1. Fix an integer $r \geq 0$ and define $f : [1, b] \to \mathbb{R}$ by $f(x) = x^r$, where $b > 1$.

   (a) Let $P_n = (1, b^{1/n}, b^{2/n}, \ldots, b^{(n-1)/n}, b)$ be a partition of $[1, b]$. Compute the lower Darboux sum $L(f, P_n)$, and show that $U(f, P_n) = b^{r/n} L(f, P_n)$.

   (b) Prove that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$, and compute their common value.

Solution. (a) Since $f(x)$ is monotone increasing, we compute that

$$m_i = \inf_{t\in[b^{i/n}, b^{(i+1)/n}]} t^r = b^{ir/n}, \qquad M_i = \sup_{t\in[b^{i/n}, b^{(i+1)/n}]} t^r = b^{(i+1)r/n}.$$

On each interval $[b^{i/n}, b^{(i+1)/n}]$ we have $\Delta x_i = b^{i/n}(b^{1/n} - 1)$, so

$$\begin{aligned} L(f, P_n) &= \sum_{i=0}^{n-1} b^{ir/n} \cdot b^{i/n}(b^{1/n} - 1) = (b^{1/n} - 1) \sum_{i=0}^{n-1} (b^{(r+1)/n})^i \\ &= (b^{1/n} - 1)\frac{b^{r+1} - 1}{b^{(r+1)/n} - 1} \\ &= \frac{b^{r+1} - 1}{b^{r/n} + b^{(r-1)/n} + \cdots + b^{1/n} + 1}. \end{aligned}$$

Similarly, we compute that

$$\begin{aligned} U(f, P_n) &= \sum_{i=0}^{n-1} b^{(i+1)r/n} \cdot b^{i/n}(b^{1/n} - 1) \\ &= b^{r/n} \cdot \sum_{i=0}^{n-1} b^{ir/n} \cdot b^{i/n}(b^{1/n} - 1) = b^{r/n} L(f, P_n). \end{aligned}$$

(b) We note that $\lim_{n\to\infty} L(f, P_n) = \dfrac{b^{r+1} - 1}{r + 1}$. In particular, the sequence $\big(L(f, P_n)\big)$ is bounded above by any single value $U(f, P_n)$, so we write $L(f, P_n) < C$ for some constant $C > 0$, and then we have

$$0 \leq U(f, P_n) - L(f, P_n) = (b^{r/n} - 1)L(f, P_n) < C(b^{r/n} - 1)$$

for all $n \geq 0$ by part (a). The right side approaches 0 as $n \to \infty$, hence so does the middle part by the squeeze theorem, and this means that

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) + \lim_{n\to\infty} L(f, P_n)$$

exists and is equal to $0 + \lim_{n\to\infty} L(f, P_n) = \dfrac{b^{r+1} - 1}{r + 1}$, by the algebra of limits.

Remark: we don't really need r to be an integer, since we can still evaluate

$$\begin{aligned} \lim_{n\to\infty} \frac{b^{r+1} - 1}{b^{r/n} + b^{(r-1)/n} + \cdots + b^{1/n} + 1} &= (b^{r+1} - 1) \lim_{n\to\infty} \frac{b^{1/n} - 1}{b^{(r+1)/n} - 1} \\ &= (b^{r+1} - 1) \lim_{x\downarrow 0} \frac{b^x - 1}{b^{(r+1)x} - 1} \\ &= \frac{b^{r+1} - 1}{r + 1} \end{aligned}$$

using l'Hôpital's rule.

2. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q}. \end{cases}$

   Prove that f is not integrable, but that f^2 is.

   Solution. Since $f(x)^2 = 1$ for all x, and constant functions are integrable, we know that f^2 is integrable. On the other hand, given any partition P of $[a, b]$ we have $\inf f(t) = -1$ and $\sup f(t) = 1$ on every interval, so that

$$L(f, P) = \sum_{i=0}^{n-1} (-1)\Delta x_i = -(b - a), \qquad U(f, P) = \sum_{i=0}^{n-1} (1)\Delta x_i = b - a$$

   independently of P. Thus $\underline{\int_a^b} f(x)\,dx = -(b - a)$ is not equal to $\overline{\int_a^b} f(x)\,dx = b - a$, and so f is not integrable.

3. Prove that any monotone increasing function $f : [a, b] \to \mathbb{R}$ is integrable, by considering its Darboux sums for partitions where every subinterval $[x_i, x_{i+1}]$ has the same length.

   Solution. Consider for all $n \in \mathbb{N}$ the partition

$$P_n = \left(a, a + \frac{b-a}{n}, a + 2\left(\frac{b-a}{n}\right), \ldots, a + (n-1)\left(\frac{b-a}{n}\right), b \right),$$

   with $x_i = a + i(\frac{b-a}{n})$ for $0 \le i \le n$ and $\Delta x_i = \frac{b-a}{n}$ for $0 \le i < n$. Since f is monotone increasing, we have

$$m_i = \inf_{x_i \le t \le x_{i+1}} f(t) = f(x_i), \qquad M_i = \sup_{x_i \le t \le x_{i+1}} f(t) = f(x_{i+1}),$$

   and so

$$L(f, P_n) = \sum_{i=0}^{n-1} m_i \Delta x_i = (f(x_0) + f(x_1) + \cdots + f(x_{n-1})) \left(\frac{b-a}{n}\right)$$
$$U(f, P_n) = \sum_{i=0}^{n-1} M_i \Delta x_i = (f(x_1) + f(x_2) + \cdots + f(x_n)) \left(\frac{b-a}{n}\right).$$

   from which we compute

$$U(f, P_n) - L(f, P_n) = (f(x_n) - f(x_0)) \left(\frac{b-a}{n}\right) = (f(b) - f(a)) \left(\frac{b-a}{n}\right).$$

   It follows that $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$, and hence that f is integrable.

4. Define the *mesh* of a partition $P = (x_0, \ldots, x_k)$ to be the maximal length of any subinterval:
$$\mathrm{mesh}(P) = \max_{0 \le i \le k-1} \Delta x_i = \max_{0 \le i \le k-1} (x_{i+1} - x_i).$$
Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\big(P_n\big)$ is any sequence of partitions of $[a, b]$ such that $\mathrm{mesh}(P_n) \to 0$ as $n \to \infty$, then
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$
The proof should follow the argument we used in lecture to show that continuous functions are integrable.

Solution. Fix $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that
$$\forall x, y \in [a, b],\ |x - y| < \delta \ \Rightarrow\ |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$
Then $\lim\limits_{n\to\infty} \mathrm{mesh}(P_n) = 0$ implies that for this value of δ, there is an $N > 0$ such that $\mathrm{mesh}(P_n) < \delta$ for all $n \ge N$. Writing $P_n = (x_0, \ldots, x_k)$, we compute that
$$U(f, P_n) - L(f, P_n) = \sum_{i=0}^{k-1} \left(\sup_{x_i \le t \le x_{i+1}} f(t) - \inf_{x_i \le t \le x_{i+1}} f(t) \right) \Delta x_i.$$
The extreme value theorem says that there are $y_i, z_i \in [x_i, x_{i+1}]$ such that
$$\sup_{x_i \le t \le x_{i+1}} f(t) = f(y_i), \qquad \inf_{x_i \le t \le x_{i+1}} f(t) = f(z_i),$$
and since $|z_i - y_i| \le x_{i+1} - x_i \le \mathrm{mesh}(P_n) < \delta$, we have $|f(z_i) - f(y_i)| < \frac{\epsilon}{b-a}$, so
$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=0}^{k-1} (f(y_i) - f(z_i)) \\ &< \sum_{i=0}^{k-1} \frac{\epsilon}{b - a}(x_{i+1} - x_i) = \frac{\epsilon}{b - a}(b - a) = \epsilon. \end{aligned}$$
Since $U(f, P_n) - L(f, P_n) < \epsilon$ for all $n \ge N$, and we can find such an N for any $\epsilon > 0$, it follows that $\lim\limits_{n\to\infty} \big(U(f, P_n) - L(f, P_n)\big) = 0$, and so
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$$
by Proposition 3.13 in the lecture notes.

5. (a) Prove for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ that if $\sin(\frac{\theta}{2}) \ne 0$, then
$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n + 1)\theta/2)}{\sin(\theta/2)}$$
using the formula $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

(b) Fix $t \in (0, \frac{\pi}{2}]$ so that $\sin(x)$ is monotone increasing on the interval $[0, t]$, and consider the partition $P_n = (0, \frac{t}{n}, \frac{2t}{n}, \dots, \frac{(n-1)t}{n}, t)$ of $[0, t]$. Compute the upper Darboux sum $U(\sin(x), P_n)$, and show that

$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\sin^2(\tfrac{t}{2}).$$

Remark: This limit is equal to $1-\cos(t)$ by the double-angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$, so problem **??** tells us that

$$\int_0^t \sin(x)\, dx = 2\sin^2\left(\frac{t}{2}\right) = 1 - \cos(t)$$

for all $t \in (0, \frac{\pi}{2}]$.

Solution. (a) If we call the sum S, then we have

$$\begin{aligned} S\sin\left(\frac{\theta}{2}\right) &= \sum_{k=1}^{n} \sin(k\theta)\sin\left(\frac{\theta}{2}\right) \\ &= \sum_{k=1}^{n} \frac{1}{2}\left[\cos\left(\left(k-\frac{1}{2}\right)\theta\right) - \cos\left(\left(k+\frac{1}{2}\right)\theta\right)\right] \\ &= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{(2n+1)\theta}{2}\right)\right) \end{aligned}$$

because the sum in the second row telescopes. By one more application of the given identity, with $\alpha = \frac{(n+1)\theta}{2}$ and $\beta = \frac{n\theta}{2}$, we conclude that

$$S\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{(n+1)\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right),$$

and we divide through by $\sin(\frac{\theta}{2})$ to solve for S.

(b) As in problem **??**, the assumption that $\sin(x)$ is monotone increasing means that

$$U(\sin(x), P_n) = \sum_{i=0}^{n-1} \sin\left(\frac{(i+1)t}{n}\right)\frac{t}{n} = \frac{t}{n}\left(\sin(\theta) + \cdots + \sin(n\theta)\right)$$

with $\theta = \frac{t}{n}$, and so by part (a) we have

$$U(\sin(x), P_n) = \frac{t}{n} \cdot \frac{\sin(\frac{t}{2})\sin(\frac{(n+1)t}{2n})}{\sin(\frac{t}{2n})} = \frac{t/n}{\sin(t/2n)}\sin\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}+\frac{t}{2n}\right).$$

We have $\displaystyle\lim_{x\to 0}\frac{tx}{\sin(tx/2)} = \lim_{x\to 0}\frac{t}{(t/2)\cos(tx/2)} = 2$ by l'Hôpital's rule, and $\frac{1}{n} \to 0$ as $x \to \infty$, so then

$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\lim_{n\to\infty}\sin\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}+\frac{t}{2n}\right) = 2\sin^2\left(\frac{t}{2}\right).$$

6. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x)$ and the product $f(x)g(x)$ are both integrable, and $f(x) \geq 0$ for all $x \in [a, b]$. If $c \leq g(x) \leq d$ for all $x \in [a, b]$, prove that
$$c \int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq d \int_a^b f(x)\,dx.$$

Solution. We claim that for any partition P of $[a, b]$, we have
$$cL(f, P) \leq L(fg, P) \leq U(fg, P) \leq dU(f, P).$$
To see this, if $P = (x_0, \ldots, x_n)$, then since $f(x)g(x) \geq cf(x)$ for all x, we have
$$\begin{aligned} L(fg, P) &= \sum_{i=0}^{n-1} \left(\inf_{t \in [x_i, x_{i+1}]} f(t)g(t) \right) \Delta x_i \\ &\geq \sum_{i=0}^{n-1} \left(\inf_{t \in [x_i, x_{i+1}]} cf(t) \right) \Delta x_i = cL(f, P) \end{aligned}$$
and the same argument with $f(x)g(x) \leq df(x)$ says that $U(fg, P) \leq dL(f, P)$.

Now we apply this claim to show that
$$c \underline{\int_a^b} f(x)\,dx = \sup_P cL(f, P) \leq \sup_P L(fg, P) = \underline{\int_a^b} f(x)g(x)\,dx,$$
so $c \int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx$ since f and fg are both integrable, and likewise
$$\overline{\int_a^b} f(x)g(x)\,dx = \inf_P U(fg, P) \leq \inf_P dU(f, P) = d \overline{\int_a^b} f(x)\,dx$$
implies that $\int_a^b f(x)g(x)\,dx \leq d \int_a^b f(x)\,dx$.

7. (*) Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/|q|, & x = \frac{p}{q} \in \mathbb{Q}. \end{cases}$

We proved in problem sheet 1 that f is discontinuous at all rational numbers.

(a) Compute the lower Darboux integral $\underline{\int_0^1} f(x)\,dx$.

(b) Consider the partition $P_n = (0, \frac{1}{n^3}, \frac{2}{n^3}, \ldots, \frac{n^3-1}{n^3}, 1)$ of $[0, 1]$. Show for n large that there are at most n^2 subintervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ on which
$$M_i = \sup_{\frac{i}{n^3} \leq t \leq \frac{i+1}{n^3}} f(t)$$
is at least $\frac{1}{n}$.

(c) Prove that $U(f, P_n) \leq \frac{2}{n}$ for n large. (Hint: break the sum into terms where $M_i \geq \frac{1}{n}$ and terms where $M_i < \frac{1}{n}$.)

(d) Conclude that f is integrable, and compute $\int_0^1 f(x)\,dx$.

Solution. (a) We have $\inf\limits_{t\in[x_i,x_{i+1}]} f(t) = 0$ on any interval, so the lower Darboux sum for any partition $P = (x_0, \dots, x_k)$ of $[0,1]$ is

$$L(f,P) = \sum_{i=0}^{k-1} 0 \cdot \Delta x_i = 0,$$

and thus $\underline{\int_0^1} f(x)\,dx = \sup\limits_P L(f,P) = 0$.

(b) If $f(t) \geq \frac{1}{n}$ then t must be a rational number of the form $\frac{p}{q}$ with $|q| \leq n$. On the interval $[0,1]$ there are at most

$$2 + 1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} + 2$$

of these: the first two counts 0 and 1, and then for each $q \geq 2$ we count at most $q-1$ additional values $\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}$ (though possibly fewer, because some of these may not be in lowest terms). And each such value of t belongs to at most two intervals, with equality iff $t = \frac{i}{n^3}$ and $0 < i < n^3$, so at most

$$2\left(\frac{n(n-1)}{2} + 2\right) = n^2 - n + 4 \leq n^2 \quad (\text{for } n \geq 4)$$

intervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ contain a point t with $f(t) \geq \frac{1}{n}$. Then $M_i \geq \frac{1}{n}$ on these intervals, and $M_i \leq \frac{1}{n+1}$ on all other subintervals of $[0,1]$.

(c) Since $M_i \leq 1$ for all i, we can write

$$\begin{aligned} U(f,P_n) &= \sum_{M_i \geq \frac{1}{n}} M_i \Delta x_i + \sum_{M_i < \frac{1}{n}} M_i \Delta x_i \\ &= \frac{1}{n^3}\left(\sum_{M_i \geq \frac{1}{n}} M_i + \sum_{M_i < \frac{1}{n}} M_i\right) \\ &\leq \frac{1}{n^3}\left(\sum_{M_i \geq \frac{1}{n}} 1 + \sum_{M_i < \frac{1}{n}} \frac{1}{n}\right). \end{aligned}$$

The first sum has at most n^2 terms, and the second sum has at most n^3 terms, so

$$U(f,P_n) \leq \frac{1}{n^3}\left(n^2(1) + n^3\left(\frac{1}{n}\right)\right) = \frac{2n^2}{n^3} = \frac{2}{n}.$$

(d) From part (c), we have

$$\overline{\int_0^1} f(x)\,dx = \inf_P U(f,P) \leq \inf_n U(f,P_n) \leq \inf_n \frac{2}{n} = 0.$$

But the upper Darboux integral is also as at least as big as $\underline{\int_0^1} f(x)\,dx = 0$, so the two are equal and we have $\int_0^1 f(x)\,dx = 0$.

8. Prove that if $f : [a,b] \to [0,\infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a,b]$, then $\int_a^b f(x)\,dx > 0$.

Solution. We can take $c \in (a,b)$ without loss of generality, since if $f(x) = 0$ for all $x \in (a,b)$ then $f(a) = f(b) = 0$ as well by continuity. Since $\frac{f(c)}{2} > 0$, we also have

$$\exists \delta > 0 \text{ such that } |x - c| < \delta \ \Rightarrow\ |f(x) - f(c)| < \frac{f(c)}{2}$$

by the continuity of f at c. Then if $|x - c| \leq \delta$ we have $f(x) \geq \frac{f(c)}{2}$. Taking a smaller δ if needed to ensure that $[c-\delta, c+\delta] \subset [a,b]$, we can write

$$\begin{aligned} \int_a^b f(x)\,dx &= \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx \\ &\geq \int_a^{c-\delta} 0 \quad dx + \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx + \int_{c+\delta}^b 0 \quad dx \end{aligned}$$

since $f(x)$ is at least 0, $\frac{f(c)}{2}$, and 0 on the intervals $[a, c-\delta]$, $[c-\delta, c+\delta]$, and $[c+\delta, b]$ respectively. We evaluate these integrals one by one to get

$$\int_a^b f(x)\,dx \geq 0 + \delta \cdot f(c) + 0 > 0.$$

9. Suppose for some $f : [a,b] \to \mathbb{R}$ and integer $n \geq 1$ that the nth power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

Solution. If n is odd then the nth root function $\sqrt[n]{\cdot} : \mathbb{R} \to \mathbb{R}$ is continuous, so if f^n is integrable then so is the composition $\sqrt[n]{f^n} = f$.

When n is even, we still have a continuous $\sqrt[n]{\cdot} : [0,\infty) \to [0,\infty)$, and $f(x)^n \geq 0$ for all x, so we can again conclude that $\sqrt[n]{f^n}$ is integrable. But in this case $\sqrt[n]{f^n} = |f|$, so we have only proved that $|f|$ is integrable. If $f(x) \geq 0$ for all x then $f = |f|$, so this proves that a nonnegative function f is integrable if f^n is.

10. Evaluate $\displaystyle\int_1^x \frac{\sqrt{t^2-1}}{t}\,dt$ for $x \geq 1$. (Hint: what is the inverse of the integrand?)

Solution. We note that $f(t) = \frac{\sqrt{t^2-1}}{t} = \sqrt{1 - t^{-2}}$ is monotone increasing, and if $y = f(t)$ with $t \geq 1$ then

$$y^2 = 1 - \frac{1}{t^2} \ \Rightarrow\ f^{-1}(y) = t = \frac{1}{\sqrt{1-y^2}}.$$

Thus we apply the formula for the integral of an inverse function to get

$$\int_0^{f(x)} \frac{1}{\sqrt{1-t^2}}\,dt + \int_1^x f(t)\,dt = xf(x) - 1f(1)$$
$$= x\sqrt{1-\frac{1}{x^2}} = \sqrt{x^2-1}.$$

We know from lecture that $\sin^{-1}(t)$ is an antiderivative of $\frac{1}{\sqrt{1-t^2}}$, so then

$$\begin{aligned}\int_1^x f(t)\,dt &= \sqrt{x^2-1} - \int_0^{f(x)} \frac{1}{\sqrt{1-t^2}}\,dt \\ &= \sqrt{x^2-1} - \sin^{-1}(f(x)) \\ &= \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right).\end{aligned}$$

We can optionally simplify the last term by writing $\theta = \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$, which lies between 0 and $\frac{\pi}{2}$ since $\frac{\sqrt{x^2-1}}{x} > 0$, and then noting that

$$\sin^2(\theta) + \cos^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \left(\frac{x^2-1}{x^2}\right) = \frac{1}{x^2},$$

so $\cos(\theta) = \frac{1}{x}$. Thus $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{x^2-1}$, or equivalently

$$\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) = \cos^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\sqrt{x^2-1}\right),$$

and we can write for example that $\int_1^x f(t)\,dt$ is equal to

$$\int_1^x \frac{\sqrt{t^2-1}}{t}\,dt = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}).$$

11. In problem sheet 4 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \to [0,\infty)$ such that $f(x) > 0$ if and only if $x \in (0,1)$.
    (a) Construct a smooth, monotone increasing function $g : \mathbb{R} \to [0,\infty)$ such that $g(x) = 0$ for all $x \le 0$ and $g(x) = 1$ for all $x \ge 1$.
    (b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \to [0,\infty)$ satisfying
    $$h(x) = 0 \text{ for all } x \notin [a,d], \qquad h(x) = 1 \text{ for all } x \in [b,c],$$
    and with h monotone increasing on $(-\infty,b]$ and decreasing on $[c,\infty)$.

Solution. (a) We construct g by integrating f. This gives us a constant function for $x \ge 1$ since $f(x) = 0$ there, but its value won't be 1, so we rescale it and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{c}\int_0^x f(t)\,dt, & x \ge -1 \end{cases} \qquad \text{where } c = \int_0^1 f(x)\,dx.$$

We claim that g is differentiable, with $g'(x) = \frac{1}{c}f(x)$. Indeed, for all $x \leq 0$ we have $g(x) = 0$ – when $-1 \leq x \leq 0$ this follows from the fact that $f(x) = 0$ for $x \leq 0$ – so $g'(x) = 0 = \frac{1}{c}f(x)$ on $(-\infty, 0)$. And for $x > -1$ the fundamental theorem of calculus tells us that g is differentiable at x with $g'(x) = \frac{1}{c}f(x)$. Since $g'(x) = f(x) \geq 0$, we know that g is monotone increasing (and hence nonnegative, since it is zero for all $x \leq 0$), and it is smooth since its derivative is infinitely differentiable. It only remains now to check that for $x \geq 1$ we have

$$\begin{aligned} g(x) = \frac{1}{c}\int_0^x f(t)\,dt &= \frac{1}{c}\left(\int_0^1 f(t)\,dt + \int_1^x f(t)\,dt\right) \\ &= \frac{1}{c}\left(\int_0^1 f(t)\,dt + \int_1^x 0\,dt\right) = 1. \end{aligned}$$

(b) The functions $h_1(x) = g\left(\frac{x-a}{b-a}\right)$ and $h_2(x) = g\left(\frac{d-x}{d-c}\right)$ are smooth since they are compositions of two smooth functions (one linear in x, and the other one g), and they satisfy

$$h_1(x) = \begin{cases} 0, & x \leq a \\ 1, & x \geq b \end{cases} \qquad h_2(x) = \begin{cases} 0, & x \geq d \\ 1, & x \leq c \end{cases}$$

with h_1 and h_2 monotone increasing and constant, respectively, on $(-\infty, b]$ and constant and monotone decreasing, respectively, on $[c, \infty)$. It follows that $h(x) = h_1(x)h_2(x)$ has the desired properties: it is smooth, zero precisely outside (a, d) and one precisely on $[b, c]$, and increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

12. (a) Given $a < b < 0$, evaluate $\displaystyle\int_a^b \frac{1}{x}\,dx$. Be careful not to take the logarithm of a negative number along the way!

(b) Check that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is strictly monotone increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, with

$$\lim_{x \downarrow -\frac{\pi}{2}} \tan(x) = -\infty \quad \text{and} \quad \lim_{x \uparrow \frac{\pi}{2}} \tan(x) = +\infty.$$

(c) Let $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse function to $\tan(x)$. Prove for all $x \in \mathbb{R}$ that

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}.$$

(d) Fix $\theta \in (0, \frac{\pi}{2})$. Find a convenient substitution which proves that

$$\int_0^\theta \tan(x)\,dx = -\log(\cos(\theta)).$$

(e) Prove for $x > 0$ that $\displaystyle\int_0^x \tan^{-1}(t)\,dt = x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2)$.

Solution. (a) We'd like to use the fundamental theorem of calculus, because we've seen that $\frac{d}{dx}\log(x) = \frac{1}{x}$, but this fact only makes sense if $x > 0$! One way to fix it is to compose log and the absolute value function, and then apply the chain rule: we have

$$\frac{d}{dx}\log|x| = \frac{1}{|x|}\cdot\frac{d}{dx}|x| = \begin{cases} +1/|x|, & x > 0 \\ -1/|x|, & x < 0. \end{cases}$$

In either case we have $\frac{d}{dx}\log|x| = \frac{1}{x}$ for all $x \neq 0$, so now we can apply the fundamental theorem of calculus:

$$\int_a^b \frac{1}{x}\,dx = (\log|x|)\big|_a^b = \log|b| - \log|a| = \log\left(\frac{|b|}{|a|}\right) = \log\left(\frac{b}{a}\right).$$

Alternatively, we can substitute $y = -x$ to get

$$\int_a^b \frac{1}{x}\,dx = \int_{-a}^{-b} -\frac{1}{y}(-1)\,dy = \int_{-a}^{-b} \frac{1}{y}\,dy = -\int_{-b}^{-a}\frac{1}{y}\,dy.$$

Since $0 < -b < -a$, this last term is equal to

$$-\left(\log(y)\big|_{-b}^{-a}\right) = -(\log(-a) - \log(-b)) = \log|b| - \log|a| = \log\left(\frac{b}{a}\right).$$

(b) We use the quotient rule to compute that as long as $\cos(x) \neq 0$, we have

$$\frac{d}{dx}\tan(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} > 0,$$

and so $\tan(x)$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. As $x \to \pm\frac{\pi}{2}$ we have $\cos(x) \to 0$ while $\sin(x) \to \pm 1$, so then $\tan(x) \to \pm\infty$, and since tan is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ it must follow that $\tan(x) \to -\infty$ as $x \downarrow -\frac{\pi}{2}$ while $\tan(x) \to +\infty$ as $x \uparrow \frac{\pi}{2}$.

(c) We wish to compute $\cos(\theta)$, where $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We square both sides of the latter equation:

$$x^2 = \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1 - \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} - 1 \implies \cos^2(\theta) = \frac{1}{1 + x^2}.$$

Now $\cos(\theta)$ is strictly positive in the given range, so we can take square roots to conclude that

$$\cos(\tan^{-1}(x)) = \cos(\theta) = \left(\frac{1}{1 + x^2}\right)^{1/2}$$

as claimed.

(d) Writing $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we notice that the numerator is (up to sign) the derivative of the denominator, so we write $y = -\cos(x)$ and then

$$\int_0^\theta \tan(x)\,dx = \int_0^\theta \left(-\frac{1}{y(x)}\right) y'(x)\,dx = \int_{-1}^{-\cos(\theta)} -\frac{1}{y}\,dy = (-\log|y|)\big|_{-1}^{-\cos(\theta)}$$

which simplifies to $-\log|\cos(\theta)|$. By hypothesis we have $0 < \cos(\theta) < 1$, so we can simplify using $|\cos(\theta)| = \cos(\theta)$ and we are done.

(e) We use the theorem from lecture about integrals of inverse functions, since tan is strictly monotone on $(0, \frac{\pi}{2})$: if $y \in (0, \frac{\pi}{2})$ then

$$\int_0^y \tan(t)\, dt + \int_0^{\tan(y)} \tan^{-1}(t)\, dt = y\tan(y) - 0\tan(0).$$

We will eventually set $y = \tan^{-1}(x)$, which is in the interval $(0, \frac{\pi}{2})$ since $x > 0$. The first term on the left is equal to $-\log(\cos(y))$ by the previous part, so we move it to the right:

$$\int_0^{\tan(y)} \tan^{-1}(t)\, dt = y\tan(y) + \log(\cos(y))$$

and then set $y = \tan^{-1}(x)$ as promised to get

$$\int_0^x \tan^{-1}(t)\, dt = x\tan^{-1}(x) + \log(\cos(\tan^{-1}(x))).$$

We use $\cos(\tan^{-1}(x)) = (1+x^2)^{-1/2}$ to simplify the right hand side further:

$$\int_0^x \tan^{-1}(t)\, dt = x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2).$$

13. (a) Check that the derivative of $x\log(x) - x$ is $\log(x)$.

(b) Use Darboux sums to prove for all integers $n \geq 1$ that

$$\log((n-1)!) \leq \int_1^n \log(x)\, dx \leq \log(n!).$$

(c) Evaluate the integral in (b) and deduce that

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$$

for all $n \geq 1$.

(d) Conclude that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Solution. (a) We have $\frac{d}{dx}(x\log x) = \log(x) + x\frac{1}{x} = \log(x) + 1$ by the product rule, so $\frac{d}{dx}(x\log(x) - x) = (\log(x)+1) - 1 = \log(x)$.

(b) We know that $\log(x)$ is integrable on $[1, n]$ since it is continuous, so if we take the partition $P = (1, 2, \ldots, n)$ of $[1, n]$ then we have

$$L(\log(x), P) \leq \int_1^n \log(x)\, dx \leq U(\log(x), P).$$

Since $\log(x)$ is monotone increasing, we compute that

$$L(\log(x), P) = \sum_{i=1}^{n-1} \log(i) \cdot 1 = \log(1 \cdot 2 \cdot \dots \cdot (n-1)) = \log((n-1)!)$$
$$U(\log(x), P) = \sum_{i=1}^{n-1} \log(i+1) \cdot 1 = \log(2 \cdot 3 \cdot \dots \cdot n) = \log(n!),$$

so we put these together to get $\log((n-1)!) \le \int_1^n \log(x)\,dx \le \log(n!)$.

(c) We use part (a) and the fundamental theorem of calculus to evaluate

$$\int_1^n \log(x)\,dx = x\log(x) - x\Big|_{x=1}^{x=n} = n\log(n) - (n-1).$$

Then part (b) says that $n\log(n) - (n-1) \le \log(n!)$, or

$$\frac{1}{n} \le \frac{\log(n!)}{n} - \log(n) + 1 = \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right)$$

after some rearranging. Similarly, if we let $m = n-1$ then the leftmost inequality from (b) tells us that

$$\log(m!) \le (m+1)\log(m+1) - m,$$

and we divide through by m and rearrange to get

$$\begin{aligned}\frac{\log(m!)}{m} - \log(m) + 1 &\le \left(1 + \frac{1}{m}\right)\log(m+1) - \log(m) \\ &= \log\left(\frac{m+1}{m}\right) + \frac{\log(m+1)}{m}.\end{aligned}$$

Relabeling the variable n gives $\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \le \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$.

(d) Applying the squeeze theorem to part (c) shows us that

$$\lim_{n\to\infty}\left(\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right)\right) = 0,$$

or equivalently

$$\lim_{n\to\infty} \log\left(\frac{\sqrt[n]{n!}}{n/e}\right) = 0 \;\Rightarrow\; \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n/e} = 1.$$

We apply the algebra of limits to conclude that $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$.

14. (*) Let $f : [N, \infty) \to [0, \infty)$ be a nonnegative, monotone decreasing function.

(a) Let $S_n = \sum_{k=N}^{n} f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that

$$S_n - f(N) \leq \int_N^n f(x)\,dx \leq S_{n-1}.$$

(b) Prove that the series $\sum_{k=N}^{\infty} f(k)$ converges if and only if the limit

$$\int_N^\infty f(x)\,dx \stackrel{def}{=} \lim_{x\to\infty} \int_N^x f(t)\,dt$$

(called an *improper integral*) exists. This is the *integral test* for convergence.

(c) Prove that if the series $S = \sum_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^\infty f(x)\,dx$ exists, then $I \leq S \leq I + f(N)$.

Solution. (a) Fix an integer $n \geq N$ and consider the partition

$$P_n = (N, N+1, N+2, \ldots, n)$$

of $[N, n)$. Since f is monotone decreasing we can compute the Darboux sums of f with respect to P:

$$L(f, P_n) = \sum_{k=N}^{n-1} f(k+1), \qquad\qquad U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

We have proved that f is integrable on $[N, n]$ since it is monotone, so we have

$$\sum_{k=N+1}^{n} f(k) = L(f, P_n) \leq \int_N^n f(x)\,dx \leq U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

The left and right sides are $S_n - f(N)$ and S_{n-1} respectively.

(b) ($\Longleftarrow$) If $I = \int_N^\infty f(x)\,dx$ exists then for any $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \;\Rightarrow\; \left|\int_N^n f(x)\,dx - I\right| < \epsilon.$$

We use this together with part (a) to deduce that

$$n \geq M \;\Rightarrow\; S_n \leq \int_N^n f(x)\,dx + f(N) < I + f(N) + \epsilon.$$

The sequence $\big(S_n\big)$ of partial sums is therefore bounded above, and it is increasing since $f(n) \geq 0$ for all n, so it converges.

($\Longrightarrow$) If $\sum_{k=N}^{\infty} f(k)$ converges to some S, then given $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \;\Rightarrow\; |S_n - S| = \left|\sum_{k=N}^{n} f(k) - S\right| < \epsilon.$$

Using part (a), we have

$$n \geq M+1 \;\Rightarrow\; \int_N^n f(x)\,dx \leq S_{n-1} < S + \epsilon.$$

Now the function $F(x) = \int_N^x f(t)\,dt$ is increasing, since for any $x < y$ we have

$$\begin{aligned} F(y) = \int_N^y f(t)\,dt &= \int_N^x f(t)\,dt + \int_x^y f(t)\,dt \\ &\geq \int_N^x f(t)\,dt + \int_x^y \;0\; \,dt = F(x), \end{aligned}$$

and it is bounded above since for any $x \geq N$ we have $F(x) \leq F(m) < S + \epsilon$ for some integer $m \geq \max(x, M+1)$. Thus the limit $\lim_{x\to\infty} F(x) = \int_N^\infty f(t)\,dt$ exists, as desired.

Remark: we can't use the fundamental theorem of calculus to assert that $F'(x) = f(x) \geq 0$ and thus prove that $F(x)$ is increasing, because we do not know that f is continuous.

(c) In part (b) we proved for any $\epsilon > 0$ and all large enough x and n that $\int_N^x f(t)\,dt < S + \epsilon$ and $S_n < I + f(N) + \epsilon$ respectively, so

$$I = \lim_{x\to\infty} \int_N^x f(t)\,dt \leq S + \epsilon, \qquad S = \lim_{n\to\infty} S_n \leq I + f(n) + \epsilon.$$

These hold for any $\epsilon > 0$, so we must actually have $I \leq S$ and $S \leq I + f(N)$.

15. Evaluate $\displaystyle\int_0^x \frac{1}{1+e^t}\,dt$. Does $\displaystyle\int_0^\infty \frac{1}{1+e^t}\,dt$ exist, and if so, what is it?

Solution. We substitute $t = \log(u)$ and decompose into partial fractions to get

$$\int_0^x \frac{1}{1+e^t}\,dt = \int_1^{e^x} \frac{1}{1+u}\cdot\frac{1}{u}\,du = \int_1^{e^x} \left(\frac{1}{u} - \frac{1}{1+u}\right)du.$$

This is equal to

$$\log(u) - \log(1+u)\big|_{u=1}^{u=e^x} = x - \log(1+e^x) + \log(2).$$

We can rewrite this as

$$\int_0^x \frac{1}{1+e^t}\,dt = \log(2) - \log\left(\frac{1+e^x}{e^x}\right) = \log(2) - \log(1+e^{-x}),$$

and this converges as $x \to \infty$ to $\displaystyle\int_0^\infty \frac{1}{1+e^t}\,dt = \log(2)$.

16. The prime number theorem says that the number $\pi(n)$ of primes between 1 and n is approximately $\int_2^n \frac{1}{\log(x)}\,dx$.

    (a) Prove that this integral equals $\frac{n}{\log(n)} + \int_2^n \frac{1}{(\log x)^2}\,dx$, up to a constant which does not depend on n.

    (b) Prove that there is a constant $C > 0$ such that $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{Cn}{(\log n)^2}$ for all sufficiently large n, by splitting the integral up into one with domain $[2, \sqrt{n}]$ and one with domain $[\sqrt{n}, n]$ and estimating each one separately.

Solution. (a) We integrate by parts, using $\frac{d}{dx}(\log x)^{-1} = -\frac{1}{x(\log x)^2}$:

$$\begin{aligned}\int_2^n \frac{1}{\log(x)}\left(\frac{d}{dx}x\right)dx &= \left.\frac{x}{\log(x)}\right|_{x=2}^{x=n} + \int_2^n x\cdot\frac{1}{x(\log x)^2}\,dx \\ &= \frac{n}{\log(n)} - \frac{2}{\log(2)} + \int_2^n \frac{1}{(\log x)^2}\,dx.\end{aligned}$$

(b) We split the domain $[2, n]$ into $[2, \sqrt{n}] \cup [\sqrt{n}, n]$ and write

$$\int_2^n \frac{1}{(\log x)^2}\,dx = \int_2^{\sqrt{n}} \frac{1}{(\log x)^2}\,dx + \int_{\sqrt{n}}^n \frac{1}{(\log x)^2}\,dx.$$

On $[2, \sqrt{n}]$ we have $\frac{1}{(\log x)^2} \le \frac{1}{(\log 2)^2}$, and on $[\sqrt{n}, n]$ we have

$$\frac{1}{(\log x)^2} \le \frac{1}{(\log(\sqrt{n}))^2} = \frac{4}{(\log n)^2},$$

so we combine these bounds to get

$$\begin{aligned}\int_2^n \frac{1}{(\log x)^2}\,dx &\le \int_2^{\sqrt{n}} \frac{1}{(\log 2)^2}\,dx + \int_{\sqrt{n}}^n \frac{4}{(\log n)^2}\,dx \\ &= \frac{\sqrt{n}}{(\log 2)^2} + \frac{4(n-\sqrt{n})}{(\log n)^2}.\end{aligned}$$

We now have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{x/(\log x)^2} = \lim_{x\to\infty} \frac{(\log x)^2}{x^{1/2}} = \lim_{y\to\infty} \frac{y^2}{e^{y/2}} = 0$$

by the substitution $y = \log(x)$, since the power series for e^x shows that $e^{y/2} \ge \frac{(y/2)^3}{3!} = \frac{y^3}{48}$ for all $y > 0$. It follows that for all large enough n we have $\frac{\sqrt{n}}{(\log 2)^2} < \frac{n}{(\log n)^2}$, and so $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{5n}{(\log n)^2}$ for $n \gg 0$.

17. Let $f : [0, \infty) \to [0, \infty)$ be uniformly continuous, and suppose that $\int_0^\infty f(x)\,dx$ exists.

(a) For each $\epsilon > 0$, prove that there is a $\delta > 0$ such that for all $y > 0$, if $f(y) \geq \epsilon$ then

$$\int_y^{y+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}.$$

(b) Prove that $\lim_{x\to\infty} f(x) = 0$.

(c) Describe a continuous function $g : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty g(x)\,dx$ exists but $\lim_{x\to\infty} g(x)$ does not. Can you make g differentiable as well?

Solution. (a) Given $\epsilon > 0$, uniform continuity says that there is a $\delta > 0$ such that

$$|y - t| < \delta \;\Rightarrow\; |f(y) - f(t)| < \frac{\epsilon}{2},$$

so if $f(y) \geq \epsilon$ then $f(t) > f(y) - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}$ for all $t \in [y, y+\delta)$. Then

$$\int_y^{y+\delta} f(t)\,dt \geq \int_y^{y+\delta} \frac{\epsilon}{2}\,dt = \frac{\epsilon\delta}{2}.$$

(b) If $\lim_{x\to\infty} f(x)$ is not zero, then there is some $\epsilon > 0$ and a sequence $x_n \to \infty$ such that $f(x_n) \geq \epsilon$ for all n. Since f is nonnegative, there is a $\delta > 0$ such that for any such x_n we have

$$\int_{x_n}^{\infty} f(t)\,dt \geq \int_{x_n}^{x_n+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}$$

by part (a).

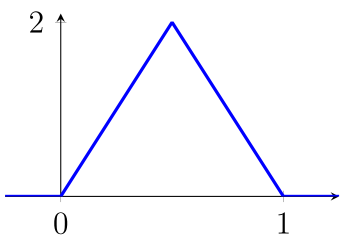
However, the convergence of $\int_0^\infty f(x)\,dx$ means that there is some $N > 0$ such that for all $x \geq N$, we have

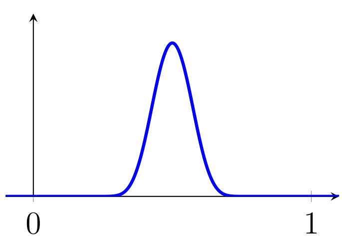
$$\left|\left(\lim_{b\to\infty}\int_0^b f(t)\,dt\right) - \int_0^x f(t)\,dt\right| < \frac{\epsilon\delta}{2},$$

hence $\int_x^\infty f(t)\,dt < \frac{\epsilon\delta}{2}$ for all $x \geq N$, and if we take x to be some $x_n \geq N$ then we have a contradiction.

(c) We take a continuous function $h : \mathbb{R} \to [0,\infty)$ satisfying $h(x) = 0$ for all $x \leq 0$ and all $x \geq 1$, $h(\frac{1}{2}) > 0$, and $\int_0^1 h(x)\,dx = 1$. Possible examples include

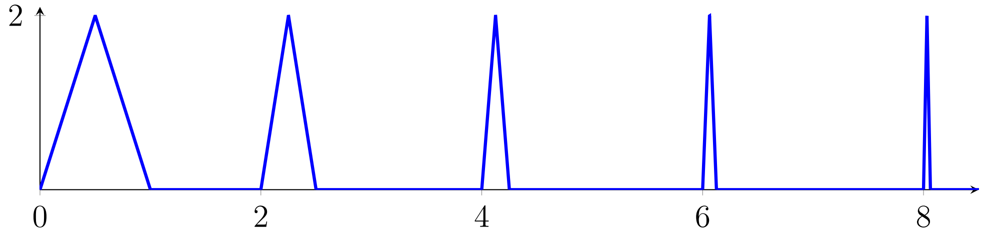
$$h_1(x) = \max(2 - |4x-2|, 0), \qquad h_2(x) = \begin{cases} ce^{-\frac{1}{x^2} - \frac{1}{(1-x)^2}}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$





where $c > 0$ is a constant chosen so that $\int_0^1 h_2(x)\,dx = 1$. Now we define

$$g(x) = \sum_{n=0}^{\infty} h(2^n(x-2n)),$$



and we find for any even integer $2n \geq 0$ that

$$\int_{2n}^{2n+2} g(x)\,dx = \int_{2n}^{2n+2} h(2^n(x-2n)) = \int_0^{2^{n+1}} h(y) \cdot \frac{1}{2^n}\,dy = \frac{1}{2^n}$$

by the substitution $x = \frac{y}{2^n} + 2n$. Since $g(x)$ is nonnegative, the integral $\int_0^t g(t)\,dt$ is increasing, and it is bounded above by

$$\sum_{n=0}^{\infty} \int_{2n}^{2n+2} g(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2,$$

so g is integrable, with $\int_0^\infty g(x)\,dx = 2$. We also know that g is continuous or differentiable iff h is. But $\lim\limits_{x\to\infty} g(x)$ does not exist, because we have

$$g\left(2n + \frac{1}{2^{n+1}}\right) = h\left(\frac{1}{2}\right) > 0$$

for all $n \geq 0$.

18. Let $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$.

 (a) Prove that this improper integral converges for all $t > 0$. (In how many ways is it improper?)

 (b) Compute $\Gamma(1)$.

 (c) Prove that $\Gamma(n+1) = n\Gamma(n)$ for all integers $n \geq 1$, and deduce that $\Gamma(n+1) = n!$ for all $n \geq 0$.

Solution. (a) This integral is improper in two ways: the upper limit is ∞, and when $t < 1$ the integrand $x^{t-1}e^{-x}$ is unbounded as $x \downarrow 0$. Thus we write

$$\Gamma(t) = \int_0^1 x^{t-1}e^{-x}\,dx + \int_1^\infty x^{t-1}e^{-x}\,dx$$

and ask for each of these integrals to exist individually. Since the integrand $x^{t-1}e^{-x}$ is positive, the two integrals

$$\int_a^1 x^{t-1}e^{-x}\,dx \qquad \text{and} \qquad \int_1^\infty x^{t-1}e^{-x}\,dx$$

both increase as $a \downarrow 0$ and $b \to \infty$ respectively, so it suffices to show that they are bounded above independently of $a \in (0,1]$ and $b \in [1,\infty)$.

For the first integral, we have $e^{-x} \leq 1$ for $0 \leq x \leq 1$, and so given $a > 0$ we have

$$\int_a^1 x^{t-1}e^{-x}\,dx \leq \int_a^1 x^{t-1}\,dx = \frac{x^t}{t}\bigg|_{x=a}^{x=1} = \frac{1-a^t}{t}.$$

This is bounded above by $\frac{1}{t}$ since $t > 0$, so the improper integral exists.

For the second integral, we have $x^{t-1} \leq e^{x/2}$ for all sufficiently large x; in fact, it suffices to take $x \geq 2^t \cdot t!$, since then

$$e^{x/2} \geq \frac{(x/2)^t}{t!} = \frac{x^t}{2^t \cdot t!} \geq x^{t-1}.$$

So we write $N = 2^t \cdot t!$ for convenience and bound this integral by

$$\begin{aligned}
\int_1^b x^{t-1}e^{-x}\,dx &= \int_1^N x^{t-1}e^{-x}\,dx + \int_N^b x^{t-1}e^{-x}\,dx \\
&\leq \int_1^N x^{t-1}e^{-x}\,dx + \int_N^b e^{-x/2}\,dx \\
&= \int_1^N x^{t-1}e^{-x}\,dx + (-2e^{-x/2})\big|_{x=N}^{x=b} \\
&= -2e^{-b/2} + \left(\int_1^N x^{t-1}e^{-x}\,dx + 2e^{-N/2}\right).
\end{aligned}$$

Thus $\int_1^b x^{t-1}e^{-x}\,dx$ is bounded above by the terms in parentheses, and so this improper integral exists as well.

(b) We have

$$\begin{aligned}
\Gamma(1) = \int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty} \int_0^b e^{-x}\,dx \\
&= \lim_{b\to\infty} -e^{-x}\big|_{x=0}^{x=b} \\
&= \lim_{b\to\infty} (1-e^{-b}) = 1.
\end{aligned}$$

(c) We integrate by parts: given $n \geq 1$ and $b > 0$, we have

$$\begin{aligned}
\int_0^b x^n e^{-x}\,dx &= \int_0^b x^n \frac{d}{dx}\left(-e^{-x}\right)\,dx \\
&= -x^n e^{-x}\big|_{x=0}^{x=b} - \int_0^b (nx^{n-1})(-e^{-x})\,dx \\
&= -\frac{b^n}{e^b} + t\int_0^b x^{n-1}e^{-x}\,dx.
\end{aligned}$$

Taking limits as $b \to \infty$ and using the algebra of limits gives us $\Gamma(n+1) = n\Gamma(n)$, since $\lim_{b\to\infty} \frac{b^n}{e^b} = 0$.
Now the claim that $\Gamma(n+1) = n!$ follows by induction: we have already proved it when $n = 0$, meaning that $\Gamma(1) = 1 = 0!$, and if $\Gamma(k+1) = k!$ for some integer $k \geq 1$ then this identity proves that

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)\cdot k! = (k+1)!,$$

as desired.
Remark: In fact, we have $\Gamma(t+1) = t\Gamma(t)$ for all real $t > 0$ by the same argument, though if $t < 1$ then we have to be careful about taking limits on the right side because $x^{t-1}e^{-t}$ is unbounded as $x \downarrow 0$.

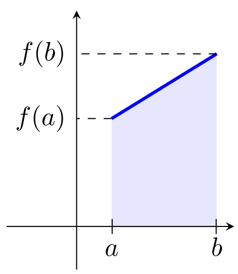
19. (*) Let $f : [a,b] \to \mathbb{R}$ be continuous, with $f''(x)$ continuous and bounded on (a,b).

(a) Use integration by parts twice to prove that

$$\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx = \int_a^b f(x)\,dx - (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

(b) If $|f''(x)| \leq M$ for all $x \in (a,b)$, prove that

$$\left|\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx\right| \leq \frac{M(b-a)^3}{12}.$$



In other words, $\int_a^b f(x)\,dx$ is the area of the trapezium shown at right, up to an error of at most $\frac{M(b-a)^3}{12}$.
(Hint: check that $(x-a)(x-b) \leq 0$ on $[a,b]$, and compute that $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$.)

(c) Apply this to $f(x) = \log(x)$ to show that

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1}\left(\frac{\log(k)+\log(k+1)}{2} + e_k\right),$$

where $|e_k| \leq \frac{1}{12k^2}$ for all k.

(d) Evaluate both the integral and the sum from part (c) to show that there is some constant $C > 0$ such that

$$\left|\log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right)\right| \leq C$$

for all n, or equivalently if $C_1 = e^{1-C}$ and $C_2 = e^{1+C}$ then

$$C_1\sqrt{n}\left(\frac{n}{e}\right)^n \leq n! \leq C_2\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Solution. (a) We integrate by parts once to get

$$\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx = \frac{(x-a)(x-b)}{2} f'(x)\Big|_{x=a}^{x=b} - \int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\,dx$$
$$= \int_a^b \left(\frac{a+b}{2} - x\right) f'(x)\,dx,$$

and then a second time to get

$$\int_a^b \left(\frac{a+b}{2} - x\right) f'(x)\,dx = \left(\frac{a+b}{2} - x\right) f(x)\Big|_{x=a}^{x=b} - \int_a^b (-1) f(x)\,dx$$
$$= \frac{a-b}{2} f(b) - \frac{b-a}{2} f(a) + \int_a^b f(x)\,dx,$$

which we rearrange slightly to get the desired answer.

(b) The triangle inequality for integrals says that

$$\left|\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx\right| \le \int_a^b \left|\frac{(x-a)(x-b)}{2} f''(x)\right| dx$$
$$\le \int_a^b \left|\frac{(x-a)(x-b)}{2}\right| \cdot M\,dx$$
$$= M \int_a^b -\frac{1}{2}(x-a)(x-b)\,dx,$$

since $\left|\frac{(x-a)(x-b)}{2}\right| = -\frac{(x-a)(x-b)}{2}$ for $x \in [a,b]$. We evaluate

$$\int_a^b (x^2 - (a+b)x + ab)\,dx = \frac{x^3}{3} - (a+b)\frac{x^2}{2} + abx\Big|_{x=a}^{x=b}$$
$$= \frac{b^3 - a^3}{3} - (a+b)\frac{b^2 - a^2}{2} + ab(b-a)$$
$$= (b-a)\left(\frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{2} + ab\right)$$
$$= (b-a)\left(-\frac{b^2}{6} + \frac{ab}{3} - \frac{a^2}{6}\right)$$
$$= -\frac{b-a}{6}(b^2 - 2ab + a^2) = -\frac{(b-a)^3}{6},$$

and so the upper bound above becomes $-\frac{M}{2}\left(-\frac{(b-a)^3}{6}\right) = \frac{M(b-a)^3}{12}$.

(c) We have $f''(x) = -\frac{1}{x^2}$, so $|f''(x)| \le \frac{1}{k^2}$ on the interval $[k, k+1]$. Thus

$$e_k = \int_k^{k+1} \log(x)\,dx - ((k+1) - k)\left(\frac{\log(k) + \log(k+1)}{2}\right)$$

satisfies $|e_k| \le \frac{1}{k^2}\left(\frac{((k+1)-k)^3}{12}\right) = \frac{1}{12k^2}$. We sum over $1 \le k \le n-1$ to get

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1}\left(\frac{\log(k) + \log(k+1)}{2} + e_k\right)$$

as claimed.

(d) On the one hand, we can evaluate the integral explicitly as

$$\int_1^n \log(x)\,dx = x\log(x) - x\big|_{x=1}^n = n\log(n) - (n-1).$$

On the other hand, if we let $E = \sum_{k=1}^{n} e_k$ then the sum can be rearranged as

$$\begin{aligned}\sum_{k=1}^{n-1}\left(\frac{\log(k)+\log(k+1)}{2} + e_k\right) &= \sum_{k=1}^{n-1}\frac{\log(k)}{2} + \sum_{k=1}^{n-1}\frac{\log(k+1)}{2} + \sum_{k=1}^{n-1} e_k \\ &= \left(\frac{\log(n!)}{2} - \frac{\log(n)}{2}\right) + \frac{\log(n!)}{2} + E \\ &= \log(n!) - \frac{\log(n)}{2} + E.\end{aligned}$$

We equate the two sides to get

$$n\log(n) - (n-1) = \log(n!) - \frac{\log(n)}{2} + E,$$

or equivalently

$$\begin{aligned}\log(n!) &= \left(n + \frac{1}{2}\right)\log(n) - (n-1) - E \\ &= \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right) - E,\end{aligned}$$

and so $\left|\log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right)\right| \leq |E|$. Since $|E| \leq \sum_{k=1}^{n-1}\frac{1}{12k^2} < \frac{1}{12}\sum_{k=1}^{\infty}\frac{1}{k^2}$, or equivalently $|E| \leq \frac{\zeta(2)}{12}$, the left side is bounded above for all n.

Remark: on the last problem sheet we saw that $\zeta(s) < \frac{s}{s-1}$ for all $s > 1$, so $\zeta(2) < 2$ and hence $|E| < \frac{1}{6}$. Thus we have

$$0.8464 < e^{-E} \leq \frac{n!}{n^{n+1/2}/e^{n-1}} \leq e^{E} < 1.1814$$

for all n, and these bounds can be further improved if we know that $\zeta(2) = \frac{\pi^2}{6}$.