(1.3.8) Proposition (Lindenbaum Lemma)

Suppose Γ is a consistent set of L-fmlas. Then there is a consistent set of L-fmlas $\Gamma^* \supseteq \Gamma$ which is complete.

Pf: The set of L-fmlas is countable, so we can list the L-fmlas as

$$\phi_0, \phi_1, \phi_2, \ldots .$$

[Why countable? Alphabet

$$\neg \quad \rightarrow \quad (\quad) \quad p_1 \quad p_2 \quad \ldots$$

is countable. ~~End~~ Formulas are finite sequences of these the set of these is countable.].

Define inductively sets of L-fmlas:

$$\Gamma \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

where $\Gamma_0 = \Gamma$ and, suppose Γ_n has been defined:

If $\Gamma_n \vdash \phi_n$ then let $\Gamma_{n+1} = \Gamma_n$

If $\Gamma_n \nvdash \phi_n$ then let

$$\Gamma_{n+1} = \Gamma_n \cup \{(\neg \phi_n)\}.$$

By an inductive argument using Prop. 1.3.7, each Γ_n is consistent.

Let $\Gamma^* = \bigcup_{n \in \mathbb{N}} \Gamma_n$

<u>Claim 1</u> Γ^* is consistent

If $\Gamma^* \vdash \psi$ and $\Gamma^* \vdash (\neg\psi)$ then as deductions are finite there is some $n \in \mathbb{N}$ with

$\Gamma_n \vdash \psi$ and $\Gamma_n \vdash (\neg\psi)$.

<u>Contradiction</u>.

<u>Claim 2</u> Γ^* is complete.

Let ϕ be any L-fmla.

Then ϕ is ϕ_n for some $n \in \mathbb{N}$.

By construction either

$\Gamma_n \vdash \phi$ or $\Gamma_{n+1} \vdash (\neg\phi)$

So either $\Gamma^* \vdash \phi$

or $\Gamma^* \vdash (\neg\phi)$. #.

(1.3.9) <u>Lemma</u>. Suppose Γ^* is a complete, consistent set of L-fmlas. Then there is a valuation v such that for every L-fmla ϕ

$$v(\phi) = T \iff \Gamma^* \vdash \phi.$$

(1.3.10) <u>Cor</u>.

(1) Suppose Δ is a consistent set of L-fmlas. Then there is a valuation v with $v(\Delta) = T$.

(2) Suppose Γ is a consistent set of L-fmlas and $\Gamma \nvdash \phi$. Then there is a valuation v with $v(\Gamma) = T$ and $v(\phi) = F$.

Pf (1) By (1.3.8) there is a complete consistent $\Delta^* \supseteq \Delta$. Then 1.3.9 gives us a valuation v with $v(\Delta^*) = T$

So $v(\Delta) = T$.

(2) Apply (1) to

$\Delta = \Gamma \cup \{(\neg \phi)\}$

(which is consistent, by 1.3.7).

Gives a val. v with $v(\Gamma) = T$

& $v((\neg\phi)) = T$, so

$v(\phi) = F$. #

(1.3.11) (Completenss / Adequacy of L)

(1) (General form) suppose ~~If~~ Γ is a consistent set of L-fmlas and ϕ is an L-fmla such that whenever v is a valuation with $v(\Gamma) = T$ then $v(\phi) = T$

THEN $\Gamma \vdash_L \phi$.

(2) (Special Case) If ϕ is a tautology, then $\vdash_L \phi$.

Pf: (1) By 1.3.10 (2).

(2) By (1) (with $\Gamma = \emptyset$).

#.

[Note: Don't need to assume consistency in (1).].

Proof of 1.3.9.

Γ^* complete, consistent

Want: a valuation v with

$v(\phi) = T \iff \Gamma^* \vdash \phi$. ---- $(\dagger)$

Each ~~of~~ variable p_i is an L-fmla so by the properties of Γ^* either $\Gamma^* \vdash p_i$ or $\Gamma^* \vdash (\neg p_i)$ (and only one happens).

Let v be the unique valuation with $v(p_i) = T \iff \Gamma^* \vdash p_i$ (for $i \in \mathbb{N}$).

Show by induction on the length of ϕ that $(\dagger)$ holds.

Base case ϕ is a variable

– this is the def. of v.

Inductive step.

Case 1 ϕ is $(\neg \psi)$

$\Rightarrow$: Suppose $v(\phi) = T$.

So $v(\psi) = F$ (as v is a val.)

By ind. hyp. $\Gamma^* \nvdash \psi$.

As Γ^* is complete, $\Gamma^* \vdash (\neg \psi)$

so $\Gamma^* \vdash \phi$.

$\Leftarrow$: Suppose $\Gamma^* \vdash (\neg \psi)$

$\Gamma^* \nvdash \psi$ (by consistency)

By ind. hyp. $v(\psi) \neq T$, so $v(\psi) = F$ + thus $v(\phi) = T$.

Case 2 $\quad \phi$ is $(\psi \to \chi)$

$\Leftarrow$: Suppose $v(\phi) = F$. Then $v(\psi) = T$ and $v(\chi) = F$.

[Show: $\Gamma^* \nvdash \phi$]

By ind. hyp. $\Gamma^* \vdash \psi$

and $\Gamma^* \nvdash \chi$.

Suppose for a contradiction

that $\Gamma^* \vdash \phi$.

Then by MP $\Gamma^* \vdash \chi$ -

contradiction.

$\Rightarrow$: Suppose $\Gamma^* \nvdash \phi$

(show $v(\phi) = F$).

So $\Gamma^* \nvdash (\psi \to \chi)$

then $\Gamma^* \nvdash \chi$

(as $\vdash (\chi \to (\psi \to \chi)))$

... ①

Also $\Gamma^* \nvdash (\neg\psi)$

(as $\vdash ((\neg\psi) \to (\psi \to \chi))$ 1.2.7 (a))

... ②

By ①, ② & ind. hyp.

$v(\chi) = F$

$v(\neg\psi) = F$ (Case 1 & ind. hyp.)

So $v(\psi) = T$. So

$v(\phi) = F$, as required.

#.