# MATH96046/MATH97073 Statistical Theory: Coursework

Due on Tuesday 14 March 2023 at 13:00 (1pm)

**Instructions**: Solutions can be either typed or handwritten. Please write legibly, since illegible scripts may be at a disadvantage. Show your working: answers without justification will not receive marks.

1. Suppose that $X_1, \ldots, X_n \sim^{iid} U[0, \theta]$, where $U[0, \theta]$ is the uniform distribution on $[0, \theta]$ having probability density $\frac{1}{\theta} 1_{[0,\theta]}(x)$ on $\mathbb{R}$. The *Pareto distribution* with parameters $\alpha, x_0 > 0$, denoted $Par(\alpha, x_0)$, is a continuous distribution on $\mathbb{R}$ with density function

$$\begin{cases} \alpha x_0^\alpha y^{-\alpha-1} & y \geq x_0, \\ 0 & y < x_0. \end{cases}$$

   (a) Do the families $\{U[0, \theta] : \theta > 0\}$ and $\{Par(\alpha, x_0) : \alpha, x_0 > 0\}$ constitute exponential families? What if we fix $x_0 > 0$ and consider the restricted family $\{Par(\alpha, x_0) : \alpha > 0\}$? Justify your answers. [2 marks]

   (b) Consider the Bayesian model where we assign the prior $\theta \sim Par(\alpha, k)$. Compute the posterior distribution of $\theta$ given the observations $X_1, \ldots, X_n$. Give an interpretation for the prior parameters $\alpha$ and $k$.
   normalise before calculating posterior mean!!!!
   Compute the Bayesian estimator for $\theta$ (also called $\pi$-Bayes decision rule) for squared error loss. Is it an admissible estimator? Justify your answer.
   *[You may find it helpful to define $M_n = \max_{0 \leq i \leq n} X_i$ with $X_0 = x_0$].* [4 marks]

   (c) Find the Jeffreys prior for $\theta$ and the corresponding posterior distribution. [2 marks]
   *[Hint: extend the usual log-likelihood function $\ell_n : \Theta \to (\infty, \infty)$ to a function taking extended values $\ell_n : \Theta \to [-\infty, \infty)$ by setting $\ell_n(\theta) = -\infty$ whenever the likelihood is zero. You can assume that for any $\theta$ such that $\ell_n(\theta) = -\infty$, we have $\ell_n'(\theta) = 0$. The resulting function should be differentiable except on a set $A$ such that $P_\theta(X_1 \in A) = 0$, so you can effectively ignore the points of non-differentiability.]*

Consider now the frequentist model where $X_1, \ldots, X_n \sim^{iid} U[0, \theta_0]$ for some 'true' $\theta_0 > 0$ and $n \geq 3$, and consider estimation of $\theta$ using *squared error loss* $L(a, \theta) = (a - \theta)^2$. Recall from the lectures that the MLE for $\theta$ in this model is $\hat{\theta}_n = \max_{1 \leq i \leq n} X_i$. In Q2 on Problem Sheet 1, it is further shown that

$$E_\theta \max_{1 \leq i \leq n} X_i = \frac{n}{n+1}\theta, \qquad \mathrm{Var}_\theta\left(\max_{1 \leq i \leq n} X_i\right) = \frac{n}{(n+1)^2(n+2)}\theta^2,$$

$$\mathrm{MSE}_\theta(\hat{\theta}_n) = \frac{2\theta^2}{(n+1)(n+2)}.$$

   (d) Compute the mean-squared error (MSE) of the estimator $\tilde{\theta}_n = \frac{n+1}{n} \max_{1 \leq i \leq n} X_i$.
   Deduce that the MLE is inadmissible for estimating $\theta$ under squared error loss. [2 marks]

(e) Is your estimator $\tilde{\theta}_n = \frac{n+1}{n} \max_{1\le i\le n} X_i$ in (d) admissible? Justify your answer.

[3 marks]

*[You may find it helpful to consider the MSE of estimators of the form*

$$T_\delta(X) = \delta \max_{1\le i\le n} X_i$$

*for $\delta = \delta_n > 0$].*

2. Consider $X_1, \ldots, X_n$ i.i.d observations from the probability density

$$f_\theta(x) = \tfrac{1}{\theta} e^{-x/\theta}, \qquad x > 0,$$

with restricted parameter space $\Theta = (0, \delta]$ for some fixed known $0 < \delta < \infty$.

(a) Find the maximum likelihood estimator (MLE) $\hat{\theta}_n$ for $\theta$. [2 marks]

(b) Derive the asymptotic distribution of (a suitably rescaled version of) $\hat{\theta}_n$ as $n \to \infty$. *[You should consider the cases $\theta \in (0, \delta)$ and $\theta = \delta$ separately. Hint: if you are stuck, consider Problem Sheet 3].* [2 marks]

(c) Consider the reparametrization $\varphi = 1/\theta$. Find the MLE $\hat{\varphi}_n$ of $\varphi$. Derive the asymptotic distribution of (a suitably rescaled version of) $\hat{\varphi}_n$ as $n \to \infty$.

[3 marks]

[Total 20 marks]