Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M3S4/M4S4

Applied Probability

Date: Tuesday, 27th May 2008   Time: 10 am–12 noon

Answer all questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Formula sheets are included on pages 6 & 7.

1. (i) (a) Write down the three axioms of the Poisson process with rate λ.

   (b) Using the axioms prove that the probability of obtaining one or more realizations in the interval $[0, t)$ for a Poisson process that starts at time 0, with rate λ, is $1 - e^{-\lambda t}$.

   (ii) Show that the deterministic approximation to the number of events by time t of a Poisson process that starts at time 0, with rate λ, is given by $D(t) = \lambda t$.
   Comment on this approximation.

   (iii) For a non-homogeneous Poisson process with rate $\lambda(t) = \sin(t) + t$, $t \geq 0$, determine the probability of obtaining fewer than two events in the interval $[u, 2u)$, $u \geq 0$.

2. (i) Define the discrete time Galton-Watson branching process.

   (ii) Consider the branching process for which the number of offspring, X, for each individual are independent with $E(X) = \frac{1}{2}$ and $\text{var}(X) = \frac{1}{8}$. Let Z_n be the number of individuals in generation n.
   Assuming that $Z_0 = 1$, use probability generating functions, or otherwise, to show that

   (a) $E(Z_n) = 2^{-n}$,

   (b) $4\,\text{var}(Z_n) = \text{var}(Z_{n-1}) + 2^{-n}$.

   (c) What is the probability of ultimate extinction for such a process?

   (iii) Now consider the branching process for which the number of offspring, X, are independently distributed with

   $$P(X = 0) = \alpha; \qquad P(X = 2) = 1 - \alpha.$$

   Find the probability of ultimate extinction, clearly distinguishing the values of α for which ultimate extinction is certain or not.

3. (i) Consider an unrestricted simple random walk with $p + q = 1$, where X_n is the position after n steps and $X_0 = 0$.

   (a) Find $\mathrm{P}(X_n = k)$, $n = 1, 2, 3, \ldots$ and $k = -n, \ldots, n$.

   (b) If the random walk is also symmetric, find an expression for an estimate of the probability that the particle is more than 20 units away from the origin after 100 steps.

   (ii) A and B play a sequence of independent games with the following rules:

   - At each play, if a player has more than £2 and loses, he pays his opponent £2; if he has £2 or less he pays his opponent £1.
   - A starts with £j and B starts with £$(a - j)$, where a is even.
   - The game ends as soon as either player loses all their money (is ruined).
   - A and B have equal probability of winning each game.

   Let q_j be the probability that A is ruined when he starts with £j.

   (a) Find an expression for q_j in terms of q_{j-2} and q_{j+2}, for $j \in \{3, 4, \ldots, a - 4, a - 3\}$.

   (b) Find alternative expressions for q_j for $j = 1, 2, (a - 2), (a - 1)$.

   (c) Show that, for $j \in \{3, 4, \ldots, a - 4, a - 3\}$,

   $$
   q_j = \begin{cases}
   1 - \dfrac{(a - 2)j/a + 2}{a + 2} & j \text{ even,} \\[2mm]
   1 - \dfrac{j + 1}{a + 2} & j \text{ odd,}
   \end{cases}
   $$

   satisfies the expression found in part (ii)(a).

   (d) Determine q_1 and q_2 in terms of a.

4. (i) What is meant by saying that a Markov chain is

   (a) irreducible?

   (b) aperiodic?

   (ii) Consider a game similar to the one described in Question 3(ii), but where neither player can be ruined, i.e. A and B play a sequence of independent games with the following rules:

   - At each play, if a player has more than £2 and loses, he pays his opponent £2; if he has £2 or less he pays his opponent £1.
   - A starts with £j and B starts with £$(a - j)$, where a is even.
   - If a player loses his last £1 then his opponent returns it on the next game.
   - A and B have equal probability of winning each game.

   (a) Write down the transition matrix for A's fortune in this game if $a = 4$.

   (b) Draw the corresponding transition diagram and identify the periodicity of any closed communicating classes.

   (c) Does a unique stationary distribution exist? Determine the stationary distribution if it exists.

   (d) What condition is needed for a stationary distribution to also be a limiting distribution? Is this condition met for this system?

   (e) In the long run, for what proportion of time does player As fortune exceed Bs by more than £2?

5. (i) For a continuous time, discrete state space Markov process with transition matrix $P(t)$ and transition rate matrix Q

   (a) write down the corresponding Kolmogorov forward equation,

   (b) write down the corresponding equations satisfying the stationary distribution $\boldsymbol{\pi}$, if it exists.

   (ii) Consider the linear birth and death process with

   $$
   \begin{aligned}
   p_{n,n+1}(\delta t) &= \beta n \delta t + o(\delta t) & n = 0, 1, 2, \ldots \\
   p_{n,n-1}(\delta t) &= \nu n \delta t + o(\delta t) & n = 1, 2, 3, \ldots \\
   p_{i,j}(\delta t) &= o(\delta t) & |i - j| > 1.
   \end{aligned}
   $$

   (a) What is the Q-matrix for this process?

   (b) From the forward equations, show that the associated differential equation for the pgf, $\Pi_i(s, t)$, of $X(t)$, the number in the population at time t given $X(0) = i$ is given by
   $$
   \frac{\partial}{\partial t} \Pi_i(s, t) = (\beta s^2 - (\nu + \beta)s + \nu) \frac{\partial}{\partial s} \Pi_i(s, t).
   $$

   (c) The solution to the differential equation in part (ii)(b) is given by
   $$
   \Pi_i(s, t) = \left(\frac{\nu(1 - s) - (\nu - \beta s)e^{(\nu - \beta)t}}{\beta(1 - s) - (\nu - \beta s)e^{(\nu - \beta)t}} \right)^i.
   $$

   When $X(0) = i$, determine the survival function $S(t) = 1 - F_T(t)$, where $F_T(t)$ is the cumulative distribution function of T, the time to extinction.

   (d) Hence or otherwise, find an expression in terms of ν and β for the mean time to extinction when $X(0) = 1$ and $\beta < \nu$.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	PGF $\Pi_X(s)$
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta s$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta s)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(s-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta s}{1-s(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta s}{1-s(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-s(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t)$$

$$\mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2 t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		$0\quad(\text{if }\nu>1)$	$\dfrac{\nu}{\nu-2}\quad(\text{if }\nu>2)$	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	