(2.5.3) Thm. (Model Existence Thm.)

Suppose $\mathcal{L}$ is a countable 1st order language and Σ is a consistent set of closed $\mathcal{L}$-fmlas.

Then there is a countable $\mathcal{L}$-str. A such that $A \models \Sigma$

(ie. $A \models \sigma$ for every $\sigma \in \Sigma$).

Proof: (Sketch; more details in notes on BB).

Step 1   Let $b_0, b_1, b_2, \ldots$ be new constant symbols. Form $\mathcal{L}^+$ by adding these to $\mathcal{L}$.

Regard Σ as a set of closed $\mathcal{L}^+$-formulas.

Check Σ is consistent

(as a set of $\mathcal{L}^+$-formulas).

— See notes.

Note: $\mathcal{L}^+$ is still a countable language.

Step 2   (Adding witnesses)

Lemma. There is a consistent set $\Sigma_\infty \supseteq \Sigma$ of closed $\mathcal{L}^+$-fmlas. such that for every $\mathcal{L}^+$-fmla $\theta(x_i)$ with one free variable there is some b_j with

$$\Sigma_\infty \vdash_{K_{\mathcal{L}^+}} \left(\left(\neg (\forall x_i) \theta(x_i) \right) \to \left(\neg \theta(b_j) \right) \right)$$

Pf: See notes.   #

Why do this?

think of $\theta(x_i)$ as $\neg \chi(x_i)$

the formula is then

$$\left((\exists x_i) \chi(x_i) \rightarrow \chi(b_j) \right)$$

So the b_j here "witnesses"
that ' there exists ... '.

Step 3. By the Lindenbaum
Lemma (2.5.2) there is
consistent set $\Sigma_i^* \supseteq \Sigma_{j,\infty} \supseteq \Sigma$
of closed $\mathcal{L}^+$-fmlas which
is complete .

Step 4. Let

$$A = \{ \bar{t} : t \text{ is a closed term} \\ \text{of } \mathcal{L}^+ \}$$

Note: ① A term is closed if it only
involves function and constant symbols
in $\mathcal{L}^+$ (no variables)

② Use $\bar{\ }$ to distinguish when
a term t is being thought of as
an elt. of A.

③ As $\mathcal{L}^+$ is countable, A is countable

$\neq$ Make A into an $\mathcal{L}^+$-str. $\mathcal{A}$

(1) Each constant symbol c of $\mathcal{L}^+$
is interpreted as $\bar{c} \in A$.

(2) Suppose f is a m-ary function symbol. Interpret f as $\bar{f} : A^m \to A$ where

$$\bar{f}(\bar{t_1}, \ldots, \bar{t_m}) = \overline{f(t_1, \ldots, t_m)}$$

(where $t_1, \ldots, t_m$ are closed terms).

(3) Suppose R is an n-ary relation symbol. Interpret R as $\bar{R} \subseteq A^n$ where

$$\bar{R}(\bar{t_1}, \ldots, \bar{t_n}) \text{ this holds}$$

iff $\quad \Sigma_i^* \vdash \underbrace{R(t_1, \ldots, t_n)}_{\text{closed } \mathcal{L}^+\text{-fmla}}$

(for $t_1, \ldots, t_n$ closed terms).

— Call this structure A.

Note: If v is a valuation in A and t a closed $\mathcal{L}^+$-term then $v[t] = \bar{t} \in A$. (by (1) + (2)). ③

Main Lemma. For every closed $\mathcal{L}^+$-fmla ϕ

$$\Sigma^* \vdash \phi \iff A \models \phi \qquad \text{④}$$

[It then follows that $A \models \Sigma$, as $\Sigma \subseteq \Sigma^*$, so the pf. of 2.5.3 is finished; we regard A as an $\mathcal{L}$-str.].

Pf: By induction on length of ϕ.

Base Case: ϕ atomic formula. By (3) of defn.

Inductive step: Assume ⊛

hold for shorter closed $\mathcal{L}^+$-fmlas.

Case 1 ϕ is $(\neg\psi)$

Case 2 ϕ is $(\psi \to \chi)$

Case 3 ϕ is $(\forall x_i)\psi$.

In Cases 1 & 2 ψ, χ are closed. Use ⊛ for these.

Ex:

Case 3 ϕ is $(\forall x_i)\psi$.

Case (3a) x_i not free in ψ so ψ is closed & ⊛ applies. // Ex.

Case 3b x_i is free in ψ ④

$\Rightarrow$: See notes.

$\Leftarrow$: Suppose for a contradiction that $A \vDash \phi$ & $\Sigma^* \nvdash \phi$.

$(\phi$ is $(\forall x_i)\psi)$.

By Step 3 $\Sigma^* \vdash (\neg\phi)$

By Step 2

$$\Sigma^* \vdash \left(\left(\underbrace{\neg(\forall x_i)\psi(x_i)}_{\phi}\right) \to (\neg\psi(b_j))\right)$$

for some constant symbol b_j .

So by MP $\Sigma^* \vdash (\neg\psi(b_j))$

⎵ induction

By ⊛ & Case 1 $A \vDash (\neg\psi(b_j))$ ⎴ closed

⑤   this contradicts

$$A \models (\forall x_i) \, \psi(x_i)$$

⌐ The term b_j is from free
for x_i in $\psi(x_i)$, so
by 2.3.7 :

$$A \models \left((\forall x_i) \, \psi(x_i) \rightarrow \psi(b_j) \right) \quad \rfloor .$$

\# Sketch.