

Q3. For  $z_1 = a_1 + ib_1$ ,  $z_2 = a_2 + ib_2$ ,

from the definition of the abs value,

$$|z_i|^2 = a_i^2 + b_i^2, \text{ for } i=1,2. \text{ Hence}$$

$$\begin{aligned} |z_1 + z_2|^2 &= (a_1 + a_2)^2 + (b_1 + b_2)^2 \\ &= a_1^2 + b_1^2 + a_2^2 + b_2^2 + 2(a_1 a_2 + b_1 b_2) \\ &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2) \end{aligned}$$

$$\begin{aligned} \operatorname{Re}(w) &\leq \sqrt{\operatorname{Re}(w)^2 + \operatorname{Im}(w)^2}, \text{ for } w \in \mathbb{C}, \\ &= |w| \end{aligned}$$

Since the diagonal of any right-angled triangle is the longest side. Hence

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots completes the first part.

For the negative triangle inequality,

$$\begin{aligned} |z_1| &= |z_2 - (z_2 - z_1)| \\ &\leq |z_2| + |z_2 - z_1| \quad (\text{from positive triangle inequality}) \\ \left| |z_1| - |z_2| \right| &\leq |z_2 - z_1| \end{aligned}$$