IMPERIAL

MATH70031

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Markov Processes

Date: Tuesday, May 20, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Give an example, with justification, of a $\mathbb{R}$-valued Markov process $(X_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} X_n = 0$ almost surely, but $\lim_{n\to\infty} \mathbb{E}(|X_n|) = \infty$. (7 marks)

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a stochastic process. State the precise condition for $(X_n)_{n\in\mathbb{N}}$ to be a stationary process. (4 marks)

(c) Let X, Y, Z be three $\mathbb{R}$-valued random variables. Assume that for any bounded measurable functions $f, g \in \mathcal{B}_b(\mathbb{R})$, we have $\mathbb{E}(g(Z)f(X) \mid Y) = \mathbb{E}(f(X) \mid Y)\mathbb{E}(g(Z) \mid Y)$.
Prove that for any bounded measurable function $g \in \mathcal{B}_b(\mathbb{R})$, we have

$$\mathbb{E}(g(Z) \mid X, Y) = \mathbb{E}(g(Z) \mid Y).$$

(9 marks)

(Total: 20 marks)

2. Let $\mathcal{X}$ be a separable complete metric space. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the natural filtration of a $\mathcal{X}$-valued time homogeneous Markov process $X = (X_n)_{n\in\mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{P^n(x,\cdot) \mid x \in \mathcal{X}, n \in \mathbb{N}\}$ be the transition functions for X.

(a) Given a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time T, show that $T+m$ is also a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time for every integer $m \geq 0$.

(3 marks)

(b) State Chapman-Kolmogorov's equation for $\{P^n(x,\cdot) \mid x \in \mathcal{X}, n \in \mathbb{N}\}$.

(3 marks)

(c) **True or False:** For each question, state whether the statement is true or false. If it is true, give a proof and if it is false give a counterexample with justification.
Let S, T be two $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping times valued in $\mathbb{N}$ satisfying $S \leq T$. Then

(i) We know that $2T - S$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time. (3 marks)

(ii) We know that $\lfloor \frac{S+3T+1}{2} \rfloor$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time (Here $\lfloor x \rfloor$ denotes the integer part of x. For example, $\lfloor 2.6 \rfloor = 2$). (3 marks)

(d) Use without proof the following equation

$$\mathbb{E}(f(X_{n+T}) \mid \mathcal{F}_T) = \int_{\mathcal{X}} f(y) P^n(X_T, dy)$$

for any $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time T valued in $\mathbb{N}$, to deduce that for every $k \geq 1$

$$\mathbb{E}\Big(\prod_{i=1}^{k} f_i(X_{T+i}) \mid \mathcal{F}_T\Big) = \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \prod_{i=1}^{k} f_i(y_i) \prod_{i=2}^{k} P(y_{i-1}, dy_i) P(X_T, dy_1).$$

Direct application of Strong Markov property is not allowed for this question. (8 marks)

(Total: 20 marks)

3. Let $\mathcal{X}$ be a finite state space. Let P be a stochastic matrix on $\mathcal{X}$. Let $\mathbb{P}_x$ be the law of the Markov process with transition probabilities P and initial distribution δ_x. Let $\mathbb{E}_x$ be the expectation with respect to $\mathbb{P}_x$. Let T_x be the first passage time to x.

(a) **True or False:** For each question, state whether the statement is true or false. If it is true, give a proof and if it is false give a counterexample with justification.

(i) If P has an eigenvalue $\lambda \neq 1$ of modulus 1, then P is reducible. (3 marks)

(ii) If P has only one invariant probability measure, and P has no eigenvalue $\lambda \neq 1$ of modulus 1, then P is irreducible. (3 marks)

(b) In this question we let $\mathcal{X} = \{1,2,3,4,5,6\}$. Find the communication classes of the following stochastic matrix P, along with their partial ordering.

$$P = \begin{bmatrix} 0.2 & 0 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.8 & 0 \\ 0.2 & 0.4 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

(3 marks)

(c) Let y be a transient state. Show that for any $x \in \mathcal{X}$, $\lim_{n\to\infty} P^n_{x,y} = 0$. Here $P^n_{x,y}$ is the n-step transition probability from x to y. (6 marks)

(d) Let $x, y \in \mathcal{X}$. Prove that $\mathbb{E}_y(T_y) \leq \mathbb{E}_y(T_x) + \mathbb{E}_x(T_y)$. (5 marks)

(Total: 20 marks)

4. (a) Let $P = (P_{i,j})_{i,j\in\mathbb{Z}}$ be a stochastic matrix on $\mathbb{Z}$. Prove that a reversible measure for P is an invariant measure for P. (4 marks)

(b) Let $Z = (Z_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}$ whose common law μ is given by $\mu(\{1\}) = 1/4$ and $\mu(\{-10\}) = 3/4$. Define an $\mathbb{R}$-valued Markov process $(X_n)_{n\in\mathbb{N}}$ by setting $X_0 = 1$, and for each $n \in \mathbb{N}$,

$$X_{n+1} = 2^{Z_n} X_n + 1.$$

(i) Prove that there exists an invariant probability measure for X. (8 marks)

(ii) Prove that there is only one invariant probability measure for X. (8 marks)

(Total: 20 marks)

5. Let $P = \{P(x, \cdot)\}_{x \in \mathcal{X}}$ be a family of transition probabilities on $\mathcal{X}$. Let $\theta : \mathcal{X}^{\mathbb{Z}} \to \mathcal{X}^{\mathbb{Z}}$ be the shift map $\theta((a_n)_{n \in \mathbb{Z}}) = (a_{n+1})_{n \in \mathbb{Z}}$.

Let π be an invariant probability measure on $\mathcal{X}$ for P. Let X be the $\mathcal{X}$-valued Markov process with transition probabilities P, and with initial distribution π. Let $\mathbb{P}_\pi$ be the law of such X, and let $\mathbb{E}_\pi$ denote the expectation with respect to $\mathbb{P}_\pi$.

(a) Assume that for every $\mathbb{R}$-valued bounded measurable functions f, g on $\mathcal{X}^{\mathbb{Z}}$ with respect to the product σ-algebra, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mathbb{E}_\pi(f \cdot g \circ \theta^n) - \mathbb{E}_\pi(f)\mathbb{E}_\pi(g)| = 0.$$

Prove that π is ergodic. (5 marks)

(b) Given an example, with justification, of a family of transition probabilities P on a finite set $\mathcal{X}$, such that there is an ergodic invariant probability measure π which does not satisfy the condition in (a). (5 marks)

(c) Let $\mathcal{X}$ be a finite set, and let P denote a stochastic matrix on $\mathcal{X}$. Assume that P is irreducible and aperiodic. Prove that the unique invariant measure π on $\mathcal{X}$ satisfies the condition in (a). (You can use without proof the fact that any bounded measurable function on $\mathcal{X}^{\mathbb{Z}}$ can be approximated in L^2 (with respect to $\mathbb{P}_\pi$) by bounded measurable functions depending only on finitely many coordinates). (10 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60031/70031

MATH60031/70031 (Solutions)

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1. (a) We let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1), \mathcal{B}, \mathrm{Leb}|_{[0,1)})$. For each integer $n \geq 0$, we divide $[0,1)$ into 2^n disjoint subintervals of form $[2^{-n}k, 2^{-n}(k+1))$, $k \in \{0, \cdots, 2^n - 1\}$. We let $X_n = 2^{2n}$ on $[0, 2^{-n})$, and let $X_n = 0$ on the remaining $2^n - 1$ intervals. (3 marks for a valid example) It is clear that $\mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = 1$ and $\mathbb{P}(X_{n+1} = 0 \mid X_n = 2^{2n}) = \mathbb{P}(X_{n+1} = 2^{2n+2} \mid X_n = 2^{2n}) = 1/2$. This gives Markov's property. For any $x \in (0,1)$, there exists $n_0 > 0$ such that for any $n > n_0$, $X_n(x) = 0$. However, for each n, $\mathbb{E}(X_n) = 2^{2n} \cdot 2^{-n} = 2^n$ which tends to infinity as n tends to infinity. (4 marks for justification) $\boxed{7, \mathrm{B}}$

(b) For any integer $m \geq 0$, the probability distributions of the stochastic processes $(X_n)_{n \in \mathbb{N}}$ and $(X_{n+m})_{n \in \mathbb{N}}$ are the same. (4 marks) $\boxed{4, \mathrm{A}}$

(c) Fix arbitrary bounded Borel measurable functions f, g and h. We have

$$\begin{aligned} & \mathbb{E}(h(Y)f(X)\mathbb{E}(g(Z) \mid X, Y)) = \mathbb{E}(f(X)h(Y)g(Z)) = \mathbb{E}(h(Y)\mathbb{E}(g(Z)f(X) \mid Y)) \\ = \; & \mathbb{E}(h(Y)\mathbb{E}(g(Z) \mid Y)\mathbb{E}(f(X) \mid Y)). \end{aligned}$$

The first equality uses Tower property. (2 marks) The second equality uses the definition of conditional expectation. (2 marks). The last equality above uses the hypothesis (2 marks). By moving the $\sigma(Y)$-measurable terms into the last conditional expectation, and by Tower property, we may write the above as

$$\mathbb{E}(\mathbb{E}(h(Y)f(X)\mathbb{E}(g(Z) \mid Y) \mid Y)) = \mathbb{E}(h(Y)f(X)\mathbb{E}(g(Z) \mid Y)).$$

(2 marks) Since h and f are arbitrary, this shows that one must have $\mathbb{E}(g(Z) \mid X, Y) = \mathbb{E}(g(Z) \mid Y)$ almost surely. (1 mark) $\boxed{9, \mathrm{A}}$

2. (a) Given integers $n, m \in \mathbb{N}$. If $n < m$ then $\{T+m=n\} = \emptyset$. If $n \geq m$, then $\{T+m=n\} = \{T = n-m\} \in \mathcal{F}_{n-m}$ as T is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time. Since $\mathcal{F}_{n-m} \subset \mathcal{F}_n$, we deduce $\{T+m=n\} \subset \mathcal{F}_n$. Hence $T+m$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time. (3 marks) $\boxed{3, \text{A}}$

(b) For every integer $n, m \geq 0$, and every Borel measurable subset $A \subset \mathcal{X}$

$$P^{n+m}(x, A) = \int_{\mathcal{X}} P^n(y, A) P^m(x, dy).$$

(3 marks) $\boxed{3, \text{A}}$

(c) (i) True. (1 mark) Let $n \geq 0$. Since $T \geq S$, $2T - S = n$ implies that $T \leq n$. Then $S \leq n$ as well. Thus we have

$$\{2T - S = n\} = \bigcup_{0 \leq m, l \leq n, 2l - m = n} \{S = m\} \cap \{T = l\}.$$

Since S, T are stopping times, we have $\{S = m\} \in \mathcal{F}_m \subset \mathcal{F}_n$ and $\{T = l\} \in \mathcal{F}_l \subset \mathcal{F}_n$. The proof follows. (2 marks) $\boxed{3, \text{C}}$

(ii) True. (1 mark) Let $n \geq 0$. Since $T \geq S$, $\lfloor \frac{S+3T+1}{2} \rfloor = n$ implies that $2S \leq n$ and $T \leq n$. Thus we have

$$\left\{ \lfloor \frac{S+3T+1}{2} \rfloor = n \right\} = \bigcup_{0 \leq m, l \leq n, \lfloor \frac{m+3l+1}{2} \rfloor = n} \{S = m\} \cap \{T = l\}.$$

Since S, T are stopping times, we have $\{S = m\} \in \mathcal{F}_m \subset \mathcal{F}_n$ and $\{T = l\} \in \mathcal{F}_l \subset \mathcal{F}_n$. The proof follows. (2 marks) $\boxed{3, \text{C}}$

(d) The statement is true for $k = 0$. We will prove by induction. Assume that the statement to prove holds for $k-1$ in place of k. (2 marks for considering induction) Let $f_1, \cdots, f_k$ be a collection of bounded Borel measurable functions on $\mathcal{X}$. By Tower property, we have

$$\mathbb{E}(\prod_{i=1}^{k} f_i(X_{T+i}) \mid \mathcal{F}_T) = \mathbb{E}(\prod_{i=1}^{k-1} f_i(X_{T+i}) \mathbb{E}(f_k(X_{T+k}) \mid \mathcal{F}_{T+k-1}) \mid \mathcal{F}_T), \quad a.s.$$

(2 marks for using Tower property) Since $T + k - 1$ is also a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping time, by hypothesis, there is a bounded Borel measurable function g on $\mathcal{X}$ such that

$$\mathbb{E}(f_k(X_{T+k}) \mid \mathcal{F}_{T+k-1}) = g(X_{T+k-1}) = \int f_k(y_k) P(X_{T+k-1}, dy_k).$$

(2 marks) By the induction hypothesis, for $k-1$ and $(f_1, \cdots, f_{k-2}, f_{k-1}g)$, we obtain

$$\mathbb{E}(\prod_{i=1}^{k} f_i(X_{T+i}) \mid \mathcal{F}_T)(\omega) = \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \prod_{i=1}^{k-2} f_i(y_i)(f_{k-1}g)(y_{k-1}) \prod_{i=2}^{k-1} P(y_{i-1}, dy_i) P(X_T(\omega), dy_1).$$

To finish the induction, it suffices to substitute the expression of g into the above equation. (2 marks) $\boxed{8, \text{B}}$

3. (a) (i) False. (1 mark) We may take $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (1 mark for counter-example)

Then P has eigenvalue -1. However, P is irreducible since $P^2 + P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. (1 mark for justification) $\boxed{3, \mathrm{B}}$

(ii) False. (1 mark) We may take $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. (1 mark for counter-example) P has eigenvalues 0 and 1. Moreover, P has a unique invariant measure supported in 2. However, P is reducible since 1 is not accessible from 2. (1 mark for justification) $\boxed{3, \mathrm{B}}$

(b) The communication classes of P are $\{1, 3, 4\}$, $\{2, 5\}$ and $\{6\}$. (2 marks) The partial orderings are $\{1, 3, 4\} \preccurlyeq \{6\}$ and $\{2, 5\} \preccurlyeq \{6\}$. (1 mark) $\boxed{3, \mathrm{A}}$

(c) Let T_x denote the first passage time to x. Let T_x^k denote the k-th passage time to x. Let N_x denote the occupation time at x. Then

$$\sum_{n=1}^{\infty} P_{x,y}^n \quad = \quad \mathbb{E}_x(N_y) = \sum_{k=1}^{\infty} \mathbb{P}_x(N_y \geq k) = \sum_{k=1}^{\infty} \mathbb{P}_x(T_y^k < \infty).$$

(1 mark for the computations) By Strong Markov's property, we may write the above as

$$\sum_{k=1}^{\infty} \mathbb{P}_x(T_y < \infty)\mathbb{P}_y(T_y^{k-1} < \infty) = \sum_{k=1}^{\infty} \mathbb{P}_x(T_y < \infty)\mathbb{P}_y(T_y < \infty)^{k-1}.$$

(2 marks for strong Markov property) Since y is transient, $\mathbb{P}_y(T_y < \infty) < 1$ and consequently the above summation converges. (2 marks for using the transient property) Then we have $\lim_{n\to\infty} P_{x,y}^n = 0$. (1 mark) $\boxed{6, \mathrm{A}}$

(d) We have

$$\begin{aligned} \mathbb{E}_y(T_y) &= \mathbb{E}_y(T_y 1_{\{T_y \leq T_x\}}) + \mathbb{E}_y(T_y 1_{\{T_y > T_x\}}) \\ &\leq \mathbb{E}_y(T_y 1_{\{T_y \leq T_x\}}) + \mathbb{E}_y(1_{\{T_x < T_y\}} \mathbb{E}_x(T_y)) \\ &\leq \mathbb{E}_y(T_x) + \mathbb{E}_x(T_y). \end{aligned}$$

The equality is clear. (2 marks for strategy) The first inequality follows from Strong Markov's property. (2 marks) The second inequality is straightforward. (1 mark) $\boxed{5, \mathrm{C}}$

4. (a) Let $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ denote a reversible measure. Then for every $x, y \in \mathcal{X}$, we have $\mu(x)P_{x,y} = \mu(y)P_{y,x}$. (2 marks) Then for every $y \in \mathcal{X}$, we have

$$\sum_{x\in\mathcal{X}} \mu(x)P_{x,y} = \sum_{x\in\mathcal{X}} \mu(y)P_{y,x} = \mu(y).$$

Hence μ is an invariant measure. (2 marks) $\boxed{4, \mathrm{A}}$

(b) (i) We will verify this using Lyapunov function. (2 marks for strategy).
For any $z \in \mathbb{R}$, the map $x \mapsto 2^z x + 1$ is clearly continuous, it follows that the transition operator T has Feller property (2 marks for Feller).
Note that

$$\gamma = \mathbb{E}(2^{Z_0}) = \mathbb{P}(Z_0 = 1) \times 2^1 + \mathbb{P}(Z_0 = -10) \times 2^{-10} = \frac{1}{4} \times 2 + \frac{3}{4} \times 2^{-10} \in (0, 1).$$

We now show that the function V on $\mathbb{R}$ given by $V(x) = |x|$ is a Lyapunov function. Clearly $V^{-1}((-\infty, c]) = [-c, c]$ is compact for every $c \geq 0$; and $V^{-1}((-\infty, c]) = \emptyset$ for every $c < 0$. We also have

$$\mathbb{E}(V(X_{n+1}) \mid X_n) \leq \mathbb{E}(2^{Z_n}|X_n| + 1) = \mathbb{E}(2^{Z_n})V(X_n) + 1 = \gamma V(X_n) + 1.$$

This proves that V is a Lyapunov function. We have seen that any transition probabilities $\{P(x, \cdot)\}_{x\in\mathcal{X}}$ with Feller property and a Lyapunov function must also have an invariant probability measure. (4 marks). $\boxed{8, \mathrm{D}}$

(ii) We argue by deterministic contraction. (3 marks for strategy)
We may write $X_{n+1} = F(X_n, Z_n)$ with $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, \xi) = 2^\xi x + 1.$$

Note that we have

$$F(x, \xi) - F(y, \xi) = 2^\xi (x - y).$$

Then

$$\int |F(x, \xi) - F(y, \xi)|\mu(d\xi) = \int 2^\xi |x - y|\mu(d\xi) = \mathbb{E}(2^{Z_0})|x - y| = \gamma|x - y|.$$

Since $\gamma < 1$, we may then conclude using Theorem 8.5.5 in the lecture note (Uniqueness due to deterministic contraction) (5 marks) $\boxed{8, \mathrm{D}}$

5. (a) If π is not ergodic, then there exists a subset $A \subset \mathcal{X}^{\mathbb{N}}$ such that $\theta(A) = A$ and $\mathbb{P}_\pi(A) \in (0,1)$. (2 marks for definition) Let $f = g = 1_A$. Then we have $f \cdot g \circ \theta^n = 1_A \cdot 1_A \circ \theta^n = 1_A^2 = 1_A$. Then for every $n \geq 0$, we have $\mathbb{E}_\pi(f \cdot g \circ \theta^n) - \mathbb{E}_\pi(f)\mathbb{E}_\pi(g) = \mathbb{E}(1_A) - \mathbb{E}(1_A)^2 = \mathbb{P}_\pi(A)(1 - \mathbb{P}_\pi(A)) > 0$. It is clear that the condition in (a) is not satisfied for such f, g. (3 marks for finishing)

$\boxed{5, \text{M}}$

(b) We make take $\mathcal{X} = \{1,2\}$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (2 marks for a valid example) Then $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ is an invariant probability measure for P. It is straightforward to verify π is ergodic, since the only P-invariant subsets of $\mathcal{X}$ are $\emptyset$ and $\mathcal{X}$.

Note that $P^2 = \mathrm{Id}$. Let $A = \{1\}$. We define f and g by $f((a_n)_{n\in\mathbb{N}}) = g((a_n)_{n\in\mathbb{N}}) = 1_A(a_0)$. Then for even n, we have $\mathbb{E}_\pi(f \cdot g \circ \theta^n) - \mathbb{E}_\pi(f)\mathbb{E}_\pi(g) = \mathbb{E}_\pi(1_A) - \mathbb{E}_\pi(1_A)^2 = 1/4$. The condition in (a) is not satisfied. (3 marks for justification)

$\boxed{5, \text{M}}$

(c) Assume that P is irreducible and aperiodic. Let $N > 0$ be an integer and let $B = \{a = (a_i)_{i\in\mathbb{Z}} \mid a_i = b_i, |i| \leq N\}$ and $C = \{a = (a_i)_{i\in\mathbb{Z}} \mid a_i = c_i, |i| \leq N\}$ be two cylinder sets. Let $f = 1_B$, $g = 1_C$. We have

$$\mathbb{E}_\pi(f) = \pi(b_{-N}) \cdot \prod_{i=-N}^{N-1} P_{b_i,b_{i+1}}, \quad \mathbb{E}_\pi(g) = \pi(c_{-N}) \cdot \prod_{i=-N}^{N-1} P_{c_i,c_{i+1}}.$$

(2 marks) For any $n > 0$, we have

$$\mathbb{E}_\pi(f \cdot g \circ \theta^n) = \int 1_B(a) \cdot 1_C(\theta^n(a))\mathbb{P}_\pi(da) = \mathbb{P}_\pi(B \cap \theta^{-n}(C)).$$

Note that when $n > 2N$, we have

$$\begin{aligned} \mathbb{P}_\pi(B \cap \theta^{-n}(C)) &= \mathbb{P}(\{a = (a_i)_{i\in\mathbb{Z}} \mid a_i = b_i, a_{i+n} = c_i \text{ for } i \in [-N,N]\}) \\ &= \pi(b_{-N}) \cdot \prod_{i=-N}^{N-1} P_{b_i,b_{i+1}} \cdot \prod_{i=-N}^{N-1} P_{c_i,c_{i+1}} \cdot P^{n-2N}_{b_N,c_{-N}}. \end{aligned}$$

(3 marks) Since P is irreducible and aperiodic, $\lim_{n\to\infty} P^{n-2N}_{x,y} = \pi(y)$. Then $\lim_{n\to\infty} |\mathbb{E}_\pi(f \cdot g \circ \theta^n) - \mathbb{E}_\pi(f)\mathbb{E}_\pi(g)| = 0$. This proves the statement for f, g being indicator functions on cylinders, which implies the statement for any f, g determined by finitely many coordinates. (3 marks for using aperiodicity)

For general f, g, it suffices to approximate f and g by bounded measurable functions f_K and g_K respectively in L^2, and use Cauchy's inequality to deduce

$$\begin{aligned} & |\mathbb{E}(f \cdot g \circ \theta^n) - \mathbb{E}(f)\mathbb{E}(g)| - |\mathbb{E}(f_K \cdot g_K \circ \theta^n) - \mathbb{E}(f_K)\mathbb{E}(g_K)| \\ \leq\ & 2\mathbb{E}(|f - f_K|^2)^{1/2}\mathbb{E}(g^2)^{1/2} + 2\mathbb{E}(|g - g_K|^2)^{1/2}\mathbb{E}(f_K^2)^{1/2}. \end{aligned}$$

This last line tends to n uniformly as $\mathbb{E}(|f - f_K|^2)$ and $\mathbb{E}(|g - g_K|^2)$ decrease. Thus we can conclude the proof for general f, g. (2 marks)

$\boxed{10, \text{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 21 of 20 marks
Total C marks: 11 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 100 marks
Total Mastery marks: 20 of 20 marks

MATH70031 Markov Processes Markers Comments

Question 1	Most students answered (a) incorrectly. Also some students got (c) wrong because they did not find the right way to apply Tower property.
Question 2	Most student did not realise how to use T is bigger or equal to S in (c). Some students did not realise how to correctly apply Tower property in (d).
Question 3	Many students use transient property the wrong way.
Question 4	Most students did well in Q4.
Question 5	Most students did not do well in Q5. Possibly because they don't have enough time to work on it.