Imperial College London

M4/5P47

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Stochastic Filtering

Date: Friday 26 May 2017

Time: 14:00 - 16:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

For the following questions, assume the set-up: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$ and V be a standard one-dimensional $\mathcal{F}_t$-adapted Brownian motion under $\mathbb{P}$. Let f and σ be bounded Lipschitz real valued functions and let X be the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f\left(X_s\right) ds + \int_0^t \sigma\left(X_s\right) dV_s. \tag{1}$$

Assume that X_0 has distribution π_0 at time 0, is independent of V and $\mathbb{E}\left[(X_0)^2\right] < \infty$. Let W be a standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X, and Y be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. The process $Y = \{Y_t,\ t \geq 0\}$ is called the observation process. Let $\{\mathcal{Y}_t,\ t \geq 0\}$ be the filtration associated with the process Y, that is $\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t])$. The filtering problem consists in determining the conditional distribution π_t of the signal X at time t given the information accumulated from observing Y in the interval $[0,t]$. That is, $\pi_t(A) = \mathbb{E}[I_A(X_t) \mid \mathcal{Y}_t]$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$ and I_A is the indicator function of the set A) and $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]$ for any bounded Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$.

1. Let $z = \{z_t, t > 0\}$ be the process defined by

$$z_t = \exp\left(\int_0^t \pi_s(h) dY_s - \frac{1}{2}\int_0^t \pi_s(h)^2\, ds\right),\ t \geq 0.$$

Let $\tilde{\mathbb{P}}$ be a probability measure which is absolutely continuous with respect to $\mathbb{P}$ under which Y is a Brownian motion.

(a) State the Novikov condition.

(b) Using Novikov's condition prove that $z = \{z_t, t > 0\}$ is a $\mathcal{Y}_t$-adapted martingale under $\tilde{\mathbb{P}}$.

(c) Deduce the evolution equation satisfied by the martingale z.

(d) Prove that $\sup_{t\in[0,T]} \mathbb{E}[z_t^4] < \infty$ for any $T > 0$.

2. Write an essay on the use of sequential Monte Carlo methods for approximating π_t, that is, the conditional distribution of the signal X at time t given the information accumulated from observing Y in the interval $[0,t]$. In your essay, you should

- describe the type of approximations that sequential Monte Carlo methods generate and how they can be deduced from the Kallianpur-Striebel's formula.
- explain why they are expected to give better extimates when compared to those produced by the (plain) Monte Carlo method.
- state whether they are theoretically justified or not.

No proofs are required.

3. Let A be the second order differential operator

$$A\varphi = f\varphi' + \frac{1}{2}\sigma^2\varphi'', \quad \varphi \in C_b^2(\mathbb{R}),$$

where $C_b^2(\mathbb{R})$ is the set of all bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$ twice differentiable with bounded first and second derivatives. Next let $\mu = \{\mu_t, t \geq 0\}$ be a measure valued process satisfying the Zakai equation. That is, assume that

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(A\varphi)\,ds + \int_0^t \mu_s(h\varphi)\,dY_s, \text{ for any } \varphi \in C_b^2(\mathbb{R}). \tag{3}$$

Let $\mu(1) = \{\mu_t(1), t \geq 0\}$ be the associated total mass process, that is $\mu_t(1) := \mu_t(\mathbb{R})$ for any $t \geq 0$ and $\nu = \{\nu_t, t \geq 0\}$ be normalized version of μ, that is, $\nu_t = \mu_t/\mu_t(1)$ for any $t \geq 0$.

In the following, you can assume, without proof, that the quadratic variation of the semi-martingale Y is t. That is, $\langle Y \rangle_t = t,\ t \geq 0$.

(a) Deduce the evolution equation satisfied by the mass process.

(b) Prove that $\nu = \{\nu_t, t \geq 0\}$, the normalized version of the process μ, satisfies the Kushner-Stratonovich equation. That is, show that

$$\begin{aligned}\nu_t(\varphi) &= \nu_0(\varphi) + \int_0^t \nu_s(A\varphi)\,ds \\ &\quad + \int_0^t (\nu_s(h\varphi) - \nu_s(h)\,\nu_s(\varphi))\,(dY_s - \nu_s(h)\,ds) \text{ for any } \varphi \in C_b^2(\mathbb{R}). \end{aligned} \tag{4}$$

(c) Prove that if the Kushner-Stratonovich equation (4) has a unique solution, then so does the Zakai equation (3).

4. Assume that the signal X is an Ornstein-Uhlenbeck process starting from $X_0 = x_0$. More precisely assume that

$$X_t = x_0 + \int_0^t aX_s ds + \sigma V_t, \tag{5}$$

where $a, \sigma \in (0, \infty)$.

(a) Solve the equation (5).

(b) Find the distribution p_t of X_t.

(c) Prove that

$$\int_0^t \mathbb{E}[|X_s|]ds < \infty.$$

(d) Deduce the equation satisfied by $p_t(\varphi)$ for any $\varphi \in C_b^2(\mathbb{R})$.

You may use any results given in the course without proof, provided that you make it clear which ones you are using.

5. In the following, consider a filtering problem where the signal process satisfies the stochastic differential equation

$$X_t = x_0 + \int_0^t f(X_s)\,ds + V_t, \quad x_0 \in \mathbb{R}. \tag{6}$$

We assume that the observation process Y satisfies the following evolution equation

$$Y_t = \int_0^t X_s\,ds + W_t. \tag{7}$$

In (6), V is a Brownian motion and the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and satisfies the Beneš condition

$$f'(x) + f^2(x) + x^2 = p^2x^2 + 2qx + r, \qquad x \in \mathbb{R}, \tag{8}$$

where f' is the derivative of f and $p, q, r \in \mathbb{R}$ are arbitrary. In (7), W is another Brownian motion independent of V. In this case $\pi_t(\varphi) := E[\varphi(X_t) \mid \sigma(Y_s,\ s \in [0,t])]$ satisfies the following explicit formula

$$\pi_t(\varphi) = \frac{1}{a_t}\int_{-\infty}^{\infty} \varphi(z)\exp\Big(F(z) + b_t z - \frac{1}{2}p\coth(tp)z^2\Big)\,dz, \tag{9}$$

where F is the anti-derivative of f, b_t is given by

$$b_t = \int_0^t \frac{\sinh(sp)}{\sinh(tp)}\,dY_s + \frac{q + p^2x_0}{p\sinh(tp)} - \frac{q}{p}\coth(tp)$$

and a_t is the corresponding normalising constant

$$a_t = \int_{-\infty}^{\infty} \exp\Big(F(z) + b_t z - \frac{1}{2}p\coth(tp)z^2\Big)\,dz. \tag{10}$$

(a) Why do we say that the solution of this filtering problem is finite dimensional?

(b) What is the connection between this filter and the Kalman-Bucy filter ?

(c) Assume that you receive the observation process corresponding to times $\{\frac{i}{n}\}_{i=1,\dots,n}$, in other words you receive $\{Y_{\frac{i}{n}}\}_{i=1,\dots,n}$. How would you use (9) to estimate π_1 ?

(d) Assume that the function f appearing in (6) is defined as $f(x) = \sqrt{3}x$, $x \in \mathbb{R}$.

  (i) Show that the function f satisfies the Beneš condition.

  (ii) Compute the normalizing constant a_t for this choice of the function f.

  (iii) Prove that π_t is a Gaussian measure for this choice of the function f.

Answers

1.(20 marks)

(a) [4 marks, seen] Novikov's condition states that if $u = \{u_t, t > 0\}$ is a process defined as $u_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ for M a continuous local martingale, then a sufficient condition for u to be a martingale is that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] < \infty, \qquad 0 \le t < \infty.$$

(b) [5 marks, seen] In this case the process $t \to \int_0^t \pi_s(h)dY_s$ is a local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by $t \to \int_0^t \pi_s(h)^2 ds$. Moreover, since h is bounded and π_s is a probability measure, it follows that $|\pi_s(h)| \le \|h\|_\infty$ and hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \pi_s(h)^2\, ds\right)\right] \le \exp\left(\frac{t\,\|h\|_\infty^2}{2}\right) < \infty, \qquad 0 \le t < \infty.$$

Hence, by Novikov's condition, the process $z = \{z_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Moreover since the process π is adapted to the filtration $\mathcal{Y}_t$ the property remains true for z as well.

(c) [5 marks, seen] Let $\xi = \{\xi_t, t > 0\}$ be the semimartingale defined by

$$\xi_t = \int_0^t \pi_s(h)dY_s - \frac{1}{2}\int_0^t \pi_s(h)^2 ds, \qquad t \ge 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned}
z_t &= \exp(\xi_t) \\
&= \exp(\xi_0) + \int_0^t \exp(\xi_s)\, d\xi_s + \frac{1}{2}\int_0^t \exp(\xi_s)\, d\langle\xi\rangle_s \\
&= 1 + \int_0^t z_s\left(\pi_s(h)dY_s - \frac{1}{2}\pi_s(h)^2 ds\right) + \frac{1}{2}\int_0^t z_s\pi_s(h)^2 ds \\
&= 1 + \int_0^t z_s\pi_s(h)dY_s.
\end{aligned}$$

(d) [6 marks, not seen] Observe that

$$\begin{aligned}
z_s^4 &= \exp\left(\int_0^t 6\pi_s(h)^2 ds\right)\bar{z}_s \\
&\le \exp\left(6t\,\|h\|_\infty^2\right)\bar{z}_s,
\end{aligned}$$

where $\bar{z} = \{\bar{z}_t, t > 0\}$ is the process defined by

$$\bar{z}_t = \exp\left(\int_0^t 4\pi_s(h)dY_s - \frac{1}{2}\int_0^t (4\pi_s(h))^2 \, ds\right), \; t \geq 0.$$

Again, by Novikov's condition, the process $\bar{z} = \{\bar{z}_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Hence

$$E[\bar{z}_s] = E[\bar{z}_0] = 1$$

and

$$\begin{aligned} \sup_{t\in[0,T]} E[z_s^4] &\leq \sup_{t\in[0,T]} \exp\left(6t \left\|h\right\|_\infty^2\right) E[\bar{z}_s] \\ &= \exp\left(6T \left\|h\right\|_\infty^2\right) < \infty \end{aligned}$$

for any $T > 0$.

2 (20 marks, seen)

[9 marks] Sequential Monte Carlo methods also known as *particle filters* or *particle methods* are algorithms which approximate π_t with discrete random measures of the form

$$\sum_i a_i(t)\delta_{v_i(t)},$$

in other words, with empirical distributions associated with sets of randomly located particles of stochastic masses $a_1(t)$,$a_2(t)$,... which have stochastic positions $v_1(t)$,$v_2(t)$,... where $v_i(t) \in \mathbb{R}$.

The basis of this class of numerical method is the representation of π_t given by the Kallianpur–Striebel formula: That is, for any φ a bounded Borel-measurable function, we have

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(\mathbf{1})},$$

where ρ_t is the unnormalised conditional distribution of X_t

$$\rho_t(\varphi) = \tilde{\mathbb{E}}\left[\varphi(X_t)\tilde{Z}_t \middle| \mathcal{Y}_t\right], \tag{1}$$

and

$$\tilde{Z}_t = \exp\left(\int_0^t h(X_s)\, dY_s - \frac{1}{2}\int_0^t (h(X_s))^2\, ds\right).$$

The expectation in (1) is taken with respect to the probability measure $\tilde{\mathbb{P}}$ under which the process Y is a Brownian motion independent of X.

One can then use a Monte Carlo approximation to compute $\tilde{\mathbb{E}}\left[\varphi(X_t)\tilde{Z}_t \middle| \mathcal{Y}_t\right]$. More precisely, let v_j, $j = 1, \ldots, n$, be n mutually independent stochastic processes and independent of Y, each of them satisfying the same SDE as that satisfied by the signal process. Also let a_j, $j = 1, \ldots, n$ be the following exponential martingales

$$a_j(t) = \exp\left(\int_0^t h(v_j(s))^\top\, dY_s - \frac{1}{2}\int_0^t \|h(v_j(s))\|^2\, ds\right),\ t \geq 0.$$

Let $\pi^n = \{\pi_t^n,\ t \geq 0\}$ be the following sequences of measure-valued processes

$$\pi_t^n \triangleq \frac{\rho_t^n}{\rho_t^n(\mathbf{1})} = \sum_{j=1}^n \bar{a}_j^n(t)\delta_{v_j(t)},\ t \geq 0, \tag{2}$$

where the normalised weights $\bar{a}_j^n$ have the form

$$\bar{a}_j^n(t) = \frac{a_j(t)}{\sum_{j=1}^n a_j(t)},\ j = 1, \ldots, n,\ t \geq 0.$$

That is, π_t^n is the empirical measure of n (random) particles with positions $v_j(t)$, $j = 1, \ldots, n$, and weights $\bar{a}_j(t)$, $j = 1, \ldots, n$. Then

$$\lim_{n\to\infty} \pi_t^n = \pi_t.$$

[9 marks] The Monte Carlo method will produce approximations for ρ_t, respectively π_t, provided enough particles (independent realizations of the signal) are used. The number of particles depends upon the magnitude of the variance of the weights a_j. This is bad news, because the variance increases very rapidly as functions of time. The particle picture makes the reason for the deterioration in the accuracy of the approximations with time clearer. Each particle has a trajectory which is independent of the signal trajectory, and its corresponding weight depends on how close its trajectory is to the signal trajectory: the weight is the likelihood of the trajectory given the observation. Typically, most particles' trajectories diverge very quickly from the signal trajectory, with a few 'lucky' ones remaining close to the signal. Therefore the majority of the weights decrease to zero, while a small minority become very large. As a result only the 'lucky' particles will contribute significantly to the sum in (2) giving the approximation for π_t. The convergence of the Monte Carlo method is therefore very slow as a large number of particles is needed in order to have a sufficient number of particles in the right area (with correspondingly large weights). To solve this problem, a wealth of methods have been proposed. In filtering theory, the generic name for these methods is particle filters or sequential Monte Carlo methods. These methods use a correction mechanism that culls particles with small weights and multiplies particles with large weights. The correction procedure depends on the trajectory of the particle and the observation data. This is effective as particles with small weights, i.e. particles with unlikely trajectories/positions are not carried forward uselessly while the most probable regions of the signal state space are explored more thoroughly. The result is a cloud of particles, with those surviving to the current time providing an estimate for the conditional distribution of the signal.
[2 marks] Unlike the Extended Kalman Filter, all Sequential Monte Carlo methods are theoretically justified, in other words they produce approximations that converge to π_t as the number of particles increases.

3. (20 marks)

(a) **[3 marks]** Since $A1 = 0$, it follows from the Zakai equation that the mass process satisfies the following

$$\begin{aligned}
\mu_t(\mathbf{1}) &= \mu_0(\mathbf{1}) + \int_0^t \mu_s(A\mathbf{1})\,ds + \int_0^t \mu_s(h\mathbf{1})\,dY_s \\
&= \mu_0(\mathbf{1}) + \int_0^t \mu_s(h)\,dY_s \\
&= \mu_0(\mathbf{1}) + \int_0^t \nu_s(h)\,\mu_s(\mathbf{1})\,dY_s.
\end{aligned}$$

(b) **[10 marks]** From the above, we deduce, by Itô's formula that

$$\begin{aligned}
\frac{1}{\mu_t(\mathbf{1})} &= \frac{1}{\mu_0(\mathbf{1})} - \int_0^t \frac{1}{\mu_s(\mathbf{1})^2} d\mu_s(\mathbf{1}) + \int_0^t \frac{1}{\mu_s(\mathbf{1})^3} d\langle \mu(\mathbf{1})\rangle_s \\
&= \frac{1}{\mu_0(\mathbf{1})} - \int_0^t \frac{1}{\mu_s(\mathbf{1})^2} \nu_s(h)\,\mu_s(\mathbf{1})\,dY_s + \int_0^t \frac{1}{\mu_s(\mathbf{1})^3}\left(\nu_s(h)\,\mu_s(\mathbf{1})\right)^2 ds \\
&= \frac{1}{\mu_0(\mathbf{1})} - \int_0^t \frac{\nu_s(h)}{\mu_s(\mathbf{1})} dY_s + \int_0^t \frac{\nu_s(h)^2}{\mu_s(\mathbf{1})} ds.
\end{aligned}$$

Hence, by stochastic integration by parts, we have

$$\begin{aligned}
\nu_t(\varphi) &= \frac{\mu_t(\varphi)}{\mu_t(\mathbf{1})} \\
&= \frac{\mu_0(\varphi)}{\mu_0(\mathbf{1})} + \int_0^t \frac{1}{\mu_s(\mathbf{1})} d\mu_s(\varphi) + \int_0^t \mu_s(\varphi)\, d\frac{1}{\mu_s(\mathbf{1})} + \left\langle \mu(\varphi), \frac{1}{\mu(\mathbf{1})}\right\rangle_s \\
&= \nu_0(\varphi) + \int_0^t \frac{1}{\mu_s(\mathbf{1})} d\left(\mu_s(A\varphi)\,ds + \mu_s(h\varphi)\,dY_s\right) \\
&\quad + \int_0^t \mu_s(\varphi)\, d\left(-\frac{\nu_s(h)}{\mu_s(\mathbf{1})} dY_s + \frac{\nu_s(h)^2}{\mu_s(\mathbf{1})} ds\right) - \int_0^t \mu_s(h\varphi)\,\frac{\nu_s(h)}{\mu_s(\mathbf{1})} ds \\
&= \nu_0(\varphi) + \int_0^t \nu_s(A\varphi)\,ds + \int_0^t \nu_s(h\varphi)\,dY_s \\
&\quad - \int_0^t \nu_s(h)\,\nu_s(\varphi)\,dY_s + \int_0^t \nu_s(h)^2\,\nu_s(\varphi)\,ds - \int_0^t \nu_s(h\varphi)\,\nu_s(h)\,ds,
\end{aligned}$$

which is exactly the Kushner-Stratonovitch equation.

(c). **[7 marks]** Let μ^1 and μ^2 be two solutions of the Zakai equation and ν^1 and, respectively, ν^2 their normalized versions. Then it follows from **(b)** that both ν^1 and ν^2 are solutions of the Kushner-Stratonovitch equation. By the assumption $\nu^1 = \nu^2 = \nu$. Their corresponding mass processes $\mu_t^i(\mathbf{1})$, $i = 1,2$ satisfy the same equation, i.e.,

$$\mu_t^i(\mathbf{1}) = \mu_0^i(\mathbf{1}) + \int_0^t \nu_s(h)\,\mu_s^i(\mathbf{1})\,dY_s, \quad i = 1,2,$$

which itself has a unique solution. Hence $\mu_t^1(\mathbf{1}) = \mu_t^2(\mathbf{1})$ for any $t \geq 0$. Therefore

$$\mu_t^1(\varphi) = \nu_t^1(\varphi)\,\mu_t^1(\mathbf{1}) = \nu_t^2(\varphi)\,\mu_t^2(\mathbf{1}) = \mu_t^2(\varphi),$$

hence the claim.

4. (20 marks, seen similar)

(a) [5 marks] We have, by Itô's formula, that

$$e^{-at}X_t = x_0 + \int_0^t X_s de^{-as} + \int_0^t e^{-as} dX_s = x_0 + \int_0^t \sigma e^{as} dV_s,$$

hence

$$X_t = x_0 e^{at} + \int_0^t \sigma e^{a(t-s)} dV_s.$$

(b) [5 marks] We use the fact that if $f : [0,t] \to \mathbb{R}$ be a Borel measurable function such that $v := \int_0^t f(s)^2 ds < \infty$, then the random variable $\int_0^t f(s) dV_s$ has a normal distribution with mean 0 and variance v. It follows that

$$\int_0^t \sigma e^{a(t-s)} dV_s \sim N\left(0, \frac{\sigma^2}{2a}(e^{2at} - 1)\right),$$

hence

$$p_t = \mathcal{L}(X_t) = N\left(x_0 e^{at}, \frac{\sigma^2}{2a}(e^{2at} - 1)\right).$$

(c) [4 marks, not seen] We have that

$$\begin{aligned}
E[|X_s|] &\leq x_0 e^{as} + E[|X_s - x_0 e^{as}|] \\
&= x_0 e^{as} + \frac{2}{\sqrt{2\pi \frac{\sigma^2}{2a}(e^{2as} - 1)}} \int_0^\infty x e^{-\frac{x^2}{2\frac{\sigma^2}{2a}(e^{2as}-1)}} dx \\
&= x_0 e^{as} - \frac{2}{\sqrt{2\pi \frac{\sigma^2}{2a}(e^{2as} - 1)}} \left(\frac{\sigma^2}{2a}(e^{2as} - 1)\right) e^{-\frac{x^2}{2\frac{\sigma^2}{2a}(e^{2as}-1)}} \Bigg|_0^\infty \\
&= x_0 e^{as} + \sqrt{\frac{2}{\pi}\frac{\sigma^2}{2a}(e^{2as} - 1)} \leq x_0 e^{at} + \sqrt{\frac{2}{\pi}\frac{\sigma^2}{2a}(e^{2at} - 1)} = C(t),
\end{aligned}$$

hence

$$\int_0^t E[|X_s|] ds \leq C(t)t < \infty.$$

(d) [6 marks] By Itô's formula we get that

$$\varphi(X_t) = \varphi(X_0) + \int_0^t A\varphi ds + \int_0^t \sigma \varphi'(X_s) dV_s, \tag{3}$$

where $A\varphi(x) = ax\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x)$. Since the process $t \to \int_0^t \sigma\varphi'(X_s)dV_s$ is a genuine martingale (it is local martingale as it is a stochastic integral with respect to a Brownian motion and

$$\int_0^t (\sigma\varphi'(X_s))^2 ds \leq \sigma^2 t||\varphi||_\infty'^2 < \infty$$

it has null expectation. Moreover, by Tonelli's theorem

$$\begin{aligned} E\left[\int_0^t |A\varphi(X_s)|ds\right] &= \int_0^t E[|A\varphi(X_s)|]ds \\ &\leq a||\varphi'||_\infty \int_0^t E[|X_s|]ds + \frac{t\sigma^2}{2}||\varphi''||_\infty \leq \infty. \end{aligned}$$

It follows that we can apply Fubini to get that

$$E\left[\int_0^t A\varphi(X_s)ds\right] = \int_0^t p_s(A\varphi)\,ds.$$

Hence, by taking expectation in (3) we get

$$p_t(\varphi) = E[\varphi(X_t)] = p_0(\varphi) + \int_0^t p_s(A\varphi)\,ds.$$

5. (20 marks)

(a) [3 marks, seen] The solution of the filtering problem is, in this case, finite dimensional as it depends only on the one-dimensional $\mathcal{Y}_t$-adapted process

$$t \mapsto \int_0^t \sinh(sp)\, dY_s.$$

(b) [3 marks, seen] The one dimensional time homogenous linear filter is a particular case of the Benes filter.

(c) [4 marks, not seen] All parts in the equation giving the formula for π_t can be computed independently of the observation values except for the stochastic integral $\int_0^1 \sinh(sp)\, dY_s$.Hence we need to replace this integral with an approximation that uses only the given data term $\{Y_{t_{i/n}}\}_{i=1,\ldots,n}$. For example we can use the approximation

$$\int_0^1 \sinh(sp)\, dY_s \approx \sum_{i=1}^n \sinh\left(\frac{(i-1)\,p}{n}\right)\left(Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}\right).$$

The approximation for π_1 is then obtained by replacing $\int_0^1 \sinh(sp)\, dY_s$ with

$$\sum_{i=1}^n \sinh\left(\frac{(i-1)\,p}{n}\right)\left(Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}\right)$$

in the equation giving the formula for π_t and a_t.

(d)

i [2 marks, not seen] We have

$$(ax)^2 + \left((ax)'\right)^2 + x^2 = (3+1)\,x^2 + 3 = 4x^2 + 3,$$

hence f satisfies the Beneš condition with $p = 2, r = 3$ and $q = 0$.

ii [4 marks, seen similar] In this case the anti-derivative F of f is given by

$$F(z) = \frac{\sqrt{3}z^2}{2},$$

$$\begin{aligned} a_t &= \int_{-\infty}^{\infty} \exp\left(b_t z - \left(\coth(tp) - \frac{\sqrt{3}}{2}\right) z^2\right) dz \\ &= \sqrt{2\pi \frac{\coth(tp) - \frac{\sqrt{3}}{2}}{2}} \exp\left(\frac{b_t^2}{4\left(\coth(tp) - \frac{\sqrt{3}}{2}\right)}\right). \end{aligned}$$

iii [4 marks, seen similar] In this case

$$\begin{aligned}\pi_t(\varphi) &= \frac{1}{a_t}\int_{-\infty}^{\infty}\varphi(z)\exp\left(F(z)+b_t z-\frac{1}{2}p\coth(tp)z^2\right)\,dz,\\ &= \frac{1}{\sqrt{2\pi p}}\int_{-\infty}^{\infty}\varphi(z)\exp\left(-\frac{(z-\nu)^2}{2p}\right)dz,\end{aligned}$$

where

$$\nu=\frac{b_t}{2\left(\coth(tp)-\frac{\sqrt{3}}{2}\right)};\quad p=\frac{1}{2\left(\coth(tp)-\frac{\sqrt{3}}{2}\right)},$$

hence

$$\pi_t = N(\nu,p).$$

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Examiner's Comments

Exam: M4P47 Session: 2016-2107

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The candidates understood this question and did it ~~v~~ very well

Marker: [signature]

Signature: D. Crisan Date: 02/06/17

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4P47 Session: 2016-2107

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Some common minor omissions: the form of the weights, the evolution of the particles

Marker: D. Crisan

Signature: [signature] Date: 02/06/17

Please return with exam marks (one report per marker)

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Department of Mathematics

Examiner's Comments

Exam: M4P47 Session: 2016-2107

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Nearly perfect answers.

Marker: D. Crisan

Signature: [signature] Date: 02/06/17

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4P47 Session: 2016-2107

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Common problem: Deducing the equation of the prior distribution.

Marker: D. Crisan

Signature: [signature]

Date: 02/06/17

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M5P47 **Session: 2016-2107**

Question 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

The candidates made little progress here. Common mistakes: justifying why the filter is finite-dimensional, computing the normalizing constant.

Marker: D. Crisan

Signature: [signature] Date: 02/06/17

Please return with exam marks (one report per marker)