MATH50001/50017/50018 - Analysis II

Complex Analysis

Lecture 4

Section: Elementary functions.

1. Exponential function.

Definition. We define exponential e^z $(z = x + iy \in \mathbb{C})$ as:

$$e^z = e^x \cos y + ie^x \sin y.$$

Properties:

a) If $y = 0$ then $e^z = e^x$.

b) e^z is entire (holomorphic for any $z \in \mathbb{C}$)

Indeed, for that we check the C-R equations. Since $u = \operatorname{Re} f = e^x \cos y$ and $v = \operatorname{Im} f = e^x \sin y$, we have

$$u'_x = e^x \cos y = v'_y \quad \text{and} \quad u'_y = e^x(-\sin y) = -v'_x.$$

c)

$$\frac{\partial}{\partial z} e^z = \frac{\partial}{\partial x} e^x \cos y + i \frac{\partial}{\partial x} e^x \sin y = e^z.$$

d) Let $g(z)$ be holomorphic. Then

$$\frac{\partial}{\partial z}\, e^{g(z)} = e^{g(z)}\, g'(z).$$

e) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) \\ &= e^{x_1+x_2}\big(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\big) \\ &= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) = e^{z_1}\, e^{z_2}. \end{aligned}$$

f) $|e^z| = |e^x|\,|e^{iy}| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x.$

The function e^z is 2π-periodic with respect to y.

g) Applying the De Moivres formula

$$(\cos y + i\ \sin y)^n = \cos ny + i\ \sin ny$$

we obtain

$$\left(e^{iy}\right)^n = e^{iny}.$$

h) Since $\arg z = \arctan y/x$

$$\arg e^z = \arctan \frac{e^x\ \sin y}{e^x\ \cos y} = \arctan(\tan y) = y + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Definition. If f is holomorphic for all $z \in \mathbb{C}$ then it calls *entire*.

Clearly the exponential function e^z is entire.

2. Trigonometric functions.

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{cases}$$

Definition. For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \qquad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Properties:

a) $\sin z$ and $\cos z$ are entire functions

b) $\frac{\partial}{\partial z}\sin z = \cos z$ and $\frac{\partial}{\partial z}\cos z = -\sin z$.

c) $\sin^2 z + \cos^2 z = 1$.

Indeed:

$$-\frac{1}{4}\left(e^{iz} - e^{-iz}\right)^2 + \frac{1}{4}\left(e^{iz} + e^{-iz}\right)^2 = \cdots = 1.$$

d)

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

3. Logarithmic functions.

Let $z = r(\cos\theta + i\sin\theta) = r\,e^{i\,\theta}$.

Definition. $\log z = \ln|z| + i\arg z = \log r + i(\theta + 2\pi k), \quad z \neq 0,$

where $\quad k = 0, \pm 1, \pm 2, \ldots.$

Clearly:

$$e^{\log z} = e^{\ln r + i\,(\theta + 2\pi k)} = r\,e^{i\,(\theta + 2\pi k)} = r\,(\cos\theta + i\,\sin\theta) = x + i\,y = z.$$

Remark. The function log is a multi-valued function.

Definition. We define $\operatorname{Log} z$ as the singe-valued function:

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal value of the argument, namely, $-\pi < \operatorname{Arg} z \leq \pi$.

Remark. The function Log is a single-valued function.

Examples.

$\operatorname{Log}(-1) = i\pi,$
$\operatorname{Log}(2i) = \ln 2 + i\pi/2,$
$\operatorname{Log}(1-i) = \ln\sqrt{2} - i\pi/4.$

Properties:

a) $\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2)$. Indeed

$$\log(z_1 \cdot z_2) = \ln|z_1 z_2| + i\arg(z_1 \cdot z_2)$$
$$= \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 = \log z_1 + \log z_2.$$

Remark. $\operatorname{Log}(z_1 \cdot z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$, because $\operatorname{Arg}(z_1 \cdot z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

b) The function $\operatorname{Log} z$ is holomorphic in $\mathbb{C} \setminus \{(-\infty, 0]\}$.

Indeed, we have already checked that the C-R equations are satisfied:

$$\operatorname{Log} z = \ln r + i\,\theta = u + iv, \qquad -\pi < \theta \leq \pi.$$

Therefore we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot 1 = \frac{1}{r}\,\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\,\frac{\partial u}{\partial \theta}.$$

Exercise. Compute $(\operatorname{Log} z)'$.

4. Powers.

Definition. For any $\alpha \in \mathbb{C}$, we define $z^\alpha = e^{\alpha \log z}$ as a multi-valued function.

Example. $i^i = e^{i \log i} = e^{i\,(i\pi/2 + i2\pi k)} = e^{-\pi/2}\, e^{-2\pi k}$, $k = 0, \pm 1, \pm 2, \ldots$.

Definition. We define the principal value of z^α, $\alpha \in \mathbb{C}$, as

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Property:

a) $z^{\alpha_1} \cdot z^{\alpha_2} = e^{\alpha_1 \operatorname{Log} z}\, e^{\alpha_2 \operatorname{Log} z} = e^{(\alpha_1 + \alpha_2) \operatorname{Log} z} = z^{\alpha_1 + \alpha_2}$.

Section: Parametrised curve.

Definition. A *parametrised curve* is a function $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parametrised curve is smooth if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as the one-sided limits

$$z'(a) = \lim_{h \to 0,\, h > 0} \frac{z(a+h) - z(a)}{h}, \quad z'(b) = \lim_{h \to 0,\, h < 0} \frac{z(b+h) - z(b)}{h}.$$

Similarly we say that the parametrised curve is piecewise - smooth if z is continuous on $[a, b]$ and if there exist a finite number of points $a = a_0 < a_1 < \cdots < a_n = b$, where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$. In particular, the righthand derivative at a_k may differ from the left-hand derivative at a_k for $k = 1, 2, \ldots, n-1$.

Two parametrisations,

$$z : [a, b] \longrightarrow \mathbb{C} \quad \text{and} \quad \tilde{z} : [c, d] \longrightarrow \mathbb{C},$$

are equivalent if there exists a continuously differentiable bijection $s \longrightarrow t(s)$ from $[c, d]$ to $[a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says precisely that the orientation is preserved: as s travels from c to d, then $t(s)$ travels from a to b.

Given a smooth curve γ in $\mathbb{C}$ parametrised by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ we define the integral of f along γ by

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

In order for this definition to be meaningful, we must show that the right-hand integral is independent of the parametrisation chosen for γ. Say that $\tilde{z}$ is an equivalent parametrisation as above. Then the change of variables formula and the chain rule imply that

$$\begin{aligned}\int_a^b f(z(t))\, z'(t)\, dt &= \int_c^d f(z(t(s)))\, z'(t(s))\, t'(s)\, ds \\ &= \int_c^d f(\tilde{z}(s))\, \tilde{z}'(s)\, ds.\end{aligned}$$

This proves that the integral of f over γ is well defined.

If γ is piecewise smooth, then the integral of f over γ is the sum of the integrals of f over the smooth parts of γ, so if $z(t)$ is a piecewise-smooth parametrisation as before, then

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\,z'(t)\,dt.$$

We can define a curve γ^- obtained from the curve γ by reversing the orientation (so that γ and γ^- consist of the same points in the plane). As a particular parametrisation for γ^- we can take $z^- : [a, b] \to \mathbb{C}$ defined by

$$z^-(t) = z(b + a - t).$$

A smooth or piecewise-smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations. A smooth or piecewise-smooth curve is simple if it is not self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$, or $s = a$ and $t = b$.

A basic example consists of a circle. Consider the circle $C_r(z_0)$ centred at z_0 and of radius r, which by definition is the set

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The positive orientation (counterclockwise) is the one that is given by the standard parametrisation

$$z(t) = z_0 + r\,e^{it}, \quad \text{where} \quad t \in [0, 2\pi],$$

while the negative orientation (clockwise) is given by

$$z(t) = z_0 + r\,e^{-it}, \quad \text{where} \quad t \in [0, 2\pi].$$

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length}\,(\gamma) = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Quizzes

Question 1: Let $f(z) = e^{iz^2}$. What is true?

Answers:

A. $|f(z)| = 1$

B. There is $C > 0$ such that $|f(z)| < C$

C. $f(z)$ is unbounded

Question 2: Which of the following functions are multivalued on the complex plane?

Answers:

A. e^z

B. $\log z$

C. $z^{1/3}$

D. z^2

Thank you