Test 2

1. (a) Derive the general form of a canonical transformation corresponding to a type 2 generating function $F_2 = F_2(q_1, \ldots, q_N, P_1, \ldots, P_N)$.

   (b) Find a generating function corresponding to the following canonical transformation
$$p = \frac{P}{q} \quad \text{and} \quad Q = \log q \, .$$

   (c) Demonstrate that this transformation is canonical by computing the fundamental Poisson brackets.

[10 marks]

2. The Lagrangian of a physical system is given by
$$L(q, \dot{q}) = \frac{m}{2} \frac{\dot{q}^2}{q^2} - \log q \, .$$

   (a) What is the corresponding Hamiltonian?

   (b) How does this Hamiltonian transform under the canonical transformation considered in 1. (b)?

   (c) Determine Hamilton's equations for the transformed Hamiltonian and solve them to deduce $q(t)$, as a function of t, using the boundary conditions
$$q(0) = 1 \quad \text{and} \quad p(0) = 1 \, .$$

   (d) How does $q(t)$ behave for small t?

[15 marks]

[Total: 25 marks]

Answers to Test 2

1. (a) As in the lecture notes, we consider the differential condition

$$\sum_{i=1}^{N} p_i dq_i - H\,dt = \sum_{i=1}^{N} P_i dQ_i - K\,dt + dF\,,$$

with $F = F_2 - \sum_{i=1}^{N} P_i Q_i$. This implies

$$\sum_{i=1}^{N} p_i dq_i - H\,dt = -\sum_{i=1}^{N} Q_i dP_i - K\,dt + dF_2\,.$$

Matching cofficients of dq_i, dP_i, and dt gives

$$p_i = \frac{\partial F_2}{\partial q_i}\,, \quad Q_i = \frac{\partial F_2}{\partial P_i}\,, \quad K = H + \frac{\partial F_2}{\partial t}$$

[4 marks]

(b) We seek a function $F_2(q, P)$ such that

$$\frac{\partial F_2}{\partial q} = \frac{P}{q} \quad \text{and} \quad \frac{\partial F_2}{\partial P} = \log q\,.$$

Integrating gives

$$F_2 = P \log q\,.$$

[3 marks]

(c) The fundamental Poisson brackets $\{P, P\}$ and $\{Q, Q\}$ are zero by antisymmetry. The remaining Poisson bracket is computed as

$$\{Q, P\} = \frac{1}{q} q - 0 \cdot p = 1\,.$$

[3 marks]

[Q1: 10 marks in total]

2. The Lagrangian of a physical system is given by

$$L(q, \dot{q}) = \frac{m}{2}\frac{\dot{q}^2}{q^2} - \log q\,.$$

(a) The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{m\dot{q}}{q^2}\,.$$

This implies that $\dot{q} = \frac{pq^2}{m}$ and the Legendre transformation is then performed as

$$H = p\dot{q} - L = \frac{p^2q^2}{m} - \frac{m}{2}\frac{p^2q^4}{m^2q^2} + \log q = \frac{p^2q^2}{2m} + \log q\,.$$

[3 marks]

(b) Since the canonical transformation is time-independent, we have

$$K = H = \frac{P^2}{2m} + Q\,.$$

[2 marks]

(c) Hamilton's equations are

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m} \quad \text{and} \quad \dot{P} = -\frac{\partial K}{\partial Q} = -1\,.$$

Integrating the second gives

$$P(t) = -t + c_1\,.$$

Substituting this into the equation for Q gives

$$\dot{Q} = \frac{-t + c_1}{m} \quad \text{implies} \quad Q(t) = \frac{1}{m}\left(-\frac{t^2}{2} + c_1 t + c_2\right)\,.$$

The boundary conditions on p and q can be stated for P and Q through the canonical transformation as

$$P(0) = 1 \quad \text{and} \quad Q(0) = 0\,.$$

Thus the constants can be deduced to give

$$Q(t) = \frac{2t - t^2}{2m}\,.$$

Finally, the expression for the old coordinate q is

$$q(t) = \exp\left(\frac{2t - t^2}{2m}\right)\,.$$

[7 marks]

(d) The expansion of $q(t)$ around $t = 0$ is

$$q(t) = 1 + \frac{t}{m} - \frac{(m-1)t^2}{2m^2} + \mathcal{O}(t^3)\,.$$

Thus, for small t the motion behaves as

$$q(t) \approx 1 + \frac{t}{m}\,.$$

[3 marks]

[Q2: 15 marks in total]

[Total: 25 marks]