Imperial College London

MATH97028 MATH70006

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Complex Analysis

Date: 31 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. The function $f(x)$, integrable over the interval $[-1,1]$, satisfies the singular integral equation

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{f(t)}{t-x}dt = \frac{1}{x^2+4}, \quad -1<x<1.$$

[here $\fint$ represents the principal value integral]

(a) Show that

$$f(x) = \frac{\sqrt{5}x}{2(x^2+4)\sqrt{1-x^2}} + \frac{A}{\sqrt{1-x^2}},$$

where A is an arbitrary constant. In doing so, evaluate the principal value integral using **contour integration**.

(16 marks)

(b) Find the solution of this equation which is finite at the point $x=1$. Is this solution also regular at $x=-1$?

(4 marks)

[The Hilbert inversion formula

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}}\fint_{-1}^{1}\frac{g(t)\sqrt{1-t^2}}{t-x}dt + \frac{A}{\sqrt{1-x^2}},$$

for the singular integral equation

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{f(t)}{t-x}dt = g(t), \quad -1<x<1,$$

may be quoted without proof. Here A is an arbitrary constant.]

(Total: 20 marks)

2. Consider the ordinary differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0, \quad -1 < x < 1,$$

where $n = 0, 1, 2, \cdots$, which admits orthogonal polynomial solutions $T_n(x)$.

(a) Show that these polynomials are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

(3 marks)

(b) Write down the Rodriguez' formula for $T_n(x)$. Use it to find the first three polynomials $T_0(x)$, $T_1(x)$ and $T_2(x)$ (up to normalisation).

(4 marks)

Now consider the generating function for $T_n(x)$ given by

$$G(x,y) = \frac{1-xy}{1-2xy+y^2},$$

which, when expanded in terms of small y, gives

$$G(x,y) = \sum_{n=0}^{\infty} T_n(x) y^n.$$

(c) Verify that this generating function produces $T_0(x)$, $T_1(x)$ and $T_2(x)$ as found in part (b) (up to normalisation).

(3 marks)

(d) Show that the polynomials $\{T_n(x)\}$ satisfy the three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1.$$

(6 marks)

A set of orthogonal polynomials $\{P_n(x)\}$ on $-1 < x < 1$ are defined by

$$P_n(x) = \cos(n\cos^{-1}(x)),$$

or in other words $P_n(\cos\theta) = \cos n\theta$.

(e) Show that, up to normalisation, $P_n(x) = T_n(x)$ (i.e these are the same set of orthogonal polynomials).

(4 marks)

(Total: 20 marks)

3. (a) Define what it means for a mapping $\zeta = f(z)$ from the complex z-plane to the complex ζ-plane to be **conformal** at a point $z = z_0$.

(2 marks)

The linear fractional transformation mapping the complex z-plane to the complex ζ-plane is given by

$$\zeta = \frac{az+b}{cz+d}, \tag{1}$$

where a, b, c, $d \in \mathbb{C}$ and $ad - bc \neq 0$.

(b) Explain why the condition $ad - bc \neq 0$ has been imposed on this conformal mapping?

(2 marks)

(c) Consider the case when $a = 1$, $b = 3$, $c = 1$ and $d = -i$. Determine the region of the ζ-plane that the interior of the circle $|z - (1+i)| = 1$ in the z-plane is mapped to via the transformation (1).

(8 marks)

(d) Find a conformal mapping $z = f(\zeta)$ from the upper-half ζ-plane to the unbounded region above a 'step' of height π in the z-plane as shaded in figure 1.

(8 marks)

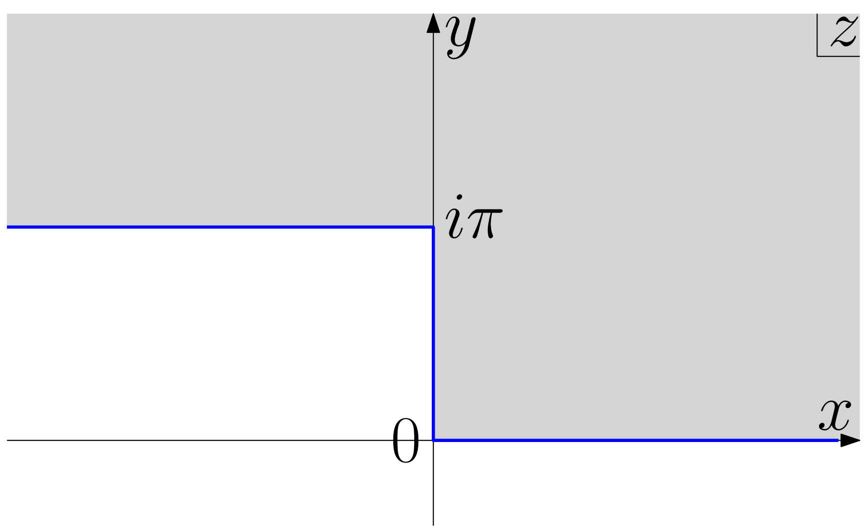


Figure 1: The region above a 'step' in the z-plane.

(Total: 20 marks)

4. The function $f(x)$ is bounded by a polynomial for $x \geq 0$, and satisfies

$$f'(x) + \frac{3}{\sqrt{7}} \int_0^\infty f(y) e^{-\sqrt{7}|x-y|} dy = 1,$$

for $x \geq 0$, with $f(0) = 0$.

(a) Using the Wiener-Hopf method, and taking the strip of analyticity to be $\{s : \alpha < \mathrm{Im}\{s\} < \beta\}$, for values α, $\beta > 0$ which you should define carefully, show that for $\mathrm{Im}\{s\} > \alpha$ the right-sided Fourier transform $F_+(s) \equiv \int_0^\infty f(x) e^{isx} dx$ is given by

$$F_+(s) = -\frac{\sqrt{7}}{3} \frac{(s + \sqrt{7}i)}{s(s+i)(s+2i)}.$$

(15 marks)

(b) Hence show that for $x \geq 0$

$$f(x) = \frac{7}{6} - \frac{\sqrt{7}}{3}(\sqrt{7} - 1)e^{-x} + \frac{\sqrt{7}}{6}(\sqrt{7} - 2)e^{-2x}.$$

(5 marks)

(Total: 20 marks)

5. The Beta function $B(z,w)$ is defined for Re$\{z\}$, Re$\{w\} > 0$ by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt.$$

(a) Show that for Re$\{z\}$, Re$\{w\} > 0$ one may write

$$B(z,w) = 2\int_0^{\pi/2} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}d\theta.$$

(2 marks)

(b) Hence, by exploiting connections between the Beta function and the Gamma function as given in lectures, find the value of

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta.$$

(6 marks)

The hypergeometric equation is given by

$$z(1-z)\frac{d^2y}{dz^2} + (c-(a+b+1)z)\frac{dy}{dz} - aby = 0,$$

for real parameters a, b and c. For $n = 0, 1, 2, \cdots$ Legendre's differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \quad -1 < x < 1. \tag{2}$$

(c) Show that the change of variable $z = \frac{1}{2}(1-x)$ transforms (2) into the hypergeometric equation with $a = -n$, $b = n+1$ and $c = 1$.

(2 marks)

For $|z| < 1$ and c not equal to zero or a negative integer, the hypergeometric series $F(a,b;c;z)$ may be represented as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n, \quad \text{where} \quad (a)_n = \prod_{k=0}^{n-1}(a+k),$$

is the Pochhammer symbol.

(d) The first three Legendre polynomials (which are solutions to Legendre's equation (2) with $n = 0, 1, 2$ respectively) are $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = 3x^2 - 1$ (up to normalisation). Verify that

$$P_n(x) = F\left(-n, n+1; 1; \frac{1}{2}(1-x)\right),$$

for $n = 0, 1, 2$. Why is this not surprising?

(5 marks)

(e) Show that

$$F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\tan^{-1}(z)}{z}.$$

(5 marks)

(Total: 20 marks)

EXAMINATION SOLUTION 21-22

Course: Applied Complex Analysis

Question: 1

Topic: SINGULAR INTEGRAL EQUATIONS

Marks & seen/unseen

Parts

(a). Applying the Hilbert inversion formula to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt = \frac{1}{x^2+4}, \quad -1<x<1,$$

we find:

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}}\underbrace{\int_{-1}^{1}\frac{\sqrt{1-t^2}}{(t^2+4)(t-x)}\,dt}_{=I(x).} + \frac{A}{\sqrt{1-x^2}}.$$

1 $\boxed{A}$ Seen

Let's evaluate $I(x)$ using contour integration.

Consider the expansion, as $z\to\infty$:

$$\frac{\sqrt{1+z^2}}{(z^2+4)(z-x)} = \frac{1}{z^2}\,\frac{\sqrt{1-\frac{1}{z^2}}}{\left(1+\frac{4}{z^2}\right)\left(1-\frac{x}{z}\right)}$$

$$= \frac{1}{z^2}\left[1-\frac{1}{2z^2}+O\left(\frac{1}{z^4}\right)\right]\left[1+O\left(\frac{1}{z}\right)\right]\left[1-O\left(\frac{1}{z^2}\right)\right]$$

$$= \frac{1}{z^2}+O\left(\frac{1}{z^3}\right) \qquad ①$$

2 $\boxed{B}$ Seen similar

Therefore, let $h(z) = \dfrac{\sqrt{z^2-1}}{(z^2+4)(z-x)}$, and consider:

$$\oint_\gamma h(z)\,dz,$$ where γ is as shown in the diagram:

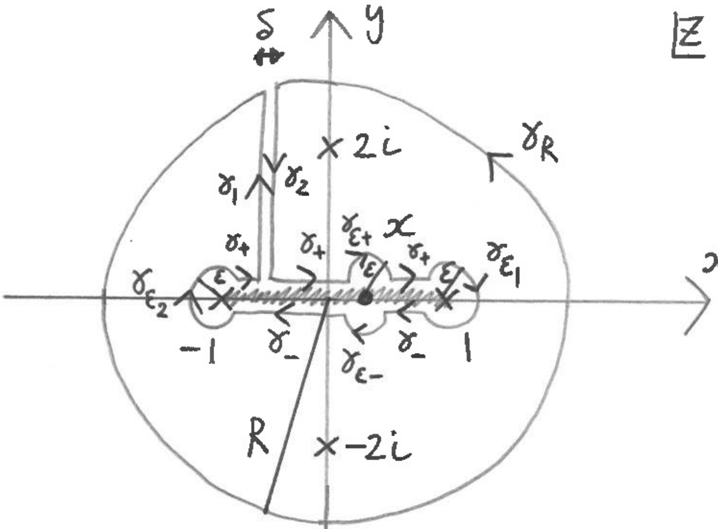
Setter's initials: SJB

Checker's initials:

	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts

(a). (continued).



We take the branch of $\sqrt{z^2-1}$ with a branch cut along $(-1,1)$ and which behaves like z as $|z| \to \infty$.

We consider the limit as $R \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$.

$h(z)$ has simple poles at $z = \pm 2i$ inside γ, with residues:

$$\operatorname{Res}\{h, 2i\} = \frac{\sqrt{5}\, i}{(2i-x)(4i)} = -\frac{\sqrt{5}}{4}\,\frac{1}{x-2i}$$

$$\operatorname{Res}\{h, -2i\} = \frac{-\sqrt{5}\, i}{(-2i-x)(-4i)} = -\frac{\sqrt{5}}{4}\,\frac{1}{x+2i}$$

By our choice of branch cut $\sqrt{z^2-1} = \sqrt{r_1 r_2}\, e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$, where.

with $-\pi \leq \theta_1, \theta_2 \leq \pi$

$\Rightarrow \sqrt{z^2-1}\,\Big|_{z=2i} = \sqrt{5}\, i$

$\sqrt{z^2-1}\,\Big|_{z=-2i} = -\sqrt{5}\, i$

1 A seen

1 A seen

1 B seen similar

2 C seen similar

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts (a). (continued).

Hence, by the residue theorem:

$$\oint_\gamma h(z)\,dz = 2\pi i\left(\frac{-\sqrt{5}}{4}\right)\left[\frac{1}{x-2i}+\frac{1}{x+2i}\right]$$
$$= -\sqrt{5}\pi i\left(\frac{x}{x^2+4}\right).$$

1 $\boxed{A}$ Seen Similar

Now evaluate integrals along the separate sections of γ.

1 $\boxed{A}$ seen

- Those along γ_1 and γ_2 cancel one another out.
- Next, for $z \in \gamma_{\varepsilon_1}$, $z = 1+\varepsilon e^{i\theta}$ and thus $|h(z)| \sim O(\sqrt{\varepsilon})$. It follows that the integral along γ_{ε_1} is zero. Similarly, one may deduce that the integral along γ_{ε_2} is also zero.

1 $\boxed{A}$ Seen

- Next, for our choice of branch we have, for $z \in \gamma_\pm$, $\sqrt{z^2-1} = \pm i\sqrt{1-x^2}$, and hence:

1 $\boxed{B}$ Seen

$$\int_{\gamma_+} h(z)\,dz = \int_{\gamma_-} h(z)\,dz = \int_{-1}^{1} \frac{i\sqrt{1-t^2}}{(t^2+4)(t-x)}\,dt,$$

taking into account that we integrate along γ_- from right to left.

1 $\boxed{A}$ Seen

- Furthermore, on $\gamma_{\varepsilon+}$ we have $z = x+\varepsilon e^{i\theta}$ where θ goes from π to 0, while on $\gamma_{\varepsilon-}$, $z = x+\varepsilon e^{i\theta}$ where θ goes from 0 to $-\pi$. Keeping in mind our choice of

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts (a). (continued).

branch, it follows that

$$\int_{\gamma_{\varepsilon+}} h(z)\,dz \to \int_{\pi}^{0} \frac{i\sqrt{1-x^2}}{(x^2+4+O(\varepsilon))\,\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\, d\theta$$

$$= \pi \frac{\sqrt{1-x^2}}{x^2+4}$$

and

$$\int_{\gamma_{\varepsilon-}} h(z)\,dz \to \int_{0}^{-\pi} \frac{-i\sqrt{1-x^2}}{(x^2+4+O(\varepsilon))\,\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\, d\theta$$

$$= -\pi \frac{\sqrt{1-x^2}}{x^2+4}.$$

1 A Seen Similar

(Note the expansion may be done here)

• Finally, for $z \in \gamma_R$, by our expansion ①:

$h(z) \sim O\left(\frac{1}{z^2}\right)$, thus the coefficient of $\frac{1}{z}$ in this expansion is zero, and so it follows that

$$\int_{\gamma_R} h(z)\,dz = 0.$$

1 A Seen similar

Thus, combining everything: $2iI(x) = -\sqrt{5}\pi i\left(\frac{x}{x^2+4}\right)$

$$\Rightarrow I(x) = -\frac{\sqrt{5}}{2}\pi\left(\frac{x}{x^2+4}\right)$$

$$\Rightarrow \boxed{f(x) = \frac{\sqrt{5}}{2}\frac{x}{(x^2+4)\sqrt{1-x^2}} + \frac{A}{\sqrt{1-x^2}}}$$

1 A Seen Similar

Total 16

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 1	Topic SINGULAR INTEGRAL EQUATIONS	Marks& seen/unseen

Parts

(b). Write $f(x) = \frac{1}{(x^2+4)\sqrt{1-x^2}} \underbrace{\left[\frac{\sqrt{5}}{2}x + A(x^2+4)\right]}$

Need to choose A so that this bracket equals zero when $x=1$. Then solution will be finite at $x=1$ (we will have removed the singularity at $x=1$)

$$\Rightarrow \frac{\sqrt{5}}{2} + 5A = 0 \Rightarrow \boxed{A = \frac{-1}{2\sqrt{5}}}$$

So the solution regular at $x=1$ is:

$$\boxed{f(x) = \frac{\sqrt{5}}{2\sqrt{1-x^2}}\left(\frac{x}{x^2+4} - \frac{1}{5}\right)}$$

3 D unseen

When $x=-1$, the curly bracket evaluates to $-\frac{2}{5} \neq 0$, and so the singularity at $x=-1$ remains.

$\Rightarrow$ NO, the solution is not also regular at $x=-1$.

1 D unseen

Total 4

Question Total 20

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(a).
$$\frac{d}{dx}\Big[\underbrace{(1-x^2)}_{\boxed{p(x)=1-x^2}}\frac{dy}{dx}\Big] + \underbrace{x}_{\boxed{q(x)=x}}\frac{dy}{dx} + n^2 y = 0$$

2 A Seen Similar

Then:
$$w(x) = \exp\left\{\int^x \frac{q(u)}{p(u)}\,du\right\} = \exp\left\{\int^x \frac{u}{1-u^2}\,du\right\}$$
$$= \exp\left\{-\frac{1}{2}\log(1-x^2)\right\} = \frac{1}{\sqrt{1-x^2}}$$

1 A Seen Similar

Total 3

(b). Rodriguez' formula:
$$\boxed{T_n(x) = \sqrt{1-x^2}\,\frac{d^n}{dx^n}\left((1-x^2)^{n-\frac{1}{2}}\right)}$$

1 A Seen Similar

$$\Rightarrow T_0(x) = \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} = 1$$
$$T_1(x) = \sqrt{1-x^2}\,\frac{1}{2}(-2x)\frac{1}{\sqrt{1-x^2}} = -x$$

1 A Seen Similar

$$T_2(x) = \sqrt{1-x^2}\,\frac{d}{dx}\left(\frac{3}{2}(-2x)(1-x^2)^{\frac{1}{2}}\right)$$
$$= \sqrt{1-x^2}\,(-3)\left[(1-x^2)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-\frac{1}{2}}\right]$$
$$= -3(1-x^2) + 3x^2$$
$$= 6x^2 - 3$$

2 A Seen Similar

Total 4

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(c).

$$G(x,y) = (1-xy)\left(1-(2xy-y^2)\right)^{-1}$$

$$= (1-xy)\left[1+(2xy-y^2)+(2xy-y^2)^2+O(y^3)\right]$$

1 B Seen Similar

$$= (1-xy)\left[1+2xy+(4x^2-1)y^2+O(y^3)\right]$$

$$= 1+xy+(2x^2-1)y^2+O(y^3)$$

$\Rightarrow T_0(x) = 1$ ✓ agrees with (b)

$T_1(x) = x$ ✓ agrees with (b). (multiply by -1)

$T_2(x) = 2x^2-1$ ✓ agrees with (b). (multiply by 3).

2 B seen similar

Total ③

(d). Differentiate $G(x,y)$ wrt y:

$$\frac{\partial G(x,y)}{\partial y} = \frac{-x}{(1-2xy+y^2)} + \frac{2(1-xy)(x-y)}{(1-2xy+y^2)^2} = \sum_{n=1}^{\infty} nT_n(x)y^{n-1}$$

1 B Seen Similar

$\Rightarrow$

$$-x+2(x-y)\left[\frac{(1-xy)}{1-2xy+y^2}\right] = (1-2xy+y^2)\sum_{n=1}^{\infty} nT_n(x)y^{n-1}$$

Now Substitute for $G(x,y)$:

$$-x+2(x-y)\sum_{n=0}^{\infty} T_n(x)y^n = (1-2xy+y^2)\sum_{n=1}^{\infty} nT_n(x)y^{n-1}$$

2 C unseen/ seen similar (unseen the $-x$ term alone)

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(d). (continued).

$$\Rightarrow -x + 2x\sum_{n=0}^{\infty} T_n(x) y^n - 2\sum_{n=0}^{\infty} T_n(x) y^{n+1}$$

$$= \sum_{n=0}^{\infty} n T_n(x) y^{n-1} - 2x \sum_{n=0}^{\infty} n T_n(x) y^n + \sum_{n=0}^{\infty} n T_n(x) y^{n+1}$$

1 B seen similar

Re-label indicies so we have y^n everywhere:

$$-x + 2x\sum_{n=0}^{\infty} T_n(x) y^n - 2\sum_{n=1}^{\infty} T_{n-1}(x) y^n$$

$$= \sum_{n=0}^{\infty} (n+1) T_{n+1}(x) y^n - 2x\sum_{n=1}^{\infty} n T_n(x) y^n + \underbrace{\sum_{n=2}^{\infty} (n-1) T_{n-1}(x) y^n}_{\text{holds for } n=1 \text{ too}}$$

1 C seen similar

For $n \geq 1$; coefficient of y^n is:

So okay for $n \geq 1$

$$2x T_n(x) - 2T_{n-1}(x) = (n+1) T_{n+1}(x) - 2nx T_n(x) + (n-1) T_{n-1}(x)$$

$$\Rightarrow (n+1) T_{n+1}(x) = 2(n+1) x T_n(x) - (n+1) T_{n-1}(x)$$

$$\Rightarrow \boxed{T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)}, \quad n \geq 1$$

1 B seen similar

Total 6

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 2	Topic ORTHOGONAL POLYNOMIALS	Marks& seen/unseen

Parts

(e). We know $\{T_n(x)\}$ are orthogonal wrt $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Let's show $\{P_n(x)\}$ are also orthogonal wrt $w(x)$.

Then $\{T_n(x)\}$ and $\{P_n(x)\}$ must be the same set of orthogonal polynomials by the property of uniqueness of orthogonal polynomials wrt $w(x)$. — 1 $\boxed{D}$ unseen

For orthogonality we require the orthogonality relation

$$\int_{-1}^{1} w(x) P_n(x) P_m(x)\, dx = 0 \quad (m \neq n).$$

— 1 $\boxed{D}$ unseen

We have:

$$\int_{-1}^{1} w(x) P_n(x) P_m(x)\, dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)}{\sqrt{1-x^2}}\, dx$$

Let $x = \cos\theta$ $\Rightarrow \frac{dx}{d\theta} = -\sin\theta$

$$= -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\, d\theta$$

$$= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta$$

$= 0$, by orthogonality of $\cos n\theta$ and $\cos m\theta$.

— 2 $\boxed{D}$ unseen

Total (4)

$\Rightarrow \{P_n(x)\} = \{T_n(x)\}$ up to normalisation.

Question Total (20)

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

Parts

(a). The mapping $\zeta = f(z)$ is called <u>conformal</u> (meaning that it preserves the angle between two different arcs) at $z = z_0$ if $f(z)$ is <u>analytic</u> at z_0 and $f'(z_0) \neq 0$.

2 $\boxed{A}$ seen

Total 2

(b). Observe we have

$$\zeta = \frac{az+b}{cz+d} = \frac{1}{c}\,\frac{az+b}{z+\frac{d}{c}}$$

$$= \frac{1}{c}\,\frac{a\left(z+\frac{d}{c}\right) - \frac{ad}{c} + b}{z+\frac{d}{c}}$$

$$= \frac{1}{c}\left[a - \frac{\frac{ad}{c} - b}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c} + \frac{bc-ad}{c}\,\frac{1}{cz+d},$$

hence if $ad - bc = 0$ then $\zeta = \frac{a}{c}$ and there is no meaningful mapping.

2 $\boxed{A}$ seen

Total 2

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

Parts

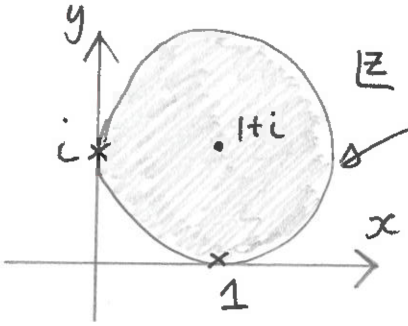
(c). $\zeta = \dfrac{z+3}{z-i} = 1 + \dfrac{(3+i)}{z-i}.$

Let $\zeta = \xi + i\eta$. Then:

$$\xi + i\eta = 1 + \frac{(3+i)}{x+i(y-1)} \times \frac{x-i(y-1)}{x-i(y-1)}$$

$$= 1 + \frac{(3x+y-1) + (x-3(y-1))i}{x^2+(y-1)^2} \quad (1)$$

2 A Seen Similar



circle equation:

$$(x-1)^2+(y-1)^2 = 1 \quad (2)$$

or: $\boxed{x^2+(y-1)^2 = 2x}$

1 A Seen Similar

Use (2) in (1):

$$\xi + i\eta = 1 + \frac{(3x+y-1)}{2x} + \frac{(x-3y+3)i}{2x}$$

1 B Seen Similar

$\Rightarrow$ Comparing Re and Im parts:

$$\xi = \frac{5}{2} + \frac{1}{2}\left(\frac{y-1}{x}\right), \quad \eta = \frac{1}{2} - \frac{3}{2}\left(\frac{y-1}{x}\right)$$

1 A Seen Similar

$\Rightarrow$ Substituting for ξ into η:

$$\eta = \frac{1}{2} - \frac{3}{2}\left(2\left(\xi - \frac{5}{2}\right)\right) = -3\xi + 8.$$

1 B Seen Similar

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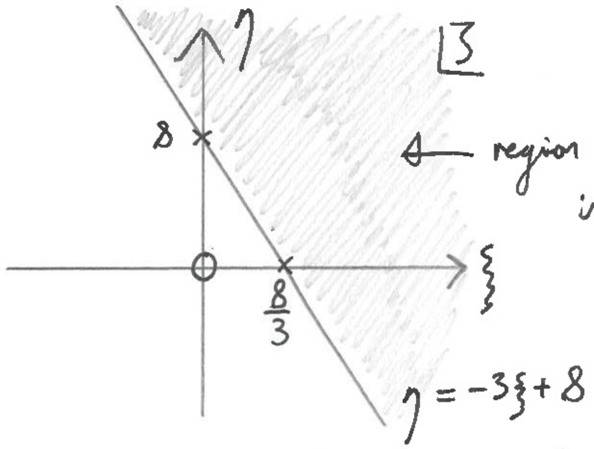
	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

Parts

(c). (continued).



We need to check which side of the line the interior of the circle maps to. When $\zeta = 0$, from $\zeta = \frac{z+3}{z-i}$ it is clear $z = -3$ which lies outside the circle, hence we want everything to the right of the line $\eta = 8 - 3\xi$ in the ζ-plane.

B 1 seen similar

B 1 seen similar

Total 8

Alternatively one can argue that the transformation $\zeta = \frac{az+b}{cz+d}$ maps circles to circles. Since $z = i$ lies on the circle then our 'circle' in the ζ-plane passes through the point at ∞; so it's a line.

Then by testing 2 more points the line can be found and testing another point determines the side of the line we want.

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EXAMINATION SOLUTION 21-22

Course: Applied Complex Analysis

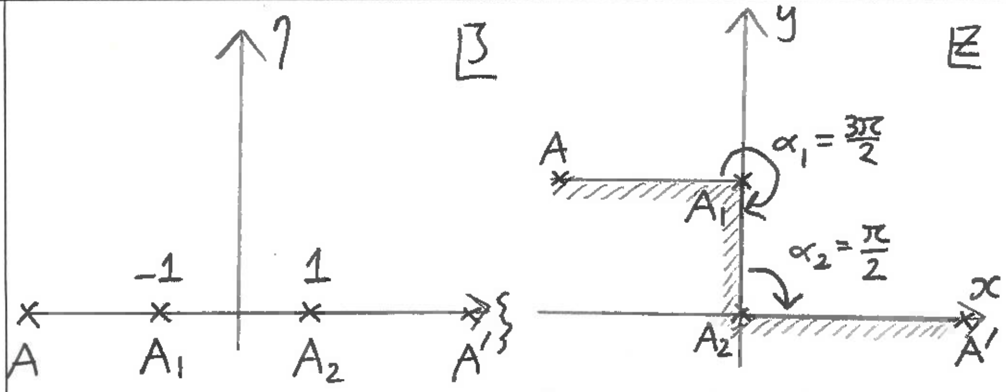
Question 3

Topic: CONFORMAL MAPPING

Marks& seen/unseen

Parts

(d).



We associate $A_1 (z = i\pi)$ with $\zeta_1 = -1$.

$A_2 (z = 0)$ with $\zeta_2 = 1$.

$A, A' (z \to \infty)$ with $\zeta \to \infty$.

2 Seen Similar $\boxed{C}$

We can apply the Schwarz-Christoffel mapping:

$$z = c_1 \int (\zeta - \zeta_1)^{\frac{\alpha_1}{\pi} - 1} (\zeta - \zeta_2)^{\frac{\alpha_2}{\pi} - 1} \, d\zeta + c_2 , \quad c_1, c_2 \in \mathbb{C} .$$

giving:
$$z = c_1 \int (\zeta + 1)^{\frac{1}{2}} (\zeta - 1)^{-\frac{1}{2}} \, d\zeta + c_2$$

$$= c_1 \int \sqrt{\frac{\zeta + 1}{\zeta - 1}} \, d\zeta + c_2$$

2 Seen Similar $\boxed{C}$

Now:
$$\int \sqrt{\frac{\zeta+1}{\zeta-1}} \times \sqrt{\frac{\zeta+1}{\zeta+1}} \, d\zeta = \int \frac{\zeta + 1}{\sqrt{\zeta^2 - 1}} \, d\zeta$$

$$= \int \frac{\zeta}{\sqrt{\zeta^2-1}} \, d\zeta + \int \frac{1}{\sqrt{\zeta^2 - 1}} \, d\zeta$$

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Checker's initials:

	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 3	Topic CONFORMAL MAPPING	Marks& seen/unseen

Parts

(d). (continued).

$$= \frac{1}{2}(2)(\zeta^2-1)^{\frac{1}{2}} + \cosh^{-1}\zeta$$

$$= \sqrt{\zeta^2-1} + \cosh^{-1}\zeta.$$

D 2 unseen

Hence: $z = c_1\left(\sqrt{\zeta^2-1} + \cosh^{-1}\zeta\right) + c_2$.

when $\zeta = 1 \rightarrow z = 0 \Rightarrow \boxed{c_2 = 0}$

$\zeta = -1 \rightarrow z = i\pi \Rightarrow i\pi = c_1 \underbrace{\cosh^{-1}(-1)}_{= i\pi}$

$\Rightarrow \boxed{c_1 = 1}$

D 2 unseen

So; $\boxed{z = \sqrt{\zeta^2-1} + \cosh^{-1}\zeta}$ is the mapping we want.

Total 8

Question Total 20

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	EXAMINATION SOLUTION 21-22	Course: Applied Complex Analysis
Question 4	Topic: WIENER-HOPF METHOD	Marks & seen/unseen

Parts

(a). Let $k(x) = \frac{3}{\sqrt{7}} e^{-\sqrt{7}|x|}$ and $p(x) = 1$.

Introduce:

$$f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$p_+(x) = \begin{cases} p(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and

$$g_-(x) = \begin{cases} 0, & x \geq 0 \\ \int_0^\infty f(y) k(x-y)\, dy, & x < 0 \end{cases}$$

1 [A] seen

Then, we have:

$$\frac{3}{\sqrt{7}} \int_0^\infty f(y) e^{-\sqrt{7}|x-y|} dy = p_+(x) - f_+'(x) + g_-(x), \quad (1)$$

for $-\infty < x < \infty$. Now, denoting by $F_+(s)$ and $F_+^{(1)}(s)$ the right-sided Fourier transforms of $f_+(x)$ and $f_+'(x)$ respectively, from lectures we know, assuming $\text{Im}\{s\} > 0$:

$$F_+^{(1)}(s) = -f(0) - isF_+(s),$$

where using the condition that $f(0) = 0$ gives:

$F_+^{(1)}(s) = -isF_+(s)$. So, taking the Fourier transform of both sides of (1) gives:

$$\hat{K}(s) F_+(s) = P_+(s) + isF_+(s) + G_-(s),$$

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EXAMINATION SOLUTION 21-22

Course: Applied Complex Analysis

Question 4

Topic: WIENER-HOPF METHOD

Marks & seen/unseen

Parts (a) (continued).

or, $\quad K(s)F_+(s) + G_-(s) = -P_+(s), \quad (2)$

where $K(s) = is - \hat{K}(s)$ and $P_+(s)$ denotes the right-sided Fourier transform of $p_+(x)$, $G_-(s)$ the left-sided Fourier transform of $g_-(x)$ and $\hat{K}(s)$ the ordinary Fourier transform of $k(x)$. One can show that: — 1 A seen

$$P_+(s) = \int_0^\infty e^{isx}\,dx = \left[\frac{e^{isx}}{is}\right]_0^\infty = \frac{-1}{is} = \frac{i}{s}, \quad \text{provided } \operatorname{Im}\{s\} > 0.$$ — 1 A seen similar

$$\hat{K}(s) = \frac{3}{\sqrt{7}}\int_{-\infty}^{\infty} e^{-\sqrt{7}|x|} e^{isx}\,dx = \frac{3}{\sqrt{7}}\left(\int_{-\infty}^{0} e^{(is+\sqrt{7})x}\,dx + \int_0^\infty e^{(is-\sqrt{7})x}\,dx\right)$$

$$= \frac{3}{\sqrt{7}}\left(\frac{1}{is+\sqrt{7}} - \frac{1}{is-\sqrt{7}}\right) = \frac{6}{s^2+7}, \quad \text{provided } -\sqrt{7} < \operatorname{Im}\{s\} < \sqrt{7}$$ — 1 A seen similar

It follows that

$$K(s) = is - \frac{6}{s^2+7} = \frac{is(s^2+7) - 6}{s^2+7} = \frac{i(s+i)(s+2i)(s-3i)}{(s+\sqrt{7}i)(s-\sqrt{7}i)}.$$ — 1 A seen similar

- We require, from lectures: $|f_+(x)| < Ae^{(\sqrt{7}-\delta)x}$ as $x \to \infty$, for some $\delta > 0$. (A constant)
  $\Rightarrow F_+(s)$ analytic in $\{s : \operatorname{Im}\{s\} > \sqrt{7} - \delta\}$
- Similarly, for $G_-(s)$, from lectures one can show that $g_-(x) = Be^{\sqrt{7}x}$ (B constant) $\Rightarrow G_-(s)$ analytic in: $\{s : \operatorname{Im}\{s\} < \sqrt{7}\}$. — 2 A seen similar

Since $f(x)$ is bounded by a polynomial as $x \to \infty$, we take the $\oplus$ and $\ominus$ regions to be:

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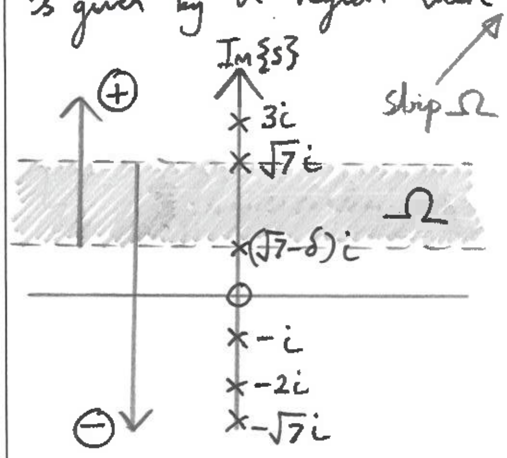
	EXAMINATION SOLUTION 21-22	Course: Applied Complex Analysis
Question 4	Topic: WIENER-HOPF METHOD	Marks& seen/unseen

Parts (a). (continued).

$\oplus = \{s : \operatorname{Im}\{s\} > \sqrt{7}-\delta\}$

$\ominus = \{s : \operatorname{Im}\{s\} < \sqrt{7}\}$, where $0 < \delta < \sqrt{7}$.

Hence the strip of analyticity where $\oplus$ and $\ominus$ overlap is given by the region where $\underline{\underline{\underbrace{\sqrt{7}-\delta}_{\alpha} < \operatorname{Im}\{s\} < \underbrace{\sqrt{7}}_{\beta}}}$



} $K(s)$ analytic provided $-\sqrt{7} < \operatorname{Im}\{s\} < \sqrt{7}$ ✓

non-zero provided $s \neq -i, -2i, 3i$ ✓

$P_+(s)$ analytic provided $\operatorname{Im}\{s\} > 0$ ✓

1 C Seen Similar

We decompose $K(s)$ as $K(s) = K_+(s)K_-(s)$, where:

$$K_+(s) = \frac{(s+i)(s+2i)}{s+\sqrt{7}i}, \quad K_-(s) = \frac{i(s-3i)}{s-\sqrt{7}i},$$

(Noting $2 < \sqrt{7} < 3$)

then ② gives:

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = R(s),$$

where $R(s) = \dfrac{-P_+(s)}{K_-(s)} = -\left(\dfrac{(s-\sqrt{7}i)}{s(s-3i)}\right)$.

2 D Seen Similar

Now observe that we can write $R(s) = R_+(s) + R_-(s)$, where:

$$R_+(s) = -\frac{\sqrt{7}}{3s}, \quad R_-(s) = \frac{\sqrt{7}-3}{3(s-3i)}.$$

2 D Seen Similar

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 4	Topic WIENER–HOPF METHOD	Marks& seen/unseen

Parts (a). (continued).

Now we have: $K_+(s)F_+(s) + \dfrac{G_-(s)}{K_-(s)} = R_+(s) + R_-(s)$,

or: $\underbrace{K_+(s)F_+(s) - R_+(s)}_{\text{analytic in } \oplus} = \underbrace{R_-(s) - \dfrac{G_-(s)}{K_-(s)}}_{\text{analytic in } \ominus}$, $s \in \Omega$

Since $\oplus, \ominus$ overlap in Ω, then:

$$E(s) = \begin{cases} F_+(s)K_+(s) - R_+(s), & s \in \oplus \\ R_-(s) - \dfrac{G_-(s)}{K_-(s)}, & s \in \ominus \end{cases}$$

is entire. Consider now $s \to \infty$ in $\oplus$:

$$K_+(s)F_+(s) - R_+(s) \sim s\underbrace{\left[\frac{if(0)}{s} + O\left(\frac{1}{s^2}\right)\right]}_{\text{from lectures, but } f(0)=0 \text{ so this } \sim O\left(\frac{1}{s^2}\right)} + \frac{\sqrt{7}}{3}\frac{1}{s}$$

$$\sim O\left(\frac{1}{s}\right)$$

$$\to 0 \text{ as } s \to \infty.$$

Hence, by Liouville's theorem, $E(s) \equiv 0$ for all s.

Therefore: $F_+(s)K_+(s) - R_+(s) = 0$

$$\Rightarrow \boxed{F_+(s) = -\frac{\sqrt{7}}{3}\frac{(s+\sqrt{7}i)}{s(s+i)(s+2i)}}$$

B 1 seen

C 2 seen similar

Total 15

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EXAMINATION SOLUTION 21-22

Course: Applied Complex Analysis

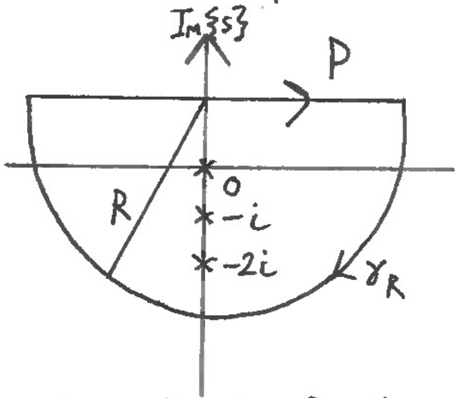
Question 4 | Topic: WIENER-HOPF METHOD | Marks & seen/unseen

Parts

(b) To retrieve $f_+(x)$, apply the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}\,ds,$$ where P is a horizontal line in the region $\oplus$.

1 B seen



• For $x<0$, from lectures we have $f_+(x)=0$ as expected.

• For $x>0$, we close P with a semi-circle γ_R below P of radius R, and take $R\to\infty$.

1 B seen similar

Let $\gamma = P+\gamma_R$. By the residue theorem:

$$\oint_\gamma -\frac{\sqrt{7}}{3}\,\frac{(s+\sqrt{7}i)e^{-isx}}{s(s+i)(s+2i)}\,ds = -2\pi i\left[\sum \text{Residues at } 0,-i,-2i\right]$$

$$= -2\pi i\left[-\frac{\sqrt{7}}{3}\left(\frac{\sqrt{7}i}{-2} + (\sqrt{7}-1)ie^{-x} + \frac{(\sqrt{7}-2)ie^{-2x}}{-2}\right)\right]$$

$$= \frac{7\pi}{3} - \frac{2\sqrt{7}\pi}{3}(\sqrt{7}-1)e^{-x} + \frac{\sqrt{7}\pi}{3}(\sqrt{7}-2)e^{-2x}.$$

2 B seen similar

Now in the limit as $R\to\infty$, $\int_{\gamma_R}\to 0$ (from lectures), hence:

$$f_+(x) = \boxed{f(x) = \frac{7}{6} - \frac{\sqrt{7}}{3}(\sqrt{7}-1)e^{-x} + \frac{\sqrt{7}}{6}(\sqrt{7}-2)e^{-2x}}$$

1 B seen similar

Total 5

Question Total 20

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	EXAMINATION SOLUTION 21-22	Course: Applied Complex Analysis
Question 5	Topic: HYPERGEOMETRIC SERIES (MASTERY)	Marks & seen/unseen

Parts

(a). $B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt$.

Let $t = \sin^2\theta$. Then $1-t = \cos^2\theta$ and $dt = 2\sin\theta\cos\theta\,d\theta$. Thus:

$t=1 \to \theta = \frac{\pi}{2}$
$t=0 \to \theta = 0$

$$B(z,w) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2z-2}(\cos\theta)^{2w-2}\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}\,d\theta.$$

2 seen

Total 2

(b). $$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}}(\cos\theta)^{-\frac{1}{2}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2(\frac{3}{4})-1}(\cos\theta)^{2(\frac{1}{4})-1}\,d\theta$$

$$= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$$, using the result from (a).

2 unseen

$$= \frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}$$, using the fact that

$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$, from lectures. Here $\Gamma(z)$ is the Gamma function.

Now $\Gamma(1) = 1$ (either calculate or from lectures), so:

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right).$$

2 unseen

Setter's initials: SJB

Checker's initials:

	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 5	Topic HYPERGEOMETRIC SERIES (MASTERY)	Marks& seen/unseen

Parts

(b). (continued). Now use the fact from lectures that:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \text{ to give:}$$

$$\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{4}) = \Gamma(\tfrac{3}{4})\Gamma(1-\tfrac{3}{4}) = \frac{\pi}{\sin(\frac{3\pi}{4})} = \frac{\pi}{\left(\frac{\sqrt{2}}{2}\right)}$$

2 unseen

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{1}{2}\cdot\frac{2\pi}{\sqrt{2}} = \boxed{\frac{\pi}{\sqrt{2}}}.$$

Total 6

(c). Put $z = \frac{1}{2}(1-x)$, then $x = 1-2z$. We have:

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\frac{1}{2}\frac{dy}{dz}, \quad \frac{d^2y}{dx^2} = \frac{1}{4}\frac{d^2y}{dz^2}, \text{ and:}$$

$$1-x^2 = 1-(1-2z)^2 = 4z(1-z).$$

Hence Legendre's equation (2) becomes:

$$4z(1-z)\frac{1}{4}\frac{d^2y}{dz^2} - 2(1-2z)\left(-\frac{1}{2}\right)\frac{dy}{dz} + n(n+1)y = 0$$

$$\Rightarrow z(1-z)\frac{d^2y}{dz^2} + (1-2z)\frac{dy}{dz} + n(n+1)y = 0,$$

2 seen similar

which is the hypergeometric equation:

$$z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0 \text{ with: } \begin{matrix} c=1 \\ a=-n \\ b=n+1 \end{matrix}$$

Total 2

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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 5	Topic HYPERGEOMETRIC SERIES (MASTERY)	Marks& seen/unseen
Parts (d).	$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=3x^2-1.$	
	For $n=0$: $$F(0,1;1;\tfrac{1}{2}(1-x)) = \frac{(0)_0(1)_0}{0!\,(1)_0}\left(\tfrac{1}{2}(1-x)\right)^0 = \underline{\underline{1}} \checkmark = P_0(x)$$	1 Seen Similar
	For $n=1$: $$F(-1,2;1;\tfrac{1}{2}(1-x)) = \frac{(-1)_1(2)_1}{1!\,(1)_1}\left(\tfrac{1}{2}(1-x)\right)^1 + \frac{(-1)_0(2)_0}{0!\,(1)_0}\left(\tfrac{1}{2}(1-x)\right)^0 = -(1-x)+1 = \underline{\underline{x}} \checkmark = P_1(x)$$	1 seen Similar
	For $n=2$: $$F(-2,3;1;\tfrac{1}{2}(1-x)) = \frac{(-2)_2(3)_2}{2!\,(1)_2}\left(\tfrac{1}{2}(1-x)\right)^2 + \frac{(-2)_1(3)_1}{1!\,(1)_1}\left(\tfrac{1}{2}(1-x)\right)^1 + \frac{(-2)_0(3)_0}{0!\,(1)_0}$$ $$= \frac{3}{2}(1-x)^2 - 3(1-x) + 1$$ $$= \frac{3}{2} - 3x + \frac{3}{2}x^2 - 3 + 3x + 1$$ $$= \underline{\underline{\frac{1}{2}(3x^2-1)}} \checkmark = P_2(x) \text{ up to normalisation.}$$	2 Seen Similar
	• This is not surprising since we know $F(a,b;c;z)$ is a solution of the hypergeometric equation; so it is very plausible it equals the polynomials which are also solutions.	1 unseen Total 5
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	EXAMINATION SOLUTION 21-22	Course Applied Complex Analysis
Question 5	Topic HYPERGEOMETRIC SERIES (MASTERY)	Marks& seen/unseen

Parts

(e).

$$F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n \cancel{(1)_n}}{(\frac{3}{2})_n} \cdot \frac{(-1)^n z^{2n}}{\cancel{n!}}, \text{ for } |z|<1$$

1 seen

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{3}{2})} \cdot (-1)^n z^{2n},$$

where we have used the fact that $(1)_n = n!$ and that $(a)_n = \dfrac{\Gamma(n+a)}{\Gamma(a)}$ from lectures.

1 unseen

$$= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{n+\frac{1}{2}} \cdot (-1)^n z^{2n},$$

where we have used the fact that $\dfrac{\Gamma(z+1)}{\Gamma(z)} = z$, from lectures.

1 unseen

$$= \frac{1}{z} \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}}_{= \tan^{-1} z}$$

$$= \frac{1}{z} \tan^{-1} z.$$

1 unseen

~~Since~~ We can use analytic continuation to deduce that this result holds for all z. (not just $|z|<1$).

1 seen similar

Total 5

Question Total 20

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Applied Complex Analysis_MATH60006 MATH97028 MATH70006	1	On the whole this was found to be the easiest question on the paper. In part (a) there were many very good solutions, only losing a few marks due to a wrong sign in the residue at -2i or lacking some details regarding the branch cut, semicircular contours or just making small mistakes. The expansion of the integrand as z ---> infinity was on the whole tackled well although some students had difficulty in this and it tended to throw them off for the rest of part (a). Part (b) was unseen and students tended to either tackle this well and score all 4 or make little to no progress.
Applied Complex Analysis_MATH60006 MATH97028 MATH70006	2	This was found to be the second easiest question on the paper. Parts (a)-(c) were generally done very well and very few marks were lost here. Part (d) was similar to a problem on coursework 2 but trickier and although most students started with a correct method, very few made it to the end. Part (e) was unseen but many students came up with good ideas although didn’t always follow them through to tie up everything nicely. Showing the same recurrence relation was satisfied for instance also requires that the first 2 polynomials are the same (which was often missing). Some students showed the first 2 polynomials for P and T were equal but didn’t suggest anything to generalise this for all polynomials

Applied Complex Analysis_MATH60006 MATH97028 MATH70006	3	This was found to be the hardest question on the paper. Parts (a) and (b) were almost always completed correctly. Part (c) caused much more difficulty than it should have. Very few students managed to finish this problem to completion, often getting bogged down in algebra if they used an algebraic method or if they broke the mapping into several maps had difficulty with working out the equation of the line in the zeta-region. Those who quoted the result that circles map to circles and spotted the critical point that z = i maps to infinity often made quick progress by realising then the result was a line and could easily find its equation by finding the images of just two points. Many students who did make good progress also then forgot to check which side of the line the interior of the circle mapped to. Part (d) was challenging. Most students realised that they needed to use Schwarz-Christoffel
Applied Complex Analysis_MATH60006 MATH97028 MATH70006	4	This was the 3rd best question on the paper in terms of scoring. Part (a) was tough and as well as being long required a lot of algebra. Many students made a small mistake and often ended up with a difficult cubic for their zeros of K(s). Those that managed to avoid making mistakes often did exceptionally well, decomposing K(s) correctly and even going on to correctly decompose R(s) using partial fractions. These were challenging decompositions.
Applied Complex Analysis_MATH60006 MATH97028 MATH70006	5	Parts (a)-(c) of this question were done well with many students scoring highly. Part (d) was done poorly on the whole, with few students correctly evaluating the hypergeometric series expressions, perhaps failing to notice almost all of the terms become 0 which simplifies the calculations. Part (e) was tough and while some students made good progress and related the series given to the Taylor series, others made little progress.