Network Science
Spring 2024
Problem sheet 7

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1. Consider $f(x,t)$ and $f_i(t)$ where the former is a solution to the 1-D diffusion equation, and the latter is a solution to the graph diffusion equation.

   (a) How does the total amount of "stuff", $F_g(t) = \sum_{i=1}^{N} f_i(t)$, vary with time for graph diffusion?
   Solution: The graph diffusion equation can be written as,

   $$df_i/dt = \alpha \sum_{j=1}^{N} A_{ij}(f_j - f_i),$$

   and summing over all nodes, we find,

   $$dF_g/dt = \sum_{i=1}^{N} df_i/dt = \alpha \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}(f_j - f_i).$$

   Since

   $$\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}f_j = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}f_i,$$

   (for an undirected graph) we find that $dF_g/dt = 0$, so the initial condition sets the total amount of stuff on the graph.

   (b) How does the total amount of "stuff", $F(t) = \int_{-\infty}^{\infty} f(x,t)dx$ vary with time? Assume that f and $\partial f/\partial x \to 0$ as $|x| \to \infty$.
   Solution: Integrating the diffusion equation, we have

   $$dF/dt = d/dt \int_{-\infty}^{\infty} f dx = \int_{-\infty}^{\infty} \partial f/\partial t dx = \alpha \int_{-\infty}^{\infty} \partial^2 f/\partial x^2 dx,$$

   Evaluating the final integral on the RHS and applying the conditions that $\partial f/\partial x \to 0$ when $|x| \to \infty$, we find that $\partial F/\partial t = 0$, so the total amount of stuff is again conserved.

2. Last year, you were introduced to the *incidence matrix* for an N-node directed graph with no self-loops or multiedges. The links are numbered from 1 to L and then the incidence matrix is defined as

$$E_{ij} = \begin{cases} 1 & \text{if link } i \text{ points to node } j \\ -1 & \text{if link } i \text{ points from node } j \text{ to another node} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

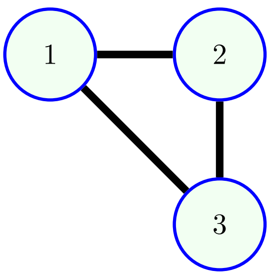
(a) Consider the "equivalent" simple undirected graph where two nodes are linked if there is a directed link from one node to the other in the original directed graph. Show that the Laplacian matrix for this undirected graph is related to the incidence matrix of the directed graph by, $\mathbf{L} = \mathbf{E}^T\mathbf{E}$
Solution: Let $\mathbf{M} = \mathbf{E}^T\mathbf{E}$. We need to show that $M_{ij} = \sum_{l=1}^{L} E_{li}E_{lj} = \delta_{ij}k_i - A_{ij}$. First consider the case where $i \neq j$. Then $E_{li}E_{lj} = -1$ if link l connects nodes i and j and is zero otherwise. Since there can only be one link between a pair of nodes, this implies that when $i \neq j$, $M_{ij} = -A_{ij}$ (*). When $i = j$, we need to consider E_{li}^2. This will be 1 for any link l that points towards or away from i. Summing over all such l will give k_i (for the equivalent undirected graph). Combining this with (*), we find that $M_{ij} = \delta_{ij}k_i - A_{ij}$ as required.

(b) Consider an arbitrary real matrix $\mathbf{F}$. Show that the eigenvalues of $\mathbf{M} = \mathbf{F}^T\mathbf{F}$ are non-negative (note that $\mathbf{M}$ is always symmetric)
Solution: $\mathbf{M}$ is symmetric ($(\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T(\mathbf{F}^T)^T = \mathbf{F}^T\mathbf{F}$), so its eigenvalues are real, and it can be orthogonally diagonalized as, $\mathbf{M} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T = \mathbf{F}^T\mathbf{F}$ (*) with real eigenvectors and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. Right-multiplying (*) with $\mathbf{V}$ and left-multiplying with $\mathbf{V}^T$ gives $\mathbf{\Lambda} = \mathbf{V}^T\mathbf{F}^T\mathbf{F}\mathbf{V}$. Define $\mathbf{G} = \mathbf{F}\mathbf{V}$ and label the columns of $\mathbf{G}$ as $\mathbf{g}_1, \mathbf{g}_2, \ldots$. Then we have, $\lambda_i = \mathbf{g}_\mathrm{i}^\mathrm{T}\mathbf{g}_\mathrm{i}$ where λ_i is the *ith* eigenvalue on the diagonal of $\mathbf{\Lambda}$, and we can see that each eigenvalue will be non-negative.

3. In lecture 10, the graph diffusion equation was introduced,

$$d\mathbf{n}/dt = -\alpha\mathbf{L}\mathbf{n},$$

where $\mathbf{L}$ is the (symmetric) Laplacian matrix for an N-node undirected graph, $\mathbf{n} \in \mathbb{R}^N$, and $\alpha > 0$. Show that this system of N coupled ODEs can be written as a system of N uncoupled ODEs. This will require a change of variable, $\mathbf{g} = \mathbf{W}\mathbf{n}$, and you will need to determine $\mathbf{W}$. Will these N uncoupled ODEs all be distinct from each other? Hint: consider the orthogonal diagonalization of $\mathbf{L}$
Solution: Since $\mathbf{L}$ is symmetric, it can be orthogonally diagonalized as $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{L}$, and $\mathbf{V}$ is an orthogonal matrix whose columns are the eigenvectors. Letting $\mathbf{g} = \mathbf{V}^T\mathbf{n}$, we find, $d\mathbf{g}/dt = -\alpha\mathbf{\Lambda}\mathbf{g}$, so we now have n equations of the form $dg_i/dt = -\alpha\lambda_i g_i$ where λ_i is the *ith* eigenvalue of $\mathbf{L}$. However, $\mathbf{L}$ may have repeated eigenvalues, so some number of these equations may be identical, and there will be m uncoupled and distinct ODEs to solve where m is the number of distinct eigenvalues of $\mathbf{L}$.



4. Consider the "triangle graph" shown above. Solve the graph diffusion equation on this graph with $\alpha = 1$, $n_1(t=0) = 1$ and $n_2(t=0) = n_3(t=0) = 0$
Solution: From the previous exercise, we know that the graph diffusion equation can be re-written as $dg_i/dt = -\lambda_i g_i, i \in \{1, 2, 3\}$ where $\mathbf{g} = \mathbf{V}^T\mathbf{n}$, λ_i is the *ith* eigenvalue of $\mathbf{L}$, and $\mathbf{V}$ is the orthogonal eigenvector matrix for $\mathbf{L}$ ($\mathbf{L}$ is the graph Laplacian). For this graph, the eigenvalues are $\lambda_1, \lambda_2, \lambda_3 = 0, 3, 3$, with orthonormal eigenvectors, $\mathbf{v}_1 = 1/\sqrt{3}[1, 1, 1]^T, \mathbf{v}_2 = 1/\sqrt{2}[1, -1, 0]^T, \mathbf{v}_3 = 1/\sqrt{6}[1, 1, -2]^T$, and $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$. Let

$\mathbf{g}_0 = \mathbf{V}^T\mathbf{n}(t=0)$. Then $g_i = g_{0,i}e^{-\lambda_i t}$ and $\mathbf{n} = \mathbf{V}\mathbf{g} = \mathbf{v}_1^T\mathbf{n}_0\mathbf{v}_1 e^{-\lambda_1 t} + \mathbf{v}_2^T\mathbf{n}_0\mathbf{v}_2 e^{-\lambda_2 t} + \mathbf{v}_3^T\mathbf{n}_0\mathbf{v}_3 e^{-\lambda_3 t}$ where $\mathbf{n}_0 = \mathbf{n}(t=0)$. What we find after more arithmetic is that the solutions for nodes 2 and 3 are identical (as we expect from the symmetry of the problem) and initially grow as $\mathbf{n}$ spreads from node 1 and then exponentially decay. Node 1 also shows exponential decay and at long times n_i for all three nodes approaches 1/3.

5. Consider the following coupled system of ODEs on a complete N-node undirected graph with Laplacian matrix, $\mathbf{L}$:

$$\frac{dx_i}{dt} = x_i - \alpha\sum_{j=1}^{N} L_{ij}x_j, i = 1, \ldots, N. \tag{2}$$

The initial condition for each node is $x_i(t=0) = y_i$.
The eigenvalues of $\mathbf{L}$ are $\lambda_1 = 0$ and $\lambda_i = N$ for $i = 2, 3, ..., N$.

(a) What is the solution of this system when $\alpha = 0$?
Solution: The equations are then decoupled, and $x_i = y_i e^t$.

(b) Show that with an appropriate choice of $\mathbf{B}$, the transformation, $\mathbf{x} = \mathbf{B}\mathbf{w}$, allows (2) to be written as a decoupled system of equations,

$$\frac{d\mathbf{w}}{dt} = (\mathbf{I} - \alpha\mathbf{\Lambda})\mathbf{w} \tag{3}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{L}$
Solution: Orthogonally diagonalize the Laplacian, $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ and set $\mathbf{B}$=$\mathbf{V}$. Multiplying both sides of (2) with $\mathbf{V}^T$ gives,

$$\frac{d(\mathbf{V}^T\mathbf{x})}{dt} = \mathbf{V}^T\mathbf{x} - \alpha\mathbf{\Lambda}\mathbf{V}^T\mathbf{x}.(*)$$

We have $\mathbf{x} = \mathbf{V}\mathbf{w}$, and due to the orthogonality of $\mathbf{V}$, this becomes $\mathbf{w} = \mathbf{V}^T\mathbf{x}(**)$. Substituting (**) into (*) leads to (3).

(c) Show that (2) synchronizes when $\alpha > 1/N$: $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$.
Solution: We have $\lambda_1 = 0$ and the corresponding orthonormal eigenvector is, $\mathbf{v}_1 = \mathbf{z}/\sqrt{N}$; all other eigenvalues are equal to N. From (3) it follows that $w_1(t) = w_1(0)e^t$ and $w_i(t) = w_i(0)e^{(1-\alpha N)t}$ for $i = 2, \ldots, N$. So

$$x(t) = \mathbf{V}\begin{pmatrix} w_1(0)e^t \\ w_1(0)e^{(1-\alpha N)t} \\ \vdots \\ w_1(0)e^{(1-\alpha N)t} \end{pmatrix} = w_1(0)e^t\mathrm{z}/\sqrt{N} + w_2(0)e^{(1-\alpha N)t}\mathbf{v_2} + \cdots + w_N(0)e^{(1-\alpha N)t}\mathbf{v_N}$$

Now consider, $x_i(t) - x_j(t)$ for distinct i and j. The first term on the far RHS of the equation above is the same for all i and will not contribute. All other terms will go to zero when $t \to \infty$ if $\alpha > 1/N$ and $|x_i(t) - x_j(t)| \to 0$ as required.