Imperial College London
Department of Mathematics

# Solutions to Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2019/20

Time allowed: 45 minutes. You should answer all questions.

**Question 1**

i) Compute the determinant of the matrix: $A = \begin{pmatrix} 1 & 3 & 4 & 10 \\ 2 & 5 & 9 & 11 \\ 6 & 8 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{R})$.

Compute the entry in the first row and column of $A^{-1}$ (i.e., the (1,1)-entry of $A^{-1}$), giving a reason for your answer. (8 marks)

ii) Let $V$ be the vector space of polynomials of degree at most 2 over $\mathbb{R}$. The linear transformation $S : V \to V$ is defined by

$$S(a + bt + ct^2) = (a + 2c)t + (b + c)t^2$$

(for $a, b, c \in \mathbb{R}$; here $t$ is the variable in the polynomial). Find the eigenvalues and eigenvectors of $S$ and hence calculate $S^{100}(t)$. (12 marks)

***Solution:*** (i) There are many possible ways of doing this. The following is by expansion along row 4:

$$\det(A) = -7\det\begin{pmatrix} 3 & 4 & 10 \\ 5 & 9 & 11 \\ 8 & 0 & 0 \end{pmatrix} = -7.8(44 - 90) = 56.46 = 2576.$$

**(4 marks; no need to do the final multiplication)**

We compute the required entry in $A^{-1}$ using the formula involving the adjugate matrix

$$A^{-1} = \mathrm{adj}(A)/\det(A).$$

The 11 entry of $\mathrm{adj}(A)$ is the 11-cofactor of $A$, $\det(A_{11})$. As $A_{11}$ has a row of zeros, this is 0. So the 11-entry of $A^{-1}$ is 0. **(4 marks)**

(ii) Consider the basis $B : 1, t, t^2$ of $V$. We compute that

$$[S]_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} = A.$$

So the characteristic polynomial is

$$\chi_S(x) = \det\begin{pmatrix} x & 0 & 0 \\ -1 & x & -2 \\ 0 & -1 & x-1 \end{pmatrix} = x(x^2 - x - 2) = x(x-2)(x+1),$$

and the eigenvalues are $0, 2, -1$.

We compute the eigenvectors of the matrix $A$:

Eigenvalue 0: Eigenvectors are non-zero multiples of $(-2, -1, 1)^T$;

Eigenvalue 2: Eigenvectors are non-zero multiples of $(0, 1, 1)^T$;

Eigenvalue -1: Eigenvectors are non-zero multiples of $(0, 2, -1)^T$.

Thus the eigenvalues of $S$ are $0, 2, -1$ and the corresponding eigenvectors are non-zero scalar multiples of $(-2 - t + t^2)$, $(t + t^2)$ and $(2t - t^2)$, respectively.

**(8 marks: lose 3 marks if only the eigenvectors of $A$ are computed)**

To compute $S^{100}(t)$ we can either use the diagonalising matrix computed using the above information, or we can proceed directly by expressing $t$ as a linear combination of the eigenvectors. We see that $t = \frac{1}{3}((t + t^2) + (2t - t^2))$ so

$$S^{100}(t) = \frac{1}{3}(S^{100}(t+t^2)+S^{100}(2t-t^2)) = \frac{1}{3}(2^{100}(t+t^2)+(-1)^{100}(2t-t^2)) = \frac{2^{100}+2}{3}t+\frac{2^{100}-1}{3}t^2.$$

**(4 marks)**

**Question 2** Suppose $n \geq 1$ is a natural number and $F$ is a field. We say that matrices $A, B \in M_n(F)$ are *similar* (denoted by $A \sim B$) if there is an invertible matrix $P \in M_n(F)$ with $B = P^{-1}AP$.

For each of the following statements, say whether it is true or false for all matrices in $M_n(F)$. If it is true, give a short proof; if it is false, give a counterexample.

i) The relation $\sim$ is an equivalence relation on $M_n(F)$.

ii) If $A \sim B$, then $A^2 \sim B^2$.

iii) If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1A_2 \sim B_1B_2$.

iv) If $A \sim B$, then $A^T \sim B^T$.

v) If $\chi_A(x) = \chi_B(x)$, then $A \sim B$.

vi) If $A, B$ are diagonalisable, then $A + B$ is diagonalisable.

vii) If $A, B$ are symmetric, then $AB$ is symmetric.

viii) If $F = \mathbb{R}$, then $A + A^T$ is diagonalisable.

(20 marks, 2.5 per part)

***Solution:*** (i) TRUE: For $A, B, C \in M_n(F)$ we have: $A \sim A$ (take $P = I_n$); If $A \sim B$ and $B = P^{-1}AP$, then $A = PBP^{-1}$ (and $P$ is invertible); If $A \sim B$ and $B \sim C$ with $B = P^{-1}AP$ and $C = Q^{-1}BQ$, then $C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$, so $A \sim C$.

(ii) TRUE: If $A \sim B$ and $B = P^{-1}AP$, then $B^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$.

(iii) FALSE: For example let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B_1 = A_2$ and $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$A_1A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B_1B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The only matrix similar to the zero matrix is the zero matrix, so $A_1A_2 \not\sim B_1B_2$.

(iv) TRUE: If $A \sim B$ and $B = P^{-1}AP$, then $B^T = P^TA^T(P^{-1})^T = P^TA^T(P^T)^{-1}$.

(v) FALSE: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ have characteristic polynomial $x^2$, but $A \not\sim B$ (as in (iii)).

(vi) FALSE: For example, in $M_2(\mathbb{R})$, let $A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. These are diagonalisable (over $\mathbb{R}$) as they each have two distinct eigenvalues ($\pm 1$). But $A + B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues.

(vii) FALSE: $(AB)^T = B^TA^T = BA$, so $AB$ is symmetric iff $A, B$ commute. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a counterexample.

(viii) TRUE: $(A + A^T)$ is a real, symmetric matrix and so is diagonalisable (over $\mathbb{R}$).

**(20 marks; 0.5 mark for each correct TRUE/ FALSE; 2 marks for each justification.)**