Applied Complex Analysis : Problem Sheet 5
Solutions

1). $k(x) = a e^{-b|x|}$, $a, b \in \mathbb{R},\ b > 0$.

Take Fourier transform: $\hat{K}(s) = \displaystyle\int_{-\infty}^{\infty} a e^{-b|x|} e^{isx}\, dx$

$$= a \left(\int_{-\infty}^{0} e^{(b+is)x}\, dx + \int_{0}^{\infty} e^{(is-b)x}\, dx \right)$$

$$= a \left(\left[\frac{e^{(is+b)x}}{is+b} \right]_{-\infty}^{0} + \left[\frac{e^{(is-b)x}}{is-b} \right]_{0}^{\infty} \right)$$

put $s = s_1 + is_2$, then
$(is+b)x = is_1 x + (b - s_2)x$
$\Rightarrow$ need $b - s_2 > 0$ for limit $x \to -\infty$ to decay

here needs $(-b-s_2)x < 0$
$\Rightarrow b + s_2 > 0$

$$= a \left(\frac{1}{is+b} - 0 + 0 - \frac{1}{is-b} \right),$$ provided $-b < \mathrm{Im}\{s\} < b$

$$= a \left(\frac{is - b - (is+b)}{(is+b)(is-b)} \right)$$

$$= \frac{-2ab}{-s^2 - b^2}$$

$$= \frac{2ab}{s^2 + b^2},$$ analytic for $-b < \mathrm{Im}\{s\} < b$.

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2).

(i). $$R(s) = \frac{1}{s^2+a^2} = \frac{i}{2a}\left(\frac{1}{s+ia} - \frac{1}{s-ia}\right).$$

Split into partial fractions

Hence we can set:

$$R_+(s) = \frac{\frac{i}{2a}}{s+ia}, \quad \text{analytic for } \operatorname{Im}\{s\} > -a \ (\text{i.e. in } \oplus)$$

$$R_-(s) = \frac{-\frac{i}{2a}}{s-ia}, \quad \text{analytic for } \operatorname{Im}\{s\} < a \ (\text{i.e. in } \ominus).$$

(ii). $$R(s) = \frac{1}{s^3-is^2-4s+4i}.$$ We want to factorise the denominator.

Notice that $s=i$ is a root. Thus:

$$R(s) = \frac{1}{(s-i)(s^2-4)} = \frac{1}{(s-i)(s-2)(s+2)}$$

$$= \frac{1}{20}\left(\frac{2-i}{s+2} + \frac{2+i}{s-2}\right) - \frac{1}{5}\frac{1}{s-i}$$

splitting into partial fractions

Hence we can set:

$$R_+(s) = \frac{1}{20}\left(\frac{2-i}{s+2} + \frac{2+i}{s-2}\right), \quad \text{analytic for } \operatorname{Im}\{s\} > \frac{1}{4} \ (\text{in } \oplus)$$

$$R_-(s) = -\frac{1}{5}\frac{1}{s-i}, \quad \text{analytic for } \operatorname{Im}\{s\} < \frac{1}{2} \ (\text{in } \ominus).$$

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3). $f(x) = \lambda \int_0^\infty e^{-4|x-y|} f(y)\,dy, \quad x \geq 0, \quad \lambda \in \mathbb{R}.$

Let $k(x) = \lambda e^{-4|x|}$. Then denoting $f_+(x)$ and $g_-(x)$ in the usual way gives:

$$\lambda \int_0^\infty e^{-4|x-y|} f(y)\,dy = f_+(x) + g_-(x), \qquad -\infty < x < \infty.$$

Taking Fourier transforms of both sides:

$$\hat{K}(s) F_+(s) = F_+(s) + G_-(s), \quad \text{where } \hat{K}(s) = \frac{8\lambda}{s^2+16}, \text{ by Q1.}$$

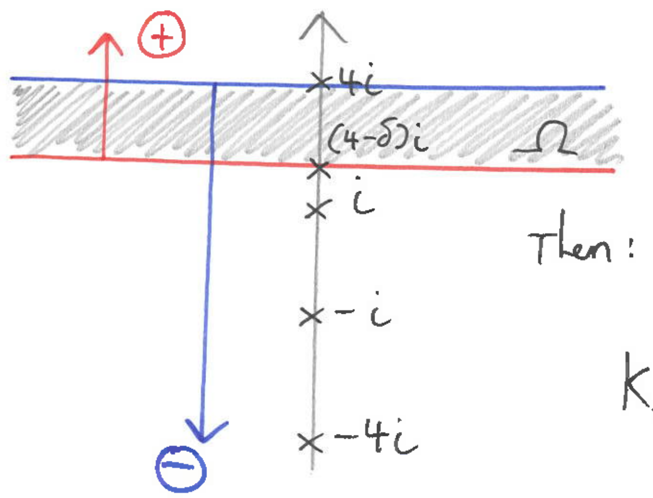
$$\Rightarrow K(s) F_+(s) + G_-(s) = 0, \quad \text{where } K(s) = 1 - \hat{K}(s)$$
$$= \frac{s^2 + 8(2-\lambda)}{s^2+16}$$

We also have: $F_+(s)$ is analytic for $\operatorname{Im}\{s\} > 4 - \delta$

$G_-(s)$ is analytic for $\operatorname{Im}\{s\} < 4$.

(i). $\lambda = \frac{15}{8}$; then $K(s) = \frac{s^2+1}{s^2+16} = \frac{(s+i)(s-i)}{(s+4i)(s-4i)}$

Take Ω and ⊕/⊖ to be:



$\Big\}\ 0 < \delta < 3$

Then: $K(s) = K_+(s) K_-(s)$, where:

$$K_+(s) = \frac{(s+i)(s-i)}{s+4i}, \quad K_-(s) = \frac{1}{s-4i}$$

$$\Rightarrow \underbrace{K_+(s)F_+(s)}_{\text{analytic in } \oplus} = \underbrace{\frac{-G_-(s)}{K_-(s)}}_{\text{analytic in } \ominus}, \quad s \in \Omega$$

Since $\oplus/\ominus$ overlap in Ω, then by analytic continuation:

$$E(s) = \begin{cases} K_+(s)F_+(s), & s \in \oplus \\ \dfrac{-G_-(s)}{K_-(s)}, & s \in \ominus \end{cases}, \quad \text{is } \underline{\text{entire}}.$$

Now, as $s \to \infty$, we have:

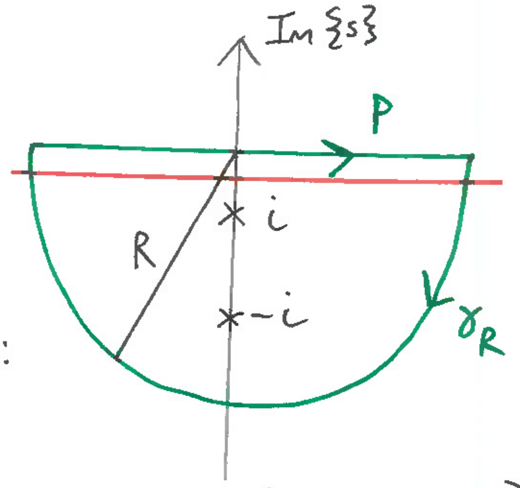
$$E(s) = (s^2+1)\frac{1}{s}\left(1 - \frac{4i}{s} + O\left(\frac{1}{s^2}\right)\right)\left[\overbrace{\frac{i f_+(0)}{s}}^{= c,\ \text{constant}} - \frac{f_+'(0)}{s^2} + O\left(\frac{1}{s^3}\right)\right]$$

$$\sim ci + O\left(\frac{1}{s}\right)$$

$$\sim ci \quad \text{as } s \to \infty.$$

Hence by Liouville's theorem: $E(s) \equiv ci$. Thus:

$$F_+(s) = \frac{ci(s+4i)}{(s+i)(s-i)}.$$



Hence $$f_+(x) = \frac{1}{2\pi}\int_P \frac{ci(s+4i)}{(s+i)(s-i)} e^{-isx}\, ds$$

Let $\gamma = P + \gamma_R$ as shown. $\int_{\gamma_R} \to 0$ as $R \to \infty$, so:

clockwise integration

$$f_+(x) = \frac{1}{2\pi} \times (-2\pi i) \times \left(\operatorname{Res}\left\{\frac{ci(s+4i)}{(s+i)(s-i)} e^{-isx}, s = i\right\} + \operatorname{Res}\left\{ \ldots, s = -i\right\}\right),$$

by the residue theorem.

$$\Rightarrow f_+(x) = -i\left(\frac{ci(5i)}{2i} e^{x} + \frac{ci(3i)}{-2i} e^{-x}\right)$$

$$\Rightarrow f(x) = \boxed{\frac{c}{2}\left(5e^{x} - 3e^{-x}\right)}, \quad x \geq 0$$

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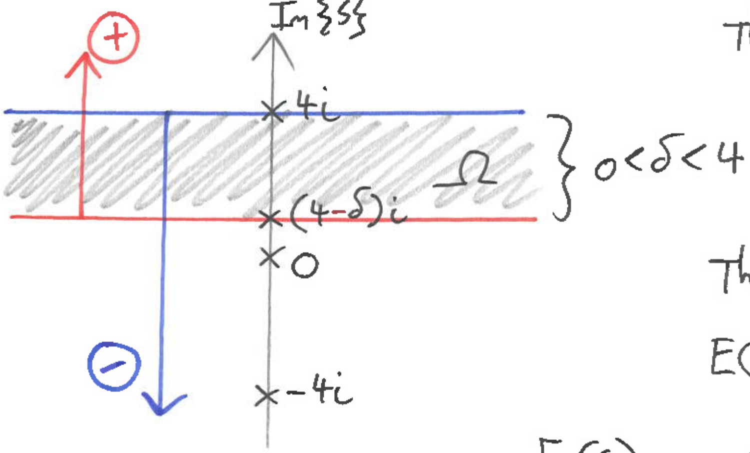
(ii). $\lambda = 2$; then $K(s) = \dfrac{s^2}{s^2+16}$. This time take Ω and $\oplus/\ominus$ to be:

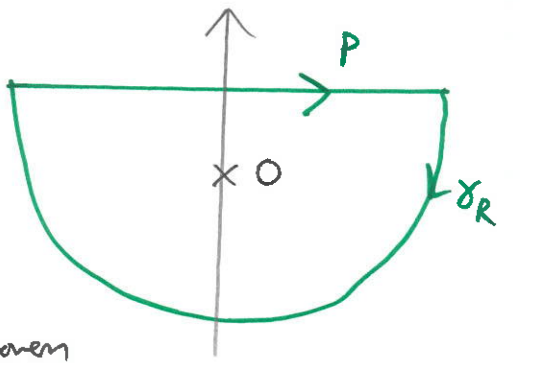


Then: $K_+(s) = \dfrac{s^2}{s+4i}$, $K_-(s) = \dfrac{1}{s-4i}$.

The analysis follows as before, we find $E(s) \equiv ci$ again. Hence we get:

$$F_+(s) = \frac{ci(s+4i)}{s^2}$$ and:

$$f_+(x) = \frac{1}{2\pi}\int_P \frac{ci(s+4i)}{s^2} e^{-isx}\, ds$$



This time instead of two simple poles we have a double pole at $s=0$ and from the residue theorem we get:

$$f_+(x) = \frac{1}{2\pi} \times (-2\pi i) \times \left(\lim_{s\to 0} \frac{d}{ds}\left[s^2 \frac{ci(s+4i)}{s^2} e^{-isx} \right] \right)$$

$$= -i\left(\lim_{s\to 0}\left(ci e^{-isx} + ci(s+4i)(-ix)e^{-isx} \right)\right)$$

$$\Rightarrow \boxed{f(x) = c(1+4x)}, \quad x \geqslant 0$$

(iii). $\lambda = \frac{17}{8}$; then $K(s) = \dfrac{s^2-1}{s^2+16} = \dfrac{(s-1)(s+1)}{(s-4i)(s+4i)}$. Take Ω and $\oplus/\ominus$ to be as in (ii). And: $K_+(s) = \dfrac{(s-1)(s+1)}{(s+4i)}$, $K_-(s) = \dfrac{1}{s-4i}$.

Again the same analysis follows, $E(s) \equiv ci$ again. We get:

$$F_+(s) = \frac{ci(s+4i)}{(s-1)(s+1)}.$$

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This time: $f_+(x) = \frac{1}{2\pi}\int_P \frac{ci(s+4i)}{(s-1)(s+1)} e^{-isx} ds$, and both $s = \pm 1$ lie inside the contour, so we get by the residue theorem:

$$f_+(x) = \frac{1}{2\pi} \times (-2\pi i)\, ci \left(\frac{(1+4i)}{2} e^{-ix} + \frac{(-1+4i)}{-2} e^{ix} \right)$$

$$= \frac{c}{2}\left[e^{-ix} + 4ie^{-ix} + e^{ix} - 4ie^{ix} \right]$$

$$= c\left[\frac{e^{ix}+e^{-ix}}{2} + 4\left(\frac{e^{ix}-e^{-ix}}{2i} \right) \right]$$

$$\Rightarrow \boxed{f(x) = c(\cos x + 4\sin x)}, \quad x \geq 0$$

4). $$1 + \alpha x = f(x) + \frac{3}{2}\int_0^\infty e^{-|x-y|} f(y)\, dy, \quad x \geq 0, \quad \alpha > 0.$$

Let $k(x) = \frac{3}{2}e^{-|x|}$, $p(x) = 1 + \alpha x$. Introducing $f_+(x), p_+(x), g_-(x)$ in the usual way gives:

$$\frac{3}{2}\int_0^\infty e^{-|x-y|} f(y)\, dy = p_+(x) - f_+(x) + g_-(x), \quad -\infty < x < \infty.$$

Take the Fourier transform of both sides:

$$\hat{K}(s) F_+(s) = P_+(s) - F_+(s) + G_-(s),$$

where $\hat{K}(s) = \frac{3}{s^2+1}$, $-1 < \text{Im}\{s\} < 1$, by Q1.

To calculate $P_+(s)$, we integrate by parts:

$$P_+(s) = \int_0^\infty (1+\alpha x) e^{isx}\, dx$$

$$= \int_0^\infty e^{isx}\,dx + \underbrace{\left[\frac{\alpha x e^{isx}}{is}\right]_0^\infty}_{\text{converges provided } \operatorname{Im}\{s\}>0} - \alpha\int_0^\infty \frac{1}{is} e^{isx}\,dx$$

$$= \left(1 - \frac{\alpha}{is}\right)\left[\frac{e^{isx}}{is}\right]_0^\infty$$

$$= \frac{-1}{is}\left(1 - \frac{\alpha}{is}\right)$$

$$= \frac{(s+\alpha i)i}{s^2}, \quad \text{provided } \operatorname{Im}\{s\}>0.$$

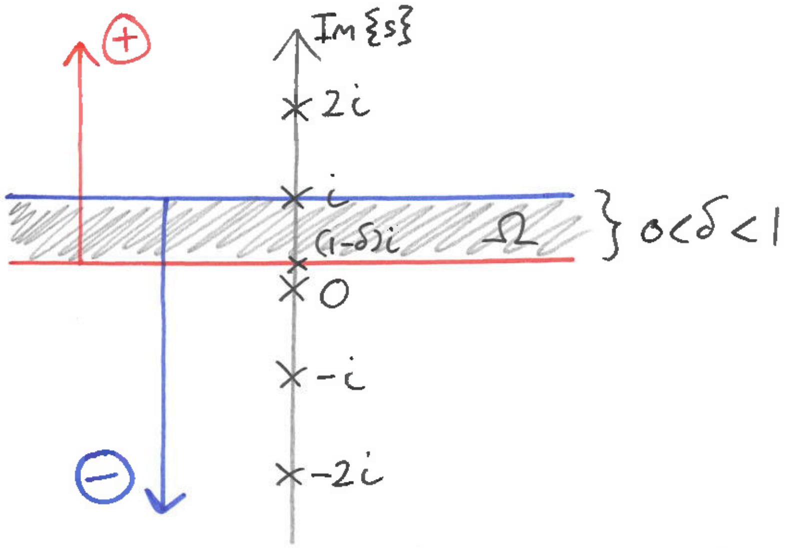
$$\Rightarrow \quad K(s)F_+(s) + G_-(s) = -P_+(s), \quad \text{where } K(s) = -1-\hat{K}(s) = -\frac{(s^2+4)}{(s^2+1)}$$

As usual we have:

$F_+(s)$ analytic in $\operatorname{Im}\{s\} > 1-\delta$,

$G_-(s)$ analytic in $\operatorname{Im}\{s\} < 1$.

So take Ω and $\oplus/\ominus$ to be as shown.



$$K(s) = -\frac{(s+2i)(s-2i)}{(s+i)(s-i)}$$

✓ non-zero + analytic in Ω.

$$P_+(s) = \frac{(s+\alpha i)i}{s^2}$$

✓ analytic in Ω.

write $K(s) = K_+(s)K_-(s)$, where $K_+(s) = \dfrac{s+2i}{s+i}$, $K_-(s) = -\dfrac{(s-2i)}{s-i}$,

then we have: $K_+(s)F_+(s) + \dfrac{G_-(s)}{K_-(s)} = \dfrac{-P_+(s)}{K_-(s)} = R(s) = \dfrac{i(s+\alpha i)(s-i)}{s^2(s-2i)}$.

Now we can write: $R(s) = R_+(s) + R_-(s)$, where:

$$R_+(s) = \frac{i}{2}\left(\frac{\left(1-\frac{\alpha}{2}\right)s + \alpha i}{s^2}\right), \quad R_-(s) = \frac{i}{2}\left(\frac{1+\frac{\alpha}{2}}{s-2i}\right).$$

This comes from splitting $R(s)$ into partial fractions. Each is analytic in ⊕/⊖ respectively. Thus, we get:

$$\underbrace{K_+(s)F_+(s) - R_+(s)}_{\text{analytic in } \oplus} = \underbrace{-\frac{G_-(s)}{K_-(s)} + R_-(s)}_{\text{analytic in } \ominus}, \quad s \in \Omega$$

Then since ⊕/⊖ overlap in Ω, by analytic continuation:

$$E(s) = \begin{cases} K_+(s)F_+(s) - R_+(s), & s \in \oplus \\ R_-(s) - \dfrac{G_-(s)}{K_-(s)}, & s \in \ominus \end{cases}, \quad \text{is } \underline{\text{entire}}.$$

Expand as $s \to \infty$ in ⊕:

$$E(s) \sim \left(1 + O\left(\frac{1}{s}\right)\right)\left(\frac{if(0)}{s} + O\left(\frac{1}{s^2}\right)\right) - O\left(\frac{1}{s}\right)$$

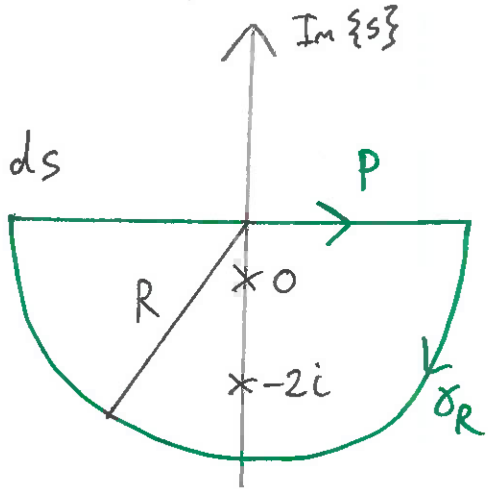
$$\sim O\left(\frac{1}{s}\right) \to 0 \text{ as } s \to \infty.$$

Hence, by Liouville's theorem: $\underline{E(s) \equiv 0}$ for all s. Therefore:

$$F_+(s) = \frac{R_+(s)}{K_+(s)} = \frac{i}{2}\left(\frac{\left(\left(1-\frac{\alpha}{2}\right)s + \alpha i\right)(s+i)}{s^2(s+2i)}\right).$$

Hence via the inversion formula:

$$f_+(x) = \frac{1}{2\pi}\int_P \frac{i}{2}\left(\frac{\left(\left(1-\frac{\alpha}{2}\right)s + \alpha i\right)(s+i)}{s^2(s+2i)}\right)e^{-isx}\, ds$$



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For $x \geq 0$, we close P with a semi-circle γ_R below P of radius R and take $R \to \infty$. Let $\gamma = P + \gamma_R$. Now:

$$\text{Res}\left\{ F_+(s)e^{-isx},\ s=0 \right\} = \lim_{s \to 0} \frac{d}{ds}\left[\frac{i\left(\alpha i + (1-\frac{\alpha}{2})s\right)(s+i)e^{-isx}}{2(s+2i)} \right]$$

$$= \frac{i}{4}(1+\alpha x).$$

$$\text{Res}\left\{ F_+(s)e^{-isx},\ s=-2i \right\} = \frac{i}{4}(1-\alpha)e^{-2x}.$$

Hence by the residue theorem:

clockwise integration

$$\oint_\gamma F_+(s)e^{-isx}\,ds = -2\pi i\left(\frac{i}{4}\left(1+\alpha x + (1-\alpha)e^{-2x}\right)\right)$$

$$= \frac{\pi}{2}\left(1+\alpha x + (1-\alpha)e^{-2x}\right)$$

Now in the limit as $R \to \infty$, $\int_{\gamma_R} \to 0$, hence:

$$\boxed{f(x) = \frac{1}{4}\left(1+\alpha x + (1-\alpha)e^{-2x}\right)}, \quad x \geq 0.$$

5). $$f'(x) + 6f(x) = 12\int_0^\infty e^{-4|x-y|} f(y)\,dy, \quad x \geq 0.$$

$f(0) = 2$. Let $k(x) = 12e^{-4|x|}$. Then:

$$12\int_0^\infty e^{-4|x-y|} f(y)\,dy = f_+'(x) + 6f_+(x) + g_-(x), \quad -\infty < x < \infty$$

Take Fourier transforms of both sides:

$$\hat{K}(s)F_+(s) = -\underbrace{f_+(0)}_{=2} - isF_+(s) + 6F_+(s) + G_-(s)$$

$$\Rightarrow K(s)F_+(s) + G_-(s) = 2, \quad \text{where } K(s) = 6 - is - \hat{K}(s).$$

From Q1). $\hat{K}(s) = \dfrac{96}{s^2+16}$ $\Rightarrow$ $K(s) = (6-is) - \dfrac{96}{s^2+16}$

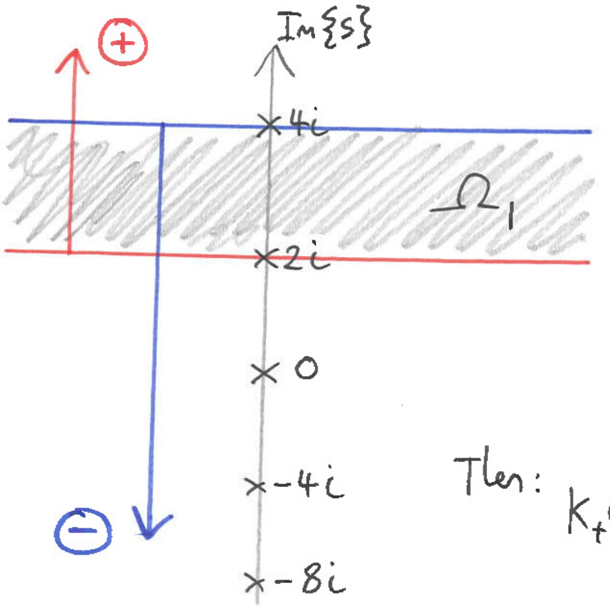
$$= \frac{(6-is)(s^2+16) - 96}{s^2+16}$$

$$= \frac{-is(s+8i)(s-2i)}{(s+4i)(s-4i)}$$

$F_+(s)$ is analytic in $\oplus$: $\mathrm{Im}\{s\} > 4-\delta$

$G_-(s)$ is analytic in $\ominus$: $\mathrm{Im}\{s\} < 4$

(i). Take $\Omega_1 = \{s: 2 < \mathrm{Im}\{s\} < 4\}$, i.e. take $0 < \delta < 2$.



write:

$$K_+(s) = \frac{-is(s+8i)(s-2i)}{s+4i}$$

$$K_-(s) = \frac{1}{s-4i}$$

Then: $K_+(s)F_+(s) + \dfrac{G_-(s)}{K_-(s)} = \dfrac{2}{K_-(s)} = R(s) = 2(s-4i)$

This is analytic (except at ∞), so we can let: $R(s) = R_+(s) + R_-(s)$, where:

$R_+(s) = 2(s-4i)$, $R_-(s) = 0$ (It doesn't matter how we split $R(s)$).

$$\Rightarrow K_+(s)F_+(s) - R_+(s) = -\frac{G_-(s)}{K_-(s)}, \quad s \in \Omega$$

$$\Rightarrow E(s) = \begin{cases} K_+F_+ - R_+, & s \in \oplus \\ -\dfrac{G_-}{K_-}, & s \in \ominus \end{cases}$$, is entire by analytic continuation.

Let $s \to \infty$, we find:

$$E(s) \sim \left(-is(s+8i)(s-2i)\frac{1}{s}\left(1+\frac{4i}{s}\right)^{-1}\right)\left[\frac{if_+(0)}{s} - \frac{\overbrace{f_+'(0)}^{=A,\text{ say}}}{s^2} + O\left(\frac{1}{s^3}\right)\right]$$

$$- (2s-8i)$$

$$\sim -i(s^2+6is+16)\left[1-\frac{4i}{s}+O\left(\frac{1}{s^2}\right)\right]\left(\frac{2i}{s}-\frac{A}{s^2}+O\left(\frac{1}{s^3}\right)\right)$$

$$-(2s-8i)$$

$$\sim \left(-is^2-4s+6s+O(1)\right)\left(\frac{2i}{s}-\frac{A}{s^2}+O\left(\frac{1}{s^3}\right)\right) -2s+8i$$

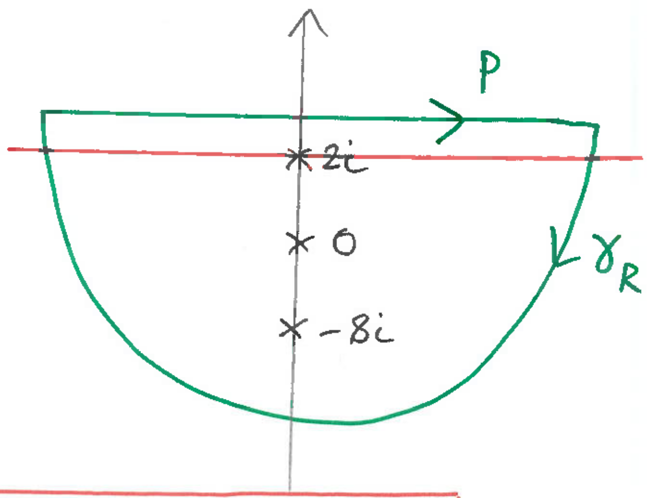
$$\sim 2s + Ai + 4i \quad -2s + 8i + O\left(\frac{1}{s}\right)$$

$$\sim \underline{\underline{(A+12)i}} \quad \text{as } s \to \infty.$$

Therefore, by Liouville's theorem: $E(s) \equiv (A+12)i$.

$$\Rightarrow \quad F_+(s) = \frac{(A+12)i + R_+(s)}{K_+(s)} = \frac{2i\left(s+\left(2+\frac{A}{2}\right)i\right)(s+4i)}{s(s+8i)(s-2i)}$$

$$\Rightarrow \quad f_+(x) = \frac{1}{2\pi}\int_P F_+(s)e^{-isx}\,ds$$



$$\text{Res}\{s=2i\} = \left(\frac{6}{5}+\frac{3A}{20}\right)e^{2x}(2i)$$

$$\text{Res}\{s=0\} = \left(-\frac{1}{2}-\frac{A}{8}\right)(2i)$$

$$\text{Res}\{s=-8i\} = \left(\frac{3}{10}-\frac{A}{40}\right)e^{-8x}(2i)$$

$$\Rightarrow \quad \boxed{f_+(x) = \left(\frac{12}{5}+\frac{3A}{10}\right)e^{2x} + \left(-1-\frac{A}{4}\right) + \left(\frac{3}{5}-\frac{A}{20}\right)e^{-8x}}$$

(ii). Take $\underline{\Omega_2 = \{s : 0 < \text{Im}\{s\} < 2\}}$, i.e. take $\underline{2 < \delta < 4}$.

In this case: $K_+(s) = \dfrac{-is(s+8i)}{s+4i}$, $K_-(s) = \dfrac{(s-2i)}{s-4i}$.

This gives: $K_+F_+ + \frac{G_-}{K_-} = \frac{2}{K_-} = R(s) = \frac{2(s-4i)}{s-2i}$.

Then we can write $R(s) = R_+(s) + R_-(s)$ with:

$$R_-(s) = \frac{2(s-4i)}{s-2i}, \quad R_+(s) = 0.$$

This leads to $E(s) = \begin{cases} K_+F_+, & s \in \oplus \\ -\frac{G_-}{K_-} + R_-, & s \in \ominus \end{cases}$, but now as $s \to \infty$ we find:

$$E(s) \sim -is(s+8i)\frac{1}{s}\left(1+\frac{4i}{s}\right)^{-1}\left[\frac{2i}{s} + O\left(\frac{1}{s^2}\right)\right] \cancel{-2(s-4i)\frac{1}{s}\left(1-\frac{2i}{s}\right)^{-1}}$$

$$\sim -i(s+8i)\left(1+O\left(\frac{1}{s}\right)\right)\left(\frac{2i}{s} + O\left(\frac{1}{s^2}\right)\right) \cancel{-2+O\left(\frac{1}{s}\right)}$$

$$\sim 2 + O\left(\frac{1}{s}\right)$$

$$\sim 2 \text{ as } s \to \infty$$

$$\Rightarrow K_+F_+ = 2 \Rightarrow \cancel{F_+(s) = \frac{2(s-4i)(s+4i)}{(s-2i)(-is)(s+8i)}}$$

$$\Rightarrow F_+(s) = \frac{2(s+4i)}{-is(s+8i)} = \frac{2i(s+4i)}{s(s+8i)}.$$

Now there are poles at $s=0$ and $s=-8i$ only, we find using the inversion formula and taking residues that:

$$f_+(x) = \frac{1}{2\pi} \times (-2\pi i)\left(\frac{(2i)(4i)}{8i} + \frac{(2i)(-4i)}{(-8i)}e^{-8x}\right)$$

$$\Rightarrow \boxed{f(x) = 1 + e^{-8x}}, \quad x \geq 0$$

Notice that by dropping the strip below $\mathrm{Im}\{s\} = 2$ in the s-plane, we have lost solutions which can grow like e^{2x} in the solution. This is equivalent to setting $\boxed{A = -8}$ in our previous answer and that recovers the solution found here.

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6). $$f''(x) + f(x) = e^{-x} + \int_0^\infty e^{-|x-y|} f(y)\,dy \,, \quad x \geq 0 .$$

$f'(0) = 0$. Let $k(x) = e^{-|x|}$, $p(x) = -e^{-x}$. Then:

$$\int_0^\infty e^{-|x-y|} f(y)\,dy = f_+''(x) + f_+(x) + p_+(x) + g_-(x) \,, \quad -\infty < x < \infty$$

Take Fourier transforms:

$$\hat{K}(s) F_+(s) = -\overbrace{f_+'(0)}^{=0} + is\overbrace{f_+(0)}^{=\varepsilon,\ \text{say}} - s^2 F_+(s) + F_+(s) + P_+(s) + G_-(s)$$

$$\Rightarrow K(s) F_+(s) + G_-(s) = -i\varepsilon s - P_+(s) \,, \quad \text{where:}$$

$$K(s) = 1 - s^2 - \hat{K}(s) \quad \text{and} \quad \hat{K}(s) = \frac{2}{s^2+1} \,, \quad P_+(s) = \frac{-i}{s+i} .$$

$$\Rightarrow K(s) = -\frac{(s^4+1)}{s^2+1} = -\frac{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}{(s+i)(s-i)} \,,$$

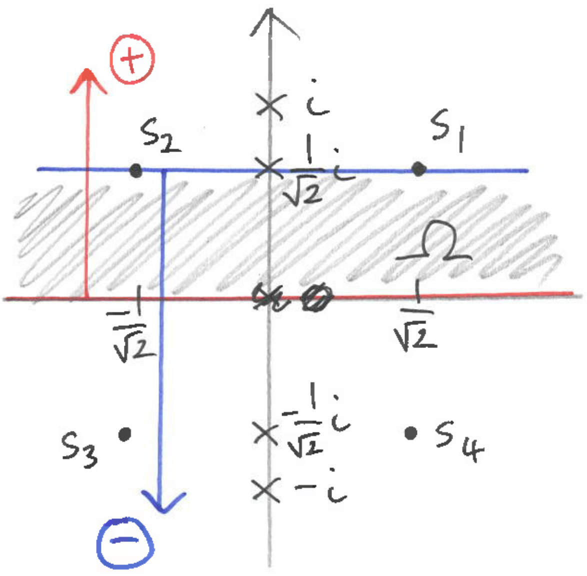
where $s_1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ $\qquad s_3 = e^{-\frac{3\pi}{4} i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

$s_2 = e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ $\qquad s_4 = e^{-\frac{\pi}{4} i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i .$

$F_+(s)$ is analytic for $\mathrm{Im}\{s\} > 1 - \delta$.

$G_-(s)$ analytic for $\mathrm{Im}\{s\} < 1$.

Take Ω to be the strip $\Omega = \{s : 0 < \mathrm{Im}\{s\} < \frac{1}{\sqrt{2}}\}$ and $\oplus / \ominus$ as shown:



Now decompose $K(s) = K_+(s) K_-(s)$ as:

$$K_+(s) = -\frac{(s-s_3)(s-s_4)}{s+i} \,,$$

$$K_-(s) = \frac{(s-s_1)(s-s_2)}{s-i}$$

And noting that $s_3 = -s_1$, $s_4 = -s_2$, we can write:

$$K_+(s) = -\frac{(s+s_1)(s+s_2)}{s+i}, \quad \text{so we have:}$$

$$K_+(s)F_+(s) + \frac{G_-(s)}{K_-(s)} = \frac{-i\varepsilon s + \frac{i}{s+i}}{K_-(s)} = R(s) = \frac{(-i\varepsilon s^2 + \varepsilon s + i)(s-i)}{(s+i)(s-s_1)(s-s_2)}$$

Now (after much algebra) we can write:

$$R(s) = \underbrace{\frac{As^2+Bs+C}{(s-s_1)(s-s_2)}}_{R_-(s)} + \underbrace{\frac{\sqrt{2}-2}{s+i}}_{=R_+(s)}, \quad \text{for some constants } A, B, C \text{ which can be calculated if desired.}$$

$$\Rightarrow K_+(s)F_+(s) - R_+(s) = -\frac{G_-(s)}{K_-(s)} + R_-(s), \quad s \in \Omega$$

$$\Rightarrow E(s) = \begin{cases} K_+F_+ - R_+, & s \in \oplus \\ -\frac{G_-}{K_-} + R_-, & s \in \ominus \end{cases} \text{ is } \underline{\text{entire}}, \text{ by analytic continuation.}$$

Let $s \to \infty$ in $\oplus$:

$$E(s) \sim -(s+s_1)(s+s_2)\frac{1}{s}\left(1+\frac{i}{s}\right)^{-1}\left[\frac{i\varepsilon}{s} + O\left(\frac{1}{s^3}\right)\right] - O\left(\frac{1}{s}\right)$$

$$\sim (-s + O(1))\left(\frac{i\varepsilon}{s} + O\left(\frac{1}{s^3}\right)\right) + O\left(\frac{1}{s}\right)$$

$$\sim -i\varepsilon + O\left(\frac{1}{s}\right) \to -i\varepsilon \text{ as } s \to \infty$$

$$\Rightarrow \underline{E(s) \equiv -i\varepsilon}$$

$$\Rightarrow F_+(s) = \frac{-i\varepsilon + R_+(s)}{K_+(s)} = \frac{i\varepsilon s - \varepsilon + 2 - \sqrt{2}}{(s+s_1)(s+s_2)}. \qquad (*)$$

At this stage we can find the value of ε. We know that as $s \to \infty$ we must have $F_+(s) = \frac{i f_+(0)}{s} - \frac{f_+'(0)}{s^2} + O\left(\frac{1}{s^3}\right)$, on expanding $(*)$:

$$F_+(s) \sim \frac{i\varepsilon}{s} + \frac{2-\sqrt{2} - \varepsilon(1-\sqrt{2})}{s^2} + O\left(\frac{1}{s^3}\right).$$

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Since we know $f_+'(0) = 0$, then comparing the two expansions gives:

$$2-\sqrt{2} - \varepsilon(1-\sqrt{2}) = 0 \Rightarrow \boxed{\varepsilon = -\sqrt{2}}$$, as given in the question.

$$\Rightarrow F_+(s) = \frac{-\sqrt{2}i(s+\sqrt{2}i)}{(s+s_1)(s+s_2)}$$

$$\Rightarrow f_+(x) = \frac{1}{2\pi}\int_P F_+(s)\, e^{-isx}\, ds$$, where now simple poles at $s=-s_1$ and $s=-s_2$ lie inside the contour.

Using the residue theorem to evaluate this we find:

$$\boxed{f(x) = -\sqrt{2}\, e^{-\frac{1}{\sqrt{2}}x}\left(\cos\frac{x}{\sqrt{2}} + \sin\frac{x}{\sqrt{2}}\right)} \quad , x \geq 0$$