Mathematical Logic (MATH6/70132;P65)
Problem Sheet 7

[1] (a) Find subsets of $\mathbb{Q}$ which (with their induced orderings from $\mathbb{Q}$) are similar to:
(i) $\mathbb{N}+\mathbb{N}+\mathbb{N}$;
(ii) $\mathbb{N}\times\mathbb{Z}$;
(iii) $\mathbb{N}+\mathbb{N}^*$ (where $\mathbb{N}^*$ is the reverse ordering on $\mathbb{N}$).
You do not need to write down the similarities involved here.

(b) Suppose $\mathcal{A}=(A;\leq)$ is any (non-empty) linearly ordered set. Prove that $\mathbb{Q}\times\mathcal{A}$ is a dense linear ordering without endpoints.

[2] We say that a set is *finite* if and only if it is equinumerous with some natural number $n\in\omega$. Otherwise it is *infinite*.
(i) Suppose β is an infinite ordinal. Using results from 3.4 of the notes, prove that $\omega\leq\beta$. Deduce that $|\beta^\dagger|=|\beta|$.
(ii) Prove that if $m,n\in\omega$ are equinumerous then $m=n$.
(iii) Suppose X is a non-empty finite set of ordinals. Prove that X has a largest element.
(iv) Suppose α is a finite ordinal. Prove that $\alpha\in\omega$.
(v) Suppose $x\subseteq n\in\omega$. Then x is finite.

[Hint: You can use results on ordinals in Section 3.4. For (ii), it suffices to prove by induction on n that if $x\subseteq n$ and x is equinumerous with n, then $x=n$.]

[3] Suppose X is a non-empty set of ordinals. From the notes, you know that $\bigcup X$ and $\bigcap X$ are ordinals and $\bigcap X\leq\alpha\leq\bigcup X$ for all $\alpha\in X$.
(i) Show that if β is an ordinal with $\alpha\leq\beta$ for all $\alpha\in X$, then $\bigcup X\leq\beta$.
(ii) Formulate and prove a similar statement about $\bigcap X$.

[4] Suppose α and β are ordinals with α similar to $\omega+\omega$ and β similar to $\omega\times\omega$ (with the orderings as defined in 3.3.3). Which of $\alpha<\beta$, $\alpha=\beta$ or $\beta<\alpha$ holds?

[5] Let β be the set of all countable ordinals.
(i) Show that β is an ordinal.
(ii) Show that β is uncountable.
(iii) Show that if γ is an uncountable ordinal then $\beta\leq\gamma$.

[6] (i) Suppose α is a ordinal and $X\subset\alpha$ is a proper initial segment of α. Prove that there is $\beta\in\alpha$ with $X=\beta$.
(ii) Suppose that $\gamma\neq\delta$ are ordinals. Prove that γ and δ are not similar.

[7] A *cardinal* is an ordinal α with the property that for all ordinals $\beta<\alpha$ we have that α and β are not equinumerous.
(i) Prove that every natural number is a cardinal and ω is a cardinal.
(ii) Prove that the ordinal β in question 5 is a cardinal.
(iii) Show that if γ is any ordinal, there is a unique cardinal α which is equinumerous with γ.