Network Science
Spring 2024
Problem sheet 9 solutions

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1. In problem sheet 8, you were asked to derive the network-SIS model. Now, use the degree-based approximation to derive a nonlinear ODE for ϕ_k, the probability that a node with degree k is infectious, for this model.
Solution: The master equation for the network-SIS model is,

$$P(x_i(t+\Delta t) = 1) = (1-\gamma\Delta t)P(x_i(t) = 1) + \beta\Delta t \sum_{j=1}^{N} A_{ij} P(x_i(t) = 0, x_j(t) = 1) + O(\Delta t^2).$$

With the degree-based approximation, we start with the same master equation but with all probabilities conditional on node i having a specified degree, $k_i = k$. The development of the term with the summation remains exactly the same as for the network-SI model, and we just need to consider the new term with γ. By definition, $P(x_i(t) = 1|k_i = k) = \phi_k$. So, the master equation becomes,

$$\phi_k(t + \Delta t) = (1 - \gamma\Delta t)\phi_k(t) + k\beta\Delta t\,(1 - \phi_k) \sum_{k'=1}^{k'_{max}} \theta(k, k')\phi_{k'-1} + O(\Delta t^2).$$

Dividing both sides by Δt and letting $\Delta t \to 0$ then gives,

$$\frac{d\phi_k}{dt} = -\gamma\phi_k + k\beta\,(1 - \phi_k) \sum_{k'=1}^{k'_{max}} \theta(k, k')\phi_{k'-1}.$$

2. In this exercise, you will analyze an improved pair-approximation for the network-SI model (on simple connected graphs).

   (a) The pair approximation assumes that:

   $$P(s_i = 1, s_j = 1, x_l = 1) \approx P(x_l = 1, s_j = 1)P(s_i = 1, s_j = 1)/P(s_j = 1),\ A_{ij} = A_{jl} = 1.$$

   Assess this approximation when $l = i$
   Solution: If $l = i$, then $P(s_i = 1, s_j = 1, x_l = 1) = 0$, however the approximation gives

   $$P(s_i = 1, s_j = 1, x_l = 1) \approx P(x_i = 1, s_j = 1)P(s_i = 1, s_j = 1)/P(s_j = 1).$$

   The RHS can be restated as,

   $$P(x_i = 1|s_j = 1)P(s_i = 1|s_j = 1)P(s_j = 1),$$

   and this is equal to,

   $$P(x_i = 1|s_j = 1)[1 - P(x_i = 1|s_j = 1)]P(s_j = 1)$$

   which will typically be non-zero with an upper bound of 1/4.

(b) Now carry out the analogous analysis for the approximation applied to the other 3rd moment:

$P(s_i = 1, x_j = 1, x_l = 1) \approx P(s_i = 1, x_l = 1)P(s_i = 1, x_j = 1)/P(s_i = 1),\ A_{ij} = A_{il} = 1.$

Assess this approximation when $l = j$.
Solution: If $l = j$, then $P(s_i = 1, x_j = 1, x_l = 1) = P(s_i = 1, x_j = 1)$, however the approximation gives,

$$P(s_i = 1, x_j = 1, x_l = 1) \approx P(s_i = 1, x_j = 1)P(x_j = 1|s_i = 1)$$

which will be less than or equal to $P(s_i = 1, x_j = 1)$. If $P(s_i = 1, x_j = 1) \approx 1$, then the approximation will also be close to 1, while if $P(s_i = 1, x_j = 1) \ll 1$, the same will be true for the approximation. Another case when the approximation will work well is when $P(s_i = 1)$ is close to 1

(c) Based on your conclusions from (a) and (b), explain why the following equation represents an "improved pair approximation:"
$\mathrm{d}\langle s_i x_j\rangle/dt = \beta\{\sum_{l=1}^{N} [A_{jl}\langle x_l s_j\rangle\langle s_i s_j\rangle/\langle s_j\rangle - A_{il}\langle s_i x_l\rangle\langle s_i x_j\rangle/\langle s_i\rangle]$
$-\langle x_i s_j\rangle\langle s_i s_j\rangle/\langle s_j\rangle - \langle s_i x_j\rangle[1 - \langle s_i x_j\rangle/\langle s_i\rangle]\}$
Solution: The "improvements" are the new terms, $-\langle x_i s_j\rangle\langle s_i s_j\rangle/\langle s_j\rangle$ and $-\langle s_i x_j\rangle[1 - \langle s_i x_j\rangle/\langle s_i\rangle]$. The first of these cancels the $l = i$ term in the sum, while the second, when added to the $l = j$ term in the sum gives, $-\langle s_i x_j\rangle$, so now the pair approximation is exact when $l = j$ and $l = i$.

3. (Barabasi 9.9.2) Consider a one-dimensional lattice with N nodes that form a circle where each node connects to its two neighbors. Partition the graph into n_c clusters of $N_c = N/n_c$ nodes where a cluster contains a "chain" of adjacent nodes, and N is an integer multiple of n_c.

(a) Find the modularity of the partition
Solution: The graph contains N links and $K = 2N$ stubs. A cluster will contain N_c nodes, $L_c = N_c - 1$ links, and $K_c = 2N_c$ stubs. The modularity of a cluster is, $M_c = 1/K(2L_c - K_c^2/K) = 1/N[N_c(1 - 1/n_c) - 1]$. The modularity of the partition is $M = n_c M_c = [1 - (n_c + N_c)/N]$.

(b) Determine how to choose n_c to maximize the modularity
Solution: If we treat n_c as a continuous variable, M is maximized when $n_c = \sqrt{N}$. If $\sqrt{N}$ is not an integer, n_c should be chosen as the integer closest to $\sqrt{N}$.

4. Show that 1 is an upper bound for the modularity of a network (as defined in lecture 14)
Solution: The modularity for a set of nodes, S_a, is:

$$M_a = \frac{1}{2L}\sum_{i\in S_a}\sum_{j\in S_a}\left(A_{ij} - k_i k_j/(2L)\right),$$

where N is the number of nodes in the graph, L is the total number of links, and A_{ij} is the adjacency matrix. We can see that

$$M_a \le \frac{1}{2L}\sum_{i\in S_a}\sum_{j\in S_a} A_{ij} = L_a/L$$

where L_a is the total number of links connecting nodes in S_a. The total modularity for the graph is the sum over all distinct sets of nodes, $M = \sum_{S_a} M_a$, and noticing that $\sum_{S_a} L_a \le L$ since L_a does not count links crossing from one set to another, we conclude that $M \le \sum_{S_a} L_a/L \le 1$.

5. In this exercise, you will derive an important property of the Laplacian matrix for simple graphs. You will show that if two nodes are in the same connected component of the graph, then the corresponding elements of an eigenvector corresponding to a zero eigenvalue must be the same.

   (a) First consider the Laplacian matrix in block-diagonal form with each block corresponding to a connected component. Construct $\mathbf{v} \in \mathcal{R}^N$ as follows. Choose one connected component. For all nodes i in this component set $v_i = 1$, and set all other elements in $\mathbf{v}$ to be zero. Show that this vector is an eigenvector of $\mathbf{L}$ which corresponds to a zero eigenvalue.
   Solution: We know $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and that $\mathbf{A}$ will have a block diagonal structure. Multiplying $\mathbf{L}$ with $\mathbf{v}$ gives $\mathbf{Lv} = \mathbf{Dv} - \mathbf{Av}$. Let $\mathbf{u} = \mathbf{Lv}$. Then $u_i = \sum_{j=1}^{N} (D_{ij} - A_{ij})\, v_j$. Using the definition of $\mathbf{v}$, we only need to consider terms in the sum where j is in the chosen component (C). Then, $u_i = \sum_{j \in C} (D_{ij} - A_{ij}) = k_i - \sum_{j \in C} A_{ij} = 0$. So $\mathbf{u} = \mathbf{Lv} = 0$ and $\mathbf{v}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 0. Note that we don't actually need $\mathbf{L}$ to be in block-diagonal form.

   (b) Given an arbitrary vector $\mathbf{x} \in \mathbb{R}^N$ and the Laplacian matrix for a simple undirected graph, show that the i-th element of $\mathbf{y} = \mathbf{Lx}$ is given by, $y_i = \sum_{j \in \mathbb{N}_i} (x_i - x_j)$ where $\mathbb{N}_i$ is the set of nodes which are neighbors of i
   Solution: We have $y_i = \sum_{j=1}^{N} (k_i \delta_{ij} - A_{ij}) x_j = k_i x_i - \sum_{j=1}^{N} A_{ij} x_j$, and then $y_i = k_i x_i - \sum_{j \in \mathbb{N}_i}^{N} x_j$ and since there are k_i terms in the sum, $y_i = \sum_{j \in \mathbb{N}_i}^{N} (x_i - x_j)$

   (c) Now show that if $a = \mathbf{x}^T \mathbf{Lx}$, then $a = 1/2 \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} (x_i - x_j)^2$. Use this result to argue that an eigenvector $\mathbf{v}$ corresponding to a zero eigenvalue for $\mathbf{L}$ must have the following property: for any 2 nodes i and j in the same connected component, $v_i = v_j$. Note that v_i is the *ith* element in $\mathbf{v}$ and not the *ith* eigenvector of $\mathbf{L}$.
   Solution: $a = \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} (x_i^2 - x_i x_j)$, and $\sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} x_i^2 = \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} x_j^2$, so $a = 1/2 \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} (x_i^2 + x_j^2) - \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} x_i x_j$ or $a = 1/2 \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} (x_i - x_j)^2$ as required.

6. Consider an adjacency matrix $\mathbf{A}$ corresponding to a simple graph.

   (a) Construct a transformation of the adjacency matrix, $\mathbf{A} \to \mathbf{A}'$, which corresponds to a renumbering of the nodes in the graph for $\mathbf{A}$ while preserving the graph structure. I.e. if the the re-numbering results in $(i, j) \to (i', j')$ then $A'_{i'j'} = A_{ij}$
   Solution: A permutation matrix, $\mathbf{P}$, which can be thought of as a re-ordering of the columns of the identity matrix, is needed here. $\mathbf{B} = \mathbf{AP}$ produces a matrix which corresponds to $\mathbf{A}$ with columns re-ordered, and $= \mathbf{P}^T\mathbf{A}$ is a matrix which corresponds to $\mathbf{A}$ with rows re-ordered. For example, if

$$\mathrm{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   and $\mathbf{A} = [\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3]$ where $\mathbf{c}_i$ is the *ith* column of $\mathbf{A}$, then $\mathbf{AP} = [\mathbf{c}_2, \mathbf{c}_1, \mathbf{c}_3]$). Applying these results, $\mathbf{A}' = \mathbf{P}^T\mathbf{AP}$

   (b) Show that the eigenvalues of $\mathbf{A}$ and $\mathbf{A}'$ are the same
   Solution: Note that $\mathbf{P}^T = \mathbf{P}^{-1}$ and $\det(\mathbf{P}) = \det(\mathbf{P}^T) = 1$ (and we are applying a similarity transformation). Then, $\det(\mathbf{P}^T\mathbf{AP} - \lambda) = \det(\mathbf{P}^T(\mathbf{A} - \lambda)\mathbf{P}) = \det(\mathbf{P}^T)\det(\mathbf{A} - \lambda)\det(\mathbf{P}) = \det(\mathbf{A} - \lambda)$.