Exercise 10.1. Let (X, d) be a metric space. Show that X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

Hint: In one direction, you have a pair of separating sets, and you can consider of the open sets in the pair. In the other direction, consider the particular set and its complement.

Exercise 10.2. Show that in the Euclidean metric space $(\mathbb{R}^1, d_1)$, the set of rational numbers $\mathbb{Q}$ is disconnected.

Hint:pick an irrational number, and consider the set of rational numbers less than that number, and the set of rational numbers larger than that set.

Exercise 10.3.* Consider the Euclidean metric space $(\mathbb{R}, d_1)$, and assume that a and b are real numbers with $a < b$.

(i) Show that the interval $[a, b)$ is connected.

   Hint: This is a special case of the proof of the connectivity of $[a, b]$

(ii) Show that the interval $(a, b]$ is connected.

   Hint: Modify the proof of the thm showing that $[a, b$ is connected; starting with b instead of a, modify I, and take the infimum of I.

(iii) Show that the interval (a, b) is connected.

   Hint: Choose $u \in U \cap (a, b)$ and $v \in V \cap (a, b)$, and consider the interval $[u, v]$ or $[v, u]$, depending on $u < v$ or $v < u$.

Exercise 10.4. Show that the following metric spaces are path connected.

(i) the Euclidean space $\mathbb{R}^n$, for any $n \geq 1$,

(ii) the open ball $B_1(0)$ in $(\mathbb{R}^n, d_2)$, for any $n \geq 2$,

(iii) the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \|(x, y)\| \leq 2\}$.

Hint: For items (i) and (ii), consider a straight line segment between any pair of points. For item (iii), write an explicit formula for a curve spiralling from x to y, using the polar coordinates.

Exercise 10.5. Consider the set of all continuous functions $f : [0, 1] \to \mathbb{R}$, that is $C([0, 1])$, with the metric d_1.

(i) Show that the space $(C([0, 1]), d_1)$ is path connected.

(ii) Conclude that the space $(C([0, 1]), d_1)$ is connected.

Hint: For arbitrary f and g in $C([0, 1])$, define an explicit map $\phi : [0, 1] \to C([0, 1])$ defined as a linear combination of f and g. You need to show that every such linear combination belongs to $C([0, 1])$, and the map Φ is continuous with respect to d_1.

Exercise 10.6.* In this exercise, we aim to show that a connected space may not be path connected.

Consider the following subset of $\mathbb{R}^2$:

$$A = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x = 0, y \in [-1, +1]\}.$$

That is, A is the union of the oscillating curve which is the graph of $\sin(1/x)$, and the vertical line segment $\{0\} \times [-1, +1]$.

(i) show that the set A is connected.

Hint: first show that each of the vertical line segment and the graph of $\sin(1/x)$ are connected. So the only way to disconnect A is to separate those two pieces by open sets. However, any open set containing the straight line segment, will also contain part of the graph.

(ii) show that the set A is not path connected.

Hint: You need to show that there is no path joining a point on the line segment to a point on the graph.

Unseen Exercise. The purpose of this exercise is to give a direct proof that a path connected space is connected.

Let us assume that there is a metric space (X, d) which is path connected, but not connected. By the definition of connected sets, there must be open sets U and V in X such that $X = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset$, and $V \neq \emptyset$.

Let us choose a point $u \in U$ and a point $v \in V$ (we can do this since U and V are not empty.). Since X is path connected, there is a continuous map $g : [0, 1] \to X$ satisfying $g(0) = u$ and $g(1) = v$. Show that the sets

$$U' = g^{-1}(U), \qquad V' = g^{-1}(V),$$

disconnect $[0, 1]$.