3.6 Dimension

Lemma 3.6.1. *Steinitz Exchange Lemma*
Let V be a vector space over F. Take $X \subseteq V$ and suppose $u \in Span(X)$ but $u \notin Span(X \setminus \{v\})$ for some $v \in X$. Let $Y = (X \setminus \{v\}) \cup \{u\}$ (i.e., we "exchange v for u"). Then $Span(X) = Span(Y)$.

Proof

Since $u \in Span(X)$ we have $\alpha_1, ..., \alpha_n \in F$ such that $v_1, ..., v_n \in X$ such $u = \alpha_1 v_1 + ... + \alpha_n v_n$. Now there is a $v \in X$ such that $u \notin Span(X \setminus \{v\})$ we may assume, WLOG, that $v = v_n$, thus $\alpha_n \neq 0$ so:

$$v = v_n = \frac{1}{\alpha_n}(u - \alpha_1 v_1 ... - \alpha_{n-1} v_{n-1})$$

Now if $w \in Span(Y)$ then for some $\beta_0, \beta_1, ..., \beta_m$ we have $v_1, ..., v_m \in X \setminus \{v\}$

$$\begin{aligned} w &= \beta_0 u + \textstyle\sum_{i=0}^{m} \beta_i v_i \\ &= \beta_0(\alpha_1 v_1 + ... + \alpha_n v_n) + \textstyle\sum_{i=0}^{m} \beta_i v_i \quad \in Span(X \setminus \{v\} \cup \{v\}) = Span(X) \end{aligned}$$

So $Span(Y) \subseteq Span(X)$.

Similarly we have that if $w \in Span(X)$ the w is a linear combination of elements of X, now we can replace v_n with $\frac{1}{\alpha_n}(u - \alpha_1 v_1 ... - \alpha_{n-1} v_{n-1}$ so we can express w as a linear combination of elements of Y. So $Span(X) \subseteq Span(Y)$, thus $Span(Y) = Span(X)$.

This lemma is essential to being able to define the dimension of a vector space - and relies on being able to invert elements in the field.

Exercise 3.6.2. Verify the Steinitz exchange lemma where:

- $V = \mathbb{R}^3$
- $X = \{e_1, e_2\}$
- $u = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

Theorem 3.6.3. Let V be a finite dimensional vector space over F. Let S, T be finte subsets of V. If S is LI and T spans V then $|S| \leq |T|$. That is, LI sets are at most as big as spanning sets.

Proof: Assume S is LI and T spans V and suppose:

$$S = \{s_1, ..., s_m\}$$
$$T = \{t_1, ..., t_n\}$$

Let $T = T_0$, since $Span(T_0) = V$ there is some I such that $s_1 \in Span(\{t_1, ..., t_i\})$, but $s_1 \notin Span(\{t_1, ..., t_{i-1}\})$.

Thus by SEL $Span(\{s_1, t_1, ..., t_{i-1}\}) = Span(\{t_1, ..., t_i\})$.

Let $T_1 = \{s_1, t_2, ..., t_{i-1}, t_{i+1}, ..., t_n\}$, then we have $Span(T_1) = Span(T_0) = V$. We continue this process inductively.

Suppose that for some j with $1 \leq j \leq m$ we have $T_j = \{s_1, ..., s_j, t_{i_1}, ..., t_{i_{n-j}}\}$, with $Span(T_j) = Span(T)$, and $t_{i_k} \in T$.

Now $s_{j+1} \in Span(T_j)$ so there is an i_k such that $s_{j+1} \in Span(\{s_1, ..., s_j, t_{i_1}, ..., t_{i_k}\})$, but $s_{j+1} \notin Span(\{s_1, ..., s_j, t_{i_1}, ..., t_{i_{k-1}}\})$.

Note S is LI so $s_{j+1} \notin Span(\{s_1, ..., s_j\})$ i.e. $t_{i_k} \in T$.

We let $T_{j+1} = \{s_1, ..., s_{j+1}, t_{i_1}, ..., t_{i_{k-1}}, t_{i_k}, ..., t_{i_{n-j}}\}$ and by SEL we have $Span(T_{j+1}) = Span(T_j) = Span(T) = V$, by relabeling the elements of T_{j+1} we can see we have a set of the form:

$$T_{j+1} = \{s_1, ..., s_{j+1}, t_{i_1}, ..., t_{i_{n-(j+1)}}$$

After j steps we have replaced j members of T with j members of S. We cannot run out of members of T before we run out of members of S; as otherwise a remaining element of S would be a linear combination of the elements of S already swapped into T, thus $m \leq n$.

Corollary 3.6.4. Let V be a finite dimensional vector space. Let S, T be bases of V, then S and T are both finite and $|S| = |T|$.

Proof: Since V is finite dimensional it has a finite basis B say. Suppose $|B| = n$. Now B is a spanning set and $|B| = n$ so by Theorem 3.6.3 any LI subset has size at most n.

Since S is LI we get $|S| \leq n$, similarly $|T| \leq n$ - so both sets are finite.

Also we have S is spanning and T is LI, so $|T| \leq |S|$, also T is spanning and S is LI, so $|S| \leq |T|$. Thus $|S| = |T|$.

Definition 3.6.5. Let V be a finite dimensional vector space. The *dimension of* V, written $\dim V$, is the size of any basis of V.

Remark 3.6.6 *Note that we needed Corollary 3.6.4 and thus the SEL to know that the size of a basis is unique (a basis certainly isn't).*

Example 3.6.7. In PS2 you were asked to describe all the subspaces of $\mathbb{R}^3$ this becomes much easier once we know about dimensions.$\mathbb{R}^3$ is an $\mathbb{R}$ vector space of dimension 3.

As subspaces are vector spaces in their own right so they also have dimensions, and these must be less than or equal to 3:

- $\dim 3$: the only subspace of dimension 3 is $\mathbb{R}^2$
- $\dim 2$: planes going through the origin
- $\dim 1$: lines going through the

- $\dim 0$: $\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

Lemma 3.6.8. Suppose that $\dim V = n$:

1. Any spanning set of size n is a basis.
2. Any linearly independent set of size n is a basis.
3. S is a spanning set if and only if it contains a basis (as a subset).
4. S is linearly independent if and only if it is contained in a basis (i.e. it's a subset of a basis).
5. Any subset of V of size $> n$ is linearly dependent.

Proof: Exercise.

3.7 More subspaces

Definition 3.7.1. Let V be a vector space, U and W be subspaces of V.

- The *intersection of* U *and* W is:

$$U \cap W = \{v \in V : v \in W \text{ and } v \in U\}$$

- The *sum of* U *and* W is:

$$U + W = \{u + w : u \in U, w \in W\}$$

Remark 3.7.2. $U \subseteq U + W$ and $W \subseteq U + W$. This is because $0 \in U$ and $0 \in W$, so for every $u \in U$, $u = u + 0 \in U + W$. Similarly, for every $w \in W$, $w = 0 + w \in U + W$

Example 3.7.3. Let $V = \mathbb{R}^2$ over $\mathbb{R}$, $U = Span\{(1, 0)\}$, $W = Span\{(0, 1)\}$. *Claim* $U+W = \mathbb{R}^2$.

Proof: Let $(\lambda, \mu) \in \mathbb{R}^2$ then $(\lambda, 0) \in U$, $(0, \mu) \in W$ so

$$(\lambda, \mu) = (\lambda, 0) + (0, \mu) \in U + W$$

Exercise 3.7.4. Let U and W be subspaces of V an F-vector space. Then $U + W$ and $U \cap W$ are subspaces of V.

Proof:

1. $U + W$ is a subspace: Clearly $U + W \subset V$, so we can apply the subspace test:
   - $0 \in U$ and $0 \in W$ so $0 + 0 = 0 \in U + W$.
   - Suppose $v_1, v_2 \in U + W$ then $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ for some $u_i \in U$ and $w_i \in W$. Consider

     $$\begin{aligned} v_1 + v_2 &= (u_1 + w_1) + (u_2 + w_2) \\ &= \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W} \quad \begin{array}{l} + \text{ in } V \text{ is commutative and associative} \\ U, W \text{ closed under } + \end{array} \end{aligned}$$

     So $v_1 + v_2 \in U + W$
   - Let $\lambda \in \mathbb{R}$ and $v \in U + W$ then $v = u + w$ for some $u \in U$ and $w \in W$. Consider

     $$\begin{aligned} \lambda v &= \lambda(u + w) \\ &= \underbrace{\lambda u}_{\in U} + \underbrace{\lambda w}_{\in W} \quad \begin{array}{l} \text{by distributivity in } V \\ U, W \text{ closed under scalar} \times \end{array} \end{aligned}$$

     So $\lambda v \in U + W$
2. $U \cap W$ is a subspace. Exercise.

Proposition 3.7.5. Let V be a vector space over F. Let U and W be subspaces of V, suppose additionally:

- $U = Span\{u_1, ..., u_s\}$
- $W = Span\{w_1, ..., w_r\}$

Then $U + W = Span\{u_1, ..., u_s, w_1, ..., w_r\}$.

Proof:

1. Show $U + W \subseteq Span\{u_1, ..., u_s, w_1, ..., w_r\}$. Let $v \in U + W$ then $u = u + w$ for some $u \in U$ and $w \in W$. Therefore:
   - $u = \lambda_1 u_1 + ... + \lambda_s u_s$
   - $w = \mu_1 w_1 + ... + \mu_r w_r$

   So $v = \lambda_1 u_1 + ... + \lambda_s u_s + \mu_1 w_1 + ... + \mu_r w_r \in Span\{u_1, ..., u_s, w_1, ..., w_r\}$
2. Show $Span\{u_1, ..., u_s, w_1, ..., w_r\} \subseteq U + W$. Suppose $v \in Span\{u_1, ..., u_s, w_1, ..., w_r\}$ then:
   $$v = \underbrace{\lambda_1 u_1 + ... + \lambda_s u_s}_{\substack{\in Span\{u_1, ..., u_s\} \\ = U}} + \underbrace{\mu_1 w_1 + ... + \mu_r w_r}_{\substack{\in Span\{w_1, ..., w_r\} \\ = W}}$$

   So $v \in U + W$.

   Alternatively:

   - $u_i \in U \subseteq U + W$ for each $i \in \{1, ..., s\}$
   - $w_i \in W \subseteq U + W$ for each $i \in \{1, ..., r\}$

   So $\{u_1, ..., u_s, w_1, ..., w_r\} \in U + W$ so $Span\{u_1, ..., u_s, w_1, ..., w_r\} \in U + W$. As $u + W$ is closed under linear combinations.

Example 3.7.6. Let $V = \mathbb{R}^2$, let $U = Span\{(0, 1)\}$, $W = Span\{(1, 0)\}$. Then by proposition 3.7.5 we have $U + W = Span\{(0, 1), (1, 0)\} = \mathbb{R}^2$. Agrees with example 3.7.3.

Example 3.7.7. Let $V = \mathbb{R}^3$ and:
Let $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$
Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -x_1 + 2x_2 + x_3 = 0\}$
Question: Find bases for $U, W, U \cap W, U + W$.

Answer:

- A general vector in $u \in U$ is of the form $u = (a, b, -a-b)$ for $a, b \in \mathbb{R}$. So $u = a(1, 0, -1) + b(0, 1, -1)$, therefore $\{(1, 0, -1), (0, 1, -1)\}$ is a spanning set for U, and as the vectors are linearly independent this is a basis for U.

- A general vector in $w \in W$ is of the form $w = (2a+b, a, b)$ for $a, b \in \mathbb{R}$. So $u = a(2,1,0) + b(1,0,1)$, therefore $\{(2,1,0),(1,0,1)\}$ is a basis for W, as they are clearly linearly independent.

- By proposition **??** we know that $\{(1,0,-1),(0,1,-1),(2,1,0),(1,0,1)\}$ is a spanning set for $U+W$, this is clearly not linearly independent, so we do row reduction to get an LI set:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

  So a linearly independent spanning set is $\{(1,0,0),(0,1,0),(0,0,1)\}$. So $dim(U+W) = 3$ so as $U+W \subseteq \mathbb{R}^3$ we have $U+W = \mathbb{R}^3$.

- We want a basis for $U \cap W$. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We have:
  $x \in U$ iff $x_1 + x_2 + x_3 = 0$
  $x \in W$ iff $-x_1 + 2x_2 + x_3 = 0$
  So $x \in U \cap W$ iff $x_1 + x_2 + x_3 = -x_1 + 2x_2 + x_3 = 0$ (i.e. $U \cap W = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \text{ and } -x_1 + 2x_2 + x_3 = 0\}$)

  That is to say $2x_1 - x_2 = 0$, so $x_2 = 2x_1$, and therefore $x_3 = -x_1 - x_2 = -3x_1$. So x is of the form $(x_1, 2x_1, -3x_1)$. So a spanning set for $U \cap W$ is $\{(1,2,-3)\}$ which is clearly a basis.

Remark 3.7.8. A neater way of finding a basis for $U+W$ would have been to use the basis for $U \cap W$. Since $U \cap W \subset U$ we can find a basis for U containing out basis for $U \cap W$ and similarly for W. The union of these bases will be a basis for U_W.

For instance, a basis for U is $\{(1,0,-1),(1,2,-3)\}$, and a basis for W is $\{(1,0,1),(1,2,-3)\}$, so a basis for $U+W$ is $\{(1,0,1),(1,0,-1),(1,2,-3)\}$. Note that this has three elements, and $dim(U+W) = 3$ so as this is a spanning set it must be a basis.

Theorem 3.7.9. Let V be a vector space over F, U and W subspaces of V. Then

$$dim(U+W) = dimU + dimW - dim(U \cap W).$$

Proof: Suppose $dim(U \cap W) = m$, $dimU = r$ and $dimW = s$ (so we need to prove that $dim(U+W) = r+s-m$).

Now as $dim(U \cap W) = m$ we have a basis $B_{U \cap W}\{v_1, ..., v_m\}$ of $U \cap W$. Now as $U \cap W \subseteq U$ and $B_{U \cap W}$ is linearly independent it is contained in a basis $B_U = \{v_1, ..., v_m, u_{m+1}, ..., u_r\} \supseteq B_{U \cap w}$. Simiarly we have a basis $B_W = \{v_1, ..., v_m, w_{m+1}, ..., w_s\}$ containing $B_{U \cap W}$.

Claim $B_U \cup B_W = \{v_1, ..., v_m, u_{m+1}, ..., u_r, w_{m+1}, ..., w_s\}$ is a basis for $V+W$.
Proof of Claim:

Span: By proposition **??** $B_U \cup B_W$ is a spanning set.

LI: Suppose we have:

$$\lambda_1 v_1 + ... + \lambda_m v_m + \mu_{m+1} u_{m+1} + ... + \mu_r u_r + \nu_{m+1} w_{m+1} + ... + \nu_s w_s = 0$$

For $\lambda_i, \mu_i, \nu_i \in F$. [We need to show $\lambda_i = \mu_j = \nu_k = 0$ for all $i.j, k$.]

Now we have

$$\underbrace{\lambda_1 v_1 + ... + \lambda_m v_m + \mu_{m+1} u_{m+1} + ... + \mu_r u_r}_{\in U} = \underbrace{-\nu_{m+1} w_{m+1} - ... - \nu_s w_s}_{\in W}$$

Thus $\lambda_1 v_1 + ... + \lambda_m v_m + \mu_{m+1} u_{m+1} + ... + \mu_r u_r \in U \cap W$. So $\lambda_1 v_1 + ... + \lambda_m v_m + \mu_{m+1} u_{m+1} + ... + \mu_r u_r = \beta_1 v_1 + ... \beta_m v_m$ for some $\beta_i \in F$. Thus

$$\beta_1 v_1 + ... \beta_m v_m + \nu_{m+1} w_{m+1} + ... + \nu_s w_s = 0$$

As $\{v_1, ..., v_m, w_{m+1}, ..., w_s\}$ is a basis for W (thus linearly independent) we have $\beta_1 = ... = \beta_m = \nu_{m+1} = ... \nu_s = 0$.

Thus $\lambda_1 v_1 + ... + \lambda_m v_m + \mu_{m+1} u_{m+1} + ... + \mu_r u_r = 0$. As $\{v_1, ..., v_m, u_{m+1}, ... u_r\}$ is a basis for U we have $\lambda_1 = ... = \lambda = \mu_{m+1} = ... \mu_r = 0$.

So $\lambda_i = \mu_j = \nu_k = 0$ for all $i.j, k$, so $B_U \cup B_W$ is linearly independent.

$B_U \cup B_W$ is a spanning set for $U + W$ and is linearly independent thus it is a basis.

Now $|B_U \cap B_W| = r + s - m$, thus $dim(U + W) = r + s - m$.

□