Imperial College London

MATH97052 MATH70059

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Differential Topology

Date: 16 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

In this exam, unless otherwise stated, all manifolds are smooth, of finite type, connected and without boundary. All maps are assumed to be smooth. When the term 'cohomology' is used without qualification it refers to de Rham cohomology.

1. (a) Briefly explain how to define the integral of a compactly supported differential form on an orientable manifold. Why is the orientability assumption necessary? (5 marks)

   (b) For a differential form $\omega \in \Omega^p(M)$, which of the following statements can be checked locally on a coordinate chart: (i) ω is zero, (ii) ω is closed, (iii) ω is exact. Briefly justify your answers. (4 marks)

   (c) Using Stokes' Theorem, prove the following. You should explicitly state any properties of the exterior derivative that you use.

      (i) Integration by parts: Let M be an n-dimensional manifold with boundary, and $\omega \in \Omega_c^p(M)$, $\eta \in \Omega_c^{n-p-1}(M)$, for some $p \in \{0, \dots, n-1\}$. Then

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

(2 marks)

      (ii) Green's theorem: Let D be a domain in $\mathbb{R}^2$ bounded by a simple closed curve, and let $f, g : D \to \mathbb{R}^2$. Then

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_{\partial D} f dx + g dy.$$

(2 marks)

   (d) (i) Consider a manifold M, written as $M = U \cup V$ with U, V open and $U \cap V$ connected. Show that M is orientable if and only if both U and V are orientable. (3 marks)

      (ii) Hence deduce that the connected sum of two manifolds $M_1 \# M_2$ is orientable if and only if both M_1 and M_2 are orientable. (4 marks)

(Total: 20 marks)

2. (a) (i) Explain briefly the following terms: cochain complex, map of cochain complexes, and homotopy operator (i.e. cochain homotopy). (4 marks)

(ii) Using the above, explain how we know that homotopy equivalent manifolds have isomorphic de Rham cohomology groups. (4 marks)

(b) Let M be expressed as a union $M = U \cup V$, with U, V open. Mentioning any important lemmas used, briefly outline the proof of the theorem that tells us how to compute the cohomology of M from knowledge of the cohomology of U, V, and $U \cap V$. (2 marks)

(c) (i) Compute the de Rham cohomology groups of $S^2 \times S^2$. Justify your answer. (3 marks)

(ii) Let $W = \mathbb{CP}^2 \setminus \{p\}$. Using Mayer-Vietoris for cohomology with compact supports, find and explicitly describe an isomorphism

$$\phi : H^2_c(W) \oplus H^2_c(W) \to H^2(\mathbb{CP}^2 \# \mathbb{CP}^2).$$

(3 marks)

(iii) Let $a \in H^2(W) \cong \mathbb{R}$ be a generator. Hence we obtain a basis $\alpha = \phi(a, 0), \beta = \phi(0, a)$ for $H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$. By considering the isomorphism

$$I : H^4(\mathbb{CP}^2 \# \mathbb{CP}^2) \to \mathbb{R}$$

$$[\omega] \longmapsto \int_M \omega$$

show that $\alpha \wedge \alpha$, and $\beta \wedge \beta$ are nonzero in $H^4(\mathbb{CP}^2 \# \mathbb{CP}^2)$, whereas $\alpha \wedge \beta = 0$. (2 marks)

(iv) Deduce that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent. You may use without proof the fact that the ring structure in de Rham cohomology is a homotopy invariant. (2 marks)

(Total: 20 marks)

3. (a) By computing cohomology groups or otherwise, show that the following pairs of manifolds are not diffeomorphic. You may use any theorems from lectures provided you state where they are used.

   (i) $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^2$. (2 marks)

   (ii) $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$, with both $n, m \geq 1$. (2 marks)

(b) Briefly outline the proof of Poincaré duality, giving the names of any important lemmas. (4 marks)

(c) Let M be compact and orientable, and let N be orientable and homotopy equivalent to M. Show using de Rham cohomology that

   (i) If N is compact then $\dim M = \dim N$ (1 mark)

   (ii) If N is not compact then $\dim N > \dim M$. (2 marks)

(d) (i) Show that the inclusion $\Omega_c(M)^p \hookrightarrow \Omega^p(M)$ of compactly supported forms induces a map of cohomology groups $i_c : H_c^p(M) \to H^p(M)$. (3 marks)

   (ii) Must this map be injective? Must it be surjective? (2 marks)

   (iii) Now let M be orientable of dimension n. Let $\widetilde{H}_c^p(M) \subset H^p(M)$ be the image of the map above. Use the Poincaré duality map to show that there is a well-defined non-degenerate pairing

$$\widetilde{H}_c^p(M) \times \widetilde{H}_c^{n-p}(M) \to \mathbb{R}.$$

(4 marks)

(Total: 20 marks)

4. (a) Let $f : M \to N$ be a map of compact orientable manifolds of dimension n. Explain what is meant by the degree of f. You do not need to show that it is an integer. (4 marks)

(b) Give, with brief justification, an example for which the degree of $f : M \to N$ is

(i) zero, (1 mark)

(ii) one, (1 mark)

(iii) neither zero nor one. (2 marks)

(c) Regarding $S^n \subset \mathbb{R}^{n+1}$ in the standard way, consider the antipodal map

$$A : S^n \to S^n$$

$$x \longmapsto -x.$$

(i) Show that the degree of the antipodal map is $(-1)^{n+1}$. (2 marks)

(ii) Let $v : S^n \to TS^n$ be a nowhere-vanishing vector field on S^n, assumed to have unit length with respect to the norm inherited from $\mathbb{R}^{n+1}$. By considering the function

$$H(x,t) = x\cos(\pi t) + v(x)\sin(\pi t),$$

show that n must be odd. (3 marks)

(d) Define $\mathbb{RP}^n$ as the space of lines through the origin in $\mathbb{R}^{n+1}$.

(i) Explain how to view $\mathbb{RP}^n$ set-theoretically as the quotient of S^n by the antipodal equivalence relation $x \sim A(x)$. (1 mark)

(ii) Show that A induces a splitting $\Omega^p(S^n) \cong \Omega^p_+(S^n) \oplus \Omega^p_-(S^n)$ into the $+1$ and -1 eigenspaces for A. *Hint: Write* $\omega = (1/2)(\omega + A^*\omega) + (1/2)(\omega - A^*\omega)$ *and use* $A^2 = $ *Id.* (2 marks)

(iii) Let $f : S^n \to \mathbb{RP}^n$ be the quotient map. Show that $f^* : \Omega^p(\mathbb{RP}^n) \to \Omega^p(S^n)$ gives an isomorphism $\Omega^p(\mathbb{RP}^n) \cong \Omega^p_+(S^n)$. (2 marks)

(iv) Prove that, for odd n, the de Rham cohomology groups of $\mathbb{RP}^n$ are given by

$$H^p(\mathbb{RP}^n) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & 1 \le p \le n-1 \\ \mathbb{R}, & p = n \text{ odd.} \end{cases}$$

(2 marks)

(Total: 20 marks)

5. (a) (i) Give, with justification, an example of a function that is Morse. (2 marks)

(ii) Using the Morse Lemma, explain why Morse functions have isolated critical points. (3 marks)

(iii) Are all functions with isolated critical points Morse? Give a proof or counterexample. (2 marks)

(b) Using a suitable Morse-Smale function, compute the Morse homology groups of S^n. (6 marks)

(c) Let $a_0 < a_1 < \cdots < a_n$ be real numbers. Consider the map $f : \mathbb{CP}^n \to \mathbb{R}$ given by

$$[z_0 : \cdots : z_n] \longmapsto \frac{\sum_{i=0}^{n} a_i |z_i|^2}{\sum_{i=0}^{n} |z_i|^2}.$$

(i) Verify that this map is well-defined. (1 mark)

(ii) Recall the standard coordinate charts $U_j = \{z = [z_0 : \ldots z_n] \in \mathbb{CP}^n \mid z_j = 1\}$. Binomially expand the numerator of f to show that the restriction of $f|_{U_j} : U_j \to \mathbb{R}$ to each coordinate chart has the form

$$f(z_0, \ldots, 1, \ldots, z_n) = a_j + \sum_{\substack{i=0 \\ i \neq j}}^{n} (a_i - a_j)|z_i|^2 + \text{ higher order terms.}$$

(1 mark)

(iii) Using (ii), show that the critical points of this map occur at

$$p_j = [0 : \cdots : 0 : 1 : 0 : \cdots : 0],$$

with z_j the only non-zero entry, for each $j = 0, \ldots, n$. *Hint: Take real coordinates $z_i = (x_i, y_i)$, so that $|z_i|^2 = x_i^2 + y_i^2$.* (1 mark)

(iv) Show that the function f is Morse, and that the Morse index of p_j is $2j$. (2 marks)

(v) Hence compute the Morse homology of $\mathbb{CP}^n$. You may assume that the function f is Morse-Smale. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH97052/M4P54

Differential Topology (Solutions)

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1. (a) *Briefly explain how to define the integral of a compactly supported differential form on an orientable manifold. Why is the orientability assumption necessary?*

seen ⇓

For a differential form $\omega \in \Omega^p_c(M)$ whose support is contained in a single coordinate chart $\phi : U \to \mathbb{R}^n$, we write $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ in local coordinates and define

$$\int_M \omega := \int_{\phi(U)} f(x)dx_1 \dots dx_n$$

where the expression on the right is the usual integral of a function familiar from the multivariable calculus of $\mathbb{R}^n$. If ω has support contained in a collection of coordinate charts U_i for $i \in I$, we take a partition of unity $\{\rho_i \mid i \in I\}$ subordinate to this cover, and define

$$\int_M \omega = \sum_{i \in I} \int_{U_i} f_i \omega,$$

noting that the sum can be taken to be finite by compactness of the support of ω. The importance of the orientability assumption is to make sure this definition is independent of choices of coordinate chart. When we transform by a transition map $\tau = \phi \circ \psi^{-1}$, the change of variables formula for real multivariable integrals show that the is multiplied by the sign of $\det D\tau$. Thus provided these signs are everywhere positive for all transition maps the integral will be well-defined. 5, A

(b) *For a differential form $\omega \in \Omega^p(M)$, which of the following statements can be checked locally on a coordinate chart: (i) ω is zero, (ii) ω is closed, (iii) ω is exact. Briefly justify your answers.* 1, A

3, B

seen ⇓

A differential form $\omega \in \Omega^p$ is the zero form if for all $x \in M$ and all p-tuples of tangent vectors $v_1 \dots v_p \in T_xM$ we have $\omega_x(v_1, \dots, v_p) = 0$. This can obviously be checked locally in coordinates, since it is a pointwise statement. Similarly, ω being closed means exactly that $d\omega$ is the zero from, which can be checked locally for the same reason. However, being exact is not a local property: by the Poincaré Lemma, for $p > 0$ all p-forms are locally exact; or in order words, the de Rham cohomology of a contractible set is trivial above degree zero. On the other hand, if all p forms were exact then all pth de Rham cohomology groups of all spaces would vanish, which is clearly not the case.

(c) *Using Stokes' Theorem, prove the following. You should explicitly state any properties of the exterior derivative that you use.*

(i) *Integration by parts: Let M be an n-dimensional manifold with boundary, and $\omega \in \Omega^p_c(M)$, $\eta \in \Omega^{n-p-1}_c(M)$, for some $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

seen ⇓

By Stokes' theorem we have

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta).$$

Applying the Leibniz rule and linearity of integration to the right side of this we obtain the desired expression. 2, A

(ii) *Green's theorem: Let D be a domain in $\mathbb{R}^2$ bounded by a simple closed curve, and let $f, g : D \to \mathbb{R}^2$. Then*

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_{\partial D} f dx + g dy.$$

seen ⇓

Problem sheet question. Stokes theorem with $\omega = f dx + g dy$ tells us that the right side of the desired equation is equal to

$$\int_D d(f dx + g dy) = \int_D d(f dx) + d(g dy) = \int_D \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy.$$

To get the first equality we have used linearity of the exterior derivative, and for the second we have used the fact that $dx \wedge dx = dy \wedge dy = 0$. Applying anticommutativity of the wedge product $dx \wedge dy = -dy \wedge dx$ then gives the result.

2, B

(d)

(i) *Consider a manifold M, written as $M = U \cup V$ with U, V open and $U \cap V$ connected. Show that M is orientable if and only if both U and V are orientable.* 3, C

Unseen but elementary. If M is orientable then so is any open submanifold of it, so one direction is easy. For the converse, if U and V are orientable, choose an orientation on each. Then $U \cap V$ inherits two orientations, one from U and one from V. Since $U \cap V$ is connected, these must either agree everywhere or be opposite everywhere. Thus flipping the orientation on one of U or V if necessary, we obtain an orientation on the union, i.e. on M. (Note: It is also possible to prove this using volume forms, which amounts to the same proof in essence.)

(ii) *Hence deduce that the connected sum of two manifolds $M_1 \# M_2$ is orientable if and only if both M_1 and M_2 are orientable.*

Unseen but straightforward consequence of the previous part. We can express the connect sum as $U_1 \cup U_2$, where $U_i = M_i \setminus \{pt\}$, and $U_1 \cap U_2$ is diffeomorphic to $(0, 1) \times S^{n-1}$. Since the latter is connected, the previous part gives the result. 4, D

2. (a) (i) *Explain briefly the following terms: cochain complex, map of cochain complexes, and homotopy operator (i.e. cochain homotopy).* 4, A

seen ⇓

A cochain complex $C_\bullet$ is a sequence of vector spaces $\ldots \to C_i \to C_{i+1} \to \ldots$ with linear maps $d_{C,i} : C_i \to C_{i+1}$ such that $d_{C,i+1} \circ d_{C,i} = 0$. (Commonly written as just $d^2 = 0$.) A map of cochain complexes $f : C_\bullet \to D_\bullet$ is a collection of linear maps $f_i : C_i \to D_i$ such that all the squares commute:

$$\begin{array}{ccc} C_i & \xrightarrow{d_{C,i}} & C_{i+1} \\ \big\downarrow{\scriptstyle f_i} & & \big\downarrow{\scriptstyle f_{i+1}} \\ D_i & \xrightarrow{d_{D,i}} & D_{i+1} \end{array}$$

A chain homotopy between maps $f, g : C_\bullet \to D_\bullet$ is a collection of maps $h_i : C_i \to D_{i-1}$ such that

$$f - g = hd_D + d_C h$$

(ii) *Using the above, briefly explain how we know that homotopy equivalent manifolds have isomorphic de Rham cohomology groups.* 4, A

seen ⇓

In lectures we showed that if $f_0, f_1 : M \to N$ are homotopic smooth maps of manifolds then there exists a homotopy operator between the corresponding maps of de Rham complexes. Furthermore, the existence of a homotopy operator guarantees that f_0, f_1 induce the same map on cohomology. If two manifolds are homotopy equivalent then there exist $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq \mathrm{id}_N$ and $g \circ f \simeq \mathrm{id}_M$. This together with the functoriality of the induced maps on cohomology means that f and g must act as isomorphisms on the level of de Rham cohomology.

(b) *Let M be expressed as a union $M = U \cup V$, with U, V open. Mentioning any important lemmas used, outline the proof of the theorem that tells us how to compute the cohomology of M from knowledge of the cohomology of U, V, and $U \cap V$.* 2, B

seen ⇓

The relevant theorem is Mayer-Vietoris. The inclusion maps

$$i_U : U \hookrightarrow M \qquad i_V : V \hookrightarrow M$$

$$j_U : U \cap V \hookrightarrow U \qquad j_V : U \cap V \hookrightarrow V$$

allow us to define maps $f^* = (i_U^*, i_V^*)$, $g^* = j_U^* - j_V^*$ fitting into an exact sequence of cochain complexes

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{f} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{g} \Omega^\bullet(U \cap V) \longrightarrow 0.$$

The non-trivial part of proving exactness is to show that g is surjective, which uses a partition of unity argument. Applying the snake lemma (also known as the zig-zag lemma) to this short exact sequence yields a long exact sequence

$$\ldots \to H^p(M) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to H^{p+1}(M) \to \ldots.$$

(c) (i) *Compute the de Rham cohomology groups of $S^2 \times S^2$. Justify your answer.* $\boxed{3, \mathrm{B}}$

Unseen but similar to example done in lectures. The easiest way is to use the Kunneth formula, though other arguments are possible. We compute

$$H^0(S^2 \times S^2) \cong \mathbb{R}$$

$$H^1(S^2 \times S^2) \cong (H^1(S^2) \otimes H^0(S^2)) \oplus (H^0(S^2) \otimes H^1(S^2)) \cong 0$$

$$H^2(S^2 \times S^2) \cong (H^0(S^2) \otimes H^2(S^2)) \oplus (H^1(S^2) \otimes H^1(S^2)) \oplus (H^2(S^2) \otimes H^0(S^2)) \cong \mathbb{R}^2$$

Combining this with Poincaré duality saves us the work of computing the remaining groups: from this we obtain $H^3(S^2 \times S^2) = 0$ and $H^4(S^2 \times S^2) = \mathbb{R}$.

(ii) *Let $W = \mathbb{CP}^2 \setminus \{p\}$. Using Mayer-Vietoris for cohomology with compact supports, find and explicitly describe an isomorphism*

$$\phi : H^2_c(W) \oplus H^2_c(W) \to H^2(\mathbb{CP}^2 \# \mathbb{CP}^2).$$

$\boxed{3, \mathrm{C}}$

We can cover the connect sum with pieces $U, V \cong W$ where $U \cap V \cong S^3 \times (0,1)$. The relevant part of the compactly supported Mayer-Vietoris sequence is thus

$$H^2_c(S^3 \times (0,1)) \to H^2_c(W) \oplus H^2_c(W) \to H^2_c(\mathbb{CP}^2 \# \mathbb{CP}^2) \to H^3_c(S^3 \times (0,1)).$$

By Poincaré duality we have $H^2_c(S^3 \times (0,1)) \cong H^2(S^3 \times (0,1))$, and $S^3 \times (0,1) \simeq S^3$ so this group is trivial. Similarly, we have

$$H^3_c(S^3 \times (0,1)) \cong H^1(S^3 \times (0,1)) \cong H^1(S^3) \cong 0.$$

This gives the desired isomorphism.

(iii) *Let $a \in H^2(W) \cong \mathbb{R}$ be a generator. Hence we obtain a basis $\alpha = \phi(a, 0), \beta = \phi(0, a)$ for $H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$. By considering the isomorphism*

$$I : H^4(\mathbb{CP}^2 \# \mathbb{CP}^2) \to \mathbb{R}$$

$$\omega \longmapsto \int_M \omega$$

show that $\alpha \wedge \alpha$, and $\beta \wedge \beta$ are nonzero in $H^4(\mathbb{CP}^2 \# \mathbb{CP}^2)$, whereas $\alpha \wedge \beta = 0$.

$\boxed{2, \mathrm{D}}$

Challenging unseen question. Here we make use of the precise nature of the isomorphism ϕ. We may choose a representative of the class α that is the extension by zero of a compactly supported differential form $\omega \in \Omega^2_c(W)$. Thus we may compute the integral there. But by Poincaré duality on W, we see that $\int_W \omega \wedge \omega \neq 0$, which means that the corresponding integral on the connect sum is non-zero, and hence the class $\alpha \wedge \alpha$ is non-zero. The same is true for β, choosing some representative η.

On the other hand, we can take ω to be supported away from the small ball around the missing point that was used to construct the connect sum, so that α and β may be represented by forms ω, η with disjoint support. This means that the form $\omega \wedge \eta = 0$, so the class in cohomology must be zero.

(iv) *Deduce that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent. You may use without proof the fact that the ring structure in de Rham cohomology is a homotopy invariant.* $\boxed{2, \mathrm{D}}$

The cohomology groups are the same, but they have different ring structure, as the question suggests. The key point is the multiplication in degree two. By the Kunneth formula calculation above, we can write $H^2(S^2) \oplus H^2(S^2) \cong H^2(S^2 \times S^2)$ where each summand corresponds to one of the factors. That is, if

$$\pi_i : S^2 \times S^2 \to S^2 \quad \text{for } i = 1, 2$$

are the projections onto each of the two factors, the Kunneth theorem tells us that the isomorphism above is given by

$$(a_1, a_2) \longmapsto \pi_1^* a_1 + \pi_2^* a_2.$$

Now choose a basis α_1, α_2 for $H^2(S^2 \times S^2)$ such that $\pi_j^* a = \alpha_j$ where a is a generator of $H^2(S^2)$. Since $H^4(S^2) = 0$ and pullback is compatible with wedge product, we see that the wedge product (i.e. the product in the cohomology ring) of $\alpha_i \wedge \alpha_j$ is zero if $i = j$ and nonzero if $i \neq j$; that is $\alpha_1 \wedge \alpha_2$ is a generator of $H^4(S^2 \times S^2)$, and $\alpha_i^2 = 0$ for each i.

Thus the intersection forms are different: there is no change of basis that will convert one to the other. This may be seen for example by comparing their determinants, looking at eigenvalues, etc.

3. (a) *By computing cohomology groups or otherwise, show that the following pairs of manifolds are not diffeomorphic. You may use any theorems from lectures provided you state where they are used.*

(i) $\mathbb{R}^2 \setminus \{0\}$ *and* $\mathbb{R}^2$. 2, A

seen ⇓

The first de Rham cohomology group of $\mathbb{R}^2 \setminus \{0\}$ is one-dimensional, because it is homotopy equivalent to S^1, and the result can be proved for S^1 using Mayer-Vietoris. (Alternatively, one can exhibit an explicit one-form seen in lectures which was shown to be closed but not exact, which shows that the H^1 must be non-trivial.) On the other hand, $\mathbb{R}^2$ is contractible so by homotopy invariance $H^1(\mathbb{R}^2)$ is trivial.

(ii) $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *for* $n \neq m$*, with both* $n, m \geq 1$. 2, A

seen ⇓

It was proved in lectures that $H^p_c(\mathbb{R}^n) = \mathbb{R}$ in degree n and 0 for $p \neq n$. This follows by invoking Poincaré duality. Alternatively, we showed that for any manifold manifold the inclusion $M \setminus \{x\} \hookrightarrow M$ induces an isomorphism on compactly supported cohomology in degree $p \geq 2$. But $S^n \setminus \{pt\} \cong \mathbb{R}^n$, and the former is compact so we know its compactly supported cohomology. This rules out all possibilities except for $n = m$ and (up to re-ordering) $n = 0$, $m = 1$, which is clear since $\mathbb{R}^0 = \{pt\}$.

(b) (i) *Briefly outline the steps in the proof of Poincaré duality, giving the names of any important lemmas.* 4, A

seen ⇓

The strategy of the proof is to use induction on the size of the good cover.

As the base of the induction, one first one shows that Poincaré duality holds for $\mathbb{R}^n$. The ordinary de Rham cohomology groups are known since $\mathbb{R}^n$ is contractible, and the compactly supported ones can be computed using the fact that the injection $\mathbb{R} \cong S^n \setminus \{pt\} \hookrightarrow S^n$ is an isomorphism on compactly supported cohomology for $p \geq 1$, and the group is zero for $p = 0$ since $\mathbb{R}^n$ is not compact. That the Poincaré duality map is an isomorphism can be checked by explicit integration, since closed 0-forms are just constants.

For the induction step, assume that M is covered by $k+1$ elements forming a good cover, and take $U = \cup_{i=1}^k U_i$, with $V = U_{k+1}$. Associated to this cover $M = U \cup V$ we have the usual Mayer-Vietoris sequence, and the Mayer-Vietoris sequence for cohomology with compact supports, which we dualise and write underneath. The Poincaré duality maps $H^p(N) \to H^{n-p}_c(N)^*$ for the corresponding terms are written vertically, giving a diagram as follows:

$$\begin{tikzcd}
H^{p-1}(U) \oplus H^{p-1}(V) \arrow[r, "g"] \arrow[d] & H^{p-1}(U \cap V) \arrow[r, "\delta"] \arrow[d] & H^p(M) \arrow[r, "f"] \arrow[d] & H^p(U) \oplus H^p(V) \arrow[d] \\
H_c^{n-(p-1)}(U)^* \oplus H_c^{n-(p-1)}(V)^* \arrow[r, "j^*"] & H_c^{n-(p-1)}(U \cap V)^* \arrow[r, "\delta_c^*"] & H_c^{n-p}(M)^* \arrow[r, "i^*"] & H_c^{n-p}(U)^* \oplus H_c^{n-p}(V)^*
\end{tikzcd}$$

One then checks (laboriously) that, if we introduce the correct signs to the vertical arrows, all squares commute. Since by induction four of the five maps are known to be isomorphisms, we conclude by the five lemma.

(c) *Let M be compact and orientable, and let N be orientable and homotopy equivalent to M. Show using de Rham cohomology that*

(i) *If N is compact then $\dim M = \dim N$* $\boxed{1, \mathrm{B}}$

Straightforward deduction from basic material. Let $m = \dim M$ and $n = \dim N$. Since M is compact and orientable, we have $H^m(M) \cong \mathbb{R}$, and the cohomology is zero in degrees above m. On the other hand, we have $H^n(N) \cong \mathbb{R}$ and N's cohomology is zero in degrees above n. If they are homotopy equivalent they have isomorphic de Rham cohomology groups, so we must have $n = m$.

(ii) *If N is not compact then $\dim N > \dim M$.* $\boxed{2, \mathrm{B}}$

Here since N is not compact we have $H^n(N) \cong 0$, so $n \neq m$. But $H^m(N)$ must be non-zero by the above, so $m < n$.

(d) (i) *Show that the inclusion $\Omega_c(M)^p \hookrightarrow \Omega^p(M)$ of compactly supported forms induces a map of cohomology groups $i_c : H^p_c(M) \to H^p(M)$.* $\boxed{3, \mathrm{C}}$

Unseen. Any class $[\omega] \in H^p_c(M)$ is represented by a closed form, so we can indeed get classes in ordinary de Rham cohomology. To show that the map is well-defined we need only observe that if a form $\omega \in \Omega^p_c(M)$ satisfies $\omega = d\eta$ with $\eta \in \Omega^{p-1}_c(M)$ then ω is exact, so gives the zero class in $H^p(M)$.

(ii) *Must this map be injective? Must it be surjective?* $\boxed{2, \mathrm{B}}$

As the examples $p = 0, n$ for M orientable and compact/non-compact show, it is neither injective nor surjective in general.

(iii) *Now let M be orientable of dimension n. Let $\widetilde{H}^p_c(M) \subset H^p(M)$ be the image of the map above. Use the Poincaré duality map to show that there is a well-defined non-degenerate pairing*

$$\widetilde{H}^p_c(M) \times \widetilde{H}^{n-p}_c(M) \to \mathbb{R}.$$

$\boxed{4, \mathrm{D}}$

Unseen, but similar to Poincaré duality as seen in lectures. As usual the map is $[\omega] \longmapsto ([\eta] \mapsto \int_M \omega \wedge \eta)$. To show that it is well-defined, note that if $\omega = d\alpha$ then $\int_M \omega \wedge \eta = \int_M d\alpha \wedge \eta = \pm \int_M d(\alpha \wedge \eta)$ because η is closed. But Stokes' theorem says that this is zero, because M has no boundary. Similarly, the pairing is well-defined in the other factor. To show that it is non-degenerate we need to show that for any $[\omega] \in \widetilde{H}^p_c M$ there exists $[\eta] \in \widetilde{H}^{n-p}_c(M)$ such that $\int_M \omega \wedge \eta \neq 0$. Since $[\omega]$ is a class in $H^p(M)$, we know there exists $[\eta] \in H^{n-p}_c(M)$ with the desired property; the image of $[\eta]$ in $\widetilde{H}^{n-p}_c(M)$ is thus the class we want. Similarly, we can reverse the factors. This shows non-degeneracy.

4. (a) *Let $f : M \to N$ be a map of compact orientable manifolds of dimension n. Explain what is meant by the degree of f. You do not need to show that it is an integer.*

We know (e.g. by Poincaré duality) that $H^n(M) = H^n(M) \cong \mathbb{R}$. Thus the pullback map $f^* : H^n(N) \to H^n(M)$ acts as multiplication by some $c \in \mathbb{R}$. This number is the degree. $\boxed{4, \mathrm{A}}$ $\boxed{\text{seen} \Downarrow}$

(b) *Give, with brief justification, an example for which the degree of $f : M \to N$ is*

(i) *zero,* For these questions any justified example that works gets full marks. Here a constant map is the obvious choice. This is not surjective when $n > 0$, so the degree can be computed as a sum over the empty set. $\boxed{1, \mathrm{A}}$

(ii) *one,* The identity map has degree one, as does any orientation preserving diffeomorphism. The reason is that choosing any point, the pre-image is a singleton, and the orientation is preserved locally, so we count it with sign $+1$ in the local degrees formula. $\boxed{1, \mathrm{A}}$

(iii) *neither zero nor one.*
The easiest example would be an orientation reversing diffeomorphism, such as the antipodal map on a sphere of even dimension. The argument is the same as the above case, except that the sign is -1. $\boxed{2, \mathrm{A}}$

(c) *Regarding $S^n \subset \mathbb{R}^{n+1}$ in the standard way, consider the antipodal map*

$$A : S^n \to S^n$$

$$x \longmapsto -x.$$

$\boxed{\text{seen} \Downarrow}$

(i) *Show that the degree of the antipodal map is $(-1)^{n+1}$.*

We consider the form $\omega = x_1 dx_2 \wedge \cdots \wedge dx_{n+1}$ on $\mathbb{R}^{n+1}$, and its restriction to the n-sphere, i.e. $i^*\omega$ where $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is the usual inclusion. Note that $d\omega = dx_1 \wedge \cdots \wedge dx_{n+1}$, so that by Stokes theorem

$$\int_{S^n} \omega = \int_D dx_1 \wedge \cdots \wedge dx_{n+1} \neq 0$$

where D is the unit disc in $\mathbb{R}^{n+1}$. Thus we can compute the degree as the real number $c \in \mathbb{R}$ such that $\int_{S^n} f^*\omega = c \int_{S^n} \omega$. It remains to observe that the pullback $f^*\omega = (-1)^{n+1}\omega$, so that the degree is $(-1)^{n+1}$. $\boxed{2, \mathrm{B}}$

(ii) *Let $v : S^n \to TS^n$ be a nowhere-vanishing vector field on S^n, assumed to have unit length with respect to the norm inherited from $\mathbb{R}^{n+1}$. By considering the function*

$$H(x,t) = x\cos(\pi t) + v(x)\sin(\pi t),$$

show that n must be odd.
As seen on problem sheet, with an extra hint given. We claim that H gives a homotopy $H : S^n \times [0,1] \to S^n$ between the identity and antipodal maps on S^n. Since degree is a homotopy invariant, this would suffice to prove that n must be odd.
Since $v(x) \in T_x S^n$ for all x, we have $\langle v(x), x\rangle = 0$ with respect to the standard inner product. Thus, since we assume that $v(x)$ has unit length, we compute

$$|H(x,t)|^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1.$$

Hence $H(x,t) \in S^n$ for all $(x,t) \in S^n \times [0,1]$. It is evidently smooth, since $v(x)$ is smooth, and we have $H(x,0) = x$ and $H(x,1) = -x$, showing that H is indeed the claimed homotopy. $\boxed{3, \mathrm{B}}$

(d) *Define $\mathbb{RP}^n$ as the space of lines through the origin in $\mathbb{R}^{n+1}$.*

(i) *Explain how to view $\mathbb{RP}^n$ set-theoretically as the quotient of S^n by the antipodal equivalence relation $x \sim A(x)$.* $\boxed{1, \mathrm{C}}$

Each line in $\mathbb{R}^{n+1}$ meets the unit sphere in exactly two places, which are antipodal. Thus after identifying antipodal points on the sphere we end up with $\mathbb{RP}^n$.

(ii) *Show that A induces a splitting $\Omega^p(S^n) \cong \Omega^p_+(S^n) \oplus \Omega^p_-(S^n)$ into the $+1$ and -1 eigenspaces for A.* Hint: Write $\omega = (1/2)(\omega + A^*\omega) + (1/2)(\omega - A^*\omega)$ and use $A^2 = \mathrm{Id}$. $\boxed{2, \mathrm{C}}$

Unseen but straightforward given the hint. Writing any $\omega = (1/2)(\omega + A^*\omega) + (1/2)(\omega - A^*\omega)$ as in the hint and applying A, we see that ω can be written as a sum of $+1$ and -1 eigenvectors for A. This decomposition is obviously unique, since no vector can lie in both eigenspaces, which gives the desired splitting.

(iii) *Let $f : S^n \to \mathbb{RP}^n$ be the quotient map. Show that $f^* : \Omega^p(\mathbb{RP}^n) \to \Omega^p(S^n)$ gives an isomorphism $\Omega^p(\mathbb{RP}^n) \cong \Omega^p_+(S^n)$.* $\boxed{2, \mathrm{D}}$

Unseen and more difficult. If $\omega \in \Omega^p(\mathbb{RP}^n)$ then applying A to the pullback we get $A^*(f^*\omega) = (f \circ A)^*\omega = f^*\omega$ since $f \circ A = f$. So $\pi^*\Omega^p(\mathbb{RP}^n) \subseteq \Omega^p_+(S^n)$. Next, given $\omega \in \Omega^p_+(S^n)$, we can define $\widetilde{\omega} \in \Omega^p(\mathbb{RP}^n)$ as follows. Given a point $x \in \mathbb{RP}^n$ and $v_1, \ldots v_p \in T_x\mathbb{RP}^n$, choose one of the two points mapping to x and call it $\widetilde{x} \in S^n$. Then in a neighbourhood of $\widetilde{x}$, the antipodal map is a local diffeomorphism, so there are unique $\widetilde{v}_1, \ldots \widetilde{v}_p \in T_{\widetilde{x}}S^n$ such that $Df_{\widetilde{x}}(\widetilde{v}_i) = v_i$. Thus we can define $\widetilde{\omega}_x(v_1, \ldots, v_p) = \omega_{\widetilde{x}}(\widetilde{v}_1, \ldots, \widetilde{v}_p)$. Since the original ω is invariant under A, this is well-defined independent of our choice of pre-image point.

(iv) *Prove that, for odd n, the de Rham cohomology groups of $\mathbb{RP}^n$ are given by*

$$H^p(\mathbb{RP}^n) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & 1 \le p \le n-1 \\ \mathbb{R}, & p = n \text{ odd.} \end{cases}$$

$\boxed{2, \mathrm{D}}$

Challenging unseen question. First observe that the exterior derivative respects the splitting

$$\Omega^p(S^n) = \Omega^p_+(M) \oplus \Omega^p_-(S^n)$$

in the sense that $d : \Omega^p_\pm(S^n) \to \Omega^{p+1}_\pm(S^n)$.

Since in degrees $p \neq 0, n$ all closed forms on S^n are exact, in particular all closed $+1$ eigenforms are exact. Because d respects the splitting into eigenspaces, and we can always find primitives for $+1$ eigenforms that are also $+1$ eigenforms. Hence all closed forms on $\mathbb{RP}^n$ are exact for $p \neq 0, n$.

As usual the group $H^0(\mathbb{RP}^n) \cong \mathbb{R}$ because $\mathbb{RP}^n$ is connected.

For the final group, observe that since by (c)(i) the degree of the antipodal map is $(-1)^{n+1}$, when n is odd a volume form for S^n lies in the $+1$ eigenspace for A. Hence it gives rise to a well-defined volume form on $\mathbb{RP}^n$ for odd n, so the latter is orientable. By Poincaré duality, we thus have $H^n(\mathbb{RP}^n) \cong \mathbb{R}$.

5. (a) (i) *Give, with justification, an example of a function that is Morse.*
Any function that has no critical points will do, for example the identity map $\mathbb{R} \to \mathbb{R}$. $\boxed{2, \mathrm{M}}$

(ii) *Using the Morse Lemma, explain why Morse functions have isolated critical points.* The Morse Lemma tells us that in the neighbourhood of a critical point $x_0 \in M$, we can choose local coordinates so that $x_0 = (0, \ldots, 0)$ and the Morse function looks like $\boxed{3, \mathrm{M}}$

$$f(x) = f(0) + (-1)\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$$

where λ is the Morse index of the critical point. We thus have $\frac{\partial f}{\partial x_i} = -2x_i$ for $i = 1, \ldots, \lambda$ and $\frac{\partial f}{\partial x_i} = 2x_i$ for $i = \lambda+1, \ldots, n$. Thus the only critical point in this coordinate patch is x_0, so the critical point is isolated.

(iii) *Are all functions with isolated critical points Morse? Give a proof or counterexample.* $\boxed{2, \mathrm{M}}$

A nice counterexample is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. The only critical point is the origin, but the Hessian there (i.e. second derivative) is also zero.

(b) *Using a suitable Morse-Smale function, compute the Morse homology groups of S^n, for $n \geq 2$.* Embed $S^n \subset \mathbb{R}^{n+1}$ in the usual way, and take the induced metric from the ambient Euclidean metric. Consider the height function $f : (x_1, \ldots x_{n+1}) \longmapsto x_{n+1}$. This has two critical points on S^n, namely $N, S \in S^n$, the North and South poles. Since these are respectively maxima and minima, their Morse indices are respectively n and 0. We have $\boxed{6, \mathrm{M}}$

$$W^s(S) = S^n \setminus N, \qquad W^u(S) = S$$

$$W^s(N) = N, \qquad W^u(N) = S^n \setminus S$$

so that the stable and unstable manifolds are transverse, and the function is Morse-Smale.

Thus the Morse-Smale-Witten chain complex looks like

$$0 \to \mathbb{Z} \to 0 \to \ldots \to 0 \to \mathbb{Z} \to 0$$

with copies of $\mathbb{Z}$ in degrees n and 0. Thus all boundary maps are zero, and the homology groups of the complex, i.e. the Morse homology groups, are: $\mathbb{Z}$ in degrees $0, n$, and zero in all other degrees.

(c) *Let $a_0 < a_1 < \cdots < a_n$ be real numbers. Consider the map $f : \mathbb{CP}^n \to \mathbb{R}$ given by*

$$[z_0 : \cdots : z_n] \longmapsto \frac{\sum_{i=0}^{n} a_i |z_i|^2}{\sum_{i=0}^{n} |z_i|^2}.$$

(i) *Verify that this map is well-defined.* If we choose a different representative of the equivalence class, $[\lambda z_0 : \ldots \lambda z_n]$ for some $\lambda \in \mathbb{C} \setminus \{0\}$, then both the numerator and denominator of f gain a factor of λ^2, so the value is unchanged. $\boxed{1, \mathrm{M}}$

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Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Differential Topology_MATH97052 MATH70059	1	This question was answered fairly well. There was some confusion as to the meaning of (b), with some incorrectly interpreting eg (i) to mean the statement 'if a differential form vanishes at any point/on any neighbourhood it must vanish everywhere'. Part (c) was straightforward. For (d) there were several approaches, either by volume forms or determinants of differentials of transition maps. The crucial thing, which not all candidates spotted, is the connectedness of the intersection of U and V.
Differential Topology_MATH97052 MATH70059	2	Part (a) was essentially a check that the definition of homotopy operator and its significance was understood, and (b) was a request to outline standard bookwork. Part (c)(i) was very straightforward, but the remainder of (c) was some of the hardest material on the exam. This was reflected in the marks. The lesson was that one needs to understand the specific form of the isomorphism from (c)(ii) and its meaning on the level of differential forms.
Differential Topology_MATH97052 MATH70059	3	Parts (a) and (b) were simple bookwork, and (c) asked for simple consequences of central facts of the course. Part (d)(iii) perhaps looked harder than it was, which may have put candidates off - the proof being asked for was exactly an invocation of Poincare duality.
Differential Topology_MATH97052 MATH70059	4	Parts (a) and (b) were standard and answered well. Part (c)(ii) was a problem sheet question, but with an extra hint; not everyone seemed to realise this. Part (d) was challenging, but there was enough scaffolding there to enable picking up a few marks without much hardship. Despite this, perhaps due to time constraints, quite a few candidates did not answer the latter part of this question.
Differential Topology_MATH97052 MATH70059	5	Performance on this question was rather worse than the other questions, most likely due to candidates running out of time, or having more difficulty with the mastery material. The second half of the question looked intimidating, but there were some easy marks there for those that made the attempt. In particular, it was possible to answer any subset of the parts of (c) without being able to do the rest.