1(a). The boundary value problem for $\phi(x)$ is

$$-\hat{c}\frac{d^2\phi}{dx^2} = \hat{f}(x) = 0, \qquad \phi(1) = 1,\ \phi(0) = 0.$$

With $\hat{c} = 1$ the general solution of this ordinary differential equation is

$$\phi(x) = Ax + B,$$

where A and B are constants. On use of the boundary conditions we find

$$\phi(x) = 1 - x.$$

(b) This is the continuous version of problem 1 on Sheet 3 where a simple random walk on a discrete line of nodes is considered. Labelling nodes $i = 0, 1, \cdots, N$, then the probability of reaching node 0 before node N starting at node i is

$$p_i = 1 - \frac{i}{N}, \qquad i = 1, 2, \cdots, N-1.$$

Setting $x = i/N$, which in the limit $N \to \infty$ takes values $x \in (0, 1)$, we see how the two solutions are related.

2(a). The boundary value problem for $\phi(x)$ is

$$-\hat{c}\frac{d^2\phi}{dx^2} = x, \qquad \phi(1) = 0,\ \phi(0) = 0.$$

(b) With $\hat{c} = 1$ the general solution of this ordinary differential equation is

$$\phi(x) = -\frac{x^3}{6} + Ax + B.$$

To satisfy the boundary condiions we must take

$$B = 0, \qquad A = \frac{1}{6}.$$

Hence

$$\phi(x) = \frac{x}{6}\left(1 - x^2\right).$$

(c) The current at any point in the wire is given by

$$-\hat{c}\frac{d\phi}{dx}.$$

From the solution, with $\hat{c} = 1$, this is

$$-\frac{d\phi}{dx} = -\frac{1}{6}(1 - 3x^2).$$

Thus the current at $x = 0$ is $-1/6$, the current at $x = 1$ is $1/3$.

(d) The total current input to the wire is the integral of the current source density over the wire:

$$\int_0^1 \hat{f}(x)dx = \int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Note that this is consistent with currents out of the wire at its two ends as computed in part (c):

$$\frac{1}{3} - \left(-\frac{1}{6}\right) = \frac{1}{2}.$$

3(a). The boundary value problem for $\phi(x)$ is

$$-\frac{d}{dx}\left[(2-x)\frac{d\phi}{dx}\right] = x(1-x), \qquad \phi(1) = 0,\ \phi(0) = 0.$$

(b) The general solution of this ordinary differential equation is

$$-\frac{d}{dx}\left[(2-x)\frac{d\phi}{dx}\right] = x(1-x), \quad \phi(1) = 0, \quad \phi(0) = 0. \tag{1}$$

Considering that $x(1-x) = -(x-2+2)(x-2+1) = -(x-2)^2 - 3(x-2) - 2$, the general solution of this ordinary differential equation is

$$\phi(x) = -\frac{1}{9}(x-2)^3 - \frac{3}{4}(x-2)^2 - 2x + A\log(2-x) + B. \tag{2}$$

To satisfy the boundary condition we must take

$$B = \frac{95}{36}, \quad A = -\frac{19}{36\log 2} \tag{3}$$

$$\phi(x) = -\frac{1}{9}(x-2)^3 - \frac{3}{4}(x-2)^2 - 2x - \frac{19}{36\log 2}\log(2-x) + \frac{95}{36}. \tag{4}$$

The current at any point in the wire is given by

$$-(2-x)\frac{d\phi}{dx}. \tag{5}$$

From the solution, this is

$$-\frac{d\phi}{dx} = -(2-x)\left(-\frac{1}{3}(x-2)^2 - \frac{3}{2}(x-2) - 2 + \frac{19}{36\log 2}\frac{1}{2-x}\right). \tag{6}$$

Thus the current at $x = 0$ is

$$+\frac{2}{3} - \frac{19}{36 \log 2}, \tag{7}$$

and the current at $x = 1$ is

$$+\frac{5}{6} - \frac{19}{36 \log 2}. \tag{8}$$

(d) The total current input to the wire is the integral of the current source density over the wire:

$$\int_0^1 \hat{f}(x) dx = \int_0^1 x(1-x) dx = \frac{1}{6}.$$

Note that this is consistent with the difference in the currents at the two ends of the wire as computed in part (c).

4(a). From lectures we saw how, for a 1D graph comprising nodes on a line, the discrete Laplacian $\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A}$ became the differential operator

$$\mathbf{K} = \mathbf{A}^T\mathbf{C}\mathbf{A} \mapsto -\frac{d}{dx}\left[\hat{c}(x)\frac{d\phi}{dx}\right],$$

where the vector of node potentials $\mathbf{x}$ becomes the continuous potential function $\phi(x)$ and the conductance matrix $\mathbf{C}$ becomes the conductivity function $\hat{c}(x)$. Therefore, by extension, if the displacement is allowed to depend on time then $\phi(x)$ will become $\phi(x,t)$ and the discrete quantity

$$\frac{d^2\mathbf{x}}{dt^2}$$

will become

$$\frac{\partial^2\phi(x,t)}{\partial t^2}.$$

Consequently, the discrete equation

$$-\mathbf{K}\mathbf{x} = \frac{d^2\mathbf{x}}{dt^2}$$

will become, with $\hat{c}(x) = 1$,

$$\frac{\partial^2}{\partial x^2}\phi(x,t) = \frac{\partial^2\phi(x,t)}{\partial t^2}$$

where the have turned d/dx into $\partial/\partial x$ since $\phi(x,t)$ also depends on t.

(b) On substitution of $\phi(x,t) = \mathbf{\Phi}(x)e^{\mathrm{i}\omega t}$ we find

$$\frac{d^2}{dx^2}\mathbf{\Phi}(x) = -\omega^2\mathbf{\Phi}(x)$$

after cancellation of $e^{i\omega t}$ on both sides. This is an ordinary differential equation for $\Phi(x)$.

(c) A general non-zero solution satisfying the given boundary conditions is given by

$$\Phi(x) = A\sin(n\pi x), \qquad n \in \mathbb{Z}, \tag{9}$$

where A is any constant and we have restricted the possible values of ω to

$$\omega = n\pi, \qquad n \in \mathbb{Z}.$$

(d) We note that if we set $x = i/(N+1)$ and evaluate $\Phi(x)$ at these values, i.e.,

$$\sin(n\pi x) \mapsto \sin(n\pi i/(N+1)), \qquad i = 1, \cdots, N$$

then we retrieve the elements of the eigenvectors of the matrix $\mathbf{K}_N$ introduced in lectures. Thus this exercise can be thought of as the continuous version of that discrete analysis. For finite N we found N eigenvectors for K_N; here, as $N \to \infty$, the functions (9) can be thought of as an infinite set of "eigenfunctions" of the operator d^2/dx^2 with vanishing boundary conditions at $x = 0, 1$.