W6, L1

2.1.2 Normed vector spaces

Def 2.5 let V be a vector space on $\mathbb{R}$, We say that a function $\|\cdot\| : V \to \mathbb{R}^1$, is a norm on V, if the following 3 properties hold:

(N1) for all $v \in V$, $\|v\| \geq 0$, and $\|v\| = 0 \Leftrightarrow v = 0$.

(N2) for all $v \in V$, $\lambda \in \mathbb{R}$, $\|\lambda v\| = |\lambda| \cdot \|v\|$.

(N3) for all $v, u \in V$, $\|u+v\| \leq \|u\| + \|v\|$.

Examples: If $V = \mathbb{R}^n$,

$$\|(x^1, x^2, \ldots, x^n)\|_1 = |x^1| + |x^2| + \cdots + |x^n|.$$

$$\|(x^1, x^2, \ldots, x^n)\|_\infty = \max\{|x^1|, |x^2|, \ldots, |x^n|\}.$$

The norm $\|\cdot\|$ on V, induces a metric on V, defined as

$$d_{\|\cdot\|} : V \times V \longrightarrow \mathbb{R}^1,$$

$$d_{\|\cdot\|}(u, v) = \|u - v\|.$$

The space $(V, \|\cdot\|)$ is called a normed vector space.

2.1.4 Open sets in metric spaces.

Def 2.6 Consider a metric space (X, d), a point $x \in X$, and a real number $\varepsilon > 0$.

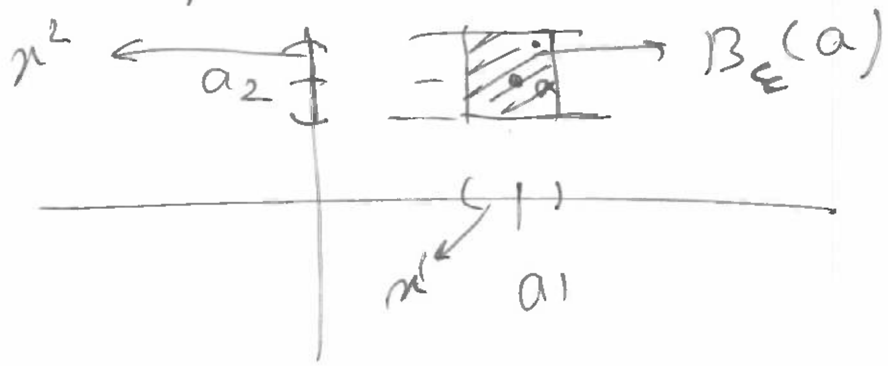
The open ball of radius ε about x is the set

$$B_\varepsilon(x) = \{ x' \in X \mid d(x', x) < \varepsilon \}.$$

Example: In $(\mathbb{R}^2, d_\infty)$ for $a = (a^1, a^2) \in \mathbb{R}^2$ & $\varepsilon > 0$

$$B_\varepsilon(a) = \{ (x^1, x^2) \in \mathbb{R}^2 \mid d_\infty((x^1, x^2), (a^1, a^2)) < \varepsilon \}$$

$$= \{ (x^1, x^2) \in \mathbb{R}^2 \mid \max\{ |x^1 - a^1|, |x^2 - a^2| \} < \varepsilon \}$$



Let $I = [0,1] \subseteq \mathbb{R}^1$, and let d_I be the metric induced on I from d_1 on $\mathbb{R}^1$.

In $(\mathbb{R}^1, d_1)$

$$B_1(1) = \{ x \in \mathbb{R}^1 \mid d_1(x,1) < 1 \}$$
$$= \{ x \in \mathbb{R}^1 \mid |x-1| < 1 \} = (0,2)$$

In (I, d_I)

$$B_1(1) = \{ x \in I \mid d_I(x,1) < 1 \}$$
$$= \{ x \in [0,1] \mid |x-1| < 1 \} = (0,1].$$

In the discrete metric space (X, d_{disc})

for $x \in X$, $\varepsilon > 0$,

if $\varepsilon \leq 1$, $B_\varepsilon(x) = \{ x' \in X \mid d_{disc}(x,x') < \varepsilon \} = \{x\}$

if $\varepsilon > 1$, $B_\varepsilon(x) = \{ x' \in X \mid d_{disc}(x,x') < \varepsilon \} = X$.

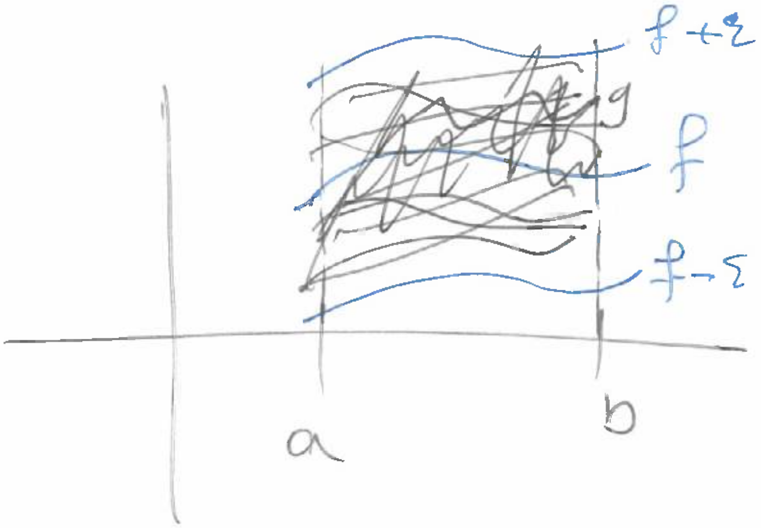
In the metric space $(C[a,b], d_\infty)$

fix $f \in C[a,b]$, $\varepsilon > 0$.

$$B_\varepsilon(f) = \{ g \in C[a,b] \mid d_\infty(g, f) < \varepsilon \}$$

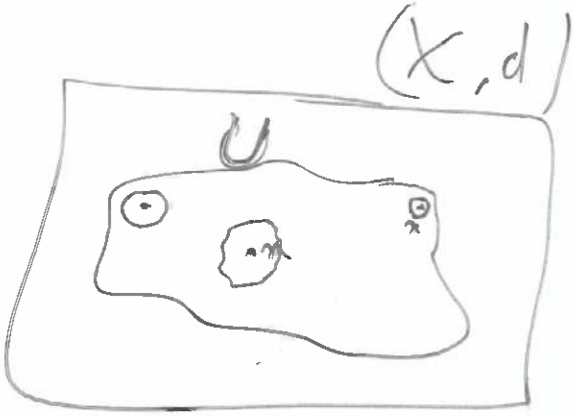
$$= \{ g \in C[a,b] \mid \max_{t \in [a,b]} |f(t) - g(t)| < \varepsilon \}.$$

= all continuous functions $g : [a,b] \to \mathbb{R}$, s.t. the graph of g lies between the graphs of $f+\varepsilon$ & $f-\varepsilon$.



Def 2.7 let (X,d) be a metric space, and $U \subseteq X$. We say that U is open in (X,d), if for every $x \in U$, there is $\varepsilon > 0$ s.t.

$$B_\varepsilon(x) \subseteq U.$$



The property of open sets U in a metric space (X,d) depends both on U, X, and d.

In $(\mathbb{R}^1, d_{disc})$ every subset is open.

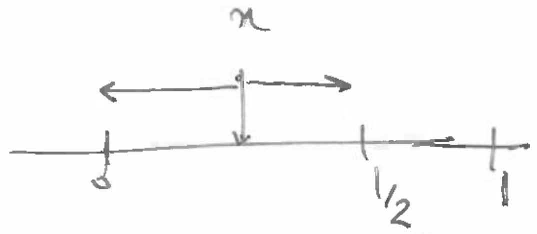
let $U \subseteq \mathbb{R}^1$, for any $x \in U$, let $\varepsilon = 1/2$.

$$B_\varepsilon(x) = \{x\} \subseteq U.$$

In $(\mathbb{R}^1, d^1)$ the set $[0,1]$ is not open.

let $I = [0,1]$, and d_I be the induced metric on I from d' on $\mathbb{R}^1$.

The set $[0, 1/2) \subseteq [0,1]$ is open in (I, d_I).



let $x \in [0, \frac{1}{2})$ be arbitrary.

If $x \in (0, 1/2)$, Let $\delta = \min\{x, 1/2 - x\}$

$$B_\delta(x) = \{x' \in I \mid d_I(x, x') < \delta\}$$

$$= \{x' \in [0,1] \mid |x - x'| < \delta\}$$

$$= (x - \delta, x + \delta) \subseteq [0, 1/2).$$

If $x = 0$, let $\delta = 1/4$.

$$B_\delta(0) = \{x' \in I \mid d_I(x, x') < 1/4\}$$

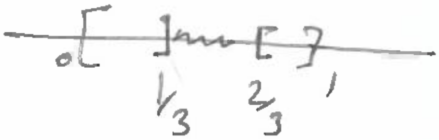
$$= [0, 1/4) \subseteq [0, 1/2).$$

In $(\mathbb{R}^1, d')$, $[0, 1/2)$ is not open.

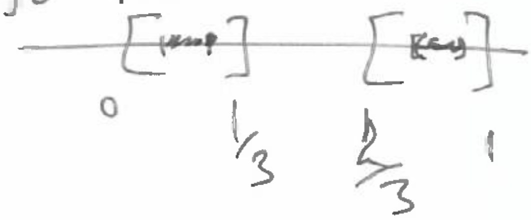
Remark. $\mathbb{R}^1$..

$I_1 = [0, 1]$.

$I_2 = [0, 1/3] \cup [2/3, 1]$.



$I_3 = [0, 1/9] \cup [\frac{2}{9}, 1/3] \cup [\frac{6}{9}, \frac{7}{9}] \cup [8/9, 1]$.



$I_4 =$ ______

$I_5 =$ ______

$I_{n+1} \subseteq I_n$

$\bigcap_{n \geq 1} I_n = C$. $0 \in C$.

2.1.4 Convergence in metric spaces

Def 2.9. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence of points in X.

We say that $(x_n)_{n\geq 1}$ converges in (X, d), if there is $x \in X$ satisfying the following property:

for all $\delta > 0$, $\exists\, N_\delta \in \mathbb{N}$, s.t.

$$\forall\, n \geq N_\delta, \quad d(x, x_n) < \delta.$$

In this case, we write $\lim_{n\to\infty} x_n = x$ in (X, d)

or $x_n \to x$ in (X, d)

Example: In $(\mathbb{R}^1, d_1)$, the sequence $(\frac{1}{n})_{n\geq 1}$ converges. Because $0 \in \mathbb{R}^1$,

$$d(\tfrac{1}{n}, 0) = |\tfrac{1}{n} - 0| = \tfrac{1}{n} \to 0.$$

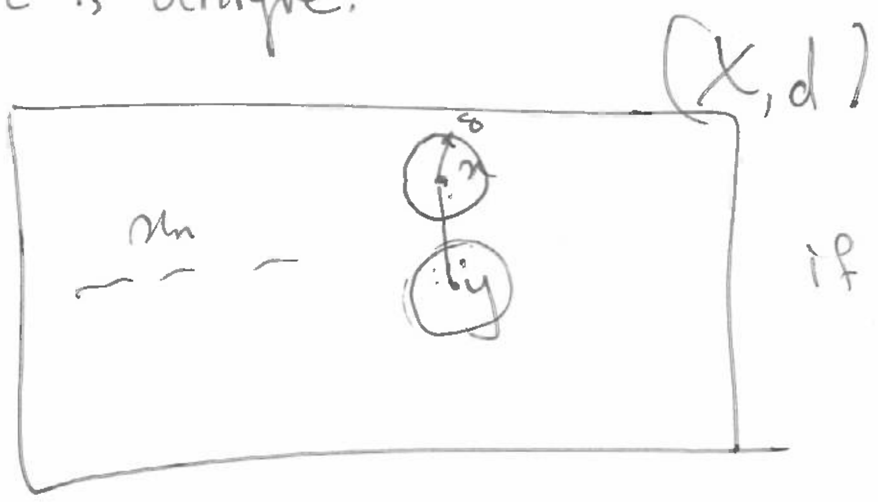
In the metric space $((0,1), d_{(0,1)})$, $(\frac{1}{n})_{n\geq 1}$, is not convergent.

W6, L2

2

lemma 2.7. let (X,d) be a metric space. If a sequence in (X,d) converges, then its limit is unique.

proof:



if $x \neq y$

$d(x,y) \neq 0$

more details in the typed notes.

2.1.5 closed sets in metric spaces.

Def 2.10 let (X,d) be a metric space, and $V \subseteq X$. We say that V is closed in (X,d), if for any sequence $(x_n)_{n\geq 1}$ in V which converges in (X,d), the limit of $(x_n)_{n\geq 1}$ belongs to V.

W6, L2

Example 2.15. $[a,b] \subseteq \mathbb{R}^1$ is closed in $(\mathbb{R}^1, d_1)$ 3

let $(x_i)_{i\geq 1}$ be a sequence in $[a,b]$, and assume that $x_i \to x \in \mathbb{R}^1$.

$\forall i, \quad a \leq x_i \leq b$

taking limits, $\lim\limits_{i\to\infty} a \leq \lim\limits_{i\to\infty} x_i \leq \lim\limits_{i\to\infty} b$.

$$\Downarrow$$

$$a \leq x \leq b.$$

$$\Longrightarrow x \in [a,b].$$

The sets (a,b) & $(a,b]$ are not closed in $(\mathbb{R}^1, d_1)$.

let $x_n = a + \frac{b-a}{n}$, $n \geq 2$.

$x_n \in (a,b) \ \forall n, \quad x_n \to a \in \mathbb{R}$,

$a \notin (a,b) \Longrightarrow (a,b)$ is not closed.

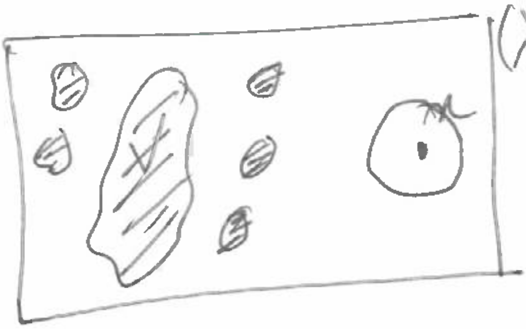
W6, L2

How about the set $(0, 1/2]$ in the metric space $((0,1), d_{(0,1)})$?

if $(x_n)_{n\geq 1}$ is a sequence in $(0, 1/2]$, and $x_n \to x \in (0,1)$, then $x \in (0, 1/2]$

Theorem 2.9. Let (X, d) be a metric space, and $V \subseteq X$. Then V is closed in (X, d) iff $X \setminus V$ is open in (X, d).

proof [idea of the proof, details are in typed notes].



(X, d) w6, L2

V is closed $\Longrightarrow$ $X \setminus V$ is open.

Fix $x \in X \setminus V$. we are looking for a ball

$B_\delta(x) \subseteq X \setminus V$.

If there is no such $\delta > 0$. then for $\delta = \frac{1}{n}$,

$$B_{1/n}(x) \cap V \neq \emptyset$$

$\exists\ x_n \in B_{1/n}(x) \cap V$. $(\Longleftarrow d(x, x_n) < \frac{1}{n})$

$x_n \in V$, $x_n \to x$. $\Rightarrow$ $x \in V$. $\divideontimes$

$X \setminus V$ is open $\Longrightarrow$ V is closed. W6, L2 <u>6</u>

Let $(x_n)_{n\geq 1}$ be a sequence in V, which converges to x in (X,d).

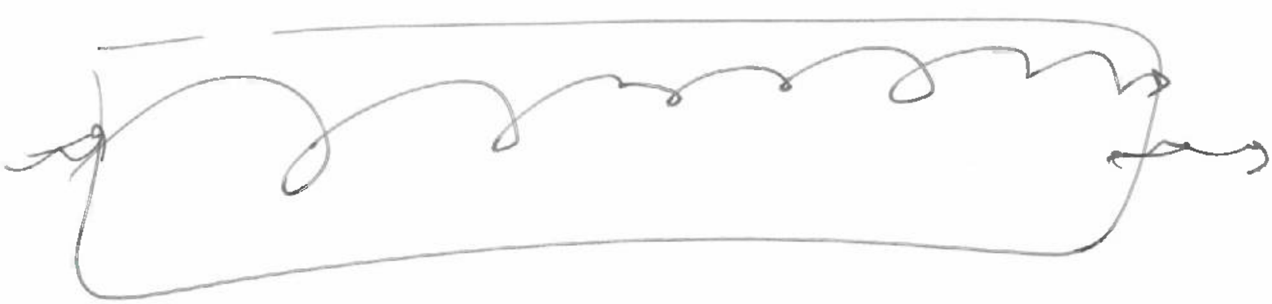
We need to show that $x \in V$. If not, $x \in X\setminus V$. Since $X\setminus V$ is open, $\exists\ \delta > 0$. s.t. $B_\delta(x) \subseteq X \setminus V$.

Since $x_n \longrightarrow x$, $\exists\ n \in \mathbb{N}$, s.t.

$$\underset{\in V}{x_n} \in B_\delta(x) \subseteq X\setminus V$$

$\times$

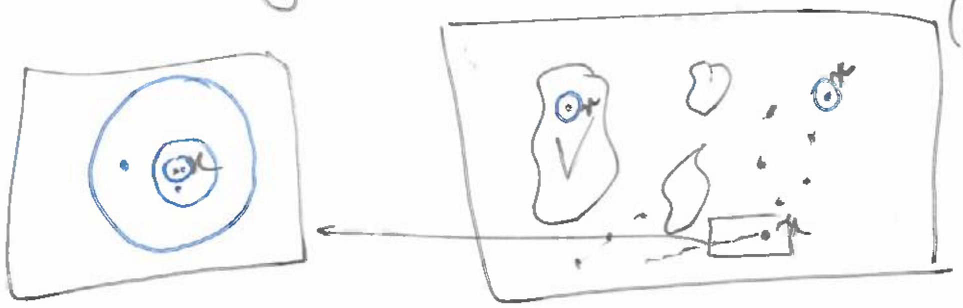
Other properties of closed sets, sequences in typed notes.



W7, L1

1

2.1.6 Interior, isolated, limit, and boundary points in metric spaces.



(X, d)

$V \subseteq X$

$x \in X$

$x \in V$

or $x \notin V$.

Def 2.11. Let (X, d) be a metric space and $V \subseteq X$.

(i) a point $x \in V$ is called an interior point of V, inner point of V, if $\exists\ \delta > 0$ s.t.

$$B_\delta(x) \subseteq V.$$

(ii) $x \in V$ is called an isolated point of V, if $\exists\ \delta > 0$ s.t.

$$B_\delta(x) \cap V = \{x\}$$

(iii) $x \in X$ is called a limit point of V, an accumulation point of V, if for any $\delta > 0$,

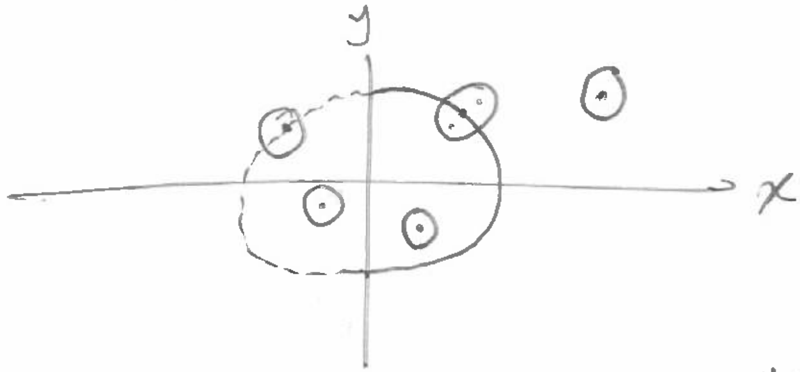
$B_\delta(x) \cap V$ has an element other than x.

(iv) $x \in X$ is called a boundary point of V, if for any $\delta > 0$,

$$B_\delta(x) \cap V \neq \emptyset, \quad B_\delta(x) \cap (X \setminus V) \neq \emptyset.$$

Example 2.16 In $(\mathbb{R}^2, d_2)$, let

$$V = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1,\ x \geq 0\}$$
$$\cup \{(x,y) \in \mathbb{R}^2 / \|(x,y)\| < 1,\ x < 0\}$$



(i) $(x,y) \in V$ is an interior point $\iff$ $\|(x,y)\| < 1$.

(ii) there are no isolated points.

(iii) $(x,y) \in \mathbb{R}^2$ is a limit point of $V \iff \|(x,y)\| \leq 1$.

(iv) (x,y) is a boundary point of $V \iff \|(x,y)\| = 1$

2.1.7 continuous maps of metric spaces

Def 2.14. let (X, d_X) and (Y, d_Y) be metric spaces, and $f: X \to Y$. We say that f is continuous at $x \in X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x' \in X$ satisfying

$$d_X(x, x') < \delta, \text{ then } d_Y(f(x), f(x')) < \varepsilon.$$

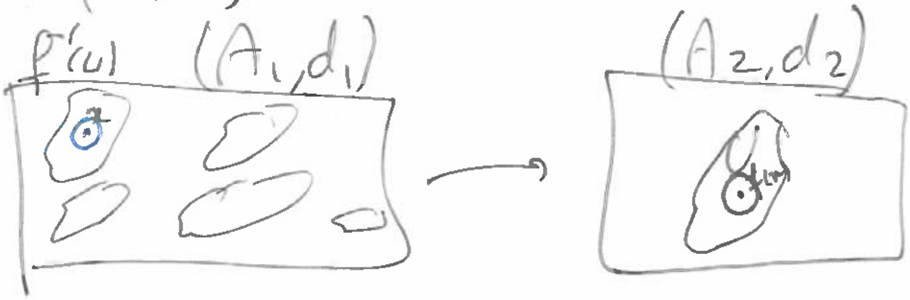
f is continuous on X if it is continuous at every $x \in X$.

W7, L1

3

Thm 2.12 let (A_1, d_1) and (A_2, d_2) be metric spaces, and $f: A_1 \longrightarrow A_2$. Then, f is continuous on A_1 iff the pre-image of any open set in (A_2, d_2) is an open set in (A_1, d_1).

proof: $\Longrightarrow$



let U be an open set is (A_2, d_2), we aim to show that $f^{-1}(U)$ is open in (A_1, d_1).

to show $f^{-1}(U)$ is open let $x \in f^{-1}(U)$. the $f(x) \in U$.

since U is open, $\exists\, \varepsilon > 0$ s.t. $\underline{B_\varepsilon(f(x)) \subseteq U}$.

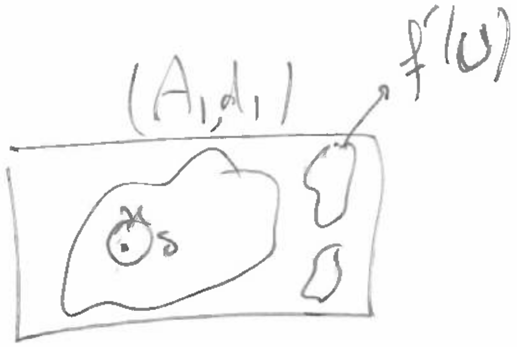
By continuity at x, $\exists\, \delta > 0$, s.t.

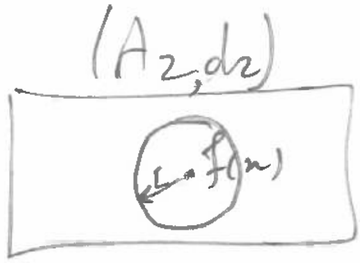
$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U.$$

$$\underline{B_\delta(x)} \subseteq f^{-1}(f(B_\delta(x))) = \underline{f^{-1}(U)}.$$

$\Longrightarrow$ as x was arbitrary $\Longrightarrow$ $f^{-1}(U)$ is open.

$\Leftarrow$





let $x \in A_1$ be arbitrary, and fix $\varepsilon > 0$.

let $U = B_\varepsilon(f(x))$ open in (A_2, d_2).

$f^{-1}(U)$ is open in (A_1, d_1).

$x \in f^{-1}(U)$. $\exists\, \delta > 0$ s.t $\underline{B_\delta(x)} \subseteq f^{-1}(U)$.

$$f(B_\delta(x)) \subseteq U.$$

$$\Updownarrow$$

$$\text{if } d_1(x', x) < \delta \Longrightarrow d_2(f(x), f(x')) < \varepsilon.$$

Def 2.15. let (X_1, d_1) and (X_2, d_2) be metric spaces.

(1) a map $f: X_1 \to X_2$ is called a homeomorphism if $f: X_1 \to X_2$ is a bijection, and both maps $f: (X_1, d_1) \to (X_2, d_2)$ and $f^{-1}: (X_2, d_2) \to (X_1, d_1)$ are continuous.

W7, L1

5

(ii) two metric spaces (X_1, d_1) and (X_2, d_2) are called homeomorphic, if there is a homeomorphism from X_1 to X_2.

the metric spaces $(\mathbb{R}^1, d_1)$ and $((-1,+1), d_{(-1,+1)})$ are homeomorphic $\quad (-\infty, +\infty) \xrightarrow[\arctan]{} (-1,+1)$

if $a < b$ then $\quad (a,b) \underset{\text{hom}}{\simeq} (0,1)$

$$[a,b] \simeq [0,1]$$

$$[a,b) \simeq [0,1)$$

$$[0,1) \simeq (0,1]$$

$$x \to -x+1$$

(but $[0,1] \not\simeq [0,1)$ and $[0,1) \not\simeq (0,1)$.)

Example $\quad [0,1] \qquad (2,3]$

$$[0,1] \xrightarrow{x \to x} [0,1], \qquad (2,3] \xrightarrow{x \to x-1} (1,2]$$

$$[0, 1 \leftarrow 2]$$

$$1 + 1/n$$

2.2 Topological Spaces

generalising Analysis:

To do analysis on X; – build a metric on X
$(d: X \times X \to \mathbb{R})$

by pass

– open balls in X

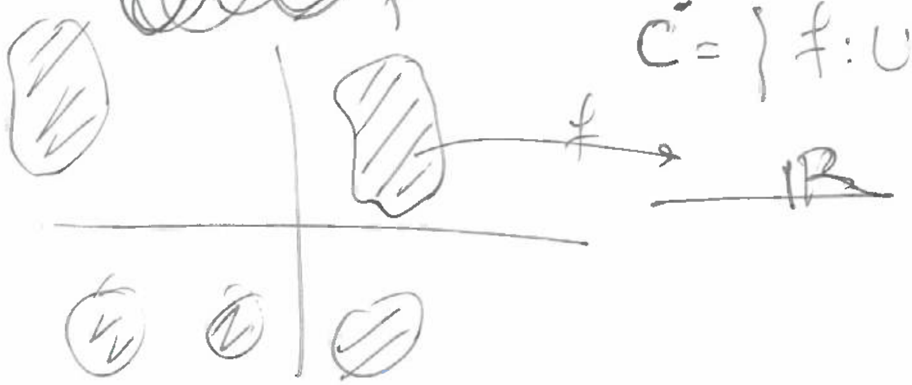
– open sets,

– Analysis on X; convergence of seq.
continuity of maps, ...

Given X, mark some subsets of X as open sets ; analysis on X.

This is more natural, because it uses objects in X, to explain events in X.

$C^0 = \left\{ f: U \to \mathbb{R} \;\middle|\; \begin{array}{l} U \text{ is open in } \mathbb{R}^2 \\ f \text{ is continuous.} \end{array} \right\}$



Def 2.17 let A be an arbitrary set, and let τ be a collection of subsets of A. We say that τ is a topology on A, if the following 3 properties hold:

(T_1) the empty set, and the set A belong to τ,

(T_2) if $G_\alpha \in \tau$, for α in some set I, then

$$\bigcup_{\alpha \in I} G_\alpha \in \tau.$$

(T_3) If $G_\alpha \in \tau$, for some finite set $\alpha \in I$, then

$$\bigcap_{\alpha \in I} G_\alpha \in \tau.$$

A togological space, denoted (A, τ), is a pair of a set A, and a topology τ on A.

Every element of τ is called an open set in (A, τ).

Given $a \in A$, we say that U is an open neighbourhood of a, if U is open in (A, τ) (i.e. $U \in \tau$), and $a \in U$.

Example let A be a set, and

$$\tau = \{\phi, A\}.$$

This is called the Coarse topology on A.

Example let $A = \{a, b\}$ where $a \neq b$,

and $\tau = \{\phi, \{b\}, \{a,b\}\}$

τ is a topology, called the Sierpinski topology on A.

Any open set which contains a, also contains b.

let $A = \mathbb{R}$, and

$$\tau = \{(a, +\infty) \mid a \in \mathbb{R} \cup \{-\infty, +\infty\}\}.$$

(T_1) If $a = -\infty$, $(-\infty, +\infty) = \mathbb{R} \in \tau$,
if $a = +\infty$, $(+\infty, +\infty) = \phi \in \tau$,

(T2) let $G_\alpha \in \tau$, $\alpha \in I$,

let $G_\alpha = (a_\alpha, +\infty)$, for $a_\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$.

- If a_α are not bounded below, then

$$\bigcup_{\alpha \in I} G_\alpha = (-\infty, +\infty).$$

- if a_α are bounded below, $\inf_{\alpha \in I} \{a_\alpha\} = a$

$$\bigcup_{\alpha \in I} G_\alpha = (a, +\infty).$$

(T3) Similar to T2.

if (X, d) is a metric space, and τ is the collection of all open sets in (X, d), then τ is a topology on X.

The topology τ on X defined as above, is called the topology induced from the metric d.

Example Let (X, τ) be a topological space, and let $Y \subseteq X$. Then

$$\tau_Y = \{ U \cap Y \mid U \in \tau \}$$

is a topology on Y.

we say (Y, τ_Y) has the subspace topology induced from (X, τ).

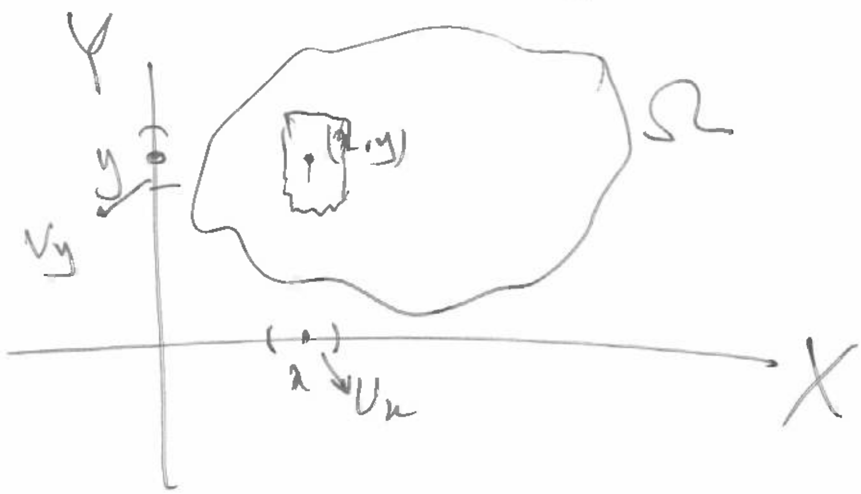
If $X = \mathbb{R}$, τ is induced from d_1, any set $E \subseteq \mathbb{R}$, τ_E on E.

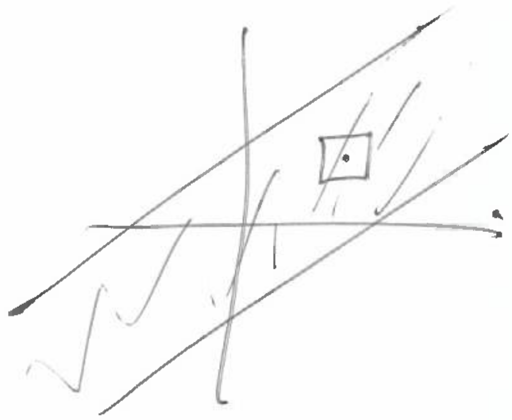
Let (X, τ_X) and (Y, τ_Y) be topological spaces, and consider the set

$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}.$$

let $\underbrace{\tau_X * \tau_Y}$ be the collection of all sets $\Omega \subseteq X \times Y$ s.t. for every $(x,y) \in \Omega$, there are $U_x \in \tau_X$, $V_y \in \tau_Y$ such that

$x \in U_x$, $y \in V_y$, and $U_x \times V_y \subseteq \Omega$.





Def 2.19. let (X, τ) be a topological space, and $\Omega \subseteq X$. A point $a \in \Omega$ is called an interior point of Ω, if there is an open set U (i.e. $U \in \tau$) s.t. $a \in U$, and $U \subseteq \Omega$.

W8, L2

2.2.3 convergence in topological spaces.

Def 2.20 let (A, τ) be a topological space, and $(x_n)_{n\geq 1}$ be a sequence in A. we say that $(x_n)_{n\geq 1}$ converges in (A, τ), if there is $x \in A$ satisfying the following. for any open set $U \ni x$ in (A, τ), there is $N \in \mathbb{N}$ s.t. for all $n \geq N$,

$$x_n \in U.$$

Example 2.34 If τ is the coarse topology on A, then any sequence in A, converges to any element in A.

let $(x_n)_{n\geq 1}$ be a sequence in A, and let $z \in A$.

If U is an open set which contains z, $U = A$, we may let $N = 1$, $\forall n$ $x_n \in A = U$.

Def 2.21. A topological space (A, τ) is called Hausdorff, if the following property holds.
For any $x \neq y$ in A, there are open set U_x and U_y in (A, τ) s.t. $x \in U_x$, $y \in U_y$, $U_x \cap U_y = \emptyset$.

W8, L2

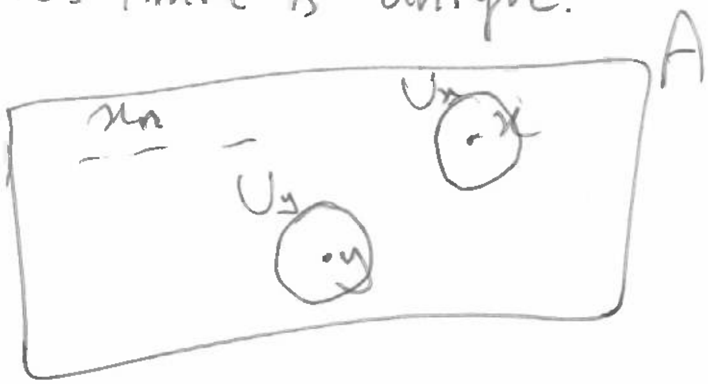
Example let $A = \{a, b, c\}$, and

$$\tau = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}.$$

τ is not Hausdorff, we cannot separate b & c.

Any open set which contains c, also contains b.

Thm 2.15 let (A, τ) be a Hausdorff topological space and $(x_n)_{n\geq 1}$ be a sequence in A. If $(x_n)_{n\geq 1}$ converges, then its limit is unique.



proof: (idea) assume $(x_n)_{n\geq 1}$ converges to distinct points x and $y \in A$. $\exists N_1 \in \mathbb{N}, \forall n \geq N_1$ $x_n \in U_x$

$\exists N_2 \in \mathbb{N}, \forall n \geq N_2$, $x_n \in U_y$.

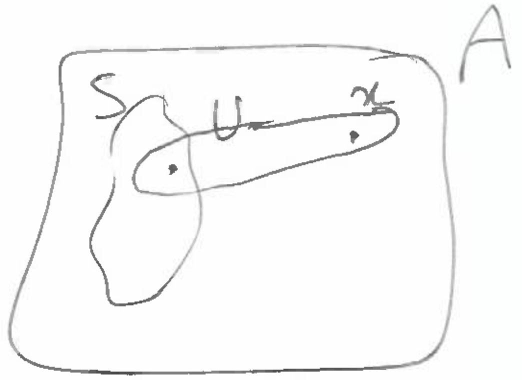
$n = \max\{N_1, N_2\}$. $x_n \in U_x$, $x_n \in U_y$ $\lightning$

2.2.4 Closed sets in topological spaces.

Def 2.23. let (A, τ) be a topological space, and $V \subseteq A$. We say that V is closed, if $A \setminus V$ is open (belongs to τ).

Example: In any topological space (A, τ), the empty set & the set A are closed.

Def 2.23. (A, τ) topological space, and $S \subseteq A$. A point $x \in A$ is called a limit point of S, if every open neighbourhood U of x, contains at least one element from S distinct from x.



The closure of S is defined as S union the set of all limit points of S. denoted by $\overline{S}$.

Example 2.36 let $A = \{a, b\}$.

$\tau = \{\phi, \{b\}, \{a, b\}\}$.

let $S = \{b\}$. What is $\overline{S}$?

$b \in \overline{S}$. Is $a \in \overline{S}$. let U be an open set in A which contains a. then $U = \{a, b\}$.

$U \cap S = \{b\}$. Then $a \in \overline{S}$.

2.2.5 Continuous maps on topological spaces.

Def 2.24 let (X, τ_X) and (Y, τ_Y) be topologicall spaces, and $f: X \longrightarrow Y$. We say that f is continuous on X, if for any open set U in (Y, τ_Y), $f^{-1}(U)$ is open in (X, τ_X).

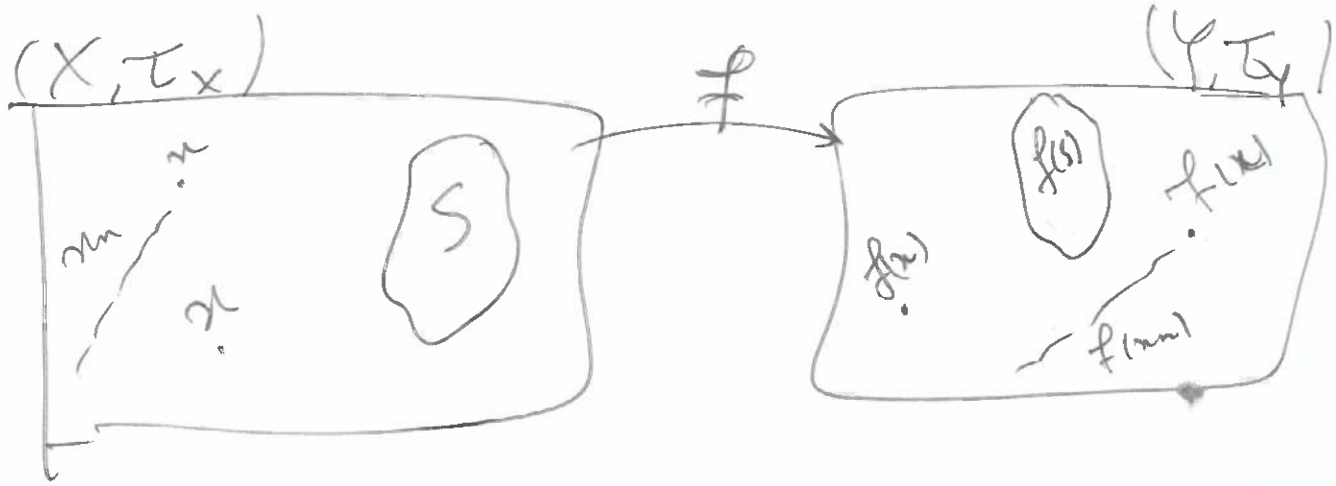
Example: (i) If τ_X is the discrete topology on X, then any $f: X \to Y$ is continuous.

(ii) If τ_Y is the coarse topology on Y. Then any $f: X \to Y$ is continuous.

$f: (X, \tau_X) \to (Y, \tau_Y)$ is called a homeomorphism if $f: X \to Y$ is a bijection and both maps

$f: (X, \tau_X) \to (Y, \tau_Y)$ & $f^{-1}: (Y, \tau_Y) \to (X, \tau_X)$

are continuous.



W8, L2

let $A = \mathbb{R}$,

and $\tau = \{ (a + \infty) \mid a \in \mathbb{R} \cup \{-\infty, +\infty\} \}$.

is (A, τ) a Hausdorff space?

let $0, 1 \in A$, $U, V \in \tau$,

$0 \in U$, $1 \in V$.

then $U = (a, +\infty)$ with $a \in (-\infty, 0) \cup \{-\infty\}$

then $1 \in U$.

let $x_n = 0$, $\forall n = 1, 2, 3 \ldots$

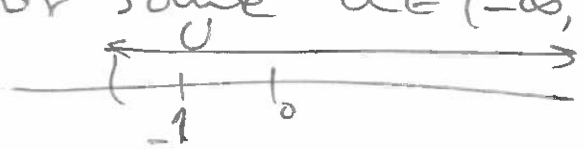
Does x_n converge to 1. let $U = (1/2, +\infty)$

$1 \in U$.

$\nexists N \in \mathbb{N}$, s.t. $x_n = 0 \in U$. X

Does x_n converge to -1. let $U \in \tau$, & $-1 \in U$.

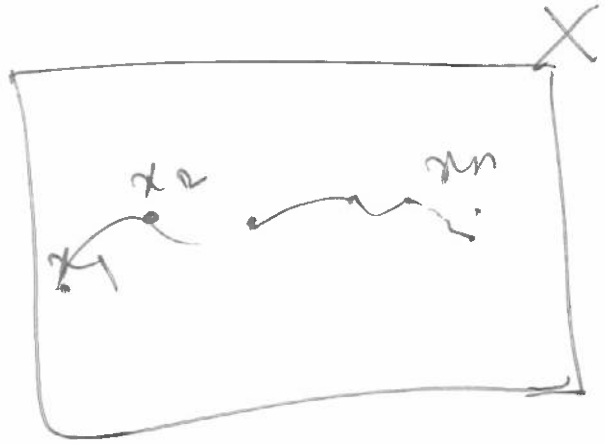
then $U = (a, +\infty)$ for some $a \in (-\infty, -1) \cup \{-\infty\}$



$x_n = 0 \in U$.

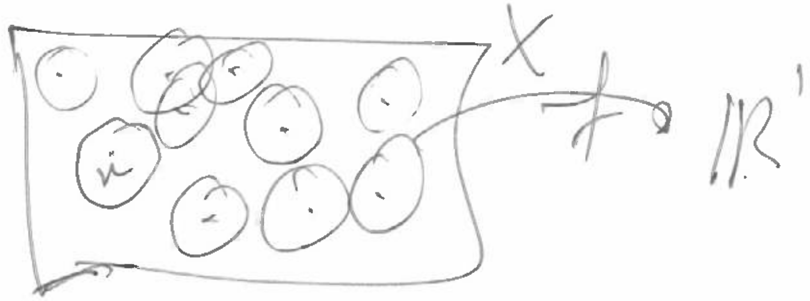
2.4 Compactness

Motivation:



$\{x_n\}_{n\geq 1}$

is there a subsequence of x_n which converges to some element in X.



Def 2.30 Let (X,d) be a metric space, and $Y \subseteq X$.

(i) A collection $\mathcal{R}$ of open subsets of X is called an open cover for Y, if

$$Y \subseteq \bigcup_{U \in \mathcal{R}} U.$$

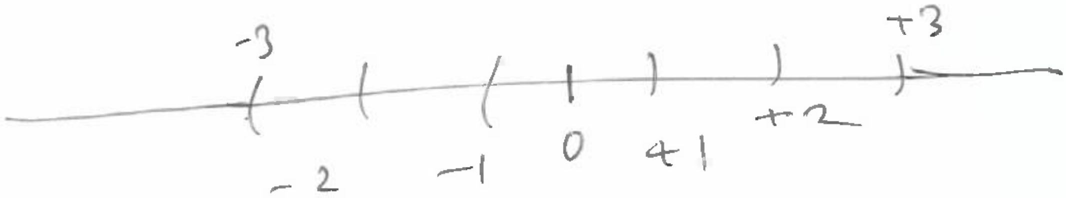
(ii) Given an open cover $\mathcal{R}$ for Y, we say that C is a subcover of $\mathcal{R}$ for Y, if

$$C \subseteq \mathcal{R} \quad \& \quad Y \subseteq \bigcup_{U \in C} U.$$

(iii) $\mathcal{R}$ is a finite cover for Y, if there are finite number of elements in $\mathcal{R}$

Example 2.43 in $(\mathbb{R}^1, d_1)$. let W9, L1 2

$$R = \{(-n, +n) \mid n \in \mathbb{N}\}$$



R is an open cover for $\mathbb{R}$.

$$C = \{(-2n, 2n) \mid n \in \mathbb{N}\}.$$

$$C \subseteq R, \quad \mathbb{R} \subseteq \bigcup_{n\geq 1}^{\infty} (-2n, 2n).$$

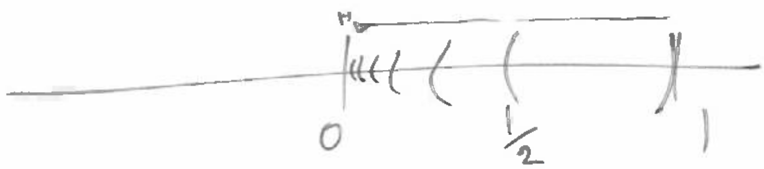
Def 2.31 let (X, d) be a metric space, and $Y \subseteq X$.

We say that Y is compact in (X, d), if any open cover for Y has a finite subcover.

Example in $(\mathbb{R}^1, d_1)$, $\mathbb{R}^1$ is not compact.

$$R = \{(-n, +n) \mid n \in \mathbb{N}\}.$$

in $((0,1), d_1)$ $\quad R = \{(\frac{1}{n}, 1) \mid n \in \mathbb{N}\}.$



W9, L1

Proposition 2.31, in $(\mathbb{R}, d_1)$, for any $a \le b$, $[a,b]$ is compact. 3

Proof (idea): Let $\mathcal{R}$ be an open cover for $[a,b]$.

Let $I = \{ s \in [a,b] \mid$ there is finite subcover of $\mathcal{R}$ for $[a,s] \}$

$I \subseteq [a,b] \Rightarrow I$ is bdd from above,

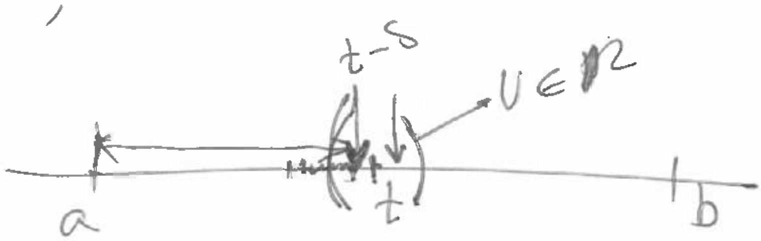
$a \in I. \Rightarrow I \neq \emptyset.$

let $t = \sup I.$

$t \neq a$

$\Rightarrow \delta \in I,$

$t \notin (a,b).$



$t = b.$

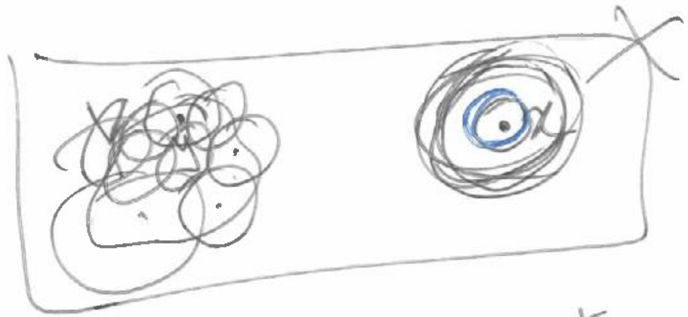
Prop 2.32 let (X,d) be a metric space, and $Y \subseteq X$.

If X is compact, and Y is closed, then Y is compact.

proof:

Theorem 2.33. (X,d) metric space, $Y \subseteq X$.

If Y is compat, then Y is closed.



let (X, d_X) and (Y, d_Y) be compact metric spaces.

Then $X \times Y$ with the metric, $\underline{d_2}$,

is compact.

$[a_1, b_1] \times [a_2, b_2] \times - — [a_n, b_n]$

in $\mathbb{R}^n$ is compact.

W9, L2

Def 2.32 let (X, d) be a metric space.

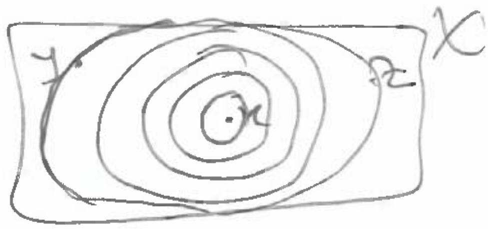
- A set $Z \subseteq X$ is called bounded, if there is $M \in \mathbb{R}$, s.t. $\forall\, x, y \in Z$, $d(x,y) < M$.
- Given a set S, $f: S \to X$ is called bounded if $f(S)$ is bounded in (X, d).

lemma 2.36. If (X, d) is a compact metric space, then X is bounded in (X, d).

Proof: if $X = \phi$, set $M = 1$.

otherwise choose $x \in X$, and consider the cover $\mathcal{R} = \{ B_n(x) \mid n \in \mathbb{N} \}$.

$\mathcal{R}$ has a finite subcover for X. $\Rightarrow X \subseteq B_m(x). \Rightarrow$



$d(y, z) \leq d(y, x) + d(x, z)$

$\leq m + m$

Theorem 2.37 (Heine-Borel) W9, L2

In the metric space $(\mathbb{R}^n, d_2)$, a set $X \subseteq \mathbb{R}^n$ is compact, iff X is closed and bounded.

proof; If X is compact, then, by lem 2.36 X is bounded. and by thm 2.33 any compact set is closed.

assume X is bdd & closed.

$\exists N \in \mathbb{N}$ s.t. $X \subseteq [-N, N]^n$

$[-N, N]^n$ is compact. by Cor 2.35 any closed set in a compact set is compact.

prop 2.32.

2.4.2 sequential compactness.

Def 2.33 A metric space (X, d) is called sequentially compact, iff every sequence in X has a convergent subsequence, which converges to some point in X.

W9, L2

Example: $(\mathbb{R}^1, d_1)$ is not seq. compact.

$$\{x_n = n\}_{n \geq 1}.$$

$((0,1), d_1)$ is not seq. compact.

$$\{x_n = 1/n\}_{n \geq 1}.$$

Lemma 2.38 Let (X, d) be a metric space, and $(x_n)_{n \geq 1}$ be a sequence in X. $(x_n)_{n \geq 1}$ has a convergent subsequence, if there $x \in X$, s.t. $\forall \varepsilon > 0$, there are infinity many $n \in \mathbb{N}$ satisfying

$$x_n \in B_\varepsilon(x).$$

(for proof see typed notes.)

Thm 2.39 If a metric space is compact, then it is sequentially compact.

proof: let us assume that X is not seq. compact.

There is a sequence $(x_n)_{n\geq 1}$, in X which has no convergent subsequence.

$\forall x \in X$, $\exists\ \varepsilon_x$ s.t. there are only finitely many n satisfying $x_n \in B_{\varepsilon_x}(x)$.

$$R = \{ B_{\varepsilon_x}(x) \mid x \in X \}.$$

R is an open cover for X. X is compact.

then, there is a finite subcover of R which covers X. $\exists\ x_1, \ldots, x_n \in X$, s.t.

$$X \subseteq B_{\varepsilon_{x_1}}(x_1) \cup B_{\varepsilon_{x_2}}(x_2) \cup \cdots \cup B_{\varepsilon_n}(x_n)$$

every $x_n \in X$, so x_n must be in one of the following.

If X is an infinite set, d_{disc} on X. W9, L2 5

2.4.3 Continuous maps & compact sets.

Thm 2.4.2. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f: X \to Y$. If $Z \subseteq X$ is compact then $f(Z) \subseteq Y$ is compact.

proof: Let $Z \subseteq X$ be compact. To show that $f(Z)$ is compact, let $R = \{U_\alpha\}_{\alpha \in I}$ be an open cover for $f(Z)$. Define

$$R' = \{\underbrace{f^{-1}(U_\alpha)}_{\text{open}} \mid \alpha \in I\}.$$

R' is an open cover for Z. Z is compact. then there is a finite set $I' \subseteq I$ s.t.

$$Z \subseteq \bigcup_{\alpha \in I'} f^{-1}(U_\alpha)$$

$$\Rightarrow f(Z) \subseteq \bigcup_{\alpha \in I'} U_\alpha.$$

let $R_0 = \{U_\alpha \mid \alpha \in I'\}$

R_0 is a finite subcover of R for $f(Z)$.

W9, L2

Recall that $f:(X,d_X)\longrightarrow(Y,d_Y)$ is uniformly continuous, if $\forall\, \varepsilon>0,\ \exists\, \delta>0$ s.t. $\forall\, x,y\in X$, with $d_X(x,y)<\delta \Rightarrow d_Y(f(x),f(y))<\varepsilon$

Thm 2.44 Any continuous map from a compact metric space to another metric space is uniformly continuous.

proof: If not, there are metric spaces (X,d_X), (Y,d_Y) and a continuous map $f:X\longrightarrow Y$ which is continuous but not uniformly continuous. (X,d_X) is compact.

$\exists\, \varepsilon>0,\ \forall\, \delta>0,\ \exists\, x,y\in X$ s.t.

$$d_X(x,y)<\delta \ \ \& \ \ d_Y(f(x),f(y))\geq\varepsilon.$$

By letting $\delta=\frac{1}{n}$, $\exists\, x_n,y_n\in X$, s.t.

$$d_X(x_n,y_n)<\frac{1}{n},\ \ \& \ \ d_Y(f(x_n),f(y_n))\geq\varepsilon.$$

Consider the sequences $(x_n)_{n\geq 1}$ & $(y_n)_{n\geq 1}$. WQ L2 7.

As (X, d_X) is compact, $(x_n)_{n\geq 1}$ has convergent subsequence, say $(x_{n_k})_{k\geq 1}$, which converges to some $x \in X$.

Consider $(y_{n_k})_{k\geq 1}$ in X, has a convergent subsequence, say $(y_{n_{k_j}})_{j\geq 1}$, which converges to some $y \in X$. obviously, $(x_{n_{k_j}})_{j\geq 1} \longrightarrow x$.

$$d(x_{n_{k_j}}, y_{n_{k_j}}) < \frac{1}{n_{k_j}}$$

$$\Longrightarrow d(x, y) = 0 \Longrightarrow x = y.$$

$$d_Y(f(x_{n_{k_j}}), f(y_{n_{k_j}})) \geq \varepsilon.$$

$$\Longrightarrow d_Y(f(x), f(y)) \geq \varepsilon.$$

2.5 Completeness

Completeness is closely related to compactness

2.5.1 Complete metric spaces & Banach spaces

Def 2.34. Let (X,d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. We say that (x_n) is a Cauchy sequence in (X,d), if for any $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$, s.t.

for all $m,n \geq N_\varepsilon$, we have $d(x_n, x_m) < \varepsilon$.

Def 2.35.

(i) a metric space (X,d) is called complete, if every Cauchy sequence in (X,d) converges to some point in X.

(iii) A normed vector space $(V, \|\cdot\|)$ is called a Banach space, if the metric space, $(V, d_{\|\cdot\|})$ is complete.

Example $((0,1), d_1)$ is not complete.

the seq. $x_n = 1/n$, $n \geq 1$, is Cauch, but does not converge in (X, d).

$(\mathbb{Q}, d_1)$ is not complete.

$(\mathbb{R}^1, d_1)$ or $(\mathbb{R}^n, d_2)$ are complete.

$$C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$$

$$d_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

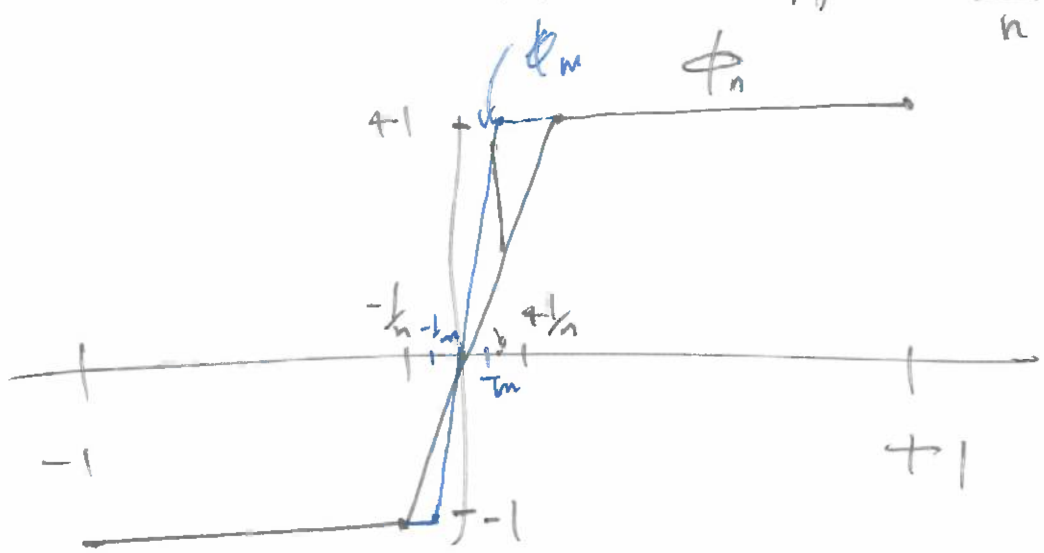
$$d_2(f,g) = \left(\int_a^b |f(t) - g(t)|^2 \, dt \right)^{1/2}$$

Prop 2.49 The metric space $(C([a,b]), d_2)$ is not complete. Equivalently, $(C([a,b]), \|\cdot\|_2)$ is not a Banach space.

Proof: to simplify the proof, let $a=-1$, $b=+1$.

For $n \geq 1$, let

$$\phi_n(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -\frac{1}{n} \\ nt & \text{if } -\frac{1}{n} \leq t \leq \frac{1}{n} \\ +1 & \text{if } \frac{1}{n} \leq t. \end{cases}$$



$(\phi_n)_{n \geq 1}$ is a sequence in $C([-1,+1])$.

W10, L1

$$d(\phi_n, \phi_m) = \left(\int_{-1}^{+1} |\phi_n(t) - \phi_m(t)|^2 \, dt \right)^{1/2}$$

$$= \left(\int_{\min\{-\frac{1}{m}, -\frac{1}{n}\}}^{\max\{\frac{1}{m}, \frac{1}{n}\}} |\phi_n(t) - \phi_m(t)|^2 \, dt \right)^{1/2}$$

$$\leq \left(1^2 \cdot 2 \cdot \max\left\{\frac{1}{m}, \frac{1}{n}\right\} \right)^{1/2}$$

$$\leq \sqrt{2} \left(\frac{1}{\min\{m, n\}} \right)^{1/2}$$

This implies that ϕ_n is Cauchy.

We claim that $(\phi_n)_{n \geq 1}$ does not converge in $(C([a,b]), d_2)$.

assume in the contrary that there $f \in C([a,b])$ s.t. $\phi_n \to f$.

consider the function

$$\varphi(t) = \begin{cases} +1 & t \in [0, 1] \\ -1 & t \in [-1, 0) \end{cases}.$$

$\varphi \notin C([a,b])$.

For $n \geq 1$,

W10, L1 5.

$$= \left(\int_{-1}^{+1} |\psi(t) - \phi_n(t)|^2 \right)^{1/2}$$

$$\leq \left(1^2 \cdot \frac{2}{n} \right)^{1/2} \longrightarrow 0$$

as $n \to \infty$.

$$0 \leq \int_0^{+1} |f(t) - \psi(t)|\, dt + \int_{-1}^{0} |f(t) - \psi(t)|\, dt$$

$$\underbrace{\int_0^{+1} |f(t) - \phi_n(t)|\, dt + \int_{-1}^{0} |f(t) - \phi_n(t)|\, dt}_{\to 0} + \underbrace{\int_0^{+1} |\phi_n(t) - \psi(t)|\, dt + \int_{-1}^{0} |\phi_n(t) - \psi(t)|\, dt}_{\to 0}.$$

$$\Rightarrow \int_0^{+1} |f(t) - \psi(t)|\, dt = 0$$

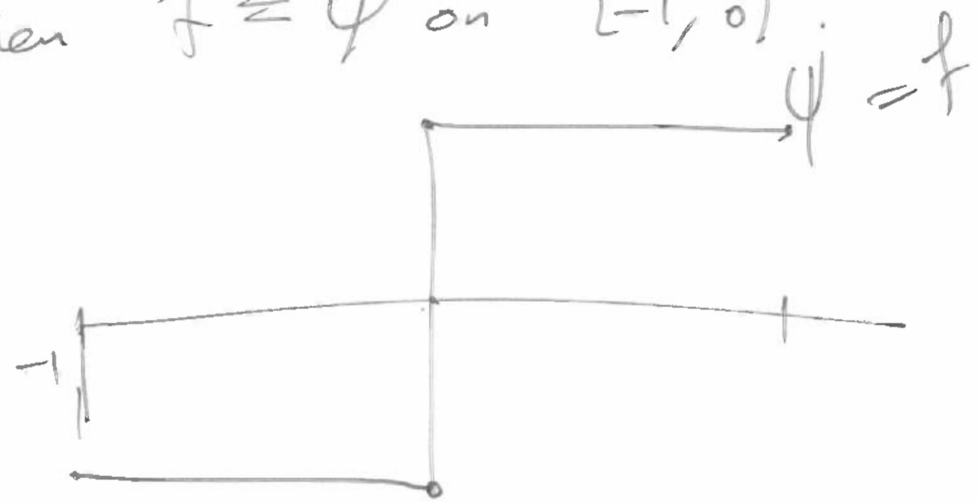
& $\displaystyle\int_{-1}^{0} |f(t) - \psi(t)|\, dt = 0$

W10, L1 6

$\Rightarrow$ As f and ψ are continuous on $[0,1]$, then $f \equiv \psi$ on $[0,1]$.

As f & ψ are continuous on $[-1, 0)$

then $f \equiv \psi$ on $[-1, 0)$.



f is not continuous. $\times$

Thm 2.5.1 The metric space $(C([a,b]), d_\infty)$ is a complete metric space.

proof: let $(\phi_n)_{n\geq 1}$ be a Cauchy sequence in $(C([a,b]), d_\infty)$.

$$d_\infty(\phi_n, \phi_m) = \sup_{t\in[a,b]} |\phi_n(t) - \phi_m(t)|$$

W10, L1 7

looking for a candidate ϕ for the limit.

for each $t \in [a,b]$, $\phi(t)$?

$$|\phi_n(t) - \phi_m(t)| \leq d_\infty(\phi_n, \phi_m) \to 0$$

$\Longrightarrow$ for fixed t, $(\phi_n(t))_{n\geq 1}$ is a Cauchy sequence in $(\mathbb{R}^1, d_1)$. By completeness of $(\mathbb{R}^1, d_1)$ the limit $\phi(t)$ exists.

We need to show that $\phi \in C([a,b])$.

$\forall \varepsilon > 0$, $\exists N_\varepsilon$, $\forall m,n \geq N_\varepsilon$.

$$d_\infty(\phi_n, \phi_m) < \varepsilon.$$

t is fixed.

$$|\phi_n(t) - \phi_m(t)| \leq d_\infty(\phi_n, \phi_m) < \varepsilon.$$

take limit as $m \to \infty$.

$$|\phi_n(t) - \phi(t)| \leq \varepsilon. \quad \forall t \in [a,b].$$

$\Longrightarrow$ ϕ_n converges uniformly to ϕ on $[a,b]$.

By a thm from analysis I, Φ is continuous $\underline{8}$ W10,L1

on $[a,b]$.

2.5.2 Arzela–Ascoli

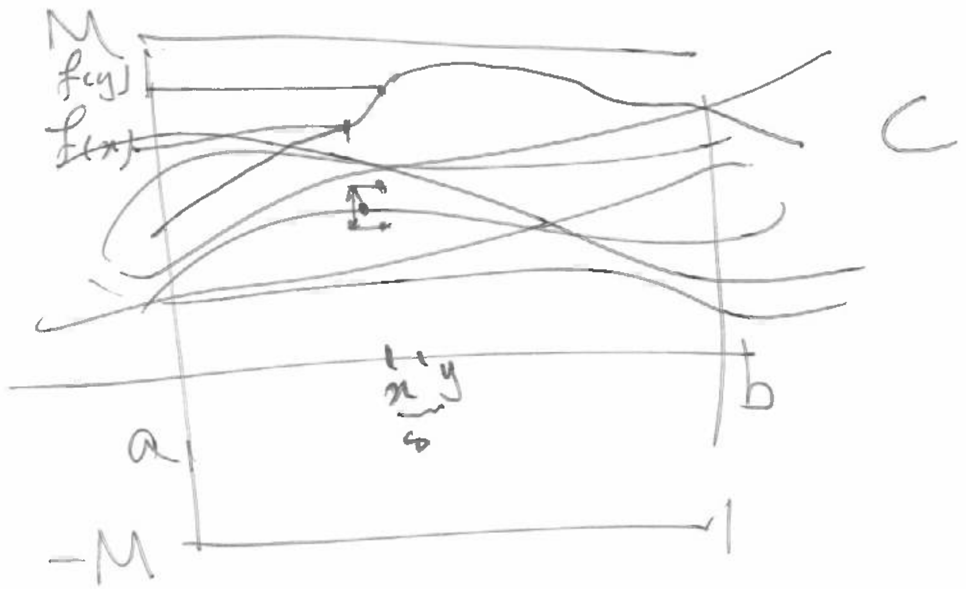
Def 2.36 Let C be a collection of functions $f : [a,b] \to \mathbb{R}$.

(i) We say that C is uniformly bounded, if there is $M \in \mathbb{R}$ s.t. for all $f \in C$, $\forall x \in [a,b]$,

$$|f(x)| \leq M.$$

(ii) We say that C is uniformly equi-continuous if for any $\varepsilon > 0$, $\exists\, \delta > 0$, s.t. for all $f \in C$, $\forall x, y \in [a,b]$ with $|x-y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$



Thm 2.53 (Arzela-Ascoli)

Assume C is a collection of continuous functions $f: [a,b] \to \mathbb{R}$. If C is uniformly bounded and uniformly equicontinuous, then every sequence in C has subsequence which converges in $(C([a,b]), d_\infty)$.

Proof: let $\{f_n\}_{n \geq 1}$ be an arbitrary sequence in C.

$$[a,b] \cap \mathbb{Q} = \{x_1, x_2, x_3, \ldots\}.$$

let $f_{0,i} = f_i, \forall i = 1, 2, 3, \ldots$

$f_{0,1}$, $f_{0,2}$, $f_{0,3}$, ..., $f_{0,i}$, ...

$f_{1,1}$, $f_{1,2}$, ..., $f_{1,i}$, ... converges at x_1

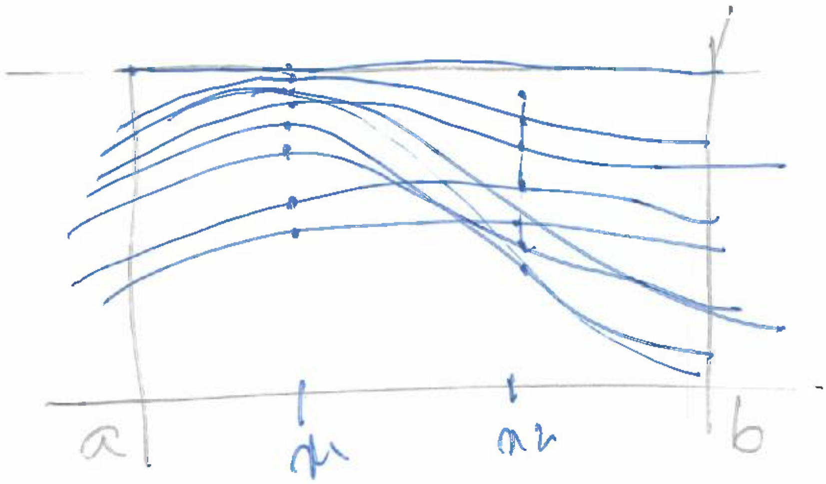
$f_{2,1}$, $f_{2,2}$, ( ), ..., $f_{2,i}$, ... converges at x_2

$\vdots$

$f_{j,1}$, $f_{j,2}$, ..., ( ), $f_{j,i}$, ...

Let g_i be the diagonal sequence

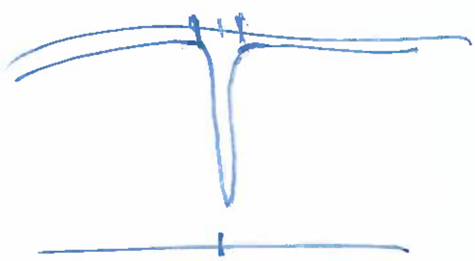
$\{g_i\}_{i\geq 1}$ converges at every x_k, $k=1,2,\ldots$,



Claim that $\{g_i\}_{i\geq 1}$ is a Cauchy sequence in $(C([a,b]), d_\infty)$.

Fix $\varepsilon > 0$. Because C is equicontinuous, there is $\delta > 0$, s.t. $\forall\, i \in \mathbb{N}$, $\forall\, x, y \in [a,b]$, if $|x-y| < \delta$,

$$|g_i(x) - g_i(y)| < \varepsilon/3.$$



$[a,b] \subseteq \bigcup_{i=1}^{k} (x_i - \delta, x_i + \delta)$ W10, L2 4

for each $m = 1, 2, \ldots, k$,

$\{g_i(x_m)\}_{i=1}^{\infty}$ converges, $\exists N_m \in \mathbb{N}$, s.t.

$\forall i, j \geq N_m$, $|g_i(x_m) - g_j(x_m)| < \varepsilon/3$.

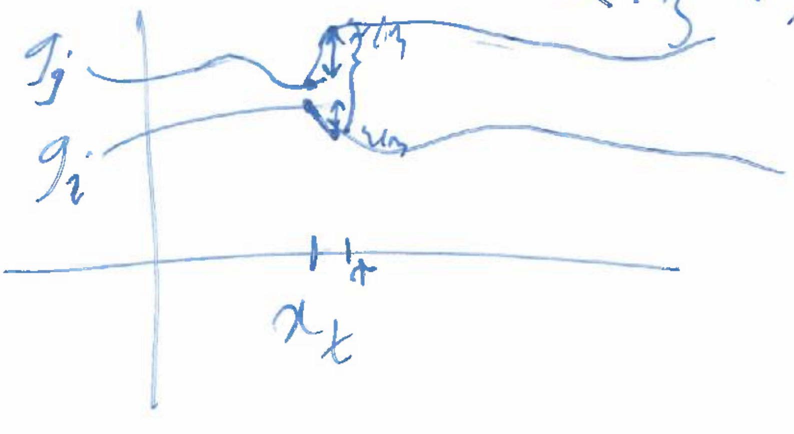
let $N = \max\{N_1, N_2, \ldots, N_k\} \in \mathbb{N}$.

let $x \in [a,b]$. there is $t \in \{1, 2, \ldots, k\}$ s.t.
$x \in (x_t - \delta, x_t + \delta)$.

if $i, j \geq N$, then.

$$|g_i(x) - g_j(x)| \leq |g_i(x) - g_i(x_t)| + |g_i(x_t) - g_j(x_t)| + |g_j(x_t) - g_j(x)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$



if $\{g_i\}_{i\geq 1}$ is Cauchy in $(C([a,b]), d_\infty)$.

Since $(C([a,b]), d_\infty)$ is a complete metric space, then g_i converges to some $g \in C([a,b])$.

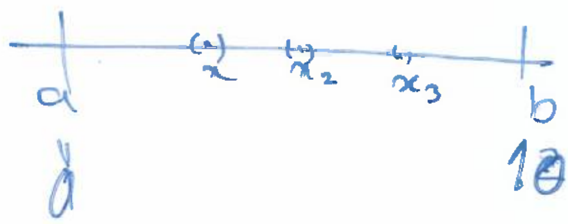
Remarks. Let $C = \{f \equiv d \mid d \in (-2,+2)\}$
$f : [a,b] \to \mathbb{R}$,

let $f_n \equiv 2 - 1/n \in C$. $f_n \xrightarrow{d_\infty} f \equiv 2. \notin C.$

$\mathbb{Q} \cap [a,b] = \{x_1, x_2, x_3 \ldots\}$
is dense in $[a,b]$,

let $r_i \in (0,+\infty)$, $\forall i = 1,2,3, \ldots$

$$\sum_{i=1}^{\infty} 2r_i < 0.01$$



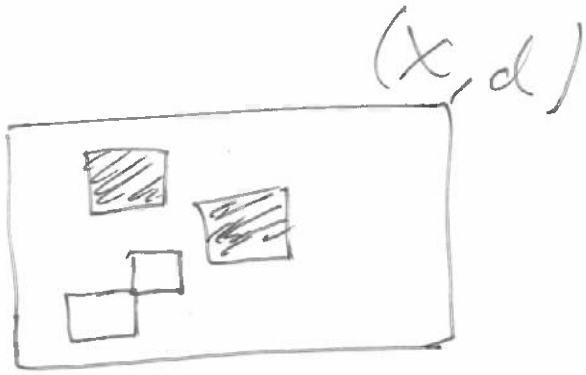
$B = \bigcup B_{r_i}(x_i) = (x_i - \delta_i, x_i + \delta_i)$

if $z \in [a,b]$, then $\exists i \in \mathbb{N}$, s.t. $x \in B_{r_i}(x_i)$?
this is not true.

~~W10,L2~~ W11, L1

1

2.3 Connectedness



2.3.1 connected sets

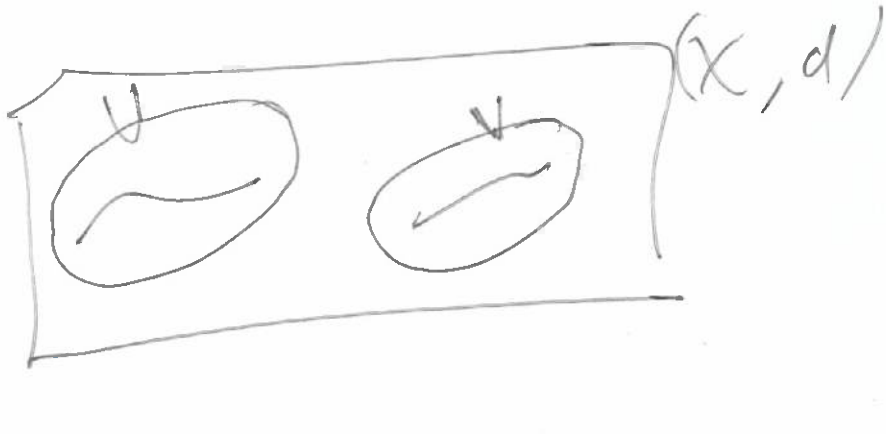
Def 2.26. let (X,d) be a metric space, and $T \subseteq X$.
We say that T is disconnected, if there are open sets U and V in (X,d) satisfying the following properties

(i) $U \cap V = \emptyset$,

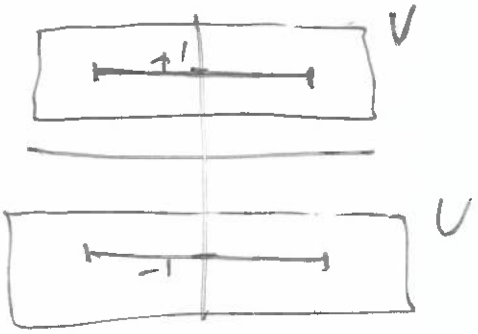
(ii) $T \subseteq U \cup V$,

(iii) $T \cap U \neq \emptyset$, $T \cap V \neq \emptyset$.

In particular, X is disconnected, if there are two open sets in X, which are not empty, disjoint, and their union is X.



Example 2.39. In $(\mathbb{R}^2, d_2)$ consider the set $T = [-1,+1] \times \{-1,+1\}$



let $U = (-2,+2) \times (\frac{1}{2}, \frac{3}{2})$

$V = (-2,+2) \times (-\frac{3}{2}, -\frac{1}{2})$

U, V are open $U \cap V = \emptyset$, $T \subseteq U \cup V$.

$U \cap T = [-1,+1] \times \{+1\} \neq \emptyset$

$V \cap T = [-1,+1] \times \{-1\} \neq \emptyset$.

Example 2.4. Let (X, d_{disc}), X has at least 2 elements. Let $x \in X$, set $U = \{x\}$, $V = X \setminus \{x\}$

U & V are open, $U \cap V = \emptyset$, $U \cup V = X$.

X is disconnected.

Def 2.27. Let (X,d) be a metric space, and $T \subseteq X$. We say that T is connected, if it is not disconnected. Equivalently, T is connected, if for any pair of open sets U and V in X satisfying $U \cap V = \emptyset$, and $T \subseteq U \cup V$, we must have either $U \cap T = \emptyset$ or $V \cap T = \emptyset$.

In particular, X is connected, if for any pair of open sets U and V in X, satisfying $U \cap V = \emptyset$, $X = U \cup V$, the either $U = \emptyset$ or $V = \emptyset$.

Lemma 2.28. Let (X,d_1) be a metric space, and $T \subseteq X$. Then T is disconnected iff there is a continuous function $f : T \longrightarrow \mathbb{R}$, such that $f(T) = \{0,1\}$.

WH, L1 4

proof, first assume that such a map exists.

$U = f^{-1}(0)$, $V = f^{-1}(1)$. , $U \cap V = \phi$

$f(T) = \{0,1\}$, $T \subseteq U \cup V$.

$U \cap T \neq \phi$. $V \cap T \neq \phi$.

$U = f^{-1}(0) = f^{-1}(-\frac{1}{2}, +\frac{1}{2}) \longrightarrow$ open

$V = f^{-1}(1) = f^{-1}(\frac{1}{2}, \frac{3}{2})$. $\longrightarrow$ open.

then T is disconnected.

∴ Assume that T is disconnected. There are open sets U & V in X s.t. $U \cap V = \phi$

$U \cup V \supseteq T$

$f: T \to \mathbb{R}$ $\qquad U \cap T \neq \phi$, $V \cap T \neq \phi$.

$$f = \begin{cases} 0 & \text{if } x \in T \cap U \\ 1 & \text{if } x \in T \cap V \end{cases}$$

$(T \cap U) \cap (T \cap V) \subseteq U \cap V = \phi$

$\Rightarrow$ f is well-defined.

$U \cap T \neq \phi \Rightarrow f(T) \ni 0$

$V \cap T \neq \phi \Rightarrow f(T) \ni 1.$ $\Rightarrow f(T) = \{0, 1\}.$

f is continuous. let $(x_n)_{n \geq 1}$ be a sequence in T converging to some $x \in T$...

$T \subset U \cup V. \Rightarrow x \in U$ or $x \in V,$

Since $U \cap V = \phi$, only one of) happens.

WLG lets assume $x \in U$. as U is open, $\exists N \in \mathbb{N},$

s.t. $\forall n \geq N,$ $x_n \in U$. $\Rightarrow$

$\forall n \geq n,$ $f(x_n) = 0.$ $\Rightarrow f(x_n) \rightarrow f(x).$

$x \in U.$ $f(x) = 0.$

Wil, L1 6

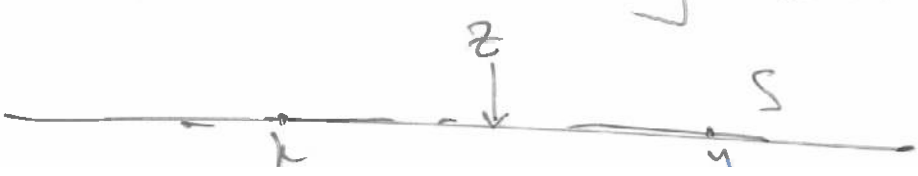
By an interval in $\mathbb{R}$, we mean any of the sets (a,b), $(a,b]$, $[a,b)$, $[a,b]$, $(-\infty, +\infty]$, $(-\infty, b)$, $[-\infty, b]$, $(a, +\infty]$, $[a, +\infty$

Lemma 2.24. Let $S \subseteq \mathbb{R}$ be non-empty. Then S is an interval iff for all $x, y \in S$, and all $z \in \mathbb{R}$ satisfying $x < z < y$, we have $z \in S$.

proof: elementary properties of sets, see typed notes.

Theorem 2.25 Consider the Euclidean metric space $(\mathbb{R}^1, d_1)$, and let $S \subseteq \mathbb{R}$. If S is connected, then S is an interval.

proof: Assume that there is a connected set which is not an interval. There are $x, y \in S$ and $z \in \mathbb{R}$, s.t. $x < z < y$ and $z \in S$.



W11, L1

let $U = (-\infty, z)$, $V = (z, +\infty)$

U and V are open in $(\mathbb{R}^1, d_1)$

- $U \cap V = \emptyset$.
- $S \subseteq U \cup V$.

$\ni x$ $\qquad\qquad\qquad\qquad$ $\ni y$

- $S \cap U \neq \emptyset$, $\quad S \cap V \neq \emptyset$.

These show that S is disconnected ※

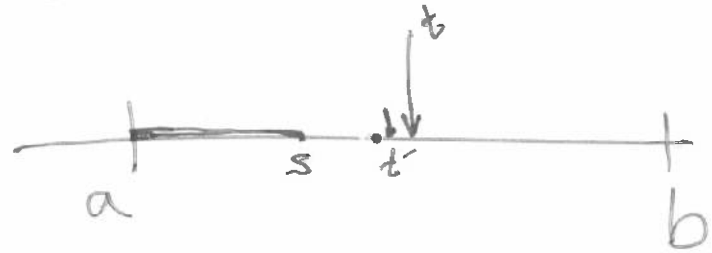
WII, L2

Thm 2.26 For every $a, b \in \mathbb{R}$, with $a \leq b$, the interval $[a,b]$ is connected in the metric space $(\mathbb{R}^1, d_1)$.

proof: let us assume that $[a,b]$ is not connected. There are open sets U and V in $(\mathbb{R}^1, d_1)$ s.t.

$[a,b] \subseteq U \cup V$, $U \cap V = \emptyset$, $U \cap [a,b] \neq \emptyset$, $V \cap [a,b] \neq \emptyset$.

$a \in [a,b]$, $a \in U \cup V$, by relabeling if necessary, we may assume that $a \in U$.



$[a,t'] \subseteq U$.

$X \subseteq Y$

$X \subset Y$ or $X = Y$

let $I = \{ s \in [a,b] \mid [a,s] \subseteq U \}$

$I \subseteq [a,b] \Rightarrow I$ is bounded from above.

$I \neq \emptyset$, $a \in I$. then $\sup I =: t$ exists.

$\boxed{a \leq t \leq b}$

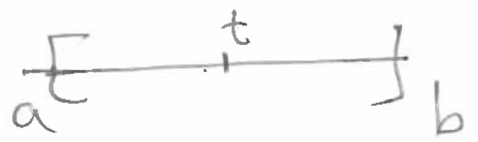
$\forall z < t$, $\exists$ $t' \in [z, t]$, s.t. $[a, t'] \subseteq U$.

$\Longrightarrow [a, t) \subseteq U$.

$t \in [a,b] \subseteq U \cup V$. either $t \in U$, or $t \in V$. 2

W11, L2

Case i) $t \in U$.



$[a,t) \subseteq U$, $t \in U$.

$\Downarrow$

$[a,t] \subseteq U$.

as U is open, $\exists\, \delta > 0$ s.t. $(t-\delta, t+\delta) \subseteq U$.

$\Downarrow$

$[a, t+\frac{\delta}{2}] \subseteq U$.

- if $t < b$, make $\delta > 0$ small so that

$t + \delta/2 \leq b$.

$[a, t+\delta/2] \subseteq U$.

$\Rightarrow t + \delta/2 \in I$ $\cancel{*}$ $t = \sup I$.

if $t = b$. $[a,b] \subseteq U$. $U \cap V = \emptyset$

$[a,b] \cap V \neq \emptyset$

case ii) $t \in V$. V is open, $\exists\, \delta > 0$ s.t.

$(t-\delta, t+\delta) \subseteq V$.

$[a, t) \subseteq U$ / $+ U \cap V = \emptyset$

$\cancel{*}$

$\square$

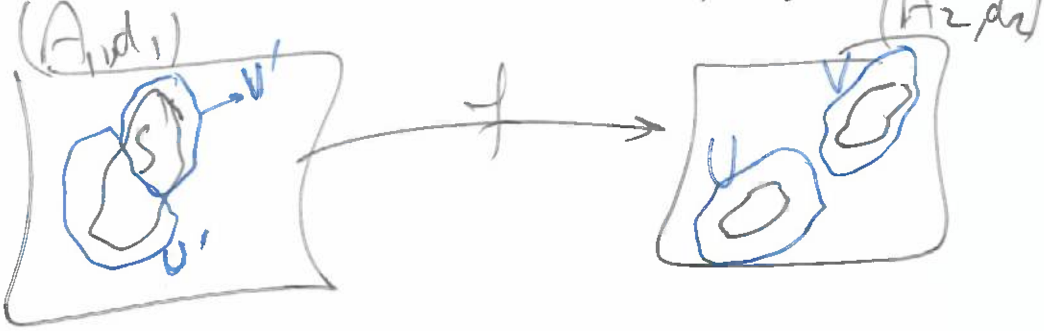
alternatively, $\nexists$ cont. function $f: [a,b] \to \{0,1\}$.

~~$f(a) \neq f(b)$~~

2.3.2 Continuous maps & Connected sets.

Thm 2.27. let (A_1, d_1) & (A_2, d_2) be metric spaces, and $f: A_1 \to A_2$ is a continuous map.

If $S \subseteq A_1$ is Connected, then $f(S)$ is connected.



proof: Assume that it is not true. there exist — — —, with S connected but $f(S)$ not connected. There are open sets U & V in (A_2, d_2) such that

$U \cap V = \phi$, $f(S) \subseteq U \cup V$, $f(S) \cap U \neq \phi$, $f(S) \cap V \neq \phi$.

let $U' = f^{-1}(U)$, $V' = f^{-1}(V)$.

$U' \cap V' = \phi$ $S \subseteq U' \cup V'$, $S \cap U' \neq \phi$, $S \cap V' \neq \phi$.

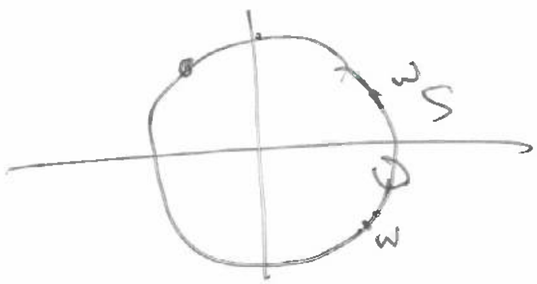
f is continuous $\Longrightarrow$ U' & V' are open. U' & V' disconnect S. $\times$

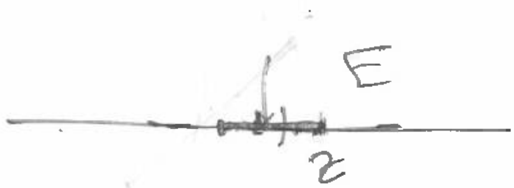
W11, L2

4

Corollary 2.28. Assume that $f:(X,d_X) \to (Y,d_Y)$ is a homeomorphism. Then, X is connected iff Y is connected.

$[0,1] \cup [3,4] \simeq [5,7]$.





$S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1$

$\downarrow \phi$

$E = \{(x,y) \in \mathbb{R}^2 \mid y=0, \quad x \in [0,1]\}$

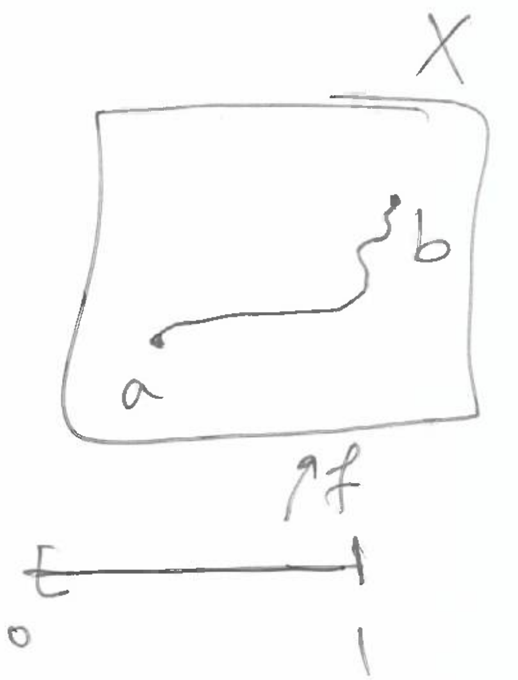
$[a,b) \not\cong (a,b)$

2.3.3 Path-connected sets.

Def 2.28. Let (X,d) be a metric space and, $a, b \in X$.

A "path" from a to b is a continuous map $f: [0,1] \to X$ s.t. $f(0)=a$, $f(1)=b$.

$\left(\forall t \in [0,1] \quad f(t) \in X. \right)$

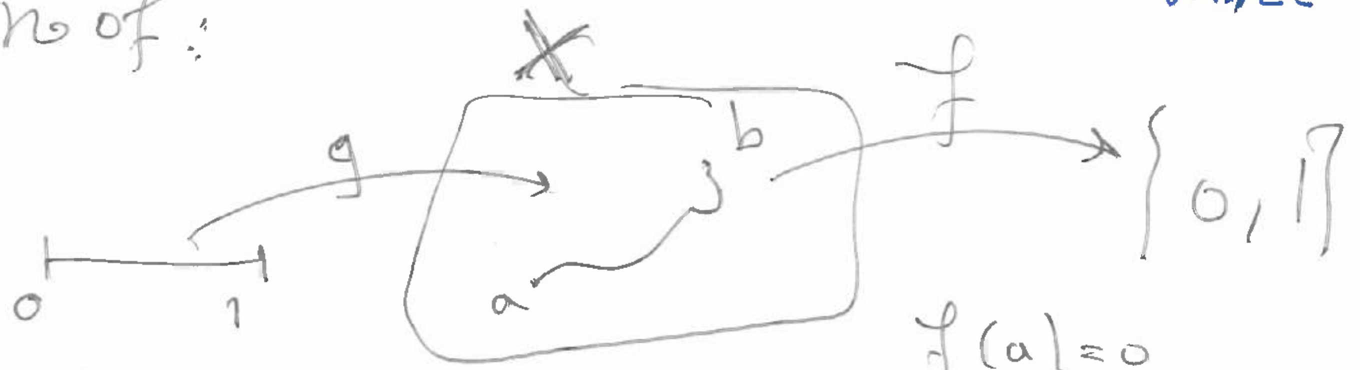


Def 2.29. A metric space (X,d) is called path-connected, if for all $a, b \in X$. there is a path from a to b in X.

Thm 2.31. If a metric space is path-connected then it is connected.

W11, L2

proof:



$f(a)=0$

$f(b)=0.$

assume there (X, d_X) which is path connected, but not Connected. $\exists\, a \in X$, $f(a) < 0$

$\exists\, b \in X$, $f(b) = 1$.

$g(0) = a$, $g(1) = b$.

$f \circ g^{-1/2} : [0,1] \longrightarrow \{-1/2, 1/2\}$

$f \circ g$ is continuous. $\cancel{*}$

Q. is $(C[a,b], d_\infty)$

f, g $\quad \phi(t) = tf + (1-t)g.$

$\phi : [0,1] \to C[a,b]$

ϕ is continuous.

$\phi(0) = g$, $\phi(1) = f$.