

Answers to Problem Sheet 5

1.

$$Q = \log p, \quad P = -qp.$$

$$\{Q, P\} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = 0 - \frac{1}{p} \cdot (-p) = 1.$$

To find a generating function write p and P as functions of q and Q

$p = e^Q$, $P = -qe^Q$. A generating function is $F(q, Q) = qe^Q$ as $p = \partial F / \partial q$, $P = -\partial F / \partial Q$.

2. The Hamiltonian for the Kepler problem in polar coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}.$$

(i) Poisson brackets:

- (a) $\{p_\theta, H\} = 0$
- (b) $\{p_r, H\} = p_\theta^2 / (mr^3) - k/r^2$
- (c) $\{r^{-1}, H\} = -r^{-2}p_r/m$
- (d) $\{e^{i\theta}, H\} = ie^{i\theta}p_\theta / (mr^2)$.

(ii)

$$\{V, H\} = \{e^{i\theta}, H\} \left(\frac{p_\theta^2}{r} - ip_r p_\theta - mk \right) + e^{i\theta} \left(p_\theta^2 \{r^{-1}, H\} - ip_\theta \{p_r, H\} \right)$$

and $\{V, H\} = 0$ follows from the results of part (i). The complex quantity $V = A_x + iA_y$ where A_x and A_y are components of the Laplace-Runge-Lenz vector, \mathbf{A} , considered in question 8 of Problem Sheet 1. $\{p_\theta, e^{i\theta}\} = -de^{i\theta}/d\theta = -ie^{i\theta}$. Accordingly, $\{p_\theta, V\} = -iV$.

(iii) Writing $V = Ae^{i\alpha}$ where A and α are real constants

$$Ae^{i(\alpha-\theta)} = \frac{p_\theta^2}{r} - ip_r p_\theta - mk.$$

Taking the real part of both sides

$$A \cos(\theta - \alpha) = \frac{p_\theta^2}{r} - mk,$$

or

$$r = \frac{p_\theta^2}{mk + A \cos(\theta - \alpha)} = \frac{L}{1 + e \cos(\theta - \alpha)},$$

where $L = p_\theta^2/(mk)$ and $e = A/(mk)$.

This shows that the orbits in the Kepler problem are *conic sections*. In polar coordinates

$$r = \frac{L}{1 + e \cos(\theta - \alpha)},$$

describes a conic section with one focus at the origin and eccentricity e .

3. Suppose that A and B are functions of q_i , p_i and t ($i = 1, \dots, N$).

(i)

$$\{A, B\} = \sum_{i=1}^N \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial t} \{A, B\} &= \sum_{i=1}^N \left(\frac{\partial A_t}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A_t}{\partial p_i} \frac{\partial B}{\partial q_i} \right) + \sum_{i=1}^N \left(\frac{\partial A}{\partial q_i} \frac{\partial B_t}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B_t}{\partial q_i} \right) \\ &= \{A_t, B\} + \{A, B_t\}, \end{aligned}$$

where $A_t = \partial A / \partial t$.

(ii)

$$\begin{aligned} \frac{d}{dt} \{A, B\} &= \{\{A, B\}, H\} + \frac{\partial}{\partial t} \{A, B\} \\ &= -\{\{B, H\}, A\} - \{\{H, A\}, B\} + \{A_t, B\} + \{A, B_t\}, \end{aligned}$$

using the Jacobi identity and the result given in part (i). If A is a constant of the motion $\{A, H\} + A_t = 0$ so that $\{A, H\} = -A_t$ and similarly $\{B, H\} = -B_t$ if B is a constant. Inserting this into the formula for $d\{A, B\}/dt$ gives the required result.

- 4.

$$\dot{q} = \frac{\partial H}{\partial p} = pq^4, \quad \dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3} - 2p^2q^3.$$

From the first equation $p = \dot{q}/q^4$. Inserting this into the second equation yields

$$\frac{d}{dt} \frac{\dot{q}}{q^4} = \frac{1}{q^3} - 2\frac{\dot{q}^2}{q^5}$$

which simplifies to

$$\ddot{q} = q + 2 \frac{\dot{q}^2}{q}.$$

(ii) Under the canonical transformation $Q = -1/q$, $P = q^2 p$ (since $\{Q, P\} = 1$) $H = K = \frac{1}{2}(Q^2 + P^2)$. $K = H$ as the canonical transformation is time-independent. Note that other canonical transformations can be used, e.g. $Q = q^2 p$, $P = 1/q$.

The general solution to the harmonic oscillator is $Q = A \cos(t + \beta)$. $q = -1/Q$ which gives

$$q = \frac{\alpha}{\cos(t + \beta)},$$

where $\alpha = -1/A$. Check that this satisfies the ODE from part (i)!

5. As the transformation is time-independent we have $H = K$ giving $p^2 + e^q = P^2$ so that

$$q = \log(P^2 - p^2) = -\frac{\partial F_4}{\partial p}.$$

Therefore

$$F_4 = - \int [\log(P + p) + \log(P - p)] dp$$

Accordingly

$$Q = \frac{\partial F_4}{\partial P} = - \int \left[\frac{1}{P + p} + \frac{1}{P - p} \right] dp = \log \frac{P - p}{P + p},$$

(or $Q = -2 \tanh^{-1}(p/P)$).

The transformation can be written directly as

$$P = \sqrt{p^2 + e^q}, \quad Q = \log \left(\frac{\sqrt{p^2 + e^q} - p}{\sqrt{p^2 + e^q} + p} \right).$$

The formula for Q can also be written as

$$Q = q - 2 \log(\sqrt{p^2 + e^q} + p) = -2 \sinh^{-1}(pe^{-q/2}).$$

Is the transformation unique?