IMPERIAL

MATH70054

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Introduction to Stochastic Differential Equations and Diffusion Processes

Date: Thursday, May 1, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Define what it means for a stochastic process to be *Gaussian*, and what it means for a stochastic process to be a *Standard Brownian motion*. (5 marks)

   (b) Explain (in words) why the law of a Gaussian process X is characterised by its mean m and covariance $C(s,t)$. (3 marks)

   (c) Let W be a standard Brownian motion.

      (i) Find the law of the process $(X_t, t > 0) := (tW_{1/t}, t > 0)$. (4 marks)

      (ii) Hence show that $X_t \to 0$ almost surely as $t \to 0$. Defining $X_0 := 0$, what is the law of $(X_t, t \geq 0)$? (3 marks)

   (d) Stating carefully any properties of Gaussians, prove that

$$Y_t := (1-t)\int_0^t \frac{dW_s}{1-s}, \quad 0 \leq t < 1$$

   is a Gaussian process. By considering $B_t := W_t - tW_1$, what is the limit of Y_t as $t \uparrow 1$? (5 marks)

(Total: 20 marks)

2. (a) Solve the SDE

$$dX_t = \lambda X_t dt + \sigma X_t dW_t \tag{1}$$

   where $X_0 = x > 0$, $\lambda, \sigma \in \mathbb{R}$, and W is a standard Brownian motion. Omitting a suitable edge case, how does X behave as $t \to \infty$?

   (10 marks)

   (b) Let $\mathcal{F}_t$ be the natural filtration for the Brownian motion W.

      (i) Define what it means for M to be a *martingale* for the filtration $\mathcal{F}_t$. (3 marks)

      (ii) Give conditions on $f_t = f(t,\omega)$ under which the Itô integral $\int_0^t f_s dW_s$ may be defined as a continuous, square integrable martingale. (2 marks)

   (c) For suitable values of λ, σ, show that X_t is a martingale in the filtration $\mathcal{F}_t$. Show that $X_t \to X_\infty$ almost surely as $t \to \infty$, for some X_∞ you should find. Is $\mathbb{E}X_0 = \mathbb{E}X_\infty$? Find $\lim_{t\to\infty} \mathbb{E}X_t^p$ for all $p \geq 1$. (5 marks)

(Total: 20 marks)

3. (a) Consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{2}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded vector field, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is a bounded matrix field, and W is a n-dimensional Brownian motion.

Write down the generator $\mathcal{L}$, and the forward and backward Fokker-Planck equations associated to (2). What quantities associated to X are governed by these equations? (6 marks)

(b) (i) State what it means for a process X to be a semimartingale. (2 marks)

(ii) State the Itô formula for $f(X_t)$, where X is a vector of semimartingales. (3 marks)

(ii) Let X and $\mathcal{L}$ be as in a). For u a smooth solution to $\partial_t u - \mathcal{L}u = 0$, with $u|_{t=0} = f$. Show that $u(T-t, X_t)$ is a martingale. Hence find an expression for $u(t,x)$ in terms of f. (6 marks)

(c) Let f be a smooth function with $f(0) = 0$, and $f(x) = 1$ if $|x| \geq 1$. Find two distinct solutions $u = u(t,x)$ to the PDE

$$\partial_t u - 3x^{1/3}\partial_x u - \frac{9}{2}x^{4/3}\partial_x^2 u = 0; \qquad u|_{t=0} = f. \tag{3}$$

You may leave your answers expressed in terms of standard stochastic processes.
(Hint: If an SDE has non-unique solutions starting at x_0, then non-unique solutions starting at $x \neq x_0$ may be constructed by running any solution until the first time it hits $X_t = x_0$, and then extending by either of the two solutions) (3 marks)

(Total: 20 marks)

4. (a) State Girsanov's theorem. (3 marks)

(b) Consider the SDE in one dimension

$$dX_t = b(X_t)dt + dW_t, \qquad X_0 = x \tag{4}$$

where W is a standard Brownian motion and $b : \mathbb{R} \to [-1, 1]$ is a measurable, but not necessarily continuous. function.

(i) For a fixed $T < \infty$, use Girsanov's theorem to construct a solution $(X_t : 0 \leq t \leq T)$ to the SDE (4). (6 marks)

(ii) Is the solution you have constructed a *strong solution*? (1 mark)

(iii) Briefly explain why all solutions to (4) have the same law. (3 marks)

(c) On a finite time interval $[0, T]$, consider the two SDEs in one dimension

$$dX_t^{(1)} = b^{(1)}(X_t^{(1)})dt + \sigma(X_t^{(1)})dW_t; \tag{5}$$

$$dX_t^{(2)} = b^{(2)}(X_t^{(2)})dt + \sigma(X_t^{(2)})dW_t; \tag{6}$$

where $b^{(1)}, b^{(2)}$ are bounded, $\inf_x \sigma(x) > 0$ and where W is a standard Brownian motion, and both start at $X_0^{(i)} = x$. Suppose both (5), (6) have uniqueness in law, and let $\mu^{(1)}, \mu^{(2)}$ be the laws of the solutions. Show that $\mu^{(2)}$ has a density, which you should find, with respect to $\mu^{(1)}$. (7 marks)

(Total: 20 marks)

5. (a) Let $b, \sigma : \mathbb{R} \to \mathbb{R}$. State what it means for b, σ to be Lipschitz continuous, and what it means for a stochastic process X to be a *solution* to

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \qquad X_0 = x \tag{7}$$

where B is a Brownian motion. In the remainder of the question, b, σ are Lipschitz.

(3 marks)

(b) The *contraction mapping* theorem asserts that, if $(\mathcal{X}, d)$ is a complete metric space and $F : \mathcal{X} \to \mathcal{X}$ satisfies $d(F(x), F(y)) \le \rho d(x, y)$ for some $\rho \in [0, 1)$ and all $x, y \in \mathcal{X}$, then there exists a unique fixed point $x_\star \in \mathcal{X} : F(x_\star) = x_\star$. For a fixed time horizon T, you may assume that the space

$$\mathcal{C}_T := \left\{ \text{Continuous, adapted } X : [0,T] \to \mathbb{R} : \|X\|_T := \mathbb{E}\left[\sup_{t \le T} |X_t|^2\right]^{1/2} < \infty \right\}$$

is complete in the norm $\|\cdot\|_T$, for functions adapted to the filtration of the Brownian motion B.

(i) Reformulate the definition of solutions to (7) as a fixed point theorem. That is, give a map $F : \mathcal{C}_T \to \mathcal{C}_T$ such that $F(X) = X$ if, and only if, X solves (7). (1 mark)

(ii) State Itô's isometry for $\int_0^t h_s dB_s$ and Doob's L^2 inequality. Hence show that, for suitable h and any $t \le T$,

$$\|I(h)\|_t^2 \le 4 \int_0^t \|h\|_s^2 \tag{8}$$

where $I(h)_t := \int_0^t h_s dB_s$. (4 marks)

(iii) Using Itô's isometry, Cauchy-Schwarz and $(x+y)^2 \le 2(x^2 + y^2)$, prove that the map F in (i) satisfies, for all $X, Y \in \mathcal{C}_T$,

$$\|F(X) - F(Y)\|_t^2 \le C \int_0^t \|X - Y\|_s^2 ds$$

for some C which you should find in terms of the Lipschitz constants of b, σ.

(7 marks)

(iv) Show by induction that, for $n \ge 1$ and $t \le T$,

$$\|F^n(X) - F^n(Y)\|_t^2 \le C^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \|X - Y\|_{t_n}^2$$

where F^n denotes the n-fold composition. The volume of $\{t \ge t_1 > t_2 > \cdots > t_n \ge 0\}$ is $t^n/n!$. Use this to show that, for sufficiently large n, F^n is a contraction on $\mathcal{C}_T$.

(3 marks)

(v) Hence conclude that, for any T, there exists a unique solution to (7) up to time T. Is the solution strong? (2 marks).

(Total: 20 marks)

1. (a) Define what it means for a stochastic process to be *Gaussian*, and what it means for a stochastic process to be a *Standard Brownian motion*. (5 marks)

(b) Explain (in words) why the law of a Gaussian process X is characterised by its mean m and covariance $C(s,t)$. (3 marks)

(c) Let W be a standard Brownian motion.

(i) Find the law of the process $(X_t, t>0) := (tW_{1/t}, t>0)$. (4 marks)

(ii) Hence show that $X_t \to 0$ almost surely as $t \to 0$. Defining $X_0 := 0$, what is the law of $(X_t, t \geq 0)$? (3 marks)

(d) Stating carefully any properties of Gaussians, prove that

$$Y_t := (1-t)\int_0^t \frac{dW_s}{1-s}, \quad 0 \leq t < 1$$

is a Gaussian process. By considering $B_t := W_t - tW_1$, what is the limit of Y_t as $t \uparrow 1$? (5 marks)

(Total: 20 marks)

Solution

(a) A stochastic process X is Gaussian if, for any $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$, the distribution of $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$ is Gaussian in $\mathbb{R}^{n+1}$. (2 A marks).
A stochastic process W is a standard Brownian motion if it is Gaussian with mean $\mathbb{E}W_t = 0$, for all $t \geq 0$, and covariance $\mathbb{E}W_sW_t = \min(s,t)$, or any equivalent characterisation by mean and covariance (2 A marks), and has continuous sample paths (1 A mark).

(b) The law of a stochastic process is uniquely determined by its finite dimensional distributions / fdds / finite dimensional marginals, and the law of a Gaussian random variable X in any arbitrary finite dimension $\mathbb{R}^n$ is determined by $\mu = \mathbb{E}X$ and covariance $\Sigma = \mathbb{E}XX^T$. Hence, for a Gaussian process X, the mean $m(t) = \mathbb{E}X(t)$ and covariance $C(s,t) = \mathbb{E}X(s)X(t)$ uniquely determine all fdds, and hence the full law. (3 A marks)

(c) (i) First, X is Gaussian, because for any n and any $0 < t_1 < \cdots < t_n$, the vector $(X_{t_1}, \ldots, X_{t_n}) = (t_1 W_{t_1^{-1}}, \ldots, t_n W_{t_n^{-1}})$ is a linear transformation of the Gaussian random variable $(W_{t_n^{-1}}, \ldots, W_{t_1^{-1}})$. (1 B mark). The mean is $\mathbb{E}X_t = t\mathbb{E}W_{t^{-1}} = 0$. For $0 < s \leq t$, the covariance is $C(s,t) = st\mathbb{E}W_{s^{-1}}W_{t^{-1}} = stt^{-1} = s$, so $C(s,t) = \min(s,t)$. (2 B marks, one for each of mean and covariance). Hence X has the law of Brownian motion for $t > 0$ by b). (1 B mark)

(ii) $W_0 = 0$ almost surely, $W_t \to W_0 = 0$ almost surely as $t \to 0$ by the continuity of paths, and $(X_t : t > 0)$ has the same law as $(W_t : t > 0)$, so $X_t \to 0$ almost surely. (2 D marks). Hence extending X by $X_0 = 0$ produces the (full, $t \geq 0$) Brownian motion. (1 C mark)

(d) The Itō integral is the limit in probability of the Riemann sums with left-endpoint evaluation, so for any tuple $0 = t_0 < t_1 \cdots < t_n < 1$, $(Y_{t_0}, \ldots Y_{t_n})$ is the limit in probability of

$$\left((1-t_i) \sum_{j=1}^{k} \frac{W_{jt_i/k} - W_{(j-1)t_i/k}}{1-(j-1)t_i/k} : 1 \leq i \leq n \right)$$

as $k \to \infty$. Each discretised sum is a linear transformation of the $\mathbb{R}^{nk}$-valued Gaussian $(W_{jt_i/k} : 1 \leq i \leq n, 0 \leq j \leq k)$, and is therefore Gaussian. (1 C mark). The limit in probability of a sequence of Gaussian random variables is still Gaussian (1 C mark, but this should be stated), so $(Y_{t_0}, \dots, Y_{t_n})$ is Gaussian for any $0 = t_0 < t_1 < \cdots < t_n$. Hence Y is Gaussian as desired.

The mean is $\mathbb{E}Y_t = 0$ and the covariance for $0 \leq s \leq t < 1$ is $\mathbb{E}Y_sY_t = s(1-t)$, which may be found either using Itō's isometry or the discrete approximation above. We can now repeat the trick of part c) to see that Y has the same mean and covariance, and hence law, as $B_t := W_t - tW_1$, and so by the same argument $B_t \to 0$ almost surely, so $Y_t \to 0$ almost surely as $t \uparrow 1$. (3 D marks)

2. (a) Solve the SDE

$$dX_t = \lambda X_t dt + \sigma X_t dW_t \tag{1}$$

where $X_0 = x > 0$, $\lambda, \sigma \in \mathbb{R}$, and W is a standard Brownian motion. Omitting a suitable edge case, how does X behave as $t \to \infty$?

(10 marks)

(b) Let $\mathcal{F}_t$ be the natural filtration for the Brownian motion W.

(i) Define what it means for M to be a *martingale* for the filtration $\mathcal{F}_t$. (3 marks)

(ii) Give conditions on $f_t = f(t, \omega)$ under which the Itô integral $\int_0^t f_s dW_s$ may be defined as a continuous, square integrable martingale. (2 marks)

(c) For suitable values of λ, σ, show that X_t is a martingale in the filtration $\mathcal{F}_t$. Show that $X_t \to X_\infty$ almost surely as $t \to \infty$, for some X_∞ you should find. Is $\mathbb{E}X_0 = \mathbb{E}X_\infty$? Find $\lim_{t\to\infty} \mathbb{E}X_t^p$ for all $p \geq 1$. (5 marks)

(Total: 20 marks)

Solution

(a) We apply Itō's formula as in lectures, to $f(x) = \log x$:

$$d\log X_t = \frac{1}{X_t} dX_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right) d\langle X\rangle_t$$

(2 A marks for correct application of Ito formula), and since $\log$ is only C^2 on $(0, \infty)$, this is valid only up until the first time T that $X_T = 0$. (1 D mark, or for any equivalent indication of localisation). From (1), the rule to discard dt-terms in the quadratic variation, squaring the integrand and $d\langle W\rangle_t = dt$, we get $d\langle X\rangle_t = \sigma^2 X_t^2 dt$. (2 B marks). Hence

$$d\log X_t = \lambda dt + \sigma dW_t - \frac{\sigma^2}{2} dt$$

which produces, up until T, $X_t = x\exp(\lambda t + \sigma W_t - \sigma^2 t/2)$. (1 A mark for solution seen in lectures) Since the right-hand side is always strictly positive, $T = \infty$ and this is the solution globally. (1 C mark).

For the long-time behaviour, we distinguish between the cases $\lambda - \sigma^2/2 < 0, = 0, > 0$. (2 A marks). In either of the cases other than the edge case, use the Brownian law of large numbers, which can be quoted without proof $W_t/t \to 0$ (1 C mark) to see that $X_t \to 0$ almost surely if $\lambda - \sigma^2/2 < 0$ and $\to \infty$ almost surely if $\lambda - \sigma^2/2 > 0$. As instructed, we discard the edge case $\lambda - \sigma^2/2 = 0$.

(b) (i) Adaptedness, or that M_t is $\mathcal{F}_t$-measuarable for each t (1 A mark); integrability, that $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$ (1 A mark); and that for all $0 \leq s \leq t, \mathbb{E}[M_t|\mathcal{F}_s] = M_s$ almost surely (1 A mark). (Definition from lectures).

(ii) The conditions are that $f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable and that $f(t, \cdot)$ is $\mathcal{F}_t$-measurable for all t (1 B mark) and that $\mathbb{E}\int_0^T f_s^2 ds < \infty$ for all T (1 B mark, allow equivalents, for instance for some fixed T and $t \leq T$).

(c) Desiring X_t to be a martingale forces the choice $\lambda = 0$ (1 B mark), so that $X_t = X_0 + \int_0^t \sigma X_s dW_s$ is a stochastic integral. To see that this is a genuine martingale, we just need to check the integrability condition in bii), which can be done by the explicit solution in a):

$$\mathbb{E}\int_0^T X_t^2 dt = x_0^2 \int_0^T \mathbb{E}e^{2\sigma W_t - \sigma^2 W_t} dt = x_0^2 \int_0^T e^{\sigma^2 t} dt < \infty$$

using $\mathbb{E}e^{\lambda W_t} = e^{\lambda^2 t/2}$. (1 C mark, available for any reasoning that establishes the integrability condition, or 1C by checking martingale property directly.)

For the final part, we get that $X_\infty = 0$ from part a), except in the trivial case $\sigma = \lambda = 0$. Immediately $\mathbb{E}X_0 = x_0 > 0 = \mathbb{E}\mathbb{X}_\infty$ (1 D mark). For $p \geq 1$,

$$\mathbb{E}X_t^p = x_0^p \mathbb{E}e^{\sigma p W_t - \sigma^2 pt/2} = e^{\sigma^2 t(p^2-p)/2}$$

which diverges if $p > 1$, and is constant if $p = 1$. (2 D marks. The marks can also be scored by an indirect argument: $p = 1$ follows from martingale property, and $p > 1$ follows from the contrapositive of the optional stopping theorem.)

3. (a) Consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{2}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded vector field, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is a bounded matrix field, and W is a n-dimensional Brownian motion.

Write down the generator $\mathcal{L}$, and the forward and backward Fokker-Planck equations associated to (2). What quantities associated to X are governed by these equations?
(6 marks)

(b) (i) State what it means for a process X to be a semimartingale. (2 marks)

(ii) State the Itô formula for $f(X_t)$, where X is a vector of semimartingales. (3 marks)

(ii) Let X and $\mathcal{L}$ be as in a). For u a smooth solution to $\partial_t u - \mathcal{L}u = 0$, with $u|_{t=0} = f$. Show that $u(T-t, X_t)$ is a martingale. Hence find an expression for $u(t,x)$ in terms of f. (6 marks)

(c) Let f be a smooth function with $f(0) = 0$, and $f(x) = 1$ if $|x| \geq 1$. Find two distinct solutions $u = u(t,x)$ to the PDE

$$\partial_t u - 3x^{1/3}\partial_x u - \frac{9}{2}x^{4/3}\partial_x^2 u = 0; \qquad u|_{t=0} = f. \tag{3}$$

You may leave your answers expressed in terms of standard stochastic processes.
(Hint: If an SDE has non-unique solutions starting at x_0, then non-unique solutions starting at $x \neq x_0$ may be constructed by running any solution until the first time it hits $X_t = x_0$, and then extending by either of the two solutions) (3 marks)

(Total: 20 marks)

Solution

(a) The generator is defined on $f \in C^2(\mathbb{R}^d)$ by

$$\mathcal{L}f(x) := b(x) \cdot \nabla f(x) + \frac{1}{2}a(x) : \nabla^2 f(x)$$

where $a_{ij}(x) := \sum_k \sigma_{ij}(x)\sigma_{jk}(x)$ - the generator can of course be written in summation notation (1 A mark). The adjoint is

$$\mathcal{L}^\star f(x) = -\nabla \cdot (b(x)f(x)) + \frac{1}{2}\nabla^2 : (a(x)f(x))$$

(1 B mark). The Forward Fokker-Planck equation is

$$\partial_t \rho = \mathcal{L}^\star f(x) = -\nabla \cdot (b(x)\rho(x)) + \frac{1}{2}\nabla^2 : (a(x)\rho(x))$$

and the backwards Fokker-Planck equation is

$$\partial_t u = \mathcal{L}b(x) \cdot \nabla u(x) + \frac{1}{2}a(x) : \nabla^2 u(x)$$

(each 1A mark)

The quantities governed by these equations are the law (or transition density or equivalent) ρ_t, i.e. $\mathbb{P}(X_t \in A) = \int_A \rho_t(x)dx$ for the forwards equation, and $u(t,x) := \mathbb{E}_x f(X_t)$ for the backwards equation. Admit, for either, any equivalent definition, e.g. in terms of the semigroup P_t for the backwards equation and its adjoint $P_t^\star$ for the forwards. (2 B marks).

(b) (i) X is a semimartingale if it can be decomposed as the sum $X = M + A$, where M is a martingale and A is a process of total variation (2 A marks: the terms need not be defined.)

(ii) For $f \in C^2(\mathbb{R}^n \to \mathbb{R})$ and semimartingales $X_t^1, X_t^2 \dots, X_t^n$, the Itō formula asserts that

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t (\partial_i f)(X_s) dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t (\partial_{ij} f)(X_s) d\langle X^i, X^j\rangle_s$$

(2 A marks), where $\langle X^i, X^j\rangle$ are the covariances (1 A mark).

(iii) First, each X^i and $T-t$ is a semimartingale, in the case of X^i from the equation (1 B mark). To apply the general Itō formula from ii), we write down the covariances $\langle T-t\rangle = \langle T-t, X^i\rangle = 0$, because $T-t$ is a process of total variation. For the covariations of the X^is, we use the rule to throw away finite variation parts / dt-terms to discard $b(X_t)dt$ and get

$$d\langle X^i, X^j\rangle_t = \sum_{k,l=1}^d d\left\langle \int_0^\cdot \sigma_{ik}(X_s)dW_s^k, \int_0^\cdot \sigma_{jl}(X_s)dW_s^l \right\rangle_t$$

Using the rule to 'take out the integrands, covariance of the integrators' and $\langle W^i, W^j\rangle_t = \delta_{ij}t$, the sum becomes

$$d\langle X^i, X^j\rangle_t = \sum_{k=1}^d \sigma_{ik}(X_t)\sigma_{jk}(X_t)dt = a_{ij}(X_t)dt.$$

(3 C marks for all covariances). Putting everything into Itō's formula, we get

$$dM_t = -\partial_t u(T-t, X_t)dt + \sum_{i=1}^d b_i(X_t)\partial_i u(T-t,X_t)dt + \sum_{i,j=1}^d a_{ij}(X_t)\partial_{ij}u(T-t,X_t)dt$$
$$+ \sum_{i,j=1}^d \partial_i u(T-t,X_t)\sigma_{ij}(X_t)dW_t^j.$$

The first line cancels by the assumption. (1 B mark). The remaining stochastic integral is a martingale because the integrand is bounded (by assumption) and progressively measurable (1 C mark) so $\mathbb{E}M_T = \mathbb{E}M_0$ (1 D mark). This yields

$$u(T,x) = \mathbb{E}_x M_0 = \mathbb{E}_x M_T = \mathbb{E}_x f(X_T).$$

(c) The given PDE (3) is the backwards Fokker-Planck equation associated to the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dW_t$$

in one dimension (1 D mark), which we saw in lectures to have non-unique solutions starting at $X_0 = 0$. Using the hint and the explicit solutions found in lectures, two distinct solutions starting at $X_0 = x$ are

$$X_t^1 = (x^{1/3} + W_t)^3; \qquad X_t^2 = X_{t \wedge T_x}^1; \qquad T_x = \inf\{t \geq 0 : W_t = -x^{1/3}\}.$$

We can construct two solutions u^1, u^2 to (3) by

$$u^i(t,x) := \mathbb{E}_x f(X_t^i).$$

(1 D mark). These differ because $X_t^2 \to 0$ almost surely, so $u^2(t,x) \to 0$ as $t \to \infty$, whereas $\mathbb{P}(|X_t^1| \leq 1) \to 0$ and so $u^1(t,x) \to 1$ as $t \to \infty$ by the choice of f (1 D mark).

4. (a) State Girsanov's theorem. (3 marks)

(b) Consider the SDE in one dimension

$$dX_t = b(X_t)dt + dW_t, \qquad X_0 = x \tag{4}$$

where W is a standard Brownian motion and $b : \mathbb{R} \to [-1, 1]$ is a measurable, but not necessarily continuous. function.

(i) For a fixed $T < \infty$, use Girsanov's theorem to construct a solution $(X_t : 0 \leq t \leq T)$ to the SDE (4). (6 marks)

(ii) Is the solution you have constructed a *strong solution*? (1 mark)

(iii) Briefly explain why all solutions to (4) have the same law. (3 marks)

(c) On a finite time interval $[0, T]$, consider the two SDEs in one dimension

$$dX_t^{(1)} = b^{(1)}(X_t^{(1)})dt + \sigma(X_t^{(1)})dW_t; \tag{5}$$

$$dX_t^{(2)} = b^{(2)}(X_t^{(2)})dt + \sigma(X_t^{(2)})dW_t; \tag{6}$$

where $b^{(1)}, b^{(2)}$ are bounded, $\inf_x \sigma(x) > 0$ and where W is a standard Brownian motion, and both start at $X_0^{(i)} = x$. Suppose both (5), (6) have uniqueness in law, and let $\mu^{(1)}, \mu^{(2)}$ be the laws of the solutions. Show that $\mu^{(2)}$ has a density, which you should find, with respect to $\mu^{(1)}$. (7 marks)

(Total: 20 marks)

Solution

(a) On a probability space with probability measure $\mathbb{P}$, let $T < \infty$, let M be a martingale satisfying $\mathbb{E}e^{\langle M\rangle_T/2} < \infty$ (or any other sufficient condition, e.g. M bounded) and W be a Brownian motion. Define a new probability measure by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := e^{M_T - \langle M\rangle_T/2}.$$

Then under $\mathbb{Q}$, $B := W - \langle W, M\rangle$ is a Brownian motion. (3 A marks, in lectures).

(b) (i) Let X be a Brownian motion defined on any probability measure $\mathbb{P}$, and take $M := \int_0^t b(X_s)dX_s$. (2 A marks). M is a martingale and $\langle M\rangle_T = \int_0^T b^2(X_s)ds \leq \|b\|_\infty T$ is bounded, so Girsanov applies. (2 B marks). Consider the new probability measure $\mathbb{Q}$ given by the Girsanov transformation, and $W := X - \langle X, M\rangle$. The covariation is $\int_0^t b(X_s)ds$, so

$$W_t = X_t - \int_0^t b(X_s)ds$$

(1 C mark). Hence X is a solution to (4) under the new probability measure $\mathbb{Q}$ (1 D mark).

(ii) No, because W is given in terms of X and not vice versa (1 A mark).

(iii) Starting from any solution, one can use a Girsanov transformation under which X is a Brownian motion. (1 B mark). The inverse of this Girsanov transformation is the same one we did in (i), which determines the density of Law(X) with respect to the Wiener measure, and hence Law(X). (2 C marks).

(c) Let $\mathbb{P}$ be a probability measure under which a solution to (5) is defined, and consider the martingale

$$M_t := \int_0^t \frac{b^{(2)}(X_s^{(1)}) - b^{(1)}(X_s^{(1)})}{\sigma(X_s^{(1)})} dW_s.$$

(1 A mark). Using the various hypotheses, this is a martingale and $\langle M \rangle_T$ is bounded, so Girsanov applies (2 B marks). The Girsanov transform sets

$$dB_t := dW_t - \frac{b^{(2)}(X_t^{(1)}) - b^{(1)}(X_t^{(1)})}{\sigma(X_t^{(1)})}$$

and substituting this into (5) shows that $X^{(1)}$ solves (6) driven by the $\mathbb{Q}$-Brownian motion B. (1A mark). Hence $\frac{d\mu^{(2)}}{d\mu^{(1)}}$ can be found by expressing

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(M_T - \frac{1}{2}\langle M \rangle_T \right)$$

as a function only of $X^{(1)}$ (1 D mark). To do this, write

$$dW_t = \frac{1}{\sigma(X_t^{(1)})}(dX_t^{(1)} - b^{(1)}(X_t^{(1)})dt)$$

and substitute it into the definition of M (1 D mark). We eventually find

$$\log \frac{d\mathbb{Q}}{d\mathbb{P}} = \int_0^T \frac{b^{(2)}(X_t^{(1)}) - b^{(1)}(X_t^{(1)})}{\sigma(X_t^{(1)})^2} dX_t^{(1)} - \int_0^T \frac{b^{(1)}(X_t^{(1)})^2 - b^{(2)}(X_t^{(1)})^2}{2\sigma(X_t^{(1)})^2} dt$$

which is indeed only a function of $X^{(1)}$ (1 D mark).

5. (a) Let $b, \sigma : \mathbb{R} \to \mathbb{R}$. State what it means for b, σ to be Lipschitz continuous, and what it means for a stochastic process X to be a *solution* to

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \qquad X_0 = x \tag{7}$$

where B is a Brownian motion. In the remainder of the question, b, σ are Lipschitz.

(3 marks)

(b) The *contraction mapping* theorem asserts that, if $(\mathcal{X}, d)$ is a complete metric space and $F : \mathcal{X} \to \mathcal{X}$ satisfies $d(F(x), F(y)) \le \rho d(x, y)$ for some $\rho \in [0, 1)$ and all $x, y \in \mathcal{X}$, then there exists a unique fixed point $x_\star \in \mathcal{X} : F(x_\star) = x_\star$. For a fixed time horizon T, you may assume that the space

$$\mathcal{C}_T := \left\{ \text{Continuous, adapted } X : [0,T] \to \mathbb{R} : \|X\|_T := \mathbb{E}\left[\sup_{t \le T} |X_t|^2\right]^{1/2} < \infty \right\}$$

is complete in the norm $\|\cdot\|_T$, for functions adapted to the filtration of the Brownian motion B.

(i) Reformulate the definition of solutions to (7) as a fixed point theorem. That is, give a map $F : \mathcal{C}_T \to \mathcal{C}_T$ such that $F(X) = X$ if, and only if, X solves (7). (1 mark)

(ii) State Itô's isometry for $\int_0^t h_s dB_s$ and Doob's L^2 inequality. Hence show that, for suitable h and any $t \le T$,

$$\|I(h)\|_t^2 \le 4 \int_0^t \|h\|_s^2 \tag{8}$$

where $I(h)_t := \int_0^t h_s dB_s$. (4 marks)

(iii) Using Itô's isometry, Cauchy-Schwarz and $(x+y)^2 \le 2(x^2 + y^2)$, prove that the map F in (i) satisfies, for all $X, Y \in \mathcal{C}_T$,

$$\|F(X) - F(Y)\|_t^2 \le C \int_0^t \|X - Y\|_s^2 ds$$

for some C which you should find in terms of the Lipschitz constants of b, σ.

(7 marks)

(iv) Show by induction that, for $n \ge 1$ and $t \le T$,

$$\|F^n(X) - F^n(Y)\|_t^2 \le C^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \|X - Y\|_{t_n}^2$$

where F^n denotes the n-fold composition. The volume of $\{t \ge t_1 > t_2 > \cdots > t_n \ge 0\}$ is $t^n/n!$. Use this to show that, for sufficiently large n, F^n is a contraction on $\mathcal{C}_T$.

(3 marks)

(v) Hence conclude that, for any T, there exists a unique solution to (7) up to time T. Is the solution strong? (2 marks).

(Total: 20 marks)

Solution

(a) Lipschitz continuity means that there exist finite constants C_b, C_σ such that, for all $x, y \in \mathbb{R}$, $|b(x) - b(y)| \le C_b|x - y|$, and similarly for σ. (1 Mark). For X to be a solution means that X is adapted to a common filtration with B, that $\mathbb{E}\int_0^T |b(X_s)|ds < \infty$, and that $\mathbb{E}\int_0^T |\sigma(X_s)|^2 ds < \infty$ for all T, and that, almost surely, for all t,

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s.$$

(2 Marks).

(i) Take the map

$$F(Y)_t := x + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$$

(1 Mark).

(ii) Itō's isometry asserts that, for any $h \in \mathcal{V}(0, T)$, $I(h)_t := \int_0^t h_s dB_s$ is a martingale, and $\mathbb{E}I(h)_T^2 = \mathbb{E}\int_0^T h_s^2 ds$. Doob's L^2 inequality states that, for a square integrable martingale and any time t,

$$\mathbb{E}\left[\sup_{s\le t}|M_s|^2\right] \le 4\mathbb{E}M_t^2.$$

(2 marks). Applying Doob's L^2 inequality to $I(h)$ yields

$$\|I(h)\|_t^2 := \mathbb{E}\left[\sup_{s\le t} I(h)_s^2\right] \le 4\mathbb{E}\left[I(h)_t^2\right] = 4\mathbb{E}\int_0^t h_s^2 ds$$

where the first inequality is Doob, and the equality is Itō. (1 mark). The last expression is equal to $4\int_0^t \mathbb{E}[h_s^2]ds$, which is at most $4\int_0^t \|h\|_s^2 ds$ by definition of $\|\cdot\|_s$. (1 mark).

(iii) We start from

$$(F(X) - F(Y))_t = \int_0^t (b(X_s) - b(Y_s))ds + \int_0^t (\sigma(X_s) - \sigma(Y_s))dW_s$$

so using the inequality in the hint,

$$\|F(Y) - F(X)\|_t^2 \le 2\mathbb{E}\left[\sup_{s\le t}\left|\int_0^s (b(X_u) - b(Y_u))du\right|^2 + \sup_{s\le t}\left|\int_0^s (\sigma(X_u) - \sigma(Y_u))dB_u\right|^2\right]$$

(2 marks). In the first term, using Cauchy-Schwarz,

$$\sup_{s\le t}\left|\int_0^s (b(X_u) - b(Y_u))du\right|^2 \le T\int_0^t |b(X_u) - b(Y_u)|^2 du \le TC_b^2\int_0^t |X_u - Y_u|^2 du$$

so the expectation of the first term is

$$2\mathbb{E}\left[\sup_{s\le t}\left|\int_0^s (b(X_u) - b(Y_u))du\right|^2\right] \le 2TC_b^2\int_0^t \mathbb{E}|X_u - Y_u|^2 du \le 2TC_b^2\int_0^t \|X - Y\|_u^2 du$$

(2 marks). In the second term, we apply (ii) to get

$$\|I(\sigma(X) - \sigma(Y))\|_t^2 \le 4\int_0^t \|\sigma(X) - \sigma(Y)\|_s^2 ds \le 4C_\sigma^2\int_0^t \|X - Y\|_s^2 ds$$

(2 marks) so all together we get

$$\|F(X) - F(Y)\|_t^2 \le (4TC_b^2 + 8C_\sigma^2)\int_0^t \|X - Y\|_s^2$$

which is a bound of the desired form (1 mark).

(iv) We proceed by induction. $n = 0$ is trivial. Given case n, plugging (iii) into the final integrand yields the case $n+1$ (1 mark). As a result, F^n satisfies

$$\|F^n(X) - F^n(Y)\|_T^2 \leq C^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \|X - Y\|_T^2 \leq \frac{C^n T^n}{n!} \|X - Y\|_T^2$$

using the volume estimate in the hint (1 mark). For n large enough, $C^n T^n / n! < 1$, because the whole sequence is summable. For such n, the estimate makes F^n a contraction (1 mark).

(v) For sufficiently large n, there is a unique fixed point of F^n, say X. Any fixed point Y of F is fixed for F^n, so there is at most X. On the other hand $X, F(X)$ are both fixed for F^n, so $F(X) = X$ is fixed for F. Hence X is a fixed point for F, so solves the SDE by (i). It is strong because all functions in $\mathcal{C}_T$ are adapted to the filtration of B.

MATH70054 Introduction to Stochastic Differential Equations and Diffusion Processes Markers Comments

Question 1	No Comments
Question 2	No comments
Question 3	(a) was done very well. (b)(i) and (ii) were bookwork and most students got full marks. (b)(iii) (c) were significantly more difficult, and only very few students scored high marks.
Question 4	(a) was done very well (b) average results overall, and there was a lot of confusion regarding (iii) (c) was difficult, and only few students got this completely right.
Question 5	No Comments