Imperial College London

Lecture 08: Confidence Intervals

Statistical Modelling I

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Outline

1. Introduction

2. Constructing confidence intervals

3. Asymptotic confidence intervals

4. Simultaneous confidence intervals

Introduction

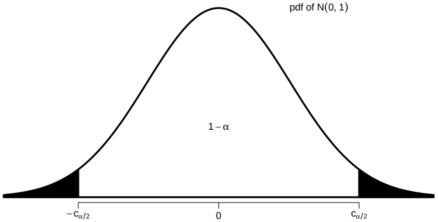
Motivation

- **(Point) estimator**: one number only (does not reflect uncertainty)
- **Confidence interval**: random interval that contains the true parameter with a certain probability

Example: $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma_0^2)$, σ_0^2 known

Want: random interval that contains μ with probability $1-\alpha$ for some $\alpha > 0$, e.g. $\alpha = 0.05$

- $\bar{Y} = \frac{1}{n} \sum Y_i \sim N(\mu, \sigma_0^2/n)$
- $\frac{\bar{Y}-\mu}{\sigma_0/\sqrt{n}} \sim N(0,1) \quad \forall \mu \in \mathbb{R}$
- $\Phi(c_{\alpha/2}) = 1 - \alpha/2$



$$1-\alpha = P\left(-c_{\alpha/2} < \frac{\bar{Y}-\mu}{\sigma_0/\sqrt{n}} < c_{\alpha/2}\right)$$
$$= P\big(\underbrace{\bar{Y} + c_{\alpha/2}\sigma_0/\sqrt{n}}_{\text{random}} > \underbrace{\mu}_{\text{non-random}} > \underbrace{\bar{Y} - c_{\alpha/2}\sigma_0/\sqrt{n}}_{\text{random}}\big)$$

Example: $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma_0^2)$, σ_0^2 known

The interval $(\bar{Y} - c_{\alpha/2}\sigma_0/\sqrt{n}, \bar{Y} + c_{\alpha/2}\sigma_0/\sqrt{n})$ is a random interval which contains the true μ with probability $1 - \alpha$.

The observed value of the random interval is $(\bar{y} - c_{\alpha/2}\sigma_0/\sqrt{n}, \bar{y} + c_{\alpha/2}\sigma_0/\sqrt{n})$.

This is called a $1 - \alpha$ **confidence interval** for μ.

Remarks

- α is usually small, often $\alpha = 0.05$.
- When speaking of a confidence interval we can either mean the realisation of the random interval or the random interval itself (this should hopefully be clear from the context).
- Could use asymmetrical values, but symmetrical values ($\pm c_{\alpha/2}$) give the shortest interval in this case.
- The value $\sigma_0/\sqrt{n}$ is exactly the **standard error** of $\bar{Y}$.

Example

In an industrial process, past experience shows it gives components whose strengths are $N(40, 1.21^2)$. The process is modified but s.d.(=1.21) remains the same.

After modification, $n = 12$ components give an average of 41.125.

New strength $\sim N(\mu, 1.21^2)$.
$n = 12$, $\sigma_0 = 1.21$, $\bar{y} = 41.125$, $\alpha = 0.05$, $c_{\alpha/2} \approx 1.96$.
$\rightarrow$ a 95% CI for μ is $(40.44, 41.81)$.

Note that our CI does not include 40 - an indication that the modification seems to have increased strength ($\rightarrow$ hypothesis testing)

This does **not** mean that we are 95% confident that the true μ lies in $(40.44, 41.81)$.
It means that if we were to take an infinite number of (indep) samples then in 95% of cases the calculated CI would contain the true value.

$1-\alpha$ Confidence interval

Definition

A $1-\alpha$ confidence interval for θ is a random interval I that contains the 'true' parameter with probability $\geq 1-\alpha$, i.e.

$$P_\theta(\theta \in I) \geq 1-\alpha \quad \forall \theta \in \Theta$$

In the above, I can be any type of interval. For example, if L and U are random variables with $L \leq U$ then I could be the open interval (L, U), the closed interval $[L, U]$, the unbounded interval $[L, \infty)$, ...

Example: $X \sim \text{Bernoulli}(\theta)$ $\theta \in [0,1]$ unknown

Want: $1-\alpha$ CI for θ (suppose $0 < \alpha < 1/2$).

Let

$$[L, U] = \begin{cases} [0, 1-\alpha], & \text{for } X = 0 \\ [\alpha, 1], & \text{for } X = 1 \end{cases}$$

This is indeed a $1-\alpha$ CI, since

$$P_\theta(\theta \in [L, U]) = \begin{cases} P_\theta(X = 0) = 1 - \theta \geq 1 - \alpha & \text{for } \theta < \alpha, \\ 1 & \text{for } \alpha \leq \theta \leq 1 - \alpha, \\ P_\theta(X = 1) = \theta \geq 1 - \alpha & \text{for } \theta > 1 - \alpha. \end{cases}$$

Example: One-sided confidence intervals

Suppose $Y_1, \ldots, Y_n$ are independent measurements of a pollutant θ, where higher values indicate worse pollution.

We want a $1-\alpha$ CI for θ of the form $(-\infty, h(y)]$.

For this h needs to be a function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$P_\theta(\theta \le h(Y)) = 1 - \alpha \quad \forall \theta.$$

Constructing confidence intervals

Definition: Pivotal quantity (pivotal statistic)

Definition

A **pivotal quantity** for θ is a function $t(Y, \theta)$ of the data and θ (and **not** any further nuisance parameters) s.t. the distribution of $t(Y, \theta)$ is known, i.e. does **not** depend on **any** unknown parameters.

This mirrors features of $\frac{\bar{Y}-\mu}{\sigma_0/\sqrt{n}}$ in the first example (where σ_0 is known)

Constructing confidence intervals via pivotal quantities

Suppose $t(Y,\theta)$ is a pivotal quantity for θ. Then we can find constants a_1, a_2 s.t.

$$P(a_1 \leq t(Y,\theta) \leq a_2) \geq 1-\alpha$$

because we know the distribution of $t(Y,\theta)$.

In many cases (as above) we can rearrange terms to give

$$P(h_1(Y) \leq \theta \leq h_2(Y)) \geq 1-\alpha$$

$[h_1(Y), h_2(Y)]$ is a random interval. The observed interval

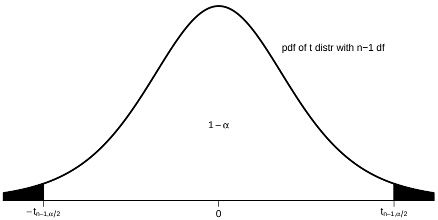
$$[\underbrace{h_1(y)}_{\text{lower confidence limit}}, \underbrace{h_2(y)}_{\text{upper confidence limit}}]$$

is a $1-\alpha$ confidence interval for θ.

Example: $Y_1, \ldots, Y_n$ i.i.d $N(\mu, \sigma^2)$, μ, σ^2 both unknown

Want: confidence interval for μ, but σ is unknown $\implies$ can't use $\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ as a pivotal quantity!

- $S^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$
- $T = \frac{\sqrt{n}}{S}(\bar{Y} - \mu)$.
- T follows a Student-t distribution with $n-1$ degrees of freedom



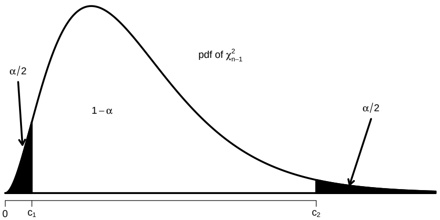
$$1 - \alpha \text{ CI is } \left(\bar{y} - \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}, \bar{y} + \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}\right)$$

Example: $Y_1, \ldots, Y_n$ i.i.d $N(\mu, \sigma^2)$, μ, σ^2 both unknown

Want: confidence interval for σ (or σ^2)

- $S^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$
- $\frac{\sum (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1}$
- c_1 and c_2 such that

$$P\left(c_1 \le \frac{\sum (Y_i - \bar{Y})^2}{\sigma^2} \le c_2\right) = 1 - \alpha$$



$$1 - \alpha \text{ CI for } \sigma \text{ is } \left(\sqrt{\frac{\sum (y_i - \bar{y})^2}{c_2}}, \sqrt{\frac{\sum (y_i - \bar{y})^2}{c_1}}\right)$$

Asymptotic confidence intervals

Definition: Asymptotic $1-\alpha$ confidence interval

Definition

A sequence of random intervals I_n is called an asymptotic $1-\alpha$ CI for θ if

$$\lim_{n\to\infty} P_\theta(\theta \in I_n) \geq 1-\alpha \quad \forall \theta \in \Theta.$$

If $\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \xrightarrow{d} N(0,1)$, then (approximately)

$$\sqrt{n}\frac{T_n-\theta}{\sigma(\theta)} \sim N(0,1)$$

and we can use the LHS as a pivotal quantity.

Simplification

Suppose $\hat{\sigma}_n$ is consistent for $\sigma(\theta)$. Thus, $\hat{\sigma}_n \xrightarrow{P_\theta} \sigma(\theta)$ for all θ.

By Slutsky's lemma and the fact that $X \sim N(0, \sigma^2(\theta))$ implies $X/\sigma(\theta) \sim N(0, 1)$,

$$\sqrt{n}\frac{T_n - \theta}{\hat{\sigma}_n} \xrightarrow{d} N(0, 1).$$

Using the LHS as the pivotal quantity leads to the approximate confidence limits

$$T_n \pm c_{\alpha/2}\hat{\sigma}_n/\sqrt{n}$$

where $\Phi(c_{\alpha/2}) = 1 - \alpha/2$.

Under mild regularity conditions, the quantity $\hat{\sigma}_n/\sqrt{n}$ estimates $\mathrm{SE}(T_n)$. Hence:

$$T_n \pm c_{\alpha/2}\,\mathrm{SE}(T_n).$$

Example: $Y \sim \text{Binomial}(n, \theta)$, $\theta \in (0, 1)$ unknown

$\sqrt{n}\frac{Y/n-\theta}{\sqrt{\theta(1-\theta)}}$ is approx. $N(0, 1)$, so

$$P(-c_{\alpha/2} \leq \frac{Y - n\theta}{\sqrt{n\theta(1-\theta)}} \leq c_{\alpha/2}) \approx 1 - \alpha$$

Using the (asymptotic) pivotal quantity

$$\sqrt{n}\frac{Y/n - \theta}{\sqrt{\frac{Y}{n}(1 - \frac{Y}{n})}}$$

leads to the confidence limits

$$\frac{y}{n} \pm \frac{c_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{y}{n}(1 - \frac{y}{n})}$$

Simultaneous confidence intervals

Extension of CIs to ≥ 1 parameter

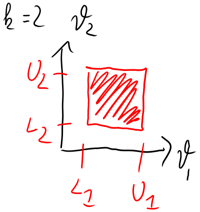
Suppose $\theta = (\theta_1, \ldots, \theta_k)^t \in \Theta \subset \mathbb{R}^k$ and we have $(L_i(Y), U_i(Y))$ such that

$$\forall \theta : \quad P_\theta(L_i(Y) < \theta_i < U_i(Y) \text{ for } i = 1, \ldots, k) \geq 1 - \alpha$$

then

$$(L_i(y), U_i(y)), \qquad i = 1, \ldots, k$$

is a $1 - \alpha$ **simultaneous confidence interval** for $\theta_1, \ldots, \theta_k$.



Can we construct simultaneous confidence intervals from one-dimensional confidence intervals?

The Bonferroni Correction

Theorem

Suppose $[L_i, U_i]$ *is a* $1-\alpha/k$ *confidence interval for* θ_i, $i = 1, \ldots, k$.

Then $[(L_1, \ldots, L_k)^t, (U_1, \ldots, U_k)^t] = (L_1, U_1) \times \cdots \times (L_k, U_k)$ *is a* $1-\alpha$ *simultaneous confidence interval for* $(\theta_1, \ldots, \theta_k)^t$.

Proof

$$P(\theta_i \in [L_i, U_i], i = 1, \ldots, k) = 1 - P(\bigcup_{i=1}^{k}\{\theta_i \notin [L_i, U_i]\}) \geq 1 - \sum_{i=1}^{k} \underbrace{P(\theta_i \notin [L_i, U_i])}_{\leq \alpha/k} \geq 1 - \alpha.$$

Example: different coverage probabilities

Suppose $[L_1, U_1]$ is a 99% confidence interval for θ_1 and $[L_2, U_2]$ is a 97% confidence interval for θ_2. Then $[L_1, U_1] \times [L_2, U_2]$ is a 96% simultaneous confidence interval for the parameter vector (θ_1, θ_2).

Example: Bonferroni corrections are conservative

Suppose $X_1, \ldots, X_n \sim N(\mu, 1)$, $Y_1, \ldots, Y_n \sim N(\theta, 1)$ independent with (μ, θ) being the unknown parameter.

One-dimensional CIs:

$$I = (\bar{X} - c_{\alpha/2}/\sqrt{n}, \bar{X} + c_{\alpha/2}/\sqrt{n}) \qquad J = (\bar{Y} - c_{\alpha/2}/\sqrt{n}, \bar{Y} + c_{\alpha/2}/\sqrt{n})$$

for $\Phi(c_{\alpha/2}) = 1 - \alpha/2$, are $1 - \alpha$ confidence intervals for μ and θ

Bonferroni correction: $I \times J$ is a $1 - 2\alpha$ confidence region for (μ, θ).

Actual coverage probability: I and J are independent, thus for $I \times J$

$$P_{(\mu,\theta)}((\mu, \theta) \in I \times J) = P_{(\mu,\theta)}(\mu \in I) P_{(\mu,\theta)}(\theta \in J) = (1 - \alpha)^2.$$

For $\alpha = 0.1$, Bonferroni guarantees coverage probability of 80%, whereas the actual probability is $0.9^2 = 0.81$.

Looking ahead

We continue to work with and generalize CIs as we look toward **hypothesis testing** and then **linear models**.