Imperial College London
MATH 50004 Multivariable Calculus
Mid-Term Examination Date: 16th November 2022
SOLUTIONS

Question One Solution

(a) We need to calculate

$$\nabla\phi = (3x^2 + 2x - 2)\mathbf{i} + (2y + \alpha)\mathbf{j} - \mathbf{k} \quad \textbf{[1 mark]}$$

and then evaluate it at $x = 1, y = 0, z = -1$:

$$\nabla\phi|_{(1,0,-1)} = 3\mathbf{i} + \alpha\mathbf{j} - \mathbf{k} \quad \textbf{[1 mark]}$$

and then convert it into a unit vector:

$$\widehat{\mathbf{n}} = \pm\frac{\nabla\phi|_{(1,0,-1)}}{\left|\nabla\phi|_{(1,0,-1)}\right|} = \pm\frac{3\mathbf{i} + \alpha\mathbf{j} - \mathbf{k}}{\sqrt{10 + \alpha^2}}. \quad \textbf{[1 mark]}$$

The magnitude of the greatest rate of change is $\left|\nabla\phi|_{(1,0,-1)}\right| = \sqrt{10 + \alpha^2}$. **[1 mark]**
(b) A vector joining the points $(1, 0, -1)$ and $(3, 4, 5)$ and directed towards $(3, 4, 5)$ is

$$\mathbf{s} = (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) - (\mathbf{i} - \mathbf{k}) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}. \quad \textbf{[1 mark]}$$

and thus the unit vector is

$$\widehat{\mathbf{s}} = \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}}. \quad \textbf{[1 mark]}$$

The rate of change of ϕ at $(1, 0, -1)$ in the direction towards $(3, 4, 5)$ is therefore

$$\nabla\phi|_{(1,0,-1)} \cdot \widehat{\mathbf{s}} = (3\mathbf{i} + \alpha\mathbf{j} - \mathbf{k}) \cdot \frac{(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}{\sqrt{14}} = \frac{2\alpha}{\sqrt{14}}. \quad \textbf{[1 mark]}$$

(c) If $\phi = 1$ and $x = 1, y = 0$ we have

$$1 = -z,$$

and so the tangent plane passes through $(x_0, y_0, z_0) = (1, 0, -1)$. **[1 mark]** The equation of the tangent plane is

$$((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) \cdot \nabla\phi|_{(x_0,y_0,z_0)} = 0$$

i.e.

$$\begin{aligned}
((x - 1)\mathbf{i} + y\mathbf{j} + (z + 1)\mathbf{k}) \cdot (3\mathbf{i} + \alpha\mathbf{j} - \mathbf{k}) &= 0 \\
3(x - 1) + \alpha y - (z + 1) &= 0 \\
z &= 3x + \alpha y - 4. \quad \textbf{[3 marks]}
\end{aligned}$$

(d) (i) Firstly

$$\mathbf{u} = \nabla\phi = (3x^2 + 2x - 2)\mathbf{i} + (2y + \alpha)\mathbf{j} - \mathbf{k},$$

as calculated in (a). It follows that

$$\frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2}\left((3x^2 + 2x - 2)^2 + (2y + \alpha)^2 + 1\right) \quad \textbf{[1 mark]}$$

and hence

$$\nabla\left(\frac{1}{2}(\mathbf{u}\cdot\mathbf{u})\right) = (6x+2)(3x^2+2x-2)\mathbf{i} + 2(2y+\alpha)\mathbf{j}. \quad \textbf{[1 mark]}$$

We also have

$$\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} &= \left((3x^2+2x-2)\frac{\partial}{\partial x} + (2y+\alpha)\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)((3x^2+2x-2)\mathbf{i} + (2y+\alpha)\mathbf{j} - \mathbf{k}) \\
&= (3x^2+2x-2)(6x+2)\mathbf{i} + (2y+\alpha)2\mathbf{j}. \quad \textbf{[1 mark]}
\end{aligned}$$

We therefore see that the RHS of the first vector identity is

$$\nabla\left(\frac{1}{2}(\mathbf{u}\cdot\mathbf{u})\right) - (\mathbf{u}\cdot\nabla)\mathbf{u} = 0, \quad \textbf{[1 mark]}$$

while the LHS is

$$\mathbf{u}\times\operatorname{curl}(\nabla\phi) = 0,$$

since the curl of a gradient is always zero. (Alternatively, the LHS can be calculated explicitly). Hence the relation is verified. **[1 mark]**

(d)(ii) Arguing as above we have that

$$\operatorname{curl}(\operatorname{curl}\mathbf{u}) = \operatorname{curl}[\operatorname{curl}(\nabla\phi)] = 0,$$

and so the LHS is zero. **[1 mark]** (Again this could be calculated explicitly instead). Considering the terms on the right hand side:

$$\begin{aligned}
\nabla(\operatorname{div}\mathbf{u}) &= \nabla(6x+4) = 6\mathbf{i}, \\
\nabla^2\mathbf{u} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\big((3x^2+2x-2)\mathbf{i} + (2y+\alpha)\mathbf{j} - \mathbf{k}\big) = 6\mathbf{i}, \quad \textbf{[2 marks]}
\end{aligned}$$

and so the RHS is also zero **[1 mark]** .

Question Two Solution

(a)

$$\operatorname{div}\mathbf{A} = \frac{\partial}{\partial x}(\beta x - y^3) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(-y^2 z) = \beta - z^2 - y^2. \quad \textbf{[2 marks]}$$

$$\operatorname{curl}\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \beta x - y^3 & -yz^2 & -y^2 z \end{vmatrix} = \mathbf{i}(-2yz + 2yz) - \mathbf{j}(0) + \mathbf{k}(3y^2) = 3y^2\mathbf{k} \quad \textbf{[2 marks]}$$

(b)

$$\widehat{\mathbf{n}} = \pm\frac{\nabla(x^2+y^2+z^2)}{|\nabla(x^2+y^2+z^2)|} = \pm\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{a},$$

since we need $\widehat{\mathbf{n}} \cdot \mathbf{k} \geq 0$ we choose the positive sign above. **[3 marks]**

(c) Since $z = 0$ on C we have

$$\mathbf{A} \cdot d\mathbf{r} = ((\beta x - y^3)\mathbf{i}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = (\beta x - y^3)dx \quad \textbf{[2 marks]}$$

We can parameterize C by writing $x = a\cos\theta, y = a\sin\theta, dx = -a\sin\theta d\theta$, with $0 \leq \theta \leq 2\pi$, **[2 marks]** so that

$$\begin{aligned} \oint_C \mathbf{A} \cdot d\mathbf{r} &= \int_0^{2\pi} (\beta a\cos\theta - a^3\sin^3\theta)(-a\sin\theta)d\theta \\ &= -\beta a^2 \int_0^{2\pi} \cos\theta\sin\theta d\theta + a^4\int_0^{2\pi} \sin^4\theta d\theta \\ &= \beta a^2\left[\frac{\cos^2\theta}{2}\right]_0^{2\pi} + 3\pi a^4/4 \\ &= 3\pi a^4/4, \end{aligned}$$

using the result provided. **[2 marks]**

(d) Using the earlier results:

$$(\operatorname{curl}\mathbf{A}) \cdot \widehat{\mathbf{n}} = 3y^2 z/a. \quad \textbf{[1 mark]}$$

Using the projection theorem,

$$I = \int_S (\operatorname{curl}\mathbf{A}) \cdot \widehat{\mathbf{n}}\, dS = \int_\Sigma \frac{3y^2 z}{a}\frac{d\Sigma}{\widehat{\mathbf{n}} \cdot \mathbf{k}} = \int_\Sigma 3y^2 dx\, dy. \quad \textbf{[2 marks]}$$

The projection Σ is a disc of radius a centre $(0,0)$ and so we can parameterize this surface by writing

$$x = r\cos\theta, y = r\sin\theta, \ (0 \leq r \leq a, 0 \leq \theta \leq 2\pi), \quad \textbf{[2 marks]}$$

so that

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^a (3r^2\sin^2\theta)\, r\, dr d\theta \\ &= \pi\left[\frac{3r^4}{4}\right]_0^a = \frac{3\pi a^4}{4}, \end{aligned}$$

where we have used that $\int_0^{2\pi}\sin^2\theta d\theta = \pi$. **[2 marks]**