IMPERIAL

MATH70034

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Algebraic Topology

Date: Thursday, May 23, 2024
Time: 10:00 – 12:30 (BST)
Time Allowed: 2.5 hours
This paper has 5 Questions.
Please Answer All Questions in 1 Answer Booklet
Candidates should start their solutions to each question on a new sheet of paper.
Supplementary books may only be used after the relevant main book(s) are full.
Any required additional material(s) will be provided.
Allow margins for marking.
Credit will be given for all questions attempted.
Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Give the definition of *covering spaces*. (2 marks)

(b) State the unique lifting property and homotopy lifting property for a covering space. (Include all the assumptions if there is any. Neither proof nor definition for any terminology is required.) (4 marks)

(c) Let $X = \bigcup_n C_n$ be the Hawaiian earring where C_n is the circle with radius $1/n$ and with centre $(1/n, 0)$ where $n \in \mathbb{N}_{\geq 1}$. Let $CX = (X \times I)/(X \times \{0\})$ be the cone over X.

(i) Prove that CX is contractible. (4 marks)

(ii) Prove that CX is *not* locally simply-connected. Recall that a space Y is locally simply-connected if for any point $y \in Y$ and any open neighborhood U containing y, there is an open neighborhood $V \subset U$ of y such that V is simply-connected. (4 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(d) Let X be a path-connected, locally path-connected, semi-locally simply connected topological space with $\pi_1(X) \cong \mathrm{D}_3 = \langle r, s \mid r^3 = s^2 = e, srs^{-1} = r^{-1} \rangle$, the dihedral group of order 6. Determine the number of path-connected covering spaces of X up to isomorphism? (6 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(Total: 20 marks)

2. (a) State the Seifert-van Kampen theorem. (Include all the assumptions if there is any. Neither proof nor definition for any terminology is required. The version for two open subsets is sufficient.) (4 marks)

(b) Let $C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be a circle.

(i) Determine the fundamental group of $X = \mathbb{R}^3 \setminus C$. (5 marks)

(ii) Let Y_n be the topological space obtained by removing n distinct points on a torus where n is a non-negative integer. Determine the fundamental group of Y_n. (7 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(c) Does there exist a covering space $f\colon Y_n \to Y_0$ for $n \geq 1$? (Either construct such a covering space or prove that it does not exist.) (4 marks)

(Total: 20 marks)

3. (a) Give the definition of *chain complexes* and their *homology groups*. (No proof is required) (3 marks)

(b) State the Excision theorem. (Include all assumptions if there is any. Neither proof nor definition for any terminology is required.) (4 marks)

(c) Let $\mathbb{RP}^n = S^n/\sim$ be the real projective space where $x \sim y$ whenever $x = \pm y \in S^n \subset \mathbb{R}^{n+1}$. Let $S^n \hookrightarrow S^{n+1}$ be the inclusion to the equator of S^{n+1}. This inclusion descends to the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$.

(i) Determine $\mathrm{H}_i(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ for all $i \geq 0$ and $n \geq 1$. (4 marks)

(ii) Determine $\mathrm{H}_i(\mathbb{RP}^n)$ for $i > n \geq 1$ and show $\mathrm{H}_n(\mathbb{RP}^n)$ is a free abelian group of rank at most 1 for $n \geq 1$. (5 marks)

(iii) Let $X = \bigcup_{n=1}^{\infty} \mathbb{RP}^n$ with the weak topology, that is, $U \subset X$ is open if and only if $U \cap \mathbb{RP}^n \subset \mathbb{RP}^n$ is open for all n.
Show that $\mathrm{H}_n(X, \mathbb{RP}^n) = 0$ for all $n \geq 1$. You can use the fact that any compact subspace of X is contained in $\mathbb{RP}^m$ for some m. (4 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(Total: 20 marks)

4. Let X be the "parachute" formed by identifying the three vertices of a 2-simplex.

(a) Compute simplicial homology groups of X using a Δ-complex structure on X. (5 marks)

(b) Explain X is homotopy equivalent to $S^1 \vee S^1$. (Drawing some clear pictures would be sufficient.) (4 marks)

(c) Prove that X is NOT homotopy equivalent to a torus. (3 marks)

(d) Give an example of a finite-dimensional CW complex X with $\mathrm{H}_0(X) \cong \mathbb{Z}$, $\mathrm{H}_1(X) \cong \mathbb{Z}/2024\mathbb{Z}$, and $\mathrm{H}_{2024}(X) \cong \mathbb{Z}$. (8 marks)

(Justify your answers. You can use any results from the course if you state them clearly.)

(Total: 20 marks)

5. (a) State the Mayer-Vietoris theorem. (Include all assumptions if there is any. Neither proof nor definition for any terminology is required.) (2 marks)

(b) Let X be a topological space and $A_1, \ldots, A_n$ be non-empty open subsets such that the union is X where $n \geq 2$ is an integer. Suppose any intersection of $A_{i_1} \cap \cdots \cap A_{i_k}$ is either empty or has trivial reduced homology groups. Show that $\mathrm{H}_i(X) = 0$ for $i \geq n-1$. (6 marks)

(c) Let X_2 be a discrete set containing two distinct points. The space X_2 shows that the inequality from 5(b) is optimal.

(i) For $n = 3$, find an example showing the inequality from 5(b) is optimal. That is, find an example satisfying the assumptions in 5(b) with $\mathrm{H}_1(X) \neq 0$. (4 marks)

(ii) Find an example showing the inequality from 5(b) is optimal, for each $n \geq 4$. That is, find an example satisfying the assumptions in 5(b) with $\mathrm{H}_{n-2}(X) \neq 0$. (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60034/MATH70034

Algebraic Topology (Solutions)

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1. (a) A covering space of a topological space X is a topological space $\widetilde{X}$ and a continuous map $p\colon \widetilde{X} \to X$ such that for each $x \in X$, there exists an open neighborhood U of x in X such that $p^{-1}U = \sqcup_{j\in J}\widetilde{U_j}$ is a disjoint union of open subsets of $\widetilde{X}$ and $p|_{\widetilde{U_j}}\colon \widetilde{U_j} \to U$ is a homeomorphism for all $j \in J$. seen $\Downarrow$ 2, A

(b) (i) (Unique lifting property) Let $p\colon \widetilde{X} \to X$ be a covering space and $f\colon Y \to X$ be a continuous map from a connected space Y. Suppose there are two lifts $\widetilde{f_1}$, $\widetilde{f_2}\colon Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$, then $\widetilde{f_1} = \widetilde{f_2}$. seen $\Downarrow$ 2, A

(ii) (Homotopy lifting property) Let $p\colon \widetilde{X} \to X$ be a covering space and $F\colon Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0}\colon Y \times \{0\} \to \widetilde{X}$ of $F|_{Y\times\{0\}}$. Then there is a unique lift $\widetilde{F}\colon Y \times I \to \widetilde{X}$ such that $\widetilde{F}|_{Y\times\{0\}} = \widetilde{f_0}$. 2, A

(c) (i) Consider $f\colon X \times I \times I \to CX$ which sends (x, t, s) to $\overline{(x, ts)}$. Note that f sends all points with $t = 0$ to a single point $\overline{x, 0}$. By the universal property of the quotient space, f descends to a map $g\colon CX \times I \to CX$. Note that g_1 is the identity map on CX and g_0 is a constant map. Thus CX can be deformation retract onto a single point. Hence, CX is contractible. sim. seen $\Downarrow$ 4, A

(ii) Let x_0 be the origin in $\mathbb{R}^2$. Consider the point $(x_0, 1) \in CX$ and an open neighborhood $U = (B_r(x_0) \cap X) \times (1/2, 1]$ where $B_r(x_0)$ is an open disk on X with centre x_0 and with radius r. Note that for any open neighborhood of the point $(x_0, 1) \in CX$, there is a smaller open neighborhood of the form $(B_r(x_0) \cap X) \times (t, 1]$ for $r > 0$ is small and $t < 1$ close to 1. Note that such open neighborhood is deformation retract onto $B_r(x_0) \cap X$, which has been shown in the lecture that it has nontrivial fundamental group. 4, B

(d) By Galois correspondence, it suffices to find the number of subgroups up to conjugation. unseen $\Downarrow$ 2, A

Let H be a subgroup of $\mathrm{D}_3 = \{e, r, r^2, s, sr, sr^2\}$. By Langrange's theorem, $|H| = 1$, 2, 3, or 6.

When $|H| = 1$ (resp. 6), $H = \{e\}$ (resp. D_3). 2, A

If $|H| = 2$, then $H = H_i := \langle sr^i\rangle$ for $i = 0$, 1, 2. They are conjugate to each other.

If $|H| = 3$, then $H = \langle r\rangle$.

Therefore, D_3 has 4 subgroups up to conjugate. 2, A

2. (a) Let X be a topological space and U_1, $U_2 \subset X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. Then $\pi_1(X, x_0)$ is isomorphic to the amalgamated product of $\pi_1(U_1, x_0)$ and $\pi_1(U_2, x_0)$ over $\pi_1(U_1 \cap U_2, x_0)$. $\boxed{\text{seen} \Downarrow}$ $\boxed{4, \text{A}}$

*Do NOT deduct any marks if the explicit construction of amalgamated product is used correctly instead of mentioning it is the amalgamated product. That is, let $j_i \colon \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_i, x_0)$ for $i = 1$, 2 and N be the normal closure of the set $\{j_1(g)j_2(g)^{-1} \mid g \in \pi_1(U_1 \cap U_2, x_0)\}$. Then

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) * \pi_1(U_2, x_0)/N.$$

$\boxed{\text{meth seen} \Downarrow}$

(b) (i) Note that X is homotopy equivalent to a sphere with a line connecting the north and south poles, which is homotopy equivalent to $S^2 \vee S^1$. By Seifert-van Kampen theorem, $\pi_1(X) \cong \pi_1(S^2 \vee S^1) \cong \mathbb{Z}$ as S^2 is simply-connected and $\pi_1(S^1) \cong \mathbb{Z}$. $\boxed{5, \text{B}}$

(ii) If $n = 0$, then Y_0 is the torus $S^1 \times S^1$ and $\pi_1(Y_0) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$.
If $n \geq 1$, then Y_n is homotopy equivalent to the wedge sum of $n+1$ circles S^1.
Thus, by van Kampen theorem, $\pi_1(Y_n) \cong F_{n+1}$, the free group generated by $n+1$ elements.
*1 mark for $n = 0$, 2 marks for $n = 1$, and 2 marks for $n = 2$. $\boxed{7, \text{C}}$

(c) If there is a covering space $p \colon Y_n \to Y_0$, then $\pi_1(Y_n)$ is isomorphic to a subgroup of $\pi_1(Y_0)$. As $\pi_1(Y_0)$ is abelian, any subgroups are abelian, which is impossible since $\pi_1(Y_n)$ is non-abelian.

*Give the full (4) marks if the reason for the case $n = 1$ is given and sound. $\boxed{4, \text{B}}$

3. (a) A chain complex $(C_\bullet, \partial)$ consists of abelian groups C_n and group homomorphisms $\partial_n \colon C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for $n \in \mathbb{Z}$. $\boxed{\text{seen} \Downarrow}$

The n-th homology group of a chain complex is $\mathrm{H}_n := \ker \partial_n / \operatorname{im} \partial_{n+1}$. $\boxed{3, \text{A}}$

(b) Let X be a topological space. Suppose there are two subspaces $Z \subset A \subset X$ such that $\overline{Z} \subset \mathring{A}$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $\mathrm{H}_n(X \setminus Z, A \setminus Z) \cong \mathrm{H}_n(X, A)$. $\boxed{\text{seen} \Downarrow}$

Equivalent version: Let X be a topological space. Suppose there are two subspaces A, $B \subset X$ such that $\mathring{A} \cup \mathring{B} = X$. Then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $\mathrm{H}_n(B, A \cap B) \cong \mathrm{H}_n(X, A)$. $\boxed{4, \text{A}}$

(c) (i) Note that $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ is a good pair. Also note that $\mathbb{RP}^n / \mathbb{RP}^{n-1}$ is homeomorphic to S^n. Then we have shown in the lecture that $\mathrm{H}_i(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \cong \widetilde{\mathrm{H}}_i(\mathbb{RP}^n / \mathbb{RP}^{n-1}) \cong \widetilde{\mathrm{H}}_i(S^n) \cong \mathbb{Z}$ for $i = n$ and 0 for $i \neq n$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{4, \text{B}}$

(ii) The long exact sequence associated to the pair $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ gives $\boxed{\text{meth seen} \Downarrow}$

$$\cdots \to \mathrm{H}_{i+1}(\mathbb{RP}^n / \mathbb{RP}^{n-1}) \to \mathrm{H}_i(\mathbb{RP}^{n-1}) \to \mathrm{H}_i(\mathbb{RP}^n) \to \mathrm{H}_i(\mathbb{RP}^n / \mathbb{RP}^{n-1}) \to \cdots$$

By (i), we get $\mathrm{H}_i(\mathbb{RP}^{n-1}) \cong \mathrm{H}_i(\mathbb{RP}^n)$ for $i > n$. By induction on n, when $i > n$, we have $\mathrm{H}_i(\mathbb{RP}^n) \cong \mathrm{H}_i(\mathbb{RP}^1) = 0$ since $\mathbb{RP}^1 = S^1$ and $i > n \geq 1$.
Next, we see again the long exact sequence

$$\cdots \to \mathrm{H}_n(\mathbb{RP}^{n-1}) \to \mathrm{H}_n(\mathbb{RP}^n) \to \mathrm{H}_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$$

As $\mathrm{H}_n(\mathbb{RP}^{n-1}) = 0$, we have a monomorphism $\mathrm{H}_n(\mathbb{RP}^n) \hookrightarrow \mathrm{H}_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$, which is isomorphic to $\mathbb{Z}$. Thus $\mathrm{H}_n(\mathbb{RP}^n)$ is free of rank at most 1 since any subgroup of a free abelian group is free. $\boxed{5, \text{C}}$

(iii) Let $X_n = \mathbb{RP}^n$. Using (i), we have $\mathrm{H}_n(X_{m+1}, X_m) = 0$ for $m \geq n$. Then from the long exact sequence associated to (X_m, X_{m-1}), we get an epimorphism $\mathrm{H}_n(X_m) \twoheadrightarrow \mathrm{H}_n(X_{m+1})$ and a monomorphism $\mathrm{H}_{n-1}(X_{m-1}) \twoheadrightarrow \mathrm{H}_{n-1}(X_m)$ for $m \geq n$. Thus, we have a sequence of epimorphisms $\boxed{\text{unseen} \Downarrow}$

$$\mathrm{H}_n(X_n) \twoheadrightarrow \mathrm{H}_n(X_{n+1}) \twoheadrightarrow \mathrm{H}_n(X_{n+2}) \twoheadrightarrow \cdots$$

and a sequence of monomorphisms

$$\mathrm{H}_{n-1}(X_n) \hookrightarrow \mathrm{H}_{n-1}(X_{n+1}) \hookrightarrow \mathrm{H}_{n-1}(X_{n+2}) \hookrightarrow \cdots$$

Using the fact with two sequences above, we get an epimorphism $\mathrm{H}_n(X_n) \twoheadrightarrow \mathrm{H}_n(X)$ and a monomorphism $\mathrm{H}_{n-1}(X_n) \hookrightarrow \mathrm{H}_{n-1}(X)$. This gives us the required vanishing from the long exact sequence of the pair (X, X_n). $\boxed{4, \text{D}}$

4. (a) The Δ-complex structure on a 2-simplex gives the Δ-complex structure on X. So on X, we have one 0-simplex v, three 1-simplices a, b, c, and one 2-simplex σ. Thus, the chain complex is seen $\Downarrow$

$$0 \to \mathbb{Z}\sigma \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}v \to 0.$$

It is clear that $\partial_1 = 0$ and $\partial_2(\sigma) = a - b + c$ if we fix the orientation. Thus, $\mathrm{H}_0(X) = \mathbb{Z}/\operatorname{Im}\partial_1 \cong \mathbb{Z}$. $\mathrm{H}_2(X) = \ker\partial_2 = 0$, and $\mathrm{H}_1(X) = \ker\partial_1/\operatorname{Im}\partial_2 \cong \mathbb{Z}\langle a,b,c\rangle/\langle a-b+c\rangle \cong \mathbb{Z}\langle a,c\rangle \cong \mathbb{Z}^2$. 5, A

(b) Pick a point in the interior of X, for instances, the barycentre. Then we connect this point to the three vertices so that 2-simplex is divided into three triangles. Thus, we have a deformation retract of each triangle onto its two edges not contained in the boundary of X. Hence after these deformation retract, we get a CW complex containing two vertices connecting by three edges. Contracting one of the edges gives a homotopy equivalence to $S^1 \vee S^1$. meth seen $\Downarrow$ 4, D

(c) Suppose they are homotopy equivalent. By (b), we have $\pi_1(X) \cong \pi_1(T)$. However, $\pi_1(X) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, which is not abelian; while $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$, which is abelian. Therefore, they are not homotopy equivalent. meth seen $\Downarrow$ 3, B

(d) In the lecture, we have shown, using van Kampen theorem, that for a given group G, there exists a two-dimensional (path-connected) CW complex X_G such that $\pi_1(X_G) \cong G$. Take $G = \mathbb{Z}/2024\mathbb{Z}$, we get a CW complex X_1. As the first homology group is abelianization of the fundamental group, we have $\mathrm{H}_1(X_1) \cong \mathbb{Z}/2024\mathbb{Z}$. Moreover, it is clear that $\mathrm{H}_i(X_1) = 0$ for $i \geq 3$. unseen $\Downarrow$

Also we consider $X_2 = S^{2024}$, the 2024-dimensional sphere. It has been shown in the lecture that, $\widetilde{\mathrm{H}}_{2024}(S^{2024}) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}}_i(S^{2024}) = 0$ for $i \neq 2024$.

Thus, $X_1 \vee X_2$ is a desired CW complex. In fact, it is clear that $X_1 \vee X_2$ is path-connected, so $\mathrm{H}_0(X_1 \vee X_2) \cong \mathbb{Z}$. Moreover, for $i \geq 1$, $\mathrm{H}_i(X_1 \vee X_2) = \widetilde{\mathrm{H}}_i(X_1 \vee X_2) \cong \widetilde{\mathrm{H}}_i(X_1) \oplus \widetilde{\mathrm{H}}_i(X_2)$, where the isomorphism is shown in the lecture. 8, D

5. (a) Let X be a topological space and A, $B \subset X$ be two subspaces such that X is the union of the interiors of A and B. Then there is a long exact sequence seen $\Downarrow$

$$\cdots \to \mathrm{H}_i(A\cap B) \to \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \to \mathrm{H}_i(X) \to \mathrm{H}_{i-1}(A\cap B) \to \cdots$$

where the second and the third arrow are induced by the corresponding inclusion.

2, M

(b) We proceed by induction on n. When $n = 2$, by (a), we have meth seen $\Downarrow$

$$\mathrm{H}_i(A_1) \oplus \mathrm{H}_i(A_2) \to \mathrm{H}_i(X) \to \mathrm{H}_{i-1}(A_1 \cap A_2)$$

is exact. By assumption, we have $\mathrm{H}_i(A_1) \oplus \mathrm{H}_i(A_2) = 0$ for $i \geq 1$. Note that $\mathrm{H}_i(\emptyset) = 0$ for all i. Thus when $A_1 \cap A_2 = \emptyset$, then we get $\mathrm{H}_i(X) = 0$ for $i \geq 1$. On the other hand, when $A_1 \cap A_2 \neq \emptyset$, then it has trivial reduced homology group. So we get $\mathrm{H}_i(X) = 0$ for $i \geq 2$. For $i = 1$, we notice that

$$0 \to \mathrm{H}_1(X) \to \mathrm{H}_0(A\cap B) \to \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \to \mathrm{H}_0(X) \to 0$$

and all $\mathrm{H}_0 \cong \mathbb{Z}$. Therefore, $\mathrm{H}_1(X) = 0$.

Now in general, we consider $X_{n-1} = \bigcup_{i=1}^{n-1} A_i$. Then the subspaces X_{n-1} and A_n satisfy the assumption for Mayer-Vietoris sequence. By induction hypothesis, we have $\mathrm{H}_i(X_{n-1}) = 0 = \mathrm{H}_i(X_{n-1} \cap A_n)$ for $i \geq n-2$. Thus, by Mayer-Vietoris sequence, we get $\mathrm{H}_i(X) = 0$ for $i \geq n-1$.

*2 marks if the case $n = 2$ is shown, another 2 marks if the cases $n = 2, 3$ are shown. 6, M

(c) (i) and (ii) unseen $\Downarrow$

Consider $[v_0, \ldots, v_{n-1}]$ be a $(n-1)$-simplex. Let X be the boundary of $[v_0, \ldots, v_{n-1}]$. Note that X is homeomorphic to S^{n-2}, which has $\mathrm{H}_{n-2}(X) \cong \mathbb{Z}$. Let A_i be a small open neighborhood of $[v_0, \ldots, \widehat{v_i}, \ldots, v_{n-1}]$ in X for $i = 1, \ldots, n-1$. Note that $A_{i_1} \cap \ldots \cap A_{i_k}$ is contractible when $k < n$, which has trivial reduced homology groups. And $\bigcap_{i=0}^{n-1} A_i = \emptyset$. Therefore, X together with all A_i give the desired example. 12, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH60034 Algebraic Topology

Question	Marker's comment
1	(b) For the unique lifting property, some people missed the assumption of connectedness on the domain of the map to the base of the covering spaces. Some stated only the properties for the path.(c) To show the cone is contractible, we need to have a deformation retract onto a point. Just retract is not enough.(d) It is great that most people used Galois correspondence and stated it correctly. However, some people are not familiar on how to study the subgroups as well as on the difference between normality and conjugacy classes.
2	(a) Some missed the assumption of path-connectedness.(b)(i) It is wonderful that people gave different valid ways to find its fundamental group. One is to find a simpler space via homotopy equivalence. Another way is to use Seifert-van Kampen theorem directly by cleverly finding two open subsets.(b)(ii) Some noticed that the fundamental group of Y_{n-1} can be an amalgamated product of the one of Y_n and the one of S_1 by Seifert-van Kampen theorem. However, it is in general not possible to find what the component is from the its amalgamated product. Besides, some missed the case when n=0.(c) One common mistake is saying that Y_n is smaller than Y_0. However, it is great that most people noted the injection property for the covering space.
3	(a) A general definition of homology group is expected. But it is okay that some gave the definition for simplicial homology groups.(b) Some are confused by the notations for the quotient and the set minus.(c)(i) The key observation is (RP^n,RP^{n-1}) is a good pair and the quotient is homeomorphic to S^n.(c)(ii) This will need the induced long exact sequence, induction on n, and RP^1=S^1.nbsp; Some tried to use cellular homology to compute directly. It is okay if you did it correctly.(c)(iii) This may be the most difficult one as we need to keep track of many subspaces and exact sequences. Some people tried to use some ideas from cellular homology. It is a valid way if you stated it and did it correctly.
4	(b) Some pictures are not clear and hard to understand to the marker.(d) Some people tried to construct a space and compute simplicial homology without providing a Delta-complex structure. Also some people try to use cellular homology but the attaching maps for CW complex are not clear, neither is computation of cellular homology.The idea is to find two path-connected spaces with prescribed H_1 and H_{2024}, respectively. Then the desired space is a wedge sum of these two.

MATH70034 Algebraic Topology

Question	Marker's comment
1	(b) For the unique lifting property, some people missed the assumption of connectedness on the domain of the map to the base of the covering spaces. Some stated only the properties for the path.(c) To show the cone is contractible, we need to have a deformation retract onto a point. Just retract is not enough.(d) It is great that most people used Galois correspondence and stated it correctly. However, some people are not familiar on how to study the subgroups as well as on the difference between normality and conjugacy classes.
2	(a) Some missed the assumption of path-connectedness.(b)(i) It is wonderful that people gave different valid ways to find its fundamental group. One is to find a simpler space via homotopy equivalence. Another way is to use Seifert-van Kampen theorem directly by cleverly finding two open subsets.(b)(ii) Some noticed that the fundamental group of Y_{n-1} can be an amalgamated product of the one of Y_n and the one of S_1 by Seifert-van Kampen theorem. However, it is in general not possible to find what the component is from the its amalgamated product. Besides, some missed the case when $n=0$.(c) One common mistake is saying that Y_n is smaller than Y_0. However, it is great that most people noted the injection property for the covering space.
3	(a) A general definition of homology group is expected. But it is okay that some gave the definition for simplicial homology groups.(b) Some are confused by the notations for the quotient and the set minus.(c)(i) The key observation is (RP^n,RP^{n-1}) is a good pair and the quotient is homeomorphic to S^n.(c)(ii) This will need the induced long exact sequence, induction on n, and $RP^1=S^1$.nbsp; Some tried to use cellular homology to compute directly. It is okay if you did it correctly.(c)(iii) This may be the most difficult one as we need to keep track of many subspaces and exact sequences. Some people tried to use some ideas from cellular homology. It is a valid way if you stated it and did it correctly.
4	(b) Some pictures are not clear and hard to understand to the marker.(d) Some people tried to construct a space and compute simplicial homology without providing a Delta-complex structure. Also some people try to use cellular homology but the attaching maps for CW complex are not clear, neither is computation of cellular homology.The idea is to find two path-connected spaces with prescribed H_1 and H_{2024}, respectively. Then the desired space is a wedge sum of these two.
5	(b) The idea is to first try the case when $n=2$. For larger n, we use induction.(c)(i) To find such example, we first find a space with non-trivial H_1 from the statement in question. The common and the first such example is S^1. Moreover, the common spaces with trivial reduced homology groups are contractible spaces. Then three suitable arcs will be sufficient.(c)(ii) Applying the similar idea, we first find a space with non-trivial H_{n-2}. And then we find some nice contractible spaces on it.