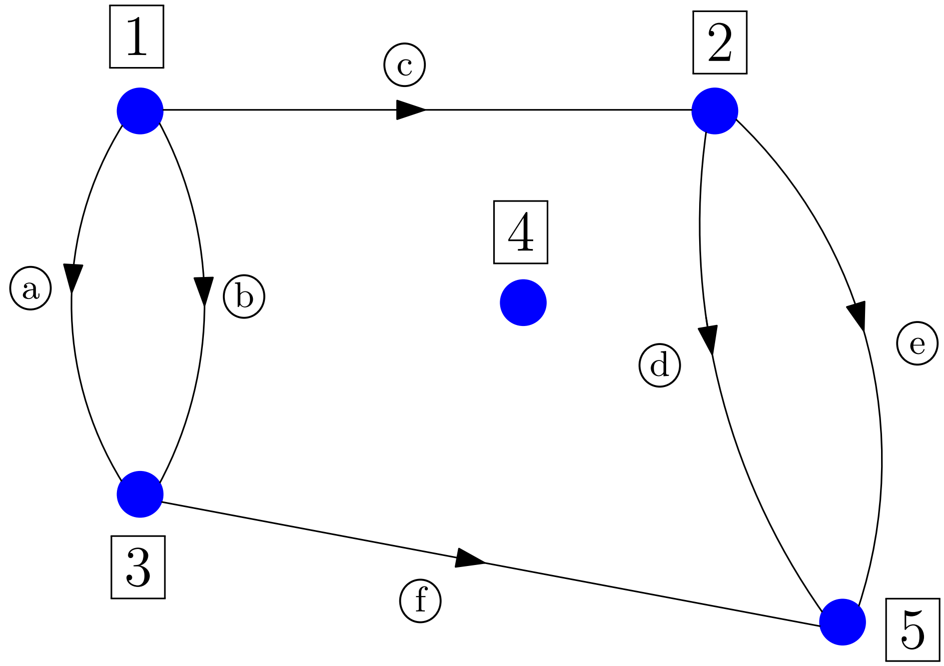
1. Consider the graph shown in the figure:



(a) The incidence matrix **A** is 6–by–5 ($m = 6$ and $n = 5$) and using the numbering and directions on the figure, is

$$\begin{array}{cc} & \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \end{array} \\ \mathbf{A} = & \begin{pmatrix} -1 & 0 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 \end{pmatrix} \begin{array}{l} \text{edge (a)} \\ \text{edge (b)} \\ \text{edge (c)} \\ \text{edge (d)} \\ \text{edge (e)} \\ \text{edge (f)} \end{array} \end{array}$$

2 marks

(b) This is a disconnected graph with 2 components. The rank r of **A** is therefore

$$r = n - 2 = 5 - 2 = 3.$$

Two linearly independent right null vectors are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

which correspond to all elements in each connected component being at the same potential. These two vectors provide a basis for the right null space. Equivalently, the right null space is

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \,,$$

where triangular brackets denote "the vector space spanned by" $\mathbf{x}_1$ and $\mathbf{x}_2$.

3 marks

(c) Let (the transpose of) the first, third and fourth row vectors be denoted by

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

If

$$0 = a_1\mathbf{r}_1 + a_3\mathbf{r}_3 + a_4\mathbf{r}_4 = a_1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

then $a_4 = 0$ by inspection of the last element, $a_1 = 0$ by inspection of the middle element, hence $a_3 = 0$. Thus the three vectors are linearly independent. Since we know the rank $r = 3$ then the row space must be

$$\langle \mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_3 \rangle \,,$$

the space spanned by the (linearly independent) $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ which form a basis of the row space.

2 marks

(d) Let the first three column vectors be denoted by

$$\mathbf{c}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{c}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

If

$$0 = \hat{a}_1 \mathbf{c}_1 + \hat{a}_2 \mathbf{c}_2 + \hat{a}_3 \mathbf{c}_3$$

then $\hat{a}_3 = 0$ by inspection of the last element, $\hat{a}_2 = 0$ by inspection of the penultimate element, hence $\hat{a}_1 = 0$ also. Hence the three vectors are linearly independent. Since we know the rank $r = 3$ then the column space must be

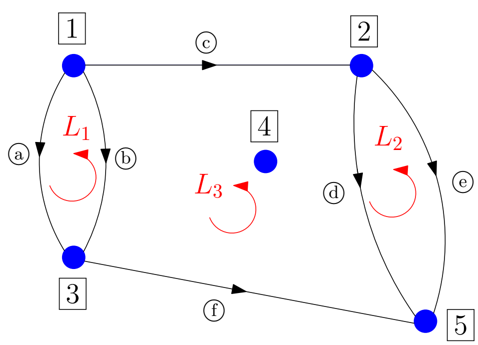
$$\langle \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \rangle ,$$

the space spanned by the (linearly independent) $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ which form a basis of the column space.

2 marks

(f) By the rank-nullity theorem the column space is orthogonal to the left null space. The left null space vectors have an interpretation as **closed loops** in the graph. By inspection of three loops L_1, L_2 and L_3 drawn in the figure a basis for the left null space can be identified as

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$



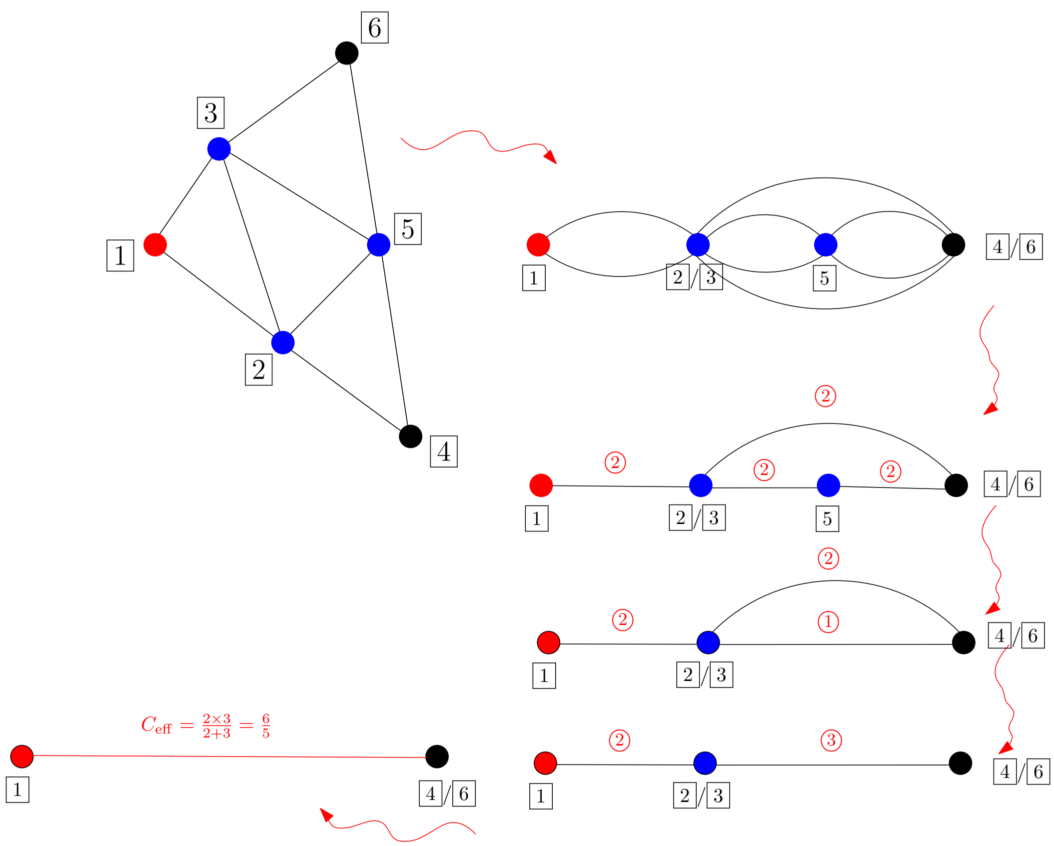
3 marks

2. There are several solution methods, some easier than others. Any successful method that is well explained is acceptable.

The first method relies on use of the symmetry of the circuit to compute the effective conductance. Notice that nodes $\boxed{2}$ and $\boxed{3}$ must be at the same potential and can be merged (while preserving all edges); so too can nodes $\boxed{4}$ and $\boxed{6}$ (since these are both grounded). Then the following sequence of "equivalent circuits" can be used to deduce the effective conductance

$$C_{\text{eff}} = \frac{6}{5}. \tag{1}$$

The equivalent circuit reductions in the figure make repeated use of the rules that two conductors with conductances c_1 and c_2 have effective conductance $c_1 + c_2$ if in parallel and $(c_1 c_2/(c_1 + c_2)$ if in series:



This is the net current into the circuit from node $\boxed{1}$. However, again by the symmetry of the circuit, the sum of the divergences of the currents at nodes $\boxed{4}$ and $\boxed{6}$ must be **equal**, and sum to $-6/5$, so that the sum of the divergences is zero.

Therefore the required divergence of currents at node $\boxed{6}$ is half of this, i.e.,

$$-\frac{3}{5}. \tag{2}$$

A second method uses linear algebra to solve

$$\mathbf{K}\mathbf{x} = \mathbf{f} \tag{3}$$

or

$$\begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \hat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \hat{\mathbf{0}} \end{pmatrix}, \tag{4}$$

where

$$\mathbf{e}_1 = (1,0,0)^T, \qquad \mathbf{f}_1 = (C_{\text{eff}}, f_4, f_6)^T, \qquad \hat{\mathbf{x}} = (x_2, x_3, x_5)^T. \tag{5}$$

On setting

$$\mathbf{K} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{R} \end{pmatrix}, \tag{6}$$

where

$$\mathbf{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \tag{7}$$

and

$$\mathbf{R} = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}, \tag{8}$$

we can show that

$$\mathbf{f}_1 = (\mathbf{P} - \mathbf{Q}^T\mathbf{R}^{-1}\mathbf{Q})\mathbf{e}_1 \tag{9}$$

or

$$C_{\text{eff}} = \mathbf{e}_1^T(\mathbf{P} - \mathbf{Q}^T\mathbf{R}^{-1}\mathbf{Q})\mathbf{e}_1 = \frac{6}{5}. \tag{10}$$

This method requires the inverse of a 3-by-3 matrix, which is do-able by hand, but is cumbersome; details omitted here (but students should show them). Finally, by the same arguments as above, the required divergence of currents at node $\boxed{6}$ is half of this, i.e.,

$$-\frac{3}{5}. \tag{11}$$

A quicker method, that avoids inverting a 3-by-3 matrix, is to deploy symmetry at this point in the calculation and notice that we expect

$$x_2 = x_3. \tag{12}$$

This means we can add columns 4 and 5, and eliminate row 5, say, since this row is redundant, to produce a different but equivalent linear system

$$\mathbf{K} = \begin{pmatrix} 2 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \tilde{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \hat{\mathbf{0}} \end{pmatrix}, \qquad \tilde{\mathbf{x}} = (x_4, x_6)^T. \tag{13}$$

This is **not** symmetric, but has the sub-block decomposition

$$\mathbf{K} = \begin{pmatrix} \mathbf{P} & \mathbf{Q}_2 \\ \mathbf{Q}_1 & \mathbf{R} \end{pmatrix}, \tag{14}$$

where

$$\mathbf{Q}_1 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \qquad \mathbf{Q}_2 = \begin{pmatrix} -2 & 0 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}, \tag{15}$$

and

$$\mathbf{R} = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}, \tag{16}$$

then we can show that we can show that

$$\mathbf{f}_1 = (\mathbf{P} - \mathbf{Q}_2\mathbf{R}^{-1}\mathbf{Q}_1)\mathbf{e}_1. \tag{17}$$

Now it is very easy to calculate

$$\mathbf{R}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \tag{18}$$

and hence to calculate the quantity

$$C_{\text{eff}} = \mathbf{e}_1^T(\mathbf{P} - \mathbf{Q}_2\mathbf{R}^{-1}\mathbf{Q}_1)\mathbf{e}_1 = \frac{6}{5}. \tag{19}$$

By the same arguments as above, the required divergence of currents at node $\boxed{6}$ is half of this, i.e.,

$$-\frac{3}{5}. \tag{20}$$

8 marks