IMPERIAL

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Bifurcation Theory

Date: Tuesday, May 20, 2025

Time: Start time 14:00 – End time 16:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Let the smooth system $\dot{x} = f(x, \varepsilon)$, $x \in \mathbb{R}^n$, have an equilibrium $x = 0$ at $\varepsilon = 0$.

    (i) What is the condition for the equilibrium at $\varepsilon = 0$ to be exponentially stable? (4 marks)

    (ii) What is the condition for the equilibrium to be persistent for all small ε? Justify your answer. (4 marks)

  (b) Let the smooth system $\bar{x} = f(x)$, $x \in \mathbb{R}^2$, have a fixed point $x = 0$.

    (i) What is the condition for the fixed point to undergo a Neimark-Sacker bifurcation? (4 marks)

    (ii) What are the conditions for strong resonances in the Neimark-Sacker bifurcation? Write down the complex normal form (up to terms of order 3) of the Neimark-Sacker bifurcation when there are no strong resonances. (6 marks)

    (iii) What happens to the normal form if there are strong resonances? (2 marks)

(Total: 20 marks)

2. Consider the system
$$\begin{cases} \frac{dx}{dt} = y + \varepsilon, \\ \frac{dy}{dt} = -x + 2zy, \\ \frac{dz}{dt} = -2z - 8\varepsilon y - 4y^2, \end{cases}$$

where $(x, y, z) \in \mathbb{R}^3$ and ε is a parameter.

  (a) Show that the origin is an equilibrium of the system at $\varepsilon = 0$ that may undergo an Andronov-Hopf bifurcation. Find the coordinate transformation $(x, y) \mapsto (w, w^*)$ that brings the system to the form
$$\begin{cases} \frac{dw}{dt} = iw + z(w - w^*), \\ \frac{dz}{dt} = -2z + (w - w^*)^2. \end{cases}$$

(4 marks)

  (b) (i) Find the center manifold up to quadratic terms for the unperturbed system. (Hint: use the normal form transformation.) (5 marks)

    (ii) Write down the equation for the restricted system on the center manifold up to cubic terms, with only the resonant ones left. (3 marks)

  (c) Find the first Lyapunov coefficient and find the number and stability of periodic orbits for all small ε. (Hint: the control parameter is the real part of the two complex conjugate eigenvalues of the equilibrium.) (8 marks)

(Total: 20 marks)

3. Study the bifurcation of the equilibrium $(0,0,0)$ of the following system:

$$\begin{cases} \frac{dx}{dt} = \varepsilon^2 x^2 + \varepsilon xy + xz + \frac{9}{2}x^3, \\ \frac{dy}{dt} = -\frac{1}{2}y + \varepsilon x^2 + y^2 z^2, \\ \frac{dz}{dt} = z + 2x^2 + \varepsilon x^4, \end{cases}$$

where $(x, y, z) \in \mathbb{R}^3$ and ε is a small parameter.

(a) Find the expression of the defining function $(y, z) = \phi(x, \varepsilon)$ of the center manifold up to order 5. (Hint: treat ε as a new variable and use the normal form transformation.)
(5 marks)

(b) Suppose that the system restricted to the center manifold assumes the form

$$\frac{dx}{dt} = f(x, \varepsilon) = \varepsilon^2 x^2 + (2\varepsilon^2 + \frac{5}{2})x^3 + o(x^3).$$

(i) Write down the normal form for the bifurcation of the equilibrium $x = 0$ in terms of control parameters, and express the control parameters as functions of ε. (6 marks)

(ii) Sketch the bifurcation diagram in the plane of control parameters (you do not need to find the exact equation of the bifurcation curve). State the number of equilibria and their stability on the center manifold in each region of the diagram. (5 marks)

(iii) Describe the dynamics near $x = 0$ on the center manifold when ε varies near 0. (4 marks)

(Total: 20 marks)

4. Consider the following one-dimensional map:

$$f(x) = a - x^3,$$

where a is a parameter.

(a) Show that the map has only one fixed point and has no periodic points of period larger than 2. (4 marks)

(b) Prove the following statements:

(i) The fixed point of the map is stable for $a \in (-\frac{4}{3\sqrt{3}}, \frac{4}{3\sqrt{3}})$. (2 marks)

(ii) The fixed point is also stable at $a = \pm\frac{4}{3\sqrt{3}}$. (4 marks)

(iii) A stable period-2 orbit is born when $|a| > \frac{4}{3\sqrt{3}}$. (2 marks)

(c) (i) What bifurcation can happen to the periodic orbit found in (b)(iii)? Find the orbit and the corresponding parameter value at the critical moment of the bifurcation. (Hint: you may use $\frac{3\sqrt{2}+\sqrt{6}}{6} + (-\frac{3\sqrt{2}+\sqrt{6}}{6})^3 = \frac{4\sqrt{6}}{9}$) (6 marks)

(ii) One notes that $(x_1, x_2) = (1, -1)$ is a periodic orbit of the map at $a = 0$. What may happen to this orbit and the one found in (b)(iii) when $|a|$ crosses $\frac{4\sqrt{6}}{9}$? (2 marks)

(Total: 20 marks)

5. (Mastery question) Consider the map

$$\begin{cases} \bar{x} = y + x^3, \\ \bar{y} = -ax + by, \end{cases}$$

where $(x, y) \in \mathbb{R}^2$ and a and b are two parameters .

(a) (i) Find the curve of parameter values for which the fixed point at $(0,0)$ undergoes a Neimark-Sacker bifurcation. (2 marks)

(ii) Consider the parameter values found in (i). Show that the multipliers of the fixed point are given by $\lambda = e^{i\omega}$ and $\lambda^* = e^{-i\omega}$, where $\tan \omega = \frac{\sqrt{4-b^2}}{b}$. (You may need Euler's formula: $e^{i\omega} = \cos\omega + i \sin\omega$.) (2 marks)

(iii) Which of the parameter values found in (i) correspond to strong resonances? (2 marks)

(b) (i) For $a = 1$ and $b \in (-1, 0) \cup (0, 2)$, show that there are coordinates (z, z^*) such that the system assumes the form

$$\bar{z} = ze^{i\omega} + \left(\frac{z + z^*}{2}\right)^3,$$

where $\omega \in (0, \pi)$ is given by (a)(i). (3 marks)

(ii) Find from the above formula the first Lyapunov coefficient of the Neimark-Sacker bifurcation as a function of parameters. (4 marks)

(c) Consider any point (a, b) on the bifurcation curve found in (a) with $b \in (-1, 0) \cup (0, 2)$. How to change a to obtain invariant curves from the Neimark-Sacker bifurcation? How many invariant curves can be born? What are their stabilities? (Hint: the control parameter is the deviation from 1 of the modulus of the multipliers). (7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

This paper is also taken for the relevant examination for the Associateship.

MATH60009/70009

Bifurcation Theory (Solutions)

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MATH60009/70009

1. (a) (i) The equilibrium is stable when all the eigenvalues of its linearisation matrix $A := \frac{\partial f}{\partial x}\big|_{x=0,\varepsilon=0}$ have negative real parts.

(ii) The equilibrium of the system corresponds to the solution of the equation $f(x,\varepsilon) = 0$. By the implicit function theorem, it always has a solution when A is invertible. Thus, the condition is that 0 is not an eigenvalue of the matrix A.

(b) (i) The condition is that $A := \frac{\partial f}{\partial x}\big|_{x=0}$ has a pair of complex conjugate multipliers $e^{i\omega}$ for some $\omega \in (0,\pi)$.

(ii) The strong resonances correspond to the cases $\omega = 2\pi/3, \pi/2$.
The normal form is given by

$$\bar{z} = e^{i\omega}(z + (L_1 + i\Omega_1)z|z|^2 + O(|z|^4)).$$

(iii) Strong resonances will produce terms with order equal or lower than 3, making the term $z|z|^2$ no longer dominant.

2. (a) At $\varepsilon = 0$ the system is

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x + 2zy, \\ \frac{dz}{dt} = -2z - 4y^2. \end{cases}$$

Its linearization matrix at zero equilibrium is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and the eigenvalues are $(-i, i, -2)$, which means that the system is at the moment of the Andronov-Hopf bifurcation.

2, B

Let $w = x - iy$. The system assumes the form

$$\begin{cases} \frac{dw}{dt} = iw + z(w - w^*), \\ \frac{dz}{dt} = -2z + (w - w^*)^2. \end{cases}$$

2, B

(b) (i) Next, we kill the powers of central variables in the equations corresponding to non-central variables. In our case, we kill powers of (w, w^*) in the equation of the stable variable z. The corresponding normal form transformation is

$$u = z - \frac{w^2}{2(1 + i)} + ww^* - \frac{(w^*)^2}{2(1 - i)}.$$

4, B

This transformation kills terms up to order 2, so the center manifold in the new coordinates takes the form $u = O(|w|^3)$. Going back to the (w, z)-coordinates, we have that the center manifold is given by

$$z = \frac{w^2}{2(1 + i)} - ww^* + \frac{(w^*)^2}{2(1 - i)} + O(|w|^3).$$

1, B

(ii) Therefore, the reduced system on W^c is given by

$$\frac{dw}{dt} = iw + \left(\frac{w^2}{2(1 + i)} - ww^* + \frac{(w^*)^2}{2(1 - i)} \right)(w - w^*) + O(|w|^4).$$

Dropping the resonant cubic terms and higher order terms, we find the normal form up to order three as

$$\frac{dw}{dt} = iw - \left(1 + \frac{1}{2(1 + i)} \right) w^2 w^*.$$

3, A

(c) The first Lyapunov coefficient is

$$L = -\operatorname{Re}(1 + \frac{1}{2(1 + i)}) = -\frac{5}{4}.$$

2, B

Since $L < 0$, the equilibrium is exponentially stable on W^c and at most one periodic orbit can be born in U for non-zero ε, which is stable on W^c and hence is stable for the full system. Denote the real part of the central eigenvalues by $\mu(\varepsilon)$. The periodic orbit appears for all small ε such that $\mu(\varepsilon) \cdot L < 0$.

3, D

The coordinates of the equilibrium are $(x = -4\varepsilon^3, y = -\varepsilon, z = 2\varepsilon^2)$ with the linearization matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 4\varepsilon^2 & -2\varepsilon \\ 0 & 0 & -2 \end{pmatrix}.$$

The characteristic polynomial is $P(\lambda) = (\lambda + 2)(\lambda^2 - 4\varepsilon^2\lambda + 1)$, which implies that the complex eigenvalues have positive real parts, i.e. $\mu > 0$, for all small $\varepsilon \neq 0$. Therefore, we have one stable periodic born from the equilibrium for all small $\varepsilon \neq 0$.

3, D

3. (a) Adding $\frac{d\varepsilon}{dt} = 0$ to the system, we find that $(x, \varepsilon), y, z$ are central, stable and unstable variables, respectively. To find the desired formula for the center manifold, we kill powers of central variables in the equations of $\frac{dy}{dt}, \frac{dz}{dt}$ up to order 5, by the coordinate transformation

$$y^{new} = y - 2\varepsilon x^2, \qquad z^{new} = z + 2x^2 + \varepsilon x^4.$$

In the new coordinates, the center manifold is given by $(y^{new}, z^{new}) = O(|x, \varepsilon|^6)$, and hence in the original coordinates, the center manifold is given by

$$y = 2\varepsilon x^2 + O(|x, \varepsilon|^6), \qquad z = -2x^2 - \varepsilon x^4 + O(|x, \varepsilon|^6).$$

`2, B`

`3, B`

(b) (i) The system on the center manifold is given by

$$\frac{dx}{dt} = f(x, \varepsilon) = \varepsilon^2 x^2 + (2\varepsilon^2 + \frac{5}{2})x^3 + o(x^3).$$

The system at $\varepsilon = 0$ is $\frac{dx}{dt} = \frac{5}{2}x^3 + o(x^3)$. So, the first non-zero Lyapunov coefficient is $\ell_3 = \frac{5}{2}$. To obtain the normal form of the bifurcation:

$$\frac{du}{dt} = \mu_0(\varepsilon) + \mu_1(\varepsilon)u + \ell_3 u^3 + o(u^3),$$

`2, C`

we need to move origin to the the solution of

$$0 = \frac{\partial^2 f(x, \varepsilon)}{\partial x^2} = 2\varepsilon^2 + 6(2\varepsilon^2 + \frac{5}{2})x,$$

which is given by

$$x = x^*(\varepsilon) = -\frac{\varepsilon^2}{3(2\varepsilon^2 + \frac{5}{2})} + o(\varepsilon^2).$$

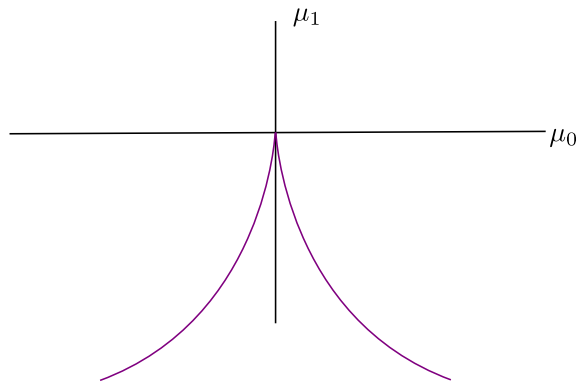The control parameters are

`2, D`

$$\mu_0 = f(x^*, \varepsilon) = \frac{\varepsilon^6}{54(2\varepsilon^2 + \frac{5}{2})^2} + o(\varepsilon^6), \qquad \mu_1 = \frac{\partial f(x^*, \varepsilon)}{\partial x} = -\frac{\varepsilon^4}{3(2\varepsilon^2 + \frac{5}{2})} + o(\varepsilon^4).$$

`2, D`

(ii) Since $\ell_3 = \frac{5}{2} > 0$, the bifurcation diagram can be sketched as below, where we have

· 3 equilibria when (μ_0, μ_1) lies inside the cusp, one stable and two unstable,

· 2 equilibria when (μ_0, μ_1) lies on the boundary of the cusp, one semi-stable and one unstable, and

· 1 unstable equilibrium when (μ_0, μ_1) lies outside the cusp.

`3, A`

$\boxed{\text{2, A}}$

(iii)   Since $\mu_0 > 0$ and $\mu_1 < 0$ for all small ε, only the dynamics in the 4th quadrant can be realized.

$\boxed{\text{4, B}}$

4. (a) There is only one fixed point since the map is decreasing. Its second iteration is monotonically increasing, and such maps cannot have periodic points other than fixed points. Therefore the original map can have only points of period at most 2.

(b) (i) The fixed point is found from the equation

$$a = x_a + x_a^3.$$

It is stable when $f'(x_a) = -3x_a^2 > -1$, or $|x_a| < \frac{1}{\sqrt{3}}$, which, combining with the above equation, yields $|a| < \frac{4}{3\sqrt{3}}$.

(ii) At $a = \pm\frac{4}{3\sqrt{3}}$, the multiplier of the fixed point $x = \pm\frac{1}{\sqrt{3}}$ is -1. Thus, the stability is determined by the sign of the Schwarzian derivative:

$$S = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right) = -\frac{6}{x^2} < 0.$$

Since the sign is negative, the fixed point is stable.

(iii) The fixed point loses stability when $|a| > \frac{4}{3\sqrt{3}}$, and hence a stable periodic orbit of period 2 is born.

(c) (i) Since the map is monotone, the derivative of its second iteration cannot be negative. Consequently, the period-2 orbit may bifurcate only when its multiplier equals to 1. This means that the bifurcating period-2 orbit (x_1, x_2) must satisfy the following system of equations:

$$a = x_1 + x_2^3, \quad a = x_2 + x_1^3, \quad 9x_1^2 x_2^2 = 1, \quad x_1 \neq x_2.$$

We obtain
$$x_1 - x_2 = x_1^3 - x_2^3, \quad x_1 x_2 = \pm\frac{1}{3}, \quad x_1 \neq x_2.$$

From the first equation one gets $1 = x_1^2 + x_2^2 + x_1 x_2$. Combining this with the second equation, yields $(x_1 - x_2)^2 = 1 \mp 1$. Since $x_1 \neq x_2$, we find

$$x_1 x_2 = -\frac{1}{3}, \quad x_1 - x_2 = \pm\sqrt{2}.$$

For certainty, assume $x_1 > x_2$, then $x_1 = \frac{3\sqrt{2}\pm\sqrt{6}}{6}$ and $x_2 = \frac{-3\sqrt{2}\pm\sqrt{6}}{6}$. The corresponding values of a are $\pm\frac{4\sqrt{6}}{9}$.

(ii) The periodic orbit is stable for $\frac{4}{3\sqrt{3}} < |a| < \frac{4\sqrt{6}}{9}$, and collide with the periodic orbit $(1, -1)$ at $|a| = \frac{4\sqrt{6}}{9}$ and disappear for $|a| > \frac{4\sqrt{6}}{9}$.

5. (a) (i) The Jocobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ -a & b \end{pmatrix}.$$

The fixed point 0 undergoes a Neimark-Sacker bifurcation if and only if

1. $\det J|_0 = 1 \Rightarrow a = 1$, and $\boxed{1, \text{M}}$

2. the characteristic equation $|J_0 - \lambda I| = \lambda^2 - b\lambda - a = 0$ has non-real solutions at $a = 1$, that is, $b^2 - 4 < 0 \Rightarrow |b| < 2$. $\boxed{1, \text{M}}$

So, the bifurcation curve is $\{a = 1, -2 < b < 2\}$.

(ii) The eigenvalues at $a = 1$ are

$$\lambda = \frac{b - \sqrt{b^2 - 4}}{2} = \frac{b}{2} - i\frac{\sqrt{4 - b^2}}{2} = e^{-i\omega},$$

and λ^* (the complex conjugate of λ) with

$$\tan \omega = \frac{\sqrt{4 - b^2}}{b} \tag{1}$$

$\boxed{2, \text{M}}$

(iii) The strong resonances correspond to $\omega = 2\pi/3$ and $\omega = \pi/2$, where ω is the argument of the multipliers of the fixed point. Thus, $\omega = 2\pi/3$ at $b = -1$ and $\omega = \pi/2$ at $b = 0$. $\boxed{2, \text{M}}$

(b) (i) The eigenvector corresponding to λ is $w = \begin{pmatrix} 1 \\ \frac{b}{2} \end{pmatrix} + i\begin{pmatrix} 0 \\ -\frac{\sqrt{4-b^2}}{2} \end{pmatrix}$. Denote

$Q = \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & -\frac{\sqrt{4-b^2}}{2} \end{pmatrix}$ with $Q^{-1} = \begin{pmatrix} 1 & \frac{b}{\sqrt{4-b^2}} \\ 0 & -\frac{2}{\sqrt{4-b^2}} \end{pmatrix}$. Applying the coordinate transformation $(u, v)^T = Q^{-1}(x, y)^T$, we obtain

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^3 \\ 0 \end{pmatrix}.$$

Further taking $z = u + iv$ and $z^* = u - iv$ leads to

$$\bar{z} = ze^{i\omega} + \left(\frac{z + z^*}{2}\right)^3.$$

$\boxed{3, \text{M}}$

(ii) Since there is no strong resonance for $b \notin \{-1, 0\}$, $z^2 z^*$ is the only non-resonant term. Thus, there exists normal form coordinates where the map assumes the form

$$\bar{z} = ze^{i\omega} + \frac{3}{8}z^2 z^* = e^{i\omega}(z + \frac{3}{8}(\cos \omega - i \sin \omega)z^2 z^*). \tag{2}$$

$\boxed{2, \text{M}}$

This along with (1) gives the first Lyapunov coefficient as

$$L_1 = \frac{3}{8}\cos\omega = \frac{3}{8}\cos\arctan\frac{\sqrt{4-b^2}}{b} \quad (\text{or } = \frac{3}{16}b).$$

<div style="text-align: right;">2, M</div>

(c) Recall that the control parameter μ of the Neimark-Sacker bifurcation is the deviation of the modulus of λ from 1, which in our case is given by $\mu = |\lambda| - 1 = \sqrt{\det J} - 1 = \sqrt{a} - 1$.

<div style="text-align: right;">2, M</div>

Hence, (2) with parameter a takes the form

$$\bar{z} = e^{i\omega}(z(1+\mu) + \frac{3}{8}(\cos\omega - i\sin\omega)z^2 z^*),$$

where ω also depends on a.

<div style="text-align: right;">1, M</div>

For $b \in (-1,0) \cup (0,2)$, we do not have strong resonances so one invariant curve can be born when $L_1 \cdot (\sqrt{a} - 1) < 0$.

For $b \in (-1,0)$, ω changes from $2\pi/3$ to $\pi/2$ by (1), which implies that $\cos\omega$ and hence L_1 is negative. Thus, a stable invariant curve is born when a increases slightly from 1.

<div style="text-align: right;">2, M</div>

For $b \in (0,2)$, ω changes from $\pi/2$ to 0 by (1), which implies that $\cos\omega$ and hence L_1 is positive. Thus, an unstable invariant curve is born when a decreases slightly from 1.

<div style="text-align: right;">2, M</div>

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 80 marks

Total Mastery marks: 20 of 20 marks

MATH60009 Bifurcation Theory Markers Comments

Question 1   Students failed the first two questions about the proofs of basic facts of dynamical system/bifurcation theory. It seems they used to taking them for granted and forgot the reasoning behind them.

Question 2   Most of the students having trouble using the normal transformation to find center manifold. This is an important method that one needs to master. Other parts were done pretty well.

Question 3   Although similar questions have appeared before, this is a complicated question requiring carefulness and clear understanding of the bifurcation, especially the part asking for the relation between true parameters and control parameters. Most of the students did well.

Question 4   It seems all the students knew how to solve the problem, but most of them failed on computations.