MATH50001/50017/50018 - Analysis II

Complex Analysis

Lecture 2

Section: Complex functions

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$.

$$f: \quad \Omega_1 \to \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$ there exists only one complex number $w = u + iv \in \Omega_2$ such that

$$w = f(z).$$

We use notations:

$$w = f(z) = u(x, y) + iv(x, y),$$

where u and v are two real functions of two real variables.

Example. Let $w = f(z) = z^2 = x^2 - y^2 + i2xy$, $z \in \mathbb{C}$. Then

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Example. Let $w = f(z) = 1/z = \bar{z}/|z|^2$, $z \in \mathbb{C} \setminus \{0\}$. Then

$$u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

Example. Möbius transformation

$$w = f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad cz + d \neq 0.$$

Definition. Let f be a function defined on a set $\Omega \subset \mathbb{C}$. We say that f is continuous at the point $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.

Definition. The function f is said to be continuous on Ω if it is continuous at every point of Ω.

Section: Complex derivative

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. We say that f is *differentiable (holomorphic)* at $z_0 \in \Omega_1$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$, $h \neq 0$ and $z_0 + h \in \Omega_1$. The limit of this quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

This means that for any $\varepsilon > 0$ there is $\delta > 0$ such that as soon $|h| < \delta$ we have

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| < \varepsilon.$$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

It should be emphasised that in the above limit that $h = h_1 + ih_2 \in \mathbb{C}$ is a complex number that may approach 0 from any direction.

Remark. The word "holomorphic" was introduced by two of Cauchy's students, Briot (1817-1882) and Bouquet (1819-1895), and derives from the Greek (holos) meaning "entire", and (morphe) meaning "form" or "appearance".

Definition. The function f is said to be holomorphic on open set Ω if f is holomorphic at every point of Ω .

If C is a closed subset of $\mathbb{C}$, we say that f is holomorphic on C if f is holomorphic in some open set containing C. Finally, if f is holomorphic in all of $\mathbb{C}$ we say that f is entire.

Example. The function $f(z) = z$ is holomorphic on any open set in $\mathbb{C}$ and $f'(z) = 1$.

Example. If $f(z) = z^n$ then $f'(z) = nz^{n-1}$. Indeed we use induction to find:

- If $n = 1$ then $(z)' = 1$.
- Assuming $(z^n)' = nz^{n-1}$ we obtain

$$(z^{n+1})' = (z \cdot z^n)' = z' \cdot z^n + z \cdot (z^n)' = z^n + z \cdot nz^{n-1} = (n+1)z^n.$$

Example. Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is holomorphic in the entire complex plane and

$$p'(z) = a_1 + \cdots + n a_n z^{n-1}.$$

Example. The function $1/z$ is holomorphic on any open set in $\mathbb{C}$ that does not contain the origin, and $f'(z) = -1/z^2$.

Proof it.

Example. The function $f(z) = \bar{z}$ is not holomorphic. Indeed, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}.$$

which has no limit as $h \to 0$, as one can see by first taking h real and then h purely imaginary.

Proposition. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = h\,\psi(h),$$

where ψ is a function defined for all small h and

$$\lim_{h \to 0} \psi(h) = 0.$$

In this case

$$a = f'(z_0).$$

Proof. The proof follow directly from the Definition. Indeed, dividing by h we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \to 0 \quad \text{as} \quad h \to 0.$$

Corollary. If a function f is holomorphic then it is continuous.

Proposition. If f and g are holomorphic in Ω then:

(i) $f+g$ is holomorphic in Ω and $(f+g)' = f' + g'$.
(ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$.
(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

(iv) Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)(z) = g'(f(z))f'(z), \qquad \forall z \in \Omega.$$

Proof. Arguing as in the case of one real variable, use the expression

$$f(z_0 + h) - f(z_0) - ah = h\,\psi(h).$$

Section: Cauchy-Riemann equations

Consider first

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}, \qquad h = h_1 + ih_2,$$

assuming that $h = h_1$ (namely that $h_2 = 0$). Then if

$$f(z_0) = f(x_0+iy_0) = u(x_0,y_0) + iv(x_0,y_0),$$

we have

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{h_1\to 0} \frac{u(x_0+h_1,y_0)+iv(x_0+h_1,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0) = u'_x(x_0,y_0) + i\,v'_x(x_0,y_0). \end{aligned}$$

Let now $h = ih_2$ (namely that $h_1 = 0$). Then

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h_2 \to 0} \frac{u(x_0, y_0 + h_2) + iv(x_0, y_0 + h2) - u(x_0, y_0) - iv(x_0, y_0)}{ih_2} \\ &= \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) = \frac{1}{i} u'_y(x_0, y_0) + v'_y(x_0, y_0) \\ &\qquad\qquad\qquad\qquad\qquad = -i\, u'_y(x_0, y_0) + v'_y(x_0, y_0). \end{aligned}$$

Thus the function u and v satisfy the following

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- *Cauchy-Riemann equations.*

Example. Let $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Then

$$u'_x = 2x = v'_y \quad \text{and} \quad u'_y = -2y = -v'_x, \qquad \text{—O'K.}$$

Example. Let $f(z) = \bar{z}$. Then $u(x, y) = x$ and $v(x, y) = -y$.

$$u'_x = 1 \neq -1 = v'_y.$$

This means that $f(z) = \bar{z}$ is not differentiable.

The Cauchy-Riemann equations link real and complex analysis.

Definition.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big).$$

Theorem. Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\,\frac{\partial u}{\partial z}(z_0).$$

Proof. Using the Cauchy-Riemann equations $u'_x = v'_y$ and $u'_y = -v'_x$ we obtain

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(u'_x - \frac{1}{i}u'_y\right) + \frac{i}{2}\left(v'_x - \frac{1}{i}v'_y\right) = \frac{1}{2}\left(u'_x + iu'_y + iv'_x - v'_y\right) = 0.$$

and

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(u'_x + \frac{1}{i}u'_y\right) + \frac{i}{2}\left(v'_x + \frac{1}{i}v'_y\right) = \frac{1}{2}\left(u'_x - iu'_y + iv'_x + v'_y\right) \\ &= \frac{1}{2}\left(2u'_x - i2u'_y\right) = u'_x + \frac{1}{i}u'_y = 2\frac{\partial u}{\partial z}.\end{aligned}$$

The fact that $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ follows from our computations before. Indeed, we have seen that

$$f'(z_0) = u'_x(x_0, y_0) + iv'_x(x_0, y_0) = u'_x(x_0, y_0) - iu'_y(x_0, y_0) = 2\frac{\partial u}{\partial z}(x_0, y_0).$$

The proof is complete.

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f(z)/\partial z$.

Quizzes

Question 1: Is the function $f(z) = 1/z$ holomorphic in the open disc of radius 1 centered at $z = 2i$?

Answers:

A. Yes

B. No

Question 2: Let $z = x + iy$. Is the function $f(z) = x^2 - y^2 - 2ixy$?

Answers:

A. Yes

B. No

Thank you