(3.1.7) Example. The following sets are equinumerous:

1) S_1 = set of all sequences of 0's & 1's $= \{0,1\}^{\mathbb{N}}$

2) $S_2 = \mathbb{R}$

3) $S_3 = \mathcal{P}(\mathbb{N})$

4) $S_4 = \mathcal{P}(\mathbb{N}\times\mathbb{N})$

5) S_5 = set of all sequences of natural number $= \mathbb{N}^{\mathbb{N}}$

Pf: Find injective functions

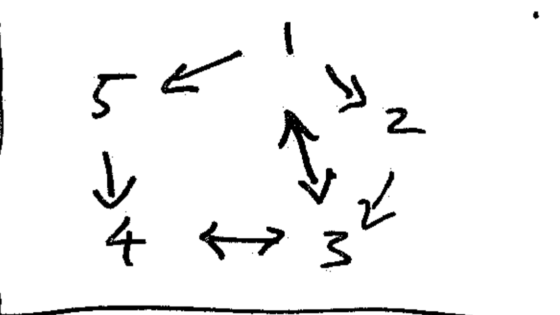
$$f_{i,j} : S_i \to S_j$$

$$(i, j \in \{1, \dots, 5\})$$

Then use 3.1.6.

As $\mathbb{N} \approx \mathbb{N}\times\mathbb{N}$ we get

$$S_3 \approx S_4$$



$$S_1 \subseteq S_5 \subseteq S_4 .$$

= Define $f_{1,2} : S_1 \to \mathbb{R}$ by

$$(a_n)_{n\in\mathbb{N}} \longmapsto 0\cdot a_0 a_1 a_2 \dots$$

= There is a bijection $f_{3,1} : \mathcal{P}(\mathbb{N}) \to S_1$

For $X \subseteq \mathbb{N}$ let

$$f_{3,1}(X) = (a_n)_{n\in\mathbb{N}}$$

$$a_n = \begin{cases} 0 & \text{if } n \notin X \\ 1 & \text{if } n \in X \end{cases}.$$

$\mathbb{Q} \approx \mathbb{N}$

So $\mathcal{P}(\mathbb{Q}) \approx \mathcal{P}(\mathbb{N}) = S_3$

Define an injective function

$$g : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$$

$$r \mapsto \{ q \in \mathbb{Q} : q < r \}$$

Obtain $f_{2,3} : \mathbb{R} \to \mathcal{P}(\mathbb{N})$ which is injective. #.

(3.2) Axioms for Set Theory

Zermelo-Fraenkel Axioms (ZF) Expressed in a 1st order language (with =) using a single 2-ary relation symbol $\in$ $(\mathcal{L}_{\in}^{=})$

ZF Axiom 1-6

ZF 1 (Extensionality)

$$(\forall x)(\forall y)((x = y) \leftrightarrow (\forall z)((z \in x) \leftrightarrow (z \in y)))$$

ZF2 (Empty set axiom)

$$(\exists x)(\forall y)(y \notin x)$$

ZF 3 (Pairing axiom)

"Given sets x, y, can form $\{x, y\}$"

$$(\forall x)(\forall y)(\exists z)(\forall w)((w \in z) \leftrightarrow ((w = x) \vee (w = y)))$$

Remarks. 1) Using ZF 1 + 2 there is a 'unique' set with the property in ZF2 : the empty set $\emptyset$.

2) Using ZF3 can form
$\{\phi, \phi\} = \{\phi\} = 1$.

and $\{\phi, 1\} = 2$

ZF4. Union Axiom

" For any set A there is a set $B = \bigcup A$ "

$B = \bigcup\{z \in A\}$.

ie.

$(\forall A)(\exists B)(\forall x)$
$\Big((x \in B) \Leftrightarrow (\exists z)\big((z \in A) \wedge (x \in z)\big)\Big)$

Eg. If $A = \{x, y\}$
then $B = x \cup y$.

Eg $\{0,1,2\} = \{0,1\} \cup \{2\}$ $= 3$

ZF5 Power Set Axiom.

" If A is any set there is a set $\mathcal{P}(A)$ whose elts. are the subsets of A "

~~Notion~~ Notation: $z \subseteq A$ means

$(\forall y)\big((y \in z) \to (y \in A)\big)$

~~Ax~~ Axiom:

$(\forall A)(\exists B)(\forall z)$
$\big((z \in B) \leftrightarrow (z \subseteq A)\big)$.

ZF6. Axiom scheme of **Specification** (or Comprehension)

Suppose $P(x, y_1, \ldots, y_r)$ is a formula (of $\mathcal{L}^{=}_{\in}$). Then we have an axiom:

$$(\forall A)(\forall y_1)\cdots(\forall y_r)(\exists B)\big((x \in B) \leftrightarrow ((x \in A) \wedge P(x, y_1, \ldots, y_r))\big)$$

i.e. "Given a set A and sets $y_1, \ldots, y_r$ we can form the set

refer to these as parameters

$$B = \{x \in A : P(x, \overbrace{y_1, \ldots, y_r}^{\text{parameters}}) \text{ holds}\} \subseteq A$$

Eg 1) Let C be a non-empty set and $A \in C$. Then

$$\bigcap C = \{x \in A : \overbrace{(\forall z)((z \in C) \to (x \in z))}^{P(x, C)}\}$$

Ex: This does not depend on the particular $A \in C$.

2) $A \times B =$

$$\{x \in \mathcal{P}(\mathcal{P}(A \cup B)) : (\exists a)(\exists b)\big((a \in A) \wedge (b \in B) \wedge (x = \{\{a\}, \{a, b\}\})\big)\}$$

Ex: Can form $B^A \subseteq \mathcal{P}(A \times B)$.

$(a, b) = \{\{a\}, \{a, b\}\}$