Imperial College London

MATH97216 MATH97220 MATH97288

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Markov Processes

Date: Tuesday, 4 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

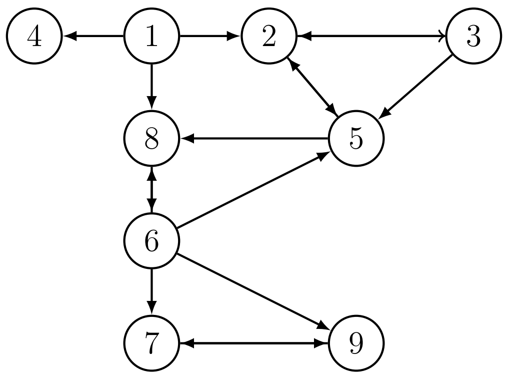
Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

All answers must be carefully justified, unless stated otherwise.

1. Consider a time homogeneous Markov chain (x_n) on the state space $\mathcal{X} = \{1,2,3,4,5,6,7,8,9\}$ with the stochastic matrix P given below.



$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & \frac{6}{7} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

(a) Indicate all communication classes of P and their partial orders (no justification is needed). (7 marks)

(c) Answer the following questions.

(i) Compute $\mathbf{P}(x_2 = 6)$, in case x_0 is distributed as $\mu_0 = \frac{1}{2}(1 - \frac{1}{\pi})\delta_1 + \frac{1}{\pi}\delta_5 + \frac{1}{2}(1 - \frac{1}{\pi})\delta_3$. (8 marks)

(ii) Compute P_{74}^{2021}. (5 marks)

(Total: 20 marks)

2. Consider a time homogeneous Markov chain (x_n) on $\mathcal{X} = \{1, 2, 3, 4, 5\}$ with stochastic matrix

$$P = \frac{1}{7}\begin{pmatrix} 3 & 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 2 & 1 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 0 & 3 \end{pmatrix}.$$

Let $T_j = \inf\{n \geq 1 : x_n = j\}$.

(a) Answer the following questions.

(i) What are the invariant probability measures?

(ii) What is the value of $\mathbf{E}_2(T_2)$?

(iii) Does $\lim_{n\to\infty} \mathbf{P}(x_n = 3 | x_0 = 5)$ exist? If yes, what is its value? (10 marks)

(b) Assuming the probability distribution of x_0 is invariant, show the following limit exists almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big[\sin(x_k)(x_{k+2} + 1)\Big].$$

What is this limit? You may treat P^2 as a known quantity. (10 marks)

(Total: 20 marks)

3. (a) Let μ_1 and μ_2 be two probability measures on $([0, 1], \mathcal{B}([0, 1]))$. Let $\mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $\mu_2(A) = 3\int_A x^2 dx$ where dx is the Lebesgue measure.
What is the value of $\|\mu_1 - \mu_2\|_{TV}$? (5 marks)

(b) Let $\mathcal{X} = [0, 2\pi]$ with the identification $0 \sim 2\pi$. Let dy denote the Lebesgue measure on $[0, 2\pi]$. Let $\{\xi_n\}$ be a sequence of i.i.d. random variables with the Gaussian distribution $\mathcal{N}(0, 1)$. Let x_n be the time homogeneous Markov process with the transition probabilities

$$P(x, dy) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbf{Z}} \exp\Big(-\frac{(y - x - 2\pi k)^2}{2}\Big)\, dy\ .$$

(i) Show that $\mu = \frac{1}{2\pi} dy$ is an invariant measure.

(ii) Let μ_n denote the probability distribution of x_n. Let $x_0 = 1$. Show that

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{TV} = 0.$$

(15 marks)

(Total: 20 marks)

4. Let dy denote the Lebesgue measure on $\mathbf{R}$. Let $a \in [0,1]$ be a constant, and $\{\xi_n\}$ be independent identically distributed real valued random variables with probability distribution Γ given below.

$$\Gamma(A) = a\,\delta_0(A) + (1-a)\,\frac{1}{\sqrt{2\pi}}\int_A e^{-\frac{y^2}{2}}\,dy, \qquad \forall\, A \in \mathcal{B}(\mathbf{R}).$$

Let

$$g(x) = \sin(x) + \pi \sin(\frac{x}{2}).$$

Let x_0 be independent of $\{\xi_n\}$. Let $c \in \mathbf{R}$ be a constant, we define for $n \geq 0$,

$$x_{n+1} = g(x_n) + c\,\xi_{n+1}.$$

(a) Let f be a bounded measurable function from $\mathbf{R} \to \mathbf{R}$. Compute

$$Tf(x) = \mathbf{E}[f(x_{n+1})|x_n = x].$$

(5 marks)

(b) Show that x_n has an invariant probability measure.

(10 marks)

(c) Identify those constants c for which the Markov chain has a *unique* invariant probability measure (as well as those c for which it does not). Justify your claim.

(5 marks)

(Total: 20 marks)

5. (a) Let $(P_t(x,\cdot), x \in \mathcal{X}, t \geq 0)$ be a transition function on a complete separable metric space $\mathcal{X}$. For any $f : \mathcal{X} \to \mathbf{R}$ bounded measurable, define

$$T_t f(x) = \int_{\mathcal{X}} f(y) P_t(x, dy).$$

Show that for any $f : \mathcal{X} \to \mathbf{R}$ is bounded and continuous,

$$\lim_{t \to 0} T_t f(x) = f(x)$$

for every $x \in \mathcal{X}$.

(5 marks)

(b) Let B_t denote a one-dimensional standard Brownian motion with $B_0 = 0$ and let $\mathcal{F}_t^+$ denote the right continuous complete filtration generated by B_t. Let T_t denote its semi-group.

(i) Show that $X_t = (B_t)^3$ is a Markov Process.
Is X_t a strong Markov process?

(10 marks)

(ii) Let x_0 be a real number. Let for $t \geq 0$, $x_t = e^{-t} x_0 + B_t - \int_0^t B_s e^{-(t-s)} ds$. Then (x_t) is an Ornstein-Uhlenbeck process. It is in particular a Markov process with respect to the filtration $\mathcal{F}_t^+$.
Let for $n \geq 1$,

$$x_n = e^{-n} x_0 + B_n - \int_0^n B_s e^{-(n-s)} ds.$$

Explain carefully whether the following sequence of random variables converges as $n \to \infty$,

$$\frac{1}{n} \sum_{k=1}^n (x_k)^2.$$

Find its value if it does.

(5 marks)

You may use freely results on the Ornstein-Uhlenbeck process $x_t, t \geq 0$.

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH96062/MATH97216/MATH97220

Markov Processes (Solutions)

Setter's signature	Checker's signature	Editor's signature
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1. (a) The minimal classes are

unseen ⇓

$$[1] = \{1\}, \quad [4] = \{4\}, \quad [6] = \{2,3,5,6,8\}, \quad [7] = \{7,9\}.$$

meth seen ⇓

The partial orders are:

$$[4] \leq [1], \qquad [7] \leq [6], [7] \leq [2] \leq [1].$$

7, A

(b) (i)

unseen ⇓

meth seen ⇓

$$\begin{aligned}\mathbf{P}(x_2 = 6) =& \sum_j \mathbf{P}(x_2 = 6 | x_0 = j)\mathbf{P}(x_0 = j) \\ =& \frac{1}{2}(1 - \frac{1}{\pi})\mathbf{P}(x_2 = 6|x_0 = 1)\mathbf{P}(x_0 = 1) + \frac{1}{\pi}\mathbf{P}(x_2 = 6|x_0 = 5)\mathbf{P}(x_0 = 5) \\ &+ \frac{1}{2}(1 - \frac{1}{\pi})\mathbf{P}(x_2 = 6|x_0 = 3)\mathbf{P}(x_0 = 3).\end{aligned}$$

$$\mathbf{P}(x_2 = 6|x_0 = 1)\mathbf{P}(x_0 = 1) = P_{18}P_{86} = \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}.$$

$$\mathbf{P}(x_2 = 6|x_0 = 5) = \sum_k P_{5k}P_{k6} = P_{58}P_{86} + P_{52}P_{26} = \frac{1}{2}\frac{2}{3} = \frac{1}{3}.$$

There is no 2-step path of positive probability from 3 to 6, so the 3rd term is zero.

$$\mathbf{P}(x_2 = 6) = \frac{1}{3\pi} + \frac{1}{2}(1 - \frac{1}{\pi}) = \frac{1}{12} + \frac{1}{4\pi}.$$

8, B

(ii) Either observe that $[7]$ is a minimal class, $[4]$ is in a separate communication class and so there is no path from 7 to 4 or we show by induction that $P_{74}^n = 0$. Firstly, $P_{74} = 0$. By Komogorov's equation,

unseen ⇓

meth seen ⇓

$$\begin{aligned}P_{74}^n =& \sum_j P_{7j}^{n-1} P_{j4} \\ =& P_{77}^{n-1}P_{74} + P_{79}^{n-1}P_{94} = 0.\end{aligned}$$

5, B

2. (a) (i) The system has a unique invariant probability $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. This can be checked by observing that there is exactly one communication class $\{1, 2, 3, 4, 5\}$ which is recurrent. This can also be directly computed (with a more lengthy computation). unseen ⇓ 3, A

(ii) Since the chain is irreducible, by the ergodic theorem $\mathbf{E}_2(T_2) = \frac{1}{\pi(2)} = 5$. 3, A

(iii) Since $P_{11} > 0$, the period of 1 is 1, and so the irreducible matrix is aperiodic. For an irreducible aperiodic stochastic matrix, we know that, by the ergodic theorem fro any initial distribution μ, μP^n converges, and so $P^n_{i,j}$ converges. They are converges to the value of the invariant measure on the target site. unseen ⇓ meth seen ⇓

$$\lim_{n\to\infty} P^n_{53} = \lim_{n\to\infty} \mathbf{P}(x_n = 3 \in A | x_0 = 5)$$

exists and its value is $\pi(3) = \frac{1}{5}$. 4, A

seen ⇓

(b) By Birkhoff's ergodic theorem, if π is an ergodic invariant probability measure, and f an integrable function, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\theta^k x.) = \int f d\mathbf{P}_\pi$$

almost surely, where θ is the shift operator.

Since π is the unique invariant probability measure it is ergodic.

We apply the theorem to $f(a.) = g(a_0, a_2)$ where

$$g(x, y) = \sin(x)(y + 1).$$

Then $f(\theta^k x.) = \sin(x_k)(x_{k+2} + 1)$. unseen ⇓ 5, C

We now compute $\int f \, d\mathbf{P}_\pi = \int g(x_0, x_2) d\mathbf{P}_\pi$. We assume that $x_0 \sim \pi$ and compute the probability distribution of (x_0, x_2).

$$\mathbf{P}(x_2 = j, x_0 = k) = \mathbf{P}(x_2 = j | x_0 = k)\pi(k) = \frac{1}{5}\mathbf{P}(x_2 = j | x_0 = k) = \frac{1}{5}P^2_{kj}.$$

where $\tilde{P}$ is the restriction of P to the set A.

$$P^2 = \frac{1}{49} \begin{pmatrix} 3 & 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 2 & 1 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 0 & 3 \end{pmatrix}^2$$

Thus, the limit is

$$\mathbf{E} f(x_0, x_2) = \frac{1}{5} \sum_{j,k=1}^{5} \sin(j)(k+1) P^2_{jk}.$$

We could also start with the invariant probability measure P_π on the path space $\{(y_0, y_1, y_2, \dot{)}\}$:

$$\int \sin(y_0)(y_2+1)dP_\pi = \mathbf{E}_\pi[\sin(x_0)(x_2+1)] = \mathbf{E}_\pi[\mathbf{E}(\sin(x_0)(x_2+1)\ |\sigma(x_0)\}]$$

$$= \mathbf{E}_\pi[\sin(x_0)\mathbf{E}((x_2+1)\ |\sigma(x_0)\}] = \frac{1}{5}\sum_i \sin(i)\sum_j (j+1)P^2_{ij}.$$

$\boxed{\text{unseen} \Downarrow}$

$\boxed{5, \text{D}}$

3. (a) The measure μ_1 is supported on $\{0,1\}$. But $\mu_2(\{0,1\}) = 0$. So they are mutually singular. Note that the two measures are mutually singular. The total variation norm between two singular probability measures is 2. meth seen $\Downarrow$

Alternative proof: that $\|\mu - \nu\|_{TV} = 2\sup_A |\mu(A) - \nu(A)| \le 2$ since $|\mu(A) - \nu(A)| \le 1$. Also take $A_0 = (0,1)$, then then $\mu(A_0) = 1$ and $\nu(A_0) = 0$, hence $|\mu(A_0) - \nu(A_0)| = 1$ which allows us to conclude that $\|\mu - \nu\|_{TV} = 2$. 5, A

(b) (i) Let us denote for $x, y \in [0, 2\pi]$,

$$p(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbf{Z}} \exp\Big(-\frac{(y-x-2\pi k)^2}{2}\Big).$$

We show that $\mu = \frac{1}{2}dx$, the normalised Lebesgue measure, is invariant. Let $A \in \mathcal{B}([0,2\pi])$, consider it as a subset of $\mathbf{R}$:

$$\begin{aligned}
T\pi(A) = \frac{1}{2\pi}\int_0^{2\pi}\int_A P(x,dy)dx &= \frac{1}{2\pi}\int_A\int_0^{2\pi} p(x,y)dxdy \\
&= \frac{1}{2\pi}\int_A\int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbf{Z}} \exp\Big(-\frac{(y-x-2\pi k)^2}{2}\Big)dxdy \\
&= \frac{1}{2\pi}\int_A \sum_{k\in Z}\int_{2\pi k}^{2\pi(k+1)} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(y-z)^2}{2}\Big)dzdy \\
&= \frac{1}{2\pi}\int_A\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(y-z)^2}{2}\Big)dzdy \\
&= \frac{1}{2\pi}\int_A dy = \pi(A).
\end{aligned}$$

seen $\Downarrow$

($P(x,dy)$ is the transition probabilities for $x_{n+1} = x_n + \xi_n \mod 2\pi$.) 7, B

(ii) For any x, y there exists a k such that $|\frac{y-x}{2\pi} - k| \le \frac{1}{2}$, hence

$$p(x,y) \ge \frac{1}{\sqrt{2\pi}} e^{-\frac{\pi^2}{2}}$$

Let c be the normalising constant so $\frac{1}{\sqrt{2\pi}}e^{-\frac{\pi^2}{2}}$ is a probability measure. Then for every x and any A,

$$\int_A P(x,dy) \ge \int_A \frac{1}{c}dy,$$

and the minorisation condition is satisfied.

Alternative proof: Let $f_n(x,y) = \sum_{k=-n}^n \exp\Big(-\frac{(y-x-2\pi k)^2}{2}\Big)$. Since for $k \ge 1$, $\frac{(y-x-2\pi k)^2}{2} \ge 2\pi^2 k^2 - 1$,

$$\sup_{x,y} |f_n(x,y) - f_{m+n}(x,y)| \le 2\sum_{k=n}^{n+m} \exp(-2\pi^2 k^2 + 1),$$

showing that f_n converges uniformly. Then $p(x,y)$ is a continuous function. Since strictly positive continuous functions on a compact set has a positive minimum. Let $p(y) = \inf_{x\in[0,2\pi]} p(x,y) > 0$. Let c be the normalising constant so $\eta = cp(y)dy$ is a probability measure. Then $P(x,dy)$ is monirosed by η. unseen $\Downarrow$

Then by the minorisation theorem, there exists a unique invariant probability measure such that μ_n converges to ν in the total variation norm.
Indeed, the minorisation theorem states that if a transition probability P is minorized by a probability measure η on $\mathcal{X}$, so there exists $\alpha > 0$ such that $P(x, \cdot) \geq \alpha\eta$ for every $x \in \mathcal{X}$. Then

(1) P has a unique invariant probability measure π.

(2) Furthermore for any initial distribution $\mu \in P(\mathcal{X})$,

$$\|T^{n+1}\mu - \pi\|_{\mathrm{TV}} \leq (1-\alpha)^n \|\mu - \pi\|_{\mathrm{TV}} \leq 2(1-\alpha)^n \to 0..$$

Apply this to $\mu = \delta_1$ to conclude the proof.

unseen ⇓

6, D

4. (a)

$$\Gamma = a\delta_0 + (1-a)p(y)dy,$$

$\boxed{\text{unseen} \Downarrow}$

Firstly if f is bounded measurable,

$\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned}
Tf(x) &= \mathbf{E}[f(x_{n+1})|x_n = x] = \mathbf{E}[f(g(x_n) + c\xi_{n+1})|x_n = x) \\
&= \int_{\mathbf{R}} f(g(x_n) + cy)\Gamma(dy) \\
&= a\int_{\mathbf{R}} f(g(x) + cy)\delta_0(dy) + (1-a)\int_{\mathbf{R}} f(g(x) + cy)p(y)dy \\
&= af \circ g(x) + (1-a)\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f(cy)e^{-\frac{(y-g(x))^2}{2}}dy.
\end{aligned}$$

If $c = 0$, this is $Tf(x) = f \circ g(x)$. $\boxed{5, \text{A}}$

(b) Let f be a bounded continuous functions, note that $f \circ g$ is continuous, so is $f(y)e^{-\frac{(y-g(x))^2}{2}}$. Taking $x_n \to x$, by the dominated convergence theorem we can passing the limit to inside the integration, and conclude that Tf is continuous when f is bounded and continuous. Feller property follows.

$\boxed{\text{unseen} \Downarrow}$ $\boxed{\text{meth seen} \Downarrow}$ $\boxed{5, \text{A}}$

Next we have $|x_{n+1}| \le |g|_\infty + c|\xi_{n+1}|$. Then $\sup_n \mathbf{E}|x_n|^2 \le 2|g|_\infty + 2c^2$, which implies that $P^n(0, \cdot)$ is tight. (Can also use Markov inequality to prove this from first principle.) $\boxed{4, \text{C}}$

By Krylov-Bogoliubov theorem, the tightness together with the Feller property of transition probabilities implies that there exist an invariant probability measure. $\boxed{1, \text{C}}$

$\boxed{\text{unseen} \Downarrow}$

(c) Set $c = 0$, uniqueness fails. Indeed, $x_{n+1} = g(x_n) = \sin(x_n) + \pi\sin(\frac{x_n}{2})$. Then both δ_0 and δ_π are invariant probability measures for $x_{n+1} = \sin(x_n) + \pi\sin(\frac{x_n}{2})$. Indeed, $\gamma = \delta_0$.

$$P(x, A) = \mathbf{1}_A(\sin(x) + \pi\sin(x/2))).$$

$$\int P(x, A)d\delta_0 = 1_A(0) = \delta_0(A).$$

$$\int P(x, A)d\delta_\pi = 1_A(\pi) = \delta_\pi(A).$$

Let $c \neq 0$. The transition probability is:

$$P(x, A) = \Gamma(A - g(x)) \ge (1-a)\frac{1}{\sqrt{2\pi}}\frac{1}{c}\int e^{-\frac{1}{2}(\frac{y}{c}-g(x))^2}dy.$$

Since g is bounded, let

$$p(y) = \min_{z\in[-1-\pi, 1+\pi]} e^{-\frac{1}{2}(\frac{y}{c}-z)^2} \neq 0.$$

Then we see that the minorisation condition is satisfied, we have uniqueness by the minorisation theorem. $\boxed{5, \text{D}}$

5. (a) Let f be bounded continuous. By the weak convergence of $P_t(x, dy) \to \delta_0$,

$$\lim_{t\to 0} T_t f(x) = \lim_{t\to 0} \int_{\mathcal{X}} f(y) P_t(x, dy) \to \int_{\mathcal{X}} f(y) d\delta_x(y) = f(x).$$

$\boxed{5, \mathrm{M}}$

(b) (i) Let $\mathcal{F}_t^X$ be the right continuous σ-algebra generated by X_t. We first show that $\mathcal{F}_t^X = \mathcal{F}_t^+$. Let $a \in \mathbf{R}$, then

$$\{\omega : (B_t(\omega))^3 \le a\} = \{\omega : B_t(\omega) \le a^{\frac{1}{3}}\}.$$

Since pre-images of $(-\infty, a)$ determines the σ-algebras $\mathcal{F}_t^+$ and $\mathcal{F}_t^X$, they are the same.
Given any $f : \mathbf{R} \to \mathbf{R}$ bounded Borel measurable, so is $f \circ g$ where $g(x) = x^3$,

$$\begin{aligned}\mathbf{E}(f(X_t)|\mathcal{F}_s^X) &= \mathbf{E}(f \circ g(B_t)|\mathcal{F}_s^+) = T_{t-s} f \circ g(B_s)\\ &= \int_{\mathbf{R}} f \circ g(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|y-B_s|^2}{2(t-s)}} dy\\ &= \int_{\mathbf{R}} f(y^3) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|y-X_s^{\frac{1}{3}}|^2}{2(t-s)}} dy.\end{aligned}$$

This shows that X_t is a Markov process. We used the Markov property for B_t. $\boxed{5, \mathrm{M}}$

From the above, we extract that

$$T_t f(x) = \int_{\mathbf{R}} f(y^3) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|y-x^{\frac{1}{3}}|^2}{2(t-s)}} dy.$$

The process (X_t) has the Feller property, as given f bounded continuous, we can show $x \mapsto T_t f(x) = \int f(y) Q_t(x, dy)$ is continuous by taking $x_n \to x$, and passing the limit inside integration with the dominated convergence theorem. We know a Markov process with Feller property has the strong Markov property.

$\boxed{5, \mathrm{M}}$

(ii)

$$x_n = e^{-n} x_0 + B_n - \int_0^n B_s e^{-(n-s)} ds.$$

Since $x_t = e^{-t} x_0 + B_t - \int_0^t B_s e^{-(n-s)} ds$ is a Markov process with respect to $\mathcal{F}_t^+$, taking $s = n-1$ and $t = n$,
$\mathbf{E}(x_{n+1} \in A|\mathcal{F}_n) = x_n$, so (x_n) is a Markov chain with transition probabilities.
In fact recall that

$$T_1 f(x_0) = \mathbf{E} f(x_1) = \int_{\mathbf{R}} f(y) \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y-e^{-t}x_0)^2}{2\tilde{\sigma}^2}} dy,$$

where $\sigma = \frac{1}{2}(1-e^{-2})$. Recall also that $\pi = N(0, \frac{1}{2})$ is the invariant probability measure for x_t. In particular $x_n \sim \pi$. (If x_0 is a constant, then x_n is Gaussian with transition probability $N(e^{-n}x_0, \frac{1}{2}(1-e^{-2n})$. If x_0 is Gaussian, so are also x_n. We can check by computing its variance, and hence verify that $var(x_0) = var(x_n)$, so π is an invariant probability distribution. On can also quote the relevant results for the continuous time Ornstein-Uhlenbeck process, instead of computing, or compute the transition probabilities and verify directly.) Hence by Birkhoff's ergodic theorem,

$$\frac{1}{n}\sum_{k=1}^{n}(x_k)^2 \to \int_{\mathbf{R}} x^2 d\pi = \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} x^2 e^{-x^2} dx = \frac{1}{2}.$$

The convergence is almost surely. $\boxed{5,\ \mathrm{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96062 MATH97216 MATH97220 MATH97228	1	Generally this was done well.
MATH96062 MATH97216 MATH97220 MATH97228	2	One can easily verify the uniform measure is invariant. Make sure to verify and state the conditions for the theorems used. In part (b), a common mistake is to replace x_{k+2} by P^2 x_k. Both the snake chain and the ergodic theorem for functions of paths can be used here, need to compute the invariant measure.
MATH96062 MATH97216 MATH97220 MATH97228	3	Using minorisation theorem for the last part of the theorem: this should work for every P(x,dy) , not just for one x.
MATH96062 MATH97216 MATH97220 MATH97228	4	A common problem is to use the deterministic contraction theorem for the uniquenss while there is no contraction. The verification for the minorisation theorem: one should show it for every measure P(x,dy) --x runs through the space.
MATH96062 MATH97216 MATH97220 MATH97228	5	Part (i): it is the definition of transition function with the weak convergence v

MATH96062 MATH97216 MATH97220 MATH97228	6	The weak convergence of T_t and the function is g(x)=x^3 is invertible to show after conditioning we have a function of X_s not of B_s and x_n is a markov chain, all essential ingredients, was hardly mentioned in the scripts. A common mistake is to claim the Markov chain in the paper is stationary.