BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Optimisation Mock Exam

Date: Wednesday, 11th May 2021
Time: 09:00-11:00
Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers
This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) Given the function

$$f\left(x_1, x_2\right) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$$

(i) Determine its stationary points. (5 marks)

(ii) Classify the stationary points found in i). (5 marks)

Answer

$$\nabla f(x) = \begin{pmatrix} 6x_1x_2 \\ 6x_2^2 - 12x_2 + 3x_1^2 \end{pmatrix}$$

which equals 0 when $x_1 = x_2 = 0$ or when $x_1 = 0$ and $x_2 = 2$. The Hessian is

$$\nabla^2 f(\bar{x}) = \begin{pmatrix} 6x_2 & 6x_1 \\ 6x_1 & 12x_2 - 12 \end{pmatrix}$$

which is negative semidefinite at $(0, 0)$ and positive definite at $(0, 2)$. Hence $(0, 2)$ is a strict local minimizer and $(0, 0)$ is a candidate to be a local maximizer, however, notice hat for any small neighborhood of $(0, 0), f$ increases in the positive x_1 direction and decreases in the positive x_2 direction, so $(0, 0)$ is a saddle point.

(b) Consider the matrix $H \in \mathbb{R}^{3\times3}$ and vector $\mathbf{g} \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which define the quadratic function $f(\mathbf{x}) = \mathbf{x}^\top H\mathbf{x} + \mathbf{g}^\top \mathbf{x}$. Does there exists a vector $\mathbf{u} \in \mathbb{R}^3$ such that $f(t\mathbf{u}) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct $\mathbf{u}$. (10 marks)

Answer. Let $u = \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ in $\mathrm{Null}(H)$ so that

$$\begin{aligned} f(tu) &= (tu)^T H(tu) + g^T(tu) \\ &= t^2 u^T Hu + tg^T u \\ &= 0t^2 - 4t \end{aligned}$$

As $t \nearrow \infty, f(tu) \searrow -\infty$. Rubric example: this is normally a type C question, 3 marks for determining H is positive semidefinite, 3 marks for the link with coercivity, 2 marks for picking u in the nullspace of H, 2 marks for calculation.

(Total: 20 marks)

2. (a) Are the following functions convex in $\mathbb{R}^n$? Justify your answer

(i) $f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$ (5 marks)

(ii) $h(\mathbf{x}) = (\|\mathbf{x}\|^2 + 1)^2$, $\mathbf{x}$ in $\mathbb{R}^n$. (5 marks)

Answer.

i The function $g(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$ is convex over $\mathbb{R}^3$ as a sum of three convex functions: the function $e^{x_1 - x_2 + x_3}$, which is convex since it is constructed by making the linear change of variables $t = x_1 - x_2 + x_3$ in the one-dimensional $\varphi(t) = e^{\top}$ For the same reason, e^{2x_2} is convex. Finally, the function x_1, being linear, is convex.

ii The function $h(x) = (\|x\|^2 + 1)^2$ is a convex function over $\mathbb{R}^n$ since it can be represented as $h(\mathbf{x}) = g(f(\mathbf{x}))$, where $g(t) = t^2$ and $f(\mathbf{x}) = \|\mathbf{x}\|^2 + 1$. Both f and g are convex, but note that g is not a nondecreasing function. However, the image of $\mathbb{R}^n$ under f is the interval $[1, \infty)$ on which the function g is nondecreasing. Consequently, the composition $h(\mathbf{x}) = g(f(\mathbf{x}))$ is convex.

(b) Consider the problem

$$
\begin{aligned}
\text{(P)} \qquad \min \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & g(\mathbf{x}) \leq 0 \\
& \mathbf{x} \in X
\end{aligned}
$$

where f, g are convex an $X \subseteq \mathbb{R}^n$ is convex. Suppose $\mathbf{x}^*$ is an optimal solution of (P) that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$
\begin{aligned}
\min \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{x} \in X
\end{aligned}
$$

(10 marks)

Rubric example: these are normally type A questions, 2 marks for stating the convexity properties to be used, and 3 marks for checking properties are satisfied for the functions.

Answer. Suppose for sake of contradiction there exists $y \in X \cap \{x : g(x) > 0\}$ such that $f(y) < f(x^*)$. The line segment $[x^*, y]$ lies in X because X is convex. Furthermore, by continuity of g, there exists a $z \in [x^*, y]$ such that $g(z) = 0$, i.e., z is feasible for the problem (P) and there exists some $\lambda \in [0,1]$ such that $z = x^* + \lambda(y - x^*)$. Observe that by convexity of f :

$$
f(z) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda \underbrace{(f(y) - f(x^*))}_{<0 \text{ by assumption}} < f(x^*)
$$

which contradicts x^* being optimal for (P).

Rubric example: this is a type D question, 2 marks for initial statement of contradiction. 2 marks for determining that segment lies in X. 3 marks for continuity argument, and 3 marks for correct inequality use.

(Total: 20 marks)

3. Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(i) Is the problem convex?. (6 marks)

(ii) Find all KKT points of the problem. (8 marks)

(iii) Find the optimal solution of the problem. (6 marks)

Answer. i) Observe the maximization problem is equivalent to the minimization problem

$$\begin{aligned} -\min - & \left(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2\right) \\ \text{s.t.} \ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The Hessian of the objective $\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$ is not psd, so the problem is not convex.

Rubric example: this is a type A question, 2 marks for expressing as minimization, 2 marks for identifying quadratic form, 2 marks for determining is not psd.

ii) The Lagrangian is

$$L(x_1, x_2, y_1, y_2, y_3) = -\left(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1\right) + x_2 + y_1(x_1 + x_2 - 1) - y_2x_1 - y_3x_2$$

defined on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2_+$. The KKT conditions are

$$\nabla_x L(x, y) = \begin{pmatrix} -2x_1 - 2x_2 + 3 + y_1 - y_2 \\ -2x_1 - 4x_2 - 1 + y_1 - y_3 \end{pmatrix} = 0 \tag{1}$$

$$y_2 x_1 = 0 \tag{2}$$

$$y_3 x_2 = 0 \tag{3}$$

$$x_1 + x_2 = 0 \tag{4}$$

$$x_1, x_2 \geq 0 \tag{5}$$

$$y_2, y_3 \geq 0 \,. \tag{6}$$

Combining (4) with (1), we find $y_1 = y_2 - 1$ and $-2x_2 - 2 + y_1 - y_3 = 0$. If $y_2 = 0$, then $y_1 = -1$ and $-2x_2 - 3 - y_3 = 0$, but this cannot happen because $y_3 = -(2x_2 + 3)$ cannot be nonnegative if $x_2 \geq 0$. Thus to satisfy (2), we must have $x_1 = 0$, so $x_2 = 1$ by (4) and $y_3 = 0$ by (3). By (1), we get that $-2 + 3 + y_1 - y_2 = 0$ and $-4 - 1 + y_1 = 0$. Putting the pieces together, $(0, 1)$ is the only KKT point with multipliers $(y_1, y_2, y_3) = (5, 6, 0)$.

Rubric example: this is a type B question, 2 marks for the Lagrangian, 3 marks for stating the right KKT system, 3 marks for solving. Last 6 marks in iii) are out of kindness from the lecturer just for repeating what was done correctly in ii).

(Total: 20 marks)

4. Consider the minimization

$$\begin{aligned}\min \quad & x_1 - 4x_2 + x_3^4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3^2 \le 2 \\ & x_1 \ge 0 \\ & x_2 \ge 0\end{aligned}$$

(i) Formulate the dual problem. (10 marks)

(ii) Solve the dual problem. (10 marks)

(Total: 20 marks)

Answer. To be solved in the revision session.

5. **Mastery question.** A community living around a lake wants to maximize the yield of fish taken out of the lake. The amount of fish at a certain time is denoted x. The growth rate of the fish is kx and fish is captured with a rate ux where u is the control variable, which is assumed to satisfy $0 \le u \le u_{\max}$. The dynamics of the fish population is then given by

$$\dot{x} = (k - u)x, \quad x(0) = x_o$$

Here $k > 0$ and $x_o > 0$. The total amount of fish obtained during a time period T is

$$J = \int_0^T uxdt$$

(i) Derive the necessary conditions given by the PMP for the problem of maximizing J. (8 marks)

(ii) Show that the necessary conditions are satisfied by a bang-bang control, that is, it only takes boundary values of the constraint set. How many switching times are there? (6 marks)

(iii) Determine an equation for calculating the switching time(s). (6 marks)

Answer.

i The problem to solve is

$$\begin{aligned}\text{minimize} \quad & \textstyle\int_0^T -uxdt \\ \text{subject to} \quad & \dot{x} = (k-u)x \\ & x(0) = x_0\end{aligned}$$

We use PMP to solve the problem. The Hamiltonian is given by

$$H(t, x, u, \lambda) = -ux + \lambda(k - u)x$$

Pointwise minimization yields

$$\tilde{\mu}(t, x, \lambda) = \underset{0 \leq u \leq u_{\max}}{\arg\min} H(t, x, u, \lambda) = \begin{cases} 0, & \lambda + 1 < 0 \\ u_{\max}, & \lambda + 1 > 0 \\ \tilde{u}, & \lambda + 1 = 0 \end{cases}$$

where $\tilde{u}$ is arbitrary in $[0, u_{\max}]$. Thus the optimal control is expressed as

$$u^*(t) \triangleq \ddot{\mu}(t, x(t), \lambda(t)) = \begin{cases} 0, & \sigma(t) < 0 \\ u_{\max}, & \sigma(t) > 0 \\ \bar{u}, & \sigma(t) = 0 \end{cases}$$

where we have defined the switching function as

$$\sigma(t) \triangleq \lambda(t) + 1$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = u - \lambda(k - u), \quad \lambda(T) = \frac{\partial \phi}{\partial x}(T, x(T)) = 0$$

Rubric example: normally, marks would be awarded for: stating the right Hamiltonian, deriving adjoint equations from here with suitable initial/terminal conditions, and working the minimization of the Hamiltonian.

ii The boundary condition in the equation above gives

$$\sigma(T) = \lambda(T) + 1 = 1 > 0$$

which gives that $u^*(T) = u_{\max}$. For finding the number of switches we consider $\dot{\sigma}(t)|_{\sigma(t)=0}$ and it follows that

$$\dot{\sigma}(t)|_{\sigma(t)=0} = \dot{\lambda}(t)\Big|_{\lambda(t)+1=0} = [u(t) - \lambda(t)(k - u(t))]|_{\lambda(t)+1=0} = k > 0$$

Hence, there can only be at most one switch, since we can pass $\sigma(t) = 0$ only once. Since $u^*(T) = u_{\max}$ is is not possible that $u^*(t) = 0$ for all $t \in [0, T]$. Thus

$$u^*(t) = \begin{cases} 0 & 0 \leq t \leq t' \\ u_{\max}, & t' < t \leq T \end{cases}$$

for some unknown switching time $t' \in [0, T]$. Note, if no switch would occur, this can still be described using $t' = 0$. Thus, we have a bang-bang control with at most one switch (from 0 to $u_{\max}$).

Rubric example: 2 marks for terminal condition for σ, 2 marks for observation of $\dot{\sigma}$, and 2 marks for conclusion.

iii The switching time $t' \in [0, T[$ occurs when

$$0 = \sigma(t') = \lambda(t') + 1$$

During the interval $t \in [t', T]$ we have that

$$\dot{\lambda}(t) = u_{\max} - \lambda(k - u_{\max}), \quad \lambda(T) = 0$$

This is a linear ODE which has the solution

$$\lambda(t) = \frac{u_{\max}}{k - u_{\max}}\left(1 - e^{(k-u_{\max})(T-t)}\right)$$

Therefore, the switching time can be found by solving

$$0 = \lambda(t') + 1 = \frac{u_{\max}}{k - u_{\max}}\left(1 - e^{(k-u_{\max})(T-t')}\right) + 1$$

3 marks for characterizing switching time, 3 marks for calculation.

(Total: 20 marks)