

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometry of Curves & Surfaces

Date: Tuesday, May 20, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Compute the length of the curve $\alpha : [2, 4] \rightarrow \mathbb{R}^3$ defined by

$$\alpha(t) = (t \cos(t), t \sin(t), \frac{1}{3}(2t)^{3/2}).$$

(6 marks)

- (b) Compute the curvature of the curve $\phi : (0, \infty) \rightarrow \mathbb{R}^2$ given by

$$\phi(t) = (t^2, t^3).$$

(6 marks)

- (c) Let $\alpha : [a, b] \rightarrow \mathbb{R}^3$ be a curve parametrized by arclength, with $\alpha''(t) \neq 0$ for all t .

(i) Write down the Frenet equations for α . (3 marks)

(ii) Prove that there exists a vector $V(t)$ such that the Frenet frame satisfies the equations

$$T' = V \times T$$

$$N' = V \times N$$

$$B' = V \times B. \quad (5 \text{ marks})$$

(Total: 20 marks)

2. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid 1 + 2x^2 + y^3 = z^2\}$, and let $p = (2, 0, 3) \in S$.

- (a) Prove that S is a regular surface, and determine the tangent plane $T_p S$. (7 marks)

- (b) Prove that the map $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$\phi(u, v) = (u, (v^2 - 2u^2 - 1)^{1/3}, v)$$

is not a chart for S at p . (5 marks)

- (c) Now let $G = \{(x, y, z) \in \mathbb{R}^3 \mid z^4 = 1 + 2x^2 + y^3\}$ and define a map $F : G \rightarrow S$ by $F(x, y, z) = (x, y, z^2)$. In the following parts you may assume without proof that G is a regular surface.

(i) Prove that $F(G) \subset S$, and that F is a smooth map. (3 marks)

(ii) Give a basis of $T_q G$, where $q = (2, 0, \sqrt{3})$, and compute the differential dF_q in terms of this basis of $T_q G$. (5 marks)

(Total: 20 marks)

3. In this problem S is always an oriented regular surface in \mathbb{R}^3 .
- (a) Define the *second fundamental form* A at p . (3 marks)
 - (b) Define $f : S \rightarrow \mathbb{R}$ by $f(x, y, z) = z$, and let $p \in S$ be a local maximum of f .
 - (i) Determine the tangent plane $T_p S$. (4 marks)
 - (ii) Show that the curvature of S at p satisfies $K(p) \geq 0$. (3 marks)
 - (c) Now let $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = \frac{1}{2}x^2 + \frac{1}{4}y^4\}$, and let $p = (0, 0, 0) \in S$.
 - (i) Compute the second fundamental form at p . (4 marks)
 - (ii) Determine the principal curvatures and principal directions of S at p . (3 marks)
 - (iii) Determine the Gaussian curvature and the mean curvature of S at p . (3 marks)

(Total: 20 marks)

4. (a) Define the *normal curvature* and *geodesic curvature* of a curve $\alpha(t)$ parametrized by arclength in a regular, oriented surface S . (4 marks)
- (b) State the *Gauss–Bonnet theorem*. (4 marks)
- (c) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 = 1\}$, which you may assume without proof to be a compact regular surface. Define $\gamma : [0, 2\pi] \rightarrow S$ by

$$\gamma(t) = \frac{(\cos(t), \sin(t), 0)}{(\cos^4(t) + \sin^4(t))^{1/4}}.$$

Prove that γ is a geodesic.

(Hint: use the fact that γ lies in the plane $\{z = 0\}$.) (6 marks)

- (d) Let S be the surface from part (c), and let $T = \{(x, y, z) \in S \mid z \geq 0\}$. Compute the integral $\int_T K dA$, where K denotes Gaussian curvature. (6 marks)

(Total: 20 marks)

5. (a) Let $\phi : U \rightarrow \mathbb{R}^3$ be a chart for a surface S , and suppose that in this chart the first fundamental form is given by the matrix

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- (i) Prove that the Christoffel symbols satisfy $\Gamma_{ij}^k = 0$ for all i, j, k . (4 marks)
(ii) Prove that the second fundamental form on $\phi(U)$ cannot be given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(4 marks)

- (b) Let $P = \{z = 0\}$ be the xy -plane in \mathbb{R}^3 , and let $\alpha : [0, L] \rightarrow P$ be a simple closed curve parametrized by arclength. Prove that $\int_0^L k(t) dt = \pm 2\pi$. (5 marks)
- (c) Let $S \subset \mathbb{R}^3$ be a regular surface that is diffeomorphic to a plane.
- (i) Let $\gamma \subset S$ be a simple closed geodesic, oriented as the boundary of a region $R \subset S$. Compute $\int_R K dA$. (3 marks)
- (ii) Prove that the saddle $S = \{z = x^2 - y^2\}$ does not contain any simple closed geodesics. You may use any results from the course, provided you state them clearly. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60032/MATH70032

Geom. of Curves and Surfaces (Solutions)

Setter's signature

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1. (a) The curve α has velocity

meth seen ↓

$$\alpha'(t) = (\cos(t) - t \sin(t), \sin(t) + t \cos(t), (2t)^{1/2}),$$

so its length squared is given by

$$\begin{aligned} |\alpha'(t)|^2 &= (\cos(t) - t \sin(t))^2 + (\sin(t) + t \cos(t))^2 + ((2t)^{1/2})^2 \\ &= (\cos^2(t) + \sin^2(t)) + t^2(\sin^2(t) + \cos^2(t)) + 2t \\ &= 1 + 2t + t^2 = (1+t)^2. \end{aligned}$$

Thus the length of α is

$$\int_2^4 |\alpha'(t)| dt = \int_2^4 (1+t) dt = \left[t + \frac{t^2}{2} \right]_{t=2}^{t=4} = 12 - 4 = 8.$$

6, A

- (b) We recall that if $\phi : [a, b] \rightarrow \mathbb{R}^2$ is a plane curve not necessarily parametrized by arclength, then

$$k(t) = \frac{\langle \phi''(t), N(t) \rangle}{|\phi'(t)|^2}$$

where $N(t)$ is a unit normal vector to $\phi(t)$. We compute that

$$\begin{aligned} \phi'(t) &= (2t, 3t^2) \\ \phi''(t) &= (2, 6t), \end{aligned}$$

so then $|\phi'(t)| = (9t^4 + 4t^2)^{1/2}$ and we can take

$$N(t) = \frac{(-3t^2, 2t)}{(9t^4 + 4t^2)^{1/2}}.$$

We plug this into the above formula to get

$$k(t) = \frac{(-6t^2 + 12t^2)/(9t^4 + 4t^2)^{1/2}}{9t^4 + 4t^2} = \boxed{\frac{6t^2}{(9t^4 + 4t^2)^{3/2}}}.$$

6, B

- (c) (i) If $(T(t), N(t), B(t))$ is the Frenet frame at $\alpha(t)$, and $k(t)$ and $\tau(t)$ are the curvature and torsion respectively, then

$$\begin{cases} T' = kN \\ N' = -kT + \tau B \quad \text{or equivalently} \\ B' = \tau N, \end{cases} \quad \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

3, A

- (ii) Since (T, N, B) is an orthonormal basis for \mathbb{R}^3 , we can try to write $V = pT + qN + rB$ for some functions p, q, r . We recall that $T \times N = B$, which also implies that $N \times B = T$ and $B \times T = N$, and then we have

$$\begin{aligned} V \times T &= (pT + qN + rB) \times T = -qB + rN \\ V \times N &= (pT + qN + rB) \times N = pB - rT \\ V \times B &= (pT + qN + rB) \times B = -pN + qT. \end{aligned}$$

unseen ↓

If each of the terms on the left equal T' , N' , and B' respectively, then we combine this with the Frenet equations to get

$$\begin{aligned} kN &= -qb + rN \\ -kT + \tau B &= pB - rT \\ -\tau N &= -pN + qT, \end{aligned}$$

which is uniquely solved by $(p, q, r) = (\tau, 0, k)$, so we have $V = \tau T + kB$. 5, D

2. (a) If we define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $f(x, y, z) = z^2 - 2x^2 - y^3$, then $S = f^{-1}(1)$. At any point of \mathbb{R}^3 we have

$$\nabla f(x, y, z) = (-4x, -3y^2, 2z),$$

which is only zero at $(0, 0, 0) \notin S$, so $\nabla f \neq 0$ on all of S . This means that S is a regular level set of the smooth function f , so it is a regular surface.

At $p = (2, 0, 3)$ the vector $\nabla f(2, 0, 3) = (-8, 0, 6)$ is normal to S , so $T_p S$ is its orthogonal complement:

$$T_p S = \{-8x + 6z = 0\} = \{4x = 3z\}.$$

meth seen ↓

- (b) We can check that $\phi(2, 3) = p$ and that $\phi(u, v) \in S$, for all (u, v) , because

$$1 + 2(u)^2 + ((v^2 - 2u^2 - 1)^{1/3})^3 = (v)^2,$$

7, A

seen/sim.seen ↓

so neither of these is a problem; it's even true that ϕ is a homeomorphism onto its image. However, the chart fails to be differentiable at $\phi^{-1}(p)$, because neither of

$$\begin{aligned}\frac{\partial \phi}{\partial u}(u, v) &= \left(1, \frac{-4u}{3(v^2 - 2u^2 - 1)^{2/3}}, 0\right), \\ \frac{\partial \phi}{\partial v}(u, v) &= \left(0, \frac{2v}{3(v^2 - 2u^2 - 1)^{2/3}}, 1\right)\end{aligned}$$

is defined when $(u, v) = (2, 3)$.

5, A

seen/sim.seen ↓

- (c) (i) Given a point $(a, b, c) \in G$, we have $c^4 = 1 + 2a^2 + b^3$ by definition, and so $(c^2)^2 = 1 + 2a^2 + b^3$, hence

$$F(a, b, c) = (a, b, c^2) \in S.$$

The map F is smooth if for every chart $\phi : U \rightarrow \mathbb{R}^3$ for G , the composition $F \circ \phi$ is smooth. But ϕ is smooth by the definition of a chart, and F is the restriction to G of a smooth map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, so $F \circ \phi$ is the composition of two smooth maps and is therefore smooth.

3, B

meth seen ↓

- (ii) We give G the chart $\phi(u, v) = (u, v, (1 + 2u^2 + v^3)^{1/4})$, with $\phi^{-1}(q) = (2, 0)$, and then $T_q S$ has basis

$$\begin{aligned}\frac{\partial \phi}{\partial u}(0, 0) &= \left(1, 0, \frac{4u}{4(1 + 2u^2 + v^3)^{3/4}}\right) \Big|_{(u,v)=(2,0)} = \left(1, 0, \frac{2}{3^{3/2}}\right), \\ \frac{\partial \phi}{\partial v}(0, 0) &= \left(0, 1, \frac{3v^2}{4(1 + 2u^2 + v^3)^{3/4}}\right) \Big|_{(u,v)=(2,0)} = (0, 1, 0).\end{aligned}$$

We then have $F(\phi(u, v)) = (u, v, (1 + 2u^2 + v^3)^{1/2})$, and so

$$\begin{aligned}dF_q \left(1, 0, \frac{2}{3^{3/2}}\right) &= dF_p(\phi_u(2, 0)) = \frac{\partial(F \circ \phi)}{\partial u} \Big|_{(u,v)=(2,0)} \\ &= \left(1, 0, \frac{4u}{2(1 + 2u^2 + v^3)^{1/2}}\right) \Big|_{(u,v)=(2,0)} = (1, 0, 4/3),\end{aligned}$$

and similarly

$$\begin{aligned}dF_q(0, 1, 0) &= \frac{\partial(F \circ \phi)}{\partial v} \Big|_{(u,v)=(2,0)} = \left(0, 1, \frac{3v^2}{2(1 + 2u^2 + v^3)^{1/2}}\right) \Big|_{(u,v)=(2,0)} \\ &= (0, 1, 0).\end{aligned}$$

5, D

3. (a) If $N : S \rightarrow S^2$ denotes the Gauss map, then the second fundamental form at p is the map $A : T_p S \times T_p S \rightarrow \mathbb{R}$ defined by

seen \Downarrow

$$A(v, w) = \langle v, -dN_p(w) \rangle.$$

- (b) (i) Fix $v \in T_p S$, and let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ be a regular curve with $\alpha(0) = p$ and $\alpha'(0) = v$. If we write

3, A

seen/sim.seen \Downarrow

$$\alpha(t) = (x(t), y(t), z(t)),$$

then $f(\alpha(t)) = z(t)$ has a local maximum at $t = 0$, so $z'(t) = 0$ and thus

$$v = \alpha'(0) = (x'(0), y'(0), 0).$$

In particular every $v \in T_p S$ has z -coordinate zero, so $T_p S$ is a plane contained in $\{z = 0\}$, hence $T_p S = \{z = 0\}$.

4, B

part seen \Downarrow

- (ii) If we had $K(p) < 0$, then any neighborhood of p would contain points of S on both sides of the tangent plane at p . But since p is a local maximum of f , we can find a neighborhood of p on which all of S lies on or below this tangent plane, so $K(p) < 0$ cannot hold and therefore $K(p) \geq 0$.

- (c) (i) We equip S with the chart $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$\phi(u, v) = (u, v, \frac{1}{2}u^2 + \frac{1}{4}v^4),$$

so $\phi(0, 0) = p$. We compute the derivatives of this chart as

$$\phi_u = (1, 0, u) \quad \phi_u(0, 0) = (1, 0, 0),$$

$$\phi_v = (0, 1, v^3) \quad \phi_v(0, 0) = (0, 1, 0),$$

$$N \circ \phi = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-u, -v^3, 1)}{\sqrt{u^2 + v^6 + 1}}, \quad N(p) = N(\phi(0, 0)) = (0, 0, 1).$$

Rather than try to directly compute $dN_p(\phi_u) = \frac{\partial}{\partial u}(N \circ \phi)$ and so on, we compute

$$\phi_{uu} = (0, 0, 1), \quad \phi_{uv} = \phi_{vu} = (0, 0, 0), \quad \phi_{vv} = (0, 0, 3v^2),$$

so that $\phi_{uu}(0, 0) = (0, 0, 1)$ and the other second derivatives all vanish at $(0, 0)$, and then

$$A = \begin{pmatrix} A(\phi_u, \phi_u) & A(\phi_u, \phi_v) \\ A(\phi_v, \phi_u) & A(\phi_v, \phi_v) \end{pmatrix} = \begin{pmatrix} \langle N, \phi_{uu} \rangle & \langle N, \phi_{uv} \rangle \\ \langle N, \phi_{vu} \rangle & \langle N, \phi_{vv} \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, we have $A((x, y, 0), (z, w, 0)) = xz$.

4, B

meth seen \Downarrow

- (ii) If $w = (a, b, 0) \in T_p S$, we have

$$A(w, w) = A((a, b, 0), (a, b, 0)) = a^2.$$

Among unit vectors in $T_p S$ (meaning $a^2 + b^2 = 1$) we have

$$\max_{|w|=1} A(w, w) = A((1, 0, 0), (1, 0, 0)) = 1,$$

$$\min_{|w|=1} A(w, w) = A((0, 1, 0), (0, 1, 0)) = 0,$$

so the principal directions and curvatures are $X_1 = (1, 0, 0)$, $\lambda_1 = 1$ and $X_2 = (0, 1, 0)$, $\lambda_2 = 0$.

3, C

meth seen ↓

(iii) We take the matrix

$$A = \begin{pmatrix} A(\phi_u, \phi_u) & A(\phi_u, \phi_v) \\ A(\phi_v, \phi_u) & A(\phi_v, \phi_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

represent the first fundamental form at p by the matrix

$$g = \begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\sigma = g^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$K(p) = \det(\sigma) = 0, \quad H(p) = \frac{1}{2} \operatorname{tr}(\sigma) = \frac{1}{2}.$$

Alternatively, these can be computed using the principal curvatures, as $K(p) = \lambda_1 \lambda_2 = 0$ and $H(p) = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}$.

3, B

4. (a) The curvature vector $\vec{k}(t) = \alpha''(t)$ is orthogonal to the tangent vector $\alpha'(t)$, so if N is the Gauss map on S then in the orthonormal basis $(\alpha', N \times \alpha', N)$ we can write

$$\alpha''(t) = k_n(t)N(\alpha(t)) + k_g(t)(N(\alpha(t)) \times \alpha'(t)).$$

The coefficients $k_n(t)$ and $k_g(t)$ are the normal and geodesic curvatures, respectively. Equivalently, we have

$$k_n(t) = \langle \alpha''(t), N(\alpha(t)) \rangle,$$

$$k_g(t) = \langle \alpha''(t), N(\alpha(t)) \times \alpha'(t) \rangle.$$

4, A

seen ↓

- (b) Let $S \subset \mathbb{R}^3$ be a regular surface with boundary. Then

$$\int_{\partial S} k_g ds + \int_S K dA = 2\pi \cdot \chi(S),$$

where k_g is geodesic curvature, K is Gaussian curvature, and χ is Euler characteristic.

- (c) We first check that $\gamma(t) \in S$, because

$$\left(\frac{\cos(t)}{(\cos^4(t) + \sin^4(t))^{1/4}} \right)^4 + \left(\frac{\sin(t)}{(\cos^4(t) + \sin^4(t))^{1/4}} \right)^4 + 0^4 = \frac{\cos^4(t) + \sin^4(t)}{\cos^4(t) + \sin^4(t)} = 1.$$

4, A

unseen ↓

Now we note that γ is *not* parametrized by arclength, but if $\phi(t) = \gamma(f(t))$ is a reparametrization by arclength then ϕ is a geodesic if and only if ϕ satisfies $k_g(t) = 0$, i.e., iff $\langle \phi''(t), N(\phi(t)) \times \phi'(t) \rangle = 0$. Now since S is a regular level set of $F(x, y, z) = x^4 + y^4 + z^4$, namely $S = F^{-1}(1)$, we have

$$N(x, y, z) = \frac{\nabla F(x, y, z)}{|\nabla F(x, y, z)|} = \frac{(4x^3, 4y^3, 4z^3)}{\sqrt{16x^6 + 16y^6 + 16z^6}} = \frac{(x^3, y^3, z^3)}{\sqrt{x^6 + y^6 + z^6}},$$

so if $\phi(t) = (x(t), y(t), 0)$ then we can compute

$$N(\phi(t)) = \frac{(x(t)^3, y(t)^3, 0)}{\sqrt{x(t)^6 + y(t)^6}}$$

$$\phi'(t) = (x'(t), y'(t), 0)$$

$$\implies N(\phi(t)) \times \phi'(t) = \left(0, 0, \frac{x(t)y'(t) - x'(t)y(t)}{\sqrt{x(t)^6 + y(t)^6}} \right)$$

while $\phi''(t) = (x''(t), y''(t), 0)$. Regardless of the actual functions x and y , it follows that

$$k_g(t) = \langle \phi''(t), N(\phi(t)) \times \phi'(t) \rangle = 0.$$

Thus ϕ is a geodesic, hence so is its reparametrization γ .

- (d) We note that $\partial T = \{(x, y, 0) \mid x^4 + y^4 = 1\}$, which is exactly the curve γ from part (c), so its geodesic curvature is zero. We also observe that T is the graph of $f(x, y) = \sqrt{1 - x^4 - y^4}$ on the domain

6, D

meth seen ↓

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 \leq 1\},$$

which is homeomorphic to a disk, so $\chi(T) = \chi(D) = 1$. Now we apply the Gauss–Bonnet theorem to get

$$\int_T K dA = 2\pi \cdot \chi(S) - \int_{\partial T} k_g ds = 2\pi \cdot 1 - 0 = \boxed{2\pi}.$$

6, C

5. (a) (i) The matrix g tells us that each of the inner products

meth seen ↓

$$\langle \phi_u, \phi_u \rangle = 1, \quad \langle \phi_u, \phi_v \rangle = 0, \quad \langle \phi_v, \phi_v \rangle = 1$$

are constant, so their derivatives in each of u and v are zero. Then we have

$$\begin{aligned} \langle \phi_{uu}, \phi_u \rangle &= \frac{1}{2} \frac{\partial}{\partial u} \langle \phi_u, \phi_u \rangle = 0, & \langle \phi_{uu}, \phi_v \rangle &= \frac{\partial}{\partial u} \langle \phi_u, \phi_v \rangle - \frac{1}{2} \frac{\partial}{\partial v} \langle \phi_u, \phi_u \rangle = 0, \\ \langle \phi_{uv}, \phi_u \rangle &= \frac{1}{2} \frac{\partial}{\partial v} \langle \phi_u, \phi_u \rangle = 0, & \langle \phi_{uv}, \phi_v \rangle &= \frac{1}{2} \frac{\partial}{\partial u} \langle \phi_v, \phi_v \rangle = 0, \\ \langle \phi_{vv}, \phi_u \rangle &= \frac{\partial}{\partial v} \langle \phi_v, \phi_u \rangle - \frac{1}{2} \frac{\partial}{\partial u} \langle \phi_v, \phi_v \rangle = 0, & \langle \phi_{vv}, \phi_v \rangle &= \frac{1}{2} \frac{\partial}{\partial v} \langle \phi_v, \phi_v \rangle = 0. \end{aligned}$$

The Christoffel symbols Γ_{ij}^k are the ϕ_u - and ϕ_v - coefficients of ϕ_{uu} , ϕ_{uv} , and ϕ_{vv} in the bases (ϕ_u, ϕ_v, N) of \mathbb{R}^3 , but since each of $\phi_{uu}, \phi_{uv}, \phi_{vv}$ is orthogonal to ϕ_u and ϕ_v they are multiples of N and so these coefficients must be zero.

Note: the Christoffel symbols are determined by the first fundamental form, so it would also suffice to show that S is locally isometric to a plane with a standard chart as in part (ii) below.

4, M

unseen ↓

(ii) Let $P = \{z = 0\}$ be the xy -plane. We define a chart

$$\psi : U \rightarrow P$$

by the formula $\psi(u, v) = (u, v, 0)$, so that $\psi_u = (1, 0, 0)$ and $\psi_v = (0, 1, 0)$ give rise to the first fundamental form

$$\begin{pmatrix} \langle \psi_u, \psi_u \rangle & \langle \psi_u, \psi_v \rangle \\ \langle \psi_v, \psi_u \rangle & \langle \psi_v, \psi_v \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We then define a map $F : \psi(U) \rightarrow S$ by

$$F(u, v, 0) = \phi(u, v).$$

Then $F \circ \psi = \phi$, so $dF(\psi_u) = \phi_u$ and $dF(\psi_v) = \phi_v$, and since we know that

$$\begin{pmatrix} \langle \phi_u, \phi_u \rangle & \langle \phi_u, \phi_v \rangle \\ \langle \phi_v, \phi_u \rangle & \langle \phi_v, \phi_v \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that F is a local isometry. Gauss's Theorema Egregium therefore tells us that F preserves Gaussian curvature, so for any point $x \in U$ we must have $K_S(\phi(x)) = K_P(\psi(x)) = 0$. But on the other hand the curvature on $\phi(U) \subset S$ is given by $K_S = \frac{\det(A)}{\det(g)} = -2$, so we have a contradiction.

(b) By the Jordan curve theorem we know that α separates P into a compact region C and an unbounded region, with $\partial C = \alpha$ up to orientation and C homeomorphic to a disk. The Gaussian curvature of $C \subset P$ is identically zero, so by Gauss–Bonnet we have

$$2\pi = 2\pi \cdot \chi(C) = \int_{\partial C} k_g ds + \int_C K dA = \int_{\partial C} k_g ds.$$

Now P has normal vector $N(p) = (0, 0, 1)$ for all p , so the curve $\alpha(t) = (x(t), y(t), 0)$ has normal curvature

$$k_n(t) = \langle \alpha''(t), N(t) \rangle = \langle (x''(t), y''(t), 0), (0, 0, 1) \rangle = 0$$

4, M

unseen ↓

and hence $k(t)^2 = k_n(t)^2 + k_g(t)^2$ becomes $k_g(t) = \pm k(t)$. We therefore compute that

$$\int_0^L k(t) dt = \pm \int_0^L k_g(t) dt = \pm \int_{\partial C} k_g(s) ds = \pm 2\pi.$$

5, M

seen ↓

- (c) (i) By the Jordan curve theorem we know that R is diffeomorphic to a disk, so we have

$$\int_R K dA = 2\pi \cdot \chi(R) - \int_{\gamma} k_g ds = 2\pi - \int_{\gamma} 0 ds = 2\pi.$$

3, M

seen/sim.seen ↓

- (ii) Certainly S is diffeomorphic to the plane $P = \{z = 0\}$, because both are graphs of smooth functions $\mathbb{R}^2 \rightarrow \mathbb{R}$. We have seen in lecture that S has strictly negative curvature – we computed that

$$K(x, y, x^2 - y^2) = \frac{-4}{(1 + 4x^2 + 4y^2)^2},$$

though we do not need the precise formula. Now if γ is a simple closed geodesic in S , then the Jordan curve theorem says that γ can be oriented as the boundary of a compact region $R \subset S$, and we have

$$\int_R K dA \leq 0$$

since $K < 0$ on all of R . At the same time we saw in the previous part that $\int_R K dA = 2\pi$, so we have a contradiction and γ cannot exist.

4, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70032 Geometry of Curves & Surfaces Markers Comments

- Question 1 This problem mostly went well, but quite a few people wrote down the definition of the Frenet frame when asked for the Frenet equations. If the equations were written down as well then I gave full marks on that part.
- Question 2 In part (b), many students computed the derivatives of the chart correctly and then said that $0^{-2/3}$ is zero rather than undefined, so they missed that it isn't even differentiable. In part (d), some students computed dF by viewing F as a map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, i.e. as a 3×3 matrix, but then did not apply it correctly (as a linear map) to the basis vectors.
- Question 3 Many students asserted the answer to b(i) without proof. Another common issue was that in c(ii), people took the matrix A from part c(i) and found its eigenvalues, even though it is not a matrix for the linear map $-dN_p$ -- you should use the matrix $\sigma = g^{-1}A$ instead, and check that $g=I$ in this case. Many answers to c(ii) also gave two-dimensional vectors $(1,0)$ and $(0,1)$ for the principal directions, but this just expresses them in the basis (ϕ_u, ϕ_v) of $T_p S$ -- the actual answers should be the vectors ϕ_u and ϕ_v themselves, which after all lie in \mathbb{R}^3 .
- Question 4 In (c), nearly everybody applied the formula from (a) for geodesic curvature to γ , but γ isn't parametrized by arclength! (Several students claimed that it was, using the incorrect formula $\cos^4(t) + \sin^4(t) = 1$). In order to use this you have to first take a reparameterization α by arclength, and then you can finish the argument using α even without finding an explicit form for it. In (d), several students said correctly that $\chi(S) = 2$, but for Gauss-Bonnet the relevant Euler characteristic is actually $\chi(T) = 1$.
- Question 5 In 5(b) nearly everyone asserted without proof that γ has index 1 or -1. This is true, but you need to prove it using the Gauss-Bonnet theorem.