1. The exponential function can be characterised in many equivalent different ways. For example, if we assume there exists a function $f : \mathbb{R} \to \mathbb{R}$, satisfying:

   - $f(x + y) = f(x) \cdot f(y), \quad \forall x, y, \in \mathbb{R}$, and
   - f is differentiable at 0 with $f'(0) = 1$,

   then we can prove such a function must be unique by following these steps:

   (a) Show that $f(0) = 1$ and $f(x) \neq 0$, for all $x \in \mathbb{R}$.

   (b) Show that f is differentiable everywhere and $f'(x) = f(x)$, for all $x \in \mathbb{R}$.

   (c) Prove that f must be unique. *Hint: assume there is another function g satisfying the two items at the beginning. Is $\frac{f(x)}{g(x)}$ well-defined? Is it differentiable? If so, what is its derivative?*

   In addition, we can derive several useful properties of f:

   (d) Argue that $f(x) > 0$, for all $x \in \mathbb{R}$, and conclude that f is strictly increasing. *Hint: for the first part use item (a) and Bolzano's theorem.*

   (e) Show that for $q \in \mathbb{Q}$, $f(x) = f(1)^q$ and conclude that the image of f is the interval $(0, \infty)$.

   Thanks to the uniqueness we can now define the constant e as $e := f(1)$. Similarly, now the notation $f(x) = e^x$, for all $x \in \mathbb{R}$, is meaningful. However, note that we do not know yet that a function like f exists. Its existence will be proven once we define the concept of integral. At this point, we are only showing that only one such function can exist.

2. This problem is borrowed from the last problem sheet: let $f : (a, b) \to \mathbb{R}$ be a monotone increasing function. Show that the following one-sided limits exist and satisfy the inequality

$$\lim_{h\to 0^-} f(x + h) \leq f(x) \leq \lim_{h\to 0^+} f(x + h).$$

3. Show that among all rectangles with a fixed perimeter, the square has the largest area.

4. Find $f'(0)$ if

$$f(x) = \begin{cases} g(x)\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   and

$$g(0) = g'(0) = 0.$$

5. Show that $e^x \geq \frac{x^n}{n!}$, for all $x \geq 0$ and $n \in \mathbb{N}$, and conclude that the exponential function grows faster than any polynomial, i.e., $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$, for all $n \in \mathbb{N}$.

6. Let
$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$
Show that f is infinitely differentiable and that $f^{(n)}(0) = 0$, for all $n \in \mathbb{N}$. (Note that this is a very special function, it interpolates smoothly between being constantly zero to being almost constantly one).

7. Let $A = (0, h)$ and $B = (p, q)$ points of $\mathbb{R}^2$, with p, q and h fixed positive numbers. Draw the two line segments joining A with $O = (x, 0)$ and then O with A. Show that the sum of both segments is minimised when the angles formed between the horizontal axis and the segments $\overline{AO}$ and $\overline{OB}$, are both equal.

8. You drive down a road whose speed limit is 60 miles per hour. An observer sees you at 12pm, and a second observer 35 miles away sees you at 12:30pm. Assuming they've watched their analysis lectures, how can they prove you were speeding?

9. Let H_n denote the harmonic sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.
   (a) Show using the mean value theorem that $\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.
   (b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n \geq 2$, where $H_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$, and deduce that $\log(n+1) < H_n < \log(n) + 1$.
   (c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty} (H_n - \log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma \approx 0.577\ldots$.)

10. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose there is a constant $C < 1$ such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. We will prove that f has exactly one fixed point, meaning there is a unique $y \in \mathbb{R}$ such that $f(y) = y$. Pick some $x_0 \in \mathbb{R}$ and let
$$x_{n+1} = f(x_n) \text{ for all } n \geq 0.$$
   (a) Prove that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all n.
   (b) Prove that the sequence $\big(x_n\big)$ converges, and that if its limit is y then $f(y) = y$.
   (c) Prove that f cannot have two different fixed points.

11. Prove using l'Hôpital's rule that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

12. The aim of this exercise is to prove $\lim_{x\to\infty} xs^{x-1} = 0$ for all $s \in (0, 1)$.
   (a) Prove that for all $c > 0$, there exists $N > 0$ such that $\log(x) < cx$ for all $x \geq N$.
   (b) Prove for $s \in (0, 1)$ that $\lim_{x\to\infty} xs^x = 0$, and that this implies the above claim.

13. (a) Prove that $f(x) = e^x$ is convex on all of $\mathbb{R}$.

(b) Let $a, b > 0$. Prove the *arithmetic mean–geometric mean inequality*

$$\frac{a+b}{2} \geq \sqrt{ab}$$

by using the convexity of e^x. (Hint: think about $\alpha = \log(a)$ and $\beta = \log(b)$.)

(c) Prove for any $a, b > 0$ and $s \in [0, 1]$ that $sa + (1-s)b \geq a^s b^{1-s}$.

(d) Prove *Young's inequality*: for any $x, y \geq 0$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$