1. Five friends arrive in London and, independently of each other, embark on Christmas shopping trips around London.

   All five friends start their shopping trips at the same time, and each of them takes an exponentially distributed amount of time with mean 1.5 hours to complete their shopping.

   What is the expected number of hours after which all five friends will have completed their shopping?

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We denote by J_i, the number of hours it takes friend $i \in \{1, 2, 3, 4, 5\}$ to complete their shopping. Using order statistics, we write $J_{(1)} < J_{(2)} < J_{(3)} < J_{(4)} < J_{(5)}$.

We define $T_1 = J_{(1)}$ to be the number of hours it takes the first friend to complete the tour, $T_2 = J_{(2)} - J_{(1)}$ is the additional time spent by the second friend to complete the tour, $T_3 = J_{(3)} - J_{(2)}$ the additional time for the third friend to complete the tour, $T_4 = J_{(4)} - J_{(3)}$ the additional time it takes the fourth friend to complete the tour and $T_5 = J_{(5)} - J_{(4)}$ the additional time it takes the fifth friend to complete the tour. By the lack of memory property of the exponential distribution and a result from lectures, we conclude that $T_1 \sim \mathrm{Exp}(5\lambda)$, $T_2 \sim \mathrm{Exp}(4\lambda)$, $T_3 \sim \mathrm{Exp}(3\lambda)$, $T_4 \sim \mathrm{Exp}(2\lambda)$ and $T_5 \sim \mathrm{Exp}(\lambda)$, where $\lambda = 2/3$. Let $T = J_{(5)}$ denote the time until all friends will have completed their shopping. Then

$$\begin{aligned}\mathrm{E}(T) &= \sum_{i=1}^{5} \mathrm{E}(T_i) = \frac{1}{5\lambda} + \frac{1}{4\lambda} + \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{10} + \frac{3}{8} + \frac{1}{2} + \frac{3}{4} + \frac{3}{2} \\ &= \frac{12 + 15 + 20 + 30 + 60}{40} = \frac{137}{40} = 3.425[\text{hours}].\end{aligned}$$

An alternative, more computationally heavy approach would be the following one: Let $X_1, \ldots, X_5$ denote the i.i.d. shopping times of the five friends, each having $\mathrm{Exp}(2/3)$ distribution. Set

$$Z = \max\{X_1, \ldots, X_5\}.$$

Then, for $z > 0$,

$$\mathrm{P}(Z \le z) = \mathrm{P}(X_1 \le z, \cdots, X_5 \le z) = (F_{X_1}(z))^5,$$

and, for $\lambda = 2/3$,

$$f_Z(z) = 5(1 - e^{-\lambda z})^4 (\lambda e^{-\lambda z}).$$

Then

$$\mathrm{E}(Z) = \int_0^\infty z f_Z(z) dz = \frac{137}{40} = 3.425[\text{hours}].$$

2. You go for a winter holiday to Northern Norway hoping that you will be able to see the Northern Lights (Aurora Borealis). Your guide books says that visitors who spend 20 days in Northern Norway during the winter will have an 85% chance of seeing the Northern lights at least once. You may assume that the arrivals of the Northern Lights can be modelled by a homogeneous Poisson process.

   You are only able to spend 5 days in Northern Norway. What is the probability that you will see the Northern lights at least once during your visit?

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We denote by $N = (N_t)_{t\geq 0}$ the homogeneous PP of rate $\lambda > 0$, which models the arrival of the Northern Lights. We denote by t the time in days. We note that

$$\mathrm{P}(N_{20} \geq 1) = \frac{85}{100} = 0.85.$$

Hence, $\mathrm{P}(N_{20} = 0) = 0.15$ and

$$\mathrm{P}(N_{20} = 0) = \exp(-20\lambda) = 0.15 \Leftrightarrow \lambda = -\frac{\log(0.15)}{20} \approx 0.095.$$

We compute

$$\mathrm{P}(N_5 = 0) = \exp(-5\lambda) \approx 0.622.$$

Hence

$$\mathrm{P}(N_5 \geq 1) = 1 - \exp(-5\lambda) \approx 0.378.$$

3. Multiple answer:

   Please select all correct statements.

   a) A compound Poisson process is a Markov chain in continuous time.
   b) Let $N = (N_t)_{t \geq 0}$ denote a Poisson process with rate $\lambda > 0$. Then $\mathrm{Var}(N_t) = \lambda t$ for $t \geq 0$.
   c) A Poisson process is continuous in probability.
   d) A Poisson process is a stationary process.
   e) The sum of two Poisson processes is a Poisson process itself.

Solution: a), b), c).

4. Let X denote an exponential random variable with mean 0.1. Define

$$\lambda := \mathrm{P}(X > 1 | X > 0.9).$$

Let $N = (N_t)_{t \geq 0}$ denote a Poisson process with rate λ. Find

$$\mathrm{P}(N_2 > \lambda).$$

Please record your answer in decimals rounding to three decimal places if needed.

Solution: Let $X \sim \mathrm{Exp}(\lambda)$. Then $\mathrm{E}(X) = 1/\lambda = 0.1$. Hence $\lambda = 10$. By the lack of memory property we know that

$$\mathrm{P}(X > 1 | X > 0.9) = \mathrm{P}(X > 0.1) = \exp(-0.1\lambda) = \exp(-1) \approx 0.368.$$

Note that

$$\mathrm{P}(N_2 > \lambda) = \mathrm{P}(N_2 > 0.368) = 1 - \mathrm{P}(N_2 = 0) = 1 - \exp(-2\lambda) \approx 0.521.$$

5. Consider a non-homogeneous Poisson process $N = (N_t)_{t \geq 0}$ with intensity function $\lambda(t) = 2 + t + t^2$ for $t \geq 0$.

   Let $M := \lfloor \mathrm{E}(N_4) \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, which returns the greatest integer less than or equal to the input, e.g. $\lfloor 4.8 \rfloor = 4$.

   Find $\mathrm{P}(N_4 = M)$.

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: We know that $N_t \sim \mathrm{Poi}(m(t))$ for

$$m(t) = \int_0^t \lambda(s)ds = \int_0^t (2 + s + s^2)ds = 2s + \frac{s^2}{2} + \frac{s^3}{3}\bigg|_0^t = 2t + \frac{t^2}{2} + \frac{t^3}{3}.$$

Note that $\mathbb{E}(N_4) = m(4) \approx 37.333$, hence, $M = 37$. Hence,

$$\mathrm{P}(N_4 = M) = \frac{m(4)^{37}}{37!} \exp(-m(4)) \approx 0.065.$$

6. Consider a non-homogeneous Poisson process $N = (N_t)_{t \geq 0}$ with intensity function $\lambda(t) = 2 + t + t^2$ for $t \geq 0$.

   Find $\mathrm{E}(N_{1.5} - N_{0.5})$.

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: As before, we know that $N_t \sim \mathrm{Poi}(m(t))$ for

$$m(t) = \int_0^t \lambda(s) ds = \int_0^t (2 + s + s^2) ds = 2s + \frac{s^2}{2} + \frac{s^3}{3}\bigg|_0^t = 2t + \frac{t^2}{2} + \frac{t^3}{3}.$$

Then

$$m(1.5) - m(0.5) \approx 4.083.$$

We note that $N_{1.5} - N_{0.5} \sim \mathrm{Poi}(m(1.5) - m(0.5))$. Hence

$$\mathrm{E}(N_{1.5} - N_{0.5}) = (m(1.5) - m(0.5)) \approx 4.083.$$

7. Consider a birth process $N = (N_t)_{t\geq 0}$ with initial condition $N_0 = 0$ with intensities $\lambda_n = 1 + n$, $n \in \{0, 1, \ldots\}$.

   Find $\mathrm{P}(N_1 = 0)$.

   Please record your answer in decimals rounding to three decimal places if needed.

Solution: Using the result from Exercise 4-38, we have that

$$\mathrm{P}(N_t = 0) = \frac{1}{\lambda_0}\lambda_0 e^{-\lambda_0 t} = e^{-\lambda_0 t}.$$

Hence,

$$\mathrm{P}(N_1 = 0) = e^{-\lambda_0} = e^{-1} \approx 0.368.$$

8. Consider a continuous-time homogeneous Markov chain with generator given by

$$\mathbf{G} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{pmatrix}.$$

Find the stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ and report the product of its components, i.e. find $\prod_{i=1}^{3} \pi_i$.

Please record your answer in decimals rounding to three decimal places if needed.

Solution: From lectures, we know that we can find the stationary distribution by solving $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$. Equivalently, we can solve $\mathbf{G}^\top \boldsymbol{\pi}^\top = \mathbf{0}^\top$, including the additional condition that the elements of $\boldsymbol{\pi}$ need to be non-negative and sum up to 1. This leads to $\boldsymbol{\pi} = (2/5, 2/5, 1/5)$. Hence

$$\prod_{i=1}^{3} \pi_i = \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} = \frac{4}{125} = 0.032.$$

9. Consider a continuous-time homogeneous Markov chain with generator given by

$$\mathbf{G} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{pmatrix}.$$

Let Z denote the corresponding jump chain. Find the stationary distribution $\boldsymbol{\pi}^Z = (\pi_1^Z, \pi_2^Z, \pi_3^Z)$ of the jump chain and report the product of its components, i.e. find $\prod_{i=1}^3 \pi_i^Z$.

Please record your answer in decimals rounding to three decimal places if needed.

Solution: The transition matrix of the jump chain Z is given by

$$\mathbf{P}_Z = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

We need to solve $\boldsymbol{\pi}^Z \mathbf{P}_Z = \boldsymbol{\pi}^Z$, where the elements of $\boldsymbol{\pi}^Z$ need to be non-negative and sum up to 1. We find that $\boldsymbol{\pi}^Z = (1/3, 1/3, 1/3)$. [We could have used the fact that the transition matrix is doubly-stochastic!] Hence

$$\prod_{i=1}^{3} \pi_i^Z = \frac{1}{3^3} = \frac{1}{27} \approx 0.037.$$

10. Multiple answer: Please choose all statements which apply.

Consider the task of modelling the number of foxes in London over a one-year time period.

a) A birth-death process might be a suitable model. In particular, for modelling the increase in the population, a birth process might be more suitable than a Poisson process since the growth rate of the population will most likely depend on the size of the population. Similarly, the death rates will most likely not be constant, but depend on the size of the population as well.

b) A Poisson process is the most suitable model out of all models considered in the course since we expect the birth rate for each female fox to be constant over time.

c) A birth-death process with constant birth and death rates is the best model choice since we expect that there is an equilibrium which cannot be surpassed.

d) The Brownian motion with drift is best suited for this task out of all models considered in the course.

Solution: a) b) FALSE: Even if the birth rate per fox stays constant, we need to look at the population overall, which would results in state-depend models for the birth rate (eg linear birth models). c) FALSE: For the same reason as above, we expect both the birth and death rate to at least partially depend on the size of the population. d) FALSE: The Brownian motion takes values in $\mathbb{R}$ and not in $\mathbb{N}$ and is hence not the most natural model choice.