Imperial College London
MATH 50004 Multivariable Calculus
Mid-Term Examination Date: 11th November 2021
Duration: 60 minutes

1(i) [**10 marks**] Given that $\mathbf{A}$ and $\mathbf{B}$ are solenoidal vector fields, use subscript notation to simplify the expression

$$\nabla(\mathbf{A}\cdot\mathbf{B}) - \operatorname{curl}(\mathbf{A}\times\mathbf{B}) - \mathbf{B}\times\operatorname{curl}\mathbf{A} - \mathbf{A}\times\operatorname{curl}\mathbf{B}. \qquad (1)$$

(ii) [**10 marks**] Consider the specific two-dimensional fields

$$\mathbf{A} = y^2\mathbf{i} + x\mathbf{j}, \;\; \mathbf{B} = x\mathbf{i} - y\mathbf{j}.$$

Verify explicitly that $\mathbf{A}$ and $\mathbf{B}$ are solenoidal and by direct calculation compute the value of (1). Demonstrate that this is indeed equal to the simplified form you derived in part (i).

You may assume the relation

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

2. Consider the two-dimensional vector field

$$\mathbf{F} = (2x - y^3 + 1)\,\mathbf{i} - xy\,\mathbf{j}$$

and let C denote the boundary of the finite region R in the upper half-plane bounded by the curves

$$x^2 + y^2 = 1, \;\; x^2 + y^2 = 9$$

and the $x-$axis.

(i) [**3 marks**] Sketch the region R and mark on the boundary C.

(ii) [**12 marks**] Show that the circulation Γ of $\mathbf{F}$ around C, where C is traversed in the positive sense, is given by

$$\Gamma = 30\pi - 52/3.$$

(iii) [**5 marks**] Verify Green's theorem for this choice of $\mathbf{F}$ and C by evaluation of the appropriate double integral over R.

You may assume the trigonometric identity:

$$\sin^4\theta \equiv \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

and that a small areal element in polar coordinates is given by $r\,dr\,d\theta$.