Expected value of a random variable X

X is discrete $E[X] = \sum_{x \in Im(X)} x P(X = x)$

X is continuous $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx$ pdf

exists if $E[|X|] < \infty$

Random variables vs observations

X - random variables UPPERCASE

x - observations lowercase

data

Revise Expected value

X - random variable

a - constant $\in \mathbb{R}$

$E[X] \in \mathbb{R}$

$E[a] = a$

$E[(X-a)] \in \mathbb{R}$

$E[\underbrace{E[X-a]}_{\in \mathbb{R}}] = E[X-a]$

X - random variable

a - constant.

squared deviation $(X-a)^2 \geq 0$

↳ random variable

$$E[(X-a)^2]$$

$$= E[(\{X \overbrace{- E[X]\} + \{E[X]}^{+0} - a\})^2]$$

$$= E[(X-E[X])^2] + \underbrace{2E[(X-E[X])(E[X]-a)]}_{=0 \text{ (linearity of } E)}$$

$$+ \underbrace{E[(E[X]-a)^2]}_{(E[X]-a)^2}$$

$$2E[(X-E[X])(\underbrace{E[X]-a}_{c \in \mathbb{R}})]$$

$$= 2(E[X]-a)\underbrace{E[(X-E[X])]}_{=0}$$

$$E[X-E[X]]$$

$$= E[X] - E[\underbrace{E[X]}_{c' \in \mathbb{R}}]$$

$$= E[X] - E[X]$$

$$= 0$$

$$E[(x-a)^2] = E[(x-E[x])^2] + 0 + \underbrace{(E[x]-a)^2}_{\geq 0}$$

$$E[(x-a)^2] \geq E[(x-E[x])^2]$$

Theorem 1.1.1

$$\min_{a \in \mathbb{R}} E[(x-a)^2] = E[(x-E[x])^2]$$

Definition 1.1.2

The variance of a random variable X is

$$Var[X] = E[(x-E[x])^2]$$

$(x-a)^2 \qquad |x-a|$

standard deviation $\sqrt{Var(X)}$

$\Rightarrow$ units the same

$x_1, x_2, \ldots, x_n$ heights in cm

Sample variance cm^2

sample standard deviation cm

Ex. 1.1.5

Show $Var[X] = E[X^2] - (E[X])^2$

Variance of bounded random variable

Prop. 1.1.6

Suppose X is known to only take values in $[a,b]$

Then $Var[X] \leq \frac{(b-a)^2}{4}$

Proof: Problem Sheet 8

Cor. 1.1.7

Suppose $X \sim Bern(p)$ $p \in (0,1)$

Then $Var[X] \leq 1/4$ $X \subset [0,1]$

Exercise Try to prove!