Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Part I – Problem Sheet 1: Logic and sets

KMB, 04/10/22

Part I of this module has four subsections; logic, sets, functions and relations. There are three problem sheets; the first is on logic and sets. There are one to three "diamonds" on each question – this is an approximate idea of question difficulty/length.

Lean: The mathematics covered in Part I of the module can be turned into a puzzle game called Lean, where solving a level of the game corresponds to proving a theorem in mathematics. Read more at this link. Here you will find links to basic problems in logic, sets and more, with instructions on how to learn how to use Lean. I (Kevin) run a Lean club called the Xena Project, which meets on Thursday evenings from 5 till 8 in the Maths Learning Centre (room 414 Huxley, the big computer room) and on Friday evenings from 5 till 7 in the Xena Project Discord (which you can join via the Imperial student hub).

Important: Lean is one of my research interests. It is an *optional* part of the IUM module, and there will be no Lean in any IUM tests or exams.

1. ✦ Draw truth tables to solve the following problems.
   (a) Prove that $\vee$ is *symmetric*. In other words, prove that if P and Q are propositions, then $P \vee Q \implies Q \vee P$.
   (b) Is $\implies$ symmetric? In other words, is it true that for all propositions P and Q we have
   $$(P \implies Q) \implies (Q \implies P)?$$
   Give a proof or a counterexample.
   (c) Is $\iff$ symmetric? In other words, is it true that for all propositions P and Q we have
   $$(P \iff Q) \implies (Q \iff P)?$$
   Give a proof or a counterexample.

2. ✦ Suppose P, Q and R are propositions, and we know that if Q is true then P is true, and that if Q is false then R is false. Can we deduce that R implies P? Give a proof or a counterexample.

3. ✦ Is it possible to find three true-false statements P, Q and R, such that
   $$(P \vee Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$
   is true?

   Background (not for exam): This sort of question is more important than it might seem. Check out this Wikipedia page on the Boolean satisfiability problem, if you want to know more. Here is a version you can play online with 9 variables and 40 equations, and this paper contains a recent application.

4. ✦✦✦ An *integer* (or a "whole number") is an element of the set $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$. Say that for every integer n we have a true/false statement P_n. Say we know that $P_n \implies P_{n+8}$ for every integer n, and also that $P_n \implies P_{n-3}$ for every integer n. Prove that the P_n are either all true, or all false. You may assume the following induction principle for integers: if $P_n \implies P_{n+1}$ for all n, and $P_n \implies P_{n-1}$ for all n, and if there exists some integer t such that P_t is true, then all the P_n are true.

5. $\boxed{\star\star}$ How could you give a formal proof that if X and Y are sets, then $X \cup Y = Y \cup X$? I know it's obvious – but if someone asked you to prove it, and wanted you to say something, what would you say? The technical term for this result is "commutativity of $\cup$". Hint: use Q1(a).

6. $\boxed{\star\star}$ Let A B and C be subsets of some large set Ω. Give proofs or counterexamples to the following statements.

   (a) Is $\cup$ distributive over $\cap$? In other words, is $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ always true?

   (b) Is $\cap$ distributive over $\cup$? In other words, is $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ always true?

   (c) Do brackets matter in statements about unions and intersections? For example, is $(A \cup B) \cap C = A \cup (B \cap C)$ always true?

7. $\boxed{\star\star\star}$ Define $A = \{x \in \mathbb{R} \,|\, x^2 < 3\}$, $B = \{x \in \mathbb{Z} \,|\, x^2 < 3\}$ and $C = \{x \in \mathbb{R} \,|\, x^3 < 3\}$. For each statement below, either prove it or disprove it! You can assume all standard facts about numbers in this question.

   (a) $\frac{1}{2} \in A \cap B$.

   (b) $\frac{1}{2} \in A \cup B$.

   (c) $A \subseteq C$.

   (d) $B \subseteq C$.

   (e) $C \subseteq A \cup B$.

   (f) $(A \cap B) \cup C = (A \cup B) \cap C$

8. $\boxed{\star\star}$ Let $P(x)$ and $Q(x)$ be propositions which depend on a variable x in a set X, and let $R(x, y)$ be a proposition which depends on two variables $x \in X$ and $y \in Y$. What are the logical negations of the following statements? Try to move the $\neg$ as far into the formulae as you can.

   (a) $\forall x \in X, P(x) \wedge \neg Q(x)$

   (b) $\exists x \in X, (\neg P(x)) \wedge Q(x)$

   (c) $\forall x \in X, \exists y \in Y, R(x, y)$.

   Background (not for exam (yet)): These questions are surprisingly important. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then, as you will learn in analysis later this year (so don't worry about it too much now!), we say that f is *continuous* at $x \in \mathbb{R}$ if $\forall \epsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$. Hence to prove that a function is *not* continuous, one has to figure out what the logical negation of the above proposition is! Can you do it? For an added challenge, try proving that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ is not continuous at $x = 0$.

   Some people find the next two questions quite confusing. Trying to solve the corresponding levels in Lean might help some of you (NB link works on a computer but not on mobile).

9. $\boxed{\star\star}$ Are the following statements true or false? Proofs or counterexamples are required!

   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 2$.

   (b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 2$.

10. $\boxed{\star\star}$ Let $\varnothing$ be the empty set. Are the following propositions true or false?

   (a) $\exists x \in \varnothing, 2 + 2 = 5$

   (b) $\forall x \in \varnothing, 2 + 2 = 5$

   Hint: think about logical negations.