ALGEBRA 3 ASSESSED PROBLEMS 2 - DUE MONDAY, 13 DECEMBER 2021

This is the second assessed coursework for Algebra 3; unlike the weekly discussion problems, all rules for assessed coursework at Imperial apply. It is due Monday, 13 November, at 11PM in Turnitin (via the course Blackboard page) and is worth 5 percent of your total marks for the course. The coursework is scored out of ten marks; each question is worth one mark.

1. Let $c \in \mathbb{R}$, and let S denote the ring $\mathbb{R}[X]/\langle X^2 - c\rangle$. Show that if $c > 0$, then S is isomorphic to $\mathbb{R} \times \mathbb{R}$, and if $c < 0$, then S is isomorphic to $\mathbb{C}$.

2. Let S be a ring that contains $\mathbb{R}$ as a subring, and such that the resulting action of $\mathbb{R}$ on S makes S into a two-dimensional $\mathbb{R}$-vector space. Show that S is isomorphic to either $\mathbb{R} \times \mathbb{R}$, $\mathbb{C}$, or $\mathbb{R}[X]/\langle X^2\rangle$.

3. Let α be a root of the polynomial $X^2 + 1$ in $\mathbb{F}_3[X]$, and let $K = \mathbb{F}_3(\alpha)$. List all elements of $K^\times$ that generate $K^\times$ as a group. (You may assume, without proof, that $X^2 + 1$ is irreducible over $\mathbb{F}_3$.)

4. Let α be as in problem 3, and let β be a root of the irreducible polynomial $X^2 + X - 1$ over $\mathbb{F}_3$. Give an explicit isomorphism of $\mathbb{F}_3(\alpha)$ with $\mathbb{F}_3(\beta)$.

5. Let L be the abelian subgroup of $\mathbb{Z}^4$ generated by the vectors $(1, 4, 0, 3)$, $(0, 3, 9, 12)$, and $(-1, -1, 3, 3)$. Express the quotient $\mathbb{Z}^4/L$ in the form

$$\mathbb{Z}^r \oplus \mathbb{Z}/\langle a_1\rangle \oplus \cdots \oplus \mathbb{Z}/\langle a_s\rangle,$$

with a_i nonzero, nonunit integers and $a_i | a_{i+1}$ for all i.

6. Let A be an abelian group of order 100. Show that if A contains no element of order 4, then A contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

7. Let R be a ring, and suppose that the ring $R[X]$ is Noetherian. Show that R is a Noetherian ring.

8. Let M and N be R-modules. Show that the set $\mathrm{Hom}_R(M, N)$ of R-module homomorphisms from M to N has the structure of an R-module, where we define $f + g$ by $(f + g)(m) = f(m) + g(m)$ and rf (for $r \in R$) by $(rf)(m) = r \cdot f(m)$.

9. Show that if M, M', and N are R-modules, and $g : M \to M'$ is an R-module homomorphism, then the map: $\mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$ that takes a map $f : M' \to N$ to the map $f \circ g$ is a homomorphism of R-modules.

10. Show that if M and N are finitely generated R-modules, and R is a Noetherian ring, then $\operatorname{Hom}_R(M, N)$ is a finitely generated R-module.