MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Quiz 3

Question 1. Correct answer: (a).

This is a differential equation that does not depend on x and can thus be solved by integration, and the initial value is solved by $t \mapsto x_0 + \int_{t_0}^{t} \sqrt{|s|}\,\mathrm{d}s$. It is clear using previous results from analysis that the solution is unique, but also the Picard–Lindelöf theorem implies this, since the right hand side is globally Lipschitz continuous with respect to x.

Question 2. Correct answer: (b).

While it is correct that the given λ_{max} solves the differential equation, we require any solution to be defined on an interval, which is a connected set (see Definition 1.2). It does not make sense to speak of solutions to initial value problems on unions of intervals such as the set $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$. Think about why it does not make sense and ask if this is unclear to you!

Question 3. Correct answer: (a).

The function f is obviously continuously differentiable as a polynomial in t and x, and thus, Proposition 2.14 implies that it is locally Lipschitz continuous.

Question 4. Correct answer: (b).

Consider $\tilde{D} = (-2, 2)$ and $f : \mathbb{R} \times \tilde{D} \to \mathbb{R}$ given by $f(t, x) = tx^2$. The partial derivative of f with respect to x is given by $2tx$ and becomes unbounded as $t \to \infty$ (for fixed $x \in (-2, 2) \setminus \{0\}$, and using the mean value theorem (note x is one-dimensional), as explained in Example 2.6, this implies that there is no global Lipschitz constant.

Question 5. Correct answer: (a).

Consider $D = \{(t, x) \in \mathbb{R} \times \mathbb{R} : t > 0 \text{ and } x < \frac{1}{t}\}$ and $f : D \to \mathbb{R}$ is given by $f(t, x) = 0$. Consider the initial value problem $x(1) = 0$. The corresponding maximal solution (which is a constant solution) converges to the boundary of the domain D as described in question, and we have $I_+(1, 0) = \infty$.