# Probability for Statistics
## Problem Sheet 1

Questions marked (r) are revision of concepts from last year.

1. Let $\Omega$ be a set.
   (a) Show that the collection $\mathcal{F} = \{\emptyset, \Omega\}$ is a sigma algebra.
   (b) Show that for any subset $E \subseteq \Omega$, $\mathcal{F}_E = \{\emptyset, E, E^c, \Omega\}$ is a sigma algebra.
   (c) Let $\mathcal{F}$ be the collection of all subsets of $\Omega$. Show that $\mathcal{F}$ is a sigma algebra.
   (d) Show that the intersection of two sigma algebras on $\Omega$ is a sigma algebra.
   (e) Give an example to show that the union of two sigma algebras on $\Omega$ need not be a sigma algebra.

2. Suppose a fair coin is flipped repeatedly, and that flips are independent. Use the continuity property of the probability function $\Pr$ to show that, with probability 1, the coin will eventually land heads up.

3. Let $\Omega = [0, 1]$, the unit interval. Define $\mathcal{F}$ to be the collection of all countable or co-countable subsets of $\Omega$, where a co-countable set is one whose complement is countable.
   (a) Show that $\mathcal{F}$ is a sigma algebra. *[Hint: Is a countable union of countable sets countable?]*
   (b) Define the function $P : \mathcal{F} \to [0, 1]$ by
   $$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is co-countable} \end{cases}.$$
   Determine whether or not $P$ is countably additive.

4. Consider the probability space $(\Omega, \mathcal{F}, \Pr)$ with $A, B \in \mathcal{F}$. Using only the Kolmogorov axioms prove
   (a) $\Pr(A) \leq 1$,
   (b) If $A \subseteq B$, then $\Pr(A) \leq \Pr(B)$,
   (c) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

5. (r) Suppose $X$ is a set containing $n$ elements. If $A$ and $B$ are randomly chosen subsets of $X$, what is the probability that $A \subseteq B$?

6. (r) Consider two coins, of which one is a normal fair coin and the other is biased so that the probability of obtaining a Head is $p > 1/2$.
   (a) Suppose $p = 1$ and a coin is selected at random and flipped $n$ times, with flips mutually independent. Evaluate the conditional probability that the selected coin is the normal one, given that the first $n$ flips are all Heads.
   (b) Now suppose $1/2 < p < 1$ and that again, one of the coins is selected randomly and flipped $n$ times. Let $E$ be the event that the $n$ tosses result in $k$ Heads and $n - k$ Tails, and let $F$ be the event that the coin is fair. Find $\Pr(F|E)$.

**Optional questions for group discussion**

*The next two questions are adapted from the work of Kahneman and Tversky, psychologists who studied subjective perceptions of probability. See the Additional Resources section on Blackboard!*

7. A city contains two hospitals, one small and one large. The long-term proportion of boys born can be taken to be 50%. On a given day, the proportion of boys born in one of the hospitals is 55%. Which hospital is this more likely to be?

8. Tom is an opera buff who enjoys touring art museums when on holiday. Growing up, he enjoyed playing chess with family members and friends. Which of the following two situations is more likely?

   (a) Tom plays trumpet for a major symphony orchestra.

   (b) Tom is a farmer.

9. Consider the data in Table 1, comparing two treatments for kidney stones.

Table 1: Success rates for the treatment of kidney stones by open surgery (Treatment A) and percutaneous nephrolithotomy (Treatment B).

| | Treatment A | Treatment B |
|---|---|---|
| Small stones | 81 / 87 (93%) | 234 / 270 (87%) |
| Large stones | 192 / 263 (73%) | 55 / 80 (69%) |
| Total | 273 / 350 (78%) | 289 / 350 (83%) |

Which treatment is better? Evaluate the following two responses:

I. 83% of the time, Treatment B was successful, whereas Treatment A was successful only 78% of the time. So Treatment B is better.

II. For patients with small stones, Treatment A was 93% successful whereas Treatment B was 87% successful. Similarly, for large stones, Treatment A was 73% successful, while treatment B was 69% successful. So Treatment A is better.