Are the following functions Convex?

(i) $f(x) = \sum_{i=1}^{n} x_i \ln(x_i) - \left(\sum_{i=1}^{n} x_i\right) \ln\left(\sum_{i=1}^{n} x_i\right)$ over $\boxed{\mathbb{R}^n_{++}}$

First: work the expression to something tractable

e.g. $n=2$ (just exploring)

$$x_1 \ln(x_1) + x_2 \ln(x_2) - \underbrace{(x_1 + x_2)} \ln(x_1 + x_2)$$

$$= \underline{x_1 \ln(x_1)} + \underbrace{x_2 \ln(x_2)} - \underline{x_1 \ln(x_1 + x_2)} - \underbrace{x_2 \ln(x_1 + x_2)}$$

$$= x_1 \left(\ln(x_1) - \ln(x_1 + x_2) \right) + x_2 \left(\ln(x_2) - \ln(x_1 + x_2) \right)$$

$$= x_1 \left(\ln\left(\frac{x_1}{x_1 + x_2}\right)\right) + x_2 \left(\ln\left(\frac{x_2}{x_1 + x_2}\right)\right) \quad \ldots$$

$$\rightarrow f(x) = \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{\sum_{k=1}^{n} x_k}\right)$$

now $f(x) = \sum_{i=1}^{n} h_i(\underline{x})$

where $h_i(\underline{x}) = x_i \ln\left(\frac{x_i}{\sum_{k=1}^{n} x_k}\right)$

$\qquad \parallel$

$\varphi(u, v) = u \ln(u/v)$

$$\underline{x} \begin{cases} \nearrow u = x_i \\ \searrow v = \sum_{k=1}^{n} x_k \end{cases} \text{ linear transformation.}$$

$$\underline{x} \xrightarrow{\varphi \circ (L.T.)} h_i(\underline{x})$$

We need to show that φ is convex and then we are ok, because f would the sum of h_i, each one of them convex (h_i's would be convex functions composed with linear transformations — quad-over-lin) $\varphi(u,v) = \frac{u^2}{v}$

We show that $\varphi(u,v) = u \ln(u/v)$ is convex

$$\varphi(u,v) = u \ln(u) - u \ln(v)$$

$$\frac{\partial \varphi}{\partial u} = \ln(u) + 1 - \ln(v)$$

$$\frac{\partial \varphi}{\partial v} = -\frac{u}{v} \quad \rightarrow \quad \frac{\partial^2 \varphi}{\partial u \, \partial v} = -\frac{1}{v}$$

$$\frac{\partial^2 \varphi}{\partial u^2} = \frac{1}{u} \qquad \frac{\partial^2 \varphi}{\partial v^2} = \frac{u}{v^2} \quad \rightarrow \quad \nabla^2 \varphi = \begin{bmatrix} 1/u & -1/v \\ -\frac{1}{v} & u/v^2 \end{bmatrix}$$

2 by 2 matrix: $\operatorname{Trace}(\nabla^2 \varphi) = \frac{1}{u} + \frac{u}{v^2}$

$$\operatorname{Det}(\nabla^2 \varphi) = \frac{1}{\cancel{u}} \cdot \frac{\cancel{u}}{v^2} - \left(\frac{1}{v}\right)^2$$

$$= 0$$

$\operatorname{Trace}(\nabla^2 \varphi)$ is positive when u i strictly positive

$\Longrightarrow \underline{x} \in \mathbb{R}^n_{++}$

$\Longrightarrow \nabla^2 \varphi \succcurlyeq 0 \Longrightarrow \varphi$ is convex

Sum of convex functions   Convex composed with L.T.

$\Longrightarrow f(\underline{x})$ is convex.