Assessed Coursework 1

You may discuss these problems with other students, but you must write up your own solutions.

NOT parametrised by arc length

Problem 1. Let $\phi : [a, b] \to \mathbb{R}^n$ (n = 2 or 3) be a regular curve, and suppose that $|\phi(t)|$ realises its maximum value at some $t_0 \in (a, b)$. Prove that

$$k(t_0) \geq \frac{1}{|\phi(t_0)|}.$$

Problem 2. Let $\phi : [a, b] \to \mathbb{R}^3$ be a regular curve parametrised by arc length, and with curvature k and torsion τ. Assume that there is a constant $c \in \mathbb{R}$ such that for all $t \in [a, b]$ we have $\tau(t) = ck(t)$ and $k(t) > 0$. Prove that the tangent vectors of ϕ make a fixed angle θ with some fixed vector $v \in \mathbb{R}^3$, for all values of t.

Hint: First find a linear combination of the vectors in the Frenet frame which is independent of t.

Problem 3. Consider the ellipse

$$\gamma(t) = (a\cos(t), b\sin(t)), \quad t \in \mathbb{R},$$

where $0 < a < b$. Calculate the curvature of this curve at each point on the curve.

Problem 4. Prove that any regular surface $S \subset \mathbb{R}^3$ is locally a regular level set, that is, for every $p \in S$ there is an open set $V \subset \mathbb{R}^3$ containing p, and a smooth function $F : V \to \mathbb{R}$ such that $S \cap V = F^{-1}(0)$ and ∇F is non-zero at every point in $V \cap S$.

Is it true that every regular surface in $\mathbb{R}^3$ is the regular level set of a smooth function on $\mathbb{R}^3$?

Problem 5. Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$. We say that S_1 and S_2 are **diffeomorphic**, if there is a bijective map $f : S_1 \to S_2$ such that both $f : S_1 \to S_2$ and $f^{-1} : S_2 \to S_1$ are smooth.

i) Are the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

and the surface

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0, z = 1/(x^2 + y^2)\}$$

diffeomorphic? Justify your answer.

ii) Are the surfaces

$$S_1 = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 < 1, z = 0\right\},$$
$$S_2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 < 1, z = \sqrt{x^2 + y^2}\right\}$$

diffeomorphic? Justify your answer.

[You do not need to prove that these sets are regular surfaces in $\mathbb{R}^3$, although it may follow from the proofs you present for the diffeomorphic property.]