IMPERIAL

**MATH70011**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

**Classical Dynamics**

**Date:** Thursday, May 29, 2025

**Time:** Start time 14:00 – End time 16:30 (BST)

**Time Allowed**: 2.5 hours

**This paper has 5 Questions.**

***Please Answer All Questions in 1 Answer Booklet***

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO**

1. (a) Demonstrate that Euler-Lagrange equations, in polar coordinates, follow from Hamilton's Principle

$$0 = \delta \int_{t_0}^{t_1} L(r, \theta, \dot{r}, \dot{\theta})\, dt\,.$$

(8 marks)

(b) A particle in a two dimensional plane has potential energy given by

$$V = \frac{K r^{\alpha}}{\alpha}\,.$$

(i) Write down a suitable Lagrangian for the dynamics of the particle. (2 marks)

(ii) Calculate the force on the particle. (1 mark)

(iii) Is this a conservative force? Is it a central force? (Yes/no answers are sufficient) (2 marks)

(iv) Determine the equations of motion. (4 marks)

(v) What is the (conserved) angular momentum of the problem? What is its associated Noether symmetry? (3 marks)

(Total: 20 marks)

2. The following question relates to canonical transformations.

(a) For a type 2 generating function $F_2(q_1, q_2, P_1, P_2, t)$, use the differential condition

$$\sum_{i=1}^{2} p_i dq_i - H\, dt = \sum_{i=1}^{2} P_i dQ_i - K\, dt + dF\,,$$

to determine the equations which express each $p_i$ and $Q_i$ in terms of $F_2$, and the 'new' Hamiltonian $K$ in terms of $H$ and $F_2$.

(6 marks)

(b) A generating function of the second kind is defined as

$$F_2(q_1, q_2, P_1, P_2, t) = \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Determine the canonical transformation corresponding to this generating function, i.e. determine the new coordinates $(Q_1, Q_2, P_1, P_2)$ in terms of $(q_1, q_2, p_1, p_2)$.

(5 marks)

(c) A Hamiltonian has the form

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{k}{2}\left(q_1^2 + q_2^2\right) - \gamma(q_1 \sin t + q_2 \cos t)\,,$$

where $k, \gamma \in \mathbb{R}$ are constants. Determine how the Hamiltonian transforms under the canonical transformation from part (b) and demonstrate that the transformed Hamiltonian $K$ is time-independent.

(5 marks)

(d) Write down the equations of motion corresponding to the Hamiltonian from part (b) in terms of the variables $(Q_1, Q_2, P_1, P_2)$.

(4 marks)

(Total: 20 marks)

3. A particle of mass $m$ in three dimensions is associated to the Lagrangian given by

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + c(x - 2y)\,. \tag{1}$$

(a) (i) What is the corresponding Hamiltonian? (3 marks)

(ii) Write down Hamilton's equations for the dynamics of the particle (there should be six equations) and identify the force on the particle. (6 marks)

(b) (i) Identify three conserved quantities of the system. (3 marks)

(ii) What is the Noether symmetry (of the Lagrangian or Hamiltonian) associated to each of the conserved quantities from question (b)(i). (3 marks)

(iii) Is the system Liouville integrable? Why? (5 marks)

(Total: 20 marks)

4. (a) Solve the Hamilton-Jacobi equation for the Hamiltonian

$$H = pq \log q$$

and use the result to determine $q(t)$ and $p(t)$. (12 marks)

(b) A particle of mass $m$ in one dimension has potential energy given by the *Morse potential* $V(q)$, defined by

$$V(q) = D\left(1 - e^{-a(q-q_0)}\right)^2 ,$$

where $D, a, q_0 \in \mathbb{R}$ are constants.

(i) What is the potential force corresponding to $V(q)$? (2 marks)

(ii) How does the motion of the particle behave when $q - q_0$ is small? (6 marks)

(Total: 20 marks)

5. A dynamical system in three dimensions is governed by the following equations, where $\boldsymbol{\chi} \in \mathbb{R}^3$ is a constant,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} + mg\boldsymbol{\Gamma} \times \boldsymbol{\chi}\,, \tag{2}$$

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}\,. \tag{3}$$

(a) Derive equations (1) and (2) by applying Hamilton's Principle to the action defined by

$$S = \int_{t_0}^{t_1} \frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} - mg\boldsymbol{\chi}\cdot\boldsymbol{\Gamma} + \mathbf{P}\cdot\left(\dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega}\times\boldsymbol{\Gamma}\right)\,dt\,,$$

where each variable $\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \mathbf{P} \in \mathbb{R}^3$ is varied arbitrarily such that the variations $(\delta\boldsymbol{\Omega}, \delta\boldsymbol{\Gamma}, \delta\mathbf{P})$ vanish at the end points $t_0$ and $t_1$.

(12 marks)

(b) A Poisson bracket is given by

$$\{f, g\} = -\boldsymbol{\Pi}\cdot\left(\frac{\partial f}{\partial\boldsymbol{\Pi}} \times \frac{\partial g}{\partial\boldsymbol{\Pi}}\right) - \boldsymbol{\Gamma}\cdot\left(\frac{\partial f}{\partial\boldsymbol{\Pi}} \times \frac{\partial g}{\partial\boldsymbol{\Gamma}} - \frac{\partial g}{\partial\boldsymbol{\Pi}} \times \frac{\partial f}{\partial\boldsymbol{\Gamma}}\right).$$

Demonstrate that equation (1) can be written in the form $\dot{\Pi}_i = \{\Pi_i, h\}$, where

$$h(\boldsymbol{\Pi}, \boldsymbol{\Gamma}) = \frac{1}{2}\boldsymbol{\Pi}\cdot\mathbb{I}^{-1}\boldsymbol{\Pi} + mg\boldsymbol{\chi}\cdot\boldsymbol{\Gamma}\,.$$

(8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

# MATH70011

# Classical Dynamics (Solutions)

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| ................ | ................ | ................ |

1. (a) Vary the action and apply Hamilton's Principle to give

sim. seen ⇓

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1} \frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial \theta}\delta \theta + \frac{\partial L}{\partial \dot r}\delta \dot r + \frac{\partial L}{\partial \dot\theta}\delta\dot\theta \, dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r}\right)\delta r + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta}\right)\delta\theta \, dt + \left.\frac{\partial L}{\partial \dot r}\delta r\right|_{t_0}^{t_1} + \left.\frac{\partial L}{\partial \dot\theta}\delta\theta\right|_{t_0}^{t_1} .
\end{aligned}
$$

Since $\delta r$ and $\delta\theta$ vanish at the end points, the endpoint terms vanish and, since $\delta r$ and $\delta\theta$ are arbitrary, this implies the Euler-Lagrange equations

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{\partial L}{\partial r}\,, \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta}\,.
$$

8, B

(b) (i) The Lagrangian in polar coordinates is

seen ⇓

$$
L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2\right) - \frac{Kr^\alpha}{\alpha}\,.
$$

The Lagrangian in cartesian coordinates is also acceptable

$$
L = \frac{1}{2}m\left(\dot x^2 + \dot y^2\right) - \frac{K(x^2+y^2)^{\alpha/2}}{\alpha}\,.
$$

2, A

(ii) The force on the particle is

sim. seen ⇓

$$
\mathbf{F} = -\nabla V = Kr^{\alpha-1}\mathbf{e}_r
$$

1, A

(iii) Yes and yes.

sim. seen ⇓

2, A

(iv) The Euler-Lagrange equations are

meth seen ⇓

$$
\frac{d}{dt}\left(m\dot r\right) = mr\dot\theta^2 - Kr^{\alpha-1}\,, \quad \text{and} \quad \frac{d}{dt}\left(mr^2\dot\theta\right) = 0\,.
$$

4, B

(v) Angular momentum is $mr^2\dot\theta$ and the associated Noether symmetry is rotation (or translation in $\theta$).

seen ⇓

1, A

2, D

2. (a) Substitute $F = F_2(q_1, q_2, P_1, P_2, t) + \sum_{i=1}^{2} Q_i P_i$ into the differential formula to obtain $\boxed{\text{seen} \Downarrow}$

$$\sum_{i=1}^{2} p_i dq_i - H dt = -\sum_{i=1}^{2} Q_i dP_i - K dt + dF_2 \,.$$

We equate the coefficients of each $dq_i$, $dP_i$ and $dt$ to give

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}, \qquad \text{and } K = H + \frac{\partial F_2}{\partial t} \,.$$

$\boxed{6, \text{A}}$

(b) Multiplying out the generating function gives $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned} F_2(q_1, q_2, P_1, P_2, t) &= \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \end{pmatrix} \begin{pmatrix} q_1 \cos t & -q_2 \sin t \\ q_1 \sin t & q_2 \cos t \end{pmatrix} \\ &= P_1 q_1 \cos t - P_1 q_2 \sin t + P_2 q_! \sin t + P_2 q_2 \cos t \\ &= (P_1 q_1 + P_2 q_2) \cos t + (P_2 q_1 - P_1 q_2) \sin t \,. \end{aligned}$$

Using the result from part (a), we have

$$\begin{aligned} p_1 &= P_1 \cos t + P_2 \sin t \,, \\ p_2 &= P_2 \cos t - P_1 \sin t \,, \\ Q_1 &= q_1 \cos t - q_2 \sin t \,, \\ Q_2 &= q_2 \cos t + q_1 \sin t \,. \end{aligned}$$

It remains to write $P_1$ and $P_2$ in terms of $p_1$ and $p_2$. Notice that

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \end{pmatrix} \,,$$

and, by inverting the matrix,

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} p_1 \\ p_1 \end{pmatrix} \,,$$

as required. $\boxed{1, \text{A}}$

(c) Using part (a), the transformed Hamiltonian is $\boxed{4, \text{B}}$

$$\begin{aligned} K &= H + \partial_t K_2 \\ &= \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{k}{2}\left(q_1^2 + q_2^2\right) - \gamma(q_1 \sin t + q_2 \cos t) \\ &\quad - (P_1 q_1 + P_2 q_2) \sin t + (P_2 q_1 - P_1 q_2) \cos t \\ &= \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{k}{2}\left(Q_1^2 + Q_2^2\right) - \gamma Q_2 \\ &\quad - [P_1(Q_1 \cos t + Q_2 \sin t) + P_2(Q_2 \cos t - Q_1 \sin t)] \sin t \\ &\quad + [P_2(Q_1 \cos t + Q_2 \sin t) - P_1(Q_2 \cos t - Q_1 \sin t)] \cos t \\ &= \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{k}{2}\left(Q_1^2 + Q_2^2\right) - \gamma Q_2 + P_2 Q_1 - P_1 Q_2 \,. \end{aligned}$$

This has no explicit time dependence. $\boxed{1, \text{A}}$

$\boxed{4, \text{B}}$

(d) The equations of motion can be easily deduced in the new variables

$$\begin{aligned}
\dot{Q}_1 &= \frac{\partial H}{\partial P_1} = P_1 - Q_2\,, \\
\dot{Q}_2 &= \frac{\partial H}{\partial P_2} = P_2 + Q_1\,, \\
\dot{P}_1 &= -\frac{\partial H}{\partial Q_1} = -kQ_1 - P_2\,, \\
\dot{P}_2 &= -\frac{\partial H}{\partial Q_2} = -kQ_2 + P_1 + \gamma\,.
\end{aligned}$$

$\boxed{4,\ \mathrm{A}}$

3. (a) (i) The generalised momenta are

sim. seen ⇓

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\,, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}\,, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}\,.$$

Applying the Legendre transformation gives

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - c(x - 2y)\,.$$

3, A

(ii) Hamilton's equations are

meth seen ⇓

$$\dot{p_x} = -\frac{\partial H}{\partial x} = c\,, \quad \dot{p_y} = -\frac{\partial H}{\partial y} = -2c\,, \quad \dot{p_z} = -\frac{\partial H}{\partial z} = 0\,,$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}\,, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}\,, \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}\,.$$

The first three of these are Newton's second law, and as such the force on the particle is

$$\mathbf{F} = c(\mathbf{i} - 2\mathbf{j})\,.$$

6, A

(b) (i) From Hamilton's equations, there are two conserved quantities

unseen ⇓

$$C_1 = p_z\,, \quad \text{and} \quad C_2 = 2p_x + p_y\,.$$

Furthermore, the Hamiltonian itself has no explicit time dependence and is the conserved energy $E = H$.

3, A

(ii) The Noether symmetry corresponding to energy conservation is time translation. For point transformations $(x, y, z) \mapsto (x', y', z')$, the symmetry for $C_1 = p_z$ is, for some $\alpha \in \mathbb{R}$, given by

sim. seen ⇓

$$(x', y', z') = (x, y, z + s\alpha)\,.$$

Similarly, the symmetry for $C_2 = 2p_x + p_y$ is

$$(x', y', z') = (x + 2s\beta, y + s\beta, z)\,,$$

for some $\beta \in \mathbb{R}$.

3, D

(iii) Yes, the system is Liouville integrable since the configuration is 3 dimensional and we have 3 independent conserved quantities in Poisson involution. That is,

sim. seen ⇓

$$\{2p_x{+}p_y, H\} = -2\cdot(-c){-}1\cdot(2x) = 2c{-}2c = 0\,, \quad \{p_z, H\} = 0\,, \quad \{p_z, 2p_x{+}p_y\} = 0\,.$$

These follow by calculating the Poisson bracket from its explicit form

$$\{F, G\} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial p_x} - \frac{\partial F}{\partial p_x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y}\frac{\partial G}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial G}{\partial p_z} - \frac{\partial F}{\partial p_z}\frac{\partial G}{\partial z}\,.$$

It is correct to either say that the three quantities from b(i) are conserved, or to say that their Poisson bracket with the Hamiltonian vanishes, i.e.

$$\{p_z, H\} = \{2p_x + p_y, H\} = \{H, H\} = 0\,.$$

5, D

4. (a) The Hamilton-Jacobi equation is

sim. seen ⇓

$$\frac{\partial S}{\partial q} q \log q + \frac{\partial S}{\partial t} = 0 \,.$$

We seek a solution in terms of Hamilton's Characteristic Function, $W(q)$, of the form $S = W(q) - \alpha t$, where $\alpha \in \mathbb{R}$ is a constant. Hamilton's Characteristic Function then satisfies

$$W'(q) q \log q = \alpha \qquad \Longrightarrow \qquad W(q) = \alpha \int \frac{dq}{q \log q} \,.$$

Computing the integral (substitution $u = \log q$ makes this simple) and dropping the additive constant gives

$$S = \alpha \log\log q - \alpha t \,.$$

Determining the 'new coordinate' as

$$\beta = \frac{\partial S}{\partial \alpha} = \log\log q - t \,,$$

and inverting this gives the solution

$$q(t) = e^{e^{t+\beta}} \,.$$

We obtain $p(t)$ by differentiating $S$ as

$$p(t) = \frac{\partial S}{\partial q} = \alpha \frac{\partial}{\partial q} (\log\log q) = \frac{\alpha}{q \log q} \,.$$

12, C

(b) (i) We calculate the force as

sim. seen ⇓

$$F = -\frac{\partial}{\partial q}\left( D\left(1 - e^{-a(q-q_0)}\right)^2 \right) = -2Dae^{-a(q-q_0)}\left(1 - e^{-a(q-q_0)}\right)$$

2, A

(ii) Notice that, for $q$ near $q_0$, we have that

unseen ⇓

$$e^{-a(q-q_0)} \approx 1 - a(q - q_0) \,.$$

Thus, expanding the force from b(i) we see that the force on the particle becomes

$$F \approx -2Da(1 - a(q - q_0))a(q - q_0) \approx -2Da^2(q - q_0) \,,$$

neglecting terms of order $(q - q_0)^2$ and smaller. Alternatively, the potential energy becomes

$$V(q) \approx Da^2(q - q_0)^2 \,.$$

Either calculation impllies that, for small $q - q_0$, the particle behaves as a simple harmonic oscillator with angular frequency $\omega = \sqrt{2Da^2/m}$. 6, D

5. (a) We calculate the following difference

sim. seen ⇓

$$
\begin{aligned}
\Delta S &= S(\boldsymbol{\Omega}+\delta\boldsymbol{\Omega},\boldsymbol{\Gamma}+\delta\mathbf{Q},\mathbf{P}+\delta\mathbf{P}) - S(\boldsymbol{\Omega},\boldsymbol{\Gamma},\mathbf{P})\\
&= \int_{t_0}^{t_1}\Big[\mathbb{I}\boldsymbol{\Omega}\cdot\delta\boldsymbol{\Omega} - mg\boldsymbol{\chi}\cdot\delta\boldsymbol{\Gamma} + \delta\mathbf{P}\cdot\Big(\dot{\boldsymbol{\Gamma}}+\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\Big)\\
&\qquad + \mathbf{P}\cdot\Big(\delta\dot{\boldsymbol{\Gamma}}+\delta\boldsymbol{\Omega}\times\boldsymbol{\Gamma}+\boldsymbol{\Omega}\times\delta\boldsymbol{\Gamma}\Big) + \mathcal{O}(\delta\boldsymbol{\Omega}^2,\delta\boldsymbol{\Gamma}^2,\delta\mathbf{P}^2)\Big]\,dt
\end{aligned}
$$

Hamilton's Principle is then

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\Big[\mathbb{I}\boldsymbol{\Omega}\cdot\delta\boldsymbol{\Omega} - mg\boldsymbol{\chi}\cdot\delta\boldsymbol{\Gamma} + \delta\mathbf{P}\cdot\Big(\dot{\boldsymbol{\Gamma}}+\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\Big)\\
&\qquad + \mathbf{P}\cdot\Big(\delta\dot{\boldsymbol{\Gamma}}+\delta\boldsymbol{\Omega}\times\boldsymbol{\Gamma}+\boldsymbol{\Omega}\times\delta\boldsymbol{\Gamma}\Big)\Big]\,dt\\
&= -\boldsymbol{\Gamma}\cdot\delta\mathbf{Q}\Big|_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1}\Big[(\mathbb{I}\boldsymbol{\Omega}-\mathbf{P}\times\boldsymbol{\Gamma})\cdot\delta\boldsymbol{\Omega}\\
&\qquad + \delta\mathbf{P}\cdot\Big(\dot{\boldsymbol{\Gamma}}+\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\Big) - \delta\boldsymbol{\Gamma}\cdot\Big(\dot{\mathbf{P}}+\boldsymbol{\Omega}\times\mathbf{P}+mg\boldsymbol{\chi}\Big)\Big]\,dt\,.
\end{aligned}
$$

Since the variations $(\delta\boldsymbol{\Omega},\delta\boldsymbol{\Gamma},\delta\mathbf{P})$ vanish at the endpoints $t_0$ and $t_1$, the endpoint term in the above equation is zero. Furthermore, since the variations are arbitrary, Hamilton's Principle implies the following equations

$$
\mathbb{I}\boldsymbol{\Omega} = \mathbf{P}\times\boldsymbol{\Gamma}\,,\quad \dot{\boldsymbol{\Gamma}} = -\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\,,\quad \text{and}\quad \dot{\mathbf{P}} = -\boldsymbol{\Omega}\times\mathbf{P} - mg\boldsymbol{\chi}\,,
$$

which are an *implicit* version of our equations (1) and (2). That is, we may assemble them as follows

$$
\begin{aligned}
\mathbb{I}\dot{\boldsymbol{\Omega}} &= \dot{\mathbf{P}}\times\boldsymbol{\Gamma} + \mathbf{P}\times\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}\times(\boldsymbol{\Omega}\times\mathbf{P}+mg\boldsymbol{\chi}) + \mathbf{P}\times(\boldsymbol{\Gamma}\times\boldsymbol{\Omega})\\
&= -\boldsymbol{\Omega}\times(\mathbf{P}\times\boldsymbol{\Gamma}) + mg\boldsymbol{\Gamma}\times\boldsymbol{\chi} = \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega} + mg\boldsymbol{\Gamma}\times\boldsymbol{\chi}\,,
\end{aligned}
$$

and the equation for $\dot{\boldsymbol{\Gamma}}$ is given directly from the variational principle as above.

12, M

(b) Calculate as

sim. seen ⇓

$$
\begin{aligned}
\dot{\Pi}_i = \{\Pi_i, h\} &= -\boldsymbol{\Pi}\cdot\left(\frac{\partial\Pi_i}{\partial\boldsymbol{\Pi}}\times\frac{\partial h}{\partial\boldsymbol{\Pi}}\right) - \boldsymbol{\Gamma}\cdot\left(\frac{\partial\Pi_i}{\partial\boldsymbol{\Pi}}\times\frac{\partial h}{\partial\boldsymbol{\Gamma}} - \frac{\partial h}{\partial\boldsymbol{\Pi}}\times\frac{\partial\Pi_i}{\partial\boldsymbol{\Gamma}}\right)\\
&= \frac{\partial\Pi_i}{\partial\boldsymbol{\Pi}}\cdot\Big(\boldsymbol{\Pi}\times\mathbb{I}^{-1}\boldsymbol{\Pi}\Big) + \frac{\partial\Pi_i}{\partial\boldsymbol{\Pi}}\cdot\Big(\boldsymbol{\Gamma}\times(mg\boldsymbol{\chi})\Big)\\
&= \delta_{ij}\Big(\boldsymbol{\Pi}\times\mathbb{I}^{-1}\boldsymbol{\Pi}+mg\boldsymbol{\Gamma}\times\boldsymbol{\chi}\Big)_j = \Big(\boldsymbol{\Pi}\times\mathbb{I}^{-1}\boldsymbol{\Pi}+mg\boldsymbol{\Gamma}\times\boldsymbol{\chi}\Big)_i
\end{aligned}
$$

8, M

**Review of mark distribution:**

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

**MATH70011 Classical Dynamics Markers Comments**

| | |
|---|---|
| Question 1 | Solutions to question 1 were generally weaker than for other questions. This was mostly due to students finding 1(a) to be challenging. Part (b) was generally better and most students did this in polar coordinates, those who did not found the question more difficult. |
| Question 2 | Question 2(a) proved challenging for many students. Many students had memorised the equations for a Type 2 generating function, but few knew how to manipulate the differential condition to derive these expressions. Answers to the remaining parts were generally good, with the most common error being neglecting to include the terms in the transformed Hamiltonian corresponding to the partial time derivative of the generating function. |
| Question 3 | Part (a) was generally good. Students who identified 3 conserved quantities generally performed well on part (b). Some students failed to notice that the Hamiltonian was not invariant under general spatial translations, and many gave a statement in favour of Liouville integrability without providing justification. |
| Question 4 | Part (a) was attempted well, with many students receiving partial credit for their solution. Some students failed to differentiate the Morse potential correctly in part (b). Amongst students who attempted a series expansion in their solution to (b)(ii), performance was generally good. |
| Question 6 | The Mastery question was challenging this year, though many good attempts were made. Students generally demonstrated that they had engaged with the assigned supplementary reading, but reproducing the methods accurately proved challenging. Students who correctly varied the action in part (a) tended to perform well on this part, though some students had difficulties with this step. Part (b) was generally better, though misinterpreting the rate of inertia tensor when varying the kinetic energy term was a common mistake. |