1. (i) Either of the following definitions, are appropriate for full marks. A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if   seen ⇓

1. $N_0 = 0$.
2. The increments are independent, that is, for any $k_1, k_2 \in \mathbb{Z}_+$, $0 < s < t$

$$\mathbb{P}(\{N_t - N_s = k_1\}|\{N_r = k_2, 0 \leq r \leq s\}) = \mathbb{P}(\{N_t - N_s = k_1\}).$$

3. The increments are stationary: for any $l > 0$, $0 < s < t$ $k \in \mathbb{Z}_+$

$$\mathbb{P}(\{N_t - N_s = k\}) = \mathbb{P}(\{N_{t+l} - N_{s+l} = k\})$$

4. There is a 'single arrival'

$$\begin{aligned} \mathbb{P}(\{N_{t+\delta} - N_t = 1\}) &= \lambda\delta + o(\delta) \\ \mathbb{P}(\{N_{t+\delta} - N_t \geq 2\}) &= o(\delta) \end{aligned}$$

Or: A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if

1. $N_0 = 0$
2. The increments are independent
3. For any $0 \leq s < t$, $k \in \mathbb{Z}_+$ we have

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}.$$

That is, the number of events in $[s, t]$ is a Poisson random variable, with mean $\lambda(t - s)$.   5

(ii) We are simply asked to calculate the PGF of a compound Poisson process:

$$\begin{aligned} G_{Y_t}(s) &= \mathbb{E}[s^{Y_t}] \\ &= \mathbb{E}[s^{\sum_{i=1}^{N_t} X_i}] \\ &= \mathbb{E}\Big[\prod_{i=1}^{n} \mathbb{E}[s^{X_i}|N_t = n]\Big] \\ &= \mathbb{E}\Big[\mathbb{E}[s^{X_i}]^{N_t}\Big] \\ &= \mathbb{E}[G_X(s)^{N_t}]. \end{aligned}$$

Now using the fact that the PGF of a Poisson random variable of parameter λ is

$$\exp\{\lambda(s-1)\}$$

the solution is easily found.   5

(iii) The random variable of interest is   unseen ⇓

$$Y = \sum_{i=1}^{N_T} X_i$$

where the X_i are i.i.d. Bernoulli and $T \sim \mathcal{U}_{[0,1]}$ (the uniform density on $[0, 1]$). Hence, using the result in (ii), one has (via the tower property)

$$G_Y(s) = \int_0^1 \exp\left\{(G_X(s) - 1)\int_0^t \frac{1}{u+1}du\right\}dt$$

where

$$G_X(s) = 1 - p + ps.$$

Hence

$$\begin{aligned} G_Y(s) &= \int_0^1 \exp\left\{p(s-1)\log(t+1)\right\}dt \\ &= \int_0^1 \left[t+1\right]^{p(s-1)}dt \\ &= \frac{1}{p(s-1)+1}\left[2^{p(s-1)+1} - 1\right]. \end{aligned}$$

To complete the question, one can differentiate w.r.t. s

$$G_Y'(s) = -\frac{1}{[p(s-1)+1]^2}p\left[2^{p(s-1)+1} - 1\right] + \frac{1}{p(s-1)+1}\left[p\log(2)2^{p(s-1)+1}\right]$$

setting $s=1$ yields:

$$\mathbb{E}[Y] = p[2\log(2) - 1].$$

10

seen ⇓

2. (i) A birth process with intensities $\lambda_0, \lambda_1, \dots$ is a counting process $\{N_t\}_{t\in[0,\infty)}$ such that

1. $N_0 = 0$
2. Conditionally independent increments: $\mathbb{P}(N_t - N_s|\{N_r\}_{0\leq r\leq s}) = \mathbb{P}(N_t - N_s|N_s)$
3. There is a 'single arrival'

$$\mathbb{P}(N_{t+\delta} = n+m|N_t = n) = \begin{cases} 1 - \lambda_n\delta + o(\delta) & \text{if } m=0 \\ \lambda_n\delta + o(\delta) & \text{if } m=1 \\ o(\delta) & o/w \end{cases}$$

4

(ii) Let $\{N_t\}$ be a birth process with positive intensities $\lambda_0, \dots$. Define the probabilities

$$p_n(t) = \mathbb{P}(N_t = n).$$

Now we can obtain the forward equations for a birth process:

$$p_n(t+\delta) = p_n(t)[1-\lambda_n\delta] + p_{n-1}(t)\lambda_{n-1}\delta + o(\delta)$$

with $n \geq 1$. That is, we have

$$\begin{aligned} p_n(t+\delta) &= \sum_{l\leq n}\mathbb{P}(N_{t+\delta} = n|N_t = l)\mathbb{P}(N_t = l) \\ &= \sum_{l\leq n}\mathbb{P}(N_{t+\delta} = j|N_t = l)p_l(t) \\ &= p_n(t)[1-\lambda_n\delta] + p_{n-1}(t)\lambda_{n-1}\delta + o(\delta) \end{aligned}$$

where the final line follows from the fact that we can only be in the state where there are $n-1$ individuals, and there is a birth, or we have jumped from n already and there is no birth. Then rearranging and taking the limit $\delta \downarrow 0$ it follows

$$\frac{dp_n(t)}{dt} = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t)$$

with the boundary condition $p_n(0) = 0$.

If $n = 0$ in a similar manner:

$$p_n(t+\delta) = p_n(t)[1 - \lambda_n \delta] + o(\delta)$$

i.e.

$$\frac{dp_n(t)}{dt} = -\lambda_n p_n(t)$$

with the boundary condition $p_n(0) = 1$. 6

(iii) (a) Since $N_0 = 1$ we have that for any $n \geq 1$ unseen ⇓

$$p_n(t+\delta) = p_n(t)[1 - n\lambda\delta] + p_{n-1}(t)(n-1)\lambda\delta + o(\delta)$$

i.e.

$$p_n'(t) = -n\lambda p_n(t) + \lambda(n-1)p_{n-1}(t).$$

Now

$$\begin{aligned}
G(s,t) &= \sum_{n=1}^{\infty} s^n p_n(t) \\
\Rightarrow \frac{\partial G(s,t)}{\partial t} &= \sum_{n=1}^{\infty} s^n p_n'(t) \\
&= \sum_{n=1}^{\infty} [\lambda(n-1)p_{n-1}(t) - n\lambda p_n(t)]s^n \\
&= \lambda s^2 \sum_{n=2}^{\infty} (n-1)s^{n-2}p_{n-1}(t) - \lambda s \sum_{n=1}^{\infty} n s^{n-1} p_n(t) \\
&= \lambda s(s-1)\frac{\partial G(s,t)}{\partial s}.
\end{aligned}$$

The solution can be verified by partial differentiation. The boundary condition $G(s,0) = s$ also holds. 5

(b) Now let us consider a power-series expansion of $G(s,t)$, this will enable us to obtain the probabilities $\mathbb{P}(N_t = j)$; summation of this latter quantity will solve the problem.

$$\begin{aligned}
G(s,t) &= \frac{se^{-\lambda t}}{1-(1-e^{-\lambda t})s} \\
&= se^{-\lambda t}\sum_{i=0}^{\infty}\left[(1-e^{-\lambda t})s\right]^i \\
&= e^{-\lambda t}\sum_{i=1}^{\infty} s^i (1-e^{-\lambda t})^{i-1} \\
&= \sum_{i=1}^{\infty} s^i \mathbb{P}(N_t = i).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{P}(N_t \geq m) &= e^{-\lambda t}\sum_{i=m}^{\infty}(1-e^{-\lambda t})^{i-1} \\
&= \frac{e^{-\lambda t}(1-e^{-\lambda t})^{m-1}}{1-(1-e^{-\lambda t})} \\
&= (1-e^{-\lambda t})^{m-1}.
\end{aligned}$$

To complete the question, as

$$\mathbb{P}(T_m \leq t) = \mathbb{P}(N_t \geq m) = (1-e^{-\lambda t})^{m-1}$$

it follows by taking differentiating the last term w.r.t t that

$$f(t_m) = (m-1)\lambda e^{-\lambda t}(1-e^{-\lambda t})^{m-2} \quad t \in \mathbb{R}_+.$$

5

seen ⇓

3. (i) Let $K(x, y)$ be a non-negative function on $E \times E$ such that:

1. For any $x \in E$, $\int_E K(x, y)dy = 1$
2. For any set A, the function $\int_A K(x, y)dy$ is *measurable*.

Then K is a **transition kernel**.

3

(ii) (a) By standard change of variables and noting that the Jacobian is $1/\sigma$ one easily obtains that

$$X_n | X_{n-1} = x_{n-1} \sim \mathcal{N}(x_{n-1}, \sigma^2).$$

That is,

$$K(x_{n-1}, x_n) = \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}(x_n - x_{n-1})^2\} \quad x_n \in \mathbb{R}.$$

Full marks are awarded if either the density or distribution are provided.

2

(b) The easiest way to verify this is via induction. For $n = 1$, one has the result by part (a). Assuming for $n-1$, via Chapman-Kolmogorov

$$\begin{aligned}
K^n(x_0, x_n) &= \int_{\mathbb{R}} K^{n-1}(x_0, x_{n-1})K(x_{n-1}, x_n)dx_{n-1} \\
&= \int_{\mathbb{R}} \phi(x_{n-1}; x_0, (n-1)\sigma^2)\phi(x_n; x_{n-1}, \sigma^2)dx_{n-1} \\
&= \phi(x_{n-1}; x_0, n\sigma^2)
\end{aligned}$$

where the second line uses the induction hypothesis and the third, the hint in the question.

5

unseen $\Downarrow$

(iii) (a) The question asks us to calculate

$$\int_{-\infty}^{\infty} e^{x_n}\phi(x_n; x_0, n\sigma^2)dx_n.$$

Using the moment generating/characteristic function, one easily concludes that this is

$$\exp\left\{x_0 + \frac{n\sigma^2}{2}\right\}$$

alternatively, one can use standard integration to complete the exercise. $\boxed{3}$

(b) This question asks for, essentially, the value of the euro call w.r.t. a special version of the Black-Scholes model. The expectation of interest is

$$\int_{\log(S)}^{\infty} (e^{x_n} - S)\phi(x_n; x_0, n\sigma^2)dx_n.$$

First consider

$$S\int_{\log(S)}^{\infty} \phi(x_n; x_0, n\sigma^2)dx_n.$$

Using the change of variable $u = (x_n - x_0)/\sigma\sqrt{n}$ one has

$$S\int_{(\log(S)-x_0)/\sigma\sqrt{n}}^{\infty} \phi(u; 0, 1)du = S\left[1 - \Phi\left(\frac{\log(S) - x_0}{\sigma\sqrt{n}}\right)\right].$$

Second consider

$$\int_{\log(S)}^{\infty} e^{x_n}\phi(x_n; x_0, n\sigma^2)dx_n.$$

The exponent in the integrand is equal to

$$-\frac{1}{2n\sigma^2}\left[(x_n - n\sigma^2 - x_0)^2 + x_0^2 - (x_0 + n\sigma^2)^2\right]$$

where we have completed the square and removed the additional terms. Hence the integral is

$$\exp\left\{x_0 + \frac{n\sigma^2}{2}\right\}\int_{\log(S)}^{\infty} \phi(x_n; x_0 + n\sigma^2, n\sigma^2)dx_n.$$

Changing variables as above yields

$$\exp\left\{x_0 + \frac{n\sigma^2}{2}\right\}\left[1 - \Phi\left(\frac{\log(S) - n\sigma^2 - x_0}{\sigma\sqrt{n}}\right)\right].$$

Using the condition on the normal C.D.F: $\Phi(-x) = 1 - \Phi(x)$ allows one to conclude. $\boxed{7}$

4. (i) A vector $\pi_{1:k}$ is a stationary distribution of a Markov chain $\{X_n\}$ on E if: $\boxed{\text{seen} \Downarrow}$

1. for each $j \in E$ $\pi_j \geq 0$ and $\sum_{j=1}^{k} \pi_j = 1$.
2. $\pi_{1:k} = \pi_{1:k}P$, that is, for each $j \in E$, $\pi_j = \sum_{i=1}^{k} \pi_i p_{ij}$. $\boxed{4}$

(ii) By definition

$$\begin{aligned}
p_{ij}(m+n) &= \mathbb{P}(X_{m+n} = j | X_0 = i) \\
&= \sum_{l=1}^{k} \mathbb{P}(X_{m+n} = j, X_m = l | X_0 = i) \\
&= \sum_{l=1}^{k} \mathbb{P}(X_{m+n} = j | X_m = l, X_0 = i)\mathbb{P}(X_m = l | X_0 = i) \\
&= \sum_{l=1}^{k} \mathbb{P}(X_{m+n} = j | X_m = l)\mathbb{P}(X_m = l | X_0 = i).
\end{aligned}$$

Here we have used

$$\mathbb{P}(X_1, X_2 | X_3) = \mathbb{P}(X_1 | X_2, X_3)\mathbb{P}(X_2 | X_3)$$

and the Markov property. $\boxed{6}$

(iii) (a) If $n = 1$, the claim is easily verified simply by adding the relevant matrices. Assume for $n = k$ and consider P^{k+1}:

$$\begin{aligned}
P^{k+1} &= P^k P \\
&= \left[\frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\alpha-\beta)^k}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}\right]\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}
\end{aligned}$$

Let us consider the first term on the R.H.S., when multiplied by P:

$$\begin{aligned}
\frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} &= \frac{1}{\alpha+\beta}\begin{pmatrix} \beta(1-\alpha)+\alpha\beta & \alpha\beta+\alpha(1-\beta) \\ \beta(1-\alpha)+\alpha\beta & \alpha\beta+\alpha(1-\beta) \end{pmatrix} \\
&= \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}.
\end{aligned}$$

Now, similarly the second part:

$$\begin{aligned}
&\frac{(1-\alpha-\beta)^k}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \\
&= \frac{(1-\alpha-\beta)^k}{\alpha+\beta}\begin{pmatrix} \alpha(1-\alpha)-\alpha\beta & \alpha^2-\alpha(1-\beta) \\ -\beta(1-\alpha)+\beta^2 & -\alpha\beta+\beta(1-\beta) \end{pmatrix} \\
&= \frac{(1-\alpha-\beta)^k}{\alpha+\beta}\begin{pmatrix} (1-\alpha-\beta)\alpha & -(1-\alpha-\beta)\alpha \\ -(1-\alpha-\beta)\beta & (1-\alpha-\beta)\beta \end{pmatrix}
\end{aligned}$$

putting the two above results together completes the result. A more constructive proof can be achieved by considering the spectral representation of P. $\boxed{7}$

(b) From the question the stationary probabilities are

$$(\pi_0, \pi_1) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$$

which gives $(\pi_0, \pi_1) = (12/21, 9/21)$. On inspection, for all practical purposes the equilibrium distribution is reached after 10 days. $\boxed{3}$