

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**May – June 2012**

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Statistical Theory I

Date: Monday, 21 May 2012. Time: 2.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (i) For a parametric family of probability density functions $\{f(x|\theta)\}$, define
- total efficient score* $U_*(\theta)$,
 - total Fisher information* $I_*(\theta)$.
- (ii) Let X_1, X_2, \dots, X_n be independent identically distributed random variables having the probability density function

$$f(x|\theta) = c(\theta) x^\theta (1-x)^{1-\theta} \quad (0 \leq x \leq 1),$$

where, for unknown parameter θ ($0 < \theta < 1$), $c(\theta)$ makes the probability density function integrate to 1.

Let $\xi(\theta) = \frac{d}{d\theta} \ln c(\theta)$.

- Find $U_*(\theta)$.
From $U_*(\theta)$ identify the unbiased estimate $\hat{\xi}$ of $\xi(\theta)$.
Find $I_*(\theta)$.
- Find $U_*(\xi)$.
Obtain the variance of $\hat{\xi}$ in terms of $\xi(\theta)$.
- Explain why there is no unbiased estimator \hat{c} of $c(\theta)$ having a variance which is the Cramér-Rao lower bound.

2. (i) (a) Explain what is meant by *complete* and *sufficient* when describing a *complete sufficient statistic*.
(b) Explain briefly why these properties are important in statistical theory.
- (ii) Let X_1, X_2, \dots, X_n be independent identically distributed random variables from the delayed exponential distribution having the probability density function

$$f(x|\theta) = \begin{cases} e^{\theta-x} & (x > \theta), \\ 0 & (\text{otherwise}), \end{cases}$$

where θ is an unknown parameter.

- Why does this probability density function not belong to the Exponential Family?
- Show that $T = \min(X_1, X_2, \dots, X_n)$ is sufficient for θ .
- Find the distribution of T .
- Show that the distribution of T is complete.
- Obtain the unique minimum variance unbiased estimator $\hat{\theta}$ for θ .
- Obtain the variance of $\hat{\theta}$.

3. (i) What is meant by an *unbiased test of size* α ($0 < \alpha < 1$)?
- (ii) Let x_1, x_2, \dots, x_n be a random sample from $\text{Normal}(0, \theta^2)$, where parameter θ ($\theta > 0$) is unknown.
- Show that there is a uniformly most powerful test of $H_0 : \theta = \theta_0$ against $H_1 : \theta < \theta_0$, where θ_0 is specified, and find the explicit form of the test.
 - Find the power function of the test in (a), expressing it in terms of a standard distribution.
 - Show that the test in (a) is biased for $H_1 : \theta \neq \theta_0$.

Give your reasoning throughout.

4. (i) (a) Given data $x \in \mathbb{X}$ from a known probability distribution having unknown parameter $\theta \in \Theta$, define a $100(1 - \alpha)\%$ confidence set $\Psi(x)$ for θ .
- (b) Illustrate (a) by considering a size α test for each $\theta_0 \in \Theta$ of the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, and the relationship between the acceptance set $\bar{R}(\theta_0)$ of values of x and the confidence set $\Psi(x)$.
- (c) What is meant by a *best confidence set*?
- (ii) Let x_1, x_2, \dots, x_n be a random sample from the distribution having the probability density function

$$f(x | \theta) = \theta x^{\theta-1} \quad (0 < \theta < 1),$$

where $\theta > 0$. The prior probability density function for θ , $\pi(\theta)$, is $\text{Exponential}(\lambda)$, where λ ($\lambda > 0$) is known.

- Find the posterior probability density function, and identify the distribution.
- Find the posterior mean (Bayes mean).
- Find the posterior mode (Bayes MLE).

You may wish to use that $\text{Gamma}(\nu, \xi)$ has expectation ν/ξ , and variance ν/ξ^2 .

Give your reasoning throughout.

