Imperial College
London

MATH97065 MATH97176 MATH97284

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamical Systems

Date: Friday, 28 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let $f : [-1,1] \to [-1,1]$ be defined as $f(x) = x^3$. List all ω-limit sets of the map f and determine which of these ω-limit sets are attractors. (4 marks)

(b) For each of the following descriptions of maps, determine whether such maps exist. Give (a sketch of) such an example or an argument why no such map exists.

(i) A continuous map of the unit interval $g : [0,1] \to [0,1]$ that is chaotic but not topologically mixing. (6 marks)

(ii) A continuous map of the unit circle $g : S^1 \to S^1$ that is chaotic but does not have periodic orbits of all periods. (4 marks)

(c) Let $T, V : [0,1] \to [0,1]$ be the "tent map" and "V map", defined as

$$T(x) = \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2-2x, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases} \qquad V(x) = \begin{cases} 1-2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2x-1, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases} \tag{1}$$

see Fig. 1(i,ii) for their graphs. Determine whether T is topologically conjugate to V. Motivate your answer. (6 marks)

(Total: 20 marks)

2. Let $h_{\text{top}}(f)$ denote the topological entropy of a continuous map $f : [0,1] \to [0,1]$, where $[0,1]$ is endowed with the metric $d(x,y) = |x-y|$.

(a) Prove that $h_{\text{top}}(f^m) = m \cdot h_{\text{top}}(f)$ for all $m \in \mathbb{N}$. (8 marks)

(b) Let f be given by

$$f(x) := \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{4}{3} - \frac{4}{3}x, & \text{if } x \in \left(\frac{1}{4}, 1\right]. \end{cases} \tag{2}$$

see Fig. 1(iii) for its graph. Determine $h_{\text{top}}(f)$. Motivate your answer. (6 marks)

(c) Prove that $h_{\text{top}}(f) = 0$ if f is invertible with a continuous inverse. [Hint: For all $n, p \in \mathbb{N}$, construct $(n, \frac{1}{p})$-spanning sets from $\{0, \frac{1}{p}, \dots, 1\}$, which is $(0, 1/p)$-spanning.] (6 marks)

(Total: 20 marks)

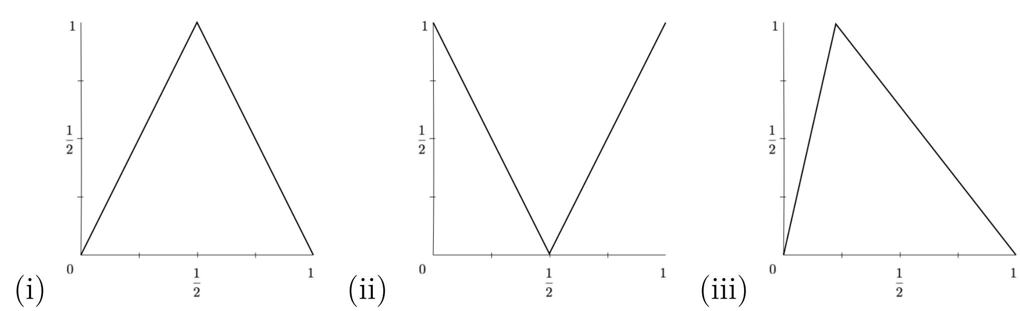

Figure 1: Graphs of the interval maps (i) T and (ii) V of Question 1(c), and (iii) f of Question 2(c).

3. Consider the map $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} \frac{2}{3} - 2x, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ 3x - 1, & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right], \\ 3 - 3x, & \text{if } x \in \left(\frac{2}{3}, 1\right], \end{cases} \tag{3}$$

see also Fig. 2.

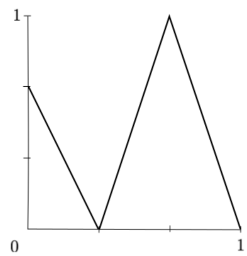

Figure 2: Graph of the map g (3).

Let σ_A denote the shift map on the topological Markov chain $\Sigma^+_{3,A}$ with connectivity matrix A (with symbols from $\{0, 1, 2\}$), where

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and $h : \Sigma^+_{3,A} \to [0,1]$ defined as

$$h(\omega_0\omega_1\omega_2\ldots) = \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} g^{-i}(I_{\omega_i})}. \tag{4}$$

where $I_0 := (0, \frac{1}{3})$, $I_1 := (\frac{1}{3}, \frac{2}{3})$ and $I_2 := (\frac{2}{3}, 1)$.

(a) (i) Determine $h(\overline{01})$ and $h(1\overline{0})$. (4 marks)

(ii) Determine $h^{-1}(0)$ and $h^{-1}(\frac{2}{3})$. (3 marks)

(b) Determine whether g has periodic orbits of all periods. Motivate your answer. (2 marks)

(c) Determine whether the following implications are true: give a proof or a counterexample.

(i) $g^n(x) \in I_j \;\Rightarrow\; \sigma_A^n(h^{-1}(x))$ starts with the symbol j, (3 marks)

(ii) $\sigma_A^n(\omega)$ starts with the symbol $j \;\Rightarrow\; g^n(h(\omega)) \in I_j$. (3 marks)

(d) Show that the map g, restricted to the set $N([\frac{1}{3}, 1])$ of points whose iterates under g do not leave $[\frac{1}{3}, 1]$, is topologically conjugate to the full shift on two symbols. (5 marks)

(Total: 20 marks)

4. Consider the map g, as defined in Question 3, equation (3) and Fig. 2. Let the measure ν be the Borel measure with Lebesgue density

$$p(x) := \begin{cases} \frac{6}{5} & \text{if } x \in [0, \frac{2}{3}], \\ \frac{3}{5} & \text{if } x \in (\frac{2}{3}, 1], \end{cases}$$

(a) Show that ν is an invariant measure for g. (4 marks)

(b) Find the Markov measure $\mu_{v,P}$ on $\mathscr{B}(\Sigma^+_{3,A})$ so that $\nu = h_* \mu_{v,P}$, with h defined as in (4). (6 marks)

(c) Let $B_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} g^i(x)$. Determine whether the following statements are true or false. Motivate your answers.

(i) The limit $\lim_{n\to\infty} B_n(x)$ exists for Lebesgue almost all $x \in [0, 1]$. (3 marks)

(ii) The set of points $x \in [0, 1]$ for which the limit $\lim_{n\to\infty} B_n(x)$ does not exist, is dense in $[0, 1]$. (4 marks)

(d) Determine $a \in [0, 1]$ such that $\lim_{n\to\infty} B_n(x) = a$ for ν-almost all $x \in [0, 1]$. Discuss the meaning of a. Are there points for which the limit exists, but is different from a? (3 marks)

(Total: 20 marks)

5. [In your answers to this question, you may quote any relevant general result from [Brin & Stuck, Section 7.1], but any such result should be explicitly referred to.]

Consider the two-parameter family of circle maps $f_{\alpha,\varepsilon} : S^1 \to S^1$ with lift

$$F_{\alpha,\varepsilon} : \mathbb{R} \to \mathbb{R},\ F_{\alpha,\varepsilon}(x) = x + \alpha + \frac{\varepsilon}{2\pi} \sin^2(2\pi x),$$

and $\alpha \in [0,1)$ and $\varepsilon \in [-1,1]$.

(a) Show that the maps $f_{\alpha,\varepsilon}$ are circle homeomorphisms. (4 marks)

Consider the rotation number

$$\rho(f_{\alpha,\varepsilon}) := \lim_{n\to\infty} \frac{1}{n} \left(F^n_{\alpha,\varepsilon}(x) - x \right) \mod 1$$

with $(\alpha,\varepsilon) \in [0,1) \times [-1,1]$ and $x \in \mathbb{R}$.

(b) Show that for each $\varepsilon \in [-1,1]$, we have $\{\rho(f_{\alpha,\varepsilon}) \mid \alpha \in [0,1)\} = [0,1)$. (4 marks)

(c) Show that $\rho(f_{\alpha,\varepsilon})$ is an increasing function of α and ε in the parameter domain $\{(\alpha,\varepsilon) \in [0,1) \times [-1,1] \mid 0 \le \alpha \le \min\{1 - \frac{\varepsilon}{2\pi}, 1\}\}$. (6 marks)

(d) Determine the values of $(\alpha,\varepsilon) \in [0,1) \times [-1,1]$ for which the rotation number satisfies $\rho(f_{\alpha,\varepsilon}) = \frac{1}{2}$. [Hint: note that $F_{\alpha,\varepsilon}(x + \frac{1}{2}) = F_{\alpha,\varepsilon}(x) + \frac{1}{2}$.] (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96040/MATH97065/MATH97176/MATH97285

Dynamical Systems (Solutions)

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1. (a) f has fixed points $\{-1,0,1\}$ and $f(x) < x$ on $(0,1)$ and $f(x) > x$ on $(-1,0)$. Hence, the fixed points are the only possible forward limit points of orbits and thus are the only ω-limit sets: $\boxed{\text{sim. seen} \Downarrow}$

$\omega(x) = \{-1\}$ if $x = -1$, $\omega(x) = \{0\}$ if $x \in (-1,1)$, $\omega(x) = \{1\}$ if $x = 1$. $\{0\}$ is an attractor, the other two ω-limits sets not. $\boxed{4, \text{A}}$

(b) (i) For example, let $\boxed{\text{sim. seen} \Downarrow}$

$$\tilde{g}(x) := \begin{cases} \tilde{g}_1(x), & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \tilde{g}_2(x), & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $\tilde{g}_1 : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ and $\tilde{g}_2 : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ are continuous, chaotic and topologically mixing satisfying $\tilde{g}_1 \circ R(x) = R \circ \tilde{g}_2(x)$, where $R(x) := 1 - x$ (the reflection in $x = \frac{1}{2}$). For instance, $\tilde{g}_2$ can be taken to be the (scaled) tent map on $[0, \frac{1}{2}]$: $\tilde{g}_2(x) = 2x - \frac{1}{2}$ if $x \in [\frac{1}{2}, \frac{3}{4})$ and $\tilde{g}_2(x) = \frac{5}{2} - 2x$ if $x \in [\frac{3}{4}, 1]$. Let $g := R \circ \tilde{g}$. g is not topologically mixing, since $g((0, \frac{1}{2})) \cap (0, \frac{1}{2}) = \emptyset$ by construction. $\boxed{2, \text{B}}$

Then, g is chaotic: it suffices to observe that

$$g^2(x) := \begin{cases} \tilde{g}_1^2(x), & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \tilde{g}_2^2(x), & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

from which it follows that g has dense periodic orbits $\boxed{2, \text{C}}$

and that g is transitive (since $\tilde{g}_1$ and $\tilde{g}_2$ are topologically mixing). $\boxed{2, \text{D}}$

(ii) For instance, $g(x) = -2x \bmod 1$ has no orbits of period 2 but is chaotic (since $g^2(x) = 4x \bmod 1$). This follows from the fact that the number of fixed points of g^n is equal to $P_n(g) = |(-2)^n - 1|$. Hence $P_1(g) = P_2(g) = 3$ so g has no period 2 orbits. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{4, \text{A}}$

(c) T and V are topologically conjugate to the same shift map on two symbols, say 0 and 1, induced from the usual Markov partition of the interval into two halfs. However, this fact alone does not imply the topological conjugacy. $\boxed{\text{unseen} \Downarrow}$ $\boxed{1, \text{B}}$

Closer examination reveals that the symbolic labelling intervals are similarly arranged in either case apart from swapping the symbols 0 and 1. This leads to the educated guess that the conjugating map should be linear (as T and V are) and permute the labelling intervals for 0 and 1. $\boxed{2, \text{D}}$

The reflection R satisfies such and by direct calculation one indeed verifies that it conjugates T to V:

$$R \circ T(x) = \begin{cases} 1 - 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1 - (2 - 2x) = 2x - 1, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and

$$V \circ R(x) = \begin{cases} 2(1 - x) - 1 = 1 - 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1 - 2(1 - x) = 2x - 1, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

$\boxed{3, \text{B}}$

2. (a) Note that for all $m, n \in \mathbb{N}_0$ we have $\max_{0\leq i<n} d(f^{mi}(x), f^{mi}(y)) \leq \max_{0\leq i<mn} d(f^i(x), f^i(y))$. Thus $\operatorname{span}(n, \varepsilon, f^m) \leq \operatorname{span}(mn, \varepsilon, f)$ which implies (by the definition of topological entropy) that $h_{\text{top}}(f^m) \leq m \cdot h_{\text{top}}(f)$. By uniform continuity of f it follows that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ (with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$) such that $d(x, y) < \delta(\varepsilon)$ implies that $d(f^i(x), f^i(y)) < \varepsilon$ for all $i \in \{0, \ldots, m\}$. Consequently, $\operatorname{span}(n, \delta(\varepsilon), f^m) \geq \operatorname{span}(mn, \varepsilon, f)$, so $h_{\text{top}}(f^m) \geq m \cdot h_{\text{top}}(f)$. $\boxed{\text{part seen} \Downarrow}$ $\boxed{8, \text{A}}$

(b) f admits the piecewise expanding Markov partition $\{(0, \frac{1}{4}), (\frac{1}{4}, 1)\}$: f is expanding on either partition element and $\overline{f((0, \frac{1}{4}))} = \overline{f((\frac{1}{4}, 1))} = [0, 1]$. Hence, f is topologically semi-conjugate to (as a factor of) the full shift $\sigma : \Sigma_2^+ \to \Sigma_2^+$. This implies that $h_{\text{top}}(f) \leq h_{\text{top}}(\sigma) = \ln(2)$. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{3, \text{A}}$

Let $h : \Sigma_2^+ \to [0, 1]$ denote the continuous map that semi-conjugates f to σ, i.e. $h \circ \sigma = f \circ h$. From the construction of labelling partitions, it follows that at most two boundaries of labelling intervals can meet at any single point in $[0, 1]$, at each partition refinement. In other words, $\operatorname{Card}(h^{-1}(x)) \leq 2 < \infty$ for all $x \in [0, 1]$. By a result from the course, this implies that $h_{\text{top}}(f) = h_{\text{top}}(\sigma) = \ln(2)$. $\boxed{3, \text{B}}$

(c) $A_{0,\frac{1}{p}} := \{0, \frac{1}{p}, \ldots, \frac{p-1}{p}, 1\}$ is $(0, \frac{1}{p})$-spanning. $\boxed{\text{unseen} \Downarrow}$
Then $A_{n,\frac{1}{p}} := \cup_{i=1}^n f^{-i}(A_{0,\frac{1}{p}})$ is $(n, \frac{1}{p})$-spanning.
Since f is invertible, it follows that $\operatorname{Card}(A_{n,\frac{1}{p}}) \leq n \cdot \operatorname{Card}(A_{0,\frac{1}{p}})$.
Hence $\operatorname{span}(n, 1/p, f) \leq n\operatorname{Card}(A_{0,\frac{1}{p}})$, which implies that $\lim_{n\to\infty} \frac{1}{n} \ln \operatorname{span}(n, 1/p, f) = 0$ for all $p \in \mathbb{N}$. Hence $h_{\text{top}}(f) = 0$. $\boxed{6, \text{D}}$

3. (a) (i) $h(\overline{01})$ is the unique 2-periodic point of g that oscillates between I_0 and I_1. The point $x \in I_0$ on this orbit satisfies $3(\frac{2}{3} - 2x) - 1 = x \Leftrightarrow x = \frac{1}{7}$. $h(\overline{0}) = \frac{2}{9}$ is the unique fixed point of g in I_0. $\frac{2}{3} - 2x = x \Leftrightarrow x = \frac{2}{9}$. $h(1\overline{0})$ is the unique preimage of $\frac{2}{9}$ in I_1: $3x - 1 = \frac{2}{9} \Leftrightarrow x = \frac{11}{27}$.

sim. seen ⇓

4, A

(ii) $h^{-1}(0) = \overline{012}$, since $g(0) = \frac{2}{3}$, $g(\frac{2}{3}) = 1$ and $g(1) = 0$. Since, say, $g((0, 1/100)) \subset I_1$, indeed the second symbol is unambiguously 1. $h^{-1}(\frac{2}{3}) = \{\overline{120}, 2\overline{201}\}$. The point $\frac{2}{3}$ lies on the same period 3 orbit as before. However, there are two ways to approach $\frac{2}{3}$, from the left and the right, giving rise to two symbolic representations, one sequence starting with 1 and the other one starting with 2.

sim. seen ⇓

3, C

(b) The map g has period 3 orbit $\{0, \frac{2}{3}, 1\}$. This implies by Sharkovskii's Theorem that g has periodic orbits of all periods.

unseen ⇓

2, B

(c) (i) True. $g^n(x) \in I_j \Rightarrow x \in g^{-n}(I_j) \Rightarrow h^{-1}(x) \in h^{-1}(g^{-n}(I_j)) \Rightarrow \sigma_A^n h^{-1}(x) \in \sigma_A^n h^{-1}(g^{-n}(I_j)) = h^{-1}(I_j) \Rightarrow I_j \ni h(\sigma_A^n h^{-1}(x)) \Rightarrow (\sigma_A^n h^{-1}(x))_0 = j$, by the definition of h and the fact that $x \notin I_i$ for all $i \neq j$ (since $I_i \cap I_j = \emptyset$ if $i \neq j$).

unseen ⇓

3, C

(ii) Not true. For instance consider $\overline{012}$ starts with 0 but $h(\overline{012}) = 0 \notin I_0$.

unseen ⇓

3, A

(d) Consider the Markov Partition $\mathcal{R} := \{I_1, I_2\}$ of $[\frac{1}{3}, 1]$. The refined Markov partitions $\mathcal{R}_n$ are such that for each $n > 2$ the closure of the union of its elements consists of 2^{n-2} disjoint intervals of width 3^{-n-1}. These intervals converge uniformly to $N(U)$. $N(U)$ is a Cantor set and the dynamics on $N(U)$ is topologically conjugate to the full shift on Σ_2^+. The conjugacy arises due to the absence of ambiguity of the coding as (eventually) the closures of the labelling intervals are disjoint.

sim. seen ⇓

5, B

4. (a) Write a measurable set A as the disjoint union $A = A_0 \cup A_1 \cup A_2$, where $A_i \subset \overline{I_i}$. Then $\nu(A) = \sum_{i=0}^{2} \nu(A_i)$. To show that $\nu(g^{-1}(A)) = \nu(A)$ it suffices to check this for each A_i. We have $\nu(A_0) = \frac{6}{5}|A_0|$, $\nu(A_1) = \frac{6}{5}|A_1|$, $\nu(A_2) = \frac{3}{5}|A_2|$. With unseen ⇓

$$\nu(g^{-1}(A_j)) = \sum_{i=0}^{2} \int_{I_i} g^{-1}(A_j)p(x)dx = \sum_{i=0}^{2} \tau_i^{-1}|A_j| \int_{I_i} p(x)dx$$

where $\tau_j = |g'(x)|$ for all $x \in I_j$, we have $\nu(g^{-1}(A_0)) = |A_0|\left(\frac{1}{2}\cdot\frac{6}{5} + \frac{1}{3}\cdot\frac{6}{5} + \frac{1}{3}\cdot\frac{3}{5}\right) = \frac{6}{5}|A_0|$, $\nu(g^{-1}(A_1)) = |A_1|\left(\frac{1}{2}\cdot\frac{6}{5} + \frac{1}{3}\cdot\frac{6}{5} + \frac{1}{3}\cdot\frac{3}{5}\right) = \frac{6}{5}|A_1|$, $\nu(g^{-1}(A_2)) = |A_2|\left(\frac{1}{3}\cdot\frac{6}{5} + \frac{1}{3}\cdot\frac{3}{5}\right) = \frac{3}{5}|A_2|$. 4, B

(b) The transition matrix for the Markov chain is derived from A by $P_{ij} = |\tau_i^{-1}|A_{ij}$, yielding meth seen ⇓

$$P := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

The unique left probability eigenvector of P for eigenvalue 1 is $v = \frac{1}{5}(2,2,1)$. P and v define the relevant Markov measure $\mu_{v,P}$. From lecture notes: for cylinder sets $C_{i_0\cdots i_{n-1}} \subset \Sigma_{3,A}^+$ we have $h_*\mu_{v,P}(C_{i_0\cdots i_{n-1}}) = \frac{v_{i_0}}{|I_{i_0}|}|h(C_{i_0\ldots,i_{n-1}})|$, where v_i denotes the ith component of v. As $|I_0| = |I_1| = |I_2| = \frac{1}{3}$, it follows that $h_*\mu_{v,P}(C_{i_0\cdots i_{n-1}}) = \frac{6}{5}|h(C_{i_0\ldots,i_{n-1}})|$ if $i_0 \in \{0,1\}$ and $h_*\mu_{v.P}(C_{i_0\cdots i_{n-1}}) = \frac{3}{5}|h(C_{i_0\ldots,i_{n-1}})|$ if $i_0 = 2$, so that $h_*\mu_{v,P} = \nu$. 6, D

(c) (i) By Birhoff's Ergodic Theorem, the statement holds for ν-almost all initial conditions. As ν is equivalent to the Lebesgue measure, the statement also applies when ν is replaced by the Lebesgue measure. unseen ⇓ 3, A

(ii) For instance, the limit does not exist if $B_n(x)$ oscillates between values below 0.4 and above 0.6, say. This would happen if an orbit remains for a sufficiently long time first in I_0, until some time n_0 at which $B_{n_0}(x) < 0.4$ and then moves to I_3 where it stays long enough (until time n_1) to find $B_{n_1}(x) > 0.6$ and then again migrates back to I_0 and stays there until time n_2 when $B_{n_2}(x) < 0.4$, etc. The existence of such an orbit is guaranteed by the symbolic dynamics, Namely, $0^{n_0}12^{n_1-1}0^{n_2}12^{n_3-1}\ldots$ is in $\Sigma_{3,A}^+$. sim. seen ⇓
The above sequence can arise in the tail of a sequence in any cylinder set of $\Sigma_{3,A}^+$, which implies that such sequences are dense in $\Sigma_{3,A}^+$ and by continuity and surjectivity of h also dense, also dense in $[0,1]$. 4, C

(d) By Birkhoff's Ergodic Theorem, $\lim_{n\to\infty} B_n(x) = \int_0^1 yp(y)dy = \frac{6}{5}\int_0^{\frac{2}{3}} ydy + \frac{3}{5}\int_{\frac{2}{3}}^1 ydy = \frac{13}{30}$, ν-almost surely. a is the average position along (typical) forward orbits of g. There are many other possible values for this limit, eg $\lim_{n\to\infty} B_n(\frac{1}{2}) = g(\frac{1}{2}) = \frac{1}{2}$. Most periodic orbits will be among the ν-measure zero set of exceptions. sim. seen ⇓ 3, A

5. (a) $F_{\alpha,\varepsilon}$ is continuous and continuously differentiable, and $F_{\alpha,\varepsilon}(x)+1 = F_{\alpha,\varepsilon}(x+1)$. $F'_{\alpha,\varepsilon}(x) = 1+\varepsilon\sin(4\pi x)$ from which it follows that as long as $|\varepsilon| \leq 1$, F' does not change sign, implying that F is invertible. $\boxed{4, \mathrm{M}}$

(b) Let $\rho(F_{\alpha,\varepsilon}) := \lim_{n\to\infty} \frac{1}{n}\left(F^n_{\alpha,\varepsilon}(x) - x\right)$ with $(\alpha,\varepsilon) \in [0,1) \times [-1,1]$ and $x \in \mathbb{R}$. Note that $\rho(f_{\alpha,\varepsilon}) = \rho(F_{\alpha,\varepsilon}) \mod 1$. $\rho(F_{\alpha,\varepsilon})$ is a continuous function of α, ε due to continuity of $F_{\alpha,\varepsilon}$ with respect to these variables.

It is convenient now to consider $\alpha \in \mathbb{R}$.

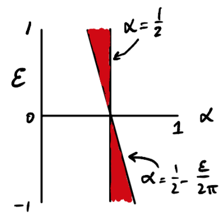
As $F_{\alpha,\varepsilon}(x+1) = F_{\alpha,\varepsilon}(x)+1 = F_{\alpha+1,\varepsilon}(x)$, it follows that $F^n_{\alpha+1,\varepsilon}(x) = F^n_{\alpha,\varepsilon}(x)+n$. In turn, it follows that $\rho(F_{\alpha+m,\varepsilon}) = \rho(F_{\alpha,\varepsilon}) + m$ for all $m \in \mathbb{Z}$.

By the continuous dependence of $\rho(F_{\alpha,\varepsilon})$ on α, it thus follows that $\{\rho(F_{\alpha,\varepsilon}) \mid \alpha \in \mathbb{R}\} \supset [0,1]$ and in particular that $\{\rho(f_{\alpha,\varepsilon}) := \rho(F_{\alpha,\varepsilon}) \mod 1 \mid \alpha \in [0,1)\} \supset [0,1)$. $\boxed{4, \mathrm{M}}$

(c) For all $\alpha \geq \alpha'$ and $\varepsilon \geq \varepsilon'$, it holds that $F_{\alpha,\varepsilon}(x) \geq F_{\alpha',\varepsilon'}(x)$ for all $x \in \mathbb{R}$. This implies (using the definition of ρ in part (b)) that $\rho(F_{\alpha,\varepsilon}) \geq \rho(F_{\alpha',\varepsilon'})$.

Since $F_{0,\varepsilon}(0) = 0$ it follows that $\rho(F_{0,\varepsilon}) = 0$. With increasing α, from 0, $\rho(F_{\alpha,\varepsilon})$ increases to reach the value of 1 when there exists x such that $F_{\alpha,\varepsilon}(x) = x+1$, or equivalently when there exists x such that $\frac{\varepsilon}{2\pi}\sin^2(2\pi x) = 1-\alpha$, at which point the rotation number jumps discontinuously to 0. This first occurs when $\alpha = 1 - \frac{\varepsilon}{2\pi}$ when $\varepsilon > 0$ or $\alpha = 1$, otherwise. This marks the domain across which $\rho(f_{\alpha,\varepsilon})$ is increasing in both α and ε. $\boxed{6, \mathrm{M}}$

(d) We have $\rho(f_{\alpha,\varepsilon}) = 1/2$ iff $F^2_{\alpha,\varepsilon}(x) = x+1$ for some x. Using the hint, this is satisfied if x is such that $F_{\alpha,\varepsilon}(x) = x + \frac{1}{2}$. It turns out that this is also a necessary condition. Namely, if $F_{\alpha,\varepsilon}(x) > x + \frac{1}{2}$ for all x then $\rho(f_{\alpha,\varepsilon}) > \frac{1}{2}$, and if $F_{\alpha,\varepsilon}(x) < x + \frac{1}{2}$ for all x then $\rho(f_{\alpha,\varepsilon}) < \frac{1}{2}$. The corresponding condition is $x + \alpha + \frac{\varepsilon}{2\pi}\sin^2(2\pi x) = x + \frac{1}{2}$, or in other words $\alpha + \frac{\varepsilon}{2\pi}\sin^2(2\pi x) = \frac{1}{2}$. Since $\sin^2(2\pi x) \in [0,1]$ this implies, taking $\alpha \in [0,1)$ that $\frac{1}{2} \geq \alpha \geq \frac{1}{2} - \frac{\varepsilon}{2\pi}$ if $\varepsilon \geq 0$ and $\frac{1}{2} \leq \alpha \leq \frac{1}{2} - \frac{\varepsilon}{2\pi}$ if $\varepsilon \leq 0$.



$\boxed{6, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total Mastery marks: 20 of 20 marks
Total marks: 100 of 100 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH96040/MATH9706	1 to 5	All of the exam questions showed a broad variety of scores by students, from full to low marks. There were no particular trends or question-specific observations.
	2	
	3	
	4	
	5	