

Probability for Statistics MATH50010

Unseen Problem 1

Let Ω be a finite set. Does there exist an event space, i.e. an algebra of sets \mathcal{F} on Ω , with precisely six elements?

More generally, for what positive integer values N does there exist an algebra of sets of size N ?

Solution

We will show the following general result: if $|\Omega| = n \geq 1$, then the sigma algebras on Ω have size 2^k for some $1 \leq k \leq n$.

Let \mathcal{F} be an algebra on Ω . For each $x \in \Omega$, define

$$A_x = \bigcap_{B \in \mathcal{F}: x \in B} B,$$

the intersection of all sets in \mathcal{F} containing x . By construction, each $A_x \in \mathcal{F}$.

Key point: Think back to the idea of an event space as representing the observable outcomes of an experiment. The experiment may not be able to distinguish between all outcomes: e.g. considering the 12 balls problem, the experiment that consists of weighing $\{1, 2, 3, 4\}$ against $\{5, 6, 7, 8\}$ cannot distinguish between outcomes 1 and 2. So, for this experiment, for any event $E \in \mathcal{F}$, either both $1, 2 \in E$ or $1, 2 \notin E$. The event A_x defined above is the collection of all outcomes that cannot be distinguished from x . Below, we will show that sets of the form A_x partition Ω .

Note that for $x, y \in \Omega$, either $A_x = A_y$ or $A_x \cap A_y = \emptyset$. To see this, suppose for contradiction that there exists $z \in A_x \cap A_y$ but $y \notin A_x$. Then for some $B \in \mathcal{F}$, we have that $x \in B$ but $y \notin B$. But then $y \in B^c$ so that $A_y \subseteq B^c$ (since $B^c \in \mathcal{F}$), giving

$$z \in A_x \cap A_y \subseteq A_y \subseteq B^c.$$

But $z \in A_x$ so $z \in B$, a contradiction. Hence $y \in A_x$ so that $A_y \subseteq A_x$, and so by symmetry in fact $A_x = A_y$.

It follows that the set Ω is partitioned by sets A_{x_1}, \dots, A_{x_k} for distinct elements $x_1, \dots, x_k \in \Omega$. This partition corresponds to the equivalence relation \sim on Ω where $x \sim y$ if and only if there is no $E \in \mathcal{F}$ that distinguishes between x and y , i.e. no $E \in \mathcal{F}$ has $x \in E$ and $y \notin E$ or $y \in E$ and $x \notin E$.

Now for any non-empty $B \in \mathcal{F}$, if $x \in B$, we must in fact have $A_x \subseteq B$, so that each non-empty B can be written as

$$B = A_{y_1} \cup \dots \cup A_{y_l}$$

for distinct elements $y_1, \dots, y_l \in \{x_1, \dots, x_k\}$. Hence there are 2^k distinct elements of \mathcal{F} , corresponding to the subsets of $\{x_1, \dots, x_k\}$.

In conclusion, only N of the form 2^k for some positive integer k can be the size of an algebra of sets. Moreover, for a set Ω with $|\Omega| = n \geq 1$, the above argument shows how to construct an algebra of sets of size 2^k for any $1 \leq k \leq n$: simply partition Ω into k equivalence classes.