = 7$ pure strategies. Company B has 2 pure strategies.

Exercise: If company B acts **after** company A (with full knowledge of company A's choices), then how many pure strategies does company B have?

2). A game consists of a fair coin being tossed showing either H (heads) or T (tails). Player A must **then** make a move from a choice of n possible moves $\{m_1, m_2, \cdots m_n\}$. How many pure strategies does player A have in this game?

Answer). A pure strategy in this game needs to tell player A what to do in either outcome of the coin toss, so it is a map:

$$\{H, T\} \longmapsto \{m_1, m_2, \cdots m_n\}.$$

For example, one possible pure strategy is: If the coin shows H, play m_1, if the coin shows T, play m_2. So, in total, player A has n^2 different possible pure strategies for the game, one corresponding to each mapping.

Notice this is very different to the set of possible **moves** for player A, where there are n, and also different to the set of possible **outcomes**, where there are $2n$ (each possible coin outcome and move pairing).

In the two examples above, the players all had a finite number of pure strategies to choose from, but this does not have to be the case - a player may have an infinite number of pure strategies, for example; a player must choose a real number from the interval $[0, 1]$. This motivates the following definition that we will use to describe games.

Definition 1.5. We call a game **finite** if **all** players in the game have a finite number of pure strategies. If at least one player has an infinite number of pure strategies then the game is an **infinite** game.

1.6 Goofspiel (The Game of Pure Strategies)

Let's look at another game; **Goofspiel**, or sometimes called the **game of pure strategies**. This is a two-player game (let's call the players A and B). At the start of the game a positive integer n is decided, which is traditionally 13 (as the game is commonly played with a deck of cards and there are 13 cards of each suit). Each player is then given n cards numbered from 1 through to n (so in a deck of cards each player takes all the cards of one suit and then the ace is considered to take value 1 and the Jack, Queen and King are considered to take values 11, 12 and 13). Let's say player A has red cards and player B has blue cards, just to help distinguish them. Then another set of n cards (again numbered 1 through n), which are coloured black, are shuffled and dealt face up into the centre, one at a time. After each black card is dealt, players A and B simultaneously bid for the card by playing one of their coloured cards into the centre. The player who places the larger bid (the higher valued card) scores a number of points equal to the value that was on the black card dealt into the centre. If the two bids are equal (for instance both players play their 1 card), then the number of points awarded to each player is equal to half the value of the black card (or, in some variants, both players score zero points here). This comprises one round of the game, after which all three cards involved are removed from the game. The game then continues with the next black card being dealt into the centre and the players bidding on its points value using their remaining coloured cards. Once all cards have been used (after n rounds) the game concludes and the player with the higher points total is declared the winner. If the players have an equal score then the game is drawn.

Activity: Play a game of 5 card ($n = 5$) Goofspiel. Discuss what seem to be 'good' strategies in this game.

Now we have seen the game of Goofspiel in action, let's use it as another example of calculating the number of pure strategies available to us. First let's consider 2-card Goofspiel and ask the following question... how many pure strategies does each player have in this game?

Well, a strategy for a player tells the player how to respond to the first dealt card. After that the rest of the game is fully determined (as each player and the face-down deck have only 1 card left in them each). So, a pure strategy is a map:

$$\begin{aligned} \{\text{dealt card}\} &\longmapsto \{\text{players first bid}\}, \\ \text{i.e. } \{1, 2\} &\longmapsto \{{\color{red}1}, {\color{red}2}\}. \end{aligned}$$

Thus there are $2^2 = 4$ different maps and hence there are 4 different pure strategies.

Remark: You will investigate more on the game of Goofspiel in problem sheet 1.

1.7 Displaying Games

Throughout this module we will encounter a wide variety of different games. Often there is a most convenient way to display them. There are two common ways to display games and we will see both throughout the course: **normal** (or **strategic**) form, or in **extensive** (or **tree**) form.

In normal form two-player games are represented in a table or matrix (for three or more players extra dimensions would be needed). A schematic is shown in figure 2 and we will look at an example game soon to explain the notation.

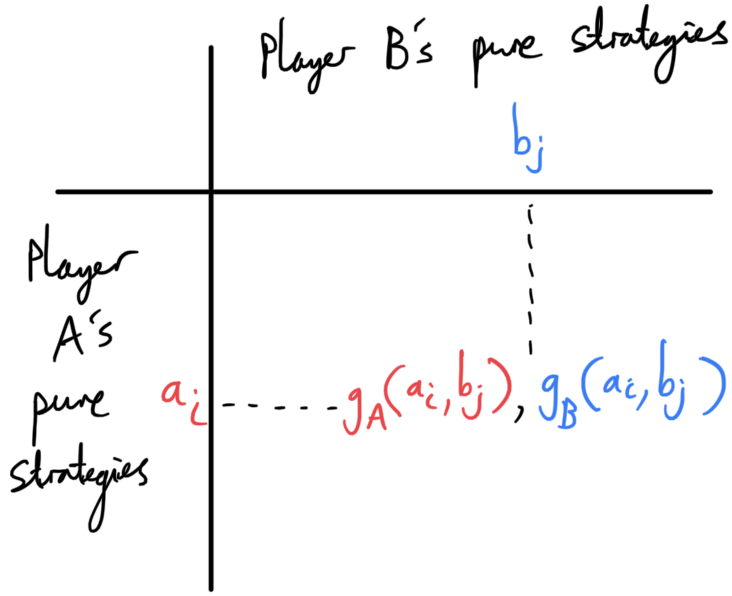

Figure 2: Schematic of a game in normal form.

In extensive form games are described using a tree, as shown in the schematic in figure 3.

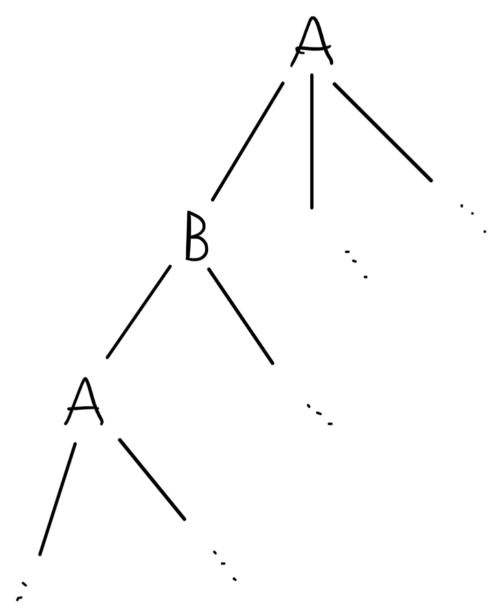

Figure 3: Schematic of a game in extensive form.

In this form strategies of the players form paths through the tree. The payoffs to each player are usually given at the bottom of each branch through the tree.

The two different approaches are equivalent, though it is often more convenient to use one approach over the other based on the game being studied. Let's look at an example to better see how a game might be displayed in these ways.

1.7.1 Example Game

Suppose players A and B are playing a game. Player A has two possible pure strategies (a_1 and a_2) and so does player B (b_1 and b_2).

- If player A chooses a_1 and player B chooses b_1, player A receives a score of 1 and player B gets 2;
- If player A chooses a_1 and player B chooses b_2, player A receives a score of 2 and player B gets nothing;
- If player A chooses a_2 and player B chooses b_1, player A receives a score of 5 and player B gets 1;
- Finally, if player A chooses a_2 and player B chooses b_2 then player A receives a score of 0 and player B gets 3.

In normal form we would display this game as shown in figure 4.

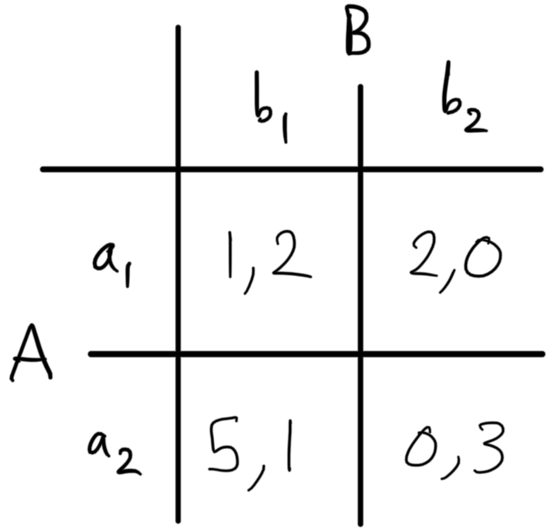

Figure 4: The example game displayed in normal form.

Player A is placed on the left and their pure strategies are represented by the different rows. Player B is placed at the top and their pure strategies are represented by the different columns. Inside the grid, each cell contains a pair of payoffs corresponding to the two strategies played; the first payoff corresponding to player A and the second to player B. If, for example, player A chooses a_2 and player B chooses b_2 the grid shows $0, 3$ representing that player A receives 0 and player B receives 3. This grid of payoffs is often referred to as the **payoff matrix** for the game.

In extensive form the game could be displayed as shown in figure 5.

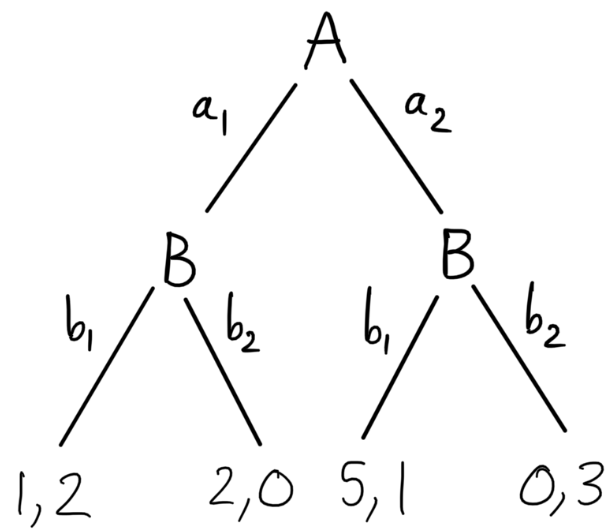

Figure 5: The example game displayed in a possible extensive form.

This tree diagram has been drawn in a way where it looks as if moves are sequential. This could indeed be the case but need not be - a tree diagram can still be drawn with pairs of simultaneous moves as in this example. A pathway through the tree, always descending, results in a payoff at the end of a branch.

Exercise: In this game we could draw the tree diagram with B's moves first then A's and this would also be an equivalent way of representing the same game. Can you draw this alternative extensive form of this game?

1.8 The Solution of a Game

Before we move on to the next chapter, let's pause for a moment to briefly discuss what we are actually trying to achieve by analysing games, i.e. what constitutes a **solution** of a game?

We know for example, if asked to **solve** a linear equation, like $2x + 4 = 0$, that we have a **unique solution** to the problem given by $x = -2$. If we were tasked with solving a quadratic, like $x^2 - 3x + 2 = 0$, then we can have either two, one (repeated) or no real solutions (or two if we consider complex solutions), here we have two solutions: $x = 1$ or $x = 2$. Maybe we are given a Diophantine equation with possibly infinitely many solutions, or a system of equations. In short, we *know* what it means to *solve* an equation.

Question: But what does it mean to **solve** a game? What should a solution to a game consist of?

A solution to a game should tell the players playing the game how to play '**optimally**', i.e a solution should provide an 'optimal' strategy for **each player** in the game (notice this is for each player). With these strategies we can deduce the resulting payoff for each player as part of the solution.

Notice that I have put quotation marks around the word optimal in the above. Indeed, what exactly we mean by a strategy being optimal is a non-trivial question and is one we will work toward answering throughout the module! For now, let's just understand that to solve a game we must provide strategies for each player that each need to fulfil some condition of optimality - precisely what this condition is we will develop in the coming lectures.

Note that, as discussed above, equations for example, can have more than one solution. So what about games? Can games have more than one solution? Or perhaps the more terrifying thought - some equations have no solutions, so do some games have no solutions? Let's eagerly press forward with these questions in mind!

Chapter 2: Dominance, Best Response and Equilibria

In this chapter we will formalise many of the concepts seen throughout chapter 1. These include the notion of dominated strategies and the concept of iteratively deleting dominated strategies. We also work towards an important definition; that of an equilibrium in a game, which will be our answer as to what constitutes 'optimal' for a collection of strategies for each player to be a solution of a game.

It will be useful to develop some standard notation to use when analysing games. In this course we will employ the following notation:

- In a game involving $N \geq 1$ players, let's refer to the players as player 1, player 2, $\cdots$, player i, $\cdots$, player N, for each $i = 1, 2, \cdots, N$. Player 1 will typically be thought of as the first player and will take their moves first in any sequential games, followed by player 2 and so on.
- Player i will have a non-empty set of pure strategies labelled S_i and a typical strategy from this set will be denoted s_i. This strategy set may be finite or infinite.
- Let's denote the payoff player i receives by playing strategy s_i against strategy s_j from player j for each $j = 1, 2, \cdots, i-1, i+1, \cdots, N$ by

$$g_i(s_1, s_2, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_N).$$

Remark: Note here that throughout this course the notation g will be used to represent payoff functions. This is for two reasons: one to represent the 'gain' of a player (hence the g), and second for sentimental value as this was the notation the lecturer learned originally as an undergraduate! However, note that in many books or papers in the subject area that this notation is not widespread. In fact it is generally more common to see the notation u used (to represent a player's utility) rather than g. Be aware if you read any textbooks or articles that these, and others, are just different ways of denoting payoff functions.

Although we have said that this will be our general notation throughout the course, typically lots of what we will encounter involves just **two** players. In such cases, this more general notation is often superfluous and so we use the following:

- In a two-player setting we will refer to the two players as player A and player B. Player A will typically be thought of as the first player and will take their moves first in any sequential games, followed by player B. Player A will have pure strategies represented by a_1, a_2, $\cdots$ etc and the set of all their pure strategies we will call A_s. Likewise B has pure strategies $B_s = \{b_1, b_2, \cdots\}$. These sets may be finite or infinite.
- We refer to the payoff player A receives by playing strategy a_i against player B's strategy b_j by $g_A(a_i, b_j)$. Likewise we denote $g_B(a_i, b_j)$ for the payoff player B receives by playing strategy b_j against player A's strategy a_i.

2.1 The Prisoner's Dilemma

Let's look at another game, the famous **Prisoner's Dilemma**. Two prisoners are currently held captive and are suspected of committing a serious crime. There has been no evidence found to prove that either is guilty, however should a prisoner testify against the other then the one testified against will be found guilty. The prisoners are taken into separate rooms and interrogated simultaneously. They can either remain silent and cooperate (strategies a_c, b_c) with each other by not testifying, or they can defect (strategies a_d, b_d) and testify against the other.

If both prisoners cooperate, then each gets a smaller sentence of one year for some minor crimes that can be pinned on them. If both defect, then each get a long sentence of five years. If one prisoner defects and the other stays silent, then whoever defects gets let free and whoever stayed silent takes all the blame and receives a double sentence of ten years. Figure 6 shows the normal form of the game.

		B	
		b_c	b_d
A	a_c	$-1, -1$	$-10, 0$
	a_d	$0, -10$	$-5, -5$

Figure 6: The Prisoner's Dilemma.

Activity: Play the game against a partner. In the classroom select two students and ask them to come to the front and to play the game. First play without communication, then allow the prisoners to talk before they make their choice and play again.

2.1.1 Discussion

Both players have an incentive to defect, so that the strategy pair (a_d, b_d) will be the predicted outcome upon playing the game. This is similar to what we saw in the sweets dilemma only stronger: strategy d (defect) gives a better payoff than c (cooperate) **regardless** of what the opponent does.

Notice however that even though the payoff pair (a_c, b_c) is better for both players, play results in the pair (a_d, b_d) occurring. This often remains the case even when we allow our prisoners to communicate - both are still incentivised to 'rat the other one out' and defect to receive a smaller sentence. Later, in chapter 5, we will investigate cooperative game theory to see how we might obtain these better outcomes for both players.

This game shows that individual incentives may jeopardise a common good. This is a common theme in game theory and we will see this concept occur a few times throughout the course. Of course this has ramifications when we apply game theory to real world problems, see problem sheet 2 for examples.

2.2 Dominated Strategies

Let us now formally define what we mean when we say a particular strategy is dominated for a player. Since most of what we will do involves two players, we begin with a simpler two-player definition.

Definition 2.6. We say that a strategy $a \in A_s$ is **strictly dominated** (or 'dominated') by another strategy $a' \in A_s$ if

$$g_A(a,b) < g_A(a',b), \quad \text{for all } b \in B_s.$$

In other words, a' is always better for player A than a, **regardless** of what B plays.

Remark: Take care with the annoying English here. In the above definition a **is dominated by** a', but, equivalently, we sometimes say a' **dominates** a. These are both correct phrasings but take care to note that in the second case the a and a' switch around. Don't mix them up!

We will consider some games with more than two players, for which it is useful for us to have the more general definition of dominance

Definition 2.7. In a game with N-players ($N \geq 2$), then we say that a strategy $s_i \in S_i$ for player i is **strictly dominated** (dominated) by another strategy $s_i' \in S_i$ if

$$g_i(s_1, s_2, \cdots, s_i, \cdots, s_N) < g_i(s_1, s_2, \cdots, s_i', \cdots, s_N),$$

for all $s_j \in S_j$, $j = 1, 2, \cdots i-1, i+1, \cdots, N$.

2.3 The Defence of Rome

It's the year 218BC and the great Carthaginian general Hannibal is preparing to lead his forces against Rome. Let's put ourselves in the shoes of the Roman consul Publius Cornelius Scipio and best prepare our forces for Hannibal's invasion.

There are two entrances to Rome; through the easy pass: the Iberian peninsula by boat, or through the arduous pass: the mountainous Alps. We can choose to position our legion to guard either of these passes. Hannibal will lead his army (which we will say is made up of two battalions worth of troops) through one of these passes to Rome. If our Roman legion meets Hannibal's army, we will destroy one of his battalions, but if our army misses his army, we won't destroy any battalions. However, if Hannibal chooses the hard mountainous pass, one of his battalions will be destroyed via attrition in the process of making the gruelling journey regardless of wether or not we meet his army.

Our payoff corresponds to the number of Hannibal's battalions destroyed and Hannibal's payoff to the number of remaining battalions he has left once in Rome. In normal form the game is shown in figure 7.

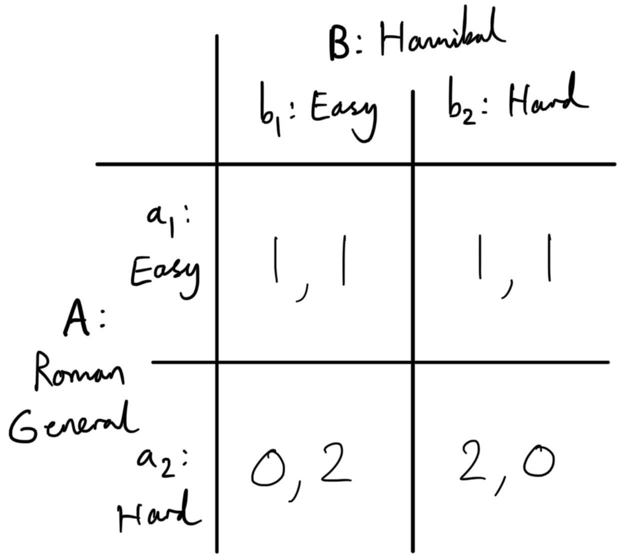

Figure 7: The Defence of Rome.

Activity: Acting as the Roman general in this game, what should you do? Play the game in class.

2.3.1 Discussion

On first glance this might be unclear as to what our best action is in this game. Taking strategy a_1 (protecting the easy pass) seems the 'safer' play (it guarantees us a payoff of 1) whereas taking strategy a_2 (protecting the hard pass) has the potential to give a higher payoff but also to give us 0, so what should we do?

The important insight comes from considering our **opponents strategies**. Take a step into the shoes of Hannibal and let's investigate his options. Hannibal's strategy b_1 (attack via the easy pass) scores at least as highly, if not better, than his alternative strategy b_2 (hard pass). Notice that this is not dominance (since b_1 only scores **at least as highly**, if not better), but something slightly weaker which we call **weak dominance**.

Based on this revelation, it seems highly plausible that Hannibal will not chose his weakly dominated strategy and thus will attack via the easy pass (b_1). Knowing this, we can make our best play (our **best response**, see shortly) and defend the easy pass, giving us a payoff of 1 rather than 0 if we defended the hard pass.

Remark: Doesn't all of this seem useless given we know historically that Hannibal did choose the hard pass? Is our game theoretic model irrelevant? Well for now note that we are still building up our intuition into how to solve games. Indeed, the pair of strategies where both players choose the easy pass (a_1, b_1) **is a solution** of this game. There are however **more** solutions which we will learn how to find later once we talk about randomised strategies and degenerate games which should ease our fears that Hannibal was not using sound tactical advice!

2.3.2 Weak Dominance

Let's formally define what we mean by weak dominance.

Definition 2.8. We say that a strategy $a \in A_s$ is **weakly dominated** by another strategy $a' \in A_s$ if

$$g_A(a,b) \leq g_A(a',b), \quad \text{for all } b \in B_s,$$

where at least one of these inequalities must be strict.

In other words, a' is never worse for A than a, and sometimes better, regardless of what B plays. For an N-player game this definition becomes

Definition 2.9. In a game with N-players ($N \geq 2$), we say that a strategy $s_i \in S_i$ for player i is **weakly dominated** by another strategy $s_i' \in S_i$ if

$$g_i(s_1, s_2, \cdots, s_i, \cdots, s_N) \leq g_i(s_1, \cdots, s_i', \cdots, s_N),$$

for all $s_j \in S_j$, $j = 1, 2, \cdots, i-1, i+1, \cdots, N$, where at least one of these inequalities must be strict.

2.3.3 Payoff Equivalence

We give one more definition in the context of N-player games. We don't expect to see this case very often, but give it for completeness. The equivalent definition would hold in our two-player notation.

Definition 2.10. In a game with N-players ($N \geq 2$), then we say that a strategy $s_i \in S_i$ for player i is **payoff equivalent** to another strategy $s_i' \in S_i$ if

$$g_i(s_1, \cdots, s_i, \cdots, s_N) = g_i(s_1, \cdots, s_i', \cdots, s_N),$$

for all $s_j \in S_j$, $j = 1, 2, \cdots, i-1, i+1, \cdots, N$.

2.4 Best Response

Let's consider the game shown in normal form in figure 8 where player A has three pure strategies.

		B	
		b_1	b_2
A	a_1	5,1	0,2
	a_2	1,3	4,1
	a_3	4,2	3,3

Figure 8: An example game to motivate the concept of best response.

Activity: Think about which strategy you would play as player A. Can you defend each strategy choice rationally?

Looking at the game in strategic form it seems there are no dominated strategies for either player, so making an informed decision here on what we should do seems harder. We can justify strategy choices a_1 and a_2 in the following way: suppose we believe that player B will play b_1, then our **best response** to this is to play a_1, since this choice gives us the largest payoff if B plays b_1. Alternatively, suppose we think it likely that B plays b_2, then we can justify choosing strategy a_2 since this is our **best response** against b_2.

But what about strategy a_3? it is neither the best response against b_1, nor against b_2, yet it seems to yield strong payoffs and is likely the choice that many of you would have made.

In the above we have made comments on our 'belief' about what player B will play. Let's put a variable to this to allow us to analyse the game in more detail. Let $q \in [0,1]$ be the probability that player B chooses strategy b_1; so then $1-q$ is the probability they choose b_2. Now we can formulate three equations representing the **expected payoffs** to player A for playing each of their pure strategies. In the next chapter we will formalise some notation when dealing with probabilities like this but for now let's refer to each of these quantities as $g_A(a_1)$, $g_A(a_2)$ and $g_A(a_3)$. Calculating these expected payoffs we find

$$
\begin{aligned}
g_A(a_1) &= 5q + 0(1-q) = 5q, \\
g_A(a_2) &= q + 4(1-q) = 4 - 3q, \\
g_A(a_3) &= 4q + 3(1-q) = q + 3.
\end{aligned}
$$

Graphing these on a pair of axes of q against g_A gives figure 9.

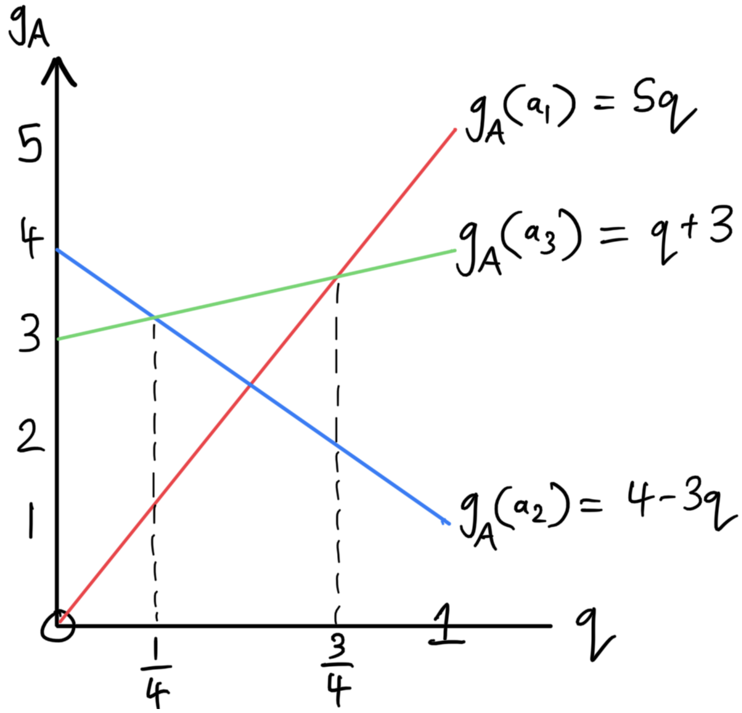

Figure 9: Expected payoffs for each of player A's pure strategies plotted against q.

This gives us a really nice visual picture of what's going on here, and allows us to better argue why a_3 is a justifiable choice, for instance:

- Since I felt the probability that B would play b_1 was less than $1/4$, then my **best response** against this is to play a_2 since it gives the highest expected payoff in this case (the blue line is the highest for $q < 1/4$).
- Since I felt the probability that B would play b_1 was more than $3/4$, then my **best response** against this is to play a_1 since it gives the highest expected payoff in this case (the red line is the highest for $q > 3/4$).
- Since I felt the probability that B would play b_1 was more than $1/4$, but less than $3/4$, then my **best response** against this is to play a_3 since it gives the highest expected payoff in this case (the green line is the highest for $1/4 < q < 3/4$).

This is the concept of best response, which we now define formally.

Definition 2.11. Consider an N-player game ($N \geq 2$). We say that strategy $s_i \in S_i$ for player i is a **best response** to the strategies $s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N$ for each of the other players if

$$g_i(s_1, s_2, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_N) \geq g_i(s_1, s_2, \cdots, s_{i-1}, s_i', s_{i+1}, \cdots, s_N), \tag{1}$$

for all $s_i' \in S_i$. That is, against the strategy choices s_j, $j = 1, 2, \cdots i-1, i+1, \cdots, N$ for each of the other players, the strategy s_i for player i results in the highest payoff.

Remark: A best response is not always unique. It is possible that for some strategies of our opponent(s) we have **multiple best responses**, hence we usually refer to a best response when found in just that way: **a** best response. When we know it is unique then we may say **the** best response or something similar.

2.5 The most important game in the world

We now come to the most important game in the world: football! In this section we will analyse the game of penalty kicks (a minigame within a game of football). A penalty kick involves the attacker, a player from team A who will run up to the ball and try to kick it into the net from 12 yards away to score a goal, and a defender, a player from team B, who in this case is called a goalkeeper, who will try to stop the ball from going into the goal by diving to block the ball.

We will model the game in a simplified manner. Player A is our attacker who will shoot to the left, the right or down the middle of the goal. Player B will be our goalkeeper who can dive to the left or to the right (you'll explore the option of staying in the middle later on problem sheet 4). We will model the game using the payoff matrix in figure 10 where the payoffs are related to the probability of a goal being scored.

		B	
		b_1 Left	b_2 Right
	a_1 Left	$4, -4$	$9, -9$
A	a_2 Middle	$6, -6$	$6, -6$
	a_3 Right	$9, -9$	$4, -4$

Figure 10: Normal form of the penalty kick game.

Activity: Acting as the attacker, player A, which way would you shoot and why?

As before, let's sketch up a graph to highlight our possible best responses. Let $q \in [0, 1]$ be the probability that the goalkeeper dives to the left (plays b_1). Then the probability they choose b_2 is just $1 - q$ and our

expected payoffs for each pure strategy are

$$g_A(a_1) = 4q + 9(1-q) = 9 - 5q,$$
$$g_A(a_2) = 6q + 6(1-q) = 6,$$
$$g_A(a_3) = 9q + 4(1-q) = 4 + 5q.$$

Graphing these gives figure 11.

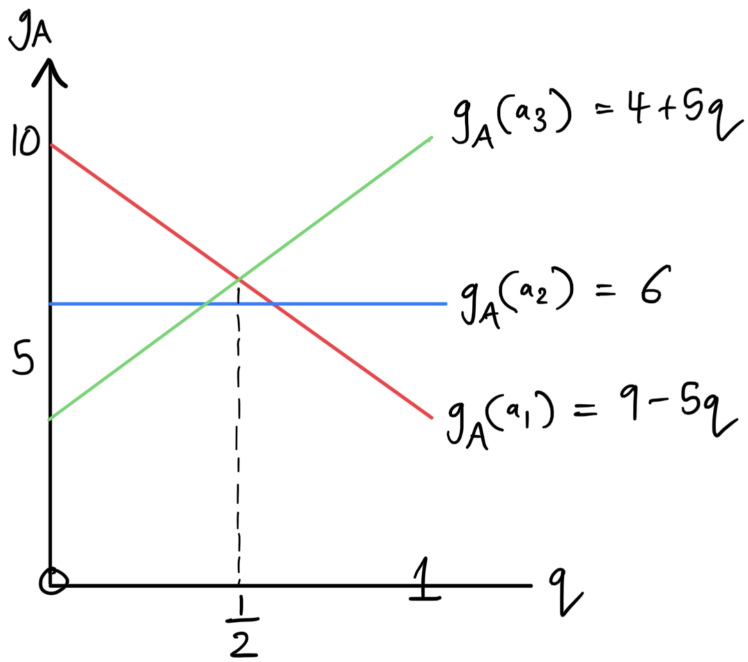

Figure 11: Expected payoffs in the penalty kick game.

Looking at the graph we can deduce some things. First something that might have seemed completely obvious, but it's good to know our analysis agrees: that if we think B will play b_1 with probability greater than 1/2, our **best response** is to play a_3, and if we think B will play b_1 with probability less than 1/2, our **best response** is to play a_1. In words, this means our best response is to shoot in the direction opposite to the direction we think the goalkeeper will dive.

Our graph tells us something else interesting though. Notice that the blue line representing our expected payoff from a_2, shoot down the middle, is **never** the highest line, for any value of q. This means that this strategy is **never** a best response to any of B's strategies. In other words, there is no reason to ever play a_2, since we could always be doing better by playing something else.

In the context of penalty kicks this gives us a valuable lesson: never shoot down the middle! So if we could just get that into the heads of the English players, then perhaps we'll beat Germany at penalty kicks one of these days...

In fact there is a strong connection between dominated strategies and those that are **never** best responses:

Proposition 2.12. *A strategy that is* ***never*** *a best response to any of the opponents strategies is a dominated strategy.*

Proof. (◇) To prove this we need to be familiar with the concept of mixed strategies (see chapter 3) as those are important here. This result can then be conceptually understood fairly easily but to prove it formally is a bit algebraically clunky. It is left as a diamond exercise if you'd like to try! □

Remark: The keen eyed reader among you may have felt uneasy about this latest proposition. Indeed, we have just argued that shooting down the middle (strategy a_2) was never a best response to any of the opponent's strategies. This latest proposition then claims that it must be a dominated strategy. However, we can spend a moment to check whether any of the pure strategies dominate one another and see that this is not the case. So what's going on here? Well it turns out that some **mixed strategies** are the strategies that dominate a_2, so that's why we'll return to this once we've seen what they are (once we've seen mixed strategies in chapter 3, come back here and consider the strategy where you shoot left or right with probability 1/2 each way for instance)!

2.6 Equilibria

We now come to the most central theme of mathematical game theory, equilibria. In fact, one might go so far to say game theory is really the study of equilibria such is their importance. Let's define what we mean by this.

Definition 2.13. An **equilibrium** of an N-player game ($N \geq 2$) is a profile of strategies, one for each player, where each strategy in the profile is a **best response** to all the other strategies in the profile. In other words the best response condition (1) holds for **all** players $i = 1, 2, \cdots, N$.

Remarks:

- An equilibrium is often called a **Nash equilibrium** named after John Nash (1951) who proved that an equilibrium **always** exists for finite strategy sets extended to include mixed strategies (we will see what all this means in Chapter 3).

- One might ask why an equilibrium is such an important concept. Indeed, we can actually go so far to say it is a **necessary** condition for a game theoretic solution. When we analyse a game to look for a solution, then what should this solution entail? Well we should determine a strategy for each player to follow which maximises their preferences, taking into account that all other players will do the same. This collection of strategies (or strategy profile) necessarily must be in equilibrium, because otherwise we would be recommending something inferior, self-defeating, to at least one of the players. If there is at least one strategy that is not a best response to the others then that player is incentivised to deviate from the strategy provided and use some other strategy which is a best response against the other strategies. This is why the concept of equilibrium is so fundamental to the study of games and a necessary condition for a game theoretic solution.

- This importance does not guarantee that such an equilibrium has to exist, however, nor does it mean that players playing the game will always necessarily play such an equilibrium. For instance the game theoretic model may be a simplification of the real world context or the game may simply be very difficult to play optimally - consider the game of chess for example.

Much of the rest of this course is about seeking equilibria in games, so let's get started with an example.

Example: Consider the game shown in figure 12 between players A and B. Find **a solution** (a pair of strategies, one for each player, that are in equilibrium) to the game.

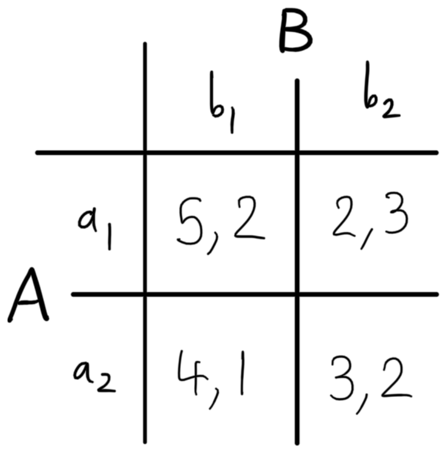

Figure 12: An example game in normal form.

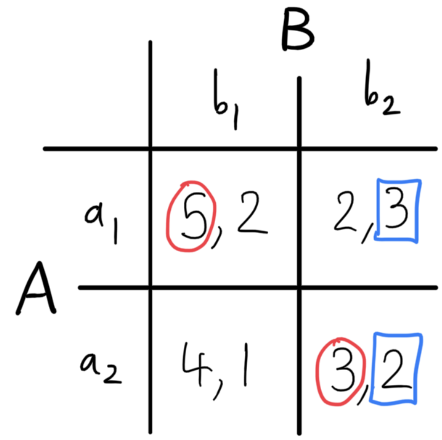

Figure 13: Best response payoffs circled and boxed for each player.

Solution: We can check each cell in the table to determine whether that pair of strategies are mutual best responses. Alternatively some people find it useful to circle the payoffs corresponding to player A's best responses against each of player B's strategies (here we circle 5 and 3 for A as a_1 is the best response against b_1 and a_2 the best response against b_2), and to box the payoffs corresponding to player B's best responses against each of player A's pure strategies (here this is b_2 in both cases - which means that b_1 is never a best response so should be a dominated strategy, indeed one can check that is the case) as shown in figure 13. Where a circle and box meet in a cell we have found a pair of strategies which are mutual best responses and hence in equilibrium. Thus (a_2, b_2) is an equilibrium and hence a solution to the game. Shortly you will be able to prove that this is the **unique** equilibrium of this game.

2.7 Games with multiple equilibria

Let's look at some more examples of games and try to find their equilibria. In the following examples we will see there are more than one equilibrium.

2.7.1 The Dating Game (The Battle of the Sexes)

Classically, the game known as the battle of the sexes is shown in figure 14. I prefer to refer to this game as the dating game since its gender stereotypes are outdated though shouldn't be taken too seriously anyway. In this game players A and B are going on a date and each decides simultaneously and independently (unrealistic nowadays with mobile phones and the ease of communication) whether to go to a fancy Shoreditch bar (a_B, b_B), afternoon tea at a posh hotel (a_T, b_T) or to the JCR (a_J, b_J).

			B	
		b_B	b_T	b_J
	a_B	3,2	1,1	1,-1
A	a_T	0,0	2,3	0,-1
	a_J	-1,0	-1,1	0,0

Figure 14: The Dating Game.

Activity: Play the game. In class select some volunteers to play the game. Then, let the volunteers have a chat beforehand and play the game again.

2.7.2 Discussion

The game has two pure strategy equilibria: (a_B, b_B) and (a_T, b_T) where both players go to the bar, or both go to tea. Their individual preferences differ however making agreement on where to go non-simple. The game is often referred to as a **coordination** game, as both players want to coordinate where they go - as a result communication before the game often helps the players find a way to coordinate. There is one thing we can all agree on - the JCR might not be the greatest place to go for a date!

2.7.3 The Class Investment Game

Let's play another game. Each member of the class is a player in this game and the idea is simple, you can either play strategy 1: invest in the class project, which costs you one sweet, or strategy 2: don't invest, which costs you nothing. If 80% or more of the class choose strategy 1 and invest, then everyone who invested receives two sweets back, netting you a sweet in profit. Otherwise nothing comes of investing and those players who invested lose their sweet.

Activity: Play the game in class with everybody raising their hand to invest or not. Everyone will close their eyes during this vote so they don't know what everyone else has done.

2.7.4 Discussion

Can we determine what are the equilibria in this game? Yes, the game is simple enough we might be able to spot them. Equilibria occur when **everyone is investing**, or where **no one is investing**. In either one of these situations there is no incentive for **any** player to change their strategy, whereas in all other situations there is.

Activity: Elect a class spokesperson. Allow them 30 seconds to give a speech to the class to 'rally the troops' and try to convince everyone to invest. Then play the game again to try to reach the 'good' equilibrium.

Note: In the past I used to play this game with 90% as the requirement and **never** managed to reach it, so I've lowered it to 80% in the hope to convince more of you to invest and win some sweets! This actually means you are in essence playing an iterated game with future years of students. If you, as the current group of students taking the course, refuse to invest at 80% then that compels me to lower this threshold even further, making it easier for future students to reach - but this comes at a cost to you as the current group as you risk losing out on sweets!

2.8 Iterative deletion of Dominated Strategies

We now come to formalising a technique we saw early on in the course. Recall when we played the everyone bids game we reached a conclusion that everyone playing '0' was an **equilibrium**. To reach this conclusion we successively removed dominated strategies from the game and then removed strategies that had become dominated when the dominated strategies were removed and so on. Let us now prove that this is justified.

We start by proving that it is valid to remove a dominated strategy from the game and not change the solutions to the game.

Proposition 2.14. *In an N-player game ($N \geq 2$), G, with strategy sets $S_1, S_2, \cdots, S_N$, let s_i and s_i' be two strategies for player i. Suppose s_i' weakly dominates or is payoff equivalent to s_i. Consider then the game G' with the identical payoffs as G but where S_i is replaced by $S_i - \{s_i\}$. Then:*

(a) Any equilibrium of G' is an equilibrium of G.

*(b) If s_i' dominates s_i, then G and G' have the **same** equilibria.*

Proof.

(a) Take an equilibrium of G'. This is a possible set of strategies in the game G and is either an equilibrium of G or, if not, then player i must benefit by switching to the extra strategy option s_i. However, if deviating to strategy s_i in G was beneficial to player i, then player i could also deviate to s_i' instead, since it weakly dominates/is payoff equivalent to s_i. But s_i' is a possible strategy in G', so this violates the fact that we initially had an equilibrium in G', since we have claimed player i can improve their payoff by switching to s_i'. Hence this deviation cannot be beneficial to player i and so this equilibrium of G' is also an equilibrium in G.

(b) Suppose s_i' dominates s_i. Then the dominated strategy s_i is **never** part of an equilibrium, since player i could always deviate to s_i' to do better. Now any equilibrium in G therefore doesn't contain s_i, so it is also a possible set of strategies in game G' where it is also an equilibrium. Now by (a), vice-versa, any equilibrium in G' is also an equilibrium in G, hence G and G' have the same equilibria.

□

This proposition means we can freely delete dominated strategies from games without changing the equilibria of the game. However, we must be cautious when weakly dominated strategies are present. Removing these will never introduce new equilibria, by (a) in the last proposition, but may result in losing equilibria from a game as seen in the following example.

Example: Consider the game shown in figure 15.

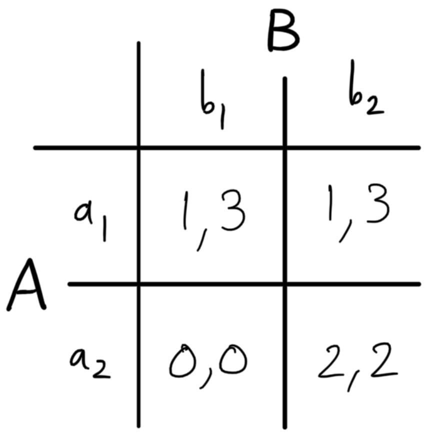

Figure 15: An example where we would lose an equilibrium by deleting a weakly dominated strategy.

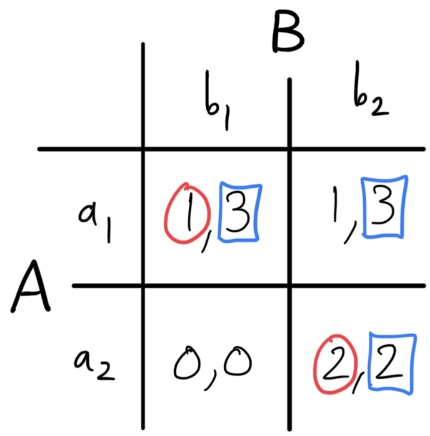

Figure 16: Best response payoffs circled and boxed for each player.

Looking for pure strategy equilibria via whichever is your favourite method shows that we have two equilibria: (a_1, b_1) and (a_2, b_2). However, the equilibria using strategy b_1 actually uses a **weakly dominated** strategy of player B (notice b_2 weakly dominates b_1). This is an example of why we should **not** delete weakly dominated strategies (here b_1) when we are seeking to find **all** equilibria in a game. However, if we want to find just **a** solution of a game, we may do this.

Let's now look at the iterated deletion of dominated strategies.

Proposition 2.15. *Consider a game G that upon performing a finite iterated deletion of dominated strategies is reduced to a single strategy profile, s (i.e. every player has only one strategy left). Then the strategy profile s is the* ***unique equilibrium*** *of G.*

Proof. After n deletions of dominated strategies, let's call the final reduced game $G^{(n)}$. Trivially this game has s as an equilibrium because no player can deviate. By proposition 2.14, the previous game, $G^{(n-1)}$, before the last dominated strategy was removed has the same **unique** equilibrium. Then, from $G^{(n-1)}$ we go backwards again, applying proposition 2.14 to claim $G^{(n-2)}$ has the same unique equilibrium. Continuing this process backward this holds true for the original game G. □

Remarks:

- **Example:** The everybody bids game falls into this category.
- If this process results in finding a unique equilibrium from a single remaining strategy profile, we often refer to the game as being **dominance solvable**.
- The order in which dominated strategies are removed **does not** matter, and several dominated strategies may be removed simultaneously (problem set 2 asks you to prove these claims).
- Based on the algorithmic form of the last proof, we may also reduce an initial game G to a state, say $G^{(3)}$, upon removal of three iterations of dominated strategies, which, although may not yet be reduced to a single strategy profile (be dominance solvable), is **simpler** and easier to solve than the original game G. For example, in the dating game (section 2.7.1); we may delete strategy J (JCR) entirely since this was a dominated strategy for both players. This then results in a 2×2 game which is easier to solve than the original 3×3 game. This means we can use this iterated deletion process partially and the resulting game will still have the same equilibria by proposition 2.14.

Let's now solve a famous game by use of this iterated deletion of dominated strategies technique.

2.9 The Cournot Duopoly of Quantity Competition

In this section we discuss a famous economic model of competition between two competing firms who will choose the **quantity** of their product they will produce, where the price decreases with the total quantity on the market. This model is called a **Cournot duopoly** named after Augustin Cournot (1838) who proposed it.

The players in the game, A and B, are two firms who choose a non-negative quantity of some product to produce, up to an upper bound M, say (beyond which their profits would become negative), so that the strategy sets of the players are $A_S = B_S = [0, M]$.

Let x and y be the strategies chosen by the players respectively. For simplicity, let's assume there are no costs of production of the product and that the total quantity produced $x + y$ is to be sold at a price $12 - (x + y)$ per unit, which is then each firm's profit per unit. Thus the payoffs to the players, which we will take as their profits, are

$$g_A(x, y) = x(12 - x - y),$$
$$g_B(x, y) = y(12 - x - y).$$

2.9.1 A finite version of the game

First let's simplify the game a little and consider just a finite version, where the players strategy sets are

$$A_S = B_S = \{0, 3, 4, 6\}.$$

This gives us the normal form of the game as shown in figure 17.

B

A	b_0	b_3	b_4	b_6
a_0	0,0	0,27	0,32	0,36
a_3	27,0	18,18	15,20	9,18
a_4	32,0	20,15	16,16	8,12
a_6	36,0	18,9	12,8	0,0

Figure 17: A finite version of Cournot duopoly.

This is a game which we can apply the iterative deletion of dominated strategies to find a unique equilibrium. First notice that strategy a_0 is dominated by both strategies a_3 and a_4, and, since the game is **symmetric** (B's strategies and payoffs are the same as for A in all cases), b_0 is dominated by b_3 and b_4. There are no other dominated strategies. This means we can safely delete strategies a_0 and b_0 from the game and our new game, see figure 18, will have the same equilibria as our original game.

B

A	b_3	b_4	b_6
a_3	18,18	15,20	9,18
a_4	20,15	16,16	8,12
a_6	18,9	12,8	0,0

Figure 18: The finite game with a_0 and b_0 deleted.

Next notice, upon deletion of a_0 and b_0, a_6 is **now** dominated by a_4 (and similarly b_6 by b_4), so we can delete these strategies resulting in the game shown in figure 19.

		B	
		b_3	b_4
A	a_3	18,18	15,20
	a_4	20,15	16,16

Figure 19: The finite game with a_0, a_6, b_0 and b_6 deleted.

Finally, now that a_6 and b_6 have been deleted, we can see that a_4 dominates a_3 (and b_4 dominates b_3), leaving (a_4, b_4) as the **unique** equilibrium of this dominance solvable game, see figure 20.

		B
		b_4
A	a_4	16,16

Figure 20: The finite game with only strategies a_4 and b_4 remaining.

Remark: Notice that, had both players **agreed** to make only 3 units of product (play a_3 and b_3), they would both have been better off (received payoffs of 18 rather than 16), but, like in the prisoners dilemma, this is not an equilibrium since in this case both firms would have incentive to deviate and make 4 units to maximise their profits. This has lots of real world economic context: in fact if the two firms agreed to partake in this deal then they would be acting as a cartel: a collusion of market users who work together to dominate the market and drive up prices to maximise profits!

2.9.2 The infinite version of the game

Let's return to the infinite version of the game and show that we can find the equilibrium of the game when we allow both players to choose any strategy from the set $[0, M]$. First observe that we can take $M = 12$, as if $M > 12$, then the price sold at $12 - x - y < 0$, so choosing our quantity to produce at 0 would give us zero profits and be better than making a loss by producing greater than 12 units.

Let's take a look now from player A's perspective. Player A's **best response** to strategy y from player B is to produce the quantity x which **maximises** A's payoff $g_A(x, y)$, where

$$\begin{aligned} g_A(x, y) &= x(12 - x - y) \\ &= -\left(x - \frac{12 - y}{2}\right)^2 + \left(\frac{12 - y}{2}\right)^2, \end{aligned}$$

which has its maximum value at $x = \frac{12-y}{2} = 6 - y/2$. Alternatively one could use calculus here to determine the value of x giving $\frac{d}{dx} g_A(x, y) = 0$. In more complicated cases this may be necessary.

A's best response to strategy y by B, namely $x = 6 - y/2$, is unique, so because the game is completely symmetric, player B's best response y, to strategy x by A, is given by $y = 6 - x/2$. In order to have an equilibrium, the strategies x and y must be **mutual best responses**, that is

$$\begin{aligned} x &= 6 - \frac{y}{2}, \\ \text{and} \quad y &= 6 - \frac{x}{2}. \end{aligned}$$

Solving this pair of linear equations gives the unique solution $x = y = 4$: the unique equilibrium of the game.

2.10 Games without a pure-strategy equilibrum

Not every game has an equilibrium in pure strategies (though we will see shortly that when we allow for mixed strategies then any finite game does!). To motivate what we are going to do in the next chapter, let's look at some simple games that fall into this category.

2.10.1 Matching Pennies

In matching pennies player A and player B each simultaneously reveal a penny which can show either heads, a_H or b_H, or tails, a_T or b_T. If the pennies match, player A wins player B's penny, otherwise player B wins the pennies. Figure 21 shows the strategic form of the game.

		B	
		b_H	b_T
A	a_H	1, −1	−1, 1
	a_T	−1, 1	1, −1

Figure 21: The game of matching pennies.

Activity: Play the game against your neighbour a few times. You can play for fun or for winning each others sweets rather than pennies if you want some stakes involved!

As alluded to by the subsection heading and as you might have seen, there is **no** pure strategy equilibrium in this game! At every outcome, one player has an incentive to deviate from their chosen strategy. So the next question to ask is what **is** a solution to this game? What strategies should we recommend our players play? Perhaps you have the intuition to guess this!

2.10.2 Rock - Paper - Scissors

This game is very well known so probably requires no introduction to you. Nevertheless the rules are simple and go as follows: each of players A and B have three pure strategies available to them: rock (a_R or b_R), paper (a_P or b_P) and scissors (a_S or b_S). The idea is that rock beats scissors which beats paper which beats rock. In other cases the game is drawn. Figure 22 gives the normal form of the game.

		B		
		b_R	b_P	b_S
A	a_R	0,0	−1,1	1,−1
	a_P	1,−1	0,0	−1,1
	a_S	−1,1	1,−1	0,0

Figure 22: Rock-Paper-Scissors.

As in matching pennies, take a moment to convince yourself that no pair of pure strategies are in equilibrium here. This time can you predict what is the equilibrium of the game?

Chapter 3: Mixed Equilibria

As seen in the games at the end of the last chapter, a game does not always have an equilibrium where every player chooses their strategies deterministically. Nash (1951) proved that any finite game has an equilibrium however if the players are allowed to use **mixed** (or sometimes called **randomised**) strategies. We will define these now and motivate their use with an example game. Following from this we will prove Nash's theorem.

3.1 Mixed Strategies

Definition 3.16. A **mixed** (or **randomised**) strategy for a player is a self-imposed (i.e. chosen actively and in control by the player) probability distribution over the player's pure strategies.

In a game where player A has n pure strategies, so $A_S = \{a_1, a_2, \cdots, a_n\}$ then a mixed strategy, α, for player A is denoted as

$$\begin{aligned} &\alpha = (p_1, p_2, \cdots, p_n), \quad \text{or} \\ &\alpha = p_1 a_1 + p_2 a_2 + \cdots + p_n a_n, \end{aligned}$$

where $0 \leq p_i \leq 1$ for all i and $\sum_i p_i = 1$ so that we have a valid set of probabilities and a valid probability distribution. It represents the idea that player A will choose to play strategy a_i with probability p_i for each i.

For a player A, we now extend the usual pure strategy set A_S to the more general **mixed strategy set**, denoted by $\mathbb{A}_S$, which is the infinite set containing **all possible** mixed strategies α for player A. Notice that we refer to this new strategy set as being 'more general'. Indeed this new set contains **all** the pure strategies in the original set, as each pure strategy is a possible valid probability distribution over the players pure strategies (it just assigns probability 1 to that particular pure strategy and probability 0 to everything else). For example the pure strategy a_3 for player A is equivalent to the mixed strategy $\alpha = (0, 0, 1, 0, \cdots, 0)$.

When we include mixed strategies in games it often makes more sense to discuss the players **expected payoffs** (since the payoff the player now gets upon a play of the game is determined by probabilistic factors). Since we will focus our attention primarily on two-player games, for simplicity of notation we give a two-player definition here.

Definition 3.17. Let player A have pure strategy set $A_S = \{a_1, a_2, \cdots, a_n\}$ and player B have pure strategy set $B_S = \{b_1, b_2, \cdots, b_m\}$. Then if player A choses to play the mixed strategy $\alpha = (p_1, p_2, \cdots, p_n) \in \mathbb{A}_S$ and player B choses to play the mixed strategy $\beta = (q_1, q_2, \cdots, q_m) \in \mathbb{B}_S$ then the **expected payoff** to player A is given by

$$g_A(\alpha, \beta) = \sum_{i=1}^{n} \sum_{j=1}^{m} g_A(a_i, b_j) p_i q_j.$$

Player B has an analogous definition.

But what if A_S and B_S are infinite sets? In this case mixed strategies α and β become, in general, continuous probability distributions over the pure strategy sets. The expected payoff for player A is then given by

$$g_A(\alpha, \beta) = \int_x \int_y g_A(x, y) f_A(x) f_B(y) dx dy,$$

where for $x \in A_S$ and $y \in B_S$, $f_A(x)$ and $f_B(y)$ are the probability density functions for α and β. Again player B has an analogous definition.

Do these new mixed strategies mess up everything we have defined and studied so far? What about equilibria for example? Well it turns out it's good news, and everything just carries over to hold for these more general mixed strategy sets. Indeed, starting with equilibria, we insist on the same definition, but extend it to take mixed strategies into account.

Definition 3.18. We define a profile of **mixed strategies** (one for each player) to be a **mixed equilibrium** (or simply equilibrium) if no player can improve their **expected payoff** by unilaterally changing their strategy.

In the context of two-players we can write this definition equivalently as

Definition 3.19. A pair of mixed strategies, α^* for A and β^* for B, are said to be in mixed equilibrium if

$$\begin{aligned} & g_A(\alpha^*, \beta^*) \geq g_A(\alpha, \beta^*), \quad \text{for all } \alpha \in \mathbb{A}_S \\ \text{and} \quad & g_B(\alpha^*, \beta^*) \geq g_B(\alpha^*, \beta), \quad \text{for all } \beta \in \mathbb{B}_S. \end{aligned}$$

If α^* and β^* are in equilibrium, neither player has any incentive to change strategies. They are mutual **best responses** to one another.

Just as we have extended our strategy sets, our calculation of payoffs and our definition of equilibrium, we can extend our definitions of dominance and best response without worry to allow for mixed strategies too. In all cases we will still use the same terminology: we may refer to a strategy as 'strategy', without specifying whether it is mixed or pure for example, or to an equilibrium without making note on whether it is mixed or pure. It should be clear in all cases what we mean, but where it is important to distinguish the difference we will do so.

We have introduced a new concept and showed that we can extend the theory previously built up without needing to start from scratch. Let's look at an example game to help us get to grips with our new tool of mixed strategies.

3.2 Compliance Inspections

Let's consider a two-player inspection game. We will see how mixed strategies naturally emerge.

Player B will be our inspectee and can either comply (strategy b_1) by fulfilling their legal obligation (e.g. paying tax, TV/driving licence, paying for transport tickets) or they can cheat (strategy b_2) and violate the obligation. player A, the inspector, can chose to inspect (strategy a_1) the inspectee to verify that they are complying, but doing so incurs an inspection cost, or they can choose to not inspect (strategy a_2). If the inspectee is caught cheating they will pay a large fine. Figure 23 gives the strategic form of the game.

		B	
		b_1 Comply	b_2 Cheat
A	a_1 Inspect	$-1, 0$	$-6, -90$
	a_2 Don't Inspect	$0, 0$	$-10, 10$

Figure 23: The inspection game.

Exercise: Check there are no pure strategy equilibria here.

A **mixed** strategy of player A is to play strategy a_1, inspect, with some probability. This makes practical sense contextually since inspecting every person incurs high costs. Inspecting only a few people reduces costs, and, even if an inspection is not certain to happen, a suitably high chance of being caught cheating should deter the inspectee from cheating.

Let p be the probability that player A plays a_1, so then $1-p$ is the probability that player A choses a_2. This ranging value of $p \in [0,1]$ then covers the whole of player A's mixed strategy set $\mathbb{A}_S$ including A's two pure strategies a_1 and a_2 when $p=1$ and $p=0$ respectively. Similarly, let q be the probability that player B plays b_1, and thus $1-q$ the probability that player B chooses b_2. Then the strategies $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ make up the players mixed strategies respectively.

The question we want to ask is when does the strategy pair (α, β) form an equilibrium? Well in this case, by definition, the strategies α and β must be best responses to each other. So, supposing player A is playing α, then let's consider player B's **pure strategy responses**. Their expected payoff for playing b_1, comply, is

$$g_B(\alpha, b_1) = p.0 + (1-p).0 = 0, \tag{2}$$

and, on the other hand, their expected payoff for b_2, cheat, is

$$g_B(\alpha, b_2) = p.(-90) + (1-p).10 = 10 - 100p. \tag{3}$$

So what can we conclude having performed these calculations? Well, when $0 > 10 - 100p$, or $p > 1/10$, then player B gets a better expected payoff from b_1, comply. But when $0 < 10 - 100p$, or $p < 1/10$, then player

B does better from b_2, cheat. Finally, note that player B is indifferent between their pure strategies **if and only if** $0 = 10 - 100p$, or $p = 1/10$.

But we've just introduced mixed strategies, so player B doesn't have to choose a pure strategy you might ask. Indeed, why can't they also mix by choosing $\beta = (q, 1-q)$, for some $0 < q < 1$? The answer is that they can, but this mixing, however, can **only** be of any benefit if they are **indifferent** between their two pure strategy choices b_1 and b_2! Why? Well, we will prove this formally soon, but indeed it is never optimal for a player to assign any positive probability to a pure strategy which is inferior to another pure strategy for a given mixed strategy of the opponent.

One way to see this concept in the inspection game is to consider what player B's best response is when p is lower than $1/10$, say $1/100$. In this case the expected payoff for cheat (b_2) from equation (3) becomes 9. Then we would be trying to weight a mixed strategy β where we assign a probability q towards the expected payoff of 0 (from b_1 in equation (2)) and a probability of $1-q$ towards the expected payoff of 9 (from b_2). Clearly, since $9 > 0$, the optimal assignment is to weight everything on the 9 (playing $q = 0$ or, in other words, playing b_2, cheat). This should help you see the concept that the only way a mixed strategy becomes relevant is if the player is impartial between the expected payoffs of their pure strategies.

We can perform a similar analysis for player A. Consider a mixed strategy of player B, $\beta = (q, 1-q)$. Then the expected payoffs for player A by playing each of their pure strategies are

$$g_A(a_1, \beta) = q.(-1) + (1-q).(-6) = 5q - 6, \tag{4}$$
$$g_A(a_2, \beta) = q.(0) + (1-q).(-10) = 10q - 10. \tag{5}$$

Thus, if player A inspects (plays a_1) with probability p, i.e. plays mixed strategy $\alpha = (p, 1-p)$, then their expected payoff is

$$\begin{aligned} g_A(\alpha, \beta) &= p(5q - 6) + (1-p)(10q - 10) \\ &= p(4 - 5q) + 10q - 10. \end{aligned}$$

This is a linear function of p, with maximum at $p = 0$ if $4 - 5q < 0$, or $q > 4/5$, and with maximum at $p = 1$ if $4 - 5q > 0$, or $q < 4/5$. Only if $q = 4/5$, does this expected payoff not depend on the value of p.

We can therefore conclude that the unique (mixed) equilibria of the inspection game occurs when $p = 1/10$ and $q = 4/5$ where the mixed strategies

$$\begin{aligned} \alpha &= \frac{1}{10}a_1 + \frac{9}{10}a_2, \\ \text{and} \quad \beta &= \frac{4}{5}b_1 + \frac{1}{5}b_2, \end{aligned}$$

are **mutual best responses**. In this case, the expected payoffs for each player are

$$g_A(a_1, \beta) = 5q - 6 = g_A(a_2, \beta) = 10q - 10 = -2,$$

for player A, and

$$g_B(\alpha, b_1) = 0 = g_B(\alpha, b_2) = 10 - 100p = 0,$$

for player B. The equilibrium probabilities p and q can be independently determined from these sets of equations.

Remarks:

- An important concept to note here is the dependence on the **opponent's payoffs** rather than the player's own in determining equilibrium. By this we mean that the equilibrium probability $p = 1/10$ is actually determined from the equations for **player B's** expected payoffs, similarly for q and player A. This is rather counter-intuitive. One might expect, for example, a more severe fine (changing -90 to -180) for being caught cheating lowers the probability of cheating in equilibrium. Surprisingly, it does not! What does change from doing this is the probability p of inspection, which is reduced until the inspectee is indifferent.

- A second important remark here is to discuss the need for mixing at all! If player A plays their mixed strategy $\alpha = (1/10, 9/10)$ and player B can then equally comply or cheat (since they are indifferent between them when A plays α), then why should they randomise at all? Indeed, they could simply comply (play b_1) and get a payoff of 0 for certain, simpler and safer, so why go through this process of mixing at all? The reason is precisely the fact that because there is no incentive to choose one strategy over the other, a player can mix, and **only then** can there be an equilibrium. If player B were to play b_1 for certain, then the best response of player A is to not inspect (play a_2), but then b_1 is not a best response, so this is **not** an equilibrium.

3.3 Finding Mixed Equilibria by considering Pure Strategies

Let us now come to proving the important point we mentioned in the inspection game; that we only need to mix over pure strategies that give maximal expected payoff.

Proposition 3.20. *For any mixed strategies α^* of player A and β^* of player B, then*

$$\begin{aligned}(a).\quad \max_{\alpha\in\mathbb{A}_S}\{g_A(\alpha,\beta^*)\} &= \max_{a\in A_S}\{g_A(a,\beta^*)\},\\ (b).\quad \max_{\beta\in\mathbb{B}_S}\{g_B(\alpha^*,\beta)\} &= \max_{b\in B_S}\{g_B(\alpha^*,b)\}.\end{aligned}$$

Proof.
(a). Clearly

$$\max_{\alpha\in\mathbb{A}_S}\{g_A(\alpha,\beta^*)\} \geq \max_{a\in A_S}\{g_A(a,\beta^*)\}, \tag{6}$$

since A_S is a subset of $\mathbb{A}_S$. Conversely, let $\alpha = (p_1, p_2, \cdots, p_n)$, $p_i \geq 0$, $\sum_i p_i = 1$. Then

$$\begin{aligned} g_A(\alpha,\beta^*) &= \sum_i p_i g_A(a_i,\beta^*), \quad \text{by definition,}\\ &\leq \sum_i p_i \max_{a\in A_S}\{g_A(a,\beta^*)\}, \quad \text{because } p_i \geq 0, \text{ for all } i,\\ &= \max_{a\in A_S}\{g_A(a,\beta^*)\}\sum_i p_i\\ &= \max_{a\in A_S}\{g_A(a,\beta^*)\}, \end{aligned}$$

so for all α, we have shown $g_A(\alpha, \beta^*) \leq \max_{a \in A_S} \{g_A(a, \beta^*)\}$, hence

$$\max_{\alpha \in \mathbb{A}_S} \{g_A(\alpha, \beta^*)\} \leq \max_{a \in A_S} \{g_A(a, \beta^*)\}. \tag{7}$$

Taking (6) and (7) together we have equality.

(b). **Exercise:** Similar proof to (a). □

Remark: The previous proof inherently assumes that A_S is a finite set, so that α has a finite number of n components. Nevertheless the proof holds true in the case where the set A_S is infinite, but we must replace max with sup (the supremum) in this case.

Proposition 3.20 allows us to consider just the player's pure strategies when trying to find maximal expected payoffs, rather than having to check an infinite set, $\mathbb{A}_S$, of mixed strategies. We now show that this indifference is important in having a mixed equilibrium.

Definition 3.21. Let c be a constant. A mixed strategy, α^*, for player A is called an **equaliser strategy** if

$$g_B(\alpha^*, b) = c, \quad \text{for all } b \in B_S.$$

Similarly for player B.

Proposition 3.22. *In a two-player game, if α^* is an equaliser strategy for A using B's payoffs and β^* is an equaliser strategy for B using A's payoffs, then (α^*, β^*) is an equilibrium.*

Proof. let c_A and c_B be constants. From the definition of equaliser strategies, we have

$$g_A(a_i, \beta^*) = c_A, \quad \text{for all } i,$$
$$g_B(\alpha^*, b_j) = c_B, \quad \text{for all } j.$$

This means that

$$g_A(\alpha^*, \beta^*) = c_A = g_A(a_i, \beta^*),$$

for all i since α^* is a mixture of a_i's. Similarly

$$g_B(\alpha^*, \beta^*) = c_B = g_B(\alpha^*, b_j),$$

for all j since β^* is a mixture of b_j's. By proposition 3.20 there are no other mixed strategies for the players that can do better, so these strategies are mutual best responses and are in equilibrium. □

Example: Consider the following game with normal form shown in figure 24.

A \ B	b_1	b_2
a_1	$2, 1$	$-1, -1$
a_2	$-3, -3$	$1, 2$

Figure 24: An example game.

There are two equilibria in pure strategies, (a_1, b_1) and (a_2, b_2), but are there any other equilibria? Let's seek a mixed strategy equilibrium. We need an equaliser strategy α for A **using B's payoffs** and an equaliser for B, β, **using A's payoffs**. A's payoffs are shown in figure 25 and B's in figure 26.

a_1	2	-1
a_2	-3	1

Figure 25: A's payoffs in a table.

b_1	b_2
1	-1
-3	2

Figure 26: B's payoffs in a table.

Using A's payoffs, setting $\beta = (q, 1-q)$, we can calculate A's expected payoffs

$$g_A(a_1, \beta) = 2q - (1-q) = 3q - 1,$$
$$g_A(a_2, \beta) = -3q + (1-q) = 1 - 4q,$$

using these, if β is an equaliser strategy for B, then we must have equality, i.e. $3q - 1 = 1 - 4q$ which gives $q = 2/7$. Thus B has the equaliser strategy

$$\beta = \frac{2}{7}b_1 + \frac{5}{7}b_2.$$

Exercise: Using B's payoffs as shown in figure 26 find an equaliser strategy for A. What is the game's mixed equilibrium?

3.4 Geometry of Games

Let's take a short detour to discuss a little bit about the geometry of games. To keep things simple and visual (at least for me who can barely handle visualising things in three dimensions let alone anything higher!), we will look at two-player games where each player has two pure strategies, though the concepts extend to more general games with more players and more strategies.

Let's take a 2×2 game in normal form as shown in figure 27.

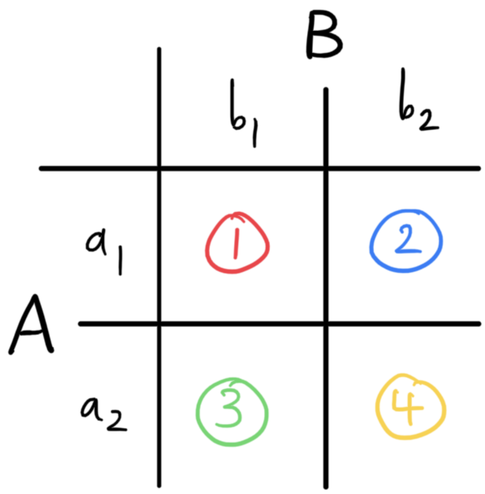

Figure 27: A general 2×2 game in normal form.

We have labelled each payoff pair with a circled number ①, ②, ③ and ④. To visualise the game geometrically, we plot each point ①, ②, ③ and ④ in the (x, y)-plane where the x coordinate is the payoff to player A and the y coordinate the payoff to B.

Let $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$, where $0 \leq p, q \leq 1$, be mixed strategies for A and B. Then we can write down expressions for any possible x and y coordinates representing possible expected payoffs of the players. For x we find

$$x = g_A(\alpha, \beta) = pq g_A(a_1, b_1) + p(1-q) g_A(a_1, b_2) + q(1-p) g_A(a_2, b_1) + (1-p)(1-q) g_A(a_2, b_2). \quad (8)$$

A similar expression holds for the y coordinate.

So now comes the pressing geometric question: which points $(x, y) \in \mathbb{R}^2$ correspond to possible payoff pairs and hence possible mixed strategy pairs for the players? This set of all such points we call S, the **payoff set** for the game.

3.4.1 The Convex Hull

Let's take a detour from that question for a moment and talk about something called the **convex hull**. Suppose we have a set of points $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n} \in \mathbb{R}^m$, then we can form their **convex hull**, C, which consists of all the points of the form

$$p_1 \mathbf{x_1} + p_2 \mathbf{x_2} + \cdots + p_n \mathbf{x_n},$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$.

For example, when $m = 2$ and using our points ①, ②, ③ and ④, then the convex hull would look something like the shaded area shown in figures 28 and 29.

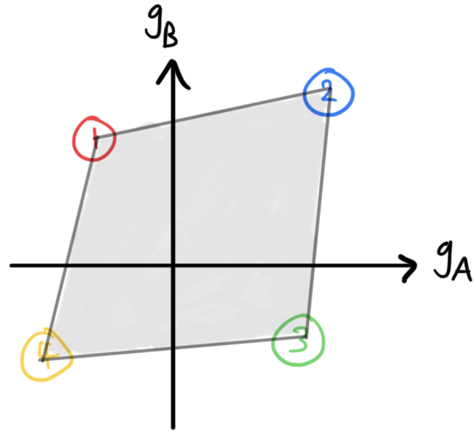

Figure 28: Possible convex hull of our game.

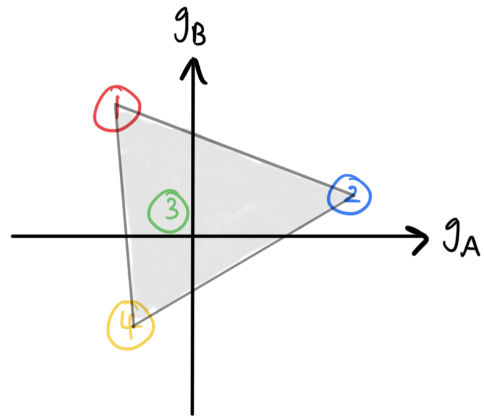

Figure 29: Another possible convex hull of our game.

The convex hull, C, is a **convex set**; which means for any two points in C, the whole of the line segment joining them is also in C. We will use this property of convex sets a few times throughout the course.

3.4.2 The Payoff set

So let's return to our question. Which points (x, y) are in our payoff set, S? Firstly, notice that any point of S must lie in the convex hull, C, of all the points (x, y) arising from the pure strategies (①, ②, ③ and ④). From (8) we can see this as the coefficients always satisfy $0 \leq pq, p(1-q), (1-p)q, (1-p)(1-q) \leq 1$ and these all sum to 1, so we can't reach a point beyond the convex hull. So now the question becomes is S just the whole of the convex hull, C, formed from ①, ②, ③ and ④?

It's not. Well not necessarily anyway. This is because the players are playing independently, so there are some convex combinations of payoff pairs that we cannot create.

Going back to our game in figure 27, the payoff set S of this game depends on the orientation of the points ①, ②, ③ and ④. What we find is that for two points in the same row or column of the normal form display, i.e. for ① and ②, ③ and ④, ① and ③, and ② and ④, then the line joining them also belongs to S. To see why this is the case (at least for the x coordinate) think about taking $p = 1$, $p = 0$, $q = 1$ and $q = 0$ respectively in (8). We can obtain all convex combinations between these pairs of points in this way.

However, S does not necessarily include the line segment joining points ① and ④ or points ② and ③. So what do the boundaries look like between these pairs of points? Referring to (8), changing the value of p can be thought of as weighting whether we are using the payoffs from row 1 or from row 2 (when $p = 1$ we use the payoffs in row 1 of the game, as p decreases to 0 we have less weighting on the first row and more weighting on the second row). Similarly changing q results in weighting on using the payoffs from column 1 or column 2. As we vary these smoothly, a curve drawn out by the envelope of the lines forming the possible convex combinations of payoffs forms **one** remaining curved boundary. Figures 30 and 31 give illustrations of a possible payoff set S shaded in grey.

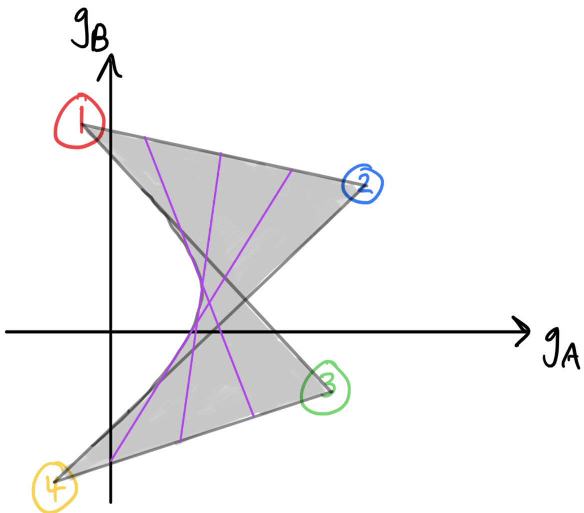

Figure 30: A schematic of a possible payoff set S for our 2×2 game showing three purple lines representing varying p for three different fixed q values.

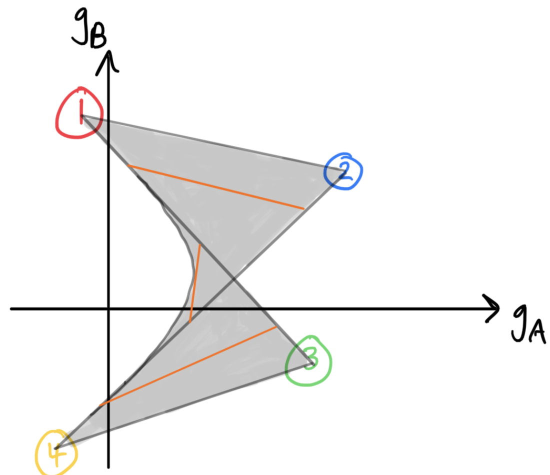

Figure 31: The same payoff set where orange lines for all q for three fixed p values are drawn.

To construct a payoff set, first the lines joining ① and ②, ③ and ④, ① and ③, and ② and ④ are drawn. By fixing a value of q and then varying through all possible p values we draw out a line of possible payoffs, three different examples of these lines are drawn on figure 30 coloured purple. Doing this for every possible value of q and then varying through all p we find that the envelope of these lines forms a curved boundary between either ① and ④ or between ② and ③. Alternatively, one can think about this process the other way around by considering the payoff pairs found by fixing a value of p and varying through all possible q values. An example of three lines drawn out by this process are shown in figure 31, coloured in orange. Either method produces the same envelope. These boundaries can be tricky to determine analytically and need to be found on a case by case basis, though if we are not too concerned by the explicit equation of such curves, then numerically they are fairly easy to produce. Everything within and on our boundaries then belongs in the payoff set S. You can see that this is in general a subset of the convex hull of the points representing the pure strategies.

We will do an example using the dating game next where we will see that by hand we can roughly sketch the payoff set, I'll then leave you the challenge of determining the equation of the curved boundary! .

Example: Consider the version of the dating game shown in figure 32.

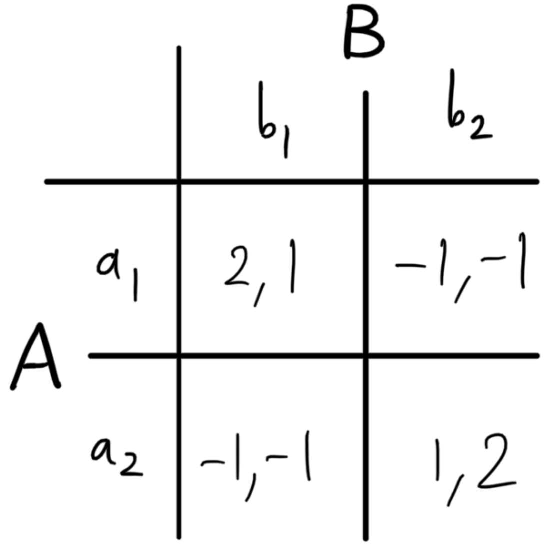

Figure 32: A dating game with two pure strategies for either player.

Let's draw its payoff set S. Well $(x, y) \in S \Leftrightarrow$ for some $p, q \in [0, 1]$ we have

$$(x, y) = pq(2, 1) + p(1 - q)(-1, -1) + (1 - p)q(-1, -1) + (1 - p)(1 - q)(1, 2). \tag{9}$$

Joining the pairs of pure strategies that we know form straight line boundaries leaves us with just the curved boundary to find. This will be between the points $(1, 2)$ and $(2, 1)$ which (as shown in figure 33) we can determine roughly by considering the envelope of the orange lines. The final payoff set is shown in figure 33.

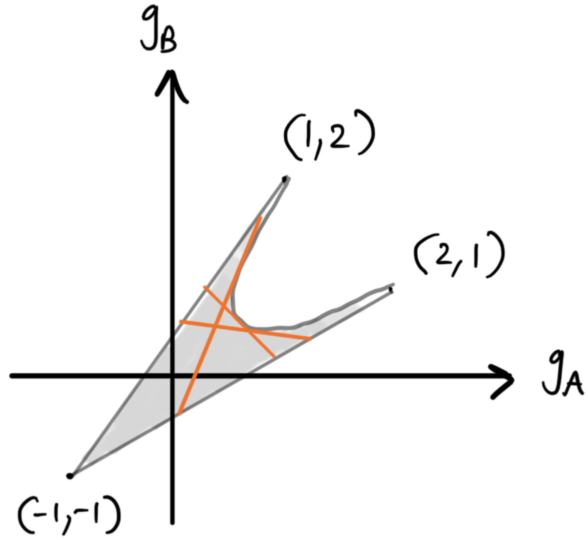

Figure 33: Payoff set S for the example dating game.

Exercise: ($\star$) Using (9) can you show that the equation of the parabolic boundary is given by

$$5(x - y + 1)^2 = 4(3x - 2y + 1).$$

3.5 Existence of an Equilibrium

We now state and prove arguably the most famous result in game theory: Nash's theorem on the existence of an equilibrium in finite games.

Theorem 3.23 (Nash (1951))**.** *Every finite game has at least one equilibrium.*

Proof. We will prove this in the case where the game is a finite (an $n \times m$) two-player game to simplify notation. The proof can be extended in the same manner to N-players. Let $C = \mathbb{A}_S \times \mathbb{B}_S$ be the product of the mixed strategy sets of our players A and B. We will use the property that this set C is compact and convex (this is because the sets $\mathbb{A}_S$ and $\mathbb{B}_S$ are convex and compact and the product of compact, convex sets is compact and convex). We have mentioned what it means for a set to be convex in the last section, compactness can be thought of as the set having no 'holes' - everything in its interior is filled with elements from the set, there are no gaps.

Our plan will be to construct a function $f : C \mapsto C$ which maps a pair of mixed strategies (α, β) to another pair $f(\alpha, \beta) = (\alpha^*, \beta^*)$ such that f has the following property. A mixed strategy of player A, i.e. $\alpha = (p_1, p_2, \cdots, p_n)$ will have a component p_i changed to p_i^* by f such that it will increase (i.e. $p_i^* > p_i$) if the pure strategy a_i does better than the current expected payoff

$$g_A(\alpha, \beta) = \sum_{i=1}^{n} \sum_{j=1}^{m} g_A(a_i, b_j) p_i q_j.$$

If the pure strategy a_i does worse, then p_i will be decreased to p_i^* unless p_i is already 0. A similar process will occur for the components of β.

Now, in equilibrium, α is a best response to β and all the sub-optimal pure strategies have probability zero. This means that through our function f, the mixed strategy α will not change, i.e. $\alpha^* = \alpha$. Similarly, $\beta^* = \beta$ if β is a best response to α. The idea of this discussion is to notice that if indeed we can create a function f with this property, then the equilibrium property is equivalent to $f(\alpha, \beta) = (\alpha^*, \beta^*) = (\alpha, \beta)$, i.e. that the function f has a **fixed-point**. Now let's go about trying to create such an f.

To define a function f with the property described, consider the following functions

$$\chi : \mathbb{A}_S \times \mathbb{B}_S \mapsto \mathbb{R}^n,$$
$$\psi : \mathbb{A}_S \times \mathbb{B}_S \mapsto \mathbb{R}^m.$$

Let $\chi_i(\alpha, \beta)$ be the ith component of $\chi(\alpha, \beta)$ and let $\psi_j(\alpha, \beta)$ be the jth component of $\psi(\alpha, \beta)$. We define these component functions by

$$\chi_i(\alpha, \beta) = \max \{0, g_A(a_i, \beta) - g_A(\alpha, \beta)\}, \quad i = 1, 2, \cdots, n,$$
$$\psi_j(\alpha, \beta) = \max \{0, g_B(\alpha, b_j) - g_B(\alpha, \beta)\}, \quad j = 1, 2, \cdots, m.$$

The difference between $g_A(a_i, \beta)$ and $g_A(\alpha, \beta)$ is positive if the pure strategy a_i gives a higher expected payoff than the mixed strategy α does against β, is zero if it gives the same payoff, and is negative if it gives less. The component $\chi_i(\alpha, \beta)$ is then exactly this difference, except it is replaced by 0 if the difference is negative. $\psi_j(\alpha, \beta)$ is defined analogously. This means that $\chi(\alpha, \beta)$ is a nonnegative vector in $\mathbb{R}^n$ and $\psi(\alpha, \beta)$

is a nonnegative vector in $\mathbb{R}^m$. These functions are also continuous.

Through our function f we will update the vectors α and β by replacing α with $\alpha+\chi(\alpha,\beta)$ to get α^*, and β by $\beta+\psi(\alpha,\beta)$ to get β^*. Both sums are non-negative, however, in general these new vectors are no longer necessarily valid mixed strategies because their components do not necessarily sum to one. Thus, we first 're-normalise' them in the following way. Let

$$p_i^* = \frac{p_i+\chi_i(\alpha,\beta)}{\sum_i p_i + \sum_i \chi_i(\alpha,\beta)} = \frac{p_i+\chi_i(\alpha,\beta)}{1+\sum_i \chi_i(\alpha,\beta)}, \tag{10}$$

$$q_j^* = \frac{q_j+\psi_j(\alpha,\beta)}{\sum_j q_j + \sum_j \psi_j(\alpha,\beta)} = \frac{q_j+\psi_j(\alpha,\beta)}{1+\sum_j \psi_j(\alpha,\beta)}, \tag{11}$$

where then we define $\alpha^* = (p_1^*, p_2^*, \cdots, p_n^*)$ and $\beta^* = (q_1^*, q_2^*, \cdots, q_m^*)$. Now α^* and β^* are valid mixed strategies. We finally define the function $f : C \mapsto C$ as

$$f(\alpha,\beta) = (\alpha^*,\beta^*),$$

with α^* and β^* as defined above with their components as given in (10) and (11).

Next we need to show that our constructed function f has the property that we have an equilibrium of the game when f has a fixed point.

Claim:

$$(\alpha,\beta) \text{ is an equilibrium} \quad \Longleftrightarrow \quad f(\alpha,\beta) = (\alpha,\beta).$$

Proof of claim:

($\Rightarrow$): Suppose (α,β) is an equilibrium. Then, by definition

$$g_A(a_i,\beta) \le g_A(\alpha,\beta),$$

for all $i = 1,2,\cdots,n$. This means that $\chi_i(\alpha,\beta) = 0$ for all $i = 1,2,\cdots,n$ and it follows that $p_i = p_i^*$ and hence $\alpha = \alpha^*$. In a similar manner we can show $\beta = \beta^*$.

($\Leftarrow$): We will prove this direction by contradiction. Assume that $\alpha^* = \alpha$ and $\beta^* = \beta$ but (α,β) is **not** an equilibrium. Then either:

$$\begin{aligned} &\text{There exists } \alpha' \text{ such that:} \quad g_A(\alpha',\beta) > g_A(\alpha,\beta), \quad \text{or} \\ &\text{There exists } \beta' \text{ such that:} \quad g_B(\alpha,\beta') > g_B(\alpha,\beta). \end{aligned}$$

Let's assume it's the first of these possibilities (the second possibility follows in a similar manner). Let $\alpha' = (p_1', p_2', \cdots, p_n')$, then, by definition

$$g_A(\alpha',\beta) = \sum_{i=1}^{n} p_i' g_A(a_i,\beta).$$

Thus, there **must** exist an i for which $g_A(a_i,\beta) > g_A(\alpha,\beta)$. Otherwise, if this were not the case, then we contradict $g_A(\alpha',\beta) > g_A(\alpha,\beta)$ because

$$g_A(\alpha',\beta) \le \sum_{i=1}^{n} p_i' g_A(\alpha,\beta) = g_A(\alpha,\beta).$$

Now for this value of i:

$$\chi_i(\alpha, \beta) = \max\{0, g_A(a_i, \beta) - g_A(\alpha, \beta)\} > 0,$$

and thus $\sum_i \chi_i(\alpha, \beta) > 0$. Now consider

$$g_A(\alpha, \beta) = \sum_{i=1}^{n} p_i g_A(a_i, \beta),$$

so $g_A(a_k, \beta) \leq g_A(\alpha, \beta)$, for some k, with $p_k > 0$. Otherwise, if this were not the case $g_A(a_k, \beta) > g_A(\alpha, \beta)$ for all k with $p_k > 0$ but then

$$\begin{aligned} g_A(\alpha, \beta) &= \sum_k p_k g_A(a_k, \beta) \\ &> g_A(\alpha, \beta) \sum_k p_k \\ &= g_A(\alpha, \beta), \end{aligned}$$

which is nonsense. For this value of k, we have

$$\chi_k(\alpha, \beta) = \max\{0, g_A(a_k, \beta) - g_A(\alpha, \beta)\} = 0,$$

and

$$p_k^* = \frac{p_k + \chi_k(\alpha, \beta)}{1 + \sum_i \chi_i(\alpha, \beta)} = \frac{p_k}{1 + \sum_i \chi_i(\alpha, \beta)} < p_k.$$

Hence $\alpha^* \neq \alpha$ (since not all p_i's are equal). But this is a contradiction since initially we assumed $\alpha^* = \alpha$. Hence (α, β) must be an equilibrium.

Thus our claim holds. This means we have shown that if our function $f(\alpha, \beta)$ has a fixed point then these pair of strategies are an equilibrium of the game.

We have one thing left to show, that our function **always** has a fixed point. For this we invoke **Brouwer's fixed point theorem**, which states that a continuous function from a closed, bounded, convex set C to itself always has at least one fixed point. Indeed, our function operating on a set which is convex and compact to itself, means it satisfies the requirements of the fixed point theorem and so we always have at least one fixed point, and hence the game always has at least one equilibrium. □

We dedicate the remainder of this chapter to developing techniques that we can use to find all the equilibria in small games.

3.6 Finding Equilibria by checking Subgames

We have seen how to find mixed equilibria in a 2×2 game (the inspection game for example) through use of ensuring indifference between the two pure strategies, but what about in slightly larger games. Let's do an example involving a 3×2 game, shown in figure 34.

		B	
		b_1	b_2
A	a_1	1,2	0,0
	a_2	0,0	2,1
	a_3	-1,4	3,1

Figure 34: A 3×2 game.

First, let's look for any pure strategy equilibria. Indeed, we find that (a_1, b_1) is a pure strategy equilibrium. This is the only one. What about mixed equilibria? How do we tackle finding any of these that might be present?

In this section we will look at how we might restrict the game into smaller 'subgames' of the original game, then find their equilibria, and check whether these are equilibria of the whole game. First, let's restrict player A to just strategies a_1 and a_2.

Exercise: In this case show that $(\alpha_1, \beta_1) = ((1/3, 2/3), (2/3, 1/3))$ and (a_2, b_2) are two new equilibria with expected payoffs for A given by $g_A(\alpha_1, \beta_1) = 2/3$ and $g_A(a_2, b_2) = 2$.

To guarantee that these equilibria are also equilibria of the whole game (the game with strategy a_3 included) we check the expected payoff from these strategies against any other pure strategies that were excluded from the full game. In this case this is just strategy a_3 for player A. So, testing strategy a_3 against strategy β_1 we find:

$$g_A(a_3, \beta_1) = (-1).\frac{2}{3} + 3.\frac{1}{3} = \frac{1}{3} < \frac{2}{3} = g_A(\alpha_1, \beta_1),$$

which is **smaller** than that found from strategy α_1. This means that α_1 **is** a best response to β_1 when considering the whole game (since we have shown a_3 performs worse than α_1, which is a mix of a_1 and a_2 and so would have **no** positive weighting in an equilibrium). Hence we can conclude $(\alpha_1, \beta_1) = ((1/3, 2/3), (2/3, 1/3))$ **is** an equilibrium of the original game.

However, checking the other subgame equilibrium found, against b_2 player A's best response is a_3, hence the pure strategy equilibrium (a_2, b_2) found in the subgame is **not** an equilibrium of the full game (we knew this already as we checked the game for all pure strategy equilibria initially). Now let's check the next restricted 2×2 subgame, where player A can chose a_1 or a_3.

Exercise: Show that in this subgame the only equilibrium is the one found already (a_1, b_1).

Lastly, let's check the final possible subgame, where player A can chose a_2 or a_3.

Exercise: Show that in this subgame the only equilibrium is $(\alpha_2, \beta_2) = ((1/4, 3/4), (1/2, 1/2))$ with $g_A(\alpha_2, \beta_2) = 1$.

Checking the removed strategy a_1 against β_2 we find

$$g_A(a_1, \beta_2) = 1.\frac{1}{2} + 0.\frac{1}{2} = \frac{1}{2} < 1 = g_A(\alpha_2, \beta_2),$$

so indeed α_2 is a best response to β_2 in the whole game, and thus is also an equilibrium of the whole game.

This means we can conclude that in the original game, all equilibria are

$$(a_1, b_1), \quad \left(\left(\frac{1}{3}, \frac{2}{3}, 0\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right) \quad \text{and} \quad \left(\left(0, \frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right).$$

Remark: One question you might have here however is how do we know that there is no equilibrium where player A mixes between **all three** of their pure strategies (since by splitting the game into all the 2×2 subgames we wouldn't find an equilibria of this form)?

This is a good question. The answer is because player B has only one probability, q, to play with, meaning that we don't have enough freedom to satisfy two equations (one being indifference between a_1 and a_2 and another indifference between a_2 and a_3, where indifference between a_1 and a_3 would then hold automatically). Nevertheless, in games where player B has just two pure strategies, there is a possibility that some value for q **does**, in fact, satisfy both indifference equations simultaneously. This leads us into the concept of degeneracy, which we will investigate in section 3.8, and on problem sheet 3 you will investigate a game of this form. In games where player B has more than two pure strategies however, then we certainly have sufficient degrees of freedom and mixing over all three pure strategies is very possible.

3.7 The Upper Envelope Method

We now discuss an alternative method for finding all the equilibria in games where one player has just two pure strategies: the **upper envelope** method. We have actually already seen this a few times earlier in the course when we introduced the concept of best response and analysed a few games, including penalty kicks. We showcase it again here using the game from the last section to show how we can use these diagrams to find all the equilibria.

From the game shown in figure 34, player A's expected payoffs against $\beta = (q, 1-q)$ for each of their pure strategies are given by

$$
\begin{aligned}
g_A(a_1, \beta) &= q, \\
g_A(a_2, \beta) &= 2 - 2q, \\
g_A(a_3, \beta) &= 3 - 4q.
\end{aligned}
$$

Graphing these against q we arrive at figure 35.

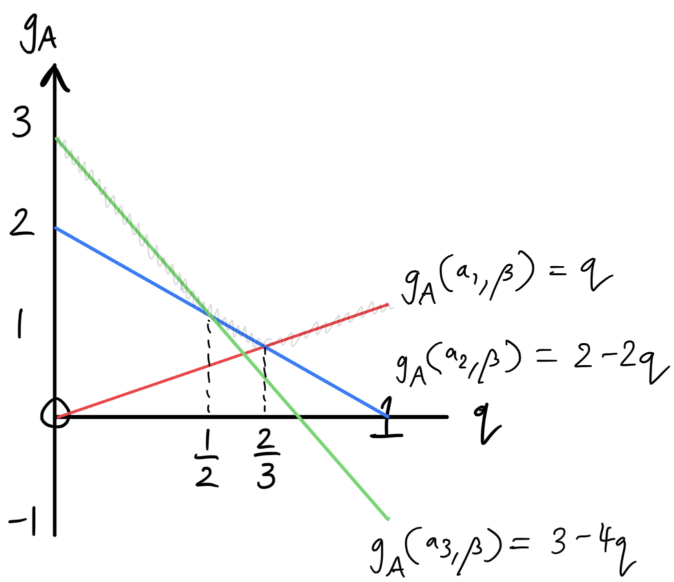

Figure 35: The upper envelope diagram for the example 3×2 game.

We call this graph an **upper envelope diagram**. The upper envelope itself is highlighted with a grey shading in figure 35. It turns out from these diagrams we can very easily determine all the game's equilibria!

The green and blue lines intersect when $q = 1/2$ and the red and blue lines intersect when $q = 2/3$ (these values appearing should make you excited when we think back to what we found the equilibria to be in this game in the last section). At each of these intersection points, as well as at both end points along the upper envelope we can find the mixed strategies of the players (these will be pure strategies at end points) and check if they are in equilibrium.

This leaves us with just **four** points to check. Indeed, when $q = 1/2$, then with player A mixing between a_2 and a_3 (since the blue and green lines relate to these two pure strategies) we find that $p = 1/4$ to make B indifferent. This is the equilibrium (α_2, β_2) we found in the last subgame earlier. When $q = 2/3$ then mixing between a_1 and a_2 we find $p = 1/3$ for indifference of player B and this gives us the equilibrium (α_1, β_1) found earlier. Lastly we check the endpoints, representing pairs of pure strategies, given by $q = 0$ so b_2 being played by B with A playing a_3 (since the green line is the upper envelope here). This turns out not to be an equilibrium, but when $q = 1$ so B is playing b_1, and when A plays a_1 (since the red line is highest) then this is indeed the final equilibrium (a_1, b_1) found previously.

The upper envelope method is particularly useful when one player has a large number of pure strategies and the other has two. On problem sheet 3 you will be tasked to find all equilibria in a 5×2 game. Considering all the separate 2×2 games would result in needing to check ten different subgames, much more time consuming than checking the intersections and end points along the upper envelope.

3.8 Degenerate Games

Let's finish this chapter by looking at a final game, a threat game, and try to determine all its equilibria. Unfortunately, just as we might feel like we have some sound methods of finding all equilibria, we will see things become slightly more complicated in this case. Figure 36 gives the normal form of the threat game we are interested in.

		B	
		b_1	b_2
A	a_1	1,3	1,3
	a_2	0,0	2,2

Figure 36: A threat game with a weakly dominated strategy for player B.

A quick check for pure strategy equilibria shows that both (a_1, b_1) and (a_2, b_2) are equilibria. But does the game have any mixed equilibria? Well, let's analyse the game in our usual manner and take a look. Let $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ be mixed strategies for either player. Then

$$g_A(a_1, \beta) = 1,$$
$$g_A(a_2, \beta) = 2 - 2q,$$

so player A is indifferent when $q = \frac{1}{2}$, and

$$g_B(\alpha, b_1) = 3p,$$
$$g_B(\alpha, b_2) = 2 + p,$$

so player B is indifferent when $p = 1$, i.e. player A plays a_1. This is not surprising, indeed, notice that strategy b_2 weakly dominates strategy b_1... so it can only be in the case that we **know** player A is playing a_1 that player B can be indifferent (otherwise they would play b_2 if there was **any** positive probability on a_2 by A).

So what have we learned? The game has two pure strategy equilibria (a_1, b_1), (a_2, b_2) and a mixed equilibria $(a_1, (1/2, 1/2))$ where only player B mixes. Is this all? What about the strategy pair $(a_1, (3/4, 1/4))$, or the strategy pair $(a_1, (7/8, 1/8))$, are these in equilibrium?

Exercise: Find the expected payoffs for each player for these pairs of strategies. Show that they too are indeed equilibria!

So it seems there are more equilibria that our previous method didn't recover. Well this is not precisely true. Observe what we concluded from the first pair of equations. We insisted that these expected payoffs were equal to find that $q = 1/2$. But, now think about what we learned from the second pair of equations, that we need to take $p = 1$, meaning that strategy a_2 is being played with 0 probability in equilibrium. This means that we no longer need to insist that $g_A(a_1, \beta) = g_A(a_2, \beta)$, since a_2 is **not** being played anyway! All that matters in equilibrium is that a_1 is the **best response** to β; i.e. we need:

$$\begin{aligned} g_A(a_1, \beta) \geq g_A(a_2, \beta), &\quad \text{or} \\ 1 \geq 2 - 2q, &\quad \text{or} \\ q \geq \frac{1}{2}. & \end{aligned}$$

So, what we have actually found is an **infinite family of equilibria** of the form $(a_1, (q, 1 - q))$, where $1/2 \leq q \leq 1$. The two extreme cases end up being two of the equilibria we found earlier.

The reason why we refer to this game as a 'threat' game is related to these equilibria. Even though b_2 weakly dominates b_1, by threatening to play b_1 with sufficiently high probability, then player A is forced into playing strategy a_1. Let's now define when a two-player game is termed degenerate and will have this issue as encountered in the threat game.

Definition 3.24. (Degenerate Game) A two-player game is called **degenerate** if some player has a mixed strategy that assigns positive probability to exactly k pure strategies so that the other player has more than k pure best responses to that mixed strategy.

Remark: This definition holds in the case when $k = 1$ (as in the threat game example) where the mixed strategy in the definition is a pure strategy (i.e. it mixes over $k = 1$ pure strategies).

On problem sheet 3 you will encounter more degenerate games including a 2×3 degenerate game.

Chapter 4: Zero-Sum Games

Zero-sum games are two-player games that are **strictly competitive**; meaning that the interests of both players are directly opposed. This means, for any two strategies, α for A and β for B, that

$$g_A(\alpha, \beta) = -g_B(\alpha, \beta) = g(\alpha, \beta). \tag{12}$$

As seen in (12), for this reason, we often drop the subscripts and let $g(\alpha, \beta)$ represent the payoff to player A where then $-g(\alpha, \beta)$ is the payoff to B (sometimes we refer to the payoff $g(\alpha, \beta)$ as the **cost** to B). Notice that (12) means that the sum of payoffs to the players in **any** outcome of the game is **zero**, hence the naming convention: zero-sum games.

4.1 The Duelist's Game

Let's play a game. This is a zero-sum game, meaning it is a game of strict competition between our two players: our two duelists, player A and player B. Our two duelists each have a gun loaded with just one bullet each. They start at a unit distance apart and simultaneously start walking towards each other. Each duelist must decide when they want to fire their gun (i.e. at what distance apart to fire).

If one duelists hits the other, then they win (let's say) one point and the other (the one who has been hit) loses a point. If a duelist fires and misses then they lose a point and the other wins a point (since the one who hasn't fired can wait until the duelists are adjacent where we will assume that they cannot miss with a point blank shot). If both duelists fire simultaneously then if both hit, or if both miss, then they each receive nothing but lose nothing.

Activity: Play the game (with pretend guns!) in class with some volunteer duelists.

4.1.1 Discussion

First note that the players have infinite pure strategy sets A_S and B_S in this game (choices of a value in the range $[0, 1]$). Nevertheless we can still try to determine what might be good strategies for the duelists. The first question to ask then, is what properties are important in determining a possible optimal strategy? Well from the position of a particular duelist, say A, what we care about at any distance is the probability that we can hit our opponent at that distance, as well as the probability of being hit by our opponent at that distance, so we need to know or have good approximations of these probability functions. Let's denote them as

$$p_A(a) = \text{Probability that player } A \text{ hits at distance } a,$$
$$p_B(b) = \text{Probability that player } B \text{ hits at distance } b,$$

where $a, b \in [0, 1]$.

Exercise: Show that the payoff to player A, if player A shoots when the distance apart is a and player B

shoots when the distance apart is b, is given by

$$g(a,b) = \begin{cases} 2p_A(a) - 1, & \text{if } a > b, \\ 1 - 2p_B(b), & \text{if } a < b, \\ p_A(a) - p_B(b), & \text{if } a = b. \end{cases}$$

Exercise: We still need to try and find an optimal strategy for each player in this game (an equilibrium). Perhaps a good first step is to try and think what would be best play if the two players were entirely equal in ability (i.e. that p_A and p_B were the same functions). Can you predict what is optimal play in this case? Show that your guess is an equaliser strategy for both players and hence gives an equilibrium and solution of the game. Can you then see how to generalise this case to when the functions $p_A(a)$ and $p_B(b)$ are not equal?

4.2 Lower Envelopes and Zero-Sum Games

Let's look at another example zero-sum game, shown in figure 37.

		B		
		b_1	b_2	b_3
A	a_1	6,-6	3,-3	1,-1
	a_2	0,0	2,-2	6,-6

Figure 37: An example zero-sum game.

		B		
		b_1	b_2	b_3
A	a_1	6	3	1
	a_2	0	2	6

Figure 38: The example zero-sum game displayed in normal form in its usual manner.

From the game shown in normal form, notice how the payoffs for player B are indeed always negative the payoff for player A in every outcome of the game. This means the game is zero-sum. Usually, instead of representing the normal form of the game as in figure 37, we do so instead as in figure 38.

This form of depicting the game uses only one payoff in each cell, that of player A. This avoids writing the superfluous second payoff for player B since, in the case of a zero-sum game, we know this payoff is simply negative the payoff shown for player A. By doing this we can also tell a game is zero-sum just by looking at its normal form by noting that only one payoff is shown in each cell.

As is usual, let's find all the equilibria of the game. First notice there are no pure strategy equilibria, so we seek mixed strategy equilibria. To help us with this, and to motivate our next insight into these zero-sum games, let's draw something very similar to an upper envelope diagram for this game. Denoting a mixed strategy for player A by $\alpha = (p, 1-p)$, then let's graph $g(\alpha, b)$ for each pure strategy (b_1, b_2 and b_3) of player B, against p, as shown in figure 39.

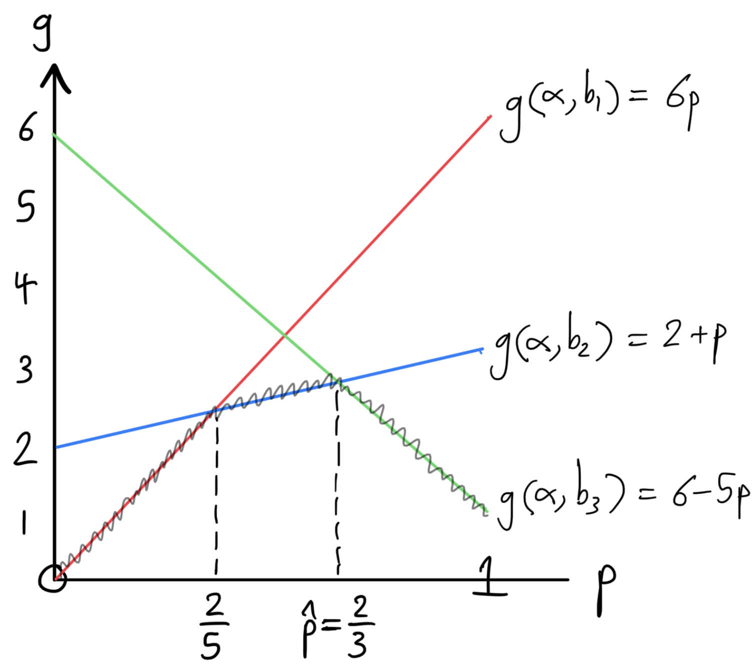

Figure 39: The lower envelope diagram for the example zero-sum game.

This diagram is slightly different to the upper envelope diagrams we have seen so far which graphed player A's expected payoffs against q (or, alternatively, player B's expected payoffs against p), since it graphs player A's expected payoffs against p. We call this a **lower envelope diagram**.

So how do we analyze this diagram? Well these payoffs to player A are essentially **costs** for player B, so the best response costs to player B are given by the **minimum** of these lines, defining the **lower envelope**, shaded with a grey squiggle on figure 39. There are two intersection points on the lower envelope, where the red and blue lines intersect, at $p = 2/5$, and where the blue and green lines intersect, at $p = 2/3$. At these points player B can mix, but when $p = 2/5$ then it turns out player A can't be made indifferent: letting $\beta = (q, 1-q)$, we find

$$g(a_1, \beta) = 3 + 3q,$$
$$g(a_2, \beta) = 2 - 2q,$$

and these can never be made equal for any $q \in [0, 1]$. However, when $p = 2/3$ then we find the unique equilibrium of the game given by (exercise):

$$(\alpha, \beta) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{5}{6}, \frac{1}{6}\right)\right).$$

Let's now take a slightly different approach and think about how player A can interpret the lower envelope in a zero-sum game. Suppose, regardless of what player B is doing, player A intends to **secure** some guaranteed (albeit expected) payoff. This can be done by playing a strategy corresponding to a point on the lower envelope (since this is the **lowest** possible expected payoff A can attain when playing this strategy). By then **maximising** over the lower envelope for some choice of p, player A maximises this 'guaranteed' expected payoff.

Referring back to figure 39, it is clear this maximum expected payoff (maximum height of the lower envelope) occurs when $p = \hat{p} = 2/3$. The corresponding strategy $\hat{\alpha} = (2/3, 1/3) = (\hat{p}, 1 - \hat{p})$ turns out to be very important in the context of zero-sum games and is given a name: a **max-min** (or **maximin**) strategy of player A. One thing you might immediately notice is that this strategy is precisely the strategy we found earlier which player A should play in equilibrium. This is by no means a coincidence and indeed we will shortly show that in a zero-sum game an equilibrium strategy for player A is a max-min strategy. Note that this property would further speed up the analysis we need to do to seek equilibria in the game: we could immediately discount the strategy with $p = 2/5$ on the lower envelope, without checking it, simply by noting it is not a maximal point on the lower envelope and hence doesn't correspond to a max-min strategy and thus is not an equilibrium strategy of the zero-sum game.

This maximum value of the lower envelope (and thus the max-min payoff) is unique (since it is nonsense for there to be two different maximum heights), but it is possible for there to be multiple choices of max-min strategies. For example, suppose we replace the payoff of 3 in the previous example at (a_1, b_2) with 2, as shown in figure 40.

		B		
		b_1	b_2	b_3
A	a_1	6	2	1
	a_2	0	2	6

Figure 40: The example zero-sum game with one payoff changed.

The corresponding lower envelope diagram for this game is shown in figure 41.

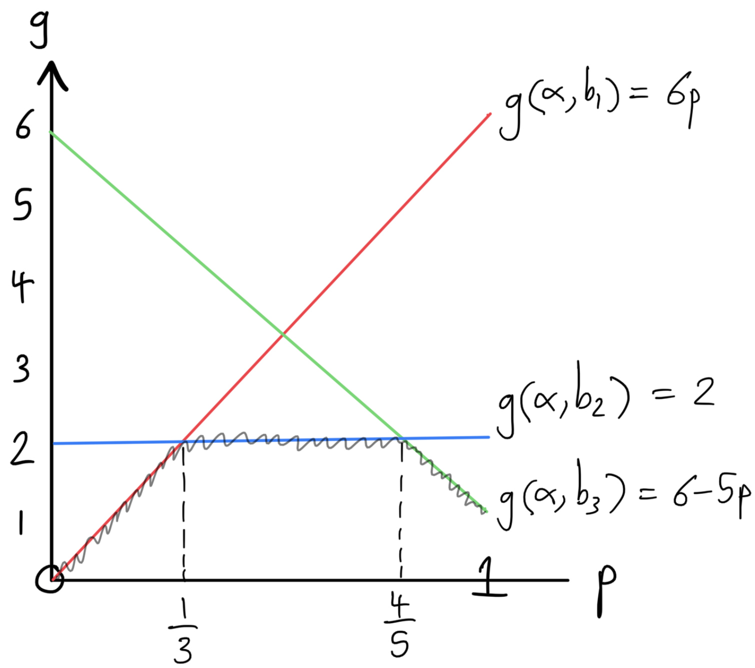

Figure 41: The lower envelope diagram for the adjusted game.

The lower envelope now has maximal value 2, but this maximum is no longer attained at just one point, but instead anywhere in the range of values $\hat{p} \in [1/3, 4/5]$. Setting $\hat{\alpha} = (\hat{p}, 1 - \hat{p})$ for any of these values of $\hat{p}$ then gives a valid max-min strategy for player A. Clearly the blue line is the lowest of all three when $p \in (1/3, 4/5)$ meaning, in these cases, playing the corresponding pure strategy b_2 is the best response for player B. It turns out that even at the intersection points, when $p = 1/3$ or $p = 4/5$ that b_2 is also the equilibrium strategy for player B.

4.3 Max-min and Min-max Strategies

A max-min strategy, as seen in the last example, is a strategy that maximises the lower envelope of player A's expected payoffs. A **min-max** (or **minimax**) strategy is the name we give to the corresponding strategy for player B: it minimises the upper envelope of player B's expected payoffs. We formally define these strategies now.

Definition 4.25. A **max-min** strategy $\hat{\alpha} \in \mathbb{A}_S$ of player A is a strategy such that

$$\min_{\beta \in \mathbb{B}_S} \{g(\hat{\alpha}, \beta)\} = \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\},$$

assuming that these maxima and minima exist. This also defines the **max-min payoff** to player A.

Definition 4.26. A **min-max** strategy $\hat{\beta} \in \mathbb{B}_S$ of player B is a strategy such that

$$\max_{\alpha \in \mathbb{A}_S} \left\{ g(\alpha, \hat{\beta}) \right\} = \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}.$$

This gives the **min-max payoff** to player B.

These strategies in strictly competitive (zero-sum) games give a **guaranteed** (albeit expected) payoff to the player **regardless** of what the opponent plays. In a game with separate payoffs for both players, like we have analysed throughout the first three chapters, player B may not always want to play a strategy minimising player A's payoffs since they care about maximising their own. In zero-sum games however these are equivalent.

Let's now remark on an important property of these strategies, that is, the fact that we don't need to minimise over all $\beta \in \mathbb{B}_S$ when seeking player A's max-min strategy (and similarly maximise over all $\alpha \in \mathbb{A}_S$ when seeking player B's min-max strategy), rather just over the opponent's pure strategies. We have seen this idea in practice in the example game where we graphed the lower envelope using just the opponent's pure strategies.

Proposition 4.27. *In a zero-sum game, for $\alpha \in \mathbb{A}_S$, then*

$$\min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} = \min_{b \in B_S} \{g(\alpha, b)\}.$$

Similarly, for $\beta \in \mathbb{B}_S$, then

$$\max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} = \max_{a \in A_S} \{g(a, \beta)\}.$$

Proof. The proof of these results almost identically follows the proof of proposition 3.20 so we omit it here. □

Let's do an example.

Example: Find max-min and min-max strategies for players A and B respectively in the zero-sum game given in normal form in figure 42.

		B	
		b_1	b_2
A	a_1	1	3
	a_2	2	1

Figure 42: An example zero-sum game.

Solution: We will do this without drawing the envelope diagrams to showcase an alternative way to find these strategies. By definition, a max-min strategy for player A is a strategy $\hat{\alpha}$ such that

$$\min_{\beta \in \mathbb{B}_S} \{g(\hat{\alpha}, \beta)\} = \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\},$$

which, via the last proposition 4.27, we can write as

$$\min_{b \in B_S} \{g(\hat{\alpha}, b)\} = \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{b \in B_S} \{g(\alpha, b)\} \right\}.$$

Writing $\alpha = (p, 1-p)$, then against B's pure strategies we have

$$\begin{aligned} g(\alpha, b_1) &= p + 2(1-p) = 2 - p, \\ g(\alpha, b_2) &= 3p + (1-p) = 1 + 2p. \end{aligned}$$

Now maximising the minimum of these two quantities over $p \in [0,1]$ it is clear this value occurs when $2 - p = 1 + 2p$, or when $p = 1/3$ and gives $5/3$ as the max-min payoff to A. The max-min strategy for A is thus $\hat{\alpha} = (1/3, 2/3)$.

For B, by definition, a min-max strategy $\hat{\beta}$ is such that

$$\max_{\alpha \in \mathbb{A}_S} \left\{ g(\alpha, \hat{\beta}) \right\} = \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}.$$

which, using proposition 4.27, we can simplify as

$$\min_{a \in A_S} \left\{ g(a, \hat{\beta}) \right\} = \min_{\beta \in \mathbb{B}_S} \left\{ \max_{a \in A_S} \{g(a, \beta)\} \right\}.$$

Writing $\beta = (q, 1-q)$, we have

$$\begin{aligned} g(a_1, \beta) &= q + 3(1-q) = 3 - 2q, \\ g(a_2, \beta) &= 2q + (1-q) = 1 + q. \end{aligned}$$

Minimising the maximum of these quantities over $q \in [0,1]$ occurs when $3 - 2q = 1 + q$, or when $q = 2/3$ giving again $5/3$ as the expected payoff to A (or $-5/3$ as the min-max payoff to B). The min-max strategy for B is $\hat{\beta} = (2/3, 1/3)$.

Remarks:

- Analysing the game to seek its equilibria using the methods we have already developed (exercise) we find that the pair of max-min and min-max strategies found here correspond to the players mixed equilibrium strategies. As alluded to earlier, this is not a coincidence.
- Notice in the last example that the max-min and min-max strategies both gave player A the **same** expected payoff of $5/3$. Again, we will see shortly that this is not a coincidence and this payoff value is given a special name.

4.4 Relationship of Equilibria and Max-min/Min-max Strategies

In this section we will prove that a pair of max-min and min-max strategies giving equal payoff to player A form an equilibrium in a zero-sum game. Before we prove this we make note of an inequality that we will need during the proof. We claim that

$$\max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\} \leq \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}, \tag{13}$$

which on first thought may seem far from trivial. However this inequality essentially says that

$$\text{max-min payoff} \leq \text{min-max payoff},$$

which is trivially true (otherwise we would have min-max strategies for B that could guarantee getting A less than A could guarantee getting using a max-min strategy, which is nonsense).

Proposition 4.28. *In a finite zero-sum game with $\hat{\alpha} \in \mathbb{A}_S$, $\hat{\beta} \in \mathbb{B}_S$ then $(\hat{\alpha}, \hat{\beta})$ is an equilibrium if and only if $\hat{\alpha}$ is a max-min strategy for A, $\hat{\beta}$ is a min-max strategy for B and*

$$\max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\} = \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}. \tag{14}$$

Proof. ($\Leftarrow$): Let $\hat{\alpha}$ be max-min and $\hat{\beta}$ be min-max and assume (14) holds. Then

$$\begin{aligned}
g(\hat{\alpha}, \hat{\beta}) &\geq \min_{\beta \in \mathbb{B}_S} \{g(\hat{\alpha}, \beta)\} \\
&= \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\}, \\
&= \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}, \quad \text{by (14)}, \\
&= \max_{\alpha \in \mathbb{A}_S} \left\{ g(\alpha, \hat{\beta}) \right\} \geq g(\hat{\alpha}, \hat{\beta}),
\end{aligned}$$

where we have used the facts that $\hat{\alpha}$ is max-min and $\hat{\beta}$ is min-max. What we have deduced means that all inequalities must be equalities. Hence we have

$$\max_{\alpha \in \mathbb{A}_S} \left\{ g(\alpha, \hat{\beta}) \right\} = g(\hat{\alpha}, \hat{\beta}) = \min_{\beta \in \mathbb{B}_S} \{g(\hat{\alpha}, \beta)\}, \tag{15}$$

which means that

$$g(\alpha, \hat{\beta}) \leq g(\hat{\alpha}, \hat{\beta}) \leq g(\hat{\alpha}, \beta),$$

for all $\alpha \in \mathbb{A}_S$, $\beta \in \mathbb{B}_S$. But this is precisely the definition that $(\hat{\alpha}, \hat{\beta})$ is in equilibrium since the left inequality means $\hat{\alpha}$ is a best response to $\hat{\beta}$ and the right inequality that $\hat{\beta}$ is a best response to $\hat{\alpha}$.

($\Rightarrow$): Suppose instead that $(\hat{\alpha}, \hat{\beta})$ is an equilibrium. Then, as just discussed, this is equivalent to (15). So, we have

$$
\begin{aligned}
g(\hat{\alpha}, \hat{\beta}) &= \min_{\beta \in \mathbb{B}_S} \{g(\hat{\alpha}, \beta)\}, \\
&\leq \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\}, \\
&\leq \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\}, \quad \text{using (13)}, \\
&\leq \max_{\alpha \in \mathbb{A}_S} \left\{ g(\alpha, \hat{\beta}) \right\} = g(\hat{\alpha}, \hat{\beta}),
\end{aligned}
$$

so we also have equalities throughout, meaning that $\hat{\alpha}$ is a max-min strategy, $\hat{\beta}$ is a min-max strategy and (14) holds. □

Let's talk about what this proposition tells us. It tells us that pairs of max-min and min-max strategies which give equal expected payoffs to player A are equilibria of the game, and vice-versa, so equilibria are indeed pairs of max-min and min-max strategies giving equal expected payoffs. This is great but we are still missing something crucial; whether these pairs of special strategies giving equal payoffs are always present in a zero-sum game, i.e. we haven't proved anything about their existence. Well the short answer is we don't need to, we have this already: we use Nash's theorem, which says that in a finite game (which can be zero-sum), that an equilibrium always exists. Hence using the last proposition 4.28, we know that these pairs of max-min and min-max strategies always exist in a zero-sum game.

Nevertheless, we now take a moment to zip back into the past, **before** Nash's theorem was proved, and consider the famous **minimax theorem of Von Neumann** (1928). This theorem says precisely that a pair of max-min and min-max strategies giving equal payoff to player A **always** exist in a zero-sum game, thus giving us the existence of an equilibrium via proposition 4.28. Now we know this is just a consequence of the stronger theorem proved by Nash, but nevertheless this theorem by Von Neumann was proved several years before Nash's theorem and hence advanced the theory on zero-sum games. Its classical importance means it is something we should be aware of at the very least!

4.5 The Minimax Theorem of Von Neumann (1928)

Theorem 4.29 (Von Neumann (1928))**.** *In a finite zero-sum game then*

$$
\max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\} \right\} = v = \min_{\beta \in \mathbb{B}_S} \left\{ \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\} \right\},
$$

*where v is the unique max-min payoff to A (and cost to player B), called the **value** of the game.*

Proof. Let the finite game have size $n \times m$. First let's consider the following two optimisation problems related to the game:

$$
\text{Minimise } v \text{ subject to } \; g(a_i, \beta) \leq v, \quad \beta \in \mathbb{B}_S, i = 1, 2, \cdots, n, \tag{16}
$$

and

$$
\text{Maximise } u \text{ subject to } \; g(\alpha, b_j) \geq u, \quad \alpha \in \mathbb{A}_S, j = 1, 2, \cdots, m. \tag{17}
$$

The constraints $g(a_i, \beta) \le v$ mean that, by playing strategy β, player B loses (or pays) at most v in every **row** of the game. Similarly, the constraints $g(\alpha, b_j) \ge u$ mean that, by playing strategy α, player A gets at least u in every **column** of the game.

Suppose now that β and v have been found optimal in problem (16). That means that at least one of the n inequalities $g(a_i, \beta) \le v$, $i = 1, 2, \cdots, n$ must hold as **equality**, since if all inequalities were strict then it must be possible to continue to reduce the value of v further. That means that v fulfills

$$v = \max_{a \in A_S} \{g(a, \beta)\} = \max_{\alpha \in \mathbb{A}_S} \{g(\alpha, \beta)\},$$

where we have used proposition 4.27 to obtain the second equality. Also note that an optimal choice of β in problem (16) is precisely a min-max strategy by definition. Similarly, in problem (17), for an optimal pair α, u we have

$$u = \min_{b \in B_S} \{g(\alpha, b)\} = \min_{\beta \in \mathbb{B}_S} \{g(\alpha, \beta)\},$$

where α is a max-min strategy. These optimal values for u and v must exist and be unique (this is because we are dealing with finite payoffs meaning u and v take possible values in non-empty compact sets).

Claim: $u = v$.
This then also means that 'max-min = min-max' and proves the theorem. So it remains to show that this claim is true.

Proof of claim: We will prove the claim by induction on the value of $n + m$.

Base Case: Let $n + m = 2$. Then it is simple to check that we must have $u = v$ in this case (exercise).

Consider now a game G of dimension $n \times m$ and let α, β, u and v be optimal solutions to (16) and (17). Note that

$$u = \min_{b \in B_S} \{g(\alpha, b)\} \le g(\alpha, \beta) \le \max_{a \in A_S} \{g(a, \beta)\} = v. \tag{18}$$

Now if all the inequalities in problems (16) and (17) hold as equalities, then

$$\begin{aligned} g(a_i, \beta) &= v \quad \text{for all } i, \\ \text{and} \quad g(\alpha, b_j) &= u \quad \text{for all } j, \end{aligned}$$

so then we also have equalities throughout (18) and hence $u = v$. So, let's assume that at least one of the inequalities in $g(a_i, \beta) \le v$, $i = 1, 2, \cdots, n$ is **strict** (it could be the case that this is not true and rather at least one of the inequalities in $g(\alpha, b_j) \ge u$, $j = 1, 2, \cdots, m$ is strict. The proof of this case follows in a similar manner). Let it be the kth row where this inequality is strict. Then, letting $\beta = (q_1, q_2, \cdots, q_m)$, we have

$$\sum_{j=1}^{m} g(a_k, b_j) q_j < v. \tag{19}$$

Now let G' be the game G with the kth row (representing pure strategy a_k for A) deleted. Let β', v' be the optimal solution to (16) using game G' instead of G and let α', u' be the optimal solution to (17) using game G'. Now it must be the case that

$$u' \leq u, \quad \text{and} \quad v' \leq v. \tag{20}$$

This is because, by removing a row of the game G, i.e. removing a pure strategy for player A, the resulting game G' can only become **less favourable** than the game G for player A. Now, by our **inductive hypothesis** we have that $u' = v'$ in G'.

We claim now however, that $v' = v$. Suppose instead that $v' < v$. Let $\varepsilon \in (0, 1]$ and consider the strategy $\beta^* = \beta(1-\varepsilon) + \varepsilon\beta'$, which is a valid mixed strategy in $\mathbb{B}_S$ because $\mathbb{B}_S$ is a convex set and β^* is a convex combination of strategies in $\mathbb{B}_S$. We will show that this strategy used in game G gives possible v^* values less than v and hence is in contradiction with with minimality of the value v in game G. First note that

$$\begin{aligned} g(a_i, \beta^*) &= g(a_i, \beta(1-\varepsilon) + \varepsilon\beta') \\ &= (1-\varepsilon)g(a_i, \beta) + \varepsilon g(a_i, \beta') \\ &\leq (1-\varepsilon)v + \varepsilon v' \\ &= v - \varepsilon(v - v') < v, \end{aligned}$$

for $i = 1, 2, \cdots, k-1, k+1, \cdots, n$, since we assumed $v' < v$. Now, writing $\beta^* = (q_1^*, q_2^*, \cdots, q_m^*)$ so that $q_j^* = q_j(1-\varepsilon) + \varepsilon q_j'$, and using strategy β^* in (19) shows that

$$\begin{aligned} \sum_{j=1}^{m} g(a_k, b_j) q_j^* &= \sum_{j=1}^{m} g(a_k, b_j) \left(q_j(1-\varepsilon) + \varepsilon q_j' \right) \\ &= \sum_{j=1}^{m} g(a_k, b_j) q_j + \varepsilon \sum_{j=1}^{m} g(a_k, b_j)(q_j' - q_j) < v, \end{aligned}$$

provided we take ε sufficiently small. This indeed means that there are strategies $\beta^* \in \mathbb{B}_S$ and values $v^* < v$ so that $g(a_i, \beta^*) \leq v^*$. This is in contradiction with the minimality of v in (16), meaning that it must be false that $v' < v$ and hence we must have $v' = v$. Finally, using this fact in (20) leads to

$$u' \leq u \leq v = v' = u',$$

and so therefore $u = v$ which was what we needed to show. $\square$

Remarks:

- This proof using induction comes from the text by Von Stengel (2021) where it originally came from Loomis (1946). The original proof of the minimax theorem by Von Neumann was done geometrically and involved considering a separating hyperplane between convex sets. We have decided to show this inductive proof to avoid needing to delve into the technical geometric aspects required for Von Neumann's proof.

- We have now witnessed the unique properties of equilibria in zero-sum games: firstly that an equilibrium strategy is the same as a max-min strategy (for player A and a min-max strategy for player B) and is therefore **independent** of what the other player does, and secondly that the equilibrium payoff to the players is unique, v, and called the **value** of the game (this is the payoff player A receives while player B gets $-v$).

4.6 Finding Solutions in Small Zero-Sum Games

We have seen examples of how we might solve small games so far throughout the course. Importantly, when the game is **not** zero-sum, it usually matters that we find **all** solutions (all equilibria) to the game, since, in general, different equilibria give different payoffs to each player. However, as we have seen, when a game is zero-sum, then it has a **unique value**, v, which is the same for **any** pair of max-min/min-max strategies. Let's first prove a proposition, then you'll see where we are going with this argument.

Proposition 4.30. *Consider two zero-sum games G and G', where G' is obtained from G by deleting a weakly dominated strategy of one of the players. Then any equilibrium of G' is an equilibrium of G, and G and G' have the **same value**.*

Proof. An equilibrium of G' defines a mixed strategy pair in G. This is also an equilibrium of G if there is no profitable deviation to a pure strategy in G. The only deviation possible which is not available in G' would be to the weakly dominated strategy, but then if that deviation was profitable then deviating to the strategy which weakly dominated this would be profitable too. But this strategy is in G' and hence deviating to it cannot be profitable otherwise this would contradict the fact we had an equilibrium initially. Hence this equilibrium must also be an equilibrium in G. By proposition 4.28 the payoff at equilibrium in a zero-sum game is the value of the game, which is therefore the same in both games. □

Remarks:

- There is an important consequence of this proposition. That if we are only interested in the value of a zero-sum game (which is usually the case, since **any** equilibrium gives this payoff, so we usually only care about finding **a solution** and **the value** when it comes to zero-sum games), then we **can delete weakly dominated strategies** (and iterate this), to help us more easily find an equilibrium and hence the value of the game. We may, of course, lose some of the possible max-min and min-max strategies by doing this however.

- In the same manner, if we encounter any groups of payoff equivalent strategies in a zero-sum game, we may also delete **all but one** of these strategies from each group and any equilibrium found in the resulting game will also be an equilibrium in the original game with the same value.

Let's see this concept in action in a game we would struggle to solve otherwise.

4.6.1 Raid on the high profile Military targets

Five high profile military targets are connected to each other as shown in figure 43.

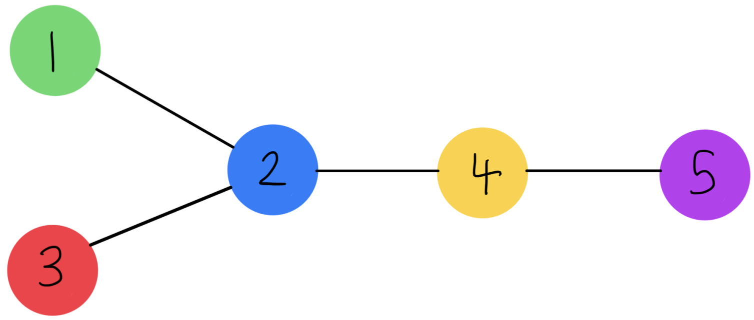

Figure 43: The connections between the high profile military targets.

Army A plans to attack one of these five targets. Army B will chose to defend one of these targets, but due to the excellent infrastructure connecting the targets, any target connected immediately (i.e. is one edge away) to army B's chosen target is also defended. These choices of each army occur simultaneously.

If A attacks a defended target they lose a point (think of the cost as a loss of forces as the defenders are well equipped to destroy the invaders) and B gains a point (some value from destruction of the enemy). If A attacks an **un**defended target, A gets a point (from the destruction of a key enemy military target) and B loses a point (from the loss of the target).

Question: Determine a pair of optimal strategies for each army and deduce the value of this attack to army A. As chief military advisor of army A should you proceed with the attack?

Solution: Let's represent pure strategy a_i as attack target i and b_j as defend target j for $i, j = 1, 2, 3, 4, 5$. Then the normal form of the game is given in figure 44.

				B		
		b_1	b_2	b_3	b_4	b_5
	a_1	-1	-1	1	1	1
	a_2	-1	-1	-1	-1	1
A	a_3	1	-1	-1	1	1
	a_4	1	-1	1	-1	-1
	a_5	1	1	1	-1	-1

Figure 44: Normal form of the military targets game.

No strategies are dominated here, but since we are only concerned with finding **a** solution to the game we can apply proposition 4.30 which says we can delete weakly dominated strategies to aid us with this. Deleting a_2 (since a_1 weakly dominates it), a_4 (since a_5 weakly dominates it), b_1 (since b_2 weakly dominates it), b_3 (since b_2 weakly dominates it) and b_5 (since b_4 weakly dominates it), then we arrive at figure 45.

		B	
		b_2	b_4
A	a_1	-1	1
	a_3	-1	1
	a_5	1	-1

Figure 45: The military targets game after the deletion of several weakly dominated strategies.

To continue our analysis we make note of our recent remark. Strategies a_1 and a_3 are **payoff equivalent**. This means we can delete one of them without changing the value of the game and any solution we find will be a solution of the original game. So, let's delete a_3, giving figure 46.

		B	
		b_2	b_4
A	a_1	-1	1
	a_5	1	-1

Figure 46: Final simplified form of the military targets game.

From here, due to the symmetry of the game, we can guess that $\alpha = (1/2, 1/2)$ and $\beta = (1/2, 1/2)$ are a pair of strategies in equilibrium (exercise: check this!). This means that in the whole game

$$\hat{\alpha} = \left(\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right),$$
$$\text{and} \quad \hat{\beta} = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0\right),$$

are a pair of optimal strategies (max-min and min-max) for army A and army B respectively. The **value** of the game is $g(\hat{\alpha}, \hat{\beta}) = 0$. This means that as army A's military advisor you would have no incentive to declare this attack: it isn't costly to you, but isn't worth anything to you.

Chapter 5: Cooperative Games

Many ‘games’ in the real world are cooperative where the players in the game can generally do better by mutually agreeing on playing specific strategies, i.e. bargaining to agree on a collective good result for all. For example, consider the situation between the buyer and seller of an asset, which could be as small as haggling over the price of a pair of sunglasses with a street vendor to as large as international treaties over weapons or foreign aid. These are game theoretic situations because it is useful to consider the position of the other players, where players generally have something to gain by reaching an agreement.

In his undergraduate thesis, later published in 1950, John Nash proposed a set of ‘axioms’ to allow players to reach a fair **bargaining solution** to a two-player cooperative game. Nash then showed these axioms always gave rise to a unique solution, which is generally straightforward to calculate. Throughout this chapter we will focus on two-player cooperative games in strategic form. We will state and discuss Nash’s bargaining axioms and prove that they indeed give rise to a unique solution to the cooperative game in question. We will then find this bargaining solution in an example cooperative game.

Players in **cooperative game theory** are assumed to be able to discuss the game and to **agree** in some way on the strategies they will play and then to adhere to that agreement (there is no option of going back on this agreement, think of all agreements as being binding). We will discuss which agreements are possible and how players benefit from them, but we will not discuss how the players might enforce that these agreements are upheld.

5.1 Bargaining Sets

Consider the example game in figure 47.

		B	
		b_1	b_2
A	a_1	2,3	0,4
	a_2	3,0	1,1

Figure 47: An example game being played cooperatively.

Player A’s strategy a_2 dominates a_1, and b_2 dominates b_1 for player B, so the game being played non-cooperatively is dominance solvable with unique equilibrium given by (a_2, b_2) with a payoff of 1 to both players. We have encountered games like this earlier in the course, consider the prisoner’s dilemma or

Cournot duopoly for example, where if players could play cooperatively then they could mutually do better than they would by playing their equilibrium strategies. By agreeing to play the strategy pair (a_1, b_1) for instance, they would receive payoffs of 2 and 3 respectively, better for both players than the equilibrium payoffs of 1.

Suppose now we allow our players to do just this: to talk to each other about the game beforehand and agree on a certain course of action that they intend to follow through. Then our first step in any cooperative game should be to determine what our players can actually achieve, or stand to gain, by collaborating. To do this, we identify a **bargaining set** of possible payoff pairs that the players can bargain over.

As usual, letting $A_S = \{a_1, a_2, \cdots, a_n\}$ and $B_S = \{b_1, b_2, \cdots, b_m\}$, then we first plot the points P_{ij}, $i = 1, \cdots, n$, $j = 1, \cdots, m$ represented by each of the pairs of pure strategy payoffs from the game in strategic form, i.e.

$$P_{ij} = (g_A(a_i, b_j), g_B(a_i, b_j)).$$

For our example game from figure 47, we plot the points $(a_1, b_1) = (2, 3)$, $(a_1, b_2) = (0, 4)$, $(a_2, b_1) = (3, 0)$ and $(a_2, b_2) = (1, 1)$. See figure 48 where these points are plotted with red crosses.

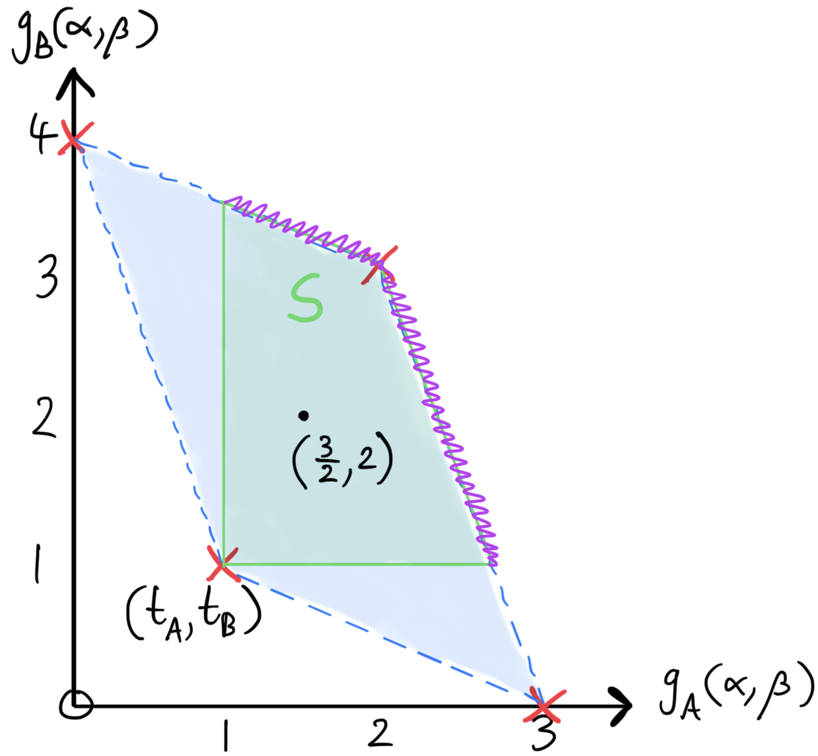

Figure 48: The construction of the bargaining set S, shaded in green, for our example game.

The **convex hull** (the set of convex combinations) of these payoff pairs is shaded in blue in figure 48. Playing cooperatively, the players can agree on **any point** within the convex hull by use of a **joint** mixed strategy that the players have decided to play. For example, the payoff pair $(3/2, 2)$ as marked in figure 48, would be the expected payoff each player would receive if they agreed to toss a fair coin, and, if it showed heads to play strategies (a_1, b_1), or if it showed tails to play the strategies (a_2, b_2). This coin toss and agreed upon retroactive action is part of the player's binding agreement.

Question: Do you think this would be a good agreement for the players? Are there better ones?

As mentioned, in this way it is possible for the players to devise some joint mixed strategy such that **any** payoff pair in the convex hull can be selected. We can thus work with payoffs and think about which payoff pair the players should agree upon, then later we can determine what strategy should be employed to achieve this.

Remark: Note that now, in a two-player **cooperative** game, the **entire convex hull** forms the possible payoff pairs available to the players. This is, in general, different to the payoff set we learned how to determine when the game is being played non-cooperatively which didn't contain the entire convex hull since convex combinations between payoff pairs not in the same row or column were not possible. Playing cooperatively however, by use of these joint mixed strategies, these convex combinations are now possible.

So is the bargaining set just the entirety of this convex hull? Not quite. Although **all** these payoff pairs are possible to attain, not all of them are reasonable. We always assume that the players have the option to not agree in advance. For example, player A would **not agree** that both players chose the strategy pair (a_1, b_2) with certainty, as this would give player A a 0 payoff, something they could **guarantee** doing better than if playing non-cooperatively!

Indeed, without any agreement with player B, player A can always just use their **max-min** strategy **using their own payoffs** (which here is to play strategy a_2) guaranteeing them a payoff of at least 1, completely irrespective of whatever player B does. Likewise, player B can use their **max-min** strategy **using their own payoffs** (which is to play b_2) to guarantee themselves a payoff of at least 1, irrespective of what player A does. Thus, it only makes sense for the players to bargain over payoff pairs that are at least as good as what they could obtain by simply playing their max-min strategies on their own payoffs. Now we can define the bargaining set.

Definition 5.31. The **bargaining set** (sometimes called the **negotiation set**), S, resulting from a two-player game in strategic form is the **convex hull** of all payoff pairs, with the added constraint that for all $(x, y) \in S$, then

$$x \geq t_A,$$
$$y \geq t_B,$$

where t_A is the max-min payoff of player A, i.e.

$$t_A = \max_{\alpha \in \mathbb{A}_S} \left\{ \min_{\beta \in \mathbb{B}_S} \{g_A(\alpha, \beta)\} \right\},$$

known as A's **security level** (or **threat level**). Similarly, t_B is the max-min payoff of player B, i.e.

$$t_B = \max_{\beta \in \mathbb{B}_S} \left\{ \min_{\alpha \in \mathbb{A}_S} \{g_B(\alpha, \beta)\} \right\},$$

known as B's **security level** (or **threat level**). The point (t_A, t_B) is called the **threat point**.

Remark: In the construction of this bargaining set, S, from a cooperative game, the threat point (t_A, t_B) need not itself be an element of S (this is a rare occurrence but possible: see problem sheet 5 for an example where this is the case). For this reason, whenever we describe a bargaining set S, we also state the associated threat point (t_A, t_B) with it, since this point turns out to be crucial in the bargaining solution we will find.

In our example, figure 48, the threat point is marked at $(1, 1)$ (as these are the max-min payoffs for each player) and the resulting bargaining set, S, is shaded in green. Note there is a final piece of information displayed on figure 48 which is the purple shading on the 'top-rightmost' border of the bargaining set. We will explain what this shading is in the next section after we see the bargaining axioms, but you might already be able to see why this region of the bargaining set is important (if not take a think about why now)!

5.2 Bargaining Axioms

Given a two-player cooperative game, we can now determine a bargaining set for the players to negotiate over. To attempt to obtain a 'solution' for such situations, we now present a list of axioms that any solution to our cooperative game should abide. We will show later that these axioms lead to a **unique bargaining solution**.

Definition 5.32 (Axioms for bargaining solution)**.** For a bargaining set S with threat point (t_A, t_B), a **Nash bargaining solution** $N(S)$ is a pair (X, Y) such that:

(a). $(X, Y) \in S$;

(b). The solution (X, Y) is **pareto-optimal**; i.e. for all $(x, y) \in S$, if $x \geq X$ and $y \geq Y$, then $(x, y) = (X, Y)$;

(c). The solution is **invariant under payoff scaling**, meaning, if $a, c > 0$ and $b, d \in \mathbb{R}$ and we define S' to be the bargaining set

$$S' = \{(ax + b, cy + d) : (x, y) \in S\},$$

with threat point $(at_A + b, ct_B + d)$, then $N(S') = (aX + b, cY + d)$;

(d). The solution is **symmetry preserving**, i.e. if $t_A = t_B$ and $(x, y) \in S$ implies $(y, x) \in S$ then we must have $X = Y$;

(e). The solution is **independent of irrelevant alternatives**. This means if S, T are bargaining sets with the same threat point and $S \subset T$, then either $N(T) \notin S$ or $N(T) = N(S)$.

Let's discuss a couple of these axioms to understand them better. Pareto-optimality means that it should not be possible to improve the solution for one player without harming the other. This means that the Nash bargaining solution is always found on the so-called **pareto-optimal frontier**, which graphically looks like the upper-right boarder of the bargaining set. The pareto-optimal frontier is shaded in purple back in figure 48 for our example game. Schematics of typical cases are given in figures 49 and 50, again shaded in purple.

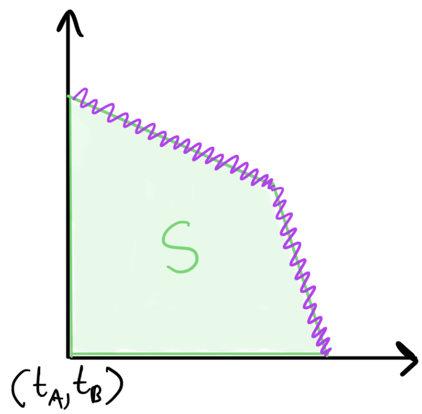

Figure 49: A typical looking bargaining set with its pareto-optimal frontier shaded purple.

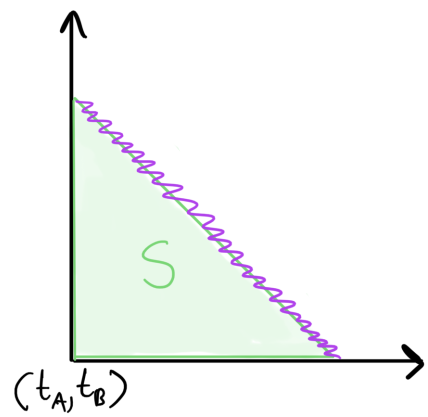

Figure 50: Another typical looking bargaining set with its pareto-optimal frontier shaded purple.

Remark: Note here that we have defined the bargaining set as the whole of the convex hull, with the added constraint that points must lie anywhere above and to the right of the threat point (including directly above or directly to the right). However, be aware that in some texts the bargaining set is defined as the pareto-optimal frontier of our bargaining set, i.e. the purple shaded regions of the green sets in the figures we have drawn so far.

Independence of irrelevant alternatives means that if our bargaining set S is extended to a new set T without changing the threat point, then either T has a solution $N(T)$ which is a new point in T and not in S, or it has the same solution as S, $N(S)$. In other words, new alternatives can't cause our solution to change to something else that we could have agreed upon before. Figure 51 shows a schematic of a bargaining set S shaded in green and its solution labelled $N(S)$. Set S is then extended to set T to include the extra region shaded in red. For set T, $N(T)$ could be a possible solution, but P could not be, since it is still within set S and not equal to $N(S)$.

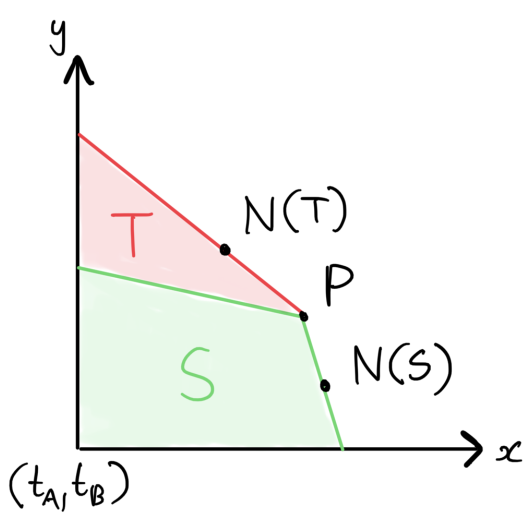

Figure 51: A schematic to illustrate the independence of irrelevant alternatives.

Remark: This axiom is generally considered to be the most contentious. The other axioms are properties that are widely accepted and in many people's opinion it would be strange should a solution **not** fulfil them. The validity of the independence of irrelevant alternatives is sometimes questioned. In this course we will assume it holds.

5.3 The Nash Bargaining Solution

We now prove that the proposed bargaining axioms give rise to a unique solution of the cooperative game.

Theorem 5.33. *Under the Nash bargaining axioms (a) - (e), every bargaining set S that contains a point (x, y) with $x > t_A$, $y > t_B$ has a* ***unique*** *solution $N(S) = (X, Y)$, which is obtained as the unique point $(x, y) \in S$ that* ***maximises*** *the* ***Nash product****, given by*

$$(x - t_A)(y - t_B).$$

Proof. Let us begin with a bargaining set S containing at least one point (x, y) where $x > t_A$ and $y > t_B$. Figure 52 gives a picture which we will follow to help us visualise the proof.

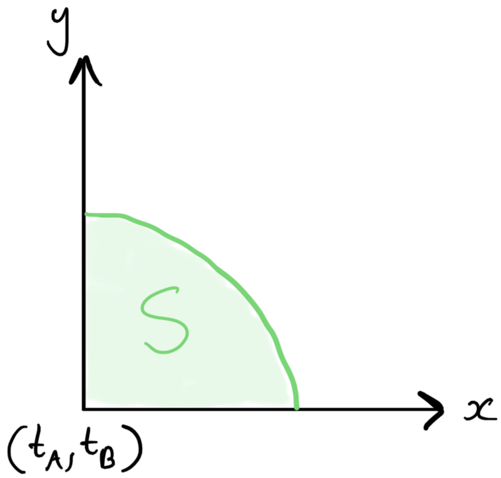

Figure 52: Our bargaining set S.

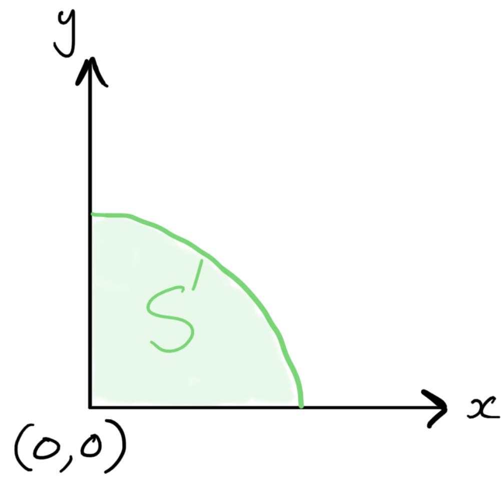

Figure 53: The translated set S'.

Now consider the set $S' = \{(x - t_A, y - t_B) : (x, y) \in S\}$, shown in figure 53, which simply translates set S so that the threat point is at the origin. In set S', maximising the Nash product then amounts to maximising xy. Let (X, Y) be a payoff pair where this maximisation occurs. Note that $XY > 0$. Figure 54 shows part of the hyperbola $xy = c$ for the maximal value c such that the hyperbola still intersects S'.

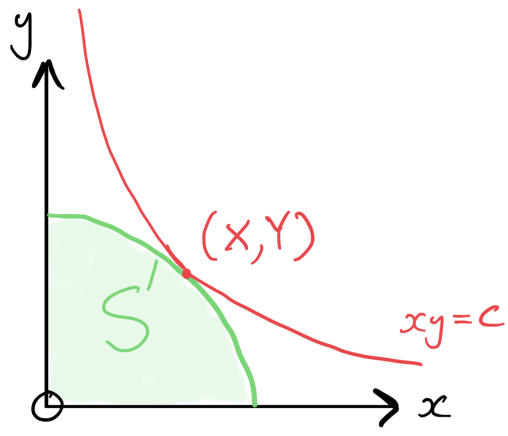

Figure 54: Set S' and the hyperbola $xy = c$.

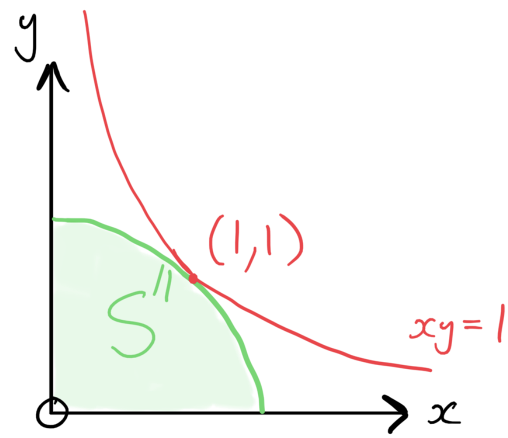

Figure 55: Set S'' and the hyperbola $xy = 1$.

Now we re-scale the payoffs so that $(X, Y) = (1, 1)$. To do this we replace set S' with set S'' given by

$$S'' = \left\{ \left(\frac{x}{X}, \frac{y}{Y} \right) : (x, y) \in S' \right\},$$

which is shown in figure 55. Note now that this 'maximal' hyperbola is rescaled to $xy = 1$. Now consider the set T given by

$$T = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 2\},$$

which we sketch along with set S'' in figure 56.

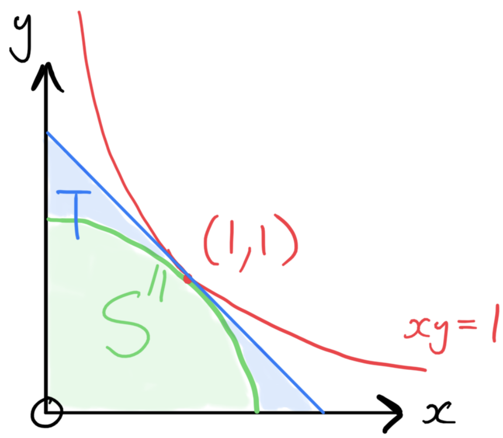

Figure 56: Set S'' and set T.

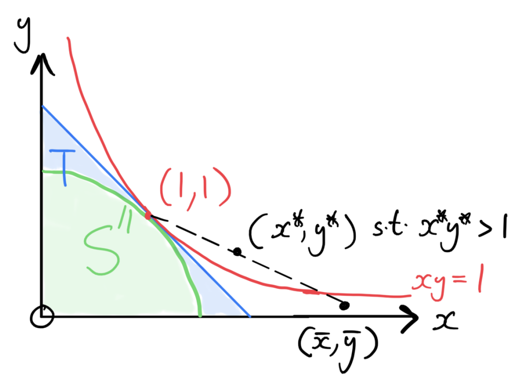

Figure 57: Some points lie to the right of the hyperbola.

For the bargaining set T, the solution is $N(T) = (1, 1)$, because T is a symmetric set and $(1, 1)$ is the only symmetric point on the pareto-frontier of T. Now we claim that $S'' \subseteq T$ (which might look obvious from figure 56, but we prove more formally now). Suppose this were **not** the case, i.e. there was a point $(\overline{x}, \overline{y}) \in S''$, but $(\overline{x}, \overline{y}) \notin T$. Well $\overline{x} \geq 0$ and $\overline{y} \geq 0$, so then it must be true that $\overline{x} + \overline{y} > 2$. Then, as shown in figure 57, even if the Nash product $\overline{xy}$ is not greater than 1, then the Nash product $x^* y^*$ of some suitable convex combination of $(1, 1)$ and $(\overline{x}, \overline{y})$, namely

$$(x^*, y^*) = (1 - \varepsilon)(1, 1) + \varepsilon(\overline{x}, \overline{y}), \tag{21}$$

for some small $\varepsilon > 0$, **is** larger than 1. This must be true since the line segment connecting $(1,1)$ and $(\overline{x},\overline{y})$ must intersect the hyperbola $xy = 1$ (otherwise the point $(\overline{x},\overline{y})$ would not lie to the right of the boundary $\overline{x}+\overline{y} = 2$ of T) and thus there are points (x^*,y^*) on this line to the right hand side of the hyperbola where $x^*y^* > 1$. But $(x^*,y^*) \in S''$ since S'' is a convex set: i.e. if $(1,1)$ and $(\overline{x},\overline{y})$ are in S'', then so is (x^*,y^*), a convex combination of these points. But this is a contradiction, since $x^*y^* > 1$ but S'' was created such that the maximal Nash product of any point in S'' was 1 (at $(1,1)$). Hence $S'' \subseteq T$, which means that $N(T) = (1,1) \in S''$, so by the independence of irrelevant alternatives, we **must** have $N(S'') = N(T)$.

We are almost done. This indeed means that our bargaining axioms have given a solution to S which comes from maximising the Nash product, but it remains to show that this solution is unique.

Uniqueness follows in precisely the same way that we just argued the point $(\overline{x},\overline{y})$ caused there to be points in S'' on the right hand side of the hyperbola. If there **is** another point $(x,1/x) \in S''$ on the hyperbola with $x \neq 1$ then, since S'' is a convex set, the line segment that connects $(1,1)$ and $(x,1/x)$ is also all part of S'', but these interior points have a larger Nash product (as they lie to the **right** of the hyperbola). Hence our bargaining solution is also unique. $\square$

Remarks:

- One can show that (21) leads to $x^*y^* > 1$ for some ε algebraically should the geometric argument we have used in the proof not satisfy them. A similar algebraic argument can also be used to show the uniqueness property.

- We make the assumption that S has a point (x,y) with $x > t_A$ and $y > t_B$ otherwise all payoffs x of player A or y of player B (or both) must be such that $x = t_A$ or $y = t_B$. The Nash product will then always be zero so it has no unique maximum (unless S is just the point (t_A,t_B)) and thus the theorem would not hold. If this is the case however, it is trivial to determine $(X,Y) = N(S)$ since the pareto-optimal frontier will just consist of a single point, which is then precisely the Nash bargaining solution.

Example: Let's find the Nash bargaining solution for the example in figure 47 from the start of the chapter. The bargaining set S along with the threat point and pareto-optimal frontier is again shown here in figure 58.

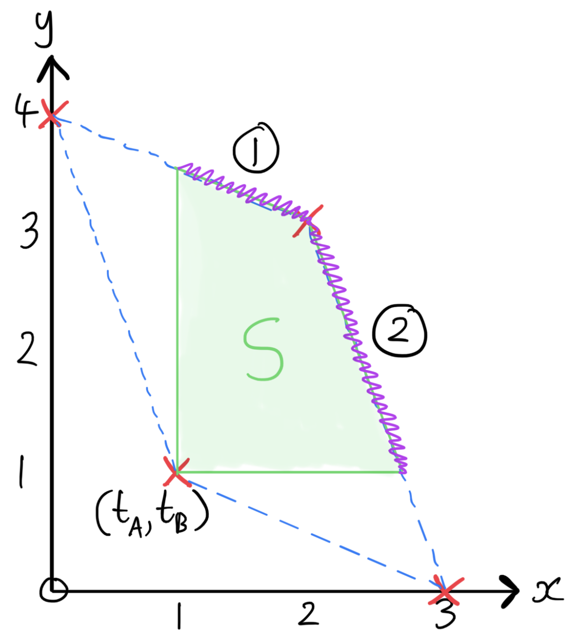

Figure 58: The bargaining set, threat point and pareto-optimal frontier for our example cooperative game.

The pareto-optimal frontier consists of two line segments, labelled as segment ① and segment ② in figure 58. We need to maximise the Nash product over both of these to determine its maximum over the whole pareto-frontier.

Line segment ① has equation $x + 2y = 8$, using this in the Nash product gives

$$\begin{aligned}(x - t_A)(y - t_B) &= (x-1)(y-1) \\ &= (x-1)(3 - \frac{1}{2}x) \\ &= -\frac{1}{2}x^2 + \frac{7}{2}x - 3,\end{aligned}$$

which, by completing the square or use of calculus, takes its maximum value when $x = 7/2$. Now we need to take care. Notice on our line segment ① we have $1 \leq x \leq 2$, but $7/2$ lies outside of this range, so the maximum of the Nash product **on** line segment ① must occur instead at an end point, which is clearly at the end point when $x = 2$, since this lies closer to $7/2$ than the other end point. This then gives maximised Nash product $(2-1)(3-1) = 2$.

Now we need to check line segment ② where $3x + y = 9$. The Nash product becomes $-3x^2 + 11x - 8$, which takes its maximum when $x = 11/6$, which again lies off our line segment where $2 \leq x \leq 8/3$. Thus the maximal Nash product must occur at the end point when $x = 2$.

Having now checked for the maximal value of the Nash product everywhere on our pareto-optimal frontier we can conclude when $x = 2$ and $y = 3$ we have our Nash bargaining solution. Going back to the game in strategic form in figure 47, we see this pair of payoffs for our players corresponds to them agreeing to play the pure strategy pair (a_1, b_1).

Remark: Note that it will not always be the case that the solution gives a pair of pure strategies. Often a joint mixed strategy for the players is required. See problem sheet 5 for examples of this.

Chapter 6: Congestion Games

6.1 An Introduction to Congestion Games

Congestion games (or routing games) arise from the context of a shared resource, such as roads or connections to a server, needing to be used by several people where the use of the resource becomes more costly as more people continue to use it. Multiple users in a congestion game will be trying to minimise this cost whilst using the resource.

Throughout this chapter we will generally consider a congestion game in the context of traffic on roads where the users of the roads (our players) will be trying to get from one location to another in as cost effective a way as possible. Figure 59 shows an example of a **congestion network**, which takes the form of a **directed graph** (or **digraph**). This congestion network has two nodes, A and B, and two **directed** edges that connect A to B.

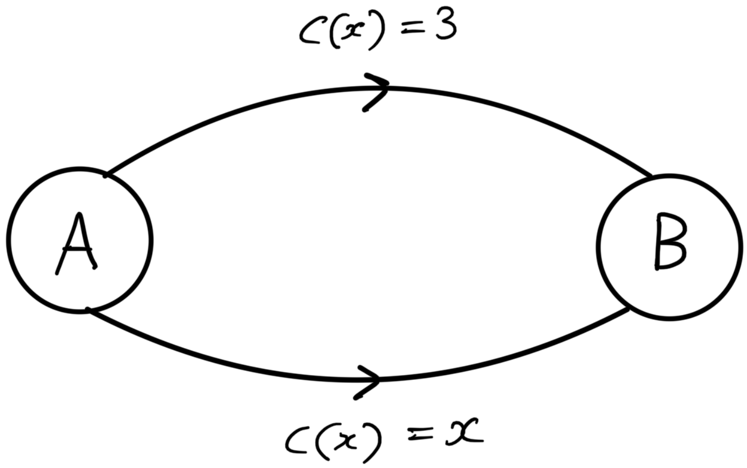

Figure 59: A congestion network of two nodes and two edges.

Suppose we now have several users of this network who all want to travel from node A to node B and they can choose any route from A to B they wish. As shown in figure 59, each edge of the network has an associated **cost function**, $c(x)$, which describes how costly it is for **each user** to use that edge when there is a **flow** (or **load**) of x users using the edge. This cost could represent the travel time, or some monetary cost from tolls, or a combination of these things. This function is the same for all users of the network.

Suppose the network in figure 59 is now used by three users, each of whom can choose to travel from A to B using the top edge, or the bottom edge. Let's look at each possibility:

- If all three use the top edge, then each pays a cost of $c(x) = 3$. Notice that the bottom edge is currently 'worth' $c(x) = 1$ to each user, i.e if they were to switch to using that edge, they would only pay a cost of 1 for the travel. For this reason (as we will define formally shortly), we say that this configuration is **not in equilibrium**, because **at least one** user can improve their cost by switching to the bottom edge (changing their current strategy). You can hopefully see where we are going with this!
- Secondly, two users could choose the top edge, with one user on the bottom edge. The users on the

top edge would pay a cost of $c(x) = 3$ each, but the user on the bottom edge pays just $c(x) = 1$ for their travel. Although the bottom user is currently happily paying $c(x) = 1$ for their journey, each top user is incentivised to switch to the bottom edge where they would then only pay $c(x) = 2$, less than what they currently pay.

- What about two users on the bottom edge and one on the top edge. Well the user on the top edge pays $c(x) = 3$, while the two users on the bottom edge each pay $c(x) = 2$. However, notice now the crucial difference to the first two cases: in this case, every user is content in the sense that no individual user can change their path to **improve** their cost. Indeed, if either user on the bottom path changes to the top path then instead they increase their cost, and if the user on the top edge changes to the bottom edge their cost remains the same (at $c(x) = 3$). There is no **strict** incentive for any individual to change their path and as so the situation is in **equilibrium** (again we will define this formally in the context of congestion games soon).
- Lastly, if all three users are on the bottom edge, all pay $c(x) = 3$, but again no individual has incentive to deviate as the top edge doesn't offer anything less, so this is again an equilibrium.

So our example network has **two equilibria** when there are three users.

Exercise: How many equilibria are there (and what are they) if our network has $N > 3$ many users?

The underlying behaviour of the users that we have been considering is known as '**selfish routing**': that is, every user of the network tries to find the route that is the least costly for them. An equilibrium is then obtained when every user is in a situation where they cannot find a better route (a route costing less) for themselves by changing just their own route.

An interesting question we will investigate throughout this chapter is the following: is selfish routing ever **socially optimal**, measured by the average cost per user of the network. In our example, if we have either one or two users on the top edge then we are in a social optimum with the average cost per user at $7/3$. Notice that the first of these cases is also an equilibrium. The second equilibrium we found, with all three users on the lower path, however leads to an average cost per user of 3, **not** a social optimum. So we see that selfish routing can lead to situations where the average user of the network is worse off than they would be in a **social optimal flow**.

6.2 Atomic and Splittable Flow

We make a remark here on the differences between two major types of flow considered in a congestion game. In the last example we considered **three** individual users and thought about their choices through the network. This is a type of **atomic flow** (think of it informally as the discrete case) through a congestion game, where we think of users being atoms that cannot be split up any further. This meant that in the last example we were not considering cases such as $5/2$ users taking the top edge with $1/2$ user taking the bottom edge, since this makes no meaningful sense if we are considering individual users.

Alternatively, we could consider the congestion networks with a continuum of users flowing throughout them, known as **splittable flow**. In this case we think of a flow unit as just some 'mass' of users of which **any** parts can be routed along different paths, rather than as just one individual user. Using our example network from figure 59 and considering a splittable flow model with 3 units of flow (or we might say 'of mass 3') we can try to find the equilibria and compare their differences/similarities with those found using atomic flow.

Indeed, by doing this, one will find now that only the case where all 3 units flow through the bottom edge is in equilibrium. Even if 2.999 units flow through the bottom edge with 0.001 units through the top edge, it is still beneficial for any one user of the 0.001 fraction of users to deviate to using the bottom edge (think of every individual user as being negligibly small in this continuum model of splittable flow).

Remarks:

- As it turns out, splittable flow tends to lead to simpler calculations because the minor variations in equilibria (that we find with discrete users in the network) don't occur. In our example we will just find the equilibria with 3 users on the bottom edge for instance.
- Throughout the rest of this chapter, unless stated otherwise, we will consider the **atomic flow** model with individual users on our congestion network to be our default case! We will explicitly state when we will consider splittable flow.

6.3 The Pigou Network

Let's now look at a very similar example to that we started with known as the Pigou network (in fact the example we started with is also a Pigou network), named after the economist Arthur Pigou, see figure 60.

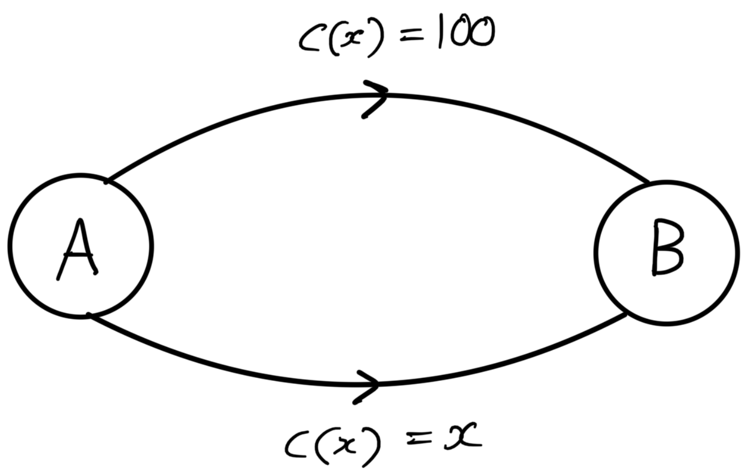

Figure 60: A Pigou network.

Assume that there are 100 users who want to travel through this network from A to B. In this case we find two equilibria: one where all 100 users are using the bottom edge, and another with 99 users on the bottom edge and one user on the top edge. In each of these cases let's find the average cost per user. Clearly, in the first case the average cost per user is 100, and in the second case the average cost is $(99 \times 99 + 1 \times 100)/100 = 99.01$.

Let's now take an alternative approach and consider what the average cost per user would be in a **social optimum**, i.e a distribution of users which gives the **lowest** possible average cost per user (rather than selfish routing). To determine this, let's suppose that y users take the top edge, so $100 - y$ users take the bottom edge. Then the average cost per user can be calculated as

$$\text{Average Cost} = \frac{100y + (100 - y)^2}{100} = \frac{1}{100}y^2 - y + 100.$$

This is a quadratic function of y, whose minimum value is found when $y = 50$ (note that if this was non-integer, then in the case of atomic flow, we would need to check the integers either side to find which integer value gave the minimum. This is unnecessary in splittable flow). When $y = 50$ then we get an average cost per user equal to 75, considerably lower than those average costs found in equilibrium cases. Notice critically however that although the average cost per user here is significantly better, the $50 - 50$ split found is not an equilibrium. The users on the path of cost $c(x) = 100$ are incentivised to switch and lower their individual costs, but in turn this will raise the average cost per user.

This has much real world context. The social optimum found here would need to be enforced somehow should we want this solution to pertain, for example, we could enforce a congestion charge or extra tolls that may maintain this. The Pigou network shows that selfish routing can easily lead to situations where the average user is worse off than in a social optimum.

6.4 The Braess Paradox

We now investigate a famous congestion game that illustrates a strange effect of users acting according to selfish routing. Figures 61 and 62 show two congestion networks, where in the second network an additional edge of zero cost has been constructed between nodes B and C. In both networks we assume that there are 100 users who want to travel from node A to node D.

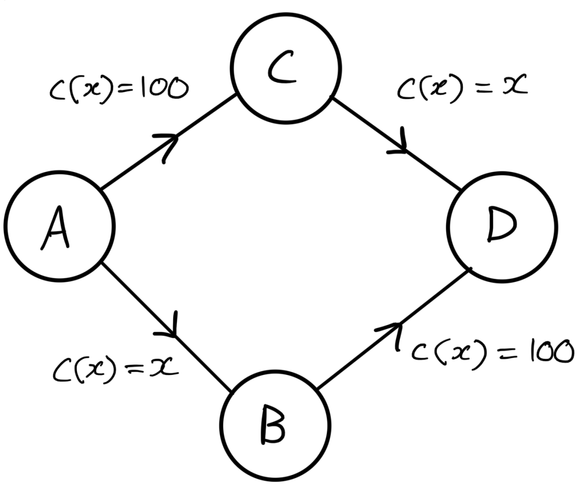

Figure 61: The original congestion network.

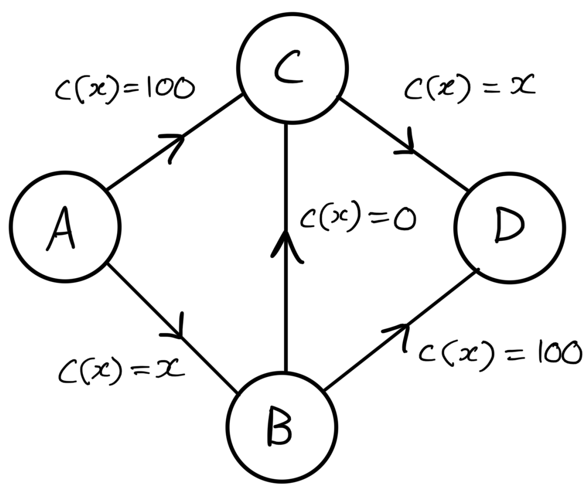

Figure 62: The modified network with an additional shortcut.

Let's start by finding the equilibria and the social optimal flow for the network in figure 61, without the extra edge. Suppose y users take the bottom path (from $A \to B \to D$), so then $100 - y$ users are taking the upper path (from $A \to C \to D$).

The cost per user of the bottom path is then $100 + y$ while the cost of the top path is $200 - y$ for each user. We have an equilibrium when these quantities are equal, or 1 apart, which here only occurs when $y = 50$, meaning that we have an equal split of 50 users on each route. Note that in this congestion network, this is also **equal** to the socially optimal flow, giving an average cost per user of 150.

Now let's consider the second network in figure 62 with the new shortcut opened up. As before, suppose y users take route AB, with z users ($z \leq y$) taking the new route BC, so then $y - z$ users take route BD. As before $100 - y$ users take route AC, but then $100 - y + z$ users will take route CD. There are now **three** possible paths through the network from A to D. They are labelled and drawn in figure 63.

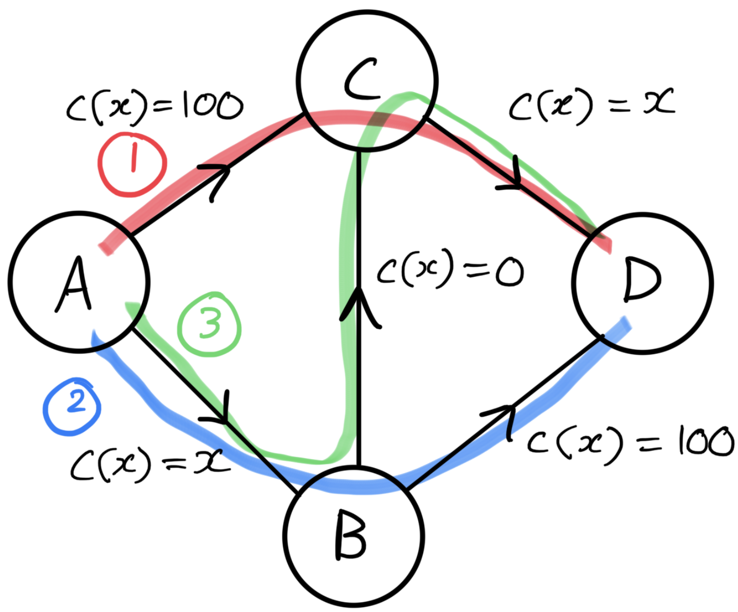

Figure 63: The three possible paths through the network from A to D.

The cost per user of each path is then:

$$\begin{aligned}
\text{cost}_{1} &= 100 + (100 - y + z) = 200 - y + z, \\
\text{cost}_{2} &= y + 100, \\
\text{cost}_{3} &= y + 0 + (100 - y + z) = 100 + z.
\end{aligned}$$

This time we have an equilibrium when either $y = z = 100$, $y = 100$ and $z = 99$, $y = z = 99$ or finally $y = 99$ and $z = 98$ (these are all the cases where the three quantities are within 1 of each other, meaning no individual has an incentive to deviate to a different path).

Note that the average cost per user in any of these equilibria becomes equal to, or just below 200 (feel free to calculate the precise values in each case if you want). Now recall, in the first congestion game in figure 61, before the extra capacity shortcut was opened, this average cost in equilibrium was 150. This has increased significantly by adding in the extra zero cost route!

This phenomenon is known as the **Braess paradox** (1968), which shows that adding extra capacity to a network can actually make congestion worse in equilibrium rather than better (as one might normally expect). Similarly, by closing down a path in some networks, congestion can be reduced. Anecdotally this paradox has been seen in the real world where, in 2002 in Seoul (Korea), a motorway was dismantled and replaced with a park. Subsequently traffic in the area improved (of course this may be due to many other different factors, but is interesting nonetheless)!

6.5 Components of a Congestion Game

Let's now formalise the concept of a congestion game and the notion of equilibrium that we have been using in this chapter so far. First we define a congestion network.

Definition 6.34. A **congestion network** has the following components:

- A finite set of nodes;
- A finite collection of directed edges where each edge, e, is an ordered pair, written AB, from some node A to some node B. Note that multiple edges from one node to another node are allowed (e.g the two edges from A to B in the Pigou network);
- Each edge, e, has an associated **cost function**, $c_e(x)$, giving a value when there are x users on edge e which corresponds to the cost each user pays for using edge e. Each cost function should be **weakly increasing**, meaning that $x \leq y \Rightarrow c_e(x) \leq c_e(y)$.

These then form a digraph (a directed graph) with cost functions for each edge.

Now we define a congestion game.

Definition 6.35. To form a **congestion game** we need the following components:

- A **congestion network**;
- A number N of users of the network (or a total 'mass' of users if using a splittable flow model). Each user $i = 1, 2, \cdots, N$ has an origin node O_i and destination node D_i which can be different or the same for all users.
- A **strategy** of user i is a **path** P_i (a sequence of distinct edges) from O_i to D_i. Given a strategy P_i for each user i, the **flow** (or **load**) through an edge, e, of the congestion network is defined as

  $$f_e = |\{i : e \in P_i\}|,$$

  i.e. the flow corresponds to the number of users on edge e.
- The **cost** to user i for their strategy choice P_i is then

  $$\text{Cost}_i(P_i) = \sum_{e \in P_i} c_e(f_e).$$

A congestion network with the above components forms a congestion game where each user tries to minimise their cost.

Now let us define the notion of a best response in this context.

Definition 6.36. We say P_i is a **best response** for user i against the strategies P_j, $j = 1, 2, \cdots, i-1, i+1, \cdots, N$ for the other users if

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i \cap Q_i} c_e(f_e) + \sum_{e \in Q_i / P_i} c_e(f_e + 1), \tag{22}$$

holds **for every** possible alternative strategy Q_i of user i.

Remarks:

- As usual, the concept of best response only concerns unilateral deviations (deviations of only one user).
- The inequality (22) simply says that

$$\mathrm{Cost}_i(P_i) \leq \mathrm{Cost}_i(Q_i),$$

  holds for all different possible strategies Q_i, i.e that P_i gives the lowest possible cost to user i. To see why this is the case, consider what happens when user i changes their strategy P_i to strategy Q_i. These are just two different paths through the network. Thus, any edges in common between the two paths, i.e. any edges in the set $P_i \cap Q_i$ will have their **flow unchanged** when player i changes from strategy P_i to strategy Q_i. Hence player i pays a cost of

$$\sum_{e \in P_i \cap Q_i} c_e(f_e),$$

  for all of these edges. However, for the remaining edges in the new path Q_i that were not part of the old path P_i then player i has now joined the flow that was previously on them, increasing it by 1 and hence paying a cost of

$$\sum_{e \in Q_i / P_i} c_e(f_e + 1),$$

  for these edges.

Finally, we can now define the concept of equilibrium in a congestion game.

Definition 6.37. In a congestion game with N users, the strategies P_1, P_2, $\cdots$, P_N of all N users define an **equilibrium** if each strategy is a best response to the other strategies, i.e. if (22) holds for all $i = 1, 2, \ldots, N$.

6.6 Existence of an Equilibrium in a Congestion Game

We will now prove that a congestion game always has an equilibrium. We will prove this for the case of atomic flow which has been the main focus of this chapter. The result also holds for splittable flow and you are encouraged to try to prove this result on problem set 6.

Theorem 6.38. *Every congestion game has at least one equilibrium.*

Proof. Let the N strategies of all users of the network be P_1 (for user 1), P_2 (for user 2), $\cdots$, P_N (for user N). These then define a flow, f_e, on each edge, e, of the network. This is the number of users i with $e \in P_i$.

Let's then denote this collection of flows on each edge by f; the flow induced across the whole network by strategies $P_1, \cdots, P_N$. Now we define the following function of this 'global' flow f:

$$\phi(f) = \sum_e \left(c_e(1) + c_e(2) + \cdots + c_e(f_e)\right). \tag{23}$$

We will understand more about what this function is doing later in the proof! Now suppose user i changes from strategy P_i to strategy Q_i. Let's call the resulting new global flow f^*.

Claim:

$$\phi(f^*) - \phi(f) = \sum_{e \in Q_i} c_e(f_e^*) - \sum_{e \in P_i} c_e(f_e), \tag{24}$$

where f_e^* refers to the values of f_e when i plays Q_i instead of P_i.

Proof of Claim: First note that the right-hand-side of (24) is precisely the difference in cost to user i between their strategies Q_i and P_i. Now the first term on the right-hand-side of (24) can be written as

$$\sum_{e \in Q_i} c_e(f_e^*) = \sum_{e \in P_i \cap Q_i} c_e(f_e) + \sum_{e \in Q_i / P_i} c_e(f_e + 1).$$

Player i's cost for the original flow, f, (i.e. the second term on the right-hand-side of (24)) can be expressed similarly as:

$$\sum_{e \in P_i} c_e(f_e) = \sum_{e \in P_i \cap Q_i} c_e(f_e) + \sum_{e \in P_i / Q_i} c_e(f_e).$$

This means we can rewrite the right-hand-side of (24) as:

$$\sum_{e \in Q_i} c_e(f_e^*) - \sum_{e \in P_i} c_e(f_e) = \sum_{e \in Q_i / P_i} c_e(f_e + 1) - \sum_{e \in P_i / Q_i} c_e(f_e).$$

Now by the definition of $\phi(f)$ in (23), we can see that this quantity above indeed equals the left-hand-side of (24). To see this, consider what happens when user i changes their strategy from P_i to Q_i. The flow on any new edge $e \in Q_i/P_i$ increases from f_e to $f_e + 1$, hence adding the term $c_e(f_e + 1)$ to the sum in the definition of $\phi(f)$ in (23). Similarly, for any edge $e \in P_i/Q_i$, the flow f_e^* is reduced from f_e down to $f_e - 1$, so the term $c_e(f_e)$ has to be subtracted from the sum in the definition of $\phi(f)$. Hence the claim is proved and (24) holds.

Now (24) says that **the change in cost to any user** i when they change their strategy (from P_i to Q_i), and hence the flow, is the same as the change in the function $\phi(f)$. So now consider a flow that achieves the **minimum value** of $\phi(f)$ for all possible flows f that result from all possible strategy combinations $P_1, P_2, \cdots, P_N$ of all users. There are finitely many such strategy combinations and thus flows, so this minimum value exists. At this minimum, no change in flow can then reduce $\phi(f)$ any further, and hence **no individual user** can reduce their cost. Thus, by definition, the players' strategies at such a minimum of $\phi(f)$ define an equilibrium. $\square$

Remarks:

- $\phi(f) = \sum_e (c_e(1) + c_e(2) + \cdots + c_e(f_e))$ is called a **potential function**. It has the remarkable property (as used in the proof) that any change of strategy by a single user incurs a cost change for that user which equals precisely the change in value of ϕ. This single function ϕ achieves this property for all the users simultaneously!

- A game for which such a potential function can be introduced is called a **potential game**, which, by the previous proof, has an equilibrium which can be found as the minimum of the potential function.

- To prove the theorem for splittable flow we need to slightly modify the potential function so it becomes an integral rather than a sum. The rest of the proof follows similarly.

6.7 The Price of Anarchy

We now return to one of our earlier questions about selfish routing and the social optimum. We give a definition of the price of anarchy (POA) and provide some bounds on its maximal value.

Definition 6.39. For a given congestion game, the **price of anarchy** (POA) is defined as

$$\text{POA} = \frac{\text{Worst average cost per user in any equilibrium}}{\text{Average cost per user in social optimum}}.$$

The use of the word 'anarchy' in this definition refers to the idea of the selfishness of the users when each user cares only about minimising their own cost.

Example: In the Pigou network with 100 users from section 6.3, the price of anarchy is $100/75 = 4/3$.

There have been some proven bounds on the price of anarchy. We mention two bounds here for the case where all cost functions in the congestion game are **affine**, that is, functions of the form $c(x) = ax + b$ for non-negative values of a and b.

Proposition 6.40. *For non-splittable, or atomic flow, the price of anarchy in* ***any*** *congestion game with affine cost functions is* ***at most*** $\frac{5}{2}$.

Proof. Problem set 6. ☐

Proposition 6.41. *For splittable flow the price of anarchy in* ***any*** *congestion game with affine cost functions is* ***at most*** $\frac{4}{3}$.

Proof. Problem set 6. ☐

6.8 Example Game: Getting lunch at Imperial

Let's look at an example game to end this chapter. It's lunchtime on the South Kensington campus at Imperial College and 10 mathematics students need to get some lunch, see the congestion game in figure 64.

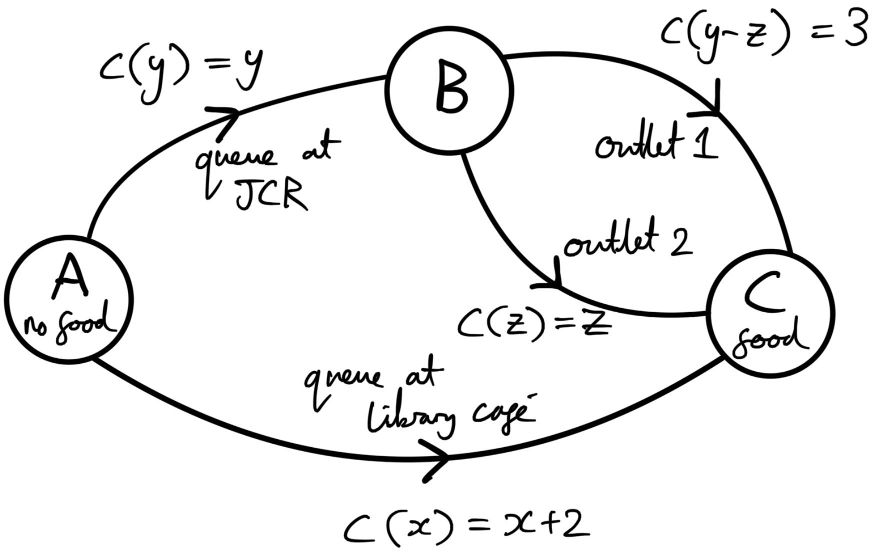

Figure 64: The possible options to get food.

Our 10 students all need to travel from node A (no food) to node C (food). They value the time it takes them to obtain their lunch (the cost of travelling from A to C) and wish to minimise this cost, shown for each edge in the network in figure 64 where $0 \leq z \leq y$.

Question:

a). Find all equilibria of this congestion game and their resulting average cost per user.

b). For the same network, find the equilibrium when x, y and z can be fractional quantities for a total flow of 10 units from A to C (i.e. find the equilibrium when the flow is **splittable**).

c). Determine the social optimum flow of this congestion game and hence the price of anarchy for the game under both atomic and splittable flow models.

Solution:

a). An equilibrium can be found by starting with some flow and then changing users who can improve their cost. A natural starting point might be an equal split between the library cafe and the JCR, i.e. setting $x = y = 5$. Now z of the y users who arrive at node B will use the lower of the two edges starting at B, where the only possibilities are $z = 2$ or $z = 3$ because for $z \leq 1$ an extra user would have a lower cost 1 or 2 rather than 3, and for $z \geq 4$ the top edge with constant cost 3 would be better.

If $x = y = 5$ and $z = 2$ then the $x + z = 7$ users would incur a cost of 7 and the remaining $y - z = 3$ users a cost of 8, and neither could improve, so this is an equilibrium with average cost 7.3.

If $x = y = 5$ and $z = 3$ then the x users would incur a cost of 7 and all y users a cost of 8, and again neither could improve, so this is an equilibrium with average cost 7.5.

If $x = 6$, $y = 4$ and $z = 2$, then the x users have a cost of 8 and the z users a cost of 6, but increasing y and z by 1 would raise the cost to 8. Hence, none of the x users have an incentive to deviate. Similarly, the route of $y - z = 2$ users via the constant cost edge has cost 7 and again does not offer an improvement for any of the x users. The resulting expected cost is $(6\times8+2\times6+2\times7)/10 = 7.4$.

If $x = 6$, $y = 4$ and $z = 3$, then this is another equilibrium with $y = 4$ users on the top route with cost 7 and $x = 6$ on the bottom edge with cost 8 and expected cost 7.6.

For $x \geq 7$ the bottom edge has cost at least 9 but the top route at most cost 6 which would be increased to 7 with another user, and one of the x users would change. Similarly, $x \leq 4$ means a cost of at most 6 on the bottom edge and at least $y + z \geq 6 + 2 = 8$ on the top route, with an incentive for any one of the y users to move to the bottom edge. So these are not part of an equilibrium.

b). With splittable flow, the edge with z users will be filled to $z = 3$ (if $y \geq 3$) because otherwise any fraction of users of the edge BC with constant cost 3 could improve their cost by switching, with a slight increase of z. So the top route always has a cost $y + 3$ and the bottom edge costs $x + 2$ which have to be equal in a flow equilibrium. Because $x + y = 10$, this means $x = 5.5$, $y = 4.5$ and $z = 3$. The resulting cost is 7.5 for all users.

c). We can write down

$$\begin{aligned}\mathrm{cost}_{\mathrm{top}} &= y + 3,\\ \mathrm{cost}_{\mathrm{middle}} &= y + z,\\ \mathrm{cost}_{\mathrm{bottom}} &= x + 2 = 12 - y,\end{aligned}$$

where we have used $x + y = 10$ in the last equation. This means that the average cost per user on the network is given by

$$\begin{aligned}&= \frac{1}{10}\left[(y - z)(y + 3) + z(y + z) + (10 - y)(12 - y)\right]\\ &= \frac{(y - \frac{19}{4})^2}{5} + \frac{(z - \frac{3}{2})^2}{10} + \frac{581}{80}.\end{aligned}$$

Clearly for **splittable** flow this quantity is minimised when $y = \frac{19}{4}$, $z = \frac{3}{2}$ and $x = \frac{21}{4}$, giving an average cost per user of 7.2625 or $\frac{581}{80}$. For atomic flow we need to try integer values close to these and we find that $x = 5$, $y = 5$ and $z = 1$ or 2, results in a minimal average cost per user of 7.3. Hence the price of anarchy for each flow type is:

$$\begin{aligned}\mathrm{POA}_{\mathrm{splittable}} &= \frac{7.5}{7.2625} \approx 1.03,\\ \mathrm{POA}_{\mathrm{atomic}} &= \frac{7.6}{7.3} \approx 1.04.\end{aligned}$$

Chapter 7: Combinatorial Games

In this chapter we focus on **combinatorial games**. These are two-player, perfect information, sequential (players alternate moves) games like Chess, Draughts (Checkers), Nim and Go. There are no chance moves like rolling dice or shuffling cards and play ends with either a win for one player and a loss for the other, or a draw. Assuming that our players are playing the game optimally, we will try to answer the question of which player will win given a specific game position.

Combinatorial games have a typically finite number of positions with well defined rules on how to arrive at the next position. They come in two distinct types:

- **Impartial Games**: where the available moves in a given game position do not depend on whose turn it is to move, e.g Nim, or
- **Partizan Games**: where the available moves in a given game position depend on whose turn it is to move, e.g Chess (players typically have different coloured pieces in games like this and can only act using their coloured pieces).

In this chapter we focus on developing a detailed theory for **impartial games**, where the game of Nim turns out to play a central role in the theory.

7.0.1 The Ending Condition

A further feature of these games is the existence of what is known as the **ending condition**: meaning that the rules are designed such that play of the game **will** come to an end due to some player being eventually unable to move. The games cannot continue indefinitely. This is an important property of these games and we will refer to it regularly.

7.0.2 The Normal Play Convention

Throughout this chapter we will consider combinatorial games played under the **normal play** convention: this means that the player who is unable to move (due to reaching the ending condition) **loses** the game. Alternatively, one can consider the convention of **misére play**, where the player who is unable to move (due to the ending condition) is declared the winner. As mentioned, we will use the normal play convention in everything we do, which turns out to synergise more easily with the theory we will develop.

7.1 Nim and Impartial Games

Let's take a look at the game of Nim. This will help us to familiarise ourselves with some of the concepts of impartial games that we have discussed so far. Later we will see that the game of Nim plays a central role in the theory of impartial games.

A game position in Nim consists of some **piles** of **tokens**, and a **move** in a given game position for either player constitutes removing some (at least one, possibly all) tokens from **one** of the remaining piles. Following our normal play convention, the last player to move (i.e. the player who takes the last token(s) away from the game) wins the game.

This means that the game can only end in a win/loss for either player. The fact that any starting game position contains a finite number of tokens, and any move removes at least one token, means that the game will last a finite number of turns and hence the ending condition (where the game ends since one player can't move as there will eventually be no tokens remaining in play) holds.

Figure 65 shows a Nim position with three piles of sizes 3, 2 and 1. We use coloured tokens (here red, blue and green) to represent the pile a given token belongs to.

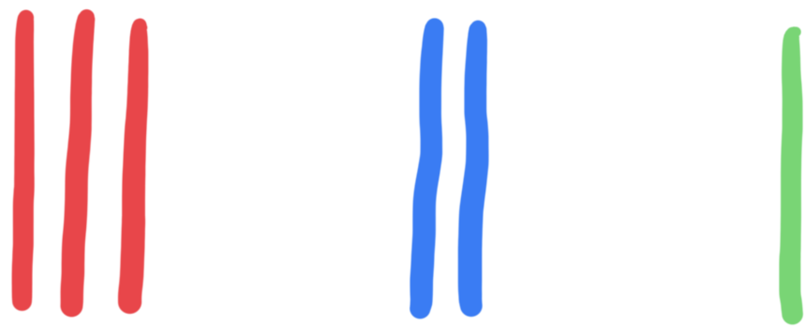

Figure 65: A Nim position with three piles of sizes 3, 2 and 1.

One possible move in this position, for example, could be to remove two red tokens (or two tokens from the pile of size 3), another possible move could be to remove both blue tokens (or all tokens from the pile of size 2). We can represent each of these moves by:

$$\begin{aligned} & 3,2,1 \longrightarrow 1,2,1, \\ \text{and} \quad & 3,2,1 \longrightarrow 3,1. \end{aligned}$$

Note that, since moves can be made in **any** pile, the order of the piles doesn't matter, so $1,2,1$ is equivalent to $1,1,2$ for example (it doesn't matter if we swapped the colour of the reds and blues).

We make a general definition:

Definition 7.42. An **option** of a game position in a combinatorial game is a position that can be reached by a single move from the player to move in the current position.

By use of this definition, we can refer to $1,2,1$ and $3,1$ as two possible **options** from the position $3,2,1$. Figure 66 shows all six possible options from the Nim game position $3,2,1$.

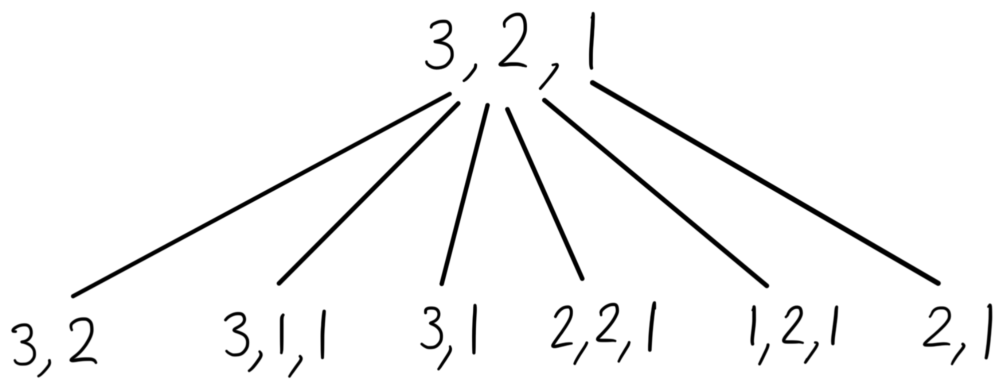

Figure 66: Nim position $3, 2, 1$ with all its options shown branching from it.

If we want to, we can represent the entire **game tree** (extensive form of the game) in this way by continuing to draw out all possible options from every position. Note that in doing this some positions will be repeated in the tree; we do not draw moves back up the tree to these positions from multiple predecessors though, we repeat the position in the game tree so that every position has a unique history of moves.

7.1.1 Winning and Losing Positions

In impartial games, game positions belong to one of two possible classes:

- **Winning positions:** meaning that the current player (with optimal play) can force a win with a suitable first 'winning move' (and subsequent 'winning moves' at all later positions); or
- **Losing positions:** meaning that **every** move from the current position leads to a winning position of the other player, who can then force a win with optimal play, so then the current player will lose.

These are the only possibilities since, in an impartial game, the available moves are independent of the player to move (by definition). If you are not yet convinced then do not fear, we will prove that this is indeed the case shortly. Moreover, although the above defines what constitutes a winning/losing position, we will take our usual definition of these positions as something slightly different once we have proved that all positions are either winning or losing (you can think of the above as our definition 'for now').

Knowing that in impartial games positions are either winning or losing, let's try to classify the game position $3, 2, 1$ in Nim. To help us do this, consider the following observation of how we can play Nim optimally when we have a Nim position with **at most two piles**:

- If there are **no** Nim piles left this **defines** a losing position under the normal play convention.
- If there is **one** Nim pile left then we are in a **winning** position; the final winning move is to remove the entire pile.
- Two Nim piles are a winning position **if and only if** the piles are of different sizes; the winning move is then to equalize their sizes by suitably reducing the larger pile, creating a losing position of two equal sized piles. This is losing since the next move either removes a pile, leaving a single pile which is winning, or it reduces one of the piles leaving two piles of different sizes, a winning position since

one can once again equalize the sizes of the piles from this position. Since all options from the original position lead to winning positions, the position with two Nim piles of the same size is losing.

Every option of $3, 2, 1$ is a winning position, because in each the player whose turn it is has a move to a two-pile losing position, see figure 67.

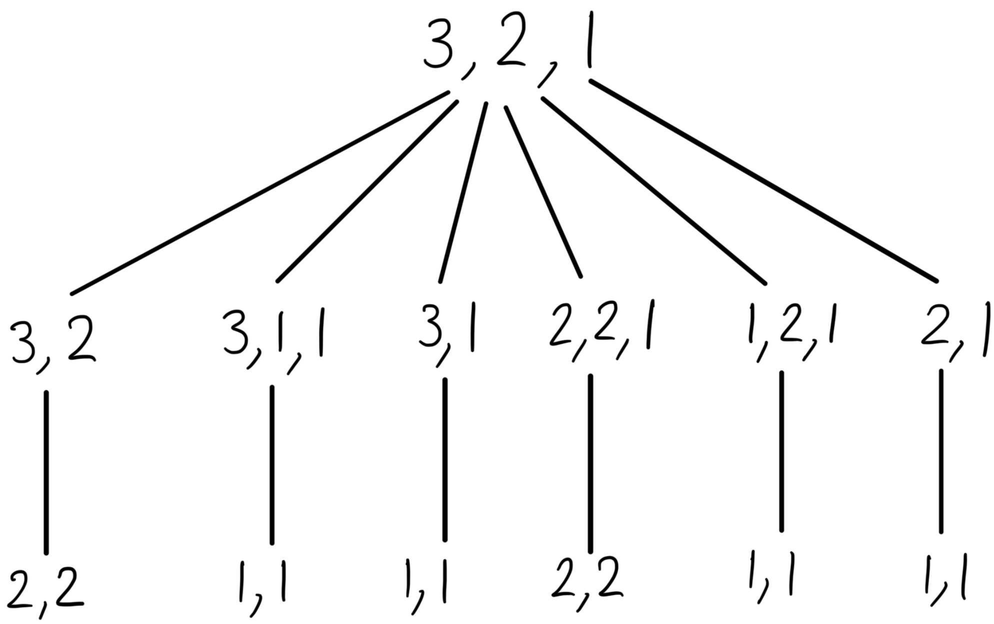

Figure 67: Nim position $3, 2, 1$ with all its options shown branching from it. A winning move from each of these options is then shown.

Figure 67 shows sequences of two moves from position $3, 2, 1$ to another position of two equal sized piles which is **losing** as argued earlier. We only show one move for each second move as, in optimal play, only such a **winning move** will be played. We see then that $3, 2, 1$ is a **losing position** in Nim. Let's formalise the argument used in showing that two Nim piles were winning if and only if they were unequal.

Proposition 7.43. *In an impartial game, a game position is losing if and only if all its options are winning positions. A game position is winning if and only if at least one of its options is a losing position; moving to that position is a winning move.*

Proof. We will return to this proof shortly once we have understood the concept of **top-down induction**. □

Remarks:

- This proposition applies 'vacuously' to the game position that has no options at all (a lost position), since every option (of which there are none) is winning, hence the position is indeed losing.
- This proposition actually acts as what we will take as our **definition** of winning and losing positions. This is not a circular definition, but a **recursive** one because the options of a game position are always **simpler** (in the sense of being closer to the end of play in the game) than the current position.

7.1.2 Bouton's solution for Nim

The game of Nim was solved by Charles Bouton, an American mathematican from Harvard, in 1901. In this subsection we demonstrate Bouton's solution strategy using the formidable Nim position $4, 5, 9, 14$. Bouton's key insight was to write the pile sizes in **binary**. This is done for our game position $4, 5, 9, 14$ in figure 68.

		8	4	2	1
4	=	0	1	0	0
5	=	0	1	0	1
9	=	1	0	0	1
14	=	1	1	1	0
Nim Sum	=	0	1	1	0

Figure 68: Binary representation of the Nim position $4, 5, 9, 14$. The **Nim sum** for the position is shown at the bottom of the table.

The bottom row of the table in figure 68 shows the **Nim sum** of the position $4, 5, 9, 14$. The Nim sum is defined to be **addition modulo 2, without carry** for each column of the table. More visually, this turns out to correspond identically to the fact that the Nim sum has a 0 in a column if the number of 1's in that column is even, or a 1 in a column if the number of 1's in the column is odd.

We now prove why the Nim sum of a position is important in determining if a position is winning or losing.

Proposition 7.44. *A Nim position is losing if and only if the Nim sum equals zero for all columns in the binary representation of the position; such a position is called a* ***zero position*** *(Note: this then means that a position is winning if and only if the Nim sum $\neq 0$ for all columns).*

Proof. To show this is the case, according to proposition 7.43, we need to show:

(a). Every move from a zero position leads to a non-zero position which is therefore winning; and

(b). From every winning position (Nim sum $\neq 0$) there is a move to reach a zero position.

To show (a) holds, note that a move changes exactly one pile, which corresponds to exactly one row in the table of binary numbers. Since the player must take some tokens, the binary number in this row must change in some way, causing at least one 0 to change to 1 or 1 to change to 0 in the row. This means that, in the respective columns of where these changes occurred, the Nim sum must change from 0 to 1. This must be the case since **only** one row had its values changed and so the odd/even parity of 1's down the columns that had values changed must have been changed.

For (b) we need to show that there is always at least one winning move for any position with non-zero Nim sum. We show that this is indeed the case via the following algorithm:

- Choose the leftmost '1' column in the Nim sum (which exists because not all columns are zero), which represents the highest power of 2 used in the binary representation of the Nim sum. In our example with position $4, 5, 9, 14$ this is $4 = 2^2$, see figure 68.

- By definition, there are an odd number of piles which use this power of two in their binary representation. In our example there are three piles of sizes $4, 5, 14$ that use the 2^2. Choose **any one** of the corresponding rows; in our example, let's choose the row of pile size 5.

- In the chosen row, 'flip' the binary digits for **each** 1 in the Nim-sum. In our example this is columns $4 = 2^2$ and $2 = 2^1$. The largest of the changed powers of two in this row was originally a digit 1 which is changed to 0, so even if all subsequent digits were changed from 0 to 1, the resulting binary number after all flips have taken place is a new, **smaller** integer (possibly zero, meaning that the move is to remove the entire pile). A winning move is then to reduce the pile to this smaller size; in our example from 5 to 3.

As a result of this process, the new Nim sum equals zero, so we have reached a zero position as required. □

Exercise: As outlined throughout the proof, one winning move in our game position was to move from $4, 5, 9, 14$ to $4, 3, 9, 14$ by removing 2 tokens from the pile of size 5. There are a further two other possible winning moves we could have made instead. Find them.

7.2 Top-down Induction

In any combinatorial game, any position could represent the starting position of another game (for example, the Nim position $3, 2, 1$ could be the start of the game, or could have been reached from the $4, 5, 9, 14$ game for instance). Because of this, when working with combinatorial games, we will often use the term **game** to mean **game position**. This means that every game, G, has finitely many options G_1, G_2, $\cdots$, G_m, which themselves are games. These are reached from G by one of the allowed moves in G, see figure 69.

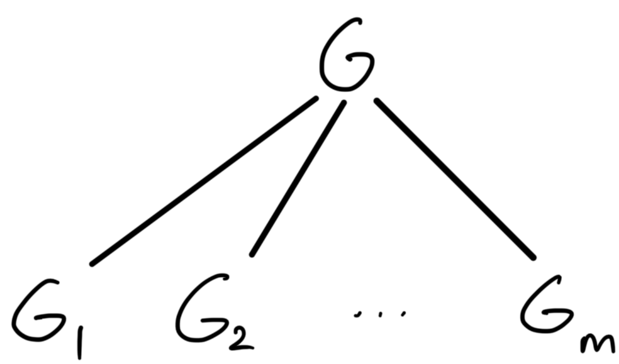

Figure 69: A game G with its m options: games G_1, $\cdots$, G_m.

The only exception to this is if $m = 0$, or, in other words, the game G has **no** options. We denote this game by 0, which, under the normal play convention, defines a losing game.

Before formalising the notion of top-down induction, we first spend a moment to take a short detour and discuss partial and total orders which will be important in our upcoming theory.

7.2.1 Partial and Total Orders

Definition 7.45. A binary relation $\simeq$ on a set S is called a **partial order** if, for all $x, y, z \in S$, we have:

- $\simeq$ is transitive: if $x \simeq y$ and $y \simeq z \Rightarrow x \simeq z$;
- $\simeq$ is refelxive: $x \simeq x$;
- $\simeq$ is antisymmetric: $x \simeq y$ and $y \simeq x \Rightarrow x = y$.

If, in addition to the above, we also have for all $x, y \in S$:

- $\simeq$ is total: $x \simeq y$ or $y \simeq x$,

then we call $\simeq$ a **total order**.

Definition 7.46. For a given partial order $\simeq$ on a set S, we define the **strict order** $\sim$ corresponding to $\simeq$ by; for all $x, y \in S$:

$$x \sim y \quad \Leftrightarrow \quad x \simeq y \text{ and } x \neq y.$$

Definition 7.47. An element $x \in S$ is called **minimal** if $\nexists y \in S$ such that $y \sim x$.

Examples:

- Any set S of real numbers has the familiar relation $\leq$ as a total order. $<$ is the corresponding strict order here.
- An important partial order is the set inclusion $\subseteq$ on a set S of sets. $A \subseteq B$ means set A is a subset of B (allowing $A = B$). In general this is a partial order and **not** a total order since two sets can be incomparable, for example, $A = \{1, 2\}$ and $B = \{2, 3\}$. Here $\subset$ is the corresponding strict order where $A \subset B$ means A is a proper subset of B (e.g $A = \{1\}$, $B = \{1, 2\}$).

7.2.2 Back to Top-Down Induction

Let's now return to developing our theory for impartial games.

Definition 7.48. Consider a set S of games, defined by a starting game and **all** the games that can be reached from it via any sequence of moves of the players. For two games; $G, H \in S$, we call H **simpler** than G, denoted with the binary relation $H \leq G$, if there is a sequence of moves that leads from G to H. We allow for $G = H$ where this sequence is empty.

Remarks:

- We will use the definition just given for the set S (i.e that S consists of some initial game and all games that can be reached from it) regularly throughout this chapter, so much so that it will become monotonous to repeat this definition without giving it some sort of name. Instead of naming it, whenever we refer to set S henceforth, take S to be the set as defined in the previous definition. We will make things clear where we change this notion.

- The corresponding strict ordering is denoted by $<$ where $H < G$ means that H is simpler than G but we do not allow for $H = G$.

Proposition 7.49. *The binary relation $\leq$ (which we say as 'is simpler than') on a set S of games (as discussed above) forms a partial order.*

Proof. Problem Set 7. □

Proposition 7.50. *Every non-empty subset, T, of S has a minimal element.*

Proof. Suppose that this was **not** the case, i.e there was a non-empty subset T of S **without** a minimal element. Take a game $G \in T$. Now because G is **not** minimal, there exists $H \in T$ such that $H < G$, so there is some sequence of moves from G to H.

Similarly, H is not minimal, so another game in T can be reached from H by some sequence of moves. Continuing in this manner means we either run out of games left in T (so the last one is minimal) or create an infinite sequence of moves which contradicts the **ending condition**. Note that we cannot re-visit any game in this process, say G, since this implies that $G < G$ which is nonsense. Thus we reach a contradiction and so T has a minimal element. □

We now prove a sweeping result which is true for all sets and partial orders when every subset of our set has a minimal element. We will proceed to use the theorem in the case where the set S is our usual set of games and the partial order is of being simpler than $\leq$.

Theorem 7.51 (Top-down Induction)**.** *Consider a set S with a partial order $\simeq$ such that every non-empty subset of S has a minimal element. Let $P(x)$ be a statement about an element $x \in S$ that may be true or false. Assume that $P(x)$ holds whenever $P(y)$ holds for all $y \in S$ such that $y \sim x$. Then $P(x)$ is true for all $x \in S$. That is*

$$(\forall x : (\forall y \sim x : P(y)) \Rightarrow P(x)) \Rightarrow (\forall x : P(x)) \,. \tag{25}$$

Proof. Suppose $P(x)$ holds whenever it holds for all elements $y \in S$ with $y \sim x$. Now consider the set $T = \{z \in S : P(z) \text{is false}\}$. Then $P(x)$ is true for all $x \in S$ if we can show that T is the empty set.

We do this by contradiction. Assume T is non-empty, which means by 7.50 that it has a minimal element, call this x. Now all $y \in S$ such that $y \sim x$ do not belong to T (since x was a minimal element of T), meaning that $P(y)$ is true. But, by our initial assumption, this implies that $P(x)$ is also true, contradicting the fact that $x \in T$. So T is empty, as claimed. □

Remarks:

- We call this method of induction **top-down induction** since we start 'from the top' (element x), and use the property that we know is true about elements y simpler than x (the 'down' part).
- Top-down induction has a very close relationship with **strong induction**. In fact, if we are working on the totally ordered natural numbers, $\mathbb{N}$, then these two methods are identical, however top-down induction only needs for $\leq$ to be partially ordered on S, hence the different naming convention to distinguish this.
- But what about the base case? Don't all induction methods need this? Yes. In the above proof this was glossed over as it turns out to simply be vacuously true: if x is a minimal element, then there are no $y \sim x$. Therefore, for minimal x, we simply have that $P(x)$ is true. This is the base case.

Now we can return to our former proposition 7.43 and finally prove this with the help of top-down induction. In fact, you'll do this on problem set 7 and we will now freely use this result: in any impartial game, a game position is losing if and only if all its options are winning positions. A game position is winning if and only if at least one of its options is a losing position. As mentioned earlier we will also extend our definitions of winning and losing positions to include these properties.

7.3 Game Sums

Combinatorial games can often be decomposed into separate parts. It turns out that this is a very useful property that we formalise in this section.

Definition 7.52. Suppose that G and H are games with options $G_1, \cdots, G_n$ and $H_1, \cdots, H_m$, respectively. We define the **game sum** of G and H, denoted $G + H$, to be the game with options:

$$G_1 + H,\, G_2 + H,\, \cdots,\, G_n + H, \quad G + H_1,\, G + H_2,\, \cdots,\, G + H_m. \tag{26}$$

Remarks:

- Note that this essentially means that the player makes a move in game G or in game H; the other part of the game sum remains untouched (there is no given option of the form $G_3 + H_4$ for example or of anything different entirely).
- This is a **recursive** definition since the game sum is defined in terms of its options, which are themselves game sums. Importantly these are simpler games, so this recursive definition makes sense as it tumbles down to the ending condition.

Example: As a short example to demonstrate this idea and its usefulness, consider the Nim game $3, 2, 1$. This, for example, is a **sum** of the games $G = 2, 1$ and $H = 3$, i.e

$$G + H = 3, 2, 1.$$

An option of the player in this game corresponds to either removing some tokens from the pile of size 1, from the pile of size 2; in either case a possible move in game G leaving game H alone, or removing some tokens from the pile of size 3; a possible move in H whilst leaving game G alone. Notice that this gives all options as in the definition of the game sum.

In fact we can go one step further with this, any Nim game is just the game sum of its individual Nim piles, because the player moves in **exactly one** of these piles.

You may also notice some other properties of the game sum from this example that we will discuss now.

Proposition 7.53. *Denoting the losing game with* ***no options*** *by* 0*, then for any games* G*,* H *and* J *we have:*

- *Commutativity of* $+$*:*

$$G + H = H + G, \tag{27}$$

- *Associativity of* $+$*:*

$$(G + H) + J = G + (H + J), \tag{28}$$

- *Zero element for* $+$*:*

$$G + 0 = G. \tag{29}$$

Proof. You'll prove these on problem set 7. □

Remarks:

- We can in fact easily generalise both (27) and (28). In a sum of several games $G_1, G_2, \cdots, G_n$ the player moves in exactly one of these games, which does not depend on how these games were arranged, so that we can write the game sum unambiguously as

$$G_1 + G_2 + \cdots + G_n.$$

- Everywhere in this proposition and in this chapter = should be read as 'is the same game as'. This feels obvious right now, but we will shortly introduce the notion of **equivalence** of games, so it is important to be clear of the difference.

The previous proposition has given us commutativity, associativity and a zero element for the sum of games. The next thing we would like to have on this list is the concept of an 'inverse' game; i.e for any game G, can we construct a game $-G$ such that $G + (-G) = 0$.

This would be great, however, we can immediately look at this equality as stated and notice there is a problem. If game G has any options, then by our definition of $+$, regardless of whatever the game $-G$ turns out to be, the LHS will have options, but the RHS is the losing game; the game defined to have **no** options, so these things cannot be equal.

This motivates our next idea. What could we do instead to ensure this 'inverse' can exist?

7.4 Equivalence of Games

We introduce the notion of **equivalence** of games via the following definition.

Definition 7.54. Two games G and H are called **equivalent**, written $G \equiv H$, if and only if for any other game J, the game sum $G + J$ is losing if and only if $H + J$ is losing.

Lemma 7.55. *The binary relation of equivalence, $\equiv$, is an equivalence relation between games, this means that it is:*

- *Reflexive: $G \equiv G$, for any game G;*
- *Symmetric: if $G \equiv H$ then $H \equiv G$; and*
- *Transitive: if $G \equiv H$ and $H \equiv K$, then $G \equiv K$.*

Proof. You'll show this on problem set 7. □

Let's discuss this definition a little bit to understand better what it is saying. First notice that if we take $J = 0$ in the definition of equivalence then we find that if the games G and H are equivalent then either both are losing games or both are winning games. So equivalence of G and H means that the games always have the same outcome class (winning/losing). However it is a stronger statement than just this, as it not only holds for when $J = 0$, but this property holds in any game sum $G + J$ and $H + J$ for any other game J. Let's take a look at an example to see this.

Example: Consider a Nim pile of size 2 and a Nim pile of size 1. We ask if these games are equivalent?

Indeed let $G =$ Nim pile of size 2, $H =$ Nim pile of size 1, then, taking $J = 0$ we have $G + J =$ Nim pile of size 2, a winning game, and $H + J =$ Nim pile of size 1, also a winning game. So this might lead us to think the games G and H are equivalent.

Instead, consider taking J as a Nim pile of size 1. Then $G + J = 2, 1$, which is a **winning** Nim position (remove one token from the pile of size two to have two equal piles), but notice that $H + J = 1, 1$, which is a **losing** Nim position. Hence we can conclude that G and H are **not** equivalent.

This shows us that there is something stronger with equivalence beyond just belonging to the same outcome class. Perhaps now, you might think, that equivalence is so strong a condition that two games being equivalent means they are identical games, i.e. does $G \equiv H$ mean the same thing as $G = H$, i.e that G and H are the same game?

Although we may struggle to show this with our theory just yet, it turns out it is true that the Nim game consisting of a single pile with 1 token **is** equivalent to the Nim game $5, 4, 3, 2, 1$, consisting of five piles of tokens of amounts 5 to 1. These are not the same game, so there is something more subtle going on with equivalence: it is a stronger statement than just belonging to the same outcome class, but not as strong as saying that the games are the same.

We continue to investigate.

Proposition 7.56. *Two Nim piles are equivalent if and only if they have equal size.*

Proof. ($\Rightarrow$): We take the contrapositive of this statement and instead prove that if two Nim piles have different sizes, then they are **not** equivalent.

Indeed, let G and H be two Nim piles of different sizes. Consider $J = G$ in the definition of equivalence. Then $G + G$ is a losing game (since this game consists of two Nim piles of the same size, which we proved was losing), but $H + G$ is winning (as it is two Nim piles of different sizes). Hence, by definition, G and H are **not** equivalent, as we needed to show.

($\Leftarrow$): If G and H are Nim piles of equal size then they are the same game and therefore equivalent. $\square$

So we have seen that equivalence of games means they belong to the same outcome class, but also that two different winning games (like in our example with the games 1 and $5, 4, 3, 2, 1$ and also from 7.56) are not necessarily equivalent.

However, that fact does not hold for two different losing games, and we will now prove that **any two losing games are equivalent**, because they are all equivalent to the game 0 that has no options.

Proposition 7.57. *G is a losing game if and only if $G \equiv 0$.*

Proof. ($\Leftarrow$): If $G \equiv 0$ then we use $J = 0$ in our definition of equivalence giving $G + 0$ losing $\Leftrightarrow$ $0 + 0$ losing, that is G losing $\Leftrightarrow$ 0 losing, which is the definition of a losing game, so G is losing.

($\Rightarrow$): Suppose G is a losing game. We need to show that $G \equiv 0$, that is, for any other game J, the game $G + J$ is losing if and only if $0 + J$ is losing. Now $0 + J = J$, so this amounts to proving:

$$\begin{aligned} &(a). \quad G + J \text{ is losing} \Rightarrow J \text{ is losing, and} \\ &(b). \quad J \text{ is losing} \Rightarrow G + J \text{ is losing.} \end{aligned}$$

Let's start with proving (b). Let J be losing (and we have that G is losing by our initial assumption). We want to show that $G + J$ is losing, meaning that every option of $G + J$ is winning.

Any such option is of the form $G' + J$ where G' is an option of G, or of the form $G + J'$, where J' is an option of J. Let's start with the first type. Now because G is losing, G' is winning, so G' has some winning move to an option G'' that is losing.

Note that G'' is a losing game which is **simpler** than G, so this motivates us to appeal to top-down induction, where we can assume from the inductive hypothesis (that (b) holds) that G'' and J are losing $\Rightarrow$ $G'' + J$ is losing. This means that the option $G' + J$ is winning because it has as an option the losing game $G'' + J$.

Note that $G' + J$ was an arbitrary option with a first move in the game G. We still need to show that the same result also holds for an arbitrary option of the form $G + J'$, but this follows in a similar manner to how we have just argued.

This means that **all** options of $G + J$ are winning games, so therefore $G + J$ is a losing game, proving (b).

Now let's prove (a). Let's do this by contradiction. Suppose that $G + J$ is losing but J is winning. That means there is a winning move to some option J' of J where J' is a losing game. Now using (b); this implies that, since G and J' are losing games, $G + J'$ is losing. But $G + J'$ is an option of $G + J$, which would represent a winning move from $G + J$, meaning $G + J$ is winning. This contradicts our assumption that $G + J$ is losing. Hence J must be losing so (a) holds. □

Corollary 7.58. *Any two losing games G and H are equivalent.*

Proof. Let G and H be losing games. By 7.57 we have $G \equiv 0$ and $H \equiv 0$. But $\equiv$ is an equivalence relation between games, meaning that it is transitive, so then $G \equiv 0$ and $0 \equiv H$ means that $G \equiv H$. □

Lemma 7.59. *For all games G, H and K we have:*

$$G \equiv H \Rightarrow G + K \equiv H + K.$$

Proof. Let $G \equiv H$. Now $G+K \equiv H+K$ means that $(G+K)+J$ is losing if and only if $(H+K)+J$ is losing, for **any** other game J. By associativity (28) this means $G + (K + J)$ is losing if and only if $H + (K + J)$ is losing. But this holds from the definition that $G \equiv H$ taking $(K + J)$ as the arbitrary added game. □

Lemma 7.60. *Let J be a losing game. Then $G + J \equiv G$ for any game G.*

Proof. By 7.57, $J \equiv 0$. Now by 7.59 we have $G + J = J + G \equiv 0 + G = G$. □

This means that **any** losing game J can take the role of a zero element when we consider game sums, i.e $G + J \equiv G$ for any game G as stated in the lemma. We can think about this conceptually in the following way: any move in the J component of the game $G + J$ has a 'counter-move' in this component back to a losing position (leaving G alone) since J is losing, eventually continuing this process game J will deplete into the 0 game with no options, so adding J to any game G doesn't affect whether game G is winning or losing.

Our next proposition finally allows us to return to our old problem about having no inverse game.

Proposition 7.61 (The Copycat Principle)**.** *$G + G \equiv 0$ for **any** impartial game G.*

Proof. Let G be an impartial game. By proposition 7.57, the condition $G + G \equiv 0$ amounts to showing that $G + G$ is a losing game. Let's prove this by top-down induction by showing that every option of $G + G$ is winning.

Any option of $G + G$ is of the form $G' + G$ for an option G' of G. But the next player then has the winning move to the game $G' + G'$ which is a losing game by the inductive assumption since $G' + G'$ is **simpler** than $G + G$. Hence $G + G$ is losing and so $G + G \equiv 0$. □

These latest results allow us to simplify complex game positions significantly. As an example to see where we are going and what we are working towards, consider the formidable Nim position $1, 2, 3, 7, 8, 8$. Is this winning or losing?

Figure 70: The Nim position $1, 2, 3, 7, 8, 8$ with braces showing parts equivalent to 0.

The game sum of two Nim piles of size 8 is a losing position, hence by proposition 7.57 these piles are equivalent to a losing, or 0 game. We also now from our earlier work in the chapter that the game $1, 2, 3$ is losing. Hence we can conclude that the Nim game $1, 2, 3, 7, 8, 8$, as shown in figure 70, is equivalent to a single Nim pile of size 7.

$$1, 2, 3, 7, 8, 8 \equiv 1, 2, 3 + 7 + 8, 8 \equiv 0 + 7 + 0 \equiv 7.$$

This is therefore a winning game with a winning move to remove the pile of size 7 leaving the sum of the two losing games; a losing position.

Lemma 7.62. *For impartial games G and H, then $G \equiv H$ if and only if $G + H \equiv 0$.*

Proof. ($\Rightarrow$): If $G \equiv H$, then by lemma 7.59 (adding H) we have:

$$G + H \equiv H + H \equiv 0,$$

by use of the copycat principle 7.61.

($\Leftarrow$): Conversely, if $G + H \equiv 0$, then

$$G \equiv G + 0 \equiv G + H + H \equiv 0 + H \equiv H,$$

where we have used the idea that $H + H \equiv 0$ by the copycat principle in the second equivalence. $\square$

7.4.1 A Remark on Misére games

We make a final remark here on misére games; games where the final player to move **loses**. A losing position in misére Nim for example is the single pile with a single token in it. Let's call this game J.

If we add this game J to another game G then this **reverses** the outcome class of G, since the player can remove the token from game J, leaving game G remaining. This certainly means we don't have $G + J \equiv G$ as in lemma 7.60.

Further, in misére Nim it turns out $1, 1$ is winning but $2, 2$ is losing, so there is no copycat principle as in proposition 7.61 for misére games.

Regardless, misére Nim **is** well understood, but general misére games are not! Normal play games however have the nice equivalence structure we have developed in the last section.

7.5 Notation for Nim piles

We will now develop some notation for single Nim piles since they will become very important for us going forward.

Definition 7.63. If G is a **single** Nim pile with $n \geq 0$ tokens in it, then we denote this game by $*n$. This game is specified by its n options, defined recursively as

$$*0, *1, *2, \cdots, *(n-1).$$

Remarks:

- $*0$ is the empty pile with no tokens in it, i.e $*0 \equiv 0$. We will alternate between these notations, but often continue to write just 0 for the losing game.
- A general Nim position is a game sum of several Nim piles. Earlier we wrote these positions by listing the pile sizes, e.g $1, 2, 3$. We now represent this game as

  $$*1 + *2 + *3,$$

  a sum of games of single Nim piles.

Definition 7.64. If $G \equiv *m$ for an impartial game G, then m is called the **Nim value** of G.

Remarks:

- This value m is **unique** as Nim piles of different sizes are **not** equivalent. We proved this in proposition 7.56.
- The Nim value is sometimes referred to as the Sprague-Grundy value, the sg-value or the Grundy value. All of these are named after Sprague (1935) and Grundy (1939) who each independently discovered the theory.

7.6 The Mex Rule

In this section we will prove a fascinating result about impartial games and the notion we mentioned that the game Nim has a central importance in the theory of understanding **any** impartial game. Indeed shortly we will prove that **every** impartial game is equivalent to some **single** Nim pile. In fact the theorem we will prove, known as the mex rule, is stronger than just this and allows us to determine precisely what size of Nim pile an impartial game is equivalent to.

Before arriving at the mex rule we first define the minimum excluded number (mex) of a set.

Definition 7.65. For a finite set of natural numbers S, the **minimum excluded number** of S, written $\text{mex}(S)$, is defined as

$$\text{mex}(S) = \min\{k \in \mathbb{N} : k \notin S\}.$$

In other words, $\text{mex}(S)$ is the smallest non-negative integer not contained in S. For example;

- $\text{mex}(\{0, 1, 3, 5\}) = 2$,

- $\text{mex}(\{1, 2, 8\}) = 0$,
- $\text{mex}(\emptyset) = 0$.

We now state and prove the mex rule.

Theorem 7.66 (The Mex Rule)**.** *Any impartial game G has* ***Nim value*** *m (that is, $G \equiv *m$), where m is uniquely determined as follows: for each option H of G, let H have Nim value s_H, and let $S = \{s_H : H \text{ is an option of } G\}$. Then $m = mex(S)$, that is, $G \equiv *(mex(S))$.*

Proof. We will prove this via top-down induction. First, if $S = \emptyset$, then $G = 0 = *0$. Now assume, as per top-down induction, that each option H of G has Nim value s_H, where s_H is itself the mex of the Nim values of the options of H. Let S be the set of these Nim values, i.e

$$S = \{s_H : H \,\text{is an option of}\, G\},$$

and denote $m = \text{mex}(S)$. We will show that $G + *m \equiv 0$, which means by lemma 7.62 that $G \equiv *m$ as required.

The game $G + *m$ is losing if all of its options are winning.

First consider an option of the form $G+*h$ for some $h < m$, i.e the player makes a move in the Nim pile $*m$ of the game sum $G+*m$. In this case, $h \in S$ because $m = \text{mex}(S)$, and based on how S was constructed, there is an option H of G such that $h = s_H$, i.e $H \equiv *h$. This means that the other player has the counter-move (by moving in G) from $G + *h$ to the losing position $H + *h \equiv *h + *h \equiv 0$, by the copycat principle 7.61. Thus all options of $G + *m$ of the form $G + *h$ ($h < m$) are indeed winning.

Second we consider the possible options of $G + *m$ of the form $H + *m$ where H is an option of G. Since H is an option of G, then $H \equiv *s_H$ and $s_H \in S$. This means that either $s_H < m$ or $s_H > m$ ($s_H \neq m$ as $m = \text{mex}(S)$ so $m \notin S$).

First let's consider $s_H < m$. In this case the other player counters by moving from $H + *m$ to the losing position $H + *s_H \equiv *s_H + *s_H \equiv 0$.

Secondly, if $s_H > m$, we appeal to the inductive hypothesis that says s_H is the mex of the Nim values of the options of H. This means that the Nim value of one of the options of H is m, call such an option $H' \equiv *m$. The other player counters by moving from $H + *m$ to the losing position $H' + *m \equiv *m + *m \equiv 0$. Thus all options of $G + *m$ of the form $H + *m$ (for H an option of G) are also winning.

But **any** option of $G + *m$ has to be of the form $H + *m$ or $G + *h$; thus **all** options of $G + *m$ are winning. Hence $G + *m$ is a losing game and so by proposition 7.57 $G + *m \equiv 0$, as required. ☐

Remark: If we find $\text{mex}(S) = 0$ for some game G, then this means that **either** all options of G are equivalent to positive Nim heaps, so **all** are winning, **or** G has no options at all. Thus, in this case, G is losing and indeed $G \equiv 0$.

Let's study some example games.

7.6.1 The Rook-move Game

Using the mex rule any impartial game can be played like Nim. To do this we need to know the Nim values of the different positions in the game. We can evaluate these **recursively** by the mex rule.

Let's see this concept via an example game; the rook-move game. Consider a rook positioned on some square of a chessboard which extends indefinitely to the right and to the bottom but has sides on the left and at the top, see figure 71.

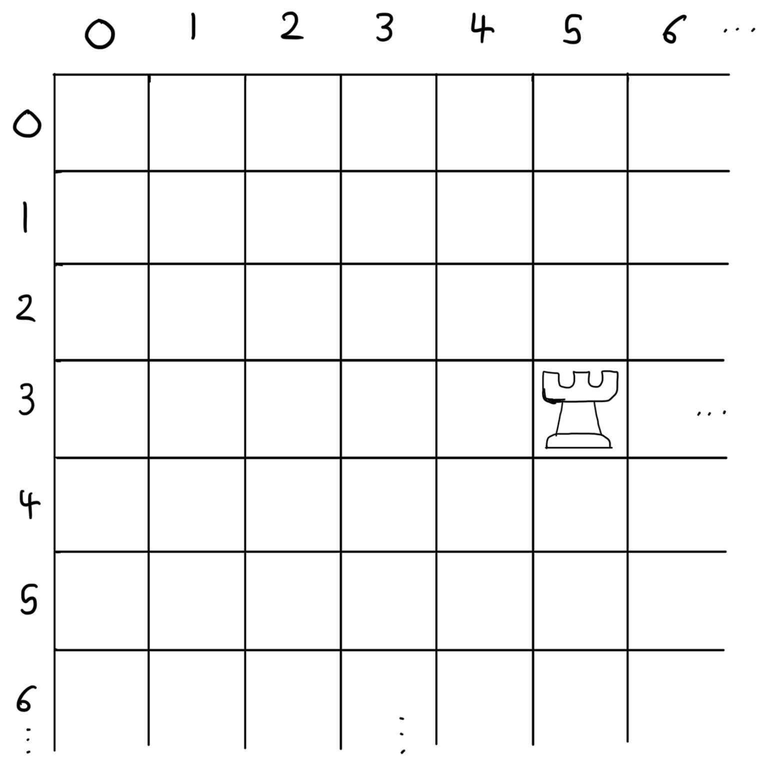

Figure 71: The Rook move Game with a single Rook on the board.

We number the rows and columns $0, 1, 2, \cdots$ starting from the top left square. On a given turn, a player must move the rook either to the left **or** upwards (and not both). The rook can be moved **any** number of squares, but must move at least one square. The first player who can no longer move the rook, because it is in the top-left square, loses (or we can think of this as the player who moves the rook into the top-left corner square wins).

Activity: Draw out a board on paper and use a small object to represent the rook. Play a few games with your neighbour. Choose a different starting square for the rook each time.

With the mex rule to aid us, let's start analysing the game. A rook in the top-left corner square is equivalent to 0 because the rook can't move, so this is precisely the game with no options.

Now consider the game where the rook is positioned just to the right of this square. This game has as its only option the 0 game (the rook can only be moved into the corner), so by the mex rule our current game has Nim value $m = \text{mex}(\{0\}) = 1$, thus this game has Nim value 1 and is equivalent to a single Nim pile of size 1. In our new notation we denote this game as $*1$. Consider now the game where the rook is in the position just to the right of this square. This game has options of 0 and $*1$, hence by the mex rule it is equivalent to the game $*2$.

In this manner we can continue to determine the Nim values of all squares along the top row, they are the games $*i$ in column i. Similarly, an analogous argument holds for all the cells in the first column, so we can determine their Nim values.

But what about when the rook is on some square in the centre of the board. Well we can move it to any square above this cell, or to its left. So in general, a position on the board is evaluated by knowing **all** Nim values for the squares to the left and those on top of it, i.e the Nim values for all options of that position. So, currently, we can determine the Nim value of the cell in row 1, column 1: it has two options: the square to its left, which is $*1$, and the square above, which is also $*1$. Therefore, by the mex rule, this square is equivalent to $\text{mex}(\{1\}) = 0$, so it is a losing position.

This process can now be continued (for example we now have the Nim values of all options of the squares row 1 column 2 and row 2 column 1) to recursively find the Nim value of any position in the rook move game. Figure 72 shows the Nim values for several of the positions of the rook.

	0	1	2	3	4	5	6 ...
0	0	*1	*2	*3	*4	*5	*6 ...
1	*1	0	*3	*2	*5	*4	*7 ...
2	*2	*3	0	*1	*6	*7	...
3	*3	*2	*1	0	*7	*6	... ...
4	*4	*5	*6	*7	0		
5	*5	*4	*7	*6		...	
6 ⋮	*6 ⋮	*7 ⋮	⋮	⋮			

Figure 72: The Rook move Game with the Nim value of several squares shown.

So, for example, consider the square in row 2 column 3 as shown in figure 72. To its left are the entries $*2$, $*3$ and 0, and above it are the entries $*3$ and $*2$. Thus, the Nim value of this square is $\text{mex}(\{0, 2, 3\}) = 1$, so the game with the rook on this square is equivalent to $*1$, a Nim pile of size 1.

Remarks:

- The losing squares on the chessboard turn out to all be along the diagonal. Some of you may have noticed the importance of the diagonal whilst playing a few games. Indeed, we can apply the copycat principle to move a player **back onto** the diagonal when the player moves off of it.
- Secondly, you might have noticed that the rook move game is actually just Nim with **two piles** in disguise! A rook in row i, column j can either reduce i by moving up, or reduce j by moving left, so is precisely the sum of the Nim piles $*i + *j$. Indeed, we can now see clearly this is a losing game if and only if $i = j$ and the Nim piles have equal size.

The Nim values of each position have more use than just helping to distinguish between winning and losing positions. They become very useful as we begin to complicate games, for example, suppose we place **multiple** rooks on the board in the rook move game...

7.6.2 The Multiple Rook-move Game

Consider now the rook-move game, but where we allow multiple rooks to be placed on the board. We allow the rooks to occupy the same space where necessary. On a given turn a player moves **one** of the rooks. As before, when a player can no longer move a rook they have lost the game.

Activity: Play a game or two with a partner. Put 2/3/4 rooks on the board. Can you see how to analyse the game?

This game corresponds to a game sum of single rook-move games. Using figure 72, there are some positions we are able to analyse, for example a game with two rooks where one is in row 2, column 3, and the other in row 0, column 1 is equivalent to the game $*1 + *1$ which we know is losing.

But what about two rooks where one is placed in row 4, column 2 and the other in row 1, column 3, equivalent to the game $*6 + *2$ ... is this winning? Losing? In fact we can ask a further question which answers our first one: by theorem 7.66, the mex rule, we know that this game $*6 + *2$ **is** equivalent to a single Nim pile of some size, but what size exactly is this?

7.7 Sums of Nim Piles

In this section we will show how to calculate the Nim value for **any** game sum of Nim piles. So shortly you will be able to deduce, for example, what game the game sum $*5 + *7 + *23$ is equivalent to, and in good time too.

With our work so far, we know some examples, so let's start there. We studied the Nim position $*1 + *2 + *3$ in detail earlier in the chapter, showing it was a losing position, hence $*1 + *2 + *3 \equiv 0$. This means by lemma 7.62 that:

$$
\begin{aligned}
&*1 + *2 \equiv *3, \\
&*2 + *3 \equiv *1, \text{ and} \\
&*1 + *3 \equiv *2.
\end{aligned}
$$

So we see from the second two above that the sizes of the Nim piles cannot simply be added together to get the equivalent Nim pile.

Definition 7.67. If $*k \equiv *m + *n$, then we call k the **Nim sum** of m and n, written $k = m \oplus n$.

Now let's observe a theorem that is going to enable us to compute whatever Nim sums we want.

Theorem 7.68. *Let $n \in \mathbb{Z}^+$, and represent n as a unique sum of powers of 2, i.e write $n = 2^a + 2^b + 2^c + \cdots$, where $a > b > c > \cdots \geq 0$. Then*

$$*n \equiv *(2^a) + *(2^b) + *(2^c) + \cdots . \tag{30}$$

Proof. Problem Set 7. □

Let's discuss the use of theorem 7.68 through an example. The RHS of (30) is a sum of games of single Nim piles whose sizes are distinct powers of 2. So, for example, let $m = 9 = 2^3 + 2^0 = 8 + 1$, thus

$$*m = *9 \equiv *8 + *1.$$

Similarly let $n = 14 = 2^3 + 2^2 + 2^1 = 8 + 4 + 2$, thus

$$*n = *14 \equiv *8 + *4 + *2.$$

Hence, we have that the game sum:

$$
\begin{aligned}
*m + *n = *9 + *14 &\equiv *8 + *1 + *8 + *4 + *2 \\
&\equiv *4 + *2 + *1 \\
&\equiv *7,
\end{aligned}
$$

where we have used the fact that we know $*8 + *8 \equiv 0$ in the second line, and applied theorem 7.68 again in the final line as all the games were powers of two. We can write this as $9 \oplus 14 = 7$ using the Nim sum. This tells us, for example, that $*7 + *9 + *14 \equiv 0$ is a losing Nim position.

The theorem therefore allows us to find the Nim sum of any set of numbers by writing each number as a sum of **distinct** powers of two, then cancelling out repetitions in pairs, and finally summing the remaining powers of two. For example, let's take the Nim position $4, 5, 9, 14$ that we saw earlier was winning and ask precisely what Nim value this position has. Well

$$
\begin{aligned}
4 \oplus 5 \oplus 9 \oplus 14 &= 4 \oplus (4 + 1) \oplus (8 + 1) \oplus (8 + 4 + 2) \\
&= 4 \oplus 2 \\
&= 6.
\end{aligned}
$$

Hence $*4 + *5 + *9 + *14 \equiv *6$. So now we can say that the game position was indeed winning and equivalent to a Nim pile of size $*6$.

Exercise: Can you see how to find what all the winning moves are from the position $4, 5, 9, 14$ using the calculation above?

7.8 Example Games

To finish our chapter, let's finally put all of our theory to the test by trying to solve some examples of impartial games.

7.8.1 The Queen-move Game

Consider now a chessboard exactly as in the rook-move game (extending indefinitely to the right and below) with a chess queen placed in some square on the board. On a given move the player must move the queen any number of squares (at least one) either vertically up, horizontally left, or **diagonally up-left**. The winner is the player who moves the queen into the top-left square of the board. Figure 73 gives a schematic of an example position.

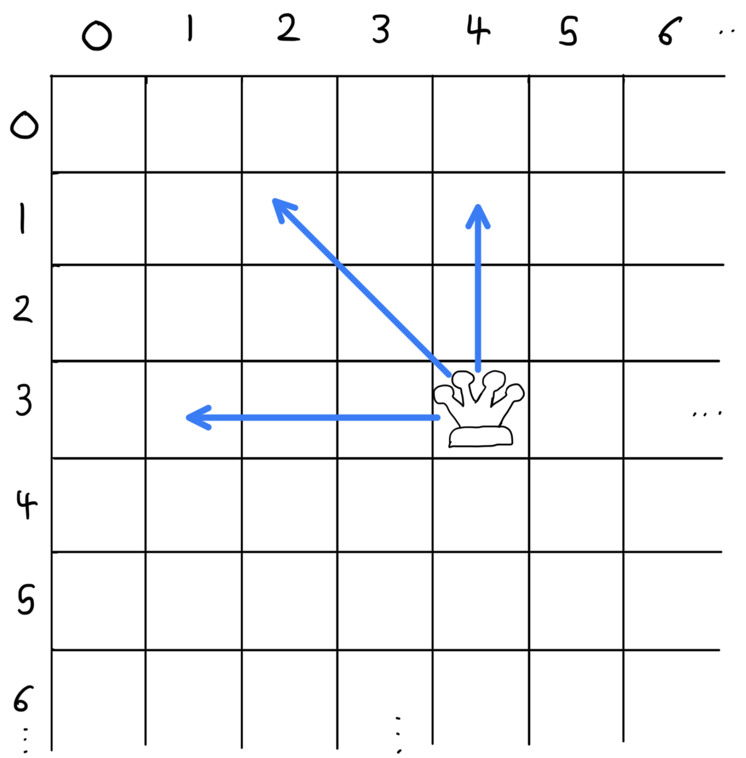

Figure 73: The Queen move Game.

This game can be played like Nim with two piles of tokens where a player may remove tokens from either pile but with the added move of being able to reduce **both** piles by the same number of tokens.

Activity: Play the game with a partner. What squares should you move the queen to?

Figure 74 shows the Nim values of some of the squares in the queen-move game. These are calculated recursively using the **mex rule** starting from the top-left corner square.

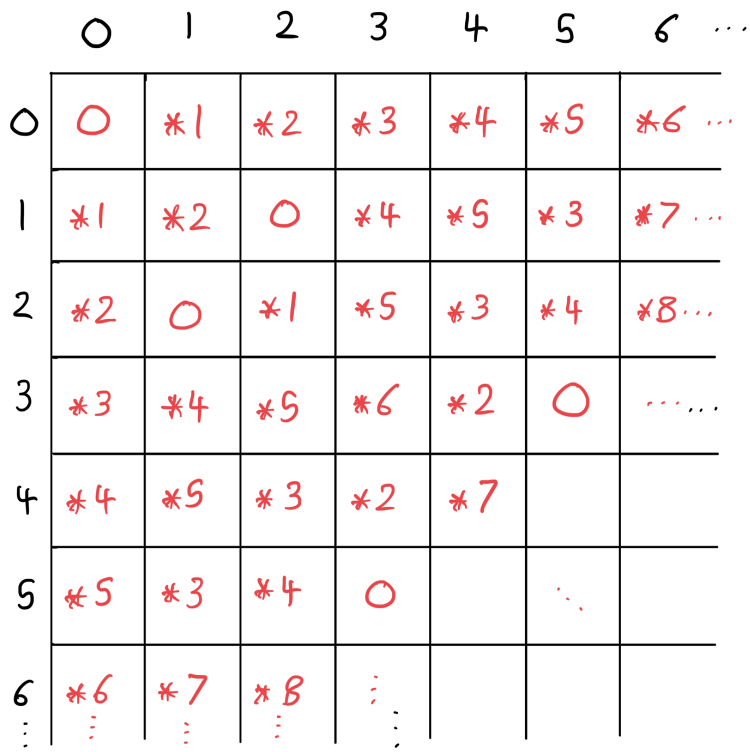

Figure 74: The Queen move Game with the Nim value of several squares shown.

We see that winning moves in this game move the queen onto one of the the 0 cells; leaving your opponent in one of the losing positions.

7.8.2 Sum of a Queen-move Game and a Nim Pile

Consider now the queen-move game with the queen initially positioned in row 3, column 4 **added to** a Nim pile of size 4, as shown in figure 75.

In this game sum the player may **either** move the queen **or** take some of the tokens from the pile. When a player is unable to do either of these, they lose the game.

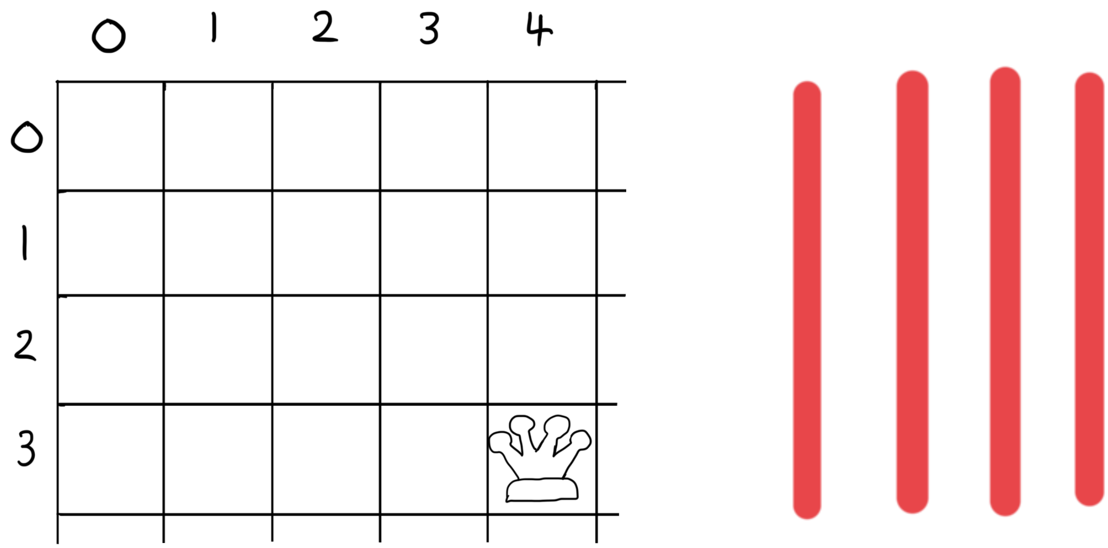

Figure 75: A game sum of a queen-move game and a Nim pile of size 4.

Question: Determine the Nim value of the game and, if winning, give **all** possible winning moves from the position.

Answer: Consulting figure 74, we see the queen occupies a square with entry $*2$. This means that currently our game has Nim value $2 \oplus 4 = 6$, which is **winning**.

We can only move in **one** of these games, so to set the Nim value of the position to 0, i.e. to play a winning move and leave our opponent in a losing position, we want to change $*2$ to $*4$ or $*4$ to $*2$. This means we can either:

- Remove 2 tokens from the Nim pile, giving the game $2 \oplus 2 = 0$;
- Move the queen to row 0, column 4, giving the game $4 \oplus 4 = 0$;
- Move the queen to row 3, column 1, giving the game $4 \oplus 4 = 0$.

These are all possible winning moves from the current position.

Chapter 8: Game Trees and Games of Imperfect Information

In this chapter we will use the extensive form representation of games, or in other words, game trees, as a way to examine **sequential** (one player moving after another) games of **imperfect information** (this is where some elements or moves made in the game are unknown to some of the players in the game).

To make it clear when we will introduce the imperfect information elements, the chapter has been split into two parts, part I, which you are about to start, which builds on the basis of representing games of perfect information as trees that we saw in chapter 1 and what we might take as solutions to such games, and part II which introduces imperfect information to such games and discusses how we may tackle this.

Part I: Game Trees of Perfect Information

8.1 Game Trees

We have seen examples of game trees in chapter 1 of the course. In the examples so far, every internal node in the tree has been what we may call a **decision** node (where one player needs to make a choice), but we can also include **chance** nodes (where something probabilistic may happen). Throughout this chapter we will represent decision nodes with black circles and chance nodes with white squares. Terminal nodes, or simply branches that end the tree, result in an ending of the game and we assign one payoff to each player at these points. We will assign player A's payoff first, followed by player B's, and so on (i.e the order of the payoffs follows the order of the naming of the players), regardless of the order these players may have moved in the game tree.

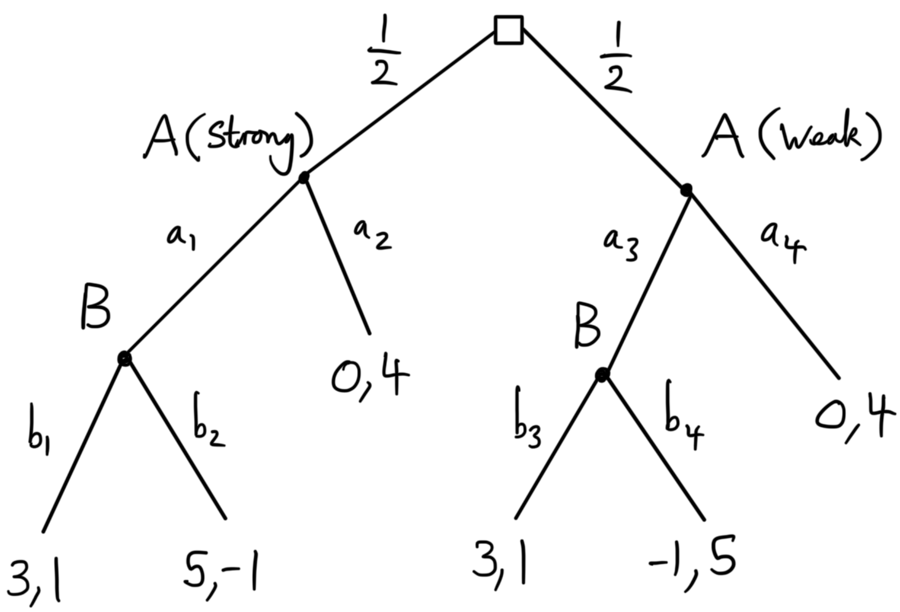

Figure 76: An example tree game we will study throughout this chapter involving firms A and B.

Figure 76 depicts a game in extensive form (a game tree). In this game firm A competes with firm B over selling their similar products in the market. Initially in this game a chance move occurs where there is a 50% chance firm A develops a strong product and a 50% chance firm A's product is weak. This strong/weak product is indicated on the game tree at A's following decision nodes. After this, firm A can either announce their product on the market (choices a_1 and a_3) or can cede the market to firm B (choices a_2 and a_4) and not release their product. After this choice from firm A, if firm A chose to announce their product (chose a_1 or a_3) then firm B can choose to be bought out by firm A (choices b_1 and b_3) taking a smaller share of the market profits, or can compete against firm A in the market (choices b_2 and b_4).

8.2 Backward Induction

When analysing a game tree, 'optimal' play for each player should aim to maximise their payoff. When a player is the last to move, this can be done irrespective of any prior players' moves. In our example game, firm B is last to move, so we can analyse what they should do. In the left hand decision node of player B (the one reached if A ended up strong and then played a_1), B should play b_1 since this gives them payoff of 1 where b_2 would yield them -1. In the right hand decision node (the one reached if A ended up weak and then played a_3), B should play b_4 (giving payoff 5 where b_3 would give 1).

Consequently, knowing that B will do this, firm A is in a position to respond accordingly. Firm A will thus choose a_1 in their left hand decision node (as this would lead to B's left hand decision node where they will play b_1 as discussed above) leading to a payoff of 3 for A which is greater than the payoff of 0 they would receive by choosing a_2. Moreover, in the right hand decision node, A chooses a_4 getting a payoff of 0 (since by choosing a_3 they know B would choose b_4 in their resulting decision node yielding A a payoff of -1).

Having analysed the game in this way, from the branch ends backwards, the chosen optimal moves are often represented on the game tree diagram using arrows, as shown below in figure 77.

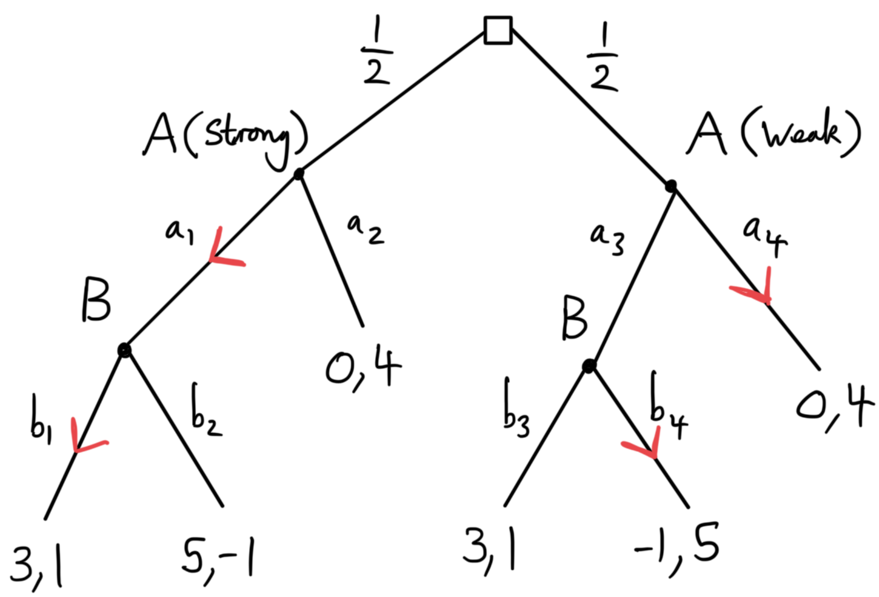

Figure 77: Arrows showing the result of backward induction on the game tree.

To deal with a chance node (like the one at the beginning of our example game), we always assume as we have throughout the course that our players will be interested in maximising their **expected payoffs**. At a chance node we can thus remove the chance node and replace it with the resulting expected payoffs for each player that would occur as a result of the subsequent 'best' play as analysed earlier. In our example game, the chance node (and thus in essence the whole tree since the chance node is the root of the tree) could be replaced with the pair of payoffs $1/2 \times 3 + 1/2 \times 0 = 3/2$ for firm A and $1/2 \times 1 + 1/2 \times 4 = 5/2$ for firm B.

This process of analysing the game we have discussed throughout this section is called **backward induction**. Provided all subsequent decision nodes have had moves decided, this process will determine a move/moves for every decision node in the game tree.

8.2.1 When it gets a little fiddly

This process of backward induction can sometimes encounter a small caveat. We illustrate this in the example game shown in figure 78.

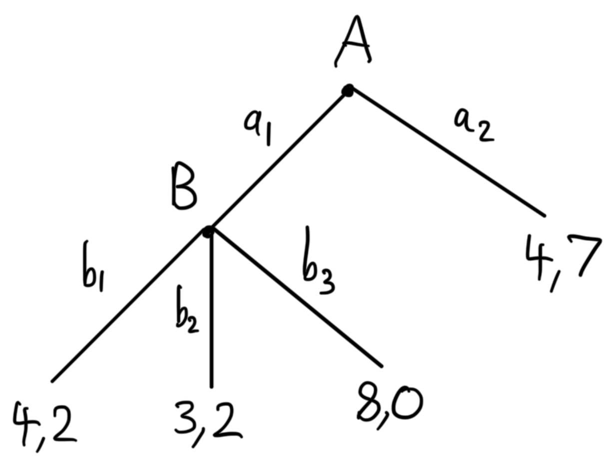

Figure 78: An example game to illustrate when backward induction gets a little tricky.

Applying backward induction, starting with player B, leads to a problem when multiple moves yield maximal payoff to the player. Both moves b_1 and b_2 are maximal here for player B. To indicate this we use two **different** arrows (to distinguish between them we use colours: here red and blue), see figure 79 below.

In turn this means that player A, when choosing between a_1 and a_2, will get payoff 4 for a_2 but the payoff when they choose a_1 depends on the choice of player B: if that was b_1, then A gets 4, and A can choose **either** of a_1 or a_2. If player B chooses b_2 however, then the payoff to player A is maximised by playing a_2.

Thus, observing figure 79 and the resulting combinations of coloured arrows, possible combinations of moves that could arise by backward induction are (listing moves for each player): (a_1, b_1), (a_2, b_1) and (a_2, b_2).

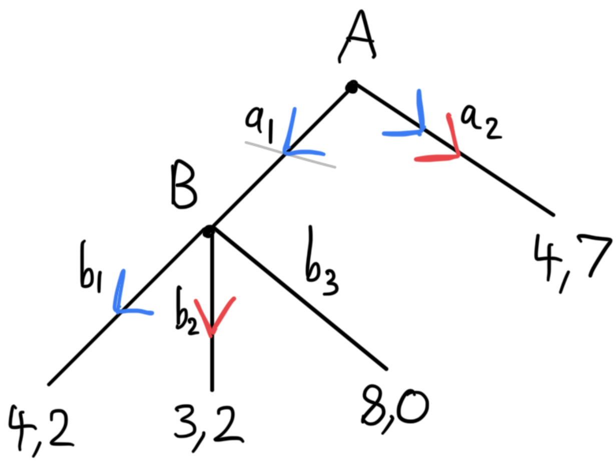

Figure 79: Backward induction arrows on the example game.

Note that, by now considering the game forwards from the beginning, by playing a_1, A risks the chance of B playing b_2, leading to the case where a_1 would have been a sub-optimal choice for A. Upon performing backward induction, we do not argue that player A should not play a_1 because of this (nor if a similar issue were to happen to another player in a game). The process of backward induction merely aims to determine lists of moves for each player (in fact these lists of moves are **pure strategies -** we will go over this in the next section) that would lead to respective maximal payoffs. We will discuss what we may take as solutions to such game trees shortly!

Remark: If there is always only one move that gives a a maximal payoff for a player at every decision node in a game tree, then it can be proved that backward induction leads to a **unique recommendation** of moves for the players (like in our primary example with forms A and B).

8.3 Strategies in Game Trees

In the first chapter of the course we discussed pure strategies but didn't talk in much detail at all about considering what pure strategies 'look like' in the context of extensive form games. In this section we dip into a little more detail.

Definition 8.69. In a game tree a **pure strategy** for a player specifies a move for **every** decision node of that player.

We will write down a pure strategy for a player as a list of moves, one for each decision node of the player. For example, in the game tree in figure 76 from earlier, the pure strategies of firm A are a_1a_3, a_1a_4, a_2a_3 and a_2a_4. Those of firm B are b_1b_3, b_1b_4, b_2b_3 and b_2b_4. This ordering is done according to a fixed order of the decision nodes in the tree so that each move is identified uniquely. In this course I will employ the following convention: the first move in a pure strategy refers to the move to be made in the **topmost**, **leftmost** decision node for that player. The second move the topmost, next leftmost node, and so on until the topmost, rightmost decision node has a move specified. Then the next lowest, leftmost decision node, and so on counting decision nodes from left to right as we descend from top to bottom.

Remarks:

- This convention I will employ depends on *how* the game tree is drawn. For example, I could have drawn the left hand decision node for firm B slightly lower than that of the right hand decision node, meaning that I would specify a move for this decision node first when writing down the pure strategies of firm B. This, of course, does not affect the analysis of the game at all, so don't worry about how your trees are drawn - just try to stick to a naming convention that is clear which move is being made where in your tree (I will order moves according to the convention above and typically try to align decision nodes at the same vertical heights when this makes sense to do so when drawing game trees).
- As mentioned at the end of the last section, the result of backward induction leads to a strategy profile(s): a sequence of moves for each player (one at each of their decision nodes), i.e a **pure strategy** for each player.

8.3.1 The strategic form of a game tree

When considering sequential games in a tree, backward induction may help us to determine some strategies which lead to pairs of maximal payoffs for the players, but sometimes more can be understood about the game by considering its strategic form.

Definition 8.70. The **strategic form of a game tree** is defined by the set of strategies for each player according to the previous definition and the **expected payoff** to each player resulting from each strategy profile.

Figure 80 below shows the **strategic form of the game tree** of our example game between firms A and B from figure 76.

		B		
	$b_1 b_3$	$b_1 b_4$	$b_2 b_3$	$b_2 b_4$
A: $a_1 a_3$	$3, 1$	$1, 3$	$4, 0$	$2, 2$
$a_1 a_4$	$\frac{3}{2}, \frac{5}{2}$	$\frac{3}{2}, \frac{5}{2}$	$\frac{5}{2}, \frac{3}{2}$	$\frac{5}{2}, \frac{3}{2}$
$a_2 a_3$	$\frac{3}{2}, \frac{5}{2}$	$-\frac{1}{2}, \frac{9}{2}$	$\frac{3}{2}, \frac{5}{2}$	$-\frac{1}{2}, \frac{9}{2}$
$a_2 a_4$	$0, 4$	$0, 4$	$0, 4$	$0, 4$

Figure 80: The strategic form of the game tree of our example game.

Note that there is a subtlety here in that this is **not** exactly another representation of the same game. The strategic form game in figure 80 is played somewhat differently to the game tree from figure 76 since play proceeds **sequentially** in the game tree but the strategic form is assumed to be played **simultaneously** (players decide their strategies at the same time - just like much of what we have discussed throughout the course).

Exercise: Determine all equilibria in the strategic form game in figure 80.

Remark: In the game tree in figure 77, backward induction specified the unique strategy profile $(a_1a_4,\, b_1b_4)$. If you have completed the previous exercise you will notice that this is **also** an equilibrium of the strategic form of the game tree in figure 80. As has been the way I have tried to introduce fascinating new results to you throughout the course, this once again is not a coincidence and this result holds more generally (see shortly)!

8.3.2 Reduced Strategies

Consider the game tree in figure 81 below.

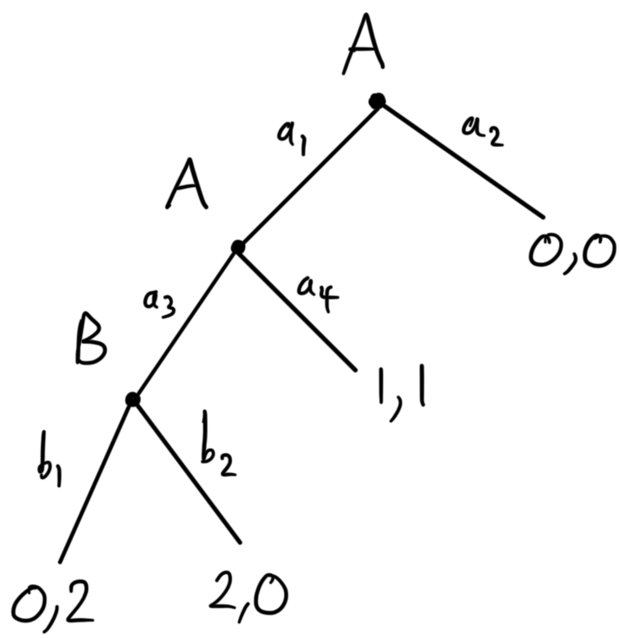

Figure 81: A game tree to showcase the concept of reduced strategies.

According to our definition of pure strategies in a game tree, player A has four pure strategies in this game tree: a_1a_3, a_1a_4, a_2a_3 and a_2a_4 (all possibilities of a choice of one of their two available moves at each of their decision nodes). Player B has two pure strategies here: b_1 and b_2.

Figure 82 shows the strategic form of the game tree for this game.

		B	
		b_1	b_2
A	a_1a_3	0,2	2,0
	a_1a_4	1,1	1,1
	a_2a_3	0,0	0,0
	a_2a_4	0,0	0,0

Figure 82: Strategic form of the game tree from figure 81.

Remark: Make sure you are comfortable with how to produce these strategic forms of game trees (more practice on the problem set).

By staring at figure 82 a little, notice that rows a_2a_3 and a_2a_4 have identical payoffs. This could just be a feature of the game, but may be due another reason: looking closer, we see that if player A selects a_2 at their first decision node, then they **never** reach their second decision node (perhaps you spotted this immediately earlier and knew this would cause some repetition in the strategic form of the game tree).

Thus, since this is entirely down to the choice of player A themself (they have complete control over whether they want to arrive at their second decision node), it makes sense to **replace** the strategies a_2a_3 and a_2a_4 with a less specific 'strategy': a_2* that prescribes move a_2 only. The $*$ is used as a placeholder to mean **any** unspecified move. In some texts this star is dropped completely, but often in a larger game tree the star offers clarity over exactly which decision nodes have unspecified moves, so I prefer to keep it (feel free to drop the star if you want so long as you feel you maintain clarity). This replacement 'strategy', a_2*, is called a **reduced strategy**.

Definition 8.71. In a game tree, a **reduced strategy** of a player specifies a move at each decision node for that player, **except** at any decision nodes that become unreachable due to an earlier **own** move of that player. We replace the move with a $*$.

In game trees it is usual practice to work with **reduced strategies** when these can be introduced and we will do the same in this course.

8.4 Subgame-Perfect Equilibrium (SPE)

Let's discuss the relationship between equilibria of the strategic form of a game tree and strategy profiles obtained via backward induction.

When considering a sequential game in the form of a game tree, a solution to such a game should in some way incorporate this sequential aspect of the game analysed by backward induction. That is, if we are to prescribe some strategy profile of the players to be a solution (an equilibrium) to the game, then the moves in this strategy profile should be optimal for **any** part of the game (any **subtree** of the game tree: a subtree of a game tree is formed by taking a node of the original game tree as the 'root' node with all the descendant nodes and moves included). This is because it would make little sense, for example, to specify a strategy for a player as part of an equilibrium that selected a sub-optimal move at some decision node of that player. Interestingly the usual convention here is to also include moves in subtrees that are no longer reachable due to earlier moves of the players (if that doesn't quite make sense now, don't worry, we'll look at an example to understand what this means).

Definition 8.72. We call a **subgame** of a game tree any **subtree** of the game tree.

Remark: Note well that this definition for a subgame of a game tree holds only for game trees with perfect information that we have been discussing so far. Shortly, when we introduce imperfect information to games, we will adjust this definition.

Definition 8.73. A strategy profile that defines an equilibrium for **every** subgame of a game tree is called a **subgame-perfect equilibrium** or SPE.

When analysing game trees we will insist that solutions to the game are SPEs. As alluded to earlier, there is a very important result linking backward induction and SPEs.

Theorem 8.74. *Backward induction leads to an SPE.*

Proof. Not examinable so this has been omitted. If you are interested in how we may prove this pages 91-92 of the text *Game Theory Basics* by Bernhard Von Stengel give an easy to digest proof. □

Remarks: This result has important consequences:

- Since backward induction chooses moves deterministically, game trees with perfect information have equilibria (SPE) in pure strategies. It is **not necessary** to consider mixed strategies.
- SPE exist in game trees with perfect information.

Remark: We have finished part I of the chapter on game trees of perfect information and are about to enter part II of the chapter and begin our discussion on how to introduce and tackle imperfect information in game trees. Now might be a good time to try problems XX - XX from problem sheet 8 which cover game trees of perfect information. Solidifying your understanding here will help with part II.

Part II: Game Trees of Imperfect Information

8.5 Information Sets

We now turn to developing methodology for tackling game trees of **imperfect information**.

Consider now the same game as in figure 76 where firm A and firm B are competing over the market share. However, suppose now that **only** firm A itself knows whether its product turns out to be strong or weak (i.e this information is not released publicly and is kept internal to firm A), so this information is **not known** by firm B.

Suppose as before that there is a 50% chance that firm A's product will be strong or weak however and this figure is still known to both firms. With all resulting payoffs and possible moves as before, the game tree with imperfect information representing this game is drawn in figure 83.

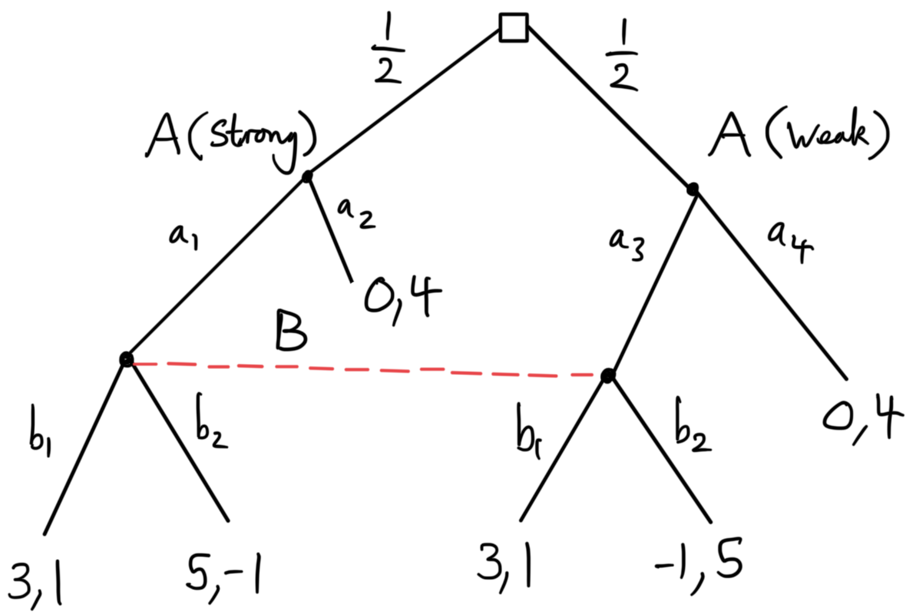

Figure 83: The game tree of imperfect information for our example game between firms A and B where now firm B does not know whether A is string or weak.

An additional feature of this game tree with imperfect information is that some nodes of the players are connected together by dashed lines (or sometimes enclosed within a blob/bubble if you prefer to use this way of connecting them) that define **information sets**. Note also the moves from each connected node have now been identically named (so for example firm B choosing to be bought out has been labelled as b_1 at both of player B's connected decision nodes rather than b_1 and b_3 as it was previously).

The concept of an information set is that during a play of the game, given their current knowledge of the game, a player only knows that they are at **some** node within the information set, but does not know at precisely which one of the connected nodes they are at. In figure 83 above, firm B does not know whether A has a strong/weak product. Hence if firm A decides to sell their product on the market (choice a_1 or a_3), firm B does not know which of their two decision nodes they are at. The nodes are thus connected with a dashed line to identify this as an information set of player B.

Remarks:

- In an information set, all nodes belong to the same player and all decision nodes in the set must have the same possible set of moves available (otherwise it would be distinguishable which node a player was at). We give these moves the same name to highlight this lack of knowledge of which precise node the player is at (hence why b_3 was relabelled as b_1 and b_4 as b_2 in our example game).
- In a game tree of imperfect information all decision nodes in the tree should be partitioned into disjoint information sets (in other words, each decision node should belong to exactly one information set). However in many cases a player may actually know precisely which node they are at. In this case the resulting information set consists of just this one node. No dashed line/bubble is used to connect this node with itself as this node plays in essence just like a game tree of perfect information.
- A further consequence of the notion that a player does not know at which node they are at in an information set means that no two nodes in an information set may share a path through the tree. This is true, at least, assuming that the players recall **all** of their prior moves in a game and **all** information they have learned during the game (any known opponents moves etc). We call this property of the players **perfect recall** and will take a brief look at this in a little more detail in section 8.7. Throughout our course we will always assume our players play games perfectly, and as such always have perfect recall.

8.6 Strategies in Game Trees of Imperfect Information

Because a player does not know at which node they are at within an information set, they should make the **same choice** of move at each node within that set. It would make no sense to try to recommend a player do a specific move at a specific node in this set, rather, based on everything they know and have learned about the game, they should make the move that they believe is best for them taking into account they they are at some node in the set.

In our example game in figure 83, firm B does not know at which node they are at when the game arrives at some node in their information set, they just know that firm A has chosen either a_1 or a_3: i.e firm A has put their product on the market. As such it no longer makes sense to try to prescribe different moves at each individual node. Instead, given firm B's current knowledge of the game, and knowing that they are at some node in the information set, we will try to determine an 'optimal' choice for the firm (which as mentioned is then this same choice at **every** node in the set). As such, we specify slightly further our definition of a pure strategy in a game tree from earlier for use in a game tree with imperfect information.

Definition 8.75. In a game tree with imperfect information a (pure) strategy for a player specifies a move for **every information set** of that player.

As before, the **strategic form** of the game tree is then defined in an analogous way.

8.6.1 A note on Reduced Strategies

As before, when we were considering game trees of perfect information, there may be some decision nodes for a player in the game tree that become unreachable due to an earlier choice of move by a player.

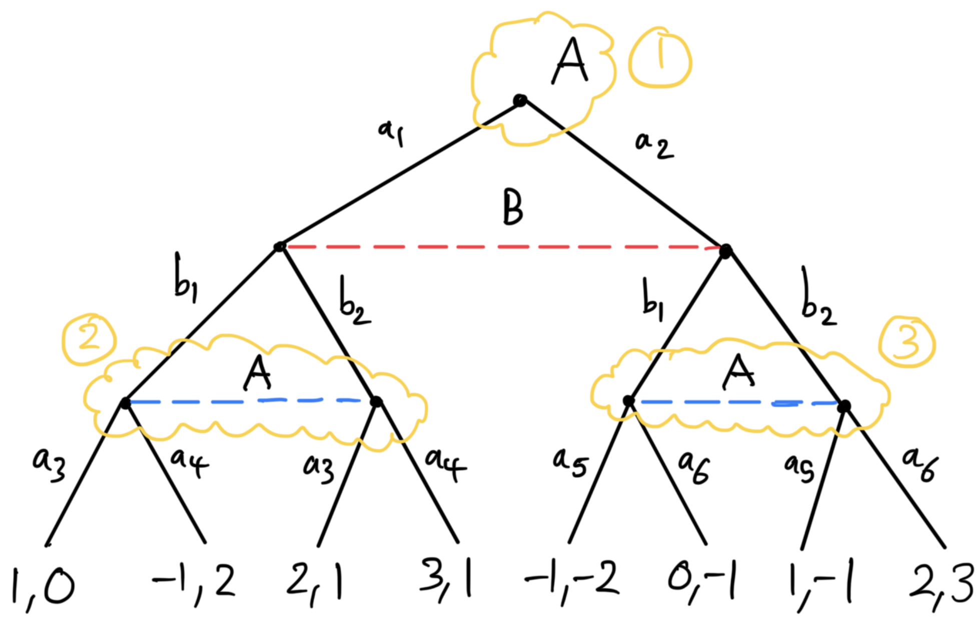

Figure 84: An example game tree of imperfect information to illustrate reduced strategies.

In the example above in figure 84, player A has 8 pure strategies; a choice of a_1 or a_2 at their first information set (the set circled and labelled ①, the root of the tree), then a choice of a_3 or a_4 at their next information set (the set labelled ②) and finally a_5 or a_6 in their last information set (labelled ③), giving $2 \times 2 \times 2 = 8$ pure strategies.

However, the initial choice of move a_1 or a_2 will leave one of the remaining two information sets **unreachable** for player A. As such we would typically consider A's **reduced strategies** in such a game; there are four:

$$a_1 a_3 *, \quad a_1 a_4 *, \quad a_2 * a_5, \quad a_2 * a_6,$$

where as usual, we don't specify the move in the part of the game tree that is unreachable, but replace it with a star. In a similar manner as before we can also produce the strategic form of the game tree and what we may call the reduced strategic form of the game tree which considers the strategic form produced from just the reduced strategies rather than all strategies.

8.7 Perfect Recall

Let's talk about what the information sets ①, ② and ③ describe in example 84. At set ① player A knows that they are at the start of the game. At set ② player A does **not know** the move of player B but does know that they are in set ②; i.e that they played a_1 at the start of the game. At set ③ again the move of player B is unknown but it is known that they played a_2 earlier in the game.

Consider instead now the slight variation of this game, represented in figure 85.

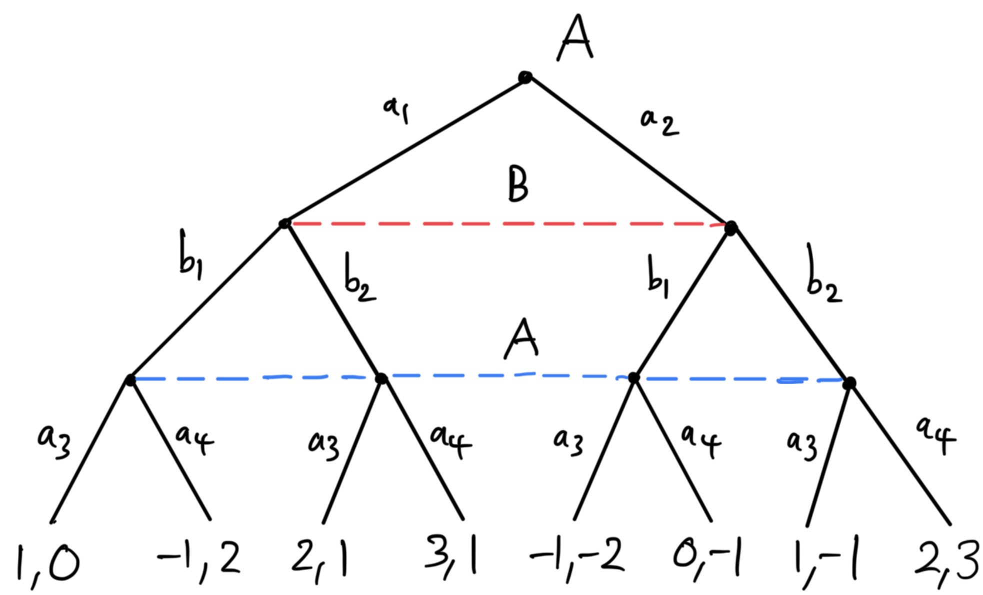

Figure 85: An example game tree of imperfect information to illustrate the concept of imperfect recall.

Information sets ② and ③ have now been merged to form one new set, we relabel moves a_5 and a_6 to a_3 and a_4 to highlight that A does not know which node in this set they are at. In this new game we say that player A has **imperfect recall**. This is because they have 'forgotten' whether they played move a_1 or a_2 at the start of the game.

Although imperfect recall is very common in many games between humans, throughout all of our analysis, to ensure our players remain rational and play optimally, we will always insist they have **perfect recall**; that is they remember all information throughout the game. Imperfect recall refers to a player that does not have perfect recall.

8.8 Analysing our Example Game

Before we introduce the concept of a behaviour strategy and re-discuss SPEs, let's go back to our example game of imperfect information between firms A and B and discuss a solution scheme. The game in figure 83 is re-shown below in figure 86 for ease of reference.

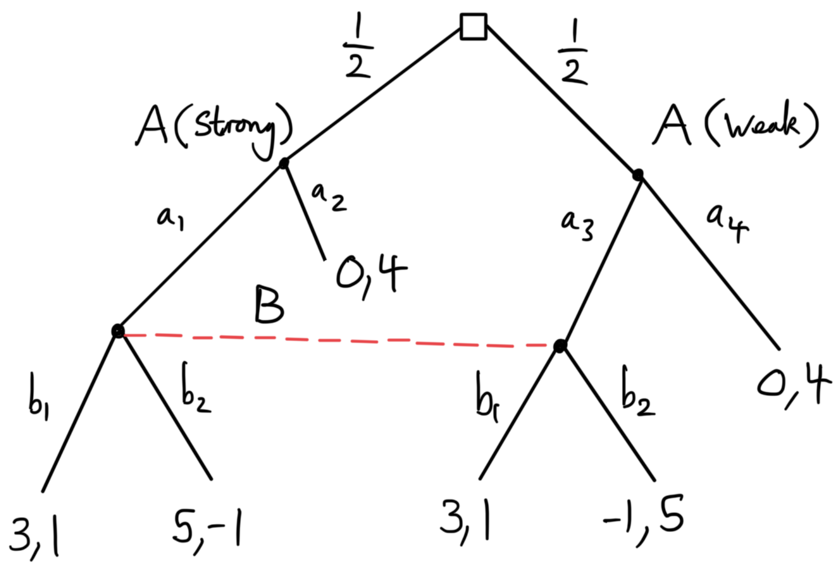

Figure 86: Our example game of imperfect information between firms A and B.

When A has a strong product, it is clear firm A should play a_1 (they will get at least a payoff of 3 regardless of B's move, which is more than A would get by playing a_2). However, studying the game tree a little longer, we see there is no deterministic (pure strategy) behaviour of the players that defines an equilibrium - if B plays b_1, A would always want to play a_1 or a_3, but then B would rather have played b_2 in the left decision node. A similar issue occurs if B plays b_2. Let's instead now consider the strategic form of the imperfect information game tree, shown in figure 87.

A \ B	b_1	b_2
a_1a_3	$3, 1$	$2, 2$
a_1a_4	$\frac{3}{2}, \frac{5}{2}$	$\frac{5}{2}, \frac{3}{2}$
a_2a_3	$\frac{3}{2}, \frac{5}{2}$	$-\frac{1}{2}, \frac{9}{2}$
a_2a_4	$0, 4$	$0, 4$

Figure 87: The strategic form of the example game of imperfect information between firms A and B.

Recall that firm B will play b_1 or b_2 at both nodes now as they are in the same information set, hence they only have two pure strategies. This game in figure 87 has two strictly dominated strategies for firm A; a_2a_3 and a_2a_4, so we delete these to arrive at the game in figure 88.

		B	
		b_1	b_2
A	a_1a_3	$3, 1$	$2, 2$
	a_1a_4	$\frac{3}{2}, \frac{5}{2}$	$\frac{5}{2}, \frac{3}{2}$

Figure 88: The strategic form of the example game of imperfect information between firms A and B with strictly dominated pure strategies removed for firm A.

This game has no pure strategy equilibria, so a sensible next step is to seek any mixed strategy equilibria of the game (**Exercise:** Do this). We find that randomising over playing a_1a_3 and a_1a_4 with probability 1/2 on each for firm A and over b_1 and b_2 with probabilities 1/4 and 3/4 respectively for firm B gives us a mixed equilibrium of the game. This gives expected payoffs of 9/4 for firm A and 7/4 for firm B.

So we have found a mixed strategy equilibrium in the strategic form for our example game. We would like to relate this in some way to the sequential game tree version of the game. To understand how we may do this we briefly discuss the concept of behaviour strategies in the next section.

8.9 Behaviour Strategies

Game trees of perfect information always have an equilibrium in pure strategies that can be found by backward induction. This is not the case when we deal with game trees of imperfect information and we need to re-introduce mixed strategies.

We have just found a mixed equilibrium in the strategic form of our example game. This was firm A mixing (1/2,1/2) over pure strategies a_1a_3 and a_1a_4 and firm B mixing (1/4, 3/4) over pure strategies b_1 and b_2. Let's discuss how to interpret these strategies in terms of the original game tree of imperfect information.

Typically, our interpretation of mixed strategies throughout the course has been that our players decide these at the beginning of a play of the game and then, upon performing some realisation of this randomness, stick to this plan of action throughout the game. When considering game trees of imperfect information we can view a mixed strategy differently, as what we may term a **behaviour strategy**. A behaviour strategy is a randomised choice of a move for each information set of the player, i.e rather than performing some realisation of the randomness at the beginning of the game and then adhering to this plan throughout (which is quite unnatural in reality), the player performs a realisation of this randomness whenever they reach a new information set (a new decision that needs to be made).

In our example game between the two firms, the behaviour strategy viewpoint of the mixed strategy equilibrium turns out to be exactly the same thing since the game is small, but take, for example, the mixed strategy $(a_1a_3*,\ a_1a_4*,\ a_2*a_5,\ a_2*a_6) = (3/10{,}4/10{,}3/10{,}0)$ for player A in the example game discussed in figure 84. Treating this as a **behaviour strategy** it would be played as follows:

- At their first information set, ①, the root of the tree, a_1 would be played with probability $3/10+4/10 = 7/10$, where a_2 played with probability $3/10+0 = 3/10$.
- Then, if arriving at information set ②, a_3 would be played with probability
$$\frac{\frac{3}{10}}{\frac{3}{10}+\frac{4}{10}} = \frac{3}{7},$$
where a_4 with probability
$$\frac{\frac{4}{10}}{\frac{3}{10}+\frac{4}{10}} = \frac{4}{7}.$$
- If arriving at information set ③, a_5 would be played with certainty.

This way of performing the randomness from the mixed strategy is considered more natural when playing the game.

Remark: There is much more than can be said about behaviour strategies - for instance it is not always actually possible to enact all possible mixed strategies as a behaviour strategy. As far as we will be concerned in this course however we will not discuss these deeper concepts and just want to understand how we may interpret any mixed strategy equilibria we find in a more natural way, as behaviour strategies.

8.10 Subgames and Subgame-Perfect Equilibria in Game Trees of Imperfect Information

Finally we come to a notion of what makes sense to deem a solution of our game trees of imperfect information. For game trees of perfect information, we argued that an SPE would suffice to be a solution of the game. Now we have introduced imperfect information to the problem, we need to incorporate a few changes to this.

When we looked at games of perfect information, we defined a subgame to be any subtree of the game, given by some node of the tree as a root for the subgame with all of its descendants. This needs to be modified for games of imperfect information so that every player has to **know** they are in this subtree.

Definition 8.76. In a game tree with imperfect information, a **subgame** is any subtree of the game such that every information set is either a subset of or disjoint from the nodes of the subtree.

In other words, each information set lies fully in the subgame, or fully outside of the subgame. Based on this updated definition we now slightly amend the definition of an SPE for a game of imperfect information.

Definition 8.77. In a game tree with imperfect information, with perfect recall of all players, a **subgame-perfect equilibrium** (SPE) is a profile of behaviour strategies that defines an equilibrium in **every** subgame of the game.

Typically, as before for games of perfect information, an equilibrium in a game tree of imperfect information can only be termed subgame-perfect if each players behaviour strategy is not specified in terms of reduced strategies and correctly defines a probability distribution for each information set of the player. It is still good practice to use reduced strategies to first determine any equilibria in the game and then retrospectively consider how to find any that are subgame-perfect (there will be examples on problem sheet 8 to help you understand this).

Finally, perhaps slightly worried by the thought of whether an SPE can always exist in behaviour strategies in a game tree of imperfect information, we state a very powerful theorem.

Theorem 8.78. *Any game tree with imperfect information where the players have perfect recall has an SPE in behaviour strategies.*

Proof. Not examinable so this has been omitted. If you are interested in how we may prove this page 294 of the text *Game Theory Basics* by Bernhard Von Stengel gives a proof but requires some deeper consideration into behaviour strategies than what we have dipped into (all details are in this text if interested however). □

So there is always at least one SPE in behaviour strategies in a game tree of imperfect information when we assume perfect recall of our players. This is what we will take as a reasonable solution to our games!