Answers to Problem Sheet 4

1.
$$\text{(i) } f(x) = e^x \qquad \text{(ii) } f(x) = -\sqrt{1-x^2} \qquad \text{(iii) } f(x) = x^4.$$

The Legendre transforms are obtained through $g(\lambda) = \lambda x - f(x)$ where $\lambda = f'(x)$. For part (i) $\lambda = e^x$ so that $x = \log \lambda$. The Legendre transforms are

$$\text{(i) } g(\lambda) = \lambda \log \lambda - \lambda \qquad \text{(ii) } g(\lambda) = \sqrt{1+\lambda^2} \qquad \text{(iii) } g(\lambda) = 3|\lambda/4|^{4/3}.$$

2.
$$\dot{q} = \frac{\partial H}{\partial p} = q, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -p,$$

which integrate to $q = q_0 e^t$, $p = p_0 e^{-t}$.

It is not possible to Legendre transform $H(q,p) = qp$ with respect to p to obtain $L(q,\dot{q})$. It does not exist as H is linear in p (much as it is not possible to Legendre transform $f(x) = x$).

3. (i)
$$L = \frac{mR^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgR\cos\theta.$$

Here $p_\theta = mR^2\dot{\theta}$ and $p_\phi = mR^2\sin^2\theta\dot{\phi}$. Therefore

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = \frac{{p_\theta}^2}{2mR^2} + \frac{{p_\phi}^2}{2mR^2\sin^2\theta} - mgR\cos\theta.$$

(ii) The Lagrangian for a charged particle in a magnetic field is (see problem sheet 3)

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\left(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}\right).$$

Here $p_x = \partial L/\partial\dot{x} = m\dot{x} + qA_x$, $p_y = m\dot{y} + qA_y$, $p_z = m\dot{z} + qA_z$ giving $m\dot{x} = p_x - qA_x$ and similarly for the other components. This gives

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = \frac{1}{2m}\left[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2\right].$$

4. It is important to keep track of what is constant in the various partial derivatives. Here $\partial H/\partial q$ is the derivative with p and t treated as constants whereas in $\partial L/\partial q$, $\dot{q}$ and t is fixed.

   $H = p\dot{q} - L(q, \dot{q}, t)$. Differentiate with respect to q (with p and t fixed and $\dot{q} = \dot{q}(q, p, t)$ is a function of q, p and t obtained by inverting $p = \partial L/\partial \dot{q}$)

$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = -\frac{\partial L}{\partial q}.$$

   A similar argument works for the t derivatives: $H = p\dot{q} - L(q, \dot{q}, t)$. Differentiate with respect to t (with q and p fixed and $\dot{q} = \dot{q}(q, p, t)$ is a function of q, p and t obtained by inverting $p = \partial L/\partial \dot{q}$)

$$\frac{\partial H}{\partial t} = p\frac{\partial \dot{q}}{\partial t} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} = -\frac{\partial L}{\partial t}.$$

5. (i) From the Lagrangian

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}.$$

   The rest of the calculation is essentially the same as Q1 part (ii).

   (ii) Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{c^2 p}{\sqrt{m^2c^4 + c^2p^2}}, \qquad \dot{p} = -\frac{\partial H}{\partial x} = F.$$

   Second equation can be integrated at once to give

$$p(t) = Ft.$$

   Inserting this into first equation

$$\dot{x} = \frac{c^2 Ft}{\sqrt{m^2c^4 + c^2F^2t^2}}.$$

   Integrating gives

$$x(t) = \frac{1}{F}\left(\sqrt{m^2c^4 + c^2F^2t^2} - mc^2\right)$$

   small t: $x(t) \sim \frac{1}{2}Ft^2/m$

   large t: $x(t) \sim ct - mc^2/F$.