Imperial College
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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Geometric Mechanics

Date: 2nd June 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. A relativistic Lagrangian

(a) For the following Lagrangian $L(\dot{\mathbf{q}}) : T\mathbb{R}^3 \to \mathbb{R}$

$$L(\dot{\mathbf{q}}) = -\Big(1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}\Big)^{1/2}$$

express the velocity $\dot{\mathbf{q}} \in T\mathbb{R}^3$ in terms of the position $\mathbf{q} \in \mathbb{R}^3$ and the fibre derivative of the Lagrangian. (6 marks)

(b) Write the Euler-Lagrange equation for this Lagrangian. (4 marks)

(c) Find the constants of the motion for the Euler-Lagrange equation and give their physical interpretations. (4 marks)

(d) Legendre transform this Lagrangian to determine its corresponding Hamiltonian and canonical equations. (4 marks)

(e) Explain the physical meaning of this motion.

Hint: suppose the Lagrangian were written as

$$L(\dot{\mathbf{q}}) = -\, m_0\Big(1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}/c^2\Big)^{1/2}$$

for particle rest mass m_0 and speed of light c. (2 marks)

(Total: 20 marks)

2. Poincaré's theorem

(a) For basis elements (dq, dp) in phase space, the contraction operation ($\lrcorner$) satisfies the **duality relations**,

$$\partial_q \lrcorner\, dq = 1 = \partial_p \lrcorner\, dp\,, \quad \text{and} \quad \partial_q \lrcorner\, dp = 0 = \partial_p \lrcorner\, dq\,, \tag{1}$$

so that differential forms are linear functions of vector fields.

Use the duality relations (1) to prove that a Hamiltonian vector field,

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_p \partial_q - H_q \partial_p = \{\,\cdot\,,\, H\,\}\,, \tag{2}$$

satisfies the following linear functional relations with the basis elements in phase space,

$$X_H \lrcorner\, dq = H_p \quad \text{and} \quad X_H \lrcorner\, dp = -\,H_q\,. \tag{3}$$

(2 marks)

(b) Prove the following theorem.

The Hamiltonian vector field $X_H = \{\,\cdot\,,\, H\,\}$ satisfies

$$X_H \lrcorner\, \omega = dH \quad \text{with} \quad \omega = dq \wedge dp\,. \tag{4}$$

(4 marks)

(c) Prove that $d^2H = 0$ for a smooth phase-space function $H(q, p)$. (6 marks)

(d) A flow is **symplectic** if it preserves the phase-space area or symplectic two-form, $\omega = dq \wedge dp$.

Prove the following theorem.

Poincaré's theorem. The flow of a Hamiltonian vector field is symplectic.

Hint; In the proof, first use the pull-back definition of the Lie derivative to derive the product rule for the Lie derivative of the differential 2-form, $\omega = dq \wedge dp$. (8 marks)

(Total: 20 marks)

3. (a) Prove that

$$[X\,,\,Y] \mathbin{\lrcorner} \alpha = \pounds_X(Y \mathbin{\lrcorner} \alpha) - Y \mathbin{\lrcorner} (\pounds_X \alpha)\,.$$

(2 marks)

(b) For a top form α and divergence free vector fields X and Y, use Cartan's formula $\pounds_X \alpha = d(X \mathbin{\lrcorner} \alpha) + X \mathbin{\lrcorner} d\alpha$ to show that

$$[X\,,\,Y] \mathbin{\lrcorner} \alpha = d\Big(X \mathbin{\lrcorner} (Y \mathbin{\lrcorner} \alpha)\Big)\,. \tag{5}$$

(7 marks)

(c) Write the equivalent of equation (5) as a formula in vector calculus. (7 marks)

(d) Show, for vector fields $u, v \in \mathfrak{X}(\mathbb{R}^3)$ and a 1-form $\alpha \in \Lambda^1(\mathbb{R}^3)$ that Palais's theorem holds,

$$\pounds_u(v \mathbin{\lrcorner} \alpha) - \pounds_v(u \mathbin{\lrcorner} \alpha) = [u, v] \mathbin{\lrcorner} \alpha + v \mathbin{\lrcorner} (u \mathbin{\lrcorner} d\alpha)\,.$$

(4 marks)

(Total: 20 marks)

4. Consider a rigid body with flywheel attached along the intermediate principal axis in the body. As for the isolated rigid body, the energy in this problem is purely kinetic; so one may define the kinetic energy Lagrangian for this system $L : TSO(3)/SO(3) \times TS^1 \to \mathbb{R}^3$ as

$$L(\mathbf{\Omega},\,\dot\phi) = \frac{1}{2}\lambda_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}\lambda_3\Omega_3^2 + \frac{1}{2}J_2(\dot\phi + \Omega_2)^2\,,$$

where (after the hat map $\mathfrak{so}(3) \to \mathbb{R}^3$) the vector $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ is the angular velocity vector of the rigid body, $\dot\phi \in TS^1$ is the rotational frequency of the flywheel about the intermediate principal axis of the rigid body, and the positive constants λ_1, I_2, λ_3 are the principal moments of inertia, $I = \mathrm{diag}(\lambda_1, I_2, \lambda_3)$ for the rigid body and J_2 for the flywheel.

The objective of this problem is derive the Hamiltonian equations by Legendre-transforming this Lagrangian as

$$H(\mathbf{\Pi}, N) = \mathbf{\Pi}\cdot\mathbf{\Omega} + N\dot\phi - L(\mathbf{\Omega},\,\dot\phi)$$

(a) Obtain the canonical equations for the angular momentum N and angular velocity $\dot\phi$ of the flywheel.

(5 marks)

(b) Eliminate the angular velocity $\dot\phi$ in favour of N and Ω_2 to find an expression for the Hamiltonian $H(\mathbf{\Pi}; N)$, in terms of only the angular momenta $\mathbf{\Pi} = \partial L/\partial\mathbf{\Omega} \in \mathbb{R}^3$ and $N = \partial L/\partial\dot\phi \in \mathbb{R}^1$ of the rigid body and flywheel. (8 marks)

(c) Explain why one may derive the motion equation for $\mathbf{\Pi} = \partial L/\partial\mathbf{\Omega} \in \mathbb{R}^3$ in the form

$$\frac{d\mathbf{\Pi}}{dt} = -\,\mathbf{\Pi}\times\nabla_{\mathbf{\Pi}}\,H(\mathbf{\Pi};N) = -\,\nabla_{\mathbf{\Pi}}\,|\mathbf{\Pi}|^2\times\nabla_{\mathbf{\Pi}}\,H(\mathbf{\Pi};N)$$

(4 marks)

(d) List the constants of motion for this problem, explain how they arose and how they influence the solution of the present gyrostat problem. (3 marks)

(Total: 20 marks)

5. Transform the Lagrangian $\widetilde{L}(y,\dot{y}) : T\mathbb{R}^n \to C^\infty(T\mathbb{R}^n)$ as $\widetilde{L}(y(t),\dot{y}(t)) = L(g_t y_0, \dot{g}_t y_0)$, where g_t is a time-dependent curve in the $n\times n$ matrix Lie group G, with $g_0 = Id$.

Assume that the Lagrangian L is invariant under linear transformations defined by the left action of the matrix Lie group G on $\mathbb{R}^n$. This means that

$$L\Big(g_t y_0, \dot{g}_t y_0\Big) = L\Big(y_0, g_t^{-1}\dot{g}_t y_0\Big) =: \ell(g_t^{-1}\dot{g}_t)\,,$$

for every initial y_0. Define $\xi := g_t^{-1}\dot{g}_t$ with variation $\delta\xi = \delta(g_t^{-1}\dot{g}_t)$ where $\xi \in \mathfrak{g}$ is an element of the left matrix Lie algebra of the matrix Lie group G.

(a) Derive the Euler-Poincaré equation resulting from Hamilton's principle with action integral

$$S = \int_{t_1}^{t_2} \ell(\xi) + \Big\langle \mu\,,\, g_t^{-1}\dot{g}_t - \xi \Big\rangle\, dt\,,$$

where the brackets $\Big\langle \cdot\,,\,\cdot \Big\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ with $\Big\langle \mu\,,\,\xi \Big\rangle := \frac{1}{2}\mathrm{tr}(\mu^T\xi)$ denote the Frobenius (trace) pairing of $n\times n$ matrices $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ and in this pairing μ^T is the transpose matrix. In particular,

$$\delta\ell(\xi) = \Big\langle \frac{\delta\ell}{\delta\xi}\,,\,\delta\xi \Big\rangle = \frac{1}{2}\mathrm{tr}\bigg(\frac{\delta\ell}{\delta\xi}^T\,\delta\xi\bigg).$$

Hint: To compute the variation of $\xi = g_t^{-1}\dot{g}_t$, begin by defining $\nu := g_t^{-1}\delta g_t$ and *proving* that

$$\delta\xi = \frac{d}{dt}\nu + \xi\nu - \nu\xi =: \frac{d}{dt}\nu + [\xi,\nu] =: \frac{d}{dt}\nu + \mathrm{ad}_\xi\nu\,,$$

where matrix multiplication is denoted by concatenation. (5 marks)

(b) Derive the Euler-Poincaré equation resulting from Hamilton's principle with the phase-space Lagrangian in the action integral

$$S = \int_{t_1}^{t_2} \Big\langle \mu\,,\, g_t^{-1}\dot{g}_t \Big\rangle - h(\mu)\, dt\,.$$

(5 marks)

(c) From the result of Hamilton's principle with the phase-space Lagrangian, derive the Lie–Poisson bracket for the Hamiltonian formulation on $n\times n$ matrices $\mu \in \mathfrak{g}^*$ as a linear functional of the matrix Lie algebra commutator. (5 marks)

(d) For the case that the matrix Lie group G is the rotation group $SO(3)$ acting on $\mathbb{R}^3$, use the hat map for its Lie algebra $\mathfrak{so}(3) \to \mathbb{R}^3$ to write the Lie–Poisson bracket for the Hamiltonian formulation on $\mathbb{R}^3$. (5 marks)

(Total: 20 marks)

Parts

(a) $\boxed{\text{C: 6 marks - Seen similar}}$ **Fibre derivative**

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\dot{\mathbf{q}}}{\sqrt{1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}}} =: \gamma \, \dot{\mathbf{q}} \quad \Longrightarrow \quad \dot{\mathbf{q}} = + \frac{\mathbf{p}}{\sqrt{1 + \mathbf{p} \cdot \mathbf{p}}} \,.$$

The positive root is taken, since the first equation requires $\mathbf{p} \cdot \dot{\mathbf{q}} > 0$. Having expressed the velocity in terms of the fibre derivative proves that this Lagrangian is hyperregular, and yields

$$\gamma := \frac{1}{\sqrt{1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}}} = \sqrt{1 + \mathbf{p} \cdot \mathbf{p}} \,.$$

(b) $\boxed{\text{A: 4 marks - Seen}}$ **Euler-Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}} \quad \Longrightarrow \quad \frac{d(\gamma \, \dot{\mathbf{q}})}{dt} = 0 \,.$$

(c) $\boxed{\text{A: 4 marks - Seen}}$ **Constants of motion**

The *momentum* $\mathbf{p}$ and *energy* $E = \gamma$ are constants of the motion.

(d) $\boxed{\text{A: 4 marks - Seen}}$ **Hamiltonian and canonical equations**

The Legendre transformation for this system produces the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{q}} - L = \sqrt{1 + |\mathbf{p}|^2} = \gamma \,,$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{\sqrt{1 + |\mathbf{p}|^2}} = \mathbf{p}/\gamma \,, \qquad \frac{d\mathbf{p}}{dt} = - \frac{\partial H}{\partial \mathbf{q}} = 0 \,.$$

(e) $\boxed{\text{C: 2 marks - Seen similar}}$ These equations represent uniform (force-free) motion in $\mathbb{R}^3$ of a relativistic particle with rest mass $m_0 = 1$, in units with $c = 1$.

In these units, the relation $H = \gamma$ is written as

$$H = \gamma \, m_0 c^2 = mc^2 = E \,.$$

Question 2 | Poincaré's theorem

Parts

(a) $\boxed{\text{A: 2 marks - Seen}}$ By inspection,
$(H_p\partial_q - H_q\partial_p) \lrcorner\, dq = H_p$ and $(H_p\partial_q - H_q\partial_p) \lrcorner\, dp = -H_q$.

(b) $\boxed{\text{A: 4 marks - Seen}}$ The proof follows from substituting the given relations.

$$X_H \lrcorner\, \omega = X_H \lrcorner\, (dq \wedge dp) = H_p dp + H_q dq = dH$$

(c) $\boxed{\text{A: 6 marks - Seen}}$ For any smooth phase-space function $H(q,p)$, we have

$$dH = H_q dq + H_p dp$$

and taking another exterior derivative yields

$$d^2H = H_{qp}\, dp \wedge dq + H_{pq}\, dq \wedge dp = (H_{pq} - H_{qp})\, dq \wedge dp = 0\,.$$

(d) $\boxed{\text{D: 8 marks - Seen}}$ Preservation of ω may be verified by using pull-back and the product rule for the Lie derivative of a differential form. Along the flow of the Hamiltonian vector field,
$X_H = (dq/dt, dp/dt) = (\dot q, \dot p) = (H_p, -H_q)$, we have

$$\begin{aligned}
\pounds_{X_H}\omega &= \pounds_{X_H}(dq \wedge dp) = \frac{d}{dt}\Big|_{t=0} g_t^*(dq \wedge dp) \\
&= \frac{d}{dt}\Big|_{t=0} (g_t^* dq \wedge g_t^* dp) = d\dot q \wedge dp + dq \wedge d\dot p \\
&= dH_p \wedge dp - dq \wedge dH_q = d(H_p\, dp + H_q\, dq) \\
&= d(X_H \lrcorner\, \omega) = d(dH) = 0\,.
\end{aligned}$$

The first two steps use the product rule for Lie derivatives of differential forms, obtained from

$$\begin{aligned}
\pounds_{X_H}(dq \wedge dp) = &\ \frac{d}{dt}\Big|_{t=0} g_t^*(dq \wedge dp) = \frac{d}{dt}\Big|_{t=0}(g_t^* dq \wedge g_t^* dp) \\
= &\ \Big[\frac{d}{dt} g_t^* dq \wedge g_t^* dp + g_t^* dq \wedge \frac{d}{dt} g_t^* dp\Big]_{t=0} \\
= &\ \pounds_{X_H} dq \wedge dp + dq \wedge \pounds_{X_H} dp
\end{aligned}$$

and the third-to-the-last and last steps use the property of the exterior derivative d that $d^2 = 0$ for continuous forms. The latter is due to the equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product $dq \wedge dp = -\, dp \wedge dq$.

Question 3

Parts

(a) $\boxed{\text{A: 2 marks - Seen}}$ Prove by pull-back and product rule, using the dynamical definition of the Lie derivative $\pounds_Y\alpha = \frac{d}{dt}|_{t=0}(\phi_t^*\alpha)$, yields

$$\frac{d}{dt}\Big|_{t=0}\phi_t^*(Y \,\lrcorner\, \alpha) \;=\; \Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*Y\Big) \,\lrcorner\, \alpha + Y \,\lrcorner\, \Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha\Big).$$

Thus the product rule for the Lie derivative yields the desired formula:

$$\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha).$$

(b) $\boxed{\text{B: 7 marks - Seen}}$

Substituting the relation $\pounds_X Y = [X, Y]$ into the product rule above in Part (a) yields $[X, Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$,for an arbitrary k-form α. From this formula we have

$$\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha &= \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)\\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + X \,\lrcorner\, d(Y \,\lrcorner\, \alpha)\big) - Y \,\lrcorner\, (\pounds_X\alpha)\\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y\alpha - Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X\alpha)\\
&= d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big). \qquad (1)
\end{aligned}$$

The last line uses $\pounds_X\alpha = (\mathrm{div}\,\mathbf{X})\alpha = 0 = \pounds_Y\alpha = (\mathrm{div}\,\mathbf{Y})\alpha$ with $X = \mathbf{X}\cdot\nabla$ and $Y = \mathbf{Y}\cdot\nabla$, and $d\alpha = 0$ for the top form α.

(c) $\boxed{\text{B: 7 marks - Seen}}$ The equivalent vector calculus formula equation to (1) may be found by writing its LHS and RHS in a coordinate basis, as

$$\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha &= (\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X})\cdot d\mathbf{S}\\
d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) &= -\,\mathrm{curl}\,(\mathbf{X}\times\mathbf{Y})\cdot d\mathbf{S}
\end{aligned}$$

I.e., for $\nabla\cdot\mathbf{X} = 0 = \nabla\cdot\mathbf{Y}$, $(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X}) = -\,\mathrm{curl}\,(\mathbf{X}\times\mathbf{Y})$.

(d) $\boxed{\text{C: 4 marks - Unseen}}$ The LHS of 1-form $\alpha = \boldsymbol{\alpha}\cdot\boldsymbol{dx} = \alpha_j dx^j$:

$$\pounds_u(v \,\lrcorner\, \alpha) - \pounds_v(u \,\lrcorner\, \alpha) = (\boldsymbol{u}\cdot\boldsymbol{\nabla})(\boldsymbol{v}\cdot\boldsymbol{\alpha}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{\alpha}),$$

with notation $\boldsymbol{u}\cdot\boldsymbol{\alpha} = \alpha_j u^j$, evaluating the right hand side yields

$$\begin{aligned}
[u, v] \,\lrcorner\, \alpha + v \,\lrcorner\, (u \,\lrcorner\, d\alpha) &= \big((\boldsymbol{u}\cdot\boldsymbol{\nabla})v^j - (\boldsymbol{v}\cdot\boldsymbol{\nabla})u^j\big)\alpha_j\\
&+\; v^j(\boldsymbol{u}\cdot\boldsymbol{\nabla})\alpha_j - u^j(\boldsymbol{v}\cdot\boldsymbol{\nabla})\alpha_j.
\end{aligned}$$

The vector form of $v \,\lrcorner\, (u \,\lrcorner\, d\alpha)$ is useful in the last step

$$v \,\lrcorner\, (u \,\lrcorner\, d\alpha) = \mathrm{curl}\boldsymbol{\alpha}\cdot\boldsymbol{u}\times\boldsymbol{v} = v^j(\boldsymbol{u}\cdot\boldsymbol{\nabla})\alpha_j - u^j(\boldsymbol{v}\cdot\boldsymbol{\nabla})\alpha_j.$$

Question 4 — Rigid body with flywheel

Parts

(a) $\boxed{\text{B: 5 marks - Seen similar}}$ The canonical part of the Legendre transform in the flywheel variables $(\phi, \dot\phi) \to (\phi, N)$ yields the flywheel Hamiltonian

$$H_W(\phi, N) = N\dot\phi - \frac{1}{2}J_2(\dot\phi + \Omega_2)^2 = \frac{N^2}{2J_2} - N\Omega_2$$

where we have substituted $N = \partial L/\partial\dot\phi = J_2(\dot\phi + \Omega_2)$, which already can be used to eliminate $\dot\phi$ in favour of N and Ω_2. However, one also has

$$\dot\phi = \frac{\partial H_W}{\partial N} = \frac{N}{J_2} - \Omega_2 \quad \text{and} \quad \dot N = -\frac{\partial H_W}{\partial \phi} = 0\,.$$

Thus, in the Hamiltonian picture, (N, ϕ) are action-angle variables coupled to the rotations. That is, ϕ may be reconstructed from its Hamiltonian equation after $\Omega_2(t)$ has been solved from the rigid body dynamics with constant N.

(b) $\boxed{\text{D: 8 marks - Seen similar}}$ The rigid body part of the Legendre transform in the angular variables then yields

$$\begin{aligned} H(\mathbf{\Pi}, N) &= \mathbf{\Pi}\cdot\mathbf{\Omega} + N\dot\phi - L(\mathbf{\Omega}, \dot\phi) \\ &= \frac{\Pi_1^2}{2\lambda_1} + \frac{\Pi_3^2}{2\lambda_3} + \underbrace{\frac{1}{2I_2}\big(\Pi_2 - N\big)^2}_{\text{offset along } \Pi_2} + \frac{N^2}{2}\Big(\frac{1}{I_2} + \frac{1}{J_2}\Big)\,. \end{aligned}$$

Proof: Upon substituting

$$\frac{\partial L}{\partial \mathbf{\Omega}} = \mathbf{\Pi} = I\mathbf{\Omega} + N\hat{\boldsymbol{e}}_2\,, \quad \mathbf{\Omega} = I^{-1}(\mathbf{\Pi} - N\hat{\boldsymbol{e}}_2)$$

into the Legendre transform, we find

$$\begin{aligned} H(\mathbf{\Pi}, N) &= \mathbf{\Pi}\cdot\mathbf{\Omega} + N\dot\phi - L(\mathbf{\Omega}, \dot\phi) \\ &= \mathbf{\Pi}\cdot\mathbf{\Omega} - \frac{1}{2}I\mathbf{\Omega}\cdot\mathbf{\Omega} + \frac{N^2}{2J_2} - N\Omega_2\,, \\ &= \frac{1}{2}(\mathbf{\Pi} + N\hat{\boldsymbol{e}}_2)\cdot I^{-1}(\mathbf{\Pi} - N\hat{\boldsymbol{e}}_2) + \frac{N^2}{2J_2} - N\frac{1}{I_2}(\Pi_2 - N) \\ &= \frac{1}{2}\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi} - \frac{N^2}{I_2} + \frac{N^2}{2J_2} - N\frac{1}{I_2}(\Pi_2 - N) \\ &= \frac{\Pi_1^2}{2\lambda_1} + \frac{\Pi_3^2}{2\lambda_3} + \underbrace{\frac{1}{2I_2}\big(\Pi_2 - N\big)^2}_{\text{offset along } \Pi_2} + \underbrace{\frac{N^2}{2}\Big(\frac{1}{I_2} + \frac{1}{J_2}\Big)}_{\text{Constant}}\,. \end{aligned}$$

Question 4+ | Rigid body with flywheel (continued)

Parts

(c) $\boxed{\text{B: 4 marks - Seen similar}}$ Since N is merely a constant parameter in the Hamiltonian $H(\mathbf{\Pi}, N)$, one may use the rigid body Poisson bracket for $\mathbf{\Pi}$ to calculate the rotational dynamics.

(d) $\boxed{\text{A: 3 marks - Seen similar}}$

(i) The two constants of motion are N and $|\mathbf{\Pi}|^2$.

(ii) N is constant by Noether's theorem: since the Lagrangian is independent of the angle ϕ.

(iii) $|\mathbf{\Pi}|^2$ is constant because it is the Casimir for the rigid body bracket.

Together, these constants reduce the dynamics to integrable motion on the sphere.

Question 5 | Euler-Poincaré equations – Mastery question

Parts

(a) $\boxed{\text{C: 5 marks - Seen similar}}$ Taking the variation of the action S yields

$$\delta S = \int_{t_1}^{t_2} \Big\langle \frac{\delta \ell}{\delta \xi} - \mu \,,\, \delta\xi \Big\rangle + \Big\langle \mu \,,\, \delta\big(g_t^{-1}\dot g_t\big) \Big\rangle \,dt\,,$$

Define $\xi := (g_t^{-1}\dot g_t)$ and $\nu := (g_t^{-1}\delta g_t) := [g_{t,\epsilon}\frac{d}{d\epsilon}g_{t,\epsilon}]_{\epsilon=0}$. To calculate $\delta(g_t^{-1}\dot g_t)$ one calculates $\delta\xi = \delta(g_t^{-1}\dot g_t)$ and $\frac{d\nu}{dt}$ then invokes equality of cross derivatives, as follows.

$$\begin{aligned} \delta\xi &= \delta(g_t^{-1}\dot g_t) = -(g_t^{-1}\delta g_t)(g_t^{-1}\dot g_t) + g_t^{-1}\delta\dot g_t = -\nu\xi + g_t^{-1}\delta\dot g_t \\ \frac{d\nu}{dt} &= \Big(\frac{d}{dt}(g_t^{-1}\delta g_t)\Big) = -(g_t^{-1}\dot g_t)(g_t^{-1}\delta g_t) + g_t^{-1}\delta\dot g_t = -\xi\nu + g_t^{-1}\delta\dot g_t \end{aligned}$$

Then $\delta\xi + \nu\xi = g_t^{-1}\delta\dot g_t = \frac{d\nu}{dt} + \xi\nu$ implies the required identity

$$\delta\xi = \frac{d\nu}{dt} + \xi\nu - \nu\xi = \frac{d\nu}{dt} + \big[\xi,\nu\big] =: \frac{d}{dt}\nu + \mathrm{ad}_\xi\nu$$

Now we have

$$\delta S = \int_{t_1}^{t_2} \Big\langle \frac{\delta \ell}{\delta \xi} - \mu \,,\, \delta\xi \Big\rangle + \Big\langle \mu \,,\, \frac{d\nu}{dt} + \xi\nu - \nu\xi \Big\rangle \,dt$$

We rearrange the second term as

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \Big\langle \mu \,,\, \frac{d\nu}{dt} + \xi\nu - \nu\xi \Big\rangle \,dt = \int_{t_1}^{t_2} \frac{1}{2}\mathrm{tr}\left(\mu^T\left(\frac{d\nu}{dt} + \xi\nu - \nu\xi\right)\right) dt \\ &= \int_{t_1}^{t_2} \frac{1}{2}\mathrm{tr}\left(\left(-\frac{d\mu^T}{dt} + \mu^T\xi - \xi\mu^T\right)\nu\right)\,dt \end{aligned}$$

Thus, one finds the Euler–Poincaré equation for μ^T,

$$\frac{d\mu^T}{dt} = \mu^T\xi - \xi\mu^T =: \mathrm{ad}^*_\xi\mu \quad \text{with} \quad \mu^T = \frac{\delta\ell}{\delta\xi}\,.$$

Question 5+ | Euler-Poincaré equations – Mastery question (continued)

Parts

(b) $\boxed{\text{C: 5 marks - Seen similar}}$ Varying the action integral $S = \int_{t_1}^{t_2} \Big\langle \mu \,,\, g_t^{-1}\dot g_t \Big\rangle - h(\mu)\,dt$ yields

$$S = \int_{t_1}^{t_2} \Big\langle \delta\mu \,,\, g_t^{-1}\dot g_t - \frac{\partial H}{\partial \mu} \Big\rangle + \Big\langle \mu \,,\, \frac{d\nu}{dt} + \xi\nu - \nu\xi \Big\rangle\,dt$$

Thus, one finds the Lie–Poisson Hamiltonian equation for μ^T,

$$\frac{d\mu^T}{dt} = \mu^T\xi - \xi\mu^T =: \mathrm{ad}^*_\xi\mu \quad \text{with} \quad \xi = g_t^{-1}\dot g_t = \frac{dH}{d\mu}\,.$$

That is,

$$\frac{d\mu^T}{dt} = \mu^T\frac{dH}{d\mu} - \frac{dH}{d\mu}\mu^T =: \mathrm{ad}^*_{dH/d\mu}\mu\,.$$

(c) $\boxed{\text{C: 5 marks - Seen similar}}$ With Frobenius pairing of $n \times n$ matrices, the time derivative of a function $F(\mu)$ is given by

$$\begin{aligned}
\frac{dF}{dt} &= \Big\langle \frac{d\mu}{dt} \,,\, \frac{dF}{d\mu} \Big\rangle = \Big\langle \mathrm{ad}^*_{dH/d\mu}\mu \,,\, \frac{dF}{d\mu} \Big\rangle \\
&= \Big\langle \mu \,,\, \mathrm{ad}_{dH/d\mu}\frac{dF}{d\mu} \Big\rangle = \Big\langle \mu \,,\, \frac{dH}{d\mu}\frac{dF}{d\mu} - \frac{dF}{d\mu}\frac{dH}{d\mu} \Big\rangle \\
&= \frac{1}{2}\mathrm{tr}\left(\mu^T\left(\frac{dH}{d\mu}\frac{dF}{d\mu} - \frac{dF}{d\mu}\frac{dH}{d\mu}\right)\right) =: \Big\{F, H\Big\}(\mu)
\end{aligned}$$

(d) $\boxed{\text{C: 5 marks - Seen similar}}$ For the case that the matrix Lie group G is the rotation group $SO(3)$ acting on $\mathbb{R}^3$, the hat map

$$\widehat{\ }: \mathfrak{so}(3) \to \mathbb{R}^3 \quad \widehat{\xi}_{ij} = -\,\epsilon_{ijk}\xi^k = (\boldsymbol{\xi}\times)_{ij}$$

for its Lie algebra $\mathfrak{so}(3) \to \mathbb{R}^3$ between skew-symmetric 3×3 matrices and vectors in $\mathbb{R}^3$ implies

$$[\xi, \nu]\widehat{\ } := \boldsymbol{\xi} \times \boldsymbol{\nu}$$

Consequently, when written for vectors on $\mathbb{R}^3$, the Lie-Poisson bracket on $\mathfrak{so}(3)$ becomes

$$\Big\{F, H\Big\}(\boldsymbol{\mu}) = \boldsymbol{\mu} \cdot \frac{dH}{d\boldsymbol{\mu}} \times \frac{dF}{d\boldsymbol{\mu}}\,.$$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	**QuestionN**	**Comments for Students**
MATH96039	1	All the marks were very successful
MATH96039	2	All the marks were very successful
MATH96039	3	Most of the marks were pretty successful
MATH96039	4	Most of the marks were pretty successful