1. (i) (a) I: P(exactly 1 event occurs in $(t, t+\delta t]) = \lambda\delta t + o(\delta t)$, $[o(\delta t)/\delta t \to 0$ as $\delta t \to 0]$. $\boxed{\text{seen} \Downarrow}$

II: P(2 or more events occur in $(t, t+\delta t]) = o(\delta t)$.

III: Occurrence of events after time t is independent of occurrence of events before t. $\boxed{3}$

(b) Let $p_0(t) = \mathrm{P}(X(t) = 0)$, where $X(t)$ is the number of events by time t,

$$\begin{aligned}
p_0(t+\delta t) &= \mathrm{P}\{X(0, t+\delta t) = 0\} \\
&= \mathrm{P}\{X(0,t) = 0 \cap X(t, t+\delta t) = 0\} \\
&= \mathrm{P}\{X(0,t) = 0\}\mathrm{P}\{X(t, t+\delta t) = 0\} \quad \text{independence} \\
&= p_0(t)(1 - \lambda\delta t + o(\delta t)) \quad \text{axioms} \\
\Rightarrow \frac{p_0(t+\delta t) - p_0(t)}{\delta t} &= -\lambda p_0(t) + p_0(t)\frac{o(\delta t)}{\delta t}.
\end{aligned}$$

$\boxed{3}$

So,

$$\begin{aligned}
\lim_{\delta t \to 0} \frac{p_0(t+\delta t) - p_0(t)}{\delta t} &= -\lambda p_0(t) \\
\frac{d}{dt} p_0(t) &= -\lambda p_0(t) \\
\Rightarrow \int \frac{1}{p_0(t)}\, dp_0(t) &= \int -\lambda\, dt \\
\log(p_0(t)) &= -\lambda t + k_1 \\
p_0(t) &= k_2 \exp(-\lambda t).
\end{aligned}$$

Initial condition $p_0(0) = 1 \Rightarrow k_2 = 1$, so

$$p_0(t) = e^{-\lambda t}$$

$\mathrm{P}(X(t) > 0) = 1 - \mathrm{P}(X(t) = 0) = 1 - e^{-\lambda t}$ as required. $\boxed{4}$

(ii) Let $X(t_1, t_2)$ be the number of events in the interval $(t_1, t_2]$. From the axioms, $\boxed{\text{meth seen} \Downarrow}$

$$\begin{aligned}
\mathrm{E}(X(t, t+\delta t)) = \mathrm{E}(X(0, \delta t)) &= 1 \times (\lambda\delta t + o(\delta t)) + 0 \times (1 - \lambda\delta t + o(\delta t)) + o(\delta t) \\
&= \lambda\delta t + o(\delta t)
\end{aligned}$$

$D(t) =$ number events by t in deterministic model. Then

$$\begin{aligned}
D(t+\delta t) &= D(t) + \mathrm{E}(X(t, t+\delta t)) \\
D(t+\delta t) &= D(t) + \lambda\delta t + o(\delta t) \\
\Rightarrow \frac{D(t+\delta t) - D(t)}{\delta t} &= \lambda + \frac{o(\delta t)}{\delta t} \\
\Rightarrow \frac{dD(t)}{dt} &= \lambda \\
\Rightarrow D(t) &= \lambda t + c
\end{aligned}$$

and since $D(0) = 0$, $D(t) = \lambda t$ as required. $\boxed{4}$

If $X(t) \sim Poisson(\lambda t)$ then $E(X(t)) = \lambda t$ in correspondence with approximation - sample trajectory approximated by expected value. $\boxed{2}$

(iii) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\mu(u,2u) &= \int_u^{2u} \lambda(t)\,dt = \int_u^{2u} (\sin(t)+t)\,dt \\ &= \left[-\cos(t)+\frac{t^2}{2}\right]_u^{2u} = \left(-\cos(2u)+\frac{(2u)^2}{2}\right) - \left(-\cos(u)+\frac{u^2}{2}\right) \\ &= \cos(u)-\cos(2u)+\frac{3u^2}{2}\end{aligned}$$

Let $X(u,2u)$ be the number of events in $(u,2u]$, then $X(u,2u) \sim Poisson(\mu(u,2u))$, giving

$$\begin{aligned}\mathsf{P}(X(u,2u)<2) &= \mathsf{P}(X(u,2u)=0)+\mathsf{P}(X(u,2u)=1) \\ &= \exp(-\mu(u,2u))+\exp(-\mu(u,2u))\mu(u,2u) \\ &= (1+\mu(u,2u))\exp(-\mu(u,2u)) \\ &= \left(1+\cos(u)-\cos(2u)+\frac{3u^2}{2}\right)\exp\left(\cos(2u)-\cos(u)-\frac{3u^2}{2}\right).\end{aligned}$$

$\boxed{4}$

2. (i) Let X = number of offspring of an individual $\boxed{\text{seen} \Downarrow}$

$$p(x) = \mathrm{P}(X = x) = \quad \text{"offspring prob. function"}$$

Assume:

(i) p same for all individuals.

(ii) individuals reproduce independently.

Assumptions (i) and (ii) define the **Galton-Watson** discrete time branching process. $\boxed{3}$

(ii) Let $\Pi(s)$ be the pgf of X, and $\Pi_n(s)$ be the pdf of Z_n. $\boxed{\text{meth seen} \Downarrow}$
Define $\mu = \mu_1 = \mathrm{E}(X) = \Pi'(1) = 1/2$ and $\mu_n = \mathrm{E}(Z_n) = \Pi_n'(1)$.
Note that $Z_0 = 1$ so $\Pi_1(s) = \Pi(s)$.
Let, X_i = number of offspring of ith member of $(n-1)$th generation,
Z_n = number in generation n.
Then

$$Z_n = X_1 + X_2 \ldots + X_{Z_{n-1}},$$

(a) Standard pgf result gives,

$$\begin{aligned}
\Pi_n(s) &= \Pi_{n-1}\left[\Pi(s)\right] \\
\Rightarrow \Pi_n'(s) &= \Pi_{n-1}'\left[\Pi(s)\right]\Pi'(s) \\
\mu_n = \Pi_n'(1) &= \Pi_{n-1}'\left[\Pi(1)\right]\Pi'(1) \\
&= \frac{1}{2}\Pi_{n-1}'(1) \qquad (\text{as } \Pi(1) = 1) \\
\Rightarrow \quad \mathrm{E}(Z_n) = \mu_n &= \frac{1}{2}\mu_{n-1} = \left(\frac{1}{2}\right)^2 \mu_{n-2} = \ldots = \left(\frac{1}{2}\right)^n = 2^{-n}
\end{aligned}$$

$\boxed{4}$

(b) Let $\sigma^2 = \mathrm{var}(X)$ and let $\sigma_n^2 = \mathrm{var}(Z_n)$.

$$\begin{aligned}
\Pi_n'(s) &= \Pi_{n-1}'\left[\Pi(s)\right]\Pi'(s) \\
\Pi_n''(s) &= \Pi_{n-1}''\left[\Pi(s)\right]\Pi'(s)^2 + \Pi_{n-1}'\left[\Pi(s)\right]\Pi''(s) \qquad (1)
\end{aligned}$$

Now $\Pi'(1) = \mu, \Pi''(1) = \sigma^2 - \mu + \mu^2$.
Also, since $\sigma_n^2 = \Pi_n''(1) + \mu_n - \mu_n^2$, we have

$$\begin{aligned}
\Pi_n''(1) &= \sigma_n^2 - \mu^n + \mu^{2n} \\
\text{and} \quad \Pi_{n-1}''(1) &= \sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2}.
\end{aligned}$$

From (1),

$$\begin{aligned}
\Pi_n''(1) &= \Pi_{n-1}''(1)\Pi'(1)^2 + \Pi_{n-1}'(1)\Pi''(1) \\
\sigma_n^2 - \mu^n + \mu^{2n} &= (\sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2})\mu^2 + \mu^{n-1}(\sigma^2 - \mu + \mu^2) \\
\Rightarrow \sigma_n^2 &= \mu^2\sigma_{n-1}^2 + \mu^{n-1}\sigma^2 \\
\Rightarrow \mathrm{var}(Z_n) &= \frac{1}{4}\mathrm{var}(Z_{n-1}) + \left(\frac{1}{2}\right)^{n-1}\frac{1}{8} = \frac{1}{4}\left(\mathrm{var}(Z_{n-1}) + \frac{1}{2^n}\right) \\
\Rightarrow 4\,\mathrm{var}(Z_n) &= \mathrm{var}(Z_{n-1}) + 2^{-n}
\end{aligned}$$

$\boxed{5}$

(c) $\mu \leq 1$ so ultimate extinction is certain. $\boxed{1}$

(iii) If $\alpha = 0$ then P(ultimate extinction) $= 0$. Otherwise, we have

part seen $\Downarrow$

$$\Pi(s) = \sum_{i=0}^{\infty} \mathrm{P}(X = i)s^i = \alpha + (1-\alpha)s^2.$$

Giving,

$$\Pi'(s) = 2(1-\alpha)s$$

So $\mu = \Pi'(1) = 2(1-\alpha)$.
Let P(ultimate extinction) $= \theta^*$, then

1. $\mu \leq 1 \Rightarrow \theta^* = 1 \Rightarrow$ ultimate extinction certain.
2. $\mu > 1 \Rightarrow \theta^* < 1 \Rightarrow$ ultimate extinction not certain.

$\mu > 1$ when $2(1-\alpha) > 1 \Rightarrow \alpha < \frac{1}{2}$.
So, when $\alpha < \frac{1}{2}$ ultimate extinction is not certain and
$\theta^* =$ smallest positive solution of $\theta = \Pi(\theta)$:

$$\begin{aligned} \theta &= \alpha + (1-\alpha)\theta^2 \\ 0 &= (1-\alpha)\theta^2 - \theta + \alpha \\ 0 &= (\theta - 1)((1-\alpha)\theta - \alpha) \end{aligned}$$

roots are 1 and $\alpha/(1-\alpha)$, as $\alpha < 1/2$ we have

$$\text{Probability of extinction} = \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha \geq 1/2 \\ \alpha/(1-\alpha) & \alpha < 1/2 \end{cases}$$

$\boxed{7}$

3. (i) (a) As $p+q=1$, $\mathsf{P}(X_n=k)=0$ if n and k are not either both even or both odd, otherwise let $\boxed{\text{seen} \Downarrow}$

W_n = number positive steps in first n steps, then $X_n = W_n - (n - W_n) = 2W_n - n$, and $W_n \sim Binomial(n,p)$.

$$\begin{aligned}
\mathsf{P}(W_n = x) &= \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, \ldots, n \\
\mathsf{P}((X_n + n)/2 = x) &= \binom{n}{x} p^x q^{n-x} \\
\mathsf{P}(X_n = 2x - n) &= \binom{n}{x} p^x q^{n-x} \\
\mathsf{P}(X_n = k) &= \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2} \quad k = -n, \ldots, n
\end{aligned}$$

$\boxed{5}$

(b) Symmetric RW, so $p = q = 1/2$.

X_n is sum of n iid rvs, Z_i, so use CLT to see large n behaviour: $\boxed{\text{sim. seen} \Downarrow}$

CLT: X_n is approx. $N(\mathsf{E}(X_n), \mathsf{var}(X_n))$ large n.

$$\begin{aligned}
\mathsf{E}(X_n) &= \sum \mathsf{E}(Z_i) = \sum [1 \times p + (-1) \times q] = n(p-q) = 0. \\
\mathsf{var}(X_n) &= \sum \mathsf{var}(Z_i) \\
&= \sum \left[\mathsf{E}(Z_i^2) - \mathsf{E}^2(Z_i)\right] \\
&= n[(1 \times p + 1 \times q) - (p-q)^2] \\
&= 4npq = n.
\end{aligned}$$

Giving (accept answers with or without continuity correction), with $Z \sim N(0,1)$,

$$\begin{aligned}
\mathsf{P}(|X_{100}| > 20) &= \mathsf{P}(X_{100} > 20) + \mathsf{P}(X_{100} < -20) \\
&= \mathsf{P}\left(Z > \frac{20}{\sqrt{100}}\right) + \mathsf{P}\left(Z < \frac{-20}{\sqrt{100}}\right) \\
&= 1 - \Phi\left(\frac{20}{10}\right) + \Phi\left(-\frac{20}{10}\right) \\
&= 2\left(1 - \Phi(2)\right).
\end{aligned}$$

$\boxed{4}$

(ii) (a) Condition on first game, where W=event A wins first game. $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} q_j &= \mathrm{P}(A \text{ ruined} \,|\, \overline{W})\mathrm{P}(\overline{W}) + \mathrm{P}(A \text{ ruined} \,|\, W)\mathrm{P}(W) \\ q_j &= \frac{1}{2}q_{j-2} + \frac{1}{2}q_{j+2} \quad \text{for} \quad j \in \{3,4,\ldots,a-4,a-3\} \end{aligned}$$

$\boxed{3}$

(b) Similarly,

$$\begin{aligned} q_1 &= \frac{1}{2}q_0 + \frac{1}{2}q_3 = \frac{1}{2}(1+q_3) \\ q_2 &= \frac{1}{2}q_1 + \frac{1}{2}q_4 \\ q_{a-2} &= \frac{1}{2}q_{a-4} + \frac{1}{2}q_{a-1} \\ q_{a-1} &= \frac{1}{2}q_{a-3} + \frac{1}{2}q_a = \frac{1}{2}q_{a-3}. \end{aligned}$$

$\boxed{3}$

(c) expression from part (ii)(a) is

$$q_j = \frac{1}{2}q_{j-2} + \frac{1}{2}q_{j+2}$$

consider even $j = 2k, k = 2, \ldots, a/2-2$, then

$$\begin{aligned} \text{LHS} &= q_j = 1 - \frac{\frac{(a-2)2k}{a} + 2}{a+2} = 1 - \frac{(a-2)2k+2a}{a(a+2)} \\ \text{RHS} &= \frac{1}{2}\left(1 - \frac{\frac{(a-2)(2k-2)}{a} + 2}{a+2}\right) + \frac{1}{2}\left(1 - \frac{\frac{(a-2)(2k+2)}{a} + 2}{a+2}\right) \\ &= 1 - \frac{(a-2)2k+2a}{a(a+2)} = \text{LHS as required.} \end{aligned}$$

consider odd $j = 2k+1, k = 1, \ldots, a/2-2$, then

$$\begin{aligned} \text{LHS} &= q_j = 1 - \frac{(2k+1+1}{a+2} = 1 - \frac{2k+2}{a+2} \\ \text{RHS} &= \frac{1}{2}\left(1 - \frac{2k-1+1}{a+2}\right) + \frac{1}{2}\left(1 - \frac{2k+3+1}{a+2}\right) \\ &= 1 - \frac{2k+2}{a+2} = \text{LHS as required.} \end{aligned}$$

$\boxed{3}$

(d)

$$\begin{aligned} q_1 &= \frac{1}{2}(1+q_3) = \frac{1}{2}\left(1+1-\frac{3+1}{a+2}\right) = \frac{a}{a+2} \\ q_2 &= \frac{1}{2}q_1 + \frac{1}{2}q_4 = \frac{1}{2}\left(\frac{a}{a+2} + 1 - \frac{(a-2)4/a+2}{a+2}\right) \\ &= \frac{1}{2}\left(\frac{2a+2}{a+2} - \frac{6a-8}{a+2}\right) = \frac{a^2-2a+4}{a(a+2)}. \end{aligned}$$

$\boxed{2}$

4. (i) (a) A Markov chain is irreducible if it has only one communicating class, i.e. there is a path of non-zero probability from state i to state j and back again for all i, j in the sample space. seen ⇓ 2

(b) A Markov chain is aperiodic if all states have period 1, i.e. if

$$\gcd\{n : p_{ij}^{(n)} > 0\} = 1.$$

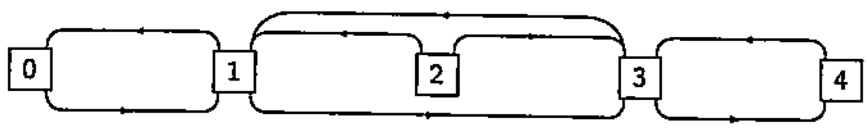
where $p_{ij}^{(n)}$ is the probability of going from i to j in n steps. 2

(ii) (a) A's fortune has state space $\{0, 1, 2, 3, 4\}$ meth seen ⇓

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

4

(b) Transition diagram:



two communicating classes: $\{0, 1, 3, 4\}$ which is closed and $\{2\}$ which is open. 2
For periodicity of the closed class $\{0, 1, 3, 4\}$ can just look at state 0 as periodicity inherited by all states in class,

$$\gcd\{n : p_{00}^{(n)} > 0\} = \gcd\{2, 4, 6, 8, \ldots\} = 2$$

so all states$\{0, 1, 3, 4\}$ have period 2. 2

(c) Only one closed communicating class, finite state space, so there is a unique stationary distribution.
Solve $\pi = \pi P, ; \sum \pi_i = 1, \ \pi_i \geq 0 \forall i$ $\boxed{1}$

$$\begin{aligned}
\pi_0 &= \frac{1}{2}\pi_1 \\
\pi_2 &= 0 \\
\pi_3 &= \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \pi_4 \\
\pi_4 &= \frac{1}{2}\pi_3 \\
1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4
\end{aligned}$$

Giving $\pi_1 = 2\pi_0, \pi_2 = 0, \pi_3 = 2\pi_0, \pi_4 = \pi_0$ and $\pi_0 = 1/6$, so

$$\pi = (1/6, 1/3, 0, 1/3, 1/6).$$

$\boxed{4}$

(d) Need aperiodicity for this to be limiting - not met in this case. $\boxed{1}$

(e) A's fortune exceeds B's by more than £2 only when he has £4, which happens in the long run a fraction $\pi_4 = 1/6$ of the time. $\boxed{2}$

5. (i) (a) Let Q have elements q_{ij} and $P(t)$ have elements $p_{ij}(t)$, then $\boxed{\text{seen} \Downarrow}$

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) = \sum_k p_{ik}(t)q_{kj} \;\; \forall i,j \;\; \text{(FORWARD EQNS)}$$

$\boxed{3}$

(b) **Stationary Distribution**

$$\pi = \pi P(t) \text{ or } \pi Q = 0; \quad \sum_j \pi_j = 1, \quad \pi_j \geq 0\, \forall j$$

$\boxed{2}$

(ii) (a) We have

$$p_{ij}(\delta t) = \begin{cases} 1 + \delta t q_{ii} + o(\delta t) & i = j \\ \delta t q_{ij} + o(\delta t) & i \neq j \end{cases}$$

So for the linear birth and death process we have

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ \nu & -(\nu+\beta) & \beta & 0 & & \\ 0 & 2\nu & -(2\nu+2\beta) & 2\beta & 0 & \\ 0 & 0 & 3\nu & -(3\nu+3\beta) & 3\beta & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix}$$

$\boxed{4}$

(b) We have

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q$$

Giving

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p_{i0}(t) &= \nu p_{i1}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) &= (j-1)\beta p_{i,j-1}(t) - j(\nu+\beta)p_{ij}(t) + (j+1)\nu p_{i,j+1}(t) \quad j \geq 1 \end{aligned}$$

Apply general method:

1. multiply by s^j.
2. sum over $0 \leq j \leq \infty$.
3. find differential equation for pgf $\Pi_i(s,t)$ of $X_i(t)$:

$$\Pi_i(s,t) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$$

To give,

$$\frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j = \beta\sum_{j=1}^{\infty}(j-1)p_{i,j-1}(t)s^j - (\nu+\beta)\sum_{j=1}^{\infty} jp_{ij}(t)s^j + \nu\sum_{j=0}^{\infty}(j+1)p_{i,j+1}(t)s^j$$

Note that,

$$\frac{\partial}{\partial s}\Pi_i(s,t) = \frac{\partial}{\partial s}\sum_{j=0}^{\infty} p_{ij}(t)s^j = \sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1}.$$

$$\begin{aligned}
\frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j &= \beta s^2\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} - (\nu+\beta)s\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} + \nu\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} \\
\frac{\partial}{\partial t}\Pi_i(s,t) &= (\beta s^2 - (\nu+\beta)s + \nu)\frac{\partial}{\partial s}\Pi_i(s,t).
\end{aligned}$$

$\boxed{6}$

(c) $\boxed{\text{unseen} \Downarrow}$

$$F_T(t) = \mathsf{P}(T \leq t) = \mathsf{P}(X(t) = 0) = \Pi_i(0,t) = \left(\frac{\nu - \nu e^{(\nu-\beta)t}}{\beta - \nu e^{(\nu-\beta)t}}\right)^i$$

$\boxed{2}$

(d) $X(0) = 1$, so

$$F_T(t) = \frac{\nu - \nu e^{(\nu-\beta)t}}{\beta - \nu e^{(\nu-\beta)t}}$$

$$\begin{aligned}
\mathsf{E}(T) &= \int_0^{\infty}(1 - F(t))\,dt \\
&= \int_0^{\infty}\left(1 - \frac{\nu - \nu e^{(\nu-\beta)t}}{\beta - \nu e^{(\nu-\beta)t}}\right)dt \\
&= \int_0^{\infty}\frac{\nu-\beta}{\nu e^{(\nu-\beta)t} - \beta}\,dt = \int_0^{\infty}\frac{(\nu-\beta)e^{-(\nu-\beta)t}}{\nu - \beta e^{-(\nu-\beta)t}}\,dt \\
&= \left[\frac{1}{\beta}\log(\nu - \beta e^{-(\nu-\beta)t})\right]_0^{\infty} = \frac{1}{\beta}\log\left(\frac{\nu}{\nu-\beta}\right) \qquad (\text{as } \beta < \nu)
\end{aligned}$$

$\boxed{3}$