Math40002 Analysis 1 **Problem Sheet 7**

1. Let a_n, b_n be sequences of real numbers such that $b_n \neq 0$ and $a_n/b_n \to r \in \mathbb{R}$.
   - Prove that if $\sum b_n$ is absolutely convergent, then so is $\sum a_n$.
   - † Give examples (for any r) for which $\sum b_n$ is convergent but $\sum a_n$ diverges.

2.† Give an example of sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ such that $a_n/b_n \to 1$ as $n \to \infty$ but $\sum_n a_n$ is convergent and $\sum_n b_n$ is divergent.

3. Suppose that $a_n \in \mathbb{C} \setminus \{0\}$ $\forall n$ and $a_{n+1}/a_n \to a \in \mathbb{C}$. What is the radius of convergence of $\sum_{n=1}^{\infty} a_n z^n$? Prove it!

   Compute the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(n!)^2 z^n}{(2n)!}$.

4. Determine the radius of convergence of the following power series.

   (i) $\sum_{n=1}^{\infty} \frac{z^n}{3^n+5^n}$, (iii) $\sum_{n=1}^{\infty} \frac{n!}{1.3.5..(2n+1)} z^n$,

   (ii) $1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$, (iv) $\sum_{n=1}^{\infty} (n!)^{1/n} z^n$.

5.* What are the possible values of the radius of convergence of a series $\sum_{n=1}^{\infty} a_n z^n$ with $en^{-\pi} < |a_n| < \pi n^e \ \ \forall n$?

6. The great Professor Martin Lietype is not very good with complex numbers, but an ace with reals. He notices that the infinite series $1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges to the function

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{1}{1+x^2},$$

   which is finite $\forall x \in \mathbb{R}$. He concludes the series converges $\forall x \in \mathbb{R}$. Is he right? If not, can you help him? Would it help if he was better with complex numbers?

7. Show the following sequence (a_n) is convergent:

$$a_n := \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}.$$

8. Suppose $a_n \geq 0$ $\forall n$. Show that if $\sum a_n$ is convergent then $\sum \frac{a_n}{1+a_n}$ is convergent. Is the converse true?

9. Let $s_n := \sum_{i=0}^{n} \frac{1}{i!}$. Show that $\frac{1}{(n+k)!} \leq \frac{1}{(n+1)^k n!}$ for all integers $n, k > 0$, and hence

$$s_N - s_n \ < \ \frac{1}{n \cdot n!} \qquad \forall N > n \geq 1 \qquad (*)$$

   Deduce (s_n) is bounded above and convergent to some $e := \sup\{s_n : n \in \mathbb{N}\} \in \mathbb{R}$ satisfying

$$0 \ < \ e - \sum_{i=0}^{n} \frac{1}{i!} \ \leq \ \frac{1}{n \cdot n!} \qquad (**)$$

   for all $n \geq 1$. If we could write $e = \frac{m}{n}$ with $m, n \in \mathbb{N}$ multiply (**) by $n!$ to get a contradiction. Conclude that e is irrational.

10.† Celebrity computer scientist Professor Buzzard has taught Thomas and Liebeck a game. They each flip a fair coin repeatedly until they get a tail. The winner is the one who got the most heads, and receives £4^n from the loser, where n is the loser's number of heads.[1]

Liebeck declares confidently "Ah ha Thomas, if you throw h heads, my expected winnings are 50p, whatever h is." Check he is right. He's pretty sure he's going to clean up.

Thomas replies "Ah but Liebeck, if *you* throw h heads, your expected winnings are -50p, whatever h is." Check he is also right.

"Lean says the game is symmetric between the pair of you, so don't you think your expected winnings should be zero?" says Buzzard. What is going on?
(Hint: we're meant to be studying absolute convergence, not coin tossing.)

Starred questions * *are good to prepare to discuss at your Problem Class.*
Questions marked † are slightly harder (closer to exam standard), but good for you.
Happy Christmas!

[1] If they flip the same number of heads it is a draw and no money changes hands.