Imperial College London

MATH70028

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: 25 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Let $\zeta, \zeta_1, \zeta_2, \dots$ be random variables.

(a) Define what it means for the sequence ζ_n to converge to ζ a.s. (5 marks)

(b) Define what it means for the sequence ζ_n to converge to ζ in probability. (5 marks)

(c) Show that a.s. convergence implies convergence in probability. (10 marks)

2. Let $\zeta, \zeta_1, \zeta_2, \dots$ be random variables.

(a) Define what it means for ζ_n to converge to ζ in distribution. (4 marks)

(b) Let ζ_n converge in distribution to a degenerate random variable ζ, that is $P(\zeta = a) = 1$ for some $a \in \mathbb{R}$. Show that ζ_n converges to a in probability. (6 marks)

(c) Using characteristic functions, prove the following law of large numbers. Let $\zeta_1, \zeta_2, \dots$ be i.i.d. (independent identically distributed) with existing finite expectation $E(\zeta_1) = m$. Then $(\zeta_1 + \cdots + \zeta_n)/n$ converges in probability to m. (10 marks)

3. (a) Define what it means for a family of random variables to be tight. (5 marks)

(b) Let $\zeta, \zeta_1, \zeta_2, \dots$ be random variables and let ζ_n converge to ζ in probability. Does it follow that the family $\{\zeta_n\}_{n=1}^{\infty}$ is tight? Give a short argument. (5 marks)

(c) For a family $\{\zeta_\alpha\}$ of random variables (α belong to some set of indices), suppose that the expectations of $\sqrt{|\zeta_\alpha|}$ are bounded uniformly in α. Prove that $\{\zeta_\alpha\}$ is tight. (10 marks)

4. (a) Formulate the strong law of large numbers for i.i.d. random variables. (5 marks)

(b) Formulate the central limit theorem for i.i.d. random variables. (5 marks)

(c) Let $\zeta_j,\ j = 1, 2, \dots,$ be i.i.d. with *characteristic* function $\phi(x) = \exp(-|x|)$. Does the strong law of large numbers hold? Give an explanation. (5 marks)

(d) Let $\zeta_j,\ j = 1, 2, \dots,$ be i.i.d. with the *density of the distribution* function $f(x) = \exp(-|x|)/2$, $x \in \mathbb{R}$. Does the central limit theorem hold? Give an explanation. (5 marks)

5. Let ζ_j, $j = 1, 2, \ldots$, be Bernoulli i.i.d. $P(\zeta_1 = 1) = P(\zeta_1 = -1) = 1/2$. Let $S_n = \zeta_1 + \cdots + \zeta_n$. Show that $\limsup S_n/\sqrt{n} = +\infty$ a.s. (20 marks)

Probability exam 2023
Solutions

1a. $P(\zeta_n \to \zeta) = 1$ [5 marks]

seen

1b. $\forall \varepsilon > 0 \quad P(|\zeta_n - \zeta| \geq \varepsilon) \to 0, \; n \to \infty$

[5 marks]

seen

1c. Let $\zeta_n \to \zeta$ a.s. Fix $\varepsilon > 0$

Then $P(|\zeta_n - \zeta| \geq \varepsilon \text{ i.o.}) = 0$

But $\{|\zeta_n - \zeta| \geq \varepsilon \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \{|\zeta_k - \zeta| \geq \varepsilon\}$

By continuity of measure

$$0 = P(|\zeta_n - \zeta| \geq \varepsilon \text{ i.o.}) =$$

$$= \lim_{n \to \infty} P\left(\bigcup_{k \geq n} \{|\zeta_k - \zeta| \geq \varepsilon\}\right)$$

$$\geq \lim_{n \to \infty} P(\{\zeta_n - \zeta| \geq \varepsilon)$$

Thus $P(|\zeta_n - \zeta| \geq \varepsilon) \to 0, \; n \to \infty$.

[10 marks]

seen

2a. $Ef(\xi_n) \to Ef(\xi)$

for any bounded continuous $f(x)$

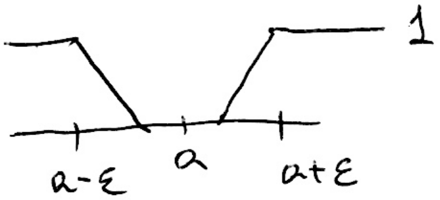
(or an equivalent definition)

[4 marks]

seen

2b. Let $\xi_n \xrightarrow{d} \xi$, $P(\xi = a) = 1$.

Fix $\varepsilon > 0$.



Take a continuous f s.t.

$f(x) = 1$, $x \in \mathbb{R} \setminus [a-\varepsilon, a+\varepsilon]$,

$f(x) = 0$, $x \in [a - \varepsilon/2, a + \varepsilon/2]$

Then $Ef(\xi_n) \to Ef(\xi) = \int_{\mathbb{R} \setminus [a-\varepsilon/2, a+\varepsilon/2]} f(x)\, dF = 0$

Since $F(a) - F(a-0) = 1$.

On the other hand,

$$Ef(\xi_n) \geq P(|\xi_n - a| \geq \varepsilon)$$

Thus $P(|\xi_n - a| \geq \varepsilon) \to 0$.

[6 marks]

unseen

2c. We have for the characteristic function $\varphi(t) = 1 + itm + o(t)$, $t \to 0$.

Thus for a fixed t,

$$\varphi\left(\frac{t}{n}\right) = 1 + \frac{itm}{n} + o\left(\frac{1}{n}\right), \quad n \to \infty$$

Since ξ_j are independent

$$\varphi_{S_n}(t) = \left(1 + \frac{itm}{n} + o\left(\frac{1}{n}\right)\right)^n \to e^{itm}$$

– a characteristic function of a degenerate r.v. ξ, $P(\xi = m) = 1$.

By continuity thm, $\frac{S_n}{n} \xrightarrow{d} m$

By 2b, it follows that

$$\frac{S_n}{n} \xrightarrow{P} m.$$

[10 marks]

unseen

3a $\{\xi_\alpha\}$ tight means

$\forall \varepsilon > 0 \ \exists K > 0$ s.t.

$\sup_\alpha P(|\xi_\alpha| > K) \leq \varepsilon$. [5 marks]

seen

3b. If $\xi_n \xrightarrow{P} \xi$ then

$\xi_n \xrightarrow{d} \xi$. By Prohorov thm

this implies that $\{\xi_n\}$ is tight.

[5 marks]

seen

3c. We have $\sup_\alpha E\sqrt{|\xi_\alpha|} < \infty$

Since $f(x) = \sqrt{x}$ is increasing,

by Markov inequality, for $K > 0$,

$$P(|\xi_\alpha| > K) = P(\sqrt{|\xi_\alpha|} > \sqrt{K})$$

$$\leq \frac{E\sqrt{|\xi_\alpha|}}{\sqrt{K}}$$

$\Rightarrow$

$$\sup_{\alpha} P(|\xi_\alpha| > K) \leq \frac{\sup_{\alpha} E\sqrt{|\xi_\alpha|}}{\sqrt{K}}$$

For a given $\varepsilon > 0$ choose K sufficiently large so that

$$\frac{\sup_{\alpha} E\sqrt{|\xi_\alpha|}}{\sqrt{K}} < \varepsilon$$

$\Rightarrow$ $\{\xi_\alpha\}$ is tight.

[10 marks]

unseen

4a. Let $\xi_1, \xi_2 \dots$ be i.i.d

s.t. $E|\xi_1| < \infty$.

Then $\frac{\xi_1 + \dots + \xi_n}{n} \to E\xi_1$ a.s.

[5 marks]

seen

4b. Let $\xi_1, \xi_2 \dots$ be i.i.d.

s.t. $E\xi_1^2 < \infty$ and ξ_j are nondegenerate.

Let $S_n = \xi_1 + \dots + \xi_n$

Then $P\left(\frac{S_n - ES_n}{\sqrt{VS_n}} \leq x\right) \to \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$

for any $x \in \mathbb{R}$.

[5 marks]

seen

4c. ~~For i.i.d. $VS_n = nV\xi_1$~~

$\frac{S_n}{n}$ has characteristic function

$$\varphi_{\frac{S_n}{n}}(x) = \left(e^{-\frac{|x|}{n}}\right)^n = e^{-|x|}$$

Therefore $\frac{S_n}{n} \xrightarrow{d} \xi_1$, so LLN does not hold.

[5 marks]

unseen

4d. Since ξ_j are nondegenerate

and $E\xi_1^2 = \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx < \infty$,

CLT holds.

[5 marks]

unseen

5. By continuity,

$$P\left(\limsup \frac{S_n}{\sqrt{n}} = +\infty\right) = \lim_{K\to\infty} P\left(\limsup \frac{S_n}{\sqrt{n}} > K\right)$$

Note that $\limsup \frac{S_n}{\sqrt{n}} > K$ is a tail event, so for the proof sufficient to show that $P\left(\limsup \frac{S_n}{\sqrt{n}} > K\right) > 0$,

but $P\left(\frac{S_n}{\sqrt{n}} > K\right) > c(K) > 0$ for n sufficiently large by CLT.

[20 marks]

unseen

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60028/70028	1	This question was generally well done
MATH60028/70028	2	Various approaches to prove 2b, mostly well done. 2c presented most difficulties.
MATH60028/70028	3	In considerable number of exams it was not noticed that solution of 3b follows by Prohorov theorem. 3c easily follows using Markov inequality
MATH60028/70028	4	This question was generally well done
MATH70028	5	Solution to this question is based on 0-1 law and central limit theorem. This was not often fully noticed.