MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Quiz 2

This is an assessed quiz. The quiz will open on Friday, 26 January at 10am and close on Saturday, 27 January at 10.30am.

Question 1 (Lipschitz continuity).
Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{1}{2}\arctan(x^2)$. Which of the following statements is true?

(a) f is Lipschitz continuous.

(b) f is not Lipschitz continuous.

Question 2 (Matrix norm).
The operator norm $\|A\|$ of the 2×2 matrix

$$A = \begin{pmatrix} \sqrt{3} & \sqrt{2} \\ -\sqrt{2} & \sqrt{3} \end{pmatrix}$$

is given by

(a) $\|A\| = \sqrt{2}$,

(b) $\|A\| = \sqrt{3}$,

(c) $\|A\| = \sqrt{5}$,

(d) $\|A\| = \sqrt{6}$.

Question 3 (Matrix norm).
Is the following statement true or false? There exist invertible matrices $A, B \in \mathbb{R}^{2\times 2}$ such that the operator norm satisfies

$$\|AB\| = \|A\|\|B\|\,.$$

(a) The statement is true.

(b) The statement is false.

Question 4 (Domain of the Picard iterates).
For an initial value problem (2.1), given by $\dot{x} = f(t, x)$, $x(t_0) = x_0$, with a continuous right hand side $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, consider the Picard iterates $\left\{\lambda_n : J \to \mathbb{R}^d\right\}_{n\in\mathbb{N}_0}$, defined on some interval $J \subset \mathbb{R}$ (see Definition 2.2). Which of the following statements is true?

(a) The Picard iterates are always well defined on the interval $J = \mathbb{R}$.

(b) There exist right hand sides f such that (at least some) Picard iterates cannot be defined on the interval $J = \mathbb{R}$.

Question 5 (Picard–Lindelöf theorem, global version).

Consider the differential equation $\dot{x} = e^{t^2} x$, the right hand side of which is globally defined on $\mathbb{R} \times \mathbb{R}$. Which of the following statements is true?

(a) The differential equation satisfies the assumptions of the global version of the Picard–Lindelöf theorem, and for every initial value problem, there exists a solution that can be defined on $\mathbb{R}$.

(b) The differential equation satisfies the assumptions of the global version of the Picard–Lindelöf theorem, and there are initial value problems to which solutions cannot be defined on $\mathbb{R}$.

(c) The differential equation does not satisfy the assumptions of the global version of the Picard–Lindelöf theorem, but for every initial value problem, there exists a solution that can be defined on $\mathbb{R}$.

(d) The differential equation does not satisfy the assumptions of the global version of the Picard–Lindelöf theorem, and there are initial value problems to which solutions cannot be defined on $\mathbb{R}$.