Imperial College London

MATH97049 MATH97160

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Tuesday, 11 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

You can use any theorem/lemma/proposition from the lectures, but if you wish to do that, you must state that result precisely. Also, if you cannot do a part of a problem, you can still use that part in order to do the later parts of that problem.

1. Consider the map $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined as

$$\gamma(t) = (\cos(4t), 3t, \sin(4t)).$$

(a) Find a reparametrisation of γ by arc length. (3 marks)

(b) Let $\eta : \mathbb{R} \to \mathbb{R}^3$ be the reparametrisation of γ by arc length you find in part (b). Find the Frenet frame (T, N, B) for the curve η. (6 marks)

(c) Find the curvature k and the torsion τ of η at each point $\eta(t)$, for $t \in \mathbb{R}$. (3 marks)

(d) Let $f : [0,1] \to (0, +\infty)$ be a smooth function. Is there a *regular curve* $\phi : [0,1] \to \mathbb{R}^2$ whose curvature k_ϕ satisfies $k_\phi(t) = f(t)$, for all $t \in [0,1]$? Justify your answer. (4 marks)

(e) Let $f : [0,1] \to \mathbb{R}$ be a smooth function whose derivatives satisfy $f^{(k)}(0) = f^{(k)}(1)$, for all $k \geq 0$. Is there a *closed regular curve* $\phi : [0,1] \to \mathbb{R}^2$ whose signed curvature κ_ϕ satisfies $\kappa_\phi(t) = f(t)$, for all $t \in [0,1]$? Justify your answer. (4 marks)

(Total: 20 marks)

2. Consider the regular surface

$$S = \{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z = e^x \sin(y)\}.$$

(a) Find the first fundamental form of S at each point $(x, y, z) \in S$. (5 marks)

(b) Find the second fundamental form of S at each point $(x, y, z) \in S$. (5 marks)

(c) Find the Gaussian curvature of S at each point $(x, y, z) \in S$. (5 marks)

(d) Find the Mean curvature of S at each point $(x, y, z) \in S$. (5 marks)

(Total: 20 marks)

3. Consider the set

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, y = 2 + \sin(x)\},$$

and let $S \subset \mathbb{R}^3$ be the set obtained from (fully) rotating the set C about the x axis.

(a) Prove that S is a regular surface. (6 marks)

(b) Find the tangent plane of S at the point $(\pi/2, 3, 0)$. (4 marks)

(c) Prove that any continuous unit normal vector $N : S \to \mathbb{S}^2$ is not surjective. (5 marks)

(d) Show that there is no local isometry from the torus $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ to the surface S.

(5 marks)

(Total: 20 marks)

4. (a) Let $\gamma : [0, L] \to \mathbb{R}^2$ be a closed regular curve parametrised by arc length. Assume that the index of γ about $(0, 0)$ is 2, and the curvature k of γ satisfies $|k(t)| \leq c$, for all $t \in [0, L]$. Prove that the length of γ, L, satisfies $L \geq 4\pi/c$. (4 marks)

(b) Let $\gamma(t) = (x(t), y(t))$, for $t \in [-1, +1]$, be a regular curve in $\mathbb{R}^2$ such that $\gamma(0) = (0, 0)$ and for all $t \in [-1, +1] \setminus \{0\}$, $y(t) > 0$. Is it true that the curvature of γ at $\gamma(0)$ is strictly positive? Justify your answer. (5 marks)

Below, assume that $\Sigma_g \subset \mathbb{R}^3$ is a regular surface of genus $g \geq 1$.

(c) Show that the Gaussian curvature of Σ_g must vanish at some point. (5 marks)

(d) Let $\mathbb{D}_1$ be the open ball of radius 1 about $(0, 0) \in \mathbb{R}^2$, and let $\overline{\mathbb{D}_1}$ be its closure. Assume that ϕ is a map from $\overline{\mathbb{D}_1}$ to Σ_g such that $\phi : \overline{\mathbb{D}_1} \to \phi(\overline{\mathbb{D}_1})$ is a diffeomorphism. What is the Euler characteristic of $\Sigma_g \setminus \phi(\mathbb{D}_1)$? Justify your answer.

(6 marks)

(Total: 20 marks)

5. Let

$$g(z) = 1 + z^2 + \sum_{i=2}^{\infty} a_i z^{2i}$$

be a an absolutely convergent series for $z \in \mathbb{R}$. Define the set

$$\Delta = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = (g(z))^2 \right\}.$$

(a) Show that there is $\delta > 0$ such that the set

$$\Delta_\delta = \{(x, y, z) \in \Delta \mid |z| < \delta\}$$

is a regular surface. (5 marks)

(b) Show that the Gaussian curvature of Δ_δ is negative at every point in the set

$$\gamma_0 = \{(x, y, z) \in \Delta \mid z = 0\}.$$

(6 marks)

(c) Show that the curve γ_0 is a geodesic on Δ_δ. (4 marks)

(d) Show that if $\epsilon > 0$ is small enough, the curve

$$\gamma_\epsilon = \{(x, y, z) \in \Delta \mid z = \epsilon\}$$

is not a geodesic on Δ_δ. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96034/MATH97049/MATH97160

Geometry of Curves and Surfaces (Solutions)

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1. (a) We have $\boxed{\text{sim.seen} \Downarrow}$

$$\gamma'(t) = (-4\sin(4t), 3, 4\cos(4t)),$$

and hence

$$\left|\gamma'(t)\right| = \sqrt{16\sin^2(4t) + 9 + 16\cos^2(4t)} = \sqrt{16+9} = 5.$$

For $t \in \mathbb{R}$, let

$$h(t) = \int_0^t |\gamma'(s)|\, ds = \int_0^t 5\, ds = 5t.$$

We have $f(s) := h^{-1}(s) = s/5$. By a result in the lectures, the curve

$$\eta(s) = \gamma \circ f(s) = (\cos(4s/5), 3s/5, \sin(4s/5))$$

is parametrised by arc length. $\boxed{3, \text{A}}$

(b) The tangent vector T to η is $\boxed{\text{sim.seen} \Downarrow}$

$$T(t) = \eta'(t) = \left(\frac{-4}{5}\sin\left(\frac{4t}{5}\right), \frac{3}{5}, \frac{4}{5}\cos\left(\frac{4t}{5}\right)\right).$$

The curvature vector is defined as

$$\vec{k}(t) = \eta''(t) = \left(\frac{-16}{25}\cos\left(\frac{4t}{5}\right), 0, \frac{-16}{25}\sin\left(\frac{4t}{5}\right)\right).$$

We note that for every $t \in \mathbb{R}$, $|\vec{k}(t)| \neq 0$, and hence the Frenet Frame for η exists at every point. The normal vector N to η is

$$\begin{aligned} N(t) = \frac{\eta''(t)}{|\eta''(t)|} &= \frac{25}{16}\left(\frac{-16}{25}\cos\left(\frac{4t}{5}\right), 0, \frac{-16}{25}\sin\left(\frac{4t}{5}\right)\right) \\ &= (-\cos(4t/5), 0, -\sin(4t/5)). \end{aligned}$$

The binormal vector B to η is defined as

$$B(t) = T(t) \times N(t) = \left(\frac{-3}{5}\sin\left(\frac{4t}{5}\right), \frac{-4}{5}, \frac{3}{5}\cos\left(\frac{4t}{5}\right)\right).$$

$\boxed{6, \text{A}}$

(c) By definition, $\boxed{\text{meth seen} \Downarrow}$

$$k(t) = \left|\eta''(t)\right| = 16/25,$$

and using the formula, $B'(t) = -\tau(t)N(t)$, we note that $\tau(t) \equiv -12/25$. $\boxed{3, \text{A}}$

(d) By a theorem in the lectures, every pair of smooth functions $k > 0$ and τ is realised as the curvature and torsion of some regular curve in $\mathbb{R}^3$. Applying this result to the functions $f > 0$ and $\tau = 0$, there is a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, with curvature f and torsion 0. Because the torsion of γ is identically equal to 0, from the Frenet equation $B' = -\tau N$ or from the lectures, we conclude that γ is contained in a plane in $\mathbb{R}^3$. We may apply a rigid motion of $\mathbb{R}^3$ to send that plane into $\mathbb{R}^2$, without changing the curvature. $\boxed{\text{sim.seen} \Downarrow}$ $\boxed{4, \text{C}}$

(e) The answer is No. By the total signed curvature theorem, the integral of the signed curvature along a closed regular curve must be an integer multiple of 2π. For example, we may let $f(t) = \sqrt{2}$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{4, \text{C}}$

2. (a) We consider the chart $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ for S defined as $\boxed{\text{meth seen} \Downarrow}$

$$\phi(x, y) = (x, y, e^x \sin y) .$$

We have

$$\phi_x(x, y) = (1, 0, e^x \sin y) , \quad \phi_y(x, y) = (0, 1, e^x \cos y) .$$

The first fundamental form at $\phi(x, y)$ is

$$\begin{aligned} g_{\phi(x,y)} &= \begin{pmatrix} \langle\phi_x(x, y)), \phi_x(x, y)\rangle & \langle\phi_x(x, y)), \phi_y(x, y)\rangle \\ \langle\phi_y(x, y)), \phi_x(x, y)\rangle & \langle\phi_y(x, y)), \phi_y(x, y)\rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 + e^{2x} \sin^2 y & e^{2x} \sin y \cos y \\ e^{2x} \sin y \cos y & 1 + e^{2x} \cos^2 y \end{pmatrix} . \end{aligned}$$

$\boxed{5, \text{A}}$

(b) We have $\boxed{\text{meth seen} \Downarrow}$

$$\phi_{xx}(x, y) = (0, 0, e^x \sin y), \quad \phi_{yy}(x, y) = (0, 0, -e^x \sin y), \quad \phi_{xy}(x, y) = (0, 0, e^x \cos y),$$

and the normal vector to the surface is

$$N(\phi(x, y)) = \frac{\phi_x(x, y) \times \phi_y(x, y)}{|\phi_x(x, y) \times \phi_y(x, y)|} = \frac{(-e^x \sin y, -e^x \cos y, 1)}{\sqrt{1 + e^{2x}}} .$$

The second fundamental form at $\phi(x, y)$ is

$$A_{\phi(x,y)} = \begin{pmatrix} \langle N, \phi_{xx}\rangle & \langle N, \phi_{xy}\rangle \\ \langle N, \phi_{yx}\rangle & \langle N, \phi_{yy}\rangle \end{pmatrix} = \frac{1}{\sqrt{1 + e^{2x}}} \begin{pmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{pmatrix}$$

$\boxed{5, \text{A}}$

(c) By a result in the lectures, the Gaussian curvature at $\phi(x, y)$ is $\boxed{\text{meth seen} \Downarrow}$

$$K(\phi(x, y)) = \frac{\det A_{\phi(x,y)}}{\det g_{\phi(x,y)}} = \frac{-e^{2x}}{1 + e^{2x}} \cdot \frac{1}{1 + e^{2x}} = \frac{-e^{2x}}{(1 + e^{2x})^2} .$$

$\boxed{5, \text{B}}$

(d) By a result in the lectures, the mean curvature of S at $\phi(x, y)$ is $H(\phi(x, y)) = \frac{1}{2} \operatorname{tr}(g^{-1}A)$, that is, $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned} &\frac{1}{2} \operatorname{tr} \left(\frac{1}{1 + e^{2x}} \begin{pmatrix} 1 + e^{2x} \cos^2 y & -e^{2x} \sin y \cos y \\ -e^{2x} \sin y \cos y & 1 + e^{2x} \sin^2 y \end{pmatrix} \frac{1}{\sqrt{1 + e^{2x}}} \begin{pmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{pmatrix} \right) \\ &= \frac{1}{2(1 + e^{2x})^{3/2}} \operatorname{tr} \begin{pmatrix} e^x \sin y & \ldots \\ \ldots & -\sin y(e^{3x} + e^x) \end{pmatrix} \\ &= \frac{-e^{3x} \sin y}{2(1 + e^{2x})^{3/2}} . \end{aligned}$$

$\boxed{5, \text{B}}$

3. (a) Consider the map

meth seen ⇓

$$\phi(u,v) = (u, (2+\sin u)\cos v, (2+\sin u)\sin v), \qquad u,v \in \mathbb{R}.$$

This is a smooth map from $\mathbb{R}^2$ into $\mathbb{R}^3$. When $v=0$, we obtain the curve C, and for each fixed u, the second and third coordinates of ϕ parametrise a circle of radius $(2+\sin u)$ about $(u,0,0)$, obtained from rotating a point on C about the x axis.

We note that if $\phi(u_1,v_1) = \phi(u_2,v_2)$, then $u_1 = u_2$, and then $\sin v_1 = \sin v_2$ and $\cos v_1 = \cos v_2$. The latter two relations imply that $v_1 - v_2 \in 2\pi\mathbb{Z}$. Thus, ϕ is injective on any set in $\mathbb{R}^2$ of vertical width less than 2π.

The partial derivatives of ϕ are

$$\phi_u(u,v) = (1, \cos u \cos v, \cos u \sin v),$$

$$\phi_v(u,v) = (0, -(2+\sin u)\sin v, (2+\sin u)\cos v).$$

These are linearly independent, and hence $D\phi_{(u,v)}$ is injective at every $(u,v) \in \mathbb{R}^2$.

For $p = (p_1,p_2,p_3) \in S$, we define the open set $V \subset \mathbb{R}^3$ as the half-plane orthogonal to the vector $(0,p_2,p_3)$, cut off by the plane $x = p_1 \pm 1$, that is,

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid p_1 - 1 < x < p_1 + 1, \langle (y,z), (p_2,p_3) \rangle > 0\},$$

We also define

$$U = \left\{(u,v) \in \mathbb{R}^2 \mid p_1 - 1 < u < p_1 + 1, \left| v - \arccos\left(p_2 / \sqrt{p_2^2 + p_3^2}\right)\right| < \pi/2 \right\}.$$

Then, we have $\phi(U) = S \cap V$. Since $V \cap S$ lies in a revolution of C of angle at most π, ϕ is injective on U. Therefore, ϕ is a chart for S at p. 6, A

(b) We have $\phi(\pi/2,0) = (\pi/2,3,0)$. By a result in the lectures, the tangent plane $T_{(\pi/2,3,0)}S$ is equal to $D\phi_{(\pi/2,0)}(\mathbb{R}^2)$. By the previous part, we have

meth seen ⇓

$$D\phi_{(\pi/2,0)} = \begin{pmatrix} 1 & 0 \\ \cos u \cos v & -(2+\sin u)\sin v \\ \cos u \sin v & (2+\sin u)\cos v \end{pmatrix}_{|(\pi/2,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{pmatrix}$$

Thus, the image of $D\phi_{(\pi/2,0)}$ is generated by the vectors $D\phi_{(\pi/2,0)}(1,0) = (1,0,0)$ and $D\phi_{(\pi/2,0)}(0,1) = (0,0,3)$. Thus, $(0,1,0)$ is orthogonal to the image of $D\phi_{(\pi/2,0)}$, and hence, the tangent plane is given by $y=0$. This is the xz-plane. 4, A

(c) Since S is connected, there are only two choices for N on S, these are,

meth seen ⇓

$$N(\phi(u,v)) = \pm \frac{\phi_u(u,v) \times \phi_v(u,v)}{|\phi_u(u,v) \times \phi_v(u,v)|}.$$

The second and third coordinates of the above vector are

$$\pm \frac{(2+\sin u)\cos v}{|\phi_u(u,v) \times \phi_v(u,v)|}, \qquad \pm \frac{(2+\sin u)\sin v}{|\phi_u(u,v) \times \phi_v(u,v)|}.$$

We can see that these components cannot be zero simultaneously, thus, the unit vector $(1,0,0)$ is never realised. 5, B

(d) Assume in the contrary that there is a local isometry ψ from the torus T to the surface S. Since T is compact, $\psi(T) \subset S$ must be compact, and hence closed. Since every local isometry is a local diffeomorphism, it follows from the inverse function theorem for surfaces that $\psi(T)$ is open in S. However, as S is connected, the only sets in S which is both open and closed are $\emptyset$ and S. We conclude that $\psi(T) = S$. Since the image of any compact set is compact, S must be compact. But, we know that S is not compact, since it is not bounded in $\mathbb{R}^3$. meth seen $\Downarrow$ 5, D

4. (a) From the lectures, the curvature k of a curve is equal to the absolute value of its signed curvature κ. By the total curvature theorem, we have **meth seen ⇓**

$$\int_0^L \kappa(t)\,dt = 2\pi\,\mathrm{ind}(\gamma).$$

Therefore,

$$2\pi \cdot 2 = \int_0^L \kappa(t)\,dt = \left|\int_0^L \kappa(t)\,dt\right| \leq \int_0^L |\kappa(t)|\,dt \leq Lc.$$

4, C

(b) The answer is No. For example, we can consider the curve **unseen ⇓**

$$\gamma(t) = (t, t^4), \quad t \in \mathbb{R}.$$

This is a smooth curve, however, it is not parametrised by arc-length.

To re-parametrise γ with arc-length, we consider the function

$$h(t) = \int_0^t |\gamma'(s)|\,ds = \int_0^t |(1, 4s^3)|\,ds = \int_0^t \sqrt{1 + 16s^6}\,ds.$$

And let $f(t) = h^{-1}(t)$. Then, by the results in the lectures, the curve $\eta = \gamma \circ f$ is parametrised by arc length. The above equation give us

$$h'(t) = \sqrt{1 + 16t^6}, \qquad h''(t) = \frac{1}{2}(1 + 16t^6)^{-1/2}(96t^5).$$

In particular, $h'(0) = 1$ and $h''(0) = 0$. Using the relation $h \circ f(t) = t$, we get

$$h'(f(t))f'(t) = 1, \qquad h''(f(t))f'(t)f'(t) + h'(f(t))f''(t) = 0.$$

Using $f(0) = h(0) = 0$, these give us $f'(0) = 1$ and $f''(0) = 0$.

On the other hand,

$$\eta'(t) = \gamma'(f(t)) \cdot f'(t), \qquad \eta''(t) = \gamma''(f(t))f'(t)f'(t) + \gamma'(f(t))f''(t).$$

Then, $\eta''(0) = 0$, which implies that the curvature of η at $t = 0$ is zero. **5, D**

(c) By the Gauss-Bonnet Theorem, we must have **meth seen ⇓**

$$\int_{\partial\Sigma_g} k_g\,ds + \int_{\Sigma_g} K\,dA = \chi(\Sigma_g).$$

Since the surface has no boundary, the first integral is 0, and by a result in the lectures, $\chi(\Sigma_g) = 2 - 2g$. Thus,

$$\int_{\Sigma_g} K\,dA = 2 - 2g \leq 0.$$

By another result in the lectures, for any compact regular surface, there is a point where $K > 0$. Then, by the above formula, K must be also negative at some point. Since K is a continuous function, and Σ_g is connected, K must be zero at some point on Σ_g. **5, B**

(d) We have seen in the lectures that there is a triangulation T of Σ_g, with V vertices, E edges, and F faces satisfying $V - E + F = 2 - 2g$. Let us first assume that $\phi(\overline{\mathbb{D}_1})$ is a small disk on Σ_g. By slightly modifying the triangulation T, we may assume that $\phi(\overline{\mathbb{D}_1})$ is one of the triangle in T. Thus, removing a face from T provides a triangulation of $\Sigma_g \setminus \phi(\overline{\mathbb{D}_1})$. Therefore,

meth seen ⇓

$$\chi\left(\Sigma_g \setminus \phi(\overline{\mathbb{D}_1})\right) = V - E + (F-1) = 2 - 2g - 1.$$

Let $\mathbb{D}_\delta$ denote the open ball of radius δ about the origin in $\mathbb{R}^2$.

For the general case, we note that $\Sigma_g \setminus \phi(\overline{\mathbb{D}_1})$ is homeomorphic to $\Sigma_g \setminus \phi(\overline{\mathbb{D}_\delta})$, for any $\delta \in (0,1)$. From the lectures we also know that homeomorphic surfaces have equal Euler characteristics. Thus, for every $\delta \in (0,1)$, the Euler characteristic of $\Sigma_g \setminus \phi(\overline{\mathbb{D}_\delta})$ is equal to the Euler characteristic of $\Sigma_g \setminus \phi(\overline{\mathbb{D}_1})$. For sufficiently small $\delta > 0$ we can use the argument in the first paragraph to conclude that

$$\chi\left(\Sigma_g \setminus \phi(\overline{\mathbb{D}_1})\right) = 2 - 2g - 1.$$

6, D

5. (a) Since $g(0) = 1$, and g is a continuous function, there is $\delta > 0$ such that for all $z \in [-\delta, +\delta]$, $g(z) > 0$. Consider the open set $\boxed{\text{meth seen} \Downarrow}$

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid z \in (-\delta, +\delta)\},$$

and the function $F : \Omega \to \mathbb{R}$, defined as

$$F(x, y, z) = x^2 + y^2 - (g(z))^2.$$

This is a smooth function on Ω, with $\nabla F = (2x, 2y, 2g'(z))$. We have

$$\Delta_\delta = \Omega \cap \Delta = F^{-1}(0).$$

For every $(x, y, z) \in \Delta_\delta$, $(x, y) \neq (0, 0)$, since $g(z) \neq 0$. Thus, for every $(x, y, z) \in \Delta_\delta$, $\nabla F(x, y, z) \neq 0$. That is, Δ_δ is a regular level set. By a result in the lectures, this implies that Δ_δ is a regular surface. $\boxed{5, \text{M}}$

(b) The map $\boxed{\text{meth seen} \Downarrow}$

$$\phi(u, v) = (g(v) \cos u, g(v) \sin u, v), \quad u \in \mathbb{R}, \quad -\delta < v < +\delta,$$

may be restricted to small open sets to give a chart around each point of Δ_δ. We have

$$\phi_u(u, v) = (-g(v) \sin u, g(v) \cos u, 0), \quad \phi_v(u, v) = (g'(v) \cos u, g'(v) \sin u, 1),$$

$$\begin{aligned}
\phi_{uu}(u, v) &= (-g(v) \cos u, -g(v) \sin u, 0)\,, \\
\phi_{vv}(u, v) &= \big(g''(v) \cos u, g''(v) \sin u, 0\big)\,, \\
\phi_{uv}(u, v) &= \big(-g'(v) \sin u, g'(v) \cos u, 0\big)\,,
\end{aligned}$$

and the unit normal to the surface at $\phi(u, v)$ is

$$N(\phi(u, v)) = \frac{\phi_u(u, v) \times \phi_v(u, v)}{|\phi_u(u, v) \times \phi_v(u, v)|} = \frac{g(v)(\cos u, \sin u, -g'(v))}{g(v)\sqrt{1 + (g'(v))^2}} = \frac{(\cos u, \sin u, -g'(v))}{\sqrt{1 + (g'(v))^2}}.$$

Thus, the first and second fundamental form of Δ_g at $\phi(u, v)$ are

$$g_{\phi(u,v)} = \begin{pmatrix} \langle \phi_u(u, v)), \phi_u(u, v) \rangle & \langle \phi_u(u, v)), \phi_v(u, v) \rangle \\ \langle \phi_v(u, v)), \phi_u(u, v) \rangle & \langle \phi_v(u, v)), \phi_v(u, v) \rangle \end{pmatrix} = \begin{pmatrix} (g(v))^2 & 0 \\ 0 & 1 + (g'(v))^2 \end{pmatrix},$$

$$A_{\phi(u,v)} = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix} = \frac{1}{\sqrt{1 + (g'(v))^2}} \begin{pmatrix} -g(v) & 0 \\ 0 & g''(v) \end{pmatrix}$$

By a result in the lectures, the Gaussian curvature at $\phi(u, v)$ is

$$K(\phi(u, v)) = \frac{\det A_{\phi(u,v)}}{\det g_{\phi(u,v)}} = \frac{-g(v) g''(v)}{1 + (g'(v))^2} \cdot \frac{1}{(g(v))^2 (1 + (g'(v))^2)}.$$

When $v = 0$, we obtain $K(\phi(u, 0)) = -2 < 0$. $\boxed{6, \text{M}}$

(c) We may parametrise the curve γ_0 as $\gamma(t) = \phi(t, 0) = (\cos t, \sin(t), 0)$, $t \in [0, 2\pi]$, which is a parametrisation by arc-length. We have $\gamma'(t) = (-\sin t, \cos t, 0)$, and hence $\boxed{\text{unseen} \Downarrow}$

$$N(\gamma(t)) \times \gamma'(t) = (\cos t, \sin t, 0) \times (-\sin t, \cos t, 0) = (0, 0, -1).$$

Thus, the geodesic curvature of γ at $\gamma(t)$ is, by definition,

$$k_g(\gamma(t)) = \langle \gamma''(t), N(\gamma(t)) \times \gamma'(t) \rangle = \langle (-\cos t, -\sin t, 0), (0, 0, -1) \rangle = 0.$$

Thus, γ_0 is a geodesic on the surface Δ_δ. $\boxed{4, \text{M}}$

(d) Let $\epsilon \in (0, \delta)$. The set $\boxed{\text{meth seen} \Downarrow}$

$$\Delta'_\epsilon = \{(x, y, z) \in \Delta_\delta \mid 0 \leq |z| \leq \epsilon\}$$

is a regular surface with boundary. We have $\partial \Delta'_\epsilon = \gamma_0 \cup \gamma_\epsilon$. We apply the Gauss-Bonnet Theorem to Δ'_ϵ, and obtain

$$\int_{\gamma_0 \cup \gamma_\epsilon} k_g \, ds + \int_{\Delta'_\epsilon} K \, dA = \chi(\Delta'_\epsilon).$$

By part (c), γ_0 is a geodesic, so k_g on $\gamma_0 = 0$. On the other hand, Δ'_ϵ is homeomorphic to a round annulus, and hence, its Euler characteristic is equal to 0. Therefore,

$$\int_{\gamma_\epsilon} k_g \, ds + \int_{\Delta'_\epsilon} K \, dA = 0.$$

By Part (b), K is negative on γ_0. Since K is continuous, K must be negative on a neighbourhood of γ_0. Thus, for $\epsilon > 0$ small enough, K is negative on Δ'_ϵ. Then, the above equation implies that k_g cannot be identically equal to 0 on γ_ϵ. Thus γ_ϵ is not a geodesic on Δ'_δ. $\boxed{5, \text{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96034 MATH97049 MATH97160	1	This was one of the easier problems, and most students did well on this.
MATH96034 MATH97049 MATH97160	2	This problem is mostly calculations, but rather long. Most students are very comfortable with this kind of problems, and can take long calculation to the final point.
MATH96034 MATH97049 MATH97160	3	I note that many student have problem with formally proving that a set is a regular surface, except few students, most students did not identify the open set V around each point, in which S is homeomorphic to an open set in R^2. It appears part d of the problem is very difficult for most, besides few smart ideas, most did not do well on this. This is mostly about analysis than geometry.
MATH96034 MATH97049 MATH97160	4	Almost everyone did part a, and almost everyone failed on part b (although this has been discussed during lectures).
MATH96034 MATH97049 MATH97160	5	This problem examines a range of skill, conceptual arguments, and calculations. Overall it was satisfactory.