MATH50010 Probability for Statistics
Unseen Problem 6
Two Envelopes

Suppose we play a game, in which I have two sealed envelopes, one with £θ and the other with £2θ. You can have the money in one of the two envelopes.

1. Suppose you randomly chose an envelope and open it to find £10. At this point in the game, you can either take the £10, or you can take the money in the other envelope (without looking inside it). What should you do?

2. More generally, let the amount in the first envelope be the random variable X, and the amount in the second envelope be Y. Show that $\mathrm{E}(Y|X = x) = \frac{5}{4}x > x$. What does this suggest that you should do?

3. Suppose we treat θ as an unknown constant, or equivalently, we consider the amount in the envelope as a random variable Θ and condition on $\Theta = \theta$. What are $\mathrm{E}(X|\theta)$ and $\mathrm{E}(Y|\theta)$? Does this contradict your answer to the previous parts?

4. Suppose now we impose a realistic prior distribution on Θ: after all, I am not infinitely generous. Let's suppose $\Theta \sim \textsc{Uniform}(0, \alpha)$ for some $\alpha > 0$. So realistically, α is a number like £20. Or perhaps if you think I am very generous maybe $\alpha =$ £100.

   Define the random variable

   $$Z = \begin{cases} 0 & \text{if } X = \min(X, Y), \\ 1 & \text{if } X = \max(X, Y). \end{cases}$$

   What is $\Pr(Z = 0)$? What is $f_X(x|Z = 0)$? *(Your second answer should depend on α).*

5. What is $\Pr(Z = 0|X = x)$?

6. What is $\mathrm{E}(Y|X = x)$ now? *(Hint: use the previous part.)*

7. Show that if, as in the earlier part $\mathrm{E}(Y|X = x) = \frac{5}{4}$, the random variables X and Z are independent. Is this a reasonable assumption?

Solution

1. *The expected amount in the other envelope is*

$$\frac{1}{2} \times \pounds 20 + \frac{1}{2} \times \pounds 5 = \pounds 12.50$$

The expected amount in the other envelope is larger than the amount you currently have, so you should (apparently!) switch.

2. $$\mathrm{E}(Y|X=x) = \frac{1}{2}\frac{1}{2}x + \frac{1}{2}2x = \frac{5}{4}x.$$

Since this amount is larger than x, you should rationally prefer the other envelope.

3. $\mathrm{E}(X|\Theta=\theta) = \frac{1}{2}\theta + \frac{1}{2}2\theta = \frac{3}{2}\theta.$

Symmetrically,

$\mathrm{E}(Y|\Theta=\theta) = \frac{1}{2}\theta + \frac{1}{2}2\theta = \frac{3}{2}\theta.$

This suggests that it shouldn't matter which envelope you choose. This is indeed at odds with the previous part.

4. $\Pr(Z=0) = \frac{1}{2}$ *by symmetry.*

$f_X(x|Z=0) = f_\Theta(x) = \begin{cases} \frac{1}{\alpha} & 0 < x < \alpha \\ 0 & \textit{otherwise} . \end{cases}$ *Since the distribution of X given $Z=0$ is equal to the distribution of X given $X \le Y$ which itself is equal to the distribution of X when $X=\theta$ and $Y=2\theta$.*

5. *Using Bayes' theorem,*

$$\begin{aligned}
\Pr(Z=0|X=x) &= \frac{f_X(x|Z=0)\Pr(Z=0)}{f_X(x|Z=0)\Pr(Z=0) + f_X(x|Z=1)\Pr(Z=1)} \\
&= \frac{f_\Theta(x)}{f_\Theta(x) + f_\Theta(2x)} = \begin{cases} \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{2\alpha}} & 0 < x < \alpha \\ 0 & \textit{otherwise} . \end{cases} \\
&= \begin{cases} \frac{2}{3} & 0 < x < \alpha \\ 0 & \textit{otherwise} . \end{cases}
\end{aligned}$$

6. $$\begin{aligned}
\mathrm{E}(Y|X=x) &= \mathrm{E}(Y|X=x, Z=0)\Pr(Z=0|X=x) + \mathrm{E}(Y|X=x,Z=1)\Pr(Z=1|X=x) \\
&= 2x\Pr(Z=0|X=x) + \frac{x}{2}\Pr(Z=1|X=x) \\
&= \begin{cases} 2x\frac{2}{3} + \frac{x}{2}\frac{1}{3} & 0 < x < \alpha \\ \frac{1}{2}x & \textit{otherwise} . \end{cases} = \begin{cases} \frac{3}{2}x & 0 < x < \alpha \\ \frac{1}{2}x & \textit{otherwise} . \end{cases}
\end{aligned}$$

So we should switch if $x < \alpha$ and not otherwise.

7. *By above,*

$$\mathrm{E}(Y|X=x) = 2x\,\Pr(Z=0|X=x) + \frac{x}{2}\Pr(Z=1|X=x).$$

Letting $\beta = \Pr(Z=0|X=x)$, *this gives*

$$\mathrm{E}(Y|X=x) = 2x\beta + \frac{x}{2}(1-\beta) = x\left(\frac{3}{2}\beta + \frac{1}{2}\right).$$

If, as in the intial reasoning, $\mathrm{E}(Y|X=x) = \frac{5}{4}x$, *then this implies* $\beta = \frac{1}{2}$. *Since* $\Pr(Z=0) = \frac{1}{2}$, $\Pr(Z=0|X=x) = \Pr(Z=0)$ *so* X *and* Z *are independent.*