Chapter 3: Special Functions

In this chapter we will consider 'special' functions! This is not a formal definition by any means, but I like to call the functions we will investigate special due to some of their magical properties as well as their uses and prevalence. We will see that these functions are important for the following reasons:

- Many elementary functions (e.g., exponentials, logarithms, trigonometric functions) can be represented in terms of these functions;
- They provide solutions to an important class of ODEs (see later).

We will start by introducing the Gamma function.

3.1 The Gamma Function

For $\text{Re}\{z\} > 0$, we can define the Gamma function, $\Gamma(z)$, via the following integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{50}$$

As mentioned, this integral representation holds for $\text{Re}\{z\} > 0$. Let's show that indeed we have convergence in this case:

We can write $t^{z-1} = e^{(z-1)\log t}$. Since, t is real, this means $|t^{z-1}e^{-t}| = t^{\text{Re}\{z\}-1}e^{-t}$.

Potential Problems:

1. For $\text{Re}\{z\} > 1, |t^{z-1}| \to \infty$ as $t \to \infty$.
2. For $\text{Re}\{z\} < 1, |t^{z-1}| \to \infty$ as $t \to 0$.

In case 1), since e^{-t} decays faster than t^{z-1} grows (exponential decay beats algebraic growth), then the integral converges.

In case 2), split the integral as follows:

$$\int_0^\infty t^{z-1}e^{-t}dt = \int_0^1 t^{z-1}e^{-t}dt + \int_1^\infty t^{z-1}e^{-t}dt.$$

Now, for this case ($\text{Re}\{z\} < 1$):

(i). The second part $\int_1^\infty t^{z-1}e^{-t}dt$ converges trivially; (ii). For the first part $\int_0^1 t^{z-1}e^{-t}dt$, see that, for $0 < t < 1$, then:

$$|t^{z-1}e^{-t}| \le |t^{z-1}| = t^{\text{Re}\{z\}-1} = t^{x-1},$$

by writing $x = \text{Re}\{z\}$. Then:

$$\left|\int_0^1 t^{z-1}e^{-t}dt\right| \le \int_0^1 t^{x-1}dt = \left[\frac{t^x}{x}\right]_{t=0}^1 = \frac{1}{x},$$

so this integral exists provided $x > 0$, i.e.Re$\{z\} > 0$.

The Gamma function as defined by (50) is analytic (one can check that for Re$\{z\} > 0$ we have no singularities $\rightarrow$ the integral converges) and in fact:

$$\begin{aligned}\frac{d\Gamma(z)}{dz} &= \int_0^\infty \frac{d}{dz}(t^{z-1}e^{-t})dt,\\ &= \int_0^\infty \frac{d}{dz}(e^{(z-1)\log t}e^{-t})dt,\\ &= \int_0^\infty (\log t)t^{z-1}e^{-t}dt.\end{aligned}$$

Now, using integration by parts, integrate (50) as follows:

$$\begin{aligned}\Gamma(z) &= \int_0^\infty \frac{1}{z}\frac{d}{dt}(t^z)e^{-t}dt\\ &= \frac{1}{z}\left[\left[e^{-t}t^z\right]_0^\infty + \int_0^\infty t^z e^{-t}dt\right]\\ &= \frac{1}{z}\int_0^\infty t^z e^{-t}dt \qquad (\text{as Re}\{z\} > 0)\\ &= \frac{1}{z}\Gamma(z+1).\end{aligned}$$

Thus we have the identity:

$$\boxed{\Gamma(z+1) = z\Gamma(z)}, \tag{51}$$

for all z such that Re$\{z\} > 0$.

Note: Suppose $z = n \in \mathbb{N}$, then, $(n-1)! = (n-1)(n-2)\cdots 2\cdot 1$

$$\begin{aligned}&\implies (n-1) = \frac{\Gamma(n)}{\Gamma(n-1)}, \qquad (n-2) = \frac{\Gamma(n-1)}{\Gamma(n-2)}, \qquad \text{etc}\cdots\\ &\implies (n-1)! = \frac{\Gamma(n)}{\Gamma(n-1)}\frac{\Gamma(n-1)}{\Gamma(n-2)}\cdots\frac{\Gamma(3)}{\Gamma(2)}\frac{\Gamma(2)}{\Gamma(1)} = \frac{\Gamma(n)}{\Gamma(1)},\end{aligned}$$

and:

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \left[-e^{-t}\right]_0^\infty = 1,$$

which gives:

$$\boxed{(n-1)! = \Gamma(n)}, n \in \mathbb{N}. \tag{52}$$

So, one may regard the Gamma function as an extension of the factorial function.

Analytic Continuation

Now let's show how we can use the identity (51) to accomplish analytic continuation, allowing us to define $\Gamma(z)$ for Re$\{z\} \leq 0$.

From (51) we have:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

By our definition of the Gamma function, we have:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt,$$

and this is defined for $\mathrm{Re}\{z\} > -1$ (for the same reason our definition of $\Gamma(z)$ holds for $\mathrm{Re}\{z\} > 0$).

Hence, by analytic continuation:

$$\Gamma(z) = \frac{\int\limits_0^\infty t^z e^{-t} dt}{z},$$

defines $\Gamma(z)$ for: $-1 < \mathrm{Re}\{z\} \leq 0$. Thus $\Gamma(z)$ is analytic in $\mathrm{Re}\{z\} > -1$, except for a singularity at $z = 0$, see the shaded region in figure 40.

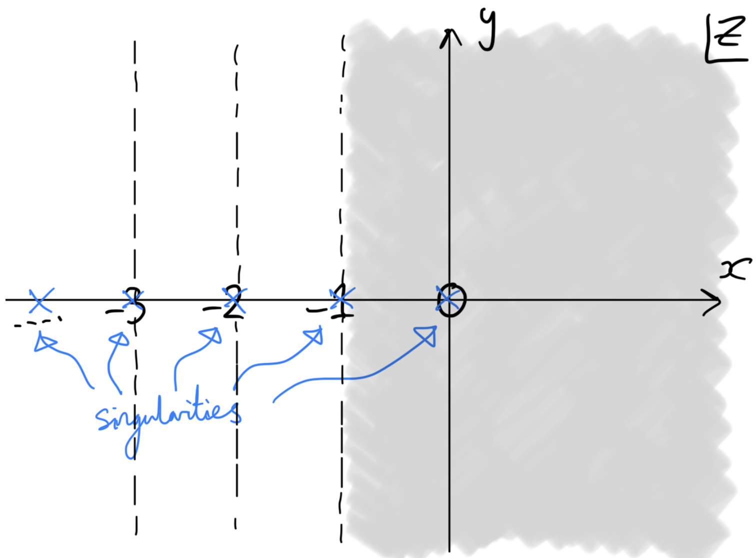

Figure 40: Analytic Continuation of the Gamma Function.

Applying the recurrence relation repeatedly, we get:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)}.$$

The right hand side provides analytic continuation of $\Gamma(z)$ to $-2 < \Re\{z\} \leq -1$, except for a singularity at $z = -1$.

$$\implies \Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \cdots = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\cdots(z+1)z}.$$

This process can be repeated indefinitely and thus provides a definition of $\Gamma(z)$ for all z. We see that $\Gamma(z)$ is analytic for all (finite)z, except for all non-positive integers $0, -1, -2, \cdots$.

Let's check the nature of these singularities:

Consider $z = -n + \delta$, where $n = 0, 1, 2, 3, \cdots$ and $\delta \to 0$. We have:

$$\begin{aligned}\Gamma(-n+\delta) &= \frac{\Gamma(-n+\delta+n+1)}{(-n+\delta+n)(-n+\delta+n-1)\cdots(-n+\delta+1)(-n+\delta)} \\ &= \frac{\Gamma(1+\delta)}{\delta(\delta-1)\ldots(\delta-n)}.\end{aligned}$$

so, as $\delta \to 0$,

$$\begin{aligned}\Gamma(-n+\delta) &= \frac{\Gamma(1)}{\delta(-1)^n n!} + O(\delta^0) \\ &= \frac{1}{\delta(-1)^n n!} + O(\delta^0).\end{aligned}$$

So we can see that $\Gamma(z)$ has a simple pole at $z = -n$ for $n = 0, -1, -2, \ldots$ with residue:

$$\text{Res}\{\Gamma(z), -n\} = \frac{(-1)^n}{n!}.$$

These are the **only** singularities of $\Gamma(z)$.

Second Identity

There is a second important identity satisfied by $\Gamma(z)$ for all z. This is:

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}} \tag{53}$$

Let's verify this as follows. We restrict ourselves to $z = \lambda \in [0, 1]$. If we can show it is true for z on this segment of the real axis, then by analytic continuation, it is true for all z. We have:

$$\begin{aligned}\Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt \\ &= 2\int_0^\infty x^{2z-1} e^{-x^2} dx.\end{aligned}$$

where we have substituted $t = x^2$, so then $dt = 2xdx$. Then, we have:

$$\begin{aligned}\Gamma(\lambda)\Gamma(1-\lambda) &= 4\left(\int_0^\infty x^{2\lambda-1} e^{-x^2} dx\right)\left(\int_0^\infty y^{1-2\lambda} e^{-y^2} dy\right) \\ &= 4\int_0^\infty\int_0^\infty \left(\frac{x}{y}\right)^{2\lambda-1} e^{-(x^2+y^2)} dxdy\end{aligned}$$

Introduce the polar coordinates: $x = r\cos\phi, y = r\sin\phi \implies dxdy = rdrd\phi$.
Then we get:

$$\begin{aligned}\Gamma(\lambda)\Gamma(1-\lambda) &= 4\int_{\phi=0}^{\frac{\pi}{2}}\int_{r=0}^\infty (\cot\phi)^{2\lambda-1} e^{-r^2} rdrd\phi \\ &= 4\int_0^{\frac{\pi}{2}} (\cot\phi)^{2\lambda-1}\left(\int_0^\infty re^{-r^2} dr\right) d\phi.\end{aligned}$$

Here $\int_0^\infty re^{-r^2}dr = \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = \frac{1}{2}$. So then:

$$\Gamma(\lambda)\Gamma(1-\lambda) = 2\int_0^{\frac{\pi}{2}} (\cot\phi)^{2\lambda-1} d\phi.$$

Now take $u = \cot\phi = 1/\tan\phi \implies du = -\mathrm{cosec}^2\phi d\phi \implies d\phi = \frac{-du}{1+\cot^2\phi} \implies d\phi = \frac{-du}{1+u^2}$. Then

$$\begin{aligned}\Gamma(\lambda)\Gamma(1-\lambda) &= -2\int_\infty^0 \frac{u^{2\lambda-1}}{1+u^2}du \\ &= 2\int_0^\infty \frac{u^{2\lambda-1}}{1+u^2}du.\end{aligned}$$

Note: $0 < \lambda < 1 \implies -1 < 2\lambda - 1 < 1$. Let $\alpha = 2\lambda - 1$, then we have:

$$\Gamma(\lambda)\Gamma(1-\lambda) = 2\int_0^\infty u^{\alpha-1}Q(u)du, \quad \text{where} \quad Q(u) = \frac{u}{1+u^2}.$$

Recall from earlier in the course, we evaluated similar integrals, we used a 'keyhole' contour of the form depicted in figure 41.

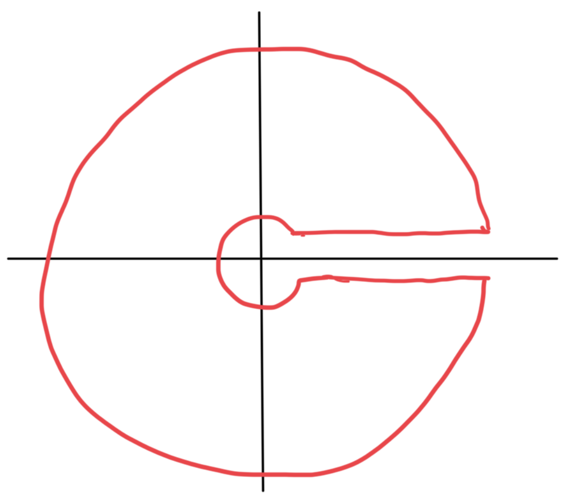

Figure 41: A depiction of a 'keyhole' contour.

One can check that:

$$\int_0^\infty u^{\alpha-1}Q(u)du = \frac{-\pi e^{-i\pi\alpha}}{\sin(\pi\alpha)} \sum \left(\text{Residues of } u^{\alpha-1}Q(u)\right).$$

Now, $Q(u)$ has poles at $\pm i$. Taking the branch cut (since we have $z^{\alpha-1}$ which is in general a non-integer power) along the positive real axis we have: $i = e^{i\frac{\pi}{2}}$ and $-i = e^{\frac{3\pi}{2}i}$ (rather than, say $-i = e^{-i\frac{\pi}{2}}$).

Thus, we get:

$$\begin{aligned}
\Gamma(\lambda)\Gamma(1-\lambda) &= \frac{-2\pi e^{-i\pi\alpha}}{\sin(\pi\alpha)} \underbrace{\left(\frac{e^{\frac{i\pi\alpha}{2}} - e^{\frac{3i\pi\alpha}{2}}}{2i}\right)}_{\text{Exercise: Check this}} \\
&= \frac{-2\pi e^{-2\pi i\lambda + i\pi}}{\sin(2\pi\lambda - \pi)} \left(\frac{e^{i\pi\lambda - \frac{i\pi}{2}} - e^{3i\pi\lambda - \frac{3i\pi}{2}}}{2i}\right) \\
&= \underbrace{\cdots}_{\text{Exercise: Check this}} \\
&= \frac{2\pi e^{-2\pi i\lambda}}{\sin(2\pi\lambda)} \left(\frac{e^{i\pi\lambda} + e^{3i\pi\lambda}}{2}\right) \\
&= \frac{2\pi\cos(\pi\lambda)}{\sin(2\pi\lambda)} \\
&= \frac{2\pi\cos(\pi\lambda)}{2\sin(\pi\lambda)\cos(\pi\lambda)} \\
&= \boxed{\frac{\pi}{\sin(\pi\lambda)}}.
\end{aligned}$$

So, $\Gamma(\lambda)\Gamma(1-\lambda)\sin(\pi\lambda) = \pi$ for $\lambda \in [0,1]$. Hence, by analytic continuation to all of z we deduce the result.
Note: Special Case: take $z = \frac{1}{2}$: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Summary of $\Gamma(z)$:

1. Integral representation, valid for z such that $\text{Re}\{z\} > 0$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

2. $\Gamma(z)$ is analytic everywhere except for simple poles at $z = 0, -1, -2, \cdots$

3. Identities:

   (i). $\Gamma(z+1) = z\Gamma(z)$;

   (ii). $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.

3.2 The Beta Function

Another related function is the Beta function, also known as the Euler integral of the first kind (the Gamma function is known as the Euler integral of the second kind).

For $\text{Re}\{z\} > 0, \text{Re}\{w\} > 0$, we define the Beta function $B(z, w)$ by the integral:

$$\boxed{B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt}. \tag{54}$$

(We require the restrictions on $\text{Re}\{z\} > 0$ and $\text{Re}\{w\} > 0$ in order for the integral in (54) to converge)

First notice that $B(z,w) = B(w,z)$. This can be shown by considering $u = 1-t$ substituted into (54).

The following is an identity relating the Beta and Gamma functions:

$$\boxed{B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}}. \tag{55}$$

Let's prove this identity.

Proof.

$$\Gamma(z) = 2\int_0^\infty x^{2z-1}e^{-x^2}dx \quad \text{(recall from earlier)}.$$

Then:

$$\begin{aligned}\Gamma(z)\Gamma(w) &= 4\left(\int_0^\infty x^{2z-1}e^{-x^2}dx\right)\left(\int_0^\infty y^{2w-1}e^{-y^2}dy\right)\\ &= 4\int_0^\infty\int_0^\infty x^{2z-1}y^{2w-1}e^{-(x^2+y^2)}dxdy\end{aligned}$$

Introduce the Polar coordinates: $x = r\cos\phi, y = r\sin\phi \implies dxdy = rdrd\phi$. Then:

$$\begin{aligned}\Gamma(z)\Gamma(w) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}}(r\cos\phi)^{2z-1}(r\sin\phi)^{2w-1}e^{-r^2}rdrd\phi\\ &= 4\left(\int_0^\infty r^{2(z+w-1)}e^{-r^2}rdr\right)\left(\int_0^{\frac{\pi}{2}}(\cos\phi)^{2z-1}(\sin\phi)^{2w-1}d\phi\right)\end{aligned}$$

Look at the r integral - take $s = r^2 \implies ds = 2rdr$. Then:

$$\int_0^\infty r^{2(z+w-1)}e^{-r^2}rdr = \frac{1}{2}\int_0^\infty s^{(z+w-1)}e^{-s}ds = \frac{1}{2}\Gamma(z+w)$$

Now, consider the ϕ integral: Substitute $t = \sin^2\phi \implies dt = 2\sin\phi\cos\phi d\phi$. Then:

$$\begin{aligned}\int_0^{\frac{\pi}{2}}(\cos\phi)^{2z-1}(\sin\phi)^{2w-1}d\phi &= \int_0^{\frac{\pi}{2}}(1-\sin^2\phi)^{z-\frac{1}{2}}(\sin^2\phi)^{w-\frac{1}{2}}d\phi\\ &= \frac{1}{2}\int_0^1(1-t)^{z-\frac{1}{2}}t^{w-\frac{1}{2}}\frac{dt}{(t(1-t))^{\frac{1}{2}}}\\ &= \frac{1}{2}\int_0^1(1-t)^{z-1}t^{w-1}dt = \frac{1}{2}B(w,z) = \frac{1}{2}B(z,w).\end{aligned}$$

Hence:

$$\Gamma(z)\Gamma(w) = \Gamma(z+w)B(z,w) \implies B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

Note, strictly speaking we have proved this only for $\mathrm{Re}\{z\}, \mathrm{Re}\{w\} > 0$ (i.e. where the Beta function integral representation holds). However, this restriction can be removed using analytic continuation of $\Gamma(z), \Gamma(w)$ and $\Gamma(z+w)$. Thus the relationship holds for all z, w. $\square$

3.3 Hypergeometric Series

We can define the hypergeometric function, $F(a,b;c;z)$, for variable z in terms of the following series:

$$\boxed{F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}}. \tag{56}$$

This series is called the hypergeometric series. Here, a,b,c are parameters and we define the notation $(a)_n$ for integer $n \geq 0$, as follows:

$$(a)_0 = 1, (a)_1 = a, (a)_2 = a(a+1), \cdots$$

Or, more generally:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

This notation is sometimes referred to as the **Pochhammer symbol** or **Pochhammer representation**.

Notice that $(1)_n = n!$. Also, recall that $z = \frac{\Gamma(z+1)}{\Gamma(z)}$.
Hence,

$$(a)_n = \frac{\Gamma(a+1)}{\Gamma(a)}\frac{\Gamma(a+2)}{\Gamma(a+1)}\cdots\frac{\Gamma(a+n)}{\Gamma(a+n-1)} = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Notice that for $n \in \mathbb{Z} \geq 0$, we have $(-n)_k = 0$ for $k \geq n+1$. So, if either a or b are negative integers (and c is not) then this series terminates after a finite number of terms (i.e., it is just a polynomial).

However, if c is a negative integer then this series is not defined (it 'blows us') unless either a or b are negative integers greater than c, so that the series terminates before reaching k such that $(c)_k = 0$.

For general a,b,c the series is infinite. We consider its convergence as follows:

Let $A_n = \frac{(a)_n(b)_n}{(c)_n n!}$, then $F(a,b;c;,z) = \sum_{n=0}^{\infty} A_n z^n$.

Apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{A_{n+1}z^{n+1}}{A_n z^n}\right| = \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}(c)_n}{(a)_n(b)_n(c)_{n+1}}\frac{n!}{(n+1)!}\frac{z^{n+1}}{z^n}\right|.$$

Now, use the fact that $\frac{(a)_{n+1}}{(a)_n} = n+a$.

$$= \lim_{n\to\infty}\left|\frac{(n+a)(n+b)}{(n+c)(n+1)}z\right| = |z|.$$

Thus, the hypergeometric series converges (absolutely) for $|z| < 1$, and thus defines an analytic function in this region.

What about outside this region? $F(a,b;c;z)$ can be analytically continued to points z such that $|z| \geq 1$ (though it may be multi-valued here). One way of performing this continuation is via the following integral representation.

3.4 Euler Integral Representation

Assume $|z| < 1$. Consider the following integral:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.$$

One can show that this integral converges for $\mathrm{Re}\{b\} > 0$ and $\mathrm{Re}\{c-b\} > 0$ (see earlier arguments for convergence of integrals in this chapter for a rough justification of these conditions).
Now insert the series:

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}(zt)^n$$

(Note, this converges since $|z| < 1$ and $0 < t < 1$ in this integral, so $|zt| < 1$). Then:

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 t^{b-1}(1-t)^{c-b-1} t^n dt \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 t^{b+n-1}(1-t)^{c-b-1} dt \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n B(b+n, c-b) \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} \\
&= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} z^n \\
&= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.
\end{aligned}$$

Now observe the definition of the Hypergeometric series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

So, provided $\mathrm{Re}\{c\} > \mathrm{Re}\{b\} > 0$, we have:

$$\boxed{F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt}. \tag{57}$$

This is called the Euler representation for the hypergeometric series. We have shown this is true for $|z| < 1$. However, it can be shown that (the details are not contained here and are not-examinable) the integral on the RHS is analytic for $|z| \geq 1$ (though possibly multi-valued). Hence, by analytic continuation this result is true for all z.

3.5 Gauss Summation Theorem

Consider the case $z = 1$. We get, using (57); provided $\text{Re}\{c\} > \text{Re}\{b\} > 0$:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}dt = \int_0^1 t^{b-1}(1-t)^{c-a-b-1}dt.$$

This last integral is $B(b, c-a-b)$, provided $\text{Re}\{b\} > 0, \text{Re}\{c-a-b\} > 0$. If this is indeed the case then this integral is equal to:

$$\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}.$$

Then we have:

$$F(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}$$

$$\implies \boxed{\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}}. \tag{58}$$

Which is called the **Gauss summation theorem** and holds provided $\text{Re}\{c\} > \text{Re}\{b\} > 0$ and $\text{Re}\{c-a-b\} > 0$ (In fact, it is possible to drop the condition that $\text{Re}\{c\} > \text{Re}\{b\} > 0$ here, we omit the details of this and this is non-examinable).

3.6 The Hypergeometric Equation

In this section we show that the hypergeometric series satisfies the following second order ODE:

$$\boxed{z(1-z)\frac{d^2F}{dz^2} + (c-(a+b+1)z)\frac{dF}{dz} - abF = 0}, \tag{59}$$

which is known as the **hypergeometric equation**.

(Aside: Why should we care about the solutions to this seemingly specific ODE? The Hypergeometric equation is important for the following reason: Differential equations can be characterised in terms of their **singular points** (points that cause parts of the ODE to blow up $\rightarrow$ we won't discuss these here). It can be shown that every second order ODE with 3 regular singular points can be written in the form of the hypergeometric equation under a change of variable. Since the study of such equations is of great interest the hypergeometric equation earns its importance.)

We will now show that the solutions of (59) can be expressed in terms of hypergeometric series.

We have, being slightly lazy and dropping the arguments:

$$F = \sum_{n=0}^{\infty} A_n z^n, \quad A_n = \frac{(a)_n(b)_n}{(c)_n n!}.$$

Thus:

$$F' = \sum_{n=0}^{\infty} nA_n z^{n-1}, \quad F'' = \sum_{n=0}^{\infty} n(n-1)A_n z^{n-2}.$$

Then:

$$\begin{aligned}
&z(1-z)F'' + (c-(a+b+1)z)F' - abF \\
&= \sum_{n=0}^{\infty} \left\{ n(n-1)A_n z^{n-1} - n(n-1)A_n z^n + cnA_n z^{n-1} - (a+b+1)nA_n z^n - abA_n z^n \right\} \\
&= \sum_{n=0}^{\infty} \left\{ n(n+1)A_{n+1} - n(n-1)A_n + c(n+1)A_{n+1} - (a+b+1)nA_n - abA_n \right\} z^n \qquad \text{(after relabelling indices)} \\
&= \sum_{n=0}^{\infty} \left\{ (n+1)(n+c)A_{n+1} - (n+a)(n+b)A_n \right\} z^n.
\end{aligned}$$

But, observe that:

$$\begin{aligned}
(n+1)(n+c)A_{n+1} &= (n+1)(n+c)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(n+1)!} \\
&= \frac{(a)_{n+1}(b)_{n+1}}{(c)_n(n)!} \\
&= (n+a)(n+b)\frac{(a)_n(b)_n}{(c)_n(n)!} \\
&= (a+n)(b+n)A_n.
\end{aligned}$$

$$\implies z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,$$

for,

$$F = \sum_{n=0}^{\infty} A_n z^n.$$

So, the hypergeometric series provides a solution of the hypergeometric equation.

Since this is a second order ODE, then it must have a second linearly independent solution. We shall seek such a solution as follows:

Consider $z^\alpha F$, for arbitrary $\alpha \neq 0$.

We have:

$$\begin{aligned}
\frac{d}{dz}(z^\alpha F) &= z^\alpha F' + \alpha z^{\alpha-1}F, \quad \text{and} \\
\frac{d^2}{dz^2}(z^\alpha F) &= z^\alpha F'' + 2\alpha z^{\alpha-1}F' + \alpha(\alpha-1)z^{\alpha-2}F.
\end{aligned}$$

Then:

$$
\begin{aligned}
&z(1-z)\frac{d^2}{dz^2}(z^\alpha F) + (c-(a+b+1)z)\frac{d}{dz}(z^\alpha F) - abz^\alpha F \\
&= z(1-z)\left[z^\alpha F'' + 2\alpha z^{\alpha-1}F' + \alpha(\alpha-1)z^{\alpha-2}F\right] + (c-(a+b+1)z)\left[z^\alpha F' + \alpha z^{\alpha-1}F\right] - abz^\alpha F \\
&= \left(z(1-z)F'' + \left[2\alpha(1-z) + (c-(a+b+1)z)\right]F' + \left[\frac{\alpha(1-\alpha)(1-z)}{z} + \frac{\alpha(c-(a+b+1)z)}{z} - ab\right]F\right)z^\alpha \\
&= \left(z(1-z)F'' + \left[c+2\alpha-(a+b+2\alpha+1)z\right]F' + \left[\frac{\alpha(\alpha+c-1)}{z} - \alpha(a+b+\alpha) - ab\right]F\right)z^\alpha.
\end{aligned}
$$

Now, assuming $c \neq 1$, take $\alpha = 1-c$. Then the above becomes:

$$= \left(z(1-z)F'' + \left[2-c-(a+b-2c+3)z\right]F' - \left[(1-c)(a+b+1-c)+ab\right]F\right)z^{1-c},$$

or

$$= \left(z(1-z)F'' + \left[\tilde{c} - (\tilde{a}+\tilde{b}+1)z\right]F' - \tilde{a}\tilde{b}F\right)z^{1-c}, \tag{60}$$

where: $\tilde{c} = 2-c, \tilde{a} = a-c+1, \tilde{b} = b-c+1$.

But, from our previous analysis, (60) has solution $F(\tilde{a},\tilde{b};\tilde{c};z)$. Hence, provided $c \neq 1$;

$$z^{1-c}F(a-c+1, b-c+1; 2-c; z),$$

is a second solution of (59).

One can check to confirm (we omit the details here) that this second solution is linearly independent of the first ($F(a,b;c;z)$).

Thus, for $|z| < 1$, the general solution of (59) is:

$$AF(a,b;c;z) + Bz^{1-c}F(a-c+1, b-c+1; 2-c; z), \tag{61}$$

for arbitrary constants A and B, provided $c \neq 1$.