Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship.

M4/5 S4

Applied Probability (Solutions to Mastery Question)

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1. This question is based on the following additional reading material:
U. N. Bhat (2008), *An Introduction to Queueing Theory*, Birkhäuser: Chapter 4, Sections 4.1, 4.2 and 4.3, pp. 29–51.
The comments "seen" and "unseen" refer to seen/unseen in lectures. All the material in that question is covered in the additional reading material and hence should have been seen by the students.

unseen ⇓

(a) The number of people in a system in a queueing model corresponds to the population size in a birth-death process. The arrivals correspond to births and the departures correspond to deaths. 2

(b) The limiting distribution $\lim_{t\to\infty} p_{in}(t) = p_n$ (for any state i in the state space) exists and is independent of the initial conditions of the process. The limits are such that they either vanish identically or all are positive and form a probability distribution. seen ⇓ 2

(c) (i) M/M/1 stands for a queue characterised by Markovian arrival, Markovian departure and one server. I.e. arrivals occur according to a Poisson process of rate $\lambda > 0$. I.e. the inter-arrival times are Exp(λ)-distributed. The service times are Exp(μ)-distributed for $\mu > 0$. unseen ⇓ 2

(ii) The traffic intensity is defined as the ratio of arrival rate to service rate, i.e. $\rho = \lambda/\mu$. 1

(iii) The generator is given by

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

1

For $n \in \mathbb{N}_0$ define $p_n(t) = \mathbb{P}(Q(t) = n)$, where $Q(t)$ is the number of people in the system at time $t \geq 0$. The Kolmogorov forward equations are given by

$$p_n'(t) = \sum_{l\in\mathbb{N}_0} p_l(t) g_{ln},$$

i.e.

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t),$$
$$p_n'(t) = \lambda p_{n-1}(t) - (\lambda+\mu) p_n(t) + \mu p_{n+1}(t), \quad n \in \mathbb{N}.$$

2

(iv) For the limiting distributions, we set $\lim_{t\to\infty} p_n(t) = \widehat{p}_n$. Also, we have that $\lim_{t\to\infty} p_n'(t) = 0$. So, we need to solve

$$0 = -\lambda \widehat{p}_0 + \mu \widehat{p}_1,$$
$$0 = \lambda \widehat{p}_{n-1} - (\lambda+\mu)\widehat{p}_n + \mu \widehat{p}_{n+1}, \quad n \in \mathbb{N}.$$

1

We find that

$$\widehat{p}_1 = \frac{\lambda}{\mu}\widehat{p}_0,$$
$$\widehat{p}_2 = (-\lambda \widehat{p}_0 + (\lambda+\mu)\widehat{p}_1)/\mu = (-\lambda\widehat{p}_0 + (\lambda+\mu)\frac{\lambda}{\mu}\widehat{p}_0)/\mu = \frac{\lambda^2}{\mu^2}\widehat{p}_0.$$

I.e. (formally by induction) we find that $\widehat{p}_n = \frac{\lambda^n}{\mu^n}\widehat{p}_0$, for $n \in \mathbb{N}$. 2

Since

$$1 = \sum_{n \in \mathbb{N}_0} \widehat{p}_n = \widehat{p}_0 \left(1 + \sum_{n \in \mathbb{N}} \left(\frac{\lambda}{\mu} \right)^n \right) = \widehat{p}_0 \frac{1}{1 - \lambda/\mu}, \text{ since } \frac{\lambda}{\mu} < 1.$$

Hence for $\rho = \lambda/\mu$, we find that $\widehat{p}_n = \rho^n(1-\rho)$ for $n \in \mathbb{N}_0$. $\boxed{1}$

(v) A *busy period* is defined as the period of time during which the server is continuously busy. When it ends, an *idle period* follows. Together they form a *busy cycle*. $\boxed{1}$

(vi) We study the underlying Markov process and notice that the busy period is the duration of time that the process starting from state 1, stays continuously away from state 0. Note that the busy period starts with an arrival, hence we need to consider the amount of time the process takes to get back to state 0. When we consider the transitions of the Markov process, transitions within a busy period can be brought about by converting state 0 into an absorbing state and all other states into an irreducible transient class. $\boxed{1}$
Then the generator matrix takes the modified form

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

$\boxed{1}$

from which we get the following Kolmogorov forward equations:

$$p_0'(t) = \mu p_1(t),$$
$$p_1'(t) = -(\lambda+\mu)p_1(t) + \mu p_2(t),$$
$$p_n'(t) = -(\lambda+\mu)p_n(t) + \mu p_{n+1}(t) + \lambda p_{n-1}(t), \text{ for } n = 2, 3, \ldots.$$

$\boxed{1}$

$\boxed{\text{unseen} \Downarrow}$

(d) The generator is given by

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

where

$$\lambda_n = \lambda, \qquad \text{for } n = 0, 1, 2, \ldots,$$
$$\mu_n = \mu n, \qquad \text{for } n = 1, 2, \ldots, s-1,$$
$$\mu_n = \mu s, \qquad \text{for } n = s, s+1, \ldots.$$

$\boxed{2}$