Introduction to University Mathematics **MATH40001/MATH40009**

Solutions: Problem Sheet II–3

1. (a) As $\mathbb{R}$ is a field, $a + x = b$ if and only if $a + x - a = b - a$ (using associativity), if and only if $x = b - a$ (using associativity, commutativity, and the additive inverse).

   (b) First we claim that negation is unique. Suppose $a+b = 0 = a+b'$. Then adding an additive inverse $-a$, we get $(-a) + a + b = (-a) + a + b'$, and applying associativity, commutativity, the additive inverse axiom, and the additive identity axiom, we get $b = b'$.

   Now, for the first equality, note that $a+(-a) = 0$ by the additive inverse axiom. This exhibits a as the additive inverse of $-a$. For the second, again note that $(-a) + (-b) + a + b = 0$ by associativity, commutativity, and the inverse axiom. Hence, applying associativity again, we have $-(a + b) = (-a) + (-b)$.

2. (a) We first claim that 0 is uniquely defined. Suppose $0'$ is another additive identity. Then $0 + 0' = 0 = 0'$.

   Now, assume $a{\cdot}b = 0$. For a contradiction, suppose $a \neq 0$ and $b \neq 0$. We prove that $a{\cdot}b \neq 0$. By the inverse axiom and associativity of multiplication, we have $a^{-1} \cdot a \cdot b = b \neq 0$. On the other hand, note that, for all c, $c \cdot 0 = c \cdot (0 + 0) = c \cdot 0 + c \cdot 0$, which implies by subtracting $c \cdot 0$ that $c \cdot 0 = 0$. Thus $a^{-1} \cdot a \cdot b = 0$ by associativity. This is a contradiction.

   (b) We proved in the previous part that 0 was unique. For 1 the proof is the same: if 1 and $1'$ are multiplicative identities, then $1 \cdot 1' = 1 = 1'$.

3. (a) If $a \neq 0$ then either $a > 0$ or $a < 0$. In the former case $|a| = a > 0$. In the latter case $|a| = -a$ and we have $0 = a - a < -a$.

   (b) Note by definition that $|a| = |-a|$. Moreover, $|a| = \pm a$, the sign uniquely determined so as to have a nonnegative result. Thus $|a \cdot b|$ and $|a| \cdot |b|$ are both equal to $\pm a \cdot b$, withe uniquely determined sign to be nonnegative, so they are equal.

   (c) If a and b have the same sign, then $|a| + |b| = \pm(a + b) = |a + b|$, by the preceding part. If they have opposite signs, so $a = \pm|a|$ and $b = \mp|a|$ (for one choice of sign now), then we have $|a + b| = |\pm(|a| - |b|)| = ||a| - |b||$. This is either $|a| - |b|$ or $|b| - |a|$, whichever is nonnegative. But both of these are $\leq |a| + |b|$, since $|a|, |b| \geq 0$.

   (d) By the preceding part, $|a + b| - |b| \leq |a|$. Now substituting $c = a + b$, we get $|c| - |b| \leq |c - b| = |b - c|$. Reversing the roles of c and b we also get $|b| - |c| \leq |b| - |c|$. Now $||b| - |c|| = \pm(|b| - |c|) \leq |b - c|$.

4. (a) i. Let $a > 0$. If $1/a = 0$ then $1 = a \cdot (1/a) = a \cdot 0 = 0 \cdot a$ which is a contradiction. If $1/a < 0$ then using axiom (O2) to multiply both sides by the positive number a we have $a \cdot (1/a) < a \cdot 0$ or $1 < 0$, again a contradiction. Now since $\leq$ is a total order we must have $\frac{1}{a} > 0$.

   ii. First we show that $a < b \Rightarrow a^2 < b^2$. If $a < b$ then by property (O2) we can multiply by a to get $a^2 < ab$ and similarly we can multiply by y to get $ab < b^2$. Now by transitivity we have $a^2 < b^2$. Now we show the reverse implication $a^2 < b^2 \Rightarrow a < y$ Suppose $a^2 < b^2$. Then using axiom (O1) to add $-a^2$ to each side we have $0 < b^2 - a^2$ or $0 < (b - a)(b + a)$. Now by axiom (O2) we can multiply both sides by $\frac{1}{b+a}$ to get $0 < b - a$ i.e. $a < b$.

   (b) Easy induction. Basis step: for $n = 0$ the statement is obviously true. Induction step: $(1 + x)^{n+1} = (1 + x)(1 + x)^{n+1} \geq (1 + x)(1 + nx) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$.

5. (a) We prove this by contradiction. Assume $\mathbb{N}$ is bounded from above. Since $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{R}$ has the least upper bound property, then $\mathbb{N}$ has a least upper bound $l \in \mathbb{R}$. Thus $n \leq l$ for all $n \in \mathbb{N}$ and it is the smallest such real number. Consequently $l-1$ is not an upper bound for $\mathbb{N}$ (if it were, since $l-1 < l$ then l would not be the least upper bound). Therefore there is some integer k with $l-1 < k$. But then $l < k+1$. This contradicts that l is an upper bound for $\mathbb{N}$.

   (b) Suppose there exist nonzero elements $x, y \in \mathbb{R}$ such that $x > 0$ and $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $\{nx | n \in \mathbb{N}\}$ has an upper bound and by the completeness axiom, it must have a least upper bound m. We claim that then $m - x$ must also be an upper bound. Indeed if $m - x$ is not an upper bound, then $nx > m - x$ for some $n \in \mathbb{N}$. Hence $(n+1)x > m$, so m is not an upper bound either. But $m - x < m$ and this contradicts the assertion that m is a least upper bound for $\{nx | n \in \mathbb{N}\}$.

6. We just give the answers; the proofs are similar to what was done in lectures: you should be able to write them down carefully.

   (a) $\sup S = \sqrt{5}, \inf S = -\sqrt{5}$.

   (b) $\sup S$ does not exist, $\inf S$ neither.

   (c) $\inf S = -1, \sup S = 0$.

7. (a) Yes: for every positive real number, it is true that the absolute value of every elements of the empty set is less than this number.

   (b) It is clear that a is a lower bound of $[a, b]$ and (a, b). It is the greatest such because if $c > a$ implies that $a < a + \frac{1}{2} \cdot (c - a) < c$, and $a + \frac{1}{2} \cdot (c - a) \in (a, b)$, so c cannot be a lower bound. The same argument works for $[a, b]$. Similarly it is clear that b is an upper bound of $[a, b]$ and (a, b), and it is the least such.

8. (a) If $A \subseteq B$, then every lower bound for B is also a lower bound of for A; thus $\inf B \leq \inf A$. Similarly $\sup A \leq \sup B$. Finally we have $\inf A \leq \sup A$ since every upper bound for A is greater than or equal to every lower bound for A.

   (b) Since A is bounded above, B is non-empty. Since $\sup A$ exists, it is the least element of B. This is therefore a lower bound for B, and the least such.