Imperial College
London

Department of Mathematics

M45A4 Mathematical Physics I: Quantum Mechanics

Question	Examiner's Comments
Q 1	This question was intended to be fairly straight forward, and seems to have in fact been perceived this way. A few people forgot the sketch in d) and a common mistake (apart from errors in calculations) was to use the ground state instead of the first excited state in part d) - I didn't deduce too many marks for this, but wasn't overly generous either, as the whole question was on the easier side. Further, in part d) things could have been very easy using the symmetry rather than calculating the probabilities, but most candidates didn't do that. Of course doing the calculations the longer way still gave full marks.
Q 2	This question caused the most problems for most candidates. In particular part c) was intended to be among the most challenging bits in the exam, but perhaps it was even a bit more challenging than I had thought. Only very few students gave a complete or almost complete solution. Even on part c) (i) which was partly discussedd in the lecture. Part a and b was fine for most. Unfortunately it seems to have been a common issue too lose a lot of time (judging from the amount that has been written) on failed attempts to part c).
Q 3	Question 3 was meant to assess the general understanding of the principles of QM without too much tedious calculations. Judging from the outcome, I hope that this was achieved and that most of the candidates did in fact understand the subtleties of the measurement principle and the time evolution etc. For some candidates part c revealed a lack of familiarity with and/or understanding of the principles of QM...
Q 4	Question Question 4 was a guided tour through the algebraic derivation of the harmonic oscillator spectrum, which had been discussed in class. Most candidates did very well here, which, I believe, is an indication that they have indeed learned and understood this important example of quantum mechanics. The most common issue in this question was not to consider the eigenvalue 0 in part b (i) or (ii) (leading to a loss of one mark only) and the details of what could be inferred from parts b (i)-(iii) in b (iv), where I was rather stingy with the marks. In part c) a common issue was to try to deduce the ground state from the eigenvalue equation, which would have been hard, rather from the fact that the operator a maps the ground state to zero.
Q 5	The mastery question was solved very smoothly and successfully by some candidates, but apparently caused problems for others. I was very generous in marking part b) with regards to the expression in p and q (as opposed to z) - as long as the expression in terms of z was fine I gave full marks. Parts a) and b) (counting for half of the whole mark) were discussed in the notes, and were intended to be straight forward. They indeed were for most, but unfortunately not all candidates. Perhaps that was an issue caused by running out of time after attempting the other questions? Part c was more challenging, but there were many good attempts and also complete solutions.

Imperial College
London

M4/5A4

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Physics 1: Quantum Mechanics

Date: Wednesday, 09 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. **The infinite square well potential.**

Consider an infinite square well potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Write down the time-independent one-dimensional Schrödinger equation with zero potential $V = 0$, and an ansatz for its solution.

(b) Use the boundary conditions to determine the eigenfunctions up to a normalisation constant, and deduce the corresponding eigenvalues.

(c) Use the normalisation condition to determine the eigenfunctions up to a phase.

(d) Assume that a particle subject to the above potential is in the first excited state. Make a sketch of the probability distribution of the particle.

(i) When measuring the position of the particle, what is the probability of finding it in the left half of the potential, i.e. in the interval $[0, \frac{L}{2}]$?

(ii) What is the probability of finding it in the centre of the potential, in the interval $[\frac{L}{4}, \frac{3L}{4}]$?

2. Angular momentum.

Consider the three components $\hat{J}_1$, $\hat{J}_2$, $\hat{J}_3$ of the angular momentum operator in three dimensions, fulfilling the commutation relations

$$[\hat{J}_j, \hat{J}_k] = \mathrm{i}\hbar\epsilon_{jkl}\hat{J}_l.$$

(a) Can $\hat{J}_2$ and $\hat{J}_3$ be measured accurately at the same time?

(b) Use the Hadamard lemma

$$\mathrm{e}^{s\hat{X}}\hat{Y}\mathrm{e}^{-s\hat{X}} = \hat{Y} + s[\hat{X}, \hat{Y}] + \frac{s^2}{2!}[\hat{X}, [\hat{X}, \hat{Y}]] + \frac{s^3}{3!}[\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]] + ...,$$

and the angular momentum commutation relations to verify that

$$\begin{aligned}
\mathrm{e}^{i\phi\hat{J}_1/\hbar}\hat{J}_1\mathrm{e}^{-i\phi\hat{J}_1/\hbar} &= \hat{J}_1 \\
\mathrm{e}^{i\phi\hat{J}_1/\hbar}\hat{J}_2\mathrm{e}^{-i\phi\hat{J}_1/\hbar} &= \hat{J}_2\cos(\phi) - \hat{J}_3\sin(\phi) \\
\mathrm{e}^{i\phi\hat{J}_1/\hbar}\hat{J}_3\mathrm{e}^{-i\phi\hat{J}_1/\hbar} &= \hat{J}_3\cos(\phi) + \hat{J}_2\sin(\phi).
\end{aligned}$$

(c) Use the relations in part (b) to

(i) prove that $\hat{J}_2$ and $\hat{J}_3$ have the same eigenvalues.

(ii) express the eigenvalues of $\hat{H} = \hat{J}_2 + \hat{J}_3$ in terms of those of $\hat{J}_3$.

3. **The principles of quantum mechanics - a two-level system.**

Consider a system described by a Hilbert space spanned by the orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$. Let the Hamiltonian $\hat{H}$ and another operator $\hat{A}$ be defined by their actions on the basis states as

$$\hat{H}|\phi_1\rangle = -E|\phi_1\rangle \qquad \hat{H}|\phi_2\rangle = E|\phi_2\rangle$$
$$\hat{A}|\phi_1\rangle = -\frac{\mathrm{i}}{2}|\phi_2\rangle \qquad \hat{A}|\phi_2\rangle = \frac{\mathrm{i}}{2}|\phi_1\rangle\,,$$

with $E \in \mathbb{R}$.

(a) Deduce the matrix representations of $\hat{H}$ and $\hat{A}$ in the basis $\{|\phi_1\rangle, |\phi_2\rangle\}$.

(b) Caclulate the eigenvalues and a set of normalised eigenvectors of $\hat{H}$ and $\hat{A}$ in the basis $\{|\phi_1\rangle, |\phi_2\rangle\}$.

(c) Assume that at time $t = 0$ the system is in the state

$$|\psi\rangle = |\phi_1\rangle.$$

(i) If we measure the observable $\hat{A}$ at time $t = 0$ with what probability do we obtain which result? What is the expectation value of $\hat{A}$?

(ii) Assume that the measurement of the observable A at time $t = 0$ has yielded the outcome $\frac{1}{2}$. What is the state at a later time $t > 0$? What is the probability that a subsequent measurement (at time $t > 0$) yields the same result $\frac{1}{2}$?

4. **The quantum harmonic oscillator.**

Consider the quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right),$$

with the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, fulfilling $[\hat{a}, \hat{a}^\dagger] = 1$, and the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$.

(a) Calculate the commutators $[\hat{N}, \hat{a}]$, and $[\hat{N}, \hat{a}^\dagger]$.

(b) Let $|\nu\rangle$ denote an eigenvector of $\hat{N}$ with corresponding eigenvalue ν.

(i) Prove that the eigenvalues of $\hat{N}$ are non-negative, by considering the norm of the vector $\hat{a}|\nu\rangle$.

(ii) Prove that $\hat{a}|\nu\rangle$ is either the zero vector or an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu - 1$.

(iii) Prove that $\hat{a}^\dagger|\nu\rangle$ is an eigenvector of $\hat{N}$ belonging to the eigenvalue $\nu + 1$.

(iv) What can you conclude about the possible eigenvalues of $\hat{N}$ and thus those of the harmonic oscillator Hamiltonian from the three statements you have just proven?

(c) Remembering that $\hat{a}$ in terms of position and momentum reads

$$\hat{a} := \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{q} + \mathrm{i}\sqrt{\frac{1}{m\omega\hbar}}\hat{p}\right),$$

find the position representation of the ground state of the harmonic oscillator (up to a normalisation constant).

5. **Mastery question - The Husimi distribution**

(a) A coherent state $|z\rangle$ can be represented as

$$|z\rangle = \hat{D}(z)|0\rangle,$$

where $|0\rangle$ is the ground state of a harmonic oscillator and $\hat{D}(z)$ is given by

$$\hat{D}(z) = \mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{e}^{z\hat{a}^\dagger}\mathrm{e}^{-z^*\hat{a}},$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the ladder operators of the harmonic oscillator, and $z \in \mathbb{C}$. Use this to show that in the harmonic oscillator basis $|z\rangle$ has the form

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle.$$

(b) Calculate the Husimi distributions

$$\rho_H(p,q) = \frac{1}{2\pi\hbar}|\langle z|\psi\rangle|^2, \quad \text{with} \quad z = \frac{1}{\sqrt{2}}(q + \mathrm{i}p),$$

of the states $|\psi_1\rangle = |0\rangle$, and $|\psi_2\rangle = |3\rangle$ (the ground states and the third excited state of the harmonic oscillator).

(c) In analogy to the coherent states of the harmonic oscillator one can define $SU(2)$ or angular momentum states for a given total angular momentum j as

$$|\zeta\rangle = N(\zeta)\mathrm{e}^{\zeta\hat{J}_+}|j,-j\rangle,$$

where $|j,-j\rangle$ is an angular momentum eigenstate with

$$\hat{J}^2|j,-j\rangle = j(j+1)|j,-j\rangle, \quad \text{and} \quad \hat{J}_3|j,-j\rangle = -j|j,-j\rangle,$$

and where $N(\zeta) = (1+|\zeta|^2)^{-j}$ is a normalisation factor. The complex variable ζ parameterises the spherical phase space. It can be related to spherical coordinates via the parameterisation $\zeta = -\mathrm{e}^{-\mathrm{i}\phi}\tan(\frac{\theta}{2})$. We can then also define a Husimi distribution on the sphere as

$$\rho_H(\theta,\phi) = \frac{1}{4\pi}|\langle\zeta|\psi\rangle|^2.$$

Consider the case $j = \frac{1}{2}$.

(i) Use that $\hat{J}_+|j,m\rangle = \sqrt{j(j+1) - m(m+1)}|j,m+1\rangle$ to express the coherent states in the standard basis $|\frac{1}{2},-\frac{1}{2}\rangle$, $|\frac{1}{2},\frac{1}{2}\rangle$.

(ii) Calculate the Husimi distributions (as functions of ϕ and θ) of the two basis states.

Quantum Mechanics 2017/18 Exam Solutions

1. The infinite square well

(Parts (a) to (c) seen in class, part (d) unseen, but standard material)

(a) The time-independent Schrödinger equation inside the well reads

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) = E\phi(x).$$

The solutions of the time-independent Schrödinger equation for vanishing potential is of the form $ae^{ikx} + be^{-ikx}$ with $k = \sqrt{2m(E)}/\hbar$, or equivalently

$$A\cos(kx) + B\sin(kx).$$

(3 points)

(b) The wave function $\phi_E(x)$ has to be continuous at the boundary, that is,

$$\phi(0) = 0 = A,$$

and

$$\phi(L) = 0 = A\cos(kL) + B\sin(kL).$$

Thus, the only non-trivial solutions are of the form

$$\phi(x) = B\sin(kL),$$

with the quantisation condition $kL = n\pi$, with integer n. Using that $k = \sqrt{2m(E)}/\hbar$ this yields the quantisation condition for the energy

$$E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2,$$

and the corresponding wave functions are

$$\phi_n(x) = B\sin\left(\frac{\pi nx}{L}\right).$$

(5 points)

(c) The norm of $\phi_n(x)$ in dependence on B is given by

$$\begin{aligned}\int_0^L |\phi_n(x)|^2 &= |B|^2\int_0^L \sin^2\left(\frac{\pi nx}{L}\right)\mathrm{d}x \\ &= \frac{|B|^2}{2}\int_0^L \left(1-\cos\left(\frac{2\pi nx}{L}\right)\right)\mathrm{d}x \\ &= \frac{|B|^2L}{2}.\end{aligned}$$

Thus, the choice $B = \sqrt{\frac{2}{L}}$ yields the normalised and real-valued wave functions

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi nx}{L}\right).$$

(4 points)

(d) Figure 1 shows a sketch of the probability distribution of the particle.

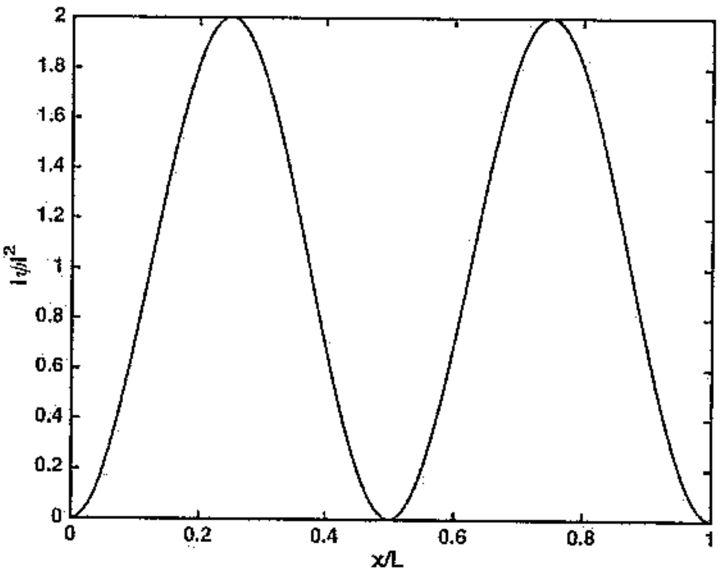

Figure 1: Sketch of the probability distribution

(2 points)

(i) Due to the symmetry of the wave function the probability is $P = \frac{1}{2}$. Alternatively one could calculate

$$\begin{aligned}
P &= \int_0^{\frac{L}{2}} |\phi_2(x)|^2 \mathrm{d}x \\
&= \frac{2}{L} \int_0^{\frac{L}{2}} \sin^2\left(\frac{2\pi x}{L}\right) \mathrm{d}x \\
&= \frac{1}{L} \int_0^{\frac{L}{2}} \left(1 - \cos\left(\frac{4\pi x}{L}\right)\right) \mathrm{d}x \\
&= \frac{1}{2}.
\end{aligned}$$

(3 points)

(ii) Again, we can deduce the probability from the symmetry of the wave function as $P = \frac{1}{2}$. Alternatively one could calculate

$$\begin{aligned}
P &= \int_{\frac{L}{4}}^{\frac{3L}{4}} |\phi_2(x)|^2 \mathrm{d}x \\
&= \frac{2}{L} \int_{\frac{L}{4}}^{\frac{3L}{4}} \sin^2\left(\frac{2\pi x}{L}\right) \mathrm{d}x \\
&= \frac{1}{L} \int_{\frac{L}{4}}^{\frac{3L}{4}} \left(1 - \cos\left(\frac{4\pi x}{L}\right)\right) \mathrm{d}x \\
&= \frac{1}{2}.
\end{aligned}$$

(3 points)

2. Angular momentum.

(Part (b) seen in class, part (c) (i) seen similar in class part (c) (ii) unseen)

(a) $\hat{J}_2$ and $\hat{J}_3$ can in general not be measured accurately at the same time, as they do not commute (their commutator is non-zero).

(2 points)

(b) The first equation holds trivially, since the $e^{\pm i\phi \hat{J}_1}$ commute with $\hat{J}_1$. For the other two expressions we need to calculate the nested commutators $[\hat{J}_1, [\hat{J}_1, [..., [\hat{J}_1, \hat{J}_{2,3}]...]]]$. This can be done iteratively. For the commutators with $\hat{J}_2$ we start from $[\hat{J}_1, \hat{J}_2] = i\hbar \hat{J}_3$ and find

$$\begin{aligned}
[\hat{J}_1, [\hat{J}_1, \hat{J}_2]] &= i\hbar[\hat{J}_1, \hat{J}_3] = \hbar^2 \hat{J}_2 \\
[\hat{J}_1, [\hat{J}_1, [\hat{J}_1, \hat{J}_2]]] &= \hbar^2[\hat{J}_1, \hat{J}_2] = i\hbar^3 \hat{J}_3 \\
[\hat{J}_1, [\hat{J}_1, [\hat{J}_1, [\hat{J}_1, \hat{J}_2]]]] &= i\hbar^3[\hat{J}_1, \hat{J}_3] = \hbar^4 \hat{J}_2 \\
&\vdots
\end{aligned}$$

And thus,

$$\begin{aligned}
e^{i\phi \hat{J}_1/\hbar} \hat{J}_2 e^{-i\phi \hat{J}_1/\hbar} &= \hat{J}_2 \left(1 + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^6}{6!} + \dots\right) + i\hat{J}_3 \left(i\phi + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^5}{5!} + \dots\right). \\
&= \hat{J}_2 \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} \pm \dots\right) - \hat{J}_3 \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \mp \dots\right). \\
&= \hat{J}_2 \cos(\phi) - \hat{J}_3 \sin(\phi).
\end{aligned}$$

Similarly we have $[\hat{J}_1, \hat{J}_3] = -i\hbar \hat{J}_2$ and find

$$\begin{aligned}
[\hat{J}_1, [\hat{J}_1, \hat{J}_3]] &= -i\hbar[\hat{J}_1, \hat{J}_2] = \hbar^2 \hat{J}_3 \\
[\hat{J}_1, [\hat{J}_1, [\hat{J}_1, \hat{J}_3]]] &= \hbar^2[\hat{J}_1, \hat{J}_3] = -i\hbar^3 \hat{J}_2 \\
[\hat{J}_1, [\hat{J}_1, [\hat{J}_1, [\hat{J}_1, \hat{J}_3]]]] &= -i\hbar^3[\hat{J}_1, \hat{J}_2] = \hbar^4 \hat{J}_3 \\
&\vdots
\end{aligned}$$

That is,

$$\begin{aligned}
e^{i\phi \hat{J}_1/\hbar} \hat{J}_3 e^{-i\phi \hat{J}_1/\hbar} &= \hat{J}_3 \left(1 + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^6}{6!} + \dots\right) - i\hat{J}_2 \left(i\phi + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^5}{5!} + \dots\right). \\
&= \hat{J}_3 \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} \pm \dots\right) + \hat{J}_2 \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \mp \dots\right). \\
&= \hat{J}_3 \cos(\phi) + \hat{J}_2 \sin(\phi).
\end{aligned}$$

(8 points)

(c) (i) Let $|\phi_\lambda\rangle$ be an eigenvector of $\hat{J}_2$ belonging to the eigenvalue λ, i.e.,

$$\hat{J}_2 |\phi_\lambda\rangle = \lambda |\phi_\lambda\rangle.$$

From (b) we have

$$e^{i\frac{\pi}{2}\hat{J}_1/\hbar} \hat{J}_3 e^{-i\frac{\pi}{2}\hat{J}_1/\hbar} = \hat{J}_2,$$

that is, we have

$$e^{i\frac{\pi}{2}\hat{J}_1/\hbar} \hat{J}_3 e^{-i\frac{\pi}{2}\hat{J}_1/\hbar} |\phi_\lambda\rangle = \lambda |\phi_\lambda\rangle.$$

Acting on this equation with $e^{-i\frac{\pi}{2}\hat{J}_1/\hbar}$ yields

$$\hat{J}_3 e^{-i\frac{\pi}{2}\hat{J}_1/\hbar}|\phi_\lambda\rangle = \lambda e^{-i\frac{\pi}{2}\hat{J}_1/\hbar}|\phi_\lambda\rangle,$$

that is, $e^{-i\frac{\pi}{2}\hat{J}_1/\hbar}|\phi_\lambda\rangle$ is an eigenvector of $\hat{J}_3$ with eigenvalue λ, for any eigenvalue of $\hat{J}_2$. Thus, every eigenvalue of $\hat{J}_2$ is also an eigenvalue of $\hat{J}_3$. The inverse transformation shows that every eigenvalue of $\hat{J}_3$ is also an eigenvalue of $\hat{J}_2$.
Alternatively one could use the cyclic property of the trace and the fact that the traces of all powers of a matrix operator determine its eigenvalues.

(5 points)

(ii) From (b) we have that

$$e^{i\frac{\pi}{4}\hat{J}_1/\hbar}\hat{J}_3 e^{-i\frac{\pi}{4}\hat{J}_1/\hbar} = \frac{1}{\sqrt{2}}\left(\hat{J}_3 + \hat{J}_2\right).$$

Thus, the operators $\hat{J}_3$ and $\frac{1}{\sqrt{2}}\left(\hat{J}_2 + \hat{J}_3\right)$ have the same spectrum (according to the same argument as in part (i)). Therefore if λ_n denote the eigenvalues of $\hat{J}_3$, the eigenvalues of $\hat{H} = \hat{J}_2 + \hat{J}_3$ are given by $\sqrt{2}\lambda_n$.

(5 points)

3. The principles of quantum mechanics

(Seen similar in example exercises and previous exams with 3×3, but I believe this is still a challenging question.)

(a) The matrix representations are

$$\hat{H} = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, \quad \text{and} \quad \hat{A} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}.$$

(2 points)

(b) Since $\hat{H}$ is already diagonal, we can read off the eigenvalues as $\lambda_H = \pm E$. The corresponding eigenvectors are $|\phi_1\rangle$ and $|\phi_2\rangle$. For $\hat{A}$ we find from the characteristic polynomial,

$$\lambda_A^2 - \frac{1}{4} = 0,$$

that is,

$$\lambda_A = \pm\frac{1}{2}.$$

For the corresponding eigenvectors $\phi_\pm = \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix}$ we find from

$$\begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} = \pm\frac{1}{2} \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix},$$

that

$$iy_\pm = \pm x_\pm.$$

Thus, the normalised eigenvectors are given by

$$\phi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \text{and} \quad \phi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

(4 points)

(c) (i) The probability to measure an eigenvalue of an observable is given by the modulus square of the coefficient of the wave function in the basis of the eigenvectors of the observable. We have from (b)

$$|\phi_+\rangle = \frac{1}{\sqrt{2}} \left(|\phi_1\rangle - i|\phi_2\rangle\right),$$

and

$$|\phi_-\rangle = \frac{1}{\sqrt{2}} \left(|\phi_1\rangle + i|\phi_2\rangle\right).$$

Thus we can express $|\phi_1\rangle$ as

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \left(|\phi_+\rangle + |\phi_-\rangle\right).$$

We can now read off the probabilities for the different eigenvalues from the wave function as

$$P(-\frac{1}{2}) = \frac{1}{2}, \quad \text{and} \quad P(\frac{1}{2}) = \frac{1}{2}.$$

(4 points)

The expectation value can either be deduced from $\langle\psi|\hat{A}|\psi\rangle$ via vector and matrix multiplications, or we calculate

$$\langle\hat{A}\rangle = \sum_j P(\lambda_j)\lambda_j = \frac{1}{2}P(\frac{1}{2}) - \frac{1}{2}P(-\frac{1}{2}) = 0.$$

(2 points)

(ii) A measurement of the value $\frac{1}{2}$ at time $t = 0$ projects the system onto the corresponding eigenstate of $\hat{A}$, the vector

$$|\psi(t=0)\rangle = |\phi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix},$$

(2 points)

which is not an eigenstate of $\hat{H}$. The time-dependent wave function is then given by

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}|\psi(0)\rangle.$$

(1 point)

We can use the method of stationary states to find $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}Et/\hbar}|\phi_1\rangle - \frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}Et/\hbar}|\phi_2\rangle,$$

(2 points)

The probability to measure the result $\frac{1}{2}$ again is given by

$$\begin{aligned} P(1) = |\langle\phi_+|\psi(t)\rangle|^2 &= \left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}Et/\hbar} - \frac{\mathrm{i}}{\sqrt{2}}\cdot\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}Et/\hbar}\right|^2 \\ &= \left|\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}Et/\hbar} + \mathrm{e}^{-\mathrm{i}Et/\hbar}\right)\right|^2 \\ &= \cos^2(Et/\hbar), \end{aligned}$$

which oscillates between zero and one.

(3 points)

4. **The harmonic oscillator.**

(Discussed in class)

(a) Recalling that the commutator fulfils $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$, and that $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$ we find for the first commutator

$$[\hat{N},\hat{a}] = [\hat{a}^\dagger\hat{a},\,\hat{a}] = \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a} = -\hat{a},$$

and for the second

$$[\hat{N},\hat{a}^\dagger] = [\hat{a}^\dagger\hat{a},\,\hat{a}^\dagger] = \hat{a}^\dagger[\hat{a},\hat{a}^\dagger] + [\hat{a}^\dagger,\hat{a}^\dagger]\hat{a} = \hat{a}^\dagger.$$

(3 points)

(b) (i) Consider the norm of the vector $\hat{a}|\nu\rangle$. We have

$$\begin{aligned} ||\hat{a}|\nu\rangle||^2 &\geq 0 \\ \langle\nu|\hat{a}^\dagger\hat{a}|\nu\rangle &\geq 0 \\ \langle\nu|\hat{N}|\nu\rangle &\geq 0 \\ \nu\langle\nu|\nu\rangle &\geq 0, \end{aligned} \tag{1}$$

from which it follows that $\nu \geq 0$. Further, for $\nu = 0$ we have that $||\hat{a}|\nu\rangle||^2 = 0$, which means that $\hat{a}|0\rangle$ is the zero vector, we write $\hat{a}|0\rangle = 0$.

(4 points)

(ii)

$$\begin{aligned} \hat{N}\hat{a}|\nu\rangle &= \left(\hat{N}\hat{a} - \hat{a}\hat{N} + \hat{a}\hat{N}\right)|\nu\rangle \\ &= \left([\hat{N},\hat{a}] + \hat{a}\hat{N}\right)|\nu\rangle \\ &= \left(-\hat{a} + \hat{a}\hat{N}\right)|\nu\rangle \\ &= \hat{a}(\hat{N}-1)|\nu\rangle \\ &= (\nu-1)\hat{a}|\nu\rangle. \end{aligned} \tag{2}$$

(3 points)

(iii)

$$\begin{aligned} \hat{N}\hat{a}^\dagger|\nu\rangle &= \left(\hat{N}\hat{a}^\dagger - \hat{a}^\dagger\hat{N} + \hat{a}^\dagger\hat{N}\right)|\nu\rangle \\ &= \left([\hat{N},\hat{a}^\dagger] + \hat{a}^\dagger\hat{N}\right)|\nu\rangle \\ &= \left(\hat{a}^\dagger + \hat{a}^\dagger\hat{N}\right)|\nu\rangle \\ &= \hat{a}^\dagger(\hat{N}+1)|\nu\rangle \\ &= (\nu+1)\hat{a}^\dagger|\nu\rangle. \end{aligned} \tag{3}$$

(3 points)

(iv) Combining the statements (i) and (ii) we conclude that $\nu = 0$ has to be in the spectrum, and no non-integer value of ν could possibly be in the spectrum (to guarantee that the series $\nu, \nu-1, \nu-2, \ldots$ does not continue to negative values). From (iii) we find that starting from $\nu = 0$ via consecutive application of $\hat{a}^\dagger$ that all integers are eigenvalues of $\hat{N}$. Thus, $\hat{N}|n\rangle = n|n\rangle$ with $n = 0, 1, 2, \ldots$. Thus, the eigenvalues of the harmonic oscillator are given by $E_n = \hbar\omega(n + \frac{1}{2})$.

(3 points)

(c) We recall the expressions for $\hat{q}$ and $\hat{p}$ in position representation,

$$\langle x|\hat{q}|\phi\rangle = x\phi(x), \quad \text{and} \quad \langle x|\hat{p}|\phi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial x}\phi(x),$$

to find

$$\langle x|\hat{a}|\phi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right)\phi(x).$$

Thus, the ground-state wave function fulfils the condition

$$\frac{\partial}{\partial x}\phi_0(x) = -\frac{m\omega}{\hbar}x\phi_0(x).$$

Thus, we find

$$\phi_0(x) = \phi_0(x) = c\,\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}, \qquad c \in \mathbb{C}.$$

(4 points)

5. **Mastery question - The Husimi distribution.**

(a) We have $\hat{a}|0\rangle = 0$, thus

$$e^{-z^*\hat{a}}|0\rangle = |0\rangle.$$

Using the Taylor expansion

$$e^{z\hat{a}^\dagger} = \sum_{n=0}^{\infty} \frac{z^n}{n!}\left(\hat{a}^\dagger\right)^n,$$

and the fact that $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$, we find

$$\hat{D}|0\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle.$$

(4 points)

(b) We need to calculate

$$\rho_H(p,q) = \frac{1}{2\pi\hbar}|\langle z|n\rangle|^2.$$

Using the harmonic oscillator basis representation of $|z\rangle$ We have

$$\langle z|n\rangle = e^{-\frac{|z|^2}{2}}\frac{z^n}{\sqrt{n!}},$$

and thus we have for the ground state

$$\rho_H(p,q) = \frac{1}{2\pi\hbar}e^{-|z|^2},$$

and for the third excited state

$$\begin{aligned}\rho_H(p,q) &= \frac{1}{2\pi\hbar}e^{-|z|^2}\frac{|z|^6}{6}\\ &= \frac{1}{96\pi\hbar}e^{-\frac{1}{2}(q^2+p^2)}\left(q^2+p^2\right)^3.\end{aligned}$$

(6 points)

(c) (i) Taylor expanding the exponential we have

$$|\zeta\rangle = N(\zeta)\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\hat{J}_+^n|\tfrac{1}{2},-\tfrac{1}{2}\rangle.$$

Since

$$\hat{J}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle,$$

and

$$\hat{J}_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0,$$

we have

$$|\zeta\rangle = \frac{1}{\sqrt{1+|\zeta|^2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \zeta|\tfrac{1}{2},\tfrac{1}{2}\rangle\right).$$

(5 points)

(ii) We first rewrite the state in terms of θ and ϕ. For this purpose we note that

$$N(\zeta) = \cos\left(\tfrac{\theta}{2}\right),$$

and thus we have

$$|\zeta\rangle = \cos\left(\tfrac{\theta}{2}\right)|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sin\left(\tfrac{\theta}{2}\right)e^{-i\phi}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Now we can calculate the Husimi distribution for the state $|\frac{1}{2}, \frac{1}{2}\rangle$ as

$$\begin{aligned}\rho_H(\theta, \phi) &= \frac{1}{4\pi}|\langle\zeta|\tfrac{1}{2}, \tfrac{1}{2}\rangle|^2 \\ &= \frac{1}{4\pi}|\sin\left(\tfrac{\theta}{2}\right) e^{-i\phi}|^2 \\ &= \frac{\sin^2\left(\frac{\theta}{2}\right)}{4\pi}.\end{aligned}$$

For the state $|\frac{1}{2}, -\frac{1}{2}\rangle$ we find

$$\begin{aligned}\rho_H(\theta, \phi) &= \frac{1}{4\pi}|\langle\zeta|\tfrac{1}{2}, -\tfrac{1}{2}\rangle|^2 \\ &= \frac{\cos^2\left(\frac{\theta}{2}\right)}{4\pi}.\end{aligned}$$

(5 points)