Imperial College London

MATH97062 MATH70029

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Functional Analysis

Date: 10 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Give a detailed proof that the space

$$\mathbb{M}_p(\mathbb{C}) \equiv \left\{ A \equiv (A_{ij} \in \mathbb{C})_{i,j\in\mathbb{N}} : \sum_{i,j} |A_{ij}|^p < \infty \right\}$$

for $p \geq 1$ with coordinate-wise defined addition and multiplication by a scalar, and with a norm

$$\|A\| \equiv (\sum_{i,j} |A_{ij}|^p)^{\frac{1}{p}}$$

is a metric linear space. (6 marks)

(b) Prove or disprove that the space in (a) is :

(i) separable; (5 marks)

(ii) complete. (4 marks)

(c) Prove or disprove that the space in (a) is a Hilbert space if $p = 6$. (5 marks)

(Total: 20 marks)

2. (a) Prove that the space $\mathcal{C}([0,1])$ of continuous real function on the unit interval is a closed subspace of the space of bounded real functions $\mathbb{B}([0,1])$on $[0,1]$ with the supremum norm $\|f\|_u \equiv \sup_{x\in[0,1]} |f(x)|$.

(7 marks)

(b) Prove that any subset in $\mathcal{C}([0,1])$ is of first category in the Banach space $(\mathbb{B}([0,1]), \|\cdot\|_u)$.
(6 marks)

(c) Consider the following linear subspace of $\mathbb{M}_2(\mathbb{C})$

$$W \equiv \{(A_{ij} \in \mathbb{C})_{ij\in\mathbb{N}} : \forall j \quad A_{jj} = 0\}$$

with the norm

$$\|A\| = \left(\sum_{ij} |A_{ij}|^2 \right)^{\frac{1}{2}}.$$

Let z be a bounded linear functional on this space. Find a Hahn-Banach extension of this functional to $\mathbb{M}(\mathbb{C})$ which has the same norm as z on the space W.

(7 marks)

(Total: 20 marks)

3. (a) For $n \in \mathbb{Z}$, let $e_n \equiv e^{inx}$ and let

$$d\nu \equiv \frac{1}{2\pi} d\lambda.$$

For $N \in \mathbb{N}$, let

$$Tf(x) = e_k(x) + K_\xi f(x) \equiv e_k(x) + \xi \sum_{n=k+1}^{k+N} e_n(x) \int e_{-n}(y) f(y) \nu(dy). \tag{*}$$

Decide for what values of $\xi \in \mathbb{C}$ this operator is contractive in $\mathbb{L}_2([0, 2\pi]), \nu)$. (5 marks)

(b) Prove that for $\xi = 1$ the operator in (*) is not strictly contractive, but nevertheless has a fixed point. (5 marks)

(c) Prove that the operator defined as follows

$$Kf \equiv \sum_{n \in \mathbb{N}} \lambda_n e_n(x) \int e_{-n}(y) f(y) \nu(dy)$$

is compact in $\mathbb{L}_2([0, 2\pi], \mu)$ if and only if $\lambda_n \to_{n \to \infty} 0$. (10 marks)

(Total: 20 marks)

4. (a) (5 marks)

Consider the normed space $\mathcal{C}([0,1])$ with supremum norm $\|f\|_u \equiv \sup_{t \in [0,1]} |f(t)|$.

Suppose $g \in \mathcal{C}([0,1])$ is such that $\inf g = g(t)$ for some $t \in (0,1)$. Find a continuous functional l on $(\mathcal{C}([0,1]), \|\cdot\|_u)$ such that

$$l(g) = \inf g.$$

(b) (6 marks)

Find a continuous linear functional l on $(\mathcal{C}([0,1]), \|\cdot\|_u)$ which vanishes on the linear subspace

$$V \equiv \{f \in \mathcal{C}([0,1]) : \ f(0) = 0\}$$

and has value 1 on $g(t) = \cos(\pi t)$.

(c) Let

$$V \equiv \{f \in \mathcal{C}([0,1]) : \ f(0) = 0\}.$$

be a linear space furnished with the supremum norm. Let $T : \mathcal{C}([0,1]) \to V$ be the linear operator defined by

$$Tv := \begin{cases} v & if\ v \in V \\ 0 & otherwise. \end{cases}$$

Prove or disprove that this operator is closed. (9 marks)

(Total: 20 marks)

5. (a) Define $W_{k,p}(\Omega)$ space and show that for $p = 2$ the corresponding norms are equivalent to the norms of the Fourier transform of a function multiplied by suitable polynomial factor.

(6 marks)

(b) State Kolmogorov-Riesz theorem.

(4 marks)

(c) Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be an open set and χ_Ω denote its characteristic function. For $x \in \Omega$ define

$$P_t f(x) \equiv \frac{1}{(2\pi t)^{d/2}} \chi_\Omega(x) \int_\Omega e^{-\frac{(y-x)^2}{2t}} f(y) \lambda(dy)$$

for any $f : \mathbb{R}^d \to \mathbb{R}$ for which the right hand side is well defined.

(i) Prove or disprove that for any $k, n \in \mathbb{N}$ and $r, s \in [1, \infty]$, the above formula defines a bounded operator

$$P_t : W_{k,r}(\mathbb{R}^d) \to W_{n,s}(\mathbb{R}^d)$$

Hint: You can use the following Young's Inequality without giving a proof. For $p, q, r \in [1, \infty]$ *satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

if $f \in L_p$ *and* $g \in L_q$*, then*

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q.$$

(ii) Prove or disprove that for $t > 0$ and $\Omega = [0, 1]^d$ the following set

$$P_t(\{f\chi_\Omega : \|f\|_2 \leq 1\})$$

is compact in $\mathbb{L}_p(\mathbb{R}^d)$ for any $p \in [1, \infty)$.

(10 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MMATH96037/MATH97062/MATH97173

Functional Analysis (Solutions)

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1. (a) Let $\boxed{\text{sim. seen} \Downarrow}$

$$\mathbb{M}_p(\mathbb{C}) \equiv \left\{ A \equiv (A_{ij} \in \mathbb{C})_{i,j\in\mathbb{N}} : \sum_{i,j} |A_{ij}|^p < \infty. \right\}$$

For $A, B \in \mathbb{M}_p(\mathbb{C})$ coordinate-wise additions is defined by

$$(A \oplus B)_{ij} = A_{ij} + B_{ij}$$

and multiplication by a scalar $\lambda \in \mathbb{C}$ is defined by

$$(\lambda \odot A)_{ij} = \lambda \cdot A_{ij}.$$

Since

$$|(A \oplus B)_{ij}|^p \leq \max(2^{p-1}, 1) \left(|A_{ij}|^p + |B_{ij}|^p\right)$$

we have $\oplus : \mathbb{M}_p(\mathbb{C}) \times \mathbb{M}_p(\mathbb{C}) \to \mathbb{M}_p(\mathbb{C})$ is well defined. Because the condition defining $\mathbb{M}_p(\mathbb{C})$ is homogeneous, it is clear that coordinate-wise multiplication by a scalar is well defined. The function $\|\cdot\| : \mathbb{M}_p(\mathbb{C}) \to \mathbb{R}^+$ given by

$$\|A\| \equiv (\sum_{i,j} |A_{ij}|^p)^{\frac{1}{p}}$$

is a norm, as it vanishes only if all coordinates of A are zero, it is homogeneous and satisfies Minkowski inequality. Since the metric given by the norm $\rho_p(A, B) \equiv \|A \ominus B\|_p$ is translation invariant the addition of vectors $\oplus$ is continuous, because

$$\begin{aligned}\rho_p(A \oplus B, A' \oplus B') &= \|(A \oplus B) \ominus (A' \oplus B')\|_p = \|(A \ominus A') \oplus (B \ominus B')\|_p \\ &\leq \|(A \ominus A')\|_p + \|(B \ominus B')\|_p = \rho_p(A, A') + \rho_p(B, B').\end{aligned}$$

On the other hand the following relation implies continuity of $\odot$

$$\begin{aligned}\rho(\lambda \odot A, \lambda' \odot A') &= \|(\lambda \odot A) \ominus (\lambda' \odot A')\|_p = \|((\lambda - \lambda') \odot A) \oplus (\lambda' \odot (A \ominus A'))\|_p \\ &\leq |\lambda - \lambda'| \cdot \|A\|_p + |\lambda'| \cdot \|A \ominus A'\|_p.\end{aligned}$$

$\boxed{6, \text{A}}$

(b)

(i) We note that the subspace $\mathbb{M}_{p,o}(\mathbb{C})$ consisting of vectors which have all but finite number of coordinates equal to zero is dense in $\mathbb{M}_p(\mathbb{C})$. Moreover the subspace $\mathbb{M}_{p,o}(\mathbb{Q} + i\mathbb{Q})$ is countable and dense in $\mathbb{M}_{p,o}(\mathbb{C})$. Thus this last subspace is also dense in $\mathbb{M}_p(\mathbb{C})$. This completes the proof that the space $\mathbb{M}_p(\mathbb{C})$ is separable. $\boxed{1, \text{A}}$

(ii) To show the completeness consider a Cauchy sequence $A^{(n)}, n \in \mathbb{N}$, in the space $\mathbb{M}_p(\mathbb{C})$. We note that for every $i, j \in \mathbb{N}$ and any $m, n \in \mathbb{N}$ $\boxed{4, \text{B}}$

$$|A_{ij}^{(n)} - A_{ij}^{(m)}| \leq \|A^{(n)} - A^{(m)}\|.$$

Thus, given ij, the corresponding coordinates $A_{ij}^{(n)}$, $n \in \mathbb{N}$, form a Cauchy sequence in $\mathbb{C}$ and since complex numbers are complete, this sequence is convergent to some number A_{ij}. Define $A \equiv (A_{ij})_{i,j\in\mathbb{N}}$ we need to show that $A \in \mathbb{M}_p(\mathbb{C})$. To this end we note that for any $L \in \mathbb{N}$ we have

$$\sum_{i,j \leq L} |A_{ij}^{(n)}|^p \leq \|A^{(n)}\|^p \leq C < \infty$$

and the constant C on the right hand side is independent of n for our Cauchy sequence (which one can see applying the definition of Cauchy sequence e.g. with $\varepsilon = 1$). Hence passing to the limit $n \to \infty$ we get

$$\sum_{i,j \le L} |A_{ij}|^p \le C < \infty$$

for any $L \in \mathbb{N}$, which implies that

$$\sum_{i,j \in \mathbb{N}} |A_{ij}|^p < \infty.$$

This ends the proof of completeness. $\boxed{4, \text{A}}$

(c) We note that the norm in (a) is equivalent to the norm given by the scalar product iff the following parallelogram identity holds.

$$\|A \oplus B\|^2 + \|A \ominus B\|^2 = 2\|A\|^2 + 2\|B\|^2$$

In case $p = 2$ this can be checked directly. For $p \neq 2$ one can provide a counterexample choosing A to have one coordinate equal to 1 and all other equal to zero and B similar except that it is non zero for a different coordinate. In this case, using the definition of the norm, we have

$$\|A \oplus B\|^2 = \|A \ominus B\|^2 = 2^{\frac{2}{p}}$$

while $\|A\|^2 = 1$ and $\|B\|^2 = 1$ and hence

$$2\|A\|^2 + 2\|B\|^2 = 4 \neq 2 \cdot 2^{\frac{2}{p}}.$$

for $p \neq 2$. Thus parallelogram identity fails for $p \neq 2$ and hence such space cannot be a Hilbert space. $\boxed{5, \text{D}}$

2. (a) Let $f_n \in \mathcal{C}([0,1])$ be a Cauchy sequence with respect to the norm $\|\cdot\|_u$. This in particular implies the pointwise convergence because for any $x \in [0,1]$ and for any $m, n \in \mathbb{N}$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u$$

which means for any $x \in [0,1]$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy and, since real line is complete, so it converges. Define a function

$$f(x) \equiv \lim_{n\to\mathbb{N}} f_n(x).$$

It follows from its definition, that f has finite supremum norm. We note that for any $n \in \mathbb{N}$

$$|f(x)-f(y)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f(y)-f_n(y)| \leq 2\|f-f_n\|_u+|f_n(x)-f_n(y)|.$$

Thus given $\varepsilon > 0$, we can choose n so that

$$\|f - f_n\|_u < \frac{1}{3}\varepsilon$$

and using continuity of f_n, we get that $\exists\delta > 0$ such that for $|x - y| < \delta$

$$|f_n(x) - f_n(y)| < \frac{1}{3}\varepsilon.$$

Hence for $|x - y| < \delta$, we get

$$|f(x) - f(y)| < \varepsilon.$$

This shows closedness of $\mathcal{C}([0,1])$ in the space $(\mathbb{B}, \|\cdot\|_u)$.

sim. seen ⇓

4, A

unseen ⇓

3, C

meth seen ⇓

(b) To show that any subset in $\mathcal{C}([0,1])$ is of first category in the Banach space $(\mathbb{B}([0,1]), \|\cdot\|_u)$, it is enough to show that $\mathcal{C}([0,1])$ in the space $(\mathbb{B}([0,1]), \|\cdot\|_u)$ does not contain any open ball. To this end, for $f \in \mathcal{C}([0,1])$ consider a ball

$$B_\varepsilon(f) \equiv \{g \in \mathbb{B}([0,1]) : \|f - g\|_u < \varepsilon\}$$

If $h \in \mathbb{B}([0,1]) \setminus \mathcal{C}([0,1])$ is such that $\|h\|_u = 1$, (such functions exist, e.g. a characteristic function of $\mathbb{Q} \cap [0,1]$ or Rademacher functions), then we have $f + \frac{\varepsilon}{2}h \in B_\varepsilon(f) \setminus \mathcal{C}([0,1])$. Since $\varepsilon > 0$ was arbitrary, this means that does not contain any open ball, and hence is nowhere dense. 3, A

(c) Since the subspace W is closed, it is a Hilbert space in itself. Then, by Riesz representation theorem there exists a unique vector $u_z \in W$ such that 3, B

2, A

$$z(v) = \langle u_z, v\rangle$$

and

$$\|z\| = \|u_z\|.$$

By Hahn-Banach theorem, there exists a functional $\tilde{z}$ on $\mathbb{M}(\mathbb{C})$ such that its restriction to W coincides with z and its norm satisfies $\|\tilde{z}\| = \|z\|$. Again invoking Riesz representation theorem on $\mathbb{M}(\mathbb{C})$, we can find a unique vector $\tilde{u}_z \in \mathbb{M}(\mathbb{C})$ representing $\tilde{z}$ and such that $\|\tilde{u}_z\| = \|\tilde{z}\| = \|z\| = \|u_z\|$. Since by our assumption 3, B

$$\langle u_z, w\rangle = \langle \tilde{u}_z, w\rangle$$

for all $w \in W$, the vector $\tilde{u}_z - u_z$ is orthogonal to W and in particular to u_z. By parallelogram identity we have

$$\|\tilde{u}_z + u_z\|^2 + \|\tilde{u}_z - u_z\|^2 = 2\|u_z\|^2 + 2\|\tilde{u}_z\|^2$$

and hence, using $\tilde{u}_z - u_z \perp u_z$, we get

$$\begin{aligned}\|\tilde{u}_z + u_z\|^2 + \|\tilde{u}_z - u_z\|^2 &= \|\tilde{u}_z - u_z + 2u_z\|^2 + \|\tilde{u}_z - u_z\|^2 \\ &= 2\|\tilde{u}_z - u_z\|^2 + 4\|u_z\|^2.\end{aligned}$$

This together with parallelogram identity implies

$$\|\tilde{u}_z - u_z\|^2 = \|\tilde{u}_z\|^2 - \|u_z\|^2 = 0.$$

Thus we conclude that

$$\tilde{u}_z = u_z.$$

$\boxed{2,\text{ C}}$

3. (a) Using orthonormality of e_j's, we have unseen $\Downarrow$

$$\|Tf - Tg\|_2^2 = \|K_\xi f - K_\xi g\|_2^2 = |\xi|^2 \sum_{n=k+1}^{k+N} \left| \int e_{-n}(y)(f-g)(y)\nu(dy) \right|^2 \leq |\xi|^2 \|f-g\|_2^2.$$

Thus T is strictly contractive for $|\xi| < 1$. 7, A

(b) For $\xi = 1$, choosing $f = e_n$, for any $n = k+1, .., N+1$ we get equality in the last inequality in (a). Since e_j's are orthonormal, by direct calculation we get $T(e_k + e_n) = e_k + e_n$, for any $n = k+1, .., N+1$. unseen $\Downarrow$ 2, B

(c) Define 3, C

$$K_n f \equiv \sum_{n=1}^{n} \lambda_n e_n(x) \int e_{-n}(y) f(y) \nu(dy).$$

unseen $\Downarrow$

Remark: Note that

$$\|K_n f\|^2 = \sum_{n=1}^{n} |\lambda_n|^2 \left| \int e_n(y) f(y) \nu(dy) \right|^2$$

Thus K is bounded iff the sequence $|\lambda_n|$ is bounded. We can also note that if

$$\sum_{n=1}^{n} |\lambda_n|^2 < \infty$$

then K is Hilbert-Schmidt and hence compact. If we define

$$K_N f = \sum_{n=1}^{N} \lambda_n e_n \int e_{-n}(y) f(y) \nu(dy)$$

then it is a finite rank operator and so it is compact. If $\lambda_n \to 0$ as $n \to \infty$, we can show that the sequence $K_N,\ N \in \mathbb{N}$, of compact operators converges in the operator norm to K as follows. First of all we note that

$$\|(K_N - K)f\|_2^2 = \sum_{n \geq N+1} |\lambda_n|^2 \left|\langle e_{-n}, f\rangle\right|^2 \leq \sup_{n \geq N+1} |\lambda_n|^2 \|f\|.$$

and hence

$$\|(K_N - K)\| \leq \sup_{n \geq N+1} |\lambda_n|^2 \to_{N \to \infty} 0$$

Since convergence in operator norm of a sequence of compact operators implies that the limit operator is also compact, we get compactness of K under the assumption that $\lambda_n \to_{n \to \infty} 0$.

On the other hand if $\lambda_n \to_{n \to \infty} 0$ does not hold, there exists $\varepsilon > 0$ and a subsequence λ_{n_k}, $k \in \mathbb{N}$ such that

$$|\lambda_{n_k}| \geq \varepsilon.$$

Then by direct calculation we can see that for a bounded sequence e_{n_k}, we have

$$\|Te_{n_k} - Te_{n'_k}\|_2^2 = |\lambda_{n_k}|^2 + |\lambda_{n'_k}|^2 \geq 2\varepsilon,$$

i.e. one cannot chose a convergent subsequence. Thus K cannot be compact. 2, B

6, D

4. (a) Suppose $g \in \mathcal{C}([0,1])$ is such that $\inf g = g(t)$ for a given $t \in (0,1)$. (Since we are having a continuous function on a compact interval the extrema are attained.) Consider the functional

unseen ⇓

5, A

$$l(f) = f(t).$$

It is continuous as for any $f \in \mathcal{C}([0,1])$ we have

$$|l(f)| = |f(t)| \le \sup_{s\in[0,1]} |f(s)|$$

For this functional we have

$$l(g) = g(t) = \inf g$$

by the choice of the point t.

unseen ⇓

(b) Consider the functional $l : \mathcal{C}([0,1]) \to \mathbb{R}$ defined by

2, B

$$l(f) = f(0).$$

2, C

It follows from (a) that the functional is continuous. Since for any $x \in V$, by definition of V, we have $x(0) = 0$. Thus $l(x) = 0$ for any $x \in V$. On the other hand $l(\cos(\pi\cdot)) = \cos(0) = 1$.

2, B

unseen ⇓

(c) We recall that for two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, a linear operator $A : X \to Y$ is called closed if its graph $\mathcal{G}(A) \equiv \{(x, Ax) : x \in X\}$ is closed with respect to the norm $\|(x, Ax)\| \equiv \|x\|_X + \|Ax\|_Y$. Equivalently, A is closed if

2, C

5, D

$$x_n \to x \;\; and \;\; Ax_n \to y \Rightarrow x \in X \;\; and \;\; Ax = y.$$

2, B

In the case of interest to us the space V is a closed subspace which can be shown in similar considerations as in 2 (b) & (c). If one considers a vector $h \notin V$ and a vector $x \in V$, then a sequence $x_n \equiv x + \frac{1}{n}h$ is contained in $\mathcal{C}([0,1]) \setminus V$. Thus $Tx_n = 0$ an so $\lim_{n\to\infty} Tx_n = 0$. On the other hand $x_n \to_{n\to\infty} x \in V$ and thus the second conditions of closedness fails.

Remark: Note that for sequences in V as well as for sequences convergent to a point in the open set $\mathcal{C}([0,1]) \setminus V$ there is no problem in the closedness condition.

5. (a) For a nonnegative integer $m \in \mathbb{Z}^+$ and $p \in [1, \infty]$, and for an open set $\Omega \subset \mathbb{R}^N$, the space $W_{m,p}(\Omega)$ is by definition the space of (equivalence classes of) functions $f \in \mathbb{L}_p(\Omega)$ such that $\nabla^\alpha f \in \mathbb{L}_p(\Omega)$ for all (weak) derivatives ∇^α of length $|\alpha| \leq m$, that is for any $\varphi \in \mathcal{C}^\infty$ compactly supported in Ω we have seen ⇓ 3, M

$$\int f \nabla^\alpha \varphi d\lambda = (-1)^{|\alpha|} \int g \varphi d\lambda$$

with some $g \in \mathbb{L}_{1,loc}(\Omega)$. It is a normed space equipped with the norm

$$\|f\| = \sum_{|\alpha| \leq m} \|\nabla^\alpha f\|_{\mathbb{L}_p(\Omega)}$$

or equivalent norm, for $p \in [1, \infty)$

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \int_\Omega |\nabla^\alpha u|^p d\lambda \right)^{\frac{1}{p}}$$

while for $p = \infty$

$$\|u\|_{m,p} = \max_{|\alpha| \leq m} \|\nabla^\alpha u\|_{\mathbb{L}_\infty(\Omega)}$$

Using similar arguments as in case of $\mathbb{L}_p(\Omega)$ based on absolutely convergent series, one can show that $W_{m,p}(\Omega)$ are Banach spaces. To prove the separability it is enough to observe that the set of m times continuously differentiable functions with bounded derivatives $\mathcal{C}_b^m(\Omega)$ is dense in $W_{m,p}(\Omega)$ and use the fact proved in the course that $\mathcal{C}_b^m(\Omega)$ is separable.

3, M

For $p = 2$ and $\Omega = \mathbb{R}^n$, we note that

$$\int |\nabla^\alpha u|^2 d\lambda = \int |q^\alpha \mathcal{F}(u)|^2 \lambda(dq) \qquad (*)$$

where $\mathcal{F}(u)$ denotes the Fourier transform of the function $u \in \mathbb{L}_2$ and $x^\alpha \equiv \prod_{j=1,..,n} (x_j)^{\alpha_j}$. One can prove this relation directly for functions in the Schwartz class $\mathcal{S}$. In this case we have $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ which allows to interchange differentiation and integration in the Fourier transform formula to get

$$\nabla^\alpha \mathcal{F}^{-1}(\mathcal{F}(u)) = \mathcal{F}^{-1}\left(q^\alpha \mathcal{F}(u)\right)$$

and the formula $(*)$ follows from the fact that $\mathcal{F}$ and its inverse are unitary in $\mathbb{L}_2(\lambda)$.

(b) **The Kolmogorov–Riesz theorem** seen ⇓

A subset $\mathcal{F} \subset \mathbb{L}_p(\mathbb{R}^n)$, $p \in [1, \infty)$ is totally bounded if and only if the following two conditions hold. 4, M

(i)(Tightness) $\forall \varepsilon > 0\ \exists R \in (0, \infty)$ such that

$$\sup_{f \in \mathcal{F}} \int_{|x|>R} |f(x)|^p d\lambda \leq \varepsilon^p$$

(ii) ($\mathbb{L}_p$ equicontinuity) $\forall \varepsilon > 0\ \exists \delta \in (0, \infty)$ such that $\forall |y| < \delta$

$$\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p d\lambda \leq \varepsilon^p$$

Remark: Recall the following facts. A metric space is called totally bounded if it admits an epsilon-cover for every positive epsilon.

A subset is pre-compact if its closure is compact.

Theorem: In metric spaces sequential compactness equals compactness.

Theorem: A subset of a complete metric space is pre-compact if and only if it is totally bounded.

Remark: The equicontinuity condition can be replaced by

$$\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} |f(x) - S_\delta f(x)|^p d\lambda \leq \varepsilon^p$$

where $S_\delta f$ denotes the Steklov average over a ball with centre x and radius δ.

unseen ⇓

(c) First of all we note that for any smooth compactly supported function f and any multi-index α, we have

$$\nabla^\alpha P_t f(x) \equiv \frac{1}{(2\pi t)^{d/2}} \int_\Omega \left(\nabla^\alpha e^{-\frac{(y-x)^2}{2t}} \right) f(y)\lambda(dy) \equiv \int_\Omega \left(p_\alpha(y-x) \frac{1}{(2\pi t)^{d/2}} e^{-\frac{(y-x)^2}{2t}} \right) f(y)\lambda(dy)$$

where p_α is a polynomial of order $|\alpha|$. Then we have

$$\|\nabla^\alpha P_t f\|_r = \|\eta * f\|_r$$

where

$$\eta(y) \equiv \frac{1}{(2\pi t)^{d/2}} p_\alpha(y) e^{-\frac{y^2}{2t}}.$$

Next recall that the following Young inequality for convolutions holds. For $\eta \in \mathbb{L}_p(\mathbb{R}^d)$ and $\xi \in \mathbb{L}_q(\mathbb{R}^d)$, if

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

with $1 \leq p, q \leq r \leq \infty$, then

$$\|\eta * \xi\|_r \leq \|\eta\|_q \|\xi\|_p.$$

Hence for any $p, q \in (0, \infty)$

$$\|\nabla^\alpha P_t f\|_r = \left(\int |\eta * f|^r \, \lambda(dx) \right)^{\frac{1}{r}} \leq \|p_\alpha e^{-\frac{w^2}{2t}}\|_q \cdot \|f\|_p$$

Since $p_\alpha e^{-\frac{w^2}{2t}} \in \mathbb{L}_q$ for any $q \in [1, \infty]$, using the definition of $W_{u,v}$ norms, for any $k, n \in [1, \infty)$, and $w, z \in \mathbb{Z}^+$, we have

$$\|P_t f\|_{k,w} \leq C \|f\|_{n,z}$$

with some constant $C \in (0, \infty)$ for all $f \in W_{n,z}$.

(ii) Hence for $f \in \mathbb{L}_r$, we have $P_t(\chi_\Omega f) \in W_{1,p}$, for any $p \leq r$ Using this, in particular with $2 \leq r$, together with the Relich-Kodrachov compact embedding theorem we conclude that

$$\{P_t \chi_\Omega f : \|f\|_2 \leq 2\}$$

is compact in $\mathbb{L}_2(\Omega)$.

Alternatively one can use the fact that for any $s > 0$ the kernel η_s decays faster than exponentially with the distance from the set $\Omega = [0, 1,]^d$. This provides tightness and smallness of the part of the integral in the region far from Ω in ($\mathbb{L}_p$ equicontinuity) while equicontinuity in the remaining part of the integral follows from continuity of the kernel η. $\boxed{10, \mathrm{M}}$

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Functional Analysis_MATH60029 MATH97062 MATH70029	1	It was not clear to some students what the question was asking, and because of this partial credit was lost on some easy parts. I checked the lecture notes, and it is fairly clear what a metric linear space is.
Functional Analysis_MATH60029 MATH97062 MATH70029	2	Fairly routine questions for the module.
Functional Analysis_MATH60029 MATH97062 MATH70029	3	This is a fairly difficult questions, but a good number of students were able to penetrate into the problem, and completely answer it.
Functional Analysis_MATH60029 MATH97062 MATH70029	4	standard problem in the module.
Functional Analysis_MATH60029 MATH97062 MATH70029	5	parts a and b of the problem were meant for a closed book exam, but since the lecturer had left Imperial, and the move to online examination occurred rather fast, this question was not adjusted. Parts c and d of the problem are rather difficult.