Imperial College London

MATH97033 MATH70036

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Group Theory

Date: 13 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

On intermediate steps you can use results from the course provided you state them clearly.

1. For the questions below provide proofs or give counterexamples.

   (a) Let G be a finite group, and let $Z(G)$ be the centre of G. For each of the following conditions on G, explain whether or not the condition implies that $Z(G)$ is non-trivial:

      (i) G is nilpotent;

      (ii) G is solvable;

      (iii) G is a p-group for a prime number p?

   (7 marks)

   (b) With G as above let $N \leq Z(G)$ and $M \geq [G, G]$. Is N necessarily normal? Is N necessarily characteristic? Is M necessarily normal? Is M necessarily characteristic?

   (5 marks)

   (c) Let H and K be finite simple groups and $G = H \times K$. Determine for which pairs (H, K) the following statements hold:

      (i) $\mathrm{Aut}\,(G) \cong \mathrm{Aut}\,(H) \times \mathrm{Aut}\,(K)$;

      (ii) $\mathrm{Aut}\,(G)$ contains an index 2 subgroup isomorphic to $\mathrm{Aut}\,(H) \times \mathrm{Aut}\,(K)$;

      (iii) neither (i) nor (ii) holds.

   (8 marks)

(Total: 20 marks)

2. (a) Find the normalisers in S_5 (which is the symmetric group of degree 5) of the subgroup generated by the element $f = (1\,2\,3\,4\,5)$ and of the subgroup generated by the elements $a = (1\,2)$ and $b = (3\,4)$. (6 marks)

   (b) Let $n \geq 5$ and $2 < k < n$. Stating clearly any results from the course that you use prove that S_n has no subgroup of index k. (4 marks)

   (c) Let G be S_5 and let Ω be the set of subgroups of order 5 in G. Let G act on Ω by conjugation. Show that this action is 3-transitive. (5 marks)

   (d) With G and Ω as defined in (c) show that

      (i) a transposition of S_5 acts on Ω as the product of three disjoint transpositions; and that

      (ii) a 3-cycle of S_5 acts on Ω as the product of two disjoint 3-cycles.

   (5 marks)

(Total: 20 marks)

3. (a) Prove that if H is a normal subgroup of a finite group G with $[G:H]$ coprime to p, where p is a prime, then H contains every Sylow p-subgroup of G. (4 marks)

(b) Let G be a group of order $294 = 2 \cdot 3 \cdot 7^2$. Show that G contains a unique subgroup of index 2. Show also that G contains either one or three subgroups of index 3, giving examples to demonstrate that both cases are possible.

(6 marks)

(c) Let p be the smallest prime divisor of G. Suppose that G has a normal cyclic Sylow p-subgroup P. Show that P is contained in the centre of G.

(3 marks)

(d) Prove that a nilpotent group is (isomorphic to) the direct product of its Sylow subgroups.

(8 marks)

(Total: 21 marks)

4. (a) Let $F = Z/3Z$ be the field with 3 elements, and let

$$G = \{(a_{ij})_{3\times 3} \in GL_3(F) \mid a_{21} = a_{31} = a_{32} = 0\}$$

be the group of upper triangular matrices in $GL_3(F)$. Let

$$H = \{(a_{ij})_{3\times 3} \in G \mid a_{11} = a_{22} = a_{33} = 1\}$$

be the group of uni-triangular matrices.

Find the orders of G and H. Which of the groups G and H are solvable and which are nilpotent? Justify your answer.

(6 marks)

(b) Let p, q and r be prime numbers. How many composition series has $C_p \times C_q \times C_r$ in the following cases:

(i) p, q and r are pairwise distinct;

(ii) $p = q = r$. (6 marks)

(c) Prove that every group of order p^2, where p is a prime number, is abelian. Let G be a non-abelian group of order p^3. Prove that $Z(G)$ coincides with $[G,G]$ and has order p. Express as a polynomial in p the number of conjugacy classes of G. (8 marks)

(Total: 20 marks)

5. Let H be a subgroup of a finite group G. Let H^{cl} be the union of the conjugates of H in G.

(a) Prove the following;

(i) $|G| - |H^{\mathrm{cl}}| \geq n - 1$, where $n = [G : H]$;

(ii) The equality in (i) is attained if and only if $H \cap g^{-1}Hg = \{e\}$ for every $g \in G \setminus H$.

(4 marks)

A pair (G, H) for which H is proper, and the equality in a(i) *is attained is called a* **Frobenius pair** *and G is called a* **Frobenius group.** *You can use without proof the fact that $N := (G \setminus H^{\mathrm{cl}}) \cup \{e\}$ is a normal subgroup in G called the* **Frobenius kernel** *while H is called a* **Frobenius complement**.

(b) Which of the symmetric and alternating groups of degree $n \geq 3$ are Frobenius groups?

(2 marks)

(c) Let G be a Frobenius group of order 18. Prove that the kernel N is of order 9 and it is either elementary abelian, or cyclic. Show that the action of $h \in H \setminus \{1\}$ on N by conjugation is via inversion. Thus conclude that up to isomorphism there are exactly two isomorphism types of Frobenius groups of order 18. (4 marks)

(d) Construct a Frobenius group of order 56 and prove that this is the unique (up to isomorphism) Frobenius group of that order. (4 marks)

(e) Let $F = \mathbb{Z}/13\mathbb{Z}$ be the field with 13 elements, let N be the direct sum of two copies of F, viewed as a 2-dimensional F-space. Let G be the semi-direct product of (the additive group) of N and a subgroup H of $GL_2(F)$ generated by the matrices $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Show that G is a Frobenius group with kernel N and complement H. What is the isomorphism type of H?

(6 marks)

(Total: 20 marks)

Group Theory 2021-22 Solutions

So.1. (a)

(i) for a nilpotent group the upper central series reaches the whole group and therefore the first term in this series, which is $Z(G)$ must be non-identity;

(ii) the symmetric group S_3 of degree 3 is solvable with composition series $S_3 \geq C_3 \geq 1$ but its centre is trivial;

(iii) a p-group P has a non-trivial centre: every conjugacy class in P either has length 1 or the length is divisible by p by the Orbit-Stabilizer Theorem; the sum of the conjugacy classes sizes is the order of P which is a p-power; the class of the identity element has size 1, so in order to maintain the divisibility by p there must be another class of size 1; the element z in this class is non-identity and its centralizer is the whole of P, meaning that $z \in Z(P)$.
(7 marks seen)

(b) The subgroup N is normal: if $n \in N$ and $g \in G$ then $g^{-1}ng = n$, since $n \in N \leq Z(G)$, so that $g^{-1}Ng = N$ for all $g \in G$.

The subgroup M is normal: since $G/[G,G]$ is abelian $g^{-1}(M/[G,G])g = M/[G,G]$ and by the Fourth Isomorphism Theorem $g^{-1}Mg = M$, demonstrating that M is normal in G.

The example when $G = C_2 \times C_2$, $N = M = C_2$ and α is an automorphism of G of order 3 shows that in general N and M are not characteristic.
(5 marks unseen)

(c) We identify H and K with the normal subgroup $\{(h, e_K) \mid h \in H\}$ and $\{(e_H, k) \mid k \in K\}$ of G. The set of automorphisms of G which normalizes H and K clearly form a subgroup in $\mathrm{Aut}\,(G)$ isomorphic to $\mathrm{Aut}\,(H) \times \mathrm{Aut}\,(K)$. There are two subjective homomorphisms

$$\varphi_H : g = (h,k) \mapsto h$$

$$\varphi_K : g = (h,k) \mapsto k$$

of G into H and K, respectively (projection on the first and second coordinates). Let $\alpha \in \mathrm{Aut}\,(G)$ and $L = \alpha(H)$ be the image of H under α. Then every $l \in L$ as an element of the direct product can be presented as $l = (l_H, l_K)$ where $l_H \in H$ and $l_K \in K$. Consider the images of L under φ_H and φ_K. Since L is normal in G (being the image under an automorphism of a normal subgroup) the images are normal subgroups in H and K, respectively. Since H and K are simple groups, each of the images is either the identity subgroup of the whole subgroup. Therefore, if $H \ncong K$ we have (i). Suppose that H and K are isomorphic non-abelian then it is possible that $\varphi_K(\alpha(H)) = K$. We claim that in this case $\varphi_H(\alpha(H)) = Id_H$. Assume the contrary, since $C_G(H) = K \cong H$, and by our assumptions $\alpha(H)$ has non-trivial projections both on H and on K. Then

$$C_G(\alpha(H)) = \{g = (h,k) \in G \mid g(l_H, l_K) = (l_H, l_K)g\},$$

and since H and K are non-abelian simple, $Z(H) = Z(K) = 1$ showing that in the considered situation $C_G(\alpha(H))$ is trivial (unlike $C_G(H)$), which is impossible. Therefore, $\alpha(H) = H$ or $\alpha(H) = K$ and (ii) holds. Finally, if $H \cong K \cong C_p$ we have $\mathrm{Aut}\,(G) \cong GL_2(p)$ and we are in the situation (iii).
(8 marks unseen)

So.2. (a) Let $N_f = N_{S_5}(< f >)$, $C_f = C_{S_5}(< f >)$. Then, since $< f >$ is abelian acting regular on 5 points, it is self-centralized in S_5. Indeed, the centralizer of a left regular representation equal to the right regular representation and for an abelian group these two representations coincide. Further, $N_f/C_f \leq \mathrm{Aut}\,(C_5)$, where C_5 is the group generated by f. The automorphism group of C_5 has order 4, since there are four generators of C_5 which are f, f^2, f^3, f^4. The element $g = (1)(2\,3\,5\,4)$ conjugates f onto $f^2 = (\,1\,3\,5\,2\,4\,)$ and since the order of g is 4 it generates the whole automorphism group of $< f >$. Thus N_f is the semidirect product of C_5 and C_4 with respect to the surjective homomorphism of C_4 onto $\mathrm{Aut}\,(C_5)$.

Let $N_a = N_{S_5}(< a, b >)$ and $C_a = C_{S_5}(< a, b >)$. Then both N_a and C_a stabilize the point "5" (stabilized by a and b) and hence they are contained in S_4. Every element of N_a preserves the imprimitivity system formed by the orbits of a and b. Furthermore, C_a stabilises each of the orbit and an element of C_a either stabilise or permute the points in the orbits, demonstrating that $C_a =< a, b >\cong C_2 \times C_2$. $N_a/C_a \leq \mathrm{Aut}\,(C_2 \times C_2) \cong S_3$, but it can not be the whole S_3, since $ab = (\,1\,2\,)(\,3\,4\,)$ has cyclic type different from that of a and b, while a and b are conjugated by $c = (\,1\,3\,)(\,2,\,4\,)$. Thus $N_a =< a, b, c >$ since the element ac has order 4, N_a is isomorphic to the dihedral group of order 8.
(6 marks)

(b) Suppose that H is a subgroup of S_n of index $2 < k <= n - 1$. Consider the action of S_n on the cosets of H. The image is a transitive permutation group of degree k and hence the order of the image is greater than 2 and strictly less than $n! = |S_n|$. Therefore, the action has a non-trivial kernel of index more than 2, and the kernel of the action can not be the alternating group. Although we know that A_n is the only proper normal subgroup in S_n when $n \geq 5$. The contradiction proves the result.
(4 marks unseen)

(c) All Sylow subgroup are conjugate by a Sylow theorem, therefore the action of S_5 on Ω is transitive. By (a) above for an element $f \in S_5$ of order 5 we have $N_f = N_{S_5}(< f >) \cong C_5 \ltimes C_4$. In particular $|\Omega| = 5!/20 = 6$. By a Sylow theorem S_5 acts transitively on Ω by conjugation. The stabiliser in S_5 of the subgroup $< f >\in \Omega$ is N_f. We are going to show that the action of N_f on $\Omega \setminus \{< f >\}$ is doubly transitive. The action is faithful because N_f contains no non-trivial normal subgroup of S_5. The subgroup $< f >$ must act transitively, since an element of order 5 needs at least 5 elements to act faithfully. The subgroup $< g >$ is cyclic of order 4 and it can only act on five points stabilising one and permuting transitively the remaining four. This proves the claimed double transitivity of N_f and by a result from the course the 3-transitivity of S_5. Notice that the action of N_f on $\Omega \setminus \{< f >\}$ is similar to its action on the original 5-element set on which S_5 acts naturally, but these two sets have different nature with respect to the whole of S_5.
(5 marks similar seen)

(d) Let τ be a transposition of S_5 and let $\omega(\tau)$ be the permutation induced by τ on Ω. Then $\omega(\tau)$ is an odd element of order 2, and therefore $\omega(\tau)$ is a transposition or a product of three disjoint transpositions. The former holds if and only if τ normalises a subgroup from Ω. Assume that this is the case and that the subgroup is $< f >$. Then τ normalizes $< f >$ and this leads to a contradiction with (a). If c is a 3-cycle then $\omega(c)$ is an element of order

3, which is a 3-cycle iff $< f >$ is normalized by c, which also contradicts (a). **(5 marks coursework)**

So.3. (a) Let p^a be the highest power of p which divides $|G|$. Since $|G| = [G : H] \cdot |H|$ and $[G : H]$ is coprime to p, p^a is also the highest power of p which divides $|H|$. Therefore a Sylow p-subgroup P of H, which exists by Sylow's theorem, is also a Sylow p-subgroup of G. Any other Sylow p-subgroup Q of G is a conjugate of P: $Q = g^{-1}Pg$ for some $g \in G$. Since $P \leq H$ and H is normal in G, $Q \leq H$ proving the assertion.
(3 marks unseen)

(b) Let P be a Sylow 7-subgroup of order 49 in G (which exists by the Sylow theorem). Then $N_G(P)$ has index equal to 1 mod 7 dividing $|G/49| = 6$. This shows that P is normal in G. The factor group G/P has order 6 and therefore either it is the cyclic group of order 6, or the dihedral group of order 6. In both cases G/P contains a normal subgroup Q of order three and its preimage in G is a subgroup of index 2. The group G/P also contains a subgroup of order 2 and index 3 (by Cauchy theorem and by the structure of G/P). If $G/P \cong C_6$ the subgroup of order 2 is unique and if $G/P \cong D_6$ there are three such subgroups. These two possibilities are realizer in the groups $C_7 \times C_7 \times C_6$ and $C_7 \times C_7 \times D_6$, respectively.
(4 marks unseen)

(c) Since P is normal in G, we have $G = N_G(P)$. Therefore, the quotient $G/C_G(P)$ is a subgroup of $\mathrm{Aut}\,(C_p) \cong C_{p-1}$ and since p is the smallest prime divisor of the order of G, $p-1$ is coprime to the order of G and therefore $G = C_G(P)$ meaning that P is in the centre of G.
(4 marks similar seen)

(d) Assume that G is nilpotent we proceed by induction on the order of G. There are two options: if $Z(G)$ is not contained in H, then $HZ(G)$ normalises H and contains it properly. If $Z(G)$ is contained in H, then $H/Z(G)$ is contained in $G/Z(G)$, which is a nilpotent group by the induction hypothesis. Therefore, by induction $G/Z(G)$ contains a subgroup properly containing and normalising $H/Z(G)$. Let $p_1, p_2, \ldots, p_s$ be the distinct primes dividing $|G|$ and let $P_i \in \mathrm{Syl}_{p_i}(G)$, for $1 \leq i \leq s$. Let $P = P_i$ for some i and let $N = N_G(P)$. Since $P \trianglelefteq N$ and two distinct Sylow p-subgroups never normalise one another by the proof of Sylow's theorem, P is the only Sylow p-subgroup in N, and hence P is characteristic in N. Since P is characteristic in N and N is a normal subgroup of $N_G(N)$, we get that $P \triangleleft N_G(N)$ by the paragraph before the theorem. This means that $N_G(N) \leq N$ and hence $N_G(N) = G$ and P is normal in G. Let $P_1, P_2, \ldots, P_s$ be the complete set of Sylow subgroups in G. Then $P_i \trianglelefteq G$. **(5 marks seen)**

We apply induction on t to prove that

$$\Pi_{t-1} = P_1 P_2 \cdots P_{t-1} \cong P_1 \times P_2 \times \cdots \times P_t$$

The assertion is clear for $t = 1$. Now, $\Pi_{t-1}P_t = \Pi_t$ and $\Pi_{t-1} \cap P_t = \{\, 1 \,\}$ by the order reason and therefore $\Pi_t = P_1 \times \cdots \times P_t$ by induction and the characterization of direct products. Therefore, the whole of G is the direct product $P_1 \times P_2 \times \cdots \times P_s$, as desired.
(4 marks unseen)

So.4. (a) An element of H has three off-diagonal entries which can contain any element of F, which gives $|H| = 3^3$. An element of G in addition can have ± 1 on the diagonal entries, which gives $|G| = 2^3 3^3$. If we assign with an element of G an element of $C_2 \times C_2 \times C_2 = \{(x, y, z) \mid x, y, z \in \{1, -1\}\}$ equal to (a_{11}, a_{22}, a_{33}) we obtain a surjective homomorphism with kernel H. Since H is a 3-group, it is nilpotent. Since G/H is a 2-group, G is solvable. If the Sylow 2-subgroup of diagonal matrices in G would be normal, the whole G would be the direct product of its Sylow subgroups. Then every diagonal matrix would commute with every upper triangular matrix, which is not the case by a direct check. Hence G is not a nilpotent group.
(6 marks part seen)

(b) (i) In this case the group has exactly three simple normal subgroup which are C_p, C_q and C_r, so there are three choices for the penultimate term in a composition series. The corresponding factor group have two normal simple subgroups which gives six composition series. **(3 marks similar seen)**

(ii) In this case the group is a 3-dimensional $GF(p)$-space. The number of simple subgroups is equal to the number of 1-dimensional subspaces which is $(p^3 - 1)/(p - 1) = p^2 + p + 1$ and the number of simple subgroups in the factor group is the number of 1-dimensional subspaces in a 2-dimensional $GF(p)$-space, which is $p + 1$. This gives the total number of composition series equal to

$$(p + p + 1)(p + 1).$$

(3 marks unseen)

(c) Let P be a p-group and suppose that $Z(P)$ has index at most p in P. We claim that in this case P is abelian. This will imply that every group of order p^2 is abelian since a centre of a p-group is non-trivial. If the index of $Z(P)$ in P is 1, P is clearly abelian. We assume that $Z(P)$ has index p and reach a contradiction. Let $g \in P \setminus Z(P)$. Then, since the factor group $P/Z(P)$ is cyclic of order p, that factor group is generated by $gZ(P)$. Therefore, the powers of g from 0 to $p - 1$ are contained in different cosets of $Z(P)$ in P and since there are only p such cosets, every coset contains a power of g. This means that every element of P is of the form zg^i for $z \in Z(P)$ and $0 \leq i \leq p - 1$. Since

$$[z_1 g^i, z_2 g^j] = [g^i, g^j] = 1,$$

the group P is abelian and $Z(P) = P$.

Let G be a non-abelian group of order p^3. By the previous part $Z(G)$ has order p and $G/Z(G)$ (of order p^2) is abelian, in particular $[G, G]$ is contained in $Z(G)$ and again since G is non-abelian the commutator subgroup is equal to the centre. Now let us turn to the conjugacy classes. Every element from $Z(G)$ forms a conjugacy class of its own which gives p classes. If $g \in G \setminus Z(G)$ then the class of g has more than one element. On the other hand, $C_G(g)$ contains both $Z(G)$ and $< g >$. Since g is not in the centre, $Z(G) < g > /Z(G)$ has order at least p and hence the order of $C_G(g)$ is p^2 and the conjugacy class of g has p elements. Therefore, there are $(p^3 - p)/p = p^2 - 1$ non-central classes giving $p^2 + p - 1$ classes altogether.
(8 marks unseen)

So.5. (a) Let X be the set of (right) cosets of H in G, so that $|X| = n$. For $x \in X$ let $G(x)$ denote the stabilizer of x in G. Then

$$H^{\mathrm{cl}} = \{e\} \cup \bigcup_{x \in G} (G(x) \setminus \{e\}.$$

Since G_x is a conjugate of H, we have $|H^{\mathrm{cl}}| \leq 1 + n(|H| - 1) = 1 + |G| - n$, where the equality holds if and only if the sets $G(x)$ are disjoint.
(4 marks)

(b) For $n \geq 5$ the group A_n is simple and it is the only proper normal subgroup of S_n. This easily shows that neither of this groups are Frobenius. Although A_4 is a Frobenius group since it is a semi-direct product of $C_2 \times C_2$ by C_3 acting fixed-point freely. Similarly, S_3 is a Frobenius group, while S_4 is not, since the actions of complements on the normal subgroups are not fixed-point free.
(2 marks)

(c) We have two Frobenius groups order 18: the semi-direct products of $N \cong C_3 \times C_3$ or C_9 with the group of order 2 inverting every element of N. On the other hand, the kernel of any Frobenius group of order 18 is easily seen to be of order 9 and there only two such groups. The complement must act fixed-point freely and the only such action is the invertion.
(4 marks)

(d) Let $N \cong C_2 \times C_2 \times C_2$ viewed as a 3-dimensional vector space and H be a subgroup of order 7 in $GL_3(2)$ (which exists by the order reason and Sylow theorem). Then H acts on N fixed-points freely (because 7 is a prime number and $|N \setminus \{e\}| = |H|$. Therefore the semi-direct product of N and H with respect to the natural action is a Frobenius group. On the other hand considering the orders of automorphism groups we conclude that a Frobenius group of order 56 must have kernel of order 8 and this kernel must be elementary abelian.
(4 marks)

(e) The generators of H are both of order 4 and their squares are equal to the scalar (-1)-scalar matrix. Checking the commutators we deduce that $H \cong Q_8$. Clearly the (-1)-scalar matrix acts fixed-point freely on F and since the centre of Q_8 is contained in every proper subgroup, we conclude that the action of H on N is fixed-point free and hence the group is a Frobenius group. The group G is the semi-direct product of N and H with respect the action determined by the generating matrices.
(6 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a seprate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH97033_MATH97141	1	This question was done well by most, difficulties with Q1 (iv), it was unseen.
MATH97033_MATH97141	2	Done well, Q2 (b) was understood by most as request of the definition of the kernel, I gave full mark for such answer. Q2 (f) was unseen but done well by most, I am pleased.
MATH97033_MATH97141	3	Quite a few students included the full proof of the simplicity for An in Q3 (f). I gave full mark for this but the expected solution was much easier.
MATH97033_MATH97141	4	Overall good response, some details were missing by some leading to partial credit.
MATH97033_MATH97141	5	Out of a few candidates only one attemted (c) and noone (d). You could do better.