MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Extra Material 3: Proof of the Poincaré–Bendixson theorem

We consider the two-dimensional differential equation

$$\dot{x} = f(x)\,, \tag{1}$$

where $f : D \to \mathbb{R}^2$ is continuously differentiable function on an open set $D \subset \mathbb{R}^2$. We denote the flow of (1) by φ. Note that one can show (and this is important for us) that the flow φ is continuously differentiable, but we will not do this here.

Theorem 1 (Poincaré–Bendixson theorem)**.** *Consider the differential equation* (1)*, and assume that for some $x \in D$, the positive half-orbit $O^+(x)$ lies in a compact subset K of D, which contains not more than finitely many equilibria. Then one of the following three statements holds for the omega-limit set $\omega(x)$.*

(i) $\omega(x)$ is a singleton consisting of an equilibrium.

(ii) $\omega(x)$ is a periodic orbit.

(iii) $\omega(x)$ consists of equilibria and non-closed orbits. The non-closed orbits in $\omega(x)$ converge forward and backward in time to equilibria in $\omega(x)$, so they are either homoclinic or heteroclinic orbits.

We start our preparations to prove the theorem of Poincaré–Bendixson and we consider so-called transversals, which are line segments through which the vector field of the right hand side points in the same half space.

Definition 2 (Transversal)**.** Let $S \subset D$ be a closed line segment with a corresponding normal vector $\bar{n} = (n_1, n_2) \in \mathbb{R}^2$. We call S a *transversal* for the differential equation (1) if

$$\big\langle \bar{n}, f(x) \big\rangle := n_1 f_1(x) + n_2 f_2(x) \neq 0 \quad \text{for all } \ x \in S\,. \tag{2}$$

It is clear from (2) that a transversal cannot contain equilibria, and the following proposition says that there exist transversals through all points that are not equilibria.

Proposition 3 (Existence of transversals)**.** *Consider the differential equation* (1)*, and let $x \in \mathbb{R}^2$ such that $f(x) \neq (0,0)$, i.e. x is not an equilibrium. Then there exists a transversal S through x, and all vectors from the vector field of* (1) *on S point into the same half space.*

Proof. Let $L = \{\bar{x} \in \mathbb{R}^2 : \langle \bar{x} - x, \bar{n} \rangle = 0\}$ be the line through x with normal vector $\bar{n} := f(x)$. Define $h : L \to \mathbb{R}$ by

$$h(\bar{x}) := \big\langle \bar{n}, f(\bar{x}) \big\rangle \quad \text{for all } \ \bar{x} \in L\,.$$

h is obviously continuous and $h(x) = \|f(x)\|^2 > 0$, so h is positive in a neighbourhood of x, and thus, (2) is satisfied on L in a neighbourhood of x, defining a transversal S. Since $h(x)$ is positive on S, all vectors from the vector field of (1) on S point into the same half space. □

We aim at establishing that orbits intersect transversals monotonically. For this purpose, we need the Jordan curve theorem which turns out surprisingly difficult to prove (and we skip the proof for this reason). Note that a so-called *Jordan curve* is given by

$$J := \big\{ \gamma(s) \in \mathbb{R}^2 : s \in [0,1] \big\}\,,$$

where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous function such that $\gamma(0) = \gamma(1)$ and the function $\gamma|_{(0,1)}$ is injective on $(0,1)$.

Theorem 4 (Jordan curve theorem)**.** *Let J be a Jordan curve. Then $\mathbb{R}^2 \setminus J$ is the disjoint union of two open and connected sets that have J as their common boundary. One of the two sets is bounded (the* interior set G_I*) and one is unbounded (the* exterior set G_E*).*

The following proposition shows that any orbit has at most countably many intersections with a transversal, and the intersection points are monotone on the transversal.

Proposition 5 (Monotonicity of orbit intersection points on transversals)**.** *Consider the differential equation (1) with a transversal S, and let $x \in D$. Then the orbit $O(x)$ has at most countably many intersection points $\varphi(t_1, x)$, $\varphi(t_2, x)$, $\varphi(t_3, x), \dots$ with S, and if $t_i < t_j < t_k$ are ordered, then $\varphi(t_i, x)$, $\varphi(t_j, x)$, $\varphi(t_k, x)$ are monotone on S in the sense that the point $\varphi(t_j, x)$ lies between $\varphi(t_i, x)$ and $\varphi(t_k, x)$ on S.*

Proof. The proof is divided into two steps.

Step 1. There are at most countably many intersection points.
We show that for any real numbers $a < b$, where $a, b \in J_{max}(x)$, the intersection $O^{[a,b]}(x) \cap S$ is finite, where $O^{[a,b]}(x) := \big\{\varphi(t, x) : t \in [a, b]\big\}$. Assume to the contrary that $O^{[a,b]}(x) \cap S$ is infinite. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $O^{[a,b]}(x) \cap S$ with $\lim_{n\to\infty} x_n = x_0$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$. We define $t_n \in [a, b]$ for $n \in \mathbb{N}$ such that $x_n = \varphi(t_n, x)$ and assume without loss of generality that $t_n \to t_0 \in [a, b]$ as $n \to \infty$. It follows that

$$\frac{x_n - x_0}{t_n - t_0} = \frac{\varphi(t_n, x) - \varphi(t_0, x)}{t_n - t_0} \to f(x_0) \quad \text{as } n \to \infty\,,$$

and with a normal vector $\bar{n}$ to the transversal S, we get

$$0 = \frac{\langle \bar{n}, x_n - x_0 \rangle}{t_n - t_0} \to \langle \bar{n}, f(x_0) \rangle \quad \text{as } n \to \infty\,,$$

which contradicts (2).

Step 2. Monotonicity of intersection points.
Assume that $\varphi(t_1, x) \in S$, and let $t_2 := \min\big\{t > t_1 : \varphi(t, x) \in S\big\}$ be the next intersection time after t_1 (we assume that such a time exists; otherwise nothing needs to be shown). Assume for simplicity that S is part of the x_1-axis, the vector field is directed towards the positive x_2-axis, and $\varphi(t_1, x)$ is left of $\varphi(t_2, x)$. Then $O^{[t_1,t_2]}(x) \cup [\varphi(t_1, x), \varphi(t_2, x)]$ is a Jordan curve. Then there are two cases: either G_I is positively invariant, or G_E is positively invariant, see Figure 1. In the first case (on the left), the inner region G_I is positively invariant, i.e. any other intersection point $\varphi(t_3, x)$ for $t_3 > t_2$ is right of $\varphi(t_2, x)$. In the second case (on the right), the exterior region G_E is positively invariant, and also any other intersection point $\varphi(t_3, x)$ for $t_3 > t_2$ is right of $\varphi(t_2, x)$. That proves monotonicity in positive time direction. Analogously, one proves monotonicity in negative time direction. □

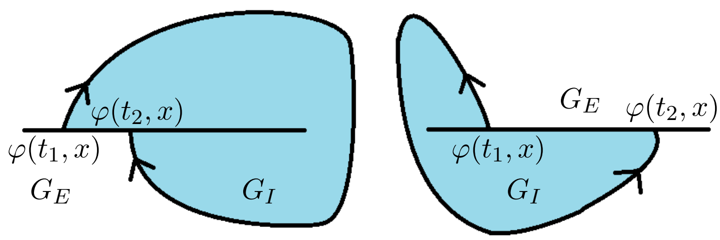

Figure 1: Two cases for proving monotonicity.

Proposition 6 (Transversal neighbourhood and transversal time)**.** *Consider the differential equation* (1) *with a transversal S, and let $x \in D$ and $t_0 \in J_{max}(x)$ such that $\varphi(t_0, x) \in S$ and $\varphi(t_0, x)$ is not an endpoint of the transversal S. Then there exists a so-called* transversal neighbourhood *V of x and a uniquely determined continuously differentiable function $\tau : V \to \mathbb{R}$, the so-called* transversal time *such that*

$$\varphi(\tau(\bar{x}), \bar{x}) \in S \quad \textit{for all } \bar{x} \in V$$

and

$$\lim_{\bar{x} \to x} \tau(\bar{x}) = t_0 \,.$$

Proof. We assume for simplicity that S is part of the x_2-axis, and $\varphi(t_0, x) = (0, 0)$ (which implies $f_1(0, 0) \neq 0$). Let φ_1 be the first coordinate (the x_1-coordinate) of the flow φ, and note that φ_1 is a continuously differentiable function, since φ is a continuously differentiable function. We have

$$\begin{aligned} \varphi_1(t_0, x) &= 0 \,, \\ \frac{\partial \varphi_1}{\partial t}(t_0, x) &= f_1(\varphi(t_0, x)) = f_1(0, 0) \neq 0 \,. \end{aligned}$$

This implies that due to the implicit function theorem, there exists a neighbourhood $V \subset \mathbb{R}^2$ of x and a continuously differentiable function $\tau : V \to \mathbb{R}$ such that

$$\varphi_1(\tau(\bar{x}), \bar{x}) = 0 \quad \text{for all } \bar{x} \in V \,.$$

This implies that $\varphi(\tau(\bar{x}), \bar{x})$ lies in the x_2-axis for all $\bar{x} \in V$, and after possibly shrinking V, it will even lie in S, since S is part of the x_2-axis with $(0, 0) \in S$. □

We now show that omega limit sets can intersect transversals only in at most one point.

Proposition 7 (Omega limit sets intersect transversals in at most one point)**.** *Consider the differential equation* (1) *with a transversal S. Then for any $x \in D$, the omega limit set $\omega(x)$ intersects S in at most one point.*

Proof. We assume that there exist two different omega limit points $z_1, z_2 \in \omega(x) \cap S$ on the transversal S. Due to Proposition 6, for the points z_1 and z_2, there exist two different transversal neighbourhoods V_1 of z_1 and V_2 of z_2, which we assume to disjoint without loss of generality (note that $t_0 = 0$ in Proposition 6 due to $z_1, z_2 \in S$). Since both z_1 and z_2 are in $\omega(x)$, we find a sequence $\{t_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} t_n = \infty$,

$$\lim_{n \to \infty} \varphi(t_{2n-1}, x) = z_1 \quad \text{and} \quad \lim_{n \to \infty} \varphi(t_{2n}, x) = z_2 \,.$$

Without loss of generality, we assume that $\varphi(t_{2n-1}, x) \in V_1$ and $\varphi(t_{2n}, x) \in V_2$ for all $n \in \mathbb{N}$, so that

$$\begin{aligned} &\varphi(\tau_1(\varphi(t_{2n-1}, x)), \varphi(t_{2n-1}, x)) \in V_1 \cap S \\ \text{and} \quad &\varphi(\tau_2(\varphi(t_{2n}, x)), \varphi(t_{2n}, x)) \in V_2 \cap S \quad \text{for all } n \in \mathbb{N} \,, \end{aligned}$$

where τ_1 and τ_2 are the transversal times defined on V_1 and V_2. Since $\tau_1(z_1) = \tau_2(z_2) = 0$, and τ_1, τ_2 are continuous, this implies that the orbit $O^+(x)$ switches infinitely often between $V_1 \cap S$ and $V_2 \cap S$, which are disjoint sets. This contradicts the monotonicity established in Proposition 5 and finishes the proof of this proposition. □

Proposition 8 (Orbits with limit points on a transversal)**.** *Consider the differential equation* (1) *with a transversal S, and assume that there exists an $x \in D$ with $\omega(x) \cap S = \{x^*\}$ such that x^* is not an endpoint of the transversal S. Then one of the two statements holds.*

(i) $O(x)$ is a periodic orbit and intersects the transversal in exactly one point.

(ii) $O(x)$ intersects S in infinitely many (but countably many) pairwise disjoint points $\varphi(\sigma_n, x)$, where $n \in \mathbb{N}$. Here the times $\{\sigma_n\}_{n\in\mathbb{N}}$ are monotonically increasing. The orbit $O(x)$ is not closed and we have

$$\lim_{n\to\infty} \varphi(\sigma_n, x) = x^* .$$

Proof. Due to Proposition 6, there exist a transversal neighbourhood V of $x^* \in S$ and a transversal time $\tau : V \to \mathbb{R}$ with

$$\varphi(\tau(\bar{x}), \bar{x}) \in S \quad \text{for all } \bar{x} \in V .$$

Since $x^* \in \omega(x)$, there exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$z_n := \varphi(t_n, x) \in V \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} z_n = x^* .$$

We thus get

$$\varphi(\tau(z_n), z_n) \in S \quad \text{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \varphi(\tau(z_n), z_n) = \varphi(\underbrace{\tau(x^*)}_{=0}, x^*) = x^* .$$

Define $\sigma_n := t_n + \tau(z_n)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \sigma_n = \infty$, we can assume without loss of generality that σ_n is strictly increasing. Then we have

$$\varphi(\sigma_n, x) = \varphi(\tau(z_n), \varphi(t_n, x)) \in S \quad \text{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \varphi(\sigma_n, x) = \lim_{n\to\infty} \varphi(\tau(z_n), z_n) = x^* .$$

Either the points $\varphi(\sigma_n, x)$ are pairwise disjoint or all points $\varphi(\sigma_n, x)$ are the same. The latter case corresponds to (i) (i.e. we have a periodic orbit). The first case corresponds to (ii), and the orbit $O(x)$ is clearly not closed, since the intersection points are monotone due to Proposition 5. □

We now have all ingredients to prove the Poincaré–Bendixson theorem.

Proof of Theorem 1. Due to Proposition 4.21, the omega limit set $\omega(x)$ is non-empty, compact and invariant. In addition, $\omega(x)$ is connected due to an exercise on a problem sheet. The remaining proof is divided into two steps.

Step 1. We show that if $\omega(x)$ contains a periodic orbit, the $\omega(x)$ is equal to this periodic orbit.
Let $O(y) \subset \omega(x)$ be a periodic orbit. We assume that $\omega(x) \setminus O(y) \neq \emptyset$. Then $\omega(x) \setminus O(y)$ is not closed, since otherwise, $\omega(x) = (\omega(x) \setminus O(y)) \cup O(y)$ is the disjoint union of two closed sets, which would mean that $\omega(x)$ is disconnected. This implies that there exists a sequence $z_n \in \omega(x) \setminus O(y)$ with

$$z^* = \lim_{n\to\infty} z_n \notin \omega(x) \setminus O(y) .$$

However, $z^* \in \omega(x)$, which implies that $z^* \in O(y)$. In particular, z^* is not an equilibrium, and due to Proposition 3, there exists a transversal S through z^*. We choose a transversal neighbourhood V of z^*, and let $\tau : V \to \mathbb{R}$ be the corresponding transversal time. There exists a $n_0 \in \mathbb{N}$ such that $z_{n_0} \in V$. Then $\varphi(\tau(z_{n_0}), z_{n_0}) \in S$. Since $z_{n_0} \in \omega(x)$ and $\omega(x)$ is invariant, we get that $\varphi(\tau(z_{n_0}), z_{n_0}) \in \omega(x)$. This means that $\omega(x)$ intersects the transversal S in at least two different points: $\varphi(\tau(z_{n_0}), z_{n_0})$ and z^*. This contradicts Proposition 7 and finishes the proof of Step 1.

Step 2. We distinguish between three cases corresponding to (i), (ii), and (iii) in the statement of the theorem.

Case (i). $\omega(x)$ contains only equilibria.
Since $\omega(x)$ can contain only finitely many equilibria and $\omega(x)$ is connected, $\omega(x)$ contains of one equilibrium.

Case (ii). $\omega(x)$ does not contain equilibria.
Let $z \in \omega(x)$. Then the invariance of $\omega(x)$ implies that

$$O(z) \subset \omega(x)\,.$$

Since $\omega(x)$ is compact, there exists an omega limit point z^* of z, and we have $z^* \in \omega(x)$, which implies in particular that z^* is not an equilibrium. Due to Proposition 3, there exists a transversal S through z^*. Proposition 8 implies that $O(z) \cap S$ is either a singleton (which implies that $O(z)$ is a periodic orbit) or $O(z) \cap S$ is countable infinite. The latter is not possible, since

$$O(z) \cap S \subset \omega(x) \cap S\,,$$

which is a singleton due to Proposition 7. This means that $\omega(x)$ contains the periodic orbit $O(z)$, and Step 1 implies that $\omega(x)$ equals the periodic orbit $O(z)$.

Case (iii). $\omega(x)$ contains both equilibria and non-equilibria.
We show that for all non-equilibria $z \in \omega(x)$, both $\alpha(z)$ and $\omega(z)$ are singletons containing equilibria. Assume to the contrary that $\alpha(z)$ or $\omega(z)$ contains a non-equilibrium points. As in Case (ii), we get the existence of a periodic orbit in $\omega(x)$. Due to Step 1, $\omega(x)$ is a periodic orbit. This is a contradiction, since $\omega(x)$ contains an equilibrium. □