Imperial College London

M3P16

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Analytic Number Theory

Date: **Wednesday 25th May 2016**

Time: **14.00 – 16.00**

Time Allowed: **2 Hours**

This paper has Five Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Recall the Euler function $\phi(n) = \#\{k \in \mathbb{N} : k \leq n, (k,n) = 1\}$, the unit function $u(n) \equiv 1$, as well as the Möbius function μ defined by $\mu(1) = 1$ and for distinct primes p_i and positive integers e_i

$$\mu(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}) = \begin{cases} (-1)^k & \text{if } e_i = 1 \text{ for all } i, \\ 0 & \text{if } e_i \geq 2 \text{ for some } i. \end{cases}$$

(a) Prove that

$$(\mu * u)(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $*$ denotes the convolution operation.

(b) Assuming the relation $\sum_{d|n} \phi(d) = n$, for $n \in \mathbb{N}$, prove that ϕ is a *multiplicative* arithmetic function.

[You may use any theorems from the lectures, provided you state them clearly.]

(c) Prove that for $n \in \mathbb{N}$ we have

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$

where the product runs over all primes p.

2. (a) State (without proof) the *Euler product formula* for the *Riemann zeta-function* $\zeta(s)$.

(b) Consider the arithmetic function

$$\nu(n) = \begin{cases} 0 & \text{if } n = 1 \\ k & \text{if } n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} \text{ with } p_i\text{'s distinct primes and all } e_i \geq 1. \end{cases}$$

Prove that for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ we have

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\nu(n)} n^{-s}.$$

3. Assuming the relation

$$\sum_{p \leq x, p \text{ prime}} \frac{\log p}{p} = \log x + O(1), \text{for } x \geq 1,$$

prove that, for some constant A,

$$\sum_{p \leq x, p \text{ prime}} \frac{1}{p} = \log(\log(x)) + A + O(\frac{1}{\log x}), \text{for all } x \geq 2.$$

4. (a) Let $\zeta(s)$ denote the Riemann zeta-function, and $\Gamma(s)$ denote the Euler gamma function. Assuming the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s), \quad \forall s \in \mathbb{C} - \{0\},$$

prove that if $\zeta(s) = 0$ then either $s \in \{-2k : k \in \mathbb{N}\}$, or s lies in the strip $0 \le \mathrm{Re}(s) \le 1$.

(b) Prove that if $\zeta(\rho) = 0$ with $0 < \mathrm{Re}(\rho) < 1$, then each of ρ, $1-\rho$, $\overline{\rho}$, and $1-\overline{\rho}$ is a zero of $\zeta(s)$.

[You are required to prove any relation about $\zeta(s)$ you need, except for the functional equation in part (a).]