Exercise 3.1. Suppose $\Omega, \Omega' \subset \mathbb{R}^n$ are open, $g : \Omega \to \Omega'$ and $f : \Omega' \to \Omega$ are functions such that g is differentiable at $p \in \Omega$ and f is differentiable at $g(p) \in \Omega'$ and moreover

$$f \circ g(x) = x, \qquad \forall\, x \in \Omega.$$
$$g \circ f(x) = x, \qquad \forall\, x \in \Omega'.$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

Solution: We need to show that

$$Df(g(p)) \circ Dg(p) = \mathrm{id} \qquad \text{and} \qquad Dg(p) \circ Df(g(p)) = \mathrm{id}.$$

By the chain rule, we know that $f \circ g$ is differentiable at p. Moreover, since the identity is differentiable with derivative also the identity by a previous question, we deduce

$$\mathrm{id} = D(f \circ g)(p) = Df(g(p)) \circ Dg(p).$$

We also know by the chain rule that $g \circ f$ is differentiable at $g(p)$, with derivative

$$\mathrm{id} = D(g \circ f)(g(p)) = Dg(f(g(p))) \circ Df(g(p)) = Dg(p) \circ Df(g(p)),$$

where we have used $f(g(p)) = p$.

Exercise 3.2 (*)**.** (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$P : (x, y) \mapsto xy$$

is differentiable at each point $p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^2$, with Jacobian:

$$DP(p) = \begin{pmatrix} \eta & \xi \end{pmatrix}.$$

Solution: Let $h = (h_1, h_2)$

$$P(p + h) - P(p) = (\xi + h_1)(\eta + h_2) - \xi\eta = h_1\eta + h_2\xi + h_1 h_2,$$

so that:

$$P(p + h) - P(p) - DP(p)\,[h] = h_1 h_2.$$

Now, by Young's inequality we know that:

$$|h_1 h_2| \le \frac{1}{2}\left(h_1^2 + h_2^2\right) = \frac{1}{2}\,\|h\|^2,$$

so we have that:

$$\frac{|P(\xi + h_1, \eta + h_2) - P(\xi, \eta) - DP(p)\,[h]|}{\|h\|} \le \frac{1}{2}\,\|h\| \to 0,$$

as $\|h\| \to 0$.

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Questions marked with $*$ are optional

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ given by:

$$Q : z \mapsto (f(z), g(z))$$

is differentiable at q and:

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

Solution: We calculate, for $h \in \mathbb{R}^n$:

$$\begin{aligned} Q(q+h) - Q(q) - DQ(q)[h] &= \begin{pmatrix} f(q+h) \\ g(q+h) \end{pmatrix} - \begin{pmatrix} f(q) \\ g(q) \end{pmatrix} - \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix} h \\ &= \begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ g(q+h) - g(q) - Dg(q)[h] \end{pmatrix}. \end{aligned}$$

Now, writing:

$$\begin{aligned} \begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ g(q+h) - g(q) - Dg(q)[h] \end{pmatrix} &= \begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ g(q+h) - g(q) - Dg(q)[h]t \end{pmatrix}, \end{aligned}$$

and applying the triangle inequality we have:

$$\begin{aligned} \|Q(q+h) - Q(q) - DQ(q)[h]\| &\leq \|f(q+h) - f(q) - Df(q)[h]\| \\ &\quad + \|g(q+h) - g(q) - Dg(q)[h]\|, \end{aligned}$$

so that :

$$\begin{aligned} \lim_{h\to 0} \frac{\|Q(q+h) - Q(q) - DQ(q)[h]\|}{\|h\|} &\leq \lim_{h\to 0} \frac{\|f(q+h) - f(q) - Df(q)[h]\|}{\|h\|} \\ &\quad + \lim_{h\to 0} \frac{\|g(q+h) - g(q) - Dg(q)[h]\|}{\|h\|} \\ &= 0. \end{aligned}$$

(c) Show that $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(z) = f(z)g(z)$ for all $z \in \mathbb{R}^n$ is differentiable at q, and:

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

[Hint: Note that $F = P \circ Q$.]

Solution: Note that $F = P \circ Q$. Since Q is differentiable at q and P is differentiable at $Q(q)$, by the chain rule we have that $P \circ Q$ is differentiable, at q, and moreover:

$$DF(q) = DP(Q(q)) \circ DQ(q) = \begin{pmatrix} g(q) & f(q) \end{pmatrix} \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix} = g(q)Df(q) + f(q)Dg(q).$$

Exercise 3.3. (a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x,y) = \begin{pmatrix} x^2 + e^{x+y} \\ x - \log y \\ 2xy + 1 \end{pmatrix}.$$

Assuming f is differentiable at a point $\begin{pmatrix} x \\ y \end{pmatrix}$, what is its derivative?

(b) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by $g(x,y,z) = x + y + z$. Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.

Solution: (a) By the general formula for the Jacobian at $p = (x,y) \in \mathbb{R}^2$

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & D_2 f^1(p) \\ D_1 f^2(p) & D_2 f^2(p) \\ D_1 f^3(p) & D_2 f^3(p) \end{pmatrix} = \left(\frac{\partial f^j}{\partial z_k}\right)_{j=1,2,3;k=1,2},$$

where we denoted $z_1 = x$, $z_2 = y$. Note that by convention in the matrix element notation a_{jk}, the first index refers to the row, and the second, to the column. Computing partial derivatives, we obtain

$$Df(p) = \begin{pmatrix} 2x + e^{x+y} & e^{x+y} \\ 1 & -\frac{1}{y} \\ 2y & 2x \end{pmatrix}.$$

(b) First, the derivative of g at a point q

$$Dg(q) = (D_1 f(q)\ D_2 f(q)\ D_3 f(q)) = (1\ 1\ 1).$$

Using Df from (a) and the chain rule, we obtain

$$D(g \circ f)(p) = (1\ 1\ 1) \times \begin{pmatrix} 2x + e^{x+y} & e^{x+y} \\ 1 & -\frac{1}{y} \\ 2y & 2x \end{pmatrix} = (2(x+y) + e^{x+y} + 1 \quad 2x + e^{x+y} - \frac{1}{y}).$$

Alternatively,

$$(g \circ f)(x,y,z) = x^2 + e^{x+y} + x - \log y + 2xy + 1,$$

and hence

$$D(g \circ f)(p) = (D_1(g \circ f)(p), D_2(g \circ f)(p)) = (2(x+y) + e^{x+y} + 1 \quad 2x + e^{x+y} - \frac{1}{y}).$$

Exercise 3.4. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable, and find the differential when:

(a) $f(x,y) = x^2 + y^2 - x - xy$,

(b) $f(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x,y) = x^5 y^2$.

Solution: (a) Computing the partial derivatives, we have (letting $p = (x, y)$):

$$D_1 f(p) = 2x - 1 - y, \qquad D_2 f(p) = 2y - x,$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce from the theorem in the lecture notes that f is everywhere differentiable and moreover:

$$Df(p) = (2x - 1 - y, 2y - x)$$

(b) Computing the partial derivatives, we have (letting $p = (x, y)$):

$$D_1 f(p) = \frac{-x}{(1 + x^2 + y^2)^{\frac{3}{2}}}, \qquad D_2 f(p) = \frac{-y}{(1 + x^2 + y^2)^{\frac{3}{2}}},$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce by the theorem in the lectures that that f is everywhere differentiable and moreover:

$$Df(p) = \frac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}} (-x, -y)$$

(c) Computing the partial derivatives, we have (letting $p = (x, y)$):

$$D_1 f(p) = 5x^4 y^2, \qquad D_2 f(p) = 2x^5 y,$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce from the theorem in the lectures that f is everywhere differentiable and moreover:

$$Df(p) = \left(5x^4 y^2, 2x^5 y\right)$$

Unseen Exercise. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} x^2 \sin(1/x) & \text{if } y = 0, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Is the map f differentiable at $(0, 0) \in \mathbb{R}^2$? Justify your answer using the definition of the derivative.

Solution: Yes, the map f is differentiable at $(0, 0)$. We claim that $Df(0, 0)$ is the linear map $\Lambda \equiv 0$. To see this, let $h = (h^1, h^2) \in \mathbb{R}^2$. We note that

$$\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\| = \|f(h^1, h^2)\| \leq |h^1|^2.$$

Also, by an inequality in the exercises, we have $|h^1| \leq \|(h^1, h^2)\|$.

Thus,

$$\frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} \leq \frac{|h^1|^2}{|h^1|} = |h^1|.$$

This implies that

$$\lim_{(h^1, h^2) \to 0} \frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} = 0.$$