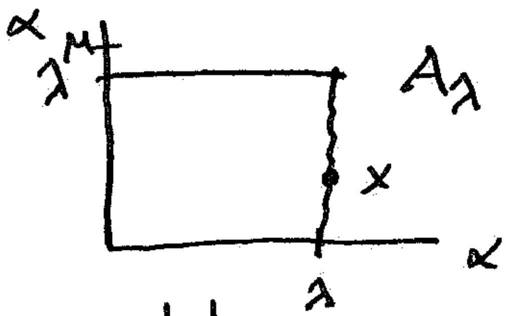
① Proving: Th 3.5.2: If α is an infinite ordinal, then $|\alpha\times\alpha| = |\alpha|$.

May assume: If $\omega \leq \beta < \alpha$ then

(1) $|\beta\times\beta| = |\beta|$ &

(2) $|\beta| < |\alpha|$ and $\beta^+ < \alpha$.

STEP 2 Find a well ordering $\leq$ on $A = \alpha\times\alpha$ such that if $x \in A$ then $|A[x]| < |\alpha|$.



For $\lambda < \alpha$ let

$$A_\lambda = \{(\theta,\zeta) \in \alpha\times\alpha : \max(\theta,\zeta) = \lambda\}$$

Define $\leq$ on A

$(\theta',\zeta') \leq (\theta,\zeta) \iff$

$\max(\theta',\zeta') < \max(\theta,\zeta)$

or:

$\max(\theta',\zeta') = \lambda = \max(\theta,\zeta)$ and either $\zeta' < \zeta$ or $\zeta' = \zeta = \lambda$ & $\theta' \leq \theta$. // (i.e. rev. lex. on A_λ).

Check: $\leq$ is a w.o. on $A = \alpha\times\alpha$.

(note: $A = \bigsqcup_{\lambda\in\alpha} A_\lambda$.)

Show: if $x = (\theta,\zeta) \in A$ then

$$|A[x]| < |\alpha|.$$

Let $\lambda = \max(\theta,\zeta) < \alpha$; may assume $\lambda \geq \omega$. Let $\mu = \lambda^+ < \alpha$.

By (2) $\mu < \alpha$ and $|\mu| < |\alpha|$.

By (1) $|\mu\times\mu| = |\mu| < |\alpha|$.

$$A[x] = \{y \in A : y < x\} \subseteq \{(\theta',\zeta') \in A : \max(\theta',\zeta') \leq \lambda\} = \mu\times\mu.$$

So $|A[x]| \leq |\mu\times\mu| < |\alpha|$.

#

(3.6) Transfinite recursion.

Allows us to construct, for ordinals α, sets $G(\alpha)$ so that $G(\alpha)$ is obtained from $G(\beta)$ for $\beta < \alpha$ by applying some operation F

$$\overbrace{G(0), G(1), \ldots, G(\beta), \ldots}^{G \restriction \alpha}$$

$$\xrightarrow{F} G(\alpha)$$

$\beta < \alpha$

$G \restriction \alpha : \alpha \to \{G(\beta) : \beta < \alpha\}$ is the function $\beta \mapsto G(\beta)$ "the restriction of G to α".

(3.6.1) Theorem (Transfinite Recursion).

Suppose F is an operation on sets. Then there is an operation G such that for all ordinals α we have

$$G(\alpha) = F(G \restriction \alpha)$$

If G' is another operation such that

$$G'(\alpha) = F(G' \restriction \alpha)$$ for all ordinals α, then $G(\alpha) = G'(\alpha)$ for all ordinals α. //. #.

Pf: Notes

Note: 1) $G(0) = F(\emptyset)$

2) In practice we do not write down F explicitly as a 1st order fmla. //.

(3.6.2) <u>Application</u> Lindenbaum Lemma (compare 1.3.7, 2.5.2) 3

Suppose $\mathcal{L}$ is a 1^{st} order language whose alphabet of symbols is <u>well-ordered</u>.

As A is well ordered we can also well order $S = \bigcup_{n \in \mathbb{N}} A^n$ (finite sequences from A): First order by length of sequence & then use a reverse lex. ordering on each A^n. Hence we obtain a w.o. on the set of closed $\mathcal{L}$-fmlas. So we can index the set of closed $\mathcal{L}$ formulas as $\{ \phi_\alpha : \alpha < \lambda \}$ for some ordinal λ (λ similar to the set with the given w.o.)

<u>Lindenbaum Lemma</u>. Suppose Σ is a consistent set of $\mathcal{L}$-fmlas. Then there is a <u>complete</u> consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas.

Construction of Σ^* is by Transfinite Recursion (using $\{\phi_\alpha : \alpha < \lambda\}$ (A))

Define for each ordinal α a set $G(\alpha) \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas.

$$G(\alpha) = \begin{cases} \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \cup \{\phi_\alpha\} & \text{if } \alpha < \lambda \text{ and } \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \vdash \phi_\alpha \\ \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \cup \{(\neg\phi_\alpha)\} & \text{if } \alpha < \lambda \text{ and } \Sigma \cup \bigcup_{\beta<\alpha} G(\beta) \nvdash \phi_\alpha \\ \Sigma \cup \bigcup_{\beta<\lambda} G(\beta) & \text{if } \alpha \geq \lambda \end{cases}$$

Note: If $\beta < \alpha$ then $G(\beta) \subseteq G(\alpha)$. Show (using transf. induction + argument as in ctble case) that every $G(\alpha)$ is consistent. Let $\Sigma^* = G(\lambda)$. By construction if ψ is closed it is ϕ_α for some $\alpha < \lambda$, then $\Sigma^* \vdash \phi_\alpha$ or $\Sigma^* \vdash (\neg\phi_\alpha)$. #.

Can use similar arguments in other parts of the Model Existence Thm (2.5.3), to get Completeness + Compactness for $\mathcal{L}$ (assuming A is well ordered).

(3.7) Axiom of Regularity / Foundation

ZF 9. $(\forall x)\Big((x \neq \phi) \to (\exists a)\big((a \in x) \wedge ((a \cap x) = \phi)\big)\Big)$

In particular there is no set b with $b \in b$.

- Consider $x = \{b\}$, by ZF 9 $b \cap \{b\} = \phi$

so $b \notin b$. //