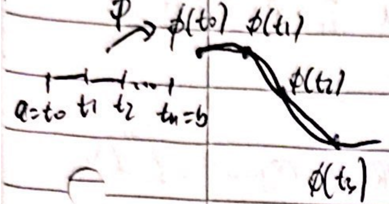
Geometry Cheat Sheet / Revision Notes

Parametrised curve in $\mathbb{R}^n$: $\phi : [a,b] \to \mathbb{R}^n$, ϕ smooth

$\begin{cases} \text{regular: } |\phi'(t)| \neq 0 \ \forall\, t \in [a,b] \\ \text{arc-length parametrisation: } |\phi'(t)| \equiv 1 \end{cases}$

If $\left\{ \begin{array}{l} f \text{ smooth } [c,d] \to [a,b] \\ |f'(x)| \neq 0 \\ f(\{c,d\}) = \{a,b\} \end{array} \right\}$ $\phi \circ f$ is reparametrisation of ϕ



length of curve

$$l(\phi) = \int_a^b |\phi'(t)|\, dt$$

$\boxed{\text{Lemma 1.1}}$ length of curve invariant under reparametrisation

Proof $\psi := \phi \circ f$ $f'(s) \neq 0$, f cont. so $f' > 0$ or $f' < 0$.

WLOG $f' > 0$,

$$|\psi'(s)| = |(\phi \circ f)'(s)| \underset{\text{rule}}{\overset{\text{chain}}{=}} \cdots = f'(s)\,|\phi'(f(s))|$$

$$l(\psi([c,d])) = \int f'(s)\,|\phi(f(s))| \underset{\text{change of variable}}{=} \cdots = l(\phi([a,b]))$$ $\square$

$\boxed{\text{Lemma 1.2}}$ Any regular curve can be parametrised by arc-length

Proof Suppose $\psi = \phi \circ f$ is an arc-length parametrisation,

aims to find $h = f^{-1}$. ($h : [a,b] \to \mathbb{R}$)

$$\int_{h(a)}^{h(t)} |\psi'(u)|\, du$$

arc-length par. $\parallel$ $\qquad\qquad$ $\parallel$ def

$$h(t) - h(a) \qquad\qquad l(\psi([h(a), h(t)])) = \int_a^t |\phi'(u)|\, dt$$

Solve $h(t) = \int_a^t |\phi'(u)|\, dt + c$ — free to choose, usually 0

$h \left\{ \begin{array}{l} \text{well-defined: } \phi'(t) \text{ exists, continuous} \\ \text{smooth: as } h'(t) = |\phi'(t)| \\ \text{non-zero derivative: same as above} \end{array} \right\} \Rightarrow f$ is well-def., smooth, $|f'(s)| \neq 0$ $\square$

curvature vector

Curvature: $k(t) := |\phi''(t)|$ $\quad \vec{k}(t) := \phi''(t)$

where $\phi : [0, L] \to \mathbb{R}^n$ is regular curve $\boxed{\text{parametrised by arc-length}}$

MUST reparametrise if not

- independent of parametrisation
- $k(t) \equiv 0 \iff \phi([a,b])$ is straight line
- $\vec{k}(t)$ perpendicular to ~~ϕ~~ tangent line at $\phi(t)$

Proof. $\langle \phi'(t), \phi'(t) \rangle = 1$, find $\frac{d}{dt} \langle \phi'(t), \phi'(t) \rangle$

Frenet Frame (T, N, B) $\quad \phi : [a,b] \to \mathbb{R}^3$ is regular, $\phi'' \not\equiv 0$

Three vectors
all unit vectors

$T := \phi'(t)$ unit tangent

$N := T'(t)/|T'(t)|$ principal normal

$B := T \times N$ binormal

Relationships: $T'(t) = k(t) N(t)$ (1) $\Rightarrow k(t) = \langle N(t), T'(t) \rangle$

$B'(t) = \cdots \overset{\text{def.}}{\underset{(1)}{=}} T(t) \times N'(t)$ so $B' \perp T$

$|B(t)|^2 = 1 \Rightarrow \langle B'(t), B(t) \rangle = 0$ so $B' \perp B$

$\Rightarrow B'(t) = -\tau(t) N(t)$ (2)

$N'(t) = (B(t) \times N(t))' = \cdots = -k(t) T(t) + \tau(t) B(t)$ $\quad \tau$: torsion (defined by (2)), unique

Differential equation:

$$\frac{d}{dt} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

$\boxed{\text{Proposition 3.1}}$ ϕ is contained in a plane $\iff \tau(t) \equiv 0$

Proof ($\Leftarrow$) $B' = -\tau N = 0$ so $B(t) = \vec{c}$ constant

$\frac{d}{dt} \langle \phi(t), \vec{c} \rangle = \langle T, B \rangle = 0$ so $\langle \phi(t), \vec{c} \rangle = d$ constant

i.e. on a plane normal to $\vec{c}$

($\Rightarrow$) $\exists \vec{v}, d$ s.t. $\langle \phi(t), \vec{v} \rangle = d$

differentiate twice yields $\langle N(t), \vec{v}\rangle = 0$, $\langle T(t), \vec{v}\rangle = 0$

so $B(t) \equiv \pm\vec{v}$ is constant $\Rightarrow \tau(t) \equiv 0$. $\square$

Fundamental Theorem of Local Theory of curves

$\boxed{\text{Theorem 3.2}}$ Given smooth $k, \tau : [a,b] \to \mathbb{R}$, $k > 0$.

$\exists$ regular curve ϕ with curvature k, torsion τ

and ϕ is unique up to rigid motion.

Proof (Uniqueness) for $g \in SO(3)$, $\vec{c}$ $\mu = g \circ \phi + \vec{c}$ is a rigid transformation.

- show it preserves arc-length, curvature, torsion using

$$T_\mu = g(T_\phi), \quad N_\mu = g(N_\phi)$$

Properties of g: $-\frac{d}{dt}(g(f(t))) = g(f'(t))$ as g is not moving

$- g(A \times B) = g(A) \times g(B)$

$- |g(v)| = |v|$ length preseriation

$-$ g is invertible

- Show if ϕ, ψ are parametrised by arc-length, has the same curvature, torsion $\underset{\text{WOLG}}{\Rightarrow}$ only differs by translation.

$-$ show $\frac{d}{dt}\left(|T_\phi - T_\mu|^2 + |N_\phi - N_\mu|^2 + |B_\phi - B_\mu|^2\right) = 0$

$-$ then $T_\phi - T_\mu =$ constant, integration yields $\phi = \psi + \underset{\text{constant}}{\vec{v}}$

(Existence) pick orthonormal basis (T_a, N_a, B_a),

Solve $\begin{pmatrix} T(a) \\ N(a) \\ B(a) \end{pmatrix} = \begin{pmatrix} T_a \\ N_a \\ B_a \end{pmatrix}$, $\begin{pmatrix} T'(t) \\ N'(t) \\ B'(t) \end{pmatrix} = \begin{pmatrix} 0 & k(t) & 0 \\ -k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}$

by existence theorem of linear equation

And prove T, N, B are orthogonal to each other, have unique size

i.e. $\frac{d}{dt}(M^T M) = 0$ where $M := \begin{pmatrix} | & | & | \\ T & N & B \\ | & | & | \end{pmatrix}$ $\square$

Corollary: regular curve $\begin{cases} \text{torsion } 0 \\ \text{constant curvature } c > 0 \end{cases} \Longleftrightarrow$ it is circle of radius $\frac{1}{c}$

For $\phi : [a,b] \to \mathbb{R}^2$, parametrised by arc-length, define **signed curvature**

$$\kappa(t) := \langle n(t), \phi''(t) \rangle \quad \text{where } n(t) := \frac{(-y'(t), x'(t))}{|\phi'(t)|}$$

for general $\phi : [a,b] \to \mathbb{R}^2$,

$$\kappa(t) = \frac{\langle n(t), \phi''(t) \rangle}{|\phi'(t)|^2}$$

Proof If $\psi = \phi \circ f$ is reparametrisation by arc-length, $f' > 0$

suppose $h = f^{-1}$

- by inverse function theorem, $f'(s) = \dfrac{1}{h'(f(s))} = \dfrac{1}{|\phi'(f(s))|}$

- ψ, ϕ are parametrisations of the same curve in the same direction ($f' > 0$)

so $\begin{cases} n_\phi(f(s)) = n_\psi(s) \\ \cancel{\kappa_\phi(f(s)) = \kappa_\psi(s)} \Rightarrow \kappa_\phi(f(s)) = \kappa_\psi(s) \end{cases}$

then use definition and formulae for $f'(s)$ to relate n_ϕ, κ_ϕ $\square$

For graph of function f, $\phi(t) = (t, f(t))$

$$\kappa(t) = \frac{f''(t)}{(1 + (f'(t))^2)^{\frac{3}{2}}}$$

Consider ϕ s.t. $\phi(a) = \phi(b)$ i.e. closed curve

Winding number around p: number of times ϕ rotates around p

usually translate p to origin, then

(1) $$w(\phi) = \frac{1}{2\pi i} \int_{\phi([a,b])} \frac{dz}{z} = \begin{matrix}\text{complex} \\ \text{integral}\end{matrix} = \frac{1}{2\pi} \int_a^b \frac{\langle \phi(t), y'(t) - x'(t) \rangle}{|\phi(t)|^2} dt$$

Index / Turning number $\mathrm{Ind}(\phi) := w(T)$ where $T = \phi'$

$\boxed{\text{Theorem 4.4}}$ $\phi \begin{cases} \text{regular} \\ \text{closed} \\ \text{parametrised by arc-length} \end{cases}$ $T(t) := \phi'(t)$

then $w(T) = \frac{1}{2\pi} \int_a^b k(t)\,dt$ where $k(t)$ is signed curvature of ϕ

i.e. integral of signed curvature must be integer multiple of 2π.

Proof: simply substitute T to equation (1)

Surfaces

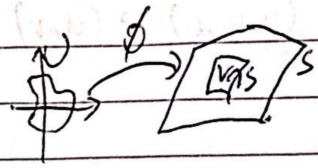
Regular Surfaces

set $S \subset \mathbb{R}^3$ s.t. $\forall p \in S$, $\exists \begin{cases} \text{open } V \subseteq \mathbb{R}^3, p \in V \\ \text{open } U \subseteq \mathbb{R}^2 \\ \phi : U \to \mathbb{R}^3 \begin{cases} \phi(U) = V \cap S \\ \phi : U \to V \cap S \text{ homeomorphism} \\ \forall q \in U, \\ d\phi_q \text{ is injective} \end{cases} \end{cases}$

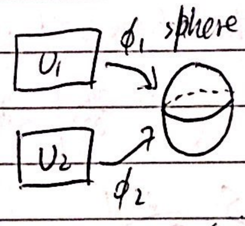
$\Updownarrow$

$\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)$ LI

(ϕ, U) chart

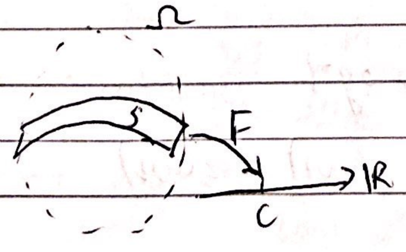


could have multiple charts to cover a complicated surface



Regular Level set

$S = \underbrace{F}_{\text{smooth } F : \underbrace{\Omega}_{\text{open set in } \mathbb{R}^3} \to \mathbb{R}}{}^{-1}(\underbrace{c}_{\text{constant in } \mathbb{R}})$, $DF(p) \neq 0 \ \forall p \in S$



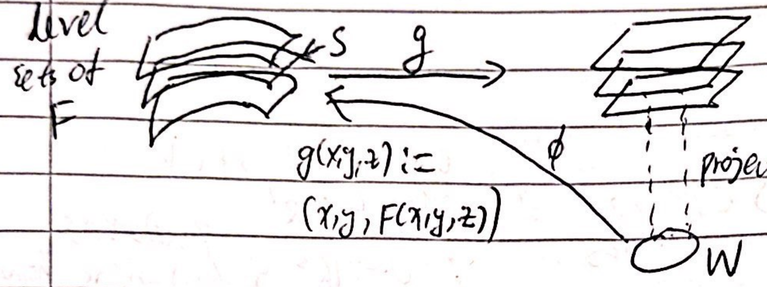
Proposition 5.2 Any regular level set is a regular surface

Proof.

IFT: $f:\Omega \to \mathbb{R}^n$, open set $\Omega \subseteq \mathbb{R}^n$, $\begin{cases} C^k \text{ map} \\ df_p \text{ invertible for some } p\in\Omega \end{cases} \Rightarrow \exists\, U$ s.t. $\begin{cases} p\in U \\ f: U \to f(U) \text{ is } C^k\text{-diffeomorphism} \end{cases}$

only applies to functions from $\mathbb{R}^n$ to $\mathbb{R}^n$

$DF(p) \neq 0$, WLOG say $\frac{\partial F}{\partial z}(p) \neq 0$



$g(x,y,z) := (x,y,F(x,y,z))$

$\det(dg_p) = \frac{\partial F}{\partial z}(p) \neq 0$

So by IFT, $g: U \to g(U)$ is diffeomorphism for some U

projection: $g(U) \mapsto W$, $(u,v,c) \to (u,v)$

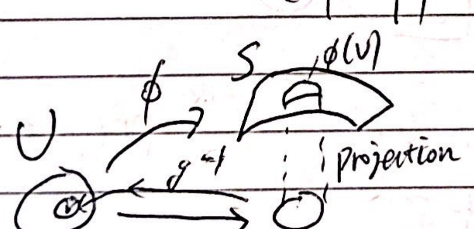
$\phi(u,v) := g^{-1}(u,v,c)$ is the required chart $\square$

Proposition 5.6 Any regular surface is locally graph of a smooth function.

Proof. $d\phi_q = \begin{pmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \end{pmatrix}$ rank 2, so any 2x2 minor is invertible.

WLOG, $\det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \neq 0$

Use IFT, $g: V \to g(V)$ is diffeomorphism for some $V \subset U$

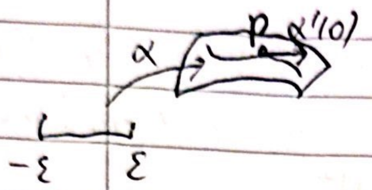


$g := (x(u,v), y(u,v))$

$f := z \circ g^{-1}$ satisfies

f smooth by chain rule $f(x(u,v), y(u,v)) = z(u,v)$ $\square$

Tangent vector at p: $\alpha'(0)$ where $\alpha: (-\varepsilon, +\varepsilon) \to S$ is a smooth path on S with $\alpha(0) = p$



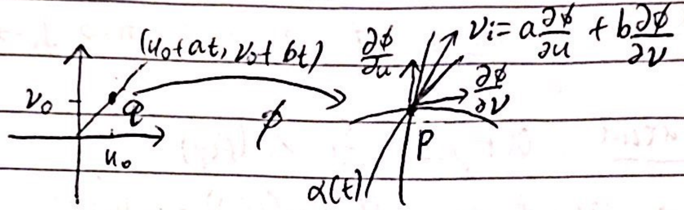
Tangent plane:

$$T_p(S) := \left\{ \alpha'(0) \;\middle|\; \begin{array}{l} \alpha: (-\varepsilon, \varepsilon) \to S \\ \text{is smooth} \\ \varepsilon > 0,\ \alpha(0) = p \end{array} \right\}$$

- Compute tangent plane:

$$T_pS = \operatorname{span}\left\{ \frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q) \right\} = d\phi_q(\mathbb{R}^2)$$

Proof: (1) $\operatorname{span}\left\{ \frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q) \right\} \subseteq T_pS$

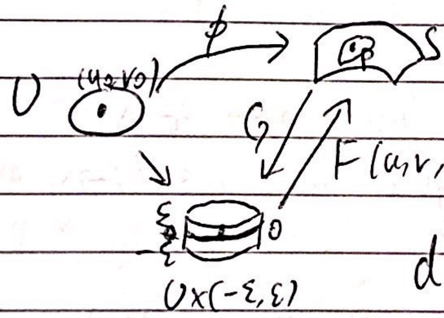


define $\alpha(t) := \phi(u_0 + at, v_0 + bt)$, then $\alpha'(0) = v$

so $v \in T_pS$

(2) $T_pS \subseteq \operatorname{span}\left\{ \frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q) \right\}$

need to bring path $\alpha(t)$ on S back to u,v-plane



S is locally graph of a function, say $z(u,v) = f(x(u,v), y(u,v))$

$F(u,v,t) := (x(u,v), y(u,v), f(x(u,v), y(u,v)) + t)$

$dF|_{(u_0, v_0, 0)}$ invertible as $d\phi_q$ is invertible

so IFT applies, say $G = F^{-1}$

$G(x, y, f(x,y)) = (u(x,y), v(x,y), 0)$ for some smooth u, v

use G to write $\alpha(t) = \phi(u(x(t), y(t)), v(x(t), y(t)))$

then $\alpha'(0) \in \operatorname{span}\left\{ \frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q) \right\}$

- Tangent Plane for regular level set $S = F^{-1}(0)$

$\forall p \in S, \quad T_pS = \{ v \in \mathbb{R}^3 : \langle v, \nabla F(p) \rangle = 0 \} = (\nabla F(p))^\perp$

Proof
- differentiating $F(\alpha(t)) = 0$ yields $T_pS \subseteq (\nabla F(p))^\perp$
- $T_pS, (\nabla F(p))^\perp$ have the same number of dimensions

so $T_pS = (\nabla F(p))^\perp$ □

$\boxed{\text{Smooth maps}}$

- $F: S_1 \to \mathbb{R}^m$ is <u>smooth</u> if $\forall$ chart $\phi: U \to S_1$, $F \circ \phi : U \to \mathbb{R}^m$ is smooth

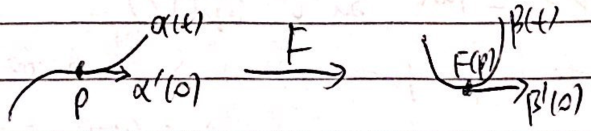
[$F: S_1 \to S_2$ <u>Smooth</u> if smooth as map $S_1 \to \mathbb{R}^3$

<u>differential</u> $dF_p : T_pS_1 \to T_{F(p)}S_2$

- defined by $dF_p(\alpha'(0)) = \beta'(0)$

where $\alpha: (-\varepsilon, +\varepsilon) \to S_1$ is s.t. $\begin{cases} \alpha(0) = p \\ \text{smooth} \end{cases}$

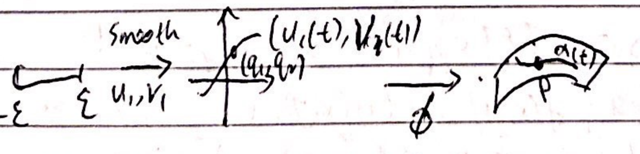
$\beta := F \circ \alpha$, $\beta: (-\varepsilon, +\varepsilon) \to S_2$ note $\beta(0) = F(p)$



- definition independent of α

Proof - Suppose $\alpha_1, \alpha_2 : (-\varepsilon, +\varepsilon) \to S_1$ are $\begin{cases} \text{smooth} \\ \alpha_1(0) = \alpha_2(0) \\ \alpha_1'(0) = \alpha_2'(0) \end{cases}$

α_1, α_2 can be written as $\phi(u_1(t), v_1(t))$, $\phi(u_2(t), v_2(t))$



then $\alpha_1'(0) = \alpha_2'(0) \Rightarrow u_1'(0) = u_2'(0)$, $v_1'(0) = v_2'(0)$

$\Rightarrow$ curves $(u_1(t), v_1(t))$, $(u_2(t), v_2(t))$ have same tangent at $0 = t$.

$$dF_p(\alpha_i'(0)) \overset{\text{def.}}{=} \cdots = \frac{\partial(F\circ\phi)}{\partial u}(q_1, q_2)\, u_i'(0) + \frac{\partial F\circ\phi}{\partial v}(q_1, q_2)\, v_i'(0)$$

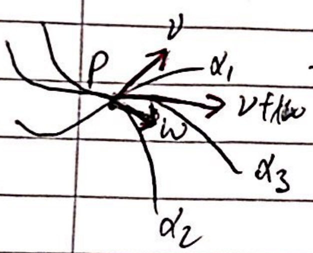
so $dF_p(\alpha_1'(0)) = dF_p(\alpha_2'(0))$ $\square$

- dF_p is linear

~~Proof~~ - Target: $dF_p(v + \lambda w) = dF_p(v) + \lambda dF_p(w)$ $\forall \lambda \in \mathbb{R}$,

$v = \alpha_1'(0)$, $w = \alpha_2'(0)$ for some smooth $\alpha_1, \alpha_2 : (-\varepsilon, +\varepsilon) \to S_1$

- again, $\alpha_i(t) = \phi(u_i(t), v_i(t))$ for some smooth $u_i, v_i : (-\varepsilon, +\varepsilon) \to \mathbb{R}$



- define $\alpha_3(t) := \phi(u_1(t) + \lambda u_2(t), v_1(t) + \lambda v_2(t))$

then $\begin{cases} \alpha_3'(0) = v + \lambda w \\ dF_p(v + \lambda w) = dF_p(\alpha_3'(0)) \underset{\text{chain rule}}{\overset{\text{def.}}{=}} \cdots = dF_p(v) + \lambda dF_p(w) \end{cases}$

- differential of $f : S \to \mathbb{R}$:

$$df_p(\alpha'(0)) = \frac{d}{dt}(f(\alpha(t)))\Big|_{t=0}$$

- also not dependent of choice of α

- finding dF_p in practice: Suppose $\phi(u_0, v_0) = p$
  - $\left\{\frac{\partial\phi}{\partial u}(u_0, v_0), \frac{\partial\phi}{\partial v}(u_0, v_0)\right\}$ is basis of T_pS
  - find $dF_p\left(\frac{\partial\phi}{\partial u}(u_0, v_0)\right) = dF_p(\alpha'(0))$ where $\alpha(t) = \phi(u_0 + t, v_0)$

$= \frac{\partial(F\circ\phi)}{\partial u}(u_0, v_0)$

similarly $dF_p\left(\frac{\partial\phi}{\partial v}(u_0, v_0)\right) = \frac{\partial(F\circ\phi)}{\partial v}(u_0, v_0)$

- Inverse of smooth map: $f: S_1 \to S_2$ smooth
  df_p invertible $\Rightarrow \exists V \subset S_1 \begin{cases} p \in V \\ f: V \to f(V) \text{ is diffeomorphism} \end{cases}$

Proof

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ U_1 \arrow[u, "\phi_1"] \arrow[r] & U_2 \arrow[u, "\phi_2"] \end{tikzcd}$$

$g := \phi_2^{-1} \circ f \circ \phi_1$ $\quad$ g from $\mathbb{R}^2 \to \mathbb{R}^2$, dg_p invertible

So IFT applies, $g: V_1 \to g(V_1)$ diffeo., so f is diffeo on $\phi(V_1)$ $\square$

$V_1 \subset U_1$

Unit Normal Vector

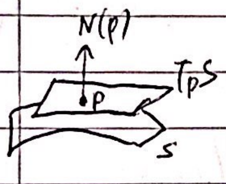
Gauss Map

$N: S \to \mathbb{S}^2$ ← unit sphere

- for regular level set

$$N(p) := \frac{\nabla F(p)}{|\nabla F(p)|}$$

- for chart



$$N(p) = \frac{\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)}{\left|\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)\right|} \quad \text{where } q = \phi^{-1}(p)$$

- this depends on chart ϕ, may not combine $N(p)$ from different charts

orientable: $\exists$ a continuous choice of $N(p)$ $\forall p \in S$

surface $\begin{cases} \text{regular} \\ \text{orientable} \end{cases} \Rightarrow N$ is smooth map

- derivative: $dN_p: T_pS \to T_{N(p)}\mathbb{S}^2 = T_pS$

  i.e. it is a map from tangent plane to itself.

- there can be 2 choices of unit normal, along the same direction, but with opposite sign: N, $-N$

└ dN_p in terms of chart ϕ: if $q = \phi^{-1}(p)$,

$$dN_p\left(\frac{\partial \phi}{\partial u}(q)\right) = \frac{\partial (N \circ \phi)}{\partial u}(q), \quad dN_p\left(\frac{\partial \phi}{\partial v}(q)\right) = \frac{\partial (N \circ \phi)}{\partial v}(q)$$

Second Fundamental Form

$$A_p : T_pS \times T_pS \to \mathbb{R}$$
$$(X, Y) \mapsto -\langle X, dN_p(Y) \rangle$$

- It is bilinear
  - prove by linearity of dN_p and inner product

- A_p is symmetric

Proof. ① $A_p\left(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\right) = A_p\left(\frac{\partial \phi}{\partial v}(q), \frac{\partial \phi}{\partial u}(q)\right)$ where $q = \phi^{-1}(p)$

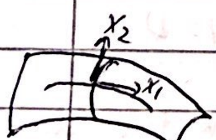
use the fact that $\langle \underbrace{\frac{\partial \phi}{\partial u}}_{\in T_pS}, \underbrace{N \circ \phi}_{\perp T_pS} \rangle = 0$

② since $\{\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q)\}$ is basis for T_pS,
A_p is symmetric over T_pS □

- obtaining A_p means obtaining dN_p (i.e. how normal vector changes along surface)
  if $\{v, w\}$ is orthonormal basis of T_pS
  $dN_p(X) = -A_p(v, X)v - A_p(w, X)w$

  - $[dN_p]_{\{v,w\}}$ is symmetric as A_p is symmetric
    ~~so matrix~~ $[dN_p]_{\{v,w\}}$ is $\begin{cases} \text{diagonalisable} \\ \text{has real Eigenvalues} \\ \text{Eigenvectors are orthogonal to each other} \end{cases}$

    i.e. $dN_p(X_1) = -\lambda_1 X_1$, $dN_p(X_2) = -\lambda_2 X_2$ for some X_1, X_2



$$\begin{cases} A_p(X_1, X_1) = \lambda_1 \\ A_p(X_2, X_2) = \lambda_2 \\ A_p(X_1, X_2) = 0 \end{cases}$$

so X_1, X_2 are principal directions

λ_1, λ_2 are principal curvatures

Lemma 10.1 Relation of A_p, principal curvatures

If $\lambda_1(p) \le \lambda_2(p)$,

$$\lambda_1(p) = \min\{A_p(X,X) \mid X \in T_pS, |X|=1\}$$
$$\lambda_2(p) = \max\{A_p(X,X) \mid X \in T_pS, |X|=1\}$$

Proof write $X = c_1X_1 + c_2X_2$ $(c_1^2+c_2^2=1)$

$$A_p(X,X) = c_1^2 A_p(X_1,X_1) + c_2^2 A_p(X_2,X_2) = c_1^2\lambda_1 + c_2^2\lambda_2$$

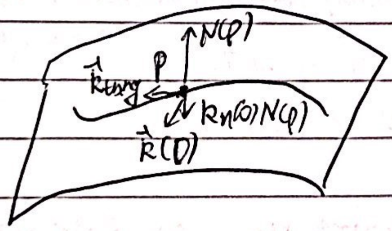
so $\lambda_1 \le A_p(X,X) \le \lambda_2$ bounds realised at X_1, X_2 $\square$

- sign of principle curvatures depends on orientation chosen
- $\lambda_1(p) = \lambda_2(p) = 0 \Rightarrow dN_p = 0 \Rightarrow N$ is constant $\Rightarrow S$ lies in a plane

Normal Curvature (in direction $c'(0)$) $k_n(p) := \langle \vec{k}(0), N(p)\rangle$ where $\vec{k}(0)$ is curvature of a path $c: (-\varepsilon, +\varepsilon) \to S$ $(c(0)=p)$

$$\vec{k}_{tang}(0) := \vec{k}(0) - k_n(p) N(p)$$

is curvature vector along tangent plane



- $k_n(p) = A_p(v,v)$ normal curvature in direction v

if $\begin{cases} \|v\|=1 \\ S \text{ orientable} \\ v \in T_pS \end{cases}$

Proof Suppose curve c satisfies $c(0)=p$, $c'(0)=v$.

differentiate $\langle c'(t), N(c(t))\rangle = 0$ w.r.t t at $t=0$ yields the result. $\square$

Relation between Principal curvature and normal curvature

- so for orientable S, $\forall p \in S$,

$\lambda_1(p), \lambda_2(p)$ are min, max of normal curvature at p along all directions in T_pS.

- If $\lambda_1 = \lambda_2$ at p, p is <u>umbilical</u>

$$S \begin{cases} \text{connected} \\ \text{regular} \\ \text{orientable} \\ \text{all points umbilical} \end{cases} \Rightarrow \begin{cases} S \text{ contained in a plane} \\ \text{or} \\ S \text{ contained in a sphere} \end{cases}$$

Proof. $\lambda_1(q) = \lambda_2(q) = \lambda(q) \ \forall q \in S$

So under basis $\{X_1, X_2\}$, $dN_q = -\lambda(q) id$

~~By smoothness of N, ϕ,~~

$$\frac{\partial^2}{\partial v \partial u}(N \circ \phi) = \frac{\partial^2}{\partial u \partial v}(N \circ \phi)$$

$$\frac{\partial}{\partial v}\left(dN_{\phi(\cdot)}\left(\frac{\partial \phi}{\partial u}\right)\right) \quad \frac{\partial}{\partial u}\left(dN_{\phi(\cdot)}\left(\frac{\partial \phi}{\partial v}\right)\right)$$

$$\Rightarrow \frac{\partial(\lambda \circ \phi)}{\partial v}\frac{\partial \phi}{\partial u} = \frac{\partial(\lambda \circ \phi)}{\partial u}\frac{\partial \phi}{\partial v} \Rightarrow \lambda \circ \phi \text{ is constant.}$$

by linear independence of $\{\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}\}$

So λ is constant on S

$\lambda \equiv 0 \Rightarrow S$ contained in a plane

$\lambda \equiv \lambda_0 \neq 0$: similar to above, one can show $\phi + \frac{1}{\lambda_0} N \circ \phi \equiv \vec{c}_0$

by $\frac{\partial}{\partial u}(\phi + \frac{1}{\lambda_0} N \circ \phi) = \frac{\partial}{\partial v}(\phi + \frac{1}{\lambda_0} N \circ \phi) = 0$

then $|\phi - \vec{c}_0| = \left|\frac{1}{\lambda_0}(N \circ \phi)\right| = \frac{1}{\lambda_0}$ i.e. $\phi(U)$ contained in sphere at $\vec{c}_0$ with radius $\frac{1}{\lambda_0}$ $\square$

does not depend on orientation

Variants from principal curvatures
$\left\{ \begin{array}{l} \underline{\text{Gassian curvature}} \quad K(p) := \lambda_1(p)\lambda_2(p) = \det(dN_p) \\ \underline{\text{Mean curvature}} \quad H(p) := \frac{\lambda_1(p)+\lambda_2(p)}{2} = -\frac{1}{2}\,tr(dN_p) \end{array} \right.$

- Local Shape and curvatures:

$K(p) > 0$: $\exists\, V \subset \mathbb{R}^3$ s.t. $\begin{cases} p \in V \\ S \cap V \text{ lies on the same side of tangent plane } p + T_pS \end{cases}$

$K(p) < 0$: $S \cap V$ lies on both sides of $p + T_pS$

~~Proof~~ pick chart ϕ s.t. $\phi(0,0) = p$ for convenience

– do Taylor expansion ~~at~~ for $\phi(u,v)$ at $\phi(0,0)$:

– $\langle \phi(u,v) - \phi(0,0), N(p) \rangle$

$$= \underbrace{\cdots}_{\text{first order terms canels}} = \frac{1}{2}\left\langle \frac{\partial^2\phi}{\partial u^2}(0,0)\,u^2 + 2\frac{\partial^2\phi}{\partial u\partial v}(0,0)\,uv + \frac{\partial^2\phi}{\partial v^2}(0,0)\,v^2,\; N(p) \right\rangle$$

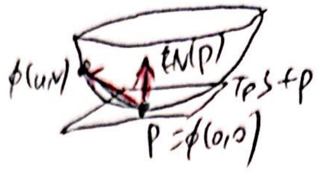
$$+ \langle R(u,v), N(p) \rangle \quad \text{where } \lim_{u,v\to 0} \frac{R(u,v)}{u^2+v^2} = 0$$

– By picking path $\gamma(t) = \phi(ct, dt)$ on S, differentiate $\langle \gamma'(t), N(\gamma(t)) \rangle$ for $t \in (-\varepsilon, \varepsilon)$

$$\Rightarrow A_{\phi(\gamma(t))}(\gamma'(t), \gamma'(t)) = \langle \gamma''(t), N(\gamma(t)) \rangle$$

$$(*) \quad \underset{\| }{} A_p\left(c\frac{\partial\phi}{\partial u}(0,0) + d\frac{\partial\phi}{\partial v}(0,0),\; c\frac{\partial\phi}{\partial u}(0,0) + d\frac{\partial\phi}{\partial v}(0,0) \right) = \left\langle c^2\frac{\partial^2\phi}{\partial u^2}(0,0) + 2cd\frac{\partial^2\phi}{\partial u\partial v}(0,0) + d^2\frac{\partial^2\phi}{\partial v^2}(0,0),\; N(p) \right\rangle$$

– so $\langle \phi(u,v) - p, N(p) \rangle = \frac{1}{2} A_p(w,w)$ where $w = u\frac{\partial\phi}{\partial u}(0,0) + v\frac{\partial\phi}{\partial v}(0,0)$



$K(p)>0$: wlog $\lambda_2(p)\ge\lambda_1(p)>0$, then

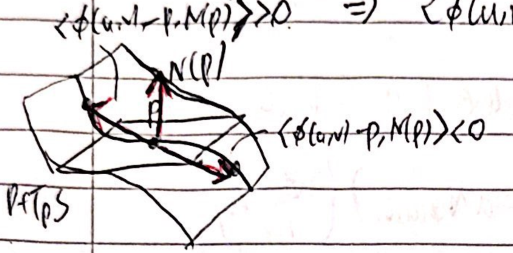
$A_p(w,w)\ge\lambda_1(p)|w|^2\ge0$

so $\langle\phi(u,v)-p,N(p)\rangle\ge0$ for small u,v

$K(p)<0$: $\lambda_1(p),\lambda_2(p)$ opposite sign

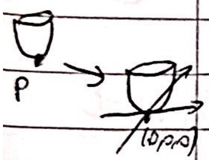
$\Rightarrow A_p(w,w)$ can take positive/negative value

$\Rightarrow \langle\phi(u,v)-p,N(p)\rangle$ can be positive/negative for small u,v.



$\square$

By transformation, make $\phi(0,0)=(0,0,0)$, $\frac{\partial\phi}{\partial u}(0,0)=(1,0,0)$,
$\frac{\partial\phi}{\partial v}(0,0)=(0,1,0)$



Taylor expansion of $\phi(u,v)$ at $(0,0,0)$ is

$$\phi(u,v)=(u,v,0)+\frac{1}{2}\left(\frac{\partial^2\phi}{\partial u^2}(0,0)u^2+2\frac{\partial^2\phi}{\partial u\partial v}(0,0)uv+\frac{\partial^2\phi}{\partial v^2}(0,0)v^2\right)+\ldots$$

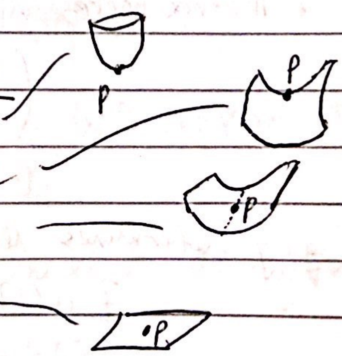
by (*), ~~(u,v,0)~~

$$\langle \text{second-order term}, N(0,0,0)\rangle = A_{(0,0,0)}\langle(u,v,0),(u,v,0)\rangle$$
$$=\lambda_1u^2+\lambda_2v^2 \quad \text{(as } (1,0,0),(0,1,0) \text{ are principal directions)}$$

$N(0,0,0)=(0,0,1)$

so $\phi(u,v)\simeq\left(u,v,\frac{1}{2}(\lambda_1u^2+\lambda_2v^2)\right)$ for small neighbourhood of $(0,0)$

$H(p)$	$K(p)$	λ_1,λ_2	Name
\	+	same sign	elliptic
\	−	opposite sign	hyperbolic
$\neq0$	$=0$	one zero	Parabolic
$=0$	$=0$	both zero	planar



- find curvatures given charts

$$K(\phi(u,v)) = \det(g^{-1}A) = \frac{\det(A)}{\det(g)}, \quad H(\phi(u,v)) = \frac{1}{2}\operatorname{tr}(g^{-1}A)$$

where $g := \begin{pmatrix} \langle \phi_u, \phi_u \rangle, & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle, & \langle \phi_v, \phi_v \rangle \end{pmatrix}$, $A := \begin{pmatrix} A_{\phi(u,v)}(\phi_u, \phi_u) & A_{\phi(u,v)}(\phi_u, \phi_v) \\ A_{\phi(u,v)}(\phi_v, \phi_u) & A_{\phi(u,v)}(\phi_v, \phi_v) \end{pmatrix}$

Proof note $A = \begin{pmatrix} -\phi_u- \\ -\phi_v- \end{pmatrix} (-dN_{\phi(u,v)}) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}$

using properties of det,

$$\det(A) = \lambda_1\lambda_2 \det(g) = K\det(g)$$

also $A = g \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}^{-1} (-dN_{\phi(u,v)}) \begin{pmatrix} | & | \\ \phi_u & \phi_v \\ | & | \end{pmatrix}$

So $-dN_{\phi(u,v)}$ is conjugate to $g^{-1}A$

$\Rightarrow \operatorname{tr}(g^{-1}A) = \operatorname{tr}(-dN_{\phi(u,v)}) = \lambda_1 + \lambda_2$ $\square$

- by finding partial derivatives of $\langle \frac{\partial\phi}{\partial u}, N \rangle = 0$ and $\langle \frac{\partial\phi}{\partial v}, N \rangle = 0$

$$A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix}$$

First Fundamental Form

$$g: T_pS \times T_pS \to \mathbb{R}$$
$$(v, w) \mapsto \langle v, w \rangle$$

- matrix representation under $\{\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\}$:

$$\begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix}$$

- g determines length: if $\alpha : [a,b] \to S$ is path on S, $\exists\ u(t), v(t)$ s.t. $\alpha(t) = \phi(u(t), v(t))$

then $l(\alpha([a,b])) = \int_a^b \sqrt{(u' \; v') \, g \begin{pmatrix} u' \\ v' \end{pmatrix}} \, dt$

local isometry $F: S_1 \to S_2$ s.t. $\forall p \in S_1$, $X, Y \in T_pS_1$,

$\langle dF_p(X), dF_p(Y) \rangle = \langle X, Y \rangle$

i.e. preserves first fundamental form

local isometry F must have bijective derivative dF_p.

Proof. $dF_p(v) = 0 \Rightarrow |v|^2 = \langle v, v \rangle = \langle dF_p(v), dF_p(v) \rangle = 0$

so injective

T_pS_1, $T_{F(p)}S_2$ same dimension dF_p is linear

so dF_p is bijective $\square$

Corollary: F is local diffeomorphism near p

local isometry iff preserves length of curves

i.e. $l(\alpha([a,b])) = l(F \circ \alpha([a,b]))$

Proof: $(\Rightarrow)$ Simply use $|(F \circ \alpha)'(t)| = |dF_{\alpha(t)}(\alpha'(t))|$

$= \sqrt{\langle dF_{\alpha(t)}(\alpha'(t)), dF_{\alpha(t)}(\alpha'(t)) \rangle}$

$(\Leftarrow)$ **Step 1.** show $|v| = |dF_p(v)|$ $\forall p \in S_1$, $v \in T_pS_1$

pick $\alpha: (-\varepsilon, \varepsilon) \to S_1$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

preserves length: $l(\alpha([-\delta, t])) = l(F \circ \alpha([-\delta, t]))$ $\forall \delta \in (0, \varepsilon)$

differentiate w.r.t. ~~t at~~ $t = 0$ yields

$|\alpha'(0)| = |(F \circ \alpha)'(0)|$

Step 2 as $\langle X+Y, X+Y \rangle = |X|^2 + 2\langle X, Y \rangle + |Y|^2$

$\langle X, Y \rangle = \dfrac{|X+Y|^2 - |X|^2 - |Y|^2}{2}$

so step 1 can be generalised to any pair of vectors X, Y $\square$

 Isometry: local isometry that is bijective

$\{\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, N\}$ forms a basis of $\mathbb{R}^3$

basis expansion $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \Gamma^1_{ij} \frac{\partial \phi}{\partial x_1} + \Gamma^2_{ij} \frac{\partial \phi}{\partial x_2} + A_{ij} N$ (*)

where Christoffel symbols

because $A = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix}$

- $\Gamma^k_{ij} = \Gamma^k_{ji}$ as ϕ is smooth
- Christoffel symbols are completely determined by g and its partial derivatives

Proof

Note $\frac{\partial}{\partial x_i}(g_{jk}) = \frac{\partial}{\partial x_i}\langle \frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial x_k} \rangle = \langle \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \frac{\partial \phi}{\partial x_k} \rangle + \langle \frac{\partial \phi}{\partial x_j}, \frac{\partial^2 \phi}{\partial x_i \partial x_k} \rangle$

use equation (*) to bring Christoffel symbols, then considering different combinations of i, j, k:

$$g \begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \frac{\partial g_{11}}{\partial x_1} \\ \frac{\partial g_{12}}{\partial x_1} - \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \end{pmatrix}$$

$$g \begin{pmatrix} \Gamma^1_{21} \\ \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \\ \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \end{pmatrix}$$

$$g \begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_{21}}{\partial x_2} - \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \\ + \frac{1}{2} \frac{\partial g_{22}}{\partial x_2} \end{pmatrix}$$

solving these systems give expression of $\Gamma^1_{ij}, \Gamma^2_{ij}$ $\square$

Corollary: local isometry preserves Christoffel symbols

$\boxed{\text{Theorema Egregium}}$ Gaussian curvature only depends on first fundamental form

Proof: $$\frac{\partial}{\partial x_2}\left(\frac{\partial^2\phi}{\partial x_1\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)$$ as ϕ is smooth

expanding in basis $\left\{\frac{\partial\phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, N\right\}$ gives

$$(*)\quad \Gamma^1_{11}\phi_{uv} + (\Gamma^1_{11})_v \phi_u + \Gamma^2_{11}\phi_{vv} + (\Gamma^2_{11})_v\phi_v + (A_{11})_v N + A_{11}N_v$$
$$= \Gamma^1_{12}\phi_{uu} + (\Gamma^1_{12})_u\phi_u + \Gamma^2_{12}\phi_{vu} + \Gamma^2_{12}\phi_v + (A_{12})_u N + A_{12}N_u$$

where $u = x_1$, $v = x_2$,

- second derivatives of ϕ can be expanded again
- $N_u =: dN(\phi_u) \in T_pS$ so $N_u = a\phi_u + b\phi_v$ for some $a,b\in\mathbb{R}$

  similarly, $N_v = c\phi_u + d\phi_v$
- $-A_{ij} = -A(\phi_{x_i},\phi_{x_j}) = \langle \phi_{x_i}, dN(\phi_{x_j})\rangle = \langle\phi_{x_i}, N_{x_j}\rangle$

  then substitute N_u, N_v gives expression of A_{ij} in terms of g.

  which is $-A = g\begin{pmatrix}a & c\\ b & d\end{pmatrix}$ so $\begin{pmatrix}a&c\\b&d\end{pmatrix} = -g^{-1}A$.

then comparing coefficients in (*), and use $K = \frac{\det A}{\det g}$ yields

$$\begin{cases} K g_{12} = \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} + (\Gamma^1_{12})_u - (\Gamma^1_{11})_v \\ K g_{11} = \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + (\Gamma^2_{11})_v - (\Gamma^2_{12})_u =: Y \end{cases}$$

then $K = \dfrac{Y g_{22} - X g_{12}}{\det(g)}$ X, Y depends on christoffel symbols which depends only on g $\square$

Corollary 15.3 : S_1, S_2 regular
Gaussian curvatures K_1, K_2

$F: S_1 \to S_2$ local isometry

$\Rightarrow \quad K_2 \circ F = K_1$

no local isometry between any two regular surfaces with different Gaussian curvatures

e.g. plane v.s. sphere

$\boxed{\text{Theorem 16.1}}$ surface S $\begin{cases} \text{compact} \\ \text{connected} \\ \text{regular} \end{cases}$, constant $K > 0$

$\Rightarrow$ S is sphere

Proof WLOG suppose $\lambda_1(x) \leq \lambda_2(x)$ are principal curvatures at $x \in S$

aim: show $\lambda_1(x) = \lambda_2(x)$ then proposition 10.6 concludes S is sphere

as compact surface cannot be contained in a plane

Compactness $\Rightarrow$ $\exists p \in S$ s.t. $\lambda_2(p) = \max_{x \in S} \lambda_2(x)$ $\quad \lambda_1\lambda_2 \equiv K > 0$

so $\lambda_1(p) = \min_{x \in S} \lambda_1(x)$

(rigid) transform surface s.t. $p = (0,0,0)$ and principal directions are $(1,0,0)$, $(0,1,0)$

then chart $\phi(u,v) = (u, v, F(u,v))$

where $F(u,v) = \dfrac{\lambda_1(p)u^2 + \lambda_2(p)v^2}{2} + O(|u|^2 + |v|^2)$ (*)

is an estimate of S near p

define $E_1(u,v) := \dfrac{\phi_u}{|\phi_u|}$, $E_2(u,v) := \dfrac{\phi_v}{|\phi_v|}$

$$\lambda_1(p) \le \lambda_1(\phi(0,t)) \overset{def}{=} \min_{\{X\in T_{\phi(0,t)},\,|X|=1\}} A_{\phi(0,t)}(X,X)$$

$$\le A_{\phi(0,t)}(E_1(0,t),E_1(0,t)) =: h_1(t)$$

similarly $\lambda_2(p) \ge A_{\phi(t,0)}(E_2(t,0),E_2(t,0)) =: h_2(t)$

so $t=0$ (point p) is local min for h_1, local max for h_2

$\Rightarrow$ $h_1''(0) - h_2''(0) \ge 0$.

using (*),

$$h_1''(0) - h_2''(0) = \underbrace{K(p)}_{>0}(\lambda_1(p) - \lambda_2(p))$$

so $\lambda_1(p) \ge \lambda_2(p)$

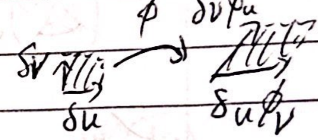
then $\lambda_1(p) = \lambda_2(p)$

$\Rightarrow$ $\lambda_1(p) \le \lambda_1(x) \le \lambda_2(x) \le \lambda_2(p)$

$\Rightarrow$ $\lambda_1(x) = \lambda_2(x)\ \forall x \in S$ □

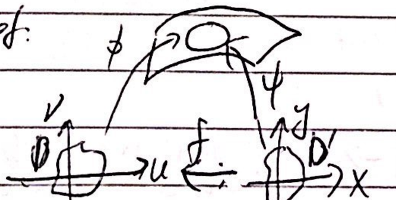
If (ϕ,U) is chart, $D\subset U$ is compact

Area of $\phi(D)$: area$(\phi(D)) := \int_D |\phi_u \times \phi_v|\,du\,dv$



└ well-defined: not depend on ϕ

Proof.



assume $\psi: U'\to S$, $\phi: U\to S$ are charts and $\phi(D) = \psi(D')$

define $f = \phi^{-1}\circ\psi$

write $f(u,v) = (x(u,v), y(u,v))$

$\psi = \phi\circ f$

$$\frac{\partial\psi}{\partial u}\times\frac{\partial\psi}{\partial v} \overset{\text{chain rule}}{=} \cdots = \left(\frac{\partial\phi}{\partial x}\times\frac{\partial\phi}{\partial y}\right)\underbrace{\det\begin{pmatrix} x_u & y_u \\ x_v & y_v\end{pmatrix}}_{\text{Jacobian}}$$

then us change of coordinates for integrals □

- area in terms of g. (first fundamental form)

$$\text{area}(\phi(D)) = \int_D \sqrt{\det(g)}\, du\, dv$$

Proof use $v \perp w - v \langle v,w\rangle / |v|^2$

and if $a \perp b$, $|a \times c| = |b \times c|$

prove $|v \times w|^2 = |v|^2 |w|^2 - \langle v,w\rangle^2$ $\square$

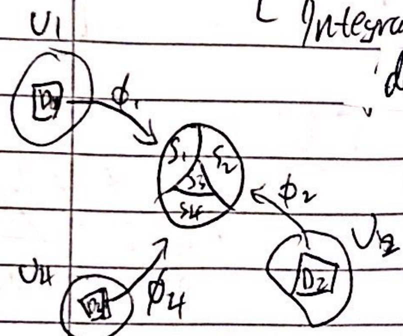
substitute. $v = \phi_u$, $v = \phi_v$ and integrate

- Integral of $f: S \to \mathbb{R}$ on $\phi(D)$:

$$\int_{\phi(D)} f\, dA := \int_D (f \circ \phi)(u,v)\, |\phi_u \times \phi_v|\, du\, dv$$

does not depend on ϕ

Integrate over whole surface S:



decompose S into $S = S_1 \cup \dots \cup S_k$

s.t. each $S_i = \phi_i(D_i)$ for chart ϕ_i, compact D_i

and ∂D_i has zero area

then $\int_S f\, dA := \sum_{i=1}^{k} \int_{S_i} f\, dA$

Geodesics

If $\gamma: [a,b] \to S$ is parametrised by arc-length

$\{\gamma', N \times \gamma', N\}$ is orthonormal basis

$\vec{k} = \gamma'' \perp \gamma'$ so

$$\vec{k} = \langle \vec{k}, N\rangle N + \langle \vec{k}, N \times \gamma'\rangle (N \times \gamma')$$

$$= k_N N + k_g (N \times \gamma')$$

normal curvature geodesic curvature

γ is <u>geodesic</u> if $k_g \equiv 0$

i.e. $\vec{k} = k_n N$, γ only curves in normal direction to S.

- geodesics on planes are straight lines
- equator is a geodesic for sphere
- local isometry maps geodesics to geodesics

<u>Proof</u> γ geodesic $\iff$ $\vec{k}(t)$ is multiple of $N(\gamma(t))$

$$\iff \langle \gamma'', \phi_u \rangle = \langle \gamma'', \phi_v \rangle = 0.$$

then write $\langle \gamma'', \phi_u \rangle$, $\langle \gamma'', \phi_v \rangle$ in terms of g.

... in fact, it is second order ODE of u, v.

local isometry preserves g, so it preserves $\langle \gamma'', \phi_u \rangle, \langle \gamma'', \phi_v \rangle$

then geodesics are preserved. $\square$

- if $\gamma : [0, L] \to S$ is { parametrised by arc-length
  { has shortest length among all curves from $\gamma(0)$ to $\gamma(L)$

$$\Rightarrow \gamma \text{ is geodesics}$$

<u>Proof</u>. <u>Variation</u> of γ:

smooth map $[0, L] \times [-\varepsilon, +\varepsilon] \to S$

$(t, s) \mapsto \gamma_s(t)$

where $\gamma_0 \equiv \gamma$, $\gamma_s(0) \equiv \gamma(0)$, $\gamma_s(L) = \gamma(L)$

For any variation γ_s of γ, as γ minimises length,

$$0 = \frac{d}{ds} l(\gamma_s) \Big|_{s=0} \overset{\text{def}}{=} \cdots = \frac{d}{ds} \int_0^L \sqrt{\langle \gamma_s'(t), \gamma_s'(t) \rangle}\, dt \Big|_{s=0}$$

$$\underset{\text{inside}}{\overset{\text{move derivative}}{=}} \cdots = \int_0^L \left\langle \frac{d}{dt} \left(\frac{d\gamma_s(t)}{ds} \right)_{s=0}, \frac{d\gamma_0(t)}{dt} \right\rangle dt$$

integration by part

$$= \cdots = -\int_0^L \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, \gamma''(t) \right\rangle dt$$

can expand into $k_n(t)\, N(\gamma(t)) + k_g(t)\,(N(\gamma(t)) \times \gamma'(t))$

$$= -\int_0^L k_g(t) \left\langle \frac{d\gamma_s(t)}{ds}\Big|_{s=0}, N(\gamma(t)) \times \gamma'(t) \right\rangle dt$$

if $k_g(t) \neq 0$ at any t, can vary γ in $N(\gamma(t)) \times \gamma'(t)$ direction at t to make integral non-zero.

so $k_g(t) \equiv 0$. □

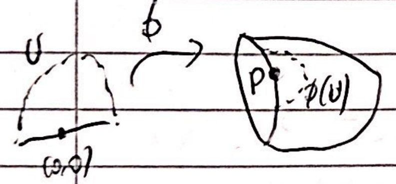
$\boxed{\text{Gauss-Bonnet}}$

Regular surface with boundary: $S \subset \mathbb{R}^3$ s.t.

$\forall p \in S$, either

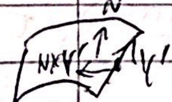
- $\exists$ chart $\phi: U \to S$ at p (interior point p)
- $\exists$ open $U \subset \mathbb{R}^2$, $(0,0) \in U$ and $\exists V \subset \mathbb{R}^3$, $p \in V$ and $\exists \phi: U \to V$ s.t.
  - $\phi(0,0) = p$
  - $\phi: \{(x,y) \in U \mid y \geq 0\} \to V \cap S$ is homeomorphism
  - $d\phi_q$ is injective $\forall q \in U$

  (boundary point p)



- Tangent space $T_pS := \{\alpha'(0) \mid \alpha: [0,\varepsilon) \to S$ or $\alpha: (-\varepsilon, 0] \to S$ is smooth w/ $\alpha(0) = p\}$
- boundary ∂S set of all boundary points
- unit normal: given parametrisation $\gamma: [a,b] \to \partial S$

  γ is positively oriented if $N \times \gamma'$ points into S



Local Gauss-Bonnet

S is $\begin{cases} \text{regular surface with boundary} \\ \text{has chart } \phi: U \to S \text{ s.t. } \begin{cases} \phi \text{ smooth on neighborhood of } \bar{U} \\ S = \phi(\bar{U}), \partial S = \phi(\partial U) \end{cases} \end{cases}$

$$\Rightarrow \int_{\partial S} k_g \, ds + \int_S K \, dA = 2\pi$$

if ∂S is positively oriented.

Proof

$\{E_1 := \frac{\phi_u}{|\phi_u|}, E_2 := N(\phi) \times E_1, N\}$ forms orthonormal basis

if $\gamma: [0,L] \to \partial S$ parametrises ∂S by arc-length with positive orientation, $\exists \, \sigma: [0,L] \to \partial U$ s.t. $\gamma = \phi \circ \sigma$ by conditions.

~~so $\exists \, \theta: [0,L] \to \mathbb{R}$ s.t.~~

$$\cancel{\gamma'(t) = \cos\theta(t) E_1(\sigma(t)) + \sin\theta(t) E_2(\sigma(t))}$$

- with help of ϕ, write the integrals into Euclidean space (U) then apply green's theorem

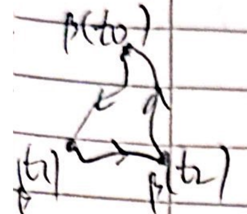
$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

- first and second fundamental forms always appear, use them to simplify equations $\square$

<u>Curvilinear Triangle</u> continuous $\beta: \mathbb{R} \to \mathbb{R}^2$ s.t.

$\begin{cases} \beta(t+3) = \beta(t) \\ \text{for } t_0, t_1, t_2 \text{ s.t. } t_0 < t_1 < t_2 < t_0 + 3 \text{ ;} \end{cases}$

$\left\{\begin{array}{l} \beta \text{ is regular curve on } (t_0, t_1), (t_1, t_2), (t_2, t_0+3) \\ \beta : [t_0, t_0+3] \to \mathbb{R}^2 \text{ injective (no self intersection)} \\ \beta'(t_i^-), \beta'(t_i^+) \text{ exists and} \end{array}\right.$



$\{\beta'(t_0^+), \beta'(t_3^-)\}, \{\beta'(t_1^+), \beta'(t_1^-)\}, \{\beta'(t_2^+), \beta'(t_2^-)\}$

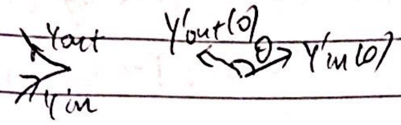
is linearly independent (three "edges")

if $\phi : U \to S$ is chart, $T \subset U$ is curvilinear triangle,

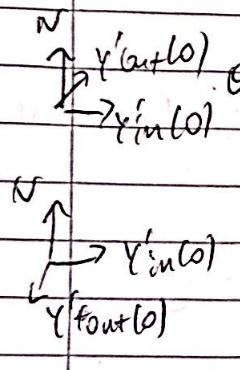
$\phi(T)$ is curvillnear triangle on S

∟ Adjacent edges of $\phi(T)$:

$\gamma_{in} : (-\varepsilon, 0] \to S \quad \gamma_{out} : [0, \varepsilon) \to S$



note $\cos(\theta) = \dfrac{\langle \gamma'_{in}(0), \gamma'_{out}(0) \rangle}{|\gamma'_{in}(0)|\,|\gamma'_{out}(0)|}$



$\left\{\begin{array}{l} \text{pick } \theta \in [0, \pi) \text{ if } \{\gamma'_{in}(0), \gamma'_{out}(0), N\} \text{ is positive basis} \\ \text{pick } \theta \in (-\pi, 0) \text{ if } \{\gamma'_{in}(0), \gamma'_{out}(0), N\} \text{ is negative basis} \end{array}\right.$

θ is exterior angle

$\pi - \theta$ is interior angle

Triangular Gauss-Bonnet

If S' is regular surface, S is curvilinear triangle on S'.

with edges γ_i meeting at exterior angles θ_i,

$$\Rightarrow \sum_{i=1}^{3} \int_{\gamma_i} k_g \, ds + \sum_{i=1}^{3} \theta_i + \int_S K \, dA = 2\pi$$

└ corollary: if edges γ_i are geodesics i.e. $k_g \equiv 0$,

$$\int_T K\,dA = \sum_{i=1}^{3} \alpha_i - \pi$$

where $\alpha_i = \pi - \theta_i$ are interior angles

Triangularisation of compact regular surface S:

Partition $\{T_i\}_{i=1,\dots,n}$ where T_i are curvilinear triangles

s.t. $S = \bigcup_{i=1}^{n} T_i$

If $T_i \cap T_j \neq \phi \Rightarrow T_i \cap T_j$ is either a vertex or edge

edges in interior: 2 triangles sharing

edges on ∂S: unique triangle

Euler characteristics:

$$\chi\left(\bigcup_{i=1}^{n} T_i\right) = V - E + F$$

V: number of distinct vertices; E: number of edges; F: number of triangles

every compact surface has triangularisation and $\chi\left(\bigcup_{i=1}^{n} T_i\right)$ does not depend on triangularisation

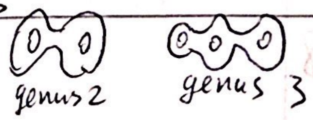
If S, S' are homeomorphic,

$$\chi(S) = \chi(S')$$

S compact, connected, orientable, no boundary $\Rightarrow S$ homeomorphic to some Σ_g (surface of genus g)

so $\chi(S) = 2-2g$

Surface of genus g: attach g torus



General Gauss-Bonnet

If S is $\begin{cases} \text{compact} \\ \text{regular} \\ \text{orientable} \\ \text{may have boundary,} \end{cases}$ ∂S is $\begin{cases} \text{parametrised by arc-length} \\ \text{positively oriented} \end{cases}$

$$\Rightarrow \int_{\partial S} k_g \, ds + \int_S K \, dA = 2\pi \chi(S)$$

Proof find triangularisation $\{T_i\}$

s.t. $\begin{cases} \text{for each } i, \ \exists \ \phi: U \to S \text{ s.t. } T_i \subset \phi(U) \\ \text{all triangles are curvilinear} \\ \text{edges } E_{ij} \quad \text{exterior edges } \theta_{ij} \\ \partial T_i \text{ positively oriented} \end{cases}$

apply Gauss - Bonnet for triangles, sum over all triangles

$$\sum_{i=1}^{n} \int_{T_i} K \, dA + \sum_{i=1}^{n} \int_{\partial T_i} k_g \, ds = \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{3} \alpha_{ij}}_{} - \sum_{i=1}^{n} \pi$$

gives the whole surface as $\bigcup_i T_i = S$

interior edges cancels

each interior vertices : 2π
boundary vertices π

$$= \int_S K \, dA + \int_{\partial S} k_g \, ds = 2\pi V - \pi(\#\text{ boundary vertices}) - \pi F$$

$3F = 2(\#\text{interior edges}) + \#\text{ boundary vertices}$
$= 2E - (\#\text{ boundary edges})$

so $\#$ boundary vertices $= 2E - 3F$

substituting yields the result $\square$

Corollary 1: if S is $\begin{cases} \text{compact} \\ \text{connected} \\ \text{regular} \\ \text{without boundary } (\partial S = \phi) \end{cases}$

has Gaussian curvature $K \geq 0 \Rightarrow S$ homeomorphic to sphere

Proof: S homeomorphic to some Σ_g (surface of genus g)

so $0 \leq \int_S K dS = 2\pi \chi(S) = 2\pi(2-2g)$

$\Rightarrow g = 0$ or 1

if $g=1$, $\int_S K dS = 0$ but $K \geq 0$

$\Rightarrow K \equiv 0$

but for compact surface, $K>0$ at some point,

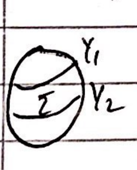
so $g=0$, S homeomorphic to Σ_0 (sphere) $\square$

Corollary 2: S is $\begin{cases} \text{compact} \\ \text{connected} \\ \text{regular} \\ \text{no boundary} \end{cases}$, $K>0$ on S

γ_1, γ_2 are $\begin{cases} \text{simple} \\ \text{closed} \\ \text{geodesics} \end{cases} \Rightarrow \gamma_1 \cap \gamma_2 \neq \phi$

Proof: $S \cong \Sigma_0$ (sphere)

Jordan curve theorem: any simple closed curve on sphere divides sphere into 2 connected components



$\Rightarrow$ if $\gamma_1 \cap \gamma_2 = \phi$, $\exists$ region Σ with $\partial \Sigma = \gamma_1 \cup \gamma_2$

By Gauss-Bonnet:

$$\underbrace{\int_{\partial \Sigma} k_g ds}_{=0} + \int_\Sigma K dA = 2\pi \chi(\Sigma)$$

$\int_\Sigma K dA > 0$

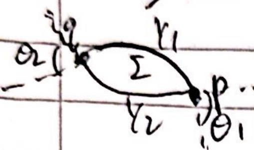
but $\Sigma \cong$ cylinder, $\chi(\Sigma) = 0$ contradiction $\square$

Corollary 3. If S is $\begin{cases} \text{regular} \\ \text{diffeomorphic to disk,} \\ K \leq 0 \end{cases}$

γ_1, γ_2 are geodesics w/ $\gamma_1(0) = \gamma_2(0) = p \in S$ $(\gamma_1 \not\equiv \gamma_2)$

$\Rightarrow \gamma_1 \cap \gamma_2 = \{p\}$ (geodesics meet at most at one point)

Proof. If $\exists$ another intersection q, say $\gamma_1(l) = \gamma_2(s) = q$



let Σ be region between $a := \gamma_1([0,l])$ and $b := \gamma_2([0,s])$

so $\partial\Sigma = a \cup b$

$\Sigma \cong$ disc $\Rightarrow$ $\chi(\Sigma) = 1$

Σ is triangle, say exterior angles are θ_1, θ_2

$$\int_{a \cup b} k_g \, ds + \theta_1 + \theta_2 + \int_\Sigma K \, dA = 2\pi$$

$= 0$ $\theta_1, \theta_2 < \pi$ as otherwise, $\gamma_1 = \gamma_2$

so $\int_\Sigma K \, dA > 0$

contradicts $K \leq 0$ $\square$