

# **Probability for Statistics MATH50010**

## **Unseen Problem 1**

Let  $\Omega$  be a finite set. Does there exist an event space, i.e. an algebra of sets  $\mathcal{F}$  on  $\Omega$ , with precisely six elements?

More generally, for what positive integer values  $N$  does there exist an algebra of sets of size  $N$ ?

# Solution

We will show the following general result: if  $|\Omega| = n \geq 1$ , then the sigma algebras on  $\Omega$  have size  $2^k$  for some  $1 \leq k \leq n$ .

Let  $\mathcal{F}$  be an algebra on  $\Omega$ . For each  $x \in \Omega$ , define

$$A_x = \bigcap_{B \in \mathcal{F}: x \in B} B,$$

the intersection of all sets in  $\mathcal{F}$  containing  $x$ . By construction, each  $A_x \in \mathcal{F}$ .

**Key point:** Think back to the idea of an event space as representing the observable outcomes of an experiment. The experiment may not be able to distinguish between all outcomes: e.g. considering the 12 balls problem, the experiment that consists of weighing  $\{1, 2, 3, 4\}$  against  $\{5, 6, 7, 8\}$  cannot distinguish between outcomes 1 and 2. So, for this experiment, for any event  $E \in \mathcal{F}$ , either both  $1, 2 \in E$  or  $1, 2 \notin E$ . The event  $A_x$  defined above is the collection of all outcomes that cannot be distinguished from  $x$ . Below, we will show that sets of the form  $A_x$  partition  $\Omega$ .

Note that for  $x, y \in \Omega$ , either  $A_x = A_y$  or  $A_x \cap A_y = \emptyset$ . To see this, suppose for contradiction that there exists  $z \in A_x \cap A_y$  but  $y \notin A_x$ . Then for some  $B \in \mathcal{F}$ , we have that  $x \in B$  but  $y \notin B$ . But then  $y \in B^c$  so that  $A_y \subseteq B^c$  (since  $B^c \in \mathcal{F}$ ), giving

$$z \in A_x \cap A_y \subseteq A_y \subseteq B^c.$$

But  $z \in A_x$  so  $z \in B$ , a contradiction. Hence  $y \in A_x$  so that  $A_y \subseteq A_x$ , and so by symmetry in fact  $A_x = A_y$ .

It follows that the set  $\Omega$  is partitioned by sets  $A_{x_1}, \dots, A_{x_k}$  for distinct elements  $x_1, \dots, x_k \in \Omega$ . This partition corresponds to the equivalence relation  $\sim$  on  $\Omega$  where  $x \sim y$  if and only if there is no  $E \in \mathcal{F}$  that distinguishes between  $x$  and  $y$ , i.e. no  $E \in \mathcal{F}$  has  $x \in E$  and  $y \notin E$  or  $y \in E$  and  $x \notin E$ .

Now for any non-empty  $B \in \mathcal{F}$ , if  $x \in B$ , we must in fact have  $A_x \subseteq B$ , so that each non-empty  $B$  can be written as

$$B = A_{y_1} \cup \dots \cup A_{y_l}$$

for distinct elements  $y_1, \dots, y_l \in \{x_1, \dots, x_k\}$ . Hence there are  $2^k$  distinct elements of  $\mathcal{F}$ , corresponding to the subsets of  $\{x_1, \dots, x_k\}$ .

In conclusion, only  $N$  of the form  $2^k$  for some positive integer  $k$  can be the size of an algebra of sets. Moreover, for a set  $\Omega$  with  $|\Omega| = n \geq 1$ , the above argument shows how to construct an algebra of sets of size  $2^k$  for any  $1 \leq k \leq n$ : simply partition  $\Omega$  into  $k$  equivalence classes.