Applied Complex Analysis - Lecture Three

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January 2025

Some admin:

- I have added *admin* notes to the Blackboard page
- Preview of MyBinder

Summary

- Last time, we will reviewed complex *functions*, analytic functions
- We started on contour integrals of not-necessarily analytic functions...
- We will see why contour integrals of analytic functions are particularly special today.

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The ML principle

For a piecewise differentiable curve γ in the complex plane and a complex function $f(z)$, we have

$$\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma}\{|f(z)|\} \times \mathrm{length}(\gamma).$$

- Proof uses definition of length of a contour.
- Example: $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz$ for $f(z) = (z+i)^{-3/2}$.
- Example: $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz$ for $f(z) = e^{iz}(z+i)^{-1/2}$.

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Jordan's Lemma

For a function of the form $f(z) = e^{iaz}g(z)$, defined on a contour $\gamma_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, such that $g(z) \leq M_R$ for $z \in \gamma_R$, $a > 0$, we have

$$\left| \int_{\gamma_R} f(z)dz \right| \leq \frac{\pi}{a} M_R.$$

- Proof is a little more involved than ML.
- When applicable, this is a stronger result than the ML principle.
- Revisit previous example: $\lim_{R\to\infty} \int_{\gamma_R} f(z)dz$ for $f(z) = e^{iz}(z+i)^{-1/2}$.

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Cauchy's Integral Theorem

Theorem:
Suppose f is analytic/holomorphic in some simply connected (no holes) open $D \subset \mathbb{C}$. For a closed contour $\gamma \subset D$, we have

$$\oint_\gamma f(z)dz = 0$$

(Proof omitted.)

Corollary:
If γ_1 and γ_2 have the same endpoints, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

- Proof of corollary
- Contour integrals of analytic functions depend only on the endpoints!
- For integrals defined over some contour γ_1, we can move the contour to another contour γ_2 which is more convenient...

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Deformation theorem

If $f(z)$ is analytic in a region D bounded by γ_1 and γ_2, with γ_2 lying completely inside γ_1, then

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- Picture
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Cauchy's Integral Formula

Let f be analytic inside and on a closed path γ bounding a simply-connected region $\mathcal{D}$. Then, at any point z interior to γ

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

- Proof
- More magic - boundary values determine all interior values!
- Distinct from *real analytic* functions, e.g. consider compactly supported bump function in $2D$.

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Derivatives via Cauchy's Integral formula

Let $f(z)$ be analytic inside and on a closed anti-clockwise path γ bounding a simply-connected region D. Then for any z within D:

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

- Proof
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Some more theorems

The maximum modulus principle states that a function f, analytic in $D \subset \mathbb{C}$, takes its maximal absolute value $|f(z)|$ on the boundary of D.

- Implies that stationary points are saddle points
- Let's just look at some plots to convince ourselves...

$\Longrightarrow$ *Louville's Theorem*: If a function is entire and bounded everywhere in $\mathbb{C}$, then it must be constant.

$\Longrightarrow$ *The fundamental Theorem of algebra:* Every non-constant polynomial must have a root in the complex plane.

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Taylor series

Suppose $f(z)$ is analytic in $|z - z_0| \le R$, for some point z_0 and $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

- Always converges for complex analytic functions - in contrast to real analytic functions, e.g. $f(z) = e^{-1/x^2}$.
- Proof
- By combining with earlier results, this implies exponential convergence of Taylor polynomials in a smaller disk with $r < R$.

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Laurent series

Suppose $f(z)$ is analytic in the annular region $r < |z - z_0| < R$, then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
$$= \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a **Laurent series** for $f(z)$ about z_0.

- Proof
- We see exponential convergence of Laurent polynomials, for similar reasons to the Taylor case.
- We have touched on the concept of *rational approximation*, a current hot topic in approximation theory.