Mathematics Year 1, Calculus and Applications I

Midterm Exam, November 2019

You have 45 minutes to complete the paper

Each question is worth 10 marks for a total of 30.

1. The motion of a particle is given by the parametric curve $x = t - \sin t$, $y = 1 - \cos t$, for $t \geq 0$.

   (a) At $t = 0$ the particle is at the origin $x = y = 0$. Show that for $t > 0$, both x and y are non-negative.

   (b) Find the times and corresponding coordinates of all points when the particle is on the $x-$axis.

   (c) Calculate dy/dx (you may leave your answer in terms of t), and hence determine all critical points of the function, stating whether they are local maxima or local minima.

   (d) Calculate $\lim_{x\to 0+}(dy/dx)$, $\lim_{x\to 2\pi-}(dy/dx)$ and $\lim_{x\to 2\pi+}(dy/dx)$, and use your results to determine whether dy/dx exists for all x.

   (e) Sketch the particle trajectory over the time interval $0 \leq t \leq 5\pi$.

2. (a) Given functions $f(x)$ and $g(x)$, what do we mean by the composite function $(f \circ g)(x)$?

   (b) Calculate $\frac{d}{dx}(f \circ g)$ and $\frac{d}{dx}(g \circ f)$. Assuming the functions are not constants, when can the two derivatives be equal?

   (c) Suppose that $\frac{d}{dx} f(x^2) = \frac{d}{dx}(f(x))^2$ at $x = 1$. Prove that $f'(1) = 0$ or $f(1) = 1$.

3. Find the following limits justifying the use of any tools you use.

   (a) $\lim_{x\to 0+}(\cos x)^{1/x}$.

   (b) $\lim_{x\to\infty}(\sin e^{-x})^{1/\sqrt{x}}$.

   (c) $\lim_{x\to 0+} \exp\left(-x^{\log x}\right)$.

SOLUTIONS

1. (a) Clearly $y \geq 0$ since $|\cos t| \leq 1$. Now $dx/dt = 1 - \cos t$ and so $dx/dt \geq 0$ implying that x is increasing almost everywhere (it is only zero at a discrete set of points). Hence $x, y \geq 0$ for all t. **1 mark**

   (b) The particle is on the $x-$axis when $y = 0$, i.e. $\cos t = 1$, hence $t = 0, 2\pi, 4\pi, ...$ At all such times, $\sin t = 0$ hence $x = 0, 2\pi, 4\pi, ...$ **1 mark**

   (c)
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$ **1 mark**

   Hence critical points when $\sin t = 0$ AND $1 - \cos t \neq 0$, i.e. $t = \pi, 3\pi, \ldots,$ implying $x = \pi - 1, 3\pi - 1, \ldots$ with the all corresponding y values being $y = 2$. These are local maxima since $1 - \cos t \leq 2$ for all t (there is no need for second derivative tests here). **2 marks**

   (d) Here I used L'Hôpital's rule since all limits are of the form $0/0$.

$$\lim_{x \to 0+} \left[\frac{dy}{dx}\right] = \lim_{t \to 0+} \left[\frac{\sin t}{1 - \cos t}\right] = \lim_{t \to 0+} \left[\frac{\cos t}{\sin t}\right] = +\infty$$ **1 mark**

$$\lim_{x \to 2\pi-} \left[\frac{dy}{dx}\right] = \lim_{t \to 2\pi-} \left[\frac{\sin t}{1 - \cos t}\right] = \lim_{t \to 2\pi-} \left[\frac{\cos t}{\sin t}\right] = -\infty$$ **1 mark**

$$\lim_{x \to 2\pi+} \left[\frac{dy}{dx}\right] = \lim_{t \to 2\pi+} \left[\frac{\sin t}{1 - \cos t}\right] = \lim_{t \to 2\pi+} \left[\frac{\cos t}{\sin t}\right] = +\infty$$ **1 mark**

   Clearly dy/dx does not exist for all x. **-1 mark if this is not stated**

   (e) This is a cycloid. Over the interval $0 \leq t \leq 5\pi$ it looks like in figure 1. **2 marks**

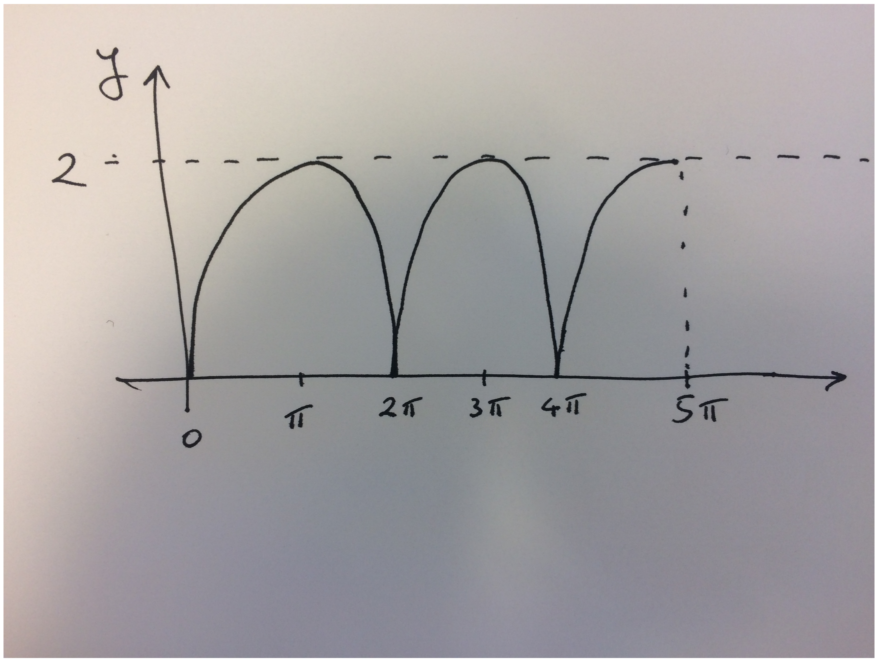

Figure 1: The sketch of question 1(e).

2. (a) $f \circ g = f(g(x))$. **2 marks**

   (b) Using the chain rule, $\frac{d}{dx}(f \circ g) = f'(g(x)).g'(x)$ and $\frac{d}{dx}(g \circ f) = g'(f(x)).f'(x)$. The two derivatives are equal if $f(x) = g(x) = x$, for example. Any $f(x) = g(x)$ will also do, but give full marks for finding one. **4 marks**

   (c) Calculate the derivatives $\frac{d}{dx} f(x^2) = 2xf'(x^2)$ and $\frac{d}{dx}(f(x))^2 = 2f(x)f'(x)$. At $x = 1$ we have

$$2f'(1) = 2f(1)f'(1), \qquad \Rightarrow \qquad f'(1) = 0 \quad \text{or} \quad f(1) = 1.$$

4 marks

3. (a) $(\cos x)^{1/x} = \exp(\log[(\cos x)^{1/x}]) = \exp\left(\frac{1}{x}\log[\cos x]\right)$. Now $\lim_{x\to 0+} \frac{\log \cos x}{x} = \lim_{x\to 0+} \frac{-\tan x}{1} = 0$, and since the exponential function is continuous we have the required limit being $\exp(0) = 1$. **2 marks**

   (b) Consider first $\frac{\log(\sin e^{-x})}{\sqrt{x}}$ which is of the form ∞/∞ as $x \to \infty$. Using L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{\log(\sin e^{-x})}{\sqrt{x}} = \lim_{x\to\infty} \left(\frac{-e^{-x}\cot e^{-x}}{(1/2)x^{-1/2}} \right) = -2 \lim_{x\to\infty} \left(\frac{x^{1/2}}{e^x \sin e^{-x}} \right)$$

$$= -2 \lim_{x\to\infty} \left(\frac{x^{1/2}}{e^x e^{-x}} \left[\frac{e^{-x}}{\sin e^{-x}} \right] \right) = -2 \lim_{x\to\infty} (x^{1/2}) = -\infty.$$

   Since $(\sin e^{-x})^{1/\sqrt{x}} = \exp\left(\frac{1}{\sqrt{x}} \log \sin e^{-x}\right)$, the required limit is $\exp(-\infty) = 0$.

5 marks

   (c) Consider $x^{\log x}$ and write it as $\exp(\log[x^{\log x}]) = \exp([\log x]^2)$ which tends to $+\infty$ as $x \to 0+$. Hence $\lim_{x\to 0+} \exp(-x^{\log x}) = 0$. **3 marks**