IMPERIAL

Department of Mathematics
Imperial College London
Academic Year 2025–2026

Introduction to University Mathematics MATH40001/MATH40009

Final Exam

Instructions: The neatness, completeness and clarity of the answers will contribute to the final mark. You may assume, without proof, any results from the lectures and lecture notes, unless you are explicitly asked to prove them.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is MATH40001 for Maths students and MATH40009 for JMC students. Each question should be answered in a separate booklet.

> In this exam, you may assume any results from the course notes or lectures (as long as you state them correctly), unless you are explicitly asked to prove them.

1. (a) Let X be non-empty. Suppose $f : X \to X$ and $h : X \to X$ are functions. Define the graph of a function $G(f)$ to be the subset of X^2 defined by the relation $y = f(x)$, i.e.

$$G(f) = \{(x, y) \in X^2 : y = f(x)\}$$

   i. Explain what it means for $f : X \to X$ to be well defined. [1 Mark]
   ii. Show that $G(f) \cup G(h)$ is not always the graph of a function. [2 Marks]
   iii. Is there a condition on f and g such that $G(f) \cup G(h)$ is the graph of a function if and only if this condition holds? Prove the if and only if statement. [4 Marks]
   iv. Is $G(f) \cap G(h)$ always the graph of a function? Give a proof or counter example. [3 Marks]

   (b) Let X be a non-empty set.
   i. Define a binary relation R on X. [1 Mark]
   ii. Is it possible for a binary relation R to be symmetric and transitive but not reflexive? [2 Marks]
   iii. Suppose in addition to being symmetric and transitive that
      A. $\exists x \in X \exists y \in X R(x, y)$
      B. $\forall x \in X \exists y \in X R(x, y)$
      C. $\exists x \in X \forall y \in X R(x, y)$

   In each case does the relation have to be reflexive? Give a proof or counter example. [7 Marks]

[Total: 20 Marks]

2. (a) Let a, b and k be natural numbers. Show that $a+k \leq b+k$ if and only if $a \leq b$. $\boxed{\text{3 Marks}}$

(Hint: one direction appears in the lecture notes. For the other direction, you may use the case split proved in the coursework: for $x, y \in \mathbb{N}$, either $x \leq y$ or $y < x$. You may also use any other facts about $<$ proved on the coursework or problem sheets.)

(b) In this problem we will define the $\leq$ relation on the integers, following the same pattern performed in lectures for the operations $-$, $+$ and $\times$.

First, we define a "pre-relation" $\preccurlyeq$ on $\mathbb{N} \times \mathbb{N}$: by definition $(a, b) \preccurlyeq (c, d)$ if $a + d \leq b + c$.

i. Prove that the pre-relation $\preccurlyeq$ respects the equivalence relation for the left argument. That is, given natural numbers a, b, c, d, a', b' with $(a, b) \sim (a', b')$, prove that $(a, b) \preccurlyeq (c, d)$ if and only if $(a', b') \preccurlyeq (c, d)$. $\boxed{\text{4 Marks}}$

Combining this with a similar check (which we skip) for the second argument, we obtain a relation $\leq$ on $\mathbb{Z}$, characterised by the property that $cl((a, b)) \leq cl((c, d))$ if and only if $(a, b) \preccurlyeq (c, d)$.

ii. (Monotonicity of addition) Given integers x, y, k with $x \leq y$, prove that $x + k \leq y + k$. $\boxed{\text{3 Marks}}$

iii. Given natural numbers a and b with $a \leq b$, prove that $i(a) \leq i(b)$. $\boxed{\text{2 Marks}}$

iv. Given integers x and y with $x \leq y$, prove that there exists a natural number n with $y = x + i(n)$. $\boxed{\text{3 Marks}}$

(c) Prove that the integers have the completeness property: Let $s \subseteq \mathbb{Z}$ be a nonempty set which is bounded above. Then s has a least upper bound. $\boxed{\text{5 Marks}}$

$\boxed{\text{Total: 20 Marks}}$

3. (a) Consider the two vectors $\mathbf{a} = 3\hat{\mathbf{e}}_1 - 1\hat{\mathbf{e}}_3$ and $\mathbf{b} = 4\hat{\mathbf{e}}_2$ in $\mathbb{R}^3$.

   i. What is the area of the parallelogram spanned by these two vectors. 3 Marks

   ii. Find an orthonormal basis for the set of vectors lying in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. 3 Marks

(b) Consider an object moving along a trajectory $\mathbf{r}(t)$, where $t \in \mathbb{R}$ denotes the time. Assume that the trajectory is given by

$$\mathbf{r}(t) = (a\cos(\omega t), b\sin(\omega t), c\sin(\omega t)),$$

where $\omega, a, b, c \in \mathbb{R}^{>}$ are real positive constants.

   i. Calculate the normalised tangent vector,

$$\hat{\mathbf{T}} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}, \text{ at } t = \frac{\pi}{2\omega}.$$

3 Marks

   ii. Provide an equation for the tangent line of the trajectory at the point with $t = \frac{\pi}{2\omega}$. 2 Marks

   iii. Assuming Newton's law $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$, deduce the force $\mathbf{F}(\mathbf{r})$ acting on the particle. 2 Marks

(c) Consider the vector space V of real polynomials of degree 4 or lower.

   i. Give an example of a basis of this vector space. What is the dimension of this vector space? 3 Marks

   ii. For each of the maps T below, state whether it is a linear map on V, and provide a reason for your answer.

      A. $T : P(x) \mapsto P(x) + 5x$ 2 Marks

      B. $T : P(x) \mapsto \frac{\mathrm{d}}{\mathrm{d}x}P(x)$ 2 Marks

Total: 20 Marks

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In this exam, you may assume any results from the course notes or lectures (as long as you state them correctly), unless you are explicitly asked to prove them.

1. (a) Let X be non-empty. Suppose $f : X \to X$ and $h : X \to X$ are functions. Define the graph of a function $G(f)$ to be the subset of X^2 defined by the relation $y = f(x)$, i.e.

$$G(f) = \{(x, y) \in X^2 : y = f(x)\}$$

i. Explain what it means for $f : X \to X$ to be well defined. 1 Mark

A function $f : X \to X$ is well defined if for every $x \in X$ there is a unique $y \in X$ such that $f(x) = y$. 1 Mark

ii. Show that $G(f) \cup G(h)$ is not always the graph of a function. 2 Marks

Let $X = \{a, b\}$ define $f : X \to X$ to be $f(x) = x$ (the identity function) and $g(a) = g(b) = a$ then $G(f) \cup G(h) = \{(a, a), (b, b), (b, a)\}$ here we have (b, a) and (b, b) so the correspondance is not a well defined function (as b has two outputs).

iii. Is there a condition on f and g such that $G(f) \cup G(h)$ is the graph of a function if and only if this condition holds? Prove the if and only if statement. 4 Marks

$G(f) \cup G(h)$ is the graph of a function if and only if $f = h$. 1 Mark

Suppose $f = h$ then $G(f) \cup G(h) = G(f)$, which is the graph of a function (namely f). Suppose conversely that $f \neq h$, then there is some $a \in X$ such that $f(a) \neq h(a)$ (otherwise they would be the same function). Therefore the distinct pairs $(a, f(a)), (a, h(a)) \in G(f) \cup G(h)$, so the correspondance is not a well defined function (as a has two outputs) 1 Mark

iv. Is $G(f) \cap G(h)$ always the graph of a function? Give a proof or counter example. 3 Marks

Let $X = \{a, b\}$ define $f : X \to X$ to be $f(x) = x$ (the identity function) and $g(a) = g(b) = a$ then $G(f) \cap G(h) = \{(a, a)\}$ here we have no value for b so the function is not well defined (as b has no output). 3 Marks

(b) Let X be a non-empty set.

i. Define a binary relation R on X. 1 Mark

A binary relation is a subset of X^2. 1 Mark

ii. Is it possible for a binary relation R to be symmetric and transitive but not reflexive? 2 Marks Yes, e.g. $X = \{a\}$ and $R = \emptyset$. 2 Marks

iii. Suppose in addition to being symmetric and transitive that

A. $\exists x \in X \exists y \in X R(x, y)$

B. $\forall x \in X \exists y \in X R(x, y)$

C. $\exists x \in X \forall y \in X R(x, y)$

In each case does the relation have to be reflexive? Give a proof or counter example.

$\boxed{\text{7 Marks}}$

A. No, e.g. $X = \{a, b\}$ and $R = \{(a, a)\}$. $\boxed{\text{2 Marks}}$

B. Yes, Let $a \in X$ then by the condition there is some $b \in X$ such that $R(a, b)$ thus by symmetry $R(b, a)$ so by transitivity $R(a, a)$, showing reflexivity. $\boxed{\text{3 Marks}}$

C. Yes. Let $a \in X$, by the condition there is some $x \in X$ such that $R(x, a)$ thus by symmetry $R(a, x)$ so by transitivity $R(a, a)$, showing reflexivity. $\boxed{\text{2 Marks}}$

$\boxed{\text{Total: 20 Marks}}$

2. (a) Let a, b and k be natural numbers. Show that $a+k \leq b+k$ if and only if $a \leq b$. $\boxed{\text{3 Marks}}$
(Hint: one direction appears in the lecture notes. For the other direction, you may use the case split proved in the coursework: for $x, y \in \mathbb{N}$, either $x \leq y$ or $y < x$. You may also use any other facts about $<$ proved on the coursework or problem sheets.)

The $\Rightarrow$ direction appears in the lecture notes (Theorem 23).

For the $\Leftarrow$ direction, suppose that $a + k \leq b + k$.

Case 1 ($a \leq b$): we are done.

Case 2 ($b < a$): Then $S(b) \leq a$, so

$$S(b + k) = S(b) + k \leq a + k \leq b + k,$$

so by definition $b + k < b + k$, which contradicts problem 4(f) on problem sheet 1.

(b) In this problem we will define the $\leq$ relation on the integers, following the same pattern performed in lectures for the operations $-$, $+$ and $\times$.

First, we define a "pre-relation" $\preccurlyeq$ on $\mathbb{N} \times \mathbb{N}$: by definition $(a, b) \preccurlyeq (c, d)$ if $a + d \leq b + c$.

i. Prove that the pre-relation $\preccurlyeq$ respects the equivalence relation for the left argument. That is, given natural numbers a, b, c, d, a', b' with $(a, b) \sim (a', b')$, prove that $(a, b) \preccurlyeq (c, d)$ if and only if $(a', b') \preccurlyeq (c, d)$. $\boxed{\text{4 Marks}}$ We are given that $(a, b) \sim (a', b')$, i.e. $a + b' = b + a'$ $(\star)$, and we want to show $(a, b) \preccurlyeq (c, d)$ if and only if $(a', b') \preccurlyeq (c, d)$, i.e. $a + d \leq b + c$ if and only if $a' + d \leq b' + c$.
By part (a) we can add b' to the first inequality and b to the second inequality; that is, it suffices to show

$$(a + d) + b' \leq (b + c) + b' \iff (a' + d) + b \leq (b' + c) + b.$$

We will show this by proving that the LHS and RHS of these inequalities are both equal.
Indeed, $(b + c) + b' = (b' + c) + b$ by associativity and commutativity of addition, and $(a+d)+b' = (a' + d)+b$ by associativity and commutativity of addition together with $(\star)$.

Combining this with a similar check (which we skip) for the second argument, we obtain a relation $\leq$ on $\mathbb{Z}$, characterised by the property that $cl((a, b)) \leq cl((c, d))$ if and only if $(a, b) \preccurlyeq (c, d)$.

ii. (Monotonicity of addition) Given integers x, y, k with $x \leq y$, prove that $x+k \leq y+k$. $\boxed{\text{3 Marks}}$
Represent x as the class of (a, b), y as the class of (c, d), and k as the class of (s, t). We are given that $x \leq y$, i.e. $a + d \leq b + c$ $(\star)$, and need to show $x + k \leq y + k$, i.e. $cl((a + s, b + t)) \leq cl((c + s, d + t))$, i.e.

$$(a + s) + (d + t) \leq (b + t) + (c + s).$$

By associativity and commutativity of addition it suffices to prove $(a + d) + (s + t) \leq (b + c) + (s + t)$, and this follows from the monoticity of addition together with $(\star)$.

iii. Given natural numbers a and b with $a \leq b$, prove that $i(a) \leq i(b)$. $\boxed{\text{2 Marks}}$ We must show that $i(a) \leq i(b)$, i.e. that $cl((a, 0)) \leq cl((b, 0))$, i.e. that $a + 0 \leq 0 + b$. This follows from our hypothesis $a \leq b$.

iv. Given integers x and y with $x \leq y$, prove that there exists a natural number n with $y = x + i(n)$. $\boxed{\text{3 Marks}}$
Represent x as the class of (a, b) and y as the class of (c, d). We are given that $x \leq y$, i.e. $a + d \leq b + c$ $(\star)$, so by the definition of $\leq$ on $\mathbb{N}$ there exists a natural number n such that $b + c = (a + d) + n$.
Rearranging using arithmetic rules, we get that $c + (b + 0) = d + (a + n)$, i.e. $(c, d) \sim (a + n, b + 0)$, i.e. $(c, d) \sim (a, b) + (n, 0)$, i.e. $y = x + i(n)$.

(c) Prove that the integers have the completeness property: Let $s \subseteq \mathbb{Z}$ be a nonempty set which is bounded above. Then s has a least upper bound. $\boxed{\text{5 Marks}}$

Since s is nonempty, there exists $x \in \mathbb{Z}$ such that $x \in s$.

Define a set $t \subseteq \mathbb{N}$ as follows: t is the set of natural numbers n, such that $x + i(n)$ is an upper bound for s.

We first prove that t is nonempty. Indeed, since s is bounded above, it has some upper bound $B \in \mathbb{Z}$. Since $x \in s$, $x \leq B$. By (b)(iv) there exists $n \in \mathbb{N}$ such that $B = x + i(n)$. So $n \in t$.

By the well-ordering principle, t has a minimal element, let's call it k. Let us show that $x + i(k)$ is a least upper bound for s.

Firstly, $k \in t$, so by the definition of t, $x + i(k)$ is an upper bound for s.

Now, given an upper bound C for s, we must show that $x + i(k) \leq C$. Indeed, since C is an upper bound for s and $x \in s$, we have that $x \leq C$, so by (b)(iv) there exists $n \in \mathbb{N}$ such that $C = x + i(n)$. By the definition of t, $n \in t$. So, since k is a minimal element of t, $k \leq n$. By (b)(ii) it follows that $i(k) \leq i(n)$, and by (b)(iii) and commutativity it follows that

$$x + i(k) = i(k) + x \leq i(n) + x = x + i(n) = C.$$

$\boxed{\text{Total: 20 Marks}}$

3. (a) Consider the two vectors $\mathbf{a} = 3\hat{\mathbf{e}}_1 - 1\hat{\mathbf{e}}_3$ and $\mathbf{b} = 4\hat{\mathbf{e}}_2$ in $\mathbb{R}^3$.

i. What is the area of the parallelogram spanned by these two vectors. $\boxed{\text{3 Marks}}$

If we note that the two vectors are orthogonal, we find the area simply by the product of the length of the two vectors as

$$A = 4\sqrt{9+1} = 4\sqrt{10}.$$

Alternatively one can calculate the vector product first and then deduce the length of the resulting vector as

$$\mathbf{A} = 4\hat{\mathbf{e}}_1 + 12\hat{\mathbf{e}}_3,$$

and $\boxed{\text{3 Marks}}$

$$A = |\mathbf{A}| = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}.$$

ii. Find an orthonormal basis for the set of vectors lying in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. $\boxed{\text{3 Marks}}$

Noting that the vectors are already orthogonal all we need to do is to normalise them as $\boxed{\text{3 Marks}}$

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{10}}\mathbf{a} = \frac{3}{\sqrt{10}}\hat{\mathbf{e}}_1 - \frac{1}{\sqrt{10}}\hat{\mathbf{e}}_3, \quad \text{and} \quad \hat{\mathbf{b}} = \hat{\mathbf{e}}_2.$$

(b) Consider an object moving along a trajectory $\mathbf{r}(t)$, where $t \in \mathbb{R}$ denotes the time. Assume that the trajectory is given by

$$\mathbf{r}(t) = (a\cos(\omega t), b\sin(\omega t), c\sin(\omega t)),$$

where $\omega, a, b, c \in \mathbb{R}^>$ are real positive constants.

i. Calculate the normalised tangent vector, $\hat{\mathbf{T}} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$, at $t = \frac{\pi}{2\omega}$. $\boxed{\text{3 Marks}}$

Taking the derivative with respect to time of all components we find

$$\dot{\mathbf{r}} = \omega(-a\sin(\omega t), b\cos(\omega t), c\cos(\omega t))).$$

which, at $t = \frac{\pi}{2\omega}$ yields $\boxed{\text{2 Marks}}$

$$\dot{\mathbf{r}}(\frac{\pi}{2\omega}) = (-a\omega, 0, 0),$$

and thus the normalised tangent vector at this point is given by $\boxed{\text{1 Mark}}$

$$\hat{\mathbf{T}} = -\hat{\mathbf{e}}_1.$$

ii. Provide an equation for the tangent line of the trajectory at the point with $t = \frac{\pi}{2\omega}$. $\boxed{\text{2 Marks}}$

The tangent line goes through the point $\mathbf{r}(\frac{\pi}{2\omega})$ and in the direction of $\hat{\mathbf{T}}$ and can be parameterised as $\boxed{\text{2 Marks}}$

$$\mathbf{x}(\lambda) = \mathbf{r}(\frac{\pi}{2\omega}) + \lambda\hat{\mathbf{T}}.$$

We have

$$\mathbf{r}(\frac{\pi}{2\omega}) = (0, b, c),$$

and thus explicitly

$$\mathbf{x}(\lambda) = (-\lambda, b, c).$$

(Full marks also for any other valid parameterisation.)

iii. Assuming Newton's law $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$, deduce the force $\mathbf{F}(\mathbf{r})$ acting on the particle. 2 Marks

We have 1 Mark

$$\mathbf{F} = m\ddot{\mathbf{r}} = m\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}} = -m\omega^2(a\cos(\omega t), b\sin(\omega t), c\sin(\omega t)))$$

And thus 1 Mark

$$\mathbf{F}(\mathbf{r}) = -m\omega^2\mathbf{r}.$$

(c) Consider the vector space V of real polynomials of degree 4 or lower.

i. Give an example of a basis of this vector space. What is the dimension of this vector space? 3 Marks

An example is the basis 2 Marks

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x \\ B_2(x) &= x^2 \\ B_3(x) &= x^3 \\ B_4(x) &= x^4. \end{aligned}$$

The dimension of this vector space is five. 1 Mark

ii. For each of the maps T below, state whether it is a linear map on V, and provide a reason for your answer.

A. $T : P(x) \mapsto P(x) + 5x$ 2 Marks

The map is not linear, we can check directly that

$$T(aP_1(x) + bP_2(x)) = aP_1(x) + bP_2(x) + 5x,$$

but this is in general not the same as

$$aT(P_1(x)) + bT(P_2(x)) = aP_1(x) + bP_2(x) + 5(a+b)x.$$

2 Marks

Alternatively you can simply notice that $T(0) = 5x \neq 0$.

B. $T : P(x) \mapsto \frac{\mathrm{d}}{\mathrm{d}x}P(x)$ 2 Marks

This is a linear map on our vector space (if $P(x)$ is in our vector space, so is $\frac{\mathrm{d}}{\mathrm{d}x}P(x)$), which we can verify directly using the properties of differentiation (note that a and b are not functions of x):

$$T(aP_1(x) + bP_2(x)) = a\frac{\mathrm{d}}{\mathrm{d}x}P_1(x) + b\frac{\mathrm{d}}{\mathrm{d}x}P_2(x) = aT(P_1(x)) + bT(P_2(x)).$$

2 Marks

Total: 20 Marks