IMPERIAL

**MATH70015**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**Quantum Mechanics 1**

Date: Thursday, May 9, 2024

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5 hours

**This paper has 5 Questions.**

**Please Answer Each Question in a Separate Answer Booklet**

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. **The postulates of Quantum Mechanics - a spin $\frac{1}{2}$ system**

Consider the matrices

$$\hat{s}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are a representation of the angular momentum operators for the total angular momentum quantum number $j = \frac{1}{2}$ in the standard basis (i.e. the eigenbasis of $\hat{s}_z$), describing, e.g. the spin degree of freedom of a spin $\frac{1}{2}$ particle.

(a) Verify that these matrices indeed fulfil the correct commutation relations to be angular momentum operators. (5 marks)

(b) Consider now a quantum system described by the matrix Hamiltonian

$$\hat{H} = \frac{E_0}{\hbar}\hat{s}_x,$$

where $E_0$ is a real-valued energy constant.

(i) Calculate the eigenvalues and normalised eigenvectors of $\hat{H}$. (5 marks)

(ii) Assume that an $\hat{s}_z$ measurement at time $t_0$ yields the outcome $\frac{\hbar}{2}$. What is the state directly after the measurement? If you were to immediately measure $\hat{H}$, with what probability would you obtain which outcome?

(4 marks)

(iii) Assume that in another experiment an $\hat{s}_z$ measurement at time $t_1$ yields the outcome $\frac{\hbar}{2}$. What is the state at some later time $t > t_1$? If you were to measure $\hat{H}$ after some time at $t > t_1$, with what probability would you obtain which outcome? (6 marks)

(Total: 20 marks)

2. **Quantum harmonic oscillator and the position operator**

Consider the harmonic oscillator Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{I}\right),$$

with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ with $[\hat{a}, \hat{a}^\dagger] = \hat{I}$.

(a) (i) Calculate the commutators of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ with $\hat{a}$ and $\hat{a}^\dagger$, respectively. (3 marks)

(ii) Show that if $|\nu\rangle$ is an eigenvector of $\hat{N}$ with eigenvalue $\nu$, then $\hat{a}|\nu\rangle$ is either an eigenvector of $\hat{N}$ with eigenvalue $\nu-1$ or the zero vector, and $\hat{a}^\dagger|\nu\rangle$ is either an eigenvector of $\hat{N}$ with eigenvalue $\nu+1$ or the zero vector. (4 marks)

(iii) Show that $\hat{a}^\dagger|\nu\rangle$ cannot be the zero vector. (4 marks)

(b) The eigenstates $|x\rangle$ of the position operator $\hat{q}$ belong to the eigenvalues $x \in \mathbb{R}$, such that $\hat{q}|x\rangle = x|x\rangle$. They fulfil the generalised orthonormality condition $\langle y|x\rangle = \delta(x-y)$, and form a resolution of the identity,

$$\int_{-\infty}^{\infty} |x\rangle\langle x|\mathrm{d}x = \hat{I}.$$

In its own eigenbasis the position operator is diagonal, that is, we have

$$\langle y|\hat{q}|x\rangle = x\delta(x-y).$$

(i) Show that the matrix elements in another basis $\{|n\rangle\}$ can be expressed as the integrals

$$\langle n|\hat{q}|m\rangle = \int_{-\infty}^{\infty} x\langle n|x\rangle\langle x|m\rangle\mathrm{d}x.$$

(4 marks)

(ii) Now consider the harmonic oscillator eigenstates $|n\rangle$, with position representation

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}\sqrt{L}}\,\mathrm{e}^{-\frac{x^2}{2L^2}}\,\mathrm{H}_n\left(\frac{x}{L}\right),$$

where $L = \sqrt{\frac{\hbar}{m\omega}}$ and $H_n(u)$ denotes the Hermite polynomial of order $n$.
The Hermite polynomials fulfil the orthogonality relation

$$\int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{H}_n(u)\mathrm{H}_m(u)\mathrm{d}u = \sqrt{\pi}2^n n!\delta_{nm},$$

and the recursion relation

$$\mathrm{H}_{n+1}(u) = 2u\mathrm{H}_n(u) - 2n\mathrm{H}_{n-1}(u).$$

Using the above, calculate an expression for the matrix elements $\langle n|\hat{q}|m\rangle$ in the harmonic oscillator basis, where $|n\rangle$ and $|m\rangle$ are eigenstates of the harmonic oscillator Hamiltonian. (5 marks)

(Total: 20 marks)

## 3. Particle hopping on a lattice

Consider a particle on a one-dimensional discrete lattice, modelled on a Hilbert space spanned by an orthonormal basis $\{|n\rangle\}$, with integer $n$, where the particle is located on the $n$-th lattice site with certainty if the system is in the state $|n\rangle$. We define the "shift" operators

$$\hat{K} = \sum_{n=-\infty}^{+\infty} |n-1\rangle\langle n|, \quad \text{and} \quad \hat{K}^\dagger = \sum_{n=-\infty}^{+\infty} |n+1\rangle\langle n|,$$

and the discrete position operator

$$\hat{N} = \sum_{n=-\infty}^{+\infty} n|n\rangle\langle n|.$$

A Hamiltonian of the form

$$\hat{H} = -\frac{E_0}{2}\left(\hat{K} + \hat{K}^\dagger\right),$$

where $E_0$ is a real positive constant, describes the system of a particle that can hop between neighbouring sites.

(a) Verify that

$$\hat{K}|m\rangle = |m-1\rangle \quad \text{and} \quad \hat{K}^\dagger|m\rangle = |m+1\rangle.$$

(2 marks)

(b) Verify that

$$|k\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-ink}|n\rangle, \quad \text{with} \quad k \in [-\pi, \pi]$$

is an eigenvector of $\hat{K}$ and of $\hat{K}^\dagger$. What are the corresponding eigenvalues? (3 marks)

(c) The vectors $|k\rangle$ are also eigenvectors of the Hamiltonian $\hat{H}$. What are the corresponding energy eigenvalues? Which range of $k$ corresponds to the energy range $[-E_0, 0]$?

(4 marks)

(d) Assume that at time $t = 0$ the system is described by the wave function

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n=0\rangle + |n=1\rangle).$$

(i) With which probability is which outcome obtained in a measurement of the discrete position operator $\hat{N}$? (2 marks)

(ii) What is the expectation value of the energy? (4 marks)

(iii) What is the probability of obtaining an outcome in the interval $[-E_0, 0]$ in an energy measurement? (5 marks)

(Total: 20 marks)

4. **Bound states in a piecewise constant potential**

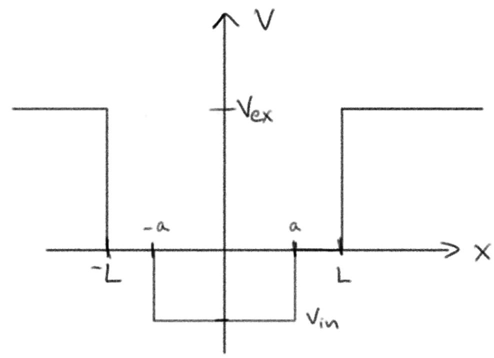
(a) Consider the time-independent Schrödinger equation in position representation for a particle moving in a constant potential $V_0$ in one dimension:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) + V_0\phi(x) = E\phi(x),$$

where $V_0$ is a positive, real energy. Write down the general solution $\phi(x)$. Are there square integrable solutions for any value of $E$? (4 marks)

(b) Now consider a potential of the form

$$V(x) = \begin{cases} V_{\text{ex}}, & x \leq -L \\ 0, & -L < x \leq -a \\ -V_{\text{in}}, & -a < x < a \\ 0, & a \leq x < L \\ V_{\text{ex}}, & x \geq L, \end{cases}$$




where $V_{\text{ex}}$, $V_{\text{in}}$, $a$, and $L$ are real and positive constants, and $a < L$.

(i) What is the possible range of energies for bound states in this potential? Is there a finite or an infinite number of bound states? (3 marks)

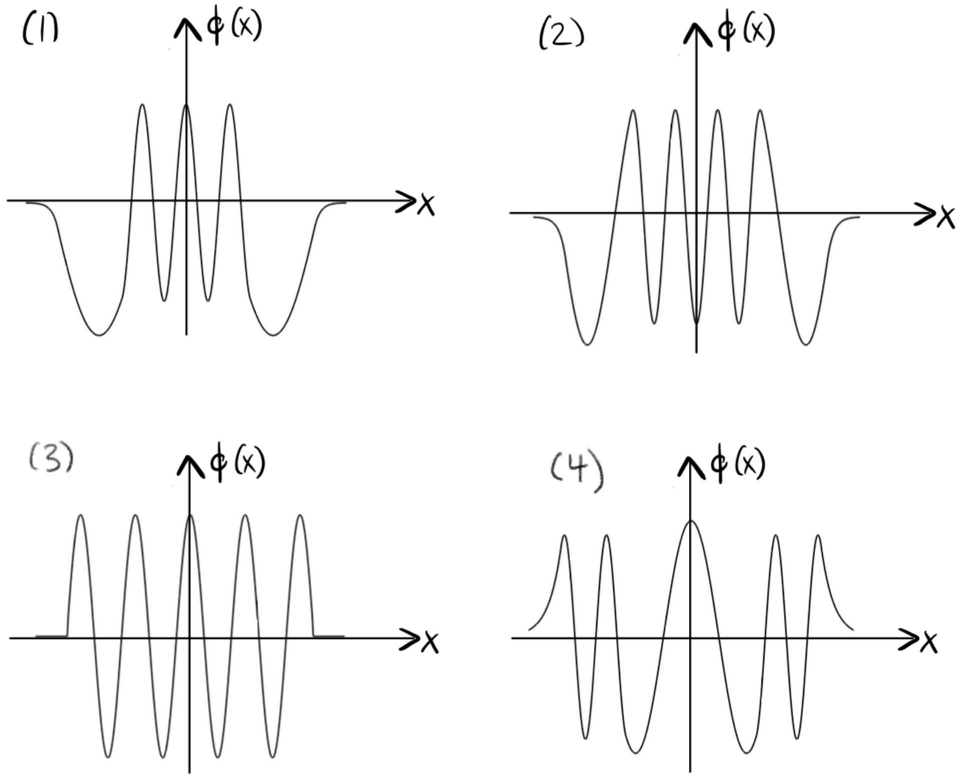
(ii) Which boundary conditions does the wavefunction (and its first derivative) of a bound state of this potential have to fulfil at $x = \pm L$, and $x = \pm a$? (3 marks)

(iii) What can you deduce from the symmetry of the potential? (2 marks)

(iv) Assume there is a bound state with energy $E \in [-V_{\text{in}}, 0]$. What is the corresponding classically allowed region? If a particle is in such a bound-state, is there a finite probability to find it outside of the classically allowed region? (4 marks)

(v) Which of the sketches on the next page might depict the wave function belonging to the bound state of this potential with the ninth highest energy? Provide a reasoning for your answer.

(4 marks)




(Total: 20 marks)

5. **Mastery - Coherent states and the Husimi distribution**

(a) The coherent states $\{|\alpha\rangle\}$ can be defined as eigenstates of the annihilation operator $\hat{a}$ of a harmonic oscillator, as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. The coherent states are not orthonormal, but instead

$$\langle\alpha_1|\alpha_2\rangle = \mathrm{e}^{-\frac{|\alpha_1|^2}{2}-\frac{|\alpha_2|^2}{2}+\alpha_1^*\alpha_2}.$$

For simplicity let us work in units where $\hbar = 1$ and $m\omega = 1$. Then $\hat{a}$ can be expressed in terms of position and momentum operators $\hat{q}$ and $\hat{p}$ as

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{q} + \mathrm{i}\hat{p}\right),$$

and we write

$$\alpha = \frac{1}{\sqrt{2}}(q + \mathrm{i}p), \quad \text{with} \quad q, p \in \mathbb{R}$$

(i) Use the position representation of the eigenvalue equation of $\hat{a}$,

$$\langle x|\hat{a}|\alpha\rangle = \alpha\langle x|\alpha\rangle,$$

to show that the position representation of a coherent state is given by

$$\phi_\alpha(x) := \langle x|\alpha\rangle = \phi_\alpha(0)\,\mathrm{e}^{-\frac{x^2}{2}+(q+\mathrm{i}p)x}.$$

(5 marks)

(ii) Use the normalisation condition to deduce $|\phi_\alpha(0)|$ and show that up to a phase the position representation of the normalised coherent state is given by

$$\phi_\alpha(x) = \frac{1}{\pi^{\frac{1}{4}}}\mathrm{e}^{-\frac{1}{2}(x-q)^2+\mathrm{i}px}.$$

You may find the integral $\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$ useful. (5 marks)

(iii) Consider a superposition of two coherent states with $\alpha_1 = 1$ and $\alpha_2 = -1$,

$$|\psi\rangle = N(|\alpha = 1\rangle + |\alpha = -1\rangle),$$

where $N$ is a real and positive normalisation factor. Calculate $N$ such that the state $|\psi\rangle$ is normalised to one. (5 marks)

(b) The Husimi distribution $Q(p, q)$ of a state $|\psi\rangle$ is defined as

$$Q(\alpha, \alpha^*) = \frac{\langle\alpha|\psi\rangle\langle\psi|\alpha\rangle}{2\pi}.$$

Calculate the Husimi distribution of the first excited harmonic oscillator eigenstate $|n = 1\rangle$. You may use that $|n = 1\rangle = \hat{a}^\dagger|n = 0\rangle$. (5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2024

This paper is also taken for the relevant examination for the Associateship.

# MATH60015/70015

# Quantum Mechanics I (Solutions)

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1. **The postulates of QM - a spin $\frac{1}{2}$ system** sim. seen ⇓

(a) Direct matrix multiplication ($[A,B]=AB-BA$) showing that

$$[\hat{s}_x,\hat{s}_y]=\mathrm{i}\hbar\hat{s}_z,$$

$$[\hat{s}_y,\hat{s}_z]=\mathrm{i}\hbar\hat{s}_x,$$

and

$$[\hat{s}_z,\hat{s}_x]=\mathrm{i}\hbar\hat{s}_y.$$

5, A

(b) (i) A straightforward calculation yields the eiegnvalues seen/sim.seen ⇓

$$\lambda_\pm=\pm\frac{E_0}{2},$$

with the corresponding normalised eigenvectors (up to an overall phase factor) 2, A

$$\phi_+=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\quad\text{and}\quad\phi_-=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

3, A

(ii) According to the state collapse postulate, after the measurement of $\hat{s}_z$ yielding the outcome $\frac{\hbar}{2}$ the state is given by the corresponding eigenstate of $\hat{s}_z$, i.e. sim. seen ⇓

$$\psi(t=t_0)=\begin{pmatrix}1\\0\end{pmatrix}.$$

The possible measurement outcomes of an energy measurement are the eigenvalues $\lambda_\pm$ of $\hat{H}$, with the probabilities 2, B

$$P(\lambda_\pm=|\langle\phi_\pm|\psi(t_0)\rangle|^2=\frac{1}{2}.$$

2, B

(iii) We again have unseen ⇓

$$\psi(t=t_1)=\begin{pmatrix}1\\0\end{pmatrix}.$$

Now, this state is not an eigenstate of the Hamiltonian, and will thus evolve in time. Expressing the state in the eigenbasis of the Hamiltonian

$$\psi(t=t_1)=\frac{1}{\sqrt{2}}\left(\phi_++\phi_-\right),$$

we deduce

$$\psi(t>t_1)=\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{E_0}{2\hbar}(t-t_1)}\phi_++\mathrm{e}^{\mathrm{i}\frac{E_0}{2\hbar}(t-t_1)}\phi_-\right),$$

The probabilities to measure the outcomes $\pm\frac{E_0}{2}$ in an energy measurement are thus given by

$$P(\pm\frac{E_0}{2})=\left|\frac{\mathrm{e}^{\mp\mathrm{i}\frac{E_0}{2\hbar}(t-t_1)}}{\sqrt{2}}\right|^2=\frac{1}{2},$$

that is, while the state is changing in time, the probabilities of the energy measurement are constant in time. 3, B

3, C

2. **Quantum harmonic oscillator and the position operator** seen $\Downarrow$

(a) (i) Using the commutator rule $[AB, C] = A[B, C] + [A, C]B$ we find

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hat{a},$$

and

$$[\hat{N}, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

3, A

(ii) We consider $\hat{N}\hat{a}|n\rangle$. and rewrite

$$\hat{N}\hat{a}|n\rangle = ([\hat{N}, \hat{a}] + \hat{a}\hat{N})|n\rangle.$$

We have

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a}] = -\hat{a}.$$

Thus

$$\hat{N}\hat{a}|n\rangle = (-\hat{a} + \hat{a}\hat{N})|n\rangle = (n-1)\hat{a}|n\rangle.$$

That is, unless the vector $\hat{a}|n\rangle$ is the zero vector, it is an eigenvector of $\hat{N}$ corresponding to the eigenvalue $n-1$. The analogous calculation shows

$$\hat{N}\hat{a}^\dagger|n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger\hat{N})|n\rangle = (n+1)\hat{a}^\dagger|n\rangle.$$

That is, unless the vector $\hat{a}^\dagger|n\rangle$ is the zero vector, it is an eigenvector of $\hat{N}$ corresponding to the eigenvalue $n+1$. 4, A

(iii) For the vector $\hat{a}^\dagger|\nu\rangle$ to be the zero vector its norm would have to vanish. Let us consider the square norm of this state, given by unseen $\Downarrow$

$$||\hat{a}^\dagger|\nu\rangle||^2 = \langle\nu|\hat{a}\hat{a}^\dagger|\nu\rangle.$$

Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{I}$, we find

$$||\hat{a}^\dagger|\nu\rangle||^2 = \langle\nu|\hat{I} + \hat{N}|\nu\rangle = (1+\nu)\langle\nu|\nu\rangle.$$

On the other hand, considering the norm of the state $\hat{a}|\nu\rangle$ we find 2, C

$$||\hat{a}|\nu\rangle||^2 = \langle\nu|\hat{N}|\nu\rangle = \nu\langle\nu|\nu\rangle \geq 0.$$

Now since $|\nu\rangle$ is an eigenstate of $\hat{N}$ it is not the zero vector, and we deduce that $\nu \geq 0$ and thus

$$||\hat{a}^\dagger|\nu\rangle||^2 = (1+\nu)\langle\nu|\nu\rangle > 0.$$

2, D

(b) (i) Inserting two identities we have unseen $\Downarrow$

$$\langle n|\hat{q}|m\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle n|x\rangle\langle x|\hat{q}|y\rangle\langle y|m\rangle \mathrm{d}x\mathrm{d}y.$$

Using that $\langle x|\hat{q}|y\rangle = x\delta(x-y)$ and integrating over $y$ yields the desired result

$$\langle n|\hat{q}|m\rangle = \int_{-\infty}^{\infty} x\langle n|x\rangle\langle x|m\rangle \mathrm{d}x.$$

4, C

(ii) Inserting

$$\langle n|x\rangle = \langle x|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}\sqrt{L}}\,\mathrm{e}^{-\frac{x^2}{2L^2}}\,\mathrm{H}_n\left(\frac{x}{L}\right),$$

into the expression from part (i) yields

$$\langle n|\hat{q}|m\rangle = \frac{1}{\sqrt{\pi}\sqrt{2^n n!}\sqrt{2^m m!}}\int_{-\infty}^{+\infty}\mathrm{d}x\, e^{-\frac{x^2}{L^2}}\,\frac{x}{L}H_n\left(\frac{x}{L}\right)H_m\left(\frac{x}{L}\right).$$

From the recursion relation we know that

$$\frac{x}{L}H_n\left(\frac{x}{L}\right) = \frac{1}{2}H_{n+1}\left(\frac{x}{L}\right) + nH_{n-1}\left(\frac{x}{L}\right)$$

That is,

$$\langle n|\hat{q}|m\rangle = \frac{1}{\sqrt{\pi}\sqrt{2^n n!}\,\sqrt{2^m m!}}\int_{-\infty}^{\infty}\mathrm{d}x\, e^{-\frac{x^2}{L^2}}\left(\frac{1}{2}H_{n+1}\left(\frac{x}{L}\right) + nH_{n-1}\left(\frac{x}{L}\right)\right)H_m\left(\frac{x}{L}\right).$$

Integrating over $u = \frac{x}{L}$ thus yields

$$\langle n|\hat{q}|m\rangle = \frac{L}{\sqrt{\pi}\sqrt{2^n n!}\sqrt{2^m m!}}\left(\frac{\sqrt{\pi}}{2}2^{(n+1)}(n+1)!\delta_{n+1,m} + \sqrt{\pi}\,n\,2^{(n-1)}(n-1)!\delta_{n-1,m}\right)$$

which simplifies to

$$\langle n|\hat{q}|m\rangle = \frac{L}{\sqrt{2}}\left(\sqrt{n+1}\delta_{n+1,m} + \sqrt{n}\delta_{n-1,m}\right).$$

$\boxed{5,\ \mathrm{D}}$

3. **Particle hopping on a lattice**

There is a question about the same model in the 2021/22 exam, but the actual questions are different. seen/sim.seen ⇓

(a)

$$\hat{K}|m\rangle = \sum_n |n-1\rangle \underbrace{\langle n|m\rangle}_{\delta_{n,m}} = |m-1\rangle$$

$$\hat{K}^\dagger|m\rangle = \sum_n |n+1\rangle\langle n|m\rangle = |m+1\rangle$$

2, A

(b) We calculate

$$\begin{aligned}\hat{K}|k\rangle &= \frac{1}{\sqrt{2\pi}} \sum_n \mathrm{e}^{-ink}|n-1\rangle \\ &= \frac{1}{\sqrt{2\pi}} \sum_m \mathrm{e}^{-i(m+1)k}|m\rangle \\ &= \frac{\mathrm{e}^{-ik}}{\sqrt{2\pi}} \sum_m \mathrm{e}^{-imk}|m\rangle \\ &= \mathrm{e}^{-ik}|k\rangle.\end{aligned}$$

That is, $|k\rangle$ is an eigenvector of $\hat{K}$ with eigenvalue $\mathrm{e}^{-\mathrm{i}k}$. Similarly we have

$$\begin{aligned}\hat{K}^\dagger|k\rangle &= \frac{1}{\sqrt{2\pi}} \sum_m \mathrm{e}^{-\mathrm{i}(m-1)k}|m\rangle \\ &= \mathrm{e}^{\mathrm{i}k}|k\rangle.\end{aligned}$$

That is, the eigenvalue for $\hat{K}^\dagger$ is $\mathrm{e}^{\mathrm{i}k}$. 3, B

(c) From part (b) we deduce

$$\hat{H}|k\rangle = \frac{-E_0}{2}\left(e^{ik} + e^{-ik}\right)|k\rangle.$$

That is, the eigenvalues of $\hat{H}$ are given by

$$\lambda(k) = -E_0\cos(k)$$

The energy range $[-E_0, 0]$ corresponds to the $k$-interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. 4, A

(d) (i) $\hat{N}$ is diagonal in the $|n\rangle$-basis with eigenvalues $n$. Thus, a measurement of $\hat{N}$ in the state $\psi\rangle$ yields the outcome $n = 0$ with probability $\frac{1}{2}$ and the outcome $n = 1$ with probbility $\frac{1}{2}$. unseen ⇓ 2, B

(ii) Expectation value of $\hat{H}$ :

$$\begin{aligned}\langle\psi|\hat{H}|\psi\rangle &= -\frac{E_0}{2}\left\langle\psi\left|\hat{K} + \hat{K}^+\right|\psi\right\rangle \\ &= -\frac{E_0}{4}\left(\langle 0| + \langle 1|\right)\left(|1\rangle + |2\rangle + |-1\rangle + |0\rangle\right) \\ &= -\frac{E_0}{2}\end{aligned}$$

4, C

(iii) We want $E \in [0, E_0]$, which means $k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ The probability to obtain an energy measurement in this range is thus given by

$$P\left(E \in [0, E_0]\right) = \int_{-\pi/2}^{\pi/2} |\langle k|\psi\rangle|^2 dk$$

We have

$$\langle k|m\rangle = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}mk}.$$

Thus

$$\begin{aligned} P\left(E \in [0, E_0]\right) &= \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \underbrace{\left|1 + \mathrm{e}^{\mathrm{i}k}\right|^2}_{2+2\cos(k)} \mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (1 + \cos(k)) \mathrm{d}k \\ &= \frac{1}{2} + \frac{1}{\pi}. \end{aligned}$$

$\boxed{5, \text{D}}$

4. **Bound states in a piecewise constant potential** seen ⇓

(a) The time-independent Schrödinger equation can be rewritten as

$$\phi''(x) = -\underbrace{\frac{2m\,(E - V_0)}{\hbar^2}}_{k^2}\,\phi(x),$$

which is solved by

$$\phi(x) = A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx},$$

where $A$ and $B$ are constants. For $E > V_0$ we have $k \in \mathbb{R}$, and $\mathrm{i}k \in \mathbb{R}$ for $E < V_0$. In neither of these cases we have any square-integrable solutions. 2, A

2, B

(b) (i) The range of energies for possible bound states is sim. seen ⇓

$$E \in [-V_{\mathsf{in}}\,, V_{\mathsf{ex}}\,]\,,$$

and there are finitely many bound states. 3, A

(ii) The wave function and its first derivative have to be continuous everywhere. 3, A

(iii) The bound-state wave functions $\phi(x)$ are either even or odd, i.e. either $\phi(x) = \phi(-x)$ or $\phi(x) = -\phi(-x)$. 2, A

(iv) The classically allowed region in this energy range is $x \in [-a, a]$. The bound-state wavefuntions are non-zero outside of this region, and thus, there is a finite probability to find a particle in a bound state in this range outside of the allowed region. unseen ⇓ 4, B

(v) In class we had briefly discussed a similar problem (where one had to pick between three potentials for a given wave function). sim. seen ⇓

Only the wavefunctions (1) or (2) could be (and are) bound states of a potential of the type considered here. (3) does not fulfill the correct boundary conditions (its first derivative is not continuous) and it has the same frequency of oscillations along the whole x-range of the potential. (4) on the other hand, belongs to a potential with a barrier rather than a dip in the middle, which we can deduce from the fact that the frequency of the oscillation is smaller in the centre, and thus the energy gap between the state and the bottom of the potential is lower there, rather than higher. Of the two sketches (1) and (2), we can deduce from the number of sign changes that (1) belongs to the seventh energy state, and (2) belongs to the ninth energy state. 4, D

## 5. Mastery - Coherent states and the Husimi distribution

(a) (i) We have for $\phi(x) = \langle x|\phi\rangle$ that

$$\langle x|\hat{q}|\phi\rangle = x\phi(x), \quad \text{and} \quad \langle x|\hat{p}|\phi\rangle = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)$$

Thus, the eigenvalue equation for $\hat{a}$ in position representation is given by

$$\frac{1}{\sqrt{2}}\left(x + \mathrm{i}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)\phi_\alpha(x) = \alpha\phi_\alpha(x),$$

or

$$\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_\alpha(x) = \sqrt{2}\alpha\phi_\alpha(x),$$

which we integrate to find

$$\ln\phi_\alpha(x) - \ln\phi_\alpha(0) = \sqrt{2}\alpha x - \frac{x^2}{2}$$

That is,

$$\phi_\alpha(x) = \phi_\alpha(0)\mathrm{e}^{-\frac{x^2}{2}+(q+\mathrm{i}p)x}.$$

$\boxed{5, \mathrm{M}}$

(ii) For the wave function to be normalised we need

$$\int_{-\infty}^{\infty}|\phi_\alpha(x)|^2\mathrm{d}x = 1.$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty}|\phi_\alpha(x)|^2\mathrm{d}x &= |\phi_\alpha(0)|^2\int_{-\infty}^{\infty}\mathrm{e}^{-x^2+2qx}\mathrm{d}x\\ &= |\phi_\alpha(0)|^2\,\mathrm{e}^{q^2}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-q)^2}\mathrm{d}x\\ &= |\phi_\alpha(0)|^2\,\mathrm{e}^{q^2}\sqrt{\pi}\end{aligned}$$

Thus, we find

$$|\phi_\alpha(0)| = \frac{\mathrm{e}^{-\frac{q^2}{2}}}{\pi^{\frac{1}{4}}}$$

and hence

$$\begin{aligned}\phi_\alpha(x) &= \frac{1}{\pi^{1/4}}\mathrm{e}^{-\frac{x^2}{2}-\frac{q^2}{2}+qx+\mathrm{i}px}\\ &= \frac{1}{\pi^{1/4}}\mathrm{e}^{-\frac{1}{2}(x-q)^2+\mathrm{i}px}\end{aligned}$$

$\boxed{5, \mathrm{M}}$

(iii) For the state to be normalised we need $\langle\psi|\psi\rangle = 1$. We have

$$\begin{aligned}\langle\psi|\psi\rangle &= N^2\left(\langle\alpha_1|\alpha_1\rangle + \langle\alpha_2|\alpha_2\rangle + \langle\alpha_1|\alpha_2\rangle + \langle\alpha_2|\alpha_1\rangle\right)\\ &= N^2\left(2 + e^{-\frac{|\alpha_1|^2}{2}-\frac{|\alpha_2|^2}{2}}\left(e^{\alpha_1^*\alpha_2} + e^{\alpha_2^*\alpha_1}\right)\right)\end{aligned}$$

With $\alpha_1 = 1$ and $\alpha_2 = -1$ this becomes

$$\langle\psi|\psi\rangle = N^2\left(2 + 2\mathrm{e}^{-2}\right),$$

and we find

$$N = \frac{1}{\sqrt{2(1+\mathrm{e}^{-2})}}.$$

$\boxed{5, \mathrm{M}}$

(b) We have

$$\langle\alpha|n=1\rangle = \langle\alpha|\hat{a}^\dagger|n=0\rangle = \alpha^*\langle\alpha|n=0\rangle.$$

Further we note that the state $|n=0\rangle$ is also a coherent state with $\alpha=0$, and thus

$$\langle\alpha|n=0\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}.$$

Combining these we find

$$Q(\alpha,\alpha^*) = \frac{|\alpha|^2}{2\pi}\mathrm{e}^{-|\alpha|^2}.$$

$\boxed{5,\ \mathrm{M}}$

**Review of mark distribution:**

Total A marks: 33 of 32 marks
Total B marks: 18 of 20 marks
Total C marks: 13 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

# MATH60015 Quantum Mechanics I

| Question | Marker's comment |
| --- | --- |
| 1 | The majorty did well on Question 1. |
| 2 | See MATH70015 |

# MATH70015 Quantum Mechanics I

| Question | Marker's comment |
| --- | --- |
| 1 | The majority did well on Question 1. |
| 2 | Straight forward question for most. 2ai and 2aii universally well answered, 2aiii the argument as to why vugt;=0 was often missing. 2bi generally well done, 2bii mostly well done, with some having difficulties with the integral or the algebra. |