Classical Dynamics **Autumn Term 2024**

Problem Sheet 8

1. What is a suitable Lagrangian or Hamiltonian for a 'rigid body' comprising two particles?

2. The Lagrangian

$$L = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + m g l \cos \theta$$

describes the motion of a simple pendulum mounted on a freely rotating turntable with moment of inertia I.

(i) Find the corresponding Hamiltonian.

(ii) Show that if $I \gg ml^2$ then

$$\frac{p_\theta^2}{2ml^2} - \frac{p_\phi^2}{2I^2} m l^2 \sin^2 \theta - m g l \cos \theta,$$

is constant (this is essentially the same result as considered in Problem sheet 2 Q6 part (iv)).

3. The Lagrangian for a symmetric top fixed at one point is

$$L = \frac{I_1}{2} \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - M g l \cos \theta,$$

where ϕ, θ and ψ are Euler angles, M is the total mass and l is the distance between the fixed point and the centre of mass.

Consider solutions where θ and $\dot{\phi}$ are constant (precession of the top)

(i) For what θ do such solutions exist if $p_\psi = 0$.

(ii) Are there solutions with $\theta = \pi/2$?

(iii) Determine $\dot{\phi}$ as a function of θ and p_ψ.

4. Euler's equations

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2,$$

describe the motion of a freely rotating body. The equation are usually derived by elementary means (see Goldstein or Landau and Lifshitz).

(i) Obtain the third Euler equation from the Lagrangian

$$L = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

where

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \quad \omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \quad \omega_3 = \dot{\phi}\cos\theta + \dot{\psi},$$

are the components of angular velocity in terms of the Euler angles.

(ii) Determine the Poisson brackets $\{p_\psi, \omega_1\}$, $\{p_\psi, \omega_2\}$, $\{\omega_1, \omega_2\}$.

5. Solve Euler's equations assuming that $I_1 = I_2$.