MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Quiz 7

Question 1. Correct answer: (a).

Fix $x \in \mathbb{R}^d$ and we show now that $O^+(O^+(x)) = O^+(x)$.
($\subset$) Let $y \in O^+(O^+(x))$. Then there exists $t \geq 0$ and $z \in O^+(x)$ with $y = \phi(t, z)$. Since $z \in O^+(x)$, there exists $\tau \geq 0$ such that $z = \phi(\tau, x)$. We get $y = \phi(t, \phi(\tau, x)) = \phi(t + \tau, x) \in O^(x)$.
($\supset$) This is clear since $x \in O^+(x)$.

Question 2. Correct answer: (a).

Consider the one-dimensional differential equation $\dot{x} = 0$. Then $O^+(0) = \{0\}$, which is obviously negatively invariant.

Question 3. Correct answer: (a).

Let $x \in M_1 \setminus M_2$, and assume that there exists $t \in \mathbb{R}$ with $\phi(t, x) \notin M_1 \setminus M_2$. Firstly, we get that $\phi(t, x) \in M_2$ (since M_1 is invariant). Then $x = \phi(-t, \phi(t, x)) \in M_2$, since M_2 is invariant, but this is a contradiction.

Question 4. Correct answer: (b).

The linearisation in the equilibrium $(0, 0)$ is given by

$$\begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix},$$

which has the eigenvalues $1 \pm ib$. The real parts of the eigenvalues are 1, so the equilibrium is repulsive, see comment on page 71 of the lecture notes.

Question 5. Correct answer: (a).

Consider the two-dimensional differential equation in Example 4.3 (ii), and let $x^* = (1, 0)$. Then there exists a homoclinic orbit connecting to this equilibrium, and it follows that $W^u(x^*) = W^s(x^*) = \mathbb{S}^1$. Hence $W^u(x^*) \cap W^s(x^*) = \mathbb{S}^1 \supsetneq \{x^*\}$.