IMPERIAL

MATH70004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Asymptotic Methods

Date: Tuesday, May 20, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the algebraic equation

$$\varepsilon x^2 = \ln x, \quad \text{as} \quad \varepsilon \searrow 0.$$

(i) Derive a two-term asymptotic expansion for the solution near unity ($x \sim 1$).

(4 marks)

(ii) Derive a leading-order asymptotic approximation for the large positive solution ($1 \ll x$). [Clue: substitute $u = \varepsilon^{1/2} x$ and try $1 \ll u$.]

(6 marks)

(b) Consider the algebraic equation

$$\mathsf{A}\mathsf{x} + \varepsilon \mathsf{x} = \mathsf{v}, \quad \text{as} \quad \varepsilon \searrow 0,$$

where A is a square matrix and v is a non-zero column vector that spans $\operatorname{null} \mathsf{A}^\dagger$, $\mathsf{A}^\dagger = \overline{\mathsf{A}}^T$ being the conjugate transpose of A. Recall that $\mathsf{A}^\dagger$ satisfies the adjoint relation

$$\langle \mathsf{A}\mathsf{x}, \mathsf{y} \rangle = \langle \mathsf{x}, \mathsf{A}^\dagger \mathsf{y} \rangle \quad \forall \mathsf{x}, \mathsf{y},$$

with respect to the inner product $\langle \mathsf{x}, \mathsf{y} \rangle = \mathsf{x}^T \bar{\mathsf{y}}$.

(i) Derive a leading-order approximation for x in terms of v, and u, a non-zero column vector that spans $\operatorname{null} \mathsf{A}$. Assume that $\langle \mathsf{u}, \mathsf{v} \rangle \neq 0$.

[Recall that $\mathsf{A}\mathsf{x} = \mathsf{b}$ has a solution if and only if $\langle \mathsf{b}, \mathsf{z} \rangle = 0$ for all $\mathsf{z} \in \operatorname{null} \mathsf{A}^\dagger$.]

(5 marks)

(ii) If $\langle \mathsf{u}, \mathsf{v} \rangle = 0$, then there exists a column vector g such that $\mathsf{A}\mathsf{g} = \mathsf{u}$. Find a leading-order approximation for x in this case, in terms of u, v and g. Assume that $\langle \mathsf{g}, \mathsf{v} \rangle \neq 0$.

(5 marks)

(Total: 20 marks)

2. (a) How do the following integrals scale as $x \to +\infty$?

$$\int_0^{\pi/2} e^{x\cos t}\,\mathrm{d}t, \qquad \int_0^{\pi/2} t^{1/2} e^{x\cos t}\,\mathrm{d}t, \qquad \int_0^{\pi/2} e^{ix\cos t}\,\mathrm{d}t.$$

(9 marks)

(b) Consider the integral

$$I(\varepsilon) = \int_0^{\pi/2} \frac{1}{\varepsilon + \sin x}\,\mathrm{d}x \quad \text{as} \quad \varepsilon \searrow 0.$$

(i) Give scaling estimates for the local and global contributions, and for the integral $I(\varepsilon)$.

(3 marks)

(ii) Find a leading-order approximation for $I(\varepsilon)$.

(3 marks)

(iii) Use the method of splitting the range of integration to derive an asymptotic expansion for $I(\varepsilon)$, to order unity.

$$\left[\text{Note the integral } \int_0^{\pi/2} \left(\frac{1}{\sin x} - \frac{1}{x}\right)\,\mathrm{d}x = \ln\frac{4}{\pi}.\right]$$

(5 marks)

(Total: 20 marks)

3. Consider the boundary-value problem

$$\varepsilon y'' + y' + y\cos x = 0, \quad y(0) = 0, \quad y(\pi) = 1, \quad \text{as} \quad \varepsilon \searrow 0.$$

(a) Seek a leading-order outer approximation, $y(x;\varepsilon) \sim y_0(x)$, and show that it is inconsistent with the boundary condition at $x = 0$. Determine $y_0(x)$ using the other boundary condition.

(4 marks)

(b) Deduce the scaling of the thickness of the boundary layer near $x = 0$.

(3 marks)

(c) Find a leading-order inner approximation, $y(x;\varepsilon) = Y(u;\varepsilon) \sim Y_0(u)$, where u is an appropriate strained coordinate.

(4 marks)

(d) Extend the inner and outer asymptotic expansions as

$$y(x;\varepsilon) \sim y_0(x) + \varepsilon y_1(x), \qquad y(x;\varepsilon) = Y(u;\varepsilon) \sim Y_0(u) + \varepsilon Y_1(u),$$

respectively, and show that

$$y_1(x) = e^{-\sin x} \int_x^\pi y_0''(s) e^{\sin s}\,\mathrm{d}s, \qquad Y_1(u) = A_1(1 - e^{-u}) + 1 - u - e^{-u}(1 + u).$$

(4 marks)

(e) Show that $y_1(x) = 2 + \pi/2 + o(1)$ as $x \searrow 0$. [Use $\int_0^\pi \cos^2 s\,\mathrm{d}s = \pi/2$.]

(1 mark)

(f) Find A_1.

(4 marks)

(Total: 20 marks)

4. (a) Consider a weakly damped oscillator with the damping coefficient increasing with time,

$$\ddot{y} + \varepsilon^2 t\dot{y} + y = 0, \quad \varepsilon \searrow 0.$$

(i) Explain briefly why the damping term is important for $t = \mathrm{ord}(1/\varepsilon)$.

[Assume a regular perturbation expansion in the form $y(t;\varepsilon) \sim y_0(t) + \varepsilon^2 y_2(t)$ and note that solutions to $\ddot{y} + y = te^{\pm it}$ are of order t^2 as $t \to \infty$.]

(2 marks)

(ii) Use the method of multiple scales to derive a leading-order approximation in the form

$$y(t;\varepsilon) \sim A(\epsilon t)e^{it} + \text{c.c.}$$

Obtain the general solution for the complex amplitude $A(\epsilon t)$, and comment on the effect of the time-dependent damping coefficient.

(9 marks)

(b) Consider the forced problem

$$\ddot{y} + \varepsilon^2 t\dot{y} + y = \varepsilon \cos \omega t, \quad \varepsilon \searrow 0.$$

(i) Explain briefly why the approximation obtained in part (a) remains consistent for $\omega \neq 1$ (here ω is considered fixed as $\varepsilon \searrow 0$).

(2 marks)

(ii) Let $\omega = 1 + \varepsilon\sigma$, with the real parameter σ held fixed as $\varepsilon \searrow 0$. Again seek a leading-order approximation in the form

$$y(t;\varepsilon) \sim A(\epsilon t)e^{it} + \text{c.c.}$$

Derive the differential equation for the complex amplitude $A(\epsilon t)$ — do not solve it.

(5 marks)

(iii) How does $A(\epsilon t)$ behave for $\epsilon t \gg 1$?

(2 marks)

(Total: 20 marks)

5. (a) The function $F_0(u)$ satisfies the differential equation

$$F_0'' + uF_0' = 1,$$

for $u \geq 0$. Show by local analysis that

$$F_0(u) = \ln u + A_0 + O(u^{-2}) \quad \text{as} \quad u \to +\infty,$$

where A_0 is a constant.

(3 marks)

(b) The function $F_1(u)$ satisfies the differential equation

$$F_1'' + uF_1' = -u^2 F_0',$$

for $u \geq 0$, where $F_0(u)$ is the function from part (a). Show by local analysis that

$$F_1(u) = -u + A_1 + O(u^{-1}) \quad \text{as} \quad u \to +\infty,$$

where A_1 is a constant.

(3 marks)

(c) Consider the boundary-value problem

$$\varepsilon^2 y'' + x(1+x)y' = 1, \quad y(0) = 1, \quad y(1) = -\ln 2, \quad \text{as} \quad \varepsilon \searrow 0.$$

(i) Assuming a boundary layer near $x = 0$, find an outer approximation to order ε.

(3 marks)

(ii) Deduce the scaling of the boundary-layer thickness.

(3 marks)

(iii) Show that the inner expansion is of the form

$$Y(u;\varepsilon) \sim \left\{\ln\frac{1}{\varepsilon}Y_{0l}(u) + Y_{00}(u)\right\} + \varepsilon\left\{\ln\frac{1}{\varepsilon}Y_{1l}(u) + Y_{10}(u)\right\} \quad \varepsilon \searrow 0,$$

with an appropriately defined strained coordinate u fixed. Formulate closed problems for $Y_{0l}(u)$, $Y_{00}(u)$, $Y_{1l}(u)$ and $Y_{10}(u)$ — do not solve these problems. [Use the results of parts (a) and (b), in conjunction with asymptotic matching. It may be initially convenient to group together terms of the same algebraic order.]

(8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60004/70004

Asymptotic Methods (Solutions)

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1. (a) (i) Let $x = 1 + \bar{x}$, where $\bar{x} \ll 1$. Then unseen $\Downarrow$

$$\varepsilon(1 + \bar{x}^2) = \ln(1 + \bar{x}) \quad \Rightarrow \quad \varepsilon(1 + O(\bar{x})) = \bar{x} + O(\bar{x}^2).$$

Thus,

$$\bar{x} \sim \varepsilon \quad \Rightarrow \quad x \sim 1 + \varepsilon.$$

4, A

(ii) Let $u = \varepsilon^{1/2} x$, so that

$$u^2 = \ln u + \frac{1}{2} \ln \frac{1}{\varepsilon}.$$

A consistent balance, with $u \gg 1$, gives

$$u^2 \sim \frac{1}{2} \ln \frac{1}{\varepsilon} \quad \Rightarrow \quad u \sim \left(\frac{1}{2} \ln \frac{1}{\varepsilon} \right)^{1/2}.$$

Thus,

$$x \sim \left(\frac{1}{2\varepsilon} \ln \frac{1}{\varepsilon} \right)^{1/2}.$$

6, B

(b) (i) With A singular, we expect a resonance-type expansion seen/sim.seen $\Downarrow$

$$\mathsf{x} \sim \varepsilon^{-1} \mathsf{x}_{-1} + \mathsf{x}_0 \quad \text{as} \quad \varepsilon \searrow 0.$$

At $O(\varepsilon^{-1})$:

$$\mathsf{A}\mathsf{x}_{-1} = 0 \quad \Rightarrow \quad \mathsf{x}_{-1} = \alpha \mathsf{u},$$

where α is a constant and u is a non-zero vector that spans $\mathrm{null}\, \mathsf{A}$.
At $O(1)$:

$$\mathsf{A}\mathsf{x}_0 = \mathsf{v} - \alpha \mathsf{u},$$

so that solvability gives

$$\langle \mathsf{v} - \alpha \mathsf{u}, \mathsf{v} \rangle = 0 \quad \Rightarrow \quad \alpha = \frac{\langle \mathsf{v}, \mathsf{v} \rangle}{\langle \mathsf{u}, \mathsf{v} \rangle},$$

where v is a non-zero vector that spans $\mathrm{null}\, \mathsf{A}^\dagger$, the null space of the adjoint of A with respect to the specified inner product. Note that $\langle \mathsf{v}, \mathsf{v} \rangle \neq 0$ and that we have used the assumption $\langle \mathsf{u}, \mathsf{v} \rangle \neq 0$.

5, C

(ii) Now

$$\mathsf{x} \sim \varepsilon^{-2} \mathsf{x}_{-2} + \varepsilon^{-1} \mathsf{x}_{-1} + \mathsf{x}_0 \quad \text{as} \quad \varepsilon \searrow 0,$$

with

$$\mathsf{x}_{-2} = \alpha \mathsf{u},$$

where α is a constant.
At $O(\varepsilon^{-1})$:

$$\mathsf{A}\mathsf{x}_{-1} = -\alpha \mathsf{u},$$

which is automatically solvable since $\langle \mathsf{u}, \mathsf{v} \rangle = 0$. We can write the general solution as

$$\mathsf{x}_{-1} = \beta \mathsf{u} - \alpha \mathsf{g},$$

where β is a constant and g is a specified vector satisfying $\mathsf{A}\mathsf{g} = \mathsf{u}$. [$\mathsf{g}$ is a generalised eigenvector associated with the zero eigenvalue of A, of second order (i.e. $\mathsf{A}^2\mathsf{g} = 0$ but $\mathsf{A}\mathsf{g} \neq 0$), which exists since $\langle \mathsf{u}, \mathsf{v} \rangle = 0$.]

At $O(1)$:

$$\mathsf{A}\mathsf{x}_0 = \mathsf{v} + \alpha\mathsf{g} - \beta\mathsf{u}.$$

Solvability gives

$$\langle \mathsf{v} + \alpha\mathsf{g} - \beta\mathsf{u}, \mathsf{v}\rangle = 0 \quad \Rightarrow \quad \alpha = -\frac{\langle \mathsf{v}, \mathsf{v}\rangle}{\langle \mathsf{g}, \mathsf{v}\rangle},$$

where we have used the assumption $\langle \mathsf{g}, \mathsf{v}\rangle \neq 0$. $\boxed{5, \text{D}}$

2. (a) (i) The level function $\phi(t) = \cos t$ has a quadratic maximum at the left boundary, $t = 0$. The dominant contribution comes from an $O(x^{-1/2})$ interval near that boundary, where the integrand is of order e^x. Thus, the integral scales as $x^{-1/2}e^x$. $\boxed{\text{unseen} \Downarrow}$ $\boxed{3,\ \mathrm{A}}$

(ii) Same as (i) however now the integrand scales as $(x^{-1/2})^{1/2}e^x$. Thus, the integral scales as $x^{-3/4}e^x$. $\boxed{3,\ \mathrm{B}}$

(iii) The phase function $\psi(t) = \cos t$ has a simple stationary point at the left boundary. The dominant contribution comes from an $O(x^{-1/2})$ interval near that boundary, where the integrand is of order unity. The integral scales as $x^{-1/2}$. $\boxed{3,\ \mathrm{A}}$

(b) (i) Local ($x = O(\varepsilon)$) contribution scales as $\varepsilon^{-1} \cdot \varepsilon = 1$. The global contribution is also of order unity. That, together with the $1/x$ behaviour of the integrand for $\varepsilon \ll x \ll 1$ suggests a leading-order overlap contribution scaling as $\ln(1/\varepsilon)$. $\boxed{\text{seen} \Downarrow}$ $\boxed{3,\ \mathrm{A}}$

(ii) Since the dominant contribution comes from the overlap domain, we can calculate a leading-order approximation as

$$I(\varepsilon) \sim \int_0^1 \frac{1}{x+\varepsilon} \sim [\ln(x+\varepsilon)]_0^1 \sim \ln(1+\varepsilon) - \ln\varepsilon \sim \ln\frac{1}{\varepsilon}.$$

$\boxed{3,\ \mathrm{B}}$

(iii) We split the range of integration,

$$I(\varepsilon) = \int_0^\lambda \frac{\mathrm{d}x}{\sin x + \varepsilon} + \int_\lambda^{\pi/2} \frac{\mathrm{d}x}{\sin x + \varepsilon}, \tag{1}$$

where $\varepsilon \ll \lambda(\varepsilon) \ll 1$. The first integral gives

$$I_1 = \int_0^\lambda \frac{\mathrm{d}x}{\sin x + \varepsilon} = \int_0^{\lambda/\varepsilon} \frac{\mathrm{d}u}{\varepsilon^{-1}\sin(\varepsilon u) + 1}. \tag{2}$$

In the integration range ($\varepsilon u \ll 1$), we can expand

$$1 + \varepsilon^{-1}\sin(\varepsilon u) = 1 + u + O(\varepsilon^2 u^3) = (1+u)(1 + O(\varepsilon^2 u^2)). \tag{3}$$

Thus,

$$\begin{aligned} I_1 &= \int_0^{\lambda/\varepsilon} \frac{\mathrm{d}u}{1+u}(1 + O(\varepsilon^2 u^2)) \\ &= \ln\frac{\lambda}{\varepsilon} + \ln\left(1 + \frac{\varepsilon}{\lambda}\right) + O(\lambda^2) = \ln\frac{\lambda}{\varepsilon} + o(1). \end{aligned} \tag{4}$$

The second integral in (1), where $\varepsilon/x \ll 1$, gives

$$\begin{aligned} I_2 &= \int_\lambda^{\pi/2} \frac{\mathrm{d}x}{\sin x + \varepsilon} = \int_\lambda^{\pi/2} \frac{\mathrm{d}x}{\sin x\,(1 + \frac{\varepsilon}{\sin x})} \\ &= \int_\lambda^{\pi} \frac{\mathrm{d}x}{\sin x}(1 + O(\varepsilon/x)) = \int_\lambda^{\pi/2} \frac{\mathrm{d}x}{\sin x} + O(\varepsilon/\lambda) \\ &= \int_\lambda^{\pi/2} \frac{\mathrm{d}x}{x} + \int_\lambda^{\pi/2} \left(\frac{1}{\sin x} - \frac{1}{x}\right)\mathrm{d}x + o(1) \\ &= -\ln\lambda + \ln\frac{\pi}{2} + \int_0^{\pi/2} \left(\frac{1}{\sin x} - \frac{1}{x}\right)\mathrm{d}x + o(1) = -\ln\lambda + \ln\frac{\pi}{2} + \ln\frac{4}{\pi} + o(1). \end{aligned} \tag{5}$$

Adding together I_1 and I_2, we find

$$I(\varepsilon) = \ln\frac{1}{\varepsilon} + \ln 2 + o(1) \quad \text{as} \quad \varepsilon \searrow 0. \tag{6}$$

$\boxed{5,\ \mathrm{D}}$

3. (a) We posit a leading-order outer approximation in the form

unseen $\Downarrow$

$$y(x;\varepsilon) \sim y_0(x) \quad \text{as} \quad \varepsilon \searrow 0, \tag{7}$$

with x fixed in some interval. The leading term $y_0(x)$ satisfies

$$y_0' + \cos x\, y_0 = 0, \tag{8}$$

with solution

$$y_0(x) = ae^{-\sin x}, \tag{9}$$

where a is a constant. Attempting to apply the boundary condition at $x = 0$ gives $a = 0$, which is inconsistent with the assumed form of the outer approximation. On the other hand, there is no difficulty applying the boundary condition at $x = \pi$, at $O(1)$, which gives

$$y_0(\pi) = 1 \quad \Rightarrow \quad a = 1. \tag{10}$$

4, A

(b) We expect that the outer expansion holds for x fixed in the interval $(0, \pi]$, with a boundary layer near $x = 0$ of thickness order $\delta(\varepsilon) \ll 1$. We expect y and it's variation Δy to both be $O(1)$ in the boundary layer, as (i) $\lim_{x\searrow 0} y_0(x) = 1$, and (ii) $y(0;\varepsilon) = 0$. The terms in the differential equation are then of order ε/δ^2, $1/\delta$ and 1, respectively; the dominant balance of the first two terms suggests

$$\delta = \varepsilon. \tag{11}$$

3, A

(c) We define $Y(u;\varepsilon) = y(x;\varepsilon)$, where $u = x/\varepsilon$. The differential equation gives

$$Y'' + Y' + \varepsilon \cos(\varepsilon u) Y = 0, \tag{12}$$

where the prime denotes differentiation with respect to u. The left boundary condition gives

$$Y(0) = 0. \tag{13}$$

We posit the inner expansion

$$Y(u;\varepsilon) \sim Y_0(u) \quad \text{as} \quad \varepsilon \searrow 0, \tag{14}$$

for $u \geq 0$ fixed. The leading approximation $Y_0(u)$ satisfies

$$Y_0'' + Y_0' = 0, \quad Y_0(0) = 0, \quad \lim_{u\to+\infty} Y_0 = 1, \tag{15}$$

where the last condition follows from straightforward leading-order matching with the outer approximation. The solution is readily found as

$$Y_0(u) = 1 - e^{-u}. \tag{16}$$

4, B

(d) We extend the outer expansion as $y(x;\varepsilon) \sim y_0(x) + \varepsilon y_1(x)$. The correction term $y_1(x)$ satisfies

$$y_1' + \cos x y_1 = -y_0'', \quad y_1(\pi) = 0. \tag{17}$$

Using an integrating factor, we find

$$y_1(x) = e^{-\sin x} \int_x^\pi y_0''(s) e^{\sin s}\, \mathrm{d}s. \tag{18}$$

We now extend the inner expansion as $Y(u;\varepsilon) \sim Y_0(u) + \varepsilon Y_1(u)$. The correction term satisfies

$$Y_1'' + Y_1' = -Y_0 \ \left\{= e^{-u} - 1\right\}, \quad Y_1(0) = 0. \tag{19}$$

Integrating once,

$$Y_1' + Y_1 = A_1 - e^{-u} - u, \tag{20}$$

where A_1 is a constant. Integrating again, using an integrating factor and the boundary condition $Y_1(0) = 0$, we find

$$Y_1(u) = e^{-u} \int_0^u \left[A_1 e^s - s e^s - 1\right] \mathrm{d}s. \tag{21}$$

Performing the integration, the solution simplifies to

$$Y_1(u) = A_1(1 - e^{-u}) + 1 - u - e^{-u} - ue^{-u}. \tag{22}$$

4, B

(e) Since $y_0 = \exp(-\sin x)$, we have $y_0''(x) = (\sin x + \cos^2 x)\exp(-\sin x)$. Substituting this result into (18), we find

$$y_1(x) = e^{-\sin x} \int_x^\pi (\sin s + \cos^2 s)\, \mathrm{d}s. \tag{23}$$

Clearly,

$$y_1(x) \sim \int_0^\pi (\sin s + \cos^2 s)\, \mathrm{d}s + o(1) \quad \text{as} \quad x \searrow 0. \tag{24}$$

The constant term is $2 + \pi/2$.

1, A

(f) We perform Van Dyke matching between the outer expansion to order ε and the inner expansion to order ε. On the one hand,

$$\begin{aligned}
\text{outer to } \varepsilon &= e^{-\sin x} + \varepsilon y_1(x) = e^{-\sin(\varepsilon u)} + \varepsilon y_1(\varepsilon u) \\
&\underset{\text{inner lim}}{=} e^{-\varepsilon u + O(\varepsilon^3)} + \varepsilon\left(2 + \frac{\pi}{2} + o(1)\right) = 1 - \varepsilon u + \varepsilon\left(2 + \frac{\pi}{2}\right) + o(\varepsilon) \\
&\underset{\text{to } \varepsilon}{\#} 1 - \varepsilon u + \varepsilon\left(2 + \frac{\pi}{2}\right).
\end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned}
\text{inner to } \varepsilon &= 1 - e^{-u} + \varepsilon\left\{A_1(1 - e^{-u}) + 1 - u - e^{-u}(1 + u)\right\} \\
&= 1 - e^{-x/\varepsilon} + \varepsilon\left\{A_1(1 - e^{-x/\varepsilon}) + 1 - x/\varepsilon - e^{-x/\varepsilon}(1 + x/\varepsilon)\right\} \\
&\underset{\text{outer lim}}{=} 1 - x + \varepsilon A_1 + \varepsilon + \text{EST} \underset{\text{to } \varepsilon}{\#} 1 - x + \varepsilon + \varepsilon A_1.
\end{aligned} \tag{26}$$

Equating the two results (by writing the latter in terms of u, say) yields

$$A_1 = 1 + \frac{\pi}{2}. \tag{27}$$

4, D

4. (a) (i) Assume a regular perturbation expansion $y(t;\varepsilon) = y_0(t) + \varepsilon^2 y_2(t) + \cdots$. The weak damping term would generate a resonant forcing term $-t\dot{y}_0$ acting on y_2, leading to a perturbation at long times scaling as $t^2\varepsilon^2$; this perturbation would become order unity for times $1/\varepsilon$. unseen ⇓ 2, A

(ii) Let $Y(\tau, T;\varepsilon) = y(t;\varepsilon)$ on $(\tau, T) = (t, \varepsilon t)$, with $Y(\tau, T;\varepsilon)$ satisfying the PDE

$$Y_{\tau\tau} + 2\varepsilon Y_{\tau T} + \varepsilon^2 Y_{TT} + \varepsilon T(Y_\tau + \varepsilon Y_T) + Y = 0 \tag{28}$$

and possessing a *regular* expansion in the form

$$Y(\tau, T;\varepsilon) \sim Y_0(\tau, T) + \varepsilon Y_1(\tau, T) \quad \text{as} \quad \varepsilon \searrow 0. \tag{29}$$

At $O(1)$,

$$Y_{0\tau\tau} + Y_0 = 0 \quad \Rightarrow \quad Y_0(\tau, T) = A(T)e^{i\tau} + \text{c.c.}, \tag{30}$$

where $A(T)$ is a complex amplitude and c.c. means complex conjugate.
At $O(\varepsilon)$:

$$Y_{1\tau\tau} + Y_1 = -2Y_{0\tau T} - TY_{0\tau} = -2iA'e^{i\tau} - iTAe^{i\tau} + \text{c.c.} \tag{31}$$

Eliminating secular terms gives the solvability condition, or amplitude equation,

$$A' = -\frac{1}{2}TA \quad \Rightarrow \quad A(T) = Ce^{-\frac{1}{4}T^2}, \tag{32}$$

where C is a complex constant. The growth with time of the damping coefficient results in super-exponential decay of initial oscillations (in contrast to exponential in the case of a constant damping coefficient). 9, A

(b) (i) With $\omega \neq 1$ fixed as $\varepsilon \searrow 0$, the forcing would not influence the solvability condition at $O(\varepsilon)$ in the above multiple-scales expansion. 2, C

(ii) Let $\omega = 1 + \varepsilon\sigma$, with σ held fixed as $\varepsilon \searrow 0$. We employ the same multiple-scales expansion as before, with the term

$$\frac{1}{2}\varepsilon e^{i\tau}e^{i\sigma T} + \text{c.c.} \tag{33}$$

added to the right-hand side of the PDE (28). The solvability condition at $O(\varepsilon)$ now gives the forced amplitude equation

$$A' = -\frac{1}{2}TA - \frac{i}{4}e^{i\sigma T}. \tag{34}$$

5, C

(iii) A dominant balance argument on the forced amplitude equation (34) gives the late-slow-time behaviour

$$A(T) \sim -\frac{i}{2T}e^{i\sigma T} \quad \text{as} \quad T \to \infty. \tag{35}$$

Thus, the magnitude of the oscillations decays algebraically with time. 2, D

5. (a) As $u \to +\infty$, the consistent dominant balance is $\boxed{\text{unseen} \Downarrow}$

$$uF_0' \sim 1 \quad \Rightarrow \quad F_0'(u) \sim u^{-1}, \tag{36}$$

so that the differential equation gives

$$F_0'(u) = u^{-1} + O(F_0''/u) \quad \text{as} \quad u \to +\infty. \tag{37}$$

As we do not expect oscillations in the limit, this suggests

$$F_0(u) = \ln u + A_0 + O(u^{-2}) \quad \text{as} \quad u \to +\infty, \tag{38}$$

where A_0 is a constant. $\boxed{3, \text{M}}$

(b) Using part (a), we have

$$F_1'' + uF_1' = -u + O(u^{-1}) \quad \text{as} \quad u \to +\infty. \tag{39}$$

The consistent dominant balance is $uF_1' \sim -u$, so that (39) gives

$$F_1' = -1 + O(u^{-2}) + O(F_1''/u) \quad \text{as} \quad u \to +\infty. \tag{40}$$

As we do not expect oscillations in the limit, this suggests

$$F_1 = -u + A_1 + O(u^{-1}) \quad \text{as} \quad u \to +\infty, \tag{41}$$

where A_1 is a constant. $\boxed{3, \text{M}}$

(c) (i) We posit the outer expansion

$$y(x;\varepsilon) \sim y_0(x) + \varepsilon y_1(x) \quad \text{as} \quad \varepsilon \searrow 0, \tag{42}$$

for $x \in (0, 1]$ fixed. At $O(1)$,

$$x(1+x)y_0' = 1, \quad y_0(1) = -\ln 2 \quad \Rightarrow \quad y_0(x) = \ln x - \ln(1+x). \tag{43}$$

At $O(\varepsilon)$,

$$x(1+x)y_1' = 0, \quad y_1(1) = 0 \quad \Rightarrow \quad y_1(x) \equiv 0. \tag{44}$$

Thus, we have the outer approximation

$$y(x;\varepsilon) = \ln x - \ln(1+x) + o(\varepsilon). \tag{45}$$

$\boxed{3, \text{M}}$

(ii) The boundary condition at $x = 0$ and matching considerations suggest that the solution is of order unity (up to logarithmic factors) and not approximately uniform in the boundary layer. Thus, if $\delta(\varepsilon)$ denotes the scaling of the boundary-layer thickness, the terms of the differential equation scale as ε^2/δ^2, 1 and 1, respectively, in the boundary layer. Since the second derivative must be included in the dominant balance, we have $\delta = \varepsilon$. $\boxed{3, \text{M}}$

(iii) Let $y(x;\varepsilon) = Y(u;\varepsilon)$, where $u = x/\varepsilon$. Then the differential equation and left boundary condition give

$$Y'' + u(1+\varepsilon u)Y' = 1, \quad Y(0) = 1. \tag{46}$$

We posit the inner expansion

$$Y(u;\varepsilon) \sim Y_0(u;\varepsilon) + \varepsilon Y_1(u;\varepsilon), \tag{47}$$

where, in anticipation of logarithmic orders (given the small-x behaviour of the outer approximation) we initially group together terms of the same algebraic order by allowing the terms to weakly (logarithmically) vary with ε.
At $O(1)$ algebraic order, the inner problem (46) and part (a) give

$$Y_0'' + uY_0' = 1, \quad Y_0(0) = 1, \quad Y_0 = \ln u + A_0(\varepsilon) + O(u^{-2}) \quad \text{as} \quad u \to +\infty, \tag{48}$$

with $A_0(\varepsilon)$ allowed to depend logarithmically upon ε. At $O(\varepsilon)$ algebraic order, the inner problem (46) and part (b) give

$$Y_1'' + uY_1' = -u^2 Y_0', \quad Y_1(0) = 0, \quad Y_1 = -u + A_1 + O(u^{-1}) \quad \text{as} \quad u \to +\infty, \tag{49}$$

with A_1 allowed to depend logarithmically upon ε.
To determine A_0 and A_1, we carry out Van Dyke matching between the inner expansion to algebraic order ε and the outer expansion to algebraic order ε. On one hand,

$$\begin{aligned}\text{outer to } \varepsilon &= \ln x - \ln(1+x) = \ln(\varepsilon u) - \ln(1+\varepsilon u) \\ &\underset{\text{inner lim}}{=} \ln\varepsilon + \ln u - \varepsilon u + O(\varepsilon^2) \underset{\text{to } \varepsilon}{\#} \ln\varepsilon + \ln u - \varepsilon u. \end{aligned} \tag{50}$$

On the other hand, using the large-u behaviours of Y_0 and Y_1,

$$\begin{aligned}\text{inner to } \varepsilon &= Y_0(u;\varepsilon) + \varepsilon Y_1(u;\varepsilon) = Y_0(x/\varepsilon;\varepsilon) + \varepsilon Y_1(x/\varepsilon;\varepsilon) \\ &\underset{\text{outer lim}}{=} \ln\frac{x}{\varepsilon} + A_0 + O(\varepsilon^2) + \varepsilon\left[-\frac{x}{\varepsilon} + A_1 + O(\varepsilon)\right] \\ &\underset{\text{to } \varepsilon}{\#} \ln\frac{x}{\varepsilon} + A_0 - x + \varepsilon A_1. \end{aligned} \tag{51}$$

Equating the two results (by writing the latter in terms of u, say) yields

$$A_0(\varepsilon) = -\ln\frac{1}{\varepsilon}, \quad A_1(\varepsilon) = 0. \tag{52}$$

The problems to be solved can be simplified by separating logarithmic orders. Thus, rewriting the inner expansion as

$$Y(u;\varepsilon) \sim \ln\frac{1}{\varepsilon}Y_{0l}(u) + Y_{00}(u) + \varepsilon\left\{\ln\frac{1}{\varepsilon}Y_{1l}(u) + Y_{10}(u)\right\} \quad \text{as} \quad \varepsilon \searrow 0, \tag{53}$$

with $u \geq 0$ fixed. From (48), (49) and (52), we find the sequence of problems

$$Y_{0l}'' + uY_{0l}' = 0, \quad Y_{0l}(0) = 0, \quad Y_{0l} \to -1 \quad \text{as} \quad u \to +\infty; \tag{54}$$

$$Y_{00}'' + uY_{00}' = 1, \quad Y_{00}(0) = 1, \quad Y_{00} = \ln u + o(1) \quad \text{as} \quad u \to +\infty; \tag{55}$$

$$Y_{1l}'' + uY_{1l}' = -u^2 Y_{0l}', \quad Y_{1l}(0) = 0, \quad Y_{00} \to 0 \quad \text{as} \quad u \to +\infty; \tag{56}$$

$$Y_{10}'' + uY_{10}' = -u^2 Y_{00}', \quad Y_{10}(0) = 0, \quad Y_{10} = -u + o(1) \quad \text{as} \quad u \to +\infty. \tag{57}$$

8, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH70004 Asymptotic Methods Markers Comments

Question 1	Most did part (a) ok. I'm surprised many did not do well on Part (b) as it was very close to notes.
Question 2	In Part (a) you were asked for scaling's, no need to calculate the asymptotic approximation in detail. In Part (b) most did not keep track of error terms systematically.
Question 3	Most of you did this question quite well, perhaps thanks to the guidance.
Question 4	Most did not do well on this question. When extending the problem to multiple time scales the 't' in the ODE should be rewritten in terms of the slow time T, rather than the fast time tau, otherwise algebraic growth in the fast time is inevitable and so is regularity of the asymptotic expansion.
Question 5	Most did not complete the last part of the question, perhaps because of time limitations.