Mathematics Year 1, Calculus and Applications I, 2022

Takehome Assessment

(a) Look for a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$. We know that $y(0) = 0$ and so $a_0 = 0$, but let's keep it and set it to 0 as it arises.

The equation becomes

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = 1 + \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$= 1 + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right)x^n \qquad \text{2 marks}$$

The question essentially asks for the first three terms - you are not required to prove convergence. Of course a good guess for the radius of convergence would be $R < \pi/2$ given that $\tan x$ diverges at $x = \pm\pi/2$. But you don't need to do this here.

Equating coefficients of x^0 gives $a_1 = 1 + a_0^2$, i.e. $a_1 = 1$.

Equating coefficients of x^1 gives

$$2a_2 = a_0 a_1 + a_1 a_0 \qquad \Rightarrow \qquad a_2 = 0.$$

Keep going:

$$\underline{x^2}: \qquad 3a_3 = a_0 a_2 + a_1^2 + a_2 a_0 = a_1^2 \qquad \Rightarrow \qquad a_3 = \frac{1}{3}.$$

$$\underline{x^3}: \qquad 4a_4 = a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 = 0.$$

and finally

$$\underline{x^4}: \qquad 5a_5 = a_0 a_4 + a_1 a_3 + a_2 a_2 + a_3 a_1 + a_4 a_0 = \frac{2}{3}, \quad a_5 = \frac{2}{15}.$$

It is easy to see that even index constants are zero, $a_{2n} = 0$, but $a_{2n+1} \neq 0$ for odd indices.

3 marks

(b) Assuming we are within the radius of convergence, the power series represents a function $y(x)$ whose derivatives are given by the formula (**) are are assumed to exist. Then we know that

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n.$$

We are given $y(0) = 0$. From (**) $y'(0) = 1 + 0^2 = 1$.

Next, $y'' = 2yy'$ hence $y''(0) = 0$.

$y^{(3)} = 2(y')^2 + 2yy''$ and so $y^{(3)}(0) = 2$.

$y^{(4)} = 6y'y'' + 2yy^{(3)}$, so $y^{(4)}(0) = 0$.

Finally, $y^{(5)} = 6y'y^{(3)} + 6(y'')^2 + 2y'y^{(3)} + 2yy^{(4)}$, hence $y^{(5)}(0) = 16$.

Putting it together gives

$$y(x) = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \qquad \text{5 marks}$$