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**Imperial College London**

# Lecture 04: Asymptotic Properties I
## Statistical Modelling I

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## Outline

# Introduction

## Evaluating estimators

Evaluating an estimator $T = T(X_1, \ldots, X_n)$ of $\theta$ depends on its *sampling distribution*

This problem often simplifies by considering $T_n \equiv T_n(X_1, \ldots, X_n)$ to be a sequence of random variables indexed by $n = 1, 2, 3, \ldots$
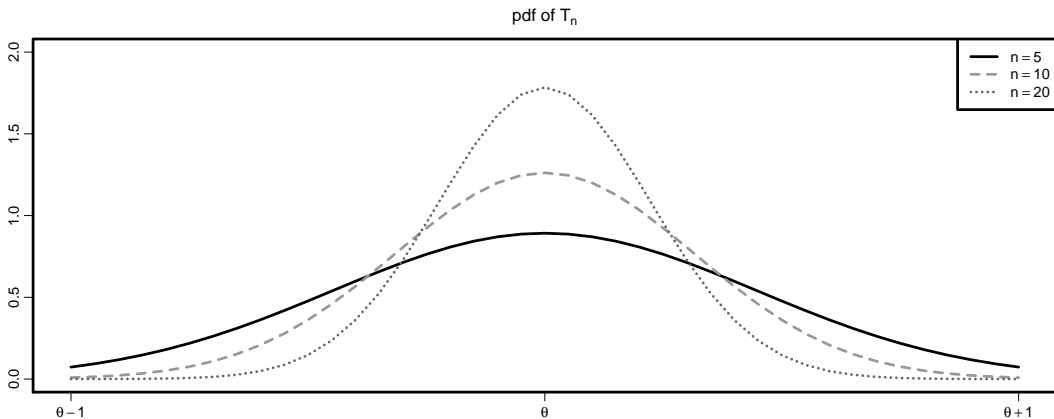
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## Example: $Y_1, Y_2, \cdots \sim N(\theta, 1)$ iid

$$T_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \qquad E(T_n) = \theta \qquad Var(T_n) = \frac{1}{n} \qquad T_n \sim N\left(\theta, \frac{1}{n}\right)$$

pdf of $T_n$

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## Stochastic Convergence

Just as sequences of numbers $x_1, x_2, \ldots$ can convergence, sequences of random variables $X_1, X_2, \ldots$ exhibit patterns of convergence

- Convergence almost surely
- Convergence in probability
- Convergence in distribution

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## Convergence almost surely and convergence in probability

$X_n$ converges almost surely to $X$ if

$$P \left( \lim_{n \to \infty} X_n = X \right) = 1$$

$X_n$ converges in probability to $X$ if, $\forall \epsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0$$

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## Convergence in distribution

$X_n$ converges in distribution to $X$ with cdf $F_X$ if

$$\lim_{n \to \infty} P(X_n \leq x) = F_X(x)$$

at all continuity points $x$ of $F_X(x)$

**Portmanteau lemma**: The following statements are equivalent:

• $X_n \overset{d}{\to} X$

• $E[f(X_n)] \to E[f(X)]$ for all bounded and continuous functions $f : \mathbb{R} \to \mathbb{R}$.

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## Relationship between modes of stochastic convergence

We have
$$X_n \to_{as} X \qquad \Rightarrow \qquad X_n \to_p X \qquad \Rightarrow \qquad X_n \to_d X$$

and if $X = c$ is a constant
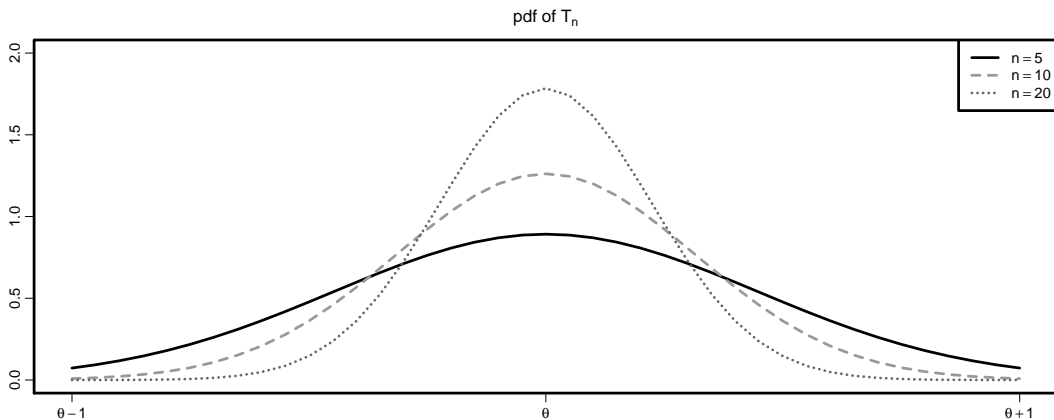$$X_n \to_p X \qquad \Leftrightarrow \qquad X_n \to_d X$$

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## Concentration

Estimating $\theta$ from $N(\theta, 1)$, we see distribution of $T_n$ concentrates around $\theta_0$



pdf of $T_n$

legend:
— $n = 5$
-- $n = 10$
⋯ $n = 20$

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# Definition: Consistency

A sequence of estimators $(T_n)_{n \in \mathbb{N}}$ for $g(\theta)$ is called **(weakly) consistent** if for all $\theta \in \Theta$:

$$T_n \overset{P_\theta}{\to} g(\theta) \quad (n \to \infty)$$

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## Definition: Asymptotically Unbiased Estimator

A sequence of estimators $(T_n)_{n \in \mathbb{N}}$ for $g(\theta)$ is called **asymptotically unbiased** if for all $\theta \in \Theta$:

$$E_\theta(T_n) \to g(\theta) \quad (n \to \infty)$$

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## Markov's Inequality

$$P(|X| \geq a) \leq \frac{E|X|}{a} \qquad \forall a > 0$$

**Proof**

$$a\mathcal{I}(|X| \geq a) \leq |X| \qquad \Rightarrow \qquad P(|X| \geq a) = E\{\mathcal{I}(|X| \geq a)\} \leq \frac{1}{a}E|X|$$

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## Lemma: MSE consistency

If $T_n$ is asymptotically unbiased for $g(\theta)$ and $\forall \theta \in \Theta$

$$Var_\theta(T_n) \to 0 \quad (n \to \infty)$$

then $T_n$ is consistent for $g(\theta)$

**Proof**

Applying Markov's inequality, we have

$$\forall \epsilon > 0: \quad P_\theta(|T_n - g(\theta)| \geq \epsilon) = P_\theta((T_n - g(\theta))^2 \geq \epsilon^2) \leq \frac{E_\theta(T_n - g(\theta))^2}{\epsilon^2} = \frac{MSE_\theta(T_n)}{\epsilon^2}$$

$$= \frac{1}{\epsilon^2}(\underbrace{Var_\theta\, T_n}_{\to 0} + (\underbrace{E_\theta\, T_n - g(\theta)}_{\to 0})^2)$$

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## Example: $X_i \sim$ Bernoulli$(\theta)$, $\theta \in \Theta = [0, 1]$

$$T_n(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\left. \begin{array}{l} E_\theta \, T_n(X_1, \ldots, X_n) = \theta \quad \forall \theta \in \Theta \\ Var_\theta(T_n) = \frac{1}{n}\theta(1 - \theta) \to 0 \quad (n \to \infty) \end{array} \right\} \implies T_n \text{ consistent}$$

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## Summary

▶ Consistency is an important property, but does not tell us all about the sampling distribution of $T_n$

▶ To establish approximations to the sampling distribution of $T_n$, we will appeal to convergence in distribution of sequences of the form $a_n(T_n - g(\theta))$ for deterministic $a_n$

▶ We typically find that $a_n$ is proportional to $SE(T_n)$

▶ Many estimators "look like" sample means, so $a_n = \sqrt{n}$ and $N(0, \sigma^2)$ limits are extremely common!