

MATH50001 Analysis II, Complex Analysis
Lecture 14

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0 . The residue of f at z_0 is

$$\text{Res}[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0 . Then

$$\text{Res}[f, z_0] = \frac{1}{2\pi i} \oint_{\gamma} f(z) dz.$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

m = 1. Then $g(z) = a_{-1} + a_0(z - z_0) + \dots$ and therefore

$$\text{Res}[f, z_0] = a_{-1} = \lim_{z \rightarrow z_0} g(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

m = 2. Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\text{Res}[f, z_0] = a_{-1} = \frac{d}{dz} g(z) \Big|_{z=z_0} = \lim_{z \rightarrow z_0} \frac{d}{dz} ((z - z_0)^2 f(z)).$$

m.

$$\text{Res}[f, z_0] = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_0)^m f(z)).$$

Example.

Evaluate

$$\oint_{\gamma} \frac{1}{z^5 - z^3} dz, \quad \gamma = \{z : |z| = 1/2\}$$

Clearly

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3(z-1)(z+1)}.$$

Since $z = \pm 1$ is outside γ we obtain

$$\begin{aligned} \oint_{\gamma} \frac{1}{z^5 - z^3} dz &= 2\pi i \operatorname{Res}[f, 0] = 2\pi i \frac{1}{2!} \lim_{z \rightarrow 0} (z^3 f(z))'' \\ &= \pi i \lim_{z \rightarrow 0} \left(\frac{1}{z^2 - 1} \right)'' = \pi i \lim_{z \rightarrow 0} \left(\frac{-2z}{(z^2 - 1)^2} \right)' \\ &= \pi i \lim_{z \rightarrow 0} \left(\frac{-2(z^2 - 1)^2 - (-2z)2(z^2 - 1)2z}{(z^2 - 1)^4} \right) = -2\pi i. \end{aligned}$$

Example.

Evaluate

$$\oint_{\gamma} \frac{1}{(z+5)(z^2-1)} dz, \quad \gamma = \{z : |z| = 2\}.$$

Because the integrand has singularities at $z = -5$ and $z = \pm 1$ only the last two are interior to γ , we have

$$\begin{aligned} & \oint_{\gamma} \frac{1}{(z+5)(z^2-1)} dz \\ &= 2\pi i \left\{ \operatorname{Res} \left[\frac{1}{(z+5)(z^2-1)}, -1 \right] + \operatorname{Res} \left[\frac{1}{(z+5)(z^2-1)}, 1 \right] \right\}. \end{aligned}$$

$$\oint_{\gamma} \frac{1}{(z+5)(z^2-1)} dz, \quad \gamma = \{z : |z| = 2\}.$$

Now $z = 1$ is a pole of order 1 and therefore

$$\begin{aligned} \text{Res} \left[\frac{1}{(z+5)(z^2-1)}, 1 \right] \\ = \lim_{z \rightarrow 1} \frac{z-1}{(z+5)(z^2-1)} = \lim_{z \rightarrow 1} \frac{1}{(z+5)(z+1)} = \frac{1}{12}. \end{aligned}$$

Similarly, $z = -1$ is a simple pole and

$$\begin{aligned} \text{Res} \left[\frac{1}{(z+5)(z^2-1)}, -1 \right] \\ = \lim_{z \rightarrow -1} \frac{z+1}{(z+5)(z^2-1)} = \lim_{z \rightarrow -1} \frac{1}{(z+5)(z-1)} = -\frac{1}{8}. \end{aligned}$$

Thus,

$$\oint_{\gamma} \frac{1}{(z+5)(z^2-1)} dz = 2\pi i \left(\frac{1}{12} - \frac{1}{8} \right) = -\frac{\pi i}{12}.$$

Section: The argument principle.

Theorem. (Principle of the Argument)

Let f be holomorphic in an open set Ω except for a finite number of poles and let γ be a simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f . Then

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i(N - P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ .

Remark. Why Principle of the Argument?

Indeed, let γ be a closed curve. Then

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \oint_{\gamma} \frac{d}{dz} \log f(z) dz = \frac{1}{2\pi i} \left. \log f(z) \right|_{z_1}^{z_2} \\ &= \frac{1}{2\pi i} \left(\ln |f(z_2)| - \ln |f(z_1)| + i(\arg f(z_2) - \arg f(z_1)) \right) = \frac{1}{2\pi} \Delta \arg f(z). \end{aligned}$$

Example. Let $f(z) = z^3$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$, then $f(z) = e^{i3\theta}$ and $\frac{1}{2\pi} \Delta_\gamma \arg f = 3$.

Example. Let $f(z) = 1/z$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi} \Delta_\gamma \arg f = -1$.

Example. Let $f(z) = z + 2$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi} \Delta_\gamma \arg f = 0$.

Thank you