Calculus I. Midterm Nov 2022

1 (a) (i) $f(x) = x^x$

$\log f = x \log x \quad \lim_{x \to 0+} x \log x = 0$

$\Rightarrow \lim_{x \to 0+} x^x = 1$ 3 marks

(a) (ii) $\frac{1}{f} f' = \log x + 1$

$f' = x^x(\log x + 1)$

$\lim_{x \to 0+} f' = \lim_{x \to 0+} (x^x \log x + x^x)$ 3

$= -\infty$ since $x^x \to 1$ and $\log x \to -\infty$

(a)(iii) From (ii) $f' = 0$ when $\log x = -1$

ie $x = 1/e \quad f(1/e) = \frac{1}{e^{1/e}} < 1$

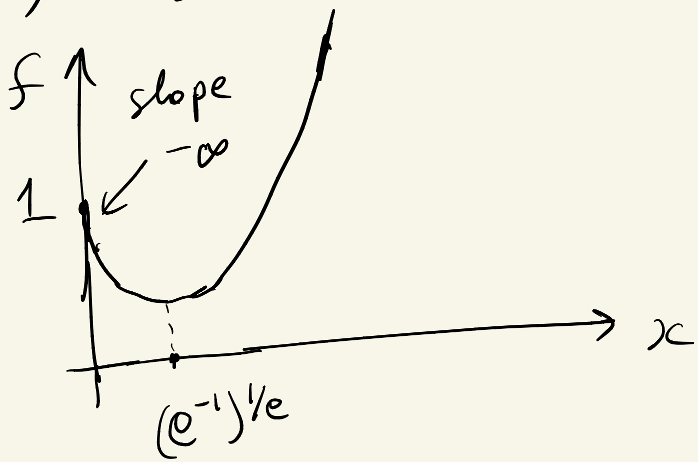
Also, $f(0) = 1$, $f \to \infty$ as $x \to \infty$ $\Rightarrow$

$(e^{-1})^{1/e}$ is the minimum value

It is a local minimum

3 marks

1(a)(iv) Sketch



3

1(b)(i) $g(x) = x^x \sin x = f(x) \sin x$

$$g' = f' \sin x + f \cos x$$

$$= x^x (\log x + 1) \sin x + x^x \cos x$$

$$\lim_{x \to 0+} g(x) = \lim_{x \to 0+} x^x \sin x \to \lim_{x \to 0+} \sin x = 0$$

since $x^x \to 1$ as $x \to 0+$

$$\lim_{x\to 0+} g' = \lim_{x\to 0+} x^x \log x \sin x + \lim_{x\to 0+} x^x \sin x$$
$$+ \lim_{x\to 0+} x^x \cos x$$

$\log x \sin x \to 0$ so only last term contributes

$$\lim_{x\to 0+} g' = 1$$

2

(b)(ii) $g(0) = 0$ (just shown)

$$g(\pi) = \pi^\pi \sin(\pi) = 0$$

$g(x)$ is continuous and diff. on $(0,\pi)$

hence by the mean value theorem

$\exists\, \xi \in (0,\pi)$ s.t. $g'(\xi) = 0$

Since $g(x) \geq 0$ for $x \in [0,\pi]$

the critical point $x = \xi$ is a maximum

3

(b) (iii) From part (i) we have

$$g'(x) = x^x(\log x + 1)\sin x + x^x \cos x$$

$$\Rightarrow \quad g'\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^{\pi/2}\left(\log\frac{\pi}{2} + 1\right) > 0$$

$$g'(\pi) = \pi^\pi \cos\pi = -\pi^\pi < 0$$

By the intermediate value theorem

$$\exists\ \xi \in \left(\frac{\pi}{2}, \pi\right) \quad \text{s.t.} \quad g'(\xi) = 0$$

3

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2 (a) $$f(x) = \frac{x(\alpha^2 - x^2)}{\sqrt{x^2-1}}$$

undefined for $|x| \le 1$

Domain $|x| > 1$.

2

$f(-x) = -f(x) \Rightarrow$ odd function

② (b) $f(x) = \dfrac{x(\alpha^2 - x^2)}{\sqrt{x^2-1}}$

$$f' = \frac{\alpha^2 - 3x^2}{\sqrt{x^2-1}} - \frac{x^2(\alpha^2 - x^2)}{(x^2-1)^{3/2}}$$

$$= \frac{(\alpha^2 - 3x^2)(x^2-1) - x^2(\alpha^2 - x^2)}{(x^2-1)^{3/2}}$$

$$= \frac{-2x^4 + 3x^2 - \alpha^2}{(x^2-1)^{3/2}}$$

b (i) $\underline{\alpha = 2}$ $\quad f' = \dfrac{-2\left(x^4 - \frac{3}{2}x^2 + 2\right)}{(x^2-1)^{3/2}}$

$$x^4 - \frac{3}{2}x^2 + 2 = \left(x^2 - \frac{3}{4}\right)^2 + 2 - \frac{9}{16} > 0$$

for $x > 1$

$\Rightarrow f'(x) < 0 \quad x > 1$

4

b (ii) $\underline{\alpha = \frac{1}{2}}$ $\quad f' = 0$ when $2x^4 - 3x^2 + \frac{1}{4} = 0$

$$\Rightarrow x^2 = \frac{3 \pm \sqrt{9-2}}{4} = \frac{3+\sqrt{7}}{4}, \frac{3-\sqrt{7}}{4}$$

Only 1st root is in $|x|>1$ so the critical point is $x = \frac{(3+\sqrt{7})^{1/2}}{2}$

It is a local maximum.
No need for a 2nd derivative

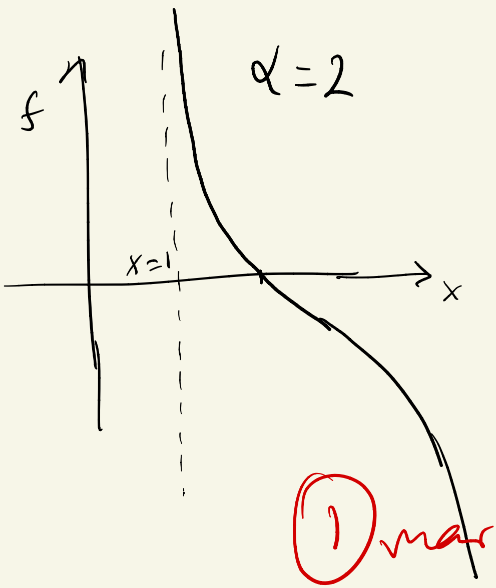
as $x \to 1+$ $f(x) \to -\infty$

as $x \to \infty$ $f(x) \to -\infty$

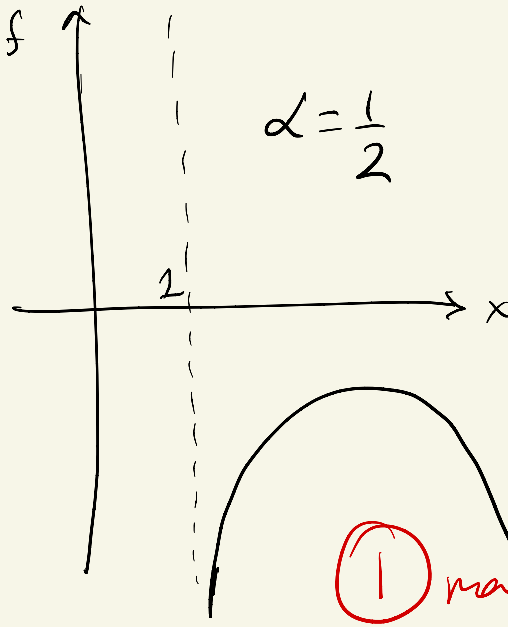
4

Also $f(x) < 0$ for all $x > 1$

b(iii)



1 mark



1 mark

② (c) $\displaystyle\int_1^{\alpha} \frac{x(\alpha^2-x^2)}{\sqrt{x^2-1}}\,dx = \lim_{\varepsilon\to 0+}\int_{1+\varepsilon}^{\alpha}\cdots$

$$= \lim_{\varepsilon\to 0+}\left[\underbrace{(x^2-1)^{1/2}(\alpha^2-x^2)}_{\to 0}\Big|_{1+\varepsilon}^{\alpha} + \int_{1+\varepsilon}^{\alpha}(x^2-1)^{1/2}\,2x\,dx\right]$$

$$= \lim_{\varepsilon\to 0} (x^2-1)^{3/2}\frac{2}{3}\Big|_{1+\varepsilon}^{\alpha} = \frac{2}{3}(\alpha^2-1)^{3/2}$$

4 marks

(d) Near $x=1$ $\quad f(x) \approx \dfrac{\alpha^2-1}{\sqrt{x-1}}$

So $f' \approx -\frac{1}{2}(x-1)^{-3/2}(\alpha^2-1)$

$f'' \approx \frac{3}{4}(x-1)^{-5/2}(\alpha^2-1) \to +\infty$ as $x\to 1$

Large x $\quad f \approx -x^2 \Rightarrow f'' \approx -2 < 0$

By IVT $\exists\, c$ s.t. $f''(c)=0$ $\qquad$ 4