Classical Dynamics **Autumn Term 2024**

Answers to Problem Sheet 2

1. $\dot{x} = \dot{u}v + u\dot{v}$, $\dot{y} = v\dot{v} - u\dot{u}$.

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(u^2 + v^2)(\dot{u}^2 + \dot{v}^2),$$

due to the cancellation of cross terms.

2. Let x be the position of the trolley and (X, Y) denote the position of the bob. The holonomic constraint is $(X - x)^2 + Y^2 = l^2$.

3.

$$x = R\cos u, \quad y = R\sin u \quad z = \alpha u,$$

where u is a real parameter.

$$T = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{m}{2}(R^2\sin^2 u\,\dot{u}^2+R^2\cos^2 u\,\dot{u}^2+\dot{z}^2) = \frac{m}{2}(R^2\dot{u}^2+\dot{z}^2) =$$

$$= \frac{m}{2}\left(\frac{R^2}{\alpha^2} + 1\right)\dot{z}^2.$$

The potential energy is $V = mgz$. The Lagrangian is

$$L = T - V = \frac{m}{2}\left(\frac{R^2}{\alpha^2} + 1\right)\dot{z}^2 - mgz.$$

The equation of motion can be written as

$$\ddot{z} = -\tilde{g}, \qquad \tilde{g} = \frac{\alpha^2 g}{R^2 + \alpha^2}.$$

This is the same as for vertical motion under constant gravity but with a reduced acceleration $\tilde{g}$.

4. The Euler-Lagrange equations are

$$\frac{d}{dt}(mR^2\sin^2\theta\dot{\phi}) = 0$$

$$\frac{d}{dt}mR^2\dot{\theta} - mR^2\sin\theta\cos\theta\dot{\phi}^2 + mgR\sin\theta = 0.$$

Setting $\theta =$ constant gives

$$-mR^2 \sin\theta\cos\theta\dot{\phi}^2 + mgR\sin\theta = 0,$$

so that

$$\dot{\phi}^2 = \frac{g}{R\cos\theta}.$$

The period is

$$T = \frac{2\pi}{\dot{\phi}} = 2\pi\sqrt{\frac{R\cos\theta}{g}}.$$

5. The Euler-Lagrange equation is

$$\frac{d}{dt}me^{\gamma t}x + e^{\gamma t}V'(x) = 0,$$

or

$$e^{\gamma t}\left(m\ddot{x} + \gamma m\dot{x} + V'(x)\right) = 0$$

so that the force is

$$F = m\ddot{x} = -\gamma m\dot{x} - V'(x)$$

which is not conservative as it depends on the velocity.

For $\gamma > 0$ the force has a frictional component $-\gamma m\dot{x}$. For $\gamma < 0$ this contribution has a negative friction which looks unphysical.

6. (i) The kinetic energy of a particle of mass m in spherical polar coordinates is

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$

Spherical polar coordinates are conventionally defined with θ as the angle between $\hat{\mathbf{r}}$ and the positive z-axis so that $z = r\cos\theta$. Now let θ be measured with respect to the negative z-axis so that $z = -r\cos\theta$ and as usual $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$. With these 'modified' spherical polar coordinates the kinetic energy formula is unchanged.

Now setting $r = l$ and $\dot{\phi} = \Omega$ gives the stated formula

$$T = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right).$$

(ii) The potential energy is the same as for a non-rotating pendulum, ie. $V = -mgl\cos\theta$. Accordingly,

$$L = T - V = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right) + mgl\cos\theta.$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

yields

$$ml^2\ddot\theta - ml^2\Omega^2 \sin\theta\cos\theta + mgl\sin\theta = 0,$$

or

$$\ddot\theta = \sin\theta\left(-\frac{g}{l} + \Omega^2\cos\theta\right).$$

(iii) Solutions of the form θ =constant require $\sin\theta = 0$ (yielding the standard equilibria $\theta = 0$ and $\theta = \pi$) or

$$\left(-\frac{g}{l} + \Omega^2\cos\theta\right) = 0$$

which has the solutions

$$\theta = \pm\cos^{-1}\frac{g}{\Omega^2 l}$$

provided $\Omega^2 > g/l$.

(iv) As L does not depend explicitly on t

$$H = \dot\theta\frac{\partial L}{\partial\dot\theta} - L = \frac{ml^2}{2}\dot\theta^2 - \frac{ml^2}{2}\Omega^2\sin^2\theta - mgl\cos\theta$$

is constant. Note that this is *not* the total energy

$$T + V = \frac{ml^2}{2}\dot\theta^2 + \frac{ml^2}{2}\Omega^2\sin^2\theta - mgl\cos\theta.$$

The total energy is not constant since

$$\frac{d}{dt}(T+V) = \frac{d}{dt}(H + ml^2\Omega^2\sin^2\theta) = 2ml^2\Omega^2\sin\theta\cos\theta\ \dot\theta.$$

(v) To find the frequency of small oscillations look at the equations of motion for θ near $\theta = 0, \pi$ and $\theta_0 = \cos^{-1}(g/\Omega^2 l)$.

Near $\theta = 0$

$$\ddot\theta = \theta\left(-\frac{g}{l} + \Omega^2\right),$$

a SHO with angular frequency $\omega = \sqrt{g/l\ - \Omega^2}$. If $\Omega^2 > g/l, \quad \theta = 0$ is an unstable equilibrium point.

For θ near θ_0

$$\frac{d^2}{dt^2}(\theta-\theta_0) = \sin[\theta_0+(\theta-\theta_0)]\left(-\frac{g}{l} + \Omega^2\cos[\theta_0+(\theta-\theta_0)]\right) \approx -\Omega^2\sin^2\theta_0\,(\theta-\theta_0),$$

using the Taylor expansion $\cos[\theta_0+(\theta-\theta_0)] = \cos\theta_0 - \sin\theta_0(\theta-\theta_0)+\ldots.$
The approximate equation of motion describes a SHO with

$$\omega^2 = \Omega^2\sin^2\theta_0 = \Omega^2(1-\cos^2\theta_0) = \Omega^2\left(1-\frac{g^2}{l^2\Omega^4}\right)$$

or

$$\omega = \Omega\sqrt{1-\frac{g^2}{l^2\Omega^4}},$$

which is always less than the frequency of the turntable.

The equilibrium at $\theta=\pi$ is unstable for any Ω. Why?