Imperial College London

MATH70003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Geophysical Fluid Dynamics

Date: 5 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

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1. Main balances and planetary geostrophy

Assume the shallow-water approximation for a single layer of fluid with depth $h = H+\eta$, consisting of the depth at rest H and dynamic perturbation η. Assume flat bottom, constant fluid density ρ_0, gravity acceleration g, and Coriolis parameter f_0.

(i) (6 marks)

Write down horizontal momentum equations in the local Cartesian coordinates and explain physical meaning of the terms involved. Define Rossby number ϵ, assume that $\epsilon \ll 1$, and describe which flows correspond to this situation. Identify the leading geostrophic balance in the momentum equations. Sketch geostrophically balanced flow around a positive Gaussian-like pressure anomaly. What is proper scaling for dynamic pressure anomaly?

(ii) (6 marks)

Prove that hydrostatic balance for geostrophic flows holds even when vertical H and horizontal L scales of motion are similar. Use geostrophic and hydrostatic balances to estimate the surface elevation anomaly η, and express its scaling in terms of Rossby number ϵ and Froude number F, thus, proving that it is $[\eta] = \epsilon F H$.

(iii) (8 marks)

Write down the shallow-water continuity equation in terms of the fluid layer depth h and velocity $\mathbf{u}$. Consider planetary-scale motions characterized by $F \gg 1$ and $\epsilon \ll 1$. Expand the momentum and continuity equations in terms of ϵ, write down the leading-order balances, and, thus, derive the "planetary geostrophic" set of equations.

(Total: 20 marks)

2. Energy equation

Consider 1.5-layer QG model on the β-plane and derive the energy equation in the form

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = 0 \,,$$

where E is the energy, and $\mathbf{S}$ is the energy flux vector.

(i) (4 marks)

Write down the governing potential vorticity equation, explain its physical meaning, and explain the meanings of the potential vorticity components.

For the following use the potential vorticity equation to obtain the energy equation.

(ii) (6 marks)

Obtain E and explain the physical meaning of its terms.

(iii) (8 marks)

By deriving the energy equation, obtain $\mathbf{S}$ and write down both components of this vector.

(iv) (2 marks)

Assuming that the flow fluctuations are zero in the far field, prove that the total energy of the flow is integral invariant.

(Total: 20 marks)

3. Planetary waves

A group of fluid dynamicists working on atmospheric circulation of some exoplanet came to the conclusion that the midlatitude atmosphere can be approximated as a two-layer fluid, in which the bottom and top layers can be described in terms of materially conserved quantities Π_1 and Π_2. In locally Cartesian coordinates:

$$\Pi_1 = \nabla^2(\psi_1 - \psi_2) + \frac{\partial^2 \psi_1}{\partial x \partial y} + S\,(\psi_1 - \psi_2) + \beta y + \alpha x\,,$$

$$\Pi_2 = \nabla^2(\psi_2 - \psi_1) + \frac{\partial^2 \psi_2}{\partial x \partial y} + S\,(\psi_2 - \psi_1) + \beta y + \alpha x\,,$$

where ψ_1 and ψ_2 are the top- and bottom-layer velocity streamfunction, respectively; x is zonal (i.e., eastward) coordinate, and y is meridional (i.e., northward) coordinate. Parameters of the problem are given as $S > 0,\ \beta > 0,\ \alpha > 0$. In this problem you have to help the fluid dynamicists to understand the linear wave properties of the exoplanet.

(i) (2 marks)

What are physical dimensions of $\psi_1,\ S,\ \beta,\ \alpha,\ \Pi_1$?

(ii) (2 marks)

In the Eulerian framework, write down the laws governing the evolution of Π_1 and Π_2, and explain the meaning of each term.

(iii) (2 marks)

Linearize the laws obtained in (ii) around the state of rest, write them down and, thus, obtain the governing equations.

(iv) (4 marks)

Prove that the governing equations can be decoupled by into vertically uniform (barotropic) and nonuniform (baroclinic) components, and write them down.

(v) (5 marks)

Obtain the dispersion relation $\omega = \omega(k, l)$ for the barotropic waves, where ω is the frequency, and (k, l) is the wavevector. Sketch the dispersion curves, first, for fixed l, then, for fixed k.

(vi) (5 marks)

Obtain the dispersion relation $\omega = \omega(k, l)$ for the baroclinic waves. Consider meridionally uniform waves, that is, $l = 0$ and find out whether these waves can propagate energy (hence, information) to the east.

(Total: 20 marks)

4. Geostrophic adjustment

Consider initially unbalanced state of fluid on the f-plane that evolves toward the final state of geostrophic balance. Stay within the framework of linear shallow-water dynamics. Use conventional notations for the velocity (u, v, w), deviation of the free surface η, and pressure p.

(i) (5 marks)

Write down the governing equations for u, v, and η. Write down the potential vorticity (PV) conservation law and the expression for PV.

(ii) (3 marks)

Linearize both PV and PV conservation law.

(iii) (6 marks)

Consider an initial state with no motion and a step-like distribution for η :

$$\eta(y,0) = +\eta_0 \qquad for \qquad y < 0\,, \qquad\qquad \eta(y,0) = -\eta_0 \qquad for \qquad y > 0\,.$$

Find the initial distribution of PV. Write down equations governing the final adjusted state. What happened with PV and energy during the adjustment process?

(iv) (6 marks)

Find the final flow assuming that it is uniform in x.

(Total: 20 marks)

5. Eddy-mean interactions in zonal flow

Consider purely 2D dynamics described by the potential vorticity material conservation law:

$$\frac{\partial \zeta}{\partial t} + J[\psi, \zeta + \beta y] = F\,,$$

assuming the β-plane approximation, with relative vorticity ζ, velocity streamfunction ψ, and external forcing F. Assume also that the eddies are transient fluctuations coexisting and interacting with a purely zonal background flow described by zonal-mean velocity component, $\langle u\rangle(t,y)$, and relative vorticity, $\langle\zeta\rangle(t,y)$ (angular brackets denote zonal averaging).

(i) (4 marks)

Prove the area integral of the Jacobian is zero:

$$\iint J[A,B]\,dx\,dy = 0\,,$$

if A and/or B describe flow fluctuations that decay to zero in the far field.

(ii) (10 marks)

Derive prognostic equations for the zonal-mean $\langle\zeta\rangle$, and the fluctuation ζ', of the relative vorticity.

(iii) (6 marks)

Derive equations for meridionally integrated enstrophies, in the form

$$\frac{d}{dt}\int \frac{1}{2}\,\langle\zeta\rangle^2\,dy = \dots\,, \qquad \frac{d}{dt}\int \frac{1}{2}\,\langle\zeta'^2\rangle\,dy = \dots\,,$$

and identify on the right-hand sides the enstrophy conversion and production terms. Use the result proven in (a) to get rid of all terms that are neither conversions nor productions.

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

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M3A28, M4A28, M5A28

Introduction to Geophysical Fluid Mechanics (Solutions)

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1. (i) Equations have material derivative, as well as Coriolis and pressure-gradient forces balancing each other: $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{Du}{Dt} - f_0 v = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{Dv}{Dt} + f_0 u = -g\,\frac{\partial \eta}{\partial y}\,.$$

Rossby number is small for large-scale, rotation-dominated, and slow flows: $\epsilon = \dfrac{U}{f_0 L} \ll 1\,.$

The leading geostrophic balance is obtained by expanding: $\mathbf{u} = \mathbf{u}_g + \epsilon\,\mathbf{u}_a + \ldots\,, \qquad \eta = \eta_g + \epsilon\,\eta_a + \ldots\,,$

so that at the leading order:

$$-f_0 v_g = -g\,\frac{\partial \eta_g}{\partial x}\,, \qquad +f_0 u_g = -g\,\frac{\partial \eta_g}{\partial y}\,. \qquad (*)$$

Therefore, a positive pressure anomaly $p = \rho_0 g \eta$ induces clockwise rotating flow.
The scaling corresponding to $(*)$ is

$$[\eta_g] = \frac{f_0 U L}{g}$$

$\boxed{6,\ \text{A}}$

(ii) Use scalings $W = UH/L$, $T = L/U$, $[p'] = \rho_0 f_0 U L$, $U = \epsilon f_0 L$ to identify the *validity bound* of the hydrostatic balance: $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{Dw}{Dt} \ll \frac{1}{\rho_0}\frac{\partial p_g}{\partial z} \quad \Longrightarrow \quad \frac{HU^2}{L^2} \ll \frac{\rho_0 f_0 U L}{\rho_0 H} \quad \Longrightarrow \quad \epsilon\left(\frac{H}{L}\right)^2 \ll 1\,,$$

so, even when $H \sim L$, there is still $\epsilon \ll 1$, ensuring that hydrostatic balance holds.

$$[\eta_g] = \frac{f_0 U L}{g} = \frac{f_0^2 L^2 \epsilon}{gH}\,H = \epsilon H\left(\frac{L}{R}\right)^2 = \epsilon F H\,, \qquad \textit{where} \qquad F \equiv \frac{L^2}{R^2}\,, \qquad R \equiv \frac{\sqrt{gH}}{f_0}$$

$\boxed{6,\ \text{A}}$

$\boxed{\text{unseen} \Downarrow}$

(iii) Let's start from the full shallow-water equations,

$$\frac{Du}{Dt} - fv = -g\,\frac{\partial h}{\partial x}\,, \qquad \frac{Dv}{Dt} + fu = -g\,\frac{\partial h}{\partial y}\,, \qquad \frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0\,,$$

and consider $F = L^2/R^2 \sim \epsilon^{-1} \gg 1$. Then, depth variations are as large as the mean depth of fluid:

$$h = H\,(1 + \epsilon F \eta) = H\,(1 + \eta)\,,$$

and asymptotic expansions $\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \ldots$, and $\eta = \eta_0 + \epsilon \eta_1 + \ldots$ yield:

$$\epsilon\Big[\frac{\partial u_0}{\partial t} + \mathbf{u}_0 \nabla u_0 - f v_1\Big] - f v_0 = -gH\,\frac{\partial \eta_0}{\partial x} - \epsilon g H\,\frac{\partial \eta_1}{\partial x} + O(\epsilon^2)\,, \qquad \ldots\ldots,$$

$$\epsilon F\Big[\frac{\partial \eta_0}{\partial t} + \mathbf{u}_0\cdot\nabla \eta_0\Big] + (1 + \epsilon F \eta_0)\nabla\cdot\mathbf{u}_0 = 0\,.$$

Thus, only geostrophic balance is retained in the momentum equation, and all terms are retained in the continuity equation, and the resulting set of equations is:

$$-fv = -g\,\frac{\partial h}{\partial x}\,, \qquad fu = -g\,\frac{\partial h}{\partial y}\,, \qquad \frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0$$

$\boxed{8,\ \text{B}}$

2. (i) Potential vorticity, Π, is materially conserved: $\boxed{\text{seen} \Downarrow}$

$$\frac{\partial \Pi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Pi}{\partial x} = 0\,, \qquad \Pi = \nabla^2\psi - \frac{1}{R^2}\,\psi + \beta y\,.$$

Potential vorticity consists of the relative vorticity, isopycnal deformation, and latitude-dependent planetary vorticity. $\boxed{4,\ \text{A}}$

(ii) Substitute Π, and multiply the governing equation by $-\psi$. Derive and use later the following identity, $\boxed{\text{sim. seen} \Downarrow}$

$$-\psi\nabla^2\psi_t = \frac{\partial}{\partial t}\frac{(\nabla\psi)^2}{2} - \nabla\cdot\psi\nabla\psi_t\,.$$

Obtain the energy equation, energy, and energy flux vector:

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{S} = 0\,.$$

The energy is found as the quadratic quantity in the tendency term:

$$E = \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] + \frac{1}{2R^2}\,\psi^2\,.$$

It consists of the kinetic and potential energies, respectively. $\boxed{6,\ \text{B}}$

(iii) The energy equation becomes $\boxed{\text{unseen} \Downarrow}$

$$\frac{\partial E}{\partial t} - \nabla\cdot\psi\nabla\psi_t - \psi\,\frac{\partial \psi}{\partial x}\left(\nabla^2\psi_y - \frac{1}{R^2}\,\psi_y + \beta\right) + \psi\,\frac{\partial \psi}{\partial y}\left(\nabla^2\psi_x - \frac{1}{R^2}\,\psi_x\right) = 0$$

After simplifications the diverging energy flux vector is obtained as

$$\mathbf{S} = \Big(-\psi\,\psi_{tx} - \frac{\beta}{2}\,\psi^2 - \frac{\psi^2}{2}\,\nabla^2\psi_y\,, \quad -\psi\,\psi_{ty} + \frac{\psi^2}{2}\,\nabla^2\psi_x \Big)\,.$$

$\boxed{8,\ \text{C}}$

(iv) Integrate the energy equation over some bounded domain stretching to the far field. By the divergence theorem (Green's theorem), the area integral of a diverging vector field is equal to the vector flux through the boundary. The later vanishes because flow fluctuations are zero in the far field, hence, time derivative of the total energy is zero. Thus, the total energy is integral invariant. $\boxed{\text{unseen} \Downarrow}$ $\boxed{2,\ \text{D}}$

3. (i) Dimensions are $\boxed{\text{seen} \Downarrow}$

$$[\psi] = L^2T^{-1}\,, \qquad [S] = L^{-2}\,, \qquad [\beta] = [\alpha] = L^{-1}T^{-1}\,, \qquad [\Pi] = T^{-1}$$

$\boxed{2,\ \text{A}}$

(ii) $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{D\Pi_1}{Dt} = 0\,, \qquad \frac{D\Pi_2}{Dt} = 0\,, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}$$

$\boxed{2,\ \text{A}}$

(iii) $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial}{\partial t}\Big[\nabla^2(\psi_1 - \psi_2) + \frac{\partial^2\psi_1}{\partial x\partial y} + S\,(\psi_1 - \psi_2)\Big] + \beta\,\frac{\partial\psi_1}{\partial x} - \alpha\,\frac{\partial\psi_1}{\partial y} = 0$$
$$\frac{\partial}{\partial t}\Big[\nabla^2(\psi_2 - \psi_1) + \frac{\partial^2\psi_2}{\partial x\partial y} + S\,(\psi_2 - \psi_1)\Big] + \beta\,\frac{\partial\psi_2}{\partial x} - \alpha\,\frac{\partial\psi_2}{\partial y} = 0$$

$\boxed{2,\ \text{A}}$

(iv) Define $\phi_1 = \psi_1 + \psi_2$ (barotropic) and $\phi_2 = \psi_1 - \psi_2$ (baroclinic) velocity streamfunctions. By summing up and subtracting the governing equations obtain: $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial}{\partial t}\Big[\frac{\partial^2\phi_1}{\partial x\partial y}\Big] + \beta\,\frac{\partial\phi_1}{\partial x} - \alpha\,\frac{\partial\phi_1}{\partial y} = 0\,,$$

$$\frac{\partial}{\partial t}\Big[2\nabla^2\phi_2 + \frac{\partial^2\phi_2}{\partial x\partial y} + 2S\phi_2\Big] + \beta\,\frac{\partial\phi_2}{\partial x} - \alpha\,\frac{\partial\phi_2}{\partial y} = 0\,.$$

$\boxed{4,\ \text{B}}$

(v) Look for wave solutions in the form $\exp[i(kx + ly - \omega t)]$ and obtain: $\boxed{\text{unseen} \Downarrow}$

$$\omega kl + k\beta - l\alpha = 0 \qquad \to \qquad \omega = \frac{\beta}{l} - \frac{\alpha}{k}$$

Dispersion curves corresponding to fixed l are hyperbolas confined to the left half of (ω, k) plane and displaced vertically by $const = \beta/l$. $\boxed{5,\ \text{D}}$

(vi) The dispersion relation is obtained similarly: $\boxed{\text{unseen} \Downarrow}$

$$-i\omega\,[-2\,(k^2 + l^2) - kl + 2S] + ik\beta - il\alpha = 0 \qquad \to \qquad \omega = \frac{-k\beta + l\alpha}{2\,(k^2 + l^2) + kl - 2S}$$

For meridionally uniform wave $(l = 0)$:

$$\omega = \frac{-k\beta}{2k^2 - 2S} \qquad \to \qquad \frac{\partial\omega}{\partial k} = \frac{-\beta\,(2k^2 - 2S) + 4k^2\beta}{(2k^2 - 2S)^2}$$

Zonal group velocity is positive (eastward) if $-k^2 + S + 2k^2 > 0$, which is always the case. $\boxed{5,\ \text{D}}$

4. (i) Linearized shallow-water dynamics: $\boxed{\text{seen} \Downarrow}$

$$\frac{\partial u}{\partial t} - f_0 v = -g\,\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v}{\partial t} + f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,,$$

Potential-vorticity description of the dynamics:

$$\frac{\partial Q}{\partial t} + \mathbf{u}\cdot\nabla Q = 0\,, \qquad Q = \frac{\zeta + f_0}{H + \eta} = \frac{(\zeta + f_0)/H}{1 + \eta/H}$$

$\boxed{5,\ \mathrm{A}}$

(ii) The linearization yields: $\boxed{\text{sim. seen} \Downarrow}$

$$Q_{lin} \approx \frac{1}{H}\,(\zeta{+}f_0)\Big(1{-}\frac{\eta}{H}\Big) \approx \frac{1}{H}\,\Big(\zeta{+}f_0{-}\frac{f_0\eta}{H}\Big) \qquad \to \qquad q = \zeta{-}f_0\,\frac{\eta}{H}\,, \qquad \frac{\partial q}{\partial t} = 0\,.$$

$\boxed{3,\ \mathrm{B}}$

$\boxed{\text{sim. seen} \Downarrow}$

(iii) According to the last equation, PV is conserved *locally*, i.e., its initial distribution remains unchanged during the adjustment process. Excessive energy is radiated away by inertia-gravity waves as part of the adjustment.

Initial PV is $q(x,y,0) = -f_0\eta_0/H$ for $y < 0$, and $q(x,y,0) = +f_0\eta_0/H$ for $y > 0$.
The final steady state is the solution of the equations:

$$\zeta - f_0\,\frac{\eta}{H} = q(x,y)\,, \qquad f_0 u = -g\,\frac{\partial \eta}{\partial y}\,, \qquad f_0 v = g\,\frac{\partial \eta}{\partial x}$$

$$\to \qquad \Psi = \frac{g\eta}{f_0} \qquad \to \qquad \Big(\nabla^2{-}\frac{1}{R_D^2}\Big)\,\Psi = q(x,y)\,, \qquad R_D \equiv \frac{\sqrt{gH}}{f_0}\,.$$

$\boxed{6,\ \mathrm{C}}$

$\boxed{\text{unseen} \Downarrow}$

(iv)

$$\frac{\partial^2 \Psi}{\partial y^2} - \frac{1}{R_D^2}\,\Psi = \frac{f_0\eta_0}{H}\,sign(y)\,,$$

$$\to \qquad \Psi = -\frac{g\eta_0}{f_0}\,(1{-}e^{-y/R_D})\,, \quad y > 0\,, \qquad \Psi = +\frac{g\eta_0}{f_0}\,(1{-}e^{+y/R_D})\,, \quad y < 0\,.$$

$$\to \qquad v = 0\,, \qquad u = -\frac{g\eta_0}{f_0 R_D}\,e^{-|y|/R_D}\,.$$

$\boxed{6,\ \mathrm{D}}$

5. (i) $\boxed{\text{seen} \Downarrow}$

$$\iint J[A,B]\,dx\,dy = \iint \Big(\frac{\partial}{\partial x}\Big[A\,\frac{\partial B}{\partial y}\Big] - \frac{\partial}{\partial y}\Big[A\,\frac{\partial B}{\partial x}\Big]\Big)\,dx\,dy = \iint \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \oint_\infty \mathbf{F}\cdot d\mathbf{r} = 0\,,$$

$$\mathbf{F} = \Big(A\,\frac{\partial B}{\partial x},\; A\,\frac{\partial B}{\partial y},\, 0\Big)\,.$$

$\boxed{4,\ \mathrm{M}}$

(ii) An eddy-mean interaction analysis for enstrophy and potential enstrophy for a zonal-mean flow, $\langle u\rangle(t,y)$: $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}
\mathbf{u} &= \langle u\rangle(t,y)\hat{\mathbf{x}} + \mathbf{u}'(t,x,y)\\
\zeta &= \langle\zeta\rangle(t,y) + \zeta'(t,x,y)\\
\langle u\rangle &= -\frac{\partial\langle\psi\rangle}{\partial y}\,, \qquad \langle v\rangle = 0\,, \qquad \langle\zeta\rangle = \frac{\partial^2\langle\psi\rangle}{\partial y^2}\\
\mathbf{u}' &= \Big(-\frac{\partial\psi'}{\partial y}, \frac{\partial\psi'}{\partial x}\Big)\,, \qquad \zeta' = \nabla^2\psi'
\end{aligned}$$

For vorticity,

$$\begin{aligned}
\frac{\partial\zeta}{\partial t} &= -J[\psi,\zeta] - \beta v + F\\
\frac{\partial\langle\zeta\rangle}{\partial t} &= -\frac{\partial\langle v'\zeta'\rangle}{\partial y} + \langle F\rangle\\
\frac{\partial\zeta'}{\partial t} &= -\langle u\rangle\,\frac{\partial\zeta'}{\partial x} - v'\,\frac{\partial\langle\zeta\rangle}{\partial y} - \beta v' - J[\psi',\zeta'] + \frac{\partial\langle v'\zeta'\rangle}{\partial y} + F'\,.
\end{aligned}$$

$\boxed{10,\ \mathrm{M}}$

$\boxed{\text{sim. seen} \Downarrow}$

(iii) For enstrophy,

$$\begin{aligned}
\frac{d}{dt}\int \frac{1}{2}\,\langle\zeta\rangle^2\,dy &= \int \langle v'\zeta'\rangle\,\frac{\partial\langle\zeta\rangle}{\partial y}\,dy + \int \langle\zeta\rangle\,\langle F\rangle\,dy\\
\frac{d}{dt}\int \frac{1}{2}\,\langle\zeta'^2\rangle\,dy &= -\int \langle v'\zeta'\rangle\,\frac{\partial\langle\zeta\rangle}{\partial y}\,dy + \int \langle\zeta' F'\rangle\,dy
\end{aligned}$$

since, by the result proven in (1):

$$\begin{aligned}
\int \langle u\rangle\,\langle\zeta'\,\frac{\partial\zeta'}{\partial x}\rangle\,dy &= 0\\
\int \beta\langle v'\zeta'\rangle\,dy &= 0\\
\int \langle\zeta' J[\psi',\zeta']\rangle\,dy &= 0\,.
\end{aligned}$$

The production terms are those that contain external forcing, and the remaining conversion terms have equal magnitudes but opposite signs. $\boxed{6,\ \mathrm{M}}$

Review of mark distribution:
Total A marks: 27 of 27 marks
Total B marks: 21 of 21 marks
Total C marks: 14 of 14 marks
Total D marks: 18 of 18 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60003/70003	1	This question was properly handled by students in line with statistics for similar questions in the past.
MATH60003/70003	2	This question was properly handled by students in line with statistics for similar questions in the past.
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MATH60003/70003	4	This question was properly handled by students in line with statistics for similar questions in the past.
MATH70003	5	No Comments Received