Introduction to Quantum Mechanics – Problem sheet 9

1. **Properties of the Pauli matrices** - *Some basic practice of calculating with Pauli matrices and using the Levi-Civita symbol, take it or leave it, this is not essential.*

   (a) Verify that the Pauli matrices have the properties
   $$\sigma_j^2 = 1, \quad \text{and} \quad \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I.$$

   (b) Use the above properties of the Pauli matrices and the commutation relation
   $$[\sigma_i, \sigma_j] = 2\mathrm{i}\sum_k \varepsilon_{ijk}\sigma_k,$$
   to derive the identity
   $$(\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma) = (\mathbf{A}\cdot\mathbf{B})\hat{I} + \mathrm{i}(\mathbf{A}\times\mathbf{B})\cdot\sigma,$$
   where $\mathbf{A}$ and $\mathbf{B}$ are vectors.

2. **Angular momentum matrices for** $j = 1$ - *Basic exercise combining an angular momentum example with the principles of quantum mechanics from Chapter 4. Definitely have a go at this one.*

   The matrices
   $$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad \hat{L}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$
   are a representation of the angular momentum operators for the total angular momentum quantum number $l = 1$ in the eigenbasis of $\hat{L}_z$.

   (a) Calculate the normalised eigenvectors of $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ corresponding to the eigenvalue $m = 0$ and show that these vectors form an orthogonal set.

   (b) Assume the system is in the state $|\phi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$.

      (i) Calculate the expectation value and the uncertainty of $\hat{L}_z$.

      (ii) Calculate the probabilities that a measurement of $\hat{L}_x$, $\hat{L}_y$ or $\hat{L}_z$ respectively yields the value zero.

3. **Dynamics of a spin $\frac{1}{2}$ system.** - *Expanding on the material from Chapter 9 and connecting back to Chapter 8. Good practice of commutators and solving dynamical equations, not too hard.*

   Consider the time evolution of the expectation values of the spin components for a spin $\frac{1}{2}$ particle generated by the Hamiltonian
   $$\hat{H} = \vec{B}\cdot\hat{\vec{\sigma}},$$
   where $\vec{B} \in \mathbb{R}^3$ and $\hat{\vec{\sigma}}$ denotes the vector consisting of the Pauli matrices. In particular, write down the Heisenberg equations of motion for the spin components. Then show that their motion is confined to a sphere. Sketch the resulting dynamics on the sphere.