Quiz 2 – Solutions

1. *Consider the random variables $X_1, ..., X_n$ with unknown parameter $\theta \in \Theta \subset \mathbb{R}$. Assume that the maximum likelihood estimator (MLE) for θ exists. Is the MLE an unbiased estimator for θ?*
   Not always. In some cases as you have seen in the problem sheets (like in the Normal distribution case) the MLE is the sample mean which is an unbiased estimator. However, this is not generally true. Indeed, unbiasedness is not a general property of the MLE (the properties of the MLE are functional invariance, consistency and asymptotic normality).

2. *Let $X_1, X_2, ...,$ be a sequence of i.i.d. random variables. Assume that the distribution of X_1 is $N(0,1)$. Then, we know that $\frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2 \xrightarrow{d} Y$ as $n \to \infty$, for some random variable Y. What is the distribution of Y? Select all that apply.*
   Using the fact that $E[X_1] = 0$ and $Var(X_1) = 1$ by the CLT we know that $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \xrightarrow{d} N(0,1)$ as $n \to \infty$. Hence, by the continuous mapping theorem (using the function $g(x) = x^2$) we have that $n\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \xrightarrow{d} \chi^2_1$ that is $\frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2 \xrightarrow{d} \chi^2_1$ hence $Y \sim \chi^2_1$.

3. *Consider an unknown paramter $\theta = (\theta_1, ..., \theta_4)^T \in \Theta \subset \mathbb{R}^4$ of a certain underlying random vector. Suppose that $[L_1, U_1]$ is a 99% confidence interval (CI) for θ_1, that $[L_2, U_2]$ is a 95% confidence interval (CI) for θ_2, that $[L_3, U_3]$ is a 95% confidence interval (CI) for θ_3, and $[L_4, U_4]$ is a 90% confidence interval (CI) for θ_4. What do we know about $P_\theta(\theta_i \in [L_i, U_i]$ for $i = 1, ..., 4)$ for all $\theta \in \Theta$? Select all that apply.*
   By the Bonferroni correction we obtain that $P_\theta(\theta_i \in [L_i, U_i]$ for $i = 1, ..., 4) \geq 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = 1 - 0.01 - 0.05 - 0.05 - 0.1 = 0.79$ for all $\theta \in \Theta$. Thus, none of the values was correct.

4. *Consider the setting of the previous question. Now, assume that $[L_1, U_1], ..., [L_4, U_4]$ are independent from each other. What do we know about $P_\theta(\theta_i \in [L_i, U_i]$ for $i = 1, ..., 4)$ for all $\theta \in \Theta$? Select all that apply.*
   By independence we obtain that $P_\theta(\theta_i \in [L_i, U_i]$ for $i = 1, ..., 4) = \prod_{i=1}^4 P_\theta(\theta_i \in [L_i, U_i]) = \prod_{i=1}^4 (1 - \alpha_i) = 0.99 \times 0.95 \times 0.95 \times 0.9 = 0.8041275$ for all $\theta \in \Theta$, which is greater than 0.795 and 0.8 and lower than 0.82.