Imperial College
London

MATH97052 MATH97163

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Differential Topology

Date: Thursday, 6 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In this exam, unless otherwise stated, all manifolds are smooth, connected and without boundary, and all maps are assumed to be smooth. When the term 'cohomology' is used without qualification it refers to de Rham cohomology.

1. (a) Prove the following, using Stokes' Theorem:

   (i) Let M be a compact orientable manifold of dimension n. Then $\int_M \omega = 0$ for any exact $\omega \in \Omega^n(M)$. (3 marks)

   (ii) Let M be a compact orientable manifold of dimension n, with boundary. Then $\int_{\partial M} \omega = 0$ for any closed $\omega \in \Omega^{n-1}(M)$. (3 marks)

   (b) Let M be an orientable manifold of dimension n, with a compact orientable submanifold $S \subset M$ of dimension k. Let $\omega \in \Omega^k_c(M)$ be closed and such that $\int_S \omega \neq 0$. Show that:

   (i) ω is not exact. (2 marks)

   (ii) S is not the boundary of an orientable compact manifold $N \subset M$. (2 marks)

   (c) Let M, N be compact orientable manifolds such that N is without boundary. Suppose $\dim N = n$, $\dim M = n + 1$, and $\partial M \neq \emptyset$. Show that any proper map $f : M \to N$ satisfies deg$f\ |_{\partial M} = 0$. (2 marks)

   (d) (i) Suppose that M, N are orientable. Show that $M \times N$ is orientable. You may use any equivalent characterisation of orientability that was given in lectures. (4 marks)

   (ii) Consider the following map $F : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2$, expressed in polar coordinates in the (r, θ) plane:

   $$F(r, \theta) = (r \cos\theta, r \sin\theta)$$

   Is F orientation preserving or orientation reversing?

   (2 marks)

   (iii) A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

   $$f(x, y) = (a(x, y), b(x, y))$$

   is called holomorphic if it satisfies the Cauchy-Riemann equations:

   $$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}.$$

   Show that any holomorphic diffeomorphism is orientation-preserving. (2 marks)

(Total: 20 marks)

2. (a) Let M be a manifold, and suppose $M = U \cup V$ with U, V open and $U \cap V \neq \emptyset$.

   (i) State the theorem from the course that relates the de Rham cohomology of M with the de Rham cohomology of U, V, and $U \cap V$. Briefly describe any maps involved.
   (2 marks)

   (ii) Suppose that U, V, and $U \cap V$ are connected. Show that there is an exact sequence

$$0 \to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to H^2(M) \to \dots$$

   (3 marks)

   (iii) Now let U, V be open in $\mathbb{R}^n$ such that $\mathbb{R}^n = U \cup V$ and $U \cap V \neq \emptyset$. Show that $U \cap V$ is connected if and only if both U and V are connected. (3 marks)

(b) Use the theorem from (a)(i) to show that for $M = \mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$ we have

$$H^0(M) \cong \mathbb{R},$$

$$H^1(M) \cong \mathbb{R}^n,$$

$$H^p(M) = 0, \quad p \geq 2.$$

(5 marks)

(c) Let M be an orientable manifold of dimension $n \geq 3$.

   (i) Compute $H^0_c(M)$ and $H^n(M)$, with brief justifications, in the cases: M is compact, and M is non-compact. (4 marks)

   (ii) Recall that the inclusion $M \setminus \{x\} \hookrightarrow M$ induces an isomorphism on pth de Rham cohomology groups for $0 \leq p \leq n-2$. Under the assumption that M is compact, show that the same is true for $p = n-1$. You may use without proof the fact that $M \setminus \{x\}$ is not compact. (3 marks)

(Total: 20 marks)

3. Let M and N be orientable manifolds of dimension n.

(a) (i) Let $f : M \to N$ be a proper map. Explain how to associate to f a real number $\deg f \in \mathbb{R}$, called the degree of f. You do not need to prove it is an integer. (2 marks)

(ii) In the proof that the degree of a proper map is an integer, we take y a regular value of f and let $f^{-1}(y) = \{x_1, \dots x_m\}$ where $x_i \in M$. Explain how to find an open set $U \subset N$ with $y \in U$, and for each $i = 1, \dots m$ disjoint open sets $U_i \subset M$ with $x_i \in U_i$, such that

$$f\mid_{U_i} : U_i \to U$$

is a diffeomorphism. (3 marks)

(iii) Suppose that there exists $\omega \in \Omega^n_c(N)$ such that $\int_M f^*\omega \neq 0$. Show that f is surjective. (2 marks)

(iv) Give an example to show that the converse need not hold. (3 marks)

(b) (i) Let $f \in \mathbb{C}[x]$ be a nonzero polynomial of degree d. Show that the induced map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ has degree d. (3 marks)

(ii) Is the same true for $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ induced from a polynomial $f \in \mathbb{R}[x]$? Justify your answer. (2 marks)

(iii) Is the degree of a proper map invariant under arbitrary smooth homotopy? Give a proof or counterexample. (2 marks)

(c) Let M and N be compact and orientable, and let $f : M \to N$ have nonzero degree. Show that for all $p \geq 0$ the pullback map on cohomology $f^* : H^p(N) \to H^p(M)$ is injective.
Hint: Poincaré duality! (3 marks)

(Total: 20 marks)

4. (a) (i) For a compact orientable manifold M, explain why the integration map

$$I_M : H^n(M) \to \mathbb{R}$$

is well-defined. (2 marks)

(ii) The manifold $\mathbb{R}^n$ has $H^0(\mathbb{R}^n) \cong \mathbb{R}$ and $H^{n-0}(\mathbb{R}^n) \cong 0$. Explain why this does not violate Poincaré duality. You may state any facts from the course without proof. (2 marks)

(iii) Carefully verify that Poincaré duality indeed holds in the case $M = S^n$. You may state the relevant cohomology groups, and use the fact that the intregration map is surjective without proof.

(2 marks)

(b) (i) Prove that the wedge product of differential forms $(\omega, \eta) \mapsto \omega \wedge \eta$ induces a well-defined map on cohomology $H^p(M) \times H^q(M) \mapsto H^{p+q}(M)$. (2 marks)

(ii) Sketch an argument that computes the cohomology groups $H^p(\mathbb{CP}^3)$. (4 marks)

(iii) Compute the cohomology groups of $S^2 \times S^4$. (3 marks)

(iv) Are $S^2 \times S^4$ and $\mathbb{CP}^3$ homotopy equivalent? Justify your answer. You may use without proof that the ring structure in cohomology is a homotopy invariant. (2 marks)

(v) Is $(S^2 \times S^4) \setminus \{p\}$ homotopy equivalent to any compact orientable smooth manifold? Justify your answer. *Hint: Look at Q2(c).* (3 marks)

(Total: 20 marks)

5. (a) Give, with brief justification, examples of:

   (i) A map $f : \mathbb{R} \to \mathbb{R}$ that is Morse. (2 marks)

   (ii) A non-constant map $f : \mathbb{R} \to \mathbb{R}$ that is not Morse. (1 mark)

   (iii) For a Riemannian manifold M of your choice, a map $f : M \to \mathbb{R}$ that is Morse but not Morse-Smale. You may use any consequence of the Morse-Smale property that was proved in the course. (3 marks)

   (iv) For a manifold N of your choice, a map $f : N \to \mathbb{R}$ to show that an isolated critical point may be degenerate. (2 marks)

(b) Show that the determinant function det : $M_{2\times 2} \to \mathbb{R}$ on the set of 2×2 matrices is Morse, and find the indices of its critical points. (3 marks)

(c) Now let M, N be Riemannian manifolds.

   (i) If $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse-Smale, show that the function

$$f + g : M \times N \to \mathbb{R}$$

$$(f + g)(x, y) = f(x) + g(y)$$

   is Morse, and that it is Morse-Smale when $M \times N$ is endowed with the product metric. (6 marks)

   (ii) If $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse, is $f \cdot g : M \times N \to \mathbb{R}$

$$(fg)(x, y) = f(x)g(y)$$

   necessarily Morse? Give a proof or counterexample. (3 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH97052/M4P54

Differential Topology (Solutions)

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1. (a) (i) If ω is exact then take $\eta \in \Omega^{n-1}(M)$ such that $d\eta = \omega$. Then by Stokes' Theorem $\boxed{\text{seen} \Downarrow}$

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$$

since M has empty boundary. $\boxed{3, \text{A}}$

(ii) If ω is closed then again by Stokes' Theorem $\int_{\partial M} \omega = \int_M d\omega = 0$. $\boxed{3, \text{A}}$

(b) (i) Suppose $\omega = d\eta$. Then the same is true of their restrictions to S. Hence by Stokes' theorem $0 \neq \int_S \omega = \int_S d\eta = \int_{\partial S} \eta = 0$ since S has empty boundary. $\boxed{\text{meth seen} \Downarrow}$

$\boxed{2, \text{B}}$

(ii) Let N be such a manifold. Then $d\omega = 0$ as ω is closed, so by Stokes' theorem ω satisfies

$$0 = \int_N d\omega = \int_S \omega \neq 0.$$

$\boxed{2, \text{A}}$

(c) Let $\omega \in \Omega^n(N)$ be such that $\int_N \omega = 1$. Then $\boxed{\text{unseen} \Downarrow}$

$$\deg(f\mid_{\partial M}) = \int_{\partial M} f^*\omega = \int_M d(f^*\omega) = \int_M f^* d\omega.$$

Where the second equality is by Stokes. But since $d\omega$ is an $n+1$ form on N, the integral on the right must be zero. $\boxed{2, \text{C}}$

(d) (i) Orientability is equivalent to the existence of an atlas such that $\det \mathrm{D}\tau_{\alpha\beta} > 0$ for all transition maps $\tau_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ of the atlas. If $\tau_{\alpha\beta}$ is a transition map for M and $\sigma_{\gamma\delta}$ is a transition map for N, then the differential of the corresponding transition map for the natural atlas on the product manifold has block form $\boxed{\text{seen} \Downarrow}$

$$\begin{pmatrix} D\tau_{\alpha\beta} & 0 \\ 0 & D\sigma_{\gamma\delta} \end{pmatrix},$$

and hence the determinant is the product of the two determinants. $\boxed{4, \text{A}}$

(ii) The differential of F is $\boxed{\text{unseen} \Downarrow}$

$$\mathrm{D}F = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

The determinant of which is $r > 0$. So it is orientation preserving. $\boxed{2, \text{B}}$

(iii) The differential of such a function looks like

$$\begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix}.$$

Substituting from the Cauchy-Riemann equations gives $\det \mathrm{D}f = (\frac{\partial a}{\partial x})^2 + (\frac{\partial a}{\partial y})^2$ which gives the result. $\boxed{2, \text{C}}$

2. (a) (i) Mayer-Vietoris: there is an exact sequence of de Rham cohomology groups $\boxed{\text{seen} \Downarrow}$

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(M) \to \dots$$

where the first map is $\omega \mapsto (i_U^*\omega, i_V^*\omega)$, the second map is $(\omega, \eta) \mapsto j_U^*\omega - j_V^*\eta$. Here $i_U : U \to M$ and $j_U : U \cap V \to U$ are the inclusions, and similarly for V. The third map is the boundary map which comes from the Snake Lemma.

$\boxed{2, \text{A}}$

(ii) By the above it suffices to prove that the last map in the sequence $\boxed{\text{meth seen} \Downarrow}$

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V)$$

is surjective. Substituting in what we know, by connectedness we obtain

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$$

We have the required surjectivity, since any constant function on the intersection can be lifted to a constant function on, say, U. $\boxed{3, \text{B}}$

(iii) Consider the sequence from the proof above: $\boxed{\text{unseen} \Downarrow}$

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0.$$

Since the alternating sum of the dimensions must be zero, and the dimension of the zeroth cohomology group is equal to the number of connected components, the result follows by a dimension count. $\boxed{3, \text{C}}$

(b) Consider the case $n = 1$. We can use Mayer-Vietoris, taking U and V contractible open sets, and $U \cap V$ a disjoint union of two contractible open sets: for example, take $\boxed{\text{meth seen} \Downarrow}$

$$U = \mathbb{R}^2 \setminus \{y = 0, x \geq 0\}$$
$$V = \mathbb{R}^2 \setminus \{y = 0, x \leq 0\}.$$

Thus we obtain

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(M) \to 0 \oplus 0 \to 0 \to H^2(M) \to 0 \dots.$$

From this it is clear that $H^2(M) = 0$, and we must have $H^1(M) \cong \mathbb{R}$ by a dimension count. Now suppose that the result holds for $n-1$. We use Mayer-Vietoris again. Let $U = \mathbb{R}^2 \setminus \{p_1, \dots p_{n-1}\}$ and $V = \mathbb{R}^2 \setminus \{p_n\}$. Then $U \cap V = \mathbb{R}^2 \setminus \{p_1, \dots p_n\}$, and $U \cup V = \mathbb{R}^2$. Thus using the induction hypothesis we obtain

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^0(\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}) \to 0 \to \mathbb{R}^{p-1} \oplus \mathbb{R} \to H^1(\mathbb{R}^2 \setminus \{p_1, \dots, p_n\}) \to 0$$

This splits up as two exact sequences at the middle zero, and applying a dimension count to each proves the claim. (One may also conclude that the zeroth cohomology group is one-dimensional directly from connectedness.) $\boxed{5, \text{D}}$

(c) (i) $H^0_c(M) = H^0(M)$ when M is compact, and since M is connected this is $\mathbb{R}$. By Poincaré duality, so is $H^n(M)$. $\boxed{\text{seen} \Downarrow}$

If M is not compact, $H^0_c(M) = 0$. This is because a class in $H^0_c(M)$ is a compactly supported locally constant function, which must therefore be zero. Poincaré duality again gives $H^n(M) = 0$.

$\boxed{4, \text{A}}$

(ii) We use Mayer-Vietoris again with open sets that up to homotopy equivalence are $U = M \setminus \{x\}$, V contractible, $U \cup V = M$ and $U \cap V = S^{n-1}$. The relevant part of the Mayer-Vietoris sequence looks like $\boxed{\text{meth seen} \Downarrow}$

$$H^{n-2}(S^{n-1}) \to H^{n-1}(M) \to H^{n-1}(M \backslash \{x\}) \to H^{n-1}(S^{n-1}) \to H^n(M) \to H^n(M \backslash \{x\}).$$

The term $H^{n-2}(S^{n-1})$ is zero, as is the term $H^n(M \setminus \{x\})$ since $M \setminus \{x\}$ is not compact. Therefore the map $H^{n-1}(S^{n-1}) \to H^n(M)$ is an isomorphism, which implies the result. $\boxed{3, \text{D}}$

3. (a) (i) Since it is proper, f induces linear pullback maps of top degree compactly supported cohomology, $f^* : H^n_c(N) \to H^n_c(M)$. By Poincaré duality we have $H^n_c(M) \cong H^n_c(N) \cong \mathbb{R}$. Hence the pullback map above corresponds to multiplication by some real number. This is the degree of f. seen ⇓ 2, A

(ii) Since y is a regular value, for each i we know that the differential Df_{x_i} is invertible. Thus by the Implicit Function Theorem there exists a small neighbourhood V_i of each x_i such that $f\,|_{V_i}$ is a diffeomorphism onto its image. We may shrink these V_i to ensure that they are disjoint. Let U be the intersection of the images of $f\,|_{V_i}$. Shrink the V_i further, by intersecting each one in turn with $f^{-1}(U)$. Then we obtain disjoint open sets $U_i \ni x_i$, such that each U_i maps diffeomorphically onto U. The preimage of U under f consists of a disjoint union of open sets, U_i, with the required property. 3, B

(iii) If f is not surjective, then take a point not in the image. This is vacuously a regular value, since the pre-image is empty. Hence the degree is zero, which gives a contradiction. 2, A

(iv) Any counterexample that works gets full marks. The easiest answer is obtained by collapsing S^1 onto the interval $[-1, 1]$, and then mapping that surjectively onto another S^1 by 'wrapping it around'. unseen ⇓ 3, D

(b) (i) Take a regular value $c \in \mathbb{C} = \mathbb{CP}^1 \setminus \infty$: this is a value such that the polynomial $f - c$ does not have a repeated root. The number of pre-images is n by the Fundamental Theorem of Calculus (since ∞ maps to ∞) and by Q1(d)(iv) they all come with a positive sign since the map is orientation-preserving. meth seen ⇓ meth seen ⇓ 3, B

(ii) No - since even degree polynomials are never surjective, and hence have degree zero. For example, $f(x) = x^2$. 2, A

(iii) No - consider homotoping the identity map on $\mathbb{R}$ to the zero map. Alternatively, consider polynomials $f : \mathbb{C} \to \mathbb{C}$. These have degree equal to their degree as polynomials, and are proper since pre-images of closed bounded sets are closed and bounded. But one can easily construct a (linear) homotopy from e.g. z^2 to a constant map. unseen ⇓ 2, C

(c) Let $0 \neq [\omega] \in H^p(N)$ be in the kernel of f^*. By Poincaré duality there exists some η such that $\int_N \omega \wedge \eta = 1$. This implies that meth seen ⇓

$$\deg(f) = \int_M f^*(\omega \wedge \eta) = \int_M f^*\omega \wedge f^*\eta.$$

But the integral on the right must be zero, if $f^*[\omega] = 0$. 3, B

4. (a) (i) The integration map is $[\omega] \to \int_M \omega$. It is well-defined since integrals of exact forms are zero. The latter fact follows from Stokes' theorem, as in Q1(a)(ii). $\boxed{\text{seen} \Downarrow}$ $\boxed{2, \text{A}}$

(ii) Poincaré duality asserts that the map

$$\Phi : H^p(M) \to H_c^{n-p}(M)^*$$

given by

$$\omega \mapsto (\eta \mapsto \int_M \omega \wedge \eta)$$

is an isomorphism- i.e. with compact supports, not in ordinary de Rham cohomology. $\boxed{2, \text{A}}$

(iii) Since S^n is compact, its compactly supported cohomology groups are isomorphic to the ordinary de Rham cohomology groups. The only nonzero cohomology groups of S^n are $H^0(S^n) \cong H^n(S^n) \cong \mathbb{R}$. Thus it remains to verify that the required duality map is an isomorphism for $p = 0, n$. First take $p = 0$. Since the groups are the same dimension, we only need to show injectivity. A non-zero class $[c] \in H^0(S^n)$ is simply a constant function, so for any $[\omega] \in H^n(S^n)$ we have

$$\int_{S^n} c \wedge \omega = c \int_{S^n} \omega.$$

To show injectivity we just need to find ω with $\int_{S^n} \omega \neq 0$, but this holds by surjectivity of the integral map. The case $p = n$ is similar. Since the integral map is surjective, it is also injective because both groups are the same dimension. Thus for any $0 \neq [\omega] \in H^n(S^n)$, we know that $\int_{S^n} \omega \neq 0$. Hence any nonzero constant function will not be in the kernel of the associated element of $H^0(S^n)^*$. $\boxed{2, \text{A}}$

(b) (i) It suffices to show that if ω is a exact then so is $\omega \wedge \eta$ for all closed η, and that the wedge of two closed forms is closed. Let $\omega = d\sigma$. Then the Leibniz rule gives $d(\sigma \wedge \eta) = d\sigma \wedge \eta \pm \sigma \wedge d\eta$. The first term is $\omega \wedge \eta$, and the second term is zero since η is closed. The wedge of two closed forms is closed by the Leibniz rule as used above. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{2, \text{A}}$

(ii) There is a homotopy equivalence $\mathbb{CP}^n \setminus \{p\} \simeq \mathbb{CP}^{n-1}$ for each n. First observe that $\mathbb{CP}^1 \cong S^2$. Then recall (as in Q2(c)(ii)) that the inclusion $\mathbb{CP}^n \setminus \{p\} \hookrightarrow \mathbb{CP}^n$ induces isomorphisms on cohomology degree less than $2n - 2$, which together with Poincaré duality and induction gives the cohomology groups as zero in all odd degrees, and $H^p(\mathbb{CP}^n) = \mathbb{R}$ for all even $p \leq 2n$. $\boxed{4, \text{B}}$

(iii) First recall that $H^p(S^n) = \mathbb{R}$ if $p = 0, n$, and the group is zero otherwise. By the Künneth formula, we have $\boxed{\text{meth seen} \Downarrow}$

$$H^k(S^2 \times S^4) \cong \bigoplus_{p+q=k} H^p(S^2) \otimes H^q(S^4).$$

Hence

$$H^0(S^2 \times S^4) \cong \mathbb{R}$$

$$H^1(S^2 \times S^4) = 0$$

$$H^2(S^2 \times S^4) \cong \mathbb{R}$$
$$H^3(S^2 \times S^4) \cong 0$$
$$H^4(S^2 \times S^4) \cong \mathbb{R}$$
$$H^5(S^2 \times S^4) \cong 0$$
$$H^6(S^2 \times S^4) \cong \mathbb{R}.$$

All higher groups are zero. $\boxed{3, \text{C}}$

(iv) No. They have the same cohomology groups, but the ring structure is different. The ring structure of $\mathbb{CP}^3$ is $\mathbb{R}[x]/(x^4)$, where x has degree 2, i.e. it corresponds to a cohomology class of degree 2. For example, any class in degree two of $S^2 \times S^4$ comes from $H^2(S^2) \otimes H^0(S^4)$, and has square zero under the wedge product. But there is a degree two class in the cohomology of $\mathbb{CP}^3$ which does not have square zero. $\boxed{\text{unseen} \Downarrow}$ $\boxed{2, \text{D}}$

(v) By Q2(c), the inclusion

$$(S^2 \times S^4) \setminus \{p\} \to (S^2 \times S^4)$$

induces an isomorphism on cohomology in degree less than $6 - 1 = 5$. Since $(S^2 \times S^4) \setminus \{p\}$ is not compact, its cohomology is zero in degree 6 and above. Suppose such a manifold M existed as in the question. Since M is compact and orientable, we must have dim$M = 4$, since that is the highest degree in which the cohomology is non-zero. But the square of the generator in degree 2 is zero, which contradicts Poincaré duality. $\boxed{3, \text{D}}$

5. (a) (i) Any justified answer that works gets full marks. The obvious answer would be the identity map, which has no critical points at all, or a polynomial function such as x^2, whose Hessian (i.e. second derivative) is identically 2. $\boxed{\text{seen} \Downarrow}$ $\boxed{2, \text{M}}$

(ii) Any justified answer that works gets full marks. Again, polynomials are a good source of examples. It needs to be a non-constant polynomial whose derivative has a repeated root, so x^3 works. The Hessian is $6x$, which vanishes at 0, a critical point. $\boxed{1, \text{M}}$

(iii) The best example is the height function on a 'torus on its side'. As proved in the course, there are two critical points of index 1, and there is a flow line between them, which cannot happen if the function is Morse-Smale. $\boxed{3, \text{M}}$

(iv) Any example that works gets full marks. The simplest example is $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^3$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{2, \text{M}}$

(b) If $\boxed{\text{unseen} \Downarrow}$ $\boxed{\text{meth seen} \Downarrow}$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the differential is $\mathrm{Ddet}_M = (d, -c, -b, a)$, which vanishes only at the zero matrix. The Hessian is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

If (w, x, y, z) is an eigenvector with eigenvalue λ then $w = \lambda z = \lambda^2 w$, so if λ is negative we have $\lambda = -1$, which implies $x = y$. Hence the negative eigenspace is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

so the index is 2. $\boxed{3, \text{M}}$

(c) $\boxed{\text{unseen} \Downarrow}$

(i) We first show that $f + g$ is Morse. Since $D(f + g)_{(x,y)} = (Df_x, Dg_y)$ we see that $\mathrm{Crit}(f + g) = \mathrm{Crit}(f) \times \mathrm{Crit}(g)$. The Hessian of $f + g$ at a critical point is block diagonal of the form

$$\begin{pmatrix} H_f & 0 \\ 0 & H_g \end{pmatrix}$$

so $f + g$ is Morse if f and g are. With respect to the Riemannian metric on $M \times N$ induced from the metrics on M and N we also have

$$\nabla(f + g) = (\nabla f, \nabla g).$$

Hence for a critical point $(x, y) \in M \times N$, we have

$$W^u_{f+g}(x, y) = W^u_f(x) \times W^u_g(y)$$

$$W^s_{f+g}(x,y) = W^s_f(x) \times W^s_g(y).$$

Thus the result follows from the observation that if $A, B \subset M$ are transverse and $C, D \subset N$ are transverse then $A \times C$ and $B \times D$ are transverse in $M \times N$. $\boxed{\text{6, M}}$

(ii) No. Again, any counterexample that works gets full marks, but the easiest way to construct one is: taking $M = N = \mathbb{R}$, with f, g any Morse functions, let x_0 be a critical point of f. Shifting f by a suitable constant we may assume that $f(x_0) = 0$. Since

$$\nabla(f \cdot g)(x,y) = (f'(x)g(y), f(x)g'(y)),$$

we see that (x_0, y) is a critical point for all $y \in N$. Since the critical points are not all isolated, $f \cdot g$ is not Morse. $\boxed{\text{unseen} \Downarrow}$

$\boxed{\text{3, M}}$

ExamModuleCode	QuestionNumber	Comments for Students
MATH97052MATH97163	1	This was generally well answered. In d(i), most students attempted to prove orientability by exhibiting a non-vanishing form in top degree, rather than directly using the Jacobian determinant of transition maps. Whilst they were usually able to write down a correct form, not many were able to convincingly show that it was non-vanishing, i.e. to provide a tuple of tangent vectors on which it did not vanish.
MATH97052MATH97163	2	This question was also answered fairly well. In defining the Mayer-Vietoris sequence for a(i), several students did not give a description of where the boundary map comes from, i.e. from the snake lemma. Parts a(ii) and (b) required a little ingenuity in working with exact sequences, but both were similar to things previously seen. Part c(ii) was a more challenging generalisation of a problem sheet question
MATH97052MATH97163	3	In Q3 the parts that posed the most difficulties were a(iv) and c. Of these, a(iv) was intended to be quite challenging, and the construction of counterexamples like these requires a good intuitive grasp of the concept of degree (a certain picture drawn in lectures was the thing to keep in mind). For part (c), in order to make use of the hint it was necessary to understand that Poincare duality states not just that certain cohomology groups are isomorphic, but that a particular geometrically meaningful map gives that isomorphism.
MATH97052MATH97163	4	This question was perhaps the most challenging. Some answers to part a(iii) seemed not to be totally clear on what it means to verify Poincare duality (i.e. unclear on precisely what Poincare duality states – see comment to Q3). Most students were able to give reasonable answers to parts a and b(i)-(iii), though not many were able to conclusively prove that the cohomology rings in (iv) were not isomorphic. Part (v) was amongst the most difficult parts of the exam, and required putting together several facts from the course. The most common error was to assume that the punctured product of spheres could only be homotopy equivalent to a manifold of dimension 6, when in fact determining what the dimension of the manifold would have to be was a non-trivial part of the question.

MATH97052MATH97163	5	Most students coped reasonably well with this question. The (counter)examples needed in parts a and c(ii) were most easily obtainable from polynomial functions, with the exception of a(iii) which was covered in the mastery material notes. Part c(i) required a lengthy verification and unwinding of definitions, but the key concept – the product metric and the gradient flow of f+g under it – had featured in a problem sheet question.