Imperial College London

MATH97039 MATH70030

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fourier Analysis and Theory of Distributions

Date: 07 June 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let the sequences of real numbers $\{a_0, a_1, \dots\}$, $\{b_1, b_2, \dots\}$ satisfy $|a_k|, |b_k| \leq \frac{1}{k^{2/3}}$, $k = 1, 2, \dots$. Does there exist $f \in L_2[-\pi, \pi]$ with these numbers as Fourier coefficients w.r.t. the trigonometric basis? Justify your answer. (5 marks)

   (b) Show that if a periodic function f such that $f(z) = f(z + 2\pi)$, $z \in \mathbb{C}$, is holomorphic on a neighbourhood of the real line then there exist $C, \epsilon > 0$ such that the Fourier coefficients of f (of the complex form of the Fourier series) satisfy $|c_n| \leq Ce^{-\epsilon|n|}$ for all n. Hint: consider a suitable rectangular contour in the complex plane. (15 marks)

2. (a) Let f be continuous on the real line and $|f(x)| \leq \frac{1}{|x|^{10}}$ for $|x|$ sufficiently large. Does the Fourier transform (in the ordinary sense: as Fourier transform of the function) exist and if yes, how many times is it at least differentiable? Justify your answer. (6 marks)

   (b) Find the Laplace transform of the function $f(x) = x^n$, where n is a positive integer. (6 marks)

   (c) Find the Fourier-Stieltjes transform of the function $f(x) = e^{-x^2/2}$. (8 marks)

3. (a) Let $c_k(f)$ be the Fourier coefficients (of the complex form of the Fourier series) of $f \in L_1[-\pi, \pi]$. Using the properties of Fourier coefficients, prove that $c_k(f) \in L_1[-\pi, \pi]^*$ and $c_k \to 0$ as $k \to \infty$ in the weak* topology. (8 marks)

   (b) Are $L_2[-\pi, \pi]$ and $L_2[-\pi, \pi]^*$ isometric? If yes, provide an isometry. (6 marks)

   (c) Give a short (3 or 4 lines) explanation why in the statement that "every *-weakly convergent sequence in E^* is bounded in norm", the original normed space E is required to be Banach. (6 marks)

4. Here $\mathcal{D}$ and $\mathcal{D}'$ denote the space of infinitely differentiable functions on the real line with compact support and the space of distributions on it, respectively.

   (a) Let $\phi \in \mathcal{D}$ and $\int_{-\infty}^{\infty} \phi(x)dx = 0$. Show that $\psi(x) = \int_{-\infty}^{x} \phi(t)dt$ belongs to $\mathcal{D}$. (10 marks)

   (b) Find a continuous function f on the real line and a number k such that in the k'th distributional derivative

$$\partial_k f = \delta(x - a),$$

   where $\delta(x - a)(g) = g(a)$ is the shifted δ-function. (10 marks)

5. For any $s = 0, 1, 2, \ldots$, define the Sobolev space $H_s(\mathbb{R})$ of order s as the space of tempered distributions u on the real line such that the distributional derivatives $u^{(k)}$, $k = 0, \ldots, s$ ($u^{(0)} = u$) are regular (i.e. identified with functions) and in $L_2(-\infty, \infty)$.

(a) Let $|x|^a F[u](x) \to 0$ as $|x| \to \infty$ for some $a > 0$, where $F[u](x)$ is continuous and is the Fourier transform of u. To which $H_s(\mathbb{R})$ does u belong? Justify your answer. (10 marks)

(b) Let $f(x) = e^{-|x|}$, $x \in \mathbb{R}$. To which $H_s(\mathbb{R})$ does f belong? Justify your answer. (10 marks)

1a. $f \in L_2[-\pi, \pi]$ with Fourier coefficients a_k, b_k exists if and only if

$$a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) < \infty$$

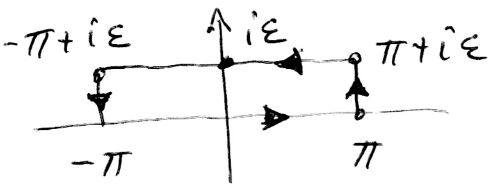
Since $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k^{2/3}}\right)^2 = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}} < \infty$, $(4/3 > 1)$,

such a function exists. [5 points]

1b. The corresponding Fourier coefficients are given by $\displaystyle C_n = \int_{-\pi}^{\pi} f(x) e^{-ixn} \frac{dx}{2\pi}$,

$n = \dots, -1, 0, 1, \dots$

For $n < 0$, consider the contour Γ



for $\varepsilon > 0$ sufficiently small so that f is analytic in a neigbourhood of the rectangle. Then

$$0 = \int_{\Gamma} f(z) e^{-izn} \frac{dz}{2\pi} = C_n + \int_{\pi}^{\pi+i\varepsilon} f(z) e^{-izn} \frac{dz}{2\pi} - \int_{-\pi}^{-\pi+i\varepsilon} f(z) e^{-izn} \frac{dz}{2\pi}$$

$$- \int_{-\pi+i\varepsilon}^{\pi+i\varepsilon} f(z) e^{-izn} \frac{dz}{2\pi} =$$

$$= c_n + i\int_0^{\varepsilon} f(\pi + iy)\, e^{-i\pi n + \varepsilon n}\,\frac{dy}{2\pi} -$$

$$- i\int_0^{\varepsilon} f(-\pi + iy)\, e^{i\pi n + \varepsilon n}\,\frac{dy}{2\pi} - \int_{-\pi}^{\pi} f(x+i\varepsilon)\, e^{-ixn+\varepsilon n}\,\frac{dx}{2\pi},$$

and using periodicity, we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+i\varepsilon)\, e^{-ixn}\,dx \cdot e^{\varepsilon n}, \quad \text{so}$$

$$|c_n| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+i\varepsilon)|\,dx \cdot e^{\varepsilon n} = C_1 e^{\varepsilon n}, \quad n<0.$$

To show that $|c_n| \le C_2 e^{-\varepsilon n}$, $n>0$, we consider similarly the contour with $\varepsilon < 0$.

Thus $|c_n| \le C e^{-\varepsilon |n|}$, $C = \max\{|c_0|, C_1, C_2\}$.

[15 points]

2a The conditions immediately imply that $f, xf, \dots, x^8 f \in L_1(-\infty, \infty)$. By a property of Fourier transform, this implies that $F[f]$ exists and 8 times differentiable.

[6 points]

2b The Laplace transform

$$\Phi(p) = \int_0^\infty x^n e^{-px} dx = [z = px] =$$

$$= \frac{1}{p^{n+1}} \int_0^\infty z^n e^{-z} dz = \frac{\Gamma(n+1)}{p^{n+1}} = \frac{n!}{p^{n+1}}$$

[6 points]

2c The Fourier-Stieltjes transform

$$g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} d e^{-x^2/2} = -\int_{-\infty}^{\infty} x e^{-x^2/2} e^{-i\lambda x} dx$$

$$= -F[x e^{-x^2/2}] = F[(e^{-x^2/2})'] =$$

$$= i\lambda F[e^{-x^2/2}] = i\lambda e^{-\lambda^2/2} \cdot \sqrt{2\pi},$$

where F denotes the Fourier transform.

[8 points]

3a The functional $c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixk} dx$ is obviously linear and moreover,

$$|c_k(f)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx = \frac{1}{2\pi} \|f\|_{L_1} \quad \forall f \in L_1[-\pi,\pi],$$

which implies that $c_k(f)$ is continuous. Thus, $c_k(f) \in L_1[-\pi,\pi]^*$.

[8 points]

For any integrable f, $c_k(f) \to 0$, $k \to \infty$

This means that $c_k \to 0$ in the weak-* topology.

3b $L_2[-\pi,\pi]$ is a Hilbert space.
Therefore it is isometric to its adjoint and the isometry is given by $f \leftrightarrow a$

$$f(x) = (x, a) = \int_{-\pi}^{\pi} x(t)\overline{a(t)}\,dt,$$

$a \in L_2[-\pi,\pi]$, $f \in L_2[-\pi,\pi]^*$.

Then $\|f\| = \|a\|_{2_2} = \left(\int_{-\pi}^{\pi} |a(t)|^2 dt\right)^{1/2}$.

[6 points]

3c Banach space is complete and completeness of E is required in the proof to use the Baire thm.

[6 points]

4a 1) $\psi(x) = \int_{-\infty}^{x} \varphi(x)\,dx$ is infinitely differentiable because $\varphi(x)$ is infinitely differentiable

2) Since $\varphi(x) = 0$ for $x \in \mathbb{R} \setminus [a,b]$ for some interval $[a,b]$ we have

$$0 = \int_{-\infty}^{\infty} \varphi(x)\,dx = \int_{a}^{b} \varphi(x)\,dx \quad \text{and also}$$

$$\psi(x) = 0 \text{ for } x \leq a, \quad \psi(x) = \int_{a}^{x} \varphi(x)\,dx, \quad x > a$$

If $x \geq b$ $\psi(x) = \underbrace{\int_{a}^{b} \varphi(x)\,dx}_{=0} + \underbrace{\int_{b}^{x} \varphi(x)\,dx}_{=0} = 0$

Thus $\psi(x) = 0$ for $x \in \mathbb{R} \setminus [a,b]$.

(1) and (2) mean that $\psi \in \mathcal{D}$. [10 points]

4b Let $g(x) = |x-a|$. Then

$$(g', \varphi) = -(g, \varphi') = -\int_{\mathbb{R}} |x-a|\, \varphi'(x)\,dx =$$

$$= \int_{-\infty}^{a} (x-a)\, \varphi'(x)\,dx - \int_{a}^{\infty} (x-a)\, \varphi'(x)\,dx =$$

$$= \underbrace{(x-a)\varphi(x)\Big|_{-\infty}^{a}}_{=0} - \underbrace{(x-a)\varphi(x)\Big|_{a}^{\infty}}_{=0}$$

$$- \int_{-\infty}^{a} \varphi(x)\,dx + \int_{a}^{\infty} \varphi(x)\,dx = (\operatorname{sgn}(x-a), \varphi)$$

Furthermore,

$$(\operatorname{sgn}(x-a)', \varphi) = \int_{-\infty}^{a} \varphi'(x)\,dx - \int_{a}^{\infty} \varphi'(x)\,dx =$$

$$= 2\varphi(a) = (2\delta(x-a), \varphi).$$

Therefore, as distributional derivative

$\left(\frac{1}{2}|x-a|\right)'' = \delta(x-a)$, i.e. $k=2$.

Note that $\frac{1}{2}|x-a|$ is continuous.

[10 points]

5a For $H_s(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} (1+x^2)^s \, |F[u](x)|^2 dx < \infty .$$

By assumption, $F = \frac{r(x)}{x^a}$, $r(x) \to 0$, $|x| \to \infty$

For $\int_{-\infty}^{\infty} \frac{r^2(x)}{x^{2(a-s)}} dx$ to converge

we must have $a-s > 1/2$, i.e. $s < a - 1/2$.

Thus $u \in H_s$, $s < a - 1/2$. [10 marks]

5b $(f', \varphi) = -(f, \varphi') = -\int e^{-|x|} \varphi' dx =$

$= -\int_{-\infty}^{0} e^{x} \varphi' dx - \int_{0}^{\infty} e^{-x} \varphi' dx = -\cancel{\varphi(0)} + \int_{-\infty}^{0} e^{x} \varphi dx$

$+ \cancel{\varphi(0)} - \int_{0}^{\infty} e^{-x} \varphi dx$,

therefore $f' = -\mathrm{sgn}(x) e^{-|x|} \in L_2(-\infty, \infty)$.

$(f'', \varphi) = (f, \varphi'') = \int_{-\infty}^{0} e^{x} \varphi' dx - \int_{0}^{\infty} e^{-x} \varphi' dx$

$= 2\varphi(0) - \int_{-\infty}^{0} e^{x} \varphi dx + \int_{0}^{\infty} e^{-x} \varphi dx$

therefore $f'' = 2\delta - e^{-|x|} \notin L_2(-\infty, \infty)$

Thus $f \in H_0, H_1$ but $f \notin H_2$,

[10 Marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Fourier Analysis and Theory of Distributions_MATH60030 MATH97039 MATH70030	1	1a was answered well, some difficulties with 1b
Fourier Analysis and Theory of Distributions_MATH60030 MATH97039 MATH70030	2	best answered question of the exam
Fourier Analysis and Theory of Distributions_MATH60030 MATH97039 MATH70030	3	Mixed results for this question. For 3c several variants of the answer are admissable
Fourier Analysis and Theory of Distributions_MATH60030 MATH97039 MATH70030	4	The main focus of 4a was to show compact support, 4b is largely a computational question. Generally good results.
Fourier Analysis and Theory of Distributions_MATH60030 MATH97039 MATH70030	5	5a uses a definition of the norm, for 5b one looks at derivatives. Mixed results.