MATH40005

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
January 2023

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Probability and Statistics

Date: 9 January 2023

Time: 10:00 – 11:00

Time Allowed: 1 hour

This paper has 2 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Throughout the exam, we assume that $(\Omega, \mathcal{F}, \mathrm{P})$ denotes a probability space.

Please remember to justify all your answers and state carefully which results from the lectures you apply in your proofs.

Statistical tables are provided on the last page of the exam paper.

1. (a) For each set $\mathcal{F}$ given below, explain whether it is a sigma-algebra on $\Omega = \{2, 4, 6, 8, 10\}$.

   (i) $\mathcal{F} = \{\emptyset, \Omega, \{2, 4, 6, 8, 10\}, \{1, 3, 5, 7, 9\}\}$ (1 mark)

   (ii) $\mathcal{F} = \{\emptyset, \{2, 4\}\}$ (1 mark)

   (iii) $\mathcal{F} = \{\emptyset, \Omega, \{2, 4\}, \{6, 8, 10\}\}$ (1 mark)

   (iv) $\mathcal{F} = \{\emptyset, \Omega, \{2, 4\}, \{6, 8, 10\}, \{2, 4, 6\}, \{8, 10\}\}$ (1 mark)

   (b) Let $\Omega = \{5, 10, 15, 20, 25\}$. Give a non-trivial example of a partition of Ω, and show why your example is a partition of Ω. (2 marks)

   (c) Let $X \sim \mathrm{Bin}(n, p)$ with $n \in \mathbb{N}$, $p \in (0, 1)$. Find $\mathrm{E}((1 + X)^{-1})$. (4 marks)

   (d) Let $X \sim \mathrm{N}(0, 1)$ with cumulative distribution function denoted by Φ. Define the random variable

$$Y = \begin{cases} -2, & \text{if } |X| \leq 1, \\ 2, & \text{if } |X| > 1. \end{cases}$$

   You may assume without proof that Y is a discrete random variable.

   (i) Find the probability mass function of Y (in terms of Φ). (3 marks)

   (ii) Show that the random variables X and Y^2 are independent. (3 marks)

   (iii) Are the random variables X and Y independent? Justify your answer. (4 marks)

(Total: 20 marks)

2. (a) Consider the following identity: For $n \in \mathbb{N}$, we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{1}$$

Prove the identity (1) using a "story-proof", where you provide a suitable interpretation of the terms on the left-hand-side and the right-hand-side of the equation. (4 marks)

(b) Let X denote a geometric random variable with $\mathrm{Im}X = \mathbb{N} \cup \{0\}$, $p \in (0,1)$ and probability mass function given by

$$\mathrm{P}(X = x) = (1-p)^x p, \quad \text{for } x = 0, 1, 2, \ldots.$$

(i) Show that the probability generating function of X is given by

$$G_X(s) = \frac{p}{1 - s(1-p)},$$

for $|s| < (1-p)^{-1}$. (3 marks)

(ii) Find the mean of X. (2 marks)

(c) Let X, Y, Z denote independent random variables with $\mathrm{U}(0,1)$-distribution.
Set $U := XY$, $V := Z^2$.

(i) Find the joint cumulative distribution function of the random variables U and V. (6 marks)

(ii) Find $\mathrm{P}(U < V)$. (5 marks)

(Total: 20 marks)

Name of distribution of X	ImX	Parameters	p.m.f. $P(X=x)$/p.d.f $f_X(x)$
Discrete distributions			
Bernoulli: $X \sim \mathrm{Bern}(p)$	$\{0,1\}$	$p \in (0,1)$	$p^x(1-p)^{1-x}$
Binomial: $X \sim \mathrm{Bin}(n,p)$	$\{0,1,\ldots,n\}$	$n \in \mathbb{N}, p \in (0,1)$	$\binom{n}{x}p^x(1-p)^{n-x}$
Hypergeometric: $X \sim \mathrm{HGeom}(N,K,n)$	$\{0,1,\ldots,\min(n,K)\}$	$N \in \mathbb{N} \cup \{0\}$, $K,n \in \{0,1,\ldots,N\}$	$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$, for $x \in \{0,1,\ldots,K\}$, $n-x \in \{0,1,\ldots,N-K\}$
Poisson: $X \sim \mathrm{Poi}(\lambda)$	$\mathbb{N}_0 = \{0,1,2,\ldots\}$	$\lambda > 0$	$\frac{\lambda^x}{x!}e^{-\lambda}$
Geometric (1): $X \sim \mathrm{Geom}(p)$	$\mathbb{N} = \{1,2,\ldots\}$	$p \in (0,1)$	$(1-p)^{x-1}p$
Geometric (2): $X \sim \mathrm{Geom}(p)$	$\mathbb{N}_0 = \{0,1,2,\ldots\}$	$p \in (0,1)$	$(1-p)^x p$
Negative binomial: $X \sim \mathrm{NBin}(r,p)$	$\mathbb{N}_0 = \{0,1,2,\ldots\}$	$r \in \mathbb{N}, p \in (0,1)$	$\binom{x+r-1}{r-1}p^r(1-p)^x$
Continuous distributions			
Uniform: $X \sim \mathrm{U}(a,b)$	(a,b)	$a,b \in \mathbb{R}, a < b$	$\frac{1}{b-a}$
Exponential: $X \sim \mathrm{Exp}(\lambda)$	$(0,\infty)$	$\lambda > 0$	$\lambda e^{-\lambda x}$
Gamma: $X \sim \mathrm{Gamma}(\alpha,\beta)$	$(0,\infty)$	$\alpha,\beta > 0$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$
Chi-squared: $X \sim \chi^2(n)$ $(X \sim \chi^2_n)$	$(0,\infty)$	$n \in \mathbb{N}$	$\frac{1}{2\Gamma(n/2)}\left(\frac{x}{2}\right)^{n/2-1}e^{-x/2}$
Beta: $X \sim \mathrm{Beta}(\alpha,\beta)$	$(0,1)$	$\alpha,\beta > 0$	$\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
Normal: $X \sim \mathrm{N}(\mu,\sigma^2)$	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Student-t: $X \sim \mathrm{Student}(\nu)$	$\mathbb{R}$	$\nu > 0$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2023

This paper is also taken for the relevant examination for the Associateship.

MATH40005

Probability and Statistics (Solutions)

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. (a) (i) $\mathcal{F}$ is not a sigma-algebra on Ω since $\{1,3,5,7,9\} \not\subseteq \Omega$. sim. seen ⇓

(ii) $\mathcal{F}$ is not a sigma-algebra on Ω since it is not closed under complements, e.g. $\emptyset^c = \Omega \notin \mathcal{F}$. 1

1

(iii) $\mathcal{F}$ is a sigma-algebra on Ω since it has the form $\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$ for $A = \{2,4\}$ which is a sigma-algebra according to lectures. 1

(iv) $\mathcal{F}$ is not a sigma-algebra on Ω since it is not closed under unions, e.g. $\{2,4\} \cup \{8,10\} = \{2,4,8,10\} \notin \mathcal{F}$. 1

[1 mark per question to be awarded when the yes/no answer is correct and a correct justification is given, otherwise 0 marks.] meth seen ⇓

(b) For instance, we can consider the partition given by $B_1 = \{5,10\}, B_2 = \{15,20,25\}$. Then $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = \Omega$. Hence, B_1, B_2 constitute a partition of Ω. 2

[1 mark for stating a correct non-trivial partition, 1 mark for checking that the example is a partition of Ω.] meth seen ⇓

(c) Using the law of the unconscious statistician (LOTUS), we get

$$\mathbb{E}((1+X)^{-1}) = \sum_{x=0}^{n} \frac{1}{x+1}\binom{n}{x} p^x (1-p)^{n-x}.$$

Note that, for $x \in \{0, \ldots, n\}$, 1

$$\begin{aligned}\frac{1}{x+1}\binom{n}{x} &= \frac{n!}{(n-x)!x!(x+1)} = \frac{n!}{(n-x)!(x+1)!} \\ &= \frac{n!(n+1)}{(n-x)!(x+1)!(n+1)} = \binom{n+1}{x+1}\frac{1}{n+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}((1+X)^{-1}) &= \sum_{x=0}^{n} \frac{1}{x+1}\binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n+1}{x+1}\frac{1}{n+1} p^x (1-p)^{n-x} \\ &= \frac{1}{p(n+1)} \sum_{x=0}^{n} \binom{n+1}{x+1} p^{x+1} (1-p)^{(n+1)-(x+1)} \\ &\overset{\text{Replace x by x}-1}{=} \frac{1}{p(n+1)} \sum_{x=1}^{n+1} \binom{n+1}{x} p^{x} (1-p)^{(n+1)-x} \\ &= \frac{1}{p(n+1)} \left(\sum_{x=0}^{n+1} \binom{n+1}{x} p^{x} (1-p)^{(n+1)-x} - \binom{n+1}{0} p^0 (1-p)^{(n+1)-0} \right) \\ &\overset{\text{binomial theorem}}{=} \frac{1}{p(n+1)} \left((p+1-p)^{n+1} - (1-p)^{n+1} \right) \\ &= \frac{1}{p(n+1)} \left(1 - (1-p)^{n+1} \right).\end{aligned}$$

[1 mark for a correct application of the LOTUS, 3 marks for simplifying the expression and reaching the stated answer.] 3

(d) (i) We note that $\mathrm{Im} Y = \{-2, 2\}$ and the p.m.f. of Y is given by $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
P(Y = -2) &= P(|X| \leq 1) = P(-1 \leq X \leq 1) = \Phi(1) - \Phi(-1), \\
P(Y = 2) &= P(|X| > 1) = 1 - P(|X| \leq 1) = 1 - P(-1 \leq X \leq 1) \\
&= 1 - (\Phi(1) - \Phi(-1)) = (1 - \Phi(1)) + \Phi(-1) = 2\Phi(-1), \\
P(Y = y) &= 0, \quad \text{for } y \notin \{-2, 2\},
\end{aligned}$$

where Φ denotes the c.d.f. of a standard normal random variable. $\boxed{3}$

[1 mark for each of the two probabilities expressed in terms of Φ, 1 mark for the case when $y \notin \{-2, 2\}$.]

(ii) We note that $Y^2 = 4$ is constant, hence Y^2 is independent of any other random variable. More precisely, for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
P(X \leq x, Y^2 \leq y) &= P(X \leq x, 4 \leq y) = P(X \leq x)\mathbb{I}_{\{z : z \geq 4\}}(y) \\
&= P(X \leq x)P(4 \leq y) = P(X \leq x)P(Y^2 \leq y),
\end{aligned}$$

hence X and Y^2 are independent. $\boxed{3}$

[1 mark for stating that $Y^2 = 4$ and 2 marks for the proof of the independence]

(iii) No, X and Y are not independent. $\boxed{1}$

To see this, note that

$$\begin{aligned}
P(X \leq 1, Y \leq -2) = P(X \leq 1, Y = -2) = P(X \leq 1, |X| \leq 1) = P(|X| \leq 1) \\
P(X \leq 1)P(Y \leq -2) = P(X \leq 1)P(Y = -2) = P(X \leq 1)P(|X| \leq 1).
\end{aligned}$$

Since $P(X \leq 1) = \Phi(1) \neq 1$, we have $P(X \leq 1, Y \leq -2) \neq P(X \leq 1)P(Y \leq 2)$. $\boxed{3}$

[1 mark for stating correctly that X and Y are not independent, 3 marks for a correct proof, various different proofs are possible, only one is stated above]

2. (a) Left hand side: Recall that $\binom{n}{k}$ is defined as the number of subsets of size k for a set of size n. Summing over $k = 0, \ldots, n$ gives the total number of all possible subsets of a set of size n.

meth seen ⇓

Right hand side: 2^n is the cardinality of the set of all possible subsets of a set of size n since for each of the elements of the set we decide whether or not to include it in a particular subset, so there are always two choices for each element, and the multiplication principle leads the result. 4

[2 marks for mentioning that we count the number of all possible subsets of a set with cardinality n (or an equivalent story), 1 mark each for a correct interpretation of the LHS and the RHS.]

sim. seen ⇓

(b) (i) Using the LOTUS, we have

$$G_X(s) = \mathrm{E}(s^X) = \sum_{x=0}^{\infty} s^x (1-p)^x p = \sum_{x=0}^{\infty} (s(1-p))^x p = \frac{p}{1-s(1-p)},$$

where we used the geometric series assuming that $|s(1-p)| < 1 \Leftrightarrow |s| < (1-p)^{-1}$. 3

[2 marks for the correct calculations, 1 mark for the justification]

(ii) Differentiating the p.g.f. leads to

$$G'_X(s) = \frac{-p(1-p)(-1)}{(1-s(1-p))^2}.$$

Hence, $\mathrm{E}(X) = G'_X(1) = p(1-p)/p^2 = (1-p)/p$. 2

[1 mark for differentiating G_X and setting $s = 1$, 1 mark for the correct result]

unseen ⇓

(c) (i) For $0 < u < 1$, we have, using the law of total probability (LTP),

$$\mathrm{P}(XY \leq u) = \mathrm{P}(XY \leq u, Y \leq u) + \mathrm{P}(XY \leq u, Y > u).$$

We note that, since $\mathrm{Im}X = (0,1)$,

$$\mathrm{P}(XY \leq u, Y \leq u) = \mathrm{P}(Y \leq u) = F_Y(u) = u.$$

Also, using the continuous law of total probability with $Y \sim U((0,1))$,

$$\begin{aligned}
\mathrm{P}(XY \leq u, Y > u) = \mathrm{P}(X \leq u/Y, Y > u) &= \int_0^1 \mathrm{P}(X \leq u/Y, Y > u | Y = y) f_Y(y) dy \\
&= \int_0^1 \mathrm{P}(X \leq u/y, y > u | Y = y) dy \overset{(*)}{=} \int_u^1 \mathrm{P}(X \leq u/y) dy \\
&= \int_u^1 F_X(u/y) dy = \int_u^1 u/y dy = -u \log(u),
\end{aligned}$$

where we used the independence between X and Y in (*). Hence, $\mathrm{P}(XY \leq u) = u(1 - \log(u))$. 2

Also, for $0 < v < 1$, since $Z \sim U((0,1))$,

$$\mathrm{P}(Z^2 \leq v) = \mathrm{P}(-\sqrt{v} \leq Z \leq \sqrt{v}) = F_Z(\sqrt{v}) - F_Z(-\sqrt{v}) = \sqrt{v}.$$

1

Let us define $U := XY, V := Z^2$. Since X, Y, Z are independent, U and V are independent. 1

Altogether, we have

1. for $u, v \in (0,1)$, we have $F_{U,V}(u,v) = u(1-\log(u))\sqrt{v}$;
2. for $u \le 0$ or $v \le 0$, we have $F_{U,V}(u,v) = 0$;
3. for $u \ge 1, v \in (0,1)$, we have $F_{U,V}(u,v) = \sqrt{v}$;
4. for $v \ge 1, u \in (0,1)$, we have $F_{U,V}(u,v) = u(1-\log(u))$;
5. for $u, v \ge 1$, we have $F_{U,V}(u,v) = 1$; $\boxed{2}$

[2 marks for correctly deriving the cdf of XY, 1 mark for correctly deriving the cdf of Z^2, 1 mark for justifying the independence between U and V, 1 mark for stating the joint cdf correctly for $u, v \in (0,1)$, 1 mark for stating cases 2.-5. correctly (if incomplete, this mark should be dropped.)]

(ii) **Method 1:** Differentiating the joint c.d.f. leads to the joint density function

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v) = [(1-\log(u)) + u/(-u)]\frac{1}{2\sqrt{v}} = \frac{-\log(u)}{2\sqrt{v}},$$

for $u, v \in (0,1)$ and 0 otherwise. Then,

$$\begin{aligned}
\mathrm{P}(U < V) &\overset{M1}{=} \int_0^1 \int_0^v f_{U,V}(u,v)dudv = \int_0^1 \frac{-1}{2\sqrt{v}} \int_0^v \log(u)dudv \\
&= -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{v}} v(\log(v)-1)dv = -\frac{1}{2}\left(\int_0^1 \sqrt{v}\log(v)dv - \int_0^1 \sqrt{v}dv\right) \\
&= -\frac{1}{2}\left(-\frac{4}{9} - \frac{2}{3}\right) = \frac{5}{9},
\end{aligned}$$

since

$$\int_0^1 \sqrt{v}\log(v)dv = \frac{2}{3}x^{3/2}\log(x)\Big|_0^1 - \frac{2}{3}\int_0^1 x^{3/2}x^{-1}dx = 0 - 0 - \frac{4}{9}x^{3/2}\Big|_0^1 = -\frac{4}{9},$$
$$\int_0^1 \sqrt{v}dv = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}.$$

[Method 1: 1 mark for joints density of U, V, 2 marks for stating M1 above for a correct choice of the bounds in the integrals, 1 mark for the integration, 1 mark for the correct result.]

Method 2: We note that X, Y, Z are independent and $\mathrm{U}(0,1)$ distributed. Hence, their joint density is given by

$$f_{X,Y,Z}(x,y,z) = 1, \quad \text{for } x, y, z, \in (0,1),$$

and 0 otherwise. Using a result from lectures, we know that, for $A := \{(x,y,z) : x, y, z \in (0,1), xy \le z^2\}$,

$$\begin{aligned}
\mathrm{P}(XY \le Z^2) &\overset{M2}{=} \int\int\int_A dzdxdy = \int_0^1\int_0^1\int_{\sqrt{xy}}^1 dzdxdy = \int_0^1\int_0^1 (1-\sqrt{xy})dxdy \\
&= \int_0^1 (1-\sqrt{y}\frac{2}{3})dy = 1 - \frac{4}{9} = \frac{5}{9}.
\end{aligned}$$

$\boxed{5}$

[Method 2: 1 mark for joint density of X, Y, Z, 2 marks for stating M2 above for a correct choice of the set A, 1 mark for the integration, 1 mark for the correct result.]

Review of mark distribution:

Total marks: 40 of 40 marks