IMPERIAL

MATH70054

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Introduction to Stochastic Differential Equations and Diffusion Processes

Date: Tuesday, May 28, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1.

(a) Let $\mathbf{W}_t := \mathbf{W}(t, \omega)$ denote a $d-$dimensional Brownian motion. Use Kolmogorov's continuity theorem to prove that for almost all ω and any $T > 0$, the sample path $t \mapsto \mathbf{W}(t, \omega)$ is uniformly Hölder continuous on $[0, T]$ for every $\gamma \in (0, \frac{1}{2})$.

(4 marks)

(b) Let $\mathbf{W}_t$ be a standard $d-$dimensional Brownian motion and let Q be an orthogonal $d \times d$ real matrix. Show that $\mathbf{B}_t = Q\mathbf{W}_t$ is also a Brownian motion.

(4 marks)

(c) Let W_t be a standard one-dimensional Brownian motion, i.e. a continuous-time Gaussian process with almost surely continuous paths, mean $\mathbb{E}W_t = 0$ and covariance $\mathbb{E}(W_t W_s) = \min(t, s)$.

(i) Show that the function $R(t, s) = \min(t, s)$ with $t, s \geq 0$ is symmetric and nonnegative definite.

(ii) Show that the one dimensional stochastic process W_t with almost surely continuous paths is a standard one-dimensional Brownian motion if and only if $W_0 = 0$ a.s. and it has independent increments with $W_t - W_s \sim \mathcal{N}(0, t - s)$ for all $t \geq s \geq 0$.

(12 marks)

(Total: 20 marks)

2. Let W_t be a standard one dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t^W$ denote the natural filtration of W_t.

(a) Use Itô's formula to show that

$$dW_t^2 = dt + 2W_t\, dW_t.$$

Use this to show that for $n = 1, 2, \ldots, N$,

$$\int_{n\Delta t}^{(n+1)\Delta t} \int_{n\Delta t}^{s} dW_\ell dW_s = \frac{1}{2}\Big(\Delta W_n^2 - \Delta t\Big),$$

where $\Delta W_n = W_{(n+1)\Delta t} - W_{n\Delta t}$.

(8 marks)

(b) Let $N\Delta t = T$, $t_n = n\Delta t$, $n = 0, \ldots N$, $\lambda \in [0,1]$ and $\tau_n^\lambda = (1-\lambda)t_n + \lambda t_{n+1}$. Define

$$I_T^\lambda := \int_0^T W_s \circ^\lambda dW_s = \lim_{N \to +\infty} \sum_{n=0}^{N-1} W(\tau_n^\lambda)\Delta W_n, \quad \text{in } L^2(\Omega),$$

where $\Delta W_k = W_{t_{n+1}} - W_{t_n}$. Calculate I_T^λ and its mean and variance. (8 marks)

(c) Let $\lambda > 0$ arbitrary. Show that $M_t^\lambda := e^{\lambda W_t - \frac{\lambda^2}{2}t}$ is a martingale with respect to the natural filtration of Brownian motion.

(4 marks)

(Total: 20 marks)

3.

(a) Define

$$B_t = W_t - tW_1, \quad t \in [0,1].$$

Show that B_t is a Gaussian process and calculate the mean and the covariance function. Show that, for $t \in [0,1)$ an equivalent definition of B_t is through the formula

$$B_t = (1-t)W\left(\frac{t}{1-t}\right).$$

(6 marks)

(b) Consider the Itô SDE

$$dX_t = \frac{1-X_t}{1-t}\,dt + dW_t, \quad X_0 = 0, \tag{1}$$

where W_t is a standard one dimensional Brownian motion and $t \in (0,1)$.

1. Use an integrating factor or otherwise to show that the solution of (1) is

$$X_t = t + (1-t)\int_0^t \frac{1}{1-s}\,dW_s. \tag{2}$$

(8 marks)

2. Calculate the mean, variance and covariance function of X_t.

(6 marks)

(Total: 20 marks)

4. Let W_t be a standard one-dimensional Brownian motion.

(a) Show that W_t is a Markov process and give the formula of the corresponding Markov semigroup.

(6 marks)

(b) Let X_t be solution of the stochastic differential equation

$$\mathrm{d}X_t = -\alpha X_t \, dt + \sqrt{2\sigma}\, dW_t, \tag{3}$$

where $\alpha, \sigma > 0$ are constants and the initial condition $X(0) = x \in \mathbb{R}$ is deterministic.

1. Show that the solution of (3) can be written in the form

$$X_t = e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t} W_{\left(\frac{1}{2\alpha}(e^{2\alpha t}-1)\right)}. \tag{4}$$

(3 marks)

2. Study the limit of X_t as $t \to +\infty$. (3 marks)
3. Obtain a formula for the Markov semigroup $(P_t f)(x) = \mathbb{E}(f(X_t)|X_0 = x)$ for all continuous, bounded functions f. Assume that f is differentiable. Use the formula for the Markov semigroup to show that

$$\frac{d}{dx}(P_t f)(x) = e^{-\alpha t} \left(P_t \frac{df}{dx} \right)(x).$$

(8 marks)

(Total: 20 marks)

5.

(a) Let X_t be a diffusion process on $[0,1]^d$ with periodic boundary conditions. The drift vector is a periodic function $a(x)$ that is divergence-free ($\nabla \cdot a(x) = 0$), and the diffusion matrix is $2DI$, where $D > 0$ and I is the identity matrix.

1. Show that X_t is an ergodic diffusion process and find the invariant distribution.

(4 marks)

2. Assume that the initial distribution, $X_0 \sim \rho_0(x)$, satisfies $\rho_0 \in L^2([0,1]^d)$. Use Poincaré's inequality to show that the probability density function $\rho(x,t)$, $X_t \sim \rho(x,t)$ converges exponentially fast to the invariant distribution in L^2. (6 marks)

(b) Let X_t be the solution of the one dimensional Stratonovich stochastic differential equation

$$dX_t = b(X_t)\,dt + \sqrt{2\Sigma(X_t)} \circ dW_t, \tag{5}$$

with smooth drift and diffusion coefficients $b(x)$ and $\Sigma(x) > 0$, respectively.

1. Transform (5) to an Itô SDE by calculating the Stratonovich-to-Itô correction. (2 marks)
2. Assume that the drift coefficient is given by the formula $b(x) = -\Sigma(x)\phi'(x) + \frac{1}{2}\Sigma'(x)$, where $\phi(x)$ is a smooth functions with $\int_{\mathbb{R}} e^{-\phi(x)}\,dx < +\infty$ and where the prime denotes differentiation with respect to x. Show that the generator of the Stratonovich SDE can be written as
$$\mathcal{L}f = e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\frac{df}{dx}\right),$$
for all $f \in C^2(\mathbb{R})$. (4 marks)
3. Show that the process X_t defined in Part (ii) is ergodic and obtain a formula for the invariant distribution.

(4 marks)

(Total: 20 marks)

Introduction to SDEs and Diffusion Processes. Solutions May 2024.

1. (a) We calculate, with $r = t - s > 0$,

$$\begin{aligned}\mathbb{E}(|W_t - W_s|^{2m}) &= \frac{1}{(2\pi r)^{d/2}} \int_{\mathbb{R}^d} |x|^{2m} e^{-\frac{|x|^2}{2r}}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} r^m \int_{\mathbb{R}^m} |y|^{2m} e^{-\frac{|y|^2}{2}}\, dy \quad \left(y = \frac{x}{\sqrt{r}}\right) \\ &= Cr^m = C|t-s|^m,\end{aligned}$$

for some (explicitly computable) constant C. we apply now Kolmogorov's theorem (in the form $\mathbb{E}|X_t - X_s|^\beta \leq C|t-s|^{1+\alpha}$) with $\beta = 2m$ and $\alpha = m-1$. We conclude that the multidimensional Brownian motion is Hölder continuous with exponent

$$0 < \gamma < \frac{\alpha}{\beta} = \frac{1}{2} - \frac{1}{2m}.$$

[4] MARKS –A

(b) (There are several different proofs of this result) We check that $\mathbf{B}_t$ and $\mathbf{W}_t$ have the same finite dimensional distributions:

$$\begin{aligned}&\mathbb{P}[B_{t_1} \in F_1, \dots, B_{t_k} \in F_k] = \mathbb{P}[W_{t_1} \in Q^T F_1, \dots, W_{t_k} \in Q^T F_k] \\ &= \int_{Q^T F_1 \cdots \times Q^T F_k} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \dots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \dots dx_k \\ =& \int_{F_1 \cdots \times F_k} p(t_1, 0, y_1) p(t_2 - t_1, y_1, y_2) \dots p(t_k - t_{k-1}, y_{k-1}, y_k)\, dy_1 \dots dy_k \\ =& \ \mathbb{P}[W_{t_1} \in F_1, \dots, W_{t_k} \in F_k].\end{aligned}$$

In the above we have made the change of variables $y_j = Qx_j$ and we have used the fact that, since Q is an orthogonal transformation, $|Qx_j - Qx_{j-1}|^2 = |x_j - x_{j-1}|^2$.

[4] MARKS –A

(c) (i) The function $R(t,s) = \min(t,s)$ is clearly symmetric. To show that it is positive definite, we calculate, using the notation $\mathbf{1}_A(s)$ for the characteristic function of the set A and for $a_1 \dots a_n \in \mathbb{R}$ and $t_1 \dots t_n \geq 0$:

$$\begin{aligned}\sum_{i,j=1}^n a_i a_j \min(t_i, t_j) &= \sum_{i,j=1}^n a_i a_j \int_0^\infty \mathbf{1}_{[0,t_i]}(s) \mathbf{1}_{[0,t_j]}(s)\, ds \\ &= \int_0^\infty \left(\sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}(s) \right)^2 ds.\end{aligned}$$

[4] MARKS –A

(ii) We start by showing that the axioms defining Brownian motion imply the second characterisation of BM. First, since $\mathbb{E}W_0 = 0$, $\mathbb{E}(W_0^2) = 0$ and W_t is a Gaussian process, we deduce that $W_0 = 0$ a.s. Second, Let $0 = t_0 \leq t_1 \leq \dots t_n$ and set $\xi_n = W_{t_{n+1}} - W_{t_n}$, $n = 0, 1, \dots$

The random vector $\boldsymbol{\xi}^n = (\xi_1, \dots \xi_n)$ is a linear transformation of Gaussian random variables and therefore it is Gaussian. To see this, take $\xi_k,\ \xi_m$ with $k,\ m \in \{1, 2 \dots n\}$ and let $\lambda_1,\ \lambda_2 \in \mathbb{R}$ arbitrary. Then

$$\begin{aligned}\lambda_1\,\xi_k + \lambda_2\,\xi_m &= \lambda_1(W_{t_{k+1}} - W_{t_k}) + \lambda_2(W_{t_{m+1}} - W_{t_m}) \\ &= \lambda_1 W_{t_{k+1}} - \lambda_1 W_{t_k} + \lambda_2 W_{t_{m+1}} - \lambda_2 W_{t_m},\end{aligned}$$

and similarly for an arbitrary number of intervals. Now we show that $\xi_k,\ \xi_m$ with $k \neq m$ are uncorrelated (take $k \geq m+1$ wlog):

$$\begin{aligned}\mathbb{E}(\xi_k\xi_m) &= \mathbb{E}\Big((W_{t_{k+1}} - W_{t_k})(W_{t_{m+1}} - W_{t_m})\Big) \\ &= \mathbb{E}(W_{t_{k+1}} W_{t_{m+1}}) - \mathbb{E}(W_{t_{k+1}} W_{t_m}) - \mathbb{E}(W_{t_k} W_{t_{m+1}}) + \mathbb{E}(W_{t_k} W_{t_m}) \\ &= t_{m+1} - t_m - t_{m+1} + t_m = 0.\end{aligned}$$

This shows that $\xi_k,\ \xi_m$ are uncorrelated and, since they are Gaussian, they are also independent. It remains to calculate the variance of the Gaussian random variable $\xi_k,\ k = 1, \dots n$. We calculate:

$$\begin{aligned}\mathbb{E}(\xi_k)^2 &= \mathbb{E}(W_{t_{k+1}} - W_{t_k}))^2 \\ &= \mathbb{E}(W_{t_{k+1}})^2 - 2\mathbb{E}(W_{t_{k+1}} W_{t_k}) + \mathbb{E}(W_{t_k})^2 \\ &= t_{k+1} - t_k.\end{aligned}$$

This completes the proof of the first part.[1]

Now we proceed with proving the converse statement. First, we show that W_t is a Gaussian process:

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k\, W_{t_k} &= \lambda_n(W_{t_n} - W_{t_{n-1}}) + (\lambda_n + \lambda_{n-1})W_{t_{n-1}} + \sum_{k=1}^{n-2} \lambda_k\, W_{t_k} \\ &= \dots \\ &= \sum_{k=1}^{n} \rho_k\,(W_{t_k} - W_{t_{k-1}}) = \sum_{k=1}^{n} \rho_k\,\xi_k,\end{aligned}$$

for appropriately chosen constants ρ_k. Therefore, it is a linear combination of the Gaussian random variables ξ_k and, hence, it is Gaussian. Since the vector $(W_{t_1}, \dots W_{t_n})$ is Gaussian for arbitrary n, it follows that W_t is a Gaussian process. Now we calculate the mean and covariance. Since $W_t - W_s \sim \mathcal{N}(0, t-s)$ and $W_0 = 0$ a.s. we have that $W_t \sim \mathcal{N}(0, t)$. We clearly have that $\mathbb{E}W_t = 0$. Furthermore, with $t \geq s$, and since the process has independent increments with variance $t - s$:

$$\begin{aligned}\mathbb{E}(W_t W_s) &= \mathbb{E}((W_t - W_s + W_s)W_s) = \mathbb{E}((W_t - W_s)W_s) + \mathbb{E}(W_s^2) \\ &= \mathbb{E}((W_t - W_s)(W_s - W_0)) + \mathbb{E}(W_s - W_0)^2 \\ &= 0 + s = \min(t, s),\end{aligned}$$

[1] We could have also shown the independence property of the increments directly by calculating the characteristic function

$$\mathbb{E}e^{(i\sum_k \lambda_k \xi_k)} = e^{-\frac{1}{2}\sum_k \lambda_k^2 (t_{k+1} - t_k)} = \prod_k \mathbb{E}e^{i\lambda_k \xi_k}.$$

and the proof is complete. **[8] MARKS –A**

[12] MARKS –D

2. (a)

(b) The generator of Brownian motion is $\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2}$.From Itô's formula we have

$$df(W_t) = (\mathcal{L}f)(W_t)\,dt + f'(W_t)\,dW_t.$$

We apply this to $f(x) = x^2$ to deduce

$$dW_t^2 = dt + 2W_t\,dW_t. \tag{1}$$

Furthermore,

$$\int_{n\Delta t}^{(n+1)\Delta t}\int_{n\Delta t}^{s} dW_\ell dW_s = \int_{n\Delta t}^{(n+1)\Delta t} W_s\,dW_s - W_{n\Delta t}\Big(W_{(n+1)\Delta t} - W_{n\Delta t}\Big).$$

From (1) we deduce that

$$\int_{n\Delta t}^{(n+1)\Delta t} W_s\,dW_s = \frac{1}{2}\Big(W_{(n+1)\Delta t}^2 - W_{n\Delta t}^2\Big) - \frac{1}{2}\Delta t.$$

We combine these two equations to obtain

$$\begin{aligned}
\int_{n\Delta t}^{(n+1)\Delta t}\int_{n\Delta t}^{s} dW_\ell dW_s &= \frac{1}{2}\Big(W_{(n+1)\Delta t}^2 - W_{n\Delta t}^2\Big) - \frac{1}{2}\Delta t - W_{n\Delta t}\Big(W_{(n+1)\Delta t} - W_{n\Delta t}\Big) \\
&= \frac{1}{2}W_{(n+1)\Delta t}^2 + \frac{1}{2}W_{n\Delta t}^2 - \frac{1}{2}\Delta t - W_{n\Delta t}\,W_{(n+1)\Delta t} \\
&= \frac{1}{2}\big(\Delta W_n^2 - \Delta t\big).
\end{aligned}$$

[8] MARKS –B

(c) We want to prove that

$$\begin{aligned}
&\lim_{n\to+\infty}\sum_k W\left((1-\lambda)t_k + \lambda t_{k+1}\right)(W_{k+1} - W_k) \\
&= \frac{W_T^2}{2} + \left(\lambda - \frac{1}{2}\right) T \text{ in } L^2(\Omega)
\end{aligned}$$

We need to show that

$$R_n^\lambda = \frac{W_T^2}{2} - \frac{1}{2}\sum_k (\Delta W_k)^2 + \sum_k \Big(W(\tau_k^\lambda) - W_k\Big)^2 + \sum_k \Big(W_{k+1} - W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda) - W_k\Big) \tag{2}$$

The second and third term of (2) converges to these limits:

$$\frac{1}{2}\sum_k (\Delta W_k)^2 \to \frac{T}{2} \text{ in } L^2(\Omega) \quad \text{(quadratic variation of Brownian motion)}$$

$$\begin{aligned}
\sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 &= \sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 \\
&\approx \sum_k \left(\tau_k^\lambda - t_k\right) \\
&= \lambda \sum_k (t_{k+1} - t_k) \\
&\to \lambda T \text{ in } L^2(\Omega)
\end{aligned}$$

For the last term of (2),

$$\mathbb{E}\left[\sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 + \sum_k \left(W_{k+1} - W(\tau_k^\lambda)\right)\left(W(\tau_k^\lambda) - W_k\right)\right]^2 \tag{3}$$

$$= \sum_k \mathbb{E}\left(W_{k+1} - W(\tau_k^\lambda)\right)^2 \mathbb{E}\left(W(\tau_k^\lambda) - W_k\right)^2 \tag{4}$$

$$= \sum_k (1-\lambda)(t_{k+1} - t_k)\lambda(t_{k+1} - t_k) \tag{5}$$

$$\leq \lambda(1-\lambda)T\,|P^n| \to 0 \text{ in } L^2(\Omega) \tag{6}$$

$$\implies \mathbb{E}\left[\sum_k \left(W(\tau_k^\lambda) - W_k\right)^2 + \sum_k \left(W_{k+1} - W(\tau_k^\lambda)\right)\left(W(\tau_k^\lambda) - W_k\right)\right]^2 \to 0 \text{ in } L^2(\Omega) \tag{7}$$

where (4) follows from the fact the Brownian motion has independent increments.
Hence,

$$\begin{aligned}
R_n^\lambda &\to \frac{W_T^2}{2} - \frac{T}{2} + \lambda T \\
&= \frac{W_T^2}{2} + \left(\lambda - \frac{1}{2}\right) T \text{ in } L^2(\Omega)
\end{aligned}$$

To show (2), we need to show that

$$R_n^\lambda = \sum_k W(\tau_k^\lambda)(W_{k+1} - W_k)$$

$$
\begin{aligned}
R_n^\lambda = \frac{W_T^2}{2} - \frac{1}{2}\sum_k (\Delta W_k)^2 + \sum_k \Big(W(\tau_k^\lambda) - W_k\Big)^2 + \sum_k \Big(W(\tau_k^\lambda) - W_k\Big)^2 \\
+ \sum_k \Big(W_{k+1} - W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda) - W_k\Big) \\
= \sum_k \left[\frac{W_{k+1}^2}{2} - \frac{W_k^2}{2} - \frac{1}{2}W_{k+1}^2 + W_k W_{k+1} - \frac{1}{2}W_k^2\right] \\
+ \sum_k \Big(W_{k+1} - W(\tau_k^\lambda)\Big)\Big(W(\tau_k^\lambda) - W_k\Big) \\
= \sum_k \Big[-W(\tau_k^\lambda)W_k + W_{k+1}W(\tau_k^\lambda)\Big] \\
= \sum_k W(\tau_k^\lambda)(W_{k+1} - W_k)
\end{aligned}
$$

The mean of I_T^λ is

$$
\mathbb{E}I_T^\lambda = \frac{T}{2} + \left(\lambda - \frac{1}{2}\right)T = \frac{\lambda}{T}.
$$

The variance is

$$
\mathbb{E}(I_T^\lambda - \mathbb{E}I_T^\lambda)^2 = \frac{\mathbb{E}W_T^4}{4} + \frac{T^2}{4} - \frac{T\mathbb{E}W_T^2}{2} = \frac{T^2}{2}.
$$

[8] MARKS –C

(d) First we immediately see that M_t^λ is $\mathcal{F}_t-$measurable, where $\mathcal{F}_t$ is the natural filtration generated by Brownian motion. Furthermore, we have that $\mathbb{E}|M_t| < +\infty$ and therefore $M_t^\lambda \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Now we calculate, using the fact that Brownian motion has independent increments with $W_t - W_s \sim \mathcal{N}(0, t-s)$ and the formula for the characteristic function of a Gaussian random variable:

$$
\begin{aligned}
\mathbb{E}\big(M_t^\lambda \,|\, \mathcal{F}_t\big) &= \mathbb{E}\big(e^{\lambda W_t - \frac{\lambda^2}{2}t} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda W_t - \frac{\lambda^2}{2}t - \lambda W_s + \lambda W_s} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda(W_t - W_s) - \frac{\lambda^2}{2}(t-s)} e^{\lambda W_s - \frac{1}{2}\lambda^2 s} \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}\big(e^{\lambda(W_t - W_s)} \,|\, \mathcal{F}_t\big) e^{\lambda W_s - \frac{1}{2}\lambda^2 s} e^{-\frac{1}{2}\lambda^2(t-s)} \\
&= e^{\frac{1}{2}\lambda^2(t-s)} M_s^\lambda e^{-\frac{1}{2}\lambda^2(t-s)} \\
&= M_s^\lambda.
\end{aligned}
$$

(e) **[4] MARKS –A**

3. (a) Gaussianity of B_t follows from the Gaussianity of $W(t)$ and the fact that $B(t)$ is constructed as a linear operation on a Gaussian process. The mean of B_t is

$$
\mathbb{E}B(t) = \mathbb{E}W(t) - t\mathbb{E}W(1) = 0.
$$

The covariance function is

$$\begin{aligned}\mathbb{E}(B(t)B(s)) &= \mathbb{E}(W(t)W(s)) - s\mathbb{E}(W(t)W(1)) - t\mathbb{E}W(1)W(s) + ts\mathbb{E}W(1)^2 \\ &= \min(t,s) - ts.\end{aligned}$$

The process

$$\hat{B}(t) = (1-t)W\left(\frac{t}{1-t}\right)$$

is Gaussian since $W(t)$ is Gaussian and $f(t) = \frac{t}{1-t}$ is a strictly increasing function on $(0,1)$. Therefore, in order to show that $\hat{B}(t)$ is equivalent to $B(t)$ it is sufficient to check that it has the same mean and covariance. We have that

$$\mathbb{E}\hat{B}(t) = 0$$

and

$$\begin{aligned}\mathbb{E}(\hat{B}(t)\hat{B}(s)) &= (1-t)(1-s)\min(\frac{t}{1-t}, \frac{s}{1-s}) \\ &= (1-\max(t,s))\min(t,s) \\ &= \min(t,s) - ts,\end{aligned}$$

since $f(t) = \frac{t}{1-t}$ is a strictly increasing function on $(0,1)$.

[6] MARKS SEEN SIMILAR–B

(b) (i) The SDE is

$$dX_t = \frac{1-X_t}{1-t}\,dt + dW_t, \quad X_0 = 0, \tag{8}$$

We rearrange terms:

$$dX_t - \frac{1-X_t}{1-t}\,dt = dW_t.$$

The integrating factor is

$$\exp\left(\int_0^t \frac{1}{1-s}\,ds\right) = \frac{1}{1-t}.$$

We multiply the equation through by $\frac{1}{1-t}$:

$$\frac{1}{1-t}\,dX_t - \frac{1-X_t}{(1-t)^2}\,dt = \frac{1}{1-t}\,dW_t.$$

The left hand side becomes an exact differential:

$$d\left(\frac{1}{1-t}X_t\right) = \frac{1}{(1-t)^2}\,dt + \frac{1}{1-t}\,dW_t.$$

We integrate from 0 to t to obtain

$$\frac{1}{1-t}X_t = \frac{1}{1-t} - 1 + \int_0^t \frac{1}{1-s}\,dW_s$$

We multiply through by $1-t$ and simplify the right hand side to obtain

$$X_t = t + (1-t)\int_0^t \frac{1}{1-s}\,dW_s. \tag{9}$$

[8] MARKS –D

(ii) We take the expectation in (9) to obtain

$$\mathbb{E}X_t = t.$$

we now calculate the variance, using Itô's formula:

$$\begin{aligned}
\text{Var}(X_t) &= \mathbb{E}(X_t - \mathbb{E}X_t)^2 = \mathbb{E}\left((1-t)\int_0^t \frac{1}{1-s}\,dW_s\right)^2 \\
&= (1-t)^2 \int_0^t \left(\frac{1}{1-t}\right)^2 dt \\
&= (1-t)t.
\end{aligned}$$

Finally, we calculate the autocorrelation function, assuming, without loss of generality, that $t \geq s$, and using Itô's formula:

$$\begin{aligned}
C(X_t, X_s) &= \mathbb{E}((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)) \\
&= \mathbb{E}\left((1-t)(1-s)\int_0^t \frac{1}{1-\ell}\,dW_\ell \int_0^s \frac{1}{1-\rho}\,dW_\rho\right) \\
&= (1-t)(1-s)\int_0^s \frac{1}{(1-\rho)^2}\,d\rho \\
&= (1-t)s = s - ts.
\end{aligned}$$

From symmetry, we deduce

$$C(X_t, X_s) = \min(s,t) - st.$$

[6] MARKS –C

4. (a) First we give the definition of a Markov process (students are not asked to do this). Let $\{\mathcal{F}_t^X\}_{t\in[0,+\infty)}$ denote the filtration generated by $\{X_t\}_{t\in[0,+\infty)}$. The process is Markov if, for all bounded Borel functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $t,\, h \geq$, we have

$$\mathbb{E}f((X_{t+h})|\mathcal{F}_h^X) = \mathbb{E}(f(X_{t+h})|X_h).$$

Let W_t be a standard Brownian motion with natural filtration $\mathcal{F}_t$. Let f be a bounded Borel function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $t,\, s \geq 0$. The Markov property of Brownian motion follows from the fact that Brownian motion has independent increments:

$$\mathbb{E}(f(W_{t+s})|\mathcal{F}_s) = \mathbb{E}(f(W_{t+s} - W_s + W_s)|\mathcal{F}_s) = \mathbb{E}(f(W_{t+s})|W_s).$$

For $x \in \mathbb{R}$, we have

$$\mathbb{E}(f(W_{t+s})|W_s = x) = \mathbb{E}(f(W_{t+s} - W_s + W_s)|W_s = x) = \mathbb{E}(f(X_t + x)),$$

where $X_t \sim \mathcal{N}(0,t)$ is a random varialbe independent from W_t. Therefore, the Markov semigroup, applied to a measurable function f is given by

$$\mathbb{E}(f(W_{t+s})|W_s = x) = \int_{\mathbb{R}} f(x+y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\,dy$$

with $P_0 = I$.

[6] MARKS –A

(b) We use the variation of constants formula to obtain the solution of the SDE:

$$X_t = e^{-\alpha t} x + \sqrt{2\sigma} \int_0^t e^{-\alpha(t-s)}\, dW_s.$$

We use the fact that, for a function $\xi(t)$ that satisfies the standard assumptions, there exists a Brownian motion B_t such that

$$\int_0^t \xi_t \, dW_t = B \left(\int_0^t \xi_s^2 \, ds \right).$$

We write the solution of the OU SDE in the form

$$\begin{aligned} X_t &= e^{-\alpha t} x + \sqrt{2\sigma} \int_0^t e^{-\alpha(t-s)}\, dW_s \\ &= e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t} \int_0^t e^{\alpha s}\, dW_s \\ &= e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t} \int_0^t e^{\alpha s}\, dW_s \\ &= e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t}\, W \left(\int_0^t e^{2\alpha s}\, ds \right) \\ &= e^{-\alpha t} x + \sqrt{2\sigma} e^{-\alpha t}\, W \left(\frac{1}{2\alpha} \left(e^{2\alpha t} - 1 \right) \right). \end{aligned}$$

[3] MARKS –D

(c) The process X_t is Gaussian, since it can be written as a linear combination of Gaussian random variables (Riemann approximation of the stochastic integral and Gaussianity of Brownian motion). The mean and variance have the limits $\lim_{t \to +\infty} \mathbb{E} X_t = 0$ and $\lim_{t \to +\infty} \mathrm{Var}(X_t) = \frac{\sigma}{\alpha}$. We conclude that in the limit as $t \to +\infty$, X_t converges to the Gaussian random variable $\mathcal{N}(0, \frac{\sigma}{\alpha})$.

[3] MARKS –A

(d) From the solution of the SDE, we deduce that

$$X_t \sim \mathcal{N} \left(e^{-\alpha t} x, \frac{\sigma}{\alpha} \left(1 - e^{-2\alpha t} \right) \right).$$

Denote by $\gamma_{\mu, \sigma^2}(y)$ the Gaussian function with mean μ and variance σ^2. The law of the process X_t starting at the deterministic point x is $\gamma_{e^{-\alpha t}x, \frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}(y)$. Consequently,

$$\begin{aligned} (P_t f)(x) &= \int f(y) \gamma_{e^{-\alpha t}x, \frac{\sigma}{\alpha}(1-e^{-2\alpha t})}(y)\, dy \\ &= \frac{1}{\sqrt{2\pi \frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}} \int f(y) e^{-\frac{|y - e^{-\alpha t} x|^2}{2 \frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int f \left(e^{-\alpha t} x + \sqrt{\frac{\sigma}{\alpha}(1 - e^{-2\alpha t})} z \right) e^{-\frac{z^2}{2}}\, dz \\ &= \int f \left(e^{-\alpha t} x + \sqrt{\frac{\sigma}{\alpha}(1 - e^{-2\alpha t})} z \right) \gamma(z)\, dz, \end{aligned}$$

where $\gamma(z)$ denotes that standard Gaussian $\gamma_{0,1}(z)$ and we have made the change of variables

$$z = \frac{y - e^{-\alpha t}x}{\sqrt{\frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}}.$$

Assume now that f is differentiable. We can interchange the derivative with respect to x and the integral to calculate, using the notation $h(x,t;z) = e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}z$

$$\begin{aligned}
\frac{d}{dx}(P_t f)(x) &= \frac{d}{dx}\int f\left(e^{-\alpha t}x + \sqrt{\frac{\sigma}{\alpha}(1 - e^{-2\alpha t})}z\right)\gamma(z)\,dz \\
&= \int f'(h(x,t;z))\partial_x h(x,t;z)\gamma(z)\,dz \\
&= e^{-\alpha t}\int f'(h(x,t;z))\gamma(z)\,dz = e^{-\alpha t}\left(P_t \frac{df}{dx}\right)(x).
\end{aligned}$$

[8] MARKS –B

5. (a) (i) We need to find the solution of the stationary Fokker-Planck equation and to show that it is unique. Since $a(x)$ is divergence-free, we have that the Fokker-Planck operator is

$$\mathcal{L}^* = -a(x)\cdot\nabla + D\Delta.$$

The stationary Fokker-Planck equation becomes

$$-a(x)\cdot\nabla\rho + D\Delta\rho = 0$$

on $[0,1]^d$ with periodic boundary conditions. Clearly, 1 is a normalized solution of this boundary value problem. To show that it is unique, we multiply the stationary FP equation by ρ, integrate over $[0,1]^d$ and integrate by parts, using the fact that $\nabla\cdot a(x) = 0$ (which implies that $\int ha\cdot\nabla h\,dx = 0$) to obtain

$$D\int_{[0,1]^d}|\nabla\rho|^2\,dx = 0,$$

from which we deduce that ρ is a constant. Hence, the diffusion process is ergodic and the invariant distribution is

$$\rho(x) = 1.$$

[4] MARKS –B

(ii) Let

$$h(x,t) = p(x,t) - 1.$$

It is a mean zero periodic function and, consequently, it satisfies Poincaré's inequality:

$$\int_{[0,1]^d}|h|^2\,dx \le C\int_{[0,1]^d}|\nabla h|^2\,dx.$$

The function $h(x,t)$ satisfies the Fokker-Planck equation

$$\frac{\partial h}{\partial t} = -a(x)\cdot\nabla h + D\Delta h.$$

We multiply the equation by h integrate by parts on the left hand side, use the fact that $a(x)$ is divergence free and use Poincaré's inequality to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{[0,1]^d}|h|^2\,dx &= -D\int_{[0,1]^d}|\nabla h|^2\,dx \\ &\leq -CD\int_{[0,1]^d}|h|^2\,dx.\end{aligned}$$

Let

$$\eta(t)=\int_{[0,1]^d}|h(x,t)|^2\,dx.$$

From the above calculation we obtain the inequality

$$\frac{d\eta}{dt}\leq -2CD\eta(t),$$

from which we deduce that

$$\eta(t)\leq \eta(0)e^{-2CDt}.$$

Consequently,

$$\int_{[0,1]^d}|p(x,t)-1|^2\,dx\leq e^{-2CDt}\int_{[0,1]^d}|p(x,0)-1|^2\,dx,$$

which shows that the stochastic process X_t converges exponentially fast to its invariant distribution.

[6] MARKS – D

(b) (i) The Stratonovich SDE is

$$\begin{aligned}dX_t &= b(X_t)\,dt+\sqrt{2\Sigma(X_t)}\circ dW_t \\ &= \Big(b(X_t)+\frac{1}{2}\sqrt{2\Sigma(X_t)}(\sqrt{2\Sigma(X_t)})'\Big)\,dt+\sqrt{\Sigma(X_t)}\,dW_t \\ &= \Big(b(X_t)+\frac{1}{2}\Sigma'(X_t)\Big)\,dt+\sqrt{\Sigma(X_t)}\,dW_t\end{aligned}$$

[2] – A

(ii) The generator of the SDE, when transformed to the Itô form, with $b(x)=-\Sigma(x)\phi'(x)+\frac{1}{2}\Sigma'(x)$, applied to a C^2 function f, is

$$\begin{aligned}\mathcal{L}f &= \Big(-\Sigma(x)\phi'(x)+\frac{1}{2}\Sigma'(x)+\frac{1}{2}\Sigma'(x)\Big)\frac{df}{dx}+\Sigma(x)\frac{d^2f}{dx^2} \\ &= \Big(-\Sigma(x)\phi'(x)+\Sigma'(x)\Big)\frac{df}{dx}+\Sigma(x)\frac{d^2f}{dx^2} \\ &= e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\right)\frac{df}{dx}+\Sigma\frac{d^2f}{dx^2} \\ &= e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\frac{df}{dx}\right) \\ &= e^{\phi}\frac{d}{dx}\left(\Sigma e^{-\phi}\frac{df}{dx}\right)\end{aligned}$$

[4] MARKS –C

(iii) First we calculate the Fokker-Planck operator, which is the L^2-adjoint of the generator. From the formula obtained in the previous part, together with two-integrations by parts, we deduce that

$$\int_{\mathbb{R}} (\mathcal{L}f)h \, dx = \int f \frac{d}{dx}\left(e^{-\phi}\Sigma \frac{d}{dx}\left(e^{\phi}h\right)\right) dx,$$

for all $f, \, h \in C_0^2(\mathbb{R})$. Consequently, the Fokker-Planck operator is

$$\mathcal{L}^* h = \frac{d}{dx}\left(e^{-\phi}\Sigma \frac{d}{dx}\left(e^{\phi}h\right)\right). \tag{10}$$

The stationary Fokker-Planck equation is

$$\mathcal{L}^* \rho = 0.$$

From (10) it immediately follows that a solution to this equation is

$$\rho = e^{-\phi}.$$

Since, by assumption, $\int e^{-\phi} \, dx < +\infty$, the solution ρ is normalizable and the invariant distribution is

$$\rho_s(x) = \frac{1}{Z} e^{-\phi}, \quad Z = \int e^{-\phi} \, dx. \tag{11}$$

To show that the process X_t is ergodic, we need to show that the invariant distribution is unique. We multiply the stationary Fokker-Planck equation by ρe^{ϕ} and integrate over $\mathbb{R}$ to obtain, after an integration by parts:

$$\begin{aligned} \int (\mathcal{L}^*\rho)\rho \, dx &= \int \frac{d}{dx}\left(e^{-\phi}\Sigma \frac{d}{dx}\left(e^{\phi}\rho\right)\right) \rho e^{\phi} \, dx \\ &= -\int \Sigma |(e^{\phi}\rho)'|^2 e^{-\phi} \, dx. \end{aligned}$$

Since $\Sigma e^{-\phi} > 0$, we have that

$$\int (\mathcal{L}^*\rho)\rho \, dx = 0$$

if and only if $e^{\phi}\rho = \text{const}$, from which we deduce that the unique normalized invariant distribution is given by (11).

[4] MARKS – D

MATH70054 Introduction to Stochastic Differential Equations and Diffusion Processes

Question	Marker's comment
1	Most students were able to answer this question. There were some issues with the proof of the equivalence between the two definitions of Brownian motion.
2	Many students did well in this question, in particular, parts (a) and (c). There were issues with the calculation of the mean and variance in part (b).
3	Most students did well in this question. Almost all students were able to solve the SDE in part (b) using an appropriate integrating factor.
4	Most students were able to use the independence of increments property of Brownian motion to prove that it is a Markov process. There were also able to obtain the solution formula for the OU SDE and to study its long time behaviour as well as the corresponding Markov semigroup.
5	Several students were able to answer correctly both parts of the Mastery question. There were issues with the correct application of Poincare's inequality for the Lebesgue measure on the torus, and also on the calculation of the invariant measure in part (b).