Mathematical Logic (MATH70132)
Mastery Material Problem Sheet; notes on solutions.

[1] Suppose $n \in \mathbb{N}$. The first-order language with equality $\mathcal{L}_n^=$ has constant symbols $c_1, \ldots, c_n$ and no other relation, function or constant symbols (apart from equality). Write down a set T_n of closed $\mathcal{L}_n^=$-formulas whose normal models are precisely infinite sets in which the constant symbols $c_1, \ldots, c_n$ are interpreted as distinct elements. Use Vaught's Test (Theorem 8.18 in Cori - Lascar) to prove that T_n is complete.

Solution: Let T_n consist of the usual formulas σ_k (saying 'at least k elements') and the formula τ_n:

$$\bigwedge_{1 \leq i < j \leq n} (c_i \neq c_j).$$

Clearly T_n has no finite models and if $\mathcal{M}_1, \mathcal{M}_2$ are countable normal models of T_n then $\mathcal{M}_1, \mathcal{M}_2$ are isomorphic: we take a bijection between M_1 and M_2 which maps the interpretation of c_i in $\mathcal{M}_1$ to the interpretation of c_i in $\mathcal{M}_2$ (note that we use τ_n here). Vaught's test then gives that T_n is complete.

[2] The language with equality $\mathcal{L}_c^=$ has equality, a single binary relation symbol R and a constant symbols c_1, c_2. How many non-isomorphic countable normal models are there in which R is interpreted as a dense linear ordering without endpoints? How many countable normal models are there which are not elementarily equivalent? What happens if the language has n constant symbols (rather than two), for $n \in \mathbb{N}$?

Solution: The proof of 8.19 in Cori - Lascar shows that if $\mathcal{M}_1, \mathcal{M}_2$ are countable dense linear orders without endpoints and $a_1 < a_2, \ldots < a_n$ in $\mathcal{M}_1$ and $b_1 < \ldots < b_n$ in $\mathcal{M}_2$, then there is an isomorphism $\mathcal{M}_1 \to \mathcal{M}_2$ sending $a_i \mapsto b_i$ for $i \leq n$.

Thus the isomorphism type of a countable normal model of the theory in the question is determined by which of $c_1 = c_2$, $c_1 < c_2$ or $c_2 < c_1$ holds in the model. All of these are possible, so there are 3 isomorphism types of countable model here. Moreover, in this case, non-isomorphic countable models are not elementarily equivalent so there are 3 elementary equivalence classes of countable models.

A similar argument can be given for the case where there are n constant symbols. If you wish you can try to derive an expression for the number as a function of n.

[3] Suppose $\mathcal{L}^=$ is a language with equality and $\mathcal{M}$ is a normal $\mathcal{L}^=$-structure with domain M. An *automorphism* of $\mathcal{M}$ is an isomorphism $\alpha : \mathcal{M} \to \mathcal{M}$. Note that in this case, if $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}^=$-formula and $a_1, \ldots, a_n \in M$, then

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \Leftrightarrow \mathcal{M} \models \phi[\alpha(a_1), \ldots, \alpha(a_n)].$$

(i) With the above notation, suppose $\mathcal{N}$ is a substructure of $\mathcal{M}$ (with domain N) having the following property. For all $a_1, \ldots, a_n \in N$ and $b \in M$ there is an automorphism α of M with $\alpha(a_i) = a_i$, for $i \leq n$ and $\alpha(b) \in N$. Using the Tarski-Vaught Test, prove that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$.

(ii) With T_n as in [1], show that if $\mathcal{M}$ is a normal model of T_n, then every infinite substructure of $\mathcal{M}$ is an elementary substructure.

(iii) Suppose $\mathcal{M}$ is a vector space (in the usual language of vector spaces over a field F) and $\mathcal{N}$ is a subspace of infinite dimension. Prove that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$.

Solution: (i) We verify the condition in the Tarski-Vaught test. Suppose $\phi(y, x_1, \ldots, x_n)$ is an $\mathcal{L}^=$-formula and $a_1, \ldots, a_n \in N$ are such that $\mathcal{M} \models (\exists y)\phi[y, a_1, \ldots, a_n]$. Let $b \in M$ be such that $\mathcal{M} \models \phi[b, a_1, \ldots, a_n]$. There is an automorphism α of $\mathcal{M}$ with $\alpha(a_i) = a_i$ for all $i \leq n$ and $\alpha(b) \in N$. By the given fact, $\mathcal{M} \models \phi[\alpha(b), a_1, \ldots, a_n]$, so the condition for Tarski-Vaught holds, and therefore $\mathcal{N} \preceq \mathcal{M}$.

(ii) An easy application of the condition in (i).

(iii) Again we verify the condition in (i). Suppose $a_1, \ldots, a_n \in N$ and $b \in M$. Let A be the subspace spanned by $a_1, \ldots, a_n$. So $A \subseteq N$ and we may assume that $b \notin N$. Without loss of generality we can assume that $a_1, \ldots, a_m$ are a basis for A (for some $m \leq n$). Then $a_1, \ldots, a_m, b$ are linearly independent. As N is infinite dimensional there is $c \in N \setminus A$ and so $a_1, \ldots, a_m, c$ are linearly independent. These l.i sets extend to bases of $\mathcal{M}$. These bases will be of the same cardinality and so we have a bijection between them sending a_i (for $i \leq m$) to itself and b to c. This bijection extends uniquely to an isomorphism (bijective linear map) $\alpha : \mathcal{M} \to \mathcal{M}$. Note that α fixes every element of A (and so all of the a_i for $i \leq n$) and $\alpha(b) \in N$, as required.

[4] Let $\mathcal{L}$ be the usual language of groups and let $(\mathcal{M}_i : i \in I)$ be a family of groups. Suppose $\mathcal{F}$ is a non-principal ultrafilter on I and consider the ultraproduct $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{F}$.

(i) Prove that $\{(a_i : i \in I) : \{i \in I : a_i = 1\} \in \mathcal{F}\}$ is a normal subgroup of $\prod_{i \in I} \mathcal{M}_i$ and that the quotient group by this is isomorphic to $\mathcal{M}$. (Here we are denoting by 1 the identity element of a group).

(ii) Let I be the set of prime numbers and $\mathcal{M}_i$ the cyclic group of order i. Prove that $\mathcal{M}$ is elementarily equivalent to $\langle \mathbb{Q}; + \rangle$ (the additive group of rational numbers).

Solutions: (i) Let $\theta : \prod_{i \in I} \mathcal{M}_i \to \prod_{i \in I} \mathcal{M}_i / \mathcal{F}$ be the map which sends each $(a_i : i \in I)$ to its equivalence class (modulo $\mathcal{F}$) in the ultraproduct. This is a group homomorphism (by definition of the group operation on the ultraproduct); it is clearly surjective, and the kernel of θ is the subset given in the question. So the result follows by the 1st isomorphism theorem for groups.

(ii) Use the Los Theorem to show that $\mathcal{M}$ is a torsion-free, divisible abelian group. For example, if $i > n > 1$ then $\mathcal{M}_i \models (\forall x)((x^n = 1) \to (x = 1))$. This shows that $\mathcal{M}$ has no element of order n.

The result then follows from 8.20 in Cori-Lascar which shows that any two divisible, torsion-free abelian groups are elementarily equivalent.

[5] Let $\mathcal{R}$ denote the structure $\langle \mathbb{R}; \leq, +, -, \cdot, 0, 1 \rangle$ in the language of rings with an ordering. Let $\mathcal{F}$ be a non-principal ultrafilter on ω and consider the ultrapower $\mathcal{R}^* = \mathcal{R}^\omega / \mathcal{F}$. Say why we can regard $\mathcal{R}^*$ as an elementary extension of $\mathcal{R}$. Decide which of the following are true, giving reasons for your answers.

(i) $\mathcal{R}^*$ is a field.

(ii) Every polynomial of odd degree with coefficients in $\mathcal{R}^*$ has a root in $\mathcal{R}^*$.

(iii) For every $r \in \mathcal{R}^*$ there is $n \in \mathbb{N}$ with $r < n$.

(iv) Every non-empty subset of $\mathcal{R}^*$ which is bounded above in $\mathcal{R}^*$ has a least upper bound in $\mathcal{R}^*$.

Solution: (i) This follows from Los' Theorem (8.31 in Cori - Lascar) and the fact that the field axioms are expressible in the first-order language.

(ii) This is as in (i): for each odd n we can write down a formula saying that every polynomial of degree n has a root. This is true in $\mathcal{R}$ and so is true in $\mathcal{R}^*$.

(iii) This is false. Let r be the class of $(m : m \in \omega)$ in the ultraproduct $\mathcal{R}^*$. By definition $r > n$ for all $n \in \omega$ (where we regard $n \in \mathcal{R}^*$ as the class of the constant sequence n).

(iv) This is false. By (iii) $\mathbb{N}$ is bounded above in $\mathcal{R}^*$, so suppose s is the supremum of $\mathbb{N}$ in $\mathcal{R}^*$. Then $s - 1$ is less than some $n \in \mathbb{N}$, so $s < n + 1$. This is a contradiction.