Imperial College London

Problem Sheet 7

MATH50011
Statistical Modelling 1

Week 9

Lecture 15: Multivariate Normal Distributions

1. Let X and B be independent random variables such that $X \sim N(0,1)$ and $B \in \{-1,1\}$ with $P(B=1) = P(B=-1) = \frac{1}{2}$. Let $Z = XB$.

   (a) Find $Cov(X,Z)$.

   (b) Show that $Z \sim N(0,1)$.

   (c) Are X and Z independent?

2. Suppose $X \sim N\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

   (a) What is the distribution of $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} X + \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$?

   (b) Are any of the components of Z independent?

   (c) Let $Y \sim N\left(\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right)$. What components of Y are independent?

3. Let
$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix}, \begin{pmatrix} \sigma_Y^2 & \rho\sigma_Y\sigma_X \\ \rho\sigma_Y\sigma_X & \sigma_X^2 \end{pmatrix}\right).$$

   (a) Find the conditional distribution of $Y|X = x$ (it will be a univariate normal distribution).

   (b) Express the conditional mean $E(Y|X = x)$ as a linear function $\beta_0 + \beta_1 x$. What are β_0 and β_1 in terms of the parameters of the bivariate normal distribution?

4. In the lecture we had the following definition:

   Let $Z \sim N(\mu, I_n)$, where $\mu \in \mathbb{R}^n$. $U = Z^T Z$ is said to have a *non-central* χ^2*-distribution* with n degrees of freedom (d.f.) and non-centrality parameter $\delta = \sqrt{\mu^T \mu}$. Notation: $U \sim \chi^2_n(\delta)$.

   (a) Show that the $\chi^2_n(\delta)$-distribution depends on μ only through δ.

   (b) Show that $E(U) = n + \delta^2$ and $Var(U) = 2n + 4\delta^2$.

   (c) Show that if $U_i \sim \chi^2_{n_i}(\delta_i)$, $i = 1, \dots, k$, and $U_1, \dots, U_k$ are independent then $\sum_{i=1}^k U_i \sim \chi^2_{\sum n_i}(\sqrt{\sum \delta_i^2})$.

   Hint: Use moment-generating functions.

5. In the lectures, we showed that for a sequence $T_n \sim t_n(0)$, $T \to_d N(0,1)$. Similar results can be derived for the χ^2_n and $F_{m,n}$ distributions.

   (a) Let $Z_1, \dots, Z_n$ be iid $N(0,1)$ and define $U_n = \sum_i Z_i^2$. Use large sample properties of U_n to derive a normal approximation to the χ^2_n distribution.

   (b) For m fixed and $n \to \infty$, show that $F_n \sim F_{m,n}$ converges in distribution to a χ^2_m random variable.

6. Revise the proofs of Lemmas 15-19 and the Fisher-Cochran theorem.