Imperial College London
Department of Mathematics

# Question Sheet 2 - Probl. Class week 4

MATH40003 Linear Algebra and Groups — Term 2, 2022/23

This is the problem sheet for the problem classes in week 4. All questions can be attempted with the material in lectures 1–5. Solutions will be released after the classes on Monday of week 4.

**Question 1** Let $n \in \mathbb{N}$, $n \geq 2$. Suppose $D : M_n(\mathbb{R}) \to \mathbb{R}$ is a function on which elementary row operations have the same effect as they do for det (for example, if $B$ is obtained from $A \in M_n(\mathbb{R})$ by interchanging two rows, then $D(B) = -D(A)$, etc.). Suppose also that $D(I_n) = 1$. Prove that $D(C) = \det(C)$ for all $C \in M_n(\mathbb{R})$.
*Harder:* What if we replace $\mathbb{R}$ by an arbitrary field $F$?

**Question 2** For each of the following linear maps $T : V \to V$, choose a basis $B$ for $V$ and compute $[T]_B$. Hence, or otherwise, compute $\det(T)$.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) $V$ is the vector space of all $2 \times 2$ matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(iii) $V$ is the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

**Question 3** Suppose $n \geq 2$ and $A \in M_n(F)$. The adjugate matrix $adj(A)$ is the transpose of the matrix of cofactors of $A$ and we showed that $adj(A)A = \det(A)I_n$. Give an expression for $adj(adj(A))$ in the case where $A$ is invertible.

**Question 4** Suppose $F$ is a field. Let $n \in \mathbb{N}$ and $a_0, ..., a_{n-1} \in F$, not all zero. Using the Vandermonde determinant, prove that the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

has at most $n-1$ distinct roots in $F$, i.e. there are at most $n-1$ distinct $\alpha \in F$ such that $f(\alpha) = 0$.

**Question 5** Suppose $U, V, W$ are vector spaces over a field $F$ and $T : U \to V$ and $S : V \to W$ are linear transformations. Show that the composition $S \circ T : U \to W$ is a linear transformation. If $U, V, W$ are finite dimensional with bases $B, C, D$, prove that

$${}_D[S \circ T]_B = {}_D[S]_C\, {}_C[T]_B.$$

**Question 6** Let $V$ be a vector space over a field $F$ and $T : V \to V$ be a linear transformation. Suppose that $\lambda \in F$ is an eigenvalue of $T$. Let $m \geq 1$ be an integer and denote by $T^m$ the composition $T \circ \ldots \circ T$ ($m$ times). Note that this is a linear transformation $V \to V$.

i) Show that $\lambda^m$ is an eigenvalue of $T^m$.

ii) If $a_0, \ldots, a_m \in F$ are such that $a_0 \operatorname{Id} + a_1 T + a_2 T^2 + \ldots + a_m T^m = 0$, show that $\lambda$ is a root of the polynomial $p(x) = a_0 + a_1 x + \ldots + a_m x^m$.

**Question 7** Suppose that $T : V \to V$ is a linear map with the property that $T(T(v)) = T(v)$ for all $v \in V$.

(i) Show that

$$V = \ker(T) + \operatorname{im}(T) \text{ and } \ker(T) \cap \operatorname{im}(T) = \{0\}.$$

*Hint: Note that if $v \in V$ then $v = (v - T(v)) + T(v)$.*

(ii) Deduce that if $V$ is of dimension $n$, then there is a basis $B$ of $V$ such that

$$[T]_B = \begin{pmatrix} I_s & 0_{s\times(n-s)} \\ 0_{(n-s)\times s} & 0_{(n-s)\times(n-s)} \end{pmatrix},$$

where $s = \dim(\operatorname{im}(T))$.