MATH50010 - Probability for Statistics
Unseen Problem 8

The transition matrix P of a Markov chain $\{X_n\}$ is:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

1. Derive the transition diagram from the transition matrix P.
2. Find the absorbing probabilities (i.e. the probability of entering a a class and never leaving) when we start at each of the recurrent states.
3. Find the stationary distributions of the chain.

1. *The transition diagram is given in Figure .*
2. *There are two recurrent classes $R1 = \{1, 6, 8\}$ and $R2 = \{4, 7, 10\}$, and the chain is absorbed in these classes once it enters them. Suppose the chain starts at a transient state $k \in T = \{2, 3, 5, 9\}$ and consider the probability it ever enters R_1. Let*

$$a_k = \mathbb{P}(\exists n > 0,\, X_n \in R_1 \mid X_0 = k).$$

Then, partitioning according to the value of X_1

$$\begin{aligned} a_k &= \mathbb{P}(\exists n > 0,\, X_n \in R_1 \mid X_0 = k) \\ &= \mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in R_2 \mid X_0 = k) \\ &\quad +\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in R_1 \mid X_0 = k) \\ &\quad +\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in T \mid X_0 = k). \end{aligned}$$

As R_2 is an absorbing class, once we enter R_2 we will never enter R_1. Thus,

$$\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in R_2 \mid X_0 = k) = 0.$$

The second probability considers the probability that we enter R_1 in the first step. By looking at the transition diagram, this is only possible when $k = 5$. Thus, for $k \in T \backslash \{5\}$,

$$\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in R_1 \mid X_0 = k) = 0$$

and

$$\begin{aligned} &\mathbb{P}(\exists n > 0, X_n \in R_1,\, X_1 \in R_1 \mid X_0 = 5) \\ &\qquad = \mathbb{P}(\exists n > 0,\, X_n \in R_1 \mid X_1 \in R_1,\, X_0 = 5)\mathbb{P}(X_1 \in R_1 \mid X_0 = 5). \end{aligned}$$

The first term is equal to one as R_1 is an absorbing class. For the second term, the only way we can enter R_1 from state 5 in one step is along the path $5 \to 8$. This occurs with probability $1/2$ so,

$$\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in R_1 \mid X_0 = 5) = 1/2.$$

Finally, we can write

$$\begin{aligned}
\mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 \in T \mid X_0 = k) &= \sum_{l \in T} \mathbb{P}(\exists n > 0,\, X_n \in R_1,\, X_1 = l, \mid X_0 = k) \\
&= \sum_{l \in T} \mathbb{P}(\exists n > 0,\, X_n \in R_1 \mid X_1 = l)\mathbb{P}(X_1 = l \mid X_0 = k) \\
&= \sum_{l \in T} \mathbb{P}(\exists n > 0,\, X_n \in R_1 \mid X_0 = l) p_{kl}.
\end{aligned}$$

Thus,

$$a_k = 1/2 \cdot \mathbf{1}\{k = 5\} + \sum_{l \in T} a_l p_{kl}$$

and substituting in values of p_{kl} we have

$$\begin{aligned}
a_2 &= a_5 \\
a_3 &= a_3/2 + a_5/4 \\
a_5 &= 1/2 + a_3/4 \\
a_9 &= a_9/4 + 3a_2/4.
\end{aligned}$$

Solving gives $a_2 = a_5 = a_9 = 4/7$ and $a_3 = 2/7$. Similar arguments can be used to find the absorption probabilities into R_2.

3. *Let π be a stationary distribution for this markov chain. We know from lectures that $\pi_2 = \pi_3 = \pi_5 = \pi_9 = 0$ as these are the transient states. Further, let $\pi(1)$ and $\pi(2)$ be the stationary distributions corresponding to the recurrent classes R_1 and R_2. These will have zero entries in any states that are not present in the recurrent class. Let $\nu(1)$ and $\nu(2)$ be the non-zero entries of $\pi(1)$ and $\pi(2)$ respectively. Then, it must be true that*

$$\nu(1)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \nu(1), \qquad \nu(2)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = \nu(2).$$

Solving these systems of equations, we have

$$\nu(1) \in (a, a, a)^T, \qquad \nu(2) = (b, 2b, b)^T$$

for $a, b \in \mathbb{R}$. As the entries in $\nu(1)$ and $\nu(2)$ must be non-negative and sum to one, we must have

$$\nu(1) = (1/3, 1/3, 1/3)^T, \qquad \nu(2) = (1/4, 2/4, 1/4).$$

Hence,

$$\begin{aligned}
\pi(1) &= (1/3, 0, 0, 0, 0, 1/3, 0, 1/3, 0, 0)^T \\
\pi(2) &= (0, 0, 0, 1/4, 0, 0, 2/4, 0, 0, 1/4)^T.
\end{aligned}$$

Any stationary distribution can be written as $\lambda_1\pi(1) + \lambda_2\pi(2)$ for some $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. Thus, any stationary distribution π can be writted as

$$\pi = (\lambda_1/3, 0, 0, \lambda_2/4, 0, \lambda_1/3, 2\lambda_2/4, \lambda_1/3, 0, \lambda_2/4)^T.$$

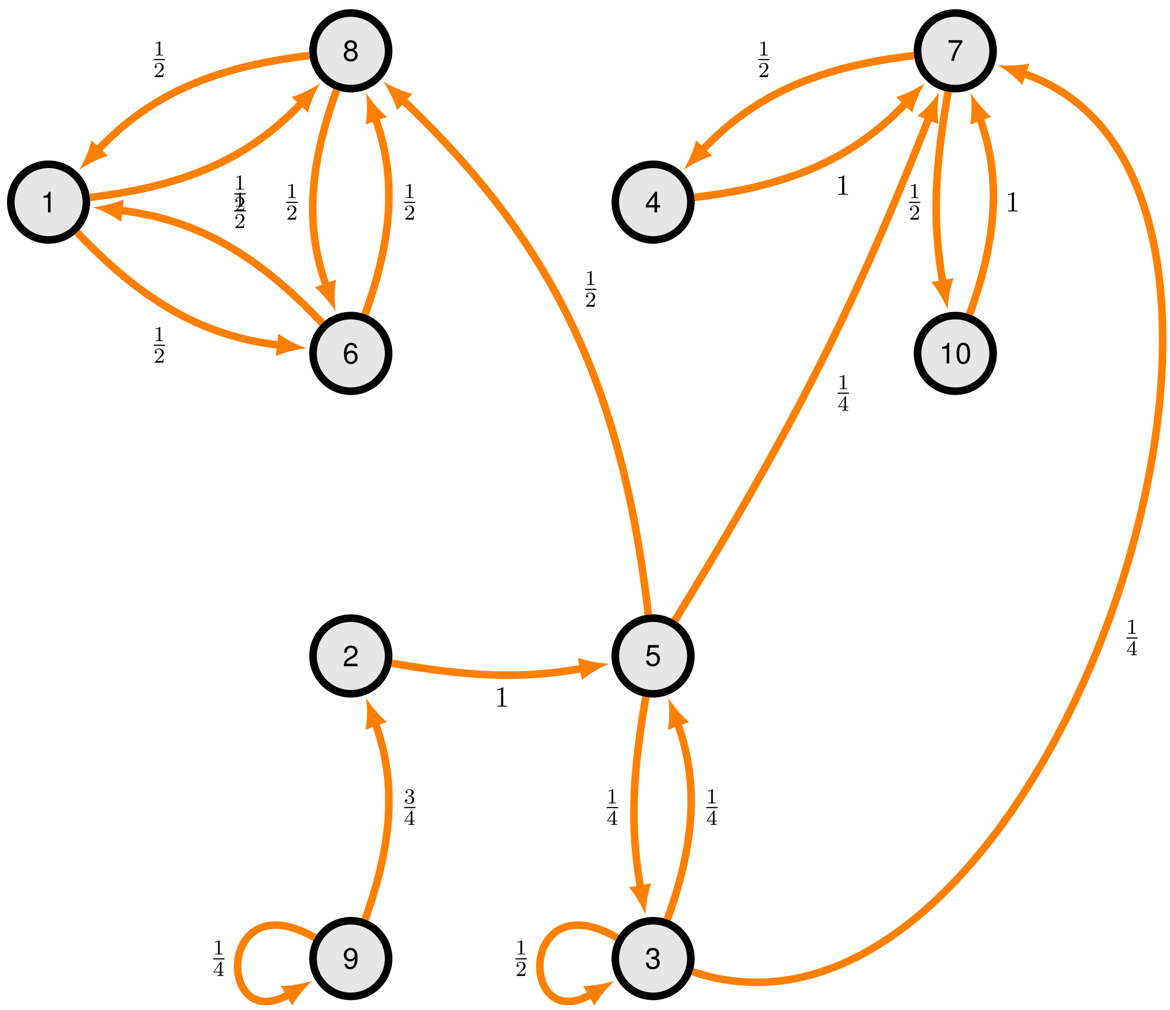

Figure 1: Transition diagram