Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2021-2022

**Partial Differential Equations in Action** **MATH50008**

## Midterm Exam

**Instructions:** The **neatness, completeness and clarity of the answers** will contribute to the final mark. **You must turn in handwritten solutions written on paper and scanned.** You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

**IMPORTANT** – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

This exam is composed of **two questions**; you must attempt both questions.

1. **Total: 20 Marks**

   (a) Consider the following quasilinear first-order PDE

   $$(u+y)\frac{\partial u}{\partial x} + uy\frac{\partial u}{\partial y} = 0 \quad \text{with} \quad x>0,\ y>0$$

   We want to find a solution satisfying $u(x,1) = 1/x$ for this:

   i. Give a parametrization of the initial curve.

   3 Marks

   ii. Give the characteristic ODEs for this equation.

   3 Marks

   iii. Show that the solution to this problem is given by

   $$u(x,y) = \frac{y}{x-\ln(y)}$$

   8 Marks

   (b) Here, we consider the following traffic flow problem

   $$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad x\in\mathbb{R}, \quad t>0$$
   $$\rho(x,0) = f(x)$$

   where the wave velocity is given by the Greenshield's law: $c(\rho) = v_m\,(1-2\rho/\rho_m)$. In the case where the maximum velocity and density are given by $v_m = 1$ and $\rho_m = 8$, give an example of initial conditions $f(x)$ which produces a solution with two expansion fans but no shock wave. You will give an explicit definition for $f(x)$, justify briefly your reasoning and sketch the associated diagram of characteristics.

   6 Marks

2. $\boxed{\textbf{Total: 20 Marks}}$ We consider the one-dimensional flow of a perfect fluid. As we have seen in lectures, the velocity $u(x,t)$ of the fluid is governed by the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

Here, we assume that the initial velocity field is given by

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 2(1-x), & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

(a) Find the equation of the characteristics for this problem. Draw them in the $(x,t)$-plane and point out any fan regions. Show that a shock forms at $t = 1/2$.

5 Marks

(b) Find an explicit solution valid for $0 < t < 1/2$.

4 Marks

(c) Find the explicit solution after the shock has formed. As this is after the shock has formed, you will need to perform shock fitting. Draw an amended diagram of characteristics including the shock path. Until when is the solution you just obtained valid? Justify your reasoning.

7 Marks

(d) Sketch on the same graph the solution for: (i) $t = 0$, (ii) $t = 1/4$, (iii) $t = 1/2$ and (iv) $t = 1$. Clearly label your plots and important values your solution takes.

4 Marks

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## Solutions to Midterm Exam

1. **Total: 20 Marks**

(a) Consider the following quasilinear first-order PDE

$$(u+y)\frac{\partial u}{\partial x} + uy\frac{\partial u}{\partial y} = 0 \quad \text{with} \quad x > 0,\ y > 0$$

We want to find a solution satisfying $u(x,1) = 1/x$.

i. A parametrization of the initial curve is given by

$$x = t, \quad y = 1, \quad z = \frac{1}{t}$$

3 Marks

ii. By definition, the characteristic ODEs for this problem are given by

$$\frac{dx}{ds} = z + y, \quad \frac{dy}{ds} = yz, \quad \frac{dz}{ds} = 0$$

subject to the following initial conditions:

$$x(0) = t, \quad y(0) = 1, \quad z(0) = \frac{1}{t}$$

3 Marks

iii. First, we solve the ODE for $z$ and write

$$\frac{dz}{ds} = 0 \Rightarrow z(s) = C$$

with $C$ an integration constant to determine. The initial conditions give us that

$$z(s) = \frac{1}{t}$$

We can now solve for $y$ and write

$$\frac{dy}{ds} = zy \Rightarrow \frac{dy}{ds} = \frac{y}{t} \Rightarrow \int \frac{dy}{y} = \int \frac{ds}{t} \Rightarrow y(s) = C\exp\left(\frac{s}{t}\right)$$

where once again $C$ is an integration constant to be determined. Initial conditions give us that $y(0) = 1$ so we conclude that $C = 1$ and

$$y(s) = \exp\left(\frac{s}{t}\right)$$

Finally, we can solve the equation for $x$ and for this, we write

$$\frac{dx}{ds} = z + y \Rightarrow \frac{dx}{ds} = \frac{1}{t} + \exp\left(\frac{s}{t}\right) \Rightarrow x(s) = \frac{s}{t} + t\exp\left(\frac{s}{t}\right) + C$$

where once again $C$ is an integration constant to be determined. The initial conditions here give us that $x(0) = t$, so we conclude that $C = 0$ and

$$x(s) = \frac{s}{t} + t \exp\left(\frac{s}{t}\right)$$

An explicit solution is a solution in terms of $(x, y)$ and not $(s, t)$. We thus need to invert these relations

$$x(s,t) = \frac{s}{t} + t \exp\left(\frac{s}{t}\right)$$
$$y(s,t) = \exp\left(\frac{s}{t}\right)$$
$$z(s,t) = \frac{1}{t}$$

$\boxed{\text{5 Marks}}$

and express $(s, t)$ as a function of $(x, y)$. First, we can write

$$\frac{s}{t} = \ln y$$

and plug this in the expression for $x$

$$x = \ln(y) + ty$$

Thus, we conclude that

$$t = \frac{x - \ln(y)}{y}$$

and as $z = 1/t$, we conclude that

$$u(x,y) = \frac{y}{x - \ln(y)}$$

$\boxed{\text{3 Marks}}$

(b) Here, we consider the following traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$\rho(x, 0) = f(x)$$

where the wave velocity is given by the Greenshield's law: $c(\rho) = v_m (1 - 2\rho/\rho_m)$. Further, we assume that the maximum velocity and density are given by $v_m = 1$ and $\rho_m = 8$.
An easy example of initial conditions $f(x)$ which would lead to a solution with two expansion fans but no shock is given by the following piecewise constant function

$$f(x) = \begin{cases} \rho_1, & x < -1 \\ \rho_2, & |x| < 1 \\ \rho_3, & x > 1 \end{cases}$$

with $\rho_1 > \rho_2 > \rho_3$. Indeed, in this problem, we know that the characteristic stemming from $x = \xi$ is a straight line with slope $1/c(f(\xi))$. Indeed, the method of characteristics gives here that

$$\frac{d\rho}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = c(\rho), \quad x(0) = \xi$$

i.e.

$$\rho = f(\xi) \quad \text{on} \quad x = c(f(\xi))t + \xi$$

According to the Greenshield's law, the wave velocity decays linearly with the density and changes sign when $\rho = \rho_m/2$ as shown on Fig.1.

$\boxed{\text{6 Marks}}$

So picking values of densities such that $\rho_1 > \rho_2 > \rho_3$ ensures that the characteristic lines do not cross (no shock formation) and fan out at the discontinuity points $x = \pm 1$.

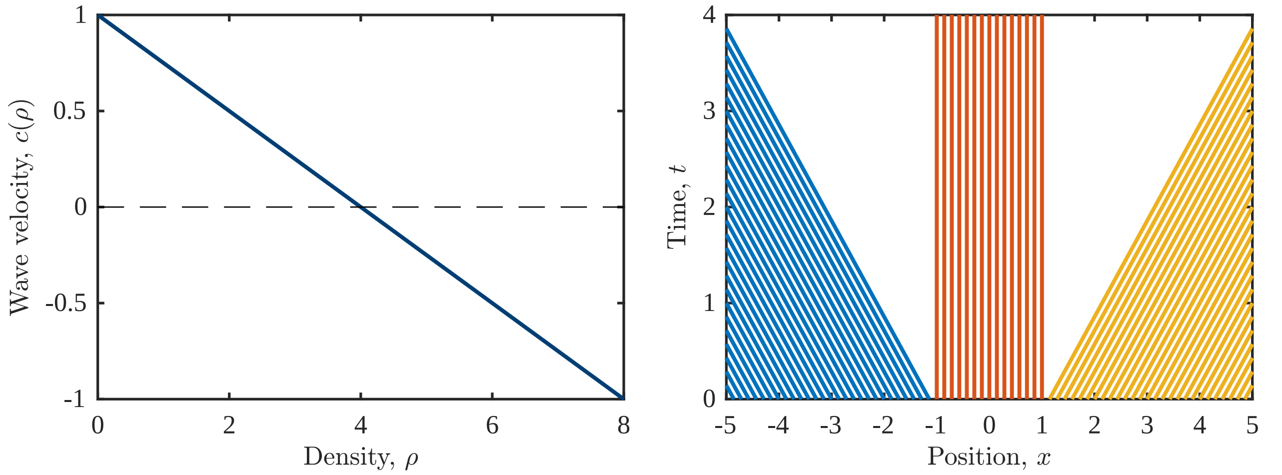


Figure 1: *(Left)* Wave velocity according to Greenshield's law with $v_m = 1$ and $\rho_m = 8$. *(Right)* Diagram of characteristics with $\rho_1 = 8$, $\rho_2 = 4$ and $\rho_3 = 0$.

2. **Total: 20 Marks** In this second problem, we consider the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

Here, we assume that the initial velocity field is given by

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 2(1-x), & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

(a) The method of characteristics gives us here that

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u, \quad x(0) = \xi$$

which means that

$$u = u(\xi, 0) \quad \text{on} \quad x = u(\xi, 0)t + \xi$$

So based on the initial conditions, we obtain the following equation for the characteristics

$$\begin{cases} \text{I} — \xi < 0: & x = t + \xi \\ \text{II} — 0 < \xi < 1: & x = 2(1-\xi)t + \xi \\ \text{III} — \xi > 1: & x = \xi \end{cases}$$

2 Marks

The diagram of characteristics is given on Fig. 2.

2 Marks

To show that a shock is forming, you can notice that all the characteristics in region II cross in a single point. Their equation reads $x = 2(1-\xi)t + \xi$, so at $t = 1/2$, we realize that all characteristics are located in $x = (1-\xi) + \xi = 1$. We conclude that a shock forms at $t = 1/2$ in $x = 1$.

1 Mark

(b) Here, we derive an explicit solution valid for $0 < t < 1/2$. Before the shock forms, we have

☐ $0 < \xi < 1$: In this region, we have

$$u = 2(1-\xi) \quad \text{on} \quad x = 2(1-\xi)t + \xi$$

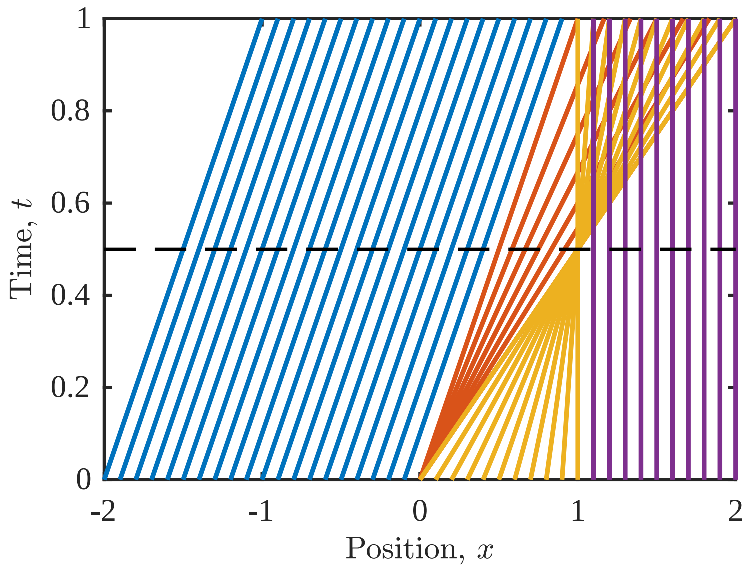


Figure 2: Diagram of characteristics with characteristics from region I (blue), region II (yellow), region III (purple). We notice that there is an expansion fan between regions I and II (orange). Characteristics from the region II all cross in $x = 1$ at $t = 1/2$, leading to shock formation.

In particular, we conclude that $\xi = (x - 2t)/(1 - 2t)$, which means that

$$u(x,t) = 2\left(1 - \frac{x-2t}{1-2t}\right) = 2\left(\frac{1-2t-x+2t}{1-2t}\right) = 2\left(\frac{1-x}{1-2t}\right)$$

and this for

$$0 < \xi < 1 \Rightarrow 0 < \frac{x-2t}{1-2t} < 1 \Rightarrow 0 < x - 2t < 1 - 2t \Rightarrow 2t < x < 1$$

- ☐ $\xi < 0$: In this region, we have

$$u = 1 \quad \text{on} \quad x = t + \xi$$

In particular, we conclude that $\xi = x - t$, which means that

$$u(x,t) = 1$$

and this for

$$\xi < 0 \Rightarrow x - t < 0 \Rightarrow x < t$$

- ☐ $\xi > 1$: In this region, we have

$$u = 0 \quad \text{on} \quad x = \xi$$

In particular, we conclude that $\xi = x$, which means that

$$u(x,t) = 0$$

and this for

$$\xi > 1 \Rightarrow x > 1$$

- ☐ Finally, we need to deal with the fan region, $t < x < 2t$. In the expansion fan, the solution varies linearly from $u = 1$ to $u = 2$, we thus know that

$$u(x,t) = 1 + \frac{x-t}{t} = \frac{x}{t}$$

We conclude that the explicit solution for $u(x,t)$ before the shock forms is given by

$$u(x,t) = \begin{cases} 1, & x < t \\ x/t, & t < x < 2t \\ 2(1-x)/(1-2t), & 2t < x < 1 \\ 0, & x > 1 \end{cases}$$

4 Marks

(c) To find the explicit solution after the shock has formed, we need to proceed to shock fitting. If we denote $s(t)$ the position of the shock, we have

After the shock: $u_+ = 0$

Before the shock: $u_- = \dfrac{s}{t}$ (which comes from the fan region)

The Rankine-Hugoniot jump condition reads

$$\frac{ds}{dt} = \frac{[u^2/2]}{[u]} = \frac{1}{2}(u_+ + u_-) = \frac{s}{2t}$$

subject to the initial condition $s(1/2) = 1$. We can integrate this equation

$$\int \frac{ds}{s} = \int \frac{dt}{2t} \Rightarrow \ln s = \frac{1}{2}\ln t + C' \Rightarrow s(t) = C\sqrt{t}$$

where $C$ is an integration constant to be determined. We use the initial condition above to write $C = \sqrt{2}$ and so conclude that

$$s(t) = \sqrt{2t}$$

2 Marks

The explicit solution is thus given by

$$u(x,t) = \begin{cases} 1, & x < t \\ x/t, & t < x < \sqrt{2t} \\ 0, & x > \sqrt{2t} \end{cases}$$

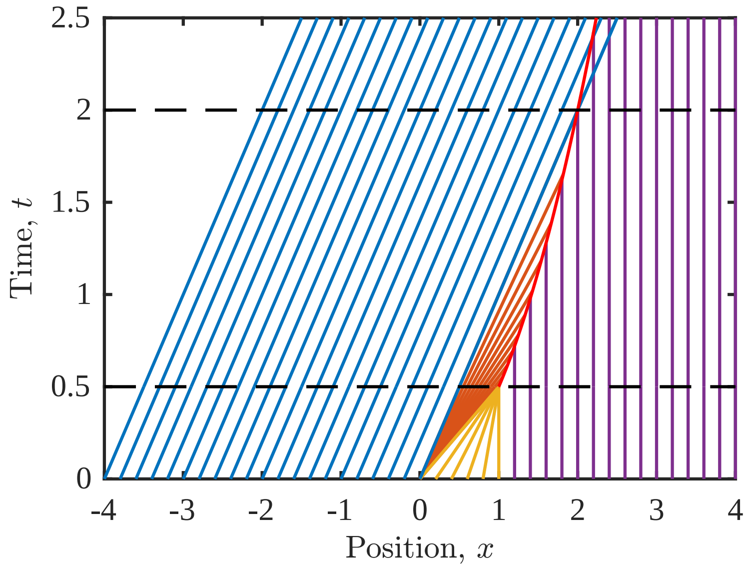


Figure 3: Amended diagram of characteristics with shock path shown in red.

1 Mark

The amended diagram of characteristics is given in Fig. 3.

2 Marks

From the diagram of characteristics, we see that the characteristics originally emanating from region I ($\xi < 0$) cross the shock path at some well defined time. The shock path is given by $s(t) = \sqrt{2t}$, while the equation of the characteristics in region I were given by $x(t) = t + \xi$. In particular, setting $\xi = 0$ (right most characteristic in the region), we find that $s(t) = x(t)$ when $t = \sqrt{2t} \Rightarrow t = 2$ which is consistent with the diagram drawn. We thus conclude that our explicit solution above is only valid in the range $1/2 < t < 2$.

2 Marks

(d) The sketch of the solutions is given in Fig. 4 (marks taken off if values of characteristic points on the solution are not clearly marked).

4 Marks

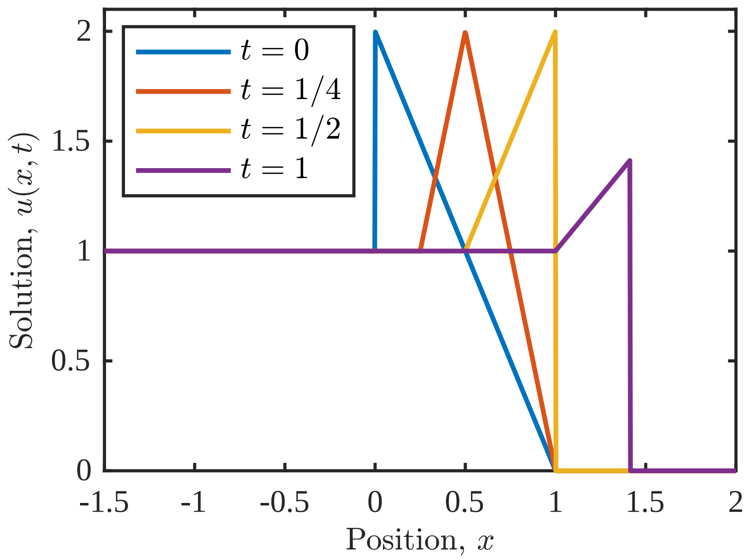


Figure 4: Sketch of the solution for various times.