18 Finite Volume Methods for Nonlinear Conservation Laws

In the previous section we studied in detail the behaviour and the general solution of a simple nonlinear PDE of hyperbolic type, namely Burger's equation with a flux function $f(u) = u^2/2$. The "*local*" wave propagation speed $u(x,t)$ too evolves, as the solution proceeds given some IVP. We used the PDE to illustrate the essential elements of solving nonlinear problems, and focussed on characteristics paths and were able to deduce the key solution properties to expect from any numerical method.

In this section, we summarise and put together the key steps required for a successful FV-based discretisation of Burgers equation. The approach we use is based on Godunov's* upwind method based on using **exact solutions** to the Riemann problem (RP) discussed previously. The approach was devised during Godunov's PhD and was one of the first workable FV-methods. Today it serves as the backbone for more advanced techniques that followed. This is a vast field, and we merely touch upon concepts and simplest of techniques – namely Godunov's method.

A reminder, some examples of more complex systems of coupled nonlinear PDEs would be, among others :

1. Navier-Stokes Equations Equations
2. Inviscid Euler Equations
3. Burger's Equation, traffic problem
4. Shallow water equations in geo-fluids

18.1 Godunov's first-order upwind method

Godunov's first-order upwind method is a conservative method of the form Eqn. (17.49), where the intercell numerical fluxes $f_{i+1/2}$ are computed by using solutions of local Riemann problems. A basic assumption of the method is that at a given time level n the data has a piece-wise constant distribution, where the cell average, at a fixed time $t = t^n = n\Delta t$ is defined as

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t^n)\,dx\ . \tag{18.1}$$

Although within cell i spatial variations of $u(x,t)$ at time $t = t^n$ arise, the integral average value u_i^n given above is constant. We shall assign that constant value at the centre of the cell, which gives rise to **cell centred conservative methods**. Computationally, we shall deal with approximations to the cell averages u_i^n, which for simplicity we shall still denote as u_i^n. The set of cell averages defines a piece-wise constant distribution of the solution at time t^n; see Fig. 18.1.

The data at time level n may be seen as pairs of constant states (u_i^n, u_{i+1}^n) separated by a discontinuity at the intercell boundary $x_{i-1/2}$. Then, locally, one can define a Riemann

*Godunov, S. K., (1959) *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*. Math. Sbornik, 47:271-306.

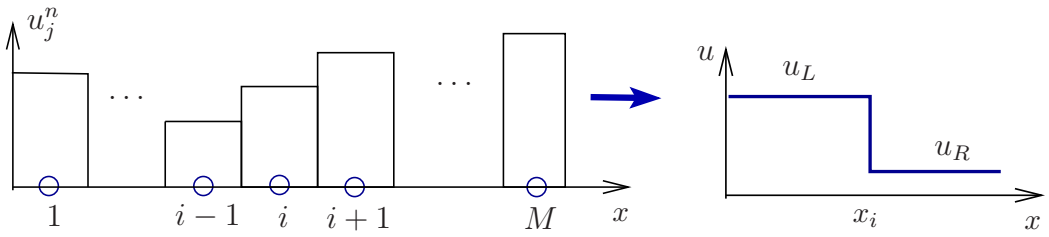

Figure 18.1: Piecewise constant reconstruction

problem

$$\begin{aligned} \text{PDE :} \quad & u_t + f(u)_x = 0 \\ \text{Initial Condition :} \quad & u(x,0) = u_o(x) = \begin{cases} u_i^n = u_L & \text{if } x - x_i < 0\,; \\ u_{i+1}^n = u_R & \text{if } x - x_i > 0\,, \end{cases} \end{aligned} \tag{18.2}$$

where $f(u)$ is the flux function.

For consistency with the **weak form** of the PDE we must use a conservative finite volume method, namely Eqn. 17.51

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left[f_{i+1/2}(u) - f_{i-1/2}(u)\right], \tag{18.3}$$

where

$$\begin{aligned} f_{i+1/2}(u) &= \tfrac{1}{k}\textstyle\int_{t^n}^{t^n+k} f(u(x_{i+1/2},t))\; dt \\ f_{i-1/2}(u) &= \tfrac{1}{k}\textstyle\int_{t^n}^{t^n+k} f(u(x_{i-1/2},t))\; dt. \end{aligned} \tag{18.4}$$

Earlier we looked at the linear advection equation and defined an **upwind** direction – we next want to define an equivalent of this to the above problem. To upwind we used a piecewise constant reconstruction, namely

$$u(x_{i-1/2} < x < x_{i+1/2}) = u_i + O(h^2).$$

At every face $x_{i+1/2}$ we need to compute a flux. If we satisfy a Courant condition, characteristics emanating from the cell faces at t^j don't reach other faces until time T^{j+1}. In this case we can focus on a single face to estimate the flux.

Suppose we could solve two separate Riemann problems

$$\text{RP}_L(u_{i-1}^n, u_i^n) \;\text{ giving }\; u_{i-1/2}(x/t)$$

solution, and

$$\text{RP}_R(u_i^n, u_{i+1}^n) \;\text{ giving }\; u_{i+1/2}(x/t).$$

It follows the overall solution would be

$$u_i^{n+1} = \frac{1}{\Delta x}\left[\int_0^{\Delta x/2} u_{i-1/2}(x/\Delta t)\; dx + \int_{-\Delta x/2}^{0} u_{i+1/2}(x/\Delta t)\; dx\right],$$

which can be re-written

$$u_i^{n+1} = \frac{1}{\Delta x}\left[\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x,\Delta t)\; dx\right],$$

where $\tilde{u}(x, \Delta t)$ is the combined $\text{RP}_L,\ \text{RP}_R$ solutions. Now from a control volume integral viewpoint, following Eqn. 17.51, taking the line integral around the boundary, we write

$$\left.\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x, \Delta t)\ dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x, 0)\ dx + \int_0^{\Delta t} f(\tilde{u}(x_{i-1/2}, t)\ dt \\ - \int_0^{\Delta t} f(\tilde{u}(x_{i+1/2}, t)\ dt\end{aligned}\right\}. \tag{18.5}$$

Hence, through usage of Eqn. 18.1, it follows the above becomes

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[f_{i-1/2}^n - f_{i+1/2}^n\right] \tag{18.6}$$

with the intercell fluxes defined as time integral averages, namely

$$f_{i-1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f(\tilde{u}(x_{i-1/2}, t)\ dt, \quad f_{i+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f(\tilde{u}(x_{i+1/2}, t)\ dt\ .$$

These are time integral averages of the physical flux $f(u)$ evaluated at the intercell boundaries. The integrand $f(\tilde{u}(x, t))$at each cell interface depends on the exact solution of the Riemann problem along the t-axis. In local coordinates this is given by

$$\tilde{u}(x_{i-1/2}, t) = u_{i-1/2}(0), \quad \tilde{u}(x_{i+1/2}, t) = u_{i+1/2}(0),$$

and the intercell fluxes are given by

$$f_{i-1/2} = f(u_{i-1/2}((0)), \quad f_{i+1/2} = f(u_{i+1/2}((0)).$$

So in the above, $u_{i-1/2}((0))$ denotes the exact solution $u_{i-1/2}(x/t)$ of the Riemann problem $\text{RP}(u_i^n,\ u_{i+1}^n)$ evaluated along the intercell boundary, $x = 0$, in local coordinates.

In summary :
The complete set of possible solutions to the RP for Burgers equation are as given in §17.8, and together with usage of Eqn. 18.6, this would form the basis of a computer code. These would be applied directly to all cells i in Fig. 18.1, for $i = 2,\ 3, \ldots, M-1$.

18.2 Boundary conditions

At the left and right boundary points $i = 1$ and $i = M$ boundary conditions would require to be imposed. The simplest is that where a boundary function $u_1(t)$ (at left boundary, say) is enforced. In this case one simply defines the intercell flux by setting $f_{1/2} = f(u_1(t))$.

Dependent upon the problem and type of condition enforced (transmissive, reflective etc.), ficticious cells u_0 or u_{M+1} may well prove useful, namely

$$u_0 = u_1, \quad u_{M+1} = u_M, \quad \text{transparent boundary},$$

or

$$u_0 = -u_1, \quad u_{M+1} = -u_M, \quad \text{reflective boundary (solid wall)},$$

among others. More sophisticated treatments based on characteristics of the mathematical problem, at hand, are clearly possible and used in practice.

18.3 CFL time step

In the linear advection case, we found that a crucial aspect was to satisfy the Courant number for stability of the numerical method, namely

$$\text{CFL} = s\frac{\Delta t}{\Delta x} \leq 1, \quad \text{where } s \text{ is the advection speed.}$$

In nonlinear problems, as we have seen, at each time level, there are multiple wave speeds and thus multiple associated Courant numbers. This implies, constraining the time step Δt such that the fastest wave travels at most one cell length Δx. Thus we define the maximum Courant number C_{CFL} at time level n by

$$C_{\text{CFL}} = s^n_{\text{max}}\frac{\Delta t}{\Delta x}.$$

One identifies wave speeds, such as those emerging from solutions of Riemann problems at the intercell boundaries, and characteristic speeds u. At time level n, there is the possibility of maximising over all u^n_i discrete characteristic speeds. The other possibility is to take solutions from the Riemann problem at each cell interface; this information is available as part of the flux computation process. For the shock, if present, the shock speed is added to the mix, while for the rarefaction situation, there will be two characteristic speeds, at the head and tail of the expansion fan (see Fig. 17.10). Thus for these an intercell speed is evaluated

$$s^n_{i+1/2} = \begin{cases} |(u^n_i + u^n_{i+1})/2| & \text{shock ,} \\ \max\left(|u^n_i|, |u^n_{i+1}|\right) & \text{rarefaction .} \end{cases} \tag{18.7}$$

We of course maximise over all intercell flux speeds (including the boundary points), namely

$$s^n_{\text{max}} = \max\left\{s^n_{i+1/2}\right\}, \quad \text{for } i = 0, \ldots, N.$$

Finally the marching time step Δt is then determined from

$$\Delta t = \frac{C_{\text{CFL}}}{s^n_{\text{max}}}\Delta x;$$

typically a $C_{\text{CFL}} \approx 0.9 < 1$ is imposed.

More details in :
Toro, E. F. (1997) *Riemann Solvers and Numerical Methods for Fluid Dynamics – A Practical Introduction*. Springer.

A good read, which discusses the topic with clarity is :
Roe, P. L. (1986) *Characteristic-based Schemes for the Euler Equations*, Ann. Rev. Fluid Mech. **18** : *pp.* 337-365. https://doi.org/10.1146/annurev.fl.18.010186.002005.

The mathematical approach is outlined in the seminal paper by Lax :
Lax, P. D. (1973) *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, SIAM. https://doi.org/10.1137/1.9781611970562.ch1