Imperial College London

MATH97054 MATH70060

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Complex Manifolds

Date: 24 May 2022

Time: 14:00 – 16:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear map and let $A_{\mathbb{R}}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding map on the underlying real vector space. Show that $\det(A_{\mathbb{R}}) = |\det(A)|^2$. (3 marks)

(b) Show that complex manifolds are oriented. (3 marks)

(c) Let $f_1, \ldots, f_m \in \mathcal{O}_n$, and suppose that the matrix $\{\partial f_i/\partial z_j\}_{i=1,j=1}^{m,n}$ has rank m when $z = 0$. Show that the analytic set $Z = \{f_1 = \cdots = f_m = 0\}$ is irreducible in a neighborhood of the origin. (4 marks)

(d) Prove the Inverse Function Theorem for holomorphic functions assuming the Implicit Function Theorem proved in class. (5 marks)

(e) Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$, with a complex structure $J \in \mathrm{End}(V_{\mathbb{R}})$ and a compatible Riemannian metric g. Prove that the decomposition $\wedge^k V_{\mathbb{C}} = \sum_{p+q=k} V^{p,q}$ is orthogonal with respect to the induced Hermitian inner product h on $\wedge^k V_{\mathbb{C}}$. (5 marks)

(Total: 20 marks)

2. (a) Denote by $U_k = \{[z_0 : \cdots : z_n] : z_k \neq 0\}$ the canonical covering of $\mathbb{CP}^n$.

(i) Prove that $\mathbb{CP}^n$ is simply connected. (Hint: apply the following consequence of Van Kampen's theorem to the covering U_k: Let X be a topological space. If $A, B \subset X$ are simply connected open sets such that $X = A \cup B$ and $A \cap B$ non-empty and path-connected, then X is simply connected.)

(6 marks)

(ii) Show that every holomorphic map from $\mathbb{CP}^n$ to a complex torus must be constant. (Hint: consider the lift to the universal covers). (5 marks)

(b) Let $\mathcal{O}(-1) = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in L\}$. Denote by $\mathcal{O}(1)$ the dual of $\mathcal{O}(-1)$ as a bundle over $\mathbb{CP}^n$. Show that there are natural hermitian metrics $h^* = h^{-1}$ on $\mathcal{O}(1)$ and h on $\mathcal{O}(-1)$, inherited from the canonical hermitian metric of $\mathbb{C}^{n+1}$, such that h in the chart U_0 is given by:

$$h = \langle (1, z_1, \ldots, z_n), (1, z_1, \ldots, z_n) \rangle_{\mathbb{C}^{n+1}} = 1 + |z_1|^2 + \cdots + |z_n|^2.$$

(4 marks)

(c) Let D be the Chern connection of the natural hermitian metric on $\mathcal{O}(1)$, from the previous exercise, and let F_D be its curvature.

(i) Show that $F_D = \partial\bar{\partial} \log(1 + \sum |z_j|^2)$. (3 marks)

(ii) Show that $\omega_{FS} = \frac{i}{2\pi} F_D$ induces a hermitian metric on $\mathbb{CP}^n$. (2 marks)

(Total: 20 marks)

3. (a) Let $D \subset \mathbb{C}$ be a polydisk. Show that $H^{p,q}(D) = 0$, for $q \geq 1$. State clearly any results you are using. (4 marks)

(b) Assume the following $\bar{\partial}$-Poincaré lemma for $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: *given a smooth function $g : \mathbb{C}^* \to \mathbb{C}$, there exists a smooth function $f : \mathbb{C}^* \to \mathbb{C}$ with $\partial f / \partial \bar{z} = g$.*

(i) Conclude that the Dolbeault cohomology groups $H^{0,1}(\mathbb{C}^*)$ and $H^{1,1}(\mathbb{C}^*)$ are both zero. (2 marks)

(ii) Compute $H^{0,0}(\mathbb{C}^*)$ and $H^{1,0}(\mathbb{C}^*)$. (3 marks)

(c) Use the $\bar{\partial}$-Poincaré Lemma to state and prove the ∂-Poincaré Lemma (Hint: $\overline{\partial \alpha} = \bar{\partial}\bar{\alpha}$). (5 marks)

(d) (A $\partial\bar{\partial}$-Poincaré Lemma) Let $D \subset \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in A^{p,q}(D)$ satisfies $d\omega = 0$, with $p, q \geq 1$. Prove that there is $\psi \in A^{p-1,q-1}(D)$ such that $\partial\bar{\partial}\psi = \omega$ (Hint: use the three Poincaré Lemmas, i.e. for d, ∂ and $\bar{\partial}$). (6 marks)

(Total: 20 marks)

4. (a) Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space with a compatible almost complex structure J. Show the following:

   (i) $\dim V$ is even. (1 mark)

   (ii) Defining multiplication by i as $iv := J(v)$ makes V into a complex vector space. (1 mark)

   (iii) $\omega(\cdot,\cdot) := \langle J\cdot,\cdot\rangle$ is an alternating 2-form. (2 marks)

   (iv) $(\cdot,\cdot) := \langle\cdot,\cdot\rangle - i\omega$ is a hermitian form on (V, J). (2 marks)

(b) Let (M, g) be a compact oriented Riemannian manifold of dimension n. Prove that the volume form $\mathrm{vol}(g) \in A^n(M)$ is harmonic. (Hint: First show that $(\mathrm{vol}(g), d\psi) = \int_M d\psi$ for any $\psi \in A^{n-1}(M)$.) (4 marks)

(c) Let f, g be holomorphic functions in some neighborhood of $0 \in \mathbb{C}^n$. Suppose that f is irreducible in $\mathcal{O}_n$, and that g vanishes on $Z(f)$, in the sense that $g(z) = 0$ for every $z \in Z(f)$. Prove that f divides g in $\mathcal{O}_n$. (4 marks)

(d) Let M be a Kahler manifold and ω its Kahler form. Show that ω^ℓ defines a non-zero homology class in $H_D^{\ell,\ell}(M)$, for all $1 \leq \ell \leq \dim M/2$. In particular, $\dim H_D^{\ell,\ell}(M) \neq 0$. (2 marks)

(e) sim Let (V, g) be a vector space with a compatible almost complex structure J and fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$ and vol_W its volume form. Show that

$$\omega^m|_W \leq m!\,\mathrm{vol}_W,$$

with equality, if and only if, W is a complex subspace, i.e. $J(W) \subset W$. Conclude that a complex submanifold of a Kahler manifold minimises the volume in its homology class. (4 marks)

(Total: 20 marks)

5. (a) Let $I \subset \mathcal{O}_n$ be a nonzero ideal. Show that if $\partial f/\partial z_j \in I$ for all $f \in I$ and $j = 1, \ldots, n$, then $I = \mathcal{O}_n$. (Hint: consider the smallest $d \geq 0$ for which there exists $f \in I$ regular of degree d). (5 marks)

(b) Let $Z \subset D$ be an analytic subset of an open set $D \subset \mathbb{C}^n$. Assume that $0 \in Z$, but Z does not contain any open neighborhood of 0.

(i) Let $k \geq 0$ be the largest integer such that there exist $f_1, \ldots, f_k \in I(Z)$ with the property that at least one $k \times k$-minor g of the matrix $J(f_1, \ldots, f_k)$ does not belong to $I(Z)$. Prove that $k \geq 1$. (3 marks)

(ii) After shrinking D we may assume that $f_1, \ldots, f_k \in \mathcal{O}(D)$. Define $D' = D \setminus Z(g)$ and $Z' = Z(f_1) \cap \cdots \cap Z(f_k) \cap D'$.

(1) Show that Z' is a submanifold of D'. (3 marks)

(2) Show that $Z \cap D'$ is a union of connected components of Z'. (6 marks)

(3) Conclude that the singular locus Z_{sing} is contained in an analytic set strictly smaller than Z. (3 marks)

(Total: 20 marks)

Module: MATH97054/MATH70060
Setter: Guaraco
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2022

MATH97054/MATH70060 COMPLEX MANIFOLDS

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a) 3 marks; 2(a)(ii) 5 marks; 2(c)(ii) 3 marks; 3(a) 4 marks; 3(d) 5 marks; 4(a)(i) 1 mark; 4(a)(ii) 1 mark; 4(a)(iii) 2 marks; 4(a)(iv) 2 marks; 4(b) 4 marks; 4(d) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(b) 3 marks; 2(a)(i) 6 marks; 2(c)(ii) 2 marks; 3(b)(i) 2 marks; 3(b)(ii) 3 marks; 4(e) 4 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
1(c) 4 marks; 2(b) 4 marks; 4(c) 4 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(d) 5 marks; 1(e) 5 marks; 3(d) 6 marks; 4(c) 4 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

COMPLEX MANIFOLDS

Date: ??

Time: ??

Time Allowed: 2.5 Hours

This paper has *4 Questions (MATH70060); 5 Questions (MATH97054)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear map and let $A_{\mathbb{R}}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the corresponding map on the underlying real vector space. Show that $\det(A_{\mathbb{R}}) = |\det(A)|^2$. (3 marks)

(b) Show that complex manifolds are oriented. (3 marks)

(c) Let $f_1, \ldots, f_m \in \mathcal{O}_n$, and suppose that the matrix $\{\partial f_i/\partial z_j\}_{i=1,j=1}^{m,n}$ has rank m when $z = 0$. Show that the analytic set $Z = \{f_1 = \cdots = f_m = 0\}$ is irreducible in a neighborhood of the origin. (4 marks)

(d) Prove the Inverse Function Theorem for holomorphic functions assuming the Implicit Function Theorem proved in class. (5 marks)

(e) Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$, with a complex structure $J \in \operatorname{End}(V_{\mathbb{R}})$ and a compatible Riemannian metric g. Prove that the decomposition $\wedge^k V_{\mathbb{C}} = \sum_{p+q=k} V^{p,q}$ is orthogonal with respect to the induced Hermitian inner product h on $\wedge^k V_{\mathbb{C}}$. (5 marks)

(Total: 20 marks)

Solution.

(1)(a) We can work on a basis $v_1, \ldots, v_n \in \mathbb{C}^n$ in which A is presented in its Jordan canonical form

$$A = \begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then $v_1, iv_1, \cdots, v_n, iv_n$ is a basis of $\mathbb{C}^n$ as a $2n$-dimensional real vector space. In this basis, $A_{\mathbb{R}}$ has the same form as before but each λ_i corresponds to the 2×2-block matrix

$$\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$$

where $a_i, b_i \in \mathbb{R}$ are given by $\lambda_i = a_i + ib_i$. It follows that

$$\det A_{\mathbb{R}} = (a_1^2 + b_1^2) \cdots (a_n^2 + b_n^2) = |\lambda_1 \cdots \lambda_n|^2 = |\det A|^2.$$

(1)(b) Working in charts, it is enough to consider a holomorphic map $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$. To talk about orientation, we must interpret f to be a map between open sets of $\mathbb{R}^{2n}$. By the previous exercise, it is enough to show that df, considered as a real map, is also $\mathbb{C}$-linear. Since the map df is already $\mathbb{R}$-linear, being $\mathbb{C}$-linear is equivalent to commutation with the canonical almost complex structure of $\mathbb{C}^n$, which we denote by J. Writing the matrix of df in one canonical coordinate $z_i = x_i + iy_i$ at a time, denoting

$v = (\cdots, a, b, \cdots)$, we obtain the expressions

$$\begin{pmatrix} & \vdots & \vdots & \\ \cdots & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \cdots \\ \cdots & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \cdots \\ & \vdots & \vdots & \end{pmatrix} v = \begin{pmatrix} \vdots \\ a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} \\ a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} \\ \vdots \end{pmatrix}.$$

Let $v = (\cdots, 1, 0 \cdots)$. Then $df \cdot J(v) = df(\cdots, 0, 1, \cdots) = (\cdots, \partial u/\partial y, \partial v/\partial y, \cdots)$ and $J \cdot df(v) = J(\cdots, \partial u/\partial x, \partial v/\partial x, \cdots) = (\cdots, -\partial v/\partial x, \partial u/\partial x, \cdots)$. It follows that J and df commute if and only if f satisfies the Cauchy-Riemann equations.

(1)(c) By the Implicit Function Theorem, Z is an m dimensional complex submanifold, i.e. there exists a biholomorphism which flattens its coordinates. Therefore, without loss of generality, we can assume that $Z = \{z_1 = \cdots = z_{n-m} = 0\}$. Remember that Z is irreducible if and only if $I(Z)$ is a prime ideal. Let $f, g \in \mathcal{O}_n$ satisfying $fg \in I(Z)$. The restrictions of f and g to Z are functions in $\mathcal{O}_m$ whose product is zero. The result follows since $\mathcal{O}_m$ is a domain.

(1)(d) We show that given a holomorphic $f : D \subset \mathbb{C}^n \to \mathbb{C}^n$ with $f(0) = 0$ and $Jf(0)$ not singular, there is $r'' \in \mathbb{R}^n_+$ and $g : \Delta(0, r'') \to \mathbb{C}^n$ holomorphic, such that $f \circ g = id$ on $\Delta(0, r'')$. In fact, consider the function $\Phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, given by $\Phi(z, w) = f(z) - w$. We use the notation $r = (r', r'') \in \mathbb{R}^n \times \mathbb{R}^n$ and $J\Phi = (J'\Phi, J''\Phi)$. By assumption $J'\Phi(0,0) = Jf(0)$ is not singular. Then, the Implicit Function Theorem implies the existence of a small positive $r = (r', r'')$ and a holomorphic map $g : \Delta(0, r'') \to \Delta(0, r')$ such that for $(z, w) \in \Delta(0, r')$, we have $\Phi(z, w) = f(z) - w = 0$ if and only if $z = g(w)$. In other words, $f \circ g(w) = w$.

(1)(e) Remember that g and J are compatible if $g(Jv, Jw) = (v, w)$ for any $v, w \in V$. This implies that J is skew-symmetric, i.e. $g(Jv, w) = g(J^2 v, Jw) = g(v, -Jw)$. By the spectral theorem there is an orthonormal basis $\{x_i, y_i\}_{i=1}^n$ where J has diagonal block form, i.e $J(x_i) = y_i$ and $J(y_i) = -x_i$. Denote by $g_{\mathbb{C}}$ the hermitian extension of g on $V_{\mathbb{C}}$. The inner product on $\wedge^k V_{\mathbb{C}}$ is defined as follows: let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ be elements of $V_{\mathbb{C}}$, then $\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det\{g_{\mathbb{C}}(\alpha_i, \beta_j)\}$. Let $z_i = x_i - iy_i \in V_{\mathbb{C}}$. The space $V^{p,q} \subset \wedge^k V_{\mathbb{C}}$ is generated by the elements of the form $z_I \wedge \bar{z}_J$, where $|I| = p$ and $|J| = q$. Therefore, to show that the spaces $\wedge^{p,q} V$ are orthogonal, it is enough to show that z_i is orthogonal to z_j, for all $j \neq i$ and to all $\bar{z}_j$. We check this now:

$$\begin{aligned} g_{\mathbb{C}}(z_i, z_j) &= g_{\mathbb{C}}(x_i - iy_i, x_j - iy_j) = g_{\mathbb{C}}(x_i, x_j) - g_{\mathbb{C}}(x_i, iy_j) - g_{\mathbb{C}}(iy_i, x_j) + g_{\mathbb{C}}(iy_i, iy_j) \\ &= g(x_i, x_j) + ig(x_i, y_j) - ig(y_i, x_j) + g(y_i, y_j) \\ &= g(x_i, x_j) + ig(x_i, y_j) - ig(Jy_i, Jx_j) + g(Jy_i, Jy_j) \\ &= g(x_i, x_j) + ig(x_i, y_j) + ig(x_i, y_j) + g(x_i, x_j) \\ &= \delta_{ij} + 0 + 0 + \delta_{ij} \\ &= 2\delta_{ij}. \end{aligned}$$

$$
\begin{aligned}
g_{\mathbb{C}}(z_i, \bar{z}_j) &= g_{\mathbb{C}}(x_i - iy_i, x_j + iy_j) = g_{\mathbb{C}}(x_i, x_j) + g_{\mathbb{C}}(x_i, iy_j) - g_{\mathbb{C}}(iy_i, x_j) - g_{\mathbb{C}}(iy_i, iy_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) - ig(y_i, x_j) - g(y_i, y_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) - ig(Jy_i, Jx_j) - g(Jy_i, Jy_j) \\
&= g(x_i, x_j) - ig(x_i, y_j) + ig(x_i, y_j) - g(x_i, x_j) \\
&= 0.
\end{aligned}
$$

2. (a) Denote by $U_k = \{[z_0 : \cdots : z_n] : z_k \neq 0\}$ the canonical covering of $\mathbb{CP}^n$.

(i) Prove that $\mathbb{CP}^n$ is simply connected. (Hint: apply the following consequence of Van Kampen's theorem to the covering U_k: Let X be a topological space. If $A, B \subset X$ are simply connected open sets such that $X = A \cup B$ and $A \cap B$ non-empty and path-connected, then X is simply connected.)

(6 marks)

(ii) Show that every holomorphic map from $\mathbb{CP}^n$ to a complex torus must be constant. (Hint: consider the lift to the universal covers). (5 marks)

(b) Let $\mathcal{O}(-1) = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in L\}$. Denote by $\mathcal{O}(1)$ the dual of $\mathcal{O}(-1)$ as a bundle over $\mathbb{CP}^n$. Show that there are natural hermitian metrics $h^* = h^{-1}$ on $\mathcal{O}(1)$ and h on $\mathcal{O}(-1)$, inherited from the canonical hermitian metric of $\mathbb{C}^{n+1}$, such that h in the chart U_0 is given by:

$$h = \langle (1, z_1, \ldots, z_n), (1, z_1, \ldots, z_n) \rangle_{\mathbb{C}^{n+1}} = 1 + |z_1|^2 + \cdots + |z_n|^2.$$

(4 marks)

(c) Let D be the Chern connection of the natural hermitian metric on $\mathcal{O}(1)$, from the previous exercise, and let F_D be its curvature.

(i) Show that $F_D = \partial\bar{\partial} \log(1 + \sum |z_j|^2)$. (3 marks)

(ii) Show that $\omega_{FS} = \frac{i}{2\pi} F_D$ induces a hermitian metric on $\mathbb{CP}^n$. (2 marks)

(Total: 20 marks)

Solution.

(a)(i) Coordinates for U_k are obtained by fixing $z_k = 1$. Therefore, U_k is homeomorphic to $\mathbb{C}^n$, which is simply connected. To proceed by induction we assume $A_k = U_1 \cup \cdots \cup U_{k-1}$ is simply connected and $k \leq n$. By the hint above, it is enough to show that $A_k \cap U_k$ is non-empty and path-connected. In the coordinates for U_k, this intersection corresponds to the complement of the set $(z_0, \cdots, \hat{z}_k, \cdots, z_n) \in \mathbb{C}^n$ satisfying $z_1 = z_2 = \cdots = z_{k-1} = 0$. In other words, $A_k \cap U_k$ is homeomorphic to the complement of a vector space of (real) codimension at least 2 inside of $\mathbb{R}^{2n}$. Therefore, it is path connected.

(a)(ii) Since $\mathbb{CP}^n$ is simply connected, any continuous map f from $\mathbb{CP}^n$ to a complex torus lifts to the universal cover of the torus, which is $\mathbb{C}^k$. The result follows since $\mathbb{CP}^n$ is compact and any holomorphic map to $\mathbb{C}^k$ must be constant.

(b) As we saw above, the fibre of $\mathcal{O}(-1)$ over $[z] \in \mathbb{CP}^n$ is the set $L([z]) \in \{([z], \lambda z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \lambda \in \mathbb{C}\}$, which is a complex subspace of $\mathbb{C}^{n+1}$. Then the restriction of the canonical hermitian inner product of $\mathbb{C}^{n+1}$ is an hermitian inner product on $L([z])$, for each $[z]$. Since the fibres vary smoothly with $[z]$ these define a hermitian metric on $\mathcal{O}(-1)$. Finally, the hermitian inner products gives a canonical isomorphism between the fibers of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$, making it also a holomorphic hermitian vector bundle. Notice $h^{-1} = h^*$. So it is enough to compute the formula for $\mathcal{O}(-1)$. In coordinates, a hermitian metric on a vector bundle of rank one is determined by a scalar function h. We compute

$$\begin{aligned}\tilde{\phi}_0^{-1}(\widehat{z}_0, z_1, \ldots, z_n, 1) &= \psi_0^{-1}(\phi_0^{-1} \times id)(\widehat{z}_0, z_1, \ldots, z_n, 1) \\ &= \psi_0^{-1}([1, z_1, \ldots, z_n], 1) \\ &= ([1, z_1, \ldots, z_n], 1 \cdot (1, z_1, \ldots, z_n)).\end{aligned}$$

So $h = \langle (1, z_1, \ldots, z_n), (1, z_1, \ldots, z_n) \rangle_{\mathbb{C}^{n+1}} = 1 + |z_1|^2 + \cdots + |z_n|^2$.

(c)(i) The metric on $\mathcal{O}(1)$ is $h^* = h^{-1} = (1+|z_1|^2+\cdots+|z_n|^2)^{-1}$. In class we saw that $F_D = \bar{\partial}\partial \log h^*$, then $F_D = \bar{\partial}\partial \log(1+|z|^2)^{-1} = -\bar{\partial}\partial \log(1+|z|^2) = \partial\bar{\partial}\log(1+|z|^2)$, as claimed.

(c)(ii)

$$\begin{aligned}\partial\bar{\partial}\log(1+|z|^2) &= \partial\left(\frac{1}{1+|z|^2}\sum_{i=1}^n \bar{\partial}(z_i\bar{z}_i)\right) \\ &= \partial\left(\frac{1}{1+|z|^2}\sum_{i=1}^n z_i d\bar{z}_i\right) \\ &= -\frac{1}{(1+|z|^2)^2}\left(\sum_{i=1}^n \bar{z}_i dz_i\right) \wedge \left(\sum_{i=1}^n z_i d\bar{z}_i\right) + \frac{1}{1+|z|^2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i \\ &= -\frac{1}{(1+|z|^2)^2}\left(\sum_{i,j=1}^n z_j\bar{z}_i dz_i \wedge d\bar{z}_j\right) + \frac{1}{1+|z|^2}\sum_{i,j=1}^n \delta_{ij} dz_i \wedge d\bar{z}_j \\ &= \frac{1}{(1+|z|^2)^2}\sum_{i,j=1}^n (-z_j\bar{z}_i + (1+|z|^2)\delta_{ij}) dz_i \wedge d\bar{z}_j\end{aligned}$$

Since $\omega = \frac{i}{2}\sum_{ij} h_{ij} dz_i \wedge d\bar{z}_j$,it is enough to show that

$$h = h_{ij}(z) := (-z_j\bar{z}_i + (1+|z|^2)\delta_{ij})$$

is a positive definite hermitian matrix.

$$\begin{aligned}\langle h(z)w, w\rangle &= \sum_{ij} -z_j\bar{w}_j\bar{z}_i w_i + (1+|z|^2)\delta_{ij} w_i\bar{w}_j \\ &= -\langle z, w\rangle\langle w, z\rangle + (1+|z|^2)|w|^2 \\ &= -\langle z, w\rangle\overline{\langle z, w\rangle} + (1+|z|^2)|w|^2 \\ &= -|\langle z, w\rangle|^2 + (1+|z|^2)|w|^2.\end{aligned}$$

From Cauchy-Scharwz inequality it follows that $|z|^2|w|^2 \geq |\langle z, w\rangle|^2$. Therefore, $\langle h(z)w, w\rangle \geq |w|^2$, so h is positive definite.

3. (a) Let $D \subset \mathbb{C}$ be a polydisk. Show that $H^{p,q}(D) = 0$, for $q \geq 1$. State clearly any results you are using. (4 marks)

(b) Assume the following $\bar{\partial}$-Poincaré lemma for $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: *given a smooth function* $g : \mathbb{C}^* \to \mathbb{C}$, *there exists a smooth function* $f : \mathbb{C}^* \to \mathbb{C}$ *with* $\partial f/\partial \bar{z} = g$.

(i) Conclude that the Dolbeault cohomology groups $H^{0,1}(\mathbb{C}^*)$ and $H^{1,1}(\mathbb{C}^*)$ are both zero. (2 marks)

(ii) Compute $H^{0,0}(\mathbb{C}^*)$ and $H^{1,0}(\mathbb{C}^*)$. (3 marks)

(c) Use the $\bar{\partial}$-Poincaré Lemma to state and prove the ∂-Poincaré Lemma (Hint: $\overline{\partial \alpha} = \bar{\partial}\bar{\alpha}$). (5 marks)

(d) (A $\partial\bar{\partial}$-Poincaré Lemma) Let $D \subset \mathbb{C}^n$ be a polydisk, and suppose that $\omega \in A^{p,q}(D)$ satisfies $d\omega = 0$, with $p, q \geq 1$. Prove that there is $\psi \in A^{p-1,q-1}(D)$ such that $\partial\bar{\partial}\psi = \omega$ (Hint: use the three Poincaré Lemmas, i.e. for d, ∂ and $\bar{\partial}$). (6 marks)

(Total: 20 marks)

Solutions.

(3)(a) Present the full statement of the $\bar{\partial}$-Poincaré lemma on a polydisk. It implies that $\bar{\partial}$-closed forms of type (p, q) on D, are exact for $q \geq 1$. It follows that the corresponding cohomomology is zero.

(3)(b)(i) Given $\alpha = g d\bar{z}$, let f be such that $\partial f/\partial \bar{z} = g$, then $\alpha = \bar{\partial} f$. It follows that every $(0, 1)$-form is exact. Similarly, given $\beta = g dz \wedge d\bar{z}$, we have $\beta = \bar{\partial}(-f dz)$, so again every $(1, 1)$-form is exact.

(3)(b)(ii) Let $\alpha = f dz$ be general form of type $(1, 0)$. Then, $\bar{\partial}\alpha = 0$ iff $\partial f/\partial \bar{z} = 0$ iff f is holomorphic. Since the only $\bar{\partial}$-exact form of type $(1, 0)$ is zero, it follows that $H^{1,0}(\mathbb{C}^*)$ is isomorphic to the vector space of holomorphic functions with domain $\mathbb{C}^*$. The same argument shows that $H^{0,0}(\mathbb{C}^*)$ is also isomorphic to the same space.

(3)(c) Let $\alpha \in A^{p+1,q}(D)$ be a ∂-closed form. Then, by the hint, $\bar{\alpha}$ is a $\bar{\partial}$-closed form in $A^{q,p+1}(D)$. From the $\bar{\partial}$-Poincaré lemma there is $\beta \in A^{q,p}(D)$ with $\bar{\alpha} = \bar{\partial}\beta$. Conjugating, $\alpha = \partial\bar{\beta}$, so α is a ∂-exact form in $A^{p,q}(D)$. In other words, we showed: Theorem: every ∂-closed form in $A^{p+1,q}(D)$ is ∂-exact.

(3)(d) By the standard Poincaré lemma, there is a form $\alpha \in A^{p+q-1}(D)$ with $d\alpha = \omega$. Since ω is a form of type (p, q) and $d = \partial + \bar{\partial}$, then $\alpha = \alpha' + \alpha'' + \eta$, with α' of type $(p-1, q)$, α'' of type $(p, q-1)$ and η a d-closed form. Without loss of generality we can assume that $\eta = 0$, i.e. $\alpha = \alpha' + \alpha''$. Moreover, $0 = d\alpha = (\partial + \bar{\partial})(\alpha' + \alpha'') = (\partial\alpha' + \bar{\partial}\alpha'') + \partial\alpha'' + \bar{\partial}\alpha'$, where the three terms have different types. We have $0 = \partial\alpha' + \bar{\partial}\alpha'' = \partial\alpha'' = \bar{\partial}\alpha'$. By the ∂- and $\bar{\partial}$-Poincaré lemmas, there are forms $\gamma'' \in A^{p-1,q-1}$ and $\gamma' \in A^{p-1,q-1}$, such that $\partial\gamma'' = \alpha'$ and $\bar{\partial}\gamma' = \alpha''$. Finally, $\partial\bar{\partial}(\gamma' - \gamma'') = \partial\alpha'' + \bar{\partial}\partial\gamma'' = \partial\alpha'' + \bar{\partial}\alpha' + (\partial\alpha' + \bar{\partial}\alpha'') = d(\alpha' + \alpha'') = \omega$.

4. (a) Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space with a compatible almost complex structure J. Show the following:

   (i) $\dim V$ is even. (1 mark)

   (ii) Defining multiplication by i as $iv := J(v)$ makes V into a complex vector space. (1 mark)

   (iii) $\omega(\cdot,\cdot) := \langle J\cdot,\cdot\rangle$ is an alternating 2-form. (2 marks)

   (iv) $(\cdot,\cdot) := \langle\cdot,\cdot\rangle - i\omega$ is a hermitian form on (V, J). (2 marks)

(b) Let (M, g) be a compact oriented Riemannian manifold of dimension n. Prove that the volume form $\mathrm{vol}(g) \in A^n(M)$ is harmonic. (Hint: First show that $(\mathrm{vol}(g), d\psi) = \int_M d\psi$ for any $\psi \in A^{n-1}(M)$.) (4 marks)

(c) Let f, g be holomorphic functions in some neighborhood of $0 \in \mathbb{C}^n$. Suppose that f is irreducible in $\mathcal{O}_n$, and that g vanishes on $Z(f)$, in the sense that $g(z) = 0$ for every $z \in Z(f)$. Prove that f divides g in $\mathcal{O}_n$. (4 marks)

(d) Let M be a Kahler manifold and ω its Kahler form. Show that ω^ℓ defines a non-zero homology class in $H_D^{\ell,\ell}(M)$, for all $1 \leq \ell \leq \dim M/2$. In particular, $\dim H_D^{\ell,\ell}(M) \neq 0$. (2 marks)

(e) sim Let (V, g) be a vector space with a compatible almost complex structure J and fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$ and vol_W its volume form. Show that

$$\omega^m|_W \leq m!\,\mathrm{vol}_W,$$

with equality, if and only if, W is a complex subspace, i.e. $J(W) \subset W$. Conclude that a complex submanifold of a Kahler manifold minimises the volume in its homology class. (4 marks)

(Total: 20 marks)

Solutions.

(a)(i) That the dimension is even follows from $0 \leq \det(J)^2 = \det(J^2) = \det(-id) = (-1)^{\dim V}$.

(a)(ii) Since V is real vector space, to see that it is a complex vector space, we just need to check $i(v+w) = J(v+w) = J(v) + J(w) = iv + iw$ and $i^2 v = i(J(v)) = J^2(v) = -v$.

(a)(iii) $\omega(v, w) = \langle J(v), w\rangle = \langle J^2(v), J(w)\rangle = \langle -v, J(w)\rangle = -\omega(w, v)$.

(a)(iv) From the definition it follows that $(\cdot,\cdot)$ is real linear. Moreover, $J(v) \perp v$ and $(v, w) = \overline{(w, v)}$, since ω is alternating. This implies that $(v, v) = \langle v, v\rangle > 0$, for $v \neq 0$. It is enough to show that the form is $\mathbb{C}$-linear on the first entry.

$$\begin{aligned}(J(v), w) &= \langle J(v), w\rangle - i\omega(J(v), w)\\ &= \langle J^2(v), J(w)\rangle - i\langle J^2(v), w\rangle\\ &= -\langle v, J(w)\rangle + i\langle v, w\rangle\\ &= \langle J(v), w\rangle + i\langle v, w\rangle\\ &= i(\langle v, w\rangle - i\omega(v, w))\\ &= i(v, w).\end{aligned}$$

(b) The Hodge Decomposition $\Omega^n(M) = \operatorname{im} d_{n-1} \oplus \operatorname{im} \delta_{n+1} \oplus \mathcal{H}^n(M)$ is an orthogonal decomposition. Since $\Omega^{n+1}(M) = 0$, we have $\operatorname{im} \delta_{n+1} = 0$. So, $\operatorname{vol}(g)$ is harmonic if it is orthogonal to all exact forms. Notice $(\operatorname{vol}(g), d\psi) = \int_M d\psi \wedge * \operatorname{vol}(g) = \int_M d\psi = 0$.

(c) Without loss of generality we can assume that f is not identically zero, otherwise the result is trivial. After a linear change of coordinates, by the Weierstrass preparation theorem, we know there exists a monic polynomial $h \in \mathcal{O}_{n-1}(z_n)$ of degree ≥ 1, and a unit $u \in \mathcal{O}_n$, such that $f = uh$. In particular $h \in \mathcal{O}_{n-1}(z_n) \cap I(Z)$, where $Z = Z(f)$. Let $p \in \mathcal{O}_{n-1}(z_n) \cap I(Z)$ be a monic polynomial with the smallest possible degree among all non-zero polynomials in $\mathcal{O}_{n-1}(z_n) \cap I(Z)$. Given $g \in I(Z)$, by Weierstrass' division, $g = qp + r$, where $r \in \mathcal{O}_{n-1}(z_n)$ has degree strictly less than that of p. Since $g, p \in I(Z)$ it follows that $r \in I(Z)$, and by the minimality of the degree of p, we must have $r \equiv 0$. In other words, $I(Z) = (p)$, so p divides every $g \in I(Z(f))$. In particular, $f = up$, for some $u \in \mathcal{O}_n$. Since f is irreducible, u must be a unit, which implies $I(Z) = (f)$.

(d) The fundamental form ω is of type $(1,1)$ and is closed since M is Kahler. Therefore, for every ℓ, the forms ω^ℓ are of type (ℓ, ℓ) and closed, so they define an element of $H^{p,p}(M)$. However ω^n is a multiple of the volume form of M (e.g. next exercise), in particular $\int_M \omega^n \neq 0$ and ω^n is not exact, i.e. $0 \neq [\omega^n] = [\omega]^n$.

(e) The restrictions of g and ω to W are also symmetric (non-degenerate) and skew-symmetric bilinear forms, respectively. By the spectral theorem, there is an orthonormal basis $\{v_i, w_i\}_{i=0}^m$ of W, for which $\omega|_W = \sum_{i=1}^m \lambda_i dv_i \wedge dw_i$. Let $\pi : V \to W$ be the orthogonal projection. Since $\omega|_W(\cdot, \cdot) = g(J\cdot, \cdot)|_W = g(\pi \circ J\cdot, \cdot)$, it follows that the basis v_i, w_i is giving diagonal blocks for the linear map $\pi \circ J : W \to W$. In particular $|\lambda_i| \leq 1$ with equality if and only if $J(v_i) = w_i$ and $J(w_i) = -v_i$. Finally, $\omega_W^m = m!(\lambda_1 \cdots \lambda_m) \sum_{i=1}^m dv_i \wedge dw_i = m!(\lambda_1 \cdots \lambda_m) \operatorname{vol}_W$, which proves the claim.

For the second part, let S and Σ be $2m$-dimensional submanifolds of a complex manifold. Assume that they are homologous i.e. $S - \Sigma = \partial U$ as singular cycles. Then $\int_S \omega^m = \int_\Sigma \omega^m + \int_U d\omega^m$. If S is complex and M is Kahler (i.e. $d\omega = 0$), then by the previous paragraph we have $m! \operatorname{vol}(S) = \int_S \omega^m = \int_\Sigma \omega^m + \int_U d\omega^m \leq m! \operatorname{vol}(\Sigma)$.

5. (a) Let $I \subset \mathcal{O}_n$ be a nonzero ideal. Show that if $\partial f/\partial z_j \in I$ for all $f \in I$ and $j = 1, \ldots, n$, then $I = \mathcal{O}_n$. (Hint: consider the smallest $d \geq 0$ for which there exists $f \in I$ regular of degree d). (5 marks)

(b) Let $Z \subset D$ be an analytic subset of an open set $D \subset \mathbb{C}^n$. Assume that $0 \in Z$, but Z does not contain any open neighborhood of 0.

(i) Let $k \geq 0$ be the largest integer such that there exist $f_1, \ldots, f_k \in I(Z)$ with the property that at least one $k \times k$-minor g of the matrix $J(f_1, \ldots, f_k)$ does not belong to $I(Z)$. Prove that $k \geq 1$. (3 marks)

(ii) After shrinking D we may assume that $f_1, \ldots, f_k \in \mathcal{O}(D)$. Define $D' = D \setminus Z(g)$ and $Z' = Z(f_1) \cap \cdots \cap Z(f_k) \cap D'$.

(1) Show that Z' is a submanifold of D'. (3 marks)

(2) Show that $Z \cap D'$ is a union of connected components of Z'. (6 marks)

(3) Conclude that the singular locus Z_{sing} is contained in an analytic set strictly smaller than Z. (3 marks)

(Total: 20 marks)

Solutions.

(a) Let $d \geq 0$ be the smallest non-negative integer such that there exists $f \in I$ regular of degree d. By the Weirstrass preparation theorem we can assume that f is a monic element of $\mathcal{O}_{n-1}(z_n)$. If $d > 1$ then $\partial f/\partial z_n \in I \cap \mathcal{O}_{n-1}(z_n)$ is a polynomial of degree $d - 1 \geq 0$ and therefore a regular function of degree exactly $d - 1$. Since this contradicts the minimality of d, we conclude that $d = 0$, i.e $f = 1$ and $I = \mathcal{O}_n$.

(b)(i) Since $0 \in Z$, we must have $I(Z) \neq \mathcal{O}_n$. Therefore, by the previous exercise, there must be $f \in I$ and $j = 1, \ldots, n$ such that $\partial f/\partial z_j \notin I(Z)$.

(b)(ii)(1) That Z' is a submanifold of D' of dimension $n - k$ follows directly from the Implicit Function Theorem applied to the function $(f_1, \ldots, f_k) : \mathbb{C}^n \to \mathbb{C}^k$.

(b)(ii)(2) Since $f_1, \ldots, f_k \in I(Z)$, we immediately have $Z \cap D' \subset Z'$. Note that given $f \in I(Z)$, by the maximality of k, the gradient Jf is a linear combination of $Jf_1, \ldots, Jf_k$ along Z, which is a basis of the normal bundle of Z'. In other words, the gradient of f as a function on Z' is zero along $Z \cap D'$. After composing with the biholomorphism given by the Implicit Function Theorem, we can assume, without loss of generality, that $Z' = \{z_1 = \cdots = z_k = 0\}$. Then, for each $f \in I(Z)$, we have $\partial f/\partial z_{k+1}, \ldots, \partial f/\partial z_n \in I(Z)$. Repeating this process we see that all of the derivatives of $f|_{Z'}$ are zero along Z. By the Identity Theorem, f must vanish on the connected components of Z' containing $Z \cap D'$. Since this is true for all $f \in I(Z)$, we conclude that $Z \cap D'$ is a union of connected components of Z'.

(b)(ii)(3) By the previous exercises $Z \cap D' = Z \cap (D \setminus Z(g))$ is non-empty and $Z_{sing} \subset Z(g)$. In other words, $Z(g)$ is a possibly empty analytic set different from Z which contains Z_{sing}.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Complex Manifolds_MATH97054 MATH70060	1	In part (b) most people did not justify why the (real Jacobian) corresponds to the determinant of a complex matrix written in the underlying real vector space.
Complex Manifolds_MATH97054 MATH70060	2	In part (a)(ii) many people failed to explain why was it important to show first that CP^n is simply connected.
Complex Manifolds_MATH97054 MATH70060	3	As we discussed, in part (b)(ii) it is not possible to use Hodge's theorem since the domain is not compact. Instead the answer follows directly from the definition of closed forms and using the hypotesis at the begininig of the question.
Complex Manifolds_MATH97054 MATH70060	4	These problems were all familiar from problem sheets and class.
Complex Manifolds_MATH97054 MATH70060	5	As expected, part(b)(ii)(2) was not fully completeded by most people since it was the most delicate argument of the exam.