CW2, Geometric Complex Analysis,

Please submit your work on BlackBoard dropbox by 13:00 on Wednesday, 12 March.

**Problem 1.** *Let $\mathcal{S}$ denote the class of all one-to-one and holomorphic maps $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$. Prove that the class of maps $\mathcal{S}$ is a normal family.*

**Problem 2.** *Let $k \geq 2$ be an integer and define*

$$\Lambda_k = \{f^{(k)}(0) : f \in \mathcal{S}\}.$$

*Prove that*

(i) *for every $k \geq 2$, there is $r_k > 0$ such that $\Lambda_k = \{w \in \mathbb{C} : |w| \leq r_k\}$;*

(ii) *there is a constant $C > 0$ such that for all $n \geq 1$ we have $r_n \leq Cn^2 \cdot n!$.*

**Problem 3.** *Let $\Omega$ be a non-empty simply connected subset of $\mathbb{C}$ which is not equal to $\mathbb{C}$. For $z \in \Omega$, the conformal radius of $\Omega$ at $z$ is defined as*

$$\mathrm{rad}_{\mathrm{conf}}(\Omega, z) = |\varphi'(0)|,$$

*where $\varphi : \mathbb{D} \to \Omega$ is the Riemann mapping with $\varphi(0) = z$.*

(i) *Prove that $\mathrm{rad}_{\mathrm{conf}}(\Omega, z)$ is independent of the choice of the Riemann map $\varphi$.*

(ii) *Define*

$$r_z = \sup\{r > 0 : B_r(z) \subset \Omega\}.$$

*Prove that*

$$r_z \leq \mathrm{rad}_{\mathrm{conf}}(\Omega, z) \leq 4r_z.$$

**Problem 4.** *Prove that there is $\lambda > 0$ such that for every map $f$ in the class $\mathcal{S}$, we have*

$$\mathrm{area}(f(B_{1/4}(-1/2))) \leq \lambda\, \mathrm{area}(f(B_{1/4}(1/2))).$$