Imperial College London

Department of Mathematics

M3P5 Geometry of Curves & Surfaces

Question	Examiner's Comments
Q 1	Parts (a) and (b) mostly went well. For part (c), many people assumed that Ind(\gamma)=1, which is only true (up to orientation) if \gamma is a simple curve; instead, the fact that k>0 implies that the index is positive.
Q 2	Most people did well on this despite a few computational mistakes. Several people got no credit on part (c) because they tried to compare first fundamental forms, which depend very much on a choice of chart, rather than Gaussian curvature (via the Theorema Egregium).
Q 3	The most common mistake here was claiming that a surface with positive Euler characteristic must be a sphere — this is true if it is compact and without boundary, but not true otherwise since a compact disc has \chi = 1. Part (c) was quite hard; one should apply Gauss-Bonnet to a compact portion of S, say x^2+y^2 \leq 1, so that integration over this piece is well-defined.
Q 4	Each part was worth 2 points for the answer and 2 for the explanation. Part (c) asks whether geodesics are global minima of the length functional, but several people only argued that they are local minima. A few people claimed in (d) that if locally S lies on one side of the tangent plane at p then K(p) > 0; this is false (consider z=x^4+y^4 at the origin) but the converse is true.

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M45P5 Geometry of Curves & Surfaces

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Q 4	Each part was worth 2 points for the answer and 2 for the explanation. Part (c) asks whether geodesics are global minima of the length functional, but several people only argued that they are local minima. A few people claimed in (d) that if locally S lies on one side of the tangent plane at p then K(p) > 0; this is false (consider z=x^4+y^4 at the origin) but the converse is true.
Q 5	Nobody noticed on part (b) that this was an isothermal chart, and hence it would be enough to check that the chart is harmonic. For (c), checking that the derivative dN_p is injective only shows that the Gauss map is an immersion, i.e. *locally* injective rather than injective.

Imperial College London

M4/5P5

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Tuesday, 22 May 2018

Time: 10:00 AM - 12:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

1. (a) We define a regular curve $\phi : (-1,1) \to \mathbb{R}^3$ by $\phi(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$. Compute the curvature $k(t)$ and the torsion $\tau(t)$ for $-1 < t < 1$.

(b) Let $\psi : [a,b] \to \mathbb{R}^3$ be a curve parametrized by arc length, with $\psi(a) = (0,0,0)$, and suppose that its curvature is nonzero and its binormal vector satisfies $B(t) = (0,0,1)$ for all t. Prove that ψ lies in the xy-plane.

(c) Let $\gamma : [0,L] \to \mathbb{R}^2$ be a closed, regular plane curve parametrized by arc length whose curvature satisfies $0 < k(t) \leq c$ for all t. Prove that the arc length L of γ satisfies $L \geq \frac{2\pi}{c}$.

2. Consider the hyperboloid $S = \{(x,y,z) \mid x^2 + y^2 - z^2 = 1\}$ in $\mathbb{R}^3$.

(a) Prove that S is a regular surface.

(b) Using the parametrization $\phi(u,v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), \sinh(v))$, find the Gaussian curvature at each point of S, and determine all points where the mean curvature is zero. (Here $\cosh(v) = \frac{1}{2}(e^v + e^{-v})$ and $\sinh(v) = \frac{1}{2}(e^v - e^{-v})$.)

(c) Is there a local isometry from S to the cylinder $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^3$? Justify your answer.

3. (a) Draw a compact orientable surface without boundary whose Euler characteristic is -4.

(b) Let $S \subset \mathbb{R}^3$ be a compact, oriented surface with boundary, such that S is diffeomorphic to the cylinder $C = \{(x,y,z) \mid x^2+y^2=1,\ -\frac{1}{2} \leq z \leq \frac{1}{2}\}$.

(i) Give an example of such an S whose Gaussian curvature satisfies $K > 0$ everywhere.

(ii) Prove that if $K > 0$ on S, then some component of the boundary ∂S is not a geodesic.

(c) Let $S \subset \mathbb{R}^3$ be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose there is a constant $R > 0$ such that $f(x,y) = 0$ whenever $x^2 + y^2 \geq R^2$. If the Gaussian curvature of S satisfies $K \geq 0$, prove that $K = 0$ everywhere.

4. Determine whether each of the following is true or false and justify your answers.

(a) A regular curve C of length 2π in the unit sphere $S^2 \subset \mathbb{R}^3$, with constant curvature $k = 1$ and torsion $\tau = 0$, must be a closed geodesic.

(b) There is a regular curve $\gamma : [-1,1] \to S$ in the hyperbolic paraboloid $S = \{z = x^2 - y^2\}$ whose curvature satisfies $k(0) = 1$ and whose normal curvature satisfies $k_n(0) = 3$.

(c) A simple (non-self-intersecting) geodesic $\gamma : [a,b] \to S$ on a compact surface S without boundary minimizes length among all paths from $\gamma(a)$ to $\gamma(b)$.

(d) There is a surface $S \subset \mathbb{R}^3$ which is unbounded (i.e., not contained in a ball of any finite radius) but whose Gaussian curvature is positive everywhere.

(e) A compact, oriented surface without boundary whose second fundamental form satisfies $A(v,v) \neq 0$ for all nonzero tangent vectors v is diffeomorphic to a sphere.

5. (a) Prove that a minimal surface is planar if and only if its Gaussian curvature is zero everywhere.

(b) Let S be a *catenoid*, obtained by rotating the catenary $x = \cosh(z)$ in the xz-plane around the z-axis, with parametrization $\phi(u, v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$. Compute the first fundamental form of S and prove that S is a minimal surface.

(c) With S as in the previous part, show that the Gauss map $N : S \to S^2$ is injective.

1. (a) We define a regular curve $\phi : (-1,1) \to \mathbb{R}^3$ by $\phi(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$. Compute the curvature $k(t)$ and the torsion $\tau(t)$ for $-1 < t < 1$.

(Solution) We compute $\phi'(t) = (\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}})$, which satisfies $|\phi'(t)|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2} = 1$, so ϕ is parametrized by arc length and $T(t) = \phi'(t)$. Then

$$\phi''(t) = \left(\tfrac{1}{4}(1+t)^{-1/2}, \tfrac{1}{4}(1-t)^{-1/2}, 0\right)$$

and since $\phi''(t) = T'(t) = k(t)N(t)$, we have

$$k(t) = |\phi''(t)| = \left(\frac{1}{16(1+t)} + \frac{1}{16(1-t)} + 0\right)^{1/2} = \left(\frac{2}{16(1-t^2)}\right)^{1/2} = \frac{1}{\sqrt{8(1-t^2)}}.$$

Then $N(t) = \frac{1}{k(t)}\phi''(t) = \left(\sqrt{\frac{1-t}{2}}, \sqrt{\frac{1+t}{2}}, 0\right)$. Since $B'(t) = -\tau(t)N(t)$, we compute

$$B(t) = T(t) \times N(t) = (-\tfrac{1}{2}\sqrt{1+t}, \tfrac{1}{2}\sqrt{1-t}, \tfrac{1}{\sqrt{2}})$$
$$B'(t) = \left(-\frac{1}{4\sqrt{1+t}}, -\frac{1}{4\sqrt{1-t}}, 0\right) = -\frac{1}{\sqrt{8(1-t^2)}}N(t)$$

so that $\tau(t) = \frac{1}{\sqrt{8(1-t^2)}} = k(t)$.

(b) Let $\phi : [a,b] \to \mathbb{R}^3$ be a curve parametrized by arc length, with $\phi(a) = (0,0,0)$, and suppose that its curvature is nonzero and its binormal vector satisfies $B(t) = (0,0,1)$ for all t. Prove that ϕ lies in the xy-plane.

(Solution) The tangent vector T is orthogonal to B, so we must have $T(t) = (x(t), y(t), 0)$ for some functions x and y, and then since $T(t) = \phi'(t)$ and $\phi(0) = (0,0,0)$ we have

$$\phi(t) = \phi(0) + \int_0^t T(s)\,ds = \left(\int_0^t x(s)\,ds, \int_0^t y(s)\,ds, 0\right),$$

which is clearly in the xy-plane.

(c) Let $\phi : [0,L] \to \mathbb{R}^2$ be a closed, regular plane curve parametrized by arc length whose curvature satisfies $0 < k(t) \leq c$ for all t. Prove that the arc length L of ϕ satisfies $L \geq \frac{2\pi}{c}$.

(Solution) We have $\frac{1}{2\pi}\int_0^L k(t)\,dt = \mathrm{Ind}(\phi)$, so that the inequalities

$$\int_0^L 0\,dt < \int_0^L k(t)\,dt \leq \int_0^L c\,dt$$

become $0 < 2\pi\,\mathrm{Ind}(\phi) \leq cL$. Then $\mathrm{Ind}(\phi)$ is positive, and since it is an integer it must be at least 1, so that $R \geq \frac{2\pi}{c}\mathrm{Ind}(\phi) \geq \frac{2\pi}{c}$.

2. Consider the hyperboloid $S = \{(x,y,z) \mid x^2 + y^2 - z^2 = 1\}$ in $\mathbb{R}^3$.

(a) Prove that S is a regular surface.

(Solution) S is the level set $F^{-1}(1)$ of the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = x^2+y^2-z^2$, with gradient $\nabla F(x,y,z) = (2x, 2y, -2z)$. If $\nabla F(x,y,z) = 0$ then $x = y = z = 0$, so $F(x,y,z) = 0$ and thus $(x,y,z) \notin S$. Thus $\nabla F \neq 0$ at all points of S, which guarantees that it is a regular surface.

(b) Using the parametrization $\phi(u,v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), \sinh(v))$, find the Gaussian curvature at each point of S, and determine all points where the mean curvature is zero. (Here $\cosh(v) = \frac{1}{2}(e^v+e^{-v})$ and $\sinh(v) = \frac{1}{2}(e^v-e^{-v})$; note that $\frac{d}{dv}\cosh(v) = \sinh(v)$, $\frac{d}{dv}\sinh(v) = \cosh(v)$, and $\cosh^2(v) - \sinh^2(v) = 1$.)

(Solution) We compute the derivatives of ϕ, and also the normal vector at each point, as

$$\begin{aligned}
\phi_u &= (-\sin(u)\cosh(v), \cos(u)\cosh(v), 0)\\
\phi_v &= (\cos(u)\sinh(v), \sin(u)\sinh(v), \cosh(v))\\
N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} &= \frac{(\cos(u)\cosh^2(v), \sin(u)\cosh^2(v), -\cosh(v)\sinh(v))}{(\cosh^4(v) + \cosh^2(v)\sinh^2(v))^{1/2}}\\
&= \frac{(\cos(u)\cosh(v), \sin(u)\cosh(v), -\sinh(v))}{(\cosh^2(v) + \sinh^2(v))^{1/2}}\\
\phi_{uu} &= (-\cos(u)\cosh(v), -\sin(u)\cosh(v), 0)\\
\phi_{uv} &= (-\sin(u)\sinh(v), \cos(u)\sinh(v), 0)\\
\phi_{vv} &= (\cos(u)\cosh(v), \sin(u)\cosh(v), \sinh(v)).
\end{aligned}$$

Then $K = \det(\sigma)$ and $H = \frac{1}{2}\operatorname{tr}(\sigma)$, where

$$g = \begin{pmatrix} \phi_u\cdot\phi_u & \phi_u\cdot\phi_v \\ \phi_v\cdot\phi_u & \phi_v\cdot\phi_v \end{pmatrix} = \begin{pmatrix} \cosh^2(v) & 0 \\ 0 & \cosh^2(v)+\sinh^2(v) \end{pmatrix}$$

$$A = \begin{pmatrix} N\cdot\phi_{uu} & N\cdot\phi_{uv} \\ N\cdot\phi_{vu} & N\cdot\phi_{vv} \end{pmatrix} = \frac{1}{\sqrt{\cosh^2(v)+\sinh^2(v)}} \begin{pmatrix} -\cosh^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma = g^{-1}A = \frac{1}{\sqrt{\cosh^2(v)+\sinh^2(v)}} \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{\cosh^2(v)+\sinh^2(v)} \end{pmatrix}$$

so $K = -\frac{1}{(\cosh^2(v)+\sinh^2(v))^2}$, and $H = 0$ iff $\operatorname{tr}(\sigma) = 0$ iff $\cosh^2(v) + \sinh^2(v) = 1$. Since $\cosh(v) - \sinh^2(v) = 1$ this is equivalent to $\sinh(v) = 0$, or $v = 0$, and so $H = 0$ at precisely the points $\phi(u,0) = (\cos(u), \sin(u), 0)$.

(c) Is there a local isometry from S to the cylinder $C = \{x^2+y^2 = 1\} \subset \mathbb{R}^3$? Justify your answer.

(Solution) A local isometry $f : S \to C$ would preserve Gaussian curvature by the Theorema Egregium, meaning that $K_S(p) = K_C(f(p))$ for all $p \in S$. Since the curvature of S is everywhere negative while C has zero curvature, this is impossible.

3. (a) Draw a compact orientable surface without boundary whose Euler characteristic is -4.

(Solution) A closed, orientable surface S of genus g has Euler characteristic $\chi(S) = 2 - 2g$, which is -4 when $g = 3$, so the picture should be a surface with 3 "holes" in it.

(b) Let $S \subset \mathbb{R}^3$ be a compact, oriented surface with boundary which is diffeomorphic to the cylinder $C = \{(x, y, z) \mid x^2 + y^2 = 1,\ -\frac{1}{2} \leq z \leq \frac{1}{2}\}$.

(i) Give an example of such an S whose Gaussian curvature satisfies $K > 0$ everywhere.

(Solution) Let S be the subset of the unit sphere where $-\frac{1}{2} \leq z \leq \frac{1}{2}$; there is a diffeomorphism $S \to C$ given by $(x, y, z) \mapsto (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z)$.

(ii) Prove that if $K > 0$ on S, then some component of the boundary ∂S is not a geodesic.

(Solution) If both components are geodesics, their geodesic curvature k_g is zero. The first term in the Gauss-Bonnet theorem

$$\int_{\partial S} k_g \, ds + \int_S K \, dA = 2\pi \cdot \chi(S)$$

is then zero, and the second term is strictly positive, so the left hand side is positive and we get $\chi(S) > 0$. But $\chi(S) = 0$ - this can be computed directly from a triangulation, or from applying Gauss-Bonnet to C (whose boundary circles are geodesics, and which has $K = 0$) – so we have a contradiction.

(c) Let $S \subset \mathbb{R}^3$ be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose there is a constant $R > 0$ such that $f(x, y) = 0$ whenever $x^2 + y^2 \geq R^2$. If the Gaussian curvature of S satisfies $K \geq 0$, prove that $K = 0$ everywhere.

(Solution) Let $\gamma : [0, 4\pi R] \to S$ be defined by $\gamma(t) = (\cos(\frac{t}{2R}), \sin(\frac{t}{2R}), 0)$: this parametrizes the circle of radius $2R$ by arc length, and it bounds a compact subsurface $D \subset S$ of the form

$$D = \{(x, y, f(x, y)) \mid x^2 + y^2 \leq (2R)^2\},$$

which is diffeomorphic to a disk. Then γ has normal curvature $k_n = 0$ since γ' is always orthogonal to the unit normal $N = (0, 0, 1)$, and thus its geodesic curvature is equal to its curvature $\frac{1}{2R}$. We apply the local Gauss-Bonnet theorem to D:

$$2\pi = \int_{\partial D} k_g \, ds + \int_D K \, dA = \int_0^{4\pi R} \tfrac{1}{2R} \, ds + \int_D K \, dA = 2\pi + \int_D K \, dA.$$

Thus $\int_D K \, dA = 0$, and since $K \geq 0$, this is only possible if $K = 0$ everywhere.

4. Determine whether each of the following is true or false and justify your answers.

(a) A regular curve C of length 2π in the unit sphere $S^2 \subset \mathbb{R}^3$, with constant curvature $k = 1$ and torsion $\tau = 0$, must be a closed geodesic.

(Solution) **True.** The fundamental theorem of the local theory of curves says the image of C is part of a unit circle, since these also have $k = 1$ and $\tau = 0$; the length 2π of C is the circumference of such a circle, so C is closed, and any unit circle in S^2 is a geodesic.

(b) There is a regular curve $\gamma : [-1,1] \to S$ in the hyperbolic paraboloid $S = \{z = x^2 - y^2\}$ whose curvature satisfies $k(0) = 1$ and whose normal curvature satisfies $k_n(0) = 3$.

(Solution) **False.** This contradicts the relation $k^2 = k_g^2 + k_n^2$ at time $t = 0$.

(c) A simple (non-self-intersecting) geodesic $\gamma : [a,b] \to S$ on a compact surface S without boundary minimizes length among all paths from $\gamma(a)$ to $\gamma(b)$.

(Solution) **False.** Let S be the unit sphere and x, y two points on the equator which are not antipodes. Then x and y divide the equator (which is a geodesic) into two arcs, both of them geodesics, but one arc is longer than the other and hence cannot minimize distance.

(d) There is a surface $S \subset \mathbb{R}^3$ which is unbounded (i.e., not contained in a ball of any finite radius) but whose Gaussian curvature is positive everywhere.

(Solution) **True.** A natural guess would be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$, since one can compute that its curvature at $(x, y, f(x,y))$ is $K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x+f_y)^2}$ and then find a suitable f. For example, taking $f(x,y) = x^2+y^2$ and chart $\phi(u,v) = (u, v, u^2+v^2)$, we compute

$$\phi_u = (1,0,2u), \quad \phi_v = (0,1,2v), \quad N = \frac{(-2u,-2v,1)}{\sqrt{1+4u^2+4v^2}},$$

and then $\phi_{uu} = \phi_{vv} = (0,0,2)$ and $\phi_{uv} = (0,0,0)$, so $K = \frac{\det A}{\det g}$ where

$$g = \begin{pmatrix} 1+4u^2 & 4uv \\ 4uv & 1+4v^2 \end{pmatrix}^{-1}, \qquad A = \frac{1}{\sqrt{1+4u^2+4v^2}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and thus $K = \frac{4/(1+4u^2+4v^2)}{1+4u^2+4v^2} = \frac{4}{(1+4u^2+4v^2)^2} > 0$.

(e) A compact, oriented surface without boundary whose second fundamental form satisfies $A(v,v) \neq 0$ for all nonzero tangent vectors v is diffeomorphic to a sphere.

(Solution) **True.** At any point p, the principal curvatures $\lambda_1 \leq \lambda_2$ are the minimum and maximum of $A(w,w)$ over all unit vectors $w \in T_pS$; if $\lambda_1 \leq 0 \leq \lambda_2$, then by the continuity of $w \mapsto A(w,w)$ there must be some unit vector $w \in T_pS$ where $A(w,w) = 0$. This means that λ_1 and λ_2 are either both positive or both negative, and the curvature $K = \lambda_1\lambda_2$ at p is then strictly positive. Thus $K > 0$ for all $p \in S$, and a corollary of the Gauss-Bonnet theorem now says that S must be diffeomorphic to a sphere.

5. (a) Prove that a minimal surface is planar if and only if its Gaussian curvature is zero everywhere.

(Solution) Certainly planar surfaces have curvature zero. For the converse, a minimal surface S with Gaussian curvature zero satisfies $H = K = 0$ everywhere, so the principal curvatures λ_1 and λ_2 are both zero everywhere, since their sum and product are $2H = 0$ and $K = 0$ respectively. Then every point of S is umbilical, so S is contained in either a plane or a sphere, and it cannot be the latter since its curvature is not positive.

(b) Let S be a *catenoid*, obtained by rotating the catenary $x = \cosh(z)$ in the xz-plane around the z-axis, with parametrization $\phi(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$. Compute the first fundamental form of S and prove that S is a minimal surface.

(Solution) We compute the partial derivatives of ϕ as

$$\phi_u = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1), \quad \phi_v = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0),$$

from which (recalling that $\cosh^2(u) - \sinh^2(u) = 1$) the first fundamental form is

$$g = \begin{pmatrix} \phi_u\cdot\phi_u & \phi_u\cdot\phi_v \\ \phi_v\cdot\phi_u & \phi_v\cdot\phi_v \end{pmatrix} = \begin{pmatrix} \sinh^2(u)+1 & 0 \\ 0 & \cosh^2(u) \end{pmatrix} = \begin{pmatrix} \cosh^2(u) & 0 \\ 0 & \cosh^2(u) \end{pmatrix}.$$

We could check that S is minimal by computing the second fundamental form, but it is easier to observe that S is isothermal, and hence H is minimal if and only if $\Delta\phi = 0$ (which follows from $2\cosh^2(u)HN = \Delta\phi$). We have

$$\phi_{uu} = (\cosh(u)\cos(v), \cosh(u)\sin(v), 0) = -\phi_{vv}$$

and so $\Delta\phi = \phi_{uu} + \phi_{vv} = 0$, as desired.

(c) With S as in the previous part, show that the Gauss map $N : S \to S^2$ is injective.

(Solution) The normal vector at a point $\phi(u,v)$ is given by

$$\begin{aligned} N(\phi(u,v)) = \frac{\phi_u\times\phi_v}{|\phi_u\times\phi_v|} &= \frac{(-\cosh(u)\cos(v), -\cosh(u)\sin(v), \sinh(u)\cosh(u))}{(\cosh^2(u)+\sinh^2(u)\cosh^2(u))^{1/2}} \\ &= \frac{(-\cos(v), -\sin(v), \sinh(u))}{(1+\sinh^2(u))^{1/2}} = \frac{(-\cos(v), -\sin(v), \sinh(u))}{\cosh(u)}. \end{aligned}$$

If $N(\phi(u_1,v_1)) = N(\phi(u_2,v_2))$, then we must have $\frac{(\cos(v_1),\sin(v_1))}{\cosh(u_1)} = \frac{(\cos(v_2),\sin(v_2))}{\cosh(u_2)}$, and taking the magnitude of both sides gives $\frac{1}{\cosh(u_1)} = \frac{1}{\cosh(u_2)}$, hence $\cosh(u_1) = \cosh(u_2)$ and $(\cos(v_1),\sin(v_1)) = (\cos(v_2),\sin(v_2))$ (from which $v_1 \equiv v_2 \pmod{2\pi}$). But cosh is even and increasing on $(0,\infty)$, so $u_1 = \pm u_2$, and then $\frac{\sinh(u_1)}{\cosh(u_1)} = \frac{\sinh(u_2)}{\cosh(u_2)}$ gives $u_1 = u_2$ since sinh is odd. Since $u_1 = u_2$ and $v_1 \equiv v_2 \pmod{2\pi}$, it follows that $\phi(u_1,v_1) = \phi(u_2,v_2)$, and so N is injective.