Imperial College
London

MATH97053 MATH97164

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Commutative Algebra

Date: Wednesday, 5 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In this paper every ring is commutative with unit. You can use, without proof, any results from the course provided you state them correctly and clearly.

1. Let R be a ring and let $\mathcal{N}(R)$ denote its *nilradical*. Let $S \subset R$ denote the subset:

$$S = \{1 + x \mid x \in \mathcal{N}(R)\}.$$

(a) Show that S is a multiplicative set. (4 marks)

(b) Prove that the localisation $S^{-1}R$ is isomorphic to R. (4 marks)

(c) Assume that $S = R^\times$, where $R^\times$ denotes the set of all invertible elements of R. Is it true that $\mathcal{N}(R)$ is equal to the Jacobson radical of R? Justify your answer. (4 marks)

(d) Assume that the quotient $R/\mathcal{N}(R)$ is a Boolean ring. Show that $S = R^\times$. Is the converse true? Justify your answer. (8 marks)

(Total: 20 marks)

2. (a) How many composition series do the following $\mathbb{Z}$-modules have? Justify your answer.

(i) the $\mathbb{Z}$-module $\mathbb{Z}$, (3 marks)

(ii) the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, (3 marks)

(iii) the $\mathbb{Z}$-module $\mathbb{Z}/(p^n)$, where p is a prime and n is a positive integer, (3 marks)

(iv) the $\mathbb{Z}$-module $\mathbb{Z}/(p_1p_2\cdots p_k)$, where the p_i are pairwise different primes. (3 marks)

(b) We say that a ring R has *property* A if an R-module M has a composition series if and only if M is finitely generated. Show that a ring R has property A if and only if R is Artinian.

(5 marks)

(c) Let R be an Artinian ring. Is it possible that every finitely generated R-module has finitely many composition series? Justify your answer. (3 marks)

(Total: 20 marks)

3. For every ring R let $\dim(R)$ denote the *Krull dimension* of R.

(a) Which of the following are true? Justify your answer.

(i) if $\phi : A \to B$ is a surjective ring homomorphism, then $\dim(B) \leq \dim(A)$, (3 marks)

(ii) if $A \cong B \times C$, then $\dim(A) = \max(\dim(B), \dim(C))$. (6 marks)

(b) Let R be a Noetherian local integral domain. Show that there is an infinite non-stationary descending chain of *primary* ideals in R if and only if $\dim(R) \geq 1$. (6 marks)

(c) Find a primary decomposition of the following ideals of $\mathbb{Z}[x, y]$:

$$(2, x), (3, x^2, y^3), (2, x^2y^2), (3x, 3x^2y^3).$$

(5 marks)

(Total: 20 marks)

4. Let A be an Artinian ring and let $R = A[[x]]$ be the *formal power series ring over* A.

(a) Prove that an element $y \in R$ is invertible if and only if its residue class in $R/(x)$ is invertible. (4 marks)

(b) Show that R is Noetherian. (5 marks)

(c) Prove that $\dim(R) = 1$. (6 marks)

(d) Let M be a finitely generated Artinian R-module. Show that $x^n M = 0$ for some positive integer n. (5 marks)

(Total: 20 marks)

5. Let $(G, +, 0, <)$ be an ordered abelian group. We define a binary relation on $G_+ = \{a \in G \mid 0 < a\}$ as follows:

$$a \sim b \text{ if and only if } \exists n \in \mathbb{N} \text{ such that } a < nb \text{ and } b < na.$$

(a) Show that $\sim$ is an equivalence relation. (4 marks)

For every multiplicative set $S \subset \mathbb{Z}$ let $S^{-1}G$ denote the localisation of G with respect S, where we consider G as a $\mathbb{Z}$-module.

(b) Let $S \subset \mathbb{Z}$ be the multiplicative set $\mathbb{Z} - \{0\}$. Show that there is a unique ordering on $S^{-1}G$ which makes it an ordered group and the canonical map $f : G \to S^{-1}G$ is order-preserving.

(8 marks)

Let F be a field equipped with a binary relation $<$ such that $(F, +, 0, <)$ is an ordered group, and for every $a, b \in F$ and $c \in F$ we have:

$$\text{if } a < b,\ 0 < c \text{ then } ac < bc.$$

For every $a \in F$ let $|a| = \max(-a, a)$ and set

$$B(F) = \{a \in F \mid \exists n \in \mathbb{N} \text{ such that } |a| < n\}.$$

(c) Show that $B(F)$ is a valuation ring in F. (5 marks)

(d) Give an example of a valuation ring which is not a field and which is not isomorphic to $B(F)$ for any F and $<$ as above. (3 marks)

(Total: 20 marks)

1. (a) Since $1 = 1 + 0$ and $0 \in \mathcal{N}(R)$, we have $1 \in S$. Since $0 = 1 - 1$ and -1 is not nilpotent, we have $0 \notin S$. If $1+x, 1+y \in S$, then $x + y + xy \in \mathcal{N}(R)$ as $x, y \in \mathcal{N}(R)$ and $\mathcal{N}(R)$ is an ideal. So $(1 + x)(1 + y) = 1 + x + y + xy \in S$. **A** (4 marks)

(b) In order to see that the canonical map $x \mapsto \frac{x}{1}$ from R to $S^{-1}R$ is an isomorphism it will be enough to show that $S \subseteq R^{\times}$. If $x \in \mathcal{N}(R)$ then the inverse of $1 + x$ is $1 - x + x^2 - x^3 + \cdots$. (This series is finite as x is nilpotent.) **A** (4 marks)

(c) Yes. By results from the lectures $\mathcal{N}(R)$ is contained by the Jacobson radical $\mathcal{J}(R)$, so we only have to show the reverse inclusion. If $x \in \mathcal{J}(R)$ then $1 + x$ is invertible (by results from the lectures), so $x \in \mathcal{N}(R)$. **B** (4 marks)

(d) Let $x \in R/\mathcal{N}(R)$ be invertible. Since $R/\mathcal{N}(R)$ is a Boolean ring, we have $x^2 = x$, so $1 = x^{-1}x = x^{-1}x^2 = x$, so the only invertible element of $R/\mathcal{N}(R)$ is 1. The image of an invertible element y of R under the quotient map $R \to R/\mathcal{N}(R)$ is invertible, so it is 1, therefore y is of the form $1 + x$ with $x \in \mathcal{N}(R)$. **D** (5 marks)

The converse is not true; set $R = \mathbb{F}_2[x]$. Then R is an integral domain, therefore $\mathcal{N}(R) = 0$, and hence $R = R/\mathcal{N}(R)$ is not Boolean. However $R^{\times} = \mathbb{F}_2^{\times} = \{1\}$, so $R^{\times} = S$. **C** (3 marks)

(Total: 20 marks)

2. (a) (i) A module has a composition series if and only if it is both Artinian and Noetherian. The the $\mathbb{Z}$-module $\mathbb{Z}$ is not Artinian, since $\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \cdots$ is a non-stationary infinite descending series, so $\mathbb{Z}$ has no composition series. **A** (3 marks)

(ii) The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is not Noetherian, since $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{4}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{8}\mathbb{Z}/\mathbb{Z} \subset \cdots$ is a non-stationary infinite descending series, so $\mathbb{Z}$ has no composition series. **A** (3 marks)

(iii) Every composition series of $\mathbb{Z}/(m)$ is of the form $0 = (m_k) \subsetneq (m_{k-1}) \subsetneq (m_{k-2}) \cdots$, where $m_i | m_{i+1}$ and $\frac{m_{i+1}}{m_i}$ is a prime. When $m = p^n$ the only possibility for these primes is p, so there is one composition series. **A** (3 marks)

(iv) Using the same notation as above, the sequence $\frac{m_k}{m_{k-1}}, \frac{m_{k-1}}{m_k}, \ldots$ is a permutation of the primes $p_1, p_2, \ldots, p_k$, and all such permutations are possible. So this module has $k!$ composition series. **B** (3 marks)

(b) First assume that R is Artinian. If M has a composition series, it is Noetherian, so it is finitely generated. If M is finitely generated, it is a quotient of R^n for some natural number n. The ring R is Artinian, so it is Noetherian, so R^n is both Artinian and Noetherian by results from the lectures, and so is its quotient M. Now assume that R has property A. The ring R is finitely generated as an R-module, so it has a composition series, so it is Artinian. **D** (5 marks)

(c) Yes, it is possible. Let R be a finite ring: then it is Artinian. Every finitely generated R-module is a quotient of R^n for some natural number n, so it is also finite, and hence only has finitely many composition series. **C** (3 marks)

(Total: 20 marks)

3. (a) (i) Yes. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in B. By results from the lectures $\phi^{-1}(\mathfrak{p}_1) \subset \phi^{-1}(\mathfrak{p}_2) \subset \cdots \subset \phi^{-1}(\mathfrak{p}_n)$ is a chain of prime ideals in A. Since ϕ is surjective, all inclusions are proper. The claim is now clear. **A** (3 marks)

(ii) Yes. Since both projections $\pi_1 : A \to B$ and $\pi_2 : A \to C$ are surjective, we get from part (i) that $\dim(A) \geq \max(\dim(B), \dim(C))$. Now let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in A. Since $(1,0) \cdot (0,1) = 0 \in \mathfrak{p}_1$, either $(1,0) \in \mathfrak{p}_1$ or $(0,1) \in \mathfrak{p}_1$, say the latter. Since $(0,1)$ generates $\mathrm{Ker}(\pi_1)$, we get that the chain above is of the form $\pi_1^{-1}(\mathfrak{q}_1) \subsetneq \pi_1^{-1}(\mathfrak{q}_2) \subsetneq \cdots \subsetneq \pi_1^{-1}(\mathfrak{q}_n)$ for some ideals $\mathfrak{q}_i \lhd B$. Since π_1 is surjective, these ideals are prime and form a chain with proper inclusions. We get that $\dim(A) \leq \max(\dim(B), \dim(C))$. **B** (6 marks)

(b) If $\dim(R) = 0$ then R is Artinian, by results from the lectures, so every infinite descending chain of ideals is stationary. Otherwise let $\mathfrak{m} \lhd R$ be the unique maximal ideal. Note that $\mathfrak{m}^n$ is primary, as its radical $r(\mathfrak{m}^n) = \mathfrak{m}$ is maximal. So we only need to show that $\mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n \supset \cdots$ is not stationary. But if $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$, then $\mathfrak{m}^n = 0$ by Nakayama's lemma. Since R is an ID, we get that $\mathfrak{m} = 0$, and hence R is a field, so it has dimension zero, a contradiction. **D** (6 marks)

(c) The ideal $(2, x)$ is prime, hence primary. The radical of $(3, x^2, y^3)$ is the maximal ideal $(3, x, y)$, so this one is primary too. The ideal $(3, x^2y^2)$ is not primary. In the ring $\mathbb{Z}[x, y]$ we have $(x^2y^2) = (x^2)(y^2) = (x^2) \cap (y^2)$. Using this and unique factorisation in polynomial rings over $\mathbb{Z}$ a small calculation shows that $(2, x^2y^2) = (2, x^2) \cap (2, y^2)$. The last two ideals are primary, so this is a primary decomposition. We have $(3x, 3x^2y^3) = (3x) = (3) \cap (x)$, and this is a primary decomposition. **C** (5 marks)

(Total: 20 marks)

4. (a) Since the image of an invertible element under a homomorphism is invertible, the condition is clearly necessary. Note that residue class in $R/(x)$ of $y \in R$ is invertible if and only if the constant term of y is invertible. In this case the coefficients of the inverse series are explicitly determined inductively, staring from the constant term. **A** (4 marks)

(b) When A is a field then $R = A[[x]]$ is a DVR, so it is Noetherian. Now let A be any Artinian ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$, and let $A = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = (0)$ be a composition series of A. Then $R = \mathfrak{a}_1 R \supset \mathfrak{a}_2 R \supset \cdots \supset \mathfrak{a}_n R = (0)$ is a finite chain of ideals of R such that each quotient $\mathfrak{a}_j R/\mathfrak{a}_{j+1} R$ is isomorphic to a quotient of $A/\mathfrak{m}_i[[x]]$ for some i, which is Noetherian by the above. Therefore by results from the lectures $A[[x]]$ is Noetherian, too.

(It is ok to use that A is Noetherian, so $A[[x]]$ is Noetherian by the Hilbert basis theorem for formal power series rings.) **C** (5 marks)

(c) Let $\mathfrak{m} \lhd A$ be maximal. Since the quotient rings $R/\mathfrak{m}R \cong A/\mathfrak{m}[[x]]$ and $R/(\mathfrak{m}, x) = A/\mathfrak{m}$ are integral domains, the sequence $\mathfrak{m}R \subsetneq (\mathfrak{m}, x)$ is a chain of prime ideals, so $\dim(R) \geq 1$. Now let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots$ be a chain of prime ideals in R. Then $\mathfrak{n} = A \cap \mathfrak{p}_1$ is a prime ideal in A, hence it is maximal, since $\dim(A) = 0$. Also $\mathfrak{n}R \subset \mathfrak{p}_i$ for each i, so $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots$ is the pre-image of a chain of prime ideals with respect to the quotient map $R \to R/\mathfrak{n}R = A/\mathfrak{n}[[x]]$. As the latter ring is a DVR for any maximal ideal $\mathfrak{n} \lhd A$, it has dimension 1, therefore $\dim(R) \leq 1$. **D** (6 marks)

(d) Since M is Artinian, the infinite descending chain $M \supset xM \supset \cdots \supset x^n M \supset \cdots$ of submodules is stationary, so $x^n M = x^{n+1} M = \cdots$ for some n. Since M is finitely generated and R is Noetherian, the module M is Noetherian, and hence $x^n M$ is finitely generated. Note that $1 + bx$ is invertible by (a) for every $b \in R$, so x is in the Jacobson radical of R. Therefore by Nakayama's lemma $x^n M = 0$. **B** (5 marks)

(Total: 20 marks)

5. (a) Since $0 < a$, we have $a < 2a$, so $\sim$ is reflexive. Since the definition of $\sim$ is symmetric, this relation is also symmetric. If $a \sim b$ and $b \sim c$ then there are $m, n \in \mathbb{N}$ such that $a < mb$ and $b < nc$. Then $mb < mnc$, so $a < mnc$. We can show the reverse the same way. Therefore $\sim$ is transitive, too. **A** (4 marks)

(b) Let m, n be positive integers and let $a, b \in G$. If $<$ is an ordering on $S^{-1}G$ of the type in the claim, and $\frac{a}{m} < \frac{b}{n}$, then $na < mb$ and vice versa, which can be seen by multiplying by mn. Since every element of $S^{-1}G$ can be written as a fraction with a positive denominator, the ordering $<$ is unique, if it exists. Next we need to see that the condition above considered as a definition is well-defined. What this means is that if m, n and a, b are as above, and $\frac{a}{m} < \frac{b}{n}$ (or equivalently $na < mb$), and r, s are positive integers, then $\frac{ra}{rm} < \frac{sb}{sn}$, too. The latter is equivalent to $(sn)(ra) < (rm)(sb)$, but this is equivalent to $na < mb$ by multiplying by rs. The axioms can be checked similarly. **A** (8 marks)

(c) Since $0 < 1$ and $-1 < 2$, $1 < 2$, we have $0, 1, -1 \in B(F)$. If there are $m, n \in \mathbb{N}$ such that $|a| < m$ and $|b| < n$ then $|a + b| \leq |a| + |b| < n + m$ and $|a \cdot b| = |a| \cdot |b| < n \cdot m$, so $B(F)$ is a ring. We need to show that if $a \in F^*$ then either $a \in B(F)$ or $a^{-1} \in B(F)$. We may assume without the loss of generality that a is positive, since $a \in B(F)$ if and only if $-a \in B(F)$. If $a \notin B(F)$ then $a \geq 2 > 1$, so $a^{-1} < 1$, and hence $a^{-1} \in B(F)$. **B** (5 marks)

(d) Consider $\mathbb{F}_p[[t]]$; this ring is not a field but a DVR by results from the lectures, so it is a valuation ring. On the other hand every field containing it has characteristic p. But every ordered group is torsion-free, so these fields cannot have an ordering with the required properties. **C**

(3 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97053MATH97164	1	This was perhaps the easiest question, all problems were fairly straightforward consequences of material covered in the course. Part (a) was a special case of an exercise, part (b) followed from the combination of two easy results from the beginning of the course, part (c) was just remembering the characterisation of elements of the Jacobson radical, and (d) was also easy, if one remembered that the only invertible element in a Boolean ring is the unit.
MATH97053MATH97164	2	This was not a particularly hard question either. Part (a) was similar to some problems in the tests, but easier: the first two did not have composition series, since they were not Artinian, Noetherian, respectively, and the last two were just easy elementary group theory. Part (b) is a variant of standard arguments in the theory of chain conditions, and for part (c) the standard example of finite rings worked.
MATH97053MATH97164	3	Part (a) was largely recalling definitions, although (ii) required to remember a basic fact about direct products. Part (b) was more challenging; it required an idea, namely taking the powers of the maximal ideal, but then it was an argument already seen. Part (c) was very similar to a problem in the class.
MATH97053MATH97164	4	This was the most challenging problem, although the two more difficult parts (b) and (c) were easy versions of problems covered in the course. Part (a) was a mild generalisation of an exercise, and part (d) needed that the student noticed it was a special case of a standard application of Nakayama's lemma.

MATH97053MATH97164	5	This problem was largely checking definitions, and did not require any original ideas, although part (b) could be done more elegantly if one pointed out that in an ordered group multiplication by a positive integer is an order-preserving homomorphism, and every finitely generated subgroup of S^{-1}G can be mapped to G by such a map. Part (c) could be done with checking the definition directly, although I saw nice solutions which actually constructed the valuation with respect to which B(F) is a valuation ring using part (a). In part (d) many students just gave an example which were good, but no argument given; I decided to give partial points in these cases.