Math40003 Linear Algebra and Groups **Problem Sheet 5**

Seen A

A.1. Using Procedure 3.8.3, Calculate the row and column ranks of the following matrices

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$

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A.2. Which of the following functions $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are linear transformations (Definition 4.1.1.)?

(a) $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 + x_2)$
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(b) $T(x_1, x_2, x_3) = (0, \sqrt{2}x_3)$.

(c) $T(x_1, x_2, x_3) = (x_1x_2, x_3)$.

(d) $T(x_1, x_2, x_3) = (0, 0)$.

A.3. For the first of these questions, try working out what a linear transformation that does the required mappings would do to the standard basis elements.

(a) Find a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which sends $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$.
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(b) Find two different linear transformations $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which send $(1, 1, 0)$ to $(1, 1)$ and $(0, 1, 1)$ to $(0, 1)$. .

Seen B

B.1. Prove that the linear transformation you found in A.3(a) is unique.

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B.2. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Which of the following functions $T : V \longrightarrow V$ are linear transformations?

(a) $T(A) = A^2$ for all $A \in V$

(b) $T(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $A \in V$

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B.3. Let V be the vector space (over $\mathbb{R}$) of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are linear transformations (thinking of $\mathbb{R}$ as $\mathbb{R}^1$ in parts (a), (c) and (d))?

(a) $T_1 : V \to \mathbb{R}$ where $T_1(f) = f(1)$ (for $f \in V$).

(b) $T_2 : V \to V$ where $T_2(f) = f \circ f$ (for $f \in V$).

(c) $T_3 : \mathbb{R} \to V$ where $T_3(\mu)$ is the function $f_\mu \in V$ given by $f_\mu(x) = \mu x$ (for $\mu, x \in \mathbb{R}$).

(d) $T_4 : V \to \mathbb{R}$ where

$$T_4(f) = \int_{-\infty}^{\infty} f(x)\,dx$$

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B.4. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly one vector $v \in \mathbb{R}^2$.

(b) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for no vector $v \in \mathbb{R}^2$.

(c) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for infinitely many vectors $v \in \mathbb{R}^2$.

(d) Show that there is no linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(v) = (1,0,0)$ for exactly two vectors $v \in \mathbb{R}^2$.

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B.5. (a) Suppose V, W are vector spaces (over a field F) and $S, T : V \to W$ are linear transformations. Prove that $S + T : V \to W$ defined by $(S+T)(v) = S(v) + T(v)$ (for $v \in V$) is a linear transformation. If $\lambda \in F$, show that $\lambda S : V \to W$ defined by $(\lambda S)(v) = \lambda S(v)$ (for $v \in V$) is a linear transformation. Explain why the set U of all linear transformations from V to W is a vector space with these operations.

(b) In the case where $V = F^2$ and $W = F^3$, what is the dimension of the vector space U? What is the dimension of U for arbitrary finite dimensional vector spaces V and W?

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