M4/M5
A41

Imperial College London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Physics II: Statistical Mechanics

Date: Monday, 18 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

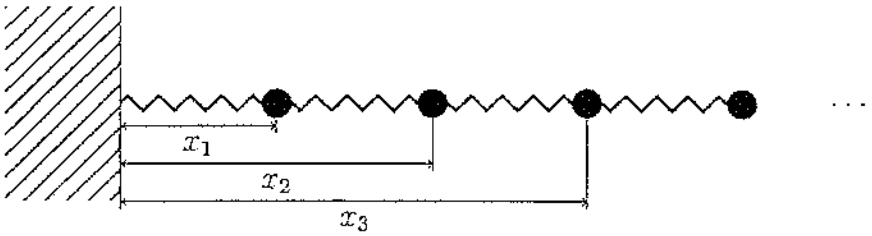
Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. In the following we consider a system of N springs and masses like the following arrangement:



Its Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N} \left(\frac{1}{2} k (x_i - x_{i-1} - a)^2 + \frac{1}{2} m \dot{x}_i^2 \right) ,$$

where k is the spring constant, a its resting length, $x_i \in \mathbb{R}$ the position of the ith mass, with $x_0 = 0$, m the mass of the masses, and $\dot{x}_i$ their velocity. For simplicity, all variables are assumed to be dimensionless in the following.

(a) Show that the free energy of the system is

$$F = -\frac{N}{\beta} \ln \left(\frac{2\pi}{\beta\sqrt{km}} \right) \tag{1}$$

where β is the inverse temperature $\beta = (k_B T)^{-1}$.

Hint: Consider $x_i - x_{i-1}$ and $\dot{x}_i$ as the independent degrees of freedom. Ignore any kind of geometrical constraint, such as $x_i \geq x_{i-1}$. Do not attempt to correct any dimensional inconsistencies.

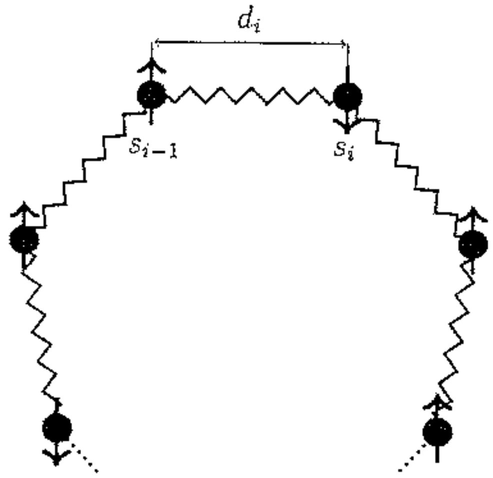
(b) Calculate the (expected) internal energy.

(c) Calculate the expected distance $\langle x_i - x_{i-1} \rangle$.

Hint: Rewrite the Hamiltonian as $\mathcal{H} = \sum_{i=1}^{N} ((1/2)k(x_i - x_{i-1} - a_i)^2 + (1/2)m\dot{x}_i^2)$ and set $a_i = a$ after differentiation.

(d) Calculate the expected variance of the distance $x_i - x_{i-1}$.

2. In the following we consider a system of N springs and spins like the following arrangement:



Its Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N} (1 - J s_{i-1} s_i)(d_i - a)^2$$

where $-1 \leq J \leq 1$ is the Ising coupling, $i = 0, \ldots, N$ is the index of the spin, $s_i \in \{-1, 1\}$ indicates the orientation of the spin, $d_i \in \mathbb{R}$ is their distance and a the resting length of the spring. The relative orientations of the spins thus determine the effective spring constants between spins. Periodic boundary conditions apply, such that $s_0 \equiv s_N$.

(a) Using the method of transfer matrices, show that the free energy density of this system in the thermodynamic limit is

$$f = \lim_{N\to\infty} \frac{F}{N} = - \lim_{N\to\infty} \frac{\ln(Z)}{N\beta} = -\frac{1}{2\beta} \ln\left(\frac{\pi}{\beta}\right) - \frac{1}{\beta} \ln\left(\sqrt{\frac{\pi}{\beta(1-J)}} + \sqrt{\frac{\pi}{\beta(1+J)}}\right) ,$$

with inverse temperature β. Note that $d_i \in \mathbb{R}$ are degrees of freedom to be integrated over.

(b) The system does not display a phase transition in β but f is singular for some values of J. By inspection, write down those two values J_c of J.

(c) Determine the exponent μ which characterises how $\partial_J f$ diverges as J approaches either of the two J_c,

$$\frac{\partial f}{\partial J} = A(J - J_c)^{-\mu} + \text{higher order terms} \tag{2}$$

with amplitude A (not to be determined).

3. The field theory to be considered in the following is defined by the Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^d r \left\{ \frac{1}{2}\tau \left(\nabla\phi(\mathbf{r})\right)^2 + \frac{1}{2}\left(\nabla^2\phi(\mathbf{r})\right)^2 + \frac{u_0}{4!}\phi(\mathbf{r})^4 + \frac{u_1}{6!}\phi(\mathbf{r})^6 \right\} .$$

(a) How does this Hamiltonian differ from the usual LGW Hamiltonian?

(b) Rewrite the Hamiltonian in k-space, i.e. use

$$\phi(\mathbf{r}) = \int \text{đ}^d k\, \phi(\mathbf{k}) e^{-\imath \mathbf{k}\mathbf{r}}$$

throughout the Hamiltonian above and perform the spatial integral over $\mathbf{r}$, so that the only remaining integrals are in k-space. Non-linearities and derivatives deserve special attention.

(c) By inspection, write down its bare propagator.

(d) Determine the upper critical dimension of the theory and remove any terms that are irrelevant just below the upper critical dimension.

4. In the following, we consider the LGW Hamiltonian for a $n = 2$ vector field theory. We do that by writing the vector field (ϕ, ψ) explicitly in its two components ϕ and ψ,

$$\mathcal{H} = \int \mathrm{d}^d r \left\{ \frac{1}{2} r_0 \phi(\mathbf{r})^2 + \frac{1}{2} \widetilde{r_0} \psi(\mathbf{r})^2 + \frac{1}{2} \left(\nabla \phi(\mathbf{r})\right)^2 + \frac{1}{2} \left(\nabla \psi(\mathbf{r})\right)^2 + \frac{u_0}{4!} \left(\phi(\mathbf{r})^2 + \psi(\mathbf{r})^2\right)^2 \right\} .$$

The bare propagators in this theory are

$$\begin{aligned} \langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle_{c,0} &= \frac{\delta\!\!\!{}^-(\mathbf{k}_1 + \mathbf{k}_2)}{\mathbf{k}_1^2 + r_0} \\ \langle \psi(\mathbf{k}_1)\psi(\mathbf{k}_2)\rangle_{c,0} &= \frac{\delta\!\!\!{}^-(\mathbf{k}_1 + \mathbf{k}_2)}{\mathbf{k}_1^2 + \widetilde{r_0}} \\ \langle \phi(\mathbf{k}_1)\psi(\mathbf{k}_2)\rangle_{c,0} &= 0 \end{aligned}$$

Notice the mass $\widetilde{r_0}$ of ψ versus the mass r_0 of ϕ. This distinction has purely notational reasons.

(a) How does the term with coupling u_0 in the present field theory differ from the scalar (standard) LGW field theory?

Hint: "Scalar LGW field theory" refers to the field theory discussed in detail in the lectures.

(b) Determine, as an integral (but do not attempt to perform the integral), the contribution to the propagator $\langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle$ to lowest order in u_0. Include all symmetry factors. Diagrammatically, the term is of the form

Hint: Take into account the different structure of the nonlinear term $\left(\phi(\mathbf{r})^2 + \psi(\mathbf{r})^2\right)^2$ compared to standard LGW field theory. You may wish to use the propagator - - - - - for $\langle \psi(\mathbf{k}_1)\psi(\mathbf{k}_2)\rangle_{c,0}$, when considering, for example,  .

(c) Determine the first-order corrections to

(i) $\langle \phi(\mathbf{k} = 0)^4 \rangle_c$,

(ii) $\langle \phi(\mathbf{k} = 0)^3 \psi(\mathbf{k} = 0) \rangle_c$ and

(iii) $\langle \phi(\mathbf{k} = 0)^2 \psi(\mathbf{k} = 0)^2 \rangle_c$,

all of which are due to diagrams of the form

Include all symmetry factors. Do not actually perform the integration.