Answers to Test 2

1. (i) (a) $p_\theta = \partial L/\partial\dot\theta = \dot\theta$, $p_\phi = \partial L/\partial\dot\phi = \sin^2\theta\dot\phi$ so that

$$H = p_\theta\dot\theta + p_\phi\dot\phi - L = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} - \frac{1}{2}\left[p_\theta^2 + \sin^2\theta\left(\frac{p_\phi}{\sin^2\theta}\right)^2\right] - g\cos\theta$$

$$= \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2\sin^2\theta} - g\cos\theta.$$

[**4 marks**]

(b) Hamilton's equations:

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = p_\theta, \quad \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\sin^2\theta},$$

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{\cos\theta}{\sin^3\theta}p_\phi^2 - g\sin\theta, \quad \dot p_\phi = -\frac{\partial H}{\partial\phi} = 0.$$

[**5 marks**]

(c) $p_\theta = 0$ requires that θ is constant. p_ϕ is constant for all solutions. If θ is constant then so is $\dot\phi = p_\phi/\sin^2\theta$. These solutions represent horizontal circular orbits. We have

$$0 = \dot p_\theta = \frac{\cos\theta}{\sin^3\theta}p_\phi^2 - g\sin\theta,$$

so that $p_\phi^2 = g\sin^4\theta/\cos\theta$ giving

$$p_\phi = \pm\sin^2\theta\sqrt{\frac{g}{\cos\theta}} \quad \text{and} \quad \dot\phi = \pm\sqrt{\frac{g}{\cos\theta}}.$$

[**5 marks**]

(ii) (a)

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \cos t\cdot\cos t - \sin t\cdot(-\sin t) = 1.$$

[**4 marks**]

(b) From the first equation

$$p = \frac{Q}{\sin t} - q\cot t.$$

Inserting this into the second equation gives

$$P = -q \sin t + \left(\frac{Q}{\sin t} - q \cot t \right) \cos t = -\frac{q}{\sin t} + Q \cot t.$$

Accordingly

$$F(q, Q, t) = \frac{qQ}{\sin t} - \frac{q^2 + Q^2}{2} \cot t.$$

[**5 marks**]

(c) It is not defined if t is a multiple of π; here the canonical transformation is the identity transformation or $Q = -q$, $P = -p$ (the identity plus a sign flip).

[**2 marks**]

[**Total: 25 marks**]