Problem Sheet 2, Geometry of Curves and Surfaces, 2022-2023

Problem 1. Let T be a real number. Consider the surface

$$H_T = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 3y^2 - z^2 = T\}.$$

Sketch the surface H_T in the three cases $T > 0$, $T = 0$, $T < 0$.

(a) Show that H_T is a regular surface if and only if T is non-zero.

(b) Find all points $P = (x, y, z)$ on H_T satisfying $x = y = 1$.

(c) For each such point P in part (b), find the equation of the tangent plane to H_T at P.

Solution: Let

$$f(x, y, z) = x^2 + 3y^2 - z^2,$$

so that $H_T = f^{-1}(T)$. The vector $\nabla f = (2x, 6y, -2z)$ is zero iff $(x, y, z) = (0, 0, 0)$. If $T \neq 0$ then $(0, 0, 0) \notin H_T$, hence H_T is a regular level set for a smooth function and hence a regular surface. If $T = 0$ then the set of tangent vectors to curves at $(0, 0, 0)$ is clearly not a plane (in fact, it is the cone H_0 itself, with span $\mathbb{R}^3$), hence H_0 is not a regular surface.

The points $P \in H_T$ with coordinates $P = (1, 1, z)$ satisfy $4 - z^2 = T$. Hence points P, with coordinates as above, exists if and only if $T \leq 4$, in which case P is given by $(1, 1, \pm\sqrt{4 - T})$ with tangent planes

$$\begin{aligned} T_{(1,1,\pm\sqrt{4-T})} H_T &= (\nabla f(1, 1, \pm\sqrt{4 - T}))^\perp \\ &= (1, 3, \mp\sqrt{4 - T})^\perp \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid x - 3y \mp \sqrt{4 - T}z = 0\}. \end{aligned}$$

Alternatively arguments suggested by students to show that H_0 is not a regular surface: removing the point $(0, 0, 0)$ disconnects H_0 but removing a single point from an open connected set in $\mathbb{R}^2$ does not make it disconnected. Another argument is to say that H_0 is not the graph of a function of (x, y), or (x, z) or (y, z), contradicting a statements proved in the lectures.

Problem 2. Let $S \subset \mathbb{R}^3$ be a regular level set of some smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, and let $G : \mathbb{R}^3 \to \mathbb{R}$ be another smooth function. We say that a point $p \in S$ is a critical point of the (restricted) map $G : S \to \mathbb{R}$ if the map $dG_p : T_pS \to \mathbb{R}$ is zero.

(a) Prove that p is a critical point of G if and only if $\nabla G(p)$ is a scalar multiple (possibly zero) of $\nabla F(p)$.

(b) Assume that S is the torus $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ in $\mathbb{R}^3$. How many critical points does the function $G(x, y, z) = y$ have on S?

Solution: Part (a): First we note that $dG_p(v) = \langle \nabla G(p), v\rangle$ for any $v \in T_pS$. To see this, consider an arbitrary smooth curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = v$, and write its coordinates as

$$\alpha(t) = (x(t), y(t), z(t)).$$

Then

$$\begin{aligned}
dG_p(\alpha'(0)) = \left.\frac{d(G\circ\alpha)}{dt}\right|_{t=0} &= \left.\frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt}\right|_{t=0} \\
&= \left\langle \left(\frac{\partial G}{\partial x}(p), \frac{\partial G}{\partial y}(p), \frac{\partial G}{\partial z}(p)\right), (x'(0), y'(0), z'(0)) \right\rangle \\
&= \langle \nabla G(p), \alpha'(0)\rangle \\
&= \langle \nabla G(p), \alpha'(0)\rangle.
\end{aligned}$$

By the above formula, $dG_p(v) = 0$ for all $v \in T_pS$ if and only if $\langle \nabla G(p), v\rangle = 0$ for all $v \in T_pS$. This means that $\nabla G(p)$ is orthogonal to T_pS, or equivalently it is a multiple of the unit normal vector $\frac{\nabla F(p)}{|\nabla F(p)|}$. Therefore, $dG_p : T_pS \to \mathbb{R}$ is zero if and only if $\nabla G(p) = \lambda \nabla F(p)$ for some $\lambda \in \mathbb{R}$. (We may recognize the constant λ as a Lagrange multiplier.)

Part (b): Consider the function

$$F(x, y, z) = \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2$$

on $\mathbb{R}^3$. This is smooth at every point $(x, y, z) \in \mathbb{R}^3$, except when $x^2 + y^2 = 0$. When $x^2 + y^2 = 0$ (i.e. $x = y = 0$), we have $F(x, y, z) = 4 + z^2 \geq 4$. Thus, F is smooth on the open set $F^{-1}((0, 2))$. Thus, the set $S = F^{-1}(+1)$ is a regular level set.

We have $\nabla G = (0, 1, 0)$, while

$$\nabla F(x, y, z) = \left(2\left(\sqrt{x^2 + y^2} - 2\right) \cdot \frac{x}{\sqrt{x^2 + y^2}}, 2\left(\sqrt{x^2 + y^2} - 2\right) \cdot \frac{y}{\sqrt{x^2 + y^2}}, 2z\right).$$

This is a multiple of (0,1,0) if and only if the first and third coordinates are zero; the latter condition means that $z = 0$, while the former means that either $x = 0$ or $\sqrt{x^2 + y^2} = 2$. Since $F(x, y, z) = 1$ along S we cannot have both $z = 0$ and $\sqrt{x^2 + y^2} = 2$, so the critical points of G happen exactly when $x = z = 0$. Then $F(0, y, 0) = (|y| - 2)^2 = 1$ when $|y| - 2 = \pm 1$, so $|y|$ is either 1 or 3. Thus G has four critical points along S, namely

$$(0, -3, 0), (0, -1, 0), (0, 1, 0), (0, 3, 0).$$

Problem 3. Let $S \subset \mathbb{R}^3$ be a non-empty, compact, and connected surface. By the Jordan-Brouwer theorem (extension of the Jordan curve theorem in the plane), S divides $\mathbb{R}^3$ into two components, so that we can talk about inside and outside components of S. Let $N(p)$ be the outward normal vector at $p \in S$. Prove that the Gauss map $N : S \to \mathbb{S}^2$ is surjective. Are there any non-compact surfaces for which the Gauss map is not surjective?

Hint: for each $v \in \mathbb{S}^2$, consider a maximum of the function $x \mapsto \langle x, v\rangle$ on S.

Solution: Fix a unit vector $v \in \mathbb{S}^2$, and consider the family of planes

$$P_c = \left\{ x \in \mathbb{R}^3 \mid \langle x, v \rangle = c \right\}, \quad c \in \mathbb{R}.$$

Since S is compact, the function $x \mapsto \langle x, v \rangle$ on S achieves its maximum value at some point $p \in S$. Define $c_0 = \langle p, v \rangle$. Then $S \cap P_c$ is empty for all $c > c_0$ whereas $S \cap P_{c_0}$ is not empty. We claim that $v = N(p)$. To see this, consider an arbitrary smooth curve $\alpha : (-\epsilon, \epsilon) \to S$, with $\alpha(0) = p$ and $\alpha'(0) = w \in T_pS$. Since $f(t) = \langle \alpha(t), v \rangle$ is maximised at $t = 0$, we must have $0 = f'(0) = \langle w, v \rangle$. Therefore, v is orthogonal to any tangent vector to S at p. In fact, v must be the outward unit normal vector, since all of S lies in the half-space $\langle x, v \rangle \leq \langle p, v \rangle$.

As $v \in \mathbb{S}^2$ was arbitrary, the Gauss map $N : S \to \mathbb{S}^2$ must be surjective.

As a counter example, one may consider a plane in $\mathbb{R}^3$, where we know that the Gauss map is constant. We also saw the example of a paraboloid $z = x^2 - y^2$ in lectures. In both cases the problem is that for many $v \in \mathbb{S}^2$, the function $x \mapsto \langle x, v \rangle$ need not achieve its maximum on S.

Problem 4. Let K and H denote the Gaussian and mean curvatures of S at the point p, respectively. Prove that $H^2 \geq K$. At which points p does equality hold?

Solution: Let λ_1, λ_2 be the principal curvatures at $p \in S$. Then $K = \lambda_1 \lambda_2$ and $H = \frac{\lambda_1 + \lambda_2}{2}$. Then,

$$H^2 - K = \left(\frac{\lambda_1 - \lambda_2}{2} \right)^2 \geq 0.$$

Therefore, $H^2 \geq K$ with equality iff $\lambda_1 = \lambda_2$ iff p is an umbilical point.

Problem 5. Let S be a nonempty, compact, oriented regular surface in $\mathbb{R}^3$ and let $p \in S$ be a point which maximises the function $f : S \to \mathbb{R}$ defined by $f(x) = |x|^2$. Prove that the normal curvature of any curve $C \subset S$ passing thought p satisfies $|k_n| \geq 1/|p|$.

Conclude that the second fundamental form A_p at p is definite, and hence that $K(p) > 0$.

The solution to Problem 5 will be released later.