3. Set Theory.

(3.0) Review of basic, informal set theory.

(0) Extensionality Sets A, B are equal $(A = B)$ iff $(\forall x)((x \in A) \leftrightarrow (x \in B))$.

(1) Natural numbers.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

think of 0 as $\emptyset$

1 as $\{0\}$

$2 = \{0, 1\}$

$\vdots$

$n+1 = \{0, 1, \dots, n\}$.

Note: For $n, m \in \mathbb{N}$

$m < n \iff m \in n$

$\iff m \subset n$.

(2) Power set If A is any set the power set of A, $\wp(A)$ is the set of subsets of A.

(3) Ordered pair the ordered pair (x, y) is the set $\{\{x\}, \{x, y\}\}$

Ex: For any x, y, z, w

$(x, y) = (w, z)$

$\iff x = w$ & $y = z$.

If A, B are sets then

$A \times B = \{(a, b) : a \in A, b \in B\}$.

Let $A^2 = A \times A$, $A^3 = A^2 \times A \dots$

$A^{n+1} = A^n \times A, \dots \quad A^0 = \{\emptyset\}$

The set of finite sequences of elts of A is $\bigsqcup_{n \in \mathbb{N}} A^n$.

(4) Functions Think of a function $f : A \to B$ as a subset of $A \times B$.

$A = \mathrm{dom}(f)$ (domain)

$B = \mathrm{ran}(f)$ (range)

If $X \subseteq A$ then

$$f[X] = \{ f(a) : a \in X \} \subseteq B.$$

Set of functions from A to B denoted by B^A $(\subseteq \mathcal{P}(A \times B))$

(3.1) Cardinality

(3.1.1) Def. Set A, B are equinumerous (or have the same cardinality) if there is a bijection $f : A \to B$.

Write $A \approx B$ or $|A| = |B|$.

(3.1.2) Def. A set A is finite if it is equinumerous with some elt. of $\mathbb{N}$. A set is countably infinite if it is equinumerous with $\mathbb{N}$.

Countable: Finite or countably infinite.

(3.1.3) Basic facts

(i) Every subset of a countable set is countable

(ii) A set A is countable iff there is an injective fn. $f : A \to \mathbb{N}$.

(iii) If A, B are countable then so is $A \times B$.

(iv) If $A_0, A_1, \ldots$ are countable sets then
$$\bigcup_{n \in \mathbb{N}} A_n \text{ is countable.}$$
(pf. uses Axiom of Choice).

Ex: $\mathbb{R}$ is not countable
(Cantor's diagonal argument.)

(3.1.4) **Thm.** (G. Cantor)
If X is any set then there is no surjective function $f : X \to \mathcal{P}(X)$
(in particular, $X \not\approx \mathcal{P}(X)$).

Pf: Suppose there is such a fn. f.
Let $Y = \{ y \in X : y \notin f(y) \} \subseteq X$.
As f is surjective, there is $z \in X$ with $f(z) = Y$.

If $z \in Y$ then
$z \notin f(z) = Y$ $\lightning$.

So $z \notin Y$. Thus $z \notin f(z)$, which means $z \in Y$. $\lightning$.

So f cannot be surjective.

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3.1.5 Def. For sets A, B write $|A| \leq |B|$ if there is an *injective* function $f : A \to B$.

[So A is equinumerous $f[A] \subseteq B$]

Ex: If $|A| \leq |B| \leq |C|$ then $|A| \leq |C|$.

Note: $|X| \leq |\mathcal{P}(X)|$

using $x \longmapsto \{x\}$.

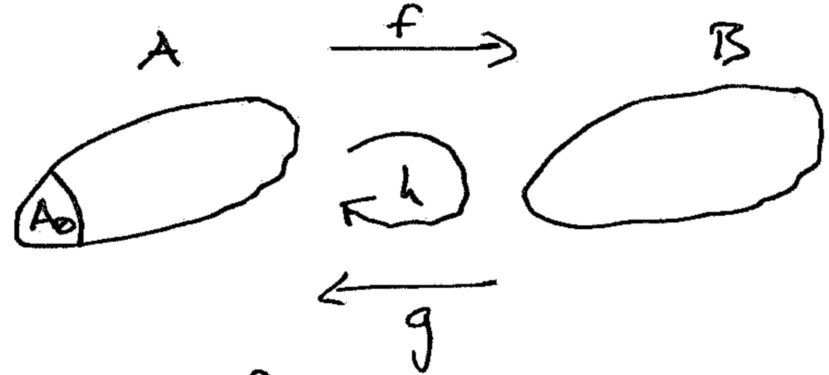
As $|X| \neq |\mathcal{P}(X)|$

we obtain $|X| < |\mathcal{P}(X)|$

// Does $|A| \leq |B|$ & $|B| \leq |A|$ imply $|A| = |B|$?

3.1.6 Thm. (Cantor-Schröder-Bernstein theorem). Suppose A, B are sets and $f : A \to B$, $g : B \to A$ are injective functions. Then there is a bijection $k : A \to B$.

Proof:



Let $h = g \circ f : A \to A$

Let $A_0 = A \setminus g[B]$

For $n > 0$ let $A_n = h[A_{n-1}]$

let $A^* = \bigcup_{n \in \mathbb{N}} A_n$.

and $B^* = f[A^*]$.

Note: $h[A^*] \subseteq A^*$

(as $h[A_{n-1}] = A_n \subseteq A^*$)

So $g[B^*] = g[f[A^*]]$

$= h[A^*] \subseteq A^*$.

Claim: $g[B \setminus B^*] = A \setminus A^*$.

Once we have this:

f gives a bijection $A^* \to B^*$

(as $f[A^*] = B^*$)

and

g gives a bijection $B \setminus B^* \to A \setminus A^*$

Then define $k : A \to B$ by

$$k(a) = \begin{cases} f(a) & \text{if } a \in A^* \\ g^{-1}(a) & \text{if } a \in A \setminus A^* \end{cases}$$

– This is a bijection!

Pf of Claim: $\supseteq$: Let $a \in A \setminus A^*$.

So $a \notin A_0 = A \setminus g[B]$. Thus there is $b \in B$ with $g(b) = a$.

Then $b \notin B^*$ as:

$b \in B^* \Rightarrow b \in f[A^*]$

$\Rightarrow g(b) \in g[f[A^*]]$

$= h[A^*] \subseteq A^*$.

Contradicts $a \notin A^*$.

So $a \in g[B \setminus B^*]$. //.

$\subseteq$: Let $b \in B$ & suppose $g(b) \in A^*$. Show $b \in B^*$.

$g(b) \notin A_0 = A \setminus g[B]$.

There is $n > 0$ with $g(b) \in A_n$

$A_n = h[A_{n-1}]$ so

$g(b) = h(a)$, some $a \in A^*$

$g(b) = g(f(a))$, so as

g is injective

$b = f(a)$, some $a \in A^*$.

thus $b \in f[A^*] = B^*$.

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