Network Science
Spring 2024
Problem class 5 solutions

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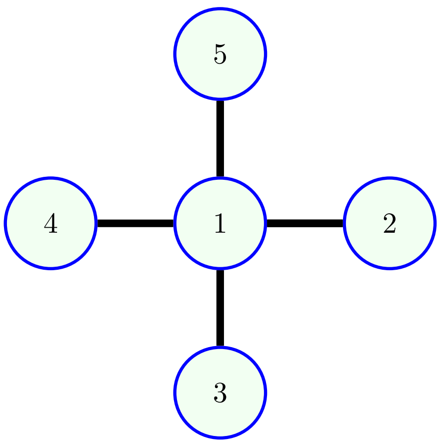
1. Let $\mathbf{L}$ be the Laplacian matrix for a simple graph.

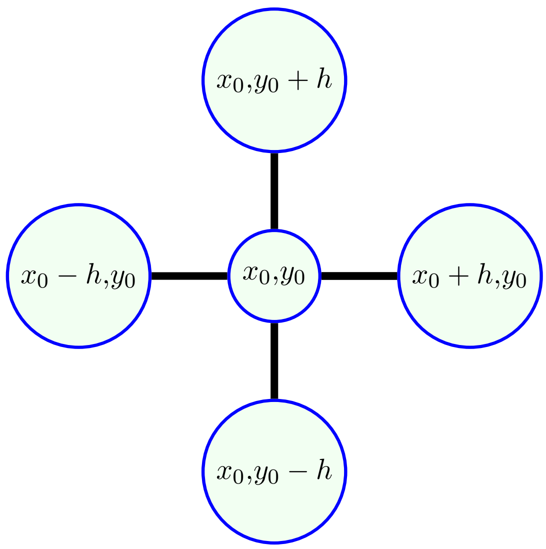
   (a) Show that $\mathbf{z}$ is always an eigenvector $\mathbf{L}$. What is the corresponding eigenvalue?
   Solution: $\mathbf{Lz} = \mathbf{Dz} - \mathbf{Az}$. $\mathbf{Dz} = \mathbf{Az} = \mathbf{k}$ where $\mathbf{k}$ is a column vector containing the degree of each node in the graph, and $\mathbf{Lz} = 0$. So $\mathbf{z}$ is an eigenvector with corresponding eigenvalue, $\lambda = 0$.

   (b) Let λ_1 be the most positive eigenvalue of $\mathbf{L}$. Show that $\lambda_1 \leq 2k_{max}$ where k_{max} is the largest degree in the graph which corresponds to $\mathbf{L}$.
   Solution: We use Gershgorin's circle theorem with $\mathbf{X} = \mathbf{I}$. Then we have $\mathbf{L} = \mathbf{H}+\mathbf{F}$ where $\mathbf{H} = \mathbf{D}$ and $\mathbf{F} = -\mathbf{A}$. The theorem tells us that λ_1 lies within the union of N discs where the i^{th} disc is defined by

$$|l - H_{ii}| \leq \sum_{j=1}^{N} |F_{ij}|.$$

   In this case, $H_{ii} = k_i$, and $|F_{ij}| = A_{ij}$, so for the i^{th} disc, $|l - k_i| \leq k_i$. So, the most positive value on the i^{th} disc is $2k_i$, and the most positive real value on the union of all discs is $2k_{max}$. It follows that $\lambda_1 \leq 2k_{max}$.





2. In this exercise, you will explore a connection between the 2-D diffusion equation and the graph diffusion equation.

(a) Consider the 5-node graph shown above on the left. What is the graph diffusion equation for node 1?:

Solution:

$$df_1/dt = \alpha\left(-4f_1 + f_2 + f_3 + f_4 + f_5\right)$$

(b) Now re-interpret the graph as 5 points in the x-y plane as shown above on the right, and consider the 2d-diffusion equation applied to $f(x,y)$. The 2-d diffusion equation is,

$$\partial f/\partial t = \alpha\left(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2\right).$$

It can be shown using Taylor series expansions that,

$$d^2f/dx^2|_{x0} \approx (1/h^2)\left(f(x_0+h) - 2f(x_0) + f(x_0-h)\right).$$

Using approximations of this form for the second derivatives in the 2-D diffusion equation, obtain an expression for $\partial f/\partial t|_{(x_0,y_0)}$ which is comparable with your result for (a).

Solution: We use, $\partial^2 f/\partial x^2|_{(x_0,y_0)} \approx (1/h^2)\left(f(x_0+h,y_0) - 2f(x_0,y_0) + f(x_0-h,y_0)\right)$ with an analogous approximation for, $\partial^2 f/\partial y^2|_{(x_0,y_0)}$. Substituting the approximations into the diffusion equation gives,

$$df/dt|_{(x_0,y_0)} = \alpha/h^2\left(-4f(x_0,y_0) + f(x_0+h,y_0) + f(x_0-h,y_0) + f(x_0,y_0+h) + f(x_0,y_0-h)\right).$$

We see a direct correspondence with the result from (a) if we set $h = 1$.