Imperial College
London

MATH70052

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Hydrodynamic Stability

Date: 18 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider convection in a horizontal layer of fluid between $\hat{z} = 0$ and d, where the temperature field $\hat{\theta}$ satisfies the equation,

$$\left[\frac{\partial}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \nabla)\right] \hat{\theta} = \kappa \nabla^2 \hat{\theta} - \hat{q},$$

in the Cartesian coordinates $\hat{\mathbf{x}} \equiv (\hat{x}, \hat{y}, \hat{z})$, where $\kappa > 0$ and $\hat{q} > 0$ are constants representing thermal diffusion and radiation heat loss, respectively. The temperatures at the lower and upper boundaries are maintained at constant values, $\hat{\theta}_0$ and $\hat{\theta}_1$, with $\hat{\theta}_1 < \hat{\theta}_0$.

(i) The basic steady state is that of pure conduction with velocity field $\hat{\mathbf{u}} = \hat{\mathbf{U}} = \mathbf{0}$. Find the vertical distribution of the temperature $\hat{\Theta}(z)$.

(3 marks)

(ii) When the Boussinesq approximation is made, the dimensional momentum and continuity equations can be written as

$$\left[\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}}\right] = -\nabla\left(\frac{\hat{p}}{\hat{\rho}_0}\right) + \nu \nabla^2 \hat{\mathbf{u}} + \alpha g \hat{\theta}\, \mathbf{k}, \qquad \nabla \cdot \hat{\mathbf{u}} = 0,$$

where ν is the kinematic viscosity, $\hat{\rho}_0$ the density, g the gravitational acceleration, and α the fluid density expansion coefficient, while $\mathbf{k}$ denotes the unit vector in the vertical direction. Deduce that the characteristic time scale and velocity of the problem are d^2/κ and κ/d, respectively. Show that in terms of the non-dimensional independent and dependent variables,

$$\mathbf{x} = \hat{\mathbf{x}}/d, \quad t = \hat{t}/(d^2/\kappa); \quad \theta = \hat{\theta}/(\hat{\theta}_0 - \hat{\theta}_1), \quad \mathbf{u} = \hat{\mathbf{u}}/(\kappa/d), \quad p = \hat{p}/(\hat{\rho}_0 \kappa^2/d^2),$$

the momentum, continuity and temperature equations assume the dimensionless form,

$$\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla p + Pr \nabla^2 \mathbf{u} + Ra Pr\, \theta\, \mathbf{k}, \qquad \nabla \cdot \mathbf{u} = 0,$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)\right] \theta = \nabla^2 \theta - q,$$

and define the parameters Ra, Pr and q in terms of the physical parameters in the problem.

(6 marks)

(iii) When the basic state is perturbed by a small-amplitude disturbance $(u', v', w', p', \theta')$, it is known that w' and θ' satisfy the equation

$$\left[\frac{\partial}{\partial t} - Pr \nabla^2\right] \nabla^2 w' = Ra Pr \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \theta'.$$

Let $(w', \theta') = \big(\tilde{w}(z), \tilde{\theta}(z)\big) f(x, y) e^{\sigma t}$, where $\sigma = \sigma_r + i\sigma_i$. Derive the equations governing $\tilde{w}(z)$ and $\tilde{\theta}(z)$. Derive further a single sixth-order equation for $\tilde{w}(z)$, and specify six boundary conditions on $\tilde{w}$ when both boundaries are free surfaces.

(6 marks)

(iv) Using the results in (iii), or otherwise, show for the case of $Pr = 1$ that $\sigma_i = 0$ when $\sigma_r = 0$ (which is referred to as the principle of exchange of stabilities).

[*Hint: Multiply $\tilde{w}^*$ to the sixth-order equation for $\tilde{w}$ and perform integration by parts.*]

(5 marks)

(Total: 20 marks)

2. Consider linear viscous stability of the viscous flow between the two concentric cylinders, which have radii R_1 and $R_2 > R_1$, and rotate with angular velocities Ω_1 and Ω_2, respectively. In a cylindrical coordinate system $(\hat{r}, \phi, \hat{z})$, the base flow velocity field is $\big(0, \hat{U}(\hat{r}), 0\big)$. When the flow is perturbed by a small-amplitude axisymmetric disturbance, the velocity $(\hat{v}', \hat{u}', \hat{w}')$ and pressure $\hat{p}'$ of the disturbance are governed by the equations,

$$\frac{\partial \hat{u}'}{\partial \hat{t}} + \frac{d\hat{U}}{d\hat{r}}\hat{v}' + \frac{\hat{U}}{\hat{r}}\hat{v}' = \nu\Big(\frac{\partial^2 \hat{u}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{u}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{u}'}{\partial \hat{r}} - \frac{\hat{u}'}{\hat{r}^2}\Big), \tag{1a}$$

$$\frac{\partial \hat{v}'}{\partial \hat{t}} - 2\frac{\hat{U}}{\hat{r}}\hat{u}' = -\frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{r}} + \nu\Big(\frac{\partial^2 \hat{v}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{v}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{v}'}{\partial \hat{r}} - \frac{\hat{v}'}{\hat{r}^2}\Big), \tag{1b}$$

$$\frac{\partial \hat{w}'}{\partial \hat{t}} = -\frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{z}} + \nu\Big(\frac{\partial^2 \hat{w}'}{\partial \hat{z}^2} + \frac{\partial^2 \hat{w}'}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{w}'}{\partial \hat{r}}\Big), \tag{1c}$$

$$\frac{\partial \hat{v}'}{\partial \hat{r}} + \frac{\hat{v}'}{\hat{r}} + \frac{\partial \hat{w}'}{\partial \hat{z}} = 0, \tag{1d}$$

where the density and kinematic viscosity of the fluid, ρ and ν, are constant.

Throughout the question, we make the narrow gap assumption $\varepsilon \equiv h/R_1 \ll 1$, where $h = R_2 - R_1$.

(a) Consider first the case where $\chi \equiv \Omega_2/\Omega_1 \neq 1$. In the narrow gap limit $\varepsilon \ll 1$, perform a scaling analysis by considering the balances of dominant terms in equations (1a)-(1d), and show that instability may occur over a short length scale of $O(h)$ and short time scale of $O(\varepsilon^{1/2}\Omega_1^{-1})$, that is,

$$\frac{\partial \hat{\phi}'}{\partial \hat{z}} = O\big(\hat{\phi}'/h\big), \quad \frac{\partial \hat{\phi}'}{\partial \hat{t}} = O\big(\phi'/(\varepsilon^{1/2}\Omega_1^{-1})\big),$$

where $\hat{\phi}'$ denotes any of $\hat{u}'$, $\hat{v}'$, $\hat{w}'$ and $\hat{p}'$. Deduce that $\hat{v}'$, $\hat{w}'$ and $\hat{p}'$ obey the following scaling relations relative to $\hat{u}'$:

$$\hat{v}' = O(\varepsilon^{1/2}\hat{u}'), \quad \hat{w}' = O(\varepsilon^{1/2}\hat{u}'), \quad \hat{p}' = O(\rho\Omega_1 h\, \hat{u}').$$

(5 marks)

(b) Consider now the case where $\chi \equiv \Omega_2/\Omega_1 = 1$, for which $\hat{U} = \Omega_1\hat{r}$.

(i) In the narrow gap limit $\varepsilon \ll 1$, perform a scaling analysis by considering the balances of dominant terms in equations (1a)-(1d), and show that instability may occur over a short length scale $O(h)$ but the time scale is $O(\Omega_1^{-1})$. Deduce that $\hat{v}'$, $\hat{w}'$ and $\hat{p}'$ obey the following scaling relations relative to $\hat{u}'$:

$$\hat{v}' = O(\hat{u}'), \quad \hat{w}' = O(\hat{u}'), \quad \hat{p}' = O(\rho\Omega_1 h\, \hat{u}').$$

[*Hint: Note that* $\dfrac{d\hat{U}}{d\hat{r}} = \dfrac{\hat{U}}{\hat{r}} = \Omega_1$ in this case.]

(3 marks)

Question continues on the next page.

(ii) Introducing the normalised independent and dependent variables through the relations,

$$\left.\begin{aligned} \hat{t} = \Omega_1^{-1}\, t, \qquad \hat{r} = R_1 + hy, \qquad \hat{z} = hz; \\ \hat{u}' = (\Omega_1 R_1)u', \quad \hat{v}' = (\Omega_1 R_1)v', \quad \hat{w}' = (\Omega_1 R_1)w', \quad \hat{p}' = \varepsilon(\rho\Omega_1^2 R_1^2)p', \end{aligned}\right\} \tag{2}$$

show that the normalised quantities, (v', u', w') and p', satisfy, at leading order in ε, the equations

$$\frac{\partial u'}{\partial t} + 2v' = \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right), \tag{3a}$$

$$\frac{\partial v'}{\partial t} - 2u' = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\right), \tag{3b}$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right), \tag{3c}$$

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{3d}$$

where Re is a non-dimensional parameter that you are expected to define. (4 marks)

(iii) Seek solutions of the normal-mode form,

$$\big(u', v', w', p'\big) = \big(\bar{u}(y), \bar{v}(y), \bar{w}(y), \bar{p}(y)\big)e^{\sigma t + i\beta z} + c.c.$$

Derive the equations satisfied by $\big(\bar{u}(y), \bar{v}(y), \bar{w}(y)\big)$ and $\bar{p}(y)$, and reduce them to two coupled equations for $\bar{u}$ and $\bar{v}$. Specify the boundary conditions on $\bar{u}$ and $\bar{v}$.

(4 marks)

(iv) Using the result in (iii), show that the flow is stable, i.e. σ_r, the real part of σ, is negative ($\sigma_r < 0$).

(4 marks)

(Total: 20 marks)

3. Consider linear inviscid stability of an exactly parallel axisymmetric shear flow with the only nonzero velocity component being in the axial direction so that the base-flow velocity field is $\big(U(r),0,0\big)$ in a cylindrical coordinate system (x,r,ϕ). The base flow is perturbed by small-amplitude axisymmetric disturbances of the normal-mode form:

$$(u',v',0)=\big(\bar{u}(r),\bar{v}(r),0\big)e^{i(\alpha x-\omega t)}+c.c.,\quad p'=\bar{p}(r)e^{i(\alpha x-\omega t)}+c.c.$$

Throughout this question, the profile $U(r)$ is assumed to be a smooth function of r.

(i) Starting from the axisymmetric Euler equations, which are given on the next page, show that $\big(\bar{u}(r),\bar{v}(r)\big)$ and $\bar{p}(r)$ satisfy the linear equations,

$$i\alpha\bar{u}+\frac{d\bar{v}}{dr}+\frac{\bar{v}}{r}=0, \tag{4a}$$

$$i\alpha(U-c)\bar{u}+U'(r)\bar{v}=-i\alpha\bar{p}, \tag{4b}$$

$$i\alpha(U-c)\bar{v}=-\frac{d\bar{p}}{dr}, \tag{4c}$$

where $U'(r)=\dfrac{dU}{dr}$, and $c=\omega/\alpha$ is the phase speed with $c=c_r+ic_i$.

(4 marks)

(ii) Show that equations (4a)-(4c) reduce to a single equation for $\bar{p}$,

$$\frac{d^2\bar{p}}{dr^2}-\Big(\frac{2U'}{U-c}-\frac{1}{r}\Big)\frac{d\bar{p}}{dr}-\alpha^2\bar{p}=0. \tag{5}$$

Assume that the flow is bounded by two rigid cylindrical surfaces at $r=r_1$ and r_2. Using this equation or otherwise, prove that if the flow is unstable, then c_r must lie in the range

$$U_{\min}<c_r<U_{\max},$$

where $U_{\max}$ and $U_{\min}$ denote the maximum and minimum of $U(r)$, respectively.
[*Hint: Multiply* (5) *by* $r\bar{p}^*/(U-c)^2$ *and perform integration by parts.*]

(8 marks)

(iii) Show that equations (4a)-(4c) also reduce to a single equation for $\bar{v}$,

$$\frac{d}{dr}\Big[\frac{1}{r}\frac{d(r\bar{v})}{dr}\Big]-\Big[\frac{d}{dr}\big(\frac{U'}{r}\big)\Big]\frac{r\bar{v}}{U-c}-\alpha^2\bar{v}=0. \tag{6}$$

Under the assumption that the flow is bounded by two rigid cylindrical surfaces at $r=r_1$ and r_2, prove a *necessary condition for inviscid shear instability*, namely, if there exists a growing mode with $c_i>0$, then

$$\frac{d}{dr}\big(\frac{U'}{r}\big)=0$$

at a radial position in the interval $r_1<r<r_2$.
[*Hint: Multiply* (6) *by* $r\bar{v}^*$ *and then perform integration by parts.*]

(8 marks)

The Euler equations in cylindrical coordinates are given on the next page.

Euler equations in cylindrical polar coordinates (x, r, ϕ) for an axisymmetric flow. Let the corresponding velocity vector be denoted by (V_x, V_r, V_ϕ) and the pressure by p. For an axisymmetrc flow, $V_\phi \equiv 0$ and the Euler equations, suitably normalised, are written as

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r} = 0, \tag{7a}$$

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_r \frac{\partial V_x}{\partial r} = - \ \frac{\partial p}{\partial x}, \tag{7b}$$

$$\frac{\partial V_r}{\partial t} + V_x \frac{\partial V_r}{\partial x} + V_r \frac{\partial V_r}{\partial r} = - \ \frac{\partial p}{\partial r}, \tag{7c}$$

where x measures the distance along the axis of symmetry of the flow, r and ϕ denote the radial distance and azimuthal angle, respectively.

(Total: 20 marks)

4. Linear stability of an exactly parallel shear flow with velocity profile $U(y)$ is studied by introducing small perturbations of the normal-mode form: $\big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\big)e^{i(\alpha x - \omega t)} + c.c.$ It follows that $\big(\bar{u}(y), \bar{v}(y), \bar{p}(y)\big)$ satisfies the equations,

$$i\alpha\bar{u} + \frac{dv}{dy} = 0, \quad i\alpha(U-c)\bar{u} + U'(y)\bar{v} = -i\alpha\bar{p} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{u}, \tag{8}$$

$$i\alpha(U-c)\bar{v} = -\frac{d\bar{p}}{dy} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{v}, \tag{9}$$

where $U'(y) = \dfrac{dU}{dy}$, $c = \omega/\alpha$ is the phase speed and Re the Reynolds number.

Apply these equations to a parallel shear flow between two infinitely large rigid flat plates located at $y = -1$ and $y = 1$ with the velocity profile $U(y)$ being modelled by

$$U(y) = \begin{cases} (y+1)/h & \text{for } \; -1 \le y < -1+h, \\ \quad 1 & \text{for } \; -1+h < y < 1-h, \\ -(y-1)/h & \text{for } \; 1-h < y \le 1, \end{cases}$$

where $0 < h < 1$ is assumed to be of $O(1)$.

(a) In the limit $Re \gg 1$, equations (8)-(9) are reduced to the Rayleigh equation for $\bar{v}$:

$$(U-c)(\frac{d^2}{dy^2} - \alpha^2)\bar{v} - U''\bar{v} = 0.$$

(i) For the $U(y)$ given above, solve the Rayleigh equation to derive the dispersion relation satisfied by the phase speed c for *symmetric modes*, for which $\dfrac{d\bar{v}}{dy} = 0$ at $y = 0$.

[*Hint: (a) Solve the Rayleigh equation in the upper half of the channel,* $0 < y \le 1$, *by using the symmetry property, and (b) you may use the fact that across a discontinuity of* U *and/or* $U'(y)$ *at* y_d*, the following (jump) relations hold,*

$$\Big[(U-c)\frac{d\bar{v}}{dy} - U'\bar{v}\Big]_{y_d^-}^{y_d^+} = 0, \qquad \Big[\frac{\bar{v}}{U-c}\Big]_{y_d^-}^{y_d^+} = 0,$$

where $[\,\cdot\,]_{y_d^-}^{y_d^+}$ *stand for the jump of the quantity across* $y = y_d$.]

(5 marks)

(ii) Find c and show that the flow is inviscidly *stable*.
Show that for $\alpha \ll 1$, c has the asymptotic approximation

$$c = c_0\alpha^2 + o(\alpha^2),$$

where c_0 depends on h, whose expression you are expected to find.
[*You may use without proof the fact that* $\tanh x = x - \frac{x^3}{3} + o(x^3)$ *for* $x \ll 1$.]

(3 marks)

(iii) Find the pressure and streamwise velocity of the perturbation at $y = 1$, $\bar{p}_w = \bar{p}(1)$ and $\bar{u}(1)$. Explain why it is necessary to introduce a viscous Stokes layer at $y = 1$.

(3 marks)

Question continues on the next page.

(b) Now analyse the disturbance in the Stokes layer by considering the continuity equation and the momentum equations with the viscous terms included, which are given in (8)–(9).

(i) Deduce that the thickness of the Stokes layer is $O\big((\omega Re)^{-1/2}\big)$, and hence introduce the local coordinate $\tilde{Y} = (y-1)/(\omega Re)^{-1/2}$, and deduce that $\bar{u}$ and $\bar{v}$ expand as

$$\bar{u} = \tilde{U}(\tilde{Y}) + \ldots, \quad \bar{v} = \alpha(\omega Re)^{-1/2}\tilde{V}(\tilde{Y}) + \ldots.$$

Derive the equations that $\tilde{U}$ and $\tilde{V}$ satisfy, and specify the appropriate boundary and matching conditions.

(4 marks)

(ii) Given that $\tilde{U} = (\alpha/\omega)\bar{p}_w\big[1 - \exp\{(-i)^{1/2}\tilde{Y}\}\big]$, show that

$$\tilde{V} \to -(i\alpha/\omega)\bar{p}_w\,\tilde{Y} + \tilde{V}_\infty \quad \text{as} \quad \tilde{Y} \to -\infty,$$

and determine the expression for $\tilde{V}_\infty$, which is independent of $\tilde{Y}$. Explain how $\tilde{V}_\infty$ affects the invisicid solution in the main layer.

(3 marks)

(iii) By examining balances in the streamwise momentum equation in the limit $\alpha \ll 1$, show that the Stokes layer solution in (i) above becomes invalid when

$$\alpha = O(Re^{-1/7}).$$

(2 marks)

(Total: 20 marks)

5. When a three-dimensional boundary layer is perturbed by a disturbance, the perturbed flow field is written as

$$(u, v, w, p) = \big(U(x, Y),\, Re^{-1/2}V(x, Y),\, W(x, Y),\, \bar{P}\big) + \epsilon(u',\, v',\, w',\, p')$$

in the Cartesian coordinate system (x, y, z), where x, y and z are non-dimensionalised by L, the distance to the leading edge, $Y = Re^{1/2}y$ and the Reynolds number $Re = \hat{V}_\infty L/\nu$ with $\hat{V}_\infty$ being the reference velocity and ν the kinematic viscosity. The parameter $\epsilon \ll 1$ measures the magnitude of the disturbance. The base-flow velocity field has the near-wall and far-field behaviours:

$$(U, W) \to (\lambda_1 Y,\, \lambda_3 Y) \quad \text{as} \quad Y \to 0; \quad (U, W) \to (1, 0) \quad \text{as} \quad Y \to \infty,$$

where λ_1 and λ_3 are functions of x. The flow field $(\mathbf{u}, p) = (u, v, w, p)$ is governed by the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}.$$

(i) Derive the linearised equations governing the perturbation (u', v', w') and p', which are functions of x, Y and z but are assumed to be independent of t, e.g. $u' = u'(x, Y, z)$. Indicate the terms which represent the non-parallel-flow effect, and explain Prandtl's parallel-flow approximation.

(5 marks)

(ii) Suppose that in the main layer (deck) where $Y = O(1)$, the solution expands as

$$(u',\, v',\, w',\, p') = \big(\bar{u}(x, Y),\, Re^{-\frac{1}{8}}\bar{v}(x, Y),\, \bar{w}(x, Y),\, Re^{-\frac{1}{8}}\bar{p}(x, Y)\big)E + c.c.,$$

where $c.c.$ stands for the complex conjugate, and $E = e^{iRe^{\frac{3}{8}}(\alpha x + \beta z)}$.

Derive the equations governing $(\bar{u},\, \bar{v},\, \bar{w})$ and $\bar{p}$, and verify that they have the solution

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{w} = A\frac{\partial W}{\partial Y}, \quad \bar{v} = -iA(\alpha U + \beta W), \quad \bar{p} = P,$$

where A and P are constants.

Explain why it is necessary to introduce an upper layer as well as a viscous wall layer.

(5 marks)

(iii) Deduce that the width of the upper layer is of $O(Re^{-3/8}L)$ and that $(u',\, v',\, w',\, p')$ expands as

$$(u',\, v',\, w',\, p') = Re^{-\frac{1}{8}}(u^\dagger, v^\dagger, w^\dagger, p^\dagger)E + c.c.$$

Introduce the variable $\bar{y} = Re^{3/8}y = Re^{-1/8}Y$, and derive the equation satisfied by $p^\dagger$ as well as the boundary condition as required by matching with the main-layer solution. Solve for $p^\dagger$ to obtain the relation between P and A.

(5 marks)

(iv) Deduce that the viscous wall layer has a width of $O(Re^{-\frac{5}{8}}L)$, and hence introduce $\tilde{y} = Re^{\frac{5}{8}}y = Re^{1/8}Y$ in this layer. Deduce that the solution expands as

$$(u',\, v',\, w',\, p') = \big(\tilde{u}(x, \tilde{y}),\, Re^{-\frac{1}{4}}\tilde{v}(x, \tilde{y}),\, \tilde{w}(x, \tilde{y}),\, Re^{-\frac{1}{8}}P\big)E + c.c.$$

Derive the equations governing $(\tilde{u}, \tilde{v}, \tilde{w})$, and specify the boundary conditions at $\tilde{y} = 0$ and the matching condition as $\tilde{y} \to \infty$.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

This paper is also taken for the relevant examination for the Associateship.

MATH97012, MATH97091

Hydrodynamic Stability (Solutions)

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1. (a) Since $\hat{\Theta} = \hat{\Theta}(z)$, the temperature equation simplifies to $\kappa \dfrac{d^2\hat{\Theta}}{d\hat{z}^2} = \hat{q}$, which is integrated to give

unseen ⇓

$$\hat{\Theta} = \hat{\theta}_0 + \hat{C}_1\hat{z} + \tfrac{1}{2}(\hat{q}/\kappa)\hat{z}^2,$$

where $\hat{C}_1$ is a constant determined by imposing the required boundary condition,

$$\hat{\theta}_0 + \hat{C}_1 d + \tfrac{1}{2}(\hat{q}/\kappa)d^2 = \hat{\theta}_1.$$

It follows that

$$\hat{C}_1 = \left(\hat{\theta}_1 - \hat{\theta}_0 - \tfrac{1}{2}(\hat{q}/\kappa)d^2\right)/d.$$

Substituting $\hat{C}_1$ back into $\hat{\Theta}$ gives the temperature distribution of the basic state,

$$\hat{\Theta} = \hat{\theta}_0 + (\hat{\theta}_1 - \hat{\theta}_0)\hat{z}/d + \tfrac{1}{2}(\hat{q}/\kappa)(\hat{z} - d)\hat{z}.$$

3, A

(b) The reference length is d. The dominant balance is between the time variation $\dfrac{\partial\hat{\theta}}{\partial\hat{t}}$ and the thermal diffusion $\kappa\nabla^2\hat{\theta}$: $\hat{\theta}/\hat{t} \sim \kappa\hat{\theta}/d^2$, which gives the time scale $\hat{t} \sim d^2/\kappa$. With the length scale d, the characteristic velocity scale is $d/(d^2/\kappa) = \kappa/d$. After substitution of the normalized variables, the equations read

meth seen ⇓

$$(\kappa^2/d^3)\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -(\kappa^2/d^3)\nabla p + \nu(\kappa/d^3)\nabla^2\mathbf{u} + \alpha g(\hat{\theta}_0 - \hat{\theta}_1)\theta\,\mathbf{k},$$

$$(\kappa/d^2)(\hat{\theta}_0 - \hat{\theta}_1)\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right]\theta = (\kappa/d^2)(\hat{\theta}_0 - \hat{\theta}_1)\nabla^2\theta - \hat{q}.$$

Dividing the two equations by factors (κ^2/d^3) and $\kappa/d^2(\hat{\theta}_0 - \hat{\theta}_1)$, respectively, we obtain the required dimensionless momentum and temperature equations with

$$Pr = \nu/\kappa, \quad Ra = \alpha g(\hat{\theta}_0 - \hat{\theta}_1)d^3/(\kappa\nu), \quad q = (d^2/\kappa)\hat{q}/(\hat{\theta}_0 - \hat{\theta}_1).$$

The continuity equation is $\nabla\cdot\mathbf{u} = 0$.

6, A

(c) (i) The non-dimensional basic temperature can be written as

sim. seen ⇓

$$\Theta \equiv \hat{\Theta}/(\hat{\theta}_0 - \hat{\theta}_1) = \theta_0 - z + \tfrac{1}{2}q(z-1)z, \qquad \frac{d\Theta}{dz} = -1 + q(z - \tfrac{1}{2}).$$

Linearisation of the non-dimensional temperature equation about $\Theta(z)$ yields

$$\frac{\partial\theta'}{\partial t} + \frac{d\Theta}{dz}w' = \nabla^2\theta'. \tag{1}$$

Substitution of the assumed form of solution into equation (1) gives

$$(D^2 + \nabla_1^2 - \sigma)f\tilde{\theta} = \frac{d\Theta}{dz}f\tilde{w}, \tag{2}$$

where the differential operators $D = \dfrac{d}{dz}$ and $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. Equation (2) can be rearranged into a 'variable separation' form,

$$\left[(D^2 - \sigma)\tilde{\theta} - \frac{d\Theta}{dz}\tilde{w}\right]/\tilde{\theta} = -(\nabla_1^2 f)/f.$$

Since the left-hand side is a function of z only while the right-hand side depends only on x and y, both sides should be a constant, a^2 say. Then

$$(D^2 - a^2 - \sigma)\tilde{\theta} = \frac{d\Theta}{dz}\tilde{w}, \tag{3}$$

and

$$\nabla_1^2 f + a^2 f = 0. \tag{4}$$

Similarly, substituting the assumed form into the equation for w' (given in the question) and making use of (4) we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma/Pr)\tilde{w} = a^2 Ra\, \tilde{\theta}. \tag{5}$$

The required equations are (3) and (5). 4, B

Let the operator $(D^2 - a^2 - \sigma)$ act on (5), and use (3) on the right-hand side of the resulting equation. This leads to the required sixth-order equation for $\tilde{w}$, unseen ⇓

$$(D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - \sigma/Pr)\tilde{w} = a^2 Ra(\frac{d\Theta}{dz})\tilde{w}. \tag{6}$$

In the case of free surface, $\tilde{w} = D^2\tilde{w} = 0$ and $\tilde{\theta} = 0$ imply that $D^4\tilde{w} = 0$. Thus the six boundary conditions are

$$\tilde{w} = D^2\tilde{w} = D^4\tilde{w} = 0 \quad \text{at} \quad z = 0, 1. \tag{7}$$

2, C

(ii) For $Pr = 1$, equation (6) can be rewritten as unseen ⇓

$$D^6\tilde{w} - (3a^2 + 2\sigma)D^4\tilde{w} + \left[(a^2+\sigma)^2 + 2a^2(a^2+\sigma)\right]D^2\tilde{w} - a^2(a^2+\sigma)^2\tilde{w}$$
$$= a^2 Ra(\frac{d\Theta}{dz})\tilde{w}. \tag{8}$$

Multiplying $\tilde{w}^*$ to (8) and integrating by parts, we obtain

$$(J_6 - I_6) - (3a^2+2\sigma)(J_4+I_4) + \left[(a^2+\sigma)^2 + 2a^2(a^2+\sigma)\right](J_2 - I_2) - a^2(a^2+\sigma)^2 I_0$$
$$= a^2 Ra\int_0^1 (\frac{d\Theta}{dz})|\tilde{w}|^2 dz. \tag{9}$$

where I_n ($n =$0, 2, 4 and 6) are positive integral, defined as

$$I_n = \int_0^1 \left|D^{n/2}\tilde{w}\right|^2 dz > 0,$$

and the boundary conditions (7) imply that the contributions from the end points all vanish, i.e.

$$J_6 = \left[\tilde{w}^* D^5\tilde{w} - D\tilde{w}^* D^4\tilde{w} + D^2\tilde{w}^* D^3\tilde{w}\right]_0^1 = 0,$$
$$J_4 = \left[\tilde{w}^* D^3\tilde{w} - D\tilde{w}^* D^2\tilde{w}\right]_0^1 = 0, \quad J_2 = \tilde{w}^* D\tilde{w}\Big|_0^1 = 0.$$

The imaginary part of(9) gives

$$\sigma_i\left\{2I_4 + \left[2(a^2+\sigma_r) + 2a^2\right]I_2 + 2a^2(a^2+\sigma_r)I_0\right\} = 0. \tag{10}$$

When $\sigma_r = 0$ (neutral mode), identity (10) implies $\sigma_i = 0$ since $I_n > 0$ ($n = 0$, 2 and 4). This proves the principle of exchange of stabilities: eigenvalues cross the imaginary axis through the real axis. (Note further that relation (10) also indicates that $\sigma_i = 0$ for $\sigma_r > 0$, that is, amplifying modes must be non-oscillatory.) 5, D

2. (a) Since it is the variation of U in the radial direction that causes the instability, while the first term in equation (1a) in the exam question describes the temporal growth of the disturbance (Taylor vortices), the dominant balance in (1a) is between the first and second terms on the left-hand side, and so

seen ⇓

$$\frac{\partial \hat{u}'}{\partial \hat{t}} \sim \frac{d\hat{U}}{d\hat{r}}\hat{v}'. \tag{11}$$

The centrifugal force, which makes fluid particles migrate in the radial direction, is represented by the second term on the left-hand side of equation (1b) in the exam question, while the first term in (1b) describes the growth with time. This suggests that the first term and the centrifugal force must balance:

$$\frac{\partial \hat{v}'}{\partial \hat{t}} \sim 2\frac{\hat{U}}{\hat{r}}\hat{u}'. \tag{12}$$

When $\chi \neq 1$, we have the estimates

$$\frac{d\hat{U}}{d\hat{r}} \sim \Omega_1 R_1/h, \qquad \frac{\hat{U}}{\hat{r}} \sim \Omega_1. \tag{13}$$

Use of these in (11) and (12) leads to

$$\frac{\hat{u}'}{\Delta\hat{t}} \sim \frac{\Omega_1 R_1}{h}\hat{v}', \qquad \frac{\hat{v}'}{\Delta\hat{t}} \sim \Omega_1\hat{u}', \tag{14}$$

where $\Delta\hat{t}$ stands for the time scale to be identified. From the two relations in (14) we find that

$$\Delta\hat{t} \sim \Omega_1^{-1}\sqrt{h/R_1} = \varepsilon^{1/2}\Omega_1^{-1}, \qquad \hat{v}' \sim \sqrt{h/R_1}\,\hat{u}' = \varepsilon^{1/2}\hat{u}'. \tag{15}$$

In equation (1c) of the exam question, the pressure gradient $\rho^{-1}\partial\hat{p}'/\partial\hat{z}$ must balance the inertia $\partial\hat{w}'/\partial\hat{t}$, i.e.

$$\frac{\partial \hat{w}'}{\partial \hat{t}} \sim \frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{z}}, \tag{16}$$

because otherwise the equation is an advection-diffusion equation for a passive scalar, and w' must be zero, or would relax to zero eventually.

In equation (1b) of the exam question, the radial pressure gradient must be comparable with the left-hand side,

$$2\frac{\hat{U}}{\hat{r}}\hat{u}' \sim \frac{1}{\rho}\frac{\partial \hat{p}'}{\partial \hat{r}}, \tag{17}$$

because otherwise $\hat{u}'$ and $\hat{v}'$ would be decoupled from the other two equations and the resulting degenerated system admits no appropriate solutions.

Finally, in the continuity equation (1d) of the exam question, the second term is smaller than the first and so the principal balance is

$$\frac{\partial \hat{v}'}{\partial \hat{r}} \sim \frac{\partial \hat{w}'}{\partial \hat{z}}. \tag{18}$$

The balances, (16), (17) and (18), give the estimates

$$\frac{\hat{w}'}{\Delta\hat{t}} \sim \frac{1}{\rho}\frac{\hat{p}'}{\Delta\hat{z}}, \qquad \Omega_1\hat{u}' \sim \frac{1}{\rho}\frac{\hat{p}'}{h}, \qquad \frac{\hat{v}'}{h} \sim \frac{\hat{w}'}{\Delta\hat{z}}. \tag{19}$$

From these relations and (15), we find that

$$\hat{p}' \sim \rho\Omega_1 h\hat{u}' = \varepsilon(\rho\Omega_1 R_1)\hat{u}', \quad \Delta\hat{z} \sim h, \quad \hat{w}' \sim (\Delta z/h)\hat{v}' = \varepsilon^{1/2}\hat{u}'. \tag{20}$$

5, A

(b) (i) For $\chi = 1$, the dominant balances (11)–(12) and (16)–(18) remain. The difference is that

unseen ⇓

$$\frac{d\hat{U}}{d\hat{r}} \sim \Omega_1, \quad \frac{\hat{U}}{\hat{r}} \sim \Omega_1. \tag{21}$$

Use of these in (11)–(12) gives $\dfrac{\hat{u}'}{\Delta\hat{t}} \sim \Omega_1 \hat{v}'$ and $\dfrac{\hat{v}'}{\Delta\hat{t}} \sim \Omega_1 \hat{u}'$, from which we found that

$$\Delta\hat{t} \sim \Omega_1^{-1}, \qquad \hat{v}' \sim \hat{u}'. \tag{22}$$

Inserting these into (19), which hold since (16)–(18) do, we find that

$$\hat{p}' \sim \rho\Omega_1 h\hat{u}' = \varepsilon(\rho\Omega_1 R_1)\hat{u}', \quad \Delta\hat{z} \sim h, \quad \hat{w}' \sim \hat{u}'.$$

3, A

(ii) Substitute the normalized variables into equations (1a)-(1d):

unseen ⇓

$$\frac{\Omega_1 R_1}{\Omega_1^{-1}}\Big[\frac{\partial u'}{\partial t} + 2v'\Big] = \frac{\nu\Omega_1 R_1}{h^2}\Big[\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} + \frac{h}{R_1 + hy}\frac{\partial u'}{\partial y} - \frac{h^2 u'}{(R_1 + hy)^2}\Big],$$

$$\frac{\Omega_1 R_1}{\Omega_1^{-1}}\Big[\frac{\partial v'}{\partial t} - 2u'\Big] = -\frac{\varepsilon\Omega_1^2 R_1^2}{h}\frac{\partial p'}{\partial y} + \frac{\nu\Omega_1 R_1}{h^2}\Big[\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} + \frac{h}{R_1 + hy}\frac{\partial v'}{\partial y} - \frac{h^2 v'}{(R_1 + hy)^2}\Big],$$

$$\frac{\Omega_1 R_1}{\Omega_1^{-1}}\frac{\partial w'}{\partial t} = -\frac{\varepsilon\Omega_1^2 R_1^2}{h}\frac{\partial p'}{\partial z} + \frac{\nu\Omega_1 R_1}{h^2}\Big(\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\Big),$$

$$\frac{\Omega_1 R_1}{h}\Big[\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} + \frac{h}{R_1 + hy}v'\Big] = 0.$$

Here we note in particular that $(d\hat{U}/d\hat{r})\hat{v}'$ and $(\hat{U}/\hat{r})\hat{v}'$ in the azimuthal momentum equation are now of the same order (actually identical). Retaining only the leading-order terms, we arrive at the required non-dimensional equations (3a)-(3d) in the question with the dimensionless parameter

$$Re = \Omega_1 h^2/\nu.$$

4, B

(iii) For the solution of the normal-mode form, the differential operators acting on the perturbation obey, to leading order accuracy, the following relations

unseen ⇓

$$\partial/\partial t \to \sigma, \quad \partial/\partial z \to i\beta, \quad \partial^2/\partial z^2 \to -\beta^2. \tag{23}$$

Substituting the local normal mode into equations (3a)-(3d) in the question, and using (23), we obtain the set of equations,

$$\sigma\bar{u} + 2\bar{v} = \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \beta^2\Big)\bar{u}, \tag{24a}$$

$$\sigma\bar{v} - 2\bar{u} = -\frac{d\bar{p}}{dy} + \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \beta^2\Big)\bar{v}, \tag{24b}$$

$$\sigma\bar{w} = -i\beta\bar{p} + \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \beta^2\Big)\bar{w}, \tag{24c}$$

$$\frac{d\bar{v}}{dy} + i\beta\bar{w} = 0. \tag{24d}$$

Eliminating $\bar{w}$ between (24c) and (24d) gives

$$\sigma\frac{d\bar{v}}{dy} = -\beta^2\bar{p} + \frac{1}{Re}\Big(\frac{d^2}{dy^2} - \beta^2\Big)\frac{d\bar{v}}{dy}.$$

Differentiating this equation with respect to y, and eliminating $d\bar{p}/dy$ between the resulting equation and (24b), we obtain

$$\sigma\big(\frac{d^2}{dy^2}-\beta^2\big)\bar{v}+2\beta^2\bar{u}=\frac{1}{Re}\big(\frac{d^2}{dy^2}-\beta^2\big)^2\bar{v}. \tag{25}$$

Equations (24a) and (25) form the required coupled system for $\bar{u}$ and $\bar{v}$. The boundary conditions are

$$\bar{u}=\bar{v}=\frac{d\bar{v}}{dy}=0 \quad \text{at } y=0,\,1,$$

where the last follows from $\bar{w}=0$ and the continuity equation (24d). $\boxed{4,\ \mathrm{C}}$

(iv) Multiplying $\bar{u}^*$ to (24a) and integrating by parts, we obtain $\boxed{\text{unseen} \Downarrow}$

$$\sigma I_u+2I_{uv}=-\frac{1}{Re}J_u, \tag{26}$$

where

$$I_u=\int_0^1|\bar{u}|^2dy>0,\;\; I_{uv}=\int_0^1\bar{u}^*\bar{v}dy,\;\; J_u=\int_0^1\Big[\big|\frac{d\bar{u}}{dy}\big|^2+\beta^2|\bar{u}|^2\Big]dy>0.$$

Similarly, multiplying $\bar{v}^*$ to (25) and integrating by parts leads to

$$-\sigma J_v+2\beta^2 I_{uv}^*=\frac{1}{Re}K_v, \tag{27}$$

where $J_v>0$ has the same expression as J_u provided that $\bar{u}$ is replaced by $\bar{v}$, and

$$K_v=\int_0^1\Big[\big|\frac{d^2\bar{v}}{d^2y}\big|^2+2\beta^2\big|\frac{d\bar{v}}{dy}\big|^2+\beta^4|\bar{v}|^2\Big]dy>0.$$

Elimination of I_{uv} between (26) and (27) gives

$$\sigma\beta^2 I_u+\sigma^* J_v=-(\beta^2 J_u+K_v)/Re,$$

the real part of which reads

$$\sigma_r(\beta^2 I_u+J_v)=-(\beta^2 J_u+K_v)/Re<0.$$

Hence $\sigma_r<0$. $\boxed{4,\ \mathrm{D}}$

3. (i) Substituting the perturbed flow field, $(V_x, V_r, V_\phi) = \big(U(r) + u', v', 0\big)$ and $p = P + p'$, into the (given) Euler equations and linearising, we obtain

sim. seen ⇓

$$\begin{aligned}\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial r} + \frac{v'}{r} &= 0, &(28a)\\ \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{dU}{dr}v' &= -\frac{\partial p'}{\partial x}, &(28b)\\ \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} &= -\frac{\partial p'}{\partial r}. &(28c)\end{aligned}$$

Inserting the normal-mode solution into (28a)-(28c), and noting that

$$\frac{\partial}{\partial t} \to -i\omega, \qquad \frac{\partial}{\partial x} \to i\alpha,$$

we obtain the required equations:

$$\begin{aligned}i\alpha\bar{u} + \frac{d\bar{v}}{dr} + \frac{\bar{v}}{r} &= 0, &(29a)\\ i\alpha(U-c)\bar{u} + U'(r)\bar{v} &= -i\alpha\bar{p}, &(29b)\\ i\alpha(U-c)\bar{v} &= -\frac{d\bar{p}}{dr}. &(29c)\end{aligned}$$

4, A

(ii) From (29c), we find that

sim. seen ⇓

$$\bar{v} = i\alpha^{-1}\frac{\frac{d\bar{p}}{dr}}{U-c}.$$

Inserting $\bar{v}$ into (29b) gives

$$i\alpha\bar{u} = -\frac{i\alpha\bar{p}}{U-c} - i\alpha^{-1}\frac{U'\frac{d\bar{p}}{dr}}{(U-c)^2}.$$

Substitute this along with $\bar{v}$ into (29a):

$$-\frac{i\alpha\bar{p}}{U-c} - i\alpha^{-1}\frac{U'\frac{d\bar{p}}{dr}}{(U-c)^2} + i\alpha^{-1}\frac{d}{dr}\left[\frac{\frac{d\bar{p}}{dr}}{(U-c)}\right] + \frac{i\alpha^{-1}}{r}\frac{\frac{d\bar{p}}{dr}}{U-c} = 0,$$

which is arranged into the required equation

$$\frac{d^2\bar{p}}{dr^2} - \Big(\frac{2U'}{U-c} - \frac{1}{r}\Big)\frac{d\bar{p}}{dr} - \alpha^2\bar{p} = 0. \qquad (30)$$

3, A

The equation for $\bar{p}$ may be rewritten as

$$\frac{(U-c)^2}{r}\frac{d}{dr}\left[\frac{r}{(U-c)^2}\frac{d\bar{p}}{dr}\right] - \alpha^2\bar{p} = 0.$$

Multiply both sides of the equation by $r\bar{p}^*/(U-c)^2$ and integrate by parts:

$$\bar{p}^*\left[\frac{r}{(U-c)^2}\frac{d\bar{p}}{dr}\right]_{r_1}^{r_2} - \int_{r_1}^{r_2}\frac{r}{(U-c)^2}\Big|\frac{d\bar{p}}{dr}\Big|^2 dr - \int_{r_1}^{r_2}\frac{\alpha^2 r|\bar{p}|^2}{(U-c)^2} = 0.$$

The impermeability boundary conditions, $\bar{v} = 0$ at $r = r_1$ and r_2, correspond to $d\bar{p}/dr = 0$ at $r = r_1$ and r_2 in view of (29c). It follows that

$$\int_{r_1}^{r_2}\frac{Q(r;\alpha)}{(U-c)^2}dr = 0, \quad \text{where} \quad Q(r;\alpha) \equiv r\Big|\frac{d\bar{p}}{dr}\Big|^2 + \alpha^2 r|\bar{p}|^2. \qquad (31)$$

Noting that $(U-c)^{-2} = \left[(U-c_r)^2 - c_i^2 + 2c_i(U-c_r)i\right]/|U-c|^2$ and splitting the integrand into the real and imaginary parts, the imaginary part of equation (31) can be written as

$$2c_i \int_{r_1}^{r_2} \tilde{Q}(r)(U-c_r)dr = 0, \tag{32}$$

where we have put

$$\tilde{Q}(r) = Q(r;\alpha)/|U-c|^2 \geq 0.$$

Since $c_i > 0$, the relation (32) indicates that $(U-c_r)$ must change its sign in the domain, i.e. c_r must lie in the range of $U(r)$,

$$U_{\min} < c_r < U_{\max}. \tag{33}$$

$\boxed{5, \mathrm{B}}$

$\boxed{\text{sim. seen} \Downarrow}$

(iii) Eliminating $\bar{u}$ between (29a) and (29b), we obtain

$$-(U-c)\left[\frac{d\bar{v}}{dr} + \frac{\bar{v}}{r}\right] + U'\bar{v} = -i\alpha\bar{p}. \tag{34}$$

On noting that $\frac{d\bar{v}}{dr} + \frac{\bar{v}}{r} = \frac{1}{r}\frac{d(r\bar{v})}{dr}$, the above equation is arranged into

$$\frac{(U-c)}{r}\frac{d(r\bar{v})}{dr} - U'\bar{v} = i\alpha\bar{p}.$$

Differentiating with respect to r and using equation (29c), we obtain

$$\frac{d}{dr}\left[\frac{(U-c)}{r}\frac{d(r\bar{v})}{dr}\right] - \frac{d}{dr}\left[\frac{U'}{r}(r\bar{v})\right] = \alpha^2(U-c)\bar{v},$$

which simplifies to

$$(U-c)\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\bar{v})}{dr}\right] - \left[\frac{d}{dr}\left(\frac{U'}{r}\right)\right](r\bar{v}) = \alpha^2(U-c)\bar{v},$$

and further to the required equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\bar{v})}{dr}\right] - \left[\frac{d}{dr}\left(\frac{U'}{r}\right)\right]\frac{r\bar{v}}{U-c} - \alpha^2\bar{v} = 0. \tag{35}$$

[This equation may alternatively be obtained from that of $\bar{p}$.] $\boxed{6, \mathrm{C}}$

Multiply $r\bar{v}^*$ to both sides of (35) and integrate with respect to r from r_1 and r_2:

$$\int_{r_1}^{r_2}(r\bar{v}^*)\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\bar{v})}{dr}\right]dr - \int_{r_1}^{r_2}\alpha^2 r|\bar{v}|^2 dr - \int_{r_1}^{r_2}\left[\frac{d}{dr}\left(\frac{U'}{r}\right)\right]\frac{r^2|\bar{v}|^2}{U-c}dr = 0.$$

Performing integration by parts in the first integral and using the boundary conditions, we have

$$\int_{r_1}^{r_2}\frac{1}{r}\left|\frac{d(r\bar{v})}{dr}\right|^2 dr - \int_{r_1}^{r_2}\alpha^2 r|\bar{v}|^2 dr - \int_{r_1}^{r_2}\left[\frac{d}{dr}\left(\frac{U'}{r}\right)\right]\frac{r^2|\bar{v}|^2}{U-c}dr = 0.$$

The imaginary part of the above equation reads

$$c_i\int_{r_1}^{r_2}\left[\frac{d}{dr}\left(\frac{U'}{r}\right)\right]\frac{r^2|\bar{v}|^2}{|U-c|^2}dr = 0,$$

from which follows the required necessary condition for inviscid instability. $\boxed{2, \mathrm{D}}$

4. (a) (i) Due to the symmetry of the flow and the fact that the profile is smooth at $y = 0$, we solve the Rayleigh equation in the upper half of the channel: $0 \le y \le 1$. Since $U'' = 0$, the Rayleigh equation reduces to $\dfrac{d^2\bar{v}}{dy^2} - \alpha^2\bar{v} = 0$, which is solved to give the solution in different regions: sim. seen ⇓

$$\bar{v} = \begin{cases} C_1^+ e^{-\alpha y} + C_2^+ e^{\alpha y} & 1-h < y < 1, \\ C_1^- e^{-\alpha y} + C_2^- e^{\alpha y} & 0 < y < 1-h. \end{cases} \tag{36}$$

Now we apply the jump conditions across the discontinuity $y_d = 1-h$,

$$\Big[(U-c)\bar{v}' - U'\bar{v}\Big]_{y_d^-}^{y_d^+} = 0, \qquad \Big[\frac{\bar{v}}{U-c}\Big]_{y_d^-}^{y_d^+} = 0. \tag{37}$$

On noticing that $U = 1$, $U' = 0$ at $y = (1-h)^-$ but $U' = -1/h$ at $y = (1-h)^+$, these conditions imply that

$$\alpha(1-c)\big[-C_1^- e^{-\alpha(1-h)} + C_2^- e^{\alpha(1-h)}\big]$$
$$= \Big[-\alpha(1-c) + \frac{1}{h}\Big]C_1^+ e^{-\alpha(1-h)} + \Big[\alpha(1-c) + \frac{1}{h}\Big]C_2^+ e^{\alpha(1-h)}, \tag{38}$$
$$C_1^- e^{-\alpha(1-h)} + C_2^- e^{\alpha(1-h)} = C_1^+ e^{-\alpha(1-h)} + C_2^+ e^{\alpha(1-h)}. \tag{39}$$

For the symmetric mode, the condition at the centre of the channel is: $\bar{v}'(0) = 0$, which gives

$$-\alpha C_1^- + \alpha C_2^- = 0. \tag{40}$$

Impose the boundary condition at the upper plate,

$$C_1^+ e^{-\alpha} + C_2^+ e^{\alpha} = 0. \tag{41}$$

Use of (40) to the left-hand sides, and (41) to the right-hand sides of (38) and (39), respectively, leads to

$$2\alpha(1-c)\sinh[\alpha(1-h)]C_1^- = \big[-2\alpha(1-c)\cosh(\alpha h) + \frac{2}{h}\sinh(\alpha h)\big]e^{-\alpha}C_1^+,$$
$$2\cosh[\alpha(1-h)]C_1^- = 2\sinh(\alpha h)e^{-\alpha}C_1^+.$$

Eliminating $C_1^{\pm}$ by taking the ratio of the above two equations, we obtain the dispersion relation

$$\alpha(1-c)\tanh[\alpha(1-h)] = -\alpha(1-c)\coth(\alpha h) + \frac{1}{h}. \tag{42}$$

5, A

(ii) From (42) we find that unseen ⇓

$$c = 1 - \frac{1/(\alpha h)}{\tanh[\alpha(1-h)] + \coth(\alpha h)}.$$

Since c is real, the flow is inviscidly stable.

When $\alpha \ll 1$, we have $\tanh[\alpha(1-h)] = \alpha(1-h)\big[1 - \frac{1}{3}\alpha^2(1-h)^2 + \ldots\big]$, and

$$\alpha h\coth(\alpha h) = \alpha h/\tanh(\alpha h) = \big[1 + \frac{1}{3}\alpha^2 h^2 + \ldots\big].$$

It follows that

$$c \approx 1 - \frac{1}{\alpha^2 h(1-h) + o(\alpha^2) + \big[1 + \alpha^2h^2/3 + o(\alpha^2)\big]} \approx (h - \frac{2}{3}h^2)\alpha^2 = c_0\alpha^2,$$

with $c_0 = h - \frac{2}{3}h^2$. 3, A

(iii) For $1-h \le y \le 1$, on noting $C_1^+ = -e^{2\alpha} C_2^+$, we have

$$\bar{v} = (-e^{2\alpha-\alpha y} + e^{\alpha y})C_2^+.$$

unseen $\Downarrow$

From the continuity equation, the streamwise velocity is found as

$$\bar{u} = i(e^{\alpha y} + e^{2\alpha-\alpha y})C_2^+.$$

The pressure follows from the streamwise momentum equation,

$$\bar{p} = -(U-c)\bar{u} + i\alpha^{-1}U'\bar{v}.$$

We have

$$\bar{u}(1) = 2ie^{\alpha}C_2^+, \quad \bar{p}_w = \bar{p}(1) = c\bar{u}(1) = 2ic\, e^{\alpha}C_2^+,$$

since $U=0$ and $\bar{v}=0$ at $y=1$. Clearly, $\bar{u}(1) \ne 0$, i.e. the no-slip condition is not satisfied and so a viscous (Stokes) layer is required.

3, B

unseen $\Downarrow$

(b) (i) In the viscous layer, the viscous diffusion balances the unsteadiness in the streamwise momentum equation

$$i\alpha(U-c)\bar{u} + \frac{dU}{dy}\bar{v} = -i\alpha\bar{p} + \frac{1}{Re}(\frac{d^2}{dy^2} - \alpha^2)\bar{u}.$$

Let the thickness, $y-1$, be of $O(\ell_s)$, which is now deduced. The viscous diffusion term and unsteadiness are

$$\frac{1}{Re}\frac{d^2\bar{u}}{dy^2} = O(\frac{\bar{u}}{Re\,\ell_s^2}), \quad -i(\alpha c)\bar{u} = O(\omega\bar{u}).$$

The balance of the two, $\dfrac{\bar{u}}{Re\,\ell_s^2} = O(\omega\bar{u})$, gives $\ell_s = O\big((\omega Re)^{-1/2}\big)$.

In the Stokes layer, $\bar{u}$ is expected to be of $O(C_2^+)$, taken to be $O(1)$. The balance in the continuity equation, $i\alpha\bar{u} = O(\bar{v}/\ell_s)$, indicates that

$$\bar{v} = O\big(\alpha(\omega Re)^{-1/2}\big).$$

Introduce $\tilde{Y} = (y-1)/(\omega Re)^{-1/2}$, which implies that $\partial/\partial y = (\omega Re)^{1/2}\partial/\partial\tilde{Y}$. With the expansion, the continuity equation becomes

$$i\tilde{U} + \frac{d\tilde{V}}{d\tilde{Y}} = 0.$$

Note that two more terms in the streamwise momemtum equation, $\alpha U\bar{u}$ and $U'\bar{v}$, are both of $O\big(\alpha(\omega Re)^{-1/2}\big)$, much smaller than $\omega\bar{u}$ provided that

$$\alpha(\omega Re)^{-1/2} \ll \omega. \tag{43}$$

The pressure gradient term $-i\alpha p = -i\alpha\bar{p}_w = -i\alpha c\bar{u}(1) = O(\omega)$. Therefore the streamwise momentum equation reduces to

$$-i\omega\tilde{U} = -i\alpha\bar{p}_w + \omega\frac{d^2\tilde{U}}{d\tilde{Y}^2}.$$

The boundary and matching conditions are:

$$\tilde{U} = \tilde{V} = 0 \quad \text{at} \quad \tilde{Y}=0; \quad \tilde{U} \to \bar{u}(1) \quad \text{as} \quad \tilde{Y} \to -\infty.$$

4, B

(ii) Inserting into the continuity equation the given solution for $\tilde{U}$,

unseen $\Downarrow$

$$\tilde{U} = (\alpha/\omega)\bar{p}_w\big[1 - \exp\{(-i)^{1/2}\tilde{Y}\}\big],$$

(which satisfies the matching condition since $\bar{p}_w = c\bar{u}(1)$), and integrating, we obtain

$$\tilde{V} = -i(\alpha/\omega)\bar{p}_w\Big[\tilde{Y} - \int_0^{\tilde{Y}} \exp\{(-i)^{1/2}\tilde{Y}\}d\tilde{Y}\Big] \to -i(\alpha/\omega)\bar{p}_w\tilde{Y} + (\alpha/\omega)\bar{p}_w(-i)^{1/2}$$

as $\tilde{Y} \to -\infty$. Thus $\tilde{V}_\infty = (\alpha/\omega)\bar{p}_w(-i)^{1/2}$, which represents the displacement effect induced by the viscous motion. The unrescaled 'transpiration velocity' is $\alpha(\omega Re)^{-1/2}\tilde{V}_\infty = (\alpha^2/\omega^{3/2})e^{-\pi i/4}\bar{p}_w Re^{-1/2}$, and it acts at the 'top' of the main layer adjacent to the upper wall. Its impact on the inviscid solution can be accounted for by replacing the impermeability boundary condition $\bar{v}(1) = 0$ by

$$\bar{v}(1) = (\alpha^2/\omega^{3/2})e^{-\pi i/4}Re^{-1/2}\bar{p}(1),$$

where the coefficient of $\bar{p}(1)$ may be interpreted as an 'impedance'. This produces a small $O(Re^{-1/2})$ correction (actually growth rate) when $\alpha = O(1)$. 3, D

(iii) When $\alpha \ll 1$, $c = O(\alpha^2)$ as was noted in Part a(ii), and so $\omega = O(\alpha^3)$. The two terms in the streamwise momentum equation, $\alpha U\bar{u}$ and $U'\bar{v}$, can no longer be neglected when

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$$\alpha(\omega Re)^{-1/2} = O(\omega), \quad \text{i.e. } \alpha(\alpha^3 Re)^{-1/2} = O(\alpha^3).$$

Thus a new regime emerges when $\alpha = O(Re^{-1/7})$, which is the well-known asymptotic scaling for the lower-branch instability of the channel flow. (The 'impedance' coefficient increases to $O(Re^{-1/7})$.) 2, D

5. (i) Substituting the expression for the perturbed flow into the Navier-Stokes equations, and neglecting all nonlinear terms, we obtain the linearized equations for the perturbation,

sim. seen $\Downarrow$

$$\frac{\partial u'}{\partial x} + Re^{1/2}\frac{\partial v'}{\partial Y} + \frac{\partial w'}{\partial z} = 0, \tag{44}$$

$$U\frac{\partial u'}{\partial x} + \underline{\frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial U}{\partial Y}v' + W\frac{\partial u'}{\partial z} = -\frac{\partial p'}{\partial x} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]u', \tag{45}$$

$$U\frac{\partial v'}{\partial x} + \underline{Re^{-\frac{1}{2}}\frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial Y} + \frac{\partial V}{\partial Y}v'} + W\frac{\partial v'}{\partial z} = -Re^{\frac{1}{2}}\frac{\partial p'}{\partial Y} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]v', \tag{46}$$

$$U\frac{\partial w'}{\partial x} + \underline{\frac{\partial W}{\partial x}u' + V\frac{\partial w'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial W}{\partial Y}v' + W\frac{\partial w'}{\partial z} = -\frac{\partial p'}{\partial z} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]w'. \tag{47}$$

The underlined terms represent the non-parallel-flow effects, caused by the streamwise variation of U and W and by the presence of a normal velocity $Re^{-1/2}V$, which is associated with the streamwise variation of U.

Parallel-flow approximation neglects the underlined terms in (45)-(47), and treats the variation of U and W with x as being parametric; the latter means that at each location, the profiles are 'frozen' so that normal-mode solutions may be sought. The viscous terms are retained (leading to the Orr-Sommerfeld type of equations). 5, M

(ii) Substituting the normal-mode form into (44)-(47), and noting that the operators

unseen $\Downarrow$

$$\frac{\partial}{\partial x} \to iRe^{3/8}\alpha, \quad \frac{\partial}{\partial z} \to iRe^{3/8}\beta, \tag{48}$$

when acting on the perturbation of the assumed form, we obtain the equations

$$i\alpha\bar{u} + \frac{\partial \bar{v}}{\partial Y} + i\beta\bar{w} = 0, \quad i\alpha U\bar{u} + \frac{\partial U}{\partial Y}\bar{v} + i\beta W\bar{u} = 0, \quad i\alpha U\bar{w} + \frac{\partial W}{\partial Y}\bar{v} + i\beta W\bar{w} = 0, \tag{49}$$

and $0 = -\frac{\partial \bar{p}}{\partial Y}$. It follows that $\bar{p} = P$ (constant). The expressions for $\bar{u}$, $\bar{v}$ and $\bar{w}$ given in the question satisfy (49) with $\bar{v}$ satisfying the required boundary condition $\bar{v} = 0$ at $Y = 0$. [Alternatively, multiplying $i\alpha$ and $i\beta$ to the second and third equations in (49), taking the sum and using the first equation in (49), one obtains

$$-i(\alpha U + \beta W)\frac{\partial \bar{v}}{\partial Y} + i\frac{\partial}{\partial Y}(\alpha U + \beta W)\bar{v} = 0.$$

This is a first-order ordinary differential equation for $\bar{v}$ and has the solution as given, substitution of which into the second and third equations in (49) gives the solution for $\bar{u}$ and $\bar{w}$.]

Note that $\bar{v} \to -i\alpha A \neq 0$ as $Y \to \infty$, and so an upper layer is required, where the perturbation attenuates to zero in the far field.

On the other hand,

$$\bar{u} \to \lambda_1 A, \quad \bar{v} \to -iA(\alpha\lambda_1 + \beta\lambda_3)Y, \quad \bar{w} \to \lambda_3 A \quad \text{as} \quad Y \to 0. \tag{50}$$

Because $\bar{u}$ and $\bar{w}$ do not satisfy the required no-slip condition, a viscous wall layer (lower deck) is required. 5, M

(iii) Let the width of the upper layer be $y = O(\ell)$ equivalent to $Y = O(Re^{1/2}\ell)$. Matching with the main-layer solution suggests that $v' = O(Re^{-1/8})$. We now deduce ℓ along with the orders of magnitude of u', w' and p'. Note that the background flow outside the boundary layer is $(U, W) = (1, 0)$, and so the momentum equations reduce to **sim. seen ⇓**

$$\frac{\partial u'}{\partial x} = -\frac{\partial p'}{\partial x}, \quad \frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y}, \quad \frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z},$$

which give

$$Re^{3/8}u' \sim Re^{3/8}p', \quad Re^{3/8}v' \sim p'/\ell, \quad Re^{3/8}w' \sim Re^{3/8}p',$$

that is,

$$u' \sim w' \sim p' \sim Re^{3/8}\ell\, v', \tag{51}$$

The continuity equation, $\dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} + \dfrac{\partial w'}{\partial z}$, implies $Re^{3/8}u' \sim v'/\ell \sim Re^{3/8}w'$, which is combined with (51) to give $Re^{3/4}\ell v' \sim v'/\ell$, and hence

$$\ell = O(Re^{-3/8}), \quad u' \sim w' \sim p' \sim v' = O(Re^{-1/8}),$$

which explains the upper-layer expansion given in the question.
Inserting the expansion and $\bar{y} = Re^{1/8}y$ into the perturbation equations, we obtain

$$i\alpha u^\dagger + \frac{\partial v^\dagger}{\partial \bar{y}} + i\beta w^\dagger = 0, \quad i\alpha u^\dagger = -i\alpha p^\dagger, \quad i\alpha v^\dagger = -\frac{\partial p^\dagger}{\partial \bar{y}}, \quad i\alpha w^\dagger = -i\beta p^\dagger.$$

From these we find the equation for $p^\dagger$: $\dfrac{\partial^2 p^\dagger}{\partial \bar{y}^2} - (\alpha^2 + \beta^2)p^\dagger = 0$. The normal momentum equation and matching ($v^\dagger \to -i\alpha A$ as $\bar{y} \to 0$) imply that $\left.\dfrac{\partial p^\dagger}{\partial \bar{y}}\right|_{\bar{y}=0} = -\alpha^2 A$. The solution for $p^\dagger$, which attenuates as $\bar{y} \to \infty$, is found as

$$p^\dagger = \alpha^2(\alpha^2 + \beta^2)^{-1/2}Ae^{-\sqrt{\alpha^2+\beta^2}\,\bar{y}}.$$

Matching the pressure in the main layer gives the required pressure-displacement relation,

$$P = \alpha^2(\alpha^2 + \beta^2)^{-1/2}A. \tag{52}$$

5, M

(iii) Let $Y = O(d)$ with $d \ll 1$ in the viscous wall layer (lower deck), where $U = \lambda_1 Y = O(d)$ and $W = \lambda_3 Y = O(d)$. It follows that the inertia term $U\dfrac{\partial u'}{\partial x} \sim O(dRe^{3/8}u')$, while the viscous diffusion $\dfrac{\partial^2 u'}{\partial Y^2} \sim O(u'/d^2)$. The balance between the two, **sim. seen ⇓**

$$dRe^{3/8}u' \sim u'/d^2,$$

suggests that $d = O(Re^{-1/8})$, which corresponds to $y = Re^{-1/2}Y = O(Re^{-5/8})$.

The asymptotic behaviour of the main-deck solution, (50), suggests that in the lower deck $u' \sim w' = O(1)$ as in the main layer, but $v' = O(Re^{-1/8}d) = O(Re^{-1/4})$ as can be deduced by the matching principle. Similarly, $p' = O(Re^{-1/8})$. Therefore, in the lower deck, the solution should expand as

$$(u', v', w', p') = (\tilde{u},\, Re^{-\frac{1}{4}}\tilde{v},\, \tilde{w},\, Re^{-\frac{1}{8}}\tilde{p})E + c.c.$$

unseen $\Downarrow$

Substituting this into (44)-(47) and using the fact that $U = Re^{-1/8}(\lambda_1 \tilde{y})$, $W = Re^{-1/8}(\lambda_3 \tilde{y})$ as well as the relations in (48), we obtain

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\tilde{y}} + i\beta\tilde{w} = 0, \tag{53}$$

$$i\alpha\lambda_1\tilde{y}\tilde{u} + \lambda_1\tilde{v} + i\beta\lambda_3\tilde{y}\tilde{u} = -i\alpha\tilde{p} + \frac{d^2\tilde{u}}{d\tilde{y}^2}, \tag{54}$$

$$i\alpha\lambda_1\tilde{y}\tilde{w} + \lambda_3\tilde{v} + i\beta\lambda_3\tilde{y}\tilde{w} = -i\beta\tilde{p} + \frac{d^2\tilde{w}}{d\tilde{y}^2}, \tag{55}$$

plus $d\tilde{p}/d\tilde{y} = 0$ so that $\tilde{p}$ is a constant, and $\tilde{p} = P$ by matching with the main-deck pressure.

The no-slip and impermeability conditions at the wall:

$$\tilde{u} = 0, \quad \tilde{w} = 0, \quad \tilde{v} = 0 \quad \text{at } \tilde{y} = 0.$$

Matching with main-layer streamwise and spanwise velocities requires that

$$\tilde{u} \to \lambda_1 A, \quad \tilde{w} \to \lambda_3 A \quad \text{as } \tilde{y} \to \infty.$$

5, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70052	1	Students did this rather well. The last part involves much algebra, and so students chose to describe the idea without working out all the algebraic detail.
MATH70052	2	Students had a decent tackle of this problem. Part b(ii) could be long if all terms in the equations are written out explicitly. In fact, it suffices only to retain the leading order terms. Some students skipped this part.
MATH70052	3	This question was quite well done.
MATH70052	4	Students did not do as well as expected.
MATH70052	5	Most students did rather poorly. It was a long paper, and students appeared to struggle with time. The overall performance by students was all right. Given this is a long paper, the E and M will be set accordingly (to ensure a fair comparison with other modules).