Appendix A

Fourier Theory

Throughout this module, we make use of Fourier theory. This topic was covered in details in **MATH40004 - Calculus and Applications**. In this Appendix, we discuss very briefly Fourier series and Fourier transforms with the aim of recapitulating important results which will be stated without proof.

A.1 Properties of functions

Orthonormal systems

A sequence of integrable functions $\{\phi_i\}_{i=1}^{\infty}$ on an interval $[a, b]$ is called **orthogonal** if

$$\int_a^b \phi_i(x)\phi_j(x)dx = 0, \quad \text{for } i \neq j \tag{A.1}$$

If in addition

$$\int_a^b (\phi_i(x))^2 dx = 1, \quad \text{for all } i, \tag{A.2}$$

the system is said to be orthonormal.
Example

> The functions
>
> $$\frac{1}{\sqrt{2\pi}}, \ \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \quad (n = 1, 2, \dots) \tag{A.3}$$
>
> form an orthonormal system over the interval $[-\pi, \pi]$ as it can easily be shown that
>
> $$\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi}}\cos(nx)dx = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi}}\sin(nx)dx = 0 \tag{A.4}$$
>
> $$\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\sin(mx)\frac{1}{\sqrt{\pi}}\cos(nx)dx = 0 \tag{A.5}$$
>
> $$\int_{-\pi}^{\pi} \left[\frac{1}{\sqrt{2\pi}}\right]^2 dx = 1 \tag{A.6}$$
>
> $$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{A.7}$$
>
> where δ_{mn} is the Kronecker delta.

Periodic functions

A function f is periodic with period T if

$$f(x + nT) = f(x) \tag{A.8}$$

for all values of x and $n \geq 1$ an integer.
Example

> The functions $\sin nx$ and $\cos nx$ are periodic with period $2\pi/n$. Further, the finite sum
>
> $$\sum_{n=1}^{N} a_n \cos nx + b_n \sin nx \tag{A.9}$$

is a sum of functions with periods 2π, $2\pi/2$, $2\pi/3$, $2\pi/4\ldots$; the overall period is therefore 2π, i.e. determined by the $n = 1$ mode.

Odd and even functions

- A function $f(x)$ is **even** about $x = a$ if $f(a + x) = f(a - x)$ for all x.
- A function $f(x)$ is **odd** about $x = a$ if $f(a + x) = -f(a - x)$ for all x.

When integrating even and odd functions over a range centered on the line of symmetry, you have the following results

If $f(x)$ is even about $x = a$, then $$\int_{a-L}^{a+L} f(x)dx = 2\int_{a}^{a+L} f(x)dx \tag{A.10}$$

If $g(x)$ is odd about $x = a$, then $$\int_{a-L}^{a+L} g(x)dx = 0 \tag{A.11}$$

A.2 Fourier series

Full-range Fourier series

First, we consider Fourier series over the range $[-\pi, \pi]$. Let $f(x)$ be a periodic function with period 2π. The Fourier series for $f(x)$ is the representation of $f(x)$ as a series in $\sin nx$ and $\cos nx$ of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} \tag{A.12}$$

where a_n and b_n are constant to be determined called Fourier coefficients. It can easily be shown by using the orthogonality of the family of functions $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}}\cos(nx)$, $\frac{1}{\sqrt{\pi}}\sin(nx)$ $(n = 1, 2, \ldots)$ that the coefficients appearing in this Fourier expansion are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{A.13}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad (n = 1, 2, \ldots) \tag{A.14}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad (n = 1, 2, \ldots) \tag{A.15}$$

Note that if $f(x)$ is even about $x = 0$, then $b_n = 0$ and conversely, if $f(x)$ is odd about $x = 0$, then $a_n = 0$ (for all values of n). To represent a function by a Fourier series, we only require the function to be piecewise continuous (i.e. to have a finite number of finite discontinuities). Further, it is also important to realize that provided you only want to represent a function by a Fourier series over a specific range, the function itself need not be periodic (but its Fourier series representation will be by definition).

Example

The Fourier series representation of $f(x) = \pi^2 - x^2$ over the range $-\pi \leq x \leq \pi$ is written

$$\pi^2 - x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} \tag{A.16}$$

with Fourier coefficients given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)dx = \frac{4}{3}\pi^2 \tag{A.17}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos(nx)dx = \frac{4(-1)^{n+1}}{n^2} \tag{A.18}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\sin(nx)dx = 0 \tag{A.19}$$

(where we have integrated by parts twice and used the fact that $\pi^2 - x^2$ is even about $x = 0$).

Parseval's theorem

If $f(x)$ is represented by the following Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}, \quad (\pi \leq x \leq \pi) \tag{A.20}$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} \{a_n^2 + b_n^2\} \tag{A.21}$$

Fourier series over a general interval

The theory of Fourier series is easily generalized to an arbitrary interval by observing that the functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ have period $2L/n$ and the set of functions

$$\frac{1}{\sqrt{2L}},\ \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, \ldots) \tag{A.22}$$

are orthonormal over the interval $[a, a+2L]$ where a is any real number, i.e. that we have

$$\int_a^{a+2L} \sin\left(\frac{n\pi x}{L}\right) dx = \int_a^{a+2L} \cos\left(\frac{n\pi x}{L}\right) dx = 0 \tag{A.23}$$

$$\int_a^{a+2L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 \tag{A.24}$$

$$\int_a^{a+2L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \int_a^{a+2L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = L\delta_{mn} \tag{A.25}$$

Using these results, we can represent the function $f(x)$ over the interval $[a, a+2L]$ in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right)\right\} \tag{A.26}$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \ldots \tag{A.27}$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \ldots \tag{A.28}$$

So it follows that if $f(x)$ is integrable over a finite interval, a Fourier series can be found $f(x)$ over this interval. Similarly, Parseval's theorem can be generalized and written on a general interval $[a, a+2L]$ as

$$\frac{1}{\pi}\int_{a}^{a+2L}[f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left\{a_n^2 + b_n^2\right\} \tag{A.29}$$

Half-range Fourier series

We have defined above the so-called full-range Fourier series (i.e. the Fourier series representation of a function over one full period of the series). Now suppose that we have a function $f(x)$ defined over the range $[-L, L]$. If we now consider the following two real functions defined over the same interval

$$f_1(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x \le 0 \end{cases} \tag{A.30}$$

$$f_2(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x \le 0 \end{cases} \tag{A.31}$$

Clearly, $f_1(x)$ is even about $x = 0$ and hence, the Fourier coefficients $b_n = 0$ in its Fourier expansion over $[-L, L]$, i.e.

$$f_1(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad (-L \le x \le L) \tag{A.32}$$

with

$$a_n = \frac{2}{L}\int_0^L f_1(x)\cos\left(\frac{n\pi x}{L}\right) \quad (n = 0, 1, 2, \dots) \tag{A.33}$$

Conversely, the function $f_2(x)$ is odd about $x = 0$ and hence, the Fourier expansion is given by

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (-L \le x \le L) \tag{A.34}$$

with

$$b_n = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, \dots) \tag{A.35}$$

Now as by definition both f_1 and f_2 are equal to f over the range $[0, L]$, we therefore have two ways of representing f over this interval:

- using what is known as the **half-range Fourier cosine series**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad (-L \le x \le L) \tag{A.36}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \quad (n = 0, 1, 2, \dots) \tag{A.37}$$

- using what is known as the **half-range Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (-L \le x \le L) \tag{A.38}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, \dots) \tag{A.39}$$

Parseval's theorem for half-range Fourier series

An analogous result to Parseval's formula can be found for half-range series. These are

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \begin{cases} a_0^2/2 + \sum_{n=1}^\infty a_n^2, & \text{(Fourier cosine series)} \\ \sum_{n=1}^\infty b_n^2, & \text{(Fourier sine series)} \end{cases} \tag{A.40}$$

Exponential form of Fourier series

Alternatively, we can represent Fourier series using exponential functions. This alternative representation can sometimes simplify calculations. It is used frequently in engineering applications and writing the formulae in this way provides a clear link to Fourier transforms. Recalling that

$$\cos(nx) = \frac{e^{inx}+e^{-inx}}{2} \quad \text{and} \quad \sin(nx) = \frac{e^{inx}-e^{-inx}}{2i} \tag{A.41}$$

we can write

$$a_n\cos(nx) + b_n\sin(nx) = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx} \tag{A.42}$$

Therefore, we can write the Fourier series representation of $f(x)$ over the interval $[-\pi,\pi]$ in the form

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{inx} + d_n e^{-inx}\right\} \tag{A.43}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad (n = 0,1,2,\dots) \tag{A.44}$$

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \quad (n = 1,2,\dots) \tag{A.45}$$

Noticing that $c_{-n} = d_n$, we can express this more succinctly as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad -\pi \le x \le \pi \tag{A.46}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \pm 2, \dots \tag{A.47}$$

Finally, for a function of period $2L$, we can easily generalize this to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad -L \le x \le L \tag{A.48}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad n = 0, \pm 1, \pm 2, \dots \tag{A.49}$$

A.3 Fourier transforms

Fourier series allows us to represent a given function in terms of sine and cosine waves of different amplitudes and frequencies, but this representation is only valid over a finite range $[-L, L]$ of the independent variable. We now wish to study what happens if we take a Fourier series defined over this interval and let $L \to \infty$. Let us consider the

Fourier series representation of a function $f(x)$ as given by (A.48)-(A.49). We define the angular frequency

$$\omega_n = n\pi/L \tag{A.50}$$

and the frequency difference

$$\delta\omega = \omega_{n+1} - \omega_n = \pi/L \tag{A.51}$$

In terms of this new notation, the Fourier series becomes

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left\{\int_{-L}^{L} f(s)e^{-i\omega_n s}ds\right\} e^{i\omega_n x}\delta\omega \tag{A.52}$$

Now as we let $L \to \infty$, $\delta\omega \to 0$ and

$$\sum_{n=-\infty}^{\infty} g(\omega_n)\delta\omega \to \int_{-\infty}^{\infty} g(\omega)d\omega \tag{A.53}$$

(think about splitting the integral up into strips of width $\delta\omega$). So in the limit $L \to \infty$, we obtain that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds\right\} e^{i\omega x}d\omega \tag{A.54}$$

We have therefore shown that for a function $f(x)$ defined over $-\infty < x < \infty$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega \tag{A.55}$$

where

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{A.56}$$

The function $\widehat{f}(\omega)$ is known as the **Fourier transform** of $f(x)$. Equation (A.55) is known as the **inverse Fourier transform** as it enables $f(x)$ to be calculated from a knowledge of the transform function $\widehat{f}(\omega)$. It is common to denote the Fourier transform $\mathcal{F}\{f(x)\}$ and the inverse Fourier transform $\mathcal{F}^{-1}\{\widehat{f}(\omega)\}$.

Fourier cosine and sine transforms

In the same way we exploited symmetry to define half-range Fourier series, we can similarly define transforms over the range $[0,\infty)$. First, suppose that $f(x)$ is even about $x = 0$, we can write

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)dx = 2\int_{0}^{\infty} f(x)\cos(\omega x)dx \tag{A.57}$$

So we define

$$\widehat{f_c}(\omega) = \int_{0}^{\infty} f(x)\cos(\omega x)dx \tag{A.58}$$

to be the **Fourier cosine transform** of $f(x)$, which is even about $\omega = 0$. Note that for an even function $f(x)$:

$$\widehat{f}(\omega) = 2\widehat{f_c}(\omega) \tag{A.59}$$

Using the inversion formula for the regular transform, we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \widehat{f_c}(\omega)e^{i\omega x}d\omega \tag{A.60}$$

which reduces to

$$f(x) = \frac{2}{\pi}\int_0^\infty \widehat{f}_c(\omega)\cos(\omega x)d\omega \tag{A.61}$$

which is the **inversion formula for the Fourier cosine transform.** Similarly, by considering $f(x)$ an odd function about $x = 0$, we can define the **Fourier sine transform** and its associated inversion formula as

$$\widehat{f}_s(\omega) = \int_0^\infty f(x)\sin(\omega x)dx \tag{A.62}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \widehat{f}_s(\omega)\sin(\omega x)d\omega \tag{A.63}$$

Properties of Fourier transforms

In this section, we list useful properties of the Fourier transforms:

- **Linearity of the Fourier transform:**

  $$\mathcal{F}\{\lambda f(x) + \mu g(x)\} = \lambda\widehat{f}(\omega) + \mu\widehat{g}(\omega) \tag{A.64}$$

  and

  $$\mathcal{F}^{-1}\{\lambda\widehat{f}(\omega) + \mu\widehat{g}(\omega)\} = \lambda f(x) + \mu g(x) \tag{A.65}$$

  with $(\lambda, \mu) \in \mathbb{R}^2$.

- If $\lambda \neq 0$, then

  $$\mathcal{F}\{f(\lambda x)\} = \frac{1}{|\lambda|}\widehat{f}\,(\omega/\lambda) \tag{A.66}$$

- Which gives us the time-reversal property

  $$\mathcal{F}\{f(-x)\} = \widehat{f}\,(-\omega) \tag{A.67}$$

- The transform of a shifted function is obtained as follows

  $$\mathcal{F}\{f(x - x_0)\} = e^{-i\omega x_0}\widehat{f}(\omega) \tag{A.68}$$

- For a shift in transform space, we also have the following relation

  $$\mathcal{F}\{e^{i\omega x_0}f(x)\} = \widehat{f}(\omega - \omega_0) \tag{A.69}$$

- **Symmetry formula:** let us denote the Fourier transform of $f(x)$ by $\widehat{f}(\omega)$, by the change of variable $\omega \to x$, we have

  $$\mathcal{F}\{\widehat{f}(x)\} = 2\pi f(-\omega) \tag{A.70}$$

- **Derivatives and Fourier transforms:**

  $$\mathcal{F}\left\{\frac{d^n f}{dx^n}\right\} = (i\omega)^n\widehat{f}(\omega) \tag{A.71}$$

- We also have that

  $$\mathcal{F}\left\{xf(x)\right\} = i\widehat{f}'(\omega) \tag{A.72}$$

- **Derivatives and Fourier cosine/sine transforms:**

$$\text{(a)} \qquad \mathcal{F}_c\left\{f'(x)\right\} = -f(0) + \omega\widehat{f}_s(\omega) \tag{A.73}$$

$$\text{(b)} \qquad \mathcal{F}_s\left\{f'(x)\right\} = -\omega\widehat{f}_c(\omega) \tag{A.74}$$

$$\text{(c)} \qquad \mathcal{F}_c\left\{f''(x)\right\} = -f'(0) - \omega^2\widehat{f}_c(\omega) \tag{A.75}$$

$$\text{(d)} \qquad \mathcal{F}_s\left\{f''(x)\right\} = \omega f(0) - \omega^2\widehat{f}_s(\omega) \tag{A.76}$$

- If $f(x)$ is a complex-valued function and $[f(x)]^*$ is its complex conjugate, then

$$\mathcal{F}\left\{[f(x)]^*\right\} = [\widehat{f}(-\omega)]^* \tag{A.77}$$

Finally, two theorems often prove useful!

Convolution theorem

We define the convolution of two functions $f(x)$ and $g(x)$ defined over $(-\infty, \infty)$ as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \tag{A.78}$$

The convolution theorem states that

$$\mathcal{F}\left\{f * g\right\} = \widehat{f}(\omega)\widehat{g}(\omega) \tag{A.79}$$

i.e. the Fourier transform of the convolution is the product of the Fourier transforms.

Energy theorem

Finally, the energy theorem is the analogous result to Parseval's theorem for Fourier series. It states that if $f(x)$ is a real-valued function, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f}(\omega)\right|^2 d\omega = \int_{-\infty}^{\infty}[f(x)]^2\, dx \tag{A.80}$$

*[This version was compiled on **March 20, 2024**.]*