Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Partial Differential Equations in Action **MATH50008**

Problem Sheet 6

1. ✦ Using the method of separation of variables, find the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 4$ subject to the following boundary conditions:

(a) $u(x, y) = 1$ on the upper side and zero on the other three sides;
(b) $u(x, y) = \sin(\pi x/2)$ on the upper side and zero on the other three sides.

2. ✦✦ Suppose we wish to find a bounded solution of the 2D Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

over the half-space $-\infty < x < \infty$, $y > 0$, subject to the Dirichlet condition

$$\phi(x, 0) = p(x)$$

Use the method of Fourier transforms to show that

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{p(\xi)}{(x - \xi)^2 + y^2} d\xi$$

3. ✦ Suppose that now we wish to find a bounded solution of the following Neumann problem

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial \phi}{\partial y}(x, 0) = q(x), \quad \phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty$$

in the half-space $-\infty < x < \infty$ and $y > 0$ Use the substitution $\psi = \partial\phi/\partial y$ to reduce this to a Dirichlet problem. Hence, show that the solution is

$$\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln((x - \xi)^2 + y^2) q(\xi) d\xi$$

4. ✦ Consider a bounded volume V with surface S_0. The volume contains N holes, each with a surface S_i, $i = 1, \ldots, N$. Suppose that

$$\nabla^2 \phi = f(\mathbf{r}) \quad \text{in } V$$

with Neumann boundary conditions

$$\frac{\partial \phi}{\partial n} = q_i(\mathbf{r}) \quad \text{on each of the surfaces } S_i, \ i = 0, \ldots, N$$

Show that the solution ϕ to this problem is unique, up to an additive constant.

5. ✦✦ We wish to solve

$$\nabla^2 \phi = f(\mathbf{r})$$

in a bounded volume V subject to

$$\frac{\partial \phi}{\partial n} = q(\mathbf{r})$$

on the boundary S of V. By considering the Green's function that satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{in } V, \quad \frac{\partial G}{\partial n} = 0 \quad \text{on } S,$$

show that the solution for ϕ can be written in the form

$$\phi(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r}) dV - \int_S q(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_0) dS + \text{constant}.$$

6. ✦✦✦ Verify that the solution for G which satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \quad \text{for } z > 0 \text{ with } \frac{\partial G}{\partial z} = 0 \quad \text{on } z = 0$$

is given by

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0'|}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{r}_0' = (x_0, y_0, -z_0)$. Use this Green's function to show that the solution of

$$\nabla^2 \phi = 0 \quad \text{for } z > 0, \quad \frac{\partial \phi}{\partial z} = q(x, y) \quad \text{on } z = 0,$$

can be written as

$$\phi(\mathbf{r}_0) = \int_{z=0} q(x, y) G(\mathbf{r}, \mathbf{r}_0) dx dy$$

If q has the specific form

$$q(x, y) = \begin{cases} q_0, & x^2 + y^2 \leq R^2 \\ 0, & x^2 + y^2 > R^2 \end{cases}$$

with q_0 and R constants, show that

$$\phi(0, 0, z) = -q_0 \left(\sqrt{R^2 + z^2} - z \right)$$

7. ✦✦✦ Show that in two dimensions, a Green's function that satisfies

$$\nabla^2 G = \delta(\mathbf{r})$$

is given by

$$G = \frac{1}{2\pi} \ln |\mathbf{r}|$$

Deduce that the corresponding solution to

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$$

that holds in the upper half-plane $y > 0$, $-\infty < x < \infty$, and satisfies

$$G = 0 \quad \text{on} \quad y = 0$$

is given by

$$G = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| - \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0'|$$

where $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{r}_0' = (x_0, -y_0)$. Use this Green's function to show that the solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{for } y > 0, \quad \phi(x, 0) = p(x)$$

is given by

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{p(\xi)}{(x - \xi)^2 + y^2} d\xi$$

Note that we have obtained the same solution using a different approach in Q2.

8. ✦✦✦ In this problem, we will solve Laplace's equation

$$\nabla^2 u(\mathbf{r}) = 0$$

in plane polar coordinates (r, θ).

(a) Using the method of separation of variables, show that the general solution to Laplace's equation in plane polar coordinates is given by

$$u(r, \theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r^n + D_n r^{-n}\right)$$

(b) A circular drumskin has is clamped on a rim at $r = a$. After dropping your drum, the rim ends up twisted so that it is displaced vertically by a small amount $\varepsilon(\sin\theta + 2\sin 2\theta)$, where θ is the azimuthal angle with respect to a given radius. Find the resulting displacement $u(r, \theta)$ over the entire drumskin.

9. ✦✦ A circular disc of radius a is heated in such a way that its perimeter $r = a$ has a steady temperature distribution given by $A + B\cos^2\theta$, where r and θ are plane polar coordinates and A and B are real constants. Find the temperature distribution $T(r, \theta)$ everywhere in the region $r < a$.