1. Fix an integer $r \geq 0$ and define $f : [1, b] \to \mathbb{R}$ by $f(x) = x^r$, where $b > 1$.
   (a) Let $P_n = (1, b^{1/n}, b^{2/n}, \dots, b^{(n-1)/n}, b)$ be a partition of $[1, b]$. Compute the lower Darboux sum $L(f, P_n)$, and show that $U(f, P_n) = b^{r/n} L(f, P_n)$.
   (b) Prove that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$, and compute their common value.

2. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q}. \end{cases}$

   Prove that f is not integrable, but that f^2 is.

3. Prove that any monotone increasing function $f : [a, b] \to \mathbb{R}$ is integrable, by considering its Darboux sums for partitions where every subinterval $[x_i, x_{i+1}]$ has the same length.

4. Define the *mesh* of a partition $P = (x_0, \dots, x_k)$ to be the maximal length of any subinterval:
$$\mathrm{mesh}(P) = \max_{0\leq i\leq k-1} \Delta x_i = \max_{0\leq i\leq k-1} (x_{i+1} - x_i).$$
   Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\big(P_n\big)$ is any sequence of partitions of $[a, b]$ such that $\mathrm{mesh}(P_n) \to 0$ as $n \to \infty$, then
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$
   The proof should follow the argument we used in lecture to show that continuous functions are integrable.

5. (a) Prove for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ that if $\sin(\frac{\theta}{2}) \neq 0$, then
$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$
   using the formula $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.
   (b) Fix $t \in (0, \frac{\pi}{2}]$ so that $\sin(x)$ is monotone increasing on the interval $[0, t]$, and consider the partition $P_n = (0, \frac{t}{n}, \frac{2t}{n}, \dots, \frac{(n-1)t}{n}, t)$ of $[0, t]$. Compute the upper Darboux sum $U(\sin(x), P_n)$, and show that
$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\sin^2(\tfrac{t}{2}).$$
   Remark: This limit is equal to $1 - \cos(t)$ by the double-angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$, so problem 4 tells us that
$$\int_0^t \sin(x)\,dx = 2\sin^2\left(\frac{t}{2}\right) = 1 - \cos(t)$$
   for all $t \in (0, \frac{\pi}{2}]$.

6. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x)$ and the product $f(x)g(x)$ are both integrable, and $f(x) \geq 0$ for all $x \in [a, b]$. If $c \leq g(x) \leq d$ for all $x \in [a, b]$, prove that
$$c \int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq d \int_a^b f(x)\,dx.$$

7. (*) Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/|q|, & x = \frac{p}{q} \in \mathbb{Q}. \end{cases}$

   We proved in problem sheet 1 that f is discontinuous at all rational numbers.

   (a) Compute the lower Darboux integral $\underline{\int_0^1} f(x)\,dx$.

   (b) Consider the partition $P_n = (0, \frac{1}{n^3}, \frac{2}{n^3}, \dots, \frac{n^3-1}{n^3}, 1)$ of $[0, 1]$. Show for n large that there are at most n^2 subintervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ on which
   $$M_i = \sup_{\frac{i}{n^3} \leq t \leq \frac{i+1}{n^3}} f(t)$$
   is at least $\frac{1}{n}$.

   (c) Prove that $U(f, P_n) \leq \frac{2}{n}$ for n large. (Hint: break the sum into terms where $M_i \geq \frac{1}{n}$ and terms where $M_i < \frac{1}{n}$.)

   (d) Conclude that f is integrable, and compute $\int_0^1 f(x)\,dx$.

8. Prove that if $f : [a, b] \to [0, \infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int_a^b f(x)\,dx > 0$.

9. Suppose for some $f : [a, b] \to \mathbb{R}$ and integer $n \geq 1$ that the nth power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

10. Evaluate $\displaystyle\int_1^x \frac{\sqrt{t^2 - 1}}{t}\,dt$ for $x \geq 1$. (Hint: what is the inverse of the integrand?)

11. In problem sheet 4 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \to [0, \infty)$ such that $f(x) > 0$ if and only if $x \in (0, 1)$.

    (a) Construct a smooth, monotone increasing function $g : \mathbb{R} \to [0, \infty)$ such that $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.

    (b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \to [0, \infty)$ satisfying
    $$h(x) = 0 \text{ for all } x \notin [a, d], \qquad h(x) = 1 \text{ for all } x \in [b, c],$$
    and with h monotone increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

12. (a) Given $a < b < 0$, evaluate $\displaystyle\int_a^b \frac{1}{x}\,dx$. Be careful not to take the logarithm of a negative number along the way!

(b) Check that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is strictly monotone increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, with

$$\lim_{x \downarrow -\frac{\pi}{2}} \tan(x) = -\infty \quad \text{and} \quad \lim_{x \uparrow \frac{\pi}{2}} \tan(x) = +\infty.$$

(c) Let $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse function to $\tan(x)$. Prove for all $x \in \mathbb{R}$ that

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}.$$

(d) Fix $\theta \in (0, \frac{\pi}{2})$. Find a convenient substitution which proves that

$$\int_0^\theta \tan(x)\, dx = -\log(\cos(\theta)).$$

(e) Prove for $x > 0$ that $\displaystyle\int_0^x \tan^{-1}(t)\, dt = x \tan^{-1}(x) - \frac{1}{2}\log(1+x^2)$.

13. (a) Check that the derivative of $x\log(x) - x$ is $\log(x)$.

(b) Use Darboux sums to prove for all integers $n \geq 1$ that

$$\log((n-1)!) \leq \int_1^n \log(x)\, dx \leq \log(n!).$$

(c) Evaluate the integral in (b) and deduce that

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$$

for all $n \geq 1$.

(d) Conclude that $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

14. (*) Let $f : [N, \infty) \to [0, \infty)$ be a nonnegative, monotone decreasing function.

(a) Let $\displaystyle S_n = \sum_{k=N}^n f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that

$$S_n - f(N) \leq \int_N^n f(x)\, dx \leq S_{n-1}.$$

(b) Prove that the series $\displaystyle\sum_{k=N}^\infty f(k)$ converges if and only if the limit

$$\int_N^\infty f(x)\, dx \overset{def}{=} \lim_{x\to\infty} \int_N^x f(t)\, dt$$

(called an *improper integral*) exists. This is the *integral test* for convergence.

(c) Prove that if the series $S = \sum_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^{\infty} f(x)\,dx$ exists, then $I \leq S \leq I + f(N)$.

15. Evaluate $\int_0^x \frac{1}{1+e^t}\,dt$. Does $\int_0^{\infty} \frac{1}{1+e^t}\,dt$ exist, and if so, what is it?

16. The prime number theorem says that the number $\pi(n)$ of primes between 1 and n is approximately $\int_2^n \frac{1}{\log(x)}\,dx$.

    (a) Prove that this integral equals $\frac{n}{\log(n)} + \int_2^n \frac{1}{(\log x)^2}\,dx$, up to a constant which does not depend on n.

    (b) Prove that there is a constant $C > 0$ such that $\int_2^n \frac{1}{(\log x)^2}\,dx < \frac{Cn}{(\log n)^2}$ for all sufficiently large n, by splitting the integral up into one with domain $[2, \sqrt{n}]$ and one with domain $[\sqrt{n}, n]$ and estimating each one separately.

17. Let $f : [0,\infty) \to [0,\infty)$ be uniformly continuous, and suppose that $\int_0^{\infty} f(x)\,dx$ exists.

    (a) For each $\epsilon > 0$, prove that there is a $\delta > 0$ such that for all $y > 0$, if $f(y) \geq \epsilon$ then
    $$\int_y^{y+\delta} f(t)\,dt \geq \frac{\epsilon\delta}{2}.$$

    (b) Prove that $\lim_{x\to\infty} f(x) = 0$.

    (c) Describe a continuous function $g : [0,\infty) \to [0,\infty)$ such that $\int_0^{\infty} g(x)\,dx$ exists but $\lim_{x\to\infty} g(x)$ does not. Can you make g differentiable as well?

18. Let $\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\,dx$.

    (a) Prove that this improper integral converges for all $t > 0$. (In how many ways is it improper?)

    (b) Compute $\Gamma(1)$.

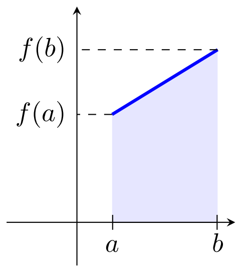
    (c) Prove that $\Gamma(n+1) = n\Gamma(n)$ for all integers $n \geq 1$, and deduce that $\Gamma(n+1) = n!$ for all $n \geq 0$.

19. (*) Let $f : [a,b] \to \mathbb{R}$ be continuous, with $f''(x)$ continuous and bounded on (a,b).

    (a) Use integration by parts twice to prove that
    $$\int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx = \int_a^b f(x)\,dx - (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

(b) If $|f''(x)| \leq M$ for all $x \in (a,b)$, prove that

$$\left| \int_a^b \frac{(x-a)(x-b)}{2} f''(x)\,dx \right| \leq \frac{M(b-a)^3}{12}.$$



In other words, $\int_a^b f(x)\,dx$ is the area of the trapezium shown at right, up to an error of at most $\frac{M(b-a)^3}{12}$. (Hint: check that $(x-a)(x-b) \leq 0$ on $[a,b]$, and compute that $\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$.)

(c) Apply this to $f(x) = \log(x)$ to show that

$$\int_1^n \log(x)\,dx = \sum_{k=1}^{n-1} \left(\frac{\log(k) + \log(k+1)}{2} + e_k \right),$$

where $|e_k| \leq \frac{1}{12k^2}$ for all k.

(d) Evaluate both the integral and the sum from part (c) to show that there is some constant $C > 0$ such that

$$\left| \log(n!) - \log\left(\frac{n^{n+1/2}}{e^{n-1}}\right) \right| \leq C$$

for all n, or equivalently if $C_1 = e^{1-C}$ and $C_2 = e^{1+C}$ then

$$C_1 \sqrt{n} \left(\frac{n}{e}\right)^n \leq n! \leq C_2 \sqrt{n} \left(\frac{n}{e}\right)^n.$$