Imperial College London

MATH70014

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Biology

Date: 31 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Consider the following model that describes the competition between two fish populations in the English Channel,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(100 - 4x - y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(60 - x - 2y).$$

(a) Suppose that population x is fished at rate Ex, where E is a positive constant, such that the model is now

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(100 - 4x - y) - Ex,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(60 - x - 2y).$$

(i) Find the fixed points. (4 marks)

(ii) Analyse the stability of the fixed points and show that two fixed points change stability at $E = 70$. What type of bifurcation occurs? (8 marks)

(b) Now suppose instead of Ex, x is fished at constant rate H, where H is a positive constant.

(i) Revise the equations to incorporate this fishing rate. (2 marks)

(ii) Find expressions for the y-values of the fixed points for $y \neq 0$. (4 marks)

(ii) Describe how these fixed points change as H increases from 0. Does a bifurcation occur? If so, what type and at what value of H? (2 marks)

(Total: 20 marks)

2. The cell cycle is the important process through which cells grow, divide and proliferate. Though it involves many chemical species, the process hinges on the interaction of two important proteins, cdc2 and cyclin. Consider the nondimensional system of equations

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = b(v-u)(\alpha+u^2) - u$$
$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = c - u$$

where u is the dimensionless concentration of the cdc2-cyclin complex and v is the dimensionless concentration of cyclin. The parameters b, α and c are all positive constants.

(a) Find and sketch the nullclines for the system. In your sketch, indicate how u and v are changing in the different regions of phase space. (5 marks)

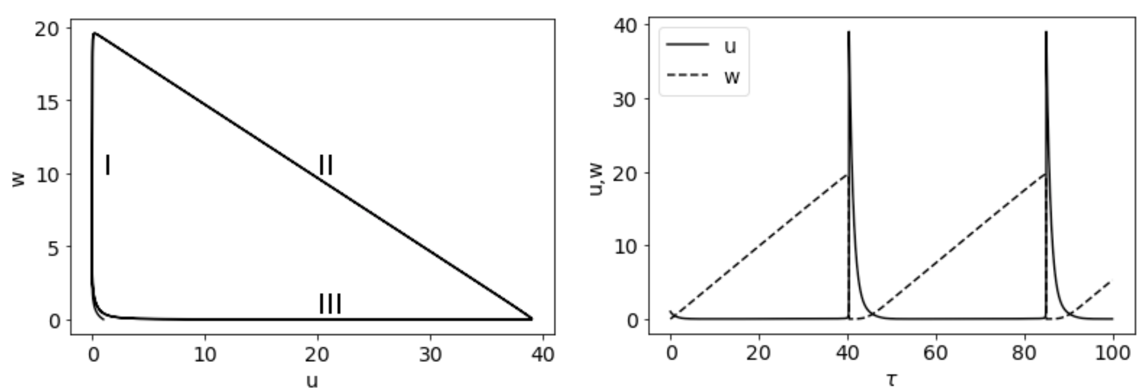
(b) Determine the fixed point. Assuming that $\alpha \ll 1$, show that the fixed point changes stability when $bc^2 = 1$. Is the fixed point stable or unstable for $bc^2 < 1$? (7 marks)

(c) Introduce $w = b(v-u)$ and hence show the system can be expressed as

$$\frac{du}{d\tau} = w(\alpha+u^2) - u$$
$$\frac{dw}{d\tau} = b\left(c - w(\alpha+u^2)\right).$$

(3 marks)

(d) The plots below show the trajectory in phase space and u and w as a function of τ, for the case $b \ll 1$.



Identify the fast and slow variables. The trajectory in phase space is divided into parts I, II, and III. Which part of the trajectory could be used to approximate the period? What is the relationship between w and u for this part of the trajectory? (5 marks)

(Total: 20 marks)

3. Consider the general system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = f(u,v) + D_1\nabla^2 u,$$
$$\frac{\partial v}{\partial t} = g(u,v) + D_2\nabla^2 v$$

and suppose there is a nontrivial spatially homogeneous solution (u_0, v_0). The conditions for a Turing instability are

$$\text{trace}\,(J) = J_{11} + J_{22} < 0, \tag{1}$$
$$\det\,(J) = J_{11}J_{22} - J_{12}J_{21} > 0, \tag{2}$$
$$D_1 J_{22} + D_2 J_{11} > 0 \tag{3}$$
$$(D_2J_{11} + D_1J_{22})^2 - 4D_1D_2\det(J) > 0, \tag{4}$$

where J is the Jacobian evaluated at (u_0, v_0).

(a) Briefly describe the origin of each of the conditions for a Turing instability. (2 marks)

(b) For the following system of reaction-diffusion equations,

$$\frac{\partial u}{\partial t} = \frac{u^2}{v} - \frac{u}{4} + \nabla^2 u,$$
$$\frac{\partial v}{\partial t} = u^2 - v + D\nabla^2 v,$$

(i) Find the **nontrivial** spatially homogeneous steady state. (2 marks)

(ii) Show that (1) and (2) are satisfied. (2 marks)

(iii) Determine the values of D for which the Turing instability conditions are met? (4 marks)

(iv) Find k_c^2, the square of the wavenumber at the bifurcation that gives rise to the Turning instability. *Hint: Recall that in general*

$$k_{\pm}^2 = \frac{(D_2J_{11} + D_1J_{22}) \pm \sqrt{(D_2J_{11} + D_1J_{22})^2 - 4D_1D_2\det(J)}}{2D_1D_2}.$$

(4 marks)

(c) Suppose

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{K}\right) - En + D\frac{\partial^2 n}{\partial x^2}$$

where r, K, and E are positive constants and $r > E$. This equation emits a travelling wave solution, $n(x,t) = N(\xi)$, where $\xi = x - ct$ with $c > 0$, i.e. right travelling wave.

(i) What values does the solution have as $\xi \to -\infty$ and $\xi \to \infty$. (2 marks)

(ii) Determine the wave speed, c, in terms of D, r, and E. *Hint: Use the fact that for the Fisher-Kolmogorov equation, $c = 2$.* (4 marks)

(Total: 20 marks)

4. Consider the general birth and death process with birth and death rates given by

$$\lambda_i = b_0 + b_1 i + b_2 i^2$$
$$\mu_i = d_1 i + d_2 i^2$$

where b_0, b_1, b_2, d_1, and d_2 are positive constants.

(a) Based on these rates, provide the infinitesimal transition probabilities and the generator matrix for the process. (3 marks)

(b) Derive the Forward Kolmogorov equation for this process. (4 marks)

(c) The partial differential equation for the probability generating function $\mathcal{P}(z,t)$ is

$$\frac{\partial \mathcal{P}}{\partial t} = b_0(z-1)\mathcal{P} + (b_1+b_2)z(z-1)\frac{\partial \mathcal{P}}{\partial z} + b_2 z^2(z-1)\frac{\partial^2 \mathcal{P}}{\partial z^2}$$
$$+ (d_1+d_2)(1-z)\frac{\partial \mathcal{P}}{\partial z} + d_2 z(1-z)\frac{\partial^2 \mathcal{P}}{\partial z^2}.$$

Derive the **last two terms** on the right hand side of this equation. (8 marks)

(d) Recall that mean and variance are given by

$$m(t) = \left.\frac{\partial \mathcal{P}}{\partial z}\right|_{z=1}, \quad \sigma^2(t) = \left.\frac{\partial^2 \mathcal{P}}{\partial z^2}\right|_{z=1} + \left.\frac{\partial \mathcal{P}}{\partial z}\right|_{z=1} - \left(\left.\frac{\partial \mathcal{P}}{\partial z}\right|_{z=1}\right)^2$$

Using these expressions, the **full** equation for $\mathcal{P}(z,t)$, and the fact that $\sum_{i=1}^{\infty} p_i(t) = 1$, derive the equation for the mean, $m(t)$, of the process. (5 marks)

(Total: 20 marks)

5. (a) Consider the cable equation

$$\frac{\partial V}{\partial t} = f(V) + \frac{\partial^2 V}{\partial x^2} \tag{5}$$

where the equation $\mathrm{d}V/\mathrm{d}t = f(V)$ has stable fixed points at 0 and 1 and an unstable fixed point at α where $0 < \alpha < 1$.

Suppose that $V(x,t) = U(\xi)$, where $\xi = x + ct$, is a travelling wave solution to the cable equation and increases monotonically from $U = 0$ as $\xi \to -\infty$ to $U = 1$ as $\xi \to \infty$, with $U'(\xi) \to 0$ as $\xi \to \pm\infty$. Derive the condition that needs to be satisfied to ensure a left moving travelling wave $(c > 0)$. (6 marks)

(b) Consider the spatial Fitzhugh-Nagumo model

$$\epsilon \frac{\partial v}{\partial t} = f(v,w) + \epsilon^2 \frac{\partial^2 v}{\partial x^2},$$
$$\frac{\partial w}{\partial t} = g(v,w)$$

where $f(v,w) = v(a-v)(v-1) - w + I_a$, $g(v,w) = \gamma(v-w)$, and $\epsilon \ll 1$. Here, a, I_a, and γ are positive constants. We seek a travelling pulse solution using a singular perturbation approach. You may leave any equations derived below in terms of $f(v,w)$ and $g(v,w)$.

(i) What are the *outer* equations? Discuss them briefly in relation to our study in lecture of the Fitzhugh-Nagumo model. (3 marks)

(ii) Introduce $\tau = t$ and $\xi = (x - y(t))/\epsilon$, where $y(t)$ is the wave location, and set $\epsilon = 0$ to derive the reduced *inner* equations for the spatial Fitzhugh-Nagumo system. (3 marks)

(iii) In seeking a travelling wave solution to the inner equations, what determines the conditions for v as $\xi \to \pm\infty$? What does the wave speed depend on? (3 marks)

(iv) Using arguments that link the solutions to the inner and outer equations, provide a short description of the propagation of a single solitary pulse. Include also a phase portrait showing the nullclines and pulse trajectory. Label the parts of the trajectory where the inner and outer solutions apply. (5 marks)

(Total: 20 marks)

Question 1 | Marks & seen/unseen

Parts

a i

THE FIXED POINTS ARE GIVEN BY

SEEN SIMILAR

$$0 = x(100 - E - 4x - y)$$

$$0 = y(60 - x - 2y)$$

THUS,

$x = 0$ OR $100 - E - 4x - y = 0$

$y = 0$ OR $60 - x - 2y = 0$.

THESE GIVE THE FIXED POINTS:

$(0,0)$, $(0,30)$, $(25 - E/4, 0)$

AND, $(20 - 2E/7, 20 + E/7)$.

4 A

a ii

STABILITY

SEEN SIMILAR

LET,

$$f(x,y) = x(100 - E - 4x - y)$$
$$g(x,y) = y(60 - x - 2y)$$

THE JACOBIAN IS GIVEN BY

$$J(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

$$= \begin{bmatrix} 100 - E - 8x - y & -x \\ -y & 60 - x - 4y \end{bmatrix}.$$

EVALUATING THE JACOBIAN AT EACH OF THE FOUR FIXED POINTS WE HAVE:

$$J(0,0) = \begin{bmatrix} 100-E & 0 \\ 0 & 60 \end{bmatrix}$$

$\lambda = 60 > 0$ UNSTABLE.

$$J(0,30) = \begin{bmatrix} 70-E & 0 \\ -30 & -60 \end{bmatrix}$$

$\lambda = -60,\ 70 - E$

UNSTABLE IF $E < 70$

STABLE FOR $E > 70$.

$$J(25 - E/4, 0) = \begin{bmatrix} E - 100 & \frac{E}{4} - 25 \\ 0 & 35 + E/4 \end{bmatrix}$$

$\lambda = 35 + E/4 > 0$ UNSTABLE

$J(20 - \frac{2E}{7}, 20 + E/7)$

$$= \begin{bmatrix} -80 + \frac{8E}{7} & \frac{2E}{7} - 20 \\ -20 - E/7 & -40 - \frac{2E}{7} \end{bmatrix}$$

$\text{TRACE}(J) = -120 + 6E/7$

$\text{TRACE}(J) > 0$ FOR $E > 140$

$$\text{DET}(J) = (80 - 8E/7)(40 + 2E/7) + (\frac{2E}{7} - 20)(20 + E/7)$$

$$= [140 - 2E][20 + E/7]$$

$\text{DET}(J) < 0$ FOR $E > 70$

THUS, STABLE FOR $E < 70$
AND UNSTABLE FOR $E > 70$.

TWO FIXED POINTS EXCHANGE STABILITY AT $E = 70$:

TRANSCRITICAL BIFURCATION

8 A

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts		
bi	THE EQUATIONS ARE NOW: $$\frac{dx}{dt} = x(100-4x-y) - H$$ $$\frac{dy}{dt} = y(60-x-2y)$$	SEEN SIMILAR 2 A
bii	FOR $y \neq 0$, WE CONSIDER $$60 - x - 2y = 0$$ $$x(100-4x-y) - H = 0$$ FROM THE FIRST EQUATION WE HAVE $$x = 60 - 2y$$ SUBSTITUTING THIS INTO THE SECOND EQUATION AND REARRANGING GIVES: $$14y^2 - 700y + (8400 + H) = 0.$$ WHICH HAS SOLUTION: $$y = 25 \pm \frac{1}{2}\sqrt{100 - \frac{2H}{7}}$$	SEEN SIMILAR 4 B
iii	AS H INCREASES, A STABLE AND UNSTABLE FIXED POINT MOVE TOWARD EACH OTHER. AT $H = 350$ A SADDLE-NODE BIFURCATION OCCURS.	SEEN SIMILAR 2 B
	Setter's initials Checker's initials	Page number 4

Question 2 | Marks & seen/unseen

Parts

a

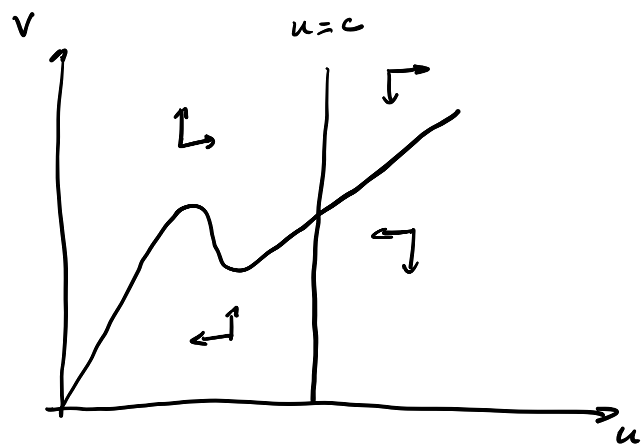
THE NULLCLINES ARE GIVEN BY

SEEN SIMILAR

$$u = c$$

AND

$$v = u + \frac{u}{b(\alpha + u^2)}$$



5 A

b

FIXED POINT:

$$u^* = c$$

$$v^* = c + \frac{c}{b(\alpha + c^2)}$$

STABILITY:

LET

$$f_1(u,v) = b(v-u)(\alpha + u^2) - u$$
$$= bv\alpha + bvu^2 - b\alpha u - bu^3 - u$$

$$f_2(u,v) = c - u$$

$$\frac{\partial f_1}{\partial u} = 2buv - b\alpha - 3bu^2 - 1$$

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts	$$\frac{\partial f_1}{\partial v} = bu^2 + b\alpha$$ $$\frac{\partial f_2}{\partial v} = 0 , \quad \frac{\partial f_2}{\partial u} = -1$$ $$J(u,v) = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ -1 & 0 \end{bmatrix}$$ $$\text{TRACE}(J) = \partial f_1/\partial u$$ $$\text{DET}(J) = \partial f_1/\partial v$$ EVALUATING DET(J) AT (u^*, v^*) GIVES $$\text{DET}(J) = bc^2 + b\alpha > 0$$ FOR THE TRACE, WE HAVE $$\text{TRACE}(J) = 2bv^*c - b\alpha - 1 - 3bc^2$$ FOR $\alpha \ll 1$, $v^* \simeq c + 1/bc$ AND $\text{TRACE}(J) \approx 2bv^*c - 1 - 3bc^2$ PUTTING THESE TOGETHER GIVES $$\text{TRACE}(J) = 1 - bc^2$$ STABLE FOR $bc^2 > 1$ UNSTABLE FOR $bc^2 < 1$	UNSEEN 7 B
	Setter's initials Checker's initials	Page number 6

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts c	BY SUBSTITUTION, WE HAVE THE FIRST EQUATION $$\frac{du}{d\tau} = w(\alpha + u^2) - u$$ DIFFERENTIATING WE SEE $$\frac{dw}{d\tau} = b\left(\frac{dv}{d\tau} - \frac{du}{d\tau}\right)$$ $$= b\left[c - u - w(\alpha + u^2) + u\right]$$ $$= b\left[c - w(\alpha + u^2)\right]$$	UNSEEN 3 B
(d)	FOR $b \ll 1$, w IS THE SLOW VARIABLE AND u IS THE FAST VARIABLE. THE PERIOD CAN BE APPROXIMATED BY THE TIME SPENT ON PART 1. 1 CORRESPONDS TO MOTION ALONG THE NULLCLINE WHICH GIVES: $$w = u/(\alpha + u^2)$$	UNSEEN 5 C
	Setter's initials Checker's initials	Page number 7

	EXAMINATION SOLUTIONS 2022-23	Course
Question 3		Marks & seen/unseen
Parts (a)	THE FIRST TWO INEQUALITIES ENSURE THAT THE SPATIALLY HOMOGENEOUS STEADY-STATE IS STABLE TO SPATIALLY HOMOGENEOUS PERTURBATION. THE REMAINING TWO INEQUALITIES ENSURE THAT $Re(\lambda) > 0$ FOR SOME $k_-^2 < k^2 < k_+^2$, i.e. STEADY STATE IS UNSTABLE TO A SPATIALLY INHOMOGENEOUS PERTURBATION.	SEEN 2 A
b	$$\frac{\partial u}{\partial t} = \frac{u^2}{v} - \frac{u}{4} + \nabla^2 u$$ $$\frac{\partial v}{\partial t} = u^2 - v + D\nabla^2 u$$	SEEN SIMILAR
bi	STEADY STATE: $\boxed{\frac{u^2}{v} - \frac{u}{4} = 0}$, $u^2 - v = 0$, $\boxed{v = u^2}$ $1 - \frac{u}{4} = 0 \Rightarrow \boxed{\begin{matrix} u = 4 \\ v = 16 \end{matrix}}$	2 A
	Setter's initials / Checker's initials	Page number 8

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen

Parts

$$f(u,v) = \frac{u^2}{v} - \frac{u}{4}, \quad g(u,v) = u^2 - v$$

bii

$$\frac{\partial f}{\partial u} = \frac{2u}{v} - \frac{1}{4} \qquad \frac{\partial g}{\partial u} = 2u$$

$$\frac{\partial f}{\partial v} = -\frac{u^2}{v^2} \qquad \frac{\partial g}{\partial v} = -1$$

THUS, THE JACOBIAN EVALUATED AT THE STEADY STATE IS:

$$J(4,16) = \begin{bmatrix} 1/4 & -1/16 \\ 8 & -1 \end{bmatrix}$$

- $\text{TRACE}(J) = -3/4 < 0$ ✓
- $\text{DET}(J) = -1/4 + \frac{1}{2} = 1/4 > 0$ ✓ — ② A

biii

- $D_1 J_{22} + D_2 J_{11} = -1 + D/4 > 0$

$$\Rightarrow D > 4$$

- $(D_1 J_{22} + D_2 J_{11})^2 - 4 D_1 D_2 \det(J)$

$$= (-1 + D/4)^2 - 4D(1/4)$$

$$= \frac{D^2}{16} - \frac{D}{2} + 1 - D > 0$$

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts	REARRANGING, WE HAVE $D^2 - 24D + 16 > 0.$ APPLYING THE QUADRATIC FORMULA WE FIND $\boxed{D > 12 + 8\sqrt{2}}$	4 B
biV	WHEN $k_+^2 = k_-^2 = k_c^2$ WE HAVE $k_c^2 = \dfrac{D_1 J_{22} + D_2 J_{11}}{2 D_1 D_2}$ AND $(D_1 J_{22} + D_2 J_{11})^2 = 4 D_1 D_2 \det(J)$. THUS $k_c^2 = \dfrac{(4 D \times 1/4)^{1/2}}{2D} = \dfrac{1}{2\sqrt{D}}$ WITH D GIVEN ABOVE.	4 D
ci	CONVERTING $\dfrac{dn}{dt} = (r-E)n - \dfrac{rn}{K}$ TO THE LOGISTIC EQUATION WE HAVE $\dfrac{dn}{dt} = (r-E)n\left[1 - \dfrac{rn}{K(r-E)}\right]$	SEEN SIMILAR
	Setter's initials Checker's initials	Page number 10

THUS $n=0$, UNSTABLE F.P.

$$n = \frac{(r-E)}{r}K, \text{ STABLE F.P.}$$

FOR A RIGHT TRAVELLING WAVE

$$n \to \frac{r-E}{r}K \quad \text{AS } \xi \to -\infty$$

$$n \to 0 \quad \text{AS } \xi \to \infty$$

2 C

cii WE CAN NONDIMENSIONALISE BY INTRODUCING

$$t = \tau/(r-E), \quad x = X\left(\frac{D}{r-E}\right)^{1/2}$$

$$n = \frac{K(r-E)}{r} v$$

WHICH YIELDS

$$\frac{\partial v}{\partial t} = v(1-v) + \frac{\partial^2 v}{\partial x^2}$$

WITH $v \to 1$ AS $\xi \to -\infty$

$v \to 0$ AS $\xi \to \infty$

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts	FROM LECTURES, WE KNOW THAT THE WAVE SPEED FOR THIS EQUATION IS $C = 2$. CONVERTING BACK TO DIMENSIONAL VARIABLES, WE HAVE $$c = 2\left[D(r - \epsilon)\right]^{1/2}$$	4 D
	Setter's initials Checker's initials	Page number 12

	EXAMINATION SOLUTIONS 2022-23	Course
Question 4		Marks & seen/unseen
Parts a	INFINITESIMAL TRANSITION PROBABILITIES	SEEN SIMILAR

$$p_{i+j,i}(\Delta t) = \begin{cases} (b_0 + b_1 i + b_2 i^2)\Delta t + o(\Delta t), & j=1 \\ (d_1 i + d_2 i^2)\Delta t + o(\Delta t), & j=-1 \\ 1 - [(b_0 + b_1 i + b_2 i^2) + (d_1 i + d_2 i^2)]\Delta t + o(\Delta t), & j=0 \\ 0, & j \neq -1, 0, 1 \end{cases}$$

GENERATOR MATRIX

$$q_{i,i} = -(b_0 + b_1 i + b_2 i^2) - (d_1 i + d_2 i^2)$$

$$q_{i+1,i} = b_0 + b_1 i + b_2 i^2$$

$$q_{i-1,i} = d_1 i + d_2 i^2$$

3 A

(b) FORWARD KOLMOGOROV EQUATION — SEEN SIMILAR

$$\frac{dp}{dt} = Q p$$

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	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts		

$$\frac{dp_i}{dt} = -\left[b_0 + b_1 i + b_2 i^2 + d_1 i + d_2 i^2\right] p_i$$
$$+ \left[b_0 + b_1 (i-1) + b_2 (i-1)^2\right] p_{i-1}$$
$$+ \left[d_1 (i+1) + d_2 (i+1)^2\right] p_{i+1}$$

4 A

(c) $\mathcal{P}(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i$ UNSEEN

MULTIPLY THE FORWARD KOLMOGOROV BY z^i AND SUM TO OBTAIN

$$\frac{\partial \mathcal{P}}{\partial t} = -b_0 \sum p_i z^i - (b_1 + d_1) \sum i p_i z^i - (b_2 + d_2) \sum i^2 p_i z^i$$
$$+ b_0 \sum p_{i-1} z^i + b_1 \sum (i-1) p_{i-1} z^i$$
$$+ b_2 \sum (i-1)^2 p_{i-1} z^i$$
$$+ d_1 \sum (i+1) p_{i+1} z^i$$
$$+ d_2 \sum (i+1)^2 p_{i+1} z^i$$

NOW WE REINDEX SUMS TO RELATE THE SUMS TO $\mathcal{P}$ AND ITS DERIVATIVES W.R.T. z.

Setter's initials Checker's initials Page number 14

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts		

THE SUMS CORRESPONDING TO THE LAST TWO TERMS ARE:

- $d_1 \left[- \sum_i i p_i z^i + \sum_i (i+1) p_{i+1} z^i \right]$

$$= d_1 \left[- z \sum_i i p_i z^{i-1} + \sum_i i p_i z^{i-1} \right]$$

$$= d_1 (1 - z) \frac{\partial P}{\partial z}$$

- $d_2 \left(- \sum_i i^2 p_i z^i \right)$

$$= - d_2 \left(z^2 \sum_i i^2 p_i z^{i-2} - z^2 \sum_i i p_i z^{i-2} + z^2 \sum_i i p_i z^{i-2} \right)$$

$$= - d_2 \left[z^2 \sum_i i(i-1) p_i z^{i-2} + z \sum_i i p_i z^{i-1} \right]$$

$$= - d_2 z^2 \frac{\partial^2 P}{\partial z^2} - d_2 z \frac{\partial P}{\partial z}$$

Setter's initials | Checker's initials

EXAMINATION SOLUTIONS 2022-23

Course

Question

Marks & seen/unseen

Parts

- $d_2 \sum_i (i+1)^2 p_{i+1} z^i$

$$= d_2 \sum_i i^2 p_i z^{i-1}$$

$$= d_2 z \sum_i i^2 p_i z^{i-2}$$

$$= d_2 z \sum_i i^2 p_i z^{i-2} - z \sum_i i p_i z^{i-2} + z \sum_i i p_i z^{i-2}$$

$$= d_2 z \sum_i i(i-1) p_i z^{i-2} + d_2 \sum_i i p_i z^{i-1}$$

$$= d_2 z \frac{\partial^2 P}{\partial z^2} + d_2 \frac{\partial P}{\partial z}$$

COMBINING THESE RESULTS,

$$d_1 (1-z) \frac{\partial P}{\partial z} - d_2 z^2 \frac{\partial^2 P}{\partial z^2} - d_2 z \frac{\partial P}{\partial z} + d_2 z \frac{\partial^2 P}{\partial z^2} + d_2 \frac{\partial P}{\partial z}$$

$$= (d_1 + d_2)(1-z) + d_2 z(1-z) \frac{\partial^2 P}{\partial z^2}$$

8 D

Setter's initials   Checker's initials

EXAMINATION SOLUTIONS 2022-23

		Course
Question		Marks & seen/unseen
Parts		
d	DIFFERENTIATING EACH TERM OF THE PDE W.R.T. z AND EVALUATING AT $z=1$ GIVES (ONLY NON-ZERO TERMS SHOWN!)	UNSEEN

$$\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial t}\right)\Bigg|_{z=1} = \frac{dm}{dt}.$$

$$\frac{\partial}{\partial z}\left[b_0(z-1)P\right]\Bigg|_{z=1} = b_0 P\Big|_{z=1}$$

$$= b_0$$

$$\frac{\partial}{\partial z}\left[(b_1+b_2)\,z(z-1)\frac{\partial P}{\partial z}\right]\Bigg|_{z=1}$$

$$= (b_1+b_2)\,z\,\frac{\partial P}{\partial z}\Bigg|_{z=1}$$

$$= (b_1+b_2)\,m$$

$$\frac{\partial}{\partial z}\left[b_2 z^2 (z-1) \frac{\partial^2 P}{\partial z^2} \right]\Big|_{z=1}$$

$$= b_2 z^2 \frac{\partial^2 P}{\partial z^2}\Big|_{z=1}$$

$$= b_2 \left[\sigma^2 + m^2 - m \right].$$

$$\frac{\partial}{\partial z}\left[(d_1 + d_2)(1-z) \frac{\partial P}{\partial z} \right]\Big|_{z=1}$$

$$= -(d_1 + d_2) \frac{\partial P}{\partial z}\Big|_{z=1}$$

$$= -(d_1 + d_2) m$$

$$\frac{\partial}{\partial z}\left[d_2 z (1-z) \frac{\partial^2 P}{\partial z^2} \right]\Big|_{z=1}$$

$$= -d_2 z \frac{\partial^2 P}{\partial z^2}\Big|_{z=1}$$

$$= -d_2 (m^2 + \sigma^2 - m)$$

EXAMINATION SOLUTIONS 2022-23

		Course
Question		Marks & seen/unseen
Parts	PUTTING ALL OF THE TERMS TOGETHER GIVES $$\frac{dm}{dt} = b_0 + (b_1 - d_1) m + (b_2 - d_2)(\sigma^2 + m^2)$$	5 C
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EXAMINATION SOLUTIONS 2022-23

Course

Question 5

Marks & seen/unseen

Parts

(a)

TAKING $V(x,t) = U(\xi)$, AND WITH

$$\frac{\partial u}{\partial t} = c\,U' \quad \text{AND} \quad \frac{\partial^2 u}{\partial x^2} = u''$$

THE CABLE EQUATION BECOMES

$$c\,u' = f(u) + u''.$$

MULTIPLYING BY u' AND INTEGRATING GIVES

$$c\int_{-\infty}^{\infty} (u')^2\, d\xi = \int_{-\infty}^{\infty} f(u)\,u'\,d\xi + \int_{-\infty}^{\infty} u''u'\,d\xi$$

WE SEE THAT

$$\int_{-\infty}^{\infty} f(u)\,u'\,d\xi = \int_0^1 f(u)\,du,$$

$$\int_{-\infty}^{\infty} u''\,u'\,d\xi = \int_{-\infty}^{\infty} \frac{1}{2}\frac{d}{d\xi}(u')^2\,d\xi$$

$$= \frac{1}{2}\left[\Big(\underbrace{\frac{du}{d\xi}(\infty)}_{=0}\Big)^2 - \Big(\underbrace{\frac{du}{d\xi}(-\infty)}_{=0}\Big)^2\right]$$

$$= 0$$

SEEN SIMILAR.

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	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts	THUS $$c\int_{-\infty}^{\infty}(u')^2 d\xi = \int_0^1 f(u)\,du$$ AND $$c = \frac{\int_0^1 f(u)\,du}{\int_{-\infty}^{\infty}(u')^2 d\xi}$$ FOR $c>0$, WE NEED $$\int_0^1 f(u)\,du > 0.$$	6
bi	TO FIND THE OUTER SOLUTION, WE SET $\epsilon=0$ TO OBTAIN $$f(v,w)=0$$ $$\frac{\partial w}{\partial t} = g(v,w)$$ THIS IS IDENTICAL TO WHAT WE ENCOUNTER WITH THE ODE MODEL WITH v BEING THE FAST VARIABLE AND w THE SLOW VARIABLE	UNSEEN 3
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	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts		
bii	WITH $\tau = t$ AND $\xi = \dfrac{x - y(t)}{\epsilon}$	UNSEEN

WE SEE THEN THAT

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \tau}\frac{d\tau}{dt}$$

$$= -\frac{y'}{\epsilon}\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \tau}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial \xi^2}$$

$$\frac{\partial w}{\partial t} = -\frac{y'}{\epsilon}\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \tau}$$

USING THESE EXPRESSIONS IN THE SPATIAL F-N GIVES

$$\epsilon\frac{\partial v}{\partial \tau} = f(v,w) + y'\frac{\partial v}{\partial \xi} + \frac{\partial^2 v}{\partial \xi^2}$$

$$\epsilon\left(\frac{\partial w}{\partial \tau} - g(v,w)\right) = y'\frac{\partial w}{\partial \xi}$$

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Question		Marks & seen/unseen

Parts

SETTING $\epsilon = 0$ GIVES

$$f(v,w) + y' \frac{\partial v}{\partial \xi} + \frac{\partial^2 v}{\partial \xi^2} = 0$$

$$y' \frac{\partial w}{\partial \xi} = 0 .$$

3

biii THE CONDITIONS AT $\xi \to \pm\infty$ CORRESPOND TO THE VALUES OF $V(w)$ GIVEN BY $f(v,w) = 0$ (i.e. THE OUTER EQUATIONS) AT THE VALUES OF w WHERE THE SOLUTION JUMPS FROM EXCITED TO RELAXED (DOWN JUMPING) OR RELAXED TO EXCITED (UPJUMPING).

DEFINING THESE VALUES AS

w_+ : UPJUMPING

w_- : DOWN JUMPING

AND $V_+(w)$: EXCITED STATE

$V_-(w)$: RELAXED STATE

	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen
Parts	THUS, WE CAN HAVE $V \to V_+(w_+)$ AS $\xi \to \infty$ AND $V \to V_-(w_+)$ AS $\xi \to -\infty$ OR $V \to V_-(w_-)$ AS $\xi \to \infty$ AND $V \to V_+(w_-)$ AS $\xi \to -\infty$ THE WAVE SPEED WILL DEPEND ON w AND WE HAVE $c(w_+) > 0$ AND $c(w_-) < 0$.	3
biv	FOR A SINGLE WAVE PULSE WE NEED A STABLE FIXED POINT IN THE RELAXED STATE WHICH THE SOLUTION ASSUMES AHEAD OF THE WAVE FRONT. THIS VALUE IS $V_-(w_+)$.	
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Course

Question

Marks & seen/unseen

Parts

THE WAVE FRONT IS GIVEN BY THE SOLUTION TO THE INNER EQUATIONS WITH

$$v \to V_+(w_+) \quad \text{AS} \quad \xi \to \infty$$

AND

$$v \to V_-(w_+) \quad \text{AS} \quad \xi \to -\infty$$

JUST BEHIND THE FRONT, THE SOLUTION WILL EVOLVE ACCORDING TO THE OUTER EQUATIONS UNTIL IT REACHES $V_+(w_-)$. AT THIS POINT, THE SOLUTION AGAIN OBEYS THE INNER EQUATION, BUT NOW WITH CONDITIONS

$$v \to V_-(w_-) \quad \text{AS} \quad \xi \to \infty$$

AND

$$v \to V_+(w_-) \quad \text{AS} \quad \xi \to -\infty$$

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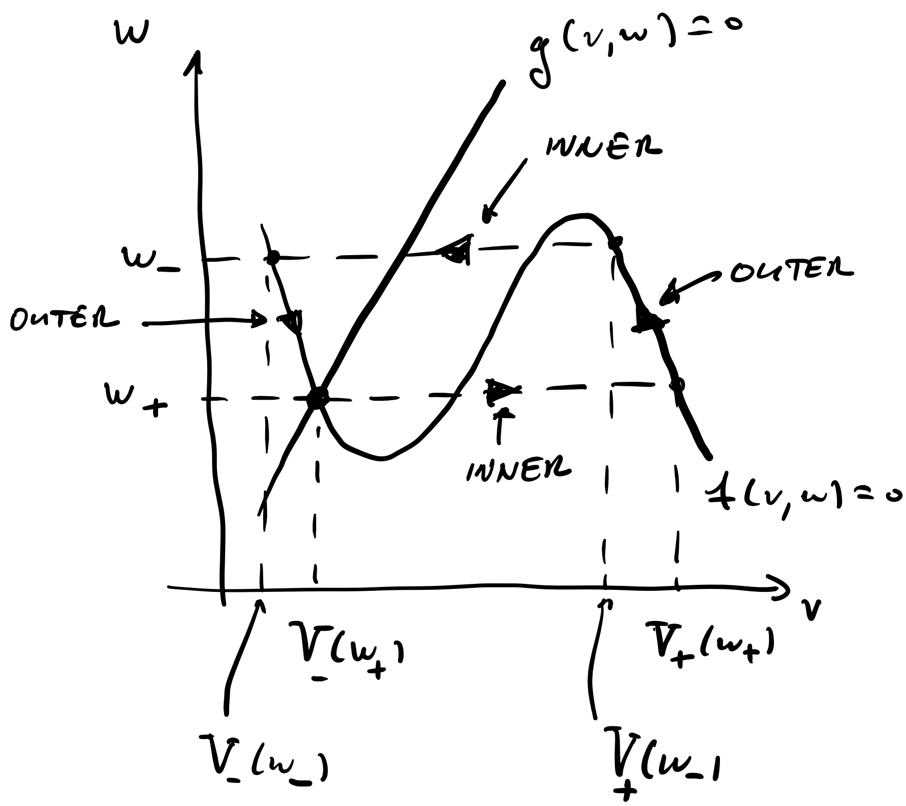
	EXAMINATION SOLUTIONS 2022-23	Course
Question		Marks & seen/unseen

Parts

FINALLY, AFTER REACHING $V_-(w_-)$, THE SOLUTION RELAXES TO $V_-(w_+)$ VIA THE OUTER EQUATIONS.

THIS CAN SUMMARISED USING THE PHASE PORTRAIT



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If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60014/70014	1	Overall good performance on this question. Students here and there lost marks by not finding all fixed points or not properly identifying the type of bifurcation.
MATH60014/70014	2	Overall, students performed well. If students lost marks, most did interpreting the plots in part (d) incorrectly identifying when the solution moves along the nullcline.
MATH60014/70014	3	The students did well on (a), (b) (i) - (iii) and (c) (i), but overall struggled with (b)(iv) and (c)(ii). For (b)(iv) most did not know that critical wavenumber corresponded to the square root in the expression being equal to zero. For (c)(ii), the students did not know how to make effective use of the hint and convert c to dimensional form.
MATH60014/70014	4	In general students struggled on all parts of this problem. In parts (a) and (b), there were many sign errors and errors that indicated that the students didn't recall precisely the definitions related to what was asked for. Performance didn't improve in (c) and (d) where students were not able to effectively manipulate the terms in the forward Kolmogorov equations or evaluate derivatives correctly to obtain an equation for the mean. It could have been that the students were running low on time at this point.
MATH70014	5	The performance on this question was not very good overall. Many students did not attempt this question, and those that did seemed overall very lost, though there were some students that impressed me and demonstrated a deep understanding of the material. Of the students that were able to make progress, they did so by correctly answering (a). As I marked all questions, I do get the feeling that time was a factor and students might have used it to improve answers to other questions rather than address this one.