Imperial College London

M4/5P62

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Random Matrices

Date: Friday 02 June 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

**This paper has 5 Questions.**

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

| Raw Mark | Up to 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| Extra Credit | 0 | ½ | 1 | 1 ½ | 2 | 2 ½ | 3 | 3 ½ | 4 |

- Each question carries equal weight.
- Calculators may not be used.

1. i. Give a definition of the Gaussian Unitary Ensemble (GUE). State the expression for the joint probability density function of the eigenvalues of GUE matrices.
ii. For GUE, derive the probability that there are no eigenvalues in a Borel set $B \subset \mathbb{R}$ in terms of Hankel determinants.

2. i. Consider the set $A_n = [n, +\infty)$.

   a) Are there any eigenvalues of (unscaled) GUE matrices in $A_n$? If yes, give a rough estimate for their density in subsets of $A_n$ for large $n$.

   b) Let $p_n(x)$ be the degree $n$ orthogonal polynomial of the orthogonal polynomial ensemble equivalent to GUE. Does $p_n(x)$ have zeros in $A_n$ for large $n$?

   Give a brief, one or two lines, justification for your answers.

   ii. Let the polynomials $q_n(x)$, $n = 0, 1, \dots,$ be orthogonal with respect to the measure on the real line

$$d\mu(x) = \begin{cases} e^{-x^4} dx, & x \in (-1, 1) \\ 0, & \text{otherwise.} \end{cases}$$

   Where are the zeros of $q_n(x)$ located (namely, give the smallest set outside of which there are no zeros of $q_n(x)$ for any $n$)? Why? What is the main asymptotic term for the density of zeros of $q_n(x)$ as $n \to \infty$? State it without derivation.

3. i. Let $\#x_j$ denote the number of points in the determinantal point process with a locally trace class kernel $K(x, y)$ on $\mathbb{R}$. Show that if $TrK = \int_{\mathbb{R}} K(x, x)dx < \infty$ then the probability $P(\#x_j < \infty) = 1$, otherwise $P(\#x_j < \infty) = 0$.
ii. Give the definition of the equilibrium measure $\mu^V$ for a $2m$'th degree polynomial $V(x) = x^{2m} + a_{2m-1}x^{2m-1} \cdots + a_0$, $m \geq 1$.

4. i. Give the definition of the sine-process on $\mathbb{R}$.
ii. Let $\zeta_n$ be the number of points of the sine-process in the interval $(0, n)$.

   a) Derive the expectation of $\zeta_n$.

   b) Show that the variance of $\zeta_n$ is $O(\log n)$ as $n \to \infty$. (You may use the fact that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \pi/2$.)

1. Starting with the variational conditions for the equilibrium measure $d\mu^V(x) = \psi(x)dx$, compute $\psi(x)$ for $V(x) = x^2$, assuming the following identity for a.e. $x$:

$$\frac{d}{dx}\left(\int \log|x-y|\psi(y)dy\right) = v.p. \int \frac{\psi(y)}{x-y}dy.$$

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 1 | | Marks & seen/unseen |
| Parts 1 | GUE is the space of $n\times n$ Hermitian matrices with probability measure $P(M)\,dM = c\cdot e^{-\mathrm{Tr}M^2}\,dM$, $dM = \prod_1^n dM_{jj} \prod_{j<k} d\,\mathrm{Re}M_{jk}\, d\,\mathrm{Im}M_{jk}$ where $c$ is s.t. $\int_{R^{n^2}} P(M)\,dM = 1$. | 4 seen |
| | Joint p.d.f. of eigenvalues $\lambda_1 \dots \lambda_n$: $p(\lambda_1 \dots \lambda_n) = c' e^{-\sum_1^n \lambda_j^2} \prod_{j<k} (\lambda_j - \lambda_k)^2$, $c'$ is s.t. $\int_R d\lambda_1 \cdots \int_R d\lambda_n\, p = 1$. | 4 |
| Setter's initials I.K. | Checker's initials DC | Page number 1 |

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
| --- | --- | --- |
| Question 1 | | Marks & seen/unseen |

Parts

2

$$\mathbb{P}(\text{no } \lambda_j\text{'s in } B) =$$

$$= \mathbb{E} \prod_{j=1}^{n} (1 - \chi_B(\lambda_j)) =$$

$$= \mathbb{E} \prod_{j=1}^{n} \chi_{R\setminus B}(\lambda_j) =$$

seen

6

$$= c' \int_{R^n} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-\lambda_j^2} \chi_{R\setminus B}(\lambda_j)\, d\lambda_j$$

$$= \frac{\int_{R^n} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-\lambda_j^2} \chi_{R\setminus B}(\lambda_j)\, d\lambda_j}{\int_{R^n} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-\lambda_j^2}\, d\lambda_j}$$

$$= \frac{D_n(e^{-\lambda^2} \chi_{R\setminus B}(\lambda))}{D_n(e^{-\lambda^2})}.$$

6

Setter's initials: L.K. Checker's initials: DC

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 2 | | Marks & seen/unseen |
| Parts | | seen/unseen |
| 1 i) | Yes, as j.p.d.f. of eigenvalues is positive on $A_n$. The density $\rho_1(x) = O(e^{-\varepsilon n})$ for some $\varepsilon > 0$, $x \in A_n$, $n \to \infty$, uniformly. | 8 |
| ii) | No, the zeros of these polynomials are inside the set $[-\sqrt{2n+2}, \sqrt{2n+2}]$ for all $n \geq n_0$ for some $n_0 > 0$. | 4 |
| 2 | All zeros of all $p_n(x)$, $n=1,2..$ are located inside $(-1,1)$ as it is the support of the orthogonality measure. The density $\rho(x) = \frac{n}{\pi} \frac{1+o(1)}{\sqrt{1-x^2}}$, $n \to \infty$, $x \in (-1,1)$ | 8 |
| | Setter's initials L.K.   Checker's initials | Page number 3 |

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 3 | | Marks & seen/unseen |
| Parts 1 | If $\int_R K(x,x)\,dx =$ <br> $= E\left(\sum_j \chi_R(x_j)\right) = E(\# x_j) < \infty$ <br> then $\# x_j < \infty$ a.s. <br><br> If $\mathrm{Tr} K = \infty$, let first <br> $B_N = [-N, N]$. Then $\# x_j$ in $B_N$, number of points in $B_N$, denote it $\Xi(B_N)$, is finite, by definition of a point process. <br> For any $c > 0$, by exponential Chebyshev inequality, <br> $\mathbb{P}(\Xi(B_N) \leq c) \leq e^{c}\, \mathbb{E}(e^{-\Xi(B_N)})$ <br> $= e^{c}\, \mathbb{E}(e^{-\sum_j \chi_{B_N}(x_j)}) =$ <br> $= e^{c} \det(I + (1 - e^{\chi_{B_N}})K)$ <br> But $\det(I+M) \leq e^{\mathrm{Tr} M}$ for a | seen <br> 15 |
| | Setter's initials L.K.    Checker's initials DC | Page number 4 |

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 3 | | Marks & seen/unseen |
| Parts | trace-class $M$, so<br>$\mathbb{P}(\Xi(B_N) \le c) \le e^{c-(e-1)\mathrm{Tr}\,K\vert_{B_N}}$<br>Since $\mathrm{Tr}\,K = \infty$, $\mathrm{Tr}\,K\vert_{B_N} \to \infty$<br>as $N \to \infty$ and therefore<br>$\mathbb{P}(\Xi(B_N) \le c) \to 0$, $N \to \infty$<br>Since $\Xi(B_k) \le \Xi(B_{k+1})$, $k=1,2\ldots$<br>it follows by monotone convergence that $\forall c > 0$<br>$\mathbb{P}(\Xi(\mathbb{R}) \le c) = \mathbb{P}(\#X_j \le c) = 0$<br>So $P(\#X_j = \infty) = 1 - \sum_{k=0}^{\infty} \mathbb{P}(\#X_j = k)$<br>$= 1$ | |
| 2 | $\mu_V$ is the unique probability measure s.t.<br>$I(\mu_V) = \inf_{\{\text{prob. measures } \mu\}} I(\mu)$, | 5 |
| | Setter's initials J.K.    Checker's initials DR | Page number 5 |

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 3 | | Marks & seen/unseen |
| Parts | | |

where

$$I(\mu) = \iint_{R^2} \log|x-y|^{-1} d\mu(x)\, d\mu(y) + \int_R V(x)\, d\mu(x) .$$

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| L.K. | [illegible] | 6 |

| | EXAMINATION SOLUTIONS 2016-17 | Course RM |
|---|---|---|
| Question 4 | | Marks & seen/unseen |

Parts

i

Sine process is a determinantal point process on $\mathbb{R}$ with correlation kernel $K = \frac{\sin \pi(x-y)}{\pi(x-y)}$ — 2 seen

2i (a)

$$\mathbb{E}\, \zeta_n = \mathbb{E}\Big( \sum_j \chi_{(0,n)}(x_j) \Big) = \int_0^n K(x,x)\, dx = n$$

3 seen

2ii (b)

unseen 15

$$\mathrm{Var}\, \zeta_n = \mathbb{E}\Big(\big(\sum_j \chi_{(0,n)}(x_j)\big)^2\Big) - \mathbb{E}\Big(\sum_j \chi_{(0,n)}(x_j)\Big)^2 =$$

$$= \mathbb{E}\Big(\sum_j \chi_{(0,n)}(x_j)\Big) + \mathbb{E}\Big(\sum_{j\neq k} \chi_{(0,n)}(x_j)\chi_{(0,n)}(x_k)\Big) - n^2$$

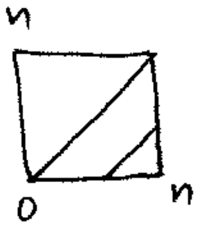
$$= n - n^2 + \int_0^n dx \int_0^n dy \left(1 - \left(\frac{\sin \pi(x-y)}{\pi(x-y)}\right)^2\right)$$

Setter's initials L.K. Checker's initials

Page number 4

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 4 | | Marks & seen/unseen |

Parts



$$u = x - y, \quad v = x + y$$

$$x = \frac{u+v}{2}; \quad y = \frac{v-u}{2}$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left|\begin{matrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{matrix}\right| .$$

$$du\,dv = \frac{1}{2}\,du\,dv$$

$u \in (-n, n)$,

$u > 0$, $v \in (u, 2n-u)$;

$u < 0$, $v \in (-u, 2n+u)$.

$$\mathrm{Var}\, \mathcal{I}_n = n - n^2 + n^2 -$$

$$-\frac{1}{2}\int_{-n}^{0}\left(\frac{\sin \pi u}{\pi u}\right)^2 2(n+u)\,du$$

$$-\frac{1}{2}\int_{0}^{n}\left(\frac{\sin \pi u}{\pi u}\right)^2 2(n-u)\,du$$

$$= n - 2\int_0^n \left(\frac{\sin \pi u}{\pi u}\right)^2 (n-u)\,du$$

$$= n - 2\int_0^{\pi n} \left(\frac{\sin x}{x}\right)^2 \left(n - \frac{x}{\pi}\right)\frac{dx}{\pi}$$

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 4 | | Marks & seen/unseen |
| Parts | $$\frac{2}{\pi}\int_0^{\pi n}\left(\frac{\sin x}{x}\right)^2 dx = 1 + \frac{2}{\pi}\int_{\pi n}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx$$ $$= 1 + O(1/n),\ n \to \infty$$ $$\frac{2}{\pi^2}\int_0^{\pi n}\frac{\sin^2 x}{x}\,dx \leq \frac{2}{\pi^2}\log n + O(1),\ n\to\infty$$ So $$\left\lvert \operatorname{Var} \mathfrak{Z}_n \right\rvert \leq \frac{2}{\pi^2}\log n + O(1),\ n \to \infty.$$ | |
| | Setter's initials A.K. | Checker's initials |
| | | Page number 9 |

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 5 | | Marks & seen/unseen |

Parts

Differentiating the variational condition

$$\int \log|x-y|^{-2}\psi(y)\,dy + x^2 = \ell, \quad x : \psi(x) > 0$$

we obtain

$$\text{v.p.} \int \frac{\psi(y)}{y-x}\,dy = -x$$

Consider $G(z) = \frac{1}{i\pi}\int \frac{\psi(y)}{y-z}\,dy$

We assume $\psi(x) \geq 0$ on $[-a,a]$, zero otherwise. On $(-a,a)$,

$$G_{\pm}(x) = \pm\psi(x) + \frac{1}{i\pi}\,\text{v.p.}\int \frac{\psi(y)}{y-x}\,dy$$

$$= \pm\psi(x) - \frac{x}{i\pi}, \quad \text{so}$$

1) $G_+ + G_- = \frac{2i}{\pi}x, \quad x \in (-a,a)$,

2) $G(z) \to 0, \quad z \to \infty$,

3) $G(z)$ - analytic in $\mathbb{C} \setminus [-a,a]$,

4) $G_{\pm}(x)$ are in $L^2$.

seen/unseen

20

| Setter's initials L.K. | Checker's initials | Page number 10 |
|---|---|---|

| | EXAMINATION SOLUTIONS 2016-17 | Course R.M. |
|---|---|---|
| Question 5 | | Marks & seen/unseen |

Parts

The solution to this b.v. problem is

$$G(z) = \frac{(z^2-a^2)^{1/2}}{\pi^2} \int_{-a}^{a} \frac{s\,ds}{\sqrt{s^2-a^2}_+\,(s-z)},$$

in particular,

$$G_\pm = \frac{\pm (x^2-a^2)_+^{1/2}}{\pi^2}\left(\pm \pi i \frac{x}{\sqrt{x^2-a^2}_+} + \text{v.p.} \int_{-a}^{a} \frac{s\,ds}{\sqrt{s^2-a^2}_+\,(s-x)}\right)$$

$$= \frac{ix}{\pi} \pm \frac{\sqrt{x^2-a^2}_+}{\pi^2}\,\text{v.p.} \int_{-a}^{a} \frac{s\,ds}{\sqrt{s^2-a^2}_+\,(s-x)}$$


By residue calculation

$$G(z) = \frac{(z^2-a^2)^{1/2}}{2\pi^2}\left(\frac{2\pi i\, z}{(z^2-a^2)^{1/2}} + I\right),$$

$$I = \oint_{\substack{|z|=R \\ \text{clockwise}}} \frac{s\,ds}{(s^2-a^2)^{1/2}(s-z)} =$$

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| | l.K. | | 11 |

| | EXAMINATION SOLUTIONS 2016-17 | Course KM. |
|---|---|---|
| Question 5 | | Marks & seen/unseen |

Parts

$$= \oint_{\substack{|z|=R \\ \text{clockwise}}} \frac{1}{s}\left(1+O\left(\tfrac{1}{s}\right)\right) ds = \\ = -2\pi i ,$$

$$G(z) = \frac{i}{\pi}\left(z - (z^2-a^2)^{1/2}\right)$$

$$\Psi(x) = \operatorname{Re} G_+(x) = \frac{1}{\pi}\sqrt{a^2-x^2}, \quad x \in (-a,a).$$

$$1 = \int \Psi(x)\,dx = \frac{1}{\pi}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx =$$

$$= \frac{a^2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{a^2}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2\varphi\,d\varphi$$

$$= \frac{a^2}{\pi}\cdot\frac{\pi}{2} = \frac{a^2}{2} \Rightarrow a = \sqrt{2}$$

$$\Rightarrow \underline{\Psi(x) = \tfrac{1}{\pi}\sqrt{2-x^2}, \ x \in (-\sqrt{2},\sqrt{2})}$$

Remains to check

$$\int_a^x \left(-iG - \tfrac{y}{\pi}\right)dy = -\int_a^x \sqrt{y^2-a^2}\,dy < 0, \quad x > a$$

$$\int_x^{-a} \left(-iG - \tfrac{y}{\pi}\right)dy = -\int_x^{-a} \sqrt{y^2-a^2}\,dy > 0, \quad x < -a.$$

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## PExaminer's Comments

**Exam:** __ **P62**______________________________ **Session: 2016-2107**

### Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Some confusion between
GUE and general
unitary ensemble.

**Marker:** ______________________________

**Signature:**______________________________ **Date:** ______________________

**Please return with exam marks (one report per marker)**

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# Examiner's Comments

**Exam: ____ P62____________________________ Session: 2016-2107**

## Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Rarely completely done

**Marker:** ______________________________

**Signature:**____________________________ **Date:** ______________________

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# Examiner's Comments

**Exam: _____ P62_________________________ Session: 2016-2107**

**Question 3**

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Standard material from lectures

**Marker:** ____________________________
**Signature:**____________________________ **Date:** ____________________

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## Examiner's Comments

**Exam:** ___ **P62**______________________________ **Session: 2016-2107**

**Question 4**

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Often the beginning of the answer
To 4b is only done, the rest of the question ok

**Marker:** ______________________________

**Signature:**______________________________ **Date:** ______________________

**Please return with exam marks (one report per marker)**

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Department of Mathematics

## Examiner's Comments

**Exam: ____ P62______________________________ Session: 2016-2107**

**Question 5**

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Standard material from lectures

**Marker:** ______________________________

**Signature:**______________________________ **Date:** ______________________

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