$$\min \quad x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 \leq 1$$
$$x_3 \leq 1$$

i) Convex? Yes, convex constraints, cost is Q.F.

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succcurlyeq 0 \quad \text{(Diag. dominant)}$$

ii) and iii) Note that $\exists\ \hat{\underline{x}} = (0,0,0)$ such that

$$\begin{matrix} 0+0+0<1 \\ 0<1 \end{matrix} \quad + \text{ convexity} \Rightarrow \text{Strong duality.}$$

$\Rightarrow$ No need to solve primal and dual separately, it is enough to solve dual and recover primal solution from there. Otherwise use KKT for primal.

$$L(\underline{x}, \underline{\lambda}) = x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 + \lambda_1(x_1 + x_2 + x_3 - 1)$$
$$+ \lambda_2(x_3 - 1)$$

$$g(\underline{\lambda}) = \underbrace{\min_{x_1, x_2, x_3} L(x_1, x_2, \underline{\lambda})}_{} \qquad (\underline{\lambda} \geq 0)$$

$$\nabla_{\underline{x}} L = 0 \iff \begin{aligned} &2x_1 + 2x_2 + 1 + \lambda_1 = 0 \\ &4x_2 + 2x_1 - 1 + \lambda_1 = 0 \\ &-1 + \lambda_1 + \lambda_2 = 0 \end{aligned}$$

$$\Rightarrow 2x_2 - 2 = 0 \Rightarrow x_2 = 1 \Rightarrow 3 + 2x_1 + \lambda_1 = 0$$
$$x_1 = -\frac{(3+\lambda_1)}{2}$$

$\hookrightarrow$ needs $1 = \lambda_1 + \lambda_2$, otherwise min is reached with $x_3 = \pm\infty$

In this case, evaluating $L(\underline{x}, \underline{\lambda})$ at $x_2 = 1$ and $x_1 = -(3+\lambda_1)/2$

$$\Rightarrow q(\underline{\lambda}) = -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4}$$

$$\Rightarrow q(\underline{\lambda}) = \begin{cases} -\infty & \text{if } \lambda_1 + \lambda_2 \neq 1 \\ -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4} & \text{otherwise} \end{cases}$$

and the dual problem is

$$\max_{\lambda_1 \geq 0} \; -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4}$$

which is maximized at $\lambda_1^* = 0 \Rightarrow \lambda_2^* = 1$

and $q^* = -9/4$.

back to the primal problem,

$x_2^* = 1$, $x_1^* = -3/2$ and $x_3^* = 1$ //