

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Asymptotic Analysis

Date: 13th May /2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD
INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION
NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. (a) Show that all solutions of the differential equation

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

decay as $x \rightarrow +\infty$. *Clue:* Write $y(x) = e^{S(x)}$. [7 pts]

- (b) Consider the boundary value problem

$$\epsilon^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0, \quad y(0) = 0, \quad y(1) = 1,$$

in the limit $\epsilon \searrow 0$.

- (i) Where is the boundary layer? What is the scaling of the boundary layer region? What is the scaling of y there? [7 pts]
- (ii) Formulate a boundary value problem whose solution determines a leading-order approximation to y in the boundary layer region. [3 pts]
- (iii) State the scaling of y in the boundary layer if the differential equation is changed to

$$\epsilon^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 1/x. \quad [3 \text{ pts}]$$

2. (a) Using integration by parts, show that

$$\int_{\lambda}^{\infty} t^{-2} e^{-t} dt \sim \frac{1}{\lambda} + \ln \lambda + \gamma - 1 + O(\lambda) \quad \text{as } \lambda \searrow 0,$$

where $\gamma = -\int_0^{\infty} e^{-t} \ln t dt$. [5 pts]

Note that $\frac{d}{dt}(t \ln t - t) = \ln t$.

- (b) Consider the integral

$$I(x) = \int_{-\infty}^{\infty} \frac{\cos(xt^2)}{1+t^2} dt$$

as $x \rightarrow +\infty$.

- (i) Use the method of stationary phase to derive a leading-order approximation. [5pts]
- (ii) Use the method of steepest descent to derive a two-term expansion. [10pts]

Useful integrals: $\int_{-\infty}^{\infty} e^{is^2} ds = \sqrt{\pi} e^{i\pi/4}$, $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$.

3. Consider the integral

$$I(\epsilon) = \int_0^1 \frac{\ln x}{\epsilon^2 + x^2} dx \quad \text{as } \epsilon \searrow 0.$$

- (a) Determine using scaling arguments whether the dominant contribution is 'local' or 'global,' as well as the order of magnitude of the integral. [5 pts]
- (b) Based on your answers to (a), obtain a leading-order approximation for $I(\epsilon)$. [5 pts]
- (c) Determine the expansion of $I(\epsilon)$ to $\text{ord}(1)$ by splitting the integral:

$$I(\epsilon) = \left(\int_0^{\lambda(\epsilon)} + \int_{\lambda(\epsilon)}^1 \right) \frac{\ln x}{\epsilon^2 + x^2} dx.$$

What constraints did you impose on $\lambda(\epsilon)$? [10pts]

Note that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$, $\frac{d}{dx} \frac{1+\ln x}{x} = -\frac{\ln x}{x^2}$ and $\int_0^\infty \frac{\ln x}{1+x^2} dx = 0$.

4. (a) Consider the nonlinear oscillator

$$\frac{d^2 y}{dt^2} + \epsilon \frac{dy}{dt} \left[\left(\frac{dy}{dt} \right)^2 - \alpha \right] + y = 0,$$

where α is a real parameter.

Use the method of multiple scales to obtain a leading-order approximation, valid for $t = O(1/\epsilon)$, in the form

$$a(\epsilon t) \cos [t + \phi_0],$$

where ϕ_0 is a constant and a is a real function of the slow time variable $T = \epsilon t$. In particular,

- (i) Derive an ordinary differential equation for $a(T)$. [8 pts]
- (ii) Briefly discuss the dynamics of the nonlinear oscillator for $\alpha < 0$ and $\alpha > 0$. [4 pts]

Useful identity: $\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta)$.

- (b) Find a three-term expansion as $x \searrow 0$ of the solution to

$$\frac{dy}{dx} = y^2 + \ln x$$

that is infinite at $x = 0$. [8pts]

Note that $\frac{d}{dx} \left[\frac{1}{3} x^3 \left(\ln x - \frac{1}{3} \right) \right] = x^2 \ln x$.

5. Consider the boundary value problem

$$\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \epsilon y \frac{dy}{dx} = 0, \quad y(1) = 0, \quad \lim_{x \rightarrow +\infty} y = 1 \quad (1)$$

in the limit $\epsilon \searrow 0$.

- (a) Find a leading-order approximation that satisfies both boundary conditions. [3 pts]
- (b) Explain why there is a 'boundary layer' at $x = \infty$ and find its scaling. [4 pts]
- (c) Use matched asymptotic expansions to obtain dy/dx at $x = 0$ to $\text{ord}(\epsilon)$. [13 pts]

Clues:

1. Write the general solution of

$$f'' + \frac{2}{x} f' = \frac{1}{x^3} - \frac{1}{x^2}$$

as

$$f(x) = \frac{a}{x} + b - \frac{\ln x}{x} - \ln x,$$

where a and b are constants.

2. You will need the result from question 2(a):

$$\int_{\lambda}^{\infty} t^{-2} e^{-t} dt \sim \frac{1}{\lambda} + \ln \lambda + \gamma - 1 + O(\lambda) \quad \text{as } \lambda \searrow 0,$$

where $\gamma = -\int_0^{\infty} e^{-t} \ln t dt$.

Solutions for M3M7 2019-20

1. (a) (SEEN SIMILAR)

Let $y(x) = e^{S(x)}$, then

$$S'^2 + S'' + xS' + 1 = 0.$$

Two possible balances lead to consistent asymptotic expansions for S' as $x \rightarrow \infty$.

The first is $S'^2 \sim -xS'$, or $S \sim -x^2/2$. This balance corresponds to an exponentially decaying solution [to find a leading-order approximation for y , the expansion for S would have to be continued to $\text{ord}(1)$.]

The second consistent balance is $xS' \sim -1$, or $S \sim -\ln x + \text{const}$. Hence there are also solutions decaying algebraically like $1/x$.

(4(A)+3(B)=7 marks)

(b) i. (SEEN SIMILAR)

Leading-order 'outer' solution is c/x , where c is a constant. This suggests a boundary layer at $x = 0$, hence $c = 1$ to satisfy the boundary condition at $x = 1$. A scaling analysis on the ODE shows that the thickness of the boundary layer near $x = 0$ is $O(\epsilon)$. From matching considerations, $y = O(1/\epsilon)$ in the boundary layer.

Can there also be a boundary layer near $x = 1$? No. The only possible dominant balance in that case is $\epsilon^2 y'' \sim -y'$, with solutions growing exponentially into the domain.

(5(B)+2(D)=7 marks)

ii. (UNSEEN)

Since the boundary layer is at $x = 0$, the outer expansion satisfies the boundary condition at $x = 1$. Hence, the outer expansion is $y(x; \epsilon) \sim 1/x$ as $\epsilon \rightarrow 0$ with x fixed.

To study the boundary layer, let $\xi = x/\epsilon$ and $Y(\xi; \epsilon) = y(x; \epsilon)$. Based on (i), we pose the expansion $Y(\xi; \epsilon) \sim \epsilon^{-1}F(\xi)$ as $\epsilon \rightarrow 0$ with ξ fixed. Since y is asymptotically larger in the boundary layer region than in the outer region, matching clearly requires $F(\xi) \rightarrow 0$ as $\xi \rightarrow +\infty$. Thus, naively, the leading-order inner problem is

$$F'' + \xi F' + F = 0; \quad F(0) = 0; \quad \lim_{\xi \rightarrow +\infty} F = 0.$$

From part (a), however, we know that all solutions of the differential equation decay as $\xi \rightarrow \infty$. The correct matching condition closing the problem governing F is derived by matching the outer expansion to $\text{ord}(1)$ and the inner expansion to $\text{ord}(1/\epsilon)$ [using an intermediate variable or Van Dyke's matching principle]. This gives

$$F \sim 1/\xi \quad \text{as} \quad \xi \rightarrow +\infty.$$

(3(B) marks)

iii. **(UNSEEN)**

Now the leading-order outer solution satisfies $xy'_0 + y_0 = 1/x$, with $y_0(1) = 1$. Hence $y_0 = (1 + \ln x)/x$. Matching considerations then imply that the inner solution is $\text{ord}(\ln \epsilon/\epsilon)$.

(3(D) marks)

2. (a) **(SEEN SIMILAR)**

Integration by parts twice yields

$$\int_{\lambda}^{\infty} t^{-2} e^{-t} dt = \left(\frac{1}{\lambda} + \ln \lambda \right) e^{-\lambda} - \int_{\lambda}^{\infty} e^{-t} \ln t dt. \quad (1)$$

The integral on the right-hand side can be written as

$$- \int_{\lambda}^{\infty} e^{-t} \ln t dt = \gamma + \int_0^{\lambda} e^{-t} \ln t dt.$$

Since

$$\int_0^{\lambda} e^{-t} \ln t dt \sim \int_0^{\lambda} \ln t dt = \lambda \ln \lambda + O(\lambda) \quad \text{as } \lambda \rightarrow 0,$$

the result follows by expanding the exponent in (1).

(5(A) marks)

(b) It is convenient to write

$$I(x) = \operatorname{Re}[J(x)], \quad J(x) = \int_{-\infty}^{\infty} \frac{e^{ixt^2}}{1+t^2} dt, \quad x > 0.$$

i. **(SEEN SIMILAR)**

Method of stationary phase gives

$$J \sim \frac{1}{x^{1/2}} \int_{-\infty}^{\infty} e^{iu^2} du = \left(\frac{\pi}{x} \right)^{1/2} e^{i\pi/4}.$$

Thus,

$$I(x) \sim \left(\frac{\pi}{2x} \right)^{1/2}.$$

(5(A) marks)

ii. **(SEEN SIMILAR)**

The integrand of the J integral exponentially decays as $t \rightarrow \infty$ in the first and third quadrants of the complex t plane. There are simple poles at $t = \pm i$. It follows that we can deform the integration contour to the steepest descent contour passing through the saddle point $t = 0$, without changing the value of the integral. The Integral can therefore be written as ($t = \xi e^{i\pi/4}$)

$$J(x) = e^{i\pi/4} \int_{-\infty}^{\infty} \frac{e^{-x\xi^2}}{1+i\xi^2} d\xi.$$

Laplace's method gives ($u = \xi x^{1/2}$)

$$J(x) = e^{i\pi/4} x^{-1/2} \int_{-\infty}^{\infty} \left(1 - \frac{iu^2}{x} + \dots \right) e^{-u^2} du + O(x^{-5/2})$$

Integrating term by term and taking the real part,

$$I(x) \sim \left(\frac{\pi}{2x} \right)^{1/2} \left\{ 1 + \frac{1}{2x} \right\}$$

($\int_{-\infty}^{\infty} u^2 e^{-u^2} du = \sqrt{\pi}/2$ by integrating by parts the second useful integral.)

(5(A) + 5(B)=10 marks)

3. (SEEN SIMILAR)

- (a) The contribution of the 'local' interval $x = \text{ord}(\epsilon)$ is $\text{ord}[\ln \epsilon / \epsilon]$; the contribution of the 'global' interval $x = \text{ord}(1)$ is $\text{ord}(1)$. We therefore anticipate a dominant local contribution on the order of $\ln \epsilon / \epsilon$.

(5(A) marks)

- (b) It readily follows from (a) that

$$I(\epsilon) \sim \frac{\ln \epsilon}{\epsilon} \int_0^\infty \frac{du}{1+u^2} = \frac{\pi \ln \epsilon}{2\epsilon}.$$

(5(C) marks)

- (c) To begin with, let $\epsilon \ll \lambda \ll 1$. Consider first the integral from λ to 1:

$$\begin{aligned} \int_\lambda^1 \frac{\ln x}{\epsilon^2 + x^2} dx &\sim \int_\lambda^1 \frac{\ln x}{x^2} \left(1 - \frac{\epsilon^2}{x^2} + \dots\right) dx \\ &\sim \left(\frac{1}{x} + \frac{\ln x}{x}\right)_\lambda^1 + O\left(\frac{\epsilon^2 \ln \lambda}{\lambda^3}\right) \sim \frac{1}{\lambda} + \frac{\ln \lambda}{\lambda} - 1 + O\left(\frac{\epsilon^2 \ln \lambda}{\lambda^3}\right). \end{aligned} \quad (2)$$

We transform the O term into $o(1)$ by further constraining λ : $\lambda \propto \epsilon^\alpha$ with $0 < \alpha < 2/3$.

The integral from 0 to λ can be written as:

$$\int_0^\lambda \frac{\ln x}{\epsilon^2 + x^2} dx = \frac{\ln \epsilon}{\epsilon} \int_0^{\lambda/\epsilon} \frac{du}{1+u^2} + \frac{1}{\epsilon} \int_0^{\lambda/\epsilon} \frac{\ln u}{1+u^2} du. \quad (3)$$

The first integral in (3) can be written as

$$\int_0^{\lambda/\epsilon} \frac{du}{1+u^2} = \left(\int_0^\infty - \int_{\lambda/\epsilon}^\infty\right) \frac{du}{1+u^2} = \frac{\pi}{2} - \int_{\lambda/\epsilon}^\infty \frac{du}{1+u^2}. \quad (4)$$

The integrand of the last integral can now be expanded giving

$$\int_0^{\lambda/\epsilon} \frac{du}{1+u^2} = \frac{\pi}{2} - \int_{\lambda/\epsilon}^\infty \left(\frac{1}{u^2} - \frac{1}{u^4} + \dots\right) du = \frac{\pi}{2} + \frac{\epsilon}{\lambda} + O(\epsilon^3/\lambda^3) \quad (5)$$

The second integral in (3) can be expanded:

$$\begin{aligned} \int_0^{\lambda/\epsilon} \frac{\ln u}{1+u^2} du &= \int_0^\infty \frac{\ln u}{1+u^2} du - \int_{\lambda/\epsilon}^\infty \ln u \left(\frac{1}{u^2} - \frac{1}{u^4} + \dots\right) du \\ &= -\frac{\epsilon}{\lambda} - \frac{\epsilon}{\lambda} \ln \frac{\lambda}{\epsilon} + O\left(\frac{\epsilon^3 \ln \lambda}{\lambda^3}, \frac{\epsilon^3 \ln \epsilon}{\lambda^3}\right) \end{aligned} \quad (6)$$

Thus,

$$\int_0^\lambda \frac{\ln x}{\epsilon^2 + x^2} dx \sim \frac{\pi \ln \epsilon}{2\epsilon} - \frac{1}{\lambda} - \frac{\ln \lambda}{\lambda} + o(1). \quad (7)$$

Summing (2) and (7):

$$I(\epsilon) \sim \frac{\pi \ln \epsilon}{2\epsilon} - 1 + o(1). \quad (8)$$

(5(C)+5(D)=10 marks)

4. (a) i. **(SEEN SIMILAR)**

Let $Y(\tau, T; \epsilon) = y(t; \epsilon)$ on the diagonal $\tau = t$, $T = \epsilon t$. The solution Y is expanded to arbitrary positive τ and T following the multiple-scales paradigm. Thus, Y satisfies

$$Y_{\tau\tau} + T + \epsilon [2Y_{\tau T} + Y_{\tau}(Y_{\tau}^2 - \alpha)] + \epsilon^2 [\dots] = 0.$$

We pose the expansion $Y \sim Y_0(\tau, T) + \epsilon Y_1(\tau, T)$. The function Y_1 satisfies

$$Y_{0\tau\tau} + Y_0 = 0,$$

with solution $Y_0 = a(T) \cos[\tau + \phi(T)]$. At the next order,

$$Y_{1\tau\tau} + Y_1 = -2Y_{0\tau T} - Y_{0\tau}(Y_{0\tau}^2 - \alpha).$$

Using the given trigonometric identity, the RHS can be written as

$$2a_T \sin(\tau + \phi) + 2a\phi_T \cos(\tau + \phi) - \alpha a \sin(\tau + \phi) + \frac{a^3}{4} (3 \sin(\tau + \phi) - \sin[3(\tau + \phi)]).$$

Secularity condition yields

$$a_T = \frac{a}{2} \left(\alpha - \frac{3}{4}a^2 \right), \quad \phi_T = 0.$$

(8(A) marks)

ii. **(UNSEEN)**

Steady-state oscillations exist only for $\alpha > 0$, with amplitude $|a| = \sqrt{4\alpha/3}$. For $\alpha < 0$ solutions decay exponentially to the trivial solution; for $\alpha > 0$ and non-trivial initial conditions, solutions approach the steady-state oscillations exponentially.

(4(B) marks)

(b) **(SEEN SIMILAR)**

The balance $y^2 \sim -\ln x$ clearly gives a contradiction where the neglected term y' is actually larger than the terms kept. The balance $y' \sim \ln x$ gives a solution bounded at $x = 0$.

The correct dominant balance is therefore $y' \sim y^2$. Integration gives

$$y \sim -1/(x - c),$$

where $c = 0$ for a singularity at $x = 0$.

Let $y = -1/x + \bar{y}$, where $\bar{y} \ll 1/x$. The differential equation becomes

$$\bar{y}' + \frac{2}{x}\bar{y} - \bar{y}^2 = \ln x. \tag{9}$$

The second term on the RHS dominates the third. Thus the new dominant balance gives the linear ODE $\bar{y}' + (2/x)\bar{y} \sim \ln x$. The $O(1/x^2)$ homogeneous solution is inconsistent, so using the given integral we find

$$\bar{y} \sim \frac{x}{3} \left(\ln x - \frac{1}{3} \right). \tag{10}$$

Note that the term neglected in (9) is algebraically small compared to the terms kept, so it is justified to retain both of the asymptotic orders in (10), which are logarithmically separated. We found the three-term expansion

$$y \sim -\frac{1}{x} + \frac{x}{3} \left(\ln x - \frac{1}{3} \right) \quad \text{as } x \searrow 0.$$

(2(C) +6(D)=8 marks)

5. (SEEN)

(a) Try $y(x; \epsilon) \sim y_0(x)$, where $y_0(x)$ satisfies

$$y_0'' + \frac{2}{x}y_0' = 0, \quad y_0(1) = 0, \quad \lim_{x \rightarrow +\infty} y_0 = 1.$$

The solution is $y_0 = 1 - 1/x$.

(3 marks)

(b) $y_0 y_0' / y_0'' = \text{ord}(x)$ as $x \rightarrow \infty$ so there is a 'boundary layer' at distances $x = O(1/\epsilon)$.

(4 marks)

(c) We extend the elementary approximation by posing the 'outer' expansion

$$y(x; \epsilon) \sim y_0(x; \epsilon) + \epsilon y_1(x; \epsilon) \quad \text{as } \epsilon \rightarrow 0 \quad (x \text{ fixed}), \quad (11)$$

where we allow y_0, y_1 to depend logarithmically upon ϵ . The function y_1 satisfies

$$y_1'' + \frac{2}{x}y_1' = -y_0 y_0' = x^{-3} - x^{-2}, \quad (12)$$

with general solution

$$y_1 = \frac{a}{x} + b - \left(1 + \frac{1}{x}\right) \ln x. \quad (13)$$

As to be expected from (ii), both boundary conditions cannot be simultaneously satisfied at this order. Applying the boundary condition at $x = 1$ yields

$$a + b = 0.$$

The appearance of the logarithmic term will introduce, through matching, terms at $\text{ord}(\epsilon \ln \epsilon)$, in addition to $\text{ord}(\epsilon)$. In what follows we group together logarithmically separated terms.

To study the boundary layer at infinity we define $\xi = \epsilon x$, $Y(\xi; \epsilon) = y(x; \epsilon)$ and consider the 'inner' limit $\epsilon \rightarrow 0$ with ξ fixed. The inner problem is

$$Y'' + \frac{2}{\xi}Y' + YY' = 0, \quad \lim_{\xi \rightarrow \infty} Y = 1.$$

It is clear from the form of the outer expansion that

$$Y(\xi; \epsilon) \sim 1 + \epsilon Y_1(\xi; \epsilon) \quad \text{as } \epsilon \rightarrow 0 \quad (\xi \text{ fixed}).$$

The function Y_1 thus satisfies the linearised problem

$$Y_1'' + \frac{2}{\xi}Y_1' + Y_1' = 0, \quad \lim_{\xi \rightarrow \infty} Y_1 = 0,$$

with solution

$$Y_1 = A \int_{\xi}^{\infty} \frac{e^{-\tau}}{\tau^2} d\tau.$$

We now apply Van Dyke's principal to match the inner expansion to $\text{ord}(\epsilon)$ with the outer expansion to $\text{ord}(\epsilon)$, the orders specified up to logarithms of ϵ as required when using Van Dyke's principle. Thus, writing the outer expansion to $\text{ord}(\epsilon)$ in terms of ξ and expanding to ord gives

$$1 - \epsilon/\xi + \epsilon b - \epsilon(\ln \xi - \ln \epsilon).$$

In the other direction, writing the inner expansion to $\text{ord}(\epsilon)$ in terms of x and expanding to $\text{ord}(\epsilon)$ gives

$$1 + A/x + A\epsilon(\ln \epsilon + \ln x) + A\epsilon(\gamma - 1).$$

Equating the above two expressions gives

$$A = -1, \quad b = \ln \frac{1}{\epsilon} + 1 - \gamma,$$

whereas $a = -b$ from before.

Thus we found the inner expansion

$$y \sim 1 - \frac{1}{x} + \epsilon \ln \frac{1}{\epsilon} \left(1 - \frac{1}{x}\right) + \epsilon \left[(1 - \gamma) \left(1 - \frac{1}{x}\right) - \ln x - \frac{\ln x}{x} \right].$$

Hence $y'(1) \sim 1 + \epsilon \ln \frac{1}{\epsilon} - \epsilon(\gamma + 1)$ as $\epsilon \searrow 0$.

Remark: A solution where logarithmic terms are considered as separate asymptotic orders is also fine, of course, but in that case matching should be carried out using an intermediate variable.

(13 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97029 MATH97106	1	<p>part a) Some of you considered only one consistent balance in the ODE satisfied by S. There were actually two consistent balances. You could have guessed the latter by noticing that the ODE for y is linear and of order two, therefore, there should be two linearly independent solutions.</p> <p>part b) The boundary value problem for the leading order term of the inner expansion is of order two. Therefore two boundary conditions were needed, yet many of you only gave one (or none). One (evident) boundary condition comes from $y(0) = 0$, the other should come from matching with the outer region.</p>	
MATH97029 MATH97106	2	Few justified why is it possible to deform the integration contour, without changing the value of the integral. This step is important since the presence of poles in the integrand might actually change the value of the integral after a contour deformation.	
MATH97029 MATH97106	3	Few stated the necessary constraints on λ for expanding to $\text{ord}(1)$. In this particular case, the initial assumption of $\epsilon \ll \lambda \ll 1$ was only enough for the leading order term.	
MATH97029 MATH97106	4		
MATH97029 MATH97106	5		