Math40003 Linear Algebra and Groups **Unseen Problem Sheet 5**

U.1. Let $\mathcal{T}_1$ be the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. What is the cardinality of $\mathcal{T}_1$?

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U.2. Let X be a set. Let $\mathcal{P}(X)$ be the set of all subsets of X (a.k.a. the power set of X), and let $A, B \in \mathcal{P}(X)$. We define $A+B$ to be $(A \setminus B) \cup (B \setminus A)$ (this is also called the *symmetric difference*, and is also written as $A \triangle B$).

(a) Prove that $\mathcal{P}(X)$, the set of all subsets of X (a.k.a. the power set of X), with $A + B$ is a vector space over $\mathbb{F}_2$. The zero vector is $\emptyset$.

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(b) Find the dimension of $\mathcal{P}(X)$ in terms of $|X|$. (You may assume that if X is an infinite set then there is no set Y such that $|X| < |Y| < \mathcal{P}(X)$, a.k.a. the Generalised Continuum Hypothesis.)

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(c) Are the following functions linear transformations?

i. Let $x \in X$, and let $f_x : \mathcal{P}(X) \to \mathbb{F}_2$ be given by $f(A) = 1$ if and only if $x \in A$.

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ii. Let $Y \subseteq X$, and let $f_Y : \mathcal{P}(X) \to \mathbb{F}_2$ be given by $f(A) = 1$ if and only if $A \cap Y \neq \emptyset$.

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(d) Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{F}_2$, with addition given by $(f + g)(x) = f(x) + g(x)$. We define $\varphi : V \to \mathcal{P}(\mathbb{R})$ to be

$$\varphi(f) = \{r \in \mathbb{R} \,:\, f(r) = 1\}$$

Is φ linear?

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