(2.5.6) Corollary (Compactness Thm.)
Suppose $\mathcal{L}$ is a countable 1^{st} order language and Σ is a set of closed $\mathcal{L}$-fmlas. Suppose that every finite subset of Σ has a model. Then Σ has a model.

Pf: Suppose Σ has no model. By 2.5.3, Σ is inconsistent.
So there is an $\mathcal{L}$-fmla χ with $\Sigma \vdash \chi$ and $\Sigma \vdash (\neg \chi)$.
Deductions in $K_{\mathcal{L}}$ are finite so there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \chi$ and $\Sigma_0 \vdash (\neg \chi)$.
So Σ_0 is inconsistent. By assumption Σ_0 has model.
Contradiction. #

2.6 Equality

2.6.1

Def: Suppose $\mathcal{L}^E$ is a 1st order language with a distinguished 2-rel. symbol E.

1. An $\mathcal{L}^E$-structure in which E is interpreted as equality $=$ is called a normal $\mathcal{L}^E$-structure.

2. The following are the axioms of equality, Σ_E

$$(\forall x_1)\, E(x_1, x_1) \wedge$$
$$(\forall x_1)(\forall x_2)(E(x_1, x_2) \to E(x_2, x_1)) \wedge$$
$$(\forall x_1)(\forall x_2)(\forall x_3)(E(x_1, x_2) \to (E(x_2, x_3) \to E(x_1, x_3)))$$

For each n-ary rel. symbol R of $\mathcal{L}^E$:

$$(\forall x_1) \cdots (\forall x_n)(\forall y_1) \cdots (\forall y_n)$$
$$(E(x_1, y_1) \wedge \cdots \wedge E(x_n, y_n) \wedge R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n))$$

For each m-ary function symbol f of $\mathcal{L}^E$:

$$(\forall x_1) \cdots (\forall x_m)(\forall y_1) \cdots (\forall y_m)$$
$$(E(x_1, y_1) \wedge \cdots \wedge E(x_m, y_m) \to E(f(x_1, \ldots, x_m), f(y_1, \ldots, y_m)))$$

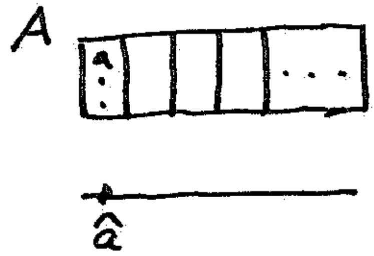
(2.6.2) Remarks + Defn.

1. If $\mathcal{A}$ is a normal $\mathcal{L}^E$-str. then $\mathcal{A} \models \Sigma_E$.

2. Suppose $\mathcal{A} = \langle A ; \bar{E}, \ldots \rangle$ and $\mathcal{A} \models \Sigma_E$.

Then $\bar{E}$ is an equivalence relation on A. For $a \in A$ let $\hat{a} = \{ b \in A : \bar{E}(a,b) \text{ holds in } \mathcal{A} \}$.

– the $\bar{E}$-equivalence class containing a.



Let $\hat{A} = \{ \hat{a} : a \in A \}$

Make $\hat{A}$ into an $\mathcal{L}^E$-str. $\hat{\mathcal{A}}$ as follows:

If R is an n-ary rel. symbol of $\mathcal{L}^E$, say that for $\hat{a}_1, \ldots, \hat{a}_n \in \hat{A}$

$\bar{R}(\hat{a}_1, \ldots, \hat{a}_n)$ holds in $\hat{\mathcal{A}}$ iff $\bar{R}(a_1, \ldots, a_n)$ in $\mathcal{A}$

this is well-defined as $\mathcal{A} \models \Sigma_E$.

Note: $\bar{E}$ in $\hat{\mathcal{A}}$ is $=$. $\quad$ ($\bar{E}(\hat{a}_1, \hat{a}_2) \Leftrightarrow E(a_1, a_2)$ holds $\Leftrightarrow \hat{a}_1 = \hat{a}_2$)

If f is an m-ary function symbol then for $\hat{a}_1, \ldots, \hat{a}_m \in \hat{A}$ let

$$\bar{f}(\hat{a}_1, \ldots, \hat{a}_m) = \widehat{\bar{f}(a_1, \ldots, a_m)}.$$

This is well-defined as $\mathcal{A} \models \Sigma_E$.

4

If c is a constant symbol of $\mathcal{L}^E$ interpreted as $\bar{c}$ in $\mathcal{A}$ interpret c in $\hat{\mathcal{A}}$ as $\hat{\bar{c}}$.

Note: $\hat{\mathcal{A}}$ is a normal $\mathcal{L}^E$-structure.

(2.6.3) Lemma. Suppose $\mathcal{A}$ is an $\mathcal{L}^E$-str. with $\mathcal{A} \models \Sigma_E$. Let $\hat{\mathcal{A}}$ be as given as above. Then for every closed $\mathcal{L}^E$-fmla. ϕ

$$\mathcal{A} \models \phi \iff \hat{\mathcal{A}} \models \phi .$$

Pf: See notes. #.

(2.6.4) Cor. Suppose $\mathcal{L}^E$ is countable. Suppose Δ is a set of closed $\mathcal{L}^E$-fmlas. Then

① Δ has a normal model ($\iff$ $\Delta \cup \Sigma_E$ is consistent)

$\iff$ $\Delta \cup \Sigma_E$ has a model.

② If Δ has a normal model, then it has a countable normal model.

Pf: ① $\Rightarrow$: OK. Σ_E hold in any normal $\mathcal{L}^E$-str.

$\Leftarrow$: If $A \models \Delta \cup \Sigma_E$ then by 2.6.3 $\hat{A} \models \Delta$ and $\hat{A}$ is a normal $\mathcal{L}^E$-str. //

② $\Rightarrow$: If Δ has a normal model then $\Delta \cup \Sigma_E$ is consistent, so by 2.5.7 there is a countable model A of $\Delta \cup \Sigma_E$.
But then $\hat{A}$ is a countable normal model of Δ. #.

(2.6.5) Thm. (Compactness Thm. for normal models)
Suppose $\mathcal{L}^E$ is a countable language with equality and Δ is a set of closed $\mathcal{L}^E$-fmlas such that every finite subset of Δ has a normal model. Then Δ has a normal model.

Pf: Consider $\Sigma = \Delta \cup \Sigma_E$

Every normal $\mathcal{L}^E$-str. is a model of Σ_E. So

Every finite subset of Σ has a ~~m~~ model. (By assumption.)

By 2.5.6 (Compactness Thm)

Σ has a model $\mathcal{A}$.

then $\hat{\mathcal{A}}$ is a normal model of A. #

(2.6.4).

Notation: $\mathcal{L}^E$ is called a language with equality.

Now on: write $\mathcal{L}^=$ instead of $\mathcal{L}^E$

and "$x_1 = x_2$" instead of $E(x_1, x_2)$ in formulas.

Denote the axioms of equality as $\Sigma_=$.