Exercise 9.1. Consider the metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$. Show that all of the intervals $(a, b]$, $[a, b)$, $[a, +\infty)$, and $(-\infty, b]$ are not compact.

Hint: For each of those intervals, you need to present an open cover of the set such which does not have a finite sub-cover.

Exercise 9.2. Show that if A and B are compact subsets of a metric space (X, d), then $A \cup B$ is a compact set.

Hint: For an arbitrary open cover for $A \cup B$, there is a finite sub-cover for A, and a finite sub-cover for B. Consider the union of those finite sub-covers.

Exercise 9.3. Show that the ball

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in the metric space $(\mathbb{R}^2, \mathrm{d}_2)$ is not compact.

Hint: consider an open cover of this set, by balls centred at $(0, 0)$ and the radii tending to 1 from below.

Exercise 9.4. Let (X, d) be a metric space, and A_1, A_2, $\dots$, A_n be a finite number of bounded sets in X. Then $\cup_{i=1}^n A_i$ is a bounded set in X.

Hint: Consider the bounds M_i for the sets A_i, for $i = 1, \dots, n$. From each i, choose a point $z_i \in A_i$, and add all the numbers M_i and $\mathrm{d}(z_i, z_j)$, over all i and j.

Exercise 9.5. Let (X, d) be a non-empty metric space, and let $Z \subseteq X$. Show that Z is bounded if and only if there is $x \in X$ and $r \in \mathbb{R}$ such that $Z \subseteq B_r(x)$.

Hint:If Z is bounded, choose a bound M, and consider the ball $B_M(x)$, for an arbitrary $x \in A$. If A is contained in a ball of radius R, work with the bound $2R$ for the set A.

Exercise 9.6. Consider the set $\mathbb{R}$ with the discrete metric $\mathrm{d}_{\mathrm{disc}}$. The set $(0, 1)$ is closed and bounded in $(\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$, but it is not compact.

Hint: Obviously, 1 provides a bound for the distance between any two points in $(0, 1)$. Use that any set in $\mathbb{R}$ with respect to the discrete metric is open, so any set is also closed (being the complement of some set in $\mathbb{R}$).

Exercise 9.7. Let (X, d) be a metric space, and assume that V_n, for $n \geq 1$, be a nest of non-empty closed sets in X.

(i) Show that if X is compact, then $\cap_{n \geq 1} V_n$ is not empty.

(ii) Give an example of a nest of non-empty closed sets V_n, for $n \geq 1$, in a metric space such that $\cap_{n \geq 1} V_n$ is empty.

Hint: If the intersection is empty, then we may consider the cover of X by the sets $X \setminus V_n$, for $n \geq 1$, and drive a contradiction. For the second part, think about closed sets in $(\mathbb{R}, \mathrm{d}_1)$.

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Questions marked with $*$ are optional

Exercise 9.8. Show that if a metric space is sequentially compact, then it is bounded.

Hint: If a set is not bounded, there are pairs of points z_n and w_n with $\mathrm{d}(z_n, w_n) \geq n$. Think about what happens if $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ converge to some points z and w, respectively. You will need to identify a subsequence, so that both sequences converge along that subsequence.

Exercise 9.9.* Let (X, d) be a sequentially compact metric space. Show that X is separable, that is, there is a countable dense set in X.

Hint: Fix an arbitrary $n \in \mathbb{N}$. Consider the open cover $\mathcal{R}_n = \{B_{1/n}(x) \mid x \in X\}$. Use the sequential compactness of X to conclude that there must be a finite sub-cover of $\mathcal{R}_n$ for X. Let A_n be the centres of the balls in that finite sub-cover of $\mathcal{R}_n$. Consider $A = \cup_{n\geq 1} A_n$, and show that A is countable and dense in X.

Exercise 9.10.* Let (X, d) be a sequentially compact metric space, and $\mathcal{R}$ be an open cover for X. Show that there is a countable sub-cover of $\mathcal{R}$ for X.

Hint: You can prove this statement in two steps. Step 1: Show that there is $n \in \mathbb{N}$ such that for every $x \in X$, $B_{1/n}(x)$ is contained in some element of $\mathcal{R}$ (assume that such n does not exist, so for every $n \in \mathbb{N}$ there is x_n such that $B_{1/n}(x_n)$ is not contained in any ball. Extract a subsequence and see what happens at the limit of that subsequence, ...). Step 2: By the previous exercise, there is a countable dense set $\{y_1, y_2, y_3, \dots\}$ in X. Let n be the number from Step 1. For each $i \in \mathbb{N}$, $B_{1/n}(y_i)$ is contained in some element $V_i \in \mathcal{R}$. Show that the collection $\{V_i \mid i \in \mathbb{N}\}$ is a countable sub-cover of $\mathcal{R}$ for X.

Exercise 9.11. Let (X, d) be a compact metric space, and assume that $f : X \to X$ is a continuous map such that for all $x \in X$, we have $f(x) \neq x$. Show that there is $\delta > 0$ such that for all $x \in X$, we have $\mathrm{d}(x, f(x)) \geq \delta$.

Hint: Work with the map $x \mapsto \mathrm{d}(x, f(x))$ on the set X, and think about if this map is continuous, and what values it may take.