Math40003 Linear Algebra and Groups **Problem Sheet 4**

Seen A

A.1. Exercise 3.6.2: Verify the Steinitz Exchange Lemma where:

- $V = \mathbb{R}^3$
- $X = \{e_1, e_2\}$
- $u = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

You may find Lemma 3.6.1. useful here.

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A.2. Prove Lemma 3.6.8: Suppose that $\dim(V) = n$. Then the following statements are true:

(a) Any spanning set of size n is a basis.

(b) Any linearly independent set of size n is a basis.

(c) S is a spanning set if and only if it contains a basis (as a subset).

(d) S is linearly independent if and only if it is contained in a basis (i.e. it's a subset of a basis).

(e) Any subset of V of size $> n$ is linearly dependent.

The definitions of span, linear independence, basis, and dimension are crucial here. You might also want to use Corollary 3.6.4..

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A.3. Exercise 3.7.4: Let U and W be subspaces of V, a vector space over F. Then $U + W$ and $U \cap W$ are subspaces of V.

Use Definition 3.7.1. for the definition of $+$ and $\cap$ of subspaces, and see how those definitions interact with what it means to be a subspace.

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A.4. Let V be an n-dimensional vector space. Prove that for all $i \leq n$ there is a subspace U of V such that U has dimension i.

Whenever you see the words "an n-dimensional vector space", you should, as a reflex, think "Let B be a basis of V. Then $|B| = n$."

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Seen B

B.1. Let $V = \mathbb{R}^4$. Let

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x_1 + x_2 = x_3 + x_4\} \\ W &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x_1 - 2x_2 = x_3 + 3x_4\}. \end{aligned}$$

Find bases for U, W. $U \cap W$, and $U + W$ such that:

(a) Your basis for U contains your basis for $U \cap W$.
(b) Your basis for W contains your basis for $U \cap W$.
(c) Your basis for $U + W$ contains your basis for U.
(d) Your basis for $U + W$ contains your basis for W.

Find a subspace X such that $U \cap X = \{0_V\} = W \cap X$.

Remember your row operations, and make sure to think about the dimension of these things.

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B.2. (a) Let U and W be 3-dimensional subspaces of $\mathbb{R}^5$, with $U \neq W$. Prove that $\dim U \cap W$ is either 1 or 2. Give examples to show that both possibilities can occur.
(b) Let U_1, U_2 and U_3 be 3-dimensional subspaces of $\mathbb{R}^4$. Give a proof that $\dim U_1 \cap U_2 \geq 2$. Deduce that $U_1 \cap U_2 \cap U_3 \neq \{0_V\}$.
(c) Now let V be the vector space of 2×3 matrices over $\mathbb{R}$. Find subspaces X and Y of V such that $\dim X = \dim Y = 4$, and $\dim X \cap Y = 2$.
(d) Let V be as in part (iii). Do there exist subspaces X and Y of V such that $\dim X = 3$, $\dim Y = 5$, and $\dim X \cap Y = 1$?

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B.3. The *rank* of an $m \times n$ matrix A is defined to be the dimension of its row space $\mathrm{RSp}(A)$ and is denoted by $\mathrm{rank} A$. Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

(a) Let v be a row vector in $\mathbb{R}^n$. Prove that vB is a linear combination of the rows of B.
(b) Prove that the row space of AB is contained in the row space of B and $\mathrm{rank} AB \leq \mathrm{rank} B$.
(c) Prove that if $m = n$ and A is invertible, then $\mathrm{rank} AB = \mathrm{rank} B$.
(d) Prove that $\mathrm{rank} AB \leq \mathrm{rank} A$.

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B.4. (a) Find the rank of the following matrices:

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ 2 & 3 \end{pmatrix}.$$

(b) Find an equation for a and b such that the following matrix has rank 2:

$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & a & -1 \\ 1 & 3 & b \end{pmatrix}.$$

(c) Find an equation for b, c and d such that the matrices

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & -3 & 0 \\ 1 & 1 & 0 & b \\ 2 & -1 & 3 & c \\ 1 & 4 & -2 & d \end{pmatrix}$$

both have the same rank.

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