IMPERIAL

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Computational Dynamical Systems

Date: Wednesday, April 30, 2025

Time: Start time 10:00 – End time 12:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Consider the system of differential equations

$$\frac{dy}{dt} = f(y(t), t)$$

with $y : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

(a) Express the scheme

$$\hat{y} = y_{n-1} + hf(y_{n-1}, t_{n-1}),$$
$$y_n = y_{n-1} + \frac{h}{2} \left(f(\hat{y}, t_n) + f(y_{n-1}, t_{n-1}) \right),$$

where h is the timestep size, as a Runge-Kutta method by finding the coefficients in its tableau. (4 marks)

(b) Now, consider the alternative scheme

$$\hat{y} = y_{n-1} + hf(y_{n-1}, t_{n-1})$$
$$y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = \frac{2}{3}hf(\hat{y}, t_n)$$

(i) By expanding about t_{n-1}, determine the order of accuracy of the scheme. (7 marks)

(ii) Provide the scalar test equation used to determine the region of absolute stability. (2 marks)

(iii) Determine the quadratic equation that provides the boundary of the stability region for the scheme. *Hint: Recall the solution to a linear difference equation, $y_n = \xi^n$.* (7 marks)

(Total: 20 marks)

2. (a) Consider the scalar boundary value problem (BVP),

$$\frac{d^4u}{dx^4} + p(x)\frac{d^2u}{dx^2} + u = 0$$

$$u(0) = 0, \quad \frac{du}{dx}(0) = 0$$

$$\frac{d^2u}{dx^2}(1) = 0, \quad \frac{d^3u}{dx^3}(1) = -1$$

for $x \in (0,1)$ and $p(x)$ being a known function.

(i) What would be the inputs and outputs of a function in a code that would be provided to a Newton solver to solve the BVP via shooting? (3 marks)

(ii) Describe the computation that this function would perform. State the equations that would need to be solved numerically and how they could be solved. (4 marks)

(b) Determine the fixed points of the one-dimensional system,

$$\frac{dy}{dt} = y(y^2 - \lambda)((\lambda - 3)^2 + y^2 - 1)$$

with parameter $\lambda \in \mathbb{R}$. Sketch the bifurcation diagram using a solid line to indicate stable branches and dashed lines for unstable branches. (7 marks)

(c) Suppose that the curve $f(y, \lambda) = 0$, where $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, can be parametrised by arclength s, and the Jacobian (entry ij given by $J_{ij} = \partial f_i / \partial y_j$) is nonsingular along the curve. Determine the unit-tangent to the curve. (6 marks)

(Total: 20 marks)

3. (a) Consider the system of differential equations

$$\frac{dy_1}{dt} = y_1 + 2e^{-\sin t}y_2$$
$$\frac{dy_2}{dt} = (\cos t)y_2$$

which has general solution $y_1 = c_1e^t - 2c_2$ and $y_2 = c_2e^{\sin t}$, where c_1 and c_2 are arbitrary constants. Determine the monodromy matrix and its eigenvalues. (8 marks)

(b) Suppose the general system of differential equations

$$\frac{dy}{dt} = f(y, \lambda)$$

has a time-periodic solution $y_p(t)$ with period T.

(i) State the differential equations whose solution is the fundamental matrix that determines the stability of the solution $y_p(t)$. State the relationship between the fundamental matrix and the monodromy matrix. (4 marks)

(ii) Show that the monodromy matrix for the system of differential equations has a unit eigenvalue and find the associated eigenvector. *Hint: Substitute $y_p(t)$ into the differential equation and differentiate with respect to time.* (8 marks)

(Total: 20 marks)

4. (a) Consider the system of equations

$$F(Y) = \begin{pmatrix} f(y, \lambda) \\ p(y, \lambda) \end{pmatrix} = 0.$$

where $Y = (y, \lambda)$. Prove that the Jacobian, $\mathcal{J}$ (entry ij is $\mathcal{J}_{ij} = \partial F_i/\partial Y_j$), is invertible at a turning point if the vector $(\partial p/\partial y)^T \notin \text{range}(J^T)$, where the ij entry of J is given by $J_{ij} = \partial f_i/\partial y_j$. (10 marks)
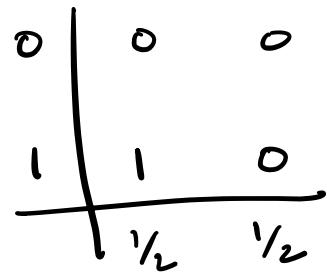
(b) Consider again $f(y, \lambda) = 0$, where $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Suppose a curve parametrised by arclength s has a simple branch point at (y_0, λ_0). Determine the expression for a tangent at this branch point as follows:

(i) Show that

$$\frac{dy}{ds} = \gamma_0(v + \alpha h_0) \tag{1}$$

where h_0 is the right null space vector of the J_0, the Jacobian (entries $J_{ij} = \partial f_i/\partial y_j$) evaluated at the branch point. Provide the meaning of the vector v, the constant γ_0, and the constant α. (5 marks)

(ii) Let g_0 denote the left null space vector of J_0. Using g_0, find the expression for α. (3 marks)

(iii) Find the expression for γ_0. You may take $\|h_0\| = 1$. (2 marks)

(Total: 20 marks)

EXAMINATION SOLUTIONS 2024-25	Course

Parts

(a)

THE SCHEME CAN BE WRITTEN AS

$$Y_1 = y_{n-1}$$

$$Y_2 = y_{n-1} + h \, f(Y_1, t_{n-1})$$

$$y_n = y_{n-1} + \frac{h}{2} \left(f(Y_1, t_{n-1}) + f(Y_2, t_n) \right)$$

THIS IN TERMS OF THE TABLEAU WE HAVE

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

SEEN

4 A

(b)
(i)

$$y(t_n) = y(t_{n-1}) + h \left. \frac{dy}{dt} \right|_{n-1}$$

$$+ \frac{h^2}{2} \left. \frac{d^2 y}{dt^2} \right|_{n-1} + \frac{h^3}{6} \left. \frac{d^3 y}{dt^3} \right|_{n-1}$$

SEEN SIMILAR

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Parts

$$y(t_{n-2}) = y(t_{n-1}) - h \frac{dy}{dt}\Big|_{n-1}$$
$$+ \frac{h^2}{2} \frac{d^2y}{dt^2}\Big|_{n-1} - \frac{h^3}{6} \frac{d^3y}{dt^3}\Big|_{n-1}$$

Thus,

$$\frac{y(t_n) - \frac{4}{3} y(t_{n-1}) + \frac{1}{3} y(t_{n-2})}{h}$$

$$= \frac{2}{3} \frac{dy}{dt}\Big|_{n-1} + \frac{2}{3} h \frac{d^2y}{dt^2}\Big|_{n-1}$$
$$+ \frac{1}{9} h^2 \frac{d^3y}{dt^2}\Big|_{n-1}$$

EXPANDING f USING INDEX NOTATION

$$f_i(\vec{y}, t_n) = f_i(y(t_{n-1}), t_{n-1}) \Big\} = \frac{dy}{dt}\Big|_{n-1}$$

$$h \frac{d^2y}{dt^2}\Big|_{n-1} = \Big\{ +h \Big(\frac{\partial f_i}{\partial y_j}\Big|_{n-1} f_j(y(t_{n-1}), t_{n-1}) $$
$$+ \frac{\partial f_i}{\partial t} \Big)$$

$$+ \frac{h^2}{2}\left(\left.\frac{\partial^2 f_i}{\partial y_j \partial y_k}\right|_{n-1} f_k \, f_j \right.$$

$$+ 2\left.\frac{\partial^2 f_i}{\partial y_j \partial t}\right|_{n-1} f_j$$

$$\left. + \frac{\partial^2 f_i}{\partial t^2} \right)$$

Thus,

$$\frac{y(t_n) - \frac{4}{3} y(t_{n-1}) + \frac{1}{3} y(t_{n-2})}{h}$$

$$- \frac{2}{3} f(\hat{y}, t_n)$$

$$= \frac{h^2}{9}\left[\left.\frac{d^3 y}{dt^3}\right|_{n-1} \right.$$

$$\left. - 3 \left(\nabla_y + \frac{\partial}{\partial t} \right)\left(\nabla_y + \frac{\partial}{\partial t} \right) f \Big|_{n-1} \right]$$

7B

EXAMINATION SOLUTIONS 2024-25		Course
Question		**Marks & seen/unseen**
Parts		

(ii)

$$\frac{dy}{dt} = \lambda y \quad, \quad \lambda \in \mathbb{C}$$

(iii)

$$\hat{y} = y_{n-1} + h\lambda y_{n-1}$$

$$= (1 + h\lambda) y_{n-1}$$

$$y_n - \frac{4}{3} y_{n-1} + \frac{1}{3} y_{n-2}$$

$$= \frac{2}{3} h\lambda (1 + h\lambda) y_{n-1}$$

USING $y_n = \bar{z}^n$ AND

DEFINING $z := h\lambda$

$$\frac{3}{2}\bar{z} - 2 + 2\bar{z}^{-1} = z^2 + z$$

SINCE THE BOUNDARY

OF THE STABILITY

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Parts

REGION CORRESPONDS TO

$|\xi| = 1$ WE HAVE

$\xi = e^{i\theta}$ FOR $\theta \in [0, 2\pi)$.

THUS,

$$\left| z^2 + z + 2(1 - e^{-i\theta}) - \frac{3}{2} e^{i\theta} \right| = 0.$$

7.

5c
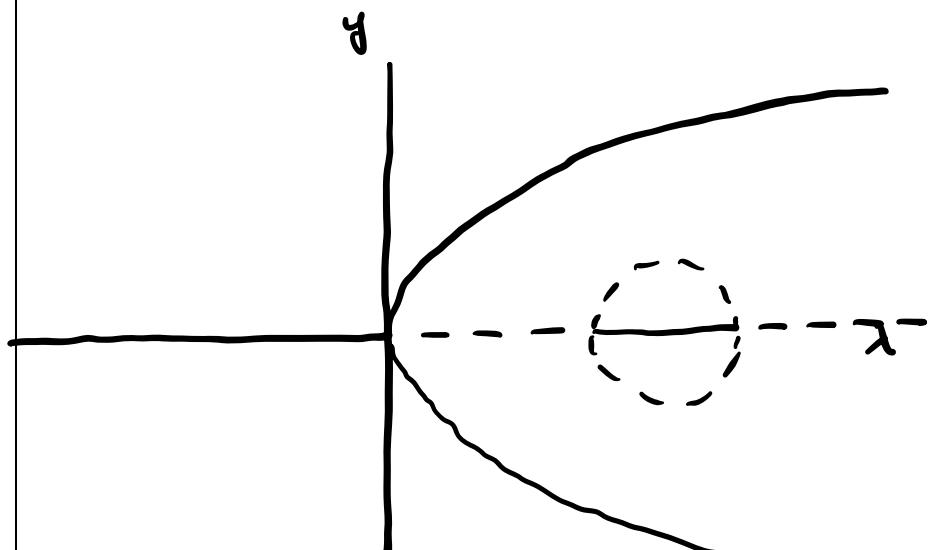
2D

Question		Marks & seen/unseen
2		

(a)

(i) THE FUNCTION WOULD TAKE AS INPUTS THE VALUES OF

$$\frac{d^2u}{dx^2} \quad \text{AND} \quad \frac{d^3u}{dx^3} \quad \text{AT}$$

$x = 0$ AND OUTPUT

$$\frac{d^2u}{dx^2}(1) \quad \text{AND}$$

$$\frac{d^3u}{dx^3}(1) + 1.$$

SEEN SIMILAR

3A

(ii) TO EVALUATE THE FUNCTION, WE MUST RECAST THE BVP AS AN IVP.

SEEN SIMILAR

Parts

WE THEREFORE CONSIDER THE SYSTEM OF ODES

$$\frac{du}{dx} = v$$

$$\frac{dv}{dx} = w$$

$$\frac{dw}{dx} = y$$

$$\frac{dy}{dx} = -f(x)w - u$$

THE FUNCTION INTEGRATES THIS SYSTEM USING AN APPROPRIATE INTEGRATOR (RK4, FOR EXAMPLE) AND RETURNS $w(1)$ AND $y(1) + 1$.

4A

Parts		
(b)		SEEN SIMILAR

FIXED POINTS

$$0 = y(y^2 - \lambda)((\lambda - 3)^2 + y^2 - 1)$$

thus,

$$y = 0 \quad , \quad \lambda = y^2$$

AND

$$(\lambda - 3)^2 + y^2 = 1$$

GIVE THE CURVES IN THE (λ, y) PLANE THAT CORRESPOND TO FIXED POINTS.

FOR STABILITY, WE CONSIDER $\left. \partial f / \partial y \right|_{y=0}$.

Parts

$$\frac{\partial f}{\partial y} = (y^2 - \lambda)((\lambda-3)^2 + y^2 - 1)$$
$$+ y [\quad \underline{\quad} \quad]$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = -\lambda((\lambda-3)^2 - 1)$$

For $\lambda < 0$, $\left.\frac{\partial f}{\partial y}\right|_{y=0} < 0$

STABLE

For $0 < \lambda < 3$ $\quad \left.\frac{\partial f}{\partial y}\right|_{y=0} > 0$

For $2 < \lambda < 4$ $\quad \left.\frac{\partial f}{\partial y}\right|_{y=0} < 0$

For $\lambda > 4$ $\quad \left.\frac{\partial f}{\partial y}\right|_{y=0} > 0.$

Question		Marks & seen/unseen
Parts		

USING THE FACT THAT THE STABILITY OF FIXED POINTS ALTERNATES (DUE TO THE CONTINUITY OF f) FOR λ FIXED, WE CAN SKETCH THE BIF. DIAGRAM:

7A

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Question		Marks & seen/unseen
Parts		

(c)

AS THE VALUE OF f DOES NOT CHANGE ALONG THE CURVE,

$$\frac{df}{ds} = 0 = \underbrace{\frac{\partial f}{\partial y}}_{J} \frac{dy}{ds} + \frac{\partial f}{\partial \lambda} \frac{d\lambda}{ds}$$

WHERE s IS THE ARCLENGTH AND

$$\hat{t} = \left(\frac{dy}{ds}, \frac{d\lambda}{ds} \right)^T$$

SOLVING FOR dy/ds GIVES

$$\frac{dy}{ds} = -\frac{d\lambda}{ds} J^{-1} \frac{\partial f}{\partial \lambda}$$

SINCE

$$\|\hat{t}\| = 1$$

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Question		Marks & seen/unseen
Parts		

$$\left(\frac{dy}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2 = 1$$

$$\left(\frac{dx}{ds}\right)^2 \left[\left\| J^{-1} \frac{\partial f}{\partial x}\right\|^2 + 1\right] = 1$$

$$\frac{dx}{ds} = \pm \left(1 + \left\| J^{-1} \frac{\partial f}{\partial x}\right\|^2\right)^{-1/2}$$

6

2C
4D

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Question **3**	Marks & seen/unseen

Parts

(a)

USING THE GENERAL SOLUTION, WE CAN FIND THE FUNDAMENTAL MATRIX $\underline{\Phi}(t)$ THAT SATISFIES:

$$\frac{d\underline{\Phi}}{dt} = A(t)\,\underline{\Phi} \quad , \quad \underline{\Phi}(0) = I$$

WHERE $\quad A(t) = \begin{bmatrix} 1 & 2e^{-\sin t} \\ 0 & \cos t \end{bmatrix}$

AND $\quad A(t + 2\pi) = A(t).$

$$y_1(t) = c_1 e^t - 2c_2$$

$$y_2(t) = c_2 e^{\sin t}$$

$$y_1(0) = c_1 - 2c_2$$

$$y_2(0) = c_2$$

Setter's initials	Checker's initials	Page number

For $y_1(0) = 1$ AND $y_2(0) = 0$

$c_2 = 0$ AND $c_1 = 1$

For $y_2(0) = 1$ AND $y_1(0) = 0$

$c_2 = 1$ AND $c_1 = 2.$

$$\Phi(t) = \begin{bmatrix} e^t & 2(e^t - 1) \\ 0 & e^{\sin t} \end{bmatrix}$$

$$M = \Phi(2\pi) = \begin{bmatrix} e^{2\pi} & 2(e^{2\pi} - 1) \\ 0 & 1 \end{bmatrix}$$

$$\mu_1 = e^{2\pi}, \quad \mu_2 = 1$$

B

Question		Marks & seen/unseen

Parts

(b)

(i)

STABILITY IS LINKED TO THE SOLUTION OF

$$\frac{d\underline{\Phi}}{dt} = J(y_p(t))\,\underline{\Phi}$$

$$\underline{\Phi}(0) = I$$

WHERE THE MATRIX $J(y_p(t))$ IS THE JACOBIAN EVALUATED AT THE PERIODIC SOLUTION.

THE MONODROMY MATRIX IS GIVEN BY

$$M = \underline{\Phi}(T).$$

4A

Setter's initials	Checker's initials	Page number

	EXAMINATION SOLUTIONS 2024-25	Course
Question		**Marks & seen/unseen**
Parts		
(ii)	THE PERIODIC SOLUTION SATISFIES	UNSEEN

$$\frac{dy_p}{dt} = f(y_p(t))$$

DIFFERENTIATING BOTH SIDES WITH RESPECT TO TIME GIVES

$$\frac{d}{dt}\left(\frac{dy_p}{dt}\right) = J(y_p(t))\frac{dy_p}{dt}$$

THUS $\frac{dy_p}{dt}(t)$ IS GIVEN BY

$$\frac{dy_p}{dt}(t) = \Phi(t)\frac{dy_p}{dt}(0)$$

Setter's initials	Checker's initials	Page number

Parts

SINCE $\dfrac{dy_P}{dt}$ IS ALSO

T- PERIODIC,

$$\frac{dy_P}{dt}(0) = \underbrace{\Phi(T)}_{M}\frac{dy_P}{dt}(0)$$

THUS M HAS $\mu = 1$

AND THE ASSOCIATED

EIGENVECTOR IS $\dfrac{dy_P}{dt}(0)$.

8
2C
6D

Parts		

(a)

$$\frac{\partial F}{\partial Y} = \begin{bmatrix} J & \partial f/\partial \lambda \\ (\partial P/\partial y)^T & \partial P/\partial \lambda \end{bmatrix}$$

WANT TO FIND

$$\begin{bmatrix} J & \partial f/\partial \lambda \\ (\partial P/\partial y)^T & \partial P/\partial \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 \in \mathbb{R}^n, \quad x_2 \in \mathbb{R}$$

SINCE WE ARE AT

A TURNING POINT,

$\partial f/\partial \lambda \notin$ RANGE (J).

THUS

$$J x_1 = - x_2 \frac{\partial f}{\partial \lambda}$$

ONLY HAS A SOLUTION

IF $x_2 = 0$ AND x_1

IS THE NON-TRIVIAL

VECTOR THAT SPANS THE

NULL SPACE OF J.

NOW, WE MUST ALSO HAVE

$$(\partial P/\partial y)^T x_1 + \cancel{x_2 \frac{\partial P}{\partial x}}^{\,0} = 0$$

HOWEVER, IF $(\partial P/\partial y)^T \notin \text{RANGE}(J^T)$

THEN $(\partial P/\partial y)^T x_1 \neq 0$.

THUS THE ONLY SOLUTION

IS $x_1 = 0$ AND $x_2 = 0$ AND

J IS INVERTIBLE.

10

Parts		UNSEEN

(b)
(i)

AT THE BRANCH POINT,

$$J_0 \frac{dy}{ds} + \frac{\partial f}{\partial \lambda} \frac{d\lambda}{ds} = 0.$$

J_0 IS SINGULAR, BUT

$\frac{\partial f}{\partial \lambda} \in \text{RANGE}(J_0).$

THE NULL SPACE OF J_0
IS SPANNED BY h_0.

THUS, WE CAN EXPRESS

$$\frac{dy}{ds} = r_0 (v + \alpha h_0)$$

WHERE $\quad J_0 v = -\frac{\partial f}{\partial \lambda}, \quad r_0 = \frac{d\lambda}{ds}$

AND $\quad \alpha$ IS AN ARBITRARY

CONSTANT.

5

Question

Marks &
seen/unseen

Parts

(ii) TO FIND α, WE INSIST

$$g_0^T \frac{dy}{ds} = 0$$

$$\Rightarrow \quad 0 = g_0^T v + \alpha g_0^T h_0$$

WHICH GIVES

$$\alpha = - \frac{g_0^T v}{g_0^T h_0}$$

3

(iii) SINCE $\hat{t} = \left(\frac{dy}{ds}, \frac{d\lambda}{ds} \right)$

AND $\|\hat{t}\| = 1$, WE HAVE

$$\left\| \frac{dy}{ds} \right\|^2 + \left(\frac{d\lambda}{ds} \right)^2 = 1.$$

THUS

$$r_0^2 \left(\|v + \alpha h_0\|^2 + 1 \right) = 1$$

Parts

SINCE $\quad v^T h_o = 0$

AND $\quad \|h_o\| = 1$,

$$r_o^2 \left[\|v\|^2 + \alpha^2 \underbrace{\|h_o\|^2}_{1.} + 1 \right] = 1$$

$$r_o = \pm \left(1 + \alpha^2 + \|v\|^2 \right)^{-\frac{1}{2}} \quad 2.$$

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Question	**Marks & seen/unseen**
Parts	

MATH70023 Computational Dynamical Systems Markers Comments

Question 1   Students did reasonable well on this question. The common errors were that the students did not use the Taylor expansion for multivariate functions and they did not provide the correct substitution for \xi to compute the boundary of the stability region.

Question 2   Students struggled to provide precise answers to part (a), often relying on a general description of shooting rather than applying it to the BVP that was given in the question.
Part (b) was handled reasonably well with the common error being that stability of the fixed points was not properly classified.
A common error for Part (c) was using expressions for the pseudo arclength rather than the arclength.

Question 3   Part (a) was handled well as was b(i). Students had more trouble with b(ii) recognising that differentiating yielding that the time derivative satisfies the differential equation linked to the monodromy matrix.

Question 4   Students did okay on (a) reflecting the fact that they might have studied the material on the problem sheet. The performance on (b) was not very good. This was the most challenging/unfamiliar material on the exam. There wasn't a common error persay for this question and many students left it unanswered.