Problem Sheet 2 - solutions

1. Our integrand is entire, so we can take the analytic strip to be as large as we like. Leave the strip pararmeter a as variable initially, and choose the optimal value later.

   Taking $f(\theta) = \exp(\cos\theta)$, we have

   $$f(z) = \exp\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right),$$

   for $|\text{Im}\theta| < a$, we have

   $$\exp\left(\frac{e^a + 1}{2}\right) = O(e^{e^a/2}),$$

   noting that as one inner exponential grows, the other shrinks. Using the theorem in the notes, we now have

   $$|I - I_N| = O(e^{e^a/2})O(e^{-aN}) = O(\exp(\phi(a))),$$

   where $\phi(a) = e^a/2 - aN$. Now we use calculus to determine the value of a which minimises a.

   $$\phi'(\tilde{a}) = 0 \implies e^{\tilde{a}}/2 = N \implies \tilde{a} = \log(2N),$$

   and

   $$\phi''(\tilde{a}) = e^{\tilde{a}}/2 > 0,$$

   proving that $\tilde{a} = \log(2N)$ is a minimium. Subbing

   $$\phi(\tilde{a}) = N - N\log(2N) = N + \log((2N)^{-N})$$

   into our convergence rate,

   $$|I - I_N| = O\left(\exp(N + \log((2N)^{-N}))\right) = O\left(\left(\frac{e}{2N}\right)^N\right)$$

2. (a) Expanding the exponential for small z,

   $$\frac{z}{e^z - 1} = \frac{z}{\sum_{n=1}^{\infty}\frac{z^n}{n!}} = \frac{1}{\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}} = \frac{1}{1 + O(z)} = \sum_{n=0}^{\infty} O(z^n) = 1 + O(z)$$

   where we have expanded as a geometric sum in the final step. Thus, we *removed the singularity* by modifying $f(0) = 1$ to make f analytic in at zero.

   (b) Aside from the removable singularity at zero, f is singular when $e^z = 1$, equivalently when $z = 2\pi i k$ for $k \in \mathbb{Z} \setminus \{0\}$. Thus, the domain of analyticity is

   $$D = \mathbb{C} \setminus (2\pi i(\mathbb{Z} \setminus \{0\}))$$

   (c) For small z, we have $e^z \approx 1$, but the difference $e^z - 1 \approx 0$. If stored with 16 digits of precision, many digits will be lost when taking the difference of these two numbers, because the size (equivalently, the exponent) of the numbers before the subtraction, and after the subtraction, is very different.

(d) Cauchy's integral theorem tells us:

$$f(x) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-x} dz,$$

for γ in a simply connected subset of D. Choosing $z = e^{i\theta}$ for $\theta \in [0, 2\pi)$,

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - x} dz,$$

and now approximating with the trapezium rule, $\theta_n = 2\pi n/N$, for $n = 1, \ldots, N$,

$$f(x) \approx f_N(x) = \frac{1}{N} \sum_{n=1}^N \frac{f(e^{i\theta_n})e^{i\theta_n}}{e^{i\theta_n} - x} dz,$$

(e) The convergence rate is determined by the width of the analytic strip. The integrand ceases to be non-analytic when

$$e^{i\theta} - x = 0,$$

or when $f(e^{i\theta})$ hits a singularity at $e^{i\theta} = 2\pi k$ for $0 \neq k \in \mathbb{Z}$. Focusing on the first case, we have for any $q \in \mathbb{Z}$

$$e^{i\theta} = x \implies i\theta = \log|x| + i\arg x + 2\pi i q, \implies \theta = -i\log|x| + \arg x + 2\pi q,$$

thus

$$|\mathrm{Im}\theta| \leq \log|x| \leq \log\epsilon.$$

Dealing with the other case, we have for $p \in \mathbb{Z}$

$$i\theta = \log|2\pi k| + i\arg(k) + 2\pi i p$$

and (following similar arguments to the first case)

$$|\mathrm{Im}\theta| \leq \log|2\pi|.$$

Thus, we choose $a = \min(\log\epsilon, \log|2\pi|)$, and the convergence rate is $O(e^{-\tilde{a}N})$ for any $\tilde{a} < a$. Note that either component of the minumum could occur, depending on if ϵ is more or less than $1/(2\pi)$.

3. By periodicity we can write

$$I = \int_0^{2\pi} f(\theta)d\theta = \int_{\mathrm{Re}z}^{2\pi + \mathrm{Re}z} f(\theta)d\theta.$$

Now consider three-sided a rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, with

- γ_1 from $\mathrm{Re}z$ to z, vertically upwards;
- γ_2 from z to $z + 2\pi$, horizontally from left to right;
- γ_3 from $z + 2\pi$ to $\mathrm{Re}z + 2\pi$, vertically downwards.

By Cauchy's theorem, we have

$$I = \int_\gamma f(\theta)d\theta,$$

but by periodicity of the integrand, the contributions from γ_1 and γ_3 cancel, thus

$$I = \int_{\gamma_2} f(\theta)d\theta = \int_z^{z+2\pi} f(\theta)d\theta,$$

as required.

2

4. Consider this stage of the proof:

$$I_N - I = \int_{\frac{\pi}{N}+ia'}^{2\pi+\frac{\pi}{N}+ia'} \left(\frac{1}{e^{iN\theta}-1}\right) f(\theta)d\theta - \int_{\frac{\pi}{N}-ia'}^{2\pi+\frac{\pi}{N}-ia'} \left(\frac{1}{e^{-iN\theta}-1}\right) f(\theta)d\theta.$$

We do not need to choose these two integrals to be equidistant from the real line, since we have the extra knowledge that our integrand is analytic in the upper-half complex plane. Hence, we can move the first integral upwards in the complex plane, where it vanishes as $a' \to \infty$, since f is bounded and the denominator will decay exponentially. This gives:

$$I_N - I = -\int_{\frac{\pi}{N}-ia'}^{2\pi+\frac{\pi}{N}-ia'} \left(\frac{1}{e^{-iN\theta}-1}\right) f(\theta)d\theta \leq \frac{2\pi M}{e^{aN}-1}$$

by the ML principle.

5. (a) This follows immediately from residue calculus,

$$\int_\gamma f(z) = 2\pi i a_{-1} = 0.$$

(b) Generalising the proof so that there are infinitely many even terms, we have

$$I_N = \frac{2\pi i}{N} \sum_{j=-\infty}^{\infty} a_{2j} r^{2j+1} \sum_{n=1}^{N} e^{2\pi(2j+1)in/N}. \tag{1}$$

By the same arguments are the original proof, the inner sum was zero unless $(2j+1)/N$ is an integer. But if N is even, we are dividing an ood number by an even number, and thus $(2j+1)/N$ cannot be an integer. Hence, the inner sum is always zero, and $I_N = 0$ as claimed.

6. (a) The discretisation error is

$$|I - I_h| = O(-e^{2\pi a/h}),$$

for any $a < 1$, since $x^2 + 1$ has a zero at $x = \pm i$.

(b) We know that $x \tanh(x) \to 1$ as $x \to \pm\infty$. Thus the truncation error, by considering the final term in the truncated trapezium rule, is

$$|I - I_h^{(N)}| = O(-e^{hN}), \quad N \to \infty.$$

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(c) Balancing the terms:

$$hN = 2\pi a/h,$$

we get

$$h = (2\pi a/N)^{1/2}$$

for any $a < 1$, so (for practical purposes) we can take

$$h = (2\pi/N)^{1/2}.$$

7. (See handwritten solution)

3

8. (a) The derivation follows similarly, except we cannot take $f \leq M$ out as a constant, instead we bound $f(Re^{i\theta}) = O(R^q)$,

$$\left| \int_{\gamma_2} f(z)e^{i\omega z^p} dz \right| \leq \int_0^{\pi/(2p)} O(R^q) \exp\left(-\omega R^p \sin(p\theta)\right) R d\theta.$$

then following similar arguments,

$$\left| \int_{\gamma_2} f(z)e^{i\omega z^p} dz \right| \leq MR \int_0^{\pi/(2p)} \exp\left(-2p\omega R^p \theta/\pi\right) d\theta$$

$$= O\left(\pi \frac{1 - e^{-\omega R^p}}{2p\omega R^{p-1-q}}\right) \to 0 \quad \text{as } R \to \infty,$$

if and only if $p - 1 - q > 0$, as claimed.

(b) Rotating ω in the complex plane means that the steepest descent path, i.e. the path which converts the integrand to pure exponential decay, will also rotate. We ultimately want to choose θ such that

$$\exp(i\omega(Re^{i\theta})^p) = \exp(-|\omega|R^p)$$