Imperial College London

M4/5A42

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Applied Stochastic Processes

Date: Tuesday 09 May 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Let $W(t)$ be a standard one dimensional Brownian motion.

(i) Let $c > 0$ arbitrary and define the stochastic processes

$$X(t) = \frac{1}{\sqrt{c}} W(ct),$$

and

$$Y(t) = W(c+t) - W(c).$$

Show that $\{X(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ and $\{Y(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ in law.

(ii) Calculate $\mathbb{E}e^{iW(t)}$ and $\mathbb{E}e^{i(W(t)+W(s))}$, for $t, s, \in (0, +\infty)$.

(iii) Let $B(t) = (1-t)W\left(\frac{t}{1-t}\right)$ for $t \in (0,1)$. Calculate the probability distribution function of $B(t)$.

2.

(i) Calculate the Karhunen-Loeve expansion of the standard Brownian motion $W(t)$ for $t \in [0,1]$.

(ii) Let W_t be a one dimensional Brownian motion and let $\mu, \sigma > 0$ and define

$$S_t = e^{t\mu + \sigma W_t}.$$

(a) Calculate the mean and the variance of S_t.

(b) Calculate the probability density function of S_t.

3. Let X_t be a one-dimensional Itô diffusion process with drift and diffusion coefficients $b(x) = -a_0 - a_1 x$ and $\Sigma(x) = b_0 + b_1 x + b_2 x^2$ where $a_i,\, b_i \geq 0,\ i = 0, 1, 2$.

(i) Write down the Itô stochastic differential equation, the generator and the backward and forward (Fokker-Planck) Kolmogorov equations for X_t.

(ii) Assume that X_0 is a random variable with probability density $\rho_0(x)$ that has finite moments. Use the forward Kolmogorov equation or Itô's formula to derive a system of differential equations for the moments of X_t.

(iii) Calculate the first two moments $M_0,\ M_1$ in terms of the moments of the initial distribution $\rho_0(x)$.

4. Let X_t be an Itô diffusion process in $\mathbb{R}$ with drift and diffusion coefficients $b(x) = \mu - ax$ and $\Sigma(x) = \sigma^2 x$. Assume that the process starts at $X_0 = x_0 > 0$, deterministic.

(i) Write down the Itô stochastic differential equation for X_t, the generator and the backward and forward (Fokker-Planck) Kolmogorov equations.

(ii) Apply Itô's formula to an appropriate function of the process X_t in order to obtain a stochastic differential equation with additive noise (Lamperti's transformation).

(iii) Repeat the previous part (i.e. apply Lamperti's transformation) to the Stratonovich SDE $dX_t = b(X_t)\,dt + \sqrt{\Sigma(X_t)} \circ dW_t$ for X_t.

5. Let X_t be a one dimensional diffusion process with drift and diffusion coefficients $b(x)$ and $\Sigma(x) > 0$, respectively.

(i) Consider the Stratonovich stochastic differential equation for X_t, $dX_t = b(X_t)\,dt + \sqrt{\Sigma(X_t)} \circ dW_t$. Transform it to an Itô SDE by calculating the Stratonovich-to-Itô correction.

(ii) Assume that the drift and diffusion coefficients are such that $b(x) = -\frac{1}{2}\Sigma(x)\phi'(x) + \sigma'(x)$, where $\Sigma(x) > 0$, $\phi(x)$ are smooth functions with $\int_{\mathbb{R}} e^{-\phi(x)}\,dx < +\infty$ and where the prime denotes differentiation with respect to x. Show that the generator of the Stratonovich SDE can be written as

$$\mathcal{L}f = \frac{1}{2}e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\frac{df}{dx}\right),$$

for all $f \in C^2(\mathbb{R})$.

(iii) Show that the process X_t defined in Part (ii) is ergodic and obtain a formula for the invariant distribution.

1. (i) Since $X(t)$ and $Y(t)$ are Gaussian, it is enough to show that the two processes have the same mean and covariance as $W(t)$. Clearly, we have $\mathbb{E}X(t) = 0$, $\mathbb{E}Y(t) = 0$. Furthermore,

$$\mathbb{E}(X(t)X(s)) = \frac{1}{c}E(W(ct)W(cs)) = \frac{1}{c}\min(ct, cs) = \min(t, s).$$

Similarly,

$$\begin{aligned}\mathbb{E}(Y(t)Y(s)) &= \mathbb{E}(W(c+t)W(c+s)) - \mathbb{E}(W(c+t)W(c)) - \mathbb{E}(W(c)W(c+s)) + \mathbb{E}W(c)^2 \\ &= \min(t+c, c+s) - c = \min(t, s).\end{aligned}$$

[7] UNSEEN

(ii) $W(t)$ is a $(0, t)$ random variable. We use the formula for the characteristic function $\phi(\lambda)$ of a Gaussian variable (with $\lambda = 0$) to obtain

$$\mathbb{E}e^{iW(t)} = e^{-\frac{1}{2}t}.$$

$W(t) + W(s)$ is Gaussian, since it is the sum of two Gaussian random variables. We have that $\mathbb{E}(W(t) + W(s)) = 0$ and

$$\mathbb{E}(W(t) + W(s))^2 = t + s + 2\min(t, s).$$

We use the formula for the characteristic function of a Gaussian random variable to obtain

$$\mathbb{E}e^{i(W(t)+W(s))} = \exp\left(-\frac{t}{2} - \frac{s}{2} - \min(t, s)\right).$$

[6] SEEN SIMILAR

(iii) $B(t)$ is a mean zero Gaussian process with variance

$$\mathbb{E}B(t)^2 = t(1-t).$$

Hence, the probability distribution function is

$$p(x, t) = \frac{1}{\sqrt{2\pi t(1-t)}}\exp\left(-\frac{x^2}{2t(1-t)}\right).$$

[7] SEEN SIMILAR

2. (i) The covariance function of Brownian motion is $\mathcal{R}(t, s) = \min(t, s)$. The eigenvalue problem $\mathcal{R}\psi_n = \lambda_n\psi_n$ becomes

$$\int_0^1 \min(t, s)\psi_n(s)\, ds = \lambda_n\psi_n(t).$$

Since 0 is not an eigenvalue we have that $\lambda_n > 0$. Upon setting $t = 0$ we obtain $\psi_n(0) = 0$. The eigenvalue problem can be rewritten in the form

$$\int_0^t s\psi_n(s)\, ds + t\int_t^1 \psi_n(s)\, ds = \lambda_n\psi_n(t).$$

We differentiate this equation once:

$$\int_t^1 \psi_n(s)\,ds = \lambda_n \psi_n'(t).$$

We set $t = 1$ in this equation to obtain the second boundary condition $\psi_n'(1) = 0$. A second differentiation yields;

$$-\psi_n(t) = \lambda_n \psi_n''(t),$$

where primes denote differentiation with respect to t. Thus, in order to calculate the eigenvalues and eigenfunctions of the integral operator whose kernel is the covariance function of Brownian motion, we need to solve the Sturm-Liouville problem

$$-\psi_n(t) = \lambda_n \psi_n''(t), \quad \psi(0) = \psi'(1) = 0.$$

The eigenvalues and (normalized) eigenfunctions are

$$\psi_n(t) = \sqrt{2}\sin\left(\frac{1}{2}(2n-1)\pi t\right), \quad \lambda_n = \left(\frac{2}{(2n-1)\pi}\right)^2, \quad n = 1, 2\dots$$

Thus, the Karhunen-Loéve expansion of Brownian motion on $[0,1]$ is

$$W_t = \sqrt{2}\sum_{n=1}^{\infty} \xi_n \frac{2}{(2n-1)\pi} \sin\left(\frac{1}{2}(2n-1)\pi t\right).$$

[10] SEEN SIMILAR

(ii) (a) We calculate

$$\begin{aligned} \mathbb{E}S_t &= e^{t\mu}\mathbb{E}e^{\sigma W_t} \\ &= e^{t\mu}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{\sigma x} e^{-\frac{x^2}{2t}}\,dx \\ &= e^{t\mu} e^{\frac{\sigma^2 t}{2}} = e^{t(\mu + \frac{\sigma^2}{2})}, \end{aligned}$$

where we calculated the integral by completing the square in the exponent, i.e. by writing $\sigma x - x^2/2t = -\frac{1}{2t}(x^2 - 2\sigma t x + \sigma^2 t^2) + \frac{\sigma^2 t}{2}$.
Similarly, we calculate

$$\mathbb{E}(S_t)^2 = e^{2t\mu}\mathbb{E}e^{2\sigma W_t} = e^{2t(\mu+\sigma^2)},$$

from which it follows that the variance is

$$\mathbb{E}(S_t - \mathbb{E}S_t)^2 = \mathbb{E}(S_t)^2 - (\mathbb{E}S_t)^2 = e^{2t(\mu+\sigma^2)} - e^{t(2\mu+\sigma^2)}.$$

(b) We calculate (remember that the S_t takes only positive values)

$$\begin{aligned} \mathbb{P}(S_t \le x) &= \mathbb{P}(e^{t\mu + \sigma W_t} \le x) \\ &= \mathbb{P}(W_t \le \frac{1}{\sigma}\ln(x e^{-t\mu})) \\ &= \frac{1}{\sqrt{2\pi t}}\int_0^{\frac{1}{\sigma}\ln(x e^{-t\mu})} e^{-\frac{z^2}{2t}}\,dz. \end{aligned}$$

Consequently,

$$\begin{aligned} p(x,t) &= \frac{\partial}{\partial x}\mathbb{P}(S_t \le x) \\ &= \frac{1}{\sqrt{2\pi t}}\frac{1}{\sigma x}e^{-\frac{(\ln(x)-t\mu)^2}{2\sigma t}}. \end{aligned}$$

[10] SEEN SIMILAR

3. (i) The generator is

$$\mathcal{L} = (-a_0 - a_1 x)\partial_x + \frac{1}{2}(b_0 + b_1 x + b_2 x^2)\partial_x^2.$$

The backward and forward Kolmogorov equations are

$$\partial_t f = \mathcal{L} f$$

and

$$\partial_t \rho = \mathcal{L}^* \rho = \partial_x((a_0 + a_1 x)\rho) + \frac{1}{2}\partial_x^2((b_0 + b_1 x + b_2 x^2)\rho).$$

[6] SEEN SIMILAR

(ii) The nth moment M_n is defined as

$$M_n(t) = \int x^n \rho(x,t)\, dx.$$

We multiply the Fokker-Planck equation by x^n, integrate over $\mathbb{R}$ and integrate by parts on the right hand side of the equation to obtain

$$\begin{aligned} \dot{M}_n &= -n\int (a_0 + a_1 x)x^{n-1}\rho\, dx + \frac{1}{2}n(n-1)\int (b_0 + b_1 x + b_2 x^2)x^{n-2}\rho\, dx \\ &= \left(-a_1 n + \frac{1}{2}b_2 n(n-1)\right)M_n + \left(-na_0 + \frac{1}{2}b_1 n(n-1)\right)M_{n-1} + \frac{1}{2}b_0 n(n-1)M_{n-2}. \end{aligned}$$

[7] SEEN SIMILAR

(iii) We set $n = 0,\ 1$ to the above equation to obtain

$$\begin{aligned} \dot{M}_0 &= 0, \\ \dot{M}_1 &= -a_1 M_1 - a_0 M_0. \end{aligned}$$

From the first equation we get

$$M_0(t) = M_0(0) = 1,$$

since $\rho_0(x)$ is a probability density. The second equation becomes

$$\dot{M}_1 = -a_1 M_1 - a_0.$$

The solution is

$$M_1(t) = e^{-a_1 t}M_1(0) + \frac{a_0}{a_1}\left(e^{-a_1 t} - 1\right).$$

[7] SEEN SIMILAR

4. (i) The SDE is

$$dX_t = (\mu - aX_t)\,dt + \sigma\sqrt{X_t}\,dW_t, \quad X_0 = x_0. \tag{1}$$

The generator of this process is

$$\mathcal{L} = (\mu - ax)\frac{\partial}{\partial x} + \frac{\sigma^2 x}{2}\frac{\partial^2}{\partial x^2}. \tag{2}$$

The Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left((\mu - ax)p\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma^2 x}{2}p\right). \tag{3a}$$

$$p(x, 0|x_0) = \delta(x - x_0). \tag{3b}$$

[4] SEEN SIMILAR

(ii) We use Lamperti's transformation with

$$\begin{aligned} h(x) &= \int^x \frac{1}{\sigma\sqrt{x}}\,dx \\ &= \frac{2}{\sigma}\sqrt{x}. \end{aligned}$$

We apply Itô's formula to the function $h(x)$ to obtain

$$\mathcal{L}h(x) = \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)x^{-1/2} - \frac{a}{\sigma}x^{1/2}.$$

Consequently:

$$dh(X_t) = \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\frac{1}{\sqrt{X_t}}\,dt - \frac{a}{\sigma}\sqrt{X_t}\,dt + dW_t,$$

from which we conclude that the CIR equation becomes, for $Y_t = \frac{2}{\sigma}\sqrt{X_t}$,

$$\begin{aligned} dY_t &= \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\frac{1}{\sqrt{X_t}}\,dt - \frac{a}{\sigma}\sqrt{X_t}\,dt + dW_t \\ &= \left[\left(\frac{2\mu}{\sigma^2} - \frac{1}{2}\right)\frac{1}{Y_t} - \frac{a}{2}Y_t\right]dt + dW_t. \end{aligned}$$

[8] SEEN SIMILAR

(iii) The Stratonovich SDE is

$$dX_t = (\mu - aX_t)\,dt + \sigma\sqrt{X_t} \circ dW_t, \quad X_0 = x_0.$$

We transform it to an Itô SDE by calculating the Stratonovich-to-Itô correction:

$$\begin{aligned} dX_t &= (\mu - aX_t)\,dt + \frac{1}{2}\sigma\sqrt{X_t}(\sigma\sqrt{X_t})' + \sigma\sqrt{X_t}\,dW_t \\ &= (\mu + \frac{1}{4}\sigma^2 - aX_t)\,dt + \sigma\sqrt{X_t}\,dW_t. \end{aligned}$$

The calculation from the previous part is still valid, with μ replaced by $\mu + \frac{1}{4}\sigma^2$:

$$dY_t = \left[\left(\frac{2\widehat{\mu}}{\sigma^2} - \frac{1}{2}\right)\frac{1}{Y_t} - \frac{a}{2}Y_t\right]dt + dW_t, \quad \widehat{\mu} = \mu + \frac{1}{4}\sigma^2.$$

[8] UNSEEN

5. (i) The Stratonovich SDE is

$$\begin{aligned} dX_t &= b(X_t)\,dt + \sqrt{\Sigma(X_t)} \circ dW_t \\ &= \Big(b(X_t) + \frac{1}{2}\sqrt{\Sigma(X_t)}(\sqrt{\Sigma(X_t)})'\Big)\,dt + \sqrt{\Sigma(X_t)}\,dW_t \\ &= \Big(b(X_t) + \frac{1}{4}\Sigma'(X_t)\Big)\,dt + \sqrt{\Sigma(X_t)}\,dW_t \end{aligned}$$

[5] SEEN SIMILAR

(ii) The generator of the SDE, when transformed to the Itô form, with $b(x) = -\sigma(x)\phi'(x) + \frac{1}{2}\sigma'(x)$, where we have introduced the notation $\Sigma(x) = 2\sigma(x)$, applied to a C^2 function f, is

$$\begin{aligned} \mathcal{L}f &= \Big(-\sigma(x)\phi'(x) + \frac{1}{2}\sigma'(x) + \frac{1}{2}\sigma'(x)\Big)\frac{df}{dx} + \sigma(x)\frac{d^2 f}{dx^2} \\ &= \Big(-\sigma(x)\phi'(x) + \sigma'(x)\Big)\frac{df}{dx} + \sigma(x)\frac{d^2 f}{dx^2} \\ &= e^{\phi}\frac{d}{dx}\left(\sigma e^{-\phi}\right)\frac{df}{dx} + \sigma\frac{d^2 f}{dx^2} \\ &= e^{\phi}\frac{d}{dx}\left(\sigma e^{-\phi}\frac{df}{dx}\right) \\ &= \frac{1}{2}e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\frac{df}{dx}\right) \end{aligned}$$

[7] UNSEEN

(iii) First we calculate the Fokker-Planck operator, which is the L^2–adjoint of the generator. From the formula obtained in the previous part, using again the notation $\Sigma(x) = 2\sigma(x)$, together with two-integrations by parts, we deduce that

$$\int_{\mathbb{R}} (\mathcal{L}f)h\,dx = \int f\frac{d}{dx}\left(e^{-\phi}\sigma\left(\frac{d}{dx}e^{\phi}h\right)\right)\,dx,$$

for all $f,\ h \in C_0^2(\mathbb{R})$. Consequently, the Fokker-Planck operator is

$$\mathcal{L}^* h = \frac{d}{dx}\left(e^{-\phi}\sigma\left(\frac{d}{dx}e^{\phi}h\right)\right). \tag{4}$$

The stationary Fokker-Planck equation is

$$\mathcal{L}^*\rho = 0.$$

From (4) it immediately follows that a solution to this equation is

$$\rho = e^{-\phi}.$$

Since, by assumption, $\int e^{-\phi}\,dx < +\infty$, the solution ρ is normalizable and the invariant distribution is

$$\rho_s(x) = \frac{1}{Z}e^{-\phi}, \quad Z = \int e^{-\phi}\,dx. \tag{5}$$

To show that the process X_t is ergodic, we need to show that the invariant distribution is unique. We multiply the stationary Fokker-Planck equation by ρe^{ϕ} and integrate over $\mathbb{R}$ to obtain, after an integration by parts:

$$\begin{aligned}\int (\mathcal{L}^* \rho)\rho\, dx &= \int \frac{d}{dx}\left(e^{-\phi}\sigma\left(\frac{d}{dx}e^{\phi}\rho\right)\right)\rho e^{\phi}\, dx \\ &= -\int \sigma |(e^{\phi}\rho)'|^2 e^{-\phi}\, dx.\end{aligned}$$

Since $\sigma e^{-\phi} > 0$, we have that

$$\int (\mathcal{L}^* \rho)\rho\, dx = 0$$

if and only if $e^{\phi}\rho = \text{const}$, from which we deduce that the unique normalized invariant distribution is given by (5).

[8] UNSEEN

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4/5A42 **Session: 2016-2107**

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most students did well in this question. There were some problems with the calculation of the characteristic functions in part (ii).

Marker: Prof. G.A. PAVLIOTIS

Signature: ________ Date: 15/05/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4/5A42 Session: 2016-2107

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most students did well in Part (i), the calculation of the KL expansion for Brownian motion. There were some difficulties in the calculation of the variance and of the PDF of S_t in Part (ii).

Marker: Prof G.A. PAVLIOTIS

Signature: [signature] Date: 16/05/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4/5A42 **Session: 2016-2107**

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Almost all students answered correctly Part (i). Most students were able to answer correctly part (ii), either by using the Fokker-Planck equation or Itô's formula. There were no difficulties in calculating M_0 in Part (iii), ~~[illegible]~~ and there were many incorrect answers in the calculation of M_1.

Marker: Prof G.A. PAVLIOTIS

Signature: [signature] **Date:** 15/05/2017

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: M4/5A42 **Session: 2016-2107**

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Almost all students answered Part (i) correctly. Many students used the correct method for answering question (ii), however quite often the answers provided were incomplete since the resulting SDE was not written explicitly for $\frac{2}{6}\sqrt{X_t}$. Similarly for Part (iii).

Marker: Prof G.A. Pavliotis

Signature: **Date:** 16/05/2017

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Department of Mathematics

Examiner's Comments

Exam: M4/5A42 **Session: 2016-2107**

Question 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Most students ~~students~~ answered Part (i) correctly. There were many correct answers to Part (ii), and most students followed the right approach. There were almost no correct ~~[illegible]~~ answers in Part (iii), in particular in the proof of the uniqueness of the invariant measure.

Marker: Prof G.A. PAVLIOTIS

Signature: [illegible] Date: 16/05/2017

Please return with exam marks (one report per marker)