MEJ Q4

$$\min \quad x_1^2 + 0.5\, x_2^2 + x_1 x_2 - 2x_1 - 3x_2$$
$$x_1 + x_2 \leq 1$$

i) Sufficient?

$$f(\underline{x}) = \underline{x}^T \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \underline{x} + [-2 \ \ -3]\, \underline{x}$$

constraint is a linear ineq (convex)

convex f? $\nabla^2 f = 2 \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \to Q \succeq 0$? $\quad \det(Q) = 0.25 > 0$, $\lambda_1(Q) = 1.5 > 0$

$\Rightarrow Q \succ 0 \Rightarrow f$ convex

$\Rightarrow$ KKT are necessary and sufficient.

KKT: $L = \underline{x}^T Q \underline{x} + \underline{b}^T \underline{x} + \lambda (x_1 + x_2 - 1)$

$\hookrightarrow \nabla_x L = 0 \iff 2Q\underline{x} + b + \lambda \mathbb{1} = 0 \quad (*)$ $\to$ $2x_1 + x_2 - 2 + \lambda = 0$, $x_1 + x_2 - 3 + \lambda = 0$

$\lambda (x_1 + x_2 - 1) = 0$

$\underline{\lambda \geq 0}$

Cases:

1) $\lambda = 0 \Rightarrow 2x_1 + x_2 - 2 = 0$, $x_1 + x_2 - 3 = 0 \quad (*)$

$\Rightarrow x_1 = -1$, $x_2 = 4$ $\to x_1 + x_2 = 3 > 1$

2) $\lambda > 0 \Rightarrow x_1 + x_2 = 1 \Rightarrow \lambda = 2 \Rightarrow x_1 = -1$, $x_2 = 2$

Suff and necessity $\Rightarrow (x_1, x_2) = (-1, 2)$ is a minimizer.

ii) Convex cost + linear constraints + Slater $(x_1 = x_2 = 0,\ x_1 + x_2 < 1)$

$\Rightarrow$ Strong duality.

<u>Dual:</u> $L(\underline{x},\lambda) = \underline{x}^T Q \underline{x} + b^T \underline{x} + \lambda(x_1 + x_2 - 1)$

$(X = \mathbb{R}^2)$

$$\min_{\underline{x} \in X = \mathbb{R}^2} L(\underline{x},\lambda) \iff \nabla_x L(\underline{x},\lambda) = 0$$

$$\iff 2Q\underline{x} + b + \lambda \mathbb{1} = 0$$

$$\underline{x}^* = -\frac{1}{2} Q^{-1}(b + \lambda \mathbb{1})$$

$$Q = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \to Q^{-1} = \frac{1}{0.25} \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\underline{x}^* = \begin{bmatrix} -1 \\ 4-\lambda \end{bmatrix}$$

<u>Dual:</u> $$\max_{\lambda \geq 0} L(\underline{x}^*, \lambda) = \max_{\lambda \geq 0} 1 + \frac{1}{2}(4-\lambda)^2$$

$$+ \lambda - 4 + 2 - 3(4-\lambda)$$

$$+ \lambda(2-\lambda)$$

$$\dots = -\frac{1}{2}\lambda^2 - 5 + 2\lambda$$

$$\max_{\lambda \geq 0} \dots \quad f' = 0$$

$$-\lambda + 2 = 0 \Rightarrow \lambda = 2$$

$$\underline{C.S.C} \Rightarrow \underline{x}^* = \begin{bmatrix} -1 \\ 4-\lambda \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Strong duality $\Rightarrow$ duality gap $= 0$