

# MATH60005/70005: Optimisation (Autumn 24-25)

## Chapter 3: exercises and solutions

Dr Dante Kalise, Dr Estefanía Loayza-Romero and Sara Bicego  
Department of Mathematics  
Imperial College London, United Kingdom  
{dkaliseb,kloayzar,s.bicego21}@imperial.ac.uk

### Polynomial fit and denoising

We will replicate the linear regression and regularized linear least squares examples from this week, with a different model. First, we will generate a noisy dataset of 200 samples coming from

$$v_i = u_i^2 + \mathcal{N}(0, 0.04), \quad i = 1, \dots, 200,$$

where the  $u_i$ 's are uniformly sampled in  $[-1, 1]$ , and  $\mathcal{N}(0, 0.04)$  means adding Gaussian noise of mean 0 and variance 0.04 for **each** sample.

1. Generate the pairs  $(u_i, v_i)$  using suitable random generators. Make a plot illustrating  $(u_i, v_i)$ . What is the model you can identify to express  $v$  as a function of  $u$ ?

```
1 Ns=200; %number of samples
2 u=2*rand(Ns,1)-1;% uniform sampling in [-1;1];
3 u=sort(u); %sorting the samples
4 v=u.^2+0.2*randn(Ns,1); %sampling with Gaussian noise of mean ...
   0 and variance 0.04
5 figure(1) %plotting the samples
6 plot(u,v,'LineStyle','none','Marker','*')
```

2. Write a linear regression problem for finding the optimal parameters in a model

$$v(u) = au^2 + bu + c$$

For a linear regression with  $N_s$  samples for a model  $v(u) = au^2 + bu + c$ , we assemble the matrices  $\mathbf{A} \in \mathbb{R}^{N_s \times 3}$  and  $\mathbf{b} \in \mathbb{R}^{N_s}$



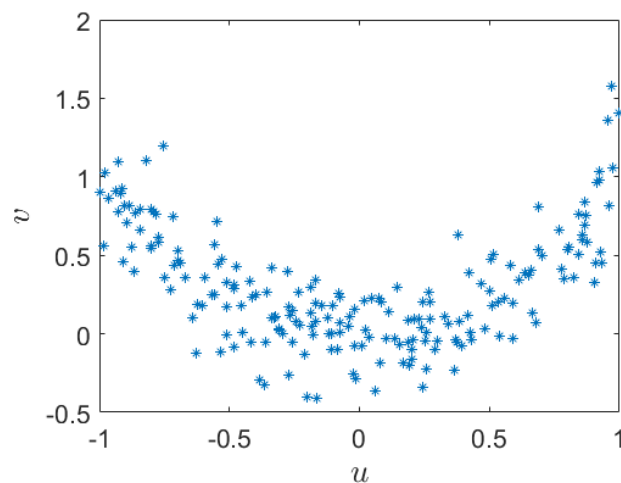


Figure 1: The sampled points suggest a quadratic dependence for  $v(u)$ .

```
1 A=[u.^2 u ones(Ns,1)]; %Ns by 3 matrix
2 b=v;
```

We can solve the normal equations

```
1 xls=(A'*A)\ (A'*b); % normal equations with backslash for the ...
   inverse
```

or, you can use backslash directly!

```
1 xls=A\ b;
```

MATLAB will recognize that  $A$  is a rectangular matrix and that you're trying to solving LLS, so instead of computing the inverse of  $A$ , it will compute the solution to the normal equations.

3. Compute the least squares solution and compute the total least squares error in the  $\ell_2$  norm.

For the error we need to compute  $\|v - Ax_{LS}\|_2$ ,

```
1 l2error=norm(v-A*xls,2);
```

4. Now, instead of solving a regression problem, use the  $v_i$  values to recover a denoised signal using regularized least squares using the same total variation regularization described the lecture notes.

We assemble the regularization matrix  $L$  as in the notes



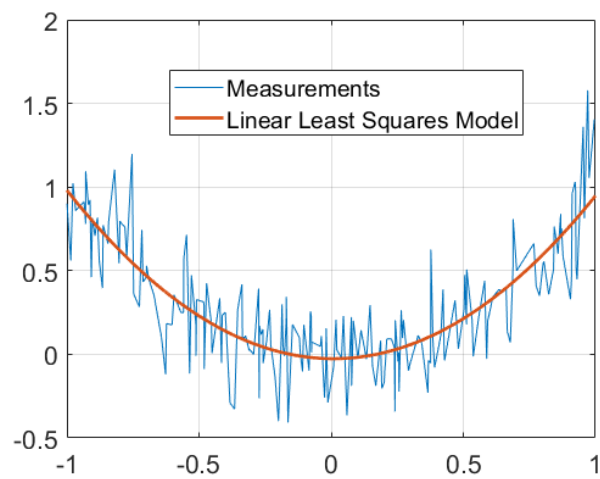


Figure 2: The measurements against linear least squares regression for  $v(u) = au^2 + bu + c$ .

```

1 L=zeros(Ns-1,Ns);
2 for i=1:Ns-1
3   L(i,i)=1;
4   L(i,i+1)=-1;
5 end

```

We set the regularization parameter  $\lambda = 100$  (just to try, you need to see the effect of different values) and solve the regularized least squares solution

```

1 lambda=100;
2 xrls=(eye(Ns)+lambda*L'*L)\b;

```



Finally, we plot the regularized signal against the noisy measurements

```
1 figure(3)
2 plot(u,v)
3 hold on
4 plot(u,xrls)
```

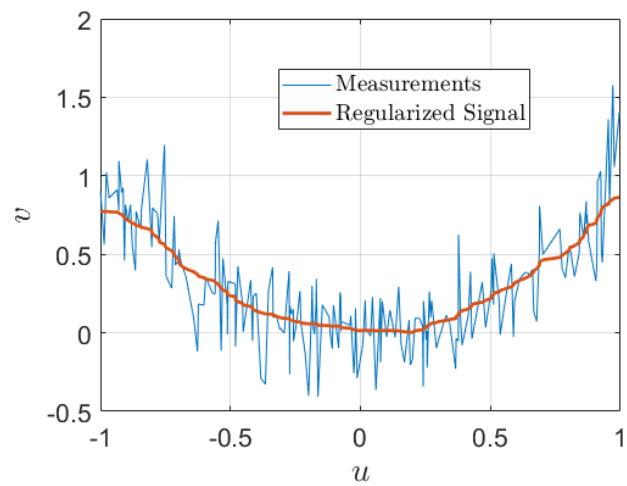


Figure 3: The regularized signal for  $\lambda = 100$ .

