Imperial College
London

MATH97022 MATH97100

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Special Relativity and Electromagnetism

Date: Monday, 17 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

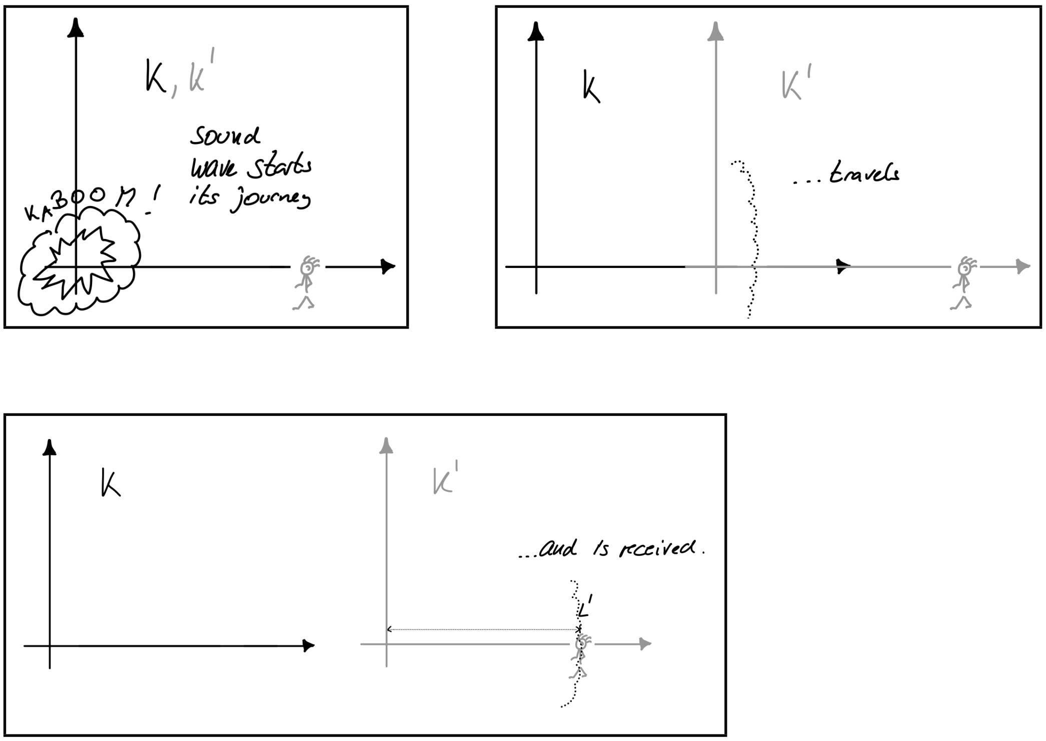
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. The following questions are concerned with events in different frames of reference. The spatial origin of each frame is chosen so that $(0,0,0,0)$ refers to the same event in every frame. All axes are parallel at all time. The origin of frame K' moves with velocity v relative to frame K along its x-axis. Dashed variables refer to measurements in the dashed frame K', undashed variables to measurements in the undashed frame K.



There is an observer situated in frame K' at distance L' from the origin in the positive direction of their parallel x-axes. At time $t = t' = 0$ a thunder (sound wave, speed $|u| > |v|$ in K) originates from the origin travelling with the same speed u in all directions. The thunder is heard by the observer in frame K' at time t'_2.

(a) The position of the observer in K' is, written as a four-vector, $(ct', L', 0, 0)$ for all t'. Using a Lorentz transform, determine the position in K of the observer in K' as a function of the time t in K.

(4 marks)

Express all answers of the following questions in terms of L', c, v and u.

(b) The thunder moves with speed u within K, the observer in K' moves with speed v within K. Given $v < u$, determine the time t_2 and the location x_2 in K of the thunder reaching the observer in K'. (4 marks)

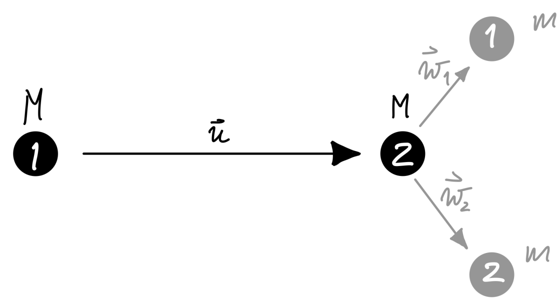
(c) By Lorentz transformation, find the time t'_2 and the location x'_2 in K' of the observer in K' at the time of them hearing the thunder (event 2). (4 marks)

(d) Derive an expression for the perceived speed of sound $u' = L'/t'_2$ in K'. (4 marks)

(e) Obtain an alternative expression for u' using the addition theorem of velocities. (4 marks)

(Total: 20 marks)

2. A particle of rest mass M is initially at rest at the origin of frame K. A second, identical particle of rest mass M collides with the first particle with initial velocity $\mathbf{u}$ parallel to the x-axis of frame K. After the collision, two identical particles of rest mass m emerge from the origin of K with velocity $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively.



(a) Write down the equation arising from momentum and energy conservation in four-vector form, using p_1^i for the energy-momentum four-vector of the first particle and p_2^i for the energy-momentum four-vector of the second particle, and correspondingly $p_1'^i$ and $p_2'^i$ for the energy-momentum four-vector after the collision. Express the components of the four-vectors in terms of the masses and velocities given in the description of the collision process above.

(4 marks)

(b) Calculate

$$(p_1^i + p_2^i)(p_{1i} + p_{2i})$$

and

$$(p_1'^i + p_2'^i)(p'_{1i} + p'_{2i}) \ .$$

Are these two expressions bound to be identical?

(4 marks)

(c) State an upper bound on m given M and $u = |\mathbf{u}|$, based on energy conservation only.

(4 marks)

(d) Determine the velocity $\mathbf{v}$ relative to K of a frame K' where the total initial momentum vanishes.

(4 marks)

(e) Show that $\mathbf{v} = -\mathbf{u}/2$ for $|\mathbf{u}|, |\mathbf{v}| \ll c$.

(4 marks)

(Total: 20 marks)

3. In the following, F^{ij} denotes the electromagnetic field tensor and A^i the four potential.

(a) Show that

$$A^i F_{ij} A^j = 0 \ .$$

(5 marks)

(b) Show that

$$\frac{\partial^2}{\partial x^i \partial x^j} F^{ij} = 0 \ .$$

(5 marks)

(c) Show that

$$F_{ij} F^{ij} = 2(H^2 - E^2) \ ,$$

where H is the magnitude of the magnetic field and E is the magnitude of the electric field.

(5 marks)

(d) Given a frame K with constant electric and magnetic fields

$$\mathbf{E} = \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{H} = \begin{pmatrix} 0 \\ 0 \\ H \end{pmatrix}$$

respectively, find the velocity $\mathbf{v}$ with which a frame K' has to move such that the magnetic field in that frame vanishes, given that $E/H = 1/2$. (5 marks)

(Total: 20 marks)

4. (a) Show that the wave equation on the four potential

$$\frac{\partial^2}{\partial x_k \partial x^k} A^i = 0$$

implies the wave equation on the electromagnetic field tensor

$$\frac{\partial^2}{\partial x_k \partial x^k} F^{ij} = 0 \ .$$

(5 marks)

(b) A monochromatic plane wave travelling along the x-axis has an electric field

$$\begin{aligned} E_x &= 0 \\ E_y &= b_1 \cos(\omega t - kx + \alpha) \\ E_z &= b_2 \sin(\omega t - kx + \alpha) \end{aligned}$$

with amplitudes b_1 and b_2, frequency ω, wave vector magnitude k and phase α.

(i) Determine the magnetic field due to the electric field. (5 marks)

(ii) Derive the charge density $\rho(\mathbf{x}, t)$ of the electric field (5 marks)

(c) Potentials are subject to a gauge condition. However, applying a gauge condition in one frame of reference, may imply another gauge condition in another frame of reference. Explain which of the two gauge conditions is Lorentz invariant:

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

or

$$\phi = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{A} = 0 \ .$$

(5 marks)

(Total: 20 marks)

5. (a) Show that the magnetic field of a charge e moving with velocity $\mathbf{v}$ is

$$\mathbf{H} = \frac{1}{c}\mathbf{v} \times \mathbf{E}$$

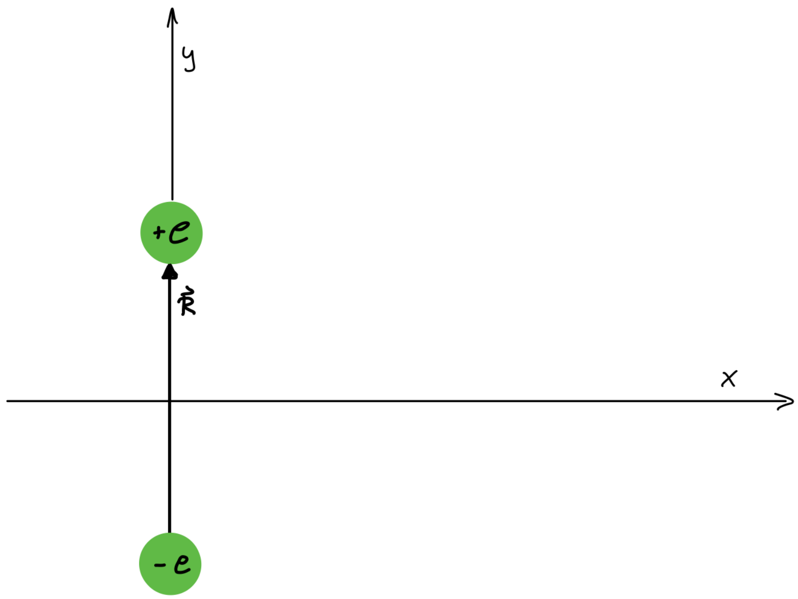
given its vector potential is $\mathbf{A} = \mathbf{v}\phi/c$, where ϕ denotes its scalar potential.

(4 marks)

(b) Express the Lorentz force acting on a charge e moving with velocity $\mathbf{v}$ in the electric and magnetic fields above in terms of the electric field $\mathbf{E}$ alone.
Hint: Use the vector triple product.

(4 marks)

(c) Given are two opposing charges e and $-e$ separated by $\mathbf{R}$ in the arrangement depicted below.



(i) Determine the electric field along the x-axis to leading order in x at large $x \gg |\mathbf{R}|$.

(4 marks)

(ii) Determine the electric field along the y-axis to leading order in y at large $y \gg |\mathbf{R}|$.

(4 marks)

(iii) In the arrangement above the electric field drops off proportional to $1/r^3$ in the distance r. How does does it behave if both charges are equal?

(4 marks)

(Total: 20 marks)

Solutions of M345A6 2019/2020

1.a *(4 marks, easy, A)*
Straight forward transformation of the four-vector $(ct', L', 0, 0)$ in K' to K, which gives $((ct' + \beta L')\gamma, (L' + \beta ct')\gamma, 0, 0)_K$, so that the required position is $(L' + \beta ct')\gamma$. This needs to be expressed in terms of t, where $ct = (ct' + \beta L')\gamma$, so that $ct' = ct/\gamma - \beta L'$ and therefore the position is $\boxed{(L' + \beta(ct/\gamma - \beta L'))\gamma = L'/\gamma + tv}$ using $1 - \beta^2 = 1/\gamma^2$ and $\beta c = v$.

1.b *(4 marks, easy, A)*
The position x of the thunder's wavefront as a function of time t in K is ut and the position of the observer is $L'/\gamma + tv$. Equating the two and solving for $t = t_2$ gives $\boxed{t_2 = L'/(\gamma(u - v))}$ and therefore $\boxed{x_2 = t_2 u = uL'/(\gamma(u - v))}$.

1.c *(4 marks, difficult, D)*
A simple inverse Lorentz transform gives

$$\boxed{ct_2' = (ct_2 - \beta x_2)\gamma = \frac{cL'}{u - v} - \frac{v}{c}\frac{u}{u - v}L' = \frac{L'c}{u - v}\left(1 - \frac{vu}{c^2}\right)}$$

and

$$\boxed{x_2' = (x_2 - \beta ct_2)\gamma = \frac{uL'}{u - v} - \frac{vL'}{u - v} = L'}$$

as it should be.

1.d *(4 marks, easy, but hinges on above, C)*
Plugging in

$$\boxed{u' = \frac{L'}{t_2'} = \frac{u - v}{1 - vu/c^2}}$$

1.e *(4 marks, medium, C)*
Addition of velocities as sound moves with u in K and K moves with $-v$ relative to K' gives

$$\boxed{\frac{u - v}{1 - vu/c^2}}$$

confirming the above.

2.a *(4 marks, easy, A)*
This is a matter of reading components off:

$$\boxed{p_1^i = \left(\frac{Mc}{\sqrt{1 - u^2/c^2}}, \frac{Mu}{\sqrt{1 - u^2/c^2}}, 0, 0\right)}$$

$$\boxed{p_2^i = (Mc, 0, 0, 0)}$$

$$\boxed{p_1'^i = \left(\frac{mc}{\sqrt{1 - w_1^2/c^2}}, \frac{m\mathbf{w}_1}{\sqrt{1 - w_1^2/c^2}}\right)}$$

$$\boxed{p_2'^i = \left(\frac{mc}{\sqrt{1 - w_2^2/c^2}}, \frac{m\mathbf{w}_2}{\sqrt{1 - w_2^2/c^2}}\right)}$$

and energy and momentum conservation means

$$\boxed{p_1^i + p_2^i = p_1'^i + p_2'^i}$$

2.b *(4 marks, medium, needs attention, B)*
Evaluating explicitly by inspection,

$$\boxed{\left(p_1^i + p_2^i\right)\left(p_{1i} + p_{2i}\right) = M^2c^2\left(1 + \frac{1}{\sqrt{1-u^2/c^2}}\right)^2 - \frac{M^2u^2}{1-u^2/c^2} = 2M^2c^2\left(1 + \frac{1}{\sqrt{1-u^2/c^2}}\right)}$$

and

$$\boxed{\left(p_1'^i + p_2'^i\right)\left(p'_{1i} + p'_{2i}\right) = m^2c^2\left(\frac{1}{\sqrt{1-w_1^2/c^2}} + \frac{1}{\sqrt{1-w_2^2/c^2}}\right)^2 - m^2\left(\frac{\mathbf{w}_1}{\sqrt{1-w_1^2/c^2}} + \frac{\mathbf{w}_2}{\sqrt{1-w_2^2/c^2}}\right)^2}$$

$$\boxed{= 2m^2c^2\left(1 + \frac{1 - \mathbf{w}_1 \cdot \mathbf{w}_2/c^2}{\sqrt{1-w_1^2/c^2}\sqrt{1-w_2^2/c^2}}\right)},$$

which $\boxed{\text{are equal}}$ according to 2.a.

2.c *(4 marks, medium, B)*
Energy conservation implies

$$Mc^2\left(1 + \frac{1}{\sqrt{1-u^2/c^2}}\right) = mc^2\left(\frac{1}{\sqrt{1-w_1^2/c^2}} + \frac{1}{\sqrt{1-w_2^2/c^2}}\right) \geq 2mc^2$$

as $\gamma \geq 1$. It follows that

$$\boxed{m \leq M\frac{1 + \frac{1}{\sqrt{1-u^2/c^2}}}{2}}.$$

2.d *(4 marks, difficult, D)*
If $V(a,b) = (a+b)/(1+ab/c^2)$ denotes the velocity in K of a particle moving with velocity a in a frame K' within which K moves with velocity $\mathbf{v}$ (so that $V(a,b) = a+b$ by Galilean transform), then we want $V(u',-v) = u$ for the velocity of particle 1 and $V(-u',-v) = 0$ for the velocity of particle 2. From the latter follows $v = -u'$ and so from the former $u = -2v/(1+v^2/c^2)$. Solving for v gives

$$\boxed{v = -\frac{c^2}{u} \pm \sqrt{\frac{c^4}{u^2} - c^2}}$$

To choose the sign, we rearrange,

$$\frac{vu}{c^2} = -\left(1 \mp \sqrt{1-u^2/c^2}\right)$$

Choosing the +ve sign results in $|vu| > c^2$ if the root is positive, and so

$$\boxed{v = -\frac{c^2}{u}\left(1 - \sqrt{1-u^2/c^2}\right) = -\frac{u}{1+\sqrt{1-u^2/c^2}}}$$

A more direct way of derivation is to determine the Lorentz transform for some frame moving with velocity v (Lorentz factor γ and ratio β) of the total four-momentum in K, which is $p_1^i + p_2^i$. That gives another four-vector, whose momentum component reads $\gamma\beta Mc(1 + 1/\sqrt{1-u^2/c^2}) + \gamma Mu/\sqrt{1-u^2/c^2}$. Demanding that this is 0 then gives $v = -u/(1 + \sqrt{1-u^2/c^2})$ as above.

2.e *(4 marks, medium, C)*
Expanding the square root

$$\boxed{v = -\frac{c^2}{u}\left(1 - \sqrt{1 - u^2/c^2}\right) = -\frac{c^2}{u}\frac{1}{2}\frac{u^2}{c^2} + \dots = -\frac{1}{2}u + \dots}$$

This is the "obvious" classical answer, which is another route to answering the question.

3.a *(5 marks, easy, A)*
From F_{ij} being antisymmetric and then relabelling,

$$\boxed{A^i F_{ij} A^j = -A^i F_{ji} A^j = -A^i F_{ij} A^j}$$

it follows that $A^i F_{ij} A^j = 0$.

3.b *(5 marks, medium, B)*
From

$$\frac{\partial}{\partial x^k} F^{ik} = -\frac{4\pi}{c} j^i$$

it follows that

$$\boxed{\frac{\partial^2}{\partial x^k \partial x^i} F^{ik} = -\frac{4\pi}{c}\frac{\partial}{\partial x^i} j^i = 0}$$

where the last identity follows from continuity. Equally one may use the same scheme as 3.a with $A_i = \partial_i$.

3.c *(5 marks, easy, A)*
This can be read off from

$$F^{ij} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}.$$

and

$$F_{ij} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}.$$

so that

$$\boxed{F_{ij}F^{ij} = 2|\mathbf{H}|^2 - 2|\mathbf{E}|^2}$$

3.d *(5 marks, difficult, D)*
This is a matter of inspecting the (inverse) transformation laws (and briefly consulting the invariants, $\mathbf{EH} = \mathbf{E}'\mathbf{H}'$ and $\mathbf{H}^2 - \mathbf{E}^2 = \mathbf{H}'^2 - \mathbf{E}'^2$, to confirm the possibility of the transform). There are only two that contain E_y and H_z, namely $E'_y = (E_y - \beta H_z)\gamma$ and $H'_z = (H_z - \beta E_y)\gamma$. All other components vanish. For $E'_y = 0$ we need $E_y - \beta H_z = 0$ and therefore $\beta = E_y/H_z = 1/2$, *i.e.* $\boxed{v = c/2}$.

4.a *(5 marks, easy, A)*
Writing the wave equation on the potential as $\Box A^i = 0$ this follows immediately from

$$F^{ij} = \frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}$$

so that $\boxed{\Box F^{ij} = 0}$.

4.b.i *(5 marks, medium, B)*

This can be shown either via $\nabla \times \mathbf{E} = -\dot{\mathbf{H}}/c$ or by noting that $\mathbf{H} = \mathbf{n} \times \mathbf{E}$ with $\mathbf{n} = (1, 0, 0)^T$ a unit vector in the direction of propagation. It follows

$$\boxed{H_x = 0}$$

$$\boxed{H_y = -b_2 \sin(\omega t - kx + \alpha)}$$

$$\boxed{H_z = b_1 \cos(\omega t - kx + \alpha)}$$

4.b.ii *(5 marks, medium, B)*

The charge density follows immediately from Gauss' law $\nabla \cdot \mathbf{E} = 4\pi\rho$, but clearly that vanishes here, $\boxed{\rho = 0}$, as $E_x = 0$, E_y independent of y and E_z independent of z. Of course, this is expected, as the wave equation is initially written down in the vacuum.

4.c *(5 marks, easy, A)*

The first (Lorenz) gauge can be written in Lorentz invariant form, $\boxed{\partial_i A^i = 0}$.
As for the second gauge, ϕ transforms like the time-component of a four vector, $\phi' = (\phi - \beta A_x)\gamma$, and vanishes in any other frame only for $\mathbf{v} = 0$ or $A_x = 0$. In general, it does not.

5.a *(5 marks, easy, A)*

From $\mathbf{H} = \nabla \times \mathbf{A}$ it follows that $\mathbf{H} = \nabla\phi \times \mathbf{v}/c$ and from $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}/c$ we therefore have

$$\boxed{\mathbf{H} = \frac{\mathbf{v}}{c} \times \left(\mathbf{E} + \frac{1}{c}\partial_t \mathbf{A}\right) = \frac{\mathbf{v}}{c} \times \mathbf{E}}$$

using that $\mathbf{A} = \mathbf{v}\phi/c$ and $\mathbf{v} \times \mathbf{v} = 0$.

5.b *(4 marks, easy, A)*

The Lorentz force is $\dot{\mathbf{p}} = e\mathbf{E} + (e/c)\mathbf{v} \times \mathbf{H}$. The magnetic field gives rise to a term $\mathbf{v} \times \mathbf{v} \times \mathbf{E} = \mathbf{v}(\mathbf{v} \cdot \mathbf{E}) - v^2\mathbf{E}$ and so

$$\boxed{\dot{\mathbf{p}} = e\left(1 - \frac{v^2}{c^2}\right)\mathbf{E} + \frac{e\mathbf{v}}{c^2}\mathbf{v} \cdot \mathbf{E}}\ .$$

5.c.i *(4 marks, medium, B)*

This is most easily done using (to leading order) $\mathbf{E} = (3(\mathbf{n} \cdot \mathbf{d})\mathbf{n} - \mathbf{d})/R_0^3$ with $\mathbf{n}$ a unit vector pointing in the direction $\mathbf{R}_0$, the position of the observer, and $\mathbf{d} = e\mathbf{R}$ the dipole moment. With $\mathbf{R}_0 = \mathbf{e}_x x$ it follows that $\mathbf{n} \perp \mathbf{R}_0$ and therefore $\boxed{\mathbf{E} = -e\mathbf{R}/x^3}$.

5.c.ii *(4 marks, difficult, D)*

Using the same expression as in 5.c.i, now $\mathbf{R}_0 = \mathbf{e}_y y$ and so $\mathbf{n} \cdot \mathbf{d} = eR$, so that $\boxed{\mathbf{E} = 2e\mathbf{R}/y^3}$.

5.c.iii *(4 marks, medium, B)*

If both charges are equal, the leading order is dominated by the electric field of two equal point charges, $\boxed{E = 2e/r^2}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH96009 MATH97022 MATH97100 Special Relativity and Electromagnetism	1	Technically fairly demanding in comparison to other questions. I was generous with follow-on errors and lack of simplification. Most damaging issue was not being able to write a four-vector for a given event.
MATH96009 MATH97022 MATH97100 Special Relativity and Electromagnetism	2	Almost no problems with 2a. Minor issues with 2b, working out the contraction, and, strangely, 2c, using \gamma>=1. 2d) was difficult, but there were a good number of creative answers.
MATH96009 MATH97022 MATH97100 Special Relativity and Electromagnetism	3	3a and 3b could be answered using the same, simple argument, when I had hoped that people would draw on the continuity equation. Despite a minor typo in the question asking for the magnetic, rather than the electric field to vanish, most worked this out (I obviously allowed for both answers).
MATH96009 MATH97022 MATH97100 Special Relativity and Electromagnetism	4	4a very straight forward, 4b a bit more demanding (even when there are some simple physics arguments), but very widely done with ease. 4c should have been simple bookwork (the first is the Lorenz gauge!), but some missed that one.

MATH96009 MATH97022 MATH97100 Special Relativity and Electromagnetism	5	5a I thought was easy, but many wrongly assumed no time-dependence of A, rather than v x A =0 as A is parallel to v. 5b was easy. 5c was easy if done by means of dipoles and a mess without it.