Solution 2

1. Decide if the following statements are true or false. Explain and justify your answers.

a) Every monotone and quasi-concave production function exhibits increasing, decreasing or constant returns to scale.

Answer: False. There are production functions that do not satisfy any of the three regimes. An example could be $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$f(x) = \begin{cases} \sqrt{x}, & x \in [0,1] \\ x^2, & x > 1 \end{cases}$$

b) The quasi-concavity of a production function implies that if we mix certain bundles of inputs we will always be able to produce not less than with any of the single bundles.

Answer: False. In formulae, the assertion means that for any two bundles $\underline{x}, \underline{x}'$ and for any $\underline{x}'' = (1-\lambda)\underline{x} + \lambda \underline{x}'$, $\lambda \in [0,1]$, we have that

$$f(\underline{x}'') \geq \max\{f(\underline{x}), f(\underline{x}')\}.$$

However, quasi-concavity only claims that

$$f(\underline{x}'') \geq \min\{f(\underline{x}), f(\underline{x}')\}.$$

So by mixing two input bundles we won't be worse off than by producing with the bundle yielding the lowest output. Indeed, the identity $f(x) = x$ on $\mathbb{R}_{\geq 0}$ is quasi-concave, but does not satisfy the claim.

2. Consider a production function $f\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{>0}$

$$f(x_1, x_2) = \frac{2}{1 + \frac{1}{x_1 x_2}}.$$

a) Show that f is a homothetic function.

Solution: Let $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $g(z) = \frac{2}{1+z^{-1}}$ and $h\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^{\geq 0}$, $h(x_1, x_2) = x_1 x_2$. Then it is obvious that g is (strictly) increasing, h is positively homogeneous (of degree 2) and $f = g \circ h$.

b) Show that f is non-decreasing and quasi-concave.

Solution: On problem sheet 1 exercise 3 we saw that h is non-decreasing and quasi-concave. So if $0 \leq (x_1, x_2) \leq (x_1', x_2')$, then $h(x_1, x_2) \leq h(x_1', x_2')$. Since g is strictly increasing. This translates into $g(h(x_1, x_2)) \leq g(h(x_1', x_2'))$.

Concerning the quasi-concavity, let $\underline{x}, \underline{x}' \in \mathbb{R}^2_{\geq 0}$ and $\lambda \in [0, 1]$. Define $\underline{x}'' = (1-\lambda)\underline{x} + \lambda \underline{x}'$. Then

$$h(\underline{x}'') \geq \min(h(\underline{x}), h(\underline{x}'))\,.$$

Since g is strictly increasing, this means that

$$g(h(\underline{x}'')) \geq \min(g(h(\underline{x})), g(h(\underline{x}')))\,.$$

□

c) Calculate the elasticity of scale of f. For which $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$ exhibits f locally increasing, decreasing or constant returns to scale.

Solution: Let $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$. Applying the chain rule, we obtain the partial derivatives

$$\partial_1 f(x_1, x_2) = \frac{2}{\left(1 + \frac{1}{x_1 x_2}\right)^2 x_1^2 x_2}\,, \qquad \partial_2 f(x_1, x_2) = \frac{2}{\left(1 + \frac{1}{x_1 x_2}\right)^2 x_1 x_2^2}\,.$$

Hence, the elasticity of scale of f at (x_1, x_2) is given by

$$\begin{aligned} e(x_1, x_2) &= \frac{\langle \nabla f(x_1, x_2), (x_1, x_2)\rangle}{f(x_1, x_2)} \\ &= \frac{\partial_1 f(x_1, x_2) x_1 + \partial_2 f(x_1, x_2) x_2}{f(x_1, x_2)} \\ &= \frac{2}{1 + x_1 x_2}\,. \end{aligned}$$

This means

$$e(x_1, x_2) \begin{cases} < 1, & \text{if } x_1 x_2 > 1, \\ = 1, & \text{if } x_1 x_2 = 1, \\ > 1, & \text{if } x_1 x_2 < 1. \end{cases}$$

Hence, f exhibits locally decreasing returns to scale on $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, x_1 x_2 > 1\}$, locally constant returns to scale on $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, x_1 x_2 = 1\}$, and locally increasing returns to scale on $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, x_1 x_2 < 1\}$.

d) Calculate the MRTS of f and show that it is positively homogeneous of degree 0.

Solution: Let $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$, $x_1 > 0$. Then the MRTS of f at (x_1, x_2) is given by

$$\mathrm{MRTS}(x_1, x_2) = -\frac{\partial_1 f(x_1, x_2)}{\partial_2 f(x_1, x_2)} = -\frac{x_2}{x_1}\,.$$

One can directly verify that for any $t > 0$ $\mathrm{MRTS}(tx_1, tx_2) = \mathrm{MRTS}(x_1, x_2)$.

e) Show that any differentiable homothetic production function has an MRTS which is homogeneous of degree 0.

Solution: Let $f = g \circ h \colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ be continuously differentiable and homothetic function. To avoid cumbersome technicalities we also assume that $h \colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, and $g \colon \mathbb{R} \to \mathbb{R}$ are continuously differentiable. Recall that g is increasing and that h is positively homogeneous of some degree $k \in \mathbb{R}$.

We first prove that the partial derivatives $\partial_i h$, $i \in \{1, 2\}$, are positively homogeneous of degree $k - 1$. Indeed, for any $\underline{x} \in \mathbb{R}^2_{\geq 0}$ and any $t > 0$ we have

$$h(t\underline{x}) = t^k h(\underline{x})\,.$$

Taking the derivative with respect to x_i on both sides yields

$$t\partial_i h(t\underline{x}) = t^k \partial_i h(\underline{x})\,.$$

Now we calculate the MRTS of f at some $\underline{x} \in \mathbb{R}^2_{\geq 0}$ such that $\partial_2 f(\underline{x}) \neq 0$. Applying the chain rule, we obtain

$$\begin{aligned}
\mathrm{MRTS}(x_1, x_2) &= -\frac{\partial_1 f(x_1, x_2)}{\partial_2 f(x_1, x_2)} \\
&= -\frac{g'(h(x_1, x_2))\partial_1 h(x_1, x_2)}{g'(h(x_1, x_2))\partial_2 h(x_1, x_2)} \\
&= -\frac{\partial_1 h(x_1, x_2)}{\partial_2 h(x_1, x_2)}\,.
\end{aligned}$$

Then, for $t > 0$

$$\begin{aligned}
\mathrm{MRTS}(tx_1, tx_2) &= -\frac{\partial_1 h(tx_1, tx_2)}{\partial_2 h(tx_1, tx_2)} \\
&= -\frac{t^{k-1}\partial_1 h(x_1, x_2)}{t^{k-1}\partial_2 h(x_1, x_2)} \\
&= \mathrm{MRTS}(x_1, x_2)\,.
\end{aligned}$$

$\square$

3. Let $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ be a non-decreasing and quasi-concave production function. Show that the following statements are true.

a) The factor demand function $\underline{x}^*\colon \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is positively homogeneous of degree 0.

Solution: The factor demand function maximises the profit function $\pi(\underline{x}, \underline{p}, \underline{w})$ in $\underline{x}$. Now, consider a rescaling of both input and output prices by a constant $t > 0$. Then

$$\pi(\underline{x}, t\underline{p}, t\underline{w}) = t\underline{p}^\top f(\underline{x}) - t\underline{w}^\top \underline{x} = t\pi(\underline{x}, \underline{p}, \underline{w}).$$

That means the profit function itself is homogeneous of degree 1 and it makes no difference whether to maximise $\pi(\underline{x}, \underline{p}, \underline{w})$ or $t\pi(\underline{x}, \underline{p}, \underline{w})$.

Observe that a rescaling of the $\underline{p}$ and $\underline{w}$ amounts to changing the currency in which prices are reported. So it makes sense that changing the currency in which prices are reported does not affect the real-term demand of products.

b) The profit function $\pi^*\colon \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is positively homogeneous of degree 1.

Solution: The profit function at prices $(\underline{p}, \underline{w}) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ is defined as

$$\pi^*(\underline{p}, \underline{w}) = \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} \pi(\underline{x}, \underline{p}, \underline{w})\,.$$

With the considerations from above we can deduce for any $t > 0$

$$\begin{aligned}
\pi^*(t\underline{p}, t\underline{w}) &= \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} \pi(\underline{x}, t\underline{p}, t\underline{w}) \\
&= \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} t\pi(\underline{x}, \underline{p}, \underline{w}) \\
&= t \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} \pi(\underline{x}, \underline{p}, \underline{w}) \\
&= t\pi^*(\underline{p}, \underline{w})\,.
\end{aligned}$$

Again, we can interpret a rescaling of $\underline{p}$ and $\underline{w}$ with $t > 0$ as simultaneously changing the currencies in which all prices are reported. Since profit is also reported in a monetary unit, also this number should change accordingly.

c) The profit function π^* is non-decreasing in $\underline{p} \in \mathbb{R}^m_{\geq 0}$ and non-increasing in $\underline{w} \in \mathbb{R}^n_{\geq 0}$.

Solution: Let $(\underline{p}, \underline{w}), (\underline{p}', \underline{w}') \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ and $\underline{p} \leq \underline{p}'$, $\underline{w} \geq \underline{w}'$. Then, since

$f \geq 0$

$$
\begin{aligned}
\pi^*(\underline{p}, \underline{w}) &= \underline{p} f\left(\underline{x}^*(\underline{p}, \underline{w})\right)^\top - \underline{w}\underline{x}^*(\underline{p}, \underline{w})^\top \\
&\leq \underline{p}' f\left(\underline{x}^*(\underline{p}, \underline{w})\right)^\top - \underline{w}\underline{x}^*(\underline{p}, \underline{w})^\top \\
&\leq \underline{p}' f\left(\underline{x}^*(\underline{p}', \underline{w})\right)^\top - \underline{w}\underline{x}^*(\underline{p}', \underline{w})^\top \\
&= \pi^*(\underline{p}', \underline{w}) .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\pi^*(\underline{p}, \underline{w}) &= \underline{p} f\left(\underline{x}^*(\underline{p}, \underline{w})\right)^\top - \underline{w}\underline{x}^*(\underline{p}, \underline{w})^\top \\
&\leq \underline{p} f\left(\underline{x}^*(\underline{p}, \underline{w})\right)^\top - \underline{w}'\underline{x}^*(\underline{p}, \underline{w})^\top \\
&\leq \underline{p} f\left(\underline{x}^*(\underline{p}, \underline{w}')\right)^\top - \underline{w}'\underline{x}^*(\underline{p}, \underline{w}')^\top \\
&= \pi^*(\underline{p}, \underline{w}') .
\end{aligned}
$$

d) The profit function π^* is convex.

Solution: Let $(\underline{p}, \underline{w}), (\underline{p}', \underline{w}') \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$, $\lambda \in [0, 1]$. Define $(\underline{p}'', \underline{w}'') = (1-\lambda)(\underline{p}, \underline{w}) + \lambda(\underline{p}', \underline{w}')$. Then

$$
\begin{aligned}
\pi^*(\underline{p}'', \underline{w}'') &= \underline{p}'' f\left(\underline{x}^*(\underline{p}'', \underline{w}'')\right)^\top - \underline{w}''\underline{x}^*(\underline{p}'', \underline{w}'')^\top \\
&= (1-\lambda)\Big[\underline{p} f\left(\underline{x}^*(\underline{p}'', \underline{w}'')\right)^\top - \underline{w}\underline{x}^*(\underline{p}'', \underline{w}'')^\top\Big] \\
&\quad + \lambda\Big[\underline{p}' f\left(\underline{x}^*(\underline{p}'', \underline{w}'')\right)^\top - \underline{w}'\underline{x}^*(\underline{p}'', \underline{w}'')^\top\Big] \\
&\leq (1-\lambda)\Big[\underline{p} f\left(\underline{x}^*(\underline{p}, \underline{w})\right)^\top - \underline{w}\underline{x}^*(\underline{p}, \underline{w})^\top\Big] \\
&\quad + \lambda\Big[\underline{p}' f\left(\underline{x}^*(\underline{p}', \underline{w}')\right)^\top - \underline{w}'\underline{x}^*(\underline{p}', \underline{w}')^\top\Big] \\
&= (1-\lambda)\pi^*(\underline{p}, \underline{w}) + \lambda\pi^*(\underline{p}', \underline{w}') .
\end{aligned}
$$

$\square$

4. (Envelope Theorem) The Envelope Theorem asserts the following. Let $\varphi\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, be some continuously differentiable function with partial derivatives $\partial_1\varphi$, $\partial_2\varphi$. Define the function $\Phi\colon \mathbb{R} \to \mathbb{R}$

$$\Phi(a) = \max_{x\in\mathbb{R}} \varphi(x, a).$$

Assume that Φ is well defined and differentiable. Let $x^*\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$x^*(a) = \arg\max\nolimits_{x\in\mathbb{R}} \varphi(x, a),$$

where we assume that the argmax is unique and x^* is differentiable and takes only values in the interior of D. Then

$$\Phi'(a) = \partial_2\varphi(x^*(a), a).$$

a) Prove the Envelope Theorem.

Proof: We can write $\Phi(a) = \varphi(x^*(a), a)$. Under the regularity assumptions from above, we can just straightforwardly calculate the derivative of Φ.

$$\Phi'(a) = \partial_1\varphi(x^*(a), a)\frac{\partial x^*(a)}{\partial a} + \partial_2\varphi(x^*(a), a).$$

Now, since $x^*(a)$ maximises the function $x \mapsto \varphi(x, a)$ and $x^*(a)$ is in the interior of D, it needs to be a critical point of that function. That means its derivative $\partial_1\varphi$ needs to vanish at $x^*(a)$. This already yields the claim. $\square$

b) Give an argument how one can use the Envelope Theorem to derive Hotelling's Lemma.

Solution: This is actually a straight forward application. The role of $\varphi(x, a)$ is played by $\pi(\underline{x}, \underline{p}, \underline{w})$. Then we only need to verify a higher dimensional version of the Envelope Theorem and we are done.