

1. (a) Give a definition of a metric and prove or disprove that the following is a metric

(i) In \mathbb{R}

$$d(x, y) \equiv |h(x) - h(y)|$$

where $h(t) = t^3 - t$;

(ii) In \mathbb{R}^2

$$d(x, y) \equiv \sup_{i=1,2} |x_i - 2y_i| ;$$

(iii) In \mathbb{R}^3

$$d(x, y) \equiv \left\{ |x_1 - y_1|^4 + |x_2 - y_2|^4 + |x_3 - y_3|^2 \right\}^{\frac{1}{4}} .$$

(b) Give a definition of a topology in a metric space explaining carefully all notions involved.

(c) Prove or disprove that the following space is a topological space.

$$(X \equiv \mathbb{N}, \mathcal{T} \equiv \{\emptyset, X, A \subsetneq X : X \setminus A \text{ is infinite}\})$$

2. (a) Give a definition of a compact space. Prove or disprove that a product space $X_1 \times \dots \times X_n$, $n \in \mathbb{N}$, $n \geq 2$, with the product topology is compact if and only if $\forall i = 1, \dots, n$ (X_i, \mathcal{T}_i) is compact.
- (b) Prove or disprove that any compact set in the space l_2 is closed.
- (c) Prove or disprove the following statements:
- (i) A closed bounded set in a metric space is compact;
 - (ii) Every compact set in a topological space is closed;
 - (iii) Let $(X, \mathcal{T}_{indisc})$ be a topological space with infinite number of points and a topology given as follows $\mathcal{T}_{indisc} \equiv \{\emptyset, X, A \subset X : X \setminus A \text{ is finite}\}$. Let $f : X \rightarrow \mathbb{R}$ be continuous function to a metric space $(\mathbb{R}, |\cdot - \cdot|)$. Then $\sup_{x \in X} |f(x)|$ may be infinite.

3. (a) State and prove the Banach Contraction Mapping Principle.

(b) Let $f : [0, 1] \rightarrow [0, 1]$, (with metric $|\cdot|$ in $[0, 1]$), be given by $f(x) \equiv \frac{1}{2} + \frac{1}{32} \sin(2\pi(\cos(2\pi x)))$. Prove or disprove that the equation $f(x) = 2x$ has a solution.

(c) State the Arzela-Ascoli theorem explaining carefully all necessary notions involved.

4. (a) Give a definition of a complete and separable metric space.

(b) Prove or disprove that the following spaces are complete

The set of sequences $\ell_{\infty} \equiv \{\mathbf{x} \equiv (x_n \in \mathbb{R})_{n \in \mathbb{N}} : \|\mathbf{x}\| \equiv \sup_{n \in \mathbb{N}} |x_n| \} < \infty$ with the metric given by the norm $\|\cdot\|$ is complete.

(c) Prove or disprove that every compact metric space is separable

5. (a) Give a definition of the disconnected topological space by providing three equivalent conditions.
- (b) Which of the following sets is connected and which is path connected ? Justify your answer.
- (i) Cantor set C_α in $[0, 1]$ with metric $|x - y|$.
 - (ii) C_α^n in \mathbb{R}^n with the Euclidean metric.
 - (iii) Set consisting of the halfline $(x, y) : x = 0, y > 1\}$ and line segments connecting the origin with the points of the form $(x, y) : x = \frac{1}{n}, y = 2$ in the space \mathbb{R}^2 with Euclidean metric.
- (c) Consider \mathbb{R}^3 and \mathbb{R} with natural metric topology in both cases. Prove or disprove the following:
- (i) They are both connected.
 - (ii) They are not homeomorphic.