

## **Partial Differential Equations in Action**

**MATH50008**

### **Guidance for the exam**

In this document, I would like to provide you with some guidance on how to prepare for the May exam. In order to prepare for the exam, I would recommend that you make sure that you understand well the lecture notes and in particular, the numerous worked examples covered in the notes. For each topic, I would encourage you then to start by practicing the problems on the problem sheets. The problem sheets include a scoring system corresponding to how difficult I think a problem is (ranging from a direct application of concepts seen in lectures to harder problems requiring more thinking about the underlying physical concepts). I hope that this scoring system will be helpful and will guide you towards the problems you should attempt first during your revision.

### **Structure of the exam and syllabus**

The exam for *MATH50008 - PDEs in Action* is a 2h exam which consists of four questions. This exam will test your understanding of Chapters 1, 2 and 3 of the lecture notes. The material of Chapter 4 is not examinable due to its computational nature.

The questions on the exam are not ordered by difficulty. This means that Q4 is not necessarily harder than Q1. All four questions will test your basic understanding of the material as well as test your deeper understanding of the more advanced part of the module. I would recommend that you take a moment to read through each of the questions before you get started. As always, I would also encourage you to take a moment to read a question entirely before starting working on it as some information provided in the later subquestions may prove useful and guide your thinking.

In this module, we put an emphasis on the analytical treatment of PDEs as mathematical objects but also discussed in great length their physical significance and applications. For the exam, my expectations in terms of your knowledge of the physical principles underlying various PDE models will be limited. To give you a concrete example, I would not expect you to derive a new PDE model from conservation principles like we did multiple times in the lectures but you should understand the physical significance of various types of boundary conditions for classical PDEs or the various terms appearing in a PDE.

Finally, as you have seen by now, this module uses many of the concepts you have been introduced to in previous modules including resolution techniques for ODEs, Fourier series and Fourier transforms etc. Thus, I recommend that you make sure that they are comfortable with these topics which were introduced in the first year module MATH40004 - Calculus and Applications and the first term of the second year module MATH50004 - Multivariable Calculus and Differential Equations. If useful to a question, I will provide you with the definition of the Fourier transform and the inversion formula as to set the convention you should use but I expect you to know the basic properties like the convolution theorem or the Fourier transform of derivatives of functions (as they have been used extensively in problems).

### **What is expected from your answers?**

As usual, the neatness, completeness and clarity of the answers will contribute to the final mark. Your answers should be clearly organized and legible. Always remember that we cannot mark what we cannot read. As you have seen throughout this module, I have generally provided you with quite thorough answers to the problems. As a consequence, I would expect at this stage that you are familiar with the level of details and justification that is expected from your answers. I would

recommend that you leave nothing to doubt, this will generally not play in your favour. Finally, the number of marks given for a question may guide you on the level of details we expect.

## Some comments about the material covered

In this module, we have introduced a variety of resolution methods which we have applied to a variety of equations in different geometry. During your revision, a useful exercise would be to summarize when each of these methods can be applied (what type of equation/which geometry). I have released a document which I hope will be helpful summarizing the steps of each of these methods. Revising this thoroughly and adding onto this document any information that you find appropriate is a useful exercise which will (hopefully) naturally emerge as a conclusion of your revisions.

Below, I do not intend to provide an exhaustive list of what we covered in the module but rather highlight what I consider to be important points.

### Chapter 1 - Introduction

- ☐ This chapter starts with very basic but important terminology and properties of PDEs. In particular, the superposition and subtraction principles often prove to be very useful.
- ☐ The next section introduces the fundamental ideas behind dimensional analysis.
- ☐ Dimensional reduction techniques are very useful in reducing the complexity of mathematical problems. We have seen that solving simple linear systems of equations emerging from dimensional arguments can drastically reduce the number of variables/parameters your problem depends on.
- ☐ We have also seen that dimensional analysis is at the heart of the nondimensionalization of equations (including PDEs).
- ☐ Using dimensional analysis, we introduced a type of solutions called similarity solutions (which were seen again in the light of space/time transformations in Chapter 3).

In this section, you may wonder how much physical intuition/knowledge is required. First, let me say that I do not expect you to know the dimensions of complicated physical quantities but it may help to know by heart some of the easy ones (like position, velocity, acceleration, energy, density). Remember that the fact that equations need to be dimensionally homogeneous may help you find the dimensions of the parameters appearing in it. I expect you to be able to nondimensionalize equations and discuss similarity solutions.

### Chapter 2 - First-order PDEs

- ☐ In this chapter, the derivation of most of the equations relied on conservation principles and the ideas behind conservation laws. This was instrumental in deriving both linear and nonlinear first-order PDEs.
- ☐ The main resolution technique which we have introduced is the method of characteristics. You should feel comfortable sketching a characteristics diagram from which you may read the solution at all times and positions. A useful exercise would be to summarize what sort of characteristics are expected depending on the problem (e.g. straight lines with equal slopes, straight lines with different slopes, general curves etc.). Remember that the solution will be constant along the characteristics if the problem is homogeneous, otherwise you need to solve an ODE.
- ☐ As we have seen in the midterm, you should think carefully about where the characteristics are emerging from (i.e. what information about the solution they carry) in initial boundary value problems.

- ☐ We also discussed in details nonlinear first-order PDEs and showed that the nonlinearity may lead to very interesting phenomena like shock formation and propagation. You should feel comfortable discussing the emergence of shocks in terms of the characteristics of the problem but also the emergence of a discontinuity in the solution.
- ☐ After a shock has formed, we have seen various methods to obtain the shock path (e.g. jump conditions and Whitham's equal area rule). These are important to discuss shock propagation.

### Chapter 3 - Classical second-order PDEs

- ☐ This third chapter deals with linear second-order PDEs. It opens up with the classification of linear second-order PDEs into parabolic, elliptic and hyperbolic equations.
- ☐ We then discussed the important topic of boundary conditions. When defining mathematically the various types of boundary conditions, we also discussed the physical interpretation of these boundary conditions in various examples. This is an example of the sort of physical intuition/understanding I would expect you to have.
- ☐ In a second time, we focused on the three classical linear second-order PDEs: the diffusion equation, the wave equation and the Laplace/Poisson equation. You should understand where these equations come from but you wouldn't be asked to re-derive them from scratch without guidance in an exam setting.
- ☐ In dealing with these three equations, we talked about uniqueness of solutions. It is important that you understand the typical method used to prove uniqueness of solutions.
- ☐ The majority of our time was spent introducing resolution methods for these equations. We introduced a variety of general methods (i.e. method of separation of variables, Fourier transform methods, solutions based on Green's functions) as well as more specific methods like d'Alembert's solution for the wave equation or the fundamental solution to the diffusion equation (which we derived as a similarity solution). Some of you may have noticed that the fundamental solution of the diffusion equation is the solution to a point source problem, i.e. this solution also called the heat kernel is the Green's function for the diffusion equation.
- ☐ You should feel comfortable applying these methods. Take a step back and practice the various examples we have seen in lecture and on the problem sheets; this will help you understand the structure of the resolution technique.
- ☐ Some of these methods can be applied on finite domains only, some can be applied on infinite domains, but others can be applied in either finite or infinite domains. During your revision, it would be useful to classify these methods and understand when they can be applied.

### Past exam papers

You may wonder where you can find resources to help you practice during your revisions. This module has been taught for a few years now, past papers can be found on Maths Central but also here: [Past exam papers](#).

To provide enough material for you to work through, I have attempted to provide you with long problem sheets. They offer more than 50 problems for you to practice. As far as I am concerned, I believe that if you develop a deep understanding of these problems you should have no problem with the final exam. The midterm exams also provides an example of the type of problems you may be asked to solve in an exam setting.

If this is not enough and you would like even more material to practice, what can you do? While this module is new, some parts of it were taught in other modules in the past years (note that I have made

significant changes to both content and style). Nevertheless, I thought it may be useful to give you a list of papers containing problems which I think are relevant. Here is a list with hyperlinks (make sure that you are authenticated to blackboard before clicking on the links below):

- Most of the concepts we introduced in Chapter 2 were previously covered in a module called *M2AM - Nonlinear waves*; here is a list of problems which I find relevant to our module and could be used to revise:
  - [May 2020](#) — See Q2 and Q3
  - [May 2019](#) — See Q2 and Q3
  - [May 2018](#) — See Q1, Q2 and Q3
  - [May 2017](#) — See Q1 and Q2
  - [May 2016](#) — See Q1 (ii) and Q2
- The part on linear second-order PDEs was treated in a much more succinct way in a module called *M2AA2 - Multivariable calculus*; here is a list of problems which I find relevant to our module and could be used to revise:
  - [January 2020](#) — See Q2
  - [May 2019](#) — See Q4
  - [May 2018](#) — See Q3 and Q4 (iii)-(v)
  - [May 2017](#) — See Q3 and Q4
  - [May 2016](#) — See Q3 and Q4

## Asking for help

During this term, the GTAs and myself made a real effort to answer your queries as quickly as possible to make sure you could get prompt feedback. Note that the Ed discussion forum will remain available and be monitored until the exam day but will be monitored less frequently. Thus, feel free to post your questions there and we will try to respond as soon as possible but expect that you may have to wait a couple of days to get a response from us.