3.3 Spanning

Definition 3.3.1. Let V be an F-vector space. Let $u_1, ..., u_m \in V$ then:

- A *Linear Combination* of $u_1, ..., u_m$ is a vector of the form $\alpha_1 u_1 + ... + \alpha_m u_m$ for scalars $\alpha_1, ..., \alpha_m \in F$. Note we can also write $\alpha_1 u_1 + ... + \alpha_m u_m$ as $\sum_{i=1}^{m} \alpha_i u_i$.
- The *span* of $u_1, ..., u_m$ is the set of linear combinations of $u_1, ..., u_m$. i.e. $Span(u_1, ..., u_m) = \{\alpha_1 u_1 + ... + \alpha_m u_m \in V : \alpha_1, ..., \alpha_m \in F\}$.

NB: there are several different notations used for Span, e.g., $Sp(X)$, $< X >$.

Lemma 3.3.2.

Let V be an F vector space, and $u_1, ..., u_m \in V$ then $Span(u_1, ..., u_m)$ is a subspace of V.

Proof: Clearly $Span(u_1, ..., u_m) \subset V$ so we do the subspace test:

SS1 $u_1 \in Span(u_1, ..., u_m)$

SS2 Suppose $v, w \in Span(u_1, ..., u_m)$ then $v = \sum_{i=1}^{m} \alpha_i u_i$ and $w = \sum_{i=1}^{m} \beta_i u_i$ so

$$v + w = \sum_{i=1}^{m} (\alpha_i + \beta_i) u_i \in Span(u_1, ..., u_m) \text{ as } F \text{ closed under addition, i.e. } \alpha_i + \beta_i \in F$$

SS3 Suppose $v \in Span(u_1, ..., u_m)$ and $\lambda \in F$ then $v = \sum_{i=1}^{m} \alpha_i u_i$ do $v = \sum_{i=1}^{m} \lambda \alpha_i u_i \in Span(u_1, ..., u_m)$ as $\lambda \alpha_i \in F$ for each $i \in \{1, ..., m\}$

Remark 3.3.3.

- By convention we take the empty sum to be 0_V, so $Span\emptyset = \{0_V\}$
- For an infinite set S we still only take finite sums i.e.

$$Span(S) = \left\{ \sum_{s_i \in S'} \alpha_i s_i : S' \subset^{finite} S, \alpha_i \in F \right\}$$

Exercise 3.3.4. Show that for an infinite subset S of an F-vector space V, $Span(S)$ is a subspace of V.

Definition 3.3.5. Let V be an F vector space, and suppose $S \subset V$ is such that $Span(S) = V$ then we say S *spans* V, or equivalently S is a *spanning set* for V.

Example 3.3.6.

- $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$.
- $\mathbb{R}^{deg \leq n}[x]$ spanned by $\{1, x, x^2 ...., x^n\}$

Exercise 3.3.7. Which of the following sets span $\mathbb{R}^3$:

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$
4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

In the above exercise we see that we sometimes have "redundant" vectors in a spanning set. If as well as spanning the set is linearly independent, then this won't happen.

3.4 Linear Independence

Definition 3.4.1. Let V be an F-vector space. We say $u_1, ..., u_m \in V$ are *linearly independent* if whenever

$$\alpha_1 u_1 + ... + \alpha_m u_m = 0_V,$$

then it must be that

$$\alpha_1 = \cdots = \alpha_m = 0.$$

We say $\{u_1, ..., u_m\}$ is a *linearly independent set*.

Alternatively, a set $\{u_1, ..., u_m\}$ is *linearly dependent* if $\alpha_1 u_1 + ... + \alpha_m u_m = 0_V$ where at least one $\alpha_i \neq 0$, and a set is linearly independent if it is not linearly dependent.

Example 3.4.2.

- The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$.
- Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be functions and suppose $f(x) = x$ and $g(x) = x^2$. The set $\{f, g\}$ is a linearly independent subset of $V = \mathbb{R}^{\mathbb{R}}$.
  Proof: Suppose $\alpha g + \beta f = 0_V$ now two functions are equal if they are equal on all of the domain. So consider $1, 2 \in \mathbb{R}$. Then we get

$$\begin{array}{lcl} 0_V(1) & = & (\alpha g + \beta f)(1) \\ 0 & = & \alpha + \beta \\ & & \\ 0_V(2) & = & (\alpha g + \beta f)(2) \\ 0 & = & 2\alpha + 4\beta \end{array}$$

  So we have $\alpha = -\beta$ and $\alpha = -2\beta$ thus $\alpha = \beta = 0$.
- The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a linearly *dependent* subset of $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- For V and F-vector space then $\{0_V\}$ is linearly *dependent*
- For V and F-vector space $v \in V$ then $\{v\}$ is linearly independent iff $v \neq 0_V$.

Exercise 3.4.3. Which of the following sets are in linearly independent subsets of $\mathbb{R}^3$:

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Lemma 3.4.4. Let $v_1, ..., v_n$ be linearly independent in an F-vector space V. Let v_{n+1} be such that $v_{n+1} \notin Span(v_1, ..., v_n)$. Then $v_1, ..., v_{n+1}$ is linearly independent.

Proof: Suppose $\alpha_1, ..., \alpha_{n+1} \in F$ are such that $\alpha_1 v_1 + ... + \alpha_{n+1} v_{n+1} = 0_V$.

If $\alpha_{n+1} \neq 0$ then $v_{n+1} = \frac{1}{\alpha_{n+1}}(\alpha_1 v_1 + ... + \alpha_n v_n) \in Span(v_1, ..., v_n)$. Contradiction.

So $\alpha_{n+1} = 0$ so $\alpha_1 v_1 + ... + \alpha_n v_n = 0_V$, but $v_1, ..., v_n$ are linearly independent, thus $\alpha_1 = ... = \alpha_n = 0$.

3.5 Bases

Definition 3.5.1.

- Let V be an F-vector space. A *basis* of V is a linearly independent spanning set of V.
- If V has a finite basis then we say V is a *finite dimensional* vector space.

Example 3.5.2.

- The set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. We have done linear independence, you must show $Span(B) = \mathbb{R}^3$.
- Let F be a field, then in F^n let e_i be the columnn vector with zeros everywhere except the i^{th} row. Then $\{e_1, ..., e_n\}$ forms a basis for F^n and is known as the *standard basis*.
- $\mathbb{R}[x]$ has basis $\{1, x, x^2, ...\}$.

Note: Not every vector space is finite dimensional. For example $\mathbb{R}[x]$ the set of real polynomials doesn't have a finite basis, but it does have infinite bases, e.g., $\{1, x, x^2, \dots\}$.

Exercise 3.5.3. Which of the following sets span $\mathbb{R}^3$:

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
3. $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$
4. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Proposition 3.5.4. Let V be an F-vector space, let $S = \{u_1, ..., u_m\} \subseteq V$. Then S is a basis of V if and only if every vector in V has a unique expression as a linear combination of elements of S.

Proof:

($\Rightarrow$) Suppose S is a basis of V. Take $v \in V$.

[AIM: there are unique $\alpha_1, ..., \alpha_n \in F$ such that $v = \sum_{i=1}^m \alpha_i u_i$]

Since V is spanned by S we have some $\alpha_1, ..., \alpha_n \in F$ such that $v = \sum_{i=1}^m \alpha_i u_i$.

Suppose for contradiction the α_i's are not unique, i.e. there exist $\beta_1, ..., \beta_n \in F$ such that $v =$

$\sum_{i=1}^{m} \beta_i u_i$.

Then we have:

$$\sum_{i=1}^{m} \alpha_i u_i = \sum_{i=1}^{m} \beta_i u_i$$
$$\sum_{i=1}^{m} (\alpha_i - \beta_i) u_i = 0$$

As S is LI we get $\alpha_i - \beta_i = 0$ thus $\alpha_i = \beta_i$

($\Leftarrow$) Suppose conversely that for every $v \in V$ has there are unique $\alpha_1, ..., \alpha_m$ such that $v = \sum_{i=1}^{m} \alpha_i u_i$

[AIM: we need to show that S is spanning and LI.]

- *Spanning:* Let $v \in V$ then $v = \sum_{i=1}^{m} \alpha_i u_i \in Span(S)$
- *LI:* First remark that $0u_1 + ...0u_m = 0_V$ so if $\sum_{i=1}^{m} \lambda_i u_i = 0_V$ then by uniqueness we get $\alpha_i = 0$

So S is a basis for V.

Remark 3.5.5 *Let $B = \{u_1, ...u_m\}$ be a basis for an F-vector space V. By Proposition 3.5.4 we see that we have a bijective map f from V to F^m, for $v = \alpha_1 u_1 + ... + \alpha_m u_m$ we define $f(v) = (\alpha_1, ..., \alpha_m)$ we call $(\alpha_1, ..., \alpha_m)$ the co-ordinates of v*

Proposition 3.5.6. Let V be a non-trivial (i.e. not $\{0\}$) F-vector space and suppose V has a finite spanning set S then S contains a linearly independent spanning set.

I.e., if V has a finite spanning set it has a basis - for cases where there is no finite spanning set we would need something called the axiom of choice to show this (see LOGIC course in year 3)

Proof:

Consider T such that T is linearly independent subset of S, and for any LI subset of S, T' we have that $|T'| \leq |T|$. We can get such a T as we have at least some $v \in V$ so $\{v\}$ is linearly indpendent (i.e. $|T| \geq 1$).

Claim T is spanning.

Proof of Claim: Suppose not then there is a $v \in S \setminus Span(T)$ but by Lemma 3.4.4 $v \cup T$ is LI. Contradiction.