Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship.

M3S4/M4S4

Applied Probability (Solutions)

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1. (i) Either of the following definitions, are appropriate for full marks. A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if **seen ⇓**

1. $N_0 = 0$.
2. The increments are independent, that is, for any $k_1, k_2 \in \mathbb{Z}_+$

$$\mathbb{P}(\{N_t - N_s = k_1\}|\{N_r = k_2, 0 \leq r \leq s\}) = \mathbb{P}(\{N_t - N_s = k_1\}).$$

3. The increments are stationary: for any $l > 0$, $k \in \mathbb{Z}_+$

$$\mathbb{P}(\{N_t - N_s = k\}) = \mathbb{P}(\{N_{t+l} - N_{s+l} = k\})$$

4. There is a 'single arrival'

$$\begin{aligned}\mathbb{P}(\{N_{t+\delta} - N_t = 1\}) &= \lambda\delta + o(\delta)\\ \mathbb{P}(\{N_{t+\delta} - N_t \geq 2\}) &= o(\delta)\end{aligned}$$

Or: A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if

1. $N_0 = 0$
2. The increments are independent
3. For any $0 \leq s < t$, $k \in \mathbb{Z}_+$ we have

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}.$$

That is, the number of events in $[s, t]$ is a Poisson random variable, with mean $\lambda(t-s)$. **5**

(ii) Here we seek to verify the properties of a Poisson process for $\bar{N}_t$. It is enough, to consider the second definition of a Poisson process.
Clearly $\bar{N}_0 = 0$, so let us consider the independence (or not) of the increments. Let $s > t$ then

$$\bar{N}_t - \bar{N}_s = N_t^1 - N_s^1 + N_t^2 - N_s^2$$

since both N_t^1 and N_t^2 have independent increments, so does $\bar{N}_t$. For full marks, one must show the following (or provide an alternative argument):

$$\begin{aligned}\mathbb{P}(\bar{N}_t - \bar{N}_s = k_1|\{\bar{N}_r\}_{0\leq r\leq s}) &= \mathbb{P}(N_t^1 - N_s^1 + N_t^2 - N_s^2 = k_1|\{N_r^1 + N_r^2\}_{0\leq r\leq s})\\ &= \sum_{A_{k_1}} \mathbb{P}(N_t^1 - N_s^1 = j_1, N_t^2 - N_s^2 = j_2|\{N_r^1 + N_r^2\}_{0\leq r\leq s})\\ &= \sum_{A_{k_1}} \mathbb{P}(N_t^1 - N_s^1 = j_1|\{N_r^1\}_{0\leq r\leq s})\mathbb{P}(N_t^2 - N_s^2 = j_2|\{N_r^2\}_{0\leq r\leq s})\\ &= \sum_{A_{k_1}} \mathbb{P}(N_t^1 - N_s^1 = j_1)\mathbb{P}(N_t^2 - N_s^2 = j_2)\\ &= \mathbb{P}(N_t^1 - N_s^1 + N_t^2 - N_s^2 = k_1)\end{aligned}$$

where $A_{k_1} = \{j_1, j_2 \geq 0 : j_1 + j_2 = k_1\}$ and we have used the independence of $\{N_t^1\}$ and $\{N_t^2\}$ and the independence of their increments.

Let us consider the last property. $\bar{N}_t = N_t^1 + N_t^2$, so we simply need to find the distribution of the sum of two independent Poisson distributions. Let us use the Laplace transform.

$$\mathbb{E}[e^{-s\bar{N}_t}] = \mathbb{E}[e^{-s[N_t^1+N_t^2]}] = \mathbb{E}[e^{-sN_t^1}]\mathbb{E}[e^{-sN_t^2}].$$

As

$$\mathbb{E}[e^{-sN_t^1}] = \exp\{\alpha t(e^{-s}-1)\} \quad \mathbb{E}[e^{-sN_t^2}] = \exp\{\beta t(e^{-s}-1)\}$$

it follows

$$\mathbb{E}[e^{-s\bar{N}_t}] = \exp\{(\alpha+\beta)t(e^{-s}-1)\}$$

i.e. a Poisson random variable of parameter $(\alpha+\beta)t$ as required. 6

(iii) Let $X_1, X_2, \ldots$ be the time that an individual spends queuing. By definition: unseen ⇓

$$\begin{aligned}
\phi_{Y_t}(s) &= \mathbb{E}\Big[\exp\{is\sum_{j=1}^{N_t} X_j\}\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\exp\{is\sum_{j=1}^{n_t} X_j\}|N_t = n_t\Big]\Big] \\
&= \mathbb{E}\Big[\phi_X(s)^{N_t}\Big].
\end{aligned}$$

Now

$$\begin{aligned}
\phi_X(s) &= \int_0^\infty e^{ixs}\rho e^{-\rho x}dx \\
&= \Big[-\frac{\rho}{\rho - is}e^{-x(\rho-is)}\Big]_0^\infty \quad s \neq \rho/i \\
&= \frac{\rho}{\rho - is}.
\end{aligned}$$

Now, letting $\bar{\alpha} = \alpha+\beta$, $\bar{\rho} = \frac{\rho}{\rho-is}$

$$\begin{aligned}
\phi_{Y_t}(s) &= e^{-\bar{\alpha}t}\sum_{n=0}^\infty \frac{(\bar{\alpha}\bar{\rho}t)^n}{n!} \\
&= \exp\{-\bar{\alpha}t + \bar{\alpha}\bar{\rho}t\} \\
&= \exp\Big\{\Big(\frac{is}{\rho - is}\Big)(\alpha+\beta)t\Big\}.
\end{aligned}$$

Differentiating w.r.t s, setting $s=0$ and dividing by i yields

$$\mathbb{E}[Y_t|\alpha,\beta] = \frac{[\alpha+\beta]t}{\rho}.$$

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(iv) As the expectation of α and β are both 1, it follows

$$\mathbb{E}[Y_t] = \int_{\mathbb{R}^+\times\mathbb{R}^+} \mathbb{E}[Y_t|\alpha,\beta]e^{-\alpha}e^{-\beta}d\alpha d\beta = \frac{2t}{\rho}.$$

As ρ increases (for fixed t) the expectation goes to zero. This makes sense, as in this case, the expected waiting time in the queue is very small. 3

2. (i) A birth process with intensities $\lambda_0, \lambda_1, \ldots$ is a counting process $\{N_t\}_{t\in[0,\infty)}$ such that seen ⇓

1. $N_0 = 0$
2. Conditionally independent increments: $\mathbb{P}(N_t - N_s|\{N_r\}_{0\leq r\leq s}) = \mathbb{P}(N_t - N_s|N_s)$
3. There is a 'single arrival'

$$\mathbb{P}(N_{t+\delta} = n+m|N_t = n) = \begin{cases} 1-\lambda_n\delta + o(\delta) & if\ m=0 \\ \lambda_n\delta + o(\delta) & if\ m=1 \\ o(\delta) & o/w \end{cases}$$

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(ii) Let $\{N_t\}$ be a birth process with positive intensities $\lambda_0, \ldots$. Define the probabilities

$$p_n(t) = \mathbb{P}(N_t = n).$$

Now we can obtain the forward equations for a birth process:

$$p_n(t+\delta) = p_n(t)[1-\lambda_n\delta] + p_{n-1}(t)\lambda_{n-1}\delta + o(\delta)$$

with $n \geq 1$. That is, we have

$$\begin{aligned} p_n(t+\delta) &= \sum_{l\leq n}\mathbb{P}(N_{t+\delta} = n|N_t = l)\mathbb{P}(N_t = l) \\ &= \sum_{l\leq n}\mathbb{P}(N_{t+\delta} = j|N_t = l)p_l(t) \\ &= p_n(t)[1-\lambda_n\delta] + p_{n-1}(t)\lambda_{n-1}\delta + o(\delta) \end{aligned}$$

where the final line follows from the fact that we can only be in the state where there are $n-1$ individuals, and there is a birth, or we have jumped from n already and there is no birth. Then rearranging and taking the limit $\delta \downarrow 0$ it follows

$$\frac{dp_n(t)}{dt} = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t)$$

with the boundary condition $p_n(0) = 0$.

If $n = 0$ in a similar manner:

$$p_n(t+\delta) = p_n(t)[1 - \lambda_n \delta] + o(\delta)$$

i.e.

$$\frac{dp_n(t)}{dt} = -\lambda_n p_n(t)$$

with the boundary condition $p_n(0) = 1$.

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(iii) This is a linear birth process, that is, that $\lambda_n = n\lambda$. Suppose $n = I$ then the forward equations yield that unseen ⇓

$$p_n'(t) = -I\lambda p_n(t)$$

with $p_n(0) = 1$. This is a separable first order ODE:

$$\int \frac{1}{p_n(t)} dp_n(t) = I\lambda t + C$$

that is,

$$p_n(t) = e^{-I\lambda t}$$

as $C = 0$ due to the boundary condition. Now assume the result for $n = k$, $k \geq I$ and consider $k+1$:

$$p_{k+1}'(t) + \quad (k+1)p_{k+1}(t) = \lambda k \binom{k-1}{I-1} e^{-\lambda It}(1 - e^{-\lambda It})^{k-I}$$

where we have used the induction hypothesis. The above ODE is a first order linear ODE and may be solved by using the integrating factor approach. The integrating factor is

$$M(t) = \exp\{\lambda(k+1)t\}.$$

Then we have that

$$p_{k+1}(t) = \exp\{-\lambda(k+1)t\}\left[\binom{k-1}{I-1}\lambda k \int_0^t (1 - e^{-\lambda s})^{k-I} e^{\lambda s(k+1-I)} ds + C\right].$$

Using the identity in the question, it follows that

$$p_{k+1}(t) = \exp\{-\lambda(k+1)t\}\left[\binom{k}{I-1} e^{-\lambda It}(1 - e^{-\lambda It})^{k+1-I} \exp\{\lambda(k+1)t\} + C\right]$$

using $p_{k+1}(0) = 0$ it follows $C = 0$ and the desired result is obtained.

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(iv) To compute the expectation, it is easiest to notice that the distribution is a negative binomial and hence

$$\mathbb{E}[N_t] = Ie^{\lambda t}.$$

Alternatively, putting $p = e^{-\lambda t}$ the objective is to calculate:

$$\sum_{k=I}^{\infty} \binom{k-1}{I-1} kp^I (1-p)^{k-I}.$$

$$
\begin{aligned}
\sum_{k=I}^{\infty}\binom{k-1}{I-1}p^I(1-p)^{k-I} &= 1 \\
\frac{d}{dp}\sum_{k=I}^{\infty}\binom{k-1}{I-1}p^I(1-p)^{k-I} &= 0 \\
\Rightarrow \sum_{k=I}^{\infty}\binom{k-1}{I-1}kp^I(1-p)^{k-I} &= I\sum_{k=I}^{\infty}\binom{k-1}{I-1}p^{I-1}(1-p)^{k-I+1}+I \\
&= I\left[\frac{1-p}{p}+1\right] \\
&= \frac{I}{p}.
\end{aligned}
$$

Substituting for p completes the manipulations. $\boxed{2}$

3. (i) A discrete-time stochastic process $\{X_n\}_{n\geq 0}$ on E is a **Markov chain** if it satisfies the **Markov condition** $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_n = s|X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = s|X_{n-1} = x_{n-1})$$

for all $n \geq 1$ and $s, x_0, \ldots, x_{n-1} \in E$. $\boxed{2}$

(ii) (a) This is a Markov chain, as consider

$$
\begin{aligned}
\mathbb{P}(Y_n|\{Y_s\}_{-\infty<s\leq n-1}) &= \mathbb{P}(X_{-n}|\{X_s\}_{-n+1\leq s<\infty}) \\
&= \frac{\mathbb{P}(\{X_s\}_{-n\leq s<\infty})}{\mathbb{P}(\{X_s\}_{-n+1\leq s<\infty})} \\
&= \frac{\mathbb{P}(X_{-n})\mathbb{P}(X_{-n+1}|X_{-n})\times\cdots}{\mathbb{P}(X_{-n+1})\mathbb{P}(X_{-n+2}|X_{-n+1})\times\cdots} \\
&= \frac{\mathbb{P}(X_{-n})\mathbb{P}(X_{-n+1}|X_{-n})}{\mathbb{P}(X_{-n+1})} \\
&= \mathbb{P}(X_{-n}|X_{-n+1}) \\
&= \mathbb{P}(Y_n|Y_{n-1})
\end{aligned}
$$

where we have used the Markov property and Bayes theorem. $\boxed{4}$

(b) A Markov chain is **time-reversible** if the transition matrices of $\{X_n\}$ and $\{Y_n\}$ are the same. $\boxed{2}$

(c) Consider

$$\pi_i p_{ij} = \pi_j p_{ji}. \tag{1}$$

Let Q be the transition matrix of $\{Y_n\}$. Then from the above arguments, we have

$$q_{ij} = p_{ji}\frac{\pi_j}{\pi_i}$$

thus $q_{ij} = p_{ij}$ iff (1) holds. $\boxed{2}$

(iii) (a) The process is a Markov chain as the number of molecules in container A at time n depends (stochastically) only on the number of molecules in container A at time $n-1$. The transition probabilities are: $\boxed{\text{unseen} \Downarrow}$

$$p_{i,i+1} = 1 - \frac{i}{m} \quad p_{i,i-1} = \frac{i}{m} \quad 0 \le i \le m.$$

$\boxed{3}$

(b) Using $\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}$ (due to the reversibility), we will write π_i, $(1 \le i \le m)$ in terms of π_0 and use the fact that the probabilities must sum to 1, to conclude. Let $i = 0$, then

$$\pi_0 = \pi_1 \frac{1}{m}$$

that is $\pi_1 = m\pi_0$. Then,

$$\pi_1\left(\frac{m-1}{m}\right) = \pi_2 \frac{2}{m}$$

i.e.

$$\pi_2 = \left(\frac{m(m-1)}{2}\right)\pi_0.$$

Then

$$\pi_2\left(\frac{m-2}{m}\right) = \pi_3 \frac{3}{m}$$

so

$$\pi_3 = \left(\frac{m(m-1)(m-2)}{2 \times 3}\right)\pi_0.$$

Thus, by induction (or simply by repeating the calculations)

$$\pi_i = \binom{m}{i}\pi_0.$$

As $\sum_{i=0}^m \pi_i = 1$:

$$\pi_0[1 + \sum_{i=1}^m \binom{m}{i}] = 1$$

that is

$$\pi_0 = \left(\frac{1}{2}\right)^m$$

i.e.

$$\pi_i = \binom{m}{i}\left(\frac{1}{2}\right)^m \quad 0 \le i \le m$$

as required. $\boxed{7}$

4. (i) Let the process be denoted $\{X_t\}$ and suppose it and satisfies: seen ⇓

1. $\{X_t\}$ is Markov chain on $E = \{0, 1, \dots\}$
2. for $n \geq 0$ the infinitesimal transition probabilities, for $\delta > 0$, are

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n) = \begin{cases} \lambda_n \delta + o(\delta) & \textit{if } m = 1 \\ \mu_n \delta + o(\delta) & \textit{if } m = -1 \\ o(\delta) & \textit{if} |m| > 1 \end{cases}$$

3. the birth rates $\lambda_0, \lambda_1, \dots$ and the death rates $\mu_0, \mu_1, \dots$ satisfy

$$\lambda_i \geq 0 \quad \mu_i \geq 0 \quad \mu_0 = 0.$$

Then the process is called a birth-death process. 4

(ii) The generator is

$$G = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

To solve the second part of the question, note that the condition allows the π, given the equality holds, to be probabilities, so our main concern is verifying the given equation. We use strong induction. Since we have $\pi G = 0$, we can say:

$$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0.$$

Suppose then, the statement holds for every $n \leq k$, let us use induction to prove that this holds for any $n \leq k + 1$; the base case has been proved. Since $\pi G = 0$ gives

$$\lambda_{k-1}\pi_{k-1} - (\lambda_k + \mu_k)\pi_k + \mu_{k+1}\pi_{k+1} = 0$$

it follows that

$$\begin{aligned} \pi_{k+1} &= \frac{1}{\mu_{k+1}} \left[(\lambda_k + \mu_k)\pi_k - \lambda_{k-1}\pi_{k-1} \right] \\ &= \frac{1}{\mu_{k+1}} \left[(\lambda_k + \mu_k) \frac{\lambda_0 \times \cdots \times \lambda_{k-1}}{\mu_1 \times \cdots \times \mu_k} \pi_0 - \lambda_{k-1} \frac{\lambda_0 \times \cdots \times \lambda_{k-2}}{\mu_1 \times \cdots \times \mu_{k-1}} \pi_0 \right] \\ &= \frac{1}{\mu_{k+1}} \frac{\lambda_0 \times \cdots \times \lambda_{k-1}}{\mu_1 \times \cdots \times \mu_{k-1}} \pi_0 \left[\frac{(\lambda_k + \mu_k)}{\mu_k} - 1 \right] \\ &= \frac{\lambda_0 \times \cdots \times \lambda_k}{\mu_1 \times \cdots \times \mu_{k+1}} \pi_0 \end{aligned}$$

so the result holds for every $n \leq k + 1$. Thus by strong induction and the comment above, the result holds

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unseen ⇓

(iii) (a)By using the result in the previous question:

$$\pi_k = \pi_0 \frac{\lambda^k \exp\{1+\cdots+k\}}{\mu^k \exp\{1+\cdots+k\}}.$$

Thus:

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k$$

where $k \in \{1,\ldots,n\}$. Thus, as

$$\pi_0 \left[1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^n\right] = 1$$

it follows ($\lambda < \mu$)

$$\pi_0 = \frac{\mu-\lambda}{\mu} \frac{1}{1-(\frac{\lambda}{\mu})^{n+1}}.$$

Hence, the probability that no investment is possible is:

$$\frac{\mu-\lambda}{\mu} \frac{1}{1-(\frac{\lambda}{\mu})^{n+1}} \left(\frac{\lambda}{\mu}\right)^n = \frac{(\mu-\lambda)\lambda^n}{\mu^{n+1}-\lambda^{n+1}}.$$

$\boxed{7}$

(b) By the lack of memory property of the exponential distribution the time until an investment position is available is an exponential random variable with parameter μe^n. Therefore the time you would expect to wait is e^{-n}/μ units. $\boxed{3}$