

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Advanced Topics in Partial Differential Equations

Date: Wednesday, May 21, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) (i) Define what it means for a function $g(x)$ to be a majorant of another function $f(x)$ in the sense of power series expansion. (4 marks)

- (ii) Let the function

$$f(x) := \frac{1}{2}(e^{x_1+\dots+x_d} - e^{-(x_1+\dots+x_d)}),$$

for all $x \in \mathbb{R}^d$. Using the corresponding Taylor series of f , construct a majorant function $g(x)$. Give the final expression for g in an analytic form. (2 marks)

- (b) Consider the wave equation in $(1+1)$ dimensions: $-\partial_t^2 u + \partial_x^2 u = 0$. Find all the characteristic surfaces. (7 marks)

- (c) Let the spatial dimension now to be 3, i.e.

$$-\partial_t^2 u + \Delta_x^2 u = 0, \quad x \in \mathbb{R}^3. \quad (1)$$

Let $u_0, u_1 : \mathbb{R}^3 \rightarrow \mathbb{R}$ be initial data assumed to be everywhere real analytic. By explicitly reforming the problem into a first order system, show that in a neighbourhood of $t = 0$, there exists a unique real analytic solution $u : \mathbb{R}^{1+3} \rightarrow \mathbb{R}$ to (1) so that $u|_{t=0} = u_0, u_t|_{t=0} = u_1$.

(7 marks)

State, without proof, all the theorems you are using.

(Total: 20 marks)

2. (a) Let $U = B_1(0) \subset \mathbb{R}^3$, i.e. $U = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < 1\}$. Find a function $u \in H^1(U)$ but $u \notin L^\infty(U)$. Justify your answer. (5 marks)

- (b) Let U be an open bounded subset of \mathbb{R}^d . Prove that if $u \in W^{k,p}(U)$, $\xi \in C^k(\bar{U})$, for some $k \in \mathbb{N}$ and $p \in [1, \infty]$, then $\xi u \in W^{k,p}(U)$. (7 marks)

- (c) Suppose U is open, bounded with Lipschitz boundary and let $u \in W^{k,p}(U)$ for $p \in [1, \infty)$. Prove that there exists a smooth sequence of functions $(u_n)_{n \geq 1} \in C^\infty(\bar{U})$, so that

$$u_n \rightarrow u \text{ in } W^{k,p}(U) \text{ as } n \rightarrow \infty.$$

[You may use without proof the smooth approximation of Sobolev functions globally away from the boundary ∂U . You may also use without proof that the translation operator is continuous in L^p , for $p \in [1, \infty)$.]

(8 marks)

(Total: 20 marks)

3. (a) Let $U \subset \mathbb{R}^2$ be a bounded open set, with C^1 boundary.
- (i) Show that if $(u_n)_{n \geq 1}$ is a sequence that converges to $u \in L^2(U)$ strongly, it also converges strongly in $L^1(U)$. (3 marks)
 - (ii) Let $(u_n)_{n \geq 1}$ be a bounded sequence in $H^1(U)$. Show that there exist a subsequence $(u_{n_k})_{k=1}^\infty$ and $u \in H^1(U)$ so that $\lim_k \|u_{n_k} - u\|_{L^p(U)} = 0$ for all $p \geq 1$.
- Hint: You may use the 2D continuous Sobolev embedding $H^1(U) \hookrightarrow L^q(U)$ for all $q \in [1, \infty)$ and the conclusion of the Rellich-Kondrachov Theorem.*
- (8 marks)
- (b) Let $U \subset \mathbb{R}^3$ be a bounded open set and fix $v_1, v_2 \in H^1(U)$. We define $T : H^1(U) \rightarrow \mathbb{R}$ so that $T(u) = \int_U (\partial_{x_1} u) v_1 v_2 dx$.
- (i) Show that the functional T is bounded and linear.
Hint: You may use without proof the inequality $\|u\|_{L^3(U)}^2 \leq \|u\|_{L^2(U)} \|u\|_{L^6(U)}$ and the Sobolev embedding $H^1(U) \hookrightarrow L^6(U)$. (3 marks)
 - (ii) Let $(u_n)_n$ be a bounded sequence in $H^1(U)$. Show that there exist a subsequence $(u_{n_k})_{k \geq 1}$ and $u \in H^1(U)$ so that

$$\lim_{k \rightarrow \infty} \int_U (\partial_{x_1} u_{n_k})(u_{n_k} v_2) dx = \int_U (\partial_{x_1} u)(uv_2) dx.$$

(6 marks)

Hint: Here, you may again use the inequality $\|u\|_{L^3(U)}^2 \leq \|u\|_{L^2(U)} \|u\|_{L^6(U)}$ and the conclusion of the Rellich-Kondrachov Theorem.

State, without proof, all the theorems you are using.

(Total: 20 marks)

4. (a) Let U be a bounded open subset of \mathbb{R}^d with C^2 boundary. Consider the operator

$$Lu := - \sum_{i,j=1}^d (a_{ij}(x) \partial_{x_i} u)_{x_j} + \sum_{i=1}^d b_i(x) u_{x_i}$$

where the coefficient matrix $(a_{ij}(x))$ is symmetric and satisfies the uniform ellipticity condition. Assume as well that the coefficients a_{ij}, b_i are bounded. Find a weak formulation for the Neumann boundary value problem:

$$\begin{cases} Lu = f, & \text{in } U, \\ \sum_{i,j=1}^d a_{ij}(x) \partial_{x_i} u \nu_j = 0, & \text{on } \partial U, \end{cases} \quad (2)$$

where ν is the outward unit normal to ∂U . In particular, show that a function $u \in C^2(\bar{U})$ is a classical solution of (2) if and only if it satisfies your weak formulation. (8 marks)

(b) Consider now the modified problem:

$$Lu - \lambda u = f \quad \text{in } U,$$

with the same homogeneous Neumann boundary condition as in (a). Show that for $\lambda > 0$ sufficiently large, the problem admits a unique weak solution. (8 marks)

- (c) (i) State the Poincaré inequality for functions in $H_0^1(U)$. When is the optimal constant attained and what does it correspond to? (2 marks)
- (ii) If $u \in H_0^1(U)$ is an eigenfunction of the Laplacian on a smooth bounded domain U , explain briefly why u is in fact smooth. (2 marks)

State, without proof, all the theorems you are using.

(Total: 20 marks)

5. Let $S_T := (0, T) \times \mathbb{R}^3$, $\Sigma_t := \{t\} \times \mathbb{R}^3$ the spatial domain at time t . We consider the nonlinear Initial Boundary Value Problem

$$\begin{cases} u_t - \Delta u = u^3, & \text{in } S_T, \\ u = \psi, & \text{on } \Sigma_0, \end{cases} \quad (3)$$

where $\psi \in H^1(\mathbb{R}^3)$. We consider the functional space $(X, \|\cdot\|_X)$ where

$$X = L^\infty([0, T]; H^1(\mathbb{R}^3))$$

equipped with the norm $\|u\|_X := \sup_{t \in (0, T)} [\|u\|_{H^1(\Sigma_t)}]$.

- (a) We consider the corresponding linear problem

$$\begin{cases} u_t - \Delta u = w^3, & \text{in } S_T, \\ u = \psi, & \text{on } \Sigma_0, \end{cases} \quad (4)$$

with $w \in X$, satisfying $\|w\|_X \leq A(\|\psi\|_{H^1(\mathbb{R}^3)}) := R$, for A a large constant. Assuming that u, w are smooth and decay sufficiently fast at infinity, show the estimates:

- (i) By multiplying (4) by u , show that for some constant $C > 0$ independent of T, u :

$$\sup_{t \in (0, T)} \|u\|_{L^2(\Sigma_t)}^2 - \|\psi\|_{L^2(\mathbb{R}^3)}^2 \leq CTR^3 [\sup_{t \in (0, T)} \|u\|_{L^2(\Sigma_t)}].$$

Hint: You may use the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. (2 marks)

- (ii) Conclude that for some constant $\tilde{C} > 0$:

$$\sup_{t \in (0, T)} \|u\|_{L^2(\Sigma_t)} \leq \tilde{C} \left(\frac{R}{A} + TR^3 \right). \quad (5)$$

(3 marks)

- (iii) Then, show that for some constant $\bar{C} > 0$:

$$\|u\|_X \leq \bar{C} \left(\frac{R}{A} + TR^3 \right). \quad (6)$$

Hint: You may use without proof the estimate from Parabolic Regularity: $\|Du\|_{L_t^\infty L_x^2} \leq C_0(\|w^3\|_{L^2(S_T)} + \|\psi\|_{H^1(\mathbb{R}^3)})$, for some constant $C_0 > 0$. (2 marks)

- (b) Let $\mathcal{F} : X \rightarrow X$ be the operator so that $w \mapsto u$, where u is the solution to (4). If $B_R := \{u : \|u\|_X \leq R\}$, show that for T sufficiently small, and A large, $\mathcal{F}(B_R) \subset B_R$. (5 marks)
- (c) (i) Let $u, v \in B_R$. Show that $\|\mathcal{F}u - \mathcal{F}v\|_X \leq CT\|u - v\|_X$, for some $C = C(R) > 0$, independent of T . (3 marks)
- (ii) Conclude that for $T < T_0 = T_0(\psi, \psi')$, (3) has a unique solution $u \in X$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

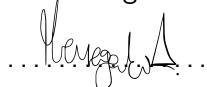
May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60021/MATH70021

Advanced Topics in PDEs (Solutions)

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1. (a) (i) Let $f = \sum_{\alpha} f_{\alpha} x^{\alpha}, g = \sum_{\alpha} g_{\alpha} x^{\alpha}$ be two formal power series. Then we say that g majorises f , or that g is a majorant function of f , and we write $g \gg f$, if $g_{\alpha} \geq |f_{\alpha}|$ for all multi-indices α .

seen ↓

- (ii) The power series of f is

$$\sum_{k \geq 0} \frac{(x_1 + \cdots + x_d)^{2k+1}}{(2k+1)!}$$

and we bound $|f(x)| \leq \sum_{k \geq 0} \frac{|x_1 + \cdots + x_d|^{2k+1}}{(2k+1)!} \leq e^{|x_1 + \cdots + x_d|}$. So a majorant is $g(x) = e^{|x_1 + \cdots + x_d|}$.

4, A

unseen ↓

- (b) Suppose we parameterise a characteristic curve by $\gamma : s \mapsto (t(s), x(s))$ where we suppose that $|\dot{\gamma}(t)| = 1$. The normal to this curve is $(-\dot{x}(s), \dot{t}(s))$. Now from the definition of characteristic surface, γ is a characteristic surface if and only if $-(\dot{x}(s))^2 + (\dot{t}(s))^2 = 0$. This means that γ is a characteristic curve iff $\dot{x}(s) = \dot{t}(s)$ or $\dot{x}(s) = -\dot{t}(s)$. Thus the characteristics are the curves $t - x =: \xi = \text{constant}$, $t + x =: \eta = \text{constant}$.

2, D

meth seen ↓

- (c) First we cast the problem into a system of first order equation. We do this since we would like to apply Cauchy-Kovalevskaya. Consider the vector $\mathbf{w} := (u, u_x, u_y, u_z, u_t) = (w_1, w_2, w_3, w_4, w_5)$. We then compute

$$\begin{aligned}\partial_t w_1 &= u_t = w_5, \quad \partial_t w_2 = u_{xt} = \partial_x w_5 \\ \partial_t w_3 &= u_{yt} = \partial_y w_5, \quad \partial_t w_4 = u_{zt} = \partial_z w_5 \\ \partial_t w_5 &= u_{tt} = u_{xx} + u_{yy} + u_{zz} = \partial_x w_2 + \partial_y w_3 + \partial_z w_4.\end{aligned}$$

7, C

sim. seen ↓

Then we bring our wave equation into a 1st order form as:

$$\partial_t \mathbf{w} = C_1 \partial_x \mathbf{w} + C_2 \partial_y \mathbf{w} + C_3 \partial_z \mathbf{w} + C \mathbf{w}$$

where C_i, C_0 are explicit 5×5 matrices. Regarding now the initial conditions, we write on $\{t = 0\}$, $\mathbf{w} = \mathbf{w}_0 = (u_0, \partial_x u_0, \partial_y u_0, \partial_z u_0, u_1)$. Then we consider $\tilde{\mathbf{w}} := \mathbf{w} - \mathbf{w}_0$ so that $\tilde{\mathbf{w}}$ satisfies

$$\partial_t \tilde{\mathbf{w}} = \sum_{i=1}^3 C_i \partial_i \tilde{\mathbf{w}} + C \tilde{\mathbf{w}} + \sum_{i=1}^3 C_i \partial_i \mathbf{w}_0 + C \mathbf{w}_0, \quad \text{with } \tilde{\mathbf{w}} = 0 \text{ on } \{t = 0\}. \quad (1)$$

where $\mathbf{w}_0(t, x, y, z) = \mathbf{w}_0(x, y, z)$. Now (1) is in the form where Cauchy-Kovalevskaya Theorem can be directly applied. This gives us the existence of a unique real-analytic solution in a neighborhood around $t = 0$ (since we have also assumed that the initial data u_0, u_1 are real-analytic).

7, B

2. (a) One example is

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$$u(x) = \begin{cases} |x|^{-\alpha}, & x \in U \setminus \{0\}, \\ \text{any value,} & x = 0, \end{cases} \quad \text{for } \alpha \in (0, \frac{1}{2}).$$

Then $u \notin L^\infty(U)$ since it blows up at $x = 0$, but we claim $u \in H^1(U)$.

For $x \neq 0$, the classical derivative is:

$$D_i u(x) = -\alpha \frac{x_i}{|x|^{\alpha+2}}.$$

We show this defines the weak derivative as well. To justify $u \in H^1(U)$, we use spherical coordinates: $u \in L^2(U) \iff 2\alpha < 3$ and $D_i u \in L^2(U) \iff 2(\alpha + 1) < 3 \iff \alpha < \frac{1}{2}$.

To confirm weak differentiability, take $\phi \in C_c^\infty(B_1(0))$ and write (with $\epsilon > 0$):

$$-\int_{U \setminus B_\epsilon(0)} u \partial_{x_i} \phi = \int_{U \setminus B_\epsilon(0)} D_i u \cdot \phi - \int_{\partial B_\epsilon} u \phi \nu_i dS_i.$$

The boundary term satisfies:

$$\left| \int_{\partial B_\epsilon} u \phi \nu_i dS \right| \leq C \epsilon^{2-\alpha} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \quad \text{if } \alpha < 2.$$

So the integration by parts formula holds, and $D_i u \in L^2(U)$ is the weak derivative.

Thus, $u \in H^1(U)$ if $\alpha \in (0, \frac{1}{2})$.

- (b) We proceed by induction on k . If $k = 0$, then clearly for $u \in L^p(U)$ we have $\xi u \in L^p(U)$, since

$$\int_U |\xi u|^p dx = \int_U |\xi|^p |u|^p dx \leq \left(\sup_{x \in U} |\xi|^p \right) \int_U |u|^p dx$$

and by assumption $\sup_{x \in U} |\xi|^p < \infty$. Suppose now that the result holds for some $k \geq 0$. It suffices to show that if $u \in W^{k+1,p}(U)$, then $D_i(\xi u) \in W^{k,p}(U)$ for all $i = 1, \dots, d$. Fix i and $\phi \in C_c^\infty(U)$. We write

$$\begin{aligned} -\int_U u \xi D_i \phi dx &= -\int_U u D_i(\xi \phi) dx + \int_U u (D_i \xi) \phi dx \\ &= \int_U D_i u \xi \phi dx + \int_U u (D_i \xi) \phi dx \\ &= \int_U (\xi D_i u + (D_i \xi) u) \phi dx. \end{aligned}$$

We used the Leibniz rule for smooth functions, and the definition of the weak derivative of u (since $\xi \phi \in C_c^\infty(U)$). Now since each term of the sum $(\xi D_i u + (D_i \xi) u)$ is a product of a $C^\infty(\bar{U})$ function with a $W^{k,p}(U)$ function, by the induction hypothesis, $(\xi D_i u + (D_i \xi) u) \in W^{k,p}(U)$, so we conclude.

- (c) This has been done identically in the lectures.

Fix $x_0 \in \partial U$. Since ∂U is $C^{0,1}$, there exists $r > 0$ and $\gamma \in C^{0,1}(\mathbb{R}^{d-1})$ so that

5, B

seen ↓

7, A

seen ↓

$$U \cap B_r(x_0) = \{(x', x_d) \in B_r(x_0) : x_d > \gamma(x')\}.$$

Set now $V := U \cap B_{r/2}(x_0)$. Then define the *shifted point* (so that we get an approximation in the interior):

$$x^\epsilon = x + \lambda\epsilon e_d, \quad x \in V, \quad \epsilon > 0.$$

Then for λ sufficiently large, since $\gamma \in C^{0,1}$, we have that $B_\epsilon(x^\epsilon) \subset U \cap B_r(x_0)$ for all ϵ small enough. Now define the translation of u :

$$u_\epsilon(x) := u(x^\epsilon), \quad x \in V.$$

Then mollify this: $v^{\epsilon, \tilde{\epsilon}} := \eta_{\tilde{\epsilon}} * u_\epsilon$ for $0 < \tilde{\epsilon} < \epsilon$, where η_ϵ is the standard mollifier. Then clearly $v^{\epsilon, \tilde{\epsilon}} \in C^\infty(\bar{V})$ and for any $\delta > 0$ we estimate:

$$\|v^{\epsilon, \tilde{\epsilon}} - u\|_{W^{k,p}(V)} \leq \|v^{\epsilon, \tilde{\epsilon}} - u_\epsilon\|_{W^{k,p}(V)} + \|u_\epsilon - u\|_{W^{k,p}(V)}.$$

Continuity of translation operator on L^p when $1 \leq p < \infty$ treats the second term: We choose ϵ small enough so that $\|u_\epsilon - u\|_{W^{k,p}(V)} \leq \frac{\delta}{2}$. Then, having fixed that ϵ , we choose $\tilde{\epsilon}$ small enough so that $\|v^{\epsilon, \tilde{\epsilon}} - u_\epsilon\|_{W^{k,p}(V)} \leq \frac{\delta}{2}$ (we can do this, since this is a smooth mollification). Thus, for $\epsilon, \tilde{\epsilon}$ small enough, we have $\|v^{\epsilon, \tilde{\epsilon}} - u\|_{W^{k,p}(V)} \leq \delta$.

Note now that these sets V for each $x_0 \in \partial U$, form an open cover and since ∂U is compact, we may cover it up with finitely many of them. Let $x_0^i \in \partial U$, radii $r_i > 0$, sets $V_i := U \cap B_{r_i/2}(x_0^i)$ and functions $v_i \in C^\infty(\bar{V}_i)$ for $i = 1, \dots, N$ so that

$$\partial U \subset \bigcup_{i=1}^N B_{r_i/2}(x_0^i) \text{ and } \|v_i - u\|_{W^{k,p}(V_i)} \leq \delta \quad \forall i.$$

To deal with the interior, we pick $V_0 \subset\subset U$, so that $U \subset \bigcup_{i=0}^N V_i$. By the Theorem of smooth approximation of Sobolev functions globally away from the boundary ∂U , we can find $v_0 \in C^\infty(\bar{V}_0)$ with $\|v_0 - u\|_{W^{k,p}(V_0)} \leq \delta$.

We finally consider a partition of unity $\{\xi_i\}_{i=0}^N$ subordinate to the cover $\{V_i\}_{i=0}^N$, and define $v := \sum_{i=0}^N \xi_i v_i$ which is in $C^\infty(\bar{U})$. This v will approximate our function. Indeed for a multi-index α with $|\alpha| \leq k$:

$$\|D^\alpha(v) - D^\alpha(u)\| = \|D^\alpha\left(\sum_{i=0}^N \xi_i v_i\right) - D^\alpha\left(\sum_{i=0}^N \xi_i u\right)\| \leq C_k \sum_{i=0}^N \|v_i - u\|_{W^{k,p}(V_i)} \leq C_k(1+N)\delta,$$

since $\sum_i \xi_i = 1$ and where $C_k = C_k(\|D^\alpha \xi_i\|_{L^\infty(U)})$. 8, A

3. (a) (i) This is a C-S inequality: $\|f\|_{L^1(U)} \leq |U|^{1/2} \|f\|_{L^2(U)}$. (Just a hint for (ii)).
(ii) First, from the hint, for any finite $q \geq 1$, the 2D Sobolev embedding yields the existence of a constant $C_q > 0$ with

$$\|f\|_{L^q(U)} \leq C_q \|f\|_{H^1(U)} \text{ for all } f \in H^1(U).$$

seen ↓
3, A
unseen ↓

Also, by weak compactness of $H^1(U)$, there exists a subsequence $(u_{n_k})_{k \geq 1}$ and $u \in H^1(U)$ so that $u_{n_k} \rightharpoonup u$ in H^1 and by Rellich-Kondrachov, $u_{n_k} \rightarrow u$ in L^2 . By (i) $u_{n_k} \rightarrow u$ in L^1 as well. Now by Hölder's:

$$\|u_{n_k} - u\|_{L^p(U)} \leq \|u_{n_k} - u\|_{L^1(U)}^{1/2p} \|u_{n_k} - u\|_{L^{2p-1}(U)}^{1-1/2p} \leq \|u_{n_k} - u\|_{L^1(U)}^{1/2p} C_p^{1-1/2p} \|u_{n_k} - u\|_{H^1(U)}^{1-1/2p},$$

which goes to 0 as $k \rightarrow \infty$ due to the L^1 convergence and as the $\|u_{n_k} - u\|_{H^1(U)}$ is bounded.

- (b) (i) Linearity is obvious. For the boundedness:

8, D
unseen ↓

$$\begin{aligned} |T(u)| &\leq \int_U |\partial_{x_1} u| |v_1| |v_2| \, dx \leq \|u\|_{H^1(U)} \left(\int |v_1|^2 |v_2|^2 \, dx \right)^{1/2} \\ &\leq \|u\|_{H^1(U)} \|v_1\|_{L^6(U)} \|v_2\|_{L^3(U)} \leq \|u\|_{H^1(U)} \|v_1\|_{H^1(U)} \|v_2\|_{L^3(U)} \\ &\leq \|u\|_{H^1(U)} \|v_1\|_{H^1(U)} \|v_2\|_{L^2(U)}^{1/2} \|v_2\|_{L^6(U)}^{1/2} \\ &\leq \|u\|_{H^1(U)} \|v_1\|_{H^1(U)} \|v_2\|_{H^1(U)} \end{aligned}$$

with a C-S in the 1st line, and then Hölder's with $p = 3/2$, $q = 3$. In the 4th inequality we used GNS Inequality which implies that $H^1(U) \subset L^6(U)$ (given in the hint!). While in the 5th Ineq. we applied the second hint to bound the L^3 norm. Then GNS again.

- (ii) Part (i) also acts as a hint for this one. First, by weak compactness of H^1 :
 $\exists u_{n_k}: u_{n_k} \rightharpoonup u$ in $H^1(U)$ and by Rellich-Kondrachov, $u_{n_k} \rightarrow u$ in $L^2(U)$.
The Hint in part (i), also implies $u_{n_k} \rightarrow u$ in $L^3(U)$. We estimate:

$$\begin{aligned} &\left| \int_U (\partial_{x_1} u_{n_k}) u_{n_k} v_2 \, dx - \int_U (\partial_{x_1} u) u v_2 \, dx \right| \leq \\ &\left| \int_U (\partial_{x_1} (u_{n_k} - u)) u v_2 \, dx \right| + \left| \int_U (\partial_{x_1} u_{n_k}) (u_{n_k} - u) v_2 \, dx \right| =: I_1 + I_2 \end{aligned}$$

3, C
unseen ↓

Now I_1 is going to 0, due to part (i), (T is linear, bounded and u_{n_k} is weakly converging in $H^1(U)$). I_2 as well since (Hölder's):

$$I_2 \leq \|Du_{n_k}\|_{L^2(U)} \|(u_{n_k} - u)v_2\|_{L^2(U)} \leq \tilde{C} \|u_{n_k} - u\|_{L^3(U)} \|v_2\|_{L^6(U)} \leq \tilde{C} \|u_{n_k} - u\|_{L^3(U)} \|v_2\|_{H^1(U)}$$

since $u_{n_k}, v_2 \in H^1(U)$, and due to the convergence in L^3 , we conclude.

6, D

4. (a) Let $u \in C^2(\bar{U})$ be a classical solution and $v \in C^2(\bar{U})$. Multiply the equation $Lu = f$, by v and integrating by parts the first term, we have

$$\int_U \left[\sum_{i,j=1}^d a_{ij} u_{x_i} v_{x_j} + \sum_i b_i u_{x_i} v \right] dx - \int_{\partial U} \sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} dS = \int_U f v dx.$$

Now due to the boundary conditions that u satisfies, we have that if $u \in C^2(\bar{U})$ is a classical solution, then for any $v \in C^2(\bar{U})$:

$$B[u, v] = (f, v)_{L^2(U)}, \text{ where } B[u, v] = \int_U \left[\sum_{i,j=1}^d a_{ij} u_{x_i} v_{x_j} + \sum_i b_i u_{x_i} v + (c + \lambda) u v \right] dx. \quad (2)$$

Now of course this expression makes sense as long as $u, v \in H^1(U)$. This shows that a classical solution is necessarily a weak solution.

For the other direction, assume that $u \in C^2(\bar{U})$, is a weak solution and take the test function $v \in C^2(\bar{U})$ in (2). Integrate by parts:

$$\int_U \left[- \sum_{i,j=1}^d (a_{ij} u_{x_i})_{x_j} + \sum_i b_i u_{x_i} + \lambda u \right] v dx - \int_{\partial U} \sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} dS = \int_U f v dx.$$

This can be rewritten as

$$\int_U (Lu - f) v dx + \int_{\partial U} \sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} dS = 0,$$

for all $v \in C^2(\bar{U})$. So in particular for all $v \in C_c^\infty(U)$, we have $\int_U (Lu - f) v dx = 0$ and so $Lu - f = 0$ in U . This means that

$$\int_{\partial U} \sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} dS = 0 \text{ for all } v \in C^2(\bar{U}).$$

We want to recover the boundary condition. Let $\chi \in C_c^\infty(B_1(0))$ so that $0 \leq \chi \leq 1$ and it has mass 1: $\int_{\mathbb{R}^d} \chi(x) dx = 1$. Assume (for contradiction) that there is $y \in \partial U$ so that $\sum_{i,j=1}^d a_{ij}(y) u_{x_i}(y) n_j(y) \neq 0$, say wlog that it is > 0 . By continuity there exists $\epsilon, \delta > 0$ so that

$$\sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} \geq \delta, \text{ in } B_\epsilon(y) \cap \partial U.$$

Define now $v(x) := \epsilon^{-d} \chi((x - y)/\epsilon)$ so that $\int_{\partial U} \sum_{i,j=1}^d a_{ij} u_{x_i} v n_{x_j} dS \geq \delta > 0$ which contradicts our initial hypothesis. So we may conclude that $\sum_{i,j=1}^d a_{ij} u_{x_i} n_j = 0$ on ∂U . Therefore, u is a strong solution.

8, A

- (b) This is an application of the 1st Existence Theorem (Lax-Milgram):

First we note that indeed $B_\lambda : H_1(U) \times H^1(U) \rightarrow \mathbb{R}$ is a bilinear form ($B_\lambda[\cdot, \cdot] := B[\cdot, \cdot] + \lambda(\cdot, \cdot)$). We will show that it is also bounded and coercive. By assumptions, there is a finite constant C : $\sup_{x \in U} |a_{ij}(x)|, \sup_{x \in U} |b_i(x)|, \sup_{x \in U} |c(x)| \leq C$. We then estimate

$$\left| \int_U \sum_{i,j=1}^d a_{ij} u_{x_i} v_{x_j} dx \right| \leq \int_U \left| \sum_{i,j=1}^d a_{ij} u_{x_i} v_{x_j} \right| dx \leq C d^2 \int_U |Du||Dv| dx \leq C d^2 \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)}.$$

meth seen ↓

Similarly

$$\left| \int_U \sum_{i=1}^d b_i u_{x_i} v dx \right| \leq C d \|Du\|_{L^2(U)} \|v\|_{L^2(U)}, \quad \left| \int_U \lambda u v dx \right| \leq \lambda \|u\|_{L^2(U)} \|v\|_{L^2(U)}.$$

Altogether:

$$|B_\lambda[u, v]| \leq \tilde{C} \|u\|_{H^1(U)} \|v\|_{H^1(U)}$$

for a constant $\tilde{C} = \tilde{C}(d, C, \lambda)$. For coercivity, we write first due to uniform ellipticity: For any $x \in U$, $\xi \in \mathbb{R}^d$:

$$\sum_{i,j=1}^d a_{ij} \xi_i \xi_j \geq \theta |\xi|^2, \text{ for some } \theta > 0.$$

We apply itt for $\xi = Du$ and we have (after also integrating over U):

$$\int_U (\theta |Du|^2 + \lambda u^2) dx \leq \int_U \left[\sum_{i,j=1}^d a_{ij} u_{x_i} u_{x_j} + \lambda u^2 \right] dx = B_\lambda[u, u] - \int_U \sum_{i=1}^d b_i u_{x_i} u dx. \quad (3)$$

We estimate as before

$$\left| \int_U \sum_{i=1}^d b_i u_{x_i} u dx \right| \leq C d \|Du\|_{L^2(U)} \|u\|_{L^2(U)} \leq \frac{\theta}{2} \|Du\|_{L^2(U)}^2 + \frac{C^2 d^2}{2\theta} \|u\|_{L^2(U)}^2.$$

In the last step we applied Young's Inequality. We conclude that

$$\theta \|Du\|_{L^2(U)}^2 + \lambda \|u\|_{L^2(U)}^2 \leq B_\lambda[u, u] + \frac{\theta}{2} \|Du\|_{L^2(U)}^2 + C' \|u\|_{L^2(U)}^2,$$

for some $C' = C'(d, C, \theta)$, independent of λ . Re-arranging we have:

$$\frac{\theta}{2} \|Du\|_{L^2(U)}^2 + (\lambda - C') \|u\|_{L^2(U)}^2 \leq B_\lambda[u, u]$$

so that for $\lambda > C'$, we have $B_\lambda[u, u]$ is coercive. Now since for all $f \in L^2(U)$, the mapping $v \mapsto (f, v)_{L^2(U)}$ is a bounded linear functional on $H^1(U)$, we can apply Lax-Milgram to deduce that for $\lambda > C'$, $\exists! u \in H^1(U)$ s.t. $B_\lambda[u, u] = (f, u)_{L^2(U)}$ for all $v \in H^1(U)$.

- (c) (i) The Poincaré inequality states that there exists a constant $C > 0$ such that for all $u \in H_0^1(U)$,

$$\|u\|_{L^2(U)}^2 \leq C \|\nabla u\|_{L^2(U)}^2.$$

8, B

seen \downarrow

The optimal constant is $C = \lambda_1^{-1}$, where λ_1 is the smallest positive eigenvalue of the negative Laplacian $-\Delta$ with Dirichlet b.c..

This constant is attained when u is an eigenfunction corresponding to e-value λ_1 , i.e., when $-\Delta u = \lambda_1 u$ in U , $u = 0$ on ∂U .

2, A

meth seen \downarrow

- (ii) Let $u \in H_0^1(U)$ satisfy the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } U, \quad u = 0 \quad \text{on } \partial U.$$

Since $\lambda u \in L^2(U)$, standard elliptic theory implies that $u \in H^2(U)$. Applying the equation again gives $\lambda u \in H^2(U)$, so $u \in H^4(U)$, and this process can be continued inductively. So, $u \in H^{2k}(U)$ for all k . By the Sobolev embedding theorem, this implies indeed that $u \in C^\infty(U)$.

2, C

5. (a) (i) Multiply by u the linear PDE and re-write

meth seen ↓

$$\left(\frac{1}{2} \frac{d}{dt} u^2 - \operatorname{div}(uD u) + |Du|^2 \right) = w^3 u.$$

Integrate over space-time to get (note that the div term vanishes)

$$\begin{aligned} \frac{1}{2} \|u\|_{L^2(\Sigma_t)}^2 - \frac{1}{2} \|\psi\|_{L^2(\mathbb{R}^3)}^2 + \|Du\|_{L^2((0,t);L_x^2)}^2 &= \int_0^t (w^3, u)_{L^2(\Sigma_\tau)} d\tau \\ &\leq T \sup_{\tau \in [0,T]} |(w^3, u)_{L^2(\Sigma_\tau)}| \\ &\leq T \sup_{\tau \in [0,T]} (\|w^3\|_{L^2(\Sigma_\tau)} \|u\|_{L^2(\Sigma_\tau)}) \\ &= T \sup_{\tau \in [0,T]} (\|w\|_{L^6(\Sigma_\tau)}^3 \|u\|_{L^2(\Sigma_\tau)}) \\ &\leq T C' \sup_{\tau \in [0,T]} (\|w\|_{H^1(\Sigma_\tau)}^3 \|u\|_{L^2(\Sigma_\tau)}) \\ &\leq T C' R^3 \sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)} \end{aligned} \tag{4}$$

where we applied C-S in the 2nd line, the Sobolev embedding in the 5th line and then that $\|w\|_X \leq R$. Taking the sup over time and disregarding the positive term $\|Du\|_{L^2((0,t);L_x^2)}^2$ on the LHS, we eventually have

$$\sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_t)}^2 \leq \|\psi\|_{L^2(\mathbb{R}^3)}^2 + T C_1 R^3 \sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)}, \tag{5}$$

as asked.

2, D

- (ii) [This part is unseen but it is not complicated given (i) - an application of Young's Inequality.]

unseen ↓

We continue by (i) and we apply Young's Inequality (with ε) in the RHS. That yields the bound

$$T C' R^3 \left[\sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)} \right] \leq \frac{(C_1)^2 T^2 R^6}{4\varepsilon} + \varepsilon \sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)}^2. \tag{6}$$

Next we re-arrange our terms in (5) to see that

$$\sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)}^2 \leq C_2 [\|\psi\|_{L^2(\mathbb{R}^3)}^2 + T^2 R^6].$$

for some constant $C_2 > 0$. Then using the inequalities $a^2 + b^2 \leq (a+b)^2 \leq 2(a^2 + b^2)$, we conclude that

$$\sup_{\tau \in [0,T]} \|u\|_{L^2(\Sigma_\tau)} \leq C_3 [\|\psi\|_{L^2(\mathbb{R}^3)} + T R^3] \leq C_4 [R/A + T R^3].$$

3, C

- (iii) By the estimates in the lectures in the Heat Equation-case (given also in the hint) we have, $\|u_t\|_{L^2(S_T)} + \|Du\|_{L_t^\infty L_x^2} \leq C_0 (\|w^3\|_{L^2(S_T)} + \|\psi\|_{H^1(\mathbb{R}^3)})$, for some constant $C_0 > 0$.

unseen ↓

From the proof of (i) we bounded $\|w^3\|_{L^2(S_T)} \leq C' T R^3$. Using then this we have:

$$\|Du\|_{L_t^\infty L_x^2} \leq C_6 (T R^3 + \|\psi\|_{H^1(\mathbb{R}^3)}) = C_6 (T R^3 + R/A).$$

Finally from the definition of the norm $\|\cdot\|_X$:

$$\begin{aligned}\|u\|_X &= \sup_{t \in (0, T)} (\|u\|_{L^2(\Sigma_t)} + \|Du\|_{L^2(\Sigma_t)}) \\ &\leq C_4[R/A + TR^3] + C_6(TR^3 + R/A) \\ &\leq C_7[R/A + TR^3].\end{aligned}$$

for some constants $C_5, C_6, C_7 > 0$.

- (b) This follows from (a): We notice that as $T \rightarrow 0$, the right-hand side tends to a quantity less than $R/2$. Thus for T small enough, $\|u\|_X \leq R$.

- (c) (i) For the contraction property: Let $u, v \in B_R$ and define $\tilde{u} := \mathcal{F}u - \mathcal{F}v$ which solves:

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = u^3 - v^3, & \text{in } S_T \\ \tilde{u} = 0, & \text{on } \Sigma_0. \end{cases}$$

We estimate as in part (a)(i) to find that

$$\|\tilde{u}\|_X \leq CT \sup_{t \in (0, T)} \|u^3 - v^3\|_{L^2(\Sigma_t)}$$

(note that the initial data now is 0). So

$$\|\mathcal{F}u - \mathcal{F}v\|_X \leq CT \sup_{t \in (0, T)} \|u^3 - v^3\|_{L^2(\Sigma_t)}$$

To estimate the RHS: we note that for $a, b > 0$: $|a^3 - b^3| = |(a - b)(a^2 + ab + b^2)| \leq 2|a - b|(a^2 + b^2)^2$. Thus

$$\begin{aligned}\|u^3 - v^3\|_{L^2(\Sigma_t)}^2 &\leq 2 \int_{\Sigma_t} (u - v)^2 (u^2 + v^2)^2 dx \\ &\leq 2 \left(\int_{\Sigma_t} (u - v)^6 dx \right)^{1/3} \left(\int_{\Sigma_t} (u^2 + v^2)^3 dx \right)^{2/3} = 2\|u - v\|_{L^6(\Sigma_t)}^2 \|(u^2 + v^2)^{1/2}\|_{L^6(\Sigma_t)}^4 \\ &\leq C\|u - v\|_{L^6(\Sigma_t)}^2 [\|u\|_{L^6(\Sigma_t)}^4 + \|v\|_{L^6(\Sigma_t)}^4] \\ &\leq C\|u - v\|_{H^1(\Sigma_t)}^2 [\|u\|_{H^1(\Sigma_t)}^4 + \|v\|_{H^1(\Sigma_t)}^4]\end{aligned}$$

where to the 2nd line we applied Hölders (1/3, 2/3), then the Sobolev embedding (GNS), and also triangle ineq. Taking the $\sup_{t \in (0, T)}$:

$$\sup_{t \in (0, T)} \|u^3 - v^3\|_{L^2(\Sigma_t)}^2 \leq C_R \|u - v\|_{H^1(\Sigma_t)}^2.$$

Combined with the estimates above, we conclude.

- (ii) For T sufficiently small, the mapping \mathcal{F} is a contraction. Indeed we can have from (c)(i) that

$$\|\mathcal{F}u - \mathcal{F}v\|_X \leq \frac{1}{2} \|u - v\|_X.$$

Together with (b), employing the contraction mapping principle we see that \mathcal{F} has a unique fixed point. This fixed point solves our original nonlinear problem.

2, C

unseen ↓

5, B

unseen ↓

3, D

meth seen ↓

5, C

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 25 of 20 marks

Total C marks: 22 of 12 marks

Total D marks: 21 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 0 of 20 marks

MATH70021 Advanced Topics in Partial Differential Equations Markers Comments

- Question 1 There was a typo in Q1(c): it was written Δ^2 instead of Δ . Both interpretations were accepted and awarded full credit accordingly. The underlying idea was the same in either case.
- Question 2 In Q2: This part was mostly made of standard material that the students had seen in the lectures, just slightly modified. Most students handled it well.
- Question 3 In Q3: This question required the application of the Rellich-Kondrachov Theorem. It involved mostly unseen material, but students were guided by the provided hints and the structure of the previous questions. In this part the average mark was lower, but it was clear that most students made a genuine attempt. Thus, I rewarded partial credit even though they might not have reached the final conclusion.
- Question 4 In Q4: Half of this question was closely related to seen material, particularly existence theorems for elliptic PDEs. The (c)(ii) part was unseen. Part (c)(ii) was unseen, requiring use of elliptic regularity. Most students handled it correctly.
- Question 5 This was a hard question on solving a nonlinear parabolic equation through a fixed point argument. The specific setting was new, but the students had seen similar arguments applied to both nonlinear elliptic and nonlinear wave equations (hyperbolic case). Many students wrote the main ideas and steps clearly. Around five students provided full technical details on the required estimates.