Tensor Calculus and General Relativity **Spring Term 2023**

Answers to Problem Sheet 1

1. Use the inverse transform

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right), \quad y = y', \quad z = z'.$$

$$\frac{dx}{dt} = \frac{\gamma(dx'/dt' + v)}{\gamma(1 + (v/c^2)dx'/dt')} = v,$$
$$\frac{dy}{dt} = \frac{dy'/dt'}{\gamma(1 + (v/c^2)dx'/dt')} = \frac{w}{\gamma}.$$

$$\text{magnitude} = \sqrt{v^2 + w^2/\gamma^2} = \sqrt{v^2 + \left(1 - \frac{v^2}{c^2}\right)w^2} = \sqrt{v^2 + w^2 - \frac{v^2w^2}{c^2}}.$$

2. K' is connected to K through the boost

$$ct' = ct\cosh\psi - x\sinh\psi, \quad x' = x\cosh\psi - ct\sinh\psi, \quad y' = y, \quad z' = z$$

where $\tanh\psi = v/c$.

K'' is connected to K' through the boost

$$ct'' = ct'\cosh\phi - x'\sinh\phi, \quad x'' = x'\cosh\phi - ct'\sinh\phi, \quad y'' = y', \quad z'' = z'$$

where $\tanh\phi = w/c$.

Combining the two transformations

$$ct'' = ct(\cosh\phi\cosh\psi + \sinh\phi\sinh\psi) - x(\sinh\phi\cosh\psi + \cosh\phi\sinh\psi)$$

$$x'' = x(\cosh\phi\cosh\psi + \sinh\phi\sinh\psi) - ct(\sinh\phi\cosh\psi + \cosh\phi\sinh\psi).$$

or

$$ct'' = ct\cosh(\phi+\psi) - x\sinh(\phi+\psi), \quad x'' = x\cosh(\phi+\psi) - ct\sinh(\phi+\psi)$$

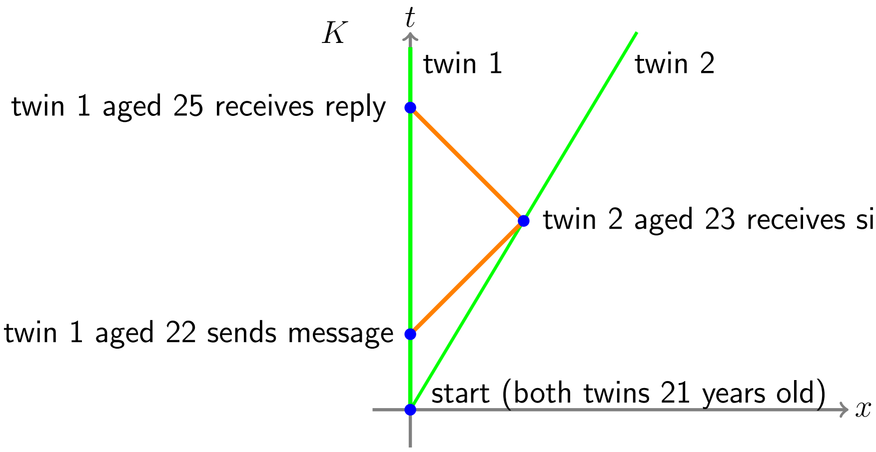
The velocity, u, of K'' with respect to K satisfies

$$\frac{u}{c} = \tanh(\phi+\psi) = \frac{\tanh\phi + \tanh\psi}{1 + \tanh\phi\tanh\psi} = \frac{w+v}{c(1 + wv/c^2)}.$$

3. (a) $\frac{3}{5}c$.

   (b) 25 years.

   (c) Space-time diagram:



   The space-time diagram helps to answer parts (a) and (b)! Here K is the inertial frame where twin 1 is at rest. The trajectory of twin 1 is $x = 0$, that of twin 2 is $x = vt$ where v is the speed of twin 2 relative to the Earth. To determine v it is sufficient to find the coordinates of the event where twin 2 receives the signal from twin 1. The trajectory of this signal is (measuring time in years) $x = c(t-1)$. The intersection satisfies $x = vt = c(t-1)$ so that $t = c/(c-v)$. Now use that twin two receives the signal when she has aged 2 years or $\sqrt{1-v^2/c^2}\; t = \sqrt{(c+v)/(c-v)} = 2$ with solution $v = \frac{3}{5}c$.

4. Do any two Lorentz boosts commute?

   No (but boosts in the same direction do commute). For example, consider boosts in the x and y direction. The matrices representing these transformations do not commute.

5. The matrix Λ corresponding to the boost quoted in question 2 is

$$\Lambda = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(\psi K)$$

   where

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To see this use $K^{2n} = J = \text{diag}(1,1,0,0)$ and $K^{2n+1} = K$ so that

$$\exp(\psi K) = I + \psi K + \frac{\psi^2 K^2}{2!} + \frac{\psi^3 K^3}{3!} + \ldots = \cosh\psi J + \sinh\psi J + \text{diag}(0,0,1,1) = \Lambda.$$

For boosts in the y and z direction $\Lambda = \exp(\psi K_y)$ and $\Lambda = \exp(\psi K_y)$, respectively with

$$K_y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

6. Write the equations as $a^0 = \alpha u^1, \quad a^1 = \alpha u^0$ where $\alpha = A/c$. This yields a system of first order ODEs for u^0 and u^1

$$\frac{du^0}{d\tau} = \alpha u^1, \quad \frac{du^1}{d\tau} = \alpha u^0.$$

General solution $u^0 = C\cosh(\alpha\tau) + D\sinh(\alpha\tau), \quad u^1 = C\sinh(\alpha\tau) + D\cosh(\alpha\tau)$ where C and D are arbitrary constants. As the particle is at rest for $\tau = 0$ we have $C = c$ and $D = 0$ so that

$$u^0(\tau) = c\cosh(\alpha\tau), \quad u^1(\tau) = c\sinh(\alpha\tau).$$

Integrating with respect to τ and using $x^0 = x^1 = 0$ for $\tau = 0$

$$x^0 = \frac{c}{\alpha}\sinh(\alpha\tau), \quad x^1 = \frac{c}{\alpha}(\cosh(\alpha\tau) - 1).$$

or

$$x^0 = \frac{c^2}{A}\sinh(A\tau/c), \quad x^1 = \frac{c^2}{A}(\cosh(A\tau/c) - 1).$$

The world-line is a branch of a hyperbola including the origin with the linear asymptotes $x^1 = \pm x^0 - c^2/A$.