Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2014

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3/4/5A4

Mathematical Physics I: Quantum Mechanics

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Gaussian wave packet in a harmonic potential

Consider the Gaussian wave packet

$$\psi(x) = \exp\left(-\alpha(x-q)^2 + \frac{\mathrm{i}}{\hbar}p(x-q) + \frac{\mathrm{i}}{\hbar}\gamma\right). \tag{1}$$

where $q, p \in \mathbb{R}$ and $\alpha, \gamma \in \mathbb{C}$ are parameters, with $\mathrm{Re}(\alpha) > 0$. Note that for arbitrary α and γ, $\psi(x)$ is in general not normalised to one.

(a) Calculate the expectation values of the position $\langle \hat{x} \rangle$ and the momentum $\langle \hat{p} \rangle$.

(b) Now consider the time evolution generated by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

with an initial wave packet of the form (1).

(i) Substitute an ansatz of the form (1) with time dependent parameters $q(t), p(t) \in \mathbb{R}$ and $\alpha(t), \gamma(t) \in \mathbb{C}$ into the time dependent Schrödinger equation and derive equations of motion for the expectation values of position and momentum.

(ii) Now deduce these equations directly from Heisenberg's equations of motion and verify that you obtain the same result, and compare to the corresponding classical dynamics.

Hint: You may use the fact that $\int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$, for $\mathrm{Re}(a) > 0$.

2. Harmonic oscillator eigenvalues.

Consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

(a) Consider the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, defined as

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} + i\left(\frac{1}{m\omega\hbar}\right)^{1/2}\hat{p}\right),$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} - i\left(\frac{1}{m\omega\hbar}\right)^{1/2}\hat{p}\right).$$

Using the fundamental commutation relation between position and momentum, $[\hat{x},\hat{p}] = i\hbar$, verify that the ladder operators fulfil the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$. Further calculate the commutators $[\hat{N},\hat{a}]$, and $[\hat{N},\hat{a}^\dagger]$, where the number operator $\hat{N}$ is defined as $\hat{N} = \hat{a}^\dagger\hat{a}$.

(b) Rewrite the Hamiltonian using the number operator $\hat{N}$.

(c) Prove that the spectrum of $\hat{N}$ consists of the non-negative integers, and deduce the spectrum of $\hat{H}$.

3. Angular Momentum

The matrices

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

are a representation of the angular momentum operators for the total angular momentum quantum number $l = 1$ in the eigenbasis of $\hat{L}_z$.

(a) Verify that the commutators between the matrices $\hat{L}_x$ and $\hat{L}_y$, and $\hat{L}_y$ and $\hat{L}_z$ yield the expected results.

(b) Calculate the normalised eigenvectors of $\hat{L}_x$ and show that these vectors are indeed orthogonal.

(c) Assume that the system is in the eigenstate of $\hat{L}_x$ corresponding to the maximal eigenvalue.

(i) With which probability does a measurement of $\hat{L}_x$ yield which result? With which probability does a measurement of $\hat{L}_z$ yield which result?

(ii) Calculate the expectation values and the uncertainties of $\hat{L}_y$ and $\hat{L}_z$.

(iii) Verify that the uncertainty relation for $\hat{L}_y$ and $\hat{L}_z$ is fulfilled.

4. **Bound state of a δ-potential**

Consider the potential $V(x) = -\lambda\delta(x)$ with $\lambda \in \mathbb{R}$, positive.

(a) Calculate the normalised bound state and the corresponding energy.

(b) Assume that the system is in the ground state. Calculate the expectation value $\langle x \rangle$, and the uncertainty Δx, and sketch the probability distribution $|\psi(x)|^2$ as a function of x.

Hint: Remember that the derivative of the wave function does not have to be continuous at the origin here. It is useful to integrate the time independent Schrödinger equation from $x = -\epsilon$ to $x = \epsilon$ and then take the limit $\epsilon \to 0$.

	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 1		Marks & seen/unseen

Parts (a)

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \bar{\psi}(x)\, x\, \psi(x)\, dx \Big/ \int_{-\infty}^{\infty} \bar{\psi}(x) \psi(x)\, dx$$

seen (lecture & HW)

$$= \frac{\int_{-\infty}^{\infty} x e^{-2\mathrm{Re}(\alpha)(x-q)^2} e^{-2\mathrm{Im}(\gamma)/\hbar}\, dx}{\int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha)(x-q)^2} e^{-2\mathrm{Im}(\gamma)/\hbar}\, dx}$$

$$= \frac{e^{-2\mathrm{Im}(\gamma)/\hbar} \int_{-\infty}^{\infty} x e^{-2\mathrm{Re}(\alpha)(x-q)^2} dx}{e^{-2\mathrm{Im}(\gamma)/\hbar} \int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha)(x-q)^2} dx}$$

substitution $z = x - q$:

$$\langle \hat{x} \rangle = \frac{\int_{-\infty}^{\infty} (z+q) e^{-2\mathrm{Re}(\alpha)\cancel{(x-q)^2} z^2} dz}{\int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha) z^2} dz}$$

$$= \frac{\int_{-\infty}^{\infty} z e^{-2\mathrm{Re}(\alpha) z^2} dz + \int_{-\infty}^{\infty} q e^{-2\mathrm{Re}(\alpha) z^2} dz}{\int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha) z^2} dz}$$

the first term in the numerator is an integral from $-\infty$ to ∞ over an odd function of z and thus vanishes.

2

$$\Rightarrow \langle \hat{x} \rangle = \frac{q \int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha) z^2} dz}{\int_{-\infty}^{\infty} e^{-2\mathrm{Re}(\alpha) z^2} dz} = q$$

For $\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \bar{\psi}(x) \hat{p}\, \psi(x) \Big/ \int_{-\infty}^{\infty} \bar{\psi}(x) \psi(x)$

we first calculate $\hat{p}\, \psi(x)$:

	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 1		Marks & seen/unseen

Parts

(a)

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x)$$
$$= -i\hbar\left(-2\alpha(x-q)+\frac{i}{\hbar}p\right)\psi$$
$$= \left(2i\hbar\alpha(x-q)+p\right)\psi$$ again subst. $z=x-q$:

seen

$$\langle\hat{p}\rangle = \frac{\int_{-\infty}^{\infty}\left(2i\hbar\alpha z e^{-2Re(\alpha)z^2}+pe^{-2Re(\alpha)z^2}\right)dz}{\int_{-\infty}^{\infty}e^{-2Re(\alpha)z^2}dz}$$

$$= p\frac{\int_{-\infty}^{\infty}e^{-2Re(\alpha)z^2}dz}{\int_{-\infty}^{\infty}e^{-2Re(\alpha)z^2}dz} = \cancel{p}$$

2

(since $\int_{-\infty}^{\infty} z e^{-2Re(\alpha)z^2}dz = 0$ as before)

(b)

(i) We need to calculate $\hat{H}\psi$ and substitute into $i\hbar\dot{\psi} = \hat{H}\psi$. $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2$

seen in homework (unassessed)

$$\frac{\partial^2}{\partial x^2}\psi(x) = \frac{\partial}{\partial x}\left[-2\alpha(x-q)+i/\hbar\, p\right]\psi(x)$$
$$= \left[-2\alpha+\left(-2\alpha(x-q)+i/\hbar\, p\right)^2\right]\psi(x)$$

$$\hat{H}\psi = \left[\frac{\hbar^2}{m}\alpha+\frac{2\hbar^2}{m}\alpha^2(x-q)^2+\frac{p^2}{2m}+2\frac{i\hbar}{m}\alpha p(x-q)\right.$$
$$\left.+\frac{1}{2}m\omega^2x^2\right]\psi$$

Setter's initials EMG	Checker's initials	Page number 2

Parts (b) (i)

that is, we find

$$\Big[\frac{\hbar^2}{m}\alpha + \frac{2\hbar^2}{m}\alpha^2(x-q)^2 + \frac{p^2}{2m} + \frac{2i\hbar}{m}\alpha p(x-q)$$
$$+\frac{1}{2}m\omega^2x^2\Big]\psi = i\hbar\dot{\psi}$$
$$= i\hbar\Big[-\dot{\alpha}(x-q)^2 + 2\alpha(x-q)\dot{q}$$
$$+\tfrac{i}{\hbar}\dot{p}(x-q) - \tfrac{i}{\hbar}p\dot{q} + \tfrac{i}{\hbar}\dot{\gamma}\Big]\psi$$

seen in HW (not assessed)

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Now this equality has to hold in each order of $(x-q)$ separately. That is, we can separate the equation into 3 independent equations for the coefficients of the different orders of $(x-q)$. First we need to rewrite $\frac{1}{2}m\omega^2x^2$ as a polynomial in $(x-q)$:

$$\tfrac{1}{2}m\omega^2x^2 = \tfrac{1}{2}m\omega^2\left((x-q)^2+2qx-q^2\right)$$
$$= \tfrac{1}{2}m\omega^2\left((x-q)^2+2q(x-q)+q^2\right)$$

We then find for the order $(x-q)^2$:

$$\boxed{\frac{2\hbar^2}{m}\alpha^2 + \frac{1}{2}m\omega^2 = -i\hbar\dot{\alpha}}$$ which doesn't concern us here.

	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question ①		Marks & seen/unseen
Parts (b) (i)	For the order $(x-q)^1$ we find $\boxed{\frac{i\hbar}{m}2\alpha p + m\omega^2 q = i\hbar 2\alpha\dot{q} - \dot{p}}$ the real and imaginary part of this equation lead to the dynamical equs for q and p: imaginary part: $2\frac{\hbar}{m}\mathrm{Re}(\alpha)p = 2\hbar\mathrm{Re}(\alpha)\dot{q}$ $\Rightarrow \boxed{\dot{q} = p/m}$ equation for exp value of $\hat{x}$ real part: $-\frac{2\hbar}{m}\mathrm{Im}(\alpha)p + m\omega^2 q = -2\hbar\mathrm{Im}(\alpha)\dot{q} - \dot{p}$ insert $\dot{q} = p/m$: $-\frac{2\hbar}{m}\mathrm{Im}(\alpha)p + m\omega^2 q = -\frac{2\hbar}{m}\mathrm{Im}(\alpha)p - \dot{p}$ $\Rightarrow \boxed{\dot{p} = -m\omega^2 q}$ equation for exp value of $\hat{p}$ that is: $\frac{d}{dt}\langle\hat{x}\rangle = \langle\hat{p}\rangle/m$ and $\frac{d\langle\hat{p}\rangle}{dt} = -m\omega^2\langle\hat{x}\rangle$	Seen in HW (not assessed) 2 1 1
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	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question ①		Marks & seen/unseen
Parts (b) (ii)	Heisenberg's equations of motion for $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$:	unseen

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle$$

since $\hat{H}=\frac{\hat{p}^2}{2m}+V(\hat{x})$:

$$[\hat{H},\hat{x}]=\frac{1}{2m}[\hat{p}^2,\hat{x}]$$

$$=\frac{1}{2m}\Big(\hat{p}\underbrace{[\hat{p},\hat{x}]}_{=-i\hbar}+[\hat{p},\hat{x}]\hat{p}\Big)$$

$$=-\frac{i\hbar}{m}\hat{p}$$

$$\Rightarrow \frac{d\langle\hat{x}\rangle}{dt}=\frac{i}{\hbar}\left(-\frac{i\hbar}{m}\langle\hat{p}\rangle\right)=\frac{\langle\hat{p}\rangle}{m}$$

as before. **2**

$$\frac{d}{dt}\langle\hat{p}\rangle=\frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle$$

$$=\frac{i}{\hbar}\langle[V(\hat{x}),\hat{p}]\rangle$$

in position representation:

$$[V(\hat{x}),\hat{p}]=[V(x),-i\hbar\frac{\partial}{\partial x}]$$

$$=i\hbar\frac{\partial V}{\partial x}$$

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EXAMINATION SOLUTIONS 2013-14

Course: M3/4/5A4

Question: 1

Marks & seen/unseen

Parts (b) (ii)

$$\frac{d}{dt}\langle\hat{p}\rangle = -\left\langle\frac{\partial\hat{V}}{\partial\hat{x}}\right\rangle = -m\omega^2\langle\hat{x}\rangle$$

just as in (b)(i).

unseen 3

These are identical to the classical equations of motion for a harmonic oscillator! 1

$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\dot{x} = \frac{\partial H}{\partial p} = p/m$$

(Ehrenfest theorem!)

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Checker's initials:

EXAMINATION SOLUTIONS 2013-14

Course M3/4/5A4

Question 2

Marks & seen/unseen

Parts

~~(a)~~ (b)

part (a) & (b) interchanged 15 April 2014

$$\hat{a}+\hat{a}^{+} = \sqrt{2}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x}\right)$$

$$\Rightarrow \hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}+\hat{a}^{+})$$

partly seen in class

1

$$\hat{a}^{+}-\hat{a} = i\left(\frac{2}{m\omega\hbar}\right)^{1/2}\hat{p}$$

$$\Rightarrow \hat{p} = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(\hat{a}-\hat{a}^{+})$$

1

$$\Rightarrow \hat{H} = -\frac{m\omega\hbar}{4m}(\hat{a}-\hat{a}^{+})^2 + \frac{\hbar m\omega^2}{4m\omega}(\hat{a}+\hat{a}^{+})^2$$

$$= -\frac{\hbar\omega}{4}(\hat{a}^2 - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} + \hat{a}^{+2})$$

$$+ \frac{\hbar\omega}{4}(\hat{a}^2 + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}^{+2})$$

unseen (only result quoted in class)

$$= \frac{\hbar\omega}{4}(\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a})$$

$$= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a})$$

now with $[\hat{a},\hat{a}^{+}] = \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$

$\hat{a}\hat{a}^{+} = 1 + \hat{a}^{+}\hat{a}$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^{+}\hat{a} + \hat{a}^{+}\hat{a} + 1) = \hbar\omega(\hat{a}^{+}\hat{a} + 1/2)$$

$$= \hbar\omega(\hat{N} + 1/2)$$

3

Setter's initials EMG

Checker's initials

Parts

~~(b)~~ (a)

$$[\hat{a},\hat{a}^{+}] = \frac{1}{2}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} + i\left(\frac{1}{m\omega\hbar}\right)^{1/2}\hat{p},\ \left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} - i\left(\frac{1}{m\omega\hbar}\right)^{1/2}\hat{p}\right]$$

partly seen in class

$$= -\frac{i}{2}\left(\frac{m\omega}{\hbar}\frac{1}{m\omega\hbar}\right)^{1/2}[\hat{x},\hat{p}] + \frac{i}{2}\left(\frac{m\omega}{\hbar}\frac{1}{m\omega\hbar}\right)^{1/2}[\hat{p},\hat{x}]$$

$$= -i\left(\frac{1}{\hbar^2}\right)^{1/2}[\hat{x},\hat{p}] = \frac{-i}{\hbar}\cdot i\hbar$$ 1

$$= 1 \quad \square$$

$$[\hat{N},\hat{a}] = [\hat{a}^{+}\hat{a},\hat{a}] = [\hat{a}^{+},\hat{a}]\,\hat{a} = -\hat{a}$$ 1

$$[\hat{N},\hat{a}^{+}] = [\hat{a}^{+}\hat{a},\hat{a}^{+}] = \hat{a}^{+}[\hat{a},\hat{a}^{+}] = \hat{a}^{+}$$ 1

(c) Spectrum of $\hat{N}$: $\hat{N}|\nu\rangle = \nu|\nu\rangle$

Seen in class

(1) We first prove that $\nu \geq 0$

for this purpose we consider

$\|\hat{a}|\nu\rangle\|^2 \geq 0$: 2

$$\|\hat{a}|\nu\rangle\|^2 = \langle\nu|\hat{a}^{+}\hat{a}|\nu\rangle = \langle\nu|\hat{N}|\nu\rangle$$
$$= \nu\langle\nu|\nu\rangle$$

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Question: 2

Marks & seen/unseen

Parts (c)

$\Rightarrow \nu\langle\nu|\nu\rangle \geq 0 \Rightarrow \nu \geq 0$ — seen in class

also $\nu = 0$ implies that $\|\hat{a}|0\rangle\| = 0$, that is $\hat{a}|0\rangle = 0$ is the zero vector.

Now we prove the following 2 lemmata:

(i) If $\hat{N}|\nu\rangle = \nu|\nu\rangle$ then $\hat{N}(\hat{a}|\nu\rangle) = (\nu-1)\hat{a}|\nu\rangle$ for $\hat{a}|\nu\rangle \neq 0$ — 1

(ii) If $\hat{N}|\nu\rangle = \nu|\nu\rangle$ then $\hat{N}(\hat{a}^+|\nu\rangle) = (\nu+1)\hat{a}^+|\nu\rangle$ — 1

Proof:

$$\hat{N}\hat{a}|\nu\rangle = ([\hat{N},\hat{a}] + \hat{a}\hat{N})|\nu\rangle = (-\hat{a} + \hat{a}\nu)|\nu\rangle = (\nu-1)\hat{a}|\nu\rangle \quad \square$$ 2

$$\hat{N}\hat{a}^+|\nu\rangle = ([\hat{N},\hat{a}^+] + \hat{a}^+\hat{N})|\nu\rangle = (\hat{a}^+ + \hat{a}^+\nu)|\nu\rangle = (\nu+1)\hat{a}^+|\nu\rangle \quad \square$$ 2

from $\nu \geq 0$ and (i) it follows that $\nu = 0$ has to be in the spectrum (such that the chain ends and does not continue to negative ν). From lemma (ii) we then deduce that all integers are in the spectr. — 2

Setter's initials: EMG

Checker's initials:

Parts

(c) From this we deduce that the spectrum of $\hat{H}$ consists of the discrete set of numbers

$$E_n = \hbar\omega(n+1/2)$$

with n non-negative integer.

seen in lecture

2

EXAMINATION SOLUTIONS 2013-14

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Question: 3

Marks & seen/unseen

Parts

(a) Commutators we expect:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$\left([\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y\right)$$

verify:

seen in HW (not assessed)

$$[\hat{L}_x, \hat{L}_y] = \frac{\hbar^2}{2}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\}$$

$$= \frac{\hbar^2}{2}\left\{\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right\}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

$$= i\hbar \, \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar \hat{L}_z \checkmark$$

$$[\hat{L}_y, \hat{L}_z] = \frac{i\hbar^2}{\sqrt{2}}\left\{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right\}$$

$$= \frac{i\hbar^2}{\sqrt{2}}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right\}$$

Setter's initials: EMG

Checker's initials

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Marks & seen/unseen

Parts

(a) $[\hat{L}_y, \hat{L}_z] = i\hbar \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = i\hbar \hat{L}_x$ ✓

3 points part a)

$$[\hat{L}_z, \hat{L}_x] = \frac{\hbar^2}{\sqrt{2}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$= \frac{\hbar^2}{\sqrt{2}} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

not asked

$$= \hbar \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= i\hbar \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = i\hbar \hat{L}_y \checkmark$$

(b) eigenvalues and eigenvectors of $\hat{L}_x$: unseen

We first need the eigenvalues, which from class we know to be $\lambda_0 = 0$, $\lambda_\pm = \pm\hbar$ 1

eigenvectors:

$|\phi_0\rangle$:

$\hat{L}_x |\phi_0\rangle = 0$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_0 = -z_0 \\ y_0 = 0 \end{array}$$

$$\Rightarrow \phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$ 1

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Checker's initials:

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Marks & seen/unseen

Parts: (b)

(up to a phase) — unseen

λ_- : $|\phi_-\rangle$

$\hat{L}_x|\phi_-\rangle = -\hbar|\phi_-\rangle$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_-\\y_-\\z_-\end{pmatrix} = -\hbar\begin{pmatrix}x_-\\y_-\\z_-\end{pmatrix}$$

$$\frac{\hbar}{\sqrt{2}}y_- = -\hbar x_- = -\hbar z_-$$

$$\frac{1}{\sqrt{2}}(x_- + z_-) = -y_-$$

$$\Rightarrow \quad z_- = x_- = \frac{-1}{\sqrt{2}}y_-$$

normalised eigenvector:

$$\phi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\-1\\1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/2\\-1/\sqrt{2}\\1/2\end{pmatrix}$$ 1

$$\phi_- = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

λ_+ : $|\phi_+\rangle$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_+\\y_+\\z_+\end{pmatrix} = \hbar\begin{pmatrix}x_+\\y_+\\z_+\end{pmatrix}$$

$$\frac{\hbar}{\sqrt{2}}y_+ = \hbar x_+ = \hbar z_+$$

$$\Rightarrow \quad z_+ = x_+ = \frac{y_+}{\sqrt{2}}$$

Setter's initials: EMG

Checker's initials:

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Question 3		Marks & seen/unseen
Parts		
(b)	normalised eigenvector	unseen
	$\phi_+ = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$	1
	orthogonality:	
	$\langle\phi_0\|\phi_-\rangle = \frac{1}{2\sqrt{2}}(1+0-1) = 0$ ✓	
	$\langle\phi_0\|\phi_+\rangle = \frac{1}{2\sqrt{2}}(1+0-1) = 0$ ✓	1
	$\langle\phi_+\|\phi_-\rangle = \frac{1}{4}(1-2+1) = 0$ ✓	
(c)	system is in state $\|\psi\rangle = \|\phi_+\rangle$	unseen
(i)	Measurement of L_x yields result $\lambda_+ = \hbar$ with probability 1.	1
	Measurement of L_z:	
	$P(\lambda_j) = \|\langle\phi_j\|\phi\rangle\|^2$ with $\hat{L}_z\|\phi_j\rangle = \lambda_j\|\phi_j\rangle$	
	eigenvalues and eigenvectors of $\hat{L}_z$: $\lambda_0 = 0$ $\lambda_\pm = \pm\hbar$	
	$\|\phi_0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\phi_+ = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\phi_- = \begin{pmatrix}0\\0\\1\end{pmatrix}$	
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	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 3		Marks & seen/unseen
Parts (c) (i)	$P(0) = \left\|\frac{\sqrt{2}}{2}\right\|^2 = \frac{1}{2}$ $P(\hbar) = \left\|\frac{1}{2}\right\|^2 = \frac{1}{4}$ $P(-\hbar) = \left\|\frac{1}{2}\right\|^2 = \frac{1}{4}$	unseen 3
(c) (ii)	expectation values of $\hat{L}_y$ and $\hat{L}_z$: & uncertainties: $\langle \hat{L}_y \rangle = \langle \phi_+ \| \hat{L}_y \| \phi_+ \rangle$ $= \frac{\hbar}{4\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ $= \frac{\hbar}{4\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}^T \begin{pmatrix} -i\sqrt{2} \\ 0 \\ i\sqrt{2} \end{pmatrix} = 0$ $\langle \hat{L}_y^2 \rangle = \frac{\hbar^2}{4\cancel{\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ $= \frac{\hbar^2}{8} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ $= \frac{\hbar^2}{8} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{pmatrix} = \frac{\hbar^2}{8} \cdot 4 = \frac{\hbar^2}{2}$ $\Rightarrow \Delta L_y = \sqrt{\frac{\hbar^2}{2}} = \frac{\hbar}{\sqrt{2}}$	unseen 3
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Parts

(c) (ii)

$$\langle \hat{L}_z \rangle = \frac{1}{4}\cdot(-\hbar) + 0\cdot\frac{1}{2} + \frac{1}{4}(\hbar) = \underline{0}$$

$$\langle \hat{L}_z^2 \rangle = \frac{1}{4}(+\hbar^2) + 0\cdot\frac{1}{2} + \frac{\hbar^2}{4} = \frac{\hbar^2}{2}$$

$$\Rightarrow \Delta L_z = \frac{\hbar}{\sqrt{2}}$$

2

unseen

(iii) the uncertainty product is $\Delta L_{\cancel{x}\,y} \Delta L_z = \hbar^2/2$

the theoretical lower bound is given by

$$\Delta L_{\cancel{x}\,y} \Delta L_{\cancel{y}\,z} \geq \frac{1}{2} \underbrace{\langle i [L_y, L_z] \rangle}$$

$$\frac{1}{2}\langle i [L_y, L_z] \rangle = \frac{1}{2}\langle -\hbar \hat{L}_x \rangle$$

we are in the eigenstate $|\phi_+\rangle$ of $\hat{L}_x \Rightarrow \langle \hat{L}_x \rangle = \hbar$

$$\Delta L_y \Delta L_z \geq \frac{\hbar^2}{2}$$

that is, the lower bound is reached here

3

	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 4		Marks & seen/unseen
Parts (a)	From the lecture we know that the possible bound states of the potential are of the form $K = \sqrt{-2mE}/\hbar$ (1) $\phi_E(x) = \begin{cases} Ae^{Kx} & , x \leq 0 \\ Be^{-Kx} & , x \geq 0 \end{cases}$ $E<0$ These have to be continuous at $x=0 \Rightarrow B=A$. Further, we have a boundary condition involving the first derivative of $\phi_E(x)$, which we obtain from the Schrödinger equation $E\phi_E(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_E(x) + V(x)\phi_E(x)$ by integrating between $-\epsilon$ and $+\epsilon$ and taking the limit $\epsilon \to 0$: $E\int_{-\epsilon}^{\epsilon} \phi_E(x)dx = -\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon} \frac{\partial^2}{\partial x^2}\phi_E(x)dx - \lambda\int_{-\epsilon}^{\epsilon}\delta(x)\phi_E(x)dx$ $= -\frac{\hbar^2}{2m}[\phi_E'(x)]_{-\epsilon}^{\epsilon} - \lambda\phi_E(0)$ taking the limit yields	Seen in HW (not assessed) 2 1
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	EXAMINATION SOLUTIONS 2013-14	Course M3/4 15A4
Question 4		Marks & seen/unseen
Parts (a)	$E \underbrace{\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\phi_E(x)dx}_{=0} = -\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\left(\phi_E'(\epsilon)-\phi_E'(-\epsilon)\right) - \lambda\,\phi_E(0)$	seen in homework (not assessed)
	that is	
	$-\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\phi_E'(\epsilon) + \frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\phi_E'(-\epsilon) = \lambda\phi_E(0)$	4
	that is with the general form of $\phi_E(x)$, (1):	
	$\frac{\hbar^2}{2m}(KA+KA) = \lambda A \Rightarrow K = \frac{m\lambda}{\hbar^2}$	2
	with $K = \sqrt{-2mE}/\hbar$ we find	
	$E = -\frac{m\lambda}{2\hbar^2}$ for the energy of the only bound state.	1
	We can deduce the value of A from the normalisation condition:	
	$\int_{-\infty}^{\infty}\lvert\phi_E(x)\rvert^2dx = \lvert A\rvert^2\left(\int_{-\infty}^{0}e^{+2Kx}dx + \int_0^{\infty}e^{-2Kx}dx\right)$	
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	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 4		Marks & seen/unseen

Parts

(a) $\int_{-\infty}^{\infty} |\psi_E(x)|^2 dx = \frac{|A|^2}{2K} + \frac{|A|^2}{2K} = \frac{|A|^2}{K} = 1$ — seen in HW (not assessed)

$A = \sqrt{K} = \left(\frac{m\lambda}{\hbar^2}\right)^{1/2}$ — 2

(up to an arbitrary phase)

(b) $\psi(x) = \begin{cases} Ae^{Kx} & x \leq 0 \\ Ae^{-Kx} & x \geq 0 \end{cases} \quad A = \sqrt{K}$ — unseen

$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \bar{\psi}(x) x \psi(x) dx$

$= K\left[\int_{-\infty}^{0} xe^{2Kx} dx + \int_0^{\infty} xe^{-2Kx} dx\right]$

$\int_0^{\infty} xe^{-2Kx} dx = +\int_0^{-\infty} xe^{2Kx} dx$

$= -\int_{-\infty}^{0} xe^{2Kx} dx$

$\Rightarrow \langle \hat{x} \rangle = 0$ — 2

uncertainty:

$\langle \hat{x}^2 \rangle = K\left[\int_{-\infty}^{0} x^2 e^{2Kx} dx + \int_0^{\infty} x^2 e^{-2Kx} dx\right]$

calculate $\int_a^b x^2 e^{\pm 2Kx}$:

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	EXAMINATION SOLUTIONS 2013-14	Course M3/4/5A4
Question 4		Marks & seen/unseen

Parts (b) — unseen

$$\int_a^b x^2 e^{\pm 2Kx}\,dx = \frac{1}{4}\frac{\partial^2}{\partial K^2}\int_a^b e^{\pm 2Kx}\,dx$$

$$= \frac{1}{4}\frac{\partial^2}{\partial K^2}\left[\pm\frac{1}{2K}e^{\pm 2Kx}\right]_a^b$$

now $\displaystyle\int_{-\infty}^{0} x^2 e^{2Kx}\,dx = \frac{1}{4}\frac{\partial^2}{\partial K^2}\left[\frac{1}{2K}\right]$

$$= \frac{1}{4K^3}$$

$$\int_0^{\infty} x^2 e^{-2Kx}\,dx = \frac{1}{4}\frac{\partial^2}{\partial K^2}\left[\frac{1}{2K}\right]$$

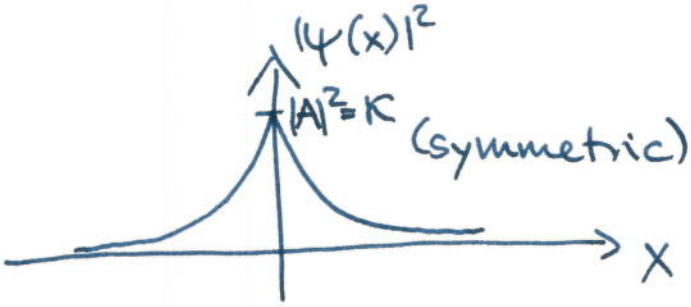
$$= \frac{1}{4K^3}$$

$$\Rightarrow \langle \hat{x}^2\rangle = \frac{1}{2K^2}$$

$$\Rightarrow \Delta x = \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2}$$

$$= \frac{1}{\sqrt{2}K} \quad \text{with } K = \frac{m\lambda}{\hbar^2}$$

4



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