

Mathematical Logic (MATH6/70132; P65)  
Problem sheet  $2\frac{1}{2}$  - for problem class

[1] Decide whether the following are true or false - give reasons. As usual,  $\Gamma$  is a set of  $L$ -formulas and  $\phi, \psi$  are  $L$ -formulas.

1. Every  $L$ -formula is a theorem.
2. If  $\phi$  is an  $L$ -formula, then one of  $\phi, (\neg\phi)$  is a theorem of  $L$ .
3. In every  $L$ -formula, the number of opening brackets ( equals the number of closing brackets ).
4. In every  $L$ -formula, the number of opening brackets ( is equal to the number of connectives in the formula.
5. If  $\Gamma \vdash_L \phi$  and  $\Gamma \vdash_L (\phi \rightarrow \psi)$ , then  $\Gamma \vdash_L \psi$ .
6. If  $\Gamma \vdash_L \phi$  and  $v$  is a valuation with  $v(\Gamma) = F$ , then  $v(\phi) = F$ .
7. If  $\Gamma \vdash_L \phi$  and  $\Delta \vdash_L (\phi \rightarrow \psi)$ , then  $\Gamma \cup \Delta \vdash_L \psi$ .
8. Suppose  $v$  is a valuation and  $\Delta = \{\phi : v(\phi) = F\}$ . Then  $\Delta$  is consistent and complete.

[2] (i) Suppose  $\phi$  is an  $L$ -formula and  $\Gamma$  is a set of  $L$ -formulas. Do the following syntactically, that is, without using the Completeness Theorem. You may use theorems of  $L$  which have already been derived in the notes or problem sheets.

(i) Express the 'law of the excluded middle'  $(\phi \vee (\neg\phi))$  as an  $L$ -formula and say why this is a theorem of  $L$ .

(ii) Show that if  $\Gamma \vdash_L ((\neg\phi) \rightarrow \psi)$  and  $\Gamma \vdash_L ((\neg\phi) \rightarrow (\neg\psi))$ , then  $\Gamma \vdash_L \phi$ .

[3] Show that the set of connectives  $\{\neg, \leftrightarrow\}$  is not adequate (There's a hint on the Ed discussion board).

[4] (For fun) The following is known as Hofstadter's MU puzzle. You can look at the Wikipedia entry, but first try the problem yourself.

The formal system  $H$  has: alphabet  $M, I, U$ ; formulas all (finite) strings of these symbols; one axiom  $MI$ ; and the following deduction rules (where  $x, y$  are any formulas):

1. from  $xI$  deduce  $xIU$ ;
2. from  $Mx$  deduce  $Mxx$ ;
3. from  $xIIIy$  deduce  $xUy$ ;
4. from  $xUUy$  deduce  $xy$ .

The problem is to decide whether  $MU$  is a theorem of  $H$ . But you could first write down some theorems of  $H$ , just to test your understanding of what a formal system is.