Math40003 Linear Algebra and Groups **Problem Sheet 3**
Seen Questions

Seen A

A.1. Exercise 3.3.7: Which of the following sets span $\mathbb{R}^3$?

(a)
$$\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}$$

(b)
$$\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(c)
$$\begin{pmatrix}3\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

(d)
$$\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$$

Definitions 3.3.1. and 3.3.5. are useful here.

A.2. For each of the following subsets of $\mathbb{R}^3$, work out if they are linearly independent or not.

(a) $\left\{\begin{pmatrix}5\\3\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$

(b) $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}$

(c) $\left\{\begin{pmatrix}1\\3\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}-1\\7\\-5\end{pmatrix}\right\}$

(d) $\left\{\begin{pmatrix}1\\-3\\2\end{pmatrix},\begin{pmatrix}2\\-1\\1\end{pmatrix},\begin{pmatrix}2\\-5\\4\end{pmatrix},\begin{pmatrix}1\\2\\5\end{pmatrix}\right\}$

Definition 3.4.1., and Section 2.3, may be useful here.

A.3. For which of the following values of $\bar{x}$ and $\bar{y}$ is the set

$$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\bar{x},\bar{y}\right\}\subseteq\mathbb{R}^4$$

a basis?

(a)
$$\bar{x}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\qquad\bar{y}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

(b)
$$\bar{x}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\qquad\bar{y}=\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

(c)

$$\bar{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

(d)

$$\bar{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} 16 \\ 32 \\ 64 \\ 128 \end{pmatrix}$$

Definition 3.5.1., and hence Definitions 3.3.5. and 3.4.1., are the key ones for this question.

A.4. Extend the following set into a basis of $\mathbb{R}^4$.

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 5 \\ 0 \end{pmatrix} \right\}$$

Put together what you've learnt from the last 3 questions for this one!

Seen B

B.1. For which a, b, c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

B.2. Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

(a) If $\{v_1, \dots, v_n\}$ is a basis, for V, and $\{x_1, \dots, x_r\}$ is a linearly independent subset of V with $r < n$, and if $v_i \notin Span\{x_1, \dots, x_r\}$ for all $i = 1, \dots, n$, then $\{x_1, \dots, x_r, v_{r+1}, \dots, v_n\}$ is a basis for V.

(b) If U is a subspace of V, then $U + U = U$.

(c) If U and W are subspaces of V, and $\dim U + \dim W = \dim V$, then $U \cap W = \{0_V\}$.

(d) If $\dim V = n$ and $v_1 \in V$, then there exist vectors $v_2, \dots, v_n$ in V such that $\{v_1, \dots, v_n\}$ spans V.

(e) If W is a subspace of V , then $\dim W \leq \dim V$ and $\dim W = \dim V$ if and only if $W = V$.

B.3. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$).

(a) Show that the functions

$$f_1(x) = 1, \quad f_2(x) = 1 + x + x^2, \quad f_3(x) = \sin x, \quad f_4(x) = \cos x$$

are linearly independent.

(b) Which of the following functions lie in $Span(f_1, f_2, f_3, f_4)$?

$$5 - 3x - 3x^2, \quad \tan x, \quad 10 - x - x^2 + \sin(x + \pi/3).$$

B.4. (a) Describe an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

(b) Find a basis for $W = Span(x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1) \leq \mathbb{R}[x]$.

B.5. Let V be the vector space of all 3×3 matrices over $\mathbb{R}$.

(a) Find a basis of V consisting of invertible matrices.

(b) Let $W = \{A \in V \colon A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

(c) Let $W \subset V$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.