MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Problem Sheet 1

Exercise 1.

We need to satisfy the solution identity $\dot\lambda(t) = a(t)\lambda(t) + g(t)$, which reads with the given ansatz for λ as

$$\dot c(t) \exp\big(\textstyle\int_{t_0}^t a(s)\,\mathrm{d}s\big) + c(t) \exp\big(\textstyle\int_{t_0}^t a(s)\,\mathrm{d}s\big) a(t) = a(t)c(t) \exp\big(\textstyle\int_{t_0}^t a(s)\,\mathrm{d}s\big) + g(t)\,.$$

This implies $\dot c(t) = g(t) \exp\big(\int_t^{t_0} a(s)\,\mathrm{d}s\big)$, so c solves the differential equation

$$\dot x = g(t) \exp\left(\int_t^{t_0} a(s)\,\mathrm{d}s\right)\,,$$

the right hand side of which does not depend on x, so the solution follows from simple integration. More precisely, using the initial condition $\lambda(t_0) = x_0$, which reads as $c(t_0) = x_0$, we get

$$\lambda(t) = \left(x_0 + \int_{t_0}^t g(s) e^{\int_s^{t_0} a(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right) e^{\int_{t_0}^t a(\tau)\,\mathrm{d}\tau} \quad \text{for all } \ t \in \mathbb{R}\,,$$

and a verification that this function solves the initial value problem can be done easily. Now assume there is another solution $\mu : \mathbb{R} \to \mathbb{R}$ of this initial value problem. Then we calculate for the difference $\nu(t) := \lambda(t) - \mu(t)$ that

$$\dot\nu(t) = \dot\lambda(t) - \dot\mu(t) = a(t)\lambda(t) + g(t) - a(t)\mu(t) - g(t) = a(t)\nu(t)\,,$$

so ν satisfies the initial value problem

$$\dot x = a(t)x\,, \qquad x(t_0) = 0\,.$$

which obviously has the zero solution $t \mapsto 0$ for all $t \in \mathbb{R}$. Assume there is another solution $\gamma : \mathbb{R} \to \mathbb{R}$ to this initial value problem. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t) e^{\int_t^{t_0} a(\tau)\,\mathrm{d}\tau}\right) = \underbrace{\dot\gamma(t)}_{=a(t)\gamma(t)} e^{\int_t^{t_0} a(\tau)\,\mathrm{d}\tau} - \gamma(t)a(t)e^{\int_t^{t_0} a(\tau)\,\mathrm{d}\tau} = 0\,,$$

hence $\gamma(t) = be^{\int_{t_0}^t a(\tau)\,\mathrm{d}\tau}$ for some constant $b \in \mathbb{R}$, and the initial condition implies $b = 0$, so $\gamma(t) \equiv 0$ is also the zero solution. It follows that $\nu(t) = 0$ for all $t \in \mathbb{R}$, and hence $\lambda(t) = \mu(t)$.

Exercise 2.

Assume that $\lambda : I \to \mathbb{R}$ is a solution to this initial value problem. Since $\lambda(0) = 0$ and $\dot\lambda(0) = -1 < 0$, there exists an $\gamma > 0$ such that $\lambda(t) < 0$ for all $t \in (0, \gamma)$ (why is this true? Ask in the problem class if this is not fully clear to you). Thus, $\dot\lambda(t) = 1$ for all $t \in (0, \gamma)$. This contradicts the mean value theorem, which says that there exists a $\tau \in (0, \gamma)$ with

$$\underbrace{\lambda(\tfrac{\gamma}{2}) - \lambda(0)}_{<0} = \underbrace{\dot\lambda(\tau)\tfrac{\gamma}{2}}_{=\frac{\gamma}{2}>0}\,.$$

Exercise 3.

(i) With $f(x) := x^2$ for all $x \in \mathbb{R}$, we get for all $t \in \mathbb{R}$ that

$$\begin{aligned}
\lambda_0(t) &= 1\,,\\
\lambda_1(t) &= 1 + \int_0^t f(\lambda_0(s))\,\mathrm{d}s = 1 + t\,,\\
\lambda_2(t) &= 1 + \int_0^t f(\lambda_1(s))\,\mathrm{d}s = 1 + \int_0^t (1+s)^2\,\mathrm{d}s = 1 + \frac{1}{3}(1+t)^3 - \frac{1}{3} = 1 + t + t^2 + \frac{1}{3}t^3\,,\\
\lambda_3(t) &= 1 + \int_0^t f(\lambda_2(s))\,\mathrm{d}s = 1 + \int_0^t \left(1 + s + s^2 + \tfrac{1}{3}s^3\right)^2\mathrm{d}s\\
&= 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7\,.
\end{aligned}$$

(ii) With $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we get for all $t \in \mathbb{R}$ that

$$\begin{aligned}
\lambda_0(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\\
\lambda_1(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t A\lambda_0(s)\,\mathrm{d}s = (\mathrm{Id} + tA)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}\,,\\
\lambda_2(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t A\lambda_1(s)\,\mathrm{d}s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ -s \end{pmatrix}\mathrm{d}s = \begin{pmatrix} t \\ 1 - \frac{1}{2}t^2 \end{pmatrix}\,,\\
\lambda_3(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t A\lambda_2(s)\,\mathrm{d}s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 - \frac{1}{2}s^2 \\ -s \end{pmatrix}\mathrm{d}s = \begin{pmatrix} t - \frac{1}{6}t^3 \\ 1 - \frac{1}{2}t^2 \end{pmatrix}\,.
\end{aligned}$$

Note that computing more Picard iterations will provide more terms from the Taylor expansion of $\sin(t)$ in the first component and $\cos(t)$ in the second component.

Exercise 4.

Define $d(t) := \alpha(t) - \lambda(t)$ for all $t \in I$ with $t \geq t_0$. The assumption implies that $d(t_0) \geq 0$. Assume for contradiction that $d(t) \leq 0$ for some $t > t_0$ and define

$$\tau := \inf\left\{t > t_0 : d(t) \leq 0\right\}.$$

We note that this implies that $d(\tau) = 0$ ($\Leftrightarrow \alpha(\tau) = \lambda(\tau)$), since d is continuous and $d(t_0) \geq 0$. We distinguish two cases.

Case 1. $\tau = t_0$.

Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, where $t_n > t_0$ with $\lim_{n\to\infty} t_n = t_0$ and $d(t_n) \leq 0$ for all $n \in \mathbb{N}$. This implies that

$$\dot{d}(t_0) = \lim_{n\to\infty} \frac{\overbrace{d(t_n)}^{\leq 0} - \overbrace{d(t_0)}^{=0}}{\underbrace{t_n - t_0}_{\geq 0}} \leq 0\,.$$

This contradicts

$$\dot{d}(t_0) = \dot{\alpha}(t_0) - \dot{\lambda}(t_0) > f(t, \alpha(t_0)) - f(t, \lambda(t_0)) \overset{\alpha(t_0)=\lambda(t_0)}{=} 0\,.$$

Case 2. $\tau > t_0$.

Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, where $t_n < \tau$ with $\lim_{n\to\infty} t_n = \tau$ and $d(t_n) > 0$ for all $n \in \mathbb{N}$. This implies that

$$\dot{d}(\tau) = \lim_{n\to\infty} \frac{\overbrace{d(t_n)}^{>0} - \overbrace{d(\tau)}^{=0}}{\underbrace{t_n - \tau}_{\leq 0}} \leq 0\,.$$

Exactly as in Case 1, we get $\dot{d}(\tau) > 0$, which is a contradiction and finishes the proof.

Exercise 5.

We show that there always exists a solution $\lambda : \mathbb{R} \to \mathbb{R}$ and a $a > 0$ such that $\lambda(t + a) - \lambda(t) \in \mathbb{Z}$ for all $t \in \mathbb{R}$. We distinguish two cases.

Case 1. $f(x^*) = 0$ for some $x^* \in \mathbb{R}$.

Due to Proposition 1.3, the constant function $\lambda(t) := x^*$ for all $t \in \mathbb{R}$ is a solution, and thus, for any $a > 0$, we have $\lambda(t + a) - \lambda(t) = 0 \in \mathbb{Z}$.

Case 2. $f(x) \neq 0$ for all $x \in \mathbb{R}$.

We use the hint and consider the unique solution $\lambda : \mathbb{R} \to \mathbb{R}$ of the initial value problem $\dot{x} = f(x)$, $x(0) = 0$. We assume without loss of generality that $f(x) > 0$ for all $x \in \mathbb{R}$ (note that f cannot change sign due to continuity), which implies that λ is strictly monotonically increasing.

Firstly, we show that

$$\lim_{t\to\infty} \lambda(t) = \infty\,. \tag{A}$$

To do so, assume this does not hold. Then monotonicity implies that there exists an $x^* \in \mathbb{R}$ with $\lim_{t\to\infty} \lambda(t) = x^*$. Due to $f(x^*) > 0$ and continuity of f, there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$f(x) \geq \delta \quad \text{for all } \ x \in (x^* - \varepsilon, x^* + \varepsilon)\,.$$

Now there exists a $\tau > 0$ such that $\lambda(t) \in (x^* - \varepsilon, x^*)$ for all $t \geq \tau$, which implies

$$\dot{\lambda}(t) = f(\lambda(t)) \geq \delta \quad \text{for all } \ t \geq \tau\,.$$

The mean value theorem implies that $\lambda(t) - \lambda(\tau) \geq \dot{\lambda}(\tilde{t})(t - \tau)$ for some $\tilde{t} = \tilde{t}(t) \in (\tau, t)$, which implies that $\lambda(t) - \lambda(\tau) \geq \delta(t - \tau)$ for all $t \geq \tau$, and hence, $\lim_{t\to\infty} \lambda(t) = \infty$, which is a contradiction and proves (A).

The intermediate value theorem implies that there exists an $a > 0$ with $\lambda(a) = 1$. We show now that $\lambda(t + a) - \lambda(t) = 1$ for all $t \in \mathbb{R}$.

To do so, we first realise that the function $\mu(t) := \lambda(t) + 1$ solves the initial value problem $\dot{x} = f(x)$, $x(0) = 1$. This follows from

$$\dot{\mu}(t) = \dot{\lambda}(t) = f(\lambda(t)) = f(\lambda(t) + 1) = f(\mu(t)) \quad \text{for all } \ t \in \mathbb{R}\,.$$

Due to translation invariance (Proposition 1.9), the function $t \mapsto \lambda(t+a)$ is also a solution of $\dot{x} = f(x)$, which satisfies the initial condition $x(0) = \lambda(a) = 1$. Due to the hint, we get $\lambda(t + a) = \mu(t)$, which finishes the proof.