1 Logic

Mathematics is less related to accounting than it is to philosophy.

- Leonard Adleman

Exercise 1.1. At which stage does this argument go wrong?

Suppose $x = 2$.

1. $\implies x - 2 = 0$
2. $\implies x^2 - 2x = 0$
3. $\implies x(x-2) = 0$
4. $\implies x = 0$ or $x = 2$.
5. Nowhere; the argument is correct. ✓ $x = 2$ *does* imply $x = 0$ or 2

Exercise 1.2. "A unless B" is the same logical statement as

1. $A \iff B$
2. $\overline{A} \iff \overline{B}$
3. $A \implies B$
4. $A \implies \overline{B}$
5. $\overline{A} \implies B$ ✓
6. $\overline{A} \implies \overline{B}$
7. None of these; something else.
8. More than one of these.

It says $\overline{B} \implies A$; equivalently $\overline{A} \implies B$. Think of "We'll go out unless it rains".

Exercise 1.3.

"*Find two real numbers x which satisfy the equation $x^2 - 3x + 2 = 0$.*"
Student solution:

$$\begin{aligned} & x^2 - 3x + 2 = 0 \\ \implies & (x-1)(x-2) = 0 \\ \implies & x = 1 \text{ or } x = 2. \end{aligned}$$

How many marks would this get in an exam?

1. Two marks – completely solved the problem.
2. One mark – partially solved the problem.
3. No marks – failed to solve the problem. ✓ Showed $x \notin \{1,2\} \implies x^2 - 3x + 2 \neq 0$.

Exercise 1.4. Is this a correct proof that $3|n^2 \implies 3|n$?
If $3|n$ then $n = 3m$ for some $m \in \mathbb{N}$ so $n^2 = 3(3m^2)$ is divisible by 3.

1. Yes.
2. No. ✓ Showed $\Longleftarrow$ instead of $\implies$
3. Uh?

Correct proof. By dividing any n by 3 and taking remainders we know it can be written as $3q$, $3q+1$ or $3q+2$ for some $q \in \mathbb{Z}$.
(Proof: set $q = \lfloor \frac{n}{3} \rfloor := \max\{Q : n - 3Q \geq 0\}$ using the Archimedean axiom and show it works – exercise!)
If $n = 3q+1$ or $n = 3q+2$ then squaring gives $n^2 = 3N+1$ for some $N \in \mathbb{Z}$. Thus $3 \nmid n^2$ ※ So $n = 3q$.

Exercise 1.5. What does $x \in \bigcup_{n=1}^{\infty} S_n$ mean?

1. $x \in S_n$ for some $n \in \mathbb{N}_{>0}$ ✓
2. Either $x \in S_n$ for some $n \in \mathbb{N}_{>0}$ or $x \in S_\infty$
3. Either $x \in S_n$ for some $n \in \mathbb{N}_{>0}$ or $x \in \lim_{n\to\infty} S_n$
4. Other

Conclusion: you need to practice your logic; please do so. Almost everything we do in this course relies on it.

Furthermore, as you move through the world some of the most valuable qualities you will be able to bring to the table are a grasp of logic and problem solving skills than most people will not have. These skills are what makes your degree in mathematics open doors.

If you make small mistakes (like confusing $\implies$ with $\Longleftarrow$) at this stage, you won't be able to solve much harder problems later on.

2 Numbers

If you spent less time asking what's examinable and more time trying to understand new maths, you would get far more marks. Worry about what's examinable in April; for now just try to think and solve and learn.

- All your lecturers, annually

2.1 Rational numbers

Recall $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}_{>0} = \{1, 2, 3, \ldots\}$, and $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with $+, \times, >$.

Recall $\mathbb{Q} := \{(p, q) \in \mathbb{Z} \times \mathbb{N}_{>0}\}/\sim$, where $\sim$ is the equivalence relation

$$(p_1, q_1) \sim (p_2, q_2) \iff p_1q_2 = p_2q_1.$$

We write the equivalence class of (p, q) as p/q or $\frac{p}{q}$. Each equivalence class has a distinguished element (p', q') such that $\nexists\, n \in \mathbb{N}$ with $n > 1$ and $n|p'$, $n|q'$. We say $\frac{p'}{q'}$ is "in lowest terms". We define

$$\begin{aligned}
\frac{p_1}{q_1} + \frac{p_2}{q_2} &:= \frac{p_1q_2 + p_2q_1}{q_1q_2}, \\
\frac{p_1}{q_1} - \frac{p_2}{q_2} &:= \frac{p_1q_2 - p_2q_1}{q_1q_2}, \\
\frac{p_1}{q_1} \times \frac{p_2}{q_2} &:= \frac{p_1p_2}{q_1q_2}, \\
\frac{p_1}{q_1} \div \frac{p_2}{q_2} &:= \frac{p_1q_2}{q_1p_2}, \quad p_2 \neq 0, \\
\frac{p_1}{q_1} \leq \frac{p_2}{q_2} &\iff p_1q_2 \leq p_2q_1.
\end{aligned}$$

These satisfy certain properties that we list next. They are sufficiently strong that you can deduce everything about $\mathbb{Q}$ just from these properties, i.e. you can treat them as axioms if you wish.

Axiom 2.1.

1. $a + b = b + a \quad \forall a, b \in \mathbb{Q}$ (+ is *commutative*)
2. $a \times b = b \times a \quad \forall a, b \in \mathbb{Q}$
3. $a + (b + c) = (a + b) + c$ (+ is *associative*)
4. $a \times (b \times c) = (a \times b) \times c$
5. $a \times (b + c) = (a \times b) + (a \times c)$ ($\times$ is *distributive* over +)

6. $\exists\, 0 \in \mathbb{Q}\colon a + 0 = a \quad \forall a \in \mathbb{Q}$
7. $\exists\, 1 \in \mathbb{Q}\colon 0 \neq 1, a \times 1 = a \quad \forall a \in \mathbb{Q}$
8. $\forall a \in \mathbb{Q},\ \exists (-a) \in \mathbb{Q}$ such that $a + (-a) = 0$
9. $\forall a \in \mathbb{Q}\backslash\{0\}\ \exists a^{-1} \in \mathbb{Q}$ such that $a \times (a^{-1}) = 1$

Axiom 2.2 (Order axioms)**.**

10. for each $x \in \mathbb{Q}$ **precisely one** of (a), (b), (c) holds:

$$\text{(a) } x > 0 \quad \text{or} \quad \text{(b) } x = 0 \quad \text{or} \quad \text{(c) } -x > 0 \qquad \text{(Trichotomy axiom)}$$

11. $x > 0,\, y > 0 \implies x + y > 0 \quad \forall x, y \in \mathbb{Q}$
12. $x > 0,\, y > 0 \implies xy > 0 \quad \forall x, y \in \mathbb{Q}$
13. $\forall x \in \mathbb{Q}\ \exists n \in \mathbb{N}$ such that $n > x$ (Archimedean axiom)

Notation: $a - b := a + (-b)$, and $a/b := a \times (b^{-1})$, while $a > b$ ($a < b$) is defined to mean $a - b > 0$ (respectively $-(a - b) > 0$).

Exercise 2.3. $x > y > z \implies x > z$.

Just write down what the LHS means:

$$\begin{aligned}
x > y > z \quad &\iff \quad \big[x - y > 0 \text{ and } y - z > 0\big] \\
&\overset{11}{\implies} \quad (x - y) + (y - z) > 0 \\
&\overset{3}{\iff} \quad x + ((-y) + y) - z > 0 \\
&\overset{8}{\iff} \quad x + 0 - z > 0 \\
&\overset{6}{\iff} \quad x - z > 0 \\
&\iff \quad x > z.
\end{aligned}$$

Exercise 2.4. Fix $a \in \mathbb{Q}$. For each $x \in \mathbb{Q}$ exactly one of the following holds

$$\text{(a) } x > a \quad \text{or} \quad \text{(b) } x = a \quad \text{or} \quad \text{(c) } x < a.$$

The real numbers $\mathbb{R}$ satisfy the exact same axioms, plus one more – the **completeness axiom** – designed to fix the problem that $\mathbb{Q}$ has holes. For instance,

Actually it replaces the Archimeddean axiom: we will see it implies it

Proposition 2.5. *There is no $x \in \mathbb{Q}$ such that $x^2 = 3$.*

Proof. Suppose $x = p/q$ in lowest terms satisfies $x^2 = 3 \iff p^2 = 3q^2$.

Thus $3|p^2$ so $3|p$.

(Proof: recall exercise 1.4.)

So writing $p = 3n$ we find $q^2 = 3n^2$ so $3|q$ as well as $3|p$, contradicting the assumption that p/q is in lowest terms. $\square$

2.2 Decimals

Finite decimals

For $a_0 \in \mathbb{Z}$ and and $a_i \in \{0, 1, \dots, 9\}$ we **define** the finite decimal $a_0.a_1 \dots a_i$ as follows. If $a_0 \geq 0$ then $a_0.a_1 \dots a_i$ is set to be

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_i}{10^i} \ \in \ \mathbb{Q}.$$

For $a_0 < 0$ we set $a_0.a_1 \dots a_i$ to be $-\big(|a_0|.a_1 \dots a_i\big)$. Putting $a_j := 0$ for $j > i$ this is a special case of an eventually periodic decimal.

Eventually periodic decimals

At school you became happy with the idea that

$$\begin{aligned}
0.\overline{3} &= 0.3333\dots \\
&= 0.3 + 0.03 + 0.003 + 0.0003 + \dots \\
&= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots \\
&= \frac{3}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \dots\right) \\
&\stackrel{?}{=} \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{3}{10}\frac{10}{9} = \frac{1}{3}\,.
\end{aligned}$$

Now I'll grant you we can certainly justify

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}\,, \qquad x \neq 1,$$

by multiplying both sides by $1 - x$. But it will be many lectures (see Example 4.1) before we know how to conclude that

$$\begin{aligned}
1 + x + x^2 + \cdots + x^n + \dots &= \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} \\
&= \frac{1}{1 - x}\,, \quad -1 < x < 1,
\end{aligned}$$

to justify the $\stackrel{?}{=}$ above. So for now we simply take it as a **definition**: for $a_0 \in \mathbb{N}$, $a_{i>0} \in \{0, 1, \dots, 9\}$ we define

$$a_0.a_1 \dots a_i\overline{a_{i+1}a_{i+2} \dots a_j}$$

to be the *rational number*

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2} \dots a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right) \tag{2.6}$$

motivated by the "fact" (which we've yet to prove) that the last fraction equals $1 + 10^{-(j-i)} + 10^{-2(j-i)} + \dots$. (There's a bit of checking that (2.6) is well-defined: if I consider the repetition to begin one period later, show I get the same rational number; similarly if I consider the period to be a multiple of $i - j$ show I get the same rational number.)[1]

Exercise 2.7. Consider two eventually periodic decimals differing in only one place:

$$a \;=\; a_0.a_1a_2 \dots a_{n-1}a_na_{n+1} \dots\,, \qquad b \;=\; a_0.a_1a_2 \dots a_{n-1}b_na_{n+1} \dots$$

Show using our definition (2.6) that $a < b$ if and only if $a_n < b_n$.

Thus any eventually periodic decimal expansion gives a **rational number** (2.6). Conversely, periodic decimals give **all** the rational numbers. Before proving this let's do an example.

Let's try to write $\frac{25}{11}$ as a decimal,

$$\frac{25}{11} \;=\; a_0.a_1a_2a_3 \dots$$

To find $a_0 = \left\lfloor \frac{25}{11} \right\rfloor$ we divide 11 into 25:

$$25 \;=\; \mathbf{2} \times 11 + 3 \quad\Longrightarrow\quad a_0 = 2. \tag{$*$}$$

To find a_1 we take $\frac{25}{11} - 2 = 0.a_1a_2a_3 \dots$ (i.e. $\frac{3}{11}$, where 3 is the remainder in $(*)$) then multiply by 10 and take integer part, i.e. $a_1 = \left\lfloor \frac{30}{11} \right\rfloor$:

We're just describing the long division algorithm here

$$30 \;=\; \mathbf{2} \times 11 + 8 \quad\Longrightarrow\quad a_1 = 2.$$

Similarly multiplying the remainder 8 by 10 and dividing 11 in gives $a_2 = \left\lfloor \frac{80}{11} \right\rfloor$:

$$80 \;=\; \mathbf{7} \times 11 + 3 \quad\Longrightarrow\quad a_2 = 7.$$

[1] E.g. show that our definition makes $0.\overline{3}$ and $0.3\overline{3}$ and $0.\overline{33}$ all the same number.

Notice we've got remainder 3 again, just as in $(*)$. So from now on everything repeats, because the remainder always determines the next step (we mutltiply it by 10 then divide 11 into the result). So $a_3 = \left\lfloor \frac{30}{11} \right\rfloor$ is just the same as a_1. And a_4 is the same as a_2. Etc. The result is the eventually periodic decimal

$$2.\overline{27}.$$

(Beware we've only shown that this decimal approximates $\frac{25}{11}$; we then need to prove they're equal. But this is clear by our definition (2.6).)

Since there are only finitely many possible remainders (i.e. nonnegative integers < 11) it was inevitable this periodicity would happen eventually. The general case is as follows.

Theorem 2.8

Any $x \in \mathbb{Q}$ is equal to an eventually periodic decimal expansion:
$x = a_0.a_1 \dots a_i\overline{a_{i+1}a_{i+2}\dots a_j}$ $(a_0 \in \mathbb{Z},\ a_\ell \in \{0,1,\dots,9\}$ for $\ell \geq 1)$.

Proof. Without loss of generality we take $x \geq 0$. It will be convenient to temporarily use a notation $\{x\} := x - \lfloor x \rfloor \in [0,1)$ for the non-integer part of x.

To write x as a decimal we let $a_0 := \lfloor x \rfloor$ and $e_0 := \{x\}$, so

$$x \ = \ a_0 + e_0, \qquad a_0 \in \mathbb{N},\ e_0 \in [0,1) \tag{2.9}$$

is the sum of an integer and a small "error". Now repeat for $10e_0 \in [0,10)$, setting $a_1 := \lfloor 10e_0 \rfloor$ and error $e_1 := \{10e_0\}$:

Illustrate with any decimal e.g. 3.14159265

$$10e_0 \ = \ a_1 + e_1, \qquad a_1 \in \{0,1,\dots,9\},\ e_1 \in [0,1).$$

Inductively, given $a_i \in \{0,1,\dots,9\}$ and $e_i \in [0,1)$ for $i < k$ we set $a_k := \lfloor 10e_{k-1} \rfloor$ and the error $e_k := \{10e_{k-1}\}$, so

$$10e_{k-1} \ = \ a_k + e_k, \qquad a_k \in \{0,1,\dots,9\},\ e_k \in [0,1). \tag{2.10}$$

Plugging each equation into the former gives

$$x \ = \ a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_k}{10^k} \ + \ \frac{e_k}{10^k}, \qquad e_k \in [0,1). \tag{2.11}$$

Now remember $x = p/q$ $(p \in \mathbb{N}, q \in \mathbb{N}_{>0})$ is rational! So $q\times$(2.9) tells us

Now illustrate with any rational number, e.g. 25/11

$$p \ = \ qa_0 + r_0,$$

where $r_0 := qe_0 = p - qa_0 \in [0, q)$ is therefore an integer (in fact the remainder when we divide q into p). Inductively $q\times$(2.10) shows that $r_k := qe_k$ is an integer in $[0, q)$.

So the remainders $r_k \in \{0, 1, \ldots, q-1\}$ lie in a finite set, so after a while they must repeat: $r_j = r_i$ for some $j > i$. Therefore the $e_k = r_k/q$ also repeat: $e_j = e_i$, so in the construction (2.10) we see the a_k repeat as well: $a_{j+1} = a_{i+1}$. Inductively then, $a_{\ell+j-i} = a_\ell$ for every $\ell \geq i+1$, and we have produced a periodic decimal expansion

$$a_0.a_1a_2 \ldots a_i\overline{a_{i+1}a_{i+2} \ldots a_j} \tag{2.12}$$

that really *ought* to be x. (It gets closer and closer to x; some might say it converges to x, but we've not defined convergence yet! That's what the rest of the course is about. So for now we have to prove it's x using our definition (2.6).) Let's check it using our convention (2.6). It says (2.12) is the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_i}{10^i} \ + \ \frac{a_{i+1}a_{i+2} \ldots a_j}{10^j} \frac{1}{1 - 10^{i-j}}\,.$$

Comparing with (2.11) we see we just need to show that

$$\frac{e_i}{10^i} \ = \ \frac{a_{i+1}a_{i+2} \ldots a_j}{10^j} \frac{1}{1 - 10^{i-j}}\,.$$

Multiplying by $(1 - 10^{i-j})$ and using the periodicity $e_i = e_j$ this is equivalent to

$$10^{-i}e_i - 10^{-j}e_j \ = \ 10^{-j}(a_{i+1}a_{i+2} \ldots a_j). \tag{2.13}$$

But adding the equalities $10^{1-k}e_{k-1} - 10^{-k}e_k = \frac{a_k}{10^k}$ of (2.10) for $k = i+1, i+2, \ldots, j$ gives

$$10^{-i}e_i - 10^{-j}e_j \ = \ \frac{a_{i+1}}{10^{i+1}} + \cdots + \frac{a_j}{10^j},$$

which is precisely (2.13), as required. □

However, not all eventually periodic decimals give *different* rational numbers: by (2.6),

$$0.\overline{9} \ = \ \left(\frac{9}{10}\right)\left(\frac{1}{1 - 10^{-1}}\right) \ = \ 1.$$

Proposition 2.14. *If $x \in \mathbb{Q}$ has two different decimal expansions then they are of the form*

$$\begin{aligned} x \ &= \ a_0.a_1a_2 \ldots a_n\overline{9} \\ &= \ a_0.a_1a_2 \ldots (a_n + 1) \quad \text{with } a_n \in \{0, 1, \ldots, 8\}. \end{aligned}$$

Proof. Suppose the two expansions are:

$$\begin{aligned} x &= a_0.a_1a_2\dots a_{n-1}a_na_{n+1}\dots \\ &= a_0.a_1a_2\dots a_{n-1}b_nb_{n+1}\dots \end{aligned}$$

with $a_n < b_n$ without loss of generality. Then by Exercise 2.7 (and some easier Exercises like $0 \le 0.c_1c_2\dots \le 1$ with equality if and only if all c_i are 0 or all c_i are 9)

$$\begin{aligned} x &= a_0.a_1a_2\dots a_na_{n+1}\dots \\ &\le a_0.a_1a_2\dots a_n999\dots \\ &= a_0.a_1a_2\dots (a_n+1)000\dots \\ &\le a_0.a_1a_2\dots b_nb_{n+1}\dots \\ &= x. \end{aligned}$$

Therefore the $\le$s must have been $=$s and the proposition follows. □

Arbitrary decimals: the real numbers

So this gives us an obvious (but ugly!) way to define the real numbers: as the set of decimal expansions which do not end in $\overline{9}$,

$$\mathbb{R} := \Big\{a_0.a_1a_2\dots \;:\; a_0 \in \mathbb{Z},\ a_{i\ge 1} \in \{0,1,\dots,9\},\ \nexists N \text{ such that } a_i = 9\ \forall i \ge N\Big\}$$

With some work one can then define $+$, $-$ $\times$, $\div$ $<$ on $\mathbb{R}$ and check they satisfy the Axioms 2.1 and 2.2.

Theorem 2.8 gives us a way to produce *many* explicit irrational numbers like

$$x = 0.1010010001\dots \notin \mathbb{Q}.$$

> **Exercise 2.15.** $\forall x, y \in \mathbb{R}$ with $x < y$ show
>
> 1. $\exists z \in \mathbb{Q} : x < z < y$, and
> 2. $\exists z \notin \mathbb{Q} : x < z < y$.
>
> Do both by (a) decimal expansions, and (b) by using only the axioms. (Hint: use Archimedean axiom to find $n \in \mathbb{N}$ to magnify the difference $y - z$ to be ≥ 1.)

In fact there is a way to make precise the fact that there are many more irrational or real numbers than rational numbers.

2.3 Countability

Definition. A set S is *countably infinite* if and only if there exists a bijection $f\colon \mathbb{N}_{>0} \to S$.

A set that is finite, or countably infinite, is said to be *countable*.

This means I can put the elements of S into a list:

$$S = \{s_1, s_2, s_3, s_4, s_5, \dots\}$$

with no repeats (all s_i distinct). Here $s_n := f(n)$.

Since the even number $2\mathbb{N} \subset \mathbb{N}$ you might think there are less of them. But $\mathbb{N} \overset{\times 4}{\hookrightarrow} 2\mathbb{N}$ so maybe there are less? Really they're the same size, in the following sense.

Proposition 2.16. *Suppose $S \subset \mathbb{N}$ is infinite. Then S is countably infinite.*

Proof. We just list the elements of S in order of size. Formally, we define $f\colon \mathbb{N} \to S$ inductively (recursively?) as follows:[2]

- $f(1) = \min S$,
- Assume $f(1), \dots, f(n-1)$ is defined already. Since S is infinite the set $S\backslash\{f(1), \dots, f(n-1)\}$ is nonempty, and all $s \in S$ are ≥ 0, so we may define

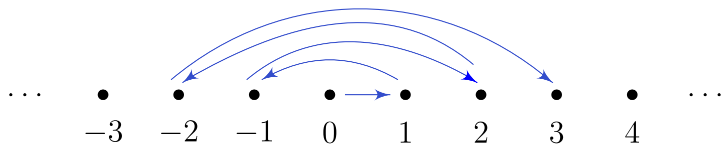
$$f(n) \ := \ \min\Big(S\backslash\{f(1), \dots, f(n-1)\}\Big).$$

This function is injective since $f(1) < f(2) < f(3) < \dots$. If f were not surjective, then $\exists$ smallest $s \in S\backslash \mathrm{im}\,(f)$. Since $s \neq \min S$ (because $f(1) = \min S$) we know $\exists s' \in S$ such that $s' < s$ – picking the largest such, then $s' = f(n)$ and by our rule $s = f(n+1)$. $\square$

$S \cap \{s', \dots, s\}$ finite nonempty so has a max

Proposition 2.17. $\mathbb{Z}$ *is countably infinite.*

Proof. We list them as $\mathbb{Z} = \{0, 1, -1, 2, -1, 3, -3, \dots\}$.



[2]Since S is infinite you should worry about taking min. Since it's nonempty pick an element $n_0 \in S$, then $S \cap \{1, 2, \dots, n_0\}$ is *finite and nonempty* so does have a minimum m say. Now check $m = \min S$.

Formally, define a bijection $f : \mathbb{N} \to \mathbb{Z}$ by declaring, for $k \geq 1$,

$$\begin{cases} f(2k-1) & := & -(k-1), \\ f(2k) & := & k. \end{cases}$$

$\square$

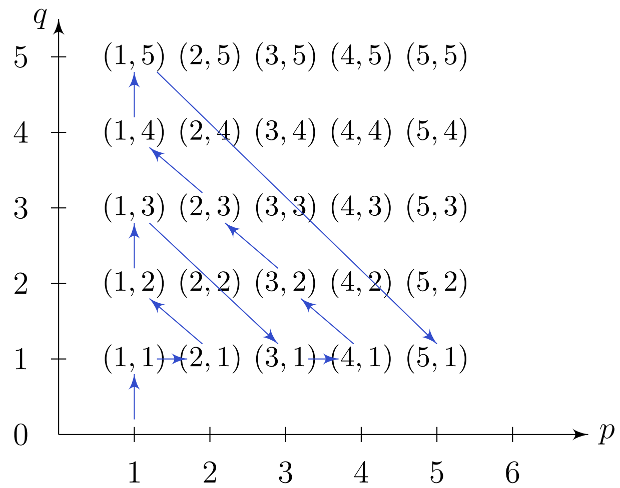
> **Exercise 2.18.** Show that f is indeed bijective.
> Show similarly than A, B countable $\Longrightarrow A \cup B$ countable.

Remarkably, $\mathbb{Q}$ is also countably infinite.

Theorem 2.19

$\mathbb{Q}$ is countably infinite.

Proof. First let's show $\mathbb{Q}_{>0}$ is countably infinite. The usual proof is a bit sketchy (but informative!): arrange the pairs $(p, q) \in \mathbb{N}^2$ in a square:



Now list the pairs according to the path shown, *missing out pairs which aren't in lowest terms*. It is not straightforward to write down an explicit formula.

A slicker proof is to define the injection Think for a minute why injection

$$f\colon \mathbb{Q}_{>0} \longrightarrow \mathbb{N}_{>0}, \qquad f(m/n) \ := \ 2^m 3^n$$

where $m, n \geq 1$ and m/n is in lowest terms. (MATH40001 Exercise: give a careful proof this is an injection. What if we'd used $f(m/n) = 2^m(2n-1)$?) So f defines a bijection between $\mathbb{Q}_{>0}$ and an infinite subset of $\mathbb{N}_{>0}$, which in turn is countably infinite by Proposition 2.16. Therefore $\mathbb{Q}_{>0}$ is also countably infinite, giving a bijection $F\colon \mathbb{N}_{>0} \to \mathbb{Q}_{>0}$.

To finish off we define $g\colon \mathbb{N}_{>0} \longrightarrow \mathbb{Q}$ by

$$g(1) := 0 \quad \text{and} \quad \begin{cases} g(2k) & := \; F(k), \\ g(2k+1) & := \; -F(k). \end{cases}$$

That is, if $q_1, q_2, \dots$ $(q_i := F(i))$ is our list of elements of $\mathbb{Q}_{>0}$ then our new list is $0, q_1, -q_1, q_2, -q_2, \dots$. □

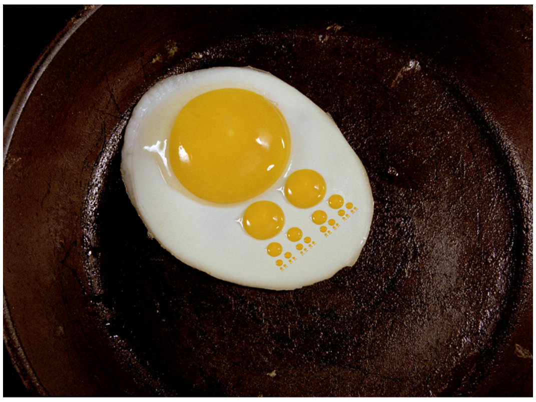
Exercise 2.20. We showed that we can list the positive rational numbers $\mathbb{Q}_{>0} = \{q_1, q_2, \dots\}$. Show this cannot be done in order of size, i.e. with $q_1 < q_2 < \dots$.

Next we see that $\mathbb{R}$ is genuinely bigger than $\mathbb{Q}$.

Theorem 2.21

$\mathbb{R}$ is uncountable.

Proof. (Cantor's Diagonal Argument)



We suppose for a contradiction that you can "list" all the real numbers. We write this as follows, using decimal expansions with no $\overline{9}$s:

$$\begin{aligned}
x_1 &= a_1.a_{11}\, a_{12}\, a_{13}\, a_{14} \dots \\
x_2 &= a_2.a_{21}\, a_{22}\, a_{23}\, a_{24} \dots \\
x_3 &= a_3.a_{31}\, a_{32}\, a_{33}\, a_{34} \dots \\
x_4 &= a_4.a_{41}\, a_{42}\, a_{43}\, a_{44} \dots \\
&\vdots \\
x_m &= a_m.a_{m1}\, a_{m2}\, a_{m3}\, a_{m4} \dots \\
&\vdots
\end{aligned}$$

As usual $a_1, a_2, a_3 \in \mathbb{Z}$ and $a_{11}, a_{12}, \ldots, a_{ij} \in \{0, 1, 2, \ldots, 9\}$.

Now we can produce a real number $x := 0.b_1\, b_2 \ldots b_n \ldots$ not on the list:

1. Pick $b_1 \in \{0, 1, \ldots, 8\}$ such that $b_1 \neq a_{11}$,
2. Pick $b_2 \in \{0, 1, \ldots, 8\}$ such that $b_2 \neq a_{22}$,

$\vdots$

n. Pick $b_n \in \{0, 1, \ldots, 8\}$ such that $b_n \neq a_{nn}$

$\vdots$

Since we don't allow 9 we don't end up with a decimal ending in $\overline{9}$.

Then $\forall i \geq 1$, we see $x \neq x_i$ because it differs in the ith decimal place x_{ii}. Therefore we have found an $x \in \mathbb{R}$ not on the list. $\square$

There's a set of numbers in between $\mathbb{Q}$ and $\mathbb{R}$ called the *algebraic numbers*: those $x \in \mathbb{R}$ which satisfy a polynomial equation $p(x) = 0$, where p has integer coefficients. E.g. any rational number $x = p/q$ satisfies an equation $p(x) := qx - p = 0$. And $x = \sqrt[n]{m}$ satisfies $p(x) := x^n - m = 0$.

On the exercise sheet you'll prove that the set of algebraic numbers is also *countable*. Therefore (by Exercise 2.18) the set of *transcendental numbers* – those which are not algebraic – is uncountable. It turns out that e and π are transcendental.

This is hard, but much easier to see they're irrational – see later.