**MATH50010 – Autumn 2022 – Midterm**

**You should state carefully any results from lectures that are used.**

Throughout, take all random variables to be defined on the probability space $(\Omega, \mathcal{F}, \Pr)$.

(a) (2 marks) State two equivalent conditions for the function $X : \Omega \to \mathbf{R}$ to be a random variable with respect to $\mathcal{F}$.

*$X$ is a a random variable if and only if $\{X \le x\} = \{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{F}$ for all $x \in \mathbf{R}$. Or $X$ is a random variable if and only if for every Borel set $B \in \mathcal{B}$, $X^{-1}(B) \in \mathcal{F}$.*

(b) (4 marks) Let $\Omega = \{1, 2, 3, 4\}$. Give an example of a function $X : \Omega \to \mathbf{R}$ and two sigma algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $X$ is a random variable with respect to $\mathcal{F}_1$ but not $\mathcal{F}_2$.

*There are many possible solutions. 1 mark each for defining $\mathcal{F}_1$ and $\mathcal{F}_2$ as valid sigma algebras. 1 mark for defining $X$ such that it is a random variable with respect to $\mathcal{F}_1$, and 1 mark for it not being a random variable with respect to $\mathcal{F}_2$.*

*One example would be to define $\mathcal{F}_1$ as the power set of $\Omega$ (which we know from lectures is a sigma-algebra), and $\mathcal{F}_2 = \{\emptyset, \Omega, \{1, 3\}, \{2, 4\}\}$. Then let*

$$X(\omega) = \begin{cases} 1 & \omega \in \{2, 4\} \\ 0 & \omega = 3 \\ -1 & \omega = 1 \end{cases}$$

*Then clearly $X$ is a random variable with respect to $\mathcal{F}_1$, but $\{0\} \in \mathcal{B}$ and $X^{-1}(\{0\}) = \{3\} \notin \mathcal{F}_2$.*

(c) (3 marks) Show that if $F_X$ is the cumulative distribution function of a random variable $X$, then $\lim_{x\to+\infty} F_X(x) = 1$.

*Take any sequence $(x_n)_{n\ge1}$ such that $x_n \uparrow -\infty$. Define the decreasing sequence of events $A_n = \{X \le x_n\}$.*

*Then, by the continuity property of $\Pr$ on decreasing sequences of events,*

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \Pr(A_n) = \Pr\left(\lim_{n\to\infty} \bigcup_{i=1}^{n} A_i\right) = \Pr(X \in \mathbf{R}) = 1.$$

In the remainder of the question, let $X$ be an absolutely continuous Beta$(1, \beta)$ random variable with probability density function given by

$$f_X(x) = \beta(1 - x)^{\beta-1}, \quad \text{for} \quad 0 < x < 1, \tag{1}$$

and zero otherwise, where $\beta \in \{1, 2, \ldots\}$.

(d) (2 marks) Write down the cumulative distribution function of $X$.

*For $x \in (0, 1)$,*

$$F_X(x) = \int_0^x \beta(1 - t)^{\beta-1}\, dt = \Big[-(1 - t)^\beta\Big]_0^x = 1 - (1 - x)^\beta,$$

*so that*

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - (1 - x)^\beta & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

(e) (1 mark) Write down the value $\Pr(X = 0.75)$ for $\beta = 2$.

$\Pr(X = 0.75) = 0$ *for any* $\beta$ *since* $X$ *is an absolutely continuous random variable, so by lecture notes (prop 2.18), it must be the case that* $\Pr(X = 0.75) = 0$.

(f) (2 marks) Determine the probability density function of the random variable $Y = 1 + 5X$.

*This represents a scale and location transform as seen in lecture notes. Therefore*

$$f_Y(y) = \frac{1}{5} f_X\left(\frac{y-1}{5}\right) = \frac{\beta}{5}(1 - \frac{y-1}{5})^{\beta-1} = \frac{\beta}{5^\beta}(6-y)^{\beta-1}$$

*This is valid for* $0 < \frac{y-1}{5} < 1$, *so* $1 < y < 5$. $f_Y(y) = 0$ *otherwise.*

*NB: This can also be calculated using the formula for 1-1 transformation or via another method. [1 mark for pdf, 1 mark for range]*

(g) (3 marks) Let $U \sim \textsc{Uniform}(0, 1)$, find a function $H$ such that $H(U)$ has the same distribution as $X$. Explain how this can be used to draw a random sample from the distribution of $X$, assuming that we can sample uniform random variables easily.

$F_X$ *is strictly increasing on* $[0, 1]$, *so we can use the probability inverse transformation to see that* $F_X^{-1}(U)$ *has distribution* $X$.

$$\Pr(F_X^{-1}(U) \le x) = \Pr(U \le F_X(x)) = F_X(x),$$

*as* $F_U(u) = u$ *for* $u \in (0, 1)$.

*Explicitly,* $H(U) = F_X^{-1}(U) = 1 - (1 - U)^{1/\beta}$ *has the same distribution as* $X$.

*Then, it is straightforward to generate samples from the uniform distribution on* $[0, 1]$. *If* $U_1, U_2, \ldots U_n$ *is such a sample, then* $X_1 = H(U_1), X_2 = H(U_2), \ldots, X_n = H(U_n)$ *is a random sample from the distribution of* $X$.

*[2 marks for finding inverse, 1 mark for describing sampling procedure]*

(h) (3 marks) Assume that $X_1, \ldots, X_n$ are i.i.d. random variables with density function $f_X$ for a fixed $\beta \in \{1, 2, \ldots\}$ as in (1). Calculate the distribution of $Y = \min\{X_1, \ldots, X_n\}$. Comment on the form of this distribution.

*For* $x \in [0, 1]$ *we have the equality of events*

$$\{\min\{X_1, X_2, \ldots, X_n\} > x\} = \bigcap_{i=1}^{n} \{X_i > x\}.$$

*Since the* $X_i$*'s are independent for* $i = 1, \ldots, n$,

$$\Pr(Y > x) = \Pr\left(\bigcap_{i=1}^{n} \{X_i > x\}\right) = \prod_{i=1}^{n} \Pr(X_i > x) = (1 - x)^{\beta n}$$

*by definition of the CDF in part (d). Hence,* $\Pr(Y \le x) = 1 - (1 - x)^{\beta n}$ *For* $x < 0$, $\Pr(Y \le x) = \Pr(\min\{X_1, \ldots, X_m\} \le x) = 0$ *and for* $x > 1$, $\Pr(Y \le x) = \Pr(\min\{X_1, \ldots, X_m\} \le x) = 1$ *since all* $X_i$*'s take values in* $(0, 1)$ *with probability 1. This has the CDF of* $X$ *with parameter* $\beta n$, *i.e. a Beta*$(1, n\beta)$ *random variable.*

*[2 marks for finding CDF, 1 mark for recognizing distribution.]*