' $\vdash_L \varphi$

Note. (1) If $\Gamma = \emptyset$

then ' $\Gamma \vdash_L \phi$ '

is the same as $\vdash_L \phi$

– ϕ is a theorem of L.

(2) If $\Delta \subseteq \Gamma$ and

$\Delta \vdash_L \phi$ then $\Gamma \vdash_L \phi$.

//

(1.2.5) **Theorem**. (Deduction Theorem, DT)

Suppose Γ is a set of L-fmlas and ϕ, ψ are L-fmlas.

Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$

then $\Gamma \vdash_L (\phi \to \psi)$.

[**Ex**: converse is also true.]

Use this:

(1.2.6) **Cor** (Hypothetical Syllogism HS).

Suppose ϕ, ψ, χ are L-formulas

and $\vdash_L (\phi \to \psi)$

and $\vdash_L (\psi \to \chi)$.

Then $\vdash_L (\phi \to \chi)$.

Pf: Show: there is a deduction of χ from $\{\phi\}$.

(So $\{\phi\} \vdash_L \chi$; then DT with $\Gamma = \emptyset$ gives $\vdash_L (\phi \to \chi)$).

Here is a deduction from $\{\phi\}$

1. ϕ (Assumption)
2. $(\phi \to \psi)$ (theorem of L)
3. ψ (1,2 + MP)
4. $(\psi \to \chi)$ (theorem of L)
5. χ (3,4 + MP).

So $\{\phi\} \vdash_L \chi$.

thus $\vdash_L (\phi \to \chi)$. #

<u>Note</u>: Allowed to use 'known' theorems of L in deductions.

(1.2.7) <u>Proposition</u>

Suppose ϕ, ψ are L-formulas. Then

(a) $\vdash_L ((\neg\psi) \to (\psi \to \phi))$

(b) $\{(\neg\psi), \psi\} \vdash_L \phi$

(c) $\vdash_L (((\neg\phi) \to \phi) \to \phi)$

Pf: (a) Problem sheet 1, qu. 6.

(b) By (a) and MP twice.

(c) Suppose χ is any formula. Then by (b) + MP

$\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L \chi$

Let α be an axiom and let χ be $(\neg\alpha)$.

So $\{(\neg\phi), ((\neg\phi)\to\phi)\} \vdash_L (\neg\alpha)$.

By DT

$\{((\neg\phi)\to\phi)\} \vdash ((\neg\phi)\to(\neg\alpha))$

Use A3 axiom

$(((\neg\phi)\to(\neg\alpha))\to(\alpha\to\phi))$

+ MP, ~~for~~ we get

$\{((\neg\phi)\to\phi)\} \vdash (\alpha\to\phi)$

We have $\vdash_L \alpha$, so by

MP: $\{((\neg\phi)\to\phi)\} \vdash \phi$.

DT then gives $\vdash_L (((\neg\phi)\to\phi)\to\phi)$.

#.

Proof of Deduction Theorem.

Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ using a deduction of length n. Prove by induction on n that

$$\Gamma \vdash_L (\phi \to \psi).$$

Base step $n = 1$. In this case ψ is either

an axiom

or is in Γ

or it is ϕ.

In the first two case

$$\Gamma \vdash_L \psi$$

Then the A1 axiom

$$\vdash_L (\psi \to (\phi \to \psi))$$

and MP gives

$$\Gamma \vdash_L (\phi \to \psi).$$

If ψ is ϕ then

$$\Gamma \vdash_L (\phi \to \phi)$$

by 1.2.3. This does the $n=1$ case.

Inductive step. Suppose the result holds for shorter deductions (ie of length $< n$).

In our deduction $\Gamma \cup \{\phi\} \vdash \psi$ either:

(a) ψ is an axiom, or in Γ or is equal to ϕ

or (b) ψ is obtained by applying MP to earlier formulas χ, $(\chi \to \psi)$.

In case (a) we argue as in the base case, to get

$$\Gamma \vdash_L (\phi \to \psi).$$

In case (b) we have

$$\Gamma \cup \{\phi\} \vdash \chi$$

and $\Gamma \cup \{\phi\} \vdash (\chi \to \psi)$

using shorter deductions ($< n$ steps)

By the inductive hypothesis

$$\Gamma \vdash (\phi \to \chi) \quad \ldots \; 1$$

& $$\Gamma \vdash (\phi \to (\chi \to \psi)) \quad \ldots \; 2$$

Use A2

$$\vdash_L ((\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))) \quad 5$$

and 2 + MP we have

$$\Gamma \vdash ((\phi \to \chi) \to (\phi \to \psi)).$$

Use 1 & MP to get

So $\Gamma \vdash (\phi \to \psi)$

This completes the inductive hypothesis. #.