Solution 5

1. Suppose Anne has a utility function $u\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ which is strictly increasing, continuous and strictly quasi-concave.

Assume has an initial budget of $m > 0$ and that the prices for the two goods are $p_1, p_2 > 0$. Now assume that there is an increase in price for good 1 such that the new price for good 1 is $q_1 > p_1$. The price for good 2 stays fixed, so $q_2 = p_2$. Of course, Anne is not happy with this increase in price.

a) Explain why it would be reasonable to pay Anne an amount of

$$e\big(\underline{q}, v(\underline{p}, m)\big) - m$$

to compensate for this increase in price.

Solution: With prices $\underline{p}$ and budget m, Anne can attain a maximal utility of $v(\underline{p}, m)$. Due to Walras' Law she then spends her whole budget m. If prices increase to $\underline{q}$ and she wants to have the same level of utility $v(\underline{p}, m)$ she needs to spend at least $e(\underline{q}, v(\underline{p}, m))$. So it is fair to compensate her with the difference $e(\underline{q}, v(\underline{p}, m)) - m$.

b) Show that:

$$\begin{aligned} e\big(\underline{q}, v(\underline{p}, m)\big) - m &= e\big(\underline{q}, v(\underline{p}, m)\big) - e\big(\underline{p}, v(\underline{p}, m)\big) \\ &= \int_{p_1}^{q_1} x^*_{H,1}\big((z, p_2), v(\underline{p}, m)\big)\mathrm{d}z \end{aligned}$$

Solution: From the lecture, we have the identity

$$e\big(\underline{p}, v(\underline{p}, m)\big) = m,$$

which proves the first equality. One can re-write this difference as an integral and use the fundamental theorem of calculus:

$$\begin{aligned} e\big(\underline{q}, v(\underline{p}, m)\big) - e\big(\underline{p}, v(\underline{p}, m)\big) &= \int_{p_1}^{q_1} \frac{\mathrm{d}\, e\big((z, p_2), v(\underline{p}, m)\big)}{\mathrm{d}z}\, \mathrm{d}z \\ &= \int_{p_1}^{q_1} x^*_{H,1}\big((z, p_2), v(\underline{p}, m)\big)\, \mathrm{d}z. \end{aligned}$$

The last step is due to Shephard's lemma.

c) Assume that good 1 is a normal good. Use Slutzky's equation to show that:

$$\int_{p_1}^{q_1} x_1^*\big((z,p_2),m\big)\,\mathrm{d}z \le \int_{p_1}^{q_1} x_{H,1}^*\big((z,p_2),v(\underline{p},m)\big)\,\mathrm{d}z. \tag{1}$$

Solution: We know that

$$x_1^*(\underline{p},m) = x_{H,1}^*\big(\underline{p},v(\underline{p},m)\big). \tag{2}$$

That means both integrands in (1) coincide at their left endpoint. Slutzkly's equation helps to compare how fast the integrands grow. We obtain that for $p_1 \le z \le q_1$

$$\frac{\mathrm{d}\,x_{H,1}^*\big((z,p_2),v(\underline{p},m)\big)}{\mathrm{d}z} - \frac{\mathrm{d}\,x_1^*\big((z,p_2),m\big)}{\mathrm{d}z} = \frac{\mathrm{d}x_1^*\big((z,p_2),m\big)}{\mathrm{d}m} x_1^*\big((z,p_2),m\big) \ge 0,$$

where we used the fact that the good is a normal good. So Hicksian demand grows faster than Marshallian demand. With (2) we obtain that

$$x_1^*\big((z,p_2),m\big) \le x_{H,1}^*\big((z,p_2),v(\underline{p},m)\big), \qquad z \ge p_1.$$

With the monotonicity of the integral, the claim finally follows. $\square$

2. Consider the market for one good. Suppose that the market demand is given by $X(p) = 10-p$, where $p \geq 0$ is the price for the good. Suppose there are two firms with the same technology. Each of the firms has a cost function of $c^*(y) = 2+y^2$ (reflecting the actual economic costs), where $y \geq 0$ is the firm's output. Suppose that we have a perfect competition, that is, consumers and firms are price takers.

a) Compute the short-run equilibrium price as well as the short-run equilibrium quantity produced.

Solution: We have to solve the profit maximisation problem for each firm to calculate their supply function y_j^*. Profit is given by

$$\varphi_j(p, y) = py - c^*(y) = py - 2 - y^2.$$

This is a concave function in y (so it has a maximum). The first order condition yields:

$$p - 2y = 0.$$

So the optimal supply is $y^*(p) = p/2$.

Industry supply is given by

$$\sum_{j=1}^{2} y_j^*(p) = p.$$

The short-run equilibrium price p_s^* is the price where market demand and industry supply meet. So

$$X(p_s^*) = Y(p_s^*) \quad \Longleftrightarrow \quad 10 - p_s^* = p_s^* \quad \Longleftrightarrow \quad p_s^* = 5.$$

The short-run equilibrium quantity will then be

$$X(p_s^*) = Y(p_s^*) = 5.$$

b) Compute the consumers' surplus, the producers' surplus and the community surplus at the short-run equilibrium.

Solution: The producers' surplus at p_s^* is given by:

$$PS(p_s^*) = \int_0^{p_s^*} Y(p)\mathrm{d}p = \frac{1}{2} p_s^* Y(p_s^*) = 12.5.$$

To calculate the consumers' surplus, beware that the market demand is always non-negative. That means we should rather consider the demand function: $X(p) = \max\{10 - p, 0\}$. So

$$CS(p_s^*) = \int_{p_s^*}^{\infty} X(p)\mathrm{d}p = \int_{p_s^*}^{10} X(p)\mathrm{d}p = 12.5.$$

So the community surplus in the short-run is:

$$PS(p_s^*) + CS(p_s^*) = 25.$$

c) Suppose there are no entry barriers for firms to the market and any new firm has the same production technology as the 2 firms that are already in the market. How many firms will operate in the long-run equilibrium?

Also determine the long-run equilibrium price and the long-run equilibrium quantity.

Solution: We first analyse whether firms are making an (abnormal) profit or not. For $j = 1, 2$ we find:

$$\pi_j(p_s^*) = p_s^* y_j^*(p_s^*) - c(y_j^*(p_s^*)) = p_s^* \cdot p_s^*/2 - 2 - (p_s^*/2)^2 = \frac{1}{4}(p_s^*)^2 - 2 = 4.25.$$

This means that the 2 firms actually make an abnormal profit. That means there is an incentive for new firms to enter the market. They will enter the market as long as there will be the possibility to make a non-negative abnormal profit. Let J^* be the number of firms acting in the market in the long-run equilibrium. Then J^* is the largest integer J such that industry supply meets market demand and the individual profits are non-negative. That is, the largest integer J such that:

$$X(p_l^*) = Y(p_l^*) \qquad \text{and} \tag{3}$$
$$\pi_j^*(p_l^*) \geq 0 \quad \forall j = 1, \ldots, J. \tag{4}$$

Consider (3). This yields:

$$10 - p_l^* = Jp_l^*/2 \quad \Longleftrightarrow \quad p_l^* = \frac{10}{J/2 + 1}.$$

The profit for each firm is then given by:

$$\begin{aligned}
\pi_j^*(p_l^*) &= p_l^* y_j^*(p_l^*) - c(y_j^*(p_l^*)) \\
&= \frac{1}{2}(p_l^*)^2 - 2 - (p_l^*/2)^2 \\
&= \frac{1}{4}(p_l^*)^2 - 2 \\
&= \frac{1}{4}\frac{100}{(J/2+1)^2} - 2 \\
&= \frac{100}{(J+2)^2} - 2
\end{aligned}$$

We need that:

$$\pi_j^*(p_l^*) = \frac{100}{(J+2)^2} - 2 \geq 0 \Leftrightarrow (J+2)^2 \leq 50 \Leftrightarrow J \leq \sqrt{50} - 2 \approx 5.07$$

where we have used that $J \geq 0$. That means in the long-run there will be $J^* = 5$ firms.

Hence, the long-run equilibrium price is:

$$p_l^* = \frac{10}{5/2+1} = \frac{20}{7} \approx 2.86.$$

The long-run equilibrium quantity is:

$$q_l^* = Y(p_l^*) = \frac{50}{7} \approx 7.14.$$

d) Compute the consumers' surplus, the producers' surplus and the community surplus at the long-run equilibrium.

Solution: The producers' surplus is now:

$$PS(p_l^*) = \int_0^{p_l^*} 5p/2 \mathrm{d}p = \frac{5}{4}(p_l^*)^2 \approx 10.20,$$

which means there is a slight decrease. In the other hand, the consumer's surplus is now

$$CS(p_l^*) = \int_{p_l^*}^{10} (10-p)\mathrm{d}p = \frac{1}{2}(50/7)^2 = \frac{1250}{49} \approx 25.51.$$

So the consumers' surplus has more than doubled. In total, the community surplus will be:

$$PS(p_l^*) + CS(p_l^*) \approx 35.71.$$

e) Sketch the situation graphically and demonstrate that there is a gain in the community surplus.

Solution:

In Figure 1, we can nicely see the different slopes the industry supply has in the short-run and in the long-run. The area in red is the deadweight loss. However, it is rather a gain in our situation.

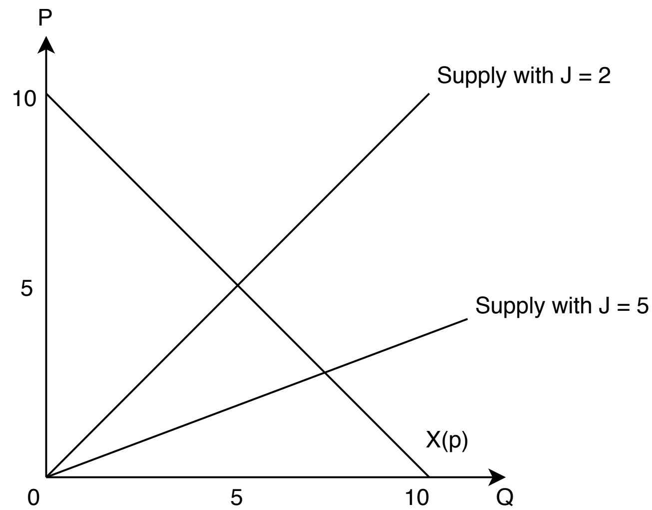

Figure 1: Solution to **2.e)**.

3. Consider a market demand that is described by the demand function:

$$X(p) = \frac{16}{p^2}.$$

On the other hand, the industry supply is given by:

$$Y(p) = 2p.$$

a) Compute the equilibrium price and quantity.

Solution: In the equilibrium, supply needs to meet demand. That is:

$$Y(p^*) = X(p^*) \quad \Longleftrightarrow \quad p^* = 2.$$

Then the equilibrium quantity is

$$Y(p^*) = X(p^*) = 2p^* = 4.$$

b) Compute the producers' surplus, the consumers' surplus and the community surplus.

Solution:

$$CS(p^*) = \int_0^{p^*} Y(p)\mathrm{d}p = (p^*)^2 = 4.$$
$$PS(p^*) = \int_{p^*}^{\infty} X(p)\mathrm{d}p = \frac{16}{p^*} = 8.$$

So the community surplus is 12.

c) Now suppose the government introduces a tax of 10% on the good. Calculate the new equilibrium price and the new equilibrium quantity.

Solution: The demand curve does not change. Also the supply curve does not change per se. However, the price 'facing the consumers' will change. Without taxes, they face the inverse supply function, which is

$$p(q) = q/2.$$

With taxes, the inverse supply is:

$$p_{tax}(q) = 1.1p(q) = 0.55q.$$

The inverse demand function is:

$$D(q) = \frac{4}{\sqrt{q}}.$$

The inverse demand and the inverse supply, facing the firm, must meet in the equilibrium quantity. That is:

$$p_{tax}(q^*_{tax}) = D(q^*_{tax})$$

This yields

$$q^*_{tax} = \left(\frac{4}{0.55}\right)^{2/3} \approx 3.75.$$

The equilibrium price with tax is

$$p^*_{tax} = D(q^*_{tax}) = p_{tax}(q^*_{tax}) \approx 2.06.$$

So we can nicely see that the equilibrium price has increased and the equilibrium quantity decreased (that's also what one would expect). However, the producers only get a price of:

$$p(q^*_{tax}) \approx 1.87.$$

d) Calculate the producers' surplus, the consumers' surplus and the revenue of the tax. How would you define the community surplus in this situation? What is the deadweight loss?

Solution: The consumer's surplus is quite straight forward. It is:

$$CS(p^*_{tax}) = \int_{p^*_{tax}}^{\infty} X(p)\mathrm{d}p = \frac{16}{p^*_{tax}} \approx 7.77.$$

For the producers' surplus, we should pay a little more attention. We must only integrate to the price they actually get. That is

$$PS = PS(p^*_{tax}/1.1) = PS(p(q^*_{tax})) = \int_0^{p(q^*_{tax})} Y(p)\mathrm{d}p = \left(p(q^*_{tax})\right)^2 \approx 3.52.$$

So we see that the producers bear the majority of the burden!

The revenue of the tax is:

$$Tax = q^*_{tax} \times \left(p^*_{tax} - p(q^*_{tax})\right) = q^*_{tax} \times 0.1p(q^*_{tax}) \approx 0.70.$$

However, if we want to calculate the community surplus, we must not only sum the consumers' and the producers' surplus. We also have to include the government into the situation. That is, the community surplus is the sum of all three components:

$$CS + PS + Tax = 7.77 + 3.52 + 0.7 = 11.9.$$

That means the deadweight loss has a size of $12 - 11.9 = 0.1$.

e) Now suppose that the government introduces a subsidy of 10%. Again, calculate the new equilibrium price and quantity, as well as consumers' surplus, producers' surplus and community surplus. What is the absolute size of the subsidy? What is the deadweight loss?

Solution: The reasoning is similar. The consumers now face an inverse supply of:

$$p_{sub}(q) = 0.9p(q) = 0.45q.$$

At the equilibrium quantity, the inverse demand function and the inverse supply facing the consumers must coincide. That is,

$$p_{sub}(q^*_{sub}) = D(q^*_{sub}).$$

This yields

$$q^*_{sub} = \left(\frac{4}{0.45}\right)^{2/3} \approx 4.29.$$

The equilibrium price with subsidies is:

$$p^*_{sub} = D(q^*_{sub}) = p_{sub}(q^*_{sub}) \approx 1.93.$$

This is the price facing the consumers. However, due to subsidies, the firms get a price of:

$$p(q^*_{sub}) \approx 2.15.$$

Let's calculate the size of the subsidy. This is:

$$Sub = q_{sub} \times \left(p(q^*_{tax}) - p^*_{sub}\right) \approx 0.94.$$

The consumers' surplus is

$$CS = CS(p^*_{sub}) = \int_{p^*_{sub}}^{\infty} X(p)\mathrm{d}p = \frac{16}{p^*_{sub}} = 8.29.$$

The producers' surplus is:

$$PS = PS(p(q^*_{sub})) = \int_0^{p(q^*_{sub})} Y(p)\mathrm{d}p = (p(q^*_{sub}))^2 \approx 4.62.$$

We see that the sum of consumers' and producers' surplus is 12.91 and exceeds the initial sum without the subsidy. However, in order to compute the community surplus, we need to include the government into the consideration. That is, we need to subtract the subsidies. We end up with a community surplus of:

$$CS + PS - Sub = 8.29 + 4.62 - 0.94 = 11.97.$$

That means, we also have a deadweight loss of $12 - 11.97 = 0.03$. That means, not only taxes cause a deadweight loss, but also subsidies.