Algebra III: Rings and Modules
In-Class Test 2, Autumn Term 2022-23

John Nicholson

In this test, a ring will be as defined in the course, i.e. a not necessarily commutative ring with a multiplicative unit 1. You may use any results from lectures provided they are clearly stated (and provided they are not the statement you are being asked to prove).

1. (a) Give the definition of:
   - (i) Algebraic integer. (1 mark)
   - (ii) Field of fractions. (1 mark)
   - (iii) $\mathbb{Z}[\alpha]$ where α is an algebraic integer. (1 mark)

   (b) Let $\beta \in \mathbb{C}$ be an algebraic integer such that $R = \mathbb{Z}[\beta]$ is a unique factorisation domain, let $\alpha \in \mathbb{C}$ be the root of a monic polynomial in $R[X]$ and let $F = \mathrm{Frac}(R)$.
   - (i) Prove that there exists a unique monic polynomial $f_{\alpha,\beta} \in F[X]$ such that, for all polynomials $f \in F[X]$ which have α as a root, we have $f_{\alpha,\beta} \mid f$ in $F[X]$. (5 marks)
   - (ii) Prove that there exists a unique monic polynomial $g_{\alpha,\beta} \in R[X]$ such that, for all polynomials $f \in R[X]$ which have α as a root, we have $g_{\alpha,\beta} \mid f$ in $R[X]$. Furthermore, show that $f_{\alpha,\beta} = g_{\alpha,\beta}$. (10 mark)
   - (iii) Let $\beta_1, \beta_2 \in \mathbb{C}$ be algebraic integers such that $R_1 = \mathbb{Z}[\beta_1]$ and $R_2 = \mathbb{Z}[\beta_2]$ are unique factorisation domains and suppose $\alpha \in \mathbb{C}$ is both a root of a monic polynomial in $R_1[X]$ and a root of a monic polynomial in $R_2[X]$. Give examples to show that f_{α,β_1} and f_{α,β_2} need not have the same degree. (2 marks)

   (Total: 20 marks)

2. (a) Define what it means for an R-module to be:
   - (i) Simple. (ii) Finitely generated. (iii) Free. (3 marks)

   (b) Determine, with proof, all implications which exist between properties (i), (ii) and (iii) for all rings R. That is, for all $a, b \in \{\mathrm{i}, \mathrm{ii}, \mathrm{iii}\}$ with $a \neq b$, prove that $(a) \Rightarrow (b)$ for all rings R or find a counterexample which demonstrates that $(a) \not\Rightarrow (b)$. (7 marks)

   (c) Give a proof or counterexample to each of the following statements:
   - (i) A non-trivial R-module M is simple and free if and only if $M \cong R$. (2 marks)
   - (ii) An R-module M is finitely generated if and only if there exists a surjective R-module homomorphism $f : R^n \twoheadrightarrow M$ for some integer $n \geq 1$. (2 marks)
   - (iii) A non-trivial R-module M is simple if and only if M is isomorphic to R/I for some prime ideal I of R. (3 marks)
   - (iv) If an R-module M has finitely many R-submodules, then M is finitely generated. (3 marks)

   (Total: 20 marks)