Exercise 5.1. For each of the following equations determine at which points one cannot find a function $y = f(x)$ which describes the graph in this neighbourhood. Sketch the graphs.

(a)
$$\frac{1}{3}y^3 - 2y + x = 1$$

(b)
$$x^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) - xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin(2\phi) + y^2\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) = 1,$$
where $a > 0$, $b > 0$, $0 \leq \phi \leq \pi/2$ are fixed parameters. Note the cases $a = b$, $\phi = 0$, $\phi = \pi/2$.

Exercise 5.2. Consider the equation

$$2x^2 + 4xy + y^2 = 3x + 4y$$

a) Show that this system of equations (implicitly) defines a function $y = f(x)$ with $f(1) = 1$.

b) Compute $f'(1)$ without knowing f explicitly.

c) Find an explicit formula for f and check your result from b).

Exercise 5.3. Let $X = \mathbb{R}^n$ and define the function $\mathrm{d}_{infty} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{d}_\infty(x, y) = \max\{|x^1 - y^1|, \ldots, |x^n - y^n|\}.$$

Show that d_∞ is a metric on $\mathbb{R}^n$.

Exercise 5.4. Show that each of the following functions is a metric on $\mathbb{R}$:

(i) $\mathrm{d}(x, y) = |x^3 - y^3|$, (here x^3 means x raised to power 3)

(ii) $\mathrm{d}(x, y) = |e^x - e^y|$,

(iii) $\mathrm{d}(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

Which property of the maps $x \mapsto x^3$, $x \mapsto e^x$, and $x \mapsto \tan^{-1}(x)$ makes these functions a metric.

Exercise 5.5. Assume that $a < b$ are real numbers, and $h : (a, b) \to (0, \infty)$ is a continuous function. For x and y in (a, b), we define

$$\mathrm{d}_h(x, y) = \int_{\min\{x,y\}}^{\max\{x,y\}} h(t)\, dt.$$

Show that d_h is a metric on (a, b).

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

Exercise 5.6. Consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(x, y) = |x - y|^2.$$

Show that g is not a metric on $\mathbb{R}$.

Exercise 5.7. Let $X = \mathbb{R}^2$, and define $\mathrm{d}_{\mathrm{rail}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\mathrm{d}_{\mathrm{rail}}(x, y) = \begin{cases} \|x - y\| & \text{if } x = ky \text{ for some } k \in \mathbb{R} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

Show that $\mathrm{d}_{\mathrm{rail}}$ is a metric on $\mathbb{R}^2$.

This is called the British rail metric. The intuition behind this metric is that if two towns are on the same rail line, then we travel between them, but if the towns are on distinct lines, we travel via London (represented as the origin in $\mathbb{R}^2$).