Network Science
Spring 2024
Problem Class 3 Solutions

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1. In Lecture 5, it is stated that:

$$\langle k_i \rangle = \sum_{G \in \Omega_N} P(G) k_i(G),$$

and

$$\langle k_i \rangle = \sum_{k=0}^{N-1} P(k_i = k) k.$$

Show that these two expressions are equivalent.
Solution: Using the law of total probability,

$$P(k_i = k) = \sum_{G \in \Omega_N} P(k_i = k, G),$$

and using the rule of conditional probability, $P(k_i = k, G) = P(k_i = k|G)P(G)$. We know that $P(k_i = k|G) = P(k_i(G) = k)$, and using these three results:

$$\sum_{k=0}^{N-1} P(k_i = k) k = \sum_{k=0}^{N-1} \sum_{G \in \Omega_N} P(k_i(G) = k) P(G) k.$$

Swapping the summations gives:

$$\sum_{k=0}^{N-1} P(k_i = k) k = \sum_{G \in \Omega_N} P(G) \sum_{k=0}^{N-1} P(k_i(G) = k) k.$$

Now, trivially, $P(k_i(G) = k) = 1$ if $k = k_i(G)$ and is zero otherwise, so

$$\sum_{k=0}^{N-1} P(k_i = k) k = \sum_{G \in \Omega_N} P(G) k_i(G).$$

2. Show that if we let $p(N) = N^{-z}$ with $z > 3/2$ then $G \in G_{N,p}$ w.h.p. has no two edges with a common vertex (or equivalently the degree at each node is at most one). In an exercise from the last problem sheet, the assumption was that $z > 2$. A different type of argument will be needed here. Hint: consider the random variable Y_{ijk} which assigns to a graph $G \in G_{Np}$ the value 1 if between the three nodes i, j, k there are two or more edges and 0 if there is at most one such edge.

Solution: $E(Y_{ijk}) = P(\text{ijk have two or more edges}) = 3p^2(1-p) + p^3 = p^2(3-2p)$ as can be seen by considering the possible edges between the nodes i, j, k. Next define $Y = \sum_{ijk} Y_{ijk}$ where the sum is taken over all distinct triples. It follows that $EY = \binom{N}{3} p^2(3-2p) \approx \frac{N^3}{2} p^2$ where $\approx$ as before means that the ratio of the left and the right hand side tends to one as $N \to \infty$. When $p = N^{-z}$ and $z > 3/2$ then $EY \to 0$ as $N \to \infty$. Hence $P(Y \geq 1) \leq EY \to 0$ as $N \to \infty$.