MATH50010: Probability for Statistics
Problem Sheet 5

1. In question 6 of Problem Sheet 4, you derived the cdfs of a number of random variables involving the minimum or maximum of a random sample. In this problem we will derive the limiting distribution (i.e. the distribution to which it converges in distribution), if it exists, of these same random variables.

   Suppose $(X_1, \ldots, X_n)$ is a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with pmf/pdf f_X and cdf F_X, let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from $X_1, \ldots, X_n$.

   (a) Suppose $X_1, \ldots, X_n \sim \mathrm{Unif}(0, 1)$, that is

   $$F_X(x) = x, \text{ for } 0 \leq x \leq 1.$$

   Find the limiting distributions of Y_n and Z_n as $n \longrightarrow \infty$.

   (b) Suppose $X_1, \ldots, X_n$ have cdf

   $$F_X(x) = 1 - x^{-1}, \text{ for } x \geq 1.$$

   Find the limiting distributions of Z_n and $U_n = Z_n^n$ as $n \longrightarrow \infty$.

   (c) Suppose $X_1, \ldots, X_n$ have cdf

   $$F_X(x) = \frac{1}{1 + e^{-x}}, \text{ for } x \in \mathbb{R}.$$

   Find the limiting distributions of Y_n and $U_n = Y_n - \log n$, as $n \longrightarrow \infty$.

   (d) Suppose $X_1, \ldots, X_n$ have cdf

   $$F_X(x) = 1 - \frac{1}{1 + \lambda x}, \text{ for } x > 0.$$

   Let $U_n = Y_n/n$ and $V_n = nZ_n$. Find the limiting distributions of Y_n, Z_n, U_n, and V_n as $n \longrightarrow \infty$.

   (a) In the limit as $n \to \infty$ we have the limit for fixed y *as*

   $$F_{Y_n}(y) = \{F_X(y)\}^n = y^n \to \begin{cases} 0, & y < 1 \\ 1, & y \geq 1 \end{cases}.$$

   This is a step function with single step of size 1 at $y = 1$. Hence the limiting random variable Y is discrete with $\Pr(Y = 1) = 1$*, that is, the limiting distribution is* degenerate *at* 1. *Also in the limit as $n \to \infty$ we have the limit for* fixed z *as*

   $$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - (1 - z)^n \to \begin{cases} 0, & z \leq 0 \\ 1, & z > 0 \end{cases}.$$

   This is a step function with single step of size 1 at $z = 0$. Hence the limiting random variable Z is a discrete variable with $\Pr(Z = 0) = 1$*, that is, the limiting distribution is* degenerate *at* 0. *The definition of convergence in distribution only refers to pointwise convergence at* *points of continuity of the limit function, and here the limit function is not continuous at zero so we do not require convergence of F_{Z_n} here.*

These results are intuitively reasonable. As the sample size gets increasingly large, we will very probably obtain a random variable arbitrarily close to each end of the range.

Remark: *We have established* convergence in distribution, *but we also have for $1 > \varepsilon > 0$ and as $n \to \infty$,*

$$\begin{aligned}
\Pr(|Y_n - 1| < \varepsilon) &= \Pr(1 - Y_n < \varepsilon) = \Pr(1 - \varepsilon < Y_n) = 1 - \Pr(Y_n < 1 - \varepsilon) = 1 - \varepsilon^n \to 1, \\
\Pr(|Z_n - 0| < \varepsilon) &= \Pr(Z_n < \varepsilon) = 1 - (1 - \varepsilon)^n \to 1.
\end{aligned}$$

So Y_n and Z_n converge in probability *to 1 and 0, respectively.*

(b) *Recall that*

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - \left(1 - \left(1 - \frac{1}{z}\right)\right)^n = 1 - \frac{1}{z^n}, \ \textit{for} \ \ z > 1.$$

In the limit as $n \to \infty$ we have for fixed z

$$F_{Z_n}(z) \to \begin{cases} 0, & z \le 1 \\ 1, & z > 1 \end{cases}.$$

This is a step function with single step of size 1 at $z = 1$. Hence the limiting random variable Z is a discrete variable with

$$\Pr(Z = 1) = 1,$$

Setting $U_n = Z_n^n$, we found that, for $u > 1$,

$$F_{U_n}(u) = \Pr(U_n \le u) = \Pr(Z_n^n \le u) = \Pr\left(Z_n \le u^{1/n}\right) = 1 - \frac{1}{\left(u^{1/n}\right)^n} = 1 - \frac{1}{u},$$

which does not depend on n. Hence the limiting distribution of U_n is

$$F_U(u) = 1 - \frac{1}{u}, \ \textit{for} \ \ u > 1.$$

For $u \le 1$, $F_U(u) = 0$ for all n.

(c) *Recall*

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left(\frac{1}{1 + e^{-y}}\right)^n, \qquad y \in \mathbb{R}.$$

In the limit as $n \to \infty$, for fixed y

$$F_{Y_n}(y) \to 0, \ \textit{for all} \ \ y.$$

Hence there is no limiting distribution. *Recall also that*

$$F_{U_n}(u) = F_{Y_n}(u + \log n) = \left(\frac{1}{1 + e^{-u - \log n}}\right)^n,$$

so that

$$F_{U_n}(u) = \left(\frac{1}{1 + \frac{e^{-u}}{n}}\right)^n = \left(1 + \frac{e^{-u}}{n}\right)^{-n} \to \exp\left\{-e^{-u}\right\}, \quad \textit{as } n \to \infty,$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-e^{-u}\right\}, \quad u \in \mathbb{R}.$$

(d) *Recall*

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left(\frac{\lambda y}{1+\lambda y}\right)^n, \ \textit{for} \ \ y > 0,$$

and so as $n \to \infty$ *for* fixed y

$$F_{Y_n}(y) \to 0, \ \textit{for all} \ \ y$$

and there is no limiting distribution. *In the limit as* $n \to \infty$ *for* fixed $z > 0$

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - \left(1 - \left(1 - \frac{1}{1+\lambda z}\right)\right)^n = 1 - \frac{1}{(1+\lambda z)^n} \to \left\{ \begin{array}{ll} 0, & z \le 0 \\ 1, & z > 0 \end{array} \right. .$$

This is a step function with single step of size 1 at $z = 0$. *Hence the limiting random variable* Z *is a discrete variable with* $P(Z = 0) = 1$*: the limiting distribution is* degenerate *at* 0.
Recall that for $u > 0$,

$$F_{U_n}(u) = \Pr(U_n \le u) = \Pr(Y_n/n \le u) = \Pr(Y_n \le nu) = F_{Y_n}(nu) = \left(\frac{\lambda nu}{1+\lambda nu}\right)^n,$$

so that

$$F_{U_n}(u) = \left(\frac{\lambda nu}{1+\lambda nu}\right)^n = \left(1 + \frac{1}{n\lambda u}\right)^{-n} \to \exp\left\{-\frac{1}{\lambda u}\right\}, \quad \textit{as } n \to \infty,$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-\frac{1}{\lambda u}\right\}, \ \textit{for} \ \ u > 0.$$

Finally, recall that for $v > 0$,

$$F_{V_n}(v) = \Pr(V_n \le v) = \Pr(nZ_n \le v) = \Pr(Z_n \le v/n) = F_{Z_n}(v/n) = 1 - \left(\frac{1}{1+\frac{\lambda v}{n}}\right)^n$$

so that

$$F_{V_n}(v) = 1 - \left(1 + \frac{\lambda v}{n}\right)^{-n} \to 1 - \exp\{-\lambda v\}, \quad \textit{as } n \to \infty,$$

which is a valid cdf. Hence the limiting distribution is

$$F_V(v) = 1 - \exp\{-\lambda v\}, \ \textit{for} \ \ v > 0.$$

Hence the limiting distribution of V *is* Exponential(λ).

2. Show that if $X_1, \ldots, X_n$ are a sequence of random variables such that $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$ for X with cdf continuous on $\mathbb{R}$, then for $a, b > 0$, $aX_n + b \xrightarrow{\mathcal{D}} aX + b$.
Let $Y_n = aX_n + b$ *and* $Y = aX + b$. *Then, for any* $y \in \mathbb{R}$,

$$F_{Y_n}(y) = \Pr(Y_n \le y) = \Pr(aX_n + b \le y) = \Pr(X_n \le \frac{y-b}{a}) = F_{X_n}(\frac{y-b}{a}).$$

By definition of convergence in distribution, we know that $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ *for any* $x \in \mathbb{R}$ *since* F_X *is continuous at all* $x \in \mathbb{R}$. *It follows that* $\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} F_{X_n}(\frac{y-b}{a}) = F_X(\frac{y-b}{a})$. *Then, by a similar argument to above,*

$$F_X(\frac{y-b}{a}) = \Pr(X \le \frac{y-b}{a}) = \Pr(aX + b \le y) = \Pr(Y \le y) = F_Y(y).$$

Hence, $\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y)$ *and so* $Y_n \xrightarrow{\mathcal{D}} Y$.

3. (a) Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Show that $X_n + Y_n \xrightarrow{P} X + Y$. Does a similar result hold for convergence in distribution?

(b) Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Show that $X_n Y_n \xrightarrow{P} XY$. Does a similar result hold for convergence in distribution?

(a) Let $\epsilon > 0$. If $|a+b| > \epsilon$ then either $|a| > \epsilon/2$ or $|b| > \epsilon/2$, since if the converse of both conditions hold we get a contradiction. Therefore,

$$\Pr(|X_n + Y_n - X - Y| > \epsilon) \leq \Pr(|X_n - X| > \epsilon/2) + \Pr(|Y_n - Y| > \epsilon/2) \to 0$$

as $n \to \infty$ since $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$.
It is not the case that iif $X_n \xrightarrow{\mathcal{D}} X$ and $Y_N \xrightarrow{\mathcal{D}} Y$ then $X_n + Y_n \xrightarrow{\mathcal{D}} X + Y$. As a counterexample, consider X_n symmetric. Then, $X_n \xrightarrow{\mathcal{D}} X$ and $-X_n \xrightarrow{\mathcal{D}} X$. However $X_n + (-X_n) \xrightarrow{\mathcal{D}} 0$ which is generally different from $2X$.

(b) Let $\epsilon > 0$. Then,

$$\begin{aligned}\Pr(|X_nY_n - XY| > \epsilon) &= \Pr(|(X_n - X)(Y_n - Y) + (X_n - X)Y + X(Y_N - Y)| > \epsilon) \\ &\leq \Pr(|X_n - X||Y_n - Y| > \epsilon/3) + \Pr(|Y_n - Y||X| > \epsilon/3) \\ &\qquad + \Pr(|X_n - X||Y| > \epsilon/3).\end{aligned}$$

For $\delta > 0$,

$$\begin{aligned}\Pr(|X_n - X||Y| > \epsilon/3) &\leq \Pr(|X_n - X||Y| > \epsilon/3 \cap |Y| \leq \delta) + \Pr(|X_n - X||Y| > \epsilon/3 \cap |Y| > \delta) \\ &\leq \Pr(|X_n - X| > \epsilon/(3\delta)) + \Pr(|Y| > \delta)\end{aligned}$$

which tends to 0 as $n \to \infty$ and $\delta \to \infty$.
The same counterexample as before also shows why if $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{\mathcal{D}} Y$ then X_nY_n does not necessarily converge in distribution to XY. Indeed $X_n(-X_n) = -(X_n)^2$ which does not necessarily converge in distribution to X^2.

4. (a) Show that if $X_1, X_2, \ldots$ are a sequence of random variable such that $X_n \xrightarrow{P} X$ as $n \to \infty$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous deterministic function, then $g(X_n) \xrightarrow{P} g(X)$.

(b) Assume the conditions of part (a), does it also hold that $g(X_n) \xrightarrow{\mathcal{D}} g(X)$?

(a) Fix some $\lambda > 0$. Then there exists some M such that $\Pr(|X| \geq M) \leq \lambda$. Note that as $\lambda \to 0$, $[-M, M]$ converges to the reals. The function g is also uniformly continuous on $[-M, M]$. Therefore, by definition of continuous functions, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|g(x) - g(y)| < \epsilon \qquad \text{if} \qquad |x - y| < \delta \text{ and } |x| < M.$$

By considering the converse, we see that if $|g(x) - g(y)| > \epsilon$ then it must be the case that either $|x - y| > \delta$ or $|x| > M$. Consequently,

$$\Pr(|g(X_n) - g(X)| \geq \epsilon) \leq \Pr(|X_n - X| \geq \delta) + \Pr(|X| \geq M)$$

As $n \to \infty$ since $X_n \xrightarrow{P} X$, $\Pr(|X_n - X| \geq \delta) \to 0$. It therefore holds that for any $\lambda, \epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|g(X_n) - g(X)| \geq \epsilon) \leq \lambda$$

and so $g(X_N) \xrightarrow{P} g(X)$ since the above condition must hold for all $\lambda > 0$.

(b) By Proposition 4.12 in lecture notes, convergence in probability implies convergence in distribution. By the previous part, we have convergence in probability of $g(X_n)$ to $g(X)$, so it follows that this also converges in distribution.

5. Show that

$$\mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0 \quad \text{as } n \to \infty \qquad \text{if and only if} \qquad X_n \xrightarrow{P} 0.$$

First assume $\mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0$ as $n \to \infty$. Note that $f(u) = u/(1+u)$ is an increasing function on $[0, \infty)$. Hence, for any $\epsilon > 0$,

$$\begin{aligned}
\Pr(|X_n| > \epsilon) = \Pr(f(|X_n|) > f(\epsilon)) &= \Pr\left(\frac{|X_n|}{1+|X_n|} > \frac{\epsilon}{1+\epsilon}\right) \\
&\leq \frac{1+\epsilon}{\epsilon}\mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0 \qquad \textit{as } n \to \infty
\end{aligned}$$

Where we have used Markov's inequality and properties of an increasing function.

Now assume $X_n \xrightarrow{P} 0$. Then, for any $\epsilon > 0$, since $f(u)$ is increasing,

$$\begin{aligned}
\mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\right] &= \mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\,\mathbb{I}\left\{|X_n| < \epsilon\right\}\right] + \mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\,\mathbb{I}\left\{|X_n| \geq \epsilon\right\}\right] \\
&\leq \frac{\epsilon}{1+\epsilon}\Pr(|X_n| < \epsilon) + 1\Pr(|X_n| \geq \epsilon) \\
&\to \frac{\epsilon}{1+\epsilon} \qquad \textit{as } n \to \infty
\end{aligned}$$

Since this holds for any $\epsilon > 0$ and $f(u) > 0 \forall u \in \mathbb{R}$, we conclude that $\mathrm{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0$ as $n \to \infty$.

For discussion

6. **Slutsky's Theorems**

   (a) Suppose that $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} c$ where c is a constant. Show that $X_nY_n \xrightarrow{\mathcal{D}} cX$ and that $\frac{X_n}{Y_n} \xrightarrow{\mathcal{D}} \frac{X}{c}$ if $c \neq 0$.
   [Hint: consider the case where $|Y_n - c| > \delta$ and $|Y_n - c| \leq \delta$ for some appropriately chosen $\delta > 0$]

   (b) Suppose that $X_n \xrightarrow{\mathcal{D}} 0$ and $Y_n \xrightarrow{P} Y$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x and $g(x, y)$ is continuous at $x = 0$ for all y. Show that $g(X_n, Y_n) \xrightarrow{P} g(0, Y)$.

(a) Suppose $c > 0$ and pick δ such that $0 < \delta < c$. Then, since $Y_N \xrightarrow{P} c$, there exists an N such that for all $n \geq N$, $\Pr(|Y_n - c| > \delta) < \delta$. For $x > 0$,

$$\Pr(X_nY_n \leq x) \leq \Pr(X_nY_n \leq x, |Y_n - c| \leq \delta) + \Pr(|Y_n - c| > \delta) \leq \Pr(X_n \leq \frac{x}{c-\delta}) + \delta.$$

Similarly,

$$\Pr(X_nY_n > x) \leq \Pr(X_nY_n > x, |Y_n - c| \leq \delta) + \Pr(|Y_n - c| > \delta) \leq \Pr(X_n \leq \frac{x}{c+\delta}) + \delta.$$

Taking the limits as $n \to \infty$ and $\delta \searrow 0$, we see that $\Pr(X_nY_n \le x) \to \Pr(X \le x/c)$ if x/c is a point of continuity of the CDF of X. A similar argument holds for $x < 0$. Hence, $X_nY_n \xrightarrow{\mathcal{D}} cX$ if $c > 0$. The cases where $c < 0$ and $c = 0$ are also similar.

For the second part, if we have that $\frac{1}{Y_n} \xrightarrow{P} \frac{1}{c}$ then the result follows from the previous case. For this, note that if $|Y_n - c| < \epsilon < |c|$, then $|Y_n^{-1} - c^{-1}| = |\frac{c-Y_n}{cy_n}| < \frac{\epsilon}{|c||Y_n|} < \frac{\epsilon}{|c|(|c|-\epsilon)}$ since $|Y_n| \ge |c| - |c - Y_n|$.

(b) *Let $\epsilon > 0$. Then, since $X_n \xrightarrow{\mathcal{D}} 0$ which is a constant, by Proposition 4.14, $X_n \xrightarrow{P} 0$. Hence, there exists a $N \in \mathbb{N}$ and $\lambda > 0$ such that for all $n \ge N$,*

$$\Pr(|X_n| > \epsilon) < \lambda, \qquad \Pr(|Y_n - Y| > \epsilon) < \lambda \quad \textit{and} \quad \Pr(|Y| > N) < \lambda.$$

By assumptions of the questions, g is uniformly continuous at points $(0, y)$ for $|y| \le N$. Therefore, there exists a $\delta > 0$ such that

$$|g(x', y') - g(0, y)| < \epsilon \quad \textit{if } |x'| < \delta, |y - y'| < \delta.$$

If $|X_n| \le \delta$, $|Y_n - Y| \le \delta$, and $|Y| \le N$, then $|g(X_n, Y_n) - g(0, Y)| < \epsilon$, so for $n \ge N$,

$$\Pr(|g(X_nY_n) - g(0, Y)| \ge \epsilon) \le \Pr(|X_n| > \delta) + \Pr(|Y_n - y| > \delta) + \Pr(|Y| > N) \le 3\lambda.$$

Hence, $g(X_n, Y_n) \xrightarrow{P} g(0, Y)$.

7. Let $X_1, X_2, \ldots$ be i.i.d. $\mathrm{N}(0, 1)$ random variables.

(a) Show that for any $x > 0$,

$$(x^{-1} - x^{-3})e^{-x^2/2} \le \int_x^\infty e^{-y^2/2}\,dy \le x^{-1}e^{-x^2/2}$$

(b) Show that with probability 1,

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1$$

(c) Show that,

$$\Pr(X_n > a_n\, i.o.) = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty \Pr(X_1 > a_n) < \infty \\ 1 & \text{if } \sum_{n=1}^\infty \Pr(X_1 > a_n) = \infty \end{cases}$$

(a) *Let $y = x + z$, then, by the transformation lemma and using the fact that $e^{-z^2/2} \le 1$,*

$$\int_x^\infty e^{-y^2/2}\,dy = \int_0^\infty e^{-(x+z)^2/2}\,dz = e^{-x^2/2}\int_0^\infty e^{-xz}e^{-z^2/2}\,dz$$
$$\le e^{-x^2/2}\int_0^\infty e^{-xz}\,dz = x^{-1}e^{-x^2/2}.$$

For the other direction, for $x > 0$,

$$(x^{-1} + x^3)e^{-x^2/2} = \int_x^\infty (1 - 3y^{-4})e^{-y^2/2}\,dy \ge \int_x^\infty e^{-y^2/2}\,dy$$

(b) *Observe that for a standard normal random variable,* $\Pr(X_i \geq x) = \int_x^\infty (2\pi)^{-1/2} e^{-y^2/2}\, dy$. *Hence by the previous part, for any* $x > 0$*,*

$$\sqrt{2\pi}(x^{-1} - x^{-3})e^{-x^2/2} \leq \Pr(X_i \geq x) \leq \sqrt{2\pi} x^{-1} e^{-x^2/2}.$$

Fix some $\epsilon > 0$*. Then,*

$$\begin{aligned}\sum_{n=1}^\infty \Pr(X_n \geq \sqrt{2(1+\epsilon)\log n}) &\leq \sum_{n=1}^\infty \sqrt{2\pi}(\sqrt{2(1+\epsilon)\log n})^{-1} e^{-(1+\epsilon)\log n} \\ &= \sum_{n=1}^\infty \sqrt{\pi} \frac{1}{\sqrt{(1+\epsilon)\log n} n^{1+\epsilon}} < \infty.\end{aligned}$$

So, by the first Borel-Cantelli Lemma, $\Pr(\{X_n/\sqrt{2\log n} > 1+\epsilon\} i.o.) = 0$*. Additionally, since the* X_n *are independent and*

$$\begin{aligned}\sum_{n=1}^\infty \Pr(X_n \geq \sqrt{2(1-\epsilon)\log n}) &\geq \sum_{n=1}^\infty \sqrt{2\pi}((\sqrt{2(1-\epsilon)\log n})^{-1} - (\sqrt{2(1-\epsilon)\log n})^{-3}) e^{-(1-\epsilon)\log n} \\ &= \sum_{n=1}^\infty \left(\frac{\sqrt{\pi}}{\sqrt{(1+\epsilon)\log n}} - \frac{\sqrt{\pi}}{\sqrt{(1+\epsilon)^3 \log^3 n}} \right) \frac{1}{n^{1-\epsilon}} = \infty,\end{aligned}$$

so by the second Borel-Cantelli Lemma, $\Pr(\{X_n/\sqrt{2\log n} > 1-\epsilon\} i.o.) = 1$.
Therefore, the sequence $X_n/\sqrt{2\log n}$ *is infinitely often above* $1-\epsilon$*, but only finitely often above* $1+\epsilon$*, in which case* $\limsup_{n\to\infty} X_n/\sqrt{2\log n}$ *must be in the interval* $(1-\epsilon, 1+\epsilon]$. *Taking* $\epsilon \to 0$*, we see that* $\Pr(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1) = 1$.

(c) *This follows easily from the Borel-Cantelli Lemmas since* $X_1, X_2, \ldots$ *are i.i.d so* $\Pr(X_1 < a_n) = \Pr(X_n > a_n)$.