Calculus and Applications I

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Contents

Preface

This set of notes is not exhaustive. Many examples will be solved outside of these pages. The student should strive to work through any "gaps" left in the notes and also to solve problems assigned here and elsewhere.

A brief roadmap and guide of use is in order. Part I is included for completeness and is cursory at best. Some of the notions may be new but they will be covered in detailed dedicated courses on Analysis. I include the brief discussion here to set the stage for the terminology that will be used later. I expect students to read and do exercises on this part on their own. It is self contained and will *not be covered in the lectures.*

I will assume throughout that students have mastered the material in Maths and Further Maths (or equivalent). In particular things like: differentiation of standard functions (including product and chain rule); curve sketching (maxima/minima, inflection points); integration of standard functions; techniques of integration (including integration by parts, integration by substitution, partial fractions); elementary mechanics. Of course we will use all of these but I will assume full good knowledge.

Part I

Preliminaries

Chapter 1

Limits of Functions, continuity

Given a function $f(x)$ we are concerned with the behaviour near a point $x = x_0$, and in particular the meaning of the statement $\lim_{x\to x_0} f(x) = \ell$, *i.e. the limit of* $f(x)$ *as* x *tends to* x_0*, exists and is equal to* ℓ*.* The precise $\varepsilon - \delta$ definition is the following.

Definition 1. $\varepsilon - \delta$ **Definition of Limit**

Let f *be a function defined at all points near* x_0*, except possibly at* x_0*, and let* ℓ *be a real number. We say that* ℓ *is the limit of* $f(x)$ *as* x *approaches* x_0*, if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - \ell| < \varepsilon$ *whenever* $|x - x_0| < \delta$ *and* $x \neq x_0$*. We write* $\lim_{x\to x_0} f(x) = \ell$*.*

Note: The definition implies that $\lim_{x\to x_0}$ is unambiguous regarding taking limits from above or below. If you see $\lim_{x\to x_0}$ then it means that we must take the limit both from above, i.e. $\lim_{x\to x_0+}$, and from below, i.e. $\lim_{x\to x_0-}$.

Example Prove that $\lim_{x\to 2}\sqrt{x} = \sqrt{2}$. *Before doing the* $\varepsilon - \delta$ *proof, we should note that what we are asked to show is trivially obvious! The function* $f(x) = \sqrt{x}$ *near* $x = 2$ *is perfectly well-behaved and so* $f(2) = \sqrt{2}$ *is immediate. I am using the example to illustrate the rigorous definition in anticipation of its use in less obvious situations.*
Solution: Here $f(x) = \sqrt{x}$, $x_0 = 2$ and we want to prove that $\ell = \sqrt{2}$. Given $\varepsilon > 0$ we need to find $\delta > 0$ so that $|\sqrt{x} - \sqrt{2}| < \varepsilon$ whenever $|x - 2| < \delta$. For all $x > 0$ we have $\sqrt{x} - \sqrt{2} = (x-2)/(\sqrt{x} + \sqrt{2})$, hence

$$|\sqrt{x} - \sqrt{2}| = \frac{|x-2|}{\sqrt{x} + \sqrt{2}} \leq \frac{|x-2|}{\sqrt{2}}.$$

Hence picking $\delta = \sqrt{2}\,\varepsilon$ will do.

In practice we do not want to be doing $\varepsilon - \delta$ proofs for every limit we encounter. Instead we use the following laws of limits which can be proven easily using the $\varepsilon - \delta$ definition. (Try some! I do analogous proofs later on also.)

Basic Properties of Limits

Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ BOTH EXIST. Then

(i) *Sum rule:*
$$\lim_{x\to x_0}[f(x)+g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x). \tag{1.1}$$

(ii) *Product rule:*
$$\lim_{x\to x_0}[f(x)\,g(x)] = \lim_{x\to x_0} f(x)\,\lim_{x\to x_0} g(x). \tag{1.2}$$

(iii) *Reciprocal rule:* If $\lim_{x\to x_0} f(x) \neq 0$ then
$$\lim_{x\to x_0}[1/f(x)] = 1/\lim_{x\to x_0} f(x). \tag{1.3}$$

(iii)$'$ *Quotient rule:* If $\lim_{x\to x_0} g(x) \neq 0$ then
$$\lim_{x\to x_0}[f(x)/g(x)] = \lim_{x\to x_0} f(x)/\lim_{x\to x_0} g(x). \tag{1.4}$$

This follows immediately from (ii) and (iii).

(iv) *Composite function rule:* If $h(x)$ is continuous at $\lim_{x\to x_0} f(x)$, then
$$\lim_{x\to x_0} h(f(x)) = h\left(\lim_{x\to x_0} f(x)\right). \tag{1.5}$$

Example 1

Calculate $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)$.

Solution. Of the form "0/0". Rationalise, i.e.

$$\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right) = \lim_{x\to 1}\left(\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\right) = \lim_{x\to 1}\left(\frac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{(x-1)}}\right) = 2.$$

Example 2

Sketch the function $f(x) = x/|x|$. Do this by considering $x > 0$ and $x < 0$ separately. What happens when $x = 0$?

Solution. If $x > 0$, then $f(x) = 1$. If $x < 0$, then $f(x) = \frac{x}{|x|} = \frac{-|x|}{|x|} = -1$.

At $x = 0$ the function is *undefined*. Hence this function is defined everywhere except at $x = 0$ (i.e. in $\mathbb{R}\setminus\{0\}$). A plot is given in Figure 1.1. The function is *discontinuous* at $x = 0$ as can be seen from the jump in going from left to right (it is called a "step function" and can be written as a linear combination of Heaviside functions - see below).

We have a problem, however. How do we define the function at $x = 0$? The answer is that we can define it to be anything we want, and irrespective of our choice we can never make the function continuous there! In figure 1.1 it is chosen to be zero (i.e. the average of the values as I approach from the right and the left - we will encounter this

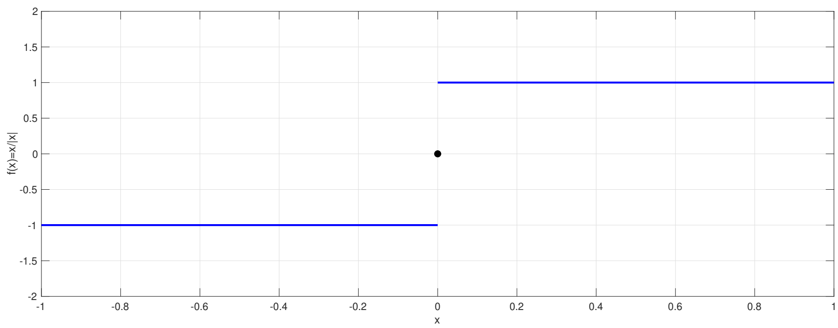


Figure 1.1: The function $f(x) = \frac{x}{|x|}$.

later when we do Fourier series). Here is a summary of what figure 1.1 depicts:

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{1.6}$$

Note that:

1. $\lim_{x\to 0+} f(x) = 1$
2. $\lim_{x\to 0-} f(x) = -1$

where $x \to 0\pm$ means *take the limit as x tends to 0 from above/below.* Of course $\lim_{x\to 0} f(x) \neq f(0)$ for this function, and so the function is not continuous. The definition of continuity is summarised below in Definition 5.

The properties given above also hold as x becomes large and positive or negative. For example if $f(x) = 1/x$ then we know that $\lim_{x\to\pm\infty} f(x) = 0$. Lets make this precise.

Definition 2. The $\varepsilon - A$ definition of $\lim_{x\to\infty} f(x) = \ell$.

Let $f(x)$ be defined on a domain containing the interval (a, ∞). A real number ℓ is the limit of $f(x)$ as x approaches ∞ if, for every $\varepsilon > 0$ there exists a $A > a$, such that $|f(x) - \ell| < \varepsilon$ whenever $x > A$. We write $\lim_{x\to\infty} f(x) = \ell$. [Similarly for $\lim_{x\to-\infty} f(x) = \ell$.]

NOTE: The limit properties (1.1)-(1.5) hold for limits of $f(x)$ as $x \to \pm\infty$, when the limits are defined.

Consider next the limits $\lim_{x\to 0} \sin(1/x)$ and $\lim_{x\to 0}(1/x^2)$. The limits do not exist (I cannot plug $x = 0$ into the functions). Sketch them and determine that they behave differently: the former is bounded, the latter is unbounded. In fact $\lim_{x\to 0}(1/x^2) = \infty$. More precisely we have:

Definition 3. $\varepsilon - B$ **definition of** $\lim_{x\to x_0} f(x) = \infty$**.**

Let $f(x)$ be a function defined in an interval containing x_0, except possibly at $x = x_0$. We say that $f(x)$ approaches ∞ as x approaches x_0 if given any real number $B > 0$, there exists $\varepsilon > 0$, so that whenever $|x - x_0| < \varepsilon$ and $x \neq x_0$, we have $f(x) > B$. We write $\lim_{x\to x_0} f(x) = \infty$. [Definition of $\lim_{x\to x_0} f(x) = -\infty$ totally analogous.]

In the example $f(x) = 1/x^2$ we found that as $x \to 0$, $f(x) \to \infty$ - the function is even, so it does not matter if I approach the limit from the right (i.e. through positive values of x) or the left (through negative x values). What about $f(x) = 1/x$? It is not hard to see that as x tends to 0 through *positive* values then $f \to +\infty$, whereas as x tends to 0 through *negative* values we have $f \to -\infty$.

Hence, we need to define one-sided limits.

Definition 4. One-Sided Limits:

Let $f(x)$ be defined for all x in an interval (x_0, a). We say that $f(x)$ approaches ℓ as x approaches x_0 from the right if, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $x_0 < x < x_0 + \delta$ we have $|f(x) - \ell| < \varepsilon$. We write $\lim_{x\to x_0+} f(x) = \ell$. [Analogous definition for the left-sided limit, i.e. $\lim_{x\to x_0-} f(x) = \ell$.]

Note: If $\lim_{x\to x_0} f(x) = +\infty$ or $-\infty$, then the line $x = x_0$ is a *vertical asymptote*. Analogously, if $\lim_{x\to\pm\infty} = \ell_\pm$ then the lines $y = \ell_\pm$ are *horizontal asymptotes*.

As an **E**xample consider $f(x) = \frac{1}{(x-1)(x-2)^2}$. There are vertical asymptotes at $x = 1$ and $x = 2$ and a horizontal asymptote $y = 0$. Sketch the graph *without using the differentiation methods of finding critical points etc., that you are familiar with. Use intuition and estimation.*

In addition to the basic properties (1.1)-(1.5), there is another powerful test which is very useful in calculations:

Comparison Test for Limits (a.k.a. Squeezing Property)

1. If $\lim_{x\to x_0} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all x near x_0 with $x \neq x_0$, then $\lim_{x\to x_0} g(x) = 0$.

2. If $\lim_{x\to\infty} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all large enough x, then $\lim_{x\to\infty} g(x) = 0$.

Example

(i) Establish Comparison Test 1 using the $\varepsilon - \delta$ definition of a limit.

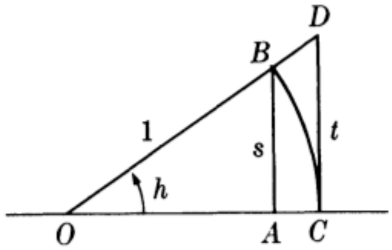


Figure 1.2: Geometrical construction in the proof of the trigonometric limits (1.7)-(1.8).

(ii) Show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Solution

(i) Since $\lim_{x\to x_0} f(x) = 0$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x)| < \varepsilon$ when $|x - x_0| < \delta$. For the same ε and δ, we also have $|g(x)| < \varepsilon$ when $|x - x_0| < \delta$, since $|g(x)| \leq |f(x)|$. Hence $\lim_{x\to x_0} g(x) = 0$ also.

(ii) Take $g(x) = x\sin(1/x)$ and $f(x) = x$. Then $|g(x)| \leq |x|$ for all $x \neq 0$, so the comparison test applies. Clearly $\lim_{x\to 0} x = 0$, hence the result follows.

Two Basic Trigonometric Limits

We will need the following results in finding derivatives of $\sin x$ and $\cos x$ from first principles.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \tag{1.7}$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \tag{1.8}$$

The proof of the former is geometrical and the construction is given in Figure 1.2. OBC is the sector of a circle of radius 1 with subtended angle h. The two triangles OAB and OCD are constructed as shown with BD the extension of OB. Considering triangles OAB and OCD we have

$$\sin h = \frac{AB}{OB} = s, \qquad \tan h = \frac{DC}{OC} = t.$$

From geometry we have the following inequality

area of triangle OAB < area of sector OCB < area of triangle OCD,

which in turn provides

$$\frac{1}{2}\sin h \cos h < \frac{h}{2} < \frac{1}{2}\tan h.$$

The middle quantity follows by noting that the area of the sector OCB is equal to $h/2\pi$ times the area of a circle of unit radius which is π. Considering the first inequality (after canceling the $1/2$ factor throughout) we have

$$\sin h \cos h < h \quad \Rightarrow \quad \frac{\sin h}{h} < \frac{1}{\cos h}.$$

The above is fine since h and $\cos h$ are positive and non-zero so I can divide by them.

The second inequality gives

$$h < \frac{\sin h}{\cos h} \quad \Rightarrow \quad \cos h < \frac{\sin h}{h}.$$

Putting these together gives

$$\cos h < \frac{\sin h}{h} < \frac{1}{\cos h}.$$

As h tends to zero $\cos h$ tends to 1, hence $\sin h/h$ is squeezed between two numbers that tend to 1. By the Squeezing Property we get the desired result.

To prove the second result we write

$$\begin{aligned}\frac{\cos h - 1}{h} &= \frac{\cos h - 1}{h}\,\frac{\cos h + 1}{\cos h + 1} = \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \frac{-\sin^2 h}{h(\cos h + 1)} = \left(-\frac{\sin h}{h}\right)\frac{\sin h}{\cos h + 1}.\end{aligned}$$

Using the product rule for limits, it follows immediately that

$$\lim_{h\to 0}\left(-\frac{\sin h}{h}\right)\frac{\sin h}{\cos h + 1} = \overset{-1}{\left(\lim_{h\to 0}\frac{-\sin h}{h}\right)}\ \overset{0}{\left(\lim_{h\to 0}\sin h\right)}\ \overset{1/2}{\left(\lim_{h\to 0}\frac{1}{\cos h + 1}\right)} = 0.$$

Continuity

Looking back at Definition 1, the $\varepsilon - \delta$ definition of a limit, we can see that it is equivalent to the statement

$$\lim_{h\to 0} f(x_0 + h) = \ell.$$

We can then define what we mean by continuity of a function $f(x)$ at a point x_0.

Definition 5. Continuity

We say that f is continuous at x_0 if $\lim_{h\to 0} f(x_0 + h) = f(x_0)$. Equivalently $\lim_{x\to x_0} f(x) = f(x_0)$.

A totally equivalent definition is: $f(x)$ is continuous at a point x_0 if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all x in the domain of f for which $|x - x_0| < \delta$.

We have seen examples of functions that are not continuous, e.g.

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad\qquad g(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

are both *not continuous* at $x = 0$. If I exclude $x = 0$, then the limits as $x \to 0$ exist, $\lim_{h\to 0+} f(h) = 1$, $\lim_{h\to 0-} f(h) = 0$, and $\lim_{h\to 0} g(h) = 0$. (Note that $\lim_{h\to 0+} = \lim_{h\to 0, h>0}$ and $\lim_{h\to 0-} = \lim_{h\to 0, h<0}$.)

Removable discontinuities

We will encounter functions that are perfectly nice everywhere *except* at a point or points. Here are some examples.

$$f(x) = \sin\left(\frac{1}{x}\right) \tag{1.9}$$

$$f(x) = x\sin\left(\frac{1}{x}\right) \tag{1.10}$$

Both of these functions are well behaved everywhere except at $x = 0$. We say that the functions *are not defined at* $x = 0$.

The question is: Can we define $f(0)$ by a number of our choice so as to make the functions (1.9) and (1.10) continuous?

For the former function this is impossible - the function oscillates infinitely many times as $x \to 0$ and takes on all values between -1 and 1. We can never satisfy the $\varepsilon - \delta$ definition of continuity.

However, for the function in (1.10) we can *remove the discontinuity* by picking $f(0) = 0$, i.e.

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{1.11}$$

Continuity at $x = 0$ is now guaranteed because

$$|x\sin(1/x) - 0| \leq |x| < \varepsilon \quad \forall \quad |x| < \varepsilon.$$

(Note that here $\delta(\varepsilon) = \varepsilon$ exactly.)

Plots of the two functions (1.9) and (1.10) are given in Figure 1.3.

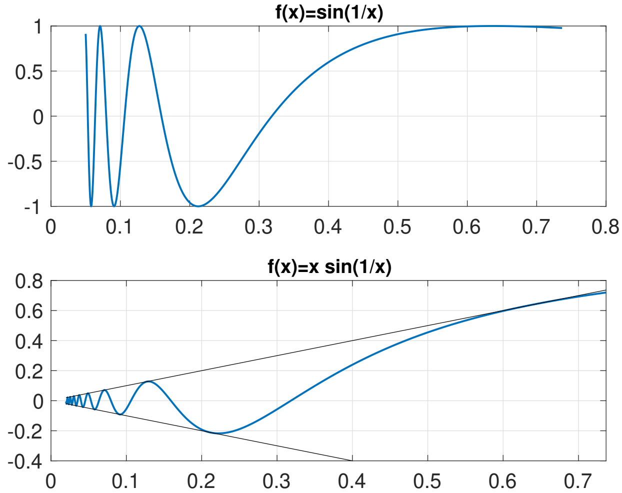


Figure 1.3: The functions $\sin(1/x)$ and $x\sin(1/x)$. Only $x > 0$ is shown, the functions are odd and even, respectively, with respect to x.

Miscellaneous Solved Examples

1. Find $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$, and interpret the result geometrically by considering a right angled triangle with base of length x and unit height.

   We calculate

$$\left(\sqrt{x^2+1}-x\right) = \left(\sqrt{x^2+1}-x\right)\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \frac{1}{\sqrt{x^2+1}+x}.$$

   As x becomes arbitrarily large then $1/(\sqrt{x^2+1}+x)$ becomes arbitrarily small, and hence $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$. [Can you prove this using the $\varepsilon - A$ definition of limit?]

   Geometrical picture for you to do. Hint: The right angled triangle suggested has hypotenuse $\sqrt{x^2+1}$. Consider a circle of radius x whose arc cuts the hypotenuse at a point, and figure out what the quantity $\sqrt{x^2+1}-x$ represents geometrically.

2. Now consider $\lim_{x\to\infty}\left(x-\sqrt{x+1}\right)$. Find the limit in this case.

   Don't need to do much here. Main thing is to notice that x is much much bigger than $\sqrt{x+1}$ when x is large. Hence, $\lim_{x\to\infty}\left(x-\sqrt{x+1}\right)=\infty$.

   A precise definition in this case (for a general function $f(x)$) would be: We say $\lim_{x\to\infty} f(x)=\infty$, if given an arbitrarily large $A>0$, there exists a number $M>0$, so that $f(x)>A$ for all $x>M$.

   In our particular example where $f(M)=M-\sqrt{M+1}$ it is easy to see that taking $M=A^2$ will do the trick. Of course it can be proven for smaller M but we are not looking for anything sharper than a proof.

3. Find (a) $\lim_{x\to 1}\frac{1}{(x-1)^2}$, and (b) $\lim_{x\to\infty}\frac{1-x^2}{x^{3/2}}$.

   For (a) as $x \to 1$ (from above or below), then $(x-1)^2$ becomes arbitrarily small. Its inverse becomes arbitrarily large, so $\lim_{x\to 1}\frac{1}{(x-1)^2} = \infty$

   Intuitive answer is: For (b) as x becomes very large then $x^2 \gg x^{3/2}$, hence the limit is $-\infty$. Can formalize as follows

$$\lim_{x\to\infty}\frac{1-x^2}{x^{3/2}} = \lim_{x\to\infty}(x^{-3/2} - x^{1/2}) = -\infty.$$

Problems

1. For the function

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ \text{c} & x = 0 \end{cases}$$

   where $c \in \mathrm{R}$, use the $\varepsilon - \delta$ definition to prove that the function is not continuous at $x = 0$.

2. Consider the function

$$f(x) = \begin{cases} x^{1/2} & x \geq 0 \\ x^2 & x < 0 \end{cases}$$

   Clearly $f(0) = 0$ and the function is continuous there. Prove continuity using $\varepsilon - \delta$. [Hint: The task here is, given a ε to find *one* δ that will work for both $x > 0$ and $x < 0$.]

Part II

Differentiation

Chapter 2

Derivative of a Function

2.1 Definition with limits, examples

Consider graphs of functions $y = f(x)$. We need to define the *derivative* or slope of the curve at a given point P.

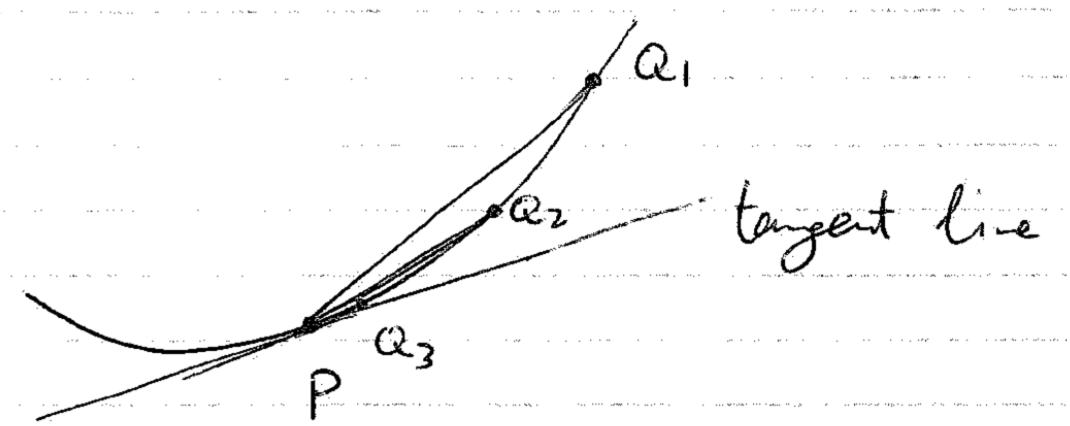


Figure 2.1: Slope of f at P is the slope of the line QP as Q tends to P. Note: The Qs are to the right of P, the definition is the same is Q_1, Q_2 etc are to the left of P.

Definition of Differentiability

The function $f(x)$ is differentiable at x if 'Newton's quotient',

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

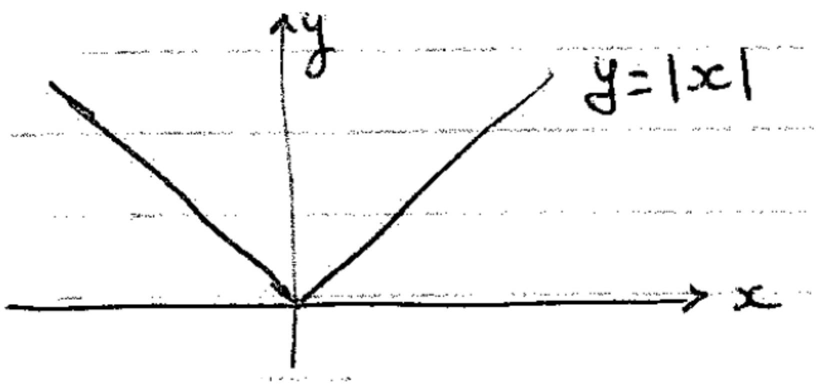
exists. We call this $f'(x)$, the derivative of f at point x.

Examples

(i) Is $f(x) = x^2$ differentiable everywhere?

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h\to 0}(2x+h) = 2x \quad \Rightarrow \textbf{YES}$$

(ii) Is $f(x) = |x|$ differentiable at $x = 0$? Draw a picture.



Need to check if the limit exists *and* the values are equal as we approach 0 from above or below.

(a)
$$\lim_{h\to 0,\, h>0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0,\, h>0} \frac{h-0}{h} = 1$$

(b)
$$\lim_{h\to 0,\, h<0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0,\, h<0} \frac{-h-0}{h} = -1$$

Right and left derivatives *exist* but are not *equal.*

A function is differentiable at x if right and left derivatives exist *and* are equal.

Exercise: Sketch the derivative of $f(x) = |x|$.

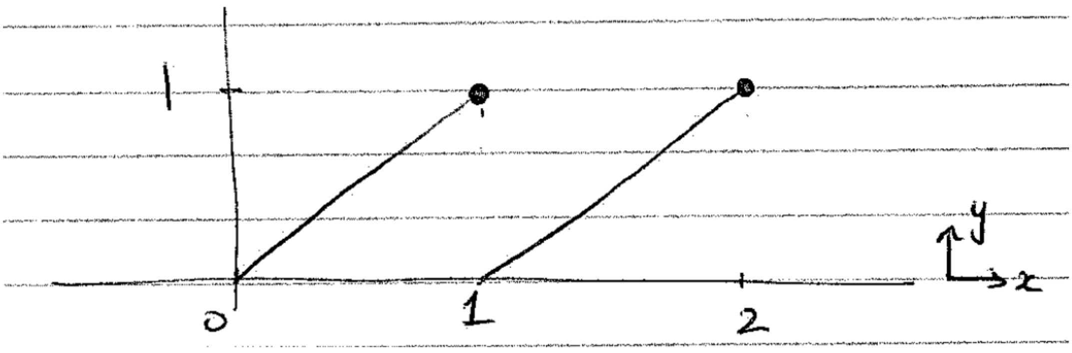
$f(x) = |x|$ is continuous at $x = 0$, but not differentiable there. Geometrically we can see this - there is a 'corner' in the graph.

Note: $f(x)$ is *not continuous* at $x = x_0$ if the limit $\lim_{x\to x_0} f(x)$ *does not exist.* We already gave the $\varepsilon - \delta$ definitions.

Example: Consider

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1 \\ x - 1 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

Here is the graph:



What is the derivative at $x = 1$?

(a) Left derivative at $x = 1$

$$\lim_{h \to 0,\, h<0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0,\, h<0} \frac{1 + h - 1}{h} = 1$$

(b) Right derivative at $x = 1$

$$\begin{aligned} \lim_{h \to 0,\, h>0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0,\, h>0} \frac{(1 + h - 1) - 1}{h} \qquad \text{as } f(1) = 1 \\ &= \lim_{h \to 0,\, h>0} (1 - \frac{1}{h}) \end{aligned}$$

which does not exist. *In fact,* $\to -\infty$. Function has no right derivative.

2.1.1 Polynomials

Theorem 1
Let n be an integer ≥ 1 and let $f(x) = x^n$. Then

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = nx^{n-1}.$$

Proof.
$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + h^2 g(x,h)$$
where $g(x,h)$ involves powers of x and h with some numerical coefficients. We don't care what it is exactly but $\lim_{h\to 0} g(x,h) =$ *some number*. Then
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + h^2 g - x^n}{h} = nx^{n-1}.$$
□

Theorem 2
Let $f(x) = x^a$, where a is any real number and $x > 0$. Then $f'(x) = ax^{a-1}$.
If a is a negative integer then this is easy. General case is different from proof above.

2.2 General rules, chain rule, rates of change

2.2.1 General rules

(i) If c is a constant, $(cf)'(x) = cf'(x)$.

(ii) If $f(x)$, $g(x)$ are given functions and $f'(x)$, $g'(x)$ exist, then
$$(f+g)'(x) = f'(x) + g'(x).$$

(iii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ *(product rule)*

(iv) Let $g(x)$ be a function that has a derivative $g'(x)$ and such that $g(x) \neq 0$. Then
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2}$$

(iv)*
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof. (iii) - *(do the rest yourselves as an exercise)*

$$\begin{aligned}
(fg)' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) \overbrace{- f(x)g(x+h) + f(x)g(x+h)}^{=0}}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x))\, g(x+h) + (g(x+h) - g(x))\, f(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$
□

2.2.2 The Chain Rule

Composition $(f \circ g)(x) = f(g(x))$ is a function constructed as follows: Take a number x, find $g(x)$ and then take the value of f at $g(x)$.

$$\begin{aligned} e.g. \quad & f(x) = x^2 \quad g(x) = \sqrt{x} \quad \text{defined for } x > 0. \\ \text{Then} \quad & (f \circ g)(x) = (\sqrt{x})^2 = x, \\ \text{and} \quad & (g \circ f)(x) = \sqrt{(x^2)} = x. \end{aligned}$$

Discussion: *In the example above we had $(f \circ g)(x) = (g \circ f)(x)$. Is this generally true? If yes prove it, if not provide a counterexample.*

Theorem 3: Chain Rule.
Let f, g be two functions having derivatives and such that f is defined for all numbers that are values of g. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \circ g)(x) = (f \circ g)'(x) = f'(g(x))g'(x)$$

Why is this useful?

Example: $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^3 + 9x^2 + \pi)^{51}\right]$

Last thing you want to do is multiply out the 51 factors and then differentiate. Of course a symbolic manipulator in Matlab, Mathematica, Python etc., would do it, but let's go green and save some computer energy!

With the chain rule we identify

$$f(x) = x^{51} \quad g(x) = (x^3 + 9x^2 + \pi).$$

So that

$$(x^3 + 9x^2 + \pi)^{51} = (f \circ g)(x)$$

$$\text{Then} \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 9x^2 + \pi)^{51} = 51(x^3 + 9x^2 + \pi)^{50}\,(3x^2 + 18x)$$

Proof.

$$\begin{aligned} (f \circ g)'(x) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \end{aligned}$$

Let $k = g(x+h) - g(x)$, *(if $h \neq 0, k \neq 0$)*, and write $u = g(x)$. Then

$$\begin{aligned} (f \circ g)'(x) &= \lim_{h \to 0} \frac{f(u+k) - f(u)}{k} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \left(\lim_{h \to 0} \frac{f(u+k) - f(u)}{k}\right)\left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right) \\ &= f'(u)g'(x) = f'(g(x))g'(x) \end{aligned}$$

□

Analogous definition of the derivative $f'(x)$ is the following:

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

To see equivalence write $y = x + h$.

Application: Particle motion (rectilinear for the moment).

Let the position of a particle at time t be $s = f(t)$, say. If the particle moved from P_1 at $t = t_1$ to P_2 at $t = t_2$, then its

$$\text{average speed} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

So instantaneous speed at any time t is

$$f'(t) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{rate of change.}$$

$f'(t)$ is also a function, call it $v(t)$. If it is differentiable then

$$v'(t) = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} \quad \text{is the acceleration.}$$

Can define higher derivatives (if they exist) by continuing this process.

Theorem 4
If $f(x)$ is differentiable at $x = x_0$, then it is also continuous there.

Question: Is the converse true?

Proof.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot x - x_0 \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \cdot \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0
\end{aligned}$$

□

2.3 Implicit differentiation, related rates of change

Recall: We saw that if n is an integer then

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}, \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^{-n}) = -nx^{-(n+1)}.$$

This also holds if,

(i) $y = x^{1/n}$ where n is an integer (and $x > 0$, *why the restriction?*),

(ii) $y = x^r$ where r is a rational number; *i.e.* $r = \frac{p}{q}$, with p, q integers.

Can prove these using *implicit differentiation*. Start with (i) $y = x^{1/n}$, n integer. Assume $x^{1/n}$ is defined.

Then

$$\begin{aligned} y^n = x \quad &\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(y^n) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \quad \Rightarrow \\ ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad &\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{n}y^{1-n} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}. \end{aligned}$$

e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{5}} = \frac{1}{5}x^{-\frac{4}{5}}$$

(ii) $y = x^{\frac{p}{q}}$. Let $g(x) = x^{\frac{1}{q}}$ where q is an integer. Then $y = (g(x))^p$ with p an integer. Use chain rule.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = pg^{p-1}\frac{1}{q}x^{\frac{1}{q}-1} = \frac{p}{q}x^{\frac{p}{q}-1}$$

These of course generalize to powers of the function.
e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x))^r = rf^{r-1}f', \quad r \text{ rational.}$$

Example of implicit differentiation:

Find the equation of the tangent line to the curve $2x^6 + y^4 = 9xy$ *at the point* $(1, 2)$.

Solution: Note that we cannot solve for y as a function of x. Hence implicit differentiation is very powerful here. Calculate the derivative

$$12x^5 + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Substitute point $(1, 2)$

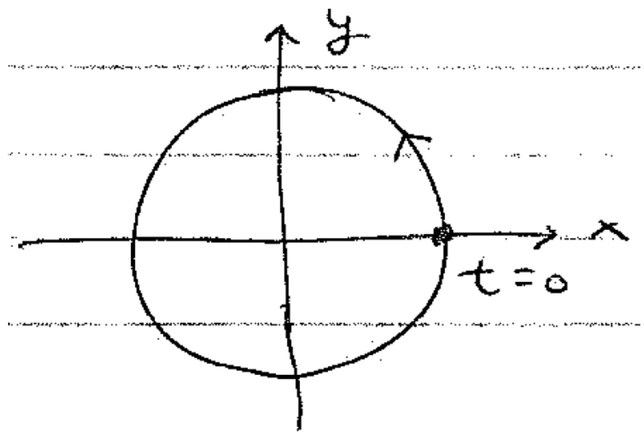
$$\begin{aligned} 12 + 32\frac{\mathrm{d}y}{\mathrm{d}x} &= 18 + 9\frac{\mathrm{d}y}{\mathrm{d}x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6}{23} \quad \text{is the slope of the tangent line} \end{aligned}$$

Its equation is $y - 2 = \frac{6}{23}(x-1)$.

Can also use implicit differentiation to obtain *related rates of change*.

If x and y are both functions of a parameter t, then we can differentiate implicitly with respect to t

$$e.g. \quad x = \cos(t),\ y = \sin(t),\ t \geq 0$$



Equation is $x^2 + y^2 = 1$, where $x = x(t)$, $y = y(t)$. Differentiate implicitly with respect to t.

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$i.e \quad \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = -\frac{x}{y}.$$

This is the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ if we think of $y = \pm\sqrt{1-x^2}$ as a function of x. (Another way is $x^2 + y^2 = 1 \quad \rightarrow \quad 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$.)

In general, if $x = f(t)$ and $y = g(t)$, describe a curve in the plane called a *parametric curve*, then the slope of the tangent line to the curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \quad \text{if} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

To prove this, note that the curve can be defined (piecewise) as the graph of a function $y = h(x)$ *or* $x = H(y)$. Chain rule gives $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ as required.

Example
The surface area of a cube is growing at a constant rate of $4\mathrm{cm}^2/\mathrm{s}$. How fast is the length of a side growing when the cube sides are 2cm long? Find the side length when the rate of change of the volume exceeds that of the area.

Solution

$$\begin{aligned} A = 6x^2 \quad &\Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}t} = 12x\frac{\mathrm{d}x}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{12x}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{4}{12x} \\ \text{If } x = 2 \quad &\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{6}\text{cm s}^{-1} \\ V = x^3 \quad &\Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3x^2}{12x}\frac{\mathrm{d}A}{\mathrm{d}t}. \end{aligned}$$

So if $x > 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} > \frac{\mathrm{d}A}{\mathrm{d}t}$ in numerical value.

Chapter 3

Mean Value and Intermediate Value Theorems

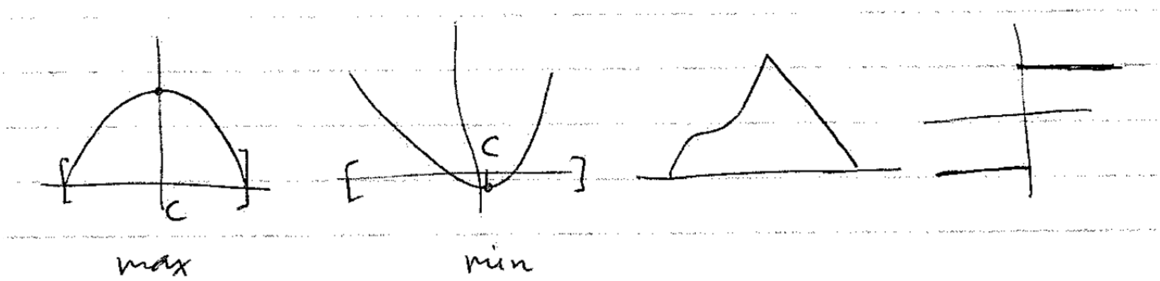
For a function $f(x)$ which is defined at a point c, we say that c is a *maximum* of f if

$$f(c) \geq f(x) \quad \forall x \text{ where } f \text{ is defined.}$$

For a *minimum* we have

$$f(c) \leq f(x).$$

e.g.

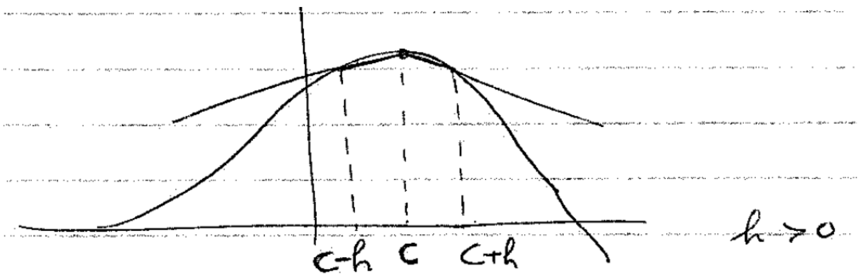


Here is a result when f is differentiable:

Theorem 1
Let f be a function which is defined and differentiable on the open interval (a, b). Let c be a number in the interval which is a maximum for the function.

Then $f'(c) = 0$. $f'(c) = 0$ also, if c is a minimum of f.

Proof. Obvious, here is a geometrical interpretation.



As $h \to 0$, the slope $\to 0$, $\Rightarrow f'(c) = 0$.

Detailed proof:

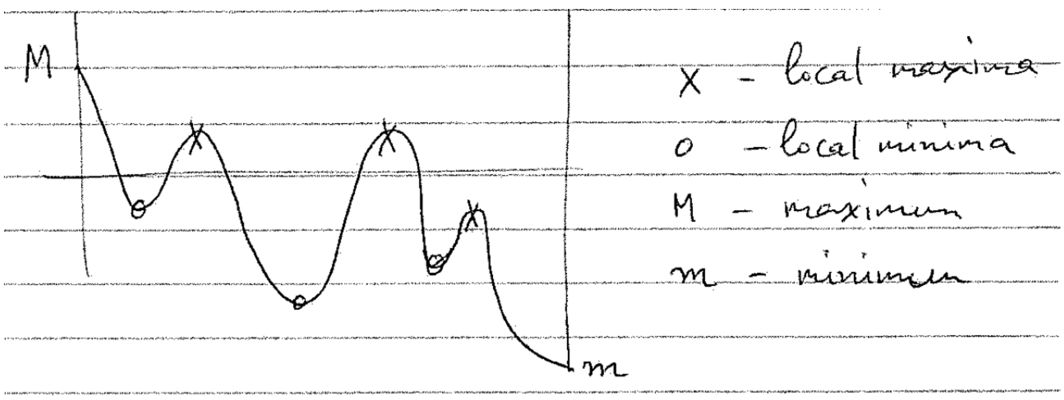
$f(c) \geq f(c+h) \Rightarrow f(c+h) - f(c) \leq 0$.

$$i.e. \quad \lim_{h \to 0,\, h>0} \frac{f(c+h) - f(c)}{h} \leq 0$$

Similarly for left limit

$$\lim_{h \to 0,\, h>0} \frac{f(c) - f(c-h)}{h} \geq 0.$$

As $h \to 0$ these can only be equal if $f'(c) = 0$ since the function is differentiable. □



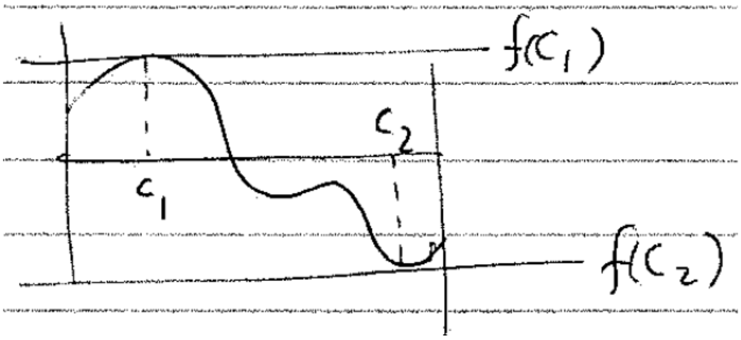
All points c such that $f'(c) = 0$ are called *critical points.*

Definition

$f(x)$ is said to be continuous on an interval $[a, b]$ if $\lim_{x \to x_0} f(x) = f(x_0)$ for all x_0 in $[a, b]$. Analogously, $\lim_{h \to 0} f(x_0 + h) = f(x_0) \quad a \leq x_0 \leq b$.

> **Theorem 2**
> Let $f(x)$ be continuous on the closed interval $[a,b]$. Then $f(x)$ has a maximum and a minimum on this interval. *I.e* there exist c_1 and c_2 so that $f(c_1) \geq f(x)$ and $f(c_2) \leq f(x)$ for all x in $[a,b]$.

e.g

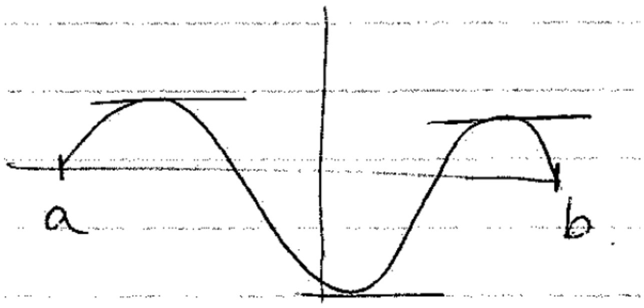


> **Theorem 3** (Combines Theorems 1 and 2)
> Let $f(x)$ be continuous over the closed interval $a \leq x \leq b$ and differentiable on the open interval $a < x < b$. Assume also that $f(a) = f(b) = 0$.
> Then there exists a point c, $a < c < b$, such that $f'(c) = 0$.

Proof. If $f(x) = 0$ then there is nothing to prove. If $f(x) \neq 0$ then it must have at least one maximum or minimum or both. Denote by c such points. By Theorem 1 we have $f'(c) = 0$.

□

Example:

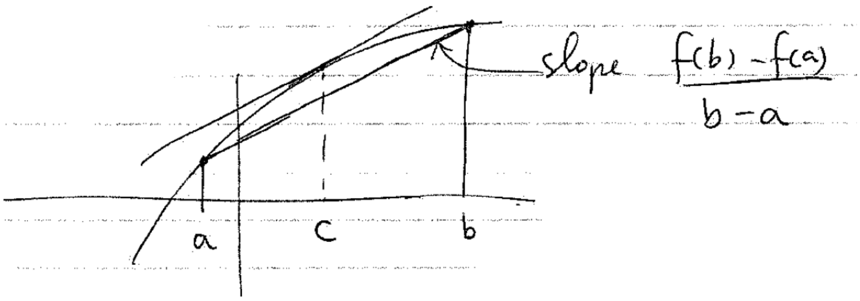


Here there are 3 such points.

Theorem 4
Suppose f is continuous on $[a, b]$ and differentiable on (a, b).
Then there exists $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation:



The point $x = c$ is where the tangent has the slope $\frac{f(b)-f(a)}{b-a}$.

Proof. The straight line joining $(a, f(a))$ and $(b, f(b))$ has equation

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a} \qquad \Rightarrow \qquad y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

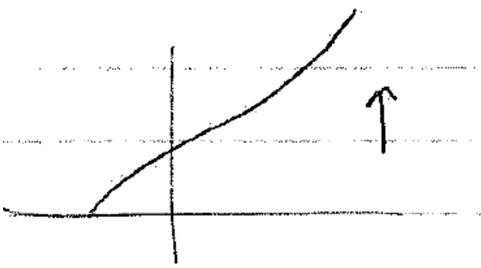
Consider $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a) - f(a)$. Then $g(a) = 0$, $g(b) = 0$, and by Theorem 3 there exists a c with $a < c < b$ such that $g'(c) = 0$.

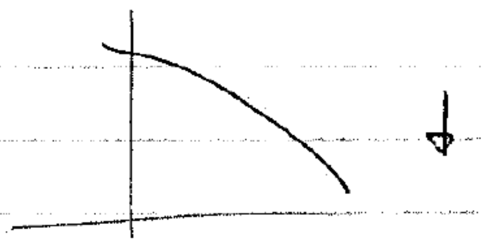
But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, and the result follows. $\square$

Definition

We say that a continuous function $f(x)$ is increasing over a given interval if given x_1, x_2 in the interval with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.

*Strictly **increasing** if $f(x_1) < f(x_2)$ when $x_1 < x_2$.*
*Strictly **decreasing** if $f(x_1) > f(x_2)$ when $x_1 < x_2$.*





Theorem 5
Let $f(x)$ be continuous in the closed interval $[a, b]$ and differentiable in the open interval $I = (a, b)$.
If $f'(x) = 0$ in I, then f is constant.
If $f'(x) > 0$ in I, then f is strictly increasing.
If $f'(x) < 0$ in I, then f is strictly decreasing.

Proof. Use the mean value theorem.

Let x_1, x_2 be points in the interval with $x_1 < x_2$. Then there exists $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_2} \quad \Longrightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

If $f'(x) = 0$ in the interval, $f'(c) = 0$ and $f(x_2) = f(x_1)$ i.e f is constant. If $f'(x) > 0$ then $f'(c) > 0$ and $f(x_2) > f(x_1)$, i.e strictly increasing. If $f'(x) < 0$ then $f'(c) < 0$ and $f(x_2) < f(x_1)$, *i.e* strictly decreasing. □

Example 1 *(Do yourself).* Determine the region of increase and decrease of the function $f(x) = x^3 - 2x + 1$.

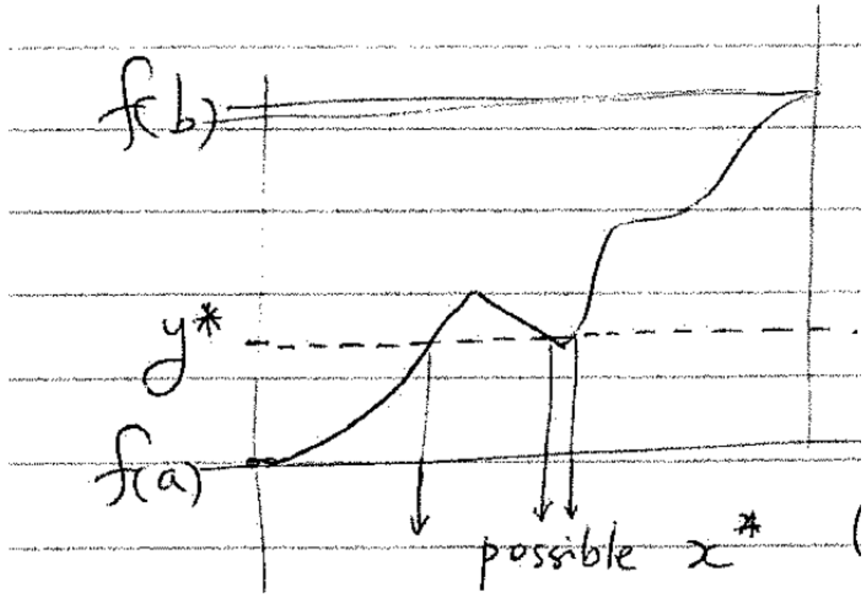
Example 2 Prove that $\sin(x) \leq x$ for $x \geq 0$.

Solution Let $f(x) = x - \sin(x)$. Then $f(0) = 0$.

$f'(x) = 1 - \cos(x) \geq 0$ for all x. Hence $f(x)$ is an increasing function $\Rightarrow f(x) \geq 0$ for all x.

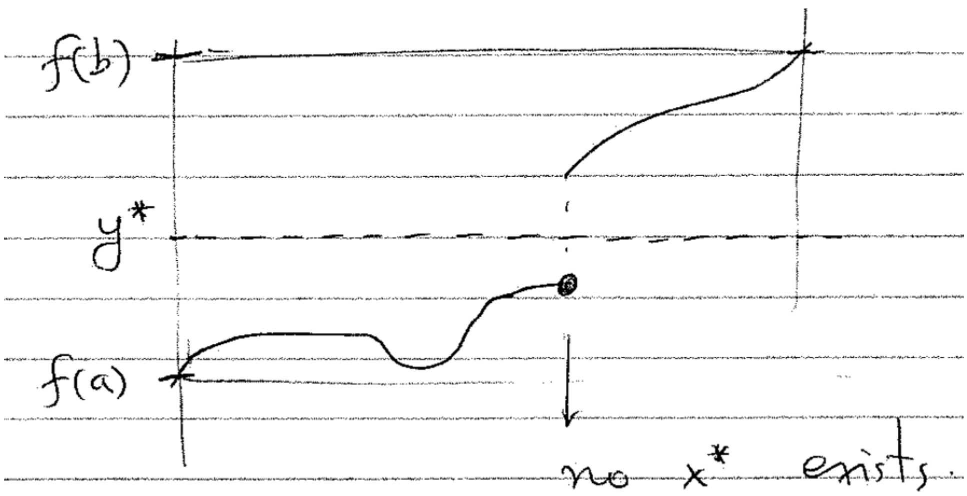
Theorem 6 - Intermediate value theorem
Let f be continuous on the closed interval $a \leq x \leq b$. Given any number y^* between $f(a)$ and $f(b)$, there exists a point x^* between a and b such that $f(x^*) = y^*$.

Picture where it works:



Note: there can be more than one x^*.

Picture where is does not work because the function is not continuous:



Chapter 4

Inverse Functions

Given y as a function of x, when can we express x as a function of y? Here is an easy case:

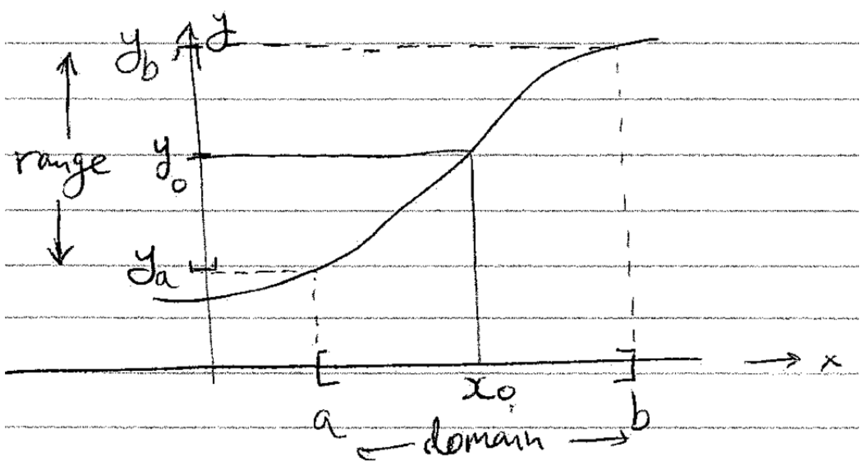
$$y = 3x + 1 \quad \Rightarrow \quad x = \frac{1}{3}(y - 1)$$

Usually we do not have a formula like this, but we can say a lot about the function $x = g(y)$.

Definition

Let $y = f(x)$ be defined on some interval. Given any y_0 in the range of f, if we can find a unique value x_0 in its domain such that $f(x_0) = y_0$, then we can define the ***inverse function***

$$x = g(y) \quad \text{(sometimes written } x = f^{-1}(y)\text{)}$$



Clearly we have

$$f(g(y)) = y \qquad \text{and} \qquad g(f(x)) = x$$

$$\text{or} \quad f(f^{-1}(y)) = y \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

Question: When can we be *certain* an inverse function exists?

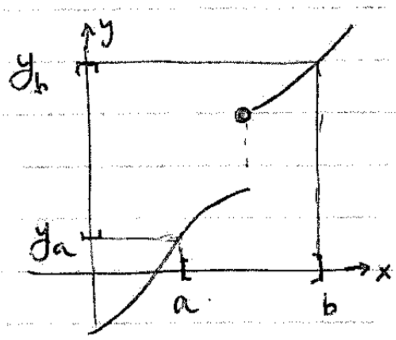
Theorem 1
Let $f(x)$ be strictly increasing or strictly decreasing. Then the inverse function exists.

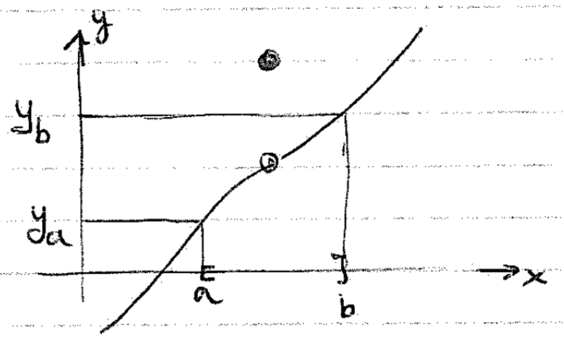
Proof. Obvious from definition of strictly increasing/decreasing. $\square$

Theorem 2
If $f(x)$ is continuous on $[a, b]$ and is strictly increasing (or decreasing), and $f(a) = y_a$ and $f(b) = y_b$, then $x = g(y)$ is defined on $[y_a, y_b]$.

Proof. Easy by the intermediate value theorem. $\square$

Here is what goes wrong if we drop continuity





4.0.1 Derivative of inverse functions

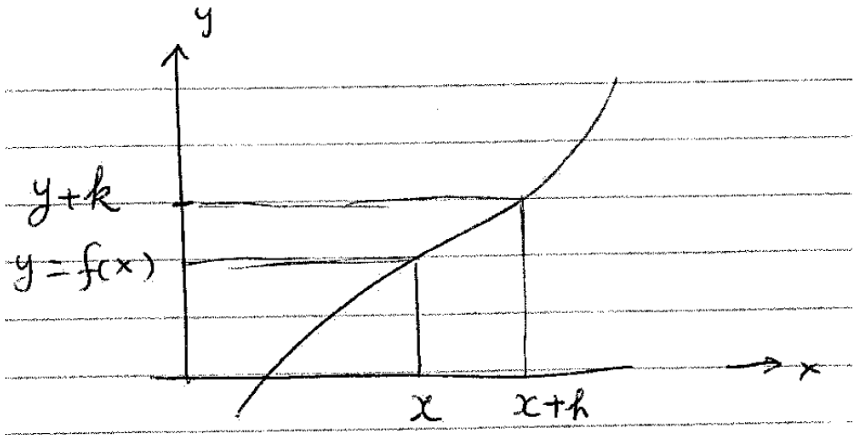
Theorem 3
Let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b). Then the inverse function exists and we have

$$g'(y) = \frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)}.$$

Proof. Need to find

$$\lim_{k \to 0} \frac{g(y+k) - g(y)}{k}$$

where we have $y = f(x)$. Here is a useful picture:



Going from y to $y+k$, increases x to $x+h$. The intermediate value theorem ensures that this is true for all k in fact.

Hence $f(x+h) = y+k \quad \Rightarrow \quad g(y+k) = x+h$. Since $f(x) = y \quad \Rightarrow \quad g(y) = x$.

Back to the limit

$$\lim_{k\to 0} \frac{g(y+k)-g(y)}{k}$$
$$\lim_{h\to 0} = \frac{h}{f(x+h)-f(x)} = \frac{1}{f'(x)}$$

□

Chain rule way: $g(y) = x$ where $y = f(x)$. *(In fact $x = g(y)$, iff $y = f(x)$). i.e*

$$\frac{\mathrm{d}g}{\mathrm{d}y} f'(x) = 1 \quad g' = \frac{1}{f'(x)}$$

Example: Consider
$$y = x^4 + 3x^3 + x - 5, \quad x > 0.$$
Find $g'(0)$ - *i.e* $\frac{\mathrm{d}}{\mathrm{d}y}\, f^{-1}(y)\big|_{y=0}$. **Note:** We will not even attempt to find $x = g(y)$.

Solution: Theorem says $\frac{\mathrm{d}g}{\mathrm{d}y} = g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$. If $y = 0$ then need to solve $0 = f(x)$ by inspection $f(1) = 0 \Rightarrow f'(1) = 14 \Rightarrow g'(0) = \frac{1}{14}$.

The solution above implicitly assumed that $x = 1$ is the *only* point where $f(x) = 0$. Need to justify this. The key is the observation that $f'(x) = 4x^3 + 9x^2 + 1$ which is > 0 for all $x > 0$. Hence the function is *strictly increasing* in the given interval and since $f(0) = -5 < 0$ and $\lim_{x\to+\infty} = +\infty$, by the intermediate value theorem there must be a unique x s.t. $f(x) = 0$. Of course this happens to be $x = 1$ by inspection.

Here are some additional things for you to consider: Suppose $x < 0$ now.

(i) Prove that there is at least one $x < 0$ such that $f(x) = 0$. Let this point(s) be $x = \xi$, i.e. $f(\xi) = 0$, $\xi < 0$.

(ii) Prove that there is at least one point $\xi < c < 0$ where $f'(c) = 0$.

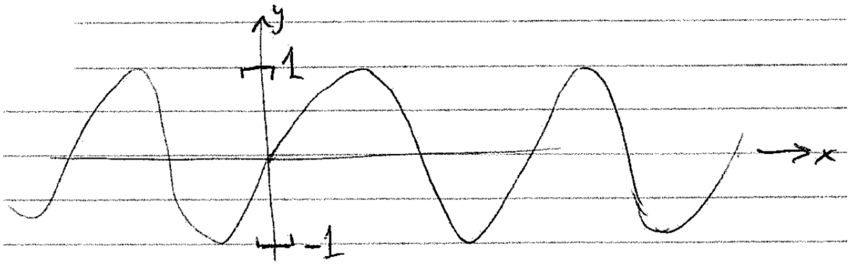
Note: Very useful in solving problems.

$x = g(y)$ *i.e.* some function of y. Our theorem really says $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\mathrm{d}y/\mathrm{d}x}$.

4.0.2 Some special inverse functions

(i) The arcsin, or $\sin^{-1}$.

Consider $y = \sin(x)$ *(shown below)*

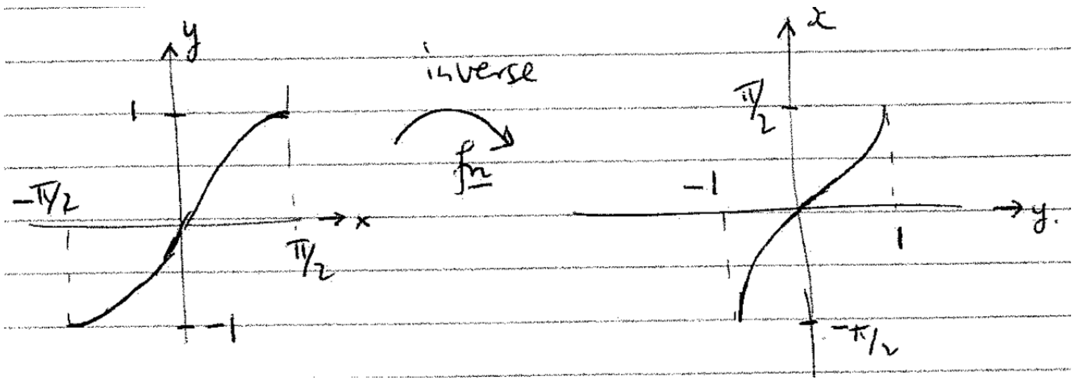


So given any $-1 \leq y \leq 1$ there are an infinite number of x such that $y = \sin(x)$. If we restrict our domain to regions where $\sin(x)$ is strictly increasing or decreasing, then we can find inverse functions - we have a theorem. By convention we take $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for the domain (the range is of course $[-1, 1]$).

Now $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) > 0$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = 0$ at $x = \pm\frac{\pi}{2}$.

Let the inverse function be $g(y) = x$. Then

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{(\sin(x))'} > 0 \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$



$$g'(y) = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-y^2}}.$$

Think of y as a dummy variable now. Then $\arcsin(y) = \sin^{-1}(y)$ is a function with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Instead of y use x, *i.e.* $y = f(x) = \sin^{-1}(x)$. Then

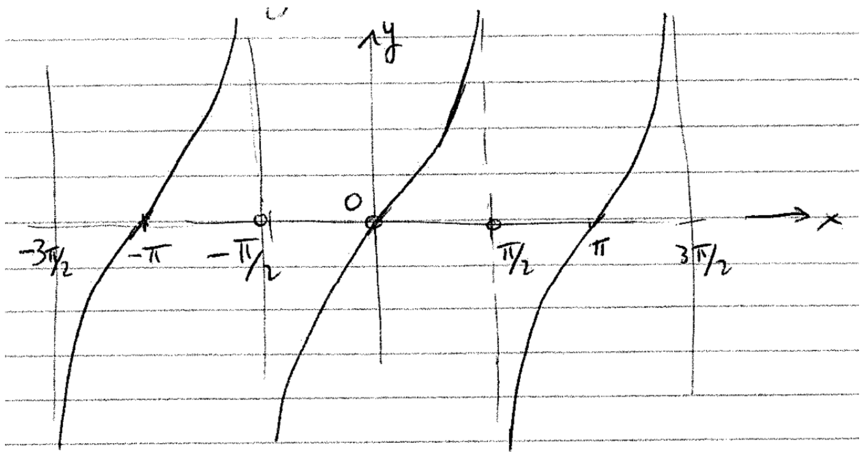
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Once you have identified the domain and range where the inverse function exists, there is an easier way (equivalent) to find derivatives.

Start with $y = \sin^{-1}(x)$ i.e y is the angle whose sin is x. Then $\sin(y) = x$. By the chain rule, $(\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$ as shown above.

(ii) arctan or $\tan^{-1}$

Consider $y = \tan(x)$

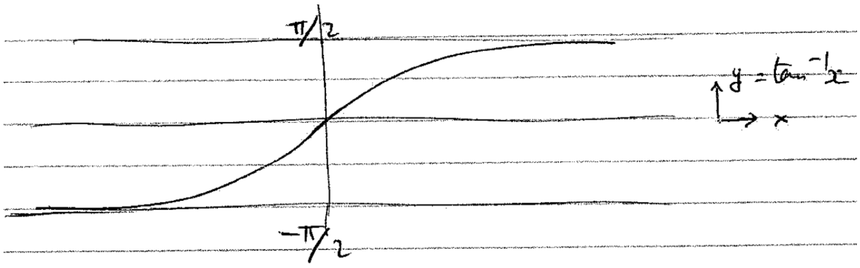


Can define $\tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = 1 + \tan^2(x) > 0$. So $x = g(y)$ the inverse function has domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Derivative of $\tan^{-1}$:

$$\begin{aligned} y &= \tan^{-1}(x) \\ \tan(y) &= x \\ (1+\tan^2(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2} \end{aligned}$$

Here is the graph of $y = \tan^{-1}(x)$



Note for later material: we showed that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$. In other words, the *anti-derivative* of $\frac{1}{1+x^2}$ is $\tan^{-1}(x)+c$ where c is a constant. Similarly, the *anti-derivative* of $\frac{1}{\sqrt{1-x^2}}$ is $\sin^{-1}(x)+c$.

Quiz Problems Week 1

1. For the following functions construct specific $\varepsilon-\delta$ definitions of continuity at $x=0$. In other words given a ε you need to find $\delta(\varepsilon)$.

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ x^2 & \text{for } x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & \text{for } x \geq 0 \\ |x|^{1/2} & \text{for } x < 0 \end{cases}$$

2. Consider the function

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

   (a) Is $f(x)$ a continuous function?

   (b) Show that $f'(0)$ exists and find its value.

   (c) Define $g(x) = f'(x)$, $x \neq 0$, and $g(0) = f'(0)$. Determine whether $g(x)$ is differentiable or not.

   (d) If instead of x^2 in the definition of $f(x)$ we had x^n where n is a positive integer. How many derivatives of $f(x)$ would exist in this case?

3. A spherical balloon is being blown up by injecting air into it at 1 liter per second. When its radius is 1 m, find the rate at which its area is increasing (pay attention to the units).

4. Sand is being piled onto a conical pile at a constant rate of $R\ cm^3/s$. As the pile grows, frictional forces between sand particles constrain the height of the pile to be equal to the radius of its base.

   (i) When the height equals 1 cm, find the rate at which it is increasing.

(ii) If the height at time t is $h(t)$, find an explicit expression for it. What happens to its rate of change as t becomes large? Explain physically/intuitively.

5. (a) Determine the regions of increase and decrease of the function $f(x) = x^3 - 2x + 1$.

   (b) Sketch functions for which the intermediate value theorem holds and:

   (i) For a chosen y^* there are *at most two* values of x^*.

   (ii) For a chosen y^* I can choose an interval $[a, b]$ to have as many x^* as I want. [Any guess as to what function this is?]

   (iii) For a chosen y^* there does not exist a x^*, i.e. Theorem 6 does not hold.

6. Find $\tan^{-1}(\tan \frac{3\pi}{4})$ and $\arctan(\tan 2\pi)$. (No computers/calculators!)