BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Optimisation Mock Exam

Date: Wednesday, 11th May 2021

Time: 09:00-11:00

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Find and classify all the stationary points of

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 1\,.$$

(10 marks)

(b) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

(ii) $f(\mathbf{x}) = \log\left(\sum_{i=1}^{k} e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)$, where $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. (5 marks)

(iii) $f(\mathbf{x}) = \|\mathbf{x}\|^4$. (5 marks)

(Total: 20 marks)

2. Consider the constrained minimization problem

$$\{\min f(\mathbf{x}) : \mathbf{x} \in \Delta_n\}\,,$$

where f is a continuously differentiable function over Δ_n. Show that $\mathbf{x}^* \in \Delta_n$ is a stationary point of this problem if and only if there exists $\mu \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) \begin{cases} = \mu, & x_i^* > 0\,, \\ \geq \mu\,, & x_i^* = 0\,. \end{cases}\,.$$

(20 marks)

(Total: 20 marks)

3. Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \geq 4 \\ & x_3 \leq 1 \end{aligned}$$

(i) Write down the KKT conditions. (8 marks)

(ii) Without solving the KKT system, prove that the problem has a unique optimal solution and that this solution satisfies the KKT conditions. (7 marks)

(iii) Find the optimal solution of the problem using the KKT system. (5 marks)

(Total: 20 marks)

4. Consider the problem

$$\begin{aligned}
\min \quad & x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \le 1 \\
& x_3 \le 1
\end{aligned}$$

(i) Is the problem convex? (4 marks)

(ii) Using KKT conditions, find an optimal solution of the primal problem. (8 marks)

(iii) Find a dual problem and solve it. (8 marks)

(Total: 20 marks)

5. **Mastery question.** Consider the problem

$$\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & \tfrac{1}{2}(x(T))^2 \\
\text{subject to} \quad & \dot{x}(t) = u(t) \\
& x(0) = x_0 \text{ given} \\
& u(t) \in [-1, 1], \ \text{for all } t \in \mathbb{R}
\end{aligned}$$

(i) Using the PMP, find an expression for the optimal control as a feedback law. (10 marks)

(ii) Find an explicit expression for the optimal value function of the problem. (10 marks)

(Total: 20 marks)