Functional Analysis

FANA 2020/21

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Lecture 1: An Overview of the Course

Linear Metric Spaces

$[Algebra]$

Linear Spaces $(V, \oplus, \otimes, (\mathbb{K}, +, \cdot))$

$[Topology]$

(Metric) Topological Spaces (V, ρ)

$\Rrightarrow$ Linear Metric Space :

$\Rightarrow$ Linear Space with a metric

where $\oplus, \otimes$ are both continuous

Examples :Function spaces $V \subset W^{\Omega}$

$$W^{\Omega} \equiv \text{All functions from } \Omega \text{ with values in } W$$

where $W = \mathbb{R}, \mathbb{C},$ or *a vector space*

For $f, g \in V$

$$(f \oplus g)(\omega) := f(\omega) +_W g(\omega)$$

$$(\lambda \odot f)(\omega) := \lambda \cdot_W f(\omega)$$

Examples of spaces

(i) $\mathbb{K}^n \equiv \{a \equiv (a_j \in \mathbb{K})_{j=1,..,n}\}$ $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}, p - adic\ numbers, ..$

(ii) $\mathbb{R}^{\mathbb{N}} \equiv \{a \equiv (a_j \in \mathbb{R})_{j \in \mathbb{N}}\}$

(iii) $l_p \equiv \{a \equiv (a_j \in \mathbb{C})_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |a_j|^p < \infty\}$, $p \in (0, \infty)$;

(iv) $l_\infty \equiv \{a \equiv (a_j \in \mathbb{C})_{j\in\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j| < \infty\}$

(v) $s_\infty \equiv \{a \equiv (a_j \in \mathbb{C})_{j\in\mathbb{N}} : \sup_{k\in\mathbb{N}} |a_k|^{\frac{1}{k}} < \infty\}$;
$\odot$ **Not continuous**.

(vi) $\mathcal{C}([a,b])$

(vii) $\mathcal{C}^{(k)}([a,b])$ k-times continuously differentiable functions

Metric $\rho : V \times V \to \mathbb{R}^+$

$$\begin{aligned} \rho(x,y) = 0 \qquad &\Longleftrightarrow \qquad x = y \\ \forall x, y \in V \qquad &\rho(x,y) = \rho(y,x) \\ \forall x, y, z \in V \qquad &\rho(x,y) \leq \rho(x,z) + \rho(z,y) \end{aligned}$$

Translation invariant Metric

$$\rho(w \oplus v, z \oplus v) = \rho(w,z)$$

Translation invariant Metric $\Rightarrow$ $\oplus$ is continuous.

Norm $\|\cdot\| : V \to \mathbb{R}^+$

$$\begin{aligned} \|x\| = 0 \qquad &\Longleftrightarrow \qquad x = 0 \\ \forall x \in V \& \forall \alpha \in \mathbb{K} \qquad &\|\alpha \odot x\| = |\alpha| \, \|x\| \\ \forall x, y \in V \qquad &\|x \oplus y\| \leq \|x\| + \|y\| \end{aligned}$$

Metric given by a Norm

$$\rho(x,y) := \|x \ominus y\|$$

Metric $\Rightarrow$ $\oplus$ and $\odot$ are both continuous.

Important Topological Properties of Spaces

(•) Closedness & Completness
[Every Cauchy Sequence Converges]
E.g. $(\mathbb{B}, \|\cdot\|)$ ***Banach Spaces***

(•) Separability
[There exists a dense countable subset]
E.g. **** Yes*** *in* l_p *for* $0 < p < \infty$. **** No*** *in* l_∞. .

(•) Compactness
[closed and bounded $\not\equiv$ compact if $dim = \infty$]
E.g. The closed unit ball is noncompact if $dim = \infty$.

Algebraic basis: **Hamel basis**

Linearly Independent Set Allowing for Finite Representations

(Provided by the Axiom of Choice.)

REM:Same axiom which allowed Banach & Tarski to deconstruct a sphere and use the parts to compose two identical copies of it.

Shauder basis : Countable Linearly Independent Set Allowing for Representation by convergent series

(•) Important Approximation Property
(vital for practical applications)

(•) Separability

Theory of Linear Operators in Normed Spaces

$$T : (\mathbb{X}_1, \|\cdot\|_1) \to (\mathbb{X}_2, \|\cdot\|_2)$$

$$T(\alpha \odot v_1 \oplus \beta \odot v_2) = \alpha \odot T(v_1) \oplus \beta \odot T(v_2)$$

Continuity condition

$$\exists C \in (0, \infty) \forall v \in \mathbb{X}_1 \qquad \|Tv\|_2 \leq C\|v\|_1$$

Remark 1. *The best such constant is called a norm of T.*

E.g.

$$Tf(x) = \kappa \int h(x, y) f(y) \mu(dy)$$

Banach contraction mapping principle
Strict contraction: with $0 < \alpha < 1$

$$\forall x, y \in X \qquad \rho(Fx, Fy) < \alpha \rho(x, y)$$

Provides unique (!) solution of fixed point problem

$$Fx = x$$

[Completness of the metric space is vital]

Application to ODEs, Probability Theory, Fourier Analysis,...
E.g. variety of integral equations involving

$$Ff(x) \equiv g(x) + \kappa \int h(x, y) f(y) \mu(dy)$$

with suitable conditions.

Hahn-Banach Theorem: Extension of Linear Functionals

$$F : \mathbb{X} \to \mathbb{K}$$

Give a continuous functional on a subspace $V \subsetneqq \mathbb{X}$, can one extend it to a continuous functional on entire space (preserving the norm of the functional) ?

[Positive answer using the Axiom of choice/Zorn lemma]

REM: Using this axiom one can show existence of non-conitnuous linear functions

Three Fundamental Theorems:

- Banach-Steinhaus Theorem
  Let $\mathbb{X}$ be a Banach space and $\mathbb{Y}$ a normed vector space.
  For a family of continuous $T_n : \mathbb{X} \to \mathbb{Y}$, $n \in \mathbb{N}$,
  if
  $$\forall x \in \mathbb{X} \qquad \sup_n \|T_n(x)\|_{\mathbb{Y}} < \infty,$$
  then
  $$\sup_n \|T_n\| < \infty.$$
  Application E.g.:

  Existence of Fourier series of a continuous periodic function which diverges at any given point.

- Open mapping Theorem
  Let $\mathbb{X}$ be a Banach space. Then a continuous linear operator $T : \mathbb{X} \to \mathbb{X}$ is an open mapping.

  Hence, if $T : \mathbb{X} \to \mathbb{X}$ is a continuous bijection, then its inverse T^{-1} is also continuous.

- Closed Graph Theorem

Hilbert Space and Spectral Theory

Eigenvalue problem :
$$Tv = \alpha v$$

Compact operators : [$T : \mathbb{X} \to \mathbb{X}$ is compact if the image of any bounded set is a pre-compact set.] Compact operators in a Hilbert space has discrete spectrum of finite multiplicity.

Applications to PDEs (Poincare and Sobolev Inequalities, Existence and Uniqueness problem), Probability Theory (Markov semigroups & processes),...

Poisson Problem

$$\Delta u = f \qquad \textbf{in } \Omega \subset \mathbb{R}^n$$

Weak solution
$$\forall \varphi \in \mathcal{C}_0^\infty \qquad -\int_\Omega \nabla\varphi \cdot \nabla u d\lambda = \int_\Omega \varphi f d\lambda$$

If f is such that $\varphi \mapsto \int_\Omega \varphi f d\lambda$ is a continuous functional on a Hilbert space H_1, then F.Riesz representation theorem gives existence of weak solution straight away.

Remark 2 (Fana Task This Year)**.** *Project: Create a theory of nonlinear topological spaces.*

(Nonlinear Metric Spaces : Nilpotent Lie Groups)

Thank you for your attention

Functional Analysis: Lecture 2.I

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Basic Examples & Tools

Content

- Examples of Linear Spaces
- Metrics and Norms
- Continuity of Linear Operations

Linear Metric Spaces

Let $(V, \oplus, \odot, (\mathbb{K}, +, \cdot))$ be a linear space over a field of numbers $(\mathbb{K}, +, \cdot)$. with modulus $|\cdot|$ which defines a metric.

Example 1. $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}, p - adic\ numbers, ..$ *with* $|\cdot|$ *suitable modulus.*
Recall that $\mathbb{Q}$ *is not complete; others are.*

Two fundamental operations:
Addition of vectors:

$$\oplus : V \times V \to V,$$

such that $(V, \oplus)$ is an abelian group and we have
Multiplication of a vector by a scalar:

$$\odot : V \times \mathbb{K} \to V,$$

With the following conditions:
• $\forall \qquad 1 \odot v = v$, where 1 is the multiplicative identity in $(\mathbb{K}, \cdot)$

• Distributivity of scalar multiplication with respect to vector addition

$$\forall a \in \mathbb{K} \quad \forall u, v \in V \qquad\qquad a \odot (u \oplus v) = a \odot u \oplus a \odot v$$

• Distributivity of scalar multiplication with respect to field addition

$$(a+b)\odot v = a\odot v \oplus b\odot v$$

Examples :Function spaces $V \subset W^{\Omega}$

$$W^{\Omega} \equiv \text{All functions from } \Omega \text{ with values in } W$$

where $W = \mathbb{R}, \mathbb{C},$ or *a vector space*

For $f, g \in V$

$$(f \oplus g)(\omega) := f(\omega) +_W g(\omega)$$

$$(\lambda \odot f)(\omega) := \lambda \cdot_W f(\omega)$$

Question: Does these operations give you a result in V ?

Example 2 (Examples of V-spaces)**.** *For a given field $\mathbb{K}$ the following are vector spaces (over any subfield of $\mathbb{K}$).*

(i) $\mathbb{K}^{\mathbb{N}} = \{f\colon \mathbb{N} \to \mathbb{K}\} \equiv$ *all functions from $\mathbb{N}$ into $\mathbb{K}$ with linear operations defined by* $(f \oplus g)(\omega) := f(\omega) + g(\omega)$ *and* $(a \odot f)(\omega) := a \cdot f(\omega)$

(ii)

$$l_{0,\mathbb{Q}} = \{(a_i \in \mathbb{Q})_{i\in\mathbb{N}} : \exists n \in \mathbb{N} \forall i \geq n \quad a_i \equiv 0\},$$

[Ex. $l_{0,\mathbb{Q}}$ is countable];

(iii)

$$l_0 = \{(a_i \in \mathbb{K})_{i\in\mathbb{N}} : \exists n \in \mathbb{N} \forall i \geq n \quad a_i \equiv 0\};$$

Example 3 (Examples of V-spaces cnd)**.** *(iv) For $1 \leq p < \infty$,*

$$l_p = \{a \in \mathbb{K}^{\mathbb{N}} \colon \sum_{i=1}^{\infty} |a_i|^p < \infty\},$$

Use $|\alpha + \beta| \leq p\,(|\alpha|^p + |\beta|^p)^{\frac{1}{p}}$.

Example 4. *(v)* $C([a,b]) = \{$*space of real valued cts functions on* $[a,b]\}$

(vi) *Riemann-integrable functions on $[a,b]$ form a vector space.*

(vii) *Let $\mathcal{F}$ be a σ-algebra of Ω. Consider the collection of simple functions $y(\omega) = \sum_i^n c_i \chi_{A_i}(\omega)$, $n \in \mathbb{N}$, $c_i \in \mathbb{R}$, $A_i \in \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$.*
https://en.wikipedia.org/wiki/Simple_function

Suppose X is a linear sub-space of V over a field $\mathbb{K}$. Define equivalence relation

$$v \sim w \iff v \ominus w \in X.$$

Let

$$V_\sim \equiv \{[v]_\sim \equiv \{w \in V : \quad w \sim v\}\}$$

Define

$$[v]_\sim \oplus_\sim [u]_\sim := [v \oplus u]_\sim$$
$$\alpha \odot_\sim [v]_\sim := [\alpha \odot v]_\sim$$

***Exercise* 1.** *Show that $\oplus_\sim$ and $\odot_\sim$ are well defined.*

Proposition 1. *$(V_\sim, \oplus_\sim, \odot_\sim, (\mathbb{K}, +, \cdot))$ is linear space.*

***Exercise* 2.** *Prove this proposition.*

Linear Metric and Normed Spaces

Definition 1 (Metric)**.**
• *Let X be a nonempty set. A function $\rho : X \times X \to \mathbb{R}_+$ satisfying the following three conditions is called a metric*

(i) $\forall x, y \in X \quad \rho(x,y) = 0 \iff x = y$

(ii) $\forall x, y \in X \quad \rho(x,y) = \rho(y,x)$

(iii) $\forall x, y, z \in X \quad \rho(x,y) \leq \rho(x,z) + \rho(z,y)$ [*Triangle Ineq*]

• *A metric is called translation invariant iff*

$$\forall x, y, z \in X \qquad \rho(x \oplus z, y \oplus z) = \rho(x,y).$$

Definition 2 (Metric space)**.** *A pair (X, ρ) consisting of a nonempty set X and a metric is called a metric space.*

Definition 3 (Norm and Normed Space)**.** *A function* $\|\cdot\|: V \to \mathbb{R}_+$ *such that*

(i) $\|x\| = 0$ *if and only if* $x = 0$*,*

(ii) $\|\lambda \odot x\| = |\lambda|\|x\|,\ \forall x \in V,\ \ \forall \lambda \in \mathbb{K}$*,*

(iii) $\|x \oplus y\| \leq \|x\| + \|y\|,\ \forall x, y \in V$*,[Minkowski Ineq]*

is called a "norm".
The space $(V, \|\cdot\|)$ *is called a normed space.*

Proposition 2. *If* $\|\cdot\|: V \to \mathbb{R}^+$ *is a norm, then* $\rho(x, y) := \|x \ominus y\|$ *is a metric on* V*. Furthermore* ρ *is translation invariant, i.e.*

$$\forall a, b, c \in V\ \ \rho(a \oplus c, b \oplus c) = \rho(a, b).$$

Example 5 (Example of Metric Space)**.**

Proposition 3. *For* $p \in [1, \infty)$*, the function*

$$l_p \ni x \mapsto \|x\|_p \equiv \left(\sum_j |x_j|^p\right)^{\frac{1}{p}}$$

defined on l_p *is a norm and*

$$\rho(a, b) := \left(\sum_{i=1}^{\infty} |a_i - b_i|^p\right)^{\frac{1}{p}}$$

is the corresponding metric.

To show Minkowski inequality we will use the following lemma.

Lemma 1 (1^{st} Maligranda)**.** *Let* $p \in (1, \infty)$*. Then* $\forall a, b \geq 0$*,*

$$(a + b)^p = \inf_{t \in (0,1)} (t^{1-p} a^p + (1 - t)^{1-p} b^p).$$

Proof. The proof uses $\frac{d}{dt}(t^{1-p}a^p + (1-t)^{1-p}b^p) = (1-p)(t^{-p}a^p - (1-t)^{-p}b^p) = 0$ has a solution only at $t = \frac{a}{a+b}$. □

Proof of Minkowski inequality. Let $\|a\|_{l_p} := (\sum_{i=1}^{\infty} |a_i|^p)^{\frac{1}{p}}$. Then $\rho(a, b) = \|a - b\|_{l_p}$. Now

$$(|a_i| + |b_i|)^p \leq t^{1-p}|a_i|^p + (1 - t)^{1-p}|b_i|^p$$

by the 1st Maligranda lemma. Taking sums

$$\sum_{i=1}^{\infty}(|a_i|+|b_i|)^p \le t^{1-p}\sum_{i=1}^{\infty}|a_i|^p + (1-t)^{1-p}\sum_{i=1}^{\infty}|b_i|^p.$$

Taking infimum over $t \in (0,1)$ gives by the 1st Maligranda lemma

$$\|a+b\|_{l_p}^p \le \||a|+|b|\|_{l_p}^p \le (\|a\|_{l_p}+\|b\|_{l_p})^p.$$

□

***Exercise* 3.** *Give an example of non-translation invariant metric on l_p, $p \in [1,\infty)$.*

E.g. (*) $|x^3 - y^3|$
(**) For $\gamma \in (1,\infty)$ consider $|\mathrm{sign}(x)x^{\frac{\gamma}{p}} - \mathrm{sign}(y)y^{\frac{\gamma}{p}}|^{\frac{1}{\gamma}}$

Definition 4. *Two norms $\|\cdot\|_k$ and two metrics $\rho_k(\cdot,\cdot)$, $k=1,2$, defined on the same linear space V are called equivalent iff we have*

$$\exists\alpha \in (0,\infty)\forall x \in V \qquad \frac{1}{\alpha}\|x\|_1 \le \|x\|_2 \le \alpha\|x\|_1$$

$$\exists\alpha \in (0,\infty)\forall x,y \in V \qquad \frac{1}{\alpha}\rho_1(x,y) \le \rho_2(x,y) \le \alpha\rho_1(x,y),$$

respectively.

***Exercise* 4.** *Give examples of equivalent and non equivalent norms and metrics on l_p and l_0, respectively.*

E.g. -add bounded coefficients
-consider Kaplan type distance and generalisation to infinite dimensions *Not all metrics are derived from norms!*

Example 6. *Let $V = \mathbb{K}^{\mathbb{N}}$ and $z > 1$. An example of a translation invariant metric in V not given by a norm is given by*

$$\rho(x,y) = \sum_{i=1}^{\infty} z^{-n}\frac{|x_m - y_m|}{1+|x_m - y_m|}.$$

Triangle inequality for this metric can be proven using the properties of concave functions.

Definition 5 (Concave and Convex functions)**.** *A function* $f\colon V \to \mathbb{R}$ *is called*

(i) "concave", if and only if $\forall s \in [0,1]\ \forall x, y \in V\ \ sf(x) + (1-s)f(y) \le f(sx + (1-s)y)$*,*

(ii) "convex", if and only if $\forall s \in [0,1]\ \forall x, y \in V\ \ sf(x) + (1-s)f(y) \ge f(sx + (1-s)y)$*.*

Proposition 4 (Triangle inequality for concave functions)**.** *Suppose* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is concave and* $f(0) = 0$*. Then*

$$f(x+y) \le f(x) + f(y).$$

Proof. Note for $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$, (using $f(0) = 0$), we have

$$\alpha f(0) + \beta f(x+y) \le f(\alpha \cdot 0 + \beta(x+y)) \tag{1}$$
$$\beta f(0) + \alpha f(x+y) \le f(\beta \cdot 0 + \alpha(x+y)). \tag{2}$$

Take $\alpha = \frac{y}{x+y}$, $\beta = \frac{x}{x+y}$, then by adding (1) and (2) we obtain

$$f(x+y) \le f(x) + f(y).$$

□

Example 7. *The following are examples of concave functions on* $\mathbb{R}_+$*,*

(i) x^p, $p \in (0,1)$

(ii) $\frac{x}{1+x}$.

Proposition 5. *If* ρ *is a metric and* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is concave vanishing at* 0*, then* $\eta \circ \rho$ *is also a metric.*

***Exercise* 5.** *Prove the above proposition.*

Remark 3. *For a bijection* $\xi : V \to V$*, and a metric* ρ *on* V*, what you can say about* $\rho \circ (\xi \times \xi)$ *?*

Example 8. *The following are examples of a metric and a norm on the space of continuous functions* *(i)* $\rho(f,g) \equiv \mathcal{R} - \int |f-g|^p dx \;\; p \in (0,1)$,

(ii) $\|f\|_p \equiv \mathcal{R} - \left(\int |f|^p dx\right)^{\frac{1}{p}}$, $p \in [1,\infty)$.

These are not norms on $\mathcal{R}$-integrable functions!
E.g. modify the value of a function at finite number of points.
This issue can be resolved by looking at equivalence classes with respect to an equivalence relation

$$f \sim g \iff \mathcal{R} - \int |f-g|^p dx = 0. \qquad (\diamond)$$

***Exercise* 6.** *Prove that $\{f : \mathcal{R} - \int |f|^p dx = 0\}$ is a linear space and $\sim$ given by $(\diamond)$ is an equivalence relation.*
Construct the corresponding factor space and introduce a metric on it.

Metrics on Product Spaces
Given a metric ρ on a vector space V, we can define a metric on $V \times V$ by

$$d((a,b),(c,d)) \equiv (\rho(a,c)^p + \rho(b,d)^p)^{\frac{1}{p}}, \; p \in [1,\infty)$$

and on $\mathbb{K} \times V$ by

$$D((\lambda,a),(\lambda',a')) = \max(|\lambda-\lambda'|, \rho(a,a')).$$

These metrics will be used when discussing continuity of $\oplus$ and $\odot$, respectively.

Definition 6 (Metric Linear Space)**.** *Let V be a vector space over $\mathbb{K}$. Let ρ be a metric in V and $|\cdot - \cdot|$ a metric in $\mathbb{K}$.*
V is called a metric linear space if and only if $\oplus$, $\odot$ are continuous.

***Exercise* 7.** *Using the metrics on product spaces given above, write explicitly what does the continuity of $\oplus$, $\odot$ mean.*

Theorem 1. *Any normed space is a linear metric space.*

***Exercise* 8.** *(i) Verify that if the metric is translation invariant, than additional of vectors is continuous;*

(ii) Verify continuity of $\oplus$ and $\odot$ in normed spaces.

***Exercise* 9.** *Verify which is the metric linear space.*

(i) l_p *with* $\rho_{l_p}(\cdot,\cdot) = \|\cdot \ominus \cdot\|_{l_p}$.

(ii) $\mathcal{R} - (\int |f|^p dx)^{\frac{1}{p}} \equiv \|f\|$ *for* $f \in C([0,1])$ *and* $p \in [1,\infty)$.

(iii) $C([0,1])$ *with* $\|f\|_\infty = \sup_{\omega\in[0,1]} |f(\omega)|$.

Example 9 (Counterexample for $\odot$ being continuous.)**.** *Let* $1 \leq p(m) \to \infty$ *as* $m \to \infty$*. Let*

$$V \equiv \{a = (a_i \in \mathbb{R})_{i\in\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j - b_j|^{\frac{1}{p(j)}} < \infty\}$$

Define

$$\rho(a,b) = \sup_{j\in\mathbb{N}} |a_j - b_j|^{\frac{1}{p(j)}},$$

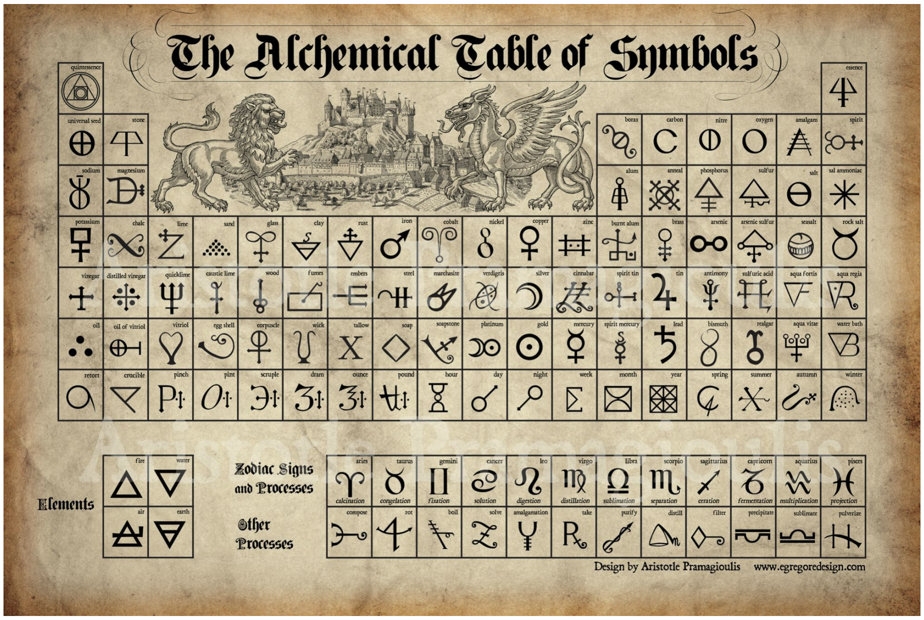
If $a^{(m)} = (a_j \equiv a_0)_{j\in\mathbb{N}}$*, for some* $a_0 > 1$*, and* $\lambda^{(n)} = \xi^n$*,* $\xi \in (0,1)$*, then* $\forall j \in \mathbb{N}$ $\lambda^{(n)} a_j^{(n)} \to 0$*, but*

$$\rho(\lambda^{(n)} \odot a^{(n)}, 0) = \sup_j \xi^{\frac{n}{p(j)}} |a_0|^{\frac{1}{p(j)}} \geq 1.$$

Thus $\rho(\lambda^{(n)} \odot a^{(n)}, 0) \nrightarrow 0$.

Remark 4. *For counter example in case of* $\oplus$*, later in the course we will discuss linear functions which are not continuous.*

When we talk about Hamel basis and Hahn-Banach theorem, Banach paper....



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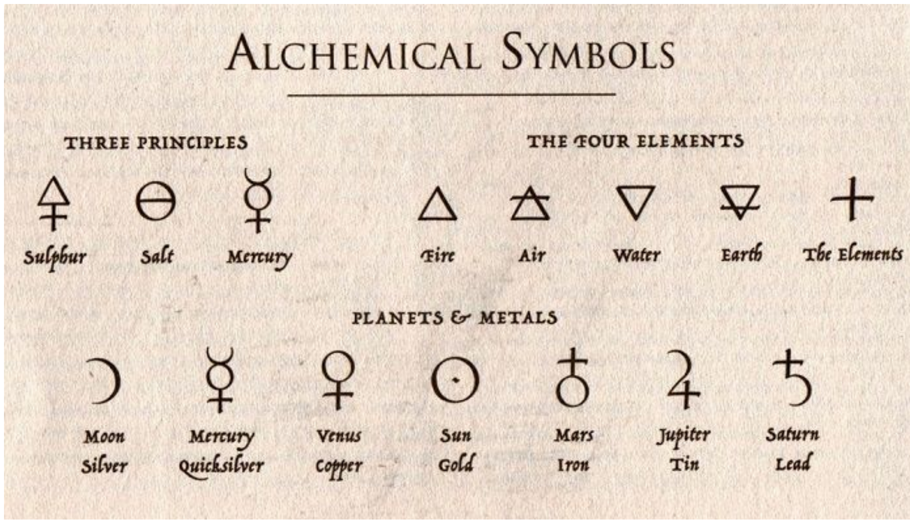

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END Lecture 2.I....

Functional Analysis: Lecture 2.II

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Basic Examples & Tools : Inequalities

Hölder Inequality

Let

$$\langle a, b\rangle \equiv \sum_j a_j b_j$$

for any $a \equiv (a_i)_{i\in\mathbb{N}}$ and $b \equiv (b_j)_{j\in\mathbb{N}}$ for which the right hand side is well defined.

Theorem 2 (Hölder Inequality)**.** *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p, q \in (1,\infty)$. *Then for* $a \equiv (a_i)_{i\in\mathbb{N}} \in l_p$ *and* $b \equiv (b_j)_{j\in\mathbb{N}} \in l_q$, *we have*

$$|\langle a, b\rangle| \leq \|a\|_{l_p}\ \|b\|_{l_q}$$

Lemma 2 (2^{nd} Maligranda)**.** $\inf\limits_{t>0}[\frac{1}{p}t^{\frac{1}{p}-1}a+\frac{1}{q}t^{\frac{1}{p}}b)] = a^{\frac{1}{p}}b^{\frac{1}{q}},\ a, b > 0$

***Exercise* 10.** *Use calculus to prove the above lemma.*

Proof of theorem 2 : Let $(a_i)_{i\in\mathbb{N}} \in l_p$ and $(b_j)_{j\in\mathbb{N}} \in l_q$, with $p, q \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$. By the 2^{nd} Maligranda lemma, we have

$$|a_j|^{\frac{1}{p}} \cdot |b_j|^{\frac{1}{q}} \leq \frac{1}{p}t^{\frac{1}{p}-1}|a_j| + \frac{1}{q}t^{\frac{1}{p}}|b_j|$$

Taking sums

$$\sum_j |a_j|^{\frac{1}{p}} \cdot |b_j|^{\frac{1}{q}} \leq \frac{1}{p}t^{\frac{1}{p}-1}\sum_j |a_j| + \frac{1}{q}t^{\frac{1}{p}}\sum_j |b_j|$$

and subsequently inf over $t \in (0,1)$, we arrive at

$$\sum_j |a_j|^{\frac{1}{p}} \cdot |b_j|^{\frac{1}{q}} \leq \left(\sum_j |a_j|\right)^{\frac{1}{p}} \cdot \left(\sum_j |b_j|\right)^{\frac{1}{q}}$$

from which the inequality in the theorem follows. $\square$

***Exercise* 11.**
(i) Prove Hoelder inequality in the complex case.
(ii) Apply this to $\mathcal{R}-$integrable functions.
(iii) Define integral of simple functions with respect to a measure μ by

$$\int \left(\sum_j c_j \chi_{A_j} \right) d\mu := \sum_j c_j \mu(A_j)$$

Prove Hölder inequality for this class of functions with such the integral.

Example 10. *(Weighted l_p spaces)*
Let $l_p(\eta)$, $\eta_j \geq 0$ and define

$$\|a\|_{l_p(\eta)} = (\sum_{j \in \mathbb{N}} \eta_j |a_j|^p)^{\frac{1}{p}}$$

The case $\sum_{j \in \mathbb{N}} \eta_j = 1$ corresponds to a probability measure on $\mathbb{N}$.

***Exercise* 12.**
Prove Hoelder inequality in weighted $l_p(\eta)$ spaces.

Jensen's Inequality

Convex functions
Let $\mathbb{I} \subseteq \mathbb{R}$.

A function $\phi : \mathbb{I} \to \mathbb{R}^+$ is convex iff

$$\forall s \in [0,1]\ \forall x, y \in \mathbb{I} \qquad \phi(sx + (1-s)y) \leq s\phi(x) + (1-s)\phi(y) \quad \text{(c*)}$$

equivalently

$$\forall y \in \mathbb{I}\ \exists \gamma \in \mathbb{R}\ \forall x \in \mathbb{I} \qquad \gamma\,(x-y) \leq \phi(x) - \phi(y). \qquad \text{(c**)}$$

∘

***Exercise* 13.** *Prove the equivalence.*
Hint: Note that $1^s t$ condition can be written as follows

$$\frac{1}{s}\left(\phi(y+s)(x-y)) - \phi(y)\right) \leq \phi(x) - \phi(y)$$

Set $\langle \alpha \rangle \equiv \sum_j \eta_j \alpha_j$ and $\langle \phi(\alpha) \rangle \equiv \sum_j \eta_j \phi(\alpha_j)$ provided both series are convergent.

Proposition 6 (Jensen's Inequality). *Suppose $\phi \geq 0$ convex function and suppose $\sum_{j \in \mathbb{N}} \eta_j = 1$*

$$|\langle \alpha \rangle| < \infty,$$

and

$$\langle \phi(\alpha) \rangle < \infty.$$

Then

$$\phi(\langle \alpha \rangle) \leq \langle \phi(\alpha) \rangle.$$

Proof. We note that by (c**) we have

$$\gamma \left(\alpha_j - \langle \alpha \rangle \right) \leq \phi(\alpha_j) - \phi(\langle \alpha \rangle)$$

Multiplying both sides by $\eta_j \geq 0$, summing over $j \in \mathbb{N}$ and using the normalisation condition $\sum_j \eta_j = 1$, we obtain the desired inequality in the form

$$0 \leq \langle \phi(\alpha) \rangle - \phi(\langle \alpha \rangle).$$

□

Remark 5. *If $\phi \colon \mathbb{R} \to \mathbb{R}$, $\int a d\eta := \sum_j a_j \eta_j$, then $\phi(\int a d\eta) \leq \int \phi(a) d\eta$.*

Example 11. *The following are examples of convex functions,*

(i) $|x|^p$, $p > 1$,

(ii) $e^{\alpha x}$, $\alpha \in \mathbb{R}$,

(iii) $x \log x$, $x \in \mathbb{R}^+$,

(iv) $x^p \log(1 + |x|)$.

Remark 6. *Note that $l_p \subset l_{p'}$, $p' > p$*
but if $\sum_{j \in \mathbb{N}} \eta_j = 1$, then $l_p(\eta) \supset l_{p'}(\eta)$.
Hint: Use Jensen inequality in the last case.

Remark 7 (Vector space associated to a convex function ϕ)**.** *Let ϕ be a convex function such that $\phi(0) = 0$, $\phi(x) \to \infty$ as $x \to \infty$. Assume it has a doubling property* $\exists C \in (0, \infty) \quad \forall x \in \mathbb{R} \quad \phi(2|x|) \le C\phi(|x|)$. *Then the following set, with coordinatewise addition of vectors and multiplication by a scalar, is a vector space.*

$$V = \{a \equiv (a_j)_{j\in\mathbb{N}} \colon \sum \eta_j \phi(|a_j|) < \infty\}.$$

Warning: One can introduce a norm on such space (generalising the l_p norms). Simple analog $\Phi^{-1}(\sum_j \Phi(a_j))$ doesn't work, if Φ is not a monomial.

Remark 8 (Relations between Inequalities)**.**
Jensen, Hölder, Minkowski ? Who is the boss ?

Figure 1: **Johan Ludvig William Valdemar Jensen**

[Mathematicians] https://en.wikipedia.org/wiki/Jensen%27s_inequality https://upload.wikimedia.org/wikipedia/commons/4/45/Johan_Ludvig_William_Valdemar_Jensen_by_Vilhelm_Rieger.jpg

Jensen, J. L. W. V. (1906). "Sur les fonctions convexes et les inégalités entre les valeurs moyennes". Acta Mathematica. 30 (1): 175–193. doi:10.1007/BF02418571 Operator Jensen Inequality https://arxiv.org/abs/math/0204049

https://en.wikipedia.org/wiki/H%C3%B6lder%27s_inequality

https://en.wikipedia.org/wiki/Otto_H%C3%B6lder

https://en.wikipedia.org/wiki/Minkowski_inequality

https://en.wikipedia.org/wiki/Hermann_Minkowski

END Lecture 2.II

Figure 2: **Otto Hoelder**

Figure 3: **Herman Minkowski**

Functional Analysis: Lecture 3

FANA 2020/21

- **Linear Metric Spaces**
- **Completeness & Separability**
- ◦ **Banach Spaces**

Recall that a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in a metric space (V,ρ) is called
• "Cauchy sequence" iff

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N \qquad \rho(x^{(n)}, x^{(m)}) < \varepsilon$$

• "convergent" to $x \in V$ iff

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad \rho(x^{(n)}, x) < \varepsilon$$

Remark 9. *Note that every Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$ is bounded, i.e.*

$$\forall z \in V \; \exists R \in (0,\infty) \; \forall n \in \mathbb{N} \quad \rho(z, x^{(n)}) < R$$

Exercise

Definition 7. *A set $A \subset V$ is called closed if and only if every convergent sequence in V has a limit in A.*

Definition 8 (Complete Metric Space)**.** *A metric space in which every Cauchy sequence is convergent is called complete.*

Theorem 3. *(Every metric space can be completed)*
For every metric space (X,ρ) there exists a complete metric space $(\tilde{X},\tilde{\rho})$ and an isometric embeding $\iota : X \to \tilde{X}$ i.e. ι is an injection s.t. $\forall x, x' \in X$ we have $\tilde{\rho}(\iota(x), \iota(x')) = \rho(x, x')$

Remark 10. *The idea is to construct $\tilde{X}$ by introducing the equivalence relation of mutually Cauchy sequences in (X,ρ) and equivalence classes become the points in $\tilde{X}$.*

Definition 9 (Banach Space)**.** *A Banach space is a normed space which is complete (with respect to the metric given by the $\|\cdot\|$).*

Classical Banach Spaces

Example 12.

Proposition 7. *l_p, $p \in [1, \infty)$, with a norm $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$ is a Banach space for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.*

Proof: Let $x^{(k)}$ be a Cauchy sequence in l_p, i.e. $\forall \varepsilon > 0$ $\exists N > 0$ such that

$$\forall k, m \geq N \qquad \left\| x^{(m)} - x^{(k)} \right\|_{l_p} < \varepsilon,$$
$$\Rightarrow \left| x_j^{(k)} - x_j^{(m)} \right| \leq \left\| x^{(m)} - x^{(n)} \right\|_{l_p}, \ \forall j \in \mathbb{N}.$$

That is $x_j^{(k)}$ is a Cauchy sequence and so $\exists x_j \in \mathbb{K}$ such that $x_j^{(k)} \to x_j$ as $k \to \infty$. Let $x = (x_j)_{j \in \mathbb{N}}$. We need to show $x \in l_p$.

Example 13 (E.g. cnd)**.** *Recall that any Cauchy sequence is bounded. One can see it by applying definition of Cauchy sequence with $\varepsilon = 1$. Then for some $N \in \mathbb{N}$, we have*

$$\|x_j^{(k)}\|_{l_p} \leq \|x_j^{(N+1)}\|_{l_p} + \|x_j^{(k)} - x_j^{(N+1)}\|_{l_p} \leq \|x_j^{(N+1)}\|_{l_p} + 1.$$

Since for any $k, L \in \mathbb{N}$, we have

$$\left(\sum_{j=1}^{L} \left| x_j^{(k)} \right|^p \right)^{\frac{1}{p}} \leq \left\| x^{(k)} \right\|_{l_p} \leq M,$$

with some $M \in (0, \infty)$, we can conclude that

$$\left(\sum_{j=1}^{L} \left| x_j^{(k)} \right|^p \right)^{\frac{1}{p}} \leq M.$$

Example 14 (E.g. cnd)**.** *Hence passing $k \to \infty$, we get*

$$\left(\sum_{j=1}^{L} |x_j|^p \right)^{\frac{1}{p}} \leq M.$$

Taking $L \to \infty$ gives $\|x\|_{l_p} < M < \infty$. Thus $x \in l_p$.
Finally one can show that $\|x^{(k)} - x\|_{l_p} \to_{k \to \infty} 0$. (Exercise)

Remark 11. *l_p, $p \in [0,1)$, with a metric*

$$\rho_p(x,y) \equiv \sup_{m\in\mathbb{N}} |x_m - y_m|^p$$

is a complete metric space.

***Exercise* 14.** *Prove that $(l_\infty, \|x\|_{l_\infty} = \sup_{m\in\mathbb{N}} |x_m|)$ is a Banach space.*

Example 15. *The following are both Banach spaces with the same norm $\|\cdot\|_\infty$,*

- $$c_0 = \{x \in l_\infty : \lim_{j\to 0} x_j = 0\}$$
- $$c = \{x \in l_\infty \colon \lim_{j\to\infty} x_j \text{ exists } \}$$

Example 16. ***Case*** c_0*: Suppose $x^{(k)}$ is Cauchy in c_0, let $x = (x_j)_{j\in\mathbb{N}}$ where $x_j = \lim_{k\to\infty} x_j^{(k)}$.*

We claim that $x \in c_0$.
Since for the Cauchy sequence $\exists N \in \mathbb{N}\ \forall k > N$

$$\|x^{(k)} - x^{(N+1)}\|_\infty < 1,$$

we get $\forall k > N$

$$\|x^{(k)}\|_\infty \leq \|x^{(N+1)}\|_\infty + \|x^{(k)} - x^{(N+1)}\|_\infty < \|x^{(N+1)}\|_\infty + 1$$

Using this and definition of x_j, we get for all sufficiently large $k \in \mathbb{N}$

$$|x_j| \leq |x_j^{(k)}| + |x_j - x_j^{(k)}| \leq \|x_j^{(k)}\|_\infty + 1 < \|x^{(N+1)}\|_\infty + 2.$$

Hence $\|x\|_\infty < \infty$.

Example 17 (E.g. cnd)**.** *Since given $\varepsilon > 0$, we have with some $N \in \mathbb{N}$, for all $n,m > N$*

$$|x_j^{(n)} - x_j^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_\infty < \varepsilon$$

Hence passing to the limit with $n \to \infty$ on the right hand side, we conclude that

$$|x_j - x_j^{(m)}| \leq \varepsilon.$$

This implies for all $m > N$

$$\sup_j |x_j - x_j^{(m)}| \leq \varepsilon$$

i.e.

$$\lim_{m\to\infty} \|x - x^{(m)}\|_\infty = 0$$

Example 18 (E.g. cnd)**.** *Finally we need to show that*

$$\lim_{j\to\infty} x_j = 0.$$

To this end we use again the property

$$|x_j| \leq \left|x_j - x_j^{(k)}\right| + \left|x_j^{(k)}\right|.$$

Then, given $\varepsilon > 0$, for large k we have that

$$\left|x_j - x_j^{(k)}\right| < \frac{\varepsilon}{2},$$

and for any $x^{(k)} \in c_0$ choosing j sufficiently large we get,

$$\left|x_j^{(k)}\right| < \frac{\varepsilon}{2}.$$

Example 19 (E.g. cnd)**.** *This implies that*

$$|x_j| < \varepsilon \Rightarrow x_j \to 0 \Rightarrow x \in c_0.$$

□

Example 20. *Consider $C([a,b])$, the space of continuous functions on $[a,b]$, with norm $\|f\|_u = \sup_{x\in[a,b]} |f(x)|$.*

Proposition 8. $(C([a,b]), \|\cdot\|_u)$ *is a Banach space.*

Proof. Let $f^{(k)} \in C([a,b])$ be a Cauchy sequence. Then

$$\left|f^{(k)}(\omega) - f^{(m)}(\omega)\right| \leq \left\|f^{(k)} - f^{(m)}\right\|.$$

Hence $\forall\omega$ $(f^{(k)}(\omega))_{k\in\mathbb{N}}$ is Cauchy and so convergent. Define $f(\omega) = \lim_{k\to\infty} f^{(k)}(\omega)$. Using the fact that convergence is uniform on the interval $[a,b]$, one can show that this is a bounded continuous function (*exercise*). □

Remark 12. *Similar result holds in $\mathbb{R}^n$, where we replace $[a, b]$ by a compact set $\Omega \subset \mathbb{R}^n$.*

Example 21. *Let $C^r([a, b])$ be the space of r times continuously differentiable functions on $[a, b]$ with the norm $\|f\|_{r,u} := \sup\limits_{\substack{\omega\in[a,b] \\ 0\le s\le r}} |D^s f(\omega)|$.*

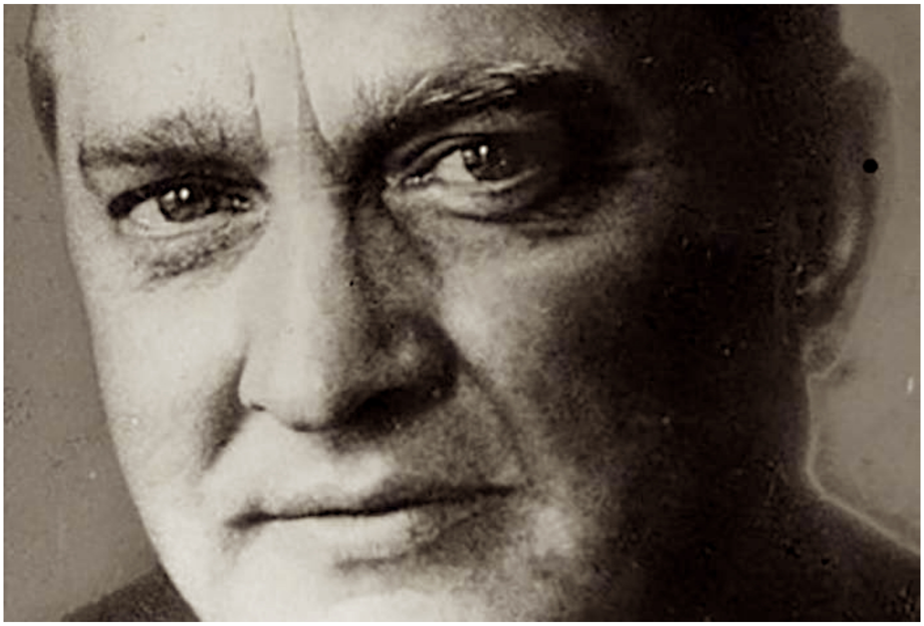
Proposition 9. $(C^r([a, b]), \|f\|_{r,u})$ *is a Banach space*

Idea of Prf: For derivative of order s we proceed similarly as in case of continuous functions to show that the limit for a Cauchy sequence exist and is continuous. This together with the fundamental theorem of calculus applied in the form

$$D^s f(x) = D^s f(a) + \int_a^x dy\ D^{s+1} f(y),$$

for every $s = 0, .., r-1$, allows as to conclude that the limit is r-times continuously differentiable.

Remark 13. *https://en.wikipedia.org/wiki/Stefan_Banach*
History http://kielich.amu.edu.pl/Stefan_Banach/e-biography.html
Scottish Book: English version: https://web.archive.org/web/20180428090844/http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka3ang.pdf
Polish version:https://web.archive.org/web/20180619155008/http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka1pol.pdf
https://newikis.com/en/Scottish_Book



Stefan Banach

END Lecture 3

Functional Analysis: Lecture 4

FANA 2020/21

Linear Metric Spaces

• Separability

• Completness cnd

◦*Absolutely convergent series and Banach spaces*

◦ L_p spaces: completness and separability.

Recall that a set $\mathcal{D}$ in a metric space (V, ρ) is called dense iff

$$\forall x \in V \quad \forall \varepsilon > 0 \quad \mathcal{D} \cap B_{x,\varepsilon} \neq \emptyset$$

Definition 10 (Separable Metric Space)**.** *A metric space which has a countable dense set is called separable.*

Theorem 4. *For $p \in [1, \infty)$, each l_p is separable.*

Proof. Recall the following inclusion relation

$$l_{0,\mathbb{Q}} \subset l_0 \subset l_p$$

from lecture 2. Note first that the subspace l_0 consisting of vectors with all but finite number of coordinates equal to zero is dense in each l_p for $p \in [1, \infty)$. This subspace can be represented as a countable union $\cup_n V_n$, where $V_n \equiv \{v : v_j \equiv 0 \qquad j > n\}$. Next we note that each $V_n \approx \mathbb{R}^n$ has countable dense set.
Hence l_p has a countable dense set $l_{0,\mathbb{Q}}$.

□

Theorem 5. *l_∞ is not separable.*

Proof. Consider a set of sequences $\{v = (v_j \in \{0, 1\})_{j \in \mathbb{N}}\}$. There are uncountably many of such sequence and two such sequences different from each other are separated by a distance ≥ 1. Since any dense set has to contain points in $B_{v,\frac{1}{2}}$, so any dense set has to be uncountable. □

***Exercise* 15.** *Show that the space of sequences with a metric $\rho(x, y) = \sup_n |x_n - y_n|^{\frac{1}{n}}$ is not separable.*

Theorem 6. *$C[a,b]$ and $C^k[a,b]$ are separable.*

Proof. The idea of the proof is to approximate continuous functions by piecewise linear functions with vertices on rational numbers with rational values. (E.g. divide the interval $[a,b]$ into intervals with rational endpoints x_l and of length $|x_{l+1} - x_l| \leq \delta$.
For a given continuous function f, chose rational numbers g_l, so that $|g_l - f(x_l)| < \varepsilon$ and $|f(x_{l+1}) - f(x_l)| < \varepsilon$ which can be achieved if $\delta > 0$ is sufficiently small. Finally consider a piecewise linear function connecting vertices (x_l, g_l).) The set of such functions is countable.

□

cnd. Alternatively, one can use Weierstrass theorem on approximation of continuous functions on an interval $[a,b]$ by polynomials (and then approximate general polynomial by a polynomial with rational coefficients).
Furthermore, we can view $C^k[a,b] \subset C[a,b] \underset{(k+1)-\text{copies}}{\times \cdots \times} C[a,b]$, so it should have a countable dense subset. □

Remark 14. *Later when we'll be discussing a basis, we will provide another constructive way to show separability for these spaces.*

Absolutely Convergent Series

Definition 11. *A series $\sum_{n\in\mathbb{N}} x_n$ in a normed space is called absolutely convergent if and only if $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.*

Example 22. *Let $x^{(n)} \in l_\infty$, $x_j^{(n)} = \frac{1}{n}\delta_{j,n}$ then $(\sum_{n\in\mathbb{N}} x^{(n)})_j = \frac{1}{j}$ but $\sum_{n\in\mathbb{N}} \left\|x^{(n)}\right\| = \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty$.*

***Exercise* 16.** *Construct examples in l_p, $p \in [1,\infty)$.*

Theorem 7. *A normed space is a Banach space if and only if every absolutely convergent series is convergent.*

Proof. ($\Rightarrow$) Suppose X is a Banach space and we have an absolutely convergent series $\sum_{j=1}^{\infty} \|a_j\| < \infty$ with $a_j \in X$. Then

$$\left\|\sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j\right\| \leq \left\|\sum_{j=n}^{m} a_j\right\| \leq \sum_{j=n}^{m} \|a_j\| \to 0 \text{ as } m,n \to, \infty$$

since the sum converges. Hence $\sum_{j=1}^{n} a_j$ is Cauchy, so in a Banach space is convergent.
($\Leftarrow$) Conversely, suppose (x_n) is Cauchy in X. Thus $\exists x_{k_l}$ $l \in \mathbb{N}$, such that

$$\left\| x_{k_{l+1}} - x_{k_l} \right\| \leq \frac{1}{2^l}.$$

$\square$

proof cnd. and so $\sum_{l=1} \left\| x_{k_{l+1}} - x_{k_l} \right\| < \infty$. By our current assumption, the sequence given as the series as follows is convergent

$$x_{k_n} = x_{k_1} + \sum_{l=1}^{n-1} \left(x_{k_{l+1}} - x_{k_l} \right).$$

Thus $\lim_{n\to\infty} x_{k_n} \in X$. Hence X is complete. $\square$

Remark 15 (L_p Spaces)**.** *Let (Ω, Σ, μ) be a measure space, where $\mu : \Sigma \to \mathbb{R}^+$ is a σ-additive and σ-finite measure defined on a σ-algebra $\Sigma \subset 2^\Omega$.*
In the class of measurable real functions $f : \Omega \to \mathbb{R}$, i.e. satisfying $f^{-1}((a,b)) \in \Sigma$, $\forall (a,b) \subset \mathbb{R}$, one introduces an equivalence relation

$$f \sim g \iff \mu\{f \neq g\} = 0$$

In Lebesgue integration theory one defines a space $L_p(\mu)$, $p \in [1, \infty)$ as a linear space of equivalence classes $[f]_\sim$
with the norm

$$\|[f]_\sim\|_{p,\mu} \equiv \left(\int |f(\omega)|^p \mu(d\omega) \right)^{\frac{1}{p}}.$$

The above theorem on absolutely convergent series can be used to show completeness of $L_p(\mu)$, $p \in [1, \infty)$.
We illustrate the idea for $p = 1$. For a Cauchy sequence $\psi_m \in L_1(\mu)$, we can choose a subsequence ψ_{n_k} such that

$$\int \left| \psi_{n_{k+1}} - \psi_{n_k} \right| < \frac{1}{2^k}.$$

Then the series

$$|\psi_{k_1}| + \sum_{k=1} |\psi_{n_{k+1}} - \psi_{n_k}|$$

converges by monotone convergence thm. Hence using Lebesgue dominated convergence thm, we conclude that the series

$$\psi_{n_N} = \psi_{k_1} + \sum_{k=1}^{N-1} \psi_{n_{k+1}} - \psi_{n_k}$$

also converges. This implies by general principle for Cauchy sequences, that ψ_j converges (to a limit which belongs the same equivalence class of functions).

Recall that the construction of Lebesgue integration theory assures that subspace of (equivalence classes) of simple functions is dense in each $L_p(\mu)$, $p \in [1, \infty)$.
As a consequence if the measure μ is separable i.e. each set $A \in \Sigma$ can be approximated in the sense of measure by sets from a countable subfamily of measurable sets, then $L_p(\mu)$, $p \in [1, \infty)$ is separable.
In particular the classical spaces $L_p(\mathbb{R}^n, \Sigma_n, \lambda)$ on Euclidean spaces with the Lebesgue measure are separable.
One can introduce also a Banach space $L_\infty(\mu)$ with a norm

$$\|[f]_\sim\|_\infty \equiv \inf\{\xi > 0 : \mu(\{f > \xi\}) = 0\}.$$

Such space is in general not separable (if Σ is large enough).
Hint: For the Lebesgue space associated to the Lebesgue measure on real line,
consider representation of $[0, 1]$ as a union of disjoint (one side close) intervals of length 2^{-k}, $k \in \mathbb{N}$,
and consider functions taking on values $\{0, 1\}$ on this intervals.
One can introduce also a Banach space $L_\infty(\mu)$ with a norm

$$\|[f]_\sim\|_\infty \equiv \inf\{\xi > 0 : \mu(\{f > \xi\}) = 0\}.$$

Such space is in general not separable.
Hint: For the Lebesgue space associated to the Lebesgue measure on real line,
consider representation of $[0, 1]$ as a union of disjoint (one side close) intervals of length 2^{-k}, $k \in \mathbb{N}$,
and consider functions taking on values $\{0, 1\}$ on this intervals.

Figure 4: **Henri Lebesgue**

[Mathematician]

https://en.wikipedia.org/wiki/Henri_Lebesgue https://en.wikipedia.org/wiki/Lebesgue_integration

End Lecture 4

Functional Analysis: Lecture 5

FANA 2020/21

Basis in Metric Linear Spaces

A natural algebraic notion of a basis in linear space is as follows.

Definition 12 (Hamel basis)**.** *A set $W \subset V$ is a Hamel basis for a linear space V iff W is linearly independent and $\forall x \in V$, $\exists! \omega_j \in W$ and $\exists! \lambda_j \in \mathbb{K}$, $1 \leq j \leq n$, for some $n \in \mathbb{N}$, such that*

$$x = \sum_{j=1}^{n} \lambda_j \omega_j.$$

If we accept axiom of choice, then each linear space has a Hamel basis.`https://en.wikipedia.org/wikiBasis_(linear_algebra){#}Hamel_basis`

Remark 16 (*The axiom of choice will be used again when discussing an important extension type theorem of Hahn-Banach. Then the existence of Hamel basis will be an exercise.*)**.** *Axiom of choice* ***`https://en.wikipedia.org/wiki/Axiom_of_choice`***

Zorn Lemma ***`https://en.wikipedia.org/wiki/Zorn%27s_lemma`***

Remark 17. *Existence of noncontinuous linear functions* ***`http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/pozn/tm/hamel.pdf`***

In metric linear spaces we have another more useful notion of basis.
`https://en.wikipedia.org/wiki/Schauder_basis`
`https://en.wikipedia.org/wiki/Juliusz_Schauder`

Definition 13 (Schauder basis)**.** *A set $W \subset V$ is a Schauder basis for $(V, \|\cdot\|)$ iff $W = \{\omega_j \in V : j \in \mathbb{N}\}$ is linearly independent and $\forall x \in V \quad \exists! \lambda_j \in W$ such that $x = \sum_{j=1}^{\infty} \lambda_j \omega_j$ where the series is convergent with respect to $\|\cdot\|$.*

Proposition 10. *If in a Banach space $(X, \|\cdot\|)$ a Schauder basis exists, then this space is separable.*

***Exercise* 17.** *Prove the above proposition.*
Hint: Consider a set of finite linear combinations of vectors from the Schauder basis with rational coefficients.

Remark 18. *Scottish book problem : Does every separable Banach space has a Schauder basis?*
https://projecteuclid.org/euclid.acta/1485889774 https://en.wikipedia.org/wiki/Per_Enflo

Example 23. *A Schauder basis for l_p, $p \in [1, \infty)$, is $W = \{e_j : j \in \mathbb{N}\}$ with $(e_j)_i = \delta_{ji}$ for $i, j \in \mathbb{N}$. Then for $x = (a_j)_{j\in\mathbb{N}}$, we have a representation $x = \sum_{i=1}^{\infty} a_i e_i$.*

Proposition 11. *If a Banach space is not separable, then it does not have a Schauder basis.*

Example 24. *l_∞, $L_\infty(\mu)$, B.V.*

Example 25 (Schauder basis in c)**.** *Consider $c = \{(a_j \in \mathbb{K})_j : \lim\limits_{j\to\infty} a_j \text{ exists}\}$.*
Claim: $\{e_0 = (1, 1, \ldots), e_j, \quad j \in \mathbb{N}\}$ is a Schauder basis for c. Let $\lambda_0 = \lim\limits_{m\to\infty} a_m$, $\lambda_j = a_j - \lambda_0$. Then

$$\sum_{j=0}^{N} \lambda_j e_j = \lambda_0 e_0 - \lambda_0 \sum_{i=1}^{N} e_i + \sum_{j=1}^{N} a_j e_j,$$

and so

$$\left\| x - \sum_{j=0}^{N} \lambda_j e_j \right\|_\infty = \left\| \sum_{j=N+1}^{\infty} (a_j - \lambda_0) e_j \right\|_\infty = \sup_{j \geq N+1} |a_j - \lambda_0| \to_{N\to\infty} 0.$$

Example 26 (Schauder basis for $C([0,1])$)**.** *The idea is to create suitable piecewise linear "spike" functions α_k.*
Let $\{\omega_k \in [0,1]\}_{k\in\mathbb{N}\cup\{0\}}$ consist of all different diadic elements in the unit interval. For the first two elements $\omega_0 \equiv 0$ and $\omega_1 \equiv 1$, we take linear functions equal to one at the given point, respectively, and zero on the other end of the interval. For a point with nominator 2^{-n}, $n > 0$, we consider a continuous piecewise linear spike with value one at the given point and equal to zero at a distance $\geq 2^{-n}$.

[Note that all spikes but the first two are obtained by squeezing and shifting a single shape.]
Idea of a Wavelet

Proposition 12. *$\{\alpha_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a Schauder basis in $C([0,1])$.*

Proof. Choose $c_0 = f(\omega_0)$, and $c_k = f(\omega_k) - \sum_{l=1}^{k-1} c_l\alpha_l(\omega_k)$. Then

$$f(\omega_k) = \sum_{i=0}^{k} c_i\alpha_i(\omega_k), \tag{*}$$

since $\alpha_k(\omega_k) = 1$. Let

$$S_N = \sum_{i=0}^{N} c_i\alpha_i.$$

Then by our construction of the spikes, we have (*) and hence for every $k \leq N$

$$S_N(\omega_k) = \sum_{i=0}^{k} c_i\alpha_i(\omega_k) + \sum_{i=k+1}^{N} c_i\alpha_i(\omega_k) = f(\omega_k) + 0$$

□

proof cnd. i.e. we get for any point ω_j

$$S_N(\omega_j) \to_{N\to\infty} f(\omega_j).$$

Since by our construction the set $\{\omega_k\}_{k\in\mathbb{N}}$ is dense in $[a,b]$, using uniform continuity property, we conclude that

$$\|f - S_N\| \to_{N\to\infty} 0.$$

□

Figure 5: **Juliusz Schauder**

[Mathematician]

https://en.wikipedia.org/wiki/Juliusz_Schauder

https://en.wikipedia.org/wiki/Schauder_basis
https://en.wikipedia.org/wiki/Schauder_estimates
https://en.wikipedia.org/wiki/Schauder_fixed-point_theorem

END Lecture 5

Functional Analysis: Lecture 6

FANA 2020/21

- **Hilbert space**

Definition 14 (Sesquilinear form)**.** *A sesquilinear form* $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ *is a function satisfying :*

(i) $\forall x, y, z \in \mathbb{H}, \qquad \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(ii) $\forall \lambda \in \mathbb{K}\ \forall x, y \in \mathbb{H}, \ \ \langle x, \lambda y \rangle = \lambda \langle x, y \rangle$

(iii) $\forall x, y \in \mathbb{H}, \qquad \langle x, y \rangle = \overline{\langle y, x \rangle}.$

A sesquillinear form is called "nondegenerate" iff
$\forall y \in \mathbb{H}\ \langle x, y \rangle = 0 \iff x = 0.$
A nondegenerate sesquilinear form is called a scalar product.
The space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$*, where* $\langle \cdot, \cdot \rangle$ *is a scalar product, is called a unitary space or a scalar product space.*

Note that in the unitary space we are given the following norm

$$\|f\|^2 \equiv \langle f, f \rangle.$$

Example 27. *(i)* l_2, $\langle a, b \rangle \equiv \sum_j \bar{a}_j b_j$ *;*
(ii)$\mathbb{H} = C([a, b]; \mathbb{C})$, $\langle f, g \rangle \equiv R - \int \bar{f} g dx$ *is a unitary space ;*
(iii) $\mathbb{X} = \{f : [a, b] \to \mathbb{C})$, $\langle f, g \rangle \equiv R - \int \bar{f} g dx$ *is not a unitary space*

Remark 19. *In any linear space* $\mathbb{X}$ *with a sesquilinear form* $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ *a subset*

$$\mathbb{X}_0 \equiv \{f : \langle f, f \rangle = 0\}$$

is a linear space. Define an equivalence relation $f \sim g \iff f - g \in \mathbb{X}_0$ *and consider a set* $\mathcal{H} \equiv \mathbb{X}/\sim$
consisting of equivalence classes

$$[f]_\sim \equiv \{g \in \mathbb{X} : g \sim f\}$$

with the following linear relations

$$[f]_\sim \bigoplus [h]_\sim \equiv [f \oplus h]_\sim$$
$$\lambda \bigodot [f]_\sim \equiv [\lambda \odot f]_\sim.$$

Remark 20 (Remark cnd)**.** *The the following sesquilinear form on $\mathbb{X}/\sim$ is non degenerate*

$$\langle [f]_\sim, [h]_\sim \rangle \equiv \langle f, h \rangle_{\mathbb{X}}$$

Ex.: Show independence of linear operations and the sesquilinear form of the choice of representant from the equivalence class.

Proposition 13. *Let $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ be a unitary space.*
(i) The function $\mathbb{X} \ni x \to \|x\| \equiv \langle x, x \rangle^{\frac{1}{2}}$ is a norm;
(ii) Paralelogram identity :

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

(iii) Polarisation identity :

$$\langle f, g \rangle = \begin{cases} \frac{1}{4}\left(\|f + g\|^2 - \|f - g\|^2\right) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4}\sum_{k=0,..,3} i^k \|f + i^k g\|^2 & \mathbb{K} = \mathbb{C} \end{cases}$$

Proof exercise

Definition 15 (Hilbert space)**.** *A unitary space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ which is complete in the norm $\| \cdot \| \equiv \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ is called a Hilbert space.*

Proposition 14 (Nearest Point Property)**.** *Every nonempty closed convex set $\mathcal{K}$ in a Hilbert space contains a vector of the smallest norm.*
Moreover given $h \in \mathcal{H}$, $\exists! h_0 \in \mathcal{K}$

$$\|h - h_0\| = dist(h, \mathcal{K}) \equiv \inf_{k \in \mathcal{K}} \|h - k\|$$

Recall that a set $\mathcal{K}$ is convex iff any convex combination of its vectors belongs to this set, in particular

$$\forall x, y \in \mathcal{K} \qquad \frac{x + y}{2} \in \mathcal{K}.$$

Proof. Let $d \equiv \inf_{k \in \mathcal{K}} \|k\|$. By definition of the infimum, there exists a sequence $k_n \in \mathcal{K}$, $n \in \mathbb{N}$, s.t.

$$d = \lim_{n \to \infty} \|k_n\|.$$

Applying parallelogram identity to the vectors k_n, k_m, we have

$$\|\frac{k_n - k_m}{2}\|^2 = \frac{1}{2}\left(\|k_n\|^2 + \|k_m\|^2\right) - \|\frac{k_n + k_m}{2}\|^2.$$

Since by our convexity assumption $\frac{k_n+k_m}{2} \in \mathcal{K}$, so

$$d \leq \|\frac{k_n + k_m}{2}\|$$

and hence

$$\|\frac{k_n - k_m}{2}\|^2 \leq \frac{1}{2}\left(\|k_n\|^2 + \|k_m\|^2\right) - d^2.$$

Using the definition of our sequence $(k_n \in \mathcal{K})_{n\in\mathbb{N}}$, we conclude that

$$\|\frac{k_n - k_m}{2}\| \to_{n,m\to\infty} 0$$

i.e. our sequence is Cauchy. Since by our assumption the set $\mathcal{K}$ is closed, the limit $\lim_{n\to\infty} k_n$ exists in $\mathcal{K}$.
For the second part consider a closed convex set $h - \mathcal{K} \equiv \{h - k : k \in \mathcal{K}\}$. □

Definition 16 (Orthogonal Systems)**.**
Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ *be a unitary space. A set of vectors* $\{e_j \in \mathbb{H}\}_{j\in J}$*, for some index set* J*, is called an orthogonal system iff*

$$\forall i, j \in J, \quad j \neq i, \qquad \langle e_i, e_j\rangle = 0.$$

In case when $\forall i \in J \qquad \|e_i\| = 1$*, the system is called orthonormal.*

Proposition 15. *Every orthogonal system is linearly independent.*

Proof. Suppose for some $n \in \mathbb{N}$ and $j_k \in J$, $k = 1, .., n$ we have

$$\sum_{k=1,..,n} a_k e_{j_k} = 0 \qquad (**)$$

with not all coefficients a_k equal to zero. For some $l \in \{1, .., n\}$, assume $a_l \neq 0$. Then using orthogonality of our vectors and (**), we get

$$a_l \|e_{j_l}\|^2 = \langle e_{j_l}, \sum_{k=1,..,n} a_k e_{j_k}\rangle = 0$$

i.e. we obtain a contradiction. □

Fact: *Using Axiom of choice (Zorn's lemma) one can show existence of the orthogonal basis in any unitary space.*

Definition 17 (Fourier Coefficients)**.**
Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ *be a unitary space and let* $\{e_j \in \mathbb{H}\}_{j\in\mathbb{N}}$ *be orthogonal. For* $f \in \mathbb{H}$, *we define its Fourier coefficients with respect to the orthogonal system by*

$$c_j \equiv \frac{\langle f, e_j\rangle}{\|e_j\|^2}$$

Proposition 16. *Suppose* $f = \sum_{k\in\mathbb{N}} \alpha_k e_k$. *Then*

$$\alpha_j = \frac{\langle f, e_j\rangle}{\|e_j\|^2}.$$

Proof. Let $n < m$ and $S_m = \sum_{k=1}^{m} \alpha_k e_k$. Then

$$\langle S_m, e_n\rangle = \overline{\alpha_n}\|e_n\|^2.$$

Hence

$$\begin{aligned}\left|\overline{\alpha_n}\|e_n\|^2 - \langle f, e_n\rangle\right| &= |\langle S_m, e_n\rangle - \langle f, e_n\rangle| \\ &= |\langle S_m - f, e_n\rangle| \\ &\le \|S_m - f\|\|e_n\| \\ &\to 0 \text{ when } m \to \infty\end{aligned}$$

□

Proposition 17 (Minimal property of Fourier projection)**.**
Suppose $(e_m)_{m\in\mathbb{N}}$ *is orthogonal and*

$$g(a_1, \ldots, a_n) := \left\| f - \sum_{j=1}^{n} a_j e_j \right\|^2.$$

Then g *attains its minimum at* $a_j = c_j$.

Proposition 18 (Bessel's inequality)**.**

$$\sum_{k=1} |c_k|^2 \|e_k\|^2 \le \|f\|^2$$

***Exercise* 18.** *Prove the last two propositions.*

Definition 18. *An orthogonal system* (e_k) *is called complete iff*

$$\langle f, e_k\rangle = 0, \ \forall k \Rightarrow f = 0.$$

Theorem 8. *The following are equivalent*

(i) $(e_k)_{k\in\mathbb{N}}$ *is complete in a Hilbert space,*

(ii) if $c_k = \frac{\langle f,e_k\rangle}{\|e_k\|^2}$*, then* $\|f - \sum_{k=1}^n c_k e_k\| \to 0$ *as* $n \to \infty$*,*

(iii) $\forall f \in \mathbb{H}$*,* $\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \|e_k\|^2$.

Proof. $(i) \Rightarrow (ii)$: Let $f \in \mathbb{H}$ with Fourier coefficients c_k. Then $y = \sum_k c_k e_k \in \mathbb{H}$ and we have

$$\langle f - y, e_k\rangle = c_k - c_k = 0, \ \forall k.$$

$(ii) \Rightarrow (iii)$: Note that, with $S_n \equiv \sum_{k=1}^n c_k e_k$, $n \in \mathbb{N}$, we have

$$\|f\|^2 = \|f - S_n + S_n\|^2 = \|f - S_n\|^2 + 2\,\mathrm{Re}\langle f - S_n, S_n\rangle + \|S_n\|^2$$

As $n \to \infty$, the first and second term goes to zero, by using (i) and Bessel's inequality for the second term. On the other hand for the last one, by orthogonality of e_k, we have

$$\|S_n\|^2 = \sum_{k=1}^{n} c_k \|e_k\|^2.$$

The sum on the r.h.s is bounded by Bessel's inequality and non-decreasing in n, so the infinite series is convergent. Thus passing to the limit with $n \to \infty$ we obtain (iii).
Finally, using the above considerations one can see that $(ii)\&(iii) \Rightarrow (i)$. □

Figure 6: **David Hilbert**

[Mathematician]

https://en.wikipedia.org/wiki/David_Hilbert
https://en.wikipedia.org/wiki/Hilbert_space
https://en.wikipedia.org/wiki/Hilbert%27s_problems
https://en.wikipedia.org/wiki/Hilbert%27s_thirteenth_problem

END Lecture 6

Functional Analysis: Lecture 7

FANA 2020/21

Finite Dimensional Spaces

Theorem 9 (Equivalence of Norms in finite dim)**.** *Let* $(\mathbb{X}, \|\cdot\|)$ *be a normed space with* $dim(\mathbb{X}) = n < \infty$*. Let* $(e_j)_{j=1,..,n}$ *be a basis for* $\mathbb{X}$*. Then there exist* $M, m \in (0, \infty)$ *s.t* $\forall\ x = \sum_{j=1}^{n} a_j e_j\ \mathbb{X}$

$$m \sum_{j=1}^{n} |a_j| \leq \|x\| \leq M \sum_{j=1}^{n} |a_j|.$$

Hence if $dim(\mathbb{X}) < \infty$ *any two norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *defined on this space are equivalent, i.e.* $\exists C \in (0, \infty)\ \forall x \in \mathbb{X}$

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1.$$

Proof. Without loss of generality we can assume the following normalisation condition $\sum_{j=1,..,n} |a_j| = 1$. Then the function $(a_j) \mapsto \| \sum_{j=1}^{n} a_j e_j\|$ attains its minimum on the closed bounded set defined by our normalisation condition. This implies left hand side inequality of the theorem. For the right hand side we note that

$$\| \sum_{j=1}^{n} a_j e_j\| \leq \sum_{j=1}^{n} |a_j| \cdot \|e_j\| \leq \max_k \|e_k\| \sum_{j=1}^{n} |a_j|$$

which ends the proof of the first part of theorem. For the second part we use corresponding inequalities for both norms

$$m_k \sum_{j=1}^{n} |a_j| \leq \|x\|_k \leq M \sum_{j=1}^{n} |a_j|$$

as follows

$$\|x\|_1 \leq M \sum_{j=1}^{n} |a_j| \leq \frac{M}{m_2}\|x\|_2 \leq \frac{M^2}{m_2} \sum_{j=1}^{n} |a_j| \leq \frac{M^2}{m_2 m_1}\|x\|_1.$$

$\square$

Theorem 10 (Completeness in finite dim)**.** *Every finite dimensional space over a complete field is complete.*

Proof. Suppose $x^{(k)} \in X$ is Cauchy. Then

$$\begin{aligned}\left\|x^{(k)} - x^{(k')}\right\| &= \left\|\sum_{j=1}^{n}(a_j^{(k)} - a_j^{(k')})e_j\right\|,\\ &\geq m\sum\left|a_j^{(k)} - a_j^{(k')}\right|,\end{aligned}$$

i.e. $\forall j\ (a_j^{(k)})_k \subset \mathbb{K}$ is Cauchy, so $\exists a_j \in \mathbb{K}$ such that $a_j^{(k)} \underset{k\to\infty}{\to} a_j$. Define $x = \sum a_j e_j$. Then

$$\left\|x^{(k)} - x\right\| \leq M\sum\left|a_j^{(k)} - a_j\right| \to 0.$$

$\square$

Next we begin a discussion of compact sets in linear spaces.

Definition 19 (Compact set)**.** *A set $\mathcal{K}$ in a metric space is compact if every sequence in K has a convergent subsequence with limit in $\mathcal{K}$.*

Theorem 11. *Every compact set $\mathcal{K}$ in a normed space $(X, \|\cdot\|_X)$ is closed and bounded.*

Proof. For closedness, if $x^{(n)} \in \mathcal{K}$ is a Cauchy sequence, then compactness implies that $x^{(n)}$ has a convergent subsequence in $\mathcal{K}$, but that means $x^{(n)}$ converges to that limit (in $\mathcal{K}$).
For boundedness, suppose $\mathcal{K}$ is compact and unbounded, i.e. $\exists x^{(n)} \in \mathcal{K}$ such that $\left\|x^{(n)}\right\| > n$; however, this sequence has no convergent subsequence. $\square$

Example 28 (Counterexample in infinite dim)**.** *Consider the canonical basis in l_2. It is bounded and closed, but not compact.*

Theorem 12 (Compactness in finite dim.I)**.** *Let $(X, \|\cdot\|_X)$ be of finite dimension $n \in \mathbb{N}$. A set $\mathcal{K} \subset \mathbb{X}$ is compact iff it is bounded and closed.*

Proof. $(\Rightarrow)$ is already proved.
$(\Leftarrow)$ Let $x^{(l)} \in \mathcal{K}$, $l \in \mathbb{N}$, be a sequence. Since $\mathcal{K}$ is bounded,

$\exists C \in (0,\infty)$ s.t. $\forall l \in \mathbb{N}$ $\quad \|x^{(l)}\| < C$. Thus, given a basis $(e_j)_{j=1,..,n}$ of X, for

$$x^{(l)} = \sum_{j=1,..,n} x_j^{(l)} e_j$$

we have

$$\sum_{j=1,..,n} |x_j^{(l)}| \leq \frac{1}{m}\|x^{(l)}\| < \frac{1}{m}C.$$

That is for each $j = 1,..,n$, the coordinate sequence $(x_j^{(l)})$ is bounded in $\mathbb{R}(or\ \mathbb{C})$, so by Bolzano-Weierstrass theorem one can chose simultaneously convergent subsequences

$$x_j^{(l_k)} \to x_j \quad as \quad k \to \infty.$$

This means for $x \equiv \sum_{j=1,..,n} x_j e_j$, we have

$$\|x^{(l_k)} - x\| \leq M \sum_{j=1,..,n} |x_j^{(l_k)} - x_j| \to 0 \quad as \quad k \to \infty.$$

□

We will prove even stronger results as follows.

Theorem 13 (Compactness in finite dim. II)**.** *A bounded closed set is compact if and only if the space is finite dimensional.*

The proof follows from the following very important lemma.

Lemma 3 (F.Riesz)**.**
Let Y be a closed proper subspace of a subspace Z in a normed space $(X, \|\cdot\|)$ (of any dimension). Then for any $\theta \in (0,1)$ $\exists z \in Z$ s.t. $\|z\| = 1$ and

$$\forall y \in Y \qquad \|y - z\| \geq \theta.$$

Proof. Let $v \in Z \setminus Y$, $v \neq 0$ and let $a = \inf_{y\in Y} \|v - y\|$. Since Y is closed $a \neq 0$ and $\exists y_0 \in Y$ s.t.

$$0 < a < \|v - y_0\| \leq \frac{a}{\theta}. \tag{***}$$

Let

$$z = \frac{v - y_0}{\|v - y_0\|} \equiv c(v - y_0).$$

Then for any $y \in Y$ we have

$$\begin{aligned} \|z - y\| &= \|c(v - y_0) - y\| \\ &= c\|v - (y_0 + \frac{1}{c}y).\| \end{aligned}$$

Since $y_1 \equiv y_0 + \frac{1}{c}y \in Y$, we have

$$\|v - y_1\| \geq a.$$

Thus, for the vector z, using our choice of y_0 in (***), we get

$$\|z - y\| = c\|v - y_1\| \geq ca = \frac{a}{\|v - y_0\|} \geq \frac{a}{a/\theta} = \theta.$$

$\square$

Figure 7: **Frigyes Riesz**

[Mathematician]

https://en.wikipedia.org/wiki/Frigyes_Riesz

https://en.wikipedia.org/wiki/Riesz%E2%80%93Fischer_theorem
https://en.wikipedia.org/wiki/Riesz_representation_theorem

END Lecture 7

Functional Analysis: Lecture 8

FANA 2020/21

Linear operators

Definition 20 (Bounded operators)**.** *A linear map $T\colon (X,\|\cdot\|_X) \to (Y,\|\cdot\|_Y)$ is said to be bounded iff $\exists C \in (0,\infty)$ such that*

$$\forall x \in X \qquad \|Tx\|_Y \le C\|x\|_X.$$

In such the case we define the operator norm of T by

$$\|T\| := \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Proposition 19. *Every linear map $T\colon X \to Y$ from a finite dimensional $(X,\|\cdot\|_X)$ to a finite dimensional $(Y,\|\cdot\|_Y)$ is bounded.*

Proof. Let $(e_i)_{i=1}^n$ be a basis of X. Then

$$\begin{aligned}
\|Tx\|_Y &= \left\|T\sum x_i e_i\right\|_Y = \left\|\sum x_i Te_i\right\|_Y \\
&\le \sum |x_i|\|Te_i\|_Y \\
&\le \max_i \|Te_i\|_Y \sum |x_i| \\
&\le \frac{1}{m}\max_i \|Te_i\|_Y \; \|x\|_X.
\end{aligned}$$

□

Theorem 14. *Suppose X and Y are Banach spaces. Let $T\colon (X,\|\cdot\|_X) \to (Y,\|\cdot\|_Y)$ be continuous. If $\mathcal{K} \subset (X,\|\cdot\|_X)$ is compact, then $T(\mathcal{K})$ is compact.*

Theorem 15. *Let $T\colon X \to \mathbb{R}$ be continuous. Then T attains* inf *and* sup *on any compact set $\mathcal{K} \subset X$.*

Theorem 16. *Let $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a linear map. Then the following are equivalent*

(i) T is continuous,

(ii) T is Lipschitz: $\exists C \in (0, \infty)$ $\forall x, z \in X$ $\quad \|Tx - Tz\|_Y \leq C\|x - z\|_X$

(iii) T is bounded,

(iv) T is continuous at one point $x_0 \in X$.

Proof. Clearly $(ii) \Leftrightarrow (iii)$.
$(iii) \Rightarrow (i)$: Suppose $\|T\| \neq 0$, and $\|T\| < \infty$. Let $\tilde{x} \in X$. Then for $\varepsilon > 0$ and any $x \in X$ such that $\|x - \tilde{x}\| < \delta$, for sufficiently small $\delta > 0$, we have

$$\|Tx - T\tilde{x}\|_Y = \|T(x - \tilde{x})\|_Y \leq \|T\|\|x - \tilde{x}\|_X \leq \|T\|\delta < \varepsilon.$$

$\square$

Proof cnd. $(iv) \Rightarrow (iii)$: Consider $x = x_0 + \frac{\delta}{\|z\|}z$, $z \neq 0$, $z \in X$.

$$\|Tx - Tx_0\|_Y < \varepsilon \quad \text{if} \quad \|x - x_0\| \leq \delta.$$

Now $\varepsilon > \|Tx - Tx_0\|_Y = \left\|\frac{\delta}{\|z\|}Tz\right\|_Y = \frac{\delta}{\|z\|}\|Tz\|$. Hence $\|Tz\| < \|z\|\frac{\varepsilon}{\delta}$. $\square$

***Exercise* 19.** *Finish the proof for the above theorem.*

Example 29. $T : l_p \to l_q$, $(Tx)_j = \sum_{i \in \mathbb{N}} T_{ji}x_i$. *Then*

$$\|Tx\|_{l_q} = \left(\sum_j |(Tx)_j|^q\right)^{\frac{1}{q}} = \left(\sum_j \left|\sum_{i \in \mathbb{N}} T_{ji}x_i\right|^q\right)^{\frac{1}{q}},$$
$$\leq \left(\sum_j \left|\left(\sum_{i \in \mathbb{N}} |T_{ji}|^r\right)^{\frac{1}{r}} \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}\right|^q\right)^{\frac{1}{q}}.$$

If $\sum_j(\sum_i |T_{ji}|^r)^{\frac{q}{r}} < \infty$, then T is bounded.

***Exercise* 20.** *What about $p, q \in \{1, \infty\}$?*

Example 30. *$T_p\colon L_p(\mu) \to L_q(\nu)$,*

$$\|f\|_p = \left(\mathcal{R} - \int |f|^p \rho_\mu dx\right)^{\frac{1}{p}}, \qquad \rho_\mu \geq 0$$
$$\|g\|_q = \left(\mathcal{R} - \int |g|^q \rho_\nu dx\right)^{\frac{1}{q}} \qquad \rho_\nu \geq 0.$$

For $f, g \in C([a,b])$, these are norms.

$$Tf(x) = \mathcal{R} - \int T(x,y) f(y) \rho_\mu(y) dy.$$

Example 31. *$P_t f(x) \coloneqq \int e^{-\frac{|x-y|^2}{2t}} f(y) \frac{\lambda(dy)}{\sqrt{2\pi t}}$, $t > 0$, λ Lebesgue measure.*
Then P_t satisfies

$$\|P_t f\|_{L_\infty} \leq \frac{1}{\sqrt{2\pi t}} \|f\|_{L_1},$$
$$\|P_t f\|_{L_1} \leq \|f\|_{L_1},$$
$$\|P_t f\|_{L_2} \leq \|f\|_{L_2} \quad \text{and} \quad \|P_t f\|_{L_p} \leq \|f\|_{L_p},\ p \in [1,\infty).$$

Additionally if P_t has the following properties

- $P_s P_t = P_{s+t}$,
- $P_t f^2 \geq 0$,
- $P_t \mathbb{1} = \mathbb{1}$,
- *$f \mapsto P_t f$ is continuous with respect to t,*

then $(P_t))_{t \geq 0}$ is called a Markov semigroup.

Example 32 (Fourier Transform)**.** *Consider $\mathcal{F}f(q) \coloneqq \int e^{iqx} f(x) \lambda(dx)$ with $f \in L_1$. Then, since $\left|\int e^{iqx} f(x) \lambda(dx)\right| \leq \int |f| \lambda(dx)$,*

$$\|\mathcal{F}f\|_{L_\infty} \leq \|f\|_{L_1}.$$

Recall

$$\|g\|_\infty = \inf\{A : \lambda(\{w\colon |g(w)| > A\}) = 0\}.$$

Schwartz functions space

$$\mathcal{S} = \{f\colon \sup_{\alpha,\beta} |x|^{\alpha} \left|\partial_x^{\beta} f\right| < \infty\}.$$

For $f \in \mathcal{S}$ *, we have*

$$\partial_q \mathcal{F} f(q) = \mathcal{F}(ix\, f)(q),$$
$$q\mathcal{F} f(q) = \mathcal{F}(-i\partial_x f)(q),$$

Hence

$$\mathcal{F}(\mathcal{S}) = \mathcal{S}$$

.

If H_m *is the* m^{th} *Hermite polynomial, then* $\{f_m = H_m e^{-\frac{x^2}{2}} \in \mathcal{S}\}$, $n \in \mathbb{N}$, *is an orthonormal basis in* $L_2(\lambda)$, *and*

$$\mathcal{F}(f_m)(q) = (-i)^m f_m(q).$$

Remark 21. *Recall the following useful formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

One can use the above relations together with a formula for H_m *to show*

$$\mathcal{F}(f_m)(q) = (-i)^m f_m(q).$$

Thus $\mathcal{F}$ *is diagonal in this basis and one has* $\|\mathcal{F}(g)\|_{L_2} = \|g\|_{L_2}$.

Remark 22 (Interpolation theory)**.** *Given bounds*

$$\|\mathcal{F}(g)\|_{\infty} \leq \|g\|_1$$

$$\|\mathcal{F}(g)\|_{L_2} = \|g\|_{L_2}$$

there is an interpolation theory which allows to obtain for bounds in intermediate spaces.

Remark 23 (**Heat Equation**). *: Using $\mathcal{F}$ one can solve the heat equation*

$$\partial_t u = \Delta_x u$$
$$u(t=0) = f$$

getting

$$\mathcal{F}(u(t,\cdot)(q) = e^{-tq^2}\mathcal{F}(f)(q)$$

and hence

$$P_t f \equiv \mathcal{F}^{-1}\left(e^{-tq^2}\mathcal{F}(f)(q)\right)$$

Definition 21. *Let* $\mathcal{L}(X,Y) \coloneqq \{T : X \to Y | T \textit{ is linear and bounded}\}$ *with norm*

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Remark 24. $(\mathcal{L}(X,Y), \|\cdot\|)$ *is a normed space.*

Theorem 17. *Let* $(X, \|\cdot\|_X)$ *be a normed space and let* $(Y, \|\cdot\|_Y)$ *a Banach space. Then* $\mathcal{L}(X,Y)$ *is a Banach space.*

Proof. Suppose $T_m \in \mathcal{L}(X,Y)$ is a Cauchy sequence. Then $\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall n, m \geq N$,

$$\|T_m - T_n\| < \varepsilon.$$

Let $x \in X$, and consider $(T_m x)_{m \in \mathbb{N}}$, then we have,

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\|. \tag{3}$$

Hence $(T_m x)_{m \in \mathbb{N}}$ is Cauchy in Y. Since Y is Banach, this implies $\exists Tx \in Y$ such that $T_m x \to Tx$ *as* $m \to \infty$. The map $T \colon X \to Y$ is linear (from the linearity of each T_m) and bounded since,

$$\|Tx\| \leq \|T_m x\| + \|Tx - T_m x\| \leq (\|T_m\| + \|T - T_m\|)\|x\|.$$

Thus T belongs to $\mathcal{L}(X,Y)$.
Finally we show that T is the limit of T_n in the operator norm.
Let $\varepsilon > 0$ be given. There exists an N such that for all $n, m > N$:

$$\|T_n - T_m\| < \varepsilon.$$

By the continuity of the norm, for every $x \in X$ and $n > N$,

$$\|T_n x - Tx\| = \lim_{m \to \infty} \|T_n x - T_m x\| \leq \lim_{m \to \infty} \|T_n - T_m\| \cdot \|x\|_X < \varepsilon \|x\|_X$$

Hence $\|T_n - T\| \varepsilon$ for all $n > N$, which prove $T_n \to T$ as $n \to \infty$.

□

Figure 8: **Joseph Fourier**

[Mathematician]
https://en.wikipedia.org/wiki/Joseph_Fourier

https://en.wikipedia.org/wiki/Fourier_transform{#}Fourier_transform_on_function_spaces

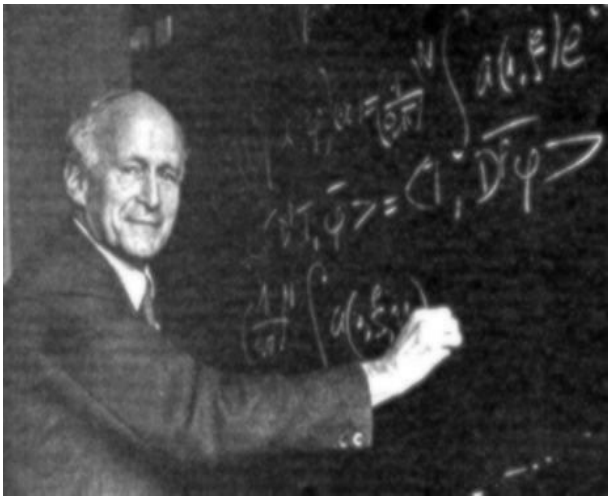


Figure 9: **Laurent Schwartz**

Schwartz said: "What are mathematics helpful for? Mathematics are helpful for physics. Physics helps us make fridges. Fridges are made to contain spiny lobsters, and spiny lobsters help mathematicians who eat them and have hence better abilities to do mathematics, which are helpful for physics, which helps us make fridges which...

http://www.ams.org/notices/199809/chandra.pdf Fields Medal in 1950
https://en.wikipedia.org/wiki/Schwartz_space https://en.wikipedia.org/wiki/Laurent_Schwartz

https://en.wikipedia.org/wiki/Hermite_polynomials https://en.wikipedia.org/wiki/Charles_Hermite

END Lecture 8

Functional Analysis: Lecture 9

FANA 2020/21

Banach Contraction Mapping Principle

Definition 22 (Contraction)**.** *A map $T\colon X \to X$ on a metric space (X, ρ) into itself is called a "contraction" iff $\exists \alpha \in (0,1]$ such that*

$$\forall x, y \in X \qquad \rho(Tx, Ty) \leq \alpha \rho(x, y) \ .$$

The map T is called strict contraction iff $\alpha \in (0,1)$.

Definition 23 (Fixed point)**.** *A vector $x \in X$ is called a "fixed point" of T iff $Tx = x$.*

Theorem 18. *If T is a strict contraction on a complete metric space, then there exists a unique fixed point.*

Proof. Let $z \in X$. For $n \in \mathbb{N}$, let $x_n =: Tx_{n-1} = T^n z$, with $x_0 = z$. Then for $n > m$

$$\begin{aligned}
\rho(x_n, x_m) &= \rho(Tx_{n-1}, Tx_{m-1}) \\
&\leq \alpha \rho(x_{n-1}, x_{m-1}) \\
&\leq \ldots \\
&\leq \alpha^m \rho(x_{n-m}, x_0) \text{ after iterating } m \text{ times.}
\end{aligned}$$

Note that

$$\begin{aligned}
\rho(x_{n-m}, x_0) &\leq \rho(x_{n-m}, x_{n-m-1}) + \rho(x_{n-m-1}, x_0) \\
&\leq \rho(x_{n-m}, x_{n-m-1}) + \rho(x_{n-m-1}, x_{n-m-2}) + \cdots + \rho(x_1, x_0) \\
&\leq \rho(x_1, x_0)(\alpha^{n-m-1} + \alpha^{n-m-2} + \cdots + \alpha + 1) \\
&\leq \frac{1}{1-\alpha} \rho(x_1, x_0).
\end{aligned}$$

Hence

$$\rho(x_n, x_m) \leq \frac{\alpha^m}{1-\alpha} \rho(x_1, x_0).$$

Since $\alpha \in (0,1)$, taking m large, the r.h.s. can be made as close to zero as we want.
Hence x_n is Cauchy.
Since (X, ρ) is complete, we have that $\exists x \in X$ such that $x_n \underset{n\to\infty}{\to} x$.
Now we show x is a fixed point

$$\begin{aligned}\rho(Tx, x) &\leq \rho(Tx, x_n) + \rho(x_n, x)\\ &= \rho(Tx, Tx_{n-1}) + \rho(x_n, x)\\ &\leq \alpha\rho(x, x_{n-1}) + \rho(x_n, x) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence $\rho(Tx, x) = 0$ and so $Tx = x$.
Uniqueness: Suppose $\exists y \in X$ such that $Ty = y$, $x \neq y$. Then

$$\rho(x, y) = \rho(Tx, Ty) \leq \alpha\rho(x, y).$$

Since $\alpha \in (0,1)$, this is only possible if $\rho(x,y) = 0$. Hence we obtain a contradiction. Thus T has a unique fixed point. □

Example 33. *Note that*

$$\frac{d}{dt}u(t) = f(t, u(t)) \Leftrightarrow u(t) = u(t_0) + \int_0^t f(s, u(s))ds.$$

Consider

$$T\omega(t) \mapsto \omega(t_0) + \int_0^t f(s, \omega(s))ds.$$

If f is Lipschitz in the second variable, i.e. $\exists C(s) \in \mathbb{R}$ such that

$$|f(s,\omega) - f(s,\nu)| \leq C(s)|\omega - \nu|.$$

Then

$$\begin{aligned}|T\omega(t) - T\nu(t)| &= \left|\int_0^t |f(s,\omega(s)) - f(s,\nu(s))|ds\right|\\ &\leq \int_0^t C(s)|\omega(s) - \nu(s)|ds\\ &\leq \int_0^t C(s)ds\|\omega - \nu\|.\end{aligned}$$

Restricting f to $[0,\tau]$, $\tau \in (0,\infty)$, we get

$$\|T\omega - T\nu\|_u \leq \int_0^\tau C(s)ds\|\omega - \nu\|_u.$$

Assuming τ is small enough and $C(s)$ is $\mathcal{R}-$integrable, we have that T is a contraction.

Example 34. *For $T(u,z) = g(z) + \lambda \int h(z,x)u(x)\mu(dx)$, with $\frac{1}{p} + \frac{1}{q} = 1$,*
we have

$$|T(u,z) - T(v,z)| \leq |\lambda| \int |h(z,x)||u(x) - \nu(x)|\mu(dx)$$
$$\leq |\lambda|(\int |h(z,x)|^q \mu(dx))^{\frac{1}{q}} \|u - v\|_{L_p(\mu)}$$

Hence

$$\|Tu - Tv\|_{L_p(\mu)} \leq |\lambda| \left(\int (\int |h(z,x)|^q \mu(dx))^{\frac{p}{q}} \mu(dz) \right)^{\frac{1}{p}} \|u - v\|_{L_p(\mu)}.$$

If $(\int(\int |h(z,x)|^q \mu(dx))^{\frac{p}{q}} \mu(dz))^{\frac{1}{p}} < \infty$ and $0 < |\lambda|$ is small enough, the operator T is a contraction $L_p(\mu)$.

***Exercise* 21.** *Prove that if for some $n \in \mathbb{N}$, $n > 1$, the operator T^n is contractive. Then T has a unique fixed point.*

Remark 25. *What if $\alpha = 1$?*
E.g. for $\mathfrak{F}(f) = \sum_n f_n (-i)^n H_n e^{-\frac{x^2}{2}}$, we have

$$\mathfrak{F}^4 = id$$

and any function

$$g \equiv f + \mathfrak{F}f + \mathfrak{F}^2 f + \mathfrak{F}^3 f$$

is a fixed point of $\mathfrak{F}$.

Remark 26. *What are the fixed points of Markov semigroups ?*

$$P_t f \equiv \mathfrak{F}^{-1} \left(e^{-t|q|^\beta} \mathfrak{F}(f) \right)$$

(It is interesting that such semigroup preserves the positivity only for $\beta \in (1, 2]$.)

[Mathematicians]

Banach fixed-point theorem

https://en.wikipedia.org/wiki/Banach_fixed-point_theorem Erik Ivar Fredholm

https://en.wikipedia.org/wiki/Erik_Ivar_Fredholm

Fredholm integral equation

https://en.wikipedia.org/wiki/Fredholm_integral_equation

Vito Volterra

https://en.wikipedia.org/wiki/Vito_Volterra

Volterra integral equation

https://en.wikipedia.org/wiki/Volterra_integral_equation

Brouwer fixed-point theorem

https://en.wikipedia.org/wiki/Brouwer_fixed-point_theorem

For any continuous function f mapping a compact convex set to itself there is a point x_0 such that $f(x_0) = x_0$.

Schauder fixed point theorem

Every continuous function from a convex compact subset K of a Banach space to itself has a fixed point.

https://en.wikipedia.org/wiki/Schauder_fixed-point_theorem

More on that here:

https://www.sciencedirect.com/topics/mathematics/schauder-fixed-point-theorem

https://people.math.aau.dk/~cornean/index.html/newindex_files/schauder.pdf

https://courses.maths.ox.ac.uk/node/view_material/36834

END Lecture 9

Functional Analysis: Lecture 10

FANA 2020/21

Dual spaces

Definition 24 (Dual Spaces)**.** *The space $X^* \coloneqq \mathcal{L}(X, \mathbb{K})$, where $(X, \|\cdot\|)$ is a normed space over $\mathbb{K}$, is called the dual space of $(X, \|\cdot\|)$. The elements of $\mathcal{L}(X, \mathbb{K})$ are called continuous functionals on $(X, \|\cdot\|)$.*

Example 35. *The following are examples of continuous functionals.*

(i) For $X = l_q$, $q \in (1, \infty)$, $(\mathbb{K} = \mathbb{R}, \mathbb{C}))$. Consider the mapping,

$$T\colon l_q \to \mathbb{R},\ x \mapsto \sum_i y_i x_i,\ y_i \in \mathbb{R}.$$

Then, if $\frac{1}{p} + \frac{1}{q} = 1$ and $y = (y_i)_i \in l_p$,

$$\left|\sum_i y_i x_i\right| \leq \left(\sum_i |y_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i|^q\right)^{\frac{1}{q}}.$$

Example 36 (Cnd)**.** *It follows that*

$$|Tx| \leq \|y\|_{l_p} \|x\|_{l_q},$$

so if $y \in l_p$, the functional T is bounded , i.e. $l_p \subset l_q^*$.

Example 37. *(ii) X = Hilbert space $\langle \cdot, \cdot \rangle$. Then, $\forall y \in X$*

$$T_y x \coloneqq \langle y, x \rangle.$$

is a bounded linear functional on X.
All linear functionals on a Hilbert space are of this form.

Theorem 19 (Riesz representation theorem)**.** *For a Hilbert space $(X, \langle \cdot, \cdot \rangle)$, let $l \in \mathcal{L}(X, \mathbb{K}) = X^*$. Then $\exists! z \in X$ such that $\forall x \in X \quad l(x) = \langle z, x \rangle$.*

Remark 27. *Riesz–Fréchet representation theorem, named after Frigyes Riesz and Maurice René Fréchet,* `https://en.wikipedia.org/wiki/Riesz_representation_theorem`

Proof. Consider $N = \ker l \neq X$. Then $\exists z_0 \in X$ such that $z_0 \perp N$. For $x \in X$, let

$$v = l(x)z_0 - l(z_0)x.$$

Then $l(v) = 0$ i.e. $v \in N$, and hence

$$\langle z_0, v\rangle = 0. \text{ (by choice of } z_0\text{)}$$

Using expression for v,

$$l(x)\|z_0\|^2 - l(z_0)\langle z_0, x\rangle = 0$$

which gives

$$l(x) = \frac{l(z_0)}{\|z_0\|^2}\langle z_0, x\rangle = l(\frac{z_0}{\|z_0\|})\ \langle\frac{z_0}{\|z_0\|}, x\rangle.$$

$\square$

Example 38. *Consider* $W_1 \equiv W_{1,2} \equiv \{f\colon \int(1+q^2)\left|\hat{f}(q)\right|^2\lambda(dq) < \infty\}$, *with* $\hat{f}(q) = \mathcal{F}f(q)$.

$$\begin{aligned}\langle f, g\rangle &= \int(1+q^2)\overline{\hat{f}(q)}\hat{g}(q)\lambda(dq),\\ &= \int(\overline{\nabla f}\cdot\nabla g + \overline{f}g)\lambda(dq),\end{aligned}$$

where $\nabla f \equiv$ *is the weak derivative, which means*

$$\exists\ \eta \in L_{1,loc}(\lambda), \forall\phi \in C_0^\infty, \qquad \int f\nabla\phi\lambda(dx) = \int\eta\phi\lambda(dx),$$

and $\nabla f \coloneqq -\eta$.

Example 39 (Example cnd)**.** *Let*

$$\begin{aligned}l_h(\phi) \coloneqq \int h\phi\lambda(dx) = \int \overline{\hat{h}}\hat{\phi}\lambda(dx)\\ &= \int\frac{\overline{\hat{h}}}{\sqrt{1+q^2}}\sqrt{1+q^2}\hat{\phi}\lambda(dq)\\ &\leq \|h\|^2_{W_{-1}}\|\phi\|_{W_1} \text{ (Cauchy-Schwarz).}\end{aligned}$$

with
$$\|h\|^2_{W_{-1}} \equiv \int \frac{\left|\hat{h}\right|^2}{1+q^2} d\lambda$$
If $\|h\|^2_{W_{-1}} < \infty$*, then* l_h *is continuous on* W_1*.*

Example 40 (Example cnd)**.** *Hence by Riesz representation theorem*
$$\int h\phi d\lambda = \int \overline{\nabla u} \cdot \nabla \phi + \overline{u}\phi d\lambda$$
for some unique $u \in W_1$*.*
Such u *is called a weak solution of*
$$-\Delta u + u = h$$

Dual spaces of l_p, $p \in [1,\infty]$

In this section we discus the dual spaces l_p^* of l_p for $p \in [1,\infty]$,

- Case $p = 2$, we have $(l_2)^* \cong l_2$.

- Case $p \in (1,\infty)$, for $y \in l_q$, $\frac{1}{q} + \frac{1}{p} = 1$,
$$f_y(x) = \sum_i y_i x_i,$$
$$|f_y(x)| \underset{\text{Hölder}}{\leq} \|y\|_{l_q} \|x\|_{l_p}.$$

Hence f_y is continuous linear function on l_p and so $l_q \subset (l_p)^*$.
To show that every $f \in (l_p)^*$, $p \in (1,\infty)$ can be represented by some $y \in l_q$, $\frac{1}{p} + \frac{1}{q} = 1$, we use canonical basis $(e_i)_{i\in\mathbb{N}}$ in l_p.
For any $x \in l_p$, $x = \sum_i a_i e_i$, and if f is linear and continuous we have
$$f(x) = \sum_i x_i f(e_i).$$

We will show $(f(e_i))_{i\in\mathbb{N}} \in l_q$.
Let $a_i = f(e_i)$. If $a_i \neq 0$ let $x_i = (\frac{a_i}{|a_i|^q})^{-1}$
In this special case
$$f(\sum_i^n x_i e_i) = \sum_i^n x_i a_i = \sum_i^n |a_i|^q \tag{4}$$

On the other hand as f is linear and continuous

$$|f(x)| \leq \|f\| \|x\|_{l_p} = \|f\| (\sum_i^n |x_i|^p)^{\frac{1}{p}} = \|f\| \left(\sum_i^n \frac{|a_i|^{qp}}{|a_i|^p} \right)^{\frac{1}{p}}$$

Now $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow pq = p + q$, so $\frac{|a_i|^{pq}}{|a_i|^p} = |a_i|^q$, and hence

$$|f(x)| \leq \|f\| (\sum_i^n |a_i|^q)^{\frac{1}{p}} \tag{5}$$

Combining (4) and (5) gives

$$\sum_i^n |a_i|^q \leq \|f\| (\sum_i^n |a_i|^q)^{\frac{1}{p}}, \quad \left(\text{divide by } \left(\sum_i^n |a_i|^q \right)^{\frac{1}{p}} \text{ using } 1 - \frac{1}{p} = \frac{1}{q} \right),$$

$$\Rightarrow (\sum_i^n |a_i|^q)^{\frac{1}{q}} \leq \|f\|.$$

Hence, after taking $n \to \infty$, we get $(a_i)_{i \in \mathbb{N}} = (f(e_i))_{i \in \mathbb{N}} \in l_q$.
We conclude that $\forall f \in (l_p)^*$, $\exists! (a_i)_{i \in \mathbb{N}} \in l_q$ such that $\forall x \in l_p$, $f(x) = \sum_i^\infty a_i x_i$.

Theorem 20. *For every $p \in (1, \infty)$ $l_p^* \cong l_q$, for $\frac{1}{p} + \frac{1}{q} = 1$.*

- What about $p = 1$?
  Then $\forall y \in l_\infty$, $f_y \in l_1^*$, with $f_y(x) = \sum_i^\infty x_i y_i$ since

$$|f_y(x)| \leq \sum_i^\infty |x_i||y_i| \leq \|y\|_\infty \|x\|_{l_1}.$$

Theorem 21. *We have $l_\infty \subset l_1^*$.*

- For $p = \infty$, $\forall y \in l_1$, $f_y(x) = \sum_i^\infty x_i y_i$ then $|f_y(x)| \leq \|x\|_\infty \|y\|_{l_1}$.

Theorem 22. *We have $l_1 \subset l_\infty^*$.*

Remark 28. *$l_1 \subsetneq l_\infty^*$ strictly contained.*
For the proof, see the Banach limit using Hahn-Banach (theorem provided later) to extend linear functional $f(x) = \lim\limits_{n \to \infty} x_n$ defined on convergent sequences to all bounded sequences in l_∞.

Figure 10: **Maurice René Fréchet**

[Mathematicians]
https://en.wikipedia.org/wiki/Maurice_Ren%C3%A9_Fr%C3%A9chet

Introduced the entire concept of metric spaces.
Fréchet spaces in functional analysis:
A Fréchet space X is defined to be a locally convex metrizable topological vector space that is complete.
The topology of such space is induced by some translation-invariant complete metric.
Topology may be induced by a countable family of semi-norms $\|\cdot\|_k$, $k \in \mathbb{N}$, and then the metric is given by

$$d(x, y) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|x-y\|_k}{1+\|x-y\|_k} \qquad x, y \in X.$$

All Banach and Hilbert spaces are Fréchet spaces.
(Fréchet was the first to use the term "Banach space".)
Important example of F-Space is the Schwartz space.
https://en.wikipedia.org/wiki/Fr%C3%A9chet_space
Riesz–Fréchet representation theorem

https://en.wikipedia.org/wiki/Riesz_representation_theorem

Banach limit
https://en.wikipedia.org/wiki/Banach_limit

END Lecture 10

Functional Analysis: Lecture 11

FANA 2020/21

Dual operators

Definition 25. *For $L \in \mathcal{L}(X, X)$ we define the dual linear operator $L^* : X^* \to X^*$ by*

$$\forall x \in X \qquad (L^* f)(x) = f \circ L(x).$$

and define its norm by

$$\|L^*\| = \sup_{f \neq 0} \frac{\|L^* f\|}{\|f\|}.$$

Proposition 20. *We have $\|L^*\| = \|L\|$.*

Proposition 21. *The following properties hold.*

(i) $(S + T)^* = S^* + T^*$

(ii) $\forall \alpha \in \mathbb{K}$, $(\alpha T)^* = \overline{\alpha} T^*$

(iii) $(T^*)^* = T$

(iv) $(ST)^* = T^* S^*$.

Dual operators in Hilbert space

Consider $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ Hilbert space, $T \colon \mathcal{H} \to \mathcal{H}$ then by the Riesz theorem,

$$(T^* f)(x) = \langle z_{T^* f}, x \rangle$$

but $T^* f = f \circ T$, so by Riesz again

$$f(Tx) = \langle u_f, Tx \rangle$$

for some $u_f \in \mathcal{H}$. So $z_{T^* f} = T^* u_f$.

Consider $(\mathcal{H}_i, \langle \cdot, \cdot \rangle_i)$, $i = 1, 2$ Hilbert spaces, $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let $f \in \mathcal{H}_2^*$, define $T^* \colon \mathcal{H}_2^* \to \mathcal{H}_1^*$

$$T^* f = f \circ T.$$

By Riesz representation theorem $\mathcal{H}_i^* \cong \mathcal{H}_i$, so we can view T^* as $\mathcal{H}_2 \to \mathcal{H}_1$,

$$\langle u_f, Tx\rangle_2 = \langle z_{T^*f}, x\rangle_1$$

$u_f \in \mathcal{H}_2$, $z_{T^*f} \in \mathcal{H}_1$.

Now write

$$\langle z_{T^*f}, x\rangle_1 = \langle T^* u_f, x\rangle_1$$

so $T^* \colon \mathcal{H}_2 \to \mathcal{H}_1$.

Sesquilinear forms and operators

Consider forms

$$h \colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{K},\ (x, y) \mapsto \langle x, Ty\rangle_1$$

or

$$k \colon \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K},\ (x, y) \mapsto \langle x, Ty\rangle_1$$

or

$$k \colon \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K},\ (x, y) \mapsto \langle Sx, y\rangle_2$$

These are sesquilinear forms.

Definition 26. *A sesquilinear form on a normed space is bounded iff*

$$\|h(\cdot,\cdot)\| = \|h\| \coloneqq \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} < \infty.$$

Theorem 23. *Let $(\mathcal{H}_i, \langle\cdot,\cdot\rangle_i)$ $i = 1, 2$ be Hilbert spaces. Suppose h is a bounded sesquilinear form*

$$h \colon \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K}.$$

Then h has representation

$$h(x, y) = \langle Sx, y\rangle_2$$

where $S \colon \mathcal{H}_1 \to \mathcal{H}_2$ bounded linear operator such that

$$\|S\| = \|h\|.$$

Proof. $\forall x \in \mathcal{H}_1$ $h(x,\cdot)$ is a bounded linear functional on $\mathcal{H}_2$. Hence by Riesz representation theorem $\exists! z_x \in \mathcal{H}_2$ such that $h(x,y) = \langle z_x, y\rangle_2$. Let $S\colon \mathcal{H}_1 \to \mathcal{H}_2$, $S(x) = z_x$. This is linear. To prove that $\|S\| = \|h\|$ we note

$$\begin{aligned}\|h\| &= \sup_{\substack{(x,y)\in\mathcal{H}_1\times\mathcal{H}_2\\(x,y)\neq(0,0)}} \frac{|h(x,y)|}{\|x\|_1\|y\|_2}\\ &= \sup \frac{|\langle Sx, y\rangle_2|}{\|x\|_1\|y\|_2}\\ &\geq \sup_{x\neq 0}\frac{|\langle Sx, Sx\rangle_2|}{\|x\|_1\|Sx\|_2}\\ &= \sup_{x\neq 0}\frac{\|Sx\|_2}{\|x\|_1} = \|S\|\end{aligned}$$

On the otherhand

$$\|h\| = \sup\frac{|\langle Sx, y\rangle_2|}{\|x\|_1\|y\|_2} \leq \sup\frac{\|Sx\|_2\|y\|_2}{\|x\|_1\|y\|_2} = \|S\|$$

Considering the inequalities above we have that $\|h\| = \|S\|$. □

Example 41. *Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Let $T\colon \mathcal{H} \to \mathcal{H}$, $u = \sum_i u_i e_i$, then*

$$Tu = \sum_i u_i T e_i.$$

Now write

$$Te_i = \sum_j T_{ij} e_j.$$

So that

$$\begin{aligned}Tu &= \sum_j (\sum_i u_i T_{ij}) e_j\\ \Rightarrow &\langle u, Tu\rangle = \langle T^* u, u\rangle\end{aligned}$$

where T^* is the conjugate of T.

Example 42. *Consider* $\nabla\colon D(\nabla) \to L_2(\lambda)$ *with* $D(\nabla) \subset L_2(\lambda)$.

(i) $D(\nabla) = C_0^\infty$ *smooth, compactly supported functions* $\langle g, \nabla\phi\rangle = \langle \nabla^* g, \phi\rangle$ *if a function on r.h.s. is locally integrable and* $D(\nabla^*) \supset C_0^\infty$. *Integrating by parts we have* $\int g\nabla\phi d\lambda = \int -\nabla g\phi d\lambda$.

(ii)

$$W_{1,2} \equiv \{f : \int (1+k^2)\left|\hat{f}(k)\right|^2 \lambda(dk) < \infty\}.$$

$$\int |\nabla f|^2 d\lambda = \int k^2 \left|\hat{f}(k)\right|^2 \lambda(dk)$$

END Lecture 11

Functional Analysis: Lecture 12

FANA 2020/21

Hahn-Banach Theorem

Classical Extension Theorem

Theorem 24 (Brower-Lebesgue-Tietze-Urysohn)**.**
If $f\colon A \to \mathbb{R}$ is continuous, for A closed in a metric space X, then $\exists \tilde{f}\colon X \to \mathbb{R}$ continuous such that $\tilde{f}|_A = f$ and $\left\|\tilde{f}\right\| = \|f\|$.

Remark 29 (Urysohn Lemma)**.** *For A,B closed with $A \cap B = \emptyset$, there exists a continuous function such that*

$$h(x) = \begin{cases} 1, & x \in A \\ 0, & x \in B \end{cases}$$

Example:

$$\tilde{h}(x) = \frac{d(x,B)}{d(x,B) + d(x,A)}.$$

Lemma 4. *Suppose $\|f\| \leq C \in (0,\infty)$. Then $\exists g$ such that $\|g\| \leq \frac{1}{3}C$ and $\|f - g\| \leq \frac{2}{3}C$.*

Proof. Let $A = f^{-1}([-C, -\frac{1}{3}C])$, $B = f^{-1}([\frac{1}{3}C, C])$. Apply inductively the same idea to

$$\begin{aligned}
&\|f - g_1\| \leq \frac{2}{3}C =: C_1, \\
&\|f - g_1 - g_2\| \leq \frac{2}{3}C_1 = \left(\frac{2}{3}\right)^2 C, \\
&\vdots \\
&\left\|f - \sum_i^n g_i\right\| \leq \left(\frac{2}{3}\right)^n C.
\end{aligned}$$

□

Prf CND. Taking $n \to \infty$, we have

$$\left\| f - \sum_{i}^{\infty} g_i \right\| = 0 \text{ on } A.$$

Then $\tilde{f} = \sum_i^\infty g_i$ is the required Tietze-Urysohn function. $\square$

Extension problem in metric linear spaces.

Definition 27. *A functional $p\colon X \to \mathbb{R}$ is called sublinear iff*

- $p(x+y) \le p(x) + p(y)$,
- $\forall \alpha \ge 0 \quad p(\alpha x) = \alpha p(x)$.

Example 43. *The following are examples of sublinear functionals.*

- $p(x) = \|x\|$,
- $p(x) = |l(x)|$ *with l a linear functional,*
- $p(x) = \sup\limits_{l \in Y} |l(x)|$ *with $Y \subset X^*$.*

Theorem 25 (Hahn-Banach)**.** *Let X be a normed space and Z a proper subspace of X.*
Let p be a sublinear functional.
Let $f\colon Z \to \mathbb{R}$ be a linear functional s.t.

$$f(x) \le p(x), \ \forall x \in Z.$$

Then $\exists \tilde{f}\colon X \to \mathbb{R}$ linear functional such that

$$\begin{aligned} & \tilde{f}|_Z = f, \\ \forall x \in X \qquad & \tilde{f}(x) \le p(x). \end{aligned}$$

Remark 30. *Hahn-Banach Thm history*
https://en.wikipedia.org/wiki/Hahn%E2%80%93Banach_theorem
Hans Hahn (mathematician) (1879–1934), Austrian mathematician https://en.wikipedia.org/wiki/Hans_Hahn_(mathematician)

Proof. Let $Z \subset X$ be a proper subspace, $v \in X \setminus Z$. Consider $W = \operatorname{span}(Z, v)$. Define $g\colon W \to \mathbb{R}$

$$g(z + \lambda v) = f(z) + \lambda\alpha.$$

with some $\alpha \in \mathbb{R}$. We want that $g(z + \lambda v) \le p(z + \lambda v)$, i.e.

$$\forall z \in Z \qquad \lambda\alpha \le p(z + \lambda v) - f(z).$$

Case $\lambda > 0$:

$$\alpha \le p(\frac{1}{\lambda}z + v) - f(\frac{1}{\lambda}z).$$

Case $\lambda < 0$:

$$-\alpha \le p(\frac{1}{-\lambda}z - v) - f(\frac{1}{-\lambda}z).$$

i.e.

$$-p(\frac{1}{-\lambda}z - v) + f(\frac{1}{-\lambda}z) \le \alpha$$

Combining, we have with any $\lambda > 0$

$$\forall z \in Z \qquad -p(\frac{1}{\lambda}z - v) + f(\frac{1}{\lambda}z) \le \alpha \le p(\frac{1}{\lambda}z + v) - f(\frac{1}{\lambda}z).$$

i.e.

$$\forall \tilde{z} \in Z \qquad -p(\tilde{z} - v) + f(\tilde{z}) \le \alpha \le p(\tilde{z} + v) - f(\tilde{z}).$$

To determine that both conditions are compatible note that $\forall z, z' \in Z$,

$$\begin{aligned} f(z) - f(z') = f(z - z') &\le p(z - z') \\ &= p(z + v - (z' + v)) \\ &\le p(z + v) + p(-z' - v). \end{aligned}$$

Rearranging,

$$-p(-z' - v)) + f(-z') \le p(z + v) - f(z),$$

since LHS independent of z, RHS independant of z',

$$\sup_{z'\in Z}(-p(-z'-v)) + f(-z')) \leq \inf_{z\in Z}(p(z+v) - f(z)).$$

Hence we may choose α such that

$$\sup_{z'\in Z}(-p(z'-v) + f(z')) \leq \alpha \leq \inf_{z\in Z}(p(z+v) - f(z)).$$

Then the extension $g\colon W \to \mathbb{R}$ satisfies on W

$$g|_Z = f \qquad \text{and} \qquad g \leq p.$$

To show that an extension to f exists we need Zorn's lemma.

Definition 28. *A set S with a partial order $\lesssim$:*

- $\forall a \in S \quad a \lesssim a$
- $a \lesssim b,\ b \lesssim a \Rightarrow\ a = b$
- $a \lesssim b,\ b \lesssim c \Rightarrow\ a \lesssim c$

is called a partially ordered set $(S, \lesssim)$.
A chain in $(S, \lesssim)$ is a set $C \subset S$ which totally ordered i.e. $\forall c_1, c_2 \in C$

$$c_1 \lesssim c_2 \text{ or } c_2 \lesssim c_1.$$

An element $\bar{c}$ s.t. $\forall c \in C,\ c \lesssim \bar{c}$ is called an upper bound.
An element m is called maximal iff $\exists c \in S\ m \lesssim c \Rightarrow m = c$.

Lemma 5 (Zorn's lemma)**.** *Let $(S, \lesssim)$ be a partially ordered set for which every totally ordered subset has an upperbound. Then $(S, \lesssim)$ contains at least one maximal element.*

We will say that two extensions (g_1, M_1), (g_2, M_2) are in relation $\lesssim$ iff

$$M_1 \subset M_2 \ \text{ and } \ g_2|_{M_1} = g_1.$$

Consider the chain $(g_\gamma, M_\gamma)_{\gamma\in T}$ where T is an index set. Let $M = \cup_{\gamma\in T} M_\gamma$, and define $G\colon M \to \mathbb{R}$ by

$$G(x) = g_\gamma(x) \text{ if } x \in M_\gamma.$$

We have $G|_{M_\gamma} = g_\gamma$, so G is an upper bound for a given chain.

Hence by Zorn's lemma $\exists$ maximal element $g\colon M \to \mathbb{R}$ such that $g|_Z = f$ and $g \le p$ on M.

Note that if $X \backslash M \neq \emptyset$, then $\exists v \in X \backslash M$ and so we can use the first part of our proof to obtain an extension of g, which contradicts the maximality of g. □

Some consequences of Hahn-Banach Theorem

Tangent functional

Corollary 1.

$$\forall x \in X \quad \exists l \equiv l_x \in X^* \qquad \|l\| = 1$$

and

$$l(x) = \|x\|$$

Proof. Consider $Z_x = \{\lambda x\colon \lambda \in \mathbb{R}\}$ and

$$f(\lambda x) = \lambda \|x\|.$$

Apply Hahn-Banach. □

***Exercise* 22.** *Find l_f for $f \in l_p, L_p(\mu)$*

Corollary 2. *Suppose $(X, \|\cdot\|)$ normed space*
$\forall x \in X$

$$\|x\| = \sup\{|l(x)|\colon l \in X^* \ \|l\| = 1\}$$

Proof. We have

$$|l(x)| \le \|l\|\|x\| = \|x\|$$

and by previous corollary

$$\exists l \in X^* \qquad l(x) = \|x\|.$$

□

Banach Limit

Let $(l_p)^*$, $p \in [1, \infty]$.
We had $(l_1)^* = l_\infty$, but $(l_\infty)^* \supset l_1$.
In fact there is no equality in this case.

Theorem 26 (Banach Limit theorem). *$\exists L \in (l_\infty)^*$ such that*

- $\|L\| = 1$
- *If $x \in c$, then $L(x) = \lim_n x_n$.*
- *If $x \in l_\infty$ with $\forall i \quad x_i \geq 0$, then $L(x) \geq 0$.*
- *If $x \in l_\infty$, and $x' = (x_{i+1})_{i \in \mathbb{N}}$, then $L(x) = L(x')$.*

Remark 31. ***Relation to LLN***

$$S(x) \equiv \frac{1}{n} \sum_{j=1,..,n} x_j$$

Müntz - Szäsz Theorem

Theorem 27 (Müntz - Szäsz). *Suppose $0 < \lambda_1 < \lambda_2 < \ldots$, and let X be the closure in $C[0,1]$
of the set of finite linear combinations of t^{λ_j}, $j \in \mathbb{N}$.
Then*

- *If $\sum_j \frac{1}{\lambda_j} = \infty$, then $X = C[0,1]$.*
-
- *If $\sum_j \frac{1}{\lambda_j} < \infty$, then for $\lambda \notin \{\lambda_j\}_j$, $t^\lambda \notin X$ and so $X \neq C[0,1]$.*

Remark 32. *Since $\sum_{p \ prime} p^{-1} = \infty$, we can choose t^p, p prime.*

Figure 11: **H. Muntz**

[Mathematicians]

[Mathematicians]

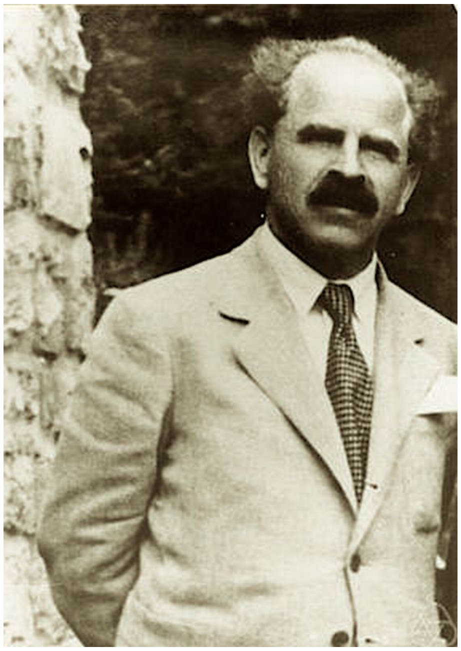


Figure 12: **Otto Szasz**

https://en.wikipedia.org/wiki/Herman_M{\unhbox\voidb@x\bgroup\accent127u\egroup}ntz

https://en.wikipedia.org/wiki/Otto_Sz/Otto_Sz{\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{a\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\relax\let\ignorespaces\relax\accent19a\egroup\spacefactor\accent@spacefactor}sz

https://en.wikipedia.org/wiki/Hahn-Banach_theorem#Partial_differential_equations

END Lecture 12

Functional Analysis: Lecture 13

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Hahn-Banach Theorem Applications CND

Theorem 28.

$$C[0,1]^* = \{F_\rho \colon F_\rho(u) = S - \int_a^b u(x)d\rho(x), \forall u \in C[0,1]$$
$$\textit{where } \rho \colon [0,1] \to \mathbb{R} \textit{ has bounded variation}\}.$$

Then

$$\|F_\rho\| = V(\rho) = \textit{ total variation of } \rho.$$

where

$$V(\rho) = \sup_{\{x_i\}} \sum |\rho(x_{i+1}) - \rho(x_i)|.$$

Functions of bounded variation and Stielties Integral

Stielties Integral : Same idea as of Riemann Integral, but replace $x_{i+1} - x_i$ with $|\omega(x_{i+1}) - \omega(x_i)|$.

Definition 29. ***Total Variation of a function*** ω

$$V(\omega) \equiv \sup_{\substack{\max\{x_{i+1}-x_i\} \\ n\to\infty}} \sum_i^n |\omega(x_{i+1}) - \omega(x_i)|$$

Remark 33 (**BV space of functions**). *https://en.wikipedia.org/wiki/Bounded_variation*
https://en.wikipedia.org/wiki/Total_variation

If $V(\omega) < \infty$ we consider lower sum

$$s_{n,\omega} \equiv \sum_{i=1,..,n-1} f(\underline{x}_i)(\omega(x_{i+1}) - \omega(x_i))$$

defined with $f(\underline{x}_i) = \min_{x \in [x_i, x_{i+1}]} f(x)$ and upper sum

$$S_{n,\omega} \equiv \sum_{i=1,..,n-1} f(\bar{x}_i)(\omega(x_{i+1}) - \omega(x_i))$$

defined with $f(\underline{x}_i) = \max_{x \in [x_i, x_{i+1}]} f(x)$
We have

$$|s_{n,\omega}|, |S_{n,\omega}| \leq \|f\|_\infty V(\omega).$$

If

$$\inf S_{n,\omega} = \sup s_{n,\omega} \text{ as } \max |x_{i+1} - x_i| \to 0$$

then we define Stielties integral of f with respect to $d\omega$ by

$$S - \int_a^b f d\omega := \inf S_{n,w}.$$

Remark 34. *If ω is differentiable then $S - \int f d\omega = R - \int f \omega' dx$.*

Remark 35. *Consider the Cantor function η, which is monotone increasing, continuous and $V(\eta) = 1$. We have*

$$S - \int_C f d\eta \neq L - \int f \rho d\lambda$$

for any finite ρ since C has measure 0 w.r.t. λ.
https://en.wikipedia.org/wiki/Cantor_function

https://en.wikipedia.org/wiki/Riemann-Stieltjes_integral https://en.wikipedia.org/wiki/Lebesgue-Stieltjes_integration

Theorem 29. *Every $l \in (C[a,b])^*$ can be represented as*

$$l(f) = l_\omega(f) \equiv S - \int f d\omega$$

for some ω with $V(\omega) < \infty$ and $\|l_\omega\| = V(\omega)$

Proof. Using Hahn-Banach theorem, we extend linear functionals on $C[a,b]$ to $B[a,b]$ (bounded functions), with same norm. By H-B, $\forall l \in (C[a,b])^*$ $\exists \tilde{l} \in (B[a,b])^*$ with

$$\|l\| = \left\|\tilde{l}\right\|$$
$$\tilde{l}|_{C[a,b]} = l.$$

In $B[a,b] \ni \eta_t := \chi_{[a,t]}$ and for $\tilde{l} \in B[a,b]$ we define

$$\omega(t) \equiv \omega_t := \tilde{l}(\eta_t)$$

First we observe that ω_t is of bounded variation as follows

$$\begin{aligned}
\sum_{i=1,..,n} |\omega(x_{i+1}) - \omega(x_i)| &= \sum_i^n \left|\tilde{l}(\eta_{x_{i+1}}) - \tilde{l}(\eta_{x_i})\right| \\
&= \sum_{i=1,..,n} \left|\tilde{l}(\eta_{x_{i+1}} - \eta_{x_i})\right| \\
&= \sum_{i=1,..,n} \varepsilon_i \tilde{l}(\eta_{x_{i+1}} - \eta_{x_i}), \text{ with } \varepsilon_i \in \{-1,1\} \\
&= \tilde{l}(\sum_{i=1,..,n} \varepsilon_i(\eta_{x_{i+1}} - \eta_{x_i})), \\
&= \left\|\tilde{l}\right\| \left\| \sum_{i=1,..,n} \varepsilon_i(\eta_{x_{i+1}} - \eta_{x_i}) \right\|_\infty, \\
&= \left\|\tilde{l}\right\| \left\| \sum_{i=1,..,n} \varepsilon_i(\chi_{[x_i,x_{i+1}]}) \right\|_\infty,
\end{aligned}$$

- Hence $V(\omega) \le \left\|\tilde{l}\right\| = \|l\|$.

For $f \in C[a,b]$ and a partition $(x_i)_{i=1,..,n}$ of $[a,b]$, define

$$g_n = \sum_i^n f(x_i)(\eta_{x_{i+1}} - \eta_{x_i}).$$

If f is continuous and $\sup |x_{i+1} - x_i| \to 0$, then

$$\|g_n - f\|_\infty \to 0 \; n \to \infty$$

Now

$$\begin{aligned}
\tilde{l}(g_n) &= \sum_i^n f(x_i)(\omega_{x_{i+1}} - \omega_{x_i}) \\
&\to S - \int_a^b f d\omega \qquad \text{as } n \to \infty \text{ and } |x_{i+1} - x_i| \to 0.
\end{aligned}$$

Next we note

$$\left|\tilde{l}(g_n - f)\right| \le \left\|\tilde{l}\right\| \|g_n - f\|_\infty \to 0 \text{ as } n \to \infty$$

So $\lim_{n\to\infty} \tilde{l}(g_n) = \tilde{l}(f) = l(f)$. Also

$$\left|\tilde{l}(g_n)\right| \leq V(\omega)\|f\|_\infty$$

So using the above limit

$$\left\|\tilde{l}\right\| = \|l\| \leq V(\omega).$$

• Hence $\|l\| = V(\omega)$. □

Other consequences of H-B theorem

Nice book :Sergei Ovchinnikov , "Functional Analysis"

Theorem 30 (Analog of Urysohn Lemma)**.** *Let Y be a closed proper subspace of a normed space X. Then there exists $f \in X^*$ such that $\|f\| = 1$ and $\forall y \in Y$ $f(y) = 0$.*

Proof: For $x_0 \in X \setminus Y$, let $W = Span\{Y, x_0\}$. Define a functional $g : W \to \mathbb{K}$ as follows : $\forall y \in Y$ and $t \in \mathbb{K}$

$$g(y + tx_0) := t.$$

We will show that the linear functional g is bounded on W. Because $X \setminus Y$ is an open set, there exists $r > 0$ such that $\|x - x_0\| \geq r$ for all $x \in Y$ and therefore, on W we have

$$\|y + tx_0\| = |t| \; \|(-1/t)y = x_0\| \geq |t|r.$$

From this, on W we have

$$|g(y + tx_0)| = |t| \leq \frac{1}{r}\|y + tx_0\|.$$

Hence $\|g\| \leq \frac{1}{r}$, so it is a bounded linear functional on W which by its definition vanishes on Y.
Hahn - Banach theorem implies that g has an extension to a bounded linear functional g on X such that $g(x) = 0$ on Y. Because $g(x_0) = 1$, the norm $\|g\|$ is nonzero. We obtain the desired result by setting $f = \frac{g}{\|g\|}$. □

Theorem 31. *If X^* is separable, then X is too.*

Sergei Ovchinnikov,"Functional Analysis"

Proof: Let X^* be separable. Consider the unit sphere

$$S^* \equiv \{\varphi \in X^* : \|\varphi\| = 1\}$$

Then S^* is separable and so there exists a subset $\{\varphi_n : n \in \mathbb{N}\}$ which is countable and dense the unit sphere. As this functionals are normalised, for each $n \in \mathbb{N}$ we can choose x_n with $\|x_n\| = 1$, such that:

$$|\varphi_n(x_n)| > 1/2$$

Let

$$\mathcal{D} \equiv \overline{span\{x_n : n \in \mathbb{N}\}}$$

By definition of $\mathcal{D}$ the following countable set set is dense in it

$$\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1,..,n} (a_j + ib_j)x_j : \; a_j, b_j \in \mathbb{Q} \right\}$$

We need to show that $\mathcal{D} = X$.
Indeed, otherwise by previous theorem, there exists a functional $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(y) = 0$ forall $y \in \mathcal{D}$. We show that in our case this is not possible. To this end we note that for any $\varphi \in S^*$, using the fact that $\{\varphi_n : n \in \mathbb{N}\}$ is dense in S^*, there exists an $n \in \mathbb{N}$ such that:

$$\|\varphi - \varphi_n\| < 1/2$$

Hence, we have

$$\begin{aligned} |\varphi(x_n)| &\geq |\varphi_n(x_n)| - |\varphi(x_n) - \varphi_n(x_n)| > 1/2 - |\varphi(x_n) - \varphi_n(x_n)| \\ &\geq 1/2 - \|\varphi - \varphi_n\| \cdot \|x_n\| = 1/2 - \|\varphi - \varphi_n\| > 0 \end{aligned}$$

This implies that φ cannot vanish on $\mathcal{D}$. Thus $\mathcal{D} = X$ and so X is also separable. $\square$

Figure 13: **Thomas Joannes Stieltjes**

[Mathematicians]
https://en.wikipedia.org/wiki/Thomas_Joannes_Stieltjes
Some other Interesting Consequences of H-B:

If you like to explore

see e.g. "BD MacCluer "Elementary Functional Analysis"
S. Kesavan "Functional Analysis"
Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, Václav Zizler, "Banach Space Theory"

Reflexive spaces

Reflexive space $X^{**} = X$.

If X is a finite dimensional, normed linear space then X is reflexive.

If $\mathcal{H}$ is a Hilbert space then $\mathcal{H}$ is reflexive.

If $1 < p < \infty$ then l_p is reflexive.

The spaces $L_p(\mathbb{R})$, $1 < p < \infty$, are also reflexive.

The spaces c_0 and l_∞ are not reflexive.

l_1 is not reflexive.

The space $C([0,1])$ is separable, but $C([0,1])^*$ is not separable.

A Banach space X is reflexive if and only if X^* is reflexive.

If X is reflexive and Y is a closed subspace of X, then Y is reflexive.

Suppose that X is a reflexive Banach space. Given $\varphi \in X^*$, there exists a unit vector $x_0 \in X$ such that $|\varphi(x_0)| = \|\varphi\|$.

Suppose $X \neq \{0\}$ is a normed linear space. Given $x_1 \neq x_2$ we may find a bounded linear functional φ on X with $\varphi(x_1) \neq \varphi(x_2)$.

Geometric Aspects
Separation Theorem
Suppose that X is a real normed space, $A \subset X$ is a non-empty, open, convex set and b lies on the boundary of A. Then there exists $h \in X^*$ such that $h(a) < h(b)$, for all $a \in A$.

If $Y \subset X$ is a proper subspace, the $\exists l \in X^*$ with $l(Y) = \{0\}$ and for $x \in X \backslash Y$, $l(x) = \operatorname{dist}(x, Y)$.

Let V and W be disjoint open convex subsets of a topological vector space X. Then there is a hyperplane H in X that separates V and W.

Suppose that X is a normed linear space, $x_0 \in X$ and M is a (not necessarily closed) subspace in X. Suppose that $d \equiv dist(x_0, M) > 0$ where $dist(x_0, M) = \inf\{\|x_0 - y\| : y \in M\}$.There exists $\varphi \in X^*$ with $\varphi(x_0) = 1$, $\varphi = 0$ on M, and $\|\varphi\| = 1/d$. In particular, x_0 is in the closure of M iff there is no bounded linear functional on X that is 0 on M and nonzero at x_0.

Functional Analysis: Lecture 14

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Baire's Category theorem

Definition 30 (Baire's Categories)**.** *Let (X, ρ) be a metric space, then a subspace $M \subset X$ is said to be*

(i) a nowhere dense set, iff $int\overline{M} = \emptyset$,

(ii) of the 1^{st} Category (or meager), iff $M = \cup_{n\in\mathbb{N}} M_n$, where M_n are nowhere dense,

(iii) of the 2^{nd} Category (or nonmeager) iff not of 1^{st} category.

Example 44. *Consider $[0,1]$ with metric $|\cdot - \cdot|$,*

(i) Cantor ternary set C_1 is nowhere dense.

(ii) $\mathbb{Q}$ is of the 1^{st} category, since $\mathbb{Q} = \cup_{n\in\mathbb{N}}\{q_n\}$ with $\{q_n\}$ nowhere dense, but

$$\overline{\mathbb{Q}} = [0,1];$$

so $\mathbb{Q}$ is not nowhere dense.

Remark 36. *Lebesgue measure zero and meager sets.*
John C. Oxtoby, Measure and Category

Example 45. *For any given $j \in \mathbb{N}$, let*

$$H_j = \{x \in l_p \colon x_j = 0\}.$$

Then $intH_j = \emptyset$ and $\overline{\cup_j H_j} \subset l_p$,

but

$$\overline{Span \cup_j H_j} = l_p.$$

Theorem 32 (Baire's Category theorem for Complete Metric Spaces)**.** *If a metric space $X \neq \emptyset$ is complete, then it is not of 1^{st} category.*

Hence if $X \neq \emptyset$ is complete and $X = \cup_{n\in\mathbb{N}} A_n$ for A_n closed,

then at least one of A_n contains a non-empty open subset.

Proof. Suppose that $X \neq \emptyset$ is a complete metric space and

$$X = \cup_{i\in\mathbb{N}} A_i,$$

with A_i a nowhere dense set.

We will construct a Cauchy sequence in X which is not convergent in this space which contradicts the completeness of X.

By definition $\overline{A_i}$ does not contain a nonempty open set.

Hence $X \backslash \overline{A_i} \neq \emptyset$ and is open.

Let $p_1 \in \overline{A_1}^c$ and let $B_{\varepsilon_1}(p_1) \subset \overline{A_1}^c$ with $0 < \varepsilon_1 < \frac{1}{2}$.

Since A_2 is nowhere dense, $\overline{A_2}^c \cap B_{\varepsilon_1}(p_1) \neq \emptyset$.

Thus $\exists B_{\varepsilon_2}(p_2) \subset \overline{A_2}^c \cap B_{\varepsilon_1}(p_1)$ with $p_2 \in B_{\varepsilon_1}(p_1)$, $0 < \varepsilon_2 < \frac{1}{4}$.

By induction, we obtain a sequence $(p_k)_{k\in\mathbb{N}}$, $p_k \notin \cup_{j\in\mathbb{N}}^k A_j$ and $B_{\varepsilon_k}(p_k)$, $0 < \varepsilon_k < 2^{-k}$, i.e.

$$\forall k \qquad B_{\varepsilon_{k+1}}(p_{k+1}) \subset B_{\frac{\varepsilon_k}{2}}(p_k).$$

Now $(p_k)_{k\in\mathbb{N}}$ is Cauchy sequence by construction.

Since X is complete and non-empty, $\exists p \in X$ such that $p_k \to p$.

Note that $\forall m, n$ with $n > m$ we have,

$$B_{\varepsilon_n}(p_n) \subset B_{\frac{\varepsilon_m}{2}}(p_m),$$

so

$$\begin{aligned}\forall m \quad d(p, p_m) \leq d(p_m, p_n) + d(p_n, p), &\leq \frac{1}{2}\varepsilon_m + d(p_n, p),\\ &\to_{n\to\infty} \frac{1}{2}\varepsilon_m,\end{aligned}$$

i.e.

$$\forall m \qquad p \in B_{\varepsilon_m}(p_m).$$

Since $B_{\varepsilon_m}(p_m) \subset \overline{A_m}^c$, we get $\forall j \quad p \notin A_j$.

Hence $p \notin X$, which contradicts the completeness of X. $\square$

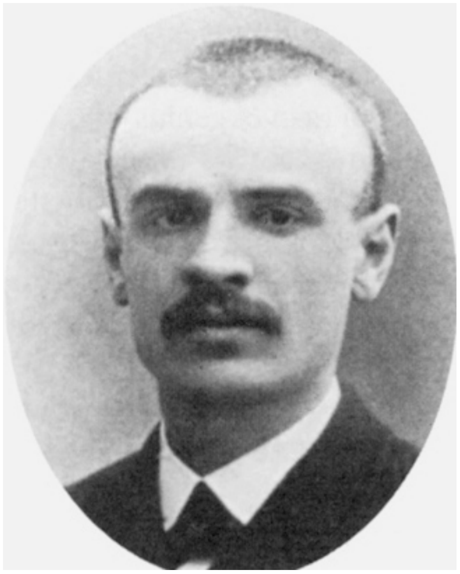

Figure 14: **René-Louis Baire**

[Mathematicians]

https://en.wikipedia.org/wiki/Rene-Louis_Baire
https://mathshistory.st-andrews.ac.uk/Biographies/Baire/
https://en.wikipedia.org/wiki/Baire_category_theorem

Functional Analysis: Lecture 15

FANA 2020/21

Banach-Steinhaus Uniform Boundedness Principle

Theorem 33 (Banach-Steinhaus)**.** *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators from a Banach space X to a normed space Y such that*
$\forall x \in X\ \exists c_x \in (0,\infty)$

$$\sup_{n\in\mathbb{N}} \|T_n x\| \leq c_x.$$

Then $\exists c \in (0,\infty)$ such that

$$\sup_{n\in\mathbb{N}} \|T_n\| \leq c.$$

Remark 37. *https://en.wikipedia.org/wiki/Uniform_boundedness_principle*

Proof. For $k \in \mathbb{N}$, define

$$A_k \coloneqq \{x \in X : \forall n \in \mathbb{N} \qquad \|T_n x\| \leq k\}.$$

Each A_k is closed and $X = \cup_k^\infty A_k$.

By Baire's category theorem, $\exists k_0 \in \mathbb{N}$ such that $\exists x_0 \in X$, $\exists r > 0$,

$$B(x_0, r) \subset A_{k_0}.$$

For any $x \in X$, $x \neq 0$, with $\gamma = \frac{r}{2\|x\|}$, let

$$z = x_0 + \gamma x.$$

Then

$$\|z - x_0\| = |\gamma| \|x\| = \frac{r}{2}$$

so $z \in B(x_0, r) \subset A_{k_0}$.

Hence

$$\begin{aligned}\|T_n x\| &= \left\| T_n \frac{1}{\gamma}(z - x_0) \right\| \\ &= \frac{1}{\gamma}\|T_n(z - x_0)\| \\ &\leq \frac{1}{\gamma}(\|T_n z\| + \|T_n x_0)\|) \\ &\leq \frac{1}{\gamma} \cdot 2k_0 = \frac{4k_0}{r}\|x\|.\end{aligned}$$

where in the last step we have used our definition $\gamma = \frac{r}{2\|x\|}$. Thus for any $n \in \mathcal{N}$, we have

$$\|T_n\| = \sup_{x \neq 0} \frac{\|T_n x\|}{\|x\|} \leq \frac{4k_0}{r}.$$

□

Some Applications of B-S Principle

Example 46. *Consider the following normed space of polynomials*

- $x(t) = \sum_j^\infty a_j t^j$ *with* $a_j = 0$ $\forall j > N$ *for some* N

- $\|x\| = \max_i |a_i|$.

Claim: The normed space of polynomials above is not complete.

Proof. Let $T_n(0) = 0$, $T_n(x) = \sum_{i=1}^n a_i$. Then we have

$$|T_n x| \leq (N_x + 1) \max |a_i| = (N_x + 1)\|x\|.$$

where N_x is the degree of the polynomial x. Let $c_x \coloneqq (N_x + 1)\|x\|$. Then $\forall n$

$$\|T_n x\| \leq c_x.$$

Let $x(t) = \sum_{k=0}^m t^k$, then $\|x\| = 1$, and

$$T_n x = \sum_{k=0}^n 1 = \min(m, n) + 1.$$

In particular, taking $m = n$,

$$T_n x = n + 1 \to \infty$$

So

$$\sup_n |T_n x| = \infty.$$

Hence the space is not complete by the Banach-Steinhaus theorem.

$\square$

Application to Fourier Series

Existence of continuous functions for which the corresponding Fourier series diverges at a given point.

For $u \in \mathcal{C}([0, 2\pi], \mathbb{R})$ its Fourier coefficients are defined by

$$a_m \equiv \frac{1}{\pi} \int_0^{2\pi} u(t) \cos(mt) dt$$
$$b_m \equiv \frac{1}{\pi} \int_0^{2\pi} u(t) sin(mt) dt$$

What are the convergence properties of the corresponding Fourier series ?

$$\mathfrak{F}_u \equiv \sum_m (a_m \cos(mt) + b_m sin(mt))$$

For $u \in \mathcal{C}([0, 2\pi], \mathbb{R})$, let

$$T_n u = \frac{1}{2} a_0 + \sum_{m=1}^{n} a_m \equiv \text{Fourier series at } t = 0 \text{ resumed only up to } n$$
$$= \frac{1}{\pi} \int_0^{2\pi} u(s) \left[\frac{1}{2} + \sum_{m=1}^{n} \cos(ms) \right] ds$$

Using trigonometric identities we have

$$\begin{aligned}
2\sin\left(\frac{1}{2}s\right)\left(\sum_{m=1}^{n}\cos(ms)\right) &= \sum_{m=1}^{n}\left(\sin\left((\frac{1}{2}+m)s\right)+\sin\left((\frac{1}{2}-m)s\right)\right) \\
&= \sum_{m=1}^{n}\left(\sin\left((\frac{1}{2}+m)s\right)-\sin\left((m-\frac{1}{2})s\right)\right) \\
&= -\sin\left(\frac{1}{2}s\right)+\sin\left((n+\frac{1}{2})s\right)
\end{aligned}$$

and hence

$$\frac{1}{2}+\sum_{m=1}^{n}\cos(ms)=\frac{1}{2}\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\left(\frac{1}{2}s\right)}\equiv\frac{1}{2}q_n(t).$$

Thus

$$T_n u=\frac{1}{2\pi}\int_0^{2\pi}u(s)q_n(s)ds,$$

and

$$|T_n u|\le\frac{1}{2\pi}\|u\|\int_0^{2\pi}|q_n(s)|ds.$$

One can show that in fact

$$\begin{aligned}
\|T_n\| &= \frac{1}{2\pi}\int_0^{2\pi}|q_n(s)|dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\left(\frac{1}{2}s\right)}\right|ds, \\
&> \frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)s}{s}\right|ds, \\
&= \frac{1}{\pi}\int_0^{2(n+\frac{1}{2})\pi}\left|\frac{\sin(v)}{v}\right|dv.
\end{aligned}$$

Splitting into intervals,

$$\begin{aligned}
\frac{1}{\pi}\sum_{k=0}^{2n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(v)|}{v}dv &\ge \frac{1}{\pi}\sum_{k=0}^{2n}\frac{1}{(k+1)}\int_0^{\pi}|\sin(v)|dv \\
&\to\infty \text{ as } n\to\infty.
\end{aligned}$$

We have shown that

$$\|T_n\| \underset{n\to\infty}{\to} \infty \ .$$

Hence by B-S theorem the family T_n, $n \in \mathbb{N}$, is not pointwise uniformly bounded. Recalling that $T_n(u)$ = value of Fourier series up to n at $t = 0$ we conclude that there exists a continuous function which Fourier series diverges at $t = 0$.
Similar result can be shown for any other $t \in [0, 2\pi]$.

Linear Functionals on l_p Again

Let $p, q \in (1, \infty)$ be conjugate exponents. Suppose that $a \equiv (a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers such that

$$|\sum_{k=1}^{\infty} a_k x_k| < \infty \qquad (*)$$

for every $x \equiv (x_k)_{k\in\mathbb{N}} \in l_p$. We show that then

$$\sum_{k=1}^{\infty} |a_k|^q < \infty$$

i.e. $a \in l_q$.
We may assume that $a = (a_n)_{n\in\mathbb{N}}$ has no zero terms. Define

$$f_n(x) = \sum_{k=1}^{n} a_k x_k.$$

By this definition each f_n is a linear functional on l_p and for any a given $x = (x_k)_{k\in\mathbb{N}} \in l_p$,

$$\sup_n \|f_n(x)\| = \sup_n \left|\sum_{k=1}^{n} a_k x_k\right| < \infty,$$

because by our assumption (*). Therefore, the family $\{f_n\}_{n\in\mathbb{N}}$ is pointwise bounded.
By Hölder inequality, we have

$$|f_n(x)| \leq \left(\sum_{k=1}^{n} |a_n|^q\right)^{\frac{1}{q}} \ \|x\|_p.$$

Hence

$$\|f_n\| \leq \left(\sum_{k=1}^{n} |a_k|^q\right)^{\frac{1}{q}}.$$

On the other hand, for

$$x_k := \begin{cases} \frac{\bar{a_k}|a_k|^{q-2}}{\left(\sum_{k=1}^{n}|a_k|^q\right)^{1-\frac{1}{q}}}, & k = 1, .., n \\ 0 & otherwise, \end{cases}$$

we have

$$|f_n(x)| = \left(\sum_{k=1}^{n} |a_k|^q\right)^{\frac{1}{q}}.$$

Hence $\|f_n\| = (\sum_{k=1}^{n}|a_k|^q)^{\frac{1}{q}}$ for $n \in \mathbb{N}$. By the B-S uniform boundedness theorem, the series

$$\left(\sum_{k=1}^{\infty} |a_k|^q\right)^{\frac{1}{q}}.$$

converges.

Figure 15: **Hugo Steinhaus**

[Mathematician]

https://en.wikipedia.org/wiki/Hugo_Steinhaus

Functional Analysis: Lecture 16

FANA 2020/21

Open Mapping Theorem

Definition 31. *Let X, Y be metric spaces, then $T\colon D(T) \to Y$, $D(T) \subset X$ is called an open mapping if $\forall U \subset D(T)$ open, the image $T(U)$ is open in Y.*

Theorem 34. *A bounded linear operator $T\colon X \to Y$ from a Banach space X onto a Banach space Y is an open mapping.*

,

Remark 38. *https://en.wikipedia.org/wiki/Open_mapping_theorem_%28functional_analysis%29*

Let $B_X(x, r)$ and $B_Y(y, r)$ denotes an open ball with radius $r > 0$ centered at $x \in X$ and $y \in Y$, respectively. For balls centered at the origin we use a simpler notation $B_X(r)$ and $B_Y(r)$, respectively. Since T is linear, it suffices to show that $T(B_X(1))$ contains an open ball centered at the origin.
Using Baire category theorem we prove first a weaker statement that

$\overline{T(B_X(1))}$ contains an open ball centered at the origin.

For this we note that by surjectivity of T we have

$$Y = \bigcup_{n\in\mathbb{N}} T(B_X(n)) = \bigcup_{n\in\mathbb{N}} \overline{T(B_X(n))}$$

and by Baire category theorem $\exists n \in \mathbb{N}$ s.t. $\overline{T(B_X(n))}$ contains an interior point. Hence by linearity of T, we conclude that $\exists y_0 \in Y$ and $\varepsilon > 0$ s.t.

$$B_Y(y_0, \varepsilon)) \subset \overline{T(B_X(1))}.$$

By definition of the closure, we can choose a $y_1 = T(x_1)$ for some $x_1 \in B_X(1)$ and

$$\|y_0 - y_1\|_Y < \frac{\varepsilon}{2}$$

Then for any $y \in B_Y(frac\varepsilon 2)$, we have that

$$y - y_1 \in \overline{T(B_X(1))},$$

and using representation

$$y = T(x_1) + y - y_1,$$

we get that

$$y \in \overline{T(B_X(2))}$$

Therefore

$$B_Y(\frac{\varepsilon}{2}) \subset \overline{T(B_X(2))},$$

and by linearity of T we get

$$B_Y(\frac{\varepsilon}{4}) \subset \overline{T(B_X(1))},$$

which completes the proof of our weaker statement.

To simplify notation, by replacing T by $frac4\varepsilon T$, we can assume that

$$B_Y(1) \subset \overline{T(B_X(1))}$$

and hence we get

$$\forall k \in \mathbb{N} \qquad B_Y(2^{-k}) \subset \overline{T(B_X(2^{-k}))}. \tag{$\circledast$}$$

Second Stage

Now we are proceed to show that in fact

$$B_Y(1/2) \subset T(B_X(1)). \tag{$⋇$}$$

To this end for an arbitrary $y \in B_Y(\frac{1}{2})$, by ($\circledast$) we can select a point $x_1 \in B_X(\frac{1}{2})$ so that

$$y - T(x_1) \in B_Y(1/2^2)$$

Applying ($\circledast$) again with $k = 2$ to the point $y - T(x_1)$ we can find $x_2 \in B_X(1/2^2)$ such that

$$y - T(x_1) - T(x_2) \in B_Y(1/2^3).$$

Proceeding by induction we can choose a sequence of points $(x_k \in B_X(1/2^k))_{k\in\mathbb{N}}$ such that $\exists x \in B_X(1)$

$$\sum_{k=1}^{n} x_k \to_{n\to\infty} x$$

and

$$\forall n \in \mathbb{N} \qquad y - \sum_{k=1}^{n} T(x_k) \in B_X(1/2^{n+1}).$$

Since by our assumption T is continuous we conclude that

$$T(x) = y$$

and so ($\circledast$) is satisfied, i.e. $T(B_X(1))$ contains an open ball centered at the origin. $\square$

Theorem 35.
Let X and Y be Banach spaces. If $T : X \to Y$ is a bounded linear and bijective map, then T^{-1} is continuous and hence bounded.

Proof:

First of all note that T^{-1} exists as a linear map, so only its boundedness remains to be shown.

By the open mapping theorem, T carries open sets to open sets.

Now T^{-1} is bounded iff T^{-1} is continuous, and $T^{-1} : Y \to X$ is continuous iff $(T^{-1})^{-1}(G)$ is open in Y for every $G \subset X$ which is open.
But

$$(T^{-1})^{-1}(G) = T(G)$$

and, by Open Mapping Theorem, $T(G)$ is open in Y for any open set G in X. Thus we conclude that T^{-1} is bounded.

$\square$

Example 47. ***Equivalence of norms***
The norms $\|\cdot\|_1, \|\cdot\|_2$ are equivalent iff

$$x_n \to x \text{ in } \|\cdot\|_1 \Leftrightarrow x_n \to x \text{ in } \|\cdot\|_2.$$

Consider $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ and $id \colon X \to X$. Then id is an open mapping.

Hence exists $C \in (0, \infty)$

$$\frac{1}{C} \|x\|_2 \leq \|x\|_1 \leq C \|x\|_2$$

Functional Analysis: Lecture 17

FANA 2020/21

Closed Graph theorem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces.

Definition 32. *For $D(T) \subset X$, let $T\colon D(T) \to Y$. We define the graph of T*

$$G(T) = \{(x, y) \in X \times Y \colon\ x \in D(T),\ y = Tx\}$$

We can introduce a norm on $G(T)$ as follows

$$\|(x, y)\| = \|x\|_X + \|y\|_Y.$$

Definition 33. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and, for $D(T) \subset X$, $T\colon D(T) \to Y$ be a linear operator.*
T is called closed iff $G(T) \subset X \times Y$ is closed with respect to the norm $\|\cdot\|$.

Theorem 36. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. For $D(T) \subset X$, let $T\colon D(T) \to Y$.*
Then the following conditions are equivalent.

- ⋇ *T is closed*
- ⊛ *If $x_m \in D(T)$ and $x_m \underset{m\to\infty}{\to} x \in X$, and $Tx_m \underset{m\to\infty}{\to} y \in Y$, then $x \in D(T)$ and $Tx = y$.*

Proof. *Exercise*

Theorem 37 (Closed Graph Theorem)**.**
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T\colon D(T) \to Y$, $D(T) \subset X$ be a linear closed operator.

If $D(T)$ is closed in X, then operator T is bounded.

Proof. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, then $X \times Y$ is a Banach space w.r.t. the norm

$$\|(x, y)\| = \|x\|_X + \|y\|_Y.$$

By our assumption T is closed, so $G(T)$ is closed. Also by our assumption $D(T)$ is closed. Hence both the graph and the domain of T are Banach spaces with corresponding norms. Consider the projection map $P\colon G(T) \to D(T)$,

$$P((x, Tx)) := x.$$

Now

$$\|P(x, Tx)\|_X = \|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|(x, Tx)\|$$

i.e. P is bounded.

By its definition P is bijective and $P^{-1}\colon D(T) \to G(T)$ is given by

$$P^{-1}(x) = (x, Tx)$$

Hence, by the bounded inverse theorem, P^{-1} is bounded and so $\exists C \in (0, \infty)$ such that

$$\forall x \in D(T) \qquad \|(x, Tx)\| \leq C\|x\|_X.$$

Therefore

$$\|Tx\|_Y \leq C\|x\|_X$$

i.e. T is bounded. □

Proposition 22. *Closedness $\not\Rightarrow$ boundedness.*

Closed Unbounded Operator

Example 48. *Let $X = C([0,1])$ be furnished with the uniform norm. Consider the following differential operators.*

$$T\colon D(T) \to X,\ T = \frac{d}{dt}$$

$D(T) \neq X$, but $\overline{D(T)} = X$.
Suppose $x_m \in D(T)$, $x_m \underset{m\to\infty}{\to} x \in X$, and $Tx_m \underset{m\to\infty}{\to} y \in X$ (in the uniform norm). With a notation $\frac{d}{dt}w \equiv w'$, we have

$$\begin{aligned}\int_0^t y(\tau)d\tau &= \int_0^t \lim_{m\to\infty} x_m'(\tau)d\tau \\ &= \lim_{m\to\infty}\int_0^t x_m'(\tau)d\tau \text{ (uniform convergence)} \\ &= \lim_{m\to\infty}(x_m(t) - x_m(0)) = x(t) - x(0).\end{aligned}$$

Thus

$$x(t) = x(0) + \int_0^t y(\tau)d\tau$$

and so, by fundamental theorem of calculus, we have

$$x \in D(T) \quad \text{and} \quad Tx = x' = y$$

Example 49. *In $L_2(\lambda)$, if $\exists g \in L_{1,loc}$, such that for any $\phi \in C_0^\infty$,*

$$\int f(-\nabla\phi)d\lambda = \int g\phi d\lambda \tag{6}$$

then g is called weak derivative of f,

and we define

$$\nabla f = (-\nabla)^* f \coloneqq g.$$

Replacing f, g with f_n, g_n. $f_n \to f$ in L_2 and $g_n \to g$ in L_2. Then (6) holds.
Hence the weak derivative is a closed operator.

Consider the following quadratic form

$$\mathcal{E}(f,g) = \int \left(\sum_i^n \nabla_i f \nabla_i g \right) \rho d\lambda$$

If $\nabla_i g \in \mathcal{D}(\nabla^*_{i,\rho})$, *then*

$$\begin{aligned} \mathcal{E}(f,g) &= \int f \left(\sum_i^n \nabla^*_{i,\rho} \nabla_i g \right) \rho d\lambda \\ &= \int f(-\Delta_\rho g) \rho d\lambda \end{aligned}$$

with $\Delta_\rho = -\sum_i^n \nabla^*_{i,\rho} \nabla_i$. *We have*

$$\Delta_\rho g = (\Delta + \nabla \log \rho \cdot \nabla) g.$$

Remark 39. *For* $0 \leq \rho \in L_1$, $\int \rho d\lambda = 1$, *then above operator plays a special role, e.g. in the general heat equation*

$$\begin{aligned} \partial_t u &= \Delta_\rho u, \\ u(t=0) &= f \end{aligned}$$

has a solution

$$u = e^{t\Delta_\rho} f = P_t f.$$

where P_t *in the right hand side is a Markov semigroup. This is similar to the special to the classical heat equation i.e. the case when* $\rho \equiv 1$, *but more involved theory is necessary to obtain the semigroup* P_t. *One of the conditions is that* Δ_ρ *is a closed operator.*

Proposition 23. *Boundedness* $\not\Rightarrow$ *closedness.*

Example 50. *Let* $id\colon D(id) \to X$, *where* $D(id) \subsetneq X$ *and* $\overline{D(id)} = X$ *i.e.* $D(id)$ *is a dense proper subset of* X. *This operator is bounded. If* $D(id)$ *would be closed, then the operator would be closed.*

Functional Analysis: Lecture 18

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Compact operators

Reminders

Remark 40. *Recall that a set C in a metric space is called compact iff every sequence in this set has a convergent subsequence to a point in this set.*

A bounded closed set is compact in a normed space iff the spaces is finite dimensional.

Compact Operator

Definition 34. *Let X, Y normed spaces. An operator $T\colon X \to Y$, is called compact if $\forall B \subset X$ bounded, $\overline{T(B)}$ is compact.*

Example 51. $T = id\colon X \to X$

(i) $\dim X < \infty$, *T compact*

(ii) $\dim X = \infty$, *T is not compact.*

Theorem 38. *Let X, Y be normed spaces.*
A linear operator $T\colon X \to Y$ is compact

if and only if it maps every bounded sequence

$(x_m)_{m\in\mathbb{N}} \subset X$ onto a sequence

$(Tx_m)_{m\in\mathbb{N}} \subset Y$ which has a convergent subsequence.

Proof. ($\Leftarrow$) Assume that for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ there is a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that $(Tx_{n_i})_{i\in\mathbb{N}}$ converges in Y.

Now let $B \subset X$ be bounded and consider an arbitrary sequence $(y_n)_{n\in\mathbb{N}} \subset T(B)$.

Then $y_n = Tx_n$ for some $x_n \in B$. Since B is bounded so is $(x_n)_{n\in\mathbb{N}}$.

Now by our assumption, $(y_n = Tx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence $(y_{n_i} = Tx_{n_i})_{i\in\mathbb{N}}$.

Hence $\overline{T(B)}$ is compact.

$\square$

***Exercise* 23.** *Finish the proof of the above theorem in the ($\Rightarrow$) direction.*

Theorem 39. *Let X, Y be normed spaces. Then compact operators from X to Y form a linear space.*

Proof. Exercise. $\square$

Theorem 40. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of compact linear operators from a normed space X into a Banach space Y.*

If $T_n \underset{n\to\infty}{\to} T$ w.r.t. the operator norm, then T is compact.

Proof: Since T_1 is compact, for a bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ $\exists (x_{1,n})_n \subset (x_n)_n$ such that $(T_1 x_{1,n})_n$ is Cauchy. Given $(x_{1,n})_n$ by compactness of T_2 $\exists (x_{2,n}) \subset (x_{1,n})$ such that $(T_2 x_{2,n})_n$ is Cauchy. By induction $\forall m$ $\exists (x_{m,n})$ such that $(T_m x_{m,n})_n$ is Cauchy. Define $\tilde{x}_n = x_{n,n}$. Then $\forall m$ $(T_m \tilde{x}_n)_n$ is Cauchy. Also bounded by construction $(\tilde{x}_n)_n$ is bounded by a constant $C \in (0,\infty)$. Let $\varepsilon > 0$, then since $T_m \to T$, $\exists p$ such that

$$\|T_p - T\| < \frac{\varepsilon}{3C}.$$

Since $(T_p\tilde{x}_n)_n$ is Cauchy $\exists N$ such that

$$\forall j,k \geq N \qquad \|T_p\tilde{x}_j - T_p\tilde{x}_k\|_Y < \frac{\varepsilon}{3}.$$

. Hence $\forall j,k \geq N$

$$\begin{aligned}
\|T\tilde{x}_j - T\tilde{x}_k\|_Y \leq & \|T\tilde{x}_j - T_p\tilde{x}_j\|_Y + \|T_p\tilde{x}_j - T_p\tilde{x}_k\|_Y \\
& + \|T_p\tilde{x}_k - T\tilde{x}_k\|_Y \\
< & \|T - T_p\|\|\tilde{x}_j\| + \frac{\varepsilon}{3} + \|T_p - T\|\|\tilde{x}_k\| \\
< & \frac{\varepsilon}{3C} \cdot C + \frac{\varepsilon}{3} + \frac{\varepsilon}{3C} \cdot C \\
= & \varepsilon.
\end{aligned}$$

Hence $(T\tilde{x}_n)_n$ is Cauchy. Since Y is complete the limit exists. Since $(x_n)_n$ is bounded, T is compact. □

Theorem 41. *Let X, Y be normed spaces, and $T\colon X \to Y$ be a linear operator. Then*

(i) If T is bounded and $\dim T(X) < \infty$, then T is compact.

(ii) If $\dim X < \infty$, then T is compact.

Proof. (i) Consider $(x_n)_n \subset X$ bounded. Then $\|Tx_n\| \leq \|T\|\|x_n\|$ i.e. (Tx_n) is bounded in finite dimensional space $T(X)$. Hence we can choose a convergent subsequence and so T is compact.

(ii) Application of (i).

□

***Exercise* 24.** *Prove part (ii).*

Simple Compact Operators in l_p.

Example 52. *By the previous theorem, for any $n \in \mathbb{N}$ the operators $T_n(x) = (a_1x_1, \ldots, a_nx_n, 0, \ldots)$ on l_p, $p \in [1, \infty)$, are compact. Suppose that $a_n \to 0$ as $n \to \infty$, then for $n > m$*

$$\begin{aligned}\|T_nx - T_mx\|_p &= \|(0, \ldots, 0, a_{m+1}x_{m+1}, \ldots, a_nx_n, 0, \ldots)\|_p \\ &= \left(\sum_{j=m+1}^{n} |a_jx_j|^p\right)^{\frac{1}{p}} \\ &\leq \sup_{j \geq m+1} |a_j| \|x\|_p.\end{aligned}$$

Hence $\|T_n - T_m\| \leq \sup_{j \geq m+1} |a_j|$ and taking $n \to \infty$

$$\|T - T_m\| \leq \sup_{j \geq m+1} |a_j|.$$

....

Example 53. ....

Thus for any sequence $(a_n)_{n \in \mathbb{N}}$, $a_n \underset{n \to \infty}{\to} 0$ the operator $T : l_p \to l_p$ given by

$$T(x) := (a_nx_n)_{n \in \mathbb{N}}$$

is compact.

***Exercise* 25.** *What about $p = \infty$?*

Compact Integral Operators in $C([a, b])$.

Example 54. *Let $T : C([a, b]) \to C([a, b])$ be defined by*

$$Tx = v + \int_a^b K(\cdot, y)x(y)dy$$

with $v \in C([a, b])$ and a jointly continuous kernel $K(\cdot, \cdot)$. Then T is bounded. Since $z \mapsto K(z, y)$ is continuous uniformly in y, so (by Arzela-Ascoli theorem) T is compact.

Finite Rank Operators in $C([a, b])$.

Example 55. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be a normed space and a Banach space (over the same field of numbers), respectively.*

For $\eta_k \in X^*$, $k \in \mathbb{N}$, *and* $y_j \in Y$, $j \in \mathbb{N}$, *define an operator* $T : X \to Y$ *by*

$$Tx := \sum_{j,k} y_j a_{jk} \eta_k(x)$$

with some $n \times m$ *matrix* $a_{j,k}$. *Then* T *is compact.*

***Exercise* 26.** *Give an example of a compact operator in a Hilbert space.*

https://en.wikipedia.org/wiki/Compact_operator

End of Lecture 18

Functional Analysis: Lecture 19

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Weak Convergence and Compact operators

Definition 35 (Weak convergence)**.** *We say a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges weakly to $x \in X$ iff*

$$\forall f \in X^* \qquad f(x_n) \underset{n\to\infty}{\longrightarrow} f(x).$$

In this case we use notation $x_n \stackrel{w}{\to} x$.

Theorem 42.

$$x_n \stackrel{\|\cdot\|}{\to} x \Rightarrow x_n \stackrel{w}{\to} x.$$

Proof. Note that for all $f \in X^*$ we have

$$|f(x_n) - f(x)| \leq \|f\| \|x_n - x\|.$$

□

Example 56 (Shift Operators in l_p)**.** *For $x \in l_p$ define Shift to the Right*

$$(Sx)_j := \begin{cases} 0 & for \quad j = 1 \\ x_{j-1} & for \quad j \geq 2 \end{cases}$$

and define Shift to the Left

$$(Tx)_j := x_{j+1} \qquad for \quad j \geq 1$$

Example 57. *For* $z = (z_j)_{j\in\mathbb{N}} \in l_p$ *consider a sequence*

$$x^{(n)} = T^n z \equiv (z_{j+n})_{j\in\mathbb{N}}, \qquad n \in \mathbb{N}.$$

Then

$$x_n \overset{\|\cdot\|_p}{\to} 0$$

and so

$$x_n \overset{w}{\to} 0.$$

Example 58. *For a sequence* $z = (z_j)_{j\in\mathbb{N}} \in l_p$ *consider*

$$x_n = S^n z = (\underbrace{0, 0, \ldots, 0}_{n^{th}\ terms}, z_1, z_2, \ldots)$$

we have

$$x_n \overset{w}{\to} 0,$$

but

$$x_n \overset{\|\cdot\|_p}{\nrightarrow} 0.$$

***Exercise* 27.** *Give analogous examples in* $L_p(\lambda)$.

Example 59. *For any* $p \in [1, \infty)$, *the sequence* $u_n = \sin(n2\pi x) \in L_p([0,1], \lambda)$, $n \in \mathbb{N}$, *converges weakly to zero, but not strongly as*

$$\inf_n \int_0^1 |\sin(n2\pi x)|^p dx > 0$$

***Exercise* 28.** *Generalise this example to any periodic function with mean value zero in* $L_p([0,1], \lambda)$.

Uniqueness of Weak Limits.

Proposition 24. *Weak limit of a sequence (x_n) is unique.*

Proof.
Suppose $x_n \overset{w}{\to} x$ and $x_n \overset{w}{\to} y$. Assume $x \neq y$. Then there is some $f \in X^*$ such that $f(x) \neq f(y)$. But for this f, we have the sequence of scalars $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$. However, limits of scalars are unique (recall,$\mathbb{K} \in \mathbb{R}, \mathbb{C}$), a contradiction.
So $x = y$. Continuity of Linear Operations.

***Exercise* 29.** *For any sequence $x_n \overset{w}{\to} x$, any sequence $y_n \overset{w}{\to} y$ and any sequence of scalars (α_n) converging to α, we have:*

(a) $x_n + y_n \overset{w}{\to} x + y$,

(b) $\alpha_n \cdot x_n \overset{w}{\to} \alpha \cdot x$.

Boundedness of W-convergent Sequences

Theorem 43. *Weakly convergent sequences are bounded.*

Proof. Consider the sequence $\omega_n \in X^{**} \equiv (X^*)^*$, defined as

$$\omega_n(f) := f(x_n), \ f \in X^*.$$

Then by definition of $x_n \xrightarrow[n\to\infty]{w} x$ it follows that $f(x_n)$ is bounded and so $\omega_n(f)$ is bounded for each $f \in X^*$. By the principle of uniform boundedness it follows that ω_n is bounded in X^{**}. The result follows from $\|\omega_n\| = \|x_n\|$. $\square$

Weak Convergence in Classical Spaces Theorem :

Let (x_n) be a sequence in any of the spaces $X = c_0, l_p$, for $p \in (1, \infty)$. Then $x_n \overset{w}{\to} x$ if and only if (x_n) is norm bounded and converges to x pointwise, i.e. $x_n(j) \to x(j)$ for all $j \in \mathbb{N}$.
Proof. For $X = c_0$ (l_p-case as an exercise): Suppose $x_n \overset{w}{\to} x$. Then

the sequence is bounded by previous theorem. Moreover, taking $f = e_j$, we obtain the pointwise convergence. Conversely, if $x_n \to x$ pointwise, and is bounded, let $M \equiv \sup_n \|x_n\|$, and let $f \in l_1 = c_0^*$. Given $\varepsilon > 0$ let $N \in \mathbb{N}$ be such that

$$\sum_{j>M} |f(j)| < \varepsilon/(2M).$$

Then for large n we have

$$\left|\sum_{j\leq M} x_n(j)f(j) - \sum_{j\leq M} x(j)f(j)\right| < \varepsilon/2$$

Hence

$$\begin{aligned}\left|\sum_{j} x_n(j)f(j) - \sum_{j} x(j)f(j)\right| &\leq \left|\sum_{j\leq M} x_n(j)f(j) - \sum_{j\leq M} x(j)f(j)\right| \\ &+ \left|\sum_{j>M} x_n(j)f(j) - \sum_{j>M} x(j)f(j)\right| < \varepsilon\end{aligned}$$

□

***Exercise* 30.** *Prove the l_p-case.*

Weak Convergence in a Hilbert Space
Theorem : If a sequence in a Hilbert space $x_n \in \mathcal{H}$, $n \in \mathbb{N}$, is s.t.

- $x_n \xrightarrow[n\to\infty]{w} x$

  and

※ $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$

then $x_n \underset{n\to\infty}{\longrightarrow} x$ in norm.

Hint: Use definition of the norm in the Hilbert space to consider $\|x_n - x\|^2$ *and recall what are bounded linear functionals in the Hilbert space.*

J. Schur Theorem 1921

Theorem 44. *In l_1, a sequence (x_n) converges weakly to x if and only if it converges to x strongly.*

J. B. Conway's A Course in Functional Analysis, where it appears as Theorem V.5.2 http://users.math.uoc.gr/~frantzikinakis/FunctionalGrad2015/Conway.pdf

More direct proof by travelling hump method: http://cms.dm.uba.ar/academico/materias/1ercuat2015/analisis_funcional/l1weakstrong.pdf
and

N.L.Carothers, A Short Course on Banach Space Theory, LMS Student Text 64.

Remark: Banach–Saks Theorem

Remark 41. ***Theorem*** *(Banach–Saks): Let $\{x_n\}$ be a sequence in a Hilbert space H that converges weakly to x. Then there exists a subsequence $\{x_{n_k}\}$ such that the arithmetic averages*

$$\frac{x_{n_1} + \ldots + x_{n_k}}{k}$$

converge strongly to x.

Kakutani Theorem **Theorem** [Kakutani]

A Banach space X is reflexive iff its unit ball is weakly compact.

Weak topology . Let X be a Banach space. We define the weak topology on X as the topology with the following base of neighborhoods: for $x \in X$, $\varepsilon_1, .., \varepsilon_n$ and $f_1, ..., f_n \in X^*$

$$U^{f_1,\ldots,f_n}_{\varepsilon_1,..,\varepsilon_n}(x) := \{y \in X : \forall i = 1, .., n \quad |f_i(y) - f_i(x)| < \varepsilon_i, \qquad \}.$$

Thus, a neighborhood is an intersection of open slabs of finite widths. We say that a sequence, or a net $\{x_\alpha\}_{\alpha\in A}$ converges weakly to x, and denote $x_\alpha \stackrel{w}{\to} x$ if it converges in the sense of the weak topology.

Exercise Show that $x_\alpha \stackrel{w}{\to} x$ if and only if $f(x_\alpha) \to f(x)$ for every functional $f \in X^*$.

Remark 42. *Theorem.*
A convex set $C \subset X$ is strongly closed iff it is weakly closed.

Weak* topology

We define a weak*-open neighborhood of f to be

$$U^{x_1,\dots,x_n}_{\varepsilon_1,..,\varepsilon_n} \equiv \{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon_i, \qquad i = 1,..,n\}$$

Identifying element of X as vectors in X^{**} we see it is a special subclass of neighborhoods defined in X^*.

It is a Hausdorff topology as pairs of distinct functionals in x^* can be separated by elements of X.

A sequence x_n converging weak* to x is necessarily bounded by the Banach-Steinhauss Theorem.

If X is reflexive, then the weak and weak*-topologies on X^* are the same.

Is weak*-topology metrizable ?

Let X be a Banach space, X^* its dual, $\mathcal{B}$ the unit ball of X and $\mathcal{B}^*$ the unit ball of X^*.

Theorem 45. *If X is separable, then $\mathcal{B}^*$ endowed with the weak*-topology*
is metrizable.

Banach-Alaoglu Theorem

Theorem 46. *The unit ball of a dual space is compact in the weak*-topology.*

Proof. Notice that for any $f \in B(X^*)$ and $x \in X$, $f(x) \in [-\|x\|, \|x\|]$. This naturally suggests to consider $B(X^*)$ as a subset of the product space $T = \prod_{x \in X}[-\|x\|, \|x\|]$. By Tihonov's theorem, this product space is compact in the product topology. It suffices to show that $B(X^*)$ is closed in T, because convergence of nets in the product topology is equivalent to pointwise convergence, which for elements of $B(X^*)$ amounts to weak* convergence. To this end, let $\{f_\alpha\}_{\alpha \in A}$ be a net in $B(X^*)$ with $\lim f_\alpha = f \in T$. By linearity of f_α's and the "pointwise" sense of the limit above, we conclude that

$$f(\lambda x + \mu u) \leftarrow f_\alpha(\lambda x + \mu u) = \lambda f_\alpha(x) + \mu f_\alpha(y) \to \lambda f(x) + \mu f(y).$$

Thus, f is linear, and since $|f_\alpha(x)| < \|x\|$, we also have for all $x \in X$ $|f(x)| < \|x\|$, which identifies f as an element of $B(X^*)$.

Figure 16: **Shizuo Kakutani**

[Mathematicians]

[Mathematicians]

https://en.wikipedia.org/wiki/Leonidas_Alaoglu
https://en.wikipedia.org/wiki/Banach-Alaoglu_theorem
https://wikivisually.com/wiki/Category:21st-century_Japanese_mathematicians
https://en.wikipedia.org/wiki/Kakutani's_theorem
https://en.wikipedia.org/wiki/Reflexive_space#Properties

END Lecture 19

LAlaoglu.jpg

Figure 17: **Leonidas Alaoglu**

Apendix (Proof of Schur theorem)

N.L.Carothers, A Short Course on Banach Space Theory, LMS Student Text 64.

Proof. Suppose that (x_n) is a bounded sequence in l_1 such that $x_n \overset{w}{\to} 0$, but $\|x_n\|_{l_1} \nrightarrow 0$. We will arrive at a contradiction by constructing an $f \in l_\infty$ such that $f(x_n) \nrightarrow 0$. We have, in particular, that $x_n(j) = e_j^*(x_n) \to 0$, as $n \to \infty$, for each $j \in \mathbb{N}$, i.e. (x_n) converges weakly "coordinatewise" to zero. By a standard gliding hump argument, we know that some subsequence of (x_n) is "almost disjoint." That is, after passing to a subsequence and relabeling, we may suppose that

(i) $\forall n \qquad \|x_n\|_{l_1} \geq 5\varepsilon > 0$,

and

(ii) for some increasing sequence of integers
$1 \leq k_1 < m_1 \leq k_2 < .. \leq k_l < m_l \leq k_{l+1}$, $l \in \mathbb{N}$, we have $\sum_{i<k_n} |x_n(i)| < \varepsilon$
and
$\sum_{i>k_n} |x_n(i)| < \varepsilon$; hence, $\sum_{i=k_n}^{m_n} |x_n(i)| \geq 3\varepsilon$.

Now we define $f \in l_\infty$ by

$$f(i) = \begin{cases} sgn(x_n(i)) & if\ k_n \leq i \leq m_n\ for\ n \in \mathbb{N}; \\ 0 & otherwise. \end{cases}$$

Then $\|f\|_\infty \le 1$ and, for any n,

$$|f(x_n)| \ge \sum_{i=m_n}^{k_n} |x_n(i)| - \sum_{i<k_n} |x_n(i)| - \sum_{i>m_n} |x_n(i)| \ge 3\varepsilon - 2\varepsilon = \varepsilon > 0$$

which is the contradiction with our assumption that $x_n \stackrel{w}{\to} 0$.

Functional Analysis: Lecture 20

FANA 2020/21

Compact Sets in Functional Spaces

Definition 36. *An $\varepsilon > 0$ cover of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called totally bounded if it admits a finite ε-cover for every $\varepsilon > 0$.*

Key lemma for many compactness results is as follows.

Lemma 6. *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists some $\delta > 0$, a metric space W, and a mapping $\Phi : X \to W$ so that $\Phi(X)$ is totally bounded, and whenever $x, y \in X$ are such that $d(\Phi(x), \Phi(y)) < \delta$, then $d(x, y) < \varepsilon$.Then X is totally bounded.*

Proof. For any $\varepsilon > 0$, pick $\delta > 0$, W and Φ as in the statement of the lemma. Since $\Phi(X)$ is totally bounded,there exists a finite δ-cover $\{V_1, .., V_n\}$ of $\Phi(X)$. Then it immediately follows from the assumptions that $\{\Phi^{-1}(V_1), .., \Phi^{-1}(V_n)\}$ is an ε-cover of X. Thus X is totally bounded.

Theorem 47. *A metric space is compact iff it is complete and totally bounded.*

Compactness in l_p, $1 \le p < \infty$

Theorem 48. *A subset $\mathcal{G} \subset l_p$, where $1 \le p < \infty$, is totally bounded iff*

(i) it is pointwise bounded, i.e. $\forall j \in \mathbb{N}$

$$\sup_{x \in \mathcal{G}} |x_j| < \infty$$

and

(ii) for every $\varepsilon > 0$ there is some n so that, for every $x \in \mathcal{G}$

$$\sum_{k>n} |x_k|^p < \varepsilon^p.$$

Proof. Assume that $\mathcal{G} \subset l_p$ satisfies the two conditions. Given $\varepsilon > 0$, pick n as in the second condition, and define a mapping $\Phi : \mathcal{G} \to \mathbb{R}^n$ defined by

$$\Phi(x) = (x_1, .., x_n).$$

By the pointwise boundedness of $\mathcal{G}$, the image $\Phi(\mathcal{G})$ is totally bounded. If $x, y \in \mathcal{G}$ with

$$|\Phi(x) - \Phi(y)|_p = \left(\sum_{j=1,..,n} |x_j - y_j|^p \right)^{\frac{1}{p}} < \varepsilon$$

then

$$\|x - y\|_p \leq |\Phi(x) - \Phi(y)|_p + \left(\sum_{j>n} |x_j - y_j|^p \right)^{\frac{1}{p}} \leq \varepsilon + 2\varepsilon$$

By Key Lemma, $\mathcal{G}$ is totally bounded. (Converse is an exercise.)

Compact Sets in l_p, $p \in [1, \infty)$ Again

Example 60. *Let $\alpha \equiv (\alpha_i > 0)_{i \in \mathbb{N}}$ be a s.t.*

$$\sum_j \alpha_j^p < \infty.$$

Then the following set is compact in l_p.

$$A_{\alpha,p} = \{x \in l_p : \quad \forall j \in \mathbb{N} \quad |x_j| \leq \alpha_j\}$$

Compact sets in Hilbert space

Theorem 49. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose $\mathcal{D} \subset \mathcal{H}$ is a bounded set.*

If for some orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ we have

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \qquad \sum_{n=N}^{\infty} |\langle f, e_n\rangle|^2 < \varepsilon^2$$

for all $f \in \mathcal{D}$, then $\mathcal{D}$ is precompact in $\mathcal{H}$.

Proof.
Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}$. By the boundedness of $\mathcal{D}$,the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded, say by 1. Let $\mathcal{V}_n \equiv Span\{e_1, .., e_n\}$ and let P_n denote the orthogonal projection onto $\mathcal{V}_n$. We shall use diagonal process to construct a convergent subsequence of $(f_n)_{n\in\mathbb{N}}$.

Using our assumption for every $\varepsilon = 1/k$ we can find $N_k \in \mathbb{N}$ s.t. $N_{k+1} > N_k$ and

$$\forall f \in \mathcal{D} \qquad \sum_{j=N_k}^{\infty} |\langle f, e_j\rangle|^2 < 1/k$$

The sequence $(P_{N_1}(f_n))$ is bounded in the finite dimensional space $\mathcal{V}_{N_1}$ and , by Heine-Borel theorem, it has a convergent subsequence $(P_{N_1}(f_{1,n}))$. We can select $(f_{1,n})$ s.t.

$$\|P_{N_1}(f_{1,n} - f_{1,m})\|_2 \leq 1/n \quad \text{ for all } n < m.$$

By induction, given a sequence $P_{N_{k-1}}(f_{k-1,n})$ in $\mathcal{V}_{N_k}$. We can choose convergent subsequence $P_{N_k}(f_{k,n})$, $n > k$, in $\mathcal{V}_{N_k}$ s.t.

$$\|P_{N_k}(f_{k,n} - f_{k,m})\|^2 \leq 1/n \quad \text{ for all } n < m.$$

Next , we claim that the sequence $(f_{k,n})$ is a Cauchy. For any large k, l with $l > k$, we have $f_{l,l} \in \{f_{k,n} : n \geq k\}$ and

$$\begin{aligned}\|f_{k,k} - f_{l,l}\|^2 &= \|P_{N_k}(f_{k,k} - f_{l,l})\|^2 + \|(\mathbb{I} - P_{N_k})(f_{k,k} - f_{l,l})\|^2 \\ &\leq \|P_{N_k}(f_{k,k} - f_{l,l})\|^2 + 2/k \leq 3/k.\end{aligned}$$

This shows that $(f_{k,k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Hence $\mathcal{D}$ is precompact in $\mathcal{H}$.

Compact sets in $L_2(\mathbb{T})$

Theorem 50. *A bounded set $\mathbb{D}$ in $H_s(\mathbb{T}) \equiv W_{2,s}(\mathbb{T})$ with $s \in (0, \infty)$ is a precompact set in $L_2(\mathbb{T})$.*

Proof.

If $f \in \mathbb{D}$, then

$$\sum_{n=N}^{\infty} |\langle f, e_n\rangle|^2 \leq \frac{1}{(1+N^2)^s} \sum_{n=N}^{\infty} (1+|n|^2)^s |\langle f, e_n\rangle|^2 \leq \frac{1}{N^{2s}} \|f\|_{H_s}^2$$

Thus the functions in $\mathbb{D}$ have uniformly small tails. Hence $\mathbb{D}$ it is precompact in $L_2(\mathbb{T})$.

Compact sets in $\mathcal{C}(\Omega)$

Theorem 51 (Arzela-Ascoli)**.** *Let Ω be a compact topological space. Then a subset $\mathcal{F} \subset \mathcal{C}(\Omega)$ is totally bounded in the supremum norm iff*

(i) *it is pointwise bounded, i.e. $\forall x \in \Omega$*

$$\sup_{f\in\mathcal{F}} |f(x)| < \infty$$

and

(ii) *it is equicontinuous, i.e. $\forall x \in \Omega$ and $\forall \varepsilon > 0$*
there exists a neighborhood V of x s.t.
$\forall\, y \in V$ $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$.

Compactness in $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, spaces.

Theorem 52 (Kolmogorov–Riesz)**.** *Let $1 \leq p < \infty$. A subset $\mathcal{F}$ of $L_p(\mathbb{R}^n)$ is totally bounded iff*
(i) $\mathcal{F}$ is bounded,

(ii) $\forall\, \varepsilon > 0\ \exists R \in (0, \infty)$ *so that* $\forall f \in \mathcal{F}$

$$\int_{|x|>R} |f(x)|^p \lambda(dx) < \varepsilon^p,$$

(iii) $\forall\, \varepsilon > 0\ \exists \rho \in (0, \infty)$ *so that* $\forall\, f \in \mathcal{F}$ *and* $y \in \mathbb{R}^n$, $|y| < \rho$

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \lambda(dx) < \varepsilon^p.$$

Harald Hanche-Olsena and Helge Holdena, The Kolmogorov-Riesz compactness theorem Expositiones Mathematicae, Vol.28 (2010) 385-394 `https://doi.org/10.1016/j.exmath.2010.03.001` With improvement removing (i) in
Harald Hanche-Olsena, Helge Holdena and Eugenia Malinnikova, An improvement of the Kolmogorov–Riesz compactness theorem Expositiones Mathematicae, Vol. 37 (2019) 84-91 `https://doi.org/10.1016/j.exmath.2010.03.001`

Theorem 53. *Assume* $1 \le p < n$ *and* $p \le q < p^*$, *where*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

and let $\mathcal{F}$ *a bounded subset of* $W_{1,p}(\mathbb{R}^n)$. *Assume that for every* $\varepsilon > 0$ *there exists some* $R > 0$ *that, for every* $f \in \mathcal{F}$,

(i) $\forall\, \varepsilon > 0\ \exists R \in (0, \infty)$ *so that* $\forall f \in \mathcal{F}$

$$\int_{|x|>R} |f(x)|^p \lambda(dx) < \varepsilon^p,$$

Then $\mathcal{F}$ *is a totally bounded subset of* $L_p(\mathbb{R}^n)$

Appendix: Compactness in Metric Spaces

Theorem 54. *Theorem Let* (X, d) *be a metric space. The following are equivalent:*

(i) X *is compact;*

(ii) X is sequentially compact;

(iii) X is complete and totally bounded.

Proof: $(i \Rightarrow ii)$

Suppose X is compact, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Suppose that the sequence (x_n) did not have a convergent subsequence, that is, (x_n) does not have a cluster point. Then for every $x \in X$, there exists some neighbourhood U_x of x such that $\{n : x_n \in U_x\}$ is finite. Then $\{U_x : x \in X\}$ is an open cover of X, so by compactness we can find a finite subcover $U_1, .., U_k$. But notice that

$$\mathbb{N} = \{n : x_n \in X\} = \bigcup_{i=1}^{k} \{n : x_n \in U_i\},$$

and the latter set is finite, which is a contradiction.

Therefore, (x_n) has a cluster point, hence a convergent subsequence.

Proof: $(ii \Rightarrow iii)$ Suppose X is sequentially compact. Since every Cauchy sequence has a convergent subsequence, it follows that X is complete. Also, if X was not totally bounded, there would exist some $\varepsilon > 0$ such that no finite collection of open balls of radius ε covers X. Let $B_1 \equiv B(x_1, \varepsilon)$ be one such ball and let $x_2 \in X \setminus B_1$. Next for $B_2 \equiv B(x_2, \varepsilon)$, let $x_3 \in X \setminus (B_1 \cup B_2)$. By induction we find a sequence (x_n) such that $x_{n+1} \in X \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. Hence $d(x_n, x_m) \geq \varepsilon$ whenever $n \neq m$, so (x_n) cannot have a convergent subsequence, which contradicts our assumption that X is sequentially compact. Thus, X is also totally bounded.

Proof: $(iii \Rightarrow i)$

Suppose that X is complete and totally bounded. In order to find a contradiction, suppose that X was not compact, i.e. there exists an open cover $U_i : i \in \mathbb{I}$, with some infinite index set $\mathbb{I}$, which does not admit a finite subcover.

Since X is totally bounded we can cover it by a finite number of sets $C^1_{i=1,..,J_1}$ of diameter ≤ 1. By our assumption $\exists k_1 \leq J_1$ s.t. $C^1 \equiv C^1_{k_1}$ cannot be covered by a finite number of sets U_i : $i\mathbb{I}$; because if all could, we would find a finite subcover for X. Now, by totall boundedness of X, C^1 can be covered by a finite number of subsets $C^2_{i=1,..,J_2}$ of diameter $\leq 1/2$. Similarly as before, $\exists k_2 \leq J_2$ s.t. $C^2 \equiv C^2_{k_2}$ cannot be covered by a finite number of sets U_i, $i\mathbb{I}$.

By induction, we find (non-empty) sets C_k of diameter $\leq 1/k$ s.t. $C_k \supset C_{k+1}$ and non of them can be covered by a finite number of sets U_i : $i\mathbb{I}$. Let $x_k \in C^k$ be an arbitrary element for every $k \in \mathbb{N}$. Then, by the condition $diam(C^k) \leq 1/k$, $(x_k)_{k\in\mathbb{N}}$ is Cauchy, so by assumed completness of X, it converges to some $x \in X$. We a have $x \in U_i$ for some U_i as U_i : $i\mathbb{I}$ covers X, so there exists some $\delta > 0$ such that $B(x,\delta) \subseteq U_i$. Hence for sufficiently large $N \in \mathbb{N}$, s.t. $d(x, x_N) < \delta/2$ and $1/N < \delta/2$, we obtain

$$C^N \subseteq B(x_N, 1/N \subseteq B(x_N, \delta/2) \subseteq B(x,\delta) \subseteq U_i$$

contradicting the construction of C^N (as a set without finite cover). Thus X is compact.

Figure 18: **Franz Rellich**

[Mathematicians]

[Mathematicians]

https://en.wikipedia.org/wiki/Franz_Rellich

https://en.wikipedia.org/wiki/Rellich-Kondrachov_theorem

with some history in Harald Hanche-Olsena and Helge Holdena, The Kolmogorov-Riesz compactness theorem Expositiones Mathematicae, Vol.28 (2010) 385-394 https://doi.org/10.1016/j.exmath.2010.03.001

https://en.wikipedia.org/wiki/Vladimir_Iosifovich_Kondrashov

END Lecture 20 Compact Sets

Figure 19: **W.I.Kondrachov**

Functional Analysis: Lecture 21

FANA 2020/21

Weak Convergence and Compact operators II

Compact Operators and Weak Convergence

Theorem 55. *Let X, Y normed and let $T \colon X \to Y$ compact. Suppose $(x_n)_n$ sequence in X such that $x_n \xrightarrow{w} x \in X$. Then $(Tx_n)_n$ is strongly convergent in Y, and $Tx_n \xrightarrow{\|\cdot\|} Tx$.*

Proof. Define $F \colon X \to \mathbb{C}$

$$F(z) = g(Tz) \text{ for } g \in Y^*$$

which is linear and bounded, so $F \in X^*$, since

$$\begin{aligned} |F(z)| &= |g(Tz)| \\ &\leq \|g\| \|Tz\| \\ &\leq \|g\| \|T\| \|z\|. \end{aligned}$$

. Now if $(x_n) \xrightarrow{w} x$, then

$$F(x_n) \to F(x)$$

as $F \in X^*$. This holds $\forall F$ defined for arbitrary g. Hence $Tx_n \xrightarrow{w} Tx$.

Assume now that $\|Tx_n - Tx\| \geq \varepsilon$, for some $\varepsilon > 0$ and sufficiently large n. Since $x_n \xrightarrow{w} x$, we have that $(x_n)_n$ is bounded. Therefore, the compactness of T implies $\exists (x_{n_i})_i \subset (x_n)_n$ such that $Tx_{n_i} \xrightarrow{\|\cdot\|} \tilde{y}$ for some $\tilde{y} \in Y$. Hence $Tx_{n_k} \xrightarrow{w} \tilde{y}$, but the weak limit is unique so $\tilde{y} = Tx$; this contradicts our assumption and thus we obtain the result. $\square$

Example 61. *If T is a compact operator on X and $(x_n)_n \subset X$ goes weakly to zero, then $Tx_n \stackrel{\|\cdot\|}{\to} 0$.*

Theorem 56. *Let X and Y be Banach spaces. Then T is compact iff T^* is compact.*

Theorem 57 (Schauder). *Let X, Y be Banach spaces and $T : X \to Y$ is bounded. Then $T^* : Y^* \to X^*$ is compact iff $T : X \to Y$ is compact.*

Theorem 58. *Let $\mathcal{H}$ be a separable Hilbert space. Then every compact operator on $\mathcal{H}$ is the norm limit of operators of finite rank.*

Proof. Let $(e_j)_{j\in\mathbb{N}}$ O-N basis in $\mathcal{H}$. Define

$$\lambda_n \equiv \sup_{f\perp\{e_j, j=1,..,n\}, \|f\|=1} \|Tf\|.$$

By definition we have $\lambda_n \geq \lambda_{n+1}$ and so it converges to some $\lambda \geq 0$. We will show that $\lambda = 0$. To this end choose a sequence $\{f_n \perp \{e_j, j = 1, .., n\}, \|f_n\| = 1\}$ s.t. $\|Tf_n\| \geq \lambda/2$. Since $f_n \stackrel{w}{\to} 0$, by because T is compact, we have also $Tf_n \stackrel{w}{\to} 0$. Hence $\lambda = 0$. Consequently

$$T^{(n)} \equiv \sum_{j=1,..,n} \langle e_j, \cdot \rangle T(e_j) \to T(\cdot)$$

in norm, because

$$\lambda_n = \|T - T^{(n)}\|$$

and as we have shown it converges to zero when $n \to \infty$.

Fredholm Alternative

Theorem 59 (Fredholm Alternative). *If A is compact operator on a Hilbert space, then either $A\Psi = \Psi$ has a solution or $(\mathbb{I} - A)^{-1}$ exists.*

REM: For noncompact ops there is no such alternative, e.g. $f \to T(f) := xf$ on $[0,2]$ has no fixed point and $(\mathbb{I} - A)^{-1}$ does not exist.

REM: Compactness and uniqueness implies existence : If for any g the equation $f = g + Af$ has at most one solution, then there exists exactly one.

Analytic Fredholm Alternative

Remark 43.

Theorem 60. *Let $D \subset \mathbb{C}$ be open and connected. Let $\Psi : D \to \mathcal{L}(\mathcal{H})$ be an operator valued - function such that $\Psi(z)$ is compact for any $z \in D$. The either*

$$(\mathbb{I} - \Psi(z))^{-1} \qquad exists\ for\ no\ z \in D$$

or

$$(\mathbb{I} - \Psi(z))^{-1} \qquad exists\ for\ all\ z \in D \setminus S$$

where S has no limit points in D. In this case $(\mathbb{I} - \Psi(z))^{-1}$ is meromorphic in D, analytic in $D \setminus S$, the residue at the poles are finite rank operators, and if $x \in S$, then $\Psi(z)f = f$ has a nonzero solution in $\mathcal{H}$.

Remark 44. *Usual Fredholm Alternative folows by choosing $\Psi(z) = zA$.*

Remark on Function of Operators

Remark 45.

$$f(A) = \frac{1}{2\pi} \oint_\gamma \frac{f(z)}{z - A} dz$$

Dirichlet Problem

Definition: u is called a solution of the Dirichlet problem in an open domain D iff $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ and

$$\begin{cases} \Delta u &= 0 \qquad\qquad in\ D \\ u &= f \qquad\qquad on\ \partial D \end{cases}$$

e.g. p.204-205 in B.Simon & M.Reed, Methods of Modern Mathematical Physics, vol.1 Functional Analysis

Note that

$$K(x,y) \equiv \langle x-y, n_y\rangle/(2\pi|x-y|^3)$$

defined with an outer normal vector n_y at $y \in \partial D$, satisfies

$$\Delta_x K(x,y) = 0 \qquad\qquad in\ D$$

Thus one should try to find a solution in the form

$$u(x) = \int_{\partial D} K(x,y)\varphi(y)dS(y)$$

Then for $x \in D$, $x \to y_0 \in \partial D$ one gets

$$u(x) \to -\varphi(y_0) + \int_{\partial D} K(y_0,y)\varphi(y)dS(y)$$

Hence, for $z \in \partial D$ one needs

$$f(z) = -\varphi(z) + (T\varphi)(z)$$

where the operator $T : \mathcal{C}(\partial D) \to \mathcal{C}(\partial D)$ is given by

$$T\varphi(z) := \int_{\partial D} K(z,y)\varphi(y)dS(y)$$

One proves that T is compact and Fredholm alternative holds.

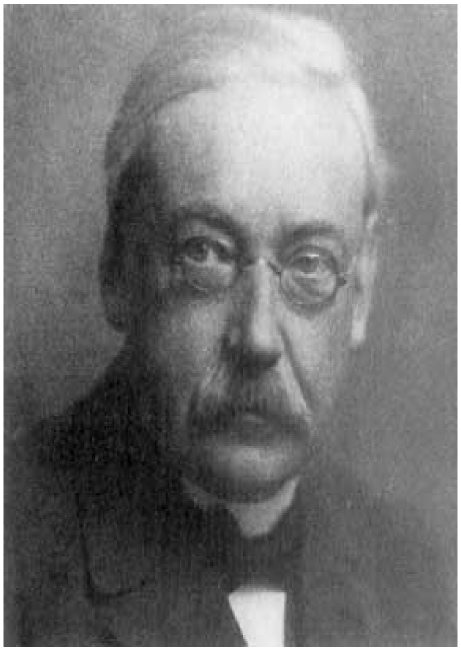

Figure 20: **Erik Ivar Fredholm**

[Mathematicians]

https://en.wikipedia.org/wiki/Erik_Ivar_Fredholm
https://en.wikipedia.org/wiki/Otto_Sz/Otto_Sz{\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{a\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\relax\let\ignorespaces\relax\accent19a\egroup\spacefactor\accent@spacefactor}sz
https://en.wikipedia.org/wiki/Fredholm_operator

END Lecture 21

Functional Analysis: Lecture 22

FANA 2020/21

The spectrum of an operator

Definition 37. *Let $(X, \|\cdot\|)$ be a complex normed space and ,*

for $D(T) \subset X$, let $T\colon D(T) \to X$ be a bounded operator.

For all $\lambda \in \mathbb{C}$, let $T_\lambda \equiv T - \lambda \mathbb{1}$.

If T_λ has an inverse, we denote it by

$$R_\lambda(T) \coloneqq T_\lambda^{-1} = (T - \lambda \mathbb{1})^{-1}$$

and call it a resolvent of T.

Definition 38. *A $\lambda \in \mathbb{C}$ is called "regular value" of T iff*

(R1) $R_\lambda(T)$ exists

(R2) $R_\lambda(T)$ is bounded

(R3) $R_\lambda(T)$ is defined on a dense subspace.

The set

$$\rho(T) = \{\lambda \in \mathbb{C} \colon \lambda \text{ regular value}\}$$

is called the "resolvent set".

Its complement,

$$\sigma(T) = \mathbb{C} \backslash \rho(T),$$

is called the "spectrum" of T.

Definition 39 (Continued)**.** *Furthermore we have the following decompositions of* $\sigma(T)$*:*

(i) The "discrete spectrum" (point spectrum) is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C}\colon R_\lambda \textit{ does not exist}\}.$$

We call $\lambda \in \sigma_p(T)$ *the eigenvalues of* T*.*

The discrete spectrum of T *on a Banach space* X *consists of* $\lambda \in \mathbb{C}$ *such that* $T - \lambda \mathbb{1}$ *fails to be injective.*

(ii) The "continuous spectrum"

$$\sigma_c(T) = \{\lambda \in \mathbb{C}\colon \textit{ (R1)\& (R3) hold \& (R2) does not.}\}$$

The continuous spectrum consists of λ *with* $T - \lambda\mathbb{1}$ *injective and with dense image, but not surjective.*

(iii) The "residual spectrum"

$$\sigma_r(T) = \{\lambda \in \mathbb{C}\colon R_\lambda(T) \textit{ exists, but (R3) is not satisfied}\}.$$

The residual spectrum consists of λ *with* $T - \lambda\mathbb{1}$ *injective but* $(T - \lambda\mathbb{1})X$ *not dense.*

Example 62. *If we consider two shift operators in* l_2*,*

$$Sx = (0, x_1, x_2, \ldots),$$
$$Tx = (x_2, x_3, \ldots)$$

we have

$$TSx = x.$$

Claim*: (R3) is not satisfied for* T*.*

We have

$$T(1, 0..0..) = 0.$$

and $R_{\lambda=0}$ *is not densely defined since* $D((0)) = \{x \in l_2\colon x_1 = 0\}$*. Thus* (R3) *is not satisfied and so* $0 \in \sigma_r(T)$*.*

Example 63. ***Claim***: *S has no eigenvalues.*

Suppose $Sx = \lambda x$, then $x_j = \lambda x_{j-1}$ for all j, and $\lambda x_1 = 0$. Thus $\forall j\ x_j = 0$ i.e. $x = 0$ which is not an eigenvector.

Example 64. *Let $X = C([a, b])$, then for $\phi \in C([a, b])$ monotone, consider the operator $A_\phi \colon X \to X$ defined as*

$$A_\phi x = \phi x.$$

Claim: *$\sigma(A_\phi) = \phi([a, b])$.*

Suppose $\lambda \in \phi([a, b])$ and B is an inverse of $A_\phi - \lambda \mathbb{1}$, then for $f \in C([a, b])$

$$(A_\phi - \lambda \mathbb{1})B(f) = f.$$

For $t_0 \in [a, b]$ such that $\phi(t_0) = \lambda$, we get

$$((A_\phi - \lambda \mathbb{1})B(f))(t_0) = 0 = f(t_0).$$

....

Example 65. *But this doesn't hold for all elements of $C([a, b])$. Thus $\lambda \in \sigma(A_\phi)$.*

On the other hand, if $\lambda \notin \phi([a, b])$, then

$$(A_\phi - \lambda \mathbb{1})^{-1} f = (\phi - \lambda)^{-1} f.$$

Theorem 61. *$\rho(T)$ is open and $\sigma(T)$ is closed.*

Proof. For $\lambda_0 \in \rho(T)$, we have

$$\begin{aligned} T - \lambda \mathbb{1} &= (T - \lambda_0 \mathbb{1}) + (\lambda_0 - \lambda)\mathbb{1} \\ &= (T - \lambda_0 \mathbb{1})(\mathbb{1} + (\lambda_0 - \lambda) R_{\lambda_0}(T)) \end{aligned}$$

Suppose λ is such that $|\lambda_0 - \lambda| \|R_{\lambda_0}(T)\| < 1$. Then

$$(\mathbb{1} + (\lambda_0 - \lambda) R_{\lambda_0}(T))^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^n(T)$$

with the RHS convergent in operator norm.

. Hence,

$$(T - \lambda 1)^{-1} = (\mathbb{1} + (\lambda_0 - \lambda) R_{\lambda_0}(T))^{-1} R_{\lambda_0}(T)$$

i.e. $\lambda \in \rho(T)$ for $|\lambda_0 - \lambda|$ sufficiently small. □

Corollary 3. *R_λ is an analytic operator function in $\rho(T)$.*

Remark 46. *For matrix M its eigenvalues are the roots of the characteristic polynomial*

$$p_M(\lambda) \equiv det(M - \lambda \mathbb{1}).$$

Hence for a matrix there exists at least one eigenvalue.

Theorem 62. *The spectrum of a bounded operator A in a Hilbert space $\mathcal{H}$ is nonempty.*

Proof. Suppose A is a bounded operator in a Hilbert space.

Then $R_\lambda \equiv (A - \lambda \mathbb{1})^{-1}$ is analytic operator function and so, for any $x, y \in \mathcal{H}$, a function

$$f(\lambda) \equiv \langle y, R_\lambda x \rangle$$

is analytic on $\rho(A)$

and $\lim_{\lambda \to \infty} f(\lambda) = 0$.
Suppose $\sigma(A) = \emptyset$.

Then f is a bounded entire function on $\mathbb{C}$ and by Liouville theorem it has to be constant.
Thus $f(\lambda) = 0$.

If however this holds for every $x, y \in \mathcal{H}$, we conclude that $R_\lambda = 0$ for all $\lambda \in \mathbb{C}$, which is impossible.

Hence we conclude that $\sigma(A)$ is not empty. $\square$

END Lecture 22

Functional Analysis: Lecture 23

FANA 2020/21

Eigenvalues of compact operators

Theorem 63. *Let $u_j \neq 0$ be solutions of $Tu_j = \lambda_j u_j$.*
Then $(u_j)_j$ associated to λ_j distinct eigenvalues are linearly independent.

Proof. Suppose the set $\{u_j\}_{j=1}^n$ is linearly independent. Let $u_{n+1} \neq 0$ be an eigenvector of T associated to an eigenvalue $\lambda_{n+1} \neq \lambda_j$, $j = 1, .., n$.

If u_{n+1} was linearly dependent on $(u_j)_{j=1}^n$, then

$$u_{n+1} = \sum_{j=1}^{n} \beta_j u_j.$$

By assumption that u_j are eigenvectors, we have

$$0 = (T - \lambda_{n+1}\mathbb{1})u_{n+1} = \sum_{j=1}^{n} \beta_j(\lambda_j - \lambda_{n+1})u_j.$$

. Since $(u_j)_{j=1}^n$ are linearly independent, so

$$\beta_j(\lambda_j - \lambda_{n+1}) = 0.$$

But $\lambda_j \neq \lambda_{n+1}$, $j = 1, .., n$, thus $\beta_j = 0$ for all j which contradicts $u_{n+1} \neq 0$. □

Theorem 64. *Let $T = T^*$ be a linear operator in a Hilbert space.*
Suppose for some $j \in \mathbb{N}$

$$Tu_j = \lambda_j u_j \text{ with } u_j \neq 0.$$

Then for $\lambda_i \neq \lambda_j$, $i \neq j$, we have

$$\langle u_j, u_i \rangle = 0.$$

Proof. Let u_i and u_j be eigenvectors of T associated to some eigenvalues $\lambda_i \neq \lambda_j$.

Then, using $T = T^*$, we have

$$\overline{\lambda_j}\langle u_j, u_i\rangle = \langle Tu_j, u_i\rangle = \langle u_j, T^* u_i\rangle = \langle u_j, \lambda_i u_i\rangle = \lambda_i\langle u_j, u_i\rangle.$$

□

Using our assumption $\lambda_i \neq \lambda_j$, this concludes the proof.

Theorem 65 (Riesz - Schauder)**.** *Suppose $T\colon X \to X$ is a compact linear operator on a normed space X. Then*

- ⊛ $\sigma(T)$ *has a countable number of eigenvalues;*
- ⊚ 0 *can be the only accumulation point of* $\sigma(T)$*;*
- ⋆ *Each of the eigenvalues has finite multiplicity.*

Proof. By contradiction, suppose $\exists(\lambda_m)_{m\in\mathbb{N}}$ with

$$|\lambda_m| > k > 0$$

(infinite sequence of distinct eigenvalues).

Let $Tx_n = \lambda_n x_n$, $(x_n \neq 0)$. (Then for different eigenvalues the eigenvectors are linearly independent.) Let $M_n = \mathrm{Span}\{x_1, \ldots, x_n\}$.

Then $\forall x \in M_n$, $x = \sum_i^n a_i x_i$ with a_i unique.

We note that

$$(T - \lambda_n \mathbb{1})M_n \subset M_{n-1} \tag{7}$$

We have $\overline{M_n} = M_n$ as M_n is finite dimensional.

Hence for any n, M_n is a closed subspace of X.

By Riesz's lemma $\exists(y_n)_n$, $y_n \in M_n$, $\|y_n\| = 1$ such that $\forall x \in M_{n-1}$, $\|y_n - x\| \geq \frac{1}{2}$.

We will show that $\forall n, m$

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}k > 0$$

and conclude that (Ty_n) has no convergent subsequence contradicting compactness of T.
For $n > m$

$$Ty_n - Ty_m = \lambda_n y_n - \tilde{x}$$

where $\tilde{x} = (\lambda_n \mathbb{1} - T)y_n + Ty_m$.

We will show that $\tilde{x} \in M_{n-1}$.
Since $m \leq n-1$ we have $y_m \in M_m \subseteq M_{n-1}$.
Hence $Ty_m \in M_{n-1}$. Using (7) we have

$$(\lambda_n \mathbb{1} - T)y_n = -(T - \lambda_n \mathbb{1})y_n \in M_{n-1}.$$

. Using $Ty_m \in M_{n-1}$ and above, we conclude that $\tilde{x} \in M_{n-1}$.
Hence

$$\begin{aligned}\|Ty_n - Ty_m\| &= |\lambda_n| \left\| y_n - \frac{1}{\lambda_n}\tilde{x} \right\| \geq |\lambda_n| \cdot \frac{1}{2} \\ &\geq \frac{k}{2} > 0.\end{aligned}$$

□

$\star$. Finite multiplicity property of a compact operator follows from

Theorem 66 (⋇)**.** *For a compact operator T and $\lambda \neq 0$, the null space Ker_λ of $T - \lambda \mathbb{1}$ is finite dimensional.*

□

Proof : ⋇. Consider a closed unit ball B_1 of the closed subspace $Ker(T - \lambda \mathbb{1})$.

Suppose there is a sequence $(x_n)_{n\in\mathbb{N}} \subset B_1$.

Then by compactness of T, there is a sub sequence $(x_{n_k})_{k\in\mathbb{N}}$ s.t. $(Tx_{n_k})_{k\in\mathbb{N}}$ converges.

But in the null space this implies that $(x_{n_k})_{k\in\mathbb{N}}$ converges to a point in B_1.

This however implies that B_1 is a compact set in the closed linear space $Ker(T - \lambda \mathbb{1})$.

That is only possible if the null space is finite dimensional. □

Resolvent

Remark 47. *Consider the following problem*

$$Tu - \lambda u = \omega, \ \omega \neq 0$$
$$\Rightarrow \frac{1}{\lambda}(Tu - \omega) = u.$$

If $\omega \in Ker(T - \lambda \mathbb{1})$, there is no solution.
Otherwise

$$u = (T - \lambda \mathbb{1})^{-1}\omega$$

and the operator $(T - \lambda \mathbb{1})^{-1}$ the resolvent (at λ).

END Lecture 23

Functional Analysis: Lecture 24

FANA 2020/21

Hilbert-Schmidt operators

Definition 40 (Hilbert-Schmidt operator)**.** *Let $(H, \|\cdot\|)$ be a Hilbert space and T a linear operator on H.*

Given $(e_j)_{j\in\mathbb{N}}$ an orthonormal basis of H, we define $\|\cdot\|_{HS}$ by

$$\|T\|_{HS} \coloneqq \left(\sum_j \|Te_j\|^2\right)^{\frac{1}{2}}, \tag{8}$$

T is called a Hilbert-Schmidt operator iff $\|T\|_{HS} < \infty$.

- The H-S norm is independent of the basis and

$$\|T^*\|_{HS}^2 = \|T\|_{HS}^2$$

Consider another O.N. basis $(f_i)_{i\in\mathbb{N}}$, then

$$\begin{aligned}\sum_{i=1}^{\infty} \|Tf_i\|^2 = \sum_{i,j=1}^{\infty} |\langle e_j, Tf_i\rangle|^2 = \sum_{j=1}^{\infty} \|T^*e_j\|^2 \\ = \sum_{i,j=1}^{\infty} |\langle e_i, T^*e_j\rangle|^2 = \sum_{i=1}^{\infty} \|Te_i\|^2 = \|T\|_{HS}^2.\end{aligned}$$

Let us define a trace of an operator A on the Hilbert space by

$$Tr(A) \equiv \sum_i \langle e_i, Ae_i\rangle$$

whenever the series on the right hand side converges.

Remark 48. *The definition of Tr is basis independent.*

⋇

$$\|T\|_{HS}^2 = \mathrm{Tr}(T^*T)$$

This is because

$$\|T\|_{HS}^2 = \sum_j \|Te_j\|^2 = \sum_j \langle Te_j, Te_j\rangle = \sum_j \langle e_j, T^*Te_j\rangle = \mathrm{Tr}(T^*T)$$

⊚ The set $\mathcal{J}_2$ of all H-S operators form a linear space with an inner product

$$\langle A, B\rangle_{HS} = \mathrm{Tr}(A^*B).$$

⊛

$$\|T\| \leq \|T\|_{HS}.$$

Since,

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \\
&= \sum_j |\langle Tx, e_j\rangle|^2 \\
&= \sum_j |\langle x, T^*e_j\rangle|^2 \\
&\leq \|x\|^2 \left(\sum_j \|T^*e_j\|^2\right) \\
&= \|x\|^2 \cdot \|T\|_{HS}^2.
\end{aligned}$$

⊛ $(\mathcal{J}_2, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

$\boxtimes$ If $a_{ij} = \langle e_i, Te_j \rangle$ then

$$\|T\|_{HS}^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2.$$

Example 66. *Let $\mathcal{H} = l_2(\mathbb{C})$.*
Suppose $(\alpha_j \in \mathbb{C})_{j \in \mathbb{N}}$ is s.t.

$$\sum_j |\alpha_j|^2 < \infty.$$

Then the operator $A : l_2 \to l_2$ given by

$$(Ax)_j := \alpha_j x_j$$

is Hilbert-Schmidt.

Theorem 67. *Hilbert-Schmidt operators are compact.*

Proof. For $N \in \mathbb{N}$, define a truncated operator T_N by

$$T_N x = \sum_{i,j=1}^{N} \langle e_i, Te_j \rangle \langle e_j, x \rangle e_i.$$

Then each T_N has a finite dimensional range and thus is compact.

One has

$$\|T_N - T\|_{HS}^2 = \sum_{i>N} \|Te_i\|^2 \to_{N \to \infty} 0.$$

Thus

$$\|T_N - T\|_{op} \to 0.$$

Since the space of compact operators form a closed normed space with the operator norm, so $T \in \mathcal{J}_2$ is compact. $\square$

Example 67. *For $\mathcal{H} = L_2(\Omega, \Sigma, \mu)$, consider the operator $K \colon \mathcal{H} \to \mathcal{H}$ defined as*

$$Ku(x) = \int k(x,y)u(y)\mu(dy).$$

Then

$$\|K\|_{HS}^2 = \int |k(x,y)|^2 \mu(dy)\mu(dx).$$

To see this, note that if $(f_i)_i$ an orthonormal basis of $L^2(\Omega)$, then the set $(f_i(x)\overline{f_j(y)})_{i,j\in\mathbb{N}}$ form an orthonormal basis of $L^2(\Omega \times \Omega)$ (exercise).

***Exercise* 31.** *Give an example of a compact operator that is not Hilbert-Schmidt. Hint: Look few lectures back for a compact operator in l_2*

◀

Suppose $T = T^*$ is a H-S operator for which the set of eigenvectors $(v_j)_{j\in\mathbb{N}}$ associated to the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$
form an O-N basis. Then

$$\|T\|_{H-S}^2 = \sum_j \lambda_j^2.$$

Example 68. .

Let Δ_Λ be the Laplacian with Dirichlet boundary conditions in a box $\Lambda \subset \mathbb{R}^n$.

Then for any $t > 0$

$$P_t \equiv e^{t\Delta_\Lambda}$$

is H-S operator. (exercise).

Remark 49. $\mathbb{L}_p(Tr)$ *spaces.*

Figure 21: **B.Simon**

[Mathematicians]
http://math.caltech.edu/simon/simon.jpg

https://en.wikipedia.org/wiki/Hilbert-Schmidt_operator

https://en.wikipedia.org/wiki/Barry_Simon

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END Lecture 24

Functional Analysis: Lecture 25

FANA 2020/21

- **Algebras of Operators: An Overview**

A Banach algebra

Definition 41. *A Banach algebra, is an associative algebra A over the real or complex numbers that at the same time is also a Banach space, that is, a normed space that is complete in the metric induced by the norm. The norm is required to satisfy*

$$\forall x, y \in A : \|x\,y\| \leq \|x\|\,\|y\|.$$

This ensures that the multiplication operation is continuous.

A Banach algebra is called unital if it has an identity element for the multiplication whose norm is 1, and is called commutative if its multiplication is commutative.

Examples of Banach algebras ⋇ $C_0(M)$, the space of (complex-valued) continuous functions on a locally compact (Hausdorff) space that vanish at infinity. $C_0(M)$ is unital if and only if M is compact.

⋇ The set of real (or complex) numbers is a Banach algebra with norm given by the absolute value.

⋇ The set of all real or complex $n \times n$ matrices becomes a unital Banach algebra if we equip it with a sub-multiplicative matrix norm.

⋇ Take the Banach space $\mathbb{R}^n$ (or $\mathbb{C}^n$) with norm

$$\|x\| := \max |x_i|$$

and multiplication defined componentwise:

$$(x_1, ..., x_n)(y_1, ..., y_n) = (x_1 y_1, ..., x_n y_n).$$

More generally for Matrices: see Hadamard product

https://en.wikipedia.org/wiki/Hadamard_product_(matrices)

⋇ The algebra of all bounded real- or complex-valued functions defined on some set (with pointwise multiplication and the supremum norm) is a unital Banach algebra.

⋇ The algebra of all bounded continuous real- or complex-valued functions on some locally compact space (again with pointwise operations and supremum norm) is a Banach algebra.

⋇ The algebra of all bounded real- or complex-valued measurable functions defined on a measure space (Ω, Σ, μ) (with pointwise multiplication and the supremum norm) is a unital Banach algebra.

⋇ The algebra of all continuous linear operators on a Banach space X (with composition of operators as multiplication operation and the operator norm as norm) is a unital Banach algebra.

⋇ The set of all compact operators on X is a Banach algebra and closed ideal. It is without identity if $dimX = \infty$.

⋇ If G is a locally compact Hausdorff topological group and μ is its Haar measure, then the Banach space $L_1(G)$ of all μ-integrable functions on G becomes a Banach algebra under the convolution

$$xy(g) := \int x(h)y(h^{-1}g)d\mu(h)$$

for $x, y \in \mathbb{L}_1(G)$

⋇ Uniform algebra: A Banach algebra that is a subalgebra of the complex algebra $\mathcal{C}(M)$ with the supremum norm and that contains the constants and separates the points of M (which must be a compact Hausdorff space).

Relation to The Stone–Weierstrass Theorem REMARK:

https://en.wikipedia.org/wiki/Stone-Weierstrass_theorem

The Stone–Weierstrass theorem generalizes the Weierstrass approximation theorem in two directions: instead of the real interval $[a, b]$, an arbitrary compact Hausdorff space M is considered, and instead of the algebra of polynomial functions, approximation with elements from more general subalgebras of $\mathcal{C}(M)$

⋇ Measure algebra: A Banach algebra consisting of all Radon measures on some locally compact group, where the product of two measures is given by convolution of measures.

Spectral theory

Unital Banach algebras over the complex field provide a general setting to develop spectral theory.

The spectrum of an element $x \in A$, denoted by $\sigma(x)$, consists of all those complex scalars λ such that $x - \lambda 1$ is not invertible in A.

The spectrum of any element x is a closed subset of the closed disc in $\mathbb{C}$ with radius $\|x\|$ and center 0, and thus is compact.

Moreover, the spectrum $\sigma(x)$ of an element x is non-empty and satisfies the spectral radius formula:

$$\sup\{|\lambda| : \lambda \in \sigma(x)\} = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Given $x \in A$, the holomorphic functional calculus allows to define $f(x) \in A$ for any function $f(x) \in A$ holomorphic in a neighborhood of $\sigma(x)$. Furthermore, the spectral mapping theorem holds:

$$\sigma(f(x)) = f(\sigma(x)).$$

Holomorphic Functional Calculus

REMARK: https://en.wikipedia.org/wiki/Holomorphic_functional_calculus

For $T \in \mathcal{B}(X)$, define

$$f(T) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - T}\, d\zeta,$$

where f is a holomorphic function defined on an open set $D \subset \mathbb{C}$ which contains $\sigma(T)$, and $\Gamma = \gamma_1, .., \gamma_m$ is a collection of disjoint Jordan curves in D bounding an "inside" set U, such that $\sigma(T)$ lies in U, and each γ_i is oriented in the boundary sense.

When the Banach algebra A is the algebra $\mathcal{B}(X)$ of bounded linear operators on a complex Banach space X the notion of the spectrum in A coincides with the usual one in operator theory.

For $f \in \mathcal{C}(M)$ (with a compact Hausdorff space M), one sees that:

$$\sigma(f) = \{f(t) : t \in M\}.$$

Let A be a complex unital Banach algebra in which every non-zero element x is invertible (a division algebra).

For every $a \in A$, there is $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible (because the spectrum of a is not empty) hence $a = \lambda 1$: this algebra A is naturally isomorphic to $\mathbb{C}$ (the complex case of the Gelfand-Mazur theorem).

C*-algebra

Definition 42. *A C^*-algebra is a Banach algebra together with an involution satisfying the properties of the adjoint. A particular case is that of a complex algebra A of continuous linear operators on a complex Hilbert space with two additional properties:*
- *A is a topologically closed set in the norm topology of operators.*
- *A is closed under the operation of taking adjoints of operators.*

Examples of C^*-algebras
- Finite-dimensional C^*-algebras
- C^* - algebras of bounded operators
- C^*- algebras of compact operators
- Commutative C^*-algebras

Includes : Important class of non-Hilbert C^*-algebra include the algebra of continuous functions $\mathcal{C}_0(X)$.
- C^*-enveloping algebra
- Convolution algebra (Measure algebra): A Banach algebra consisting of all Radon measures on some locally compact group, where the product of two measures is given by convolution of measures.

A von Neumann algebra

Definition 43. *A von Neumann algebra or W^*-algebra is a $*$-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator.*

Examples of von Neumann algebras

• The albegra $L_\infty(\mathbb{R})$ of essentially bounded measurable functions on the real line is a commutative von Neumann algebra, whose elements act as multiplication operators by pointwise multiplication on the Hilbert space $L_2(\mathbb{R})$ of square-integrable functions.

• The algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is a von Neumann algebra, non-commutative if the Hilbert space has dimension at least 2.

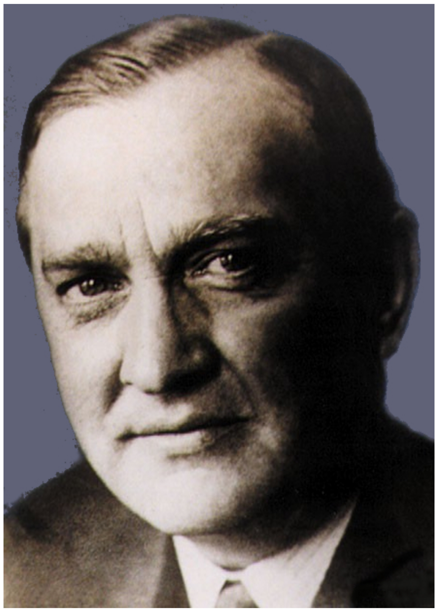

Figure 22: **Stefan Banach**

30 March 1892-31 August 1945

Remark 50. *https://en.wikipedia.org/wiki/Stefan_Banach*
History *http://kielich.amu.edu.pl/Stefan_Banach/e-biography.html*
Scottish Book: English version: *https://web.archive.org/web/20180428090844/http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka3ang.pdf*
Polish version: *https://web.archive.org/web/20180619155008/http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka1pol.pdf*
https://newikis.com/en/Scottish_Book

Budapest, 28 December 1903 – Washington, 8 February 1957

Figure 23: **John von Neumann**

https://www.ams.org/journals/bull/1958-64-03/S0002-9904-1958-10189-5/S0002-9904-1958-10189-5.pdf
https://it.wikipedia.org/wiki/John_von_Neumann

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https://en.wikipedia.org/wiki/Banach_algebra
https://en.wikipedia.org/wiki/C*-algebra
https://en.wikipedia.org/wiki/Von_Neumann_algebra

John von Neumann contribution to the Scottisch book problem 163 on p.72-73, 4 July, 1937 http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka3ang.pdf
https://en.wikipedia.org/wiki/Scottish_Book
https://www.maa.org/press/maa-reviews/pearls-from-a-lost-city-the-lvov-school-of-mathematics

Offer Von Neumann to Banach:

https://www.sciencedirect.com/topics/mathematics/von-neumann

" Von Neumann made a strange offer of a professorship at the Advanced Study Institute to Stefan Banach from the John Casimir University in Lwów. He handed him a cheque with a handwritten figure "1" and asked Banach to add as many zeros as he wanted. "This is not enough money to persuade me to leave Poland" -answered Banach."

END Lecture 25

Functional Analysis: Lecture 26

FANA 2020/21

- **Appendix: Integration and Measures - A Review**

Metric Linear Spaces

Examples of linear spaces (over a field of numbers $\mathbb{K}$)

(i) **Space of simple functions with sup norm**:
Suppose a family $\mathcal{F}$ of subsets of a metric space (M, ρ), is closed with respect to finite (or countable) set theoretical operations.
For $\alpha_i \in \mathbb{K}$ and pairwise disjoint $A_i \in \mathcal{F}$, $i = 1, .., n$, $(n \in \mathbb{N})$, we define simple functions

$$\xi \equiv \sum_{i=0}^{n} \alpha_i \, \chi_{A_i}$$

(ii) **Continuous functions** $\mathcal{C}(\Omega)$ on a (compact) metric space Ω.

(ii.a) with a metric

$$\rho(f, g) \equiv \sum_j a_j \frac{|f(q_j) - g(q_j)|}{1 + |f(q_j) - g(q_j)|} \qquad where (q_j)_{j\in\mathbb{N}} = \mathbb{Q}$$

(ii.b) sup norm

(iii) **Bounded operators** $\mathcal{B}(\mathcal{H})$, on a Hilbert space $\mathcal{H}$, with op norm it is a normed space.

An analog of simple functions

$$\boldsymbol{\xi} \equiv \sum_{i=0}^{n} \alpha_i \, P_{A_i}$$

where P_{A_i} are orthogonal projectors onto a subspace $A_i \subset \mathcal{H}$

Remark: They are all algebras.

Positive Cone
A set $C \subset V$ is called a cone in a linear space V iff

$$\forall x \in C \quad \forall \alpha \geq 0 \quad \text{we have } \alpha x \in C.$$

Positive cone C_+

(i)& (ii) chose e.g. functions s.t. $x(\omega) \geq 0$

(iii) E.g. $C \equiv \{x \in \mathcal{B}(\mathcal{H}) : \forall v \in \mathcal{H} \;\; \langle v, xv\rangle \geq 0\}$.

Alternatively

$$C \equiv \{x \in \mathcal{B}(\mathcal{H}) : \exists T \in \mathcal{B}(\mathcal{H}) \quad x = T^*T\}.$$

Positive Functionals
A functional $F : V \to \mathbb{K}$ from a linear space V to the field $\mathbb{K}$ is called *positive* iff

$$\forall x \in C_+ \qquad F(x) \geq 0.$$

Example 69. *(i)& (ii) For real valued functions the following functionals are bounded and positive:*

Evaluation of function at a point ω

$$\delta_\omega(f) = f(\omega)$$

Then the following functional defined with $a_j > 0$, $j \in \mathbb{N}$

$$\nu \equiv \sum_j a_j \delta_{\omega_j} \qquad (*)$$

is positive and bounded iff $\sum_j a_j < \infty$.

Example 70. *(iii) $\eta : \mathcal{B}(\mathcal{H}) \to \mathbb{K}^+$*

$$\eta(x) \equiv Tr(\rho x)$$

where $\rho \in C^+$ is in the trace class $\mathcal{J}_1$, i.e. $Tr\rho < \infty$.

η is called a state iff $\eta(\mathbb{1}) = 1$

Approximation Property

⋇ For $\mathcal{C}([0,1])$, functionals (*) approximate the functional $R - \int \cdot dx$,

i.e. there exists sequences $a_j^{(n)} > 0$ and $\omega_j^{(n)}$, $n \in \mathbb{N}$, $j \in J_n$, s.t. we have

$$\sum_{j \in J_n} a_j^{(n)} \delta_{\omega_j^{(n)}} \underset{n \to \infty}{\longrightarrow} R - \int_{[a,b]} \cdot dx$$

in norm of dual space.

Remark 51. *Note that R-integral cannot be defined even simplest simple functions, e.g. $\chi_{[0,1] \setminus \mathbb{Q}}$ is not R-integrable, because upper and lower sums is 1 and 0, respectively.*

⋇ Representation of $\mathcal{C}([0,1])^*$

Using Hahn-Banach Theorem first extending the functional to a larger space, (while preserving the norm of the functional), which included characteristic functions of a set, we have shown that

$$\forall F \in \mathcal{C}([0,1])^* \ w \in BV \quad \exists F(g) = S - \int g dw$$

with

$$\|F\| = Var(w).$$

Remark 52. *A special case of the continuous functionals includes the point measures given by*

$$\sum_{i=1,..,n} \alpha_i \delta_{\omega_i}$$

with $\alpha_i > 0$, $\sum_{i=1,..,n} \alpha_i = 1$ *corresponds to a function*

$$w(t) = \sum_{\{j:\sum_{i\leq j} \alpha_i \leq t\}} \chi_{[\omega_j,t]}$$

Riesz–Markov–Kakutani representation theorem

Theorem 68 (The topological dual space of $C_0(X)$)**.** *Let X be a locally compact Hausdorff space. For any continuous linear functional Ψ on $C_0(X)$, there is a unique regular countably additive Borel measure μ on X s.t. $\forall f \in C_0(X)$*

$$\psi(f) = \int_X f(x)\, d\mu(x).$$

The norm of an extension Ψ (to a larger space containing characteristic functions of Borel sets $\mathcal{B}_X$) as a linear functional is the total variation of μ, that is

$$\|\Psi\| = |\mu|(X) \equiv \sup_{A\in\mathcal{B}_X} \left(\mu(A) + \mu(X \setminus A)\right).$$

Ψ is positive iff the measure μ is non-negative and for $A \in \mathcal{B}_X$

$$\mu(A) := \Psi(\chi_A)$$

Remark: An approach to measure theory is to start with a *Radon measure*, defined as a positive linear functional on $C_c(X)$ where X is a locally compact space. [Nicholas Bourbaki]

A measure space

Definition 44. *A measure space is a triple* (Ω, Σ, μ)
• $\Omega \neq \emptyset$ *is a set;*

• Σ *is a* σ*-algebra of subsets in* Ω*, i.e. a family containing* Ω *and closed w.r.t. complements and countable unions;*

• μ *is a measure on* (Ω, Σ)*, i.e.* σ *additive function that is* $\mu : \Sigma \to \mathbb{R}^+$ *s.t.*

$$\forall A_j \in \Sigma \quad A_i \cap A_j = \emptyset \quad for \quad i \neq j,\ i, j \in \mathbb{N}$$

$$\mu(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} \mu(A_j).$$

Remark 53. *The* σ*-additivity condition together with translation invariance precludes a posibility of defining a natural measure on all subsets in* $\mathbb{R}$*. Using Axiom of Choice one constructs a Vitali set choosing one element from each equivalence classes* $[x]_\sim \equiv \{y : x-y \in \mathbb{Q}\}$ *for which the natural Lebesgue measure cannot be defined.*

Lebesgue Integral Let (Ω, Σ, μ) be a measure space . For simple functions

$$s(\omega) = \sum_{i=1}^{n} a_i \chi_{A_i}(\omega)$$

where the $A_i \in \Sigma$ and $a_i \in \mathbb{K}$, $i = 1, .., n$, $n \in \mathbb{N}$, are pairwise disjoint sets and numbers, respectively,

define the integral by

$$\int_\Omega \left[\sum_{i=1}^{n} a_i \chi_{A_i}(x) \right] d\mu := \sum_{i=1}^{n} a_i \mu(A_i)$$

Let $f : \Omega \to \mathbb{K}$ be measurable function, i.e. $\forall C \in \mathcal{B}_\mathbb{K} \equiv$ *Borel sets in* $\mathbb{K}$ we have $f^{-1}(C) \in \Sigma$.

Since any nonnegative measurable function can be approximated from below uniformly on compact sets by a monotone sequence of simple functions $s_n \leq f$, we define

$$\int f d\mu \equiv \lim_{n\to\infty} \int s_n d\mu.$$

For $f = f_+ - f_-$ such that $f_\pm \geq 0$ and at least one has finite integral, we define

$$\int f d\mu \equiv \int f_+ d\mu - \int f_- d\mu$$

A function is called integrable iff $\int f_\pm d\mu$ are both finite, or equivalently

$$\int |f| d\mu < \infty$$

The integral is a a linear functional on the space of integrable functions.

Basic Properties of Lebesgue Integral https://en.wikipedia.org/wiki/Lebesgue_integration

If f, g are non-negative measurable functions s.t. $f = g$ almost everywhere (a.e.), then

$$\int f \, d\mu = \int g \, d\mu.$$

i.e. the integral respects the equivalence relation of a.e.-equality.

Linearity: If f and g are Lebesgue integrable functions and $a, b \in \mathbb{K}$, then $af + bg$ is Lebesgue integrable and

$$\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu.$$

Monotonicity: If $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu.$$

A Measure Defined by an Integral

Let (Ω, Σ, μ) be a measure space. Denote $\mathcal{B}_{\mathbb{R}_{\geq 0}}$ the σ-algebra of Borel sets on $[0, +\infty]$. (By definition, $\mathcal{B}_{\mathbb{R}_{\geq 0}}$ contains the set $\{+\infty\}$

and all Borel subsets of $\mathbb{R}_{\geq 0}$. Consider a $(\Sigma, \mathcal{B}_{\mathbb{R}_{\geq 0}})$-measurable non-negative function $\rho : \Omega \to [0, +\infty]$. For a set $S \in \Sigma$, define

$$\nu(S) = \int_S \rho \, d\mu.$$

Then ν is a measure on (Ω, Σ).

Fundamental Theorems of Lebesgue Integration

Monotone convergence theorem: Suppose $\{f_k \geq 0\}_{k\in\mathbb{N}}$ is a sequence measurable functions s.t.

$$\forall k \in \mathbb{N} \, \forall x \in \Omega \qquad f_k(x) \leq f_{k+1}(x)$$

Then, the pointwise limit f of f_k is Lebesgue measurable and

$$\lim_k \int f_k \, d\mu = \int f \, d\mu.$$

The value of any of the integrals is allowed to be infinite.

Fatou's lemma: If $\{f_k \geq 0\}_{k\in\mathbb{N}}$ are measurable, then

$$\int \liminf_k f_k \, d\mu \leq \liminf_k \int f_k \, d\mu.$$

Dominated convergence theorem: Suppose $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of measurable functions with pointwise limit f, and $\exists g$ s.t. $\int |g| d\mu < \infty$ and $\forall k \in \mathbb{N} \quad |f_k| \leq g$. Then, $f = \lim_k f_k$ is Lebesgue integrable and

$$\lim_k \int f_k \, d\mu = \int \lim_k f_k \, d\mu.$$

L_p **Spaces** On the space of equivalence classes of measurable functions

$$[f]_\sim \equiv \{g : \mu(\{\omega \in \Omega : \; g(\omega) \neq f(\omega)\}) = 0\}$$

one can introduce a structure of linear space

$$[f_1]_\sim \oplus [f_2]_\sim := [f_1 + f_2]_\sim$$
$$\alpha \odot [f]_\sim := [\alpha \cdot f]_\sim$$

and the norms, for $p \in [1,\infty)$

$$\|[f]_\sim\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|[f]_\sim\|_p := \inf\{\lambda \in (0,\infty) : \mu(\{\omega \in \Omega : |f(\omega)| \geq \lambda\} = 0\}).$$

Remark 54. *Note that neither Riemann nor Lebesgue integral does not provide a norm directly on the space of R- and L-integrable functions.*

Theorem 69. $\forall p \in [1,\infty]$ $L_p(\mu)$ *space of vectors for which the* $\|\cdot\|_p$ *is finite is a Banach space. If* $p \in [1,\infty)$*, then* $L_p(\mu)$ *is separable if the measure* μ *is separable.*

Remark 55 (Dense Subspaces)**.** *The space of constinuous functions with the corresponding norm defined with Riemann integral is isometrically isomorphic to a subspace*

$$\{[f]_\sim \in L_p([a,b], \Sigma_{Leb} \cap [a,b], \lambda) : f \in \mathcal{C}([a,b])\}$$

which is dense in $L_p([a,b]\lambda)$ *(where* λ *is the Lebesgue measure). Note that there is another model of a complete normed space containing isometric copy of the space continuous functions with the* $p-th$ *norm (*$p \in [1,\infty)$*), which is constructed by use of equivalence classes of Cauchy sequences.*

Conditional Expectations as Projections

Suppose (Ω, Σ, μ) is a measure space and let Σ_1 be a proper sub-σ-algebra in Σ. Then the linear space

$$V_1 \equiv \{[f]_\sim \in \mathbb{L}_2(\Omega,\Sigma,\mu) : f \text{ is } \Sigma_1 - \text{ measurable}\}$$

is a closed subspace of $\mathbb{L}_2(\Omega,\Sigma,\mu)$. Let E_1 denote the orthogonal projector on this subspace. Then for any $[g]_\sim \in \mathbb{L}_2(\Omega,\Sigma,\mu)$ and $[f]_\sim \in V_1$ we have

$$\langle [f]_\sim, [g]_\sim\rangle_{\mathbb{L}_2(\Omega,\Sigma,\mu)} = \langle [f]_\sim, E_1[g]_\sim\rangle_{\mathbb{L}_2(\Omega,\Sigma,\mu)} = \langle [f]_\sim, E_1[g]_\sim\rangle_{\mathbb{L}_2(\Omega,\Sigma_1,\mu_{\Sigma_1})}.$$

Remark 56. *If μ is a probability measure, i.e. $\mu(\Omega) = 1$, then E_1 is called the conditional expectation w.r.t. the σ-algebra Σ_1.*

Bochner Integral https://en.wikipedia.org/wiki/Bochner_integral

Let (Ω, Σ, μ) be a measure space . Simple functions with values in a Banach space X:

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) v_i$$

where the $A_i \in \Sigma$ and $v_i \in X$, $i = 1, .., n$, $n \in \mathbb{N}$, are pairwise disjoint sets and vectors, respectively.

Define

$$\int_\Omega \left[\sum_{i=1}^{n} \chi_{A_i}(x) v_i \right] d\mu := \sum_{i=1}^{n} \mu(A_i) v_i$$

A measurable function: A function $f : (\Omega, \Sigma, \mu) \to X$ is called Bochner-measurable if it is equal μ-almost everywhere to a function g taking values in a separable subspace X_0 of X, and such that the pre-image $g^{-1}(U)$ of every open set $U \subset X$ belongs to Σ. Equivalently, f is limit μ-almost everywhere of a sequence of simple functions.

A measurable function $f : (\Omega, \Sigma, \mu) \to X$ is Bochner integrable if there exists a sequence of integrable simple functions ξ_n such that

$$\lim_{n \to \infty} \int_\Omega \|f - \xi_n\|_X \, d\mu = 0,$$

where the integral on the l.h.s. is an ordinary Lebesgue integral.

In this case, the Bochner integral is defined by

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} \int_\Omega \xi_n \, d\mu.$$

Bochner integrability criterion A Bochner-measurable function $f : (\Omega, \Sigma, \mu) \to X$ is Bochner integrable iff

$$\int_\Omega \|f\|_X \, d\mu < \infty.$$

If A is a continuous linear operator, and f is Bochner-integrable, then Af is Bochner-integrable and integration and A may be interchanged:

$$\int_X Af \, d\mu = A \int_X f \, d\mu.$$

Holomorphic Functional Calculus

REMARK: https://en.wikipedia.org/wiki/Holomorphic_functional_calculus

For $T \in \mathcal{B}(X)$, define

$$h(T) = \frac{1}{2\pi i} \int_\Gamma \frac{h(\zeta)}{\zeta - T} \, d\zeta,$$

where h is a holomorphic function defined on an open set $D \subset \mathbb{C}$ which contains $\sigma(T)$, and $\Gamma = \gamma_1, .., \gamma_m$ is a collection of disjoint Jordan curves in D bounding an "inside" set U, such that $\sigma(T)$ lies in U, and each γ_i is oriented in the boundary sense.

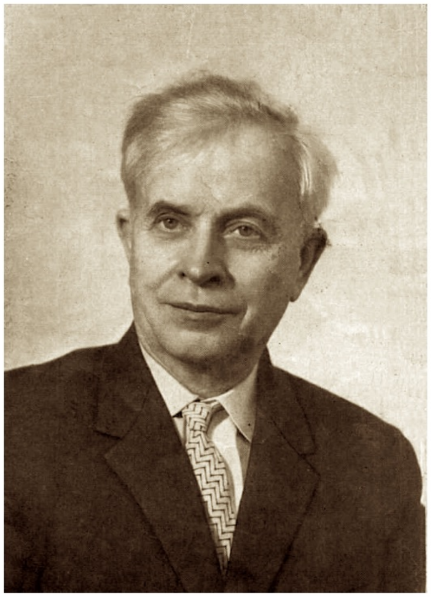

Figure 24: **Andrey A. Markov Jr.**

St. Petersburg, September 22, 1903 – Moscow, October 11, 1979

https://en.wikipedia.org/wiki/Andrey_Markov_Jr.

Wiki-cit"In 1960, Markov obtained fundamental results showing that the classification of four-dimensional manifolds is undecidable: no general algorithm exists for distinguishing two arbitrary manifolds with four or more dimensions. This is because four-dimensional manifolds have sufficient flexibility to allow us to embed any algorithm[clarification needed] within their structure, so that classification of all four-manifolds would imply a solution to Turing's halting problem. This result has profound implications for the limitations of mathematical analysis. " https://en.wikipedia.org/wiki/4-manifold

[Riesz–Markov–Kakutani representation theorem]
https://en.wikipedia.org/wiki/Riesz-Markov-Kakutani_representation_theorem

20 August 1899 – 2 May 1982

https://en.wikipedia.org/wiki/Salomon_Bochner

Figure 25: **Salomon Bochner**

https://www.nytimes.com/1982/05/05/obituaries/dr-salomon-bochner-of-princeton-is-dead.html

1934–...

https://en.wikipedia.org/wiki/Nicolas_Bourbaki

The Artist and the Mathematician: The Story of Nicolas Bourbaki, the Genius Mathematician Who Never Existed, Amir D. Aczel

END Lecture 26

NBourbaki.jpg

Figure 26: **Nicolas Bourbaki**

Functional Analysis: Lecture 27

FANA 2020/21

- **Unbounded Operators**

Weak Derivative

A function $f \in \mathbb{L}_{1,loc}(\mathbb{R}^n)$, i.e. integrable on any compact subset of $\mathbb{R}^n$, is weakly differentiable iff $\forall \varphi \in \mathcal{D}(\mathbb{R}^n) \equiv \mathcal{C}_0^\infty(\mathbb{R}^n)$ $\exists g \in \mathbb{L}_{1,loc}(\mathbb{R}^n)$ s.t.

$$\int f \partial_i^\alpha \varphi d\lambda = -1 \int g \varphi d\lambda$$

where ∂_i denotes a partial derivative in direction x_i, $i = 1, .., n$.

Then g is called the weak derivative of f (in direction x_i) and is denoted by $\partial_i f$.

The gradient operator is denoted by $\nabla \equiv (\partial_i)_{i=1,..,n}$ and for a weakly differentiable function f we set

$$|\nabla f|^2 \equiv \sum_{i=1,..,n} |\partial_i f|^2.$$

Let $\partial^\alpha \equiv \prod_{i=1,..,n} \partial_i^{\alpha_i}$ where $\alpha \equiv (\alpha_i \in \mathbb{N} \cup \{0\})_{i=1,..,n}$, with a convention $\partial_i^0 \equiv \mathbb{1}$. The number $|\alpha| \equiv \sum_{i=1,..,n} \alpha_i$ is called the order of the derivative.

We say that $f \in \mathbb{L}_{1,loc}(\mathbb{R}^n)$ is α-weakly differentiable iff for $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ $\exists g_\alpha \in \mathbb{L}_{1,loc}(\mathbb{R}^n)$ s.t.

$$\int f \partial^\alpha \varphi d\lambda = (-1)^{|\alpha|} \int (g_\alpha) \varphi d\lambda$$

Then g_α is called the weak derivative of higher order of f (of order $|\alpha|$) and is denoted by $\partial^\alpha f$.

Closedness of Weak Derivative

The graph of the weak derivative is closed in $\mathbb{L}_{1,loc} \times \mathbb{L}_{1,loc}$

Lemma 7. *Let $(f_n) \subset \mathbb{L}_{1,loc}(\Omega)$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ be s.t.*

$$f_n \to f, \quad \partial^\alpha f_n \to g.$$

Then $g = \partial^\alpha f$.

Proof. For every $\varphi \in \mathcal{D}(\Omega)$, we have by definition that

$$\int f_n \partial^\alpha \varphi d\lambda = (-1)^{|\alpha|} \int (\partial^\alpha f_n) \varphi d\lambda$$

Since by our assumption

$$\left| \int (f_n - f) \partial^\alpha \varphi d\lambda \right| \leq \|\partial^\alpha \varphi\|_\infty \int_{supp\varphi} |f_n - f| \, d\lambda \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\left| \int (\partial^\alpha f_n - g)) \varphi d\lambda \right| \leq \|\varphi\|_\infty \int_{supp\varphi} |\partial^\alpha f_n - g| \underset{n\to\infty}{\longrightarrow} 0$$

we obtain

$$\int f \partial^\alpha \varphi d\lambda = (-1)^{|\alpha|} \int g \varphi d\lambda$$

□

Example 71. *Weak Derivatives For $-n < \beta \leq 1$ a function $f(x) \equiv |x|^\beta$ is in $\mathbb{L}_{1,loc}(\mathbb{R}^n)$. For $\varepsilon > 0$, consider*

$$f_\varepsilon(x) \equiv (|x|^2 + \varepsilon)^{\frac{\beta}{2}} \in \mathcal{C}_\infty(\mathbb{R}^n).$$

One has

$$\partial_k f_\varepsilon(x) = \beta x_k (|x|^2 + \varepsilon)^{\frac{\beta}{2} - 1}$$

and

$$|\partial_k f_\varepsilon(x)| \leq |\beta| \, |x|^{\beta - 1}.$$

If $\beta > 1-n$, we get in $\mathbb{L}_{1,loc}(\mathbb{R}^n)$

$$f_\varepsilon(x) \to_{\varepsilon\to 0} f(x) = |x|^\beta$$
$$\partial_k f_\varepsilon(x) \to_{\varepsilon\to 0} g \equiv \beta\ x_k|x|^{\beta-2}$$

Hence $\partial_k f(x) = \beta\ x_k|x|^{\beta-2}$.

Sobolev spaces $W^{k,p}(\Omega)$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall|\alpha| \leqslant k\}.$$

The number $k \in \mathbb{N}$ is called the order of the Sobolev space $W^{k,p}(\Omega)$.

Density of $\mathcal{D}$ in Lebesgue and Soblev Spaces

The set of smooth compactly supported functions is dense in $\mathbb{L}_p(\Omega)$ and $W^{k,p}(\Omega)$, $p \in [1,\infty)$

For $f \in W^{k,p}(\Omega)$ consider its mollification

$$f_\varepsilon(x) \equiv \int \eta_\varepsilon(x-y)f(y)dy$$

with a compactly supported smooth function $\eta \geq 0$, $\int \eta dx = 1$ and $\eta_\varepsilon(x) \equiv \varepsilon^{-n}\eta(\frac{x}{\varepsilon})$.

Note that, for $p \in [1,\infty)$, by Jensen inequality, we have

$$\left|\int \eta_\varepsilon(x-y)f(y)dy\right|^p \leq \int \eta_\varepsilon(x-y)\,|f(y)|^p\,dy$$

and hence

$$\|f_\varepsilon\|_p \leq \|f\|_p.$$

cnd...

Since

$$\nabla(f_\varepsilon)(x) = (\nabla f)_\varepsilon(x)$$

we also have

$$\||\nabla f_\varepsilon|\|_p \le \||\nabla f|\|_p.$$

Hölder inequality

Let $1 \le p, q \le \infty,\ 1/p + 1/q = 1$. If $u \in \mathbb{L}_p(\mu),\ v \in \mathbb{L}_q(\mu)$, then

$$\int |uv|\, d\mu \le \|u\|_p\, \|v\|_q$$

Proof: Apply Young inequality

$$xy \le \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$$

to functions of norm one, respectively.

Iterated Hölder inequality

Let $1 \le p_i < \infty,\ \sum_{i=1,..,n} 1/p_i = 1$. If $u_i \in \mathbb{L}_p(\mu)$, then

$$\int \left| \prod_{i=1,..,n} u_i \right| d\mu \le \prod_{i=1,..,n} \|u\|_{p_i}$$

Proof: By induction, suppose it holds for the case $n-1$, i.e. with $\sum_{i=1,..,n-1} \frac{1}{\tilde{p}_i} = 1$ we have

$$\int \left| \prod_{i=1,..,n-1} u_i \right| d\mu \le \prod_{i=1,..,n-1} \|u\|_{\tilde{p}_i}$$

Let $1 \le p_i < \infty,\ \sum_{i=1,..,n} 1/p_i = 1$. With $1/p_n + 1/q_n = 1$, by original Hölder inequality, we have

$$\int \left| \left(\prod_{i=1,..,n-1} u_i \right) u_n \right| d\mu \le \left(\int \prod_{i=1,..,n-1} |u_i|^{q_n} d\mu \right)^{\frac{1}{q_n}} \|u_n\|_{p_n}$$

Next, since we have

$$\sum_{i=1,..,n-1} \frac{1}{p_i/q_n} = 1$$

By inductive assumption applied to the integral in the first term on the r.h.s. with $\tilde{p}_i \equiv p_i/q_n$, $i = 1,..,n-1$, we have

$$\left(\int \prod_{i=1,..,n-1} |u_i|^{q_n} d\mu \right)^{\frac{1}{q_n}} \leq \left(\prod_{i=1,..,n-1} \||u_i|^{q_n}\|_{\tilde{p}_i} \right)^{\frac{1}{q_n}} = \prod_{i=1,..,n-1} \|u_i\|_{p_i}.$$

$\square$

Corollary 4. *For any $n \geq 2$ and $u_i \in \mathbb{L}_{n-1}$, $i = 1,..,n-1$*

$$\int \prod_{i=1,..,n-1} u_i d\mu \leq \prod_{i=1,..,n-1} \|u_i\|_{n-1}$$

Gagliardo-Nirenberg Inequality

Theorem 70. *There exists $c_1 \in (0, \infty)$ s.t.*

$$\|f\|_{\frac{n}{n-1}} \leq c_1 \|\nabla f\|_1.$$

for every f for which the r.h.s. is finite.

Proof:
Suppose $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Then, for every $i = 1,..,n$, by FTC we have

$$f(x) = \int_{-\infty}^{x_i} \partial_i f(x_{\{i\}^c} \bullet y_i) dy_i$$

This with similar relation with integration from x_i to infinity, implies

$$|f(x)| \leq \frac{1}{2} \int |\partial_i f(x_{\{i\}^c} \bullet y_i)| dy_i$$

cnd...

Taking a product of all this relations with root of order $n-1$, after integration one gets.

$$\int |f(x)|^{\frac{n}{n-1}} dx_1..dx_n \leq 2^{-\frac{n}{n-1}} \int \prod_{i=1,..,n} \left(\int |\partial_i f(x_{\{i\}^c} \bullet y_i)| dy_i \right)^{\frac{1}{n-1}} dx_1..dx_n$$

We will perform iterated integration on the r.h.s., each time excluding a factor which does not depend on the integration variable. In the first step we exclude the term $\int |\partial_1 f(x_{\{1\}^c} \bullet y_1)| dy_1$ and apply the Hoelder inequality to the integral of remaining $n-1$ factors as follows

$$\int \prod_{i=2,..,n} \left(\int |\partial_i f(x_{\{i\}^c} \bullet y_i)| dy_i \right)^{\frac{1}{n-1}} dx_1 \leq \prod_{i=2,..,n} \left(\int |\partial_i f(x_{\{i\}^c} \bullet y_i)| dy_i dx_1 \right)^{\frac{1}{n-1}}$$

cnd...

We insert the result into the r.h.s. of our problem and apply similar arguments based on Hölder inequality for the integral w.r.t. the second variable. After finite induction, taking a root $n/n-1$ one arrives at the following inequality,

$$\left(\int |f(x)|^{\frac{n}{n-1}} dx_1..dx_n \right)^{\frac{n-1}{n}} \leq \frac{1}{2} \left(\prod_{i=1,..,n} \int |\partial_i f(x)| dx_1..dx_n \right)^{\frac{1}{n}}$$

Applying Geometric-Arithmetic Inequality to the r.h.s., we obtain

$$\left(\int |f(x)|^{\frac{n}{n-1}} dx_1..dx_n \right)^{\frac{n-1}{n}} \leq \frac{1}{2} \int \frac{1}{n} \sum_i |\partial_i f(x)| dx_1..dx_n.$$

□

Theorem 71. *For any* $1 < p < n$

$$\|f\|_{\frac{pn}{n-p}} \leq c_p \|\nabla f\|_p.$$

Proof. Let $1 < p < n$. We substitute a function $g := |u|^\gamma$ with a constant γ to be chosen later. Then $\partial_i g = \gamma (u^2)^{\frac{\gamma}{2}-1} u \partial_i u$ and hence $|\nabla g| = \gamma |u|^{\gamma-1} |\nabla u|$. Applying the basic Gagliardo-Niremberg inequality to g, we get

$$\||u|^\gamma\|_{\frac{n}{n-1}} \leq c \int \gamma |u|^{\gamma-1} |\nabla u| dx$$

Applying Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\||u|^\gamma\|_{\frac{n-1}{n}} \leq c\gamma \left(\int \int |u|^{q(\gamma-1)} dx \right)^{\frac{1}{q}} \left(\int |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

Choosing $q(\gamma - 1) = \frac{n}{n-1}\gamma$, i.e. setting $\gamma = \frac{q}{q - \frac{n}{n-1}}$, we get $\frac{n}{n-1}\gamma = \frac{np}{n-p}$. In this way we arrive at the following inequality

$$\|u\|_{\frac{pn}{n-p}} \leq c_p \|\nabla u\|_p.$$

□ □

The Sobolev Embedding Theorem

Theorem 72. *The following embedding is compact:*

$$W^{1,p}(\mathbf{R}^n) \subset L^{p^*}(\mathbf{R}^n).$$

with $1/p^* = 1/p - 1/n$.

Figure 27: **Sergei Lvovich Sobolev**

St. Petersburg, 6 October 1908 – Moscow, 3 January 1989

https://en.wikipedia.org/wiki/Sergei_Sobolev
http://www.math.nsc.ru/LBRT/g2/english/ssk/sobolev-schwartz_e.html

14 May 1875 Turin, Italy – 28 August 1961 Rosario, Argentina

https://en.wikipedia.org/wiki/Beppo_Levi

https://en.wikipedia.org/wiki/Beppo-Levi_space

https://en.wikipedia.org/wiki/Sobolev_inequality

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Figure 28: **Beppo Levi**

END Lecture 27

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