Department of Mathematics
Imperial College London
Academic Year 2023-2024

**Partial Differential Equations in Action** **MATH50008**

## Problem Sheet 1

1. ✦ Show that for any constant $k$, the function $u(x,y) = e^{kx}\cos(ky)$ is a solution of Laplace's equation, i.e. $\Delta u = 0$.

2. ✦✦ As we will see later in the module, the one-dimensional heat equation can be used to describe the temperature $u(x,t)$ inside a conducting rod of length $L$. This equation reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

Find:

(a) its steady-state solution with the boundary conditions $u(0,t) = T_1$ and $u(L,t) = T_2$

(b) its steady-state solution with the boundary conditions $u(0,t) = T_1$ and $\frac{\partial u}{\partial x}(L,t) = 0$

in both cases, provide a physical intuition for the boundary conditions and the solution.

3. ✦✦ The propagation of sound in air in governed by the so-called wave equation. Find a solution of the following wave equation

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \sin t + x^3$$

4. ✦ Show that the following equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} = 0$$

has a solution of the form $u(x,y) = e^{\alpha x + \beta y}$. Find the constants $\alpha$ and $\beta$.

5. ✦✦ Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

(a) Write this equation in the coordinates $s = x$ and $t = x - y$.

(b) Find the general solution of the equation.

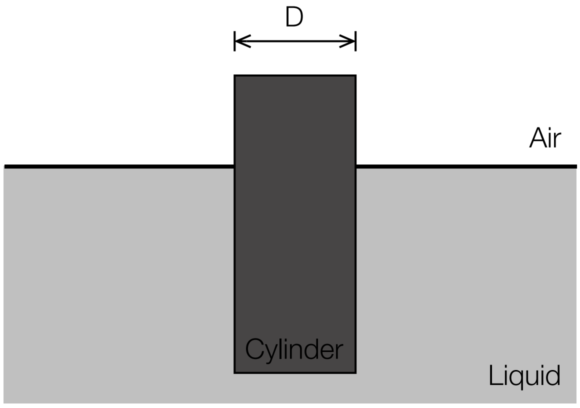
6. ✦ Assume that $u$ represents a mass density. Give the fundamental dimensions of the following quantities:

(a) $\dfrac{\partial}{\partial x}\left(u\dfrac{\partial u}{\partial t}\right)$

(b) $\dfrac{\partial^2}{\partial x^2}\left(u^3\dfrac{\partial u}{\partial t}\right)$

(b) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial t}(u^7)\right)$

7. ✦✦ We consider a cylinder floating upright in a liquid. When the cylinder is displaced slightly along its vertical axis it starts oscillating around its equilibrium position at frequency $\omega$. How does the frequency change if we increase the mass of the cylinder?




8. (✦✦✦) Consider that dominoes of width $w$ and height $h$ are placed standing in a straight line, evenly spaced at a distance $d$ of each other. At time $t = 0$, we knock over the first domino. Everybody who has ever played with dominoes knows what happens then, as the dominoes fall a wave propagates along the domino line. Use dimensional analysis to show that the wave velocity can be dimensionally reduced to the following expression

$$v = \sqrt{gh}F(\Pi_1, \Pi_2) \tag{2}$$

where $\Pi_1$ and $\Pi_2$ are two dimensionless groups to be determined. What assumption (usually valid) would allow you to simplify this problem further?

9. (✦✦✦) In industry, an important problem is to understand the flow of viscous liquids in pipes as it is crucial to efficient distribution across factories for instance. Say that an oil of dynamic viscosity $\nu$ flows down the factory through a long straight pipe of cross-sectional area $A$, driven by a pressure drop per unit length, $p$. Due to increased demands, your boss wants to double the volume flow rate $Q$ (i.e., volume passing through the pipe per unit time) and leaves you with the task of finding the new cross-sectional area $A$ needed knowing that all other parameters need to remain unaltered. To answer this question, you can either solve a complex fluid dynamics problem or you can use dimensional analysis, get the answer easily and go get a well-deserved break. Find $A$.

10. (✦✦✦) **The Barbenheimer strikes again...** and I have taken the side of Oppenheimer! Those of you who have seen the movie know that in 1945, the US army conducted the first nuclear test, the so called 'Trinity' test. While this explosion opened the nuclear age, calculating the actual energy released by the bomb proved to be a very complex problem because of the large number of physical and chemical processes involved in the detonation. The British government (which cooperated with the US army in the Manhattan project) asked GI Taylor, one of the most brilliant British applied mathematician of the time, to work on predicting the bomb's yield. For security reasons, GI Taylor did not have access to classified information and so worked independently from the US army. In 1950, photos of the explosion were released (see Fig. 1); as you can see below, the pictures contained a timestamp (time spent since detonation) and a lengthscale. With just these photographs and dimensional analysis, GI Taylor (and others) independently estimated the energy of the bomb to be 17 kilotons of TNT; no need to say that the US army wasn't thrilled to see these results published in the public domain!

   Taylor made only very few assumptions to come to that conclusion. He knew that the energy would be released from a small volume and assumed that it would expand as a spherical shock wave. From this, Taylor gauged that the only relevant variables should be: the radius of the shock wave $R$, the blast energy $E$, the time since detonation $t$ and the density of the air $\rho$.

   (a) Use dimensional analysis to determine how the radius depends on the other relevant physical variables.

   (b) It was shown by G.I. Taylor that if $E = 1$ J and $\rho = 1\ \mathrm{kg/m^3}$, then $R = t^{2/5}\ \mathrm{m/s^{2/5}}$. Use this information to find the exact formula for $R$.

(c) Using Fig. 1 and your results to the previous question, estimate the energy released. The air density is $\rho = 1\mathrm{kg/m^3}$.

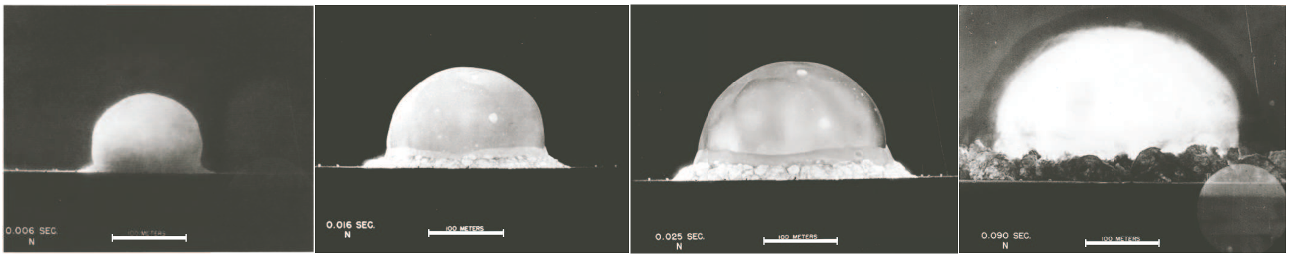



Figure 1: Snapshots of the shock wave produced by the Trinity nuclear explosion at $t = 6\ \mathrm{ms}$, $t = 16\ \mathrm{ms}$, $t = 25\ \mathrm{ms}$ and $t = 90\ \mathrm{ms}$. The width of the white bar on each of the snapshot is $100\ \mathrm{m}$