Applied Complex Analysis: Problem Sheet 3 Solutions

1). Replacing z with $-z$ in the first identity:

$$\Gamma(1-z) = -z\Gamma(-z).$$

Combining this with the second identity:

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z\sin(\pi z)}.$$

This is valid for all z, in particular, taking $z = iy$ $(y \in \mathbb{R})$, and using the fact $\overline{\Gamma(z)} = \Gamma(\bar{z})$ for all z, we have:

$$|\Gamma(iy)|^2 = \Gamma(iy)\Gamma(-iy) = \frac{-\pi}{iy\sin(\pi iy)}$$

$$\Rightarrow |\Gamma(iy)| = \left(\frac{-\pi}{iy\sin(\pi iy)}\right)^{\frac{1}{2}}$$

$$= \left(\frac{\pi}{y\sinh(\pi y)}\right)^{\frac{1}{2}}$$

Since $\sinh(y) \to \pm\infty$ as $y \to \pm\infty$ then this result shows us that $|\Gamma(iy)| \to 0$ as $y \to \pm\infty$.

2).

(a). Introduce $u=1-x$. Then $dx=-du$ and $x=0$ and $x=1$ correspond respectively to $u=1$ and $u=0$. Hence one may also write:

$$I=\int_0^1 \log(\Gamma(1-u))\,du\,.$$

Then:

$$I=\frac{1}{2}\left(\int_0^1 \log(\Gamma(x))\,dx+\int_0^1 \log(\Gamma(1-x))\,dx\right)$$

$$=\frac{1}{2}\int_0^1 \log(\Gamma(x)\Gamma(1-x))\,dx$$

$$=\frac{1}{2}\int_0^1 \log\left(\frac{\pi}{\sin(\pi x)}\right)dx$$

$$=\frac{1}{2}(\log\pi-J)\,,\quad \text{where } J=\int_0^1 \log(\sin(\pi x))\,dx\,.$$

(b). One may write:

$$J=\frac{1}{\pi}\int_0^{\pi}\log(\sin u)\,du$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\log(\sin 2v)\,dv$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\log(2\sin v\cos v)\,dv$$

(3)

$$= \log 2 + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(\sin v)\, dv + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(\cos v)\, dv$$

$$= \log 2 + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(\sin v)\, dv + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \log(\sin w)\, dw$$

$$= \log 2 + \frac{2}{\pi}\int_0^{\pi} \log(\sin v)\, dv$$

$$= \log 2 + 2J,$$

where the first equality follows from the change of variables $u = \pi x$, the second from $v = \frac{u}{2}$ and the fifth from $w = v + \frac{\pi}{2}$.

Hence:

$$\underline{\underline{J = -\log 2.}}$$

Hence:

$$2I = \log(2\pi)$$

$$\Rightarrow \underline{\underline{I = \log\sqrt{2\pi}}}$$

3). Let $I = \int_0^4 x^2\sqrt{16-x^2}\,dx = 4\int_0^4 x^2\sqrt{1-\frac{x^2}{16}}\,dx$.

Let $u = \frac{1}{16}x^2$, so $du = \frac{1}{8}x\,dx \Rightarrow dx = \frac{2}{\sqrt{u}}\,du$. $\quad x=0 \rightarrow u=0$
$\qquad x=4 \rightarrow u=1$

$$\Rightarrow I = 4\int_0^1 (16u)(1-u)^{\frac{1}{2}}\,2u^{-\frac{1}{2}}\,du$$

$$= 128\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du$$

$$= 128\int_0^1 u^{\frac{3}{2}-1}(1-u)^{\frac{3}{2}-1}\,du$$

$$= 128\,B\left(\tfrac{3}{2},\tfrac{3}{2}\right)$$

$$= 128\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{128\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{2!}$$

$$= 64\left(\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\right)^2$$

$$= 64\left(\tfrac{1}{2}\sqrt{\pi}\right)^2$$

$$= \underline{\underline{16\pi}}.$$

4).

(a). $\text{RHS} = F(a,b;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(b)_n} z^n, \quad |z|<1$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

$$= 1 + az + \frac{1}{2}a(a+1)z^2 + \cdots$$

Let's check the n^{th} Taylor coefficient of the LHS:

$$\frac{1}{n!}\frac{d}{dz^n}(1-z)^{-a}\Big|_{z=0} = \frac{1}{n!}\left(a(a+1)(a+2)\cdots(a+n-1)(1-z)^{-a-n} \right)\Big|_{z=0}$$

$$= \frac{1}{n!}\left(a(a+1)(a+2)\cdots(a+n-1)\right)$$

$$= \frac{(a)_n}{n!}.$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \quad \text{for } |z|<1.$$

By analytic continuation the result holds for all z.

(b). For $|z|<1$; $F(1,1;2;z) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{n!\,(2)_n} z^n$

$$= \sum_{n=0}^{\infty} \frac{n!\,n!}{n!\,(n+1)!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n+1}$$

(6)

$$= \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

$$= -\frac{1}{z} \log(1-z).$$

Then, by analytic continuation, the result holds for all z.