Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Partial Differential Equations in Action **MATH50008**

Midterm Exam

1. Total: 20 Marks

Let us consider the linear boundary value problem

$$\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \quad t > 0$$
$$u(t,1) = 0; \quad u(0,x) = \phi(x), \quad \text{where } \phi \in C[0,1]$$

(a) Plot the characteristics (do not forget about the one passing through $x = 0, t = 0$). 5 Marks

(b) Find the region where $u(t,x) = 0$, no matter what ϕ is. Explain why the boundary conditions at $x = 0$ are not needed. 5 Marks

(c) Write down the explicit formula for the solution of the problem. 5 Marks

(d) Using this formula, show that the quantity

$$E_p(t) = \left(\int_0^1 |u(t,x)|^p dx\right)^{1/p}$$

decays exponentially in time for $1 \leq p < \infty$. Prove that the decay rate depends on p. 5 Marks

2. Total: 20 Marks

Let us consider the Riemann problem for the quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{1}{1+u^2}\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(0,x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x < 0. \end{cases}$$

(a) Find the characteristics of this equation. 6 Marks

(b) Show that there are no shocks for it. 4 Marks

(c) Find the explicit solution for the solution. 10 Marks

MATH50008 TEST: SOLUTIONS

Problem 1.

a) The characteristics solve the system of ODEs $\frac{dt}{ds} = 1$, $\frac{dx}{ds} = -x$. Solving this system gives

$$x(s) = x_0 e^{-s},\ t(s) = t_0 + t.$$

The curves are $xe^t = const$ and the characteristic passing through the origin is a vertical line $x = 0$. See the picture below

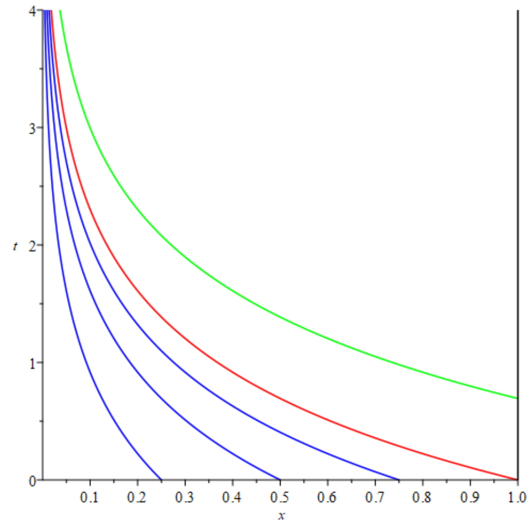

FIGURE 1

b) The solution $u(t, x)$ is identically zero in the region above the curve $xe^t = 1$ (red curve on the graph) due to zero boundary condition at $x = 1$. The condition at $x = 0$ is not needed since the value of the solution $u(t, 0) = u(0, 0) = \phi(0)$ (transported along the vertical characteristic line).

c) The general solution is $u(t, x) = \psi(xe^t)$ and our boundary and initial conditions give

$$u(t,x) = \begin{cases} \phi(xe^t),\ xe^t \le 1, \\ 0,\ xe^t > 1. \end{cases}$$

d) From the obtained formula we have

$$E_p(t)^p = \int_0^{e^{-t}} |\phi(xe^t)|^p\, dx = e^{-t} \int_0^1 |\phi(y)|^p\, dy = e^{-t} E_p(0)^p$$

where we change the variable $y = xe^t$. Thus, $E_p(t) = e^{-t/p} E_p(0)$ and the L^p-norm of the solution decays as $e^{-t/p}$.

Problem 2.

a) The equations for the characteristics are

$$\frac{dt}{ds} = 1,\ \frac{dx}{dt} = \frac{1}{1+u^2},\ \ \frac{du}{dt} = 0$$

and the solutions are

$$u = const, \quad x = \frac{1}{1+u^2}t + \xi.$$

b) We see that bigger values of u are transported slower than the smaller ones, so starting from the step function $\phi(x) = 0$ for $x > 0$ and $\phi(x) = 1$ for $x < 0$, we will never get the intersection of characteristics for positive time and the shock cannot be formed.

c) We know that $u = 0$ for $x = t + \xi$, $\xi > 0$, $u = 1$ for $x = \frac{1}{2}t + \xi$, $\xi > 0$ and $x = \frac{1}{1+u^2}t$ for $\xi = 0$ and $0 < u < 1$. Thus, $u(t, x) = 0$ if $x > t$ and $u(t, x) = 1$ if $x < t/2$. For the region $t/2 \le x \le t$, we need to use the characteristics which correspond to $\xi = 0$. This means, we have the relation

$$x = \frac{1}{1+u^2}t \quad \text{or} \quad u = \sqrt{\frac{t}{x} - 1}$$

and finally

$$u(t,x) = \begin{cases} 1, & x < t/2, \\ \sqrt{\frac{t}{x} - 1}, & t/2 \le x \le t, \\ 0, & x > t. \end{cases}$$