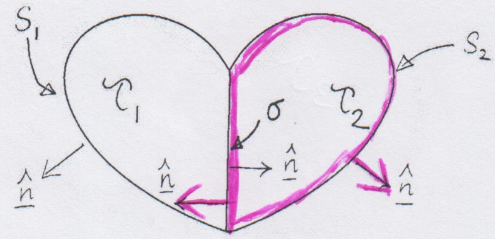


Figure 17: The divergence theorem applied to a non-convex surface.

1.8.7 The divergence theorem in more-complicated geometries

(i) Non-convex surfaces

If the surface is non-convex, the divergence theorem still holds. This can be established for a given surface by slicing the enclosed volume into sub-volumes, the boundaries of which can be described by single-valued functions of (x, y), (y, z) and (x, z).

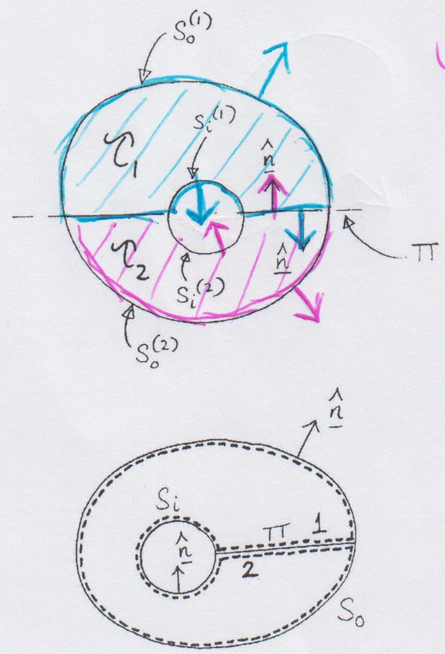
As an example consider the volume with 'heart-shaped' cross-section in figure 17. In this case the non-convex surface S can be divided by a surface σ into two parts S_1 and S_2 which, together with σ, form convex surfaces $S_1 + \sigma, S_2 + \sigma$ (figure 17). We can then apply the divergence theorem to $S_1 + \sigma, S_2 + \sigma$ with τ_1, τ_2 being the respective enclosed volumes, where $\tau_1 + \tau_2 = \tau$. On adding the results, the surface integrals over σ cancel out, and since $S = S_1 + S_2$ we have

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}} \, dS = \int_\tau \mathrm{div}\mathbf{A} \, d\tau$$

as before.

More complicated geometries require further slicing. For example, for the case of a torus see `https://www.math.uci.edu/~ndonalds/math2e/16-9divergence.pdf` .

$$\int_{S_o^{(1)}} \underline{A}\cdot\hat{\underline{n}}\,dS + \int_{S_i^{(1)}} \underline{A}\cdot\hat{\underline{n}}\,dS + \int_{\Pi} \underline{A}\cdot\hat{\underline{n}}\,dS = \int_{\tau_1} \mathrm{div}\,\underline{A}\,d\tau$$



$$\int_{S_o^{(2)}} + \int_{S_i^{(2)}} - \int_{\Pi} \underline{A}\cdot\hat{\underline{n}}\,dS = \int_{\tau_2} \mathrm{div}\,\underline{A}\,d\tau$$

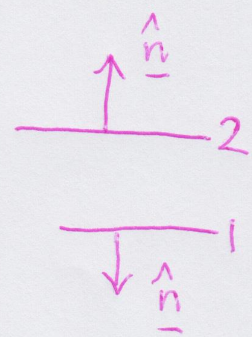


Figure 18: Diagrams for the proof of the divergence theorem in (top): a simply-connected domain; (bottom): a multiply-connected region.

(ii) A region with internal boundaries

(a) Simply-connected regions (top diagram in figure 18)

For example this could be the space between concentric spheres. Suppose we have an interior surface S_i and outer surface S_o. Draw a plane Π that cuts both S_o and S_i. This divides S_o into two open surfaces $S_o^{(1)}, S_o^{(2)}$. S_i is similarly divided into $S_i^{(1)}, S_i^{(2)}$. We then apply the divergence theorem to the volume τ_1 which is bounded by the closed surface $S_o^{(1)} + S_i^{(1)} + \Pi$, and we then apply the divergence theorem to the volume τ_2 which is bounded by $S_o^{(2)} + S_i^{(2)} + \Pi$. We add these results together. The contributions over Π cancel, leaving the result:

$$\int_{S_o+S_i} \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = \int_{\tau_1} \mathrm{div}\,\underline{A}\,d\tau + \int_{\tau_2} \mathrm{div}\,\underline{A}\,d\tau = \int_{\tau} \mathrm{div}\,\underline{A}\,d\tau$$

with the normal to S_i drawn inwards, i.e. out of τ.

(b) Multiply-connected regions (bottom diagram in figure 18)

For example this could be the region between two cylinders. Again let S_o and S_i be the outer and inner surfaces, linked by the plane Π. Label the two sides of the plane 1 and 2. Consider the surface

S_i + side 1 of Π + S_o + side 2 of Π

This is closed and encloses a simply-connected region τ. We then apply the divergence theorem to τ. The contributions along the two sides of Π cancel, giving

$$\int_{S_o+S_i} \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = \int_{\tau} \mathrm{div}\mathbf{A}\,d\tau.$$

1.8.8 Green's identities in 3D

Let ϕ and ψ be two scalar fields with continuous second derivatives. Consider the quantity

$$\mathbf{A} = \phi\nabla\psi.$$

$$\underline{\nabla}\cdot(\varphi\underline{F}) = \varphi\,\underline{\nabla}\cdot\underline{F} + \underline{\nabla}\varphi\cdot\underline{F}$$

It follows that

$$\begin{aligned} \operatorname{div}\mathbf{A} &= \varphi\nabla^2\psi + \underline{\nabla}\varphi\cdot\underline{\nabla}\psi \\ \widehat{\mathbf{n}}\cdot\mathbf{A} &= \varphi(\underline{\nabla}\psi\cdot\underline{\hat{n}}) = \varphi\,\partial\psi/\partial n \end{aligned}$$

"normal derivative"

Applying the divergence theorem we obtain

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n}\right\} dS = \int_\tau \varphi\nabla^2\psi + \underline{\nabla}\varphi\cdot\underline{\nabla}\psi \, d\tau \tag{1}$$

which is known as **Green's first identity**. Interchanging ϕ and ψ we have

$$\int_S \left\{\psi\frac{\partial\phi}{\partial n}\right\} dS = \int_\tau \psi\nabla^2\varphi + \underline{\nabla}\psi\cdot\underline{\nabla}\varphi \, d\tau \tag{2}$$

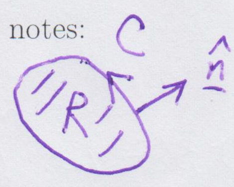
Subtracting (2) from (1) we obtain

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right\} dS = \int_\tau \varphi\nabla^2\psi - \psi\nabla^2\varphi \, d\tau \, .$$

which is known as **Green's second identity**. These identities are very useful when constructing solutions to partial differential equations (see for example 'PDEs in action' in term 2).

1.8.9 Green's identities in 2D

If we use the divergence theorem in 2D derived in the first section of the notes:



$$\int_R \operatorname{div}\mathbf{F}\, dx\, dy = \oint_C \mathbf{F}\cdot\widehat{\mathbf{n}}\, ds.$$

then we can calculate down the corresponding Green identities. These are

$$\oint_C \phi\frac{\partial\psi}{\partial n}\, ds = \int_R \left[\phi\nabla^2\psi + (\nabla\psi)\cdot(\nabla\phi)\right]\, dx\, dy$$

and

$$\oint_C \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]\, ds = \int_R \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]\, dx\, dy.$$

These formulae are the generalisation of integration by parts to two dimensions.

$$\int_R \varphi\nabla^2\psi\, dx\, dy = \oint_C \varphi\frac{\partial\psi}{\partial n}\, ds - \int_R (\underline{\nabla}\psi)\cdot(\underline{\nabla}\varphi)\, dx\, dy$$

cf $\displaystyle\int_a^b f\frac{dg}{dx}\, dx = [fg]_a^b - \int_a^b g\frac{df}{dx}\, dx$

1.8.10 Gauss' flux theorem

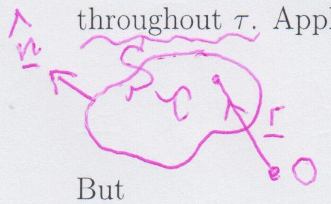
Let S be a closed surface with outward unit normal $\widehat{\mathbf{n}}$, and let O be the origin of the coordinate system. Then:

$\underline{A} = \underline{r}/r^3$

$$\oint_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \begin{cases} 0, & \text{if } O \text{ is exterior to } S \\ 4\pi, & \text{if } O \text{ is interior to } S. \end{cases}$$

Proof

First suppose O is exterior to S and that S encloses a volume τ. Then we have $r \neq 0$ throughout τ. Applying the divergence theorem:



$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \int_\tau \mathrm{div}\left(\frac{\underline{r}}{r^3}\right) d\tau$$

But

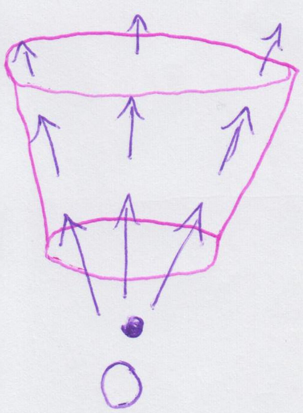
$$\mathrm{div}\left(\frac{\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\,\mathrm{div}\,\underline{r} + \underline{r}\cdot\underline{\nabla}\left(\frac{1}{r^3}\right) = \frac{3}{r^3} + \underline{r}\cdot\left(\frac{-3\underline{r}}{r^5}\right) = \underline{\underline{0}}$$

Hence we have that

$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \int_\tau \mathrm{div}\left(\frac{\mathbf{r}}{r^3}\right) d\tau = 0,$$

as required.

$$\begin{aligned}\underline{\nabla}\left(\frac{1}{r^3}\right) &= \hat{i}\,\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} \\ &\quad + \hat{j}\,\frac{\partial}{\partial y}(\quad)^{-3/2} \\ &\quad + \hat{k}\,\frac{\partial}{\partial z}(\quad)^{-3/2} \\ &= -3x\hat{i}(\quad)^{-5/2} \\ &\quad -3y\hat{j}(\quad)^{-5/2} \\ &\quad -3z\hat{k}(\quad)^{-5/2} \\ &= -3\underline{r}/r^5\end{aligned}$$



What goes in must come out

— overall flux through surface is zero.

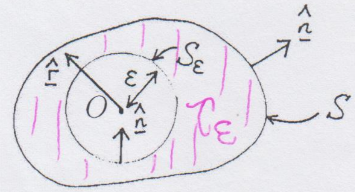


Figure 19: Diagram for the proof of Gauss theorem with O interior to S.

Now suppose O is interior to S (figure 19). We surround O with a small sphere radius ε, with surface S_ε, lying entirely within S. We consider the volume τ_ε enclosed between S and S_ε Then, applying the divergence theorem and proceeding as above we have

$$\int_{S+S_\varepsilon} \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \int_{\tau_\varepsilon} \operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) d\tau = 0$$

$r \neq 0$ in τ_ε

by previous calc.

Breaking up the surface integral into two parts:

$$0 = \int_{S+S_\varepsilon} \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = \int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS + \int_{S_\varepsilon} \frac{\mathbf{r}}{r^3}\cdot(-\hat{\mathbf{r}})\,dS$$

$\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ on S_ε

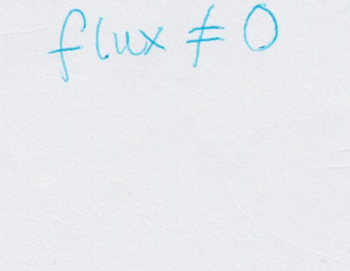
However (since $r = \varepsilon$ on S_ε):

$$\int_{S_\varepsilon} \frac{\widehat{\mathbf{r}}\cdot\mathbf{r}}{r^3}\,dS = \int_{S_\varepsilon} \frac{1}{r^2}\,dS = \frac{1}{\cancel{\varepsilon^2}}\underbrace{\int_{S_\varepsilon} dS}_{4\pi\cancel{\varepsilon^2}}$$

S.A. of a sphere of radius ε

Thus it follows that

$$\int_S \frac{\widehat{\mathbf{n}}\cdot\mathbf{r}}{r^3}\,dS = 4\pi$$



flux $\neq 0$

Applications

LHS electric/gravitational flux

RHS enclosed charge/mass.

Concept of solid angle in astronomy.

Recall G-T
$\oint_\gamma \underline{A}\cdot d\underline{r} = \int_R (\text{curl}\,\underline{A})\cdot\hat{\underline{k}}\, dS$

// R //

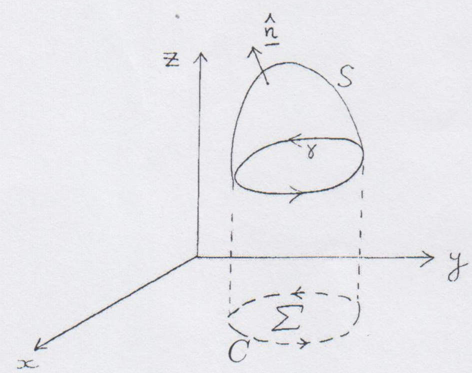


Figure 20: Diagram for the proof of Stokes' theorem.

1.8.11 Stokes theorem

Suppose S is an **open** surface with a simple closed curve γ forming its boundary, and let $\mathbf{A}$ be a vector field with continuous partial derivatives. Then:

$$\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{A}\cdot\hat{\mathbf{n}}\, dS,$$

where the direction of the unit normal to S and the sense of γ are related by a right-hand rule (i.e. $\hat{\mathbf{n}}$ is in the direction a right-handed screw moves when turned in the direction of γ).

Proof

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Consider

$$\operatorname{curl}(A_1\mathbf{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}$$

Then we have

$$\int_S [\operatorname{curl}(A_1\mathbf{i})]\cdot\hat{\mathbf{n}}\, dS = \int_S \frac{\partial A_1}{\partial z}(\hat{j}\cdot\hat{n}) - \frac{\partial A_1}{\partial y}(\hat{k}\cdot\hat{n})\, dS$$

If we now project onto the $x-y$ plane, S becomes Σ say, and γ becomes C (figure 20). Let the equation of S be $z = f(x,y)$. Then we have

$$\hat{\mathbf{n}} = \frac{\nabla(z - f(x,y))}{|\nabla(z-f(x,y))|} = \pm\left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right)\Big/\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right)^{1/2}$$

Choose $\hat{n}\cdot\hat{k} > 0$

Therefore, on S :

$$\mathbf{j}\cdot\hat{\mathbf{n}} = -\frac{\partial f}{\partial y}(\hat{k}\cdot\hat{n})$$
$$= -\frac{\partial z}{\partial y}(\hat{k}\cdot\hat{n})$$

Thus:

Chain rule for partial diffⁿ

G-T: $\oint_C B_1\,dx + B_2\,dy = \int_\Sigma \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\,dx\,dy$

$$\int_S [\mathrm{curl}\,(A_1\mathbf{i})]\cdot\hat{\mathbf{n}}\,dS = \int_S\left(-\left.\frac{\partial A_1}{\partial y}\right|_{x,z} - \left.\frac{\partial A_1}{\partial z}\right|_{x,y}\frac{\partial z}{\partial y}\right)(\hat{k}\cdot\hat{n})\,dS \quad \leftarrow \text{fixed}$$

$$= -\int_S \left.\frac{\partial}{\partial y}\right|_x A_1(x,y,f(x,y))\,(\hat{k}\cdot\hat{n})\,dS$$

$$= -\int_\Sigma \frac{\partial A_1}{\partial y}(x,y,f)\,dx\,dy \quad (\text{since } \hat{k}\cdot\hat{n}>0)$$

$$= \oint_C A_1(x,y,f)\,dx$$

with the last line following by using Green's theorem. However on γ we have $z = f$ and so

$$\oint_C A_1(x,y,f)\,dx = \oint_\gamma A_1(x,y,z)\,dx.$$

We have therefore established that

$$\int_S (\mathrm{curl}\,A_1\mathbf{i})\cdot\hat{\mathbf{n}}\,dS = \oint_\gamma A_1\,dx$$

In a similar way we can show that

$$\int_S (\mathrm{curl}\,A_2\mathbf{j})\cdot\hat{\mathbf{n}}\,dS = \oint_\gamma A_2\,dy$$

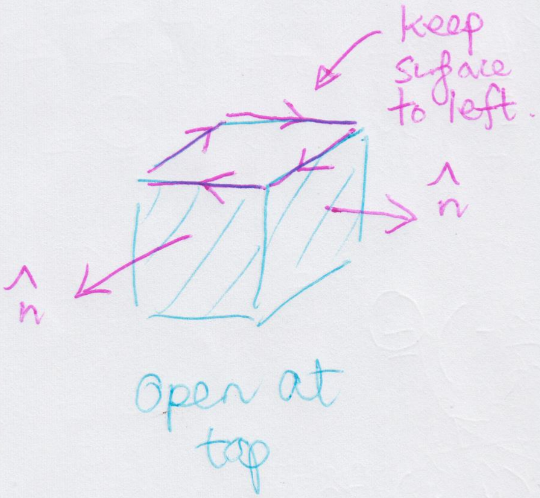
and

$$\int_S (\mathrm{curl}\,A_3\mathbf{k})\cdot\hat{\mathbf{n}}\,dS = \oint_\gamma A_3\,dz$$

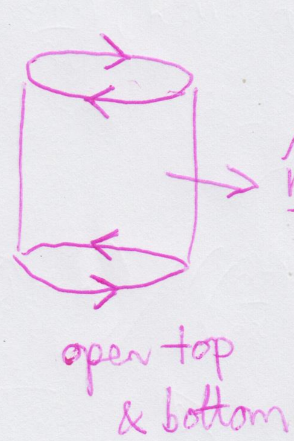
Check!

and so the theorem is proved by adding all three results together.

Note that although S must be open, it is not necessarily smooth. For example it could be in the shape of a box without a lid.



Open at top



open top & bottom

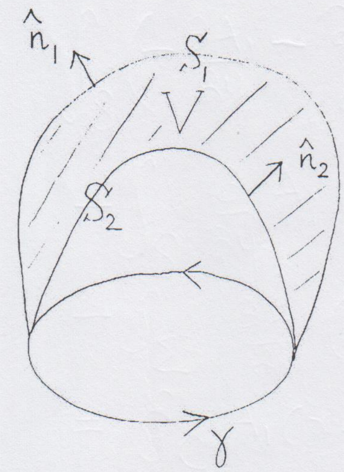


Figure 21: Two different open surfaces, both with the closed curve γ as boundary.

The theorem is actually true for **any** open surface with γ as boundary. To see this consider figure 21. The normal to S_1 is $\widehat{\mathbf{n}}_\mathbf{1}$ and to S_2 is $\widehat{\mathbf{n}}_\mathbf{2}$. The surface $S_1 + S_2$ is closed: let it enclose a volume V. Applying the divergence theorem to curl $\mathbf{A}$ over this region gives

$$\oint_{S_1+S_2} \operatorname{curl} \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_V \operatorname{div}(\operatorname{Curl} \underline{A})\, dV = 0$$

In the divergence theorem the normal must always point out of V and hence

$$0 = \int_{S_1+S_2} \operatorname{curl} \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_{S_1} (\operatorname{Curl} \underline{A}) \cdot \hat{\underline{n}}_1\, dS + \int_{S_2} (\operatorname{Curl} \underline{A}) \cdot (-\hat{\underline{n}}_2)\, dS$$

implying that

$$\int_{S_1} (\operatorname{Curl} \underline{A}) \cdot \hat{\underline{n}}_1\, dS = \int_{S_2} (\operatorname{Curl} \underline{A}) \cdot \hat{\underline{n}}_2\, dS$$
$$= \oint_\gamma \underline{A} \cdot d\underline{r}$$

Theorem

A necessary and sufficient condition that $\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = 0$ for any simple closed curve γ is that curl $\mathbf{A} = 0$ throughout the region in which γ is drawn (assuming $\mathbf{A}$ is continuously differentiable and the region is simply-connected).

$\underline{A}$ is conservative

Proof

We already know that if $\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = 0$ then there exists a potential ϕ such that $\mathbf{A} = \nabla\phi$. Therefore we see that curl $\mathbf{A} = 0$ since the curl of a gradient is always zero.

Conversely, if curl $\mathbf{A} = 0$ then by Stokes' theorem we have $\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = 0$ for any simple closed curve γ.

$\underline{A} = \underline{\nabla}\varphi \iff \underline{A}$ conservative $\iff \operatorname{Curl} \underline{A} = 0$

$\therefore$ we can use $\operatorname{Curl} \underline{A} = 0$ as a test for conservative field.

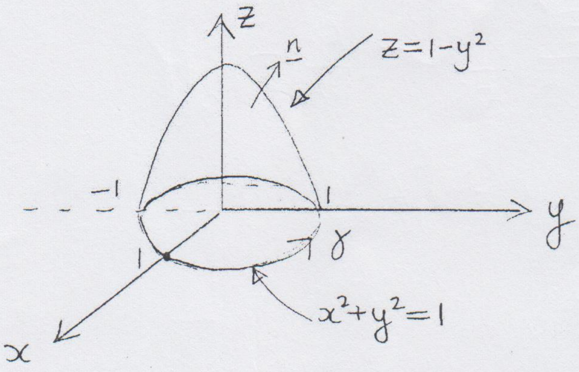


Figure 22: The parabolic surface $z = 1 - x^2 - y^2$ with $z \geq 0$.

Example

$= y\hat{i} + z\hat{j} + x\hat{k}$

Verify Stokes theorem for the vector field $\mathbf{A} = (y, z, x)$ and the surface S given by $z = 1 - x^2 - y^2$ with $z \geq 0$.

Let's do path integral

$\underline{A} \cdot d\underline{r} = y\,dx + \cancel{z\,dy} + \cancel{x\,dz}$. $\therefore \oint_\gamma \underline{A} \cdot d\underline{r} = \int y\,dx$

on γ : $z = dz = 0$

on γ $\left.\begin{matrix} x = \cos\theta \\ y = \sin\theta \end{matrix}\right\} 0 \leq \theta \leq 2\pi$

γ y x θ

$-\int_0^{2\pi} \sin^2\theta \, d\theta = \underline{\underline{-\pi}}$

Now for the surface integral:

Curl $\underline{A} = \ldots = -(\hat{i} + \hat{j} + \hat{k})$ (Check).

$\hat{\underline{n}} = \pm \underline{\nabla}(z - (1 - x^2 - y^2)) / |\qquad|$

$= \pm (2x\hat{i} + 2y\hat{j} + \hat{k}) / \sqrt{(4x^2 + 4y^2 + 1)}$

$\int_S (\text{Curl}\,\underline{A}) \cdot \hat{\underline{n}}\, dS = -\int_S \frac{2x + 2y + 1}{\sqrt{(4x^2 + 4y^2 + 1)}}\, dS$

$\hat{\underline{n}} \cdot \hat{k} > 0$ by right hand rule

project onto $z = 0$ $= -\int_{\substack{x^2+y^2 \leq 1 \\ z=0}} \cancel{\frac{2x+2y+1}{\sqrt{(\;)}}} \frac{dx\,dy}{\cancel{|\hat{\underline{n}} \cdot \hat{k}|}} \quad |\hat{\underline{n}} \cdot \hat{k}| = \frac{1}{\sqrt{(\;)}}$

$= -\int_{\text{unit disc}} (2x + 2y + 1)\, dx\, dy$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \begin{array}{l} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 1 \end{array} \qquad dx\,dy = r\,dr\,d\theta$$

$$- \int_0^{2\pi}\int_0^1 (\underbrace{2r\cos\theta}_{} + \underbrace{2r\sin\theta}_{} + 1)\, r\,dr\,d\theta$$

integrate to zero

$$= \underline{\underline{-\pi}} = \text{LHS}.$$

Stokes thm verified.

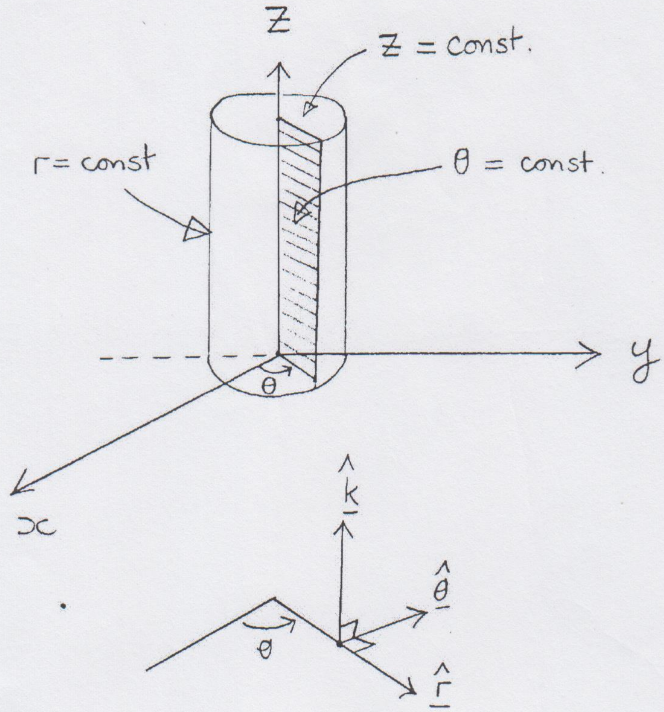


Figure 23: The surfaces $r =$ constant, $\theta =$ constant, $z =$ constant, for the cylindrical polar coordinate system, and the orientation of the unit vectors.

1.9 Curvilinear coordinates

1.9.1 Introduction & definition

Often it is more convenient, depending on the geometry of the problem under consideration, to use coordinates other than Cartesians. An example is cylindrical polar coordinates (r, θ, z) which are related to Cartesian coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \qquad \begin{pmatrix} 0 \leqslant \theta \leqslant 2\pi, \\ r \geqslant 0 \end{pmatrix}$$

$$-\infty < z < \infty$$

from which we can deduce that

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

The equation $r =$ constant therefore defines a family of circular cylinders with axes along the z-axis, while the equation $\theta =$ constant defines a family of planes, as does the equation $z =$ constant (figure 23). Cylindrical polar coordinates are an example of **curvilinear coordinates**. The unit vectors $\widehat{\mathbf{r}}, \widehat{\theta}, \widehat{\mathbf{k}}$ at any point P are perpendicular to the surfaces $r =$ constant, $\theta =$ constant, $z =$ constant through P in the directions of increasing r, θ, z. Note that the direction of the unit vectors $\widehat{\mathbf{r}}, \widehat{\theta}$ vary from point to point, unlike the corresponding Cartesian unit vectors.

More generally now, let us suppose that our Cartesian coordinates $(x, y, z) \equiv (x_1, x_2, x_3)$ can be expressed as single-valued differentiable functions of the new coordinates (u_1, u_2, u_3), i.e.

$$x_i = x_i(u_1, u_2, u_3) \quad \text{for } i = 1, 2, 3.$$

We would like to know what the conditions are under which we can invert these expressions and write the u_i as single-valued differentiable functions of the x_i. First let's differentiate the above expression with respect to x_j :

$$\partial x_i / \partial x_j = \delta_{ij} = \frac{\partial x_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial x_i}{\partial u_2}\frac{\partial u_2}{\partial x_j} + \frac{\partial x_i}{\partial u_3}\frac{\partial u_3}{\partial x_j} \quad \begin{array}{l} i=1,2,3 \\ j=1,2,3 \end{array}$$

Writing this out for each i and j we have the matrix equation

$$\begin{pmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \partial x_1/\partial u_3 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \partial x_2/\partial u_3 \\ \partial x_3/\partial u_1 & \partial x_3/\partial u_2 & \partial x_3/\partial u_3 \end{pmatrix} \begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{pmatrix} = I,$$

where I is the identity matrix. We can express this more succinctly as

$$J(x_u)\,J(u_x) = I$$

where $J(x_u)$ is the **Jacobian matrix** for the (x_1, x_2, x_3) system and $J(u_x)$ is the corresponding Jacobian for (u_1, u_2, u_3). We therefore see that $J(u_x)$ exists (i.e. the u_i are differentiable functions of the x_i provided $(J(x_u))^{-1}$ exists, i.e. we require

$$\det(J(x_u)) \neq 0$$

It turns out that this condition is sufficient to guarantee that our transformation can be inverted. More precisely, the **inverse function theorem** states that around any point where $\det(J(x_u))$ is nonzero, there exists a neighbourhood in which the u_i can be expressed as single-valued differentiable functions of the x_i. There is more on this theorem in the Differential Equations course next term.

Note also that the result $J(x_u)J(u_x) = I$ implies that

⚠ $\frac{\partial x_1}{\partial u_1} \neq 1 \Big/ \frac{\partial u_1}{\partial x_1}$

$$\det(J(x_u)) = 1 / \det(J(u_x))$$

a useful result that we will exploit later when we consider the transformation of integrals. From now on we will assume we are in a region where $\det(J(x_u)) \neq 0$ and so our transformations can indeed be inverted.

Example

Consider cylindrical polar coordinates (r, θ, z) again. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so the determinant is equal to $r(\cos^2\theta + \sin^2\theta) = r$. So provided $r \neq 0$, the transformation can be inverted.

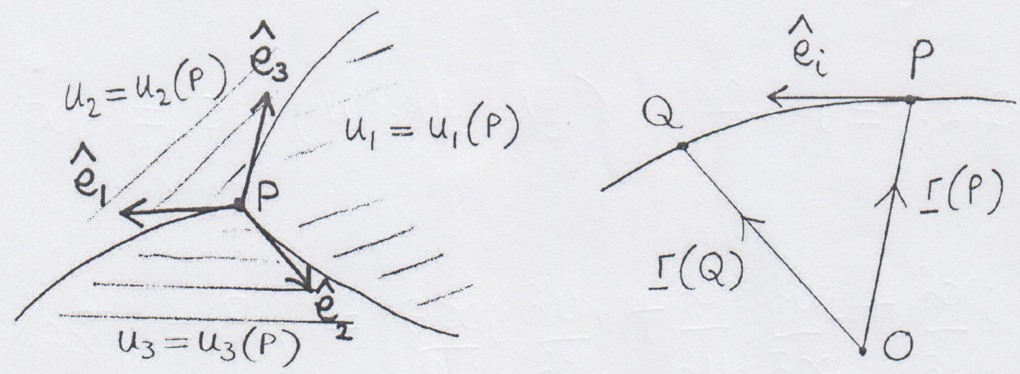


Figure 24: Left: the intersection of the surfaces $u_i = u_i(P)$; right: P and Q are points on a curve along which only one component u_i varies.

Given that we can now write $u_i = u_i(x_1, x_2, x_3)$, the equations u_1 = constant, u_2 = constant, u_3 = constant define three families of surfaces, and (u_1, u_2, u_3) is said to be a **curvilinear coordinate system**. Through each point $P(x, y, z)$ there passes one member of each family. Let $(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)$ be unit vectors at P in the directions normal to $u_1 = u_1(P), u_2 = u_2(P), u_3 = u_3(P)$ respectively, such that u_1, u_2, u_3 increase in the directions $\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3$. Clearly we must have

$$\widehat{\mathbf{a}}_i = \underline{\nabla} u_i / |\underline{\nabla} u_i|$$

If $(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)$ are mutually orthogonal, the coordinate system is said to be an **orthogonal curvilinear coordinate system**.

The surfaces $u_2 = u_2(P)$ and $u_3 = u_3(P)$ intersect in a curve, along which only u_1 varies. Let $\widehat{\mathbf{e}}_1$ be the unit vector tangential to the curve at P. Let $\widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$ be unit vectors tangential to curves along which only u_2, u_3 vary. For an orthogonal system we must have $\widehat{\mathbf{e}}_i = \widehat{\mathbf{a}}_i$ (left diagram in figure 24). Let Q be a neighbouring point to P on the curve along which only u_i varies (right diagram of figure 24). We have

$$\frac{\partial \mathbf{r}}{\partial u_i} = \lim_{P \to Q} \left(\underline{r}(Q) - \underline{r}(P)\right) / \delta u_i$$

$$= \lim_{Q \to P} \frac{\underline{r}(Q) - \underline{r}(P)}{|\overrightarrow{PQ}|} \lim_{Q \to P} \frac{|\overrightarrow{PQ}|}{\delta u_i}$$

$$= \underbrace{\lim_{Q \to P} \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}}_{\hat{e}_i} \underbrace{\lim_{Q \to P} \frac{|\overrightarrow{PQ}|}{\delta u_i}}_{h_i} = h_i \hat{e}_i$$

where we have defined $h_i = |\partial \mathbf{r} / \partial u_i|$. The quantities h_i are often known as the **length scales** for the coordinate system.

or "scale factors"

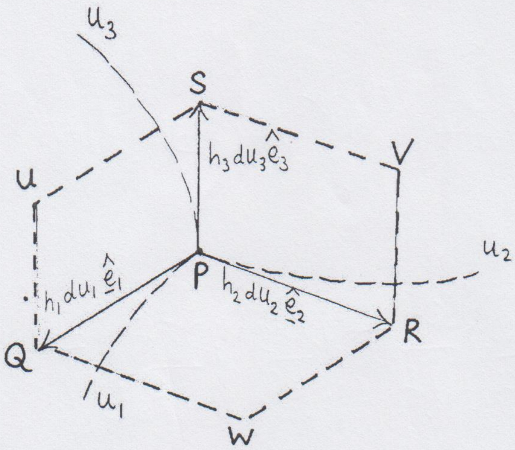


Figure 25: A volume element in an orthogonal curvilinear coordinate system.

1.9.2 Path element

Since $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$, the **path element** $d\mathbf{r}$ is given by

$$d\mathbf{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 = h_1 \hat{e}_1 du_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$(\hat{e}_i \cdot \hat{e}_j = \delta_{ij})$

If the system is orthogonal then it follows that

$$(ds)^2 = (d\underline{r} \cdot d\underline{r}) = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$$

In what follows we will assume we have an orthogonal system so that

$$\hat{\mathrm{e}}_i = \hat{\mathrm{a}}_i = \frac{\partial \underline{r}/\partial u_i}{|\partial \underline{r}/\partial u_i|} = \frac{\underline{\nabla} u_i}{|\underline{\nabla} u_i|} \qquad i = 1, 2, 3$$

In particular, path elements along curves of intersection of u_i surfaces have lengths $h_1 du_1, h_2 du_2, h_3 du_3$ respectively.

1.9.3 Volume element

Since the volume element is approximately rectangular (figure 25) we can take

$$d\tau = (h_1 du_1)(h_2 du_2)(h_3 du_3) = h_1 h_2 h_3 \, du_1 du_2 du_3$$

1.9.4 Surface element

Also from figure 25, by looking at the areas of the faces of the volume element, we can see that the surface element for a surface with u_1 constant is

$$dS = h_2 h_3 \, du_2 du_3$$

and similarly for $u_2 =$ constant, $u_3 =$ constant.

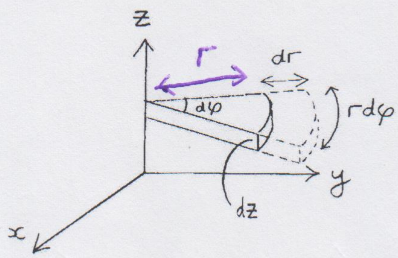


Figure 26: An element of volume in cylindrical polar coordinates.

1.9.5 Properties of various orthogonal coordinate systems

(i) Cartesian coordinates (x, y, z)

$$d\tau = dx\,dy\,dz \qquad d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$(ds)^2 = (d\underline{r})\cdot(d\underline{r}) = (dx)^2 + (dy)^2 + (dz)^2$$

and so $h_1 = h_2 = h_3 = 1$ in this case.

(ii) Cylindrical polar coordinates (r, ϕ, z)

See figure 26. The coordinates are related to Cartesians by

$x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$

To show that this is an orthogonal system we calculate

$$\partial\mathbf{r}/\partial r = (\partial x/\partial r)\hat{i} + (\partial y/\partial r)\hat{j} + (\partial z/\partial r)\hat{k} = (\cos\varphi)\hat{i} + (\sin\varphi)\hat{j}$$

$$\partial\mathbf{r}/\partial\phi = (\partial x/\partial\varphi)\hat{i} + (\partial y/\partial\varphi)\hat{j} + (\partial z/\partial\varphi)\hat{k} = (-r\sin\varphi)\hat{i} + (r\cos\varphi)\hat{j}$$

$$\partial\mathbf{r}/\partial z = \hat{k}$$

Orthogonality then follows from the fact that

$$(\partial\underline{r}/\partial r)\cdot(\partial\underline{r}/\partial\varphi) = -r\cos\varphi\sin\varphi + r\cos\varphi\sin\varphi = 0 \qquad \left(\frac{\partial\underline{r}}{\partial r}\right)\cdot\left(\frac{\partial\underline{r}}{\partial z}\right) = 0$$

$$\&\ \left(\frac{\partial\underline{r}}{\partial\varphi}\right)\cdot\left(\frac{\partial\underline{r}}{\partial z}\right) = 0$$

The lengthscales are

$$h_1 = \left|\frac{\partial\underline{r}}{\partial r}\right| = 1,\ h_2 = \left|\frac{\partial\underline{r}}{\partial\varphi}\right| = r,\ h_3 = \left|\frac{\partial\underline{r}}{\partial z}\right| = 1$$

and so the elements of length and volume are

$$(ds)^2 = (dr)^2 + r^2(d\varphi)^2 + (dz)^2, \quad d\tau = r\,dr\,d\varphi\,dz$$

The surface elements can also be calculated, e.g. an element of the surface along which r is constant (i.e. a cylinder) is

$$dS = h_2 h_3\,du_2\,du_3 = r\,d\varphi\,dz = a\,d\varphi\,dz$$

$r = a$ say

$0 \leqslant \theta \leqslant \pi$

$0 \leqslant \varphi \leqslant 2\pi$

$0 \leqslant r < \infty$

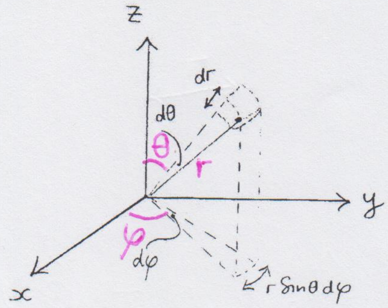


Figure 27: An element of volume in spherical polar coordinates.

(iii) Spherical polar coordinates (r, θ, ϕ)

See figure 27. In this case the relationship between the coordinates is

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

Then

$$\partial\mathbf{r}/\partial r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\partial\mathbf{r}/\partial\theta = r\cos\theta\cos\varphi\,\hat{i} + r\cos\theta\sin\varphi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\partial\mathbf{r}/\partial\phi = -r\sin\theta\sin\varphi\,\hat{i} + r\sin\theta\cos\varphi\,\hat{j} + 0\hat{k}$$

It can then be seen that

$$(\partial\mathbf{r}/\partial r)\cdot(\partial\mathbf{r}/\partial\theta) = r\sin\theta\cos\theta\cos^2\varphi + r\sin\theta\cos\theta\sin^2\varphi - r\sin\theta\cos\theta = 0$$

Similarly:

$$(\partial\underline{r}/\partial r)\cdot(\partial\underline{r}/\partial\varphi) = 0 \quad \& \quad \left(\frac{\partial\underline{r}}{\partial\varphi}\right)\cdot\left(\frac{\partial\underline{r}}{\partial\theta}\right) = 0$$

and so the system is orthogonal. Then

$$h_1 = \left|\frac{\partial\underline{r}}{\partial r}\right| = \sqrt{(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)} = 1$$

$$h_2 = \left|\frac{\partial\underline{r}}{\partial\theta}\right| = \sqrt{(r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta)} = r$$

$$h_3 = \left|\frac{\partial\underline{r}}{\partial\varphi}\right| = \sqrt{(r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi)} = r\sin\theta$$

(We have assumed here that $\sin\theta > 0$, which is OK since the range of θ is 0 to π). The volume element is

$$d\tau = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

Also, an element of the surface $r =$ constant $= a$ (i.e. a sphere of radius a) is:

$$r = u_1 \quad \theta = u_2 \quad \varphi = u_3$$

$$dS = h_2 h_3\,du_2\,du_3 = a^2\sin\theta\,d\theta\,d\varphi$$

Example

Find the volume and surface area of a sphere of radius a, and also find the surface area of a cap of the sphere that subtends an angle 2α at the centre of the sphere.

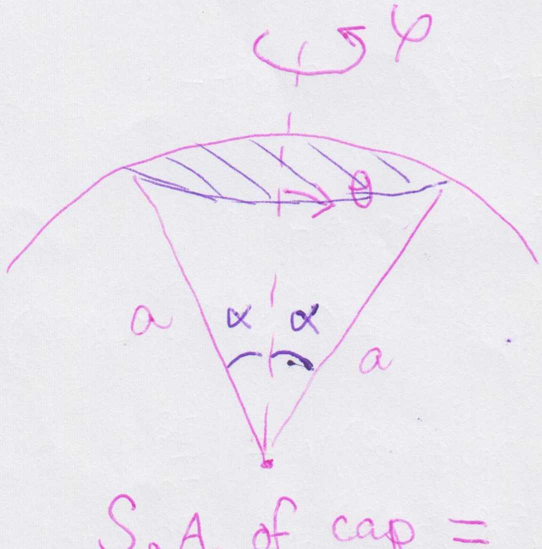
$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$\therefore \text{total volume} = \int_\tau d\tau = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= 2\pi \left[-\cos\theta\right]_0^{\pi} \left[\frac{r^3}{3}\right]_0^{a} = \frac{4}{3}\pi a^3$$

$$\text{surface area} = \int_S dS = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} a^2 \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi a^2 \left[-\cos\theta\right]_0^{\pi} = 4\pi a^2$$



$$\text{S.A. of cap} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\alpha} a^2 \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi a^2 \left[-\cos\theta\right]_0^{\alpha}$$

$$= 2\pi a^2 (1 - \cos\alpha)$$

1.9.6 Gradient in orthogonal curvilinear coordinates

Let

$$\nabla\Phi = \lambda_1\widehat{\mathbf{e}}_1 + \lambda_2\widehat{\mathbf{e}}_2 + \lambda_3\widehat{\mathbf{e}}_3$$

in a general coordinate system, where $\lambda_1, \lambda_2, \lambda_3$ are to be found. Recall that the element of length is given by

$$d\mathbf{r} = h_1\,du_1\,\hat{e}_1 + h_2\,du_2\,\hat{e}_3 + h_3\,du_3\,\hat{e}_3$$

Now

$$\begin{aligned} d\Phi &= \left(\frac{\partial\Phi}{\partial u_1}\right)du_1 + \left(\frac{\partial\Phi}{\partial u_2}\right)du_2 + \left(\frac{\partial\Phi}{\partial u_3}\right)du_3 \\ &= \left(\frac{\partial\Phi}{\partial x}\right)dx + \left(\frac{\partial\Phi}{\partial y}\right)dy + \left(\frac{\partial\Phi}{\partial z}\right)dz \\ &= (\underline{\nabla}\Phi)\cdot d\underline{r} \end{aligned}$$

But, using our expressions for $\nabla\Phi$ and $d\mathbf{r}$ above:

$$(\nabla\Phi)\cdot d\mathbf{r} = \lambda_1 h_1\,du_1 + \lambda_2 h_2\,du_2 + \lambda_3 h_3\,du_3$$

and so we see that

$$h_i\lambda_i = \partial\Phi/\partial u_i \qquad i=1,2,3$$

Thus we have the result that

$$\nabla\Phi = \hat{e}_1\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

This result now allows us to write down ∇ easily for other coordinate systems.

(i) Cylindrical polars (r, ϕ, z)

Recall that $h_1 = 1, h_2 = r, h_3 = 1$. Thus

$$\nabla =$$

(ii) Spherical polars (r, θ, ϕ)

We have $h_1 = 1, h_2 = r, h_3 = r\sin\theta$, and so

$$\nabla =$$

1.9.7 Expressions for unit vectors

From the expression for ∇ we have just derived it is easy to see that:

$$\widehat{\mathbf{e}}_i =$$

Alternatively, since the unit vectors are orthogonal, if we know two unit vectors we can find the third from the relation

$$\widehat{\mathbf{e}}_1 =$$

and similarly for the other components, by permuting in a cyclic fashion.

1.9.6 Gradient in orthogonal curvilinear coordinates

Let

$$\nabla\Phi = \lambda_1\widehat{\mathbf{e}}_1 + \lambda_2\widehat{\mathbf{e}}_2 + \lambda_3\widehat{\mathbf{e}}_3$$

in a general coordinate system, where $\lambda_1, \lambda_2, \lambda_3$ are to be found. Recall that the element of length is given by

$$d\mathbf{r} = h_1\,du_1\,\hat{e}_1 + h_2\,du_2\,\hat{e}_3 + h_3\,du_3\,\hat{e}_3$$

Now

$$\begin{aligned} d\Phi &= \left(\frac{\partial\Phi}{\partial u_1}\right)du_1 + \left(\frac{\partial\Phi}{\partial u_2}\right)du_2 + \left(\frac{\partial\Phi}{\partial u_3}\right)du_3 \\ &= \left(\frac{\partial\Phi}{\partial x}\right)dx + \left(\frac{\partial\Phi}{\partial y}\right)dy + \left(\frac{\partial\Phi}{\partial z}\right)dz \\ &= (\underline{\nabla}\Phi)\cdot d\underline{r} \end{aligned}$$

But, using our expressions for $\nabla\Phi$ and $d\mathbf{r}$ above:

$$(\nabla\Phi)\cdot d\mathbf{r} = \lambda_1 h_1\,du_1 + \lambda_2 h_2\,du_2 + \lambda_3 h_3\,du_3$$

and so we see that

$$h_i\lambda_i = \partial\Phi/\partial u_i \qquad i=1,2,3$$

Thus we have the result that

$$\nabla\Phi = \hat{e}_1\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

This result now allows us to write down ∇ easily for other coordinate systems.

(i) Cylindrical polars (r, ϕ, z)

Recall that $h_1 = 1, h_2 = r, h_3 = 1$. Thus

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\varphi}}{r}\frac{\partial}{\partial\varphi} + \hat{k}\frac{\partial}{\partial z}$$

(ii) Spherical polars (r, θ, ϕ)

We have $h_1 = 1, h_2 = r, h_3 = r\sin\theta$, and so

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

1.9.7 Expressions for unit vectors

From the expression for ∇ we have just derived it is easy to see that:

$$\widehat{\mathbf{e}}_i = h_i\,\underline{\nabla}u_i$$

Alternatively, since the unit vectors are orthogonal, if we know two unit vectors we can find the third from the relation

$$\widehat{\mathbf{e}}_1 = \hat{e}_2\times\hat{e}_3 = h_2h_3(\underline{\nabla}u_2\times\underline{\nabla}u_3)$$

and similarly for the other components, by permuting in a cyclic fashion.

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1.9.8 Divergence in orthogonal curvilinear coordinates

$$\boxed{\nabla\cdot(\varphi\mathbf{B}) = \varphi\nabla\cdot\mathbf{B} + (\nabla\varphi)\cdot\mathbf{B}}$$

Suppose we have a vector field

$$\mathbf{A} = A_1\widehat{\mathbf{e}}_1 + A_2\widehat{\mathbf{e}}_2 + A_3\widehat{\mathbf{e}}_3.$$

First consider

$$\nabla u_2 \times \nabla u_3 = \frac{\widehat{\mathbf{e}}_1}{h_2h_3}$$

$$\begin{aligned}\nabla\cdot(A_1\widehat{\mathbf{e}}_1) &= \nabla\cdot\left[A_1h_2h_3(\nabla u_2\times\nabla u_3)\right] \\ &= A_1h_2h_3\,\nabla\cdot(\nabla u_2\times\nabla u_3) + \nabla(A_1h_2h_3)\cdot\frac{\widehat{\mathbf{e}}_1}{h_2h_3}\end{aligned}$$

using the results established just above. Also we know that

$$\nabla\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot\operatorname{curl}\mathbf{B} - \mathbf{B}\cdot\operatorname{curl}\mathbf{C},$$

and so it follows that

$$\nabla\cdot(\nabla u_2\times\nabla u_3) = (\nabla u_3)\cdot\operatorname{Curl}(\nabla u_2) - (\nabla u_2)\cdot\operatorname{Curl}(\nabla u_3) = 0$$

since the curl of a gradient is always zero. Thus we are left with

$$\nabla\cdot(A_1\widehat{\mathbf{e}}_1) = \nabla(A_1h_2h_3)\cdot\frac{\widehat{\mathbf{e}}_1}{h_2h_3} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(A_1h_2h_3)$$

We can proceed in a similar fashion for the other components, and establish that

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right\}$$

It is now easy to write down div in other coordinate systems.

(i) Cylindrical polars (r,ϕ,z)

Recall that $h_1 = 1, h_2 = r, h_3 = 1$. Thus using the above formula:

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial\varphi}A_2 + \frac{\partial}{\partial z}(rA_3)\right\} \\ &= \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial\varphi} + \frac{\partial A_3}{\partial z}\end{aligned}$$

(ii) Spherical polars (r,θ,ϕ)

We have $h_1 = 1, h_2 = r, h_3 = r\sin\theta$. Hence

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\,A_1) + \frac{\partial}{\partial\theta}(r\sin\theta\,A_2) + \frac{\partial}{\partial\varphi}(rA_3)\right\}$

1.9.9 Curl in orthogonal curvilinear coordinates

$$\boxed{\underline{\nabla}\times(\varphi\underline{B}) = \varphi(\underline{\nabla}\times\underline{B}) + \underline{\nabla}\varphi\times\underline{B}}$$

Again just consider the curl of the first component of $\mathbf{A}$:

$$\begin{aligned}
\nabla\times(A_1\widehat{\mathbf{e}}_1) &= \underline{\nabla}\times(A_1 h_1 \underline{\nabla} u_1) \\
&= A_1 h_1 \underline{\nabla}\times(\underline{\nabla} u_1) + \underline{\nabla}(A_1 h_1)\times\underline{\nabla} u_1 \\
&= \text{zero} + \underline{\nabla}(A_1 h_1)\times(\hat{e}_1/h_1) \\
&= \left\{\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_1) + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_1 h_1)\right\}\times\frac{\hat{e}_1}{h_1} \\
&= \frac{\hat{e}_2}{h_1 h_3}\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\hat{e}_3}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1)
\end{aligned}$$

(since $\widehat{\mathbf{e}}_1\times\widehat{\mathbf{e}}_1 = 0$, $\widehat{\mathbf{e}}_2\times\widehat{\mathbf{e}}_1 = -\widehat{\mathbf{e}}_3$, $\widehat{\mathbf{e}}_3\times\widehat{\mathbf{e}}_1 = \widehat{\mathbf{e}}_2$). We can obviously find $\mathrm{curl}\,(A_2\widehat{\mathbf{e}}_2)$ and $\mathrm{curl}\,(A_3\widehat{\mathbf{e}}_3)$ in a similar way. These can be shown to be

$$\begin{aligned}
\nabla\times(A_2\widehat{\mathbf{e}}_2) &= \frac{\widehat{\mathbf{e}}_3}{h_2h_1}\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\widehat{\mathbf{e}}_1}{h_2h_3}\frac{\partial}{\partial u_3}(h_2A_2), \\
\nabla\times(A_3\widehat{\mathbf{e}}_3) &= \frac{\widehat{\mathbf{e}}_1}{h_3h_2}\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\widehat{\mathbf{e}}_2}{h_3h_1}\frac{\partial}{\partial u_1}(h_3A_3).
\end{aligned}$$

Adding the three contributions together, we find we can write this in the form of a determinant as

$$\mathrm{curl}\,\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

in which form it is probably easiest remembered. It's then straightforward to write down curl in various orthogonal coordinate systems.

(i) Cylindrical polars

$$\mathrm{curl}\,\mathbf{A} = \frac{1}{r}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\phi} & \widehat{\mathbf{k}} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_1 & rA_2 & A_3 \end{vmatrix}.$$

(ii) Spherical polars

$$\mathrm{curl}\,\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\theta} & r\sin\theta\,\widehat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_1 & rA_2 & r\sin\theta A_3 \end{vmatrix}.$$

1.9.10 The Laplacian in orthogonal curvilinear coordinates

From the formulae already established for grad and div, we can see that

Φ is a scalar

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi)$$

$$= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(h_2h_3\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3h_1\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(h_1h_2\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right\}$$

This formula can then be used to calculate the Laplacian for various coordinate systems.

(i) Cylindrical polars (r,ϕ,z) $h_1 = 1$ $h_2 = r$ $h_3 = 1$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}.\end{aligned}$$

(ii) Spherical polars (r,θ,ϕ) $h_1 = 1$ $h_2 = r$ $h_3 = r\sin\theta$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.\end{aligned}$$

→ Laplacian of a vector

$$\nabla^2\underline{A} = \underbrace{\underline{\nabla}(\text{div}\,\underline{A}) - \text{Curl}\,(\text{Curl}\,\underline{A})}_{\text{use formulae for these derived on previous pages}}$$

1.9.11 Alternative definitions for grad, div, curl (not examinable)

Let τ be a region enclosed by a surface S and let P be a general point of τ. We established earlier that

$$\int_\tau \nabla\phi \, d\tau = \int_S \widehat{\mathbf{n}}\, \phi \, dS.$$

3

This result is a consequence of the divergence theorem (see problem sheet). It follows that

$$\int_\tau \mathbf{i}\cdot\nabla\phi \, d\tau = \int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi \, dS.$$

Now the left-hand-side above can be written as $\tau\{\overline{\mathbf{i}\cdot\nabla\phi}\}$ where the bar denotes the mean value of this quantity over τ. Since we are assuming that ϕ has continuous derivatives throughout τ, we can write

$$\{\overline{\mathbf{i}\cdot\nabla\phi}\} = \{\mathbf{i}\cdot\nabla\phi\}_Q$$

for some point Q of τ. Thus we have that

$$\{\mathbf{i}\cdot\nabla\phi\}_Q = \frac{1}{\tau}\int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi \, dS.$$

Now let $\tau \to 0$ about P. Then $P \to Q$ and we have that at any point P of τ :

$$\mathbf{i}\cdot\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\mathbf{i}\cdot\widehat{\mathbf{n}})\phi \, dS.$$

Similar results can be established for $\mathbf{j}\cdot\nabla\phi$ and $\mathbf{k}\cdot\nabla\phi$. Taken together, these imply that

$$\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\oint_S \widehat{\mathbf{n}}\,\phi \, dS.$$

This can be regarded as an alternative way of defining $\nabla\phi$, rather than defining it as $(\partial\phi/\partial x)\mathbf{i} + (\partial\phi/\partial y)\mathbf{j} + (\partial\phi/\partial z)\mathbf{k}$.

We can similarly establish that

$$\begin{aligned} \mathrm{div}\mathbf{A} &= \lim_{\tau\to 0}\frac{1}{\tau}\oint_S (\widehat{\mathbf{n}}\cdot\mathbf{A})\, dS, \\ \mathrm{curl}\,\mathbf{A} &= \lim_{\tau\to 0}\frac{1}{\tau}\oint_S (\widehat{\mathbf{n}}\times\mathbf{A})\, dS, \end{aligned}$$

which are alternative definitions of the divergence and curl, and are clearly independent of the choice of coordinates, which is one of the advantages of this approach. In particular we can see that the divergence is a measure of the flux of a quantity.

Equivalence of definitions

Let's show that the definition of divergence given here is consistent with the curvilinear formula given earlier. Consider $\delta\tau$ to be the volume of a curvilinear volume element located at the point P, with edges of length $h_1\delta u_1, h_2\delta u_2, h_3\delta u_3$, and unit vectors aligned as shown in the picture (figure 28). The volume of the element $\delta\tau \simeq h_1h_2h_3\delta u_1\delta u_2\delta u_3$. We start with our definition

$$\mathrm{div}\mathbf{A} = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\mathbf{n}}\cdot\mathbf{A})\, dS,$$

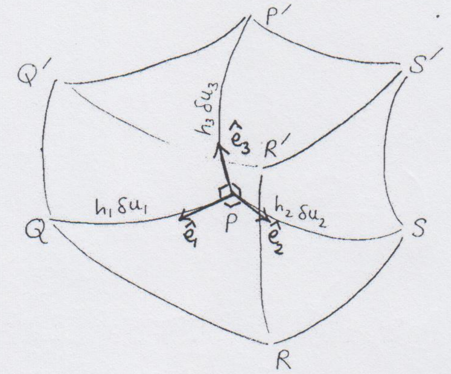


Figure 28: A curvilinear volume element.

and aim to compute explicitly the right-hand-side. This involves calculating the contributions to $\int_S$ arising from the six faces of the volume element. If we start with the contribution from the face $PP'S'S$, this is:

$$-(A_1 h_2 h_3)_P \, \delta u_2 \delta u_3 + \text{ higher order terms.}$$

The contribution from the face $QQ'R'R$ is

$$(A_1 h_2 h_3)_Q \, \delta u_2 \delta u_3 + \text{h.o.t.} \;=\; \left[(A_1 h_2 h_3) + \frac{\partial}{\partial u_1}(A_1 h_2 h_3) \delta u_1 \right]_P \delta u_2 \delta u_3 + \text{h.o.t.},$$

using a Taylor series expansion. Adding together the contributions from these two faces we get

$$\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t.}$$

Similarly, the sum of the contributions from the faces $PSRQ$, $P'S'R'Q'$ is

$$\left[\frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t.},$$

while the combined contributions from $PQQ'P'$, $SRR'S'$ is

$$\left[\frac{\partial}{\partial u_2}(A_2 h_3 h_1) \right]_P \delta u_1 \delta u_2 \delta u_3 + \text{ h.o.t..}$$

If we then let $\delta\tau \to 0$ we have that

$$\lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \int_S \widehat{\mathbf{n}} \cdot \mathbf{A} \, dS = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right\},$$

and so we can see that the integral expression for $\operatorname{div} \mathbf{A}$ is consistent with the formula in curvilinear coordinates derived earlier.

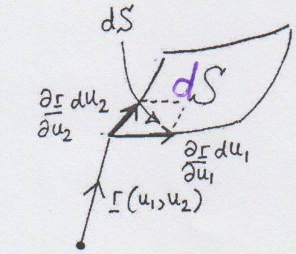


Figure 29: A surface S parameterized by u_1 and u_2.

1.10 Changes of variable in surface integration

Suppose we have a surface S which is parameterized by the quantities u_1, u_2. We can therefore write that on S:

$$x = x(u_1, u_2), \quad y = y(u_1, u_2), \quad z = z(u_1, u_2).$$

[For example, if S is the surface of a sphere of unit radius we have $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$ and so we can take $u_1 = \theta$, $u_2 = \phi$.]

We can consider the surface to be comprised of arbitrarily small parallelograms whose sides are obtained by keeping either u_1 or u_2 constant: see figure 29, i.e.

$$\begin{aligned} dS &= \text{Area of parallelogram with sides } \frac{\partial \mathbf{r}}{\partial u_1} du_1 \text{ and } \frac{\partial \mathbf{r}}{\partial u_2} du_2 \\ &= |\mathbf{J}|\, du_1\, du_2, \end{aligned}$$

where the **vector Jacobian J** is given by $\mathbf{J} = \frac{\partial \underline{r}}{\partial u_1} \times \frac{\partial \underline{r}}{\partial u_2}$

This result is particularly useful when using a substitution in a surface integral, as we can write

$$\int_S f(x, y, z)\, dS = \int_S F(u_1, u_2)\, |\underline{J}|\, du_1\, du_2$$

where $F(u_1, u_2) = f(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$.

If S is a region R in the $x - y$ plane, (i.e. $z = 0$ on R), the result reduces to

$$\int_R f(x, y)\, dx\, dy = \int_R F(u_1, u_2)\, |\det(J(x_u))|\, du_1\, du_2$$

see next page

where $J(x_u)$ is the Jacobian matrix we met earlier, i.e.

$$J(x_u) = \begin{pmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 \\ \partial y/\partial u_1 & \partial y/\partial u_2 \end{pmatrix}$$

Note that since $dx\, dy = |\det(J(x_u))|\, du_1 du_2$ it follows that $du_1 du_2 = (1/|\det(J(x_u))|) dx\, dy$, and hence

$$1/|\det(J(x_u))| = |\det(J(u_x))|$$

⚠ $\partial u_1/\partial x \neq \dfrac{1}{\partial x/\partial u_1}$

which is a result we found earlier by a different method. These formulae apply for both orthogonal and non-orthogonal transformations.

$$\underline{J} = \frac{\partial \underline{r}}{\partial u_1} \times \frac{\partial \underline{r}}{\partial u_2}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u_1 & \partial y/\partial u_1 & \partial z/\partial u_1 \\ \partial x/\partial u_2 & \partial y/\partial u_2 & \partial z/\partial u_2 \end{vmatrix}$$

if we are in the x-y plane

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u_1 & \partial y/\partial u_1 & 0 \\ \partial x/\partial u_2 & \partial y/\partial u_2 & 0 \end{vmatrix}$$

$$= \hat{k}\left(\frac{\partial x}{\partial u_1}\frac{\partial y}{\partial u_2} - \frac{\partial x}{\partial u_2}\frac{\partial y}{\partial u_1}\right)$$

Hence

$$|\underline{J}| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{pmatrix} \right|$$

$$= \left| \det\left(J(x_u)\right) \right|$$

Suppose a surface is described by $z = f(x, y)$. Then $u_1 = x$, $u_2 = y$ and $\mathbf{r} = (x, y, f(x, y))$. It follows that

$$\frac{\partial \mathbf{r}}{\partial u_1} = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial u_2} = \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

so then

$$\left|\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}\right| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{matrix} \right\|$$
$$= \left| -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right|$$
$$= \sqrt{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)}$$
$$= \sqrt{(1 + |\underline{\nabla} f|^2)}$$

Therefore the area of surface is

$$\int_\Sigma \sqrt{(1 + |\nabla f|^2)}\, dx\, dy,$$

where Σ is the projection of S onto the $x - y$ plane. We will use this expression in the next section.

We'll use this expression in Part 2 on the Calculus of Variations.

Could also see this by using projection theorem

$$\left(\frac{1}{|\hat{n} \cdot \hat{k}|} = \sqrt{(1 + |\underline{\nabla} f|^2)} \right)$$

Check this.

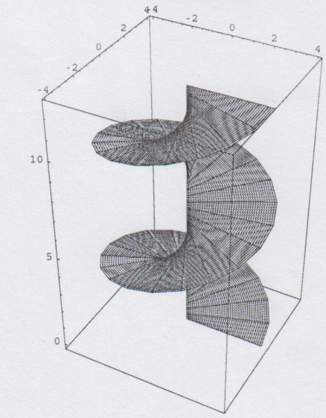


Figure 30: A section of a helicoid.

Example

Evaluate the integral

$$\int_S \sqrt{(1+x^2+y^2)}\, dS$$

where S is the surface of the helicoid (shown in figure 30):

$$x = u\cos v,\ \ y = u\sin v,\ \ z = v,$$

with $0 \leqslant u \leqslant 4$ and $0 \leqslant v \leqslant 4\pi$.

$$\underline{J} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \hat{i}(\sin v) - \hat{j}(\cos v) + \underbrace{(u\cos^2 v + u\sin^2 v)}_{u}\hat{k}$$

$$\therefore |\underline{J}| = \sqrt{(\sin^2 v + \cos^2 v + u^2)} = \sqrt{(1+u^2)}$$

$$\& \sqrt{(1+x^2+y^2)} = \sqrt{(1+u^2\cos^2 v + u^2\sin^2 v)} = \sqrt{(1+u^2)}$$

N.B. not usually the same!

$$\therefore \int_S \sqrt{(1+x^2+y^2)}\,dS = \int \sqrt{(1+u^2)}\,|\underline{J}|\,du\,dv$$

$$= \int_0^4 \int_0^{4\pi} (1+u^2)\,dv\,du$$

$$= 4\pi\left[u + u^3/3\right]_0^4 = \ldots = 304\pi/3$$

2 The Calculus of Variations

2.1 Preliminary motivational examples

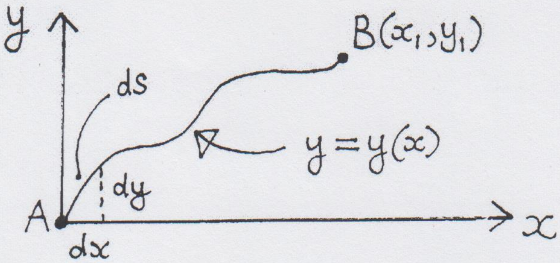


Figure 1: The figure for Example 1.

Example 1. *Shortest path between 2 points*
Suppose we have two points $A(0,0)$ and $B(x_1, y_1)$. The length l of a curve $y(x)$ joining the two points is (see figure 1):

$$l = \int_A^B ds = \int_0^{x_1} \left(1 + (dy/dx)^2\right)^{1/2} dx$$

The shortest path can be found by finding the $y\,(x)$ which minimizes this integral. Intuition suggests that it is a straight line. We will return to this problem later.

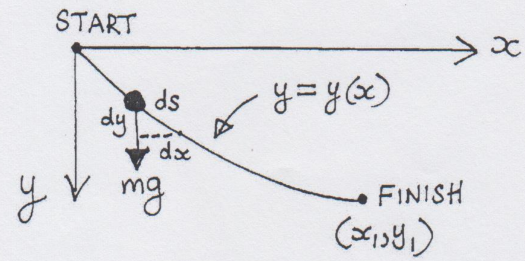


Figure 2: The brachistochrone problem

Example 2. *Curve of quickest descent ('brachistochrone')*
A slightly less trivial example is the following. A particle starts from rest at the origin and travels under gravity along a smooth curve until it reaches the point (x_1, y_1). What shape of curve should it travel along in order that the time of descent is a minimum?
If s is distance along the curve then as in the first example

$$ds = \left(1 + (dy/dx)^2\right)^{1/2} dx,$$

where $y(x)$ is the path. As the particle travels, it converts potential energy into kinetic energy while respecting the overall conservation of energy principle:

$$\frac{1}{2}mv^2 - mgy = \text{const} = 0 \quad \left(\begin{array}{c}\text{Since}\\ v=0\\ \text{when}\\ y=0\end{array}\right)$$

where y is measured vertically downwards from the origin, $v(x)$ is the velocity at location $(x, y(x))$ and m is the mass of the particle. Therefore we have

$$v = ds/dt = (2gy)^{1/2}$$

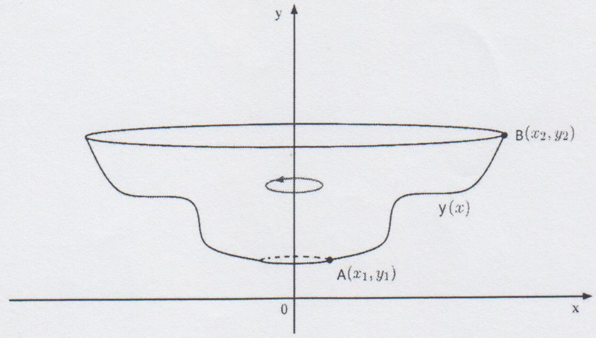
Rearranging: $$dt = \frac{ds}{(2gy)^{1/2}} = (2gy)^{-1/2}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

Thus, the time τ taken to travel to x_1 along $y(x)$ is

$$\tau = \frac{1}{(2g)^{1/2}} \int_0^{x_1} y^{-1/2}\left(1+(dy/dx)^2\right)^{1/2} dx$$

The curve of quickest descent is found by minimizing this integral. This time the answer is far from obvious.

$y \neq 0$ except at $x=0$



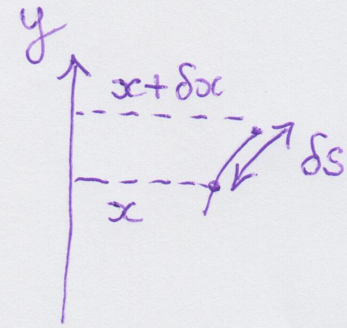


Figure 3: Surface of revolution

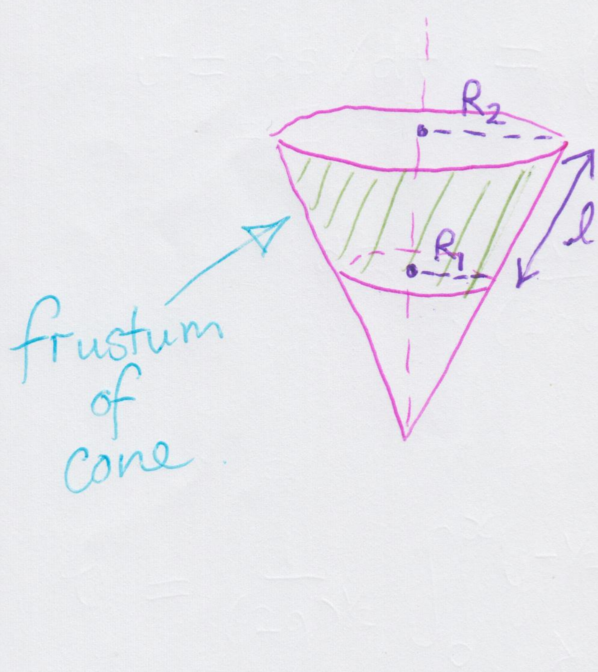
Example 3. *Minimal surface of revolution*
Consider a curve $y = y(x)$ joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. We now consider the surface formed by rotating this curve about the $y-$axis. The surface area is given by

$$\mathcal{A} = \int_A^B 2\pi x \, ds$$

Using the expression for arclength as in the first two examples, this can be rewritten as

$$\mathcal{A} = 2\pi \int_{x_1}^{x_2} x\left(1 + (dy/dx)^2\right)^{1/2} dx$$

It is of interest to find the curve $y(x)$ which minimizes $\mathcal{A}$. Again the answer is not obvious.



Area of surface is $\pi(R_1 + R_2)l$

2.2 'The Vanishing Lemma'

Before we proceed with the general theory we need the following result. If g is a continuous function such that

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx = 0$$

for all smooth functions $\eta(x)$, with $\eta(x_1) = \eta(x_2) = 0$, then $g(x) \equiv 0$.

Proof

Assume for a contradiction that there is a point $x_0 \in [x_1, x_2]$ for which $g(x_0) \neq 0$. Let's assume without loss of generality that $g(x_0) > 0$. Since g is continuous there is a neighbourhood of x_0 in which g remains positive. Denote this neighbourhood by NH.

If x_0 is not equal to x_1 or x_2 then we can take $NH = (x_0 - \epsilon, x_0 + \epsilon)$ with $\epsilon > 0$. If $x_0 = x_1$ then $NH = [x_1, x_1 + \epsilon)$ and if $x_0 = x_2$ then $NH = (x_2 - \epsilon, x_2]$. In each case $g(x) > c > 0$ for all $x \in NH$.

Consider now a smooth function $h(x)$ on $[x_1, x_2]$ with the following properties†

(i) $h(x) = 0$ for all x outside the neighbourhood;

(ii) $\int_{x_1}^{x_2} h(x)\,dx = \int_{NH} h(x)\,dx > 0$.

It follows then that

$$\int_{x_1}^{x_2} g(x)h(x)\,dx = \int_{NH} g(x)h(x)\,dx > c\int_{NH} h(x)\,dx > 0$$

and hence leads to a contradiction.

†For an example of such a function $h(x)$ see problem sheet 5.

2.3 General theory for 1D integrals

The examples mentioned above are special cases of the integral

$$I = \int_{x_1}^{x_2} L(x, y, y')\, dx$$

where $y' = dy/dx$. In example 1, $L = (1 + (y')^2)^{1/2}$. L is known as a *functional.*

Suppose $y = y(x)$ passes through $A(x_1, y_1)$ and $B(x_2, y_2)$. What is the particular $y(x)$ which minimizes/maximizes (extremizes) the integral I? If $y = Y(x)$ is the extremal curve, how do we find it?

Consider the family of curves

$$y(x,\varepsilon) = Y(x) + \varepsilon\eta(x)$$

where ε is any real number and η is a smooth curve with $\eta(x_1) = \eta(x_2) = 0$. Each member of the family passes through A and B. It follows that

$$I(\varepsilon) = \int_{x_1}^{x_2} L(x, Y+\varepsilon\eta, Y'+\varepsilon\eta')\, dx$$

The integral I takes on its extreme value when $\varepsilon = 0$ (since then $y = Y$). Therefore we must have

(Necessary condition) $$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

Now

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial L}{\partial y}\overbrace{\frac{dy}{d\varepsilon}}^{\eta} + \frac{\partial L}{\partial y'}\overbrace{\frac{dy'}{d\varepsilon}}^{=\eta'} \right) dx$$

When $\varepsilon = 0$ we have $y = Y$ and $y' = Y'$, and so

$$0 = I'(0) = \int_{x_1}^{x_2} \left(\eta\frac{\partial L}{\partial Y} + \eta'\frac{\partial L}{\partial Y'} \right) dx$$

We now integrate by parts to get

$$0 = \int_{x_1}^{x_2} \eta(x)\frac{\partial L}{\partial Y}\, dx + \left[\eta\frac{\partial L}{\partial Y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) dx$$

The integrated term vanishes since $\eta(x_1) = \eta(x_2) = 0$ and we are left with

$$0 = \int_{x_1}^{x_2} \eta(x)\left\{ \frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) \right\} dx$$

Since $\eta(x)$ is an arbitrary smooth curve we can use the Vanishing Lemma above to deduce that Y satisfies

$$\frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) = 0 \qquad (1)$$

which is known as the **Euler-Lagrange equation** in one dimension.

2.3.1 Remarks

(i) In order to integrate by parts we have assumed that the curve $Y(x)$ is of the class C^2 (i.e. the derivatives Y' and Y'' exist and are continuous).
(ii) $Y(x)$ renders I stationary, not necessarily a maximum or minimum, so the Euler-Lagrange equation is a necessary but not sufficient condition for $Y(x)$ to minimize I. In order to prove it definitely gives a (local) minimum we have to show that $I''(0) > 0$ (which is complicated to establish except for very simple examples).
(iii) We usually refer to $Y(x)$ as an *extremal curve* of I.
(iv) The Euler-Lagrange equation is an equation to determine $Y(x)$; the functional L is known for a given problem and is referred to as the *Lagrangian*.
(v) From now on we will replace Y by y, i.e. we will denote the extremal curve by $y(x)$.

2.3.2 Short forms of the 1D Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

The equation simplifies if the functional L is independent of one or more of the variables x, y, y'.

Case 1. L is explicitly independent of y.
Here $L = L(x, y')$ and so $\partial L/\partial y = 0$. Thus the E-L equation reduces to

$$-\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

and hence

$$\frac{\partial L}{\partial y'} = \text{const.}$$

Case 2. $L = L(x, y)$ so that $\partial L/\partial y' = 0$. In this case the E-L equation reduces to

$$\frac{\partial L}{\partial y} = 0$$

Case 3. $L = L(y, y')$ so that $\partial L/\partial x = 0$, but $dL/dx \neq 0$. Using the chain rule

$$\begin{aligned} \frac{dL}{dx} &= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{dy'}{dx} \\ &= y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'}. \end{aligned}$$

Using the E-L equation, the RHS can be rewritten as

$$y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) + y''\frac{\partial L}{\partial y'} \equiv \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

Therefore we see that

$$\frac{dL}{dx} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

and hence the E-L equation reduces in this case to

$$L - y'\frac{\partial L}{\partial y'} = \text{constant}.$$

It's useful to remember the short forms, but the most important equation to remember is the original Euler-Lagrange equation (1). Now that we have this we can revisit our motivational examples.

2.4 Revisiting our examples

Example 1 revisited: *shortest path between 2 points.*
Here the integral to minimize is

$$I = \int_0^{x_1} \left(1 + (y')^2\right)^{1/2} dx.$$

and hence $L = \left(1 + (y')^2\right)^{1/2}$, explicitly independent of x and y. Therefore the E-L equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

reduces to

$$\frac{\partial L}{\partial y'} = \text{const.}$$

Substituting for L we find:

$$\frac{1}{2}(2y')(1 + (y')^2)^{-1/2} = A, \text{ say}$$

This implies

$$(y')^2 = A^2(1 + (y')^2)$$

and hence

$$y' = \text{const.}$$

Therefore the extremal curve is of the form

$$y = mx + C$$

with m, C found from the conditions that ~~y~~ the curve passes through $(0,0)$ and (x_1, y_1). In this case:

$$y = \left(\frac{y_1}{x_1}\right)x$$

Thus the answer is a straight line as expected. In this case we can check explicitly that $I''(0) > 0$ and hence demonstrate rigorously that this is a minimum rather than a maximum (although here of course it is obvious there is no maximal curve).

Example 2 revisited: brachistochrone
Here the integral to minimize is

$$\tau = \frac{1}{(2g)^{1/2}} \int_0^{x_1} y^{-1/2} \left(1 + (y')^2\right)^{1/2} dx$$

and so we can take

$$L = y^{-1/2}(1 + (y')^2)^{1/2}.$$

Since this is independent of x we can use the appropriate short form (case 3) of the E-L equation, namely:

$$L - y'\frac{\partial L}{\partial y'} = \text{ constant.}$$

Substituting for L :

$$y^{-1/2}(1+(y')^2)^{1/2} - (y')\, y^{-1/2}(y')(1+(y')^2)^{-1/2} = \text{const.}$$

Putting over a common denominator:

$$\frac{1 + \cancel{(y')^2} - \cancel{(y')^2}}{y^{1/2}(1+(y')^2)^{1/2}} = \frac{1}{y^{1/2}(1+(y')^2)^{1/2}} = \text{const.}$$

$$\Rightarrow y(1+(y')^2) = \alpha^2 \Rightarrow (y')^2 = \alpha^2/y - 1$$

where α is an arbitrary constant. We now separate the variables and integrate, setting $y = 0$ when $x = 0$ as this is the initial location of the particle. This gives

$$x = \pm \int_0^y \frac{dy}{(\alpha^2/y - 1)^{1/2}} = \pm \int_0^y \frac{y^{1/2}}{(\alpha^2 - y)^{1/2}}\, dy$$

To solve the integral we make the substitution $y = \alpha^2 \sin^2\theta, dy = 2\alpha^2 \sin\theta\cos\theta$. Thus:

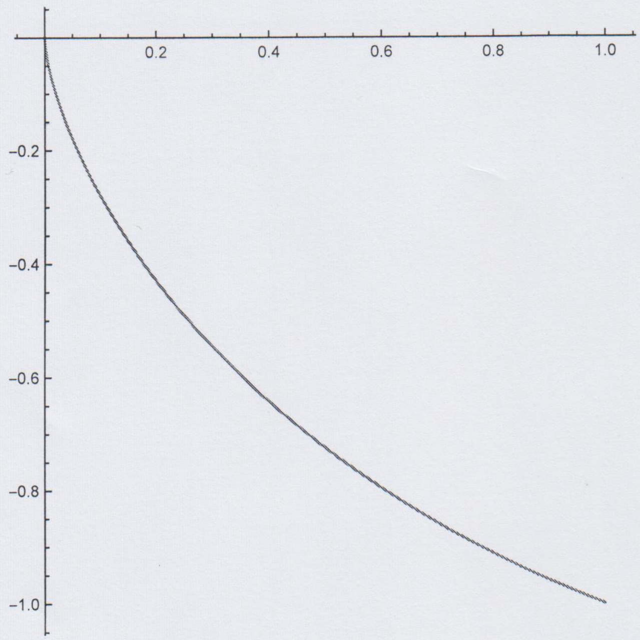
$$x = \pm \int_0^\theta 2\alpha^2 \sin^2\theta\, d\theta = \pm\alpha^2 \int_0^\theta (1 - \cos 2\theta)\, d\theta = \pm\alpha^2\left(\theta - \frac{\sin 2\theta}{2}\right)$$

We take the positive sign so that x increases as θ increases (i.e. the parameter θ increases as the particle moves along the curve from left to right). Thus the parametric form of the mimimizing curve is:

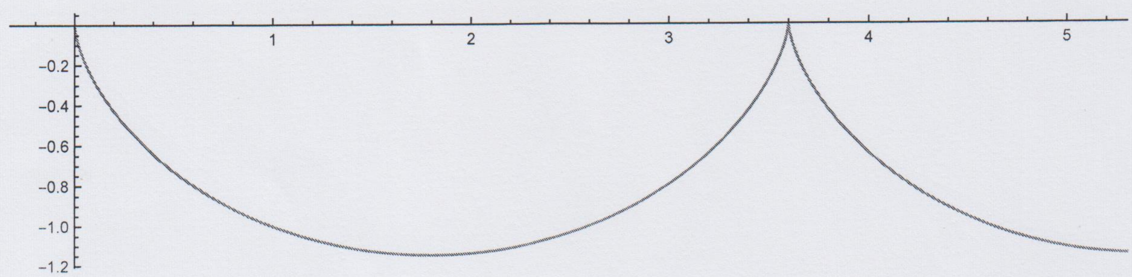
$$x = \alpha^2(\theta - \frac{1}{2}\sin 2\theta), \quad y = \frac{1}{2}\alpha^2(1 - \cos 2\theta), \quad (0 \le \theta \le \theta_1),$$

where α and θ_1 can be expressed in terms of x_1 and y_1 from the condition that $x = x_1, y = y_1$ when $\theta = \theta_1$. The solution is the arc of a cycloid. A sketch is shown in figure 4. Recall that y is measured downwards. The resulting shape is a compromise between travelling the shortest distance (a straight line) and achieving the highest speed (moving vertically downwards and then horizontally).

```
In[7]:= θ1 = θ /. FindRoot[2 (θ - (1 / 2) Sin[2 θ]) - (1 - Cos[2 θ]) == 0, {θ, 0.5, 1.5}]

Out[7]= 1.20601

In[8]:= x[θ_] := (θ - (1 / 2) Sin[2 θ]) / ((θ1 - (1 / 2) Sin[2 θ1]))

In[9]:= y[θ_] := (1 / 2) (1 - Cos[2 θ]) / ((θ1 - (1 / 2) Sin[2 θ1]))

In[11]:= ParametricPlot[{x[θ], -y[θ]}, {θ, 0, θ1}]

Out[11]=
```




```
In[12]:= ParametricPlot[{x[θ], -y[θ]}, {θ, 0, 5}]

Out[12]=
```



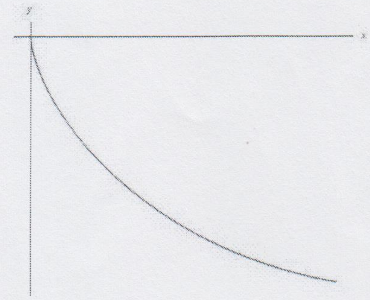


Figure 4: The curve of quickest descent under gravity

Example 3 revisited: *minimal surface of revolution*
Here we want to minimize the area

$$\mathcal{A} = 2\pi \int_{x_1}^{x_2} x\left(1 + (y')^2\right)^{1/2} dx.$$

We take $L = x\left(1 + (y')^2\right)^{1/2}$, which is explicitly independent of y (case 1). Hence the E-L equation is $\partial L/\partial y' =$ constant, i.e.

$$x \tfrac{1}{2}(2y')(1 + (y')^2)^{-1/2} = \beta$$

This can be rearranged into the form

$$y' = \pm \frac{\beta}{(x^2 - \beta^2)^{1/2}}$$

which can be integrated to give

$$y = \pm\beta\cosh^{-1}(x/\beta) + \gamma.$$

When written in the form $x = x(y)$ this curve is known as a **catenary**. The curve has the shape shown on the left in figure 5. On the right we show a sample surface of revolution linking two circles of different radii - the surface is known as a **catenoid**.

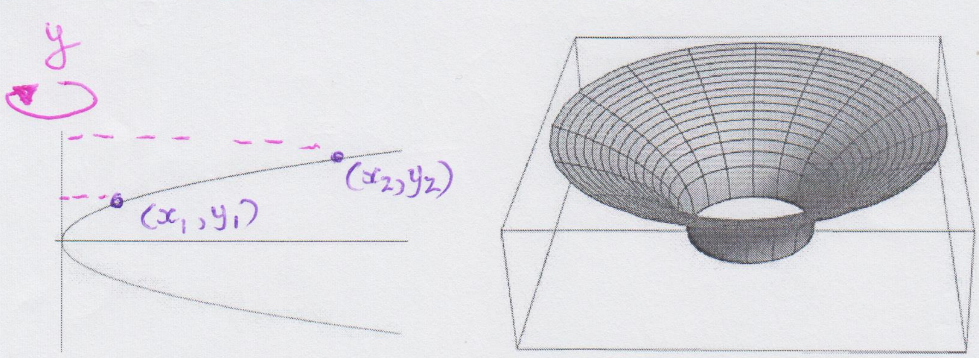
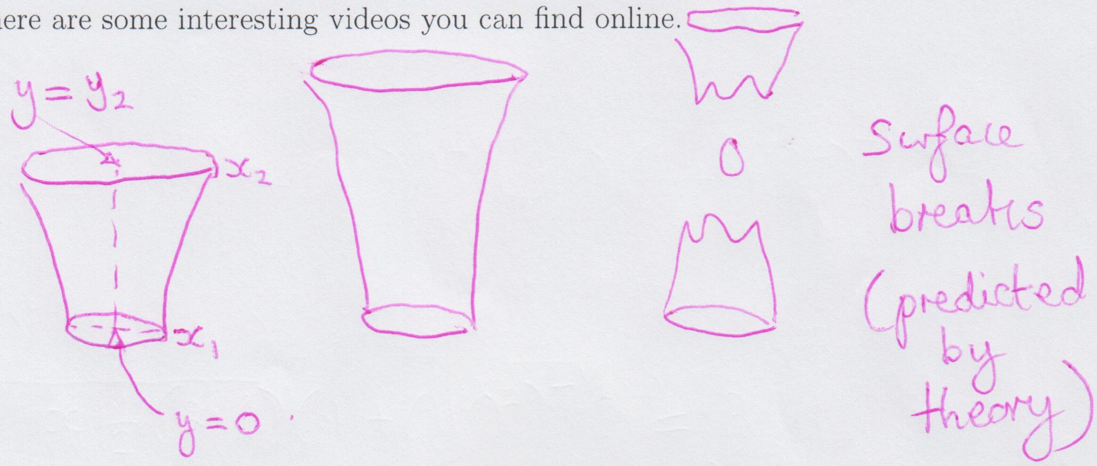


Figure 5: Left: the catenary curve $x = \cosh y$. Right: a surface of revolution formed from a section of a catenary.

Recall that the boundary conditions are such that $y(x_1) = y_1, y(x_2) = y_2$ and we can take $y_1 = 0$ without loss of generality so that one of our rings lies in the plane $y = 0$. We therefore need to choose β and γ such that

$$x_1 = \beta \cosh\left(\frac{\gamma}{\beta}\right), \quad x_2 = \beta \cosh\left(\frac{y_2 - \gamma}{\beta}\right).$$

However for some boundary conditions this is not possible: in particular if x_1 and x_2 are small, but y_2 is large. This means that there is no continuous minimal surface between small rings a large distance apart. This has applications to soap films among other things and there are some interesting videos you can find online.



2.5 Extension of the Euler-Lagrange equation to more variables

Suppose we now have an integral of the form

$$I = \int_{t_1}^{t_2} L(t, x_1(t), x_2(t), \ldots, x_n(t), x_1'(t), x_2'(t), \ldots x_n'(t))\, dt$$

so that L is a scalar function of $(2n+1)$ variables. For simplicity let's write

$$\mathbf{x} = (x_1(t), x_2(t), \ldots, x_n(t)), \;\; \mathbf{x}' = (x_1'(t), x_2'(t), \ldots x_n'(t))$$

If we suppose that the extremal solution is

$$\mathbf{X} = (X_1(t), X_2(t), \ldots, X_n(t)),$$

then in a similar way to our earlier proof we can consider a perturbation to this solution of the form

$$\mathbf{x}(t, \varepsilon) = \mathbf{X}(t) + \varepsilon \boldsymbol{\eta}(t)$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots \eta_n)$ is a smooth $n-$dimensional vector function of t, with $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0$. We then seek a solution for which

$$dI/d\varepsilon = 0 \text{ when } \varepsilon = 0.$$

Thus $$0 = \int_{t_1}^{t_2} \frac{d}{d\varepsilon} L(t, \underline{X} + \varepsilon \underline{\eta}, \underline{X}' + \varepsilon \underline{\eta}')\Big|_{\varepsilon=0} dt$$
$$= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\eta_i \frac{\partial L}{\partial X_i} + \eta_i' \frac{\partial L}{\partial X_i'} \right) dt$$

using the chain rule. We can integrate by parts to get

$$0 = \sum_{i=1}^{n} \left(\int_{t_1}^{t_2} \eta_i \frac{\partial L}{\partial X_i}\, dt + \cancel{\left[\eta_i \frac{\partial L}{\partial X_i'} \right]_{t_1}^{t_2}} - \int_{t_1}^{t_2} \eta_i \frac{d}{dt}\left(\frac{\partial L}{\partial X_i'} \right) dt \right)$$

Since $\eta_i(t_1) = \eta_i(t_2) = 0$ for all i, this reduces to

$$0 = \sum_{i=1}^{n} \int_{t_1}^{t_2} \eta_i \left(\frac{\partial L}{\partial X_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial X_i'} \right) \right) dt$$

Since the η_i are arbitrary smooth functions, the Vanishing Lemma implies that

$$\frac{\partial L}{\partial X_i} - \frac{d}{dt} \frac{\partial L}{\partial X_i'} = 0 \qquad (2)$$

for all $i = 1, 2, \ldots, n$. Thus rather than having one E-L equation we now have a set of n simultaneous E-L equations to solve for the function $\mathbf{X} = (X_1, X_2, \ldots, X_n)$.

Example 4. A trivial example of this is to consider the area $\mathcal{A}$ enclosed by a simple closed curve in the $x-y$ plane. In Part 1 on Green's theorem we showed that if the boundary is denoted by C, then

$$\mathcal{A} = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Writing this in parametric form:

$$A = \frac{1}{2}\int_{t_1}^{t_2}\left(x(t)y'(t) - y(t)x'(t)\right)dt$$

So here we have $\mathbf{x} = (x, y)$ and we can apply the theory above to find the closed curve which extremizes the area. We therefore need to solve the simultaneous E-L equations

$x_1 \equiv x$
$x_2 \equiv y$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'} = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'} = 0,$$

where

$$L(t, x, y, x', y') = \frac{1}{2}xy' - \frac{1}{2}yx'.$$

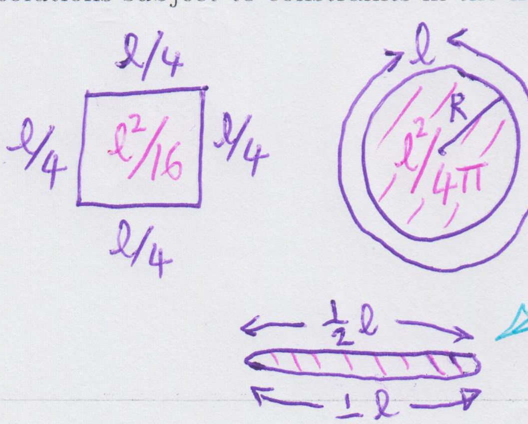
Substituting for L the equations become

$$\frac{1}{2}y' - \frac{d}{dt}\left(-\frac{1}{2}y\right) = y' = 0 \,; \quad -\frac{1}{2}x' - \frac{d}{dt}\left(\frac{1}{2}x\right) = -x' = 0$$

In this case we can see that the only solution is that x and y are both constant. in other words the E-L equation has led us to the minimum area of zero which is obtained by shrinking the curve C to a point. This of course is self-evident but the problem becomes more interesting if we restrict our attention to closed curves that have a fixed length l say. This is equivalent to imposing the arclength constraint

$$\int_{t_1}^{t_2}\left((x'(t))^2 + (y'(t))^2\right)^{1/2} dt = l$$

We would then hope to obtain a non-trivial answer to our problem of maximising/minimizing $\mathcal{A}$. We will return to this problem later. This example motivates our study of finding extremal solutions subject to constraints in the next section.



$l = 2\pi R$
$A = \pi R^2$
$= \pi\left(\frac{l}{2\pi}\right)^2$

looks like $A_{min} = 0$
but is there a max
and what
shape gives it?

2.6 Variational problems involving constraints

We will start with the 1D case again as it is easier to visualize before generalizing to vector functions. Suppose we wish to find the curve $y(x)$ with $y(x_1) = y_1, y(x_2) = y_2$ such that

$$I = \int_{x_1}^{x_2} L(x, y, y')\, dx$$

is stationary, and

$$J = \int_{x_1}^{x_2} g(x, y, y')\, dx$$

is a fixed constant, J_0 say. As usual, L and g are known functionals. As before we consider a family of functions

$$y(x,\varepsilon) = Y(x) + \varepsilon\eta(x)$$

where $Y(x)$ is the desired solution to the problem and η is a smooth function which satisfies $\eta(x_1) = \eta(x_2) = 0$ so that each member of the family passes through the end points. We therefore have

$$I(\varepsilon) = \int_{x_1}^{x_2} L(x, Y+\varepsilon\eta, Y'+\varepsilon\eta')\, dx$$

and

$$J(\varepsilon) = \int_{x_1}^{x_2} g(x, Y+\varepsilon\eta, Y'+\varepsilon\eta')\, dx = J_0$$

We want I to be stationary and so

$$I'(0) = 0$$

J is a constant and so in particular

$$J'(0) = 0$$

Calculating $I'(0)$ and $J'(0)$ by the same method as in the unconstrained case we arrive at the following conclusion:

{①} is orthogonal to everything that is orthogonal to {②}

$$\int_{x_1}^{x_2} \eta(x) \overbrace{\left\{ \frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) \right\}}^{①} dx = 0$$

for all smooth functions $\eta(x)$ vanishing at the end points which satisfy

$$\int_{x_1}^{x_2} \eta(x) \overbrace{\left\{ \frac{\partial g}{\partial Y} - \frac{d}{dx}\left(\frac{\partial g}{\partial Y'}\right) \right\}}^{②} dx = 0.$$

If follows (see problem sheet 5) that there exists a scalar λ (a **Lagrange multiplier**) such that

{①} & {②} are co-linear.

$$\frac{\partial L}{\partial Y} - \frac{d}{dx}\left(\frac{\partial L}{\partial Y'}\right) = -\lambda\left(\frac{\partial g}{\partial Y} - \frac{d}{dx}\left(\frac{\partial g}{\partial Y'}\right)\right)$$

(− sign for convenience)

and hence we have

$$\frac{\partial}{\partial Y}(L+\lambda g) - \frac{d}{dx}\left(\frac{\partial}{\partial Y'}(L+\lambda g)\right) = 0. \qquad (3)$$

We therefore retain the familiar Euler-Lagrange equation but with L simply replaced by $L+\lambda g$. As before we will now use y rather than Y to denote the (constrained) extremal curve.

The solution procedure is as follows: if we solve equation (3) we obtain $y = y(x, \lambda, C_1, C_2)$ where C_1, C_2 are constants of integration. Then applying the boundary conditions we can reduce this to $y = y(x, \lambda)$. Finally, substituting into the integral constraint will give us the value of λ.

Example 5

Find the form of $y(x)$ which extremizes the integral

$$I = \int_0^{\pi/2} (y')^2 - y^2 + 2xy\, dx$$

subject to $y(0) = y(\pi/2) = 0$ and the constraint $\int_0^{\pi/2} y\, dx = \pi^2/8$.

We have $L = (y')^2 - y^2 + 2xy$; $g = y$

E-L for $L + \lambda g$:

$$\frac{\partial}{\partial y}\left((y')^2 - y^2 + 2xy + \lambda y\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left((y')^2 - y^2 + 2xy + \lambda y\right)\right) = 0$$

$$-2y + 2x + \lambda - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow y'' + y = x + \tfrac{1}{2}\lambda \Rightarrow y = \overbrace{A\cos x + B\sin x}^{\text{Homo soln.}} + \overbrace{x + \tfrac{1}{2}\lambda}^{\text{P.S.}}$$

End condns. $y(0) = 0 \Rightarrow A = -\frac{1}{2}\lambda$

$y(\frac{\pi}{2}) = 0 \Rightarrow B = -\frac{\pi}{2} - \frac{1}{2}\lambda$

$$\therefore y = -\tfrac{1}{2}\lambda(\cos x - 1) - \left(\tfrac{\pi}{2} + \tfrac{1}{2}\lambda\right)\sin x + x$$

To find λ subst into $\int_0^{\pi/2} y\, dx = \pi^2/8$

$$\Rightarrow \int_0^{\pi/2} -\tfrac{1}{2}\lambda(\cos x - 1) - \left(\tfrac{\pi}{2} + \tfrac{\lambda}{2}\right)\sin x + x\, dx = \pi^2/8$$

$$\Rightarrow \left[-\tfrac{1}{2}\lambda(\sin x - x) + \left(\tfrac{\pi}{2} + \tfrac{\lambda}{2}\right)\cos x + \tfrac{x^2}{2}\right]_0^{\pi/2} = \pi^2/8$$

$$-\tfrac{1}{2}\lambda\left(1 - \tfrac{\pi}{2}\right) - \left(\tfrac{\pi}{2} + \tfrac{\lambda}{2}\right) + \cancel{\tfrac{\pi^2}{8}} = \cancel{\tfrac{\pi^2}{8}}$$

$$\Rightarrow \lambda = -\pi/(2 - \pi/2) \quad \text{(Check)}.$$

$$\therefore y = \frac{\pi}{4-\pi}(\cos x - 1) - \left(\frac{\pi}{2} - \frac{\pi}{4-\pi}\right)\sin x + x$$

2.7 Extension of the constrained case to more variables

As in the unconstrained case the method can easily be extended to problems in which we want to find the extremal solution $\mathbf{x}(t)$ (where $\mathbf{x}$ is an $n-$dimensional vector) of an integral

$$I = \int_{t_1}^{t_2} L(t, \mathbf{x}(t), \mathbf{x}'(t))\, dt$$

subject to the constraint

$$J = \int_{t_1}^{t_2} g(t, \mathbf{x}(t), \mathbf{x}'(t))\, dt = J_0.$$

As before we need to solve n simultaneous E-L equations, but now they are for the functional $L + \lambda g$, i.e.

$$\frac{\partial}{\partial X_i}(L + \lambda g) - \frac{d}{dt}\frac{\partial}{\partial X_i'}(L + \lambda g) = 0$$

for $i = 1, \ldots, n$.

Example 4 revisited.

Let's return to example 4 where we computed the area enclosed by a simple closed curve but now let us impose the constraint that the length of the curve is fixed. Our problem is to find a relation between $x(t), y(t)$ such that the area

$$\mathcal{A} = \frac{1}{2}\int_{t_1}^{t_2} (x(t)y'(t) - y(t)x'(t))\, dt$$

is rendered stationary, subject to

$$\int_{t_1}^{t_2} (x'(t)^2 + y'(t)^2)^{1/2} dt = l,$$

where l is a constant representing the length of the closed curve. For this problem the minimum area of zero is clearly achieved if the curve collapses to a straight line. We might hope that a variational approach to the constrained problem leads to the determination of the curve that encloses the *maximum* area. We apply the Euler-Lagrange equations

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0, \quad \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0$$

to the functional $f = L + \lambda g$ where

$$L = \frac{1}{2}xy' - \frac{1}{2}yx' \; ; \quad g = \left((x')^2 + (y')^2\right)^{1/2}$$

y'

The equations become

$\equiv -x'$

$$\frac{1}{2}y' - \frac{d}{dt}\left(-\frac{1}{2}y\right) - \lambda\frac{d}{dt}\left(x'\left((x')^2+(y')^2\right)^{-1/2}\right) = 0 \; ;$$

$$-\frac{1}{2}x' - \frac{d}{dt}\left(\frac{1}{2}x\right) - \lambda\frac{d}{dt}\left(y'\left((x')^2+(y')^2\right)^{-1/2}\right) = 0$$

Integrating we obtain

$$y - \lambda\left(\frac{x'}{((x')^2+(y')^2)^{1/2}}\right) = b \; ; \quad -x - \lambda\frac{y'}{((x')^2+(y')^2)^{1/2}} = -a$$

where a and b are constants. Squaring and adding we find that

$$(y-b)^2 + (x-a)^2 = \frac{\lambda^2(x')^2 + \lambda^2(y')^2}{(x')^2+(y')^2} = \lambda^2$$

and so the extremal curve is a circle of radius λ. Since the perimeter is fixed equal to l then we must have $\lambda = l/2\pi$ and therefore $\mathcal{A} = l^2/4\pi$. From what we have said earlier we expect this curve maximizes (rather than minimizes) the area enclosed and this is indeed the case: the circle gives the largest area for a fixed perimeter l. Thus for any simple closed curve we have the **isoperimetric inequality**

$$4\pi\mathcal{A} \leq l^2,$$

where equality holds only when the curve is a circle.

2.8 The Euler-Lagrange equation for higher-dimensional integrals

In the final part of Chapter 1 we showed that the area of surface of a function $z = f(x, y)$ is given by the integral

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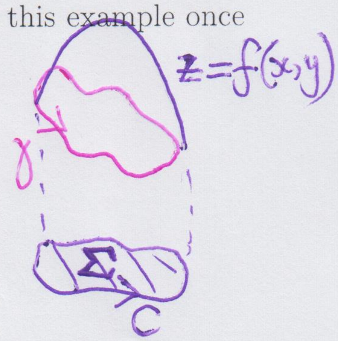
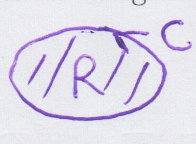
$$I = \int_{\Sigma} (1 + |\nabla f|^2)^{1/2}\, dx\, dy$$

where Σ is the projection of the surface onto the $x - y$ plane. Suppose that the surface is bounded by a closed curve γ lying in 3D space. If a wire loop is bent into this shape and dipped into a soap solution, a film will form. It turns out that the soap film will assume a shape which has the least surface area, at least locally, compared to all other surfaces that span the wire loop. If we want to find this shape we need to find the function f which minimizes I. Since I is a surface integral, if we want to use a variational approach we need to extend our Euler-Lagrange formulation. We will return to this example once we have derived the general theory.



2.8.1 Euler-Lagrange theory for surface integrals

We consider integrals of the form



$$I = \int_R L(\mathbf{r}, f(\mathbf{r}), \nabla f(\mathbf{r}))\, dx\, dy$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector in $\mathbb{R}^2$. Let C denote the boundary of R and suppose f is prescribed on C. Suppose $F(\mathbf{r})$ is the extremal function we are trying to find. Consider a family of functions

$$f(\mathbf{r}) = F(\mathbf{r}) + \varepsilon\eta(\mathbf{r}),$$

where η is a smooth function which vanishes on C so that all members of the family take on the same prescribed values on the boundary. We write

$$I(\varepsilon) = \int_R L(\mathbf{r}, F + \varepsilon\eta, \nabla F + \varepsilon\nabla\eta)\, dx\, dy.$$

Since we require I to be stationary when $\varepsilon = 0$ we have

$$I'(0) = 0$$

as in our earlier formulations. Using the chain rule:

$$\frac{dI}{d\varepsilon} = \int_R \left(\eta\frac{\partial L}{\partial f} + \nabla\eta \cdot \nabla_{\nabla f} L\right) dx\, dy. \tag{4}$$

Here we adopt the notation

$$\nabla_{\mathbf{p}} \equiv \mathbf{i}\frac{\partial}{\partial p_1} + \mathbf{j}\frac{\partial}{\partial p_2}$$

for any vector $\mathbf{p}$ in $\mathbb{R}^2$ and we have used the result from early in the course (Sheet 1 Q3) that

$$\frac{d}{d\varepsilon} f(\mathbf{g}(\varepsilon)) = \mathbf{g}'(\varepsilon) \cdot \nabla_{\mathbf{g}} f.$$

Setting $\varepsilon = 0$ in (4) we therefore have

$$0 = \int_R \left(\eta \frac{\partial L}{\partial F} + \boldsymbol{\nabla}\eta \cdot \boldsymbol{\nabla}_{\boldsymbol{\nabla} F} L \right) dx\, dy. \tag{5}$$

Now since η vanishes on the boundary C of R, the divergence theorem tells us that

$$\int_R \boldsymbol{\nabla}\eta \cdot \mathbf{A}\, dx\, dy = -\int_R \eta\, \mathrm{div}\mathbf{A}\, dxdy$$

for any vector field $\mathbf{A}$ (see Problem Sheet 3, Q1). Thus choosing

$$\mathbf{A} = \boldsymbol{\nabla}_{\boldsymbol{\nabla} F} L,$$

(5) can be rewritten in the form

$$\int_R \eta \left(\frac{\partial L}{\partial F} - \mathrm{div}(\boldsymbol{\nabla}_{\boldsymbol{\nabla} F} L) \right) dxdy = 0.$$

Since η is arbitrary, and using an appropriate extension of the Vanishing Lemma to higher dimensions, we conclude that

$$\frac{\partial L}{\partial F} - \mathrm{div}(\boldsymbol{\nabla}_{\boldsymbol{\nabla} F} L) = 0, \tag{6}$$

which is the generalization of the Euler-Lagrange equation we derived for 1D integrals. Again, henceforth we use f rather than F to denote the extremal function.

2.8.2 Remarks

(i) The equation holds for volume integrals and in fact also for n-dimensional integrals.
(ii) Constraints can be accommodated in a similar way to before.

Example 6
We conclude by revisiting the minimal surface area (soap film) example. Here we wish to minimize the integral

$$I = \int_{\Sigma} (1 + |\nabla f|^2)^{1/2}\, dx\, dy$$

and so

$$L = (1 + |\nabla f|^2)^{1/2},$$

which is explicitly independent of position $\mathbf{r}$ and the function f. The E-L equation (6) therefore becomes

$$\operatorname{div}\left(\underline{\nabla}_{\underline{\nabla} f} L\right) = 0$$

$\frac{\partial f}{\partial x} = f_x$ etc.

Writing $\nabla f = (f_x, f_y)$ we have

$$\underline{\nabla}_{\underline{\nabla} f} L = \left(\hat{\imath}\frac{\partial}{\partial f_x} + \hat{\jmath}\frac{\partial}{\partial f_y}\right)(1 + \underbrace{f_x^2 + f_y^2}_{|\underline{\nabla} f|^2})^{1/2}$$

$$= (f_x\hat{\imath} + f_y\hat{\jmath})(1 + f_x^2 + f_y^2)^{-1/2} = \frac{\underline{\nabla} f}{(1 + |\underline{\nabla} f|^2)^{1/2}}$$

and so the minimal surface equation is

$$\operatorname{div}\left(\frac{\underline{\nabla} f}{(1 + |\underline{\nabla} f|^2)^{1/2}}\right) = 0$$

After some algebra (problem sheet 5) the equation can be written as the following non-linear second order partial differential equation:

$$(1 + f_y^2) f_{xx} + (1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} = 0.$$

2nd order nonlinear $\Rightarrow$ multiple solutions

Some solutions to this equation are investigated on sheet 5.

Take a look at youtube for minimal surface videos.

Classical Dynamics in Y3 uses E-L.

trivially $f = ax + by$ satisfies this equation. (flat solution).