

Math40002 Analysis 1

Problem Sheet 2

1. Fix $S \subset \mathbb{R}$ with an upper bound, and suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Give proofs or counterexamples to each of the following statements.

- (a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.
- (b) If $S \subset \mathbb{R} \setminus \mathbb{Q}$ then $\sup S \in \mathbb{R} \setminus \mathbb{Q}$.
- (c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.
- (d) $S \cap \left\{ \frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}_{>0}, m \leq 10^{100} \right\}$ has a minimum if it is nonempty.
- (e) There exists a $\max S$ if and only if $\sup S \in S$.
- (f) $\sup S = \inf(\mathbb{R} \setminus S)$.
- (g) $\sup S = \inf(\mathbb{R} \setminus S) \iff S$ is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

2. Fix nonempty sets $S_n \subset \mathbb{R}$, $n = 1, 2, 3, \dots$. Prove that

$$\sup \{\sup S_1, \sup S_2, \sup S_3, \dots\} = \sup \left(\bigcup_{n=1}^{\infty} S_n \right),$$

in the sense that if either exists then so does the other, and they are equal.

3. Take bounded, nonempty $S, T \subset \mathbb{R}$. Define $S + T := \{s + t : s \in S, t \in T\}$. Prove

$$\sup(S + T) = \sup S + \sup T.$$

- 4.* Fix $a \in (0, \infty)$ and $n \in \mathbb{N}_{>0}$ with $n > 1$. We will prove $\exists x \in \mathbb{R}$ such that $x^n = a$. Set

$$S_a := \{s \in [0, \infty) : s^n < a\}$$

and show S is nonempty and bounded above, so we may define $x := \sup S_a$.

For $\epsilon \in (0, 1)$ show $(x + \epsilon)^n \leq x^n + \epsilon[(x + 1)^n - x^n]$. (Hint: multiply out.) Hence show that if $x^n < a$ then $\exists \epsilon \in (0, 1)$ such that $(x + \epsilon)^n < a$. (*)

If $x^n > a$ deduce from (*) that $\exists \epsilon \in (0, 1)$ such that $(\frac{1}{x} + \epsilon)^n < \frac{1}{a}$. (**)

Deduce contradictions from (*) and (**) to show that $x^n = a$.

5. Suppose $0 < q \in \mathbb{Q}$ and $a \in (0, \infty)$. Write $q = \frac{m}{n}$ with $m, n \in \mathbb{N}_{>0}$ and define

$$a^q := x^m,$$

where $x =: a^{1/n}$ is defined in the last question. Show this is well defined, and make a definition of a^{-q} .

Show that $(ab)^q = a^q b^q$ and $(a^{q_1})^{q_2} = a^{q_1 q_2}$ for any $a, b \in (0, \infty)$ and $q, q_1, q_2 \in \mathbb{Q}$.

6. For real numbers x, y, z , consider the following inequalities.

(a) $|x + y| \leq |x| + |y|$
(b) $|x + y| \geq |x| - |y|$
(c) $|x + y| \geq |y| - |x|$
(d) $|x - y| \geq ||x| - |y||$

(e) $|x| \leq |y| + |x - y|$
(f) $|x| \geq |y| - |x - y|$
(g) $|x - y| \leq |x - z| + |y - z|$

Prove (a) from first principles. Why is it called the “triangle inequality”?

Deduce (b,c,d,e,f,g) from (a).

*Starred questions ** are good to prepare to discuss with your Personal Tutor.