Applied Complex Analysis - Lecture Twelve

Andrew Gibbs

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Trapezium rule for periodic analytic functions

Trapezium rule for periodic functions

Thm: For

$$I = \int_0^{2\pi} f(\theta)d\theta \approx I_N = \frac{1}{N}\sum_{n=1}^{N} f(\theta_n), \quad \theta_n = 2\pi n/N,$$

if f is 2π-periodic and analytic in the strip $S_a := \{\theta : -a < \mathrm{Im}\{\theta\} < a\}$ for $a > 0$, then

$$|I - I_N| \leq \frac{4\pi M}{e^{aN} - 1},$$

where $M := \sup_{\theta \in S_a} |f(\theta)|$.

- Proved last Lecture
- More flexibility - consequences of different contour choices
- Theoretical examples

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Exactness of trapezium rule for periodic functions

Thm: If a function f has a Laurent expansion of the form

$$f(z) = \sum_{j=-N}^{N-2} a_j (z - z_0)^j,$$

for z in some annulus D, then an N-point trapezium rule I_N can exactly approximate

$$I = \oint_\gamma f(z) dz,$$

where γ is a closed anti-clockwise-oriented contour in D.

- Analagous to N-point Gauss quadrature integrating degree $(2N-1)$-degree integrals exactly
- Not particularly useful - but a helpful way to check your code!
- Proof
- Numerical examples

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The argument principle

For f meromorphic (all non-analytic points are are poles) and g analytic in Ω,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}g(z)dz = \sum_{a\in\{\text{zeros of } f\}} g(a)m_a - \sum_{b\in\{\text{poles of } f\}} g(b)m_b.$$

where m_a and m_b represent the order of the zeros and poles respectively, γ is a closed contour in Ω with no loops, such that $f(z) \neq 0$ for $z \in \gamma$.

- Proof
- Consequences and applications
- Approximation trick when $g = 1$
- Example: Root-finding

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