Imperial College London

Department of Mathematics
Imperial College London
Academic Year 2023–2024

Introduction to University Mathematics **MATH40001/MATH40009**

Final Exam

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You may assume, without proof, any results from the lectures, lecture notes and videos, unless you are explicitly asked to prove them.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

> In this exam, you may assume any results from the course notes or lectures (as long as you state them correctly), unless you are explicitly asked to prove them.

1. **Total: 20 Marks**

   (a) Let X be a set.

   i. Give the definition of a binary relation R on X. 1 Mark

   ii. What does it mean for a binary relation R on X to be antisymmetric? 1 Mark

   iii. Assume that R is a reflexive binary relation on X, such that for all $x, y, z \in X$, $R(x, y) \wedge R(x, z) \implies R(y, z)$. Show that R is symmetric and transitive. 3 Marks

   (b) i. Let R be a binary relation on a set X, and define another binary relation R^{-1} on X by $R^{-1}(x, y) = R(y, x)$ for $x, y \in X$. Show that R^{-1} is an equivalence relation if and only if R is. 3 Marks

   ii. Let R and S be two equivalence relations on a set X. Define the following binary relation on the set of equivalence relations:

   $$R \leq S \quad \text{if and only if} \quad \forall x, y \in X, R(x, y) \Rightarrow S(x, y).$$

   Show that $\leq$ is a partial order. 3 Marks

   (c) Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.

   i. Prove that if f is surjective and g is not bijective, then $g \circ f$ is not bijective. 3 Marks

   ii. Find sets X, Y and Z, and functions f and g such that neither f nor g is bijective, but $g \circ f$ is bijective. 1 Mark

   (d) Let $\sim$ be an equivalence relation on a set X, and $V := \{cl(x)|x \in X\}$ the set of equivalence classes of $\sim$.

   i. Show that the function $f : X \to V$ sending x to $cl(x)$ is surjective. 3 Marks

   ii. Is f always injective? Prove or disprove. 2 Marks

2. **Total: 20 Marks**

   (a) In this part, the only things you can assume about addition on the naturals are that $x+0 = x$ and $x + S(n) = S(x + n)$. You can also assume that $1 = S(0)$.

   i. Prove that if $n \in \mathbb{N}$ then $S(n) = n + 1$. 2 Marks

   ii. Prove that if $x, y \in \mathbb{N}$ then $S(x) + y = S(x + y)$. 2 Marks

   (b) In this part you can assume all standard results about the naturals and integers.

i. What is the *greatest common divisor* of two positive integers a and b? 2 Marks

ii. Using any method you like, find $\gcd(24, 10)$. 2 Marks

iii. Using any method you like, find integers λ and μ such that $24\lambda + 10\mu = \gcd(24, 10)$. 3 Marks

(c) In this part, you can assume all standard results about the real numbers. Consider the empty subset $S = \emptyset \subseteq \mathbb{R}$.

i. Prove or disprove: the empty set is bounded above in $\mathbb{R}$. 1 Mark

ii. Prove or disprove: the empty set has a least upper bound in $\mathbb{R}$. 3 Marks

(d) In this part, you can assume all standard results about the real numbers, but nothing about the complex numbers.

i. Give the definition of the complex numbers, and the definitions of addition and multiplication on the complex numbers. 3 Marks

ii. Prove that if p, q, r are complex numbers, then $p(q + r) = pq + pr$. 2 Marks

3. **Total: 20 Marks**

(a) Show that for any $k \in \mathbb{N}$ such that $k \geq 1$

$$\left|\sum_{i=1}^{k} \mathbf{r}_i\right| \leq \sum_{i=1}^{k} |\mathbf{r}_i|$$

where $\mathbf{r}_i \in \mathbb{R}^n$ for all $i \in [1, k]$. 4 Marks

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^3$, prove that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$. 3 Marks

(c) In $\mathbb{R}^3$, we define $\mathcal{L}_1$ to be the line parallel to $\mathbf{u}_1$ and passing through the point $\mathbf{r}_1$, and $\mathcal{L}_2$ to be the line parallel to $\mathbf{u}_2$ and passing through the point $\mathbf{r}_2$.

i. Show that, if the two lines are not parallel, the smallest distance between the two lines is given by

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|}$$

4 Marks

ii. Compute the shortest distance between the line $\mathcal{L}_1$ passing through the point $(-1, -1, 1)$ and $(0, 0, 0)$ and the line $\mathcal{L}_2$ passing through the points $(0, -2, 0)$ and $(2, 0, 5)$. 2 Marks

(d) In $\mathbb{R}^3$, the path of a particle is given by the following time-dependent vector function $\mathbf{r}(t) = (\cos t, \sin t, t^2)$.

i. Find the speed of the particle at $t = 4\pi$. 2 Marks

ii. Is $\mathbf{r}'(t)$ ever orthogonal to $\mathbf{r}(t)$? If so, give all values of t for which this is realized. 2 Marks

iii. Determine a parametrization of the tangent line to the particle trajectory at $t = 4\pi$. 3 Marks

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Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Final Exam

1. **Total: 20 Marks**

   (a) Let X be a set.

   i. Write the definition of a binary relation R on X. 1 Mark
   Solution: A binary relation on a set X is a function $X \times X \to \mathrm{Prop}$.

   ii. What does it mean for a binary relation R on X to be antisymmetric? 1 Mark
   Solution: A binary relation R is antisymmetric if and only if

   $$\forall x, y \in X, R(x,y) \wedge R(y,x) \implies x = y.$$

   iii. Assume that R is a reflexive binary relation on X, such that for all $x, y, z \in X$, $R(x,y) \wedge R(x,z) \implies R(y,z)$. Show that R is symmetric and transitive. 3 Marks
   Solution: Since $R(x,y) \wedge R(x,z) \implies R(y,z)$, in particular setting $z = x$ we have $R(x,y) \wedge R(x,x) \implies R(y,x)$. But since R is reflexive, $R(x,x)$ is always true, hence $R(x,y) \implies R(y,z)$ and R is symmetric.
   Now if $R(x,y) \wedge R(x,z) \implies R(y,z)$, since R is symmetric we have $R(x,y) \iff R(y,x)$, hence $R(y,x) \wedge R(x,z) \implies R(y,z)$, which is exactly transitivity.

   (b) i. Let R be a binary relation on a set X, and define another binary relation R^{-1} on X by $R^{-1}(x,y) = R(y,x)$ for $x, y \in X$. Show that R^{-1} is an equivalence relation if and only if R is. 3 Marks
   Solution: Assume R is an equivalence relation. Reflexivity: $R^{-1}(x,x) = R(x,x)$, so clearly since R is reflexive, i.e. $R(x,x)$ is true for all x, so is $R^{-1}(x,x)$.
   Symmetry: Since R is symmetric, $R^{-1}(x,y) = R(y,x) = R(x,y) = R^{-1}(y,x)$ and R^{-1} is symmetric.
   Transitivity: Since R is transitive for all $x, y \in X$, $R(x,y) \wedge R(y,z) \implies R(x,z)$. Hence

   $$R^{-1}(x,y) \wedge R^{-1}(y,z) = R(y,x) \wedge R(z,y) = R(z,y) \wedge R(y,x) \implies R(z,x) = R^{-1}(x,z).$$

   Similarly one can reverse all implications to go show the converse.

   ii. Let R and S be two equivalence relations on a set X. Define the following binary relation on the set of equivalence relations:

   $$R \leq S \text{ if and only if } \forall x, y \in X, R(x,y) \Rightarrow S(x,y).$$

   Show that $\leq$ is a partial order. 3 Marks
   Solution: Reflexivity: Obviously $R \leq R$, as $R(x,x) \Rightarrow R(x,x)$ is always true.
   Antisymmetry: We have to show that if $R \leq S$ and $S \leq R$, then $R = S$. Assume that $R \leq S$ and $S \leq R$, i.e. $\forall x, y \in X, (R(x,y) \Rightarrow S(x,y)) \wedge (S(x,y) \Rightarrow R(x,y))$. Then $(R(x,y) \iff S(x,y))$ by a proposition from lecture about logical implications and $R = S$.
   Transitivity: Assume $R \leq S$ and $S \leq T$ for some equivalence relations, then $\forall x, y \in X, (R(x,y) \Rightarrow S(x,y)) \wedge (S(x,y) \Rightarrow T(x,y))$, hence $\forall x, y \in X, (R(x,y) \Rightarrow T(x,y)$ by transitivity of the logical implication.

   (c) Let X and Y be sets, $f : X \to Y$ and $g : Y \to Z$ be functions.

i. Prove that if f is surjective and g is not bijective, then $g \circ f$ is not bijective. $\boxed{\text{3 Marks}}$
Solution: Assume f is surjective. If f is not injective, neither is $g \circ f$, so the implication holds. Assume therefore that f is injective, so it is bijective. Then, given g is not bijective, $g \circ f$ cannot be bijective either, because $g \circ f, f$ bijective imply g is bijective.

ii. Find sets X and Y, and functions f and g such that neither f nor g is bijective, but $g \circ f$ is bijective. $\boxed{\text{1 Mark}}$
Solution: Let $X = \{0\}$, $Y = \{0, 1\}$, $Z = \{0\}$, and $f : X \to Y$, such that $f(0) = 0$ (f injective, not surjective), $g : Y \to Z$, such that $g(0) = g(1) = 0$ (g surjective, not injective). But $g \circ f : X \to Z$ is bijective.

(d) Let $\sim$ be an equivalence relation on a set X, and $V := \{cl(x)|x \in X\}$ the set of equivalence classes of $\sim$.

i. Show that the function $f : X \to V, x \mapsto cl(x)$ is surjective. $\boxed{\text{3 Marks}}$
Solution Take an arbitrary equivalence class $cl(x) \in V$. We have that necessarily $x \in cl(x)$ as $x \sim x$, hence $f(x) = cl(x)$ and we are done.

ii. Is f always injective? Prove or disprove. $\boxed{\text{2 Marks}}$.
Solution f is not always injective. Indeed assume that for some $x, y \in X, cl(x) = cl(y)$. If f were injective, we would have to show that $x = y$. But this is not true for all x and y unless there is only one representative in each equivalence class. In fact assume $X = \mathbb{Z}$ and take the equivalence class of integers modulo 2. Then $[2] = \{z \in \mathbb{Z}|z = 2k, k \in \mathbb{Z}\}$, hence all even integers are sent to $[2]$ via the map f.

2. $\boxed{\textbf{Total: 20 Marks}}$

(a) i. $n + 1 = n + S(0) = S(n + 0) = S(n)$ (and they don't have to say why, but it's definition of 1, definition of $x + S(y)$ and definition of $x + 0$ respectively). $\boxed{\text{2 Marks}}$

ii. Induction on y. Base case says $S(x) + 0 = S(x + 0)$ and using the fact that $t + 0 = t$ twice this reduces to $S(x) = S(x)$, which is true. Inductive step: we assume $S(x)+d = S(x + d)$ and we need to deduce $S(x) + S(d) = S(x + S(d))$. But the left hand side $S(x)+S(d) = S(S(x)+d)$ (by definition of $+$) which equals $S(S(x+d))$ (by the inductive hypothesis). And the right hand side is $S(x + S(d))$ which is also $S(S(x + d))$ (again by definition of $+$). $\boxed{\text{2 Marks}}$

(b) i. It is the largest number which is a factor of both a and b. $\boxed{\text{2 Marks}}$

ii. Euclid says $24 = 2 \times 10 + 4$, $10 = 2 \times 4 + 2$, and $4 = 2 \times 2 + 0$, so the last non-zero remainder is 2 which must be the GCD. Alternatively just factor both sides and solve it manually. $\boxed{\text{2 Marks}}$

iii. The Euclid approach: the first equation gives $4 = 24 - 2 \times 10$ and so the second gives $2 = 10 - 2 \times 4 = 10 - 2 \times (24 - 2 \times 10) = 5 \times 10 - 2 \times 24$, so $\lambda = -2$ and $\mu = 5$ works (and of course there are infinitely many other solutions too, for example $\lambda = 3$ and $\mu = -7$). Alternatively just mess around until you spot a solution. I will give full marks to solutions which give values which work but with no working, but of course a solution which has values which don't work and has no working should get 0. $\boxed{\text{3 Marks}}$

(c) i. This is true. For example 37 is an upper bound for the empty set. To check this we need to check that for all reals x, if x is in the empty set then $x \leq 37$. But this is true because a false statement implies any statement. $\boxed{\text{1 Mark}}$

ii. This is false, there is no least upper bound. Indeed the same argument as in the first part shows that every real number is an upper bound for the empty set, so if B is an upper bound, then it's not the least upper bound, because $B - 1$ is a smaller upper bound. $\boxed{\text{3 Marks}}$

(d) i. The complex numbers are defined to be $\mathbb{R} \times \mathbb{R}$. Addition is defined by $(a, b) + (c, d) = (a+c, b+d)$ and multiplication is defined by $(a, b) \times (c, d) = (ac-bd, ad+bc)$. $\boxed{\text{3 Marks}}$

ii. Say $p = (a, b)$, $q = (c, d)$ and $r = (e, f)$. Then $p(q + r) = (a, b)(c + e, d + f) = (a(c+e)-b(d+f), a(d+f)+b(c+e)) = (ac+ae-bd-bf, ad+af+bc+be)$, and $pq+pr =$

$(ac-bd, ad+bc)+(ae-bf, af+be) = (ac-bd+ae-bf, ad+bc+af+be)$ and these are easily checked to be equal (using commutativity and associativity lots of times, of course, but you're allowed to do this without explicitly mentioning it). 2 Marks

3. **Total: 20 Marks**

(a) We want to show that for any $k \in \mathbb{N}$

$$\left|\sum_{i=1}^{k} \mathbf{r}_i\right| \leq \sum_{i=1}^{k} |\mathbf{r}_i|$$

where $\mathbf{r}_i \in \mathbb{R}^n$ for all $i \in [1, k]$.
We will do this by induction on k. Let's start by the base case.

- ☐ For $k = 1$, we have that indeed $|\mathbf{r}_1| \leq |\mathbf{r}_1|$ as we have in particular equality.
- ☐ We can build an intuition about how to proceed by considering the case $k = 2$, we have then $|\mathbf{r}_1 + \mathbf{r}_2| \leq |\mathbf{r}_1| + |\mathbf{r}_2|$ given by the triangle inequality.
- ☐ Now assume that for an arbitrary k, we have

$$\left|\sum_{i=1}^{k} \mathbf{r}_i\right| \leq \sum_{i=1}^{k} |\mathbf{r}_i| \quad \text{(induction hypothesis)}$$

We want to show that the statement is true for $k+1$; we write

$$\left|\sum_{i=1}^{k+1} \mathbf{r}_i\right| = \left|\sum_{i=1}^{k} \mathbf{r}_i + \mathbf{r}_{k+1}\right| \leq \left|\sum_{i=1}^{k} \mathbf{r}_i\right| + |\mathbf{r}_{k+1}| \leq \sum_{i=1}^{k} |\mathbf{r}_i| + |\mathbf{r}_{k+1}| = \sum_{i=1}^{k+1} |\mathbf{r}_i|$$

where we have used first the triangle inequality and then the induction hypothesis leading to the result.

4 Marks

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors of $\mathbb{R}^3$, we want to prove that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = 0$.
Let us assume that

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &\iff (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \quad \text{(Scalar TP)} \\
&\iff (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\
&\iff (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = \mathbf{0} \\
&\iff \mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = \mathbf{0} \quad \text{(Scalar TP)} \\
&\iff (\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}
\end{aligned}$$

All steps were equivalences so we are done with the proof. 3 Marks

(c) In $\mathbb{R}^3$, we define $\mathcal{L}_1$ to be the line parallel to $\mathbf{u}_1$ and passing through the point $\mathbf{r}_1$ and $\mathcal{L}_2$ to be the line parallel to $\mathbf{u}_2$ and passing through the point $\mathbf{r}_2$.

i. Assume the two lines are not parallel, parametric equations for the two lines are given by

$$\begin{aligned}
\mathcal{L}_1: &\quad \mathbf{p}_1 = \mathbf{r}_1 + \lambda \mathbf{u}_1 \\
\mathcal{L}_2: &\quad \mathbf{p}_2 = \mathbf{r}_2 + \mu \mathbf{u}_2
\end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. The shortest distance is to be found on the line connecting the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ and perpendicular to both. The vector directing that cross line is $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2$. Take any two points P and Q respectively on the lines $\mathcal{L}_1$ and $\mathcal{L}_2$, the shortest distance between the lines is thus the norm of the vector projection of $\mathbf{PQ}$ onto $\mathbf{n}$, this leads to

$$d(\mathcal{L}_1, \mathcal{L}_2) = |\mathrm{proj}_{\mathbf{n}} \mathbf{PQ}| = \frac{|(\mathbf{r}_2 + \mu \mathbf{u}_2 - \mathbf{r}_1 - \lambda \mathbf{u}_1) \cdot \mathbf{n}|}{|\mathbf{n}|}$$

but as $\mathbf{n} \cdot \mathbf{u}_1 = \mathbf{n} \cdot \mathbf{u}_2 = 0$, we obtain

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|}$$

$\boxed{\text{4 Marks}}$

ii. The distance between the line $\mathcal{L}_1$ passing through the point $(-1, -1, 1)$ and $(0, 0, 0)$ and the line $\mathcal{L}_2$ passing through the points $(0, -2, 0)$ and $(2, 0, 5)$ is given by

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|}$$

with $\mathbf{r}_1 = (0, 0, 0)$, $\mathbf{u}_1 = (1, 1, -1)$, $\mathbf{r}_2 = (2, 0, 5)$ and $\mathbf{u}_2 = (2, 2, 5)$. We easily obtain

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|(-2, 0, -5) \cdot (7, 7, 0)|}{|7\sqrt{2}|} = \sqrt{2}$$

$\boxed{\text{2 Marks}}$

(d) In $\mathbb{R}^3$, the path of a particle is given by the following time-dependent vector function $\mathbf{r}(t) = (\cos t, \sin t, t^2)$.

i. The speed of the particle is the norm of its velocity vector

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = (-\sin t, \cos t, 2t)$$

We thus obtain

$$v = |\mathbf{v}(t)| = \sqrt{1 + 4t^2}$$

which takes the value

$$v = \sqrt{1 + 64\pi^2}$$

at $t = 4\pi$. $\boxed{\text{2 Marks}}$

ii. To check whether $\mathbf{r}'(t)$ is ever orthogonal to $\mathbf{r}(t)$ we take the scalar product and obtain

$$\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = (\cos t, \sin t, t^2) \cdot (-\sin t, \cos t, 2t) = 2t^3 = 0 \iff t = 0$$

So yes, they are orthogonal for $t = 0$ only.

$\boxed{\text{2 Marks}}$

iii. The tangent line to the particle trajectory at $t = 4\pi$ has for parametric equation

$$\mathbf{p}(t) = \mathbf{r}(4\pi) + t\dot{\mathbf{r}}(4\pi) \Rightarrow \mathbf{p}(t) = \begin{pmatrix} 1 \\ 0 \\ 16\pi^2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 8\pi \end{pmatrix}, \quad \text{with } t \in \mathbb{R}.$$

$\boxed{\text{3 Marks}}$