IMPERIAL

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Vortex Dynamics

Date: Tuesday, May 28, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

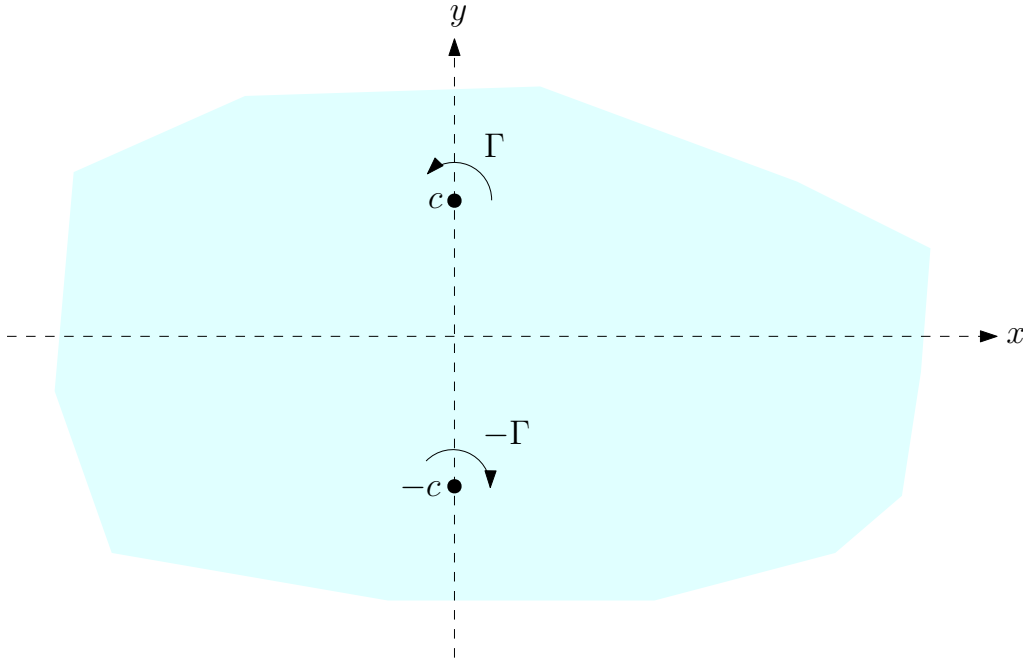Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider two point vortices of circulation $\pm\Gamma$ with $\Gamma > 0$ located, at some initial instant $t = 0$, at $(0, \pm c)$ in an (x, y) plane where $c > 0$ as shown in the figure.

(a) Suppose the flow around the point vortices is irrotational. On introducing the variable $z = x + iy$ the velocity field (u, v) in the vicinity of the vortex at $z = ic$ has the local form

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{1}{z - ic} + \alpha + \beta(z - ic) + \cdots$$

Find the values of the constants α and β. (4 marks)

(b) Find the principal axes of the local linear straining flow at ic and the principal strain rates. Sketch the streamlines associated with this local linear straining flow. (6 marks)

(c) Show that both vortices move with uniform speed α for $t > 0$. (4 marks)

(d) Find any stagnation points of the flow in a frame of reference cotravelling with the two vortices (i.e. in the frame in which the flow is steady). (4 marks)

(e) Suppose instead that the flow around the vortices shown in the figure above is now instantaneously in solid body rotation with angular velocity Ω. Find the instantaneous speeds of the two vortices in this situation. (2 marks)

(Total: 20 marks)

2. Consider the problem of two-dimensional uniform shear flow in an $\mathbf{x} = (x, y)$ plane with velocity $\mathbf{u} = (u, v) \to (y, 0)$ as $|\mathbf{x}| \to \infty$, past an impenetrable elliptical cylinder centred at the origin with semi-axes a and b as shown in the figure. There is no circulation around the cylinder.
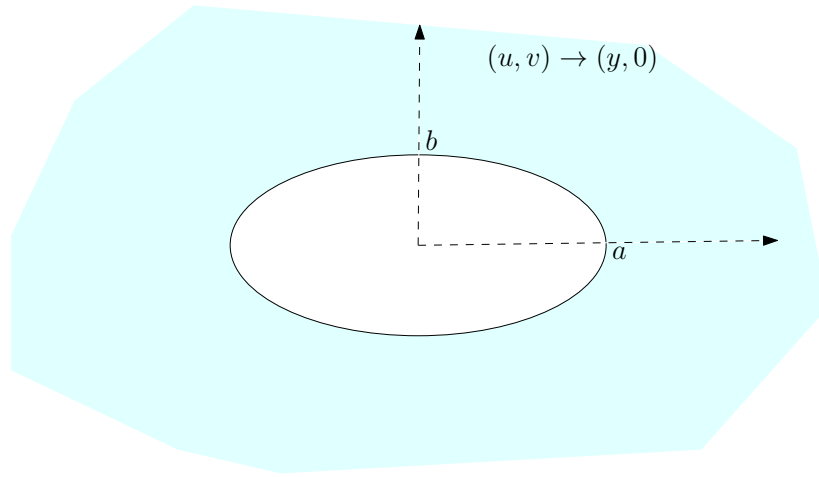
(a) Find the conformal mapping $z = Z(\zeta)$ from the interior of the unit disc $|\zeta| < 1$ to the unbounded fluid region exterior of the elliptical cylinder (shaded in the Figure). (2 marks)

(b) To find the streamfunction ψ for the flow as a function of $z = x + iy$ and $\bar{z}$ let

$$\psi(z, \bar{z}) = \frac{z\bar{z}}{4} + \text{Im}[w(z)],$$

where $w(z)$ is analytic in the fluid region but with a singularity as $z \to \infty$. Find the required behaviour of $w(z)$ as $z \to \infty$. (4 marks)

(c) Define the function of ζ given by

$$W(\zeta) \equiv w(Z(\zeta)).$$

  (i) Identify the singularities of $W(\zeta)$ inside the unit disc $|\zeta| < 1$. (2 marks)

  (ii) Write down the boundary condition satisfied by $W(\zeta)$ on $|\zeta| = 1$. (2 marks)

(d) Find a function, $W_s(\zeta)$ say, with the same singularities as $W(\zeta)$ inside the unit disc $|\zeta| < 1$ and satisfying the condition $\text{Im}[W_s(\zeta)] = 0$ on $|\zeta| = 1$. (4 marks)

(e) By writing
$$W(\zeta) = W_s(\zeta) + \hat{W}(\zeta),$$

where $\hat{W}(\zeta)$ is analytic inside the unit disc $|\zeta| < 1$, solve the boundary value problem for $W(\zeta)$ following from part (c)(ii) by finding $\hat{W}(\zeta)$. (6 marks)

(Total: 20 marks)

3. At time $t = 0$ a point vortex of circulation Γ is situated at $(1, 0)$ in a region of otherwise irrotational, incompressible, two-dimensional flow (u, v) occupying the (x, y) plane exterior to an infinite impenetrable wall along the entire negative x-axis as shown in the figure.
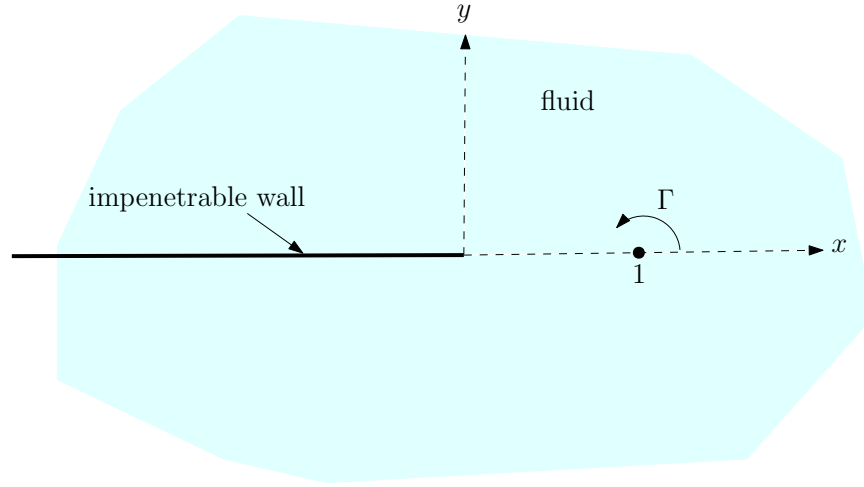
(a) Find the conformal mapping, $z = f(\zeta)$, from the interior of the unit disc $|\zeta| < 1$ in a parametric complex ζ plane to the fluid region shown in the figure. (6 marks)

(b) Find the instantaneous fluid velocity (u, v) at $t = 0$ on the upper side of the wall at $(-1, 0)$. (6 marks)

(c) Show that if the position of the point vortex at later times $t > 0$ is $z_a(t) = f(a(t))$ then $a(t)$ is a solution of the nonlinear equation

$$(1 - a(t)\overline{a(t)})^2(a(t) - 1)(\overline{a(t)} - 1) = (a(t) + 1)^3(\overline{a(t)} + 1)^3.$$

(8 marks)

Hint: You may use the following relations between Hamiltonians for N point vortex motion in a fluid domain in a z-plane and a corresponding domain in a ζ-plane:

$$H^{(z)}(\{z_j(t)\}) = H^{(\zeta)}(\{\zeta_j(t)\}) + \sum_{j=1}^{N} \frac{\Gamma_j^2}{4\pi} \log|f'(\zeta_j)|, \qquad f'(\zeta) = \frac{df}{d\zeta}$$

$$H^{(\zeta)}(\{\alpha_j\}) = \mathrm{Im}\left[\sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{k=1}^{N} \frac{\Gamma_j \Gamma_k}{2} G_0(\alpha_j, \alpha_k) + \sum_{j=1}^{N} \frac{\Gamma_j^2}{2} g(\alpha_j, \alpha_j) \right],$$

$$G_0(\zeta, \alpha) = -\frac{i}{2\pi} \log(\zeta - \alpha) + g(\zeta, \alpha),$$

where $z = f(\zeta)$ is a conformal mapping from $|\zeta| < 1$ to the fluid domain and $G_0(\zeta, \alpha)$ is the complex potential for a single point vortex of unit circulation at $\zeta = \alpha$ in the unit ζ disc. Here $\mathrm{Im}[.]$ denotes the imaginary part of the complex quantity in square brackets.

(Total: 20 marks)

4. This question concerns the motion of point vortices on the surface of a sphere of unit radius where (r, θ, ϕ) are the usual spherical polar coordinates about an axis through the north and south poles.

(a) Describe the stereographic projection from the surface of the sphere to a complex ζ plane through the equator of the sphere and give a formula for ζ in terms of θ and ϕ. (4 marks)

(b) Consider a configuration in which four point vortices each of circulation Γ are situated at equi-spaced longitudinal distances around a latitude circle at $\theta = \theta_a$. Show that this configuration rotates at constant angular velocity Ω about an axis through the north and south poles, and find the value of Ω in terms of θ_a. (8 marks)

Hint: The streamfunction for solid body rotation with angular velocity Ω about an axis through the north and south poles of a unit-radius sphere is $-2\Omega/(1 + \zeta\bar{\zeta})$.

(c) Show that a configuration of 6 point vortices, all having circulation Γ, with 4 of them equally spaced around the equator, one at the north pole and another at the south pole, is a stationary equilibrium. (8 marks)

(Total: 20 marks)

5. Consider a symmetric point vortex street as shown in the figure where an upper row of point vortices of circulation Γ and period a is distance $2b$ above a lower row of point vortices of circulation $-\Gamma$ with the same period a and with the x-axis running midway between the two vortex rows:
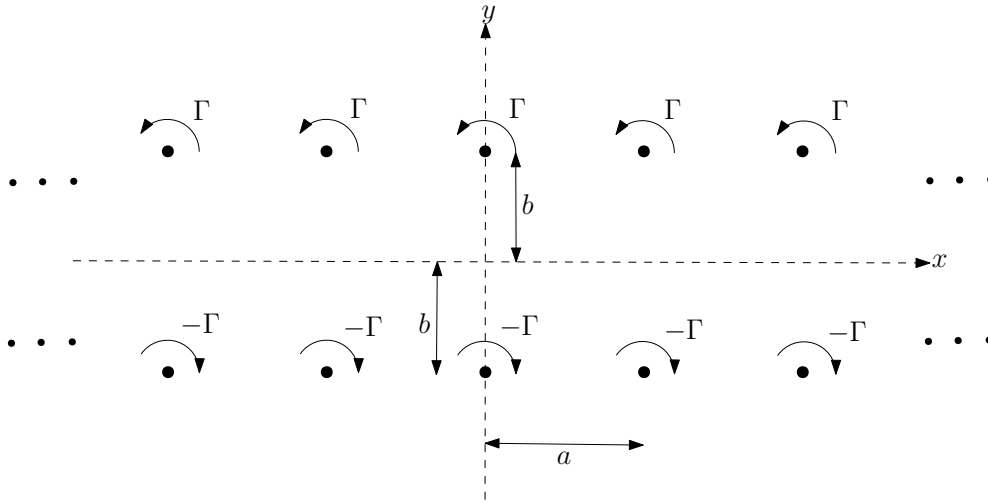
(a) Let $z = x + \mathrm{i}y$. Write down the complex potential $w(z)$ for the flow shown in the figure.

Hint: This convergent infinite product representation may be useful:

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2} \right).$$

(3 marks)

(b) Show that the configuration of vortices shown in the figure moves at uniform speed U parallel to the x direction and find U as a function of Γ, a and b. (3 marks)

(c) Show that the x axis is a streamline of the flow. (2 marks)

(d) Find the image in a complex z plane of the interior of the unit ζ disc, $|\zeta| < 1$, under the conformal mapping

$$z = -\frac{\mathrm{i}a}{2\pi} \log \zeta,$$

where the branch cut of the logarithm is chosen between 0 and ∞ along the negative real ζ axis (you should identify the image of the circle $|\zeta| = 1$ as well as the images of the two sides of the branch cut along the negative real ζ axis inside the unit disc). (4 marks)

(e) Write down the complex potential $\mathcal{G}_0(\zeta, \alpha)$ for a point vortex of circulation Γ at a point $\zeta = \alpha$ inside the unit ζ disc (i.e., $|\alpha| < 1$). (2 marks)

(f) Use your answers to parts (c), (d) and (e) to rederive your answer to part (a). (6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

MATH70051

Vortex dynamics (Solutions)

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1. (a) Using the well known complex potential for a point vortex, the complex velocity field is
seen ⇓

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{1}{z - ic} + \frac{i\Gamma}{2\pi}\frac{1}{z + ic}.$$

Let $Z = z - ic$ then this is

$$
\begin{aligned}
u - iv &= -\frac{i\Gamma}{2\pi}\frac{1}{Z} + \frac{i\Gamma}{2\pi}\frac{1}{Z + 2ic} \\
&= -\frac{i\Gamma}{2\pi}\frac{1}{Z} + \frac{i\Gamma}{2\pi}\frac{1}{2ic(1 + Z/(2ic))} \\
&= -\frac{i\Gamma}{2\pi}\frac{1}{Z} + \frac{i\Gamma}{2\pi}\frac{1}{2ic}\left(1 - \frac{Z}{2ic} + \cdots\right) \\
&= -\frac{i\Gamma}{2\pi}\frac{1}{Z} + \frac{\Gamma}{4\pi c} + \frac{i\Gamma}{8\pi c^2}Z + \cdots
\end{aligned}
$$

where the expansion of a geometric series for small $|Z|$ has been used. Hence, the required constants are

$$\alpha = \frac{\Gamma}{4\pi c}, \qquad \beta = \frac{i\Gamma}{8\pi c^2}.$$

4, A

(b) The linear straining flow component, using $Z = z - ic = X + iY$ is

seen ⇓

$$U - iV = \frac{i\Gamma}{8\pi c^2}(X + iY) = \frac{\Gamma}{8\pi c^2}(-Y + iX)$$

or

$$(U, V) = \frac{\Gamma}{8\pi c^2}(-Y, -X)$$

In matrix form this is

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{\Gamma}{8\pi c^2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

To find the principal axes of strain, this 2-by-2 matrix must be diagonalized. The eigenvalues are $\lambda = \pm\Gamma/(8\pi c^2)$ giving eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = -\frac{\Gamma}{8\pi c^2} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda = \frac{\Gamma}{8\pi c^2}.$$

The eigenvectors are the principal axes of strain, the eigenvalues are the principal strain rates. The streamlines (which are hyperbolae) are as shown in the following sketch (centred at $z = ic$):

6, B

(c) Suppose the velocity (u, v) in a frame cotravelling with speed α is

seen ⇓

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{1}{z - ic} + \frac{i\Gamma}{2\pi}\frac{1}{z + ic} - \alpha.$$

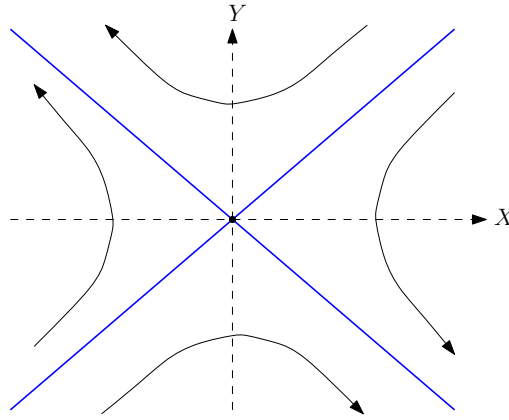By the usual non-self-induction rule the vortex at ic moves with speed

$$\left(+\frac{i\Gamma}{2\pi}\frac{1}{z + ic} - \alpha\right)\Bigg|_{z=ic} = 0$$

and the vortex at $-ic$ moves with speed

$$\left(-\frac{i\Gamma}{2\pi}\frac{1}{z - ic} - \alpha\right)\Bigg|_{z=-ic} = 0.$$

Hence both vortices are steady in this cotravelling frame confirming that both vortices move with speed α.

4, A

(d) The stagnation points of the flow in the cotravelling frame are solutions of

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{1}{z - ic} + \frac{i\Gamma}{2\pi}\frac{1}{z + ic} - \alpha = 0.$$

Equivalently,

$$-\frac{i\Gamma}{2\pi}\frac{2ic}{z^2 + c^2} = \alpha = \frac{\Gamma}{4\pi c}$$

or

$$4c^2 = z^2 + c^2, \qquad z = \pm\sqrt{3}c.$$

Therefore there are two stagnation points of the flow at

$$(\sqrt{3}c, 0), \qquad (-\sqrt{3}c, 0).$$

4, C

(e) The complex velocity $u - iv$ associated with solid body rotation with angular velocity Ω is $-i\Omega\bar{z}$. Therefore the instantaneous velocity field in the presence of the two point vortices is now

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{1}{z - ic} + \frac{i\Gamma}{2\pi}\frac{1}{z + ic} - i\Omega\bar{z}.$$

By the non-self-induction rule, the vortex at ic now moves with speed

$$\left(+\frac{i\Gamma}{2\pi}\frac{1}{z + ic} - i\Omega\bar{z}\right)\Bigg|_{z=ic} = \frac{\Gamma}{4\pi c} - \Omega c.$$

Similarly, the vortex at $-ic$ moves with speed

$$\left(-\frac{i\Gamma}{2\pi}\frac{1}{z - ic} - i\Omega\bar{z}\right)\Bigg|_{z=-ic} = \frac{\Gamma}{4\pi c} + \Omega c.$$

2, A

2. (a) The required conformal mapping is well-known from lectures to be

$$z = Z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta, \qquad \alpha, \beta \in \mathbb{R},$$

where

$$\alpha + \beta = a, \qquad \alpha - \beta = b.$$

(b) The complex velocity is given by

$$u - iv = 2i\frac{\partial\psi}{\partial z}.$$

Since

$$\psi(z, \overline{z}) = \frac{z\overline{z}}{4} + \frac{w(z) - \overline{w(z)}}{2i}$$

then

$$u - iv = 2i\left(\frac{\overline{z}}{4} + \frac{w'(z)}{2i}\right) = \frac{1}{2i}\left(2iw'(z) - \overline{z}\right).$$

Since we require $u - iv \to y = (z - \overline{z})/(2i) + \mathcal{O}(1/z)$ then we must have

$$2iw'(z) \to z + \mathcal{O}(1/z), \qquad \text{as } |z| \to \infty.$$

Therefore, the far-field condition on $w(z)$ is

$$w(z) \to -\frac{iz^2}{4} + \text{constant} \qquad \text{as } |z| \to \infty.$$

(c) (i) Let $W(\zeta) = w(Z(\zeta))$. By part (b), $w(z)$ is analytic everywhere in the fluid except at infinity where $w(z) \to -iz^2/4 +$constant as $|z| \to \infty$. Since $\zeta = 0$ is the preimage of the point at infinity in the physical plane under the mapping $Z(\zeta)$ then this implies that

$$W(\zeta) \to -\frac{i}{4}\left(\frac{\alpha}{\zeta} + \beta\zeta\right)^2 + \cdots = -\frac{i\alpha^2}{4\zeta^2} + \text{constant}, \qquad \text{as } \zeta \to 0.$$

Therefore, $W(\zeta)$ must be analytic inside the unit disc except for this second-order pole at $\zeta = 0$.

(ii) The boundary of the ellipse must be a streamline. Since this corresponds to $|\zeta| = 1$, then we require $\psi = 0$ on $|\zeta| = 1$ or

$$\text{Im}[W(\zeta)] = -\frac{Z(\zeta)\overline{Z(\zeta)}}{4}, \qquad \text{on } |\zeta| = 1.$$

(d) Let

$$W_d(\zeta) = -\frac{i\alpha^2}{4\zeta^2}$$

The function

$$W_s(\zeta) = W_d(\zeta) + \overline{W_d(\zeta)}$$

has the required singularity at $\zeta = 0$, and it is real everywhere including on $|\zeta| = 1$, so it appears to satisfy all requirements. But it is not an analytic function. However, modifying it to

$$W_s(\zeta) = W_d(\zeta) + \overline{W_d(1/\overline{\zeta})},$$

where $\zeta = 1/\overline{\zeta}$ makes it an analytic function which is real on $|\zeta| = 1$ and with the correct singularity at $\zeta = 0$. Therefore,

$$\boxed{W_s(\zeta) = -\frac{i\alpha^2}{4}\left(\frac{1}{\zeta^2} - \zeta^2\right)}$$

is the required function.

(e)   On setting

$$W(\zeta) = W_s(\zeta) + \hat{W}(\zeta)$$

then $\hat{W}(\zeta)$ must be analytic inside the disc with boundary conditions

$$\text{Im}[\hat{W}(\zeta)] = -\frac{Z(\zeta)\overline{Z(\zeta)}}{4}, \qquad \text{on } |\zeta| = 1,$$

where we have used the fact that $\text{Im}[W_s(\zeta)] = 0$ on $|\zeta| = 1$. But, using part (a), this can be written as

$$\text{Im}[\hat{W}(\zeta)] = -\frac{1}{4}\left(\frac{\alpha}{\zeta} + \beta\zeta\right)\left(\alpha\zeta + \frac{\beta}{\zeta}\right)$$

$$= -\frac{1}{4}(\alpha^2 + \beta^2) - \frac{\alpha\beta}{4}\left(\zeta^2 + \frac{1}{\zeta^2}\right), \qquad \text{on } |\zeta| = 1.$$

Notice that the right hand side is

$$\text{Im}\left[-\frac{i\alpha\beta\zeta^2}{2} - \frac{i}{4}(\alpha^2 + \beta^2)\right], \qquad \text{on } |\zeta| = 1.$$

But the function in square brackets in this expression is analytic in $|\zeta| < 1$ therefore

$$\hat{W}(\zeta) = -\frac{i\alpha\beta\zeta^2}{2} - \frac{i}{4}(\alpha^2 + \beta^2) + c$$

where $c \in \mathbb{R}$ is a suitable solution for $\hat{W}(\zeta)$. The required streamfunction, written as a function of z and ζ, is then

$$\psi(z,\overline{z}) = \frac{z\overline{z}}{4} + \text{Im}\left[-\frac{i\alpha^2}{4}\left(\frac{1}{\zeta^2} - \zeta^2\right) - \frac{i\alpha\beta\zeta^2}{2} - \frac{i}{4}(\alpha^2 + \beta^2)\right].$$

3. (a) The Cayley mapping

$$\eta = \frac{1 - \zeta}{1 + \zeta}$$

transplants the unit disc $|\zeta| < 1$ to the right half η plane. Therefore the mapping

$$z = f(\zeta) = \eta^2 = \left(\frac{1 - \zeta}{1 + \zeta}\right)^2$$

transplants the unit disc to the unbounded region exterior to the negative x axis and where $|\zeta| = 1$ maps to the negative x axis.

6, B

(b) The complex potential $w(z)$ for a point vortex of circulation Γ at z_a is given parametrically by $W(\zeta) = w(f(\zeta))$ where

$$W(\zeta) = \mathcal{G}_0(\zeta, a) = -\frac{i\Gamma}{2\pi} \log\left(\frac{\zeta - a}{|a|(\zeta - 1/\bar{a})}\right),$$

where $z_a = f(a)$. Under the mapping found in part (a) the preimage of $z_a = f(a) = 1$ is $a = 0$ thus

$$W(\zeta) = \mathcal{G}_0(\zeta, a) = -\frac{i\Gamma}{2\pi} \log \zeta.$$

Therefore, by the chain rule,

$$u - iv = \frac{dw}{dz} = \frac{W'(\zeta)}{f'(\zeta)} = \frac{\mathcal{G}_0'(\zeta, a)}{f'(\zeta)}.$$

Now

$$z = f(\zeta) = \eta^2 = \left(\frac{1 - \zeta}{1 + \zeta}\right)^2 = \left(1 - \frac{2}{\zeta + 1}\right)^2 = 1 - \frac{4}{\zeta + 1} + \frac{4}{(\zeta + 1)^2}$$

hence

$$f'(\zeta) = \frac{4}{(\zeta + 1)^2} - \frac{8}{(\zeta + 1)^3} = \frac{4(\zeta - 1)}{(\zeta + 1)^3}.$$

Therefore,

$$u - iv = \frac{\mathcal{G}_0'(\zeta, a)}{f'(\zeta)} = -\frac{i\Gamma}{2\pi\zeta} \times \frac{(\zeta + 1)^3}{4(\zeta - 1)}.$$

The preimage of $(-1, 0)$ on the upper wall is $\zeta = -i$: this is because at this point

$$\eta = \frac{1 + i}{1 - i} = \frac{(1 + i)^2}{2} = i$$

is on the positive y axis where $\arg[\eta] = \pi/2$ hence $z = \eta^2 = -1$ on the upper side where $\arg[z] = \pi$. Therefore the velocity there is

$$u - iv = -\frac{\Gamma}{2\pi} \frac{(1 - i)^3}{4(1 + i)} = -\frac{\Gamma}{2\pi} \frac{(1 - i)^4}{8} = -\frac{\Gamma}{2\pi} \frac{(1 - 4i + 6(i)^2 - 4(i)^3 + 1)}{8} = \frac{\Gamma}{4\pi}.$$

The required fluid velocity at this point is therefore

$$\left(\frac{\Gamma}{4\pi}, 0\right).$$

6, D

(c)   This question is asking for the trajectory of a vortex initially at $(1,0)$. By the results of Kirchhoff-Routh theory (given in the hint), since this is a single vortex problem in a simply connected domain, the point vortex trajectories are the contours of the Hamiltonian $H^{(z)}(z_a)$ which are given by

$$(1 - |a|^2)|f'(a)| = \text{constant},\tag{1}$$

where $z_a = f(a)$ is the point vortex position in the z domain. Here we have used the hint, together with the fact that

$$G_0(\zeta, a) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{(\zeta - a)}{|a|(\zeta - 1/\bar{a})}\right) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(\zeta - a) + g(\zeta, a)$$

so that

$$g(\zeta, a) = \frac{\mathrm{i}}{2\pi}\log(|a|(\zeta - 1/\bar{a}))$$

and consequently that, for a single vortex,

$$H^{(\zeta)}(a) = \frac{\Gamma^2}{4\pi}\log\left||a|(a - 1/\bar{a})\right| = \frac{\Gamma^2}{4\pi}\log(1 - |a|^2).$$

Hence,

$$H^{(z)}(z_a) = H^{(\zeta)}(a) + \frac{\Gamma^2}{4\pi}\log|f'(a)| = \frac{\Gamma^2}{4\pi}\log(1 - |a|^2) + \frac{\Gamma^2}{4\pi}\log|f'(a)|.$$

Contours of $H^{(z)}(z_a)$ therefore coincide with (1). This general result is familiar from lectures and coursework. In this case, it follows on squaring (1) that the trajectories are given by

$$(1 - |a|^2)^2|f'(a)|^2 = c,$$

where the constant c is set by initial conditions. Hence

$$(1 - |a|^2)^2\frac{4(a - 1)}{(a + 1)^3}\frac{4(\bar{a} - 1)}{(\bar{a} + 1)^3} = c,$$

where we have used $f'(\zeta)$ as calculated in part (b). The initial location corresponds to $a = 0$ therefore $c = 16$. The trajectories are then given by $z_a(t) = f(a(t))$ where $a(t)$ is the solution of

$$(1 - a(t)\overline{a(t)})^2(a(t) - 1)(\overline{a(t)} - 1) = (a(t) + 1)^3(\overline{a(t)} + 1)^3.$$

4. (a) Consider a (complex) ζ-plane through the equator of the sphere. Pick a point P sim. seen ⇓
on the spherical surface. Draw a straight line between P and the north pole N
of the sphere of unit radius. Extrapolate this line if necessary so that it hits the
plane through the equator. This construction gives a one-to-one mapping (the
stereographic projection) of points P on the spherical surface and points ζ in the
plane. The north pole N corresponds to the point at infinity in the plane. The
south pole S corresponds to $\zeta = 0$. If (θ, ϕ) are the usual angles in spherical polar
coordinates, simple geometrical considerations lead to the relation

$$\zeta = \cot(\theta/2)\, e^{i\phi}.$$

4, B

(b) Let $a = \cot(\theta_a/2)$ so that one of the vortices is at longitudinal angle $\phi = 0$. The sim. seen ⇓
streamfunction for 4 circulation-Γ vortices equally spaced around the latitude circle
at $\theta = \theta_a$, in a corotating frame with angular velocity Ω (from the hint) is

$$\psi = -\frac{\Gamma}{4\pi}\log\left(\frac{(\zeta^4 - a^4)\overline{(\zeta^4 - a^4)}}{(1 + \zeta\bar\zeta)^4 (1 + a^2)^4}\right) + \frac{2\Omega}{1 + \zeta\bar\zeta}.$$

Here we have subtracted off the solid body rotation in order to be in a frame
corotating with the vortices but Ω is not yet known. We have also used the fact
that the streamfunction associated with a point vortex of circulation Γ at projected
point ζ_0 is

$$-\frac{\Gamma}{4\pi}\log\left(\frac{(\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}}{(1 + \zeta\bar\zeta)(1 + \zeta_0\bar\zeta_0)}\right). \tag{2}$$

By the rotational symmetry of the configuration, to find Ω, it is enough to consider
the vortex at $\zeta = a$. Note that

$$\psi = -\frac{\Gamma}{4\pi}\log\left(\frac{(\zeta - a)\overline{(\zeta - a)}}{(1 + \zeta\bar\zeta)(1 + a^2)}\right) - \frac{\Gamma}{4\pi}\log\left(\frac{|\zeta^3 + \zeta^2 a + \zeta a^2 + a^3|^2}{(1 + \zeta\bar\zeta)^3 (1 + a^2)^3}\right) + \frac{2\Omega}{1 + \zeta\bar\zeta}.$$

Therefore, the (complex) non-self-induced velocity at $\zeta = a$ is proportional to

$$-\frac{\Gamma}{4\pi}\left[\frac{3\zeta^2 + 2a\zeta + a^2}{\zeta^3 + \zeta^2 a + \zeta a^2 + a^3} - \frac{3\bar\zeta}{1 + \zeta\bar\zeta}\right] - \frac{2\Omega\bar\zeta}{(1 + \zeta\bar\zeta)^2}$$

where we have omitted a contribution from the first (self-induced) term. By the
equations of motion for point vortices on a sphere this must vanish when evaluated
at $\zeta = a$ implying

$$\frac{2\Omega a}{(1 + a^2)^2} = -\frac{\Gamma}{4\pi}\left[\frac{6a^2}{4a^3} - \frac{3a}{1 + a^2}\right].$$

Or rearrangement, this gives

$$\Omega = \frac{3\Gamma}{16\pi a^2}(a^4 - 1).$$

This can also be written as

$$\Omega = \frac{3\Gamma}{4\pi}\frac{\cos(\theta_a)}{\sin^2(\theta_a)}$$

on use of the fact that $a = \cot(\theta_a/2)$. 8, C

(c) This configuration corresponds to that in part (b), but with $\theta_a = \pi/2$, and with two additional point vortices at the north pole and south pole. By the symmetry of this 6-vortex configuration, it is enough to consider the vortex at $\zeta = 1$. From part (b) the 4 vortices on the equator do not influence each other since $\Omega = 0$ when $\theta_a = \pi/2$ according to the result of part (b). Therefore it only remains to determine the effect on the vortex at $\zeta = 1$ of the vortices at the north and south poles. The streamfunction, $\hat{\psi}$ say, associated with those two vortices is

$$\hat{\psi} = -\frac{\Gamma}{4\pi} \left[\log\left(\frac{\zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}\right) + \log\left(\frac{1}{1 + \zeta\bar{\zeta}}\right) \right],$$

where we have used (2) and considered $\zeta_0 = 0$ (the south pole) and $\zeta_0 \to \infty$ (the north pole). The streamfunction $\hat{\psi}$ induces a velocity at $\zeta = 1$ proportional to

$$\left.\frac{\partial \hat{\psi}}{\partial \zeta}\right|_{\zeta=1} = -\frac{\Gamma}{4\pi} \left[\frac{1}{\zeta} - \frac{\bar{\zeta}}{1 + \zeta\bar{\zeta}} - \frac{\bar{\zeta}}{1 + \zeta\bar{\zeta}}\right]\Bigg|_{\zeta=1} = 0.$$

The vortex at $\zeta = 1$ is therefore in equilibrium, as are all the other 5 vortices (by the rotational symmetry of the configuration).

5. (a) From lecture notes the complex potential for a single row of point vortices, circulation Γ, with period a and with one vortex at $z = 0$ is

$$-\frac{i\Gamma}{2\pi} \log \sin\left(\frac{\pi z}{a}\right)$$

This can be quoted, or it can be derived directly using the hint, if needed, as was done in the lectures by adding up an infinite number of point vortex contributions and using the fact that a sum of logs is the log of a product. Therefore the required complex potential for the vortex street comprising two such point vortex rows is

$$w(z) = -\frac{i\Gamma}{2\pi} \log \sin\left(\frac{\pi(z - ib)}{a}\right) + \frac{i\Gamma}{2\pi} \log \sin\left(\frac{\pi(z + ib)}{a}\right).$$

(b) The complex velocity, in a frame of reference cotravelling with the vortices moving at speed U, is

$$u - iv = \frac{dw}{dz} - U = -\frac{i\Gamma}{2a} \cot\left(\frac{\pi(z - ib)}{a}\right) + \frac{i\Gamma}{2a} \cot\left(\frac{\pi(z + ib)}{a}\right) - U,$$

where we have subtracted off the (as yet unknown) velocity U in order to move with the street. The vortex at $z = ib$ will not move under the influence of other vortices in its same row, but it will move under the influence of the other non-self-induced terms, i.e. it will move at speed

$$+\frac{i\Gamma}{2a} \cot\left(\frac{\pi(ib + ib)}{a}\right) - U = \frac{i\Gamma}{2a} \cot\left(\frac{2\pi ib}{a}\right) - U.$$

This speed must be zero in the cotravelling frame implying

$$U = \frac{i\Gamma}{2a} \cot\left(\frac{2\pi ib}{a}\right) = \frac{i\Gamma}{2a} \times i\left(\frac{e^{-2\pi b/a} + e^{2\pi b/a}}{e^{-2\pi b/a} - e^{2\pi b/a}}\right) = \frac{\Gamma}{2a} \coth\left(\frac{2\pi b}{a}\right).$$

The same arguments at $z = -ib$ lead to the same velocity, hence the configuration moves to the right with this speed U.

(c) On the x axis where $z = x$,

$$w(z) = -\frac{i\Gamma}{2\pi} \log \sin\left(\frac{\pi(x - ib)}{a}\right) + \frac{i\Gamma}{2\pi} \log \sin\left(\frac{\pi(x + ib)}{a}\right)$$

but the second term on the right hand side is the complex conjugate of the first term, so $w(z)$ is purely real on the x axis. Since the streamfunction is the imaginary part of $w(z)$ then this is zero on the x axis implying it is a streamline.
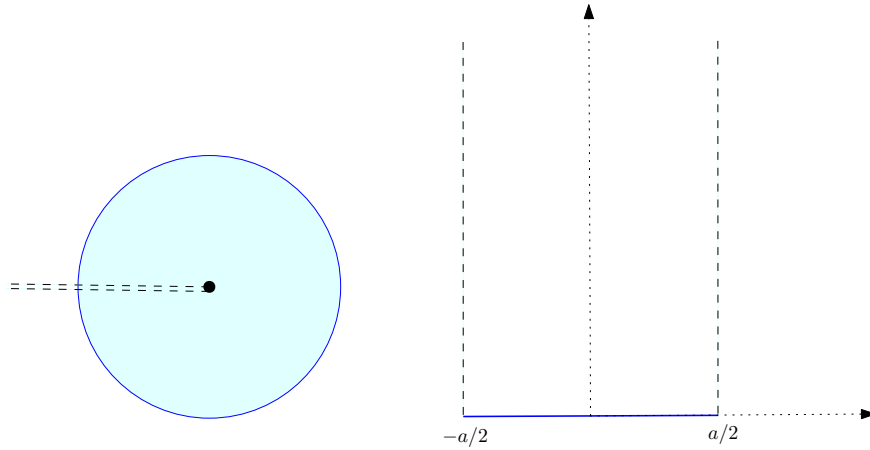
(d) Since if $\zeta = re^{i\theta}$ then

$$z = x + iy = -\frac{ia}{2\pi}(\log r + i\theta) = \frac{a\theta}{2\pi} - \frac{ia}{2\pi} \log r$$

then the image of the unit circle $-\pi < \theta < \pi$ is the real interval $x \in [-a/2, a/2]$ while the two sides of the branch cut $\theta = \pm\pi, 0 < r < 1$ map to the two vertical lines $x = \pm a/2, y > 0$ The image of the cut unit disc is therefore the upper semi-strip shown in the figure:

(e) The complex potential for a point vortex of circulation Γ at a point $\zeta = \alpha$ inside the unit ζ disc is well-known from lectures to be

$$\mathcal{G}_0(\zeta, \alpha) = -\frac{i\Gamma}{2\pi} \log\left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right).$$

(f) Since the real axis is a streamline then the flow in the upper semi-strip $x \in [-a/2, a/2], y > 0$ contains a single vortex of circulation Γ with the real axis being a streamline and periodicity of the velocity field required across the vertical edges of the period window. Since dw/dz is periodic with period a, i.e.

$$\frac{dw}{dz}(z + a) = \frac{dw}{dz}(z)$$

then, on integration,

$$w(z + a) = w(z) + \text{constant}.$$

But the function

$$\mathcal{G}_0(\zeta, \alpha) = -\frac{i\Gamma}{2\pi} \log\left(\frac{\zeta - \alpha}{|\alpha|(\zeta - 1/\overline{\alpha})}\right)$$

with α chosen so that

$$ib = -\frac{ia}{2\pi} \log \alpha, \qquad \text{or} \qquad \alpha = e^{-2\pi b/a}$$

has all the required properties: it has a logarithmic singularity of the correct strength at $\zeta = \alpha$ corresponding to $z = ib$, it jumps by a constant (in fact, by Γ) as any full circle of $\zeta = 0$ is made (corresponding to $z \mapsto z + a$) and it has constant imaginary part on $|\zeta| = 1$ corresponding to the real axis $x \in [-a/2, a/2]$. This complex potential must therefore coincide with part (a), to within a constant. Now with

$$\zeta = e^{2\pi i z/a}, \qquad \alpha = e^{-2\pi b/a}$$

then

$$\mathcal{G}_0(\zeta, \alpha) = -\frac{i\Gamma}{2\pi} \log\left(\frac{e^{2\pi i z/a} - e^{-2\pi b/a}}{e^{2\pi i z/a} - e^{2\pi b/a}}\right) + \text{constant}.$$

Note that

$$\frac{e^{2\pi i z/a} - e^{-2\pi b/a}}{e^{2\pi i z/a} - e^{2\pi b/a}} = \frac{e^{\pi i z/a} - e^{-2\pi b/a}e^{-\pi i z/a}}{e^{\pi i z/a} - e^{2\pi b/a}e^{-\pi i z/a}}$$

$$= \frac{e^{-\pi b/a}}{e^{\pi b/a}} \times \frac{e^{\pi b/a}e^{\pi i z/a} - e^{-\pi b/a}e^{-\pi i z/a}}{e^{-\pi b/a}e^{\pi i z/a} - e^{\pi b/a}e^{-\pi i z/a}}$$

$$= e^{-2\pi b/a}\frac{\sin(\pi(z - ib)/a)}{\sin(\pi(z + ib)/a)}.$$

Therefore,

$$w(z) = \mathcal{G}_0(\zeta, \alpha) = -\frac{\mathrm{i}\Gamma}{2\pi} \log \sin \left(\frac{\pi(z - \mathrm{i}b)}{a} \right) + \frac{\mathrm{i}\Gamma}{2\pi} \log \sin \left(\frac{\pi(z + \mathrm{i}b)}{a} \right) + \text{constant}$$

which is the same complex potential as derived in part (a), to within a constant. $\boxed{6, \text{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70051   Vortex Dynamics

Question	Marker's comment
1	This was meant as a warm-up question of moderate difficulty and was done quite well. Part (b) was found the most challenging part.
2	This question involving adapting ideas from the course to a new setting: it was quite a challenging question although students performed quite well on the first few parts of the question.
3	A fairly standard question on Kirchhoff-Routh theory. All parts were done quite well. If students failed to get the correct conformal mapping in part (a), errors in subsequent parts were "carried forward" and marks awarded for the correct approach.
4	A fairly standard question akin to many from previous exams. Students performed quite well, although the algebra for part (b) caused problems for some (this was graded generously).nbsp;
5	This Mastery question was answered surprisingly well, with a number of students getting close to full marks.