Network Science
Spring 2024
Problem sheet 4

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1. Assume that $f(N)$ tends to zero as $N \to \infty$ and take $p(N) = f(N)/N$. Prove that w.h.p. $G \in G_{Np}$ has no "squares", i.e. no (i, j, k, l) with links $i - j - k - l - i$ and i, j, k, l all distinct.
   Solution: Let $Q_N(G)$ be the number of squares. Each square has probability p^4. Summing over all squares:
   $$\langle Q_N \rangle = \binom{N}{4} 3p^4 = \frac{N(N-1)(N-2)(N-3)}{4} (\frac{f(N)}{N})^4 \to 0.$$
   Hence, by Markov's inequality $P(Q_N \geq 1) \leq \langle Q_N \rangle \to 0$, and:
   $$P(Q_N = 0) \to 1.$$
   Note that there are 3 distinct squares that can be constructed from 4 distinct nodes.

2. Show that if we let $p(N) = N^{-z}$ with $z > 2$ then w.h.p. all nodes in graphs generated by the G_{Np} model will be isolated (they will not have any links).
   This is a modified version of the last question on problem sheet 3
   Solution: Let X_i be the degree of node i and $K = X_1 + \cdots + X_N$. Then $\langle X_i \rangle = p(N-1)$. So $\langle K \rangle = p(N-1)N$. It would be sufficient to show that $P(K \geq 1) \to 0$ because this implies $P(K = 0) \to 1$ as $N \to \infty$. By the Markov inequality we have $P(K \geq 1) \leq \langle K \rangle = p(N-1)N$ and this goes to zero when $p(N) = N^{-z}$ when $z > 2$.