1(a). Graph I:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \tag{1}$$

Graph II:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Graph III:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{3}$$

1(b). In this question, any vector in these spaces can be written as a *linear combination* of the following basis vectors:

Graph I ($\mathbf{A}$):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4}$$

Graph I ($\mathbf{A}^T$): Note that a right null vector of $\mathbf{A}^T$ corresponds to a left null vector of $\mathbf{A}$ (to see this, just take a transpose) which correspond to loops in the graph.

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{5}$$

Graph II ($\mathbf{A}$):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{6}$$

Graph II ($\mathbf{A}^T$):

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{7}$$

Graph III ($\mathbf{A}$):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{8}$$

Graph III ($\mathbf{A}^T$):

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

1(c). The degree matrix $\mathbf{D}$ for each graph:

Graph I

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{10}$$

Graph II

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{11}$$

Graph III

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{12}$$

1(d). The adjacency matrix $\mathbf{W}$ for each graph:

Graph I

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{13}$$

Graph II

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{14}$$

Graph III

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{15}$$

1(e). The graph Laplacian is given by $\mathbf{K} \equiv \mathbf{A}^T\mathbf{A} = \mathbf{D} - \mathbf{W}$. It is therefore easy to deduce these from the previous answers.

1(f). **Completeness** of graphs can be checked by examining if all nodes are connected to each other.

Graph I: Complete

Graph II: Not complete

Graph III: Complete

2(a). We define incidence matrices of Graph I, II, III to be $\mathbf{A}^{\mathrm{I}}$, $\mathbf{A}^{\mathrm{II}}$, and $\mathbf{A}^{\mathrm{III}}$, respectively. Because these graphs are disconnected, the incidence matrix $\mathcal{A}$ of this new single graph is

$$\mathcal{A} = \begin{pmatrix} \mathbf{A}^{\mathrm{I}} & & O \\ & \mathbf{A}^{\mathrm{II}} & \\ O & & \mathbf{A}^{\mathrm{III}} \end{pmatrix}, \tag{16}$$

where O is short-hand for filling in the rest of the matrix (off the diagonal sub-blocks) with zero elements.

The rank of $\mathbf{A}^{\mathrm{I}}$, $\mathbf{A}^{\mathrm{II}}$, and $\mathbf{A}^{\mathrm{III}}$ is 2, 3, and 3 respectively, so the rank of $\mathcal{A}$ is **8**.

2(b). From 1(b), we collect all vectors in the null spaces of $\mathbf{A}$ of each graph I–III and pad them with an appropriate set of zeros (to make up an 11-dimensional vector). The right null vectors of $\mathcal{A}$ are

$$\boldsymbol{x} = \begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\1\end{pmatrix}. \tag{17}$$

2(c). We define the i-th right null vector of graph j ($j = \mathrm{I}, \mathrm{II}, \mathrm{III}$) as $\boldsymbol{v}_i^j$ $(i = 1, 2, \cdots)$. For example,

$$\boldsymbol{v}_1^{\mathrm{I}} = \begin{pmatrix}1\\1\\-1\end{pmatrix}, \qquad \boldsymbol{v}_1^{\mathrm{II}} = \begin{pmatrix}1\\0\\0\\1\\-1\end{pmatrix}, \qquad \boldsymbol{v}_1^{\mathrm{III}} = \begin{pmatrix}1\\0\\0\\-1\\1\\0\end{pmatrix}. \tag{18}$$

Because graphs I, II, and III are disconnected, linearly independent solutions (the left null vectors of $\mathcal{A}$) are

$$\boldsymbol{w} = \begin{pmatrix}\boldsymbol{v}_1^{\mathrm{I}}\\ \mathbf{O}_5\\ \mathbf{O}_6\end{pmatrix}, \begin{pmatrix}\mathbf{O}_3\\ \boldsymbol{v}_1^{\mathrm{II}}\\ \mathbf{O}_6\end{pmatrix}, \begin{pmatrix}\mathbf{O}_3\\ \boldsymbol{v}_2^{\mathrm{II}}\\ \mathbf{O}_6\end{pmatrix}, \begin{pmatrix}\mathbf{O}_3\\ \mathbf{O}_5\\ \boldsymbol{v}_1^{\mathrm{III}}\end{pmatrix}, \begin{pmatrix}\mathbf{O}_3\\ \mathbf{O}_5\\ \boldsymbol{v}_2^{\mathrm{III}}\end{pmatrix}, \begin{pmatrix}\mathbf{O}_3\\ \mathbf{O}_5\\ \boldsymbol{v}_3^{\mathrm{III}}\end{pmatrix}, \tag{19}$$

where $\mathbf{O}_n$ denotes a zero vector with n elements. There are 6 of these, in accordance with the rank-nullity theorem $(14 - 8 = 6)$.

3(a). We know that the diagonal element of the Laplacian matrix K_{ii} is the number of edges at each node, and the off-diagonal element K_{ij} is -1 if node i is connected to node j, all other elements are zero.

The quickest way to compute the number of zero elements is to count the non-zero elements as follows. Notice that there are 9 nodes and 12 edges. All diagonal elements will be non-zero, and there are 9 of these; each edge produce 2 non-zero elements because a -1 will appear in K_{ij} (for $i \neq j$) as well as in K_{ji}. Hence the total number of non-zero elements is

$$9 + 12 \times 2 = 33. \tag{20}$$

The number of zero elements is therefore

$$81 - 33 = 48. \tag{21}$$

Another argument goes as follows. For each row of $\mathbf{K}$, corresponding to a given node, the number of zeros in that row is the number of nodes that are **not** connected to the given node by an edge. By the symmetry of this graph there are 3 types of nodes: 4 corner nodes, 4 nodes in the middle of each side, and one central node. Each corner node is **not** connected to 6 other nodes; each middle-side node is **not** connected to 5 other nodes; the central node is **not** connected to 4 nodes. The total number of zeros in the Laplacian is therefore

$$\underbrace{4 \times 6}_{\text{from corner nodes}} + \underbrace{4 \times 5}_{\text{from middle}-\text{side nodes}} + \underbrace{1 \times 4}_{\text{from middle node}} = 48. \tag{22}$$

3(b). The degree matrix is the diagonal matrix containing the number of nodes connected to each node. Using the node numbering given in the figure:

$$D_0 = \text{diag}(2, 3, 2, 3, 4, 3, 2, 3, 2). \tag{23}$$

4. Consider a graph having n nodes and n edges. It must have at least one connected subgraph and hence the graph's n-by-n incidence matrix $\mathbf{A}$ will have a corresponding right null vector with all ones in components corresponding to the nodes in this connected subgraph (and zeros elsewhere) meaning that the rank is $n - 1$ or less. By rank-nullity there must therefore be at least one left null-vector, or equivalently, a vector in the nullspace of $\mathbf{A}^T$. This corresponds to a loop.

5(a). Graphs **(a)** and **(b)** are connected so the dimension of the right null space of their incidence matrices is 1. Hence the rank of those matrices is 7 (since there are $n = 8$ nodes). For graph **(c)** the graph is disconnected, with two connected components, so the dimension of the right null space of its incidence matrix is 2 making its rank equal to 6 (= $n - 2$).

5(b). The rank-nullity theorem says that the dimension of the left null space of the incidence matrix $\mathbf{A}$ is

$$m - r,$$

where m is the number of rows of $\mathbf{A}$ and r is its rank. We therefore need to compute m for each graph. Graph **a** is connected, and complete, with $n = 8$ nodes; each node is connected to $n - 1 = 7$ other nodes making a total of

$$8 \times 7 = 56 \tag{24}$$

connections but this is *twice* the number of edges (since each one is double counted) hence $m = 28$ for graph **(a)**.

For graph **(b)** we find $m = 14$ (just by counting).

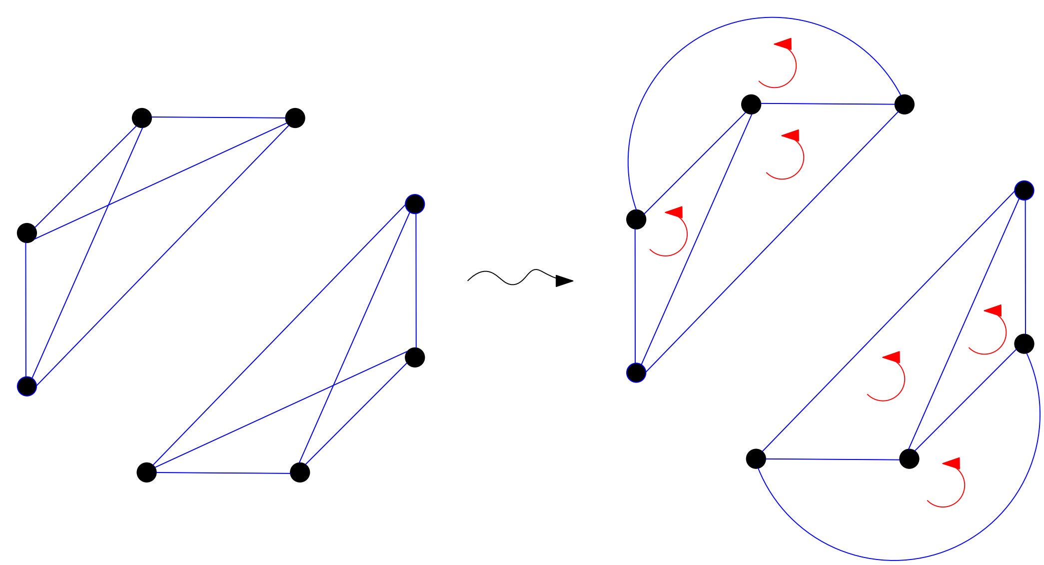
Similarly, for graph **(c)** we find $m = 12$ (just by counting).

The required dimensions of the left null space are therefore

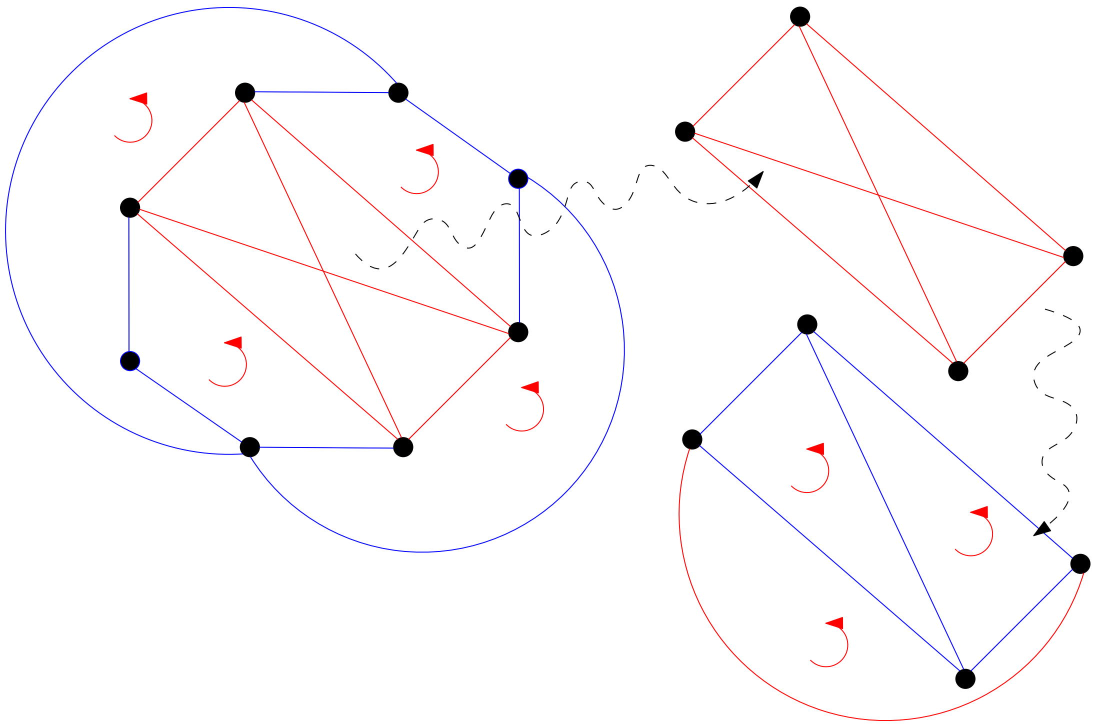
$$\begin{array}{rr} \text{graph } (\mathbf{a}): & 21 \\ \text{graph } (\mathbf{b}): & 7 \\ \text{graph } (\mathbf{c}): & 6 \end{array} \tag{25}$$

5(c). We know that elements of the left null space of an incidence matrix correspond to closed loops in a graph. A basis for the left null space can therefore be constructed geometrically by identifying such (independent) loops (with an obvious interpretation of "adding" loops).

It is easiest to start with graph **(c)** and to redraw the graph as follows so that the choice of the 6 independent loops representing the left null space of the incidence matrix become obvious:



For graph **(b)** by redrawing as in the following figure (on the left) we can identify a subgraph (shown red) looking like the two subgraph components making up graph **(c)**:



Therefore, the 4 loops shown on the left, together with the 3 loops associated with the subgraph shown on the right, form an independent basis of $4 + 3 = 7$ loops representing the left null space of the incidence matrix.

6. For this question it is useful to note that $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$ and that for $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$.

(a)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{26}$$

(b)

$$\mathbf{K}\boldsymbol{x_n} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix} = (2 - \omega^n - \omega^{2n}) \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix}. \tag{27}$$

Therefore, $\lambda_n = 2 - \omega^n - \omega^{2n}$:

$$\lambda_0 = 2 - \omega^0 - \omega^0 = 0, \qquad (n = 0), \tag{28}$$

$$\lambda_1 = 2 - \omega - \omega^2 = 3, \qquad (n = 1), \tag{29}$$

$$\lambda_2 = 2 - \omega^2 - \omega^4 = 3, \qquad (n = 2). \tag{30}$$

(c) For any integer n, $n \geq 0$, we can classify $n = 3k$, $n = 3k+1$, $n = 3k+2$. It is easy to check $\lambda_{3k} = \lambda_0$, $\lambda_{3k+1} = \lambda_1$, and $\lambda_{3k+2} = \lambda_2$. Therefore it is only necessary to consider $n = 0, 1, 2$.

(d)

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}. \tag{31}$$

It is easy to check that both $\overline{\boldsymbol{x}_0}^T \boldsymbol{x}_1$ and $\overline{\boldsymbol{x}_0}^T \boldsymbol{x}_2$ equal zero because $1 + \omega + \omega^2 = 0$. Furthermore,

$$\overline{\boldsymbol{x}}_1^T \boldsymbol{x}_2 = 1 + \bar{\omega}\omega^2 + \overline{\omega^2}\omega = 1 + \omega + \frac{1}{\omega} = \frac{1}{\omega}(1 + \omega + \omega^2) = 0, \tag{32}$$

where the relations of $\overline{\omega}\omega = \overline{\omega^2}\omega^2 = 1$ have been used.

7(a). It is easy to check, by direct computation of the characteristic equation, i.e., evaluation of the determinant

$$\det(\mathbf{C} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 & -1 \\ -1 & 3-\lambda & 0 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0, \tag{33}$$

that the eigenvalues λ of $\mathbf{C}$ are

$$\lambda_n = 3 - \omega^n, \qquad n = 0, 1, 2, \tag{34}$$

where

$$\omega = e^{2\pi \mathrm{i}/3} \tag{35}$$

is a third root of unity. The eigenvector corresponding to $\lambda = 2$ is $(1, 1, 1)^T$; the eigenvector corresponding to $\lambda = 3 - \omega$ is $(1, \omega^2, \omega^4)^T$; the eigenvector corresponding to $\lambda = 3 - \omega^2$ is $(1, \omega, \omega^2)^T$.

(b) The eigenvectors are the same as those of the Laplacian matrix in part 6(b).

(c) Any matrix of the form

$$\begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} \tag{36}$$

has the same eigenvectors (later in the course, we will see why - these are called *circulant matrices*).

8. This question is based on the same idea as question 6 except that the number of nodes, N, in the "ring" of nodes changes from $N = 4$ up to $N = 6$. Students should see a pattern emerging. For example, when $N = 4$, the 4 eigenvalues are

$$\lambda_n = 2 - \omega^n - \omega^{3n}, \qquad n = 0, 1, 2, 3, \tag{37}$$

where $\omega = e^{2\pi \mathrm{i}/4}$ and we can use $1 + \omega + \omega^2 + \omega^3 = 0$. The corresponding vectors $\mathbf{x}_n$ ("eigenvectors") can be verified to be

$$\boldsymbol{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \omega^{3n} \end{pmatrix} \qquad n = 0, 1, 2, 3. \tag{38}$$

For general N we find

$$\lambda_n = 2 - \omega^n - \omega^{(N-1)n} = 2 - \omega^n - \frac{1}{\omega^n}, \qquad n = 0, 1, 2, \ldots, N-1, \tag{39}$$

where $\omega = e^{2\pi \mathrm{i}/N}$. The corresponding vectors $\mathbf{x}_n$ ("eigenvectors") are

$$\boldsymbol{x}_n = \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \cdot \\ \cdot \\ \omega^{(N-1)n} \end{pmatrix} \qquad n = 0, 1, 2, \cdots, N-1. \tag{40}$$

9(a). The general form of $\mathbf{K}$ is the n-by-n matrix

$$\mathbf{K} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdots & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}. \tag{41}$$

(b) Notice that $\mathbf{K}$ can be written

$$\mathbf{K} = n\mathbf{I} - \mathbf{J}, \tag{42}$$

where $\mathbf{I}$ is the n-by-n identity matrix and $\mathbf{J}$ is the rank-one n-by-n matrix of all ones:

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{43}$$

By rank-nullity, since $\mathbf{J}$ has rank 1, it has $n-1$ right null vectors which are easy to work out. They are the following n-dimensional vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ . \\ . \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{x}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \\ -1 \end{pmatrix}. \tag{44}$$

These satisfy

$$\mathbf{J}\mathbf{x}_j = 0, \qquad j = 1, \cdots, n-1. \tag{45}$$

Moreover the vector

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \tag{46}$$

clearly satisfies

$$\mathbf{J}\mathbf{x}_0 = n\mathbf{x}_0. \tag{47}$$

By (42) the vectors $\{\mathbf{x}_j | j = 0, 1, \cdots, n-1\}$ just found satisfy

$$\mathbf{K}\mathbf{x}_0 = (n\mathbf{I} - \mathbf{J})\mathbf{x}_0 = n\mathbf{x}_0 - n\mathbf{x}_0 = 0, \tag{48}$$

and

$$\mathbf{K}\mathbf{x}_j = (n\mathbf{I} - \mathbf{J})\mathbf{x}_j = n\mathbf{x}_j, \qquad j = 1, \cdots, n-1. \tag{49}$$

Hence we have found the required vectors and associated values of $\lambda = 0, \underbrace{n, n, \cdots, n}_{n-1 \text{ times}}$.

(c) The general form of $\mathbf{K}_0$ is the same as $\mathbf{K}$ except you can remove the last column and row making it now an $(n-1)$-by-$(n-1)$ matrix.

(d) The arguments here are exactly as in part (b). We write

$$\mathbf{K}_0 = n\hat{\mathbf{I}} - \hat{\mathbf{J}}, \tag{50}$$

where $\hat{\mathbf{I}}$ is the $(n-1)$-by-$(n-1)$ identity matrix and $\hat{\mathbf{J}}$ is the rank-one $(n-1)$-by-$(n-1)$ matrix of all ones. Now it can be shown that the modified $(n-1)$-dimensional vectors given by

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ . \\ . \\ 0 \end{pmatrix}, \cdots, \hat{\mathbf{x}}_{n-2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \\ -1 \end{pmatrix} \tag{51}$$

(there are $(n-2)$ of these) satisfy

$$\hat{\mathbf{J}}\hat{\mathbf{x}}_j = 0, \qquad j = 1, \cdots, n-2 \tag{52}$$

and the $(n-1)$-dimensional vector

$$\hat{\mathbf{x}}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}, \tag{53}$$

satisfies

$$\hat{\mathbf{J}}\hat{\mathbf{x}}_0 = (n-1)\hat{\mathbf{x}}_0. \tag{54}$$

Hence these vectors $\{\hat{\mathbf{x}}_j | j = 0, 1, \cdots, n-1\}$ satisfy

$$\mathbf{K}_0\hat{\mathbf{x}}_0 = (n\hat{\mathbf{I}} - \hat{\mathbf{J}})\hat{\mathbf{x}}_0 = n\hat{\mathbf{x}}_0 - (n-1)\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0, \tag{55}$$

and

$$\mathbf{K}_0\hat{\mathbf{x}}_j = (n\hat{\mathbf{I}} - \hat{\mathbf{J}})\hat{\mathbf{x}}_j = n\hat{\mathbf{x}}_j, \qquad j = 1, \cdots, n-1. \tag{56}$$

We have therefore found the required vectors and associated values of $\lambda = 1, \underbrace{n, n, \cdots, n}_{n-2 \text{ times}}$.

(e) By explicitly computing the inverses of $\mathbf{K}_0$ for $n = 2, 3, 4$ and spotting a pattern we can guess that, for general n,

$$\mathbf{K}_0^{-1} = \frac{1}{n} \begin{bmatrix} 2 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 2 & \cdots & \cdots & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & \cdots & \cdots & 1 & 2 \end{bmatrix} \tag{57}$$

and it is easily verified that this is the correct inverse: i.e., check directly that $\mathbf{K}_0 \mathbf{K}_0^{-1} = \mathbf{I}$.