Math40003 Linear Algebra and Groups **Problem Sheet 3**
Unseen Questions

U.1. Prove that if V is a finite dimensional vector space, then V does not have an infinite basis.

U.2. Find a basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

U.3. Find a basis of the vector spaces introduced in C.2 on Problem Sheet 2:

(a) $\mathbb{R}$ with $x \oplus y = xy$ and $r \odot x = x^r$.

(b) The game "Lights Out" consists of a 5×5 grid of lights, which be be either on or off. If you press one of the lights, it and its 4 adjacent neighbours switch state (i.e. if on, they become off, and vice versa). Let V be the set of all possible states of the grid.

(c) Let P be a set of propositions, and let $\delta : P \to \{T, \perp\}$ be a function that assigns each proposition a true/false value. Let $\mathcal{S}(P)$ be the set of all sentences we can build from P using $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and $\neg$. Using truth tables, we can uniquely extend δ to $\mathcal{S}(P)$. Let Δ be the set of extensions of all the possible δ. This Δ is our candidate for a vector space. For the operations, consider how you can apply $\neg$ and $\wedge$ to these functions.

U.4. Does every vector space have a basis?

WARNING: I'm an exceptionally cruel person to set this question with no guidance or advice. I'm willing to argue that this question is the most subtle piece of mathematics you'll see in your first and second year.