Fluid Dynamics II, Chapter -1: Some useful vector identities

Most of this sheet is revision from 2nd year multivariable calculus. We begin by recalling the suffix notation for vectors and tensors, and then present some vector identities, which we write both in vector notation and suffix (tensor) notation. In different circumstances each is useful, and you should try to be at ease switching from one to another.

A vector, $\mathbf{u}$, has by default three components u_1, u_2 and u_3. We write u_i as an alternative representation of $\mathbf{u}$, where the index i is allowed to take the values 1, 2 and 3 in turn. We represent three-dimensional space by the Cartesian coordinates (x_1, x_2, x_3) so that $\mathbf{x}$ or equivalently x_i is a general position vector. Thus $\nabla\phi$ can be written $\partial\phi/\partial x_i$ in suffix notation.

The summation convention: in any product of quantities any index can appear once or twice. If only once, it takes all three values in turn. If it appears twice, then it is treated as a **dummy index to be summed over**. Thus $a_ib_i = a_1b_1 + a_2b_2 + a_3b_3 = \mathbf{a}\cdot\mathbf{b}$, $\partial u_i/\partial x_i = \nabla\cdot\mathbf{u}$ and $a_ib_ja_i$ is equivalent to $|\mathbf{a}|^2\mathbf{b}$. You must have this idea clear in your mind. Never repeat an index three times in a single product. **In any equation each product must have exactly the same 'free' indices.**

Two special tensors: Recall the Kronecker delta δ_{ij} and the alternating tensor ε_{ijk}.

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if} \quad ijk = 123,\ 231,\ 312 \\ -1 & \text{if} \quad ijk = 213,\ 132,\ 321 \\ 0 & \text{otherwise} \end{cases} \tag{-1.1}$$

We can then see by direct calculation that, for example,

$$(\nabla\times\mathbf{u})_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \delta_{ij}u_j = u_i \qquad \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\ . \tag{-1.2}$$

Some vector identities:

$$\nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \nabla\phi\cdot\mathbf{F} \qquad \frac{\partial}{\partial x_i}(\phi F_i) = \phi\frac{\partial F_i}{\partial x_i} + \frac{\partial\phi}{\partial x_i}F_i \tag{-1.3}$$

If we write

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} \qquad \omega_i = \varepsilon_{imn}\frac{\partial u_n}{\partial x_m}, \tag{-1.4}$$

then

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\tfrac{1}{2}|\mathbf{u}|^2\right) + \boldsymbol{\omega}\times\mathbf{u} \qquad u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_ju_j\right) + \varepsilon_{ijk}\omega_ju_k\,, \tag{-1.5}$$

$$\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\boldsymbol{\omega} \qquad \frac{\partial^2 u_i}{\partial x_j\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \varepsilon_{ijk}\frac{\partial\omega_j}{\partial x_k} \tag{-1.6}$$

and

$$\mathbf{u}\cdot(\nabla\times\boldsymbol{\omega}) = \nabla\cdot(\boldsymbol{\omega}\times\mathbf{u}) - |\boldsymbol{\omega}|^2 \qquad u_i\varepsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\varepsilon_{ijk}\omega_ku_i\right) - \omega_k\omega_k\ . \tag{-1.7}$$

The divergence theorem: this is perhaps the most important result in all mathematics. We shall use it in the following form. If V is a volume enclosed by a surface S with outward unit normal $\mathbf{n}$, and T is **any** tensor (i.e. T could be a scalar, a vector, a second order tensor…), then

$$\boxed{\int_V \frac{\partial T}{\partial x_j}\, dV = \int_S n_j T\, dS} \qquad (-1.8)$$

If we replace T by the vector u_j, for example, we obtain the ordinary divergence theorem,

$$\int_V \nabla\cdot\mathbf{u}\, dV = \int_S \mathbf{n}\cdot\mathbf{u}\, dS$$

If we replace T by the scalar p, we find

$$\int_V \nabla p\, dV = \int_S p\,\mathbf{n}\, dS \qquad (-1.9)$$

A few facts about tensors: We say a second order tensor σ_{ij} is symmetric if $\sigma_{ij} = \sigma_{ji}$, or antisymmetric if $\sigma_{ij} = -\sigma_{ji}$.

Any second order tensor a_{ij} can be written as the sum of a symmetric tensor e_{ij} and an antisymmetric tensor Ω_{ij}. In particular, the **velocity gradient tensor**

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \Omega_{ij} \qquad \text{where} \quad e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \Omega_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \quad (-1.10)$$

The symmetric part e_{ij} is known as the strain rate tensor. Note that $e_{ii} = \nabla\cdot\mathbf{u}$. The antisymmetric part Ω_{ij} can be related to the vorticity, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$.

A **symmetric** tensor has real eigenvalues and its eigenvectors are mutually orthogonal. By a change of basis (rotation of the coordinate axes) it can be **diagonalised**, i.e. in a suitable coordinate system

$$e_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{ij} \qquad (-1.11)$$

The axes are then said to point along the **principal axes** of the tensor e_{ij}. These axes always have physical significance.

An antisymmetric tensor has only 3 independent components, and can be related to a vector. Thus the antisymmetric part of the velocity gradient is related to the vorticity, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$:

$$-2\Omega_{ij} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}_{ij} = \varepsilon_{ijk}\omega_k \ . \qquad (-1.12)$$

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We do not require knowledge of "Fluid Dynamics I" in this module, but we will take some of its results as given. We assume that at every point $\mathbf{x}$ of the fluid, and at all times t, we can define properties like density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, and pressure $p(\mathbf{x}, t)$, and that these vary smoothly (differentiably) over the fluid. We do not deal with the dynamics of individual molecules. A small volume δV thus has mass $\delta V \rho$ and momentum $\delta V \rho \mathbf{u}$.

The material derivative: A *fluid particle*, sometimes called a *material element*, is one that moves with the fluid, so that its velocity is $\mathbf{u}(\mathbf{x}, t)$ and its position $\mathbf{x}(t)$ satisfies $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$. The rate of change of a quantity as seen by a fluid particle is called the *material derivative* and written D/Dt. It is given by the chain rule as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{0.1}$$

Mass conservation:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0. \tag{0.2}$$

For an *incompressible* fluid, the density of each material element is constant, and

Incompressible flow:

$$\frac{D\rho}{Dt} = 0 \qquad \Longrightarrow \qquad \nabla\cdot\mathbf{u} = 0. \tag{0.3}$$

In this course we shall concentrate on fluids that are incompressible and have uniform density, so that ρ is independent of both $\mathbf{x}$ and t.

Momentum equation: we need to include both volume forces $\mathbf{F}$ (perhaps gravity, $\mathbf{F} = \rho\mathbf{g}$) and surface forces. Surface forces depend on the stress tensor:

The stress tensor: A fluid exerts a force on a surface element of area δS and normal (n_1, n_2, n_3) of size $\sigma_{ij}n_j$ where σ_{ij} is called the **stress tensor**. (In Fluid Dynamics I, Prof Wu may have used p_{ij}). Conservation of momentum then takes the form

$$\rho\frac{Du_i}{Dt} = F_i + \frac{\partial\sigma_{ij}}{\partial x_j}. \tag{0.4}$$

For a **Newtonian Fluid**,

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij} \qquad \text{where} \quad e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{0.5}$$

The constant μ is known as the **viscosity** of the fluid, while $p(\mathbf{x}, t)$ is the fluid pressure.

The Navier-Stokes Equations for an incompressible fluid

$$\rho\frac{D\mathbf{u}}{Dt} \equiv \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{F} + \mu\nabla^2\mathbf{u}\ . \tag{0.6}$$

$$\nabla\cdot\mathbf{u} = 0\ . \tag{0.7}$$

In this course, we shall take these equations as our starting point.

Boundary Conditions

In order to determine the velocity $\mathbf{u}(\mathbf{x},\, t)$ and pressure $p(\mathbf{x},\, t)$ in some region V bounded by a surface S, we need to know what boundary conditions to apply on S. The appropriate conditions to apply are that the velocity and the total stress should be continuous across any interface. Here 'total stress' includes stresses from other physical processes such as **surface tension** (see below.)

(a) Fluid/solid boundaries: A solid boundary can provide whatever stress is needed to support the fluid motion, so it is sufficient to require that the fluid velocity $\mathbf{u}$ be the same as the velocity of the boundary. Thus for a stationary boundary

$$\mathbf{u} = 0\ . \tag{0.8}$$

Note that (0.8) requires that the tangential velocity components be zero as well as the normal component. In **inviscid flow** only the normal velocity need be continuous at an interface, and a 'slip velocity' must be permitted. The presence in the Navier-Stokes equation of the second derivative $\mu\nabla^2\mathbf{u}$ requires an extra boundary condition.

The continuity in stress enables the force on the solid boundary due to the fluid to be calculated, by integrating the stress over the boundary

$$F_i = -\int_S \sigma_{ij} n_j \, dS = \int_S \big(p n_i - 2\mu\, e_{ij} n_j\big) dS. \tag{0.9}$$

The pressure acts normal to the boundary, while the viscous stress has both normal and tangential components.

(b) Fluid/fluid boundaries: These are more complicated, because the interface can move. Furthermore, it is a physical fact that an extra normal stress, due to **surface tension**, acts on the interface. This extra stress takes the form $\gamma K(\mathbf{x})$ where γ is the positive surface tension constant, and K is the curvature of the fluid surface, which can be defined by $K = \nabla\cdot\widehat{\mathbf{n}}$ where $\widehat{\mathbf{n}}$ is the unit normal to the interface.

If one of the fluids is dynamically negligible, as often happens with a liquid/gas interface, then we can treat one fluid as having a constant pressure p_0 and neglect its motion. If the interface is stationary, then the appropriate boundary conditions to apply on the other fluid are zero normal velocity and zero tangential stress. (So if the surface is $y = 0$ and velocity $(u,\, v,\, 0)$ then we have $v = 0$ and $\mu\,\partial u/\partial y = 0$. For inviscid flow, $\mu = 0$ and the tangential stress condition is trivially satisfied.) If the interface moves and we describe its position at time t by the function $\zeta(\mathbf{x},\, t) = 0$, then the **kinematic boundary condition** for the normal velocity can be written

$$\frac{D\zeta}{Dt} = 0. \tag{0.10}$$

This condition essentially states that if a fluid particle lies in the interface between two fluids, then it remains in the interface as the fluids move.

Most of the time we will be dealing with solid boundaries. We will revisit the fluid/fluid interface conditions if needed later.

Fundamentals 1: Dynamic Similarity and the Reynolds Number

Consider incompressible viscous fluid flowing steadily around a stationary solid sphere of radius a. In terms of **dimensional** Cartesian coordinates $\widehat{\mathbf{x}} = (\widehat{x},\, \widehat{y},\, \widehat{z})$ let the dimensional velocity be $\widehat{\mathbf{u}} = (\widehat{u},\, \widehat{v},\, \widehat{w})$. Then the governing equations are

$$\widehat{\nabla} \cdot \widehat{\mathbf{u}} = 0, \qquad \widehat{\mathbf{u}} \cdot \widehat{\nabla}\widehat{\mathbf{u}} = -\frac{1}{\rho}\,\widehat{\nabla}\widehat{p} + \nu\widehat{\nabla}^2\widehat{\mathbf{u}}, \tag{1.1}$$

where ρ and $\nu \equiv \mu/\rho$ are respectively the fluid density and kinematic viscosity, and $\widehat{\nabla}$ denotes differentiation with respect to $\widehat{\mathbf{x}}$. A long way from the sphere, the pressure should approach the free stream pressure, $\widehat{p} \to p_\infty$ and the velocity should become uniform. On the sphere's surface the velocity must obey the no-slip condition, so that

$$\widehat{\mathbf{u}} \to (V_\infty,\, 0,\, 0) \quad \text{as} \quad |\widehat{\mathbf{x}}| \to \infty, \qquad\qquad \widehat{\mathbf{u}} = 0 \quad \text{on} \quad |\widehat{\mathbf{x}}| = a. \tag{1.2}$$

Engineers often like to work with dimensional variables, but mathematicians MUCH prefer dimensionless quantities. So let us introduce **dimensionless** coordinates $(x,\, y,\, z)$, defined by

$$\widehat{x} = ax, \quad \widehat{y} = ay, \quad \widehat{z} = az, \qquad \widehat{\nabla} = \frac{1}{a}\nabla. \tag{1.3}$$

We can also use the dimensional quantities, ρ, ν, p_∞ and V_∞ in the problem to define a dimensionless pressure p and a dimensionless velocity $\mathbf{u}$,

$$\widehat{\mathbf{u}} = V_\infty\mathbf{u}, \qquad \widehat{p} = p_\infty + \rho V_\infty^2\, p. \tag{1.4}$$

We now rewrite the problem in terms of the new variables

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{V_\infty^2}{a}\,\mathbf{u} \cdot \nabla\mathbf{u} = -\frac{V_\infty^2}{a}\,\nabla p + \frac{\nu V_\infty}{a^2}\,\nabla^2\mathbf{u}, \tag{1.5}$$

or

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u} \cdot \nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}, \tag{1.6}$$

where **the Reynolds number**

$$Re = \frac{V_\infty a}{\nu}. \tag{1.7}$$

Our boundary conditions are now

$$p \to 0, \quad \mathbf{u} \to (1,\, 0,\, 0) \qquad \text{as} \qquad |\mathbf{x}| \to \infty, \qquad\qquad \mathbf{u} = 0 \quad \text{on} \quad |\mathbf{x}| = 1. \tag{1.8}$$

Our problem now depends on a single non-dimensional parameter, Re.

We assume that the boundary-value problem (1.6) with (1.8) has a unique solution. Suppose two spheres with different radii are placed into two flows with different V_∞ and ν, such that the Reynolds number Re is the same. These two flows will then be identical in terms of the non-dimensional variables. This concept is known as **dynamic similarity**. It may be extended in an obvious way to arbitrary body shapes. Provided two bodies are geometrically similar, and the Reynolds number is the same, then the resulting flows are mathematically identical and one only needs to rescale the velocity and pressure fields by a constant factor.

This similarity property of viscous flows is widely used by experimentalists. In particular, wind tunnel tests are essential in aircraft design, and these are performed on scaled down models of the entire aircraft or its elements. To achieve dynamic similarity of the flow in a wind tunnel with real flight conditions, one needs only to ensure the same Reynolds number applies.

Note that we can nondimensionalise problems in more than one way. For example, we could have chosen the diameter rather than the radius of the sphere as our reference length-scale in (1.3), or we could have used some average velocity rather than the value at infinity. For more general flows, the ambiguity is greater. But once we choose a reference velocity and length-scale, a Reynolds number can be defined for any flow.

To do this, we introduce the concept of "scales of variation." If we have a differentiable function $f(x)$ on an interval $0 < x < a$ we wish to estimate its typical size and the lengthscale over which it varies. We can define a typical value, $||f||$, in many ways. For example, we could set $||f|| = \frac{1}{a}\int_0^a |f(x)|\,dx$. Similarly we could find a typical value of its derivative $||f'||$. From these two quantities we could construct a lengthscale, $L = ||f||/||f'||$, which estimates the scale of variation of $f(x)$. Alternatively, if $f' \neq 0$, we could define $L = ||f/f'||$. Thus for example if $f = e^{-kx}$, then in either case $L = 1/k$, and $||f|| = 1/ka$. This definition of $||f||$ may not be best, especially when a is large. Then the average value of f is small, but we may be more interested in a typical value in regions where f isn't negligible. We must be guided by common sense and intuition when choosing appropriate scales. Usually, we don't know the solution f at the time we wish to estimate L. We might then set L to be a typical dimension of the boundary.

From our estimates for u and x we can then estimate an order of magnitude for any x-derivative, for example $u_x \sim U/L$, and $u_{xx} \sim U/L^2$. In doing this, as often in applied mathematics, we are making implicit assumptions about physical quantities behaving "sensibly". Note we are not saying $u_x \simeq U/L$. We are saying $u_x = \alpha U/L$ where we expect α to be $O(1)$. Certainly we would not be surprised if say $\alpha = 2$, at some point.

We can now define the Reynolds number as **the ratio between estimates of the inertial and viscous terms** in the Navier-Stokes equation:

$$Re = \frac{||\rho \mathbf{u}\cdot\nabla\mathbf{u}||}{||\mu\nabla^2\mathbf{u}||} \simeq \frac{\rho U^2/L}{\mu U/L^2} = \frac{UL}{\nu}. \tag{1.9}$$

The numerical value of the Reynolds number so defined depends on how we define U and L and so is not a precise quantity in general parlance. The way to think of it is that at low Reynolds number inertial terms are small and may be negligible, whereas at high Reynolds number the viscous terms are small and may be negligible.

We note that individual components of velocity may be of different sizes, while the x-, y- and z-derivatives may vary on different scales – this will be crucial later in the module. Also, just as the magnitude of the velocity may be much greater in some places than others, so may it vary on a small scale in some regions of the flow but on a larger scale in others.

We shall begin our study of Fluid Dynamics by considering low-Reynolds-number flows, which are also known as Stokes flows, or creeping flows. If inertia terms can be neglected, then our equations become linear and may be easier to solve.

Fundamentals 2: The Energy Equation

Let us now consider the kinetic energy of the fluid within a volume V which moves with the fluid,

$$E = \int_V \tfrac{1}{2}\rho|\mathbf{u}|^2 dV. \tag{1.10}$$

The rate of change of this energy is given by the mass and momentum equations (0.3) & (0.4)

$$\frac{dE}{dt} = \int_V \frac{D}{Dt}\left(\tfrac{1}{2}\rho u_i u_i\right) dV = \int_V \rho u_i \frac{Du_i}{Dt}\, dV = \int_V u_i F_i\, dV + \int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j}\, dV. \tag{1.11}$$

The final term here may be written

$$\begin{aligned}
\int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j}\, dV &= \int_V \left[\frac{\partial}{\partial x_j}(u_i\sigma_{ij}) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}\right] dV \\
&= \int_S u_i \sigma_{ij} n_j\, dS - \int_V \sigma_{ij} e_{ij}\, dV \\
&= \int_S u_i \tau_i\, dS - 2\mu \int_V e_{ij} e_{ij}\, dV,
\end{aligned}$$

where $\tau_i = \sigma_{ij} n_j$ is the surface force or *traction*. We have used here the divergence theorem (0.8), the symmetry of the stress tensor ($\sigma_{ij} = \sigma_{ji}$) and the incompressibility relation $\nabla \cdot \mathbf{u} \equiv e_{ii} = 0$. We have therefore shown that

$$\frac{dE}{dt} = \int_V u_i F_i\, dV + \int_S u_i \tau_i\, dS - 2\mu \int_V e_{ij} e_{ij}\, dV. \tag{1.12}$$

This equation has a clear physical interpretation in terms of the energy balance within the fluid. The first two terms on the right represent the rate of working by body and surface forces on the fluid within V, and the final term is therefore the rate of energy dissipation due to viscosity, which we can think of as a kind of friction. The rate of viscous heating Φ per unit volume is thus

$$\Phi = \sigma_{ij} e_{ij} = 2\mu e_{ij} e_{ij}. \tag{1.13}$$

The second law of thermodynamics demands that Φ and hence μ must be positive.

This heating can change the temperature T in the fluid. If then the density or viscosity depend on temperature, a further equation involving convection and diffusion of heat, is needed to determine $T(\mathbf{x}, t)$. We shall not pursue this (interesting) complication in this course.

Exercise 1: Suppose a flow is such that $e_{ij} = 0$ at every point, so that the energy dissipation is zero. What can you say about the velocity? Interpret the result physically.

Chapter 1. Low Reynolds number: Stokes flows

The Stokes equations

In this chapter we consider the low-Reynolds-number limit, $Re \ll 1$, and we neglect inertial terms in the Navier-Stokes equations (0.6) or (1.6) to obtain

$$\mu \nabla^2 \mathbf{u} = \nabla p - \mathbf{F}^{\text{body}}, \qquad \nabla \cdot \mathbf{u} = 0. \tag{2.1}$$

These are the (forced) Stokes equations. Natural boundary conditions are that at each point of S either $\mathbf{u}$ or the traction $\sigma_{ij} n_j$ is given. Equation (2.1) is equivalent to

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -F_i^{\text{body}}, \qquad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \qquad e_{ii} = 0. \tag{2.2}$$

Properties of the Stokes equations:

The Stokes equations have several properties, some of them counterintuitive. Without even solving the equations, we can deduce a lot about their solutions:

(a) Instantaneity

There are no time derivatives in (2.1). Thus $\mathbf{u}$ responds instantaneously to the boundary motion and the force $\mathbf{F}^{\text{body}}$. For instance, a body falling under gravity achieves its terminal velocity at once in Stokes flow. At all times, forces must balance exactly.

(b) Linearity

There is no $\mathbf{u} \cdot \nabla \mathbf{u}$ term in (2.1); therefore $\mathbf{u}$, p and σ_{ij} are linearly forced by any boundary motion or body force. If for instance we have a falling sphere, doubling the velocity will double σ_{ij} and thus double the drag. More generally, force $\propto$ velocity rather than acceleration.

Consider the drag force $-\mathbf{F}$ acting on a solid body moving with velocity $\mathbf{U} = (U_1, U_2, U_3)$. We assume the body does not rotate and so there may also be a torque $-\mathbf{G}$ acting on the body to maintain this motion. We have

$$\mathbf{F} = \int_S \boldsymbol{\tau}\, dS, \qquad \mathbf{G} = \int_S \mathbf{x} \times \boldsymbol{\tau}\, dS \qquad \text{where} \quad \tau_i = \sigma_{ij} n_j. \tag{2.3}$$

Because of the linearity of the problem, we can say $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ where $\mathbf{F}_1$ is the force when the body moves with velocity $(U_1, 0, 0)$ and similarly for $\mathbf{F}_2$ and $\mathbf{F}_3$. In general $\mathbf{F} = M\mathbf{U}$, where M is some matrix. The torque will also be linear in the velocity, $\mathbf{G} = N\mathbf{U}$ where N is some matrix.

Now let the body be a cube aligned with the coordinate axes. By symmetry, $\mathbf{F}_1 = (\alpha U_1, 0, 0)$ for some constant α. Furthermore, $\mathbf{G}_1 = 0$ as there is no reason why the torque should act one way or the other. Similarly, $\mathbf{F}_2 = (0, \alpha U_2, 0)$ for the same constant α and hence in general $M = \alpha I$ where I is the identity matrix and $\mathbf{F} = \alpha \mathbf{U}$. It also follows that $N = 0$ and $\mathbf{G} = 0$. Surprisingly, the Stokes drag force for a cube is the same in any orientation, and it moves without rotating. The same is not true at higher Reynolds numbers.

(c) Reversibility

Suppose no body force acts and a Stokes flow is driven by the motion of a boundary. If the velocity on the boundary is reversed then so is the velocity everywhere in the fluid. If a prescribed boundary motion is reversed over time then each material point retraces its history and ends up exactly where it started. This will be beautifully illustrated in a video. It is not even necessary that the forwards motion proceed at the same rate as the backwards motion for this to be true. This has implications for swimming at low-Reynolds number. No time-reversible motion can lead to a net swimming speed. This is known as "the scallop theorem."

Does a sphere falling by a wall migrate towards or away from the wall? Neither: on reversal of $\mathbf{g}$, $\mathbf{u}$ must reverse and so if the sphere were to move towards the wall under $\mathbf{g}$ then it would move away from the wall under $-\mathbf{g}$.

(d) Uniqueness Theorem for Stokes Flows

Theorem: There exists at most one Stokes flow in a volume V for which $\mathbf{u}$ is specified on the boundary, S.

Proof: Suppose $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are two such flows. Let $\mathbf{u}^* = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$, $\sigma^*_{ij} = \sigma^{(1)}_{ij} - \sigma^{(2)}_{ij}$ and $e^*_{ij} = e^{(1)}_{ij} - e^{(2)}_{ij}$. Then $\mathbf{u}^* = 0$ on S while (2.3) gives that $\frac{\partial \sigma^*_{ij}}{\partial x_j} = 0$ and $\frac{\partial u^*_i}{\partial x_i} = 0$. Now consider

$$2\mu \int_V e^*_{ij} e^*_{ij}\, dV = \int_V \sigma^*_{ij} \frac{\partial u^*_i}{\partial x_j}\, dV = \int_V \frac{\partial}{\partial x_j}\left(\sigma^*_{ij} u^*_i\right)\, dV = \int_S \sigma^*_{ij} u^*_i n_j\, dS = 0. \qquad (2.4)$$

Thus since $e^*_{ij} e^*_{ij} \geq 0$ we must have $e^*_{ij} = 0$, so that for example $\partial u^*_1/\partial x_1 = 0$. But since $u^*_1 = 0$ on S we have $u^*_1 = 0$ and hence $\mathbf{u}^* = 0$ everywhere.

A more sophisticated argument (including the $\mathbf{u}\cdot\nabla\mathbf{u}$ term) proves uniqueness if $Re < \pi\sqrt{3}$, but for large Re there may be more than one solution.

(e) Minimal Dissipation Theorem

Theorem: Suppose $\mathbf{u}(\mathbf{x})$ is the unique unforced Stokes flow in V satisfying $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ on S. Let $\overline{\mathbf{u}}(\mathbf{x})$ be another 'kinematically possible' flow in V such that $\nabla\cdot\overline{\mathbf{u}} = 0$ and $\overline{\mathbf{u}} = \mathbf{u}_0$ on S. Then

$$2\mu \int_V \overline{e}_{ij}\overline{e}_{ij}\, dV \geq 2\mu \int_V e_{ij} e_{ij}\, dV, \qquad (2.5)$$

with equality only if $\mathbf{u} = \overline{\mathbf{u}}$, i.e. energy dissipation is minimum in Stokes flow.

Proof: Let $\mathbf{u}^* = \mathbf{u} - \overline{\mathbf{u}}$ and $e^*_{ij} = e_{ij} - \overline{e}_{ij}$, so that $\mathbf{u}^* = 0$ on S and $e^*_{ii} = 0$. Consider

$$\int_V \left(\overline{e}_{ij}\overline{e}_{ij} - e_{ij}e_{ij}\right) dV = -\int_V e^*_{ij}\left(\overline{e}_{ij} + e_{ij}\right) dV = \int_V e^*_{ij}e^*_{ij}\, dV - 2\int_V e^*_{ij} e_{ij}\, dV\ . \qquad (2.6)$$

The first term is clearly positive; we now show the last term is zero.

$$2\mu \int_V e^*_{ij} e_{ij}\, dV = \int_V \sigma_{ij} e^*_{ij}\, dV = \int_V \frac{\partial u^*_i}{\partial x_j}\sigma_{ij}\, dV = \int_S u^*_i \sigma_{ij} n_j\, dS = 0. \qquad (2.7)$$

As an example of this theorem, we consider the drag $\mathbf{F}$ on a rigid particle of arbitrary shape moving with velocity $\mathbf{U}$ in unbounded fluid. The rate of working of this force is

$$U_i F_i = \int_S u_i \sigma_{ij} n_j \, dS = 2\mu \int_V e_{ij} e_{ij} \, dV \leq 2\mu \int_V \overline{e}_{ij} \overline{e}_{ij} \, dV, \tag{2.8}$$

where $\overline{e}_{ij}$ is the strain rate for any kinematically admissable flow field. If we choose $\overline{\mathbf{u}} = \mathbf{U}$ inside a region $\overline{V}$ enclosing the body and $\overline{\mathbf{u}}$ to be a Stokes flow outside $\overline{V}$, then $\overline{e}_{ij} = 0$ inside $\overline{V}$, while the RHS is the rate of working of the drag of a solid body occupying $\overline{V}$. It follows that the drag in Stokes flow on any body is less than the drag of any larger body.

(f) The biharmonic equation

Taking the divergence of the Stokes equations (2.6) (with $\mathbf{F} = 0$) we see that p is harmonic, $\nabla^2 p = 0$. Taking the curl we see similarly that the vorticity vector is harmonic, $\nabla^2 \boldsymbol{\omega} = 0$.

$$\nabla^2 p = 0 \qquad \text{and} \quad \nabla^2 \boldsymbol{\omega} = 0 \tag{2.9}$$

For a planar flow $\mathbf{u} = \nabla \times (0, 0, \psi(x,\, y))$ and $\boldsymbol{\omega} = (0, 0, -\nabla^2 \psi)$. Thus

$$\nabla^2(\nabla^2 \psi) = \nabla^4 \psi = 0 \tag{2.10}$$

so that ψ satisfies the **biharmonic equation.** We can first solve $\nabla^2 \omega = 0$ and then $\nabla^2 \psi = -\omega$. However, usually we don't have a boundary condition for ω but instead have two boundary conditions on ψ. This often renders the biharmonic equation harder to solve than two harmonic equations.

A similar result holds in axisymmetry. Let $(R,\, \phi,\, z)$ be cylindrical polar coordinates where R is the distance from the axis of symmetry. The axisymmetric flow $\mathbf{u} = (u,\, 0,\, w)$ can be written in terms of the Stokes streamfunction $\Psi(R, z)$, where

$$\mathbf{u} = \nabla \times \left(0,\, \frac{\Psi}{R},\, 0\right) = \frac{1}{R}\left(-\Psi_z,\, 0,\, \Psi_R\right). \tag{2.11}$$

Then $\nabla \times \mathbf{u} = \boldsymbol{\omega} = (0,\, \omega,\, 0)$ where ω is given by

$$\omega = -\frac{1}{R} D^2 \Psi \qquad \text{where} \quad D^2 \Psi = R^2 \nabla \cdot \left(\frac{\nabla \Psi}{R^2}\right) = \Psi_{RR} - \frac{1}{R}\Psi_R + \Psi_{zz}\,. \tag{2.12}$$

D^2 is known as the Stokes operator – it closely resembles the Laplacian. This time $\nabla^2 \boldsymbol{\omega} = 0$ leads to the modified biharmonic equation for Ψ

$$D^4 \Psi \equiv D^2(D^2 \Psi) = 0\,. \tag{2.13}$$

Alternatively we could use spherical polar coordinates, $(r,\, \theta,\, \phi)$, where $R = r \sin\theta$ and $z = r\cos\theta$. In this case the Stokes operator takes the form

$$D^2 \Psi = \Psi_{rr} + \frac{\sin\theta}{r^2}\left(\frac{\Psi_\theta}{\sin\theta}\right)_\theta \qquad \text{and} \quad D^4 \Psi \equiv D^2(D^2 \Psi) = 0\,. \tag{2.14}$$

Stokes flow due to a translating sphere

We consider the inertialess flow generated by a sphere of radius a and velocity $\mathbf{U}$ immersed in unbounded fluid of viscosity μ which is at rest at infinity. In particular we want to calculate the force $\mathbf{F}$ exerted by the sphere on the fluid.

The linearity of the Stokes equations requires that F is proportional to both U and μ. Dimensional arguments therefore give $F = \alpha\mu a U$, where α is a positive dimensionless constant. The isotropy of the sphere's shape then implies that $\mathbf{F} = \alpha\mu a\mathbf{U}$. We must do some work to find the constant α.

We take spherical polars (r, θ, ϕ) with $\theta = 0$ parallel to $\mathbf{U}$. The flow is then axisymmetric with no ϕ dependence and so admits a Stokes streamfunction $\Psi(r, \theta)$ such that the components of the velocity $\mathbf{u}$ can be written

$$\mathbf{u} = \nabla \times \left(0,\, 0,\, \frac{\Psi}{r\sin\theta}\right) = \left(\frac{\Psi_\theta}{r^2\sin\theta},\, \frac{-\Psi_r}{r\sin\theta},\, 0\right). \tag{2.15}$$

It follows from the Stokes equations that $D^2\left(D^2\Psi\right) = 0$ as in (2.14), where

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{2.16}$$

The no-slip condition on the sphere surface $\mathbf{u} = (U\cos\theta,\, -U\sin\theta,\, 0)$ gives

$$\Psi = \frac{1}{2}Ua^2\sin^2\theta \quad \text{and} \quad \frac{\partial\Psi}{\partial r} = Ua\sin^2\theta \quad \text{on } r = a.$$

Finally, for $r \to \infty$, $\Psi = o(r^2)$. By inspection of the geometry and boundary conditions, we look for a solution $\Psi = f(r)\sin^2\theta$ so that

$$\Omega = -D^2\Psi = -F(r)\sin^2\theta \qquad \text{where} \quad F(r) = f'' - \frac{2f}{r^2} \qquad \text{and} \quad F'' - \frac{2F}{r^2} = 0 \tag{2.17}$$

as $D^2\Omega = 0$. Solving for F and f we have

$$f = Ar^4 + Br^2 + Cr + \frac{D}{r} \tag{2.18}$$

and the boundary conditions

$$f(a) = \tfrac{1}{2}Ua^2, \quad f'(a) = Ua, \quad f'' \to 0 \text{ as } r \to \infty \tag{2.19}$$

give $A = B = 0$, $C = \frac{3}{4}Ua$ and $D = -\frac{1}{4}Ua^3$. Substituting back we obtain

$$\Psi = \frac{Ua^2}{4}\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta \qquad \text{and} \quad \Omega = \tfrac{3}{2}U\frac{a}{r}\sin^2\theta \tag{2.20}$$

and if we write $\mathbf{u} = (u_r,\, u_\theta,\, 0)$, then

$$u_r = \frac{U}{2}\left(\frac{3a}{r} - \frac{a^3}{r^3}\right)\cos\theta \quad \text{and} \quad u_\theta = -\frac{U}{4}\left(\frac{3a}{r} + \frac{a^3}{r^3}\right)\sin\theta. \tag{2.21}$$

The force on a sphere in Stokes flow

Last time we found the flow resulting from a sphere in motion. It is still some effort to calculate the force on the sphere. To do this we must calculate the stress on the spherical surface.

We can obtain the pressure (to within an arbitrary constant, p_∞) from

$$\nabla p = -\mu \nabla \times \boldsymbol{\omega} = -\mu \nabla \times \left(0,\, 0,\, \frac{\Omega}{r \sin\theta}\right) = \left(\frac{-\mu}{r^2 \sin\theta} \frac{\partial \Omega}{\partial \theta},\, \frac{\mu}{r \sin\theta} \frac{\partial \Omega}{\partial r},\, 0\right) \tag{2.22}$$

so that

$$p - p_\infty = \frac{3 U a \mu \cos\theta}{2 r^2}. \tag{2.23}$$

The stress can now be determined from (2.2) although care must be taken in evaluating e_{ij} in this curvilinear co-ordinate system. The thing to remember is that the unit vectors $\widehat{r}$ and $\widehat{\theta}$ vary as θ varies. In particular $\partial/\partial\theta(\widehat{\theta}) = -\widehat{r}$. The normal $\mathbf{n}$ is in the $(-r)$-direction, and the traction components in the r and θ-directions are

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r}\ , \qquad \sigma_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right)\ . \tag{2.24}$$

By symmetry, the net force exerted by the sphere on the fluid must point in the direction of $\mathbf{U}$, and the component of the stresses in that direction is $-\sigma_{rr}\cos\theta + \sigma_{r\theta}\sin\theta$. Substituting in from (2.21), we find that on $r = a$

$$\frac{\partial u_r}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial u_r}{\partial \theta} = -\frac{U \sin\theta}{a} \qquad \text{and} \quad r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = \frac{5U}{2a} \sin\theta. \tag{2.25}$$

so that using (2.23), (2.24) and (2.25),

$$-\sigma_{rr} \cos\theta + \sigma_{r\theta} \sin\theta = p_\infty \cos\theta + \frac{3 U \mu}{2a}. \tag{2.26}$$

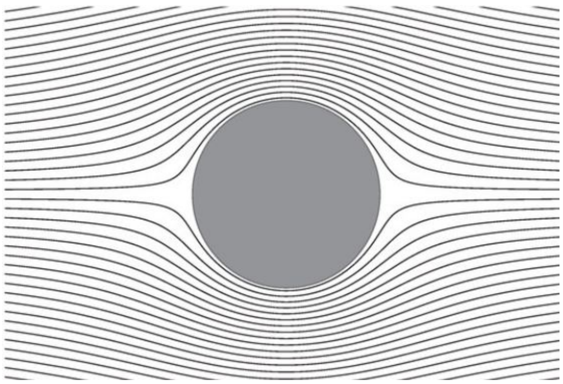
This is the force on the fluid; the force on the body is equal and opposite. We now need to integrate this force over the entire surface, which means multiply by $2\pi a^2 \sin\theta$ and integrate from $\theta = 0$ to $\theta = \pi$. The $\cos\theta$ term integrates to zero by symmetry; as expected, the uniform pressure at infinity does not exert a net force on the sphere. Conveniently, the other term is constant, so we can just multiply by the surface area $4\pi a^2$.

Finally, we obtain the force on the fluid as

$$\mathbf{F} = 6\pi\mu a \mathbf{U}, \tag{2.27}$$

a result known as Stokes' law. The drag force on the particle is $-6\pi a\mu\mathbf{U}$.

The flow at infinity; Stokeslets:



In the figure, we plot the streamlines for the flow in the frame of the moving sphere, which we obtain by subtracting $\frac{1}{2}Ur^2\sin^2\theta$ from Ψ in (2.20). The left/right flow symmetry only happens in practice at low Reynolds number; we know we can shelter behind objects in a strong wind. Note also that the velocity in (2.21) decays only slowly: as $r \to \infty$, $\mathbf{u} \sim \frac{1}{r}$. As a result, the presence of distant boundaries or other bodies may affect the flow somewhat.

We can calculate the drag force $\mathbf{F}$ more easily by moving the integral to a sphere at infinity using the divergence theorem;

$$F_i = \int_{r=a} \sigma_{ij} n_j \, dS = -\int_{S_\infty} \sigma_{ij} n_j dS.$$

Only terms of order r^{-2} in σ_{ij} (r^{-1} in $\mathbf{u}$ or r in Ψ) therefore contribute to the force. In the far field $\Psi \sim Cr\sin^2\theta$ and $p - p_\infty \sim 2\mu C\cos\theta r^{-2}$. Thus for general shapes of particle in unbounded fluid exerting a force $\mathbf{F}$ on the fluid, at large distances

$$\Psi \sim \frac{Fr}{8\pi\mu}\sin^2\theta \quad p - p_\infty \sim \frac{F}{4\pi r^2}\cos\theta \quad \mathbf{u} \sim \frac{F}{8\pi\mu r}\,(2\cos\theta,\, -\sin\theta,\, 0) \tag{2.28}$$

This solution for $\mathbf{u}$ and p satisfies the Stokes equations everywhere except at $r = 0$ and corresponds to a point force $\mathbf{F}$ acting at $r = 0$. It is called a **Stokeslet velocity field**.

Sedimentation:

A spherical particle of radius a and density ρ_p feels a gravitational force $\frac{4}{3}\pi a^3(\rho_p - \rho)\mathbf{g}$. This must balance the Stokes drag, so that we deduce that its sedimentation velocity is

$$\mathbf{U} = \frac{2}{9}\frac{a^2}{\mu}\,(\rho_p - \rho)\,\mathbf{g}. \tag{2.29}$$

If we have two spherical particles of radius a, separated by a distance $L \gg a$, we can consider each as lying in the far field of the other. Looking at (2.21), we can see that when $\theta = \frac{1}{2}\pi$ we would predict that two spheres in a horizontal line would fall faster than one by a factor $(1 + \frac{3}{4}(a/L))$. If we have two particles one above the other, ($\theta = 0$) then they fall faster by a factor $(1 + \frac{3}{2}(a/L))$.

If instead of a solid sphere we have a gas bubble, the solution is slightly different. We must assume that surface tension is strong enough to keep the shape spherical, but the appropriate boundary condition is now $e_{r\theta} = 0$ rather than $u_\theta = -U\sin\theta$.

Flow past a cylinder: Stokes' paradox

Another feature of the slow $1/r$ decay gives rise to a phenomenon knows as **Stokes' paradox.** Instead of the three-dimensional sphere, let us consider the problem of a cylinder moving sideways, which in two-dimensions is just a circle moving with velocity $\mathbf{U} = U(\cos\phi,\, -\sin\phi,\, 0)$ using cylindrical polar coordinates $(R,\, \phi,\, z)$. We can represent the z-independent flow with a streamfunction $\psi(R,\, \phi)$, such that the velocity at a general point is $\mathbf{u} = (\frac{1}{R}\psi_\phi,\, -\psi_R, 0)$. Then the Stokes equations reduce to the biharmonic equation

$$\nabla^2(\nabla^2\psi) = 0 \qquad \text{where} \quad -\omega \equiv \nabla^2\psi \equiv \psi_{RR} + \tfrac{1}{R}\psi_R + \tfrac{1}{R^2}\psi_{\phi\phi}. \tag{2.30}$$

Taking the cylinder radius as 1 for the simplicity, the boundary conditions on $R = 1$ are $\mathbf{u} = \mathbf{U}$ or $\psi = U\sin\phi$ and $\psi_R = U\sin\phi$. We therefore look for a solution

$$\psi = \sin\phi f(R) \quad \Longrightarrow \quad -\omega = \nabla^2\psi = \sin\phi\left(f'' + \tfrac{1}{R}f' - \tfrac{1}{R^2}f\right). \tag{2.31}$$

Solving $\nabla^2\omega = 0$ we find solutions $\omega \propto R^n\sin\phi$ where $n = \pm 1$. Then solving (2.31), we find the general solution for f and

$$\psi = \sin\phi\left(AR^3 + BR\log R + CR + D/R\right). \tag{2.32}$$

As well as our two conditions on $R = 1$, we would like to impose $\mathbf{u} \to 0$ as $R \to \infty$. This requires $\psi_R \to 0$ as $R \to \infty$, which would require $A = B = C = 0$ which would only leave one constant for the other two boundary conditions. There is **no solution** to this problem, a result known as Stokes' paradox. One way of thinking about this is to break the cylinder into sections in a 3D flow. A section with coordinate z generates a flow which behaves like $1/\sqrt{R^2 + z^2}$ as $R \to \infty$. Integrating or summing over z leads to an answer which diverges logarithmically.

So what has gone wrong? The Navier-Stokes equations are correct; the only subsequent assumption we made is that inertia is negligible compared to the viscous term. Let us examine this assumption at large R. If we discard the highly singular term AR^3 in (2.32), we still have $u \sim \log R$ and hence $\omega \sim 1/R$ at large R. The inertial term $\sim \rho U^2(\log R)^2/R$ while the viscous term $\sim \mu U/R^2$. Clearly, for large enough R we can not ignore inertia any more, and our equations are incorrect. The paradox was resolved by Oseen by partially reintroducing the inertia terms. He argued that the correct equation to use was

$$+\rho U\mathbf{u}_x + \nabla p = \mu\nabla^2\mathbf{u} \tag{2.33}$$

which regularises the solution.

In contrast, for the 3-D problem of a moving sphere, we have $u \sim 1/r$ and $\omega \sim 1/r^2$. Both $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\nabla^2\mathbf{u}$ scale as $1/r^3$ for large r and so the Stokes approximation is uniform in r. Nevertheless, inertial terms introduce a correction into the Stokes drag law,

$$\mathbf{F} = 6\pi a\mu\mathbf{U}\left(1 + \tfrac{3}{8}Re\right). \tag{2.34}$$

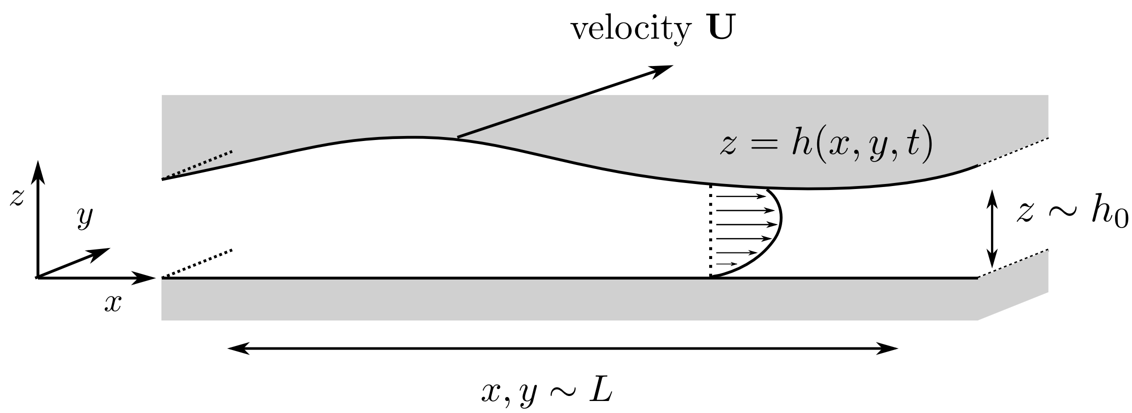
Chapter 2: Lubrication Theory – Flow in Thin Films

It is an observed fact that thin layers of fluid can prevent solid bodies from contact. The theory of these flows can be referred to in several ways: **lubrication theory, thin film theory,** or **the slowly-varying approximation**.

We begin with a simple demonstration, which you can also see performed in G.I. Taylor's magnificent film on low-Reynolds-number flows, which is in the "Aren't Fluids Wonderful" video folder, which you are encouraged (but not required) to watch. We take an uncluttered desk, and slide a piece of paper along it. While moving, the paper does not actually touch the desk but floats on a thin cushion of air. Without this effect, most machinery would find it much harder to function. However, the ideas are very useful in practice and are applicable to far more than "lubrication"!

Crucial to our analysis is the idea that the scales of variation may be different in different directions. Derivatives across the thin layer we expect to be larger than derivatives along the layer. This enables us to simplify the governing equations considerably.

Consider a solid body with surface $z = h(x, y, t)$ close to a stationary, solid plane at $z = 0$. The top surface may be moving at a velocity $\mathbf{U} = (U, V, W)$.



We assume that x and y vary on similar scales, say L, but we consider a thin gap in the z-direction and so z varies on the scale h_0, a typical value of h.. We can regard the gap as 'thin' provided $h_0 \ll L$. We write the velocity $\mathbf{u} = (u, v, w)$ in Cartesian components, where u and v have typical scales U_0 and w has typical scale W_0. Then the incompressibility condition

$$u_x + v_y + w_z = 0 \quad \Longrightarrow \quad W_0 \sim U_0 h/L \ll U_0 \ . \tag{3.1}$$

We now consider the x-component of the momentum equation,

$$\rho(u_t + uu_x + vu_y + wu_z) = -p_x + \mu(u_{xx} + u_{yy} + u_{zz}).$$

A suitable time-scale follows from $\partial h/\partial t \simeq W_0$. Then all terms on the LHS have typical magnitudes $\rho U_0^2/L$, while the viscous term scales as $\mu U_0/h_0^2$, noting that the z-derivatives dominate. In lubrication theory, we assume the inertia terms are negligible compared to the viscous terms, which means

$$\rho U_0^2/L \ll \mu U_0/h_0^2 \quad \Longrightarrow \quad Re(h_0/L)^2 \ll 1 \qquad \text{where} \quad Re = \rho U_0 L/\mu. \tag{3.2}$$

We don't need the Re based on the large lengthscale L to be small, only the "reduced Reynolds number", $Re(h_0/L)^2$. In fact, Re could be large, as long as h_0/L is small enough.

Balancing the pressure term with the dominant viscous stress scale $\mu U_0/h_0^2$ suggests a pressure scale $P_0 \sim \mu U_0 L/h_0^2$. Then moving to the z-momentum equation, the pressure term has scale $p_z \sim P_0/h_0 \sim \mu U_0 L/h_0^3$, which dominates over the largest viscous term, $\mu w_{zz} \sim \mu U_0/(h_0 L)$. If we only keep the dominant terms, we are left with the lubrication equations

$$\left.\begin{aligned} 0 &= -p_x + \mu u_{zz} \\ 0 &= -p_y + \mu v_{zz} \\ 0 &= -p_z \end{aligned}\right\} \quad \Longrightarrow \quad \begin{cases} p = p(x,y,t) \\ u = \dfrac{p_x}{\mu}(z^2 + Az + B) \\ v = \dfrac{p_y}{\mu}(z^2 + Cz + D) \end{cases} \tag{3.3}$$

Importantly, the z-momentum equation states that the pressure does not vary across the layer. The LHS of the x- and y-momentum equations are thus independent of z and the equations may be integrated trivially. Imposing the solid body boundary conditions $\mathbf{u} = 0$ on $z = 0$ and $\mathbf{u} = (U, V, W)$ on $z = h$, we have

$$u = -\frac{p_x}{2\mu} z(h-z) + \frac{Uz}{h(x,y,t)} \qquad v = -\frac{p_y}{2\mu} z(h-z) + \frac{Vz}{h} \; . \tag{3.4}$$

We could now calculate w from $\nabla \cdot \mathbf{u} = 0$ and impose the boundary conditions. Instead, we impose continuity of mass by integrating $\nabla \cdot \mathbf{u} = 0$ across the layer. We first observe that

$$\frac{\partial}{\partial x} \int_0^{h(x,y,t)} u(x,y,z,t)\, dz = \int_0^h \frac{\partial u}{\partial x}\, dz + u(x,y,h,t)\frac{\partial h}{\partial x} = \int_0^h u_x\, dz + U h_x \tag{3.5}$$

and a similar result holds for v and y. Thus

$$W = \int_0^h w_z\, dz = -\int_0^h (u_x + v_y)\, dz = -\frac{\partial}{\partial x}\int_0^h u\, dz - \frac{\partial}{\partial y}\int_0^h v\, dz + U h_x + V h_y \; . \tag{3.6}$$

Substituting (3.5) into (3.6) and evaluating the z-integrals we obtain **Reynolds' Lubrication Equation:**

$$\frac{1}{12\mu}\left(\frac{\partial}{\partial x}(h^3 p_x) + \frac{\partial}{\partial y}(h^3 p_y)\right) = W - \tfrac{1}{2}U h_x - \tfrac{1}{2} V h_y \tag{3.7a}$$

We can rewrite this relation using the **kinematic boundary condition.** In general, suppose that the surface of the fluid is given by $\Phi(x,\, y,\, z,\, t) = 0$. Now a fluid particle at the interface between two fluids remains on the surface as the surface moves. It follows that the value of Φ evaluated on the surface remains zero, and hence that the material derivative (i.e. the time-derivative following the fluid as it moves) is zero. So on the surface $z = h$

$$0 = \frac{D}{Dt}\big(z - h(x,y,t)\big) = -h_t + w - uh_x - vh_y = -h_t + W - Uh_x - Vh_y$$

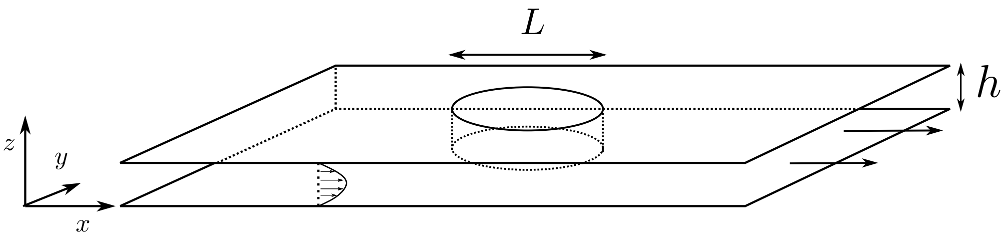
which implies

$$\boxed{\frac{1}{6\mu}\nabla \cdot (h^3 \nabla p) = h_t + W \qquad (3.7b)}$$

If h is small but not constant, we can see that a typical value for the pressure in the gap is $p \sim \mu W L^2/h_0^3$. We note that if h_0 is small this pressure can be very large indeed. It is this feature which is very important in a variety of practical problems. We also note that in the important cases when the surface $z = h$ is planar or axisymmetric, (3.7) is just an ODE, which we can hope to solve exactly.

Lubrication flows: Examples

(a) Hele-Shaw flow



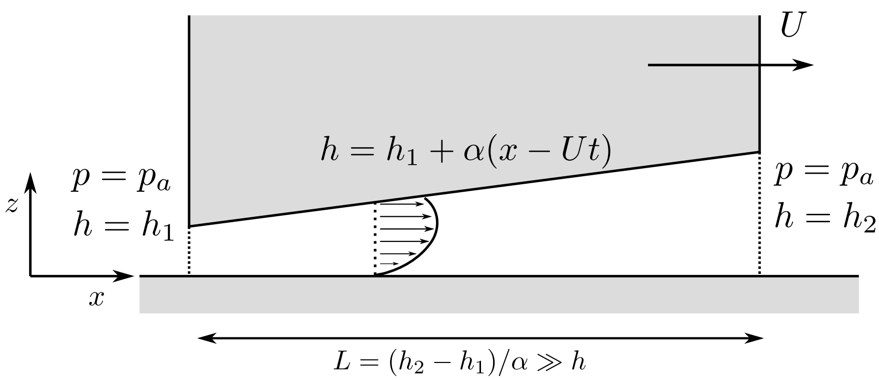
If h is constant and the top boundary is not moving, $U = V = W = 0$, then (3.7) reduces to

$$\nabla^2 p = 0 \quad \text{and} \quad (\overline{u},\, \overline{v},\, 0) = -(h^2/12\mu)\nabla p, \tag{3.8}$$

where $\overline{u}$ and $\overline{v}$ are the values of u and v averaged over z. This flow between two close rigid plates is called Hele-Shaw flow, with the large-scale variation L in x and y directions due to obstacles or inclusions. Curiously, this highly viscous flow is the easiest way to achieve two-dimensional potential flow, beloved of inviscid theory.

(b) Slider bearing

Consider a finite plane sliding over a stationary plane, with velocity $(U,\, 0,\, 0)$ so that $h = h_1 + \alpha(x - Ut)$ where $\alpha \ll 1$ and $V = W = 0$.



Equation (3.7b) thus reduces to

$$(h^3 p_x)_x = -6\mu\alpha U \qquad \text{or} \quad (h^3 p_h)_h = -6\mu U/\alpha\ . \tag{3.9}$$

where we may think of p depending on h instead of x. Integrating between the two ends of the plane $h = h_1$ and $h = h_2$ say, where we assume the pressure is atmospheric, $p = p_a$, we find

$$p = \frac{6\mu U}{\alpha h} + \frac{A}{h^2} + B = p_a + \frac{6\mu U}{\alpha(h_1 + h_2)h^2}(h - h_1)(h_2 - h)\ . \tag{3.10}$$

We see that $p > p_a$ if $U > 0$ for $h_1 < h < h_2$ so we expect a force to act separating the planes. We now find the force on the sliding plane. First we note that $||\mu e_{ij}|| \sim \mu U/h \ll$

$p \sim \mu U/(\alpha h)$ Thus the pressure part of the stress dominates. This acts normally to the plate, and the normal to the plane is

$$\mathbf{n} = \frac{\nabla(z-h)}{|\nabla(z-h)|} = \frac{1}{(1+\alpha^2)^{1/2}}(-\alpha,\, 0,\, 1) \sim (-\alpha,\, 0,\, 1) \quad \text{as } \alpha \ll 1,$$

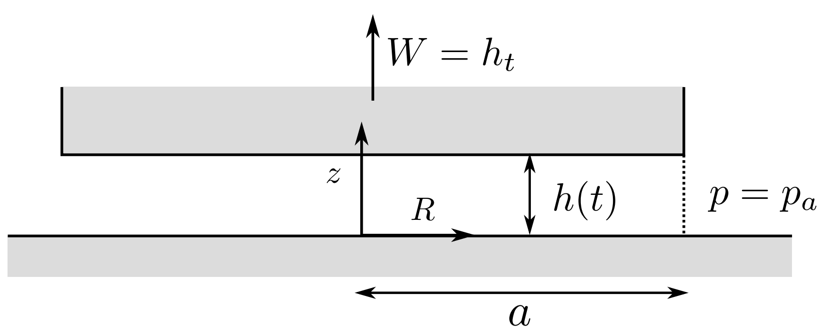
so the normal force is $(p - p_a)\mathbf{n} \sim (p - p_a)(-\alpha, 0, 1)$. Thus the x-component of the force is $O(\alpha)$ times the z-component. As the motion is in the x-direction we can interpret this as meaning the drag force is much smaller than the lift force. The total force per unit length in the y-direction is

$$\int (p - p_a)\, dx = \int_{h_1}^{h_2} \frac{(p - p_a)\, dh}{\alpha} = \frac{6\mu U}{\alpha^2}\left[\ln\left(\frac{h_2}{h_1}\right) - 2\frac{h_2 - h_1}{h_1 + h_2}\right]. \tag{3.11}$$

The quantity in square brackets can be shown to be positive for $h_2 > h_1$ as it should be. Note that the lift force, which is $O(1/\alpha^2)$, is very large for small α. It increases with μ and U. Physically, fluid is being dragged into the region between the planes, keeping them apart. If U were negative however, then the reverse would apply, and the solid planes would soon come into contact.

(c) Squeeze films – viscous adhesion

Consider now a circular disc radius a, a distance $h(t)$ above the plane $z = 0$. Then $U = V = 0$ and $W = h_t$.



Using cylindrical polars $(R,\, \phi,\, z)$, (3.7b) becomes

$$\frac{h^3}{R}(Rp_R)_R = 12\mu h_t \quad \Longrightarrow \quad p(R,\, t) = \frac{3\mu h_t}{h^3}(R^2 + A\ln R + B)\ . \tag{3.12}$$

If the disc includes $R = 0$ then we must have $A = 0$ and imposing $p = p_a$ at $R = a$ we have

$$p - p_a = \frac{3\mu h_t}{h^3}(R^2 - a^2)\ . \tag{3.13}$$

The total force exerted on the disc in the z-direction is therefore

$$F_{\text{tot}} = \int_0^a (p - p_a) 2\pi R\, dR = -\frac{3\mu\pi a^4 h_t}{2h^3}$$

The sign indicates that the thin film resists the motion, so that if $h_t > 0$ the force is in the negative z-direction. If a constant force $(0,\, 0,\, F)$ is applied to the disc for $t > 0$ when $h = h_0$, and the plane is held fixed, we can find the time T needed to separate the surfaces, i.e. for h to reach ∞:

$$FT = -\int_0^T F_{\mathrm{tot}} dt = \tfrac{3}{2}\mu\pi a^4 \int_0^T \frac{h_t}{h^3} dt = \tfrac{3}{2}\mu\pi a^4 \int_{h_0}^{\infty} \frac{dh}{h^3} = \frac{3\mu\pi a^4}{4h_0^2} \; . \tag{3.14}$$

Once more, we see the strong influence of the thin gap $h_0 \ll a$ so that T is large. Also T increases as μ increases. Formally, if we try to squeeze the fluid out of the gap now with a force $(0, 0, -F)$, the time to reach a smaller height $\epsilon < h_0$ is

$$FT = \int_0^T F_{\mathrm{tot}} dt = \tfrac{3}{2}\mu\pi a^4 \int_{\epsilon}^{h_0} \frac{dh}{h^3} = \tfrac{3}{4}\mu\pi a^4 \left(\frac{1}{\epsilon^2} - \frac{1}{h_0^2} \right), \tag{3.15}$$

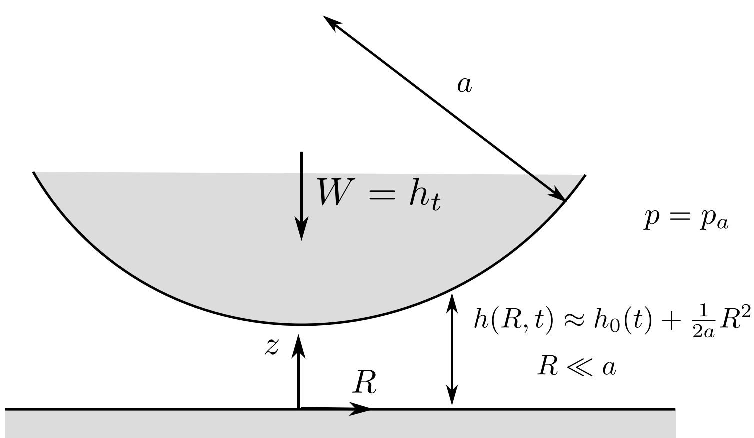
which diverges as $\epsilon \to 0$. Thus it takes an infinite time for the surfaces to touch. Although in practice, surface roughness (or if necessary molecular scales) limit the minimum separation.

Suppose now the disc has a small hole in it at $R = b \ll a$. Equation (3.14) is then

$$p - p_a = \frac{3\mu h_t}{h^3} \left[(R^2 - a^2) + (a^2 - b^2) \frac{\ln(R/a)}{\ln(b/a)} \right] \; . \tag{3.16}$$

Now calculating the total force, and using that $a \ll b$, the result is approximately $(1 - 1/\ln(a/b))$ times the earlier result. This can make quite a difference – even if $b = 0.01a$ this factor equals 0.78, reducing the required force by 22%. The difference is that fluid can now flow through the hole, reducing the pressure where it was highest: at the centre of the disc $(R = 0)$.

(d) Sphere falling towards a plane



When a sphere of radius a is a distance h_0 from the plane, we can show that at a radial distance R

$$h(R,\, t) \simeq h_0(t) + \tfrac{1}{2}\frac{R^2}{a} \qquad \text{for} \quad R \ll a. \tag{3.17}$$

Once again, taking $U = V = 0$, $W = h_t$, we have

$$\frac{1}{R}\frac{\partial}{\partial R}\left[Rh^3\frac{\partial p}{\partial R}\right] = 12\mu h_t \quad \Longrightarrow \quad \frac{\partial p}{\partial R} = \frac{6\mu h_t R}{h^3}, \tag{3.18}$$

using that p is finite at $R = 0$. Noting that

$$\frac{R}{h^3} = \frac{a}{h^3}\frac{\partial h}{\partial R} = -\frac{1}{2}a\frac{\partial}{\partial R}\left(\frac{1}{h^2}\right),$$

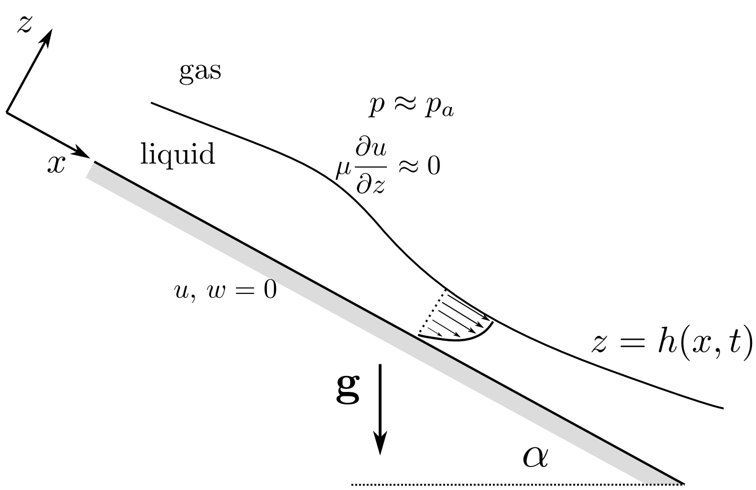
the integration for p can be done exactly, giving

$$p = p_a - \frac{3\mu a h_t}{h^2} \quad \Longrightarrow \quad \int_0^\infty (p - p_a)2\pi R\, dR = -\frac{6\mu h_t a^2}{h_0} \,. \tag{3.19}$$

Note we have integrated to $R = \infty$. By this we mean $R \gg h_0$, but with $R \ll a$ still! We can incorporate this force in an equation of motion for a solid sphere of mass M a distance $h_0(t)$ above a table: $Mg = -6\mu a^2 h_{0,t}/h_0$. We find that, formally, it takes an infinite time to reach the table! In practice, balls can bounce because the large lubrication pressures cause the sphere or surface to deform. See the youtube video in the Videos folder on Blackboard.

Lubrication Theory: Flow of a thin layer down a slope

We have seen that a thin film between two solid bodies can have a very large pressure. However, if the thin film is a liquid with a free surface (i.e. a gas) on one side, the pressure is constrained by continuity of the normal stress to be close to atmospheric and so cannot be large. We have ignored gravity so far, as the pressures generated between two solid boundaries are usually much larger than hydrostatic pressures. But if gravity is the driving force then it must clearly be included in equations (3.3) and (3.7). For example, we will consider a thin layer of thickness $h(x,\, t)$, of infinite extent in the y-direction. The layer flows under gravity down a plane inclined at an angle α to the horizontal. The x-direction points down the plane and z-direction is normal to it, and the velocity is 2D, $(u(x, z, t),\, 0,\, w(x, z, t))$. There is no variation nor flow in the y-direction.



Thin film equations: Gravity acts as a body force $\mathbf{F} = \rho\mathbf{g} = (\rho g \sin\alpha,\, 0,\, -\rho g\cos\alpha)$. Again, the film has a typical thickness scale h_0, and variations parallel to the slope occur over a scale $L \gg h_0$. Then we neglect the same terms as before, but now we include the gravitational terms. We have mass conservation,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{3.18}$$

the x-momentum equation

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu\frac{\partial^2 u}{\partial z^2}, \tag{3.19}$$

and z-momentum

$$0 = -\frac{\partial p}{\partial z} - \rho g\cos\alpha. \tag{3.20}$$

From the gravity terms, (3.20) suggests a pressure scale $p \sim P_0 = \rho g h_0 \cos\alpha$ and (3.19) suggests the velocity scale $u \sim U_0 = \rho g h_0^2 \sin\alpha/\mu$. Then, as usual, (3.18) gives the other velocity scale $w \sim W_0 = U_0 h_0/L$. Equation (3.20) integrates to give

$$p = -\rho g z\cos\alpha + A(x, t). \tag{3.21}$$

To determine A, we need the stress condition at the free surface.

Free-surface stress balance: Continuity of stress at the surface $z = h$ is

$$\sigma_{ij} n_j = -p_a n_i \quad \text{(negligible gas viscosity)} \tag{3.22}$$

where the only force exerted on the liquid is a constant pressure from the gas above (and surface tension neglected). The normal component is

$$-p + \mu n_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j = -p_a \tag{3.23}$$

but $\mathbf{n} \approx (-\partial h/\partial x, 0, 1)$, and the largest viscous stress term scales like

$$2\mu \frac{\partial w}{\partial z} \sim \frac{\mu U_0}{L} = \rho g \sin \alpha \frac{h_0^2}{L} \ll P_0 \sim p. \tag{3.24}$$

So the pressure dominates the normal stress, giving that p must be atmospheric

$$p = p_a \quad \text{on } z = h. \tag{3.25}$$

The tangential component of the stress only has viscous terms,

$$\mu t_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j = 0 \tag{3.26}$$

and using that the tangent $\mathbf{t} \approx (1, 0, \partial h/\partial x)$, the dominant term turns out to be $\mu \partial u/\partial z$. So the stress-free tangential component becomes

$$\frac{\partial u}{\partial z} = 0 \quad \text{on } z = h. \qquad (3.27).$$

Solution: Applying the atmospheric pressure condition to (3.21) results in

$$p(x, z, t) = p_a + \rho g \cos \alpha (h(x, t) - z). \tag{3.28}$$

This states that the pressure is **hydrostatic** in z, based only on the local film thickness $h(x, t)$. Substituting this into (3.19) we have

$$\rho g \left(\cos \alpha \frac{\partial h}{\partial x} - \sin \alpha \right) = \mu \frac{\partial^2 u}{\partial z^2} \tag{3.29}$$

We can integrate this twice. On the solid wall ($z = 0$) the velocity is zero, $u = 0$. On the free surface ($z = h$), we have the tangential stress condition (3.27). Integrating and imposing these boundary conditions, we have

$$u = \frac{\rho g}{2\mu} \left(\sin \alpha - \cos \alpha \frac{\partial h}{\partial x} \right) (2zh - z^2). \tag{3.30}$$

We could now calculate w from mass conservation (3.17), but it's less algebra first to consider the kinematic condition

$$0 = \frac{D}{Dt}\big(h(x,t) - z\big) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - w \qquad \text{on } z = h. \tag{3.31}$$

Then, integrating $\partial u/\partial x + \partial w/\partial z = 0$ across the film,

$$w(x,h) = -\int_0^h \frac{\partial u}{\partial x}\,dz = -\frac{\partial}{\partial x}\int_0^h u(x,z,t)\,dz + u(x,h,t)\frac{\partial h}{\partial x}, \tag{3.32}$$

and combining (3.32) and (3.31), we arrive at

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,dz = 0. \tag{3.33}$$

which can be interpreted as the depth-averaged form of conservation of mass. Substituting our parabolic solution (3.30) we therefore have

$$\boxed{\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\left(\sin\alpha - \cos\alpha\,\frac{\partial h}{\partial x}\right)\right) = 0.} \tag{3.34}$$

This equation describes the evolution of a non-uniform layer as it falls down the plane. It is a nonlinear wave equation (with diffusion) for $h(x,t)$, permitting many wave-like solutions, e.g. travelling waves, solitary waves, periodic waves (in x and/or t). Next, we will consider a travelling wave solution to this equation, while on problem sheet 2 a similarity solution for the draining of honey off a vertical knife is considered.

Including surface tension: Earlier we commented that we neglected surface tension. It is not hard to include. Surface tension contributes an additional term γK to the normal stress balance, where K is the curvature of h. In general, the curvature of $h(x,t)$ is

$$K = \frac{1}{(1 + (\partial h/\partial x)^2)^{3/2}}\frac{\partial^2 h}{\partial x^2} \tag{3.35}$$

but this reduces to $K \approx \partial^2 h/\partial x^2$ under the thin film assumption, since $\partial h/\partial x \ll 1$. Including this gives the modified pressure condition $p = p_a - \gamma\partial^2 h/\partial x^2$ at the free surface. This extra term follows through to the evolution equation for h, giving

$$\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\left(\sin\alpha - \cos\alpha\,\frac{\partial h}{\partial x} + \frac{\gamma}{\rho g}\frac{\partial^3 h}{\partial x^3}\right)\right) = 0. \tag{3.36}$$

with now up to 4 spatial derivatives present! Formally, we need the ratio of surface tension effects to gravitational effects be large, i.e. $\gamma/(\rho g h_0^2) \gg 1$ (this is often true in practice) so that the new term is not negligible compared to the other terms.

Lubrication Theory: Gravity currents

We have previously showed that a viscous layer of thickness $h(x,t)$ travelling under gravity down a slope of angle α satisfies the equation (neglecting surface tension)

$$\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\left(\sin\alpha - \cos\alpha\,\frac{\partial h}{\partial x}\right)\right) = 0. \qquad (3.37)$$

This equation can permit many kinds of solution, depending on the boundary conditions and initial conditions. Here we will consider two types: a *travelling wave solution* and a *similarity solution.*

(a) Travelling wave

The idea is that a variety of initial conditions can converge upon a solution of fixed shape that travels down the slope with a constant speed. In other words, we shall seek a solution of (3.37) in the form

$$h = f(\eta) \qquad \text{where} \quad \eta = x - Vt \qquad \text{for some constant } V. \qquad (3.38)$$

We shall be particularly interested in solutions which describe a uniform height as $x \to -\infty$ and which have a “leading edge” where the thickness shrinks to zero, i.e. $h = 0$. This kind of solution might be relevant say to the flows of lava or mud down a mountain.

An advantage of seeking travelling wave solutions is that they reduce the number of independent variables by one. The partial derivatives reduce to

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d}{d\eta} = -V\frac{d}{d\eta} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{d}{d\eta} = \frac{d}{d\eta} \qquad (3.39)$$

and thus the PDE (3.37) becomes the ODE

$$-Vf' + \frac{\rho g}{3\mu}(f^3(\sin\alpha - \cos\alpha f'))' = 0, \qquad \text{where } f'(\eta) = df/d\eta. \qquad (3.40)$$

This integrates to give

$$-Vf + \frac{\rho g}{3\mu}f^3(\sin\alpha - \cos\alpha f') = A, \qquad \text{with } A \text{ a constant.} \qquad (3.41)$$

If we want to match onto a state where $f \to 0$ for some value of η, then we must choose the constant $A = 0$. Suppose that a long way upstream, the layer approaches a uniform height H so that

$$f \to H \quad \text{and} \quad f' \to 0 \qquad \text{as } \eta \to -\infty. \qquad (3.42)$$

We can then deduce the wave speed V as we must have

$$-VH + \frac{\rho g}{3\mu}\sin\alpha H^3 = 0 \quad \Longrightarrow \quad V = \frac{\rho g}{3\mu}\sin\alpha H^2. \qquad (3.43)$$

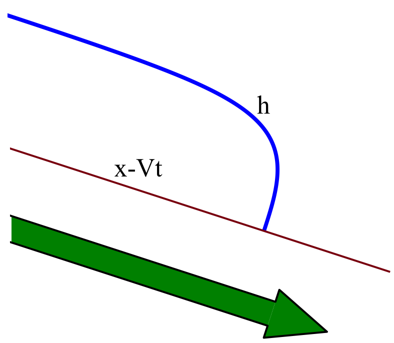
(3.41) then reduces to

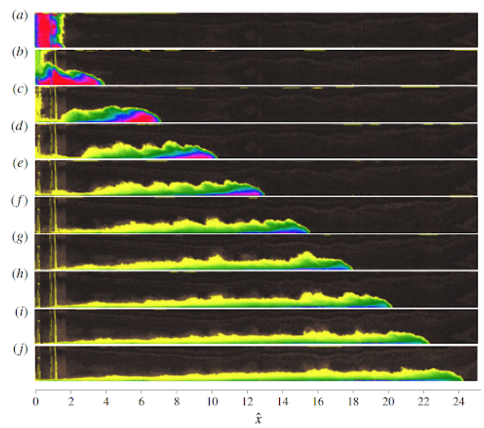
$$f^2(1 - \cot\alpha f') = H^2. \qquad (3.44)$$

We can simplify the problem further by defining $f(\eta) = HF(\xi)$ where $\xi = \eta \tan\alpha/H$. The ODE reduces to

$$F^2\left(1 - \frac{dF}{d\xi}\right) = 1 \quad \Longrightarrow \quad \xi = \int \frac{F^2 dF}{F^2 - 1} = F + \tfrac{1}{2}\log\frac{1-F}{1+F} + C. \tag{3.45}$$

The constant C can be regarded as time translation, essentially defining the location of the travelling wave in x at the initial time $t = 0$. We can choose $C = 0$, then $\xi = 0$ at $F = 0$. We then have F given implicitly in terms of ξ.





Our solution for $F(\xi)$ (or $f(\eta)$, or $h(x, t)$) is plotted in the left hand diagram above. We note that at the leading edge where $F \to 0$ and the film thickness vanishes, we see the slope become vertical, i.e. $F' \to -\infty$. Indeed, considering the behaviour of (3.45) for small F, we find

$$\xi \sim -\frac{1}{3}F^3 \quad \Longrightarrow \quad F \sim -3^{1/3}\xi^{1/3} \quad \text{as } \xi \to 0.$$

However, this might lead us to question whether the lubrication approximation remains valid in the region near $F = 0$. The travelling wave solution agrees reasonably with experiment, although in many problems of interest the basic lubrication equations do not really hold. One such experiment, taken from a paper by Sher and Woods (Journal of Fluid Mechanics, 2015), is illustrated in the right-hand diagram above. In it, a partition between regions of dense and light fluid is released, and the resulting flow is recorded at equal time intervals. Once the barrier is lifted, the heavier fluid advances to the right in what becomes approximately a travelling wave pattern, while the light fluid moves to the left on top of it. Initially, the layer of heavy fluid is certainly not thin, but it arguably becomes thin as time evolves. We can see several differences with our theory. Firstly, we see that the top surface of the heavier fluid is far from flat, for reasons we'll discuss later in the course. Furthermore, the leading edge of the gravity current appears to be angled rather than vertical. There are many differences between this experiment and our theory, for example the density (and viscosity) differences between the two fluids are much smaller in the experiment. And of course the solid boundary is horizontal in the experiment. https://www.youtube.com/watch?v=HALHkKcFbg8 is a slightly less controlled experiment, which underlines the importance of understanding gravity currents down slopes.

(b) Horizontal spreading: similarity solution

Another type of gravity current is the spreading of a finite drop on a horizontal surface. For a 2D liquid, this corresponds to (3.37) with $\alpha = 0$, giving

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right). \tag{3.46}$$

The fluid will not keep its shape in this case, but instead spread symmetrically out due to gradients in hydrostatic pressure. The fluid has a fixed volume V (per unit length in y, in which there is no variation or spreading) and the condition at the edges is

$$h = 0 \quad \text{at } x = \pm x_0(t). \tag{3.47}$$

with conservation of volume, where by symmetry we only need to consider $0 < x < x_0(t)$,

$$\int_0^{x_0(t)} h\,dx = \frac{1}{2}V. \tag{3.48}$$

Solutions are observed to converge to one that is *self similar*, where the solution at all times can be collapsed to a single shape under a stretch of coordinates. Hence, we look for a solution of the form

$$h(x,t) = t^b f(\eta) \quad \text{where } \eta = xt^a, \tag{3.49}$$

and a and b are numerical values that we must find. Then, we seek an ODE for the shape $f(\eta)$. Substituting into the volume conservation first, we find

$$t^{b-a}\int_0^{\eta_0} f(\eta)\,d\eta = \frac{1}{2}V \qquad \Longrightarrow \qquad \int_0^{\eta_0} f(\eta)\,d\eta = \frac{1}{2}V \tag{3.50}$$

and hence for this to not depend on t we have chosen $a = b$, but their value is not yet known. The derivatives of h become

$$\frac{\partial h}{\partial t} = at^{a-1}\left(f + f'\eta\right), \quad \frac{\partial h}{\partial x} = t^{2a}f', \quad \frac{\partial^2 h}{\partial x^2} = t^{3a}f'' \tag{3.51}$$

which substituted into (3.46) gives

$$at^{a-1}\left(f + \eta f'\right) = t^{6a}\frac{\rho g}{3\mu}\left(f^3 f'' + 3(ff')^2\right) \quad \Longrightarrow \quad -\frac{1}{5}\left(\eta f\right)' = \frac{\rho g}{3\mu}\left(f^3 f'\right)' \tag{3.52}$$

and we must have $a = -1/5$ for the time dependence to drop out. This ODE for f can be integrated, using the edge condition $f(\eta_0) = 0$ and the volume constraint (3.50). Integrating once, the integration constant is zero since $f(\eta_0) = 0$ so

$$-\frac{1}{5}\eta f = \frac{\rho g}{3\mu}f^3 f' \quad \text{or} \quad -\frac{1}{5}\eta = \frac{\rho g}{9\mu}(f^3)' \tag{3.53}$$

We can define $f(\eta) = HF(\eta)$ where $H = (9\mu/10\rho g)^{1/3}$ to remove the parameters from the problem. Then integrating and applying $F(\eta_0) = 0$, we have

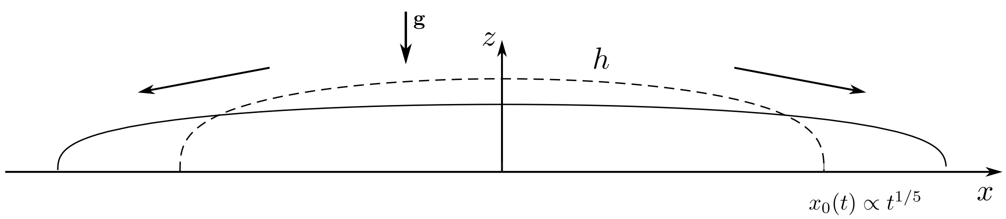
$$F(\eta) = (\eta_0^2 - \eta^2)^{1/3}. \tag{3.54}$$

The value of η_0 is found from the volume constraint (3.50), which reduces to (after a change of integration variable $\eta = \eta_0 s$)

$$\eta_0^{5/3} \int_0^1 (1 - s^2)^{1/3} ds = \frac{V}{2H}. \tag{3.55}$$

The integral can not be done analytically, but is easily calculated numerically to be $\approx 0.84...$ Rearranging and using the definition of H gives

$$\eta_0 = 0.73.. \left(\frac{10\rho g V^3}{9\mu} \right)^{1/5}$$



The shape of the drop (h at two different times) is shown in the figure, and the edge then evolves according to $x_0(t) = \eta_0 t^{1/5}$. This growth with time is rather slow. If it takes 10 seconds for it to reach a given width, then it will take 6 minutes to double in width, and 3 hours to double again! For an axisymmetrically spreading drop, the radius $R_0(t)$ grows even slower, $R_0 \propto t^{1/8}$. These scaling laws with time generally agree rather well with experiment.

This concludes the chapter on Lubrication Theory. The arguments behind lubrication flows are very powerful, and often enable "analytical" solutions to be found. In conclusion, recall that these viscosity dominated, thin-layer flows can occur even when the Reynolds number is large $Re \gg 1$ provided $Re(h_0/L)^2 \ll 1$. This might lead us to wonder what happens in layers of thickness h when $Re(h_0/L)^2 \simeq 1$ and inertia is no longer negligible? This leads us onto the important topic of **boundary layers** in the next chapter.

Chapter 3: Boundary Layer Theory

The Navier-Stokes equations behave well for small Reynolds numbers. It can be shown that a solution exists, and we proved that this solution was unique. Furthermore, the solutions are smooth and regular. At high Reynolds number the nonlinear $\mathbf{u}\cdot\nabla\mathbf{u}$ term gains in significance and the situation is very different. Existence is hard to prove, and there may be more than one possible solution. Smooth, steady, symmetric flows may suddenly become unsteady and asymmetric for no obvious reason. Laminar (smooth) flows may become unstable and **turbulence** may develop. Furthermore, singular regions may form, especially near solid boundaries. These thin layers near boundaries are known as "boundary layers."

Introduction to Boundary Layer Theory

To understand why this occurs, consider the following simple ODE for $y(x)$ in terms of a small positive parameter ε

$$\varepsilon y''(x) + y'(x) = -1 \qquad \text{with} \quad y(0) = 0 = y(1). \tag{4.1}$$

This problem has the exact solution

$$y = -x + \frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}. \tag{4.2}$$

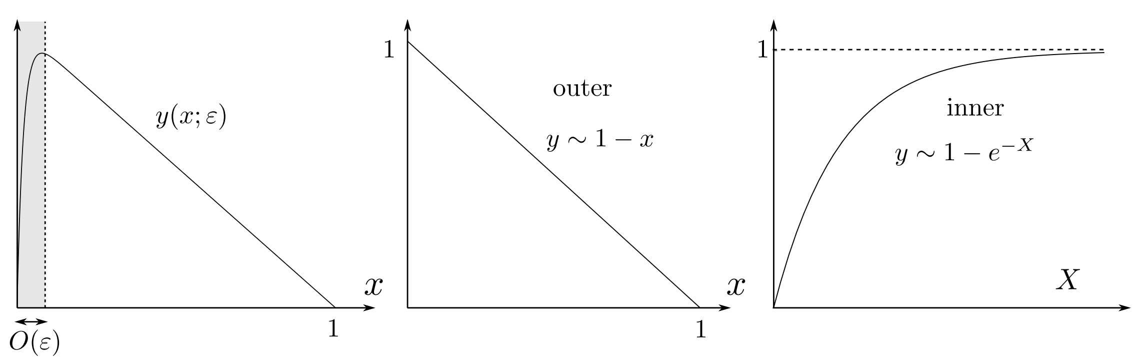
In the limit $\varepsilon \to 0$, this function essentially appears like

$$y \sim 1 - x \qquad \text{if } \varepsilon \to 0 \text{ with } x \text{ fixed}, \tag{4.3}$$

except when $x \sim \varepsilon$. Over a layer of thickness $O(\varepsilon)$ near $x = 0$, the solution falls from 1 to 0. This is an example of a boundary layer, and arises here because settting $\varepsilon = 0$ in (4.1), reduces the ODE from 2nd order to 1st order. A 1st order ODE cannot satisfy both boundary conditions, so large gradients appear near one of the boundaries, causing the $\varepsilon y''$ term to no longer be negligible there. This is an example of a **singular perturbation.**

The approximation (4.3) is the so-called "outer" solution. A corresponding "inner" solution in the thin region where $x = O(\varepsilon)$ can be found by keeping $x/\varepsilon = X = O(1)$ fixed as $\varepsilon \to 0$ in (4.2), giving

$$y \sim 1 - e^{-X} \qquad \text{if } \varepsilon \to 0 \text{ with } X \text{ fixed}. \tag{4.4}$$



Matched Asymptotic Expansions

Suppose we didn't have the exact solution, only the ODE. We may find the above "inner" and "outer" solutions by using the technique of "matched asymptotic expansions." First, we expand look for an expansion

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \quad \varepsilon \to 0$$

given x fixed, which we substitute into (4.3) and equate powers of ε. At leading order, $O(\varepsilon^0)$, we obtain $y_0' = -1$ or $y_0 = A - x$ for some constant A. We notice we can satisfy only one of our two boundary conditions. Therefore this solution will break down close to a boundary. If we impose $y_0(1) = 0$ then $A = 1$, but then $y_0(0) = 1$ whereas we want $y(0) = 0$. This is the "outer" expansion. Close to $x = 0$ we need to use a different expansion.

Close to the boundary $x = 0$, we look for a solution with x/ε fixed as $\varepsilon \to 0$, i.e. we can define the inner or stretched coordinate $X = x/\varepsilon$ where $X = O(1)$. Then rewriting the equation in terms of X and $Y(X) = y(x)$, noting $d/dX = \varepsilon d/dx$, (4.3) becomes

$$Y''(X) + Y'(X) = -\varepsilon. \tag{4.5}$$

If we now expand $Y(X)$ for $\varepsilon \to 0$, and look at the leading order problem,

$$Y = \sum_{n=0}^{\infty} \varepsilon^n Y_n(X) \quad \Longrightarrow \quad Y_0'' + Y_0' = 0 \quad \Longrightarrow \quad Y_0 = C + De^{-X}. \tag{4.6}$$

We impose $Y_0(0) = 0$ now so that $Y_0 = C(1 - e^{-X})$. To determine the constant C we cannot impose the condition at $x = 1$, as that boundary is at $X = 1/\varepsilon \to \infty$, and outside the inner domain. Instead, C is found by *matching* this inner solution with the outer solution,

$$\lim_{X \to \infty} Y_0(X) = \lim_{x \to 0} y_0(x). \tag{4.7}$$

This says that the limit leaving the inner region ($X \to \infty$) must equal the limit leaving the outer region ($x \to 0$). Here this gives $C = 1$. Our (approximate) solution throughout the whole domain is (written in terms of the original coordinate x only)

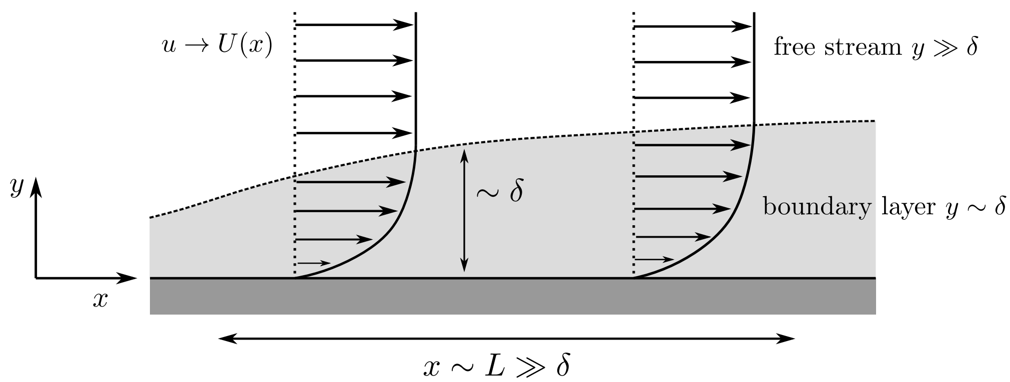
$$y \sim \begin{cases} 1 - x & \text{if} \quad x = O(1) \\ 1 - e^{-x/\varepsilon} & \text{if} \quad x = O(\varepsilon) \end{cases} \tag{4.8}$$

Derivation of the Boundary Layer Equations

The incompressible Navier-Stokes equations,

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{4.9}$$

are similar to our ODE model for very small kinematic viscosity $\nu = \mu/\rho$ (high Reynolds numbers). The kinematic viscosity ν multiplies the highest derivative in the equation, so that if we set $\nu = 0$ (giving the Euler equations for inviscid flow), one of the boundary conditions cannot be satisfied. Inviscid flows may have slip velocities over solid walls. When the viscosity is small but non-zero ($0 < \nu \ll 1$) we therefore anticipate that thin layers may develop near the walls across which the tangential velocity adjusts to zero from its inviscid value.



We will consider a steady 2D boundary layer at a solid wall. Let the layer thickness be of asymptotic size δ, and let x and y be coordinates locally tangential and parallel to the wall, with typical scales of variation L in x and δ in y, with $L \gg \delta$. Let the corresponding velocity components u and v have typical scales U_0 and V_0 in the boundary layer. Then from the continuity condition

$$u_x + v_y = 0 \qquad \Longrightarrow \qquad V_0 \sim U_0\delta/L\ , \tag{4.10}$$

so that just as in lubrication theory, the normal velocity is small. Taking now the x-momentum equation,

$$uu_x + vu_y = -\frac{1}{\rho}p_x + \nu(u_{xx} + u_{yy}), \tag{4.11}$$

we see that the V_0 scale implies that the inertial terms uu_x and vu_y are of similar size, $\sim U_0^2/L$. Normal derivatives ($\partial^2/\partial y^2$) in the viscous term will be much larger than the tangential derivatives ($\partial^2/\partial x^2$) , so that $\nu u_{xx} \ll \nu u_{yy} \sim \nu U_0/\delta^2$.

"Prandtl's Boundary Layer Hypothesis" states that in the boundary layer the inertial terms and the viscous terms should balance. This determines the **boundary layer thickness scale** δ,

$$uu_x \sim \nu u_{yy} \qquad \Longrightarrow \qquad \frac{U_0^2}{L} \sim \frac{\nu U_0}{\delta^2} \qquad \Longrightarrow \qquad \delta \sim \left(\frac{\nu L}{U_0}\right)^{1/2}. \tag{4.12}$$

Now as y increases and we leave the boundary layer, we expect the viscous terms to become negligible. The flow should approach the inviscid flow solution, for which there is a slip velocity, i.e. $u \to U(x)$ as $y/\delta \to \infty$. So at the top of the boundary layer ($y/\delta \to \infty$), x-momentum reduces to

$$UU'(x) = -\frac{1}{\rho}p_x, \tag{4.13}$$

which gives a scale $P_0 \sim \rho U_0^2$ for the pressure in the layer. Turning to the y-momentum equation, this means the pressure term is $p_y/\rho \sim U_0^2/\delta$, which is $O(L^2/\delta^2)$ larger than either the inertial or viscous terms (which balance each other):

$$\nu v_{yy} \sim u v_x \sim \left(\frac{\delta}{L}\right)\left(\frac{U_0^2}{L}\right) \ll \frac{U_0^2}{\delta} \sim p_y/\rho$$

It follows, just as for lubrication theory, that $p_y = 0$, so that the pressure does not vary across the layer. Putting all this together, we obtain the **boundary layer equations**

$$uu_x + vu_y = -\frac{1}{\rho}p_x + \nu u_{yy}, \qquad p_y = 0, \qquad u_x + v_y = 0. \tag{4.14}$$

As p and hence p_x are constant across the layer, we can evaluate p_x using its value at the top, given by (4.12). The boundary layer equations then take the simpler form

$$\boxed{uu_x + vu_y = UU'(x) + \nu u_{yy} \qquad u_x + v_y = 0. \qquad (4.15)}$$

An appropriate set of boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0, \qquad u \to U(x) \quad \text{as} \quad y \to \infty. \tag{4.16}$$

A single equation can be obtained by introducing a streamfunction $\psi(x, y)$ where $u = \psi_y$ and $v = -\psi_x$. Equation (4.15) then takes the form

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = UU'(x) + \nu\psi_{yyy}\ . \tag{4.17}$$

Remarks on the boundary layer equations

(1) **Nondimensionalise:** Often it is convenient to nondimensionalise the problem. For example, one can rescale x by L, y by δ, u by U_0 and v by $\delta U_0/L$. We can write the thickness

$$\delta = L(Re)^{-1/2} \quad \text{where} \quad Re = U_0 L/\nu$$

is the Reynolds number. Most boundary layers have thickness $O(Re^{-1/2})$, but some problems have other rational powers, e.g. $Re^{-1/3}$, Re^{-1}, $Re^{-3/8}$, $Re^{-1/7}$, and there are more esoteric examples.

(2) **Normal velocity:** As $y/\delta \to \infty$ we cannot require that $v \to 0$. There is in fact a small velocity ($v \sim \delta$) flowing out of the boundary layer—see the upcoming Blasius flow example.

(3) **Parabolic type:** The boundary layer equation (4.15) is **parabolic** in x and y, with x taking the part of the time-like variable (compare $u_t = u_{xx}$). This means we must integrate downstream, in the direction of increasing x if $u > 0$. There is practically no upstream influence in boundary layers, whereas the full steady Navier-Stokes equations are **elliptic.**

(4) **Separation:** As we integrate the equations downstream, the equations can behave well, or they can develop a singularity. In the latter case, the boundary layer assumptions break down, it is not longer confined to a thin layer close to the wall and **the boundary layer separates**. This alters the external flow structure. A large wake may form, for example, allowing us to shelter behind objects in a strong wind. This is the resolution of d'Alembert's paradox, the fact that potential flows exert no force on an object in the flow, contrary to our every day experience.

(5) **Pressure gradient:** Generally speaking, separation will not occur while $UU' > 0$, a situation described as a 'favourable pressure gradient' or 'accelerating external flow'. Once U begins to decrease, and the pressure gradient is unfavourable, then separation is likely. For example, irrotational flow around a cylinder of radius a has the streamfunction $\psi = \sin\phi(R - a^2/R)$, which has the slip velocity $U(x) = 2\sin(x/a)$, where $x = a\phi$ denotes length around the circumference of the cylinder. Now $UU' > 0$ until $x/a = \frac{1}{2}\pi$, when the surface slip-velocity begins to slow down. The layer separates off shortly after that point. The design of bodies intended to move at high Reynolds numbers (e.g. fish, aeroplanes) is governed by the need to inhibit or minimise separation of the boundary layer, a process called "streamlining."

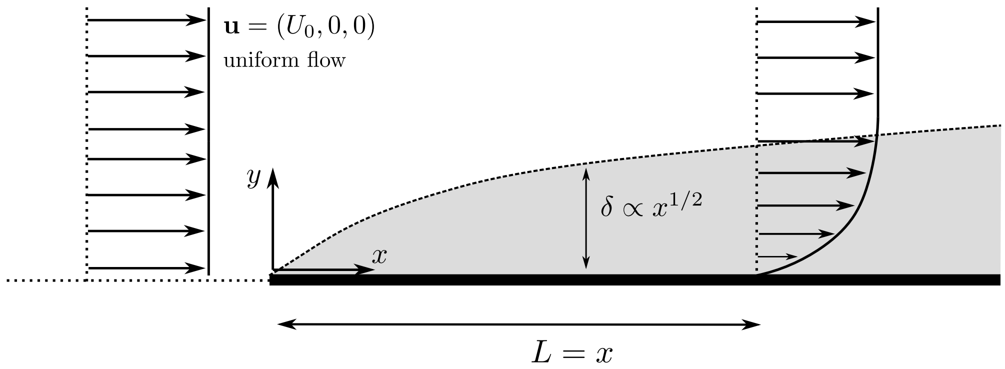
(6) **Vorticity confinement:** The vorticity for steady 2D flow satisfies $\mathbf{u} \cdot \nabla\omega = \nu\nabla^2\omega$, and in the boundary layer this simplifies to

$$u\,\omega_x + v\,\omega_y = \nu\omega_{yy}. \tag{4.18}$$

The outer inviscid flow is often potential flow (zero vorticity), and vorticity is only generated at the solid wall and is advected/diffuses into the flow. Thus, the boundary layer can be interpreted as the confinement of vorticity to a thin layer. Separation then corresponds to the failure to confine this vorticity close to the boundary, e.g. due to the advection normal to the wall becoming too strong.

(7) **Dissipation:** The vorticity and strain-rate tensor e_{ij} within the boundary layer have typical magnitude U/δ. Recall the rate of energy dissipation is $\int 2\mu e_{ij}e_{ij}\,dV$. In the main body of the fluid this has order of magnitude $\mu(U/L)^2L^3 = \mu U^2 L$. The volume of the boundary layer is $O(\delta L^2)$, so that the energy dissipated within the boundary layer is of order $\mu(U/\delta)^2L^2\delta = \mu U^2L^2/\delta$, which is a factor of L/δ larger. Most of the energy dissipation occurs within the boundary layers on solid boundaries.

Flow over a flat plate: the Blasius solution



Consider uniform flow (U_0, 0, 0) over a thin, planar solid plate at $y = 0$, $x > 0$. It is clear that were it not for the no-slip condition $u = 0$ on $y = 0$ the flow would be undisturbed. We therefore expect a boundary layer to be formed over the plate. We would expect this to be governed by (4.7) for which $U(x) = U_0$ and hence $UU' = 0$. We therefore have the problem

$$uu_x + vu_y = \nu u_{yy}, \qquad u_x + v_y = 0, \tag{4.19}$$

with boundary conditions

$$u = v = 0 \qquad \text{on } y = 0, \qquad\qquad u \to U_0 \qquad \text{as } y \to \infty. \tag{4.20}$$

This problem has an interesting feature: because the plate has zero thickness and infinite length there is no natural spatial length-scale. This suggests that

- x itself, the distance from the leading edge, may be an appropriate (variable) length-scale to use for L. This would imply the boundary layer thickness is $\delta = \sqrt{\nu x/U_0}$, which grows like $\sqrt{x}$ and therefore widens as you move downstream.
- the problem may have a **self similar** structure.

By self similar we mean that, under a stretch of our coordinates and variables, the solution at different x locations collapse to the same shape. To put it another way, suppose we make the linear transformation for some constant c

$$x \to c^2 x, \qquad y \to cy, \qquad u \to u \qquad \text{and} \quad v \to v/c. \tag{4.21}$$

Then equation (4.19) and boundary conditions are unchanged. Assuming our problem has a unique solution, then the functional form of $u(x, y)$ must be the same, therefore $u(x, y) = u(c^2 x,\, cy)$ for any choice of c. In particular, we can choose it to be $c = 1/\sqrt{x}$, then

$$u(x, y) = u\left(1,\, \frac{y}{\sqrt{x}}\right)$$

which is a function of the combination $y/\sqrt{x}$ only.

If we seek a solution in terms of a streamfunction ψ such that $u = \psi_y$, then ψ is $x^{1/2}$ times a function of $y/x^{1/2}$. So formally, let us seek a solution to (4.15) (with $U' = 0$) of the form

$$\psi = Ax^{1/2}\phi(\eta) \qquad \text{where} \quad \eta = B\frac{y}{x^{1/2}}, \tag{4.22}$$

where A and B are suitable (dimensional) constants which we will choose later for our convenience. We now calculate the partial derivatives (writing $'$ for $d/d\eta$)

$$\psi_y = AB\phi',$$
$$\psi_x = \tfrac{1}{2}Ax^{-1/2}\phi + Ax^{1/2}(-\tfrac{1}{2})Byx^{-3/2}\phi' = \tfrac{1}{2}Ax^{-1/2}(\phi - \eta\phi'),$$

and

$$\psi_{xy} = \tfrac{1}{2}Ax^{-1/2}(\phi - \eta\phi')'Bx^{-1/2} = -\tfrac{1}{2}ABx^{-1}\eta\phi'',$$
$$\psi_{yy} = AB\phi''Bx^{-1/2}$$

and so the LHS of (4.15) is

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = -\tfrac{1}{2}A^2B^2x^{-1}\eta\phi'\phi'' - \tfrac{1}{2}A^2B^2x^{-1}(\phi - \eta\phi')\phi'' = -\tfrac{1}{2}A^2B^2x^{-1}\phi\phi''.$$

The RHS of (4.15) is simply

$$\nu\psi_{yyy} = \nu Ax^{1/2}B^3x^{-3/2}\phi''' = \nu AB^3x^{-1}\phi''',$$

and so (4.15) reduces to

$$-\tfrac{1}{2}\phi\phi'' = \nu B/A\phi'''. \tag{4.23}$$

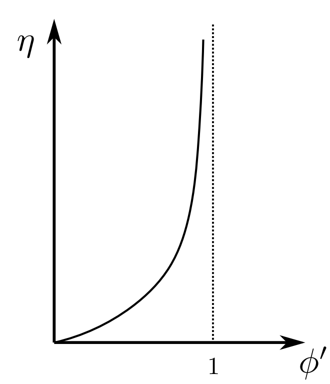
For convenience, we choose the dimensional constants B and A such that $\nu B/A = 1$. We also have the boundary condition as $y \to \infty$,

$$\psi_y \to U_0 \quad \Longrightarrow \quad AB\phi' \to U_0$$

so we choose $AB = U_0$. Therefore we have chosen $B = \sqrt{U_0/\nu}$ and $A = \sqrt{\nu U_0}$, and our problem for $\phi(\eta)$, $\eta > 0$ reduces to the **Blasius equation**,

$$\boxed{\phi''' + \tfrac{1}{2}\phi\phi'' = 0 \qquad \text{with } \phi(0) = \phi'(0) = 0 \quad \text{and } \phi' \to 1 \text{ as } \eta \to \infty.} \tag{4.24}$$

This is a 3rd order ODE which must be solved numerically. The shape of $\phi'(\eta) = u$ is shown in the figure.



The solution for the streamfunction is then

$$\psi = \sqrt{\nu U_0 x}\,\phi(\eta) \qquad \text{where} \quad \eta = \sqrt{\frac{U_0}{\nu x}}y. \tag{4.25}$$

and notice that $\eta = y/\delta(x)$ where we recall $\delta = \sqrt{\nu x/U_0}$.

The velocity v at the top of the boundary layer then follows from this solution,

$$v = -\psi_x = \frac{1}{2}\sqrt{\frac{U_0\nu}{x}}(-\phi + \eta\phi') \to 0.86\sqrt{\frac{U_0\nu}{x}} = 0.86\,U_0\delta(x) \quad \text{as } \eta \to \infty. \tag{4.26}$$

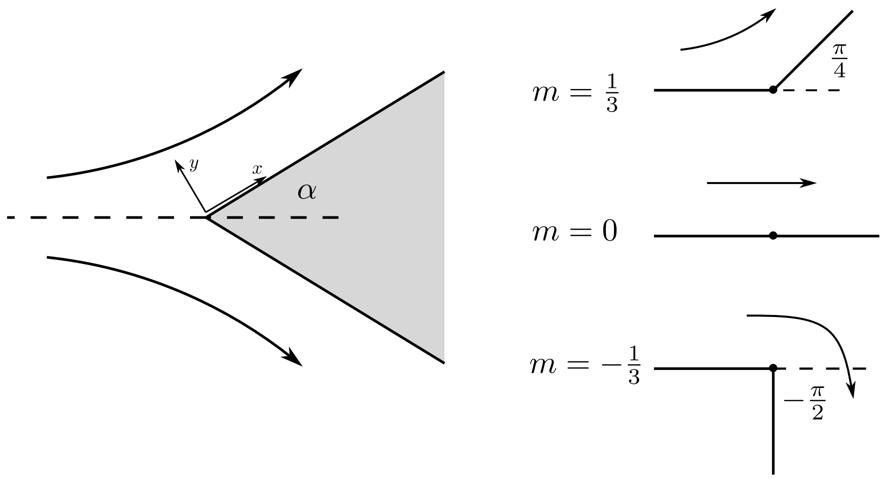
which is small, $O(\delta)$, but nonzero. Since the layer widens and the flow decelerates as you advance downstream, fluid must leave the boundary layer by conservation of mass. This induces a small correction to the outer inviscid flow.

The Falkner-Skan equations

We now try to generalise the Blasius differential equation as follows: Assume that outside the layer the velocity is

$$U(x) = U_0 \left(\frac{x}{L}\right)^m \tag{4.18}$$

where m is a given parameter (not necessarily an integer) and U_0 and L are suitable velocity and length scales. It can be considered as the inviscid slip flow on the surface of a wedge of angle 2α placed symmetrically in an oncoming flow. The potential flow about such a wedge can be found by considering the conformal mapping $\zeta = z^{m+1}$, where $m = \alpha/(\pi - \alpha)$. This gives a slip velocity on the wedge proportional to R^m in terms of polar coordinates based on the origin. Which translates into x^m in terms of the local boundary layer coordinates we are using.



We seek a **similarity solution** for the streamfunction in the boundary layer of the form

$$\psi(x, y) = Ax^a \phi(\eta) \qquad \text{where} \quad \eta = Byx^b. \tag{4.19}$$

We note $\eta_y = Bx^b$ and $\eta_x = b\eta/x$. Thus

$$\psi_y = ABx^{a+b}\phi', \qquad \psi_x = Aax^{a-1}\phi + Abx^{a-1}\eta\phi' = Ax^{a-1}(a\phi + b\eta\phi'). \tag{4.20}$$

Now as $\eta \to \infty$, we want to match with the slip velocity U, so that

$$U_0 \left(\frac{x}{L}\right)^m = U = \lim_{\eta\to\infty} \psi_y = ABx^{a+b}\phi'(\infty). \tag{4.21}$$

Assuming ϕ' is of course finite, this requires

$$a + b = m \qquad \text{and} \quad AB = U_0/L^m. \tag{4.22}$$

Continuing,

$$\psi_{yy} = AB^2x^{a+2b}\phi'', \quad \psi_{yyy} = AB^3x^{a+3b}\phi''', \quad \psi_{xy} = ABx^{a-1+b}(a\phi + b\eta\phi')'. \tag{4.23}$$

So the LHS of (4.9) is

$$\begin{aligned}\psi_y\psi_{xy} - \psi_x\psi_{yy} &= ABx^{a+b}\phi' ABx^{a-1+b}(a\phi + b\eta\phi')' - Ax^{a-1}(a\phi + b\eta\phi')AB^2x^{a+2b}\phi'' \\ &= A^2B^2x^{2a+2b-1}\left(a\phi'^2 + b\phi'^2 + b\eta\phi'\phi'' - a\phi\phi'' - b\eta\phi'\phi''\right).\end{aligned} \tag{4.24}$$

We note the 'worst' term $\eta\phi'\phi''$ cancels. The RHS of (4.9) is

$$UU' + \nu\psi_{yyy} = \frac{mU_0^2}{L^{2m}}x^{2m-1} + \nu AB^3x^{a+3b}\phi'''. \tag{4.25}$$

Now the powers of x must balance for the solution to be consistent. Thus we must have

$$2a + 2b - 1 = 2m - 1 = a + 3b. \tag{4.26}$$

One of these equations reduces to $a + b = m$ (again) and the other gives

$$b = (m-1)/2 \qquad \text{and} \quad a = (m+1)/2.$$

We note these scalings agree with the Blasius scalings when $m = 0$, which is reassuring. Putting this together, we have

$$A^2B^2\left(m\phi'^2 - \tfrac{1}{2}(m+1)\phi\phi''\right) = \frac{mU_0^2}{L^{2m}} + \nu AB^3\phi'''. \tag{4.27}$$

We now choose the constants A and B to simplify the algebra, recalling we have already fixed AB. For example, we could define

$$A^2B^2 = \nu AB^3 \quad \implies \quad A = \left(\frac{\nu U_0}{L^m}\right)^{1/2}, \quad B = \left(\frac{U_0}{\nu L^m}\right)^{1/2} \tag{4.28}$$

which leads to the equation

$$\boxed{\phi''' + \tfrac{1}{2}(m+1)\phi\phi'' + m(1-\phi'^2) = 0.} \tag{4.29}$$

This is called the **Falkner-Skan equation**; sometimes it is scaled slightly differently. It should be solved in tandem with the boundary conditions:

$$\phi(0) = 0 = \phi'(0), \qquad \phi' \to 1 \qquad \text{as} \quad \eta \to \infty. \tag{4.30}$$

We have to solve this ODE numerically, but have a sensible solution for $m > 0$. In fact, we find a solution exists for $-0.0904 < m$, despite the fact that we have an **unfavourable pressure gradient**, $UU' < 0$ for $m < 0$. **Special case** $m = -1/3$**:** In this case, the nonlinear terms combine nicely and we can integrate analytically.

$$\phi''' + \tfrac{1}{3}(\phi\phi'' + \phi'^2 - 1) = 0 \implies \phi'' + \tfrac{1}{3}\phi\phi' - \tfrac{1}{3}\eta = C \implies \phi' + \tfrac{1}{6}\phi^2 = \tfrac{1}{6}\eta^2 + C\eta + D. \tag{4.31}$$

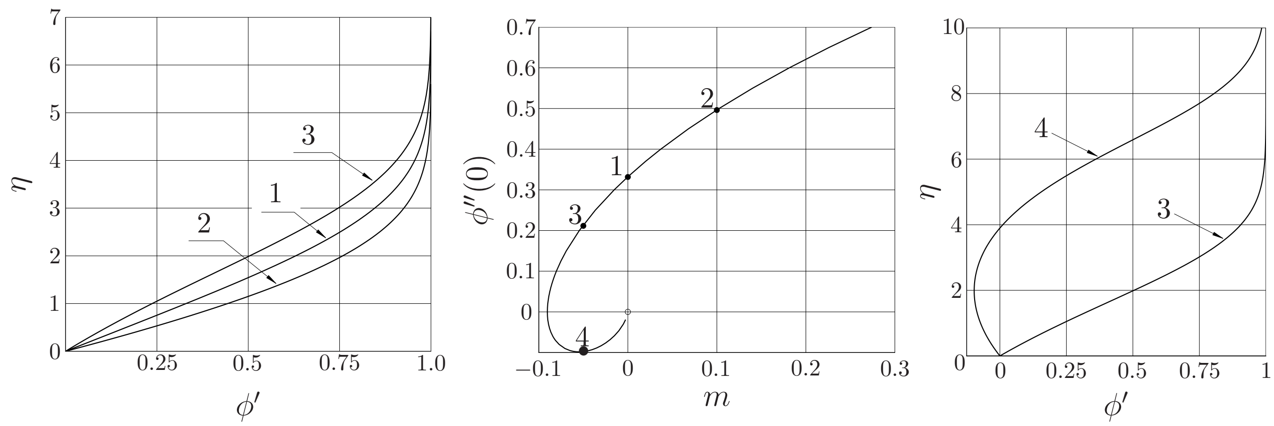
Imposing the conditions $\phi(0) = 0 = \phi'(0)$ requires $D = 0$. Now as $\eta \to \infty$ we want $\phi' \to 1$ so we expect $\phi \sim \eta + E$. But substituting this into (4.31) requires

$$1 + \tfrac{1}{6}(\eta + E)^2 = \tfrac{1}{6}\eta^2 + C\eta + D \quad \implies E = 2C, \quad 1 + \tfrac{1}{6}E^2 = D = 0. \tag{4.32}$$

We find that E has to be imaginary for this to work, and we conclude that the ODE does not have a real solution consistent with the boundary conditions (4.30) when $m = -\frac{1}{3}$. This is consistent with numerical findings that there is no solution for $m < -0.0904$. Provided a solution exists, we see numerically that the shape of $\phi'(\eta)$ does not vary much as m changes. However, we should remember that η varies in a different way with the downstream coordinate x for different m. The boundary layer thickness varies as $x^{-b} = x^{(1-m)/2}$ so that for the Blasius flow it increases as $x^{1/2}$. When $m = 1$, the boundary layer has constant thickness – this corresponds to stagnation point flow.

Numerical solutions of the Falkner-Skan equation

The Falkner-Skan equation (4.29) with (4.30) cannot usually be solved exactly, but it is not difficult to find a numerical solution, for example using the MATLAB routine BVP5c for Boundary Value problems. Alternatively one can use an initial value approach. If one guesses the value of $\phi''(0)$, along with the known values $\phi(0) = 0 = \phi'(0)$, one can integrate forwards until some value $\eta = \eta_\infty$ at which we want to impose $\phi'(\eta_\infty) = 1$. Normally, this will not be satisfied. The guessed value of $\phi''(0)$ can then be adjusted until the solution is found. The following curves were taken from the series of books on Fluid Dynamics by Ruban and Gajjar. Prof Ruban is an expert on boundary layer theory and retired from this department a few years ago.



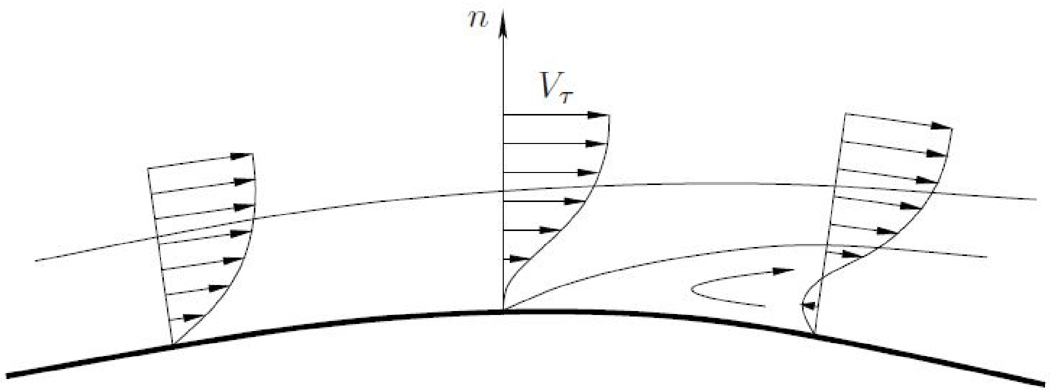
The figure on the left shows the velocity profile, $\phi'(\eta)$, for three different values of m. The curve labelled "1" is the Blasius solution ($m = 0$) while "2" and "3" correspond to $m = 0.1$, and $m = -0.05$ respectively. The shapes do not vary greatly. If $m > 0$ the pressure gradient is favourable ($UU' > 0$). This accelerates the flow, and so it is not a surprise that the velocity ϕ' is greater. In contrast, curve "3" experiences an adverse pressure gradient, and the flow is reduced. Despite the adverse pressure gradient, a solution is found to exist for some values.

The middle figure plots the scaled wall shear stress, $\phi''(0)$ against m for every solution. Each point on the curve represents a solution of (4.29). The points "1", "2" and "3" are as before, so that "1" corresponds to the Blasius solution. To the right of this point, $\phi''(0)$ increases with m, and a unique solution exists for all m.

To the left of the Blasius point the graph loops round. Firstly, we find that a solution exists only for $m > m_c \approx -0.0904$ and, secondly, there are two solutions for all $m \in (m_c, 0)$. For $m = -.05$ the velocity profiles for the upper ("3") and lower ("4") branch solutions are drawn in the right-hand figure above. Curve "4" is noticeably different from those for "1–3". As $\phi''(0) < 0$, the lower branch solution develops a negative velocity region ($\phi' < 0$) near the wall, where the fluid moves in the direction opposite to the main stream. When boundary layers do this, they tend to be close to separation. The width of this region becomes progressively larger as the limit point (shown by an open circle in the middle figure) is approached.

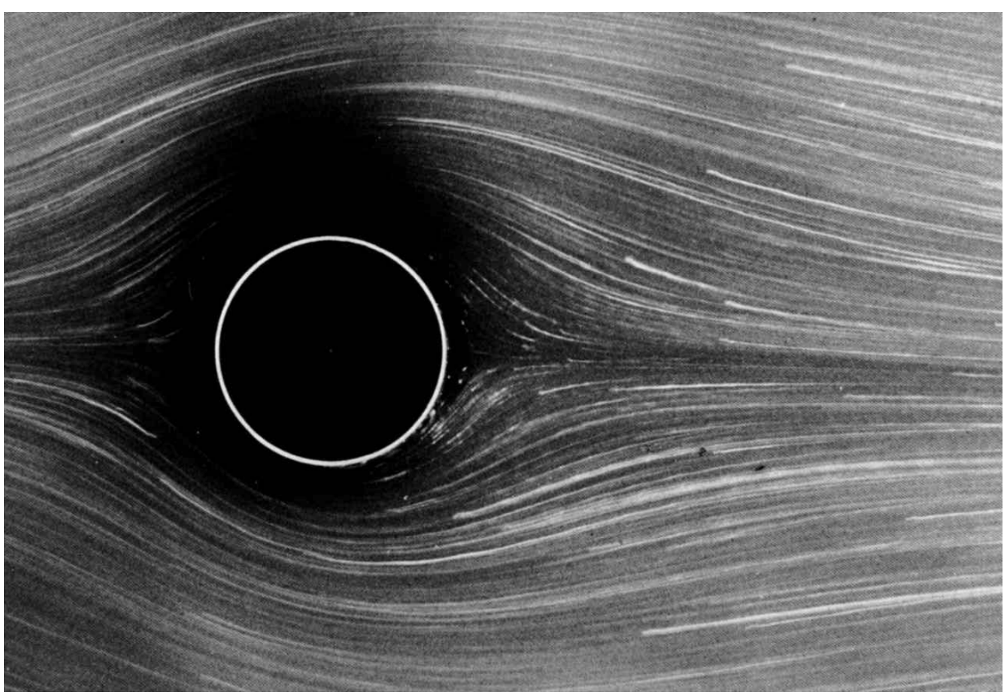
Separation Behind a Cylinder: Effect of the Reynolds Number

We have seen how boundary layer separation off a body begins. The wall shear u y falls to zero and then a region of reversed flow develops. This region grows as we advance downstream. See the sketch below by A. Ruban:

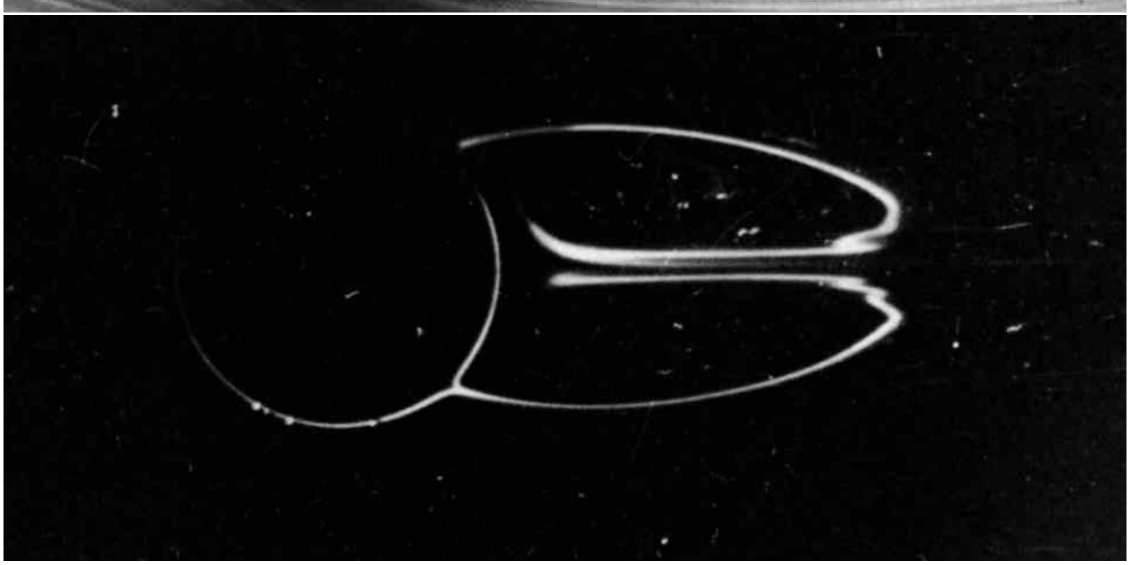


Sometimes the separated region reattaches to the body, leaving a small "bubble region" on the surface, but very often the boundary layer detaches totally forming a wake and shedding vorticity into the main flow. We include below various experimental pictures, which were taken from "Album of Fluid Motion" by M. van Dyke. See that book for the proper citations of the following figures. In all cases the flow is from left to right, around a stationary cylinder.

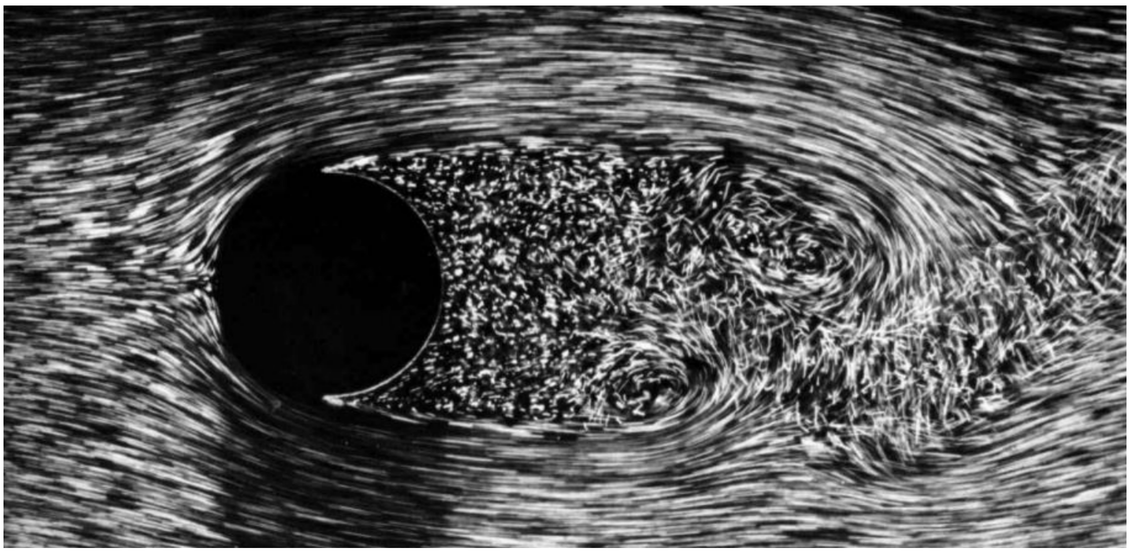
(***Re* ∼ 1.5**) First here is a fairly low Reynolds number case. At first glance the flow is left-right symmetric, but that is not exactly the case, and you can see some widening of the streamlines as the flow passes the cylinder:



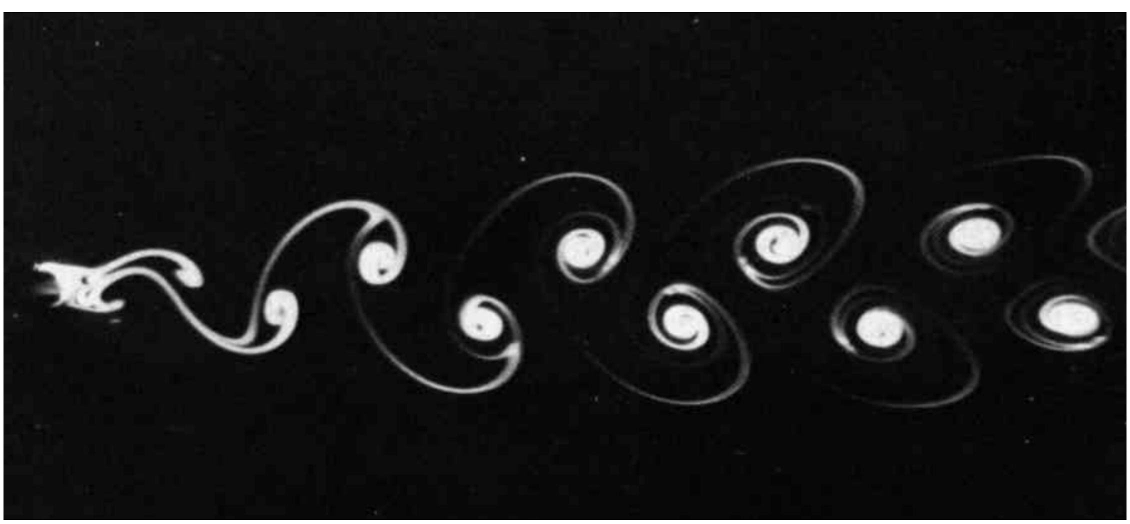
($\boldsymbol{Re} \sim \mathbf{20-30}$) Increasing the Reynolds number a little, we see that some separation has occurred. Two vortex regions have formed within the wake. The wake narrows afterwards. The flow is top-down symmetric and steady. The second photo shows just the boundary of the recirculation region via a liquid leeching from the cylinder surface. The length of the wake grows linearly with the Reynolds number (for a sphere it grows logarithmically).



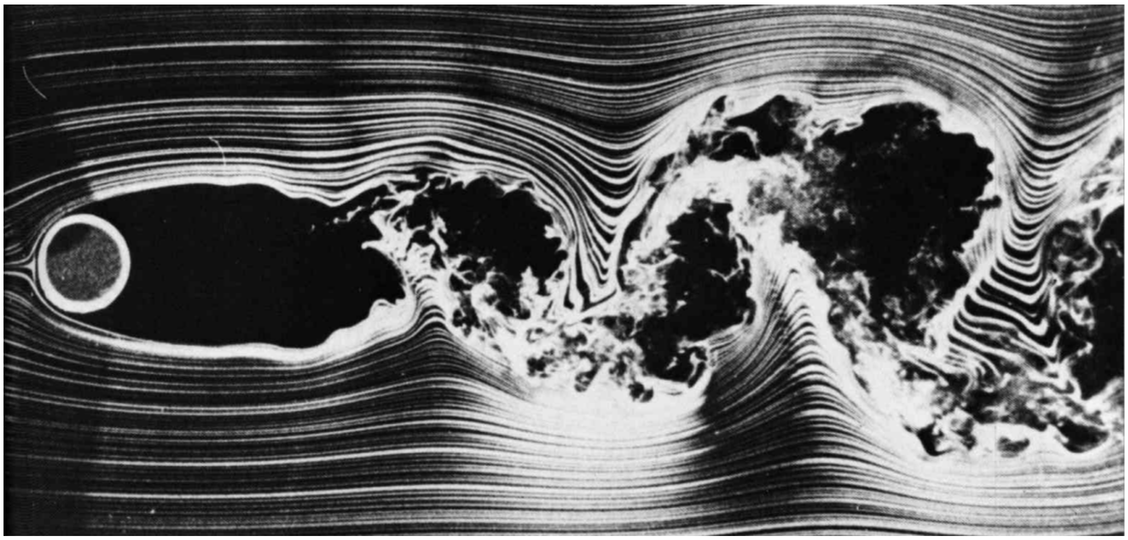
(***Re*** $\sim$ **100**) At a larger Reynolds number we see something different has happened. The two vortices are clearly visible but are no longer symmetric. In fact the flow is no longer steady. The vortices themselves have separated from the cylinder, but they have done so at different times. One can imagine that it is easier for one vortex to separate from the cylinder rather than two as in the above case, and this has driven a symmetry-breaking Hopf bifurcation (to be technical) to this state:



(***Re*** $\sim$ **100−150**) As the Reynolds number increases further, we encounter this beautiful time-periodic state known as the "von Karman vortex street". Vortices are shed off the cylinder alternately from the top and the bottom. This leads to an oscillating sideways force on the cylinder, which has been attributed as the cause of the Tacoma bridge collapse – see video in "Aren't Fluids Wonderful". In fact, this was probably not the cause of that disaster, but this effect can have important environmental consequences. Also in that folder is a video of a numerical simulation of Navier–Stokes showing the development of this instability.



(**$\boldsymbol{Re \sim 10,000}$**) Finally, at high Reynolds number, the ordered time-periodic nature of the wake is lost, although the remnants of the vortex street can be seen. The main flow is still laminar as visible by the smooth streaks, but chaotic motion is visible within the wake. Full turbulence will develop at higher Reynolds number.



Within the near wake, the fluid is almost stationary – we know that sheltering behind an obstacle makes sense in a storm. This shear layer structure can be analysed analytically. However, it can be shown to be unstable. You can see a wavy pattern beginning to develop on the separation line.

The Wall Jet

So far our boundary layer solutions have been driven by a flow at infinity, by which we mean a tangential flow, usually potential, outside the boundary layer. But the equations will apply whenever there is a **thin layer** in which **both inertia and viscosity are important**. We are now going to consider a flow where a jet of some sort is formed near a wall and parallel to it. We will then consider how it evolves downstream.

We will ignore for the moment the source of the flow, and investigate whether we can find solutions to the boundary layer equations with zero tangential velocity at infinity ($U = 0$). That is, we want to find non-zero solutions in $x > 0$, $y > 0$ to

$$uu_x + vu_y = \nu u_{yy}, \qquad u_x + v_y = 0, \qquad \text{with the boundary conditions} \tag{4.33}$$

$$u = v = 0 \qquad \text{on } y = 0 \qquad \text{and} \quad u \to 0 \qquad \text{as } y \to \infty. \tag{4.34}$$

We write $u = \psi_y$, $v = -\psi_x$ where $\psi = Ax^a\phi(\eta)$ and $\eta = Byx^b$. Then in the usual way

$$u = ABx^{a+b}\phi', \qquad v = -x^{a-1}A(a\phi + b\eta\phi'). \tag{4.35}$$

Substituting in the boundary layer equation and dividing by A^2B^2 we get

$$x^{2a+2b-1}[(a+b)\phi'^2 - a\phi\phi''] = \frac{\nu B}{A}x^{a+3b}\phi''', \tag{4.36}$$

For the powers of x to balance we must have $2a + 2b - 1 = a + 3b$ or $a = b + 1$. Choosing $\nu B = A$ to reduce the algebra, (4.36) takes the form

$$(2b+1)\phi'^2 - (b+1)\phi\phi'' = \phi''', \qquad \text{with} \quad \phi(0) = 0 = \phi'(0), \quad \phi' \to 0 \qquad \text{as } \eta \to \infty. \tag{4.37}$$

This is like an eigenvalue problem; we want to find a value of b for which there is a non-zero solution. We now perform a series of ingenious and nonobvious manipulations of this equation which will enable us to integrate it entirely.

Determining b: First, we note that $\phi\phi'' = (\phi\phi')' - \phi'^2$, so that integrating from η to ∞, we have

$$(3b+2)\phi'^2 - (b+1)(\phi\phi')' = \phi''' \qquad \Longrightarrow \qquad (3b+2)\int_\eta^\infty \phi'^2(\widehat{\eta})d\widehat{\eta} + (b+1)\phi\phi' = -\phi'', \tag{4.38}$$

where have used that $\phi', \phi'' \to 0$ as $\eta \to \infty$. We now multiply by ϕ' and integrate over $(0, \infty)$.

$$(3b+2)\int_0^\infty \phi'(\eta)\left(\int_\eta^\infty \phi'^2(\widehat{\eta})d\widehat{\eta}\right)d\eta + (b+1)\int_0^\infty \phi\phi'^2 d\eta = -\left[\tfrac{1}{2}\phi'^2\right]_0^\infty = 0. \tag{4.39}$$

Integrating the first term by parts, we obtain

$$(3b+2)\left[\phi\int_\eta^\infty \phi'^2(\widehat{\eta})d\widehat{\eta}\right]_0^\infty + (3b+2)\int_0^\infty \phi\phi'^2\,d\eta + (b+1)\int_0^\infty \phi\phi'^2\,d\eta = 0. \tag{4.40}$$

Now the boundary term is zero because the integral is zero at $\eta = \infty$ and $\phi(0) = 0$. So we have shown that

$$(4b+3)\int_0^\infty \phi\phi'^2\,d\eta = 0. \tag{4.41}$$

If we want a solution in which u remains positive, so that $\phi' > 0$ for all η, it follows that $\phi > 0$ also and the integral is non-zero. Then the only possibility is that $4b+3=0$, i.e. $b = -\frac{3}{4}$ and $a = \frac{1}{4}$. Our equation becomes

$$\phi''' + \tfrac{1}{2}\phi'^2 + \tfrac{1}{4}\phi\phi'' = 0. \tag{4.42}$$

Conserved quantity: If we define the quantity M for a given x

$$M \equiv \int_0^\infty \psi u^2\,dy = \int_0^\infty (Ax^a\phi)(ABx^{a+b}\phi')^2\left(\frac{d\eta}{Bx^b}\right) = A^3B\int_0^\infty \phi\phi'^2\,d\eta > 0, \tag{4.43}$$

then we find M does not vary with x. We can therefore use it as a (dimensional) measure of the strength of the flow and define A and B in terms of M. It is not easy to associate a natural physical attribute with M, but it represents a conserved quantity of the flow as it evolves.

Solution for $\phi(\eta)$: The last two terms in (4.42) are an exact derivative if we multiply by ϕ,

$$\phi\phi''' + \tfrac{1}{4}(\phi^2\phi')' = 0. \tag{4.44}$$

Also the first term can be written $\phi\phi''' = (\phi\phi'')' - \phi'\phi''$, also an exact derivative. Integrating, we get

$$\phi\phi'' - \tfrac{1}{2}\phi'^2 + \tfrac{1}{4}\phi^2\phi' = C. \tag{4.45}$$

On $y=0$ we have $\eta = 0$ and $\phi = 0$ and $\phi' = 0$. It follows that $C = 0$.

If we multiply (4.45) by ϕ and integrate over $(0,\infty)$, we can deduce something about M:

$$\tfrac{1}{2}\int_0^\infty \phi\phi'^2 d\eta = \int_0^\infty \phi^2\phi'' d\eta + \tfrac{1}{4}\int_0^\infty \phi^3\phi' d\eta = \left[\phi^2\phi'\right]_0^\infty - \int_0^\infty (2\phi\phi')\phi' d\eta + \tfrac{1}{16}\left[\phi^4\right]_0^\infty. \tag{4.46}$$

Thus, as $\phi(0) = 0 = \phi'(\infty)$, and defining $\phi \to \phi_\infty$ as $\eta \to \infty$,

$$\tfrac{5}{2}\int_0^\infty \phi\phi'^2\,d\eta = \tfrac{1}{16}\phi_\infty^4 \qquad \Longrightarrow \qquad \phi_\infty^4 = \frac{40M}{A^3B}, \tag{4.47}$$

using (4.44). We can choose A in such a way that $\phi_\infty = 1$.

Continuing our inspired manipulations, we multiply (4.45) by $\phi^{-3/2}$ so that the first two terms are an exact differential and we can integrate again

$$\phi^{-1/2}\phi'' - \tfrac{1}{2}\phi^{-3/2}\phi'^2 + \tfrac{1}{4}\phi^{1/2}\phi' = 0 \qquad \Longrightarrow \qquad \phi^{-1/2}\phi' + \tfrac{1}{6}\phi^{3/2} = \tfrac{1}{6}\phi_\infty^{3/2} = \tfrac{1}{6}. \tag{4.48}$$

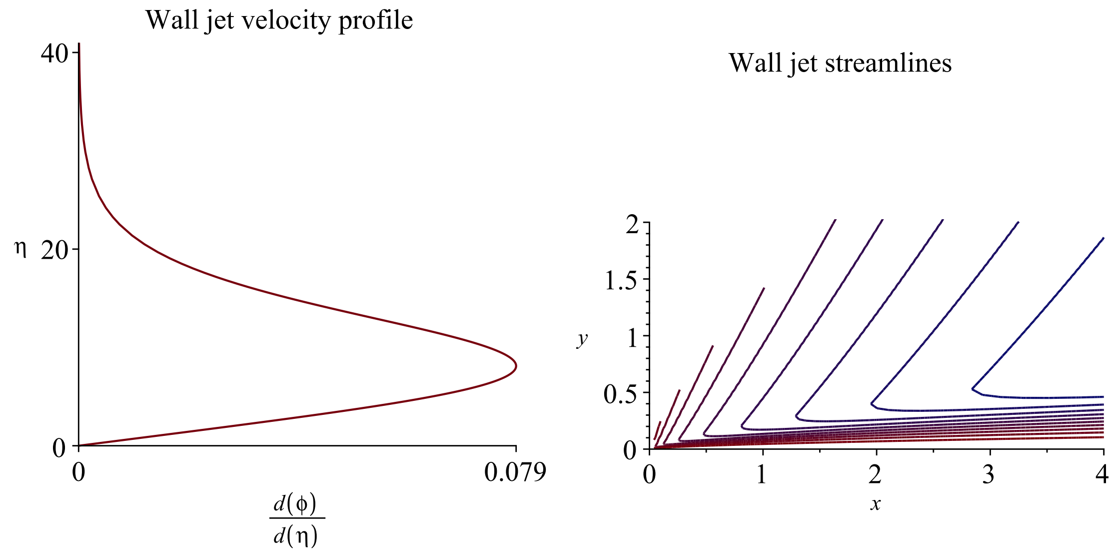
Putting $\phi = f^2$ so that $\phi' = 2ff'$ and using $f(0) = 0$ we get

$$f' = \tfrac{1}{12}(1-f^3) \qquad \Longrightarrow \qquad \eta = 2\log\frac{1+f+f^2}{(1-f)^2} + 4\sqrt{3}\tan^{-1}\left(\frac{2f+1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\pi. \tag{4.49}$$

With ϕ determined, the final streamfunction takes the form

$$\psi = (40\nu Mx)^{1/4}\phi(\eta), \qquad \eta = (40\nu M)^{1/4}\frac{y}{(\nu x)^{3/4}}. \tag{4.50}$$

We are very fortunate to be able to integrate the problem in full, but still in order to visualise the flow, we need to plot the solution.



The left-hand figure plots $\phi'(\eta)$, which is essentially the x-component of the velocity. As expected it is zero on the wall, but quickly rises to a maximum within the boundary layer before falling away to zero. The y-velocity does not, of course. As $\eta \to \infty$, we have $\psi \to Ax^{1/4} = (40M\nu x)^{1/4}$. This drives a weak flow outside the jet, satisfying

$$\nabla^2\psi = 0, \qquad \psi = 0 \text{ on } x = 0,\ \psi = Ax^{1/4} \text{ on } y = 0. \tag{4.51}$$

This problem has the solution $\psi = AR^{1/4}\sin(\varphi/4 + 7\pi/8)/\sin(7\pi/8)$ in terms of cylindrical coordinates (R, φ, z). The global "composite" asymptotic solution involves a superposition of the jet and the weak core flow. The streamlines of which are plotted in the right-hand diagram above. We can see that the jet entrains fluid as it progresses and widens.

Wall stress: For this solution the stress on the wall $\sim u_y \propto x^{-5/4}$ on $y = 0$. This is infinite at $x = 0$, but also it is a non-integrable singularity, as $\int_0^1 u_y\,dx = \infty$. Whereas $\int_1^\infty$ is finite. Thus if this flow applies near $x = 0$ it exerts a very large force on the wall. As x increases, the jet widens and the stress diminishes so that the force on the rest of the wall is manageable. For the Blasius boundary layer, we had $u_y \propto x^{-1/2}$ and the reverse applies; the singularity at the leading edge is integrable, whereas the total force $\int^\infty$ diverges.

Wall-jets appear in many practical circumstances, although they are often turbulent, when the exact solution we have derived does not apply. We can also find a laminar solution for free jets without a wall (see Problem Sheet 3), where the scaling is different. We can also find solutions for axisymmetric jets.

Chapter 4: Introduction to Flow Stability

Once we have found a solution to our equations that may not be the end of the story. Really, we should also investigate whether our flow is **stable**. If it is not, then it will not be realised in practice.

For a given problem, we solve the governing equations and obtain a solution which we assume is steady, $\mathbf{u} = \mathbf{U}(\mathbf{x})$ with a corresponding pressure distribution $p = P(\mathbf{x})$ with an interface given by $S(\mathbf{x}) = 0$. We then make a small perturbation to the flow, so that

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) + \varepsilon\mathbf{u}'(\mathbf{x},\, t), \qquad p = P(\mathbf{x}) + \varepsilon p'(\mathbf{x},\, t) \qquad S(\mathbf{x}) + \varepsilon S'(\mathbf{x},\, t) = 0, \tag{5.1}$$

where ε is a small positive constant. **Note** that a prime $'$ denotes "perturbation" rather than differentiation in this chapter. We then consider the behaviour of the perturbation $(\varepsilon\mathbf{u}', \varepsilon p', \varepsilon S')$. If this perturbation remains small for all time, we say that the underlying flow is **stable**, whereas if it eventually becomes large no matter how small ε is, we say the flow is **unstable.**

Substituting into the Navier–Stokes equations, and using that $\mathbf{U}(\mathbf{x}), P(\mathbf{x})$ are steady solutions, the exact equations for $\mathbf{u}'$ and p' are (cancelling the factor of ε)

$$\rho\left(\frac{\partial\mathbf{u}'}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\mathbf{U} + \varepsilon\mathbf{u}'\cdot\nabla\mathbf{u}'\right) = -\nabla p' + \mu\nabla^2\mathbf{u}', \qquad \nabla\cdot\mathbf{u}' = 0 \tag{5.2}$$

Linear stability theory neglects the last term on the LHS, coming from products of perturbations, as ε is arbitrarily small. The resulting **linear equation** has solutions of the form

$$\mathbf{u}'(\mathbf{x}, t) = \widehat{\mathbf{u}}(\mathbf{x})e^{st}, \qquad p'(\mathbf{x}, t) = \widehat{p}(\mathbf{x})e^{st}$$

for some functions $\widehat{\mathbf{u}}(\mathbf{x}), \widehat{p}(\mathbf{x})$ and constant s. This is because none of the coefficient depends on t as $\mathbf{U}$ is steady. The general solution to this problem will be a linear combination of this type—see the theory of Laplace and Fourier Transforms. Hence, (5.2) reduces to the time-independent system

$$\rho\left(s\widehat{\mathbf{u}} + \mathbf{U}\cdot\nabla\widehat{\mathbf{u}} + \widehat{\mathbf{u}}\cdot\nabla\mathbf{U}\right) = -\nabla\widehat{p} + \mu\nabla^2\widehat{\mathbf{u}}, \qquad \nabla\cdot\widehat{\mathbf{u}} = 0 \tag{5.3}$$

The possible values of s can be regarded as **eigenvalues** of the system. These can be real, but are in general complex

$$s = s_r + is_i, \qquad e^{st} = e^{s_r t + is_i t} = e^{s_r t}\left[\cos(s_i t) + i\sin(s_i t)\right]$$

(a) If for all possible values of s we have $s_r < 0$ we say the flow is **stable**.
(b) If there is at least one eigenvalue s for which $s_r > 0$, the flow is **unstable**.
(c) If $s_r \leqslant 0$ but $s_r = 0$ for some eigenvalue, we say the flow is **neutrally stable**.

Instability is closely connected with solution uniqueness. Low Re flows tend to be stable, while very high Re flows are usually unstable. However, free surfaces can be unstable even at very low Re.

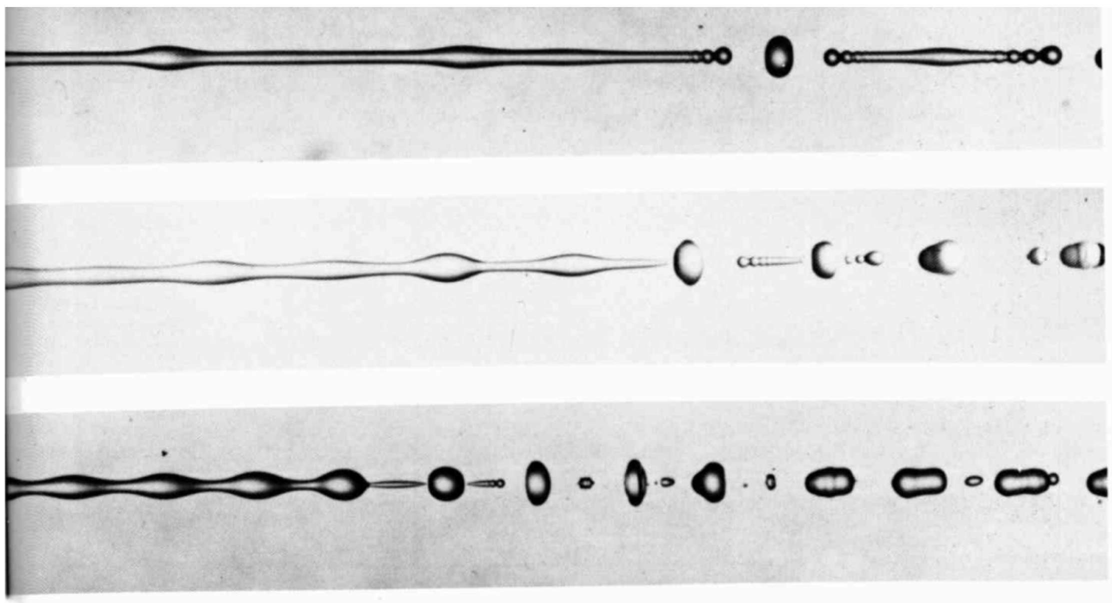
Examples of instabilities

(a) Parallel shear flow: $\mathbf{u} = (U(y), 0, 0)$ is a solution to the inviscid equations for all functions U, whereas for viscous flow $U(y)$ must be quadratic. This will be considered in depth in the module "Hydrodynamic Instability."

(b) Kelvin-Helmholtz instabilities: If two fluid layers have different speeds, the interface can go unstable. Clouds often exhibit this classic "cats-eyes" pattern.

(c) Rayleigh-Taylor instabilities: Heavy fluid over light fluid tends to be unstable. These configurations are connected with **Convection**.

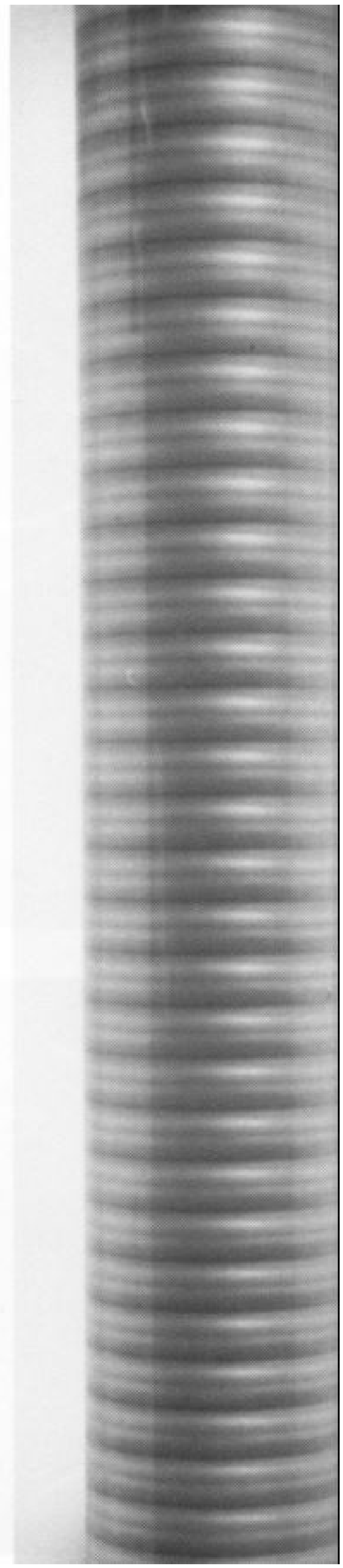
(d) Surface Tension instabilities: e.g. a dripping tap, Rayleigh–Plateau liquid jet.

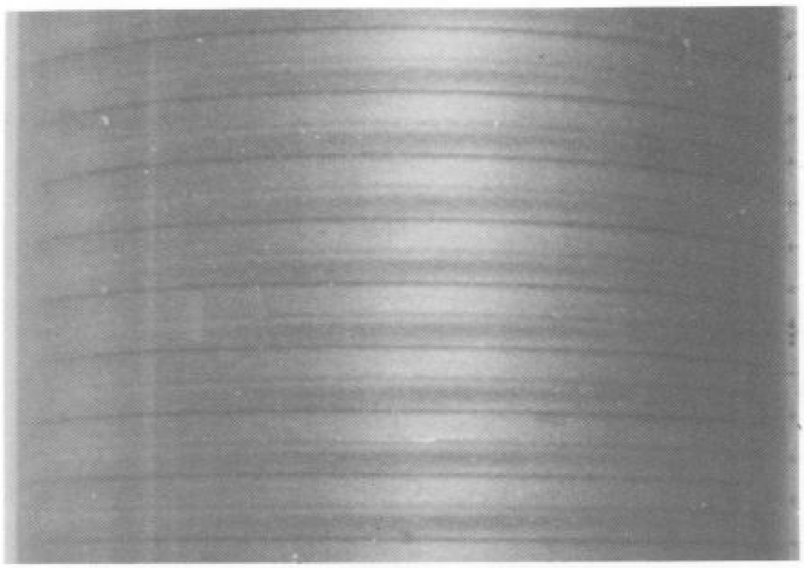


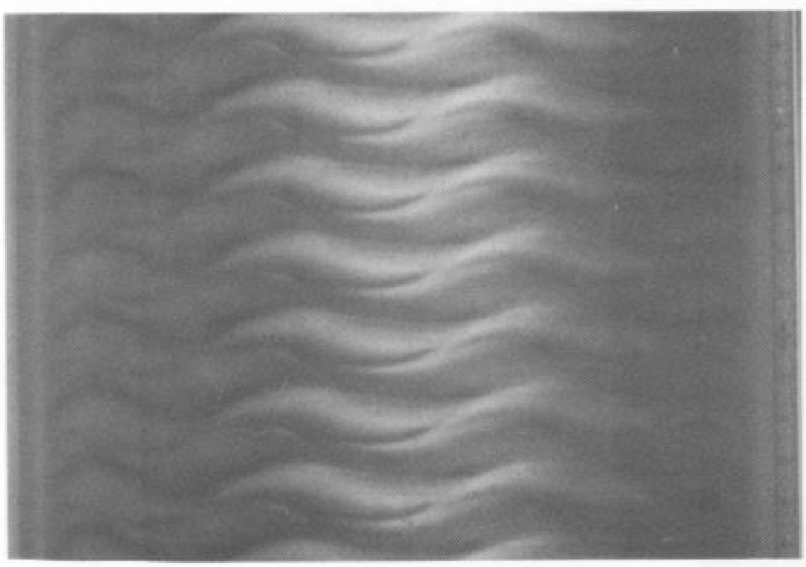
122. Capillary instability of a liquid jet. Water forced from a 4-mm tube is perturbed at various frequencies by a loudspeaker. The wavelength is 42, 12.5, and 4.6 diameters, the last being nearly Rayleigh's value for maximum growth of disturbances. The top two photographs show secondary swellings between the primary crests. *Rutland & Jameson 1971*

(e) Circular flows: $\mathbf{u} = (0, V(r), 0)$ in cylindrical coordinates (r, θ, z). See overleaf.

127. Axisymmetric laminar Taylor vortices. Machine oil containing aluminum powder fills the gap between a fixed outer glass cylinder and a rotating inner metal one, of relative radius 0.727. The top and bottom plates are fixed. The rotation speed is 9.1 times that at which Taylor predicts the onset of the regularly spaced toroidal vortices seen here. The flow is radially inward on the heavier dark horizontal rings and outward on the finer ones. The motion was started impulsively, giving narrower vortices than would result from a smooth start. *Burkhalter & Koschmieder 1974*







128. Laminar Taylor vortices in a narrow gap. A larger inner cylinder in the apparatus to the right gives a radius ratio of 0.896. Again only the inner cylinder rotates. The upper photograph shows the center section of axisymmetric vortices at 1.16 times the critical speed. In the lower, at 8.5 times the critical speed, the flow is doubly periodic, with six waves around the circumference, drifting with the rotation. *Koschmieder 1979*

Kelvin-Helmholtz and Rayleigh-Taylor Instabilities

Consider two fluid regions, $y > 0$ and $y < 0$. In $y > 0$ let the fluid have uniform velocity $\mathbf{u} = (U_1,\, 0,\, 0)$ and constant density $\rho = \rho_1$, whereas in $y < 0$ let $\rho = \rho_2$ and $\mathbf{u} = (U_2,\, 0,\, 0)$. Gravity is assumed to act in the negative y-direction, $\mathbf{g} = (0,\, -g,\, 0)$. This configuration could represent wind blowing over a lake, or a model of separated flow over a step.

We will assume the flow is inviscid. It can therefore tolerate the tangential velocity discontinuity at $y = 0$. The vortex sheet associated with this discontinuity would diffuse outwards if $\mu \neq 0$, but we will neglect this spreading and assume that the interface has the shape $y = \varepsilon h(x,\, t)$. As there is no vorticity elsewhere in the flow initially, we expect the flow to be irrotational, i.e.

$$\mathbf{u} = \nabla\phi \qquad \text{where} \quad \nabla^2\phi = 0. \tag{5.3}$$

Then we write

$$\phi = U_1 x + \varepsilon\phi_1 \qquad \text{in} \quad y > \varepsilon h, \qquad \phi = U_2 x + \varepsilon\phi_2 \qquad \text{in} \quad y < \varepsilon h\ . \tag{5.4}$$

ϕ_1 and ϕ_2 satisfy Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$. We want $\phi_1 \to 0$ as $y \to \infty$ and $\phi_2 \to 0$ as $y \to -\infty$. The kinematic boundary condition takes the form on the interface

$$0 = \frac{D}{Dt}(y - \varepsilon h) = \frac{\partial\phi}{\partial y} - \varepsilon\frac{\partial h}{\partial t} - \varepsilon\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}\ . \tag{5.5}$$

or neglecting terms of $O(\varepsilon^2)$,

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial h}{\partial t} + U_1\frac{\partial h}{\partial x} \qquad \text{and} \quad \frac{\partial\phi_2}{\partial y} = \frac{\partial h}{\partial t} + U_2\frac{\partial h}{\partial x}\ . \tag{5.6}$$

To leading order we can evaluate (5.6) on $y = 0$ rather than $y = \varepsilon h$. The other condition to apply is that the pressure must be continuous across the interface or if we ignore surface tension for the moment $p_1 = p_2$ on $y = \varepsilon h$. The time-dependent Bernoulli condition is

$$p + \rho\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + gy\right) = \text{constant},$$

so that to leading order on the interface $y = \varepsilon h$,

$$\rho_1\left(\frac{\partial\phi_1}{\partial t} + U_1\frac{\partial\phi_1}{\partial x} + gh\right) = \rho_2\left(\frac{\partial\phi_2}{\partial t} + U_2\frac{\partial\phi_2}{\partial x} + gh\right)\ . \tag{5.7}$$

Once more, we can evaluate (5.7) on $y = 0$ rather than $y = \varepsilon h$.

As neither x or t appears in the coefficients of the problem, we can seek a solution proportional to e^{ikx+st}. Here $k > 0$ is a real **wave-number**, and s is possibly complex. If we can find a value of k for which the corresponding s has a positive real part ($\Re e(s) > 0$), then the interface $y = 0$ is **unstable.** We write

$$h = h_0 e^{ikx+st}, \qquad \phi_1 = \Phi_1(y)e^{ikx+st}, \qquad \phi_2 = \Phi_2(y)e^{ikx+st}\ .$$

Then $\Phi_1(y)$ satisfies the ODE and boundary condition (5.6)

$$\Phi_1'' - k^2\Phi_1 = 0, \qquad \Phi_1 \to 0 \quad \text{as} \quad y \to \infty \qquad \Phi_1'(0) = h_0(s + ikU_1) \tag{5.8}$$

from which it follows that

$$\Phi_1(y) = -\frac{h_0}{k}(s + ikU_1)e^{-ky} \qquad \text{and similarly} \quad \Phi_2(y) = \frac{h_0}{k}(s + ikU_2)e^{ky} \ . \tag{5.9}$$

The pressure constraint (5.6) requires that

$$\rho_1 \left[(s + ikU_1)\Phi_1(0) + gh_0\right] = \rho_2 \left[(s + ikU_2)\Phi_2(0) + gh_0\right] \tag{5.10}$$

or combining (5.10) and (5.9),

$$\rho_1 \left[gk - (s + ikU_1)^2\right] = \rho_2 \left[gk + (s + ikU_2)^2\right] \ . \tag{5.11}$$

This is a quadratic equation for $s(k)$, which is in general known as the **dispersion relation**. Solving this equation we obtain

$$s = -ik\,\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + kg\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right]^{1/2} \ . \tag{5.12}$$

For instability, s must have a positive real part. It is clear that this will occur if and only if the quantity in square brackets is positive, that is if

$$\boxed{\textbf{Instability iff:} \qquad k^2\rho_1\rho_2(U_1 - U_2)^2 > kg(\rho_2^2 - \rho_1^2) \ .} \tag{5.13}$$

Instability of a vortex sheet: If $U_1 \neq U_2$, we see from (5.13) that instability always occurs for large enough k, i.e. for short wavelengths. The smaller the wavelength the larger the growth rate. This is known as the Kelvin-Helmholtz instability.

Water wave generation: For example, consider air moving over a stationary lake, so that $U_2 = 0$, $\rho_2 \gg \rho_1$. Instability occurs if

$$k > \frac{g}{U_1^2}\frac{\rho_2}{\rho_1} \tag{5.14}$$

Heavy fluid over light fluid: If $\rho_1 > \rho_2$ we see that every value of k leads to instability (recall $k > 0$). This is known as the Rayleigh-Taylor instability.

Wave frequencies: If $U_1 = U_2 = 0$ and $\rho_1 < \rho_2$ then the interface can support surface waves of frequency ω ($s = i\omega$) where

$$\omega = \sqrt{gk}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)^{1/2} \ . \tag{5.15}$$

The phase velocity $c = \omega/k$ for small water waves ($\rho_1 = 0$) is therefore $c = (g/k)^{1/2}$.

Three dimensional disturbances:

So far we have only considered 2-D perturbations. If instead we allow the surface to take the form

$$y = \varepsilon h_0 e^{ikx+ilz+st} \qquad \text{where} \quad \kappa = (k^2 + l^2)^{1/2} \ , \tag{5.16}$$

then the analysis is very similar, with for example $\Phi_1 \propto e^{\kappa y}$. The instability condition (5.13) remains the same with the "k" on the RHS replaced by "κ." As $\kappa \geqslant k$, we can infer from this that if a mode with a particular $(k,\, l)$ is unstable so is the mode with $(k,\, 0)$, and indeed the growth rate is **larger** for the 2-D disturbance. This idea resembles **Squires' theorem,** in Hydrodynamic Stability. The 2D disturbances are the first to appear and for many purposes 3-D disturbances can be neglected.

If a configuration is unstable, we have shown that the largest growth rates have large k, and are formally infinite. In practice, some physical effect we have neglected will become important. Two processes we might expect to limit the size of the wave-number and growth-rates are **viscosity** and **surface tension.**

The Effect of Surface Tension

The curvature of the surface (5.16) is

$$K = \nabla \cdot \widehat{\mathbf{n}} = \nabla \cdot [(-\varepsilon ikh_0 e^{ikx+ilz+st},\, 1,\, -\varepsilon ilh_0 e^{ikx+ilz+st})] = \varepsilon\kappa^2 h_0\, e^{ikx+ilz+st} \ . \tag{5.17}$$

The normal stress condition $p_1 = p_2$ is now replaced by $p_2 = p_1 + \gamma K$. Making this modification, the dispersion relation (5.12) takes the form

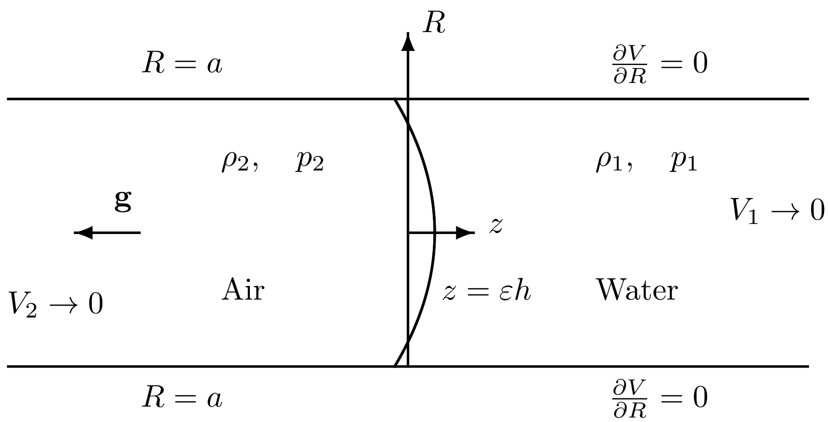
$$s = -ik\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{\kappa g(\rho_1 - \rho_2) - \kappa^3\gamma}{\rho_1 + \rho_2}\right]^{1/2} \ . \tag{5.18}$$

The cubic term $\gamma\kappa^3$ will clearly dominate the large wave-numbers irrespective of the other parameters, so as we might expect, surface tension strongly resists high-curvature perturbations. Likewise, if $\kappa \ll 1$, the linear gravitational term will be largest, so that the long wavelengths will be unstable if $\rho_1 > \rho_2$. If $\rho_2 > \rho_1$, however, it is possible for the flow to be stable for all k and l. Once more we can show that the most unstable case has $l = 0$ so we can replace κ by k. The term in brackets is negative for all $k > 0$, giving stability, if

$$(U_1 - U_2)^2 < \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2}\, [\gamma g(\rho_2 - \rho_1)]^{1/2} \ . \tag{5.19}$$

As $|U_1 - U_2|$ increases, the first wave to go unstable has $k^2 = g(\rho_2 - \rho_1)/\gamma$ We can now calculate the wind speed necessary to drive waves on the surface of a lake. If we put $g = 9.8$, $U_2 = 0$, $\rho_2 = 1000$ $\rho_1 = 1.25$ and $\gamma = 0.074$ appropriate for air above water, we find $U_1 = 6.6$m/s. At this critical wind-speed, the wavelength of the waves is $2\pi/k = 1.7$cm.

Two fluids in a rigid cylinder



We have seen that a vortex sheet can be stabilised by the combined action of gravity and surface tension. Is it possible for the gravitational instability of a heavy fluid being held over a light fluid to be nullified? Let us consider two fluids in a vertical, rigid cylinder of radius a. We shall use cylindrical polar coordinates $(R,\, \phi,\, z)$, as in the figure, with gravity in the negative z-direction (to the left). Once more $z > 0$ contains a fluid of density ρ_1 and $z < 0$ has density ρ_2. We perturb the interface $z = 0$ to $z = \varepsilon h(R,\, \phi,\, t)$ and consider what will happen.

Once more the velocity is irrotational, $\mathbf{u} = \varepsilon \nabla V$ with $\nabla^2 V = 0$, or

$$\mathbf{u} = \varepsilon \left(\frac{\partial V}{\partial R},\, \frac{1}{R}\frac{\partial V}{\partial \phi},\, \frac{\partial V}{\partial z} \right), \qquad \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R} \right) + \frac{1}{R^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0\ . \tag{5.20}$$

Now the separable solutions to Laplace's equation in this geometry are

$$V_1 = A_1 e^{-kz} u(R) e^{im\phi + st}, \qquad V_2 = A_2 e^{+kz} u(R) e^{im\phi + st} \tag{5.21}$$

where $u(R) = J_m(kR)$. The **Bessel function** J_m is a is a suitably normalized solution, finite at $R = 0$, to the equation

$$\frac{1}{R}(Ru')' + u\left(k^2 - \frac{m^2}{R^2} \right) = 0. \tag{5.22}$$

The possible values of k are determined by the rigid wall at $R = a$, where the normal velocity, $\partial V/\partial R = 0$ or $J_m'(ka) = 0$. This gives a set of possible values $k = k_{mn}$, for $m = 0,\, 1 \ldots,$ $n = 1,\, 2 \ldots$ Note that $m = 0$ is an axisymmetric disturbance.

The kinematic constraint $D(z - \varepsilon h)/Dt = 0$ implies that

$$\frac{\partial V}{\partial z} = \frac{\partial h}{\partial t} \qquad \text{on } z = 0. \tag{5.23}$$

Once more, we can evaluate the boundary condition on $z = 0$ rather than $z = \varepsilon h$ to leading order. This suggests we look at perturbations

$$z = \varepsilon h \equiv \varepsilon h_0 u(R) e^{im\phi + st} \quad \Longrightarrow \quad \widehat{\mathbf{n}} = \left(-\frac{\varepsilon u' h}{u}, -\frac{\varepsilon imh}{R}, 1 \right). \tag{5.24}$$

Now (5.23) and (5.21) apply in both fluids 1 and 2, so we have

$$-kV_1 = sh = kV_2 \qquad \text{on } z = 0. \tag{5.25}$$

From (5.24) the surface curvature is

$$K = \nabla \cdot \widehat{\mathbf{n}} = \varepsilon h_0 e^{im\phi + st} \left(\frac{1}{R}(Ru')' + \frac{m^2 u}{R^2} \right) = \varepsilon k^2 h, \tag{5.26}$$

using (5.23).

Thus the pressure boundary condition $p_2 = p_1 + \gamma K$ on $z = \varepsilon h$, together with the time-dependent Bernoulli theorem gives

$$\rho_1 \left(\frac{\partial V_1}{\partial t} + gh \right) = \rho_2 \left(\frac{\partial V_2}{\partial t} + gh \right) + \gamma K \qquad \text{on } z = 0. \tag{5.27}$$

Putting all this together, we obtain the dispersion relation

$$s^2 = \frac{k[g(\rho_1 - \rho_2) - \gamma k^2]}{\rho_1 + \rho_2} \qquad \text{where} \quad J_m'(ka) = 0. \tag{5.28}$$

We see that we get stability provided the smallest admissable value of ka satisfies

$$(ka)^2_{\mathbf{min}} > \frac{ga^2(\rho_1 - \rho_2)}{\gamma}. \tag{5.29}$$

Plotting the turning points of the Bessel function J_m, it turns out that the smallest value of ka occurs when $m = 1$, and $(ka)_{\mathbf{min}} \simeq 1.8$. So the largest pipe radius for which surface tension can support water above air is given by

$$a_{max} \simeq \left(\frac{(1.8)^2 \times 0.074}{(9.81 \times 1000)} \right)^{1/2} \simeq 0.005, \tag{5.30}$$

that is, about $a_{max} \simeq 5$mm. This confirms that what we saw at the start of the lecture should not happen. It could not have happened. Your eyes were playing tricks on you. Or something.

The theory predicts that the first mode to go unstable has $m = 1$ so is, say, proportional to $\cos\phi$. If the interface is unstable the air will rush in one side of the pipe (say near $\phi \sim 0$) and the water will fall out on the other ($\phi \sim \pi$).

Does the nonaxisymmetric motion agree with your physical intuition, or did you expect the fluid to fall out in an axisymmetric manner, perhaps down in the middle and up near the boundary?

Inviscid instability of a cylindrical jet

When we turn on a tap a jet of water emerges, whose cylindrical surface quickly becomes kinked and then breaks up into drops. As we shall see, this is a **capillary** or **surface tension** instability. Gravity may be neglected during the evolution of the jet.

Consider a liquid cylinder $0 < R < a$ in terms of cylindrical polar coordinates (R, ϕ, z) moving with constant velocity $(0, 0, U)$. The outside we consider to be dynamically negligible, so that the external pressure is constant p_0 on the surface. We envisage a surface perturbation of the form

$$R = a(1 + \varepsilon\zeta) \equiv a\left(1 + \varepsilon e^{ikz+im\phi+st}\right) \; . \tag{5.31}$$

Here m must be an integer, but k can be any positive number. The curvature of this surface $K = \nabla \cdot \widehat{\mathbf{n}}$ takes the form (ignoring terms proportional to ε^2)

$$\widehat{\mathbf{n}} = (1,\, -\varepsilon im\zeta,\, -ika\varepsilon\zeta) \qquad K = \frac{1}{R} + \frac{\varepsilon}{a}(m^2 + k^2a^2)\zeta = \frac{1}{a} + \frac{\varepsilon\zeta}{a}(m^2 + k^2a^2 - 1) \; . \tag{5.32}$$

Taking the velocity $\mathbf{u} = (0, 0, U) + \varepsilon\nabla V$, the kinematic boundary condition on $R = a$ is

$$0 = \frac{D}{Dt}(R - a - a\varepsilon\zeta) \quad \Longrightarrow \frac{\partial V}{\partial R} = a\zeta(s + ikU) \; , \tag{5.33}$$

and the pressure condition (Bernoulli) is

$$p_0 + \gamma K + \varepsilon\rho\,(sV + UikV) = \text{constant on } R = a. \tag{5.34}$$

The solutions to Laplace's equation proportional to ζ take the form

$$V = A\,u(R)\zeta \qquad \text{where} \quad u(R) = I_m(kR) \qquad \text{satisfies} \quad u'' + \frac{u'}{R} - \left(k^2 + \frac{m^2}{R^2}\right)u = 0 \; . \tag{5.35}$$

I_m is called a **modified Bessel function** (compare (5.22).) It behaves like R^m near $R = 0$ and increases monotonically, behaving like e^{kR} as $R \to \infty$.

Combining (5.35), (5.34) and (5.33), we find that

$$(s + ikU)^2 = \frac{\gamma}{\rho a^3}(1 - m^2 - k^2a^2)\left(\frac{kaI'_m(ka)}{I_m(ka)}\right) \; . \tag{5.36}$$

The last factor involving Bessel functions is always positive, so doesn't affect the stability. We see the RHS is negative if $m \geqslant 1$ or if $ka \geqslant 1$, However, the long, axisymmetric waves with $m = 0$ and $0 < ka < 1$ are **unstable.** The jet is unstable to all axial wavelengths larger than its circumference ($2\pi a$). Calculation shows that the greatest value of s occurs at $ka \simeq 0.7$. This was shown by Lord Rayleigh in 1879. Try it in your kitchen! The theory predicts a wavelength of about 4.5 jet diameters.

Circular flow: Rayleigh's circulation criterion

In cylindrical polar coordinates $(R,\, \phi,\, z)$, the Navier-Stokes equations for the velocity $\mathbf{u} = (u,\, v,\, w)$ are

$$\left.\begin{aligned} \rho\left(\frac{Du}{Dt} - \frac{v^2}{R}\right) &= -\frac{\partial p}{\partial R} + \mu\left(\nabla^2 u - \frac{u}{R^2} - \frac{2}{R^2}\frac{\partial v}{\partial \phi}\right) \\ \rho\left(\frac{Dv}{Dt} + \frac{uv}{R}\right) &= -\frac{1}{R}\frac{\partial p}{\partial \phi} + \mu\left(\nabla^2 v - \frac{v}{R^2} + \frac{2}{R^2}\frac{\partial u}{\partial \phi}\right) \\ \rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \mu\nabla^2 w \end{aligned}\right\} \tag{5.37}$$

together with the incompressibility relation

$$0 = \frac{1}{R}\frac{\partial (Ru)}{\partial R} + \frac{1}{R}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} \tag{5.38}$$

where

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial R} + v\frac{1}{R}\frac{\partial f}{\partial \phi} + w\frac{\partial f}{\partial z} \quad \text{and} \quad \nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{5.39}$$

A few lectures back, we saw that an instability develops when the boundary of a cylinder of uniformly rotating fluid is brought to rest, leading to a vertical pattern of "Taylor vortices". We now generalise this and consider the stability of the flow $\mathbf{u} = (0,\, V(R),\, 0)$ between two infinite cylinders, $R_1 < R < R_2$.

This flow satisfies (5.38) and substituting (5.37), we require

$$\frac{\rho V^2}{R} = \frac{\partial p}{\partial R}, \qquad \frac{\partial p}{\partial z} = 0, \qquad \frac{1}{R}\frac{\partial p}{\partial \phi} = \mu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) - \frac{V}{R^2}\right) \tag{5.40}$$

As V is a function of R only, then so is $\partial p/\partial \phi$. But p has to be 2π-periodic in ϕ for the solution to make sense. It follows that $p = p(R)$, and balances the centrifugal force of the rotation. For the inviscid problem ($\mu = 0$) we therefore have that any $V(R)$ is admissible. If we have viscosity then we must have $V = AR + B/R$ for some constants A and B.

We now consider the stability of this flow. We shall concentrate on the inviscid problem, $\mu = 0$ and so $V(R)$ is arbitrary. The experiments suggest that the first instability is axisymmetric. Writing $\zeta = e^{ikz+st}$, we therefore perturb the flow in the form

$$p = p_0(R) + \varepsilon p_1(R)\zeta \qquad \mathbf{u} = (0,\, V(R),\, 0) + \varepsilon[u_1(R),\, v_1(R),\, w_1(R)]\zeta, \tag{5.41}$$

where $0 < \varepsilon \ll 1$. Neglecting terms of $O(\varepsilon^2)$, we have

$$\rho\left(su_1 - \frac{2Vv_1}{R}\right) = -p_1' \tag{5.42}$$

$$\rho\left(sv_1 + u_1 V' + \frac{u_1 V}{R}\right) = 0 \tag{5.43}$$

$$\rho s w_1 = -ikp_1 \tag{5.44}$$

$$\frac{1}{R}(Ru_1)' + ikw_1 = 0. \tag{5.45}$$

Eliminating p from (5.42) and (5.44),

$$\rho ik \left(su_1 - \frac{2Vv_1}{R} \right) = \rho s w_1'. \tag{5.46}$$

Substituting for w_1 from (5.45)

$$(ik)^2 \left(su_1 - \frac{2Vv_1}{R} \right) = -s \left[u_1' + \frac{u_1}{R} \right]' = -s \left(u_1'' + \frac{u_1'}{R} - \frac{u_1}{R^2} \right). \tag{5.47}$$

Finally, eliminating v_1 using (5.43),

$$-k^2 \left[su_1 + \frac{2Vu_1}{Rs} \left(V' + \frac{V}{R} \right) \right] = -s \left(u_1'' + \frac{u_1'}{R} - \frac{u_1}{R^2} \right) \tag{5.48}$$

giving

$$u_1'' + \frac{u_1'}{R} - \frac{u_1}{R^2} - k^2 u_1 = \frac{k^2}{s^2} \Phi u_1 \tag{5.49}$$

where

$$\Phi(R) = \frac{2V}{R^2} \frac{d}{dR}(RV) = \frac{1}{R^3} \frac{d}{dR}(R^2 V^2). \tag{5.50}$$

The boundary conditions are that $u_1 = 0$ on $R = R_1, R_2$. This is an eigenvalue problem of "Sturm-Liouville" type.

Multiplying (5.50) by Ru_1 and integrating from $R = R_1$ to $R = R_2$,

$$\int_{R_1}^{R_2} \left(u_1 (Ru_1')' - \frac{u_1^2}{R} - Rk^2 u_1^2 \right) dR = \frac{k^2}{s^2} \int_{R_1}^{R_2} R \Phi u_1^2 \, dR. \tag{5.51}$$

Integrating the first term by parts using $u_1 = 0$ on the boundary, all terms on the LHS are negative. It follows that the RHS must be negative. We see that s^2 is real and for instability it must be positive. It follows that the integral on the RHS must be negative for instability to occur, which can only happen if the integrand is negative somewhere, which requires $\Phi < 0$ somewhere. This is **Rayleigh's circulation criterion**: flows for which the circulation RV decreases outwards are prone to instability. A physical argument leading to the same conclusion is on Problem Sheet 4. Although we haven't shown this, for the inviscid problem the flow is in fact unstable unless RV increases monotonically outwards.

A circular flow with a stationary outer wall is therefore inviscidly unstable. Including viscosity in the problem reveals that there is a critical Reynolds number (or "Taylor number" as it is known for this problem) at which the instability first occurs, and determines the critical value of k. But we shall not pursue that here.

This concludes the stability chapter in this course. There is much more which could be said on the subject, as anyone attending the "Hydrodynamic Stability" module can attest.

Locomotion: Swimming and Flight

We now consider in more detail solid bodies moving through fluids. Let's begin with some basic questions:

(a) What is locomotion? Typically, it is deliberate motion from A to B. We exclude passive effects like Brownian motion or being blown by the wind. Marine animals which alter their buoyancy do not technically swim. But a hovering bird or insect is flying.

(b) What is the difference between swimming and flying? Clearly, flight requires support of the body against gravity. Swimming we will define to be self-driven motion of a neutrally buoyant body through a fluid. Some sharks are denser than water, and would naturally sink if they stopped moving they are really flying not swimming.

(c) What is the difference between fish and birds and ships and planes? Obviously the fuel sources are different, but there is a more fundamental fluid mechanical difference. Man-made devices are usually almost rigid, whereas animals generate their forward thrust by altering the shape of their bodies. Moving boundary problems are generally hard, and we anticipate the need for some approximations in analysing the associated flows. A further point is that the study of ships and planes is partly motivated by the desire to improve design. The animal kingdom has evolved to the stage where it is very good at efficient locomotion in its natural environment. We do not always know what it is trying to optimise, but we can learn from it.

When a (rigid) body moves relative to the surrounding fluid it does work for two reasons. Firstly, it must move the fluid out of its way, imparting kinetic energy to the fluid. Secondly, it must overcome the internal fluid friction, as measured by the **viscous** term. The relative importance of these two effects depends crucially on the Reynolds number, R_e. Flows which are small-scale, slow and sticky have **Low Reynolds Number**, whereas motions which are large, fast and momentum dominated have a **High Reynolds Number.** For swimming in water, $R_e \sim 10^{-5}$ for a bacterium, $R_e \sim 0.1$ for a protozoan, $R_e \sim 3 \times 10^5$ for a medium sized fish and $R_e \sim 2 \times 10^6$ for a human. We recall he idea of **dynamic similarity**: for the fluid dynamics, only the value of R_e is important. A protozoan swimming in water is just like a human swimming through syrup at 0.1m/s, apart from shape differences. Our fluid dynamical knowledge leads us to expect very different propulsion mechanisms at high and low Reynolds numbers.

Note that although we are interested in a body moving in stationary fluid, it is mathematically equivalent to consider the body at rest and the fluid moving past it. This is because the Navier-Stokes equations are Galilean invariant. Note also that if the body translates at some average velocity, then the average force it exerts on the fluid must be zero. Of course, this doesn't mean the body does no work. Overcoming viscous drag (at low R_e) or imparting kinetic energy to the fluid (at high R_e) may involve considerable effort.

Drag at Low Reynolds Number:

When the nonlinear inertial term can be neglected, the problem is relatively easy to solve. A body moving with speed $\mathbf{U}_0$ through a fluid when $R \ll 1$ experiences a **drag** force

$$\mathbf{D} = 6\pi L\mu\, M\mathbf{U}_0 \qquad \text{where } M \text{ is a dimensionless geometrical tensor.} \tag{6.1}$$

If the direction of $\mathbf{U}_0$ is along a principal axis of M, then $M\mathbf{U}_0 = \beta\mathbf{U}_0$, and the drag acts in the opposite direction to the velocity. For a sphere of radius L, we found that $\beta = 1$. β does not vary too much for other shapes. Neither does it matter very much in which orientation the body lies; the drag force tends to be governed by the greatest linear dimension of the body. This is in marked contrast to flows at higher R_e, where a streamlined shape is very important. (Try pushing a boat sideways!) Despite this, we will see that low Reynolds number swimming exploits the difference in resistance coefficients for motion parallel and normal to flagella.

The crucial point is that the drag force varies **linearly** with the velocity. We know also that the entire flow is reversible: if we replace $\mathbf{u}$ by $-\mathbf{u}$, then $\mathbf{D}$ goes to $-\mathbf{D}$. The "scallop theorem" is a consequence – no time-reversible motion can result in swimming. Note also a general body will rotate in response to a net fluid torque.

Drag at high Reynolds number:

The situation is much more complicated when $R_e \gg 1$. We have seen that the boundary layer on the body may separate leading to a marked increase in drag (when flying, this is called **stall**). We also saw that vortices may be shed, the force may be time-dependent and the flow may be turbulent. Nevertheless, at high Reynolds numbers, the average drag on a body moving with speed U_0 is approximately quadratic in U_0, and may be summarised

$$D = \tfrac{1}{2} C_D \, \rho U_0^2 S \qquad \text{where } S \text{ is the cross-sectional area normal to } \mathbf{u} \; . \tag{6.2}$$

C_D is a dimensionless number known as the **drag coefficient**, which varies with both shape and Reynolds number. In this case, for the drag to be small it is very important for the shape to be **streamlined** and to present a small cross-section to the oncoming flow.

As R_e increases, C_D behaves like R_e^{-1} for low R_e, then becomes constant, although a sudden drag reduction can occur when the boundary layer becomes turbulent, as first observed by Gustave Eiffel when he wasn't building towers in Paris or the Statue of Liberty in New York.

Lift forces:

At high R_e also the total force may not be aligned with the velocity. (Imagine carrying a large sheet of wood slightly inclined to a strong wind.) A large force normal to the velocity can be generated, due to an asymmetric pressure distribution. This **lift** force, G, is usually written

$$G = \tfrac{1}{2} C_L \, \rho U_0^2 W \qquad \text{where } W \text{ is the wing area} \tag{6.3}$$

and C_L is the dimensionless **lift coefficient**. It is proportional to $\sin\alpha$ where α is the angle between wing and oncoming flow. This lift can balance gravity and enable flight. It does however increase the drag force, by an amount known as the **induced drag**, D_I. We shall examine later the reasons for the induced drag. The total drag acting in the presence of lift is given by the sum of D and D_I.

Flagellar swimming – resistive-force theory.

We shall first consider low-Reynolds-number swimming. There is a fairly clear distinction between bacteria, spermatazoa and ciliates which swim at low R_e, the Stokesian regime, and birds and fishes which live in the Eulerian regime at high R_e. Relatively few creatures, small fishes, molluscs and insects, have to cope with both inertia and viscosity.

An important result for swimming at low-R_e is called **The Scallop Theorem:** *No time reversible sequence of boundary configurations can swim at low R_e.* This is more general than saying no time-reversible motion can swim, because the rate at which the 'forward' stroke occurs can be different from the 'return' stroke. A scallop is a rigid bivalve, which opens and closes by rotating its two halves about an axis. Suppose the opening is slow and the closing is rapid. At $R_e \geqslant O(1)$ this will drive a jet of water backwards during the closing stroke, but while it opens, fluid is sucked in from all directions. Nevertheless if $R_e \ll 1$ no net motion can occur. Time appears as a parameter in the Stokes problem

$$\nabla p = \mu \nabla^2 \mathbf{u}, \qquad \mathbf{u} = \mathbf{U}_b(\mathbf{x}, t) \qquad \text{on the boundary,} \tag{6.4}$$

the speed of motion does not affect the argument. If we run the film backwards, we still end up where we started, even if we run the camera slowly. Scallops must therefore open or close quickly enough if they are to transport themselves.

It is important to note that a travelling wave is not time-reversible. This is a very common method of propulsion over a wide range of Reynolds numbers.

Resistive Force Theory

Let us now consider an organism propelled by a single, thin flagellum, which undergoes prescribed motion. We would like to calculate the force the moving flagellum exerts on the fluid; if this is non-zero on average, then it will move, and it will swim. Furthermore, if it is attached to an inert head, as is the case for spermatazoa, the entire organism will swim, but at a lower speed than would the flagellum by itself.

A useful approximation, known as **resistive force theory,** was introduced by Gray and Hancock. They argued that as the flagellum undulates, provided its radius of curvature is large compared to its diameter, the forces corresponding to the normal and tangential motion would be approximately given by the local flagellum velocity and the coefficients K_N and K_T which would apply for finite straight cylinders. On Problem sheet 1, we noted that $K_N \simeq 2K_T$.

Let the flagellum have total length L. Observation shows that sending travelling waves down the flagellum is a popular way to swim. We assume that the flagellum oscillates about the x-axis, sending a travelling wave of some shape in the positive x-direction with speed c. and that it swims with speed U in the negative x-direction. We suppose the flagellum as a result propels the entire organism in the negative x-direction.

We further assume that the solid flagellum is inextensible and incompressible. This means that its tangential velocity must be a constant, say Q. With respect to a frame moving in the x-direction with the wave speed c, the flagellum appears to be at rest. In fact the flagellum is moving tangential to itself with speed $-Q$ in the tangential direction. Thus each part of the flagellum then moves relative to the fluid with velocity

$$\mathbf{u} = (c - U,\, 0,\, 0) - Q\widehat{\mathbf{T}} = (c - U)\sin\theta\, \widehat{\mathbf{N}} + [(c - U)\cos\theta - Q]\, \widehat{\mathbf{T}} \tag{6.5}$$

where $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ are the unit tangential and normal vectors to the flagellum shape, while θ is the angle between the x-axis and $\widehat{\mathbf{T}}$.

Now that we have the normal and tangential components we can use resistive force theory, claiming that the local resistance to the velocity $\mathbf{u} = U_N\widehat{\mathbf{N}} + U_T\widehat{\mathbf{T}}$ is $K_N U_N \widehat{\mathbf{N}} + K_T U_T \widehat{\mathbf{T}}$. We then take the component in the x-direction and integrate along the flagellum to obtain the total force on the fluid

$$F = \int_0^L \big(K_T[(c-U)\cos\theta - Q]\cos\theta + K_N(c-U)\sin^2\theta\big)\, ds. \tag{6.6}$$

Now for a given wave shape we can define

$$\int_0^L \cos\theta\, ds = \alpha L \qquad \text{and} \qquad \int_0^L \cos^2\theta\, ds = \beta L. \tag{6.7}$$

Also, considering the time taken for the tip of the flagellum to reach the front, we have $c = \alpha Q$. Equation (6.6) then can be written

$$F = K_T L[(c-U)\beta - c] + K_N L(c-U)(1-\beta). \tag{6.8}$$

Now if we have included everything in the problem, the flagellum should be in equilibrium, with no net force acting, i.e. $F = 0$. Alternatively, the force F may be used to propel an inert head with the swimming velocity U. We assume the drag force for the head motion is independent of the flagellar motion, and for convenience we write this drag as $K_N L U \delta$, so that

$$\delta = \frac{\text{resistance to forward motion of head}}{\text{resistance to sideways motion of entire flagellum}}. \tag{6.9}$$

Equating the head drag to F and solving for the swimming speed U gives

$$\frac{U}{c} = \frac{(1-\beta)(1-\rho)}{1-\beta(1-\rho)+\delta} \qquad \text{where} \quad \rho = \frac{K_T}{K_N}. \tag{6.10}$$

Note no swimming occurs ($U = 0$) if $\rho = 1$, so the differential resistance is important. If $\rho > 1$, $U < 0$ and the swimming is in the same sense as the wave, but for Stokes flow $\rho < 1$ and the swimming is in the opposite sense. Obviously $\beta = 1 = \alpha$ means the flagellum isn't moving at all. In (6.10) U increases as β decreases towards zero, so we can say

$$U < \frac{c(1-\rho)}{1+\delta}. \tag{6.11}$$

Of course if the flagellum swims at all, it can propel an arbitrarily large head at a slow speed (until its energy source runs out.)

Gray and Hancock obtained reasonable agreement with observations assuming a value $\rho = 0.5$. Unfortunately, the more realistic value of $\rho = 0.7$ gives worse results. Lighthill suggested that the proximity of the sperms in their experiment to a solid boundary, with resultant interactions with images, should arguably lead to a lower value of ρ.

Flights of fancy: 2-D steady flight:

In Fluid Dynamics I, the basics of 2-D flight were introduced. Today we discuss how this theory must be modified for finite wings. First, we summarise the theory of 2-D lift generation.

We can solve for potential flow around 2-D bodies of any shape using conformal mapping techniques. A typical aerofoil shape has a pointed trailing edge, at which the potential flow speed may be infinite, with a corresponding very low pressure. The vorticity-laden boundary layer is unable to cope with the associated large adverse pressure gradient, and it separates, carrying a vortex away from the wing. Circulation, Γ, of an opposite sense to the shed vortex remains around the wing, and if it is of suitable magnitude it can ensure that the potential flow is finite at the trailing edge. This is known as the Kutta condition. For example, consider a flat plate from $x = -2a$ to $x = 2a$ in a flow of speed U. The mapping

$$Z = \zeta + \frac{a^2}{\zeta}, \tag{6.21}$$

where $Z = x + iy$, takes the flat plate $-2a < x < 2a$ to the circle $|\zeta| = a$ in the ζ-plane. If we have a uniform flow inclined at an angle α, the complex potential around the circular cylinder is

$$w(\zeta) = U\left(e^{-i\alpha}\zeta + \frac{a^2 e^{i\alpha}}{\zeta}\right) - \frac{i\Gamma}{2\pi}\log\zeta, \tag{6.22}$$

The velocity about the flat plate is

$$u - iv = \frac{dw}{dZ} = \frac{dw}{d\zeta}\frac{d\zeta}{dZ} = \left[U\left(e^{-i\alpha} - \frac{a^2 e^{i\alpha}}{\zeta^2}\right) - \frac{i\Gamma}{2\pi\zeta}\right]\frac{\zeta^2}{\zeta^2 - a^2}. \tag{6.23}$$

The velocity at the trailing edge $\zeta = a$ can be made finite by choosing

$$\Gamma = -4\pi U a \sin\alpha. \tag{6.24}$$

This then gives a lift force

$$G = \rho\Gamma U = 4\pi\rho a U^2 \sin\alpha, \tag{6.25}$$

perpendicular to the incoming flow velocity. The flat plate flow still has a singularity at the leading edge ($\zeta = -a$). This can be avoided by considering circles in the ζ-plane which enclose $\zeta = -a$ but still pass through $\zeta = a$. These map onto shapes in the Z-plane called Joukowski aerofoils, which are very convenient analytic wing shapes.

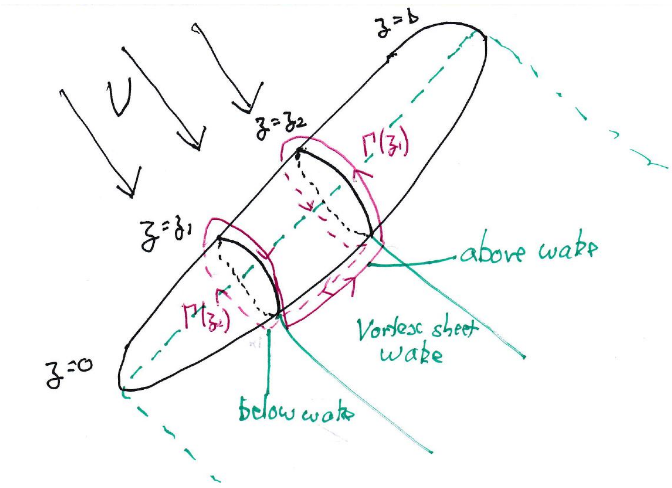
The lift expression (6.25) is valid so long as the boundary layer remains attached, which it does for small angle of attack, say $\alpha \leqslant 15°$. Higher values of α lead to *stall* and a massive increase of drag. This is generally avoided when flying, although pigeons deliberately stall when landing.

3-D steady flight:

Once the starting vortex has been shed from a 2-D aerofoil, it can be forgotten about. The remaining flow is potential, as the circulation is about an apparent vortex line inside the wing, which extends to infinity. When the wing has a finite extent in the 3rd dimension, however, the situation is more complex. Vortex lines cannot stop abruptly and must close or head off to infinity. A 3-D potential flow can provide no lift force, so during flight vorticity must continually be shed into the wake.

Real wings leave a trail of vorticity behind them. The edges of this vortex sheet roll up into wing-tip vortices, as illustrated in the figure. These vortices are not the same as the vapour trails we see behind aircraft which are caused by condensation from the engines.

The magnitude of the vorticity shed into the wake may be calculated using Prandtl's 'lifting-line theory.' We assume a thin wing cross-section which varies slowly in the z-direction, so that the 2-D theory is approximately correct, with $a = a(z)$. We suppose that the circulation at point $z = z_1$ will be some function $\Gamma(z_1)$, which is given by (6.24) with some effective angle of attack $\alpha(z_1)$.



Now the circulation around any closed curve in a 3-D irrotational flow is zero. We integrate around the curve in the diagram, which passes around the wing at two locations z_1 and z_2, closing bypassing above and below the thin vortical wake as shown. The flow round the wing we suppose is given locally by the 2-D theory overleaf. We therefore have

$$0 = \int \mathbf{u} \cdot d\mathbf{l} = \Gamma(z_2) - \Gamma(z_1) + \int \omega dA, \tag{6.26}$$

where ω is the streamwise vorticity in the wake, and A is a rectangle of length $(z_2 - z_1)$ and a small height. Letting $z_2 \to z_1$, we deduce that a vorticity $-d\Gamma/dz$ is shed into the wake at each point along the wing (see sketch).

Now a vortex line of strength $-\Gamma'$ drives a flow of size $-\Gamma'/2\pi R$ around it, where R is the radial distance from the vortex. It follows by symmetry, that at the end of a semi-infinite vortex line, the velocity should be half of this. We can then superpose the effect of all the vortex lines shed into the wake to provide the downward velocity at the leading edge as the principle value integral

$$v(z) = -\frac{1}{4\pi}\int_0^b \frac{\Gamma'(z_1)}{z - z_1}\, dz_1. \tag{6.27}$$

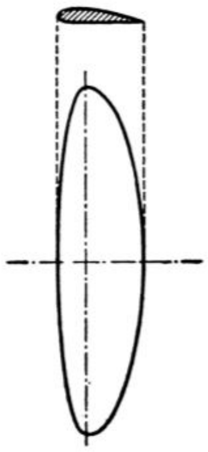
This additional velocity alters the effective angle of attack. Putting all this together with (6.25) leads to an integral equation for Γ. If the attack angle α is small, then $\Gamma(z)$ satisfies

$$\Gamma(z) = -4\pi a(z)\left[\alpha U + \frac{1}{4\pi}\int_0^b \frac{\Gamma'(z_1)}{z - z_1}\, dz_1\right].$$

Once this is solved, the total lift can be found by integrating along the wing. Details can be found in Childress (1981) or Batchelor (1967). As the effective velocity (including the effect of the wake) at the leading edge is not parallel to the oncoming flow, there is a component of the 'lift' in the downstream direction proportional to the integral of $\rho v\Gamma$ along the wing. This increase in the drag force, known as the induced drag, is usually written

$$D_I = \frac{kG^2}{\frac{1}{2}\rho U^2 b^2}, \tag{6.28}$$

where b is the wing length and k is a dimensionless constant equal to $1/\pi$ for an elliptic wing profile, and which doesn't vary much from this value.



Incidentally, the theory suggests that the most efficient wing shape, maximising lift for a given drag, should be elliptical. Spitfire aircraft in the 2nd world war were designed with elliptical wings, which arguably gave a greater manoeuvrability. In fact for structural reasons they were built from two semi-ellipses, as shown in the diagram (copied from B. Cantwell, at Stanford University).

Birds in Flight

When thou seest an eagle, thou seest a portion of genius; lift up thy head! – William Blake

Birds have been flying for a very long time and they do it very well.

We will assume the wings are thin with total wing area W and length $2b$. Their shapes are streamlined so that the boundary layers remain attached up to the trailing edge. Experiment shows that as the wing moves through fluid it feels forces: a profile drag, D_P and lift, G, which can be written

$$D_P = \tfrac{1}{2} C_D \, \rho S U^2, \qquad G = \tfrac{1}{2} C_L \, \rho W U^2, \tag{6.29}$$

where S is the area from the front and W is the wing area from on top. The drag and lift coefficients, C_D and C_L are $O(1)$ and determined by the geometry and Reynolds number. The induced drag we calculated in (6.28) is in addition to the profile drag.

Steady Horizontal Flight: Suppose our bird exerts a mean thrust T. Then for steady horizontal flight the lift must balance the weight, $G = mg$, and we have the simple force balance

$$T = D = D_P + D_I = \tfrac{1}{2} C_D \rho S U^2 + \frac{k(mg)^2}{\frac{1}{2}\rho b^2} \frac{1}{U^2}. \tag{6.30}$$

Viewed as a function of U, the drag has a minimum at the value $U = U_m$ when the profile drag equals the induced drag, where

$$U_m = \left[\frac{4k(mg)^2}{C_D \rho^2 b^2 S} \right]^{1/4}. \tag{6.31}$$

At this optimal flight speed, the bird can fly a given distance for minimal work. If instead we want to find the speed which permits the longest time airborne for given work expenditure, we should consider the power, TU, and minimise that, whch will give a different optimising speed. However, flight actually at $U = U_m$ would not be stable. A slight decrease in the flight speed would lead to an increased drag and hence a further decrease in flying speed. So for steady stable flight we expect a speed slightly above U_m, so that a decrease in U leads to a decrease in D and hence a restoring acceleration.

Gliding:

Now suppose the bird exerts no thrust T, and so begins to descend. While we might have doubts about ignoring unsteady effects during flapping flight, the bird can clearly adopt this steady configuration. It can glide at a downwards angle θ to the horizontal if

$$G = mg\cos\theta, \qquad D = mg\sin\theta \quad \Longrightarrow \quad \tan\theta = D/G. \tag{6.32}$$

The glide angle θ affording minimum energy loss is again attained at $U = U_m$. If the bird wishes to glide at a slower speed (smaller θ) then it should increase $b^2 S$ which means extending its wings. In contrast if it wishes to dive, it pulls in its wing tips. Hunting birds often glide at zero velocity relative to the ground near cliffs where an appropriate upward wind can be found.

Flying for Free:

Under suitable circumstances, a bird may extract energy from variations in the wind strength and direction. The simplest way of doing this is to find a *thermal*, a vertically rising air column. Within this thermal, it glides downwards relative to the air, while banking (altering its inclination to the vertical), which causes it to follow a helical path within the extent of the thermal. If the updraft in the thermal is greater than the vertical descent relative to the air, the bird will gain height. From its new vantage point it can leave the thermal, gliding downwards until it finds a new thermal and repeat the process. Vultures and migrating birds are observed to do this.

Soaring:

In the absence of land topography, more sophisticated manoeuvres are used by the albatross to extract energy from a horizontal shear. Suppose the wind is a 1-D shear, $\mathbf{u} = (U(z), 0, 0)$, where z is the vertical coordinate. Let a bird flying in this shear be at position $(X(t), Y(t), Z(t))$ with velocity (u, v, w) relative to the wind. We work in a non-inertial frame travelling with the air velocity at the instantaneous position of the bird. This frame has the acceleration in the x-direction

$$U'(Z)\frac{dZ}{dt} = wU'. \tag{6.33}$$

If we apply Newton's laws in such an accelerating frame, we must include a fictitious force $-mwU'$ in the equation of motion of the bird. This force does work at a rate $-muwU'$. All the bird need do is make sure that this is on average positive and it can extract energy from the mean flow. In other words, if $U' > 0$, it must arrange that $\overline{uw} < 0$. The albatross arranges to fly upwind ($u < 0$) when going up ($w > 0$) and downwind when going down. This soaring motion is very efficient. There are similarities with the way turbulent fluctuating velocities extract energy from the mean flow – $\overline{uw}$ can be thought of as a Reynolds stress.

Why can't we fly?

The largest flying birds are about 12kg. We can easily understand why there is an upper limit for bird mass using **allometric** arguments. If the lift G balances the body mass, an animal of characteristic length-scale L must generate a lift $G \propto L^2U^2 \propto mg \propto L^3$. Thus the speed must scale as $L^{1/2}$. It follows that the rate at which birds must do work as they fly, TU, scales as $L^{7/2}$. Now experiment shows that the metabolic rate of all animals scale very closely as $L^{9/4}$ (see the "mouse to elephant curve" online.) It follows that for small enough animals the metabolic rate will be lower than the power needed to fly, but that as L increases, eventually the two curves will cross, and the power needed will be less than that available by the standard chemical processes in the body. This shows an upper mass limit must exist for animals of similar design. So if you took this module hoping to learn how to fly, you will need to evolve proportionally larger wings than birds.

High-Reynolds-number flows with closed streamlines

Often we consider fluid flow in an unbounded domain, avoiding the complication of a far boundary. But sometimes the effects of the outer boundary are vital. In two-dimensions (or axisymmetry) the streamlines of such a bounded flow are necessarily closed, and this has some interesting consequences.

Consider steady, two-dimensional flow, described by a streamfunction $\psi(x,\, y)$, with $\mathbf{u} = \nabla \times (0,\, 0,\, \psi)$. The vorticity equation then takes the form

$$\mathbf{u} \cdot \nabla\omega = \nu\nabla^2\omega \qquad \text{where} \qquad \omega = -\nabla^2\psi. \tag{6.1}$$

From (6.1) we can see that in the high Reynolds number limit ($\nu \to 0$) the vorticity does not vary in the direction of the flow (along streamlines $\psi = \text{const}$), which means that

$$\omega = \omega(\psi), \tag{6.2}$$

but with the functional variation of ω across the streamlines still to be determined. When $\nu = 0$ any function is possible. We saw this before when considering unidirectional flows $(U(y),\, 0,\, 0)$. If there is no viscosity then any vorticity U' is possible.

How then do we determine the function $\omega(\psi)$? The key point is that as a fluid particle goes round and round a closed streamline, the small viscous term has a lot of time in which to act, and it can redistribute vorticity between the streamlines as required. If we write the Navier-Stokes equation in the form

$$\nabla\left(p/\rho + \tfrac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u} \times \boldsymbol{\omega} = \nu\nabla^2\mathbf{u} \tag{6.3}$$

and then take the line-integral around a closed streamline $\psi = \psi_0$ giving a contour C, then the $\mathbf{u} \times \boldsymbol{\omega}$ term disappears as it is perpendicular to $\mathbf{u}$ and so we get

$$\oint_C \nabla\left(p/\rho + \tfrac{1}{2}|\mathbf{u}|^2\right) \cdot d\mathbf{l} = \nu \oint_C \nabla^2\mathbf{u} \cdot d\mathbf{l}. \tag{6.4}$$

The LHS vanishes because it is an exact differential and C encloses only fluid. Hence, cancelling the factor of $\nu > 0$ we find,

$$0 = \oint_C \nabla^2\mathbf{u} \cdot d\mathbf{l}. \tag{6.5}$$

This is true for any $\nu > 0$, but we can also consider the limit as $\nu \to 0$. First, we use Stokes' theorem followed by the two-dimensional version of the divergence theorem to write the above as

$$0 = \oint_C \nabla^2\mathbf{u} \cdot d\mathbf{l} = \int_S \nabla^2(\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}\; dS = \int_S \nabla^2\omega\; dS = \oint_C \nabla\omega \cdot \widehat{\mathbf{n}}\; dl. \tag{6.6}$$

Now, in the limit $\nu \to 0$, to leading order we may use the fact that ω is only a function of ψ, i.e. $\omega \sim \omega(\psi)$ as $\nu \to 0$, so

$$\oint_C \nabla\omega \cdot \widehat{\mathbf{n}}\; dl \sim \oint_C \frac{d\omega}{d\psi}\nabla\psi \cdot \widehat{\mathbf{n}}\; dl = \frac{d\omega}{d\psi} \oint_C \nabla\psi \cdot \widehat{\mathbf{n}}\; dl. \tag{6.7}$$

But, with another application of the divergence theorem and Stokes theorem,

$$\oint_C \nabla\psi \cdot \widehat{\mathbf{n}}\ dl = \int_S \nabla^2\psi\ dS = -\int_S \nabla^2\omega\ dS = -\oint_C \mathbf{u}\cdot d\mathbf{l} \tag{6.8}$$

and so (6.7) and (6.6) give, to leading order in $\nu \to 0$,

$$0 = \frac{d\omega}{d\psi} \oint_C \mathbf{u}\ d\mathbf{l}. \tag{6.9}$$

Now the circulation $\oint_C \mathbf{u}\cdot d\mathbf{l}$ will not vanish in general, and so we deduce that $d\omega/d\psi = 0$ or ω is constant inside a closed streamline. This is known as the **Prandtl-Batchelor theorem.**

[There is a corresponding result for axisymmetric flows, where it can be shown that ω/R must be constant, where R is the distance from the axis of symmetry.]

Von Mises coordinates

Suppose we have flow within a unit cylinder with a given surface velocity $U_s(\theta)$ in the θ-direction on $R = 1$. At high Reynolds number we might expect a boundary layer to develop near $r = 1$. Outside this boundary layer, but in the interior of the cylinder, the Prandtl-Batchelor theorem tells us (since all streamlines will be closed) that the vorticity should be constant. By symmetry, we may expect the inviscid velocity in the interior to be "rigid body motion", i.e. $\mathbf{u} = (0, RU, 0)$ for a constant U. But we do not know U and it must be found by matching with flow in the boundary layer. We do at least know that pressure gradient in the boundary layer is zero, $UU' = 0$.

We note that since ψ is constant on the wall $R = 1$, we could consider using ψ as a normal coordinate. So let us transform the boundary layer equation (where x and y are the local and tangential coordinates to the boundary)

$$uu_x + vu_y = UU' + u_{yy} \qquad \text{to new coordinates} \qquad (x,\, y) \to (\theta,\ \psi) \tag{6.10}$$

Now $u = \psi_y$ and $v = -\psi_x$, and since x is actually the arc length along the cylinder, we set $\theta = x$. Using the chain rule for partial derivatives

$$\frac{\partial}{\partial x} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial \psi} = \frac{\partial}{\partial \theta} - v\frac{\partial}{\partial \psi} \tag{6.11}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial \psi} = u\frac{\partial}{\partial \psi}. \tag{6.12}$$

Then (6.10) becomes

$$u\frac{\partial u}{\partial \theta} - uv\frac{\partial u}{\partial \psi} + vu\frac{\partial u}{\partial \psi} = u\frac{\partial}{\partial \psi}\left(u\frac{\partial u}{\partial \psi}\right) \tag{6.13}$$

A factor of u cancels from (6.13), giving

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \psi}\left(u\frac{\partial u}{\partial \psi}\right) = \frac{\partial^2}{\partial \psi^2}\left(\tfrac{1}{2}u^2\right). \tag{6.14}$$

The parabolic nature of the boundary layer equation is clear, and we are looking for solutions which are periodic in θ. If we integrate around a closed streamline (from $\theta = 0$ to $\theta = 2\pi$) the LHS disappears. The variables θ and ψ are independent and so we conclude

$$0 = \frac{\partial^2}{\partial \psi^2}\int_0^{2\pi} u^2 d\theta \qquad \Longrightarrow \qquad \int_0^{2\pi} u^2 d\theta = A\psi + B. \tag{6.15}$$

Now if we go out of the layer $\psi \to \infty$ we have $u \to U$. It follows that the constant $A = 0$. Finally we conclude $\oint u^2 d\theta$ is constant across the boundary layer and so we must have

$$U^2 = \frac{1}{2\pi}\int_0^{2\pi} U_s^2\, d\theta. \tag{6.16}$$