MATH60005/70005: Optimisation (Autumn 24-25)

Solutions to Additional Exercises

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1. a) Find and classify all the stationary points of

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 1\,.$$

   b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

      i) $f(\mathbf{x}) = \log\left(\sum_{i=1}^{k} e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)$, where $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

      iii) $f(\mathbf{x}) = \|\mathbf{x}\|^4$.

Answer. The handwritten solution is included in the next pages.

Find stationary points of

$$f(x_1, x_2) = \frac{x_1^3}{3} + \frac{x_1^2}{2} + 2x_1x_2 + \frac{x_2^2}{2} - x_2 + 1$$

$\nabla f = 0$

$$\frac{\partial f}{\partial x_1} = x_1^2 + x_1 + 2x_2 = 0$$

$$\frac{\partial f}{\partial x_2} = 2x_1 + x_2 - 1 = 0$$

$$\hookrightarrow x_2 = 1 - 2x_1$$

$$\Rightarrow x_1^2 + x_1 + 2(1 - 2x_1) = 0$$

$$(x_1 - 1)(x_1 - 2) = 0$$

$x_1 = 1 \quad\quad x_1 = 2$

$\downarrow \quad\quad\quad \downarrow$

$x_2 = -1 \quad\quad x_2 = -3$

$$\nabla^2 f = \quad \frac{\partial f}{\partial x_1^2} = 2x_1 + 1 \qquad \frac{\partial^2 f}{\partial x_2^2} = 1$$

$$\frac{\partial f}{\partial x_1 \partial x_2} = \frac{\partial f}{\partial x_2 \partial x_1} = 2 \qquad \nabla^2 f = \begin{bmatrix} 2x_1+1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\nabla^2 f(1,-1) = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{cases} \det < 0 \\ \text{trace} > 0 \end{cases} \Longrightarrow \text{Saddle point}$$

$$\nabla^2 f(2,-3) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{cases} \det > 0 \\ \text{trace} > 0 \end{cases} \Longrightarrow \underline{\underline{\text{minimum.}}}$$

local or global? check convexity

$$\frac{x_1^3}{3} + \frac{x_1^2}{2} + \ldots \text{ lower-order terms}$$

Acke $x_1 > 0 \longrightarrow \infty \implies f(x_1, x_2) \longrightarrow +\infty$

$x_1 < 0$, take $x_1 \longrightarrow -\infty$ $f(x_1, x_2) \longrightarrow -\infty$

$\implies$ We can only obtain local optimality.

Are these functions convex?

(i) $f(\underline{x}) = \ln\left(\sum_{i=1}^{k} e^{\underline{a_i}^T \underline{x} + b_i}\right)$, $\underline{a_i} \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

Answer: We know that log-sum-exp is convex (lecture notes convex)

$$h(\underline{x}) = \ln\left(e^{x_1} + \cdots + e^{x_n}\right)$$

but we indentify

$$f(\underline{x}) = h(A\underline{x} + \underline{b}) = \ln\left(e^{\underline{a_1}^T \underline{x} + b_1} + \cdots + e^{\underline{a_m}^T \underline{x} + b_m}\right)$$

$$= \ln\left(\sum_{i=1}^{k} e^{\underline{a_i}^T \underline{x} + b_i}\right)$$

where $A = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_k^T \end{bmatrix} \in \mathbb{R}^{k \times n}$ and $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \in \mathbb{R}^k$

Convex h composed with linear transformation

$\Rightarrow$ $f(\underline{x}) = h(A\underline{x} + \underline{b})$ is convex.

ii) $f(\underline{x}) = \|\underline{x}\|^4$

$f(\underline{x}) = h \circ g(\underline{x})$, where $h(y) = y^4$ and $g(\underline{x}) = \|\underline{x}\|$

h is convex and non-decreasing in $\mathbb{R}_+$:

$h'(y) = 4y^3 \geq 0$ for $y \in \mathbb{R}_+$

$h''(y) = 12y^2 \geq 0$

$g(\underline{x}) = \|\underline{x}\|$ is convex (Convexity of norms)

and goes from $\mathbb{R}^n$ to $\mathbb{R}_+$

$\Rightarrow$ Composition of $h \circ g(\underline{x})$ with

h convex and non-decreasing on Image(g)

g convex in $\mathbb{R}^n$ $\Rightarrow f = h \circ g(\underline{x})$ convex //

2. a) Consider the function

$$f(\mathbf{x}) = f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2, \quad \mathbf{x} \in \mathbb{R}^2,$$

with

$$\begin{aligned} f_1(\mathbf{x}) &= -13 + x_1 + ((5 - x_2)x_2 - 2)x_2\,, \\ f_2(\mathbf{x}) &= -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2\,. \end{aligned}$$

Knowing that there exists an $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = 0$, find a minimizer for this function, and discuss whether it is local or global.

b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

i) $f(\mathbf{x}) = \sum_{i=1}^n x_i \ln(x_i) - \left(\sum_{i=1}^n x_i\right) \ln\left(\sum_{i=1}^n x_i\right)$ over $\mathbb{R}^n_{++}$

ii) $f(\mathbf{x}) = \sqrt{\mathbf{x}^T Q \mathbf{x} + 1}$ over $\mathbb{R}^n$, where $Q \succeq 0$ is an $n \times n$ matrix.

Answer a) Since $f(\mathbf{x})$ is the sum of two positive functions, the information about an $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = 0$ requires that both $f_1(\mathbf{x}^*) = 0$ and $f_2(\mathbf{x}^*) = 0$, which is found by solving the system of equations, obtaining $x_1 = 5$ and $x_2 = 4$. It is clear that for $f(\mathbf{x}) = 0 \leq f(\mathbf{y})$ for any $\mathbf{y} \in \mathbb{R}^2$, which is the definition of a global minimizer. $\mathbf{x}^* = (5, 4)^\top$ is thus a global minimizer of $f(\mathbf{x})$.

Answer b) The handwritten solution is included in the next pages.

Are the following functions Convex?

(i) $f(x) = \sum_{i=1}^{n} x_i \ln(x_i) - \left(\sum_{i=1}^{n} x_i\right) \ln\left(\sum_{i=1}^{n} x_i\right)$ over $\boxed{\mathbb{R}^n_{++}}$

First: work the expression to something tractable

e.g. $n=2$ (just exploring)

$$x_1 \ln(x_1) + x_2 \ln(x_2) - (x_1 + x_2) \ln(x_1 + x_2)$$

$$= x_1 \ln(x_1) + x_2 \ln(x_2) - x_1 \ln(x_1 + x_2) - x_2 \ln(x_1 + x_2)$$

$$= x_1 \left(\ln(x_1) - \ln(x_1 + x_2)\right) + x_2 \left(\ln(x_2) - \ln(x_1 + x_2)\right)$$

$$= x_1 \left(\ln\left(\frac{x_1}{x_1 + x_2}\right)\right) + x_2 \left(\ln\left(\frac{x_2}{x_1 + x_2}\right)\right) \ \ldots$$

$$f(x) = \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{\sum_{k=1}^{n} x_k}\right)$$

now $f(x) = \sum_{i=1}^{n} h_i(\underline{x})$

where $h_i(\underline{x}) = x_i \ln\left(\frac{x_i}{\sum_{k=1}^{n} x_k}\right)$

$$\varphi(u, v) = u \ln(u/v)$$

$$\underline{x} \nearrow u = x_i, \quad \searrow v = \sum_{k=1}^{n} x_k \quad \Big\} \text{ linear transformation.}$$

$$\underline{x} \xrightarrow{\varphi \circ (L.T.)} h_i(\underline{x})$$

We need to show that φ is convex and then we are ok, because f would the sum of h_i, each one of them convex (h_i's would be convex functions composed with linear transformations — $\varphi(u,v) = \frac{u^2}{v}$ quad-over-lin)

We show that $\varphi(u,v) = u \ln(u/v)$ is convex

$$\varphi(u,v) = u \ln(u) - u \ln(v)$$

$$\frac{\partial \varphi}{\partial u} = \ln(u) + 1 - \ln(v)$$

$$\frac{\partial \varphi}{\partial v} = -\frac{u}{v} \quad \rightarrow \quad \frac{\partial^2 \varphi}{\partial u \, \partial v} = -\frac{1}{v}$$

$$\frac{\partial^2 \varphi}{\partial u^2} = \frac{1}{u} \qquad \frac{\partial^2 \varphi}{\partial v^2} = \frac{u}{v^2} \quad \rightarrow \nabla^2 \varphi = \begin{bmatrix} 1/u & -1/v \\ -\frac{1}{v} & u/v^2 \end{bmatrix}$$

2 by 2 matrix: $\text{Trace}(\nabla^2 \varphi) = \frac{1}{u} + \frac{u}{v^2}$

$$\text{Det}(\nabla^2 \varphi) = \frac{1}{\cancel{u}} \cdot \frac{\cancel{u}}{v^2} - \left(\frac{1}{v}\right)^2$$

$$= 0$$

$\text{Trace}(\nabla^2 \varphi)$ is positive when u i strictly positive

$\Longrightarrow \underline{x} \in \mathbb{R}^n_{++}$

$\Longrightarrow \nabla^2 \varphi \succcurlyeq 0 \Longrightarrow \varphi$ is convex

Sum of convex functions $\longrightarrow$ Convex composed with L.T.

$\Longrightarrow f(\underline{x})$ is convex.

ii) $f(\underline{x}) = \sqrt{1 + \underline{x}^T Q \underline{x}}$ is convex, given $Q \succcurlyeq 0$

In previous weeks, we have shown that

$h(y) = \sqrt{1 + y^2}$ is convex. ✓

$h'(y) = \frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{1+y^2}} > 0$ for $y > 0 \rightarrow$ non-decreasing

$$h''(y) = \frac{\sqrt{1+y^2} - \frac{y^2}{\sqrt{1+y^2}}}{(1+y^2)} = \frac{1}{(1+y^2)^{3/2}} > 0$$

$$f(\underline{x}) = \sqrt{1 + \underline{x}^T Q \underline{x}} = \sqrt{1 + y^2}, \text{ where } y^2 = x^T Q x$$

$$\Rightarrow y = g(\underline{x}) = \sqrt{x^T Q x}$$

but $Q \succcurlyeq 0$, and we have seen in Week 2 $\quad d_i \geq 0$

$$Q = U^T \underbrace{D}_{\text{Diagonal}} U \qquad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$Q = U^T \sqrt{D} \sqrt{D} U \Longrightarrow \underline{x}^T U^T \sqrt{D} \sqrt{D} U \underline{x}$$

$$\| \qquad \underline{x}^T Q \underline{x}$$

$$\sqrt{\underline{x}^T Q \underline{x}} = \sqrt{\underline{x}^T U^T \sqrt{D} \sqrt{D} U \underline{x}}$$

$$= \| L x \|$$

$\hookrightarrow$ where $L = \sqrt{D} U$

But we know that $\| \cdot \|$ is convex function and

$x \longrightarrow Lx$ is a linear transformation

$\Longrightarrow \| L x \|$ is convex in $\underline{x}$

$g(\underline{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ $g := \| L x \|$ is convex

$$f(\underline{x}) = \sqrt{\underline{x}^T Q \underline{x} + 1} = h \circ g(\underline{x}) \quad \text{convex}$$

$$h = \sqrt{1+\underline{y}^2}, \quad g = \|L x\|$$

because of composing $h \circ g$ w/ g convex and h convex and non-decreasing.

$$\min \; - x_1 x_2 x_3$$

$$\text{s.t.} \quad x_1 + 3x_2 + 6x_3 \leq 48$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

$$-x_3 \leq 0$$

3. a) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x_1, x_2) := x_2^4 - 2x_2^2 + 1 + \left(x_1^2 + x_2^2 - 1\right)^2$$

i) Find all the stationary points of f.

ii) Classify the stationary points found in i).

b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex as well as concave function. Show that f is an affine function, that is, there exist $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$.

Answer a.i) The gradient of f is given by

$$\nabla f(\mathbf{x}) = 4 \begin{pmatrix} \left(x_1^2 + x_2^2 - 1\right) x_1 \\ \left(x_1^2 + x_2^2 - 1\right) x_2 + \left(x_2^2 - 1\right) x_2 \end{pmatrix}$$

The stationary points are those satisfying

$$\begin{aligned} \left(x_1^2 + x_2^2 - 1\right) x_1 &= 0 \\ \left(x_1^2 + x_2^2 - 1\right) x_2 + \left(x_2^2 - 1\right) x_2 &= 0\,. \end{aligned}$$

By the first equation, there are two cases: either $x_1 = 0$, and then by the second equation x_2 is equal to one of the values $0, 1, -1$; the second option is that $x_1^2 + x_2^2 = 1$, and then by the second equation we have that $x_2 = 0, \pm 1$ and hence x_1 is $\pm 1, 0$ respectively. Overall, there are 5 stationary points: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$.

Answer a.ii) The Hessian of the function is

$$\nabla^2 f(\mathbf{x}) = 4 \begin{pmatrix} 3x_1^2 + x_2^2 - 1 & 2x_1 x_2 \\ 2x_1 x_2 & x_1^2 + 6x_2^2 - 2 \end{pmatrix}.$$

Since

$$\nabla^2 f(0, 0) = 4 \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \prec 0$$

it follows that (0,0) is a strict local maximum point. Since $f(x_1, 0) = (x_1^2 - 1)^2 + 1 \longrightarrow \infty$ as $x_1 \longrightarrow \infty$, the function is not bounded above and thus (0,0) is not a global maximum point. Also,

$$\nabla^2 f(1, 0) = \nabla^2 f(-1, 0) = 4 \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

which is an indefinite matrix and hence (1,0) and (-1,0) are saddle points. Finally,

$$\nabla^2 f(0, 1) = \nabla^2 f(0, -1) = 4 \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \succeq 0\,.$$

The fact that the Hessian matrices of f at (0,1) and (0,-1) are positive semidefinite is not enough in order to conclude that these are local minimum points; they might be saddle points. However, in this case it is not difficult to see that (0,1) and (0,-1) are in fact global minimum points since $f(0,1) = f(0,-1) = 0$, and the function is lower bounded by zero. Note that since there are two global minimum points, they are nonstrict global minima, but they actually are strict local minimum points since each has a neighborhood in which it is the unique minimizer.

In summary: $(0,0)$ is a strict local maximum point, $(1,0)$ and $(-1,0)$ are saddle points, $(0,1)$ $(0,-1)$ are strict local minimum points (or non-strict global minima).

Answer b) Since f is concave and convex, it follows immediately that $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. It is straightforward (though a bit tedious) to use this to check that the function $g(x) = f(x) - f(0)$ is linear, hence $g(x) = a^T x$, so letting $f(0) = b$, $f(x) = a^T x + b$ is affine.

4. a) Let $f(x_1, x_2)$ be a twice-differentiable convex function in $\mathbb{R}^2$ such that $f(0,0) = f(1,0) = f(0,1) = 0$. What do you know about:

   i) $f\left(\frac{1}{2}, \frac{1}{2}\right)$?

   ii) $a = \frac{\partial^2 f}{\partial x_1^2}$, $b = \frac{\partial^2 f}{\partial x_2^2}$, and $c = \frac{\partial^2 f}{\partial x_1 \partial x_2}$?

   b) Consider the function

$$g(x_1, x_2, x_3) = 59x_1^2 + 52x_2^2 + 17x_3^2 + 80x_1x_2 - 24x_1x_3 + 8x_2x_3 + 27x_1 - 84x_2 + 20x_3 \, .$$

   i) Is $g(\mathbf{x})$ convex?

   ii) Solve

$$\begin{array}{ll} \max & g(x_1, x_2, x_3) \\ \text{s.t.} & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Answer a.i) Noting that $\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$, using the definition of convexity we have

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}(1,0) + \frac{1}{2}(0,1)\right) \leq \frac{1}{2} f(0,1) + \frac{1}{2} f(1,0) \leq 0 \, .$$

Answer a.ii) For a twice-differentiable smooth convex function we have that its Hessian is positive semidefinite, which applying determinant and trace criterion means $a + b \geq 0$ and $ab - c^2 \geq 0$.

Answer b.i) The function g is a quadratic function associated matrix

$$\mathbf{A} = \begin{bmatrix} 59 & 40 & -12 \\ 40 & 52 & 4 \\ -12 & 4 & 17 \end{bmatrix}$$

which is positive definite (diagonally dominant with positive entries in the diagonal). Therefore g is convex.

Answer b.ii) The function g is non-constant, continuous and convex over the convex set of the constraints, which corresponds to the unit simplex Δ_3. Therefore, there exists at least one maximizer of g over Δ_3 that is an extreme point of Δ_3. The extreme points of Δ_3 are given by the canonical basis in $\mathbb{R}^3$. It suffices to evaluate $g(1,0,0) = 86$, $g(0,1,0) = -32$, and $g(0,0,1) = 37$ to conclude that the maximizer is $g(1,0,0) = 86$.

5. a) Given the function

$$f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$$

i) Determine its stationary points.

ii) Classify the stationary points found in i).

b) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{g} \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which define the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top H\mathbf{x} + \mathbf{g}^\top \mathbf{x}$. Does there exists a vector $\mathbf{u} \in \mathbb{R}^3$ such that $f(t\mathbf{u}) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct $\mathbf{u}$.

Answer a.i) The gradient of $f(x_1, x_2)$

$$\nabla f(x_1, x_2) = \begin{pmatrix} 6x_1x_2 \\ 6x_2^2 - 12x_2 + 3x_1^2 \end{pmatrix}$$

which equals 0 when $x_1 = x_2 = 0$ or when $x_1 = 0$ and $x_2 = 2$.

Answer a.ii) The Hessian is

$$\nabla^2 f(\bar{x}) = \begin{pmatrix} 6x_2 & 6x_1 \\ 6x_1 & 12x_2 - 12 \end{pmatrix}$$

which is negative semidefinite at $(0, 0)$ and positive definite at $(0, 2)$. Hence $(0, 2)$ is a strict local minimizer and $(0, 0)$ is a candidate to be a local maximizer, however, notice hat for any small neighborhood of $(0, 0)$, f increases in the positive x_1 direction and decreases in the positive x_2 direction, so $(0, 0)$ is a saddle point.

Answer b) Let $u = \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ in $\text{Null}(H)$ so that

$$\begin{aligned} f(tu) &= (tu)^\top H(tu) + g^\top (tu) \\ &= t^2 u^\top Hu + tg^\top u \\ &= 0t^2 - 4t \end{aligned}$$

As $t \nearrow \infty$, $f(tu) \searrow -\infty$.

6. a) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

i) $f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$

ii) $h(\mathbf{x}) = \left(\|\mathbf{x}\|^2 + 1\right)^2$, $\mathbf{x}$ in $\mathbb{R}^n$.

b) Consider the problem

$$
\begin{array}{lll}
(\mathrm{P}) & \min & f(\mathbf{x}) \\
 & \text{s.t.} & g(\mathbf{x}) \leq 0 \\
 & & \mathbf{x} \in X
\end{array}
$$

where f, g are convex an $X \subseteq \mathbb{R}^n$ is convex. Suppose $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$
\begin{array}{ll}
\min & f(\mathbf{x}) \\
\text{s.t.} & \mathbf{x} \in X
\end{array}
$$

Answer a.i) The function $f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$ is convex over $\mathbb{R}^3$ as a sum of three convex functions: the function $e^{x_1 - x_2 + x_3}$, which is convex since it is constructed by making the linear change of variables $t = x_1 - x_2 + x_3$ in the one-dimensional function $\varphi(t) = e^t$. For the same reason, e^{2x_2} is convex. Finally, the function x_1, being linear, is convex.

Answer a.ii) The function $h(x) = \left(\|x\|^2 + 1\right)^2$ is a convex function over $\mathbb{R}^n$ since it can be represented as $h(\mathbf{x}) = g(f(\mathbf{x}))$, where $g(t) = t^2$ and $f(\mathbf{x}) = \|\mathbf{x}\|^2 + 1$. Both f and g are convex, but note that g is not a nondecreasing function. However, the image of $\mathbb{R}^n$ under f is the interval $[1, \infty)$ on which the function g is nondecreasing. Consequently, the composition $h(\mathbf{x}) = g(f(\mathbf{x}))$ is convex.

Answer b) Suppose for sake of contradiction there exists $y \in X \cap \{x : g(x) > 0\}$ such that $f(y) < f(x^*)$. The line segment $[x^*, y]$ lies in X because X is convex. Furthermore, by continuity of g, there exists a $z \in [x^*, y]$ such that $g(z) = 0$, i.e., z is feasible for the problem (P) and there exists some $\lambda \in [0, 1]$ such that $z = x^* + \lambda(y - x^*)$. Observe that by convexity of f:

$$
f(z) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda \underbrace{(f(y) - f(x^*))}_{<0 \text{ by assumption}} < f(x^*)
$$

which contradicts x^* being optimal for (P).

7. Consider the constrained minimization problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \Delta_n\},$$

where f is a continuously differentiable function over Δ_n. Show that $\mathbf{x}^* \in \Delta_n$ is a stationary point of this problem if and only if there exists $\mu \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) \begin{cases} = \mu, & x_i^* > 0\,, \\ \geq \mu\,, & x_i^* = 0\,. \end{cases}.$$

Answer. The handwritten solution is included in the next pages.

$$\min f(\underline{x}), \quad \underline{x} \in \Delta_n = \left\{ \underline{x} \in \mathbb{R}^n : \sum_{i=1}^{n}{}' x_i = 1, \quad x_i \geq 0 \right\}$$

We need to show that stationarity is equiv. to

$$(*) \quad \frac{\partial f}{\partial x_i}(\underline{x}^*) = \begin{cases} = \mu, & x_i^* > 0 \\ \geq \mu, & x_i^* = 0 \end{cases}, \quad \text{for some } \mu \in \mathbb{R}.$$

1) $(*) \overset{?}{\Longrightarrow}$ Stationarity

We assume $\exists\ \underline{x}^* \in \Delta_n$ and μ such that $(*)$ holds.

We want to show stationarity, that is

$$\nabla f(\underline{x}^*)^T(\underline{x} - \underline{x}^*) \geq 0 \quad \forall\, x \in \Delta_n.$$

Canonical vector, j-coord. $\rightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$

Because of $(*)$, $\nabla f(\underline{x}^*)$ is a vector $\sum'_{j,\, x_j^* > 0} \mu \underline{e}_j + \sum'_{j,\, x_j^* = 0} (\mu + \delta_j)\underline{e}_j$ with $\delta_j \geq 0$.

Now, $\nabla f(\underline{x}^*)^T \underline{x} = \sum'_{j,\, x_j^* > 0} \mu x_j + \sum'_{j,\, x_j^* = 0} (\mu + \delta_j) x_j \geq \sum_{j=1}^{n}{}' \mu \cdot x_j = \mu \sum_{j=1}^{n} x_j \overset{x \in \Delta_n}{=} \mu.$

Then, $\nabla f(\underline{x}^*)^T(\underline{x} - \underline{x}^*) \geq \mu - \underbrace{\nabla f(\underline{x}^*)^T(\underline{x}^*)}_{=}$

$$\mu - \mu \cdot \sum'_{j,\, x_j^* > 0} x_j^* = \mu - \mu = 0. \quad (\underline{x}^* \in \Delta_n)$$

2) Stationarity $\Rightarrow (*)$

Assume $\underline{x}^*$ stationary and pick $i \neq j$ such that $x_i^* > 0$ and $x_j^* > 0$. For sufficiently small δ, take

$$\left.\begin{aligned} \underline{x}^+ &= \underline{x}^* + \delta \underline{e}_i - \delta \underline{e}_j \\ \underline{x}^- &= \underline{x}^* - \delta \underline{e}_i + \delta \underline{e}_j \end{aligned}\right\} \underline{x}^+ \text{ and } \underline{x}^- \in \Delta_n$$

Using the def. of stationarity with $\underline{x}^+$ and $\underline{x}^-$

$$\nabla f(\underline{x}^*)^T(\underline{x}^+ - \underline{x}^*) \geq 0$$

$$\frac{\partial f}{\partial x_i}(\underline{x}^*) \geq \frac{\partial f}{\partial x_j}(\underline{x}^*)$$

and

$$\nabla f(\underline{x}^*)^T(\underline{x}^- - \underline{x}^*) \geq 0$$

$$\frac{\partial f}{\partial x_j}(\underline{x}^*) \geq \frac{\partial f}{\partial x_i}(\underline{x}^*)$$

Implying that $\frac{\partial f}{\partial x_i}(\underline{x}^*) = \frac{\partial f}{\partial x_j}(\underline{x}^*)$ for all positive coordinates

Now, we assume that $x_i^* > 0$ and $x_j^* = 0$.
The stationarity condition with $\underline{x}^-$ reads

$$\nabla f(\underline{x}^*)^T(\underline{x}^- - \underline{x}^*) = \nabla f(\underline{x}^*)^T(-\delta \underline{e}_i + \delta \underline{e}_j) \geq 0$$

$$= \delta\left(\frac{\partial f}{\partial x_j}(\underline{x}^*) - \frac{\partial f}{\partial x_i}(\underline{x}^*)\right) \geq 0$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x_j}(\underline{x}^*) \geq \frac{\partial f}{\partial x_i}(\underline{x}^*), \text{ whenever } x_j^* = 0.$$

//

8. Consider the problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n,\ \mathbf{e}^\top \mathbf{x} = 1\},$$

where f is a continuously differentiable function. Show that $\mathbf{x}^*$ is a stationary point of this problem if and only if

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}^*).$$

Answer. The handwritten solution is included in the next pages.

Consider The problem

$$\min f(\underline{x}) \quad \text{s.t.} \quad \underbrace{e^T \cdot \underline{x}}_{x_1+x_2+\dots+x_n = 1} = 1, \quad \underline{x} \in \mathbb{R}^n$$

$$e = [\underbrace{1\ 1\ 1\ 1\ 1 \dots}_{n}] \qquad \sum_{i=1}^{n} x_i = 1$$

where f is a continuously differentiable function. Show that $\underline{x}^*$ is a stationary point $\boxed{\text{if and only if}}$

$$\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$$

ANSWER: iif

Part (I): Stationary point $\Longrightarrow \frac{\partial}{\partial x_1} f(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$

Part (II): $\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots \frac{\partial f}{\partial x_n}(\underline{x}^*) \Rightarrow \underline{x}^*$ is stationary.

We begin showing part II:

Stationarity def.

$$\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*) \Longrightarrow \nabla f(\underline{x}^*)^T(\underline{x} - \underline{x}^*) \geq 0$$

$\forall\ \underline{x}$ such that $\underline{e}^T \cdot \underline{x} = 1$

$$\nabla f(\underline{x}^*)^T \cdot (\underline{x} - \underline{x}^*) \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}^*)(x_i - x_i^*),$$

but recall

$$= \frac{\partial f}{\partial x_1}(\underline{x}^*) \sum_{i=1}^{n}(x_i - x_i^*)$$

$$= \frac{\partial f}{\partial x_1}(\underline{x}^*) \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^* \right)$$

but, $\underline{x}$ and $\underline{x}^*$ are such that $\underbrace{e^T \cdot \underline{x}}_{x_1 + \dots + x_n} \overset{\text{or } \underline{x}^*}{=} 1$

$$\sum_{i=1}^{n} x_i = 1$$

$$= \frac{\partial f}{\partial x_1}(\underline{x}^*)\left(1 - 1\right) = 0 \geq 0 \qquad \blacksquare$$

Part (I): Assume $\underline{x}^*$ such that $\nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) \geq 0$

$\forall\ \underline{x}$ s.t. $\underset{(n)}{\underline{e}^T} \cdot \underline{x} = 0$

$\sum^i x_i = 0$

$$\Longrightarrow \frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$$

By Contradiction. Assume $\underline{x}^*$ stationary, but where $\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*)$ does NOT hold,

or equivalently, that there exist two indexes i, j, $i \neq j$ such that $\frac{\partial f}{\partial x_i}(\underline{x}^*) \neq \frac{\partial f}{\partial x_j}(\underline{x}^*)$

or $\frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$

We construct the following vector from the stationary point

$$x_k = \begin{cases} x_k^* & k \notin \{i, j\} \\ x_i^* - 1 & k = i \\ x_j^* + 1 & k = j \end{cases}$$

$\frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$

$$\underline{x}^* = [x_1^* \dots x_i^* \dots x_j^* \dots x_n^*]$$

$$\underline{x} = [x_1^* \dots x_i^* - 1 \dots x_j^* + 1 \dots x_n^*]$$

Note that if $\underline{x}^*$ is such that $e^T \underline{x}^* = 1$

$\Longrightarrow \underline{x}$ is such that $e^T \underline{x} = 1$

$\Longrightarrow$ $\underline{x}$ is feasible (inside the constraint set)

But then,

$$\nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) = \frac{\partial f}{\partial x_i}(\underline{x}^*)\underbrace{(x_i - x_i^*)}_{-1} \quad = x_i^*-1-x_i^*$$

$$+ \frac{\partial f}{\partial x_j}(\underline{x}^*)\underbrace{(x_j - x_j^*)}_{1} \quad = x_j^*+1-x_j^*$$

$$+ \sum_{\substack{k=1 \\ k\neq i,j}}^{m} \frac{\partial f}{\partial x_k}(\underline{x}^*)\underbrace{(x_k - x_k^*)}_{0} \quad x_k = x_k^*$$

but recall the construction of x_k

$$\Longrightarrow \nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) = \frac{\partial f}{\partial x_i}(\underline{x}^*)(-1) + \frac{\partial f}{\partial x_j}(\underline{x}^*)(1) + 0$$

$$= \frac{\partial f}{\partial x_j}(\underline{x}^*) - \frac{\partial f}{\partial x_i}(\underline{x}^*) < 0$$

because we assumed $\frac{\partial f}{\partial x_i}(\underline{x}^*) > \frac{\partial f}{\partial x_j}(\underline{x}^*)$

$\Longrightarrow$ This contradicts the hypothesis that $\underline{x}^*$

is a stationary point such that

$$\nabla f(\underline{x}^*)^T(\underline{x}-\underline{x}^*) \geq 0 \quad \forall \underline{x}$$

such that

$$\underline{e}^T \cdot \underline{x} = 1.$$

9. Find the optimal solution of the problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^3} \quad & 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - 3x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Answer. The handwritten solution is included in the next pages.

Solve $\max \; 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - 3x_2 + 4x_3$

s.t. $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$.

A] The cost is a quadratic function

$$\underline{x}^T \begin{bmatrix} 2 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \underline{x} + [2 \;\; -3 \;\; 4] \, \underline{x}$$

$\hookrightarrow Q \succ 0 \implies$ Cost is convex.

The constraint is the unit simplex in 3D

Δ_3, convex. $\implies$ Convex cost + convex constraint

$\implies$ Maximizer is one of the extreme points!

Extreme points of $\Delta_3 = \begin{cases} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{cases}$

$$\Longrightarrow f(1,0,0) = 2\cdot 1^2 + 2\cdot 1 = 4$$
$$f(0,1,0) = 1\cdot 1^2 - 3\cdot 1 = -2$$
$$f(0,0,1) = 1\cdot 1^2 + 4\cdot 1 = 5$$

Hence, the maximizer is $(0,0,1)$ //.

10. Consider the problem

$$
\begin{array}{ll}
\min & x_1^2 + x_2^2 + x_3^2 \\
\text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 4 \\
 & x_3 \leq 1
\end{array}
$$

i) Write down the KKT conditions.

ii) Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions.

iii) Find the optimal solution of the problem using the KKT system.

Answer. The handwritten solution is included in the next pages.

$$\min \quad x_1^2 + x_2^2 + x_3^2$$
$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 \geq 4$$
$$x_3 \leq 1$$

i) KKT system.

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda_1 (4 - x_1 - 2x_2 - 3x_3) + \lambda_2 (x_3 - 1)$$

$$\nabla_{\underline{x}} L = 0 \iff \begin{aligned} 2x_1 - \lambda_1 &= 0 \\ 2x_2 - 2\lambda_1 &= 0 \\ 2x_3 - 3\lambda_1 + \lambda_2 &= 0 \end{aligned} \quad + \quad \begin{aligned} &\lambda_1 (4 - x_1 - 2x_2 - 3x_3) = 0 \\ &\lambda_2 (x_3 - 1) = 0 \\ &\lambda_1 \geq 0, \ \lambda_2 \geq 0 \end{aligned}$$

ii) Convex constraint + strictly convex cost $\underline{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ & & 1 \end{bmatrix} \underline{x}$ $Q > 0$

* Also need to mention coercivity for existence

$\exists!$ optimal solution and KKT is sufficient.

iii) a) $\lambda_1 = \lambda_2 = 0 \implies x_1 = x_2 = x_3 = 0$, unfeasible

$$x_1 + 2x_2 + 3x_3 \cancel{\geq} 4$$

b) $\lambda_1 > 0, \ \lambda_2 = 0 \implies$
$$2x_1 = \lambda_1 \qquad 4 - x_1 - 2x_2 - 3x_3 = 0$$
$$x_2 = \lambda_1$$
$$2x_3 = 3\lambda_1$$

$$\implies \begin{aligned} 2x_1 &= x_2 \\ 2x_3 &= 3x_2 \end{aligned} \implies 4 - \frac{x_2}{2} - 2x_2 - \frac{9x_2}{2} = 0$$

$$x_2 = \frac{4}{7}, \quad x_1 = \frac{2}{7}, \quad x_3 = \frac{6}{7}$$

and $\lambda_1 = 4/7 > 0$ ✓.

The optimal solution is

$$\underline{x}^* = \frac{1}{7} (2, 4, 6)^T$$

11. Consider the problem

$$
\begin{aligned}
\min \quad & -x_1 x_2 x_3 \\
\text{s.t.} \quad & x_1 + 3x_2 + 6x_3 \leq 48 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(i) Write the KKT conditions for the problem.

(ii) Find the optimal solution of the problem.

Answer. The handwritten solution is included in the next pages.

$$f(\underline{x}) = \sqrt{\underline{x}^T Q \underline{x} + 1} = \underset{\underset{\sqrt{1+\underline{y}^2}}{\shortparallel}}{h} \circ \underset{\underset{\|L x\|}{\shortparallel}}{g}(\underline{x}) \quad \text{convex}$$

because of composing $h \circ g$ w/ g convex and h convex and non-decreasing.

$$\min \; - x_1 x_2 x_3$$

$$\text{s.t. } x_1 + 3x_2 + 6x_3 \leq 48$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$
$$-x_3 \leq 0$$

i) KKT System

$$\mathcal{L} = -x_1 x_2 x_3 + \lambda_1 (x_1 + 3x_2 + 6x_3 - 48) - \lambda_2 x_1 - \lambda_3 x_2 - \lambda_4 x_3$$

$$\nabla_{\underline{x}} \mathcal{L} = 0 \iff \begin{aligned} -x_2 x_3 + \lambda_1 - \lambda_2 &= 0 \\ -x_1 x_3 + 3\lambda_1 - \lambda_3 &= 0 \\ -x_1 x_2 + 6\lambda_1 - \lambda_4 &= 0 \end{aligned}$$

$$\lambda_1 (x_1 + 3x_2 + 6x_3 - 48) = 0$$

$$\lambda_2 x_1 = 0 \qquad \lambda_i\text{'s} \geq 0$$

$$\lambda_3 x_2 = 0 \qquad i = 1, \ldots, 4$$

$$\lambda_4 x_3 = 0$$

1) $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$

$$\left.\begin{aligned} -x_2 x_3 &= 0 \\ -x_1 x_3 &= 0 \\ -x_1 x_2 &= 0 \end{aligned}\right\} \begin{aligned} &\to x_2 = 0 \quad \text{or} \quad x_3 = 0 \\ &\to x_1 = 0 \quad \text{or} \quad x_3 = 0 \\ &\to x_1 = 0 \quad \text{or} \quad x_2 = 0 \end{aligned}$$

We need at least 2 out of 3 x_i's to be 0

$\Longrightarrow$ That all the KKT points that follow from this system have an optimal value equal to 0.

2) $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$

$\lambda_1 = x_2 x_3$ but $\lambda_1 > 0$, so $\Rightarrow x_1, x_2, x_3 \neq 0$

$3\lambda_1 = x_1 \cdot x_3$

$6\lambda_1 = x_1 \cdot x_2$ $\Longrightarrow 3 = x_1/x_2$

$\Rightarrow x_1 = 3x_2$

$x_2 = 2x_3$

and plug into $x_1 + 3x_2 + 6x_3 - 48 = 0$

$\Rightarrow x_3 = 8/3$, $x_2 = 16/3$,

$x_1 = 16$

$\Longrightarrow$ the value is $-x_1 x_2 x_3$

$-16 \cdot 16/3 \cdot 8/3$

Now, for the remaining cases, if ~~any~~ of $\lambda_2, \lambda_3, \lambda_4$ is >0 $\Rightarrow$ there exists at least one coordinate $= 0$ in the solution $\Rightarrow$ its cost is 0
$(-x_1 \cdot x_2 \cdot x_3)$

$\Rightarrow$ That the optimal solution is the only KKT point with 3 coordinates $\neq$ from 0.
(case 2))

12. Consider the minimization problem for $\mathbf{x} \in \mathbb{R}^n$

$$\min_{\mathbf{x} \in C} \|\mathbf{x} - \mathbf{y}\|^2,$$

where $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$, with $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$. Assume that the rows of $\mathbf{A}$ are linearly independent.

i) Determine the KKT conditions for this problem. Are these sufficient?.

ii) Find the optimal solution of the problem using the KKT system.

iii) Given the problem

$$\min_{(x_1,x_2) \in C} x_1^2 + 2x_2^2 - 3x_1,$$

where $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$, write the explicit gradient descent iteration for this problem with a constant stepsize $t = 1$. You can help yourself using the result in part ii)

Answer i) This is a convex optimization problem, so the KKT conditions are necessary and sufficient. The Lagrangian function is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x} - \mathbf{y}\|^2 + 2\lambda^\top(\mathbf{A}\mathbf{x} - \mathbf{b}) = \|\mathbf{x}\|^2 - 2\left(\mathbf{y} - \mathbf{A}^\top\lambda\right)^\top \mathbf{x} - 2\lambda^\top\mathbf{b} + \|\mathbf{y}\|^2, \quad \lambda \in \mathbb{R}^m$$

Therefore, the KKT conditions are

$$\begin{aligned} 2\mathbf{x} - 2\left(\mathbf{y} - \mathbf{A}^\top\lambda\right) &= 0, \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \end{aligned}$$

Answer ii) The first equation can be written as

$$\mathbf{x} = \mathbf{y} - \mathbf{A}^\top\lambda$$

Substituting this expression for $\mathbf{x}$ in the second equation yields the equation

$$\mathbf{A}\left(\mathbf{y} - \mathbf{A}^\top\lambda\right) = \mathbf{b}$$

which is the same as

$$\mathbf{A}\mathbf{A}^\top\lambda = \mathbf{A}\mathbf{y} - \mathbf{b}$$

Thus,

$$\lambda = \left(\mathbf{A}\mathbf{A}^\top\right)^{-1}(\mathbf{A}\mathbf{y} - \mathbf{b})$$

where here we used the fact that $\mathbf{A}\mathbf{A}^\top$ is nonsingular since the rows of $\mathbf{A}$ are linearly independent. Using the latter expression for λ, we obtain that the optimal solution is

$$\mathbf{x}^* = \mathbf{y} - \mathbf{A}^\top\left(\mathbf{A}\mathbf{A}^\top\right)^{-1}(\mathbf{A}\mathbf{y} - \mathbf{b})$$

Answer iii) Solving the problem in ii) corresponds to computing the orthogonal projection of a vector $\mathbf{y}$ onto the affine space C. Applying this directly to the problem in c) with $\mathbf{A} = [1\ 1]$ and $\mathbf{b} = 1$, the orthogonal projection of a vector $\mathbf{y}$ onto C is given by

$$P_C(\mathbf{y}) = \mathbf{y} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (y_1 + y_2 - 1)\,.$$

The projected gradient descent with $t = 1$ reads

$$\mathbf{x}^{k+1} = P_C(\mathbf{x}^k - \nabla f(\mathbf{x}^k))\,,$$

where

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - 3 \\ 4x_2 \end{bmatrix}\,.$$

13. Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t. } & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

i) Is the problem convex?.

ii) Find all KKT points of the problem.

iii) Find the optimal solution of the problem.

Answer i) Observe the maximization problem is equivalent to the minimization problem

$$\begin{aligned} -\min - & \left(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2\right) \\ \text{s.t. } & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The Hessian of the objective $\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$ is not psd, so the problem is not convex.

Answer ii) The Lagrangian is

$$L(x_1, x_2, y_1, y_2, y_3) = -\left(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1\right) + x_2 + y_1(x_1 + x_2 - 1) - y_2x_1 - y_3x_2$$

defined on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+^2$. The KKT conditions are

$$\nabla_x L(x, y) = \begin{pmatrix} -2x_1 - 2x_2 + 3 + y_1 - y_2 \\ -2x_1 - 4x_2 - 1 + y_1 - y_3 \end{pmatrix} = 0 \tag{1}$$

$$y_2x_1 = 0 \tag{2}$$

$$y_3x_2 = 0 \tag{3}$$

$$x_1 + x_2 = 0 \tag{4}$$

$$x_1, x_2 \geq 0 \tag{5}$$

$$y_2, y_3 \geq 0\,. \tag{6}$$

Combining (4) with (1), we find $y_1 = y_2 - 1$ and $-2x_2 - 2 + y_1 - y_3 = 0$. If $y_2 = 0$, then $y_1 = -1$ and $-2x_2 - 3 - y_3 = 0$, but this cannot happen because $y_3 = -(2x_2 + 3)$ cannot be nonnegative if $x_2 \geq 0$. Thus to satisfy (2), we must have $x_1 = 0$, so $x_2 = 1$ by (4) and $y_3 = 0$ by (3). By (1), we get that $-2 + 3 + y_1 - y_2 = 0$ and $-4 - 1 + y_1 = 0$. Putting the pieces together, $(0, 1)$ is the only KKT point with multipliers $(y_1, y_2, y_3) = (5, 6, 0)$.

Answer iii) Since the problem is not convex but linearly constrained, the KKT-conditions are only necessary. Thus, the minimiser must satisfy the KKT-conditions, i.e., the optimal solution of the problem is $(x_1^*, x_2^*)^\top = (0, 1)$.

14. Consider the minimization

$$
\begin{aligned}
\min \quad & x_1 - 4x_2 + x_3^4 \\
\text{s.t.} \quad & x_1 + x_2 + x_3^2 \leq 2 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

i) Formulate the dual problem.

ii) Solve the dual problem.

Answer. The handwritten solution is included in the next pages.

min $\quad x_1 - 4x_2 + x_3^4$ $\qquad$ ME4 Q4

s.t. $\quad x_1 + x_2 + x_3^2 \le 2$

$x_1 \ge 0$ ✓ $x_2 \ge 0$ ✓

i) Cost is convex $\quad \nabla^2 f = \begin{bmatrix} \; & \\ & 12x_3^2 \end{bmatrix} \succeq 0$

$$\boxed{\begin{aligned} x_3^* &= 0 \quad x_2^* = 2 \\ x_1^* &= 0 \end{aligned}}$$

Constraints are convex $\quad \begin{bmatrix} \; & \\ & 2 \end{bmatrix} \succeq 0$

(Do KKT).

ii) Duality: $\quad \mathcal{X} = \mathbb{R}^3$

$$L(\underline{x}, \lambda) = x_1 - 4x_2 + x_3^4 + \lambda_1(x_1 + x_2 + x_3^2 - 2)$$
$$+ \lambda_2(-x_1)$$
$$+ \lambda_3(-x_2)$$

$$\min_{x \in \mathbb{R}^3} L(\underline{x}, \lambda) = x_1(1 + \lambda_1 - \lambda_2)$$
$$+ x_2(-4 + \lambda_1 - \lambda_3)$$
$$+ x_3^4 + \lambda_1 x_3^2 - 2\lambda_1$$

$$\Rightarrow \min L = \begin{cases} \min_{\underline{x} \in \mathbb{R}^3} x_3^4 + \lambda_1 x_3^2 - 2\lambda_1 & 1 + \lambda_1 - \lambda_2 = 0 \text{ and } -4 + \lambda_1 - \lambda_3 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_3} \; x_3^4 + \lambda_1 x_3^2 - 2\lambda_1$$

$$f' = 0 \Rightarrow 4x_3^3 + 2\lambda_1 x_3 = 0$$

$$2x_3(\underbrace{2x_3^2 + 2\lambda_1}_{\overset{?}{=}0}) = 0$$

$\overset{?}{=} 0$, not possible because $\lambda_1 \geq 0$

$\boxed{x_3 = 0}$

$$\min_{x \in \mathbb{R}^2} L = \begin{cases} -2\lambda_1 & 1 + \lambda_1 - \lambda_2 = 0 \\ & -4 + \lambda_1 - \lambda_3 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual:

$$\max \; -2\lambda_1$$
s.t. $\underline{\lambda} \geq 0$

$1 + \lambda_1 - \lambda_2 = 0$

$-4 + \lambda_1 - \lambda_3 = 0$

$\rightarrow \lambda_1 = 4$

$\lambda_2 = 5$

$\lambda_3 = 0$

, $\boxed{x_3 = 0}$

$$\lambda_1(\underbrace{x_1 + x_2 + x_3^2 - 2}_{=0}) = 0$$

$\lambda_2(-x_1) = 0$

$\lambda_3(-x_2) = 0$

$\lambda_1 = 4 \Rightarrow x_1 + x_2 - 2 = 0$

$\lambda_2 = 5 \Rightarrow \boxed{x_1 = 0}$

$\Rightarrow \boxed{x_2 = 2}$

Extra: Using KKT for the primal.

$$\mathcal{L}(\underline{x}, \underline{\lambda}) = x_1 - 4x_2 + x_3^4 + \lambda_1(x_1 + x_2 + x_3^2 - 2) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

$$\nabla_x L = 0 \iff \begin{array}{ll} 1 + \lambda_1 - \lambda_2 = 0 & (1) \\ -4 + \lambda_1 - \lambda_3 = 0 & (2) \\ 4x_3^3 + 2\lambda_1 x_3 = 0 & (3) \end{array}$$

$$\lambda_1(x_1 + x_2 + x_3^2 - 2) = 0 \quad (4)$$

$$\lambda_2(-x_1) = 0 \quad (5)$$

$$\lambda_3(-x_2) = 0 \quad (6)$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Case $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$

$\lambda_1 \neq 0 \Rightarrow x_1 + x_2 + x_3^2 - 2 = 0$

$\lambda_2 \neq 0 \Rightarrow x_1 = 0$

$\Rightarrow x_2 + x_3^2 - 2 = 0$

Also, (3) $\iff 2x_3(2x_3^2 + \lambda_1) = 0$

$\hookrightarrow \lambda_1 \neq 0 \Rightarrow x_3 = 0$

$x_2 = 2$

Therefore $\underline{x}^* = (0, 2, 0)$ solves KKT + convexity + Slater ($\hat{x} = (0,0,0)$; $0 + 0 + 0^2 < 2$)

$\Rightarrow$ KKT are nec. and sufficient $\Rightarrow \underline{x}^*$ is optimal.

15. Consider the problem

$$\begin{array}{ll} \min & x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array}$$

i) Solve this problem using KKT conditions. Are these sufficient?

ii) Find the solution of the dual problem. What is the duality gap?

Answer. The handwritten solution is included in the next pages.

ME3 Q4

$$\min \; x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2$$

$$x_1 + x_2 \leq 1$$

i) Sufficient?

$$f(\underline{x}) = \underline{x}^T \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \underline{x} + [-2 \;\; -3]\underline{x}$$

constraint is a linear ineq (convex)

convex f? $\nabla^2 f = 2\begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \rightarrow Q \succeq 0$? $\det(Q) = 0.25 > 0$, $\lambda_1(Q) = 1.5 > 0$

$\Rightarrow Q \succ 0 \Rightarrow f$ convex

$\Rightarrow$ KKT are necessary and sufficient.

<u>KKT</u>: $L = \underline{x}^T Q \underline{x} + \underline{b}^T \underline{x} + \lambda(x_1 + x_2 - 1)$

$\hookrightarrow \nabla_x L = 0 \iff 2Q\underline{x} + b + \lambda \mathbb{1} = 0 \;\; (*)$

$2x_1 + x_2 - 2 + \lambda = 0$

$x_1 + x_2 - 3 + \lambda = 0$

$\lambda(x_1 + x_2 - 1) = 0$

$\lambda \geq 0$

<u>Cases</u>:

1) $\lambda = 0 \Rightarrow 2x_1 + x_2 - 2 = 0 \quad (*)$

$x_1 + x_2 - 3 = 0$

$\Rightarrow x_1 = -1, \; x_2 = 4 \rightarrow x_1 + x_2 = 3 > 1$ X

2) $\lambda > 0 \Rightarrow x_1 + x_2 = 1 \Rightarrow \lambda = 2 \Rightarrow x_1 = -1, \; x_2 = 2$

Suff and necessity $\Rightarrow (x_1, x_2) = (-1, 2)$ is a minimizer.

ii) Convex cost + linear constraints + Slater ($x_1 = x_2 = 0$, $x_1 + x_2 < 1$)

$\Rightarrow$ Strong duality.

<u>Dual:</u> $\mathcal{L}(\underline{x},\lambda) = \underline{x}^T Q \underline{x} + b^T \underline{x} + \lambda(x_1 + x_2 - 1)$

$(\mathcal{X} = \mathbb{R}^2)$

$$\min_{\underline{x} \in \mathcal{X} = \mathbb{R}^2} \mathcal{L}(\underline{x},\lambda) \iff \nabla_x \mathcal{L}(\underline{x},\lambda) = 0$$

$$\iff 2Q\underline{x} + b + \lambda \mathbb{1} = 0$$

$$\underline{x}^* = -\frac{1}{2} Q^{-1}(b + \lambda \mathbb{1})$$

$$Q = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \to Q^{-1} = \frac{1}{0.25} \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\underline{x}^* = \begin{bmatrix} -1 \\ 4-\lambda \end{bmatrix}$$

<u>Dual:</u> $$\max_{\lambda \geq 0} \mathcal{L}(\underline{x}^*, \lambda) = \max_{\lambda \geq 0} 1 + \frac{1}{2}(4-\lambda)^2$$

$$+ \lambda - 4 + 2 - 3(4-\lambda)$$

$$+ \lambda(2-\lambda)$$

$$\dots = -\frac{1}{2}\lambda^2 - 5 + 2\lambda$$

$$\max_{\lambda \geq 0} \dots \quad f' = 0$$

$$-\lambda + 2 = 0 \implies \lambda = 2$$

$$\underline{C.S.C} \implies \underline{x}^* = \begin{bmatrix} -1 \\ 4-\lambda \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Strong duality $\implies$ duality gap $= 0$

16. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \le 1 \\ & x_3 \le 1 \end{aligned}$$

i) Is the problem convex?

ii) Using KKT conditions, find an optimal solution of the primal problem.

iii) Find a dual problem and solve it.

Answer. The handwritten solution is included in the next pages.

$$\min \quad x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 \leqslant 1$$
$$x_3 \leqslant 1$$

i) Convex? Yes, convex constraints, cost is Q.F.

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succcurlyeq 0 \quad \text{(Diag. dominant)}$$

ii) and iii) Note that $\exists\ \hat{\underline{x}} = (0,0,0)$ such that

$$0+0+0<1$$
$$0<1$$
+ convexity $\Rightarrow$ Strong duality.

$\Rightarrow$ No need to solve primal and dual separately, it is enough to solve dual and recover primal solution from there. Otherwise use KKT for primal.

$$L(\underline{x}, \underline{\lambda}) = x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 + \lambda_1(x_1+x_2+x_3-1) + \lambda_2(x_3-1)$$

$$g(\underline{\lambda}) = \underbrace{\min_{x_1,x_2,x_3} L(x_1, x_2, \underline{\lambda})} \qquad (\underline{\lambda} \geqslant 0)$$

$$\nabla_{\underline{x}} L = 0 \iff \begin{aligned} 2x_1 + 2x_2 + 1 + \lambda_1 &= 0 \\ 4x_2 + 2x_1 - 1 + \lambda_1 &= 0 \\ -1 + \lambda_1 + \lambda_2 &= 0 \end{aligned}$$

$$\Rightarrow 2x_2 - 2 = 0 \Rightarrow x_2 = 1 \Rightarrow 3 + 2x_1 + \lambda_1 = 0$$
$$x_1 = -\frac{(3+\lambda_1)}{2}$$

$\hookrightarrow$ needs $1 = \lambda_1 + \lambda_2$, otherwise min is reached with $x_3 = \pm\infty$

In this case, evaluating $L(\underline{x}, \underline{\lambda})$ at $x_2 = 1$ and $x_1 = -(3+\lambda_1)/2$

$$\Longrightarrow q(\underline{\lambda}) = -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4}$$

$$\Rightarrow q(\underline{\lambda}) = \begin{cases} -\infty & \text{if } \lambda_1 + \lambda_2 \neq 1 \\ -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4} & \text{otherwise} \end{cases}$$

and the dual problem is

$$\max_{\lambda_1 \geq 0} \; -\frac{9}{4} - \frac{\lambda_1}{2} - \frac{\lambda_1^2}{4}$$

which is maximized at $\lambda_1^* = 0 \Rightarrow \lambda_2^* = 1$ and $q^* = -9/4$.

back to the primal problem,

$x_2^* = 1$, $x_1^* = -3/2$ and $x_3^* = 1$ //

17. **Mastery question.** Consider the problem

$$\begin{array}{ll} \underset{u(\cdot)}{\text{minimize}} & \frac{1}{2}(x(T))^2 \\ \text{subject to} & \dot{x}(t) = u(t) \\ & x(0) = x_0 \text{ given} \\ & u(t) \in [-1, 1], \text{ for all } t \in \mathbb{R} \end{array}$$

i) Using the PMP, find an expression for the optimal control as a feedback law.

ii) Find an explicit expression for the optimal value function of the problem.

Answer i) The Hamiltonian is given by

$$H(t, x, u, \lambda) = \lambda u$$

Pointwise minimization yields

$$\mu(t, x) = \underset{|u| \leq 1}{\arg\min}\, \lambda u = \begin{cases} 1, & \lambda < 0 \\ -1, & \lambda > 0 \\ \tilde{u}, & \lambda = 0 \end{cases} = -\operatorname{sign}(\lambda)$$

where $\tilde{u} \in [-1, 1]$ is arbitrary. The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x, u, \lambda) = 0, \quad \lambda(T) = \frac{\partial \phi}{\partial x}(x(T)) = x(T)$$

which has the solution $\lambda(t) = x(T)$. We now have to cases:

- $x(T) \neq 0$: In this case $\lambda(t) \neq 0$ for all t and we can write

$$\mu(t, x) = -\operatorname{sign}(\lambda) = -\operatorname{sign} x(T) = -\operatorname{sign} x$$

  The last equality holds since x will have the same sign as $x(T)$ during the whole state trajectory.

- $x(T) = 0$: In this case $\lambda = 0$ for all t and we may use any control signal $\tilde{u} \in [-1, 1]$, which obeys the constraint $x(T) = 0$. One such control signal is

$$\mu(t, x) = -\operatorname{sign} x$$

  since this will drive x to zero and stay there.

Consequently, one optimal control is

$$\mu^*(t, x) = -\operatorname{sign}(x)$$

Answer ii) Since $J^*(t,x) = \frac{1}{2}\left(x^*(T)\right)^2$, we need to find $x^*(T)$. It holds that

$$x(T) - x(t) = \int_t^T \dot{x}(\tau)d\tau = \int_t^T u(\tau)d\tau$$

which can be written as

$$x(T) = x(t) - \int_t^T \operatorname{sign}\{x(\tau)\}d\tau \tag{SE}$$

There are two cases:

- $x(t) > 0$: In this case the controller will decrease $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$\begin{aligned} x(T) &= \max\{0, \overbrace{x(t) - (T-t)}^{\text{from (SE)}} \\ &= \max\{0, |x(t)| - (T-t)\} \end{aligned}$$

- $x(t) < 0$: In this case the controller will increase $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$\begin{aligned} x(T) &= \min\{0, \overbrace{x(t) + (T-t)\}}^{\text{from (SE)}} = -\max\{0, -x(t) - (T-t)\} \\ &= -\max\{0, |x(t)| - (T-t)\} \end{aligned}$$

Thus, the only difference between the two cases are the sign in front of the max and the optimal value function becomes

$$V(t,x) = J^*(t,x) = \frac{1}{2}\left(x^*(T)\right)^2 = \frac{1}{2}(\max\{0, |x| - (T-t)\})^2 .$$

18. **Mastery question.** We consider a set of N robots with state (position,velocity) $(x_i(t), v_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ interacting under second-order dynamics of the form

$$\frac{dx_i}{dt} = v_i\,, \tag{7}$$

$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i) + u_i(t)\,, \tag{8}$$

$$x_i(0) = x_0\,, \quad v_i(0) = v_0\,, \qquad i = 1, \ldots, N\,, \tag{9}$$

where $u_i(t) \in \{u : \mathbb{R}_+ \longrightarrow \mathbb{R}^2\}$ correspond to control signals for each robot, and $a(r)$ is a communication kernel of the type

$$a(r) = \frac{1}{(1+r^2)}\,.$$

Our goal is to drive the system to consensus, that is, to converge towards a configuration in which

$$v_i = \bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j \quad \text{for all } i.$$

For this, we write a finite horizon control problem of the form

$$\min_{\mathbf{u}(\cdot)} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \left(\|\bar{v} - v_j\|^2 + \gamma\|u_j\|^2\right)\, dt\,,$$

with $\gamma > 0$, subject to the dynamics (7)-(9).

Write the necessary optimality conditions for this problem, giving an explicit expression of the optimal control as a function of the adjoint variable.

Answer While existence of a minimiser $\mathbf{u}^*$ follows from the smoothness and convexity properties of the system dynamics and the cost, the Pontryagin Minimum Principle yields first-order necessary conditions for the optimal control. Let $(p_i(t), q_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ be adjoint variables associated to (x_i, v_i), then the optimality system consists of a solution $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*, \mathbf{p}^*, \mathbf{q}^*)$ satisfying the system dynamics along with the adjoint equations

$$-\frac{dp_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \frac{a'\left(\|x_j - x_i\|\right)}{\|x_j - x_i\|}\left\langle q_j - q_i, v_j - v_i\right\rangle \left(x_j - x_i\right)\,, \tag{10}$$

$$-\frac{dq_i}{dt} = p_i + \frac{1}{N}\sum_{j=1}^{N} a\left(\|x_j - x_i\|\right)\left(q_j - q_i\right) - \frac{2}{N}\left(\bar{v} - v_i\right), \tag{11}$$

$$p_i(T) = 0\,, \quad q_i(T) = 0\,, \qquad i = 1, \ldots, N\,, \tag{12}$$

and the optimality condition

$$\mathbf{u}(t) = \underset{\mathbf{w}\in\mathbb{R}^{2N}}{argmin} \sum_{j=1}^{N} \left(\left\langle q_j, \frac{dv_j}{dt}\right\rangle + \frac{\gamma}{N}\left\|w_j\right\|^2\right) = -\frac{N}{2\gamma}\mathbf{q}^t\,. \tag{13}$$

19. **Mastery question.** The dynamics

$$\dot{x}(t) = -x(t) + u(t) \qquad |u| \leq 1$$

are to be controlled so that $x(1) = 0$ while minimizing the cost

$$J = \int_0^1 |u(t)|\, dt\,.$$

Show that the control

$$u(t) = \begin{cases} 0 & 0 \leq t < 0.5 \\ -1 & 0.5 \leq t \leq 1 \end{cases}$$

satisfies Pontryagin's necessary optimality conditions for some $x(0)$.

Answer The Hamiltonian is

$$H(x, u, \lambda) = |u| + \lambda(u - x) \quad \text{and} \quad \Phi(x) = 0, \quad \Psi(x) = x$$

The multipliers λ shall fulfill

$$\begin{aligned} \dot{\lambda}(t) &= -H_x\left(x^*, u^*, \lambda\right) = \lambda(t) \\ \lambda(1) &= \Phi_x\left(x^*(1)\right) + \mu \Psi_x\left(x^*(1)\right) = \mu \end{aligned}$$

This means that $\lambda(t) = \mu e^{t-1}$, this means positive and increasing or negative and decreasing. We will minimize the Hamiltonian with respect to u for each time instance. Is is equivalent to minimizing

$$H_1(u, \lambda) = |u| + \lambda u = \underbrace{(\operatorname{sign}(u) + \lambda)}_{\sigma} u$$

If $\sigma > 0$ we want to choose u as the smallest feasible negative value and if $\sigma < 0$ we want to choose u as the largest feasible positive value. Consequently, the following cases minimize the Hamiltonian: Positive λ

$$\begin{cases} u = -1 \text{ and } \lambda > 1 \\ u = 0 \text{ and } 0 \leq \lambda \leq 1 \end{cases}$$

Negative λ

$$\begin{cases} u = 1 \text{ and } \lambda < -1 \\ u = 0 \text{ and } -1 \leq \lambda \leq 0 \end{cases}$$

Our control candidate contains the control signals $0, -1$ which requires a positive λ. Thus, we have to find a μ such that λ starts with a positive value less than 1 and passes 1 at the time 0.5. This results in

$$1 = \mu e^{-\frac{1}{2}} \Rightarrow \mu = e^{\frac{1}{2}}$$

This μ fulfills the requirements and the optimality conditions are satisfied for the suggested control.

20. **Mastery question.** A community living around a lake wants to maximize the yield of fish taken out of the lake. The amount of fish at a certain time is denoted x. The growth rate of the fish is kx and fish is captured with a rate ux where u is the control variable, which is assumed to satisfy $0 \leq u \leq u_{\text{max}}$. The dynamics of the fish population is then given by

$$\dot{x} = (k-u)x, \quad x(0) = x_o$$

Here $k > 0$ and $x_o > 0$. The total amount of fish obtained during a time period T is

$$J = \int_0^T uxdt$$

i) Derive the necessary conditions given by the PMP for the problem of maximizing J.

ii) Show that the necessary conditions are satisfied by a bang-bang control, that is, it only takes boundary values of the constraint set. How many switching times are there?

iii) Determine an equation for calculating the switching time(s).

Answer i) The problem to solve is

$$\begin{array}{ll} \text{minimize} & \int_0^T -uxdt \\ \text{subject to} & \dot{x} = (k-u)x \\ & x(0) = x_0 \end{array}$$

We use PMP to solve the problem. The Hamiltonian is given by

$$H(t, x, u, \lambda) = -ux + \lambda(k-u)x$$

Pointwise minimization yields

$$\tilde{\mu}(t, x, \lambda) = \underset{0 \leq u \leq u_{\text{max}}}{\arg\min} H(t, x, u, \lambda) = \begin{cases} 0, & \lambda + 1 < 0 \\ u_{\text{max}}, & \lambda + 1 > 0 \\ \tilde{u}, & \lambda + 1 = 0 \end{cases}$$

where $\tilde{u}$ is arbitrary in $[0, u_{\text{max}}]$. Thus the optimal control is expressed as

$$u^*(t) \triangleq \tilde{\mu}(t, x(t), \lambda(t)) = \begin{cases} 0, & \sigma(t) < 0 \\ u_{\text{max}}, & \sigma(t) > 0 \\ \bar{u}, & \sigma(t) = 0 \end{cases}$$

where we have defined the switching function as

$$\sigma(t) \triangleq \lambda(t) + 1$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = u - \lambda(k-u), \quad \lambda(T) = \frac{\partial \phi}{\partial x}(T, x(T)) = 0$$

Answer ii) The boundary condition in the equation above gives

$$\sigma(T) = \lambda(T) + 1 = 1 > 0$$

which gives that $u^*(T) = u_{\max}$. For finding the number of switches we consider $\dot{\sigma}(t)|_{\sigma(t)=0}$ and it follows that

$$\dot{\sigma}(t)|_{\sigma(t)=0} = \dot{\lambda}(t)\Big|_{\lambda(t)+1=0} = [u(t) - \lambda(t)(k - u(t))]|_{\lambda(t)+1=0} = k > 0$$

Hence, there can only be at most one switch, since we can pass $\sigma(t) = 0$ only once. Since $u^*(T) = u_{\max}$ is is not possible that $u^*(t) = 0$ for all $t \in [0, T]$. Thus

$$u^*(t) = \begin{cases} 0 & 0 \leq t \leq t' \\ u_{\max}, & t' < t \leq T \end{cases}$$

for some unknown switching time $t' \in [0, T]$. Note, if no switch would occur, this can still be described using $t' = 0$. Thus, we have a bang-bang control with at most one switch (from 0 to $u_{\max}$).

Answer iii) The switching time $t' \in [0, T[$ occurs when

$$0 = \sigma\left(t'\right) = \lambda\left(t'\right) + 1$$

During the interval $t \in [t', T]$ we have that

$$\dot{\lambda}(t) = u_{\max} - \lambda\left(k - u_{\max}\right), \quad \lambda(T) = 0$$

This is a linear ODE which has the solution

$$\lambda(t) = \frac{u_{\max}}{k - u_{\max}}\left(1 - e^{(k-u_{\max})(T-t)}\right)$$

Therefore, the switching time can be found by solving

$$0 = \lambda\left(t'\right) + 1 = \frac{u_{\max}}{k - u_{\max}}\left(1 - e^{(k-u_{\max})(T-t')}\right) + 1$$