Mathematics Year 1, Calculus and Applications I

Solutions Quizzes 1-6

The number corresponds to the Recording number

1. For the following functions construct specific $\varepsilon - \delta$ definitions of continuity at $x = 0$. In other words given a ε you need to find $\delta(\varepsilon)$.

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ x^2 & \text{for } x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & \text{for } x \geq 0 \\ |x|^{1/2} & \text{for } x < 0 \end{cases}$$

Solution: Both functions are continuous at $x = 0$.

Given $\varepsilon > 0$ we need to find a $\delta > 0$ so that $|f(x)| < \varepsilon$ when $|x| < \delta$. Now $f(\delta) = \delta$, $f(-\delta) = \delta^2$, and so $|f(x)| < \varepsilon$ for $|x| < \varepsilon$.

For $g(x)$, we have $g(\delta) = \delta$, $g(-\delta) = \delta^{1/2}$, hence $|g(x)| < \varepsilon$ for $|x| < \varepsilon^2$.

2. Consider the function

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

   (a) Is $f(x)$ a continuous function?

   (b) Show that $f'(0)$ exists and find its value.

   (c) Define $g(x) = f'(x)$, $x \neq 0$, and $g(0) = f'(0)$. Determine whether $g(x)$ is differentiable or not.

   (d) If instead of x^2 in the definition of $f(x)$ we had x^n where n is a positive integer. How many derivatives of $f(x)$ would exist in this case?

Solution: (a) Yes, the function is continuous ($\varepsilon - \delta$ proof with $\delta = \varepsilon^{1/2}$).

(b) Function is continuous and $f(0) = 0$. Now

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} |h| = 0.$$

(c) It follows from (a) that

$$g(x) = f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ -2x & x < 0 \end{cases}$$

Clearly $g(x)$ is differentiable everywhere except possibly at $x = 0$. Calculate

$$g'(0) = \lim_{h \to 0} \frac{g(h)}{h} = \begin{cases} \lim_{h \to 0, h>0} \frac{2h}{h} = 2 \\ \lim_{h \to 0, h<0} \frac{2(-h)}{h} = -2 \end{cases}$$

[Could have seen this by directly differentiating from above and below, but the above is the formal way of doing it.] Since $\lim_{x \to 0} g'(x)$ does not exist, $g(x)$ is not differentiable everywhere.

(d) There would exist $n - 1$ derivatives, again the problematic point is $x = 0$.

3. A spherical balloon is being blown up by injecting air into it at 1 liter per second. When its radius is 1 m, find the rate at which its area is increasing (pay attention to the units).

   Solution: Let the radius of the ballon at time t be equal to $x(t)$. We have the familiar formulas for the volume V and area A:

$$V = \frac{4}{3}\pi x^3, \qquad A = 4\pi x^2.$$

   We are given $dV/dt = 1$ liter/s. But

$$\frac{dV}{dt} = 4\pi x^2 \frac{dx}{dt}, \qquad \frac{dA}{dt} = 8\pi x \frac{dx}{dt},$$

   and so $dx/dt = (1/4\pi x^2) dV/dt$ where $dV/dt = 1000\, cm^3/s = 1000 \times 10^{-6}\, m^3/s$, hence $dx/dt = \frac{10^{-3}}{4\pi}\, ms^{-1}$. This gives finally $\frac{dA}{dt} = 2 \times 10^{-3}\, m^2 s^{-1}$.

4. Sand is being piled onto a conical pile at a constant rate of $R\ cm^3/s$. As the pile grows, frictional forces between sand particles constrain the height of the pile to be equal to the radius of its base.

   (i) When the height equals 1 cm, find the rate at which it is increasing.

   (ii) If the height at time t is $h(t)$, find an explicit expression for it. What happens to its rate of change as t becomes large? Explain physically/intuitively.

   Solution: The volume of a cone of base radius a and height h is $V = \frac{1}{3}\pi a^2 h$. Here, $a = h$ and so $V(t) = \frac{1}{3}\pi h^3$, and $dV/dt = \pi h^2 dh/dt$.

   (i) From above and using $h = 1$ we have $R = \pi \frac{dh}{dt}$, i.e. $dh/dt = (R/\pi)\, cm\, s^{-1}$.

   (ii) For general $h(t)$ we have

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \quad \Rightarrow \quad R = \pi h^2 \frac{dh}{dt}.$$

   Integrate using $h(0) = 0$ (i.e. no sand at $t = 0$) to find

$$h = \left(\frac{3Rt}{\pi}\right)^{1/3}, \qquad \frac{dh}{dt} = \frac{1}{3}(3R/\pi)^{1/3}\, t^{-2/3},$$

   hence $dh/dt \to 0$ as $t \to \infty$.

   Intuition/physics: The volume increases at a constant rate and since the shape is kept a cone of equal height and base radius, we have $h^2 dh/dt = constant$. The height *has to increase* with time since I am adding more and more sand, and the only way to make $h^2 dh/dt$ a constant is for $dh/dt \to 0$ as $t \to \infty$.

5. (a) Determine the regions of increase and decrease of the function $f(x) = x^3 - 2x + 1$.

   (b) Sketch functions for which the intermediate value theorem holds and:

      (i) For a chosen y^* there are *at most two* values of x^*.

      (ii) For a chosen y^* I can choose an interval $[a, b]$ to have as many x^* as I want. [Any guess as to what function this is?]

      (iii) For a chosen y^* there does not exist a x^*, i.e. Theorem 6 does not hold.

Solution:

(a) Calculate $f'(x) = 3x^2 - 2$, hence $f'(x) < 0$ for $|x| < (2/3)^{1/2}$, i.e. decreasing, and $f'(x) > 0$ for $|x| > (2/3)^{1/2}$, i.e. increasing. At $|x| = (2/3)^{1/2}$ it is neither decreasing nor increasing and is equal to 0.

(b)(i) The parabolic function $y = x^2$ will do as long as $y^* > 0$ and the interval $[a, b]$ is big enough.

(b)(ii) For example $y = \sin x$ or $\cos x$ or $\tan x$. The bigger the interval size the more values of x^* there are.

(b)(iii) A discontinuous function will do, for example $y = 1, x > 0$, and $y = -1, x < 0$. The only possible y^* are ± 1.

6. Find $\tan^{-1}(\tan \frac{3\pi}{4})$ and $\arctan(\tan 2\pi)$. (No computers/calculators!)

   Solution:

   Let $\tan^{-1}(\tan \frac{3\pi}{4}) = \alpha$, hence $\tan \frac{3\pi}{4} = \tan \alpha$. Then, $\alpha = \frac{3\pi}{4} + n\pi$, where n is any integer.

   Similarly $\tan 2\pi = \tan \alpha$ hence $\alpha = n\pi$, where n is any integer.