ExamModuleCode	Question Number	Comments for Students
M3B	1	•(a) That was mostly straight forward. (iv) was the one with most complications. •(d)(i) It is important to check when the converse of the shut-down condition is satisfied. Otherwise, you risk to state that the optimal profit will be negative. And also the optimal output would be negative... Moreover, you should remember to check the second order condition as well. •(d)(ii) The firm can only adjust their output, not their price. Moreover, the important thing to realise is that the first order condition changes if the market price depends on the output. So the question amounted to: Are the optimal outputs different for the situation when a firm is a price taker as opposed to the situation when the output influences the price. •(e) The problem describes that the two suppliers can sell as much as they like at the respective prices given. So their challenge is to determine an optimal level of supply. This was very often misunderstood.
M3B	2	•(a)(i) The minus sign belongs to the income effect. This was a common mistake to forget it. •(a)(ii) Please recall that in the definitions of a normal good and an ordinary good, there is a non-strict inequality. •(a)(iii) Here it is essential that one gives an argument why the substitution effect is negative for i=j. Moreover, it should be stated that a good is either a normal good or an inferior good. Similarly, it is either an ordinary or a Giffen good. •(a)(iv) Here I was fairly flexible with your answers. •(b)(i)-(iii) This was usually answered quite well. •(b)(iv) The key observation is that the indirect utility remains fixed to compute the substitution effect. Hardly anyone provided a correct answer. •(b)(v) Also here the answers were very often incorrect.
M3B	3	•(a) Again, it was necessary to also check whether the SOC and whether the converse of the shutdown hold. •(b), (c), (d) I appreciated that these questions where generally answered quite well. •(e) An alternative to the solution provided is that, in the long run, marginal costs must equal average costs (then there are no (abnormal) profits). This implies that each firm produces y=1. Some of you started with this ansatz, which was fine as well. A correct solution was provided in approximately 50% of the cases. •(h) A lot of people struggled to find the right solution. •(i) There was a correct solution provided by only a few (this was also one of the most challenging problems of the entire paper). •(j) This was done quite well. I was very liberal what I counted as a correct solution.
M3B	4	•(a)(i) We discussed in class that there are always two circuits: one for money and the second one for goods and services. It was sufficient to denote only the one related to money, as we have done it in the lecture. Overall it was answered rather well. •(a)(ii) There should have been some sort of explanation... That was sometimes a bit weak. •(c) Some provided the answer that Anne's utility is u(x,y) = x 1{x=y}. That means she only has a positive utility if she has exactly the same amount of each good. However, if she had, say (2,3), she could get rid of one egg without decreasing her utility. This is not reflected in the proposed utility function. Moreover, one can verify that the proposed utility function is also not monotone. •(c)(ii) Quite a number attempted to add the two utility functions. However, we discussed several times in the lecture that there is no sensible way of adding or comparing ordinal utilities. •(c)(iii) One common error was to give the definition of a Pareto improvement rather than the definition of a Pareto efficient allocation. •(c)(iv) Some people only gave one Pareto optimal allocation, which earned 1 mark out of 2.

Imperial College London

M3B

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematics of Business and Economics

Date: Friday 31 May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. (a) Let $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ be a production function. State the following definitions.

   (i) The function f is monotone.

   (ii) The function f is concave.

   (iii) The function f is quasi-concave.

   (iv) The function f has diminishing marginal productivity (you may assume that f is differentiable).

   (v) The function f is positively homogeneous of degree $k \in \mathbb{R}$.

   (b) Either prove or give a counterexample.

   (i) Any concave function is quasi-concave.

   (ii) Any quasi-concave function is concave.

   (c) Let $\pi(\underline{x}, p, \underline{w}) = pf(\underline{x}) - \underline{w}\underline{x}^\top$ be the profit at output price $p > 0$, input bundle $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and input price $\underline{w} \in \mathbb{R}^n_{\geq 0}$ where $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ denotes the production function. Assume that f is increasing, quasi-concave and that $f(0) = 0$. Show that if f has constant returns to scale, then the maximal profit is either 0 or infinite.

   (d) Consider a firm with a cost function $c^*(y) = y^2 + 3y + 2$ for output $y \geq 0$. Assume the output price of the product is $p_0 \geq 0$.

   (i) Determine the optimal output quantity y^* of the firm as a function of the price p_0.

   (ii) Suppose the firm produces at the optimal output $y_0 = y^*(p_0)$. Now, the market research department of the firm discovers that the market price reacts to a change in supply. That is, if the firm were to produce more than $y^*(p_0)$, they can sell their good only for less than p_0. Vice versa, if the firm were to produce less than $y^*(p_0)$, they can sell their good for more than p_0.

   Assuming the firm want to maximise their profit, should they adjust their output, and if so how?

   Explain and justify your answer.

   (e) Consider two electricity suppliers A and B, using the same technology, having the same input prices, and both being price takers (their level of output does not influence the market price). Suppose that supplier A can sell electricity at a price of 14 pence per kwh during the entire day. On the other hand, supplier B can sell electricity at a price of 16 pence per kwh from 6am to 10pm and 10 pence per kwh from 10pm to 6am.

   Suppose both suppliers try to maximise their profits and that they are able to change their production instantaneously and without any transaction costs.

   (i) Establish whether the two suppliers achieve the same profit during a day. If you conclude that their profits are different, establish which of the suppliers achieves a higher profit. You may use all facts about profit functions established in the course without proving them.

   (ii) Briefly discuss how realistic the result established in part (i) is.

2. Let $u\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ be an increasing quasi-concave utility function. We denote the derived quantities as $\underline{x}^*$ for Marshallian demand, $\underline{x}^*_H$ for Hicksian demand, and v for the indirect utility function. *Slutsky's Equation* states that, under sufficient regularity conditions, for any strictly positive price vector $\underline{p} \in \mathbb{R}^n_{>0}$, any strictly positive budget $m > 0$ and any $i, j \in \{1, \ldots, n\}$ it holds that

$$\frac{\partial x^*_j(\underline{p}, m)}{\partial p_i} = \frac{\partial x^*_{H,j}(\underline{p}, v(\underline{p}, m))}{\partial p_i} - \frac{\partial x^*_j(\underline{p}, m)}{\partial m} x^*_i(\underline{p}, m)\,. \tag{1}$$

(a) (i) Identify the Substitution Effect and the Income Effect in Slutsky's Equation.

(ii) State the definitions of a normal good, an ordinary good, a Giffen good and an inferior good.

(iii) Establish logical relations between these four types of goods using Slutsky's Equation.

(iv) Give a real-world example of a Giffen good. Explain and justify your example.

(b) Now assume that $n = 2$ and u takes the form $u(x_1, x_2) = \sqrt{x_1 x_2}$ for $x_1, x_2 \geq 0$.

(i) Suppose that a consumer is endowed with a budget of $m = 4$ and that the prices for the two goods are $\underline{p} = (p_1, p_2) = (1, 1)$. Draw a graph visualising the budget line $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, p_1 x_1 + p_2 x_2 = m\}$, the indifference curve $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, u(x_1, x_2) = v(\underline{p}, m)\}$ and Marshallian demand $\underline{x}^*(\underline{p}, m)$. Explicitly state your graphically determined result for Marshallian demand $\underline{x}^*(\underline{p}, m)$.

(ii) Now, suppose the price of good 1 increases and the consumer is faced with the new price vector $\underline{q} = (4, 1)$, while their budget remains constant at $m = 4$. Compute the new Marshallian demand $\underline{x}^*(\underline{q}, m)$.

(iii) Replicate your graph from part (i) and visualise the new situation by adding the new budget line $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, q_1 x_1 + q_2 x_2 = m\}$, the new indifference curve $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, u(x_1, x_2) = v(\underline{q}, m)\}$ and the new Marshallian demand $\underline{x}^*(\underline{q}, m)$.

(iv) Decompose the change in Marshallian demand for good 1, $x^*_1(\underline{q}, m) - x^*_1(\underline{p}, m)$, into a Substitution Effect and an Income Effect, generalising the differential form of Slutsky's Equation given at (1). Give an explicit formula for the Substitution Effect. You may define the Income Effect implicitly.

(v) Replicate your graph from part (iii) and visualise the corresponding Substitution Effect and Income Effect.

(vi) Is good 1 a normal, ordinary, inferior or Giffen good? Briefly justify your answer.

3. Consider the market for coal. Suppose market demand for coal is given by $X^*(p) = \frac{8}{p^2}$, where $p \geq 0$ is the price for one unit of coal. Suppose that, in the short-run, there is one coal mine in the market with a cost function of $c^*(y) = \frac{1}{2}y^2 + \frac{1}{2}$.

(a) Determine the market supply $Y^*(p)$ as a function of the price $p \geq 0$.

(b) Determine the equilibrium price and the equilibrium quantity.

(c) Compute the producers' surplus, the consumers' surplus and the community surplus.

(d) Draw a graph (quantity on the horizontal axis, price on the vertical axis) and depict the market supply, market demand, equilibrium price and equilibrium quantity as well as producers' and consumers' surplus.

(e) Now consider the long-run scenario. How many coal mines, each having the same cost function $c^*(y) = \frac{1}{2}y^2 + \frac{1}{2}$ will operate in the long run?

(f) What is the long-run equilibrium price and equilibrium quantity?

(g) What is the long-run producers' and consumers' surplus? What is the community surplus.

Now, the government introduces a new environmental policy and wants to reduce the usage of coal. Its long-run goal is a reduction of (at least) 50% compared to the long-run equilibrium. One suggestion is to issue licences for coal mines: Each mine needs to purchase such a licence as a prerequisite to run their business.

(h) How many licences can the government issue to guarantee a reduction of the usage of coal of (at least) 50%?

(i) If the government decides to sell an unbounded amount of licences for production, how high does it have to set the price per licence to achieve its goal of a 50% reduction?

(j) Give another suggestion of how the government could economically incentivise a reduction of the usage of coal.

4. (a) (i) Sketch and describe the circuit flow of income of a national economy comprising 5 sectors.

   (ii) How would you change the circuit flow of income described in part (i) if you were to describe the global economy. Justify your answer.

(b) (i) State the definition of the 'Gross Domestic Product (GDP)'.

   (ii) State the definition of the 'Gross National Product (GNP)'.

(c) Anne and Bob both consume toast (good 1) and eggs (good 2) for breakfast. However, their behaviour differs considerably: While Anne insists on having exactly as many slices of toast as eggs, Bob does not care too much about the proportion of eggs and toast. Both of them prefer to have more of any good than less.

   (i) Give ordinal utility functions $u_A\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ and $u_B\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ which represent Anne's and Bob's preferences, respectively.

   (ii) Suppose you are Anne and Bob's parents. You have only four slices of toast and three eggs left to prepare their breakfast. Is there a way to distribute the toast and eggs which maximises the total utility? If so, describe and justify the allocation. If not, justify and explain your answer.

   (iii) Give the definition of a Pareto efficient allocation.

   (iv) Determine all Pareto efficient allocations of toast and eggs, given Anne's and Bob's preferences.

M3B SOLUTIONS

5 marks (A) seen

1. (a) Let $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ be a production function. State the following definitions.

   (i) The function f is monotone.
   Solution:
   For any $\underline{x}, \underline{x}' \in \mathbb{R}^n_{\geq 0}$ such that $\underline{x} \leq \underline{x}'$ componentwise it holds that $f(\underline{x}) \leq f(\underline{x}')$.

   (ii) The function f is concave.
   Solution:
   For any $\underline{x}, \underline{x}' \in \mathbb{R}^n_{\geq 0}$ and any $\lambda \in [0,1]$

$$f((1-\lambda)\underline{x} + \lambda \underline{x}') \geq (1-\lambda) f(\underline{x}) + \lambda f(\underline{x}').$$

   (iii) The function f is quasi-concave.
   Solution:
   For any $\underline{x}, \underline{x}' \in \mathbb{R}^n_{\geq 0}$ and any $\lambda \in [0,1]$

$$f((1-\lambda)\underline{x} + \lambda \underline{x}') \geq \min\{f(\underline{x}), f(\underline{x}')\}.$$

   (iv) The function f has diminishing marginal productivity (you may assume that f is differentiable).
   Solution:
   For $i \in \{1, \dots, n\}$ the partial derivative $\partial_i f$ is decreasing in its ith argument.

   (v) The function f is positively homogeneous of degree $k \in \mathbb{R}$.
   Solution:
   For any $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and any $t > 0$

$$f(t\underline{x}) = t^k f(\underline{x}).$$

2 marks (B) seen

(b) Either prove or give a counterexample.

   (i) Any concave function is quasi-concave.
   Solution:
   This is true. Indeed, let f be concave and $\underline{x}, \underline{x}' \in \mathbb{R}^n_{\geq 0}$, $\lambda \in [0,1]$. Then we obtain that

$$f((1-\lambda)\underline{x} + \lambda \underline{x}') \geq (1-\lambda) f(\underline{x}) + \lambda f(\underline{x}') \geq \min\{f(\underline{x}), f(\underline{x}')\}.$$

   (ii) Any quasi-concave function is concave.
   Solution:
   This is false. Indeed, consider the function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, with $f(0) = 1$ and $f(x) = 0$ for all $x > 0$.

(c) Let $\pi(\underline{x}, p, \underline{w}) = pf(\underline{x}) - \underline{w}\underline{x}^\top$ be the profit at output price $p > 0$, input bundle $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and input price $\underline{w} \in \mathbb{R}^n_{\geq 0}$ where $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ denotes the production function. Assume that f is increasing, quasi-concave and that $f(0) = 0$. Show that if f has constant returns to scale, then the maximal profit is either 0 or infinite.

3 marks
(C)
not
seen

Solution:

First, we notice that $\pi(0, p, \underline{w}) = 0$. That means that the maximal profit cannot be negative. For any $\underline{x} \in \mathbb{R}^n_{\geq 0}$ with $\underline{x} \neq 0$ and any $t > 0$ we have that

$$\pi(t\underline{x}, p, \underline{w}) = pf(t\underline{x}) - \underline{w}t\underline{x}^\top = t\pi(\underline{x}, p, \underline{w}).$$

That means if, for fixed $p, \underline{w}$, there is some $\underline{x} \in \mathbb{R}^n_{\geq 0}$, $\underline{x} \neq 0$, such that $\pi(\underline{x}, p, \underline{w}) > 0$ then any arbitrarily large profit can be generated by producing with the input bundle $t\underline{x}$ for sufficiently large $t > 0$.

(d) Consider a firm with a cost function $c^*(y) = y^2 + 3y + 2$ for output $y \geq 0$. Assume the output price of the product is $p_0 \geq 0$.

(i) Determine the optimal output quantity y^* of the firm as a function of the price p_0.

3 marks
(B)
similar
seen

Solution:

We need to check the FOC, SOC and the converse of the shutdown condition.

- The FOC is $p_0 = c^{*\prime}(y) = 2y + 3$ which yields that $y(p_0) = p_0/2 - 3/2$.
- To check the SOC, observe that $c^{*\prime\prime}(y) = 2 > 0$. So it is satisfied for any output.
- The converse of the shutdown condition is that the average variable costs are $\leq p_0$. That means $y + 3 \leq p_0$. Using the FOC for the quantity, we arrive at $p_0/2 - 3/2 + 3 \leq p_0$. This yields that $p_0 \geq 3$.

In summary, we get that

$$y^*(p_0) = \begin{cases} 0, & \text{if } p_0 < 3 \\ p/2 - 3/2, & \text{if } p_0 \geq 3. \end{cases}$$

(ii) Suppose the firm produces at the optimal output $y_0 = y^*(p_0)$. Now, the market research department of the firm discovers that the market price reacts to a change in supply. That is, if the firm were to produce more than $y^*(p_0)$, they can sell their good only for less than p_0. Vice versa, if the firm were to produce less than $y^*(p_0)$, they can sell their good for more than p_0.

Assuming the firm wants to maximise their profit, should they adjust their output, and if so how?

Explain and justify your answer.

3 marks
(D)
unseen

Solution:
Let $p(y)$ be the inverse demand function. If the firm produces y_0, they make a profit of $p(y_0)y_0 - c^*(y_0) = p_0y_0 - c^*(y_0)$. We know that y_0 is the unique optimal output quantity under the assumption that $p(y) \equiv p_0$. That means that $p_0y_0 - c^*(y_0) > p_0y - c^*(y)$ for all $y \geq 0$, $y \neq y_0$. We show that the firm will not produce more than y_0.
Indeed, assume it were optimal for the firm to produce $y > y_0$. That means they would realise a profit of $p(y)y - c^*(y)$. However,

$$p(y)y - c^*(y) < p_0y - c^*(y) < p_0y_0 - c^*(y_0) = p(y_0)y_0 - c^*(y_0),$$

which is a contradiction.
On the other hand, we cannot generally establish that the firm will necessarily reduce their output. If, for example, $y_0 = 0$, it is just not possible. (And also if $y_0 > 0$ and $p'(y_0) = 0$, the FOCs coincide and there won't be a change in output.)

(e) Consider two electricity suppliers A and B, using the same technology, having the same input prices, and both being price takers (their level of output does not influence the market price). Suppose that supplier A can sell electricity at a price of 14 pence per kwh during the entire day. On the other hand, supplier B can sell electricity at a price of 16 pence per kwh from 6am to 10pm and 10 pence per kwh from 10pm to 6am.
Suppose both suppliers try to maximise their profits and that they are able to change their production instantaneously and without any transaction costs.

(i) Establish whether the two suppliers achieve the same profit during a day. If you conclude that their profits are different, establish which of the suppliers achieves a higher profit. You may use all facts about profit functions established in the course without proving them.

3 marks
(D)
unseen

Solution:
First notice that since the two suppliers use the same technology, their maximised profit functions π^* coincide. Let $\underline{w} \in \mathbb{R}^n_{\geq 0}$ be the price vector of the input prices. Then the maximal profit of supplier A is given by

$$\pi^*(14, \underline{w}).$$

On the other hand, supplier B faces a price of $p = 16$ during $2/3$ of the day and $p = 10$ during $1/3$ of the day. Notice that $14 = 2/3 \times 16 + 1/3 \times 10$. Using the *convexity* of the profit function π^* in the arguments $(p, \underline{w})$, we obtain for the average profit of supplier B

$$\frac{2}{3}\pi^*(16, \underline{w}) + \frac{1}{3}\pi^*(10, \underline{w}) \geq \pi^*(\frac{2}{3} \times 16 + \frac{1}{3} \times 10, \underline{w}) = \pi^*(14, \underline{w}).$$

Hence, the profit of supplier B is at least as high as the profit of supplier A.

1 mark
(C)
unseen

(ii) Briefly discuss how realistic the result established in part (i) is.

Solution: *Any of the following points would earn a mark (or another unforeseen reasonable issue):*

- It would be more realistic to assume that changing the output causes some transaction costs.
- Moreover, adapting output usually does not happen instantaneously, such that supplier B faces the possibility of producing with a non-optimal output for a while.
- In the long-run, it would be plausible if the two supplier adapt their production technology. Supplier A needs less production capacity and thus a smaller amount of machinery etc. than supplier B.

Overall, it would be plausible if supplier A had a higher profit, taking into account the above considerations.

2. Let $u\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ be an increasing quasi-concave utility function. We denote the derived quantities as $\underline{x}^*$ for Marshallian demand, $\underline{x}^*_H$ for Hicksian demand, and v for the indirect utility function. *Slutsky's Equation* states that, under sufficient regularity conditions, for any strictly positive price vector $\underline{p} \in \mathbb{R}^n_{>0}$, any strictly positive budget $m > 0$ and any $i, j \in \{1, \ldots, n\}$ it holds that

$$\frac{\partial x^*_j(\underline{p}, m)}{\partial p_i} = \frac{\partial x^*_{H,j}(\underline{p}, v(\underline{p}, m))}{\partial p_i} - \frac{\partial x^*_j(\underline{p}, m)}{\partial m} x^*_i(\underline{p}, m)\,. \tag{1}$$

(a) (i) Identify the Substitution Effect and the Income Effect in Slutsky's Equation.

1 mark (A) seen

Solution:

The Substitution effect is $\frac{\partial x^*_{H,j}(\underline{p}, v(\underline{p},m))}{\partial p_i}$. The Income Effect is the remainder $-\frac{\partial x^*_j(\underline{p},m)}{\partial m} x^*_i(\underline{p}, m)$.

(ii) State the definitions of a normal good, an ordinary good, a Giffen good and an inferior good.

2 marks (A) seen

Solution:

- Good j is a *normal good* if
$$\frac{\partial x^*_j(\underline{p}, m)}{\partial m} \geq 0.$$
- Good j is an *inferior good* if
$$\frac{\partial x^*_j(\underline{p}, m)}{\partial m} < 0.$$
- Good j is an *ordinary good* if
$$\frac{\partial x^*_j(\underline{p}, m)}{\partial p_j} \leq 0.$$
- Good j is a *Giffen good* if
$$\frac{\partial x^*_j(\underline{p}, m)}{\partial p_j} > 0.$$

(iii) Establish logical relations between these four types of goods using Slutsky's Equation.

3 marks (B) seen

Solution:

- We see that a good can be either normal or inferior. Similarly, it is either an ordinary good or a Giffen good.
- We know from the lecture that the expenditure function is concave in the price. That means
$$\frac{\partial x^*_{H,j}(\underline{p}, v(\underline{p}, m))}{\partial p_j} = \frac{\partial^2 e(\underline{p}, v(\underline{p}, m))}{(\partial p_j)^2} \leq 0.$$
Since $x^*_j(\underline{p}, m) \geq 0$, Slutsky's Equation yields that $\frac{\partial x^*_j(\underline{p},m)}{\partial m} \geq 0$ implies that $\frac{\partial x^*_j(\underline{p},m)}{\partial p_j} \leq 0$. That is, any normal good is always an ordinary good.
- The contraposition of this assertion is that any Giffen good is an inferior good.

(iv) Give a real-world example of a Giffen good. Explain and justify your example.

2 marks
(B)
seen

Solution:

The above considerations (and also our discussion in the lecture) yield three necessary conditions for a Giffen good:

1. It must be an inferior good.
2. The substitution effect must be relatively small – i.e. there is no close substitute good.
3. A substantial part of income is spent on the respective good, but not all of it.

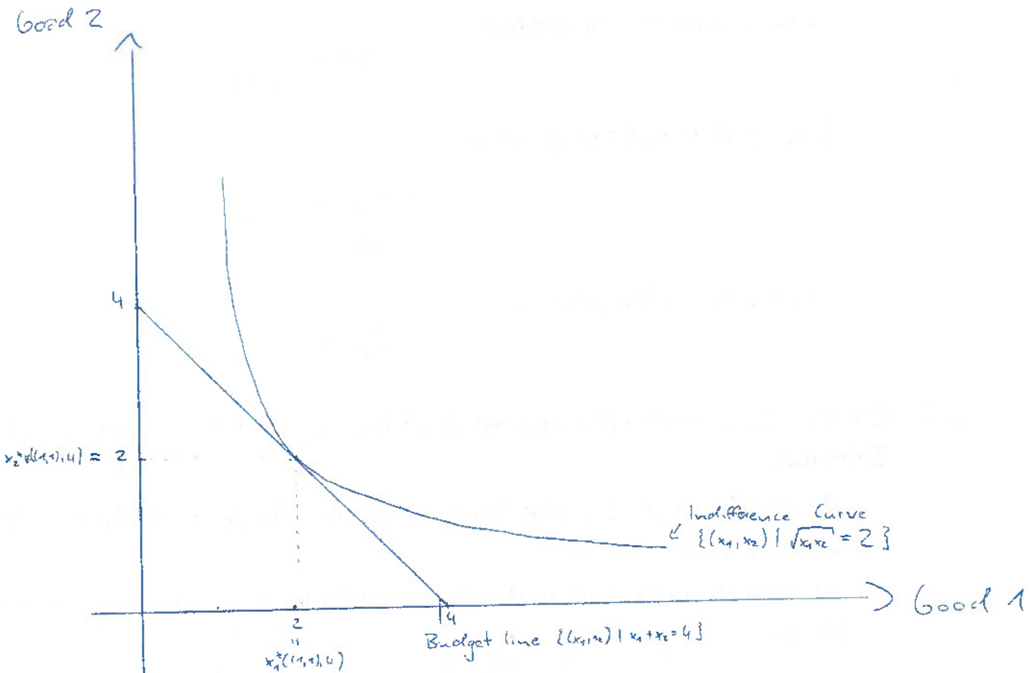
Example: Staple food in relatively poor countries.

(b) Now assume that $n = 2$ and u takes the form $u(x_1, x_2) = \sqrt{x_1 x_2}$ for $x_1, x_2 \geq 0$.

(i) Suppose that a consumer is endowed with a budget of $m = 4$ and that the prices for the two goods are $\underline{p} = (p_1, p_2) = (1, 1)$. Draw a graph visualising the budget line $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, p_1 x_1 + p_2 x_2 = m\}$, the indifference curve $\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \,|\, u(x_1, x_2) = v(\underline{p}, m)\}$ and Marshallian demand $\underline{x}^*(\underline{p}, m)$. Explicitly state your graphically determined result for Marshallian demand $\underline{x}^*(\underline{p}, m)$.

3 marks
(A)
seen

Solution:



Marshallian demand is

$$\underline{x}^*((1,1),4) = (2,2).$$

(ii) Now, suppose the price of good 1 increases and the consumer is faced with the new price vector $\underline{q} = (4, 1)$, while their budget remains constant at $m = 4$. Compute the new Marshallian demand $\underline{x}^*(\underline{q}, m)$.

2 marks
(A)
seen

Solution:
Marshallian demand at price $\underline{q}$ and budget m is defined as $\arg\max_{q_1x_1+q_2x_2 \le m} u(x_1, x_2)$. Due to Walras' Law, the maximum is attained on the budget line. Hence,

$$x_1^*(\underline{q}, m) = \arg\max_{x_1 \ge 0} x_1(m/q_2 - (q_1/q_2)x_1).$$

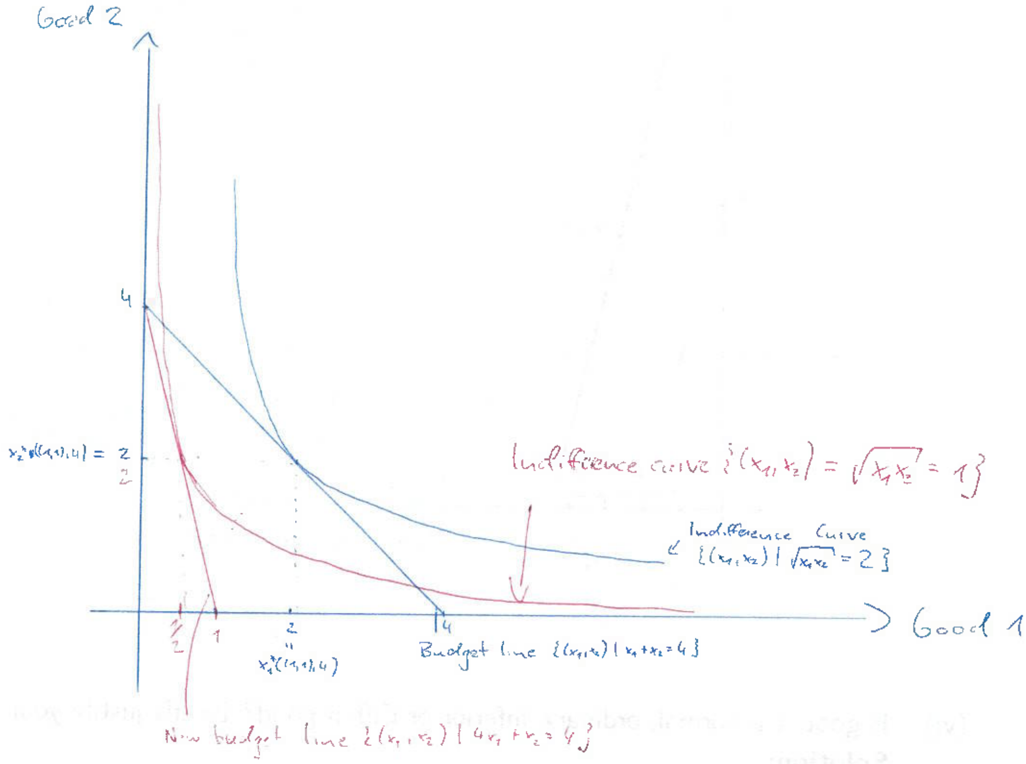
This implies the FOC and the final solution of $x_1^*(\underline{q}, m) = m/(2q_1)$. Similarly, $x_2^*(\underline{q}, m) = m/(2q_2)$. Therefore,

$$\underline{x}^*((4,1),4) = (1/2, 2)$$

(iii) Replicate your graph from part (i) and visualise the new situation by adding the new budget line $\{(x_1, x_2) \in \mathbb{R}^2_{\ge 0} \,|\, q_1x_1 + q_2x_2 = m\}$, the new indifference curve $\{(x_1, x_2) \in \mathbb{R}^2_{\ge 0} \,|\, u(x_1, x_2) = v(\underline{q}, m)\}$ and the new Marshallian demand $\underline{x}^*(\underline{q}, m)$.

2 marks
(B)
seen

Solution:



(iv) Decompose the change in Marshallian demand for good 1, $x_1^*(\underline{q}, m) - x_1^*(\underline{p}, m)$, into a Substitution Effect and an Income Effect, generalising the differential form of Slutsky's Equation given at (1). Give an explicit formula for the Substitution Effect. You may define the Income Effect implicitly.

2 marks
(D)
seen

Solution:
What defines the Substitution Effect is that one keeps the initial level of indirect utility fixed. Using this idea, the difference $x_1^*(\underline{q}, m) - x_1^*(\underline{p}, m)$ can be decomposed into a Substitution Effect given by

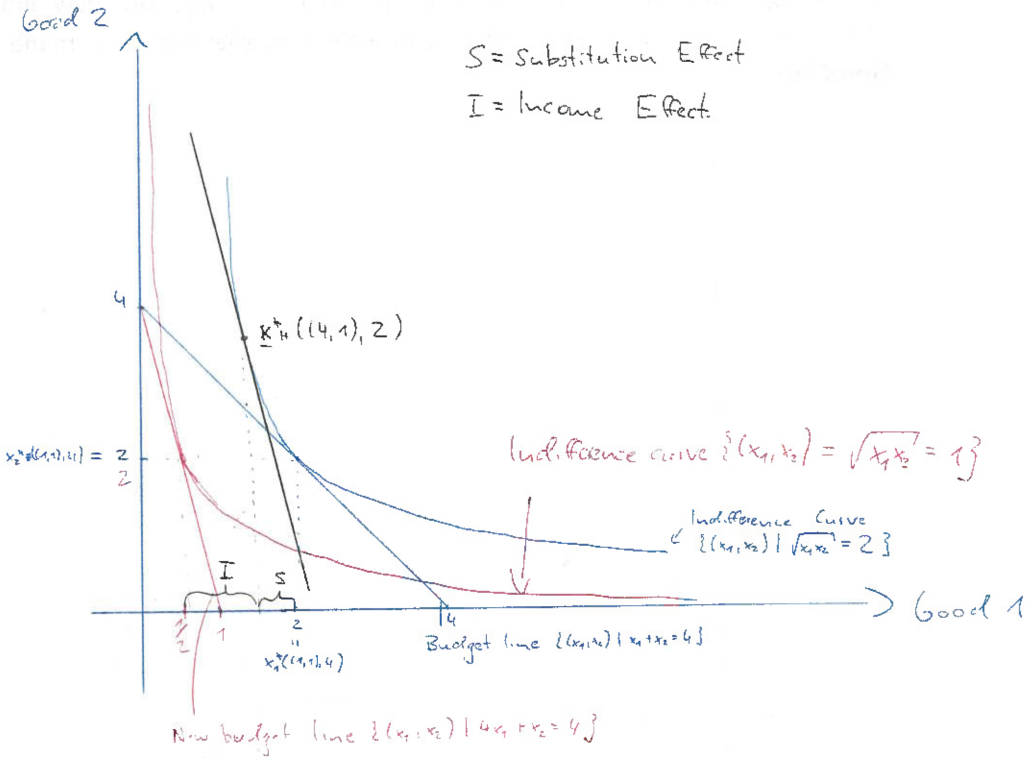
$$x_{H,1}^*(\underline{q}, v(\underline{p}, m)) - x_{H,1}^*(\underline{p}, v(\underline{p}, m))$$

and an Income Effect, which is simply the remainder.

(v) Replicate your graph from part (iii) and visualise the corresponding Substitution Effect and Income Effect.

2 marks
(C)
seen

Solution:



(vi) Is good 1 a normal, ordinary, inferior or Giffen good? Briefly justify your answer.

1 mark
(B)
seen

Solution:
Good 1 is a normal good, since both the substitution effect and the income effect are negative.

3. Consider the market for coal. Suppose market demand for coal is given by $X^*(p) = \frac{8}{p^2}$, where $p \geq 0$ is the price for one unit of coal. Suppose that, in the short-run, there is one coal mine in the market with a cost function of $c^*(y) = \frac{1}{2}y^2 + \frac{1}{2}$.

(a) Determine the market supply $Y^*(p)$ as a function of the price $p \geq 0$.

3 marks (A) similar seen

Solution:

If the mine produces $y \geq 0$ units of coal, they make a profit of $\pi^*(y) = py - c^*(y)$. The FOC of the profit maximisation yields that $p = c^{*\prime}(y) = y$. Since $c^{*\prime\prime}(y) > 0$, the SOC is also satisfied. The converse of the shutdown condition is that $\frac{1}{2}y \leq p$. Inserting the output quantity derived via the FOC, we obtain that $\frac{1}{2}p \leq p$, which holds for all $p \geq 0$. Hence the market supply is $y^*(p) = Y^*(p) = p$.

(b) Determine the equilibrium price and the equilibrium quantity.

1 mark (A) similar seen

Solution:

The equilibrium price is obtained by equating $X^*(p) = Y^*(p)$. This yields an equilibrium price of $p^* = 2$ and an equilibrium quantity of $X^*(p^*) = Y^*(p^*) = 2$.

(c) Compute the producers' surplus, the consumers' surplus and the community surplus.

2 marks (A) similar seen

Solution:

The producers' surplus at $p^* = 2$ is given by

$$PS(p^*) = \int_0^{p^*} Y^*(p)\mathrm{d}p = \int_0^2 p\mathrm{d}p = 2.$$

The consumers' surplus at $p^* = 2$ is given by

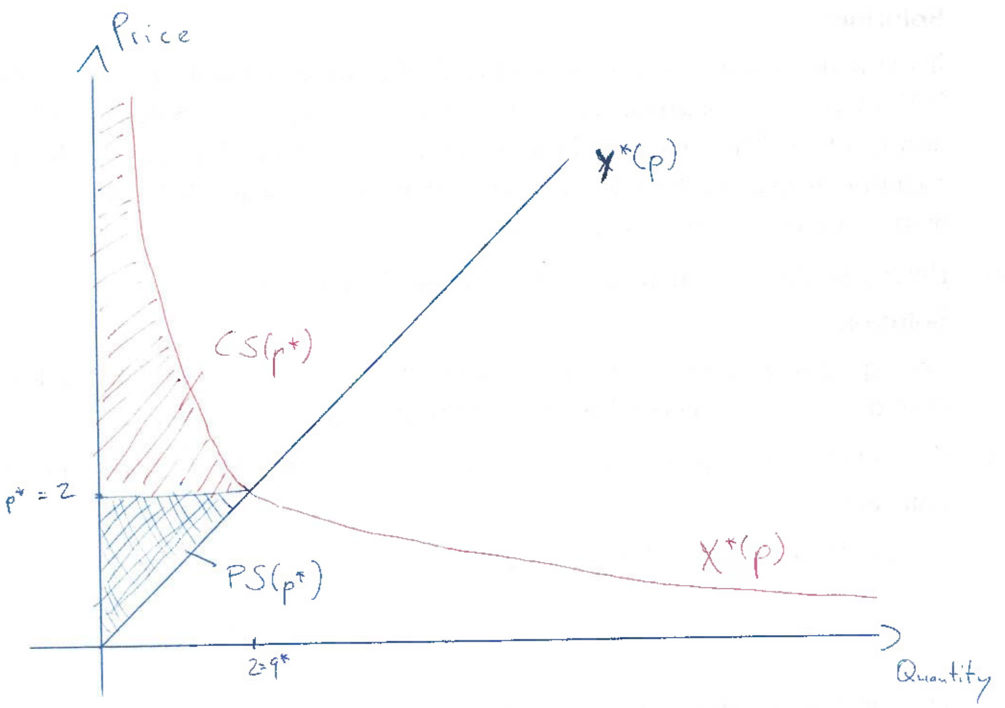
$$CS(p^*) = \int_{p^*}^{\infty} X^*(p)\mathrm{d}p = \int_2^{\infty} \frac{8}{p^2}\mathrm{d}p = 4.$$

The community surplus is the sum of the two, that is, 6.

(d) Draw a graph (quantity on the horizontal axis, price on the vertical axis) and depict the market supply, market demand, equilibrium price and equilibrium quantity as well as producers' and consumers' surplus.

2 marks (B) similar seen

Solution:



(e) Now consider the long-run scenario. How many coal mines, each having the same cost function $c^*(y) = \frac{1}{2}y^2 + \frac{1}{2}$ will operate in the long run?

3 marks (C) similar seen

Solution:

In the lecture, we have discussed that, so long as there is a possibility for firms to make some profit, they will enter the market. That means in the long-run the number J of mines will be the largest integer such that each single mine will make a non-negative profit.

Since each firm has the same cost function, resulting in the same optimal output $y^*(p) = p$, market supply is given by $Y^*(p) = Jp$. Again, one determines the equilibrium price equating $X^*(p) = Y^*(p)$. This results in a equilibrium price of

$$p^* = \frac{2}{J^{1/3}}. \tag{2}$$

Then, each firm operates with a profit of

$$p^*y(p^*) - c(y(p^*)) = \frac{1}{2}(p^*)^2 - \frac{1}{2} = \frac{2}{J^{2/3}} - \frac{1}{2}. \tag{3}$$

This is 0 for $J = 8$. That means, in the long run there are $J^* = 8$ mines on the market.

(f) What is the long-run equilibrium price and equilibrium quantity?

2 marks (C) similar seen

Solution:

From above, we obtain that

$$p^* = \frac{2}{J^{1/3}} = 1.$$

This results in an equilibrium quantity of $X^*(p^*) = Y^*(p^*) = 8$.

(g) What is the long-run producers' and consumers' surplus? What is the community surplus.

1 mark (A) similar seen

Solution:

The long-run producers' surplus is given by

$$PS(p^*) = \int_0^{p^*} Y^*(p)\mathrm{d}p = \int_0^1 8p\mathrm{d}p = 4.$$

The long-run consumers' surplus is given by

$$CS(p^*) = \int_{p^*}^{\infty} X^*(p)\mathrm{d}p = \int_1^{\infty} \frac{8}{p^2}\mathrm{d}p = 8.$$

Hence, the community surplus doubles and attains 12 in the long-run.

Now, the government introduces a new environmental policy and wants to reduce the usage of coal. Its long-run goal is a reduction of (at least) 50% compared to the long-run equilibrium. One suggestion is to issue licences for coal mines: Each mine needs to purchase such a licence as a prerequisite to run their business.

(h) How many licences can the government issue to guarantee a reduction of the usage of coal of (at least) 50%?

2 marks (D) similar seen

Solution:

We are looking for the largest integer J such that the equilibrium quantity with J mines is 4 or smaller than 4. From (2) we yield a general equilibrium price for J mines. Hence, the equilibrium quantity is generally given via

$$Y^*(p^*) = J\frac{2}{J^{1/3}} = 2J^{2/3}.$$

Hence, the ansatz $Y^*(p^*) \leq 4$ amounts to $J \leq 2^{3/2} \in (2,3)$. Therefore, the solution is $J = 2$. So the government can issue 2 licences only.

(i) If the government decides to sell an unbounded amount of licences for production, how high does it have to set the price per licence to achieve its goal of a 50% reduction?

3 marks (D) similar seen

Solution:

Let $t \geq 0$ be the price per licence. The price for the licence is reflected in increased fixed costs for the mines. That means the mines operate with a new cost function of

$$c_t^*(y) = \frac{1}{2}y^2 + \frac{1}{2} + t.$$

The marginal costs remain unchanged. That means that the FOC, SOC and the shutdown condition remain unchanged. Hence, each firm has still the output $y^*(p) = p$. We already know from the previous part that the costs t per licence must be large enough such that we end up with 2 mines only. The profit for a single mine is determined, similarly to (3) with

$$p^*y(p^*) - c_t(y(p^*)) = \frac{1}{2}(p^*)^2 - \frac{1}{2} - t = \frac{2}{2^{2/3}} - \frac{1}{2} - t.$$

Equating this to 0, we can determine the price of the licence as $t = 2^{1/3} - 1/2$.

(j) Give another suggestion of how the government could economically incentivise a reduction of the usage of coal.

1 mark (A) similar seen

Solution:

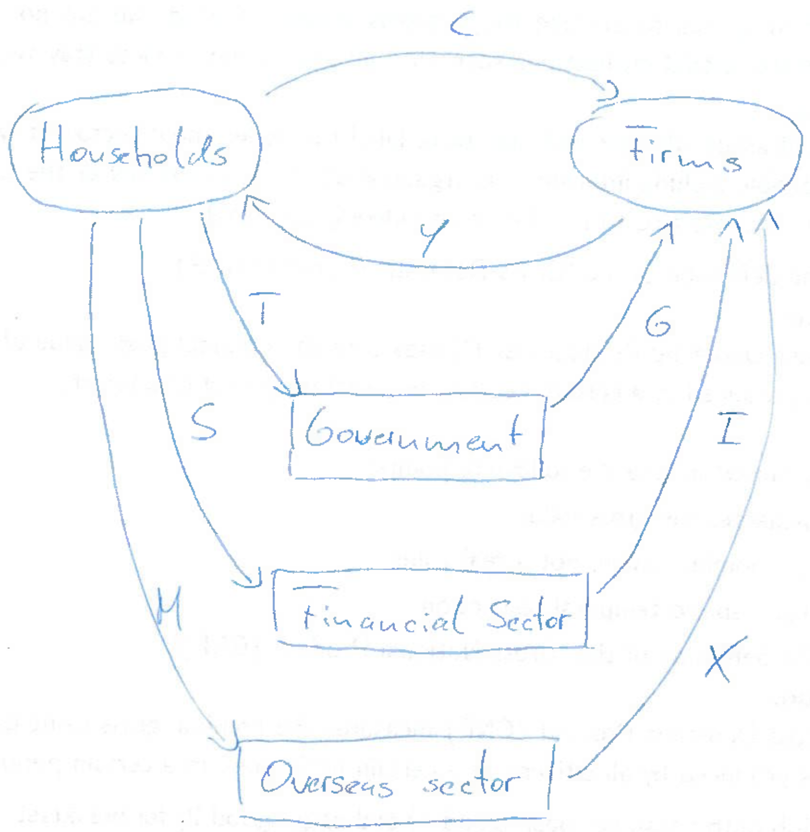
There are several other possibilities. One is to introduce an overall production cap. Another is to introduce a special tax on coal.

4. (a) (i) Sketch and describe the circuit flow of income of a national economy comprising 5 sectors.

5 marks
(A)
seen

Solution:

A sketch of the 5 sector model can be made as follows:



The arrows describe flows of *money* rather than goods. The legend is as follows:

- C: Consumption expenditure
- Y: Income from labour and capital (wages, rents, dividents)
- T: Taxes
- G: Government spending on goods and services
- S: Savings
- I: Investment spending
- M: Import spending
- X: Export spending

2 marks (B) seen

(ii) How would you change the circuit flow of income described in part (i) if you were to describe the global economy. Justify your answer.

Solution:
One definitely has to exclude the overseas sector. Indeed, we are not (yet) trading with extraterrestrial civilisations such that all goods and services stay within the world economy.
One could argue whether and how to re-label the government sector. It would probably be reasonable include international organisations such as the UN or the EU. But in the end this amounts also to a political and ideological debate.

3 marks (A) seen

(b) (i) State the definition of the 'Gross Domestic Product (GDP)'.

Solution:
The Gross Domestic Product (GDP) measures the nominal gross value of all goods and services produced in a certain country in a certain period of interest.

What is important are the following points:

- It measures the gross value.
- It is a nominal value, not a real value.
- It has a spatio-temporal restriction.

2 marks (A) seen

(ii) State the definition of the 'Gross National Product (GNP)'.

Solution:
The Gross Domestic Product (GNP) measures the nominal gross value of all goods and services produced by all citizens of a certain nationality in a certain period of interest.

(c) Anne and Bob both consume toast (good 1) and eggs (good 2) for breakfast. However, their behaviour differs considerably: While Anne insists on having exactly as many slices of toast as eggs, Bob does not care too much about the proportion of eggs and toast. Both of them prefer to have more of any good than less.

2 marks (C) unseen

(i) Give ordinal utility functions $u_A\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ and $u_B\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ which represent Anne's and Bob's preferences, respectively.

Solution:
$u_A(x_1, x_2) = \min\{x_1, x_2\}$ and $u_B(x_1, x_2) = x_1 + x_2$. Any strictly increasing transformation of the respective utilities works as well.

2 marks (C / D) similar seen

(ii) Suppose you are Anne and Bob's parents. You have only four slices of toast and three eggs left to prepare their breakfast. Is there a way to distribute the toast and eggs which maximises the total utility? If so, describe and justify the allocation. If not, justify and explain your answer.

Solution:
There is no way how to aggregate the two ordinal utility functions. The underlying problem is that one can transform the utilities u_A and u_B with any strictly increasing transformation and they still represent the same preferences.

2 marks
(A)
seen

(iii) Give the definition of a Pareto efficient allocation.

Solution:

A Pareto improvement is a change that makes at least one person better off, without making any other person worse of. An allocation is Pareto efficient if no Pareto improvement is possible.

2 marks
(D)
unseen

(iv) Determine all Pareto efficient allocations of toast and eggs, given Anne's and Bob's preferences.

Solution:

The definition of Pareto efficiency implies that one should give all slices of toast and all eggs to Bob and Anne. Indeed, if something is left, one could always give it to Bob and increase his utility without decreasing Anne's utility. Moreover, Anne should get exactly as many slices of toast as eggs. Indeed, if she has strictly more of one of the goods, one could take away the excess and give it to Bob. This would not decrease her utility, but increase his utility. That means the Pareto optimal allocations are of the form that Anne gets x slices of toast and x eggs for $0 \leq x \leq 3$ and Bob gets $4 - x$ slices of toast and $3 - x$ eggs. Indeed, one can see that these allocations are Pareto efficient, and we have seen that no other allocation can be Pareto efficient.

[I do not expect such a detailed answer – even though it would be nice, of course. Moreover, I do not mind them only stating the integer valued Pareto efficient allocations.]