Imperial College London

Department of Mathematics
Imperial College London

Computational PDEs MATH60025/70025, 2024-2025
Released : 4 February 2025
Upload Deadline : 1.00 pm, 17 February 2025
The project mark, will be weighted to comprise 25% of the overall Module.

You are required to investigate and attempt all Questions below and summarise your findings in form of a well written project report – on which you will be assessed.

Please name your files in following way:

- Technical report : **CPDES_yourCID.pdf** (limit your report to 16 pages or less (including plots). **Anything beyond the 16 page limit will NOT be marked!**
- All your code(s), label as follows :
  CPDES_Q1_of__X_yourCID.m (Matlab scripts example) or
  CPDES_Q1_of__X_yourCID.py (Python scripts).
  Zip all program files and call your zipped folder: **CPDES_programs_yourCID.zip**

Where in the above **yourCID** will be your College ID number.

Notes : *Important*

1. Marking will consider both the correctness of your code as well as the soundness of your analysis and clarity and legibility of the technical report.
2. Marks will primarily be based on contents of your written technical report. You are warned that if you ONLY submit the codes for the work with **NO** technical report, you can **NOT** expect a pass mark.
3. Do **NOT** include source code listing in your technical report.
4. All figures created by your code should be well-made and properly labelled in the technical pdf report.
5. The codes **must** be submitted.

Question 1: Nonlinear ODE Problem, Total mark: 45%

Use 2^{nd}-order accurate finite differences (FDs) and Newton linearisation to develop a code to solve the following nonlinear ordinary differential equation :

$$\epsilon \frac{d^2u}{dx^2} + u\left(\frac{du}{dx} - 1\right) = 0 \qquad 0 \leq x \leq 1\,, \tag{1.1}$$

given $u(0) = -1$ and $u(1) = 1/2$ and ϵ is a arbitrary constant parameter.

1. Describe the key steps used to solve Eqn. 1.1.
2. Compute solutions for $\epsilon = (0.1;\ 0.01;\ 0.001;\ 0.0001)$. Present your results concisely in form of plots and discuss how you ascertain solutions are correct and accurate, through usage of an appropriate error norm measure(s). Make plots and discuss how and why you think your solution errors behaves as they do, when the number of points used in the discretisation increase.
3. Modify your code to 4^{th}-order accurate FD accuracy. Hence investigate numerical accuracy and compare results with those in Part 1. Present your results concisely which highlight differences or detriments of using the 4^{th}-order FDs. Use the $x = 0.72$, $x = 0.75$ points, together with an appropriate error norm, as performance measures.
4. In a plot show how errors are reduced as h the discretisation step size in x is reduced, comparing the 2^{nd} and 4^{th}-order results.
5. Can you explain the solutions and error behaviour (or accuracy) you obtain as ϵ varies?

Question 2: Nonlinear Diffusion, Total mark: 55%

The function u satisfies the nonlinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 (u^p)}{\partial x^2}, \;\; 0 \leq x \leq 1\,, \;\; t > 0, \tag{2.1}$$

with the initial condition $u = 4x(1-x),\ 0 \leq x \leq 1,\ t = 0$, and boundary conditions $u = 0$ at $x = (0, 1)$ for $t \geq 0$.

1. For $p = 1$, at large time $u \to 0$. Write a fully explicit 2^{nd}-order accurate FD code, and estimate the time it takes for the solution to tend to $u = 0$. Quote the time you estimate to 4 decimal places. Discuss accuracy aspects, step size limitations based on Lecture material.
2. For $p = 2$, assuming that $u = u(x - vt)$, v a constant, is a solution of the equation, deduce the exact analytic solution in this case.

3. Discretise and solve Eqn. 2.1 for $p = 2$ using a fully explicit 2^{nd}-order accurate FD approach. In your report discuss the numerical solution and numerical stability aspects. Use Maximum Principle analysis from Lecture notes as a guide.
4. Discretise and solve the equation $(p = 2)$ using the lagged approach to treat the nonlinear term.
5. Discretise and solve using the fully implicit Newton linearisation approach to treat the nonlinear term $(p = 2)$.
6. Investigate and report through appropriate plots and discussion, on the accuracy of your solutions and on step size stability constraints ($r = k/h^2$ where k is the time step and h the spatial step size) with the three approaches you have investigated in the $p = 2$ case.
7. Make plots of your solutions for all x at time values $t = (0.2,\ 0.5,\ 0.75)$ for the $p = 2$ case.
8. Finally in your report, make a table of values of your numerical and exact solutions at values of $x = (0.1,\ 0.3\ 0.5\ 0.7,\ 0.9)$ at $t = (0.2,\ 0.5,\ 0.75)$ for the different approaches investigated $(p = 2)$.

If you are unable to deduce the analytic solution in part 2, compare between the different approaches you devise (either will do).

You may use any tridiagonal solver available in Python (NumPy, Scipy), Matlab or the example codes provided.

You may contact the lecturer for guidance/advice and clarification
email: s.mughal@imperial.ac.uk