IMPERIAL

MATH70028

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Probability Theory

Date: Friday, May 9, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

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In all the questions below, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and, unless said otherwise, all random variables are defined on this space and are almost surely finite.

1. (a) (i) Let $\mathcal{D}$ be a collection of subsets of Ω. Carefully show from first principles (using **only** the definition of σ-algebras) that there exists a σ-algebra, denoted $\sigma(\mathcal{D})$, such that $\sigma(\mathcal{D}) \supset \mathcal{D}$ and any σ-algebra $\mathcal{G} \supset \mathcal{D}$, satisfies $\mathcal{G} \supset \sigma(\mathcal{D})$. (5 marks)

   (ii) Let $\mathcal{D} \subset \mathcal{F}$ and suppose that $A \in \sigma(\mathcal{D})$. Prove that there exists $\mathcal{D}' \subset \mathcal{D}$ such that $\mathcal{D}'$ is countable and $A \in \sigma(\mathcal{D}')$. (6 marks)

   (b) (i) Given a sequence of random variables $(X_n)_{n=1}^{\infty}$ and another random variable X, all defined on the same probability space, carefully state the definition of "$X_n \to X$ in probability as $n \to \infty$". (3 marks)

   (ii) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $X_n \to X$ in probability as $n \to \infty$. Carefully prove that $f(X_n) \to f(X)$ as $n \to \infty$ in probability. (6 marks)

(Total: 20 marks)

2. (a) Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables, all with finite second moment, and with the properties that, for $i, j \geq 1$, $\mathbb{E}[X_i] = 1 + 1/i$ and $\mathbb{E}[X_i X_j] = (1 + 1/i)(1 + 1/j) + 2^{-|i-j|}$. Let $S_n = \sum_{j=1}^{n} X_j$. Prove that $S_n/n \to 1$ in L^2 as $n \to \infty$ (note, you should actually prove this rather than just applying a single result from the module). (9 marks)

   (b) (i) Given a sequence of random variables $(X_n)_{n=1}^{\infty}$ and another random variable X, carefully state the definition of "X_n converges to X in distribution", or "$X_n \Rightarrow X$", as $n \to \infty$. (3 marks)

   (ii) Prove that if $X_n \to X$ in probability then $X_n \Rightarrow X$. (5 marks)

   (iii) Give an example, with justification, of a sequence of random variables $(X_n)_{n=1}^{\infty}$ and another random variable X all defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \Rightarrow X$, but one does not have $X_n \to X$ in probability. (3 marks)

(Total: 20 marks)

3. (a) Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables, define the corresponding σ-algebra of tail events $\mathcal{T}$. (2 marks)

(b) (i) Let X be a non-negative random variable. Prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(X > n) \leq \mathbb{E}[X] \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(X > n) \ .$$

(6 marks)

(ii) Let $(X_n)_{n=1}^{\infty}$ be a sequence of i.i.d random variables with $\mathbb{E}\Big[|X_i|\Big] = \infty$. Prove that

$$\mathbb{P}\Big(\limsup_{n\to\infty} |X_n|/n = \infty \Big) = 1 \ .$$

(6 marks)

(iii) Let $(Y_n)_{n=1}^{\infty}$ be an i.i.d random variables with $\mathbb{E}\Big[|Y_n|\Big] < \infty$. Define the sequence of random variables $(T_n)_{n=1}^{\infty}$ by setting $T_n = Y_n \mathbf{1}\{|Y_n| \leq n\}$ and suppose that there exists $c \in \mathbb{R}$ such that

$$\mathbb{P}\Big(\frac{1}{n} \sum_{j=1}^{n} T_j \to c \text{ as } n \to \infty \Big) > \frac{1}{2} \ .$$

Prove that $\frac{1}{n}\sum_{j=1}^{n} Y_j \to c$ almost surely as $n \to \infty$. (6 marks)

(Total: 20 marks)

4. (a) State (both parts of) the Lévy Continuity Theorem. (5 marks)

(b) Prove the following from "first principles" (that is, only use the definition, existence, and/or uniqueness of conditional expectations).

If X, Y are two integrable random variables with $X \geq Y$, and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, then

$$\mathbb{E}[X \mid \mathcal{G}] \geq \mathbb{E}[Y \mid \mathcal{G}] \text{ almost surely} \ .$$

(6 marks)

(c) (i) Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables and $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration. Give the (three) definitions for $(X_n)_{n=1}^{\infty}$ being a submartingale, supermartingale, and martingale. (3 marks)

(ii) Suppose $(X_n)_{n=1}^{\infty}$ is a martingale for a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ with $\mathbb{E}[X_1] = 0$, and with X_n having finite second moment for every $n \in \mathbb{N}$. Prove that for any $n, k \in \mathbb{N}$, one has

$$\mathbb{E}\Big[(X_{n+k} - X_n)^2\Big] = \sum_{j=1}^{k} \mathbb{E}\Big[(X_{n+j} - X_{n+j-1})^2\Big] \ .$$

(6 marks)

(Total: 20 marks)

5. (a) Let $\mu = \bigotimes_{n=0}^{\infty} \mu^{(n)}$ and $\nu = \bigotimes_{n=0} \nu^{(n)}$ be product measures on $\left(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}})\right)$ with the property that, for each n, $\mu^{(n)} \ll \nu^{(n)}$. Let $X_n(x) = \prod_{j=0}^{n} \frac{\mathrm{d}\mu^{(j)}}{\mathrm{d}\nu^{(j)}}(x_j)$, where we write $x \in \mathbb{R}^{\mathbb{N}}$ as $x = (x_j : j \in \mathbb{N})$. Prove that X_n converges ν-almost surely to some limit X (note, don't just apply a theorem from mastery material but argue this using non-mastery material yourself)

(7 marks)

(b) (i) State Doob's inequality on maximums of non-negative parts of submartingales.

(5 marks)

(ii) Let $(X_n)_{n=1}^{\infty}$ be a positive submartingale for the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. Prove that, for any $\lambda > 0$,

$$\lambda \mathbb{P}\Big(\max_{1 \le j \le n} X_j > 2\lambda \Big) \le \mathbb{E}[X_n \mathbf{1}\{X_n > \lambda\}] \, .$$

(8 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60028/70028

MATH60028/70028 (Solutions)

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1. (a) (i) Let $\mathfrak{F}$ be the collection of all σ-algebras on Ω that contain $\mathcal{D}$. Note that $\mathfrak{F}$ is non-empty as $2^\Omega \in \mathfrak{F}$. Let $\mathcal{C} = \bigcap_{\mathcal{A}\in\mathfrak{F}} \mathcal{A}$. Since $\mathcal{D} \subset \mathcal{A}$ for every $\mathcal{A}$ in the above intersection, clearly $\mathcal{D} \subset \mathcal{C}$. Now, let $\mathcal{G}$ be a σ-algebra on Ω that contains $\mathcal{D}$. Then $\mathcal{G} \in \mathfrak{F}$ and so $\mathcal{C} \subset \mathcal{G}$.

It suffices to show that $\mathcal{C}$ is a σ-algebra. Let $A \in \mathcal{C}$. Then $A \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$. Since each $\mathcal{A}$ is a σ-algebra, $A^c \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$. Thus $A^c \in \mathcal{C}$. Let $A_1, A_2, \ldots \in \mathcal{C}$. Then $A_n \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$. Since each $\mathcal{A}$ is a σ-algebra, $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$. Thus $\bigcup_{n=1}^\infty A_n \in \mathcal{C}$. Therefore $\mathcal{C}$ is a σ-algebra that contains $\mathcal{D}$ and is contained in every σ-algebra that contains $\mathcal{D}$. 5, A

(ii) Let $\mathcal{A}$ be the collection of all elements of 2^Ω such that there exists some countable set $\mathcal{D}' \subset \mathcal{D}$ (possibly depending on A) with $A \in \sigma(\mathcal{D}')$. It suffices to show that (i) $\mathcal{D} \subset \mathcal{A}$, (ii) $\mathcal{A}$ is a σ-algebra. This would then imply that $\sigma(\mathcal{D}) \subset \mathcal{A}$ as desired.

For (i), for any $D \in \mathcal{D}$ we clearly have $D \in \sigma(\{D\})$ and $\{D\}$ is countable (finite even!) so we have $D \in \mathcal{A}$. For (ii), suppose that $A \in \mathcal{A}$. Then there exists a countable set $\mathcal{D}' \subset \mathcal{D}$ such that $A \in \sigma(\mathcal{D}')$. It follows that $A^c \in \sigma(\mathcal{D}')$, so $A^c \in \mathcal{A}$.

Next, let $A_1, A_2, \ldots \in \mathcal{A}$. Then there exist countable sets $\mathcal{D}_1, \mathcal{D}_2, \ldots \subset \mathcal{D}$ such that $A_n \in \sigma(\mathcal{D}_n)$. Let $\mathcal{D}' = \bigcup_{n=1}^\infty \mathcal{D}_n$. We note that that, for every n, $\sigma(\mathcal{D}') \supset \sigma(\mathcal{D}_n)$ since $\mathcal{D}_n \subset \mathcal{D}' \subset \sigma(\mathcal{D}')$ and $\sigma(\mathcal{D}')$ is a σ-algebra.

We also note that $\mathcal{D}'$ is countable and $A_n \in \sigma(\mathcal{D}')$ for every n. Thus $\bigcup_{n=1}^\infty A_n \in \sigma(\mathcal{D}')$ and so $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$. 6, D

(b) (i) For every $\epsilon > 0$, one has

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0 \,.$$

3, A

(ii) Let $\epsilon, \eta \in (0,1)$. It suffices to show that there exists N such that $n \geq N$ implies that

$$\mathbb{P}(|f(X_n) - f(X)| > \epsilon) < \eta \,.$$

Now first note that, thanks to continuity from below, $\lim_{K\uparrow\infty} \mathbb{P}(|X| \leq K) = 1$. It follows that there exists $\bar{K} > 0$ such that $\mathbb{P}(|X| > \bar{K}) < \eta/2$. Since f is continuous, it is uniformly continuous on $[-\bar{K}-1, \bar{K}+1]$. Then there exists $\delta \in (0,1)$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in [-\bar{K}-1, \bar{K}+1]$. Now, since $X_n \to X$ in probability, there exists N such that $n \geq N$ implies that $\mathbb{P}(|X_n - X| > \delta/2) < \eta/2$. Thus, for $n \geq N$, we have

$$\begin{aligned}
\mathbb{P}(|f(X_n) - f(X)| > \epsilon) &\leq \mathbb{P}(|f(X_n) - f(X)| > \epsilon, |X| \leq \bar{K}) + \mathbb{P}(|X| > \bar{K}) \\
&\leq \mathbb{P}(|X_n - X| > \delta) + \mathbb{P}(|X| > \bar{K}) \\
&< \eta/2 + \eta/2 = \eta \,.
\end{aligned}$$

In the second inequality above, we used that if $|X| \leq \bar{K}$ and $|X_n - X| \leq \delta/2$, then we have $X, X_n \in [-\bar{K}-1, \bar{K}+1]$ and $|f(X_n) - f(X)| < \epsilon$. 6, B

2. (a) Let $\bar{X}_n = X_n - \mathbb{E}[X_n]$. Let $\bar{S}_n = \sum_{j=1}^n \bar{X}_n$. Then we have

$$\frac{1}{n}\bar{S}_n = \frac{1}{n}S_n - \frac{1}{n}\sum_{j=1}^{n}(1+1/j)\ .$$

Since $\frac{1}{n}\sum_{j=1}^{n}(1+1/j) \to 1$ as $n \to \infty$, it suffices to show that

$$\frac{1}{n}\bar{S}_n \to 0 \text{ in } L^2 \text{ as } n \to \infty\ .$$

Now, we note that

$$\mathbb{E}[\bar{X}_i\bar{X}_j] = 2^{-|i-j|}\ .$$

Thus, we have that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{n}\bar{S}_n\right)^2\right] &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[\bar{X}_i\bar{X}_j] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}2^{-|i-j|} \\
&\leq \frac{2}{n^2}\sum_{i=1}^{n}\sum_{r=0}^{\infty}2^{-r} \\
&= \frac{2}{n^2}\sum_{i=1}^{n}2 \\
&= \frac{4n}{n^2} \to 0 \text{ as } n \to \infty\ .
\end{aligned}$$

In the inequality above, we used that, for fixed i, each value of 2^{-r} appears at most twice in the sum $\sum_{j=1}^n 2^{-|i-j|}$. $\boxed{9, \mathrm{B}}$

(b) (i) Let F_n be the distribution function of X_n, and F be the distribution function of X. We say $X_n \Rightarrow X$ if, for every point of continuity x of F, one has $F_n(x) \to F(x)$ as $n \to \infty$. $\boxed{3, \mathrm{A}}$

(ii) To show that $X_n \Rightarrow X$, it suffices to show that, for any continuous bounded function f, one has $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ as $n \to \infty$. Now, since $X_n \to X$ in probability, it follows that every subsequence of X_n has a further subsequence converging to X_n almost surely. Recall that if a subsequence $X_{n_k} \to X$ converges almost surely as $k \to \infty$, then $f(X_{n_k}) \to f(X)$ almost surely as $k \to \infty$. By bounded convergence, we have $\mathbb{E}[f(X_{n_k})] \to \mathbb{E}[f(X)]$ as $k \to \infty$. It follows that every subsequence of $\mathbb{E}[f(X_n)]$ has a subsequence converging to $\mathbb{E}[f(X)]$, therefore $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_n)]$ as desired. $\boxed{5, \mathrm{B}}$

(iii) Let X be a Bernoulli ± 1 random variable with mean 0. Let $X_n = (-1)^n X$. Since X has a symmetric distribution, it follows that each X_n is equal in distribution to X. Therefore, $F_{X_n} = F_X$, and so $X_n \Rightarrow X$. On the other hand, for $\epsilon \in (0, 1/2)$,

$$\mathbb{P}(|X_n - X| > \epsilon) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we do not have $X_n \to X$ in probability. $\boxed{3, \mathrm{A}}$

3. (a)

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots) \ .$$

$\boxed{2,\ \mathrm{A}}$

(b) (i) We recall that we have, for any $X \geq 0$,

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, dx = \sum_{j=0}^{\infty} \int_j^{j+1} \mathbb{P}(X > x)\, dx \ .$$

Since $\mathbb{P}(X > x)$ is decreasing in x, for each j we have

$$\mathbb{P}(X > j) \geq \int_j^{j+1} \mathbb{P}(X > x) \geq \mathbb{P}(X > j+1) \ .$$

It follows that

$$\sum_{j=0}^{\infty} \int_j^{j+1} \mathbb{P}(X > x)\, dx \geq \sum_{j=0}^{\infty} \mathbb{P}(X > j+1) = \sum_{j=1}^{\infty} \mathbb{P}(X > j) \ .$$

On the other hand,

$$\sum_{j=0}^{\infty} \int_j^{j+1} \mathbb{P}(X > x)\, dx \leq \sum_{j=0}^{\infty} \mathbb{P}(X > j) \leq 1 + \sum_{j=1}^{\infty} \mathbb{P}(X > j) \ .$$

$\boxed{6,\ \mathrm{A}}$

(ii) Without loss of generality we can take $X_n \geq 0$ by setting $X_n' = |X_n|$. It then follows, that, for each $c > 0$,

$$\infty = \mathbb{E}[X_1/c] \leq \sum_{j=1}^{\infty} \mathbb{P}(X_1 > cj) \ .$$

By the second Borel-Cantelli lemma, we have that $\mathbb{P}(X_j/j > c \text{ i.o.}) = 1$. Let $A_c = \{\limsup_n X_n/n > c\}$, it then follows that $\mathbb{P}(A_c) = 1$ for all $c > 0$. We then have $A = \bigcap_{c=1}^{\infty} A_c$ has probability 1, but

$$A = \{\limsup_n X_n/n = \infty\} \ .$$

$\boxed{6,\ \mathrm{D}}$

(iii) We first note that

$$\sum_{n=1}^{\infty} \mathbb{P}(\{Y_n \neq T_n\}) = \sum_{n=1}^{\infty} \mathbb{P}(|Y_n| > n) = \sum_{n=1}^{\infty} \mathbb{P}(|Y_1| > n) < \mathbb{E}[|Y_1|] < \infty \ .$$

Therefore, by the first Borel-Cantelli lemma, $\mathbb{P}(\{Y_n \neq T_n \text{ i.o.}\}) = 0$. It follows that it suffices to show that

$$\frac{1}{n} \sum_{j=1}^{\infty} T_n \to c \text{ almost surely as } n \to \infty \ .$$

Next, we note that the event $\{\frac{1}{n}\sum_{j=1}^{\infty} T_n \to c\}$ is in $\mathcal{T}$. By the Kolmogorov 0-1 law, we know that $\mathbb{P}(\{\frac{1}{n}\sum_{j=1}^{\infty} T_n \to c\}) = 0$ or 1. Since we assume that this event has probability greater than $1/2$, it must have probability 1. $\boxed{6,\ \mathrm{C}}$

4. (a) Part 1: Suppose that we have a sequence of probability measures μ_n and a probability measure μ such that $\mu_n \Rightarrow \mu$, then $\phi_{\mu_n} \to \phi_\mu$ in probability where ϕ_ν is the characteristic function of μ.

Part 2: Suppose that we have a sequence of probability measures μ_n such that $\phi_{\mu_n} \to \phi$ pointwise where ϕ is continuous at 0. Then, there exists a probability measure μ with $\phi_\mu = \phi$ and $\mu_n \Rightarrow \mu$. $\boxed{5, \text{A}}$

(b) Suppose by contradiction that $H = \{\mathbb{E}[X \mid \mathcal{G}] - \mathbb{E}[Y \mid \mathcal{G}] < 0\}$ satisfies $\mathbb{P}(H) > 0$. Note that $H \in \mathcal{G}$ since both $\mathbb{E}[X \mid \mathcal{G}]$ and $\mathbb{E}[Y \mid \mathcal{G}]$ are $\mathcal{G}$-measurable.

Now, by the definition of conditional expectation we have

$$\mathbb{E}[\mathbf{1}_H \mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[\mathbf{1}_H X] \ , \quad \mathbb{E}[\mathbf{1}_H \mathbb{E}[Y \mid \mathcal{G}]] = \mathbb{E}[\mathbf{1}_H Y] \ .$$

Therefore we have

$$\mathbb{E}\Big[\mathbf{1}_H \big(\mathbb{E}[X \mid \mathcal{G}]] - \mathbb{E}[Y \mid \mathcal{G}]]\big)\Big] = \mathbb{E}[\mathbf{1}_H (X - Y)] \ .$$

However, if $\mathbb{P}(H) > 0$ we must have

$$\mathbb{E}\Big[\mathbf{1}_H \big(\mathbb{E}[X \mid \mathcal{G}]] - \mathbb{E}[Y \mid \mathcal{G}]]\big)\Big] < 0 \ .$$

This contradicts the fact that $\mathbb{E}[\mathbf{1}_H(X - Y)] \geq 0$ since $X \geq Y$. $\boxed{6, \text{C}}$

(c) (i) We say $(X_n)_{n=1}^\infty$ is a sub-martingale if the following three conditions hold. (I) X_n is integrable for all n. (II) X_n is $\mathcal{F}_n$-measurable for all n. (III) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ for all n.

We say $(X_n)_{n=1}^\infty$ is a super-martingale if $(-X_n)_{n=1}^\infty$ is a sub-martingale. We say $(X_n)_{n=1}^\infty$ is a martingale if it is both a sub-martingale and a super-martingale.

$\boxed{3, \text{A}}$

(ii) We have that

$$\begin{aligned}
\mathbb{E}[(X_{n+k} - X_n)^2] &= \mathbb{E}\Big[\Big(\sum_{j=1}^{k} (X_{n+j} - X_{n+j-1})\Big)^2\Big] \\
&= \sum_{j=1}^{k} \sum_{l=1}^{k} \mathbb{E}\Big[(X_{n+j} - X_{n+j-1})(X_{n+l} - X_{n+l-1})\Big] \ .
\end{aligned}$$

It suffices to show that, for $j < l$,

$$\mathbb{E}\Big[(X_{n+j} - X_{n+j-1})(X_{n+l} - X_{n+l-1})\Big] = 0 \ .$$

To see this, we note that

$$\begin{aligned}
&\mathbb{E}\Big[(X_{n+j} - X_{n+j-1})(X_{n+l} - X_{n+l-1})\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[(X_{n+j} - X_{n+j-1})(X_{n+l} - X_{n+l-1}) \mid \mathcal{F}_{n+l-1}\big]\Big] \\
&= \mathbb{E}\Big[(X_{n+j} - X_{n+j-1})\mathbb{E}\big[(X_{n+l} - X_{n+l-1}) \mid \mathcal{F}_{n+l-1}\big]\Big] \\
&= \mathbb{E}\Big[(X_{n+j} - X_{n+j-1})(X_{n+l-1} - X_{n+l-1})\Big] = 0 \ ,
\end{aligned}$$

where we used the tower property in the first equality above, then the fact that $X_{n+j} - X_{n+j-1}$ is $\mathcal{F}_{n+l-1}$-measurable in the second equality since $j < l$, and finally we used the martingale property in the third equality. $\boxed{6, \text{C}}$

5. (a) Let $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$ where, for $j \in \mathbb{N}$, $\xi_j : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is projection onto the j-th component. Then $(\mathcal{F}_n)_{n=0}^{\infty}$ is a filtration. We claim that X_n is a martingale for this filtration.

Since X_n depends only on $\xi_0, \ldots \xi_n$, it is adapted to $\mathcal{F}_n$. In particular we have $X_n = \dfrac{\mathrm{d}\mu_n}{\mathrm{d}\nu_n}$ where μ_n and ν_n are the restrictions of μ and ν to $\mathcal{F}_n$, respectively - this is because the product of Radon-Nikodym derivatives is the Radon Nikodym derivative between product measures (they agree on rectangles in $\mathcal{F}_n$). We have that $X_n \geq 0$, and $\mathbb{E}[X_n] = \mu_n(\mathbb{R}^{\mathbb{N}}) = \mu(\mathbb{R}^{\mathbb{N}}) = 1$.

We now show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$. Let $A \in \mathcal{F}_n$. Then we have that $A = B \times \mathbb{R}^{\mathbb{N}\setminus\{0,\ldots,n\}}$ for some $B \in \mathcal{B}(\mathbb{R}^{n+1})$. It follows that

$$\begin{aligned}
\mathbb{E}[X_{n+1}\mathbf{1}_A] = \mu_{n+1}(A) &= \mu_{n+1}(B \times \mathbb{R}^{\mathbb{N}\setminus\{0,\ldots,n\}}) \\
&= \Big(\bigotimes_{j=0}^{n+1} \mu^{(j)}\Big)\big(B \times \mathbb{R}\big) \\
&= \Big(\bigotimes_{j=0}^{n} \mu^{(j)}\Big)\big(B\big) \times 1 \\
&= \mu_n(B \times \mathbb{R}^{\mathbb{N}\setminus\{0,\ldots,n\}}) \\
&= \mathbb{E}[X_n\mathbf{1}_A]\ .
\end{aligned}$$

Since X_n is also $\mathcal{F}_n$-measurable, it follows that by uniqueness of conditional expectations that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$. Since $X_n \geq 0$, convergence then follows by the martingale convergence theorem. $\boxed{7, \text{M}}$

(b) (i) Let $(X_n)_{n=1}^{\infty}$ be a sub-martingale, then for each n and $\lambda > 0$,

$$\lambda\mathbb{P}\big(\sup_{1\leq k\leq n} X_k^+ \geq \lambda\big) \leq \mathbb{E}\Big[X_n\mathbf{1}\big\{\sup_{1\leq k\leq n} X_k^+ \geq \lambda\big\}\Big]\ .$$

where, for a random variable Y, we write $Y^+ = \max(Y, 0)$. $\boxed{5, \text{M}}$

(ii) Let $N = \inf\{k : X_k \geq 2\lambda \text{ or } k = n+1\}$. Let $\bar{X}_n = \sup_{1\leq k\leq n} X_k$. Then, by Doob's inequality, we have

$$\mathbb{E}[X_n\mathbf{1}\{N \leq n\}] \geq 2\lambda\mathbb{P}(\bar{X}_n \geq 2\lambda)\ .$$

On the other hand,

$$\mathbb{E}[X_n\mathbf{1}\{X_n \leq \lambda, N \leq n\}] \leq \lambda\mathbb{P}(X_n \leq \lambda, N \leq n) \leq \lambda\mathbb{P}(N \leq n) = \lambda\mathbb{P}(\bar{X}_n \geq 2\lambda)$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}[X_n\mathbf{1}\{X_n > \lambda\}] &\geq \mathbb{E}[X_n\mathbf{1}\{X_n > \lambda, N \leq n\}] \\
&\geq \mathbb{E}[X_n\mathbf{1}\{N \leq n\}] - \mathbb{E}[X_n\mathbf{1}\{X_n \leq \lambda, N \leq n\}] \\
&\geq 2\lambda\mathbb{P}(\bar{X}_n \geq 2\lambda) - \lambda\mathbb{P}(\bar{X}_n \geq 2\lambda) = \lambda\mathbb{P}(\bar{X}_n \geq 2\lambda)\ .
\end{aligned}$$

$\boxed{8, \text{M}}$

Review of mark distribution:

Total A marks: 30 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 18 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 100 of 100 marks

Total Mastery marks: 20 of 20 marks

MATH70028 Probability Theory Markers Comments

Question 1	Q1) Students overall did well on 1)a)(i) and 1)b)(i). Students found 1)a)(ii) difficult but a reasonable portion of students got it correct. On 1)b)(ii), many students lost points because their proof implicitly assumed that f is uniformly continuous, rather than only continuous. The quickest way to argue for this question was to use subsequence's converging almost surely, but the solutions guide gives a brute force proof.
Question 2	Q2) Most students got 2)a), although some students didn't complete the estimates correctly. Nearly everyone got over half credit though. Overall students did well on 2)b) as well, although a good number of people made errors on 2)b)(ii).
Question 3	Q3) Some students made minor errors on 3)a). Overall students did well on 3)b)i) and 3)b)iii), but many had difficulty with 3)b)ii).
Question 4	Q4) Students did well on 4)a) and 4)c(i). Many students gave only partial answers for 4)b). Many students had difficulty with 4)c)(ii).
Question 5	Q5) Many students didn't write much for 5)a), or wrote something that was about Kakutani's Dichotomy theorem in general rather than the specific step they were asked to prove. On 5)b)(ii), many students noticed a simple monotonicity argument that worked However, the majority of students had difficulty with 5)b)(ii). Most students got 5)b)(i),