# M345P65: Notes on Theorem 2.5.3

(2.5.3) Theorem: Suppose $\mathcal{L}$ is a countable first-order language and $\Sigma$ is a consistent set of closed $\mathcal{L}$-formulas. Then there exists a countable $\mathcal{L}$-structure $\mathcal{A}$ such that $\mathcal{A} \models \phi$ for every $\phi \in \Sigma$.

We use:

(2.5.8) Three (Fairly Technical) Lemmas:
(1) Suppose $\mathcal{L}$ and $\mathcal{L}^+$ are first-order languages and $\mathcal{L}^+$ only differs from $\mathcal{L}$ by having extra constant symbols $(b_j : j \in \mathbb{N})$. Suppose $\Sigma$ is a consistent set of $\mathcal{L}$-formulas. Then $\Sigma$ is consistent as a set of $\mathcal{L}^+$-formulas.

*Proof.* Suppose for a contradiction that there is an $\mathcal{L}^+$-formula $\psi$ with

$$\Sigma \vdash_{K_{\mathcal{L}^+}} \psi \text{ and } \Sigma \vdash_{K_{\mathcal{L}^+}} (\neg\psi).$$

Without loss, we may assume that $\psi$ is an $\mathcal{L}$-formula.

Take these deductions and replace (systematically) each $b_j$ occurring in them by a variable $x_{j(i)}$ not occuring anywhere else in the deduction. The results are deductions in $K_{\mathcal{L}}$ of $\psi$ and $(\neg\psi)$: contradiction. $\square_{(1)}$.

(2) (Renaming bound variables) Suppose $\phi(x_1)$ is an $\mathcal{L}$-formula and $x_2$ is a variable free for $x_1$ in $\phi$ and $x_2$ is not free in $\phi(x_1)$. Then

$$\vdash_{K_{\mathcal{L}}} ((\forall x_1)\phi(x_1) \to (\forall x_2)\phi(x_2)).$$

*Proof.* By axiom K1:

$$(\forall x_1)\phi(x_1) \vdash \phi(x_2).$$

Now use Generalization and the Deduction Theorem. $\square_{(2)}$

(3) Suppose $\Sigma$ is a consistent set of $\mathcal{L}$-formulas and $\phi, \psi$ are $\mathcal{L}$-formulas. If $\Sigma \cup \{(\phi \to \psi)\}$ is inconsistent, then $\Sigma \vdash \phi$ and $\Sigma \vdash (\neg\psi)$.

*Proof.* If $\Sigma \not\vdash \phi$ then by 2.5.2, $\Sigma \cup \{(\neg\phi)\}$ is consistent. As $\vdash ((\neg\phi) \to (\phi \to \psi))$, we have $\Sigma \cup \{\neg\phi\} \vdash (\phi \to \psi)$. Thus $\Sigma \cup \{(\phi \to \psi)\}$ is consistent - contradiction.
Similarly if $\Sigma \not\vdash (\neg\psi)$ then $\Sigma \cup \{(\neg\neg\psi)\}$ is consistent. As $\vdash (\neg\neg\psi \to \psi)$, it follows that $\Sigma \cup \{(\neg\neg\psi)\} \vdash \psi$. But $\vdash (\psi \to (\phi \to \psi))$ so $\Sigma \cup \{(\phi \to \psi)\}$ is consistent - contradiction. $\square_{(3)}$

Proof of Theorem 2.5.3: The proof is in a series of steps/ lemmas: the notation is cumulative.

Step 1: Let $b_0, b_1, \ldots$ be new constant symbols and $\mathcal{L}^+$ be the language obtained from $\mathcal{L}$ by including these as extra constant symbols (note that $\mathcal{L}^+$ is still countable). Consider $\Sigma$ as a set of $\mathcal{L}^+$-formulas. Note that by 2.5.8 (1), $\Sigma$ is still consistent when we do this. For the rest of the proof, $\vdash$ denotes $\vdash_{K_{\mathcal{L}^+}}$.

Step 2: We prove the following:

*Lemma (Adding Witnesses):* There is a consistent set of closed $\mathcal{L}^+$-formulas $\Sigma_\infty \supseteq \Sigma$ such that for every $\mathcal{L}^+$-formula $\theta(x_i)$ with one free variable, there is some $b_j$ with

$$\Sigma_\infty \vdash (\neg(\forall x_i)\theta(x_i)) \to (\neg\theta(b_j)).$$

(To understand this, it's best to think of $\theta$ as $\neg\chi$; then the above formula is equivalent to $(\exists x_i)\chi(x_i) \to \chi(b_j)$. The $b_j$ here is called a *witness*: it witnesses the fact that 'there exists $x_i \ldots$' . )
*Proof of Lemma:* Enumerate the $\mathcal{L}^+$-formulas containing just one free variable as

$$\phi_0(x_{i_0}), \phi_1(x_{i_1}), \ldots$$

(This uses countability of $\mathcal{L}^+$.) There exists a subsequence $a_0, a_1, \ldots$ of $b_0, b_1, \ldots$ such that
(1) $a_0$ is not in $\phi_0$; and for all $n$
(2) $a_{n+1}$ does not occur in $\phi_0, \ldots, \phi_n$
and the $a_i$ are distinct.
Let $\psi_k$ be $((\neg(\forall x_{i_k})\phi_k(x_{i_k})) \to (\neg\phi_k(a_k)))$. Let $\Sigma_0$ be $\Sigma$ and $\Sigma_{n+1} = \Sigma \cup \{\psi_0, \ldots, \psi_n\}$. We show by induction that $\Sigma_n$ is consistent. Suppose $\Sigma_k$ is consistent but $\Sigma_{k+1}$ is not. Then by 2.5.8 (3) we have:

$$\Sigma_k \vdash \neg(\forall x_{i_k})\phi_k(x_{i_k}) \text{ and } \Sigma_k \vdash \phi_k(a_k).$$

Take a deduction of $\phi_k(a_k)$ from $\Sigma_k$ and let $x_j$ be a variable not appearing in this. Let $\phi_k(x_j)$ be the result of substituting $x_j$ for all free occurrences of $x_{i_k}$ in $\phi_k(x_{i_k})$ and note that $x_{i_k}$ is then free for $x_j$ in this. Recall that $a_k$ does not appear in $\phi_0, \ldots, \phi_{k-1}$ so does not appear in $\psi_0, \ldots, \psi_{k-1}$ (or in any formula in $\Sigma$). Moreover, if in any axiom we replace $a_k$ by $x_j$ then we get an axiom. So performing this substitution in our deduction gives a deduction from $\Sigma_k$ of $\phi_k(x_j)$. Applying Gen we get

$$\Sigma_k \vdash (\forall x_j)\phi_k(x_j).$$

But the we can rename the bound variable (2.5.8 (2)) to get

$$\Sigma_k \vdash (\forall x_{i_k})\phi_k(x_{i_k}).$$

Thus $\Sigma_k$ is inconsistent: contradiction.
So each $\Sigma_n$ is consistent. Thus if we let $\Sigma_\infty = \Sigma \cup \{\psi_0, \psi_1, \ldots\}$ we get our required consistent set of closed $\mathcal{L}^+$-formulas. $\square_{Lemma}$

Step 3: As in the Lindenbaum Lemma (2.5.2) there is a consistent set $\Sigma^* \supseteq \Sigma_\infty$ of closed $\mathcal{L}^+$ formulas with the property that for every closed $\mathcal{L}^+$-formula $\psi$ we have $\Sigma^* \vdash \psi$ or $\Sigma^* \vdash (\neg\psi)$ (but not both).

Step 4: (Building a structure) Let

$$A = \{\bar{t} : t \text{ is a closed term of } \mathcal{L}^+\}.$$

Note that a term is closed if it has no variables in it, so it is made up of constant and function symbols. We use the bars to distinguish between when we are thinking of terms as part of the language and when we are thinking of them as elements of the set $A$. Note also that $A$ is a non-empty, countable set.
Now we make $A$ into an $\mathcal{L}^+$-structure $\mathcal{A}$ by saying how to interpret the relation, function and constant symbols of $\mathcal{L}^+$:
(1) Each constant symbol $c$ of $\mathcal{L}^+$ is interpreted as $\bar{c} \in A$.
(2) Suppose $R$ is an $n$-ary relation symbol. We define the relation $\bar{R} \subseteq A^n$ by saying

$$(\bar{t}_1, \ldots, \bar{t}_n) \in \bar{R} \Leftrightarrow \ \Sigma^* \vdash R(t_1, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are closed terms.
(3) Suppose $f$ is an $m$-ary function symbol. We define a function $\bar{f} : A^m \to A$ by

$$\bar{f}(\bar{t}_1, \ldots, \bar{t}_m) = \overline{f(t_1, \ldots, t_m)}$$

for closed terms $t_1, \ldots, t_m$.
By parts 1 and 3 here, if $v$ is any valuation in $\mathcal{A}$ and $t$ is a closed term, then $v(t) = \bar{t}$ (proof is by induction on the length of $t$).
*Main Lemma:* For every closed $\mathcal{L}^+$-formula $\phi$

$$\Sigma^* \vdash \phi \Leftrightarrow \mathcal{A} \models \phi.$$

*Proof of Main Lemma:* This is by induction on the number of connectives and quantifiers in $\phi$.
*Base step:* $\phi$ is atomic. So $\phi$ is $R(t_1, \ldots, t_n)$ for some terms $t_i$ and relation symbol $R$. As $\phi$ is closed, the $t_i$ are closed terms. Thus the lemma follows immediately from part (2) of the definition in this case.
*Inductive step:* Assume the result holds for closed formulas involving fewer connectives and quantifiers than in $\phi$. The proof splits into various cases:
*Case 1:* $\phi$ is $\neg\psi$;
*Case 2:* $\phi$ is $(\psi \to \chi)$;
*Case 3:* $\phi$ is $(\forall x_i)\psi$.
In cases 1 and 2, $\psi$ and $\chi$ are closed so the Lemma is assumed to hold for these. So for example, for Case 1 we have

$$\mathcal{A} \models \phi \overset{2.3.3}{\Leftrightarrow} \mathcal{A} \not\models \psi \overset{\textit{induction}}{\Leftrightarrow} \Sigma^* \not\vdash \psi \overset{\textit{Step3}}{\Leftrightarrow} \Sigma^* \vdash \neg\psi.$$

We leave Case 2 as an exercise.
We subdivide Case 3 according to whether $x_i$ is free in $\psi$ or not.
*Case 3a:* $x_i$ is not free in $\psi$. So $\psi$ is closed and the inductive hypothesis applies to it:

$$\Sigma^* \vdash \psi \Leftrightarrow \mathcal{A} \models \psi.$$

In general, we have

$$\mathcal{A} \models \psi \Leftrightarrow \mathcal{A} \models (\forall x_i)\psi,$$

and using axiom K1 and Gen we get

$$\Sigma^* \vdash \psi \Leftrightarrow \Sigma^* \vdash (\forall x_i)\psi.$$

(Note that we can use Gen in $\Rightarrow$, because all formulas in $\Sigma^*$ are closed.)
Putting these together gives Case 3a.
*Case 3b:* $x_i$ is free in $\psi$. Note that as $\phi$ is closed, $x_i$ is the only free variable in $\psi$, so we write it as $\psi(x_i)$.
*Proof of $\Leftarrow$ in Lemma:* Suppose for a contradiction that $\mathcal{A} \models \phi$ and $\Sigma^* \not\vdash \phi$. Then $\Sigma^* \vdash \neg\phi$, by Step 3. By Step 2 there is a constant symbol $b_j$ such that

$$\Sigma^* \vdash ((\neg(\forall x_i)\psi(x_i)) \to (\neg\psi(b_j))),$$

that is

$$\Sigma^* \vdash ((\neg\phi) \to \neg\psi(b_j)),$$

So, using Modus Ponens,

$$\Sigma^* \vdash \neg\psi(b_j).$$

Now, $\neg\psi(b_j)$ is closed as $x_i$ was the only free variable in $\psi$, so the induction hypothesis applies (given that we have already completed case 1), and the above gives:

$$\mathcal{A} \models \neg\psi(b_j).$$

But $\mathcal{A} \models \phi$, so $\mathcal{A} \models (\forall x_i)\psi(x_i)$. This implies $\mathcal{A} \models \psi(b_j)$ (the term $b_j$ is free for $x_i$ in $\psi(x_i)$, so this follows from 2.3.6). This is a contradiction.

*Proof of $\Rightarrow$ in Lemma:* Suppose $\Sigma^* \vdash \phi$ and, for a contradiction, suppose $\mathcal{A} \not\models \phi$. So $\mathcal{A} \not\models (\forall x_i)\psi(x_i)$. Let $v$ be a valuation in $\mathcal{A}$ not satisfying $\psi(x_i)$ and let $v(x_i) = \bar{t}$, where $t$ is some closed term. Then $v$ does not satisfy $\psi(t)$ (by definition of $\mathcal{A}$), so $v$ satisfies $\neg\psi(t)$. As $\psi(t)$ is a closed formula we therefore have $\mathcal{A} \models \neg\psi(t)$ (for example, by 2.3.3 again). Moreover, we can then use the induction hypothesis to get

$$\Sigma^* \vdash \neg\psi(t).$$

But by $\Sigma^* \vdash \phi$, Axiom K1 and the fact that $t$ is free for $x_i$ in $\psi(x_i)$ (as it is a closed term), we get $\Sigma^* \vdash \psi(t)$: a contradiction. $\square_{Lemma}$

Step 5: Proof of the Theorem. As $\Sigma \subseteq \Sigma^*$ the Main Lemma in Step 4 gives $\mathcal{A} \models \phi$ for all $\phi \in \Sigma$. Of course $\mathcal{A}$ is an $\mathcal{L}^+$-structure, but if we simply 'forget' the interpretation of the new constant symbols (the $b_j$) we obtain an $\mathcal{L}$-structure, which, of course, is still a model of $\phi$. $\square$