Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2016

This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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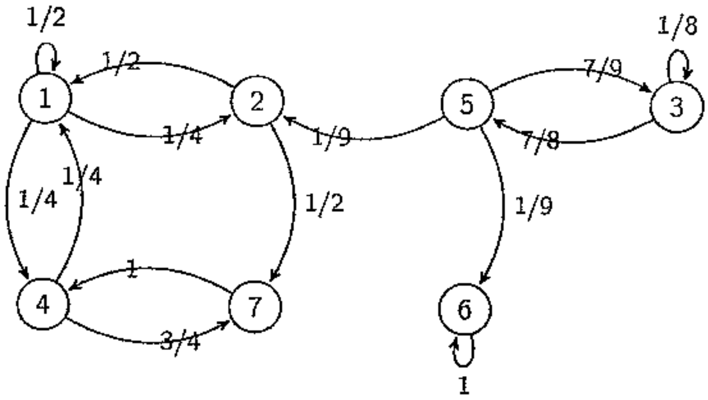

$\boxed{\text{meth seen} \Downarrow}$

1. (1) $\boxed{3}$

   (2) We have three communicating classes: $\{1,2,4,7\}$, $\{6\}$, and $\{3,5\}$. $\boxed{3}$

   (3) From the graph, we have two recurrent classes: $\{1,2,4,7\}$ and $\{6\}$, and one transient class $\{3,5\}$. $\boxed{3}$

   (4) It's a reducible chain: the chain has two recurrent classes. $\boxed{2}$

   (5) Using the formula: $\mathbb{P}_i(X_2 = j) = Q^2(i,j) = \sum_k Q(i,k)Q(k,j)$, we have

$$\mathbb{P}_3(X_2 = 6) = Q(3,5)Q(5,6) = \frac{7}{8} \cdot \frac{1}{9} = \frac{7}{72}.$$

$$\mathbb{P}_1(X_2 = 7) = Q(1,2)Q(2,7) + Q(1,4)Q(4,7) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} = \frac{5}{16}.$$

$\boxed{4}$

2. (1) $\mathbb{E}(X_t - X_s|\mathcal{F}_s) = \mathbb{E}(N_t - N_s - \lambda(t-s)|\mathcal{F}_s) = \mathbb{E}(N_t - N_s|\mathcal{F}_s) - \lambda(t-s) = \lambda(t-s) - \lambda(t-s) = 0$. $\boxed{\text{seen} \Downarrow}$

Therefore $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$. $\boxed{3}$

(2) By definition,

$$\mathbb{E}[e^{-uN_t}] = \sum_{n=0}^{\infty} e^{-un}\mathbb{P}(N_t = n) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t e^{-u})^n}{n!} = \exp\{\lambda t[e^{-u} - 1]\}.$$

Therefore,

$$\phi(u) = \mathbb{E}[e^{-uX_t}] = \exp\{\lambda t[u + e^{-u} - 1]\}.$$

$\boxed{3}$

(3) The Laplace transform of S_t is $\boxed{\text{part seen} \Downarrow}$

$$\begin{aligned}
\mathbb{E}(e^{-uS_t}) &= \mathbb{E}(e^{-u\sum_{i=1}^{N_t} Y_i}) \\
&= \sum_{n=0}^{\infty}\mathbb{E}(e^{-u\sum_{i=1}^{n} Y_i})\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty}\prod_{i=1}^{n}\mathbb{E}(e^{-uY_i})\mathbb{P}(N_t = n) \quad (\textit{the r.v } Y_i \textit{ are iid}) \\
&= \sum_{n=0}^{\infty} G(u)^n\mathbb{P}(N_t = n) \\
&= e^{-\lambda t}\sum_{n=0}^{\infty} G(u)^n\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}e^{\lambda t G(u)}
\end{aligned}$$

$\boxed{3}$

Since $\mathbb{E}(S_t) = \psi'(0)$ and $\mathbb{E}(Y_t) = G'(0)$, we have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\mathbb{E}(S_t) &= \psi'(0) \\
&= \lambda t G'(0)\underbrace{\exp(-\lambda t + \lambda t G(0))}_{=1} \\
&= \lambda t \mathbb{E}(Y_t)
\end{aligned}$$

$\boxed{3}$

Also, we have $\mathbb{E}(Y_t^2) = G''(0)$

$$\begin{aligned}
\mathrm{Var(S_t)} &= \mathbb{E}(S_t^2) - \mathbb{E}(S_t)^2 \\
&= [\lambda t G''(0) - \lambda t G'(0)^2]\exp(-\lambda t + \lambda t G(0)) \\
&= \lambda t G''(0) - \lambda t G'(0)^2 \\
&= \lambda t \mathbb{E}(Y_t^2) - \lambda t \mathbb{E}(Y_t)^2 \\
&= \lambda t \mathrm{Var(Y_t)}.
\end{aligned}$$

$\boxed{3}$

3. (1) Let $\{P_t\}_{t\geq 0}$ denote the standard stochastic semigroup associated with the Markov chain. The generator is defined as

$$G = \lim_{h\to 0}\frac{1}{h}(P_h - I).$$

$\boxed{3}$

(2) From the lectures, we know that the transition probabilities , denoted by p_{ij} for $i,j\in E$, of the corresponding jump chain are given by $p_{ij} = g_{ij}/(-g_{ii})$ for $i\neq j$ if $-g_{ii}>0$. If $-g_{ii}=0$, then the state i is absorbing. Noting that the row elements in the transition have to sum to one we get

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 \\ 2/5 & 0 & 0 & 3/5 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\boxed{4}$

(3) (i) The generator is

$$\mathbf{P} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(2\lambda+\mu) & 2\lambda & 0 & 0 & \cdots \\ 0 & 4\mu & -(3\lambda+4\mu) & 3\lambda & 0 & \cdots \\ 0 & 0 & 9\mu & -(4\lambda+9\mu) & 4\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\boxed{4}$

(ii) As in the lectures, we use the result that $\pi G = 0$, then

$$\begin{aligned} -\lambda_0\pi_0 + \mu_1\pi_1 &= 0 \\ -\lambda_{n-1}\pi_{n-1} - (\lambda_n+\mu_n)\pi_n + \mu_{n+1}\pi_{n+1} &= 0 \quad n\geq 1. \end{aligned}$$

This leads to

$$\pi_n = \frac{\lambda_0\times\cdots\times\lambda_{n-1}}{\mu_1\times\cdots\times\mu_n}\pi_0$$

for any $n\in\mathbb{N}$. Such a vector π is stationary distribution if and only if $\sum_n \pi_n = 1$; that is

$$\sum_{n=0}^{\infty}\frac{\lambda_0\times\cdots\times\lambda_{n-1}}{\mu_1\times\cdots\times\mu_n} < \infty,$$

with the term $(n=0)$ defined to be 1 (i.e. $\lambda_0\lambda_{-1}/\mu_1\mu_0 = 1$). Given this condition, it follows that

$$\pi_0 = \left(\sum_{n=0}^{\infty}\frac{\lambda_0\times\cdots\times\lambda_{n-1}}{\mu_1\times\cdots\times\mu_n}\right)^{-1}.$$

Here, we get

$$\pi_0 = \left(\sum_{n=0}^{\infty}\frac{\lambda_0\times\cdots\times\lambda_{n-1}}{\mu_1\times\cdots\times\mu_n}\right)^{-1} = \left(\sum_{n=0}^{\infty}\frac{\lambda^n n!}{\mu^n (n!)^2}\right)^{-1} = \left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\mu^n n!}\right)^{-1} = e^{\lambda/\mu},$$

and for $n\in\mathbb{N}$, we have

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}e^{-\lambda/\mu}.$$

$\boxed{4}$

4. (1) For $0 \le s < t$, from the lectures, we have $\boxed{\text{seen} \Downarrow}$

$$B_t - B_s \sim N(0, t-s).$$

$\boxed{2}$

(2) For $s < t$, and using the fact that $B_t - B_s$ and B_s are independent, we have $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}\mathrm{Cov}(B_t, B_s) &= \mathbb{E}(B_t B_s) \\ &= \mathbb{E}((B_t - B_s)B_s) + \mathbb{E}(B_s^2) \\ &= 0 + s = s\end{aligned}$$

For $t < s$, and using the fact that $B_s - B_t$ and B_t are independent, we have $\boxed{3}$

$$\begin{aligned}\mathrm{Cov}(B_t, B_s) &= \mathbb{E}(B_t B_s) \\ &= \mathbb{E}((B_s - B_t)B_t) + \mathbb{E}(B_t^2) \\ &= 0 + t = t\end{aligned}$$

Therefore , $\boxed{3}$

$$\mathrm{Cov}(B_t, B_s) = \min(t, s).$$

$\boxed{\text{part seen} \Downarrow}$

(3) The stochastic process W is a Gaussian process with zero mean $\mathbb{E}(W_t) = a\mathbb{E}(B_{t/a^2}) = 0$, and

$$\mathrm{Cov}(W_t, W_s) = a^2\mathbb{E}(B_{t/a^2} B_{s/a^2}) = a^2 \min(t/a^2, s/a^2) = \min(t, s).$$

The process W has also continuous sample paths. Therefore, from the lectures, W is a standard Brownian motion.

$\boxed{4}$

(4) No, since for $0 \le s \le t$, $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\mathrm{Var}(X_t - X_s) &= \mathrm{Var}(\sqrt{t}Z_t - \sqrt{s}Z) \\ &= (\sqrt{t} - \sqrt{s})^2 \mathrm{Var}(Z) \\ &= (\sqrt{t} - \sqrt{s})^2 \neq t - s.\end{aligned}$$

$\boxed{3}$

Mastery Question

part seen ⇓

5. (1) The process X is the sum of two Gaussian processes. Therefore it is a Gaussian process with continuous sample paths. We have

$$\mathbb{E}(X_t) = \rho\mathbb{E}(W_t) + \sqrt{1-\rho^2}\mathbb{E}(\widetilde{W}_t) = 0.$$

Moreover,

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= \mathbb{E}(X_t X_s) \\
&= \mathbb{E}\left[(\rho W_t + \sqrt{1-\rho^2}\widetilde{W}_t)(\rho W_s + \sqrt{1-\rho^2}\widetilde{W}_s)\right] \\
&= \mathbb{E}\left[\rho^2 W_t W_s + \rho\sqrt{1-\rho^2}W_t\widetilde{W}_s + \rho\sqrt{1-\rho^2}W_s\widetilde{W}_t + (1-\rho^2)\widetilde{W}_t\widetilde{W}_s\right] \\
&= \rho^2(\min(s,t)) + 0 + 0 + (1-\rho^2)(\min(s,t)) \\
&= \min(s,t)
\end{aligned}$$

Therefore, X is a Brownian motion. 5

(2)

unseen ⇓

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}(e^{\sigma W_t - \sigma^2 t/2}) \\
&= e^{-\sigma^2 t/2}\mathbb{E}(e^{\sigma W_t}) \\
&= e^{-\sigma^2 t/2}\int_{\mathbb{R}} e^{\sigma x}\cdot\frac{1}{\sqrt{2\pi t}}e^{\frac{-x^2}{2t}}dx \\
&= e^{-\sigma^2 t/2}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}e^{\sigma^2 t/2}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}dz = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-z^2/2}dz \\
&= 1
\end{aligned}$$

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