Imperial College
London

MATH70045

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Applied Probability

Date: 5 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

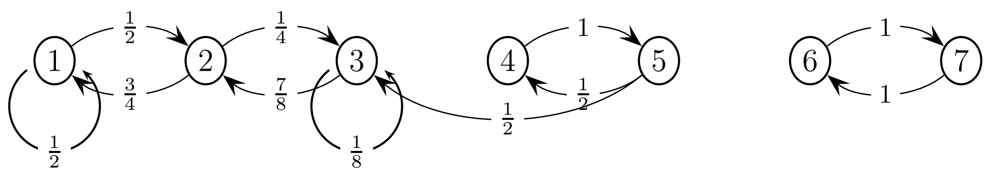
Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.

1. (a) Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$ on the state space $E = \{1, 2, 3, 4, 5, 6, 7\}$. We denote the (one-step) transition matrix by $\mathbf{P} = (p_{ij})_{i,j \in E}$. The corresponding transition diagram is given below.



(i) Find the transition matrix $\mathbf{P}$ of this Markov chain. (2 marks)

(ii) Specify the communicating classes and, for each class, determine whether it is transient, null recurrent or positive recurrent. (4 marks)

(iii) Determine the period of each state. (3 marks)

(iv) Find all possible stationary distributions. (3 marks)

(b) Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$ on the state space $E = \mathbb{Z}$ with one-step transition probabilities for $i, j \in E$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1, \\ 1 - p & \text{if } j = i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $p \in (0, 1)$.

(i) Draw the transition diagram of this Markov chain. (2 marks)

(i) Explain whether or not this Markov chain is irreducible. (2 marks)

(ii) Prove that this Markov chain cannot have any positive recurrent states. (4 marks)

(Total: 20 marks)

Recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.

2. Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$ on the state space $E = \{0, 1, \ldots, M\}$, for $M \in \mathbb{N}$, with (one-step) transition matrix denoted by $\mathbf{P} = (p_{ij})_{i,j \in E}$. Suppose that $p_{ij} \geq 0$ if $|i - j| = 0$, $p_{ij} > 0$ if $|i - j| = 1$ and $p_{ij} = 0$ if $|i - j| \geq 2$. We will refer to this Markov chain as a *birth-death chain*.

   (a) Show that the birth-death chain is irreducible. (1 mark)

   (b) Find the stationary distribution of the birth-death chain and show that the birth-death chain is time-reversible (once its marginal distribution is given by its stationary distribution).

   (5 marks)

   (c) Define a birth-death process in continuous time on the state space $E = \{0, 1, \ldots, M\}$, state its generator and derive its stationary distribution.

   (5 marks)

   (d) Find the transition matrix of the associated jump chain of the birth-death process on E defined in (c).

   (3 marks)

   (e) Write a short paragraph comparing the birth-death chain with the jump chain associated with a continuous-time birth death process with regards to the behaviour of their sample paths. Formulate suitable (non-trivial) assumptions which ensure that both processes have the same probabilistic properties.

   (6 marks)

(Total: 20 marks)

3. (a) Consider a continuous-time, time-homogeneous, minimal Markov chain $X = (X_t)_{t\geq 0}$ on the state space $E = \{1, 2, 3\}$ with generator given by

$$\mathbf{G} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} & \frac{1}{8} & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 5 & -5 \end{pmatrix}.$$

(i) Draw the transition diagram of this Markov chain. (2 marks)

(ii) Define what it means for a state $i \in E$ to be recurrent or transient. (2 marks)

(ii) For each state in the state space, justify whether it is recurrent or transient. (5 marks)

(b) Let $N = (N_t)_{t\geq 0}$ denote a homogeneous Poisson process with rate $\lambda > 0$. For $n \in \mathbb{N}$, let J_n denote the time of the nth jump of the Poisson process N, where $J_0 = 0$. Define the excess lifetime process as $Z = (Z_t)_{t\geq 0}$, where $Z_t = J_{N_t+1} - t$.

(i) For $t \geq 0$, show that $J_{N_t} \leq t < J_{N_t+1}$. (2 marks)

(ii) Show that, for $x > 0$,

$$\mathrm{P}(Z_t > x) = e^{-\lambda(t+x)} + \int_0^t \mathrm{P}(Z_{t-u} > x)\lambda e^{-\lambda u} du.$$

Hint: Condition on $J_1 = u$ and consider the three cases when 1) $u > t + x$, 2) $t < u \leq t + x$, 3) $u \leq t$. (8 marks)

(iii) Verify (using 3(b)(ii) or otherwise) that Z_t follows an exponential distribution with parameter λ. (1 mark)

(Total: 20 marks)

4. (a) Let $W = (W_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ denote two independent Brownian motions. Let $0 < \rho < 1$ and define a new process $X = (X_t)_{t\geq 0}$ by

$$X_t = \rho W_t + \sqrt{1-\rho^2} B_t.$$

(i) Show that X is a standard Brownian motion. (5 marks)

(ii) Calculate $\mathrm{Cor}(X_t, W_t)$ and $\mathrm{Cor}(X_t, B_t)$ for $t \geq 0$. (4 marks)

(b) Suppose that the temperature of the River Thames is measured continuously at a location in London (and recorded in degree Celsius). Measurement errors are assumed to be normally distributed. As the precision of the measurements decreases over time, a scientist suggests to model the measurement errors by a standard Brownian motion $B = (B_t)_{t\geq 0}$, where the time t is recorded in years.

(i) How could you justify the scientist's modelling choice? (3 marks)

(ii) For how many years would you have a probability of at least 90% that all measurement errors are smaller than 3 degrees? (8 marks)
Hint: You might find the table given below useful.

Partial table showing values of z for $\mathrm{P}(Z < z)$, where

Z has a standard normal distribution

z	$\mathrm{P}(Z < z)$
1.281	0.900
1.645	0.950
1.960	0.975
2.326	0.990
2.576	0.995

(Total: 20 marks)

5. This question refers to the additional reading material on "Martingales" as described in the book by Robert Dobrow, Introduction to Stochastic Processes with R (2016), see Section 8.6, p.356–371.

(a) Define a martingale in continuous time. (3 marks)

(b) Show that every continuous-time martingale has constant expectation. (3 marks)

(c) Let $N = (N_t)_{t \geq 0}$ denote a homogeneous Poisson process with rate $\lambda > 0$. Is N a martingale? Justify your answer. (3 marks)

(d) Let $B = (B_t)_{t \geq 0}$ denote a standard Brownian motion. Define the stochastic process $Y = (Y_t)_{t \geq 0}$ by

$$Y_t = B_t^3 - 3tB_t.$$

Show that Y is a martingale with respect to B. (6 marks)

(e) Consider independent and identically distributed random variables $X_1, X_2, \ldots$ and define a random walk as $S_n = \sum_{i=0}^{n} X_i$, where $X_0 = c$, for a constant $c \in \mathbb{R}$. Formulate suitable conditions which ensure that $S = (S_n)_{n \in \mathbb{N}_0}$ is a martingale in discrete time and prove this result.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60045/70045

Applied Probability (Solutions)

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1. (a) (i) The transition matrix of this Markov chain is given by

sim. seen ⇓

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

2, A

(ii) From the transition matrix (or the diagram), we read off that there are three communicating classes: $C_1 = \{1,2,3\}, T = \{4,5\}, C_2 = \{6,7\}$.

sim. seen ⇓

We note that T is not closed, hence transient.
Both C_1 and C_2 are finite and closed and hence positive recurrent (by a result from lectures).

4, A

(iii) We recall that period is a class property and we denote by $d(i) = \gcd\{n : p_{ii}(n) > 0\}$ the period of state i. Since $p_{11} > 0$, we get that $d(1) = d(2) = d(3) = 1$. Also, $d(4) = d(5) = 2$ and $d(6) = d(7) = 2$.

sim. seen ⇓

3, A

(iv) We know from a result from lectures that the stationary distribution is not unique in this case since we have two positive recurrent communication classes. We recall that the elements of the stationary distribution associated with transient states are 0.

sim. seen ⇓

Next, we solve two systems of equations to find the stationary distributions associated with C_1 and C_2.
First, we solve

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{7}{8} & \frac{1}{8} \end{pmatrix} = (\pi_1, \pi_2, \pi_3), \quad \sum_{i=1}^{3} \pi_i = 1,$$

which implies the following system of equations: $\frac{1}{2}\pi_1 + \frac{3}{4}\pi_2 = \pi_1, \frac{1}{2}\pi_1 + \frac{7}{8}\pi_3 = \pi_2, [\frac{1}{4}\pi_2 + \frac{1}{8}\pi_3 = \pi_3,]\pi_1 + \pi_2 + \pi_3 = 1$. This can be solved by solving for π_1: $\pi_1 = \frac{3}{2}\pi_2, \pi_1 = 2\pi_2 - \frac{7}{4}\pi_3, \pi_1 = 1 - \pi_2 - \pi_3$. Then $\frac{3}{2}\pi_2 = 2\pi_2 - \frac{7}{4}\pi_3, \frac{3}{2}\pi_2 = 1 - \pi_2 - \pi_3$, which implies $\pi_2 = \frac{7}{2}\pi_3, \pi_2 = \frac{2}{5} - \frac{2}{5}\pi_3$. Then $\pi_3 = \frac{4}{39}, \pi_2 = \frac{13}{39}, \pi_1 = \frac{7}{13}$.

Next, we solve $(\pi_6, \pi_7) = (\pi_6, \pi_7)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\pi_6 + \pi_7 = 1$, which implies $\pi_6 = \pi_7 = \frac{1}{2}$.

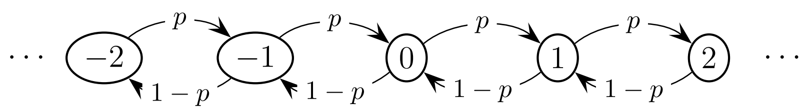
Finally, we need to ensure that the components of the stationary distribution are non-negative and sum up to one. Hence, we obtain that all stationary distributions of the Markov chain are of the form

$$w\left(\frac{7}{13}, \frac{14}{39}, \frac{4}{39}, 0, 0, 0, 0\right) + (1-w)\left(0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}\right),$$

for $0 \leq w \leq 1$. [Alternative representations of the above solution are possible.]

3, A

(b) (i) The transition diagram is given by: seen $\Downarrow$



2, A

(ii) Yes, this Markov chain is irreducible since all states communicate with each other. This can be read off from the transition diagram (and is also a result from lectures). seen $\Downarrow$

2, A

(iii) We observe that this Markov chain has a doubly stochastic transition matrix and is irreducible. Since furthermore the state space is infinite, we know from a result from the problem class that all states have to be either null recurrent or transient. sim. seen $\Downarrow$

4, B

Students can prove the result as follows if they don't recall it: Suppose the chain is positive recurrent and let $\mathbf{P}$ denote the transition matrix. Then according to a theorem from lectures, there exists a positive root of the equation $\mathbf{xP} = \mathbf{x}$, which is unique up to a multiplicative constant. Since $\mathbf{P}$ is doubly stochastic, we can take $\mathbf{x} = \mathbf{1}$ (the vector of 1's). Since the root $\mathbf{x}$ is unique, there cannot exist a stationary *distribution* and therefore the chain is null or transient.

2. (a) Since $p_{ij} > 0$ for all $|i-j| = 1$, all states communicate with each other; hence the chain is irreducible. sim. seen ⇓

1, A

(b) Since the Markov chain is irreducible and the state space is finite, we know that there exists a unique stationary distribution which we denote by $\boldsymbol{\pi}$ (with all entries being strictly positive).

Assuming its marginal distribution is given by its stationary distribution, then time-reversibility is equivalent to $\boldsymbol{\pi}$ satisfying the detailed balance equations, i.e. $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in E$. I.e. we have

$$\begin{aligned}
\pi_1 p_{10} = \pi_0 p_{01} \Leftrightarrow \pi_1 &= \frac{p_{01}}{p_{10}}, \\
\pi_2 p_{21} = \pi_1 p_{12} \Leftrightarrow \pi_2 &= \frac{p_{12}}{p_{21}} \pi_1 = \frac{p_{01} p_{12}}{p_{21} p_{10}} \pi_0, \quad \ldots, \\
\pi_j &= \frac{p_{01} \cdots p_{j-1,j}}{p_{j,j-1} \cdots p_{10}} \pi_0, \quad 1 \leq j \leq M.
\end{aligned}$$

We choose π_0 such that $\sum_{j=0}^{M} \pi_1 = 1$, hence we require that

$$\pi_0 \left(1 + \sum_{j=1}^{M} \frac{p_{01} \cdots p_{j-1,j}}{p_{j,j-1} \cdots p_{10}} \right) = 1 \Leftrightarrow \pi_0 = \left(1 + \sum_{j=1}^{M} \frac{p_{01} \cdots p_{j-1,j}}{p_{j,j-1} \cdots p_{10}} \right)^{-1}.$$

Since $\boldsymbol{\pi}$ satisfies the detailed-balance equations, it is the stationary distribution of the Markov chain and the chain is time-reversible (by a result from lectures). 5, C

(c) We adapt the definition from the lectures to a finite state space $E = \{0, 1, \ldots, M\}$: Let $\{X_t\}_{t \geq 0}$ be a birth-death process on E satisfying the following conditions:

1. $\{X_t\}_{t \geq 0}$ is Markov chain on E
2. The infinitesimal transition probabilities are (for $t \geq 0$, $\delta > 0$, $n \in \mathbb{N}_0, m \in \mathbb{Z}$):

$$\mathrm{P}(X_{t+\delta} = n+m | X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \textit{if } m = 0, \\ \lambda_n \delta + o(\delta), & \textit{if } m = 1, \\ \mu_n \delta + o(\delta), & \textit{if } m = -1, \\ o(\delta), & \textit{if} |m| > 1. \end{cases}$$

3. The birth rates $\lambda_0, \lambda_1, \ldots$ and the death rates $\mu_0, \mu_1, \ldots$ satisfy

$$\lambda_i \geq 0 \quad \mu_i \geq 0 \quad \mu_0 = 0,$$

and $\lambda_n = 0$ for all $n \geq M$ [$\mu_n = 0$ for all $n > M$].

[It would be reasonable to assume here that $\lambda_0, \ldots, \lambda_{M-1}, \mu_1, \ldots, \mu_M > 0$, to align the definition with the discrete-time one. Either one will be given full marks.]

3, B

The generator is given by

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \mu_{M-1} & -(\lambda_{M-1}+\mu_{M-1}) & \lambda_{M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \mu_M & -\mu_M \end{pmatrix}.$$

$\boxed{1, \text{B}}$

The stationary distribution of the birth-death process can be derived by solving $\boldsymbol{\pi}\mathbf{G} = 0$ (as in lectures), which leads to

$$\pi_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_j \cdots \mu_1}\pi_0, \quad 1 \le j \le M, \quad \text{with } \pi_0 = \left(1 + \sum_{j=1}^{M} \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_j \cdots \mu_1}\right)^{-1}.$$

$\boxed{1, \text{B}}$

(d) The transition matrix of the associated jump chain, denoted by (Z_n), is given by

$$\mathbf{P}_Z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{\mu_1}{\lambda_1+\mu_1} & 0 & \frac{\lambda_1}{\lambda_1+\mu_1} & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \frac{\mu_2}{\lambda_2+\mu_2} & 0 & \frac{\lambda_2}{\lambda_2+\mu_2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \frac{\mu_{M-1}}{\lambda_{M-1}+\mu_{M-1}} & 0 & \frac{\mu_{M-1}}{\lambda_{M-1}+\mu_{M-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & 0 \end{pmatrix},$$

provided that $\lambda_i + \mu_i > 0$ for all $i = 0, \ldots, M$. $\boxed{3, \text{B}}$

(e) The sample paths of the birth-death chain are characterised by the fact that, in each time step, the process will either decrease by one, increase by one or stay at the same level.

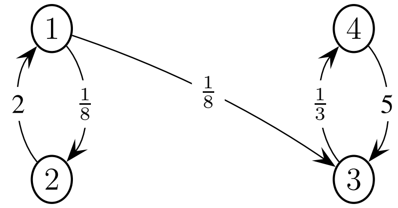
The sample path of the jump chain defined above, however, does not allow for the feature that the process stays in the same state (assuming $\lambda_i + \mu_i > 0$ for all $i = 0, \ldots, M$). $\boxed{3, \text{C}}$

We observe that the transition matrix of the birth-death chain and the jump chain are identical if we choose

$$\begin{aligned} p_{00} = 0, p_{01} = 1, \\ p_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i},\ p_{ii} = 0,\ p_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}, \quad 1 \le i \le M-1, \\ p_{M,M-1} = 1, p_{MM} = 0. \end{aligned}$$

$\boxed{3, \text{B}}$

3. (a) (i) The transition diagram is given by: sim. seen ⇓



2, A

(ii) We say that state $i \in E$ is *recurrent* if $\mathrm{P}(\{t \geq 0 : X_t = i\}$ is unbounded $|X_0 = i) = 1$. We say that state $i \in E$ is *transient* if $\mathrm{P}(\{t \geq 0 : X_t = i\}$ is unbounded $|X_0 = i) = 0$. seen ⇓ 2, A

(iii) We know that, if a state is recurrent (transient) for the corresponding jump chain, then it is recurrent (transient) for the continuous-time Markov chain. According to lectures, the transition probabilities of the corresponding embedded jump chain are given by $p_{ij} = -g_{ij}/g_{ii}$ for all $i, j \in E$, provided that $g_{ii} \neq 0$. Hence, the transition matrix of the embedded jump chain is given by sim. seen ⇓ 1, A

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We observe that the jump chain has two communicating classes: The class $\{1, 2\}$ is not closed, hence transient for both the jump chain and for X. The class $\{3, 4\}$ is finite and closed, hence recurrent for both the jump chain and for X. 2, A 2, A

(b) (i) We recall that, according to the third definition of the Poisson process introduced in lectures, $N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}$, $\forall t \geq 0$. This implies that $J_{N_t} \leq t < J_{N_t+1}$. unseen ⇓ 2, A

(ii) We use the continuous law of total probability and follow the hint of conditioning on J_1, which according to lectures follows an Exp(λ) distribution. Hence, for $x > 0$, we get

$$\mathrm{P}(Z_t > x) = \int_0^\infty \mathrm{P}(Z_t > x | J_1 = u) f_{J_1}(u) du = \int_0^\infty \mathrm{P}(Z_t > x | J_1 = u) \lambda e^{-\lambda u} du.$$

We follow the hint and consider three cases: 1, D

1. For $u > t + x$, we have $\mathrm{P}(Z_t > x | J_1 = u) = \mathrm{P}(J_{N_t+1} - t > x | J_1 = u) = \mathrm{P}(J_{N_t+1} > x + t | J_1 = u) = 1$. 2, D
2. For $u \in (t, t + x]$, we have $\mathrm{P}(Z_t > x | J_1 = u) = \mathrm{P}(J_{N_t+1} - t > x | J_1 = u) = \mathrm{P}(J_{N_t+1} > x + t | J_1 = u) = 0$. 2, D
3. For $u \leq t$, we have $\mathrm{P}(Z_t > x | J_1 = u) = \mathrm{P}(J_{N_t+1} - t > x | J_1 = u) = \mathrm{P}(J_{N_{t-u}+1} - (t - u) > x) = \mathrm{P}(Z_{t-u} > x)$, where we used the time-homogeneity of the process. 2, D

Hence

$$
\begin{aligned}
\mathrm{P}(Z_t > x) &= \int_0^t \mathrm{P}(Z_t > x | J_1 = u)\lambda e^{-\lambda u} du \\
&\quad + \int_t^{t+x} \mathrm{P}(Z_t > x | J_1 = u)\lambda e^{-\lambda u} du + \int_{t+x}^{\infty} \mathrm{P}(Z_t > x | J_1 = u)\lambda e^{-\lambda u} du \\
&= \int_0^t \mathrm{P}(Z_{t-u} > x)\lambda e^{-\lambda u} du + 0 + \int_{t+x}^{\infty} \lambda e^{-\lambda u} du \\
&= e^{-\lambda(t+x)} + \int_0^t \mathrm{P}(Z_{t-u} > x)\lambda e^{-\lambda u} du.
\end{aligned}
$$

$\boxed{1, \mathrm{D}}$

(iii) We show that, for $x > 0$, $P(Z_t > x) = \exp(-\lambda x)$ (which is the survival function of the Exp(λ)-distribution) satisfies the integral equation in (ii): The right hand side is given by

$$
\begin{aligned}
&e^{-\lambda(t+x)} + \int_0^t \mathrm{P}(Z_{t-u} > x)\lambda e^{-\lambda u} du \\
&= e^{-\lambda(t+x)} + \int_0^t e^{-\lambda x}\lambda e^{-\lambda u} du e^{-\lambda(t+x)} + e^{-\lambda x}(-e^{-\lambda t} + 1) \\
&= e^{-\lambda x} = P(Z_t > x),
\end{aligned}
$$

which is equal to the left hand side. Also, for $x \leq 0$, we get from (ii), that $\mathrm{P}(Z_t > x) = 1$. $\boxed{1, \mathrm{C}}$

4. (a) (i) We note that B and W are standard Brownian motions. Hence

meth seen $\Downarrow$

1. $B_0 = 0$, $W_0 = 0$ almost surely, hence $X_0 = \rho W_0 + \sqrt{1-\rho^2} B_0 = 0$ almost surely;
2. B, W have independent increments and are independent of each other, hence their linear combination X inherits the independent increments;
3. B, W has stationary increments, hence their linear combination X inherits the stationary increments;
4. For $0 \leq s < t$, $B_t - B_s, W_t - W_s \sim \mathrm{N}(0, (t-s))$; since W and B are independent $X_t \sim \mathrm{N}(\rho 0 + \sqrt{1-\rho^2}0, \rho^2 t + (1-\rho^2)t) = \mathrm{N}(0, t)$.
5. The sample paths of W and B are almost surely continuous, hence their linear combination X has almost surely continuous sample paths.

5, B

[An alternative solution showing that X is a Gaussian process, starting at 0, having continuous sample paths, mean zero and $\mathrm{Cov}(X_s, X_t) = \min\{s, t\}$ for all $s, t \geq 0$ is also acceptable for full marks.]

(ii) Let $t \geq 0$. Then

$$\begin{aligned}\mathrm{Cov}(X_t, W_t) &= \mathrm{E}(X_t W_t) - \mathrm{E}(X_t)\mathrm{E}(W_t) = \mathrm{E}(X_t W_t) = \mathrm{E}((\rho W_t + \sqrt{1-\rho^2} B_t) W_t) \\ &= \rho \mathrm{E}(W_t^2) + \sqrt{1-\rho^2}\mathrm{E}(B_t)\mathrm{E}(W_t) = \rho t,\end{aligned}$$

where we used that $\mathrm{E}(X_t) = \mathrm{E}(W_t) = 0$ and $\mathrm{E}(W_t^2) = t$, that the expectation is linear, and that W and B are independent. Hence $\mathrm{Cor}(X_t, W_t) = \frac{\rho t}{\sqrt{t^2}} = \rho$. Similarly, $\mathrm{Cor}(X_t, B_t) = \sqrt{1-\rho^2}$.

4, A

(b) (i) Suppose that $B = (B_t)_{t \geq 0}$ denotes a standard Brownian motion. We know that $B_t \sim \mathrm{N}(0, t)$, hence the measurement errors follow a Gaussian distribution in this model. We recall that $\mathrm{Var}(B_t) = t$. Hence the variance of a Brownian motion increases with time, which can potentially describe the decreasing precision of the measurements observed by the scientist.

unseen $\Downarrow$

3, C

(ii) For $t \geq 0$, define $M_t^+ := \max_{0 \leq s \leq t} B_s$. The question asks us to find the time t such that $\mathrm{P}(M_t^+ < 3) \geq 0.9$. We recall the reflection principle from lectures: For $x > 0$, $\mathrm{P}(M_t^+ \geq x) = 2\mathrm{P}(B_t > x)$. Hence, we have

2, D

1, D

$$\mathrm{P}(M_t^+ < 3) = 1 - \mathrm{P}(M_t^+ \geq 3) = 1 - 2\mathrm{P}(B_t > 3) = 2\mathrm{P}(B_t \leq 3) - 1.$$

We need to find t such that

2, D

$$\mathrm{P}(M_t^+ < 3) = 2\mathrm{P}(B_t \leq 3) - 1 \geq 0.9 \Leftrightarrow \mathrm{P}(B_t \leq 3) \geq 0.95.$$

Let $Z \sim \mathrm{N}(0, 1)$. Then

$$\mathrm{P}(B_t \leq 3) \geq 0.95 \Leftrightarrow \mathrm{P}(\sqrt{t}Z \leq 3) \geq 0.95 \Leftrightarrow \mathrm{P}(Z \leq 3/\sqrt{t}) \geq 0.95.$$

Using the table, we read off that we need that $3/\sqrt{t} = 1.645 \Leftrightarrow t = 9/1.645^2$. For $t = 9/1.645^2[\approx 3.3]$ years, there would be a probability of at least 90% that all measurement errors are smaller than 3 degrees.

3, D

5. The Mastery material covers the topic of "Martingales" as described in the book by Robert Dobrow, Introduction to Stochastic Processes with R (2016), see Section 8.6, p.356–371. seen ⇓

(a) A stochastic process $(Y_t)_{t\geq 0}$ is a martingale, if for all $t \geq 0$, $\mathrm{E}(Y_t|Y_r, 0 \leq r \leq s) = Y_s$, for all $0 \leq s \leq t$, and $\mathrm{E}(|Y_t|) < \infty$. 3, M

[Note that the more general definition of a martingale including a general filtration with an adaptedness condition is, of course, valid.] seen ⇓

(b) Let $(Y_t)_{t\geq 0}$ be a martingale. Using the law of total expectation, for any $t \geq s \geq 0$, $\mathrm{E}(Y_t) = \mathrm{E}(\mathrm{E}(Y_t|Y_r, 0 \leq r \leq s)) = \mathrm{E}(Y_s)$. Hence $\mathrm{E}(Y_t) = \mathrm{E}(Y_0)$ for all $t \geq 0$. 3, M

(c) No, N is not a martingale, since $\mathrm{E}(N_t) = \lambda t$, for all $t \geq 0$. Hence it does not have a constant expectation and cannot be a martingale. sim. seen ⇓

3, M

unseen ⇓

(d) Since $B_t \sim \mathrm{N}(0, t)$, we have that $\mathrm{E}(|Y_t|) \leq \mathrm{E}(|B_t^3|) + 3t\mathrm{E}(|B_t|) < \infty$ for all $t \geq 0$. We want to show that, for all $0 \leq s \leq t$, $\mathrm{E}(Y_t|B_r, 0 \leq r \leq s) = Y_s$, which is equivalent to showing that $\mathrm{E}(Y_t - Y_s|B_r, 0 \leq r \leq s) = 0$. Let $0 \leq s \leq t$. Then 1, M

$$\mathrm{E}(Y_t - Y_s|B_r, 0 \leq r \leq s) = \mathrm{E}(B_t^3 - B_s^3|B_r, 0 \leq r \leq s) - 3\mathrm{E}(tB_t - sB_s|B_r, 0 \leq r \leq s).$$

Recall the binomial formula $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$. Hence

$$\begin{aligned}\mathrm{E}(B_t^3 - B_s^3|B_r, 0 \leq r \leq s) = {}& \mathrm{E}((B_t - B_s)^3|B_r, 0 \leq r \leq s) \\ & + \mathrm{E}(3B_tB_s(B_t - B_s)|B_r, 0 \leq r \leq s).\end{aligned}$$

Since B has independent and stationary increments, we have that

$$\mathrm{E}((B_t - B_s)^3|B_r, 0 \leq r \leq s) = \mathrm{E}((B_t - B_s)^3) = \mathrm{E}(B_{t-s}^3) = 0,$$

since $B_{t-s} \sim \mathrm{N}(0, t - s)$. Also,

$$\begin{aligned}&\mathrm{E}(3B_tB_s(B_t - B_s)|B_r, 0 \leq r \leq s) = \mathrm{E}(3B_t^2B_s|B_r, 0 \leq r \leq s) - \mathrm{E}(3B_tB_s^2|B_r, 0 \leq r \leq s) \\ &= 3B_s\mathrm{E}(B_t^2|B_r, 0 \leq r \leq s) - 3B_s^2\mathrm{E}(B_t|B_r, 0 \leq r \leq s) = 3B_s(B_s^2 + (t - s)) - 3B_s^3 \\ &= 3B_s(t - s),\end{aligned}$$

since $(B_t^2 - t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are martingales with respect to B (according to the reading material). Finally,

$$\begin{aligned}-3\mathrm{E}(tB_t - sB_s|B_r, 0 \leq r \leq s) &= -3[\mathrm{E}(tB_t|B_r, 0 \leq r \leq s) - \mathrm{E}(sB_s|B_r, 0 \leq r \leq s)] \\ &= -3(tB_s - sB_s) = -3(t - s)B_s,\end{aligned}$$

which concludes the proof. 5, M

(e) For $n \in \mathbb{N} \cup \{0\}$, we have $\mathrm{E}(S_{n+1}|S_0, \ldots, S_n) = \mathrm{E}(X_{n+1} + S_n|S_0, \ldots, S_n) = \mathrm{E}(X_{n+1}|S_0, \ldots, S_n) + \mathrm{E}(S_n|S_0, \ldots, S_n) = \mathrm{E}(X_{n+1}) + S_n$, where we used the independence of X_{n+1} from $X_0, \ldots, X_n$ and, hence, S_n for the first term, and properties of the conditional expectation for the second term. From this result and the definition of a martingale, we can read off the following sufficient conditions which ensure that S is a martingale: We require that $\mathrm{E}(|S_n|) < \infty$, $c = 0$ and $\mathrm{E}(X_n) = 0$ for all $n \in \mathbb{N}$ [or $\mathrm{E}(X_n) = 0$ for all $n \in \mathbb{N} \cup \{0\}$]. Then $\mathrm{E}(|S_n|) < \infty$ and, for $n \in \mathbb{N} \cup \{0\}$, we have $\mathrm{E}(S_{n+1}|S_0, \ldots, S_n) = S_n$. sim. seen ⇓ 5, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 80 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60045/70045	1	This question was generally answered well by the majority of students. Surprisingly, many students struggled with the last part of Q1b iii), which followed directly from one of the questions on the problem sheet.
MATH60045/70045	2	No Comments Received
MATH60045/70045	3	Part (b-ii) of Q3 was the one students found more difficult. Question (a) was generally well addressed.
MATH60045/70045	4	This question is well designed at the right level, and students performed reasonably well.
MATH70045	5	Q5 was generally answered well by most students. In particular, parts a), b), c) did not cause any problems.Some students lost marks in part d) for not providing all the details of the proof , i.e. by jumping to the conclusion without sufficient justifications.Part e) was also answered well by the majority of students, but some students forgot to mention that c=0 or E(X_n)=0 for all n in {0, 1, 2, ...}.