Problem Sheet 2 Solutions:

1). $$I(x) = \int_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x}\,dt$$

(a). Introduce the Cauchy transform $C(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-z}\,dt$, defined for complex z off $[-1,1]$.

From the Plemelj formulae we have:

$$I(x) = \pi i\left(C_+(x) + C_-(x)\right),$$

where, as usual, $C_\pm(x)$ denote the limits of $C(z)$ as we approach a point x between -1 and 1 from either above or below respectively. Hence if we can evaluate these limits then we can compute $I(x)$. We do so as follows.

First take $z = x \in \mathbb{R}$ where $x > 1$. Substituting $t = \cos\theta$, we find:

$$C(x) = \frac{1}{2\pi i}\int_0^{\pi} \frac{\cos^3\theta - \cos\theta}{x - \cos\theta}\,d\theta.$$

Next, one may write:

$$\cos^3\theta = (\cos\theta - x)(\cos^2\theta + x\cos\theta + x^2) + x^3,$$

and:

$$\cos\theta = (\cos\theta - x) + x.$$

Hence:

$$\frac{\cos^3\theta - \cos\theta}{x - \cos\theta} = 1 - \cos^2\theta - x\cos\theta - x^2 + \frac{x(x^2-1)}{x - \cos\theta}.$$

Then, using the fact that $\cos\theta$ is an even function of θ; $\int_0^{\pi}\cos\theta\,d\theta = 0$ and $\int_0^{\pi}\cos^2\theta\,d\theta = \frac{\pi}{2}$, one finds that:

$$C(x) = \frac{1}{2i}\left(\frac{1}{2} - x^2\right) + \frac{x(x^2-1)}{4\pi i}\int_{-\pi}^{\pi} \frac{d\theta}{x - \cos\theta}.$$

To evaluate this final integral, introduce $w = e^{i\theta}$. Then integrating with respect to θ from $-\pi$ to π corresponds to integrating with respect to w in an anti-clockwise direction around the unit circle $|w| = 1$, and one finds that

$$\int_{-\pi}^{\pi} \frac{d\theta}{x - \cos\theta} = \oint_{|w|=1} \frac{\left(\frac{-i}{w}\right)}{x - \frac{1}{2}\left(w + \frac{1}{w}\right)} dw = \oint_{|w|=1} \frac{2i}{w^2 - 2xw + 1} dw . \quad (1)$$

The zeros of $w^2 - 2xw + 1$ are at $x \pm \sqrt{x^2 - 1}$. Both are real and positive. Furthermore, their product must be 1. Hence, since they are non-equal, the smaller of the two, namely $x - \sqrt{x^2-1}$, must be contained in the interior of the unit circle $|w| = 1$. Then, by the residue theorem it follows from (1) that

$$\int_{-\pi}^{\pi} \frac{d\theta}{x - \cos\theta} = 2\pi i \left(\frac{2i}{\left(x - \sqrt{x^2-1}\right) - \left(x + \sqrt{x^2-1}\right)} \right) = \frac{2\pi}{\sqrt{x^2-1}} .$$

Hence, one arrives at

$$C(x) = \frac{1}{2i}\left(\frac{1}{2} - x^2 + x\sqrt{x^2-1} \right) , \quad x \in \mathbb{R}, x > 1 .$$

Then, by analytic continuation we have:

$$C(z) = \frac{1}{2i}\left(\frac{1}{2} - z^2 + z\sqrt{z^2-1} \right) \quad (2) , \quad z \in \mathbb{C} .$$

Let us now consider the sum of the limits $C_\pm(x)$ of (2). With a branch cut of $\sqrt{z^2-1}$ along the interval $(-1, 1)$ of the real axis, whichever of the two branches we pick, its values on either side of the cut are minus one another and hence cancel when we add $C_+(x)$ and $C_-(x)$. It follows that for $-1 < x < 1$,

$$C_+(x) + C_-(x) = i\left(x^2 - \frac{1}{2}\right) .$$

Finally, from the Plemelj formulae:

$$I(x) = \frac{\pi}{2} - \pi x^2 , \qquad \text{as required.}$$

3

Take branch of $\sqrt{z^2-1} \sim z$ as $z \to \infty$ rather than the one which $\sim -z$.

(b). Consider $\dfrac{z\sqrt{z^2-1}}{z-x}$ as $z \to \infty$:

$$\frac{z\sqrt{z^2-1}}{z-x} = \frac{z\sqrt{1-\frac{1}{z^2}}}{1-\frac{x}{z}}$$

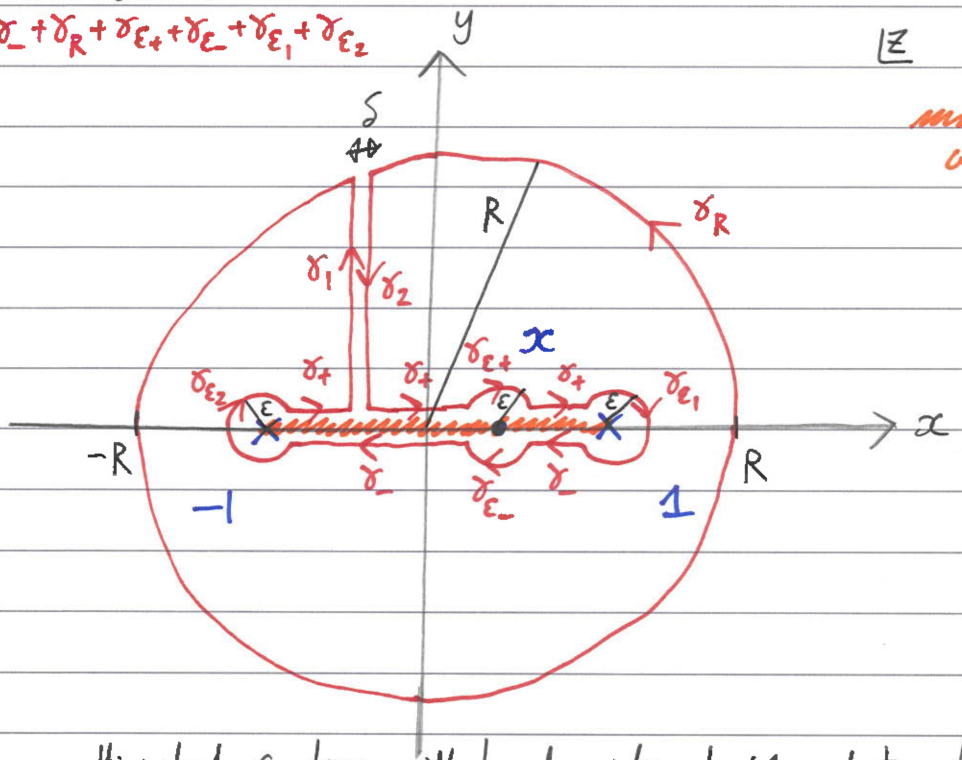
$$= z\left(1-\frac{1}{2z^2}+O\left(\frac{1}{z^4}\right)\right)\left(1+\frac{x}{z}+\frac{x^2}{z^2}+O\left(\frac{1}{z^3}\right)\right)$$

$$= z\left(1+\frac{x}{z}+\frac{x^2}{z^2}-\frac{1}{2z^2}+O\left(\frac{1}{z^3}\right)\right)$$

$$= z+x+\frac{\left(x^2-\frac{1}{2}\right)}{z}+O\left(\frac{1}{z^2}\right) \quad 3$$

Consider now $\oint_\gamma f(z)\,dz$, where $f(z) = \dfrac{z\sqrt{z^2-1}}{z-x} - (z+x)$ and:

$\gamma = \gamma_1+\gamma_2+\gamma_++\gamma_-+\gamma_R+\gamma_{\varepsilon+}+\gamma_{\varepsilon-}+\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}$



~~~~ : branch cut for $\sqrt{z^2-1} \sim z$ as $z \to \infty$

$f(z)$ is a multi-valued function with branch points at ± 1 and two branches. We take as a branch cut the section of the real axis between ± 1.

Take the contour of integration γ to be the closed contour shown above, consisting of the sections $\gamma_R, \gamma_\pm, \gamma_{\varepsilon\pm}, \gamma_{\varepsilon_{1,2}}$ and $\gamma_{1,2}$. γ_R is a circle of radius R centred on the origin, while $\gamma_{\varepsilon\pm}$ are semi-circles of radius ε centred on x.
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Also $\gamma_{\varepsilon_{1,2}}$ are circles of radius ε centred on ± 1, and $\gamma_{1,2}$ are separated by a gap of width δ. We consider the limit as $R \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$.

$f(z)$ is analytic everywhere inside γ, and thus by Cauchy's theorem:

$$\oint_{\gamma} f(z)\,dz = 0.$$

Now consider the integrals along the separate sections of γ. Consider first γ_R. For $z \in \gamma_R$, by 3 we have:

$$f(z) = \frac{(x^2 - \frac{1}{2})}{z} + O\left(\frac{1}{z^2}\right)$$

Thus, as $R \to \infty$, we have; on γ_R: $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$. Then:

$$\begin{aligned}
\lim_{\substack{R\to\infty \\ \delta \to 0}} \int_{\gamma_R} f(z)\,dz &= \lim_{R\to\infty} \int_{\gamma_R} \left(\frac{(x^2 - \frac{1}{2})}{z} + O\left(\frac{1}{z^2}\right)\right) dz \\
&= \lim_{R\to\infty} \int_0^{2\pi} \left(\frac{x^2 - \frac{1}{2}}{Re^{i\theta}} + O\left(\frac{1}{R^2}\right)\right) iRe^{i\theta}\,d\theta \\
&= \lim_{R\to\infty} \int_0^{2\pi} \left(i(x^2 - \tfrac{1}{2}) + O\left(\tfrac{1}{R}\right)\right) d\theta \\
&= 2\pi i \left(x^2 - \tfrac{1}{2}\right).
\end{aligned}$$

Next consider the contributions due to $\gamma_\pm$ and $\gamma_{\varepsilon\pm}$. On γ_+ and $\gamma_{\varepsilon+}$: $\sqrt{z^2-1} = i\sqrt{1-x^2}$. On $\gamma_{\varepsilon+}$: $z = x + \varepsilon e^{i\beta}$ where $\pi \geq \beta \geq 0$. So, we have:

$$\lim_{\substack{\varepsilon\to 0 \\ \delta \to 0}} \int_{\gamma_+ + \gamma_{\varepsilon+}} f(z)\,dz = \lim_{\varepsilon \to 0} \Bigg[\int_{-1}^{x-\varepsilon} \frac{ti\sqrt{1-t^2}}{t-x}\,dt + \int_{x+\varepsilon}^{1} \frac{ti\sqrt{1-t^2}}{t-x}\,dt$$

$$+ \int_{\pi}^{0} \frac{i\sqrt{1-\cancel{x}^2}}{\varepsilon e^{i\beta}} (x + \varepsilon e^{i\beta})\, i\varepsilon e^{i\beta}\,d\beta - \int_{-1}^{1} (t+x)\,dt \Bigg]$$

$$= i\int_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x}\,dt + \lim_{\varepsilon\to 0}\int_0^{\pi} \sqrt{1-x^2}\,(x+O(\varepsilon))\,d\beta - \left[\frac{t^2}{2}+xt\right]_{-1}^{1}$$

$$= iI(x) + \pi x\sqrt{1-x^2} - 2x$$

Similarly, on γ_- and γ_{ε_-}: $\sqrt{z^2-1} = -i\sqrt{1-x^2}$. On γ_{ε_-}: $z = x+\varepsilon e^{i\beta}$ where $0 \geqslant \beta \geqslant -\pi$. So, we have:

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}} \int_{\gamma_- + \gamma_{\varepsilon_-}} f(z)\,dz = \lim_{\varepsilon\to 0}\Bigg[\int_1^{x+\varepsilon} \frac{-it\sqrt{1-t^2}}{t-x}\,dt + \int_{x-\varepsilon}^{-1} \frac{-it\sqrt{1-t^2}}{t-x}\,dt$$

$$+ \int_0^{-\pi} \frac{-i\sqrt{1-x^2}}{\varepsilon e^{i\beta}}(x+\varepsilon e^{i\beta})\,i\varepsilon e^{i\beta}\,d\beta - \int_1^{-1}(t+x)\,dt\Bigg]$$

$$= i\int_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x}\,dt + \lim_{\varepsilon\to 0}\int_0^{-\pi} \sqrt{1-x^2}\,(x+\varepsilon e^{i\beta})\,d\beta - \left[\frac{t^2}{2}+xt\right]_{1}^{-1}$$

$$= iI(x) - \pi x\sqrt{1-x^2} + 2x$$

Finally, one can check that as $\varepsilon\to 0$, $\int_{\gamma_{\varepsilon_{1,2}}} f(z)\,dz = 0$, and also that as $\delta \to 0$ $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 0$.

Thus, combining the above we arrive at:

$$2\pi i\left(x^2 - \frac{1}{2}\right) + 2iI(x) = 0$$

and hence:

$$I(x) = \frac{\pi}{2} - \pi x^2, \qquad \text{as required.}$$

6

2). $$I(x) = \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-2)(t-x)}\, dt .$$

We can evaluate this as follows. First note that:

$$\frac{1}{(t-2)(t-x)} = \frac{1}{(x-2)}\left(\frac{1}{t-x} - \frac{1}{t-2}\right)$$

Then:

$$I(x) = \int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-2}\left(\frac{1}{t-x} - \frac{1}{t-2}\right) dt$$

$$= \frac{1}{x-2}\left(\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt - \int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-2}\,dt\right),$$

where we have removed the principle value from one integral since this integral is non-singular.

Now from lectures (section 2.5), we have: $\displaystyle\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt = -\pi x.$

To evaluate the second integral we substitute $t = \cos\theta$, we get:

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-2}\,dt = \int_{0}^{\pi}\frac{1-\cos^2\theta}{\cos\theta - 2}\,d\theta$$

$$= -\int_{0}^{\pi}\frac{(\cos\theta-2)(2+\cos\theta)+3}{\cos\theta - 2}\,d\theta$$

$$= -\int_{0}^{\pi}\left(\cos\theta + 2 + \frac{3}{\cos\theta-2}\right)d\theta$$

$$= -2\pi - \frac{3}{2}\int_{-\pi}^{\pi}\frac{d\theta}{\cos\theta-2}.$$

Let $w = e^{i\theta}$ in this final integral gives $-2i\displaystyle\oint_{|w|=1}\frac{dw}{w^2-4w+1}$, which can be computed using the residue theorem and gives $\dfrac{-2\pi}{\sqrt{3}}$.

7

Hence $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-2}\,dt = -2\pi - \frac{3}{2}\left(\frac{-2\pi}{\sqrt{3}}\right) = -2\pi + \sqrt{3}\pi.$

Hence we have: $\displaystyle I(x) = \frac{1}{x-2}\left(-\pi x - \left(-2\pi + \sqrt{3}\pi\right)\right)$

$$= \frac{1}{x-2}\pi\left(2-\sqrt{3}-x\right)$$

$$= \underline{\underline{\frac{-\sqrt{3}\pi}{x-2} - \pi}}.$$

3) We have: $\displaystyle A\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt + \sqrt{3}\int_{-1}^{1}\frac{dt}{(t-2)\sqrt{1-t^2}} = 0$

Let $t = \cos\theta$, then we have:

$$A\int_{\pi}^{0}\frac{(-\sin\theta)\,d\theta}{\sin\theta} + \sqrt{3}\int_{\pi}^{0}\frac{(-\sin\theta)\,d\theta}{(\cos\theta-2)\sin\theta} = 0$$

$$\Rightarrow \pi A + \frac{\sqrt{3}}{2}\int_{-\pi}^{\pi}\frac{d\theta}{\cos\theta - 2} = 0$$

But from question 2; $\displaystyle\int_{-\pi}^{\pi}\frac{d\theta}{\cos\theta-2} = \frac{-2\pi}{\sqrt{3}}.$

Using this, we find:

$$\pi A + \frac{\sqrt{3}}{2}\left(\frac{-2\pi}{\sqrt{3}}\right) = 0$$

$$\Rightarrow \underline{\underline{A = 1}}.$$

8

4). $$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt = \frac{x}{\sqrt{1-x^2}}, \qquad -1<x<1$$

Applying the (Hilbert) inversion formula gives:

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{t}{t-x}\,dt + \frac{A}{\sqrt{1-x^2}} \quad (4), \quad A \text{ constant.}$$

writing $\frac{t}{t-x} = 1 + \frac{x}{t-x}$, and using the result from lectures that

$$\int_{-1}^{1}\frac{1}{t-x}\,dt = \log\left(\frac{1-x}{1+x}\right),$$

then it follows from (4) that:

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\left(x\log\left(\frac{1-x}{1+x}\right) + 2\right) + \frac{A}{\sqrt{1-x^2}}.$$

Then from the condition $f(0)=1$, we have:

$$1 = \frac{-1}{\pi}(2) + A \quad \Rightarrow \quad A = \frac{2}{\pi}+1.$$

Hence, the solution is given by:

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\left(x\log\left(\frac{1-x}{1+x}\right) + 2 - \pi\left(\frac{2}{\pi}+1\right)\right)$$

$$\Rightarrow \quad f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\left(x\log\left(\frac{1-x}{1+x}\right) - \pi\right).$$

9

5). $$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|\,dt = 1 + \arcsin(x), \quad (5) \qquad -1<x<1$$

Differentiating (5) with respect to x gives:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(t)}{t-x}\,dt = \frac{-1}{\sqrt{1-x^2}}, \qquad -1<x<1$$

Then applying the (Hilbert) inversion formula gives:

$$f(x) = \frac{-1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{-1}{t-x}\,dt + \frac{A}{\sqrt{1-x^2}}, \qquad A \text{ constant.}$$

Again, using the result from lectures that $\int_{-1}^{1}\frac{1}{t-x}\,dt = \log\left(\frac{1-x}{1+x}\right)$, we have:

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}\log\left(\frac{1-x}{1+x}\right) + \frac{A}{\sqrt{1-x^2}}. \quad (6)$$

To determine A we substitute (6) into (5) and evaluate this at a particular value of x, say $x=0$, to get:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\pi\sqrt{1-t^2}}\left(\log\left(\frac{1-t}{1+t}\right)+\pi A\right)\log|t|\,dt = 1$$

$$\Rightarrow \underbrace{\frac{1}{\pi^2}\int_{-1}^{1}\frac{\log\left(\frac{1-t}{1+t}\right)\log|t|}{\sqrt{1-t^2}}\,dt}_{=0} + \frac{A}{\pi}\int_{-1}^{1}\frac{\log|t|}{\sqrt{1-t^2}}\,dt = 1$$

$$\Rightarrow A\underbrace{\int_{0}^{1}\frac{\log t}{\sqrt{1-t^2}}\,dt}_{= -\frac{\pi}{2}\log 2 \text{ (see lectures)}} = \frac{\pi}{2}$$

$$\Rightarrow \underline{\underline{A = \frac{-1}{\log 2}}}.$$

$$\Rightarrow f(x) = \frac{1}{\pi\sqrt{1-x^2}}\log\left(\frac{1-x}{1+x}\right) - \frac{\frac{1}{\log 2}}{\sqrt{1-x^2}}.$$

6). $$\frac{1}{\pi}\int_{-1}^{1} f(t)\log|t-x|\,dt + \lambda\int_{-1}^{1} f(t)\,dt = 1, \quad (7) \qquad -1<x<1$$

Differentiating (7) with respect to x gives:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(t)}{t-x}\,dt = 0, \qquad -1<x<1.$$

Then the solution for $f(x)$ follows from the (Hilbert) inversion formula as

$$f(x) = \frac{A}{\sqrt{1-x^2}}, \quad (8) \qquad A \text{ constant.}$$

To determine A substitute (8) into (7) and evaluate this at a particular value of x, say $x=0$, to get:

$$\frac{A}{\pi}\int_{-1}^{1} \frac{(\log|t| + \lambda)}{\sqrt{1-t^2}}\,dt = 1,$$

or equivalently:

$$A\int_{0}^{1} \left(\frac{\log|t| + \lambda}{\sqrt{1-t^2}}\right) dt = \frac{\pi}{2}.$$

Then again using the result used in question 5 that $\int_0^1 \frac{\log|t|}{\sqrt{1-t^2}}\,dt = -\frac{\pi}{2}\log 2$

and the fact that $\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}$, it follows that:

$$A\left(\frac{-\pi}{2}\log 2 + \lambda\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad A = \frac{1}{(\lambda - \log 2)}.$$

Hence no solution when $\lambda = \lambda_0 = \log 2$. Otherwise:

$$f(x) = \frac{1}{(\lambda - \log 2)\sqrt{1-x^2}}.$$