Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

June 2008

This paper is also taken for the relevant examination for the Associateship.

M3/4PA45

Tilings and Patterns

Date: Monday, 2nd June 2008 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) State the definition of

    (i) A $\mathbb{Z}$-module in $\mathbb{E}^n$

    (ii) The rank of a $\mathbb{Z}$-module

    (iii) A basis of a $\mathbb{Z}$-module

    (iv) A lattice.

   (b) Recall that a $\mathbb{Z}$-module can always be considered as the projection of a lattice in a higher dimensional space. Let $\mathbb{Z}^n \subset \mathbb{E}^n$ be the orthogonal lattice and Π a surjective linear mapping (projection) from $\mathbb{E}^n \mapsto \mathbb{E}^m$.

   State a theorem that gives the structure of $\Pi(\mathbb{Z}^n)$.

   (c) Let $\mathbb{Z}^3$ be the standard lattice:

$$\mathbb{Z}^3 = \{(x, y, z) \in \mathbb{E}^3 | x, y, z \in \mathbb{Z}\}$$

   and

$$E_\lambda = \{(x, y, z) \in \mathbb{E}^3 | x + \lambda y + \sqrt{\lambda} z = 0\}$$

   where $\lambda \in \mathbb{R}^+$. Let $\Pi_\lambda : \mathbb{E}^3 \to \mathbb{E}_\lambda$ be the orthogonal projection of $\mathbb{E}^3$ onto E_λ (orthogonal with respect to the standard inner product on $\mathbb{E}^3$), and $Z_\lambda = \Pi_\lambda(\mathbb{Z}^3)$.

    (i) Prove that Z_λ is a $\mathbb{Z}$-module.

    (ii) Determine the rank of Z_λ as a function of λ.

    (iii) Prove that Z_λ is not dense in E_λ if $\lambda \in \mathbb{Q}$.

    (iv) Prove that Z_λ is dense in E_λ if $\lambda \notin \mathbb{Q}$. [you may use the result stated in (b)].

2. (a) State the definition of:

    (i) A regular polygon

    (ii) A monohedral tiling

    (iii) A uniform tiling

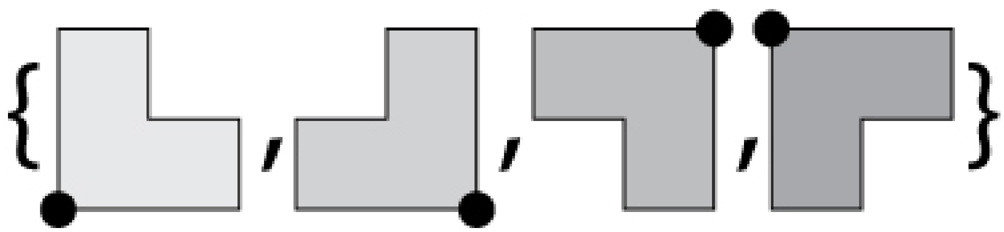
   (b) Give the equation for the interior angle (in radians) of a regular polygon.

   (c) Prove that there are exactly three monohedral tilings with regular polygons.

   (d) Recall from the course that there are exactly 11 uniform tilings with regular polygons. These all have just a single vertex configuration.

   Prove that there are an infinite number of distinct (not equivalent up to translation) tilings with regular polygons with 3 vertex configurations. [Hint: Consider uniform tilings to construct an infinite number of examples with 3 vertex configurations.]

3. Consider a protoset P of tiles of the following form:

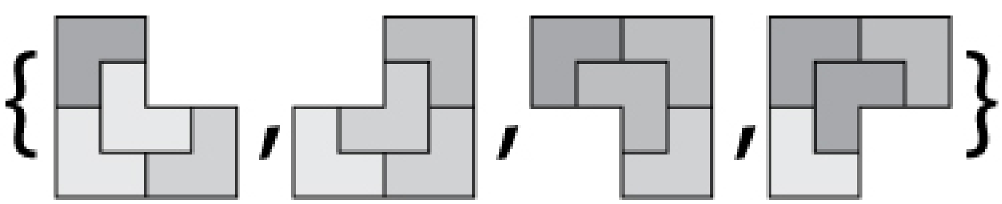


We will refer to this as the set $P = \{A, B, C, D\}$.

Two tiles are considered equivalent if they are equal up to translation.

(a) Prove that P is not an aperiodic proto-set.

We can give the following decompositions for the tiles:



Use these decompositions to define a substitution rule σ.

(b) (i) Give the scaling factor for σ.

(ii) Let the control points for the tiles be the black circles shown in the first picture. Give the sets $S_{A,A}$, $S_{A,B}$, $S_{A,C}$, $S_{A,D}$ that give the translations of tiles in the replacement rule for A in σ. Explain how this can be used to find the sets for the replacement rule of B, C and D.

(c) Prove that this substitution rule generates a tiling of the plane.

(d) Calculate the incidence matrix for σ. Use this to find the relative frequencies of tiles A, B, C, D in the tilings generated by σ. Hint: The incidence matrix has 4 as an eigenvalue.

4. Let A be an alphabet and A^* the set of finite words in A. Let $\sigma_1 = a \to ab,\ b \to ba$, $\sigma_2 = a \to ab,\ b \to abb$ and $\sigma_3 = a \to abaa,\ b \to a$ be letter substitution rules for the alphabet $A = \{a, b\}$.

(a) State the definition of a letter substitution rule, and its action on A^*.

(b) Find the incidence matrices for σ_1, σ_2 and σ_3.

(c) Prove that σ_1 is not invertible.

(d) Find the inverses of σ_2 and σ_3.

(e) The letter substitution rule σ_3 can be used to construct a geometric substitution rule σ_2^G on the line $\mathbb{E}$. Give the scaling for σ_3^G and the lengths of the tiles in the substitution tilings.

(f) State a theorem that characterises which letter substitution rules generate Canonical Projection tilings, and use this to determine which of the letter substitutions σ_1, σ_2 and σ_3 generate Canonical Projection Tilings.

5. (a) State the definition of:

(i) Canonical Projection Tiling in terms of tiling space V and window space W and the projections Π_V and Π_W.

(ii) Local isomorphism of tilings and local isomorphism class.

(b) State for which subspaces V and W the Canonical projection tilings are not periodic.

(c) Let P be a patch of a non-periodic Canonical Projection Tiling T. Consider the set of all occurrences of P in T. In particular consider the set $T_P \subset vert(T)$ the set of vertices of T followed by an occurrence of P. By the definition of Canonical Projection tiling there is a set $V_P \subset \mathbb{Z}^2$ such that $\Pi_V(V_P) = T_P$.

Prove that

$$\Pi_W(V_P) = \Omega_P \cap \Pi_W(\mathbb{Z}^2)$$

where Ω_P is an interval in W.

Hint: You may assume the Lemma that states that every vertex in a Canonical Projection staircase has a unique successor.

(d) Let T and T' be two Canonical Projection tilings with the same spaces V and W. Prove that they are locally isomorphic.