Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Revision Problem Sheet

1. Consider the following PDE problem

$$3x\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad y > 0$$
$$u(x,0) = 1, \quad x > 0 \quad ; \quad u(0,y) = 0, \quad y > 0$$

We will try to obtain a similarity solution to this problem. We consider the transformation: $\{\tilde{x} = ax, \tilde{y} = a^\beta y, \tilde{u} = a^\gamma u\}$ where a is a positive real constant.

(a) For what value of β is the PDE invariant under this transformation?

(b) If our PDE problem is invariant under the above transformation, we know that a similarity solution is of the form $u(x,y) = y^{\gamma/\beta} f(xy^{-1/\beta})$. What value γ needs to take to leave the boundary conditions unchanged?

(c) Show that this PDE problem can be reduced to solving the following ODE boundary value problem

$$f''(\eta) + \eta^2 f'(\eta) = 0$$
$$f(0) = 0 \quad \text{and} \quad f(\eta) \to 1, \eta \to +\infty$$

where $\eta = x/y^{1/3}$.

(d) Show that a similarity solution to this problem can be written

$$u(x,y) = A\int_0^{x/y^{1/3}} e^{-s^3/3} ds$$

where $A = \left[\int_0^\infty e^{-s^3/3} ds\right]^{-1} = 3^{2/3}/\Gamma(1/3)$.

2. Here, we consider the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the initial condition

$$u(x,0) = u_0(x) = \begin{cases} 0, & x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

(a) Sketch the initial condition. Find the equations of the characteristics for this problem. Draw them in the (x,t)-plane and point out any fan regions. Show that a shock forms at $t = 1$.

(b) Find an explicit solution valid for $0 < t < 1$.

(c) Find an explicit solution valid for $t > 1$.

(d) Sketch the solution when the shock strength is $1/3$.

3. Assume that the function $G(x,t)$ satisfies the following problem

$$\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2} = \delta(x - x_0)\delta(t - t_0), \quad x \in \mathbb{R}, t > 0$$

with $G = 0$ for $t = 0$, δ is the Dirac delta function (whose properties you can assume), x_0 and t_0 are constants. Find the solution to this initial value problem.

4. Solve the following initial value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0 \\
&u(x,0) = f(x) \\
&\frac{\partial u}{\partial t}(x,0) = g(x)
\end{aligned}$$

with

$$f(x) = \begin{cases} 0, & |x| \geq 1 \\ x^2 - x^4, & |x| < 1 \end{cases} \qquad \text{and} \qquad g(x) = 0$$

5. In this problem, we use Green's functions to find a solution to a 2D Poisson equation.

(a) Show that in 2D the free-space Green's function for the Laplacian is given by

$$G_f = \frac{1}{2\pi} \ln |\mathbf{r}|$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. You may assume the 2D divergence theorem which gives

$$\int_R \nabla \cdot \mathbf{F} dxdy = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds$$

(b) Use the method of images to find the solution to

$$\nabla^2 H = \delta(\mathbf{r} - \mathbf{r}_0)$$

in the upper-half plane $y > 0$, $-\infty < x < \infty$ with the following boundary condition

$$\frac{\partial H}{\partial y} = 0 \quad \text{on} \quad y = 0$$

6. The state of a quantum particle is described by what is called a wavefunction $\psi(x,t)$. A wavefunction is a complex valued function and its square modulus $|\psi(x,t)|^2$ gives the probability density of measuring a particle as being at position x at time t. The time evolution of the wave function, say of an electron, inside an external time-independent potential $V(x)$ is governed by the following Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$

where $m \in \mathbb{R}$ is the mass of the particle and $\hbar$ is the reduced Planck constant (a real constant). Here, we will be looking for bounded solutions $\psi(x,t)$.

(a) Show that if the Schrödinger equation admits separated solutions of the form $\psi(x,t) = \tau(t)\phi(x)$, then $\tau(t) = e^{-iE\hbar t}$ where E is an energy and $\phi(x)$ is a solution to the following equation

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi(x) = (V(x) - E)\phi(x)$$

(b) Suppose that

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq L \\ V_0, & \text{if } x < 0 \text{ or } x > L \end{cases}$$

such that V defines a square potential energy well which tries to confine the electron in the interval $[0; L]$. Assume that $E < V_0$, so that the electron does not have sufficient energy to escape the interval. Find $\phi(x)$ for a given value of E. Sketch the solution.

(c) Show that the electron has a non-zero probability of being found out of the interval $[0; L]$. This is called *quantum tunnelling* and is impossible in a classical system!

(d) Show that E can only take certain values, those correspond to discrete energy levels for the electron. This is called *quantization of energy*.