M40007: Introduction to Applied Mathematics

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1 A matrix

Here is a matrix:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{1}$$

It is not square. It is an *m-by-n* matrix with m rows where $m = 5$ and n columns where $n = 4$. The symbols m and n will be used throughout to denote the number of rows and columns of a matrix respectively.

In a linear algebra course matrices are studied, and concepts such as the *rank* of a matrix are defined. The *rank* of a matrix is the dimension of its *column space*. By the so-called *rank-nullity theorem*, the rank is also the dimension of its *row space*.

What is the rank r of $\mathbf{A}$?

An important observation, which is easy to make given that the elements of $\mathbf{A}$ are either $0, 1$ or -1, is that the vector

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

is such that

$$\mathbf{A}\mathbf{x}_0 = 0. \tag{3}$$

The vector $\mathbf{x}_0$ is therefore said to be in the *right null space*, or *kernel*, of the matrix $\mathbf{A}$. $\mathbf{x}_0$ is a *right null vector*.

If the 4 columns of $\mathbf{A}$ are denoted by $\mathbf{a}_j$ for $j = 1, 2, 3$ and 4 so that

$$\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4] \tag{4}$$

then the existence of the right null-vector $\mathbf{x}_0$ means that

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = 0. \tag{5}$$

This is just another way to write (3). In the form (5) it is easy to see that $\mathbf{a}_4$ is a linear combination of the first three columns of $\mathbf{A}$, indeed,

$$\mathbf{a}_4 = -\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3. \tag{6}$$

Therefore $\mathbf{a}_4$ is linearly dependent on $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$.

It is natural to ask: are the first three columns $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ of the matrix $\mathbf{A}$ *linearly independent*?

In linear algebra one learns the usual test for linear independence of a set of

vectors is to suppose

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_3 \mathbf{a}_3 = 0 \tag{7}$$

for some set of coefficients c_1, c_2 and c_3 or, in this case,

$$c_1 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0. \tag{8}$$

The vectors are linearly independent if all the coefficients c_1, c_2 and c_3 are zero. From the last three elements of the vector equation (8) it is clear that

$$c_1 = c_2 = c_3 = 0. \tag{9}$$

Therefore $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are indeed linearly independent.

We have now established that the dimension of the column space of $\mathbf{A}$ is 3; recall that the column space is defined to be the subspace of $\mathbb{R}^m = \mathbb{R}^5$ spanned by the columns of $\mathbf{A}$. In other words, the rank of $\mathbf{A}$

$$r = 3. \tag{10}$$

Another question is to ask how many vectors $\mathbf{w}$ satisfy

$$\mathbf{A}^T \mathbf{w} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \mathbf{w} = 0? \tag{11}$$

Any such vector $\mathbf{w}$ must be 5-dimensional, $\mathbf{w} \in \mathbb{R}^5$. Consider the choice

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}. \tag{12}$$

It is easy to check that

$$\mathbf{A}^T \mathbf{w}_1 = 0. \tag{13}$$

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Consider also the two vectors

$$\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}. \tag{14}$$

Again, it is easy to check that

$$\mathbf{A}^T \mathbf{w}_2 = 0, \qquad \mathbf{A}^T \mathbf{w}_3 = 0. \tag{15}$$

These vectors $\mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{w}_3$ have been plucked from the air, but we will see later how to generate them.

It is also easy to check that

$$\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_2. \tag{16}$$

So $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Are $\mathbf{w}_1$ and $\mathbf{w}_2$ linear independent?

To examine this, we do the usual test. Suppose that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = 0 \tag{17}$$

for some coefficients c_1 and c_2. From the first two elements of this vector equality it is clear that

$$c_1 = c_2 = 0. \tag{18}$$

This confirms that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent vectors both satisfying (11).

Can we expect to find any other, linearly independent, solutions to (11)?

The answer is no. This is guaranteed for us by the *rank-nullity theorem*. Note that, on taking a transpose,

$$\mathbf{A}^T \mathbf{w} = 0 \iff \mathbf{w}^T \mathbf{A} = 0. \tag{19}$$

We have used the fact that

$$(\mathbf{A}^T \mathbf{w})^T = \mathbf{w}^T (\mathbf{A}^T)^T = \mathbf{w}^T \mathbf{A}, \tag{20}$$

because transposing a matrix twice leaves the matrix unchanged. It follows that

$$\mathbf{w}_1^T \mathbf{A} = \mathbf{w}_2^T \mathbf{A} = 0. \tag{21}$$

Since $\mathbf{w}_1$ and $\mathbf{w}_2$ (or rather their transposes in order that the matrix multiplication makes sense) appear on the left of $\mathbf{A}$, we say that $\mathbf{w}_1$ and $\mathbf{w}_2$ lie in the *left null space* of $\mathbf{A}$; the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are called *left null vectors*.

The rank-nullity theorem says that

$$\text{dim(left nullspace)} + \text{rank}(\mathbf{A}) = m. \tag{22}$$

But $m = 5$ and we established earlier in (10) that $\text{rank}(\mathbf{A}) = r = 3$ so (22) says that

$$\text{dim(left nullspace)} = 5 - 3 = 2. \tag{23}$$

The two linearly independent vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ therefore form a two-dimensional basis for the left nullspace of $\mathbf{A}$.

2   A graph

Consider the diagram of *nodes* linked by *edges* shown in Figure 1.
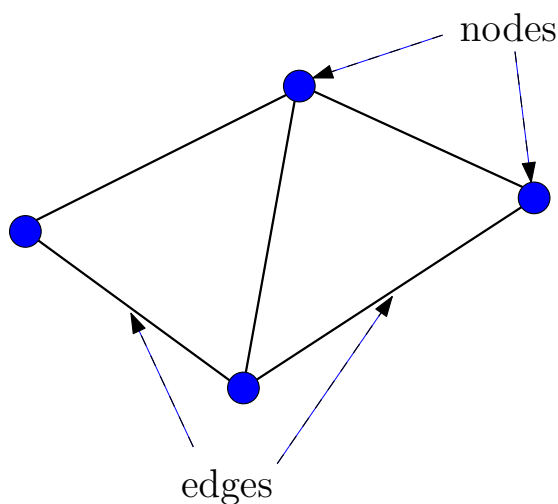

Figure 1: A simple graph with $m = 5$ edges and $n = 4$ nodes.

Such an arrangement of nodes joined by edges will be called a *graph*. This graph has 4 nodes, with 5 edges linking some of them. The letter m will be used to denote the number of edges of a graph; n will be the number of nodes. For the graph in Figure 1 it is clear that $m = 5$ and $n = 4$.

It is useful to number the nodes as 1,2,3 and 4. The edges will be labelled with the letters a, b, c, d and e. Another thing that is useful is to create a *direction* on each edge: we will do this arbitrarily – any choice will do. Let us pick a direction on each edge as illustrated in Figure 2.
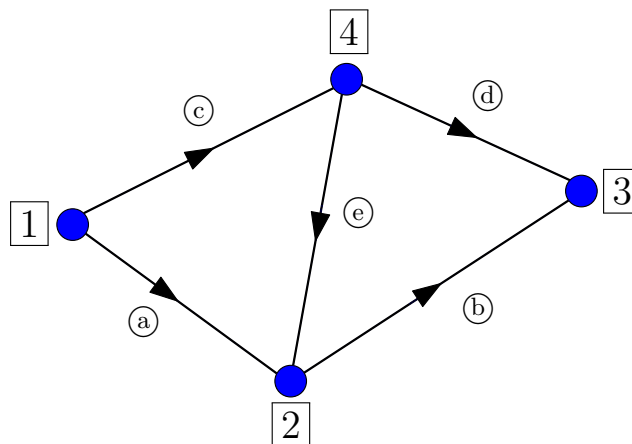
Figure 2: Labelling of nodes and edges with edge directions selected.

Constructing a matrix: A matrix associated with this simple graph can now be constructed according to the following prescription:

Step 1: set the number of rows of the matrix, m, equal to the number of edges and assign a row to each edge;

Step 2: set the number of columns of the matrix, n, equal to the number of nodes and assign a column to each node;

Step 3: set all elements of the m-by-n matrix equal to zero;

Step 4: for each edge alter the corresponding row of the matrix by putting -1 in column i and $+1$ in column j if the edge connects nodes i and j with the arrow pointing as shown here:
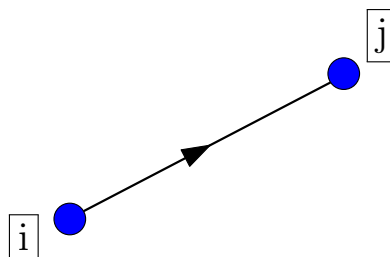
Let us carry out Steps 1–4 for the graph in Figure 2.
Steps 1–3 result in a 5-by-4 matrix of zeros given by

$$
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\end{array}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{array}{l}
\text{edge } \text{\textcircled{a}} \\
\text{edge } \text{\textcircled{b}} \\
\text{edge } \text{\textcircled{c}} \\
\text{edge } \text{\textcircled{d}} \\
\text{edge } \text{\textcircled{e}}
\end{array}
\tag{24}
$$

It is natural to order the rows and columns as shown.

Looking first at edge $\text{\textcircled{a}}$, corresponding to row 1, Step 4 leads to

$$
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\end{array}
$$

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{array}{l}
\text{edge } \text{\textcircled{a}} \\
\text{edge } \text{\textcircled{b}} \\
\text{edge } \text{\textcircled{c}} \\
\text{edge } \text{\textcircled{d}} \\
\text{edge } \text{\textcircled{e}}
\end{array}
\tag{25}
$$

Looking next at edge $\text{\textcircled{b}}$, corresponding to row 2, Step 4 leads to

$$
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\end{array}
$$

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{array}{l}
\text{edge } \text{\textcircled{a}} \\
\text{edge } \text{\textcircled{b}} \\
\text{edge } \text{\textcircled{c}} \\
\text{edge } \text{\textcircled{d}} \\
\text{edge } \text{\textcircled{e}}
\end{array}
\tag{26}
$$

Repeating Step 4 for edges $\text{\textcircled{c}}$, $\text{\textcircled{d}}$ and $\text{\textcircled{e}}$ leads to

$$
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\end{array}
$$

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 0 & -1
\end{pmatrix}
\begin{array}{l}
\text{edge } \text{\textcircled{a}} \\
\text{edge } \text{\textcircled{b}} \\
\text{edge } \text{\textcircled{c}} \\
\text{edge } \text{\textcircled{d}} \\
\text{edge } \text{\textcircled{e}}
\end{array}
\tag{27}
$$

This is exactly the matrix **A**!

By virtue of these steps we have now associated a matrix **A** to the simple graph in Figure 2. On the other hand, the simple graph in Figure 2 provides a *geometrical* interpretation to the matrix **A** as a graph.

3 Incidence matrix

A matrix constructed in this way from a graph is called the *incidence matrix* of the graph.

Suppose we construct the incidence matrices for the graphs in Figure 3.
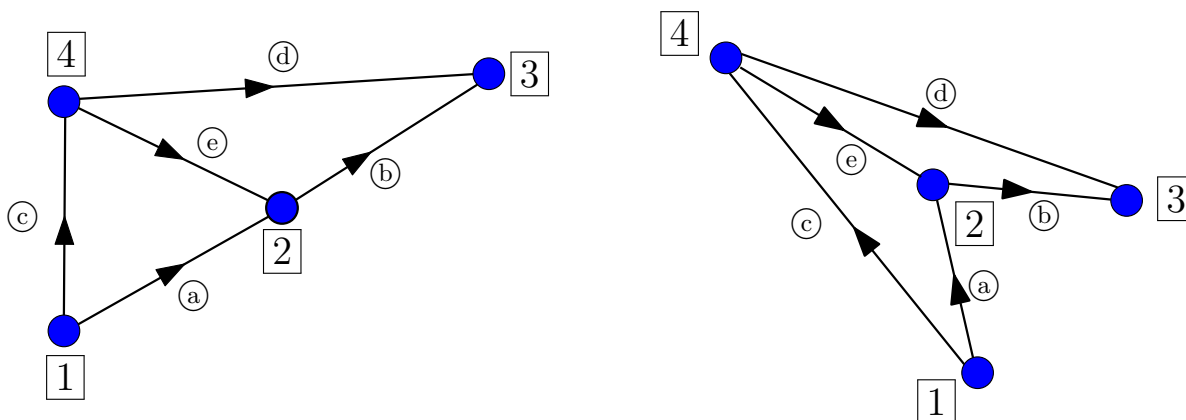
Figure 3: Two other graphs equivalent to that in Figure 2.

At a quick glance these two graphs look different from that in Figure 2, and also from eachother. But the incidence matrix for each graph is also **A**. The only differences between the three graphs in Figures 2 and 3 are that the nodes have been moved around the page, and the edges have been shortened or lengthened accordingly; the *topology* of all three graphs is the same.

Suppose an edge joining nodes 1 and 3 , call it edge ⓕ, is added to the graph in Figure 2.
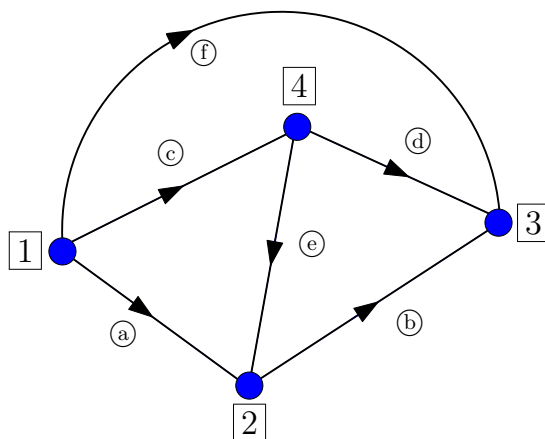
Figure 4: Adding an edge f between nodes 1 and 3 produces a complete graph.

The corresponding incidence matrix will now have an extra row

$$
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}
\end{array}
$$
$$
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\begin{array}{l}
\text{edge } ⓐ \\
\text{edge } ⓑ \\
\text{edge } ⓒ \\
\text{edge } ⓓ \\
\text{edge } ⓔ \\
\text{edge } ⓕ
\end{array}
\tag{28}
$$

Notice now that each node is connected by an edge to every other node. Any such graph is called a *complete graph*.

Complete graph: A *complete graph* is one in which each node is connected by a unique edge to every other node.

Figure 5 shows a more complicated graph, complete with node and edge labels, and directions assigned to each edge.
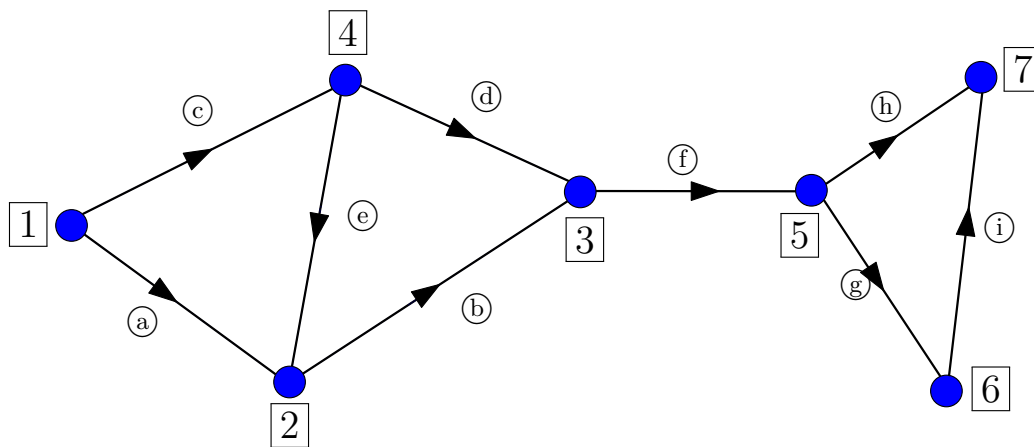

Figure 5: A graph with $m = 9$ edges and $n = 7$ nodes.

The associated incidence matrix **A** is the 9-by-7 matrix

$$
\mathbf{A} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
\begin{matrix}
\text{edge } ⓐ \\
\text{edge } ⓑ \\
\text{edge } ⓒ \\
\text{edge } ⓓ \\
\text{edge } ⓔ \\
\text{edge } ⓕ \\
\text{edge } ⓖ \\
\text{edge } ⓗ \\
\text{edge } ⓘ
\end{matrix}
\tag{29}
$$

(columns labelled $\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}$)

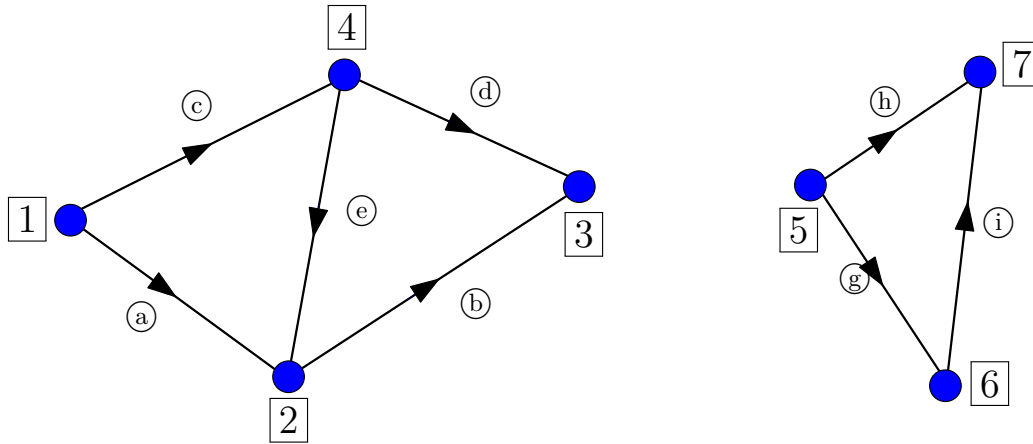Suppose now that we remove edge ⓕ so that the graph becomes that shown in Figure 6.

Figure 6: The graph in Figure 5 but with edge ⓕ removed.

The associated incidence matrix **A** is now the 8-by-7 matrix

$$
\mathbf{A} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
\begin{matrix}
\text{edge } ⓐ \\
\text{edge } ⓑ \\
\text{edge } ⓒ \\
\text{edge } ⓓ \\
\text{edge } ⓔ \\
\text{edge } ⓖ \\
\text{edge } ⓗ \\
\text{edge } ⓘ
\end{matrix}
\tag{30}
$$

(columns labelled $\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}$)

The removal of edge ⓕ has led to an important difference between the graphs in Figures 5 and 6. Starting at any node in the graph in Figure 5 it is possible to reach *every* other node by following a path of edges. There are lots of choices of this path. However, this is no longer possible for the graph in Figure 6. For example, starting at node 7 it is impossible to reach any of the nodes $1, 2, 3$ or 4 by a path of edges.

Connected graph: A *connected graph* is one for which, starting at any node, it is possible to reach *every* other node by following a path of edges.

4   The matrix meets the graph

Earlier we studied a matrix $\mathbf{A}$ from the point of view of the rank-nullity theorem from linear algebra. That same matrix $\mathbf{A}$ has also been seen to be associated to a simple graph.

It is natural to ask the following questions.

How do all the objects and ideas appearing in the rank-nullity theorem show themselves in the context of the simple graph? What does the right null vector $\mathbf{x}_0$ correspond to? What do the two left null vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ correspond to?

A central fact from linear algebra is that matrices operate on vectors. Consider the 5-by-4 matrix $\mathbf{A}$ and take a typical vector

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4. \tag{31}
$$

Since each column of $\mathbf{A}$ corresponds to a node in the associated graph then we can think of x_j as a value, or a *potential*, assigned to node j. The vector $\mathbf{x}$ is the 4-dimensional vector containing these 4 values.

Potentials: The scalar values assigned to the nodes of a graph will be called the *potentials* at the nodes. Collecting these potentials into an n-dimensional vector $\mathbf{x}$ gives the vector of potentials.

Suppose we operate on this vector with the matrix $\mathbf{A}$. This means that we compute

$$
\mathbf{e} \equiv \mathbf{A}\mathbf{x} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \\ x_2 - x_4 \end{pmatrix} \tag{32}
$$

The vector $\mathbf{e} = \mathbf{A}\mathbf{x}$ is the 5-dimensional vector of the *differences* in these node values across the 5 edges.

Potential differences: If the potentials are the elements of the n-dimensional vector $\mathbf{x}$ then the m-dimensional vector

$$\mathbf{e} = \mathbf{A}\mathbf{x}, \tag{33}$$

where $\mathbf{A}$ is the incidence matrix of the graph, is the vector of *potential differences* across the edges of the graph.

Interpretation of the right null space: It is obvious that if the potential at every node in a graph has the *same* value c then the vector of potential differences is the zero vector. For the matrix $\mathbf{A}$ considered in Lecture 1 the corresponding vector of potentials must then have the form

$$\mathbf{x} = c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = c\mathbf{x_0}, \tag{34}$$

where $\mathbf{x}_0$ is the right null vector introduced in Lecture 1. The corresponding vector of potential differences is

$$\mathbf{e} = \mathbf{A}\mathbf{x} = c\mathbf{A}\mathbf{x}_0 = 0. \tag{35}$$

This gives us an interpretation of the right null vector $\mathbf{x}_0$ as one corresponding to all nodes having the same potential.

Provided a graph is connected – by which we mean that every node can be reached by some path of edges to any other node – then there will always be an n-dimensional right null vector, call it $\mathbf{x}_0$, of the form

$$\mathbf{x}_0 = \left. \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \right\} n \tag{36}$$

For a connected graph, it can be argued that this is the *only* right null vector.

Consider the graph in Figure 6. This graph is not connected. It is easy to check

that the two vectors

$$\mathbf{x}_0^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{37}$$

satisfy

$$\mathbf{A}\mathbf{x}_0^{(1)} = 0, \qquad \mathbf{A}\mathbf{x}_0^{(2)} = 0. \tag{38}$$

Clearly both $\mathbf{x}_0^{(1)}$ and $\mathbf{x}_0^{(2)}$ are right null vectors. And it is easily checked that they are linearly independent. Note also that

$$\mathbf{x}_0 = \mathbf{x}_0^{(1)} + \mathbf{x}_0^{(2)}, \tag{39}$$

where $\mathbf{x}_0$ is the vector in (36) with $n = 7$ so this vector still sits in the right null space. However, it is not the *only* vector in this right null space when the graph is not connected.

The dimension of the right null space of an incidence matrix $\mathbf{A}$ corresponds to the number of connected components making up a disconnected graph.

Interpretation of the left null space: Returning to the graph in Figure 7, is there a geometrical way to interpret the two left null vectors $\mathbf{w}_1$ and $\mathbf{w}_2$? Yes – as *closed loops* in the graph as shown in Figure 7.
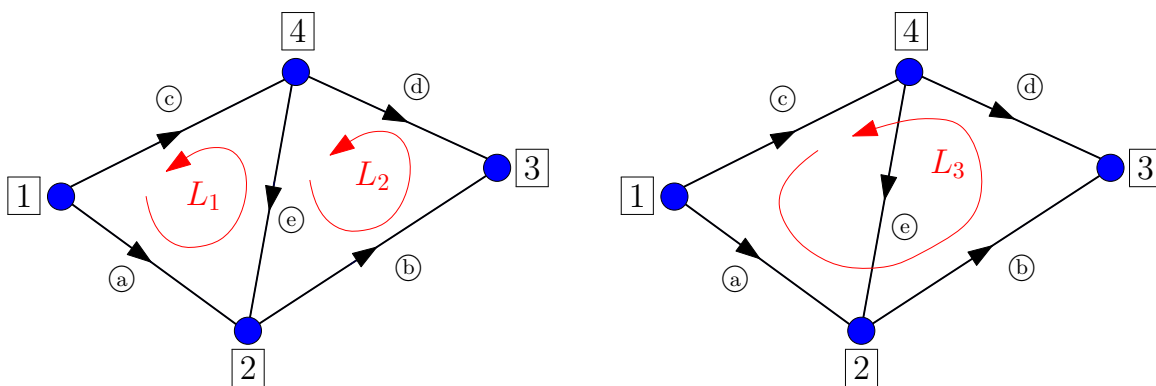

Figure 7: The closed loops L_1, L_2 and L_3. "Adding" loops L_1 and L_2 is equivalent to loop L_3 since edge (e) is effectively not traversed at all.

The left null vector $\mathbf{w}_1$ is associated with the loop L_1 in Figure 7. To see how, since edge (a) is traversed in the positive direction (as determined by the arrow) we put $+1$ in the corresponding row; since edges (c) and (e) are both traversed in the

negative direction we put -1 in the corresponding rows. The result is

$$
\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}
\begin{array}{l} \leftarrow \text{edge } \textcircled{a} \\ \leftarrow \text{edge } \textcircled{b} \\ \leftarrow \text{edge } \textcircled{c} \\ \leftarrow \text{edge } \textcircled{d} \\ \leftarrow \text{edge } \textcircled{e} \end{array}
\tag{40}
$$

which is precisely the vector $\mathbf{w}_1$.

A similar construction shows that loop L_2 in Figure 7 corresponds to the vector $\mathbf{w}_2$.

It is also easy to check that the bigger loop also shown in Figure 7 corresponds to $\mathbf{w}_3 = \mathbf{w}_1 + \mathbf{w}_2$. Notice that in loop L_1 edge $\textcircled{e}$ is traversed in a negative direction while in in loop L_2 it is traversed in a positive direction. Therefore the net effect of "adding" both loops is that edge $\textcircled{e}$ is effectively not traversed at all, as occurs for loop L_3.
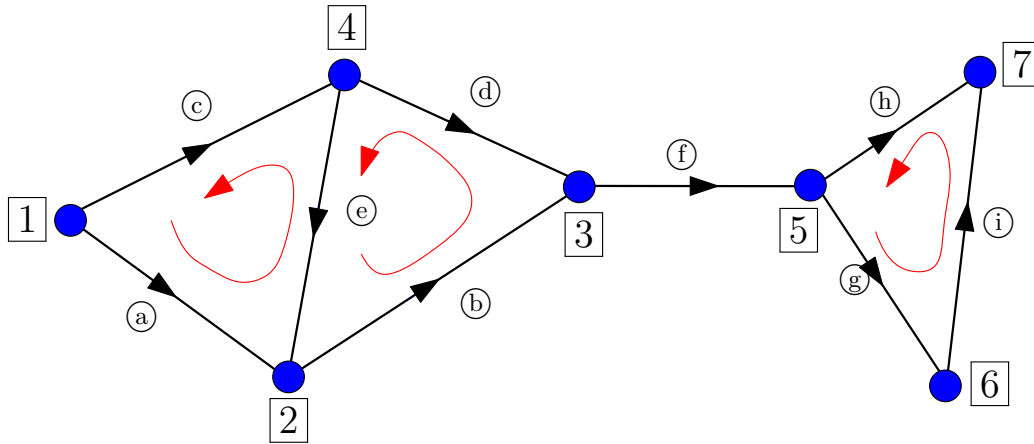

Figure 8: This graph with $m = 9$ edges and $n = 7$ nodes and 3 closed loops.

Let us return to the graph in Figure 8. What is the rank r of its incidence matrix? Since this graph is connected we know that a 7-dimensional vector $\mathbf{x}_0$ as defined in (36) with $n = 7$ is the only right null vector of the incidence matrix. By the rank-nullity theorem,

$$
\dim(\text{right null space}) + r = n = 7.
\tag{41}
$$

Since dim(right null space) = 1 then

$$
r = 6.
\tag{42}
$$

Let us check that this is consistent with the other statement in the rank-nullity

theorem:

$$\dim(\text{left null space}) + r = m = 9. \tag{43}$$

This means that dim(left null space) = 3. We therefore expect to find 3 independent closed loops in the graph. These are easy to find and are indicated in Figure 8.
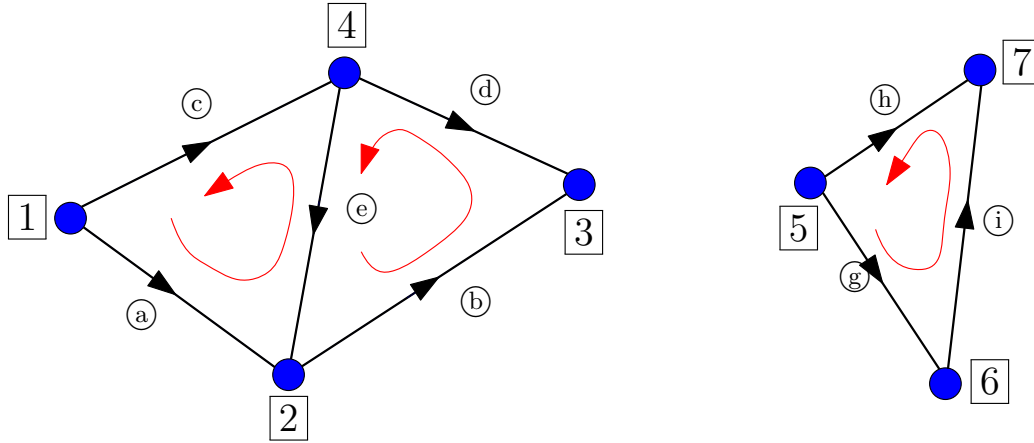

Figure 9: The graph in Figure 8 with edge $\textcircled{f}$ removed: there are now $m = 8$ edges, $n = 7$ nodes and 3 closed loops.

Suppose now that edge $\textcircled{f}$ is removed as shown in Figure 9. The dimension of the right null space of the corresponding incidence matrix increases from 1 to 2 because the two linearly independent vectors (37) are in its right null space. These two vectors correspond to node potentials in the two disconnected graphs making up the full graph being set equal independently; because the graph is disconnected you can set all potentials equal in each disconnected component without affecting the potentials in the other components. This means the rank of the incidence matrix changes according to (41):

$$r = n - \dim(\text{right null space}) = 7 - 2 = 5. \tag{44}$$

Since the number of closed loops is still equal to 3, (43) now implies that

$$m = \dim(\text{left null space}) + r = 3 + 5 = 8 \tag{45}$$

which is consistent with the fact that, by removing edge $\textcircled{f}$, the value of m has decreased by 1 from 9 to 8.

5 Potentials and fluxes

Given any two nodes labelled i and j in a graph each has an associated node value, or potential, denoted by x_i and x_j. The edge $\textcircled{a}$ joining them is also assigned an
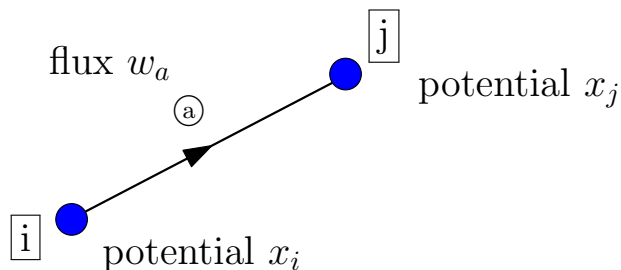
edge value, called a *flux*, denoted by w_a.

Figure 10: There is a potential x_i at node i, a potential x_j at node j and a flux w_a along edge (a).

Many interesting problems in applied mathematics can be described by this simple model of a graph with node potentials and edge fluxes. Using it we will build up a framework for applied mathematics. The potentials and fluxes associated with the graph considered earlier can be assigned as follows:
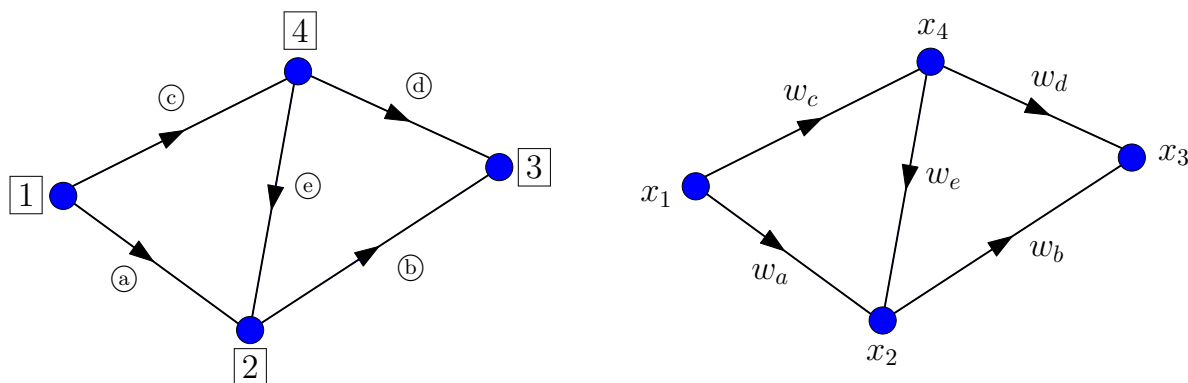

Figure 11: The graph with node potentials x_1, x_2, x_3 and x_4 and edge fluxes w_a, w_b, w_c, w_d and w_e.

Let us examine how the incidence matrix **A** can help us in manipulating these potentials and fluxes.

Potential differences across the edges: Since we have defined potentials at the nodes, it is of interest to ask about the differences in these potentials across the edges joining the nodes. The vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \tag{46}$$

represents the potentials at the $n = 4$ nodes. It has already been seen that the m-by-n incidence matrix $\mathbf{A}$ can be used to produce the 5-dimensional vector of potential differences across the edges:

$$\mathbf{e} \equiv \mathbf{Ax} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \\ x_2 - x_4 \end{pmatrix} \in \mathbb{R}^5. \qquad (47)$$

Net fluxes out of the nodes: Another 5-dimensional vector is the vector of edge fluxes defined by

$$\mathbf{w} = \begin{pmatrix} w_a \\ w_b \\ w_c \\ w_d \\ w_e \end{pmatrix} \in \mathbb{R}^5. \qquad (48)$$

It is not possible to operate on $\mathbf{w}$ with the incidence matrix $\mathbf{A}$ because their dimensions are not consistent with matrix multiplication. On the other hand, we can operate on $\mathbf{w}$ with $\mathbf{A}^T$:

$$\mathbf{A}^T\mathbf{w} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} w_a \\ w_b \\ w_c \\ w_d \\ w_e \end{pmatrix} = \begin{pmatrix} -w_a - w_c \\ w_a - w_b + w_e \\ w_b + w_d \\ w_c - w_d - w_e \end{pmatrix} \in \mathbb{R}^4. \qquad (49)$$

What is the meaning of the resulting 4-dimensional vector? From inspection of Figure 11 its elements correspond to adding up the fluxes *converging into* each node. For example, the first element

$$-w_a - w_c \qquad (50)$$

is the net flux into node $\boxed{1}$; we have added minus signs since the arrows shown in Figure 11 define the positive direction along the respective edges. Actually, it is more common practice to add a minus sign and to define the vector

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w} \qquad (51)$$

which is the n-dimensional vector containing the total *flux diverging out of each node,*

also known as the *divergence of the fluxes* at the nodes. For the graph in Figure 11,

$$\mathbf{f} = -\mathbf{A}^T \mathbf{w} = \begin{pmatrix} w_a + w_c \\ -w_a + w_b - w_e \\ -w_b - w_d \\ -w_c + w_d + w_e \end{pmatrix} \in \mathbb{R}^4 \tag{52}$$

which is the vector containing the net flux diverging *out* of each node.

6 Ohm meets Kirchhoff

One application of the framework is to *electric circuits*. In this context each node potential x_i represents the *voltage* at node i. The flux w_a in edge ⓐ represents the *current* in that edge.

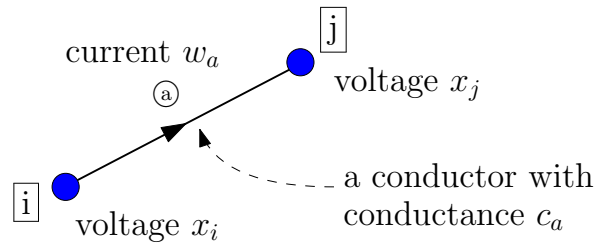Figure 12: In an electric circuit each edge is a conductor, with a positive conductance c_a, joining nodes at voltages x_i and x_j producing a current w_a along edge ⓐ joining them.

Suppose the graph in Figure 11 is an electric circuit with voltages x_1, x_2, x_3 and x_4 at the nodes and currents w_a, w_b, w_c, w_d and w_e in the edges.

Ohm's law: What is the link between $\mathbf{x}$ and $\mathbf{w}$? That is, between the voltages at the nodes and the currents in the edges? Some additional physics is required to tell us this. In resistive circuit theory each edge is assumed to be linear *conductor* for which *Ohm's Law* states that, with respect to the graph shown in Figure 12,

$$w_a = -c_a(x_j - x_i), \qquad c_a > 0, \qquad \boxed{\text{Ohm's Law}} \tag{53}$$

where the positive constant c_a is the *conductance* of edge ⓐ. The conductance is a property of the material of which the edge is made up. It is a measure of the ease with which current flows along that edge. Some materials conduct more easily than others and will have correspondingly higher values of conductance.

The *resistance* R_a of edge ⓐ is defined as the inverse of the conductance

$$R_a = \frac{1}{c_a}. \tag{54}$$

Ohm's law (53) states that the current in the conductor is proportional to the *voltage drop* across it. It is important to note from (53) that *current flows from high voltage to low voltage*: this is the reason for the minus sign in (53). If $\phi_2 > \phi_1$ then, since $c_a > 0$, the current w_a along the conductor in Figure 12 is negative meaning that current flows in the opposite direction to that indicated by the arrow.

It is convenient to assume first that all edges have unit conductance, i.e.,

$$c_a = c_b = c_c = c_d = c_e = 1. \tag{55}$$

This assumption will be relaxed later.

It turns out that Ohm's law (53) can be conveniently stated using the incidence matrix $\mathbf{A}$ associated with the circuit. Operating on $\mathbf{x}$ with the incidence matrix $\mathbf{A}$ of the graph produces the 5-dimensional vector of potential differences

$$\mathbf{e} \equiv \mathbf{Ax} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \\ x_2 - x_4 \end{pmatrix}. \tag{56}$$

Under the assumption (55) Ohm's Law (53) in each of the 5 conductors can therefore be expressed as

$$\mathbf{w} = -\mathbf{e} = -\mathbf{Ax}. \tag{57}$$

Suppose we take

$$\mathbf{x} = \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{58}$$

so that all the nodes have the same voltage. We know that $\mathbf{x}_0$ is a right null vector of $\mathbf{A}$ therefore there is no potential difference across any of the edges, hence $\mathbf{e} = 0$, and by Ohm's law (57), $\mathbf{w} = 0$. All currents are zero. This makes sense because there are no voltage drops to drive any current through the circuit.

Next current out of the nodes: The vector

$$\mathbf{f} = -\mathbf{A}^T\mathbf{w} \tag{59}$$

is the divergence of the currents at each node. It is the n-dimensional vector containing the net current flowing out of each node of the graph.

On the other hand, Ohm's law (57) tells us that the vector of currents in the edges is given by

$$\mathbf{w} = -\mathbf{Ax}. \tag{60}$$

Equations (59) and (60) can be combined to produce the vector equation

$$\mathbf{f} = -\mathbf{A}^T(-\mathbf{Ax}) = \mathbf{A}^T\mathbf{Ax}. \tag{61}$$

Each side of this equation is an n-dimensional vector. The matrix $\mathbf{A}^T\mathbf{A}$ appearing in (61) is important so it will be denoted by

$$\mathbf{K} = \mathbf{A}^T\mathbf{A}. \tag{62}$$

The matrix $\mathbf{K}$ is called the *Laplacian matrix* of the graph; it is also sometimes called the *discrete Laplacian* or the *graph Laplacian*. It will be studied in detail soon.

Equation (61) tells us that the Laplacian matrix relates the currents out of each node to the voltage potentials at those nodes:

$$\mathbf{f} = \mathbf{Kx}. \tag{63}$$

The vector $\mathbf{f}$ is another vector of node variables. It is a potential defined at the nodes that is different from the voltage potential $\mathbf{x}$. Equation (63) shows us how these two node potentials are related by the Laplacian matrix $\mathbf{K}$.

Kirchhoff's current law (KCL): A second law of resistive circuit theory is that, unless a node in the circuit is connected to a source of current such as a battery, the net current out of the node must be zero. In other words, in the absence of a source of current at a node, all current flowing into a node must flow out of it. This is indicated in Figure 13 where, at the node shown, we must have

$$-w_a - w_b - w_c + w_d = 0. \tag{64}$$

This says that the net current out of the node vanishes. It can also be rewritten as

$$w_a + w_b + w_c = w_d \tag{65}$$

and the natural way to interpret this equation, given the assignment of arrows in Figure 13, is to say that *current in equals current out*. This principle is known as Kirchhoff's current law, or KCL for short.

7 Laplacian, degree and adjacency matrices

For the graph in Figure 11 the incidence matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ are

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \qquad \mathbf{A}^T = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \tag{66}$$
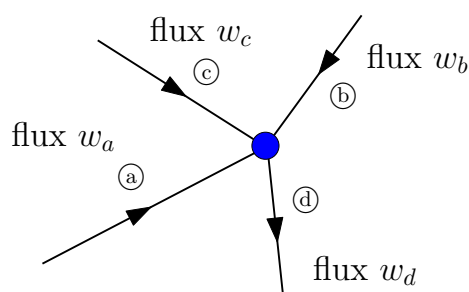
$$-w_a - w_b - w_c + w_d = 0$$

$$\boxed{\text{Kirchhoff's current law (KCL)}}$$

Figure 13: A typical node with four attached conductors. Unless the node is connected to a source of current, such as a battery, or is grounded, the net current out of it must vanish.

The Laplacian matrix – or *Laplacian* for short – is easily computed as follows:

$$
\mathbf{K} = \mathbf{A}^T \mathbf{A} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.
\tag{67}
$$

This Laplacian matrix has an interesting structure:

- It is an *n*-by-*n* square matrix.

- It is a symmetric matrix: $\mathbf{K}^T = \mathbf{K}$.

- All its non-zero off-diagonal elements are equal to -1.

- The sum of the elements in every row and every column vanishes.

Let us try a different graph to see if we can spot any patterns. For the graph the incidence matrix, and its transpose, are

$$
\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \qquad \mathbf{A}^T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.
\tag{68}
$$
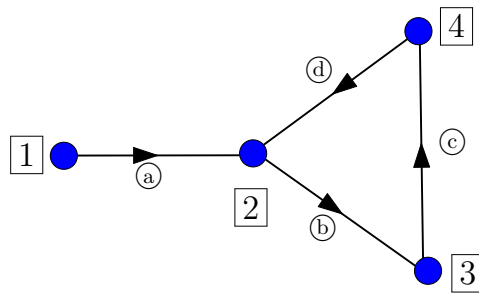
The corresponding Laplacian matrix is readily computed to be

$$
\mathbf{K} = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.
$$

(69)

Again, this new Laplacian matrix $\mathbf{K}$ has all the same properties noticed earlier for the Laplacian associated with a different graph. A pattern is beginning to emerge. Inspection of the diagonal elements shows that

$$
K_{ii} \tag{70}
$$

is equal to the number of edges connected to node i. Moreover the element

$$
K_{ij} \tag{71}
$$

is equal to -1 if there is an edge connecting nodes i and j and 0 otherwise.

It is an easy exercise to prove these statements are true in general.

The Laplacian matrix can therefore always be decomposed as follows:

$$
\mathbf{K} = \mathbf{D} - \mathbf{W}, \tag{72}
$$

where $\mathbf{D}$ is the *degree matrix* and $\mathbf{W}$ is the *adjacency matrix*. The degree matrix $\mathbf{D}$ is a diagonal matrix with elements

$$
D_{ii} = \text{number of edges connected to node } i, \qquad i = 1, \ldots, n. \tag{73}
$$

The adjacency matrix $\mathbf{W}$ has zeros on its diagonal and has the off-diagonal elements

given by

$$W_{ij} = \begin{cases} 1 & \text{if node } j \text{ connected to node } i, \\ 0 & \text{if node } j \text{ not connected to node } i, \end{cases} \quad i \neq j, \; i,j = 1, \ldots, n. \quad (74)$$

For the Laplacian matrix just computed, namely,

$$\mathbf{K} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \quad (75)$$

the corresponding degree and adjacency matrices are

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad (76)$$

These observations are useful since it means that the Laplacian matrix $\mathbf{K}$ for a graph can be written down directly and without necessarily writing down the incidence matrix $\mathbf{A}$.