Imperial College London

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Analysis 1A

Lecture 4 - Supremums and infimums and the Completeness Axiom

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Definition

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Good questions to ask:

Existence

- Does every non-empty set $S \subset \mathbb{R}$ have a maximum?
- Is it possible for more than one element of S to be a maximum of S?.

Uniqueness

Exercise 2.22

Show if a subset $S \subset \mathbb{R}$ has a maximum then it is *unique*.

If S has a maximum then we denote it $\max S$. Show if $\max S$ exists then $-S := \{-s : s \in S\}$ has a minimum, $\min(-S) = -\max S$.

Exercise 2.23

What is the maximum of the interval $(0, 1)$?

1. 0
2. 0.5
3. $0.\overline{9}$.
4. 1
5. Something else.
6. More than one of these.
7. It has no maximum.

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Exercise 2.24

Show that S is bounded if and only if

$$\exists\ R > 0 \text{ such that } \forall x \in S,\ |x| \leq R.$$

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Exercise 2.28

Suppose $\exists \sup S$. Then show that $\inf(-S)$ exists too, and equals

1. $\sup S$
2. $-\sup S$ ✓
3. $\inf S$
4. $-\inf S$
5. None of these

Example 2.29

Let $S = (0, 1)$. Let's find $\sup(S)$ and $\inf(S)$.

Claim $1 = \sup(S)$

- **Claim: 1 is an upper bound** ✓
  $\forall x \in (0,1),\ x < 1$

- Claim: 1 is the least upper bound

  Spse by contradiction, $\exists y$ an upper bound with $y < 1$.

  Since y is an upper bound, $y \geq 1/2 \in (0,1)$

  Then $\underbrace{\frac{y+1}{2} \in (0,1)}_{\substack{y<1 \\ y \geq 1/2}}$. But $\frac{y+1}{2} > y$

  So y is not an upper bound.

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Let $S = (0,1)$. Let's find $\sup(S)$ and $\inf(S)$.

Exercise 2.30

Show that $\sup S \in S \iff S$ has a maximum and $\max S = \sup S$.

Theorem 2.31 - Completeness Axiom

Suppose that $S \subset \mathbb{R}$ is nonempty and bounded above. Then S has a supremum.

Not true over $\mathbb{Q}$

$$S = \{ x \in \mathbb{Q} : x^2 < 2 \}$$

$\sup(S)$ doesn't exist in $\mathbb{Q}$

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Remark 2.32

- We have defined supremums for sets $S \subset \mathbb{R}$ that are non-empty and bounded above.
- Many textbooks you will sometimes see people set $\sup S = \infty$ if S is not bounded above, and $\sup \emptyset = -\infty$.
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Exercise 2.33

Apply Theorem 2.31 to $-S$ to deduce if $\emptyset \neq S \subset \mathbb{R}$ is bounded below then S has an inf.

Proposition 2.34

There exists $0 < x \in \mathbb{R}$ such that $x^2 = 3$. We call $x =: \sqrt{3}$.

Proof

Set $S = \{x \in \mathbb{R} : x^2 < 3\}$

- S is non-empty, $1 \in S$
- 2 is an upper bound (Exercise)

So by Completeness, there exists a $\sup(S) =: x$

Want to show $x^2 = 3$

Will show $\underbrace{x^2 \not< 3}_{(i)}$, and $\underbrace{x^2 \not> 3}_{(ii)}$

To prove (1)

Spse $x^2<3$

$3-x^2$

$\varepsilon < \frac{3-x^2}{10}$

Compute

$(x+\varepsilon)^2 = x^2+2\varepsilon x+\varepsilon^2 < x^2+(2x+1)\varepsilon \le x^2+\boxed{5\varepsilon} < 3$

$0<\varepsilon\le 1$ 2 upper bound $\frac{3-x^2}{5} <$

$\varepsilon^2\le\varepsilon$ $x\le 2$

$\varepsilon\le 1$ and $\varepsilon\le\frac{3-x^2}{10}$ and $\varepsilon>0$

Then $(x+\varepsilon)^2 = x^2+2\varepsilon x+\varepsilon^2 < 3$

So $x+\varepsilon\in S$

If $x^2<3$, we can pick any $\varepsilon = \min\left(1, \frac{3-x^2}{10}\right)$

$x+\varepsilon\in S$ but $x+\varepsilon > x$ so x is not an upper bound

(ii) Suppose, by contradiction, that $x^2 > 3$

Compute
$(x-\varepsilon)^2 = x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - 2\varepsilon x \geq x^2 - 4\varepsilon$

Set $\varepsilon_0 = \frac{x^2-3}{4} > 0$

If $y \in (x-\varepsilon_0, x)$ then $y^2 > 3$ but $y < x$ so y is an upper bound but $y < x$ ⨳

Conclusion: $x^2 = 3$

Exercise 2.35

Show $\sqrt[3]{2}$ exists.