

Chapter 1

Introduction

1.1 Overview of the module

Ironically, as a university student, **one often does not understand the real reason for studying a subject until one knows it already!** This is particularly true as you are faced with your first choice of elective modules in your university career. Keeping this in mind, I will attempt to give you **a short introduction that will (hopefully) motivate the material covered in this module and give you a sense for the philosophy behind it.**

You may be wondering "**Why should I study partial differential equations? Where do they arise and why are they useful?**". If so, I could simply answer: **if you are interested in understanding the world surrounding you, you are in the right place, read on!** Indeed, the language of science is mathematics. As a mathematician, you possess the tools to build mathematical models to describe important phenomena in the physical, biological and social sciences, as well as in engineering. What I mean by a **mathematical model** is a set of equations and/or other mathematical relations capable of capturing the essential features of a complex natural or artificial system. Mathematical models can be **discrete** (like in the Ising model for magnetism or the Schelling model of residential moves) or **continuous** (like in fluid mechanics), **deterministic** (like wave propagation in an elastic medium) or **stochastic** (like most biochemical processes). In this module, we will be interested in **continuous and deterministic models**. While mathematical models can take many forms, the vast majority of mathematical models take the form of differential equations, i.e. equations quantifying the change of a quantity in terms of other quantities.

For instance, Newton's second law of motion expresses the acceleration of an object in terms of the forces on the object. The study of several interacting particles would translate to studying a **system of ordinary differential equations** when the force law is known. For instance, studying particles interacting via linear springs would give rise to a second-order system of differential equations, which can be resolved into its normal modes. It turns out, in practice, that **many important problems involve interaction between a large number of objects**, for which using directly the fundamental laws of physics is impractical. For instance, if you consider the flow of a gas in a pipe. Each molecule of the gas obeys Newton's laws of motion but in practice, a macroscopic volume of gas contains an Avogadro number of individual molecules (i.e. $\sim 10^{23}$ molecules). **It is then clear that one cannot solve such a large system of ODEs.** In many applications, it quickly becomes necessary to develop simpler models.

The basic strategy to derive these simpler models is generically to:

- **define new quantities that describe average macroscopic values of the fundamental microscopic quantities** (like the density, the temperature, the pressure in the example of the gas);

- **assume fundamental principles** which often appear as conservation (or balance) laws, e.g. conservation of mass, momentum or energy;
- **introduce constitutive relations which connect different macroscopic entities.**

The outcome of this strategy is generally a **partial differential equation or a system of partial differential equations**. In theory, one should start from the fundamental laws and then average them to achieve simpler models; this is called **coarse-graining**. In practice though, it is often very hard to do so, and, instead, **we shall sometimes use experimental observations to supplement the basic principles**. Thus, the constitutive relations can be of **experimental nature or even simply ad-hoc assumptions** (they depend strongly on the features of the system under study (e.g. Fick's law for the diffusion of a substance or the way the speed of a driver depends on the density of cars ahead).

In practice, **PDEs have historically been used to model phenomena in physics and chemistry**. Progress in the theory of PDEs originally meant to develop analytical methods for finding explicit solutions. In the nineteenth century, **advances in the field of PDEs were often triggered by advances in different branches of physics**. Indeed, the method of characteristics invented by Hamilton led to major advances in optics and in analytical mechanics, the Fourier method enabled the solution of heat transfer and wave propagation, and Green's method was instrumental in the development of the theory of electromagnetism. More recently, **PDE models have become more and more prominent in diverse branches of sciences including in finance, biology, ecology, medicine, sociology**. In this module, **the development of the theory will be motivated by examples and applications of PDEs**.

The module will be organized as follows: in the rest of Chapter 1, we will define formally partial differential equations, give a few examples of PDEs and introduce a number of basic properties. Chapter 2 will mostly deal with first-order PDEs, while Chapter 3 will cover important results on second-order PDEs. All throughout Chapters 2 and 3, we will provide a physical derivation for the equations we will study, provide examples of applications and important solution strategies. In Chapter 4, we will our knowledge of first- and second-order PDEs to introduce more advanced examples of applications. Motivated by the computational progress made in the last decades, Chapter 4 will also briefly introduce some numerical methods which allow the use of computers to solve PDEs (of virtually any kind).

1.2 What is a PDE?

1.2.1 Definition

A partial differential equation (PDE) is a functional relation between an unknown function and its derivatives. PDEs appear naturally in physics and engineering but in recent years, PDE models have also become very popular in biological and social sciences – as we will see. In all cases, we will consider systems where there is some sort of interaction between n independent variables, which we denote vectorially as $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. In most cases, these independent variables will be spatial variables, which we will denote for simplicity (x, y) (for $n = 2$) or (x, y, z) (for $n = 3$). In some cases, there will be a particular variable that plays an important role, called time that we will denote $t \geq 0$. In modelling the system under study, we will define functions of these independent variables and construct equations relating these functions. When the value of the unknown function(s) at a certain point depends only on what happens in the vicinity of this point, we shall, in general, obtain a PDE.

In most cases, we will consider real-valued functions $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ defined typically on an open set $\Omega \subset \mathbb{R}^n$, so that we are allowed to talk about the derivatives of these

functions with respect to each component. Through your reading, you may find different notations for the partial derivative of u with respect to x_i ,

$$\frac{\partial u}{\partial x_i} = \partial_{x_i} u = u_{x_i} \quad (1.1)$$

In general, given a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad (1.2)$$

is the partial derivative of u of order $|\alpha| = \sum_i \alpha_i$ obtained by taking α_1 derivatives with respect to x_1 , α_2 derivative with respect to x_2 , etc.

Definition 1.2.1: Partial differential equation

A partial differential equation (PDE) is a functional relation of the form

$$F(\mathbf{x}, u, Du, D^2u, \dots, D^m u) = 0, \mathbf{x} \in \Omega \subset \mathbb{R}^n \quad (1.3)$$

where the unknown is the function $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ and

$$D^m u = \left\{ \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \mid \alpha \in \mathbb{N}^n, |\alpha| = m \right\} \quad (1.4)$$

is the set of all derivatives of order m .

A solution of the PDE is a function $u(x_1, \dots, x_n)$ which possesses all required derivatives and that satisfies Eq. (1.3) for all points $\mathbf{x} \in \Omega$. This kind of solutions will be referred to as **classical solutions**.

The equation is, in general, supplemented by additional conditions such as initial conditions or boundary conditions. The equation and its associated conditions are then called a **problem** (e.g. initial-value problem, boundary-value problem). We will discuss the **various types of initial and boundary conditions** as we introduce various PDE models.

1.2.2 Basic properties of PDEs

Order

There exists various classifications for PDEs (and we shall see some of them in the next Chapters). One way to classify PDEs is according the **order** of the equation. The order is defined to be the order of the highest derivative appearing in the equation. In Definition 1.2.1, the highest derivative is of order m and so the equation is said to be of order m .

Example

For example, the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t, x) \quad (1.5)$$

is a *second-order equation*, while

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0 \quad (1.6)$$

| is a *fourth-order equation*.

Linearity

PDEs are also classified in two groups: **linear PDEs** and **nonlinear PDEs**. Eq. (1.3) is called linear if the function F is a linear function of the unknown function and its derivatives. We can also describe the linearity of partial differential equations in the context of a differential operator \mathcal{L} applied to a function u .

Example

| Here are some examples of differential operators:

$$\mathcal{L}u = \frac{\partial u}{\partial x}, \quad \mathcal{L}u = 5u^2 + \sin(x)\frac{\partial u}{\partial x}, \quad \mathcal{L}u = u\frac{\partial^2 u}{\partial x^2}. \quad (1.7)$$

More precisely, a PDE of order m is called **linear** if it is written in the following form:

$$\sum_{|\alpha| \leq m} \lambda_\alpha(\mathbf{x}) D^\alpha u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \quad (1.8)$$

or equivalently

$$\mathcal{L}u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \quad (1.9)$$

where \mathcal{L} is a linear differential operator, i.e. an operator $\mathcal{L} : \mathcal{C}^m(\Omega) \mapsto \mathcal{C}(\Omega)$ which associates to each function $u \in \mathcal{C}^m(\Omega)$ a continuous function $(\mathcal{L}u)(\mathbf{x})$ such that

$$\forall u, v \in \mathcal{C}^m(\Omega), \forall \alpha, \beta \in \mathbb{R}, \mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}u + \beta \mathcal{L}v \quad (1.10)$$

Further, the PDE is called **homogeneous** if $f(\mathbf{x}) \equiv 0$ and **non-homogeneous** otherwise.

Example

| For example, the equation

$$x^7 \frac{\partial u}{\partial x} + e^{xy} \frac{\partial u}{\partial y} + \sin^2(x^2 + y^2)u = x^3 \quad (1.11)$$

is a *linear* equation, while

$$\left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 = 0 \quad (1.12)$$

is a *nonlinear* equation.

We often further classify the nonlinear equations in subclasses according to the type of nonlinearity. The nonlinearity of an equation is more pronounced when it appears on a higher derivative. We commonly call:

- **semilinear equations**, equations where F is nonlinear only with respect to u but is linear with respect to all its derivatives;
- **quasilinear equations**, equations where F is linear with respect to the highest order derivatives of u only;
- **fully nonlinear equations**, equations where F is nonlinear with respect to the highest order derivatives of u .

Example

For example, we can say that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^3 \quad (1.13)$$

is a *semilinear* equation, while

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = |\nabla u|^2 u \quad (1.14)$$

is a *quasilinear* equation. The following equation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 = 0 \quad (1.15)$$

is *fully nonlinear*.

Well-posedness

The french mathematician Jacques Hadamard (1865-1963) introduced the notion of **well-posedness**. According to his definition, a problem is called well-posed if it satisfies all the following criteria

1. **Existence** — The problem has a solution.
2. **Unicity** — There is no more than one solution.
3. **Stability** — A small change in the equation or in the associated conditions gives rise to a small change in the solution.

Taking a mathematical modelling standpoint, the last condition can be understood as the need to obtain a solution which depends continuously on the data, i.e. that a **small error in the data would lead to a small error in the solution**. This last property is particularly important when one thinks of using numerical methods to approximate the solution. If the solution was too sensitive to small variations in the data, numerical methods, which entail approximation errors of various types, would produce unacceptable results.

If one or more of these criteria is not fulfilled, we say that the problem is **ill-posed**. While in practice the fundamental problems arising in mathematical physics are all well-posed, one may encounter ill-posed problems in mathematical modelling, e.g. in engineering applications. **As an ill-posed problem is in practice unsolvable, one should start by modifying it appropriately in order to render it well-posed before trying to tackle it.**

1.2.3 Examples of PDEs

As for ODEs, one can discuss single PDEs or systems of PDEs. Single PDEs are generally called scalar equations because the unknown is a scalar function. While we will mainly talk in this module about scalar PDEs, some of the most famous PDEs are actually systems of equations. Let us have a look at a variety of important examples of both scalar PDEs and systems of PDEs:

1. **Transport equation** (first order, linear)

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = 0 \quad (1.16)$$

This equation describes for instance the transport of a pollutant along a channel; here, u is the concentration of the pollutant and \mathbf{v} is the stream speed. We consider a one-dimensional version of this equation in Chapter 2.

2. Burgers' equation (first order, quasilinear)

$$\frac{\partial u}{\partial t} + cu \frac{\partial u}{\partial x} = 0 \quad (1.17)$$

This equation is used to model traffic dynamics and the one dimensional flux of a non viscous fluid. This particular form of Burger's equation is called the *inviscid* Burger's equation. One can generalize it to include a so-called viscous term,

$$\frac{\partial u}{\partial t} + cu \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad (1.18)$$

This variant is an important equation in fluid dynamics; it constitutes a basic example of competition between *dissipation* (the $\nu \partial^2 u / \partial x^2$ term) and *steepening* (i.e. shock formation due to the $cu \partial u / \partial x$ term). We will examine Burger's equation in more details in Chapter 2.

3. Diffusion/heat equation (second order, linear)

$$\frac{\partial u}{\partial t} - D \Delta u = 0 \quad (1.19)$$

where $\Delta = \partial / \partial x_1^2 + \cdots + \partial / \partial x_n^2$ is the Laplace operator. It describes either the diffusion of a substance in an homogeneous and isotropic bath or the conduction of heat through a homogeneous and isotropic medium; in the heat conduction problem, u is a temperature and D encodes the thermal properties of the medium. We consider the heat/diffusion equation in details in Chapter 3.

4. Wave equation (second order, linear)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad (1.20)$$

This equation describes for instance the propagation of transversal waves of small amplitude in an elastic string (e.g. a violin or guitar string) if $d = 1$ or an elastic membrane (e.g. in a drum) if $d = 2$. In the case of $d = 3$, this equation governs the propagation of electromagnetic waves in vacuum or small amplitude sound waves. Here, u represents the wave amplitude and c is the propagation speed. We consider the wave equation in details in Chapter 3.

5. Poisson/Laplace equation (second order, linear)

$$\Delta u = 0 \quad (1.21)$$

This equation is Laplace's equation; it can be seen as the *steady state* limit of the diffusion or wave equation, i.e. it describes the case where solutions of the diffusion or wave equation do not depend on time any more. Its nonhomogeneous version is called Poisson's equation

$$\Delta u = f(\mathbf{r}) \quad (1.22)$$

These equations play an important role in electrostatics. We consider Laplace's and Poisson's equations in details in Chapter 3.

6. Schrödinger equation (second order, linear)

$$-\frac{\hbar^2}{2m} \nabla^2 u + V(\mathbf{r})u = i\hbar \frac{\partial u}{\partial t} \quad (1.23)$$

This equation is fundamental in quantum mechanics and governs the evolution of a particle subject to a potential $V(\mathbf{r})$. Here, $u(\mathbf{r}, t)$ is a complex function called the quantum mechanical wavefunction of a non-relativistic particle of mass m , \hbar is Planck's constant divided by 2π and i is the complex unit. The function $|u|^2$ represents a probability density.

7. Fisher-Kolmogorov-Petrovsky-Piskunov equation (second order, semilinear)

$$\frac{\partial u}{\partial t} - D\Delta u = ru(M - u) \quad (1.24)$$

This is one of the simplest examples of what are called *reaction-diffusion* equations. It was originally derived to describe the evolution of a population of density u , subject to diffusion and logistic growth. It is one of the simplest examples of front propagation in biology.

8. Black-Scholes equation (second order, linear)

$$\frac{\partial u}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0 \quad (1.25)$$

This equation is fundamental in mathematical finance; it governs the evolution of the price u of a so-called derivative (e.g. an European option), based on the underlying asset (e.g. a stock or a currency) whose price is x .

9. Minimal surface equation (second order, quasilinear)

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \quad \mathbf{r} \in \mathbb{R}^2 \quad (1.26)$$

The graph of a solution u of this equation provides the surface which minimizes area and whose boundary is a given curve. Soap bubbles are examples of minimal surfaces.

10. Maxwell's equations (system of linear scalar first order equations)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1.27)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.28)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.29)$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad (1.30)$$

This system describes the evolution of the electric field \mathbf{E} and the magnetic field \mathbf{B} in the presence of an electric charge density ρ and current \mathbf{J} . These equations also involve ϵ_0 the permittivity of free space and μ_0 the permeability of free space; two universal physical constants related to the speed of light in vacuum $c = 1/\sqrt{\epsilon_0 \mu_0}$.

11. Navier-Stokes equations for incompressible flow (system of three quasilinear scalar second order equations and one linear first order equation)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} \quad (1.31)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (1.32)$$

where $\mathbf{u} = (u_x(\mathbf{r}, t), u_y(\mathbf{r}, t), u_z(\mathbf{r}, t))$ and $p = p(\mathbf{r}, t)$, $\mathbf{r} \in \mathbb{R}^3$. This equation governs the motion of a viscous, homogeneous and incompressible fluid. Here, \mathbf{u} is the fluid velocity, p its pressure, ρ its density and ν its kinematic viscosity (i.e. the ratio between the dynamic viscosity and the density of the fluid).

1.2.4 General form of solutions

Before we introduce the methods by which we may hope to solve PDEs such as the ones introduced in Section 1.2.3, let us think about how PDEs emerge from a set of possible solutions. This will hopefully inform us on how equations obtained not from possible solutions but from physical arguments may be solved!

Let us suppose that we have a set of functions involving two independent variables x and y . Without further specification, this set of functions is way too wide to expect to find a useful equation that they would all satisfy. Instead, consider a particular type of functions $u_i(x, y)$ in which x and y appear in a particular way, such that u_i can be written as a function of a **single variable** η , itself a function of x and y .

To illustrate this, we can consider the following three functions

$$u_1(x, y) = x^4 + 4(x^2y + y^2 + 1) \quad (1.33)$$

$$u_2(x, y) = \sin x^2 \cos 2y + \cos x^2 \sin 2y \quad (1.34)$$

$$u_3(x, y) = \frac{x^2 + 2y + 2}{3x^2 + 6y + 5} \quad (1.35)$$

It may not be obvious at the moment that these three functions are all solutions to the same differential equation. However, if we observe that each of these functions can be expressed as a function of the variable $\eta = x^2 + 2y$ alone, then a great deal of simplification takes place. Let us write these functions in terms of η

$$u_1(x, y) = (x^2 + 2y)^2 + 4 = \eta^2 + 4 = f_1(\eta) \quad (1.36)$$

$$u_2(x, y) = \sin(x^2 + 2y) = \sin \eta = f_2(\eta) \quad (1.37)$$

$$u_3(x, y) = \frac{(x^2 + 2y) + 2}{3(x^2 + 2y) + 5} = \frac{\eta + 2}{3\eta + 5} = f_3(\eta) \quad (1.38)$$

For each u_i , we form the following partial derivatives $\partial u_i / \partial x$ and $\partial u_i / \partial y$ and obtain

$$\frac{\partial u_i}{\partial x} = \frac{df_i(\eta)}{d\eta} \frac{\partial \eta}{\partial x} = 2x f'_i \quad (1.39)$$

$$\frac{\partial u_i}{\partial y} = \frac{df_i(\eta)}{d\eta} \frac{\partial \eta}{\partial y} = 2f'_i \quad (1.40)$$

for $i = 1, 2, 3$. We can eliminate f'_i from these equations by cross-multiplication, obtaining

$$\frac{\partial \eta}{\partial y} \frac{\partial u_i}{\partial x} = \frac{\partial \eta}{\partial x} \frac{\partial u_i}{\partial y} \quad (1.41)$$

which gives for our specific case where $\eta = x^2 + 2y$

$$\frac{\partial u_i}{\partial x} = x \frac{\partial u_i}{\partial y} \quad (1.42)$$

It is clear that not only are the three functions u_1 , u_2 and u_3 solutions of Eq. (1.42) but so is any **arbitrary function** $f(\eta)$ whose argument η has the form $x^2 + 2y$.

Remark. What we have just seen points to the following idea: we can try to solve specific PDEs by seeking combinations of variables in terms of which the solutions may be expressed as arbitrary functions.

1.2.5 Superposition principle

Consider the second-order equation $\partial^2 u / \partial x^2 = 0$ for the unknown function $u(x, y)$. Integrating this equation once leads to

$$\frac{\partial u}{\partial x} = f(y), \quad (1.43)$$

while a second integration leads to

$$u(x, y) = xf(y) + g(y), \quad (1.44)$$

where f and g are **arbitrary functions**. It is thus clear that there are infinitely many different choices for f and g . A general solution of this equation cannot be specified in terms of a finite number of arbitrary constants like you were used to for ordinary differential equations. Said differently, the space of solutions is here infinite-dimensional! Nevertheless, we can write down some rules to construct solutions.

Proposition 1.2.1: Superposition principle for homogeneous equations

If u_1, \dots, u_n are solutions of the same **linear homogeneous PDE** $\mathcal{L}u = 0$, and $\lambda_1, \dots, \lambda_n$ are constants, then $\lambda_1 u_1 + \dots + \lambda_n u_n$ is also a solution of the PDE.

Proof. By definition,

$$\mathcal{L}u_i = 0, \quad i = 1, \dots, n \quad (1.45)$$

By linearity of the differential operator \mathcal{L} , we have

$$\mathcal{L}(\lambda_1 u_1 + \dots + \lambda_n u_n) = \lambda_1 \mathcal{L}u_1 + \dots + \lambda_n \mathcal{L}u_n = 0 \quad (1.46)$$

■

Example

You can verify that for any constant k , the function $u(x, y) = e^{kx} \cos ky$ is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad (1.47)$$

Therefore, we can conclude, by the superposition principle, that the function $u(x, y) = e^{-x} \cos y + 2e^{-3x} \cos 3y - 5e^{-\pi x} \cos \pi y$ is also a solution of Laplace's equation.

Note that the superposition principle does not apply to nonhomogeneous equations. For instance, if u_1 and u_2 are solutions of the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1 \quad (1.48)$$

then the function $u_1 + u_2$ is solution of the following Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2. \quad (1.49)$$

Proposition 1.2.2: Subtraction principle for nonhomogeneous equations

If u_1 and u_2 are solutions of the same **linear nonhomogeneous PDE** $\mathcal{L}u = g$, then

the function $u_1 - u_2$ is a solution of the associated homogeneous PDE $\mathcal{L}u = 0$.

Proof. This is obvious by linearity of the differential operator

$$\mathcal{L}(u_1 - u_2) = \mathcal{L}u_1 - \mathcal{L}u_2 = g - g = 0 \quad (1.50)$$

■

The subtraction principle thus allows us to find the general solution of a nonhomogeneous equation $\mathcal{L}u = g$ once we know: (1) a particular solution of the equation and (2) the general solution of the related homogeneous equation $\mathcal{L}u = 0$.

Corollary 1.2.1

The general solution of the linear partial differential equation $\mathcal{L}u = g$ is of the form

$$u = U + v \quad (1.51)$$

where U is a particular solution of the equation $\mathcal{L}u = g$ and v is the general solution of the related homogeneous equation $\mathcal{L}v = 0$.

Example

Find the general solution $u(x, y)$ of the equation

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad (1.52)$$

First, we realize that $u(x, y) = x^2$ is a solution of the given equation. The general solution of the associated homogeneous equation was obtained above and reads $u(x, y) = xf(y) + g(y)$. Therefore, the general solution of the nonhomogeneous equation is

$$u(x, y) = x^2 + xf(y) + g(y) \quad (1.53)$$

1.2.6 Initial and boundary conditions

The exact functional form of the solution for any particular situation must be determined by specifying additional constraints. This is analogous to the case of ordinary differential equations (ODE) for which we need initial values or boundary values to single out solutions. For the following first order ODE

$$\frac{dx}{dt} = f(t, x(t)) \quad (1.54)$$

we needed to specify an initial condition, e.g. $x(0) = x_0$ and for the second order ODE

$$\frac{d^2x}{dt^2} = g(t, x(t), x'(t)) \quad (1.55)$$

we needed to provide two constraints, e.g. $x(0) = x_0$ and $x'(0) = v_0$ or $x(0) = x_0$ and $x(1) = x_1$.

Now, imposing as condition for a PDE the knowledge of the unknown function at a point is not enough to single out a solution. In general, one needs to impose infinitely many conditions.

Example

By direct integration, a solution $u : \mathbb{R}^2 \mapsto \mathbb{R}$ of the PDE

$$\frac{\partial u}{\partial y} = 0 \quad (1.56)$$

is given by $u(x, y) = g(x)$ where g is an arbitrary function of x . This is telling us that u is independent of y or, said differently, that u is constant on vertical lines.

The number of solutions depends on the additional constraints that we set, in particular:

1. The following problem

$$\begin{cases} \partial u / \partial y = 0 \\ u(0, y) = f(y) \end{cases} \quad (1.57)$$

has **no solution**.

2. The following problem

$$\begin{cases} \partial u / \partial y = 0 \\ u(x, 0) = f(x) \end{cases} \quad (1.58)$$

has **a unique solution**.

3. The following problem

$$\begin{cases} \partial u / \partial y = 0 \\ u(0, 0) = 0 \end{cases} \quad (1.59)$$

has **infinitely many solutions**.

For instance, one considers a PDE involving two independent variables x and y , the complete determination of its solution $u(x, y)$ will require the specification of $u(x, y)$ along a curve in the xy -plane.

1.3 Dimensional analysis

On the surface, the ideas we will develop in this section may seem trivial, but they are actually quite profound. These ideas have to do with the dimensions of the physical variables, or parameters, in a problem. To illustrate this, suppose that we know that the speed s of an object is function of its radius r and the length of time t it has been moving

$$s = f(r, t) \quad (1.60)$$

where we implicitly assume that the speed does not depend on any other physical quantity. Now, f is not specified, all we know is that there is some expression connecting s with r and t . The only way to combine r and t to produce the dimension of speed is through their ratio r/t . Indeed, we cannot write $s = \alpha r + \beta t$ without α and β having dimensions (indeed, s is a speed but r is a length and t is a time). But this would mean that α and β are physical parameters and we have assumed that there are no other in this problem! We can conclude based on this observation that the only function we can have is $s = \alpha r/t$ where α is a number.

1.3.1 The goal of dimensional analysis

What we just saw in our simple example is that the dimensions of the variables in the problem dictated the form of the function. This very useful idea is at the core of what we call **dimensional analysis**. The first person to introduce the concept of dimensional

reasoning in physical relations is Leonhard Euler in 1765. Dimensional analysis is a way to simplify complex physical problems by reducing the number of relevant variables; dimensional analysis makes use of the concept of **dimensional homogeneity**.

The general purpose of dimensional analysis is to reduce variables and group them in dimensionless form, it has several benefits:

- (A) it is useful when establishing physical models of natural phenomena by establishing the relevance and importance of some variables;
- (B) it allows to present and interpret experimental data and compare it to the behavior of a physical model;
- (C) it provides scaling laws which allow to obtain results on small-scale model and predict the behavior of a full-scale prototype;
- (D) it provides a way to nondimensionalize equations;
- (E) last but not least, it is very useful for checking equations.

Let us first establish some terminology.

1.3.2 Dimensions and units

The **dimension** of a physical quantity is its type, while the **unit** of a physical quantity is a way to assign a numerical value to this quantity. One must be careful not to be confused between **dimensions** and **units**; indeed, the former provides a global idea of the nature of the physical quantity, while the latter is not unique and in general is chosen based on orders of magnitude following a measure of the physical quantity (**e.g.** one would probably not expressed the distance earth-moon in micrometers).

Physical variables like force, density or velocity can be broken down into length L , time T and mass M . Length, time and mass are independent in the sense that one of them cannot be written in terms of the other two. We call these *fundamental dimensions*; they are given together with their SI units as follows:

- Mass, M (kilogram, kg)
- Length, L (meter, m)
- Time, T (second, s)

For problems which involve thermodynamics, we will expand this list to include the **temperature** θ and for electrical problems, we add **current** I . In Table 1.1, we give the fundamental dimensions of common physical quantities.

Definition 1.3.1: Dimensions notation

Given a physical quantity q , the fundamental dimensions of q will be denoted $[q]$. If q is a dimensionless quantity, we will denote $[q] = 1$.

1.3.3 The principle of dimensional homogeneity

The principle of **dimensional homogeneity** was first introduced by Joseph Fourier in 1822 in his book *Analytical Theory of Heat*; it states that to properly express a relationship between variables in a physical process, an equation needs to be dimensionally homogeneous, *i.e.* all additive terms in an equation must have the same dimensions. This forms the basis for the formal procedure of dimensional analysis that we outline below. This principle is particularly useful for you to check your equations; thus, it is always a good idea to keep the algebraic symbols for as long as possible in your equations and substitute for their numerical values only at the end. Note that dimensional analysis cannot determine numerical prefactors as those are dimensionless!

Quantity	Dimensions	Quantity	Dimensions
Mass	M	Strain	1
Time	T	Stress, Pressure	$ML^{-1}T^{-2}$
Frequency	T^{-1}	Elastic modulus	$ML^{-1}T^{-2}$
Length	L	Surface tension	MT^{-2}
Area	L^2	Viscosity (dynamic)	$ML^{-1}T^{-1}$
Volume	L^3	Viscosity (kinematic)	L^2T^{-1}
Velocity	LT^{-1}	Concentration	L^{-3}
Acceleration	LT^{-2}	Mass density	ML^{-3}
Momentum	MLT^{-1}	Wavelength	L
Angle	1	Temperature	θ
Angular velocity	T^{-1}	Internal energy	ML^2T^{-2}
Angular acceleration	T^{-2}	Entropy	$ML^2T^{-2}\theta^{-1}$
Angular momentum	ML^2T^{-1}	Enthalpy	ML^2T^{-2}
Force	MLT^{-2}	Specific heat	$L^2T^{-2}\theta^{-1}$
Torque	ML^2T^{-2}	Thermal conductivity	$MLT^{-3}\theta^{-1}$
Energy/work	ML^2T^{-2}	Thermal diffusivity	L^2T^{-1}
Power	ML^2T^{-3}		

Table 1.1 Fundamental dimensions of common physical quantities. You are not expected to know all of these by !

1.3.4 Examples of dimensional reduction

Here, we will introduce the ideas behind dimensional reduction via two simple examples which will set the stage for the general concepts introduced by the Buckingham Pi theorem.

Motion of a projectile

Consider the motion of an object thrown directly upwards in the earth gravitational field. In particular, we are interested in the maximum height y_M that the projectile reaches. In a uniform gravitational field and neglecting air resistance, Newton's second law of motion gives

$$m \frac{d^2y}{dt^2} = -mg \quad (1.61)$$

Provided initial conditions for the position and the velocity of the projectile

$$y(t=0) = 0 \quad \text{and} \quad \frac{dy}{dt}(t=0) = v_0 \quad (1.62)$$

one could integrate the ODEs in (1.61) to obtain the trajectory and find the maximum height by looking at when the vertical velocity of the projectile is zero. We can do simpler by using dimensional analysis.

If one assumes that the only physical parameters that y_M depends on are g , v_0 and m , we can write that

$$y_M = f(g, m, v_0) \quad (1.63)$$

We are thus interested in finding the unknown function f . The only way to combine g , m and v_0 to produce the correct dimensions (i.e. a length) is through products or ratios. Our hypothesis thus leads to the fact that there are numbers a , b and c such that

$$[y_M] = [m^a v_0^b g^c] \quad (1.64)$$

Using the fundamental dimensions of these variables, we can write

$$L = M^a(L/T)^b(L/T^2)^c = M^a L^{b+c} T^{-b-2c} \quad (1.65)$$

Equating the exponents of the respective terms, we obtain the following system of equations

$$\begin{cases} L : & b + c = 1 \\ T : & -b - 2c = 0 \\ M : & a = 0 \end{cases} \quad (1.66)$$

Solving these equations, we obtain $a = 0$, $b = 2$ and $c = -1$. So we conclude that

$$ra \quad (1.67)$$

where α is an arbitrary (dimensionless) number. While we have not fully determined the functional dependence of the maximal height (α is unknown), we have obtained important information via a minimal effort:

- We conclude that if the initial velocity is increased by a factor of 2 then the maximum height will increase by a factor of 4. Through very easy experimentations, we can thus check that our modelling assumptions were correct.
- The constant α can also be determined by running a single experiment. Just pick a particular v_0 and measure the experimentally observed y_M , then $\alpha = gy_M/v_0$ (all of which you now know).
- The maximum height does not depend on the mass of the projectile.

This example was particularly easy and what it did not show is how to handle situations where you may have several parameters.

Drag on a sphere

When designing racing cars, racing bicycles, and aircrafts, one of the overall goals is to reduce as much as possible the drag on the object, i.e. the resistance force of the air opposing motion of the object. To simplify the discussion, here we will consider the drag on a sphere.

Our modelling assumption is that the drag force F_D on the sphere depends on the radius of the sphere R , the velocity of the sphere v , the density of the air ρ and the dynamic viscosity of the air μ (i.e. a property of the fluid characterizing the resistance of the air to motion).

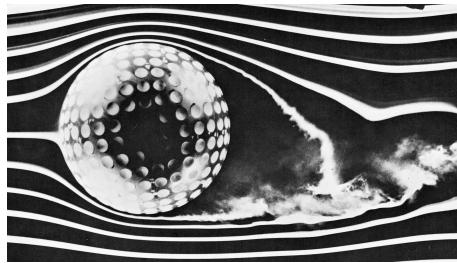


Figure 1.1 Airflow around a golf ball visualized using smoke.

Mathematically, we wrote that

$$F_D = f(R, v, \rho, \mu) \quad (1.68)$$

Proceeding as before, we ask whether we can find numbers a, b, c and d such that

$$[F_D] = [R^a v^b \rho^c \mu^d] \quad (1.69)$$

Expressing these in the fundamental dimensions, we obtain

$$MLT^{-2} = L^a (L/T)^b (M/L^3)^c (M/LT)^d = L^{a+b-3c-d} T^{-b-d} M^{c+d} \quad (1.70)$$

which, by equating exponents, leads to the following system

$$\begin{cases} L : & a + b - 3c - d = 1 \\ T : & -b - d = -2 \\ M : & c + d = 1 \end{cases} \quad (1.71)$$

Here, we have four unknowns and three equations, so we can anticipate that one of the constants will be undetermined. From the equation in T , we find that $b = 2 - d$ and from the equation in M , $c = 1 - d$. This leads to $a = 2 - d$. We finally have that

$$F_D = \alpha R^{2-d} v^{2-d} \rho^{1-d} \mu^d = \alpha \rho R^2 v^2 \left(\frac{\mu}{R v \rho} \right)^d \quad (1.72)$$

where d and α are arbitrary numbers. Thus, we conclude more generally that

$$F_D = \rho R^2 v^2 F(\Pi) \quad (1.73)$$

where

$$\Pi = \frac{\mu}{R v \rho} \quad (1.74)$$

is a dimensionless quantity (check this!), called a **dimensionless group** and F is an arbitrary function.

Remark. To understand how to pass from (1.72) to (1.74), one can realize that if you are given two sets of values (α, d) , say (α_1, d_1) and (α_2, d_2) , then their sum

$$F_D = \alpha_1 \rho R^2 v^2 \Pi^{d_1} + \alpha_2 \rho R^2 v^2 \Pi^{d_2} = \rho R^2 v^2 (\alpha_1 \Pi^{d_1} + \alpha_2 \Pi^{d_2}) \quad (1.75)$$

is also a solution. This stands for any arbitrary number of sets (α, d) . In other words,

$$F_D = \rho R^2 v^2 (\alpha_1 \Pi^{d_1} + \alpha_2 \Pi^{d_2} + \alpha_3 \Pi^{d_3} + \dots) \quad (1.76)$$

where d_1, d_2, d_3, \dots and $\alpha_1, \alpha_2, \alpha_3, \dots$ are arbitrary numbers. We write it more compactly as $F_D = \rho R^2 v^2 F(\Pi)$.

Dimensional reduction thus allowed us to determine the drag on a sphere up to an arbitrary function of the dimensionless group Π . Those of you, who will take modules in fluid dynamics, will be quickly introduced an important dimensionless parameters: the Reynolds number Re . The Reynolds number helps predict flow patterns in various flow situations; it measures the ratio of the inertial forces to the viscous forces. By definition, the Reynolds number for this problem is defined as

$$Re = \frac{R v \rho}{\mu} \quad (1.77)$$

Note that the dimensionless group we exhibited is such that $Re = \Pi^{-1}$. So another representation for the general solution is

$$F_D = \rho R^2 v^2 G(Re) \quad (1.78)$$

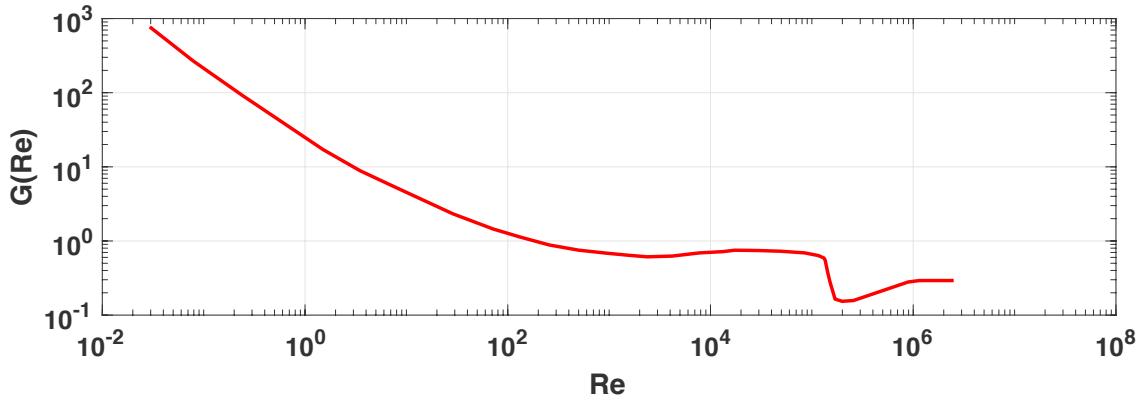


Figure 1.2 Experimentally measured values for $G(\text{Re})$ the unknown function appearing in (1.78).

where $G(\text{Re}) = F(\Pi)$. Because of its importance in fluids, G has been experimentally measured for a wide range of Reynolds numbers, producing the curve in Fig. 1.2.

There are in this problem four unknowns but only three equations, so one of the unknowns will be used in the general solution. Here, we used d but we could have as well picked one of the others. Using one of the other unknowns would lead to a different looking but mathematically equivalent expression. Thus, we have here multiple representations of the drag force. Depending on the context, some will be more useful than others. Note that although there are multiple ways to write the general solution, they all involve the same number of dimensionless groups.

1.3.5 Buckingham Pi Theorem

Following early developments from Euler (1765) and Fourier (1822), Rayleigh published his book *The theory of Sound* (1877) in which he proposed a "method of dimensions" and gave numerous examples of applications of what is now called dimensional analysis. In 1914, E. Buckingham¹ put this method on a solid theoretical basis and proved the so-called *Buckingham Pi Theorem*. This theorem is based on ideas of matrix algebra and the rank-nullity theorem for non-square matrices²; these are concepts you are familiar with from your algebra modules. Although this theorem is usually credited to E. Buckingham, we now know that A. Vaschy (France, 1892) and D. Riabouchinsky (Russia, 1911) had independently derived and published equivalent results.

Assume that a physical quantity q depends on n physical parameters or variables p_1, p_2, \dots, p_n . Assume as well that each can be expressed in terms of m fundamental dimensions. So far, we have used $m = 3$ with fundamental dimensions $\{M, T, L\}$. In this case, we write

$$[q] = L^{\ell_0} T^{t_0} M^{m_0} \quad (1.79)$$

and

$$\forall i \in [1, n], [p_i] = L^{\ell_i} T^{t_i} M^{m_i} \quad (1.80)$$

Our modelling assumption is that $q = f(p_1, p_2, \dots, p_n)$ and that this relation is **dimensionally homogeneous**. To dimensionally reduce the equation, we want to determine if

¹On Physically Similar Systems; Illustrations of the Use of Dimensional Equations, Phys. Rev. 4, 345 (1914).

²The formal proof can be found in the book *Scaling, Self Similarity and Intermediate Asymptotics* by GI Barenblatt, Cambridge University Press, Cambridge, UK (1996)

there are numbers a_1, a_2, \dots, a_n such that

$$[q] = [p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}] \quad (1.81)$$

Introducing (1.79) and (1.80) in the expression above and equating the exponents leads to the following general system of equations

$$\begin{cases} L : & \ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_n a_n = \ell_0 \\ T : & t_1 a_1 + t_2 a_2 + \cdots + t_n a_n = t_0 \\ M : & m_1 a_1 + m_2 a_2 + \cdots + m_n a_n = m_0 \end{cases} \quad (1.82)$$

which can be expressed in matrix form as

$$\mathbf{A}\mathbf{a} = \mathbf{b} \quad (1.83)$$

where

$$\mathbf{A} = \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \\ t_1 & t_2 & \cdots & t_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix} \quad (1.84)$$

is a $m \times n$ matrix known as the dimension matrix. In other words, the number of rows of this matrix corresponds to the number of fundamental dimensions needed and the number of columns equals the number of physical parameters. We have thus transformed our dimensional reduction question into a linear algebra problem. If (1.83) has no solution, it means that our assumption was wrong and that our list of relevant variables p_1, p_2, \dots, p_n must be incomplete.

Definition 1.3.2: Dimensionally complete

We say that the set p_1, p_2, \dots, p_n is **dimensionally complete** for q if it is possible to combine the p_i 's to form a quantity with the same dimension as q . If it is not possible, the set is dimensionally incomplete for q .

We thus assume that the set $\{p_i\}_{i \in [1, n]}$ is complete for q . Consider the associated homogeneous equation, $\mathbf{A}\mathbf{a} = \mathbf{0}$. The set of solutions of this equation form what is called the null space of \mathbf{A} . Letting k be the dimension of this subspace, then the general solution of $\mathbf{A}\mathbf{a} = \mathbf{0}$ can be written as

$$\mathbf{a} = \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \cdots + \gamma_k \mathbf{a}_k \quad (1.85)$$

where $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$ form a basis of the null space of \mathbf{A} and $\gamma_1, \gamma_2, \dots, \gamma_k$ are arbitrary numbers. So the general solution of (1.83) can be written as

$$\mathbf{a} = \mathbf{a}_p + \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \cdots + \gamma_k \mathbf{a}_k \quad (1.86)$$

where \mathbf{a}_p is any vector that satisfies (1.83).

We can now take our linear algebra conclusions and apply them to the dimensional reduction problem. Each of the γ_i coefficient in (1.86) gives rise to a dimensionless group in the general solution. To be specific, writing the i -th basis vector \mathbf{a}_i in component form

$$\mathbf{a}_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \vdots \\ \gamma_i \end{pmatrix} \quad (1.87)$$

then the corresponding dimensionless product is

$$\Pi_i = p_1^{\alpha_i} p_2^{\beta_i} \cdots p_n^{\gamma_i} \quad (1.88)$$

As the \mathbf{a}_i 's are independent vectors (they form a basis of the null space of \mathbf{A}), the dimensionless groups $\Pi_1, \Pi_2, \dots, \Pi_k$ are independent. If we assume that the particular solution \mathbf{a}_p has components

$$\mathbf{a}_p = \begin{pmatrix} a \\ b \\ \vdots \\ c \end{pmatrix} \quad (1.89)$$

then the quantity

$$Q = p_1^a p_2^b \cdots p_n^c \quad (1.90)$$

has the same dimensions as q . Based on this, we can write the general product solution as

$$q = \alpha Q \Pi_1^{\kappa_1} \Pi_2^{\kappa_2} \cdots \Pi_k^{\kappa_k} \quad (1.91)$$

where $\alpha, \kappa_1, \kappa_2, \dots, \kappa_k$ are arbitrary numbers. This leads to the following theorem.

Theorem 1.3.1: Buckingham Pi Theorem

Let q, p_1, \dots, p_n be $n + 1$ dimensional variables in a given physical problem. Assuming that the relation

$$q = f(p_1, p_2, \dots, p_n) \quad (1.92)$$

is dimensionally homogeneous and dimensionally complete, then it is possible to reduce it to one of the form

$$q = Q F(\Pi_1, \Pi_2, \dots, \Pi_k), \quad (1.93)$$

where $\Pi_1, \Pi_2, \dots, \Pi_k$ are **independent dimensionless groups** (also called Pi groups) formed by products and ratios of the elements of the set $\{p_1, p_2, \dots, p_n\}$. The quantity Q is a dimensional group of $\{p_1, p_2, \dots, p_n\}$ with the same dimension as q . Thus, we can reduce the formula for q from a function with n variables down to one with k variables, where k is the nullity of the dimension matrix ($0 \leq k \leq n - 1$) which depends on the set $\{p_1, p_2, \dots, p_n\}$. In the case where $k = 0$, then the function F reduces to a constant and the conclusion is that

$$q = \alpha Q \quad (1.94)$$

where α is an arbitrary number.

Procedure in practice

One can obviously proceed along the linear algebra route as we devised above. Now, in practice, the Buckingham Pi theorem provides the theoretical grounding for the following procedure

Step 1 Devise the n relevant variables involved in the problem $\{p_1, p_2, \dots, p_n\}$ given a physical quantity q . You thus have in total $n + 1$ physical variables in your problem.

Step 2 Express each variable in terms of the m fundamental dimensions (in general, $\{M, L, T\}$ will be sufficient but one may need to include θ in problems where temperature is involved).

Step 3 Select m dimensionally distinct primary scaling variables (also called repeating variables), where m is the number of independent dimensions. We can always order the list of variables such that the primary variables are the m first variables $\{p_1, p_2, \dots, p_m\}$.

Step 4 Form a dimensional group Q with the m primary scaling variables with the same dimensions as q .

Step 5 For each of the $n - m$ remaining variables, construct a non-dimensional Π by taking one of the non-repeating variables and dividing it by a product of the repeating variables raised to the proper powers such that this Π is made non-dimensional. The Π s will have the form

$$\Pi_i = \frac{p_{m+i}}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}}, \text{ for } i = 1, \dots, n - m. \quad (1.95)$$

Step 6 Express the problem as a functional relationship between the Π groups and think about physically what this relationship implies.

At this point, a couple observations need to be made:

- The hardest part in dimensional analysis is often to **determine which are actually the relevant variables**; this is the part that requires a bit of practice and physical common sense.
- To help **ensure dimensional independence** when using the $\{M, L, T\}$ system, it is common to pick the repeating variables as one purely geometric quantity, one kinematic quantity (not involving mass) and one dynamic quantity (containing mass or force).
- If $\Pi_1, \Pi_2, \dots, \Pi_{n-m}$ represent dimensionless groups for the problem at hand, **one can replace any Π group by any powers or products of Π groups** granted that one keeps the right number of independent dimensionless groups!
- It may seem like it is possible to derive useful information about the system under study without getting involved with the laws of physics or potentially difficult mathematical problems. **A word of caution:** the method relies heavily on knowing the fundamental laws for the problem under study, particularly when devising the relevant variables. For instance, without the knowledge of the equations of motion for fluids, we would not have known to include in our relevant variables the dynamic viscosity of the fluid.

So how can dimensional analysis be relevant to PDEs? In what follows, we provide two examples.

1.3.6 Similarity variables

Dimensional analysis can be used to simplify complex mathematical problems. For instance, consider a semi-infinite thin pipe filled with stagnating water parametrized by $0 < x < \infty$. We will assume that at $x = 0$ a source of (blue) pollutant maintains the mass density of pollutant at a fixed value $u_0 > 0$ (see Fig. 1.3). For $t < 0$, the pollutant is not in contact with the water in the channel but at $t = 0$, we open the gate at $x = 0$. Can we determine the mass density of pollutant $u(x, t)$ in the channel over time?

We will see in Chapter 3 that the mass density of pollutant satisfies the so-called diffusion equation, which is written in this 1D problem

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \infty, t > 0 \quad (1.96)$$

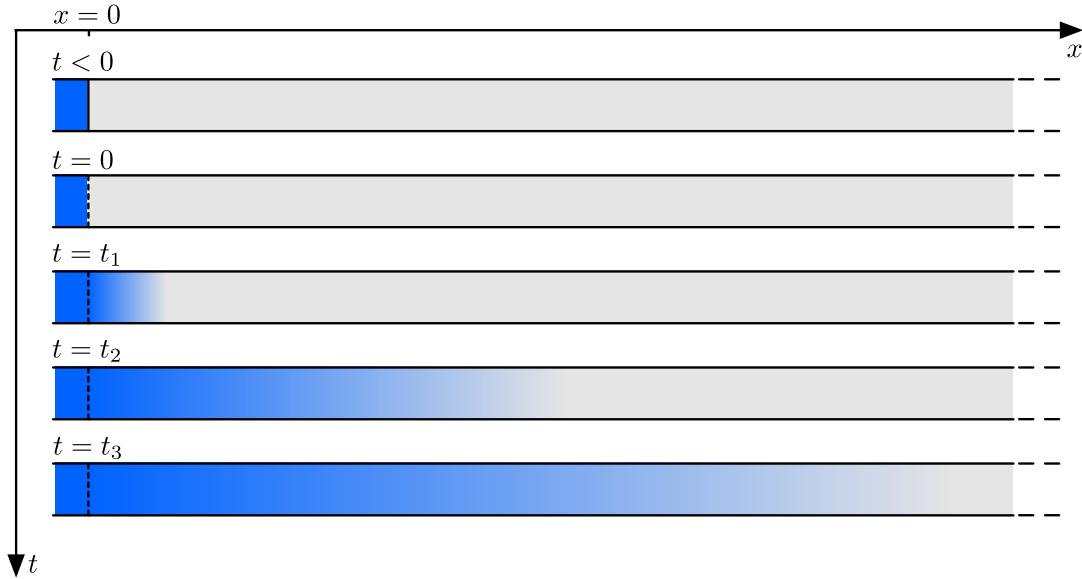


Figure 1.3 Evolution of the mass density of pollutant in a semi-infinite channel.

where D is the diffusion constant of the pollutant. The boundary and initial conditions associated to our problem are given by

$$u(0, t) = u_0 \quad (1.97)$$

$$u(\infty, t) = 0 \quad (1.98)$$

$$u(x, 0) = 0 \quad (1.99)$$

We can use dimensional reduction to find u in this specific problem. Given $u(x, t)$, we have

$$\left[\frac{\partial u}{\partial t} \right] = \left[\frac{[u]}{[t]} \right], \quad \left[\frac{\partial u}{\partial x} \right] = \left[\frac{[u]}{[x]} \right], \quad \text{and} \quad \left[\frac{\partial^2 u}{\partial x^2} \right] = \left[\frac{[u]}{[x]^2} \right], \quad (1.100)$$

which comes directly from the definition of the derivatives.

Now, for (1.96) to be dimensionally homogeneous, LHS and RHS must have the same dimensions; this means

$$\left[\frac{\partial u}{\partial t} \right] = \left[D \frac{\partial^2 u}{\partial x^2} \right] \quad (1.101)$$

Because $[u] = ML^{-3}$, we can therefore determine that the dimensions of the diffusion coefficient are $[D] = L^2T^{-1}$.

The only dimensional variables in this problem are u , u_0 , D , x and t . Thus, it must be true that

$$u = f(x, t, D, u_0) \quad (1.102)$$

Let us find (a, b, c, d) such that

$$[u] = [x^a t^b D^c u_0^d] \quad (1.103)$$

Using the fundamental dimensions $\{M, L, T\}$ and equating their exponents, we get the following system

$$\begin{cases} L : & a + 2c - 3d = -3 \\ T : & b - c = 0 \\ M : & d = 1 \end{cases} \quad (1.104)$$

It is easy to find the solution of this system as $d = 1$ and $b = c = -a/2$. We finally obtain that the general product solution is

$$u = \alpha u_0 \left(\frac{x}{\sqrt{Dt}} \right)^a \quad (1.105)$$

Thus the general solution has the form

$$u = u_0 F(\eta) \quad \text{with} \quad \eta = \frac{x}{\sqrt{Dt}} \quad (1.106)$$

The dimensionless group η is called a **similarity variable** in this case as it is a dimensionless product that involves the independent variables in the problem. Interestingly, we have determined that the mass density $u(x, t)$ does not depend independently on x and t but on a single intermediate variable η . We still need to determine the unknown function F . We substitute (1.106) back into the diffusion equation to find an equation for F . Using the chain rule, we obtain

$$\frac{\partial u}{\partial t} = u_0 F'(\eta) \frac{\partial \eta}{\partial t} = -u_0 F'(\eta) \frac{\eta}{2t} \quad (1.107)$$

and

$$\frac{\partial^2 u}{\partial x^2} = u_0 F''(\eta) \frac{1}{Dt} \quad (1.108)$$

In summary, we obtain the following equation for $F(\eta)$

$$F'' = -\frac{1}{2} \eta F'(\eta) \quad (1.109)$$

with $0 < \eta < \infty$ as $0 < x < \infty$ and $t > 0$. We must also transform the boundary and initial conditions:

- Letting $x = 0$ in (1.106) yields $u_0 F(0) = u_0$ so we conclude that $F(0) = 1$.
- Letting $x \rightarrow \infty$ in (1.106) yields $u_0 F(\infty) = 0$ so we conclude that $F(\infty) = 0$.
- As $\eta = x/\sqrt{Dt}$, we must deal with the initial condition using the following limit

$$\lim_{t \rightarrow 0^+} u_0 F \left(\frac{x}{\sqrt{Dt}} \right) = 0 \quad (1.110)$$

which gives $F(\infty) = 0$ for $0 < x < \infty$.

We have thus replaced our original complicated PDE problem with the following ODE problem

$$F'' = -\frac{1}{2} \eta F', \quad 0 < \eta < \infty \quad (1.111)$$

$$F(0) = 1 \quad (1.112)$$

$$F(\infty) = 0 \quad (1.113)$$

Letting $G = F'$, (1.111) takes the form $G' = -\eta G/2$ whose general solution reads

$$G(\eta) = \alpha \exp(-\eta^2/4) \quad (1.114)$$

Now, we can integrate G to obtain

$$F(\eta) = \beta + \alpha \int_0^\eta \exp(-\xi^2/4) d\xi \quad (1.115)$$

Using (1.112), we obtain $\beta = 1$ and from (1.113), we get

$$1 + \alpha \int_0^\infty \exp(-\xi^2/4) d\xi = 0. \quad (1.116)$$

Given that $\int_0^\infty e^{-\xi^2/4} d\xi = \sqrt{\pi}$, we finally obtain

$$F(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int_0^\eta \exp(-\xi^2/4) d\xi \quad (1.117)$$

We recognize here the definition of the **complementary error function** defined as

$$\text{erfc}(z) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2} dr \quad (1.118)$$

Therefore, we finally obtain the following solution for our diffusion problem (see Fig. 1.4).

$$u(x, t) = u_0 \text{erfc}\left(\frac{x}{2\sqrt{Dt}}\right) \quad (1.119)$$

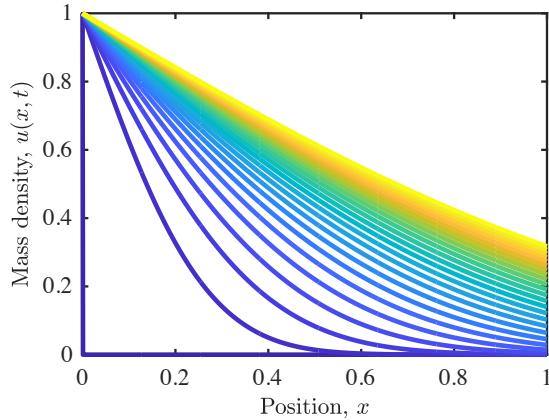


Figure 1.4 Mass density of pollutant $u(x, t)$ in the channel, for $0 < t < 1$ with time increasing from blue to yellow. Here, we used $u_0 = 1$ and $D = 1$.

Using similarity variables and dimensional analysis provides a powerful tool for solving PDEs, including nonlinear PDEs. However, note that the problem must have a specific form to work; for instance, this method would not work in a finite channel. We will see other examples of this technique being applied in further chapters.

1.3.7 Nondimensionalization of equations

Finally, another use of dimensional analysis is to transform a problem into dimensionless form. This is particularly important when trying to devise well-controlled approximation methods (which are in general based on comparisons). We will highlight this concept via a simple example. Consider the motion of a projectile thrown directly upwards in the earth gravitational field. We quickly wrote earlier that the motion of the projectile in the absence of air resistance is governed by the following equation

$$\frac{d^2y}{dt^2} = -g \quad (1.120)$$

where g is the gravitational acceleration constant. In reality, this is already an approximation and is only valid sufficiently close to the surface of the earth. Let's try to understand

why. Newton's law of universal gravitation states that there is a gravitational force between two massive objects with masses m_1 and m_2 and that this force has for magnitude

$$F = G \frac{m_1 m_2}{r^2} \quad (1.121)$$

where r^2 is the distance between the two objects. If one considers that *object 1* is the earth and that *object 2* is our projectile (of mass m), then using Newton's second law, we can write that

$$m \frac{d^2y}{dt^2} = G \frac{mM}{(y+R)^2} = -m \frac{gR^2}{(y+R)^2} \quad (1.122)$$

where we have introduced $g = -GM/R^2$, where M is the mass of the earth and R is the radius of the earth. This shows that the **actual** problem we are trying to solve is given by

$$\frac{d^2y}{dt^2} = -g \frac{R^2}{(y+R)^2} \quad (1.123)$$

$$\frac{dy}{dt}(0) = v_0 \quad (1.124)$$

$$y(0) = 0 \quad (1.125)$$

It is tempting to just assume $R \gg y$ to write that $y+R \approx R$. While this is easy to do here, in many cases, it will not be this simple and so we may want to find a more general procedure.

We will do this by scaling the variables in the problem using characteristic values. The first step in nondimensionalizing a problem is to introduce a change of variable. Here, this takes the form

$$t = t_c \tau \quad \text{and} \quad y = y_c u \quad (1.126)$$

In this change of variables, y_c and t_c are constants representing characteristic values for x and t respectively (so characteristic lengthscale and timescales). In some problems, it is quite clear from the outset what these characteristics values are, in other not so. Here, we will assume that we have no clue and proceed very generally. These characteristic scales will be determined using the physical parameters in the problem, i.e. here g , R and v_0 . Using the chain rule, we notice that

$$\frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = \frac{1}{t_c} \frac{d}{d\tau} \quad (1.127)$$

and

$$\frac{d^2}{dt^2} = \frac{d}{dt} \left(\frac{d}{dt} \right) = \frac{1}{t_c^2} \frac{d^2}{d\tau^2} \quad (1.128)$$

With this in mind, the projectile equation takes the form

$$\frac{1}{t_c^2} \frac{d^2}{d\tau^2} (y_c u) = -g \frac{R^2}{(y_c u + R)^2} \quad (1.129)$$

We will collect the parameters in dimensionless groups once again. There is no unique way to do this; for instance, one could nondimensionalize the the denominator in the RHS of (1.130) as either $R(1 + y_c u / R)$ or $y_c(u + R/y_c)$. Using the former, we can cancel the R in the numerator and so we will adopt this nondimensionalization which yields

$$\Pi_1 \frac{d^2u}{d\tau^2} = -\frac{1}{(1 + \Pi_2 u)^2} \quad (1.130)$$

with the following initial conditions

$$u(0) = 0 \quad \text{and} \quad \frac{du}{d\tau}(0) = \Pi_3 \quad (1.131)$$

where we have defined the following dimensionless groups

$$\Pi_1 = \frac{y_c}{gt_c^2}, \quad \Pi_2 = \frac{y_c}{R}, \quad \Pi_3 = \frac{t_c v_0}{y_c} \quad (1.132)$$

Note that the three dimensionless groups we defined do not involve the variables u or τ , they only depend on the physical parameters of the problem. Furthermore, the Π 's are independent in the sense that it is not possible to write any of them in terms of the other two. These can be physically understood as:

- Π_1 measures the characteristic acceleration y_c/t_c^2 compared to the acceleration due to gravity g ;
- Π_2 measures the characteristic height of the projectile y_c compared to the radius of the earth R ;
- Π_3 measures the characteristic velocity x_c/t_c compared to the velocity the projectile starts with v_0 .

Now, we need to decide what to take for our characteristic scales y_c and t_c . We have two parameters to determine, one way to go about this is by setting two of the dimensionless groups equal to 1. We just need to decide which two to pick out of the three. To guide our choice, we can follow the following rules of thumb.

Rule of Thumb 1 — Pick the Π 's that appear in the initial and/or boundary conditions.

In this problem, the only dimensionless group involved in our initial conditions is Π_3 . Setting $\Pi_3 = 1$, we conclude that $y_c = v_0 t_c$.

Rule of Thumb 2 — Pick the Π 's that appear in the reduced problem.

In our problem, we want to assume that the projectile height is small compared to the radius of the earth, which is equivalent to saying that Π_2 is small. The reduced problem is the one obtained by taking $\Pi_2 \rightarrow 0$. Taking this limit,

$$\Pi_1 \frac{d^2 u}{d\tau^2} = -1, \quad u(0) = 0, \quad \frac{du}{d\tau}(0) = 1 \quad (1.133)$$

and the only dimensionless group appearing in the reduced problem is Π_1 . Thus, we set $\Pi_1 = 1$ and obtain $y_c = v_0^2/g$, which should not surprise you if you remember that we obtained earlier that the maximum height of the projectile was $y_M \propto v_0^2/g$.

Combining these results, we obtain $y_c = v_0^2/g$ and $t_c = v_0/g$. With this scaling, our projectile problem takes the form

$$\frac{d^2 u}{d\tau^2} = -\frac{1}{(1 + \varepsilon u)^2} \quad (1.134)$$

with

$$u(0) = 0 \quad \text{and} \quad \frac{du}{d\tau}(0) = 1 \quad (1.135)$$

and ε a dimensionless parameter defined as

$$\varepsilon = \frac{v_0^2}{gR} \quad (1.136)$$

We can evaluate this parameter: $R \approx 6.4 \times 10^6$ m and $g \approx 9.8 \text{m/s}^2$, then $\varepsilon \approx 1.6 \times 10^{-8} v_0^2$. So for all everyday situations, where v_0 is not particularly large, ε is very small and the approximation we crudely wrote earlier is valid. This small parameter plays a crucial role when constructing accurate approximations of the solution to the projectile motion. Here, we only use it to simplify the equation of motion in a controlled way.