Imperial College London

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# Analysis 1A

## Lecture 8 - More on convergence of sequences

Ajay Chandra

## How to prove $a_n \to a$

$$\boxed{\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}_{>0} \text{ such that } |a_n - a| < \epsilon \ \forall n \geq N_\epsilon}$$

(I) Fix $\epsilon > 0$.

(II) Calculate $|a_n - a|$.

(II′) Find a good estimate $|a_n - a| \leq b_n$.

(III) Try to solve $b_n < \epsilon$. $\quad (*)$

(IV) Find $N_\epsilon \in \mathbb{N}_{>0}$ such that $(*)$ holds whenever $n \geq N_\epsilon$.

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see the examples next lecture!).

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Notice you only have to do this for **one** $\epsilon > 0$, so long as it is **arbitrary**; that way you've done it for **any** $\epsilon > 0$.

## Example 3.7

Prove that $a_n = \dfrac{n+5}{n+1} \to 1.$

**Rough work:**

$$\left|a_n - 1\right| = \left|\frac{n+5}{n+1} - 1\right| = \left|\frac{n+5}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{4}{n+1}\right| = \frac{4}{n+1} < \frac{4}{n}$$

$\frac{4}{n} < \varepsilon \iff n > \frac{4}{\varepsilon}$ so take $N_\varepsilon > 4/\varepsilon$, $N_\varepsilon \in \mathbb{N}_{>0}$

**Proof**

Let $\varepsilon > 0$, fix $N_\varepsilon \in \mathbb{N}_{>0}$ with $N_\varepsilon > 4/\varepsilon$.

Then $\forall n \geq N_\varepsilon$

$$|a_n - 1| = \frac{4}{n+1} \underset{\substack{\uparrow \\ n \geq N_\varepsilon}}{\leq} \frac{4}{N_\varepsilon + 1} \leq \frac{4}{N_\varepsilon} < \varepsilon. \quad \blacksquare$$

## Example 3.8

Define $(a_n)$ by setting $a_1 = a_2 = 0$ and $a_n = \dfrac{n+2}{n-2}$ for $n \geq 3$. Prove $a_n \to 1$.

**Roughwork**

$$\underbrace{|a_n - 1|}_{\text{"}n\geq 3\text{"}} = \left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2} \underset{n\geq 4}{\leq} \frac{4}{n/2} = \frac{8}{n} < \varepsilon$$

$\frac{4}{n} < \varepsilon$ will not imply $\frac{4}{n-2} < \varepsilon$

$\frac{8}{n} < \varepsilon \iff n > \frac{8}{\varepsilon}$

$N_\varepsilon > \max(4, 8/\varepsilon)$

$n - 2 \geq n - n/2 = n/2$

Doesn't work for $n=3$ but does for $n \geq 4$

**Proof** Let $\varepsilon > 0$, choose $N_\varepsilon \in \mathbb{N}_{>0}$ with $N_\varepsilon > \max(4, 8/\varepsilon)$.

For $n \geq N_\varepsilon$

$$|a_n - 1| \underset{n\geq 3}{=} \frac{4}{n-2} \underset{n\geq 4}{\leq} \frac{8}{n} \leq \frac{8}{N_\varepsilon} < \varepsilon \qquad N_\varepsilon > 8/\varepsilon$$

∎

We now say what it means for a sequence to **converge**

### Definition

We say $a_n$ *converges* if and only if $\exists\ a \in \mathbb{R}$ such that $a_n \to a$, i.e.

$\exists\ a$ such that $\forall \epsilon > 0\ \exists\ N \in \mathbb{N}_{>0}$ such that $n \geq N \implies |a_n - a| < \epsilon$.

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### Definition

We say $a_n$ *diverges* if and only if it does not converge (to any $a \in \mathbb{R}$), i.e.

$\underline{\forall a}\ \underline{\exists\ \epsilon >} 0$ such that $\underline{\forall N} \in \mathbb{N}_{>0}$, $\underline{\exists n \geq}\ N$ such that $\underline{|a_n - a| \geq \epsilon}$.

We now say what it means for a sequence to **converge**

### Definition

We say $a_n$ *converges* if and only if $\exists\ a \in \mathbb{R}$ such that $a_n \to a$, i.e.

$$\exists\ a \text{ such that } \forall \epsilon > 0\ \exists\ N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n - a| < \epsilon.$$

By negating the above, we have a notion for what it means for a sequence to **diverge**.

### Definition

We say $a_n$ *diverges* if and only if it does not converge (to any $a \in \mathbb{R}$), i.e.

$$\forall a\ \exists\ \epsilon > 0 \text{ such that } \forall N \in \mathbb{N}_{>0},\ \exists n \geq N \text{ such that } |a_n - a| \geq \epsilon.$$

### Remark 3.9

Notice *diverge* does not mean $\to \pm\infty$, for instance a later exercise will convince you that $a_n = (-1)^n$ diverges.

## Example 3.10

Fix a sequence of real numbers $(a_n)_{n \geq 1}$. Consider

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This means?

$\varepsilon = |a_n| + 1$

$\uparrow$

$\varepsilon_n$

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded
3. Precisely nothing $\leftarrow$
4. More than one of these
5. None of these

## Example 3.11

What about

$$\exists\, \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon\,?$$

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded $\longleftarrow$ $\{a_n : n \in \mathbb{N}_{>0}\}$ is a bounded set
3. Precisely nothing
4. More than one of these
5. None of these

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What about

$$\exists\, \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon\,?$$

1. $a_n \to 0$
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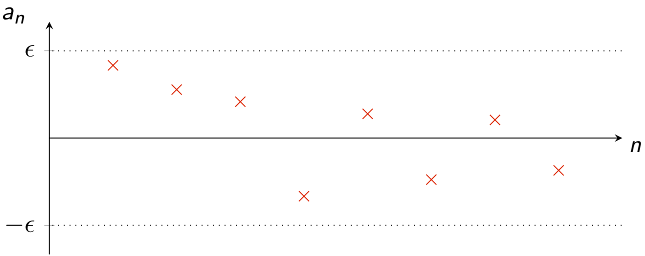

We can also define limits for *complex sequences*.
Let $|z| := \sqrt{(\Re z)^2 + (\Im z)^2}$.

$z = Re(z) + i\,(Im(z))$

$Re(z),\ Im(z) \in \mathbb{R}$

## Definition

$a_n \in \mathbb{C},\ \forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ if and only if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n - a| < \epsilon \,.$$

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**Definition**

$a_n \in \mathbb{C},\ \forall n \geq 1.$ We say $a_n \to a \in \mathbb{C}$ if and only if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}_{>0} \text{ such that } n \geq N \implies |a_n - a| < \epsilon\ .$$

On Problem Sheet 4 you'll prove that this definition is equivalent to

$$(\Re a_n) \to \Re a \text{ and } (\Im a_n) \to \Im a\ .$$

## Example 3.12

Prove $a_n = \dfrac{e^{in}}{n^3 - n^2 - 6} \to 0$ as $n \to \infty$.

$|e^{in}| = 1$

**Rough work**

$$|a_n - 0| = \left|\frac{e^{in}}{n^3-n^2-6}\right| = \left|\frac{1}{n^3-n^2-6}\right| \underset{n\geq 4}{\leq} \frac{1}{n^3 - n^3/2} = \frac{2}{n^3} \leq \frac{2}{n} < \varepsilon$$

$n^2 + 6 \leq \frac{n^3}{2} \Leftarrow n \geq 4$

$n > \frac{2}{\varepsilon}$

$N_\varepsilon > \max(4, 2/\varepsilon)$

__Proof__

Let $\varepsilon > 0$, pick $N_\varepsilon \in \mathbb{N}$ with $N_\varepsilon > \max(4, 2/\varepsilon)$

Then $\forall n \geq N_\varepsilon$,

$$|a_n - 0| = \left|\frac{1}{n^3-n^2-6}\right| \underset{n\geq 4}{\leq} \frac{2}{n^3} \underset{n\geq 1}{\leq} \frac{2}{n} \overset{n \geq N_\varepsilon}{\leq} \frac{2}{N_\varepsilon} \underset{N_\varepsilon > \frac{2}{\varepsilon}}{<} \varepsilon$$

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