Imperial College London

MATH97007 MATH97087

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Fluid Dynamics 2

Date: Wednesday, 12 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Fluid of density ρ and viscosity μ spreads out horizontally in a thin layer under gravity (g) over a solid plane at $y = 0$. The surface of the layer is at $y = h(x,\, t)$ and there is no variation in the z-direction. The height h is an even function of x, and at time t the fluid has reached a value $x_0(t)$, so that $h = 0$ for $|x| \geqslant x_0$. Writing the velocity as $\mathbf{u} = (u,\, v,\, 0)$, the boundary conditions are $u = v = 0$ on $y = 0$ and $u_y = 0$ and $p = p_0$ on $y = h$, where p_0 is constant.

(a) Explain how the assumptions of Lubrication Theory lead to the equation

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

(8 marks)

(b) Show that the total volume of fluid per unit length in the z-direction,

$$V = \int_{-\infty}^{\infty} h(x,t)\, dx,$$

is constant in time. (2 marks)

Seek a similarity solution of the form

$$h = At^a f(\eta) \qquad \text{where} \qquad \eta = Bxt^b.$$

Determine a and b and choose convenient values for the constants A and B. (3 marks)

Find $f(\eta)$, recalling that $f(\eta)$ is an even function, and that there exists a value η_0 such that $f = 0$ for $|\eta| \geqslant \eta_0$. (4 marks)

Determine η_0 from the integral condition in terms of the constant

$$\beta = \int_{-1}^{1} (1-\eta^2)^{1/3}\, d\eta,$$

and deduce that at time t the fluid has spread up to

$$x = x_0 = \left(\frac{10\rho g V^3 t}{9\mu\beta^3}\right)^{1/5}.$$

(3 marks)

(Total: 20 marks)

2. A sphere of radius a moves in a constant direction with time-dependent speed $U(t)$ through inviscid fluid which is at rest at infinity. The fluid has density ρ and its velocity is irrotational, so that $\mathbf{u} = \nabla\chi$ where χ is the velocity potential.

(a) State the equation and boundary conditions for χ and verify that in terms of spherical polar coordinates $(r,\, \theta,\, \phi)$ these are satisfied by

$$\chi = -\frac{Ua^3}{2r^2}\cos\theta. \qquad \text{(4 marks)}$$

(b) **EITHER:** Find the time-dependent pressure field and deduce that the total force on the sphere is $m\,dU/dt$ for a value of m which you should determine. (You may assume d'Alembert's result that steady potential flow exerts no net force and so only the time-derivatives contribute.)

OR: Show that the total kinetic energy of the fluid is

$$E = \tfrac{1}{2}mU^2,$$

for a value of m which you should find. (4 marks)

What is the physical significance of m? (1 mark)

(You may assume the two values of m are the same, but you need only find one of them.)

(c) A spherical gas bubble of negligible density rises under gravity through the fluid at HIGH Reynolds number. Surface tension is sufficient to keep the bubble spherical but is otherwise negligible, and no boundary layer separation occurs. Use the results of part (b) to show that the initial acceleration of the bubble is $2g$, where g is the gravitational acceleration. (3 marks)

(d) The local rate of viscous dissipation is $2\mu\, e_{ij}e_{ij}$, where e_{ij} is the strain rate tensor and μ is the viscosity. Show that for an **irrotational** flow in a volume V bounded by a surface S with unit normal $\mathbf{n}$

$$2\mu\int_V e_{ij}e_{ij}\,dV = 2\mu\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,dV = \mu\int_S n_j\frac{\partial}{\partial x_j}\left(u_iu_i\right)\,dS. \qquad \text{(3 marks)}$$

Assuming that the dissipation within the free surface boundary layer can be neglected, deduce that the total rate of energy dissipation for the flow in part (a) is $12\pi\mu aU^2$. (3 marks)

By considering the work done by gravity on a bubble rising at a constant speed U_0, deduce that at high Reynolds number the steady rise speed is

$$U_0 = \frac{a^2\rho g}{9\mu}. \qquad \text{(2 marks)}$$

$\Big[$*Note that in spherical coordinates the volume element* $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$, *while the axisymmetric gradient and Laplacian take the forms*

$$\nabla\chi = \left(\frac{\partial\chi}{\partial r},\, \frac{1}{r}\frac{\partial\chi}{\partial\theta},\, 0\right), \qquad \nabla^2\chi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right). \Big]$$

(Total: 20 marks)

3. The region $0 > y > -h$ is an ocean of stationary water, with a solid boundary at $y = -h$. Dynamically negligible air at pressure p_0 occupies $y > 0$. A surface tension of strength γ acts at the interface.

(a) The surface is perturbed to $y = \varepsilon \sin(kx - \omega t)$, where $0 < \varepsilon \ll 1$, k is known and ω is to be found. Find the irrotational flow and show that

$$\omega^2 = \left(gk + \frac{\gamma}{\rho}k^3\right)\tanh(kh).$$

(12 marks)

(b) The phase speed of the wave is defined as $c = \omega/k$. When the ocean is very deep, show that $c^2(k)$ has a minimum value as k varies and find it. By considering the behaviour near $kh = 0$, show that there exists a minimum somewhere in $k > 0$ when

$$\frac{\gamma}{\rho g h^2} < \tfrac{1}{3}.$$

(4 marks)

(c) An underwater event perturbs the surface at $t = 0$ to some shape $y = \varepsilon f(x)$, where f is a superposition of a range of wavenumbers k. Describe qualitatively what you would expect to see some time later and some distance away from the event when (i) $kh \ll 1 \sim \rho g h^2/\gamma$ and (ii) kh is very large. (4 marks)

(Total: 20 marks)

4. Fluid flows towards an axisymmetric body at high Reynolds number. We denote by s and n coordinates which are tangential and normal to the body. The cylindrical radius of the body is described by $R = R_0(s)$, and the inviscid flow has a slip velocity $\overline{U}(s)$ as in the figure. Inside the boundary layer the velocity has a tangential component $\overline{u}$ and a normal component $\overline{v}$ which satisfy the equations

$$\overline{u}\frac{\partial \overline{u}}{\partial s} + \overline{v}\frac{\partial \overline{u}}{\partial n} = \overline{U}\,\frac{d\overline{U}}{ds} + \frac{\partial^2 \overline{u}}{\partial n^2}, \qquad \frac{1}{R_0}\frac{\partial}{\partial s}(R_0\overline{u}) + \frac{\partial \overline{v}}{\partial n} = 0. \tag{1}$$

(a) Consider the coordinate transformation

$$y = R_0(s)n, \qquad x = \int_0^s R_0^2(\sigma)\, d\sigma.$$

Show that the equations (1) transform into the 2-D boundary layer equations

$$uu_x + vu_y = UU' + u_{yy}, \qquad u_x + v_y = 0, \tag{2}$$

where $u(x, y) = \overline{u}(s, n)$, $U(x) = \overline{U}(s)$ and v is related to $\overline{u}$ and $\overline{v}$ in a manner you should determine. (12 marks)

(b) In terms of cylindrical coordinates (R, ϕ, z), an inviscid flow has the velocity $(R, 0, -2z)$. An axisymmetric boundary layer develops over a solid plane wall at $z = 0$. Write down the appropriate functions $R_0(s)$ and $\overline{U}(s)$ for this problem.

You may assume without proof that when $U = Cx^m$, equation (2) has the Falkner-Skan solution for which $u = Cx^m F(yx^b)$, where C is a constant and $b = \frac{1}{2}(m - 1)$. You need not determine the function F.

Show from part (a) that the solution to (1) can be related to a member of the Falkner-Skan family of solutions to (2) for suitable values of C and m. Hence find $\overline{u}(R, z)$ in terms of the function F. (8 marks)

(Total: 20 marks)

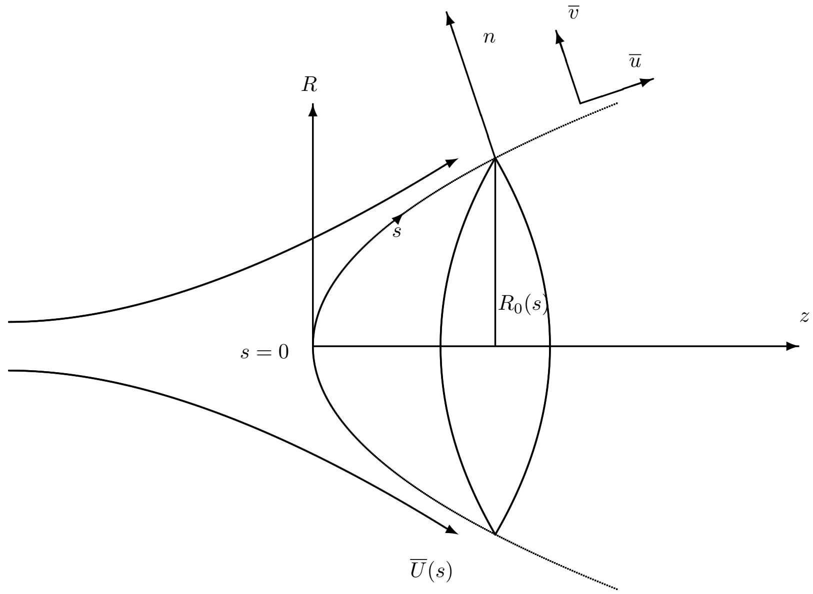

Diagram for question 4

5. Discuss the effects of viscosity in incompressible flows of Newtonian fluid. (20 marks)

(Total: 20 marks)

1. (a) We assume derivatives in the y-direction are large, so that $\nabla^2 \sim \frac{\partial^2}{\partial y^2}$ and that inertia terms are negligible in comparison. Pressure is scaled so that it drives flow in the x-direction. The governing equations are then

seen ⇓

$$p_x = \mu u_{yy}, \qquad p_y = -\rho g, \qquad u_x + v_y = 0.$$

Since $p = p_0$ on $y = h$, we have $p = p_0 + \rho g(h - y)$ and hence $\rho g h_x = \mu u_{yy}$ which integrates to

$$u = \frac{\rho g}{2\mu} h_x (y^2 - 2yh),$$

where we have imposed $u = 0$ on $y = 0$ and $u_y = 0$ on $y = h$. The kinematic boundary condition is

$$0 = \frac{D}{Dt}(y - h) = v - h_t - u h_x \qquad \text{on} \qquad y = h.$$

Now integrating $u_x + v_y = 0$ across the layer

$$\Big[v\Big]_0^h = -\int_0^h u_x \, dy = -\frac{\rho g}{2\mu} \int_0^h \Big(h_{xx}(y^2 - 2yh) - 2y(h_x)^2\Big) \, dy = \frac{\rho g}{2\mu} \left(\tfrac{2}{3} h^3 h_{xx} + h^2 h_x^2\right)$$

Combining the last three equations, using $v = 0$ on $y = 0$,

$$h_t = \frac{\rho g}{2\mu} \left(h^2 h_x^2 + \tfrac{2}{3} h^3 h_{xx} + h^2 h_x^2\right) = \frac{\rho g}{3\mu} (h^3 h_{xx} + 3h^2 h_x^2)$$

or as required

$$h_t = \frac{\rho g}{3\mu} \left(h^3 h_x\right)_x.$$

4, A

(b) We have

4, B

$$\frac{dV}{dt} = \int_{-\infty}^{\infty} \frac{\partial h}{\partial t} \, dx = \frac{\rho g}{3\mu} \left[h^3 \frac{\partial h}{\partial x}\right]_{-\infty}^{\infty} = 0.$$

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So V is conserved. Writing $h = A t^a f(\eta)$ where $\eta = B x t^b$, we have

2, A

$$V = \int_{-\infty}^{\infty} A t^a f(\eta) dx = \frac{A}{B} t^{a-b} \int_{-\infty}^{\infty} f(\eta) \, d\eta.$$

So we require $a = b$ and for convenience choose $A/B = V$, so that

$$\int_{-\infty}^{\infty} f(\eta) \, d\eta = 1.$$

Substituting in the PDE,

$$A a t^{a-1} f + A t^a f' \frac{b\eta}{t} = \left(\frac{\rho g}{3\mu}\right) A^4 t^{4a} (f^3 f')' (B^2 t^{2b}).$$

Balancing the powers of t requires $a - 1 = 4a + 2b$ giving

$$a = -\tfrac{1}{5}, \qquad b = -\tfrac{1}{5}.$$

For convenience, we choose

$$A = A^4 B^2 \frac{\rho g}{3\mu} \qquad \Longrightarrow \qquad A = \left(\frac{3\mu V^2}{\rho g}\right)^{1/5}, \qquad B = \left(\frac{3\mu}{\rho g V^3}\right)^{1/5},$$

giving

$$-\tfrac{1}{5}(\eta f' + f) = (f^3 f')'.$$

3, B

Integrating we have

$$-\tfrac{1}{5}\eta f = f^3 f' + C.$$

Either by imposing symmetry $f'(0) = 0$ or imposing $f(\eta_0) = 0$, we have $C = 0$. Cancelling f and integrating again

$$-\tfrac{1}{10}\eta^2 = \tfrac{1}{3}f^3 + D \qquad \Longrightarrow \tfrac{3}{10}(\eta_0^2 - \eta^2) = f^3,$$

using $f(\eta_0) = 0$. Imposing the integral constraint, noting that $f = 0$ for $|\eta| > \eta_0$, we have

4, D

$$1 = \int_{-\eta_0}^{\eta_0} \left(\tfrac{3}{10}\right)^{1/3} (\eta_0^2 - \eta^2)^{1/3} d\eta = \left(\tfrac{3}{10}\right)^{1/3} \eta_0^{5/3} \int_{-1}^{1} (1 - \xi^2)^{1/3} d\xi,$$

substituting $\eta = \eta_0 \xi$, giving

$$\eta_0 = \left(\frac{10}{3\beta^3}\right)^{1/5} = Bx_0(t)t^{-1/5} \implies x_0 = \left(\frac{10\rho g V^3 t}{9\mu\beta^3}\right)^{1/5}.$$

3, D

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2. (a) $\chi(r, \theta)$ satisfies

$$\nabla^2\chi = 0 \qquad \text{in } r > a, \quad \chi_r = U\cos\theta \qquad \text{on } r = a, \quad \nabla\chi \to 0 \qquad \text{as } \chi \to \infty.$$

Seeking a solution $\chi = f(r)\cos\theta$ gives

$$f'' + 2\frac{f'}{r} - 2\frac{f}{r^2} = 0 \implies f = Ar + \frac{B}{r^2}.$$

Applying the boundary conditions, we have

$$\chi = -\frac{Ua^3}{2r^2}\cos\theta, \qquad \text{as required.}$$

(Verifying the solution is ok.)

4, A

(b) By symmetry the force must be parallel to the velocity. The time-dependent Bernoulli theorem states $\boxed{\text{unseen} \Downarrow}$

$$p + \rho\frac{\partial \chi}{\partial t} + \tfrac{1}{2}\rho|\nabla\chi|^2 = \text{constant}, \quad \text{and} \quad F = \int p\cos\theta\, dS$$

is the force component on the fluid in the direction of motion. Now as steady potential flow exerts no force, only the time derivative can contribute and on $r = a$ $\chi_t = -\frac{1}{2}U'(t)a\cos\theta$. We therefore have

$$F = -2\pi a^2\rho\int_0^\pi \sin\theta\frac{\partial\chi}{\partial t}\cos\theta d\theta = \pi a^3\rho U'\int_0^\pi \sin\theta\cos^2\theta d\theta = \tfrac{2}{3}\pi a^3\rho U' = mU'.$$

Alternatively, the kinetic energy density of the fluid is $\boxed{4, \text{C}}$

$$\tfrac{1}{2}\rho|\nabla\chi|^2 = \tfrac{1}{2}\rho(\chi_r^2 + \frac{\chi_\theta^2}{r^2}) = \tfrac{1}{2}\rho U^2\frac{a^6}{r^6}\left(\cos^2\theta + \tfrac{1}{4}\sin^2\theta\right).$$

$$E = 2\pi\int_a^\infty\int_0^\pi \tfrac{1}{2}\rho|\nabla\chi|^2 r^2\sin\theta\, d\theta\, dr = \pi\rho a^6 U^2\int_a^\infty\frac{dr}{r^4}\int_0^\pi [\tfrac{1}{4}+\tfrac{3}{4}\cos^2\theta]\sin\theta\, d\theta,$$

$$= \pi a^6\rho U^2\left(\frac{1}{3a^3}\right)(1) = \tfrac{1}{2}mU^2$$

where $m = \frac{2}{3}\pi a^3\rho$.

The two values of m are the same. m represents the "added mass" of fluid, which the sphere apparently carries with it as it moves. $\boxed{1, \text{A}}$

(c) The bubble accelerates from rest because of the Archimedean force acting on it, $\frac{4}{3}\pi a^3\rho g$. As the mass of the bubble is negligible, all the effort goes in accelerating the added fluid mass, $m\, dU/dt$. Equating these, we find $dU/dt = 2g$. $\boxed{3, \text{A}}$

(d) e_{ij} is the symmetric part of the velocity gradient tensor $\partial u_i/\partial x_j$, and the antisymmetric part is related to the vorticity (e.g. $\partial u_1/\partial x_2 - \partial u_2/\partial x_1$ is the x_3-component of vorticity). Thus for potential flows, $e_{ij} = \partial u_i/\partial x_j$. Now

$$2\mu\int\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV = 2\mu\int\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_i}{\partial x_j}\right)dV = 2\mu\int n_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}u_i u_i\right)dS,$$

as $\nabla^2\mathbf{u} = 0$ for potential flows $\mathbf{u} = \nabla\chi$. $\boxed{3, \text{C}}$

For the above flow, $u_i u_i = U^2a^6/r^6(\cos^2\theta + \frac{1}{4}\sin^2\theta)$, while $n_j\partial/\partial x_j = -\partial/\partial r$. Thus the dissipation rate is

$$\mu U^2\frac{6}{a}2\pi a^2\int_0^\pi(\cos^2\theta + \tfrac{1}{4}\sin^2\theta)\sin\theta\, d\theta = 12\pi a\mu U^2.$$

(we have already evaluated this integral above.) $\boxed{3, \text{D}}$

In equilibrium, the energy dissipation must be balanced by the work done by the bubble as it rises. This is the drag force times the rise velocity U_0 so we infer that the drag force is $12\pi a\mu U_0$. Equating this with the Archimedean upthrust $\frac{4}{3}\pi a^3\rho g$ we have

$$U_0 = \frac{a^2\rho g}{9\mu}.$$

$\boxed{2, \text{A}}$

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3. (a) We write $\mathbf{u} = \varepsilon\nabla\phi$, where $\nabla^2\phi = 0$. If the surface is $y = \varepsilon\sin(kx - \omega t)$ then the kinematic condition is

$$\phi_y = -\omega\cos(kx - \omega t) + O(\varepsilon) \qquad \text{on } y = \varepsilon\sin(kx - \omega t)$$

and we can evaluate this condition on $y = 0$ to leading order. We look for a solution $\phi = f(y)\cos(kx - \omega t)$, which requires

$$f'' - k^2 f = 0 \implies f = Ae^{ky} + Be^{-ky}$$

We want $\phi_y = 0$ on $y = -h$ so $f'(-h) = 0$ and $f'(0) = -\omega$ so that

$$f(y) = -\frac{\omega\cosh[k(y+h)]}{k\sinh kh}.$$

Now the unit normal to the surface is

$$\widehat{\mathbf{n}} = (-k\varepsilon\cos(kx - \omega t),\, 1,\, 0) + O(\varepsilon^2) \implies K = \nabla\cdot\widehat{\mathbf{n}} = k^2\varepsilon\sin(kx - \omega t)$$

is the curvature of the surface, which to leading order we can evaluate on $y = 0$, so that

$$p = p_0 + \varepsilon\gamma k^2\sin(kx - \omega t) \qquad \text{on } y = 0.$$

Thus the time-dependent Bernoulli condition

$$p + \rho\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\rho|\mathbf{u}|^2 + \rho gh = \text{constant}$$

becomes

$$\gamma k^2 - \rho\frac{\omega^2\coth kh}{k} + \rho g + O(\varepsilon) = 0$$

or

$$\omega^2 = \left(gk + \frac{\gamma k^3}{\rho}\right)\tanh kh,$$

as required. $\boxed{6,\ \mathrm{A}}$

(b) When $kh \gg 1$ $\tanh(kh) \simeq 1$ and then $\boxed{6,\ \mathrm{D}}$

unseen $\Downarrow$

$$c^2 = \frac{g}{k} + \frac{\gamma k}{\rho} \implies \frac{d(c^2)}{dk} = 0 \qquad \text{when} \quad k^2 = \frac{\rho g}{\gamma}$$

and then $c = (4g\gamma/\rho)^{1/4}$. This is a minimum as it is the only stationary point, while $c^2 \to +\infty$ as $k \to 0$ and also as $k \to \infty$ (or look at the second derivative.) Now when $kh \ll 1$, we have $\tanh kh \simeq kh - (kh)^3/3 + O(kh)^5$. So

$$c^2 = \left(gh + \frac{\gamma}{\rho h}(kh)^2\right)\frac{\tanh kh}{kh} = gh + (kh)^2\left(\frac{\gamma}{\rho h} - \frac{gh}{3}\right) + O(kh)^4.$$

If $\gamma/(\rho gh^2) < 1/3$, this is a decreasing function for small kh, and as $c^2 \to \infty$ as $k \to \infty$ there must be a minimum somewhere. (We haven't shown that there is no minimum if $\gamma/(\rho gh^2) > 1/3$, but we're not asked to.) $\boxed{4,\ \mathrm{A}}$

(c) Now suppose there is an event some distance away, which generates a variety of waves with speeds which may depend on k. If $kh \ll 1$, all waves travel with the same speed, $c = \sqrt{gh}$. Therefore the disturbance propagates without change of shape. In a deep ocean, however, the waves travel at different speeds, and so the disturbance shape changes. The last waves to reach us have the minimum wavespeed and corresponding wavenumber. $\boxed{4,\ \mathrm{B}}$

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4. (a) We have

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} = R_0^2\frac{\partial}{\partial x} + nR_0'\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial n} = R_0\frac{\partial}{\partial y}.$$

Thus

$$\overline{U}\frac{d\overline{U}}{ds} = R_0^2 U\frac{dU}{dx}$$

The boundary layer equation becomes

$$\overline{u}\left(R_0^2\frac{\partial\overline{u}}{\partial x} + nR_0'\frac{\partial\overline{u}}{\partial y}\right) + \overline{v}R_0\frac{\partial\overline{u}}{\partial y} = R_0^2 U\frac{dU}{dx} + R_0\frac{\partial}{\partial y}\left(R_0\frac{\partial\overline{u}}{\partial y}\right)$$

or dividing by R_0^2, noting $\partial(R_0(s(x)))/\partial y = 0$ and replacing $\overline{u}$ by u

$$u\frac{\partial u}{\partial x} + \left(\frac{nuR_0'}{R_0^2} + \frac{\overline{v}}{R_0}\right)\frac{\partial u}{\partial y} = UU' + \frac{\partial^2 u}{\partial y^2}.$$

This suggests we identify

$$v = \frac{n\overline{u}R_0'}{R_0^2} + \frac{\overline{v}}{R_0},$$

which will give us the standard boundary layer equation. Using this value in the incompressibility condition, it becomes

$$\frac{R_0'}{R_0}\overline{u} + R_0^2\frac{\partial\overline{u}}{\partial x} + nR_0'\frac{\partial\overline{u}}{\partial y} + \frac{\partial}{\partial n}\left(R_0 v - \frac{nR_0'}{R_0}u\right) = 0$$

or

$$R_0^2\frac{\partial\overline{u}}{\partial x} + \frac{R_0'}{R_0}\overline{u} + nR_0'\frac{\partial\overline{u}}{\partial y} + R_0^2\frac{\partial v}{\partial y} - \frac{R_0'}{R_0}\overline{u} - \frac{nR_0'}{R_0}R_0\frac{\partial\overline{u}}{\partial y} = 0$$

which simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

We note $y = 0$ corresponds to $n = 0$ and $\overline{u} = \overline{v} = 0$ imply $u = v = 0$ there, so we have the usual boundary conditions

$$u = v = 0 \qquad \text{on } y = 0, \quad u \to U \qquad \text{as } y \to \infty.$$

5, B

(b) If the surface S is $z = 0$ then $n = -z$ and $s = R$. The surface $R = R_0$ requires $R_0(s) = s$. We have the given inviscid flow $\overline{U} = R = s$. Then the transformed variables are 2, A 5, C

$$x = \int_0^s s^2\,ds = \tfrac{1}{3}s^3, \qquad y = n.$$

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The function

$$U(x) = \overline{U}(s) = (3x)^{1/3},$$

which is of Falkner-Skan type with $m = 1/3$. The solution has $U = (3x)^{1/3}F(yx^{-1/3})$ (given). The boundary layer has a thickness which increases with x as $x^{1/3} = s/3^{1/3} = R/3^{1/3}$. Transforming back the thickness in the (R, z)-plane therefore scales with R. We could therefore expect a solution of the form

$$\overline{u} = RF(-3^{1/3}z/R).$$

(Or define z in the opposite direction and change its sign.) 4, A

4, B

5. Essay question. Anything relevant will earn credit, unless copied verbatim from notes. 20, M

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96001 MATH97007 MATH97087	1	The exam as a whole was found to be very challenging this year. It has been a difficult time for everyone, with unfamiliar teaching delivery and practice. There was clear evidence of rustiness in the candidates' solutions - maybe I should have set more prescribed practice material. It seems that many people had forgotten or missed the problem class sessions, where I went over the problem sheets. With particular reference to question 1, I was surprised by how difficult this question was found. The derivation of the equations is in the notes for a slope of angle alpha, and setting alpha=0 in that discussion solves the first part. Many people just wrote down p_y=0 forgetting gravity, or assumed gravity acted in the x-direction, as for a vertical plate. But I was astonished by how few people managed to derive the similarity equations. We did many such examples in the boundary layer section. Almost nobody appreciated that the integral of h dx being constant gave a constraint on a and b, even though there was a coursework in which this happened, and also the given answer involved V. The answer also gave the power of t with which x varied. A very common error occurred when working out h_t there are two terms, one involving d/d(eta). Many people tried to equate powers of t while leaving x in the ODE for eta, which wouldn't make sense. Again, we did this so many times in the Boundary Layer similarity solutions, but people forgot that. I didn't expect many people to reach the end of this question, but in the event nobody got anywhere close. As a rsult I shifted the marks distribution towards the start of the question

MATH96001 MATH97007 MATH97087	2	This question was also poorly done. What baffles me about this is that most of the question appears on Problem Sheet 5, to which I supplied written solutions. I can only conclude that most people did not look at the Locomotion problem sheet, or they would surely have recognised this. A disappointing trend was that many did not appreciate that when solving potential flow there is a slip velocity - you can only impose continuity on the normal velocity. The solution was given in the question - so anyone who realised that in potential flow del^2(chi)=0 was able to verify that it worked. Some tried to use the Stokes equations, despite the stress on HIGH Reynolds number in the question.
MATH96001 MATH97007 MATH97087	3	The suffering continues. Many copied the result for an infinitely deep ocean from notes. Very few managed to solve Laplace's equation with a solid boundary at y=-h, although I twice mentioned this in the problems classes. In fact not many imposed that phi must satisfy Laplace's equation at all. I was surprised that people could not find the minimum of c^2 in the limit of large h, preferring to differentiate for arbitrary h. I regret not giving the two-term expansion of tanh x for small x in the question, which might have helped a few. Although in lectures I often explained what "dynamically negligible" meant, I should perhaps have stated explicitly that air density could be treated as zero, as in question 2.
MATH96001 MATH97007 MATH97087	4	This question also was found difficult. The first part requires change of variables, but many were unable to cope with the chain rule. A few could not simplify d/ds (int_0^s f(t) dt) - I think this is indicative of how out of practice with basic calculus people have become over the past year. Nobody managed the last part - the plane at z=0 can be regarded as an axisymmetric body with R_0=s. The x=int(s^2)ds=s^3/3, and this gives the appropriate value of m in the Falkner-skan solution. So once again I increased slightly the marks awarded to the first part.
MATH96001 MATH97007 MATH97087	5	The essay question was answered well, which is perhaps not so surprising due to its open-ended, "tell me anything you know" structure. Nevertheless, a few candidates may have been pressed for time and omitted entire topics, such as Lubrication, Boundary Layers, Dynamic similarity...