IMPERIAL

MATH70016

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Special Relativity & Electromagnetism

Date: Monday, June 2, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

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1. **Lorentz transforms**

In the following the spatial origins of frames K, K' and K'' are chosen so that $(0,0,0,0)$ refers to the same event in all frames. All axes are parallel at all time. The origin of frame K' moves with velocity V relative to frame K along its x-axis. The origin of frame K'' moves with velocity W relative to frame K' along its x-axis. Double dashed variables refer to measurements in the double dashed frame K'', single dashed variables refer to measurements in the single dashed frame K', undashed variables to measurements in the undashed frame K.

To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$. The transformation law from frame K' to frame K for contravariant four-vectors may then be written as

$$x^i = \mathcal{L}^i_j x'^j \quad \text{with} \quad \mathcal{L}^i_j = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) For any real β with $|\beta| < 1$ there exists a real ψ such that $\sinh\psi = \beta\gamma$. Show that $\cosh\psi = \gamma$.
Hint: $\cosh^2\psi - \sinh^2\psi = 1$.

(5 marks)

(b) Rewriting the contravariant transformation law from K' to K as

$$x^i = \mathcal{L}^i_j(\psi) x'^j \quad \text{with} \quad \mathcal{L}^i_j(\psi) = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the contravariant transformation law from K'' to K' as

$$x'^i = \mathcal{L}^i{}_j(\phi) x''^j$$

show that the transformation law from K'' to K is

$$x^i = \mathcal{L}^i{}_j(\psi + \phi) x''^j$$

Hint: $\cosh\psi\cosh\phi + \sinh\psi\sinh\phi = \cosh(\psi+\phi)$
and $\sinh\psi\cosh\phi + \cosh\psi\sinh\phi = \sinh(\psi+\phi)$.

(5 marks)

(c) The above entails that if $\beta\gamma = \sinh\psi$ and $(W/c)/\sqrt{1 - W^2/c^2} = \sinh\phi$ then $(U/c)/\sqrt{1 - U^2/c^2} = \sinh(\psi + \phi)$ where U is the velocity of frame K'' relative to frame K. Use this to derive the velocity addition theorem for velocities parallel to the x-axes.

(5 marks)

(d) A rocket (frame K') moves with speed V within observer frame K. Inside the rocket, a ball (frame K'') moves with speed W along its x-axis, passing through the rocket's origin at time 0 and arriving at some target at time τ', as measured in the rocket.

(5 marks)

(i) State in four-vector form the two events in frame K' of the ball passing through the rocket's origin and of the ball arriving at the target at time τ'.

(ii) Transform the two events to the observer frame K.

(iii) State the time of arrival of the ball at the target as observed in frame K.

(iv) State the total distance travelled by the ball from passing through the origin until the arrival at the target as observed in frame K.

(v) Derive the (total) speed of the ball as observed in frame K.
Hint: The result should be consistent with that obtained in Part (c).

(Total: 20 marks)

2. **Compton scattering**

Photons are characterised by an energy-momentum four-vector $p^i_\gamma = (p, \mathbf{p})$ with $p = |\mathbf{p}|$.

(a) Calculate $p^i_\gamma p_{\gamma\, i}$.

(4 marks)

(b) A photon with momentum $\mathbf{p} = (p, 0, 0)$ travelling in the positive direction of the x-axis, $p > 0$, hits an electron at rest with mass m. After the collision, the electron travels with speed V along the x-axis and the photon is deflected, travelling with energy $cp' > 0$ in the negative direction of the x-axis and thus with momentum $-p'$.

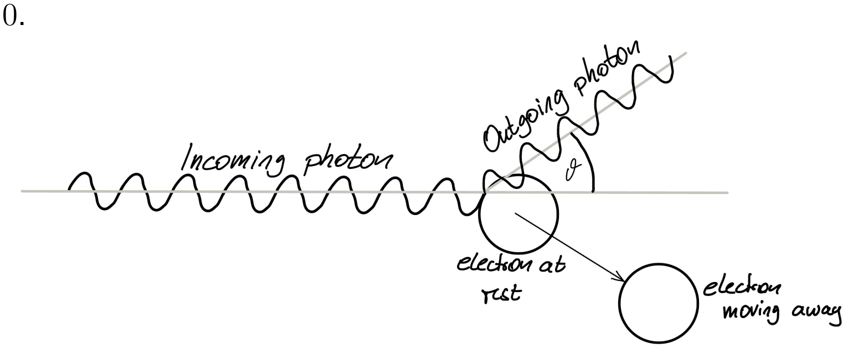
(8 marks)

(i) State the energy-momentum four-vectors of the photon and the electron before and after the collision.

(ii) State the equations following from energy and momentum conservation.

(iii) Derive an expression for p' in terms of m, c and p.
Hint: Subtract $(p + p')^2$ *from* $(p + mc - p')^2$.

(c) A photon with momentum $\mathbf{p} = (p, 0, 0)$ travelling in the positive direction of the x-axis, $p > 0$, hits an electron at rest with mass m. After the collision, the electron travels with speed V in the $x - y$ plane and the photon is deflected at an angle θ as shown, travelling with energy $cp' > 0$.



(8 marks)

(i) State the energy-momentum four-vectors of the photon and the electron before and after the collision.

(ii) Derive

$$p' = p \frac{1}{1 + \frac{p}{mc}(1 - \cos\theta)} ,$$

consistent with Part (b) with $\theta = \pi$.
Hint: Use energy and momentum conservation in four-vector form. Rearrange for the energy-momentum four-vector of the electron after the collision and project each side onto itself. Use that $p^i p_i = mc$ *for the electron before and after the collision and similarly for the photon, which has no rest mass.*

(Total: 20 marks)

3. **Particles in fields**

In the following the spatial origins of frames K and K' are chosen so that $(0,0,0,0)$ refers to the same event in both frames. All axes are parallel at all time. The origin of frame K' moves with velocity V relative to frame K along its x-axis. Dashed variables refer to measurements in the dashed frame K', undashed variables to measurements in the undashed frame K.

To ease notation, use $\gamma = 1/\sqrt{1 - V^2/c^2}$ and $\beta = V/c$.

In frame K', a constant and uniform electric field

$$\mathbf{E}' = \begin{pmatrix} 0 \\ 0 \\ E' \end{pmatrix}$$

is applied. There is no magnetic field, $\mathbf{H}' = \mathbf{0}$. A charge e with rest mass m is at rest at the origin of frame K' at time $t' = 0$.

The general formula for the Lorentz force is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H} \ .$$

(a) Calculate the momentum $\mathbf{p}'$ of the particle in the dashed frame as a function of time t' in the dashed frame.
Hint: $p'_y = p'_x = 0$.

(5 marks)

(b) Calculate the kinetic energy of the particle in the dashed frame, $\mathcal{E}'(t)$, as a function of time.
Hint: In general $\mathcal{E}^2 = m^2c^4 + p^2c^2 = m^2c^4\gamma^2$.

(5 marks)

(c) Show that the velocity component v'_z as a function of time is given by

$$v'_z = \frac{eE't}{\sqrt{m^2 + \frac{(eE't)^2}{c^2}}} \ .$$

(5 marks)

(d) Transform the electric field $\mathbf{E}'$ and the magnetic field $\mathbf{H}'$ from frame K' to frame K.
Hint: In the undashed frame, $\mathbf{E}$ *will remain parallel to the* z*-axis and* $\mathbf{H}$ *will only have a* y*-component. The transformation may be read off from*

$$F^{ik} = \mathcal{L}^i{}_j \mathcal{L}^k{}_\ell F'^{j\ell}$$

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -H'_z & H'_y \\ E'_y & H'_z & 0 & -H'_x \\ E'_z & -H'_y & H'_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5 marks)

(Total: 20 marks)

4. **Electromagnetic waves**

(a) In four-vector form, the inhomogeneous Maxwell equations in the vaccuum are

$$\frac{\partial F^{ik}}{\partial x^k} = 0$$

where F^{ik} is the electromagnetic field tensor

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \ .$$

Using the Lorenz gauge

$$\frac{\partial A^k}{\partial x^k} = 0$$

derive the wave equation for A^i in four-vector form.

(5 marks)

(b) If A^i obeys the Lorenz gauge and f obeys the wave-equation, $c^{-2}\partial_t^2 f = \nabla^2 f$, show that

$$A'^k = A^k + \left(\frac{1}{c}\partial_t f, -\nabla f\right)$$

also obeys the Lorenz gauge.

(5 marks)

(c) State any non-trivial solution of the wave-equation in the form

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

(5 marks)

(d) Assuming that the vector potential is a travelling wave, $\mathbf{A} = \mathbf{A}(ct - \mathbf{r}\cdot\mathbf{n})$, in the direction $\mathbf{n}$, show that $\partial_t \mathbf{A}$ is orthogonal to $\nabla \times \mathbf{A}$.

(5 marks)

(Total: 20 marks)

5. **Electric dipoles**

The electric dipole moment is

$$\mathbf{d} = \sum_a e_a \mathbf{r}_a$$

for point charges indexed by a located at $\mathbf{r}_a$ with charge e_a.

(a) Assuming $\sum_a e_a = 0$, show that the dipole moment is independent of the location of the origin, $i.e.$ if $\mathbf{r}'_a = \mathbf{r}_a + \mathbf{b}$ and $\sum_a e_a \mathbf{r}_a = \mathbf{d}$ then $\sum_a e_a \mathbf{r}'_a = \mathbf{d}$.

(2 marks)

(b) Calculate the dipole moment of two point charges, e^+ at $\mathbf{b}^+$ and e^- at $\mathbf{b}^-$, where $e^+ = e = -e^-$ and $\mathbf{b}^+ = \mathbf{b} = -\mathbf{b}^-$.

(2 marks)

(c) The potential $\phi(\mathbf{R})$ at position $\mathbf{R}$ of the field of a dipole moment $\mathbf{d}$ is

$$\phi(\mathbf{R}) = \frac{\mathbf{d} \cdot \mathbf{R}}{R^3} \ ,$$

where $R = |\mathbf{R}|$. Calculate or state the electric field of the dipole in the absence of a vector potential.

(4 marks)

(d) State the electric field at $\mathbf{R}$ of the two point charges in Part (b). Expand to first order in $\mathbf{b}$ and verify that the result coincides with the finding in Part (c).

(4 marks)

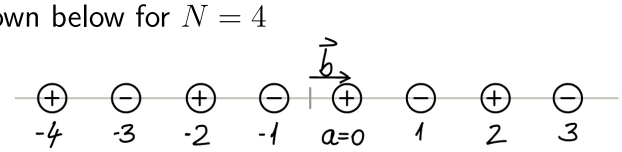
(e) Sketch the electric field around a dipole moment.

(4 marks)

(f) Calculate the dipole moment of point charges located at $\mathbf{b}_a = (1 + 2a)\mathbf{b}$ with $a = -N, -N + 1, \ldots, N - 1$ for even N and

(4 marks)

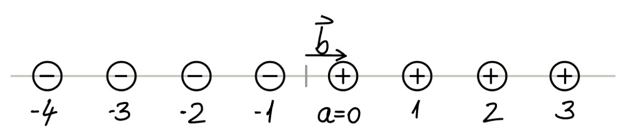
(i) charges $e_a = (-1)^a e$ as shown below for $N = 4$



(ii) charges

$$e_a = \begin{cases} -e & \text{for a<0} \\ e & \text{otherwise} \end{cases}$$

as shown below for $N = 4$



(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60016

Special Relativity and Electromagnetism (Solutions)

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1. **Lorentz transforms**

unseen $\Downarrow$

(a) This is a matter of knowing that $(1-\beta^2)\gamma^2 = 1$, which, given that $\beta^2\gamma^2 = \sinh^2\psi$, immediately produces $\gamma^2 = 1 + \sinh^2\psi = \cosh^2\psi$, so that $\boxed{\gamma = \cosh\psi \geq 1}$ and γ real.

5, A

unseen $\Downarrow$

(b) With the two transformations given, what needs to be shown is $\mathcal{L}^i_{\ j}(\psi)\mathcal{L}^j_{\ k}(\phi) = \mathcal{L}^i_{\ k}(\psi+\phi)$, which amounts to showing that

$$\boxed{\mathcal{L}^i_{\ j}(\psi)\mathcal{L}^j_{\ k}(\phi) = \begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\boxed{= \begin{pmatrix} \cosh\psi\cosh\phi + \sinh\psi\sinh\phi & \cosh\psi\sinh\phi + \sinh\psi\cosh\phi & 0 & 0 \\ \sinh\psi\cosh\phi + \cosh\psi\sinh\phi & \sinh\psi\cosh\phi + \cosh\psi\cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\boxed{= \begin{pmatrix} \cosh(\psi+\phi) & \sinh(\psi+\phi) & 0 & 0 \\ \sinh(\psi+\phi) & \cosh(\psi+\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathcal{L}^i_{\ k}(\psi+\phi)}$$

where the simplification in the last line follows from the hint.

5, B

unseen $\Downarrow$

(c) Using that $\beta\gamma = \sinh\psi$ and $\gamma = \cosh\psi$, it follows that $\beta = V/c = \tanh\psi$ and similarly $W/c = \tanh\psi$ and $U/c = \tanh\psi + \phi$. From the hint in (b) or otherwise,

$$\tanh(\psi+\phi) = \frac{\tanh\psi + \tanh\phi}{1 + \tanh\psi\tanh\phi}$$

which produces, after rearranging,

$$\boxed{U = \frac{V+W}{1+\frac{VW}{c^2}}}$$

5, D

sim. seen $\Downarrow$

(d) (i) By inspection $\boxed{(0,0,0,0)}$ and $\boxed{(c\tau', W\tau', 0, 0)}$.

(ii) By inspection $\boxed{(0,0,0,0)}$ and $\boxed{\big((c\tau' + \beta W\tau')\gamma, (W\tau' + c\tau\beta)\gamma, 0, 0\big)}$.

(iii) The time of travel can be read off from above, $\boxed{(\tau' + \beta W\tau'/c)\gamma}$.

(iv) The distance travelled can be read off from (ii), $\boxed{(W\tau' + \beta c\tau')\gamma}$.

(v) The ratio of distance over time gives observed speed,

$$U = \boxed{\frac{(W\tau' + \beta c\tau')\gamma}{(\tau' + \beta W\tau'/c)\gamma} = \frac{W + V}{1 + \frac{WV}{c^2}}}$$

using that $\beta c = V$. $\boxed{5, \mathrm{C}}$

2. **Compton scattering**

seen ⇓

(a) From $p^i_\gamma = (p, \mathbf{p})$ it follows that $\boxed{p^i_\gamma p_{\gamma\, i} = p^2 - \mathbf{p}\cdot\mathbf{p} = 0}$.

4, A

sim. seen ⇓

(b.i) Electron before $\longrightarrow$ electron after: $\boxed{(mc, 0, 0, 0) \longrightarrow (mc\gamma, mV\gamma, 0, 0)}$

Photon before $\longrightarrow$ photon after: $\boxed{(p, p, 0, 0) \longrightarrow (p', -p', 0, 0)}$.

For what follows, the sign of the momentum of the photon is crucial.

2, A

sim. seen ⇓

(b.ii) Energy conservation: $\boxed{mc^2 + pc = mc^2\gamma + p'c}$

Momentum conservation: $\boxed{p = mV\gamma - p'}$.

2, A

unseen ⇓

(b.iii) Shuffling terms as suggested gives $\boxed{(p + mc - p')^2 = m^2\gamma^2c^2}$ and $\boxed{(p + p')^2 = m^2\gamma^2V^2}$. Subtracting one from the other gives $\boxed{(p + mc - p')^2 - (p + p')^2 = m^2c^2 + 2mc(p - p') - 4pp' = m^2c^2}$, using that $\gamma^2(1-\beta^2) = 1$. Rearranging for p' then produces $\boxed{p' = pmc/(mc + 2p)}$.

4, D

sim. seen ⇓

(c.i) The added difficulty compared to (b.i) is that the photon is scattered at a given angle and the details of the electron's trajectory have to be filled in.

Electron before $\longrightarrow$ electron after: $\boxed{(mc, 0, 0, 0) \longrightarrow (mc\gamma, mV_x\gamma, mV_y\gamma, 0)}$

Photon before $\longrightarrow$ photon after: $\boxed{(p, p, 0, 0) \longrightarrow (p', p'\cos\theta, p'\sin\theta, 0)}$.

Neither V_x nor V_y are given in the question and $\gamma = 1/\sqrt{1 - (V_x^2 + V_y^2)/c^2}$ is introduced implicitly.

4, B

sim. seen ⇓

(c.ii) From energy-momentum conservation in four-vector notation, $(mc, 0, 0, 0) + (p, p, 0, 0) = (mc\gamma, mV_x\gamma, mV_y\gamma, 0) + (p', p'\cos\theta, p'\sin\theta, 0)$, it follows that $\boxed{\left((mc, 0, 0, 0) + (p, p, 0, 0) - (p', p'\cos\theta, p'\sin\theta, 0)\right)^2 = m^2c^2}$ using the square notation to indicate projection onto itself and $(mc\gamma, mV_x\gamma, mV_y\gamma, 0)^2 = m^2c^2$. On the left, the cross terms are the difficult ones,

$$\boxed{\left((mc, 0, 0, 0) + (p, p, 0, 0) - (p', p'\cos\theta, p'\sin\theta, 0)\right)^2}$$

$$\boxed{= m^2c^2 + 0 + 0 + 2mcp - 2mcp' - 2(pp' - pp'\cos\theta)}$$

Subtracting m^2c^2 makes this expression vanish by energy-momentum conservation and re-arranging produces the desired

$$\boxed{p' = p\frac{1}{1 + \frac{p}{mc}(1 - \cos\theta)}} \,.$$

4, D

3. **Particles in fields**

sim. seen ⇓

(a) There is no magnetic field, so $\dot{p}'_z = eE'$ and, together with the initial condition $\boxed{p'_z = eE't}$. All other components vanish. $\boxed{5,\ \mathrm{A}}$

sim. seen ⇓

(b) From the hint $\boxed{\mathcal{E} = \sqrt{m^2c^4 + p'^2c^2} = \sqrt{m^2c^4 + (ecE't')^2}}$. $\boxed{5,\ \mathrm{A}}$

sim. seen ⇓

(c) The momentum is related to the velocity via $p'_z = v'_z m\gamma$ and the energy to the mass, according to the hint in Part (b), via $\mathcal{E} = mc^2\gamma$, so that

$$\boxed{v'_z = \frac{p'_z}{\mathcal{E}/c^2} = \frac{eE't'}{\sqrt{m^2 + \left(\frac{eE't'}{c}\right)^2}}}\ .$$

$\boxed{5,\ \mathrm{C}}$

seen ⇓

(d) At first this looks pretty messy, but according to the hint, one has to calculate only

$$F^{ik} = \mathcal{L}^i{}_j \mathcal{L}^k{}_\ell F'^{j\ell}$$
$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -H'_z & H'_y \\ E'_y & H'_z & 0 & -H'_x \\ E'_z & -H'_y & H'_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further, in the present case,

$$\begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -H'_z & H'_y \\ E'_y & H'_z & 0 & -H'_x \\ E'_z & -H'_y & H'_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -E'_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E'_z & 0 & 0 & 0 \end{pmatrix}$$

so that one can pretty easily derive $\boxed{E_z = E'_z\gamma}$ and $\boxed{H_y = -E'_z\beta\gamma}$. $\boxed{5,\ \mathrm{B}}$

4. (a) The Maxwell equation in the vacuum, expressed in terms of the four-potential are

unseen $\Downarrow$

$$0 = \frac{\partial^2 A^k}{\partial x_i \partial x^k} - \frac{\partial^2 A^i}{\partial x_k \partial x^k} \ .$$

The first term on the right vanishes by the gauge, so that

$$\boxed{0 = \frac{\partial^2 A^i}{\partial x_k \partial x^k}} \ ,$$

which is the desired wave equation.

5, A

unseen $\Downarrow$

(b) The challenge is to get the signs right in the following. Firstly $(\partial_t f, -\nabla f) = \partial^k f$, as differentiation with respect to the contravariant coordinates gives $\partial f/\partial x^k = (\partial_t f, \nabla f) = \partial_k f$ and is covariant. So, A' can be written as $A'^k = A^k + \partial^k f$. Using the same notation, the fact that f obeys the wave equation can be written as $\partial_k \partial^k f = 0$. It follows that

$$\boxed{\frac{\partial A'^k}{\partial x^k} = \partial_k A'^k = \partial_k A^k + \partial_k \partial^k f = 0 + 0}$$

because $\partial_k A^k = 0$ from the Lorenz gauge, and $\partial_k \partial^k f = 0$ from f obeying the wave equation.

5, C

seen $\Downarrow$

(c) Anything goes, such as the plane-wave solution $\boxed{f(t,x) = A\exp(-\mathring{\imath}(ctk - kx))}$ or even just any travelling wave, $\boxed{f(t,x) = g(ct \pm x)}$.

5, A

sim. seen $\Downarrow$

(d) Writing $\boxed{\partial_t \mathbf{A} = c\mathbf{A}'}$, the curl is correspondingly

$$\boxed{\nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} -n_y A'_z + n_z A'_y \\ -n_z A'_x + n_x A'_z \\ -n_x A'_y + n_y A'_x \end{pmatrix} = -\mathbf{n} \times \mathbf{A}'}$$

which is orthogonal to $\mathbf{A}'$ itself and thus to $\partial_t \mathbf{A}$.

5, B

5. (a) This follows immediately from the definition, $\boxed{\text{unseen} \Downarrow}$

$$\boxed{\mathbf{d}' = \sum_a e_a \mathbf{r}'_a = \sum_a e_a(\mathbf{r}_a + \mathbf{b}) = \mathbf{b}\left(\sum_a a_a\right) + \sum_a e_a \mathbf{r}_a = 0 + \mathbf{d}}$$

$\boxed{\text{2, A}}$

$\boxed{\text{unseen} \Downarrow}$

(b) Again, this follows immediately from the definition,

$$\boxed{\mathbf{d} = e\mathbf{b} + (-e)(-\mathbf{b}) = 2e\mathbf{b}}$$

$\boxed{\text{2, A}}$

$\boxed{\text{unseen} \Downarrow}$

(c) Using $\mathbf{E} = -\nabla\phi$ this is a matter of multivariable calculus, producing

$$\boxed{\mathbf{E} = -\frac{\mathbf{d}}{R^3} + 3\frac{\mathbf{b}\cdot\mathbf{R}}{R^5}\mathbf{R}}\,,$$

as expected for the field of a dipole. $\boxed{\text{4, B}}$

$\boxed{\text{unseen} \Downarrow}$

(d) The two fields are $e(\mathbf{R} - \mathbf{b})/|\mathbf{R} - \mathbf{b}|^3$ and $-e(\mathbf{R} + \mathbf{b})/|\mathbf{R} + \mathbf{b}|^3$, so that $\boxed{\mathbf{E} = e\mathbf{R}/|\mathbf{R} - \mathbf{b}|^3 - e\mathbf{R}/|\mathbf{R} + \mathbf{b}|^3}$. Expanding this expression to first order in $\mathbf{b}$ gives

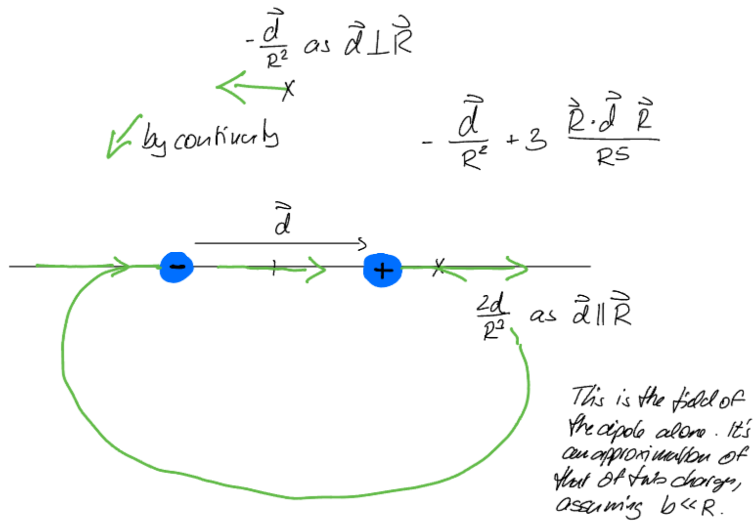
$$\boxed{\mathbf{E} = \left(\frac{e\mathbf{R}}{R^3} - \mathbf{b}\cdot\frac{e}{R^3} + 3e\frac{\mathbf{b}\cdot\mathbf{R}}{R^5}\mathbf{R} + \mathcal{O}(b^2)\right) + \left(\frac{-e\mathbf{R}}{R^3} + \mathbf{b}\cdot\frac{-e}{R^3} + 3e\frac{\mathbf{b}\cdot\mathbf{R}}{R^5}\mathbf{R} + \mathcal{O}(b^2)\right)}$$

$$\boxed{= -\frac{2e\mathbf{b}}{R^3} + 3\frac{2e\mathbf{b}\cdot\mathbf{R}}{R^5}\mathbf{R} + \mathcal{O}(b^2) = -\frac{\mathbf{d}}{R^3} + 3\frac{\mathbf{d}\cdot\mathbf{R}}{R^5}\mathbf{R} + \mathcal{O}(b^2)}$$

consistent with Part (c). $\boxed{\text{4, D}}$

(e) What is important are the directions of the field ahead of the dipole and to its side, for example unseen $\Downarrow$

Either



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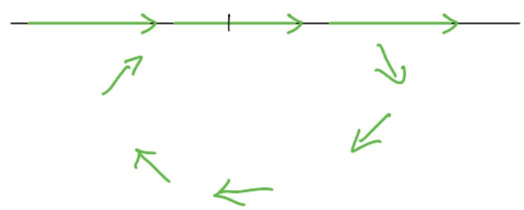
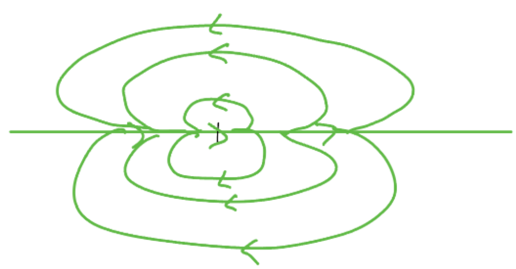


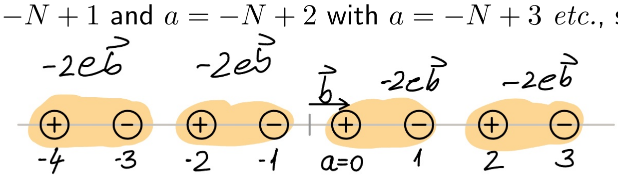
Or



... but variations are allowed. 4, C

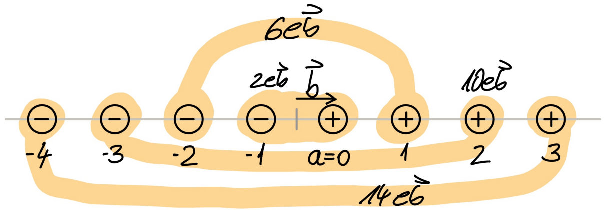
unseen $\Downarrow$

(f.i) This is a matter of pairing the charges up in a smart way, for example $a = -N$ with $a = -N+1$ and $a = -N+2$ with $a = -N+3$ *etc.*, such as



Each such pair contributes $-2e\mathbf{b}$ to the sum $\sum_a e_a \mathbf{r}_a$. Doing this repeatedly gives N such contributions, so that $\boxed{\mathbf{d} = -2Ne\mathbf{b}}$. As an aside: If N was odd, the pairs would have dipole moment $2e\mathbf{b}$ each, producing $\mathbf{d} = (-)^{N+1} 2Ne\mathbf{b}$ in general. 2, B unseen ⇓

(f.ii) Again, some smart pairing helps, as shown,



producing $2e\mathbf{b}$ for the charges at $\pm\mathbf{b}$ and correspondingly $6e\mathbf{b}$ for those at $\pm 3\mathbf{b}$ and $10e\mathbf{b}$ for those at $\pm 5\mathbf{b}$. Summing those up gives

$$\boxed{\mathbf{d} = 2e\mathbf{b}\underbrace{(1+3+5+\cdots)}_{N \text{ terms}} = 2e\mathbf{b}\left(N + 2\sum_{i=1}^{N-1} i\right) = 2e\mathbf{b}\left(N + 2\frac{N(N-1)}{2}\right) = 2e\mathbf{b}N^2}$$

Of course, the same result is obtained by direct evaluation,

$$\boxed{2e\mathbf{b}(1+3+5+\ldots+N) = 2e\mathbf{b}N^2}$$

2, A

MATH60016 Special Relativity & Electromagnetism Markers Comments

Question 1	Pretty easy question, in 1c there we as confusion about how to extract U, as some students essentially derived the velocity addition theorem from first principles. I allowed one mark for stating it, a mark for kicking things off using hyperbolic trigonometric functions and a mark once a tanh is reached.
Question 2	Only significant hurdle were b.iii and c.ii, but that was expected. The hint in b.iii helped a lot, but some students didn't use it. Occasional algebraic mistakes. Some students found very elegant solutions for c.ii.
Question 3	Most or all of this question had been covered in similar form in classes. 3d resulted in a lot of writing for some students, even when I had provided most of the writing already. There was some confusion of t vs t' in 3c, but I allowed for that in the marking. I should have asked for functions of t' rather than t.
Question 4	Well answered in general, some confusion about signed of the derivatives in 4b. The biggest hurdle was in 4d the curl. Maybe some student misunderstood the notation.