ALGEBRA 3 ASSESSED PROBLEMS 1 - DUE FRIDAY, 12 NOVEMBER 2021

This is the first assessed coursework for Algebra 3; unlike the weekly discussion problems, all rules for assessed coursework at Imperial apply. It is due Friday, 12 November, at 11PM in Turnitin (via the course Blackboard page) and is worth 5 percent of your total marks for the course. The coursework is scored out of ten marks; each question is worth one mark.

1. Let R be a ring, $S \subset R$ a subring, and $I \subset R$ an ideal. Let $S + I$ be the subset of R consisting of elements of the form $s + i$, with $s \in S$ and $i \in I$. Show that $S + I$ is a subring of R, and that the inclusion of S in $S + I$ induces an isomorphism:

$$S/(S \cap I) \cong (S + I)/I.$$

2. Let K be a field, and $P(X) \in K[X]$ a nonconstant polynomial. Show that $P(X)$ is squarefree if, and only if, the quotient $K[X]/\langle P(X)\rangle$ is a product of fields.

3. Let K be a field, and let S denote the subring $K[X^2, X^3]$ of $K[X]$. Show that X^2 and X^3 are irreducible in S, and are not associates. Conclude that S is not a Euclidean domain.

4. Find a greatest common divisor, in $\mathbb{Z}[\sqrt{-2}]$, of 11 and $2 + 3\sqrt{-2}$. (You may use, without proof, the fact that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean.)

5. Find rational numbers x and y such that $\frac{1}{4+3\sqrt{2}} = x + y\sqrt{2}$ in $\mathbb{Q}[\sqrt{2}]$.

6. Let n be a squarefree integer and $P(X) \in \mathbb{Q}[X]$ an irreducible polynomial of odd degree. Show that there does not exist $\alpha \in \mathbb{Q}[X]/\langle P(X)\rangle$ such that $\alpha^2 = n$.

7. Show that if a and b are integers with no common factor, then the inclusion of $\mathbb{Z}$ in $\mathbb{Z}[\sqrt{-2}]$ induces an isomorphism:

$$\mathbb{Z}/(a^2 + 2b^2)\mathbb{Z} \cong \mathbb{Z}[\sqrt{-2}]/\langle a + b\sqrt{-2}\rangle.$$

8. Let a_1 and a_2 be distinct complex numbers, and $n_1, n_2 \geq 1$ integers. Show that for any two polynomials $Q_1(X), Q_2(X) \in \mathbb{C}[X]$, there exists a polynomial $P(X) \in \mathbb{C}[X]$ such that $P(X) \equiv Q_1(X) \pmod{(X - a_1)^{n_1}}$, and $P(X) \equiv Q_2(X) \pmod{(x - a_2)^{n_2}}$.

9. Show that $\sqrt{1+\sqrt{3}}$ has degree 4 over $\mathbb{Q}$, and find its minimal polynomial over $\mathbb{Q}$.

10. Let $a, b \in \mathbb{Q}$ such that $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{ab}$ are irrational. Show that $\sqrt{a}+\sqrt{b}$ has degree 4 over $\mathbb{Q}$, and find its minimal polynomial.