Imperial College London

Autumn Term, 2022

Analysis 1A

Lecture 14
Series

Ajay Chandra

In preparation for talking about series, we'll want to define diverging to infinity.

Definition

We say $a_n \to +\infty$ if and only if

$$\forall R > 0 \ \exists \ N \in \mathbb{N}_{>0} \text{ such that } a_n > R \ \forall n \geq N.$$

In preparation for talking about series, we'll want to define diverging to infinity.

Definition

We say $a_n \to +\infty$ if and only if

$$\forall R > 0 \ \exists \ N \in \mathbb{N}_{>0} \text{ such that } a_n > R \ \forall n \geq N.$$

Remark 3.41

Recall this is not the same as (but it does imply) a_n being divergent!

In preparation for talking about series, we'll want to define diverging to infinity.

Definition

We say $a_n \to +\infty$ if and only if

$$\forall R > 0 \; \exists \; N \in \mathbb{N}_{>0} \text{ such that } a_n > R \; \forall n \geq N.$$

Remark 3.41

Recall this is not the same as (but it does imply) a_n being divergent!

Exercise 3.42

Suppose $a_n > 0 \; \forall n$. Show $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$.

Now let's get started with series:

Definition

An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots,$$

where $(a_i)_{i \geq 1}$ is a sequence.

Now let's get started with series:

Definition

An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots ,$$

where $(a_i)_{i \geq 1}$ is a sequence.

For now, it is **not** a real number. It is just a **formal expression** - we haven't given this a mathematical meaning yet.

Now let's get started with series:

Definition

An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots ,$$

where $(a_i)_{i \geq 1}$ is a sequence.

For now, it is **not** a real number. It is just a **formal expression** - we haven't given this a mathematical meaning yet.

We could write the series

$$\sum_{n=1}^{\infty} \sin(n) \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n n \, .$$

but these only make sense as **formal expressions**.

Key Idea: Given a sequence a_n, in addition to the infinite series $\sum_{n=1}^{\infty} a_n$, we also get a sequence of **partial sums**

$$s_n := \sum_{i=1}^{n} a_i$$

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$

$\sum_{n=1}^{\infty} a_i$ makes sense iff S_n converges

Key Idea: Given a sequence a_n, in addition to the infinite series $\sum_{n=1}^{\infty} a_n$, we also get a sequence of **partial sums**

$$s_n := \sum_{i=1}^{n} a_i$$

While the infinite series is just a formal expression, the sequence (s_n) actually makes sense as a legitimate sequence of real numbers.

Key Idea: Given a sequence a_n, in addition to the infinite series $\sum_{n=1}^{\infty} a_n$, we also get a sequence of **partial sums**

$$s_n := \sum_{i=1}^{n} a_i$$

While the infinite series is just a formal expression, the sequence (s_n) actually makes sense as a legitimate sequence of real numbers.

Recall in Exercise 3.2 you proved that a_n and s_n determine each other – they are equivalent information since $a_n = s_n - s_{n-1}$.

Definition - Convergence of series

We say that the series $\sum_{n=1}^{\infty} a_n$ "converges to $A \in \mathbb{R}$" if and only if the associated sequence of partial sums converges to A:

$$\sum_{n=1}^{\infty} a_n = A \iff s_n \to A.$$

We can obviously let the sum be from $n = 0$, or over even natural numbers, or any other "list" (countable set) ...

$$\sum_{i=0}^{\infty} b_i, \quad \sum_{j \in \mathbb{N} \text{ even}} c_j, \quad \sum_{k=102}^{\infty} d_k, \quad \cdots$$

Example 4.1

Consider $a_n = x^n, \ n \geq 0$, so that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

Investigate whether this series converges (this will depend on the value of x).

$$S_n = \sum_{i=0}^{n} x^i$$

$S_0 = x^0 = 1$

$S_1 = x^0 + x^1 = 1 + x$

$S_2 = x^0 + x^1 + x^2 = 1 + x + x^2$

$\vdots$

x is a fixed real number

Does S_n converge and if so, what is its limit?

$S_n = 1 + x + \cdots + x^n$, $xS_n = x + x^2 + x^3 + \cdots + x^{n+1}$

$\to 0$ if $|x|<1$

$S_n - xS_n = 1 - x^{n+1}$

$(1-x)S_n = 1 - x^{n+1}$

as long as $x \neq 1$ $\quad S_n = \dfrac{1 - x^{n+1}}{1 - x}$

If $x = 1$ $\quad S_n = n+1$ diverges

If $x = -1$, $S_n = \dfrac{1 - (-1)^{n+1}}{2}$ diverges

If $|x| < 1$, $S_n \longrightarrow \dfrac{1}{1-x}$

If $|x| > 1$ $\quad S_n$ is unbounded!

Theorem 4.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$.

Proof

$s_0 = a_0,\ s_1 = a_0 + a_1, \ldots$

Define $s_n = \sum_{i=0}^{n} a_i$, $s_n \longrightarrow L$

Note $a_n = s_n - s_{n-1}$

So by Algebra of Limits

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = \emptyset$$

■

Theorem 4.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$.

Remark 4.3

The converse of Theorem 4.2 is *not true*.

You can have $a_n \to 0$

but $\sum_{n=0}^{\infty} a_n$ fail to be convergent.

Example 4.4

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent. Harmonic Series

Example 4.5

Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Example 4.4

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Example 4.4

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Rough work

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \dots + \frac{1}{7}\right)$$

$$+ \left(\frac{1}{8} + \dots + \frac{1}{15}\right) + \left(\frac{1}{16} + \dots + \frac{1}{31}\right) + \dots$$

$\geq$

$\frac{2}{3}$

$\frac{1}{3} + \frac{1}{3}$

$\frac{4}{7}$

$\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$

$\geq 1 +$ infinitely many $1/2$'s

$\frac{1}{15} + \dots + \frac{1}{15}$

$\frac{8}{15}$

$\downarrow$

$\frac{16}{31}$

Example 4.4

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Proof

Example continued: We can bound the k th bracketed term from below:

$$\left(\frac{1}{2^k} + \cdots + \frac{1}{(2^{k+1}-1)}\right) > \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} = \frac{2^k}{2^{k+1}} = \frac{1}{2}.$$

In particular then

$$s_{2^{k+1}-1} > 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{k \text{ terms}} = 1 + \frac{k}{2}$$

$k=1$

$s_{2^2-1} = s_3 > 1 + \frac{1}{2}$

S_n monotone increasing

S_n is unbounded, can't converge!

Can also say S_n diverges to ∞.

Example 4.5

Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof (Using a trick)

Look at $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, then if $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$

Telescoping Series

Then $s_n = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$+ \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

$s_n \to 1$

Example 4.5

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Example continued:

Let $\displaystyle \sigma_n = \sum_{j=1}^{n} \frac{1}{j^2} = 1 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2}$

$$\leq 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} = 1 + S_{n-1}$$

S_n is bounded, so σ_n is bounded, σ_n is monotone increasing so convergent!

∎