**Classical Dynamics** **Autumn Term 2021**

**Answers to Test 1**

1. (a) $F_x = xe^{y^2} = -\partial V/\partial x$, which integrates to $V = -\frac{1}{2}x^2 e^{y^2} + C(y)$. $F_y = -\partial V/\partial y = yx^2 e^{y^2}$+arbitrary function of $y$.

(4 marks)

(b) Equation of motion

$$\ddot{x} = \frac{1}{2}e^x.$$

(i) To show that $\dot{x}^2 - e^x$ is a constant of the motion differentiate with respect to $t$ or use $T + V$ is a constant of the motion.

(ii) From part (i) $C = \dot{x}^2 - e^x$ is a constant of the motion. For the given initial conditions $C = -1$. Therefore $\dot{x}^2 = e^x - 1$ or $dx = \sqrt{e^x - 1}dt$ so that the time take to reach $x = +\infty$ is

$$T = \int_0^\infty \frac{dx}{\sqrt{e^x - 1}}.$$

This is finite as the integrand behaves like $x^{-1/2}$ for $x$ small and $e^{-x/2}$ for $x$ large.

Alternatively, one can compute the integral explicitly

$$\int_0^\infty \frac{dx}{\sqrt{e^x - 1}} = \int_0^\infty \frac{e^{-x/2}dx}{\sqrt{1 - (e^{-x/2})^2}} = -\,2\sin^{-1}(e^{-x/2})\Big|_0^\infty = -2(0 - \pi/2) = \pi.$$

(iii) If the particle has unit mass $T = \frac{1}{2}\dot{x}^2$, $V = -\frac{1}{2}e^x$ giving $L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}e^x$. Taking a different mass (not specified in the question) leads to a Lagrangian proportional to $\dot{x}^2 + e^x$.

(9 marks)

(c)

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}(y\dot{x} - x\dot{y}).$$

(i) The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\dot{x} + \frac{1}{2}y\right) + \frac{1}{2}\dot{y} = \ddot{x} + \dot{y} = 0.$$

Similarly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \ddot{y} - \dot{x} = 0.$$

The force acting on the particle is not conservative as it is velocity-dependent.

(ii)

$$x = R\cos t, \qquad y = R\sin t$$

is a solution of the equation of motion as $\ddot{x} = -R\cos t$ and $\dot{y} = R\cos t$. Similarly for the $y$ equation.

(iii) As only derivatives enter the equations of motion one can add constants to the solution from part (ii). As the equations are time-independent one can shift $t$ by a constant

$$x = R\cos(t+\gamma) + A, \qquad y = R\sin(t+\gamma) + B.$$

(12 marks)

(Total: 25 marks)