Problem Sheet 2

1. Show that the trapezium rule approximation I_N of the integral

$$I = \int_0^{2\pi} \exp\left(\cos(\theta)\right) d\theta$$

converges super-exponentially, like

$$|I - I_N| = O\left(\left(\frac{e}{2N}\right)^N\right), \quad N \to \infty.$$

2. (a) Show that the function

$$f(z) = \frac{z}{e^z - 1}$$

has a removable singularity at $z = 0$, and modify the definition of the function at zero, so that it is analytic at zero.

(b) What is the domain of analyticity of your modified version of f?

(c) Why might rounding errors occur in practice, when evaluating $f(x)$ with $x \approx 0$?

(d) Use Cauchy's integral theorem with a circular contour of radius one centered at zero, to write down an approximation for $f(x)$ which will not suffer from rounding errors when $x \approx 0$.

(e) Given that $x < \epsilon < 1$, determine an estimate for the convergence rate of your approximation. Do not attempt to bound the maximum value of f over a complex strip, just determine the convergence *rate*.

3. If f is 2π-periodic and entire, show that

$$\int_0^{2\pi} f(\theta) d\theta = \int_z^{2\pi + z} f(\theta) d\theta, \quad \text{for all } z \in \mathbb{C}.$$

4. Suppose f is 2π-periodic, and is both bounded analytic in the upper-half complex plane, $\text{Im}\{z\} > a$, where $a < 0$. Modify the proof of Theorem 2.4 to reduce the constant in the final estimate from 4π to 2π.

5. Suppose f is analytic in in some annulus D, and has an even Laurent expansion, i.e.

$$f(z) = \sum_{j=-\infty}^{\infty} a_{2j}(z - z_0)^{2j}.$$

(a) For a contour $\gamma \subset D$, show that

$$\int_\gamma f(z) dz = 0$$

(b) Let I_N be the N-point trapezium rule approximation for a circular contour γ in D. By modifying the proof of Theorem 2.7, show that

$$I_N = 0, \quad \text{for } N \text{ even.}$$

6. Consider the integral
$$\cancel{I = \int_{-\infty}^{\infty} \frac{x \exp(\tanh x)}{(1+x^2)} dx},$$
and the trapezium rule approximation by I_h and $I_h^{(N)}$.

   (a) Estimate the discretisation error (do not attempt to bound the integrand in the analytic strip).

   (b) Estimate the truncation error.

   (c) Determine the optimal meshwidth h.

7. Recall the integral component of the *complementary error function* as defined in the notes:
$$I(z) = \int_{-\infty}^{\infty} \frac{e^{-z^2 t^2}}{t^2+1} dt, \quad z \in \mathbb{R}.$$

   (a) Using an appropriate theorem in the notes, give an estimate of the convergence rate of the trapezium rule approximation I_h for this integral?

   (b) By considering the truncation error, determine the optimal meshwidth h as $N \to \infty$.

   (c) By using residue calculus, derive a version of the trapezium rule $\tilde{I}_h$, such that the convergence rate is $\tilde{I}_h - I = O(e^{-2\pi\tilde{a}/h})$ for any $\tilde{a} > 0$.

   (d) By considering the component M in the error estimate, find the $\tilde{a}$ which gives the optimal convergence rate as $N \to \infty$, and find this rate.

8. (a) Following similar ideas to §2.7, use contour deformation to show that if f is analytic for $\arg z \in [0, \pi/4]$ with $f(z) = O(z^q)$ as $z \to \infty$, then
$$\int_0^{\infty} f(z) e^{i\omega z^p} dz = e^{i\pi/(2p)} \int_0^{\infty} f(te^{i\pi/(2p)}) e^{-\omega t^p} dt, \quad \text{for } \omega > 0,$$
if $p \geq 2$ and $0 \leq q < p-1$.

   (b) How does the result change if we generalise to $\omega \in \mathbb{C}$, for $\mathrm{Re}\{\omega\} > 0$, and how does this change analyticity requirements about f?