Classical Dynamics **Autumn Term 2024**

Problem Sheet 4

1. Find the Legendre transform for each of the following functions

$$\text{(i) } f(x) = e^x \quad \text{(ii) } f(x) = -\sqrt{1-x^2} \quad \text{(iii) } f(x) = x^4.$$

2. Solve Hamilton's equation for the Hamiltonian $H(q,p) = qp$. Is there a Lagrangian for this problem?

3. (i) Find the Hamiltonian, $H(\theta, \phi, p_\theta, p_\phi)$, for Q4 on Problem Sheet 2.

   (ii) The Lagrangian for a charged particle in a magnetic field is (see problem sheet 3)

$$L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

   where q is the electric charge of the particle. What is the corresponding Hamiltonian?

4. Let $H(q,p,t)$ be the Legendre transform of $L(q,\dot{q},t)$ with respect to $\dot{q}$. Show that

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

   Do not use the equation of motion!

5. (i) The motion of a relativistic particle of rest mass m is described by the Lagrangian

$$L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}},$$

   where c is the speed of light. Show that the corresponding Hamiltonian is

$$H = \sqrt{m^2c^4 + c^2p^2}.$$

   (ii) Consider the Hamiltonian

$$H = \sqrt{m^2c^4 + c^2p^2} - Fx,$$

   where F is a constant (this Hamiltonian describes the motion of a charged relativistic particle in a constant electric field). Write down Hamilton's equations and find the solution for the initial conditions $x(0) = 0$ and $p(0) = 0$. How does $x(t)$ behave for small t? How does $x(t)$ behave for large t?