MVC Sheet 1 Hints, tips, answers

1/ (i) We have $\underline{A}\cdot\underline{r} = xA_1 + yA_2 + zA_3$. Then $\underline{\nabla}(\underline{A}\cdot\underline{r})$
$= \hat{\imath}A_1 + \hat{\jmath}A_2 + \hat{k}A_3 = \underline{A}$.

(ii) Write $r^n = (x^2+y^2+z^2)^{n/2}$
Then $\underline{\nabla}(r^n) = \cdots = (n/2)(x^2+y^2+z^2)^{n/2-1}(2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k})$
$= n r^{n-2}\underline{r}$

(iii) $\underline{r}\cdot\underline{\nabla}(x+y+z) = \cdots = x+y+z$. & $\underline{\nabla}(x+y+z) = \hat{\imath}+\hat{\jmath}+\hat{k}$

2/ Find $\underline{\nabla}\varphi|_{(1,1,2)} = 6\hat{\imath}+\hat{\jmath}+4\hat{k}$. $\hat{\underline{s}} = (\hat{\imath}+2\hat{\jmath}+3\hat{k})/\sqrt{14}$
Then directional deriv $= \hat{\underline{s}}\cdot(\underline{\nabla}\varphi)_{(1,1,2)} = \cdots = 20/\sqrt{14}$.

[For Q3 see next page]

4/ (i) Let $\varphi = x^2+2y^2-z^2-8$; find $(\underline{\nabla}\varphi)_P = 2\hat{\imath}+8\hat{\jmath}-2\hat{k}$
then tgt plane is $(\underline{r}-\underline{r}_P)\cdot(\underline{\nabla}\varphi)_P = 0$
$\Rightarrow \cdots \Rightarrow x+4y-z = 8$.

(ii) Let $\varphi = z - 3x^2y\sin(\pi x/2)$. Point P is $(1,1,3)$.
$(\underline{\nabla}\varphi)_P = -6\hat{\imath}-3\hat{\jmath}+\hat{k}$
tgt plane is $6x+3y-z = 6$.

5/ Write $r^2 = x^2+y^2+z^2$
(i) $\underline{\nabla}\varphi = \hat{\imath}(3x^2+y^2+z^2) + \hat{\jmath}\,2xy + \hat{k}\,2zx$
(ii) write $\underline{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$. Then $\underline{\nabla}\cdot(\varphi\underline{r}) = \cdots = 6xr^2$.
(iii) Expand out curl & use $\partial r/\partial x = x/r$ etc.
Each component vanishes.

6/ (i) $\underline{u}\times\underline{v} = \cdots = -z^3\hat{\jmath} + z^2y\hat{k} \Rightarrow \underline{\nabla}\cdot(\underline{u}\times\underline{v}) = 2zy$
on RHS: $\underline{v}\cdot(\underline{\nabla}\times\underline{u}) = 2zy$ while $\underline{\nabla}\times\underline{v} = 0$.
(ii) $\underline{\nabla}\cdot(\psi\underline{u}) = \cdots = 2z^2x$
on RHS: $(\underline{\nabla}\psi)\cdot\underline{u} = 2z^2x$ while $\underline{\nabla}\cdot\underline{u} = 0$

7/ (i) Start on LHS; use tensor notation for $(\underline{a}\times\underline{b})_i$ and then use $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
(ii) as above
(iii) First write out $[(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})]_i = \varepsilon_{ijk}(\underline{a}\times\underline{b})_j(\underline{c}\times\underline{d})_k$
and then use tensor notation for each cross product.

MVC Sheet 1 Hints, tips, answers (ctd.)

8. (i) $\delta_{ij} \dfrac{\partial x_i}{\partial x_j} = \dfrac{\partial x_i}{\partial x_i} = 3$

(ii) only contribution when $i=j$ & $i=k$ $\Rightarrow x_i x_i = |\underline{r}|^2$

(iii) " " when $i=j$ $\Rightarrow \dfrac{\partial^2 \varphi}{\partial x_i^2} = \underline{\nabla}^2 \varphi$

(iv) simplifies to $\delta_{ik}\delta_{ki} = \delta_{ii} = 3$

(v) Write $\varepsilon_{ijk} \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial A_k}{\partial x_j}\right) = \dfrac{1}{2}\varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial A_k}{\partial x_j}\right) + \dfrac{1}{2}\varepsilon_{jik}\dfrac{\partial}{\partial x_j}\left(\dfrac{\partial A_k}{\partial x_i}\right)$

$= \cdots = 0$

9. (i) Start on LHS & write $[\mathrm{curl}(\varphi \underline{A})]_i = \varepsilon_{ijk} \dfrac{\partial}{\partial x_j}(\varphi A_k)$

and expand out.

(ii) $\underline{\nabla}\cdot(\underline{A}\times\underline{B}) = \dfrac{\partial}{\partial x_i}(\underline{A}\times\underline{B})_i = \dfrac{\partial}{\partial x_i}(\varepsilon_{ijk} A_j B_k)$ and expand out.

(iii) $[\underline{A}\times \mathrm{curl}\,\underline{A}]_i = \varepsilon_{ijk} A_j \underbrace{(\mathrm{curl}\,\underline{A})_k}_{\varepsilon_{klm} \frac{\partial A_m}{\partial x_l}}$

and then write product of ε's in terms of δ.

3. We have $\underline{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

Then calculate $\underline{r}'(t)\cdot\underline{\nabla}\varphi$ and use chain rule.

For the verification, both sides are equal to $\cos^3 t - 2\cos t \sin^2 t + 2t\cos t - t^2 \sin t$

For last part use chain rule on $\varphi = \varphi(g_1(t), g_2(t), g_3(t))$.