## 2.7 Geometric Interpretation

As you have seen in the introductory module vectors in $\mathbb{R}^2/\mathbb{R}^3$ can be represented as points in 2 or 3 dimensional space. In this section we will look geometric interpretations of some of the things we have seen so far.

A system of linear equations in $n$ unknowns specifies a set in $n$-space.

**Example 2.7.1.**

Consider:

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ 2x_1 + x_3 &= 1 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

Using row reduction we get $x_1 =$ , $x_2 =$ $x_3 =$ , which specifies a point. Whereas:

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ 2x_1 + x_3 &= 1 \end{aligned}$$

Using row reduction we get $x_1 = -2.5 - 0.5x_3$ and $x_2 = 1.5 - 0.5z$ giving the line

$$\left( \quad \right) + \lambda \left( \quad \right) \quad \text{for } \lambda \in \mathbb{R}$$

We have seen that we can apply matrices to vectors via matrix multiplication. So we can see a matrix $A \in M_{m\times n}(\mathbb{R})$ as a map:

$$\begin{aligned} A: \quad \mathbb{R}^n &\mapsto \mathbb{R}^m \\ A(v) &= Av \end{aligned}$$

We can represent many different operations using matrices.

**Example 2.7.2.**

Consider $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Then $A \begin{pmatrix} x \\ y \end{pmatrix} =$ .

This is a stretch by a factor of .

**Definition 2.7.3.** Let $T$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then we say $T$ is a *linear transformation* if for every $v_1, v_2 \in \mathbb{R}^n$ and every $\alpha, \beta \in \mathbb{R}$ we have:

$$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$$

**Proposition 2.7.4.** Let $A \in M_{n\times m}(\mathbb{R})$ then seen as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ $A$ is a linear transformation.

**Proof**:

**Proposition 2.7.5.** Let $A \in M_{n\times n}(\mathbb{R})$. The following are equivalent:

(i) $A$ is invertible with inverse $A^{-1} = A^T$

(ii) $A^T A = I_n = AA^T$.

(iii) $A$ preserves inner products (i.e. for all $x, y \in \mathbb{R}^n$ $(Px) \cdot (Py) = x \cdot y$.

**Proof:**

**Definition 2.7.6.** A matrix $A \in M_{n\times n}$ is called *Orthogonal* if it is such that $A^{-1} = A^T$

**Example 2.7.7.**

1. Consider the matrix
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
This matrix is orthogonal as $A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
If we apply it to $\begin{pmatrix} x \\ y \end{pmatrix}$ we get $\begin{pmatrix} -y \\ x \end{pmatrix}$. This is a rotation through $\frac{\pi}{2}$ radians anti clockwise.

2. Consider the matrix
$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
This matrix is orthogonal as $A^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
If we apply it to $\begin{pmatrix} x \\ y \end{pmatrix}$ we get $\begin{pmatrix} -y \\ -x \end{pmatrix}$. This is a reflection through the line $y = -x$.

**Exercise 2.7.8.** Watch the linear Algebra video to help you.

1. Let $R_\theta$ be the anticlockwise rotation of $\mathbb{R}^2$ about the origin through $\theta$ radians. Using any school geometry or trigonometry you like, find the matrix representing $R_\theta$.

## 2.8 Fields

So far, for both matrices and linear equations, we have only been using entries in $\mathbb{R}$. However, we could have taken entries from any field.

Every field has distinguished elements $0$ (additive identity) and $1$ (multiplicative identity).

**Fact 2.8.1.** Over any field $F$ we can define:

1. The null matrix (i.e. the additive identity matrix) for $M_{n\times m}(F)$ as

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

2. The (multiplicative) identity matrix for $M_{n\times n}(F)$ as

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

**Remark 2.8.2** *It is important to know what field we are working in, and that we don't say take scalars from a different field to the one matrix entries are from. (e.g. the set of matrices $M_{n\times m}(\mathbb{Q})$ is not closed under scalar multiplication by elements from $\mathbb{R}$).*

Being able to work over a general field allows us to use finite fields.

**Theorem 2.8.3.** Let $\mathbb{F}_p = \{0, 1, ..., p-1\}$, consider $\mathbb{F}_p$ with addition defined by addition modulo $p$ and multiplication as multiplication modulo $p$. Then the structure $(\mathbb{F}_p, +_{\pmod p}, \times_{\pmod p})$ is a field.

**Proof:**

**Example 2.8.4.** $\mathbb{F}_6$ defined as above is not a field. For example $3 \neq 0$ does not have an inverse.

# 3 Vector Spaces

## 3.1 Intro to Vector Spaces

**Definition 3.1.1.** Let $F$ be a field. A **vector space** over $F$ is a non-empty set $V$ together with the following maps:

1. **Addition**
$$\begin{array}{rccl} \oplus: & V \times V & \mapsto & V \\ & (v_1, v_2) & \mapsto & v_1 \oplus v_2 \end{array}$$

2. **Scalar Multiplication**
$$\begin{array}{rccl} \odot: & F \times V & \mapsto & V \\ & (f, v_2) & \mapsto & f \odot v_2 \end{array}$$

$\oplus$ and $\odot$ must satisfy the following *Vector Space axioms:*

For Vector Addition:

A1 Associative law: $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$.

A2 Commutative law: $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$.

A3 Additive identity: $0_V \oplus \mathbf{v} = \mathbf{v}$, where $0_V$ is called the IDENTITY vector (or sometimes the *zero vector*).

A4 Additive inverse: $\mathbf{v} \oplus (\ominus \mathbf{v}) = 0_V$.

For scalar multiplication:

A5 Distributive law: $r \odot (\mathbf{v} \oplus \mathbf{w}) = (r \odot \mathbf{v}) \oplus (r \odot \mathbf{w})$.

A6 Distributive law: $(r + s) \odot \mathbf{v} = (r \odot \mathbf{v}) \oplus (s \odot \mathbf{v})$.

A7 Associative law: $r \odot (s \odot \mathbf{v}) = (rs) \odot \mathbf{v}$.

A8 Identity for scalar mult: $1 \odot \mathbf{v} = \mathbf{v}$.

From now on we will drop the $\oplus$ and $\odot$, and use $+$ and $\cdot$ the point was to emphasise that these are not the same as field addition and multiplication.

**Definition 3.1.2.** Let $V$ be a vector space over $F$ we call:

- Elements of $V$ are called *vectors*.
- Elements of $F$ are called *scalars*.
- We sometimes refer to $V$ as an $F$-vector space.

**Example 3.1.3.** The following are examples of vector spaces over $\mathbb{R}$:

- The canonical example is the set of vectors $\mathbb{R}^n$ over $\mathbb{R}$, where $\oplus$ is normal vector addition and $\odot$ is multiplication by a scalar. The additive inverse of $\mathbf{v}$ is simply $-\mathbf{v}$
- The set $\mathsf{M}_{\mathsf{mn}}$ of all $m \times n$ matrices. This is because addition of two $m \times n$ matrices produces an $m \times n$ matrix and multiplication of an $m \times n$ matrix by a scalar also produces a $m \times n$ matrix. The zero vector is the zero matrix, and for any matrix $\mathsf{A}$, the matrix $-\mathsf{A}$ is the additive inverse. Properties of matrix arithmetic covered in Chapter 1, show that all properties in Definition 3.1.2 required of a vector space are satisfied. We will see this later in the course in detail.
- Define $\mathbb{R}^X$ to be the set of real valued functions on $X$ (i.e. $\mathbb{R}^X := \{f : f \text{ a function}, f : X \to \mathbb{R}\}$). Then for $f, g \in \mathbb{R}^X$ and $\alpha \in \mathbb{R}$ define:

$$\begin{aligned} f \oplus g :& X \to \mathbb{R} & (\alpha \odot f) :& X \to \mathbb{R} \\ (f \oplus g)(x) &= f(x) + g(x) & (\alpha \odot f)(x) &= \alpha(f(x)) \end{aligned}$$

**Exercise 3.1.4.** Which of the following examples of vector spaces over $\mathbb{R}$ :

1. The set of vectors

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Z} \right\} \text{ with the usual vector addition and multiplication}$$

2. The set of vectors:

$$V = \left\{ \begin{pmatrix} a+1 \\ 2 \end{pmatrix} : a \in \mathbb{R} \right\} \text{ with the usual vector addition and multiplication}$$

3. $V = \mathbb{R}^2$ with the following addition and scalar multiplication operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix} \quad \text{and} \quad r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ ry \end{pmatrix}$$

## 3.2 Subspaces

**Definition 3.2.1.** A subset $W$ of a vector space $V$ is a **subspace** of $V$ if

S1 $W$ is not empty (i.e. $e \in W$)

S2 for $\mathbf{v}, \mathbf{w} \in W$, then $\mathbf{v} \oplus \mathbf{w} \in W$ *closed under vector addition*

S3 $\mathbf{v} \in W$ and $r \in \mathbb{R}$, then $r \odot \mathbf{v} \in W$ *closed under scalar multiplication.*

*N.B.* Sometimes we use the notation $U \leq V$ to mean $U$ is a subspace of $V$.

**Remark 3.2.2** *Note that $V$ and the zero subspace,* **0** *are always subspaces of $V$. Any other subspace of $V$ is called a* ***proper subspace*** *of $V$.*

**Proposition 3.2.3.** Every subspace of an $F$-vector space $V$ must contain the zero vector.

**Proof:**

**Example 3.2.4.** Show that the set $X = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} ; x \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$.

**Exercise 3.2.5.** All subspaces of a vector space over $F$ are vector spaces over $F$ in their own right.

**Theorem 3.2.6.** Let $U, W$ be subspaces of $V$. Then $U \cap W$ is a subspace of V. In general, the intersection of any set of subspaces of a vector space $V$ is a subspace of $V$.

**Proof**

**Example 3.2.7.** Note that in general if $U$ and $W$ are subspaces of $V$, then $U \cup W$ is not a subspace of $V$. For example, let

$$U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}, W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\} \quad \text{and} \quad V = \mathbb{R}^2.$$

Then

$$\in U \cup W$$

but

$$\begin{pmatrix} \quad \end{pmatrix} \oplus \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix} \notin U \cup W.$$