Imperial College London

MATH70051

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Vortex Dynamics

Date: 25 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. The streamfunction $\psi(x, y)$ associated with a two-dimensional, incompressible flow in the (x, y) plane is given by

$$\psi(x, y) = \frac{y^2}{2} - \log((x^2 + y^2)^2 - 2(x^2 - y^2) + 1).$$

(a) Find the circulation around the circle centred at the origin and of radius $1/2$.

(6 marks)

(b) Find the circulation around the circle centred at the origin and of radius 2.

(2 marks)

(c) Let $\omega(x, y)$ denotes the vorticity field associated with this two-dimensional flow. Calculate

$$\int\int_S \omega(x, y) dx dy,$$

where S is the square region $0 \leq x \leq 2, -1 \leq y \leq 1$.

(2 marks)

(d) Show that the flow has 3 stagnation points on the y-axis and find their locations.

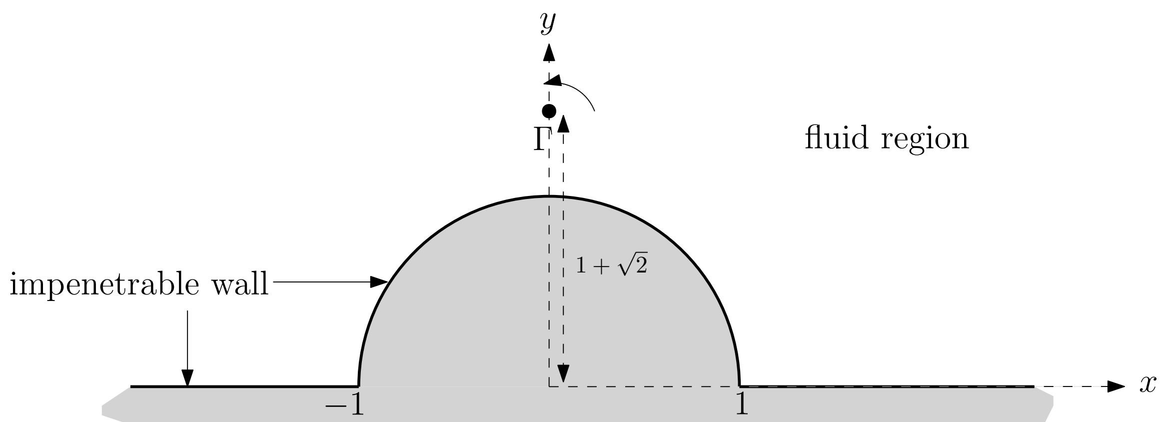
(6 marks)

(e) At some instant, a point vortex of unit circulation is added at the point $(2, 0)$ in the flow associated with the streamfunction $\psi(x, y)$ given above. Find the velocity of this vortex assuming it is free of any external forces.

(4 marks)

(Total: 20 marks)

2. At time $t = 0$ a point vortex of circulation Γ is situated at $(0, 1+\sqrt{2})$ in a region of otherwise irrotational flow occupying the upper half (x, y) plane above an impenetrable wall along the x axis with a semicircular hump of unit radius and centred at the origin as shown in the Figure:



(a) Find the image of the right-half η plane under the conformal mapping

$$\eta + (\eta^2 + 1)^{1/2}.$$

Hence find the conformal mapping, $z = f(\zeta)$, from a unit disc $|\zeta| < 1$ in a parametric complex ζ plane to the fluid region shown in the figure.

(6 marks)

(b) Find the fluid velocity induced by the point vortex at the top of the hump, i.e. at $(0, 1)$.

(7 marks)

(c) Show that if the position of the point vortex at later times $t > 0$ is $z_a(t) = f(a(t))$ then $a(t)$ is a solution of the nonlinear equation

$$\left| \frac{1}{(1 + a(t))^{5/2}} \left(\sqrt{2}(a(t) - 1) - 2(1 + a(t)^2)^{1/2} \right) \right| = \frac{2 - \sqrt{2}}{1 - |a(t)|^2}.$$

(7 marks)

Hint: You may use the following relations between Hamiltonians for N point vortex motion in a domain in a z-plane and a corresponding domain in a ζ-plane:

$$H^{(z)}(\{z_j(t)\}) = H^{(\zeta)}(\{\zeta_j(t)\}) + \sum_{j=1}^{N} \frac{\Gamma_j^2}{4\pi} \log |f'(\zeta_j)|, \qquad f'(\zeta) = \frac{df}{d\zeta}$$

$$H^{(\zeta)}(\{\alpha_j\}) = \mathrm{Im}\left[\sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} \frac{\Gamma_j \Gamma_k}{2} G_0(\alpha_j, \alpha_k) + \sum_{j=1}^{N} \frac{\Gamma_j^2}{2} g(\alpha_j, \alpha_j) \right],$$

$$G_0(\zeta, \alpha) = -\frac{\mathrm{i}}{2\pi} \log(\zeta - \alpha) + g(\zeta; \alpha),$$

where $z = f(\zeta)$ is a conformal mapping between the two domains and $G_0(\zeta, \alpha)$ is the complex potential for a single point vortex of unit circulation at $\zeta = \alpha$ in the unit ζ disc.

(Total: 20 marks)

3. A solid elliptical particle is held in place at the origin in a simple shear flow in an (x, y) plane with its principal axes, of length a and $b < a$, aligned with the x and y axes as indicated in the figure. Let $\psi(x, y)$ denote the streamfunction associated with the incompressible flow $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$ with $\mathbf{u} \to (y, 0)$ as $|\mathbf{x}| \to \infty$. Its boundary is impenetrable to the fluid. The vorticity everywhere in the fluid is constant.

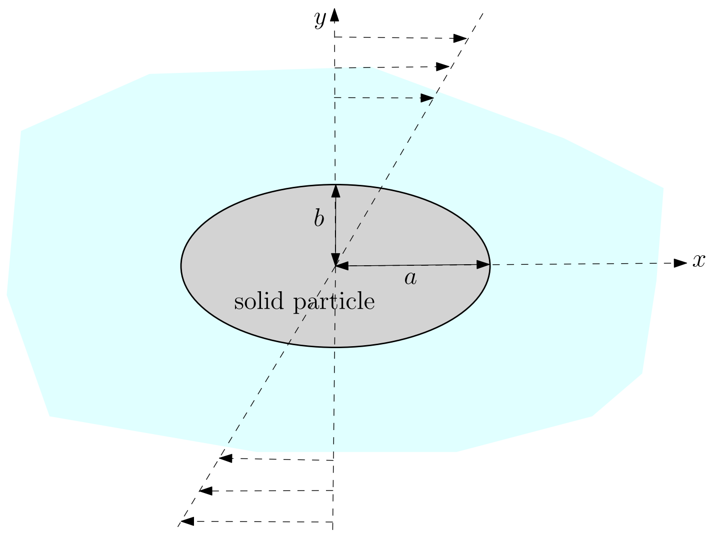


Figure 1: Elliptical particle in an incompressible simple shear.

(a) Find the value of the vorticity in the fluid.

(2 marks)

(b) Introduce the change of variable $(x, y) \mapsto (z, \overline{z})$ where $z = x + \mathrm{i}y$. Show that $\Psi(z, \overline{z}) = \psi(x, y)$ can be written as

$$\Psi(z, \overline{z}) = -\frac{1}{8}\left(z^2 + \overline{z}^2 - 2z\overline{z}\right) + \mathrm{Im}[w(z)],$$

where $w(z)$ is an analytic function of z in the fluid region and is bounded as $|z| \to \infty$.

(4 marks)

(c) Write down the condition that $w(z)$ must satisfy on the particle boundary. (2 marks)

(d) Find a conformal mapping from the unit disc in a complex ζ plane to the fluid region exterior to the particle.

(2 marks)

(e) Find the inverse function of the conformal mapping function in part (d). (2 marks)

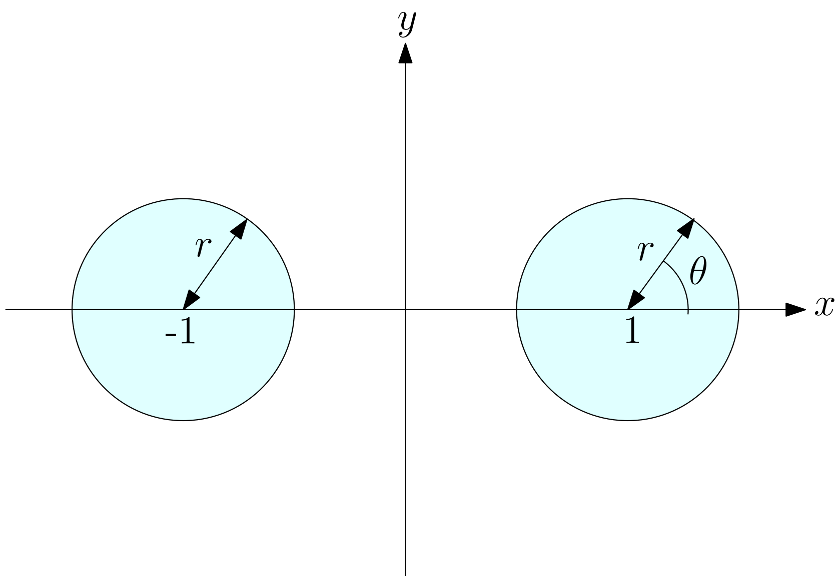
(f) Use your answers to parts (d) and (e), or otherwise, to show that

$$w(z) = \frac{\mathrm{i}b^2 z(z - \sqrt{z^2 - (a^2 - b^2)})}{2(a - b)^2}$$

is a solution. (8 marks)

(Total: 20 marks)

4. Two circular vortex patches, each of uniform vorticity ω_0 and of radius r where $0 < r < 1$, are centred at $(\pm 1, 0)$ in an unbounded region of incompressible fluid in an (x, y) plane as shown in the Figure. The flow outside the two vortex patches is irrotational.



(a) Let $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$. Find an expression for $\overline{z}$ as a function of z on the boundary of each vortex patch. (2 marks)

(b) Assuming that the fluid velocity decays as $|z| \to \infty$, find the velocity field everywhere in the fluid. (5 marks)

(c) Show that the normal fluid velocity, into the irrotational fluid region, on the boundary of the patch centred at $(1, 0)$ is

$$\frac{\omega_0 r^2 \sin\theta}{4 + r^2 + 4r\cos\theta},$$

where θ is the angle shown in the figure. (8 marks)

(d) Assume now that there is a straining flow as $|z| \to \infty$, so that

$$(u, v) \to (x, -y) \qquad \text{as } |z| \to \infty.$$

Find the velocity field everywhere in the fluid. (5 marks)

(Total: 20 marks)

5. Let $0 < \rho < 1$ be a real parameter. Consider the conformal mapping from the annulus $\rho < |\zeta| < 1$ given by

$$z = Z(\zeta) = R\frac{P(\zeta,\rho)}{P(-\zeta,\rho)},$$

where R is another real parameter and

$$P(\zeta,\rho) \equiv (1-\zeta)\hat{P}(\zeta,\rho) \qquad \text{with} \qquad \hat{P}(\zeta,\rho) \equiv \prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}/\zeta).$$

You may assume that this infinite product is convergent. In addition, define the function

$$K(\zeta,\rho) \equiv \frac{\zeta}{P(\zeta,\rho)}\frac{\partial P}{\partial \zeta}(\zeta,\rho).$$

(a) Show that $P(\zeta,\rho)$ satisfies the two functional relations

$$P(1/\zeta,\rho) = -(1/\zeta)P(\zeta,\rho), \qquad P(\rho^2\zeta,\rho) = -(1/\zeta)P(\zeta,\rho).$$

(4 marks)

(b) Use part (a) to show that $K(\zeta,\rho)$ satisfies the two functional relations

$$K(1/\zeta,\rho) = 1 - K(\zeta,\rho), \qquad K(\rho^2\zeta,\rho) = K(\zeta,\rho) - 1.$$

(4 marks)

(c) Use part (a) to show that, under the conformal mapping $z = Z(\zeta)$ given above, the boundary circle $|\zeta| = 1$ of the annulus $\rho < |\zeta| < 1$ is transplanted to the imaginary axis of the z plane while the image of the circle $|\zeta| = \rho$ lies on the real z axis.

(4 marks)

(d) In fact, under the mapping $z = Z(\zeta)$, the annulus $\rho < |\zeta| < 1$ is transplanted in a one-to-one fashion to the unbounded region in the right-half z plane exterior to a slit of finite length on the real axis. Given this fact, suppose now that this image domain is a region of incompressible irrotational flow. Use the results of parts (a) and (b) to show that the complex potential $w(z)$ defined by $w(z) = w(Z(\zeta)) = W(\zeta)$ where

$$W(\zeta) = \mathrm{i}K(-\zeta,\rho)$$

corresponds to a solution for uniform flow, with some speed V in the positive y direction, past this impenetrable slit where the y-axis is also an impenetrable boundary and determine V in terms of R and ρ. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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M70051

Vortex dynamics (solutions)

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1. (a) It is easy to see that the first term in the streamfunction, $y^2/2$, is associated with a simple shear having uniform vorticity -1. The easiest way to proceed is to rewrite the logarithmic part of the streamfunction in the variables $(z, \overline{z})$. After some algebra, using $\boxed{\text{sim. seen} \Downarrow}$

$$x = \frac{z + \overline{z}}{2}, \qquad y = \frac{z - \overline{z}}{2\mathrm{i}}$$

this is found to be

$$-\log((z^2 - 1)(\overline{z}^2 - 1)) = -2\mathrm{Re}\left[\log(z^2 - 1)\right] = -2\mathrm{Im}\left[\mathrm{i}\log(z^2 - 1)\right]$$

which is

$$\mathrm{Im}\left[-2\mathrm{i}\log(z - 1) - 2\mathrm{i}\log(z + 1)\right].$$

Comparing with the well-known complex potential for a point vortex of circulation Γ at $z = a$, namely,

$$-\frac{\mathrm{i}\Gamma}{2\pi}\log(z - a)$$

we read off that this part of the streamfunction corresponds to two point vortices, one each at $z = \pm 1$, both of circulation

$$\frac{\Gamma}{2\pi} = 2, \qquad \text{or} \qquad \Gamma = 4\pi.$$

Now Stokes theorem says that if $\omega(x, y)$ is the two-dimensional vorticity field associated with the velocity $\mathbf{u} = (u, v)$ then

$$\int\int_D \omega(x, y) dx dy = \oint_{\partial D} \mathbf{u}.\mathbf{dx}$$

for any bounded simply connected region D with boundary ∂D.
With these observations, the circulation around the circle centred at the origin of radius $1/2$, that is,

$$\oint_{|x|=1/2} \mathbf{u}.\mathbf{dx},$$

is, by Stokes theorem, just the area of the circle times the uniform vorticity value -1 or

$$\boxed{-\frac{\pi}{4}}$$

$\boxed{6, \mathrm{A}}$

(b) This part is similar to part (a), except that the circle is now sufficiently large that, in addition to the uniform vorticity of -1 occupying the area 4π of this larger circle, the circulations associated with the two point vortices is also collected. The associated circulation is therefore $\boxed{\text{sim. seen} \Downarrow}$

$$-4\pi + 4\pi + 4\pi = \boxed{4\pi}$$

$\boxed{2, \mathrm{A}}$

(c) For this part, the uniform vorticity -1 in the square S of area 4 is now added to the circulation $\Gamma = 4\pi$ associated with the point vortex at $z = 1$ (which is inside the square). The required area integral is just $\boxed{\text{sim. seen} \Downarrow}$

$$\boxed{-4 + 4\pi}$$

$\boxed{2, \mathrm{A}}$

(d) The velocity components are given by

sim. seen ⇓

$$u = \frac{\partial \psi}{\partial y} = y - \frac{4y(x^2+y^2)+4y}{(x^2+y^2)^2 - 2(x^2-y^2)+1}$$

and

$$v = -\frac{\partial \psi}{\partial x} = \frac{(4x(x^2+y^2)-4x)}{(x^2+y^2)^2 - 2(x^2-y^2)+1}.$$

It is clear that $v = 0$ on $x = 0$. For a stagnation point, we need to ensure $u = 0$ too. This requires

$$u = y\left(1 - \frac{4y^2+4}{y^4+2y^2+1}\right) = 0$$

which is satisfied if $y = 0$ or if

$$1 - \frac{4y^2+4}{y^4+2y^2+1} = 1 - \frac{4}{y^2+1} = 0, \qquad \text{or} \qquad y = \pm\sqrt{3}.$$

There are therefore 3 stagnation points at

$$(0,0),\ (0,\pm\sqrt{3}).$$

6, B

(e) If a point vortex is added instantaneously at $(2,0)$ then, if it is free of external force, it will move with the velocity associated with ψ at that point, namely,

sim. seen ⇓

$$u = 0, \qquad v = \left.\frac{(4x^3-4x)}{x^4-2x^2+1}\right|_{x=2} = \frac{24}{9}.$$

4, A

2. (a) The question gives the conformal mapping seen $\Downarrow$

$$z = \eta + (\eta^2 + 1)^{1/2}.$$

The image, in a complex z plane, of the right half plane $\mathrm{Re}[\eta] > 0$ of a complex η plane under this mapping is the region in a right half z plane exterior to a semicircular hump of unit radius centred at the origin and projecting into the right half plane. To see this, let $\eta = \mathrm{i}r$ then

$$z = \mathrm{i}r + (1 - r^2)^{1/2}.$$

If $r \leq 1$,

$$z = x + \mathrm{i}y = (1 - r^2)^{1/2} + \mathrm{i}r$$

so that

$$x = (1 - r^2)^{1/2}, \qquad y = r$$

and

$$x^2 + y^2 = 1.$$

Notice also that, with the positive branch of the square root chosen, $\eta = 0$ maps to $z = 1$ which is on the unit circle $|z| = 1$ in the right half plane. If $r > 1$ then

$$z = x + \mathrm{i}y = (1 - r^2)^{1/2} + \mathrm{i}r = \mathrm{i}\left(r + \sqrt{r^2 - 1}\right)$$

which lies on the positive y axis with $y > 1$. Letting $\eta = -\mathrm{i}r$ for $r > 1$ similarly lies on the negative y axis with $y < -1$. Finally, with the positive square root taken, as $|\eta| \to \infty$, we have $z \sim 2\eta$ so the right half η plane maps to the right half z plane. (Strictly speaking, we should check the mapping is a 1-1 map, but this is not expected).

Then, to find the required conformal mapping $f(\zeta)$ consider the following sequence of maps:

$$\zeta \mapsto \eta = \frac{1 - \zeta}{1 + \zeta} \mapsto \chi = \eta + (\eta^2 + 1)^{1/2} \mapsto z = \mathrm{i}\chi$$

leading to

$$z = f(\zeta) = \mathrm{i}\left[\frac{1 - \zeta + \sqrt{2(1 + \zeta^2)}}{1 + \zeta}\right].$$

The first mapping is the familiar Cayley mapping taking the disc to the right half η plane; the second mapping takes the right half plane to the right half plane exterior to the semicircular hump (as described above), the final mapping simply rotates this configuration by $\pi/2$. 6, A

(b) From the lectures it is known that the streamfunction is given by sim. seen $\Downarrow$

$$\psi = \mathrm{Im}\left[G_0(\zeta; a)\right],$$

where

$$G_0(\zeta; a) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{(\zeta - a)}{|a|(\zeta - 1/\bar{a})}\right).$$

The function $G_0(\zeta; a)$ is the complex potential for a point vortex of circulation Γ in a flow in the unit ζ disc. By the conformal invariance of the boundary value problem for the flow generated by the point vortex. This means the complex potential

$w(z; z_a)$ for the flow given is $w(z; z_a) = G_0(\zeta; a)$ where $z = f(\zeta)$. Hence, by the chain rule,

$$u - \mathrm{i}v = \frac{dw}{dz} = \frac{dG_0(\zeta; a)/d\zeta}{df/d\zeta}.$$

But

$$\frac{dG_0(\zeta; a)}{d\zeta} = -\frac{\mathrm{i}\Gamma}{2\pi}\left(\frac{1}{\zeta - a} - \frac{1}{\zeta - 1/\overline{a})}\right)$$

and

$$\begin{aligned} f'(\zeta) &= \frac{\mathrm{i}}{1+\zeta}\left(-1 + \frac{\sqrt{2}\zeta}{(1+\zeta^2)^{1/2}}\right) - \frac{\mathrm{i}}{(1+\zeta)^2}\left(1 - \zeta + \sqrt{2(1+\zeta^2)}\right) \\ &= \frac{\mathrm{i}}{(1+\zeta)^2}\left(\frac{\sqrt{2}(\zeta - 1)}{(1+\zeta^2)^{1/2}} - 2\right) \qquad (1) \\ &= \frac{\mathrm{i}}{(1+\zeta)^{5/2}}\left(\sqrt{2}(\zeta - 1) - 2(1+\zeta^2)^{1/2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} u - \mathrm{i}v &= -\frac{\mathrm{i}\Gamma}{2\pi}\left(\frac{1}{\zeta - a} - \frac{1}{\zeta - 1/\overline{a})}\right)\frac{(1+\zeta)^{5/2}}{\mathrm{i}(\sqrt{2}(\zeta-1) - 2(1+\zeta^2)^{1/2})} \\ &= -\frac{\Gamma}{2\pi}\left(\frac{1}{\zeta - a} - \frac{1}{\zeta - 1/\overline{a})}\right)\frac{(1+\zeta)^{5/2}}{\sqrt{2}(\zeta-1) - 2(1+\zeta^2)^{1/2}}. \end{aligned}$$

If the vertex is initially at $(0, \sqrt{2}+1)$ then, from the form of the conformal mapping, we notice that $z_a(0) = f(0)$, or $a(0) = 0$. Hence the induced velocity field is

$$u - \mathrm{i}v = -\frac{\Gamma}{2\pi\zeta}\frac{(1+\zeta)^{5/2}}{\sqrt{2}(\zeta-1) - 2(1+\zeta^2)^{1/2}}.$$

Again, by inspection of the mapping function, the preimage of the top of the semi-circular hump is $\zeta = 1$. Hence the velocity there, (U, V) say, is given by

$$U - \mathrm{i}V = -\frac{\Gamma}{2\pi\zeta}\frac{(1+\zeta)^{5/2}}{\sqrt{2}(\zeta-1) - 2(1+\zeta^2)^{1/2}}\bigg|_{\zeta=1} = \frac{\Gamma}{\pi}.$$

Hence the required velocity at the top of the hump is

$$(U, V) = (\Gamma/\pi, 0).$$

7, C

(c) By the results of Kirchhoff-Routh theory (given in the hint), since this is a single vortex problem in a simply connected domain, the point vortex trajectories are the contours of the Hamiltonian $H^{(z)}(z_a)$ which are given by

sim. seen ⇓

$$(1 - |a|^2)|f'(a)| = \text{constant}, \qquad (2)$$

where $z_a = f(a)$ is the point vortex position in the z domain. Here we have used the hint, together with the fact that

$$G_0(\zeta; a) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{(\zeta - a)}{|a|(\zeta - 1/\overline{a})}\right) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(\zeta - a) + g(\zeta; a)$$

so that

$$g(\zeta; a) = \frac{\mathrm{i}}{2\pi}\log(|a|(\zeta - 1/\overline{a}))$$

and consequently that, for a single vortex,

$$H^{(\zeta)}(a) = \frac{\Gamma^2}{4\pi}\log\left||a|(a-1/\overline{a})\right| = \frac{\Gamma^2}{4\pi}\log(1-|a|^2).$$

Hence,

$$H^{(z)}(z_a) = H^{(\zeta)}(a) + \frac{\Gamma^2}{4\pi}\log|f'(a)| = \frac{\Gamma^2}{4\pi}\log(1-|a|^2) + \frac{\Gamma^2}{4\pi}\log|f'(a)|.$$

Contours of $H^{(z)}(z_a)$ therefore coincide with (2). This general result is familiar from lectures and coursework. In this case, from (1), the condition (2) becomes

$$(1-|a|^2)\left|\frac{1}{(1+a)^{5/2}}\left(\sqrt{2}(a-1) - 2(1+a^2)^{1/2}\right)\right| = \text{constant}. \qquad (3)$$

If the vortex is initially at $(0, \sqrt{2}+1)$ then $a(0) = 0$. This sets the constant in (3) for the corresponding trajectory to be

$$(1-|a|^2)\left|\frac{1}{(1+a)^{5/2}}\left(\sqrt{2}(a-1) - 2(1+a^2)^{1/2}\right)\right| = 2-\sqrt{2}$$

which, on rearrangement, gives the required result. $\boxed{7, \mathrm{B}}$

3. (a) For a two-dimensional flow (u, v) the vorticity is sim. seen $\Downarrow$

$$\omega(x, y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

From the far-field condition $(u, v) \to (y, 0)$ and the fact that the vorticity everywhere in the fluid is constant we deduce that the vorticity in the fluid is -1. 2, A

(b) Since sim. seen $\Downarrow$

$$u = \frac{\partial \psi}{\partial y} \to y$$

then

$$\psi \to \frac{y^2}{2}, \qquad \text{hence} \qquad \Psi \to \frac{1}{2}\left(\frac{z - \overline{z}}{2\mathrm{i}}\right)^2 = -\frac{1}{8}\left(z^2 + \overline{z}^2 - 2z\overline{z}\right).$$

Since this has constant vorticity -1 any correction to this must be irrotational and, thus, can be written as the imaginary part of some complex potential $w(z)$. Hence, we can write

$$\Psi = -\frac{1}{8}\left(z^2 + \overline{z}^2 - 2z\overline{z}\right) + \mathrm{Im}[w(z)].$$

4, A

(c) The boundary of the particle is impenetrable, hence Ψ must be constant on the particle boundary since it must be a streamline (a line of constant Ψ). Thus, by part (b), the boundary condition on $w(z)$ is sim. seen $\Downarrow$

$$\mathrm{Im}[w(z)] = \frac{1}{8}\left(z^2 + \overline{z}^2 - 2z\overline{z}\right) + \text{constant},$$

on the particle boundary. 2, A

(d) The required conformal mapping is well-known from lectures to be sim. seen $\Downarrow$

$$z = Z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta, \qquad \alpha, \beta \in \mathbb{R},$$

where

$$\alpha + \beta = a, \qquad \alpha - \beta = b.$$

It follows that

$$\alpha = \frac{a + b}{2}, \qquad \beta = \frac{a - b}{2}.$$

2, A

(e) On multiplying the mapping by ζ, it follows that sim. seen $\Downarrow$

$$\beta\zeta^2 - z\zeta + \alpha = 0$$

which is a quadratic equation for ζ whose solution is

$$\zeta = f^{-1}(z) = \frac{z - \sqrt{z^2 - 4\alpha\beta}}{2\beta},$$

where the minus sign is taken because $\zeta \to 0$ as $|z| \to \infty$. Using the relations between (α, β) and (a, b) in part (c), this can also be written as

$$\zeta = f^{-1}(z) = \frac{z - \sqrt{z^2 - (a^2 - b^2)}}{a - b}.$$

2, B

(f) To find $w(z)$, consider $W(\zeta) \equiv w(f(\zeta))$. By part (c), this function must satisfy

sim. seen $\Downarrow$

$$\mathrm{Im}[W(\zeta)] = \mathrm{Im}[w(f(\zeta))] = \frac{1}{8}\left(z^2 + \overline{z}^2 - 2z\overline{z}\right) + \text{constant},$$

on the particle boundary. This can be written in terms of the conformal mapping $f(\zeta)$ as follows:

$$\mathrm{Im}[W(\zeta)] = \frac{1}{8}\left(\left(\frac{\alpha}{\zeta} + \beta\zeta\right)^2 + \left(\alpha\zeta + \frac{\beta}{\zeta}\right)^2 - 2\left(\frac{\alpha}{\zeta} + \beta\zeta\right)\left(\alpha\zeta + \frac{\beta}{\zeta}\right)\right) + \text{constant},$$

where we have used the facts that α and β are real, and the fact that $\overline{\zeta} = 1/\zeta$ on the particle boundary (because the latter is the image of $|\zeta| = 1$). On expanding out the brackets, and after some simple algebra, we find

$$\mathrm{Im}[W(\zeta)] = \frac{(\alpha - \beta)^2}{8}\left(\zeta^2 + \frac{1}{\zeta^2} - 2\right) + \text{constant}, \tag{4}$$

Now $W(\zeta)$ must be analytic in $|\zeta| < 1$ because this corresponds to the fluid region under the conformal mapping and $w(z)$ must be analytic in the fluid region, including as $z \to \infty$ where it tends to a constant (or vanishes). By inspection of the boundary condition (4), which can be written as

$$\frac{W(\zeta)}{2\mathrm{i}} - \frac{\overline{W(\zeta)}}{2\mathrm{i}} = \frac{(\alpha - \beta)^2}{8}\left(\zeta^2 + \frac{1}{\zeta^2} - 2\right) + \text{constant},$$

it is clear that

$$W(\zeta) = \frac{\mathrm{i}(\alpha - \beta)^2}{4}\zeta^2 = \frac{\mathrm{i}b^2}{4}\zeta^2$$

is a possible solution. But from part (e),

$$\begin{aligned}\zeta^2 &= \frac{1}{(a-b)^2}\left(z^2 + z^2 - (a^2 - b^2) - 2z\sqrt{z^2 - (a^2 - b^2)}\right)\\ &= \frac{2}{(a-b)^2}\left(z^2 - z\sqrt{z^2 - (a^2 - b^2)}\right) + \text{constant}.\end{aligned}$$

Hence,

$$w(z) = W(\zeta) = \frac{\mathrm{i}b^2}{2}\frac{1}{(a-b)^2}\left(z^2 - z\sqrt{z^2 - (a^2 - b^2)}\right)$$

is a solution (the constant term does not matter in determining the flow) which tends to zero as $|z| \to \infty$.

8, D

4. (a) The equations of the two boundaries of the vortex patches are $\boxed{\text{sim. seen} \Downarrow}$

$$|z \pm 1|^2 = r^2$$

which can be written as

$$(z \pm 1)(\overline{z} \pm 1) = r^2$$

hence on the patch centred at $z = 1$ ("patch 1" henceforth)

$$\overline{z} = 1 + \frac{r^2}{z-1}$$

while on the patch centred at $z = -1$ ("patch 2" henceforth),

$$\overline{z} = -1 + \frac{r^2}{z+1}.$$

$\boxed{2, \text{A}}$

(b) Using the usual construction, $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial \psi}{\partial z} = \left(\begin{array}{rl} -\frac{\omega_0}{4}\left(\overline{z} - C_i^{(1)}(z)\right), & \text{patch 1} \\ -\frac{\omega_0}{4}\left(\overline{z} - C_i^{(2)}(z)\right), & \text{patch 2} \\ \frac{\omega_0}{4} C_0(z) & \text{outside the patches} \end{array} \right.$$

where $C_0(z)$ is analytic outside both patches and decays like $1/z$ as $|z| \to \infty$ while $C_i^{(k)}(z)$ is analytic inside the k-th patch. Continuity of velocity on the boundary of each patch requires, using part (a), that

$$\overline{z} = 1 + \frac{r^2}{z-1} = C_i^{(1)}(z) - C_0(z) \tag{5}$$

and

$$\overline{z} = -1 + \frac{r^2}{z+1} = C_i^{(2)}(z) - C_0(z), \tag{6}$$

where $C_i^{(j)}(z)$ is analytic in patch j and $C_0(z)$ is analytic outside both patches and decays like $1/z$ as $|z| \to \infty$. By inspection, we see that

$$C_0(z) = -\frac{r^2}{z-1} - \frac{r^2}{z+1}, \qquad C_i^{(1)}(z) = 1 - \frac{r^2}{z+1}, \qquad C_i^{(2)}(z) = -1 - \frac{r^2}{z-1}$$

where it can be checked that these functions satisfy (5)–(6) and have their required analyticity properties. Therefore

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial \psi}{\partial z} = \left(\begin{array}{rl} -\frac{\mathrm{i}\omega_0}{2}\left(\overline{z} - \left(1 - \frac{r^2}{z+1}\right)\right), & \text{patch 1} \\ -\frac{\mathrm{i}\omega_0}{2}\left(\overline{z} - \left(-1 - \frac{r^2}{z-1}\right)\right), & \text{patch 2} \\ \frac{\mathrm{i}\omega_0}{2}\left(-\frac{r^2}{z-1} - \frac{r^2}{z+1}\right) & \text{outside the patches.} \end{array} \right.$$

$\boxed{5, \text{C}}$

(c) The outward complex **unit** normal on patch 1 (into the irrotational fluid region) is $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{(z-1)}{r}$$

hence we are required to calculate

$$\mathbf{u.n} = \mathrm{Re}\left[\frac{(z-1)}{r}(u - \mathrm{i}v)\right],$$

where we have used the fact that $\mathbf{a}.\mathbf{b} = \mathrm{Re}[a\bar{b}]$ where $\mathbf{a} = (a_x, a_y)^T \mapsto a_x + \mathrm{i}a_y$. From part (b), using the velocity expression for the outside of the patches as it tends to the boundary, this is

$$\mathbf{u}.\mathbf{n} = \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{2\pi r}\left[\frac{1}{z-1} + \frac{1}{z+1}\right](z-1)\right], \qquad \Gamma \equiv \omega_0 \pi r^2.$$

This can be rewritten

$$\begin{aligned}
\mathbf{u}.\mathbf{n} &= \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{2\pi r}\left[1 + \frac{z-1}{z+1}\right]\right] \\
&= \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{2\pi r}\frac{z-1}{z+1}\right] \\
&= \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{2\pi r}\frac{re^{\mathrm{i}\theta}}{2+re^{\mathrm{i}\theta}}\right] \\
&= \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{2\pi r}\frac{re^{\mathrm{i}\theta}(2+re^{-\mathrm{i}\theta})}{(2+re^{\mathrm{i}\theta})(2+re^{-\mathrm{i}\theta})}\right] \\
&= \mathrm{Re}\left[-\frac{\mathrm{i}\Gamma}{\pi}\frac{(\cos\theta + \mathrm{i}\sin\theta)}{4+r^2+4r\cos\theta}\right] = \frac{\Gamma}{\pi}\frac{\sin\theta}{4+r^2+4r\cos\theta} = \frac{\omega_0 r^2 \sin\theta}{4+r^2+4r\cos\theta}.
\end{aligned}$$

$\boxed{8, \mathrm{D}}$

(d) The imposed far-field flow is now such that $\boxed{\text{sim. seen} \Downarrow}$

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial \psi}{\partial z} \to z.$$

Note that direct superposition of this irrotational velocity field on that computed in part (b) will not change the vorticity distribution, or disrupt the continuity of velocity. Hence the required velocity field is just

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial\psi}{\partial z} = \left(\begin{array}{cc} -\frac{\mathrm{i}\omega_0}{2}\left(\overline{z} - \left(1 - \frac{r^2}{z+1}\right)\right) + \frac{z}{2\mathrm{i}}, & \text{patch 1} \\ -\frac{\mathrm{i}\omega_0}{2}\left(\overline{z} - \left(-1 - \frac{r^2}{z-1}\right)\right) + \frac{z}{2\mathrm{i}}, & \text{patch 2} \\ \frac{\mathrm{i}\omega_0}{2}\left(-\frac{r^2}{z-1} - \frac{r^2}{z+1}\right) + \frac{z}{2\mathrm{i}} & \text{outside the patches.} \end{array}\right. \tag{7}$$

Alternatively, the same construction used earlier can be adapted. By direct integration of the streamfunction-vorticity relation we find

$$\frac{\partial\psi}{\partial z} = \left(\begin{array}{cc} -\frac{\omega_0}{4}\left(\overline{z} - C_i^{(1)}(z)\right), & \text{patch 1} \\ -\frac{\omega_0}{4}\left(\overline{z} - C_i^{(2)}(z)\right), & \text{patch 2} \\ \frac{z}{2\mathrm{i}} + \frac{\omega_0}{4}C_0(z) & \text{outside the patches} \end{array}\right. \tag{8}$$

where, again, $C_0(z)$ is analytic outside both patches and decays like $1/z$ as $|z| \to \infty$ while $C_i^{(k)}(z)$ is analytic inside the k-th patch. Continuity of velocity on the boundary of each patch requires, using part (a), that

$$-\frac{\omega_0}{4}\left(\overline{z} - C_i^{(1)}(z)\right) = -\frac{\omega_0}{4}\left(1 + \frac{r^2}{z-1} - C_i^{(1)}(z)\right) = \frac{z}{2\mathrm{i}} + \frac{\omega_0}{4}C_0(z) \tag{9}$$

and

$$-\frac{\omega_0}{4}\left(\overline{z} - C_i^{(2)}(z)\right) = -\frac{\omega_0}{4}\left(-1 + \frac{r^2}{z+1} - C_i^{(2)}(z)\right) = \frac{z}{2\mathrm{i}} + \frac{\omega_0}{4}C_0(z), \tag{10}$$

where $C_i^{(j)}(z)$ is analytic in patch j and $C_0(z)$ is analytic outside both patches and decays like $1/z$ as $|z| \to \infty$. By inspection, we see that

$$C_0(z) = -\frac{r^2}{z-1} - \frac{r^2}{z+1}, \qquad C_i^{(1)}(z) = 1 - \frac{r^2}{z+1} + \frac{2z}{\mathrm{i}\omega_0}, \tag{11}$$

and

$$C_i^{(2)}(z) = -1 - \frac{r^2}{z-1} + \frac{2z}{\mathrm{i}\omega_0},$$

where it can be checked that these functions satisfy (9)–(10) and have their required analyticity properties. This gives the same result (7) found earlier. $\boxed{5, \mathrm{B}}$

5. (a) Clearly,

unseen $\Downarrow$

$$P(\zeta^{-1},\rho) = (1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta) = \frac{(\zeta-1)}{\zeta}\frac{P(\zeta,\rho)}{1-\zeta} = -\zeta^{-1}P(\zeta,\rho).$$

2, M

Also

$$\begin{aligned} P(\rho^2\zeta,\rho) &= (1-\rho^2\zeta)\prod_{k=1}^{\infty}(1-\rho^{2(k+1)}\zeta)(1-\rho^{2(k-1)}\zeta^{-1}) \\ &= (1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta) \\ &= -\frac{(1-\zeta)}{\zeta}\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta) \\ &= -\zeta^{-1}P(\zeta,\rho). \end{aligned}$$

2, M

(b) On taking a derivative of the first identity with respect to ζ we find

unseen $\Downarrow$

$$-\frac{1}{\zeta^2}P'(1/\zeta,\rho) = \frac{1}{\zeta^2}P(\zeta,\rho) - \frac{1}{\zeta}P'(\zeta,\rho).$$

On division by the original identity, and multiplication by ζ, we find

$$-\frac{(1/\zeta)P'(1/\zeta,\rho)}{P(1/\zeta,\rho)} = -1 + \frac{\zeta P'(\zeta,\rho)}{P(\zeta,\rho)}$$

which implies

$$K(1/\zeta,\rho) = 1 - K(\zeta,\rho).$$

2, M

The two identities in part (a) imply

$$P(\rho^2\zeta,\rho) = P(1/\zeta,\rho).$$

On taking a derivative of this, we find

$$\rho^2 P'(\rho^2\zeta,\rho) = -\frac{1}{\zeta^2}P'(1/\zeta,\rho).$$

On division by the original identity, and multiplication by ζ, we find

$$K(\rho^2\zeta,\rho) = -K(1/\zeta,\rho)$$

which implies, using the identity just derived, that

$$K(\rho^2\zeta,\rho) = K(\zeta,\rho) - 1.$$

2, M

(c) We need to show the image of the circle $|\zeta| = 1$ lies on the imaginary z axis. For this, for $|\zeta| = 1$, consider

unseen $\Downarrow$

$$\overline{Z(\zeta)} = R\frac{P(\overline{\zeta},\rho)}{P(-\overline{\zeta},\rho)} = R\frac{P(1/\zeta,\rho)}{P(-1/\zeta,\rho)} = R\frac{-(1/\zeta)P(\zeta,\rho)}{(1/\zeta)P(-\zeta,\rho)} = -Z(\zeta),$$

where we have used the fact that $\overline{\zeta} = 1/\zeta$ on this circle and the first identity in (a). This means the image of $|\zeta| = 1$ under the conformal mapping lies on the imaginary axis in the z plane. $\boxed{2,\ \mathrm{M}}$

To show the image of the circle $|\zeta| = \rho$ is the real z axis, on $|\zeta| = \rho$, consider

$$\overline{Z(\zeta)} = R\frac{P(\overline{\zeta},\rho)}{P(-\overline{\zeta},\rho)} = R\frac{P(\rho^2/\zeta,\rho)}{P(-\rho^2/\zeta,\rho)} = R\frac{P(\zeta,\rho)}{P(-\zeta,\rho)} = Z(\zeta),$$

where we have used the fact that $\overline{\zeta} = \rho^2/\zeta$ on this circle and both identities in (a) which imply that $P(\rho^2\zeta,\rho) = P(1/\zeta,\rho)$. This means the image of $|\zeta| = \rho$ under the conformal mapping lies on the real axis in the image plane. $\boxed{2,\ \mathrm{M}}$

(d) To show that $W(\zeta)$ is the complex potential for uniform flow parallel to the y-axis we need to show that the complex potential $w(z)$ defined by $W(\zeta) = w(Z(\zeta))$ is $\boxed{\text{unseen} \Downarrow}$

(i) analytic everywhere in the fluid except for $z = \infty$ where $w(z) \to -\mathrm{i}Vz$ for some real V (to be found);

(ii) that its imaginary part is constant on both the imaginary z axis and on the flat plate lying on the real axis.

First note that by direct calculation, and noticing a logarithmic derivative of a product,

$$K(\zeta,\rho) = \zeta\left[-\frac{1}{1-\zeta} + \sum_{k=1}^{\infty}\left[-\frac{\rho^{2k}}{1-\rho^{2k}\zeta} + \frac{\rho^{2k}}{\zeta^2(1-\rho^{2k}/\zeta)}\right]\right].$$

Hence $K(\zeta,\rho)$ has a simple pole at $\zeta = 1$ but is analytic everywhere else in the annulus. Hence $W(\zeta) = \mathrm{i}K(-\zeta,\rho)$ has a simple pole of residue $-\mathrm{i}$ at $\zeta = -1$, on the boundary of the annulus, but is analytic everywhere inside the annulus; since the interior of the annulus corresponds to the fluid region, then $w(z)$ is analytic inside the fluid region. Now the mapping is

$$z = Z(\zeta) = R\frac{P(\zeta,\rho)}{P(-\zeta,\rho)} = \frac{RP(\zeta,\rho)}{(\zeta+1)\hat{P}(-\zeta,\rho)}$$

which means that near $\zeta = -1$,

$$z = \frac{RP(\zeta,\rho)}{(\zeta+1)\hat{P}(-\zeta,\rho)} = \frac{RP(-1,\rho)}{\hat{P}(1,\rho)}\frac{1}{\zeta+1} + \text{locally analytic part} \tag{12}$$

which also has a simple pole at $\zeta = -1$. Hence, near $\zeta = -1$,

$$\begin{aligned} w(z) = W(\zeta) &= -\frac{\mathrm{i}}{\zeta+1} + \text{locally analytic part} \\ &= -\frac{\mathrm{i}\hat{P}(1,\rho)}{RP(-1,\rho)}z + \text{locally analytic part} \end{aligned}$$

where, in the last equality, we have substituted for $1/(\zeta+1)$ from (12). This means that $w(z)$ corresponds to uniform flow with speed

$$V = \frac{\hat{P}(1,\rho)}{RP(-1,\rho)}.$$

To show (ii), note that on $|\zeta| = 1$, corresponding to the impenetrable wall along the imaginary z axis,

$$\overline{W(\zeta)} = -\mathrm{i}K(-\overline{\zeta},\rho) = -\mathrm{i}K(-1/\zeta,\rho) = -\mathrm{i}(1 - K(-\zeta,\rho)) = W(\zeta) - \mathrm{i}$$

hence on this circle the streamfunction, or the imaginary part of $W(\zeta)$, satisfies

$$\frac{W(\zeta)-\overline{W(\zeta)}}{2\mathrm{i}}=\frac{1}{2}$$

which is constant, showing that the imaginary z axis is a streamline.
On the other hand, on $|\zeta|=\rho$, corresponding to the impenetrable flat plate along the real z axis,

$$\begin{aligned}\overline{W(\zeta)}&=-\mathrm{i}K(-\overline{\zeta},\rho)=-\mathrm{i}K(-\rho^2/\zeta,\rho)=-\mathrm{i}(K(-1/\zeta,\rho)-1)=-\mathrm{i}(1-K(-\zeta,\rho)-1)\\&=W(\zeta),\end{aligned}$$

where we have used both functional relations in part (b). Hence on this circle the streamfunction, or the imaginary part of $W(\zeta)$, satisfies

$$\frac{W(\zeta)-\overline{W(\zeta)}}{2\mathrm{i}}=0$$

which is constant, showing that the flat plate is also a streamline. $\boxed{8,\ \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70051	1	This was reasonably well done. The key here was to rewrite the given real streamfunction in complex variables.
MATH70051	2	A fairly standard exercise in Kirchhoff-Routh theory that was generally well done. The mapping had been seen before in the coursework exercise.
MATH70051	3	This question was a "spin" on the usual vortex patch questions that have appeared in previous exams (the Kirchhoff ellipse). This time the ellipse was a solid particle. The test was to see if students can work out how to adapt the Kirchhoff ellipse analysis to this case.
MATH70051	4	A vortex patch question, requiring use of standard ideas presented in the course, with the twist that there are two patches. Some students got the idea and delivered excellent answers.
MATH70051	5	This question was done quite well actually. It involved mathematical ideas that had appeared in past mastery questions, so well-prepared students (those who had attempted many past papers) fared well here.