Imperial College London

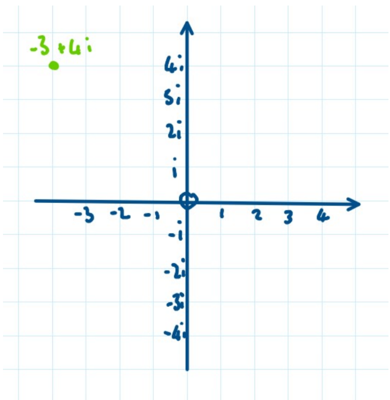
DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022-2023

Mathematics Pre-arrival course

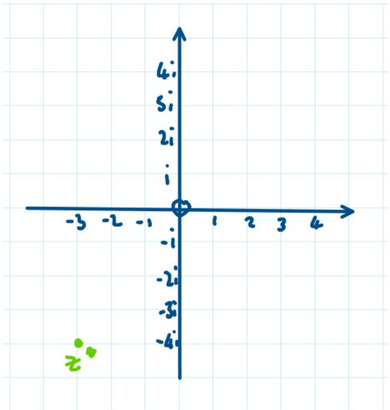
Solutions to Problem Sheet 4 – Complex Numbers

The starred questions on this problem sheet are for you to think about — we will not be giving solutions to them in the pre-arrival course. Instead these will form the basis of discussion in your first *MATH40001/MATH40009 - Introduction to University Mathematics* session once you arrive at Imperial.

1. Let $z = 1 + 2i$ and $w = 3 - 4i$. Find:
   (a) $z + w = 4 - 3i$
   (b) $zw = 11 + 2i$
   (c) $z^2 + w^* = 8i$
   (d) $\frac{w}{z} = -1 - 2i$
2. Let $z = -3 + 4i$
   (a) Sketch z in the complex plane.



   (b) Calculate the modulus and argument of z.
   $\arg(z) = 2.214$ (to 4 s.f.), $\mod(z) = 5$
   (c) Sketch z^* in the complex plane.



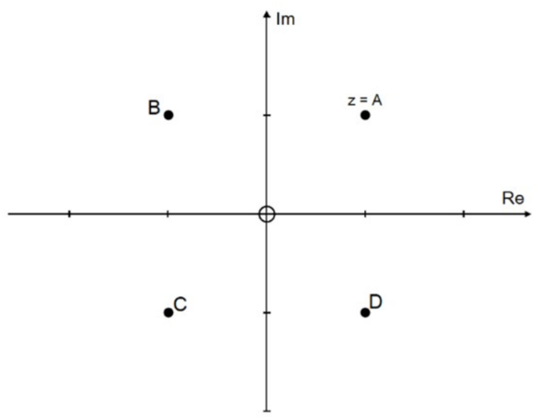
   (d) Find $z \cdot z^*$.
   $z \cdot z^* = 25$

Note: some books/courses use $\bar{z}$ for the complex conjugate of z.

3. Given the complex number z represented by the point A on the Argand diagram below which point represents:

   (a) z^* — z^* represents the point D.

   (b) iz — iz represents the point B.



4. Express the following in modulus-argument form:

   (a) $z_1 = \sqrt{3} + i$ — modulus: $|z_1| = 4$, argument: $\arg(z_1) = \frac{\pi}{6}$

   (b) $z_2 = -1 - \sqrt{3}i$ — modulus: $|z_2| = 4$, argument: $\arg(z_2) = \frac{2\pi}{3}$

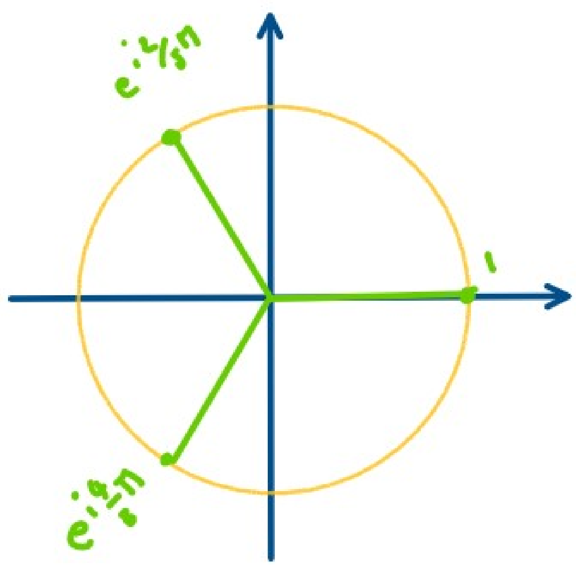
5. Given that $\arg(z) = \frac{\pi}{4}$, find:

   (a) $\arg(iz) = \frac{3\pi}{4}$

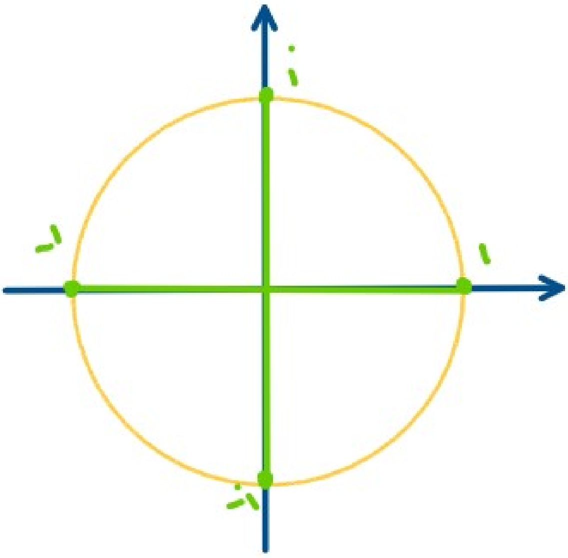
   (b) $\arg(-z) = \frac{5\pi}{4}$

6. Find all the complex roots of $x^3 - 1 = 0$. Do the same for $x^4 - 1 = 0$, try and sketch these roots on the complex plane. Can you guess where the roots of $x^n - 1 = 0$ will be located in the complex plane?

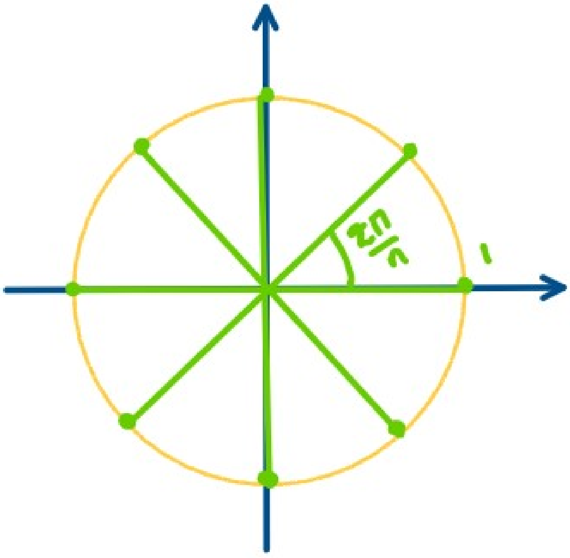
   Solutions to $x^3 - 1 = 0$ are $-1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. On an Argand diagram:



   Solutions to $x^4 - 1 = 0$ are $-1, i, -1, -i$. On an Argand diagram:



Solutions to $x^n - 1 = 0$ will be of the form $e^{i\frac{2k}{n}}$ for $k \in \{1, ..., n\}$. On an Argand diagram:



7. By considering the real and imaginary parts of $(e^{i\theta})^3$, derive the triple angle formulae for $\sin$ and $\cos$:

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$
$$\cos(3\theta) = 3\sin\theta - 4\sin^3\theta$$

We start by writing

$$\begin{aligned}
\cos(3\theta) + i\sin(3\theta) &= e^{i3\theta} \\
&= (e^{i\theta})^3 \\
&= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)
\end{aligned}$$

So looking at the real part, we get:

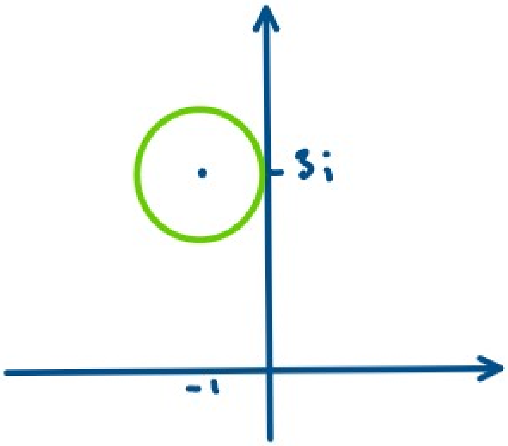
$$\begin{aligned}
\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta(1 - cos^2\theta) \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

and looking at the imaginary part, we get:

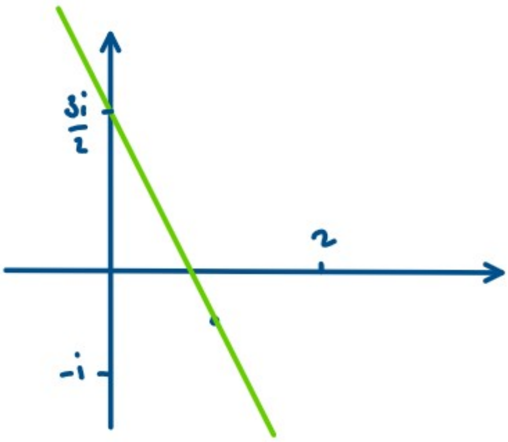
$$\begin{aligned}\sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

8. On the complex plane, find:

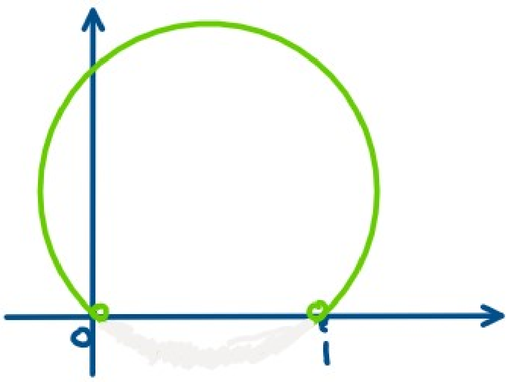
   (a) all the points z such that $|z+1-3i| = 1$.



   (b) all the points z such that $|z-2| = |z+i|$.



   (c) all the points z such that $\arg\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$



9. ★ Prove the *conjugate root theorem*: for any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ with real coefficients $a_0 \ldots a_n$, if z is a root of P, then so is $\bar{z}$.

   This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.

10. ★ Show that if $|z| = 1$, then

$$\operatorname{Im} \frac{z}{(z+1)^2} = 0.$$

Is there a nice geometric interpretation of this equation? Find all the points on the complex plane such that $\operatorname{Im} \frac{z}{(z+1)^2} = 0$ — there are more of them than just the ones on the unit circle.

This problem will be discussed during *MATH40001/MATH40009 - Introduction to University Mathematics*.