IMPERIAL

MATH70028

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Probability Theory

Date: Friday, May 10, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Let $(\Omega, P, \mathcal{F})$ be a probability space. Give the definition of a random variable ζ, its distribution function, and its characteristic function. (5 marks)

   (b) What properties does a distribution function $F(x)$, $x \in \mathbb{R}$, satisfy? (5 marks)

   (c) Let η be $N(0,1)$ (normal random variable with mean 0 and variance 1), and ν be a random variable independent of η and taking the values 1 and -1 each with the probability $1/2$. Show that the product $\eta\nu$ is $N(0,1)$. (10 marks)

2. (a) State the strong law of large numbers. (4 marks)

   (b) State and prove the weak law of large numbers for uncorrelated random variables with uniformly bounded variance. (6 marks)

   (c) Let ζ_j, $j = 1, \ldots, n$, be independent random variables uniformly distributed over $(-1, 1)$. Show that the random variable $\zeta_1 + \cdots + \zeta_n$, $n \geq 2$, has density

$$f(x) = \frac{1}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^n \cos(tx) dt.$$

(10 marks)

3. (a) Define what it means for a family of probability measures to be tight. (5 marks)

   (b) Let ζ_j, $j = 1, 2, \ldots$, be a tight sequence of random variables, and a sequence of positive real numbers $c_j > 0$, $j = 1, 2, \ldots$ be such that $c_n \to 0$ as $n \to \infty$. Show that the sequence $c_n\zeta_n$ converges to zero in probability. (10 marks)

   (c) Show that for any positive random variable ζ and any $p > 0$,

$$\mathbb{E}\left(\frac{1}{\zeta^p}\right) \geq \frac{1}{\mathbb{E}(\zeta)^p}.$$

(5 marks)

4. (a) State the central limit theorem for i.i.d. random variables. (5 marks)

   (b) Let $\zeta_1, \zeta_2, \ldots$ be an infinite sequence of i.i.d. random variables. What values can the probability of the event $\{\sum_{j=1}^\infty \zeta_j$ converge $\}$ assume? No justification is needed. (5 marks)

   (c) Let $\zeta, \zeta_1, \zeta_2, \ldots$ be random variables such that ζ_n is independent with ζ for each n and ζ_n converges to ζ in probability. Prove that ζ is degenerate, that is $\mathbb{P}(\zeta = a) = 1$ for some constant a. (10 marks)

5. Let random variables ζ_n, $n = 1, 2, \ldots$, have characteristic functions ϕ_n. Prove the following version of the continuity theorem. Suppose $\phi_n(t) \to f(t)$ as $n \to \infty$ for all $t \in \mathbb{R}$, where $f(t)$ is continuous at zero. Then ζ_n, $n = 1, 2, \ldots$, has a subsequence converging in distribution to a random variable.

(20 marks)

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Solutions

1a. A function $f:\Omega\to\mathbb{R}$ is called a random variable if it is $\mathbb{P}$-measurable.

The distribution function

$$F(x) = P(\omega\in\Omega : f(\omega)\le x) \quad \forall x\in\mathbb{R}.$$

The characteristic function

$$\varphi(t) = \mathbb{E}e^{itf} = \int_{-\infty}^{\infty} e^{itx}\,dF$$

[5 marks]

1b F(x) is

nondecreasing;

continuous on the right $\forall x\in\mathbb{R}$;

$$\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to+\infty} F(x) = 1$$

[5 marks]

1c. $P(\vartheta\eta \le x)$

$= P(\vartheta = 1, \eta \le x) + P(\vartheta = -1, \eta \ge -x)$

$= \frac{1}{2}\left(P(\eta \le x) + P(\eta \ge -x)\right)$

[By independence]

$= P(\eta \le x)$

[By the symetry of the normal distribution].

Thus, $\vartheta\eta$ has the same distribution as η, i.e. $N(0,1)$.

[10 marks]

2a. Let $\xi_1, \xi_2 \dots$ be i.i.d.
with $\mathbb{E}|\xi_1| < \infty$.
Then $S_n/n \xrightarrow{a.s.} E\xi_1$. [4 marks]

2b. Thm Let $\xi_1, \xi_2 \dots$ be uncorrelated integrable r.v. with $V(\xi_j) \le C$, $j=1,2\dots$ for some $C>0$.
Then $\frac{S_n^c}{n} = \frac{1}{n}\sum_1^n (\xi_j - E\xi_j) \xrightarrow{prob} 0$.

Proof Fix $\varepsilon > 0$.

$$P\left(\left|\frac{S_n^c}{n}\right| \ge \varepsilon\right) \le \frac{V\left(\frac{S_n^c}{n}\right)}{\varepsilon^2}$$

(By chebyshev ineq.)

$$= \frac{1}{n^2\varepsilon^2}\sum_1^n V(\xi_j)$$

(Since ξ_j are uncorrelated)

$$\le \frac{C}{n\varepsilon^2} \to 0, \quad n \to \infty.$$

[6 marks]

2c. The characteristic function of ξ_j

$$\varphi_j(t) = \frac{1}{2}\int_{-1}^{1} e^{itx}\,dx = \frac{e^{it} - e^{-it}}{2it}$$

$$= \frac{\sin t}{t}.$$

Since ξ_j are independent the characteristic function of $\xi = \xi_1 + \dots + \xi_n$

$$\varphi(t) = \prod_1^n \varphi_j(t) = \left(\frac{\sin t}{t}\right)^n.$$

By the inversion thm, since

$$\int_{-\infty}^{\infty} |\varphi(t)|\,dt = \int_{-\infty}^{\infty} \left|\frac{\sin t}{t}\right|^n dt < \infty$$

for $n \geq 2$,

ξ is a.c. with density

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\,\varphi(t)\,dt =$$

$$= \frac{1}{2\pi}\int_0^{\infty} (e^{itx} + e^{-itx})\,\varphi(t)\,dt$$

$$= \frac{1}{\pi}\int_0^{\infty} \cos(tx)\,\varphi(t)\,dt.$$

[10 marks]

3a. A family of probability measures $\{P_\alpha, \alpha \in A\}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called tight if $\forall \varepsilon > 0$ there is a compact set $K \subset \mathbb{R}$ such that $\sup_{\alpha \in A} P_\alpha(\mathbb{R} \setminus K) \leq \varepsilon$. [5 marks]

3b. Let $\varepsilon > 0$. Since $\{\xi_n\}$ is tight, there is $R = R(\varepsilon)$ s.t. $P(|\xi_n| > R) \leq \varepsilon$ $\forall n$

Fix $\delta > 0$. Consider

$$P(|c_n \xi_n| \geq \delta) = P(|\xi_n| \geq \delta / c_n)$$

$$\leq P(|\xi_n| > R) \leq \varepsilon$$

for all $n > N$, where $N = N(R(\varepsilon))$ is chosen s.t. $\delta / c_N > R$ (possible since $c_n \to 0$).

Thus $P(|c_n \xi_n| \geq \delta) \to 0$, $n \to \infty$.

[10 marks]

3c. $(x^{-p})'' = -p\,(x^{-p-1})'$

$$= p(p+1)\,x^{-p-2} > 0, \quad x > 0$$

So $\frac{1}{x^p}$ is convex.

Therefore, by Jensen inequality,

$$\frac{1}{(E\xi)^p} \leqslant E\left(\frac{1}{\xi^p}\right).$$

[5 marks]

4a. Let $\xi_1, \xi_2 \dots$ – i.i.d.

with $E\xi_1^2 < \infty$ and nondegenerate.

Let $S_n = \xi_1 + \dots + \xi_n$.

[5 marks]

Then $\frac{S_n - ES_n}{\sqrt{VS_n}} \xrightarrow{d} N(0,1)$.

4b. 0 or 1. [5 marks]

4c. We show that ξ is independent with itself, i.e. $P(\xi \leq x)^2 = P(\xi \leq x)$.

Then $F_\xi(x)$ is either 1 or 0 so that ξ is degenerate.

Let $\varepsilon > 0$

$$P(\xi_n \leq x, \xi \leq x+\varepsilon) \overset{\text{by independence}}{=} P(\xi_n \leq x) P(\xi \leq x+\varepsilon)$$

$$\longrightarrow P(\xi \leq x) P(\xi \leq x+\varepsilon)$$

$\forall x \in C_{F_\xi}$ since convergence in probability implies convergence in distribution.

On the other hand,

$$P(\xi_n \le x, \xi \le x+\varepsilon) + P(\xi_n \le x, \xi > x+\varepsilon)$$

$$= P(\xi_n \le x) \to P(\xi \le x)$$

and

$$P(\xi_n \le x, \xi > x+\varepsilon) \leqslant P(|\xi_n - \xi| > \varepsilon) \to 0.$$

We conclude that

$$P(\xi \le x) P(\xi \le x+\varepsilon) = P(\xi \le x)$$

Taking $\varepsilon \to 0$ s.t. $x+\varepsilon \in C_{F_\xi}$,

we obtain the result.

[10 marks]

5. We show that $\{\xi_n\}$ is tight. Then the statement immediatly follows by Prokhorov's thm.

By Fubini's thm,

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt = \int_{-\infty}^{\infty}\frac{1}{u}\int_{-u}^{u}(1-e^{itx})\,dt\,dF_n$$

$$= 2\int_{-\infty}^{\infty}\left(1-\frac{\sin ux}{ux}\right)dF_n$$

$$\geq 2\int_{|x|\geq 2/u}\left(1-\frac{1}{|ux|}\right)dF_n$$

$$\geq P(|\xi_n|\geq \tfrac{2}{u})$$

Since $f(t)$ is continuous at zero and $f(0)=1$, for an $\varepsilon>0$ there is u s.t.

$$\frac{1}{u}\int_{-u}^{u}(1-f(t))\,dt < \varepsilon .$$

Since $\varphi_n(t)\to f(t)$ $\forall t$, by dominated converpence, there exists N s.t.

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt < 2\varepsilon\ ,\quad \forall n>N$$

Thus

$$P(|\xi_n| \geq \tfrac{2}{u}) < 2\varepsilon \quad \forall n > N$$

Decreasing u if needed, we obtain

$$P(|\xi_n| \geq 2/u) < 2\varepsilon \quad \forall n.$$

Thus $\{\xi_n\}$ is tight.

[20 marks]

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Question	Marker's comment
1	sometimes product mixed up with sum while applying characteristic functions (which is not necessary) in 1c
2	2a,b : care should be taken formulating conditions for theorems, 2c - conditions for applying inversion theorem should be mentioned
3	some problems formulating proof of 3b; in 3c convexity should be used
4	proof of 4c is the highlight of the question, sometimesnbsp; there were deficiencies

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Question	Marker's comment
1	Comments for Q1-Q4 see the non-master exam
5	In the second part of the proof more care should be given to the issue of how to use continuity of characteristic functionsnbsp;