Imperial College London

MATH97065 MATH97176 MATH97284

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Dynamical Systems

Date: 12th May 2020

Time: 13.00pm - 15.30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1.

(a) Let X be a metric space with metric $d : X \times X \to \mathbb{R}$ and $f : X \to X$ be continuous.

(i) Give definitions of the following properties: (4 marks)

- f is topologically mixing.
- f has sensitive dependence (on initial conditions).

(ii) Show that f has sensitive dependence if f is topologically mixing (unless X consists of a single point). (4 marks)

(b) Let Σ_3^+ denote the set of half-infinite sequences $\omega = \omega_0\omega_1 \dots$ with $\omega_i \in \{0, 1, 2\}$ for all $i \in \mathbb{N}_0$, endowed with the metric

$$D(\omega, \tilde{\omega}) := \sum_{i=0}^{\infty} \frac{\delta_{\omega_i,\tilde{\omega}_i}}{3^i}, \text{ where } \delta_{\omega_i,\tilde{\omega}_i} := \begin{cases} 0 & \text{if } \omega_i = \tilde{\omega}_i. \\ 1 & \text{if } \omega_i \neq \tilde{\omega}_i. \end{cases} \tag{1}$$

Show that the cylinder set

$$C_{\omega_0\omega_1\dots\omega_{n-1}} := \{\tilde{\omega} \in \Sigma_3^+ \mid \omega_i = \tilde{\omega}_i,\ i = 0, \dots, n-1\}.$$

is an open ball in Σ_3^+ around any point $\omega \in C_{\omega_0\omega_1\dots\omega_{n-1}}$ of radius 3^{1-n}. (4 marks)

(c) Let X, Y be metric spaces and $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be the associated Borel-measurable spaces. Let $f : X \to X$, $g : Y \to Y$ and $h : X \to Y$ be measurable such that $h \circ f = g \circ h$. Suppose that μ is an invariant measure of f.

(i) Show that $h_*\mu := \mu \circ h^{-1}$ is an invariant measure of g. (4 marks)

(ii) Show that $h_*\mu$ is ergodic if μ is ergodic. (4 marks)

(Total: 20 marks)

Questions 2,3 and 4 are largely concerned with the following two dynamical systems:

- A piecewise affine map $f : [0,1] \to [0,1]$:

$$f(x) = \begin{cases} 2 + \sqrt{2}(x-1) & \text{if } 0 \le x \le c, \\ \sqrt{2}(1-x) & \text{if } c < x \le 1, \end{cases} \tag{2}$$

where $c = \frac{1}{2}(2-\sqrt{2})$. The graph of f is sketched below:

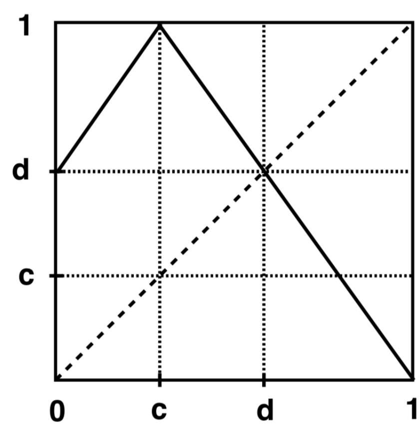


Figure 1: Sketch of the graph of f (2). $d = 2c$.

- A subshift of finite type $\sigma : \Sigma^+_{3,A} \to \Sigma^+_{3,A}$, where

$$\Sigma^+_{3,A} := \{\omega \in \Sigma^+_3 \mid \omega_i\omega_{i+1} \neq jk \text{ if } a_{jk} = 0, \forall i \in \mathbb{N}_0\} \text{ with } A := (a_{jk})_{j,k=0,1,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{3}$$

Σ^+_3 is as defined above in Q1(b), and the subshift σ acts as follows

$$\sigma(\omega_0\omega_1\ldots) = \omega_1\omega_2\ldots. \tag{4}$$

2. Consider the maps f and σ, as introduced above, after Q1.

In order to compare the topological dynamics of f with that of σ, we consider $[0,1]$ as a union of labelling intervals

$$[0,1] = I_0 \cup I_1 \cup I_2,$$

where $I_0 = [0,c]$, $I_1 = [c,d]$, $I_2 = [d,1]$ and $d = 2c = 2 - \sqrt{2}$, see also Figure 1.

Let

$$h_n(\omega) := \overline{\cap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(I_{\omega_i}))},$$

where $\omega = \omega_0\omega_1\dots$ and $\mathrm{Int}(I_{\omega_i})$ denotes the interior of I_{ω_i}.

You may use without proof that $h := \lim_{n\to\infty} h_n : \Sigma^+_{3,A} \to [0,1]$ is well-defined, continuous and surjective.

(a) (i) Determine the partition $\{h_2(\omega) \mid \omega \in \Sigma^+_{3,A}\}$. (4 marks)

(ii) Determine $h^{-1}(d)$ and $h^{-1}(c)$. (4 marks)

(iii) Show that f and σ are topologically semi-conjugate through h, i.e. (4 marks)

$$h \circ \sigma = f \circ h.$$

(b) Answer the following questions briefly, with motivation but no detailed proofs:

(i) Are the periodic orbits of f dense in $[0,1]$? (4 marks)

(ii) Is f topologically mixing? (4 marks)

(Total: 20 marks)

3. Consider a compact metric space X and a continuous map $g : X \to X$.

(a) (i) Give the definition of an (n,ε)-separated set for g in X. (4 marks)

(ii) Give the definition of topological entropy $h_{\mathrm{top}}(g)$ of g in terms of the maximal cardinality of (n,ε)-separated sets. (4 marks)

(iii) Let $P_m(g) := \#\{x \in X \mid g^m(x) = x\}$. Suppose that $\limsup_{m\to\infty} \frac{1}{m} \ln P_m(g) = b$. Show that this implies that $h_{\mathrm{top}}(g) \geq b$. (4 marks)

Consider the maps f and σ, as introduced above, after Q1.

(b) Show that $h_{\mathrm{top}}(\sigma) = \frac{1}{2}\ln 2$. What does this imply for $h_{\mathrm{top}}(f)$? HINT: Use the results in Q2(a)(iii) and Q3(a)(iii). (You may quote any results obtained in the course without proof, but any such result should be precisely stated.) (8 marks)

(Total: 20 marks)

4. Consider the maps f and σ, as introduced above, after Q1. You may use without proof that these maps are measurable with respect to the Borel σ-algebras $\mathcal{B}([0,1])$ and $\mathcal{B}(\Sigma^+_{3,A})$.

Consider the measure μ on $\mathcal{B}([0,1])$ defined by

$$\mu(A) := d^{-1} \int_A \left(\frac{\mathbb{1}_{[0,d]}(x)}{2} + \frac{\mathbb{1}_{(d,1]}(x)}{\sqrt{2}} \right) dx$$

with $d = 2 - \sqrt{2}$ and the characteristic function

$$\mathbb{1}_C(x) := \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

(a) (i) Show that μ is a probability measure and sketch the graph of its density. (5 marks)

(ii) Show that μ is an invariant measure for f. HINT: write $\mu(A) = \mu(A_0) + \mu(A_1)$, where $A = A_0 \cup A_1$ with $A_0 = A \cap [0,d]$ and $A_1 = A \cap (d,1]$. (5 marks)

(b) Apply Birkhoff's Ergodic Theorem to show that ergodicity of μ implies that

$$\int_0^1 (\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f^i(x))dx = \frac{5}{4} - \frac{1}{\sqrt{2}}.$$

You are not asked to establish the ergodicity of μ, and should use this fact without proof. (The identity $d^2 + \sqrt{2}(1-d^2) = 2d^2 + d$ may be useful.) (5 marks)

Recall the semi-conjugacy between the maps f and σ, $h \circ \sigma = f \circ h$, as discussed in Q2(a)(iii). You may use without proof that h is measurable with respect to the Borel σ-algebras $\mathcal{B}(\Sigma^+_{3,A})$ and $\mathcal{B}([0,1])$.

(c) Determine the invariant Markov measure ν of σ on $\mathcal{B}(\Sigma^+_{3,A})$ such that $\mu = h_*\nu$. Motivate your answer. (5 marks)

(Total: 20 marks)

5. Consider the two-parameter family of lifts

$$F_{\alpha,\varepsilon} : \mathbb{R} \to \mathbb{R},\ F_{\alpha,\varepsilon}(x) = x + \alpha + \frac{\varepsilon}{2\pi}\cos^2(2\pi x),\ \text{with } \alpha \in [0,1) \text{ and } \varepsilon \in [-1,1],$$

of circle homeomorphisms $f_{\alpha,\varepsilon} : S^1 \to S^1$.

(a) Show that $F_{\alpha,\varepsilon}$ is not a lift of a circle homeomorphism if $\varepsilon \notin [-1,1]$. (2 marks)

Consider $\rho(F_{\alpha,\varepsilon}) := \lim_{n\to\infty} \frac{1}{n}\left(F^n_{\alpha,\varepsilon}(x) - x\right)$ with $(\alpha,\varepsilon) \in [0,1) \times [-1,1]$.

(b) Show that $\rho(F_{\alpha,\varepsilon})$ is independent of x. You may assume without proof that the limit in the definition of $\rho(F_{\alpha,\varepsilon})$ exists. (4 marks)

(c) Show that $\rho(F_{\alpha,\varepsilon})$ is a non-decreasing function of α. (4 marks)

(d) Determine $\{(\alpha,\varepsilon) \in [0,1) \times [-1,1] \mid \lim_{n\to\infty} f^n_{\alpha,\varepsilon}(x) \text{ exists } \forall x \in S^1\}$.
(You may quote any relevant general result on circle maps without proof, but any such result should be precisely stated.) (6 marks)

(e) Show that there exists an $\alpha \in [0,1)$ such that $\rho(F_{\alpha,\frac{1}{2}}) \notin \mathbb{Q}$.
(You may quote any relevant general result on circle maps without proof, but any such result should be precisely stated.) (4 marks)

(Total: 20 marks)

1. [The entire question concerns basic theory; all results were mentioned in class but (b) and (c)(ii) not spelled out in detail.]

(a) (i) - f is *topologically mixing* if for any two nonempty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that

$$f^n(U) \cap V \neq \emptyset, \forall n \geq N\,.$$

(2 marks)

- f has *sensitive dependence (on initial conditions)* if there exists a $\delta > 0$ such that for all $x \in X$ and all $\varepsilon > 0$, there exists a $y \in B_\varepsilon(x)$ and an $n \in \mathbb{N}$ with

$$d(f^n(x), f^n(y)) \geq \delta\,.$$

(2 marks)

(ii) Take $\delta > 0$ such that there are points x_1, x_2 with $d(x_1, x_2) > 4\delta$. Let $V_i = B_\delta(x_i)$. Suppose $x \in X$ and U is a neighbourhood of x. Then by topological mixing there are N_1, N_2 such that $f^n(U) \cap V_1 \neq \emptyset\ \forall n > N_1$ and $f^n(U) \cap V_2 \neq \emptyset\ \forall n > N_2$. For $n > \max(N_1, N_2)$ there are points $y_1, y_2 \in U$ such that $f^n(y_1) \in V_1$ and $f^n(y_2) \in V_2$; hence $d(f^n(y_1), f^n(y_2)) \geq 2\delta$. By the triangle inequality this implies that $d(f^n(y_1), f^n(x)) \geq \delta$ or $d(f^n(y_2), f^n(x)) \geq \delta$.

(4 marks)

(b) Let $\omega = \omega_0\omega_1 \ldots \omega_{n-1} \ldots$ and $\tilde{\omega} \in C_{\omega_0\omega_1\ldots\omega_{n-1}}$ then

$$D(\omega, \tilde{\omega}) = \sum_{i=0}^{\infty} \frac{\delta_{\omega_i, \tilde{\omega}_i}}{3^i} = \sum_{i=n}^{\infty} \frac{\delta_{\omega_i, \tilde{\omega}_i}}{3^i} = \frac{1}{3^n} \sum_{i=0}^{\infty} \frac{\delta_{\omega_{i+n}, \tilde{\omega}_{i+n}}}{3^i} \leq \frac{1}{3^n}\frac{3}{3-1} = \frac{1}{2 \cdot 3^{n-1}} < 3^{1-n}.$$

In addition, if $\tilde{\omega} \notin C_{\omega_0\omega_1\ldots\omega_{n-1}}$ then $D(\omega, \tilde{\omega}) \geq 3^{1-n}$ since $\omega_i = \tilde{\omega}_i$ for some $i \in \{0, \ldots, n-1\}$.

(4 marks)

(c) (i) μ is an invariant measure of f: all $A \in \mathcal{B}(X)$, $f_*\mu(A) := \mu(f^{-1}(A)) = \mu(A)$. For all $B \in \mathcal{B}(X)$, $h_*\mu = \mu(h^{-1}(B))$. Then $h_*\mu(g^{-1}(B)) = \mu(h^{-1} \circ g^{-1}(B)) = \mu((g \circ h)^{-1}(B)) = \mu((h \circ f)^{-1}(B)) = \mu(f^{-1} \circ h^{-1}(B)) = \mu(h^{-1}(B)) = h_*\mu(B)$. (It may be noted that $h : X \to Y$ need not be invertible for $h^{-1} : \mathcal{B}(Y) \to \mathcal{B}(X)$ to be well defined as $h^{-1}(B) = \{A \in \mathcal{B}(X) \mid h(A) = B\}$.) (4 marks)

(ii) μ is ergodic iff $\forall A \in \mathcal{B}(X)$ satisfying $f^{-1}(A) = A$, we have $\mu(A) \in \{0, 1\}$. If $g^{-1}(B) = B$ then $f^{-1}(h^{-1}(B)) = (h \circ f)^{-1}(B) = (g \circ h)^{-1}(B) = h^{-1} \circ g^{-1}(B) = h^{-1}(B)$. The latter implies that $h_*\mu(B) := \mu(h^{-1}(B)) \in \{0, 1\}$ and thus that $h_*\mu$ is ergodic. (4 marks)

(Total: 20 marks)

2. (Symbolic dynamics for similar examples has been dealt with in class and on problem sheet. (a)(i) standard, (a)(ii) seen similar in test for point d but situation for point c unseen, (a)(iii) basic theory; (b) unseen, all examples dealt with in class have been topologically mixing.)

(a) (i) The matrix A symbolizes the possible transitions between the interiors of the partition elements I_i in orbits of the map f. $h_2(\omega)$ only depends on the first two symbols of ω: $\omega_0\omega_1$. Let $I_{\omega_0\omega_1} := h_2(\omega)$. Then $I_{02} = I_0$, $I_{12} = I_1$, $I_{20} = [(d+1)/2, 1]$ and $I_{21} = [d, (d+1)/2]$. (4 marks)

(ii) d is a fixed point of f, but points near f oscillate between I_1 and I_2. Hence the labelling intervals accumulating to d from the right have the asymptotic symbolic code $\overline{21}$ and those converging from the left have code $\overline{12}$. Hence $h^{-1}(d) = \{\overline{21}, \overline{12}\}$. The point c is such that $f^3(c) = f^2(1) = f(0) = d$. We note that $1 \in I_2$, $0 \in I_0$ and c lies on the boundary of I_0 and I_1. Moreover, the code of the orbit of c is the limit point of codes of orbits approaching d always first from the right (this is also reflected in the fact that the subsequence "01" is not admissible). Hence $h^{-1}(c) = \{020\overline{21}, 120\overline{21}\}$. (4 marks)

(iii) $h \circ \sigma(\omega) = \lim_{n\to\infty} \overline{\cap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(I_{\omega_{i+1}}))} = \lim_{n\to\infty} \overline{\cap_{i=1}^{n} f^{1-i}(\mathrm{Int}(I_{\omega_i}))}$ and $f \circ h(\omega) = f(\lim_{n\to\infty} \overline{\cap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(I_{\omega_i}))}) \stackrel{(1)}{=} \lim_{n\to\infty} \overline{\cap_{i=0}^{n-1} f^{1-i}(\mathrm{Int}(I_{\omega_i})))} \stackrel{(2)}{=} \lim_{n\to\infty} \overline{\cap_{i=1}^{n-1} f^{1-i}(\mathrm{Int}(I_{\omega_i}))})$, using in (1) the continuity of f and in (2) the fact that $f(I_{\omega_i}) \supset I_{\omega_{i+1}}$ if $\omega_i\omega_{i+1}$ is an admissible subsequence in $\Sigma_{3,A}^+$. (4 marks)

(b) (i) Yes. Labelling intervals $h_n(\omega)$ shrink exponentially with n since $h_{n+2}(\omega) = \frac{1}{2}h_n(\omega)$. Hence, since the labelling intervals partition $[0,1]$, every open set contains such a labelling interval. Every labelling interval contains a periodic orbit since every admissible subsequence in $\Sigma_{3,A}^+$ has an admissible periodically continuation. (4 marks)

(ii) No. Any non-empty open set $U \subset (0,d)$ satisfies $f^n(U) \cap (d,1) = \emptyset$ for all n odd, since $f([0,d]) = [d,1]$ and $f([d,1]) = [0,d]$. (4 marks)

(Total: 20 marks)

3. ((a)(i)&(ii) basic bookwork, (iii) argument used in problem sheet example; (b)(i) algorithm developed in course, (ii)&(iii) used similar in other example on problem sheet.)

(a) (i) Let $n \in \mathbb{N}$, $\varepsilon > 0$, and consider the metric $d_n : X \times X \to \mathbb{R}$ defined by $d_n(x,y) := \max_{i \in \{0,\ldots,n-1\}} d(f^i(x), f^i(y))$. Then a set $A \subset X$ is called (n,ε)-separated if for all $x, y \in A$ with $x \neq y$, we have $d_n(x,y) \geq \varepsilon$. (4 marks)

(ii) Let $\mathrm{Sep}(n,\varepsilon)$ denote the largest possible cardinality of an (n,ε)-separated set. Then $h_{\mathrm{top}}(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln \mathrm{Sep}(n,\varepsilon)$. (4 marks)

(iii) For any fixed m, the set $P_m(g)$ is finite and invariant and hence an (n,ε)-separated set for some ε sufficiently small and all n. Hence, for such ε and n, $\mathrm{Sep}(n,\varepsilon) \geq P_m(g)$. One can now choose a sequence $\varepsilon_i \to 0$ so that this holds for a monotonically increasing sequence $m_i \to \infty$. Consequently, $h_{\mathrm{top}}(g) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln \mathrm{Sep}(n,\varepsilon) \geq \lim_{m \to \infty} \frac{1}{m} \ln P_m(g) = b$. (4 marks)

Consider the maps f and σ, as introduced above, after Q1.

(b) We have $P_m(\sigma) = \mathrm{Tr}(A^m)$. As $A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we have for all $m \geq 1$, $\mathrm{Tr}(A^{2m}) = 2 \cdot 2^{m-1} + 1$. Moreover, $\mathrm{Tr}(A^{2m+1}) = 0$ as there are no periodic orbits with odd period. Hence $\limsup_{m \to \infty} \frac{1}{m} \ln P_m(\sigma) = \frac{1}{2} \ln 2$. By Q3(a)(iii) this implies that $h_{\mathrm{top}}(\sigma) \geq \frac{1}{2} \ln 2$.

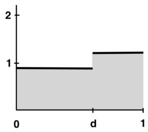
To show that $h_{\mathrm{top}}(\sigma) \leq \frac{1}{2} \ln 2$, we note that the cylinder sets $C_{\omega_0 \ldots \omega_{k-1}}$ are $\varepsilon = 3^{1-k}$ balls covering $\Sigma^+_{3,A}$. The number of such (nonempty) sets serves as an upper bound of the smallest cardinality $\mathrm{Span}(n,\varepsilon)$ of an (n,ε) spanning set, for all n. For each k the number of these (nonempty) sets in $\Sigma^+_{3,A}$ is equal to the number of admissible sequences of length k which is equal to the sum of the entries of A^k. For A^{2m} this is equal to $4 \cdot 2^m + 1$. So $h_{\mathrm{top}}(\sigma) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathrm{Span}(n,\varepsilon) \leq \lim_{m \to \infty} \frac{1}{2m} \ln(4 \cdot 2^m + 1) = \frac{1}{2} \ln 2$.

Finally, as f is a factor of σ it follows (by a result in the lectures) that $h_{\mathrm{top}}(f) \leq h_{\mathrm{top}}(\sigma)$. (This observation will be rewarded with 2/8 marks.) Since - it is easy to see that - the (supremal) exponential rate of growth of $P_m(f)$ is the same as that of $P_m(\sigma)$, in fact also $h_{\mathrm{top}}(f) \geq \frac{1}{2} \ln 2$. Hence $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(\sigma)$. (8 marks)

(Total: 20 marks)

4. ((a) unseen but elementary; (b) unseen but application of BET on the birkhoff sum inside the integral appeared in a class test; (c) discussed for other piecewise affine example in class)

(a) (i) $\int_0^1 d\mu(x) = d^{-1}\left(\frac{1}{2}\int_0^d dx + \frac{1}{\sqrt{2}}\int_d^1 dx\right) = d^{-1}\left(\frac{1}{2}d + \frac{1}{\sqrt{2}}(1-d)\right) = (2d)^{-1}\left(d + \sqrt{2}(1-d)\right) = (2d)^{-1}\left(2-\sqrt{2}+\sqrt{2}(\sqrt{2}-1)\right) = 1.$



(5 marks)

(ii) We need to show that $\mu(f^{-1}(A)) = \mu(A)$, $\forall A \in \mathcal{B}([0,1])$. Note that $\mu(A) = \frac{1}{2}\lambda(A_0) + \frac{1}{\sqrt{2}}\lambda(A_1)$, where λ denotes the Lebesgue measure. Also, $\mu(f^{-1}(A)) = \mu(f^{-1}(A_0)) + \mu(f^{-1}(A_1))$ since $f(\mathrm{Int}(A_0)) = \mathrm{Int}(A_1)$ and $f(\mathrm{Int}(A_1)) = \mathrm{Int}(A_0)$. (3/5 marks will be awarded for this first part of the answer). As f is uniformly expanding, with $|f'(x)| = \sqrt{2}$, we have $\lambda(f^{-1}(A_0)) = \frac{1}{\sqrt{2}}\lambda(A_0)$ and $\lambda(f^{-1}(A_1)) = 2\cdot\frac{1}{\sqrt{2}}\lambda(A_1)$ (since f^{-1} has two branches in $(d,1]$.) We have $\mu(f^{-1}(A_0)) = \frac{1}{\sqrt{2}}\lambda(f^{-1}(A_0))$ since $f^{-1}(A_0) \subset [d,1]$ and $\mu(f^{-1}(A_1)) = \frac{1}{2}\lambda(f^{-1}(A_1))$ since $f^{-1}(A_1) \subset [0,d]$. Hence $\mu(f^{-1}(A)) = \frac{1}{\sqrt{2}}\lambda(f^{-1}(A_0)) + \frac{1}{2}\lambda(f^{-1}(A_1)) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\lambda(A_0) + \frac{1}{2}\cdot 2\cdot\frac{1}{\sqrt{2}}\lambda(A_1) = \mu(A)$.

(5 marks)

(b) By Birkhoff's Ergodic Theorem, $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f^i(x)$ is μ-a.s. equal to $\int_{[0,1]} x d\mu(x) = d^{-1}\left(\frac{1}{2}\int_0^d x dx + \frac{1}{\sqrt{2}}\int_d^1 x dx\right) = (4d)^{-1}(d^2 + \sqrt{2}(1-d^2)) = \frac{2d+1}{4} = \frac{5}{4} - \frac{1}{\sqrt{2}}$ using $d^2 - \sqrt{2}(d^2 - 1) = 2d^2 + d$. As μ is a density with full support, the Birkhoff sum is also Lebesgue-a.s equal to $\frac{5}{4} - \frac{1}{\sqrt{2}}$. The given integral is the average of the Birkhoff sum with respect to the Lebesgue measure on $[0,1]$ and thus precisely equal to $\frac{5}{4} - \frac{1}{\sqrt{2}}$. (5 marks)

(c) The Markov measure related to μ can be obtained from the connectivity matrix A associated to the symbolic dynamics, by transforming (A) into the (irreducible) stochastic matrix $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, with the matrix entries representing the (relative) transition probabilities between the labelling intervals. The left eigenvector π for eigenvalue 1 of P is $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. The corresponding Markov measure is defined as $\nu(C_{\omega_0\ldots\omega_{n-1}}) := \pi_{\omega_0} p_{\omega_0,\omega_1}\ldots p_{\omega_{n-2},\omega_{n-1}}$. In the lecture it was derived that $h_*\nu(C_{\omega_0\cdots\omega_{n-1}}) = \frac{\pi_{\omega_0}}{|I_{\omega_0}|}|h_n(\omega)|$, where π_i denotes the ith component of π. As $|I_0| = |I_1| = d/2$, $|I_2| = 1 - d = d/\sqrt{2}$, we have

$$h_*\nu(C_{\omega_0\cdots\omega_{n-1}}) = \frac{\pi_{\omega_0}}{|I_{\omega_0}|}|h_n(\omega)| = \frac{1}{4}\frac{2}{d}|h_n(\omega)| = \frac{1}{2}d^{-1}|h_n(\omega)|, \text{ if } \omega_0 \in \{0,1\}$$

$$h_*\nu(C_{\omega_0\cdots\omega_{n-1}}) = \frac{\pi_{\omega_0}}{|I_{\omega_0}|}|h_n(\omega)| = \frac{1}{2}\frac{\sqrt{2}}{d}|h_n(\omega)| = \frac{1}{\sqrt{2}}d^{-1}|h_n(\omega)|, \text{ if } \omega_0 = 2$$

so that indeed $\mu = h_*\nu$. (5 marks)

(Total: 20 marks)

5. (Students have self-studied notes on circle homeomorphism by Hasselblatt & Katok and Brin & Stuck. (b) and (c) are standard theory questions. (a), (d) and (e) link theory to the specific example.)

(a) $F_{\alpha,\varepsilon}$ is continuous and continuously differentiable, and $F_{\alpha,\varepsilon}(x)+1=F_{\alpha,\varepsilon}(x+1)$. $F'_{\alpha,\varepsilon}(x)=1-\varepsilon\sin(4\pi x)$ from which it follows that if $|\varepsilon|>1$, F' changes its sign somewhere, implying that F is not invertible. (2 marks)

(b) Since $F(x+1)=F(x)+1$, it follows that if $|x-y|\leq k\in\mathbb{N}$ then also $|F_{\alpha,\varepsilon}(x)-F_{\alpha,\varepsilon}(y)|\leq k$ and $|F^n_{\alpha,\varepsilon}(x)-F^n_{\alpha,\varepsilon}(y)|\leq k$. Hence,

$$|(F^n_{\alpha,\varepsilon}(x)-x)-(F^n_{\alpha,\varepsilon}(y)-y)|\leq|F^n_{\alpha,\varepsilon}(x)-F^n_{\alpha,\varepsilon}(y)|+|x-y|\leq 2k,$$

implying that

$$\lim_{n\to\infty}\frac{F^n_{\alpha,\varepsilon}(x)-x}{n}=\lim_{n\to\infty}\frac{F^n_{\alpha,\varepsilon}(y)-y}{n}.$$

(4 marks)

(c) If $\alpha>\alpha'$ then $F_{\alpha,\varepsilon}(x)>F_{\alpha',\varepsilon}(x)\ \forall x\in\mathbb{R}$. Hence, $F^2_{\alpha,\varepsilon}(x)>F_{\alpha,\varepsilon}(F_{\alpha',\varepsilon}(x))>F^2_{\alpha',\varepsilon}(x)$, where the first inequality is due to monotonicity of $F_{\alpha,\varepsilon}$. Similarly, $F^n_{\alpha,\varepsilon}(x)>F^n_{\alpha',\varepsilon}(x)$, $\forall n\in\mathbb{N}$, from which it follows by taking limits that $\rho(F_{\alpha,\varepsilon})\geq\rho(F_{\alpha',\varepsilon})$. (4 marks)

(d) From the notes, it is known that orbits converge (to a fixed point) if and only if $\rho(f_{\alpha,\varepsilon})=0$ or equivalently if $\rho(F_{\alpha,\varepsilon})=k\in\mathbb{Z}$. The latter is equivalent to the existence of $x\in\mathbb{R}$ such that $F_{\alpha,\varepsilon}(x)=x+k$, with $k\in\mathbb{Z}$, which yields

$$\alpha+\frac{\varepsilon}{2\pi}\cos^2(2\pi x)=k.$$

With $(\alpha,\varepsilon)\in[0,1)\times[-1,1]$ this implies that $k\in\{0,1\}$.
$k=0$, $\varepsilon\geq 0$: solution exists iff $0\leq-\alpha\leq\frac{\varepsilon}{2\pi}\Leftrightarrow\alpha=0$.
$k=0$, $\varepsilon\leq 0$: solution exists iff $\frac{\varepsilon}{2\pi}\leq-\alpha\leq 0\Leftrightarrow\alpha\geq 0$ and $\varepsilon\leq-2\pi\alpha$.
$k=1$, $\varepsilon\geq 0$: solution exists iff $0\leq 1-\alpha\leq\frac{\varepsilon}{2\pi}\Leftrightarrow\alpha\leq 1$ and $\varepsilon\geq 2\pi(1-\alpha)$.
$k=1$, $\varepsilon\leq 0$: solution exists iff $\frac{\varepsilon}{2\pi}\leq 1-\alpha\leq 0\Leftrightarrow$ no solution in domain. (6 marks)

(e) $\rho(F_{\alpha,\varepsilon})$ continuous in α as $F_{\alpha,\varepsilon}$ is continuous in α. It is also a non-decreasing function of α, cf Q5(c). Moreover, $\rho(F_{0,\frac{1}{2}})=0$ and $\rho(F_{0.99,\frac{1}{2}})=1$, see Q5(d). [There are many other ways to show that $\rho(F_{\alpha,\frac{1}{2}})$ is not constant, which suffices.] Hence for any $a\in[0,1]$, there exists $\alpha\in[0,1)$ such that $\rho(F_{\alpha,\frac{1}{2}})=a$. (4 marks)

(Total: 20 marks)

No comments provided