Imperial College London

MATH70018

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Quantum Mechanics 2

Date: 2 June 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. First order time-independent perturbation theory.

(a) Suppose we have a solution to the time-independent Schrödinger equation $\hat{\mathcal{H}}_0 \left|\phi^{(0)}\right\rangle = \varepsilon \left|\phi^{(0)}\right\rangle$. Explain carefully but briefly the meaning of the statement: ε is non-degenerate.

(2 marks)

(b) Take ε of the previous part to be a non-degenerate eigenvalue of $\hat{\mathcal{H}}_0$. Now consider adding a Hermitian perturbation to the Hamiltonian so that the new Hamiltonian is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{V}$. Here $\hat{V}$ is a Hermitian operator and λ is our usual real perturbative parameter. Let $|\phi\rangle$ be the eigenstate of $\hat{\mathcal{H}}$ that reduces to $\left|\phi^{(0)}\right\rangle$ when $\lambda \to 0$ and let E be the associated eigen-energy, $\hat{\mathcal{H}}|\phi\rangle = E|\phi\rangle$. Show that to first order in λ,

$$E = \varepsilon + \lambda \left\langle \phi^{(0)} \middle| \hat{V} \middle| \phi^{(0)} \right\rangle .$$

(5 marks)

(c) Consider the infinite square-well Hamiltonian $\hat{\mathcal{H}}_0 = \frac{\hat{p}^2}{2m} + U(\hat{x})$ where $U(x) = 0$ for $0 \le x \le L$ and $U(x) = \infty$ otherwise. The ground-state wavefunction can be found to be $\phi(x) = \sqrt{\frac{2}{L}} \sin(\pi x/L)$.

Suppose a perturbation $\lambda V(\hat{x})$ is added to the Hamiltonian where $V(x) = \gamma$ for $0 \le x \le L/2$ and zero otherwise. What is the ground state energy to first order in λ?

(5 marks)

(d) What can go wrong with the treatment described in (b) if the eigen-energy of $\hat{\mathcal{H}}_0$ is degenerate?

(3 marks)

(e) Determine the eigen-energies, to first order in λ of

$$H = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & 1 & \lambda \\ 0 & \lambda & 2 \end{pmatrix}.$$

You may quote and use without derivation any results from lecture or elsewhere.

(5 marks)

(Total: 20 marks)

2. Heisenberg equations of motion.

Consider a particle in one dimension with motion governed by the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}$.

(a) Show that the Heisenberg equations of motion are given by

$$m\frac{d}{dt}\hat{x}_H = \hat{p}_H$$
$$\frac{d}{dt}\hat{p}_H = 0.$$

(4 marks)

(b) Solve the equations of (a) to express $\hat{x}_H$ and $\hat{p}_H$ in terms of $\hat{x}$ and $\hat{p}$.

(4 marks)

(c) Suppose at $t = 0$, the normalised state in the position basis is given by $\bar{\psi}(x) = e^{ikx}\phi(x)$ where k is real and $\phi(x)$ is an unspecified function of x. Further suppose that $\langle\phi|\,\hat{x}\,|\phi\rangle = \langle\phi|\,\hat{p}\,|\phi\rangle = 0$. Determine the expectation value of the position and momentum operators for later times.

(6 marks)

(d) Further suppose that $\phi(x)$ of the previous part can be taken to be real. Let $\sigma_x^2(t)$ denote the time-dependent variance in position. That is, $\sigma_x^2(t) = \langle\psi(t)|\,\hat{x}^2\,|\psi(t)\rangle - \langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle^2$. Let $\bar{\sigma}_x^2$ and $\bar{\sigma}_p^2$ be the variance in position and momentum, respectively, at $t = 0$. Determine an expression for $\sigma_x^2(t)$ at later times.

(6 marks)

(Total: 20 marks)

3. Quantum circuits and measurement.

(a) Consider a system composed of two qubits. Write down the computational basis for this system.

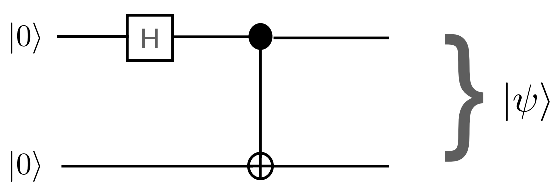
(2 marks)

(b) Suppose that the system is in the state $|\Psi\rangle = |00\rangle$. Suppose that it is measured in the two-qubit computational basis. Determine the probabilities of each of the four possible experimental outcomes. What is the state after the measurement?

(3 marks)

(c) Alice and Bob carefully take the two qubits described in (b), so they each have a qubit in the state $|0\rangle$. They then send the qubits through the quantum circuit shown below. Alice's qubit enters and exits the top wire of the circuit while Bob's is processed by the bottom wire. Show that the resulting state exiting the circuit is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We say that this state is *entangled*. Briefly describe the meaning of "entanglement" here.

Comments: The components of the circuit follow the notation used in the module. The circuit contains a Hadamard and a CNOT gate.



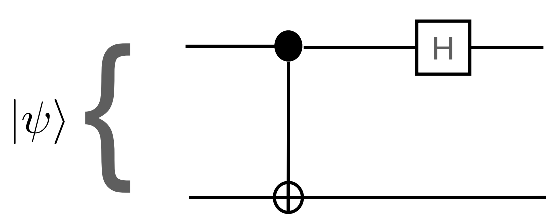
(5 marks)

(d) Upon being processed by the quantum circuit of (c), Alice performs a measurement of her qubit in the computational basis while Bob sits idly. What is the probability that Alice will measure $|0\rangle$ for her qubit? What is the probability that she will measure $|1\rangle$?

Suppose that Alice determines her qubit is in state $|0\rangle$. What is the two-qubit state of the system after this measurement? Suppose now that Bob measures his qubit in the computational basis. What is the probability that he measures his state to also be in $|0\rangle$?

(5 marks)

(e) Finally we send the state output from the gate described in (c), $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, through the circuit described below. What is the resulting state?



(5 marks)

(Total: 20 marks)

4. Shorter problems covering different aspects of the module.

(a) Evaluate and simplify the following commutators.

(i) $[\hat{x}, \hat{p}^2]$

(ii) $[\hat{a}, \hat{a}^\dagger \hat{a}]$ where $\hat{a}$ is a bosonic operator

(iii) $[\hat{c}, \hat{c}^\dagger \hat{c}]$ where $\hat{c}$ is a fermionic operator

(4 marks)

(b) Consider the Harmonic oscillator Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. Write down (but don't solve) the corresponding time-independent Schrödinger equation in the <u>momentum basis</u>.

(3 marks)

(c) Consider the following transformation on a qubit: $|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|1\rangle \to |0\rangle$. Show that this transformation cannot be accomplished by a unitary quantum gate.

(4 marks)

(d) Suppose we have two linearly-independent eigenstates of a Hamiltonian $\hat{\mathcal{H}}$ corresponding to a dengenerate eignenvalue ε: $\hat{\mathcal{H}}\,|\phi_1\rangle = \varepsilon\,|\phi_1\rangle$ and $\hat{\mathcal{H}}\,|\phi_2\rangle = \varepsilon\,|\phi_2\rangle$ where $\langle\phi_1|\,\phi_2\rangle = 0$. Construct two Hermitian operators that both commute with the Hamiltonian but not with each other. That is find Hermitian $\hat{A}$ and $\hat{B}$ such that $[\hat{A}, \hat{\mathcal{H}}] = [\hat{B}, \hat{\mathcal{H}}] = 0$ and $[\hat{A}, \hat{B}] \neq 0$.

(5 marks)

(e) Suppose that the time-dependent Schrödinger equation is satisfied by $|\psi\rangle$: $i\hbar\partial_t\,|\psi\rangle = \hat{\mathcal{H}}\,|\psi\rangle$. Now introduce another state related to $|\psi\rangle$ by a unitary transformation: $|\psi'\rangle = \hat{U}\,|\psi\rangle$ where $\hat{U}$ is a time-dependent unitary operator. Show that $|\psi'\rangle$ satisfies the Schrödinger equation $i\hbar\partial_t\,|\psi'\rangle = \hat{\mathcal{H}}'\,|\psi'\rangle$ where the Hermitian operator $\hat{\mathcal{H}}'$ should be expressed in terms of $\hat{\mathcal{H}}'$ and $\hat{U}$.

(4 marks)

(Total: 20 marks)

5. In this problem, we consider a collection of Bosons in a one-dimensional lattice with N lattice sites. We take $\hat{a}_n^\dagger$ to create a boson on site n. The bosonic operators satisfy the usual algebra $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}$. For notational simplicity, we will use the convention $\hat{a}_n = \hat{a}_{n+N}$.

In this problem we will also introduce the Fourier transformed operators $\hat{a}_k$ where the values of k are always restricted to be integer multiples of $2\pi/N$. For these we use a similar convention: $\hat{a}_{k+2\pi} = \hat{a}_k$.

(a) Introduce the Fourier transformed operators $\hat{a}_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} \hat{a}_n$. Show that these operators satisfy

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = f(k - k')$$

where $f(k) = \frac{1}{N} \sum_{n=1}^{N} e^{ikn}$.

(3 marks)

(b) Show that $f(k) = 1$ when k is an integer multiple of 2π and zero otherwise. Hint: recall or derive an expression for the geometric series $x + x^2 + \ldots + x^N$. Hence $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$.

(4 marks)

(c) A "Bogoliubov" Hamiltonian describing collective excitations of a superfluid is given by

$$\hat{\mathcal{H}} = \sum_{n=1}^{N} \Big((g + 2w)\hat{a}_n^\dagger \hat{a}_n - w(\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n) + \frac{g}{2}(\hat{a}_n \hat{a}_n + \hat{a}_n^\dagger \hat{a}_n^\dagger) \Big)$$

where w and g are real positive parameters. Show that this Hamiltonian written in terms of the Fourier transformed operators takes on the form

$$\hat{\mathcal{H}} = \sum_k \left(\varepsilon_k \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2} \hat{a}_k \hat{a}_{-k} + \frac{g}{2} \hat{a}_{-k}^\dagger \hat{a}_k^\dagger \right).$$

In the summation above, k is restricted to the range $0 < k \leq 2\pi$. Determine an explicit expressions for ε_k.

(5 marks)

(d) Consider the operator $\hat{\alpha}_k = u_k \hat{a}_k + v_k \hat{a}_{-k}^\dagger$ where u_k and v_k are unspecified real constants. Determine a condition on u_k and v_k that ensures $[\hat{\alpha}_k, \hat{\alpha}_k^\dagger] = 1$.

(3 marks)

(e) Determine the Heisenberg equations of motion for $\hat{\alpha}_k$. For a special choice of u_k and v_k (which you are not asked to find), the equations of motion simplify to $i\hbar \frac{d}{dt}(\hat{\alpha}_k)_H = E_k (\hat{\alpha}_k)_H$ where $E_k > 0$. This $\hat{\alpha}_k^\dagger$ creates a collective excitation of the superfluid which has energy E_k. Determine E_k.

(5 marks)

(Total: 20 marks)

Solutions for Quantum Mechanics II Exam, 2023

1. First order time-independent perturbation theory.

   (a) Suppose we have a solution to the time-independent Schrödinger equation $\hat{\mathcal{H}}_0 \left|\phi^{(0)}\right\rangle = \varepsilon \left|\phi^{(0)}\right\rangle$. Explain carefully but briefly the meaning of the statement: ε is non-degenerate.

   Seen. This means that there is only one linearly-independent eigenstate with eigen-energy ε.

   (b) Take ε of the previous part to be a non-degenerate eigenvalue of $\hat{\mathcal{H}}_0$. Now consider adding a Hermitian perturbation to the Hamiltonian so that the new Hamiltonian is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda \hat{V}$. Here $\hat{V}$ is a Hermitian operator and λ is our usual real perturbative parameter. Let $|\phi\rangle$ be the eigenstate of $\hat{\mathcal{H}}$ that reduces to $\left|\phi^{(0)}\right\rangle$ when $\lambda \to 0$ and let E be the associated eigen-energy, $\hat{\mathcal{H}}|\phi\rangle = E|\phi\rangle$. Show that to first order in λ,

   $$E = \varepsilon + \lambda \left\langle \phi^{(0)} \right| \hat{V} \left| \phi^{(0)} \right\rangle$$

   Similar seen. Expand E and $|\phi\rangle$ in λ: $E = E^{(0)} + E^{(1)}\lambda + \ldots$ and $|\phi\rangle = \left|\phi^{(0)}\right\rangle + \left|\phi^{(1)}\right\rangle \lambda + \ldots$. Insert into SE and look at λ^1 term:

   $$\hat{\mathcal{H}}_0 \left|\phi^{(1)}\right\rangle + \hat{V} \left|\phi^{(0)}\right\rangle = E^{(0)} \left|\phi^{(1)}\right\rangle + E^{(1)} \left|\phi^{(0)}\right\rangle.$$

   Now apply $\left\langle \phi^{(0)} \right|$ to this equation. One finds $E^{(1)} = \left\langle \phi^{(0)} \right| \hat{V} \left| \phi^{(0)} \right\rangle$. Also note that $\varepsilon = E^{(0)}$. Therefore $E = \varepsilon + \lambda \left\langle \phi^{(0)} \right| \hat{V} \left| \phi^{(0)} \right\rangle$

   (c) Consider the infinite square-well Hamiltonian $\hat{\mathcal{H}}_0 = \frac{p^2}{2m} + U(\hat{x})$ where $U(x) = 0$ for $0 \le x \le L$ and $U(x) = \infty$ otherwise. The ground-state wavefunction can be found to be $\phi(x) = \sqrt{\frac{2}{L}} \sin(\pi x/L)$.

   Suppose a perturbation $\lambda V(\hat{x})$ is added to the Hamiltonian where $V(x) = \gamma > 0$ for $0 \le x \le L/2$ and zero otherwise. What is the ground state energy to first order in λ?

   Unseen. The unperturbed ground state energy can be seen to be (e.g. take second derivative of wavefunction)

   $$\varepsilon = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2.$$

   Now we use the formula found in (b) to determine the second order correction. One finds

   $$E^{(1)} = \frac{2\gamma}{L} \int_0^{L/2} \sin^2(\pi x/L) dx = \frac{\gamma}{L} \int_0^{L/2} (1 - \cos(2\pi x/L)) dx = \frac{\gamma}{2}.$$

(d) What can go wrong with the treatment described in (b) if the eigen-energy of $\hat{\mathcal{H}}_0$ is degenerate?

Similar seen. (Answers may vary.) If the eigen-energy is degenerate, there is ambiguity in the basis to use to describe the degenerate subspace. It is easy to see that you can get different answers by naively trying to use the result of (b) for different bases, for instance.

(e) Determine the eigen-energies, to first order in λ of

$$H = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & 1 & \lambda \\ 0 & \lambda & 2 \end{pmatrix}.$$

You may quote and use without derivation any results from lecture or elsewhere.

Unseen. To obtain results that are correct to first order in λ, one may diagonalise H within degenerate subspaces (degenerate perturbation theory). One finds $E_{1,2} = 1 \pm \lambda$ and $E_3 = 2$ for the three eigenvalues.

2. Heisenberg equations of motion.

Consider a particle in one dimension with motion governed by the Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}$.

(a) Show that the Heisenberg equations of motion are given by

$$m\frac{d}{dt}\hat{x}_H = \hat{p}_H$$
$$\frac{d}{dt}\hat{p}_H = 0.$$

Seen. This follows from a direct calculation.

(b) Solve the equations of (a) to express $\hat{x}_H$ and $\hat{p}_H$ in terms of $\hat{x}$ and $\hat{p}$.

Seen. $\hat{x}_H = \hat{x} + \frac{1}{m}\hat{p}t$, $\hat{p}_H = \hat{p}$.

(c) Suppose at $t = 0$, the normalised state in the position basis is given by $\bar{\psi}(x) = e^{ikx}\phi(x)$ where k is real and $\phi(x)$ is an unspecified function of x. Further suppose that $\langle\phi|\,\hat{x}\,|\phi\rangle = \langle\phi|\,\hat{p}\,|\phi\rangle = 0$. Determine the expectation value of the position and momentum operators for later times.

Unseen. One needs to use key relations from the Heisenberg EOM: $\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle = \langle\bar{\psi}|\,\hat{x}_H\,|\bar{\psi}\rangle$ and $\langle\psi(t)|\,\hat{p}\,|\psi(t)\rangle = \langle\bar{\psi}|\,\hat{p}_H\,|\bar{\psi}\rangle$. We can now use the results of (b). We need to then evaluate expectation values of the position and momentum operators.

$$\langle\bar{\psi}|\,\hat{x}\,|\bar{\psi}\rangle = \int dx\phi^*(x)e^{-ikx}x\phi(x)e^{ikx} = \langle\phi|\,\hat{x}\,|\phi\rangle = 0.$$

$$\langle\bar{\psi}|\,\hat{p}\,|\bar{\psi}\rangle = \int dx\phi^*(x)e^{-ikx}(-i\hbar\partial_x)\phi(x)e^{ikx} = \langle\phi|\,\hat{x}\,|\psi\rangle = k\,\langle\phi|\,\phi\rangle + \langle\phi|\,\hat{p}\,|\phi\rangle = \hbar k.$$

Using this, $\langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle = \hbar kt/m$ and $\langle\psi(t)|\,\hat{p}\,|\psi(t)\rangle = \hbar k$.

(d) Further suppose that $\phi(x)$ of the previous part can be taken to be real. Let $\sigma_x^2(t)$ denote the time-dependent variance in position. That is, $\sigma_x^2(t) = \langle\psi(t)|\,\hat{x}^2\,|\psi(t)\rangle - \langle\psi(t)|\,\hat{x}\,|\psi(t)\rangle^2$. Let $\bar{\sigma}_x^2 = \sigma_x^2(t=0)$ be the variance at $t=0$. Determine an expression for $\sigma_x^2(t)$ at later times.

Unseen. The problem is greatly simplified by realising $(\hat{x}_H)^2 = (\hat{x}^2)_H$. Using this, we have

$$\sigma_x^2 = \langle\bar{\psi}|\,(\hat{x}^2 + t(\hat{x}\hat{p} + \hat{p}\hat{x})/m + t^2\hat{p}^2/m^2\,|\bar{\psi}\rangle\,.$$

With some work, one can show that due to the reality of ϕ, the expectation value of $\hat{x}\hat{p} + \hat{p}\hat{x}$ vanishes (details are in notes in Chap. 2). Finally, using the results of (c) we have

$$\sigma_x^2 = \bar{\sigma}_x^2 + \frac{t^2}{m^2}(\bar{\sigma}_p^2 + \hbar^2 k^2).$$

3. Quantum circuits and measurement.

(a) Consider a system composed of two qubits. Write down the computational basis for this system.

Seen. $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$.

(b) Suppose that the system is in the state $|\Psi\rangle = |00\rangle$. Suppose that this system is measured in the two-quibit computational basis. Determine the probabilities of each of the four possible experimental outcomes. What is the state after the measurement?

Unseen. The probability of measuring $|00\rangle$ is one while the others are zero. The state after the measurement is $|00\rangle$.

(c) Alice and Bob carefully take the two qubits described in (b), so they each have a quibit in the state $|0\rangle$. They then send the qubits through the quantum circuit shown below. Alice's qubit enters and exits the top wire of the circuit while Bob's is processed by the bottom wire. Show that the resulting state exiting the circuit is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We say that this state is *entangled*. Briefly describe the meaning of "entanglement" here.

Comments: The components of the circuit follow the notation used in the module. The circuit contains a Hadamard and a CNOT gate.

Similar seen. Let's look at how the initial state transforms as it goes through the circuit. After the Hadamard, the state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$. Sending this state through the CNOT gives $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Entanglement here means that the state is not a computational basis element. Entanglement is a basis-dependent property.

(d) Upon being processed by the quantum circuit of (c), Alice performs a measurement of her qubit in the computational basis while Bob sits idly. What is the probability that Alice will measure $|0\rangle$ for her quibit? What is the probability that she will measure $|1\rangle$?

Suppose that Alice determines her qubit is in state $|0\rangle$.What is the two-qubit state of the system after this measurement? Suppose now that Bob measures his qubit in the computational basis. What is the probability that he measures his state to also be in $|0\rangle$?

Similar seen. The probability that she will measure 0 is $P_0 = \sum_{n=0}^{1} |\langle 0n|\psi\rangle)^2 = 1/2$. A similar calculation shows that the probability of her measuring 1 is also 1/2 as it must be.

Assuming she measures 0, the state collapses to $|\psi\rangle = |00\rangle$. So then Bob is assured to also measure 0 with probability 1.

(e) Finally suppose that the state output from the gate described in (c), $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, is sent through the circuit described below. What is the resulting state?

Unseen. To tackle this part, one can directly work out what the circuit does to the *entangled* state of (b) for full makrs. Perhaps errors are difficult to avoid with this direct approach.

Instead, it is useful to consider this circuit in succession with that of (b). One can note that two CNOTs in succession is the identity. Furthermore two Hadamards in succession is also the identity. So the resulting state will be $|00\rangle$. The gate serves to disentangle the input state.

4. Shorter problems covering different aspects of the module.

(a) Evaluate and simplify the following commutators.

i. $[\hat{x}, \hat{p}^2]$

ii. $[\hat{a}, \hat{a}^\dagger \hat{a}]$ where $\hat{a}$ is a bosonic operator

iii. $[\hat{c}, \hat{c}^\dagger \hat{c}]$ where $\hat{c}$ is a fermionic operator

Mostly seen. $[\hat{x}, \hat{p}^2] = i\hbar\hat{p}$. $[\hat{a}, \hat{a}^\dagger \hat{a}] = \hat{a}.[\hat{c}, \hat{c}^\dagger \hat{c}] = \hat{c}$.

(b) Consider the Harmonic oscillator Hamiltonian $\hat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. Determine (but don't solve) the corresponding time-independent Schrödinger equation in the <u>momentum basis</u>.

Similar seen. $(\frac{p^2}{2m} - \frac{\hbar^2}{2}m\omega^2\partial_p^2)\tilde{\phi}(p) = E\tilde{\phi}(p)$

(c) Consider the following transformation on a qubit: $|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|1\rangle \to |0\rangle$. Show that this transformation cannot be accomplished by a unitary quantum gate.

Unseen. For instance, we can determine the matrix corresponding to this transformation is

$$M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{pmatrix}.$$

A quick check shows that this matrix is not unitary: $M^\dagger M \neq \mathbb{1}$. So this does not correspond to an acceptable quantum gate which need to be unitary.

(d) Suppose we have a two linearly-independent eigenstates of a Hamiltonian $\hat{\mathcal{H}}$ corresponding to a dengenerate eignenvalue ε: $\hat{\mathcal{H}} |\phi_1\rangle = \varepsilon |\phi_1\rangle$ and $\hat{\mathcal{H}} |\phi_2\rangle = \varepsilon |\phi_2\rangle$ where $\langle \phi_1 | \phi_2 \rangle = 0$. Construct two Hermitian operators that both commute with the Hamiltonian but not with each other. That is find Hermitian $\hat{A}$ and $\hat{B}$ such that $[\hat{A}, \hat{\mathcal{H}}] = [\hat{B}, \hat{\mathcal{H}}] = 0$ and $[\hat{A}, \hat{B}] \neq 0$.

Unseen. Through experimentation, one can find that the following satisfy the requirement (answer is not unique):

$$\begin{aligned} \hat{A} &= |\phi_1\rangle \langle \phi_1| - |\phi_2\rangle \langle \phi_2| \\ \hat{B} &= |\phi_1\rangle \langle \phi_2| + |\phi_1\rangle \langle \phi_2| . \end{aligned}$$

Need to demonstrate this for full marks. It is interesting to compare this with the theorem we proved in Chap 3 addressing conditions for degeneracies.

(e) Suppose that the time-dependent Schrödinger is satisfied by $|\psi\rangle$: $i\hbar\partial_t |\psi\rangle = \hat{\mathcal{H}} |\psi\rangle$. Now introduce another state related to $|\psi\rangle$ by a time-dependent unitary transformation: $|\psi'\rangle = \hat{U} |\psi\rangle$ where $\hat{U}$ is a time-dependent unitary transformation. Show that $|\psi'\rangle$ satisfies the Schrödinger equation $i\hbar\partial_t |\psi'\rangle = \hat{\mathcal{H}}' |\psi'\rangle$ where the Hermitian operator $\hat{\mathcal{H}}'$ is to be determined.

Seen. This was done explicitly in Chap 3 and lecture.

5. In this problem, we consider a collection of Bosons in a one-dimensional lattice with N lattice sites. We take $\hat{a}_n^\dagger$ to create a boson on site n. The bosonic operators satisfy the usual algebra $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}$. For notational simplicity, we will use the convention $\hat{a}_n = \hat{a}_{n+N}$.

In this problem we will also introduce the Fourier transformed operators $\hat{a}_k$ where the values of k are always restricted to be integer multiples of $2\pi/N$. For these we use a similar convention: $\hat{a}_{k+2\pi} = \hat{a}_k$.

(a) Introduce the Fourier transformed operators $\hat{a}_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} \hat{a}_n$. Show that these operators satisfy

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = f(k - k')$$

where $f(k) = \frac{1}{N} \sum_{n=1}^{N} e^{ikn}$.

Seen. Carefully taking the adjoint and working on the expression we have:

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = \frac{1}{N}\sum_{n=1}^{N}\sum_{n'=1}^{N} e^{ikn}e^{-ikn'}[\hat{a}_n, \hat{a}^\dagger_{n'}] \quad = \frac{1}{N}\sum_{n=1}^{N}\sum_{n'=1}^{N} e^{ikn}e^{-ikn'}\delta_{n,n'} = \frac{1}{N}\sum_{n=1}^{N} e^{i(k-k')n}.$$

(b) Show that $f(k) = 1$ when k is an integer multiple of 2π and zero otherwise. Hint: recall or derive an expression for the geometric series $x + x^2 + \ldots + x^N$. Hence $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}$.

Unseen. First we note that if k is a multiple of 2π, then each term in the sum is one. So we have $f(k) = 1$. If k is not an integer multiple of 2π we follow the hint and find

$$f(k) = e^{ik}\frac{1 - e^{ikN}}{1 - e^{ik}}.$$

Since k is always a multiple of $2\pi/N$, we have e^{ikN}. This means that $f(k) = 0$. The the $\hat{a}_k$ operators satisfy Bosonic commutations.

(c) A "Bogoliubov" Hamiltonian describing collective excitations of a superfluid is given by

$$\hat{\mathcal{H}} = \sum_{n=1}^{N}\Big((g + 2w)\hat{a}^\dagger_n\hat{a}_n - w(\hat{a}^\dagger_n\hat{a}_{n+1} + \hat{a}^\dagger_{n+1}\hat{a}_n) + \frac{g}{2}(\hat{a}_n\hat{a}_n + \hat{a}^\dagger_n\hat{a}^\dagger_n)\Big)$$

where w and g are real positive parameters. Show that this Hamiltonian written in terms of the Fourier transformed operators takes on the form

$$\hat{\mathcal{H}} = \sum_k\left(\varepsilon_k\hat{a}^\dagger_k\hat{a}_k + \frac{g}{2}\hat{a}_k\hat{a}_{-k} + \frac{g}{2}\hat{a}^\dagger_{-k}\hat{a}^\dagger_k\right).$$

In the summation above, k is restricted to the range $0 < k \leq 2\pi$. Determine an explicit expressions for ε_k.

Unseen. Working on the various bits of the Hamiltonian,

$$\sum_n \hat{a}^\dagger_n\hat{a}_{n+1} = \sum_{k,k'} e^{ik'}f(-k + k')\hat{a}^\dagger_k\hat{a}_{k'} = \sum_k e^{ik}\hat{a}^\dagger_k\hat{a}_k$$
$$\sum_n \hat{a}^\dagger_n\hat{a}_n = \sum_{k,k'} f(-k + k')\hat{a}^\dagger_k\hat{a}_{k'} = \sum_k \hat{a}^\dagger_k\hat{a}_k.$$
$$\sum_n \hat{a}_n\hat{a}_n = \sum_{k,k'} f(k + k')\hat{a}_k\hat{a}_{k'} = \sum_k \hat{a}_k\hat{a}_{-k}.$$

Using these relations and their adjoints, we obtain the required result with

$$\varepsilon_k = g + 4w\sin^2(k/2).$$

(d) Consider the operator $\hat{\alpha}_k = u_k \hat{a}_k + v_k \hat{a}^\dagger_{-k}$ where u_k and v_k are unspecified real constants. Determine a condition on u_k and v_k that ensures $[\alpha_k, \alpha_k^\dagger] = 1$.

Unseen.

An explicit calculation shows that we need $u_k^2 - v_k^2 = 1$.

(e) Determine the Heisenberg equations of motion for $\hat{\alpha}_k$. For a special choice of u_k and v_k (which you are not asked to find), the equations of motion simplify to $i\hbar \frac{d}{dt}(\hat{\alpha}_k)_H = E_k (\hat{\alpha}_k)_H$ where $E_k > 0$. This $\hat{\alpha}_k^\dagger$ creates a collective excitation of the superfluid which has energy E_k. Determine E_k.

Unseen. Turn the crank to find the Heisenberg EOM for $\hat{a}_k$ and $\hat{a}^\dagger_{-k}$. Combine to find:

$$i\hbar \frac{d}{dt}(\hat{\alpha}_k)_H = (u_k \tilde{\varepsilon}_k - v_k g)(\hat{a}_k)_H + (u_k g - v_k \tilde{\varepsilon}_k)(\hat{a}^\dagger_{-k})_H$$

where $\tilde{\varepsilon}_k = \varepsilon_k + g$. Now, if this equation is equal to $E_k \hat{\alpha}_k$ it must be that

$$\begin{pmatrix} \tilde{\varepsilon}_k & -g \\ g & -\tilde{\varepsilon}_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}.$$

This means that E_k is an eigenstate of this matrix. A short calculation gives $E_k = \sqrt{(g + \varepsilon_k)^2 - g^2}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60018/70018	1	Most did well on this problem on perturbation theory. Part (d) was more challenging, but a number of students persevered.
MATH60018/70018	2	Heisenverg EOM question. Vast majority got full marks on (a),(b). Many could work (c) and (d), others had difficulty.
MATH60018/70018	3	Pleasantly surprised by results on this question on quantum circuits. Students were well prepared.
MATH60018/70018	4	This problem covered various topics from module through the sub-problems. Most difficult was (d).
MATH70018	5	Mastery was more challenging, may require scaling. Still some excellent solutions!