Midterm Solutions

08 February 2021 22:09

Q1 $f(x) = \sin x \cos^2 x = \sin x \dfrac{(1+\cos 2x)}{2}$

$$= \frac{1}{4i}(e^{ix} - e^{-ix}) + \frac{1}{8i}[e^{3ix} + e^{-ix} - e^{ix} - e^{-3ix}]$$

$$= \frac{1}{8i}[e^{3ix} - e^{-ix} + e^{ix} - e^{-3ix}]$$

$\frac{3}{6}$

Given the result $\delta(w - \omega_0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{\pm i(w-w_0)x}\, dx$

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

$$= \frac{\pi}{4i}\left[\delta(w-3) - \delta(w+1) + \delta(w-1) - \delta(w+3)\right]$$

Note $\delta(x) = \delta(-x)$

so $[\delta(3-w) - \delta(-1-w) + \delta(1-w) \cdots]$ also correct.

$\frac{3}{6}$

Q2. This is a third order Euler-Cauchy ODE (section 5.5 in notes). So $x = e^z$ works

We have:

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \qquad \text{(from notes)}$$

$$x^3\frac{d^3y}{dx^3} = x^3\frac{d}{dx}\left(\frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]\right)$$

$$= x^3\frac{1}{x^2}\left[\frac{d}{dx}\frac{d^2y}{dz^2} - \frac{d}{dx}\frac{dy}{dz}\right] + x^3\left(\frac{-2}{x^3}\right)\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$= x\frac{dz}{}\left[\frac{d^3y}{} \quad \frac{d^2y}{}\right] - 2\left[\frac{d^2y}{} - \frac{dy}{}\right]$$

$$= x \frac{dz}{dx}\left[\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right] - 2\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$= \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$$ $\frac{3}{7}$

So we have in terms of $z = \ln x$

$$\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = e^{-z}$$

$\Rightarrow (\lambda-1)^3 = 0$
$\Rightarrow \lambda = 1$ repeated 3 times

$$y_{CF} = c_1 e^{z} + c_2 z e^{z} + c_3 z^2 e^{z}$$ $\frac{2}{7}$

Ansatz $y_{PI} = Ae^{-z} \Rightarrow A = -\frac{1}{8}$ $\frac{1}{7}$

$$y_{GS}(x) = c_1 x + c_2 x \ln x + c_3 x(\ln x)^2 - \frac{1}{8x}$$ $\frac{1}{7}$

Q3 we define $u = \frac{dx}{dt}$, we have

$$\frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}}_{A}\begin{pmatrix} x \\ u \end{pmatrix}$$ $\frac{3}{7}$

Eigen values of $A \Longrightarrow \lambda^2 + 3\lambda - 4 = 0$ $\lambda = 1, -4$

$\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -4 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ $\frac{2}{7}$

$$\begin{pmatrix} x_{GS} \\ u_{GS} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$$ $\frac{2}{7}$

please note if u is defined differently the system of please note if u is defined differently the system of ODEs would be different but solution for x should be independent of choice of u. Give full mark if done correctly for a diffrent choice of u