Imperial College London
Department of Mathematics

Solutions to Blackboard quiz 3

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

You should enter your answers on Blackboard by 1pm on Wednesday 1 March 2023. The test is worth 1.5 percent of the marks for the module.

(A) (The following text refers to Question 1 - 5) In each of the examples below, determine whether $(A, *)$:

i) has an identity element.

ii) is associative.

iii) is commutative.

iv) defines a group.

v) none of the above.

Qu 1: A the set of all $n \times n$ matrices over $\mathbb{R}$ with determinant 1 (for $n \in \mathbb{N}$).
$*$ is the matrix multiplication.

Solution: This set is a group because matrices with determinant 1 are invertible and the determinant is multiplicative. The group is not abelian

$$M_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $n > 3$ take the block matrices

$$M_1' = \begin{pmatrix} M_1 & 0 \\ 0 & I_{n-3} \end{pmatrix}, M_2' = \begin{pmatrix} M_2 & 0 \\ 0 & I_{n-3} \end{pmatrix}.$$

Qu 2: $A = \{\sigma \in S_3 \mid \sigma = id \text{ or } \sigma(i) \neq i \text{ for } i = 1, 2, 3\}$.
$*$ is the composition of functions (that is, the multiplication in S_n).

Solution: $(A, *)$ is a cyclic group of order 3. (If one does not know that, one can check the group axioms pretty easily or use the subgroup test.)

Qu 3: $A = M_n(\mathbb{R})$ where $n \geq 2$.
$*$ defined by $a * b = ab - ba$ for $a, b \in A$.

Solution: The binary operation is non-associative and non-commutative. There is no identity element because $a * b = -b * a$, so there cannot be $e \in A$ such that $ae - ea = ea - ae = a$ if $a \neq 0$.

Qu 4: $A = \mathbb{Z}$.
$*$ defined by $a * b = a - b$ for $a, b \in A$.

Solution: The operation is not associative as $a - (b - c) \neq (a - b) - c$ if $c \neq 0$, and is not commutative. There is no identity because $a * e = a$ implies that $e = 0$ but $0 * a = -a \neq a$ if $a \neq 0$.

Qu 5: $A = \{a \in \mathbb{R} \mid a \leq 0\}$.
$*$ defined by $a * b = \min(a, b)$.

Solution: The binary operation is associative and commutative and there is an identity element, namely 0. However, A is not a group because non-zero elements have no inverses.

(B) (The following text refers to Question 6 - 10) For each of the following statements, determine whether it is true or false.

Qu 6: Let G be a finite group and let $g \in G$. Then there is $m \in \mathbb{N}$, such that $g^m = e$ where e is the identity of G.

Solution: True. Let $H = \langle g \rangle = \{g^i \mid i \in \mathbb{N}\}$. Then the fact that H is finite, implies that an m as above exists. Indeed, since H is finite there will be $a, b \in \mathbb{N}$ such that $g^a = g^b$. Then $g^{a-b} = e$; thus, we may take $m = a - b$ (if this is non-negative) otherwise $m = b - a$.

Qu 7: There is a cyclic group of order n for all $n \geq 1$.

Solution: True. Take the remainder classes module n with addition or equivalently the group of n-th roots of unity in $\mathbb{C}$ with the complex multiplication.

Qu 8: There is an infinite cyclic group which is uncountable.

Solution: False. Every infinite cyclic group $\langle g \rangle$ is in bijection with $\mathbb{Z}$ via the map $i \mapsto g^i$ (check the details).

Qu 9: Let $n \in \mathbb{N} \setminus \{0\}$ and

$$H = \{\sigma \in S_n \mid \sigma(i) \neq i, \forall i \in \{1, \ldots, n\}\}.$$

Then H is a subgroup of S_n.

Solution: False. H does not contain the identity.

Qu 10: Let $n \in \mathbb{N} \setminus \{0\}$ and

$$H = \{\sigma \in S_n \mid \sigma(1) = 1\}.$$

Then H is a subgroup of S_n.

Solution: True. The identity fixes everything, so $id \in H$. If $\sigma_1, \sigma_2 \in H$, then $\sigma_1\sigma_2(1) = \sigma_1(\sigma_2(1)) = \sigma_2(1) = 1$. Finally if σ fixes 1 also σ^{-1} fixes 1.