L30 Revision lecture : (1)

// Postscript to Lectures 1-29 :

If we work in ZFC then
Gödel's Completeness Thm + Compactness Thm .
hold for arbitrary $\mathcal{L}$
(Also versions of Löwenheim-Skolem, ...)

// Exam : Past papers '18, '19, '20 (Different), '23, '24 on Maths Central (BB)
- used to be called P65 .

MATH60132
pass mark 40

4 qus

3 on logic 1 on set th .

MATH70132 + 1 qu.
pass : 50 (on Mastery)

Examinable : Lectures, problem sheets (inc. prob. classes)

Non-examinable : Certain long/complicated proofs :
Th. 2.3.3, Lemma 2.3.8, Th. 2.3.10, Th 2.5.3, Lemma 2.6.3
Th. 3.1.6, Th. 3.5.2, Th. 3.6.1. (Numbering as in typed notes.)

Propositional logic.

Formulas ; truth tables / propositional valuations

i.e. of formulas ; adequacy of sets of connectives

'disjunctive normal form' (1.1.9 /10)

Formal system L : axioms, proof, theorem $\vdash_L \psi$.

- a few examples

Deductions, consequences $\Sigma \vdash_L \phi$

Deduction theorem

$\rightsquigarrow$ more theorems of L.

Soundness (+ generalisation)

if $\Sigma \vdash_L \phi$ then for every val. v

with $v(\Sigma) = T$, we have $v(\phi) = T$.

Completeness / Adequacy Thm. for L. Converse of this.

<u>Key Lemma</u>: If Σ is consistent + $\Sigma \nvdash_L \phi$
then $\Sigma \cup \{\neg\phi\}$ is consistent.

}

<u>Lindenbaum Lemma</u>. ... there is $\Sigma^* \supseteq \Sigma$ which is consistent
and <u>complete</u>. .... .

// <u>1st-order logic</u> $\mathcal{L}$

$\mathcal{L}$-structures — common examples
- groups, fields, ...
- Equivalence relations
- Linear orders
- Graphs

$\mathcal{L}$-formulas
- terms, atomic formulas, ...
- valuation v in a $\mathcal{L}$-structure $\mathcal{A}$
- 'v satisfies ϕ in $\mathcal{A}$', "$v[\phi] = T$".

Examples
- formal use of defns.
- 'informal' "is $\mathcal{A} \models \phi$" ?
  (particularly if ϕ is closed).

<u>Def</u>: 't is free for x_i in ϕ'.

<u>Formal system</u> $K_{\mathcal{L}}$ — Axioms, Deduction rules

MP + Gen

$\vdash_{K_{\mathcal{L}}} \phi$ $\quad$ $\Sigma \vdash_{K_{\mathcal{L}}} \phi$

(restriction on use of Gen).

Deduction thm : as for L

Soundness + generalisation

Gödel Completeness Th + Generalisation

<u>Hard part</u> : 2.5.3 Model Existence Thm.

<u>Compactness thm</u>.

<u>Equality</u> <u>Normal</u> $\mathcal{L}^{=}$-structures ; Axioms for equality

Compactness Thm. for <u>normal</u> $\mathcal{L}^{=}$-structures.

Downward Lowenheim-Skolem Thm!

<u>Application</u> : Axiom<u>a</u>tising $Th((\mathbb{Q}; \leq))$ d.l.o.w.e.

Set Theory. Cardinality $\approx$

Cantor - Schröder - Bernstein

Examples

ZF axioms (in $=$ $\in$)

Orderings + operations : $+$ $\times$ reverse lexicographic

$\cong$ similar (isomorphic).

Well orderings .

ordinals - generalise natural numbers.

Ordinals α, β $\alpha < \beta \iff \alpha \in \beta \iff \alpha \subsetneq \beta$.

Transfinite induction + recursion

treat informally.

$\alpha \geq \omega \implies |\alpha \times \alpha| = |\alpha|$.

AC : WO ; ZL ; Cardinality of X ; $|X| = \lambda$ cardinal ; Cardinal arithmetic.