Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Part I – Problem Sheet 2: Functions

KMB, 8/10/22

Note: I am releasing this on Saturday 8th October because someone has asked for it on Ed Stem. I will not cover the material for this problem sheet until the lecture at 9am on Monday 10th October, so in particular this sheet might look intimidating if it's not Monday yet. My suggestion would be that people start work on this problem sheet after that lecture, and if you're really looking for some maths to do over the weekend then you should instead try doing some of the Lean problem sheets which are an optional part of the course ;-)

1. ✦✦ Say X, Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions. In the course notes we proved that if f and g are injective, then $g \circ f$ is also injective, and that if f and g are surjective, then $g \circ f$ is surjective. But what about the other way?

   (a) If $g \circ f$ is injective, then is f injective? Give a proof or a counterexample.
   (b) If $g \circ f$ is injective, then is g injective? Give a proof or a counterexample.
   (c) If $g \circ f$ is surjective, then is f surjective? Give a proof or a counterexample.
   (d) If $g \circ f$ is surjective, then is g surjective? Give a proof or a counterexample.

2. ✦✦✦ For each of the following functions, decide whether or not they are injective, surjective, bijective. Proofs required!

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$ if $x \neq 0$ and $f(0) = 0$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$.
   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - x$ if the Riemann hypothesis is true, and $f(x) = 2 + x$ if not. [NB the Riemann Hypothesis is a hard unsolved problem in mathematics; nobody currently knows if it is true or false.]
   (e) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^3 - 2n^2 + 2n - 1$.

3. ✦ For each of the following "functions", explain why I just lost a mark.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x}$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = (n + 1)^2/2$.
   (d) $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is a solution to $y^3 - y = x$.
   (e) $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$, $f(x) = 1 + x + x^2 + x^3 + \cdots$.

4. ✦✦ Prove that if $f : X \to Y$ is a function, and $g : Y \to X$ is a two-sided inverse of f, then f is a two-sided inverse for g. Deduce that if X and Y are sets, and there exists a bijection from X to Y, then there exists a bijection from Y to X.

5. ✦✦✦ Let Z be a set. If $f : X \to Z$ and $g : Y \to Z$ are injective functions, let's say that f *is friends with* g if there is a bijection $h : X \to Y$ such that $f = g \circ h$. Prove that f is friends with g if and only if the range of f equals the range of g. NB: by the *range* of $f : X \to Z$ I mean the subset of Z consisting of things "hit" by f, in other words the set $\{z \in Z : \exists x \in X, f(x) = z\}$. Some people call this the "image" of f, and some people use "range" to mean the same thing as "codomain" :-/

6. ✦ If T is a set, let $\mathcal{P}(T)$ denote the set of all subsets of T. Say T is a finite set of size n. Then $\mathcal{P}(T)$ is also finite. What is its size?

7. ✦✦ Notation as in Q6. Let α be any set (it could be infinite).

   (a) Say $f : \alpha \to \mathcal{P}(\alpha)$ is an arbitrary function. Just to be clear, this means that f is a function which sends an element of α to a subset of α. Define X to be the following subset of α: $X = \{x \in \alpha \,|\, x \notin f(x)\}$. Prove that X is not in the range of f.

   (b) Deduce that if α is any set then there does not exist a surjection from α to $\mathcal{P}(\alpha)$.

   (c) Deduce that if $n \in \{0, 1, 2, \ldots\}$ then $n < 2^n$.

8. ✦✦ Let α and β be sets, and let $f : \alpha \to \beta$ be a function.

   If $X \subseteq \alpha$ is a subset, then its *image* in β is, informally, the "set of things in β which f sends the elements of X to". Formally we say that the image of X in β is the subset $\{b \in \beta \,|\, \exists a \in X, f(a) = b\}$ of β. Write $f_*(X)$ to mean the image of X in β. So f_* is a function from $\mathcal{P}(\alpha)$ to $\mathcal{P}(\beta)$.

   Now let's go the other way. We still have $f : \alpha \to \beta$. Now say $Y \subseteq \beta$ is a subset of β. Define its *preimage* in α to be, informally, the "things in α which get mapped into an element of Y by f". Formally we say that the preimage of Y is the subset $\{a \in \alpha \,|\, f(a) \in Y\}$. Write $f^*(Y)$ to mean the preimage of Y in α. So f^* is a function from $\mathcal{P}(\beta)$ to $\mathcal{P}(\alpha)$.

   Summary so far: given a function $f : \alpha \to \beta$ we have made two new functions: we have f_*, a function from subsets of α to subsets of β, and f^*, a function from subsets of β to subsets of α.

   What kind of questions come into your head at this point? Do you think these functions $f_* : \mathcal{P}(\alpha) \to \mathcal{P}(\beta)$ and $f^* : \mathcal{P}(\beta) \to \mathcal{P}(\alpha)$ are two-sided inverses of each other, for example? In this question we do some experiments with a concrete example, to try and figure things like this out.

   So, to make things concrete, for the rest of this question, let $\alpha = \{1, 2\}$, let $\beta = \{8, 9, 10\}$, and let's define $g : \alpha \to \beta$ by $g(1) = g(2) = 8$.

   (a) What is $g_*(\{1\})$? What is $g_*(\{1, 2\})$?

   (b) Is g_* injective? Prove it.

   (c) Is g_* surjective? Prove it.

   (d) What is $g^*(\{8\})$? What is $g^*(\{8, 9\})$?

   (e) Is g^* injective? Prove it.

   (f) Is g^* surjective? Prove it.

   (g) Are g_* and g^* two-sided inverses of each other? Prove it.

9. ✦ Notation as in Q8. Let α and β be any sets, and let $f : \alpha \to \beta$ be any function. Prove that if $X \subseteq \alpha$ and $Y \subseteq \beta$ are any subsets, then then $f_*(X) \subseteq Y \iff X \subseteq f^*(Y)$.

10. ✦ Notation as in Q9. Prove that if $X \subseteq \alpha$ then $X \subseteq f^*(f_*(X))$. Then prove that if $Y \subseteq \beta$ then $f_*(f^*(Y)) \subseteq Y$.