EXAM Tips

Think of special cases
- when two points A, B are the same points
- or two curves overlap

carefully justify the differentiability of functions you constructed
- inverse of a smooth function may NOT be differentiable (counter-example: $f(x) = x^3$)

In some proofs, including
- a particular parametrisation for a curve
- a particular chart/level set for a surface is used.
- the regular curve α used to find DF_p

You need to state why your arguments do not change if a different parametrisation/chart is used.
This also has an advantage of generalisation (i.e. your derived result may hold for all other parametrisations/charts)

proving a map is linear: prove $T(v + \lambda w) = T(v) + \lambda T(w)$ for all v, w and any real number λ

IFT(inverse function theorem) can only be used when the domain and codomain has the same dimensions.
- if you need IFT when going from e.g. R^2 to R^3, try to extend it to a map $R^3 \to R^3$

local surjectivity: open neighbourhood of any point in the image belongs to the image.

local surjectivity + closed $\Rightarrow$ surjective

Curves

You need **parameterisation by arc-length** when:
- calculating curvature, torsion
- working on Frenet Frame
- theorem 4.4: when calculating the turning number (winding number of tangent) using signed curvature
- finding geodesic curvature

If $|r(t)| = 1$ (imagine a circle), $r'(t)\, r(t) = 0$. So tangent is orthogonal to the vector.

Frenet frame is construct only when **$\phi'' \neq 0$** and in this case, N(t), B(t), T(t) are not 0

- torsion $\tau(t)$ describes how the plane formed by N(t), T(t) twists in R^3
- curvature $\kappa(t)$ describes how the curve twists in the plane formed by N(t), T(t)
- N(t) leads the direction of change of B(t) and T(t)
  - therefore, direction of change of N(t) is affected by B(t), T(t) together

Cross product relationships of Frenet frame:
$B(t) = T(t) \times N(t)$,
$T(t) = N(t) \times B(t)$
$N(t) = B(t) \times T(t)$
- so knowing two vectors of Frenet's Frame gives you the third one

rigid motions preserve arc-length, curvature, and the torsion
- these are transformations in SO(n) combined with translation

if g is in SO(n) and ϕ represents some regular curve:
- $g(A \times B) = g(A) \times g(B)$ where A, B are vector functions
- $|g(\phi)| = |\phi|$
  - for derivatives: $|(g \circ \phi)'| = |g(\phi')| = |\phi'|$ this holds for derivatives of all orders
- $g(kA) = k\, g(A)$ where k is scalar or scalar function
- g must be invertible

For curves on R^2:
there are two choices of unit normal vectors
When calculating signed curvature, remember to divide by $|\phi'|^2$ when the curve is not parametrised by arc length.
Curvature of curves in R^2 is absolute value of the signed curvature

For closed regular curve: $\gamma : [a, b] \to R^2$(or R^3)
- $\gamma(a) = \gamma(b)$
- $\gamma^{(n)}(a) = \gamma^{(n)}(b)$ derivative of any order aligns

Surfaces

Ways to define a regular surface S
- chart (ϕ, U)

    - Must clearly state sets U, V s.t. $\phi(U) = V \cap S$
    - show ϕ and $d\phi$ are injective ($d\phi$ is injective if its columns are LI)
- level set $S = F^{-1}(c)$
    - F is smooth
    - $\nabla F(p)$ non zero for all p in S
- build a diffeomorphism from f: $R^3 \to R^3$, show S = f(R) for some surface R.
    - e.g. if you have a map from an open set in R^2 to R^3, but struggle to show it is a chart. You can thicken the set to embed it into R^3, and also thicken the surface in a way s.t. it is diffeomorphic to the thickened open set.

if WOLG assume $S = F^{-1}(0)$, this means that: if a point is on the surface, assign 0, otherwise, assign some distance measure (which could be negative)

- if assigning distance to the surface is difficult, do not use level set approach to show a surface is regular

some regular surfaces requires more than one chart to cover

Prove a surface is not regular:

- show it cannot be written as graph of a smooth function
    - either all the possible functions are not smooth, or there is no such a function

for maps F between surfaces (which cannot be assumed to be Euclidean), chain rule cannot be applied directly (but you can use prop 7.4)

Finding dF_p in practice:
Chart
If $\phi(u_0, v_0) = p$, then $\{\phi_u(u_0, v_0), \phi_v(u_0, v_0)\}$ is a basis for T_pS. So it is enough to find dF for the basis.

$$dF_p\left(\frac{\partial \phi}{\partial u}(u_0, v_0)\right) = \frac{\partial (F \circ \phi)}{\partial u}(u_0, v_0)$$

there is a similar relation for ϕ_v.

Level set
Method 1: find a chart at p, use the method above
Method 2: Compute tangent space, directly find a basis for T_pS (does not have to be orthogonal), find dF of the basis using dF_x, dF_y, dF_z.

Jordan Curve Theorem: on a plane, any simple closed curve divides the plane into two connected components
Jordan Brouwer theorem (General Version): If S is non-empty, compact, connected, S divides R^3 into two components

non-empty, compact, oriented regular surface must have a point with positive K(p)

tangent planes T_pS are defined as vector spaces (so they are always passing 0)

changing the sign of N causes sign change in the following quantities
- second fundamental form
- principal curvatures
- normal curvature
- mean curvature z

Properties preserved under homeomorphism:
- Euler Characteristics
- Fundamental groups

Intrinsic Properties of Surfaces (preserved under isometry)
- First fundamental form (definition)
- length of curves
- Gaussian Curvature (Theorema Egregium)
- area of sections on the surface
- exterior, interior angles of curvilinear triangles on the surface
- geodesics
    - length of all closed geodesics

local isometry is defined to preserve inner products between all pairs of vectors X, Y in the tangent plane, but in practice, you can use
$<X + Y, X + Y> = <X, X> + 2 <X, Y> + <Y, Y>$
so checking inner product with itself (i.e. norm of vectors) is enough.

proposition 12.1 (calculate Gaussian and mean curvatures using g, A) can be used not only on partial derivatives of charts ϕ_u ϕ_v, but also any orthonormal basis of the tangent plane T_pS.

Gaussian curvature K is **continuous** along any continuous path on the surface

Gauss-Bonnet
Common Euler characteristics:
- Sphere: 2
- Surface of genus Σ_g: 2 - 2g
    - note any compact, connected, orientable surface without boundary is homeomorphic to some Σ_g
- disk: 1

- cylinder: 0