Introduction to Quantum Mechanics – Problem sheet 1

1. Constants of motion

(a) Verify that in Hamiltonian dynamics the energy indeed stays constant in time.

(b) Consider a Hamiltonian system where the two phase-space variables C_1 and C_2 are constants of motion. Show that the variable defined by $C_3 = \{C_1, C_2\}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket, is also a constant of motion (i.e. show that $\frac{\mathrm{d}}{\mathrm{d}t}C_3 = \{H, C_3\} = 0$). You can assume here that $\frac{\partial}{\partial t}C_{1,2} = 0$.

3. Pendulum

The Hamiltonian of a pendulum of mass m attached to a rigid rod of length L can be written as

$$H = \frac{p^2}{2mL^2} - mgL\cos(q),$$

where p and q are the canonically conjugate variables (the angular momentum and the angle between the rod and the vertical), and g is the gravitational constant.

(a) Deduce the equations of motion for p and q.

(b) Combine them to a single differential equation for q.

(c) Assume that the angle is small and make a Taylor expansion of lowest order in the differential equation for q. Does the resulting equation look familiar?

(d) Make a sketch of the phase space portrait of the pendulum (without the small angle approximation), and explain the different regions in phase space.

2. Two-dimensional harmonic oscillator

Consider a classical particle confined to two spatial dimensions with position coordinates $q_{1,2} \in \mathbb{R}$ and canonically conjugate momenta $p_{1,2} \in \mathbb{R}$, moving under the influence of a harmonic potential $V(q_1, q_2) = \frac{1}{2}m\left(\omega_1^2 q_1^2 + \omega_2^2 q_2^2\right)$, where $\omega_{1,2}$ are real-valued frequencies. and m denotes the mass of the particle. The Hamiltonian function of the system is given by

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m\left(\omega_1^2 q_1^2 + \omega_2^2 q_2^2\right).$$

(a) Deduce the equations of motion for p_j and q_j, and solve them for arbitrary initial conditions $p_j(0)$ and $q_j(0)$.

(b) Sketch the trajectories in position space $\mathbb{R}^2$ (spanned by $q_{1,2}$) for the initial conditions
   - (i) $q_1(0) = 1$, $q_2(0) = 1$, $p_1(0) = 1$, and $p_2(0) = 1$ for $\omega_2 = \omega_1$,
   - (ii) $q_1(0) = 1$, $q_2(0) = 1$, $p_1(0) = 1$, and $p_2(0) = -1$ for $\omega_2 = \omega_1$,
   - (iii) $q_1(0) = 1$, $q_2(0) = 1$, $p_1(0) = 1$, and $p_2(0) = -1$ for $\omega_2 = 2\omega_1$,

   You may let $m = \omega_1 = 1$ for plotting purposes. Use a computer if you wish.

(c) Verify that the two energies $H_1 = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_1^2 q_1^2$ and $H_2 = \frac{p_2^2}{2m} + \frac{1}{2}m\omega_2^2 q_2^2$ are constants of the motion.

(d) In the special case $\omega_1 = \omega_2$ the potential is symmetric with respect to rotations in position space. A consequence of this is that in addition to the energies H_1 and H_2 the *angular momentum* $L = \frac{1}{2}\left(q_2 p_1 - q_1 p_2\right)$ is conserved in time.
   - (i) Verify that L is indeed a constant of motion.
   - (ii) Use the result from Question 1 (b) to identify another independent constant of motion.