Geometric Complex Analysis

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Introduction

Complex variables appear in many areas of mathematics as those have been truly the ancestor of many subjects, from algebraic geometry, number theory, dynamical systems, to quantum field theory. Basic examples and techniques in complex analysis have been developed over many years into sophisticated methods in analysis, algebra, and geometry. On the other hand, as the real and imaginary parts of any analytic function satisfy the Laplace equation, complex analysis is widely employed in the study of two-dimensional problems in physics, for instance in, hydrodynamics, thermodynamics, ferromagnetism, percolation, etc.

In complex analysis one often starts with a rather weak requirement (regularity) of differentiability. That is, a map $f : U \to \mathbb{C}$ is called *holomorphic* on Ω, if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists at every point in the open set $U \subseteq \mathbb{C}$. Then with little effort one concludes that f is infinity many times differentiable, and indeed it has locally convergent power series. This is in direct contrast with the notions of C^k regularities we have seen for real maps of Euclidean spaces. That is, there are C^k real maps which are not C^{k+1}, for any $k \geq 1$. Or, there are C^∞ real maps which have no convergent power series. The difference is rooted in the fact that here h tends to 0 in all directions, and there is a multiplication operation on the plane that interacts nicely with the addition. Due to this difference, complex analysis is not merely “extending the calculus to complex-valued functions”; rather it is an area of mathematics on its own.

Let Ω be an open set in $\mathbb{C}$ which is bounded by a piece-wise smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. For any C^1 simple closed curve γ in Ω, if we know the values of f on γ, the Cauchy Integral Formula provides a simple formula for the values of f inside γ:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz.$$

Also, there is a similar formula for the higher order derivatives of f at any point inside γ. On the other hand, if we know all derivatives of f at some point $z_0 \in \Omega$, then the infinite series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

is convergent for z close enough to z_0, and the value of the series is equal to $f(z)$.

When the domain Ω enjoys some form of symmetry, for example, when Ω is the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

with rotational symmetry, the objects of interest in complex analysis often find simple algebraic forms. In Chapters 2 we prove some results of this nature, including,

Theorem 0.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a one-to-one and onto holomorphic mapping. Then, there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}.$$

Although the above type of results point to the restrictive nature of holomorphic property, there are also statements that go in the other direction. For example, in Chapter 5, we prove the Riemann mapping theorem, which, as a special case, implies the following.

Theorem 0.2. *Let $\Omega \subset \mathbb{C}$ be an open set which is bounded by a continuous simple closed curve, and let $z_0 \in \Omega$. Then, there is a one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$ with $f(0) = z_0$.*

The domain Ω in the above theorem may have a very complicated shape (geometry), or may have a highly irregular boundary (analysis) obtained from a randomly generated curve. See Figure 1.

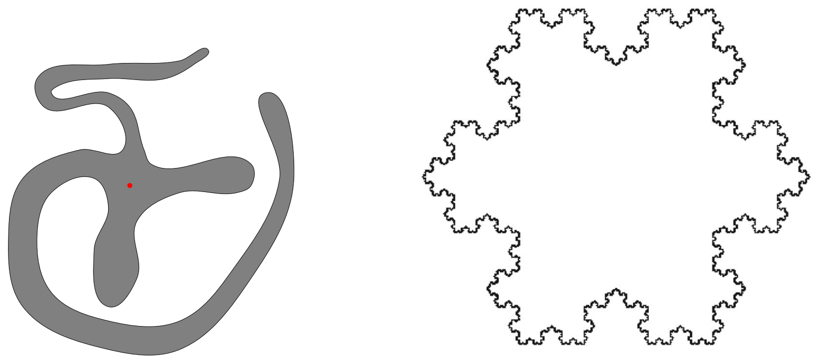

Figure 1: An arbitrary open set Ω bounded by a continuous simple closed curve on the right hand side, and the Koch snowflake in the right hand side.

The map f in the above theorem is called the *uniformlization* of the domain Ω. One aim of this course is to study the behaviour of such maps in connection with the geometric shape of Ω and its boundary. We also look for such geometric quantities that remain invariant under conformal mappings.

One often comes across domains Ω with very complicated shapes, or very irregular boundaries. Although the above theorem provides us with a seemingly nice behaving map, there are little chances that we know the higher order derivatives of f at some $z_0 \in \Omega$ or the behaviour of the map f near the boundary of $\mathbb{D}$ in order to use the Taylor series or the Cauchy Integral Formula to study the behaviour of f. But, is it still possible to say something about the map f? As we shall see in Chapter 6 there are some *universal laws* that every one-to-one holomorphic map must obey. Let us give an example of this. For an arbitrary $\theta \in [0, 2\pi)$, one may ask how fast the curve $r \mapsto f(re^{i\theta})$, for $r \in [0, 1)$, move away from 0, or how fast it may spiral about 0? In Chapter 6 we prove some results of the following type.

Theorem 0.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be an arbitrary one-to-one and holomorphic map normalized with $f(0) = 0$ and $f'(0) = 1$. Then, for every $\theta \in [0, 2\pi]$ and $r \in [0, 1)$ we have*

$$|\arg f'(re^{i\theta})| \leq 2 \log \frac{1+r}{1-r}. \tag{1}$$

While Theorem 0.2 is strong and general, its proof is far from constructive. On rare occasions we are able to provide a formula for the map f (a list of such examples appear in Chapter 5). This is a rather general theme in holomorphic mappings that we often know that a holomorphic function with some prescribed conditions exists, but we don't have a constructive approach to it.

In Chapter 7 we introduce the concept of quasi-conformal maps, a generalization of conformal maps. Roughly speaking, these are homeomorphisms whose first partial derivatives exist almost everywhere, and the Cauchy-Riemann condition is nearly satisfied (being small instead of 0). These maps naturally come up in complex analysis in several ways. It turns out that such maps still enjoy many properties of conformal maps, while having a more constructive nature. Many problems related to the behavior of conformal maps through quasi-conformal maps reduce to the study of a certain type of partial differential equation, where there are constructive approaches to the solutions.

Although the above method turns out to be unexpectedly powerful, we must remain humble. It is easy to pose simple looking open (and probably extremely hard) questions in complex analysis, for instance,

Brennan's conjecture: For every one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$, and every real p with $-2 < p < 2/3$, we have

$$\int_{\mathbb{D}} |f'(re^{i\theta})|^p \, dxdy < \infty.$$

We will see in this course that geometric complications of Ω and irregularities of its boundary result in large and small values for $|f'|$. The above conjecture suggests some bounds on the average values of $|f'|$. This is part of a set of conjectures knows as *universal integral means spectrum*, and covers some conjectures of Littlewood on the extremal growth rate of the length of the closed curves $f(r \cdot e^{i\theta})$, $\theta \in [0, 2\pi]$, as r tends to 1. These questions are motivated by important problems in statistical physics.

The actual prerequisite for this course is quite minimal. We assume that the students taking this class are familiar with the notions of holomorphic maps and their basic properties. This is a concise math course with ε-δ proofs, and so precise forms of definitions and statements appear in the notes. To rectify the challenge of where we start, we have summarized in Chapter 1 (in three pages) the basic results from complex analysis that we will rely on.

I prepared these notes for the course Geometric Complex Analysis, M3/4/5P60, for the autumn term of 2016 at Imperial College London. I am very pleased with the maths department for agreeing to offer this course for the first time. Complex analysis with its surprises is one of the most beautiful areas of mathematics. A small list of literature appears at the end of these notes, if you are interested to further pursue the topic. You may help me to improve these notes by emailing me any comments or corrections you have.

Contents

Chapter 1

Preliminaries from complex analysis

1.1 Holomorphic functions

In this section we recall the key concepts and results from complex analysis.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and $\mathbb{C}$ denote the set of complex numbers. It is standard to write a point $z \in \mathbb{C}$ as $z = x + iy$, where x and y are real, and $i \cdot i = -1$. Here $x = \operatorname{Re} z$ is called the *real part* of z and $y = \operatorname{Im} z$ is called the *imaginary part* of z. With this correspondence $z \mapsto (x, y)$ is a bijection from $\mathbb{C}$ to $\mathbb{R}^2$. Through this bijection, we may define open sets, closed sets, closure of a set, etc, for subsets of $\mathbb{C}$. That is, a set Ω is open is $\mathbb{C}$, if it is open when viewed as a subset of $\mathbb{R}^2$.

Recall the modulus function $|\cdot|$ on $\mathbb{R}$ and the norm function $\|\cdot\|$ on $\mathbb{R}^2$. We may extend the modulus function onto $\mathbb{C}$ by defining $|x + iy| = \|(x, y)\|$.

Let Ω be an open set in $\mathbb{C}$, and $f : \Omega \to \mathbb{C}$ be a map. There are functions u and v from $\mathbb{R}^2$ to $\mathbb{R}$ such that $f(x + iy) = u(x, y) + iv(x, y)$. When we view f as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, we recall from analysis that f is real differentiable at a point $(p_1, p_2) \in \Omega$, if there is a linear map $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{(h_1,h_2)\to 0} \frac{\|(u(p_1 + h_1), v(p_2 + h_2)) - (u(p_1, p_2), v(p_1, p_2)) - \Lambda(h_1, h_2)\|}{\|(h_1, h_2)\|} = 0$$

In the above definition, we view $\mathbb{C}$ as a vector space over $\mathbb{R}$, and define the differentiability accordingly. However, we may also view $\mathbb{C}$ as a vector space over $\mathbb{C}$, which is one dimensional. Then any linear map from $\mathbb{C}$ to $\mathbb{C}$ is of the from $z \mapsto az$ for some fixed complex number $a \in \mathbb{C}$. Now, we may say that f is complex differentiable at $p = p_1 + ip_2 \in \Omega$, if there is a linear map $\Lambda : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{|f(p + h) - f(p) - \Lambda(h)|}{|h|} = 0.$$

It is equivalent, and customary, to present the above definition in the following form we present as the definition of complex differentiability.

Definition 1.1. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then f is called *complex differentiable* at a point $z \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is a finite complex number. This limit is denoted by $f'(z)$. The map f is called *holomorphic* (*analytic*) on Ω, if f is complex differentiable at every point in Ω.

In these notes we are mostly dealing with complex differentiability, so for convenience, we shall simply use differentiable, to mean complex differentiable.

It follows that if $f : \Omega \to \mathbb{C}$ is differentiable at $z \in \Omega$, then it is continuous at z.

It is important to note that in Definition 1.1 h tends to 0 in the complex plane. In particular, h may tend to 0 in any direction. Using the notation $z = x + iy$ and $f(x+iy) = u(x,y) + iv(x,y)$, we obtain the following relations. When h tends to 0 in the horizontal direction, then

$$f'(z) = \lim_{x' \to 0} \frac{f(z+x') - f(z)}{x'} = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial f}{\partial x}(z). \tag{1.1}$$

On the other hand, if h tends to 0 in the vertical direction, that is, in the y direction, then

$$f'(z) = \lim_{y' \to 0} \frac{f(z+iy') - f(z)}{iy'} = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) = -i\frac{\partial f}{\partial y}(z) \tag{1.2}$$

Then, if $f'(z)$ exists, we must have

$$\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z) \tag{1.3}$$

In terms of the coordinate functions u and v, we must have

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \quad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y). \tag{1.4}$$

The above equations are known as the *Cauchy-Riemann* equations. On the other hand, if u and v are real-valued functions on Ω which have continuous first partial derivatives satisfying Equation (1.4), then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on Ω.

Theorem 1.2 (Cauchy-Goursat theorem-first version)**.** *Let Ω be an open set in $\mathbb{C}$ which is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any piece-wise C^1 simple closed curve γ in Ω we have*

$$\int_\gamma f(z)\, dz = 0.$$

There is an important corollary of the above theorem, that we state as a separate statement for future reference.

Theorem 1.3 (Cauchy Integral Formula-first version)**.** *Let Ω be an open set in $\mathbb{C}$ which is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any C^1 simple closed curve γ in Ω and any point z_0 in the region bounded by γ,*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \, d\zeta.$$

The condition Ω bounded by a smooth simple closed curve is not quite necessary in the above two theorems. Indeed, you may have only seen the above theorems when Ω is a disk or a rectangle. We shall see a more general form of these theorems later in this course, where a topological feature of the domain Ω comes into play.

Theorem 1.3 reveals a remarkable feature of holomorphic mappings. That is, if we know the values of a holomorphic function on a simple closed curve, then we know the values of the function in the region bounded by that curve, provided we *a priori* know that the function is holomorphic on the region bounded by the curve.

There is an analogous formula for the higher derivatives of holomorphic maps as well.[1] Under the assumption of Theorem 1.3, and every integer $n \geq 1$, the n-th derivative of f at z_0 is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta. \tag{1.5}$$

In Definition 1.1, we only assumed that the first derivative of f exists. It is remarkable that this seemingly weak condition leads to the existence of higher order derivatives. Indeed, an even stronger statement holds.

Theorem 1.4 (Taylor-series)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on an open set $\Omega \subseteq \mathbb{C}$. For every $z_0 \in \Omega$, the infinite series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

is absolutely convergent for z close to z_0, with the value of the series equal to $f(z)$.

The above theorems are in direct contrast with the regularity properties we know for real maps on $\mathbb{R}$ or on $\mathbb{R}^n$. That is, we have distinct classes of differentiable functions,

[1] Chauchy had proved Theorem 1.2 when the complex derivative $f'(z)$ exists and is a continuous function of z. Then, Édouard Goursat proved that Theorem 1.2 can be proven assuming only that the complex derivative $f'(z)$ exists everywhere in Ω. Then this implies Theorem 1.3 for these functions, and from that deduce these functions are in fact infinitely differentiable.

C^1 functions, C^2 functions, C^∞ functions, real analytic functions (C^ω). For any k, it is possible to have a function that is C^k but not C^{k+1} (Find an example if you already don't know this). There are C^∞ functions that are not real analytic. For example, the function defined as $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$. But these scenarios don't exist for complex differentiable functions.

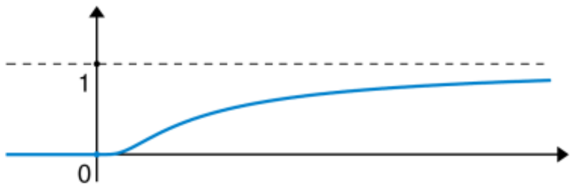

Figure 1.1: The graph of the function f.

Since a holomorphic function $f : \Omega \to \mathbb{C}$ is infinitely differentiable, higher order partial derivatives of u and v exist and are continuous. Differentiating Equations (1.4) with respect to x and y, and using $\partial_x \partial_y v = \partial_y \partial_x v$ and $\partial_x \partial_y u = \partial_y \partial_x u$, we conclude that the real functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are harmonic, that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

hold on Ω. We state this as a separate theorem for future reference.

Theorem 1.5 (Harmonic real and imaginary parts)**.** *Let $f(x + iy) = u(x, y) + iv(x, y)$ be a holomorphic function defined on an open set Ω in $\mathbb{C}$. Then, $u(x, y)$ and $v(x, y)$ are harmonic functions on Ω.*

A pair of harmonic functions u and v defined on the same domain $\Omega \subseteq \mathbb{C}$ are called *harmonic conjugates*, if they satisfy the Cauchy-Riemann equation, in other words, the function $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic.

Theorem 1.6 (maximum principle)**.** *If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function defined on an open set Ω, then its absolute value $|f(z)|$ has no maximum in Ω. That is, there is no $z_0 \in \Omega$ such that for all $z \in \Omega$ we have $|f(z)| \leq |f(z_0)|$.*

On the other hand, under the same assumptions, either f has a zero on Ω or $|f(z)|$ has no minimum on Ω.

Let K be an open set in Ω such that the closure of K is contained in Ω. If $f : \Omega \to \mathbb{C}$ is an analytic function, $|f(z)|$ is continuous on K and by the extreme value theorem, $|f|$ has a maximum on the closure of K. But by the above theorem, $|f|$ has no maximum on K. This implies that the maximum of $|f|$ must be realized on the boundary of K. Similarly, the minimum of $|f|$ is also realized on the boundary of K.

Chapter 2

Schwarz lemma and automorphisms of the disk

2.1 Schwarz lemma

We denote the disk of radius 1 about 0 by the notation $\mathbb{D}$, that is,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given $\theta \in \mathbb{R}$ the rotation of angle θ about 0, i.e. $z \mapsto e^{i\theta} \cdot z$, preserves $\mathbb{D}$. Due to the rotational symmetry of $\mathbb{D}$ most objects studied in complex analysis find special forms on $\mathbb{D}$ that have basic algebraic forms. We study some examples of these in this section, and will see more on this later on.

A main application of the maximum principle (Theorem 1.6) is the lemma of Schwarz. It has a simple proof, but has far reaching applications.

Lemma 2.1 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,*

(i) for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|$;

(ii) $|f'(0)| \leq 1$;

(iii) if either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then f is a rotation about 0.

Proof. Since f is holomorphic on $\mathbb{D}$, we have a series expansion for f centered at 0,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

which is convergent on $\mathbb{D}$. Since $f(0) = 0$, $a_0 = 0$, and we obtain

$$f(z) = a_1 z + a_2 z^2 + \cdots = z(a_1 + a_2 z + a_3 z^2 + \dots).$$

The series in the above parenthesis is convergent on $\mathbb{D}$. In particular, the function $g(z) = f(z)/z = a_1 + a_2 z + a_3 z^2 + \dots$ is defined and holomorphic on $\mathbb{D}$. Note that $g(0) = a_1 = f'(0)$.