MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Quiz 3

Question 1 (Unique solution).
Consider the one-dimensional initial value problem

$$\dot{x} = \sqrt{|t|}\,, \qquad x(t_0) = x_0\,.$$

Is the following statement true? For all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$, this initial value problem has a unique solution defined on $\mathbb{R}$.

(a) The statement is true.

(b) The statement is false.

Question 2 (Maximal solution).
Is the following statement true or false? The maximal solution to the initial value problem

$$\dot{x} = 1 + x^2\,, \qquad x(0) = 0$$

is given by $\lambda_{max}(t) = \tan(t)$, defined for all $t \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$.

(a) The statement is true.

(b) The statement is false.

Question 3 (Local Lipschitz continuity).
Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$f(t,x) := tx(t+x) \quad \text{for all } \ t, x \in \mathbb{R}\,.$$

Which of the following two statements is true?

(a) f is locally Lipschitz continuous (in the sense of Definition 2.12 (ii)).

(b) f is not locally Lipschitz continuous (in the sense of Definition 2.12 (ii)).

Question 4 (Global Lipschitz continuity).
Consider a bounded and open set $\tilde{D} \subset \mathbb{R}^d$, define $D := \mathbb{R} \times \tilde{D}$, and let $f : D \to \mathbb{R}^d$ be locally Lipschitz continuous (with respect to x). Which of the following two statements is true?

(a) f is globally Lipschitz continuous (in the sense of Definition 2.12 (i)).

(b) It does not follow that f is globally Lipschitz continuous (in the sense of Definition 2.12 (i)).

Question 5 (Convergence to the boundary of D).
Consider an open set $D \subsetneq \mathbb{R} \times \mathbb{R}^d$ and the maximal solution $\lambda_{max} : I_{max}(t_0, x_0) \to \mathbb{R}^d$ to an initial value problem

$$\dot{x} = f(t,x)\,, \quad x(t_0) = x_0\,,$$

under the assumption that $f : D \to \mathbb{R}^d$ is continuous and locally Lipschitz continuous with respect to x. Is it possible (i.e. does there exist an example of D, such a function f and an initial pair (t_0, x_0)) such that $I_+(t_0, x_0) = \infty$ and $\lim_{t\to\infty} \operatorname{dist}((t, \lambda_{max}(t)), \partial D) = 0$.

(a) This is possible.

(b) This is not possible.