Imperial College London

MATH70002

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Fluid Dynamics 2

Date: 31 May 2023

Time: 14:00 – 16:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

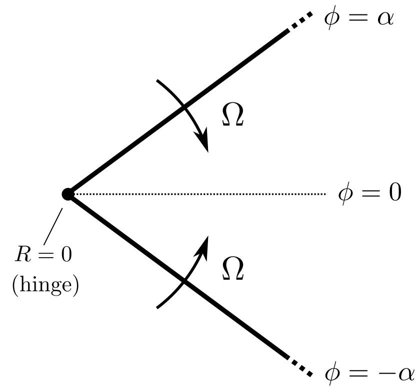
Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. In cylindrical coordinates (R, ϕ, z), consider viscous fluid in the region $-\alpha < \phi < \alpha$, between two semi-infinite plates at $\phi = -\alpha$ and $\phi = +\alpha$. The plates are on a hinge at the origin $R = 0$ and rotate towards each other at a constant angular velocity Ω.



The instantaneous flow velocity, $\mathbf{u} = (u_R(R, \phi), u_\phi(R, \phi), 0)$, is 2D and independent of z, with the boundary conditions on the plates:

$$u_R = 0 \quad \text{and} \quad u_\phi = \begin{cases} -\Omega R & \phi = +\alpha \\ \Omega R & \phi = -\alpha \end{cases}$$

Assume the flow is governed by the Stokes equations, $\nabla p = \mu \nabla^2 \mathbf{u}$ and $\nabla \cdot \mathbf{u} = 0$.

(a) Using the formulation in terms of the streamfunction $\psi(R, \phi)$ and vorticity ω given by

$$\mathbf{u} = \nabla \times (\psi \hat{\mathbf{z}}) = \left(\frac{1}{R} \frac{\partial \psi}{\partial \phi}, -\frac{\partial \psi}{\partial R}, 0 \right), \qquad \omega = -\nabla^2 \psi,$$

show that ψ satisfies $\nabla^2 \nabla^2 \psi = 0$. (2 marks)

(b) Derive boundary conditions on ψ, making the choice $\psi = 0$ at $R = 0$. (4 marks)

(c) Assuming ψ takes the form $\psi = \Omega R^n f(\phi)$ for an appropriate choice of n that you should state, show that $f(\phi)$ satisfies

$$f'''' + 4f'' = 0 \quad \text{for } -\alpha < \phi < \alpha$$
$$f(\pm\alpha) = \pm 1/2, \quad f'(\pm\alpha) = 0$$

where $f'''' = d^4 f / d\phi^4$. (6 marks)

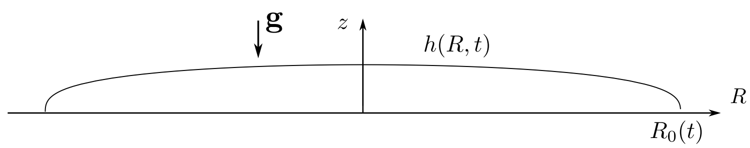
(d) Find the solution for $f(\phi)$. Is there a value of α for which the solution breaks down? (8 marks)

[*You may use that the Laplacian in cylindrical coordinates, for χ independent of z, is*

$$\nabla^2 \chi = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \chi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \chi}{\partial \phi^2}. \quad]$$

(Total: 20 marks)

2. A fluid of density ρ and viscosity μ spreads axisymmetrically in a thin layer under gravity on a horizontal plate $z = 0$. In cylindrical coordinates (R, ϕ, z), the height of the liquid surface is $z = h(R, t)$, and there is no variation in the ϕ direction. The leading edge of the layer where $h = 0$ is at radius $R = R_0(t)$ at time t.



The velocity $\mathbf{u} = (u(R, z, t), 0, w(R, z, t))$ and pressure p satisfy the lubrication equations

$$0 = -\frac{\partial p}{\partial R} + \mu \frac{\partial^2 u}{\partial z^2}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g, \qquad \frac{1}{R}\frac{\partial}{\partial R}(Ru) + \frac{\partial w}{\partial z} = 0$$

with $u = w = 0$ on $z = 0$ and $\mu \partial u/\partial z = 0$, $p = p_0$ (a constant) on $z = h$.

(a) Find p and u in terms of the layer thickness $h(R, t)$. (5 marks)

(b) By integrating the mass conservation equation from $z = 0$ to $z = h$, show that h satisfies

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{1}{R} \frac{\partial}{\partial R}\left(R h^3 \frac{\partial h}{\partial R} \right).$$

[*You may use that the kinematic condition at $z = h$ in cylindrical coordinates is*

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial R} = w. \quad]$$

(5 marks)

(c) Consider a self-similar solution of the form

$$h(R, t) = A t^a f(\eta) \quad \text{where } \eta = B R t^b$$

where a, b, A, B are constants.

(i) Assuming the fluid volume $V = \int_0^{R_0(t)} 2\pi R h \, \mathrm{d}R$ is constant in time, show that this implies $a = 2b$ and, under a suitable choice of A and B, that

$$\int_0^{\eta_0} \eta f(\eta) \, \mathrm{d}\eta = 1,$$

where η_0 is the value of η at $R = R_0$. (4 marks)

(ii) You may assume there is a solution with $a = -1/4$, $b = -1/8$ (and some choice of A and B) such that $f(\eta)$ satisfies

$$-\frac{1}{4}\eta f - \frac{1}{8}\eta^2 f' = \left(\eta f^3 f'\right)' \quad \text{and} \quad f(\eta_0) = 0.$$

Integrate this equation and show that $f(\eta) = \left(\frac{3}{16}\right)^{1/3} (\eta_0^2 - \eta^2)^{1/3}$. Derive an expression for η_0 (which you do not have to write in its simplest form). (6 marks)

(Total: 20 marks)

3. Viscous fluid occupies the region $y > 0$ above a solid boundary at $y = 0$. Fluid is sucked out via a sink at $x = 0$, $y = 0$, with a constant volume flow rate. Assuming it is a 2D high Reynolds number flow, away from $y = 0$ the flow is approximately inviscid with the slip velocity as $y \to 0$ for $x > 0$ given by

$$U(x) = -\frac{U_0}{x}$$

for a constant $U_0 > 0$. Consider the viscous boundary layer near $y = 0$ on one side of the sink, $x > 0$, governed by the boundary layer equations

$$\begin{aligned} uu_x + vu_y &= U(x)U'(x) + \nu u_{yy}, \\ u_x + v_y &= 0. \end{aligned}$$

(a) Briefly describe the assumptions needed to derive these equations. What are the boundary conditions? (5 marks)

(b) Seek a similarity solution for the streamfunction $\psi(x, y)$ of the form

$$\psi(x, y) = (U_0\nu)^{1/2} x^a \phi(\eta) \quad \text{where} \quad \eta = \left(\frac{U_0}{\nu}\right)^{1/2} y\, x^b$$

Show that

$$a = 0, \qquad b = -1,$$

and hence that ϕ satisfies

$$\phi''' + (\phi')^2 - 1 = 0,$$

and state the boundary conditions. (9 marks)

(c) You can assume that there is a solution to the above ODE which takes the form

$$\phi' = 2 - 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + C\right)$$

for a constant $C > 0$. Write down an equation for C. (2 marks)

Using this solution for ϕ', sketch carefully the velocity profile u for 2 different values of $x > 0$. (4 marks)

(Total: 20 marks)

4. An inviscid fluid of density ρ occupying $-\infty < y < \infty$ has a vortex sheet at $y = 0$. That is, in region 1 ($y > 0$), the fluid moves with uniform speed U_1 to the right, and in region 2 ($y < 0$) it moves with uniform speed U_2 to the right, with $U_2 \neq U_1$. As the fluid in both regions is the same, there is no surface tension on the boundary separating them.

Consider a small perturbation of the boundary between the two regions to $y = \varepsilon h(x, t)$ where ε is a small constant.

(a) Assume the velocity in each region is 2D, irrotational, decays as $y \to \pm\infty$, and takes the form

$$\mathbf{u} = \begin{cases} (U_1, 0) + \varepsilon \nabla \phi_1 & \text{region 1} \\ (U_2, 0) + \varepsilon \nabla \phi_2 & \text{region 2.} \end{cases}$$

Write down governing equations and the far field boundary conditions for ϕ_1, ϕ_2. By linearising in ε, show that the conditions on the boundary reduce to (now on $y = 0$):

$$\frac{\partial h}{\partial t} + U_1 \frac{\partial h}{\partial x} = \frac{\partial \phi_1}{\partial y}, \qquad \frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x},$$

$$\frac{\partial h}{\partial t} + U_2 \frac{\partial h}{\partial x} = \frac{\partial \phi_2}{\partial y}.$$

(6 marks)

(b) Consider interface perturbations of the form $h = \mathrm{e}^{\mathrm{i}kx + st}$ (where it is understood that only the real part is taken).

(i) Show that s satisfies

$$(s + U_1 \mathrm{i}k)^2 + (s + U_2 \mathrm{i}k)^2 = 0.$$

(6 marks)

(ii) If k is real and positive, show that the flow is unstable as long as $U_1 \neq U_2$. (4 marks)

(iii) Set $U_1 = 1$, $U_2 = 0$ for simplicity. Now consider s to be purely imaginary, $s = \mathrm{i}\omega$, meaning k is complex. This corresponds to an oscillation of frequency ω applied to the interface at the location $x = 0$. From the dispersion relation you found in part (i), find $\mathrm{Im}(k)$ in terms of ω. What happens to the flow as you move downstream, $x \to +\infty$?

(4 marks)

[*You may use that the unsteady Bernoulli equation is*

$$p + \rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gy \right) = f(t), \quad f \text{ an arbitrary function of } t \,]$$

(Total: 20 marks)

5. Discuss how the "thin layer" assumption, where the scales of variation are significantly different in different directions, can be used to analyse viscous flow. Consider the cases where inertia is negligible and where inertia is important. (20 marks)

(Total: 20 marks)

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MATH60002/70002

Fluid Dynamics 2 (Solutions)

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1. **Seen Stokes flow in a corner with a sink at $R = 0$, not for moving plates.** $\boxed{\text{seen} \Downarrow}$

(a) Taking the curl of $\nabla p = \mu \nabla^2 \mathbf{u}$, and using that $\nabla \times \nabla p = 0$ and $\nabla \times \mathbf{u} = \omega\,\hat{\mathbf{z}}$, we have

$$\nabla^2 \omega = 0.$$

Substituting $\omega = -\nabla^2 \psi$ gives

$$\nabla^2 \nabla^2 \psi = 0$$

$\boxed{2, \text{A}}$

(b) Using $\partial\psi/\partial\phi = Ru_R$ and $\partial\psi/\partial R = -u_\phi$, the no-slip boundary conditions that are given become $\boxed{\text{sim. seen} \Downarrow}$

$$\frac{\partial \psi}{\partial \phi} = 0 \qquad \text{at } \phi = \pm\alpha,$$
$$\frac{\partial \psi}{\partial R} = \pm R\Omega \qquad \text{at } \phi = \pm\alpha,$$

but the derivative $\partial\psi/\partial R$ is directed along, i.e. tangent to, the boundary so those conditions can be integrated. Integrating the latter conditions with respect to R and choosing $\psi = 0$ at $R = 0$ to fix the constant, the full set of conditions are $\boxed{2, \text{A}}$

$$\frac{\partial \psi}{\partial \phi} = 0 \qquad \text{at } \phi = \pm\alpha,$$
$$\psi = \pm\frac{1}{2}\Omega R^2 \quad \text{at } \phi = \pm\alpha.$$

$\boxed{2, \text{A}}$

(c) Assuming $\psi = \Omega R^n f(\phi)$, choose $n = 2$ so that the boundary conditions for f will be independent of R. $\boxed{\text{sim. seen} \Downarrow}$

Substituting into $\omega = -\nabla^2\psi$ and using the Laplacian in cylindrical coordinates, $\boxed{2, \text{C}}$

$$\begin{aligned}\omega &= -\Omega\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial(R^2 f)}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2(R^2 f)}{\partial \phi^2}\right] \\ &= -\Omega(4f + f'').\end{aligned}$$

since $f(\phi)$ is only a function of ϕ. Notice ω is independent of R, so substituting into $\nabla^2\omega = 0$ gives

$$0 = -\frac{\Omega}{R^2}(4f + f'')'' \qquad \Longrightarrow \qquad 0 = 4f'' + f''''.$$

$\boxed{2, \text{B}}$

On the boundaries, $\psi(R, \pm\alpha) = \pm\frac{1}{2}\Omega R^2 = \Omega R^2 f(\pm\alpha)$ implies $f(\pm\alpha) = \pm 1/2$, and $\partial\psi/\partial\phi = \Omega R^2 f'(\pm\alpha) = 0$ implies $f'(\pm\alpha) = 0$. $\boxed{2, \text{B}}$

(d) Integrating the f equation twice gives $f'' + 4f = c\phi + d$ where c and d are constants. A particular solution to this equation is $f = (c\phi + d)/4 = C\phi + D$, and hence the general solution takes the form $\boxed{\text{sim. seen} \Downarrow}$

$$f = A\cos 2\phi + B\sin 2\phi + C\phi + D, \quad \text{with } A, B, C, D \text{ constants.}$$

$\boxed{4, \text{C}}$

Noticing that the boundary conditions for f imply an odd solution in ϕ, set $A = D = 0$. Then B, C determined by $f(\alpha) = 1/2$ and $f'(\alpha) = 0$, which are

$$B \sin 2\alpha + C\alpha = 1/2, \qquad 2B \cos 2\alpha + C = 0.$$

Solving for B and C,

$$B = \frac{1}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)}, \qquad C = \frac{-2\cos 2\alpha}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)}.$$

giving the solution for f,

$$f = B \sin 2\phi + C\phi = \frac{\sin 2\phi - 2\phi \cos 2\alpha}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)}.$$

The solution is singular and hence breaks down when the denominator vanishes, i.e. $\sin 2\alpha - 2\alpha \cos 2\alpha = 0$, giving the equation $\tan 2\alpha = 2\alpha$ for the problematic angle α. $\boxed{4, \mathrm{A}}$

2. **Seen 2D layer spreading under gravity, but not axisymmetric case.** sim. seen $\Downarrow$

(a) Integrating the z-momentum equation $\partial p/\partial z = -\rho g$ and applying the boundary condition $p = p_0$ on the surface $z = h$ gives the pressure

$$p = p_0 + \rho g[h(R,t) - z].$$

Then substituting p into the R-momentum equation gives 2, A

$$\mu \frac{\partial^2 u}{\partial z^2} = \rho g \frac{\partial h}{\partial R}.$$

The RHS is independent of z, so this can be integrated easily with respect to z. Integrating once and applying the condition $\partial u/\partial z = 0$ at $z = h$ gives

$$\frac{\partial u}{\partial z} = \frac{\rho g}{\mu}\frac{\partial h}{\partial R}(z - h).$$

Integrating again and applying no-slip $u = 0$ on $z = 0$ gives

$$u = \frac{\rho g}{2\mu}\frac{\partial h}{\partial R}(z^2 - 2hz).$$

3, A

(b) Integrating the conservation of mass equation over the layer in z gives sim. seen $\Downarrow$

$$0 = \frac{1}{R}\int_0^h \frac{\partial}{\partial R}(Ru)\,\mathrm{d}z + [w]_{z=0}^{z=h}$$

But $w = 0$ on $z = 0$, and substituting the (given) kinematic condition for w on $z = h$, we find

$$0 = \frac{1}{R}\int_0^h \frac{\partial}{\partial R}(Ru)\,\mathrm{d}z + \frac{\partial h}{\partial t} + u|_{z=h}\frac{\partial h}{\partial R}.$$

But using the Leibniz integral rule,

$$\frac{\partial}{\partial R}\int_0^h Ru\,\mathrm{d}z = \int_0^h \frac{\partial}{\partial R}(Ru)\,\mathrm{d}z + Ru|_{z=h}\frac{\partial h}{\partial R}$$

which allows the preceding equation to be written

$$\begin{aligned} 0 &= \frac{1}{R}\frac{\partial}{\partial R}\int_0^h Ru\,\mathrm{d}z + \frac{\partial h}{\partial t} \\ &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\int_0^h u\,\mathrm{d}z\right) + \frac{\partial h}{\partial t} \end{aligned}$$

We can now substitute the lubrication approximation for u from part (a) which has

$$\int_0^h u\,\mathrm{d}z = -\frac{\rho g}{3\mu}h^3\frac{\partial h}{\partial R}$$

giving the required equation for h. 5, B

(c) (i) Substituting the solution of the form sim. seen $\Downarrow$

$$h = At^a f(\eta), \qquad \eta = BRt^b$$

into the volume constraint, and using that

$$\mathrm{d}R = \frac{\mathrm{d}\eta}{Bt^b}$$

we find

$$V = \int_0^{R_0} 2\pi R h \, \mathrm{d}R = \frac{2\pi A}{B^2} t^{a-2b} \int_0^{\eta_0} \eta f \, \mathrm{d}\eta$$

where $\eta_0 = BR_0(t)t^b$. Since V is constant, we require the RHS to be independent of t, and hence need $a = 2b$. Also we can choose A,B such that $V = 2\pi A/B^2$, then this integral reduces to

$$\int_0^{\eta_0} \eta f \, \mathrm{d}\eta = 1$$

as required. $\boxed{4, \mathrm{C}}$

(ii) Notice that the LHS of the equation can be written $-\frac{1}{4}\eta f - \frac{1}{8}\eta^2 f' = -\frac{1}{8}(\eta^2 f)'$, which is an exact derivative. Then integrating the equation once, $\boxed{\text{sim. seen} \Downarrow}$

$$-\frac{1}{8}\eta^2 f = \eta f^3 f' + C$$

but since we need $f = 0$ at the leading edge η_0 (and assuming that $f^3 f' \to 0$ there), we must have $C = 0$. Cancelling factors,

$$-\frac{1}{8}\eta = f^2 f' \qquad \Longrightarrow \qquad -\frac{1}{8}\eta = \left(\frac{1}{3} f^3\right)'$$

which if we integrate again and impose that $f(\eta_0) = 0$, we arrive at $f = \left(\frac{3}{16}\right)^{1/3} (\eta_0^2 - \eta^2)^{1/3}$, as required. $\boxed{3, \mathrm{D}}$

Determine η_0 from the integral constraint,

$$\begin{aligned} 1 = \int_0^{\eta_0} \eta f \, \mathrm{d}\eta &= \left(\frac{3}{16}\right)^{1/3} \int_0^{\eta_0} \eta(\eta_0^2 - \eta^2)^{1/3} \, \mathrm{d}\eta \\ &= \frac{3}{4} \left(\frac{3}{16}\right)^{1/3} \left(-\frac{1}{2}\right) \left[(\eta_0^2 - \eta^2)^{4/3}\right]_0^{\eta_0} \\ &= \frac{3}{8} \left(\frac{3}{16}\right)^{1/3} \eta_0^{8/3} \end{aligned}$$

or

$$\eta_0 = \left(\frac{8}{3}\right)^{3/8} \left(\frac{16}{3}\right)^{1/8}$$

(Any correct expression for η_0 will get full credit, even if not simplified.) $\boxed{3, \mathrm{D}}$

3. **Seen Falkner-Skan derivation. Not seen for the outer flow due to a sink.** seen $\Downarrow$

(a) We assume the boundary layer to be thin, so that if the coordinates scale as $y \sim \delta$ and $x \sim L$ then $\delta \ll L$. Hence, we assume that $\partial/\partial y \gg \partial/\partial x$ and we can neglect the viscous term $\nu u_{xx} \ll \nu u_{yy}$. Also, $v \sim \delta u/L \ll u$ from mass conservation. The layer thickness δ is such that inertia and viscosity effects balance (Prandtl's boundary layer hypothesis). Pressure is approximately constant across the layer, and so we can use the pressure gradient $\rho U U'$ from the free stream. 3, A

The boundary conditions are no-slip on the wall,

$$u = v = 0 \quad \text{on } y = 0,\, x > 0$$

and a matching condition to the slip flow,

$$u \to U(x) \quad \text{as } y \to \infty$$

2, A

(b) If sim. seen $\Downarrow$

$$\psi = (U_0\nu)^{1/2} x^a \phi(\eta) \qquad \text{where } \eta = \left(\frac{U_0}{\nu}\right)^{1/2} y x^b$$

then the derivatives we need are

$$\begin{aligned}
u = \psi_y &= (U_0\nu)^{1/2} x^{a+b} \left(\frac{U_0}{\nu}\right)^{1/2} \phi' = U_0 x^{a+b}\phi' \\
u_x = \psi_{yx} &= U_0 x^{a+b-1}\left((a+b)\phi' + b\eta\phi''\right) \\
u_y = \psi_{yy} &= U_0 \left(\frac{U_0}{\nu}\right)^{1/2} x^{a+2b}\phi'' \\
u_{yy} = \psi_{yyy} &= \frac{U_0^2}{\nu} x^{a+3b}\phi''' \\
v = -\psi_x &= -(U_0\nu)^{1/2} x^{a-1}(a\phi + \eta b\phi')
\end{aligned}$$

Substituting these into the x-momentum boundary layer equation, 4, B

$$\begin{aligned}
U_0^2 x^{2a+2b-1}\phi'((a+b)\phi' + b\eta\phi'') - U_0^2 x^{2a+2b-1}(a\phi + \eta b\phi')\phi'' \\
= -U_0^2 x^{-3} + U_0^2 x^{a+3b}\phi'''
\end{aligned}$$

In order for the powers of x to cancel out and the equation be independent of x, we must have $2a + 2b - 1 = -3 = a + 3b$, which simplify to $a + b = -1$ and $a + 2b = -3$. Solving this pair of equations gives that

$$b = -1 \qquad \text{and} \qquad a = 0$$

Then the above equation for ϕ simplifies (and the factors of U_0^2 cancel), giving

$$-\phi'(\phi' + \eta\phi'') + \eta\phi'\phi'' = -1 + \phi'''$$

or rearranging

$$\phi''' + (\phi')^2 - 1 = 0.$$

2, C

The no-slip boundary conditions become

$$\phi' = \phi = 0 \quad \text{on } \eta = 0$$

and the matching condition $u = U_0 x^{-1}\phi' \to U = -U_0 x^{-1}$ gives

$$\phi' \to -1 \quad \text{as } \eta \to \infty$$

$\boxed{3, \mathrm{B}}$

(c) We are told that the given ϕ' satisfies the differential equation. It also satisfies the matching condition since $\boxed{\text{unseen} \Downarrow}$

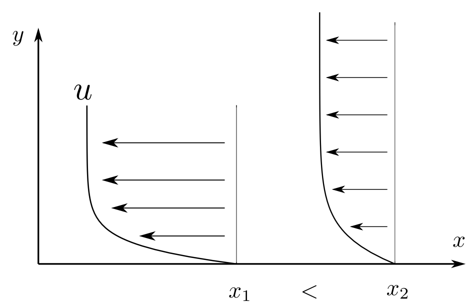
$$\phi' = 2 - 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + C\right) \to 2 - 3 = -1 \quad \text{as } \eta \to \infty$$

To satisfy $\phi'(0) = 0$ we must have $2/3 = \tanh^2 C$, or taking the positive square root since $C > 0$, we find $\tanh C = \sqrt{2/3}$, determining C. $\boxed{2, \mathrm{D}}$

The velocity u in terms of the solution above is

$$u = \psi_y = \frac{U_0}{x}\phi'\left(\frac{y}{x}\right)$$

Notice ϕ' is monotonic and goes from 0 (at $y = 0$) to -1 (at $y = \infty$). Going towards the sink, the profile is squished in y (the layer thickness scales as $\sim x$), but the velocity increases. Sketches at two positive x values are below (any drawings showing these qualitative features are accepted).



$\boxed{4, \mathrm{D}}$

4. **Seen Kelvin-Helmholtz instability. Mentioned but not seen spatial instability in detail.** $\boxed{\text{seen} \Downarrow}$

(a) If the velocity is given by $\mathbf{u} = (U_j, 0, 0) + \varepsilon \nabla \phi_j$ in each region $j = 1, 2$, then $\nabla \cdot \mathbf{u} = 0$ gives that each ϕ_j is harmonic,

$$\begin{cases} \nabla^2 \phi_1 = 0 & \text{region 1} \\ \nabla^2 \phi_2 = 0 & \text{region 2.} \end{cases}$$

If the pertubation velocity $\nabla \phi_j$ should decay far away from the interface, then we require (picking the arbitrary constants to be zero)

$$\begin{cases} \phi_1 \to 0 & y \to \infty \\ \phi_2 \to 0 & y \to -\infty \end{cases}$$

On the interface $y = \varepsilon h$, the kinematic condition(s) on the flow in each region $j = 1, 2$ is:

$$0 = \frac{D}{Dt}(\varepsilon h - y) = \varepsilon \frac{\partial h}{\partial t} + \varepsilon U_j \frac{\partial h}{\partial x} - \varepsilon \frac{\partial \phi_j}{\partial y} + O(\varepsilon^2)$$

Taylor expanding terms onto $y = 0$, and only keeping terms leading order in ε, we have

$$\text{on } y = 0 \begin{cases} \dfrac{\partial h}{\partial t} + U_1 \dfrac{\partial h}{\partial x} = \dfrac{\partial \phi_1}{\partial y} \\ \dfrac{\partial h}{\partial t} + U_2 \dfrac{\partial h}{\partial x} = \dfrac{\partial \phi_2}{\partial y} \end{cases}$$

The pressure perturbation in each region, ϵp_j, should also match at $y = \varepsilon h$, meaning $p_1 = p_2$. Using the unsteady Bernoulli equation in each region, evaluating at $y = \varepsilon h$ (and choosing the arbitrary function to be zero),

$$\varepsilon p_j + \rho \left(\varepsilon \frac{\partial \phi_j}{\partial t} + \varepsilon U_j \frac{\partial \phi_j}{\partial x} + O(\epsilon^2) + \varepsilon g h \right) = 0, \quad j = 1, 2.$$

Neglect terms $O(\epsilon^2)$, the condition $p_1 = p_2$ can be written

$$\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x}$$

To leading order in ε, this can also be imposed on $y = 0$ instead of $y = \varepsilon h$. $\boxed{6, \text{A}}$

(b) (i) Considering $h = \mathrm{e}^{\mathrm{i}kx+st}$, then look for ϕ_1, ϕ_2 in the form $\boxed{\text{seen} \Downarrow}$

$$\phi_1(x, y, t) = \Phi_1(y)\mathrm{e}^{\mathrm{i}kx+st},$$
$$\phi_2(x, y, t) = \Phi_2(y)\mathrm{e}^{\mathrm{i}kx+st},$$

Laplace equation for ϕ_1 reduces to

$$\frac{d^2\Phi_1}{dy^2} - k^2\Phi_1 = 0 \qquad \Longrightarrow \qquad \Phi_1 = \Phi_1(0)\mathrm{e}^{-ky} \quad \text{(satisfying decay as } y \to \infty\text{)}$$

Similarly, $\Phi_2 = \Phi_2(0)\mathrm{e}^{ky}$. The kinematic conditions become

$$s + U_1\mathrm{i}k = -k\Phi_1(0), \qquad s + U_2\mathrm{i}k = k\Phi_2(0),$$

and the pressure continuity condition,

$$(s + U_1\mathrm{i}k)\Phi_1(0) = (s + U_2\mathrm{i}k)\Phi_2(0).$$

Eliminating $\Phi_1(0), \Phi_2(0)$, we arrive at $(s + U_1\mathrm{i}k)^2 + (s + U_2\mathrm{i}k)^2 = 0.$ $\boxed{6, \text{A}}$

(ii) Expanding the dispersion relation, $\boxed{\text{sim. seen} \Downarrow}$

$$s^2 + (U_1 + U_2)\mathrm{i}ks - (U_1^2 + U_2^2)k^2/2 = 0$$

and solving for s we find two solutions,

$$s = -\frac{(U_1 + U_2)}{2}\mathrm{i}k \pm \frac{(U_1 - U_2)}{2}k.$$

For $k > 0$ real and $U_1 \neq U_2$, then one of the roots always has $\mathrm{Re}(s) > 0$, meaning the flow is unstable. $\boxed{4, \text{B}}$

(iii) Setting $U_1 = 1$, $U_2 = 0$ and $s = \mathrm{i}\omega$, rearranging the above relation between s and k, $\boxed{\text{unseen} \Downarrow}$

$$k = \frac{2\mathrm{i}\omega}{-\mathrm{i} \pm 1} = \sqrt{2}\omega(-1 \pm \mathrm{i})$$

So $\mathrm{Im}(k) = k_i = \pm\sqrt{2}\omega$. This means $\mathrm{e}^{\mathrm{i}kx+\mathrm{i}\omega t} = \mathrm{e}^{-k_i x}\mathrm{e}^{\mathrm{i}k_r x+\mathrm{i}\omega t}$, and thus one of the solutions (and hence the perturbation) grows exponentially downstream, $x \to \infty$. $\boxed{4, \text{D}}$

5. Open ended essay question. Anything relevant from the course or even outside the course material will earn credit. For example, discussion of the lubrication approximation (with or without a free surface), and the reduction to single one-dimensional models (inertia negligible if the layer is thin enough, “reduced Reynolds number”, etc.). Also discussion of boundary layers where inertia is important and balances viscosity. $\boxed{20, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60002/70002	1	Generally answered well. Boundary conditions were not always simplified in (b) but full marks were given. Many did not spot immediately that n=2 from the boundary conditions, but from the equation for f instead, making the calculation longer. Small errors common in (d) when determining constants, missing factors of 2 or alpha, but most found the solution.
MATH60002/70002	2	Overall the students found this question difficult. Part (a) answered well, but in part (b) almost no one used the Leibniz rule (used in the lectures), which lengthened the calculation considerably. Order of integration and differentiation were commonly swapped incorrectly. Part (c)(i) some struggled, differentiating V in time unnecessarily, or deriving the equation for f (which was given). Part (c)(ii) done well, but some missed that last part.
MATH60002/70002	3	Parts (a) and (b) were answered well, as they were basically Falkner-Skan derivation done in lectures. But part (b) likely took them too long. Part (c) was difficult. Students struggled understanding what to sketch, but some students did it correctly.
MATH60002/70002	4	Even though most of the question was bookwork from the lecture notes, many students struggled, perhaps due to lack of time. Mostly part (a) was answered well, but part (b) was difficult. Part (b)(i): Many did not attempt to solve Laplace's equation for the potentials, and tried just manipulating the boundary conditions. Few understood the stability criterion in (b)(ii). Part (b)(iii) was unseen material and many could do the calculation, but not interpret the result.
MATH70002	5	Well answered, students understood the topic being asked about and the level of detail required in their answers.