Solution 4

1. Consider the lexicographic order $\preceq$ on $\mathbb{R}^2$. That means for any $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\underline{y} = (y_1, y_2) \in \mathbb{R}^2$ it holds that $\underline{x} \preceq \underline{y}$ if and only if

$$(x_1 < y_1) \qquad \text{or} \qquad (x_1 = y_1 \quad \text{and} \quad x_2 \leq y_2).$$

a) Check which properties of preferences defined in the lecture (completeness, transitivity, continuity, strong monotonicity, local nonsatiation, strict convexity) the lexicographic order satisfies.

Solution: All properties but continuity are satisfied.

Completeness: Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. If $x_1 < y_1$ then $x \preceq y$. If $y_1 < x_1$ then $y \preceq x$. Consider the case $x_1 = y_1$. If $x_2 \leq y_2$ then $x \preceq y$. If $y_2 \leq x_2$ then $y \preceq x$.
Remark: It is interesting to observe that $x \sim y$ if and only if $x = y$.

Transitivity: Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$. Assume that $x \preceq y$ and $y \preceq z$. We need to show that $x \preceq z$. We have to consider 4 cases: (i) If $x_1 < y_1$ and $y_1 < z_1$, then $x_1 < z_1$. (ii) If $x_1 = y_1$ and $y_1 < z_1$, then $x_1 < z_1$. (iii) If $x_1 < y_1$ and $y_1 = z_1$ then $x_1 < z_1$. (iv) If $x_1 = y_1 = z_1$, then we necessarily have that $x_2 \leq y_2$ and $y_2 \leq z_2$. So $x_2 \leq z_2$.

Continuity: To see that the continuity property is not satisfied, observe that

$$(-1/n, 1) \prec (0, 0) \qquad \forall n \in \mathbb{N}.$$

However, $(-1/n, 1)$ converges to $(0, 1)$ and $(0, 0) \prec (0, 1)$. Thus, the set $\{x \in \mathbb{R}^2 \colon x \preceq (0, 0)\}$ is not closed.

Strong monotonicity: Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. If $x \leq y$ and $x \neq y$, then, either $x_1 < y_1$ such that $x \prec y$, or $x_1 = y_1$ and $x_2 < y_2$ such that $x \prec y$.

Local nonsatiation: This is implied by the strong monotonicity. Indeed, for any $x \in \mathbb{R}^2$ and any $\varepsilon > 0$ there is some y such that $x < y$ and $\|x - y\| < \varepsilon$.

Strict convexity: Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$. Suppose that $x \succeq z$ and $y \succeq z$ with $x \neq y$. Since, $x \sim y$ iff. $x = y$, we can assume (WLOG) that $x \prec y$. Now, let $a = (1 - \lambda)x + \lambda y$ for some $\lambda \in (0, 1)$. Then we have to show that $a \succ z$. Again we have to consider 4 cases:

(i) If $x_1 > z_1$ and $y_1 > z_1$ then $a_1 > z_1$. (ii) If $x_1 = z_1$ and $y_1 > z_1$ then $a_1 > z_1$. (iii) If $x_1 > z_1$ and $y_1 = z_1$ then $a_1 > z_1$. (iv) If $x_1 = y_1 = z_1$, then $z_2 \leq x_2 < y_2$. This implies that $a_2 > z_2$. So in all 4 cases we get that $a \succ z$.

$\square$

b) Show that if there is a utility function $u\colon \mathbb{R}^2 \to \mathbb{R}$ representing $\preceq$, u is an injection.

Solution: If $u\colon \mathbb{R}^2 \to \mathbb{R}$ represents $\preceq$ that means $u(x) \leq u(y)$ if and only if $x \preceq y$ for all $x, y \in \mathbb{R}^2$. Therefore, $u(x) = u(y)$ if and only if $x \sim y$. However, one can easily see that $x \sim y$ if and only if $x = y$. $\square$

2. Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a convex and closed set. Let $u\colon X \to \mathbb{R}$ be a continuous, strictly monotone, strictly quasi-concave utility function.

a) Show that for any k such that there is some $x \in X$ with $u(x) = k$ the expenditure function $e(\cdot, k)\colon \mathbb{R}^n_{\geq 0} \to [0, \infty)$ is concave (so it is concave in the prices).

Solution: Let k be fixed and consider the prices $\underline{p}, \underline{p}' \in \mathbb{R}^n_{\geq 0}$. Let $\underline{x}^*_H$ be the corresponding Hicksian demand. Then we have for $\underline{p}'' = (1-\lambda)\underline{p} + \lambda \underline{p}'$, $\lambda \in [0,1]$:

$$\begin{aligned} e(\underline{p}'', k) &= \underline{p}'' \underline{x}^*_H(\underline{p}'', k)^\top = (1-\lambda)\underline{p}\underline{x}^*_H(\underline{p}'', k)^\top + \lambda \underline{p}' \underline{x}^*_H(\underline{p}'', k)^\top \\ &\geq (1-\lambda)\underline{p}\underline{x}^*_H(\underline{p}, k)^\top + \lambda \underline{p}' \underline{x}^*_H(\underline{p}', k)^\top \\ &= (1-\lambda) e(\underline{p}, k) + \lambda e(\underline{p}', k). \end{aligned}$$

$\square$

b) Let $n = 2$, $u(x_1, x_2) = x_1^a x_2^b$ with $a, b > 0$. Calculate the indirect utility function v, expenditure function e, Marshallian demand x^* and Hicksian demand x^*_H.

Solution:

Assume that the prices $\underline{p} = (p_1, p_2)$ are strictly positive. Let $m \geq 0$ be the budget.

Marshallian demand: We can use Walras' law and already assume that the budget line is binding.

$$
\begin{aligned}
\underline{x}^*(\underline{p}, m) &= \operatorname*{arg\,max}_{p_1 x_1 + p_2 x_2 = m} \ u(x_1, x_2) \\
&= \operatorname*{arg\,max}_{x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1} \ u(x_1, x_2) \\
&= \operatorname*{arg\,max}_{x_1 \geq 0} u\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2} x_1\right) \\
&= \operatorname*{arg\,max}_{x_1 \geq 0} (x_1)^a \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right)^b .
\end{aligned}
$$

We know that the Marshallian demand exists (and is unique). That means we only need to check for first order conditions and can leave out second order conditions.

FOC:

$$
\begin{aligned}
& 0 = a x_1^{a-1} \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right)^b + x_1^a b \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right)^{b-1} \left(-\frac{p_1}{p_2}\right) \\
\iff \quad & 0 = \underbrace{x_1^{a-1} \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right)^{b-1}}_{\neq 0} \left\{ a \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right) - \frac{p_1}{p_2} b x_1 \right\} \\
\iff \quad & 0 = a \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1\right) - \frac{p_1}{p_2} b x_1 \\
\iff \quad & x_1 = \frac{ma}{p_1(a+b)}.
\end{aligned}
$$

Using the budget constraint (or a symmetry argument), we obtain

$$
\underline{x}^*(\underline{p}, m) = \left(\frac{ma}{p_1(a+b)}, \frac{mb}{p_2(a+b)}\right).
$$

Indirect utility:

$$
v(\underline{p}, m) = u(\underline{x}^*(\underline{p}, m)) = \left(\frac{ma}{p_1(a+b)}\right)^a \left(\frac{mb}{p_2(a+b)}\right)^b .
$$

Hicksian demand: Let $k \geq 0$ be some level of utility. Then

$$
\underline{x}_H^*(\underline{p}, k) = \operatorname*{arg\,min}_{\underline{x} \in u^{-1}([k, \infty))} \ p_1 x_1 + p_2 x_2 .
$$

Some graphical illustration will show how it works, see figure 1.

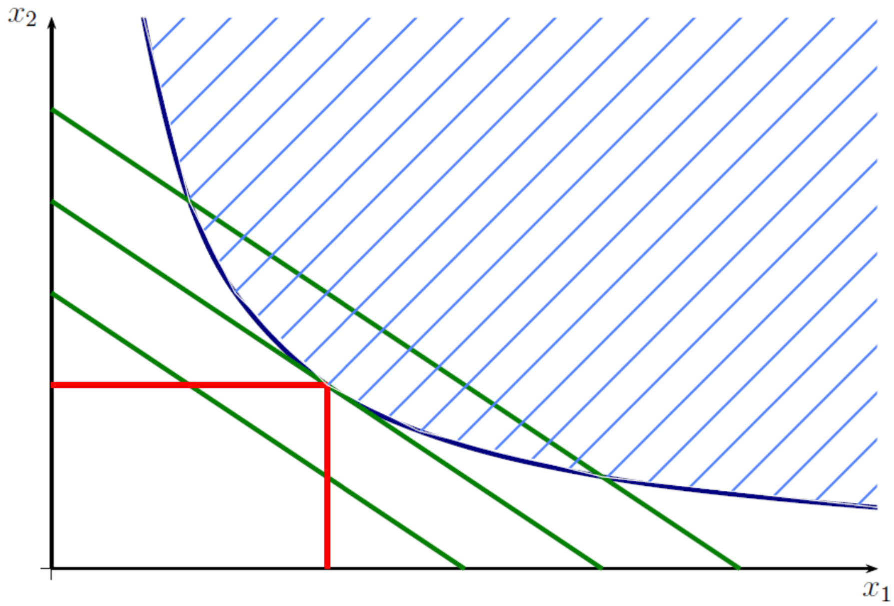

Figure 1: Several lines (green) with slope $-p_1/p_2$ and different intersections (total costs). The region in blue is $u^{-1}([k, \infty))$ for some $k > 0$.

Since the situation for $k = 0$ is not interesting ($\underline{x}^*_H(\underline{p}, 0) = (0, 0)$) we assume that $k > 0$. The pre-image $u^{-1}([k, \infty))$ is depicted in this picture. So we have to choose some bundle $\underline{x} \in u^{-1}([k, \infty))$ to attain at least utility k. Recall that – due to the quasi-concavity of u – $u^{-1}([k, \infty))$ is a convex set. Moreover, let's consider 'iso-expenditure lines'. That is, all possible bundles of inputs such that the expenditure to purchase them is the same at prices (p_1, p_2). If expenditure is $e \geq 0$ that means all such bundles satisfy:

$$e = p_1 x_2 + p_2 x_2.$$

Those lines can also be represented as graphs of functions of the form:

$$x_2(x_1) = \frac{e}{p_2} - \frac{p_1}{p_2} x_1.$$

That means they have slope $-p_1/p_2$ and intersection e/p_2. Now we have to determine the minimal e such that the corresponding iso-expenditure line has a non-empty intersection with $u^{-1}([k, \infty))$.

Indeed, one can see that this expenditure is given by the iso-expenditure line that is tangential to $u^{-1}(\{k\})$. If it intersected $u^{-1}([k, \infty))$, then – due to the convexity of $u^{-1}([k, \infty))$ – a part of the iso-expenditure line would be in the interior of $u^{-1}([k, \infty))$. But that means one could reduce the expenditure and still obtain a utility of k.

The derivative of the iso-utility set $u^{-1}(\{k\})$ is given in terms of the marginal rate of substitution:

$$MRS(x_1, x_2) = -\frac{\frac{\partial u(\underline{x})}{\partial x_1}}{\frac{\partial u(\underline{x})}{\partial x_2}}.$$

So the Hicksian demand $\underline{x}_H^*$ needs to satisfy:

$$u(\underline{x}_H^*) = k \quad \text{and} \quad -\frac{\frac{\partial u(\underline{x}_H^*)}{\partial x_1}}{\frac{\partial u(\underline{x}_H^*)}{\partial x_2}} = -\frac{p_1}{p_2}.$$

We have that:

$$-\frac{\frac{\partial u(\underline{x})}{\partial x_1}}{\frac{\partial u(\underline{x})}{\partial x_2}} = \frac{a}{b}\frac{x_2}{x_1} = -\frac{p_1}{p_2}$$

If and only if $x_2 = \frac{p_1}{p_2}\frac{b}{a}x_1$. Plugging that into the utility function yields:

$$u\left(x_1, \frac{p_1}{p_2}\frac{b}{a}x_1\right) = x_1^a\left(\frac{p_1}{p_2}\frac{b}{a}x_1\right)^b = x_1^{a+b}\left(\frac{p_1}{p_2}\frac{b}{a}\right)^b = k$$

if and only if:

$$x_1 = \left(\frac{k}{\left(\frac{p_1}{p_2}\frac{b}{a}\right)^b}\right)^{1/(a+b)}.$$

So – using $x_2 = \frac{p_1}{p_2}\frac{b}{a}x_1$ or symmetry considerations –

$$x_{H,1}^*(\underline{p}, k) = k^{1/(a+b)}\left(\frac{ap_2}{bp_1}\right)^{b/(a+b)} \qquad x_{H,2}^*(\underline{p}, k) = k^{1/(a+b)}\left(\frac{bp_1}{ap_2}\right)^{a/(a+b)}$$

Expenditure function: This is – at least in principle – an easy task now.

$$\begin{aligned}
e(\underline{p}, k) =& p_1 x_{H,1}^*(\underline{p}, k) + p_2 x_{H,2}^*(\underline{p}, k) \\
&= k^{1/(a+b)}\{(a/b)^{b/(a+b)} p_2^{b/(a+b)} p_1^{a/(a+b)} + (a/b)^{-a/(a+b)} p_2^{b/(a+b)} p_1^{a/(a+b)}\} \\
&= k^{1/(a+b)}\{(a/b)^{b/(a+b)} + (a/b)^{-a/(a+b)}\} p_2^{b/(a+b)} p_1^{a/(a+b)} \\
&= \theta(k)\, p_2^{b/(a+b)} p_1^{a/(a+b)}.
\end{aligned}$$

c) Verify that the expenditure function you obtain in **(b)**, as a function in the prices (so for fixed utility level) is nondecreasing, homogeneous of degree 1 and concave.

Solution: Let $k > 0$ be fixed. Then

$$\frac{\partial}{\partial p_1} e(\underline{p}, k) = \theta(k)\frac{a}{a+b} p_2^{b/(a+b)} p_1^{-b/(a+b)} > 0.$$
$$\frac{\partial}{\partial p_2} e(\underline{p}, k) = \theta(k)\frac{b}{a+b} p_2^{-a/(a+b)} p_1^{a/(a+b)} > 0.$$

So the expenditure function is even increasing.

For the homogeneity, let $t > 0$. Then:

$$e(t\underline{p}, k) = \theta(k)(tp_2)^{b/(a+b)}(tp_1)^{a/(a+b)} = t^{(b+a)/(a+b)}e(\underline{p}, k) = t\, e(\underline{p}, k)\,.$$

This shows that $e(\cdot, k)$ is positively homogeneous of degree 1.

We will dispense with the concavity. In principle, one can compute the Hessian checks that it is negative definite.

d) Now suppose you have an alternative representation of the ordinal utility which is induced by u given by $u_{\log}\colon X \to \mathbb{R}$, $u_{\log}(x_1, x_2) = \log(u(x_1, x_2))$. Compute the associated quantities: indirect utility function $v_{\log}$, expenditure function $e_{\log}$, Marshallian demand $x^*_{\log}$ and Hicksian demand $x^*_{\log,H}$.

Solution: First recall that u and $u_{\log}$ represent the same preference relation. We could indeed do the same sort of calculations for $u_{\log}(x_1, x_2) = a\log(x_1) + b\log(x_2)$. And you can realise that these calculations are in fact easier than the previous ones due to the more appealing form (so it would have been worth doing the calculations in terms of $u_{\log}$ at first and then do the argumentation for u). So we will confine ourselves to some arguments.

For the Marshallian demand, we obtain:

$$\underline{x}^*_{\log}(\underline{p}, m) = \operatorname*{arg\,max}_{p_1x_1+p_2x_2=m} \log(u(x_1, x_2)) = \operatorname*{arg\,max}_{p_1x_1+p_2x_2=m} u(x_1, x_2) = \underline{x}^*(\underline{p}, m).$$

For the indirect utility, we obtain

$$\begin{aligned}
v_{\log}(\underline{p}, m) &= u_{\log}(\underline{x}^*_{\log}(\underline{p}, m)) \\
&= \log(u(\underline{x}^*(\underline{p}, m))) \\
&= \log(v(\underline{p}, m)) \\
&= a\log(ma) - a\log(p_1(a+b)) + b\log(mb) - b\log(p_2(a+b)).
\end{aligned}$$

For the Hicksian demand, we obtain:

$$\begin{aligned}
\underline{x}^*_{\log,H}(\underline{p}, k) &= \operatorname*{arg\,min}_{\underline{x}\in u_{\log}^{-1}([k,\infty))} p_1x_1 + p_2x_2 \\
&= \operatorname*{arg\,min}_{\underline{x}\in(\log\circ u)^{-1}([k,\infty))} p_1x_1 + p_2x_2 \\
&= \operatorname*{arg\,min}_{\underline{x}\in u^{-1}(\log^{-1}([k,\infty))} p_1x_1 + p_2x_2 \\
&= \operatorname*{arg\,min}_{\underline{x}\in u^{-1}([\exp(k),\infty))} p_1x_1 + p_2x_2 \\
&= \underline{x}^*_H(\underline{p}, \exp(k))
\end{aligned}$$

Finally, the expenditure function is:

$$e_{\log}(\underline{p}, k) = p_1\underline{x}^*_{H,1}(\underline{p}, \exp(k)) + p_2\underline{x}^*_{H,2}(\underline{p}, \exp(k)).$$

3. Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a convex and closed set. Let $u\colon X \to \mathbb{R}$ be a continuous, strictly monotone, strictly quasi-concave utility function.

a) Let $v\colon \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the indirect utility function.

- Prove that for any $\underline{p} > 0$ the function $v(\underline{p}, \cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is strictly increasing.
- Prove that for any $m \geq 0$ the function $v(\cdot, m)\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ quasi-convex. Recall that a function f is quasi-convex if $-f$ is quasi-concave; see question 3 on Problem Sheet 1.

Solution: For the first assertion, let $\underline{p} > 0$. Recall that for a budget $m \geq 0$ the indirect utility is given as:

$$v(\underline{p}, m) = \max_{\underline{x} \in B_{\underline{p},m}} u(\underline{x})$$

where $B_{\underline{p},m} = \{\underline{x} \in X \colon \underline{p}\underline{x}^\top \leq m\}$ is the budget set. Now suppose that $0 \leq m < m'$. Since prices are strictly positive, one can easily see that

$$B_{\underline{p},m} \subsetneq B_{\underline{p},m'} .$$

We have seen in the lecture that under the given conditions Walras' Law holds. That is, the utility maximising consumption bundle (given in terms of Marshallian demand) is necessarily on the budget line. Equivalently, one can say that a consumer needs to spend all their budget in order to maximise their utility. Hence

$$v(\underline{p}, m') = \max_{\underline{x} \in B_{\underline{p},m'}} u(\underline{x}) = u\big(\underline{x}^*(\underline{p}, m')\big) > u(\underline{x}) \qquad \forall \underline{x} \in B_{\underline{p},m} .$$

Therefore:

$$u\big(\underline{x}^*(\underline{p}, m')\big) > u\big(\underline{x}^*(\underline{p}, m)\big) = v(\underline{p}, m)$$

$\square$

Now let $m \geq 0$. Let $\underline{p}, \underline{p}' \geq 0$, set $t \in [0, 1]$ and define the price-vector $\underline{p}'' = t\underline{p} + (1-t)\underline{p}'$. Let $k \in \mathbb{R}$ and assume that $v(\underline{p}, m) \leq k$ and $v(\underline{p}', m) \leq k$. We define the three budget sets

$$\begin{aligned} B &:= B_{\underline{p},m} = \{\underline{x} \in X \colon \underline{p}\underline{x}^\top \leq m\}, \\ B' &:= B_{\underline{p}',m} = \{\underline{x} \in X \colon \underline{p}'\underline{x}^\top \leq m\}, \\ B'' &:= B_{\underline{p}'',m} = \{\underline{x} \in X \colon \underline{p}''\underline{x}^\top \leq m\}. \end{aligned}$$

The central step is to show that:

$$B'' \subseteq B \cup B' .$$

To show this inclusion, suppose that $\underline{x} \notin B \cup B'$. That means $\underline{p}\underline{x}^\top > m$ and $\underline{p}'\underline{x}^\top > m$. Therefore also:

$$\underline{p}''\underline{x}^\top = t\underline{p}\underline{x}^\top + (1-t)\underline{p}'\underline{x}^\top > tm + (1-t)m = m.$$

Using this inclusion, we obtain:

$$\begin{aligned} v(\underline{p}'', m) &= \max_{\underline{x} \in B''} u(\underline{x}) \\ &\leq \max_{\underline{x} \in B \cup B'} u(\underline{x}) \\ &\leq \max\{\max_{\underline{x} \in B} u(\underline{x}), \max_{\underline{x} \in B'} u(\underline{x})\} \\ &\leq \max\{v(\underline{p}, m), v(\underline{p}', m)\} \\ &\leq k\,. \end{aligned}$$

This is what we wanted to show. □

b) Assume that the prices for the goods are strictly positive, $\underline{p} > 0$, and income is positive, $m > 0$. Is it possible that all goods are inferior? Prove your claim.

Solution: Since the standard assumptions hold (in particular the local nonsatiation of the underlying preferences) Walras' Law holds. That is, in order to maximise utility, one needs to spends the entire budget. That means that the Marshallian demand $\underline{x}^*(\underline{p}, m)$ lies on the budget line. In formulae, this means that:

$$m = \underline{p}\underline{x}^*(\underline{p}, m)^\top = \sum_{i=1}^{n} p_i x^*(\underline{p}, m).$$

If we take the derivative with respect to m on both sides, we obtain

$$1 = \sum_{i=1}^{n} p_i \frac{\partial x_i^*(\underline{p}, m)}{\partial m}.$$

Since prices p_i are strictly positive, there must be at least one $i \in \{1, \ldots, n\}$ such that $\frac{\partial x_i^*(\underline{p},m)}{\partial m} > 0$. That means that this good is a normal good and not an inferior good. □