IMPERIAL

MATH70008

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamical Systems

Date: Tuesday, May 21, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

In all your answers, you may quote results derived in the course (lecture notes inclusive of exercises) without proof (unless where a proof is explicitly asked for in the question), but any such results must be carefully stated or referenced.

The exam questions feature the following three continuous maps of the interval $[0, 1]$ to itself:

$$\begin{aligned}
U(x) &:= 4x^2 - 4x + 1,\\
V(x) &:= \begin{cases} 1 - 2x & \text{if } x \in [0, \frac{1}{2}],\\ 2x - 1 & \text{if } x \in (\frac{1}{2}, 1].\end{cases}\\
W(x) &:= \begin{cases} 1 - 4x & \text{if } x \in [0, \frac{1}{4}],\\ x - \frac{1}{4} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}],\\ \frac{3}{4} - x & \text{if } x \in (\frac{1}{2}, \frac{3}{4}],\\ 4x - 3 & \text{if } x \in (\frac{3}{4}, 1].\end{cases}
\end{aligned}$$

Their graphs are sketched in Figure 1.

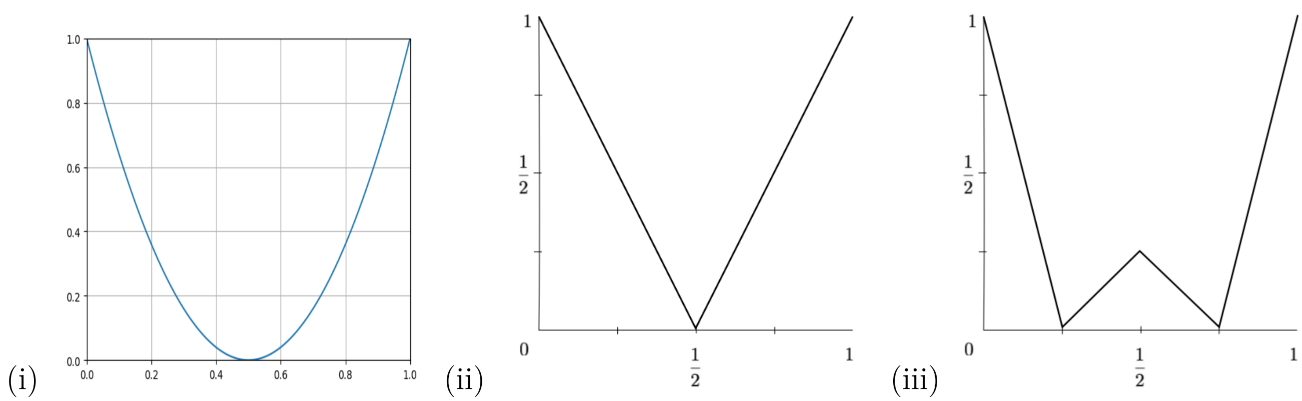

Figure 1: Graphs of the interval maps (i) U (ii) V, and (iii) W.

1. (a) Define the properties of topological transitivity and topological mixing for a continuous interval map $f : [0,1] \to [0,1]$. (4 marks)

(b) Give the definition of a topological conjugacy between two continuous interval maps $f, g : [0,1] \to [0,1]$. (4 marks)

Consider the interval maps U, V and W as defined on Page 2:

(c) Determine whether W is topologically transitive and/or topologically mixing. Motivate your answer. (4 marks)

(d) Show that U and V are topologically conjugate. (4 marks)

(e) Show that U and W are not topologically conjugate. (4 marks)

(Total: 20 marks)

2. Consider the interval map W, as defined on Page 2, and let $I_0 = (0, \frac{1}{4})$, $I_1 = (\frac{1}{4}, \frac{1}{2})$, $I_2 = (\frac{1}{2}, \frac{3}{4})$ and $I_3 = (\frac{3}{4}, 1)$.

Let Σ_4^+ denote the set of half-infinite sequences in the symbols $0, 1, 2, 3$.

Let

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and let $\Sigma_{4,A}^+ \subset \Sigma_4^+$ consist of those sequences in Σ_4^+ that do not contain any subsequence ij, when the matrix element $A_{ij} = 0$.

(a) Show that $h : \Sigma_{4,A}^+ \to [0,1]$, with

$$h(\omega_0\omega_1\omega_2\ldots) := \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} W^{-i}(I_{\omega_i})}, \tag{1}$$

is well-defined (i.e. that $h(\omega) \in [0,1]$ for all $\omega \in \Sigma_{4,A}^+$) and surjective, but not injective. (5 marks)

(b) Determine $h^{-1}(\frac{1}{2})$, $h^{-1}(\frac{1}{3})$, and $h(\overline{02})$. (6 marks)

(c) Let the shift $\sigma_A : \Sigma_{4,A}^+ \to \Sigma_{4,A}^+$ be defined as $\sigma_A(\omega_0\omega_1\omega_2\ldots) := \omega_1\omega_2\ldots$ Show that $W \circ h = h \circ \sigma_A$. (4 marks)

(d) Let $P_n(W) := \#\{x \in [0,1] \mid W^n(x) = x\}$. Show that $\lim_{n\to\infty} \frac{1}{n}\ln(P_n(W)) = \ln(1+\sqrt{3})$. (5 marks)

(Total: 20 marks)

3. Consider the interval map W, as defined on Page 2.

(a) Find the invariant measure μ of W that is absolutely continuous with respect to the Lebesgue measure. Explain why this measure is unique. (Hint: recall Question 2(c).) (6 marks)

(b) Let $DW(x) := \frac{\partial W}{\partial x}(x)$, if $\frac{\partial W}{\partial x}(x)$ exists and $DW(x) := 1$, otherwise. Define the *finite-time Lyapunov exponent*

$$\lambda_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} \ln |DW(W^k(x))|,$$

and let $X \subset [0,1]$ denote the set of points in $[0,1]$ for which the (infinite-time) *Lyapunov exponent* $\lambda(x) := \lim_{n\to\infty} \lambda_n(x)$ exists, i.e. $X := \{x \in [0,1] \mid \lambda(x) \text{ exists}\}$.

(i) Show that the range of observable Lyaponov exponents is given by $\{\lambda(x) \mid x \in X\} = [\ln 2, 2\ln 2]$. (Hint: recall Question 2(c).) (4 marks)

(ii) Show that the Lyapunov exponent is Lebesgue almost surely constant, i.e. that $\lambda(x) = \overline{\lambda}$ for Lebesgue almost all $x \in [0,1]$. Determine $\overline{\lambda}$. (5 marks)

(c) Consider the interval map U, as defined on Page 2. Show that this map has a unique absolutely continuous invariant probability measure and determine its density. (Hint: recall Question 1(d).) (5 marks)

(Total: 20 marks)

4. Let $f : [0,1] \to [0,1]$ be continuous and consider the dynamical system generated by f. Let the topological entropy of f be defined as

$$h_{\text{top}}(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \ln \operatorname{sep}(n, \varepsilon, f). \tag{2}$$

(a) (i) Define $\operatorname{sep}(n, \varepsilon, f)$. (4 marks)

(ii) Show that $h_{\text{top}}(f)$ is well-defined, in the sense that the limit in (2) exists. (6 marks)

(b) Consider the shift σ_A on the topological Markov chain $\Sigma^+_{4,A}$, as defined in Question 2(c). Let $\Sigma^+_{4,A}$ be endowed with the metric

$$d^{\Sigma^+}(\omega, \nu) := \sum_{i=0}^{\infty} \frac{\delta(\omega_i, \nu_i)}{3^i}, \text{ where } \delta(\omega_i, \nu_i) = \begin{cases} 0 & \text{if } \omega_i = \nu_i, \\ 1 & \text{if } \omega_i \neq \nu_i. \end{cases}$$

(i) Determine the topological entropy of σ_A, i.e. $h_{\text{top}}(\sigma_A)$. (4 marks)

(ii) Determine the topological entropy of ${\sigma_A}^2 := \sigma_A \circ \sigma_A$, i.e. $h_{\text{top}}({\sigma_A}^2)$. Motivate your answer. (6 marks)

(Total: 20 marks)

5. (a) Let $f : S^1 \to S^1$ be an orientation preserving circle homeomorphism with continuous lift $F : \mathbb{R} \to \mathbb{R}$.

The rotation number $\rho(f)$ of f is defined as follows:

$$\rho(f) := \rho(F) \quad \mathrm{mod}\ 1, \text{ where } \rho(F) := \lim_{n\to\infty} \frac{F^n(x) - x}{n}. \tag{3}$$

(i) Show that the limit in the definition of $\rho(F)$ in Eqn. (3) exists. (4 marks)

(ii) Show that $\rho(f)$ does not depend on the choice of lift F, nor on the choice of the initial point x. (4 marks)

(b) Let $f, g : S^1 \to S^1$ be orientation preserving circle homeomorphisms. Determine whether the following statements are true or false (give a proof or a counter example).

(i) $\rho(f \circ g) = \rho(f) + \rho(g) \bmod 1$. (4 marks)

(ii) $\rho(f) + \rho(f^{-1}) = 0 \bmod 1$. (4 marks)

(iii) If $\rho(f) = \alpha \in [0, 1)$ and $g \circ f = r_\alpha \circ g$, where $r_\alpha(x) := x + \alpha \bmod 1$, then $f = r_\alpha$. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH96040/MATH97065/MATH97176/MATH97285

Dynamical Systems (Solutions)

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1. (a) A continuous map $f : [0,1] \to [0,1]$ is *topologically transitive* if for any pair of open sets $A, B \subset X$ there exists $n \in \mathbb{N}_0$ such that $f^n(A) \cap B \neq \emptyset$. A continuous map $f : [0,1] \to [0,1]$ is *topologically mixing* if for any pair of non-empty open sets $A, B \subset X$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $f^n(A) \cap B \neq \emptyset$.

seen ⇓

4, A

(b) Two continuous maps $f, g : X \to X$ are *topologically conjugate* if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g \circ h$.

seen ⇓

4, A

(c) W is both topologically mixing and topologically transitive. It suffices to proof topological mixing as this implies topological transitivity. Let P_n be the partition of $[0,1]$ in 4^n (closed) intervals of size 4^{-n}. For each of these intervals $I \in P_n$, we have that either $W(I) \in P_{n-1}$ (if $I \subset [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$), or that $W(I) \in P_n$ (if $I \subset [\frac{1}{4}, \frac{3}{4}]$). However, in the latter case $W^2(I) \in P_{n-1}$. Hence there exists $k \leq 2n$ such that $W^k(I) = [0,1]$. As every open subset of $[0,1]$ contains a partition element of P_n for some value of n (sufficiently large), it follows that for every open U, there exists $N > 1$ such that $W^k(U) = [0,1]$ for all $k \geq N$. This implies that W is topologically mixing.

sim. seen ⇓

4, B

(d) Let $R(x) := 1 - x$, then $R \circ U \circ R^{-1} = f_4(x) := 4x(1-x)$ (logistic map) and $R \circ V \circ R^{-1} = T$, the "tent map". The logistic map and the tent map are known to be topologically conjugate by the transformation $h(x) = \sin^2(\frac{\pi}{2}x)$, in the sense that $f_4 \circ h = h \circ T$. Hence $U \circ (R \circ h \circ R^{-1}) = (R \circ h \circ R^{-1}) \circ V$. We compute $(R \circ h \circ R^{-1})(x) = 1 - \sin^2(\frac{\pi}{2}(1-x)) = 1 - \cos^2(\frac{\pi}{2}x) = \sin^2(\frac{\pi}{2}x) = h(x)$, Hence h also conjugates U to V: $U \circ h = h \circ V$. (Direct computation of course also suffices.)

sim. seen ⇓

4, B

(e) As U is topologically conjugate to V, it suffices to show that V is not topologically conjugate to W. As topological conjugacy implies equal numbers of fixed points, it suffices to show that V^2 has a different number of fixed points than W^2. (It turns out that V and W have the same number of fixed points, so hence we resort to consider the second iterate.) To compare we sketch the graphs of V and W and examine the intersection of these graphs with the diagonal. From the figure it shows that V^2 has 4 fixed points and W^2 has 8 fixed points. This is an obstruction for V and W to be topologically conjugate.

unseen ⇓

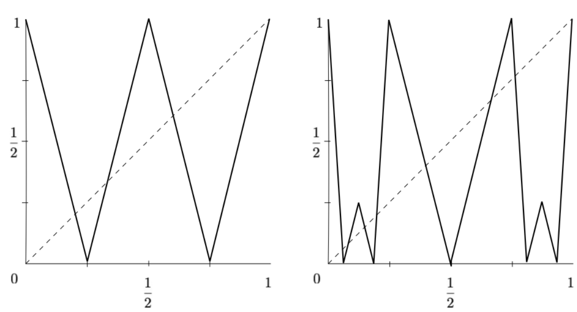

Figure 1: Graphs of the interval maps V^2 (left) and W^2 (right). Diagonals "$x = y$"are also depicted.

4, B

2. (a) $\{I_0, I_1, I_2, I_3\}$ is a Markov partion of $[0,1]$ for the map W, i.e. if $W(I_i) \cap I_j \neq \emptyset \Rightarrow I_j \subset W(I_i)$. This implies that for each fixed $n \geq 1$, for each admissible subsequence $\omega_0 \dots \omega_{n-1}$ in $\Sigma^+_{4,A}$ sim. seen ⇓

$$h(\omega_0\omega_1\omega_2 \dots \omega_{n-1}) := \overline{\bigcap_{i=0}^{n-1} W^{-i}(I_{\omega_i})}, \tag{1}$$

is an interval. W is eventually expanding on this partition, and $|h(\omega_0\omega_1\omega_2 \dots \omega_{n-1})| \leq 2^{-n}$. As result, $\lim_{n\to\infty} h(\omega_0\omega_1\omega_2 \dots \omega_{n-1}) \in [0,1]$.

h is surjective since, due to the Markov partition property, the union of the intervals $h(\omega_0 \dots \omega_{n-1})$ over all $\Sigma^+_{4,A}$-admissible subsequences is equal to $[0.1]$ for any n. This thus also holds in the limit as $n \to \infty$, resulting in surjectivity of h.

h is not injective since two symbol sequences may represent the same point through h: boundary points of the partition elements I_1 and I_2 and their preimages under W correspond to exactly 2 codes in $\Sigma^+_{4,A}$. For instance, $h^{-1}(\frac{1}{4}) = \{00\overline{3}, 10\overline{3}\}$. 5, B

(b) $W^3(\frac{1}{2}) = W^2(\frac{1}{4}) = W(0) = 1$ and $W(1) = 1$. Hence $h^{-1}(\frac{1}{2}) = \{100\overline{3}, 200\overline{3}\}$, the different codes reflect ambiguity of the left and right limits of coding intervals near $\frac{1}{2}$. Note that there is no ambiguity at the level of the second symbol, since both limits approach $\frac{1}{4}$ from the left. $W^3(\frac{1}{3}) = W^2(\frac{1}{12}) = W(\frac{2}{3}) = \frac{1}{12}$. Hence $h^{-1}(\frac{1}{3}) = 1\overline{02}$. There is no ambiguity in the code as $\frac{1}{3} \neq \frac{p}{4^n}$ for any positive integer values of p, n (this set contains all W-preimages of boundary point of I_1, I_2. $h(\overline{02}) = \frac{1}{12}$, as by the previous calculation. unseen ⇓ 6, A

(c) By continuity of W, for any $\omega \in \Sigma^+_{4,A}$ sim. seen ⇓

$$W \circ h(\omega) \;=\; \lim_{n\to\infty} W\left(\overline{\bigcap_{i=0}^{n-1} W^{-i}(I_{\omega_i})}\right) = \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} W^{-i+1}(I_{\omega_i})}.$$

The Markov property of the partition implies that $W(I_{\omega_i}) \supset I_{\omega_{i+1}}$ so that

$$\begin{aligned}\lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} W^{-i+1}(I_{\omega_i})} \;&\supset\; \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} W^{-i}(I_{\omega_{i+1}})} \\ &=\; \lim_{n\to\infty} \overline{\bigcap_{i=0}^{n-1} W^{-i}(I_{\sigma_A(\omega)_i})} = h \circ \sigma_A(\omega).\end{aligned}$$

Hence, $W \circ h(\omega) \supset h \circ \sigma_A(\omega)$. This implies that $W \circ h(\omega) = h \circ \sigma_A(\omega)$, since left- and right-hand side are singletons in $[0,1]$. 4, A

(d) Let $P_n(\sigma_A) := \#\{\omega \in \Sigma^+_{4,A} \mid \sigma^n_A(\omega) = \omega\}$, then $P_n(\sigma_A) = \mathrm{Tr}(A^n)$ (by result from the course) and $\lim_{n\to\infty} \frac{1}{n} \ln(P_n(\sigma_A)) = \lim_{n\to\infty} \frac{1}{n} \ln(\mathrm{Tr}(A^n)) = \ln r(A)$, where $r(A)$ is the spectral radius of A. To determine the eigenvalues of A, we first note that as has two pairs of equal rows, A has two eigenvalues equal to 0. The remaining eigenvectors are of the form $(a, b, b, a)^T$ satisfing the characteristic equations $2a + 2b = \lambda a$ and $a = \lambda b$, leading to the quadratic equation $2\lambda + 2 = \lambda^2$, resulting in $\lambda = 1 \pm \sqrt{3}$. Hence $r(A) = 1 + \sqrt{3}$. It remains to be verified that $\lim_{n\to\infty} \frac{1}{n} \ln(P_n(\sigma_A)) = \lim_{n\to\infty} \frac{1}{n} \ln(P_n(W))$. We observe that ambiguities in the coding arise only in the set $S := \cup_{n=0}^{\infty} W^{-n}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}) = \cup_{n=1}^{\infty} W^{-n}(1)$. So for all $x \in S$, there exists n such that $W^n(x) = 1$. Recall that $W(1) = 1$. Hence, S does not contain any periodic points and hence the equality holds. unseen ⇓ 5, D

3. (a) It is known (from the course notes) that for an eventually piecewise linear expanding map with a finite Markov partition, like W - see Question 2, a stationary measure can be found that is equivalent to Lebesgue and constant on each element of the Markov partition. It is derived from the finite state Markov process with connectivity matrix A, with transition probabilities corresponding to the transport of Lebesgue measure. In this case, we derive from A in Question 2, meth seen ⇓

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

The (normalized) left eigenvector v of P for eigenvalue 1 is calculated to be $v = \frac{1}{6}(3,1,1,1)$. As the intervals of the partition all have equal size, the corresponding invariant measure has density $p(x) := \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{6} & \text{if } x \in (\frac{1}{4}, 1]. \end{cases}$ This invariant measure is equivalent to Lebesgue measure in the sense that its density is $[0,1]$. From course notes it follows that this measure is ergodic (since A is irreducible). If there was another absolutely continuous invariant measure ν, its support would intersect the support of μ in a positive Lebesgue measure set. If ν's support is $[0,1]$, it would be need to be ergodic, but this would imply that $\nu = \mu$ (see course notes). Otherwise, a smaller support for ν would contradict ergodicity of μ, since it would imply the existence of an invariant set A with $\mu(A) \in (0,1)$. $\boxed{6, \text{A}}$

(b) If $x \in I_1 \cup I_2$ then $\ln DW(x) = 0$. Let S be equal to the pre-image set of 1, i.e. $S := \cup_{n=0}^{\infty} W^{-n}(1)$. S contains all points x where $\frac{\partial W}{\partial x}$ does not exist. $\ln DW(x) = 2\ln 2$ if $x \in (I_0 \cup I_3) \setminus S$. S is countable. unseen ⇓

(i) According to the symbolic dynamics (Question 2(c)), once $x \in I_1 \cup I_2$, $W(x) \in I_0 \cup I_3$, where orbits can stay arbitrarily long. For all $x \in S$, $\lambda(x) = 2\ln 2$ (since such points land in a finite number of iterates on the fixed point 1, and $DW(1) = 2\ln 2$). As sequences in $\Sigma_{4,A}^{+}$ consist of arbitrary concatenations of symbol blocks 0, 3, 10 and 20. Where the first blocks contribute $2\ln 2$ to the average, the latter two blocks contribute $\ln 2$. As sequences exist with any frequency of these blocks arising, the range of observable Lyapunov exponents is equal to $\{p \ln 2 + (p-1) 2\ln 2 \mid p \in [0,1]\} = [\ln 2, 2\ln 2]$. $\boxed{4, \text{D}}$

(ii) By ergodicity of μ, Birkhoff's ergodic theorem establishes that for μ almost all $x \in [0,1]$, $\lambda(x) = \int_0^1 \ln |DW(x)| d\mu = \frac{1}{2}\ln 4 + \frac{1}{3}\ln 1 + \frac{1}{6}\ln 4 = \frac{2}{3}\ln 4 = \frac{4}{3}\ln 2 =: \overline{\lambda}$. As μ is equivalent to Lebesgue (Question 3(a)) , $\lambda(x) = \overline{\lambda}$ for Lebesgue almost all $x \in [0,1]$. unseen ⇓ $\boxed{5, \text{C}}$

(c) As V is conjugate to U by a homeomorphism h (see Question 1(d)), if ν is the unique absolutely continuous ergodic invariant measure of V, then $\tilde{\nu} = h_* \nu$ is the unique absolutely continuous ergodic invariant measure for U. As V is a piecewise expanding full branch map of the interval, Lebesgue measure is an ergodic invariant measure of V (lecture notes). We have $d\tilde{\nu}(x) = d\nu(h^{-1}(x)) = \frac{d}{dx}(h^{-1}(x))dx = \frac{1}{h'(h^{-1}(x))}dx$. As $h = \sin^2(\frac{\pi}{2}x)$, $h'(x) = \pi\sqrt{\sin^2(\frac{\pi}{2}x)(1-\sin^2(\frac{\pi}{2}x))}$, so that the lebesgue density p of μ is given by $p(x) = (h'(h^{-1}(x)))^{-1} = (\pi\sqrt{x(1-x)})^{-1}$. part seen ⇓ $\boxed{3, \text{B}}$

$\boxed{2, \text{C}}$

4. (a) (i) For a continuous map $f:[0,1]\to[0,1]$ with (Euclidean) metric d, for any $n\in\mathbb{N}$, we define $d_n(x,\tilde{x}) := \max_{0\le k\le n-1} d(f^k(x), f^k(\tilde{x}))$,the maximum distance between the first n iterates of f of points $x,\tilde{x}\in[0,1]$.
A subset $A\subset X$ is (n,ε)-*separated* if any two distinct points in A are at least ε apart in the metric d_n. $\operatorname{sep}(n,\varepsilon,f)$ is the maximum cardinality of an (n,ε)-separated set.

seen ⇓

4, A

(ii) $\operatorname{sep}(n,\varepsilon,f)$ is finite due to compactness of $[0,1]$, Namely, as $[0,1]$ is compact wrt to the (Euclidean) metric d then so is $[0,1]$ wrt the metric d_n, since they are topologically equivalent. Thus $h_\varepsilon(f) := \limsup_{n\to\infty}\frac{1}{n}\ln\operatorname{sep}(n,\varepsilon,f)$ exists. If $\tilde{\varepsilon}<\varepsilon$ then $\operatorname{sep}(n,\varepsilon,f)\le\operatorname{sep}(n,\tilde{\varepsilon},f)$ since every (n,ε)-separated set is also $(n,\tilde{\varepsilon})$-separated. Hence $h_\varepsilon(f)\le h_{\tilde{\varepsilon}}(f)$, and because of this monotonicity $\lim_{\varepsilon\to0} h_\varepsilon(f)$ exists (in $\mathbb{R}_{\ge0}\cup\{\infty\}$).

unseen ⇓

6, C

(b) (i) By results from the notes, $h_{\text{top}}(\sigma_A)=\ln r(A)$ where $r(A)$ denotes the spectral radius of A. Hence $h_{\text{top}}(\sigma_A)=\ln(1+\sqrt{3})$ (For a calculation of $r(A)$, see the answer to Question 2(d).)

meth seen ⇓

4, A

(ii) In the proof of the topological entropy for a shift on a topological Markov chain $\Sigma^+_{k,A}$ in the lecture notes, the topological entropy corresponds to the exponential growth rate of the number of cylinder sets corresponding to admissible paths of length $m-1+\tilde{n}(n)$, $||A^{m-1+\tilde{n}(n)}||_1$, where $||\cdot||_1$ denotes the matrix 1-norm. Here $\tilde{n}(n)$ is number of symbols shifted in n iterations and the limit $m\to\infty$ corresponds to the limit $\varepsilon\to0$. In particular, for the shift σ_A we have $\tilde{n}(n)=n$, and

unseen ⇓

$$h_{\text{top}}(\sigma_A)=\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\ln||A^{m-1+n}||_1.$$

For the shift ${\sigma_A}^2$ we have $\tilde{n}(n)=2n$, so that

$$\begin{aligned} h_{\text{top}}({\sigma_A}^2) &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\ln||A^{m+2n-1}||_1,\\ &= 2\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{2n}\ln||A^{m+2n-1}||_1,\\ &= 2\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\ln||A^{m+n-1}||_1 = 2h_{\text{top}}(\sigma_A). \end{aligned}$$

Hence $h_{\text{top}}({\sigma_A}^2)=2\ln(1+\sqrt{3})$.

6, D

5. (a) (i) Suppose $F^q(x) = x + p$ for some $x \in [0,1)$ and some $p, q \in \mathbb{N}$ (i.e. f has a q-periodic point). For $n \in \mathbb{N}$, write $n = kq + r$, $0 \le r < q$. Then $F^n(x) = F^r(F^{kq}(x)) = F^r(x + kp) = F^r(x) + kp$. Since $|F^r(x) - x|$ is bounded for $0 \le r < q$, $\lim_{n\to\infty} \frac{F^n(x)-x}{n} = \lim_{k\to\infty} \frac{F^r(x)+kp-x}{kp+r} = \frac{p}{q}$. Thus the rotation number exists and is rational whenever f has a periodic point. seen ⇓

Suppose that $F^q(x) \neq x + p$ for all $x \in \mathbb{R}$ and $p, q \in \mathbb{N}$. By continuity, for each pair $p, q \in \mathbb{N}$, either $F^q(x) > x + p$ for all $x \in \mathbb{R}$, or $F^q(x) < x + p$ for all $x \in \mathbb{R}$. For $n \in \mathbb{N}$, choose $p_n \in \mathbb{N}$ so that $p_n - 1 < F^n(x) - x < p_n$ for all $x \in \mathbb{R}$. Then for any $m \in \mathbb{N}$,

$$m(p_n - 1) < F^{mn}(x) - x < \sum_{k=0}^{m-1} F^n(F^{kn}(x)) - F^{kn}(x) < mp_n,$$

implying that $\frac{p_n}{n} - \frac{1}{n} < \frac{F^{mn}(x)-x}{mn} < \frac{p_n}{n}$. Interchanging the roles of m and n, we also have $\frac{p_m}{m} - \frac{1}{m} < \frac{F^{mn}(x)-x}{mn} < \frac{p_m}{m}$. Thus, $|\frac{p_m}{m} - \frac{p_n}{n}| < |\frac{1}{m} + \frac{1}{n}|$, so $\{\frac{p_n}{n}\}$ is a Cauchy sequence, and $\frac{F^n(x)-x}{n}$ converges as $n \to \infty$. 4, M

(ii) Let $F : \mathbb{R} \to \mathbb{R}$ (continuous) denote a lift of f: $\pi \circ F = f \circ \pi$ with $\pi : \mathbb{R} \to S^1$, $\pi(x) = x \bmod 1$. Hence $F(x+1) = F(x) + 1$. By continuity of F, the lift F is unique up to an integer additive constant. If $\tilde{F} = F + k$ ($k \in \mathbb{Z}$) then $\rho(\tilde{F}) = \rho(F) + k$. Hence $\rho(f)$ is independent on the choice of the lift.

Since F maps any interval of length 1 to an interval of length 1, it follows that $|F^n(x) - F^n(y)| \le 1$ if $|x - y| \in [0,1)$. Thus $|(F^n(x) - x) - (F^n(y) - y)| \le |F^n(x) - F^n(y)| + |x - y| \le 2$, so $\lim_{n\to\infty} \frac{F^n(x)-x}{n} = \lim_{n\to\infty} \frac{F^n(y)-y}{n}$. Hence $\rho(F)$ is independent of the initial condition x. 4, M

(b) (i) False. For instance, let $f(x) = 0.01\sin^2(2\pi x) + x \bmod 1$ and $g(x) = 0.01\cos^2(2\pi x) + x \bmod 1$. Then $\rho(f) = \rho(g) = 0$, since both maps have fixed points. But $f \circ g(x) = 0.01\sin^2(0.02\pi\cos^2(2\pi x) + 2\pi x) + 0.01\cos^2(2\pi x) + x \bmod 1$ has no fixed point since for all $x \in [0,1)$ we have $0 < 0.01\sin^2(0.02\pi\cos^2(2\pi x) + 2\pi x) + 0.01\cos^2(2\pi x) < 1$. unseen ⇓

4, M

(ii) True. $0 = \rho(F \circ F^{-1}) = \lim_{n\to\infty} \frac{F^n \circ F^{-n}(x) - x}{n} = \lim_{n\to\infty} \frac{F^n(F^{-n}(x)) - F^{-n}(x)}{n} + \lim_{n\to\infty} \frac{(F^{-1})^n(x) - x}{n} = \rho(F) + \rho(F^{-1})$. Hence $\rho(f) + \rho(f^{-1}) = 0 \bmod 1$. unseen ⇓

4, M

(iii) False. For instance, Let $g(x) = 0.1\sin(2\pi x) + x$ and $f = g^{-1} \circ r_{\frac{1}{2}} \circ g$. Then $f \neq r_{\frac{1}{2}}$ since $g \circ r_{\frac{1}{2}}(x) = 0.1\sin(2\pi x + \pi) + x + \frac{1}{2} = -0.1\sin(2\pi x) + x + \frac{1}{2} \neq r_{\frac{1}{2}} \circ g(x) = 0.1\sin(2\pi x) + x + \frac{1}{2}$. unseen ⇓

4, M

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks