

Intro to ODEs - First Order ODEs

Quiz

Are the following first ODEs linear (A), separable (B), Homogenous (C) or Bernoulli (D)?

$$x \frac{dy}{dx} - y = x^2 + 1$$

$$\frac{dy}{dx} = \frac{\sin(x)y^3}{1+x}$$

$$\frac{dy}{dx} - xy = \frac{x^3}{y^2}$$

$$e^x \frac{dy}{dx} - \cos(x)y = 0$$

$$\frac{dy}{dx} = \frac{x+y+2}{2x+y+3}$$

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- ▶ Implicit form:

$$G(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

- ▶ Explicit form:

$$\frac{d^2y}{dx^2} = F(x, y, \frac{dy}{dx})$$

This is common in Mechanics, Newton's second law with independent variable t . Difficult to solve for general F but there are some special cases that can be solved as described in the following.

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1. F only depends on x

$$\frac{d^2y}{dx^2} = F(x)$$

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2. F only depends on x and $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = F(x, \frac{dy}{dx})$$

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3. F only depends on y

$$\frac{d^2y}{dx^2} = F(y) \quad \text{let } u = \frac{dy}{dx} \quad \Rightarrow \quad \frac{du}{dx} = F(y)$$

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Example: Mechanics Harmonic Oscillator

Hooke's law states if $x(t)$ is displacement relative to an ideal spring relaxed position, the spring force is: $F = -kx$ Using second Newton Law we have: $ma = F \implies m\frac{d^2x}{dt^2} = -kx$

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$$u = \frac{dx}{dt} = \pm \sqrt{\frac{2E - kx^2}{m}} \implies \int \frac{dx}{\pm \sqrt{\frac{2E - kx^2}{m}}} = \int dt$$

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4. F only depends on y and $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = F(y, \frac{dy}{dx})$$

let $u = \frac{dy}{dx} \implies \frac{du}{dx} = F(y, u)$. So we have

$$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy} = \frac{d}{dy} \left(\frac{1}{2} u^2 \right).$$

Therefore we have the following first order ODE for $u(y)$ to solve

$$\frac{d}{dy} \left(\frac{1}{2} u^2 \right) = F(y, u).$$

Given $u_{GS}(y; c_1)$ being a general solution for the above ODE, we have the following first order ODE for $y(x)$:

$$\frac{dy}{dx} = u_{GS}(y; c_1).$$

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Quiz: Find the family of curves with constant radius of curvature R .

$$R(x, y) = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}.$$