IMPERIAL

MATH70017

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Tensor Calculus and General Relativity

Date: Tuesday, April 30, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Formula Sheet

Equation of Parallel Transport:

$$\frac{dv^a}{d\lambda} + \Gamma^a_{bc} v^b \frac{dx^c}{d\lambda} = 0.$$

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b,$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad \mathcal{R} = g^{bd} R_{bd}.$$

Einstein tensor:

$$G^{ab} = R^{ab} - \frac{1}{2} \mathcal{R} g^{ab}.$$

Schwarzschild metric:

$$ds^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right),$$

where R is the Schwarzschild radius. In the equatorial plane, $\theta = \pi/2$:

$$ds^2 = c^2 \left(1 - \frac{R}{r}\right) dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2 \; d\phi^2.$$

1. (a) Consider a version of the twins paradox where twin 1 remains on Earth. Twin 2 departs from Earth with uniform velocity at 80 percent light speed before returning to Earth at 60 percent light speed to reunite with twin 1. How much does twin 2 age during the round trip if twin 1 has aged 7 years while twin 2 was away?

Hint: a space-time diagram may be helpful. (6 marks)

(b) (i) What is the four-momentum of a particle? (distinguish massive and massless particles).

(ii) A particle of mass M is initially at rest. The particle emits a photon in the negative x^1-direction. The recoiling particle has mass $m < M$.
Determine the four-momentum of the recoiling particle and show that the energy of the emitted photon is

$$E = \frac{(M^2 - m^2)c^2}{2M}.$$

(iii) Determine the speed of the recoiling particle. Express your answer as a function of the ratio $r = m/M$.

(14 marks)

(Total: 20 marks)

2. The standard metric on the unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta \; d\phi^2,$$

where θ and ϕ are spherical polar coordinates. The non-zero Christoffel symbols are

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

(a) Verify that $\Gamma^{\phi}_{\theta\phi} = \cot\theta$. (5 marks)

(b) Show that

$$h = \sin^2\theta \frac{d\phi}{ds},$$

is constant along geodesics. (5 marks)

(c) Consider the covariant vector field with components $v_\theta = 0$, $v_\phi = \sin^2\theta$. Determine the four components of $\nabla_c v_b$. (5 marks)

(d) Show that the covariant vector field defined in part (c) satisfies

$$\nabla_a v_b + \nabla_b v_a = 0,$$

and comment on the result. (5 marks)

(Total: 20 marks)

3. (a) The metric on a torus is

$$ds^2 = a^2 d\theta^2 + (r + a\sin\theta)^2 d\phi^2.$$

The constants r and a are the external radius and inner radius, respectively. Here θ and ϕ are periodic angles; shifting θ or ϕ by 2π gives the same point.

Write down the components of the metric and the inverse metric. (5 marks)

(b) Suppose that the components of a metric, $g_{ab}(x)$, do not depend on a coordinate x^p. Show that $g_{pb}dx^b/ds$ is constant along geodesics. (5 marks)

(c) The metric on a two-dimensional surface (a semi-infinite cigar) with coordinates ρ and θ is

$$ds^2 = d\rho^2 + \tanh^2\rho \; d\theta^2.$$

Here $\rho \geq 0$ and θ is an angle; shifting θ by 2π gives the same point.

Using the result of part (b) we find that $h = \tanh^2\rho \; d\theta/ds$ is constant along geodesics.

(i) What are the possible values of h?

(ii) What is the minimum value of ρ for a geodesic with a given h?

(5 marks)

(d) Show that Einstein's equation,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

can be written in the alternative form

$$R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2}\mathcal{T} g^{\mu\nu}\right),$$

where $\mathcal{T} = g_{\alpha\beta}T^{\alpha\beta}$. (5 marks)

(Total: 20 marks)

4. (a) In the Schwarzschild spacetime,

$$h = r^2 d\phi/ds,$$

is constant along geodesics in the equatorial plane, $\theta = \pi/2$.

Show that geodesics in the equatorial plane satisfy

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{R}{r}\right) - \frac{R}{r} = \text{constant}.$$

(4 marks)

(b) Derive the orbit equation

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}Ru^2 = \frac{R}{2h^2},$$

where $u = 1/r$. (5 marks)

(c) Consider circular orbits of the form, $u = u_0$, where u_0 is a constant. Use the orbit equation to show that if $h^2 > 3R^2$ there are two circular orbits. (3 marks)

(d) For the two circular orbits obtained in part (c) show that one is stable and the other unstable. Hint: Consider $u = u_0 + u_1(\phi)$ where u_1 is small. (8 marks)

(Total: 20 marks)

5. The Kerr metric is

$$ds^2 = c^2\left(1 - \frac{Rr}{\Sigma}\right)dt^2 + \frac{2Racr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$$
$$-\left(r^2 + a^2 + \frac{Ra^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\ d\phi^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - Rr + a^2$. Here R and a are constants.

(a) Show that if $R > 2a$, g_{rr} is positive for $r_- < r < r_+$, where

$$r_\pm = \frac{R \pm \sqrt{R^2 - 4a^2}}{2}.$$

Explain briefly the significance of this result. (7 marks)

(b) Consider a freely falling massive particle moving on the axis of rotation $\theta = 0$.

(i) Show that

$$\left(\frac{dr}{ds}\right)^2 - \frac{Rr}{r^2 + a^2} = C,$$

where C is a constant.

(ii) Suppose that C is positive. How large should C be so that the trajectory includes all positive and negative values of r?

(13 marks)

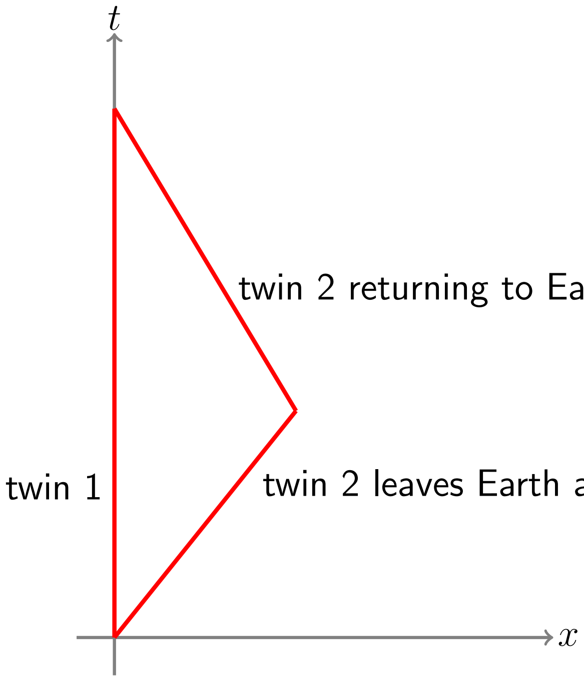
(Total: 20 marks)

Answers to April 2024 Examination

1. **(a)** In an inertial frame where twin 1 is stationary twin 2 travels at $v = \frac{4}{5}c$ for 3 years and $v = \frac{3}{5}c$ for 4 years. The ageing of twin 2 in years is

$$\tau = 3\sqrt{1 - \frac{16}{25}} + 4\sqrt{1 - \frac{9}{25}} = 3 \cdot \frac{3}{5} + 4 \cdot \frac{4}{5} = 5.$$

Space-time diagram:



(6 marks, A, seen similar)

(b) (i) Four-momentum of a particle is defined through $p^\mu = mu^\mu$ where m is the mass and $u^\mu = dx^\mu/d\tau$ is the four-velocity. For a massive particle $u = \gamma(c, \mathbf{v})$, and $p = m\gamma(c, \mathbf{v})$ where $\gamma = (1 - v^2/c^2)^{-1/2}$. For massless particles u is not defined - as massive particles satisfy $p \cdot p = m^2c^2$ four momentum of photons is defined to be null $p \cdot p = 0$. Identifying $p^0 = E/c$ where E is the energy the remaining components are fixed by the direction of the photon.

(5 marks, A, bookwork)

(ii) Initial four-momentum of stationary particle $p_M = (Mc, 0, 0, 0)$

Four-momentum of emitted photon $p_\gamma = (E/c, -E/c, 0, 0)$

By conservation of momentum the four-momentum of the recoiling particle is $p = p_M - p_\gamma = (Mc - E/c, E/c, 0, 0)$. Now

$$m^2c^2 = p \cdot p = (Mc - E/c)^2 - E^2/c^2 = M^2c^2 - 2ME,$$

hence the result.

(5 marks, B, seen similar)

(iii) Use

$$\frac{v}{c} = \frac{p^1}{p^0} = \frac{E/c}{Mc - E/c} = \frac{ME}{M^2c^2 - EM} = \frac{M^2 - m^2}{M^2 + m^2} = \frac{1 - r^2}{1 + r^2},$$

where $r = m/M$

(4 marks, D, unseen)

2. **(a)** $\Gamma^{\phi}_{bc} = \frac{1}{2}g^{\phi d}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right)$. Only $d = \phi$ contributes so that $\Gamma^{\phi}_{\theta\phi} = \frac{1}{2}g^{\phi\phi}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\theta\phi} - \partial_\phi g_{\theta\phi}\right) = \frac{1}{2}\frac{1}{\sin^2\theta}(\partial_\theta \sin^2\theta + 0 - 0) = \cot\theta$ as required. (5 marks, A, seen)

(b) As g_{ab} does not depend on ϕ, $h = g_{\phi b}dx^b/ds = g_{\phi\phi}d\phi/ds = \sin^2\theta d\phi/ds$ is constant along geodesics. An alternative method is to use the geodesic equation for ϕ. (5 marks, A, seen)

(c) $v_\theta = 0$, $v_\phi = \sin^2\theta$.

$\nabla_\theta v_\theta = \partial_\theta v_\theta - \Gamma^a_{\theta\theta}v_a = 0 - 0 = 0$

$\nabla_\theta v_\phi = \partial_\theta v_\phi - \Gamma^a_{\phi\theta}v_a = \partial_\theta \sin^2\theta - \Gamma^{\phi}_{\phi\theta}v_\phi = 2\sin\theta\cos\theta - \cot\theta\cdot\sin^2\theta = \sin\theta\cos\theta$.

$\nabla_\phi v_\theta = \partial_\phi v_\theta - \Gamma^a_{\theta\phi}v_a = 0 - \Gamma^{\phi}_{\theta\phi}v_\phi = -\sin\theta\cos\theta. = 0$

$\nabla_\phi v_\phi = \partial_\phi v_\phi - \Gamma^a_{\phi\phi}v_a = 0 - \Gamma^{\theta}_{\phi\phi}v_\theta = 0.$

(5 marks, B, seen similar)

(d) $\nabla_a v_b + \nabla_b v_a = 0$ holds as $\nabla_\theta v_\theta = 0$ and $\nabla_\phi v_\phi = 0$. In addition $\nabla_\theta v_\phi + \nabla_\phi v_\theta = \nabla_\theta v_\theta = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$.

Therefore v_a is a Killing vector and $v_b dx^b/ds = \sin^2\theta d\phi/ds$ is constant along geodesics. This result is the same as in part (b)

(5 marks, D, unseen)

3. **(a)** $g_{\theta\theta} = a^2, \quad g_{\phi\phi} = (r + a\sin\theta)^2, \quad g_{\theta\phi} = g_{\phi\theta} = 0.$

$g^{\theta\theta} = 1/a^2, \quad g^{\phi\phi} = (r + a\sin\theta)^{-2}, \quad g^{\theta\phi} = g^{\phi\theta} = 0.$

(5 marks, A, seen similar)

(b)

$$\frac{d}{ds}\left(g_{pb}\frac{dx^b}{ds}\right) = \partial_a g_{pb}\frac{dx^a}{ds}\frac{dx^b}{ds} + g_{pb}\frac{d^2x^b}{ds^2}$$

$$= \frac{1}{2}\partial_a g_{pb}\frac{dx^a}{ds}\frac{dx^b}{ds} + \frac{1}{2}\partial_b g_{pa}\frac{dx^a}{ds}\frac{dx^b}{ds} + g_{pb}\frac{d^2x^b}{ds^2},$$

as $\frac{dx^a}{ds}\frac{dx^b}{ds}$ is symmetric in the a and b indices. Assuming that $\partial_p g_{ab} = 0$ this can be written as

$$\frac{d}{ds}\left(g_{pb}\frac{dx^b}{ds}\right) = \frac{1}{2}(\partial_a g_{pb} + \partial_b g_{pa} - \partial_p g_{ab})\frac{dx^a}{ds}\frac{dx^b}{ds} + g_{pb}\frac{d^2x^b}{ds^2} = \Gamma_{pab}\frac{dx^a}{ds}\frac{dx^b}{ds} + g_{pb}\frac{d^2x^b}{ds^2}$$

$$= g_{pd}\left(\frac{d^2x^d}{ds^2} + \Gamma^d_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds}\right) = 0,$$

by the geodesic equation.

(5 marks, C, seen)

(c) (i)

$$g_{ad}\frac{dx^a}{ds}\frac{dx^b}{ds} = \left(\frac{d\rho}{ds}\right)^2 + \tanh^2\rho\left(\frac{d\theta}{ds}\right)^2 = \left(\frac{d\rho}{ds}\right)^2 + \frac{h^2}{\tanh^2\rho} = 1,$$

which gives

$$h^2 = \tanh^2\rho\left[1 - \left(\frac{d\rho}{ds}\right)^2\right].$$

As $-1 < \tanh\rho < 1$ it follows that $-1 < h < 1$.

(ii) Setting $d\rho/ds = 0$ gives $h^2 = \tanh^2\rho$. The minimum value of ρ is $\tanh^{-1}|h|$.

(5 marks, C, seen similar)

(d) Einstein's equation

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\mathcal{R}g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}.$$

Lower ν index

$$R^{\mu}{}_{\nu} - \frac{1}{2}\mathcal{R}\delta^{\mu}_{\nu} = \frac{8\pi G}{c^4}T^{\mu}{}_{\nu}.$$

Contract μ and ν indices

$$(1 - \tfrac{1}{2}4)\mathcal{R} = \frac{8\pi G}{c^4}T^{\mu}_{\mu}.$$

Inserting this formula for R into Einstein's equation gives

$$R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2}\mathcal{T}g^{\mu\nu}\right).$$

where $\mathcal{T} = g_{\alpha\beta}T^{\alpha\beta} = T^{\alpha}_{\alpha}$.

(5 marks, B, seen)

4. **(a)**

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 = 1$$

As the metric does not depend on t, $k = \left(1-\frac{r}{R}\right)dt/ds$ is constant along geodesics. This with the given result for $h = r^2 d\phi/ds$ yields

$$\frac{c^2k^2}{1-\frac{r}{R}} - \left(1-\frac{r}{R}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

Multiplying by $1-r/R$ gives

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{r}{R}\right) - \frac{R}{r} = c^2k^2 - 1 = \text{constant.}$$

(4 marks, A, seen)

(b) Now

$$\frac{dr}{ds} = -\frac{1}{u^2}\frac{du}{ds} = -\frac{1}{u^2}\frac{d\phi}{ds}\frac{du}{d\phi} = -h\frac{du}{d\phi}.$$

Accordingly,

$$h^2\left(\frac{du}{d\phi}\right)^2 + h^2(u^2 - Ru^3) - Ru = \text{constant.}$$

Differentiating with respect to ϕ gives the orbit equation

$$h^2\frac{du}{d\phi}\left(2\frac{d^2u}{d\phi^2} + 2u - 3Ru^2\right) = R\frac{du}{d\phi},$$

so that $du/d\phi = 0$ (a circular orbit) or

$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}Ru^2 = \frac{R}{2h^2}.$$

(5 marks, A, seen)

(c) Setting $u = u_0$ where u_0 is constant yields

$$u_0 - \frac{3}{2} R u_0^2 = \frac{R}{2h^2},$$

a quadratic equation with two roots

$$u_0 = \frac{1 \pm \sqrt{1 - 3R^2/h^2}}{3R},$$

which are real and positive if $h^2 > 3R^2$.

(3 marks, B, unseen)

(d) Writing $u = u_0 + u_1(\phi)$ where u_0 is a root from part (i) the orbit equation is

$$\frac{d^2 u_1}{d\phi^2} + u_1 - \frac{3}{2} R(u_1^2 + 2u_0 u_1) = 0.$$

Neglecting the u_1^2 term gives the linear equation

$$\frac{d^2 u_1}{d\phi^2} + (1 - 3Ru_0) u_1 = 0.$$

$\underline{3Ru_0 = 1 - \sqrt{1 - 3R^2/h^2}}$

$$\frac{d^2 u_1}{d\phi^2} + \sqrt{1 - 3R^2/h^2}\, u_1 = 0,$$

with trigonometric solutions $u_1 = A \cos[(1 - 3R^2/h^2)^{1/4} \phi + \beta]$ giving a stable orbit

$\underline{3Ru_0 = 1 + \sqrt{1 - 3R^2/h^2}}$

$$\frac{d^2 u_1}{d\phi^2} - \sqrt{1 - 3R^2/h^2}\, u_1 = 0,$$

with exponential (or hyperbolic) solutions showing that the (lower in r) circular orbit is unstable

(8 marks, D, unseen)

5. **(a)** $g_{rr} = -\Sigma/\Delta$ where Σ is non-negative. For $R > 2a$, $\Delta = (r - r_+)(r - r_-)$ where

$$r_\pm = \frac{R \pm \sqrt{R^2 - 4a^2}}{2}$$

are the roots. Therefore Δ is negative for $r_- < r < r_+$ so that g_{rr} is positive if $r_- < r < r_+$. This means that for $r_- < r < r_+$, r is a time coordinate and dr/ds cannot be zero.

(7 marks, seen)

(b) (i) If $\theta = 0$, $\Sigma = r^2 + a^2$, $g_{\phi t} = 0$. As the metric is independent of t

$$c^2 k = g_{t\nu}\frac{dx^\nu}{ds} = g_{tt}\frac{dt}{ds} = c^2\left(1 - \frac{Rr}{r^2+a^2}\right)\frac{dt}{ds} = c^2\frac{r^2 - Rr + a^2}{r^2+a^2}\frac{dt}{ds},$$

is constant along the axis of rotation.

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = c^2k^2\frac{r^2+a^2}{r^2 - Rr + a^2} - \frac{r^2+a^2}{r^2 - Rr + a^2}\left(\frac{dr}{ds}\right)^2 = 1.$$

Multiplying by $(r^2 - Rr + a^2)/(r^2 + a^2)$

$$c^2k^2 - \left(\frac{dr}{ds}\right)^2 = \frac{r^2 - Rr + a^2}{r^2 + a^2} = 1 - \frac{Rr}{r^2+a^2},$$

or

$$\left(\frac{dr}{ds}\right)^2 - \frac{Rr}{r^2+a^2} = c^2k^2 - 1 = C.$$

(7 marks, seen similar)

(ii) Assuming $dr/ds < 0$ requires

$$C + \frac{Rr}{r^2+a^2} > 0$$

or

$$C > -\frac{Rr}{r^2+a^2}$$

which is satisfied for all positive r. We require the maximum value of $f(r) = -r/(r + a^2)$. The maximum occurs at $r = -a$ using $f'(r) = -1/(r^2 + a^2) + 2r^2/(r^2 + a^2)^2$. Accordingly

$$C > \frac{Ra}{a^2 + a^2} = \frac{R}{2a}.$$

(6 marks, unseen)

Category A marks

1 (a)(b)(i) 11 marks, 2 (a)(b) 10 marks, 3 (a) 5 marks, 4(a)(b) 9 marks

(35 marks)

Category B marks

1 (b)(ii) 5 marks, 2 (c) 5 marks, 3 (d) 5 marks, 4 (c) 3 marks

(18 marks)

Category C marks

3 (b)(c) 10 marks

(10 marks)

Category D marks

1 (b)(iii) 4 marks, 2 (d) 5 marks, 4 (d) 8 marks

(17 marks)

MATH60017 Tensor Calculus & General Relativity

Question	Marker's comment
1	1 (a) was well answered. In question 1(b) (ii) students were expected to use p.p=m^2c^2 (as in the similar Problem Sheet question). Many students attempted to obtain the given result through some rather messy algebra. Part (b) (iii) was (as expected) difficult - in fact this is the same result as for the photon rocket considered in the 2023 paper! One can consider the 2024 problem as a `photon rocket' which emits a single photon.
2	This question was very well answered. While most students correctly noted that the result in the last part is the Killing equation, few students noted that the result in part (d) is identical to that from part (b).
3	This question comprised 4 short exercises similar to those on the Problem Sheets. 1 (a) was straightforward. 1 (b) proved to be the most difficult.
4	Parts (a) (c) were well answered. Some students skipped part (b). Part (d) proved more difficult - this is essentially the same calculation as in the discussion of the orbit of Venus in the lectures.

MATH70017 Tensor Calculus & General Relativity

Question	Marker's comment
1	see MATH60017 comments
2	see MATH60017 comments
3	see MATH60017 comments
4	see MATH60017 comments
5	My impression is that students did slightly better on this focused Mastery question than in previous years where there was no specific Mastery content. Few students completed the last part but most candidates answered the question to a good standard.