Exercise 6.1. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a, b])$:

(i)
$$\|f\|_1 = \int_a^b |f(t)| \, dt$$

(ii)
$$\|f\|_\infty = \max_{t \in [a,b]} |f(t)|$$

(iii)
$$\|f\|_2 = \left(\int_a^b |f(t)|^2 \, dt \right)^{1/2}$$

Hint: to show that $\|\cdot\|_2$ is a norm, you need to use the Cauchy-Schwarz inequality and the definition of the integral as the limit of certain sums.

Exercise 6.2. Show that if V is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2 \, \mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric $\mathrm{d}_{\mathrm{disc}}$ on $\mathbb{R}^2$.

Exercise 6.3. Let (X, d) be a metric space.

(i) Show that for every x, y, and z in X, we have

$$|\mathrm{d}(x, z) - \mathrm{d}(y, z)| \leq \mathrm{d}(x, y).$$

(ii) Show that for all x, y, z and t in X, we have

$$|\mathrm{d}(x, y) - \mathrm{d}(z, t)| \leq \mathrm{d}(x, z) + \mathrm{d}(y, t).$$

(iii) Show that for all x_1, x_2, ..., x_n in X, we have

$$\mathrm{d}(x_1, x_n) \leq \mathrm{d}(x_1, x_2) + \mathrm{d}(x_2, x_3) + \cdots + \mathrm{d}(x_{n-1}, x_n).$$

Exercise 6.4. Let (X, d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By an example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n \in \mathbb{N}} B_{1/n}(x) = \{x\}.$$

Exercise 6.5. (i) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x, y) \leq \mathrm{d}_2(x, y) \leq \sqrt{n} \cdot \mathrm{d}_\infty(x, y).$$

(ii) Show that for all x and y in $\mathbb{R}^n$, we have

$$d_\infty(x, y) \leq d_1(x, y) \leq n \cdot d_\infty(x, y).$$

(iii) Show/conclude that for all x and y in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}} d_2(x, y) \leq d_1(x, y) \leq n \, d_2(x, y).$$

(iv) Conclude that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.

Exercise 6.6. Let (X, d_{disc}) be a discrete metric space, and $(x_n)_{n \geq 1}$ be a sequence in X. Then, $(x_n)_{n \geq 1}$ converges in (X, d_{disc}) if and only if the sequence $(x_n)_{n \geq 1}$ is eventually constant.

Exercise 6.7. Let (X, d) be a metric space, and $(x_n)_{n \geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n \geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.
  Hint: U can be the ball $B_r(x)$.

Exercise 6.8. Let (X, d_{disc}) be a discrete metric space. Show that every set in X is closed.
  Hint: First show that every set in X is open with respect to d_{disc}.