Geometry III Course Work 2

Problem 1. Suppose $\phi : I \to \mathbb{R}^2$ ~~is a~~ ($I \subseteq \mathbb{R}$) is a parametrisation of curve C by arc length. (~~this is~~ ϕ always exists by lemma 1.2) Say $\phi(t) = (x(t), y(t))$ for $t \in I$

Let $U := \{(x, y) \in \mathbb{R}^2 \mid x \in I, y \in \mathbb{R}\}$

Define $F : U \to S$ by

$$F(u, v) = (x(u), y(u), v)$$

Fix ~~let~~ $p = (u_1, v_1)$ and suppose ~~tangent of curve $\alpha(t)$ in U~~ ~~is at $t=0$ is~~ $(u, v) =: T$ is tangent of $\alpha(t) := (u_1 + ut, v_1 + vt)$ at $t = 0$,

$$\langle T, T \rangle = u^2 + v^2$$

$$dF_p(T) = \frac{d}{dt}(F \circ \alpha)\Big|_{t=0} = \frac{d}{dt}\big((x(u_1 + ut), y(u_1 + ut), v_1 + vt)\big)\Big|_{t=0}$$

$$= (u x'(u_1 + ut), u y'(u_1 + ut), v)\Big|_{t=0} = (u x'(u_1), u y'(u_1), v)$$

so $\langle dF_p(T), dF_p(T) \rangle = u^2(x'(u_1)^2 + y'(u_1)^2) + v^2$

Since ϕ is parametrisation by arc length,

$$x'(u_1)^2 + y'(u_1)^2 = |\phi'(u_1)|^2 = 1$$

So $\langle T, T \rangle = \langle dF_p(T), dF_p(T) \rangle = u^2 + v^2$

By the formulae $\langle X+Y, X+Y \rangle = \langle X, X \rangle + \langle Y, Y \rangle + 2\langle X, Y \rangle$,

we have $\langle T_1, T_2 \rangle = \langle dF_p(T_1), dF_p(T_2) \rangle$ for any two T_1, T_2 in U,

So F is local isometry.

Suppose $(x(u_1), y(u_1), v_1) = (x(u_2), y(u_2), v_2)$

then $v_1 = v_2$, $u_1 = u_2$ as parametrisation ϕ satisfies $|\phi'(t)| \neq 0$ $\forall t$ and C is not self-intersecting.

$\forall (x, y, z) \in S$, let $v = z$, let u be s.t. $\phi(u) = (x, y)$ (u must exists by definition of ϕ), so then $F(u, v) = (x, y, z)$

So F is bijection, that means F is isometry.

By corollary 15.3,

if K_U is Gaussian curvature of U and K_S is Gaussian curvature of S

~~$K_S = F\circ$~~ $K_S \circ F = K_U$ or $K_U \circ F^{-1} = K_S$

but for plane, $K_U = 0$ so $K_S = 0$.

Problem 2. (a). By proposition 14.1

$$g\begin{pmatrix}\Gamma^1_{11}\\ \Gamma^2_{11}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\lambda_u e^\lambda \\ -\frac{1}{2}\lambda_v e^\lambda\end{pmatrix} = \begin{pmatrix}\Gamma^1_{11} e^\lambda \\ \Gamma^2_{11} e^\lambda\end{pmatrix}$$

$$g\begin{pmatrix}\Gamma^1_{21}\\ \Gamma^2_{21}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\lambda_v e^\lambda \\ \lambda_u e^\lambda\end{pmatrix} = \begin{pmatrix}\Gamma^1_{21} e^\lambda \\ \Gamma^2_{12} e^\lambda\end{pmatrix}$$

$$g\begin{pmatrix}\Gamma^1_{22}\\ \Gamma^2_{22}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\lambda_u e^\lambda \\ \lambda_v e^\lambda\end{pmatrix} = \begin{pmatrix}\Gamma^1_{22} e^\lambda \\ \Gamma^2_{22} e^\lambda\end{pmatrix}$$

$$\Rightarrow \Gamma^1_{11} = \frac{\lambda_u}{2}, \quad \Gamma^2_{11} = -\frac{\lambda_v}{2}$$
$$\Gamma^1_{21} = \frac{\lambda_v}{2}, \quad \Gamma^2_{12} = \frac{\lambda_u}{2}$$
$$\Gamma^1_{22} = -\frac{\lambda_u}{2}, \quad \Gamma^2_{22} = \frac{\lambda_v}{2}$$

Note by symmetry of double derivative,

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{\lambda_v}{2}$$

this gives the required form of in the question

(b). From the proof of Theorema Egregium,

$$K(g_{11}g_{22}-g_{12}^2)=Yg_{22}-Xg_{12}$$

where $X=\Gamma_{12}^2\Gamma_{12}^1-\Gamma_{11}^2\Gamma_{22}^1+(\Gamma_{12}^1)_u-(\Gamma_{11}^1)_v$

$$=\frac{\lambda_u}{2}\frac{\lambda_v}{2}-\left(-\frac{\lambda_v}{2}\right)\left(-\frac{\lambda_u}{2}\right)+\frac{\lambda_{uv}}{2}-\frac{\lambda_{vu}}{2}$$

λ is smooth so $\lambda_{uv}=\lambda_{vu}$

$\Rightarrow X=0$

$$Y=\Gamma_{11}^1\Gamma_{21}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-\Gamma_{12}^2\Gamma_{12}^2$$
$$+(\Gamma_{11}^2)_v-(\Gamma_{12}^2)_u$$
$$=\frac{\lambda_u}{2}\frac{\lambda_u}{2}+\left(-\frac{\lambda_v}{2}\right)\frac{\lambda_v}{2}-\frac{\lambda_v}{2}\left(-\frac{\lambda_v}{2}\right)-\left(\frac{\lambda_u}{2}\right)\frac{\lambda_u}{2}$$
$$+\left(-\frac{\lambda_v}{2}\right)_v-\left(\frac{\lambda_u}{2}\right)_u=-\frac{1}{2}(\lambda_{uu}+\lambda_{vv})=-\frac{1}{2}\Delta\lambda$$

~~so~~ $g_{11}g_{22}-g_{12}^2=e^{2\lambda}$ and so

$$Ke^{2\lambda}=-\frac{1}{2}\Delta\lambda\cdot e^{\lambda}\Rightarrow 2Ke^{\lambda}+\Delta\lambda=0 \quad \square$$

Problem 3.

$\phi_u=(-\sin u\cos v,\ \cos u\cos v,\ 0)$ Note $v\neq\frac{\pi}{2},\frac{3}{2}\pi$ on $S^1\times\{(0,0,1)\}$

$\phi_v=(-\cos u\sin v,\ -\sin u\sin v,\ \cos v)$ i.e. $\cos v\neq 0$

so $g=\begin{pmatrix}\sin^2u\cos^2v+\cos^2u\cos^2v & g_{12}\\ g_{12} & \cos^2u\sin^2v+\sin^2u\sin^2v+\cos^2v\end{pmatrix}$

where $g_{12}=\langle\phi_u,\phi_v\rangle=\sin u\sin v\cos u\cos v-\sin u\sin v\cos u\cos v+0$
$=0$

so $g=\begin{pmatrix}\cos^2v & 0\\ 0 & 1\end{pmatrix}$

again we can use formulas in prop 14.1 to get

$$\begin{cases} \Gamma_{11}^1 \cos^2 v = \frac{1}{2}\frac{\partial}{\partial u}(g_{11}) = 0 \\ \Gamma_{11}^2 = -\frac{1}{2}\frac{\partial}{\partial v}(g_{11}) = \cos v \sin v \end{cases}$$

so $\Gamma_{11}^1 = 0$, $\Gamma_{11}^2 = \cos v \sin v$

$$\begin{cases} \Gamma_{21}^1 \cos^2 v = \frac{1}{2}\frac{\partial}{\partial v}(g_{11}) = -\cos v \sin v \\ \Gamma_{12}^2 = \frac{1}{2}\frac{\partial}{\partial u}(g_{22}) = 0 \end{cases}$$

so $\Gamma_{12}^2 = 0$, $\Gamma_{12}^1 = -\tan v$ (well-defined as $v \neq \frac{\pi}{2}, \frac{3}{2}\pi$)

$$\begin{cases} \Gamma_{22}^1 \cos^2 v = -\frac{1}{2}\frac{\partial}{\partial u}(g_{22}) = 0 \\ \Gamma_{22}^2 = \frac{1}{2}\frac{\partial}{\partial v}(g_{22}) = 0. \end{cases}$$

To sum up, $\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos v \sin v \end{pmatrix}$, $\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} -\tan v \\ 0 \end{pmatrix}$

$\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Problem 4. (a) $\phi_u = (\cos v, \sin v, \frac{1}{u})$ Note $u > 0$

$\phi_v = (-u\sin v, u\cos v, 0)$

$$\text{so } g_\phi = \begin{pmatrix} \cos^2 v + \sin^2 v + \frac{1}{u^2} & -u\cos v \sin v + u\cos v \sin v \\ -u\cos v \sin v + u\cos v \sin v & u^2\sin^2 v + u^2\cos^2 v \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{u^2} & 0 \\ 0 & u^2 \end{pmatrix}$$

Use equations in prop 14.1, (implicitly assume Γ_{ij}^k are christoffel symbols for S_1)

$$\begin{cases} \Gamma_{11}^1 (1 + \frac{1}{u^2}) = \frac{1}{2}\frac{\partial}{\partial u}(1 + \frac{1}{u^2}) = -\frac{2}{u^3}\cdot\frac{1}{2} = -\frac{1}{u^3} \\ \Gamma_{11}^2 u^2 = -\frac{1}{2}\frac{\partial}{\partial v}(g_{11}) = 0 \end{cases}$$

so $\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{u(u^2+1)} \\ 0 \end{pmatrix}$

$\begin{cases} \Gamma_{21}^1 \left(1+\frac{1}{u^2}\right) = \frac{1}{2}\frac{\partial}{\partial v}\left(1+\frac{1}{u^2}\right) = 0 \\ \Gamma_{12}^2 u^2 = \frac{1}{2}\frac{\partial}{\partial u}(u^2) = u \end{cases}$

so $\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \Gamma_{21}^1 \\ \Gamma_{21}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{u} \end{pmatrix}$

$\begin{cases} \Gamma_{22}^1 \left(1+\frac{1}{u^2}\right) = -\frac{1}{2}\frac{\partial}{\partial u}(u^2) = -u \\ \Gamma_{22}^2 u^2 = \frac{1}{2}\frac{\partial}{\partial v}(u^2) = 0 \end{cases}$

so $\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} -\frac{u^3}{u^2+1} \\ 0 \end{pmatrix}$

~~Now on~~

Using formulas in theorem 15.1,

$K_\phi = \frac{Y g_{22} - X g_{12}}{g_{11}g_{22} - g_{12}^2}$

where $X = 0 \cdot \frac{1}{u} - 0 \cdot \left(-\frac{u^3}{u^2+1}\right) + \frac{\partial}{\partial u}(0) - \frac{\partial}{\partial v}(\Gamma_{11}^1)$

$= 0$

$Y = -\frac{1}{u(u^2+1)} \cdot \frac{1}{u} + 0 \cdot 0 - 0 \cdot 0 - \frac{1}{u}\frac{1}{u}$

$+ \frac{\partial}{\partial v}(\Gamma_{11}^2) - \frac{\partial}{\partial u}\left(\frac{1}{u}\right)$

$= -\frac{1}{u^2(u^2+1)} - \frac{1}{u^2} + \frac{1}{u^2} = -\frac{1}{u^2(u^2+1)}$

so $K_\phi = \frac{-\frac{1}{u^2+1}}{u^2+1} = -\frac{1}{(u^2+1)^2}$

Now consider S_2

$\psi_u = (\cos v, \sin v, 0)$

$\psi_v = (-u\sin v, u\cos v, 1)$

so $g_\psi = \begin{pmatrix} \cos^2 v + \sin^2 v & -u\sin v\cos v + u\cos v\sin v \\ -u\sin v\cos v + u\cos v\sin v & u^2\sin^2 v + u^2\cos^2 v + 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix}$

use equations in prop 14.1 (assume Γ_{ij}^k are christoffel symbols for S_2)

$\begin{cases} \Gamma_{11}^1 = \frac{1}{2}\frac{\partial}{\partial u}(g_{11}) = 0 \\ \Gamma_{11}^2(u^2+1) = -\frac{1}{2}\frac{\partial}{\partial v}(g_{11}) = 0 \end{cases}$

so $\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} \Gamma_{21}^1 = \frac{1}{2}\frac{\partial}{\partial v}(g_{11}) = 0 \\ \Gamma_{12}^2(u^2+1) = \frac{1}{2}\frac{\partial}{\partial u}(u^2+1) = u \end{cases}$

so $\begin{pmatrix} \Gamma_{21}^1 \\ \Gamma_{21}^2 \end{pmatrix} = \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{u}{u^2+1} \end{pmatrix}$

$\begin{cases} \Gamma_{22}^1 = -\frac{1}{2}\frac{\partial}{\partial u}(u^2+1) = -u \\ \Gamma_{22}^2 = \frac{1}{2}\frac{\partial}{\partial v}(u^2+1) = 0 \end{cases}$

so $\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} -u \\ 0 \end{pmatrix}$

$K_\psi = \frac{Y(u^2+1)}{u^2+1} = Y = 0 \cdot \frac{u}{u^2+1} + 0 \cdot 0 - 0 \cdot 0$

$$-\frac{u}{u^2+1}\cdot\frac{u}{u^2+1} + \frac{\partial}{\partial v}\left(\Gamma_{11}^{2}\right) - \frac{\partial}{\partial u}\left(\frac{u}{u^2+1}\right)$$

$$= -\frac{u^2}{(u^2+1)^2} - \frac{u^2+1-2u\cdot u}{(u^2+1)^2}$$

$$= -\frac{1}{(u^2+1)^2}$$

so $K_\psi = K_\phi$ □

(b) By Corollary 15.3, if F is ~~isom~~ local isometry,

$K_\psi \circ F = K_\phi$ from (a), $K_\psi = K_\phi$

but F is clearly not id, otherwise,

~~$S_1 = S_2$~~ $F(\phi(u,v)) = \phi(u,v) = \psi(u,v)$

~~but $\phi($~~ which is false by definition of ϕ, ψ. □

so F cannot be local isometry.