IMPERIAL

MATH70020

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Function Spaces and Applications

Date: Tuesday, May 7, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

3. Consider the subspace of $L^2(\mathbb{R})$

$$E_M = \left\{ f \in L^2,\ \int_{-M}^{M} f\,dx = 0 \right\}, \qquad \text{where } M \in \mathbb{R}.$$

(a) Is the set E_M closed? (5 marks)

(b) Show that the intersection $\cap_{M>0} E_M$ is equal to the set $\mathcal{O}$ of odd functions in L^2. Is the set $\mathcal{O}$ closed? (5 marks)

(c) Determine the orthogonal of E_M, in other words the set $E_M^{\perp}$ of elements of L^2 whose scalar product with elements of E_M is zero. (10 marks)

(Total: 20 marks)

4. Consider the Volterra operator on $L^2([0,1])$

$$f \mapsto Vf, \qquad \text{where} \qquad [Vf](x) = \int_0^x f(t)\,dt$$

(a) Show that V is bounded and compact. (6 marks)

(b) Show that 0 is in the spectrum of V (i.e. that there does not exist a bounded operator which inverts V) (6 marks)

(c) Show that 0 is the only point in the spectrum of V (i.e. that there does exist a bounded operator S which inverts $V - \lambda\,\mathrm{Id}$ if $\lambda \neq 0$) (8 marks)

(Total: 20 marks)

1. Are the following subsets of $\ell^2(\mathbb{N})$ bounded, or compact, or both? Justify your answers.

(a) $E_1 = \{(x_n) \in \ell^2,\ \sum |x_n|^2 \le 1\}$ (5 marks)

(b) $E_2 = \{(x_n) \in \ell^2,\ \sum |x_n|^4 \le 1\}$ (7 marks)

(c) $E_3 = \{(x_n) \in \ell^2,\ |x_n| \le n^{-1}\}$. (8 marks)

(Total: 20 marks)

2. We consider the operator T defined through the kernel K

$$T : f \mapsto \int K(x-y) f(y)\, dy.$$

(a) Show that T is bounded on $L^p(\mathbb{R})$ if $K \in L^1(\mathbb{R})$. (5 marks)

(b) For which $K \in L^1$ is T compact on $L^p(\mathbb{R})$? (5 marks)

(c) We want to give an analogous criterion for the boundedness over $L^2(\mathbb{R})$ of a more general class of operators. Let

$$U : f \mapsto \int L(x,y) f(y)\, dy.$$

Show first that U is bounded on L^2 if and only if there exists a constant C such that for any $f, g \in L^2$,

$$\iint L(x,y) f(y) g(x)\, dx\, dy \le C \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

(4 marks)

(d) With the help of the previous question, show Schur's criterion: the operator U is bounded on L^2 if

$$\sup_x \int L(x,y)\, dy + \sup_y \int L(x,y)\, dx < \infty,$$

which gives the desired result. (6 marks)

(Total: 20 marks)

5. In this question, we examine the question of the metrizability of weak convergence in a separable Hilbert space H. In other words, we want to understand whether weak convergence can be expressed through a metric.

(a) We fix a Hilbert basis (e_n) and define for x, y in the unit ball B of H

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |\langle x - y \, , \, e_n \rangle|.$$

Show that d is a distance on B. (6 marks)

(b) Consider a sequence (x_n) in B and an element x in B. Show that $x_n \rightharpoonup x$ (i.e. the sequence x_n converges weakly to x) if $d(x_n, x) \to 0$. (7 marks)

(c) Conversely, show that $d(x_n, x) \to 0$ if $x_n \rightharpoonup x$. (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

MATH 60020 - 70020

Function Spaces and Applications (Solutions)

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1. (a) E_1 is nothing but the closed unit ball, hence its boudedness is obvious. By Riesz' lemma, the unit ball is not compact in an infinite-dimensional vector space. $\boxed{5, \text{A}}$

(b) For any $N \in \mathbb{N}$, consider the sequence $(x_n^N)_{n\in\mathbb{N}}$ defined by

$$x_n^N = \begin{cases} N^{-1/4} & \text{if } n \leq N \\ 0 & \text{if } n > N. \end{cases}$$

It is such that

$$\sum_{n=1}^{\infty} |x_n^N|^4 = N \cdot N^{-1} = 1$$

but

$$\sum_{n=1}^{\infty} |x_n^N|^2 = N \cdot N^{-1/2} = N^{1/2}.$$

Therefore, E_2 is not bounded, and hence not compact. $\boxed{7, \text{B}}$

(c) Consider a sequence (x_n^N) in E_3; we want to show that there exists a convergent subsequence. For any M, we can extract a subsequence $(x_n^{\varphi(N)})$ such that the M first coordinates, $x_1^{\varphi(N)}, \dots x_M^{\varphi(N)}$ converge as $N \to \infty$.

By a diagonal argument, we can find a subsequence such that $x_n^{\varphi(N)}$ converges for any n to a limit y_n. Then (y_n) is in ℓ^2 and furthermore

$$\sum_n |x_n^{\varphi(N)} - y_n|^2 = \sum_{n=1}^{M} |x_n^{\varphi(N)} - y_n|^2 + \sum_{n=M}^{\infty} |x_n^{\varphi(N)} - y_n|^2.$$

Fix $\epsilon > 0$. First, we can choose M such that

$$\sum_{n=M}^{\infty} |x_n^{\varphi(N)} - y_n|^2 \leq \sum_{n=M}^{\infty} 4n^{-2} \leq \epsilon.$$

Second, by the convergence of each coordinate, we can find N_0 such that, for any $N \geq N_0$,

$$\sum_{n=1}^{M} |x_n^{\varphi(N)} - y_n|^2 \leq \epsilon.$$

This implies that for $N \geq N_0$,

$$\sum_{n=1}^{\infty} |x_n^{\varphi(N)} - y_n|^2 \leq 2\epsilon.$$

In other words, $(x_n^{\varphi(N)})$ converges in ℓ^2 to (y_n). This gives compactness in ℓ^2, and hence boundedness. $\boxed{4, \text{B}}$

$\boxed{4, \text{C}}$

2. (a) First observe that T, after a change of variables, can be written as

$$(Tf)(x) = \int K(y)f(x-y)\,dy.$$

By the generalized Minkowski inequality,

$$\begin{aligned}\left\|\int K(y)f(x-y)\,dy\right\|_{L^p_x} &\leq \int \|K(y)f(x-y)\|_{L^p_x}\,dy \\ &= \int |K(y)|\,\|f(x-y)\|_{L^p_x}\,dy = \|f\|_{L^p}\|K\|_{L^1}.\end{aligned}$$

$\boxed{5, \mathrm{A}}$

(b) T is only compact when it is identically zero. This is a consequence of the invariance by translation of this operator: for any $f \in L^2$,

$$T[f(x-n)] = [Tf](x-n).$$

The set $([Tf](x-n))_{n\in\mathbb{N}}$ is only precompact in $L^2(\mathbb{R})$ if $Tf = 0$. $\boxed{5, \mathrm{A}}$

(c) By duality in $L^2(\mathbb{R})$, the norm of Uf can be written as

$$\|Uf\|_{L^2} = \sup_{g\in L^2} \frac{\langle Uf\,,\,g\rangle}{\|g\|_{L^2}} = \sup_{g\in L^2} \frac{\iint L(x,y)f(y)g(x)\,dx\,dy}{\|g\|_{L^2}}.$$

Therefore, $\|Uf\|_{L^2} \leq C\|f\|_{L^2}$ for a constant C if and only if

$$\iint L(x,y)f(y)g(x)\,dx\,dy \leq C\|f\|_{L^2}\|g\|_{L^2}.$$

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$\boxed{4, \mathrm{B}}$

(d) By the previous question, it suffices to prove that, for a constant C.

$$\iint L(x,y)f(y)g(x)\,dx\,dy \leq C\|f\|_{L^2}\|g\|_{L^2}$$

In this inequality, we can divide by the norms of f and g to see that it suffices to prove for a constant C

$$\iint L(x,y)f(y)g(x)\,dx\,dy \leq C \qquad \text{if } \|f\|_{L^2} = \|g\|_{L^2} = 1$$

Assuming $\|f\|_{L^2} = \|g\|_{L^2} = 1$ and using the elementary inequality $ab \leq \frac{1}{2}(a^2+b^2)$,

$$\begin{aligned}\left|\iint L(x,y)f(x)g(y)\,dx\,dy\right| &\leq \iint |L(x,y)|\left[\frac{1}{2}|f(x)|^2 + \frac{1}{2}|g(y)|^2\right]\,dx\,dy \\ &= \frac{1}{2}\iint |L(x,y)||f(x)|^2\,dx\,dy + \frac{1}{2}\iint |L(x,y)||g(y)|^2\,dx\,dy \\ &\leq \frac{1}{2}\left[\sup_x \int L(x,y)\,dy + \sup_y \int L(x,y)\,dx\right]\left[\|f\|_{L^2}^2 + \|g\|_{L^2}^2\right] \\ &= \frac{1}{2}\left[\sup_x \int L(x,y)\,dy + \sup_y \int L(x,y)\,dx\right]\end{aligned}$$

$\boxed{6, \mathrm{D}}$

3. (a) The map

$$\Phi : f \mapsto \int_{-M}^{M} f\,dx$$

is a bounded linear form on $L^2(\mathbb{R})$. Indeed,

$$|\Phi(f)| \lesssim (2M)^{1/2}\|f\|_{L^2}$$

by the Cauchy-Schwarz inequality.
Therefore, $E_M = \Phi^{-1}(\{0\})$ is closed. $\boxed{5, \text{A}}$

(b) Note that f belongs to E_M if and only if

$$\int_0^M [f(x) + f(-x)]\,dx = 0.$$

This implies that, for any $0 < a < b$,

$$\int_a^b [f(x) + f(-x)]\,dx = 0,$$

which means that $f(x) + f(-x) = 0$ for almost any x, or in other words that f is odd.
The set $\mathcal{O}$ is closed as the intersection of closed sets. $\boxed{5, \text{B}}$

(c) For $c \in \mathbb{R}$, define the functions g_c by

$$g_c(x) = \begin{cases} c & \text{if } x \in [-M, M] \\ 0 & \text{if } x \notin [-M, M] \end{cases}.$$

We claim that $E_M^\perp$ is made up of the functions g_c:

$$E_M^\perp = \{g_c,\ c \in \mathbb{R}\}.$$

First, it is easy to check that $g_c \in E_M^\perp$:

$$\langle g_c\,,\, f\rangle = 0 \qquad \text{if } f \in E_M \text{ and } c \in \mathbb{R}.$$

Second, if $g \in E_M^\perp$, then for any $f \in E_M$,

$$0 = \langle f\,,\, g\rangle = \int_{-\infty}^{-M} f(x)g(x)\,dx + \int_{-M}^{-M} f(x)g(x)\,dx + \int_M^\infty f(x)g(x)\,dx$$

Since $f \in E_M$ can take arbitrary values on the complement of $[-M, M]$, the above implies that $g = 0$ on the complement of $[-M, M]$. On $[-M, M]$, we decompose g into

$$g = c + h, \qquad \text{where } c \in \mathbb{R} \text{ and } \textstyle\int_{-M}^{M} h(x)\,dx = 0.$$

Then, for any $f \in E_M$

$$0 = \langle f\,,\, g\rangle = \int_{-M}^{M} f(x)g(x)\,dx = \int_{-M}^{M} f(x)h(x)\,dx$$

(where we used in the last equality the definition of E_M). Since h has mean zero, this implies that, for any $F \in L^2(-M, M)$,

$$\int_{-M}^{M} F(x)h(x)\,dx = 0$$

which finally implies that $h = 0$, or in other words $g = g_c$. $\boxed{8, \text{C}}$

 $\boxed{2, \text{D}}$

4. (a) An application of the Cauchy-Schwarz inequality shows that V maps $L^2([0,1)$ to $L^\infty([0,1)$:

$$\|Vf\|_{L^\infty([0,1])} = \sup_x \left| \int_0^x f(t)\, dt \right| \leq \|f\|_{L^1([0,1)} \leq \|f\|_{L^2([0,1)}$$

Since $L^2([0,1)$ is contained in $L^\infty([0,1])$, this shows that V is bounded on L^2. Furthermore, we observe that, by another application of the Cauchy-Schwarz inequality, if $x < y \in [0,1]$,

$$|Vf(x) - Vf(y)| = \left| \int_x^y f(t)\, dt \right| \leq \|f\|_{L^1([x,y])} \leq |x-y|^{1/2} \|f\|_{L^2}.$$

This implies that the set $V(B) = \{Vf,\, f \in L^2,\, \|f\|_{L^2} \leq 1\}$ is uniformly equicontinuous on $[0,1]$; and we saw earlier that it is also bounded in L^∞. As a consequence, we can apply the Arzela-Ascoli theorem and obtain that the set $V(B)$ is precompact in the set of continuous functions $\mathcal{C}([0,1])$. Since $\mathcal{C}([0,1])$ is contained in $L^2([0,1])$, this shows that $V(B)$ is precompact in $L^2([0,1])$, hence that V is a compact operator on L^2. $\boxed{6, \mathrm{A}}$

(b) To show that 0 is not in the spectrum, we need to show that V is not invertible on L^2. But this is clear from our findings in (a). Indeed, we saw that the image of an L^2 function by V is continuous. Hence, elements of L^2 which are not continuous are not in the image of V, and V cannot be invertible. $\boxed{6, \mathrm{A}}$

(c) For $\lambda \neq 0$, we want to show that $V - \lambda Id$ is invertible. This means that, for any $g \in L^2$, there exists $f \in L^2$ such that

$$Vf(x) - \lambda f(x) = g(x).$$

We assume first that g is smooth, so that we can use integrodifferential calculus without worrying about the regularity of the functions involved. Setting $F(x) = \int_0^x f(t)\, dt$, this is equivalent to solving

$$F(x) - \lambda F'(x) = g(x), \qquad F(0) = 0.$$

This can be rephrased as $\frac{d}{dx}[e^{-\lambda^{-1}x} F(x)] = \lambda^{-1} e^{-\lambda^{-1}x} g(x)$, which leads to the solution

$$F(x) = \lambda^{-1} \int_0^x e^{\lambda^{-1}(x-y)} g(y)\, dy,$$

or

$$f(x) = \lambda^{-1} g(x) + \lambda^{-2} \int_0^x e^{\lambda^{-1}(x-y)} g(y)\, dy.$$

Denoting S for the map $g \mapsto f$, it is bounded on L^2 and provides a solution of $Vf(x) - \lambda f(x) = g(x)$.

We can now treat the case where g is only taken in L^2 and not assumed to be smooth by approximation: if g_n is a sequence of smooth functions converging to g, then Sg_n converge to Sg, which provides an inverse to V. $\boxed{8, \mathrm{D}}$

5. (a) Observe first that the series is convergent since, for any $x, y \in B$,

$$|\langle x - y\,,\, e_n\rangle| \leq \|x - y\|\|e_n\| \leq (\|x\| + \|y\|) \leq 2.$$

The triangle inequality and the positive homogeneity are immediately checked since they hold for each $|\langle x - y\,,\, e_n\rangle|$, hence for the sum.
There remains to check that $d(x, y) = 0$ if and only if $x = y$. But $d(x, y) = 0$ is equivalent to $|\langle x - y\,,\, e_n\rangle| = 0$ for any n. Since (e_n) is a Hilbertian basis,

$$\|x - y\|^2 = \sum |\langle x - y\,,\, e_n\rangle|^2 = 0,$$

hence $x = y$.

(b) Assume that $d(x_k, x) \to 0$ as $k \to \infty$. This implies that, for any n,

$$\langle x_k - x\,,\, e_n\rangle \to 0.$$

Next, we choose $y \in H$ and expand it in the Hilbert basis as

$$y = \sum c_n e_n,$$

We can assume without loss of generality that $\|y\|^2 = \sum |c_n|^2 = 1$. Then

$$\langle x - x_k, y\rangle = \sum_n c_n \langle x_k - x\,,\, e_n\rangle.$$

We now fix $\epsilon > 0$ and choose M such that $\sum_{n=M}^{\infty} |c_n|^2 < \epsilon^2$ and N such that, if $k > N$, then $\sum_{n=0}^{M} |\langle x_k - x\,,\, e_n\rangle| < \epsilon$. As a consequence, if $k > N$,

$$\begin{aligned}
|\langle x - x_k, y\rangle| &\leq \left|\sum_{n=0}^{M} c_n \langle x_k - x\,,\, e_n\rangle\right| + \left|\sum_{n=M+1}^{\infty} c_n \langle x_k - x\,,\, e_n\rangle\right| \\
&\leq \sum_{n=0}^{M} |\langle x_k - x\,,\, e_n\rangle| + \left[\sum_{n=M}^{\infty} |c_n|^2\right]^{1/2} \|x - x_k\| \\
&\leq \epsilon + 2\epsilon.
\end{aligned}$$

(c) Assume that $x_k \rightharpoonup x$. Then, for any n, $|\langle x_k - x\,,\, e_n\rangle| \to 0$. Fix $\epsilon > 0$. We can choose M such that $\sum_{n=M}^{\infty} 2^{-n} \cdot 2 < \epsilon$ and N such that, for any $k > N$ and any $n \leq M$, $\sum_{n=0}^{M} |\langle x_k - x\,,\, e_n\rangle| < \epsilon$.
Then, if $k > N$,

$$d(x_k, x) = \sum_{n=0}^{M} |\langle x_k - x\,,\, e_n\rangle| + \sum_{n=M}^{\infty} |\langle x_k - x\,,\, e_n\rangle \leq 2\epsilon,$$

which proves that $d(x_k, x) \to 0$.

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 0 of 20 marks