IMPERIAL

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Finite Elements: Numerical Analysis & Implementation

Date: Tuesday, May 27, 2025

Time: Start time 10:00 – End time 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. Which of the following are unisolvent finite elements? In each case, justify your answer.

(a) K is the interval $[a, b]$. $\mathcal{P}$ are the degree 3 polynomials on K. The nodal variables are

$$N_1(v) = v(a), \ N_2(v) = v(b), \ N_3(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(a), \ N_4(v) = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}(b),$$

for all polynomials $v \in \mathcal{P}$. (6 marks)

(b) K is a triangle. $\mathcal{P}$ are the degree 5 polynomials on K. The nodal variables are

$$N_1(v) = v(z_1), \ N_2(v) = v(z_2), \ N_3(v) = v(z_3), \ N_4(v) = \frac{\partial v}{\partial x_1}(z^*), \ N_5(v) = \frac{\partial v}{\partial x_2}(z^*),$$

where x_1, x_2 are the coordinates in the triangle, z_1, z_2, z_3 are the vertices and $z^* = (z_1 + z_2 + z_3)/3$. (6 marks)

(c) K is the unit square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$. $\mathcal{P} = \mathrm{span}(1, x, y, x^2, y^2, xy, x^2y, y^2x, y^2x^2)$. The nodal variables are point evaluations at the points $(i/2, j/2)$ for integers $0 \leq i, j \leq 2$. (8 marks)

(Total: 20 marks)

2. Let K be a triangle. For $f \in C^{k,\infty}(K)$, the averaged Taylor polynomial of degree k is defined as

$$(Q_{k,B}f)(x) = \frac{1}{|B|} \int_B \sum_{|\alpha| \leq k} D^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}\, d\,y,$$

where: $B \subset K$ is a disk in $\mathbb{R}^2$ of area $|B|$, the sum is over multi-indices α of degree $|\alpha|$, $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2}}$, $\alpha! = \alpha_1! \alpha_2!$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$.

(a) Let β be a multi-index with $|\beta| \leq k$. Show that

$$D^\beta (Q_{k,B} f)(x) = (Q_{k-|\beta|,B} D^\beta f)(x).$$

(8 marks)

(b) For a triangle K, we assume that for $f \in C^{k+1,\infty}$, there exists C (depending on K and k), such that

$$\|f - Q_{k,B} f\|_{L^2(K)} \leq C|f|_{H^{k+1}(K)}.$$

Show that for a multi-index β with $|\beta| \leq k+1$,

$$\|D^\beta(f - Q_{k,B} f)\|_{L^2(K)} \leq C|f|_{H^{k+1}(K)}.$$

(8 marks)

(c) Briefly, given the use of rescaling arguments to obtain error estimates that depend on the triangle sizes for the global interpolation operator, explain why it is critical that the semi-norm $|f|_{H^{k+1}(K)}$ appears in these formulae instead of the norm $\|f\|_{H^{k+1}(K)}$. (4 marks)

(Total: 20 marks)

3. In this question we consider the following boundary value problem solved on the interval $[0, 1]$,

$$\frac{d^4 u}{d x^4} - \frac{d^2 u}{d x^2} + u = f, \ u(0) = u(1) = \frac{d^2 u}{d x^2}(0) = \frac{d^2 u}{d x^2}(1) = 0. \tag{1}$$

(a) Formulate (1) as a linear variational problem, seeking $u \in V$ where

$$b(u_h, v) = F(v), \quad \forall v \in V, \tag{2}$$

where $b(\cdot, \cdot)$ is a symmetric bilinear form, and V is an appropriate subspace of $H^2([0, 1])$.
(8 marks)

(b) Propose an appropriate finite element space $V_h \subset V$ to obtain a Galerkin approximation of (2). (6 marks)

(c) Assume that $\|f\|_{L^2([0,1])} < \infty$. Show that the Galerkin approximation has a unique solution u_h, and that there exists $C > 0$ such that

$$\|u_h\|_{H^2([0,1])} \leq C\|f\|_{L^2(\Omega)}.$$

You may make use of any of the stated results in the course. (6 marks)

(Total: 20 marks)

4. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Given a known C^∞ function $\alpha : \Omega \to \mathbb{R}$ with $0 < a < \alpha < b < \infty$ for given constants $a, b \in \mathbb{R}$, we consider the PDE

$$u - \nabla \cdot (\alpha \nabla u) = f, \tag{3}$$

with boundary conditions $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial \Omega$ of Ω, and known $f \in L^2(\Omega)$. The variational form of this PDE seeks $u \in H^1(\Omega)$ such that

$$\int_\Omega uv + \alpha \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x, \quad \forall u \in H^1(\Omega).$$

(a) Let u_h be the finite element approximation of the solution of (3) obtained using a quadratic Lagrange finite element space of degree 2. Assuming that the solution u of (3) is in $H^2(\Omega)$, show that

$$\|u_h - u\|_{H^1(\Omega)} \leq Ch^2 \|u\|_{H^3(\Omega)},$$

for some constant $C > 0$, independent of the mesh parameter h. You may assume any results stated in lectures without proof. (10 marks)

(b) Let w solve the equation

$$w - \nabla \cdot (\alpha \nabla w) = g, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega.$$

We assume that there exists $C_2 > 0$ such that for any $g \in L^2(\Omega)$,

$$|w|_{H^2(\Omega)} \leq C_2 \|g\|_{L^2(\Omega)}.$$

Under this assumption, show that

$$\|u_h - u\|_{L^2(\Omega)} \leq C_3 h^3 \|u\|_{H^3(\Omega)},$$

for some constant $C_3 > 0$. (You may assume the Galerkin orthogonality result.) (10 marks)

(Total: 20 marks)

5. We consider the following abstract mixed linear variational problem. Find $u \in V$ and $p \in Q$ such that

$$a(u, v) + b(v, p) = F[v], \; b(u, q) = G[q], \tag{4}$$

for continuous bilinear forms a on $V \times V$ and b on $V \times Q$ respectively, and continuous linear forms $F \in V'$ and $G \in Q'$ respectively.

(a) Assume that there exists $\beta > 0$ such that

$$\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta.$$

Consider the operator $B : V \to Q'$ given by

$$Bv[p] = b(v, p), \quad \forall v \in V, \, p \in Q.$$

Show that we can always find $u_g \in V$ such that $Bu_g = G$. (You may assume that B^* injective is equivalent to B surjective, according to the Closed Range Theorem.) (6 marks)

(b) Define $Z \subset V$ as the kernel of B. Assume that a is coercive on V. Show that there exists $u_Z \in Z$ such that

$$a(u_Z, v) + b(v, p) = F[v] - a(u_g, v), \; \forall v \in Z.$$

(6 marks)

(c) Using the previous parts, show that there exists (u, p) solving (4).

(You may assume that $\text{Im}(B^*) = (\text{Ker}(B))^0$, where ^{0} indicates the polar space, according to the Closed Range Theorem.) (8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

M70022

Finite Elements (Solutions)

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1. (a) This triple is unisolvent, because if $N_i(v) = 0$, $i = 1, 2, 3, 4$, then v is a degree 3 polynomial with two double roots, and hence is zero from Fundamental Theorem of Algebra.

   (b) This triple is not unisolvent, because the space of degree 5 polynomials is 6 dimensional and there are only 5 nodal variables.

   (c) Let v be a polynomial in $\mathcal{P}$, which vanishes at all the specified points. In particular v vanishes at 3 points on the line $x = 0$. Since v restricted to that line is a quadratic polynomial in y only, v vanishes on that line. Hence, $v(x, y) = xq_3(x, y)$ where q is a cubic polynomial. Similarly, $v(x, y)$ vanishes on $y = 0$, and hence by continuity $q_3(x, y)$ vanishes there, so $q_3(x, y) = yq_2(x, y)$ where q_2 is a quadratic polynomial. Iterating this argument two more times, considering the lines $x = 1$, and $y = 1$ in succession, we obtain that $v(x, y) = cxy(x - 1)(y - 1)$, where $c \in \mathbb{R}$. Finally, $v(x, y)$ vanishes at $(1/2, 1/2)$, so $c = 0$, and thus $v \equiv 0$. Hence, the triple is unisolvent.

2. (a)

$$D^\beta(Q_{k,B}f)(x) = D^\beta \frac{1}{|B|} \int_B \sum_{|\alpha|\leq k} D^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}\, \mathrm{d}\,y,$$

$$= \frac{1}{|B|} \int_B \sum_{|\alpha|\leq k} D_y^\alpha f(y) \frac{(x-y)^\alpha}{(\alpha-\beta)!}\, \mathrm{d}\,y,$$

$$= \frac{1}{|B|} \int_B \sum_{|\alpha|\leq k} D_y^\alpha f(y) D_x^\beta \frac{(x-y)^\alpha}{\alpha!}\, \mathrm{d}\,y,$$

$$= \frac{1}{|B|} \int_B \sum_{|\alpha|\leq k} D_y^\alpha f(y) \frac{(x-y)^{\alpha-\beta}}{(\alpha-\beta)!}\, \mathrm{d}\,y,$$

$$= \frac{1}{|B|} \int_B \sum_{|\alpha'|\leq k-|\beta|} D^{\alpha'+\beta} f(y) \frac{(x-y)^{\alpha'}}{(\alpha')!}\, \mathrm{d}\,y,$$

$$= (Q_{k-|\beta|} D^\beta f)(x).$$

(b)

$$\|D^\beta(f-Q_{k,B}f)\|^2_{L^2(K)} = \|D^\beta f - Q_{k-|\beta|,B} D^\beta f\|^2_{L^2(K)},$$

$$\leq C|D^\beta f|^2_{H^{k-|\beta|+1}(K)},$$

$$\leq C \sum_{|\gamma|=|\beta|} |D^\gamma f|^2_{H^{k-|\beta|+1}(K)},$$

$$= C \sum_{|\gamma|=|\beta|} \sum_{|\delta|=k-|\beta|+1} |D^{\gamma+\delta} f|^2_{L^2(K)},$$

$$= C \sum_{|\gamma|=k+1} |D^\gamma f|^2_{L^2(K)},$$

$$= C|f|^2_{H^{k+1}(K)}.$$

(c) The rescaling arguments involve changing variables from K to K_1 in the integral defining the (squared) seminorm. Since this only involves derivatives of the same degree, we get a common factor of h^{k+1}. If the full norm is used, the derivatives are a mixture of degrees of $k+1$ or less, and the common factor is only h, which spoils the estimate.

3. (a) Multiplication by a test function v with $v(0) = v(1) = 0$, and integrate:

$$\int_0^1 v\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - v\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + uv \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x.$$

Integration by parts twice in the first term and once in the second:

$$\int_0^1 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + uv \, \mathrm{d}x - \left[v\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^1 - \left[v\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}\right]_0^1 + \left[\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right]_0^1 = \int_0^1 fv \, \mathrm{d}x.$$

The first two boundary terms vanish since $v(0) = v(1) = 0$, and the third one vanishes due to $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(1) = 0$. Hence we seek $u \in \mathring{H}^2([0,1])$ such that

$$\int_0^1 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + uv \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x, \quad \forall v \in \mathring{H}^2([0,1]),$$

where

$$\mathring{H}^2([0,1]) = \{u \in H^2([0,1]) : u(0) = u(1) = 0\}.$$

(b) One choice is the cubic Hermite element with nodal variables being function values and derivatives at cell endpoints, thus ensuring that the finite element functions are continuous and have continuous derivatives (since these values will agree either side of the cell endpoint). Then $V_h \subset C^1([0,1]) \subset H^2([0,1])$.

(c) To satisfy the conditions of the Lax-Milgram theorem, we need to show that $a : V \times V \to \mathbb{R}$ is continuous and coercive in H^2, and $F : V \to \mathbb{R}$ is bounded in H^2, where a and F are the bilinear and linear forms from the left and right hand side of the variational problem, respectively.

$$F[v] \leq \|v\|_{L^2}\|f\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^2},$$

from the Schwarz inequality and the definition of L^2 and H^2 norms, so F is bounded. The bilinear form a is exactly the H^2 inner product, so it has continuity and coercivity constants both equal to 1. Hence the solution exists and is unique, and the required bound holds with $C = 1/1 = 1$.

4. (a) From the notes, Céa's Lemma says that

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma} \min_v \|u - v\|_{H^1},$$

where M is the continuity constant of the bilinear form and γ is the coercivity constant, both using the H^1 norm. For this problem, we have

$$\int_\Omega uv + \alpha \nabla u \cdot \nabla v \, \mathrm{d}x \leq \|u\|_{L^2}\|v\|_{L^2} + \max(1,b)|u|_{L^2}|v|_{L^2} \leq (1 + \max(1,b))\|u\|_{H^1}\|v\|_{H^1},$$

so an upper bound for the continuity constant is $\max(1, b)$. Similarly, a lower bound for the coercivity constant is $\min(1, a)$. Hence,

$$\|u - u_h\|_{H^1} \leq \frac{\max(1,b)}{\min(1,a)} \min_v \|u - v\|_{H^1}.$$

Choosing $v = \mathcal{I}_h u$, we use the error estimate derived in lectures,

$$\|u - u_h\|_{H^1} \leq \frac{\max(1,b)}{\min(1,a)}\|u - \mathcal{I}_h u\|_{H^1} \leq \leq \frac{\max(1,b)}{\min(1,a)}h^2\|u - \mathcal{I}_h u\|_{H^3}.$$

10, B

(b) Following the steps of the Aubin-Nitsche trick, we can choose $g = u - I_h u$. Then,

$$\|u - I_h u\|_{L^2} = \langle u - u_h, u - u_h \rangle_{L^2}$$

$$\text{weak form of } w \text{ equation} = a(w, u - u_h),$$

$$\text{Galerkin orthogonality} = a(w - I_h w, u - u_h),$$

$$\text{Continuity of } a \quad \leq M\|u - u_h\|_{H^1}\|w - I_h w\|_{H^1},$$

$$\text{error estimate for } w \quad \leq M\|u - u_h\|_{H^1}Ch|w\|_{H^2},$$

$$\text{elliptic regularity for } w \quad \leq M\|u - u_h\|_{H^1}ChC_1\|u - u_h\|_{L^2},$$

$$\text{error estimate for } u \quad \leq MC_2h^2|u|_{H^2}ChC_1\|u - u_h\|_{L^2},$$

and dividing by $\|u - u_h\|_{L^2}$ gives the result.

10, C

5. (a) We need to show that B^* is injective. The inf-sup condition says

$$\beta \le \inf_{0 \ne q \in Q} \sup_{0 \neq v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} = \inf_{0 \ne q \in Q} \frac{1}{\|q\|_Q} \sup_{0 \neq v \in V} \frac{b(v,q)}{\|v\|_V} = \inf_{0 \ne q \in Q} \frac{\|B^* q\|_{V'}}{\|q\|_Q}.$$

Hence, $\|B^* q\|_{V'} \ge \beta \|q\|_Q$ for all $q \in Q$. If there exist q_1, q_2 such that $B^* q_1 = B^* q_2$, then

$$0 = \|B^*(q_1 - q_2)\|_{V'} \ge \beta \|q_1 - q_2\|_Q,$$

so $q_1 = q_2$, i.e. B^* is injective. Therefore B is surjective, so for all G there exists u_g such that $B u_g = G$.

6, M

(b) The right hand side is continuous as it is the sum of two continuous forms. a is continuous on coercive on V and therefore is continuous and coercive on $Z \subset V$. Hence, Lax-Milgram gives a unique solution u_Z.

6, M

(c) We have $B u_g = G$ and $B u_z = 0$ (because $u_z \in Z$). Therefore, $Bu = B(u_g + u_z) = G$, i.e. $b(u,q) = G[q]$ for all $q \in Q$, which is the second of the two equations in the mixed system. Further, if we define

$$L[v] = F[v] - a(u,v), \quad \forall v \in V,$$

we have that

$$L[v] = F[v] - a(u_g, v) - a(u_z, v) = 0, \quad \forall v \in Z.$$

Hence, $L \in (\mathrm{Ker} B)^0$ i.e. $L \in \mathrm{Im}(B^*)$. This means that there exists p such that $B^* p = L$, i.e.

$$b(v,p) = F[v] - a(u,v), \quad \forall v \in V.$$

Rearranging, we have

$$a(u,v) + b(v,p) = F[v], \quad \forall v, \in V,$$

which is the first of the two equations in the mixed system.

8, M

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 26 of 20 marks
Total C marks: 10 of 12 marks
Total D marks: 12 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH70022 Finite Elements: Numerical Analysis & Implementation Markers Comments

Question 1   Common issues in this question: considering univariate polynomials in part b when it was in 2D resorting to algebraic computations in the Vandermonde matrix when the geometric "line based" technique taught in lectures is much simpler, resulting in getting stuck/errors.

Question 2   Most students did reasonably well with this question. Common error in part a was mixing up derivatives between x (the range of the function on the LHS) and y (the variable that is integrated over in the average). Only a small number of students were able to do part c.

Question 3   In part a, several candidates were nonspecific about the space the variational formulation is posed on (including boundary conditions on the space), and relatedly were nonspecific about the reasons for the various vanishing boundary terms in the formulation, whilst others just got this wrong. In part b, several candidates did not specify a particular finite element space to use, just gave some properties of it. Most candidates could do part c.

Question 4   Most candidates were able to do part (a), but some did not find coercivity/continuity constants for the bilinear form. Only very few candidates could recall how to use the Aubin-Nitsche trick for part (b).

Question 5   A range of marks in this question depending on whether candidates could recall this bookwork argument.