Exercise 6.1. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a,b])$:

(i)
$$\|f\|_1 = \int_a^b |f(t)|\,dt$$

(ii)
$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$$

(iii)
$$\|f\|_2 = \left(\int_a^b |f(t)|^2\,dt\right)^{1/2}$$

Hint: to show that $\|\cdot\|_2$ is a norm, you need to use the Cauchy-Schwarz inequality and the definition of the integral as the limit of certain sums.

Solution: (i) By the properties of the Riemann integral, $\|f\|_1 \geq 0$. By a lemma in the lecture notes, $\|f\|_1 = 0$ iff $f \equiv 0$. For every $\lambda \in \mathbb{R}$, we have

$$\|\lambda f\|_1 = \int_a^b |\lambda f(t)|\,dt = \int_a^b |\lambda||f(t)|\,dt = |\lambda| \int_a^b |f(t)|\,dt = |\lambda|\,\|f\|_1\,.$$

Moreover, for all f and g in $C([a,b])$, we have

$$\|f+g\|_1 = \int_a^b |f(t)+g(t)|\,dt \leq \int_a^b (|f(t)|+|g(t)|)\,dt = \int_a^b |f(t)|\,dt + \int_a^b |g(t)|\,dt,$$

which implies that $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.

(ii) For every f in $C([a,b])$, the maximum of f on $[a,b]$ is realised, so $\|f\|_\infty$ is well-defined, and a real number. Evidently, $\|f\|_\infty \geq 0$, and $\|f\|_\infty = 0$ iff $f \equiv 0$. Moreover, for all $\lambda \in \mathbb{R}$, we have

$$\|\lambda f\|_\infty = \max_{t\in[a,b]} |\lambda f(t)| = \max_{t\in[a,b]} (|\lambda||f(t)|) = |\lambda| \max_{t\in[a,b]} |f(t)| = |\lambda|\,\|f\|_\infty\,.$$

Finally, for all f and g in $C([a,b])$, we have

$$\begin{aligned}\|f+g\|_\infty &= \max_{t\in[a,b]} |f(t)+g(t)| \\ &\leq \max_{t\in[a,b]} (|f(t)|+|g(t)|) \\ &\leq \max_{t\in[a,b]} |f(t)| + \max_{t\in[a,b]} |g(t)| \\ &= \|f\|_\infty + \|g\|_\infty\,.\end{aligned}$$

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Questions marked with $*$ are optional

(iii) Fix arbitrary functions f and g in $C([a,b])$. We note that for all $\lambda \in \mathbb{R}$, we have

$$\int_a^b (f(t) - \lambda g(t))^2\, dt \geq 0.$$

This implies that

$$\int_a^b f(t)^2\, dt - 2\lambda \int_a^b f(t)g(t)\, dt + \lambda^2 \int_a^b g(t)^2\, dt \geq 0.$$

One may think of the expression on the left hand side of the above equation as a quadratic polynomial in λ. We know that if a quadratic polynomial of the above form is non-negative, then the discriminant ("$b^2 - 4ac$") must be non-positive, that is,

$$4\left(\int_a^b f(t)g(t)\, dt\right)^2 \leq 4\int_a^b f(t)^2\, dt \cdot \int_a^b g(t)^2\, dt.$$

This implies that for all f and g in $C([a,b])$, we have

$$\left|\int_a^b f(t)g(t)\, dt\right| \leq \|f\|_2 \|g\|_2 .$$

The above inequality is known as the Cauchy–Schwarz inequality. It is also possible to prove the above inequality, using the definition of the integral as limits of sums, and using the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

Using the Cauchy-Schwarz inequality, we can see that for all f and g in $C([a,b])$, we have

$$\|f+g\|_2^2 = \int_a^b |f(t)+g(t)|^2\, dt = \int_a^b f(t)^2 dt + 2\int_a^b f(t)g(t) dt + \int_a^b g(t)^2 dt \leq (\|f\|_2 + \|g\|_2)^2,$$

which implies that $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

The other properties for $\|\cdot\|_2$ can be proved by arguments similar to the ones for $\|\cdot\|_1$.

Exercise 6.2. Show that if V is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2\,\mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric $\mathrm{d}_{\mathrm{disc}}$ on $\mathbb{R}^2$.

Solution: Since for every norm function, any $v \in V$ and any $\lambda \in \mathbb{R}$, we have $\|\lambda v\| = |\lambda|\,\|v\|$, we must have

$$\mathrm{d}_{\|\|}(0, 2v) = \|2v\| = 2\,\|v\| = 2\,\mathrm{d}_{\|\|}(0, v).$$

For the discrete metric, we have

$$\mathrm{d}_{\mathrm{disc}}((0,0),(1,1)) = \mathrm{d}_{\mathrm{disc}}((0,0),(2,2)) = 1,$$

which does not satisfy the above relation when $v = (1,1)$.

Exercise 6.3. Let (X, d) be a metric space.

(i) Show that for every x, y, and z in X, we have

$$|\,\mathrm{d}(x,z) - \mathrm{d}(y,z)| \leq \mathrm{d}(x,y).$$

(ii) Show that for all x, y, z and t in X, we have

$$|\,\mathrm{d}(x,y) - \mathrm{d}(z,t)| \leq \mathrm{d}(x,z) + \mathrm{d}(y,t).$$

(iii) Show that for all x_1, x_2, $\dots$, x_n in X, we have

$$\mathrm{d}(x_1,x_n) \leq \mathrm{d}(x_1,x_2) + \mathrm{d}(x_2,x_3) + \cdots + \mathrm{d}(x_{n-1},x_n).$$

Solution: (i) Using the triangle inequalities

$$\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z), \qquad \mathrm{d}(y,z) \leq \mathrm{d}(x,y) + \mathrm{d}(x,z),$$

we obtain

$$-\,\mathrm{d}(x,y) \leq \mathrm{d}(x,z) - \mathrm{d}(y,z) \leq \mathrm{d}(x,y),$$

which is equivalent to the the desired inequality.

(ii) Using the triangle inequality two times, we obtain

$$\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + \mathrm{d}(y,z) \leq \mathrm{d}(x,z) + \mathrm{d}(z,t) + \mathrm{d}(y,t),$$

and

$$\mathrm{d}(z,t) \leq \mathrm{d}(z,x) + \mathrm{d}(x,t) \leq \mathrm{d}(z,x) + \mathrm{d}(x,y) + \mathrm{d}(y,t).$$

By adding and subtracting appropriate terms, we obtain

$$\mathrm{d}(x,y) - \mathrm{d}(z,t) \leq \mathrm{d}(x,z) + \mathrm{d}(y,t),$$

and

$$-\,(\mathrm{d}(x,z) + \mathrm{d}(y,t)) \leq \mathrm{d}(x,y) - \mathrm{d}(z,t).$$

These two inequalities imply the desired inequality in part (ii).

(iii) We prove the desired statement by induction on the number of points, n. For $n=2$ the inequality is obvious. Assume that the inequality holds for n points. For any collection of $n+1$ points, x_1, x_2, $\dots$, x_{n+1}, we have

$$\begin{aligned}\mathrm{d}(x_1,x_{n+1}) &\leq \mathrm{d}(x_1,x_n) + \mathrm{d}(x_n,x_{n+1})\\ &\leq \mathrm{d}(x_1,x_2) + \mathrm{d}(x_2,x_3) + \cdots + \mathrm{d}(x_{n-1},x_n) + \mathrm{d}(x_n,x_{n+1}).\end{aligned}$$

Exercise 6.4. Let (X,d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By an example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n\in\mathbb{N}} B_{1/n}(x) = \{x\}.$$

Solution: (i) If $y \in B_\epsilon(x)$, then $\mathrm{d}(x,y) < \epsilon$, and hence $\mathrm{d}(x,y) < \delta$. Therefore, $y \in B_\delta(x)$. In the discrete metric on $\mathbb{R}$, $B_2(0) = B_3(0) = \mathbb{R}$.

(ii) It is enough to show that $\{x\} \subseteq \cap_{n\in\mathbb{N}} B_{1/n}(x)$ and $\cap_{n\in\mathbb{N}} B_{1/n}(x) \subseteq \{x\}$. Since for all $n \geq 1$ we have $x \in B_{1/n}(x)$, we conclude that $x \in \bigcap_{n\in\mathbb{N}} B_{1/n}(x)$.

Fix an arbitrary $y \in \cap_{n\in\mathbb{N}} B_{1/n}(x)$. Then, for every $n \geq 1$ we have $\mathrm{d}(x,y) < 1/n$. This implies that $\mathrm{d}(x,y) = 0$, and by the property of the metrics, we obtain $y = x$. Therefore, $y \in \{x\}$.

Exercise 6.5. (i) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x,y) \leq \mathrm{d}_2(x,y) \leq \sqrt{n} \cdot \mathrm{d}_\infty(x,y).$$

(ii) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x,y) \leq \mathrm{d}_1(x,y) \leq n \cdot \mathrm{d}_\infty(x,y).$$

(iii) Show/conclude that for all x and y in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}} \mathrm{d}_2(x,y) \leq \mathrm{d}_1(x,y) \leq n\, \mathrm{d}_2(x,y).$$

(iv) Conclude that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.

Solution: (i) This is the statement in Exercise 1.2, formulated in a different form.

(ii) If $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$, we have

$$\max_{j=1,\dots,n} |x^j - y^j| \leq \sum_{j=1}^{n} |x^j - y^j| \leq n \max_{j=1,\dots,n} |x^j - y^j|.$$

(iii) These immediately follow from the inequalities in part (i), (ii).

(iv) We need to show that for any set $U \subseteq \mathbb{R}^n$, U is open with respect to d_1, if and only if U is open with respect to d_2, if and only if U is open with respect to d_∞. Let us assume that U is open with respect to d_1.

Fix an arbitrary $x \in U$. Since U is open with respect to d_1, there is $r > 0$ such that

$$B_r(x, \mathbb{R}^n, \mathrm{d}_1) \subseteq U.$$

By the right-hand side of the inequality in part (iii), we have

$$B_{r/n}(x, \mathbb{R}^n, \mathrm{d}_2) \subseteq B_r(x, \mathbb{R}^n, \mathrm{d}_1).$$

Therefore,

$$B_{r/n}(x, \mathbb{R}^n, \mathrm{d}_2) \subseteq U.$$

Because $x \in U$ was arbitrary, this implies that U is open with respect to d_2.

Similarly, by the right-hand side of the inequality in part (ii), we have

$$B_{r/n}(x, \mathbb{R}^n, \mathrm{d}_\infty) \subseteq B_r(x, \mathbb{R}^n, \mathrm{d}_1).$$

This implies that

$$B_{r/n}(x, \mathbb{R}^n, \mathrm{d}_\infty) \subseteq U.$$

As $x \in U$ was arbitrary, this implies that U is open with respect to d_∞.

All the other implications can be proved in a similar fashion using the other sides of the inequalities in part (ii) and (iii).

Exercise 6.6. Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\text{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

Solution: Assume $(x_n)_{n\geq 1}$ converges to $x \in (X, \mathrm{d}_{\text{disc}})$. Then $\forall \epsilon > 0$ there is N s.t. $\forall n > N$, $x_n \in B_\epsilon(x)$. Take for example $\epsilon = 1/2$. Since in our space $B_{1/2}(x) = \{x\}$, we have $x_n = x$, $\forall n > N$. In other words, the sequence is eventually constant.

For the opposite implication, assume that there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n = x_N$. Then, for all $\epsilon > 0$, $x_n \in B_\epsilon(x_n)$. Thus, for all $\epsilon > 0$, and all $n \geq N$, $x_n \in B_\epsilon(x_N)$. This implies that the sequence $(x_n)_{n\geq 1}$ converges to x_N.

Exercise 6.7. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

Hint: U can be the ball $B_r(x)$.

Solution: Assume that $(x_n)_{n\geq 1}$ converges to $x \in X$. Let U be an arbitrary open set which contains x. Since U is open and $x \in U$, there is $\delta > 0$ such that $B_\delta(x) \subset U$. Since $(x_n)_{n\geq 1}$ converges to x, for δ there is $N = N(\delta)$ such that for all $n \geq N$ we have $x_n \in B_\delta(x)$. Since $B_\delta(x) \subset U$, for all $n \geq N$ we have $x_n \in U$.

For the opposite implication assume that $(x_n)_{n\geq 1}$ is a sequence in X and for any open set $U \subset X$ with $x \in U$, there is N such that for all $n \geq N$, we have $x_n \in U$. Fix an arbitrary $\epsilon > 0$ and define $U = B_\epsilon(x)$ (recall that any ball is an open set). By the hypothesis, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U = B_\epsilon(x)$. As ϵ was arbitrary, we conclude that $(x_n)_{n\geq 1}$ converges to x.

Exercise 6.8. Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space. Show that every set in X is closed.

Hint: First show that every set in X is open with respect to d_{disc}.

Solution: We first show that every set in X is open. Let us fix an arbitrary set $A \subset X$. For any $x \in A$, we have $x \in B_{1/2}(x) = \{x\} \subset A$. By definition, this means that A is open. Thus any set in X is open. Now take an arbitrary set $B \in X$. We have just shown that $X \setminus B$ is open. Therefore, by a theorem in the lectures, B is closed.

Unseen Exercise. Let $E = \{1, 2, 3, 4, 5, 6\}$, and let $\mathcal{P}(E)$ be the set of all subsets of E. Consider the metric d_{card} on $\mathcal{P}(E)$ (see typed lecture notes). Let $e = \{1, 2, 3\} \in \mathcal{E}$. What is $B_{1/2}(e)$? What is $B_1(e)$? What is $B_{3/2}(e)$?

Solution: By definition, $B_\epsilon(e)$ is the set of all points $y \in \mathcal{P}(E)$ such that $\mathrm{d}_{\text{card}}(e, y) < \epsilon$. By definition, $\mathrm{d}_{\text{card}}(x, y) = \mathrm{Card}(x \Delta y)$.

Fix an arbitrary $r \in (0, 1)$. If $y \in \mathcal{P}(E)$, and $\mathrm{d}_{\text{card}}(e, y) < r$, we must have

$$\mathrm{Card}((e \setminus y) \cup (y \setminus e)) = \mathrm{Card}(e \setminus y) + \mathrm{Card}(y \setminus e) < r.$$

This is because the sets $e \setminus y$ and $y \setminus e$ are disjoint sets. The above inequality implies that

$$\mathrm{Card}(e \setminus y) < 1, \text{ and } \mathrm{Card}(y \setminus e) < 1.$$

The inequality on the left hand side implies that $e \setminus y = \emptyset$ and hence $e \subseteq y$. Similarly, the inequality on the right hand side implies that $y \subseteq e$. Therefore, $y = e$. On the other hand, since $r > 0$, we have $e \in B_r(e)$. Combining these together, we obtain $B_r(e) = \{e\}$. In particular, $B_{1/2}(e) = B_1(e) = e$.

By the definition of the metric $\mathrm{d}_{\mathrm{card}}$, is $y \in B_{3/2}(e)$, y may have at most one more element than the set e or at most one element less than e. Therefore,

$$B_{3/2}(e) = \{e, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,6\}, \{1,2\}, \{1,3\}, \{2,3\}\}.$$