Module Notes

Applied Complex Analysis

MATH60006/70006/97028, Chapter One

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Contents

1. Review of Complex Analysis

In this first chapter we will review some of the fundamental results of complex analysis that will be relevant to our later chapters. Sometimes we will omit proofs from this chapter as we are interested in using the results to solve problems in applied mathematics/mathematical physics rather than studying how they are obtained.

For those in year 3/4 of their undergraduate studies at Imperial, it may help to refer back to your complex analysis notes from year 2 if you want to refresh your memory on any of the proofs of these results/see more examples. Similarly for those on Masters programmes it might be useful to look back at any complex analysis you did during your undergraduate degree. For those without access to any previous complex analysis notes, it may help to refer to any introductory text on complex analysis (see the reading list) to look up the proofs/see more examples.

That being said, these lecture notes are entirely self-contained in the sense that no external knowledge outside of these notes is necessary to study this course and for the exam/assessments. A knowledge of the basic results and properties of complex numbers as well as first/second year undergraduate knowledge in calculus and differential equations will be assumed throughout this course however.

Although I (Andrew Gibbs) will be teaching this course, a huge amount of credit should be given to Samuel Brzezicki, who taught the course previously, and created these fantastic course notes, alongside the accompanying problem sheets.

1.1 Complex Numbers and their Properties

Let us begin by reviewing a few basic results about complex numbers before moving on to recap the complex derivative.

A complex number has the form $z = x + iy$, where $i = \sqrt{-1}$. We denote the **real** and **imaginary** parts respectively as $x = \mathrm{Re}\{z\}$ and $y = \mathrm{Im}\{z\}$. We may also think of z as a point in the complex plane, which in polar coordinates has **absolute value** $r = |z| = \sqrt{x^2 + y^2}$ and **argument** θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Note that θ is defined periodically with period 2π and may also be denoted $\theta = \arg\{z\}$.

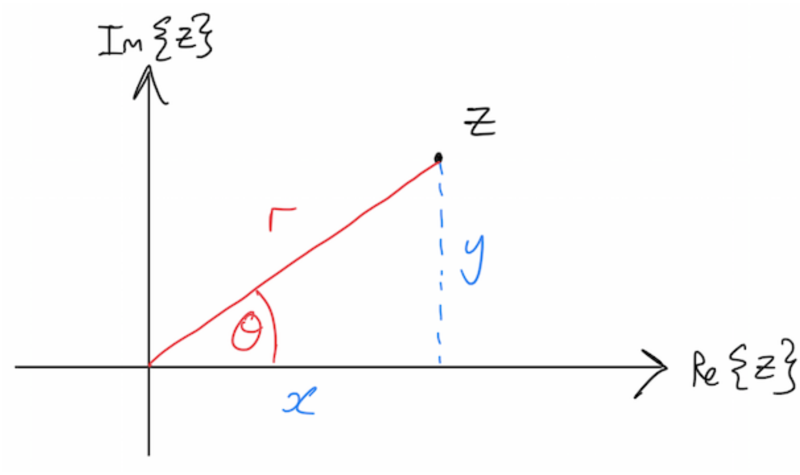

Figure 1: A complex number z visualised on the complex plane.

The complex number may then be represented in polar form $z = r(\cos\theta + i\sin\theta)$ or exponential form $z = re^{i\theta}$.

The exponential form is known as *Euler's identity*, and can be proved by Taylor expanding the exponential function (Taylor expansion is justified a little later on). Since the sum is absolutely convergent, we can split it into two sums, over odd and even terms, like so:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \cos\theta + i\sin\theta,$$

in the final line we have used the Taylor expansions for sin and cos.

Complex numbers obey the following algebraic properties:

$$\begin{aligned} z_1 + z_2 =& (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 =& (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned}$$

i.e. they have the same algebra as the real numbers, with the convention $i^2 = -1$.

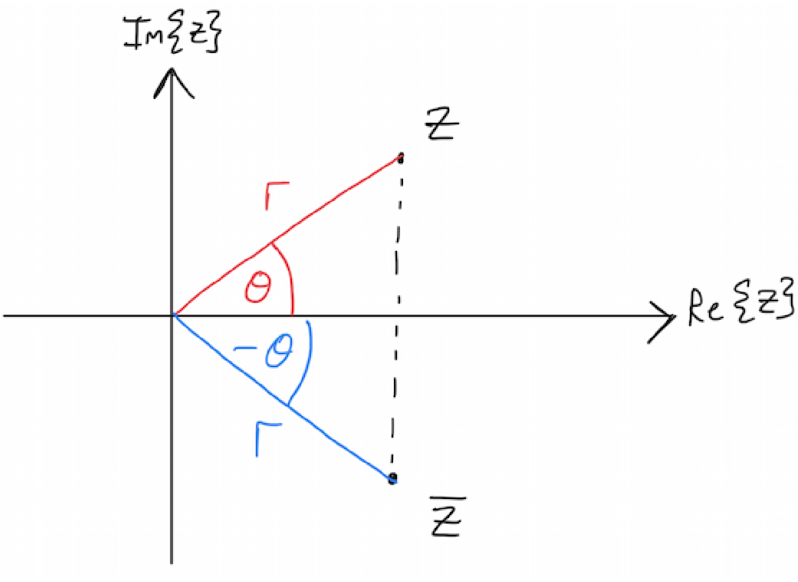

Figure 2: Complex conjugate $\overline{z} = x - iy$ is defined by reflecting in the x-axis.

If $z = x + iy$ then throughout this course the **complex conjugate** of z will be defined as $\overline{z} = x - iy$, which is the symmetric image of z with respect to the x-axis. Note the following properties of the complex conjugate:

$$\begin{aligned} |z| =& \sqrt{x^2 + y^2} = |\overline{z}| \\ \arg\{z\} =& - \arg\{\overline{z}\} \\ z\overline{z} =& (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \end{aligned}$$

This helps us to obtain a useful formula for the division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}.$$

We may also do multiplication in polar coordinates:

$$\begin{aligned} z_1 z_2 =& (r_1 \cos\theta_1 + ir_1 \sin\theta_1)(r_2 \cos\theta_2 + ir_2 \sin\theta_2) \\ =& r_1 r_2 \left[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)\right] \\ =& r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right], \end{aligned}$$

where we have used the double-angle formulae for sin and cos. Then we see that

$$|z_1 z_2| = |z_1|\,|z_2|$$

and

$$\arg\{z_1 z_2\} = \arg\{z_1\} + \arg\{z_2\} \pmod{2\pi}.$$

Thus, we may take the nth power and, after that, nth root of z as:

$$\begin{aligned} z^n =& r^n(\cos(n\theta) + i \sin(n\theta)) \\ \sqrt[n]{z} =& r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \qquad \text{for} \qquad k = 0, ..., n - 1 \end{aligned}$$

Each non-zero z has n different nth roots and these are related by rotations in the complex plane, see Figure 3.

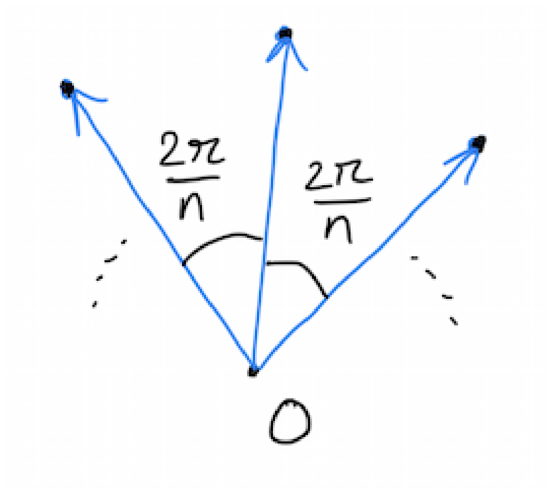

Figure 3: The nth roots are related by rotation through angle $2\pi/n$ in the complex plane.

The definition of the complex conjugate $\overline{z}$ also allows us to write

$$\mathrm{Re}\{z\} = \frac{z+\overline{z}}{2}, \quad \mathrm{Im}\{z\} = \frac{z-\overline{z}}{2i}.$$

We will use these results throughout the course.

The Triangle Inequality

A final result we will use throughout the course is the so called **triangle inequality**, which states for complex numbers z_1, $z_2 \in \mathbb{C}$, then

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

The proof of this statement is left as an exercise (see problem sheet 1).

1.2 Functions of Complex Variables

A standard notation we use for a complex function is

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real valued functions of x and y.

Examples

1. Some very simple complex functions are $f(z) = a$, where a is a constant and $f(z) = z$. By doing multiplication and addition with these, we obtain

2. Polynomials:

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

Employing the division operation, we obtain

3. Rational functions:

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are both polynomials.

4. Consider also the exponential function:

$$\exp\{z\} = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The coefficients of this series are exactly the same as for the real exponential function, so, because the algebraic operations on real and complex numbers obey the same rules, the exponential function defined on complex numbers obeys the same algebra as the exponential function on real numbers.

5. The formula $e^{i\theta} = \cos\theta + i\sin\theta$ gives $e^{-i\theta} = \cos\theta - i\sin\theta$ from which we can deduce

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

One can show that all the regular rules for $\sin x$ and $\cos x$ apply for these complex valued functions too.

6. From the formula $e^{i\theta} = \cos\theta + i\sin\theta$, we see that $e^{i\theta}$ is 2π-periodic in θ:

$$e^{i\theta + 2\pi k i} = e^{i\theta} \qquad \text{for} \qquad k \in \mathbb{Z}.$$

The periodicity of the complex exponential function implies that its inverse function is **multi-valued**:

$$\log z = \log\left(re^{i\theta}\right) = \log|z| + i\arg\{z\} = \log r + i\left(\theta + 2k\pi\right) \qquad \text{for} \qquad k \in \mathbb{Z}.$$

More on multi-valued functions later!

7. We may use the logarithm to define arbitrary complex powers of a complex number z:

$$z^\alpha = e^{\alpha \log z}$$

This function is also multivalued (when α is not an integer).

The previous examples are 'good' functions, in the sense that they are analytic, as we will see soon. However, we may also define 'bad' functions of complex variables, such as:

$$\begin{aligned} f(z) &= \mathrm{Re}(z) \\ g(z) &= |z| \\ h(z) &= \overline{z} \end{aligned}$$

These functions are non-analytic (how do we know why at a glance?).

The analyticity is the main topic here. While a typical complex function is not analytic, many important functions are analytic and, as a consequence, possess non-trivial and useful properties. The analyticity can be defined in several equivalent ways. We start with the notion of the derivative.

1.3 The Complex Derivative

Definition 1.1. A complex function $f(z)$ is said to be **differentiable** at a point $z \in \mathbb{C}$ if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \tag{1.1}$$

exists, and is **independent** of the path along which $h \to 0$ (note $h = h_1 + ih_2 \in \mathbb{C}$).

In particular, we may write $h = re^{i\theta}$, so the limit $h \to 0$ must be independent of θ, the direction along which h approaches zero.

Definition 1.2. A function $f(z)$ is **analytic** (or holomorphic) at the point z_0 if it has a derivative at all points close to z_0.

Note, for example, that $f(z) = |z|^2$ is not analytic at $z = 0$, even though it has a derivative at $z = 0$, because one can show that it does not have a derivative for any $z \neq 0$.

Definition 1.3. A function $f(z)$ is **analytic** in an open region D if the derivative $f'(z)$ exists at every point $z \in D$. A function that is analytic everywhere in $\mathbb{C}$ is called **entire**.

Examples

1. Let $f(z) = z^2$. Then at $z = 0$ we have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{(0+h)^2 - 0^2}{h} \\ &= \lim_{h \to 0} h \to 0. \end{aligned}$$

   So the derivative at $z = 0$ exists.

2. Let $f(z) = \overline{z}$. Then at $z = 0$ we have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{\overline{(0+h)} - \overline{0}}{h} \\ &= \lim_{h \to 0} \frac{\overline{h}}{h}. \end{aligned}$$

   Now if we let $h = h_1$ (so set $h_2 = 0$), then in the limit we find $f'(0) = 1$. On the other hand, if we let $h = ih_2$ (so $h_1 = 0$), then in the limit we find $f'(0) = -1$. So the limits are not equal and so the function is not differentiable at $z = 0$.

Theorem 1.4 (Cauchy-Riemann Conditions)**.** *Let $f(z) = u(x, y) + iv(x, y)$. The derivative $f'(z)$ exists at $z = x + iy$ if and only if u and v are differentiable and satisfy the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{1.2a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.2b}$$

Proof. Suppose $f'(z)$ exists. Then the limit in eq. (1.1) exists and is independent of the direction in which the limit is taken. Therefore set $h = h_1$ and we see that:

$$f'(z) = \lim_{h_1 \to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - u(x,y) + iv(x,y)}{h_1} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Similarly, if we take $h = ih_2$ then we find:

$$f'(z) = \lim_{h_2 \to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - u(x,y) + iv(x,y)}{ih_2} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts gives eq. (1.2).

Suppose conversely that u and v satisfy eq. (1.2). We keep a general form for $h = h_1 + ih_2$ and then use Taylor expansions of functions of two-variables:

$$\begin{aligned}
f(z+h) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2)\\
u(x+h_1, y+h_2) &= u(x,y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + o\left(\sqrt{h_1^2 + h_2^2}\right)\\
v(x+h_1, y+h_2) &= v(x,y) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + o\left(\sqrt{h_1^2 + h_2^2}\right)\\
\implies \qquad f(z+h) - f(z) &= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2\right) + o\left(|h|\right).
\end{aligned}$$

By eq. (1.2), the right-hand side of this formula is

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) h + o\left(|h|\right)$$

and therefore $f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$ exists. □

Properties of the Complex Derivative

The derivative satisfies the same rules as in the real case: for any analytic functions f and g, we have

1. $(f+g)' = f' + g'$,
2. $(fg)' = f'g + fg'$,
3. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$, $\quad$ ($g \neq 0$ at these points),
4. (chain rule) for any z in the domain of g, $(f(g(z)))' = f'(g(z))g'(z)$.

Remark 1.5. *If g is the inverse of f, i.e. for any z in the domain of f, $g(f(z)) = z$, then $g'(f(z)) = \frac{1}{f'(z)}$, and $g'(z) = \frac{1}{f'(g(z))}$.*

Some examples of how to derive the derivatives are given in what follows.

Some Examples

1). Suppose $f(z) = c$ for any $z \in \mathbb{C}$, where c is a constant complex number. Since for any $z, h \in \mathbb{C}$

$$\frac{f(z+h) - f(z)}{h} = \frac{c-c}{h} = 0,$$

the derivative of f is 0.

2). Suppose $f(z) = z$ for any $z \in \mathbb{C}$. Since for any $z, h \in \mathbb{C}$,

$$\frac{f(z+h) - f(z)}{h} = \frac{z+h-z}{h} = 1,$$

we get $f'(z) = 1$.

3). Using the product rule, for any $z \in \mathbb{C}$, we get

$$(z^n)' = nz^{n-1}$$

4). Now, for any $z \in \mathbb{C}$, letting $P(z) = a_0 + a_1 z + ... + a_n z^n$, we get the derivative of P as follows:

$$P'(z) = a_1 + ... + na_n z^{n-1}.$$

5). Since any rational function can be written as $\frac{P}{Q}$, where P and Q are polynomials, and since $(\frac{P}{Q})' = \frac{P'Q - PQ'}{Q^2}$, we obtain that rational functions are analytic everywhere except for the points where $Q(z) = 0$.

6). For $e^z = \sum_{i=0}^{\infty} \frac{z^n}{n!}$, its derivative is

$$(e^z)' = 1 + z + ... + \frac{nz^{n-1}}{n!} + ... = \sum_{i=0}^{\infty} \frac{z^n}{n!} = e^z.$$

The inverse function of e^z is $\log z$ (recall that $\log(re^{i\theta}) = \log r + i(\theta + 2k\pi)$). Using the rule for the derivative of the inverse function, we find

$$(\log z)' = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

7). For any $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$(z^\alpha)' = (e^{\alpha \log z})' = \frac{\alpha}{z} e^{\alpha \log z} = \alpha z^{\alpha - 1}.$$

1.4 Harmonic Functions

Definition 1.6. Let $\phi = \phi(x, y)$ be a real function ($x, y \in \mathbb{R}$). ϕ is called **harmonic** if

$$\Delta\phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Theorem 1.7. *Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then u and v are harmonic.*

Proof. Indeed, using the Cauchy-Riemann equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

So $\nabla^2 u = 0$ (similarly for v). □

Definition 1.8. Let u be a harmonic function. Then we say that another harmonic function v is the **harmonic conjugate** of u if the complex function $f(z) = u + iv$ is analytic.

1.5 Integrals over paths in $\mathbb{C}$

Let's introduce the notion of an integral over a path in the complex plane.

Definition 1.9. Given a smooth curve $\gamma \in \mathbb{C}$ and a continuous function f defined on $\gamma = \{z = z(t) : t \in [a, b]\}$ $(a, b \in \mathbb{R})$, we define the integral of f over γ as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

The integral has the following basic properties.

1. If $\tilde{\gamma}$ is the same curve as γ, just with opposite orientation, then

$$\int_\gamma f(z)dz = -\int_{\tilde{\gamma}} f(z)dz.$$

2. Let the end point of γ_1 be the starting point of γ_2. Then

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

We can also deduce the useful **ML-Inequality**:

Theorem 1.10 (ML-Inequality)**.** *For a piecewise differentiable curve γ in the complex plane and a complex function $f(z)$, we have*

$$\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma}\{|f(z)|\} \times \textit{length}(\gamma). \tag{1.3}$$

Proof. Let $z(t)$ be a parameterisation of γ with $t \in [a, b]$. Then

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq \max_{t\in[a,b]}\{|f(z(t))|\} \int_a^b |z'(t)|dt \\ &\leq \max_{z\in\gamma}\{|f(z)|\} \times \text{length}(\gamma),\end{aligned}$$

where we have used the fact that $\int_\gamma dz$ = length of the curve. ☐

Throughout this course, we will consider integrals on complex arcs, and we will let the radii $R \to \infty$. If the above ML-inequality is applied to such an integral, then we must address length$(\gamma) \to \infty$. The following result allows us to on the ML-inequality in this case, by yielding a result which does not contain length$(\gamma) \to \infty$.

Theorem 1.11 (Jordan's Lemma)**.** *For a function of the form $f(z) = e^{iaz}g(z)$, defined on a contour $\gamma_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, such that $g(z) \leq M_R$ for $z \in \gamma_R$, we have*

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \frac{\pi}{a} M_R.$$

Proof. Converting to polar coordinates by changing variable $z = Re^{i\theta}$:

$$\int_{\gamma_R} f(z)dz = \int_0^{\pi} g(Re^{i\theta})e^{iaR(\cos\theta + i\sin\theta)}iRe^{i\theta}d\theta = iR\int_0^{\pi} g(Re^{i\theta})e^{aRi\cos\theta - aR\sin\theta}e^{i\theta}d\theta,$$

bounding, by taking the absolute value inside of the integral:

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \int_{\gamma_R} |f(z)|\, dz = R\int_0^{\pi} \left|g(Re^{i\theta})\right| e^{-aR\sin\theta}d\theta \leq RM_R\int_0^{\pi} e^{-aR\sin\theta}d\theta.$$

Now we can use the symmetry of sin about $\pi/2$ to write

$$\left|\int_{\gamma_R} f(z)dz\right| \leq 2RM_R\int_0^{\pi/2} e^{-aR\sin\theta}d\theta,$$

and the concavity of $\sin\vartheta \geq 2\vartheta/\pi$ for $\vartheta \in [0, \pi/2]$, to write

$$\left|\int_{\gamma_R} f(z)dz\right| \leq 2RM_R\int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta = 2RM_R\frac{\pi(1-e^{aR})}{2aR} = M_R\frac{\pi(1-e^{aR})}{a} \leq \frac{\pi}{a}M_R,$$

as claimed. □

Example 1.12. *Consider the integral*

$$\int_{\gamma_R} f(z)dz$$

for $\gamma_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, and $f(z) = g(z)e^{iz}$, where $g(z) \leq C|z|^{-1/2}$ in the upper-half complex plane, for some C.

If we apply the ML principle, we get

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \max_{z\in\gamma_R}\{|f(z)|\} \times \text{length}(\gamma_R) = CR^{-1/2} \times \pi R = C\pi R^{1/2} \to \infty \quad \text{as } R \to \infty,$$

which is not particularly instructive - this essentially tells us that the limiting contour integral tends to a number which is at most infinity.

If we apply Jordan's lemma instead, we get

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \pi CR^{-1/2} \to 0 \quad \text{as } R \to \infty,$$

which is much more informative - we now know that the limiting contour integral tends to zero.

To summarise: the ML principle makes intuitive sense, and as we will see, is very useful. But in situations where Jordan's Lemma can be applied, it may give a stronger result.

1.6 Cauchy's Theorem and Cauchy's Integral Formula

Analytic functions have many wonderful, or even magical properties. For example, let $\mathcal{D}$ be a simply connected open region in $\mathbb{C}$, as in figure 4.

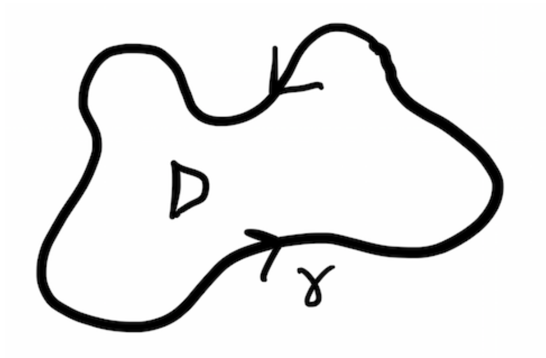

Figure 4: A simply connected region $\mathcal{D}$, bounded by a contour γ.

Then the following is true:

Theorem 1.13 (Cauchy's Theorem)**.** *If γ bounds a simply connected region $\mathcal{D}$, and f is analytic inside $\mathcal{D}$ and on γ, then*

$$\oint_\gamma f(z)dz = 0. \tag{1.4}$$

This theorem implies the following corollary: the path independence of integrals of analytic functions.
Corollary. *If $f(z)$ is analytic in a simply-connected domain $\mathcal{D}$, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

*for any curves γ_1 and γ_2 in $\mathcal{D}$ which have the same end points (i.e. the contour integral of $f(z)$ is **independent** of path in $\mathcal{D}$).*

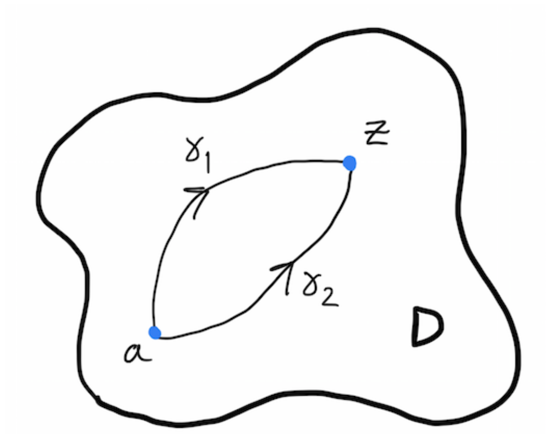

Figure 5: The integral of an analytic function in the simply-connected domain $\mathcal{D}$ can depend only on the end points of the path of integration, and does not depend on the path itself.

Proof. To prove this, consider the curve γ obtained by the concatenation of the paths γ_1 and γ_2 reversed ($\tilde{\gamma}_2$). We have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\tilde{\gamma}_2} f(z)dz \\ &= \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz.\end{aligned}$$

The path γ is closed, so $\int_\gamma f(z)dz = 0$ by Cauchy's theorem, hence

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

as claimed. $\square$

We can also deduce the so called **deformation theorem**.

Theorem 1.14 (Deformation Theorem)**.** *If $f(z)$ is analytic in a region D bounded by γ_1 and γ_2, with γ_2 lying completely inside γ_1, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

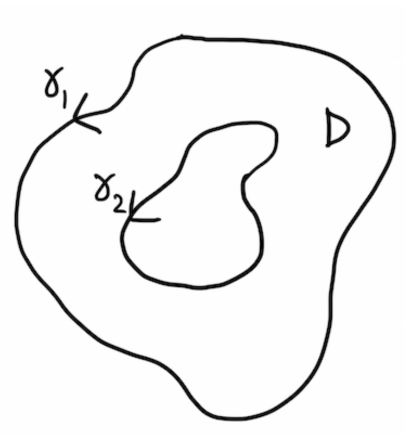

Figure 6: Curve γ_2 lying within curve γ_1.

Proof. To see this, we take a cut (shown by blue in figure 7) connecting the outer boundary γ_1 and the inner boundary γ_2.

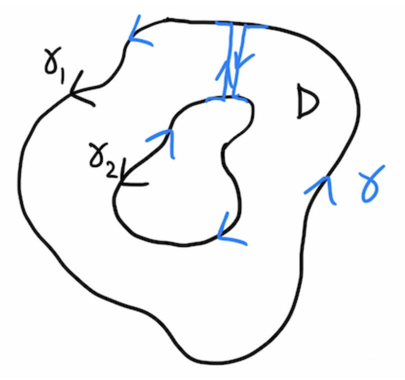

Figure 7: The integral over the outer boundary equals to the integral over the inner boundary.

The closed path starting with the end point of the blue cut, then going along γ_2 in reverse, then along the blue cut towards γ_1, then along γ_1, then back along the blue cut, bounds a simply-connected domain (the region D minus the blue cut). Therefore, the integral of $f(z)$ over this path is zero, according to Cauchy's Theorem. Since the blue cut is traversed twice in opposite directions, the contributions of the "blue parts" of the path cancel each other, which gives us that the integral over γ_1 plus the integral over $\tilde{\gamma}_2$ equals zero, which gives the result of the theorem. $\square$

Next comes one of the most central theorems to the topic of complex analysis.

Theorem 1.15 (Cauchy's Integral Formula)**.** *Let f be analytic inside and on a closed path γ bounding a simply-connected region $\mathcal{D}$. Then, at any point z interior to γ*

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi. \tag{1.5}$$

Proof. Since $f(\xi)$ is analytic for $\xi \in \mathcal{D}$, it follows that $\frac{f(\xi)}{\xi - z}$ is analytic in $\mathcal{D} \backslash \{z\}$. Therefore, by the Deformation theorem:

$$\oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \oint_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z} d\xi,$$

where γ_ε is a circle of small radius ε with centre at the point z, see figure 8.

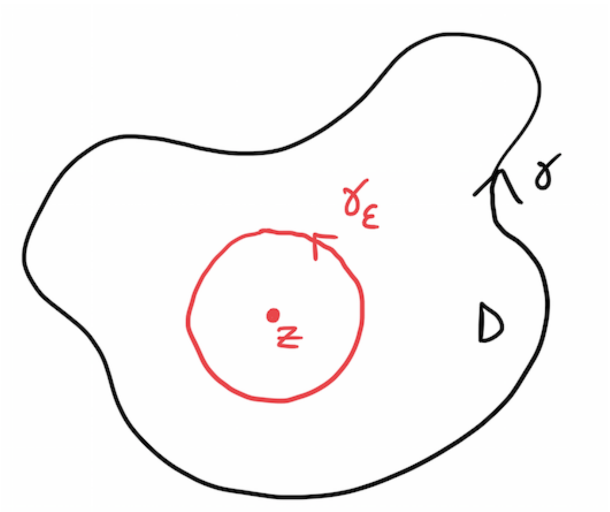

Figure 8: The integral over γ equals to the integral over a small circle around z.

This circle is given by equation $\xi = z + \varepsilon e^{i\theta}$, where $\theta \in [0, 2\pi]$. After making this substitution into the integral, we obtain

$$\oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \oint_{\gamma_\varepsilon} \frac{f(\xi)}{\xi - z} d\xi = \int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = i \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta = i \int_0^{2\pi} (f(z) + O(\varepsilon)) d\theta.$$

Now, taking the limit $\varepsilon \to 0$, we obtain

$$\oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z),$$

which gives the required result upon re-arrangement. $\square$

Note that this is indeed a magical formula: the values of the analytic function f at any point inside the domain $\mathcal{D}$ are determined by the values of f on the boundary of the domain only!

Magical formulas have magical consequences: using the Cauchy Integral formula, we will also show that f has derivatives of **all orders** and the Taylor series of f converges to f. Nothing similar happens for real functions of a real variable, they may have a first derivative, but not second derivative, or first and second derivatives, but no third derivative, etc; or derivatives of all orders may exist, but the Taylor series may diverge, or converge to a wrong function - all these complications disappear for analytic functions of complex variables.

For example, consider the bump function for $x \in \mathbb{R}^n$:

$$f(x) = \begin{cases} e^{-1/(|x|^2-1)}, & |x| < 1, \\ 0, \quad |x| \geq 1. \end{cases}$$

This is compactly supported, i.e. zero, outside of the unit ball, despite all derivatives existing on the circumference of the ball. No amount of data outside of the ball can tell us about the behaviour inside of the ball, as the data outside would equally suggest $f \equiv 0$ everywhere.
Let us now show that an analytic function $f(z)$ has derivatives of all orders and that these are also analytic.

Theorem 1.16. *Let $f(z)$ be analytic inside and on a closed path γ bounding a simply-connected region D. Then for any z within D:*

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi. \tag{1.6}$$

Proof. We prove this via induction on n. The case $n = 0$ is the Cauchy Integral formula. When $n = k$, let $h \in \mathbb{C}$ such that $z + h$ lies inside γ. Then let

$$I = \frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} = \frac{(k-1)!}{2\pi i} \oint_\gamma \frac{1}{h} f(\xi) \left[\frac{1}{(\xi - (z+h))^k} - \frac{1}{(\xi - z)^k} \right] d\xi.$$

Now on the right hand side apply the formula:

$$A^k - B^k = (A - B)(A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1}), \quad \text{with:} \quad A = \frac{1}{(\xi - (z+h))}, \quad B = \frac{1}{\xi - z},$$

and

$$A - B = \frac{(\xi - z) - (\xi - (z+h))}{(\xi - (z+h))(\xi - z)} = \frac{h}{(\xi - (z+h))(\xi - z)},$$

giving

$$I = \frac{(k-1)!}{2\pi i} \oint_\gamma f(\xi) \left[\frac{1}{(\xi - (z+h))(\xi - z)} [A^{k-1} + \cdots + B^{k-1}] \right] d\xi.$$

Finally take the limit $h \to 0$ of both sides:

$$\begin{aligned} \lim_{h \to 0} I = \frac{d^k}{dz^k} f(z) &\longrightarrow \frac{(k-1)!}{2\pi i} \oint_\gamma f(\xi) \left[\frac{1}{(\xi - z)^2} \right] \left(\frac{k}{(\xi - z)^{k-1}} \right) d\xi \\ &\longrightarrow \frac{k!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi. \end{aligned}$$

□

1.7 Estimates on the derivatives of analytic functions and Liouville's Theorem

Let us now derive some important results using the derivatives of the Cauchy Integral formula (1.6). In particular we will be interested in Liouville's theorem. Letting $\gamma = \{|\xi - z| = r\}$ in the Cauchy Integral formula (1.5) and its derivatives (1.6), we get $\xi = z + re^{i\theta}$ and $d\xi = ire^{i\theta}d\theta$, which gives

$$\begin{aligned} |f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi\right| \\ &\leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{|f(\xi)|}{r^{n+1}}rd\theta \\ &\leq \frac{n!}{2\pi}\max_{\xi\in\gamma}(f(\xi)) \times \frac{2\pi}{r^n} \\ &= n!\frac{M}{r^n}, \end{aligned} \tag{1.7}$$

where M is the maximal value of $|f(z)|$ on γ. This provides estimates for the absolute value of the function $f(z)$ (the case $n = 0$) and its derivatives ($n \geq 1$).

Theorem 1.17 (Maximum Modulus Principle)**.** *The absolute value of an analytic function takes its maximum on the boundary of the analyticity domain.*

We do not give a proof of the maximum modulus principle.

Theorem 1.18 (Liouville's theorem)**.** *If an entire function $f(z)$ is bounded everywhere in the complex plane, including at ∞, it is a constant.*

Proof. Entire functions are holomorphic everywhere, by definition, therefore we have a Taylor expansion about zero which converges for all $z \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = f^{(n)}(0)/n!$. Combining this definition of the Taylor coefficient with (1.7), we have $|a_n| \leq \frac{M}{r^n}$, where r represents the radius around which the Cauchy Integral is taken. The maximal value on the Cauchy Integral M may depend on r, but we have assumed that it is bounded above, so we will denote this upper bound by $\tilde{M}$, thus we can write $|a_n| \leq \frac{\tilde{M}}{r^n}$ where $\tilde{M}$ does not depend on r. As f is analytic everywhere, we can take this radius to be as large as we like,

$$|a_n| \leq \lim_{r\to\infty}\frac{\tilde{M}}{r^n} = 0, \quad \text{for } n \geq 1.$$

Thus $f \equiv a_n$, completing the proof. □

This can be extended by allowing for algebraic growth at infinity. But we will not need this at this stage. From Liouville's theorem we can also get the Fundamental theorem of Algebra.

Theorem 1.19 (The Fundamental Theorem of Algebra)**.** *Every non-constant polynomial has a root in the complex plane.*

Proof. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$, where $a_n \neq 0$. Indeed, suppose $p(z)$ has no zeros, then the function $\frac{1}{p(z)}$ would be analytic everywhere. Now

$$\left|\frac{1}{p(z)}\right| = \frac{1}{|z^n|}\left(\frac{1}{a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}}\right).$$

Then as $|z| \to \infty$, $|1/p(z)| \to 0$, so $1/p(z)$ is bounded. Therefore by Liouville's Theorem, $1/p(z) =$ constant, but this contradicts our assumption that $p(z)$ was a non-constant polynomial. $\square$

1.8 Taylor Series

Definition 1.20. Suppose $f(z)$ is analytic in $|z - z_0| \leq R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

for $|z - z_0| < R$, is called the **Taylor series** of $f(z)$ about z_0.

The existence of derivatives of all orders allows one to write the Taylor expansion of an analytic function $f(z)$ at any point z_0 of the analyticity domain. We know that for real functions of real variables the Taylor series can diverge, however it is not the case for functions of complex variables.

Theorem 1.21. *The Taylor series* $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ *of an analytic function $f(z)$ at any point z_0 converges to $f(z)$ for all z such that $|z - z_0| < R$, where the radius of convergence R equals the distance from z_0 to the nearest non-analytic point of f.*

Proof. Let $\gamma = \{\xi : |\xi - z_0| = R\}$. Then, using the Cauchy Integral formula (see (1.5)) we have:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0)}\left(\frac{1}{1 - (\frac{z-z_0}{\xi-z_0})}\right) d\xi \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{(\xi - z_0)}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}}\right)(z - z_0)^n \\
&= f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots,
\end{aligned}$$

where we used formula (1.6) for the derivatives of $f(z)$. As we see, the Taylor series converges to f if $|z - z_0| < |\xi - z_0|$ for all $\xi \in \gamma$.
The path γ is taken to be as close as we want to the boundary of the analyticity domain of f, hence R cannot be smaller than the distance from z_0 to the nearest singularity. Of course, R cannot also be larger than this, since the sum of a convergent power series is analytic (non-singular) everywhere within the radius of convergence. Thus R equals the distance from z_0 to the closest singularity of f. $\square$

Not only does Taylor series converge inside of the unit disk, it converges exponentially fast inside any smaller disk also centred at z_0, of radius $r < R$. To see why, consider truncating after the first $N-1$ terms:

$$\begin{aligned}\left| f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \right| &= \left| \sum_{n=N}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \right|, \\ &\leq \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \right|.\end{aligned}$$

We approximating inside a smaller disk, thus $|z-z_0| < r$, and we have from (1.7) that $\frac{f^{(n)}(z_0)}{n!} \leq M/R^n$, thus

$$\begin{aligned}\left| f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \right| &\leq \sum_{n=N}^{\infty} \frac{M}{R^n} r^n = M \sum_{n=N}^{\infty} \left(\frac{r}{R}\right)^n = M \left(\frac{r}{R}\right)^N \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \\ &= \frac{M\left(\frac{r}{R}\right)^N}{1-\frac{r}{R}}.\end{aligned}$$

If we want to write this in terms of the exponential function, we can simply write

$$\left(\frac{r}{R}\right)^N = \exp\left(N \log\left(\frac{r}{R}\right)\right) = \exp\left(-\alpha N\right)$$

where $\alpha = -\log\left(\frac{r}{R}\right) > 0$.

1.9 Laurent Series

There is another type of series expansion that is widely used when dealing with complex functions, that is a Laurent series.

Definition 1.22. Suppose $f(z)$ is analytic in the annular region $r < |z-z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

Theorem 1.23 (Laurent expansion Theorem)**.** *Let $f(z)$ be analytic in the annular region $D = \{z : r < |z-z_0| < R\}$. Then $f(z)$ can be expressed in the form of a Laurent series:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n; \quad \textit{where} \quad a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \tag{1.8}$$

where γ is a closed curve in D that contains z_0 in its interior.

Proof. For simplicity of writing we set $z_0 = 0$ here. Then we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\text{blue}} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \oint_{\text{red}} \frac{f(\xi)}{\xi - z} d\xi \\ &= \underbrace{\frac{1}{2\pi i} \oint_{|\xi|=R'} \frac{f(\xi)}{\xi - z} d\xi}_{=I_1} + \underbrace{\frac{-1}{2\pi i} \oint_{|\xi|=r'} \frac{f(\xi)}{\xi - z} d\xi}_{=I_2},\end{aligned}$$

where the blue and red paths and the radii r' and R' are as shown in figure 9. Note that the integral over the red contour is simply a closed contour of a function analytic in the interior, and thus evalutes to zero. In the second line we group the red and blue contours together into two circle, noting that the non-circular parts of the red and blue contours will cancel.

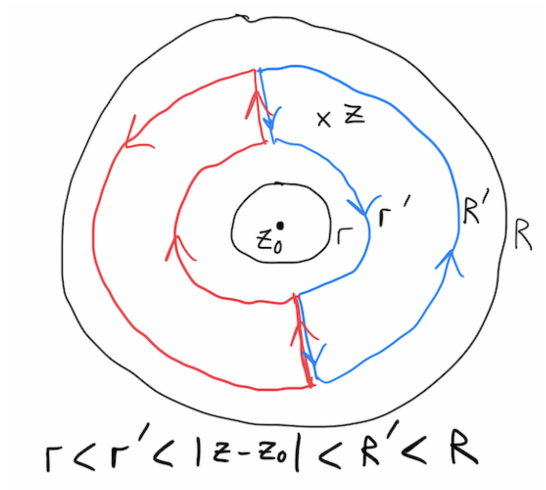

Figure 9: Diagram of the annular region $r < |z - z_0| < R$ and the blue and red contours.

Then

$$I_1 = \frac{1}{2\pi i}\oint_{|\xi|=R'} \frac{1}{\xi}\frac{f(\xi)}{1 - z/\xi} d\xi = \frac{1}{2\pi i}\oint_{|\xi|=R'} \sum_{n=0}^{\infty} \frac{f(\xi)}{\xi}\left(\frac{z}{\xi}\right)^n d\xi = \frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n \oint_{|\xi|=R'} \frac{f(\xi)}{\xi^{n+1}} d\xi,$$

and

$$\begin{aligned} I_2 &= \frac{-1}{2\pi i}\oint_{|\xi|=r'} \frac{f(\xi)}{z(\xi/z - 1)} d\xi = \frac{1}{2\pi i}\oint_{|\xi|=r'} \frac{f(\xi)}{z(1 - \xi/z)} d\xi = \frac{1}{2\pi i}\oint_{|\xi|=r'} \frac{f(\xi)}{z}\sum_{n=0}^{\infty}\left(\frac{\xi}{z}\right)^n d\xi \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \oint_{|\xi|=r'} f(\xi)\xi^n d\xi = \frac{1}{2\pi i}\sum_{k=-\infty}^{-1} z^k \oint_{|\xi|=r'} \frac{f(\xi)}{\xi^{k+1}} d\xi, \end{aligned}$$

where we have used the substitution $n + 1 = -k$ in the last line. The sum of I_1 and I_2 is equal to $f(z)$ and gives the Laurent series expansion. □

The Taylor series is valid in a circle who's radius R grows outwards from an analytic point z_0 and is valid until the radius (of convergence) R reaches the first singularity/point of non-analyticity of f. The Laurent series coincides with the Taylor series if an analytic point z_0 is chosen, however the Laurent series's power is that it works when z_0 is not analytic - thus it gives us a series representation of our function f about points of non-analyticity. We'll talk about this more soon.

1.10 Zeros and Singularities of Complex Functions

Let's talk briefly about zeros and singularities of complex functions.

Definition 1.24. We say that a function $f(z)$ has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots, m - 1$ and $f^{(m)}(z_0) \neq 0$.

Theorem 1.25. *A function $f(z)$ has a zero of order m if and only if it can be written in the form $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.*

Proof. Assume $f(z)$ has a zero of order m. Then Taylor expanding we have

$$\begin{aligned} f(z) &= 0 + 0 + \cdots + 0 + \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \cdots \\ &= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \cdots \right] \\ &= (z - z_0)^m g(z). \end{aligned}$$

Conversely, if $f(z) = (z - z_0)^m g(z)$, with $g(z_0) \neq 0$, then clearly $f^{(k)}(z_0) = 0$ if $k < m$ and $f^{(m)}(z_0) = m! g(z_0) \neq 0$. □

Definition 1.26. A point z_0 is called a **singularity** of a complex function $f(z)$ if $f(z)$ is not analytic at z_0 but every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.

Definition 1.27. A singularity z_0 of $f(z)$ is said to be **isolated** if there exists a neighbourhood of z_0 at which z_0 is the only singularity of $f(z)$.

Definition 1.28. Suppose an analytic function $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < R$, gives its Laurent series representation about z_0. Then:

- If $a_n = 0$ for all $n < 0$, then z_0 is called a **removable** singularity.
- If $a_n = 0$ for all $n < -m$, where m is a fixed positive integer, but $a_{-m} \neq 0$, then z_0 is called a **pole of order** m.
- If $a_n \neq 0$ for infinitely many negative n, then z_0 is an **essential** singularity.

For a large number of calculations within this course, we will be interested in isolated singularities, in particular, poles, as this is where we can apply residue theory, which we will cover shortly.
An example of a non-isolated singularity would be a point which is a limit point of a sequence of other singularities of a function. For instance $f(z) = 1/\sin(1/z)$ has singularities at the points $z_n = 1/\pi n$ and the sequence $\{z_n\}$ of singular points has a limit point at $z = 0$.
Examples: $\frac{\sin z}{z}$, $\frac{1}{z^3(z+2)^2}$ and $e^{1/z}$ are an example of each type of singularity respectively.

Theorem 1.29. *A function $f(z)$ has a pole of order m at z_0 if and only if*

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Proof. First let $f(z) = \frac{g(z)}{(z-z_0)^m}$. If $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$, then $g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$. Substituting this into our expression for $f(z)$ we find

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \cdots ,$$

in other words, $f(z)$ has a pole of order m. On the other hand, if $f(z)$ has a pole of order m at z_0, then

$$\begin{aligned}
f(z) &= \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots + a_0 + a_1(z-z_0) + \cdots \\
&= \frac{1}{(z-z_0)^m}\left[a_{-m} + a_{-(m-1)}(z-z_0) + \cdots\right] \\
&= \frac{g(z)}{(z-z_0)^m},
\end{aligned}$$

where $g(z)$ has the required properties. □

Before collecting relevant results from residue theory, it is worth noting that there is a class of singularity which we have not yet covered: branch points. These result in non-isolated singularities which require additional care, and will occur frequently in the course. Branch points are discussed in §1.16.

1.11 Residue Theory

Let's consider $f(z)$ represented by its Laurent series about z_0, that is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

where $0 < |z-z_0| < R$. The coefficient a_{-1} in this expansion is special and has a name given to it.

Definition 1.30. The coefficient a_{-1} in the Laurent series expansion is called the **residue** of $f(z)$ at z_0. We use the notation

$$a_{-1} = \mathrm{Res}(f, z_0).$$

Examples:

1. for $f = \frac{3}{z}$, we have $\mathrm{Res}(f, 0) = 3$;
2. for $f = \frac{1}{z^2}$, we have $\mathrm{Res}(f, 0) = 0$;
3. for $f = \cos(\frac{1}{z}) = 1 - \frac{1}{2z^2} + \cdots$, we have $\mathrm{Res}(f, 0) = 0$;
4. for $f = \sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{6z^3} + \cdots$, we have $\mathrm{Res}(f, 0) = 1$.

Computing Residues of poles

When the function $f(z)$ contains a pole of order m at z_0 there are some techniques we can use to find the residue of f at z_0. Suppose

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + \cdots,$$

so that $f(z)$ has a pole of order m at z_0. Then:

- If $m = 1$: $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z)$,
- If $m = 2$: $\mathrm{Res}(f, z_0) = \lim_{z\to z_0}\frac{d}{dz}[(z-z_0)^2 f(z)]$,

- For a pole of order m: $\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$.

There are a few more useful tricks to know: for instance, suppose $f(z)$ is of the form

$$f(z) = \frac{A(z)}{(z-z_0)^m},$$

where $A(z)$ is analytic at $z = z_0$ (and that $A(z_0) \neq 0$), then

$$\text{Res}(f, z_0) = \frac{A^{(m-1)}(z_0)}{(m-1)!}. \tag{1.9}$$

Alternatively, if $f(z)$ contains a simple pole (pole of order $m = 1$) and $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0 and B has a simple zero at z_0 ($m = 1$), with $A(z_0) \neq 0$, then (**exercise:** derive this!)

$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}. \tag{1.10}$$

Example 1.31. *Let $f(z) = \frac{1}{z^4+1}$. Then the poles of f are the points such that $z^4 + 1 = 0$, which is equivalent to $z = e^{\frac{i\pi}{4} + \frac{i\pi k}{2}} = \pm\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$. By (1.10), the residue of f at $z = z_j$ is:*

$$Res(f, z_j) = \frac{1}{4z_j^3} = \frac{z_j}{4z_j^4} = \frac{-z_j}{4}$$

(we use that $z_j^4 = -1$ here).

The next theorems explain why we are interested in the residues of a function.

Theorem 1.32. *Let γ be a closed curve that contains z_0 and lies within $0 < |z - z_0| < R$ (R is the radius of convergence), then*

$$Res(f, z_0) = \frac{1}{2\pi i} \oint_\gamma f(z)dz.$$

Proof. Let $0 < r < R$ and $\gamma_r = \{z : |z - z_0| = r\}$. Then using the deformation theorem

$$\frac{1}{2\pi i} \oint_\gamma f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_r} f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_r} \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n dz,$$

upon substituting the Laurent series for $f(z)$. Now if we set $z - z_0 = re^{i\theta}$, then

$$\frac{1}{2\pi i} \oint_{\gamma_r} \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n dz = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{in\theta} ire^{i\theta} d\theta = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{n+1} a_n \int_0^{2\pi} e^{i(n+1)\theta} d\theta.$$

But

$$\int_0^{2\pi} e^{i(n+1)\theta} d\theta = \begin{cases} \left[\frac{1}{i(n+1)} e^{i(n+1)\theta}\right]_0^{2\pi} = 0, & \text{if } n \neq -1 \\ 2\pi, & \text{if } n = -1. \end{cases}$$

So putting everything together

$$\frac{1}{2\pi i} \oint_\gamma f(z)dz = \frac{1}{2\pi} a_{-1}(2\pi) = a_{-1} = \text{Res}(f, z_0).$$

$\square$

Theorem 1.33 (Residue Theorem)**.** *Let $f(z)$ be a single-valued analytic function inside a domain $\mathcal{D}$ bounded by a closed path γ except at the isolated non-branching singularities (either essential singularities, or poles) $z_1, z_2, \dots z_n$ which are lying inside γ. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f, z_j). \tag{1.11}$$

Proof. Figure 10 shows a domain $\mathcal{D}$ and z_j are the isolated singularities of f.

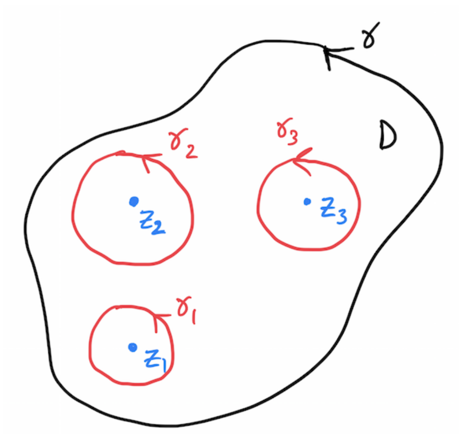

Figure 10: A domain $\mathcal{D}$, bounded by γ, with $\{z_j\}$ as singularities of f. The paths γ_j are small circles of radius r_j around z_j.

Consider a small circle around z_j, denoted by $\gamma_j = \{z : |z - z_j| = r_j\}$, where r_j is chosen sufficiently small so that γ_j lies inside γ and γ_j contains only z_j and no other singularities of $f(z)$. Then, using the deformation theorem

$$\begin{aligned}\oint_\gamma f(z)dz &= \sum_{j=1}^{n} \oint_{\gamma_j} f(z)dz \\ &= \sum_{j=1}^{n} (2\pi i \mathrm{Res}(f, z_j)) \\ &= 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j),\end{aligned}$$

where we have used the previous theorem 1.32. □

Example 1.34. *Let $f(z) = \frac{1}{1+z^4}$. We already know that it has four poles: $z_j = \pm\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$ and the corresponding residues are*

$$Res(f, z_j) = \frac{-z_j}{4}.$$

We want to compute the contour integral of f over different paths, for example, γ_1 and γ_2 as shown in figure 11; the points z_1 and z_4 are inside γ_1, and z_3 and z_4 are inside γ_2.

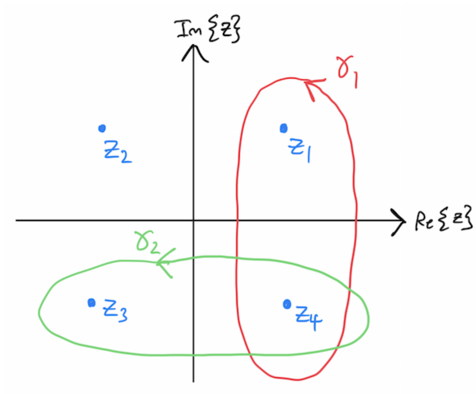

Figure 11: The poles $z_1, ..., z_4$ of f, and two paths γ_1 and γ_2.

We have:

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_1} \frac{1}{1+z^4}dz = 2\pi i\,[Res(f, z_1) + Res(f, z_4)] = -\frac{2\pi i}{4}(z_1 + z_4) = -\pi i\frac{\sqrt{2}}{2},$$

and

$$\oint_{\gamma_2} f(z)dz = \oint_{\gamma_2} \frac{1}{1+z^4}dz = 2\pi i\,[Res(f, z_3) + Res(f, z_4)] = -\frac{2\pi i}{4}(z_3 + z_4) = \frac{-\pi\sqrt{2}}{2}.$$

1.12 Analytic Continuation

We saw earlier that the values of an analytic function at any point z inside the analyticity domain are completely determined by its values on the boundary of the domain only (by the Cauchy Integral formula). Well, the somewhat opposite is also true: information about the local behaviour of an analytic function near any point inside the analyticity domain completely determines the function globally, i.e., everywhere in the domain.

Theorem 1.35 (Analytic Continuation). *If f and g are analytic in a connected domain $\mathcal{D}$ and $f = g$ in some common region D' (this may be a line segment/contour or a patch or even a collection of points on a convergent sequence) within $\mathcal{D}$, then $f \equiv g$ throughout $\mathcal{D}$.*

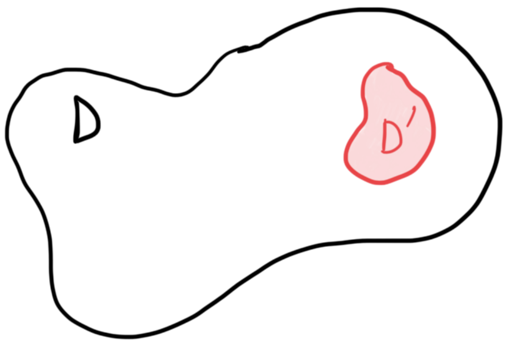

Figure 12: A region D with a patch D' inside indicated in red.

Example: Consider the function

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This is convergent for $|z| < 1$. So we have a representation for this analytic function everywhere inside the unit disc. However consider now the function

$$g(z) = \frac{1}{1-z}.$$

This function coincides with $f(z)$ inside the unit disc, and is analytic everywhere in the complex plane except at $z = 1$, hence $g(z)$ is an analytic continuation of $f(z)$ to all points outside and on the unit disc (except $z = 1$).

Consequence of this theorem

When an analytic function $f(z)$ in D' is analytically continued to a region D such that $D' \subset D$, this continuation is unique.

The main observation of the theory of complex analytic functions is that local data of an analytic function in a neighbourhood of any given point contains all the information about this function anywhere else in the complex plane, however far away from the initial point. In many situations this allows one to replace computations performed in one part of the complex plane by simpler computations at a different part. We will see examples of this approach in the following sections. The powerful idea is that whenever we have to work with a function of real variables which admit an analytic continuation to complex numbers, we continue the function to as large a domain as possible in the complex plane, and then perform computations anywhere in this domain in order to obtain (with luck) as much information as possible about the behaviour for real values of the variables.

1.13 Examples of Contour Integrals

Throughout this course we will be interested in evaluating many real integrals. A key technique to enable us to do this is to analytically continue the real integrand into the complex plane, where we will then close the contour (path of integration) and utilise the residue/Cauchy's theorem to enable us evaluate the original real integral. Let's look at some examples:

1). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

We introduce $f(z) = \frac{z^2}{z^4+1}$, and consider

$$\oint_\gamma \frac{z^2}{z^4+1} dz,$$

where $\gamma = \gamma_1 + \gamma_R$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$ and $\gamma_R = \{z : z = Re^{i\theta}, \theta \in [0, \pi]\}$ as shown in figure 13.

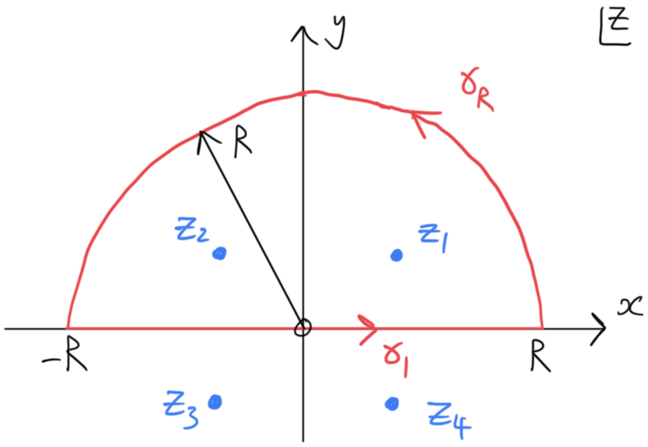

Figure 13: A semi-circular contour in the UHP encapsulating two of the four poles.

The function $f(z)$ has simple poles at $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$ and $e^{i7\pi/4}$, shown in blue in figure 13. Now, as $R \to \infty$, by the residue theorem

$$\begin{aligned}\oint_\gamma f(z)dz &= 2\pi i\left[\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2)\right]\\ &= 2\pi i\left(\left(\frac{z^2}{4z^3}\right)\bigg|_{z_1} + \left(\frac{z^2}{4z^3}\right)\bigg|_{z_2}\right)\\ &= \frac{\pi i}{2}\left[\frac{1}{z_1} + \frac{1}{z_2}\right]\\ &= \frac{\pi i}{2}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right)\\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Now consider integrals around the separate components of γ as $R \to \infty$.

On γ_1:

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = I.$$

On γ_R: $z = Re^{i\theta}$:

$$\int_{\gamma_R} f(z)dz = \int_0^\pi \frac{R^2e^{2i\theta}}{R^4e^{4i\theta}+1}iRe^{i\theta}d\theta = \int_0^\pi \frac{iR^3e^{3i\theta}}{R^4e^{4i\theta}+1}d\theta.$$

As $R \to \infty$:

$$\begin{aligned}\left|\int_{\gamma_R} f(z)dz\right| &\leq \max_{\theta\in\gamma}\left\{\frac{iR^3e^{3i\theta}}{R^4e^{4i\theta}+1}\right\} \times \pi\\ &\leq \frac{R^3}{R^4-1} \to 0 \text{ as } R \to \infty,\end{aligned}$$

where we have used the ML-inequality in the first line and the fact that $|R^4e^{4i\theta}+1| \geq ||R^4e^{4i\theta}|-1| \geq R^4 - 1$ by the triangle inequality to get to the second line. Hence

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0.$$

So, putting everything together gives

$$\begin{aligned}\lim_{R\to\infty}\oint_\gamma f(z)dz &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz + \lim_{R\to\infty}\int_{\gamma_1} f(z)dz\\ \Rightarrow \frac{\pi}{\sqrt{2}} &= 0 + I\\ \Rightarrow \int_{-\infty}^{\infty}\frac{x^2}{x^4+1}dx &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Remark: We could have chosen to close γ_1 with a semicircular contour γ_R in the LHP (lower-half plane) if we had wished. Instead we would've picked up the residues of f at z_3 and z_4 in the residue theorem and, since we would be integrating clockwise now, would pick up a minus sign. All in all we should recover the same answer upon doing this (**exercise:** check this!).

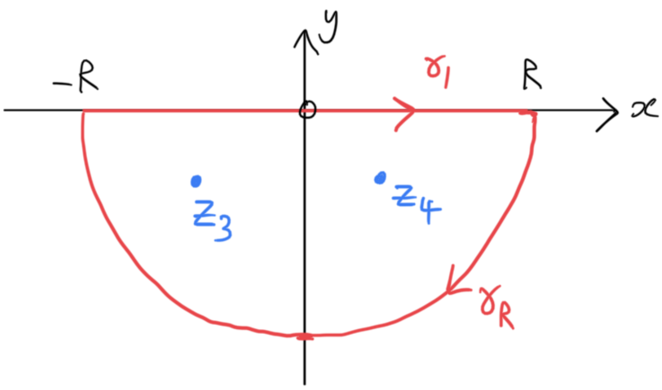

Figure 14: A semi-circular contour in the LHP.

2). Evaluate

$$I = \int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+a^2}dx, \quad a, k > 0.$$

Let $f(z) = \frac{e^{ikz}}{(z-ia)(z+ia)}$. Note we have an exponential in the integrand now - so we must take care with the sign in the exponential. Observe

$$e^{ikz} = e^{ik(x+iy)} = e^{ikx}e^{-ky}.$$

Now for $k > 0$, as was given, we need $-ky < 0$ giving $y > 0$ to have exponential decay of our integrand (and to avoid exponential growth). Thus we should choose to close our contour with γ_R as a semi-circle in the **upper-half plane**, see figure 15.

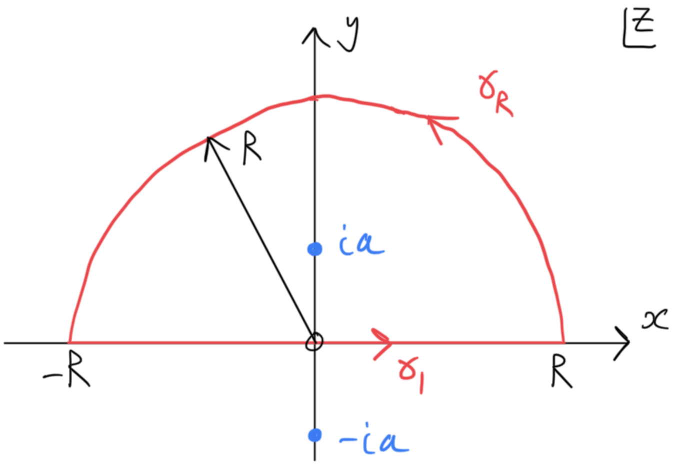

Figure 15: A semi-circular contour in the UHP containing the pole at ia.

Here $\gamma = \gamma_1 + \gamma_R$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$ and $\gamma_R = \{z : z = Re^{i\theta}, \theta \in [0, \pi]\}$. $f(z)$ has simple poles at $z = \pm ia$. By (1.10) the residue of $f(z)$ at $z = ia$ is $e^{ik(ia)}/(ia + ia) = e^{-ka}/(2ia)$. Thus by the residue theorem

$$\oint_\gamma \frac{e^{ikz}}{z^2 + a^2} dz = 2\pi i \left(\frac{e^{-ka}}{2ia} \right) = \frac{\pi}{a} e^{-ka}.$$

Now look at the integrals around the separate components of γ as $R \to \infty$. As before

$$\lim_{R \to \infty} \int_{\gamma_1} f(z) dz = I.$$

On γ_R, choosing $R > a$, we have $f(z) = g(z)e^{ikz}$, with $|g(z)| \leq 1/(R^2 - a^2), \quad z \in \gamma_R$, so we can apply Jordan's Lemma to obtain

$$\int_{\gamma_R} f(z) dz \leq \frac{\pi}{k(R^2 - a^2)} \implies \lim_{R \to \infty} \int_{\gamma_R} f(z) dz = 0.$$

Thus we have $I = \frac{\pi}{a} e^{-ka}$.

3). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2} dx, \quad k > 0.$$

Note that $\cos kx = \mathrm{Re}\{e^{ikx}\}$. Hence, using the result from the last example

$$\begin{aligned} I &= \mathrm{Re} \left\{ \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx \right\} \\ &= \frac{\pi}{a} e^{-ka}. \end{aligned}$$

4). Evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx, \quad \text{where } 0 < a < 1.$$

Let $f(z) = e^{az}/(1+e^z)$. Can we choose the contour with a semi-circle as before?

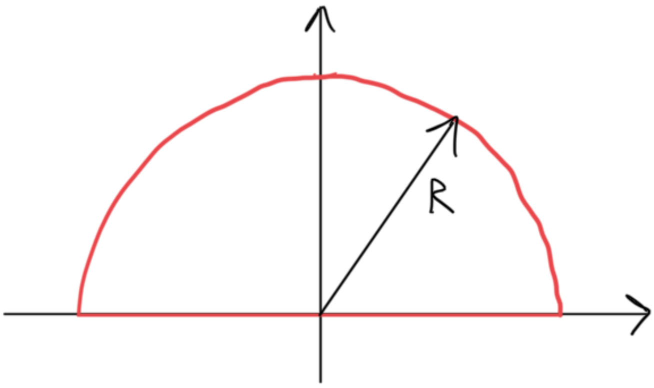

Figure 16: A semi-circular contour in the UHP.

On such a semi-circle $z = Re^{i\theta}$. Then

$$f(z) = \frac{e^{aR(\cos\theta + i\sin\theta)}}{1 + e^{R(\cos\theta + i\sin\theta)}} = \frac{e^{aRi\sin\theta}e^{aR\cos\theta}}{1 + e^{R(\cos\theta + i\sin\theta)}}.$$

Bounding the denominator via the negative triangle inequality, as we have done previously,

$$|f(z)| \leq \frac{e^{aR\cos\theta}}{|1 - e^{R\cos\theta}|},$$

which is unbounded as $\theta \to 0$. Our previous approach (from the preceding three examples), of bounding the integrand along a circlular arc, has not worked here. As we will see, this isn't just an over-cautious over-estimate (which can sometimes occur), the integrand is genuinely unbounded for arbitrarily large values of R, due to infinitely many poles along $\mathrm{Re}\{\theta\} = 0$.

Instead, consider

$$\oint_\gamma f(z)dz,$$

where $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ with $\gamma_1 = \{z : z = x, x \in [-R, R]\}$, $\gamma_2 = \{z : z = R + iy, y \in [0, 2\pi]\}$, $\gamma_3 = \{z : z = x + 2\pi i, x \in [R, -R]\}$ and $\gamma_4 = \{z : z = -R + iy, y \in [2\pi, 0]\}$.

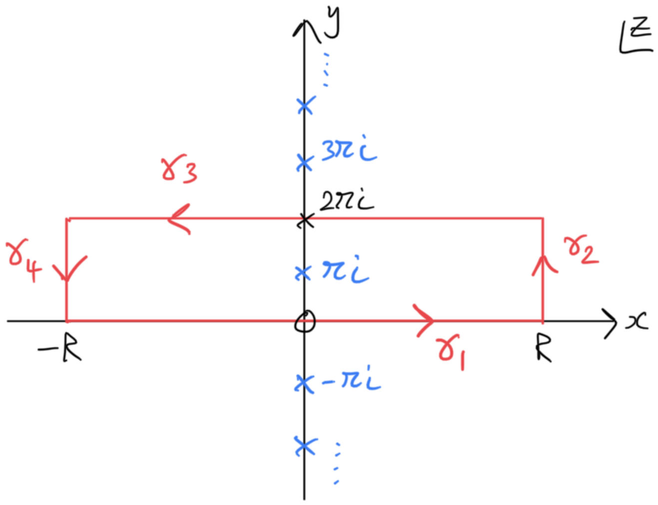

Figure 17: A rectangular contour containing one of the infinitely many poles.

Now $f(z) = e^{az}/(1+e^z)$ has singularities where $e^z = -1$. Putting $z = x + iy$ into this leads to $z = (2k+1)\pi i$, $k \in \mathbb{Z}$ (**exercise:** check this!). The singularities are shown in blue in figure 17. Only the singularity at πi is within γ. It turns out that these singularities are all simple poles - let's show this for the singularity at πi: let's look at $f(z)$ local to πi; let $z = \pi i + \varepsilon$, for small ε. Then

$$
\begin{aligned}
f(z) = f(\pi i + \varepsilon) &= \frac{e^{a(\pi i+\varepsilon)}}{1+e^{(\pi i+\varepsilon)}} \\
&= \frac{e^{\pi i a} e^{a\varepsilon}}{1-e^{\varepsilon}} \\
&= \frac{e^{\pi i a}(1+a\varepsilon+O(\varepsilon^2))}{1-(1+\varepsilon+O(\varepsilon^2))} \\
&= \frac{e^{\pi i a}(1+a\varepsilon+O(\varepsilon^2)}{-\varepsilon(1+O(\varepsilon))} \\
&= -\frac{e^{\pi i a}}{\varepsilon}(1+a\varepsilon+O(\varepsilon^2))(1+O(\varepsilon)) \\
&= -\frac{e^{\pi i a}}{\varepsilon} + O(1) \\
&= -\frac{e^{\pi i a}}{z-\pi i} + O(1),
\end{aligned}
$$

where we have used the expansions $e^x = 1 + x + x^2/2! + \cdots$ and $1/(1-z) = 1 + z + \cdots$ for $|z| < 1$ in the third and fifth lines respectively. Thus we see there is a simple pole at πi with residue $-e^{\pi i a}$. Hence by the residue theorem

$$\oint_\gamma f(z)dz = 2\pi i(-e^{\pi i a}) = -2\pi i e^{\pi i a}.$$

Consider now the integrals around separate components of γ.

On γ_2:

$$\begin{aligned}\left|\int_{\gamma_2} f(z)dz\right| &= \left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}} idy\right| \\ &\leq 2\pi \cdot \max_{y\in\gamma_2}\left\{\left|\frac{e^{aR}e^{aiy}}{1+e^Re^{iy}}\right|\right\} \\ &\leq 2\pi \cdot \frac{e^{aR}}{e^R-1} \\ &\to 0 \text{ as } R\to\infty \text{ since } 0<a<1,\end{aligned}$$

where in the second line we have used the ML-inequality and in the third line have used $|1+e^R| \geq ||e^R|-1| = e^R - 1$ by the triangle inequality.

Similarly, one can show that

$$\lim_{R\to\infty}\int_{\gamma_4} f(z)dz = 0.$$

On γ_3:

$$\int_{\gamma_3} f(z)dz = \int_{\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{(x+2\pi i)}}dx = -\int_{-\infty}^{\infty}\frac{e^{2\pi ia}e^{ax}}{1+e^x}dx = -e^{2\pi ia}I.$$

Thus, putting everything together, we have

$$\oint_\gamma f(z)dz = -2\pi i e^{\pi ia} = (1-e^{2\pi ia})I.$$

$$\begin{aligned}\Rightarrow I &= \frac{-2\pi i e^{\pi ia}}{1-e^{2\pi ia}} \times \frac{e^{-\pi ia}}{e^{-\pi ia}} \\ &= \pi\left(\frac{2i}{e^{\pi ia}-e^{-\pi ia}}\right) \\ &= \frac{\pi}{\sin(\pi a)}.\end{aligned}$$

General Method to Evaluate $I = \int_a^b f(x)dx$:

1). Add a suitable contour, γ', to $[a,b]$ to get a **closed** contour γ.

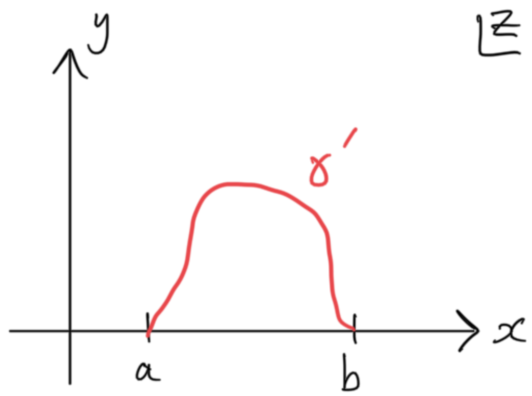

Figure 18: A contour γ' added to $[a,b]$.

2). Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$ (for example $\mathrm{Re}\{g(x)\} = f(x)$ as in example 3). $g(z)$ is sometimes referred to as the **auxiliary** function.

3). Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.

4). If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

When might these integrals arise?

Integrals over the real line, where the integrals have poles, may arise when using Fourier transforms to solve differential equations. Consider the general form of a linear ODE

$$\sum_{n=0}^{N} \frac{d^n f}{dx^n}(x) = g(x).$$

After taking Fourier transforms of both sides, we can use the standard trick of converting the derivatives to algebraic growth:

$$\sum_{n=0}^{N} (2\pi i\omega)^n \hat{f}(\omega) = \hat{g}(\omega).$$

Noting that $\hat{f}$ doesn't depend on n, we can divide through by the polynomial in ω, and take inverse Fourier transforms to obtain

$$f(x) = \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)e^{2\pi i x\omega}}{\sum_{n=0}^{N}(2\pi i\omega)^n} d\omega,$$

which is similar in many ways to the examples we've seen:

- Poles in the denominator
- The integral is over $\mathbb{R}$
- Under suitable assumptions on $\hat{g}$, we have exponential decay in the upper-half plane

1.14 Riemann sphere and analyticity at infinity

Definition 1.36. The Riemann Sphere is the compactification of $\mathbb{C}$:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Without delving on the details, we can define an open set $D \subset \bar{\mathbb{C}}$ on the Riemann sphere, where $\infty \in D$ implies that there exists an R such that $\{z : |z| > R\} \subset D$.
A function $f(z)$ defined on an open set $D \subset \bar{\mathbb{C}}$ such that $\infty \in D$ is *analytic at* ∞ if $f(z^{-1})$ is analytic at zero.

1.15 The residue at ∞

Suppose that an analytic function $f(z)$ is analytic at $z = \infty$, and further that $f(\infty) = 0$. Then we can write

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \frac{a_{-3}}{z^3} + \cdots$$

Definition 1.37. a_{-1} is called the residue of $f(z)$ at ∞. Denoted by $\text{Res}(f,\infty) = a_{-1}$.

One can show that, in this case

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\gamma_R} f(z)dz &= \lim_{R\to\infty}\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{a_{-1}}{Re^{i\theta}} + \frac{a_{-2}}{R^2e^{2i\theta}} + \cdots\right) iRe^{i\theta}d\theta \\ &= a_{-1},\end{aligned}$$

where γ_R is a circle centred at the origin of radius R on which $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$.

Example: In the previous example at the end of section 1.15, we could have computed the integral over γ_R as follows. First, expand $f(z)$ as

$$\begin{aligned}f(z) &= \frac{1}{\sqrt{z^2-1}} \\ &= \frac{1}{\sqrt{z^2}}\frac{1}{\sqrt{1-\frac{1}{z^2}}} \\ &= \frac{1}{z}\frac{1}{\sqrt{1-\frac{1}{z^2}}} \\ &= \frac{1}{z}\left(1+\frac{1}{2z^2}+O(1/z^4)\right) \\ &= \frac{1}{z}+\frac{1}{2z^3}+\cdots\end{aligned}$$

where in the third line we used the fact that $\sqrt{z^2} = +z$ by the choice of branch cut used and in the fourth line the general binomial expansion was used since on γ_R, $|z|$ is large so $1/z^2$ is small. Now we have $f(z)$ expanded in the form where we can use the above result. So we see that

$$\text{Res}(f,\infty) = 1,$$

leading to

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 2\pi i\text{Res}(f,\infty) = 2\pi i,$$

as found earlier.

1.16 Branch Cuts for Multi-valued Functions

Throughout this course we'll encounter multi-valued functions such as non-integer powers of z (e.g $z^{1/2}$, $z^{1/3}$, etc) and logarithms.

Definition 1.38. A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., analytically continuing along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

A way to deal with multi-valued analytic functions is to decompose them into single-valued '**branches**', by removing from the complex plane the so-called '**branch cuts**'.

Definition 1.39. A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

A branch cut must pass through all branching points; often it has an arc which extends to infinity. There is much freedom in the choice of branch cuts and we can choose the cut to suit our needs.

Examples

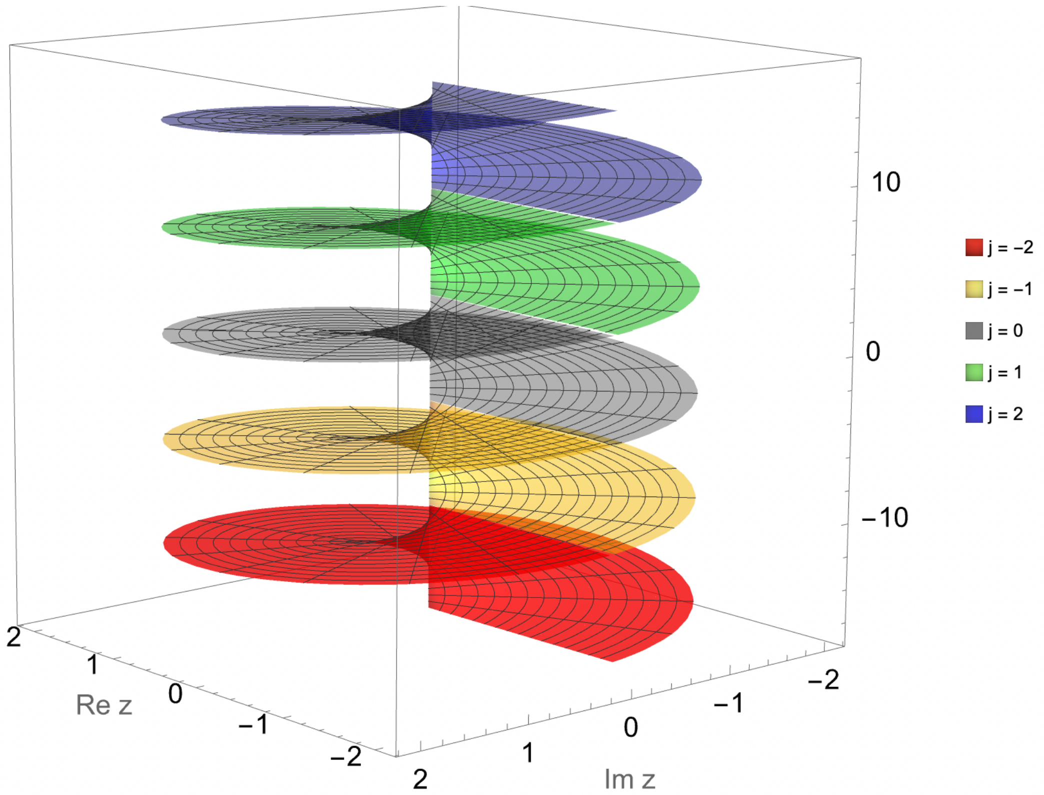

Figure 19: Complex 'sheets' of $\mathrm{Im} \log z$, on branches $j = -2, \ldots, 2$. Produced using Mathematica, code is available online.

1. Let's start off with the complex logarithm. We define

$$f(z) = \log z = \log |z| + i \arg\{z\}. \tag{1.12}$$

   This is a multi-valued function, since the argument of z is not unique. We can add or take away any integer multiple of $2\pi i$ and we have a different value for $\log z$ for the same point z.

   The logarithm has an infinite number of **branches** (one corresponding to each integer multiple of $2\pi i$ we can add to the argument). This is visualised in Figure 19. To enable us to work with this function, we need to make it single-valued. To do this, let's start by finding its branch points.

   First, clearly $z = 0$ is a branch point, since on traversing a small circuit around $z = 0$, $\arg\{z\} = \theta$ changes by $2\pi \Rightarrow f(z) = \log z = \log |z| + i\theta$ changes its value by $2\pi i$. It is clear no other finite point can be a branch point for this function, but what about complex infinity? Well as it turns out $z=\infty$ (the point at infinity or complex infinity) is a branch point. To see this, introduce $w=1/z$. Then $z=\infty$ corresponds to $w=0$, and $f(z)=\log z=-\log w$. Hence on traversing a small circuit around $w=0$, by the argument used for $z=0$ above, $\log w$ changes value and hence $\log z$ changes value.

To construct a single-valued function from $\log z$ we introduce a branch cut joining together the branch points 0 and ∞. Figure 20 shows two possibilities.

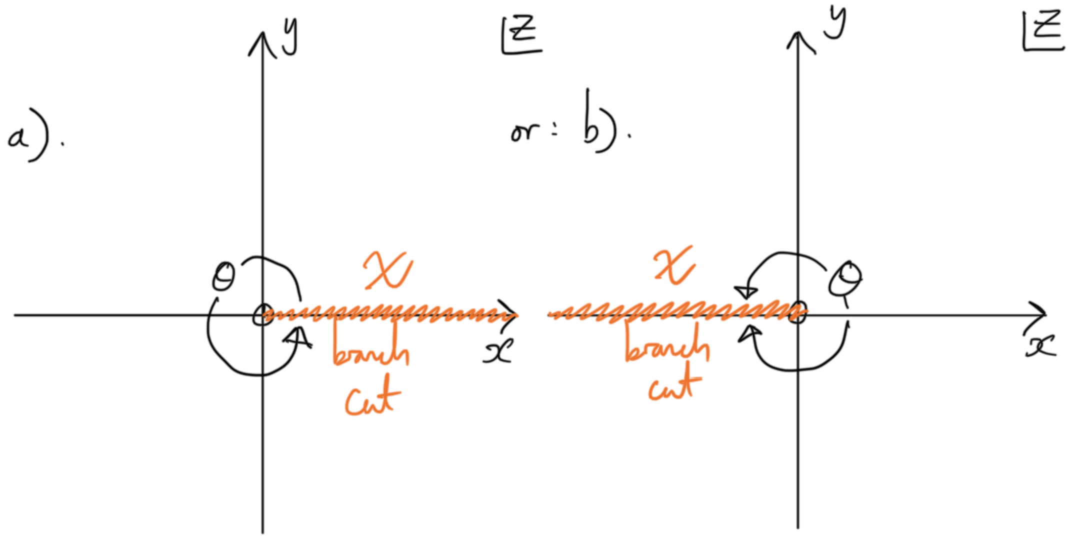

Figure 20: a). corresponds to restricting $0\leq\theta\leq 2\pi$. b). corresponds to restricting $-\pi\leq\theta\leq\pi$.

Any line joining $z=0$ to $z=\infty$ with θ appropriately restricted would work as a branch cut for this function, not just the cases a) and b) above.

One thing to note is that on either side of the branch cut, the value of $\arg\{z\}=\theta$ differ by 2π (they approach 0 and 2π in a). and they approach $-\pi$ and π in b), so the branch cut is a discontinuity for the function $f(z)$.

2. Let's look at an example of a non-integer power of z. Let $f(z)=z^{1/2}=\sqrt{z}$. If we let $z=re^{i\theta}$, then $f(z)=(re^{i\theta})^{1/2}=\sqrt{r}e^{i\theta/2}$. We can now see that this function is multi-valued since if we let $\theta\mapsto\theta+2\pi$, then $f(z)=\sqrt{r}e^{i\left(\frac{\theta+2\pi}{2}\right)}=-\sqrt{r}e^{i\theta/2}$.

   In fact we can see that this function has **two branches**, a usual way to describe each is to say one is where $f(z)\sim\sqrt{x}$ as $x\to\infty$ and the other where $f(x)\sim-\sqrt{x}$ as $x\to\infty$. There are only two branches, since a second circling of the origin takes us back to the same output as where we started (imagine adding 4π to theta in the calculation above and see what happens).

   $f(z)=\sqrt{z}$ also has branch points at $z=0$ and $z=\infty$. To see $z=0$ is a branch point, consider $z_0=\varepsilon e^{i\theta}$, where ε is small. At this point $f(z_0)=\sqrt{\varepsilon}e^{i\theta_0/2}$.

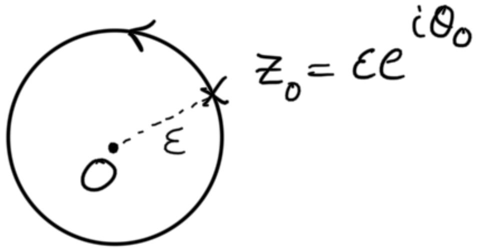

Figure 21: A small circular path around $z = 0$.

But after circling around $z = 0$ one time we have $f(z_0) = \sqrt{\varepsilon} e^{i\frac{(\theta_0+2\pi)}{2}} = -\sqrt{\varepsilon} e^{i\theta_0/2}$. i.e. $\sqrt{z_0}$ has changed its value $\Rightarrow z = 0$ is a branch point.

To show $z = \infty$ is a branch point, in a similar manner to as in example 1, let $w = 1/z$ and then consider $f(w)$ upon making a small circle around $w = 0$. We have

$$f(z) = \sqrt{z} = \sqrt{\frac{1}{w}} = \frac{1}{\sqrt{w}}$$

Letting $w = re^{i\theta}$,

$$f(z) = f\left(\frac{1}{re^{i\theta}}\right) = e^{-i\theta/2}/\sqrt{r}$$

Now replacing θ with $\theta + 2\pi$, i.e. one full rotation around $w \approx 0$, we see

$$f\left(\frac{1}{re^{i(\theta+2\pi)}}\right) = e^{-i(\theta+2\pi)/2}/\sqrt{r} = -e^{-i\theta/2}/\sqrt{r}.$$

We have changed sign after one rotation *around infinity.* This is the case for arbitrarily small r, equivalently, arbitrarily large $|z|$.

To construct a single-valued branch of this function we introduce a branch cut connecting $z = 0$ to $z = \infty$. As in example 1, figure 20, cases a) and b) would both work.

Note in any case one can check which branch of the function we have implicitly chosen (the choice of restriction on our range of angles θ implicitly chooses the branch of the function we have). It turns out both cases a) and b) from figure 20 with the restrictions on θ as given select the branch where $f(z) \sim \sqrt{x}$ as $x \to \infty$. For example in case b) for $z = x \in \mathbb{R}_{>0}$, we have $\arg\{z\} = 0$, so $f(z) = \sqrt{z}e^{i0/2} = \sqrt{z} \sim \sqrt{x}$. To choose the branch which $\sim -\sqrt{x}$ as $x \to \infty$ the easiest thing to do is to add 2π to our range for θ (you can check this is what would happen).

3. More generally than example 2, regularly throughout this course we'll encounter functions of the form

$$f(z) = \sqrt{(z - z_1)(z - z_2)} = ((z - z_1)(z - z_2))^{\frac{1}{2}}, \quad (z_1, z_2 \in \mathbb{C}).$$

To analyze the behaviour of functions like this better, we introduce local coordinates: for $j = 1, 2$ let $r_j = |z - z_j|$ and $\theta_j = \arg\{z_j\}$, then $z - z_1 = r_1 e^{i\theta_1}$ and $z - z_2 = r_2 e^{i\theta_2}$, see figure 22.

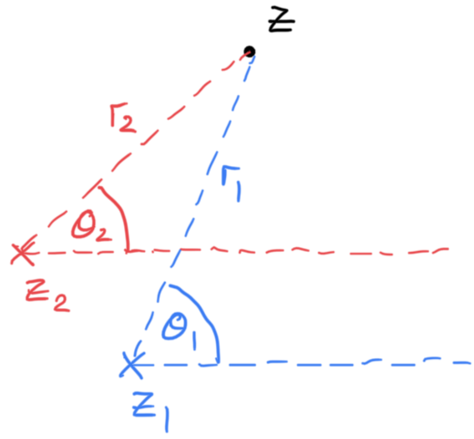

Figure 22: Schematic of the local coordinates.

Then

$$f(z) = \left[(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})\right]^{\frac{1}{2}} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{(\theta_1+\theta_2)}{2}} = (r_1 r_2)^{\frac{1}{2}} e^{i\Theta/2},$$

where $\Theta = \theta_1 + \theta_2$.

On traversing a **small** circuit around z_1 (small meaning it doesn't contain z_2), then θ_1 changes by 2π, but θ_2 remains unchanged $\Rightarrow$ Θ changes by 2π $\Rightarrow$ $f(z)$ is multiplied by -1, i.e. $f(z)$ has changed, so z_1 is a branch point. Similarly, one can show that z_2 is a branch point.

Consider any other point, say z_3, in the finite plane (not ∞). On traversing a small circuit around z_3, neither θ_1 nor θ_2 changes $\Rightarrow$ $f(z)$ doesn't change, so z_3 is **not** a branch point of $f(z)$.

Finally, consider the point at ∞. Looking at figure 23, as $R \to \infty$, the only point left **outside** of γ_R is the point at ∞.

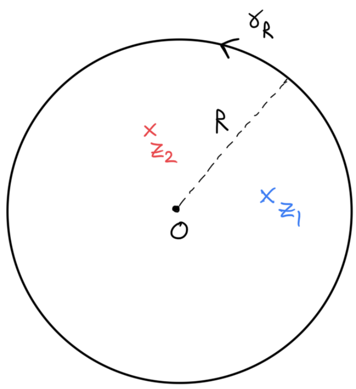

Figure 23: A circular path of radius $R \to \infty$ centered at the origin.

So we can think of moving around γ_R as moving around a small circuit about ∞. Since γ_R is large, it contains **both** z_1 and z_2, so on traversing it, both θ_1 and θ_2 change by 2π $\Rightarrow$ Θ changes by 4π, i.e.

$f(z)$ is multiplied by 1 and doesn't change, so ∞ is **not** a branch point.

We introduce a branch cut to make the function single-valued by connecting the branch points z_1 and z_2 together. There are two usual choices:

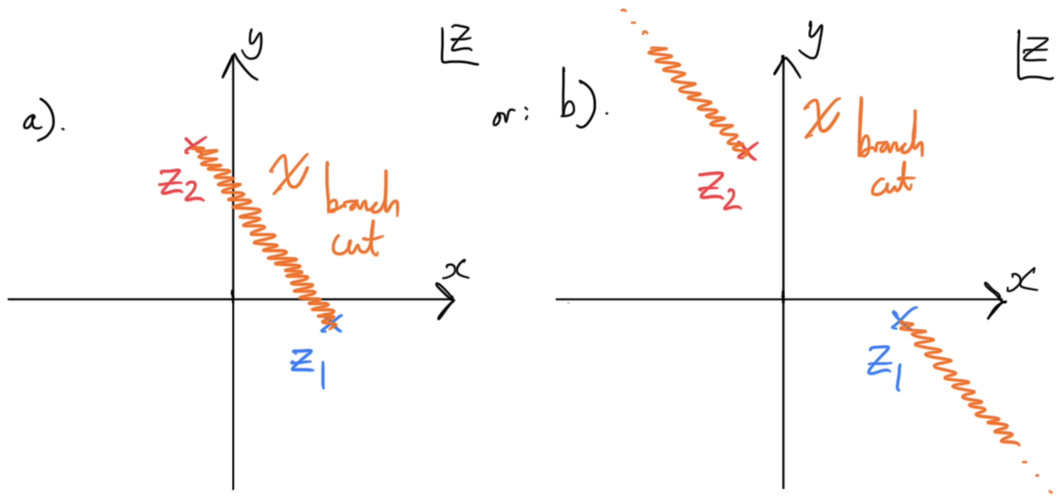

Figure 24: a). A branch cut joining z_1 and z_2 directly. b). A branch cut joining z_1 to z_2 by a straight line passing through the point at infinity.

Note that in case b) the branch cut passes through the point at ∞. Let's take a look at case a) in a little more detail. Consider the values of $f(z)$ at z_+ and z_-, two points just either side of the branch cut as shown in figure 25.

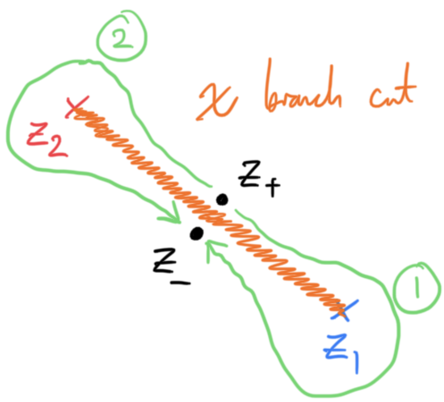

Figure 25: Two points on either side of the branch cut with two paths drawn to move from z_+ to z_-.

On moving from z_+ to z_- along the path labelled 1 in green, θ_1 changes by 2π but θ_2 stays the same (z_+ and z_- are **just** on either side of the cut). So Θ changes by $2\pi \Rightarrow f(z_+) = -f(z_-)$. Similarly the same thing happens with θ_2 changing if we move along the path labelled 2 in green. So $f(z)$ takes different values at points on opposite sides of the branch cut. This is a general feature of multi-valued functions and is crucial to their use in applied mathematics!

Cautionary Remark: In the next section and in other areas of the course we will be considering contour integration when branch cuts are present. So long as the branch cut remains **entirely outside** of the closed contour γ, the residue theorem and Cauchy's theorem remain valid and the same value of the integral will be found regardless of the branch cut chosen. What is not okay however is to let the contour under consideration cross over the branch cut or contain the entire branch cut in its inside.

1.17 Integrals involving multi-valued functions

When the real integrals we are interested in calculating contain multi-valued functions like $\sqrt{z}$ or $\log z$ we have to take a little more care using contour integration. Consider the following examples:

1). Evaluate

$$I = \int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \text{where } 0 < \alpha < 1.$$

Let

$$f(z) = \frac{z^{\alpha-1}}{z+1}.$$

This function is multi-valued (due to the power $\alpha - 1 \notin \mathbb{Z}$) with branch points at $z = 0$ and $z = \infty$. Let's take the branch cut connecting $z = 0$ to ∞ along the positive real axis, choosing $0 \leq \theta \leq 2\pi$. Note that $f(z)$ has a simple pole at $z = -1$. Consider

$$\oint_\gamma f(z)dz,$$

where $\gamma = \gamma_1 + \gamma_R + \gamma_2 + \gamma_\varepsilon$ given by $\gamma_1 = \{z : z = xe^{i0}; 0 < x \leq R\}$, $\gamma_R = \{z : z = Re^{i\theta}; 0 \leq \theta \leq 2\pi\}$, $\gamma_2 = \{z : z = xe^{2\pi i}; R \geq x > 0\}$ and $\gamma_\varepsilon = \{z : z = \varepsilon e^{i\theta}; 2\pi \geq \theta \geq 0\}$ as shown in figure 26 (note how I emphasize that the argument of the points in γ_1 and γ_2 are 0 and 2π respectively here).

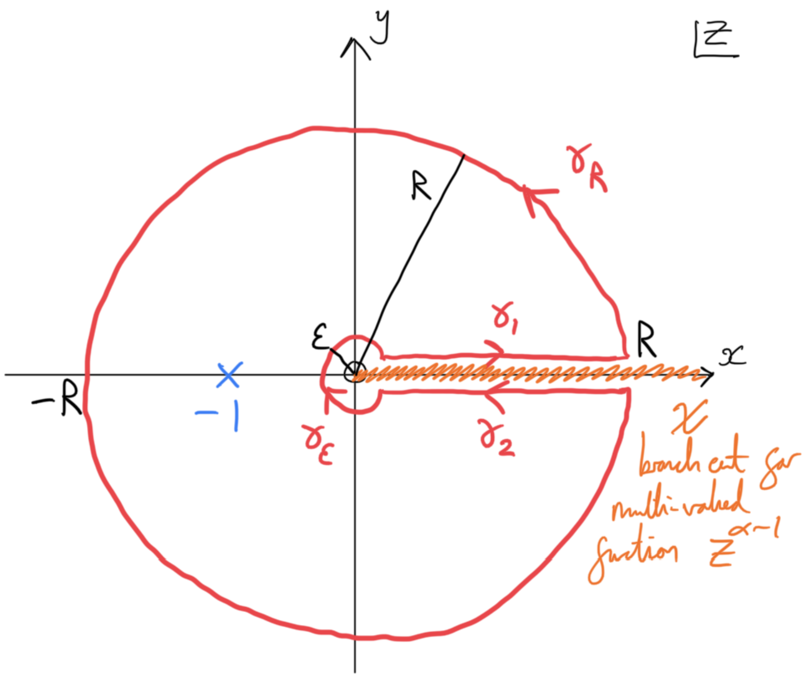

Figure 26: A 'keyhole' contour carefully avoiding the branch cut.

Then, by the residue theorem

$$\begin{aligned}\oint_\gamma f(z)dz &= 2\pi i \mathrm{Res}(f,-1)\\ &= 2\pi i(-1)^{\alpha-1}\\ &= 2\pi i(e^{\pi i})^{\alpha-1}\\ &= -2\pi i e^{\pi i\alpha},\end{aligned}$$

where we have used formula (1.9) with $m = 1$ in the second line to calculate the residue and the fact that $\arg\{-1\} = \pi$ based on our choice of branch cut ($0 \le \theta \le 2\pi$) in the third line. Now let's consider the integrals along the separate components of γ.

On γ_R: $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$.

$$\begin{aligned}\Rightarrow \left|\int_{\gamma_R} f(z)dz\right| &\le \int_0^{2\pi}\left|\frac{(Re^{i\theta})^{\alpha-1}}{1+Re^{i\theta}} iRe^{i\theta}\right| d\theta\\ &= \int_0^{2\pi}\left|\frac{iR^\alpha e^{\alpha i\theta}}{1+Re^{i\theta}}\right| d\theta\\ &\le 2\pi\frac{R^\alpha}{R-1} \sim R^{\alpha-1} \to 0 \text{ as } R\to\infty,\end{aligned}$$

since $\alpha - 1 < 0$. Thus

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0.$$

On γ_ε: $z = \varepsilon e^{i\theta}$, $\theta \in [2\pi, 0]$.

$$\begin{aligned}\left|\int_{\gamma_\varepsilon} f(z)dz\right| &= \left|\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{\alpha-1}}{1+\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta\right|\\ &\le \int_0^{2\pi}\left|\frac{\varepsilon^\alpha e^{\alpha i\theta}}{1+\varepsilon e^{i\theta}}\right| d\theta\\ &\le 2\pi\frac{\varepsilon^\alpha}{-\varepsilon+1} \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)dz = 0.$$

On γ_1: $z = xe^{i0}$; $\varepsilon < x \le R$.

$$\begin{aligned}\int_{\gamma_1} f(z)dz &= \int_\varepsilon^R \frac{(xe^{i0})^{\alpha-1}}{1+xe^{i0}}dx\\ &= \int_\varepsilon^R \frac{x^{\alpha-1}}{1+x}dx,\end{aligned}$$

noting that $e^{i0(\alpha-1)} = e^{i0} = 1$ to reach the second line (this may seem over the top when the argument is 0, but when dealing with branch cuts it is very important to take care the correct values of the function are being calculated). Hence

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\gamma_1} f(z)dz = I.$$

On γ_2: $z = xe^{2\pi i}$; $R \geq x > \varepsilon$.

$$\begin{aligned}\int_{\gamma_2} f(z)dz &= \int_R^\varepsilon \frac{(xe^{2\pi i})^{\alpha-1}}{1+xe^{2\pi i}}dx \\ &= -\int_\varepsilon^R \frac{e^{2\pi i\alpha}x^{\alpha-1}}{1+x}dx,\end{aligned}$$

noting that $e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha}e^{-2\pi i} = e^{2\pi i\alpha}$ to reach the second line (notice now how we would've missed this crucial extra factor along γ_2 had we not been careful and used 2π for the argument here). Hence

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\gamma_2} f(z)dz = -e^{2\pi i\alpha}I.$$

So, putting everything together:

$$\begin{aligned}&\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_R} + \int_{\gamma_\varepsilon}\right) \\ &\Rightarrow -2\pi i e^{\pi i\alpha} = (1-e^{2\pi i\alpha})I + 0 \\ &\qquad \Rightarrow I = \pi\left(\frac{2ie^{\pi i\alpha}}{e^{2\pi i\alpha}-1}\right) \times \frac{e^{-\pi i\alpha}}{e^{-\pi i\alpha}} \\ &\qquad\qquad = \pi\left(\frac{2i}{e^{\pi i\alpha}-e^{-\pi i\alpha}}\right) \\ &\qquad\qquad = \frac{\pi}{\sin(\alpha\pi)}.\end{aligned}$$

Remark on this example: Recall the example 4) from section 1.13, namely

$$I = \int_{-\infty}^\infty \frac{e^{\alpha x}}{1+e^x}dx, \quad 0 < \alpha < 1.$$

We found that

$$I = \frac{\pi}{\sin(\alpha\pi)}.$$

If we take $e^x = t$, then $e^{\alpha x} = t^\alpha$ and $dx = \frac{dt}{t}$. Also $x = -\infty \mapsto t = 0$ and $x = \infty \mapsto t = \infty$. Hence under this substitution we find

$$I = \int_0^\infty \frac{t^{\alpha-1}}{1+t}dt,$$

which is precisely the integral we just solved in the last example. So under this correspondence

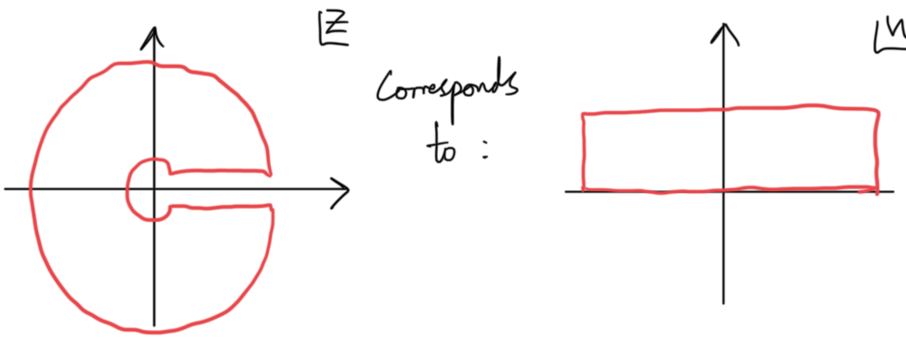

Figure 27: Correspondence between 'keyhole' and rectangle under the mapping $w = \log z$.

This can be seen from $w = \log z = \log|z| + i\theta$.

2). Evaluate

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}.$$

Here again we have a multi-valued integrand, with branch points at $x = \pm 1$. We consider

$$\oint_\gamma f(z)dz,$$

where

$$f(z) = \frac{1}{\sqrt{z^2-1}},$$

and $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_R$ as shown in figure 28 (note well here the slight difference between $f(z)$ and the function in the integrand of I).

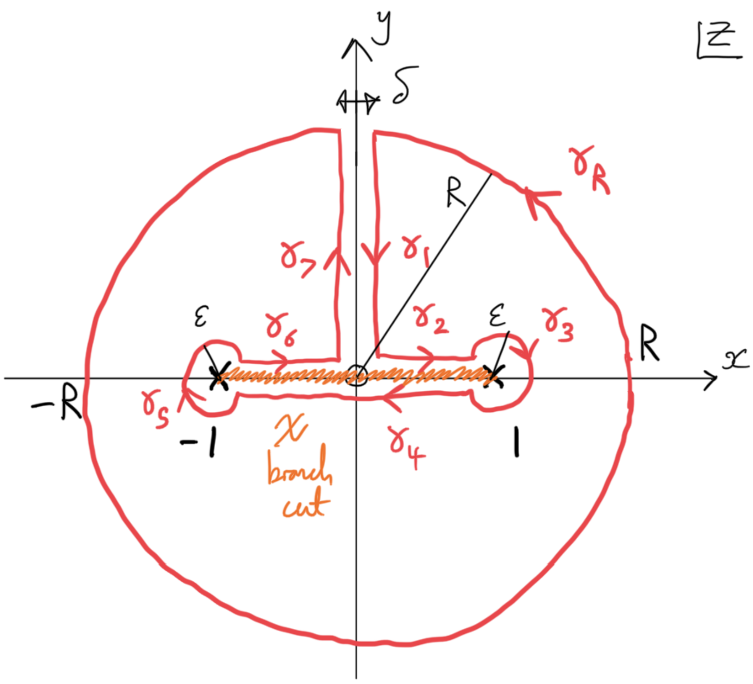

Figure 28: A contour with a cut that carefully excludes the branch cut along $[-1, 1]$.

The branch cut for $f(z)$ is chosen to connect the points $z = \pm 1$ by a straight line along the segment $[-1, 1]$ of the real axis. Our contour carefully excludes this.

With our choice of contour γ, $f(z)$ is analytic everywhere inside γ, hence by Cauchy's theorem

$$\oint_\gamma f(z)dz = 0.$$

Now to study the portions of γ near to the branch cut we introduce local coordinates: $r_1 = |z-1|$, $r_2 = |z+1|$ and lets choose $-\pi \leq \theta_1, \theta_2 \leq \pi$ (where $\theta_1 = \arg\{z-1\}$, $\theta_2 = \arg\{z+1\}$, see figure 29.

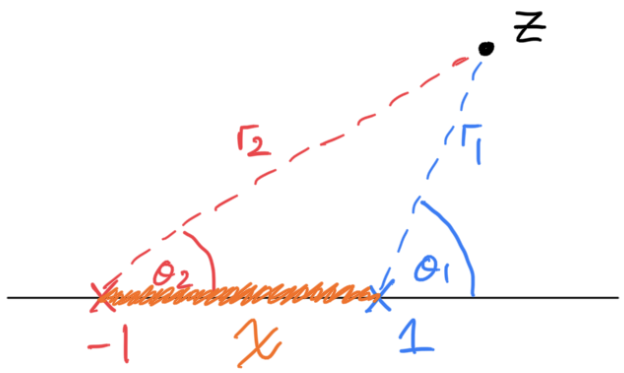

Figure 29: Local coordinates around the branch cut.

Note: The choices of the ranges for θ_1 and θ_2 are free to be chosen by us, however this choice will implicitly decide which branch of the multi-valued function we are taking. The choice I've made here corresponds to the branch of $f(z)$ that $\sim \sqrt{x^2-1}$ as $x \to \infty$ along the real axis - you'll see why when we consider the integral along γ_R soon.

Now let's examine the components of γ near the branch cut first. Let's start with γ_2.
On γ_2: $\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = |z-1| = 1-x$ and $r_2 = |z+1| = x+1$. See figure 30 for an illustration of where these values come from.

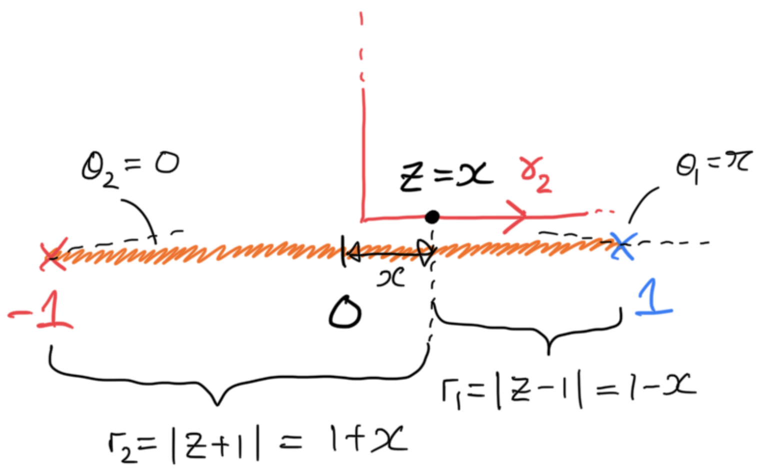

Figure 30: Values of the local coordinates when $z \in \gamma_2$.

Hence we find on γ_2:

$$\begin{aligned}\sqrt{z^2-1} &= (r_1 r_2)^{1/2} e^{i\frac{(\theta_1+\theta_2)}{2}} \\ &= \sqrt{1-x^2}\, e^{i\pi/2} \\ &= i\sqrt{1-x^2}.\end{aligned}$$

On γ_4: $\theta_1 = -\pi$, $\theta_2 = 0$, $r_1 = 1 - x$ and $r_2 = 1 + x$, so now we find

$$\begin{aligned}\sqrt{z^2-1} &= (r_1 r_2)^{1/2} e^{i\frac{(\theta_1+\theta_2)}{2}} \\ &= \sqrt{1-x^2}e^{-i\pi/2} \\ &= -i\sqrt{1-x^2}.\end{aligned}$$

As expected with multi-valued functions, we have a jump in the function values on either side of the branch cut.
Similarly to γ_2, **on γ_6:** $\theta_1 = \pi$, $\theta_2 = 0$, $r_1 = 1 - x$ and $r_2 = 1 + x$, so again $\sqrt{z^2-1} = i\sqrt{1-x^2}$ here.

Hence, as $\varepsilon \to 0$ and $\delta \to 0$, we have

$$\begin{aligned}\int_{\gamma_2+\gamma_6} f(z)dz &= \int_{-1}^{1} \frac{1}{i\sqrt{1-x^2}}dx = -i\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = -iI \\ \int_{\gamma_4} f(z)dz &= \int_{1}^{-1} \frac{1}{-i\sqrt{1-x^2}}dx = -i\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = -iI\end{aligned}$$

What about the other components of γ?
First note that

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_7} f(z)dz = 0,$$

since along this portion of the plane there is no branch cut and hence the function varies continuously, so as $\delta \to 0$ this is equivalent to traversing one contour forwards and then backwards again giving no contribution.

On γ_3: $z = 1 + \varepsilon e^{i\theta}$; $\pi \geq \theta \geq -\pi$. Here

$$\begin{aligned}1 - z^2 &= 1 - (1 + \varepsilon e^{i\theta})^2 \\ &= -2\varepsilon e^{i\theta}(1 + O(\varepsilon)),\end{aligned}$$

hence

$$\begin{aligned}|f(z)| &= \left|\frac{1}{\sqrt{-2\varepsilon e^{i\theta}(1+O(\varepsilon))}}\right| \\ &= \left|\frac{1}{\sqrt{2\varepsilon}}\right|\left|\frac{1}{\sqrt{1+O(\varepsilon)}}\right| \\ &\approx \left|\frac{1}{\sqrt{2\varepsilon}}\right|(1+O(\varepsilon)),\end{aligned}$$

where to reach the final line the general binomial expansion ($(1+x)^n = 1 + nx + n(n-1)x^2/2! + \cdots$) was used giving $(1 + O(\varepsilon))^{-1/2} = 1 + O(\varepsilon)$. This means that

$$\begin{aligned}\left|\int_{\gamma_3} f(z)dz\right| &\leq \left|\frac{1}{\sqrt{2\varepsilon}}\right|(1+O(\varepsilon)) \times 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0. \\ &\Rightarrow \lim_{\varepsilon\to 0}\int_{\gamma_3} f(z)dz = 0.\end{aligned}$$

Similarly, one can show that $\int_{\gamma_5} f(z)dz = 0$.

Finally, consider γ_R. First we need to check which branch of our multi-valued function $f(z)$ we've chosen implicitly by our choice of restrictions for θ_1 and θ_2. Consider $x \in \mathbb{R}$, $x > 1$. Then $\theta_1 = \theta_2 = 0$, $r_1 = |x-1| = x-1$ and $r_2 = |x+1| = x+1$. So we get

$$\sqrt{z^2-1} = (r_1 r_2)^{\frac{1}{2}} e^{i\frac{(\theta_1+\theta_2)}{2}} = +\sqrt{x^2-1}.$$

Note: For a different range of θ_1, θ_2 we may have chosen the other branch and this red $+$ sign would've been a $-$ sign; for instance setting $\pi \leq \theta_1 \leq 3\pi$ and keeping the range for θ_2 the same would've done this.

Hence, for z on γ_R: $z = Re^{i\theta}$; $\sqrt{z^2-1} \approx \sqrt{z^2} = +z$ (positive due to our choice of branch cut).

$$\begin{aligned}\Rightarrow \lim_{R\to\infty} \int_{\gamma_R} f(z)dz &= \int_0^{2\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= i\int_0^{2\pi} d\theta \\ &= 2\pi i.\end{aligned}$$

Hence, putting everything together

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0 \\ \delta\to 0}} \left(\int_\gamma = \sum_{j=1}^{7} \int_{\gamma_j} + \int_{\gamma_R} \right)$$

$$\begin{aligned}&\Rightarrow 0 = -2iI + 2\pi i \\ &\Rightarrow I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.\end{aligned}$$

Remark: Indeed we could have arrived at this result in seconds using techniques of real calculus; setting $x = \sin\theta$ in I gives

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

Nevertheless this example teaches us how to deal with these types of functions so that when we do encounter integrals of this form that we can't do easily, we have a procedure to deal with them.

References