Mathematical Logic (MATH60132 and MATH70132)
2024-25, Coursework 1

**This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 10 February 2025. The coursework is marked out of 20 and the marks per question are indicated below.**
**The work which you submit should be your own, unaided work. Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet, generative AI agent or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.**

**[1]**

(a) Use results from the notes to show that, if $\Gamma \cup \{\phi\}$ is a set of $L$-formulas, then $\Gamma \vdash_L \phi$ if and only if for every valuation $v$ with $v(\Gamma) = T$ we have $v(\phi) = T$. In the case where $\Gamma$ is a finite set, how does this result allow you to check using truth tables (involving only the propositional variables in $\Gamma$) whether or not $\Gamma \vdash_L \phi$?

(b) Is $((p_2 \to (p_1 \to p_3))$ a consequence of $\{(p_1 \to ((\neg p_2) \to p_3)), ((\neg p_3) \to (p_1 \to p_2))\}$? Give reasons for your answers.

(c) Suppose $\Gamma_n$ is a set consisting of $n$ $L$-formulas (where $n \in \mathbb{N}$).

(i) In the case $n = 4$, show that there exists a set $\Gamma_n$ with the property that $\Gamma_n$ is inconsistent and every subset of $\Gamma_n$ of size $n-1$ is consistent. You do not necessarily need to say explicitly what the formulas in $\Gamma_n$ are.
[Hint: we may take the formulas in $\Gamma_4$ to involve only variables $p_1, p_2$.]

(ii) Do (i) for general $n \in \mathbb{N}$. [The hint in (i) does not apply in general.]

*Solution:* (a) ($\Rightarrow$:) This is the Generalised Soundness Theorem (1.3.3) (also on problem sheet 2).

($\Leftarrow$:) If $\Gamma$ is assumed consistent, this is by Theorem 1.3.10(2); if $\Gamma$ is inconsistent, then $\Gamma \vdash \phi$ for any formula $\phi$ (e.g. by 1.2.7(2)).

This means that we can check check whether $\Gamma \vdash \phi$ by: constructing the the truth tables of $\phi$ and the formulas in $\Gamma$ (using the variables $p_1, \ldots, p_n$ as other variables do not appear in the formulas and therefore their truth value does not affect the truth value of the formulas); looking to see whether there is a value of of the variables which gives all formulas in $\Gamma$ value T and $\phi$ value F; then use the above result.

(b) No. If $v$ is a valuation which gives $(p_1, p_2, p_3)$ truth values $(T, T, F)$ then $v(((p_2 \to (p_1 \to p_3))) = F$. But in this case $v(\{(p_1 \to ((\neg p_2) \to p_3)), ((\neg p_3) \to (p_1 \to p_2))\}) = T$, so (a) gives what we claim.

(c) Here's one possible solution. (i) By adequacy of the set of connectives used for $L$, there exist $L$-formulas $\phi_1, \ldots, \phi_4$ with the following truth table:

| $p_1$ | $p_2$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

Take $\Gamma_4$ to consist of these $\phi_i$. As there's no valuation $v$ with $v(\Gamma_4) = T$, the set $\Gamma_4$ is inconsistent (by the proof of the Completeness Theorem). For any three of the $\phi_i$, there is a valuation giving all of them value T (by the table). So any 3 of them are consistent.

(ii) Use the same argument. Take some $k$ with $2^k \geq n$. Consider truth functions $F_1, \ldots, F_n$ of $k$ variables where $F_i$ has value $T$ in row $j$ if and only if $j \leq n$ and $j \neq i$ (with respect to some fixed listing of the rows). For $i \leq n$, let $\phi_i$ be an $L$-formula in variables $p_1, \ldots, p_k$ with truth function $F_i$. Take $\Gamma_n = \{\phi_i : i \leq n\}$.

**[2]** We say that $L$-formulas $\phi, \psi$ are *$L$-equivalent* if $\vdash_L (\phi \to \psi)$ and $\vdash_L (\psi \to \phi)$. In this question you should give syntactic arguments, not involving truth tables, valuations, or use of the Completeness Theorem for $L$. You may use results from the notes and problem sheets about theorems of $L$.

(a) Show that if $\beta$ is an $L$-formula, then $(\neg(\neg\beta))$ is $L$-equivalent to $\beta$.

(b) Prove that if $\chi, \eta$ are $L$-formulas, then $((\chi \to \eta) \to ((\neg\eta) \to (\neg\chi)))$ is a theorem of $L$.

(c) If $\alpha$ is a subformula of the $L$-formula $\phi$, then there exist $L$-formulas $\phi_1$, ..., $\phi_k$ such that: $\phi_1$ is $\alpha$; $\phi_k$ is $\phi$; and (for $i < k$) we have that $\phi_{i+1}$ is one of $(\neg\phi_i)$, $(\phi_i \to \chi_i)$ or $(\chi_i \to \phi_i)$, for some formula $\chi_i$.

Give a **syntactic** proof of the following result:

Suppose $\alpha$ is a subformula of $\phi$. Let $\hat{\alpha}$ be an $L$-formula which is $L$-equivalent to $\alpha$ and let $\hat{\phi}$ be the $L$-formula obtained by replacing $\alpha$ by $\hat{\alpha}$ in $\phi$. Then $\phi$ and $\hat{\phi}$ are $L$-equivalent. [Hint: Prove this by induction on $k$, treating the cases $k = 1, 2$ as the base case.]

*Solution:* (a) This follows immediately from Question 1 on Problem sheet 2.

(b) As $\vdash \eta \to \neg\neg\eta$, we have $(\chi \to \eta) \vdash (\chi \to \neg\neg\eta)$ by HS. Similarly, as $\vdash \neg\neg\chi \to \chi$, we then obtain $(\chi \to \eta) \vdash (\neg\neg\chi \to \neg\neg\eta)$. As application of an A3 axiom, MP and DT then gives what we want.

(c) We prove the result by induction on $k$. The base case is $k = 1$ and in this case $\phi$ is equal to $\alpha$ and $\hat{\phi}$ is equal to $\hat{\alpha}$, so there is nothing to prove.

We also consider the case $k = 2$. There are then 3 cases depending on whether (i) $\phi$ is $\neg\alpha$; (ii) $\phi$ is $\alpha \to \chi$; (iii) $\phi$ is $\chi \to \alpha$. So $\hat{\phi}$ is (respectively) $\neg\hat{\alpha}$, $\hat{\alpha} \to \chi$; $\chi \to \hat{\alpha}$. We show $\vdash \phi \to \hat{\phi}$ ($\vdash \hat{\phi} \to \phi$ follows by symmetry):

(i) By (b) $\vdash ((\hat{\alpha} \to \alpha) \to (\phi \to \hat{\phi}))$. Then MP gives what we want. (ii), (iii) are just applications of transitivity of implication (HS).

Suppose $k > 2$ and the result holds for smaller $k$. Define $L$-formulas $\hat{\phi}_1, \ldots, \hat{\phi}_k$ such that $\hat{\phi}_1$ is $\hat{\alpha}$ and (for $i < k$) we have that $\hat{\phi}_{i+1}$ is obtained from $\hat{\phi}_i$ in the same way as $\phi_{i+1}$ is obtained from $\phi_i$. So each $\hat{\phi}_j$ is obtained from $\phi_j$ by substituting $\hat{\alpha}$ in place of $\alpha$. In particular $\hat{\phi}_k$ is $\hat{\phi}$. By induction hypothesis $\phi_{k-1}$ is $L$-equivalent to $\hat{\phi}_{k-1}$; by the case $k = 2$, we then have the inductive step.

**[3]** In the following, you should give syntactic arguments, not involving truth tables, valuations, or use of the Completeness Theorem for $L$. You may use results from the notes and problem sheets about theorems of $L$. Suppose $\phi, \psi$ are $L$-formulas. In $L$ we define $(\phi \wedge \psi)$ to be $(\neg(\phi \to (\neg\psi)))$. Prove the following (you may use 2(b) if you wish):

(a) $\vdash_L ((\phi \wedge \psi) \to \phi)$;

(b) $\{\psi, \phi\} \vdash_L (\phi \wedge \psi)$.

*Solution:* Miss out some brackets. (a) We have (1.2.7) $\vdash (\neg\phi \to (\phi \to \neg\psi))$. From 2(b) and MP we then obtain $\vdash (\phi \wedge \psi) \to \neg\neg\phi$. As $\vdash (\neg\neg\phi \to \phi)$ (Problem 1, sheet 2), HS then gives what we want.

(b) By MP we have $\phi, \psi, (\phi \to \neg\psi) \vdash \neg\psi$. So by DT we obtain $\phi, \psi \vdash ((\phi \to \neg\psi) \to \neg\psi)$. Using 2(b) and MP then gives $\phi, \psi \vdash (\neg\neg\psi \to \neg(\phi \to \neg\psi))$. As $\psi \vdash \neg\neg\psi$ we obtain (using MP) $\phi, \psi \vdash \neg(\phi \to \neg\psi)$, as required.