Mathematical Logic (MATH6/70132; P65)
Notes on Solutions, Problem Sheet 1

1. Let p denote 'I will pass this course,' q denote 'I do my homework regularly' and r denote 'I am lucky.'

(a) I will pass this course only if I do my homework regularly: $(p \to q)$.

(b) Doing homework regularly is a necessary condition for me to pass this course: $(p \to q)$

(c) If I do my homework regularly and I do not pass this course then I am unlucky:

$((q \wedge (\neg p)) \to (\neg r))$.

(d) If I do not do my homework regularly and I pass this course then I am lucky:
$(((\neg q) \wedge p) \to r)$.

2. (a) The formula $\phi : ((p \to q) \to ((\neg p) \wedge q))$ has truth table

p	q	ϕ
T	T	F
T	F	T
F	T	T
F	F	F

So the disjunctive normal form is $((p \wedge (\neg q)) \vee ((\neg p) \wedge q))$.

(b) $(\neg((p \to q) \to r))$. This has truth value T when $((p \to q) \to r)$ has value F. This happens when r has value F and $(p \to q)$ has value T. So the d.n.f. is (omitting brackets)

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

3. (a) Observe that $(\neg \phi)$ is logically equivalent to $(\phi \mid \phi)$ and $(\phi \wedge \psi)$ is logically equivalent to $((\phi \mid \psi) \mid (\phi \mid \psi))$.

(b) Suppose we have a binary connective $*$. There are 16 possibilities for the truth table for $*$. Half of these give $p * q$ truth value T when p, q have truth value T, and such a connective cannot express $(\neg p)$ (any formula involving only such a $*$ would always take truth value T when the propositional variables in it all took value T). Of the remaining 8, half give $p * q$ truth value F when p and q have truth value F, and these also cannot express $(\neg p)$. Two of the remaining cases are $\mid$ and $\downarrow$, and we know that these are adequate. It remains to eliminate the other two possibilities. But in these cases $(p * q)$ is logically equivalent either to $(\neg p)$ or to $(\neg q)$, and clearly this cannot be adequate. (To see this more formally, suppose in one of these cases that ϕ is a formula obtained using this connective and propositional variables from $p_1, \ldots, p_n$. Then ϕ is logically equivalent to p_i or $(\neg p_i)$, for some $i \leq n$. Thus there are at most $2n$ possibilities for the truth function of ϕ.)

4. There are 2^{2^n} truth functions of n variables (1.1.7 in the notes). Half of these take value T at $(T, T, \ldots, T)$, so the number of such truth functions of n variables is $2^{2^n - 1}$. (Alternaltively, argue as in the proof of 1.1.7, noting that to specify the function we need to say what value it has at the remaining $2^n - 1$ inputs apart from $(T, \ldots, T)$.)

If $f(F, \ldots, F) = T$ then f cannot be expressed as the truth function of a formula constructed using connectives $\wedge, \vee$ as such a formula always takes value F when the variables have value F.

5. (i) Either construct a truth table or argue as follows. If a valuation v gives the formula truth value F, then we have $v(((p_3 \to p_2) \to p_1)) = F$ and $v((p_1 \to ((\neg p_2) \to p_3)) = T$. From the first of these, $v(p_1) = F$, $v(p_3 \to p_2) = T$, and any such valuation also satisfies

$v((p_1 \to ((\neg p_2) \to p_3)) = T$. Thus the possible values for (p_1, p_2, p_3) which make the original formula F are

$$(F, T, T), (F, T, F), (F, F, F).$$

$\neg\theta$ has truth value T iff θ has truth value F, so a formula in dnf which is logically equivalent to $\neg\theta$ can be obtained from a disjunction of formulas which are true precisely at the above values, ie

$$((\neg p_1) \wedge p_2 \wedge p_3) \vee ((\neg p_1) \wedge p_2 \wedge (\neg p_3)) \vee ((\neg p_1) \wedge (\neg p_2) \wedge (\neg p_3)).$$

(ii) We take χ to be the conjunction $p_1 \wedge (\neg p_2) \wedge p_3$. This has truth value T iff each of the conjuncts has value T: ie iff p_1, p_2, p_3 have the indicated values.

6.

1. $((\neg\psi) \to ((\neg\phi) \to (\neg\psi)))$ (Axiom A1)
2. $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi))$ (Axiom A3)

Denote this formula by χ

3. $(\chi \to ((\neg\psi) \to \chi))$ (Axiom A1)
4. $((\neg\psi) \to \chi)$ (2, 3 and Modus Ponens)

Denote this formula by θ

5. $(\theta \to (((\neg\psi) \to ((\neg\phi) \to (\neg\psi))) \to ((\neg\psi) \to (\psi \to \phi))))$ (Axiom A2)
6. $((\neg\psi) \to ((\neg\phi) \to (\neg\psi))) \to ((\neg\psi) \to (\psi \to \phi))$ (4, 5 and Modus Ponens)
7. $((\neg\psi) \to (\psi \to \phi))$ (1, 6 and Modus Ponens).