Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Solutions to Coursework 1

1. $\boxed{\textbf{Total: 15 Marks}}$ In this problem, we assume that the mass density $u(x,t)$ of a chemical substance in steady water in a long and thin channel is governed by the following 1D diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0,$$

where D is the diffusion coefficient. The boundary and initial conditions associated to our problem are given by

$$\begin{aligned} u(x,0) &= 0, \quad x > 0 \\ u(\infty,t) &= 0, \quad t > 0 \\ \frac{\partial u}{\partial x}(0,t) &= -q, \quad t > 0 \end{aligned}$$

where $q > 0$.

(a) The initial conditions set the value of the mass density for all $x > 0$, here the fact that $u(x,0) = 0$ means that there is originally no chemical substance in the long and thin channel. Further, we can rewrite the diffusion equation in the form of a conservation law $u_t + q_x = 0$, with $q(u) = Du_x$. Setting the value of u_x is equivalent to setting the flux of chemical on the LHS of the domain $[0,\infty)$. The final boundary condition sets that the mass density of chemical is always zero in $x \to \infty$. $\boxed{\text{2 Marks}}$
[NB: Note that we have used here the notation u_x for $\partial u/\partial x$.]

(b) Dimensional analysis leads us to find

$$u = f(x,t,D,q)$$

with $[u] = ML^{-3}$, $[D] = L^2T^{-1}$ and $[q] = ML^{-4}$. At this stage, we want to find $(a,b,c,d) \in \mathbb{Q}$ such that

$$[u] = [x^a t^b D^c q^d]$$

Using the fundamental dimensions M, L, T and equating exponents, we get the following system of equations

$$\begin{cases} L: & a + 2c - 4d = -3 \\ T: & b - c = 0 \\ M: & d = 1 \end{cases}$$

This system is solved by $d = 1$, $b = c$ and $a + 2c = 1$, leading to $d = 1$ and $b = c = \frac{1-a}{2}$. This translates to write

$$u = \alpha q x^a t^{\frac{1-a}{2}} D^{\frac{1-a}{2}} \Rightarrow \alpha q\sqrt{Dt}\left(\frac{x}{\sqrt{Dt}}\right)^a$$

where α and a are arbitrary numbers and so more generally we have

$$u = q\sqrt{Dt}F(\eta), \quad \text{with} \quad \eta = \frac{x}{\sqrt{Dt}}$$

$\boxed{\text{2 Marks}}$

We have the following

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{Dt}} \quad \text{and} \quad \frac{\partial \eta}{\partial t} = -\frac{x}{2\sqrt{D}t^{3/2}}$$

By the chain rule, we have

$$\frac{\partial u}{\partial x} = q\sqrt{Dt}F'(\eta)\frac{\partial \eta}{\partial x} = q\sqrt{Dt}F'(\eta)\frac{1}{\sqrt{Dt}} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{q}{\sqrt{Dt}}F''(\eta)$$
$$\frac{\partial u}{\partial t} = q\sqrt{D}\left[\frac{t^{-1/2}}{2}F(\eta) - \sqrt{t}\frac{x}{2\sqrt{D}t^{3/2}}F'(\eta)\right] = \frac{q\sqrt{D}}{2\sqrt{t}}\left[F(\eta) - \eta F'(\eta)\right]$$

So we conclude that F is solution to the following ODE

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{q\sqrt{D}}{2\sqrt{t}}\left[F(\eta) - \eta F'(\eta)\right] - \frac{qD}{\sqrt{Dt}}F''(\eta) = 0$$

which reduces after simplification to

$$2F''(\eta) + \eta F'(\eta) - F(\eta) = 0 \quad (\star)$$

$\boxed{\text{3 Marks}}$

(c) Let's now write the original boundary and initial conditions in terms of the similarity variable.

$$\begin{cases} \text{BCs:} & u(\infty, t) = 0 \quad \Rightarrow F(\infty) = 0 \\ \text{BCs:} & u_x(0, t) = -q \quad \Rightarrow q\sqrt{Dt}\frac{1}{\sqrt{Dt}}F'(0) = -q \Rightarrow F'(0) = -1 \end{cases}$$

$\boxed{\text{3 Marks}}$

(d) Finally, we solve the ODE problem to obtain the solution to the original PDE problem. We realize that $F(\eta) = \lambda\eta$ with $\lambda \in \mathbb{R}$ is solution to the ODE $(\star)$ so we look for solutions of the form $F(\eta) = \eta\lambda(\eta)$.

This gives us

$$F'(\eta) = \lambda(\eta) + \eta\lambda'(\eta)$$
$$F''(\eta) = 2\lambda'(\eta) + \eta\lambda''(\eta)$$

So substituting this in $(\star)$, we obtain

$$4\lambda'(\eta) + 2\eta\lambda''(\eta) + \eta\lambda(\eta) + \eta^2\lambda'(\eta) - \eta\lambda(\eta) = 0$$

which means

$$2\eta\lambda''(\eta) + (4 + \eta^2)\lambda'(\eta) = 0$$

This is a separable equation which we can write as

$$\frac{\lambda''}{\lambda'} = -\frac{2}{\eta} - \frac{\eta}{2}$$

which after integration leads to

$$\ln|\lambda'| = -2\ln\eta - \frac{\eta^2}{4} + C = \ln\frac{1}{\eta^2} - \frac{\eta^2}{4} + C$$

where C is an integration constant. This finally leads to

$$\lambda'(\eta) = \frac{A}{\eta^2}\exp\left(-\frac{\eta^2}{4}\right)$$

and so by integration

$$\lambda(\eta) = A\int^{\eta} s^2 \exp\left(-\frac{s^2}{4}\right) ds + B = A\left[-\frac{e^{-\eta^2/4}}{\eta} - \frac{1}{2}\int_0^{\eta} e^{-s^2/4} ds\right] + B$$

with A and B integration constants to be determined. Finally, we obtain that

$$F(\eta) = A'\left[e^{-\eta^2/4} + \frac{\eta}{2}\int_0^{\eta} e^{-s^2/4} ds\right] + B\eta$$

Taking a derivative, we have

$$F'(\eta) = \frac{A'}{2}\int_0^{\eta} e^{-s^2/4} ds + B$$

Using the BCs, we then find that

$$F'(0) = -1 \Rightarrow B = -1$$

and as $\eta \to \infty$, we have that

$$F(\eta) \sim \eta\left[\frac{A'}{2}\int_0^{\infty} e^{-s^2/4} ds + B\right] \Rightarrow 0 = \frac{A'}{2}\int_0^{\infty} e^{-s^2/4} ds + B = \frac{A'}{2}\sqrt{\pi} + B$$

leading to $B = -1$ and $A' = 2/\sqrt{\pi}$. and finally, we have

$$\begin{aligned}
F(\eta) &= \frac{2}{\sqrt{\pi}}\left[e^{-\eta^2/4} + \frac{\eta}{2}\int_0^{\eta} e^{-s^2/4} ds\right] - \eta \\
&= \frac{2}{\sqrt{\pi}}\left[e^{-\eta^2/4} + \frac{\eta}{2}\int_0^{\eta} e^{-s^2/4} ds - \frac{\sqrt{\pi}}{2}\eta\right] \\
&= \frac{2}{\sqrt{\pi}}\left[e^{-\eta^2/4} + \frac{\eta}{2}\int_0^{\eta} e^{-s^2/4} ds - \frac{\eta}{2}\int_0^{\infty} e^{-s^2/4} ds\right] \\
&= \frac{2}{\sqrt{\pi}}\left[e^{-\eta^2/4} - \frac{\eta}{2}\int_{\eta}^{\infty} e^{-s^2/4} ds\right]
\end{aligned}$$

and finally, we obtain

$$u(x,t) = q\sqrt{\frac{4Dt}{\pi}}\left[e^{-x^2/(4Dt)} - \frac{x}{\sqrt{Dt}}\int_{x/\sqrt{4Dt}}^{\infty} e^{-\xi^2} d\xi\right]$$

$\boxed{\text{5 Marks}}$

2. $\boxed{\textbf{Total: 10 Marks}}$ In this question, we consider the following boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0, \quad x \in \mathbb{R},\ y \in \mathbb{R} \\
&u(x,0) = f(x), \quad x \in \mathbb{R}
\end{aligned}$$

where f is a real function.

(a) First, we find the equation of the characteristics. The method of characteristics for this PDE gives us that

$$\frac{du}{dx} = 0 \quad \text{on} \quad \frac{dy}{dx} = x$$

So the equation for the characteristics is

$$y = \frac{x^2}{2} + \xi$$

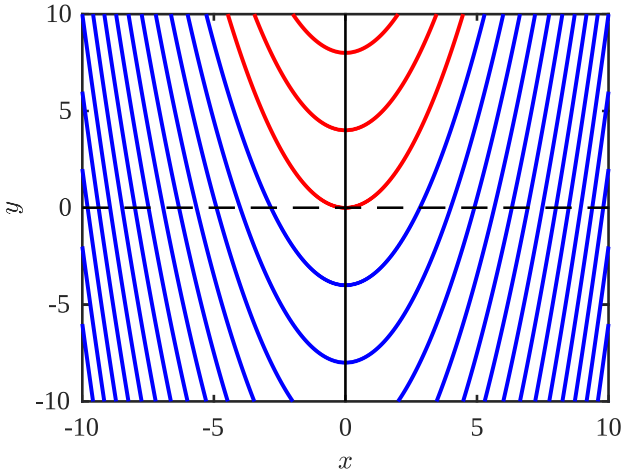

Figure 1: Characteristics associated with $u_x + xu_y = 0$.

where ξ is the label for each of the characteristics. The characteristics are plotted on Fig. 1. Note that importantly the only characteristics we are interested in are the ones plotted in blue and corresponding to $\xi < 0$ because the red characteristics do not cross our boundary conditions, they thus do not carry any information in our problem.

4 Marks

(b) To answer this question, we need to observe the diagram of characteristics. In particular, if we set a particular value of $\xi < 0$, then we can see that the characteristic associated to ξ crosses the x-axis (i.e. $y = 0$) exactly twice in $x = \pm\sqrt{-2\xi}$. As the solution needs to be constant along the characteristics, we need to ensure that $f(\sqrt{-2\xi}) = f(-\sqrt{-2\xi})$ for all $\xi < 0$. We conclude that for this problem to have a solution, we need to constrain $f(x)$ to be an even real function. 3 Marks

(c) Finally, going back to the comment made in (a), we can see that the characteristics only cross the line $y = 0$ (dashed line on Fig 1) for $\xi \leq 0$ so using the method of characteristics we can only determine the solution to this problem for $y \leq x^2/2$. 3 Marks

3. **Total: 5 Marks** Finally, we want to solve the following boundary value problem using the method of characteristics

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0, \quad x > 0,\ t > 0 \\
&u(x, 0) = 0, \quad x > 0 \\
&u(0, t) = \tanh(t), \quad t > 0
\end{aligned}
$$

Let's start by drawing a diagram of characteristics. The method of characteristics tells us that

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = t$$

which means that

$$u = \text{cst.} \quad \text{on} \quad x = \frac{t^2}{2} + \xi$$

A diagram of characteristics is shown on Fig. 2. 2 Marks

In particular, we can easily see that the characteristic labelled by $\xi = 0$ divides the (x, t)-plane in two regions.

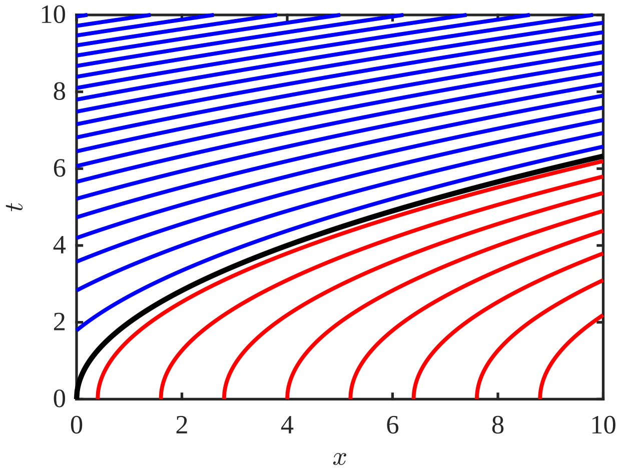

Figure 2: Characteristics associated with $u_t + tu_x = 0$.

For $x > t^2/2$, the solution is identically equal to 0 as the characteristics in this region (the red characteristics) carry the information from the initial conditions which are given by $u(x, 0) = 0,\ x > 0$.

Conversely for $x < t^2/2$, u is not identically zero. We need to use information from the boundary condition in $x = 0$ to solve the problem in this region. To do so, we parametrize as follows $t = \tau$ when $x = 0$ leading to $x = \frac{1}{2}(t^2 - \tau^2)$, i.e. $\tau = \sqrt{t^2 - 2x}$.

Using the boundary conditions, which stipulate that $u(0, t) = \tanh(t),\ t > 0$, we finally obtain

$$u(x,t) = \begin{cases} 0, & 0 < x \leq t^2/2 \\ \tanh\sqrt{t^2 - 2x}, & x > t^2/2 \end{cases}$$

3 Marks