Network Science
Spring 2024
Problem sheet 9

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1. In problem sheet 8, you were asked to derive the network-SIS model. Now, use the degree-based approximation to derive a nonlinear ODE for ϕ_k, the probability that a node with degree k is infectious, for this model.

2. In this exercise, you will analyze an improved pair-approximation for the network-SI model (on simple connected graphs).

   (a) The pair approximation assumes that:

   $$P(s_i = 1, s_j = 1, x_l = 1) \approx P(x_l = 1, s_j = 1)P(s_i = 1, s_j = 1)/P(s_j = 1),\ A_{ij} = A_{jl} = 1.$$

   Assess this approximation when $l = i$

   (b) Now carry out the analogous analysis for the approximation applied to the other 3rd moment:

   $$P(s_i = 1, x_j = 1, x_l = 1) \approx P(s_i = 1, x_l = 1)P(s_i = 1, x_j = 1)/P(s_i = 1),\ A_{ij} = A_{il} = 1.$$

   Assess this approximation when $l = j$.

   (c) Based on your conclusions from (a) and (b), explain why the following equation represents an "improved pair approximation:"
   $\mathrm{d}\langle s_i x_j\rangle / dt = \beta\{\sum_{l=1}^{N} [A_{jl} \langle x_l s_j\rangle \langle s_i s_j\rangle / \langle s_j\rangle - A_{il} \langle s_i x_l\rangle \langle s_i x_j\rangle / \langle s_i\rangle]$
   $- \langle x_i s_j\rangle \langle s_i s_j\rangle / \langle s_j\rangle - \langle s_i x_j\rangle [1 - \langle s_i x_j\rangle / \langle s_i\rangle]\}$

3. (Barabasi 9.9.2) Consider a one-dimensional lattice with N nodes that form a circle where each node connects to its two neighbors. Partition the graph into n_c clusters of $N_c = N/n_c$ nodes where a cluster contains a "chain" of adjacent nodes, and N is an integer multiple of n_c.

   (a) Find the modularity of the partition

   (b) Determine how to choose n_c to maximize the modularity

4. Show that 1 is an upper bound for the modularity of a network (as defined in lecture 14)

5. In this exercise, you will derive an important property of the Laplacian matrix for simple graphs. You will show that if two nodes are in the same connected component of the graph, then the corresponding elements of an eigenvector corresponding to a zero eigenvalue must be the same.

(a) First consider the Laplacian matrix in block-diagonal form with each block corresponding to a connected component. Construct $\mathbf{v} \in \mathcal{R}^N$ as follows. Choose one connected component. For all nodes i in this component set $v_i = 1$, and set all other elements in $\mathbf{v}$ to be zero. Show that this vector is an eigenvector of $\mathbf{L}$ which corresponds to a zero eigenvalue.

(b) Given an arbitrary vector $\mathbf{x} \in \mathbb{R}^N$ and the Laplacian matrix for a simple undirected graph, show that the i-th element of $\mathbf{y} = \mathbf{L}\mathbf{x}$ is given by, $y_i = \sum_{j \in \mathbb{N}_i} (x_i - x_j)$ where $\mathbb{N}_i$ is the set of nodes which are neighbors of i

(c) Now show that if $a = \mathbf{x}^T \mathbf{L} \mathbf{x}$, then $a = 1/2 \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} (x_i - x_j)^2$. Use this result to argue that an eigenvector $\mathbf{v}$ corresponding to a zero eigenvalue for $\mathbf{L}$ must have the following property: for any 2 nodes i and j in the same connected component, $v_i = v_j$. Note that v_i is the *ith* element in $\mathbf{v}$ and not the *ith* eigenvector of $\mathbf{L}$.

6. Consider an adjacency matrix $\mathbf{A}$ corresponding to a simple graph.

(a) Construct a transformation of the adjacency matrix, $\mathbf{A} \rightarrow \mathbf{A}'$, which corresponds to a renumbering of the nodes in the graph for $\mathbf{A}$ while preserving the graph structure. I.e. if the the re-numbering results in $(i, j) \rightarrow (i', j')$ then $A'_{i'j'} = A_{ij}$

(b) Show that the eigenvalues of $\mathbf{A}$ and $\mathbf{A}'$ are the same