Cantor sets in Euclidean and Metric Spaces

Mini Project 1

In this mini-project, we look at a classic example of a set in analysis. The set is built in a constructive fashion, and it is flexible enough to get a range of outcomes. We first start with a special construction, which is easy to describe, and then generalise it. The key features of these sets are broken into several steps, which can be done using the material we study in Analysis II.

Most of the problems presented below are independent of each other, and can be done independently. It is not necessary to do all of them in order to benefit from working on them. Some of the problems below may require material from the lectures which we will cover later in the module.

Middle third Cantor set in $\mathbb{R}$ (also called ternary Cantor set). Let

$$I_0 = [0,1] \subset \mathbb{R}.$$

We partition I_0 into 3 equal intervals, and remove the open middle interval, so that we get

$$I_1 = [0, 1/3] \cup [2/3, 1].$$

Let $I_{1,1} = [0, 1/3]$ and $I_{1,2} = [2/3, 1]$. We partition each of $I_{1,1}$ and $I_{1,2}$ into 3 equal parts, and remove the middle ones. We will be left with the set

$$I_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9].$$

We inductively repeat this process, each time partitioning the subintervals into three equal parts, and removing the middle ones. So we obtain the sets I_n, for $n \geq 1$, so that

- $I_0 \supset I_1 \supset I_2 \supset \dots,$
- each I_n is the union of 2^n subintervals,
- the length of each subinterval in I_n is $1/3^n$.

The *middle third Cantor set* is defined as

$$I = \bigcap_{n=1}^{\infty} I_n.$$

Problem 1. Show that I is a closed set in the metric space $(\mathbb{R}, d_1)$, where $d_1(x, y) = |x - y|$.

Problem 2. Show that I has infinitely many elements. Show that I is uncountable.

A set $S \subset \mathbb{R}^n$ is called a *perfect set*, if for every $x \in S$ there is a sequence of distinct points $x_i \in S$ such that $x_i \to x$, as $i \to \infty$.

Problem 3. Show that I is a perfect set.

Problem 4. Show that there is no interval of the form $[c, d] \subset I$ with $c \neq d$.

Recall that a set $S \subset \mathbb{R}^n$ is said to have *measure* 0, if for every $\epsilon > 0$ there is a countable collection of balls $B(x_i, r_i) \subset \mathbb{R}^n$, for $i \geq 1$, such that

$$S \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \quad \sum_{i=1}^{\infty} (2r_i)^n < \epsilon.$$

In other words, S may be covered by a collection of balls, with area less than any positive constant.

Problem 5. Show that the total length of the intervals removed from I_0 to get I is equal to 1. Conclude that I has measure 0.

Now we generalise the above construction of the Cantor set. Let $\alpha_i \in (0,1)$, for $i \geq 1$, be an arbitrary sequence. Let

$$J_0 = [0,1] \subset \mathbb{R}.$$

We partition J_0 into 3 subintervals so that the right and left subintervals have equal length, and the middle interval has length α_1, and remove the open middle interval, so that we get

$$J_1 = \left[0, \frac{1-\alpha_1}{2}\right] \cup \left[\frac{1+\alpha_1}{2}, 1\right].$$

The set J_1 consists of 2^1 intervals $J_{1,1} = [0, (1-\alpha_1)/2]$ and $J_{1,2} = [(1+\alpha_1)/2, 1]$. We partition each of $J_{1,1}$ and $J_{1,2}$ into 3 subintervals so that in each one the left and right subintervals have equal length, but the length of the middle subintervals are $\alpha_2 |J_{1,1}|$, where $|\cdot|$ denotes the length of an interval. By removing those middle intervals we obtain the set J_2. Inductively repeating this process, we obtain the sets J_n satisfying the following properties:

- each J_n is the union of 2^n subintervals, each of length $2^{-n} \prod_{k=1}^{n} (1-\alpha_k)$,
- $J_0 \supset J_1 \supset J_2 \supset \dots$.

Let us define

$$J = \bigcap_{n=1}^{\infty} J_n.$$

Problem 6. Show that for any choice of the sequence α_n as above, J is closed and perfect set. Moreover, if $[c,d] \subset J$ we must have $c = d$.

Problem 7. Show that if $\prod_{k=1}^{\infty}(1-\alpha_k) \neq 0$, for example if $\alpha_k \to 0$ very fast, then J is not measure 0.

Two sets A and B in $\mathbb{R}^n$ are called ambiently homeomorphic, if there is a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(A) = B$. In particular, ϕ is a one to one and onto continuous map from A to B.

Problem 8. Show that for any Cantor set J as above, I and J are ambiently homeomorphic subsets of $\mathbb{R}^1$.

Hint for the above problem: Build a sequence of piece-wise linear maps $\phi_n : \mathbb{R}^1 \to \mathbb{R}^1$ which maps the set I_n to the set J_n, and show that the sequence of maps ϕ_n is convergent.

Problem 9. Show that a homeomorphism $\phi : \mathbb{R}^1 \to \mathbb{R}^1$ may send a set of measure 0 to a set of non-measure 0.

A set $S \subset \mathbb{R}^n$ is called *totally disconnected*, if the only connected subsets of S are single points. You will learn about connected sets in Part 2 of Analysis 2, so the following problem can be done after we learn about connected sets.

Let (X, d) be a metric space, and let $C \subset X$. We say that C is a Cantor set in (X, d) if the following 3 properties hold:

- C is compact,
- C is totally disconnected,
- C is perfect.

In particular, if $C = X$ satisfies the above properties, we say that (X, d) is a Cantor metric space.

Problem 10. Show that any two Cantor sets in $\mathbb{R}^1$ are ambiently homeomorphic.

Hint: first prove the above statement in $\mathbb{R}^1$.

Problem 11. Show that any totally disconnected, perfect, compact metric space is homeomorphic to (I, d_1), where I is the middle third Cantor set, and d_1 is the induced metric on I.