Temporary front page

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Introduction to Game Theory Mock Paper A

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH60 paper; 2.5 Hours for MATH70 paper

This paper has *4 Questions (MATH60 version); 5 Questions (MATH70 version)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Throughout this question consider a finite, two-player, simultaneous move game being played between players A and B. Let the pure strategy sets of the players be A_S and B_S respectively and let $\mathbb{A}_S$ and $\mathbb{B}_S$ be the respective mixed strategy sets of the players. Denote $g_A(\alpha, \beta)$ to represent the payoff to player A when A plays strategy $\alpha \in \mathbb{A}_S$ and B plays strategy $\beta \in \mathbb{B}_S$. Similarly denote $g_B(\alpha, \beta)$ as the payoff to player B in this case.

(a) Define what it means to have an **equilibrium** of the game. (3 marks)

(b) Prove that for any mixed strategy β^* of player B

$$\max_{\alpha \in \mathbb{A}_S} \{g_A(\alpha, \beta^*)\} = \max_{a \in A_S} \{g_A(a, \beta^*)\}.$$

(4 marks)

(c) Define what it means for a strategy α^* of player A to be an equaliser strategy. (2 marks)

(d) Prove that if α^* is an equaliser strategy for player A and β^* is an equaliser strategy for player B then (α^*, β^*) forms an equilibrium of the game. (4 marks)

(e) Consider the game of **Twisted Twos and Threes**: a two-player, simultaneous move game where player A must decide to declare 'odd' or 'even' and player B must choose to declare $3n-1$, $3n$ or $3n+1$ (in other words, to declare 'one less than a multiple of three', 'a multiple of three' or 'one more than a multiple of three').

Player A then receives payoff equal to the unique positive integer less than 7 for which the two conditions apply, for example, if A chooses 'even' and B chooses 'a multiple of three', then A gets a payoff of 6. Player B receives payoff equal to 7 minus the payoff of player A.

(i) Draw a normal (strategic) form representation of the game. (2 marks)

(ii) Find all equilibria of the game. (5 marks)

(Total: 20 marks)

2. (a) In a two-player zero-sum game the (i, j) entry of the matrix

$$\begin{pmatrix} 1 & x \\ x & 3 \end{pmatrix}$$

where $x \in \mathbb{R}$, gives the payoff to player A when A plays pure strategy a_i and player B plays pure strategy b_j.

(i) Find the value(s) of x for which the game has a pure strategy equilibrium. (3 marks)

Suppose now that the value of the game is equal to $\frac{11}{3}$.

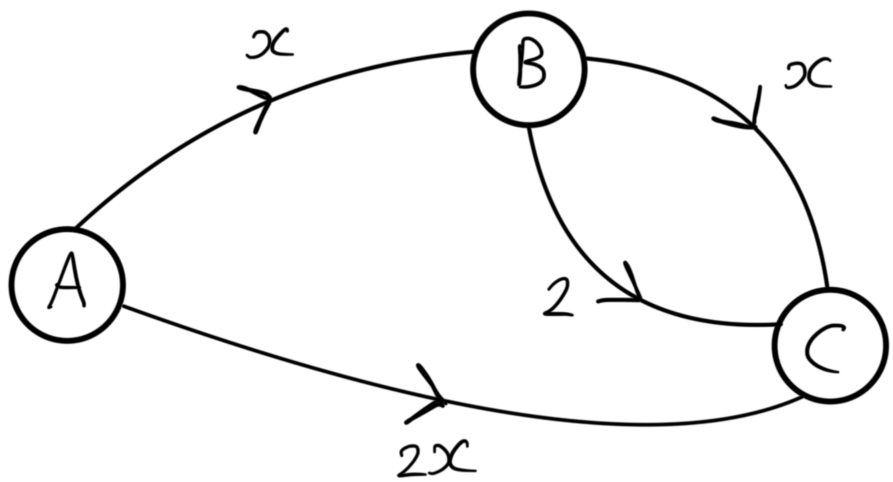
(ii) Why must x be greater than 3? (1 mark)

(iii) Find x. (5 marks)

(b) In the following question we consider the atomic model of flow through a congestion game.

(i) Define what it means to have an **equilibrium** in a congestion game. (3 marks)

Consider the congestion game played on the network below with three nodes A, B and C and 10 users who all want to travel from node A to node D. Each edge in the diagram shows the cost function for a flow of x users on that edge, so, for example, $2x$ is short for $c(x) = 2x$.



(ii) Find all equilibria of the game. (4 marks)

(iii) Find the socially optimal flow of the game (i.e the distribution of users which minimises the average cost per user of the network). (4 marks)

(Total: 20 marks)

3. Consider the two-player game shown in normal form below where the first entry in each ordered pair is the payoff to player A and the second entry is the payoff to player B where A has pure strategies a_i $(i = 1, 2)$ and B has pure strategies b_j $(j = 1, 2, 3)$.

		B		
		b_1	b_2	b_3
A	a_1	1,2	2,2	3,2
	a_2	3,1	1,0	2,4

(a) First, consider the game being played **cooperatively** by the players.

(i) Determine the threat point for the game. (5 marks)

(ii) Sketch the payoff set and identify the bargaining set. Indicate the pareto-optimal frontier of this set. (4 marks)

(iii) Find the Nash bargaining solution for the game and show how the players can implement this solution. (5 marks)

(b) Now consider the game being played **non-cooperatively**. Show that the game has an infinite number of equilibria given by

$$(a_1, (q_1, q_2, 1 - q_1 - q_2)),$$

where $0 \leq q_1 \leq \frac{1}{3}$, and $0 \leq q_2 \leq 1 - q_1$. (6 marks)

(Total: 20 marks)

4. [Throughout this question you may assume any results about impartial games and the game of Nim unless you are asked to prove them.]

(a) Define the **Nim value** of an impartial game G. (2 marks)

(b) Prove that an impartial game is losing if and only if all its options are winning and that an impartial game is winning if and only if at least one of its options is losing. (4 marks)

(c) Consider the following variant of the game Nim called **Split-Nim**. This is an impartial game played with piles of tokens as in Nim and, like in Nim, on their turn a player can remove some tokens from one of the piles, or otherwise can split a pile into two (not necessarily equal) new piles.

For example, a pile of size 4 can be reduced to size 3, 2, 1 or 0 as in Nim, or can be split into two piles of sizes 1 and 3, or two piles of sizes 2 and 2.

The game is played with the normal play convention, where the last player to move wins.

(i) Find the Nim values of the single Split-Nim piles of size 1, 2, 3 and 4 respectively. (5 marks)

(ii) Find all winning moves, if any, when Split-Nim is played with three piles of sizes 1, 2 and 3. What about with three piles of sizes 1, 2 and 4? (5 marks)

(iii) Prove or disprove the following claim: "Two piles in Split-Nim are equivalent if and only if they have equal size". (4 marks)

(Total: 20 marks)

5. In a two-player game, players A and B each simultaneously choose an integer from the set $G_n = \{1, 2, \cdots, n\}$, $n \geq 1$. If their choices are equal the game is drawn, if their choices differ by ± 1 then player A wins the game and in all other cases player B wins the game.

(a) Find a solution to the game when $n = 3$. (5 marks)

(b) When $n = 4$, the strategy for player A in which A chooses 2 and 3 with equal probability and never chooses 1 or 4 is a max-min strategy. Find a min-max strategy for player B and find the value of the game. (5 marks)

(c) When $n = 5$, the strategy for player A in which A chooses 1, 2, 3, 4 and 5 with probabilities 0, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{2}{5}$ and 0 respectively is a max-min strategy. Show that any strategy where B chooses 1, 2, 3, 4 and 5 with probabilities p, $\frac{4}{5}-2p$, 0, $2p-\frac{2}{5}$ and $\frac{3}{5}-p$ respectively where $\frac{4}{15} \leq p \leq \frac{5}{15}$ is a min-max strategy for B and find the value of the game. (8 marks)

(d) For what value(s) of n is the game a fair game? Justify your answer (you need not prove this). (2 marks)

(Total: 20 marks)