Throughout the exam you must clearly state any result you use, and justify your answers fully.

1. (a) Let $c \in \mathbb{R}$. By finding the augmented matrix, find all solutions to the following system of equation:

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 1 \\ z &= c \end{aligned}$$

(3 marks)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & c \end{array}\right)$$

(2 marks)

So for any $c \in \mathbb{R}$ we have $x = 1$ and $z = c$ $y = 1 - c$. **(1 mark)**

(b) Let $a, b, c, \alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} x + \alpha y + \beta z &= a \\ y + \alpha z &= b \\ z &= c \end{aligned}$$

How many solutions does the above system of linear equations have? Give all possible solutions in terms of a, b, c, α, β. (5 marks)

$$\left(\begin{array}{ccc|c} 1 & \alpha & \beta & a \\ 0 & 1 & \alpha & b \\ 0 & 0 & 1 & c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & \alpha & \beta & a \\ 0 & 1 & 0 & b - \alpha c \\ 0 & 0 & 1 & c \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & \alpha & 0 & a - \beta c \\ 0 & 1 & 0 & b - \alpha c \\ 0 & 0 & 1 & c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & a - \beta c - \alpha(b - \alpha c) \\ 0 & 1 & 0 & b - \alpha c \\ 0 & 0 & 1 & c \end{array}\right)$$

(3 marks)

So there is a unique solution for any values of a, b, c, α, β. That is to say:

$$\begin{aligned} x &= a - \beta c - \alpha b - \alpha^2 c \\ y &= b - \alpha c \\ z &= c \end{aligned}$$

(2 marks)

(c) Let $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$.

$$\begin{aligned} x + \alpha y + \beta z &= a \\ y + \alpha z &= b \\ \gamma z &= c \end{aligned}$$

Give conditions on $a, b, c, \alpha, \beta, \gamma$ to have:

i. One solution.

ii. No solutions.

iii. Infinitely many solutions.

In each case find all solutions in terms of a, b, c, α, β. For part (iii.) describe the shape of the set of solutions (e.g. plane), and give the solution in the standard form for this shape.

(9 marks)

We get the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & \alpha & \beta & a \\ 0 & 1 & \alpha & b \\ 0 & 0 & \gamma & c \end{array}\right)$$

i. If $\gamma \neq 0$ then we can row reduce to:

$$\left(\begin{array}{ccc|c} 1 & \alpha & \beta & a \\ 0 & 1 & \alpha & b \\ 0 & 0 & 1 & \frac{c}{\gamma} \end{array}\right)$$

By letting $c' = \frac{c}{\gamma}$ we are in the same situation as above. Thus we get one solution:

$$\begin{aligned} x &= a - \beta c' - \alpha b - \alpha^2 c' &&= a - \beta\frac{c}{\gamma} - \alpha b - \alpha^2\frac{c}{\gamma} \\ y &= b - \alpha c' &&= b - \alpha\frac{c}{\gamma} \\ z &= c' &&= \frac{c}{\gamma} \end{aligned}$$

(3 marks)

ii. If $\gamma = 0$ and $c \neq 0$ then the final equation is $0 = c \neq 0$ so the system is inconsistent and thus has no solutions. **(2 marks)**

iii. If $\gamma = 0$ and $c = 0$ we get:

$$\left(\begin{array}{ccc|c} 1 & \alpha & \beta & a \\ 0 & 1 & \alpha & b \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & \beta-\alpha^2 & a-b \\ 0 & 1 & \alpha & b \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case we get infinitely many solutions of the form:

$$x \ = \ (a-b)-(\beta-\alpha^2)zy \ = \ b-\alpha z$$

(2 marks)

This is a line:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -(\beta-\alpha^2) \\ \alpha \\ 1 \end{pmatrix} + \begin{pmatrix} a-b \\ b \\ 0 \end{pmatrix}$$

(2 marks)

(d) Can you find a system of linear equations with exactly two solutions? *Justify any answer you give carefully.* (3 marks)

A reasonable solution explaining why systems of linear equations over infinite fields cannot have exactly two solutions should be given 2 marks.

Let $F = \mathbb{F}_2$ then $x+y=0$ is a system of linear equations and has two solutions ($x=y=0$ and $x=y=1$).

(3 marks)

(Total: 40 marks)

Throughout the exam you must clearly state any result you use, and justify your answers fully.

2. (a) Let

$$A = \begin{pmatrix} 6 & -2 & 6 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

Let $V = M_{3\times3}(\mathbb{R})$ be the $\mathbb{R}$-vector space with standard addition and scalar multiplication. Define:

$$T : V \to V \text{ where } T(M) = AM$$

i. Find the rank of A. (2 marks)

To find the rank of A put A into REF:

$$\begin{pmatrix} 6 & -2 & 6 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So A has rank 2. **(2 marks)**

ii. Show that T is a linear transformation. (3 marks)

- ***Closed under addition:*** **Let $V_1, V_2 \in M_{3\times3}(\mathbb{R})$**

$$\begin{aligned} T(V_1 + V_2) &= A(V_1 + V_2) \\ &= AV_1 + AV_2 \textbf{by distributivity of matrix multiplication} \\ &= T(V_1) + T(V_2) \end{aligned}$$

- ***Closed under scalar multiplication:*** **Let $V \in M_{3\times3}(\mathbb{R})$ and $\lambda \in \mathbb{R}$**

$$\begin{aligned} T(\lambda) &= A(\lambda V) \\ &= \lambda AV \textbf{by commutativity of scalar multiplication for matrices} \\ &= \lambda T(V) \end{aligned}$$

So T is a linear transformation. **(3 marks)**

iii. Find $\ker T$. (6 marks)

$$\ker T = \{V \in M_{3\times3}(\mathbb{R}) : AV = 0\}$$

Let $V \in M_{3\times3}(\mathbb{R})$ then $V = (a_{ij})$, if $AV = 0$ then we have

$$\begin{pmatrix} 6 & -2 & 6 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 6a_{11} - 2a_{21} + 6a_{31} & 6a_{12} - 2a_{22} + 6a_{32} & 6a_{13} - 2a_{23} + 6a_{33} \\ 3a_{11} + 3a_{31} & 3a_{12} + 3a_{32} & 3a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have the following systems of linear equations:

$$\begin{aligned} 6a_{11} - 2a_{21} + 6a_{31} &= 0 \\ 3a_{11} + 3a_{31} &= 0 \\ a_{21} &= 0 \end{aligned} \qquad \begin{aligned} 6a_{12} - 2a_{22} + 6a_{32} &= 0 \\ 3a_{12} + 3a_{32} &= 0 \\ a_{22} &= 0 \end{aligned} \qquad \begin{aligned} 6a_{13} - 2a_{23} + 6a_{33} &= 0 \\ 3a_{13} + 3a_{33} &= 0 \\ a_{23} &= 0 \end{aligned}$$

$$A \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now by part (i.) we know $A\bar{x} = 0$ if and only if

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{x} = 0$$

So the above reduces to:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So

$$\ker T = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : \begin{array}{l} a_{12} = a_{22} = a_{32} = 0 \textbf{ and} \\ a_{11} + a_{13} = a_{21} + a_{23} = a_{31} + a_{33} = 0 \\ \textbf{for } a_{ij} \in \mathbb{R} \end{array} \right\}$$

i.e.

$$\ker T = \{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ -a_{11} & -a_{12} & -a_{13} \end{pmatrix} : a_{ij} \in \mathbb{R} \}$$

(6 marks)

iv. Find a basis for $\ker T$, and deduce the nulity of T. (3 marks)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Clearly forms a linearly independent set of matrices. It spans $\ker T$ because:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ -a_{11} & -a_{12} & -a_{13} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} + a_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So as $\ker T$ has a basis of size 3 the nulity of T is 3. (3 marks)

v. Deduce the rank of T. (2 marks)

The Rank nulity theorem states that the dimension of V is equal to the rank of T plus the nulity of T. Thus the rank of T is the dimension of V (i.e. 9) minus the nulity of T (i.e. 3). So the rank of T is 6. (2 marks)

You may use standard properties of matrix multiplication/addition/scalar multiplication but these must be clearly stated. You may use the rank-nulity theorem if properly stated.

(b) Now let B be some matrix in $M_{3\times3}(\mathbb{R})$, $V = M_{3\times3}(\mathbb{R})$. Define:

$$T' : V \to V \text{ where } T'(M) = BM$$

What are the possibilities for:

i. Rank of B? (1 mark) **As B is a 3×3 matrix its row space is a subset of $\mathbb{R}^3$, thus could be dimension (and therefore rank) $3, 2, 1, 0$. (1 mark)**

ii. Rank of T'? (3 marks)

To calculate this it is easier to find the Kernel of T' and then use the rank nulity theorem.

To calculate $\ker T$ we use the technique above, i.e. solve:

$$B\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We get the following system:

$$B\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$B\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$B\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As each of these is the same set of linear equations repeated each system will have solution space of dimension either $0, 1, 2$ or 3. So the possibilities for the nulity are 0, 3, 6 or 9.

So, using $\dim V = \text{rank}T' + \text{nulity}T'$ possibilities for the rank are 0, 3, 6 or 9. **(3 marks)**

Give justification for any answer you give.

(Total: 40 marks)