Module:	MATH40004/MATH40011
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2022

MATH40004/MATH40011 Calculus and Applications

The following information must be completed:

Is the paper suitable for resitting students from previous years:

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
eg 1(a) 8 marks; 2(a) 5 marks; 2(b) (ii) 3 marks; 3(a) 7 marks; 3(b) 3 marks; 4(a) 4 marks; 4(c) 2 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
eg 1(b) 5 marks; 2(b) (i), (iii) 6 marks; 3(b) 2 marks; 3(d) 3 marks; 4(b) 4 marks

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
eg 1(c) 3 marks; 2(b)(ii) 2 marks; 3(d) 2 marks; 4(c) 3 marks, 4(d) 2 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
eg 1(c) 4 marks; 2(c) 4 marks; 3(b) computing the limit, 3 marks; 4(d) final part, 5 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Calculus and Applications

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

4. (a) Show that the following functions are linearly independent

$$\{x, x+2, x^2\}$$

(3 marks)

(b) Consider the following ODE:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 12\cosh 3x$$

(i) Find the general solution to the corresponding homogenous ODE using an appropriate basis for the solution space. (3 marks)

(ii) Find the solution to the full non-homogenous ODE given initial condition of

$$y(0) = \frac{dy}{dx}(0) = 0.$$

(7 marks)

(c) Find the inverse Fourier transform of the following function assuming $a, b \in \mathbb{R}$ and $a \neq 0$. Express your answer in real form if possible.

$$\hat{f}(\omega) = \delta(a\omega + b) + \delta(a\omega - b)$$

(7 marks)

(Total: 20 marks)

5. (a) Consider the function:

$$u(x, y) = (x + y - 4)(xy - x - y + 1)$$

(i) Find the set of points that form the zero contour, i.e. $u(x, y) = 0$. (2 marks)

(ii) Find the stationary points of $u(x, y)$ and classify them stating clearly the conditions used for the classification (6 marks)

(iii) Sketch the contour plot of $u(x, y)$ indicating the zero contour ($u = 0$) and a few other representative contours. Indicate in your sketch the positions of the stationary points and the regions of positive and negative u. (3 marks)

(b) Consider the following first order nonlinear ODE

$$\frac{dy}{dt} = y^2 - 2y - r + 3$$

where $r \in \mathbb{R}$.

(i) Find the fixed points of the system, denoted by y^* and determine their stability for representative values of r. (3 marks)

(ii) Sketch the possible values of the fixed points (y^*) for the system against r to produce the bifurcation diagram, and classify any bifurcation found. (4 marks)

(iii) Using the bifurcation diagram find for what values of r the asymptotic behaviour of $y(t)$ as $t \to \infty$ does not depend on the initial value of $y(0)$. (2 marks)

(Total: 20 marks)

6. (a) Consider the following system of linear ODEs:

$$\frac{dx}{dt} = -\frac{1}{2}x + y$$
$$\frac{dy}{dt} = -x - 3y$$

(i) Find the general solution of this system. Describe the fixed point(s) of the system and their stability. (6 marks)

(ii) Sketch the phase portrait for this system in the (x, y) plane. (3 marks)

(iii) Now consider the perturbed system

$$\frac{dx}{dt} = -\frac{1}{2}x + (1+\epsilon)y$$
$$\frac{dy}{dt} = -x - 3y$$

where $\epsilon \in \mathbb{R}$ is a parameter. Either by sketching the trace-determinant diagram or otherwise, find the value(s) of ϵ at which the system changes stability. (3 marks)

(b) Consider the transformation:

$$x = u^2 - v^2$$
$$y = 2uv$$

(i) Compute the Jacobian of this transformation. What is the infinitesimal element of area in the (u, v) coordinate system. (4 marks)

(ii) Obtain $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial u}{\partial y}\right)_x$ in terms of u and v.

(4 marks)

(Total: 20 marks)