

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**  
**May-June 2022**

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Algebraic Curves**

Date: 30 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD  
WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS  
ANSWERED AND PAGE NUMBERS PER QUESTION.**

You may use the results and exercises from the lecture notes, problem sheets and coursework, but make sure to clearly indicate what you are using in your answers. Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.

1. (a) Let  $C$  be the cubic curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_1^2 x_2 - x_0^3 - x_2^3 = 0$ .
  - (i) Show that  $C$  is smooth. (2 marks)
  - (ii) Compute the tangent line of  $C$  at  $[0, 1, 0]$ . (2 marks)
  - (iii) Show that  $[0, 1, 0]$  is an inflection point of  $C$ . (4 marks)
  - (iv) We denote by  $L$  the line in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0 + x_1 = 0$ . For every point  $p$  of  $C$ , we denote by  $M_p$  the unique line in  $\mathbb{P}_{\mathbb{C}}^2$  through  $p$  and  $[1, 1, 0]$ , and by  $h(p)$  the unique intersection point of  $L$  with  $M_p$ . This defines a morphism of Riemann surfaces

$$h: C \mapsto L, p \mapsto h(p).$$

What is the ramification degree of  $h$  at  $q = [0, 1, 0]$ ? Justify your answer. (4 marks)

- (b) Prove that every isomorphism of Riemann surfaces  $\Phi: \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a projective transformation of  $\mathbb{P}_{\mathbb{C}}^1$ . *Hint: first reduce to the case where  $\Phi$  has a fixed point.* (8 marks)

(Total: 20 marks)

2. (a) Let  $C$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^3 x_2^3 + x_1^6 - x_2^6 = 0$ . Let  $D$  be the line in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_1 - x_2 = 0$ . Compute the intersection points of  $C$  and  $D$ , and the intersection multiplicity of  $C$  and  $D$  at each intersection point. (6 marks)
- (b) Let  $C$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^2 + x_1^2 + x_2^2 = 0$ . Let  $D$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^3 + x_1^3 + x_2^3 = 0$ . Determine the cardinality of  $C \cap D$ , and justify your answer. (6 marks)
- (c) We say that an affine plane curve  $C$  in  $\mathbb{C}^2$  is *closed under complex conjugation* if, for every point  $(u, v)$  on  $C$ , the complex conjugate  $(\bar{u}, \bar{v})$  also lies on  $C$ . Show that an affine plane curve  $C$  in  $\mathbb{C}^2$  is closed under complex conjugation if and only if it has an equation of the form  $P(x, y) = 0$  where  $P \in \mathbb{R}[x, y]$ . (8 marks)

(Total: 20 marks)

3. A circle in  $\mathbb{C}^2$  is an affine plane curve defined by an equation of the form

$$(x - a)^2 + (y - b)^2 - c = 0$$

where  $a, b, c$  are real numbers and  $c > 0$ . The point  $(a, b)$  in  $\mathbb{R}^2$  is then called the *centre* of  $C$ . You may freely use that this equation is always irreducible.

- (a) Let  $C$  be an irreducible affine plane curve of degree 2 in  $\mathbb{C}^2$ . Show that the following are equivalent:
- (1) the curve  $C$  is a circle;
  - (2) the points at infinity of  $C$  are  $[1, i, 0]$  and  $[1, -i, 0]$ , the curve  $C$  contains a point of  $\mathbb{R}^2$ , and  $C$  is closed under complex conjugation (see Question 2(c)).
- (6 marks)
- (b) Let  $p, q, r$  be non-collinear points in  $\mathbb{R}^2$ . Show that there exists a unique circle  $C$  in  $\mathbb{C}^2$  that passes through  $p, q, r$ .
- (4 marks)
- (c) Let  $C_1$  and  $C_2$  be two circles in  $\mathbb{C}^2$ , and denote by  $\overline{C}_1$  and  $\overline{C}_2$  their respective projectivizations.
- (i) Show that  $\overline{C}_1$  and  $\overline{C}_2$  are tangent at a point at infinity if and only if  $C_1$  and  $C_2$  have the same center.
  - (ii) Assume that  $C_1$  and  $C_2$  intersect at a point  $p \in \mathbb{R}^2$  and that they are not tangent at  $p$ . Prove that  $C_1$  and  $C_2$  intersect in precisely one point  $q$  different from  $p$ .
  - (iii) Show that the point  $q$  from part (ii) has real coordinates.
- (4 marks)  
(2 marks)

(Total: 20 marks)

4. (a) For every divisor  $D$  in  $\mathbb{P}_{\mathbb{C}}^2$ , we denote its support by  $\text{supp}(D)$ . Consider a pencil of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^2$ , for some positive integer  $d$ , and let  $D_0, D_1$  and  $D_2$  be three distinct members of this pencil. Show that  $\text{supp}(D_0) \cap \text{supp}(D_1) = \text{supp}(D_0) \cap \text{supp}(D_2)$ .
- (6 marks)
- (b) Let  $\mathcal{L}_2$  be the complete linear system of effective divisors of degree 2 in  $\mathbb{P}_{\mathbb{C}}^2$ . Let  $p, q$  be two distinct points in  $\mathbb{P}_{\mathbb{C}}^2$ . Compute the dimension of the linear system of divisors  $D$  in  $\mathcal{L}_2$  such that  $\text{mult}_p D \geq 2$  and  $\text{mult}_q D \geq 2$ . Justify your answer.
- (6 marks)
- (c) We refer to Question 3 for the definition of a circle in  $\mathbb{C}^2$ . Let  $p, q, r$  be non-collinear points in  $\mathbb{R}^2$ . Let  $p', q', r'$  be points in the interiors of the line segments  $[qr]$ ,  $[pr]$  and  $[pq]$ , respectively. Let  $C_1, C_2$  and  $C_3$  be the unique circles through the sets  $\{p, q', r'\}$ ,  $\{p', q, r'\}$  and  $\{p', q', r\}$ . Assume that no two among these circles are tangent. Use the Cayley-Bacharach theorem and the results in Question 3 to show that  $C_1 \cap C_2 \cap C_3$  contains a point in  $\mathbb{R}^2$ . *Hint: consider cubics of the form  $\overline{C}_i \cup L_i$  where  $\overline{C}_i$  is the projectivization of  $C_i$  and  $L_i$  is a carefully chosen line.*
- (8 marks)

(Total: 20 marks)

5. In this question you may freely use that every connected compact Riemann surface satisfies the Riemann-Roch theorem from the mastery material. Let  $X$  be a connected compact Riemann surface of genus  $g$ .
- (a) Let  $p$  be a point of  $X$ . For every integer  $n \geq 0$ , compute the dimension of the vector space of meromorphic differential forms on  $X$  that have a pole of order at most  $n$  at  $p$  and that are holomorphic at all other points of  $X$ . (4 marks)
  - (b) Let  $p$  be a point of  $X$ . Show that there exists a non-constant meromorphic function on  $X$  that is holomorphic at all points of  $X \setminus \{p\}$ . (4 marks)
  - (c) Show that there exists a non-constant morphism of Riemann surfaces  $X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  of degree at most  $g + 1$ . (6 marks)
  - (d) Assume that  $g = 2$ . Show that there exists a meromorphic function on  $X$  that has two distinct poles of order 1 and that is holomorphic at all other points of  $X$ . (6 marks)

(Total: 20 marks)

Module: MATH60033/MATH70033/MATH97041  
Setter: Nicaise  
Checker: Corti  
Editor: Pal  
External: external  
Date: April 26, 2022  
Version: Final version

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2022

MATH60033/MATH70033/MATH97041 course name

*The following information must be completed:*

**Is the paper suitable for resitting students from previous years: Yes**

**Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):**

1(a)(i,ii,iii) 8 marks; 2(a) 6 marks; 3(a) first part 2 marks; 3(c) 10 marks; 4(a) 6 marks.

**Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):**

1(a)(iv) 4 marks; 2(b) 6 marks; 3(a) second part 4 marks; 4(b) 6 marks.

**Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):**

3(b) 4 marks; 2(c)(iii) 8 marks.

**Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):**

1(b) 8 marks; 4(c) 8 marks.

*Signatures are required for the final version:*

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science.

coursename

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has 4 Questions (*MATH96 version*); 5 Questions (*MATH97 versions*).

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

You may use the results and exercises from the lecture notes, problem sheets and coursework, but make sure to clearly indicate what you are using in your answers. Even if you are not able to solve one of the problems, you are still allowed to use the result to solve the other questions.

1. (a) Let  $C$  be the cubic curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_1^2 x_2 - x_0^3 - x_2^3 = 0$ .

(i) Show that  $C$  is smooth. (2 marks)

**Solution: Seen similar, category A.** The partial derivatives of our equation are given by  $-3x_0^2$ ,  $2x_1x_2$  and  $x_1^2 - 3x_2^2$ . Therefore, any singular point on  $C$  must satisfy  $x_0 = x_1 = x_2 = 0$ , which is impossible.

(ii) Compute the tangent line of  $C$  at  $[0, 1, 0]$ . (2 marks)

**Solution: Seen similar, category A.** Evaluating the partial derivatives of our equation at  $[0, 1, 0]$  yields 0, 0 and 1, respectively. Therefore, the tangent line of  $C$  at  $[0, 1, 0]$  is defined by  $x_2 = 0$ .

(iii) Show that  $[0, 1, 0]$  is an inflection point of  $C$ . (4 marks)

**Solution: Seen similar, category A.** Let  $M_q$  be the tangent line to  $C$  at  $q = [0, 1, 0]$ , defined by  $x_2 = 0$ . The intersection multiplicity of  $C$  and  $M_q$  at  $q$  is given by

$$\mathbf{I}(q, C, M_q) = \text{mult}_0(-t^3) = 3.$$

This means that  $q$  is an inflection point of  $C$ .

(iv) We denote by  $L$  the line in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0 + x_1 = 0$ . For every point  $p$  of  $C$ , we denote by  $M_p$  the unique line in  $\mathbb{P}_{\mathbb{C}}^2$  through  $p$  and  $[1, 1, 0]$ , and by  $h(p)$  the unique intersection point of  $L$  with  $M_p$ . This defines a morphism of Riemann surfaces

$$h: C \mapsto L, p \mapsto h(p).$$

What is the ramification degree of  $h$  at  $q = [0, 1, 0]$ ? Justify your answer. (4 marks)

**Solution: Seen similar, category B.** This ramification degree is equal to  $\mathbf{I}(q, C, M_q)$  (we have proved this in the lecture notes for the projection from  $[0, 0, 1]$  onto the line given by  $x_2 = 0$ , but we can reduce to this case by means of a projective transformation of  $\mathbb{P}_{\mathbb{C}}^2$ ). The line  $M_q$  through  $[1, 1, 0]$  and  $q$  is defined by  $x_2 = 0$ ; this is the tangent line to  $C$  at  $q$ . We have already computed in part (iii) that  $\mathbf{I}(q, C, M_q) = 3$ .

(b) Prove that every isomorphism of Riemann surfaces  $\Phi: \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a projective transformation of  $\mathbb{P}_{\mathbb{C}}^1$ . *Hint: first reduce to the case where  $\Phi$  has a fixed point.* (8 marks)

**Solution: Unseen, category D.** Composing  $\Phi$  with a projective transformation, we may assume that it sends  $[1, 0]$  to  $[1, 0]$ . Then the restriction of  $\Phi$  to the chart  $U_1$  on  $\mathbb{P}_{\mathbb{C}}^1$  is a bijective holomorphic function  $h: \mathbb{C} \rightarrow \mathbb{C}$  that is meromorphic at infinity. We have seen that any such function is rational, that is, of the form

$$h: \mathbb{C} \rightarrow \mathbb{C}, z \mapsto P(z)/Q(z)$$

where  $P$  and  $Q$  are polynomials with complex coefficients and  $Q$  is not the zero polynomial. We may assume that  $P$  and  $Q$  have no non-constant common factors. Since  $h$  is holomorphic,

$Q$  does not have any zeros; this means that  $Q$  is constant, and we may assume that  $Q = 1$ . Since  $h$  does not have any ramification points, the derivative of  $P$  is nowhere vanishing so that  $P$  is a polynomial of degree 1. Writing  $P(z) = az + b$  with  $a, b \in \mathbb{C}$  and  $a \neq 0$ , we finally see that  $\Phi$  is the projective transformation given by

$$\mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1, [x_0, x_1] = [ax_0 + bx_1, x_1].$$

(Total: 20 marks)

2. (a) Let  $C$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^3 x_2^3 + x_1^6 - x_2^6 = 0$ . Let  $D$  be the line in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_1 - x_2 = 0$ . Compute the intersection points of  $C$  and  $D$ , and the intersection multiplicity of  $C$  and  $D$  at each intersection point. (6 marks)

**Solution: Seen similar, category A.** There are two intersection points, namely,  $p = [1, 0, 0]$  and  $q = [0, 1, 1]$ . We have  $\mathbf{I}(p, C, D) = \text{mult}_0 P(1, t, t) = 3$  and similarly  $\mathbf{I}(q, C, D) = \text{mult}_0 P(t, 1, 1) = 3$ . You can also deduce this result from the axiomatic characterization of intersection multiplicities.

- (b) Let  $C$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^2 + x_1^2 + x_2^2 = 0$ . Let  $D$  be the projective plane curve in  $\mathbb{P}_{\mathbb{C}}^2$  defined by  $x_0^3 + x_1^3 + x_2^3 = 0$ . Determine the cardinality of  $C \cap D$ , and justify your answer. (6 marks)

**Solution: Seen similar, category B.** By the weak Bézout theorem, the number of intersection points is at least 1 and at most 6. Let  $p = [p_0, p_1, p_2]$  be an intersection point of  $C$  and  $D$ . It is straightforward to check that  $p_0, p_1$  and  $p_2$  must all be distinct; otherwise, there would exist complex numbers  $a$  and  $b$ , not both zero and satisfying  $a^2 = -2b^2$  and  $a^3 = -2b^3$ , which is impossible. The equations for  $C$  and  $D$  are symmetric under permutation of the variables; therefore, permuting the homogeneous coordinates  $(p_0, p_1, p_2)$ , we find 6 distinct intersection points of  $C$  and  $D$ .

Alternatively, you can argue that  $C$  and  $D$  are nowhere tangent: if they had the same tangent line at an intersection point  $[p_0, p_1, p_2] \in \mathbb{P}_{\mathbb{C}}^2$  then  $(p_0, p_1, p_2)$  would be proportional to  $(p_0^2, p_1^2, p_2^2)$ , which implies that all the non-zero values among  $p_0, p_1, p_2$  are equal. Imposing the equation for  $C$  then yields  $p_0 = p_1 = p_2 = 0$ , which is impossible. Now the strong Bézout theorem implies that there are precisely 6 intersection points.

- (c) We say that an affine plane curve  $C$  in  $\mathbb{C}^2$  is *closed under complex conjugation* if, for every point  $(u, v)$  on  $C$ , the complex conjugate  $(\bar{u}, \bar{v})$  also lies on  $C$ . Show that an affine plane curve  $C$  in  $\mathbb{C}^2$  is closed under complex conjugation if and only if it has an equation of the form  $P(x, y) = 0$  where  $P \in \mathbb{R}[x, y]$ . (8 marks)

**Solution: Unseen, category C.** If  $P$  is a polynomial in  $\mathbb{R}[x, y]$  and  $(u, v) \in \mathbb{C}^2$  satisfies  $P(u, v) = 0$ , then taking the complex conjugate of this expression, we get  $P(\bar{u}, \bar{v}) = 0$ . Thus an affine plane curve defined by a polynomial with real coefficients is closed under complex conjugation. (2 marks)

Conversely, let  $C$  be an affine plane curve in  $\mathbb{C}^2$  that is closed under complex conjugation, defined by a non-constant polynomial  $P(x, y) \in \mathbb{C}[x, y]$  with no repeated factors. Rescaling the equation, we may assume that at least one of the non-zero coefficients of  $P(x, y)$  is real. Denote by  $\bar{P}(x, y)$  the polynomial we obtain by replacing the coefficients of  $P$  by their complex conjugates. By assumption,  $P$  and  $\bar{P}$  have the same zero set. The Nullstellensatz implies that  $\bar{P} = \lambda P$  for some  $\lambda \in \mathbb{C}^*$ . Since  $P$  has a non-zero real coefficient, it follows that  $\lambda = 1$ , so that  $P$  is preserved by complex conjugation and therefore lies in  $\mathbb{R}[x, y]$ . (6 marks)

(Total: 20 marks)

3. A *circle* in  $\mathbb{C}^2$  is an affine plane curve defined by an equation of the form

$$(x - a)^2 + (y - b)^2 - c = 0$$

where  $a, b, c$  are real numbers and  $c > 0$ . The point  $(a, b)$  in  $\mathbb{R}^2$  is then called the *centre* of  $C$ . You may freely use that this equation is always irreducible.

(a) Let  $C$  be an irreducible affine plane curve of degree 2 in  $\mathbb{C}^2$ . Show that the following are equivalent:

- (1) the curve  $C$  is a circle;
- (2) the points at infinity of  $C$  are  $[1, i, 0]$  and  $[1, -i, 0]$ , the curve  $C$  contains a point of  $\mathbb{R}^2$ , and  $C$  is closed under complex conjugation (see Question 2(c)).

(6 marks)

**Solution: Seen similar, category A-B.** First, assume that  $C$  is a circle defined by  $(x - a)^2 + (y - b)^2 - c = 0$  as above. Then  $C$  is closed under complex conjugation by Question 3(c). The projectivization of  $C$  is defined by  $(x_0 - ax_2)^2 + (x_1 - bx_2)^2 - cx_2^2 = 0$ . Setting  $x_2 = 0$  we find that the points at infinity are  $[1, i, 0]$  and  $[1, -i, 0]$ . Moreover,  $C$  contains the real point  $(a, b + \sqrt{c})$ . **(category A, 2 marks)**

Conversely, assume that  $C$  satisfies the conditions in (2). The general equation for an affine plane curve of degree 2 is

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \zeta = 0$$

where  $\alpha, \beta, \gamma$  are not all zero. Since  $C$  is closed under complex conjugation, we may assume that the coefficients in this equation are real numbers by Question 2(c). The points at infinity for this equation are  $[1, \pm i, 0]$  if and only if  $\alpha - \gamma \pm \beta i = 0$ , which implies that  $\beta = 0$  and  $\alpha = \gamma$ . Dividing the equation by  $\alpha$ , we obtain

$$\left(x + \frac{\delta}{2}\right)^2 + \left(y + \frac{\varepsilon}{2}\right)^2 + \zeta - \frac{\delta^2 + \varepsilon^2}{4} = 0.$$

Now we can set  $a = -\delta/2$ ,  $b = -\varepsilon/2$  and  $c = (\delta^2 + \varepsilon^2)/4 - \zeta$  to put our equation in the form  $(x - a)^2 + (y - b)^2 - c = 0$ . If the equation has at least one real solution for  $(x, y)$ , then  $c$  is a sum of squares in  $\mathbb{R}$  and therefore non-negative. If the equation is irreducible, then we moreover have  $c > 0$ . **(category B, 4 marks)**

(b) Let  $p, q, r$  be non-collinear points in  $\mathbb{R}^2$ . Show that there exists a unique circle  $C$  in  $\mathbb{C}^2$  that passes through  $p, q, r$ . (4 marks)

**Solution: Seen similar, category C.** We have seen that there is a unique conic through  $p, q, r, [1, i, 0]$  and  $[1, -i, 0]$ . This conic must be closed under complex conjugation because its complex conjugate is again a conic through these five points. No three of these points are collinear, because the point at infinity of a line through two distinct points in  $\mathbb{R}^2$  has real homogeneous coordinates. It follows that our conic is non-degenerate. Restricting to the affine chart  $U_2$ , we obtain a unique circle through  $p, q, r$  by part (a).

(c) Let  $C_1$  and  $C_2$  be two circles in  $\mathbb{C}^2$ , and denote by  $\overline{C}_1$  and  $\overline{C}_2$  their respective projectivizations.

- (i) Show that  $\overline{C}_1$  and  $\overline{C}_2$  are tangent at a point at infinity if and only if  $C_1$  and  $C_2$  have the same center. (4 marks)

**Solution: Seen similar, category A.** Let  $C$  be a circle defined by  $(x-a)^2 + (y-b)^2 - c = 0$  for some real numbers  $a, b, c$  with  $c > 0$ . Direct calculation show that the projective tangent line at  $[1, \pm i, 0]$  of the projectivization of  $C$  is defined by  $x_0 \pm ix_1 - (a \pm bi)x_2 = 0$ ; thus each of these tangent lines completely determines the center  $(a, b)$ .

- (ii) Assume that  $C_1$  and  $C_2$  intersect at a point  $p \in \mathbb{R}^2$  and that they are not tangent at  $p$ . Prove that  $C_1$  and  $C_2$  intersect in precisely one point  $q$  different from  $p$ . (4 marks)

**Solution: Seen similar, category A.** The projectivizations  $\overline{C}_1$  and  $\overline{C}_2$  intersect transversally at  $p$  by assumption, and also at  $[1, i, 0]$  and  $[1, -i, 0]$  by part (i) (if  $C_1$  and  $C_2$  had the same center, they would be equal and therefore tangent at  $p$ ). The strong Bézout theorem now implies that they intersect at a unique point  $q \in \mathbb{C}^2$  different from  $p$ .

- (iii) Show that the point  $q$  from part (ii) has real coordinates. (2 marks)

**Solution: Unseen, category A.** This follows from the uniqueness of  $q$ : if  $q$  were not real, then its complex conjugate would be a third intersection point of  $C_1$  and  $C_2$ .

(Total: 20 marks)

4. (a) For every divisor  $D$  in  $\mathbb{P}_{\mathbb{C}}^2$ , we denote its support by  $\text{supp}(D)$ . Consider a pencil of degree  $d$  in  $\mathbb{P}_{\mathbb{C}}^2$ , for some positive integer  $d$ , and let  $D_0$ ,  $D_1$  and  $D_2$  be three distinct members of this pencil. Show that  $\text{supp}(D_0) \cap \text{supp}(D_1) = \text{supp}(D_0) \cap \text{supp}(D_2)$ . (6 marks)

**Solution: Seen similar, category A.** Let  $P_0$  and  $P_1$  be non-zero homogeneous polynomials in  $\mathbb{C}[x_0, x_1, x_2]$  such that  $D_0 = \text{div}(P_0)$  and  $D_1 = \text{div}(P_1)$ . Then every member of the pencil distinct from  $D_0$  and  $D_1$  is of the form  $\text{div}(\lambda P_0 + \mu P_1)$  for some  $\lambda, \mu \in \mathbb{C}^*$ . A point  $p$  in  $\mathbb{P}_{\mathbb{C}}^2$  satisfies  $P_0(p) = P_1(p) = 0$  if and only if it satisfies  $P_0(p) = (\lambda P_0 + \mu P_1)(p) = 0$ .

- (b) Let  $\mathcal{L}_2$  be the complete linear system of effective divisors of degree 2 in  $\mathbb{P}_{\mathbb{C}}^2$ . Let  $p, q$  be two distinct points in  $\mathbb{P}_{\mathbb{C}}^2$ . Compute the dimension of the linear system of divisors  $D$  in  $\mathcal{L}_2$  such that  $\text{mult}_p D \geq 2$  and  $\text{mult}_q D \geq 2$ . Justify your answer. (6 marks)

**Solution: Seen similar, category B.** The expected dimension is  $5 - 3 - 3 = -1$  but the conditions are *not* independent. The actual dimension is 0: there is a unique divisor  $D$  satisfying the condition, namely, the double line through  $p$  and  $q$ .

- (c) We refer to Question 3 for the definition of a circle in  $\mathbb{C}^2$ . Let  $p, q, r$  be non-collinear points in  $\mathbb{R}^2$ . Let  $p', q', r'$  be points in the interiors of the line segments  $[qr]$ ,  $[pr]$  and  $[pq]$ , respectively. Let  $C_1$ ,  $C_2$  and  $C_3$  be the unique circles through the sets  $\{p, q', r'\}$ ,  $\{p', q, r'\}$  and  $\{p', q', r\}$ . Assume that no two among these circles are tangent. Use the Cayley-Bacharach theorem and the results in Question 3 to show that  $C_1 \cap C_2 \cap C_3$  contains a point in  $\mathbb{R}^2$ . *Hint: consider cubics of the form  $\overline{C}_i \cup L_i$  where  $\overline{C}_i$  is the projectivization of  $C_i$  and  $L_i$  is a carefully chosen line.* (8 marks)

**Solution: Unseen, category D.** We write  $\overline{C}_1, \overline{C}_2, \overline{C}_3$  for the projectivizations of  $C_1, C_2, C_3$ . Then  $\overline{C}_1, \overline{C}_2$  and  $\overline{C}_3$  are non-degenerate conics that all contain the points  $s_1 = [1, i, 0]$  and  $s_2 = [1, -i, 0]$ . Let  $L_1$  be the line in  $\mathbb{P}_{\mathbb{C}}^2$  through  $p', q$  and  $r$ ; let  $L_2$  be the line through  $p, q'$  and  $r$ ; and let  $L_3$  be the line through  $p, q$  and  $r'$ . Consider the projective plane cubic  $D_i = \overline{C}_i \cup L_i$  for  $i = 1, 2, 3$ .

Since the circles  $C_1$  and  $C_2$  are not tangent, they intersect in  $r'$  and a second point  $t$  in  $\mathbb{R}^2$  different from  $r'$ , by Question 3(c). If  $t \in \{p', q'\}$  then it is a common intersection point of  $C_1, C_2$  and  $C_3$ , so we may assume that this is not the case. Then  $D_1 \cap D_2$  consists of the nine distinct points  $p, q, r, p', q', r', s_1, s_2, t$ . The cubic  $D_3$  contains the first eight of these points, and therefore also the ninth point  $t$ , by the Cayley-Bacharach theorem. The point  $t$  does not lie on  $L_3$ , because otherwise, this line would intersect  $C_1$  in three distinct points, contradicting the weak Bézout theorem. It follows that  $t$  lies on  $\overline{C}_3$ , and therefore on  $C_3$  since it is not a point at infinity.

(Total: 20 marks)

5. In this question you may freely use that every connected compact Riemann surface satisfies the Riemann-Roch theorem from the mastery material. Let  $X$  be a connected compact Riemann surface of genus  $g$ .

- (a) Let  $p$  be a point of  $X$ . For every integer  $n \geq 0$ , compute the dimension of the vector space of meromorphic differential forms on  $X$  that have a pole of order at most  $n$  at  $p$  and that are holomorphic at all other points of  $X$ . (4 marks)

**Solution: Unseen.** The problem is to compute  $\ell(K + np)$ , where  $K$  denotes a canonical divisor of  $X$ . If  $n = 0$  this is the genus  $g$  of  $X$ . For  $n > 0$ , we have  $\ell(-np) = 0$  so that the Riemann-Roch theorem implies that

$$\ell(K + np) = \deg(K + np) + 1 - g = 2g - 2 + n + 1 - g = g + n - 1.$$

- (b) Let  $p$  be a point of  $X$ . Show that there exists a non-constant meromorphic function on  $X$  that is holomorphic at all points of  $X \setminus \{p\}$ . (4 marks)

**Solution: Unseen.** Pick a positive integer  $n$  such that  $n > g$ . Then the Riemann-Roch theorem implies that

$$\ell(np) \geq \deg(np) + 1 - g = n + 1 - g \geq 2.$$

Any non-constant element of  $\mathcal{L}(np)$  satisfies the requirements.

- (c) Show that there exists a non-constant morphism of Riemann surfaces  $X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  of degree at most  $g + 1$ . (6 marks)

**Solution: Unseen.** By (b) there exists a non-constant meromorphic function  $f$  on  $X$  with a pole of order  $m \leq g + 1$  at  $p$  and that is holomorphic everywhere else. We denote by  $\bar{f}: X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  the corresponding morphism to the Riemann sphere. Then  $p$  is the only point of  $X$  mapped to the point at infinity  $[1, 0]$ , and the ramification degree of  $\bar{f}$  at  $p$  is equal to  $m$ . Therefore,  $\bar{f}$  has degree  $m \leq g + 1$ .

- (d) Assume that  $g = 2$ . Show that there exists a meromorphic function on  $X$  that has two distinct poles of order 1 and that is holomorphic at all other points of  $X$ . (6 marks)

**Solution: Unseen.** By Riemann-Roch,  $\ell(K) = 2$  so that we may assume that  $K$  is effective. It has degree  $2g - 2 = 2$  and therefore can be written as  $p + q$  with  $p$  and  $q$  not necessarily distinct points on  $X$ . Since  $\ell(p + q) = 2$ , the vector space  $\mathcal{L}(p + q)$  contains a non-constant function  $f$ . There are no non-constant functions in  $\mathcal{L}(p)$  and  $\mathcal{L}(q)$  because these would induce isomorphisms  $X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ , contradicting  $g = 2$ .

Therefore,  $f$  has poles of order 1 at  $p$  and  $q$  if  $p \neq q$ , and a pole of order 2 at  $p$  if  $p = q$ . In the former case we are done, so we may assume that  $p = q$ . We denote by  $\bar{f}: X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  the morphism corresponding to  $f$ ; it has degree 2. Let  $y$  be a point of  $\mathbb{P}_{\mathbb{C}}^1$  such that  $\bar{f}$  is unramified over  $y$ . Composing  $\bar{f}$  with a projective transformation of  $\mathbb{P}_{\mathbb{C}}^1$  that maps  $y$  to  $[1, 0]$ , we find a morphism  $h: X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  of degree 2 that is unramified over  $\infty = [1, 0]$ . This morphism corresponds to a meromorphic function on  $X$  with a simple pole at each of the two points in  $h^{-1}(\infty)$  and no other poles.

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
<u>Algebraic Curves_MATH60033 MATH97041 MATH70033</u>	1	(a)(i-iii) were routine calculations and handled well by almost everyone. Question (a)(iv) required some thought to reduce to the setting in the lecture notes. Question (b) was more challenging, the reduction to the case where $\Phi$ maps the point at infinity to itself was found by most students but the remainder of the argument was often wobbly.
<u>Algebraic Curves_MATH60033 MATH97041 MATH70033</u>	2	Question (a) was done well by most students. For (b), some students tried to compute intersection points explicitly, which was not the right way to go. The "if" implication in (c) was generally handled well. A recurrent mistake in the "only if" implication was that polynomials with no repeated factors defining the same affine plane curve need not be equal (only up to scaling).
<u>Algebraic Curves_MATH60033 MATH97041 MATH70033</u>	3	Question (a) was done well. "Three non-collinear points in $\mathbb{R}^2$ determine a unique circle" was not an acceptable answer to (b), this is precisely what you were asked to prove. No one checked in (b) that no two points in $\{p,q,r\}$ were collinear with $[1,i,0]$ or $[1,-i,0]$ . Similarly, very few people explained in (c)(ii) why the circles cannot be tangent at one of the points at infinity. Still, most students performed well on this question overall.
<u>Algebraic Curves_MATH60033 MATH97041 MATH70033</u>	4	Question (a) was a minor variation on an exercise for the lecture notes, but several students had poorly understood the definition of the support of a divisor: it is not the set of irreducible components, and the intersection of the supports of two divisors is often not a divisor but a finite collection of points. In (b) the main source of mistakes was that the expected dimension is different from the actual dimension. Question (c) was done well by many students, although some picked the wrong lines leading to incorrect applications of Cayley-Bacharach.
<u>Algebraic Curves_MATH60033 MATH97041 MATH70033</u>	5	Few people attempted this question, even though (a) and (b) were direct applications of the Riemann-Roch theorem.