

Analysis 1A

Lecture 15
Series continued:
Absolute convergence

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Example 4.4

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Example 4.5

Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proposition 4.6

Suppose $a_n \geq 0 \ \forall n$ ($\iff s_n = \sum_{i=1}^n a_i$ is monotonically increasing), Then the following two facts are true:

- 1 $\sum_{n=1}^{\infty} a_n$ converges if and only if (s_n) is bounded above.
- 2 Similarly $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ (i.e. $\forall M > 0 \ \exists N \in \mathbb{N}$ such that $s_n > M \ \forall n \geq N$) if and only if (s_n) is unbounded.

Proof ① s_n is monotonically increasing

s_n converges $\iff s_n$ bounded above

" $\sum a_n$ converges"

s_n is not bounded
if and only if

② s_n is unbounded $\iff \forall M > 0, \exists N \in \mathbb{N}$ such that $s_N > M$

$\iff \forall M > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, s_n > s_N > M$

$\iff s_n \rightarrow \infty$

" $\sum a_n$ diverges to ∞ "

■

Theorem 4.7 Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Proof Let $A_n = \sum_{i=1}^n a_i$, $B_n = \sum_{i=1}^n b_i$

$0 \leq A_n \leq B_n$ and B_n is bounded above (since $\sum b_n$ is convergent)
so $B_n \leq M$

Therefore A_n is bounded above by M , and so

$\sum_{n=1}^{\infty} a_n$ converges. ■

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Exercise 4.8 - Converse of Comparison Test

If $0 \leq a_n \leq b_n$ then

$\sum a_n$ diverges to $+\infty \implies \sum b_n$ diverges to $+\infty$.

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Remark 4.9

So from the fact that $\sum \frac{1}{n^2}$ is convergent, we can now deduce that $\sum \frac{1}{n^\alpha}$ converges for $\alpha \geq 2$ by the Comparison Test. In fact we can improve on this.

Example 4.10

Let $\alpha > 1$, then show that $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is convergent.

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$$\begin{aligned} 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \dots &= 1 + \left(\frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} \right) + \left(\frac{1}{4^{\alpha}} + \dots + \frac{1}{7^{\alpha}} \right) \\ &\quad + \left(\frac{1}{8^{\alpha}} + \dots + \frac{1}{15^{\alpha}} \right) + \left(\frac{1}{16^{\alpha}} + \dots + \frac{1}{31^{\alpha}} \right) + \dots \end{aligned}$$

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Rough work

$$\begin{aligned} 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots &= 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha} \right) + \left(\frac{1}{4^\alpha} + \dots + \frac{1}{7^\alpha} \right) \\ &\quad + \left(\frac{1}{8^\alpha} + \dots + \frac{1}{15^\alpha} \right) + \left(\frac{1}{16^\alpha} + \dots + \frac{1}{31^\alpha} \right) + \dots \end{aligned}$$

Bound the k th bracketed term:

$$\left(\frac{1}{(2^k)^\alpha} + \dots + \frac{1}{(2^{k+1}-1)^\alpha} \right) \leq \frac{1}{2^{k\alpha}} + \dots + \frac{1}{2^{k\alpha}} = \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}.$$

Example 4.10

Let $\alpha > 1$, then show that $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent.

Example continued:

Proof Let $s_n = \sum_{k=1}^n \frac{1}{2^k}$

Since $(\frac{1}{2^1} + \dots + \frac{1}{2^{m-1}}) \leq \frac{2^k}{2^m} = \frac{1}{2^{m+1}}$

So $s_n \leq 1 + \frac{1}{2^m} + \frac{1}{2^{m+1}}$

For any $n \leq 2^m - 1$

$s_n \leq \sum_{j=0}^k \frac{1}{2^{mj}} \leq \sum_{j=0}^m r^j = \frac{1}{1-r}$

where $r = 2^{(1-\alpha)}$ ($|r| < 1$)

s_n is bounded above, so convergent. \blacksquare

Theorem 4.11 - Algebra of Limits for Series

If $\sum a_n$, $\sum b_n$ are convergent then so is $\sum(\lambda a_n + \mu b_n)$, to

$$\sum_{n=1}^{\infty} (\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

Proof

$$\sum_{j=1}^{\infty} \lambda a_j + \mu b_j = \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j \rightarrow \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j$$

by Algebra of limits for sequences.

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Definition - Absolute Convergence

For $a_n \in \mathbb{R}$ or \mathbb{C} , we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

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Remark 4.12

It is possible for a series to be convergent, but not absolutely convergent!

Example 4.13

We note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent (remember the harmonic series), but show that it is convergent.

Rough Working:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \dots$$

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with k -th bracket $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$.

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This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$.

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This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$.

We saw this is convergent in Example 4.5. So cancellation between consecutive terms is enough to make series converge by comparison with $\sum \frac{1}{k(k-1)}$.

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Proof We know $\sum \frac{1}{2k+1}$ is convergent to L .

and we know $\frac{(-1)^{n+1}}{n} \rightarrow 0$, $\sum \frac{1}{2k+1} = s_n$

for $\epsilon > 0$ (i) $\exists N_1$ st $\forall n \geq N_1$, $|s_n - L| < \epsilon$

(ii) $\exists N_2$ st $\forall n \geq N_2$, $\left| \frac{(-1)^{n+1}}{n} \right| < \epsilon$

Let $\sigma_n = \sum_{j=1}^n \frac{(-1)^{j+1}}{j}$, $\sigma_n = s_{\lfloor n/2 \rfloor} + s_n$ $s_n = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

Let $N = 2 \max(N_1, N_2) + 1$

Then for $n \geq N$ $\lfloor n/2 \rfloor \geq N_1$ \downarrow $\lfloor n/2 \rfloor \geq N_2$

$|\sigma_n - L| \leq |s_{\lfloor n/2 \rfloor} - L| + |s_n| < \epsilon + \epsilon = 2\epsilon$. So $\sigma_n \rightarrow L$.

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