Question 1

Marks & seen/unseen

Parts

(i) Suppose T_1 and T_2 are unbiased & have minimum variances σ^2 and correlation coefficient ρ — Seen

then $T_3 = \frac{1}{2}T_1 + \frac{1}{2}T_2$ is unbiased and

$var(T_3) = \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 + \frac{1}{2}\rho\sigma^2 \leq \sigma^2$ since $\rho \leq 1$

- If $\rho < 1$, $var(T_3) < \sigma^2$ so T_1 & T_2 are not MV

so, if T_1 & T_2 are MVU, $\rho = 1$

- If $\rho = 1$, $T_1 = a + cT_2$ w.p. 1 (a, c consts, $c > 0$)

so $\mu = a + c\mu$ and $\sigma^2 = c^2\sigma^2$, so $c = 1$ & $a = 0$

i.e. $T_1 = T_2$ (w.p. 1) so MVUE is unique. — 3

(ii)(a) Statistic a is ancillary for θ if the conditional sampling distribution of a for given θ is the same for all θ. — 1

(b) If a minimal sufficient statistic t for θ takes the form $t = \{a, s\}$ where a is anc. for θ, then s is called quasi-suff (or conditional-suff). — 1

(c) The completeness of statistic s for θ guarantees that it is independent of a. — 1

(d) $R = X_{max} - X_{min}$. Let $Z_i = X_i - \theta$.

Then $R = Z_{max} - Z_{min}$

$$F_{R|\theta}(r) = P\{\max(Z_i+\theta) - \min(Z_i+\theta) \leq r\}$$
$$= P\{(Z_{max}+\theta) - (Z_{min}+\theta) \leq r\}$$
$$= P(Z_{max} - Z_{min} \leq r)$$ which does not depend on θ

so $r = x_{max} - x_{min}$ is anc. for θ. — 3

(e) $f(x|\theta) = \frac{1}{2}$ $(\theta - 1 < x < \theta + 1)$ i.e. $(-1 < x - \theta < 1)$

so θ is a location parameter

ctd on p2

Course: M3S1

Question: 1 ctd

Marks & seen/unseen

Parts (ii)(e) ctd

$$f(\underset{\sim}{x}|\theta) = \left(\tfrac{1}{2}\right)^n H(x_{min} > \theta - 1) H(x_{max} < \theta + 1)$$
$$= \left(\tfrac{1}{2}\right)^n H(x_{max} - 1 < \theta < x_{min} + 1)$$

so by Neyman Factorization $\{x_{min}, x_{max}\}$ is sufficient for θ, and is clearly minimal sufficient.

unseen

Linear transformation $r = x_{max} - x_{min}$, $s = x_{max} + x_{min}$ gives sufficient statistic $\{r, s\}$ where r is ancillary and s is quasi-sufficient.

$u = x_{max} - 1$, $v = x_{min} + 1$; $a = u - v + 2 = r$, $s = u + v$. **6**

(iii)

$$f(\underset{\sim}{x}|\theta) = e^{-n(\bar{x} - \theta)} H(x_{min} \geq \theta)$$
$$\pi(\theta) = 1 \quad (0 < \theta < 1)$$
$$\pi(\theta|\underset{\sim}{x}) = e^{-n(\bar{x} - \theta)} H(0 < \theta \leq t) \quad \text{where } t = \min(x_{min}, 1)$$

i.e. $\pi(\theta|\underset{\sim}{x}) \propto e^{n\theta} \quad (0 < \theta \leq t)$

so $\pi(\theta|\underset{\sim}{x}) = \dfrac{ne^{+n\theta}}{e^{nt} - 1} \quad (0 < \theta \leq t)$

$\pi(\theta|\underset{\sim}{x}) \uparrow$ as $\theta \uparrow$ in $(0 < \theta \leq t)$

so max when $\theta = t$ i.e. $\hat{\theta}_{BMLE} = \min(x_{min}, 1)$ **5**

Setter's initials: RC

Checker's initials: AY

M3S1/M4S1 EXAMINATION SOLUTIONS 2010–11

Course: M3S1

Question: 2

Marks & seen/unseen: Unseen

Parts

(i)
$$f = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(x-\theta)^2}$$
$$\ln f = \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln\theta - \frac{1}{2\theta}(x^2 - 2x\theta + \theta^2)$$
$$U(\theta) = \frac{\partial \ln f}{\partial\theta} = -\frac{1}{2\theta} + \frac{X^2}{2\theta^2} - \frac{1}{2} = \frac{1}{2\theta^2}\{X^2 - \theta(1+\theta)\}$$
$$U_\cdot(\theta) = \frac{1}{2\theta^2}\{\Sigma X_i^2 - n\theta(1+\theta)\} = \frac{n}{2\theta^2}\{\overline{X^2} - \theta(1+\theta)\}$$ 2
$$I(\theta) = E\left(-\frac{\partial^2 \ln f}{\partial\theta^2}\right) = -E\left(\frac{1}{2\theta^2} - \frac{X^2}{\theta^3}\right)$$
$$= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\{\underbrace{var(X)}_{\theta} + \underbrace{[E(X)]^2}_{\theta}\} = \frac{1}{2\theta^2}(1+2\theta)$$ 3
$$I_\cdot(\theta) = \frac{n}{2\theta^2}(1+2\theta)$$

(ii) MLE $\hat\theta$ for θ is given by $\hat\theta(1+\hat\theta) = \overline{X^2}$

i.e. $\hat\theta = +\sqrt{\overline{X^2} + \frac{1}{4}} - \frac{1}{2}$ $\quad(\hat\theta > 0)$ 3

By the asympt normality of MLEs $\hat\theta$ is $AN(\theta, CRLB(\hat\theta))$

i.e. $\hat\theta$ is $AN\left(\theta, \frac{1}{I_\cdot(\theta)}\right) = AN\left(\theta, \frac{2\theta^2}{n(1+2\theta)}\right)$ 2

(iii) The asympt relative efficiency of $\bar X$ to $\hat\theta$

is $\dfrac{1/var(\bar X)}{1/\text{asympt } var(\hat\theta)} = \dfrac{1/\frac{\theta}{n}}{1/\frac{2\theta^2}{n(1+2\theta)}} = \dfrac{2\theta}{1+2\theta}$ 3

(iv) For $\psi = \ln(\theta)$, MLE $\hat\psi = \ln(\hat\theta)$

$I_\cdot(\psi) = \frac{1}{\psi'(\theta)^2} I_\cdot(\theta)$ so $CRLB(\hat\psi) = \frac{1}{I_\cdot(\psi)} = \frac{\psi'(\theta)^2}{I_\cdot(\theta)} = \frac{1}{\theta^2}\cdot\frac{2\theta^2}{n(1+2\theta)}$ 3

so $\hat\psi$ is $AN\left(\psi, \frac{2}{n(1+2\theta)}\right)$

(v) For $\xi = \Phi(\theta)$ $\quad \xi'(\theta)^2 = \varphi(\theta)^2 = \left\{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2}\right\}^2 = \frac{1}{2\pi} e^{-\theta^2}$

MLE $\hat\xi = \Phi(\hat\theta)$ and $CRLB(\hat\xi) = \frac{\xi'(\theta)^2}{I_\cdot(\theta)} = \frac{1}{2\pi} e^{-\theta^2} \cdot \frac{2\theta^2}{n(1+2\theta)}$

so $\hat\xi$ is $AN\left(\xi, \frac{\theta^2 e^{-\theta^2}}{\pi n(1+2\theta)}\right)$ 4

Setter's initials: RC

Checker's initials: AY

	M3S1/M4S1 EXAMINATION SOLUTIONS 2010–11	Course M3S1
Question 3		Marks & seen/unseen

Parts

Unseen

(i)

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta) = \prod_{k=1}^{n} \frac{e^{k\theta}(k\theta)^{x_k}}{x_k!} = \left(\prod_k \frac{k^{x_k}}{x_k!}\right) \underbrace{e^{(\sum_1^n k)\theta}\, \underbrace{\theta^{\sum_1^n x_k}}_{e^{(\sum x_k)\ln\theta}}}_{}$$

$$= h(\underset{\sim}{x}) \,.\, g(\Sigma x_k, \theta)$$

By Neyman Factorization, $S = \sum_1^n X_k$ is sufficient for θ — 4

(ii) S has a Poisson$\left(\sum_{k=1}^{n} k\theta\right)$ distribution i.e. Poisson$(a\theta)$ where $a = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$, which is (full) Exponential Family & so is complete for θ. — 2

(iii)

$$E(T) = P(X_1 + X_n = 0) = P(X_1 = 0 \cap X_n = 0)$$
$$= P(X_1=0)P(X_n=0) \text{ by independence}$$
$$= e^{-\theta}e^{-n\theta} = e^{-(n+1)\theta} = \xi$$

— 2

So T is unbiased for ξ

By Rao-Blackwell we obtain an unbiased estr which is a function of S alone

$$\varphi(s) = E(T|S=s) = P(X_1=0 \cap X_n = 0 | S=s)$$

$$= \frac{P(X_1=0 \cap X_n=0 \cap \sum_{k=2}^{n-1} X_k = s)}{P(S=s)}$$

$$= \frac{e^{-\theta}\,.\,e^{-n\theta}\,.\,e^{-\sum_{k=2}^{n-1}k\theta}\left(\sum_{k=2}^{n-1}k\theta\right)^s/s!}{e^{-\sum_{k=1}^{n}k\theta}\left(\sum_{k=1}^{n}k\theta\right)^s/s!}$$

$$= \left(\frac{\sum_{k=2}^{n-1}k}{\sum_{k=1}^{n}k}\right)^s = \left(\frac{a-1-n}{a}\right)^s = \left(1 - \frac{n+1}{\frac{1}{2}n(n+1)}\right)^s$$

$$= \left(1-\frac{2}{n}\right)^s.$$

— 10

This is a function of suff. stat s only, so by sufficiency and completeness, it is the unique MVUE of its expectation $\xi = e^{-(n+1)\theta}$ (Lehmann-Scheffé Thm) — 2

Setter's initials: RC	Checker's initials: AM	Page number: 4

	M3S1/M4S1 EXAMINATION SOLUTIONS 2010−11	Course M3S1
Question 4		Marks & seen/unseen

Parts

(i) The likelihood ratio test for $H_0: \theta=\theta_0$ v. $H_1: \theta=\theta_1>\theta_0$ has lik ratio — Unseen

$$\lambda(\underset{\sim}{x}) = \frac{\theta_1^n}{\prod_i (1+x_i)^{\theta_1+1}} \Big/ \frac{\theta_0^n}{\prod_i (1+x_i)^{\theta_0+1}} = \left(\frac{\theta_1}{\theta_0}\right)^n \prod_i (1+x_i)^{\theta_0-\theta_1}$$

The most powerful test, by the Neyman-Pearson Lemma, and we reject H_0 if $\lambda(\underset{\sim}{x})$ is too large, i.e. if

$$n\ln\left(\frac{\theta_1}{\theta_0}\right) - (\theta_1-\theta_0)\sum_i \ln(1+x_i) > \kappa \text{ for some } \kappa \text{, a const.}$$

i.e. if $z = \sum_1^n \ln(1+x_i) < c$ for some constant c.

The test does not depend on the value of θ_1, so the test is UMP, with respect to $H_1: \theta>\theta_0$ — 7

(ii) Let $Y=\ln(1+X)$ $\quad e^y = 1+x \quad x = e^y-1 \quad dx = e^y dy$

$x=0 \Rightarrow y=0 \quad x=\infty \Rightarrow y=\infty$

$$f_Y(y|\theta) = \frac{\theta}{(e^y)^{\theta+1}} \cdot e^y H(0<y<\infty)$$

$$= \theta e^{-\theta y} \quad (0<y<\infty) \text{ i.e. Exponential}(\theta)$$

$Z = \sum Y_i = \sum \ln(1+X_i)$ is $Gamma(n,\theta)$ — 5

so $T = 2\theta Z$ is χ^2_{2n}

~~Under H_0, $P(2\theta_0 Z \le \kappa_0) = \alpha$ where κ_0 is the α-quantile of χ^2_{2n}~~

~~i.e. $\alpha = P(Z \le c|\theta_0) = P(2\theta_0 Z < 2\theta_0 c | \theta = \theta_0) = F_{\chi^2_{2n}}(2\theta_0 c)$~~

$$\text{Size } \alpha = P(Z<c|\theta_0) = P(T<c_0|\theta_0)$$

$$= F_T(c_0) \text{ where } c_0 = 2\theta_0 c \text{ \& } T \text{ is } \chi^2_{2n}$$ — 3

$$\text{Power } \beta(\theta) = P(T<c_0|\theta) = P(2\theta_0 Z < c_0|\theta)$$

$$= P\left(2\theta Z < \frac{\theta}{\theta_0}c_0 \Big| \theta\right) = F_T\left(\frac{\theta}{\theta_0}c_0\right)$$

where T is χ^2_{2n} — 5

Note: Results in terms of Gamma will be accepted.

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Setter's initials: RC

Checker's initials: AY