Quiz Solutions 25 - 30.

(25.) $a_n = \dfrac{1}{n^2 - \log n} < \dfrac{2}{n^2}$ for n large enough.

Since $\sum \frac{2}{n^2} < \infty \Rightarrow \sum a_n < \infty$ also

$$\left[\ \frac{1}{n^2-\log n} = \frac{1}{n^2-\log n} + \frac{2}{n^2} - \frac{2}{n^2} = \frac{2}{n^2} - \underbrace{\left(\frac{2}{n^2} - \frac{1}{n^2-\log n}\right)}_{>0 \text{ for } n \text{ large enough.}}\ \right]$$

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(26)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+5}}{2n+5} \cdot \frac{2n+1}{x^{2n+1}}\right| = |x|^4 \frac{2n+1}{2n+5}$$

$$\to |x|^4 \Rightarrow |x| < 1$$

$x = +1$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} < \infty$ alternating series test.

$x = -1$ $\displaystyle\sum \frac{(-1)^n(-1)}{2n+1} < \infty$ alt. series test.

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(27) (a) Write as $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$ and think of this as $\sum_{k\geq 0} c_k x^k$, ie we need all multiples from each sum that make up x^k, ie need to have

$$c_k = \sum_{i+j=k} a_i b_j = \sum_{i=0}^{\infty} a_i b_{k-i}$$ so the result is $\displaystyle\sum_{k=0}^{\infty}\left(\sum_{i=0}^{\infty} a_i b_{k-i}\right)x^k$ as required

is

27 (b) $\frac{\log(1+x^2)}{1+x^2}$ (converges for $x^2<1$)

Set $t=x^2$

$$\frac{\log(1+t)}{1+t} = \left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}\cdots\right)(1+t)^{-1}$$

$$= \left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}\cdots\right)\left(1-t+t^2-t^3+t^4-t^5+\cdots\right)$$

$$= t+t^2\left(-1-\tfrac{1}{2}\right)+t^3\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)+t^4\left(-1-\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{4}\right)$$

$$+t^5\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}\right)+\cdots+t^{2n}\left(-1-\tfrac{1}{2}\cdots-\tfrac{1}{2n}\right)$$

$$+t^{2n+1}\left(1+\tfrac{1}{2}+\cdots+\tfrac{1}{2n+1}\right)-\cdots$$

28

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(iv)}(x)\cdots$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(iv)}(x)+\cdots$$

Add. $f(x+h)+f(x-h)=2f(x)+h^2f''(x)+O(h^4)$

$$\Rightarrow f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+O(h^2)$$

So the formula $f''(x)\approx(\quad\downarrow\quad)$ is 2nd order accurate.

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29 $1/e=e^{-1}$ $\quad e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$e^{-1}=1-1+\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\cdots+=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$

Alternating $\Rightarrow$ need n terms where $1/n!<10^{-3}$, ie $n\geq 7$

29 cont. 6 terms are.

$$e^{-1} \approx \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$$

Same terms as in the recording. Remainder was $\frac{e}{(n+1)!}$ but did not have enough size to affect accuracy of three decimals. It would play a role for 1, and 2 ~~digits~~ decimals accuracy.

30.

$$\log(1+\sqrt{\sin x}) = \sqrt{\sin x} - \frac{\sin x}{2} + \frac{(\sin x)^{3/2}}{3} - \frac{(\sin x)^2}{4} + \frac{(\sin x)^{5/2}}{5} - \frac{(\sin x)^3}{6} + \cdots$$

$$\sqrt{\sin x} = x^{1/2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)^{1/2}$$

$$= x^{1/2}\left(1 - \frac{x^2}{2\times 3!} + \frac{1}{2}\frac{x^4}{5!} - \frac{1}{2}\frac{x^6}{7!} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\frac{x^4}{(3!)^2} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(-\frac{2x^2 \cdot x^4}{3!\,5!}\right.\right.$$

$+$ higher order terms) (h.o.t.)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

so combining $\sqrt{\sin x} - \frac{\sin x}{2} \to x^{1/2} - \frac{x}{2} - \frac{x^{5/2}}{2\cdot 3!} - \frac{x^3}{3!} + \text{h.o.t.}$

$$\frac{1}{3}(\sin x)^{3/2} = \frac{x^{3/2}}{3}\left(1 - \frac{x^2}{3!} + \cdots\right)^{3/2} = \frac{x^{3/2}}{3} + O(x^{7/2})$$

$$(\sin x)^2 = x^2 + O(x^4)$$

$$(\sin x)^{5/2} = x^{5/2} + \text{h.o.t.}$$

$$(\sin x)^3 = x^3 + \text{h.o.t.}$$

and this is as far as I need to go.

$$\log(1+\sqrt{\sin x}) \approx x^{1/2} - \frac{x}{2} - \frac{x^{5/2}}{2\cdot 3!} - \frac{x^3}{3!} + \frac{x^{3/2}}{3} - \frac{1}{4}x^2 + \frac{1}{5}x^{5/2} - \frac{1}{6}x^3 + \cdots$$

i.e. 4 largest terms $= x^{1/2} - \frac{x}{2} + \frac{x^{3/2}}{3} - \frac{x^2}{4} + \cdots$