CONFIDENCE INTERVALS

Case	Distribution	σ^2 known?	Method
1	F_X unknown	✓	Chebyshev's inequality
2	F_X normal	✓	"Z-scores" $N(0,1)$ table provided
3	F_X normal	✗	sample variance S^2: Student's t-dist.

Suppose $X_1, X_2, \ldots, X_n$ are known/assumed to be independent and follow normal distribution with unknown mean θ and known variance σ^2.

Goal: $(1-\alpha)$ - confidence interval for θ

Corollary: $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$$Z = \frac{\theta - \overline{X}}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$E[Z] = E\left[\frac{\theta - \overline{x}}{\sigma/\sqrt{n}}\right] = \frac{1}{\frac{\sigma}{\sqrt{n}}} E[\theta - \overline{X}]$$

$$= \frac{\sqrt{n}}{\sigma}\left(E(\theta) - E(\overline{X})\right)$$

$$= \frac{\sqrt{n}}{\sigma}(\theta - \theta)$$

$$\Rightarrow E[Z] = 0$$

$$Var(Z) = Var\left[\frac{\theta - \overline{x}}{\sigma/\sqrt{n}}\right] = \frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)^2} Var(\theta - \overline{X})$$

$$= \frac{n}{\sigma^2}\left[Var(-\overline{X})\right]$$

$$= \frac{n}{\sigma^2}\left[Var(\overline{X})\right]$$

$$= \frac{n}{\sigma^2}\left(\frac{\sigma^2}{n}\right)$$

$$\Rightarrow Var[Z] = 1$$

$$\boxed{Z = \frac{\theta - \overline{x}}{\sigma/\sqrt{n}}} \sim N(0,1)$$

(alternative : $W = \frac{\overline{X} - \theta}{\sigma/\sqrt{n}} \sim N(0,1)$)

$$P\left(L(x) < \theta < U(x)\right) = 1-\alpha$$

$$P\left(Z < z_{\alpha/2}\right) = \alpha/2$$

random variable

number

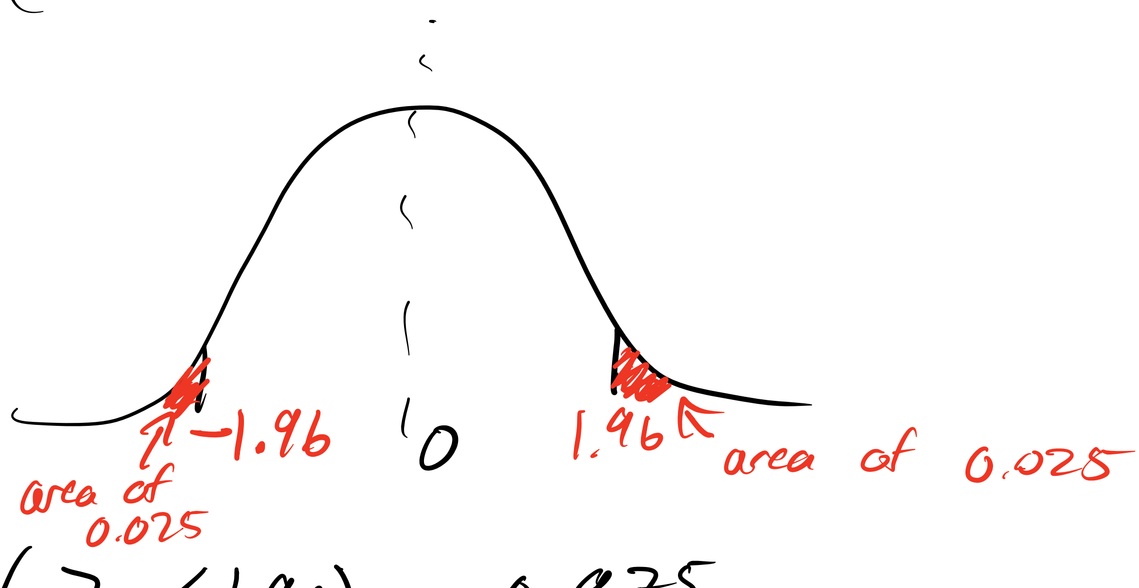
$$P\left(Z < z_{1-\alpha/2}\right) = 1-\alpha/2$$

Using table, $P(Z < 1.96) = 0.975$

(assuming we choose $\alpha = 0.05$)

$\Rightarrow 1-\alpha/2 = 0.975$

$$P(Z < -1.96) = \alpha/2 = 0.025$$



$$P(Z < 1.96) = 0.975$$

$$P(Z \leq 1.96) = 0.975$$

$$\Rightarrow P(Z > 1.96) = 0.025$$
$$(1 - 0.975)$$

$$P(-1.96 < Z < 1.96) = 0.975 - 0.025$$
$$= 0.95$$

$$P(z_{\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha$$

$$P\left(z_{\alpha/2} < \frac{\theta - \bar{X}}{\sigma/\sqrt{n}} < z_{1-\alpha/2}\right) = 1 - \alpha$$

$$P\left(z_{\alpha/2} \cdot \sigma/\sqrt{n} < \theta - \bar{X} < z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(\bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \theta < \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

Recall: $z_{\alpha/2} = -z_{1-\alpha/2}$

$$\Rightarrow P\left(\bar{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \theta < \bar{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$$
$$= 1 - \alpha$$

equivalent

$$\boxed{P\left(\bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \theta < \bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha}$$

$$P(Z < z_{1-\alpha/2}) = 1 - \alpha/2$$

$$P(Z < z_{\alpha/2}) = \alpha/2$$

$$P(z_{\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha$$

Example

$X_1, \dots, X_{25} \sim N(\theta, \sigma^2)$ $\quad \sigma^2 = 8$

We want a 99% confidence interval.

Suppose for $x_1, \dots, x_{25}$

$\bar{x} = 175$

$\Rightarrow 1-\alpha = 0.99$

$\Rightarrow \alpha = 0.01$

$\Rightarrow \alpha/2 = 0.005$

$\Rightarrow 1-\alpha/2 = 0.995$

$z_{1-\alpha/2} = z_{0.995} = 2.57$

(or 2.576)

$z_{\alpha/2} = -2.57$

$$P(z_{\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha$$

$$P\left(\bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \theta < \bar{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$\sigma^2 = 8$

$n = 25$

$\alpha = 0.01$

$z_{\alpha/2} = -2.57$; $z_{1-\alpha/2} = 2.57$

$$P\left(\bar{X} - 2.57\frac{\sqrt{8}}{\sqrt{25}} < \theta < \bar{X} + 2.57\frac{\sqrt{8}}{\sqrt{25}}\right) = 0.99$$

Realised confidence interval, using $\bar{x} = 175$

$$\left(175 - 2.57\frac{2\sqrt{2}}{5},\ 175 + 2.57\frac{2\sqrt{2}}{5}\right)$$