BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
January 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: 10 January 2023

Time: 14:00 – 15:00

Time Allowed: 1 hour

This paper has 2 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Mathematics Year 1, Calculus and Applications I

Instructor: Prof. D.T. Papageorgiou

January Exam, 2023
You have 60 minutes to complete the paper

Each question carries 20 marks. Marks allocated to individual parts are also indicated.

1. (a) Define what we mean when we say the function $f(x)$ is differentiable at the point $x = x_0$. **[2 marks]**

    (b) Consider the function $f(x) = \int_a^b \sin(xt^2)dt$, where $a \neq b$ are constants. Use the definition in 1(a) to show that $f'(x) = \int_a^b t^2 \cos(xt^2)dt$. **[6 marks]**
    [You may quote, without proof, the power series expansions of $\sin \xi$ and $\cos \xi$.]

    (c) Consider the function $F(k) = \int_0^\infty e^{-kx^2} dx$, $k > 0$. Given that $F(1) = \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$, find $F(k)$. **[6 marks]**

    (d) Let $I = \int_0^\infty \frac{e^{-\alpha x^2}}{x^\alpha} dx$, where α is a non-negative constant.
      (i) For what values of α does I exist? **[3 marks]**
      (ii) Show that when I exists then it satisfies
      $$I < \frac{1}{1-\alpha} + \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$ **[3 marks]**

2. (a) The region A is bounded above by the function $y = 1 - x^2$ and below by the $x-$axis.
      (i) Find the centre of mass of the region A. **[4 marks]**
      (ii) A mass m is placed at the vertex $(0, 1)$ of A. Assuming that A has density $\rho = 1$ per unit area, find m such that the centre of mass of the resulting system is at $(0, 1/2)$. **[2 marks]**

    (b) Starting from $y = \tan^{-1} x$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$. **[2 marks]**
    Hence find a rational expression for $\tan^{-1}(1/2)$ that is guaranteed to be accurate to two decimals. **[4 marks]**

    (c) The function
    $$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \leq x < \pi \end{cases}$$

    is extended to $(-\pi, 0]$ according to $f(-x) = -f(x)$.
      (i) Find the Fourier series of the resulting $2\pi-$periodic function. **[4 marks]**
      (ii) If $\Sigma_1 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots$ and $\Sigma_2 = 1 - 1/3 + 1/5 - 1/7 + \cdots$, use your results in 2(c)i to show that
      $$\pi = \Sigma_2 + (4\Sigma_1 + \Sigma_2^2)^{1/2}.$$ **[4 marks]**

1. (a) From the notes, $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists.   **2 marks**

   (b) By definition

   $$f'(x) = \lim_{h\to 0} \frac{\int_a^b \sin[(x+h)t^2]dt - \int_a^b \sin[xt^2]dt}{h}$$

   $$= \lim_{h\to 0} \frac{\int_a^b [\sin(xt^2)\cos(ht^2) + \cos(xt^2)\sin(ht^2) - \sin(xt^2)]dt}{h}$$

   $$= \lim_{h\to 0} \frac{\int_a^b [\sin(xt^2)\{1 + h^2t^4/2 + \cdots\} + \cos(xt^2)\{ht^2 - h^3t^6/3! + \cdots\} - \sin(xt^2)]dt}{h}$$

   $$= \int_a^b t^2 \cos(xt^2)dt.$$ **6 marks**

   (c) By the result above we have

   $$F'(k) = \int_0^\infty [-x^2 e^{-kx^2}]dx.$$

   Integrate the right hand side by parts

   $$F'(k) = \left[x\frac{e^{-kx^2}}{2k}\right]_0^\infty - \int_0^\infty \frac{e^{-kx^2}}{2k}dx = -\frac{1}{2k}F(k).$$

   Separate variables and integrate

   $$\frac{dF}{F} = -\frac{1}{2k}dk \quad \Rightarrow \quad \log(F) = -\frac{1}{2}\log k + C, \quad \sqrt{k}F = C.$$

   But $F(1) = \frac{1}{2}\sqrt{\pi}$ is given, hence $C = \sqrt{\pi}/2$ and

   $$F(k) = \frac{1}{2}\sqrt{\frac{\pi}{k}}.$$ **6 marks**

   (d) (i) Need to consider I as an improper integral as $x \to 0$ and $x \to \infty$.

   First observe that for any $x_0 > 0$, $\lim_{M\to\infty} \int_{x_0}^M \frac{e^{-\alpha x^2}}{x^\alpha}dx$ exists for all $\alpha > 0$.
   [If $\alpha = 0$ the integral clearly does not exist.]
   Near $x = 0$, the integral is dominated by $1/x^\alpha$. For any $0 < x_1 < \infty$ we
   have $\lim_{\epsilon\to 0} \int_\epsilon^{x_1} \frac{1}{x^\alpha}dx < \infty$ as long as $0 < \alpha < 1$.
   Combining, we have existence of I as long as $0 < \alpha < 1$.   **3 marks**

   (ii) Write

   $$I = \left(\int_0^1 + \int_1^\infty\right)\frac{e^{-\alpha x^2}}{x^\alpha}dx < \int_0^1 \frac{dx}{x^\alpha} + \int_1^\infty e^{-\alpha x^2}dx$$

   $$< \int_0^1 \frac{dx}{x^\alpha} + \int_0^\infty e^{-\alpha x^2}dx$$

   $$= \frac{1}{1-\alpha} + \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$ **3 marks**

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2. (a) (i) The area of A is $R = \int_{-1}^{1}(1-x^2)dx = 4/3$.

By symmetry the centre of mass lies on the $y-$axis. Hence, enough to take moments about the $x-$axis. At any x the moment about the $x-$axis of a vertical strip of width dx and height y is $\frac{1}{2}y \cdot y dx$, hence for the whole area we have the moment

$$M = \int_{-1}^{1} \frac{1}{2}(1-x^2)^2 dx = \int_{0}^{1}(1-x^2)^2 dx = \int_{0}^{1}(1-2x^2+x^4)dx = 1-\frac{2}{3}+\frac{1}{5} = \frac{8}{15}.$$

If the centre of mass has coordinates $(0, \bar{y})$, then by balancing moments we have

$$R\bar{y} = M \quad \Rightarrow \quad \bar{y} = \frac{8/15}{4/3} = \frac{2}{5}. \qquad \textbf{4 marks}$$

(ii) Add the mass m at the vertex $(0,1)$ and balance the system at the pivot $(0, 1/2)$. For a balance we must have

$$m \cdot \frac{1}{2} = R \cdot (\frac{1}{2} - \frac{2}{5}) \quad \Rightarrow \quad m = \frac{4}{15}. \qquad \textbf{2 marks}$$

(b) We have $\tan y = x$ and implicit differentiation gives $\sec^2 y \frac{dy}{dx} = 1$, i.e. $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$ as required. **2 marks**

Integrate, using the fact that $\tan^{-1} 0 = 0$ to find $y = \tan^{-1} x = \int_{0}^{x} \frac{dt}{1+t^2}$. Using the binomial theorem to expand the integrand gives

$$\tan^{-1} x = \int_{0}^{x}(1 - t^2 + t^4 - t^6 + \cdots)dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The series is alternating so the error is bounded above by the absolute value of the first term in the remainder. For $x = 1/2$ we have $(1/2)^5/5 = 1/(32 \cdot 5) < 1/100$, hence for 2 decimals accuracy we have

$$\tan^{-1}(1/2) = \frac{1}{2} - \frac{1}{24} \qquad \textbf{4 marks}$$

(c) (i) The function is extended to be odd, hence we need the Fourier sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$. Calculate

$$b_n = \frac{2}{\pi} \left(\int_{0}^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} \frac{\pi}{2} \sin nx \, dx \right)$$

$$= \frac{2}{\pi} \left(\left[-\frac{x \cos nx}{n} \right]_{0}^{\pi/2} + \int_{0}^{\pi/2} \frac{\cos nx}{n} dx + \frac{\pi}{2} \left[-\frac{\cos nx}{n} \right]_{\pi/2}^{\pi} \right)$$

$$= \frac{2}{\pi} \left(-\frac{(\pi/2)\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} + \frac{\pi}{2} \left\{ -\frac{\cos n\pi}{n} + \frac{\cos(n\pi/2)}{n} \right\} \right)$$

$$= \frac{2}{\pi} \frac{\sin(n\pi/2)}{n^2} - \frac{\cos(n\pi)}{n} \qquad \textbf{4 marks}$$

(ii) Pick $x = \pi/2$. The function is continuous there, hence

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \frac{\sin(n\pi/2)}{n^2} - \frac{\cos(n\pi)}{n} \right) \sin(n\pi/2)$$

$$= \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) + \left(1 - \frac{1}{3} + \frac{1}{5} + \cdots \right)$$

$$= \frac{2}{\pi} \Sigma_1 + \Sigma_2 \qquad\qquad \textbf{2 marks}$$

Rearrange this into a quadratic equation for π,

$$\pi^2 - 2\pi\Sigma_2 - 4\Sigma_1 = 0, \qquad \Rightarrow \qquad \pi = \frac{2\Sigma_2 \pm \sqrt{4\Sigma_2^2 + 16\Sigma_1}}{2}$$

Since $\pi > 0$, we have $\pi = \Sigma_2 + \sqrt{\Sigma_2^2 + 4\Sigma_1}$. $\qquad\qquad$ **2 marks**