MATH70062

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Lie Algebras

Date: 24 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

All Lie algebras in this paper are finite-dimensional complex Lie algebras.

1. Let L be a Lie algebra and let V be a vector subspace of L. Let us define the *normaliser* $N_L(V)$ of V in L as the set of elements $l \in L$ such that $[lv] \in V$ for all $v \in V$.

   (a) (i) Prove that $N_L(V)$ is a Lie subalgebra of L. (2 marks)

   (ii) Suppose that V is a Lie subalgebra of L. Prove that $N_L(V)$ is the largest Lie subalgebra of L such that V is an ideal of $N_L(V)$. (2 marks)

   (b) Let $L = \mathfrak{gl}(3)$. Let $E_{i,j}$ be the 3×3-matrix all of whose entries are 0 except the (i, j)-entry which is 1. Let $V_{i,j} = \mathbb{C}E_{i,j}$ be the span of $E_{i,j}$. Showing your working, determine $N_L(V_{i,j})$ in the following cases:

   (i) $i = j = 1$; (3 marks)

   (ii) $i = 1$, $j = 2$; (3 marks)

   (iii) $i = 1$, $j = 3$. (3 marks)

   (c) Let $\mathfrak{h} \subset \mathfrak{gl}(n)$ be the set of diagonal matrices, where $n \geq 2$. Determine the normaliser of $\mathfrak{h}$ in $\mathfrak{gl}(n)$. (You are asked to justify your answer.) (7 marks)

(Total: 20 marks)

2. Let L be a Lie algebra.

   (a) (i) Give the definition of a *representation* of L. (1 mark)

   (ii) Give the definition of an *irreducible* representation of L. (1 mark)

   (iii) Give the definition of the *adjoint* representation of L. (1 mark)

   (b) For each of the following Lie algebras determine whether the adjoint representation is irreducible.

   (i) The Lie algebra of upper-triangular $n \times n$-matrices $\mathfrak{t}(n)$, $n \geq 2$. (2 marks)

   (ii) The Lie algebra of strictly upper-triangular $n \times n$-matrices $\mathfrak{u}(n)$, $n \geq 3$. (2 marks)

   (iii) The Lie algebra $\mathfrak{gl}(n)$, $n \geq 2$. (2 marks)

   (iv) The Lie algebra $\mathfrak{sl}(n)$, $n \geq 2$. (2 marks)

   (Justify your answers. You can use any results from lectures if you state them clearly.)

   (c) Let $x \in L$ and let $V \subset L$ be the span of all the eigenvectors of $\mathrm{ad}(x)$ in L.

   (i) Prove that V is a Lie subalgebra of L. (4 marks)

   (ii) Is it true that if $V \neq L$, then V cannot be an ideal of L? Give a proof or a counterexample. (5 marks)

(Total: 20 marks)

3. (a) (i) Give the definition of the *centre* of a Lie algebra. (1 mark)

(ii) Is it true that the centre of a solvable Lie algebra is always non-zero? Give a proof or a counterexample. (5 marks)

(iii) Let L be a nilpotent Lie algebra and let $K \subset L$ be a Lie subalgebra such that $K \neq L$. Show that there is an element $x \in L$, $x \notin K$, such that $[x, K] \subset K$. (8 marks)

(b) (i) Give the definition of the *Killing form*. (1 mark)

(ii) Compute the Killing form of the Lie algebra of upper-triangular 2×2-matrices $\mathfrak{t}(2)$ in the basis $E_{1,1}$, $E_{2,2}$, $E_{1,2}$. (Here $E_{i,j}$ is the 2×2-matrix all of whose entries are 0 except the (i, j)-entry which is 1.) (5 marks)

(Total: 20 marks)

4. (a) (i) Give the definition of a *Cartan subalgebra* of a semisimple Lie algebra. (2 marks)

(ii) Prove that every semisimple Lie algebra has a non-zero Cartan subalgebra. (You can use all other results from lectures if you state them clearly.) (6 marks)

(b) (i) Give the definition of a *root system*. (3 marks)

(ii) Let R be a root system and let $\alpha, \beta \in R$. Let V be the real vector space spanned by α and β. Prove that $R \cap V$ is a root system in V. (5 marks)

(iii) Let R be a root system and let $\alpha \in R$. Let R' be the set of elements $\beta \in R$ such that $(\beta, \beta) = (\alpha, \alpha)$. Let V be the real vector space spanned by R'. Prove that R' is a root system in V. (4 marks)

(Total: 20 marks)

5. Let

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{so}(4)$ be the set of matrices $X \in \mathfrak{gl}(4)$ such that $X^T S + SX = 0$.

(a) Prove that $\mathfrak{so}(4)$ is a Lie algebra. (3 marks)

(b) Determine the general form of elements of $\mathfrak{so}(4)$. (2 marks)

(c) Show that the set of diagonal matrices of $\mathfrak{so}(4)$ is a Cartan subalgebra, find the root decomposition and list all the roots. (11 marks)

(d) Decompose $\mathfrak{so}(4)$ into a direct sum of two non-zero ideals, hence show that $\mathfrak{so}(4)$ is isomorphic to the direct sum of Lie algebras $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. (4 marks)

(Justify your answers. You can use any results from lectures if you state them clearly.)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH70062

Lie algebras (Solutions)

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Question 1

(a) (i) Prove that $N_L(V)$ is a Lie subalgebra of L.

(ii) Suppose that V is a Lie subalgebra of L. Prove that $N_L(V)$ is the largest Lie subalgebra of L such that V is an ideal of $N_L(V)$.

(b) Let $L = \mathfrak{gl}(3)$. Let E_{ij} be the 3×3-matrix all of whose entries are 0 except the (i,j)-entry which is 1. Let $V_{ij} = \mathbb{C}E_{ij}$ be the span of E_{ij}. Showing your working, determine $N_L(V_{ij})$ in the following cases:

(i) $i = j = 1$; (ii) $i = 1$, $j = 2$; (iii) $i = 1$, $j = 3$.

(c) Let $\mathfrak{h} \subset \mathfrak{gl}(n)$ be the set of diagonal matrices, where $n \geq 2$. Determine the normaliser of $\mathfrak{h}$ in $\mathfrak{gl}(n)$. (You are asked to justify your answer.)

$\boxed{\text{unseen} \Downarrow}$

1. (a) (i) Let $l_1, l_2 \in N_L(V)$. The Jacobi identity gives $[[l_1 l_2]v] = [l_1[l_2 v]] - [l_2[l_1 v]] \in V$, hence $[l_1 l_2] \in N_L(V)$ so $N_L(V)$ is a subalgebra. $\boxed{2, \text{A}}$

(ii) This is immediate from part (i) and the definition of the normaliser. $\boxed{2, \text{A}}$

(b) (i) The normaliser of $E_{1,1}$ is $\mathfrak{gl}(1) \times \mathfrak{gl}(2)$ (the matrices x such that $x_{12} = x_{13} = x_{21} = x_{31} = 0$). The inclusion $\mathfrak{gl}(1) \times \mathfrak{gl}(2) \subset N(V_{1,1})$ is clear since $V_{1,1} = \mathfrak{gl}(1)$ is an ideal of $\mathfrak{gl}(1) \times \mathfrak{gl}(2)$. We have $[E_{1,2}E_{1,1}] = -E_{1,2}$ and $[E_{1,3}E_{1,1}] = -E_{1,3}$, hence this inclusion is an equality. $\boxed{3, \text{A}}$

(ii) We have $[E_{1,1}E_{1,2}] = E_{1,2}$, $[E_{2,2}E_{1,2}] = -E_{1,2}$. Next, $[E_{1,2}E_{1,2}] = [E_{1,3}, E_{1,2}] = [E_{3,2}, E_{1,2}] = [E_{3,3}, E_{1,2}] = 0$, so all these matrices are in $N(V_{1,2})$. Finally, we have $[E_{2,1}E_{1,2}] = E_{2,2} - E_{1,1}$, $[E_{2,3}E_{1,2}] = -E_{1,3}$, $[E_{3,1}E_{1,2}] = E_{3,2}$. Thus the normaliser of $E_{1,2}$ is the set of matrices x such that $x_{21} = x_{31} = x_{23} = 0$.

$\boxed{3, \text{A}}$

(iii) The normaliser of $E_{1,3}$ is $\mathfrak{t}(3)$ (upper-triangular matrices). Indeed, $V_{1,3}$ is an ideal of $\mathfrak{t}(3)$, hence $V_{1,3} \subset \mathfrak{t}(3)$. On the other hand, we have $[E_{2,1}E_{1,3}] = E_{2,3}$, $[E_{3,1}E_{1,3}] = E_{3,3} - E_{1,1}$, $[E_{3,2}E_{1,3}] = -E_{1,2}$, so this inclusion is an equality. $\boxed{3, \text{A}}$

(c) The normaliser of $\mathfrak{h}$ is $\mathfrak{h}$. Indeed, let d be a diagonal matrix with entries $d_1, \ldots, d_n$, where $d_i \neq d_j$ for $i \neq j$. If x is a matrix such that $x_{ij} \neq 0$ for some $i \neq j$, then the (i,j)-entry of $[dx]$ is $(d_i - d_j)x_{ij} \neq 0$. Thus x is not in the normaliser of $\mathfrak{h}$. On the other hand, $[\mathfrak{h}\mathfrak{h}] = 0$, hence $\mathfrak{h}$ is contained in its normaliser. $\boxed{7, \text{C}}$

Question 2

(a) (i) Give the definition of a representation of L.

(ii) Give the definition of an irreducible representation of L.

(iii) Give the definition of the adjoint representation of L.

(b) For each of the following Lie algebras determine whether the adjoint representation is irreducible.

(i) The Lie algebra of upper-triangular $n \times n$-matrices $\mathfrak{t}(n)$, $n \geq 2$.

(ii) The Lie algebra of strictly upper-triangular $n \times n$-matrices $\mathfrak{u}(n)$, $n \geq 3$.

(iii) The Lie algebra $\mathfrak{gl}(n)$, $n \geq 2$.

(iv) The Lie algebra $\mathfrak{sl}(n)$, $n \geq 2$.

(Justify your answers. You can use any results from lectures if you state them clearly.)

(c) Let $x \in L$ and let $V \subset L$ be the span of all the eigenvectors of $\mathrm{ad}(x)$ in L.

(i) Prove that V is a Lie subalgebra of L.

(ii) Is it true that if $V \neq L$, then V cannot be an ideal of L? Give a proof or a counterexample.

seen ⇓

2. (a) (i) A representation of L in a vector space V is a homomorphism of Lie algebras $\rho\colon L \to \mathfrak{gl}(V)$. 1, A

(ii) A representation $\rho\colon L \to \mathfrak{gl}(V)$ is irreducible if V does not contain a vector subspace $W \neq 0$, $W \neq V$, such that $\rho(L)W \subset W$. 1, A

(iii) The adjoint representation is the representation $L \to \mathfrak{gl}(L)$ which sends x to the linear map $\mathrm{ad}(x) : L \to L$ defined by $\mathrm{ad}(x)(y) = [xy]$. 1, A

(b) The adjoint representation is reducible if and only if L contains an ideal $I \neq 0$, $I \neq L$. meth seen ⇓

(i), (ii) These Lie algebras are solvable and non-abelian, so the derived subalgebra is a non-trivial ideal. 2, B 2, B

(iii) The centre of $\mathfrak{gl}(n)$ is the 1-dimensional ideal of scalar matrices. 2, B

(iv) By lectures, $\mathfrak{sl}(n)$ is a simple Lie algebra, so it has no non-trivial ideals.

Thus the adjoint representation is irreducible in (iv), and reducible in (i), (ii), (iii). 2, B

(c) (i) It is enough to prove if a and b are eigenvectors of $\mathrm{ad}(x)$, then so is $[ab]$. Indeed, we have $[xa] = \lambda a$ and $[xb] = \mu b$ for some $\lambda, \mu \in \mathbb{C}$. Then $[x[ab]] = [[xa]b] + [a[xb]] = (\lambda + \mu)[ab]$, so we are done. 4, C

(ii) This is false. For example, if $x = E_{1,2} \in \mathfrak{t}(2)$, then $V = \mathbb{C}x$. This is a non-trivial ideal of $\mathfrak{t}(2)$. 5, D

Question 3

(a) (i) Give the definition of the centre of a Lie algebra.

(ii) Is it true that the centre of a solvable Lie algebra is always non-zero? Give a proof or a counterexample.

(iii) Let L be a nilpotent Lie algebra and let $K \subset L$ be a Lie subalgebra such that $K \neq L$. Show that there is an element $x \in L$, $x \notin K$, such that $[x, K] \subset K$.

(b) (i) Give the definition of the Killing form.

(ii) Compute the Killing form of the Lie algebra of upper-triangular 2×2-matrices $\mathfrak{t}(2)$ in the basis E_{11}, E_{22}, E_{12}.

3. (a) (i) The centre of L is the set of elements $x \in L$ such that $[xy] = 0$ for all $y \in L$. $\boxed{\text{seen} \Downarrow}$

(ii) This is not true. Let $L = \mathbb{C}x \oplus \mathbb{C}y$ with $[xy] = x$. Clearly, $L' = \mathbb{C}x$, so L is solvable. If $ax + by$ is in the centre of L, then $[ax + by, x] = -bx = 0$, hence $b = 0$, and $[ax + by, y] = ax = 0$, hence $a = 0$. Thus the centre of L is 0. $\boxed{\text{unseen} \Downarrow}$ $\boxed{1, \text{A}}$

(iii) We have seen that every nilpotent Lie algebra has a non-zero centre (because the centre contains the last non-zero term of the lower central series). The proof goes by induction on $n = \dim(L)$. The case $n = 1$ is obvious. Suppose that the statement is proved for nilpotent Lie algebras of dimension less than n. Let $Z \neq 0$ be the centre of L. If Z is not a subset of K, take any $x \in Z \setminus K$. Thus we can assume that $Z \subset K$. In this case we consider the quotient algebras $L_1 = L/Z$ and its subalgebra $K_1 = K/Z$. Since $\dim(L_1) < \dim(L)$, by induction assumption there is an element $y \in L_1$, $y \notin K_1$, such that $[yK_1] \subset K_1$. Let $x \in L$ be an element that maps to y. Then $[xK] \subset K$. $\boxed{5, \text{B}}$ $\boxed{8, \text{D}}$

(b) (i) Let $x, y \in L$. The Killing form $K(x, y)$ is defined as $\mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$. $\boxed{\text{seen} \Downarrow}$

(ii) We have $[E_{1,1}E_{1,2}] = E_{1,2}$, $[E_{2,2}E_{1,2}] = -E_{1,2}$, $[E_{1,1}E_{2,2}] = 0$. Thus in the basis E_{11}, E_{22}, E_{12} we have $\boxed{1, \text{A}}$ $\boxed{\text{meth seen} \Downarrow}$

$$\mathrm{ad}(E_{1,1}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathrm{ad}(E_{2,2}) = -\mathrm{ad}(E_{1,1}), \ \mathrm{ad}(E_{1,2}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

Thus the matrix of the Killing form is

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{5, \text{B}}$

Question 4

(a) (i) Give the definition of a Cartan subalgebra of a semisimple Lie algebra.

(ii) Prove that every semisimple Lie algebra has a non-zero Cartan subalgebra. (You can use all other results from lectures if you state them clearly.)

(b) (i) Give the definition of a root system.

(ii) Let R be a root system and let $\alpha, \beta \in R$. Let V be the real vector space spanned by α and β. Prove that $R \cap V$ is a root system in V.

(iii) Let R be a root system and let $\alpha \in R$. Let R' be the set of elements $\beta \in R$ such that $(\beta, \beta) = (\alpha, \alpha)$. Let V be the real vector space spanned by R'. Prove that R' is a root system in V.

seen ⇓

4. (a) (i) A Cartan subalgebra is a maximal abelian Lie subalgebra consisting of semisimple elements. (Other equivalent definitions are OK.) 2, A

(ii) If a semisimple Lie algebra L does not contain semisimple elements, then the Jordan decomposition of every $x \in L$ has zero semisimple part, so $\mathrm{ad}(x)$ is nilpotent. By Engel's theorem, L is a nilpotent Lie algebra, which contradicts semisimplicity of L. Thus L contains a semisimple element $x \neq 0$, hence contains non-zero abelian subalgebras consisting of semisimple elements, e.g., $\mathbb{C}x$. Such a subalgebra of maximal dimension is a Cartan subalgebra. 6, A

(b) (i) Let E be a finite-dimensional real vector space with a positive-definite bilinear form (x, y). A finite set of vectors $R \subset E$ is called a root system if seen ⇓

(1) R spans E, $0 \notin R$;

(2) if $\alpha \in R$, then $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$;

(3) $s_\alpha(R) = R$, where $s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$ is a reflection in α, for every $\alpha \in R$;

(4) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for any $\alpha, \beta \in \mathbb{Z}$. 3, A

(ii) Axioms (1), (2), (4) trivially hold for $R \cap V$. For any $\alpha \in R \cap V$ the reflection s_α preserves V. Since $s_\alpha(R) = R$, we see that s_α preserves $R \cap V$, hence is a permutation of this set. 5, B

(iii) Axioms (1), (2), (4) trivially hold for R'. Reflections preserve the scalar product, hence $s_\alpha(R') \subset R'$ for every $\alpha \in R$. In particular, we have $s_\alpha(R') = R'$. 4, C

Question 5

Let

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{so}(4)$ be the set of matrices $X \in \mathfrak{gl}(4)$ such that $X^T S + SX = 0$.

(a) Prove that $\mathfrak{so}(4)$ is a Lie algebra.

(b) Determine the general form of elements of $\mathfrak{so}(4)$.

(c) Show that the set of diagonal matrices of $\mathfrak{so}(4)$ is a Cartan subalgebra, find the root decomposition and list all the roots.

(d) Decompose $\mathfrak{so}(4)$ into a direct sum of two non-zero ideals, hence show that $\mathfrak{so}(4)$ is isomorphic to the direct sum of Lie algebras $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

(Justify your answers. You can use any results from lectures if you state them clearly.)

sim. seen ⇓

5. (a) The condition $X^T S + SX = 0$ is linear, so we just need to prove that $\mathfrak{so}(4)$ is closed under the Lie bracket. Indeed, $(XY - YX)^T S = Y^T X^T S - X^T Y^T S = -Y^T SX + X^T SY = -S(XY - YX)$. 3, A

(b) Any (4×4)-matrix X can be written as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are (2×2)-matrices. Then one immediately checks that $X^T S + SX = 0$ if and only if $D = -A^T$, $B^T = -B$ and $C^T = -C$. 2, B

(c) By part (b) we can choose the following basis of $\mathfrak{so}(4)$:
$H_1 = E_{1,1} - E_{3,3}$, $H_2 = E_{2,2} - E_{4,4}$, $X_{e_1-e_2} = E_{2,1} - E_{3,4}$, $X_{e_2-e_1} = E_{1,2} - E_{4,3}$, $X_{e_1+e_2} = E_{3,2} - E_{4,1}$, $X_{-e_1-e_2} = E_{2,3} - E_{1,4}$. 4, A

Define $\mathfrak{h} = \mathbb{C}H_1 \oplus \mathbb{C}H_2$. Let $e_i \colon \mathfrak{h} \to \mathbb{C}$ be linear forms given by $e_i(x_1 H_1 + x_2 H_2) = x_i$, for $i = 1, 2$. A direct verification using the standard formulae for $[E_{i,j}, E_{k,l}]$ as in Q1(b) and Q3(b) shows that each element X_α, as defined above, is a common eigenvector of $\mathfrak{h}$ of weight α. 4, A

Thus the roots are $\pm e_1 \pm e_2$. Choosing $x_1 H_1 + x_2 H_2 \in \mathfrak{h}$ with $x_1 \neq \pm x_2$ we deduce that $\mathfrak{h}$ is equal to its centraliser in $\mathfrak{so}(4)$. Every element of $\mathfrak{h}$ is semisimple, hence, by a result in lectures, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{so}(4)$. 3, D

unseen ⇓

(d) Let I_- be the span of $X_{e_1-e_2}$, $X_{e_2-e_1}$, and $[X_{e_1-e_2}, X_{e_2-e_1}] = H_2 - H_1$. Let I_+ be the span of $X_{e_1+e_2}$, $X_{-e_1-e_2}$, and $[X_{e_1+e_2}, X_{-e_1-e_2}] = -H_1 - H_2$. Thus we have a direct sum of vector spaces $\mathfrak{so}(4) = I_+ \oplus I_-$. Using results from lectures, or directly, one checks that I_+ and I_- are ideals. Comparing the structure constants, we deduce that $I_+ \cong I_- \cong \mathfrak{sl}(2)$. 4, D

Review of mark distribution:
Total A marks: 40 of 40 marks
Total B marks: 25 of 25 marks
Total C marks: 15 of 15 marks
Total D marks: 20 of 20 marks
Total marks: 100 of 100 marks
Total Mastery marks: 0 of 0 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	QuestionNumber	Comments for Students
MATH70062	1	This was a rather easy question.
MATH70062	2	A medium strength question, done reasonably well. In part (c) (ii) there is a typo: V is spanned by x and the identity matrix.
MATH70062	3	A slightly harder question. Only a handful of student did (a)(iii) correctly. There were three different solutions, all of them different from my own solution.
MATH70062	4	An easy question, done very well.
MATH70062	5	Not many people did correctly the final part. The rest is relatively straightforward though requires computational skills.