MVC: Sheet 2 Hints, tips, answers

1/ Write out curl $\underline{v}$ in determinant form. Then show it is zero.
The potential φ satisfies $\varphi_x = 2xy + z^2$, $\varphi_y = 2yz + x^2$
& $\varphi_z = 2xz + y^2$

Integrating each equation & putting the info together:
$\varphi = yx^2 + xz^2 + zy^2$ (since $\varphi = 0$ when $x = y = z = 0$)
Then $\int_P \underline{v} \cdot d\underline{r} = \varphi(1,2,3) - \varphi(0,0,0) = \dots = 23$

2/ Parametrization $x = t, y = 2t, z = 3t$ $(0 \leq t \leq 1)$
Then $I = \dots = \int_0^1 23t^2\,dt = 23/3$.

3/ $\underline{F} \cdot d\underline{r} = \dots = 3x^2\,dx + (2xz - y)\,dy + z\,dz$.
(i) pzn is $x = 2t, y = t, z = 3t$ $(0 \leq t \leq 1)$
Then $I = \int_0^1 (36t^2 + 8t)\,dt = 16$.

(ii) $I = \dots = \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)\,dt = 71/5$.
(iii) $I = \dots = \int_0^2 \left(\frac{51}{64}s^5 - \frac{s^3}{8} + 3s^2\right) ds = 16$.

4/ (a) Integration regions for (i), (ii) & (iv) are triangles
for (iii) the region is that in the 1st quadrant between
$y = x$ & $y = \sqrt{x}$.
(i) (b) $I = a^2/2$; (c) & (d): $I = \int_0^a \int_0^{a-y} dx\,dy = \dots = a^2/2$.
(ii) (b) $I = a^4/3$; (c) & (d): $I = \int_0^a \int_y^a (x^2 + y^2)\,dx\,dy$
$= \dots = a^4/3$.
(iii) (b) $I = 1/35$; (c) & (d): $I = \int_0^1 \int_{y^2}^y (xy^2)\,dx\,dy$
$= \dots = 1/35$.
(iv) (b) $I = \frac{1}{2}(1 - e^{-1})$.
(c) & (d): $I = \int_0^1 \left(\int_y^1 e^{-x^2}\,dx\right) dy$. Inner integral cannot be evaluated.

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5/ First calculate $\hat{\underline{n}} = \underline{\nabla}\varphi / |\underline{\nabla}\varphi|$ where $\varphi = y^2 - 8x$

$= (4\hat{\imath} - y\hat{\jmath})/\sqrt{(16+y^2)}$ taking -ve sq. root (why?)

Thus $\underline{F}\cdot\hat{\underline{n}} = (8y + yz)/\sqrt{(16+y^2)}$

Project integral onto $\cancel{z}\, x = 0$ (i.e. $dS \rightarrow \dfrac{dy\,dz}{|\hat{\underline{n}}\cdot\hat{\underline{\imath}}|}$)

Hence ... $I = \frac{1}{4}\int_0^6 \int_0^4 (8y + yz)\,dy\,dz = \ldots = 132.$

6/ First Curl $\underline{F} = \ldots = x\hat{\imath} + y\hat{\jmath} - 2z\hat{k}$

& $\hat{\underline{n}} = \ldots \pm (x\hat{\imath} + y\hat{\jmath} + z\hat{k})/a$ Choose +ve sign (why?)

Project onto $z = 0$ where projected surface Σ will be a circle of radius a.

Hence, after some work $I = \int_\Sigma (3x^2 + 3y^2 - 2a^2)(a^2 - x^2 - y^2)^{-1/2}\, d\Sigma$

Then use plane polars to simplify to

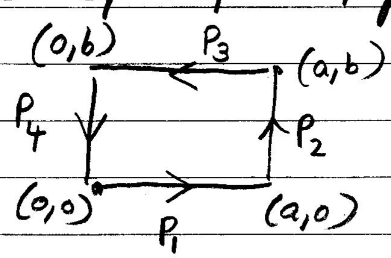
[For Q7 see next page]

$$I = 2\pi \int_0^a -3r(a^2 - r^2)^{1/2} + a^2 r (a^2 - r^2)^{-1/2}\, dr$$

$$= \ldots = 0.$$

8/ RHS is $\int_0^b \left(\int_0^a (2y - a)\,dx \right) dy = \ldots = ab^2 - a^2 b.$

LHS: Split up into 4 parts around sides of rectangle



we find $\int_{P_1} = 0$; $\int_{P_2} = ab^2$

$\int_{P_3} = -a^2 b$; $\int_{P_4} = 0.$

9/ G-T with $L = -y$, $M = x$. (seen in notes)

Cycloid: Let C_1 be path along x-axis from $x = 0$ to $x = 2\pi a$ ($y = 0$)

Let C_2 be path along cycloid with t starting at 2π & ending at zero. Then $\int_{C_1} = 0$

& $\int_{C_2} (x\,dy - y\,dx) = \ldots = 6\pi a^2$. Hence area $= 3\pi a^2$.

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7. $\hat{n} = \nabla(x^2+y^2)/|\underline{\nabla}(x^2+y^2)| = \dots = \pm(x\hat{\imath}+y\hat{\jmath})/a$

$dS = dx\,dz/|\hat{n}.\hat{\jmath}| = a(a^2-x^2)^{-1/2}\,dx\,dz$

Surface area $= \displaystyle\int_{H_1}^{H_2}\int_{-a}^{a} \frac{a}{(a^2-x^2)^{1/2}}\,dx\,dz = \dots = \pi a(H_2-H_1)$