MATH40005 Probability and Statistics

Midterm [20 points] **24 November 2022**

Please remember to define your notation and justify all your answers.

Question 1: Consider the experiment, where we first flip a fair coin and then roll a fair die.

(a) (2 points) Find the sample space for this experiment.

(b) (2 points) What is the probability that one obtains heads followed by either a 2 or a 4?

Solution:

(a) We write H for heads and T for tails. The sample space of the compound experiment is given by

$$\begin{aligned}\Omega &= \{H, T\} \times \{1,2,3,4,5,6\} \\ &= \{\omega = (\omega_1, \omega_2) : \omega_1 \in \{H, T\}, \omega_2 \in \{1,2,3,4,5,6\}\}.\end{aligned}$$

[**2 points** for any correct representation of the sample space]

(b) From lectures (by the multiplication principle) we know that card$(\Omega) = 2 \times 6 = 12$. Define the event A:="one obtains heads followed by either a 2 or a 4". Then

$$A = \{(H,2), (H,4)\}, \quad \mathrm{card}(A) = 2.$$

Using the naive/classical interpretation of probability, we get

$$\mathrm{P}(A) = \frac{\mathrm{card}(A)}{\mathrm{card}(\Omega)} = \frac{2}{12} = \frac{1}{6}.$$

[**1 point for the justification and 1 point for the correct probability**]

Question 2: If the letters I, I, I, I, I, N, N, C, O, M, P, R, E, E, E, H, S, S, B, L, T are arranged at random, what is the probability that

(a) (2 points) the arrangements spell the word INCOMPREHENSIBILITIES?

(b) (2 points) the arrangements have three adjacent E's?

Solution:

(a) Let Ω denote the sample space consisting of all possible arrangements of the letters I, I, I, I, I, N, N, C, O, M, P, R, E, E, E, H, S, S, B, L, T. Using the multiplication principles, there are 21! possibilities of arranging the 21 letters. Recall that we need to adjust for overcounting for letters which appear multiple times. Here we note that I appears five times, these letters can be arranged in 5! possible ways; also, N appears twice, these letters can be arranged in 2! possible orders; also, E appears three times, these letters can be arranged in 3! possible orders; finally, S appears twice, these letters can be arranged in 2! possible orders; Altogether we have

$$\mathrm{card}(\Omega) = \frac{21!}{5!2!3!2!}.$$

We define the event E= the arrangements spell the word INCOMPREHENSIBILITIES. Since only one arrangement spells the word INCOMPREHENSIBILITIES, the corresponding probability is given by

$$\mathrm{P}(E) = \frac{\mathrm{card}(E)}{\mathrm{card}(\Omega)} = \frac{5!2!3!2!}{21!}.$$

[**1 point for the justification and 1 point for the correct probability**]

(b) Consider removing the E's: _ I_ I_ I_ I_ I_ N_ N_ C_ O_ M_ P_ R_ H_ S_ S_ B_ L_ T_. We can arrange the remaining letters in $\frac{18!}{5!2!2!}$ possible ways, where we adjust for the fact that the letter I appears five times, N twice and S twice.

In order to ensure that we have three adjacent E's, we must choose one of the blank spaces _ to put the three E's. We can do this in $\binom{19}{1} = 19$ possible ways. We define the event E= the arrangements have three adjacent E's. Then

$$\mathrm{card}(E) = \frac{18!}{5!2!2!} \cdot 19 = \frac{19!}{5!2!2!}.$$

Hence,

$$\mathrm{P}(E) = \frac{\mathrm{card}(E)}{\mathrm{card}(\Omega)} = \frac{19!}{5!2!2!}\frac{5!2!3!2!}{21!} = \frac{3!}{21 \times 20} = \frac{1}{70}.$$

[**1 point for the justification and 1 point for the correct probability**]

Question 3: (4 points) Consider a standard well-shuffled 52-card deck comprising of 13 ranks in the four suits clubs, diamonds, hearts and spades. Suppose that four players play a game, where, at the start, each player gets 13 cards. What is the probability that each player has one ace?

Solution: Let Ω denote the sample space consisting of all possible allocations when dealing 52 cards into four equal hands. Then

$$\mathrm{card}(\Omega) = \frac{52!}{(13!)^4},$$

since there are 52! permutations of the 52 cards, which we split into four groups of 13 each. And for each of the four groups, the order of the cards does not matter, so be adjust for overcounting by dividing by 13! possible permutations for 13 cards for each of the four players. Let E denote the event that each player gets one ace. There are 4! ways of distributing the aces so that each player holds one, and there are $\frac{48!}{(12!)^4}$ of dealing the remaining cards.

Hence

$$\mathrm{P}(E) = \frac{\mathrm{card}(E)}{\mathrm{card}(\Omega)} = \frac{4!48!}{(12!)^4}\frac{(13!)^4}{52!} = \frac{4!13^4}{52 \times 51 \times 50 \times 49}[\approx 0.105].$$

[**2 points for the probability, 2 points for the justification**]

Alternative representation of the solution:

$$\mathrm{P}(E) = \frac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}\binom{1}{1}\binom{12}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}.$$

Justification for the numerator: We first choose 1 out of 4 aces for the first player and 12 out of the remaining 48 (non-ace) cards. For the second player, we choose 1 out of the remaining 3 aces and 12 from the remaining 36 (non-ace) cards. For the third player, we choose 1 out of the remaining 2 aces and 12 from the remaining 24 (non-ace) cards. The fourth player gets the remaining ace and 12 (non-ace) cards. Justification for the denominator: For the first player, we choose 13 out of 52 cards. For the second player, we choose 13 out of the remaining 39 cards. For the third player, we choose 13 from the remaining 26 cards, and the four player gets the remaining 13 cards.

In all instances, the order of the cards do not matter and we sample without replacement, hence the binomial coefficient(s) can be used here.

Question 4: (4 points) Consider a standard domino game. Each domino is a rectangular tile, usually with a line dividing its face into two square ends. Each of the two ends is marked with a number of n spots, where $n \in \{0\,(\text{blank}), 1, 2, 3, 4, 5, 6\}$. In a standard set of dominos, each possible combination of spots occurs on exactly one domino, e.g. there is exactly one domino that has an end with 5 spots and the other end with 3 spots, and exactly one domino that has 6 spots on each end. How many dominos are in a standard set of dominos?

Solution: We can view this as an experiment where we sample k=2 times with replacement from the n=7 spots in $\{0, 1, 2, 3, 4, 5, 6\}$, where the order in which we draw the number of spots does not matter. Hence, using a result from lectures, we conclude that there are $\binom{n+k-1}{k} = \binom{8}{2} = 28$ different dominos.

[**2 points for the result, 2 points for the justification**]

Question 5: (4 points) Let Ω be a non-empty set and let $\mathcal{F}$ and $\mathcal{G}$ denote σ-algebras on Ω. Is $\mathcal{H} := \mathcal{F} \cup \mathcal{G}$ a σ-algebra on Ω? If so, prove it, otherwise, provide a counterexample.

Solution: $\mathcal{H}$ is in general not a σ-algebra. To see this, consider the following counterexample. Let $\Omega = \{1, 2, 3\}$, $\mathcal{F} = \{\{1\}, \{2, 3\}, \emptyset, \Omega\}$ and $\mathcal{G} = \{\{1, 2\}, \{3\}, \emptyset, \Omega\}$. We note that both $\mathcal{F}$ and $\mathcal{G}$ are valid σ-algebras, which are of the form $\{\emptyset, \Omega, A, A^c\}$ for $A = \{1\}$ and $A = \{1, 2\}$, respectively.

$\mathcal{H} = \mathcal{F} \cup \mathcal{G} = \{\{1\}, \{2, 3\}, \{1, 2\}, \{3\}, \emptyset, \Omega\}$ is not closed under union. E.g. take $\{1\}, \{3\} \in \mathcal{H}$, but $\{1\} \cup \{3\} = \{1, 3\} \notin \mathcal{H}$. Hence, $\mathcal{H}$ is not a σ-algebra.

[**2 points for a correct counterexample, 2 points for the justification**]