Imperial College London

MATH97023 MATH97005

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Tensor Calculus and General Relativity

Date: Thursday, 27 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Formula Sheet

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b.$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad R = g^{bd} R_{bd}.$$

1. In this question K is an inertial frame with coordinates (t, x, y, z).

(a) What is meant by proper time? (5 marks)

(b) In frame K a particle accelerates along the x-axis with trajectory

$$x = \frac{c^2}{A}\left(\sqrt{1 + \frac{A^2 t^2}{c^2}} - 1\right), \qquad y = z = 0,$$

where A is a non-zero constant.

(i) At what time $t > 0$ does the particle reach half the speed of light? (5 marks)

(ii) Determine the proper time between when the particle is at rest and when the particle reaches half the speed of light. (5 marks)

(c) A spacecraft is at rest ($x = y = z = 0$) for $t < 0$. At $t = 0$ the spacecraft accelerates in the x-direction following the trajectory given in part (b) until it reaches half the speed of light. The spacecraft then shuts down its propulsion system and moves uniformly at half the speed of light. Sketch the trajectory of the spacecraft on a space-time diagram. (5 marks)

(Total: 20 marks)

2. (a) What is meant by parallel transport? (5 marks)

(b) A paraboloid $z = \frac{1}{2}(x^2 + y^2)$ is embedded in three dimensional space with the standard metric $ds^2 = dx^2 + dy^2 + dz^2$.

(i) Show that the metric on the surface is

$$ds^2 = (1 + \rho^2) d\rho^2 + \rho^2 d\theta^2,$$

where ρ and θ are cylindrical coordinates defined through

$$x = \rho \cos\theta \quad \text{and} \quad y = \rho \sin\theta.$$

(5 marks)

(ii) Determine $\Gamma^{\rho}_{\rho\rho}$. (5 marks)

(iii) The remaining non-zero Christoffel symbols are

$$\Gamma^{\rho}_{\theta\theta} = -\frac{\rho}{1 + \rho^2}, \quad \Gamma^{\theta}_{\rho\theta} = \Gamma^{\theta}_{\theta\rho} = \frac{1}{\rho}.$$

Obtain a set of coupled ODES describing the parallel transport of a vector along the circles $\rho =$constant. Find the general form of $v^{\theta}(\theta)$. (5 marks)

(Total: 20 marks)

3. A torus with inner radius a and external radius $r > a$ is embedded in three dimensional space through the parametrisation

$$x = (r + a\sin\theta)\cos\phi, \quad y = (r + a\sin\theta)\sin\phi, \quad z = a\cos\theta.$$

Here θ and ϕ are periodic angles; shifting θ or ϕ by 2π gives the same point. The metric is

$$ds^2 = a^2 d\theta^2 + (r + a\sin\theta)^2 d\phi^2,$$

and the non-zero Christoffel symbols are

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\theta\phi} = \frac{a\cos\theta}{r + a\sin\theta}, \quad \Gamma^{\theta}_{\phi\phi} = -\frac{\cos\theta(r + a\sin\theta)}{a}.$$

(a) Show that

$$R^{\theta}_{\phi\theta\phi} = \frac{(r + a\sin\theta)\sin\theta}{a}.$$

(8 marks)

(b) Use the result quoted in part (a) to deduce $R^{\phi}_{\theta\phi\theta}$. (4 marks)

(c) Determine $R_{\theta\theta}$ and $R_{\phi\phi}$. (4 marks)

(d) Compute the scalar curvature R. (4 marks)

(Total: 20 marks)

4. The Schwarzschild metric is

$$ds^2 = c^2\left(1-\frac{R}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{R}{r}\right)} - r^2 d\phi^2,$$

where R is the Schwarzschild radius and $\theta = \pi/2$.

(a) Show that $h = r^2 d\phi/ds$ is constant along geodesics and

$$c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1. \tag{1}$$

(5 marks)

(b) Show that $k = (1-R/r)dt/ds$ is constant along geodesics and use this result to rewrite (1) in the form

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{R}{r}\right) - \frac{R}{r} = \text{constant}.$$

(5 marks)

(c) A mass is released from rest ($dr/ds = 0$ and $h = 0$) at $r = R_1 > R$ and falls directly towards the black hole. Use the result of part (b) to determine the proper time, $\tau = s/c$, taken to reach the event horizon at $r = R$. Give an approximation to τ valid for $R_1 \gg R$.

Hint: use the integral

$$\int \sqrt{\frac{x}{1-x}}dx = \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + c.$$

(10 marks)

(Total: 20 marks)

5. (a) Photon trajectories in the Schwarzschild spacetime satisfy the ODE

$$\frac{d^2u}{d\phi^2} + u = \frac{3Ru^2}{2}, \tag{1}$$

where $u = 1/r$. Equation (1) has the trivial solution, $u = \frac{2}{3}R^{-1}$, representing the photon sphere $r = \frac{3}{2}R$. Write u in the form

$$u(\phi) = \frac{2}{3R} + \tilde{u}(\phi),$$

and solve equation (1) assuming $\tilde{u}$ is small (neglect terms of order $\tilde{u}^2$). Comment on the result. (10 marks)

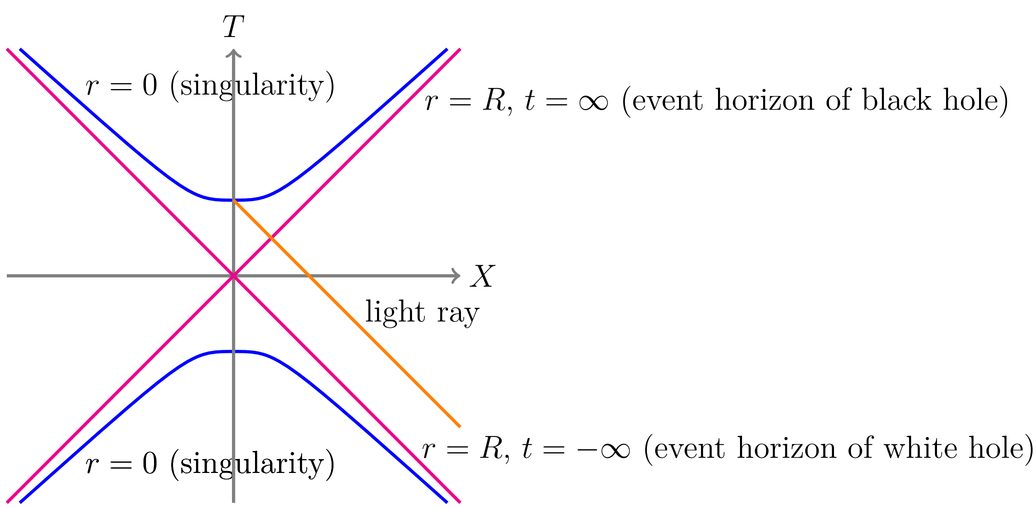
(b) Kruskal coordinates X, T are defined though

$$X = \left(\frac{r}{R} - 1\right)^{\frac{1}{2}} e^{\frac{1}{2}r/R}\cosh(\tfrac{1}{2}ct/R), \quad T = \left(\frac{r}{R} - 1\right)^{\frac{1}{2}} e^{\frac{1}{2}r/R}\sinh(\tfrac{1}{2}ct/R),$$

for $r > R$ (the exterior region), and

$$X = \left(1 - \frac{r}{R}\right)^{\frac{1}{2}} e^{\frac{1}{2}r/R}\sinh(\tfrac{1}{2}ct/R), \quad T = \left(1 - \frac{r}{R}\right)^{\frac{1}{2}} e^{\frac{1}{2}r/R}\cosh(\tfrac{1}{2}ct/R),$$

for $r < R$ (inside the black hole). A light ray passes through the event horizon of the black hole and reaches the singularity at $(T, X) = (1, 0)$ as depicted in the diagram below.



Use the exterior Kruskal coordinates to determine t as a function of r for the light ray before it reaches the event horizon. What is r as a function of t for t large and positive?

(10 marks)

(Total: 20 marks)

Tensor Calculus and General Relativity

Answers to May 2021 Examination

1. (a) Proper time τ is the time measured by a moving clock

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}dt,$$

where v is the speed of the clock. [5 marks, A, bookwork]

(b) (i) Differentiating with respect to t

$$\dot{x} = \frac{At}{\sqrt{1 + \frac{A^2t^2}{c^2}}}.$$

Setting $\dot{x}^2 = \frac{1}{4}c^2$ gives $A^2t^2 = \frac{1}{4}(1 + A^2t^2/c^2)c^2$ or $\frac{3}{4}A^2t^2 = \frac{1}{4}c^2$. Therefore $v = \frac{1}{2}c$ when $t = c/(\sqrt{3}A)$.

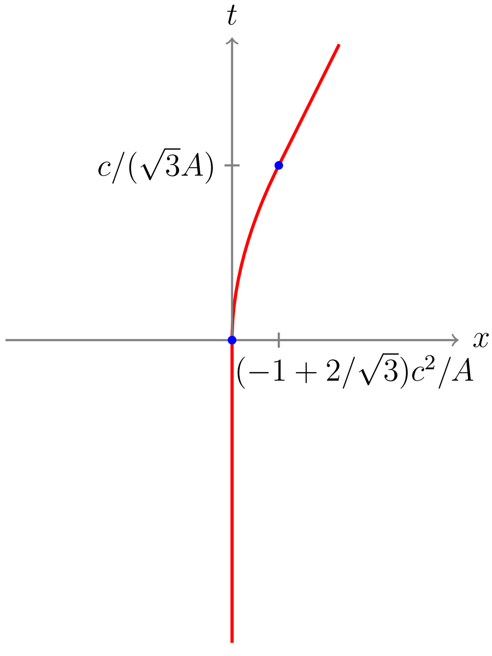
[5 marks, A, unseen]

(ii)

$$\Delta\tau = \int_0^{c/\sqrt{3}A} \sqrt{1 - \frac{\dot{x}^2}{c^2}} = \int_0^{c/\sqrt{3}A} \frac{dt}{\sqrt{1 + \frac{A^2t^2}{c^2}}} = \frac{c}{A}\int_0^{1/\sqrt{3}} \frac{dp}{\sqrt{1+p^2}}$$
$$= \frac{c}{A}\sinh^{-1}\frac{1}{\sqrt{3}}.$$

Also accept $\frac{1}{2}\log 3\; c/A$. [5 marks, C, unseen]

(c)



[5 marks, B, unseen]

[Total: 20 marks]

2. (a) Parallel transport allows one to perform vector and tensor algebra with vectors defined at different points. A vector v^a at one point can be transported along a curve to another point. Along the curve $x^a = x^a(\lambda)$ where λ is a parameter

$$\frac{dv^a}{d\lambda} + \Gamma^a_{bc} v^b \frac{dx^c}{d\lambda} = 0.$$

[5 marks, A, bookwork]

(b) (i) $dx^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2$. $z = \frac{1}{2}\rho^2$ so that $dz = \rho d\rho$. $ds^2 = dx^2 + dy^2 + dz^2 = (1+\rho^2)d\rho^2 + \rho^2 d\theta^2$.

[5 marks, B, seen similar]

(ii)

$$\Gamma^\rho_{bc} = \frac{1}{2} g^{\rho d} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right).$$

$$\Gamma^\rho_{\rho\rho} = \frac{1}{2} g^{\rho\rho} \left(1 + 1 - 1\right) \partial_\rho g_{\rho\rho} = \frac{1}{2} \cdot \frac{1}{1+\rho^2} \cdot \partial_\rho (1+\rho^2) = \frac{\rho}{1+\rho^2}.$$

[5 marks, B, seen similar]

(iii) Using the equation of parallel transport as in part (a) use $\lambda = \theta$ as a parameter on the circle ρ =constant.

$$\frac{dv^\rho}{d\theta} + \Gamma^\rho_{b\theta} v^b = 0, \qquad \frac{dv^\theta}{d\theta} + \Gamma^\theta_{b\theta} v^b = 0.$$

Using the given Christoffel symbols

$$\frac{dv^\rho}{d\theta} - \frac{\rho}{1+\rho^2} v^\theta = 0, \qquad \frac{dv^\theta}{d\theta} + \frac{1}{\rho} v^\rho = 0.$$

Differentiation the second equation with respect to θ yields

$$\frac{d^2 v^\theta}{d\theta^2} + \frac{1}{1+\rho^2} v^\theta = 0,$$

which has the general solution

$$v^\theta(\theta) = A \cos\left(\frac{\theta + \beta}{\sqrt{1+\rho^2}}\right),$$

where A and β are arbitrary constants.

[5 marks, C, seen similar]

[Total: 20 marks]

3. (a)

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}.$$

$$\begin{aligned} R^\theta_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\phi\theta} + \Gamma^\theta_{e\theta}\Gamma^e_{\phi\phi} - \Gamma^\theta_{e\phi}\Gamma^e_{\phi\theta} \\ &= \partial_\theta \Gamma^\theta_{\phi\phi} - 0 + 0 - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} \\ &= \partial_\theta \left[-\frac{(r + a\sin\theta)}{a}\cos\theta \right] + \cos^2\theta \\ &= \frac{(r + a\sin\theta)\sin\theta}{a}. \end{aligned}$$

[8 marks, A, seen similar]

(b)

$$R^\phi_{\theta\phi\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = g^{\phi\phi} g_{\theta\theta} R^\theta_{\phi\theta\phi} = \frac{1}{(r + a\sin\theta)^2} a^2 R^\theta_{\phi\theta\phi} = \frac{a\sin\theta}{r + a\sin\theta}.$$

[4 marks, B, seen similar]

(c) $R_{\phi\phi} = R^a_{\phi a\phi} = R^\theta_{\theta\phi\theta\phi} = (r + a\sin\theta)\sin\theta/a$

$R_{\theta\theta} = R^\phi_{\theta\phi\theta} = a\sin\theta/(r + a\sin\theta)$.

[4 marks, C, seen similar]

(d)

$$R = g^{ab} R_{ab} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2\sin\theta}{a(r + a\sin\theta)}.$$

[4 marks, D, seen similar]

[Total: 20 marks]

4. (a) The component of the metric do not depend on ϕ. Hence $g_{\phi\mu}dx^\mu/ds = -r^2 d\phi/ds = -h$ is constant along geodesics.

The equation $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1$ yields

$$c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2$$
$$= c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

[5 marks, A, bookwork]

(b) As the components of the metric do not depend on t $g_{t\mu}dx^\mu/ds = c^2(1 - R/r)^{-1}dt/ds = c^2k$ is constant along geodesics

$$c^2k^2\left(1-\frac{R}{r}\right)^{-1} - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

Multiplying by $(1 - R/r)$ gives

$$c^2k^2 - \left(\frac{dr}{ds}\right)^2 - h^2\left(\frac{1}{r^2} - \frac{R}{r^3}\right) = 1 - \frac{R}{r}.$$

Therefore

$$\left(\frac{dr}{ds}\right)^2 + h^2\left(\frac{1}{r^2} - \frac{R}{r^3}\right) - \frac{R}{r} = \text{constant}.$$

[5 marks, B, bookwork]

(c) Using the initial condition to fix the constant from part (b)

$$\left(\frac{dr}{ds}\right)^2 + -\frac{R}{r} = -\frac{R}{R_1},$$

so that

$$\frac{dr}{ds} = -\sqrt{\left(\frac{R}{r} - \frac{R}{R_1}\right)},$$

which is negative as the particle is infalling. Accordingly,

$$ds = -\frac{\sqrt{r}dr}{\sqrt{R}\sqrt{1-\frac{r}{R_1}}}$$

and

$$c\tau = -\frac{1}{\sqrt{R}}\int_{R_1}^{R}\frac{\sqrt{r}dr}{\sqrt{1-\frac{r}{R_1}}} = \frac{R_1^{3/2}}{R^{1/2}}\int_{R/R_1}^{1}\frac{\sqrt{x}dx}{\sqrt{1-x}},$$

using the substitution $x = r/R_1$. Using the given integral

$$c\tau = \frac{R_1^{3/2}}{R^{1/2}}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{R}{R_1}} + \sqrt{\frac{R}{R_1}\left(1-\frac{R}{R_1}\right)}\right].$$

If $R_1 \gg R$,

$$\tau \approx \frac{\pi R_1^{3/2}}{2cR^{1/2}}.$$

[10 marks, D, unseen]

[Total: 20 marks]

5. (a) Inserting $u = \frac{2}{3}R^{-1} + \tilde{u}$ into the differential equation yields

$$\frac{d^2\tilde{u}}{d\phi^2} + \frac{2}{3R} + \tilde{u} = \frac{3R}{2}\left(\frac{4}{9R^2} + \frac{4}{3R}\tilde{u} + \tilde{u}^2\right)$$

which simplifies to

$$\frac{d^2\tilde{u}}{d\phi^2} - \tilde{u} = \frac{3R}{2}\tilde{u}^2.$$

Neglecting the quadratic term yields

$$\frac{d^2\tilde{u}}{d\phi^2} - \tilde{u} = 0,$$

with solutions

$$\tilde{u} = Ae^{\phi} + Be^{-\phi},$$

where A and B are constants. This shows that a deviation from the circular orbit is unstable as $\tilde{u}$ grows with increasing (or decreasing) ϕ.

[10 marks, unseen]

(b) As light rays are 45 degrees to the axes in Kruskal coordinates the equation of the light ray is $T = -X + 1$. Outside the black hole

$$T + X = \left(\frac{r}{R} - 1\right)^{1/2} e^{\frac{1}{2}(r+ct)/R} = 1,$$

so that

$$\frac{1}{2}\log\left(\frac{r}{R} - 1\right) + \frac{r + ct}{2R} = 0,$$

or

$$t = -\frac{r}{c} - \frac{R}{c}\log\left(\frac{r}{R} - 1\right).$$

For large t, $r \approx R$ giving

$$t \approx -\frac{R}{c} - \frac{R}{c}\log\left(\frac{r}{R} - 1\right).$$

or

$$\frac{r}{R} = 1 + e^{1-ct/R}.$$

[10 marks, unseen]

[Total: 20 marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH96010 MATH97023 MATH97005	1	This question differed most from those in questions from previous exam. The question comprised bookwork and three straightforward but unseen exercises in Special Relativity. The marks were more spread out than for questions 2,3 and 4.
MATH96010 MATH97023 MATH97005	2	This question on parallel transport was well answered by most candidates.
MATH96010 MATH97023 MATH97005	3	This question is a continuation of test 2 and was very well answered. Note that on setting r=0 the standard results for a sphere of radius a are recovered (even though the question specified r>a).
MATH96010 MATH97023 MATH97005	4	This standard question on the Schwarzschild space-time was well answered.
MATH96010 MATH97023 MATH97005	5	This question with two unseen part proved difficult. In part (b) the diagram was intended as a hint to note that the equation of the light ray is T=-X+1.