Imperial College London

MATH97029 MATH70004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Asymptotic Methods

Date: 30 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the integral

$$I(\epsilon) = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \epsilon}\, dx \quad \text{as} \quad \epsilon \searrow 0.$$

(a) Give a scaling estimate for $I(\epsilon)$. (5 marks)

(b) Briefly explain why

$$\int_0^7 \frac{dx}{x + \epsilon}$$

gives the correct leading-order approximation for $I(\epsilon)$. (5 marks)

(c) Find an asymptotic expansion for $I(\epsilon)$ to order unity.

Use the integral $\int_0^{\pi/2} \left(\frac{\cos^2 x}{\sin x} - \frac{1}{x}\right) dx = \ln \frac{4}{\pi} - 1$.

(10 marks)

(Total: 20 marks)

2. Consider the integral

$$I(x) = \int_0^{\pi} (\pi - t)^2 \, e^{ix\cos t} \, dt \quad \text{as} \quad x \to +\infty.$$

(a) Derive a leading-order approximation using the method of stationary phase. Why is only the stationary point at zero relevant?

Use the integral $\int_0^\infty e^{-i\tau^2/2} \, d\tau = \sqrt{\frac{\pi}{2}} e^{-i\pi/4}$.

(7 marks)

(b) Derive a two-term expansion using the method of steepest descent. Sketch the deformed contour in the complex-t plane and explain why only the saddle point at the origin is relevant.

Use the identity $i\cos(u + iv) = \sin u \sinh v + i\cos u \cosh v$ and the integrals

$$\int_0^\infty e^{-p^2/2} \, dp = \sqrt{\frac{\pi}{2}}, \quad \int_0^\infty p e^{-p^2/2} \, dp = 1.$$

(13 marks)

(Total: 20 marks)

3. (a) Consider the boundary value problem

$$\epsilon y'' - (2 - x^2)y = -1, \quad y'(0) = 0, \quad y(1) = 0,$$

in the limit $\epsilon \searrow 0$.

(i) Find a leading-order outer approximation. (2 marks)

(ii) Determine the location and scaling of the boundary layer. (4 marks)

(iii) Find a leading-order inner approximation. (4 marks)

(b) Consider the problem from (a) but with the differential equation replaced by

$$\epsilon y'' - (1 - x^2)y = -1.$$

(i) Find a leading-order outer approximation. (2 marks)

(ii) Determine the location and scaling of the boundary layer.

Clue: The solution is not order unity in the boundary layer.

(4 marks)

(iii) Formulate a parameter-free boundary value problem whose solution determines a leading-order inner approximation.

Clue: Note the asymptotic behaviour of solutions to the ordinary differential equation

$$F''(X) + 2XF(X) = -1 \quad \text{as} \quad X \to -\infty.$$

There is a homogeneous solution that decays exponentially, another homogeneous solution that grows exponentially, and a particular solution $\sim -1/(2X)$.

(4 marks)

(Total: 20 marks)

4. (a) Consider the delay-differential equation

$$x''(t) + \epsilon \alpha x'(t) + x(t-\epsilon) = 0, \tag{1}$$

where α is a real parameter and $0 < \epsilon \ll 1$ represents a small delay.

Use the method of multiple scales to derive a leading-order approximation in the form

$$a(\epsilon t)\cos\{t + \phi(\epsilon t)\},$$

valid for long times $t = \text{ord}(1/\epsilon)$. Derive a pair of first-order differential equations governing the slow dynamics of the amplitude $a(\epsilon t)$ and phase $\phi(\epsilon t)$. Briefly describe the solutions to (1) depending on the value of the parameter α.

Clue: $x(t-\epsilon) = x(t) - \epsilon x'(t) + O(\epsilon^2 x''(t))$.

(12 marks)

(b) Repeat the tasks in (a) for the nonlinear delay-differential equation

$$x''(t) + \epsilon \alpha x'(t) x^2(t) + x(t-\epsilon) = 0. \tag{2}$$

Use the identity $\sin\theta\cos^2\theta = \frac{1}{4}(\sin\theta + \sin 3\theta)$.

(8 marks)

(Total: 20 marks)

5. Consider the boundary value problem

$$y'' + \frac{1}{x}y' - \epsilon^2 y = \frac{2}{x^3}, \quad y(1) = 1, \quad \lim_{x\to+\infty} y = 0,$$

in the limit $\epsilon \searrow 0$.

(a) Determine the scaling of the boundary layer at infinity.

Clue: The function $2/x$ is a particular solution of $y'' + y'/x = 2/x^3$.

(6 marks)

(b) Using the method of matched asymptotic expansions, find an approximation for $y'(1)$ to leading algebraic order. Collect together terms that are of the same algebraic order.

Clue: Consider the ordinary differential equation

$$Y'' + \frac{1}{X}Y' - Y = 0.$$

Solutions that decay as $X \to +\infty$ are proportional to a Bessel function, denoted by $K(X)$, that possesses the asymptotic behaviour

$$K(X) = \ln\frac{1}{X} + \ln 2 - \gamma + O(X^2 \ln X) \quad \text{as} \quad X \searrow 0,$$

where $\gamma \doteq 0.5722\ldots$ is the Euler-Gamma constant.

(14 marks)

(Total: 20 marks)

	EXAMINATION SOLUTIONS ~~2017-18~~ 202~~1~~-22	Course
Question 1		Marks & seen/unseen
Parts		
(a)	• local contribution $\simeq \frac{1}{\varepsilon} \cdot \varepsilon = 1$ $(x = \mathrm{ord}(\varepsilon))$	5 seen similar
	• global contribution $\simeq 1 \cdot 1 = 1$	
	• integrand in overlap is $\sim \frac{1}{x}$ suggesting l.o. contribution comes from overlap and $I \simeq \ln 1/\varepsilon$.	(A)
(b)	integrand $\frac{1}{x+\varepsilon}$ has correct behaviour in overlap, while local and global contributions have same scalings as before.	5 unseen (B)
(c)	$I(\varepsilon) = \underbrace{\int_0^{\lambda} \frac{\cos^2 x}{\sin x + \varepsilon}\,dx}_{*} + \underbrace{\int_{\lambda}^{\pi/2} \frac{\cos^2 x}{\sin x + \varepsilon}\,dx}_{**},$	10 seen similar
	where $\varepsilon \ll \lambda \ll 1$.	(6C+4D)
	$* \underset{u = x/\varepsilon}{=} \int_0^{\lambda/\varepsilon} \frac{1}{1+u}\left[1 + O(\varepsilon^2 u^2)\right] du$	
	$= \ln(1+u)\Big\rvert_0^{\lambda/\varepsilon} + O\left(\varepsilon^2 \times \lambda^2/\varepsilon^2\right)$	
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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 1		Marks & seen/unseen
Parts		
(c) contin...	$\cancel{\alpha} = \ln \lambda/\varepsilon + o(1)$.	

$$\cancel{\alpha}\alpha = \int_{\lambda}^{\pi/2} \frac{\cos^2 x}{\sin x} \left(1 + O(\varepsilon/x)\right) dx$$

$$= \int_{\lambda}^{\pi/2} \frac{\cos^2 x}{\sin x}\, dx + O(\varepsilon/\lambda)$$

$$= \int_{\lambda}^{\pi/2} \frac{1}{x}\, dx + \int_{\lambda}^{\pi/2} \left(\frac{\cos^2 x}{\sin x} - \frac{1}{x}\right) dx + o(1)$$

$$= \ln\frac{\pi}{2} - \ln\lambda + \int_{0}^{\pi/2} \left(\frac{\cos^2 x}{\sin x} - \frac{1}{x}\right) dx + o(1)$$

$$= -\ln\lambda + \ln 2 - 1 + o(1)$$

Thus,

$$I(\varepsilon) = \ln 1/\varepsilon + \ln 2 - 1 + o(1)$$

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OS		2

	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 2		Marks & seen/unseen
Parts (a)	▫ $\psi = \cos t \Rightarrow \psi' = -\sin t,\ \psi'' = -\cos t, \ldots$ ▫ there are simple stationary points at $t = 0, \pi$, which are boundary points. ▫ Since $(\pi - t)^2 \simeq 1$ as $t \to 0$, $(\pi - t)^2 \simeq \lvert\pi - t\rvert^2$ as $t \to \pi$, and since the local regions for a simple stationary point are $\Delta t \simeq 1/x^{1/2}$, the local contribution from 0 is $O(1/x^{1/2})^{\dagger}$ while that from π is $O(1/x^{3/2})$. $$I \sim \pi^2 e^{ix} \int_0^\infty e^{-ixt^2/2}\,dt$$ $$= \frac{\pi^2 e^{ix}}{x^{1/2}} \int_0^\infty e^{-i\tau^2/2}\,d\tau = \frac{\pi^{5/2}}{2^{1/2} x^{1/2}} e^{ix - i\pi/4}.$$ $\dagger$ in fact $\mathrm{ord}(1/x^{1/2})$ as the calculation confirms.	7 seen similar (A)
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EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22

Course

Question 2

Marks & seen/unseen

Parts

(b)

$$I(x) = \int_{\mathcal{C}} h(t)\, e^{x\rho(t)}\, dt,$$

B seen similar

where $\mathcal{C}: 0 \to \pi$ on real axis,

$h(t) = (\pi - t)^2$, and

(11B+2D)

$\rho(t) = i\cos t = \sin u \,\mathrm{sh}\, v + i \cos u \,\mathrm{ch}\, v$,

breakdown: 7pts: sketch and choose new contour, 2(D)pts: scaling argument to determine dominant contrubtions, 4pts: calculation of asymptotic expansion

wherein $t = u + iv$. $\phi = \sin u \,\mathrm{sh}\, v$

$\psi = \cos u \,\mathrm{ch}\, v$

$\rho'(t) = -i\sin t$

$\rho''(t) = -i\cos t \Rightarrow$ simple saddle points at $t_s = n\pi$, $n = 0, \pm 1, \ldots$

$\phi = 0$ at all saddle points.

Const. phase via $t_s^{(0)} = 0$: $(\psi \equiv 1)$

$\cos u \,\mathrm{ch}\, v = 1 \Rightarrow u \sim \pm v$ as $t \to 0$

$|v| \to \infty$ as $u \to \pm\pi/2$

const. phase via other saddle points follow from periodicity.

We deform $\mathcal{C}$ into $\mathcal{C}'$ as shown in sketch:

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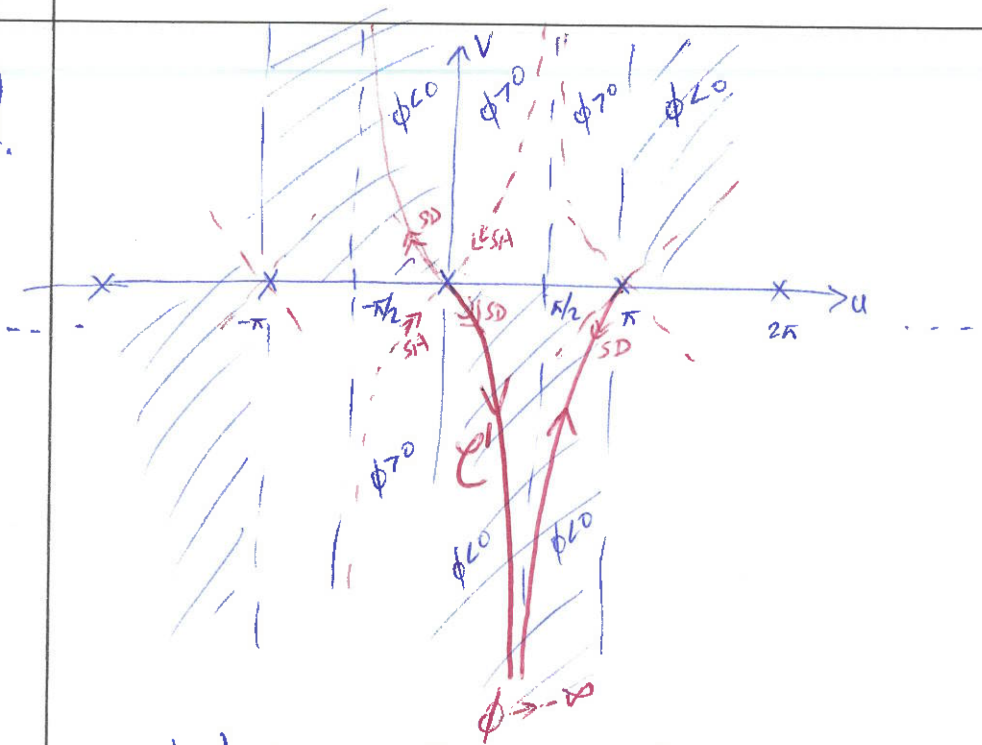
	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 2		Marks & seen/unseen

Parts (b) cont.



- contribution from $t_s^{(0)}=0$ is $\simeq 1/x^{1/2}$ (local scaling $\times$ scaling of integrand),
- there will be a correction $\simeq 1/x$ because $(\pi-t)^2 \sim \pi^2 - \underbrace{2\pi t}_{\text{ord}(1/x^{1/2})}$
- contribution from $t_s^{(1)}=\pi$ is only $\simeq 1/x^{3/2}$ because of the $(\pi-t)^2$ factor.

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EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22

Course

Question 2

Marks & seen/unseen

Parts

(b) cont..

- to calculate the contribution from $t_3^{(0)} = 0$, we locally deform to a contour which is tangent to the steepest descent path at 0.

Let $t = p e^{-i\pi/4}/x^{1/2}$,

$$I \simeq e^{-i\pi/4} x^{-1/2} \int_0^\infty \left(\pi - \frac{p e^{-i\pi/4}}{x^{1/2}}\right)^2 e^{ix\cos\left(\frac{p e^{-i\pi/4}}{x^{1/2}}\right)} dp$$

$$+ O(x^{-3/2})$$

$$= e^{-i\pi/4} x^{-1/2} \int_0^\infty \left(\pi^2 - \frac{2\pi p e^{-i\pi/4}}{x^{1/2}}\right) e^{ix} e^{-ip^2 e^{-i\pi/2}}$$

$$\cdot (1 + O(1/x))\, dp$$

$$= \frac{e^{ix - i\pi/4}}{x^{1/2}} \left[\pi^2 \int_0^\infty e^{-p^2/2} dp - \frac{2\pi e^{-i\pi/4}}{x^{1/2}} \int_0^\infty p e^{-p^2/2} dp\right]$$

$$+ O(1/x^{3/2}) \sim \frac{\pi e^{ix - i\pi/4}}{x^{1/2}} \left(\frac{\pi^{3/2}}{2^{1/2}} - \frac{2e^{-i\pi/4}}{x^{1/2}}\right)$$

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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 3		Marks & seen/unseen
Parts		
(a)(i)	$y(x;\varepsilon) \sim y_0(x), \quad y_0(x) = \frac{1}{2-x^2}$	2 seen sim. (A)
(a)ii	• outer approximation happens to satisfy $y_0'(0)=0$, but not $y_0(1)=0$, suggesting a boundary layer at $x=1$. • Scaling? Note that $y_0(1) = 1$ thus expect $y \simeq \Delta y \simeq 1$ in boundary layer. Let Δx be the boundary-layer width. Then $$\underbrace{\varepsilon y''}_{\simeq \varepsilon/\Delta x^2} - \underbrace{(2-x^2)y}_{\simeq 1} = \underbrace{-1}_{\simeq 1}$$ $\Rightarrow \Delta x \simeq \varepsilon^{1/2}$. • part (iii) confirms that l.o. boundary-layer solution can be matched to outer approximation.	4 unseen (C)
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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 3		Marks & seen/unseen

Parts

(a)iii

Let $y(x;\varepsilon) = Y(X;\varepsilon)$, where

$$X = \frac{x-1}{\varepsilon^{1/2}}.$$

Inner problem is:

$$\begin{cases} Y'' - \left(2 - (1+\varepsilon^{1/2}X)^2\right)Y = -1 \\ Y(0) = 0 \end{cases}$$

Inner expansion:

$Y(X;\varepsilon) \sim Y_0(X)$ as $\varepsilon \searrow 0$ with $X<0$ fixed.

O(1)

$$\begin{cases} Y_0'' - Y_0 = -1 \\ Y_0(0) = 0 \text{ Boundary condition} \\ Y_0(-\infty) = 1 \text{ matching condition} \end{cases} \Rightarrow \boxed{Y_0 = 1 - e^X}$$

4 seen sim.

(B)

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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 3		Marks & seen/unseen
Parts (b)(i)	• Now, $$y_0(x) = \frac{1}{1-x^2}.$$	2 seen sim. (A)
(b)(ii)	• Again $y_0'(0)=0$ suggesting boundary layer at $x=1$. • Note however that $y_0 \sim \frac{1}{2(1-x)}$ as $x \to 1$, thus if ΔX is the boundary-layer scaling then $y, Y \simeq \frac{1}{\Delta X}$ here. • ODE scaling balance gives $$\underbrace{\varepsilon y''}_{\simeq \frac{\varepsilon \Delta y}{\Delta X^2} \simeq \varepsilon/\Delta X^3} - \underbrace{(1-x)(1+x)y}_{\simeq \Delta X \cdot y \simeq 1} = \underbrace{-1}_{\simeq 1}$$ $$\Rightarrow \Delta X \simeq \varepsilon^{1/3}, \quad y \simeq \varepsilon^{-1/3}.$$	4 unseen (D)
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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 3		Marks & seen/unseen
Parts (b) iii	Let $y(x;\varepsilon) = Y(X;\varepsilon)$, where $X = \frac{x-1}{\varepsilon^{1/3}}$. Inner expansion is $Y(X;\varepsilon) \sim \varepsilon^{-1/3} Y_0(X)$ as $\varepsilon \to 0$ with $X < 0$ fixed. $\underline{O(1)}$ $\begin{cases} Y_0'' + 2XY_0 = -1 \\ Y_0(0) = 0 \quad \text{boundary condition} \\ Y_0(X) \to 0 \quad \text{as } X \to -\infty \\ \qquad\qquad (\text{matching}) \end{cases}$ * the decay condition is enough since there is a particular solution $\sim -\frac{1}{2X}$ as $X \to -\infty$ and the homogeneous solutions are growing/decaying exponentially as $X \to -\infty$.	4 unseen (D)
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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 4		Marks & seen/unseen

Parts

(a)

- $x(t;\varepsilon) = y(\tau, T; \varepsilon)$ on $\tau = t$, $T = \varepsilon t$,
- $x(t-\varepsilon) = x(t) - \varepsilon x'(t) + O(\varepsilon^2 x'')$
- $y(\tau, T;\varepsilon) \sim y_0(\tau, T) + \varepsilon y_1(\tau, T) + \ldots$ required to be regular.
- y satisfies

$$y_{\tau\tau} + 2\varepsilon y_{\tau T} + \varepsilon^2 y_{TT} + \varepsilon\alpha(y_\tau + \varepsilon y_T)$$
$$+ y - \varepsilon(y_\tau + \varepsilon y_T) + O(\varepsilon^2 y_\tau, \varepsilon^3 y_T) = 0$$

$\underline{O(1)}$: $\quad y_{0\tau\tau} + y_0 = 0$

$$\Rightarrow \quad y_0 = a(T)\cos(\tau + \phi(T))$$

$\underline{O(\varepsilon)}$: $\quad y_{1\tau\tau} + y_1 = -2y_{\tau T} - \alpha y_\tau + y_{0\tau}$

$$\text{RHS} = 2\left(a_T \sin(\tau+\phi) + a\phi_T \cos(\tau+\phi)\right)$$
$$+ (\alpha - 1)\, a \sin(\tau+\phi)$$

12

seen sim. (delay term unseen)

(10A+2D)

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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 4		Marks & seen/unseen

Parts

(a) cont...

to avoid secular terms

$$\begin{cases} a_T = \frac{1}{2}(1-\alpha)a \\ \phi_T = 0 \end{cases}$$

- oscillations grow for $\alpha < 1$
- oscillations decay for $\alpha > 1$
- for $\alpha = 1$, oscillations steady

on ord$(1/\varepsilon)$ time scales.

(b) Same as (a) but now

8

seen sim./unseen

$$RHS = -2y_{0\tau T} - \alpha y_{0\tau} y_0^2 + y_{0\tau}$$

$$= 2(a_T \sin(\tau+\phi) + a\phi_T \cos(\tau+\phi))$$

$$+\alpha a^3 \sin(\tau+\phi)\cos^2(\tau+\phi) - a\sin(\tau+\phi)$$

(6A+2C)

use: $\sin\theta\cos^2\theta = \frac{1}{4}(\sin\theta + \sin 3\theta)$

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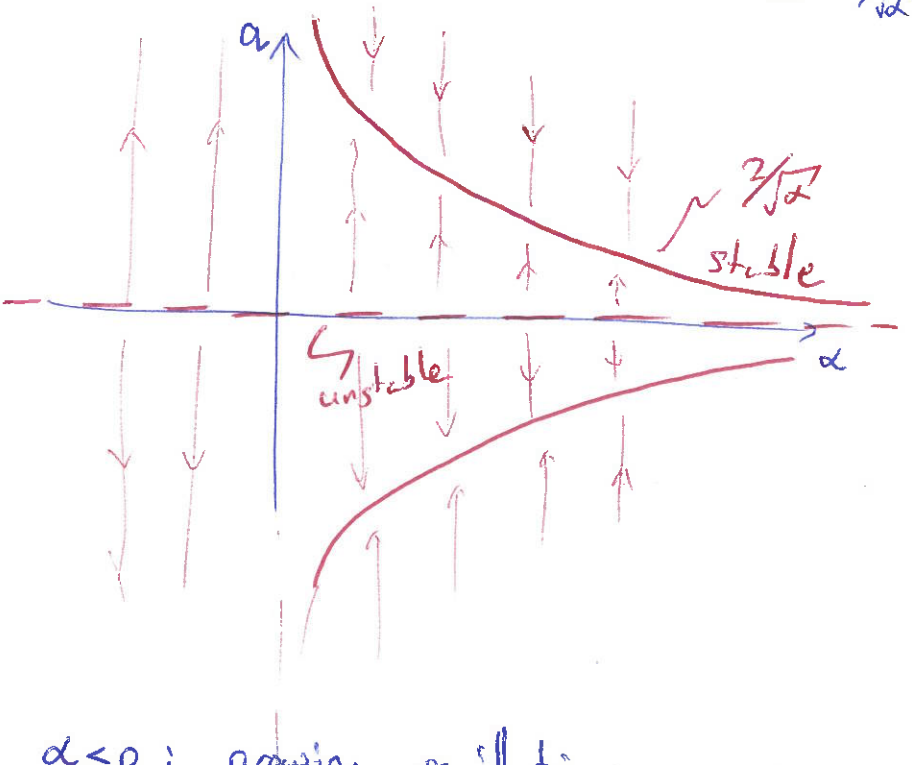
	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 4		Marks & seen/unseen
Parts (b) cont...	to remove secular terms: $$\begin{cases} 2a_T + \frac{1}{4}\alpha a^3 - a = 0 \\ \phi_T = 0 \end{cases}$$ $$a_T = \frac{1}{2} a \left(1 - \frac{\alpha}{4} a^2\right)$$ fixed points: $a = 0$; $a = 2/\sqrt{\alpha}$ for $\alpha > 0$  $\alpha \leq 0$: growing oscillations $\alpha > 0$: approach limit cycle of magnitude $2/\sqrt{\alpha}$.	
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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 5		Marks & seen/unseen
Parts (a)	outer:	6 seen sim.

$y(x;\varepsilon) \sim \tilde{y}_0(x)$ as $\varepsilon \downarrow 0$.

$O(1)$

$$\begin{cases} \tilde{y}_0'' + \frac{1}{x}\tilde{y}_0' = \frac{2}{x^3} \\ \tilde{y}_0(1) = 1 \end{cases}$$

$\Rightarrow \tilde{y}_0 = \tilde{a} \ln x + \frac{2}{x} - 1$, $\tilde{a}$ = const

Cannot satisfy condition as $x \to \infty$.

Note that, for large x,

$$\frac{\varepsilon^2 \tilde{y}_0}{\tilde{y}_0''} \simeq \frac{\varepsilon^2}{1/x^2} \simeq \varepsilon^2 x^2$$

up to log factors

suggesting boundary layer at ∞ where $x \simeq 1/\varepsilon$.

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	EXAMINATION SOLUTIONS ~~2017/18~~ 2021-22	Course
Question 5		Marks & seen/unseen
Parts		
(b)	• inner limit: $\varepsilon \searrow 0$ with $0 < X = \varepsilon x$ fixed. $y(x;\varepsilon) = Y(X;\varepsilon)$. • Inner problem: $\begin{cases} Y'' + \frac{1}{X} Y' - Y = \varepsilon \frac{2}{X^3} \\ \lim_{X\to\infty} Y = 0, \\ \text{matching as } X \searrow 0. \end{cases}$ • Going back to outer, $\tilde{a}$ must be zero to avoid Y being asymptotically large. {This would result in contradiction when using the clue and matching back to the outer.} Then, however, $\tilde{y}_0 \to -1$ as $X \to +\infty$, which also leads to a similar contradiction. • Given the form of the outer problem, and the need to eliminate the approach to -1 of the outer solution,	14 unseen/ seen sim.
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	EXAMINATION SOLUTIONS ~~2017/18~~ 2021-22	Course
Question 5		Marks & seen/unseen
Parts (b) cont.		

the outer expansion should be extended:

$$y(x;\varepsilon) \sim \frac{2}{x} - 1 + \frac{\tilde{a}_1}{\ln 1/\varepsilon} \ln x,$$

where $\tilde{a}_1 = 1$ $\left(\text{then } -1 + \frac{\tilde{a}_1}{\ln 1/\varepsilon} \ln 1/\varepsilon = 0 \right)$

- The $\frac{1}{\ln 1/\varepsilon}$ term in the outer suggests $Y \sim \frac{1}{\ln 1/\varepsilon} K(X)$, using the clue, which will result in a $\frac{1}{\ln^2 1/\varepsilon}$ term in the outer expansion etc.
- We proceed by collecting together terms which are of the same algebraic order. That is

$$y(x;\varepsilon) \sim a(\varepsilon) \ln x + \frac{2}{x} - 1,$$

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Question 5		Marks & seen/unseen

Parts

(b) cont...

$$Y(X;\varepsilon) \sim A(\varepsilon) K(X),$$

where $a(\varepsilon)$, $A(\varepsilon)$ depend logarithmically upon ε, and the errors are algebraically small. (implied by form of equations and matching below...)

- Van Dyke matching $O(1):O(1)$, to log factors:

$$a(\varepsilon) \ln X/\varepsilon - 1 = A(\varepsilon)\left[\ln \frac{1}{X} + \ln 2 - \gamma\right]$$

thus
- $A(\varepsilon) = -a(\varepsilon)$
- $-1 + a \ln 1/\varepsilon = -a(\ln 2 - \gamma)$

$$\Rightarrow \quad a(\varepsilon) = \frac{1}{\ln 1/\varepsilon + \ln 2 - \gamma}$$

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	EXAMINATION SOLUTIONS ~~2017-18~~ 2021-22	Course
Question 5		Marks & seen/unseen
Parts (b) cont.	Based on our outer approximation: $$y'(1) \sim -2 + \frac{1}{\ln 1/\varepsilon + \ln 2 - \gamma},$$ with algebraically small error.	
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Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Asymptotic methods MATH60004 MATH97029 MATH70004	1	A question on asymptotic evaluation of an integral using the method of splitting the range of integration. Some did well but many worked without systematically evaluating errors, leading to incorrect results.
Asymptotic methods MATH60004 MATH97029 MATH70004	2	Standard question on asymptotic evaluation of integrals (method of stationary phase and method of steepest descent). Many students did fairly well, though some had technical difficulties with contour deformation and others did not estimate correctly the dominant contributions.
Asymptotic methods MATH60004 MATH97029 MATH70004	3	A basic question on the method of matched asymptotic expansions, with parts (b)(ii) and (b)(iii) testing deeper understanding. Many students did well.

Asymptotic methods MATH60004 MATH97029 MATH70004	4	A rather straightforward question based on the method of multiple scales, concerned with an oscillator featuring weak delay. Many students did this question well, both using the method to derive the slow-time dynamics and interpreting the solutions. Some students did not describe the dynamics correctly owing to algebraic mistakes, which could have been identified based on a basic intuitive understanding of the governing ODE.
Asymptotic methods MATH60004 MATH97029 MATH70004	5	This mastery question required a more deeper understanding of asymptotic matching — the students had to set up their asymptotic expansions systematically in order to be able to make use of the information provided in the hint. Several students did very well.