Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 18
Rearrangements

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Before, we discuss rearrangements, one theorem we didn't get to last week:

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Set $\widetilde{r} := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \widetilde{r}^{\,n}$.

Since $\sum_{n=N}^{\infty} \tilde{r}^n$ is convergent (geometric series), by comparison so is $\sum_{n=N}^{\infty} |a_n|$.

Therefore $\sum a_n$ is absolutely convergent.

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Example 4.28

$\sum(-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots$

Exercise 4.30

Let $a_n = \frac{(-1)^{n+1}}{n}$, then $\sum_{n=1}^{\infty} a_n > \frac{1}{2}$

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Example 4.29

We know that $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent by the Alternating Series Test. Show that one can get different values for the series $\sum a_n$ by rearranging terms.

$$1 \quad 1/3 \quad 1/5 \quad 1/7$$
$$-1/2 \quad -1/4 \quad -1/6$$

$$= 1 + 1/3 + 1/5 + 1/7$$
$$-1/2 \left[1 + 1/2 + 1/3 + \cdot \quad \right]$$
$$= \text{"} \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{"}$$

"Exercise"

There is a rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ that converges to $\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

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1. Take only odd terms $a_{2n+1} > 0$ until their sum is $> \pi$.
2. Now take only even terms $a_{2n} < 0$ until sum gets $< \pi$.
3. Repeat 1 and 2 to fade.

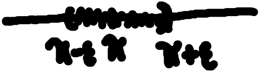
The s_n you build converges to π

Let $\varepsilon > 0$. Then $\exists N$ s.t. $\forall n \geq N$ $\left|\frac{(-1)^{n+1}}{n}\right| < \varepsilon$

For any k big enough (and $\{a_1, \ldots, a_N\}$ are in s_k)

Then $|s_k - \pi| < \varepsilon$

and you cross afterwards



Definition - Rearrangement of a sequence

Given a bijection $n\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$, define $b_i := a_{n(i)}$. Then $(b_i)_{i\geq 1}$ is a *rearrangement* or *reordering* of $(a_n)_{n\geq 1}$.

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Then the method of Example 4.31 shows that if (a_n) is any sequence such that

- $a_n \to 0$,
- $\displaystyle\sum_{n\colon a_n \geq 0} a_n \to +\infty$,
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Exercise 4.32

Show that, under the three conditions above, it is also possible to make the sum diverges to $+\infty$ and to $-\infty$.

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These two conditions combined: imply $a_n \to 0$ and are also equivalent to $\sum_n a_n$ being absolutely convergent. **For absolutely convergent series, any reordering gives the same sum.**

Theorem 4.34

$\sum a_n$ is absolutely convergent $\iff$ (1) + (2) $\Rightarrow$ (3) + (4), where

(1) $\sum_{a_n \geq 0} a_n$ is convergent (to A say),

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Similarly the partial sums of the n_i are monotonically decreasing, bounded below and so convergent (to B say), proving (2).

$$-R \leq -\sum_{i=1}^{N} |a_i| \leq \sum_{i=1}^{n} n_i$$

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We have shown that absolute convergence of $\sum a_n \Rightarrow (1) + (2)$.