Imperial College London

Department of Mathematics
Imperial College London
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Partial Differential Equations in Action **MATH50008**

Problem Sheet 3

1. ✦ Consider the following kinematic wave equation governing the field $u(x,t)$

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0$$

where k is a constant. The general solution is given implicitly by the following expression

$$u = f(x - (k+u)t)$$

where f is an arbitrary function. Verify directly that $u(x,t)$ as defined by this implicit formula is a solution of the kinematic wave equation.

2. ✦ Here, we consider the same kinematic wave equation as in Q1. In this problem, we want to solve this equation for $t > 0$ in $-\infty < x < \infty$, given the following initial condition

$$u(x,0) = \begin{cases} 1, & x < -1 \\ (1-x)/2, & |x| < 1 \\ 0, & x > 1 \end{cases}$$

   (a) Plot the characteristics for this equation in the (x,t)-plane.
   (b) Show that a shock forms at some critical time t_s and find the value of t_s.
   (c) Use the result from Q1 to find an explicit analytical expression for the solution $u(x,t)$ and hence verify your answer to part (b).

3. ✦ For the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = -x, \quad x \in \mathbb{R}$$

Sketch the characteristic diagram and find the solution.

4. ✦✦ In this problem, we want to solve the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0$$

subject to the following initial conditions

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

   (a) Find a continuous solution for $t < 1$.
   (b) For $t > 1$, fit a shock and find the form of the solution.
   (c) Sketch snapshots of the wave for $t = 0$, $t = 1/2$, $t = 1$ and $t = 3/2$.

5. ✦✦ In the lecture notes, we have modelled traffic flow using the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

where $q(\rho)$ is the traffic flux. Empirical data obtained for the Lincoln Tunnel (a tunnel joining NY and NJ) shows that the traffic flux can be well described by the following functional form

$$q(\rho) = \alpha \rho \ln(\rho_m/\rho)$$

where α and ρ_m are positive constants. Let the initial density of cars $\rho(x, 0)$ be a bumper-to-bumper traffic (with value ρ_m) for $x < -x_0$, no traffic for $x > 0$ and a linear variation in between.

(a) Give a mathematical expression for the initial conditions and sketch them.

(b) Determine the position on the road where $\rho = \rho_m/2$ after two hours.

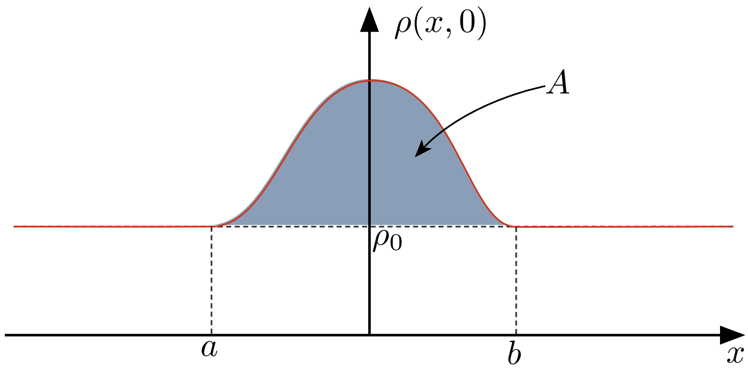
6. ✦✦ We finish by going back to the kinematic wave equation from Q1, i.e.

$$\frac{\partial \rho}{\partial t} + (\rho + k)\frac{\partial \rho}{\partial x} = 0$$

where k is a constant. In this problem, we will solve this equation with $k = 0$ and the general initial condition

$$\rho(x,0) = \begin{cases} \rho_0, & x \le a \\ g(x), & a \le x \le b \\ \rho_0, & x \ge b \end{cases}$$

where $g(x)$ is some continuous function with $g(a) = g(b) = \rho_0$ with the general form shown in the figure below.



The area of the shaded region in the figure is known to be A. With this kind of initial condition, we expect a shock to form at some finite time t_s. We are here interested in the solution at large times $t \gg t_s$ (i.e., in the large-t asymptotic solution).

(a) Verify that a solution to the equation is given by $\rho(x,t) = \tilde{\rho}(x,t)$ where

$$\tilde{\rho}(x,t) = \begin{cases} \rho_0, & x \le \rho_0 t \\ x/t, & \rho_0 t \le x \le s(t) \\ \rho_0, & x \ge s(t) \end{cases}$$

and where $s(t)$ is some arbitrary function. Draw a sketch of this solution as a function of x for a fixed value of t.

(b) Assume that at large times $t \gg t_s$, the solution of the original initial value problem looks like the solution $\tilde{\rho}(x,t)$ with $s(t)$ determined by the usual shock condition. Show that the ordinary differential equation for $s(t)$ is

$$\frac{ds}{dt} = \frac{1}{2}\left(\rho_0 + \frac{s}{t}\right)$$

(c) Finally, we want to find the general solution of this equation. Show that

$$s(t) = \rho_0 t + \sqrt{2At}$$

Draw a labelled sketch of this solution as a function of x at some large time t. This large-t asymptotic solution has entirely forgotten about the details of the original initial condition except for A and ρ_0.

7. ✦✦✦ In this problem, we consider the viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

subject to the following boundary conditions

$$\begin{aligned} u &\to u_1, \quad \text{as } x \to -\infty \\ u &\to u_2, \quad \text{as } x \to +\infty \end{aligned}$$

with $u_1 > u_2$. Show that this equation admits a nonlinear travelling wave solution of the form

$$u(x,t) = \frac{u_2 + u_1 e^{-\frac{(u_1-u_2)(x-ct)}{2\nu}}}{1 + e^{-\frac{(u_1-u_2)(x-ct)}{2\nu}}}$$