Course: M3S1

Question: 1

Marks & seen/unseen

Parts

(i) Total efficient score $U_{\cdot}(\theta) = \sum_j U_j(\theta) = \sum_j \frac{\partial}{\partial\theta} \ln f_{X_j|\theta}(X_j|\theta)$ — bookwork 1.

Total Fisher information

$$I_{\cdot}(\theta) = \sum_j I_j(\theta) = \sum_j E_{X_j|\theta}\left\{-\frac{\partial U_j(\theta)}{\partial\theta}\right\} \text{ or } \sum_j E_{X_j|\theta}\{U_j(\theta)^2\}$$ 1.

rest unseen

(ii)(a) $\ln f(x|\theta) = \ln c(\theta) + \theta \ln x + (1-\theta)\ln(1-x)$

$\frac{d}{d\theta} \ln f(x|\theta) = \xi(\theta) - \ln\left(\frac{1-x}{x}\right)$

so $U(\theta) = \xi(\theta) - \ln\left(\frac{1-X}{X}\right)$

$$U_{\cdot}(\theta) = n\Big\{\xi(\theta) - \underbrace{\frac{1}{n}\sum_j \ln\left(\frac{1-X_j}{X_j}\right)}_{=\hat{\xi}}\Big\} = -n(\hat{\xi} - \xi) \quad \text{—}(*)$$

From $(*)$, $\hat{\xi}$ is unbiased for $\xi(\theta)$ (since $E\{U_{\cdot}(\theta)\} = 0$) 6.

$\frac{d^2}{d\theta^2} \ln f(x|\theta) = \frac{d\xi}{d\theta} = \xi'$ say

$I(\theta) = E\left(-\frac{d^2}{d\theta^2} \ln f_{X|\theta}(X|\theta)\right) = -\xi'$ so $I_{\cdot}(\theta) = -n\xi'$ —$(**)$ 5.

(b) $U_{\cdot}(\xi) = \frac{1}{\xi'} U_{\cdot}(\theta)$ & $I_{\cdot}(\xi) = \frac{1}{\xi'^2} I_{\cdot}(\theta) = -\frac{n}{\xi'}$ from $(**)$

$\text{var}(\hat{\xi}) = \frac{1}{I_{\cdot}(\xi)} = -\frac{\xi'}{n}$ 5.

[Note: From $(*)$ $I_{\cdot}(\theta) = E\{U_{\cdot}(\theta)^2\} = n^2 \text{var}(\hat{\xi})$]

(c) $U_{\cdot}(\theta) = n\{\ln c(\theta) - \hat{\xi}(\underset{\sim}{X})\}$

so $U_{\cdot}(c) = \frac{n}{\frac{dc(\theta)}{d\theta}}\{\ln c - \hat{\xi}(\underset{\sim}{X})\}$

which cannot be written after non-linear transformation in the form $\kappa(c)\{c - \hat{c}(\underset{\sim}{X})\}$ necessary for $\text{var}(\hat{c})$ to be the CRLB 2.

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Question 2		Marks & seen/unseen

Parts

(i)(a) • A statistic $t = t(\underset{\sim}{x})$ is sufficient for θ if $f_{Z|T,\theta}(z|t,\theta)$ does not depend on θ for any statistic $z = z(\underset{\sim}{x})$. — bookwork; $1\frac{1}{2}$

• A family of distributions $\{f_{T|\theta}\}$ of a statistic t is complete iff the only unbiased estimator of 0 that is a function of t is the statistic that is 0 with probability 1. — $1\frac{1}{2}$

• If $t = t(\underset{\sim}{x})$ is sufficient and its distribution is complete, it is a complete sufficient statistic.

(b) For a complete sufficient statistic $t = t(\underset{\sim}{x})$, any function of $t(\underset{\sim}{X})$ is a minimum variance unbiased estimator (MVUE) of its expectation. — 1.

(ii)(a) The range (support) of $f_{X|\theta}$ depends on θ, so $f_{X|\theta}$ cannot be an Exponential Family. — rest unseen; 2.

(b)
$$
\begin{aligned}
f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta) &= \prod_1^n e^{(\theta - x_i)} . H(x_{min} > \theta) \\
&= e^{-\Sigma x_i} e^{n\theta} H(x_{min} > \theta) \\
&= h(\underset{\sim}{x}) g(x_{min}, \theta)
\end{aligned}
$$

so $t = x_{min}$ is sufficient for θ by Neyman Factorisation. — 3.

(c)
$$
\begin{aligned}
P_T(X_{min} > t|\theta) = P(\text{each } X_i > t) &= \prod_i e^{-(t-\theta)} \\
&= e^{-n(t-\theta)} \qquad (t > \theta)
\end{aligned}
$$

$$
f_{T|\theta}(t|\theta) = ne^{-n(t-\theta)} \qquad (t > \theta)
$$
— 3.

(d) To show that $E\{h(T)\} = 0 \Rightarrow h(t) \equiv 0$ w.p.1:

$$
\int_\theta^\infty h(t) n e^{n(\theta - t)} dt = e^{n\theta} \int_\theta^\infty h(t) n e^{-nt} dt
$$

$ne^{n\theta} > 0$ so to show $\int_\theta^\infty h(t) e^{-nt} dt = 0 \Rightarrow h(t) \equiv 0$

Differentiate wrt θ, $-h(\theta) e^{-n\theta} = 0$ whatever θ $\Rightarrow h(\theta) = 0$ since $-e^{-n\theta} < 0$ i.e. $\neq 0$ — 3.

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Question 2 ctd | Marks & seen/unseen

Parts

(ii)(e) Let $Z = n(T-\theta)$, $z = n(t-\theta)$ — unseen

$$P(Z > n(t-\theta)) = e^{-n(t-\theta)}$$

i.e. $P(Z > z) = e^{-z}$

so Z is Exponential (1) with $E(Z)=1$, $var(Z)=1$

Then $E\{n(T-\theta)\}=1$ so $E(T) = \frac{1}{n}+\theta$

So $W = T - \frac{1}{n} = X_{min} - \frac{1}{n}$ is unbiased for θ,

and is a function only of the sufficient statistic

so $X_{min} - \frac{1}{n}$ is UMVU for θ (by Lehmann-Scheffé) — 4.

(f) $var\{n(T-\theta)\} = 1$ so $n^2 var(T) = 1$

so $var(T) = \frac{1}{n^2}$ — 1.

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Course: M3S1

Question: 3

Marks & seen/unseen

Parts

(i) A test of size α is <u>unbiased</u> if its power function $\beta(\theta) = P(\underset{\sim}{X} \in R \mid \theta)$, where R is the rejection set, satisfies

$$\beta(\theta) \begin{cases} \leq \alpha & \theta \in \Theta_0 \quad (\text{size } \alpha) \\ \geq \alpha & \theta \in \overline{\Theta}_0 . \end{cases}$$

bookwork — 2.

(ii)(a) Consider $H_0: \theta = \theta_0$ v. $H_1: \theta = \theta_1 < \theta_0 \quad (\theta_1 > 0)$

~~test unseen~~ similar seen in exercise

These are 2 simple hypotheses so the most powerful test by the Neyman-Pearson Lemma is the likelihood ratio test, to reject H_0 if the likelihood ratio $\lambda(\underset{\sim}{x}) > k$ for some k to be determined. — 1.

Here

$$\lambda(\underset{\sim}{x}) = \frac{\left(\frac{1}{\sqrt{2\pi}\,\theta_1}\right)^n e^{-\frac{1}{2}\frac{1}{\theta_1^2}\sum x_i^2}}{\left(\frac{1}{\sqrt{2\pi}\,\theta_0}\right)^n e^{-\frac{1}{2}\frac{1}{\theta_0^2}\sum x_i^2}} = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-\frac{1}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)t}$$

where $t = \sum x_i^2$ is sufficient for θ.

unseen

We reject H_0 if $t < c_\alpha$ since $\theta_1 < \theta_0$ so $\frac{1}{\theta_1^2} > \frac{1}{\theta_0^2}$ — 3.

where c_α is s.t.

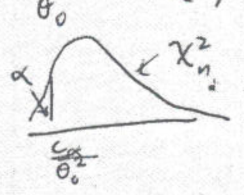
$$\alpha = P(T < c_\alpha \mid \theta = \theta_0) = P\left(Z = \frac{T}{\theta_0^2} < \frac{c_\alpha}{\theta_0^2} \,\middle|\, \theta = \theta_0\right)$$

$$= F_Z\left(\frac{c_\alpha}{\theta_0^2}\right) \quad \text{so} \quad c_\alpha = \theta_0^2 F_Z^{-1}(\alpha) \quad \text{i.e. on } \{\underset{\sim}{x} \mid t(\underset{\sim}{x}) < c_\alpha\}$$ — 3.

Since this test does not depend on the value of θ_1 in $(0, \theta_0)$, it is uniformly most powerful (UMP) for all θ in $(0, \theta_0)$ i.e. for $H_1: \theta < \theta_0$. — 2.

Under H_0,

$\frac{X_i}{\theta_0}$ is $N(0,1)$ so $Z = \sum_i \frac{X_i^2}{\theta_0^2}$ is χ^2_n — 2.



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Course: M3S1

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Parts

(ii)(b) For θ, $\frac{X_i}{\theta}$ is $N(0,1)$ & $Z = \sum_i \frac{X_i^2}{\theta^2}$ is χ^2_n

The power function

$$\beta(\theta) = P(T < c_\alpha \mid \theta)$$

$$= P\left(Z < \frac{c_\alpha}{\theta^2} \,\Big|\, Z \text{ is } \chi^2_n\right)$$

$$= F_Z\left(\frac{c_\alpha}{\theta^2}\right) \quad \text{where } c_\alpha = \theta_0^2 F_Z^{-1}(\alpha)$$

4.

(c) $\beta(\theta) \downarrow$ as $\theta \uparrow$

$\beta(\theta_0) = \alpha$

$$\beta(\theta) \begin{cases} > \alpha & (\theta < \theta_0) \\ = \alpha & (\theta = \theta_0) \\ < \alpha & (\theta > \theta_0) \end{cases}$$

The test is thus biased for testing
$H_0: \theta = \theta_0$ v. $H_1: \theta \neq \theta_0$

3.

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Question 4		Marks & seen/unseen

Parts

(i)(a) A $100(1-\alpha)\%$ confidence set $\Psi(\underset{\sim}{x})$ for θ is a random set (an interval or union of intervals for one-dimensional parameter θ) that contains the true, fixed unknown θ with probability $1-\alpha$. — bookwork — 2.

(b) Let $\Psi(\underset{\sim}{x})$ be the values in Θ containing those for which an acceptance set is $\underset{\sim}{x} \in \bar{R}(\theta)$, then

$$P(\theta \in \Psi(\underset{\sim}{X}) \mid \theta) = P(\underset{\sim}{X} \in \bar{R}(\theta) \mid \theta) = 1-\alpha.$$

2.

(c) A confidence set is <u>best</u> if the probability of its containing a value of θ other than the true one is a small as possible. — 1.

unseen

(ii)(a)

$$\pi(\theta|\underset{\sim}{x}) \propto f(\underset{\sim}{x}|\theta)\pi(\theta) = \theta^n \left(\prod_i x_i\right)^{\theta-1} \cdot \lambda e^{-\lambda\theta}$$

$$\propto \theta^n z^\theta e^{-\lambda\theta} \quad \text{where } z = \prod_i x_i$$

$$= \theta^n e^{\theta \ln z - \lambda\theta} = \theta^n e^{-\theta t}$$

$$\text{where } t = \lambda - \ln z = \lambda - \sum_i \ln x_i$$

6.

$$\propto \frac{t^{n+1}}{\Gamma(n+1)} \theta^{(n+1)-1} e^{-\theta t} \quad \text{i.e. Gamma}(n+1,t)$$

2.

(b) The posterior mean is the expectation of Gamma$(n+1,t)$

so $$E(\Theta|t) = \frac{n+1}{t} = \frac{n+1}{\lambda - \sum \ln x_i}$$

3.

(c) $$\ln \pi(\theta|\underset{\sim}{x}) = \text{const.} + n\ln\theta - \theta t$$

$$\frac{d}{d\theta} \ln \pi(\theta|\underset{\sim}{x}) = \frac{n}{\theta} - t$$

so $\pi(\theta|\underset{\sim}{x})$ is a maximum when $\theta = \frac{n}{t}$

i.e. the posterior mode is at $\theta = \frac{n}{t}$

4.

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