Imperial College London
Department of Mathematics

Mid-term test

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

You should answer all questions. Time allowed: 40 minutes.

Question 1

(a) Suppose F is a field and $n \in \mathbb{N}$. Suppose $A = (a_{ij}) \in M_n(F)$ is such that $a_{ij} = 0$ if $j > n - i + 1$.

(i) In the case $n = 3$, prove that $\det(A) = -a_{13}a_{22}a_{31}$. (3 marks)

(ii) What is $\det(A)$ in the cases $n = 5$ and $n = 6$? Explain your answer. (3 marks)

(iii) Write down an expression for $\det(A)$ for general n (you need not prove your statement). (2 marks)

(b) Suppose V is a vector space over $\mathbb{R}$ with a basis B consisting of vectors v_1, v_2. Suppose $T : V \to V$ is the linear map with

$$T(v_1) = -10v_1 - 6v_2 \quad \text{and} \quad T(v_2) = 18v_1 + 11v_2.$$

(i) Write down the matrix $[T]_B$. (1 mark)

(ii) Find the eigenvectors of T (show the details of your calculation). (6 marks)

(iii) Express v_1 as a linear combination of eigenvectors of T and hence write down an expression for $T^{50}(v_1)$ as a linear combination of v_1 and v_2. (5 marks)

Question 2

(a) (7 marks) Suppose V is a vector space over $\mathbb{R}$ and $T : V \to V$ is a linear map. Suppose $v_1, v_2, v_3 \in V$ are eigenvectors of T with eigenvalues $1, 2, 3$ respectively. Without quoting a result from your notes, prove that v_1, v_2, v_3 are linearly independent.

If you had been allowed to quote a result from your notes, what would it have said?

(b) (13 marks, 2 or 3 per part) For each of the following statements, say whether it is true or false. If it is true, give a short proof; if it is false, give a counterexample.

(i) Suppose A is a 3×3 matrix of even integers. Then there is no matrix of integers B with $AB = I_3$.

(ii) There is a matrix $A \in M_2(\mathbb{R})$ with $A \neq I_2$ and $A^{17} = I_2$.

(iii) If $A, B \in M_2(\mathbb{R})$ are diagonalisable over $\mathbb{R}$, then so is AB.

(iv) The matrix $\begin{pmatrix} 0 & i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \in M_3(\mathbb{C})$ is diagonalisable over $\mathbb{C}$.

(v) If $A \in M_n(\mathbb{C})$ and the characteristic polynomial of A has no repeated roots in $\mathbb{C}$, then A is diagonalisable over $\mathbb{C}$.