Imperial College London

MATH97012_MATH97091

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Hydrodynamic Stability

Date: Tuesday, 11 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider the modified Eckhaus equation,

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}=\frac{1}{R}\frac{\partial^2 u}{\partial y^2}-\frac{\partial^4 u}{\partial x^4}+a,$$

which may be taken as a model for studying fluid motion and its stability, where R and a are real positive parameters. The boundary conditions for the sought function $u(t,x,y)$ are

$$u\Big|_{y=0}=0,\qquad \frac{\partial u}{\partial y}\Big|_{y=1}=b,$$

with $b>0$ being a constant.

(a) Consider the basic state and perturbation.

(i) Find the steady basic state $u=U(y)$.
The basic state is perturbed by a small-amplitude disturbance $u'(t,x,y)$, i.e. $u=U(y)+u'(t,x,y)$. Write down the linearised equation and boundary conditions satisfied by u'.

(4 marks)

(ii) Assume that $u'(t,x,y)$ is periodic in x, that is, $u'(t,x+L,y)=u'(t,x,y)$ for arbitrary x, y and t. Let

$$E=\frac{1}{2}\int_0^1\int_0^L (u')^2 dxdy.$$

Show that

$$\frac{dE}{dt}=\int_0^1\int_0^L \frac{dU}{dy}\Big(\frac{\partial u'}{\partial x}\Big)^2 dxdy-\int_0^1\int_0^L\Big\{\frac{1}{R}\Big(\frac{\partial u'}{\partial y}\Big)^2+\Big(\frac{\partial^2 u'}{\partial x^2}\Big)^2\Big\}dxdy,$$

and comment on the stabilizing and destabilizing roles of the terms in the perturbation equation.

(5 marks)

(b) For the case $a=0$, seek temporal normal-mode solutions of the form

$$u'=\bar{u}(y)e^{\sigma t+i\alpha x}+c.c.,$$

where $c.c.$ stands for the complex conjugate, and show that the growth rate

$$\sigma=b\,\alpha^2-\alpha^4-(n+\tfrac{1}{2})^2\pi^2/R,$$

where $n=0,\ 1,\ 2,\ldots$. Find the corresponding eigenfunctions.
For the case of $n=0$, show that the basic state becomes unstable for

$$R>R_c=\pi^2/b^2.$$

(8 marks)

(c) For the case $a=0$ again, show that the basic state is *spatially unstable* to normal mode perturbations of the form

$$u'=\bar{u}(y)e^{i(\alpha x-\omega t)}+c.c.$$

with $\omega\neq 0$.

(3 marks)

(Total: 20 marks)

2. (a) Consider convection in a horizontal layer of fluid, where the temperature field $\hat{\theta}$ satisfies the equation,

$$\Big[\frac{\partial}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \nabla)\Big]\hat{\theta} = \kappa \nabla^2 \hat{\theta} - \hat{q},$$

in the Cartesian coordinates $\hat{\mathbf{x}} \equiv (\hat{x}, \hat{y}, \hat{z})$, where $\kappa > 0$ and $\hat{q} > 0$ are constants representing thermal diffusion and radiation heat loss, respectively. The lower boundary at $\hat{z} = 0$ is a rigid plate subject to heating with a fixed heat flux $\hat{h}_s$, while the upper boundary is a free surface in contact with the air with a mean position $\hat{z} = d$, where the temperature is maintained at a constant value $\hat{\theta}_1$ so that the boundary conditions are

$$\kappa \frac{\partial \hat{\theta}}{\partial \hat{z}} = -\hat{h}_s \quad \text{at} \quad \hat{z} = 0; \qquad \hat{\theta} = \hat{\theta}_1 \quad \text{at} \quad \hat{z} = d.$$

The basic steady state is that of pure conduction with velocity field $\hat{\mathbf{u}} = \hat{\mathbf{U}} = \mathbf{0}$. Find the vertical distribution of the temperature $\hat{\Theta}(z)$ and the temperature $\hat{\theta}_0 = \hat{\Theta}(0)$ at the lower boundary. (3 marks)

(b) When the Boussinesq approximation is made, the dimensional momentum and continuity equations can be written as

$$\Big[\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}}\Big] = -\nabla(\frac{\hat{p}}{\hat{\rho}_0}) + \nu \nabla^2 \hat{\mathbf{u}} + \alpha g \hat{\theta}\, \mathbf{k}, \qquad \nabla \cdot \hat{\mathbf{u}} = 0,$$

where ν is the kinematic viscosity, $\hat{\rho}_0$ the density, g the gravitational acceleration, and α the fluid density expansion coefficient, while $\mathbf{k}$ denotes the unit vector in the vertical direction.

Deduce that the characteristic time scale and velocity of the problem are d^2/κ and κ/d, respectively. Show that in terms of the non-dimensional independent and dependent variables,

$$\mathbf{x} = \hat{\mathbf{x}}/d, \quad t = \hat{t}/(d^2/\kappa); \quad \theta = \hat{\theta}/(\hat{\theta}_0 - \hat{\theta}_1), \quad \mathbf{u} = \hat{\mathbf{u}}/(\kappa/d), \quad p = \hat{p}/(\hat{\rho}_0 \kappa^2/d^2),$$

the momentum, continuity and temperature equations assume the dimensionless form,

$$\Big[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\Big] = -\nabla p + Pr \nabla^2 \mathbf{u} + Ra Pr\, \theta\, \mathbf{k}, \qquad \nabla \cdot \mathbf{u} = 0,$$

$$\Big[\frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)\Big]\theta = \nabla^2 \theta - q,$$

and define the parameters Ra, Pr and q in terms of the physical parameters in the problem. (6 marks)

(c) When the basic state is perturbed by a small-amplitude disturbance $(u', v', w', p', \theta')$, it is known that w' and θ' satisfy the equation

$$\frac{\partial}{\partial t}\nabla^2 w' = Pr \nabla^4 w' + Ra Pr \big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)\theta'.$$

Let $(w', \theta') = \big(\tilde{w}(z), \tilde{\theta}(z)\big) f(x, y) e^{\sigma t}$.

(i) Derive the equations governing $\tilde{w}(z)$ and $\tilde{\theta}(z)$, and specify the boundary conditions on $\tilde{w}$ and $\tilde{\theta}$. (6 marks)

(ii) Show that σ is real when $q = 0$ and $Ra > 0$. What does a real-valued σ imply physically? (5 marks)

(Total: 20 marks)

3. Inviscid instability of a parallel shear flow with velocity profile $U(y)$ is known to be governed by the Rayleigh equation

$$(U-c)(\frac{d^2}{dy^2}-\alpha^2)\bar{v}-\frac{d^2U}{dy^2}\bar{v}=0,$$

where $\bar{v}(y)$ is the eigenfunction of the perturbation velocity in the y-direction.

Consider a parallel shear flow above an infinitely large rigid flat plate located at $y=-H$ with the velocity profile $U(y)$ being given by

$$U(y)=\begin{cases} 0 & \text{for } -H\le y<-h, \\ \frac{1}{2}(1+y/h) & \text{for } -h<y<h, \\ 1 & \text{for } y>h, \end{cases}$$

a model that is sometimes taken as a representation of a separated boundary layer.

(i) Solve the Rayleigh equation for the parallel shear flow described above to find the solution for $\bar{v}$ in the different ranges of y: $y>h$, $-h<y<h$ and $-H<y<-h$.

(3 marks)

(ii) Impose appropriate boundary and far-field conditions as well as the jump conditions at $y=\pm h$, where the gradient dU/dy of the basic flow profile is discontinuous; write down the equations that the unknown constants in the solution must satisfy. You may use the fact that across a discontinuity at y_d, the following relations hold,

$$\Big[(U-c)\frac{d\bar{v}}{dy}-\frac{dU}{dy}\bar{v}\Big]_{y_d^-}^{y_d^+}=0, \quad \Big[\frac{\bar{v}}{(U-c)}\Big]_{y_d^-}^{y_d^+}=0,$$

where $[\ \cdot\ ]_{y_d^-}^{y_d^+}$ stand for the jump of the quantity across y_d.

(6 marks)

(iii) Derive the dispersion relation satisfied by the phase speed c and wavenumber α. You are not required to solve c in terms of α.

(4 marks)

(iv) Show that when $H/h\gg 1$ and $\alpha h=O(1)$, the dispersion relation simplifies to

$$(2\alpha h-1)^2-(2\alpha h)^2(2c-1)^2=e^{-4\alpha h},$$

and hence find c in terms of α. Give a brief interpretation of the result.

(5 marks)

Find the expression for c in the long-wavelength limit $\alpha h\ll 1$, and give the condition on α for this expression to be valid.

(2 marks)

(Total: 20 marks)

4. When a boundary layer over a concave wall is perturbed by an unsteady three-dimensional small-amplitude disturbance, the suitably normalised velocity (u', v', w') and pressure p' of the disturbance are governed by the equations,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} + \frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial y} = \Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)u', \tag{1a}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + v'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial y} + 2GUu' = -\frac{\partial p'}{\partial y} + \Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)v', \tag{1b}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + V\frac{\partial w'}{\partial y} = -\frac{\partial p'}{\partial z} + \Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)w', \tag{1c}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{1d}$$

where

- $U(x, y)$ denotes the streamwise velocity and $V(x, y)$ the (rescaled) wall-normal velocity of the basic flow, with y being the wall-normal coordinate normalised by the boundary-layer thickness so that $y = O(1)$ in the boundary layer;
- G is the Görtler number, which controls the stability.

(i) Perform a scaling analysis by considering the balances of dominant terms in equations (1a)-(1d) in the limit of $G \gg 1$, and show that instability may occur over a short length scale $\Delta = O(G^{-1/2})$ and time scale of $O(G^{-1/2})$, that is,

$$\frac{\partial \phi'}{\partial x} = O(\phi'/\Delta), \quad \frac{\partial \phi'}{\partial t} = O(\phi'/\Delta),$$

where ϕ' denotes any of u', v', w' and p'. Deduce that v', w' and p' obey the following scaling relations to u':

$$v' = O(G^{1/2}u'), \quad w' = O(G^{1/2}u'), \quad p' = O(G\,u').$$

(7 marks)

(ii) Let $X = x/G^{-1/2}$ and $T = t/G^{-1/2}$, and seek solutions of the local normal-mode form,

$$(u', v', w', p') = \Big(\bar{u}(x, y), G^{1/2}\bar{v}(x, y), G^{1/2}\bar{w}(x, y), G\bar{p}(x, y)\Big)e^{\sigma X + i\beta z - i\omega T} + c.c.$$

to (1a)-(1d), where $c.c.$ denotes the complex conjugate. Derive the equations satisfied by $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$. (4 marks)

(iii) Show that the set of equations governing $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$ can be reduced to a single equation for $\bar{v}$. Specify the appropriate boundary and far-field conditions. (5 marks)

(iv) Show that for steady perturbations with $\omega = 0$, σ^2 must be real. Define linear stability and instability in terms of σ.

(4 marks)

(Total: 20 marks)

5. A wall jet is a two-dimensional flow of boundary layer type. When it is perturbed by a three-dimensional disturbance, the perturbed flow field is written as

$$(u, v, w, p) = \Big(U(x, Y),\, Re^{-1/2}V(x, Y),\, 0,\, P\Big) + (u',\, v',\, w',\, p')$$

in the Cartesian coordinate system (x, y, z), where the coordinates x, y and z are non-dimensionalised by L, the distance to the leading edge, $Y = Re^{1/2}y$ with the Reynolds number $Re = \hat{V}_\infty L/\nu$. The basic flow velocity has the far-field and near-wall behaviours that

$$U \to 0 \quad \text{as} \quad Y \to \infty, \qquad U \to \lambda Y \quad \text{as} \quad Y \to 0,$$

with λ being a function of x. The flow field $(\mathbf{u}, p) = (u, v, w, p)$ is, in the Cartesian coordinate system (x, y, z), governed by the three-dimensional Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}.$$

(i) Derive the linearised equations governing the perturbation u', v', w' and p', which are functions of x, Y, z and t, e.g. $u' = u'(x, Y, z, t)$. Indicate the terms which represent the non-parallel-flow effect, and explain Prandtl's parallel-flow approximation that leads to the Orr-Sommerfeld equation.

(4 marks)

(ii) Suppose that in the main layer (deck) where $Y = O(1)$, the solution expands as

$$(u',\, v',\, w',\, p') = \Big(\bar{u}(x, Y),\, Re^{-\frac{1}{14}}\bar{v}(x, Y),\, Re^{-\frac{1}{7}}\bar{w}(x, Y),\, Re^{-\frac{1}{7}}\bar{p}(x, Y)\Big)E + c.c.,$$

where $c.c.$ stands for the complex conjugate, and

$$E = e^{i\{Re^{\frac{3}{7}}(\alpha x + \beta z) - Re^{\frac{2}{7}}\omega t\}}\ .$$

Derive the equations governing $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$, and verify that they have the solution

$$\bar{u} = AU', \quad \bar{v} = -i\alpha AU, \quad \bar{w} = -(\beta/\alpha)\bar{p}/U, \quad \bar{p} = -\alpha^2 A \int_\infty^Y U^2 dY.$$

Explain why it is necessary to introduce a viscous sublayer (i.e lower deck) but an upper layer is not required. (5 marks)

(iii) Deduce that the lower deck has a width of $O(Re^{-\frac{9}{14}}L)$, and hence introduce $\tilde{y} = Re^{\frac{9}{14}}y = Re^{1/7}Y$ in the lower deck. Deduce that the solution expands as

$$(u',\, v',\, w',\, p') = \Big(\tilde{u}(x, \tilde{y}),\, Re^{-\frac{3}{14}}\tilde{v}(x, \tilde{y}),\, \tilde{w}(x, \tilde{y}),\, Re^{-\frac{1}{7}}\tilde{p}(x, \tilde{y})\Big)E + c.c.$$

Write down the equations governing $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$, and determine the pressure $\tilde{p}$. Comment on the orders of magnitude of the viscous effect and the non-parallelism. (6 marks)

Question continues on the next page.

Show that $\tilde{v}_{\tilde{y}\tilde{y}} \equiv \dfrac{d^2\tilde{v}}{d\tilde{y}^2}$ satisfies the equation

$$i(\alpha\lambda\tilde{y} - \omega)\tilde{v}_{\tilde{y}\tilde{y}} = (\tilde{v}_{\tilde{y}\tilde{y}})_{\tilde{y}\tilde{y}},$$

and specify the boundary conditions at $\tilde{y} = 0$ and the matching condition as $\tilde{y} \to \infty$.

Solve the above equation subject to the boundary and matching conditions to determine $\tilde{v}_{\tilde{y}} \equiv \dfrac{d\tilde{v}}{d\tilde{y}}$ by introducing

$$\zeta = (i\alpha\lambda)^{1/3}\tilde{y} + \zeta_0 \ , \quad \zeta_0 = -i\omega(i\alpha\lambda)^{-2/3},$$

and derive the dispersion relation

$$\int_{\zeta_0}^{\infty} A\mathrm{i}(\zeta)d\zeta + i\alpha^{-1}\lambda I^{-1}(\alpha^2 + \beta^2)^{-1}(i\alpha\lambda)^{2/3}\, A\mathrm{i}'(\zeta_0) = 0,$$

where $A\mathrm{i}$ is the Airy function and

$$I = \int_0^{\infty} U^2 dY.$$

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH97012, MATH97091

Hydrodynamic Stability (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. (a) (i) The basic state satisfies the equation $\frac{1}{R}\frac{d^2U}{dy^2} = a$, and so

sim. seen ⇓

$$U = c_1 y + \tfrac{1}{2}aRy^2.$$

The boundary condition at $y = 1$ requires that $c_1 + aR = b$ and so $c_1 = b - aR$. Hence

$$U(y) = (b - aR)y + \tfrac{1}{2}aRy^2.$$

The linearised equation for the disturbance:

$$\frac{\partial u'}{\partial t} + \frac{dU}{dy}\frac{\partial^2 u'}{\partial x^2} = \frac{1}{R}\frac{\partial^2 u'}{\partial y^2} - \frac{\partial^4 u'}{\partial x^4}. \qquad (1)$$

The boundary conditions are

$$u' = 0 \quad \text{at} \quad y = 0, \qquad \frac{\partial u'}{\partial y} = 0 \quad \text{at} \quad y = 1.$$

4, A

(ii) Multiplying u' to both sides of the perturbation equation, and integrating with respect to y from 0 to 1 and with respect to x from 0 and L, we have

$$\frac{dE}{dt} + I_1 = \frac{1}{R}I_2 - I_3, \qquad (2)$$

where

$$I_1 = \int_0^1\int_0^L \frac{dU}{dy}\Big(u'\frac{\partial^2 u'}{\partial x^2}\Big)dxdy = \int_0^1 \frac{dU}{dy}\Big(u'\frac{\partial u'}{\partial x}\Big)\Big|_{x=0}^{L} dy - \int_0^1\int_0^L \frac{dU}{dy}\Big(\frac{\partial u'}{\partial x}\Big)^2 dxdy$$

$$= -\int_0^1\int_0^L \frac{dU}{dy}\Big(\frac{\partial^2 u'}{\partial x^2}\Big)^2 dxdy,$$

$$I_2 = \int_0^1\int_0^L u'\frac{\partial^2 u'}{\partial y^2}dxdy = \int_0^L u'\frac{\partial u'}{\partial y}\Big|_{y=0}^{1} dx - \int_0^1\int_0^L \Big(\frac{\partial u'}{\partial y}\Big)^2 dxdy = -\int_0^1\int_0^L \Big(\frac{\partial u'}{\partial y}\Big)^2 dxdy,$$

$$I_3 = \int_0^1\int_0^L u'\frac{\partial^4 u'}{\partial x^4}dxdy = \int_0^1 u'\frac{\partial^3 u'}{\partial x^3}\Big|_{x=0}^{L} dy - \int_0^1\int_0^L \frac{\partial u'}{\partial x}\frac{\partial^3 u'}{\partial x^3}dxdy$$

$$= -\int_0^1 \frac{\partial u'}{\partial x}\frac{\partial^2 u'}{\partial x^2}\Big|_{x=0}^{L} dy + \int_0^1\int_0^L \Big(\frac{\partial^2 u'}{\partial x^2}\Big)^2 dxdy = \int_0^1\int_0^L \Big(\frac{\partial^2 u'}{\partial x^2}\Big)^2 dxdy;$$

in the above either the periodic condition in x or boundary conditions have been used. The required equation follows from rearranging equation (2) into

$$\frac{dE}{dt} = -I_1 + \frac{1}{R}I_2 - I_3.$$

The expressions above indicate that $I_2 < 0$ and $-I_3 < 0$ while $(-I_1 > 0)$. Hence the two terms on the right-hand side of the perturbation equation, $\frac{1}{R}\frac{\partial^2 u'}{\partial y^2}$ and $\frac{\partial^4 u'}{\partial x^4}$, play a stabilising role (akin to viscous diffusion (dissipation)). The second term on the left-hand side, $\frac{\partial U}{\partial y}\frac{\partial^2 u'}{\partial x^2}$, which somewhat resembles 'shear', causes amplification of disturbances.

5, B

(b) When $a = 0$, $dU/dy = b$. For the assumed normal-mode form,

sim. seen $\Downarrow$

$$\frac{\partial}{\partial t} \to \sigma, \quad \frac{\partial}{\partial x} \to i\alpha.$$

Substitution of the normal mode perturbation into the linearised perturbation equation yields

$$\frac{d^2\bar{u}}{dy^2} + R\big[b\,\alpha^2 - \alpha^4 - \sigma\big]\bar{u} = 0.$$

The solution is

$$\bar{u} = Ae^{\lambda y} + Be^{-\lambda y}.$$

with

$$\lambda^2 + R(b\,\alpha^2 - \alpha^4 - \sigma) = 0. \tag{3}$$

The boundary conditions,

$$\bar{u}(0) = A + B = 0, \quad \frac{d\bar{u}}{dy}\Big|_{y=1} = \lambda Ae^{\lambda} - \lambda Be^{-\lambda} = 0,$$

can be satisfied only when $e^{2\lambda} = -1 (= e^{(2n+1)\pi i})$, i.e. when $\lambda = (n + 1/2)\pi i$ with $n = 0, 1, 2, \ldots$. It follows from (3) that the growth rates are given by

$$\sigma = b\,\alpha^2 - \alpha^4 - (n + \tfrac{1}{2})^2\pi^2/R, \tag{4}$$

and the corresponding eigenfunctions are

$$\bar{u} = \sin\big((n + \tfrac{1}{2})\pi\big)y.$$

5, A

Since σ is real, the neutral curve corresponds to $\sigma = 0$:

$$R = \frac{(n + \frac{1}{2})^2\pi^2}{b\alpha^2 - \alpha^4}.$$

For each n, the minimum critical value of R occurs when $2b\,\alpha - 4\alpha^3 = 0$, i.e. at $\alpha = \alpha_c = \sqrt{b/2}$, and for $n = 0$,

$$R_c = \pi^2/b^2.$$

3, A

unseen $\Downarrow$

(c) For spatial instability, the dispersion relation corresponds to (4) with σ being replaced by $-i\omega$, that is,

$$\alpha^4 - b\alpha^2 + \Big[(n + \tfrac{1}{2})^2\pi^2/R - i\omega\Big] = 0,$$

from which α is found as

$$\alpha = \pm\frac{1}{\sqrt{2}}\Big[b \pm \sqrt{b^2 - 4(2n+1)^2\pi^2/R + 4i\omega}\Big]^{1/2}.$$

Among 4 roots, all of which are complex for $\omega \neq 0$, two of them must have $-\alpha_i > 0$. Thus the basic state ('flow') is always *spatially unstable*, and the eigenfunctions are

$$\bar{u} = \sin\big((n + \tfrac{1}{2})\pi\big)y$$

as in the temporal case.

3, D

2. (a) Since $\hat{\Theta} = \hat{\Theta}(z)$, the temperature equation simplifies to $\kappa \dfrac{d^2\hat{\Theta}}{d\hat{z}^2} = -\hat{q}$, which is integrated to give

unseen ⇓

$$\hat{\Theta} = \hat{\theta}_0 + \hat{C}_1\hat{z} - \tfrac{1}{2}(\hat{q}/\kappa)\hat{z}^2,$$

where $\hat{C}_1$ is a constant. To determine $\hat{C}_1$ and $\hat{\theta}_0$, we impose the required boundary conditions, which give

$$\hat{\theta}_0 + \hat{C}_1 d - \tfrac{1}{2}(\hat{q}/\kappa)d^2 = \hat{\theta}_1, \quad \kappa\hat{C}_1 = -\hat{h}_s.$$

It follows that $\hat{C}_1 = -\hat{h}_s/\kappa$, and

$$\hat{\theta}_0 = \hat{\theta}_1 + (\hat{h}_s/\kappa)d + \tfrac{1}{2}(\hat{q}/\kappa)d^2.$$

and so the temperature distribution of the basic state is

$$\hat{\Theta} = \left[\hat{\theta}_1 + (\hat{h}_s/\kappa)d + \tfrac{1}{2}(\hat{q}/\kappa)d^2\right] - (\hat{h}_s/\kappa)\hat{z} - \tfrac{1}{2}(\hat{q}/\kappa)\hat{z}^2.$$

3, A

(b) The reference length is d. The dominant balance is between the time variation $\frac{\partial\hat{\theta}}{\partial t}$ and the thermal diffusion $\kappa\nabla^2\hat{\theta}$:

meth seen ⇓

$$\hat{\theta}/\hat{t} \sim \kappa\hat{\theta}/d^2,$$

which gives the time scale $\hat{t} \sim d^2/\kappa$. With the length scale d, the characteristic velocity scale is κ/d.

After substitution of the normalized variables, the equations read

$$(\kappa^2/d^3)\Big[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big] = -(\kappa^2/d^3)\nabla p + \nu(\kappa/d^3)\nabla^2\mathbf{u} + \alpha g(\hat{\theta}_0 - \hat{\theta}_1)\,\theta\,\mathbf{k},$$

$$(\kappa/d^2)(\hat{\theta}_0 - \hat{\theta}_1)\Big[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\Big]\,\theta = (\kappa/d^2)(\hat{\theta}_0 - \hat{\theta}_1)\nabla^2\theta + \hat{q}.$$

After dividing the two questions by factors (κ^2/d^3) and $\kappa/d^2(\hat{\theta}_0 - \hat{\theta}_1)$, respectively, we obtain the required form of dimensionless momentum and temperature equations with

$$Pr = \nu/\kappa, \quad Ra = \alpha g(\hat{\theta}_0 - \hat{\theta}_1)d^3/(\kappa\nu), \quad q = (d^2/\kappa)\hat{q}/(\hat{\theta}_0 - \hat{\theta}_1).$$

The continuity equation is $\nabla\cdot\mathbf{u} = 0$.

6, A

sim. seen ⇓

(c) (i) The non-dimensional basic temperature can be written as

$$\Theta = \hat{\theta}/(\hat{\theta}_0 - \hat{\theta}_1) = \Big[\frac{\hat{\theta}_1}{(\hat{h}_s/\kappa)d + \frac{1}{2}(\hat{q}/\kappa)d^2} + 1\Big] - \frac{(\hat{h}_s d/\kappa)}{\hat{h}_s/d + \frac{1}{2}\hat{q}d^2}z - \tfrac{1}{2}qz^2$$

$$\equiv \theta_0 - C_1 z - \tfrac{1}{2}qz^2.$$

Linearisation of the nondimensional temperature equation about $\Theta(z)$ yields

$$\frac{\partial\theta'}{\partial t} + (\mathbf{U}\cdot\nabla)\theta' + (\mathbf{u}'\cdot\nabla)\Theta = \nabla^2\theta'.$$

As the temperature profile Θ is quadratic and independent of x, the temperature perturbation equation becomes

$$\frac{\partial \theta'}{\partial t} + w'\frac{d\Theta}{dz} = \nabla^2\theta', \tag{5}$$

where

$$\frac{d\Theta}{dz} = -C_1 - qz.$$

Substitution of the assumed form of solutions into equation (5) gives

$$(D^2 + \nabla_1^2 - \sigma)f\tilde{\theta} = -f(C_1 + qz)\tilde{w}, \tag{6}$$

where the differential operators D and ∇_1^2 are defined as

$$D = \frac{d}{dz}, \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Equation (6) can be rearranged into the 'variable separation' form,

$$\Big[(D^2 - \sigma)\tilde{\theta} + (C_1 + qz)\tilde{w}\Big]/\tilde{\theta} = -(\nabla_1^2 f)/f.$$

Since the left-hand side is a function of z only while the right-hand side depends on only x and y, both sides should be a constant, a^2 say. Then

$$(D^2 - a^2 - \sigma)\tilde{\theta} = -(C_1 + qz)\tilde{w}, \tag{7}$$

and

$$\nabla_1^2 f + a^2 f = 0. \tag{8}$$

Similarly, substituting the assumed form into a slightly re-arranged form of the equation for w' (in the question), we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma/Pr)\tilde{w} = a^2 Ra\,\tilde{\theta} \tag{9}$$

after use is made of (8). The required equations are (7) and (9).
The boundary conditions at $z = 0$ (rigid surface):

$$\tilde{w} = D\tilde{w} = 0, \quad D\tilde{\theta} = 0;$$

while the boundary conditions at $z = 1$ (free surface),

$$\tilde{w} = D^2\tilde{w} = 0, \quad \tilde{\theta} = 0.$$

The system of coupled equations of (7) and (9) together with these boundary conditions defines the eigenvalue problem. $\boxed{6,\ \mathrm{B}}$

(ii) Multiplying $\tilde{w}^*$ to (9) and integrating by parts, we obtain

$$(\sigma/Pr)J_1 + J_2 + J_B = a^2 Ra \int_0^1 \tilde{w}^*\tilde{\theta}dz, \tag{10}$$

where J_1 and J_2 are positive,

$$J_1 = \int_0^1 (|D\tilde{w}|^2 + a^2|\tilde{w}|^2)dz, \quad J_2 = \int_0^1 (|D^2\tilde{w}|^2 + 2a^2|D\tilde{w}|^2 + a^4|\tilde{w}|^2))dz,$$

and the boundary conditions imply that the contributions from the end points vanish, i.e.

$$J_B = \Big[\tilde{w}^* D^3 \tilde{w} - D\tilde{w}^* D^2 \tilde{w} - (2a^2 + \sigma/Pr)\tilde{w}^* D\tilde{w}\Big]_0^1 = 0.$$

unseen $\Downarrow$

Multiplying $\tilde{\theta}^*$ to (7) and integrating by parts,

$$-\int_0^1 \Big[|D\tilde{\theta}|^2 + (a^2 + \sigma)|\tilde{\theta}|^2\Big] dz + \Big[\tilde{\theta}^* D\tilde{\theta}\Big]_0^1 = -\int_0^1 (C_1 + qz)\tilde{w}\tilde{\theta}^* dz. \quad (11)$$

The contributions from $z = 0$ and $z = 1$ are both zero due to the boundary conditions, and hence

$$\sigma I_0 + I_1 = C_1 \int_0^1 \tilde{w}\tilde{\theta}^* dz + q \int_0^1 z\tilde{w}\tilde{\theta}^* dz, \quad (12)$$

where I_0 and I_1 are both positive.
If $q = 0$, the right-hand sides of (10) and (12) can be eliminated to give

$$-\sigma^* a^2 Ra I_0 + \sigma(C_1/Pr)J_1 + C_1 J_2 - a^2 Ra I_1 = 0.$$

the imaginary part of which is

$$\sigma_i \Big[(C_1/Pr)J_1 + a^2 Ra I_0\Big] = 0.$$

Since $C_1 > 0$, the factor multiplying σ_i is strictly positive, and hence $\sigma_i = 0$. A real-valued σ means that disturbances would either amplify (if $\sigma > 0$), or decay (if $\sigma < 0$), monotonically without exhibiting oscillation. 5, D

3. (i) Since $d^2U/dy^2 = 0$, the Rayleigh equation reduces to $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{d^2\bar{v}}{dy^2} - \alpha^2\bar{v} = 0, \tag{13}$$

which is solved to give the solution in different regions:

$$\bar{v} = \begin{cases} C^+e^{-\alpha y} & \text{if} \quad y > h, \\ C_1e^{-\alpha y} + C_2e^{\alpha y} & \text{if} \quad -h < y < h, \\ C_1^-e^{-\alpha y} + C_2^-e^{\alpha y} & \text{if} \quad -H < y < -h. \end{cases} \tag{14}$$

$\boxed{3, \text{A}}$

(ii) The constants in the solution are to be determined by imposing the boundary condition as well as applying the jump conditions, $\boxed{\text{meth seen} \Downarrow}$

$$\Big[(U-c)\bar{v}' - U'\bar{v}\Big]_{y_d^-}^{y_d^+} = 0, \quad \Big[\frac{\bar{v}}{(U-c)}\Big]_{y_d^-}^{y_d^+} = 0.$$

At $y = h$, where $U = 1$, we have

$$\begin{aligned} -\alpha(1-c)e^{-\alpha h}C^+ &= \Big[-\alpha(1-c) - 1/(2h)\Big]e^{-\alpha h}C_1 + \Big[\alpha(1-c) - 1/(2h)\Big]e^{\alpha h}C_2, \\ C^+e^{-\alpha h} &= C_1e^{-\alpha h} + C_2e^{\alpha h}, \end{aligned}$$

elimination of C^+ from which gives

$$\Big[-1/(2h)\Big]e^{-\alpha h}C_1 + \Big[2\alpha(1-c) - 1/(2h)\Big]e^{\alpha h}C_2 = 0. \tag{15}$$

At $y = -h$, where $U = 0$, we have

$$\begin{aligned} -\alpha(-c)e^{\alpha h}C_1^- + \alpha(-c)e^{-\alpha h}C_2^- &= \Big[-\alpha(-c) - 1/(2h)\Big]e^{\alpha h}C_1 \\ &\quad + \Big[\alpha(-c) - 1/(2h)\Big]e^{-\alpha h}C_2, \qquad (16) \\ C_1^-e^{\alpha h} + C_2^-e^{-\alpha h} &= C_1e^{\alpha h} + C_2e^{-\alpha h}. \qquad (17) \end{aligned}$$

The impermeability condition at $y = -H$ requires that

$$C_1^-e^{\alpha H} + C_2^-e^{-\alpha H} = 0,$$

which gives

$$C_1^- = -C_2^-e^{-2\alpha H}.$$

This is inserted to (16)-(17) to give

$$\begin{aligned} \Big[\alpha c e^{\alpha h}(-e^{-2\alpha H}) - \alpha c e^{-\alpha h}\Big]C_2^- &= \Big[-\alpha(-c) - 1/(2h)\Big]e^{\alpha h}C_1 \\ &\quad + \Big[\alpha(-c) - 1/(2h)\Big]e^{-\alpha h}C_2, \qquad (18) \end{aligned}$$

$$[-e^{-2\alpha H + \alpha h} + e^{-\alpha h}]C_2^- = C_1e^{\alpha h} + C_2e^{-\alpha h}. \tag{19}$$

$\boxed{6, \text{B}}$

(iii) Eliminating C_2^- from equations (18) and (19), we have

$$-(\alpha c)\coth\big(\alpha(H-h)\big)\Big[C_1 e^{\alpha h}+C_2 e^{-\alpha h}\Big] = \Big[-\alpha(-c)-1/(2h)\Big]e^{\alpha h}C_1 + \Big[\alpha(-c)-1/(2h)\Big]e^{-\alpha h}C_2,$$

unseen ⇓

which is simplified to

$$\Big\{\alpha c\big[\coth\big(\alpha(H-h)\big)+1\big]-\frac{1}{2h}\Big\}e^{\alpha h}C_1+\Big\{\alpha c\big[\coth\big(\alpha(H-h)\big)-1\big]-\frac{1}{2h}\Big\}e^{-\alpha h}C_2=0. \tag{20}$$

The requirement for non-zero solutions means that the determinant of the coefficient matrix of (15) and (20) must vanish, that is,

$$-\frac{1}{2h}\Big\{\alpha c\big[\coth\big(\alpha(H-h)\big)-1\big]-\frac{1}{2h}\Big\}e^{-2\alpha h} - \Big[2\alpha(1-c)-\frac{1}{2h}\Big]\Big\{\alpha c\big[\coth\big(\alpha(H-h)\big)+1\big]-\frac{1}{2h}\Big\}e^{2\alpha h}=0, \tag{21}$$

which gives the dispersion relation,

$$\Big\{2\alpha hc\big[\coth\big(\alpha(H-h)\big)-1\big]-1\Big\}e^{-2\alpha h} + \Big[4\alpha h(1-c)-1\Big]\Big\{2\alpha hc\big[\coth\big(\alpha(H-h)\big)+1\big]-1\Big\}e^{2\alpha h}=0.$$

4, D

unseen ⇓

(iv) For $H\gg h$ with $O(\alpha h)=O(1)$, $\coth\big(\alpha(H-h)\big)\approx 1$ and so the dispersion relation reduces to

$$-e^{-2\alpha h}+\Big[4\alpha h(1-c)-1\Big]\Big\{4\alpha hc-1\Big\}e^{2\alpha h}=0,$$

which can be rewritten as

$$\Big[(2\alpha h-1)-(4\alpha hc-2\alpha h)\Big]\Big[(2\alpha h-1)+(4\alpha hc-2\alpha h)\Big]=e^{-4\alpha h},$$

or

$$(2\alpha h-1)^2-(2\alpha h)^2(2c-1)^2=e^{-4\alpha h}.$$

Solving for c, we find that

$$c=\frac{1}{2}\pm\frac{1}{4}(\alpha h)^{-1}\Big[(2\alpha h-1)^2-e^{-4\alpha h}\Big]^{1/2}.$$

The result becomes independent of H, implying that the rigid surface has negligible effect on disturbances with wavelengths of order $h\ll H$.

5, C

Taking the limit $\alpha h\ll 1$ further, we have

$$c\to\frac{1}{2}\pm\frac{1}{4}(\alpha h)^{-1}\Big[1-4\alpha h+4(\alpha h)^2-1+4\alpha h-8(\alpha h)^2+\dots\Big]^{1/2}\approx\frac{1}{2}\pm i\frac{1}{2}.$$

This approximation requires $\alpha h\ll 1$ but also $\alpha H\gg 1$ so that $\coth\big(\alpha(H-h)\big)\approx 1$ still holds, and hence the result is valid for

$$\frac{1}{H}\ll\alpha\ll\frac{1}{h}.$$

2, C

4. (i) Since it is the variation of U in the normal direction that causes the instability, while the second term in (1a) in the exam question describes the spatial growth of the disturbance (Görtler vortices), the dominant balance in (1a) is between the second and third terms on the left-hand side, and so sim. seen ⇓

$$u'/\Delta \sim v'. \tag{22}$$

Furthermore, a distinguished scaling is that the first term balances the second and third terms as well, and so the time scale is of $O(\Delta)$.

The centrifugal force, that makes fluid particles migrate across the boundary layer, is represented by the last term on the left-hand side of equation (1b) in the exam question, while the second term in describes the growth in space. This suggests that the second term and the centrifugal force must balance, yielding

$$v'/\Delta \sim Gu'. \tag{23}$$

The pressure must also be comparable with the centrifugal force, and thus

$$Gu' \sim p'. \tag{24}$$

It follows from (22) and (23) that

$$\Delta \sim G^{-1/2}, \quad v' \sim G^{1/2}u'. \tag{25}$$

In (1c) of the exam question, the pressure gradient $\partial p/\partial z$ must balance the inertia (i.e. the first and second terms on the left-hand side), because otherwise the equation is an advection-diffusion equation for a passive scalar, and w' must be zero, or would relax to zero downstream. The required balance gives

$$w'/\Delta \sim p',$$

use of which in (24) gives

$$w' \sim G^{1/2}u', \quad p' \sim G\,u', \tag{26}$$

a result that could also have been deduced by balancing the third term with the first two terms in (1d) of the exam question. 7, A

(ii) For the solution of the local normal-mode form, the differential operators acting on the perturbation obey, to leading order accuracy, the following relations unseen ⇓

$$\partial/\partial x \to G^{1/2}\sigma, \quad \partial/\partial t \to G^{1/2}(-i\omega), \quad \partial/\partial z \to i\beta. \tag{27}$$

Substituting the local normal mode into equations (1a)-(1d) and using (27), we obtain the set of equations,

$$(\sigma U - i\omega)\bar{u} + \frac{dU}{dy}\bar{v} = 0, \tag{28a}$$

$$(\sigma U - i\omega)\bar{v} + 2U\bar{u} = -\frac{d\bar{p}}{dy}, \tag{28b}$$

$$(\sigma U - i\omega)\bar{w} = -i\beta\bar{p}, \tag{28c}$$

$$\sigma\bar{u} + \frac{d\bar{v}}{dy} + i\beta\bar{w} = 0. \tag{28d}$$

As the equations indicate, both viscous and non-parallel-flow effects are negligible. 4, B

(iii) In order to reduce the set of equations (28) to a single equation for $\bar{v}$, we start by eliminating function $\bar{w}$. It follows from (28d) that unseen ⇓

$$\bar{w} = \frac{i\sigma}{\beta}\hat{u} + \frac{i}{\beta}\frac{d\bar{v}}{dy},$$

which is substituted into (28c) to give

$$\bar{p} = -\frac{\sigma}{\beta^2}(\sigma U - i\omega)\bar{u} - \frac{1}{\beta^2}(\sigma U - i\omega)\frac{d\bar{v}}{dy}. \tag{29}$$

From equation (28a), we have

$$\bar{u} = -\frac{1}{(\sigma U - i\omega)}\frac{dU}{dy}\bar{v}, \tag{30}$$

use of which in (29) yields

$$\bar{p} = \frac{\sigma}{\beta^2}\frac{dU}{dy}\bar{v} - \frac{1}{\beta^2}(\sigma U - i\omega)\frac{d\bar{v}}{dy}. \tag{31}$$

Finally, substituting (30) and (31) into (28b), and rearranging, we obtain the equation for $\bar{v}$,

$$(U - i\omega/\sigma)\Big[\frac{d^2}{dy^2} - \beta^2\Big]\bar{v} - \frac{d^2U}{dy^2}\bar{v} = -2\frac{\beta^2}{\sigma^2}\frac{U}{U - i\omega/\sigma}\frac{dU}{dy}\bar{v}. \tag{32a}$$

The boundary and far filed conditions are

$$\bar{v} = 0 \quad \text{at} \quad y = 0, \qquad \bar{v} \to 0 \quad \text{as} \quad y \to \infty.$$

5, C

unseen ⇓

(iv) When $\omega = 0$, the equation for $\bar{v}$ reduces to

$$\Big[\frac{d^2}{dy^2} - \beta^2\Big]\bar{v} - \frac{\frac{d^2U}{dy^2}}{U}\bar{v} = -2\frac{\beta^2}{\sigma^2}\frac{\frac{dU}{dy}}{U}\bar{v}.$$

Multiplying both sides of the equation by $\bar{v}^*$, the complex conjugate of $\bar{v}$, and integrating with respect to y, we obtain

$$-\int_0^\infty\Big[|\frac{d\bar{v}}{dy}|^2 + \beta^2|\bar{v}|^2 + \frac{\frac{d^2U}{dy^2}}{U}|\bar{v}|^2\Big]dy = -2\frac{\beta^2}{\sigma^2}\int_0^\infty\frac{\frac{dU}{dy}}{U}|\bar{v}|^2 dy.$$

Since the integrals are real, σ^2 must be real. When $\sigma^2 > 0$, σ would have two real values, one positive and one negative, and so the flow is unstable. When $\sigma^2 < 0$, both σ values are purely imaginary and the flow is inviscidly stable. 4, D

5. (i) Substituting the expression for the perturbed flow into the Navier-Stokes equations, and neglecting all nonlinear terms, we obtain the linearized equations for the perturbation, sim. seen ⇓

$$\frac{\partial u'}{\partial x} + Re^{1/2}\frac{\partial v'}{\partial Y} + \frac{\partial w'}{\partial z} = 0, \tag{33}$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \underline{\frac{\partial U}{\partial x}u' + V\frac{\partial u'}{\partial Y}} + Re^{\frac{1}{2}}\frac{\partial U}{\partial Y}v' = -\frac{\partial p'}{\partial x} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]u', \tag{34}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + \underline{Re^{-\frac{1}{2}}\frac{\partial V}{\partial x}u' + V\frac{\partial v'}{\partial Y} + \frac{\partial V}{\partial Y}v'} = -Re^{\frac{1}{2}}\frac{\partial p'}{\partial Y} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]v', \tag{35}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + \underline{V\frac{\partial w'}{\partial Y}} = -\frac{\partial p'}{\partial z} + \Big[\frac{\partial^2}{\partial Y^2} + \frac{1}{Re}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\big)\Big]w'. \tag{36}$$

The underlined terms represent the non-parallel-flow effects, caused by the streamwise variation of the streamwise velocity U and the presence of a normal velocity $Re^{-1/2}V$, which is associated with the streamwise variation of U.

Parallel-flow approximation neglects the underlined terms in (34)-(35), and treats the variation of U with x as being parametric; the latter means that at each location, the profile is 'frozen' so that normal-mode solutions may be sought. The viscous terms are retained and so the analysis leads to the usual Orr-Sommerfeld equation.

[4 marks]

unseen ⇓

(ii) Substituting the normal-mode form into (33)-(36), and noting that the operators

$$\frac{\partial}{\partial t} \to -iRe^{2/7}\omega, \quad \frac{\partial}{\partial x} \to iRe^{3/7}\alpha, \quad \frac{\partial}{\partial z} \to iRe^{3/7}\beta, \tag{37}$$

when acting on the perturbation, we obtain the equations

$$i\alpha\bar{u} + \frac{\partial \bar{v}}{\partial Y} = 0, \quad i\alpha U\bar{u} + \frac{\partial U}{\partial Y}\bar{v} = 0, \quad i\alpha U\bar{w} = -i\beta\bar{p}, \quad i\alpha U\bar{v} = -\frac{\partial \bar{p}}{\partial Y}, \tag{38}$$

with $\bar{v}$ having to satisfy the boundary condition $\bar{v} = 0$ at $Y = 0$. Elimination of $\bar{u}$ between the first two equations gives

$$U\frac{\partial \bar{v}}{\partial Y} - \frac{\partial U}{\partial Y}\bar{v} = 0,$$

which is a first-order ordinary differential equation for $\bar{v}$ and has the solution

$$\bar{v} = -i\alpha AU,$$

where A is a constant yet to be determined, and the pre-factor $-i\alpha$ is introduced for convenience. It follows that

$$\bar{u} = A\frac{\partial U}{\partial Y}, \quad \bar{w} = -(\beta/\alpha)\bar{p}/U,$$

and

$$\bar{p} = -\alpha^2 A\int_{\infty}^{Y} U^2 dY, \tag{39}$$

where the integration constant is taken such that $\bar{p}$ vanishes as $Y \to \infty$.

[3 marks]

Note that

$$\bar{u} \to \lambda A, \quad \bar{v} \to -i\alpha A\lambda Y, \quad \bar{w} = -(\beta/\alpha)\bar{p}\,(\lambda Y)^{-1} \quad \text{as} \quad Y \to 0. \tag{40}$$

Because $\bar{u}$ does not satisfy the required no-slip condition (and $\bar{w}$ is singular as the wall is approached), a viscous lower deck is required. On the other hand, since $U \to 0$ as $Y \to \infty$ for a wall jet, $\bar{v} \to 0$ as $Y \to \infty$ and so an upper layer is not required.

[2 marks]

sim. seen ⇓

(iii) Let $Y = O(d)$ with $d \ll 1$ in the lower deck, where $U = \lambda Y = O(d)$. It follows that the inertia term $U\dfrac{\partial u'}{\partial x} \sim O(dRe^{3/7}u')$, while the viscous diffusion $\dfrac{\partial^2 u'}{\partial Y^2} \sim O(u'/d^2)$. The balance between the two,

$$dRe^{3/7}u' \sim u'/d^2,$$

suggests that $d = O(Re^{-1/7})$, which corresponds to $y = Re^{-1/2}Y = O(Re^{-9/14})$.

The asymptotic behaviour of the main-deck solution, (40), suggests that in the lower deck $u' = O(1)$ as in the main layer, but $v' = O(Re^{-1/14}d) = O(Re^{-3/14})$ as can be deduced by the matching principle. Similarly, $p' = O(Re^{-1/7})$. The result for $\bar{w}$ in (40) suggests that $w' = O(Re^{-1/7}d^{-1}) = O(1)$. Therefore, in the lower deck, the solution should expand as

$$(u',\, v',\, w',\, p') = (\tilde{u},\, Re^{-\frac{3}{14}}\tilde{v},\, \tilde{w},\, Re^{-\frac{1}{7}}\tilde{p})E + c.c.$$

[2 marks] unseen ⇓

Substituting this into (33)-(35) and using the fact that $U = Re^{-1/7}(\lambda\tilde{y})$ as well as the relations in (37), we obtain

$$i\alpha\tilde{u} + \frac{d\tilde{v}}{d\tilde{y}} + i\beta\tilde{w} = 0, \tag{41}$$

$$-i\omega\tilde{u} + i\alpha\lambda\tilde{y}\tilde{u} + \lambda\tilde{v} = -i\alpha\tilde{p} + \frac{d^2\tilde{u}}{d\tilde{y}^2}, \tag{42}$$

$$-i\omega\tilde{w} + i\alpha\lambda\tilde{y}\tilde{w} = -i\beta\tilde{p} + \frac{d^2\tilde{w}}{d\tilde{y}^2}, \tag{43}$$

plus $d\tilde{p}/d\tilde{y} = 0$ so that $\tilde{p}$ is a constant.
Note that the main-deck solution (39) has the asymptote

$$\bar{p} \to -\alpha^2 A \int_{\infty}^{0} U^2 dY \quad \text{as} \quad Y \to 0.$$

Matching $\tilde{p}$ with $\bar{p}$ gives

$$\tilde{p} = \alpha^2 I A \quad \text{where} \quad I = \int_0^{\infty} U^2 dY; \tag{44}$$

this is the pressure-displacement relation.

Note that the leading order terms, including the viscous terms, in the x-momentum equation are of $O(Re^{2/7})$ while the terms representing non-parallelism, e.g. $\frac{\partial U}{\partial x}u'$, is $O(Re^{-1/7})$, and hence non-parallelism contributes an $O(Re^{-3/7})$ correction.

[4 marks] sim. seen $\Downarrow$

Multiplying $i\alpha$ and $i\beta$ to (42) and (43) respectively, and adding the resulting equation and using (41), we obtain

$$-i\omega(-\tilde{v}_{\tilde{y}}) + i\alpha\lambda\tilde{y}(-\tilde{v}_{\tilde{y}}) + i\alpha\lambda\tilde{v} = (\alpha^2+\beta^2)\tilde{p} + (-\tilde{v}_{\tilde{y}\tilde{y}\tilde{y}}), \tag{45}$$

which is differentiated with respect to $\tilde{y}$ to obtain the required equation for $\tilde{v}$,

$$i(\alpha\lambda\tilde{y}-\omega)\frac{d^2\tilde{v}}{d\tilde{y}^2} = \frac{d^4\tilde{v}}{d\tilde{y}^4}. \tag{46}$$

Now consider the boundary conditions. The no-slip condition $\tilde{u}=0$ and $\tilde{w}=0$ at $\tilde{y}=0$ implies that

$$\frac{d\tilde{v}}{d\tilde{y}} = 0 \quad \text{at} \quad \tilde{y}=0, \tag{47}$$

after use has been made of (41). Putting $\tilde{y}=0$ in (45) and using the fact that $\tilde{v}=\tilde{v}_{\tilde{y}}=0$ at $\tilde{y}=0$, we find that

$$\frac{d^3\tilde{v}}{d\tilde{y}^3} = (\alpha^2+\beta^2)\tilde{p} \quad \text{at} \quad \tilde{y}=0. \tag{48}$$

Matching $\tilde{u}$, or equivalently $\frac{d\tilde{v}}{d\tilde{y}}$, with the main-deck solution (40) requires that

$$\frac{d\tilde{v}}{d\tilde{y}} \to -i\alpha\lambda A \quad \text{as} \quad \tilde{y}\to\infty. \tag{49}$$

In terms of ζ, equation (46) can be written as

$$\frac{d^4\tilde{v}}{d\zeta^4} - \zeta\frac{d^2\tilde{v}}{d\zeta^2} = 0,$$

which is the Airy equation for $\frac{d^2\tilde{v}}{d\zeta^2}$, and so the solution

$$\frac{d^2\tilde{v}}{d\zeta^2} = C\,\mathrm{Ai}(\zeta), \tag{50}$$

where C is a constant. Integrating and using the boundary condition (47) taking into account the fact that $\tilde{y}=0$ corresponds to $\zeta=\zeta_0$, we obtain

$$\frac{d\tilde{v}}{d\zeta} = C\int_{\zeta_0}^{\zeta}\mathrm{Ai}(\zeta)d\zeta$$

and so

$$\frac{d\tilde{v}}{d\tilde{y}} = (i\alpha\lambda)^{1/3}\frac{d\tilde{v}}{d\zeta} = (i\alpha\lambda)^{1/3}C\int_{\zeta_0}^{\zeta}\mathrm{Ai}(\zeta)d\zeta.$$

Inserting the above expression into the matching condition (49) gives

$$(i\alpha\lambda)^{1/3}C\int_{\zeta_0}^{\infty}\mathrm{Ai}(\zeta)d\zeta = -i\alpha\lambda A. \tag{51}$$

Use of (50) in (48) shows that

$$(i\alpha\lambda)C\,\mathrm{Ai}'(\zeta_0) = (\alpha^2+\beta^2)\tilde{p}. \tag{52}$$

Finally, eliminating A and $\tilde{p}$ from (51), (52) and the (pressure-displacement) relation (44), we arrive at the dispersion relation,

$$\int_{\zeta_0}^{\infty} \mathrm{Ai}(\zeta) d\zeta + i\lambda\alpha^{-1} I^{-1} (\alpha^2 + \beta^2)^{-1} (i\alpha\lambda)^{2/3}\, \mathrm{Ai}'(\zeta_0) = 0,$$

for the instability.

[5 marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97012MATH97091	1	Almost all of students did well on this.
MATH97012MATH97091	2	Only a small number of students was able to do Part (c). This is disappointing as this part is rather like what was seen before.
MATH97012MATH97091	3	Most students were able to approach the questions and apply the correct the jump conditions. The algebra in (iii) is lengthy, and no one was able to handle it entirely correctly. Students would have done better on Part (iv) if they realised by the physical intuition that the problem reduces to that of the vortex sheet.
MATH97012MATH97091	4	The average score on this question was rather low. The attention to the scaling analysis is adequate.
MATH97012MATH97091	5	Some good attempts, but most only managed the basic parts. The core algebra is actually very similar to what was done in lectures and exercises. Probably the tight time was a factor.