Simplify $\nabla^2(\underline{A}.\underline{B}) - 2\,\mathrm{div}\left\{(\underline{B}.\underline{\nabla})\underline{A} + \underline{B}\times \mathrm{Curl}\,\underline{A}\right\}$

$$\nabla^2(\underline{A}.\underline{B}) = \frac{\partial^2}{\partial x_j^2}(A_i B_i) = A_i\frac{\partial^2 B_i}{\partial x_j^2} + B_i\frac{\partial^2 A_i}{\partial x_j^2} + 2\frac{\partial A_i}{\partial x_j}\frac{\partial B_i}{\partial x_j}$$

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$$2\,\mathrm{div}\left\{(\underline{B}.\underline{\nabla})\underline{A} + \underline{B}\times\mathrm{Curl}\,\underline{A}\right\} = 2\frac{\partial}{\partial x_i}\left\{\quad\right\}_i$$

$$\left\{\quad\right\}_i = B_j\frac{\partial A_i}{\partial x_j} + \varepsilon_{ijk}B_j(\mathrm{Curl}\,\underline{A})_k$$

$$= B_j\frac{\partial A_i}{\partial x_j} + \varepsilon_{ijk}\varepsilon_{klm}B_j\frac{\partial}{\partial x_l}A_m$$

$$= B_j\frac{\partial A_i}{\partial x_j} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(B_j\frac{\partial A_m}{\partial x_l}\right)$$

$$= \cancel{B_j\frac{\partial A_i}{\partial x_j}} + B_j\frac{\partial}{\partial x_i}A_j - \cancel{B_j\frac{\partial A_i}{\partial x_j}}$$

$$\therefore 2\,\mathrm{div}\left\{\quad\right\} = 2\frac{\partial}{\partial x_i}\left(B_j\frac{\partial A_j}{\partial x_i}\right)$$

$$= 2\frac{\partial B_j}{\partial x_i}\frac{\partial A_j}{\partial x_i} + 2B_j\frac{\partial^2 A_j}{\partial x_i^2} \qquad 2$$

swap i & j

$$= 2\frac{\partial B_i}{\partial x_j}\frac{\partial A_i}{\partial x_j} + 2B_i\frac{\partial^2 A_i}{\partial x_j^2} \qquad 2'$$

$$1 - 2' = A_i\frac{\partial^2 B_i}{\partial x_j^2} - B_i\frac{\partial^2 A_i}{\partial x_j^2} = \underline{\underline{\underline{A}.\nabla^2\underline{B} - \underline{B}.\nabla^2\underline{A}}}$$

$S: x^2+y^2=a^2 \quad 0 \leq z \leq H$ 2022 Exam

$\underline{B} = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath} + h(x,y,z)\hat{k}$

Question Two Solution [(a)–(e)(i) seen similar; (e)(ii)–(f) unseen]

(a) S is open (it is the curved surface of a cylinder). **[1 mark (A)]**
(b) On S we have $x = a\cos\theta, y = a\sin\theta$. Hence $x = dy/d\theta$ and $y = -dx/d\theta$. **[1 mark (A)]**
(c) The unit normal is

$$\widehat{\mathbf{n}} = \pm\nabla(x^2+y^2)/\left|\nabla(x^2+y^2)\right| = \pm(2x\mathbf{i}+2y\mathbf{j})/\sqrt{4x^2+4y^2} = (x\mathbf{i}+y\mathbf{j})/a,$$

where the + sign has been chosen to ensure that $\widehat{\mathbf{n}}\cdot\mathbf{i} > 0$ for $x > 0$. **[2 marks (A)]**
(d) We calculate

$$\operatorname{curl}\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x,y) & g(x,y) & h(x,y,z) \end{vmatrix} = \frac{\partial h}{\partial y}\mathbf{i} - \frac{\partial h}{\partial x}\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}. \text{ [2 marks (A)]}$$

Using the result for the normal from (c) we find

$$(\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}} = \frac{1}{a}\left(x\frac{\partial h}{\partial y} - y\frac{\partial h}{\partial x}\right), \text{ as required. [1 mark (A)]}$$

(e) (i) First close the cylinder by adding circular discs of radius a at $z = 0, H$. **[1 mark (B)]**. Call these surfaces S_0 and S_H. We then form the closed surface $S_C = S_0 \cup S_H \cup S$. Applying the divergence theorem to S_C we have

$$\int_{S_C} (\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}}\, dS = \int_V \operatorname{div}(\operatorname{curl}\mathbf{B})\, dV = 0, \text{ [1 mark (B)]}$$

where V is the volume enclosed by S_C and the divergence of a curl is always zero. Thus we have

$$0 = \int_S (\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}}\, dS + \int_{S_0} (\operatorname{curl}\mathbf{B})\cdot(-\mathbf{k})\, dx\, dy + \int_{S_H} (\operatorname{curl}\mathbf{B})\cdot\mathbf{k}\, dx\, dy, \text{ [1 mark (B)]}$$

where we have taken care to use the outward normal to S_C in the second and third integrals. Now since

$$(\operatorname{curl}\mathbf{B})\cdot\mathbf{k} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

is independent of z, the contributions from the second and third integrals are equal and opposite and cancel each other out. **[2 marks (B)]**. Hence we conclude that

$$\int_S (\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}}\, dS = 0.$$

(e)(ii) Using Stokes theorem we can relate I to integrals around the boundary curves at $z = 0$ and $z = H$, call these curves γ_0 and γ_H, say. According to the right hand rule we need to traverse γ_H clockwise and γ_0 anti-clockwise. Applying the theorem

$$\int_S (\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}}\, dS = \oint_{\gamma_0} \mathbf{B}\cdot d\mathbf{r} + \oint_{\gamma_H} \mathbf{B}\cdot d\mathbf{r}, \text{ [2 marks (C)]}$$

with $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$, since z is constant on both curves. Given that

$$\mathbf{B}\cdot d\mathbf{r} = f(x,y)dx + g(x,y)dy$$

is independent of z, and γ_0 ,γ_H are traversed in opposite directions, the contributions from these integrals are equal and opposite and hence

$$\int_S (\operatorname{curl}\mathbf{B})\cdot\widehat{\mathbf{n}}\, dS = 0. \text{ [2 marks (C)]}$$

2nd order ODEs , const. coeffs + P.S.
also $x^2y'' + xy' + y = 0$ subst $x = e^t$
1st order separable, integrating factor, homogeneous y/x

2022 Resit Exam.

properties of $\cosh x$, $\sinh x$ remember

3. (a) Show that the extremal curve of the integral

$$I = \int_0^{2\pi} \left(y^2 - (y')^2\right) dx,$$

satisfying the conditions

$$y(0) = 1, \; y'(2\pi) = 1,$$

is given by

$$y = \cos x + \sin x.$$

explicitly indept of x could use $L - y'\frac{\partial L}{\partial y'} = \text{const}$

(8 marks)

(b) If the constraint

$$\int_0^{2\pi} y(x) \cos x \, dx = \frac{\pi}{2}$$

is added to the problem, find the new extremal curve of I. (12 marks)

(Total: 20 marks)

Full E-L

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

$$\Rightarrow 2y - \frac{d}{dx}(-2y') = 0 \Rightarrow \cancel{2}y + \cancel{2}y'' = 0$$

$$\Rightarrow y = A\cos x + B\sin x.$$

Short form.

$$y^2 - (y')^2 - y'(-2y') = \text{const}$$

$$\Rightarrow y^2 + y'^2 = \beta^2$$

$$\int \frac{dy}{(\beta^2 - y^2)^{1/2}} = \int dx$$

$y = \beta \sin u$

$$\Leftrightarrow \sin^{-1}\left(\frac{y}{\beta}\right) = x + C. \Rightarrow y = \beta\sin(x+C)$$

If it's better to use * the short form I will tell you

Question Three Solution [seen similar]

(a) Taking $L = y^2 - (y')^2$ in the Euler-Lagrange equation we have

$$\frac{\partial}{\partial y}\left(y^2 - (y')^2\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(y^2 - (y')^2\right)\right) = 0$$

$$\Rightarrow 2y - \frac{d}{dx}\left(-2y'\right) = 0$$

$$\Rightarrow y'' + y = 0. \text{ [\textbf{3 marks (A)}]}$$

The general solution is

$$y = A\cos x + B\sin x. \text{ [\textbf{2 marks (A)}]}$$

Applying the boundary conditions:

$$y(0) = 1 \Rightarrow A = 1, \ \ y'(2\pi) = 1 \Rightarrow B = 1. \text{ [\textbf{2 marks (A)}]}$$

Thus the solution is

$$y = \cos x + \sin x, \text{ as required. [\textbf{1 mark (A)}]}$$

OR

$$L - y'\frac{\partial L}{\partial y'} = K$$

$$y^2 - (y')^2 - y'(-2y') = K$$

$$y^2 + y'^2 = \beta^2$$

etc.

$$\int \frac{dy}{\sqrt{(\beta^2 - y^2)}} = \int dx$$

$[y = \beta \sin u]$

$$\sin^{-1}\left(\frac{y}{\beta}\right) = x + C$$

$$y = \beta \sin(x + C)$$

$$1 = \beta \sin C$$

$$1 = \beta \cos(2\pi + C)$$

$$\Rightarrow \tan C = 1$$

$$\Rightarrow C = \pi/4$$

$$\& \ \beta = \sqrt{2}$$

$$\therefore y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sin x + \cos x$$

I WILL TELL YOU TO USE THE SHORT FORM IF APPROPRIATE

(b) We add the constraint by applying the E-L equation to $L + \lambda g$, where $g = y\cos x$:

$$\frac{\partial}{\partial y}\left(y^2 - (y')^2 + \lambda y\cos x\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(y^2 - (y')^2 + \lambda\cos x\right)\right) = 0$$

$$\Rightarrow 2y + \lambda\cos x - \frac{d}{dx}\left(-2y'\right) = 0$$

$$\Rightarrow y'' + y = -(\lambda/2)\cos x. \text{ [\textbf{3 marks (B)}]}$$

The homogeneous solution (as before) is

$$y_H = A\cos x + B\sin x. \text{ [\textbf{1 mark (B)}]}$$

For the particular solution try

$$y_P = Cx\cos x + Dx\sin x. \text{ [\textbf{1 mark (C)}]}$$

Then

$$y_P'' + y_P = -2C\sin x + 2D\cos x = -(\lambda/2)\cos x \Rightarrow C = 0, D = -\lambda/4. \text{ [\textbf{2 marks (C)}]}$$

Therefore the solution is

$$y = y_H + y_P = A\cos x + B\sin x - (\lambda/4)x\sin x.$$

Applying the boundary conditions:

$$y(0) = 1 \Rightarrow A = 1, \ \ y'(2\pi) = 1 \Rightarrow B - (\lambda/4)(2\pi) = 1 \Rightarrow B = 1 + \lambda\pi/2. \text{ [\textbf{2 marks (C)}]}$$

To find λ substitute into integral constraint:

$$\begin{aligned}
\frac{\pi}{2} &= \int_0^{2\pi} \cos^2 x + (1 + \lambda\pi/2)\sin x\cos x - (\lambda/4)x\sin x\cos x\,dx \\
\Rightarrow &\ \ \frac{\pi}{2} = \pi - \frac{\lambda}{8}\int_0^{2\pi} x\sin 2x\,dx \\
\Rightarrow &\ \ \frac{\lambda}{8}\left\{[-\frac{x}{2}\cos 2x]_0^{2\pi} + \int_0^{2\pi}\frac{1}{2}\cos 2x\,dx\right\} = \frac{\pi}{2} \\
\Rightarrow &\ \ \lambda = -4.
\end{aligned}$$

So the final solution is

$$y = \cos x + (1 - 2\pi)\sin x + x\sin x. \text{ [\textbf{3 marks (D)}]}$$