4 Classification of 2nd-order Quasi-linear PDEs in Two Variables

Why didn't our first finite difference program work very well? This was especially the case with using Eqn. (3.16), while using Eqn. (3.18) gave a much more accurate result. There must be something wrong with (a) the mathematical problem itself, (b) our solution algorithm, or (c) our computational implementation. Usually when this happens one suspects that there must be a bug in our program – failing that, our solution method may be at fault. But sometimes, the problem we set out to solve is not well-posed. In this lecture we will introduce some theory and illustrate some ways in which a PDE problem might be insoluble.

Most physical systems are governed by second order PDEs. The majority of problems fall into three main categories: *equilibrium, eigenvalue and propagation problems*. In this course we shall not deal with eigenvalue problems*.

Equilibrium problems are generally steady state ones in which the equilibrium, state ϕ in a domain D is to be determined by solving the differential equation

$$L[\phi] = f \tag{4.1}$$

within D, subject to certain boundary conditions $B_i[\phi] = g_i$ on the boundary of D. Often the integration domain D is closed and bounded. Fig. 4.1 illustrates the general equilibrium problem - such problems are generally known as *boundary value problems (BVPs)*. Examples are steady viscous flow, steady temperature distributions and equilibrium stresses in structures among others. Generally the governing equations for equilibrium problems are *elliptic*.

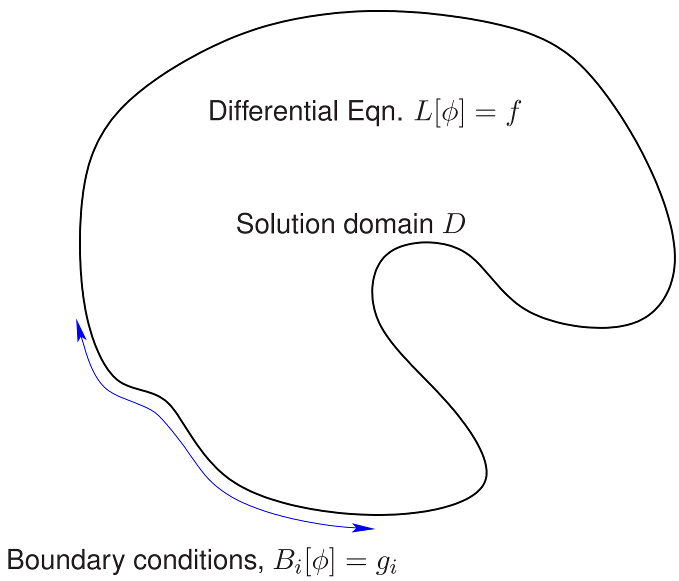

Figure 4.1: Equilibrium or bounded boundary value problem.

Propagation problems are *initial value problems (IVPs)* that have an unsteady state or a transient nature. In such problems one seeks the subsequent behaviour of the system

*The finite difference techniques we discuss for solving equilibrium problems, are generally similar to those used to discretise eigenvalue problems.

given an initial state. This is done by solving some differential equation

$$L[\phi] = f \tag{4.2}$$

within the domain D, when the initial state is prescribed as

$$I_i[\phi] = h_i \tag{4.3}$$

and subject to prescribed conditions

$$B_i[\phi] = g_i \tag{4.4}$$

on some open boundaries. Fig. 4.2 illustrates a typical propagation or *initial value problem*. Examples include propagation of pressure waves in a fluid, propagation of stresses and displacements in elastic systems, heat propagation and self excited vibrations amongst others. There are though two distinct classes of problems here, namely **parabolic** and **hyperbolic** types.

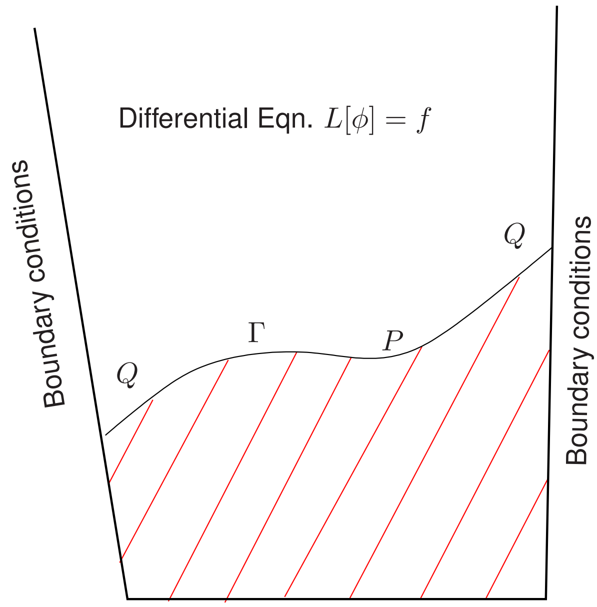

Figure 4.2: Propagation or initial value problem.

A distinction between equilibrium and propagation problems is that in the former, the entire solution is dependent upon satisfaction of all the boundary conditions and all internal requirements. In propagation problems the solution is marched out from the initial state guided during the marching process by the side boundary conditions and governing PDE.

In this course we discuss **Finite Difference Methods (FDMs)** for solving such equations. We want our algorithms to be able to reproduce the physics and so to begin with, we must understand the physical background. We consider the equation for $u(x, y)$

$$a u_{xx} + b u_{xy} + c u_{yy} = f \; . \tag{4.5}$$

This equation is called **quasi-linear** provided the functions a, b, c and f do not depend on u_{xx}, u_{xy} or u_{yy}. They may, however depend on x, y, u, u_x and u_y, so that (4.5) is not necessarily **linear** – for example it is perfectly allowable in the discussion that follows that

$$f = du_x + eu_y + hu + g, \tag{4.6}$$

provided d, e, h, g functions retain the *quasi-linear* requirement.

Classification of equations is best accomplished by developing the concept of **characteristics**. Suppose we know u, u_x and u_y along some curve Γ in (x, y)-space. From a point P on Γ we move a small vector displacement $(dx,\ dy)$ to a new point Q not on Γ, as shown in Fig. 4.3. Under what circumstances can we determine uniquely the values of u, u_x and u_y at Q? We denote the change in these variables by du, $d(u_x)$ and $d(u_y)$. Then by the chain rule for partial derivatives, the total derivative

$$du = u_x dx + u_y dy,$$

which is known because u_x and u_y are known along Γ. Similarly we may define

$$\left.\begin{aligned} d(u_x) &= u_{xx}dx + u_{xy}dy \\ d(u_y) &= u_{xy}dx + u_{yy}dy \end{aligned}\right\}. \tag{4.7}$$

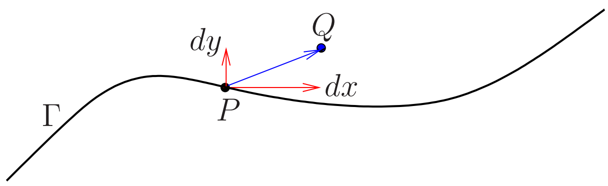

Figure 4.3: Problem description.

We combine (4.5) and (4.7) in matrix form:

$$\begin{pmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix}, \tag{4.8}$$

a, b and c are known locally because u, u_x and u_y are known, and so the 3×3 matrix is known. Equation (4.8) will have a unique solution for u_{xx}, u_{xy} and u_{yy} *if* the determinant is **not** zero – then the derivatives have the same values above and below the curve Γ. If the determinant of the matrix vanishes, that is

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = a(dy)^2 - b\,dxdy + c(dx)^2 = 0, \tag{4.9}$$

the equation (4.8) will have either no solution or infinitely many solutions. The condition for solutions to exist, which we will use later in the course, is that

$$a\frac{d(u_x)}{dx} + c\frac{d(u_y)}{dy} = f\ . \tag{4.10}$$

This is surprising. If we can choose a direction (dx, dy) which satisfies (4.9) we have the possibility that the second derivatives u_{xx} etc. may not be uniquely defined. In other words, the solution may have discontinuities across the line PQ, with u_{xx} taking different values on each side.

It is very important to know whether our solution can have this property. Equation (4.9) is called the **Characteristic equation** of (4.5). It is a quadratic in dy/dx with solution

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \; . \tag{4.11}$$

Equation (4.5) is classified as hyperbolic, parabolic or elliptic according to whether these roots are real. Note the sign of b in the formula.

$$\text{If} \left\{ \begin{array}{lll} b^2 - 4ac > 0, & 2 \text{ real roots (4.5) is} & \textbf{hyperbolic} \\ b^2 - 4ac = 0, & 1 \text{ real roots (4.5) is} & \textbf{parabolic} \\ b^2 - 4ac < 0, & 0 \text{ real roots (4.5) is} & \textbf{elliptic} \end{array} \right\} . \tag{4.12}$$

For hyperbolic equations, (4.11) is an ODE for $y(x)$ which can be integrated to define two sets of curves (one for the $(+)ve$ sign, one for the $(-)ve$ sign), called the **characteristics** of (4.5).

4.1 Example: Significance of characteristics

Consider the one-dimensional wave equation for $u(x, t)$ with constant c

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad t > 0, \qquad \text{with} \quad u(x,0) = F(x) \;\text{ and }\; u_t(x,0) = G(x) \; . \tag{4.13}$$

This equation has the general solution $u = f(x - ct) + g(x + ct)$ for any functions f and g and with the above boundary conditions we have d'Alembert's solution:

$$u(x, t) = \tfrac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi) d\xi \; . \tag{4.14}$$

Using (4.11) the characteristics of (4.13) are two families of straight lines

$$\frac{dt}{dx} = \pm\frac{1}{c} \qquad \text{or} \quad x \pm ct = \text{constant.} \tag{4.15}$$

From the actual solution, we see that the solution at some point P or (x_0, t_0) with $t_0 > 0$ depends only on some of the initial data, that for which

$$x_0 - ct_0 \le x \le x_0 + ct_0.$$

Only points from which characteristics going forwards in time can reach the point P can influence the solution at P. The set of such points is called the **domain of dependence** of P. In exactly the same way not all points with $t > t_0$ can be affected by the solution at P. The collection of such points is called the **domain of influence** of P.

This behaviour is easy to understand physically, if one interprets characteristics as **curves along which information travels at a finite speed**. If something happens at P it takes a certain time before news of it reaches another point. For (4.13), the characteristics are parallel lines, but for more general hyperbolic systems they will be curved and may meet. Since the higher derivatives are indeterminate along these curves they provide paths for the propagation of discontinuities.

In such cases discontinuities may form. If neighbouring characteristics touch, conflicting information arrives at the same point, leading to the creation of **shock waves** (such as sonic booms, or pressure fronts); see Figs. 4.4–4.7. Such discontinuities then propagate into the medium along the characteristics.

It is clear from this example that whether or not characteristics exist is vital for the understanding and therefore the numerical modelling of a problem. They are associated with *"time-like"* behaviour, and have a characteristic speed associated with them, defining the rate at which information travels. In contrast elliptic problems have no "time-like" variable; x and y behave like space coordinates.

4.1.1 General solution of the wave equation D'Alembert's solution

The one-dimensional wave equation is defined

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{4.16}$$

Changing variables to $r,\ s$ where,

$$r = x - ct, \quad s = x + ct \tag{4.17}$$

Eqn. 4.16 becomes

$$\frac{\partial^2 u}{\partial r \partial s} = 0.$$

The general solution to this is

$$u = f(r) + g(s),$$

i.e.

$$u = f(x - ct) + g(x + ct); \tag{4.18}$$

Moreover note we can differentiate these expressions to give

$$\frac{\partial u}{\partial t} = -cf(x - ct) + cg(x + ct) \tag{4.19}$$

The arbitrary functions $f,\ g$ can be described completely if u and $\partial u/\partial t$ are given at $t = 0$, *i.e.* if

$$u = \xi(x), \quad \frac{\partial u}{\partial t} = \eta(x) \ \text{ at } \ t = 0,$$

where $\xi(x),\ \eta(x)$ are given functions of x, then, by substituting into Eqns. 4.18 and 4.19 a pair of equations is obtained for f and g,

$$\begin{aligned} \xi(x) &= f(x) + g(x) \\ \eta(x) &= -cf'(x) + cg'(x). \end{aligned} \tag{4.20}$$

Integrating the second expression gives

$$\int_{x_o}^{x} \eta(z)\ dz = -cf(x) + cg(x). \tag{4.21}$$

Solving Eqn. 4.20 and 4.21 gives

$$\begin{aligned} f(x) &= \frac{1}{2}\left[\xi(x) - \frac{1}{c}\int_{x_o}^{x} \eta(z)\ dz\right] \\ g(x) &= \frac{1}{2}\left[\xi(x) + \frac{1}{c}\int_{x_o}^{x} \eta(z)\ dz\right]. \end{aligned} \tag{4.22}$$

Hence

$$\begin{aligned} f(x-ct) &= \frac{1}{2}\left[\xi(x-ct) - \frac{1}{c}\int_{x_o}^{x-ct} \eta(z)\ dz\right] \\ g(x+ct) &= \frac{1}{2}\left[\xi(x+ct) + \frac{1}{c}\int_{x_o}^{x+ct} \eta(z)\ dz\right]. \end{aligned} \tag{4.23}$$

Finally substituting into Eqn. 4.18 gives

$$u = \frac{1}{2}\left[\xi(x-ct) + \xi(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \eta(z)\ dz\right]. \tag{4.24}$$

4.2 Boundary conditions for well-posed problems

A problem involving a PDE is said to be 'well-posed' if three conditions hold :

1. A solution exists.
2. The solution is unique.
3. The solution is continuous in the boundary conditions, *i.e.* small changes in the boundary conditions do not lead to large changes in the local solution.

The first requirement states the obvious fact that we cannot possibly think of finding a numerical approximation of the solution if it does not exist. The second states that the numerical scheme should always give one solution – if we get a numerical solution that oscillates between two or more, then clearly the solution is not unique. The last condition is the most important : a solution of the PDE is physically meaningful if small changes in the data cause small changes to the solution. In other words, a mathematical model of a physical phenomenon in general will not be useful if small errors in the measured data would lead to drastically different solutions. If this last condition fails to hold, the problem is non-physical and a disaster for numerical modelling.

Whether or not a problem is well-posed depends critically on whether its boundary conditions are appropriate. Typical boundary conditions are:

(a) boundary value problems (BVP); the PDE holds in some closed region and the solution is constrained all over the boundary;

(b) initial value problems (IVP); One or more constraints are given on some curve (usually $t = 0$) only partially bounding the region in which the PDE holds.

4.3 Hyperbolic equations

From our discussion of characteristics, it is clear that hyperbolic systems should have initial value conditions. Information spreads out from the initial values at a finite rate.

An example of an **ill-posed** hyperbolic BVP for $u(x,\, y)$ with a non-unique solution is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \text{ in } \left\{ \begin{array}{l} 0 < x < 1, \\ 0 < y < 1, \end{array} \right\} \text{ with } \left\{ \begin{array}{l} u(x,0) = u(x,1) = 0 \\ u(0,y) = u(1,y) = 0. \end{array} \right\}. \tag{4.25}$$

This problem has the solution $u = A \sin n\pi x \sin n\pi y$ for any constant A and integer n.

4.4 Elliptic equations

These have no characteristics; no lines along which information travels, which suggests that IVPs are inappropriate. A typical elliptic equation is **Laplace's equation**

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \in D, \tag{4.26}$$

where D is some region of (x, y)-space. It can be shown that this equation together with one boundary condition (say $u = f$) on the boundary ∂D can give a well-posed problem, with a smooth solution. In contrast, the IVP

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \in y > 0 \qquad \text{with} \quad u(x, 0) = u_y(x, 0) = 0 \tag{4.27}$$

is ill-posed, even though the solution $u(x, y) = 0$ is unique! Suppose we perturb the initial conditions so that $u(x,\, 0) = \varepsilon \sin nx$, and $u_y(x,\, 0) = 0$, where $0 < \varepsilon \ll 1$ and n is arbitrary but large. No matter how big n is, $|\sin nx| \leq 1$, so we are not altering the boundary condition by more than ε. The (unique) solution to this new problem is

$$u = \varepsilon \sin nx \cosh ny \simeq \varepsilon \sin nx\, \tfrac{1}{2} e^{|ny|} \qquad \text{when} \quad |ny| \gg 1\ . \tag{4.28}$$

So a small distance away from the initial line $y = 0$, the solution is now exponentially large, whereas for the unperturbed problem it was zero. If a tiny (albeit very wiggly) perturbation to the boundary conditions can lead to a vast difference in the solution the problem is physically meaningless and impossible to model numerically. This example, due to Hadamard, shows that IVPs for elliptic equations are discontinuous in the boundary conditions.

4.5 Parabolic equations

A typical example is the **diffusion equation** for $u(x, t)$ with constant diffusivity K:

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad u(x,\, 0) = f(x)\ . \tag{4.29}$$

As we know $u(x, 0)$, we can calculate $u_t(x, 0) = K f''(x)$ from the equation, and so we would expect to be able to step away from $t = 0$. From (4.12), the characteristics are given by the repeated root $dt/dx = 0$ or $t =$ constant. This corresponds to an infinite speed of propagation of information. Is the solution stable? Once more we consider the boundary condition $f(x) = \varepsilon \sin nx$, so that the unique solution of (4.29) is

$$u(x,\, t) = \varepsilon \sin nx\, e^{-n^2 K t}\ . \tag{4.30}$$

When $\varepsilon = 0$ the solution is $u = 0$. When $\varepsilon > 0$ the solution decays away provided $Kt > 0$, but if $Kt < 0$ and n is large, the perturbed solution blows up once more.

Thus, parabolic equations require one initial condition and it is vital that we move “forwards in time." Physically, parabolic equations describe the smoothing out of an initial configuration towards an equilibrium. Many different initial conditions give rise to almost the same final state. This is why running the process backwards in time is an ill-posed problem. You **can not** un-stir a cup of tea!

4.6 Change of type : fluid flow example

Consider the partial differential equation, the so-called Prandtl-Glauert equation for subsonic or supersonic flow

$$(1 - M_\infty^2)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{4.31}$$

as M_∞ varies from values of $M_\infty < 1$ to $M_\infty > 1$, a change in PDE-type and behaviour of physical phenomenon occurs. A slightly more complicated form is the transonic small disturbance equation, namely

$$[K - (\gamma + 1)\phi_x]\,\phi_{xx} + \phi_{yy} = 0, \tag{4.32}$$

where $K = c(1 - M_\infty^2)$ with (c, γ) are constant parameters. This equation is nonlinear due to the $\phi_x \phi_{xx}$ term and on changing sign *i.e.* $K > 0$ and $K < 0$ the equation switches from elliptic to hyperbolic type.

An alternative to the above is the steady compressible potential equation :

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial^2 \phi}{\partial x^2} - \frac{2uv}{a^2}\frac{\partial^2 \phi}{\partial x \partial y} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4.33}$$

where

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y} \tag{4.34}$$

and a is the speed of sound,

$$\frac{a^2}{\gamma - 1} + \frac{u^2 + v^2}{2} = const. \tag{4.35}$$

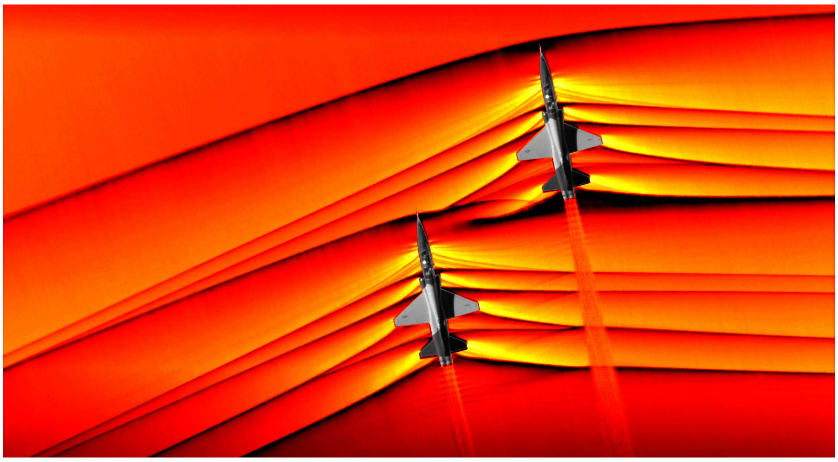

Figure 4.4: (A) Navier-Stokes Equations in action : Supersonic flow

Another example of mixed character of PDEs and complexity of solution to expect is the following steady form of the incompressible Navier-Stokes equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}\right\}. \tag{4.36}$$

Figure 4.5: (B) Navier-Stokes Equations in action : Transonic flow (1)

where ν is the kinematic viscosity, (u, v) are velocities and p is the pressure. These equations (4.38) can be shown to be elliptic. However, in a certain domain of the field of interest, it can be shown that a subset of the above operates, namely in the immediate neighbourhood of fluid flow over a solid surface, the following PDEs describe the flow behaviour:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial p}{\partial y} &= 0 \end{aligned}\right\} . \tag{4.37}$$

These equations adequately describe the so-called "viscous boundary-layer" and can be shown to be of *nonlinear* parabolic type, while far away from the surface the flow is adequately described by the inviscid Euler *elliptic* form:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= 0 \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= 0 \end{aligned}\right\} . \tag{4.38}$$

This complexity of solutions is shown in figure 4.6.

Figure 4.7 shows the typical complexity of solutions possible in various domains in the flow over an aerofoil, varying from elliptic, hyperbolic and parabolic type behaviours, all in the same equations set, and remarkably nature is effortlessly able to slip from one type to another!

Clearly, an understanding of each type of equation is essential, prior to undertaking a "*numerical attack*". On the computer, we shall consider and use the **Finite Difference Method**

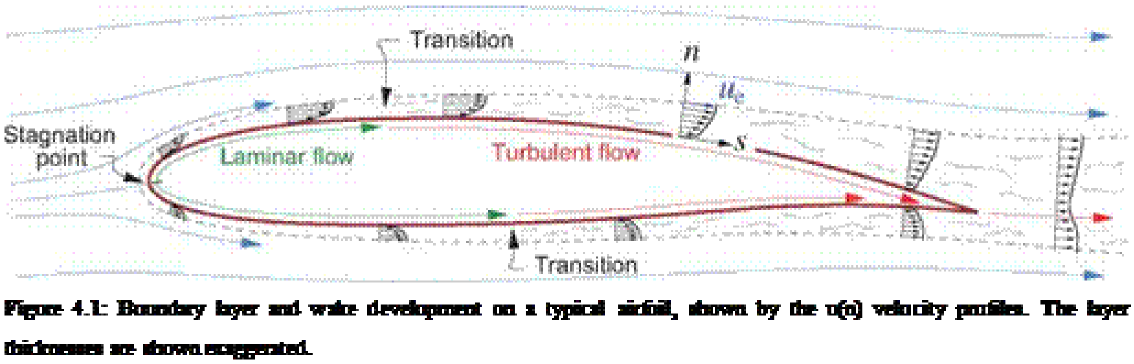

Figure 4.6: (D) Navier-Stokes Equations in action : Boundary Layers

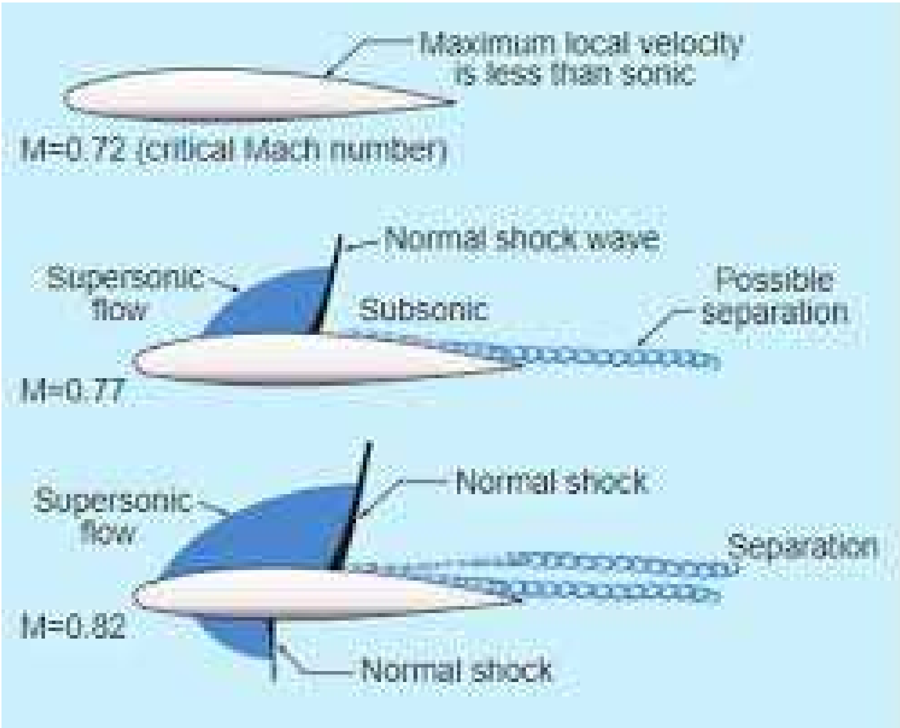

Figure 4.7: (C) Navier-Stokes Equations in action : Transonic flow (2)

(FDM) to solve the PDEs that we discuss. We will find, that the most appropriate numerical solve strategy involves an *apriori* insight into the mathematical and physical character of the problem, and the associated properties of the PDE – **together** with the correct application of boundary conditions whether on some surface or at some so-called far-field unbounded domain.

Summary

Equation type	Appropriate B.C.	Method of solution
Hyperbolic	Initial	Step in either direction from initial line
Parabolic	Initial	Step in one direction only from initial line
Elliptic	Boundary	Must solve everywhere simultaneously

Examples of various equation types

Hyperbolic	Parabolic	Elliptic
Solutions may be discontinuous	Smooth solutions	Smooth solutions
(characteristic speed c)	(diffusivity K)	
Maxwell's equations	Heat equation	Electrostatics (Poisson)
Unsteady 1-D compressible	Unsteady incompressible N-S	Potential flow
Steady 2-D supersonic	Steady boundary layer	Steady Navier-Stokes

4.7 Classification of 1st-order quasi-linear PDEs in two variables

We next consider a coupled first-order quasi-linear systems of equations, namely

$$\left.\begin{array}{rcl} a_1u_x + b_1u_y + c_1v_x + d_1v_y & = & g_1 \\ a_2u_x + b_2u_y + c_2v_x + d_2v_y & = & g_2 \end{array}\right\}, \tag{4.39}$$

where the coefficients $a_1,\ a_2,\ b_1,\ldots,\ g_1,\ g_2$ may be functions of $x,\ y,\ u$ and v (satisfying the quasi-linear requirement). Using the chain rule, *i.e.*

$$\left.\begin{array}{rcl} du & = & u_x\,dx\ +\ u_y\,dy \\ dv & = & v_x\,dx\ +\ v_y\,dy \end{array}\right\},$$

the coupled system (4.39) may then again be written in matrix form :

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ du \\ dv \end{pmatrix}, \tag{4.40}$$

with u, v and the coefficient functions $a_1,\ a_2,\ b_1,\ldots,\ g_1,\ g_2$ known along Γ in Fig. 4.3. Again a unique solution for $u_x,\ u_y,\ v_x$ and v_y exists if the determinant of (4.40) is **not** zero – in which case the directional derivatives have the same value above and below Γ.

A zero value of the determinant implies that a multiplicity of solutions is possible, and thus the partial derivatives $u_x,\ u_y,\ v_x$ and v_y can not be determined uniquely, and thus discontinuities in these derivatives may occur on crossing Γ. Hence the characteristic equation can be shown to be

$$dy^2(a_2c_1 - a_1c_2) + dxdy(b_1c_2 - b_2c_1 - a_2d_1 + a_1d_2) + dx^2(b_2d_1 - b_1d_2) = 0, \tag{4.41}$$

which is a quadratic in dy/dx. It follows that the characteristics may be real, distinct, identical or complex according to whether the discriminant

$$(b_1c_2 - b_2c_1 - a_2d_1 + a_1d_2)^2 - 4(a_2c_1 - a_1c_2)(b_2d_1 - b_1d_2) \tag{4.42}$$

is positive, zero or negative. Hence again allowing classification of (4.39), into whether they are hyperbolic, parabolic or elliptic.

In this section we have essentially been looking at whether locally unique solutions exist for analytic quasi-linear partial differential equations associated with initial value problems. Further details of the formal proof may be found in the **Cauchy-Kovalevskaya** theorem, which concerns the existence, continuity and "*smoothness*" of solutions to a system of m differential equations in n dimensions when the coefficients are analytic functions. The theorem and its proof are valid for analytic functions of either real or complex variables.