Math40002 Analysis 1 — Problem Sheet 4

1. Consider the following properties of a sequence of real numbers $(a_n)_{n\geq 0}$:

   (i) $a_n \to a$, or

   (ii) "a_n eventually equals a" – i.e. $\exists N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ a_n = a$, or

   (iii) "(a_n) is bounded" – i.e. $\exists\, R \in \mathbb{R}$ such that $|a_n| < R \ \ \forall n \in \mathbb{N}_{>0}$.

   For each statement (a-e) below, which of (i-iii) is it equivalent to? Proof?

   (a) $\exists\, N \in \mathbb{N}_{>0}$ such that $\forall n \geq N,\ \forall \epsilon > 0,\ |a_n - a| < \epsilon$.

   (b) $\forall \epsilon > 0$ there are only finitely many $n \in \mathbb{N}_{>0}$ for which $|a_n - a| \geq \epsilon$.

   (c) $\forall N \in \mathbb{N}_{>0},\ \exists\, \epsilon > 0$ such that $n \geq N \ \Rightarrow\ |a_n - a| < \epsilon$.

   (d) $\exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}_{>0},\ |a_n - a| < \epsilon\ \forall n \geq N$.

   (e) $\forall R > 0\ \exists\, N \in \mathbb{N}_{>0}$ such that $n \geq N \ \Rightarrow\ a_n \in (a - \frac{1}{R}, a + \frac{1}{R})$.

   (a) $\iff$ (ii) because "$\forall \epsilon > 0,\ |a_n - a| < \epsilon$" is the same statement as "$a_n = a$".

   (Proof: if $a_n \neq a$ then set $\epsilon := |a_n - a| > 0$ so that $|a_n - a| < \epsilon$ is not true.)

   (b) $\iff$ (i). Suppose (b) is true. Fix any $\epsilon > 0$ and let $n_1, \ldots, n_r$ be the finite number of n_i with $|a_{n_i} - a| \geq \epsilon$.

   Set $N := \max\{n_1, \ldots, n_r\} + 1$. Then $\forall n \geq N$ we have $|a_n - a| < \epsilon$, so $a_n \to a$.

   Suppose (i) is true. Fix any $\epsilon > 0$, then $\exists\, N \in \mathbb{N}_{>0}$ such that $|a_n - a| < \epsilon\ \forall n \geq N$. In particular if $|a_n - a| \geq \epsilon$ then $n < N$ so there are only finitely many such $n \in \mathbb{N}_{>0}$.

   (c) $\iff$ (iii). Suppose (c) is true and take $N = 1$. Then $\exists \epsilon > 0$ such that $|a_n - a| < \epsilon\ \forall n \geq 1$. So, by the triangle inequality, $|a_n| < |a| + \epsilon$. Putting $R := |a| + \epsilon$ gives (iii).

   Suppose (iii) is true, i.e. $\exists R \in \mathbb{R}$ such that $|a_n| < R \quad \forall n \in N$. By the triangle inequality, $|a_n - a| < R + |a|\ \ \forall n \geq N$. Putting $\epsilon := R + |a|$ proves (c).

   (d) $\iff$ (iii). Suppose (d) is true and take $N = 1$. Then $|a_n - a| < \epsilon\ \forall n \geq 1$. So, by the triangle inequality, $|a_n| < |a| + \epsilon$. Putting $R := |a| + \epsilon$ gives (iii).

   Suppose (iii) is true, i.e. $\exists R \in \mathbb{R}$ such that $|a_n| < R \quad \forall n \in N$. By the triangle inequality, $|a_n - a| < R + |a|\ \ \forall n \geq N$. Putting $\epsilon := R + |a|$ proves (d).

   (e) $\iff$ (i): just replace ϵ by $1/R$ in the definition of convergence.

2. Given a sequence $(a_n)_{n\geq 1}$ of *complex* numbers, define what $a_n \to a$ means. For $x, y \in \mathbb{R}$ and $z := x + iy \in \mathbb{C}$ show $\max(|x|, |y|) \ \leq\ |z| \ \leq\ \sqrt{2}\max(|x|, |y|)$, and

$$a_n \to a + ib \in \mathbb{C} \quad \iff \quad \mathrm{Re}(a_n) \to a \quad \text{and} \quad \mathrm{Im}(a_n) \to b.$$

   The inequalties

$$\max(x^2, y^2) \ \leq\ x^2 + y^2 \ \leq\ \max(x^2, y^2) + \max(x^2, y^2)$$

   give

$$\max(|x|, |y|)^2 \ \leq\ |z|^2 \ \leq\ 2\max(|x|, |y|)^2.$$

   Suppose $a_n \to a + ib$ and fix any $\epsilon > 0$. Then $\exists\, N \in \mathbb{N}_{>0}$ such that

$$n \geq N \ \Rightarrow\ |a_n - (a + ib)| < \epsilon \ \Rightarrow\ \max(|\mathrm{Re}(a_n) - a|, |\mathrm{Im}(a_n) - b|) < \epsilon,$$

   using the first stated inequality. Therefore $|\mathrm{Re}(a_n) - a| < \epsilon$ and $|\mathrm{Im}(a_n) - b| < \epsilon$ as required.

   Conversely, suppose $\mathrm{Re}(a_n) \to a \quad \text{and} \quad \mathrm{Im}(a_n) \to b$ and fix any $\epsilon > 0$. Then $\exists\, N \in \mathbb{N}_{>0}$ such that $n \geq N \ \Rightarrow\ |\mathrm{Re}(a_n) - a| < \epsilon/\sqrt{2}$ and $|\mathrm{Im}(a_n) - b| < \epsilon/\sqrt{2}$. Thus

$$|a_n - (a + ib)| \ <\ \sqrt{2}\max(|\mathrm{Re}(a_n) - a|, |\mathrm{Im}(a_n) - b|) \ <\ \sqrt{2}\epsilon/\sqrt{2} \ =\ \epsilon,$$

   using the second stated inequality.

3. Suppose that $a_n \leq b_n \leq c_n \; \forall n$ and that $a_n \to a$ and $c_n \to a$. Prove that $b_n \to a$.

$\exists\, N_1 \in \mathbb{N}_{>0}$ such that $n \geq N_1 \;\Rightarrow\; |a_n - a| < \epsilon \;\Rightarrow\; a_n > a - \epsilon$.

$\exists\, N_2 \in \mathbb{N}_{>0}$ such that $n \geq N_2 \;\Rightarrow\; |c_n - a| < \epsilon \;\Rightarrow\; c_n < a + \epsilon$.

Set $N := \max(N_1, N_2)$. Then $n \geq N \;\Rightarrow\; a - \epsilon < a_n \leq b_n \leq c_n < a + \epsilon$. Therefore $|b_n - a| < \epsilon$.

4. Suppose that $a_n \to 0$ and (b_n) is bounded. Prove that $a_n b_n \to 0$.

$\exists\, B > 0$ such that $|b_n| \leq B \; \forall n$.

Given $\epsilon > 0$, $\exists\, N \in \mathbb{N}_{>0}$ such that $n \geq N \;\Rightarrow\; |a_n| < \epsilon/B$.

Therefore $|a_n b_n| = |a_n||b_n| \leq (\epsilon/B)B = \epsilon$, as required.

5. * Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \to a$ and $b_n \to b \neq 0$. Prove that the set $\{a_n : n \in \mathbb{N}_{>0}\}$ is bounded and that

$$\exists\, N \in \mathbb{N}_{>0} \quad \text{such that} \quad n \geq N \;\Rightarrow\; |b_n| > |b|/2.$$

Set $\epsilon = |b|/2 > 0$. Then $\exists\, N \in \mathbb{N}_{>0}$ such that

$$n \geq N \;\Rightarrow\; |b_n - b| < \epsilon \;\Rightarrow\; |b| < |b_n| + \epsilon \;\Rightarrow\; |b_n| > |b| - \epsilon = |b|/2.$$

Therefore $(a_n/b_n)_{n \geq N}$ is a sequence of real numbers; prove it tends to a/b.

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b b_n}\right| = \left|\frac{(a_n - a)b + a(b - b_n)}{b b_n}\right| \leq \left|\frac{(a_n - a)}{b_n}\right| + \left|\frac{a(b - b_n)}{b b_n}\right|.$$

From above we can find $N_1 \in \mathbb{N}_{>0}$ such that $n \geq N_1 \;\Rightarrow\; |b_n| \geq |b|/2$, which in turn implies that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|a_n - a|}{|b|/2} + |a|\frac{|b - b_n|}{|b|.|b|/2} = \frac{2}{|b|}|a_n - a| + \frac{2|a|}{b^2}|b - b_n|.$$

Now fix any $\epsilon > 0$. There exists $N_2 \in \mathbb{N}_{>0}$ such that $n \geq N_2 \;\Rightarrow\; |a_n - a| < |b|\epsilon/4$. And there exists $N_3 \in \mathbb{N}_{>0}$ such that $n \geq N_3 \;\Rightarrow\; |b_n - b| < |b|^2\epsilon/4(1 + |a|)$.

Therefore if we set $N := \max\{N_1, N_2, N_3\}$ then

$$n \geq N \;\Rightarrow\; \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{2|b|\epsilon/4}{|b|} + \frac{2|a|}{b^2}\frac{b^2\epsilon}{4(1 + |a|)} < \epsilon/2 + \epsilon/2 = \epsilon.$$

6. We call a sequence *sorta-Cauchy* if it satisfies the condition

$$\forall \epsilon > 0 \;\exists\, N \in \mathbb{N}_{>0} \;\; n \geq N \;\Rightarrow\; |a_n - a_{n+1}| < \epsilon.$$

Give an example of a sorta-Cauchy sequence which diverges to $+\infty$. Conclude that sorta-Cauchy is not as strong as Cauchy.

Any a_n that increases so slowly to infinity that $a_{n+1} - a_n$ converges to zero. Eg $a_n = \sqrt{n}$ or $a_n = \log n$ or $a_n = \sum_{i=1}^{n} \frac{1}{i}$.

7. Give an example of a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$.

In lectures we show that in $\mathbb{R}$, a sequence is Cauchy if and only if it is convergent. Show that it is impossible to prove this using only the arithmetic and order axioms of $\mathbb{R}$ (i.e. all the axioms except the completeness axioms – the one about the existence of least upper bounds).

Let a_n be $\sqrt{2}$ to n decimal places (so $a_1 = 1.4,\ a_2 = 1.41,\ a_3 = 1.414$, etc).
Or let $a_n = 0.101001000100001....1$ where there are n 1s.
I.e. any sequence of rational numbers which converges to an irrational number. By the uniqueness of limits it cannot converge to any other limit, so it cannot converge to a rational number.

If the proof of "Cauchy $\Rightarrow$ convergent" didn't use the completeness axiom, then the same proof would work in $\mathbb{Q}$ (where all the same axioms hold) to show that this sequence converged in $\mathbb{Q}$, which is a contradiction.

8. Let $(a_n)_{n\in\mathbb{N}_{>0}}$ be a bounded sequence.

   (a) For each $n \in \mathbb{N}_{>0}$, define the set $S_n = \{a_j : j \geq n\}$. Prove that, for every $n \in \mathbb{N}_{>0}$, there exists some $b_n \in \mathbb{R}$ such that $b_n = \sup(S_n)$. **Let M be an upper bound for $(a_n)_{n\in\mathbb{N}_{>0}}$, then the set S_n is non-empty with M as an upper bound, so by the completeness axiom S_n has a supremum.**

   (b) Let $B = \{b_n : n \in \mathbb{N}_{>0}\}$ where b_n is defined as above. Prove that there exists some $l \in \mathbb{R}$ such that $l = \inf(B)$. (Remark: l is called the limit supremum of the sequence $(a_n)_{n\in\mathbb{N}_{>0}}$, and the usual notation is $l = \limsup_{n\to\infty} a_n$).

   The set B is clearly non-empty. Let $\tilde{M}$ be a lower bound for $(a_n)_{n\in\mathbb{N}_{>0}}$. Then $b_n = \sup(S_n) \geq \tilde{M}$. This means $\tilde{M}$ is a lower bound for B and so by the completeness axiom, B has an infimum.

   (c) For each of the sequences below, find the value of $\limsup_{n\to\infty} a_n$ and give justification for your answer.

      i. $a_n = (-1)^n$ $\limsup_{n\to\infty} a_n = 1$. **Note that, for every $j \in \mathbb{N}_{>0}$, one has $S_j = \{-1, 1\}$ so that $b_j = \sup(S_j) = 1$. Therefore $\limsup_{n\to\infty} a_n = \inf\{1\} = 1$.**

      ii. $a_n = \frac{(-1)^n}{n}$

      $\limsup_{n\to\infty} a_n = 0$. **Note that, for every $j \in \mathbb{N}_{>0}$, one has $b_j = \sup(S_j) = \frac{1}{j}$ for j even and $\frac{1}{j+1}$ for j odd. Therefore we argue that $\inf(B) = 0$. Clearly 0 is a lower bound. To show that it is an infimum, we use Proposition 2.38 and show that for any $\epsilon > 0$, there exists $b \in B$ such that $b - \epsilon < 0$. Note that, by the Archimedean Axiom, we can find an $j \in \mathbb{N}_{>0}$ such that $j \geq \frac{1}{\epsilon}$, so that $\frac{1}{j} < \epsilon$. It then follows that $b_j - \epsilon < 0$, which finishes the proof since $b_j \in B$.**

*Starred questions * are good to prepare to discuss at your Problem Class.*