M40007: Introduction to Applied Mathematics

Imperial College London

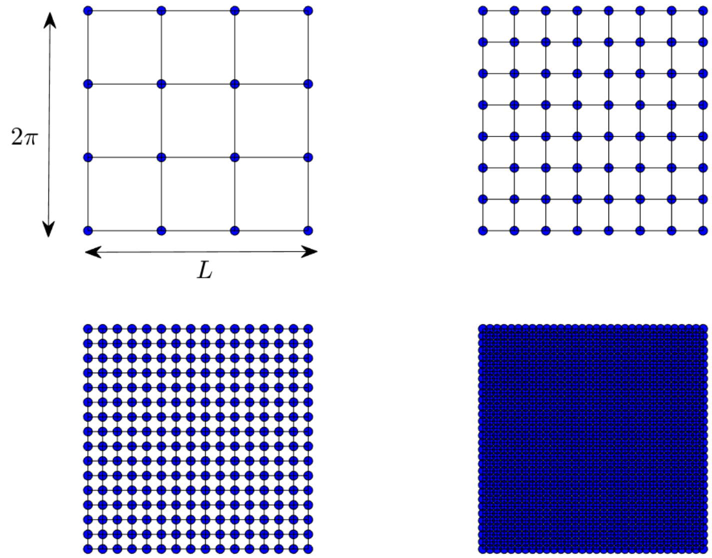

Figure 1: Increasing the number of nodes N in a rectangular grid graph to fill out a rectangular region: N nodes are placed on each horizontal line $y = -\pi + 2\pi j/(N-1)$ for $j = 0, \ldots, N-1$.

1 Two-terminal conduction in a rectangle

Figure 1 shows a selection of $(N+1)$-by-$(N+1)$ regular grid graphs, for increasing values of N, with a node at (x_j, y_k) for any choice of $j, k = 0, \ldots, N$ where

$$x_j = L\left(-1 + \frac{j}{N}\right), \qquad y_k = \pi\left(-1 + \frac{2k}{N}\right), \qquad j, k = 0, \ldots, N. \tag{1}$$

meaning that $(N+1)^2$ nodes eventually fill out the rectangular region

$$-L \leq x \leq 0, \qquad -\pi \leq y \leq \pi \tag{2}$$

as $N \to \infty$.

Suppose now that such a grid graph is an electric circuit where, as shown in Figure 2 for $N = 9$, the left-most nodes at $x = -L$ are all set to unit voltage and all the right-most nodes at $x = 0$ are grounded. By the symmetry of this circuit arrangement we expect all nodes having the same x-coordinate to be at the same potential. It is therefore sufficient to consider any one row of the grid parallel to the x axis, indeed, without loss of generality, we will study the single-row subgraph of N nodes located on the x axis, between the dashed red lines in Figure 2 and reproduced under the grid graph in this figure.

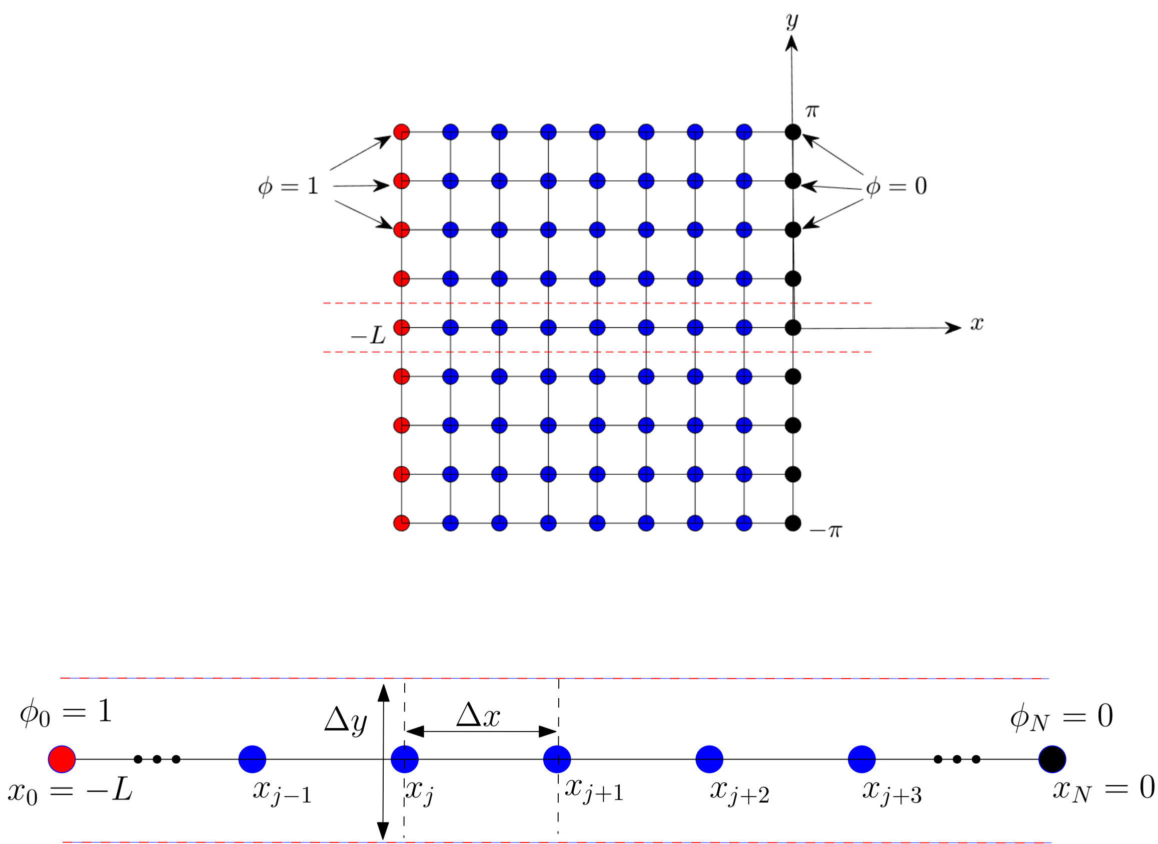

Figure 2: Typical $(N+1)$-by-$(N+1)$ grid with $N = 8$. All red notes on $x = -L$ are at unit potential, all black nodes at $x = 0$ are grounded and KCL holds at all blue nodes. By the symmetry of the arrangement and the boundary conditions, all nodes with the same x coordinate are expected to be at the same potential. It is therefore enough to consider the single row of nodes on the x axis between dashed red lines.

Let x_j for $0 \leq j \leq N$ be the location of node $\boxed{\text{j}}$ in the interval $-L \leq x \leq 0$ with $\Delta x = L/N$ being the length of each edge between nodes; let $\Delta y = 2\pi/N$ denote the length of the edges joining this single-row subgraph to its neighbouring rows. It is clear that

$$x_j = -L + j\Delta x, \qquad j = 0, \dots, N. \tag{3}$$

Let ϕ_j be the potential at node $\boxed{\text{j}}$ of this single-row subgraph for $0 \leq j \leq N$. Let the conductance c_j of the edge between x_j and x_{j-1} then

$$c_j = \frac{\hat{c}\Delta y}{\Delta x}, \tag{4}$$

where $\hat{c}$ is the uniform conductivity constant. Then by KCL at interior nodes, and Ohm's law,

$$(\phi_{j+1} - \phi_j)c_{j+1} = (\phi_j - \phi_{j-1})c_j \tag{5}$$

or

$$(\phi_{j+1} - \phi_j)\frac{\hat{c}\Delta y}{\Delta x} = (\phi_j - \phi_{j-1})\frac{\hat{c}\Delta y}{\Delta x} \tag{6}$$

Defining

$$p_j = \phi_j - \phi_{j-1}, \qquad j = 1, \dots, N \tag{7}$$

then (6) implies

$$p_{j+1} = p_j \tag{8}$$

which, by inspection, has the solution

$$p_j = D, \qquad j = 1, \dots, N \tag{9}$$

for some constant D. With $\phi_0 = 1$,

$$\begin{aligned}
\phi_1 &= \phi_0 + D = 1 + D, \\
\phi_2 &= \phi_1 + D = 1 + D + D \\
\phi_3 &= \phi_2 + D = 1 + +D + D + D \\
&\vdots
\end{aligned} \tag{10}$$

so that

$$\phi_j = 1 + jD, \qquad j = 0, \dots, N. \tag{11}$$

But $\phi_N = 0$ implies

$$D = -\frac{1}{N}. \tag{12}$$

Hence the solution is

$$\phi_j = 1 - \frac{j}{N} = -\frac{x_j}{L}, \qquad j = 0, \dots, N. \tag{13}$$

Consequently, we infer that the solution over the whole grid graph is

$$\phi(x_j, y_k) = -\frac{x_j}{L}, \qquad j, k = 0, \ldots, N \tag{14}$$

where we use the notation $\phi(x_j, y_k)$ to denote the value of the potential at each of the node points (1).

It is clear that the continuum limit of this solution as $N \to \infty$ when $x_j \mapsto x, y_k \mapsto y$ and $\phi(x_j, y_k) \mapsto \phi(x, y)$ is

$$\phi(x, y) = -\frac{x}{L}. \tag{15}$$

It is of interest to compute the effective conductance, C_{eff}. By definition, this is the total current into the circuit from the unit-voltage nodes. There are N such nodes, all with the same current divergence equal to

$$c_1(1 - \phi_1), \tag{16}$$

by the symmetry of the arrangement. The effective conductance is therefore

$$Nc_1(1-\phi_1) = N\left(\frac{\hat{c}\Delta y}{\Delta x}\right)(1 - (1 + D)) = N\left(\frac{\hat{c}\Delta y}{\Delta x}\right)\left(1 - \left(1 - \frac{1}{N}\right)\right) = \frac{\hat{c}\Delta y}{\Delta x}. \tag{17}$$

Since $\Delta x = L/N$ and $\Delta y = 2\pi/N$ then, as $N \to \infty$,

$$C_{\text{eff}} \to \frac{2\pi\hat{c}}{L}. \tag{18}$$

2 Two-terminal conduction in an annulus

Figure 3 shows a selection of $(N+1)$-by-$(N+1)$ annular grid graphs in an (X, Y) plane, for increasing values of N. Introducing the polar coordinates (R, Θ) where

$$R^2 = X^2 + Y^2, \qquad \Theta = \tan^{-1}(Y/X) \tag{19}$$

then nodes are placed at (R_j, Θ_k) for any choice of $j, k = 0, \ldots, N$ where

$$R_j = \rho + \frac{(1-\rho)j}{N}, \qquad \Theta_k = \frac{2\pi k}{N+1}, \qquad j, k = 0, \ldots, N, \tag{20}$$

meaning that $(N+1)^2$ nodes eventually fill out the annular region

$$\rho \le R \le 1, \qquad -\pi \le \Theta \le \pi \tag{21}$$

as $N \to \infty$.

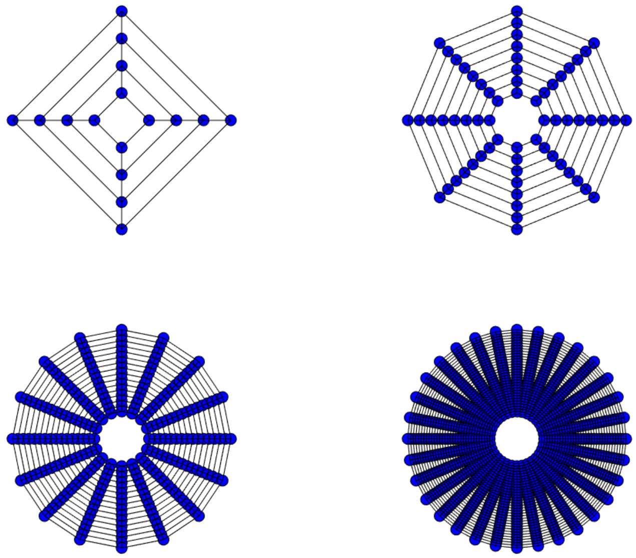

Figure 3: Increasing the number of nodes N to fill out an annular region: N nodes are placed on each ray at angle $2\pi j/N$ for $j = 0, \ldots, N-1$

Suppose now that this graph is an electric circuit with all the nodes at radial distance ρ from the origin being at unit voltage and all those at unit radial distance from the origin being grounded as shown in Figure 4 for $N = 8$. By the symmetry of this graph and the chosen boundary conditions, we expect all nodes at the same radial distance from the origin to be at the same potential. It is then sufficient to consider the subgraph of $N+1$ nodes on the X axis; this corresponds to $k = 0$ so that $\Theta = \Theta_0$ and we can write $R_j = X_j$ where

$$X_j = \rho + j\Delta X, \qquad \Delta X = \frac{1-\rho}{N}. \tag{22}$$

Note that we have taken the nodes on this axis to be equally spaced. Let the conductance of the small annular wedge between X_j and X_{j-1} be

$$c_j = \frac{\hat{c} X_j \Delta\Theta}{\Delta X}, \qquad \Delta\Theta = \frac{2\pi}{N}, \tag{23}$$

where $\hat{c}$ is the uniform conductivity value. Let Φ_j be the voltage at X_j. Enforcing KCL at the interior nodes, and using Ohm's law, we find

$$(\Phi_{j+1} - \Phi_j)c_{j+1} = (\Phi_j - \Phi_{j-1})c_j, \tag{24}$$

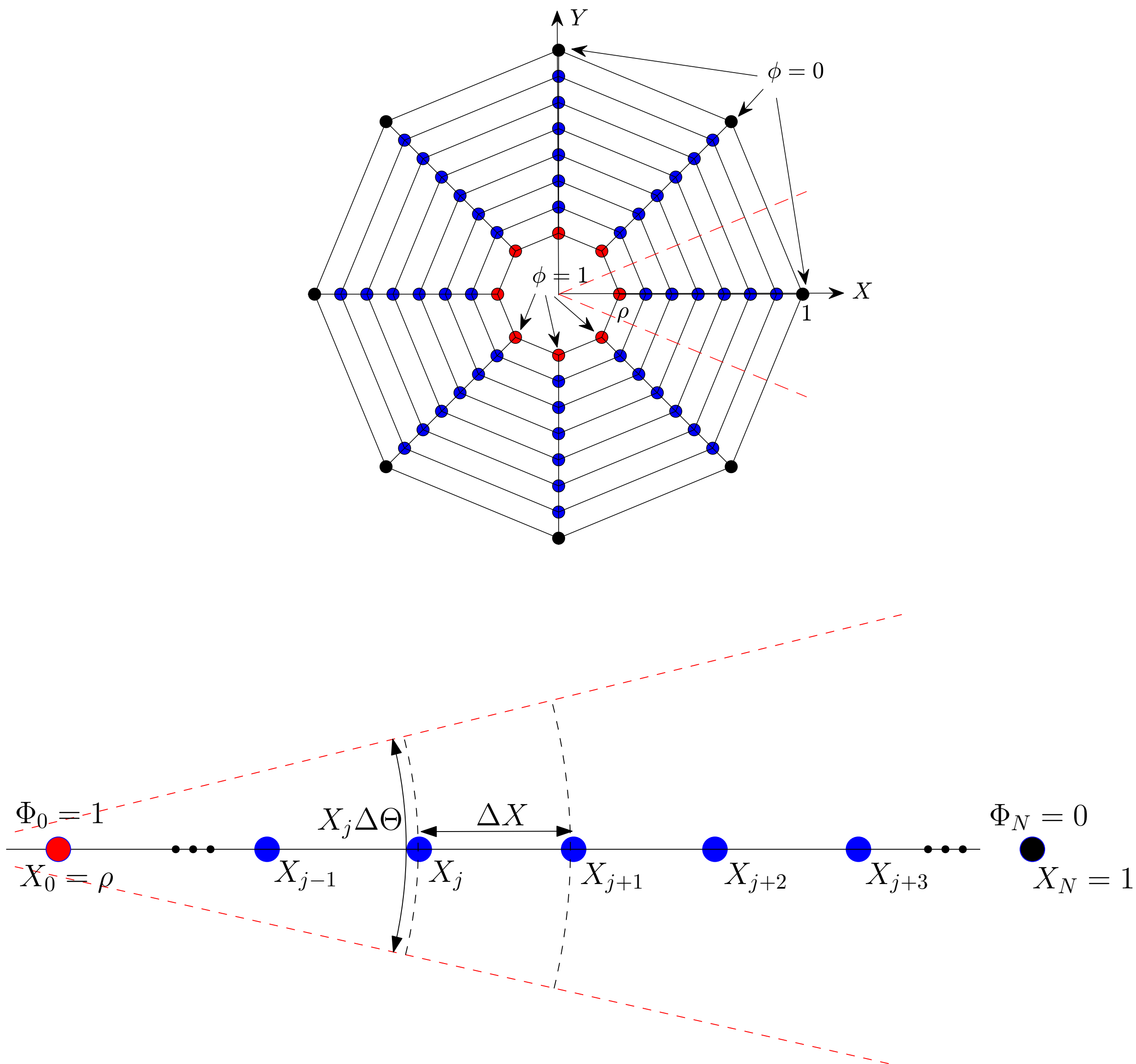

Figure 4: Typical N-by-N wheel graph with $N = 8$. Red notes at radius $r = \rho$ are at unit potential, black nodes at radius $r = 1$ are grounded. KCL holds at all blue nodes. By the symmetry of the arrangement all nodes at the same radius are expected to be at the same potential. It is therefore enough to consider the set of nodes along the x axis between the dashed red lines.

or

$$(\Phi_{j+1} - \Phi_j)\frac{\hat{c}X_{j+1}\Delta\Theta}{\Delta X} = (\Phi_j - \Phi_{j-1})\frac{\hat{c}X_j\Delta\Theta}{\Delta X} \tag{25}$$

which, after cancelling common terms, gives

$$(\Phi_{j+1} - \Phi_j)X_{j+1} = (\Phi_j - \Phi_{j-1})X_j. \tag{26}$$

Defining

$$P_j = \Phi_j - \Phi_{j-1} \tag{27}$$

then (26) implies that

$$P_{j+1}X_{j+1} = P_jX_j. \tag{28}$$

By inspection, this has the solution

$$P_j = \frac{D}{X_j} \tag{29}$$

for some constant D. Now with $\Phi_0 = 1$,

$$\begin{aligned}
\Phi_1 &= \Phi_0 + \frac{D}{X_1} = 1 + \frac{D}{x_1}, \\
\Phi_2 &= \Phi_1 + \frac{D}{X_2} = 1 + \frac{D}{X_1} + \frac{D}{X_2} \\
\Phi_3 &= \Phi_2 + \frac{D}{X_3} = 1 + \frac{D}{X_1} + \frac{D}{X_2} + \frac{D}{X_3} \\
&\vdots \qquad\qquad \vdots
\end{aligned} \tag{30}$$

from which the pattern becomes clear and we infer that

$$\Phi_j = 1 + \sum_{n=1}^{j} \frac{D}{X_n}, \qquad j = 1, \ldots, N. \tag{31}$$

Enforcing the condition $\Phi_N = 0$ requires the choice

$$D = -\frac{1}{\sum_{n=1}^{N} \frac{1}{X_n}}. \tag{32}$$

Hence, on substitution back into (31),

$$\Phi_j = 1 - \frac{\displaystyle\sum_{n=1}^{j} \frac{1}{X_n}}{\displaystyle\sum_{n=1}^{N} \frac{1}{X_n}} = \frac{\displaystyle\sum_{n=j+1}^{N} \frac{1}{X_n}}{\displaystyle\sum_{n=1}^{N} \frac{1}{X_n}}, \qquad j = 0, \ldots, N. \tag{33}$$

On multiplication of both numerator and denominator by ΔX, this solution becomes

$$\Phi_j = \frac{\displaystyle\sum_{n=j+1}^{N} \frac{\Delta X}{X_n}}{\displaystyle\sum_{n=1}^{N} \frac{\Delta X}{X_n}}, \tag{34}$$

which we recognize this as a ratio of the lower Riemann sum approximations of the integrals

$$\int_{X_j}^{1} \frac{dx}{x}, \qquad \text{and} \qquad \int_{\rho}^{1} \frac{dx}{x}. \tag{35}$$

Therefore, in the limit $N \to \infty$ where $\Delta X \to 0$,

$$\Phi_j \to \int_{X_j}^{1} \frac{dx}{x} \bigg/ \int_{\rho}^{1} \frac{dx}{x} = \frac{\log X_j}{\log \rho} \tag{36}$$

Introducing the continuous function Φ with the property that $\Phi_j = \Phi(X_j)$ then we deduce that

$$\Phi(X) = \frac{\log X}{\log \rho}. \tag{37}$$

By the rotational symmetry of the configuration, this solution will be the same for other values of Θ_k, but the variable X will now be the radial coordinate R. Consequently, the continuum limit of the solution as $N \to \infty$ over the entire annular grid graph is

$$\phi(X, Y) = \frac{\log R}{\log \rho}, \qquad R = \sqrt{X^2 + Y^2}. \tag{38}$$

It is of interest to compute the effective conductance, C_{eff}. By definition, this is the total current into the circuit from the unit-voltage nodes. There are N such nodes, all with the same current divergence equal to

$$c_1(1 - \Phi_1), \tag{39}$$

by the symmetry of the arrangement. The effective conductance is therefore

$$\begin{aligned} Nc_1(1 - \Phi_1) &= N\left(\frac{\hat{c}X_1\Delta\Theta}{\Delta X}\right)\left(1 - (1 + \frac{D}{X_1})\right) \\ &= N\left(\frac{\hat{c}\Delta\Theta}{\Delta X}\right)\frac{1}{\sum_{n=1}^{N}\frac{1}{X_n}} = N\left(\hat{c}\Delta\Theta\right)\frac{1}{\sum_{n=1}^{N}\frac{\Delta X}{X_n}}. \end{aligned} \tag{40}$$

But, the denominator is again the lower Riemann sum approximation to

$$\int_{\rho}^{1} \frac{dx}{x}, \tag{41}$$

while, on use of $\Delta\Theta = 2\pi/N$, it is clear that, as $N \to \infty$,

$$C_{\text{eff}} \to -\frac{2\pi\hat{c}}{\log \rho}. \tag{42}$$

The solution to the two-terminal conduction problem in the rectangle

$$-L < x < 0, \qquad -\pi < y < \pi \tag{43}$$

considered in the previous chapter was found to be

$$\textbf{Rectangular conductor}: \quad \phi(x, y) = -\frac{x}{L}. \tag{44}$$

And the solution for the potential for the two-terminal conduction problem in the annulus

$$\rho < R < 1, \qquad R^2 = X^2 + Y^2, \tag{45}$$

was

$$\textbf{Annular conductor}: \quad \Phi(X, Y) = \frac{\log R}{\log \rho}. \tag{46}$$

3 Geometric view of complex functions

By elementary complex analysis we can introduce the following two complex variables

$$z = x + \mathrm{i}y = |z|e^{\mathrm{i}\arg[z]} := re^{\mathrm{i}\theta} \tag{47}$$

and

$$Z = X + \mathrm{i}Y = |Z|e^{\mathrm{i}\arg[Z]} := Re^{\mathrm{i}\Theta}. \tag{48}$$

Consequently,

$$\log Z = \log R + \mathrm{i}\Theta. \tag{49}$$

It is of interest to rewrite the solution (44) in terms of the complex variable z, namely,

$$\phi(x, y) = \mathrm{Re}\left[-\frac{z}{L}\right]. \tag{50}$$

This can also be written as

$$\textbf{Rectangular conductor}: \quad \phi(x, y) = \mathrm{Re}\left[\frac{\log e^{z}}{\log e^{-L}}\right]. \tag{51}$$

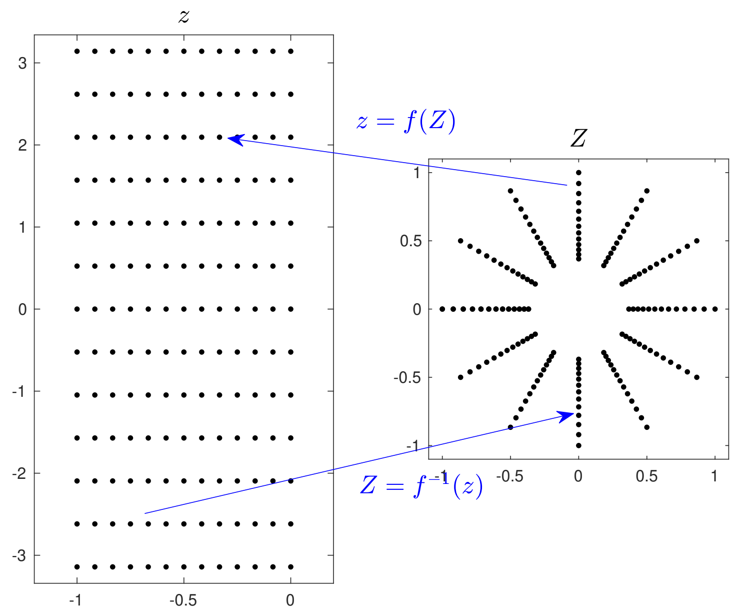

Figure 5: Distribution of points, in the Z plane, of points on a regular grid in the z plane under the correspondence $Z = f^{-1}(z) = e^z$.

For the annular conductor, the solution (46) can be written in terms of the complex variable Z as

$$\textbf{Annular conductor}: \quad \Phi(X, Y) = \mathrm{Re}\left[\frac{\log Z}{\log \rho}\right]. \tag{52}$$

Written like this, the two solutions (51) and (52) are not dissimilar. Indeed, they suggest making the identifications

$$Z = e^z, \qquad \rho = e^{-L}. \tag{53}$$

These identifications have an interesting geometrical interpretation. For any complex point $z = x + \mathrm{i}y$ lying inside the rectangular conductor

$$-L < x < 0, \qquad -\pi < y < \pi \tag{54}$$

the corresponding point $Z = X + \mathrm{i}Y = e^z = e^{x+\mathrm{i}y}$ lies in the annulus $\rho = e^{-L} < |Z| < 1 = e^0$. Moreover, it can be checked that there is a single point Z in this annulus corresponding to every point z in the rectangle. In other words, $Z = e^z$ provides a one-to-one "mapping" between points z in the rectangular conductor and points Z in the annular conductor. Complex functions which transplant one region of the complex plane to another region are called *conformal mappings*.

A more useful way to think about this is to notice that the solution in the rectangular conductor can be expressed *parametrically*, in terms of an intermediate complex Z variable, as

$$\begin{aligned} \textbf{Rectangular conductor}: \quad \phi(x,y) &= \mathrm{Re}\left[\frac{\log Z}{\log \rho}\right], \\ z = x + \mathrm{i}y &= \log Z \end{aligned} \tag{55}$$

provided that we make the special choice of ρ given by

$$\log \rho = -L. \tag{56}$$

But that choice of ρ in terms of the geometrical parameter L of the rectangular conductor is precisely the one that ensures that the identification

$$z = x + \mathrm{i}y = \log Z \tag{57}$$

furnishes a one-to-one correspondence, or mapping, between the annulus and the given rectangular conductor.

Also, the effective conductance of the two-terminal conductance problem in the rectangular conductor is

$$C_{\mathrm{eff}} = \frac{2\pi\hat{c}}{L}. \tag{58}$$

and the effective conductance of the two-terminal conductance problem in the annular conductor is

$$C_{\mathrm{eff}} = -\frac{2\pi\hat{c}}{\log \rho}. \tag{59}$$

Using formula (56) for ρ in terms of L in (59) leads exactly to formula (58) for the effective conductance in the rectangular conductor.

We are naturally led to conjecture that the generalized parametric representation given by

$$\begin{aligned} \text{Conductor } D: \quad \phi(x,y) &= \mathrm{Re}\left[\frac{\log Z}{\log \rho}\right], \\ x + \mathrm{i}y = z &= f(Z) \end{aligned} \tag{60}$$

for *other* choices of the function $f(Z)$, or conformal mapping, providing a one-to-one correspondence between the annulus $\rho < |Z| < 1$ and some *other* conductor, D say, provides the solution to the two-terminal conductance problem in that conductor too. Again, we expect the value of ρ to be dictated by the requirement that $f(Z)$ provides the required one-to-one mapping.

A good way to explore this conjecture is by studying some explicit examples where we pose some forms of $f(Z)$, corresponding to different conductor shapes D, write down the potential according to (60) and verify that it constitutes the solution to the two-terminal conductance problem in that conductor.

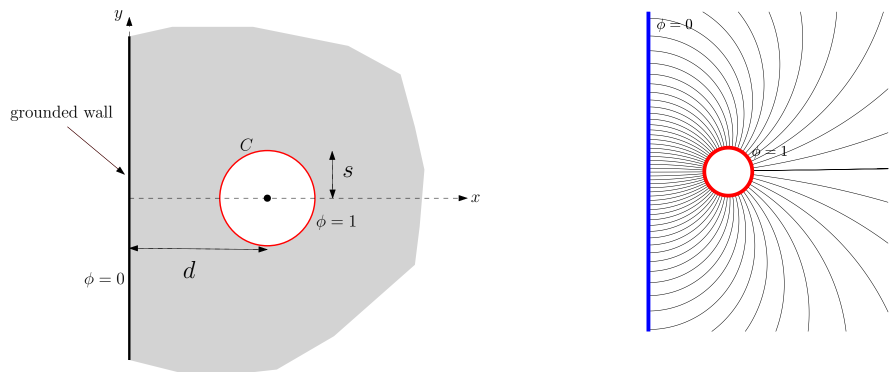

Figure 6: Two-terminal conduction problem in a half-plane conductor: a circular electrode is set to unit voltage near a grounded semi-infinite wall. Current lines for the case $d = 1, s = 0.3$ are shown.

4 Circular electrode near a wall

Suppose we take the conductor D to be the right half-plane region exterior to a circular electrode centred at $(d, 0)$ and of radius s. The wall at $x = 0$ will be grounded, while the circular electrode will be set to unit voltage. Figure 6 shows a schematic. To test the conjecture just described we need to identify both the functional form of the mapping

$$z = f(Z), \tag{61}$$

as well as a suitable value of ρ, that will provide a one-to-one correspondence between the annulus $\rho < |Z| < 1$ and the conductor D.

Consider the choice

$$z = f(Z) = R\left(\frac{1-Z}{1+Z}\right), \qquad R \in \mathbb{R}. \tag{62}$$

The reason for this choice will be made clearer from more general considerations in the next section. A little algebra reveals that

$$Z = \frac{R-z}{R+z}. \tag{63}$$

Consequently,

$$|Z|^2 = Z\overline{Z} = \left(\frac{R-z}{R+z}\right) \times \left(\frac{R-\overline{z}}{R+\overline{z}}\right) \tag{64}$$

and this expression can be used to identify the curves in the z plane that correspond to the two circles $|Z| = 1$ and $|Z| = \rho$. For example, if $|Z| = 1$ then (64) implies

$$1 = \left(\frac{R - z}{R + z} \right) \times \left(\frac{R - \overline{z}}{R + \overline{z}} \right) \tag{65}$$

which, after simplification, becomes

$$z + \overline{z} = 2x = 0 \tag{66}$$

which is the imaginary axis in the complex z-plane. On the other hand, if $|Z| = \rho$ then (64) implies

$$\rho^2 = \left(\frac{R - z}{R + z} \right) \times \left(\frac{R - \overline{z}}{R + \overline{z}} \right) \tag{67}$$

which, after simplification, becomes

$$z\overline{z} - R \left(\frac{1 + \rho^2}{1 - \rho^2} \right) (z + \overline{z}) + R^2 = 0. \tag{68}$$

But a circle in the z plane with centre at $(d, 0)$ and with radius s is given by the equation

$$|z - d|^2 = s^2, \qquad \text{or} \qquad z\overline{z} - d(z + \overline{z}) + d^2 - s^2 = 0. \tag{69}$$

Formulas (68) and (69) will coincide if we choose R and ρ so that

$$d = R \left(\frac{1 + \rho^2}{1 - \rho^2} \right), \qquad R = \sqrt{d^2 - s^2}, \tag{70}$$

which, after some rearrangement, means that we must choose

$$\rho = \left(\frac{d - \sqrt{d^2 - s^2}}{d + \sqrt{d^2 - s^2}} \right)^{1/2}. \tag{71}$$

This formula gives the required value of ρ in terms of the geometrical parameters of the domain, in this case, d and s.

The potential (60) is then given by

$$\begin{aligned} \phi(x, y) &= \mathrm{Re} \left[\frac{1}{\log \rho} \log \left(\frac{R - z}{R + z} \right) \right] \\ &= \frac{1}{2 \log \rho} \log \left(\frac{(R - x)^2 + y^2}{(R + x)^2 + y^2} \right). \end{aligned} \tag{72}$$

It is easy to check that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{1}{2\log\rho}\left[\frac{-2(R-x)}{(R-x)^2+y^2} - \frac{2(R+x)}{(R+x)^2+y^2}\right],\\
\frac{\partial^2 \phi}{\partial x^2} &= \frac{1}{2\log\rho}\left[\frac{2y^2-2(R-x)^2}{((R-x)^2+y^2)^2} - \frac{2y^2-2(R+x)^2}{((R+x)^2+y^2)^2}\right],\\
\frac{\partial \phi}{\partial y} &= \frac{1}{2\log\rho}\left[\frac{2y}{(R-x)^2+y^2} - \frac{2y}{(R+x)^2+y^2}\right],\\
\frac{\partial^2 \phi}{\partial y^2} &= \frac{1}{2\log\rho}\left[\frac{2(R-x)^2-2y^2}{((R-x)^2+y^2)^2} - \frac{2(R+x)^2-2y^2}{((R+x)^2+y^2)^2}\right],
\end{aligned} \tag{73}$$

which confirms that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{74}$$

in the conducting region D. Moreover, since

$$\phi(x,y) = \mathrm{Re}\left[\frac{\log Z}{\log\rho}\right] = \frac{\log|Z|}{\log\rho} \tag{75}$$

then $\phi = 0$ on the imaginary z axis since it has been shown above that such points on this line correspond to $|Z| = 1$. Also, $\phi = 1$ on the circular boundary C since points z on this circle correspond to $|Z| = \rho$. Both the partial differential equation (74) and boundary conditions for this problem of an electrified circular electrode near a grounded straight wall are therefore satisfied. Typical current lines for the case $d = 1, s = 0.3$ based on this solution are shown in Figure 6 and appear as expected.

5 Harmonic functions

In the case of a discrete graph, we defined the notion of a harmonic potential; these were node potentials whose values were the average of all the node potentials at adjacent nodes. If KCL holds inside at points inside a two-dimensional domain D then we have seen that the mathematical statement of this is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{76}$$

everywhere in D. In analogy with the notion of a harmonic potential, functions ϕ satisfying (76) are called *harmonic functions*. Equation (76) is an example of a *partial differential equation*: it is called *Laplace's equation*. Strictly speaking, when discussing a partial derivative we should write

$$\left.\frac{\partial \phi}{\partial x}\right|_y, \qquad \left.\frac{\partial \phi}{\partial y}\right|_x \tag{77}$$

to highlight the fact that in taking a partial x derivative, one keeps y fixed, and vice versa. However, this extra notation is often dropped in practice and in writing just

$$\frac{\partial \phi}{\partial x}, \qquad \frac{\partial \phi}{\partial y} \tag{78}$$

it is understood implicitly which variable is kept fixed.

6 Brief introduction to analytic functions

Let p be a complex-valued function of x and y. Assume that p is differentiable at some point (x, y) as a function of x and y then by the usual rules of calculus,

$$\Delta p = \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\Delta y, \tag{79}$$

where Δp denotes the change in the value of p due to small changes Δx and Δy in x and y respectively. Now if we introduce the notation

$$\Delta z = \Delta x + \mathrm{i}\Delta y, \qquad \Delta \overline{z} = \Delta x - \mathrm{i}\Delta y, \tag{80}$$

then

$$\Delta x = \frac{\Delta z + \Delta \overline{z}}{2}, \qquad \Delta y = \frac{\Delta z - \Delta \overline{z}}{2\mathrm{i}}. \tag{81}$$

We can write (79) as

$$\begin{aligned}\Delta p &= \frac{\partial p}{\partial x}\left(\frac{\Delta z + \Delta \overline{z}}{2}\right) + \frac{\partial p}{\partial y}\left(\frac{\Delta z - \Delta \overline{z}}{2\mathrm{i}}\right) \\ &= \frac{1}{2}\left(\frac{\partial p}{\partial x} - \mathrm{i}\frac{\partial p}{\partial y}\right)\Delta z + \frac{1}{2}\left(\frac{\partial p}{\partial x} + \mathrm{i}\frac{\partial p}{\partial y}\right)\Delta \overline{z}.\end{aligned} \tag{82}$$

Now introduce the new complex derivatives[1] defined by

$$\frac{\partial p}{\partial z} \equiv \frac{1}{2}\left[\frac{\partial p}{\partial x} - \mathrm{i}\frac{\partial p}{\partial y}\right], \qquad \frac{\partial p}{\partial \overline{z}} \equiv \frac{1}{2}\left[\frac{\partial p}{\partial x} + \mathrm{i}\frac{\partial p}{\partial y}\right]. \tag{83}$$

[1] They are called Wirtinger derivatives.

These definitions are very natural since (82) then becomes

$$\Delta p = \frac{\partial p}{\partial z}\Delta z + \frac{\partial p}{\partial \bar{z}}\Delta \bar{z} \tag{84}$$

which is what one would be inclined to write down if z and $\bar{z}$ are treated as independent variables, cf. (79).

We can ask if it is possible to treat the complex-valued function p as a function of the complex variable $z = x + \mathrm{i}y$ just as we would a real function f of the single real variable x where, for example, the familiar derivative df/dx, often also denoted by $f'(x)$, is defined by

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}, \qquad \Delta f = f(x + \Delta x) - f(x). \tag{85}$$

In analogy we would ideally like to define the derivative of the complex-valued function p with respect to z as

$$\frac{dp}{dz} \equiv \lim_{\Delta z \to 0} \frac{\Delta p}{\Delta z}. \tag{86}$$

From (84) this means the quantity

$$\lim_{\Delta z \to 0} \left(\frac{\partial p}{\partial z} + \frac{\partial p}{\partial \bar{z}} \frac{\Delta \bar{z}}{\Delta z} \right) \tag{87}$$

must be well-defined. However, a problem immediately arises. If $\Delta z = \Delta x$ where $\Delta x \in \mathbb{R}$ then

$$\frac{\Delta \bar{z}}{\Delta z} = 1. \tag{88}$$

On the other hand, if $\Delta z = \mathrm{i}\Delta y$ where $\Delta y \in \mathbb{R}$ then

$$\frac{\Delta \bar{z}}{\Delta z} = -1. \tag{89}$$

The only way (87) will be well-defined for a general $\Delta z \to 0$ is if

$$\frac{\partial p}{\partial \bar{z}} = 0, \tag{90}$$

in which case, by (87),

$$\frac{dp}{dz} \equiv \lim_{\Delta z \to 0} \frac{\Delta p}{\Delta z} = \frac{\partial p}{\partial z}. \tag{91}$$

A function p for which equations (90) and (91) hold is called an *analytic function* of the variable z.

In a similar way a function q, say, is called an *anti-analytic function* of z if

$$\frac{\partial q}{\partial z} = 0, \tag{92}$$

and

$$\frac{dq}{d\overline{z}} = \frac{\partial q}{\partial \overline{z}}. \tag{93}$$

Note: If one imagines z and $\overline{z}$ to be independent variables an interpretation of equation (90) is to say that the function p is independent of the variable $\overline{z}$, but depends only on z. For this reason it is common to write a general analytic function as

$$p = p(z) \tag{94}$$

to reflect the fact that the function does not depend on $\overline{z}$. Similarly, it is common to write a general anti-analytic function of z as

$$q = q(\overline{z}). \tag{95}$$

Note: Given an analytic function $p(z)$ of z, the easiest way to generate an anti-analytic function is to take its complex conjugate:

$$\overline{p(z)}, \tag{96}$$

where the overline means that one takes a complex conjugate of every complex quantity defining the function. By way of example, if

$$p(z) = e^{\mathrm{i}z} + 2\mathrm{i}z^2 \tag{97}$$

then, on taking a complex conjugate of all quantities on the right hand side,

$$\overline{p(z)} = e^{-\mathrm{i}\overline{z}} - 2\mathrm{i}\overline{z}^2, \tag{98}$$

which is clearly just a function of $\overline{z}$, indeed one can write (98) as

$$\overline{p(z)} = q(\overline{z}), \tag{99}$$

where the function q is defined as

$$q(u) = e^{-\mathrm{i}u} - 2\mathrm{i}u^2. \tag{100}$$

This function q, which is generally different from the function p but is clearly related to it, is known as the *Schwarz conjugate* of p.

Returning to the electric conductor problem suppose the harmonic potential Φ represents the values of the voltage at a given (x, y) position where $z = x + \mathrm{i}y$.

Consider the expression

$$\Phi = \mathrm{Re}[h(z)] = \frac{h(z) + \overline{h(z)}}{2}, \qquad z = x + \mathrm{i}y, \tag{101}$$

where $h(z)$ is some analytic function. Since $h(z)$ is an analytic function, and $\overline{h(z)}$ is an anti-analytic function, it is easy to check using the properties of such functions just explained that

$$\frac{\partial^2 \Phi}{\partial z \partial \overline{z}} = 0. \tag{102}$$

But on use of the definitions (83), equation (102) is equivalent to

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \tag{103}$$

Thus Φ satisfies Laplace's equation. What we have indicated here is that a general real-valued solution of Laplace's equation can be written in terms of a general analytic function $h(z)$ as in (101).

The upshot of all this is that the problem of finding Φ can be reduced to finding the analytic function $h(z)$; the *real part* of $h(z)$ will be the required voltage potential. This analytic function $h(z)$ producing the required voltage distribution according to (101) is called the *complex potential* associated with the voltage distribution.

An important point is that one treats the complex potential $h(z)$ much as one does a function of a single real variable, certainly as far as taking derivatives is concerned. This means that all the rules of differentiation for simple functions of z are the same as their real counterparts.

Note: If we write

$$h(z) = \mathrm{Re}[h(z)] + \mathrm{i}\mathrm{Im}[h(z)] = \Phi + \mathrm{i}\Psi, \tag{104}$$

where Φ is the voltage potential and $\Psi = \mathrm{Im}[h(z)]$ where $\mathrm{Im}[.]$ means take the imaginary part of the quantity in brackets, then (90) can be written

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(\Phi + \mathrm{i}\Psi) = 0. \tag{105}$$

Collecting together the real and imaginary parts of the left hand side gives

$$\left(\frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y}\right) + \mathrm{i}\left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x}\right) = 0. \tag{106}$$

Reading off the real and imaginary parts of this complex equation gives the two real equations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}. \tag{107}$$

These are known as the *Cauchy-Riemann equations*.[2]

7 Current density as a derivative of $h(z)$

We know that the current density vector is

$$\mathbf{j} = \begin{pmatrix} J^{(x)} \\ J^{(y)} \end{pmatrix} = -\hat{c} \begin{pmatrix} \partial\Phi/\partial x \\ \partial\Phi/\partial y \end{pmatrix}. \tag{108}$$

Consider the quantity

$$J^{(x)} - \mathrm{i}J^{(y)} = -\hat{c}\left(\frac{\partial\Phi}{\partial x} - \mathrm{i}\frac{\partial\Phi}{\partial y}\right). \tag{109}$$

Using (83) it is clear that we can write

$$J^{(x)} - \mathrm{i}J^{(y)} = -2\hat{c}\frac{\partial\Phi}{\partial z}. \tag{110}$$

And since, from (101), we have

$$\Phi = \mathrm{Re}[h(z)] = \frac{1}{2}\left(h(z) + \overline{h(z)}\right) \tag{111}$$

then, on substitution into (110) we conclude

$$J^{(x)} - \mathrm{i}J^{(y)} = -\hat{c}\frac{\partial h(z)}{\partial z}. \tag{112}$$

Since $h(z)$ is an analytic function, the right hand side of (112) can also be written as

$$-\hat{c}\frac{dh(z)}{dz} \qquad \text{or} \qquad -\hat{c}h'(z), \tag{113}$$

where, as we saw earlier, the prime notation denotes a derivative just as it does in real analysis.

8 Dot product in complex notation

Given two vectors in $\mathbb{R}^2$, $\mathbf{a}$ and $\mathbf{b}$ say, where

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}, \tag{114}$$

[2]This section is a short preview of material you will study more deeply next year in your complex analysis course.

the dot product is defined by

$$\mathbf{a}.\mathbf{b} \equiv \mathbf{a}^T\mathbf{b} = a_x b_x + a_y b_y. \tag{115}$$

There is a natural association of a vector $\mathbf{a}$ in $\mathbb{R}^2$ with a complex number $a \in \mathbb{C}$ given by

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} \mapsto a = a_x + \mathrm{i}a_y, \qquad \mathbf{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} \mapsto b = b_x + \mathrm{i}b_y. \tag{116}$$

The symbol "$\mapsto$" is used here to mean "the complex form of" the quantity. It is easy to check that, in terms of these complex numbers a and b, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a}.\mathbf{b} \mapsto \mathrm{Re}[a\overline{b}] = \mathrm{Re}[\overline{a}b], \tag{117}$$

where the complex conjugates are the usual

$$\overline{a} = a_x - \mathrm{i}a_y, \qquad \overline{b} = b_x - \mathrm{i}b_y. \tag{118}$$

9 Two-terminal conductance problem

For circuits made up of discrete graphs, to generate a non-trivial current we assigned conditions at a chosen selection of *boundary nodes* and assumed that KCL holds at all other nodes. We must now do the same for a given two-dimensional conductor in $\mathbb{R}^2$.

The analogue of the "2-point source-sink" circuits studied in the discrete case is the "2-terminal conductance problem", sometimes also called the "2-electrode conductance problem". It is most common to place the boundary nodes on the *physical* boundary of the conductor. An example is shown in Figure 7: a continuous portion of the boundary of a two-dimensional conductor D is grounded, $\Phi = 0$ there, and another continuous portion of the domain boundary is set to unit voltage $\Phi = 1$. These boundary portions, shown in red and black respectively in Figure 7, are the two *terminals*, or *electrodes*, and one imagines these are connected to a battery. On the rest of the physical boundary of the conductor, the condition that KCL holds there becomes the condition

$$\mathbf{j}.\mathbf{n} = 0. \tag{119}$$

This says that there is no current flowing *normal to the boundary*, or *out of* the domain D at those parts of the boundary. Recall that $\mathbf{j}.\mathbf{n}$ has the interpretation as the component of the current density vector $\mathbf{j}$ in the direction of $\mathbf{n}$, i.e., in the normal direction to the boundary. One can think of the condition $\mathbf{j}.\mathbf{n} = 0$ on the boundary of the conductor to be the statement that no current can flow into a non-conducting region (the region exterior to the conductor). Of course there is no reason, in this continuous limit, that current cannot flow tangent to, that is, *around* the boundary of D.

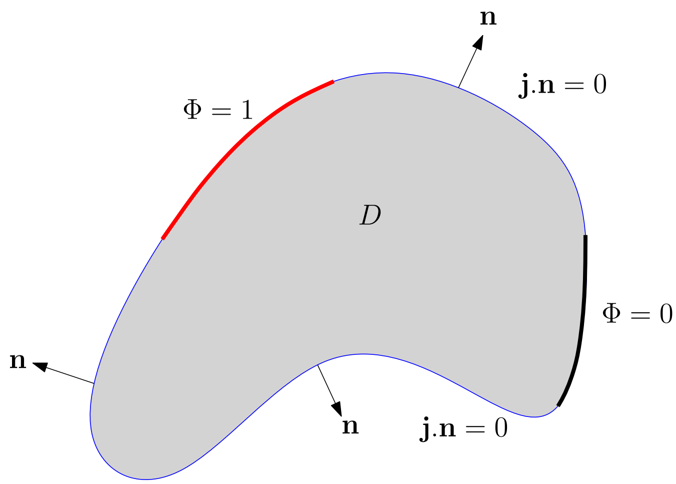

Figure 7: Two terminal conductance problem for a general domain D.

With these boundary conditions one can then ask:

- What is the voltage distribution $\Phi(x, y)$ inside the conductor D?
- What is the distribution of current density entering/leaving the conductor at the two terminals?
- What is the *effective conductance* of this circuit? In analogy to the discrete circuit problem, this is defined to be the total current entering the conductor through the boundary portion (terminal, or electrode) that is set at unit voltage.

Note that, in general, to find the total current entering the conductor through some boundary portion we need to integrate the normal component of the current *density* vector $\mathbf{j}$ with respect to *arclength* along the boundary, i.e., it is necessary to compute

$$\int_{\text{boundary}} \mathbf{j}.\mathbf{n} \, ds, \tag{120}$$

where $\mathbf{n}$ denotes the unit normal vector on the boundary and, as is usual in vector calculus, we denote an infinitesimal element of arclength by ds defined by

$$ds = \sqrt{dx^2 + dy^2}. \tag{121}$$

The best way to illustrate how to answer such questions is by a series of examples.

10 Rectangular conductor

Consider the two-terminal conductance problem in a rectangular conductor, with uniform conductivity $\hat{c}$ of width L and height H as shown in Figure 8. The conductor occupies the region

$$-L < x < 0, \qquad -\frac{H}{2} < y < \frac{H}{2}. \tag{122}$$

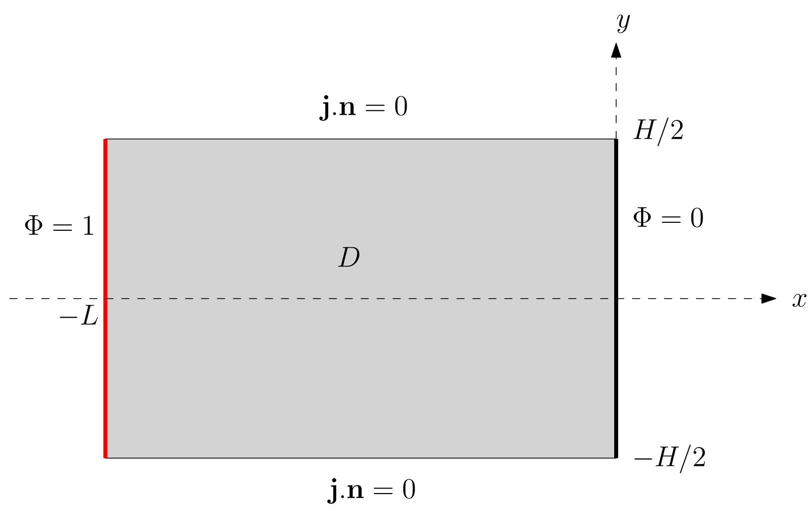

Figure 8: Two terminal conductance problem in a rectangular conductor. The edge of the conductor at $x = -L$ is set to unit voltage and the edge on $x = 0$ is grounded.

The edge $x = -L, -H/2 < y < H/2$ is held at unit voltage, i.e.,

$$\Phi = 1, \qquad x = -L, \ -H/2 < y < H/2 \tag{123}$$

and the edge $x = 0, -H/2 < y < H/2$ is grounded, i.e.,

$$\Phi = 0, \qquad x = 0, \ -H/2 < y < H/2. \tag{124}$$

On the upper and lower walls $-L < x < 0, y = \pm H/2$ the condition $\mathbf{j}.\mathbf{n} = 0$ holds, i.e.,

$$\mathbf{j}.\mathbf{n} = 0, \qquad -L < x < 0, \ y = \pm H/2. \tag{125}$$

This last condition says that current does not flow out of the conductor through these edges. Inside the conductor the potential Φ satisfies

$$\nabla^2 \Phi = 0 \tag{126}$$

which is the continuous version of the KCL condition at points inside the conductor.

Hence we know we can write the solution as

$$\Phi = \mathrm{Re}[h(z)] \tag{127}$$

for some analytic function $h(z)$.

What is $h(z)$? The geometry, and the boundary conditions, are sufficiently simple in this case that we can guess the answer. Let us try

$$h(z) = -\frac{z}{L}. \tag{128}$$

This is clearly an analytic function in D since it is purely a function of z, moreover it has no singularities in the conductor. From (128) we have

$$\Phi = \mathrm{Re}[h(z)] = -\frac{x}{L} = \begin{cases} 1, & x = -L, \\ 0, & x = 0. \end{cases} \tag{129}$$

Therefore the choice (128) satisfies the boundary conditions on those two walls. Also,

$$J^{(x)} - \mathrm{i}J^{(y)} = -\hat{c}\frac{\partial h(z)}{\partial z} = \frac{\hat{c}}{L}. \tag{130}$$

The right hand side is real everywhere implying, in particular, that

$$J^{(y)} = 0, \qquad y = \pm H/2. \tag{131}$$

Since on the upper and lower walls, $y = \pm H/2$, the (outward) normal vectors are $\mathbf{n} = (0, \pm 1)$, then the conditions

$$\mathbf{j}.\mathbf{n} = \pm J^{(y)} = 0, \qquad y = \pm H/2 \tag{132}$$

are clearly satisfied. We have therefore checked that (136) with the complex potential (128) meets all the conditions of the problem.

Note: It turns out that, just as there was a *uniqueness theorem for harmonic potentials* in the discrete circuit problem, there is an analogous *uniqueness theorem for harmonic functions*. This guarantees that the solution just found is, in fact, the required solution.

What is the effective conductance of this 2-terminal conductance problem? As in the discrete case, this is defined to be the total current entering the circuit through the terminal set at unit voltage. Since for this boundary $ds = dy$, the effective conductance is

$$C_{\mathrm{eff}} = \int_{-H/2}^{H/2} J^{(x)}\Big|_{x=-L} dy = \frac{\hat{c}H}{L}, \tag{133}$$

where we have used (130) and integrated the current density in the *inward normal* direction, i.e. $J^{(x)}$, along the boundary $x = -L, -H/2 < y < H/2$.

11 Annular conductor

Consider now the 2-terminal conductance problem in an annular conductor shown in Figure 9. The conductor D is the annular region

$$\rho < |z| < 1. \tag{134}$$

The inner circular boundary $|z| = \rho$ is set to unit voltage, the outer circular boundary $|z| = 1$ is grounded. Since current flows from high to low voltage, by the axisymmetry of the boundary conditions – namely, the radial voltage drop across the conductor looks the same for every angle – we expect current to flow purely radially.

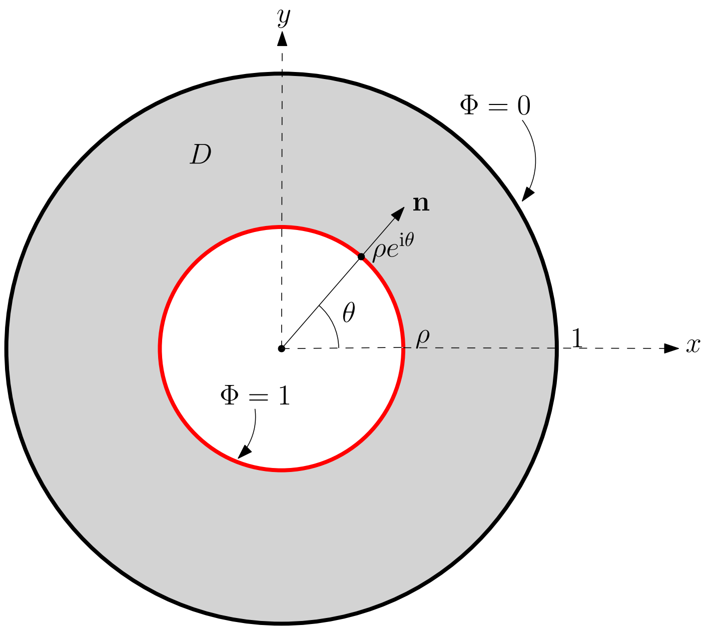

Figure 9: Two terminal conductance problem in a concentric annulus. The voltage on the inner circular boundary is specified to be unity and the outer circular boundary is grounded.

Inside the conductor the potential Φ satisfies

$$\nabla^2 \Phi = 0 \tag{135}$$

which is the continuous version of the KCL. Hence we know we can write the solution as

$$\Phi = \mathrm{Re}[h(z)] \tag{136}$$

for some analytic function $h(z)$. Let us try

$$h(z) = \frac{\log z}{\log \rho}, \tag{137}$$

where the complex logarithm is defined, as usual, as

$$\log z = \log |z| + \mathrm{i}\arg[z], \tag{138}$$

where arg[.] denotes the argument of a complex number. This is purely a function of z which is non-singular in D; it is therefore a candidate analytic function. With the choice (137), and using properties (138) of the complex logarithm,

$$\Phi = \mathrm{Re}[h(z)] = \frac{\log |z|}{\log \rho} = \begin{cases} 1, & |z| = \rho, \\ 0, & |z| = 1. \end{cases} \tag{139}$$

The choice (137) therefore satisfies the boundary conditions on the two boundaries of the conductor. The function $h(z)$ in (137) is the required complex potential.

What is the effective conductance of this 2-terminal conductance problem? First note that

$$J^{(x)} - \mathrm{i}J^{(y)} = -\hat{c}\frac{\partial h(z)}{\partial z} = -\hat{c}\frac{dh(z)}{dz} = -\frac{\hat{c}}{z \log \rho}. \tag{140}$$

As shown in Figure 9 the complex form of the unit normal vector $\mathbf{n}$ to either boundary is

$$\mathbf{n} \mapsto e^{\mathrm{i}\theta}, \tag{141}$$

where θ is the argument of the point on the boundary as shown in Figure 9. Hence, using an earlier result on the complex form of the dot product,

$$\mathbf{j}.\mathbf{n} \mapsto \mathrm{Re}\left[(J^{(x)} - \mathrm{i}J^{(y)})e^{\mathrm{i}\theta}\right] = \mathrm{Re}\left[-\frac{\hat{c}e^{\mathrm{i}\theta}}{z \log \rho}\right], \tag{142}$$

where we have used (140). On the boundary $|z| = \rho$, where we can write $z = \rho e^{\mathrm{i}\theta}$, this is

$$\mathbf{j}.\mathbf{n} = -\frac{\hat{c}}{\rho \log \rho} \tag{143}$$

which is uniform around this boundary, as might be expected given the axisymmetry of the problem: there is no reason for more current to flow out of the conductor in one direction than in any other.

For this problem the arclength element around the internal circular boundary is most conveniently written as $ds = \rho d\theta$ so that θ is used as the parameter. The total current entering the circuit through the boundary $|z| = \rho$ is therefore

$$C_{\text{eff}} = \int_{\text{inner boundary}} \mathbf{j}.\mathbf{n}\, ds = \int_0^{2\pi} \mathbf{j}.\mathbf{n}(\rho d\theta) = \int_0^{2\pi} \left(-\frac{\hat{c}}{\rho \log \rho}\right) \rho d\theta = -\frac{2\pi\hat{c}}{\log \rho}, \tag{144}$$

where we have used (143).

12 A point source of current

Hence the effective conductance in the annular conductor is

$$C_{\text{eff}} = -\frac{2\pi\hat{c}}{\log\rho}. \tag{145}$$

This is positive because $0 < \rho < 1$ and hence $\log\rho < 0$. In the limit $\rho \to 0$ the inner boundary disappears. In this limit we therefore find

$$C_{\text{eff}} \to 0. \tag{146}$$

This means that the net current into the conductor from the inner terminal, or electrode, vanishes. This makes sense because it is the unit voltage on the inner boundary that is driving current through the conductor. If this terminal vanishes, as it does as $\rho \to 0$, then there is eventually nothing left to drive any current into the conductor.

Suppose however that the voltage on the inner boundary is taken to be V instead of 1. This just requires a rescaling of the solution just found by a factor V so that

$$\Phi = \text{Re}[h(z)], \qquad h(z) = V\,\frac{\log z}{\log\rho}. \tag{147}$$

The rescaled effective conductance (145) is

$$C_{\text{eff}} = -\frac{2\pi V\hat{c}}{\log\rho}. \tag{148}$$

If we now consider a different *double* limit where, as $\rho \to 0$, the constant voltage $V \to \infty$ in such a way that the ratio $V/\log\rho$ remains fixed, i.e.,

$$\frac{V}{\log\rho} = -\frac{m}{2\pi} \tag{149}$$

for some real constant m. It is then clear from (148) and (149) that

$$C_{\text{eff}} = m\hat{c}. \tag{150}$$

That is, the net current into the conductor, or C_{eff} by definition, is equal to $m\hat{c}$. The corresponding complex potential (147) is then

$$h(z) = -\frac{m}{2\pi}\log z, \tag{151}$$

where we have used (149) in (147). We then say that there is a *point source* of current of strength m at the origin $z = 0$. What is happening here is that as the inner electrode gets smaller and smaller, the voltage on it becomes larger and larger, and in such a way that there is always a net current $m\hat{c}$ into the conductor.

More generally, if *any* complex potential can be written in the form

$$h(z) = -\frac{m}{2\pi}\log(z - z_0) \underbrace{\quad + \quad \hat{h}(z),}_{\text{not singular at } z_0} \tag{152}$$

where the analytic function $\hat{h}(z)$ is *not* singular when evaluated at $z = z_0$ then we say there is a *point source* of current of strength m at the point z_0. This is because the current distribution associated with a complex potential of the form (152) is similar, near $z = z_0$, to the current distribution associated with (151).

In summary, logarithmic singularities in the complex potential $h(z)$ at points inside a conductor are signals of the presence of point sources of current at those points. The coefficient in front of the logarithm tells us about the *strength* of the point source of current. Again, this is best illustrated by some examples.

13 Half-space conductor with a point current source

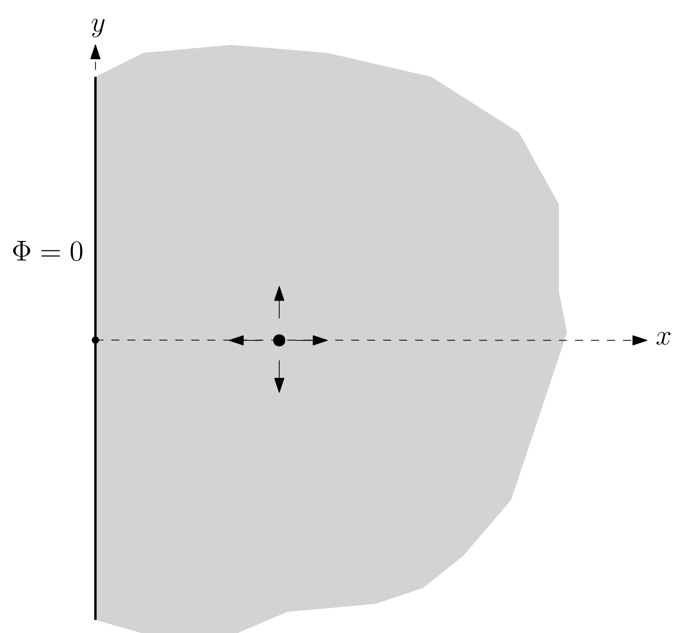

Figure 10: A point current source of strength m inside a half-space conductor with a grounded boundary.

Consider a point source of current of strength m located at the origin inside a conductor occupying the right-half space

$$x > 0, \qquad -\infty < y < \infty \tag{153}$$

with uniform conductivity $\hat{c}$. Suppose that the edge of the conductor is grounded:

$$\Phi = 0, \qquad x = 0. \tag{154}$$

We will check that the voltage distribution associated with this problem is

$$\Phi = \mathrm{Re}[h(z)], \tag{155}$$

where

$$h(z) = -\frac{m}{2\pi}\log f(z), \qquad f(z) \equiv \frac{1-z}{1+z}. \tag{156}$$

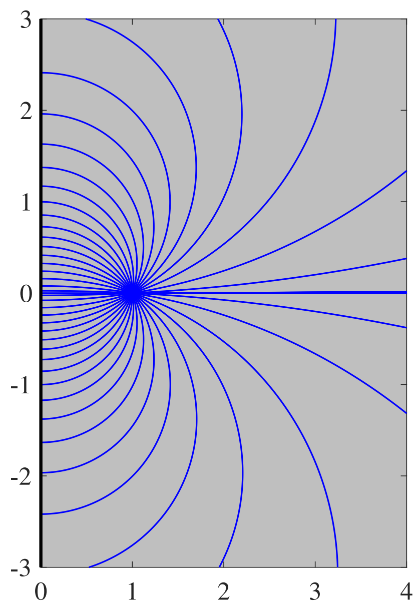

Figure 11: Current lines for a point current source of strength m at $(1, 0)$ inside a half-space conductor with a grounded boundary.

First note that $h(z)$ is a well-defined function of z everywhere in the conductor (153) except at $z = 1$. This is because the zeros and poles of $f(z)$ which are at ± 1 will both give logarithmic singularities of $h(z)$. However only the point $z = 1$ is in the right-half plane (153). Since we can write

$$h(z) = -\frac{m}{2\pi}\log\left[(z-1)\left(-\frac{1}{1+z}\right)\right] = -\frac{m}{2\pi}\log(z-1)\underbrace{-\frac{m}{2\pi}\log\left(-\frac{1}{1+z}\right)}_{\text{not singular at z=1}} \quad (157)$$

then we recognize, in accordance with the prescription laid out earlier, that there is a point current source of strength m at $z = 1$. It only remains to verify if the boundary condition on $x = 0$ is satisfied. First note that on the line $x = 0$,

$$x = \frac{z+\bar{z}}{2} = 0, \qquad \text{or} \qquad \bar{z} = -z. \quad (158)$$

This implies that, when $x = 0$,

$$\overline{f(z)} = \frac{1-\bar{z}}{1+\bar{z}} = \frac{1+z}{1-z} = \frac{1}{f(z)}. \quad (159)$$

Therefore $|f(z)|^2 = 1$ on the line $x = 0$ and we find

$$\Phi = \mathrm{Re}[h(z)] = -\frac{m}{2\pi}\log|f(z)| = 0, \qquad \text{when } x = 0. \quad (160)$$

The boundary at $x = 0$ is therefore grounded.

Note: In (159), in order to confirm that the boundary condition holds, we took a complex conjugate of $f(z)$ and then tried to relate the result back to the original function $f(z)$ when z is located on the boundary. This is a good strategy and is often the simplest way to confirm that boundary conditions are satisfied.

14 Strip conductor with a point current source

Consider a point source of current of strength m located at the origin inside the strip conductor

$$-\infty < x < \infty, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}, \quad (161)$$

with uniform conductivity $\hat{c}$. Suppose that the two edges of the conductor are grounded:

$$\Phi = 0, \qquad y = \pm\frac{\pi}{2}. \quad (162)$$

We will verify that the solution to this problem is

$$\Phi = \mathrm{Re}[h(z)], \quad (163)$$

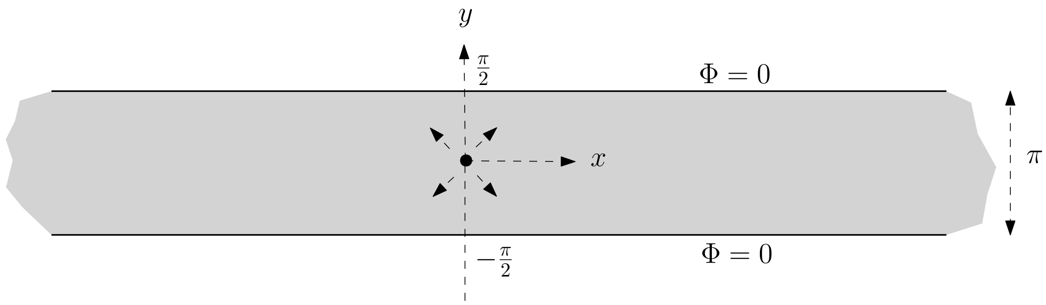

Figure 12: A point current source of strength m inside a strip conductor of width π with grounded boundaries $y = \pm\pi/2$.

where

$$h(z) = -\frac{m}{2\pi}\log f(z), \qquad f(z) \equiv \frac{1-e^z}{1+e^z}. \tag{164}$$

First note that $h(z)$ is a well-defined function of z everywhere in the strip (161) except at $z = 0$. This is because the zeros and poles of $f(z)$ are at the roots of

$$1 = e^{2\pi p \mathrm{i}} = e^z, \qquad \text{and} \qquad -1 = e^{\mathrm{i}\pi + 2\pi q \mathrm{i}} = e^z, \qquad p, q \in \mathbb{Z} \tag{165}$$

respectively. That is, at

$$z = 2\pi p \mathrm{i}, \qquad \text{and} \qquad z = \mathrm{i}\pi + 2\pi q \mathrm{i} \qquad p, q \in \mathbb{Z}. \tag{166}$$

Since $h(z)$ in (164) involves taking a logarithm of $f(z)$ then each of these poles and zeros of $f(z)$ would be logarithmic singularities of $h(z)$. However only the case $p = 0$, corresponding to $z = 0$, lies inside the strip conductor. Notice that we can write

$$f(z) = \frac{1-e^z}{1+e^z} = z \times \left\{ \frac{1}{1+e^z}\left[\frac{1-e^z}{z}\right]\right\}. \tag{167}$$

Hence

$$h(z) \sim -\frac{m}{2\pi}\log z \underbrace{-\frac{m}{2\pi}\log\left\{\frac{1}{1+e^z}\left[\frac{1-e^z}{z}\right]\right\}}_{\text{not singular at } z=0}, \tag{168}$$

where, on use of the Taylor expansion of e^z, i.e,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots, \tag{169}$$

it can be checked that the second term on the right hand side of (168) is not singular as $z \to 0$. This means, again in accordance with the earlier prescription, that there is a source of strength m at $z = 0$.

It only remains to verify if the boundary conditions on $y = \pm\pi/2$ are satisfied.

On $y = \pi/2$ we have

$$y = \frac{z - \overline{z}}{2\mathrm{i}} = \frac{\pi}{2} \tag{170}$$

implying that, on this boundary,

$$\overline{z} = z - \mathrm{i}\pi. \tag{171}$$

We repeat the strategy adopted in the previous example and take a complex conjugate of $f(z)$. On the line $y = \pi/2$,

$$\overline{f(z)} = \frac{1 - e^{\overline{z}}}{1 + e^{\overline{z}}} = \frac{1 - e^{z-\mathrm{i}\pi}}{1 + e^{z-\mathrm{i}\pi}} = \frac{1 + e^{z}}{1 - e^{z}} = \frac{1}{f(z)}. \tag{172}$$

Therefore $|f(z)|^2 = 1$ on $y = \pi/2$ and therefore

$$\Phi = \mathrm{Re}[h(z)] = -\frac{m}{2\pi} \log|f(z)| = 0, \qquad \text{on } y = \pi/2. \tag{173}$$

A similar analysis shows that on $y = -\pi/2$ where

$$\overline{z} = z + \mathrm{i}\pi \tag{174}$$

that $|f(z)| = 1$, and hence $\Phi = 0$ there too.

Since the point current source is on the center line of the conductor, by the symmetry of the configuration, we expect that the net current leaving the conductor through the top and bottom boundary must be equal, and since KCL holds everywhere away from the source, current $m/2$ must leave the conductor across each boundary. To check this, note that the current density is

$$J^{(x)} - \mathrm{i}J^{(y)} = -\hat{c}\frac{\partial h}{\partial z} = \frac{m}{2\pi}\left[-\frac{e^{z}}{1 - e^{z}} - \frac{e^{z}}{1 + e^{z}}\right]. \tag{175}$$

Hence, on the upper wall $y = \pi/2$,which can be parametrized by

$$z = x + \frac{\mathrm{i}\pi}{2} \tag{176}$$

it is clear that $e^{z} = \mathrm{i}e^{x}$ and hence that

$$J^{(x)} - \mathrm{i}J^{(y)} = -\frac{m}{2\pi}\left[\frac{\mathrm{i}e^{x}}{1 - \mathrm{i}e^{x}} + \frac{\mathrm{i}e^{x}}{1 + \mathrm{i}e^{x}}\right] = -\frac{\mathrm{i}m}{2\pi}\mathrm{sech}x. \tag{177}$$

Therefore, on $y = \pi/2$,

$$J^{(x)} = 0, \qquad J^{(y)} = \frac{m}{2\pi}\mathrm{sech}x. \tag{178}$$

Evaluation of an integral then leads to

$$\int_{-\infty}^{\infty} \mathbf{j}.\mathbf{n}\Big|_{y=\pi/2} dx = \int_{-\infty}^{\infty} J^{(y)} dx = \frac{m}{2}. \tag{179}$$

This is consistent with our expectations based on the symmetry of the configuration. Figure 13 shows some typical current lines associated with this point source of current. The symmetry of these current lines about the center line of the conductor is clear.

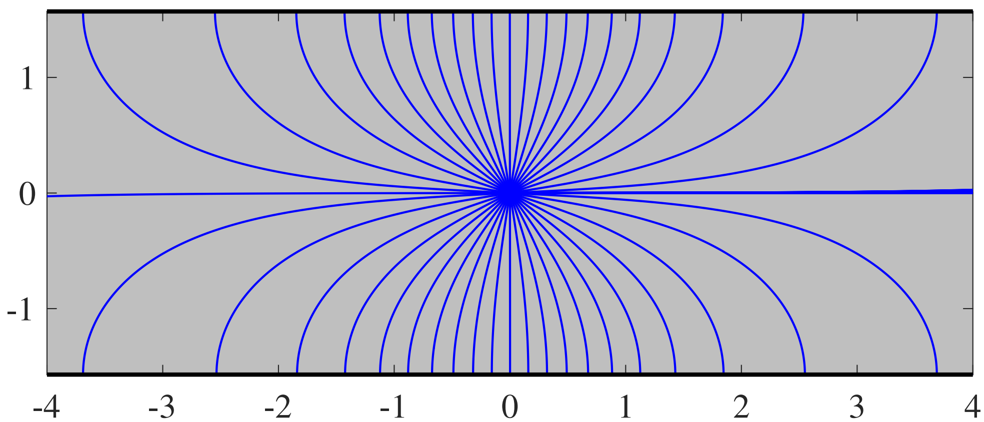

Figure 13: Current lines for a point current source of strength m inside a strip conductor at $(0, 0)$ with grounded boundaries.

15 A geometrical observation

So far three examples involving a point current source of strength m have been considered: a point source in a disc conductor, a half-space conductor and a strip conductor. All the complex potentials describing the associated voltage distribution had the form

$$h(z) = -\frac{m}{2\pi}\log f(z), \tag{180}$$

where $f(z)$, the argument of the logarithm, is

$$f(z) = \begin{cases} z, & \text{(disc)}, \\[2ex] \dfrac{1-z}{1+z}, & \text{(half}-\text{space)}, \\[2ex] \dfrac{1-e^z}{1+e^z}, & \text{(strip)}. \end{cases} \tag{181}$$

It is instructive to introduce a new complex variable ζ defined by

$$\zeta = f(z), \tag{182}$$

where $f(z)$ takes the various forms given in (181). One can then ask the following geometrical question: suppose the point z takes value inside the various conductors just considered. What is the corresponding region in a complex ζ plane when points ζ and z are related by (182)?

Disc conductor: For the disc, for which $\zeta = f(z) = z$, the answer is easy. It is clear that the unit disc in the complex z plane corresponds to the unit disc in the complex ζ plane.

Half-space conductor: For the half-space conductor the question is more interesting. Suppose that the variable z takes values on the boundary of the half-space conductor $x > 0$ and suppose (182) holds. On taking a complex conjugate of (182) it is certainly true that

$$\overline{\zeta} = \overline{f(z)} = \frac{1-\overline{z}}{1+\overline{z}}, \tag{183}$$

however on $x = 0$ we know that $\overline{z} = -z$ and consequently, on that line,

$$\overline{\zeta} = \frac{1+z}{1-z} = \frac{1}{\zeta}, \qquad \text{or} \qquad |\zeta|^2 = 1. \tag{184}$$

Points on the boundary of the half-space conductor in the z plane, i.e., points $(0, y)$, therefore correspond to points on the unit circle in a complex ζ plane. We also note that $z = 1$ corresponds to $\zeta = 0$ under the correspondence given by (182).

Let us check a few interior points. The x-axis in the z plane, or $z = x$, corresponds to the set of real-valued points

$$\zeta_x = \frac{1-x}{1+x} \tag{185}$$

in the complex ζ plane where we have set $\zeta = \zeta_x + \mathrm{i}\zeta_y$. It is an easy exercise to verify that there is a one-to-one correspondence between the positive real x axis and the real diameter $-1 < \zeta_x < 1$. Indeed, it can be checked that there is a one-to-one correspondence between any point $x > 0$ in the right-half space in the complex $z = x + \mathrm{i}y$ plane and the interior of the unit disc $|\zeta| < 1$ in the complex $\zeta = \zeta_x + \mathrm{i}\zeta_y$ plane. This is left as an exercise.

Strip conductor: Suppose now that z takes values on the upper wall $y = \pi/2$ of the strip conductor and suppose (182) holds with $f(z)$ given by the formula in (181) for the strip. It certainly true that

$$\overline{\zeta} = \overline{f(z)} = \frac{1-e^{\overline{z}}}{1+e^{\overline{z}}}. \tag{186}$$

However, if $y = \pi/2$, we have

$$\overline{z} = z - \mathrm{i}\pi \tag{187}$$

hence on this line,

$$\overline{\zeta} = \frac{1 - e^{z-\mathrm{i}\pi}}{1 + e^{z-\mathrm{i}\pi}} = \frac{1 + e^z}{1 - e^z} = \frac{1}{\zeta}, \qquad \text{or} \qquad |\zeta|^2 = 1. \tag{188}$$

That is, points on this boundary in the z plane correspond to points on the unit circle in a complex ζ plane. It is easy to check that, on $y = -\pi/2$,

$$\overline{z} = z + \mathrm{i}\pi \tag{189}$$

and, again,

$$\overline{\zeta} = \frac{1 - e^{z-\mathrm{i}\pi}}{1 + e^{z-\mathrm{i}\pi}} = \frac{1 + e^z}{1 - e^z} = \frac{1}{\zeta}. \tag{190}$$

Points on the boundaries of the strip conductor in the z plane therefore correspond to points on the unit circle in a complex ζ plane. Notice also that $z = 0$ corresponds to $\zeta = 0$ under the correspondence given by (182).

Let us check a few interior points. The x-axis in the z plane or $z = x$ corresponds to the set of real-valued points

$$\zeta_x = \frac{1 - e^x}{1 + e^x} \tag{191}$$

in the complex ζ plane where we have set $\zeta = \zeta_x + \mathrm{i}\zeta_y$. It is an easy exercise to verify that there is a one-to-one correspondence between the positive real x axis and the real diameter $-1 < \zeta_x < 1$. Indeed, it can be checked that there is a one-to-one correspondence between any point in the strip conductor in the complex $z = x + \mathrm{i}y$ plane and the interior of the unit disc $|\zeta| < 1$ in the complex $\zeta = \zeta_x + \mathrm{i}\zeta_y$ plane. This is left as an exercise.

General result: It turns out that the examples considered above are just three special cases of the following general geometrical result.[3] If we can find an analytic function $f(z)$ such that

$$\zeta = f(z) \tag{192}$$

transplants, in a one-to-one fashion, points inside any given conductor D in a complex z plane to points inside the unit disc in a complex ζ plane then

$$h(z) = -\frac{m}{2\pi} \log f(z) \tag{193}$$

is the complex potential associated with current in the conductor due to a point current source of strength m located at the single root z_* inside the conductor D of

$$f(z_*) = 0. \tag{194}$$

[3] This is stated here without proof.

16 Point source of current in a quarter-plane conductor

In view of this geometrical observation, let us consider finding the complex potential $h(z)$ associated with the current generated by a point current source of strength m located at

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{195}$$

in a conductor occupying the quarter plane, or first quadrant,

$$x, y > 0. \tag{196}$$

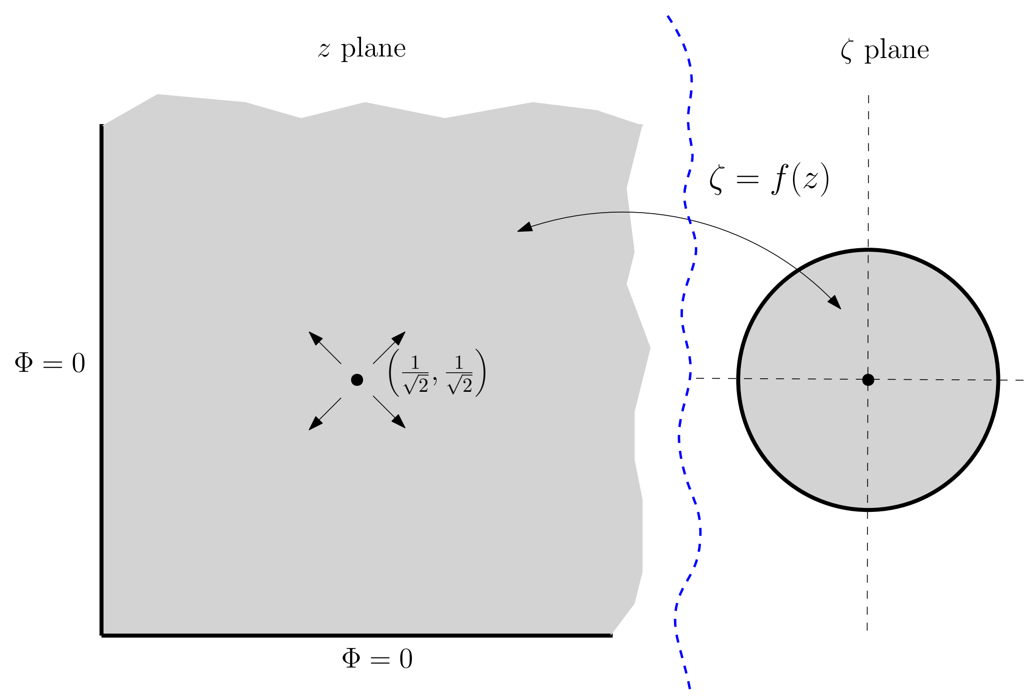

Figure 14: A point current source of strength m at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ inside a quarter-plane conductor with grounded boundaries. To find $h(z)$, we can equivalently ask for the function $\zeta = f(z)$ that transplants the conductor to the unit disc $|\zeta| < 1$ in a complex ζ plane.

From the result stated in the previous section we need to find a function

$$\zeta = f(z) \tag{197}$$

such that points in this quarter-plane conductor (196) to the unit disc $|\zeta| < 1$. Let $z = x + \mathrm{i}y$ be some point in quarter plane conductor (196). We will construct the required function $f(z)$ by composing a sequence of functions each of which have simple geometrical interpretations using well-known properties of complex numbers.[4]

Since squaring a complex number multiplies its argument by 2 then the point

$$\eta = z^2, \tag{198}$$

where the point z is taken to lie in the conductor in the first quadrant (196), will lie in the upper half plane $\mathrm{Im}[\eta] > 0$. This is because, when squared, any point z in the first quadrant (196) must have positive imaginary part. Now consider a second transformation of this upper half complex η plane. Using the fact that multiplication of a complex number by i adds $\pi/2$ to its argument or, conversely, multiplication by $-\mathrm{i}$ will subtract $\pi/2$ from the argument, then the point defined by

$$\chi = -\mathrm{i}\eta \tag{199}$$

will lie in the half space $\mathrm{Re}[\chi] > 0$. This is because if one subtracts $\pi/2$ from the argument of any complex number in the upper half plane, the result lies in the right half plane.

There is one final step needed, a third transformation to add to this sequence. We already know, from our earlier analysis of the half-space conductor, that

$$\zeta = \frac{1-z}{1+z} \tag{200}$$

transplants points in the right half-space $x = \mathrm{Re}[z] > 0$ to points inside the unit disc so

$$\zeta = \frac{1-\chi}{1+\chi} \tag{201}$$

will do the same thing to points in the right half space $\mathrm{Re}[\chi] > 0$. Under the transformation (201) the right half χ plane will correspond to the interior of the unit disc $|\zeta| < 1$.

Put together, the successive application of these three transformations (198), (199) and (201), i.e.,

$$z \longrightarrow \eta \longrightarrow \chi \longrightarrow \zeta \tag{202}$$

will therefore perform the desired task of transplanting any point in the quadrant conductor to the interior of the unit disc in a complex ζ plane. The geometrical effect of this sequence of transformations is illustrated in Figure 15. On composing the functions (198), (199) and (201), we arrive at the required $f(z)$:

$$\zeta = f(z) \equiv \frac{1+\mathrm{i}z^2}{1-\mathrm{i}z^2}. \tag{203}$$

[4] These transformations are simple examples of *conformal mappings*.

Notice also that $\zeta = 0$ corresponds to $1 + \mathrm{i}z^2 = 0$ or

$$z^2 = e^{\mathrm{i}\pi/2}, \qquad z = \pm e^{\mathrm{i}\pi/4} = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \tag{204}$$

Therefore

$$z_* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{205}$$

is the point in the conductor satisfying (194). This is the location of the point source of current.

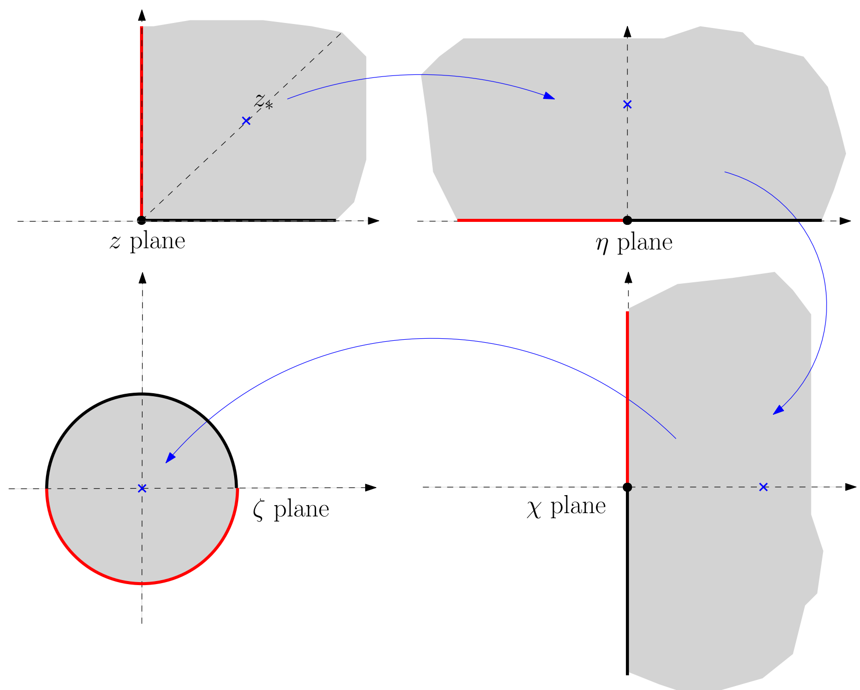

Figure 15: Sequence of geometrical transformations from the z plane, to the η plane, to the χ plane, and finally to the ζ plane. Composition of the functions (198), (199) and (201) produces the required function $f(z)$ in (203).

Based on the geometrical observation made earlier we assert that the complex potential for a source at (195) in the quarter plane conductor (196) is

$$h(z) = -\frac{m}{2\pi}\log f(z) = -\frac{m}{2\pi}\log\left[\frac{1+\mathrm{i}z^2}{1-\mathrm{i}z^2}\right]. \tag{206}$$

Irrespective of this geometrical derivation, the reader can now confirm, following steps akin to those used in previous examples, that this complex potential has all the required properties. Figure 16 shows typical current lines associated with (206).

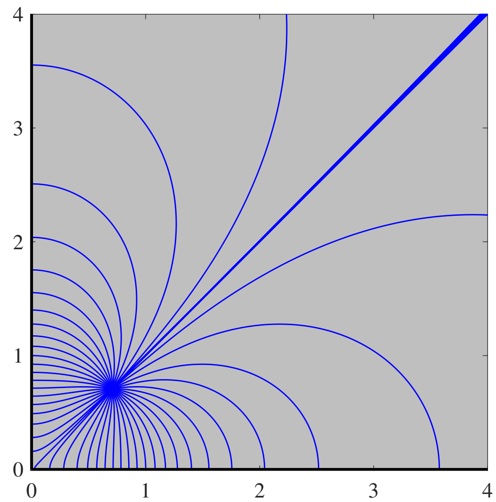

Figure 16: Current lines for a point current source of strength m at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ inside a quarter-plane conductor with grounded boundaries.

17 Final remarks

This final part of the course, in this set of notes, is intended to give you a glimpse of more sophisticated mathematical ideas used in applications, including how complex analysis comes into play in solving problems arising from the two-dimensional continuum limit of the mathematical framework we have developed. You will learn much more about analytic functions and their properties in your Second Year complex analysis course, and in higher level courses.

For *this* course, you will not be expected to *find* the complex potential $h(z)$ for any given conductor arrangement. The geometrical derivation of (206) has been included only to give you a flavour of how the geometry of analytic functions ("conformal geometry") can be deployed to solve problems in applications. However, once presented with the relevant $h(z)$ whose real part gives the voltage distribution in some given conductor, you might be expected, as has been done in these lecture notes, to use it to deduce things about a given conductor problem, for example, to check the nature of any boundary conditions, to compute current densities, or effective conductances, or to identify the presence of any point sources of current. This will give you practice at manipulating, and understanding, analytic functions that will be useful next year.