Imperial College London

MATH97021 MATH97099

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 2

Date: Thursday, 13 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. This question involves several sub-problems covering the material appearing throughout the module. If any computations are required to complete these subproblems, they should be fairly short. Unless stated otherwise, assume these subproblems are unrelated to each other.

(a) Let $\hat{a}$ be a bosonic annihilation operator. Evaluate and simplify the following commutator: $[\hat{a}, \hat{a}^\dagger \hat{a}^\dagger]$. (3 marks)

(b) Show that the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \gamma \cos\left(\frac{2\pi\hat{x}}{a}\right)$ commutes with the discrete translation operator $\hat{\mathcal{T}} = e^{-i\hat{p}a/\hbar}$. (3 marks)

(c) Your friend, late at night, considers the Hamiltonian of (b) in the infinite-mass limit $\hat{\mathcal{H}}_{m\to\infty}$ and observes that the position eigenkets $|x\rangle$ are eigenstates of this Hamiltonian. Then your friend has the following thought: "Since I know that $[\hat{\mathcal{T}}, \hat{\mathcal{H}}_{m\to\infty}] = 0$ and that $|x\rangle$ are eigenstates of $\hat{\mathcal{H}}_{m\to\infty}$, the position eigenkets will also be eigenstates of $\hat{\mathcal{T}}$." Is your friend's reasoning correct? Please explain. (4 marks)

(d) Consider the Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \left|\phi\right\rangle \left\langle\phi\right|$, where $|\phi\rangle$ is an arbitrary state and $\lambda > 0$. Show that the ground state energy of $\hat{\mathcal{H}}$ cannot be smaller than the ground state energy of $\hat{\mathcal{H}}_0$. (4 marks)

(e) Let $\hat{S}_x$ and $\hat{S}_y$ be spin operators for a spin-s particle. Let $\hat{\mathcal{H}} = \alpha\hat{S}_x\hat{S}_y + \beta\hat{S}_y\hat{S}_x$ where α and β are constants. Suppose $\hat{\mathcal{H}}$ is Hermitian *and* time-reversal invariant. What constraints does this place on α and β? (3 marks)

(f) Let $\hat{c}$ be a fermionic annihilation operator and let $\hat{n} = \hat{c}^\dagger\hat{c}$. For constant α, the unitary operator $\hat{U} = e^{i\alpha\hat{n}}$ can be written as $\hat{U} = 1 + f(\alpha)\hat{n}$. Determine $f(\alpha)$. (3 marks)

(Total: 20 marks)

2. In this question we consider the following Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}(\hat{p}_y - B\hat{x})^2$$

which describes a (charged) particle in two dimensions under the presence of a magnetic field. In this, B is a positive constant and the position and momentum operators satisfy the usual canonical commutation relations.

(a) Show that the above Hamiltonian commutes with $\hat{p}_y$. Explain why this means that we can write eigenstates of the Hamiltonian in the form $\phi_k(x, y) = e^{iky} f_k(x)$ where k is a real parameter and $f_k(x)$ is some function that depends on x but not y. (4 marks)

(b) Substitute $|\phi_k\rangle = e^{ik\hat{y}} |f_k\rangle$ into the time-independent Schrödinger equation for the above Hamiltonian. Solve this to determine the ground state energy. Hint: Think of the 1d harmonic oscillator. In particular, show that the ground state energy is independent of the parameter k. This massively degenerate collection of ground states is known as the lowest Landau level. (6 marks)

(c) In addition to $\hat{p}_y$, the operator $\hat{\pi}_x = \hat{p}_x - B\hat{y}$ commutes with the above Hamiltonian. Evaluate $[\hat{\pi}_x, \hat{p}_y]$. Show that $|\phi_k\rangle$ will *not* be an eigenstate of $\hat{\pi}_x = \hat{p}_x - B\hat{y}$. (To do this, it is not necessary to use an explicit expression for $|\phi_k\rangle$.) (4 marks)

(d) From $\hat{\pi}_x$, one can construct the so-called magnetic translation operator as $\hat{\mathcal{T}} = e^{-i\hat{\pi}_x\xi/\hbar}$ where ξ is a real constant. Show that $\hat{\mathcal{T}}$ will transform between different states within the lowest Landau level. That is, show that

$$\hat{\mathcal{T}} |\phi_k\rangle = |\phi_{k'}\rangle ,$$

explicitly finding k'. (6 marks)

(Total: 20 marks)

3. Let $\hat{a}$ and $\hat{b}$ be bosonic annihilation operators satisfying the usual commutation relations. For instance, $[\hat{a}, \hat{a}^\dagger] = 1, [\hat{a}, \hat{b}^\dagger] = 0$. In this question, we focus on the Hamiltonian

$$\hat{\mathcal{H}} = -\hbar\omega(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \tag{1}$$

where $\omega > 0$. This is an effective model for bosons in a two-well system. That is, $\hat{a}^\dagger$ creates a particle in the left well, while $\hat{b}^\dagger$ creates a particle in the right well.

(a) Evaluate and simplify $[\hat{a}, \hat{\mathcal{H}}]$ and $[\hat{b}, \hat{\mathcal{H}}]$ and hence determine the Heisenberg equations of motion for $\hat{a}_{\mathrm{H}}$ and $\hat{b}_{\mathrm{H}}$. (6 marks)

(b) Solve the Heisenberg equations of motion you found in part (a) to determine $\hat{a}_{\mathrm{H}}(t)$ and $\hat{b}_{\mathrm{H}}(t)$. Hint: one of the solutions should be of the form $\hat{a}_{\mathrm{H}}(t) = \cos(\omega t)\hat{a} + if(t)\hat{b}$ where $f(t)$ is a real function of time to be determined. (6 marks)

(c) The operator $\hat{n} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$ denotes the relative particle number between the two wells. Determine $\hat{n}$ in the Heisenberg picture (which is $\hat{n}_{\mathrm{H}} = \hat{\mathcal{U}}^\dagger\hat{n}\hat{\mathcal{U}}$) and simplify. (4 marks)

(d) The system starts in an initial state and then evolves according to the Hamiltonian in equation (1). We are interested in the expectation value of $\hat{n}$ at later times. Determine $\langle\psi(t)|\,\hat{n}\,|\psi(t)\rangle$ as a function of time for the case of the initial states: (i) $|\psi(t=0)\rangle = \frac{1}{\sqrt{N!}}(\hat{a}^\dagger)^N\,|0\rangle$ and (ii) $|\psi(t=0)\rangle = e^{-|z_1|^2/2}e^{-|z_2|^2/2}e^{z_1\hat{a}^\dagger + z_2\hat{b}^\dagger}\,|0\rangle$. In these expressions, N is a positive integer, $z_{1,2}$ are complex numbers, and $|0\rangle$ is the vacuum state. (4 marks)

(Total: 20 marks)

4. In this problem we will consider the driven harmonic oscillator described by the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\tilde{\gamma}\cos(\Omega t)\hat{x}$$

where $\tilde{\gamma}, \lambda,$ and Ω are all positive parameters.

(a) Using the harmonic oscillator ladder operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}$, show that this Hamiltonian can be rewritten as

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda\gamma\cos(\Omega t)(\hat{a} + \hat{a}^\dagger)$$

where $\gamma = \sqrt{\frac{\hbar}{2m\omega}}\tilde{\gamma}$. (4 marks)

(b) Under the condition $|\omega + \Omega| \gg |\omega - \Omega|$, the dynamics given by

$$\hat{\mathcal{H}}_{\mathrm{rwa}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda\frac{\gamma}{2}(\hat{a}e^{i\Omega t} + \hat{a}^\dagger e^{-i\Omega t})$$

is a good approximation to the dynamics given by $\hat{\mathcal{H}}$ from (a). This is known as the rotating wave approximation. Using the perturbative result derived in lecture for the transition probability $P_{n\to n'}$ explain why such an approximation is plausible. For the remainder of this question we will take the dynamics to be governed by $\hat{\mathcal{H}}_{\mathrm{rwa}}$. (5 marks)

(c) At time $t = 0$, the system is in the harmonic oscillator ground state $|0\rangle$ where $\hat{a}\,|0\rangle = 0$. Using perturbation theory, determine the probability $P_{0\to 1}$, to second order in λ, that the system will transition to the first excited state after time t. (6 marks)

(d) Now take the condition of resonance: $\omega = \Omega$. Show that for this case, $P_{0\to 1}$ computed in (c) will grow without bound as t increases. Therefore this perturbative result will become inaccurate for sufficiently large t. As a better approach, find a unitary operator $\hat{U}$ such that $\hat{\mathcal{H}}'_{\mathrm{rwa}} = \hat{U}\hat{\mathcal{H}}_{\mathrm{rwa}}\hat{U}^\dagger - i\hbar\hat{U}\partial_t\hat{U}^\dagger$ is time independent (taking $\omega = \Omega$ throughout). (5 marks)

(Total: 20 marks)

5. In this problem we will consider fermionic systems having a Hilbert space of finite dimension.

(a) Let $\hat{c}$ be a fermionic annihilation operator such that $\{\hat{c}, \hat{c}^\dagger\} = 1$. Define the following so-called Majorana operators: $\hat{\gamma} = \hat{c} + \hat{c}^\dagger$ and $\xi = -i(\hat{c} - \hat{c}^\dagger)$. (Notice the analogy with expressing the position and momentum operators in terms of ladder operators.) Show that $\hat{\xi}^2 = \hat{\gamma}^2 = 1$ and $\xi\hat{\gamma} = -\hat{\gamma}\xi$. Thus, these operators square to one and anti-commute with one another. Furthermore, show that $\hat{c}^\dagger\hat{c} = \frac{1}{2}(1 + i\hat{\gamma}\hat{\xi})$.

(6 marks)

(b) Next consider the two-site Hamiltonian

$$\hat{\mathcal{H}} = -w(\hat{c}_1^\dagger\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_1 + \hat{c}_1\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_1^\dagger)$$

where w is a positive constant and $\hat{c}_1$ and $\hat{c}_2$ are separate fermionic annihilation operators. Remember $\{\hat{c}_1, \hat{c}_2^\dagger\} = \{\hat{c}_1, \hat{c}_2\} = 0$. Evaluate and simplify $\hat{\mathcal{H}}^2$. The resulting expression should be sufficiently simple that you will not need to do any additional calculations to determine its eigenvalues. Determine the values of the squares of the eigenenergies of $\hat{\mathcal{H}}$. With this calculation we have determined the eigenenergies of $\hat{\mathcal{H}}$ up to a sign. (6 marks)

(c) Introduce separate Majorana operators for these fermionic operators, similar to what was done in (a): $\hat{c}_1 = \frac{1}{2}(\hat{\gamma}_1 + i\hat{\xi}_1)$, $\hat{c}_2 = \frac{1}{2}(\hat{\gamma}_2 + i\hat{\xi}_2)$. With these, show that the Hamiltonian of (b) can be written as

$$\hat{\mathcal{H}} = -iw\hat{\gamma}_1\hat{\xi}_2.$$

(4 marks)

(d) Determine the eigenenergies of the Hamiltonian from this problem. Hint: consider re-writing (c) in terms of strategically-chosen fermionic operators. (4 marks)

(Total: 20 marks)

Solutions for Quantum Mechanics II Exam, 2021

1. This question involves several sub-problems covering the material appearing throughout the module. If any computations are required to complete these subproblems, they should be fairly short. Unless stated otherwise, assume these subproblems are unrelated to each other.

   (a) Let $\hat{a}$ be a bosonic annihilation operator. Evaluate and simplify the following commutator: $[\hat{a}, \hat{a}^\dagger \hat{a}^\dagger]$.

   Seen. $[\hat{a}, \hat{a}^\dagger \hat{a}^\dagger] = 2\hat{a}^\dagger$.

   (b) Show that the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \gamma \cos\left(\frac{2\pi\hat{x}}{a}\right)$ commutes with the discrete translation operator $\hat{\mathcal{T}} = e^{-i\hat{p}a/\hbar}$.

   Similar seen. From lecture we have that $\hat{\mathcal{T}}^\dagger \hat{x} \hat{\mathcal{T}} = \hat{x} + a$. Spatial translation acts trivially on momentum. Note that $\hat{\mathcal{T}}^\dagger \cos\left(\frac{2\pi\hat{x}}{a}\right) \hat{\mathcal{T}} = \cos\left(\frac{2\pi(\hat{x}+a)}{a}\right) = \cos\left(\frac{2\pi\hat{x}}{a}\right)$. Therefore $\hat{\mathcal{T}}^\dagger \hat{\mathcal{H}} \hat{\mathcal{T}} = \hat{\mathcal{H}}$ which means that $[\hat{\mathcal{T}}, \hat{\mathcal{H}}] = 0$.

   (c) Your friend, late at night, considers the Hamiltonian of (b) in the infinite-mass limit $\hat{\mathcal{H}}_{m\to\infty}$ and observes that the position eigenkets $|x\rangle$ are eigenstates of this Hamiltonian. Then your friend has the following thought: "Since I know that $[\hat{\mathcal{T}}, \hat{\mathcal{H}}_{m\to\infty}] = 0$ and that $|x\rangle$ are eigenstates of $\hat{\mathcal{H}}_{m\to\infty}$, the position eigenkets will also be eigenstates of $\hat{\mathcal{T}}$." Is your friend's reasoning correct? Please explain.

   Unseen. The corresponding eigenvalues will be $\gamma \cos\left(\frac{2\pi x}{a}\right)$. The reasoning is incorrect. This is because the states $|x + an\rangle$ for integer n will have the same eigenenergies. So we would need to diagonalise the translation operator within the degenerate subspace. One can explicitly check that a position ket is not an eigenstate of the translation operator.

   (d) Consider the Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda |\phi\rangle \langle\phi|$ where $|\phi\rangle$ is an arbitrary state and $\lambda > 0$. Show that the ground state energy of $\hat{\mathcal{H}}$ cannot be smaller than the ground state energy of $\hat{\mathcal{H}}_0$.

   Similar seen. The Let $|\phi\rangle$, $|\phi_0\rangle$ be the ground states of $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_0$ respectively. Let E and E_0 be the corresponding energies. By expanding $|\psi\rangle$ in a basis of eigenstates of $\hat{\mathcal{H}}_0$ it follows that $\langle\psi| \hat{\mathcal{H}}_0 |\psi\rangle \geq \langle\psi_0| \hat{\mathcal{H}}_0 |\psi_0\rangle$. We also note that the piece multiplying λ is positive definite. Therefore we have $\langle\psi| \hat{\mathcal{H}} |\psi\rangle = \langle\psi| \hat{\mathcal{H}}_0 |\psi\rangle + \lambda |\langle\psi|\phi\rangle|^2 \geq \langle\psi_0| \hat{\mathcal{H}}_0 |\psi_0\rangle + \lambda |\langle\psi|\phi\rangle|^2 \geq \langle\psi_0| \hat{\mathcal{H}}_0 |\psi_0\rangle = E_0$.

   (e) Let $\hat{S}_x$ and $\hat{S}_y$ be spin operators for a spin-s particle. Let $\hat{\mathcal{H}} = \alpha \hat{S}_x \hat{S}_y + \beta \hat{S}_y \hat{S}_x$ where α and β are constants. Suppose $\hat{\mathcal{H}}$ is Hermitian *and* time-reversal invariant. What constraints does this place on α and β?

   Unseen. Requiring $\hat{\mathcal{H}}$ to be Hermitian tells us that $\alpha = \beta^*$. Requiring $\hat{\mathcal{H}}$ to be TRI requires α and β to be real. So together, we have $\alpha = \beta$ and α is real.

   Possibly similar seen. Directly diagonalising seems hard! Instead, we recognise that this Hamiltonian is TRI and describes a half-inter spin. So Kramer's theorem applies. So all states, including the ground state, will be degenerate.

(f) Let $\hat{c}$ be a fermionic annihilation operator and let $\hat{n} = \hat{c}^\dagger \hat{c}$. For constant α, the unitary operator $\hat{U} = e^{i\alpha\hat{n}}$ can be written as $\hat{U} = 1 + f(\alpha)\hat{n}$. Determine $f(\alpha)$.

Unseen. One can, for instance, expand the exponent, or (easier) act with $\hat{U}$ on eigenstates. One finds $f = e^{i\alpha} - 1$.

2. In this question we consider the following Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}(\hat{p}_y - B\hat{x})^2$$

which describes a (charged) particle in two dimensions under the presence of a magnetic field. In this, B is a positive constant and the position and momentum operators satisfy the usual canonical commutation relations.

(a) Show that the above Hamiltonian commutes with $\hat{p}_y$. Explain why this means that we can write eigenstates of the Hamiltonian in the form $\phi_k(x, y) = e^{iky} f_k(x)$ where k is a real parameter and $f_k(x)$ is some function that depends on x but not y.

Unseen. We first note that $\hat{y}$ does not appear in the above Hamiltonian. In other words it is translationally invariant in the y-direction. It will therefore commute with $\hat{p}_y$. Because of this symmetry $\hat{p}_y$ and $\hat{\mathcal{H}}$ can be simultaneously diagonalised. The expression written in the question is the general form of eigenstates of $\hat{p}_y$ (with eigenvalue $\hbar k$).

(b) Substitute $|\phi_k\rangle = e^{ik\hat{y}} |f_k\rangle$ into the time-independent Schrödinger equation for the above Hamiltonian. Solve this to determine the ground state energy. Hint: 1d harmonic oscillator. In particular, show that the ground state energy is independent of the parameter k. This massively degenerate collection of ground states is known as the lowest Landau level.

Unseen. Subbing in, one finds

$$\left(\frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}(\hbar k - B\hat{x})^2\right)|f_k\rangle = E_k |f_k\rangle .$$

We recognise this as the (displaced) harmonic oscillator with effective frequency $\omega = B/m$. The ground state energy will be $\hbar\omega/2 = \hbar B/(2m)$, independent of k.

(c) In addition to $\hat{p}_y$, the operator $\hat{\pi}_x = \hat{p}_x - B\hat{y}$ commutes with the above Hamiltonian. Evaluate $[\hat{\pi}_x, \hat{p}_y]$. Show that $|\phi_k\rangle$ will *not* be an eigenstate of $\hat{\pi}_x = \hat{p}_x - B\hat{y}$. (To do this, it is not necessary to use an explicit expression for $|\phi_k\rangle$.)

Unseen. Direct evaluation gives $[\hat{\pi}_x, \hat{p}_y] = -i\hbar B$. Suppose that $|\phi_k\rangle$ is a simultaneous eigenstate of these operators. Then it follows that $[\hat{\pi}_x, \hat{p}_y]|\phi_k\rangle = 0$. But from the above commutation relation this would mean $|\phi_k\rangle = 0$.

(d) From $\hat{\pi}_x$, one can construct the so-called magnetic translation operator as $\hat{\mathcal{T}} = e^{-i\hat{\pi}_x\xi/\hbar}$ where ξ is a real constant. Show that $\hat{\mathcal{T}}$ will transform between different states within the lowest Landau level. That is, show that

$$\hat{\mathcal{T}}\,|\phi_k\rangle = |\phi_{k'}\rangle\,,$$

explicitly finding k'.

Unseen. Noting the form of the Hamiltonian found in (b), one can find that the LLL solutions can be written as $|\phi_k\rangle = e^{ik\hat{y}}e^{-i\hat{p}_x k/B}\,|f_{k=0}\rangle$. This is an important realisation. Next, applying the magnetic translation we have

$$\hat{\mathcal{T}}\,|\phi_k\rangle = e^{i(k+B\xi/\hbar)\hat{y}}e^{-i\hat{p}_x(k+B\xi/\hbar)/B}\,|f_{k=0}\rangle = |\phi_{k'}\rangle$$

with $k' = k + B\xi/\hbar$.

3. Let $\hat{a}$ and $\hat{b}$ be bosonic annihilation operators satisfying the usual commutation relations. For instance, $[\hat{a},\hat{a}^\dagger] = 1$, $[\hat{a},\hat{b}^\dagger] = 0$. In this question, we focus on the Hamiltonian

$$\hat{\mathcal{H}} = -\hbar\omega(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$$

where $\omega > 0$. This is an effective model for bosons in a two-well system. That is, $\hat{a}^\dagger$ creates a particle in the left well, while $\hat{b}^\dagger$ creates a particle in the right well.

(a) Evaluate and simplify $[\hat{a},\hat{\mathcal{H}}]$ and $[\hat{b},\hat{\mathcal{H}}]$ and hence determine the Heisenberg equations of motion for $\hat{a}_{\mathrm{H}}$ and $\hat{b}_{\mathrm{H}}$.

Similar seen. The EOM are

$$\frac{d}{dt}\hat{a}_{\mathrm{H}} = i\omega\hat{b}_{\mathrm{H}}$$
$$\frac{d}{dt}\hat{b}_{\mathrm{H}} = i\omega\hat{a}_{\mathrm{H}}.$$

(b) Solve the Heisenberg equations of motion you found in part (a) to determine $\hat{a}_{\mathrm{H}}(t)$ and $\hat{b}_{\mathrm{H}}(t)$. Hint: one of the solutions should be of the form $\hat{a}_{\mathrm{H}}(t) = \cos(\omega t)\hat{a} + if(t)\hat{b}$ where $f(t)$ is a real function of time to be determined.

Similar seen. This solution is

$$\hat{a}_{\mathrm{H}}(t) = \cos(\omega t)\hat{a} + i\sin(\omega t)\hat{b}$$
$$\hat{b}_{\mathrm{H}}(t) = \cos(\omega t)\hat{b} + i\sin(\omega t)\hat{a}.$$

(c) The operator $\hat{n} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}$ denotes the relative particle number between the two wells. Determine $\hat{n}$ in the Heisenberg picture and simplify.

Unseen. It is very useful to note that it is unnecessary to construct and solve the Heisenberg EOM for $\hat{n}$. Instead this can be found from what we have already figured out in the previous parts. In particular,

$$\begin{aligned}\hat{n}_{\mathrm{H}} &= \hat{a}^\dagger_{\mathrm{H}} \hat{a}_{\mathrm{H}} - \hat{b}^\dagger_{\mathrm{H}} \hat{b}_{\mathrm{H}} \\ &= \cos(2\omega t)\hat{n} + i\sin(2\omega t)(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}).\end{aligned}$$

(d) The system starts in an initial state and then evolves according to the Hamiltonian of this problem. We are interested in the expectation value of $\hat{n}$ at later times. Determine $\langle\psi(t)|\,\hat{n}\,|\psi(t)\rangle$ as a function of time for the case of the initial states: (i) $|\psi(t=0)\rangle = \frac{1}{\sqrt{N!}}(\hat{a}^\dagger)^N |0\rangle$ and (ii) $|\psi(t=0)\rangle = e^{-|z_1|^2/2} e^{-|z_2|^2/2} e^{z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger} |0\rangle$. In these expressions, N is a positive integer, $z_{1,2}$ are complex numbers, and $|0\rangle$ is the vacuum state.

Unseen. For the first case we have $\langle\psi(t)|\,\hat{n}\,|\psi(t)\rangle = N\cos(2\omega t)$ while for the second case we have $\langle\psi(t)|\,\hat{n}\,|\psi(t)\rangle = \cos(2\omega t)(|z_1|^2 - |z_2|^2) + i\sin(2\omega t)(z_1^* z_2 - z_2^* z_1)$.

4. In this problem we will consider the driven harmonic oscillator described by the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\tilde{\gamma}\cos(\Omega t)\hat{x}$$

where $\tilde{\gamma}$, λ, and Ω are all positive parameters.

(a) Using the harmonic oscillator ladder operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}$, show that this Hamiltonian can be rewritten as

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda\gamma\cos(\Omega t)(\hat{a} + \hat{a}^\dagger)$$

where $\gamma = \sqrt{\frac{\hbar}{2m\omega}}\tilde{\gamma}$.

Similar seen. First one needs to express the position and momentum operators in terms of the ladder operators. Then, upon substitution into the given Hamiltonian, and upon utilising the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, one will arrive at the result.

(b) Under the condition $|\omega + \Omega| \gg |\omega - \Omega|$, the dynamics given by

$$\hat{\mathcal{H}}_{\mathrm{rwa}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda\frac{\gamma}{2}(\hat{a}e^{i\Omega t} + \hat{a}^\dagger e^{-i\Omega t})$$

is a good approximation to the dynamics given by $\hat{\mathcal{H}}$ from (a). This is known as the rotating wave approximation. Using the perturbative result derived in lecture for the transition probability $P_{n\to n'}$ explain why such an approximation is plausible. For the remainder of this question we will take the dynamics to be governed by $\hat{\mathcal{H}}_{\text{rwa}}$.

Unseen. For this one needs to carefully inspect the two terms that are present in $\hat{\mathcal{H}}$ but absent in $\hat{\mathcal{H}}_{\text{rwa}}$. Upon evaluating the perturbative result for $P_{n\to n'}$, one discovers that these terms are suppressed by a factor of $|\omega - \Omega|/|\omega + \Omega|$ in comparison to the result arising from the RWA Hamiltonian alone. The RWA amounts to dropping these small terms.

(c) At time $t = 0$, the system is in the harmonic oscillator ground state $|0\rangle$ where $\hat{a}\,|0\rangle = 0$. Using perturbation theory, determine the probability $P_{0\to 1}$, to second order in λ, that the system will transition to the first excited state after time t.

Similar seen. After correctly setting this up, and evaluating the integral one finds $P_{0\to 1} = \frac{\lambda^2\gamma^2}{\hbar^2(\omega-\Omega)^2}\sin^2\left(\frac{\omega-\Omega}{2}t\right)$.

(d) Now take the condition of resonance: $\omega = \Omega$. Show that for this case, $P_{0\to 1}$ computed in (c) will grow without bound as t increases. Therefore this perturbative result will become inaccurate for sufficiently large t. As a better approach, find a unitary operator $\hat{U}$ such that $\hat{\mathcal{H}}'_{\text{rwa}} = \hat{U}\hat{\mathcal{H}}_{\text{rwa}}\hat{U}^\dagger - i\hbar\hat{U}\partial_t\hat{U}^\dagger$ is time independent (taking $\omega = \Omega$ throughout).

Similar seen. Taking the limit of the two frequencies approaching each other one finds $P_{0\to 1} = \frac{\lambda^2\gamma^2}{4\hbar^2}t^2$. This will grow without bound as t increases and so this perturbation theory will inevitably break down. By experimentation, one can find that the unitary transformation given by $\hat{U} = e^{i\omega\hat{a}^\dagger\hat{a}t}$ will give $\hat{\mathcal{H}}'_{\text{rwa}} = \hbar\omega/2 + \frac{\gamma\lambda}{2}(\hat{a} + \hat{a}^\dagger)$ which is time independent.

5. In this problem we will consider fermionic systems having a Hilbert space of finite dimension.

(a) Let $\hat{c}$ be a fermionic annihilation operator such that $\{\hat{c}, \hat{c}^\dagger\} = 1$. Define the following so-called Majorana operators: $\hat{\gamma} = \hat{c} + \hat{c}^\dagger$ and $\xi = -i(\hat{c} - \hat{c}^\dagger)$. Notice the analogy with expressing the position and momentum operators in terms of ladder operators. Show that $\hat{\xi}^2 = \hat{\gamma}^2 = 1$ and $\xi\hat{\gamma} = -\hat{\gamma}\xi$. Thus, these operators square to one and anti-commute with one another. Furthermore, show that $\hat{c}^\dagger\hat{c} = \frac{1}{2}(1 + i\hat{\gamma}\hat{\xi})$.

Unseen. Working on this, $\hat{\gamma}^2 = \hat{c}\hat{c} + \hat{c}^\dagger\hat{c}^\dagger + \{\hat{c}, \hat{c}^\dagger\} = 0 + 0 + 1 = 1$. $\hat{\xi}^2 = 1$ follows from a similar calculation. Multiplying out, we find $\hat{\gamma}\hat{\xi} = i(\hat{c}\hat{c}^\dagger - \hat{c}^\dagger\hat{c})$ and $\hat{\xi}\hat{\gamma} = -i(\hat{c}\hat{c}^\dagger - \hat{c}^\dagger\hat{c})$ and so they anticommute. To evaluate the final expression, it is useful to solve for $\hat{c} = \frac{1}{2}(\hat{\gamma} + i\hat{\xi})$. Then using the previous parts of the problem, one finds the result.

(b) Next consider the two-site Hamiltonian

$$\hat{\mathcal{H}} = -w(\hat{c}_1^\dagger\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_1 + \hat{c}_1\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_1^\dagger)$$

where w is a positive constant and $\hat{c}_1$ and $\hat{c}_2$ are separate fermionic annihilation operators. Remember $\{\hat{c}_1, \hat{c}_2^\dagger\} = \{\hat{c}_1, \hat{c}_2\} = 0$. Evaluate and simplify $\hat{\mathcal{H}}^2$. The resulting expression should be sufficiently simple that you will not need to do any additional calculations to determine its eigenvalues. Determine the values of the squares of the eigenenergies of $\hat{\mathcal{H}}$. With this calculation we have determined the eigenenergies of $\hat{\mathcal{H}}$ up to a sign.

Unseen. After some algebra, one finds the impressively simple result $\hat{\mathcal{H}}^2 = w^2$. So the eigenenergies of $\hat{\mathcal{H}}$ will either be $+w$ or $-w$.

(c) Introduce separate Majorana operators for these fermionic operators, similar to what was done in (a): $\hat{c}_1 = \frac{1}{2}(\hat{\gamma}_1 + i\hat{\xi}_1)$, $\hat{c}_2 = \frac{1}{2}(\hat{\gamma}_2 + i\hat{\xi}_2)$. With these, show that the Hamiltonian of (b) can be written as

$$\hat{\mathcal{H}} = -iw\hat{\gamma}_1\hat{\xi}_2.$$

Unseen. This calculation is aided by the results of part (a). Looking at the various pieces:

$$\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1 = \frac{i}{2}(\hat{\gamma}_1\hat{\xi}_2 + \hat{\gamma}_2\hat{\xi}_1)$$
$$\hat{c}_1 \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1^\dagger = \frac{i}{2}(\hat{\xi}_1\hat{\gamma}_2 + \hat{\gamma}_1\hat{\xi}_2).$$

Putting together, one finds the result.

(d) Determine the eigenenergies of the Hamiltonian from this problem. Hint: consider re-writing (c) in terms of strategically-chosen fermionic operators.

Unseen. Choosing $\hat{d} = \frac{1}{2}(\hat{\gamma}_1 + i\hat{\xi}_2)$ and $\hat{f} = \frac{1}{2}(\hat{\gamma}_2 + i\hat{\xi}_1)$, one can see that these satisfy correct commutation relations. The Hamiltonian becomes

$$\hat{\mathcal{H}} = -w(2\hat{d}^\dagger\hat{d} - 1).$$

Notice that the other fermionic operator does not appear (meaning that there will be a degeneracy). So the eigenenergies will be w (2 states) and $-w$ (2 states).

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96008 MATH97021 MATH97099 Quantum Mechanics 2	1	This question had a similar format to Q1 from previous years. It is composed of several sub-problems covering aspects throughout the module. Some were very straightforward (e.g. everyone got full marks on a). Part(c) required some serious thought. I was delighted that a number of you were able to do this.
MATH96008 MATH97021 MATH97099 Quantum Mechanics 2	2	Q2 I thought was the most difficult out of the four (non-mastery) problems. However, some of the parts were designed to be easier (e.g. part a). Some received full marks or close to full marks on Q2 which I found remarkable.
MATH96008 MATH97021 MATH97099 Quantum Mechanics 2	3	I think everyone expected a question like Q3 (Heisenberg EOM) and indeed most did very well on this question.
MATH96008 MATH97021 MATH97099 Quantum Mechanics 2	4	Q4 was a problem on time-dependent perturbation theory which probably was also expected. However, we hadn't done anything like part b which many found tricky. Part a and c are bread and butter QM2 calculations and most did very well on these.