Imperial College London
Department of Mathematics

Solutions to Blackboard quiz 2

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

You should enter your answers on Blackboard by 1pm on Wednesday 8 February 2023. The test is worth 1.5 percent of the marks for the module.

(A) (The following text refers to Questions 1 - 2) In each of the following questions, find $\det(A)$, where $A, B \in M_{3\times 3}(\mathbb{R})$.

Qu 1: $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det(B) = 5$, and $B - A = \begin{pmatrix} 2a & 0 & 0 \\ 2d & 0 & 0 \\ 2g & 0 & 0 \end{pmatrix}$.

Solution: A has been obtained from B by changing the sign to its first column. Hence $\det(A) = -5$.

Qu 2: $A^3 = A^t B A^{-1}$ where $\det(B) = 125$.

Solution: Taking determinants, the relation in the question gives $\det(A)^3 = \det(B)$. Hence $\det(A)^3 = 125$, which gives $\det(A) = 5$.

(B) (The following text refers to Questions 3 - 5) Say whether each of the following statements is true for all $A, B \in M_5(\mathbb{R})$.

Qu 3: If A has one eigenvalue and 5 distinct eigenvectors, then it is diagonalisable.

Solution: False, they may be distinct but still be linearly dependent.

Qu 4: If A has a unique eigenspace of dimension 5, then it is diagonalisable.

Solution: True. This follows from Definition 6.3.1.

Qu 5: If $\chi_A(X) = \chi_B(X)$, then there is an invertible matrix $P \in M_5(\mathbb{R})$, such that $P^{-1}AP = B$.

Solution: False. Take, for example,

$$A = I_5, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the eigenspace relative to 1 has dimension 5 for A and dimension 4 for B. Hence A is diagonalisable, while B is not. This means that a matrix P with the required properties cannot exist.

(C) (The following text refers to Questions 6 - 10) For each of the matrices below, determine whether it is:

i) Diagonalisable over $\mathbb{R}$ and not over $\mathbb{Q}$.

ii) Diagonalisable over $\mathbb{C}$ and not over $\mathbb{R}$.

iii) Diagonalisable over $\mathbb{R}$ and $\mathbb{Q}$.

iv) None of the above.

Qu 6: $\begin{pmatrix} 3 & 10 \\ -1 & -3 \end{pmatrix}$;

Solution: Diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$.

Qu 7: $\begin{pmatrix} 3 & -7 \\ 1 & -3 \end{pmatrix}$;

Solution: Diagonalisable $\mathbb{R}$ but not over $\mathbb{Q}$.

Qu 8: $\begin{pmatrix} 0 & 1 & -2 \\ -1 & -2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$;

Solution: Not diagonalisable. The eigenspace relative to -1 has dimension 1 but the multiplicity of -1 as a root of the characteristic polynomial is 2.

Qu 9: $\begin{pmatrix} 0 & -2 & -2 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$.

Solution: Not diagonalisable. This has two eigenvalues 1 and 2, the latter with multiplicity two as a zero of $\chi_A(x)$. The eigenspace relative to 2 has dimension 1, so the matrix is diagonalisable.

Qu 10: $\begin{pmatrix} -4 & 7 & -7 \\ -3 & 5 & -4 \\ -1 & 1 & 0 \end{pmatrix}$.

Solution: Diagonalisable over $\mathbb{R}$ and not $\mathbb{Q}$. This has characteristic polynomial $(X^2 - 2)(X - 1)$. Thus it has three distinct eigenvalues over $\mathbb{R}$ but not over $\mathbb{Q}$.