

ExamModuleCode	Question Number	Comments for Students
M45S17	1	Generally students did well on this question. No systematic errors.
M45S17	2	Generally students did well on this question, although few students made progress with part (g). This was the hard part of the question, of course.
M45S17	3	Students generally performed poorly on this question. Even with the basic material, eg definitions of terms, students made errors. Surprisingly, many students showed confidence answering the harder parts (g) to (j) and were able to make up marks.
M45S17	4	I was pleased that many year 4/5 students attempted this question and marks were good given this is Mastery-level. Many students were able to get full marks for part (c) which requires attention to detail.

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**

**May-June 2019**

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science

**Quantitative Methods in Retail Finance**

Date: Tuesday 21 May 2019

Time: 14.00 - 15.30

Time Allowed: 1 Hour 30 Minutes

**This paper has 3 Questions.**

**Candidates should use ONE main answer book.**

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**

**May-June 2019**

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science

**Quantitative Methods in Retail Finance**

Date: Tuesday 21st May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

**This paper has 4 Questions.**

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

Statistical tables will not be provided

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Credit will be given for all questions attempted
- Each question carries equal weight
- Calculators may not be used.

1. (a) Give four reasons why a survival model would be used for credit risk modelling, rather than logistic regression.
- (b) What are time-varying covariates and for what purposes are they used in survival models for credit risk modelling?
- (c) Derive the relationship between survival probability  $S(t)$  and hazard  $h(t)$ :

$$S(t) = \exp \left( - \int_0^t h(u) du \right).$$

A Cox proportional hazard model is built for a loan product with default as the failure event and  $t$  as the duration of the loan, in months. Two predictor variables are included in the model:-

- Employed,  $X_1(t)$ , taking the value 1 if the borrower was employed at loan application, otherwise it takes the value 0, fixed for all  $t$ .
- Delinquency,  $X_2(t)$ , a non-negative integer giving number of months delinquent on repayments of the loan, with a 3 month lag.

The following coefficient estimates were computed:-

Variable	Coefficient estimate $\beta_1$	P-value
Employed, $X_1(t)$	-0.5	0.001
Delinquency, $X_2(t)$	0.1	0.002

The baseline hazard is estimated as

$$h_0(t) = \begin{cases} 0 & \text{if } t \leq 3 \\ 1/30 & \text{if } 3 < t \leq 6 \\ 1/60 & \text{if } t > 6 \end{cases}$$

*QUESTION CONTINUED ON NEXT PAGE.*

- (d) Interpret the association of each predictor variable to the hazard of default.
- (e) Give the hazard ratio for Employed, supposing the value of Delinquency is fixed.
- (f) How far ahead can the model be used for forecasting without having to estimate values of the predictor variables? Explain why.
- (g) Use the model to compute the survival probability at  $t = 12$  for an unemployed borrower with the following values of Delinquency:  $X_2(t) = 0$  for  $t \leq 6$  and  $X_2(t) = 1$  for  $t > 6$ .
- (h) What is the probability of default for the borrower described in part (g) within the first 12 months of the loan?
- (i) The model is rebuilt with an additional macroeconomic predictor variable, Unemployment Rate,  $X_3(t)$ , lag 3 months. You are told that the coefficient estimate on this new variable is  $-0.2$  with p-value 0.001. What is your response to this information? Would you go ahead and deploy this model in operation?

Make use of the following table of exponentials (to 3 significant figures):-

$x$	$\exp(x)$	$x$	$\exp(x)$
-0.5	0.607	0.1	1.11
-0.211	0.810	0.211	1.23
-0.1	0.905	0.5	1.65

2. For this question, use the following notation for a fixed term loan:

- $L_0 > 0$  is initial loan amount.
- $l_D > 0$  is loss given default (LGD).
- $e_D > 0$  is the fraction of loan exposed at default.
- $r \geq 0$  is interest rate across the term of the loan.
- $x > 0$  is a credit score for a prospective customer.
- $p_T(x) \in (0, 1)$  is probability of default (PD) for an individual with score  $x$ .
- $q(r, o, x) = (1 + rx)^{-2}$  is a response rate function giving the response rate for individuals with score  $x$  accepting a loan with interest rate  $r$  and other factors given by  $o$ .
- $P_E(r, x) = a(x)r - b(x)$  is expected profit of the borrower taking PD into account, where  $a(x) = (1 - p_T(x))L_0$  and  $b(x) = p_T(x)l_De_DL_0$ .
- $P_R(r, x) = q(r, o, x)P_E(r, x)$ .
- Let

$$\hat{r} = \arg \max_r P_R(r, x)$$

be the optimal interest rate for an individual with score  $x$ .

- (a) Compare and contrast the traditional lending strategy based on a fixed price against the risk-based pricing strategy.
- (b) Explain how response to an interest rate offer changes with changing credit score for the response rate function given above.
- (c) Briefly explain why the risk-based pricing strategy involves maximizing  $P_R(r, x)$  with respect to  $r$  to compute the optimal interest rate,  $\hat{r}$ . Contrast to maximizing  $P_E(r, x)$  with respect to  $r$ .
- (d) Find a closed form solution for  $\hat{r}$ .  
*Ensure that you show your solution is a maximum, rather than a minimum.*
- (e) Considering all other parameters fixed, how does an increase in LGD,  $l_D$ , change the optimal interest rate  $\hat{r}$ ? Is the relationship between LGD and the optimal interest rate correct? Explain why.
- (f) Show that

$$P_R(\hat{r}, x) = \frac{a(x)/x + b(x)}{4(1 + xb(x)/a(x))^2}.$$

- (g) Suppose the lender sets a minimum expected profit  $p_{\min}$ . That is, a loan will only be offered if  $P_E(r, x) \geq p_{\min}$ . How does this constraint alter the solution for  $\hat{r}$ ?

3. (a) Define Value-at-Risk (VaR), in general.  
 (b) Why are financial institutions interested in using VaR as a measure of risk?  
 (c) Why might VaR be much greater than expected loss for financial applications?  
 (d) What is the definition of Regulatory Capital?

Vasicek's formula for VaR in terms of probability of default, based on Merton's one factor model, is

$$K(p_i) = \Phi \left( \frac{\Phi^{-1}(p_i) + \Phi^{-1}(q)\sqrt{\rho}}{\sqrt{1-\rho}} \right) \quad (1)$$

where

- $p_i$  is probability of default for a loan  $i$ ;
  - $\Phi$  is the cumulative standard normal function;
  - $q$  is the VaR level (typically  $q = 0.999$ );
  - $\rho$  is a correlation term.
- (e) Which assumption leads to there being only one correlation term  $\rho$  in formula (1)?  
 (f) Define  $\rho$  precisely, giving a formula.  
 (g) If a credit product has  $\rho = 0$ , then  $K(p_i) = p_i$ . Interpret this result in terms of the impact of economic conditions on the credit risk of this product?  
 (h) Why did the Bank for International Settlements (BIS) set a higher value of  $\rho$  for mortgages (0.15) compared to credit cards (0.04)? What is the consequence for calculation of Regulatory Capital, comparing the two types of credit?  
 (i) The normal functions  $\Phi$  in formula (1) are a consequence of normal distributional assumptions on the terms used to derive it. Suppose one of these assumptions is changed to allow for a t-distribution on the systematic term in Merton's one factor model. How would that change formula (1)?

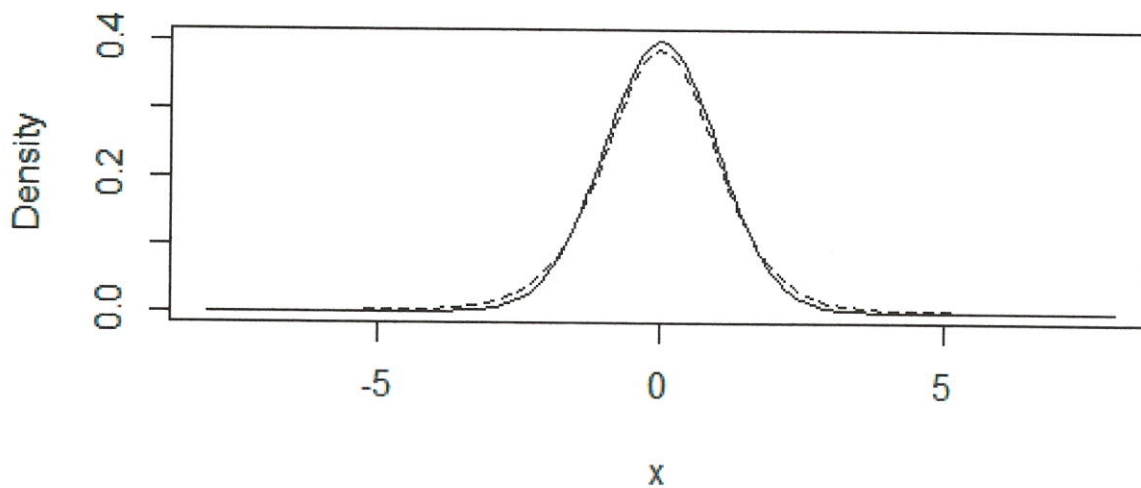
*QUESTION CONTINUED ON NEXT PAGE.*

- (j) For a mortgage product, use  $\rho = 0.15$ ,  $q = 0.999$  for VaR and  $p_i = 0.005$ . Using the standard formula (1) with normality assumption on the systematic term, the Regulatory Capital per unit of loan value is 0.0624. When a t-distribution with degrees of freedom=8 is used instead, the Regulatory Capital per unit is 0.178.

Explain the large discrepancy between these two values.

Does this suggest any problems with the method for calculating Regulatory Capital?

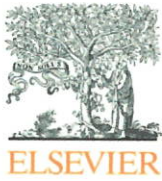
The following graph shows distributions for the standard normal (solid line) and t-distribution with degrees of freedom=8 (dashed line):-



4. This question is based on the article,

– Transition matrix models of consumer credit ratings, by Madhur Malik and Lyn C. Thomas, *International Journal of Forecasting*, 28 (2012) 261–272.

- (a) What is meant by behavioural scores, as discussed in the article?
- (b) What is meant by stating that  $U_t^i$  is *latent*, in Section 3?
- (c) Derive Equation (9) from the definition of  $U_t^i$  given in Equation (8) and the text of Section 3.
- (d) Looking at Table 2, interpret the transitions to default state from the current and previous states. In particular, does Table 2 suggest any additional information about transition to default beyond that given by the first order model reported in Table 1?
- (e) Interpret the association of variables *Interest rate* and *Months on books* with state transitions, as reported in Table 6.
- (f) Explain how the Intercept/barrier estimates shown in Table 6 are affecting the transition probabilities.
- (g) Briefly, what are the main conclusions of the article?



## Transition matrix models of consumer credit ratings

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### ABSTRACT

Although the corporate credit risk literature includes many studies modelling the change in the credit risk of corporate bonds over time, there has been far less analysis of the credit risk for portfolios of consumer loans. However, behavioural scores, which are calculated on a monthly basis by most consumer lenders, are the analogues of ratings in corporate credit risk. Motivated by studies of corporate credit risk, we develop a Markov chain model based on behavioural scores for establishing the credit risk of portfolios of consumer loans. Although such models have been used by lenders to develop models for the Basel Accord, nothing has been published in the literature on them. The model which we suggest differs in many respects from the corporate credit ones based on Markov chains — such as the need for a second order Markov chain, the inclusion of economic variables and the age of the loan. The model is applied using data on a credit card portfolio from a major UK bank.

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### 1. Introduction

Since the mid 1980s, banks' lending to consumers has exceeded that to companies (Crouhy, Galai, & Mark, 2001). However, not until the subprime mortgage crisis of 2007 and the subsequent credit crunch was it realised what an impact such lending had on the banking sector, and also how under-researched it is compared to corporate lending models. In particular, the need for robust models of the credit risk of portfolios of consumer loans has been brought into sharp focus by the failure of the ratings agencies to accurately assess the credit risks of the Mortgage Backed Securities (MBS) and collateralized debt obligations (CDO) which are based on such portfolios. Many reasons for the subprime mortgage crisis and the subsequent credit crunch have been put forward (Demyanyk & van Hemert, 2008; Hull, 2009), but, clearly, one reason why the former led to the latter was the lack of an easily updatable model of the credit risk of portfolios of consumer loans. This lack of a suitable model of portfolio level consumer risk

was first highlighted during the development of the Basel Accord, when a corporate credit risk model was used to calculate the regulatory capital for all types of loans (Basel Committee on Banking Supervision, 2005), even though the basic idea of such a model — that default occurs when debts exceed assets — is not the reason why consumers default.

This paper develops a model for the credit risk of portfolios of consumer loans based on the behavioural scores of the individual consumers whose loans make up that portfolio. Such a model is attractive to lenders, since almost all lenders calculate behavioural scores for all of their borrowers on a monthly basis. The behavioural score is usually translated into the default probability over a fixed time horizon (usually one year) in the future for that borrower, but one can also consider it as a surrogate for the unobservable creditworthiness of the borrower. We build a Markov chain credit risk model based on behavioural scores for consumers which has similarities with the reduced form mark to market corporate credit risk models based on the rating agencies' grades (Jarrow, Lando, & Turnbull, 1997). Such behavioural score based Markov chain models have been developed by lenders for their Basel modelling, but no analysis has appeared in the literature; in this paper, we discuss the features which

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should be included in such models and compare a standard and a more sophisticated version of the model. The methodology constructs an empirical forecasting model for deriving a multi-period distribution of the default rate for long time horizons based on migration matrices built from a historical database of behavioural scores. Although it is possible to calibrate the scores to the long run probability of default if one has data for a sufficiently long outcome period, such data are not available in practice. The transition matrix approach allows one to undertake such a calibration using much shorter data series. In our case study we use the lenders' behavioural scores, but we can also use the same methodology on generic bureau scores.

The approach also helps lenders to make long term lending decisions by estimating the risk associated with the change in the quality of portfolio of loans over time. Since the model includes economic conditions, the approach allows banks to stress test their retail portfolios, as required by the Basel Accord and other banking regulations. In addition, the model provides insights on portfolio profitability, the determination of appropriate capital reserves, and the creation of estimates of portfolio value by generating portfolio-level credit loss distributions.

There have recently been several papers which have looked at modelling the credit risk in consumer loan portfolios. Rosch and Scheule (2004) take a variant of the one factor Credit Metrics model, which is the basis of the Basel Accord. They use empirical correlations between different consumer loan types and try to build in economic variables to explain the differences during different parts of the business cycle. Perli and Nayda (2004) also take the corporate credit risk structural model and seek to apply it to consumer lending, assuming that a consumer defaults if his assets are below a specified threshold. However, consumer defaults are usually more about cash flow problems, financial naiveté or fraud, and thus such a model misses some aspects of consumer defaults.

Musto and Souleles (2005) use equity pricing as an analogy for changes in the value of consumer loan portfolios. They use behavioural scores, but take the monthly differences in behavioural scores as the return on assets when applying their equity model.

Andrade and Thomas (2007) describe a structural model for the credit risk of consumer loans, where the behavioural score is a surrogate for the creditworthiness of the borrower. A default occurs if the value of this reputation for creditworthiness, in terms of access to further credit, drops below the cost of servicing the debt. Using a case study based on the Brazilian credit bureau, they found that a random walk was the best model for the idiosyncratic part of creditworthiness. Malik and Thomas (2010) developed a hazard model of the time to default for consumer loans, where the risk factors were the behavioural score, the age of the loan, and economic variables, and used it to develop a credit risk model for portfolios of consumer loans. Bellotti and Crook (2009) also used proportional hazards to develop a default risk model for consumer loans. They investigated which economic variables might be the most appropriate, though they did not use behavioural scores in their model. Thomas (2009b)

reviewed the existing consumer credit risk models and pointed out the analogies with some of the established corporate credit risk models.

Since the seminal paper by Jarrow et al. (1997), the Markov chain approach has proved popular in modelling the dynamics of the credit risk in corporate portfolios. The idea is to describe the dynamics of the risk in terms of the transition probabilities between the different grades the rating agencies award to the firm's bonds. There are papers which look at how both economic conditions and the industry sector of the firm affects the transition matrices (Nickell, Perraudin, & Varoli, 2001), while others generalise the original idea of Jarrow et al. by using Affine Markov chains (Hurd & Kuznetsov, 2006) or continuous time processes (Lando & Skodeberg, 2002). However, none of these suggest increasing the order of the Markov chain or considering the age of the loan, which are two of the features which we introduce here, in order to model the consumer credit risk using Markov chains. This is surprising, because there has been work on downgrading by rating agencies which suggests that there is a momentum effect where, once a company has been downgraded, it is more likely to be further downgraded than to be subsequently upgraded (Bangia, Diebold, & Schuermann, 2002).

Markov chain models have been used in the consumer lending context before, but none of the published papers have used the behavioural score as the state space, nor has the objective of the models been to estimate the credit risk at the portfolio level. The first such application was by Cyert, Davidson, and Thompson (1962), who developed a Markov chain model of customers' repayment behaviours. Subsequently, more complex models have been developed by Ho (2001), Thomas, Ho, and Scherer (2001) and Trench et al. (2003). Schniederjans and Loch (1994) used Markov chain models to model the marketing aspects of customer relationship management in the banking environment.

Behavioural score based Markov chain models are sometimes used in the industry (see Scallan, 1998), but mainly as ways of assessing provisioning estimates, and they do not include the economic drivers and months on the books effects presented in this paper. Moreover, the introduction of economic factors into the model allows one to deal with the correlations between defaults on individual loans in a portfolio, since they are affected by common economics. One can obtain the mean default rate in a portfolio from the long run distributions, while a Monte Carlo simulation using the transitions of individual loans would give the distribution of the default rate.

In Section 2, we review the properties of behavioural scores and Markov chains, while in Section 3 we describe the Markov chain behavioural score based consumer credit risk model developed. This is parameterised by using cumulative logistic regression to estimate the transition probabilities of the Markov chain. The motivation behind the model and the accuracy of the model's forecasts are shown by means of a case study, and Section 4 describes the data used in the case study. Sections 5–7 give the reasons why the model includes higher order transition matrices (Section 5); economic variables for explaining the non-stationarity of the chain (Section 6); and the age of

the loan (Section 7). Section 8 describes the full model used, while Section 9 reports the results of out-of-sample forecasts, and out-of-time and out-of-sample forecasts, using the model. The final section draws some conclusions, including how the model could be used. It also identifies one issue – which economic variables drive consumer credit risk – where further investigation would benefit all models of consumer credit risk.

## 2. Behaviour score dynamics and Markov chain models

Consumer lenders use behavioural scores updated every month to assess the credit risk of individual borrowers. The score is considered to be a sufficient indication of the probability that a borrower will be “Bad”, and so default within a certain time horizon (normally taken to be the next twelve months). Borrowers who are not Bad are classified as “Good”. Thus, at time  $t$ , a typical borrower with characteristics  $x(t)$  (which may describe the recent repayment and usage performance, the current information available on the borrower at a credit bureau, and socio-demographic details) has a score  $s(x(t), t)$ , so

$$p(B|x(t), t) = p(B|s(x(t), t)). \quad (1)$$

Some lenders obtain a Probability of Default (PD), as required under the Basel Accord, by taking a combination of behavioural and application scores. New borrowers are scored using only the application score to estimate PD, then once there is sufficient history for a behavioural score to be calculated, a weighted combination of the two scores is used to calculate PD; eventually, the loan is sufficiently mature that only the behavioural score is used to calculate PD. The models described hereafter can also be applied to such a combined scoring system.

Most scores are log odds score (Thomas, 2009a), and thus the direct relationship between the score and the probability of being Bad is given by

$$\begin{aligned} s(x(t), t) &= \log \left( \frac{P(G|s(x(t), t))}{P(B|s(x(t), t))} \right) \Leftrightarrow P(B|s(x(t), t)) \\ &= \frac{1}{1 + e^{s(x(t), t)}}, \end{aligned} \quad (2)$$

though in reality this may not hold exactly. Applying the Bayes theorem to Eq. (2) gives the expansion where if  $p_G(t)$  is the proportion of the population who are Good at time  $t$  ( $p_B(t)$  is the proportion who are Bad), one has

$$\begin{aligned} s(x(t), t) &= \log \left( \frac{P(G|s(x(t), t))}{P(B|s(x(t), t))} \right) \\ &= \log \left( \frac{p_G(t)}{p_B(t)} \right) + \log \left( \frac{P(s(x(t), t)|G, t)}{P(s(x(t), t)|B, t)} \right) \\ &= s_{\text{pop}}(t) + \text{woe}_t(s(x(t), t)). \end{aligned} \quad (3)$$

The first term is the log of the population odds at time  $t$  and the second term is the weight of evidence for that score (Thomas, 2009a). This decomposition may not hold exactly in practice, and is likely to change as a scorecard ages. However, it shows that the term  $s_{\text{pop}}(t)$ , which is common to the scores of all borrowers, can be thought to play the role of a systemic factor which affects the default risk of all

of the borrowers in a portfolio. Normally, though, the time dependence of a behavioural score is ignored by lenders. Lenders are usually only interested in ranking borrowers in terms of risk, and they believe that the second term (the weight of evidence) in Eq. (3), which is the only one which affects the ranking, is more stable over time than  $s_{\text{pop}}(t)$ , particularly over horizons of two or three years. In reality, the time dependence is important because it describes the dynamics of the credit risk of the borrower. Given the strong analogies between behavioural scores in consumer credit and the credit ratings used for corporate credit risk, one obvious way of describing the dynamics of behavioural scores is to use a Markov chain approach similar to the reduced form mark to market models of corporate credit risk (Jarrow et al., 1997). To use a Markov chain approach with behavioural scores, we divide the score range into a number of intervals, each of which represents a state of the Markov chain; hereafter, when we mention behavioural scores we are thinking of this Markov chain version of the score, where the states are intervals of the original score range.

Markov chains have proved ubiquitous models of stochastic processes because their simplicity belies their power to model a variety of situations. Formally, we define a discrete time  $\{t_0, t_1, \dots, t_n, \dots: n \in N\}$  and a finite state space  $S = \{1, 2, \dots, s\}$  first order Markov chain as a stochastic process  $\{X(t_n)\}_{n \in N}$ , with the property that for any  $s_0, s_1, \dots, s_{n-1}, i, j \in S$ :

$$\begin{aligned} P[X(t_{n+1}) = j | X(t_0) = s_0, X(t_1) = s_1, \dots, X(t_{n-1}) \\ = s_{n-1}, X(t_n) = i] &= P[X(t_{n+1}) = j | X(t_n) = i] \\ &= p_{ij}(t_n, t_{n+1}), \end{aligned} \quad (4)$$

where  $p_{ij}(t_n, t_{n+1})$  denotes the transition probability of going from state  $i$  at time  $t_n$  to state  $j$  at time  $t_{n+1}$ . The  $S \times S$  matrix of elements  $p_{ij}(\cdot, \cdot)$ , denoted  $P(t_n, t_{n+1})$ , is called the first order transition probability matrix associated with the stochastic process  $\{X(t_n)\}_{n \in N}$ . If  $\pi(t_n) = (\pi_1(t_n), \dots, \pi_s(t_n))$  describes the probability distribution of the states of the process at time  $t_n$ , the Markov property implies that the distribution at time  $t_{n+1}$  can be obtained from that at time  $t_n$  by  $\pi(t_{n+1}) = \pi(t_n)P(t_n, t_{n+1})$ . This extends to a  $m$ -stage transition matrix, so that the distribution at time  $t_{n+m}$  for  $m \geq 2$  is given by

$$\pi(t_{n+m}) = \pi(t_n)P(t_n, t_{n+1}) \dots P(t_{n+m-1}, t_{n+m}).$$

The Markov chain is called time homogeneous or stationary, provided that

$$p_{ij}(t_n, t_{n+1}) = p_{ij} \quad \forall n \in N. \quad (5)$$

Assume that the process  $\{X(t_n)\}_{n \in N}$  is a nonstationary Markov chain, which is the case with the data we examine later. If one has a sample of the histories of previous customers, let  $n_i(t_n)$ ,  $i \in S$ , be the number who are in state  $i$  at time  $t_n$ , whereas let  $n_{ij}(t_n, t_{n+1})$  be the number who move from state  $i$  at time  $t_n$  to state  $j$  at time  $t_{n+1}$ . The maximum likelihood estimator of  $p_{ij}(t_n, t_{n+1})$  is then

$$\hat{p}_{ij}(t_n, t_{n+1}) = \frac{n_{ij}(t_n, t_{n+1})}{n_i(t_n)}. \quad (6)$$

If one assumes that the Markov chain was stationary, then, given the data for  $T + 1$  time periods  $n = 0, 1, 2, \dots, T$ , the transition probability estimates become

$$\hat{p}_{ij} = \frac{\sum_{n=0}^{T-1} n_{ij}(t_n, t_{n+1})}{\sum_{n=0}^{T-1} n_i(t_n)} \quad (7)$$

Note that the Markov property means that previous transitions do not affect the current probabilities of transition, and thus in these calculations we do not need to be concerned that transitions coming from the same customer are dependent. All transitions are essentially independent, even those from the same customer. One can weaken the Markov property so that the information required to estimate the future of the chain is the current state and the previous state of the process. This is called a second order Markov chain, which is equivalent to the process being a first order Markov chain, but with state space  $S \times S$ . The concept can be generalized to defining  $k$ th order Markov chains for any  $k$ , though of course, the state space and the size of the transition probability matrices go up exponentially as  $k$  increases.

### 3. Behavioural score based Markov chain model of consumer credit risk

The behavioural score  $B_t$  of a borrower is an observable variable given by a scorecard. It is related to the underlying unobservable “creditworthiness” of the borrower,  $U_t$ , which also depends on the length of time the loan has been running and the current economic situation. Our model is constructed by assuming that the borrower's behavioural score is in one of a finite number of states, namely  $\{s_0 = D, s_1, \dots, s_n, C\}$ , where  $s_i$  ( $i > 0$ ) describes an interval in the behavioural score range;  $s_0 = D$  means that the borrower has defaulted, and  $C$  is the state when the borrower closed his loan or credit card account, having repaid everything (an absorbing state). The Markov property means that the dynamics of the behavioural score from time  $t$  onwards are conditional on the realization of the score state at time  $t - 1$ ,  $B_{t-1}$ , or at least that its movement between the score range intervals depends only on which interval it is currently in. Given that the behavioural score is in state  $s_i$ ,  $i = 1, \dots, n$ , at time  $t - 1$ , we write the latent variable  $U_t$  at time  $t$  as  $U_t^i$ . For the active accounts,  $U_t^i$  is defined so that the relationship between  $B_t$  and  $U_t^i$  is

$$B_t = s_j \Leftrightarrow \mu_j^i \leq U_t^i \leq \mu_{j+1}^i, \quad j = 0, 1, \dots, n$$

with  $\mu_0 = -\infty$ ,  $\mu_{n+1} = \infty$ , (8)

where  $\mu_j^i$  are the values in the unobservable creditworthiness which correspond to the end points of the behavioural score intervals  $s_i$ . Moreover, one chooses  $\mu_1^i$  so that if the consumer defaults, one must have  $U_t^i \leq \mu_1^i$ . The dynamics of the underlying variable  $U_t^i$  are assumed to be related to the explanatory variable vector  $x_{t-1}$  by a linear regression of the form  $U_t^i = -\beta_i' x_{t-1} + \varepsilon_t^i$ , where  $\beta_i$  is a column

vector of regression coefficients and  $\varepsilon_t^i$  are random error terms. If the  $\varepsilon_t^i$  are standard logistic distributions, then this is a cumulative logistic regression model, and the transition probabilities of  $B_t$  are given by

$$\begin{aligned} \text{Prob}(B_t = D | B_{t-1} = s_i) &= \text{logit}(\mu_1^i + \beta_i' x_{t-1}), \\ \text{Prob}(B_t = s_1 | B_{t-1} = s_i) &= \text{logit}(\mu_2^i + \beta_i' x_{t-1}) \\ &\quad - \text{logit}(\mu_1^i + \beta_i' x_{t-1}), \end{aligned} \quad (9)$$

$\vdots$

$$\text{Prob}(B_t = s_n | B_{t-1} = s_i) = 1 - \text{logit}(\mu_n^i + \beta_i' x_{t-1}).$$

Estimating the cumulative logistic model using usual maximum likelihood means that, conditional on the realization of the time dependent covariate vector  $x_{t-1}$ , transitions to various states for different borrowers in the next time period are independent, both cross-sectionally and through time. Thus, the dynamics of the behavioural scores are driven by the explanatory variable  $x_{t-1}$ . In the model presented here, we assume three types of drivers: economic variables, the age of the loan and the previous behaviour of the score. We justify these choices in Sections 5–7 by looking at their effect on the simple first order Markov chain model. Note that states  $C$  and  $D$  are absorbing states, and thus there are no transitions from them; we will discuss the modeling of movements to the closed state,  $C$ , in Section 8

This has parallels with some of the corporate credit risk models. In credit metrics, for example (Gordy, 2000), the transitions in corporate ratings are given by changes in the underlying “asset” variables in a similar fashion, but with quite different drivers.

Since the behaviour scores are only calculated monthly, the calendar time  $t$  needs to be discrete; then, the creditworthiness at time  $t$  of a borrower, whose creditworthiness at time  $t - 1$  was in state  $i$ , is given by the latent variable  $U_t^i$ , which satisfies the relationship

$$U_t^i = - \sum_{k=2}^K a_{ik} \text{State}_{t-k} - \mathbf{b}_i' \text{EcoVar}_{t-1} - c_i \text{MoB}_{t-1} + \varepsilon_t^i \quad (10)$$

where  $\text{State}_{t-k}$  is a vector of indicator variables denoting the borrower's state at time  $t - k$ ,  $\text{EcoVar}_{t-1}$  is a vector of economic variables at time  $t - 1$ , and  $\text{MoB}_{t-1}$  is a vector of indicator variables denoting the length of time (in months) the loan has been on the books (Months on Books) at time  $t - 1$ . One could smooth this latter effect by using a continuous variable of the age of the loan, but we instead describe the effect using more predictive binary variables for different age bands.  $a$ ,  $\mathbf{b}$ , and  $c$  are coefficients in the expression, and  $\varepsilon_t^i$  is a random variable representing a logit error term. Since  $U_t^i$  depends on  $i$ , the underlying creditworthiness at time  $t$  depends on the state at  $t - 1$ , and thus the behavioural score at time  $t$  will also depend on the state, and hence the behavioural score, at time  $t - 1$ . If  $a_{ik} \neq 0$ , then the creditworthiness at time  $t$  also depends on the state at time  $t - k$ , and thus the Markov chain model of the corresponding behavioural scores  $B_t$  will be of order  $k$ .

**Table 1**  
First order average transition matrix.

Initial state	Transition state						
	13–680	681–700	701–715	716–725	726–high	Closed	Default
13–680	49.0 (0.2)	22.1 (0.2)	9.6 (0.1)	4.0 (0.1)	4.0 (0.1)	4.7 (0.1)	6.7 (0.1)
681–700	15.7 (0.1)	34.7 (0.2)	25.1 (0.2)	9.6 (0.1)	11.2 (0.1)	2.8 (0.1)	0.8 (0.0)
701–715	6.0 (0.1)	13.6 (0.1)	35.9 (0.2)	18.1 (0.1)	23.4 (0.1)	2.6 (0.1)	0.5 (0.0)
716–725	3.0 (0.1)	6.1 (0.1)	15.7 (0.1)	28.3 (0.2)	44.1 (0.2)	2.5 (0.1)	0.3 (0.0)
726–high	0.7 (0.0)	1.2 (0.0)	2.7 (0.0)	4.3 (0.0)	88.4 (0.0)	2.4 (0.0)	0.2 (0.0)

The transitions also depend on economic variables and on the length of time the loan has been being repaid. Since the coefficients depend on  $i$ , the impact of these other factors will vary from state to state. If the score band intervals were of equal length and the decomposition in Eq. (3) really held, then one would expect  $a_{ik} = 0$ ,  $c_i = 0$ ,  $b_i = \mathbf{b}$ , and thus this model allows for more complex dynamics in the behavioural scores.

The Months on books term does not occur in any corporate credit model, but is of real importance in consumer lending (Breedon, 2007; Stepanova & Thomas, 2002). Similarly, it is rare to have higher order Markov chains models in corporate credit, although the state space is sometimes extended to include whether there have recently been upgrades or downgrades in the ratings. Thus, although corporate credit models may have more complex factors affecting their dynamics, such as the industry type, geographical area and seniority of the debt, they are not affected so much by recent changes of state or the age of the loan, which are important in consumer credit risk models.

#### 4. Data description

The data set used for the case study in this paper contains records of the credit card customers of a major UK bank who were on the books as of January 2001, together with all those who joined between January 2001 and December 2005. The data set consists of customers' monthly behavioural scores, along with the information on their time since account opened, time to default or time when the account was closed within the above period. We randomly selected approximately 50,000 borrowers to form a training data set which contained their histories over the period January 2001–December 2004. We tested our Markov models using the customer's performance during 2005 from a subsample of the 50,000 and also from a holdout sample of approximately 15,000 customers. Anyone who became 90 days delinquent (even if this was subsequently cured), was charged off, or was declared bankrupt, is considered as having defaulted.

The bank reported that there were no major changes in credit limit setting or minimum repayment levels during the period under consideration, nor were there any changes to the scorecard or intentional attempts to change the mix of the portfolio of borrowers through portfolio acquisition or marketing campaigns. To analyse

the changes in the distribution of behavioural score, we first coarsely divide the behavioural scores into various segments. Initially, we segment the behavioural score into deciles of the distribution of the score among all of the borrowers in the sample over all of the months in the sample. We use the chi-square statistic to decide whether to combine adjacent deciles if their transition probabilities are sufficiently similar. This technique of coarse classifying is standard in scorecard building (Thomas, 2009a) for dealing with continuous variables where the relationship with default is nonlinear. In this case, it led to a reduction to five scorebands, namely  $s_1 = \{13\text{--}680\}$ ,  $s_2 = \{681\text{--}700\}$ ,  $s_3 = \{701\text{--}715\}$ ,  $s_4 = \{716\text{--}725\}$  and  $s_5 = \{726\text{ and above}\}$ . In addition to these five states, there are two more special states corresponding to Default and Account Closed. If there are too many states in the chain, the parameter estimates lose robustness, while if there are too few one loses structure and does not have enough segments to validate the model according to the Basel Accord requirements.

Behavioural scores are generated or updated every month for each individual, so it would be possible to estimate a 1-month time step transition matrix. Since transitions between some states will have very few 1 month transitions, such a model may lead to less than robust estimates of the parameters. Hence, we use 3-month time steps. Longer time steps, say six or twelve months, make it harder to include the impact of the changes in economics and the months on books effect. In the following sections we will justify the use of higher order Markov chains and provide an analysis of the effects of the time varying macroeconomic and months on books covariates on behavioural score transitions.

#### 5. Order of the transition matrix

We first estimate the average transition matrix, assuming that the Markov chain is stationary and first order, using the whole duration of the sample from January 2001 to December 2004. Table 1 shows the 3-month time step transition matrix for that sample, where the figures in brackets are the standard sampling errors. As one might expect, once a borrower is in the least risky state ( $s_5$ ), there is a high probability, 88%, that they will stay there in the next quarter. More surprisingly, the state with the next highest probability of the borrower staying there is  $s_1$ , the riskiest behavioural score state, while the borrowers in the

**Table 2**  
Second order average transition matrix.

(Previous state, current state)	Terminal state						
	13–680	681–700	701–715	716–725	726–high	Closed	Default
(13–680, 13–680)	58.0	19.2	6.9	2.3	1.6	5.0	7.0
(681–700, 13–680)	42.2	27.8	12.2	4.2	3.2	3.8	6.6
(701–715, 13–680)	36.7	28.3	13.0	6.5	5.2	4.2	6.1
(716–725, 13–680)	34.7	23.8	15.4	8.4	7.0	3.8	6.9
(726–high, 13–680)	22.8	18.9	16.0	9.5	19.9	5.2	7.7
(13–680, 681–700)	24.5	36.7	21.3	7.0	6.6	3.1	0.8
(681–700, 681–700)	14.0	40.4	25.7	8.2	7.9	3.1	0.7
(701–715, 681–700)	12.4	34.4	29.4	10.1	10.3	2.7	0.7
(716–725, 681–700)	13.8	27.7	26.8	12.9	15.5	2.5	0.8
(726–high, 681–700)	9.3	20.9	23.0	15.0	28.5	2.4	1.0
(13–680, 701–715)	14.2	19.0	28.2	17.6	17.0	3.6	0.5
(681–700, 701–715)	7.6	19.8	36.6	15.8	17.1	2.5	0.6
(701–715, 701–715)	4.7	12.2	45.7	17.7	16.7	2.6	0.4
(716–725, 701–715)	4.2	11.0	36.6	22.5	22.6	2.6	0.5
(726–high, 701–715)	4.3	8.9	24.1	18.3	41.3	2.6	0.6
(13–680, 716–725)	9.9	11.8	16.7	20.9	37.1	3.2	0.6
(681–700, 716–725)	4.9	11.3	19.8	22.6	37.7	3.4	0.2
(701–715, 716–725)	3.0	7.5	21.6	28.9	36.0	2.7	0.3
(716–725, 716–725)	2.4	4.5	15.5	42.1	32.9	2.4	0.3
(726–high, 716–725)	1.8	4.1	12.3	23.6	55.4	2.5	0.3
(13–680, 726–high)	5.5	5.6	7.9	8.5	69.3	3.1	0.2
(681–700, 726–high)	3.1	6.4	10.2	12.1	64.7	3.2	0.3
(701–715, 726–high)	2.1	4.1	9.6	12.2	68.8	2.9	0.3
(716–725, 726–high)	1.5	3.0	6.6	12.1	73.8	2.8	0.2
(726–high, 726–high)	0.5	0.8	2.0	3.4	90.7	2.4	0.2

other states move around more. The probabilities of defaulting in the next quarter are monotonic, with, as one would expect, 13–680 being the most risky state, with a default probability of 6.7%, and 726–high the least risky state, with a default probability of 0.2%. Note that there is obvious stochastic dominance ( $\sum_{j \geq k} p_{ij} \leq \sum_{j \geq k} p_{i+1j}$ ) for all of the active states, which shows that the behavioural score correctly reflects future score changes, as well as future defaults.

This first order Markov chain model assumes that the current state has all of the information needed in order to estimate the probability of the transitions next quarter, and thus these are not affected by the borrower's previous states. If this is not true, one should use a second or higher order Markov chain model. This might seem surprising, in that a behavioural score is considered to be a sufficient statistic of the credit risk. However, this is a very specific credit risk – the chance of default in the next 12 months – whereas the Markov chain describes the dynamics of the credit risk estimates over a different 12 month interval each period. Thus, it is quite possible that the score does not include all of the information needed to estimate how this risk is likely to change. Table 2 displays the estimates of the transition matrix for such a second order chain, obtained in a similar way as Table 1. Analysing Table 2 shows that there are substantial changes in the transition probabilities based on the previous state of the borrower. Consider, for example, if the current state is the risky one  $s_1 = \{13-680\}$ . If the borrowers were also in the risky state last quarter, then their chance of either staying in it or defaulting in the next quarter is  $58\% + 7\% = 65\%$ ; if they were in the least risky state in the last quarter  $\{726+\}$  but are now in  $s_1$ , the chance of being in  $s_1$  or defaulting next quarter is  $22.8\% + 7.7\% = 30.5\%$ .

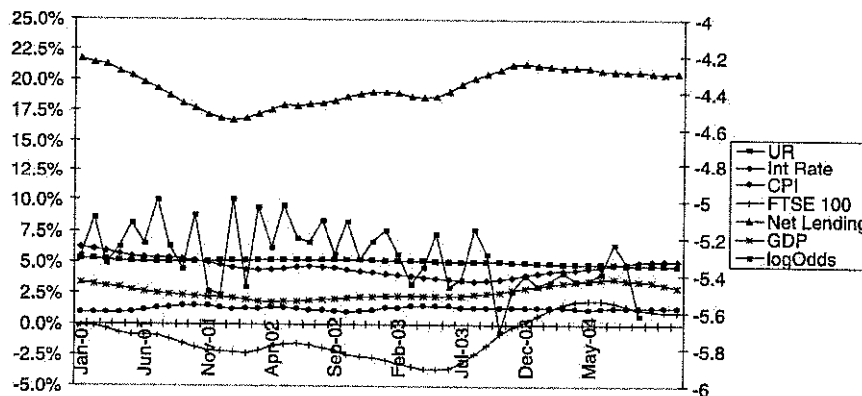
Thus, there is a propensity to reverse direction and return in the direction one came. This effect is seen in all five of the behavioural score interval states in the model. These results do not support the “momentum” idea, that borrowers whose score has dropped are more likely to drop further (see Bangia et al., 2002, for examples in corporate credit), but suggests there may be an event of very short duration which appears and then is reversed in the next quarter, such as being put in arrears due to some misunderstanding. This effect, seen in all five states, could be due to using score bands rather than the scores themselves, and thus the previous score band might suggest where in the interval the score is. However, the same result was seen when a finer classification, i.e. more states with smaller intervals, was used. One could investigate whether higher order models are even more appropriate, but for third and higher order Markov chains, data sparsity and robustness of predictions become problems, and so we use a second order chain to model the dynamics of the behavioural scores.

## 6. Macroeconomic variables

Traditionally, behavioural score models are built on customers' performances with the bank over the previous twelve months, using characteristics like the average account balance, number of times in arrears and current credit bureau information. Thus, the behavioural score can be considered as capturing the borrower's specific risk. However, it was shown for corporate credit risk models (Das, Duffie, Kapadia, & Saita, 2007) that although the borrower-specific risk is a major factor, systemic risk factors emerge during economic slowdowns, and have had a substantial effect on the default risk in a portfolio of loans.

**Table 3**  
Correlation matrix of macroeconomic factors.

	Interest rate	% change in CPI	% change in GDP	% change in net lending	Unemployment rate	Return on FTSE 100
Interest rate	1	<b>−0.51</b>	<b>0.34</b>	0.14	0.01	<b>0.39</b>
% change in CPI	<b>−0.51</b>	1	<b>−0.11</b>	<b>−0.23</b>	<b>−0.45</b>	<b>−0.09</b>
% change in GDP	<b>0.34</b>	<b>−0.11</b>	1	<b>0.85</b>	<b>−0.71</b>	<b>0.87</b>
% change in net lending	0.14	<b>−0.23</b>	<b>0.85</b>	1	<b>−0.49</b>	<b>0.70</b>
Unemployment rate	0.01	<b>−0.45</b>	<b>−0.71</b>	<b>−0.49</b>	1	<b>−0.73</b>
Return on FTSE 100	<b>0.39</b>	<b>−0.09</b>	<b>0.87</b>	<b>0.70</b>	<b>−0.73</b>	1



**Fig. 1.** 3-month observed log(default odds) and macroeconomic variables.

The decomposition of the behavioural score in Eq. (3) suggests that this is also the case in consumer lending, since the population log odds  $s_{pop}(t)$  must be affected by such systemic changes in the economic environment. The question is, which economic variables affect the default risk of consumers? We investigate five variables which have been suggested in consumer finance as being important (Liu & Xu, 2003; Tang, Thomas, Thomas, & Bozzetto, 2007), together with one variable which reflects market conditions in consumer lending. The variables considered are:

- Percentage change in the consumer price index over 12 months: reflects the inflation felt by customers; high levels may cause an increase in the customer default rate.
- Monthly average sterling inter-bank lending rate: higher values correspond to a general tightness in the economy, as well as increases in debt service payments.
- Annual return on FTSE 100: gives the yield from the stock market and reflects the buoyancy of industry.
- Percentage change in GDP compared with the equivalent quarter in the previous year.
- UK unemployment rate.
- Percentage change in net lending over 12 months: this gives an indication of the funds being made available for consumer lending.

There is a general perception (Figlewski, Frydman, & Liang, 2007) that changes in economic conditions do not have an instantaneous effect on the default rate. To allow for this, we use lagged values of the macroeconomic covariates in the form of a weighted average over a six month period, with an exponentially declining weight of 0.88.

This choice is motivated by the recent study by Figlewski et al. (2007). Since macroeconomic variables represent the general health of the economy, they are expected to show some degree of correlation. Table 3 shows the pairwise correlation matrix for the six macroeconomic variables above, with no lags considered. The entries in bold are the correlations which are considered to be statistically significant at the 5% level. Thus, at the 5% significance level, the interest rate is negatively correlated with the percentage change in CPI and positively correlated with the percentage change in the GDP and the return on the FTSE 100. Similarly, the percentage change in net lending is negatively correlated with the unemployment rate and positively correlated with the percentage change in GDP and return on the FTSE 100 at the 5% significance level. The presence of a non-zero correlation between the variables does not invalidate the model, but the degree of association between the explanatory variables can affect the parameter estimation. Moreover, the variables used are chosen in order to avoid long run trends, and the fact that three of the variables are percentage changes is akin to already taking differences in order to avoid non-stationarity.

Fig. 1 shows the variation of the observed log(Default Odds) over 3-month windows, compared with the lagged macroeconomic factor values used in the analysis for the sample duration of January 2001 to December 2004. The macroeconomic factor values are represented by the primary y-axis, and the log(Default Odds) by the secondary y-axis.

We plot the lagged economic values for each month, though of course we only use the values every quarter in the Markov chain model, since it is quarterly. In the benign

**Table 4**  
Comparison of transition matrices at different calendar times.

Comparison of transition matrices at different calendar times								
Initial state	Terminal state						Default	Number in state
	13-680	681-700	701-715	716-725	726--	Closed		
January-December 2001								
13-680	52.90	21.77	9.24	3.62	3.67	3.31	5.50	24,015
681-700	17.80	35.56	23.86	9.51	10.40	2.14	0.72	25,235
701-715	8.74	14.84	35.25	17.90	22.72	2.16	0.40	31,477
716-725	3.28	6.99	16.84	27.85	42.64	2.12	0.29	27,781
726--	0.72	1.35	2.86	4.30	88.39	2.10	0.28	220,981
October 03-September 04								
13-680	46.24	22.68	9.30	4.03	4.18	5.35	8.22	24,060
681-700	14.79	35.62	23.25	9.80	10.99	2.74	0.82	25,235
701-715	5.42	13.42	37.30	18.20	22.89	2.33	0.43	42,200
716-725	2.68	5.63	16.17	29.34	43.79	2.05	0.33	38,932
726--	0.62	1.14	2.65	4.69	88.80	1.90	0.19	289,814

**Table 5**  
Comparison of transition matrices for loans of different ages.

Comparison of transition matrices for loans of different ages.								
Initial state	Terminal state						Default	Number in state
	13-680	681-700	701-715	716-725	726-	Closed		
1-12 months (new obligors)								
13-680	51.0	22.3	8.1	3.1	2.0	5.8	7.6	24,858
681-700	18.2	35.6	24.2	9.3	8.7	3.2	0.8	22,019
701-715	8.1	15.9	30.5	17.8	25.6	2.7	0.5	21,059
716-725	4.5	8.2	14.7	21.4	48.6	2.2	0.3	18,050
726-	1.8	3.0	5.7	7.6	79.3	2.3	0.2	59,767
49-high (mature obligors)								
13-680	44.1	23.5	11.3	4.9	7.0	4.0	5.3	28,604
681-700	13.6	32.5	25.6	10.7	14.4	2.5	0.6	39,835
701-715	4.7	11.8	37.2	18.8	24.8	2.5	0.3	66,389
716-725	2.1	5.0	14.9	30.4	44.7	2.6	0.3	67,660
726-	0.4	0.9	2.1	3.7	90.4	2.4	0.2	698,782

environment of 2001–04 there are no large swings in any variable, and the log of the default odds –  $s_{\text{pop}}(t)$  – is quite stable.

To convince ourselves that changes in economic conditions do affect the transition matrix, we look at transition matrices based on data from two different time periods, which have slightly different economic conditions. In order not to complicate matters, we show the differences that occur even in the first order Markov chain. In Table 4, we estimate the first order transition probability matrices for two different twelve-month periods between January 2001 and December 2004, in order to determine the effect of calendar time on transition probabilities. The first matrix is based on a sample of customers who were on the books during the period January–December 2001, and uses their transitions each quarter during that period, while the second is based on those in the portfolio during the period September 03–October 04 and their performance over that period. Both transition matrices are quite similar to the whole sample average transition matrix in Table 1, with the probability of moving into default decreasing as the behavioural score increases and the stochastic dominance effect still holds. However, there are some significant differences between the transition probabilities of the two matrices in Table 4. For example, borrowers who were in a current state of  $s_1 = \{13-680\}$  during the period January–December 2001 have a lower probability of defaulting in the next quarter – 5.5% –

than those who were in the same state during the period September 03–October 04, where the value is 8.22%. We test the differences between the corresponding transition probabilities in the two matrices in Table 4 using the two-proportion z-test with unequal variances. The entries in bold in Table 4 identify the transition probabilities where the differences between the corresponding terms in the two matrices are significant at the 5% level. Note that there are 35 transition probabilities being compared, and thus one might expect 2 significant comparisons at the 5% level if there were really no difference. In actual fact, there are 20 significant differences, which suggests that this calendar effect is real.

## 7. Months on books effects

As is well known in consumer credit modeling (Breen, 2007; Stepanova & Thomas, 2002), the age of the loan (number of months since the account was opened) is an important factor in the default risk. To investigate this, we split the age into seven segments, namely 0–6 months, 7–12 months, 13–18 months, 19–24 months, 25–36 months, 37–48 months, and more than 48 months. The effect of age on behavioural score transition probabilities can be seen in Table 5, which shows the first order probability transition matrices for borrowers who had been on the books for between one and twelve months (upper section) or more than 48 months (lower section). Again, the overall structure is similar to that of Table 1,

Table 6

Parameters for a second order Markov chain with age and economic variables.

Parameter estimates	Initial behavioural score									
	13–680	Std error	681–700	Std error	701–715	Std error	716–725	Std error	726–high	Std error
Interest rate	0.0334	(0.0161)	0.092	(0.0143)	0.0764	(0.0123)	0.0834	(0.0134)	0.0778	(0.00885)
Net lending					0.0129	(0.00489)				
Months on books										
0–6	–0.027	(0.0351)	0.0161	(0.0347)	–0.2182	(0.0368)	–0.1637	(0.0448)	–0.0849	(0.0315)
7–12	0.2019	(0.0241)	0.1247	(0.0225)	0.2051	(0.0226)	0.2317	(0.0261)	0.3482	(0.018)
13–18	0.2626	(0.0262)	0.2663	(0.0236)	0.2301	(0.0228)	0.2703	(0.0268)	0.2554	(0.0193)
19–24	–0.07	(0.0275)	–0.0796	(0.0251)	–0.1001	(0.0241)	–0.0873	(0.0284)	0.031	(0.0206)
25–36	–0.0015	(0.0244)	–0.0521	(0.0223)	0.00191	(0.0198)	–0.00487	(0.0229)	–0.0254	(0.0162)
37–48	–0.0703	(0.0262)	–0.0519	(0.0243)	0.019	(0.0206)	–0.0801	(0.0241)	–0.00709	(0.0166)
49–high	–0.2957		–0.2235		–0.13781		–0.16603		–0.51721	
SecState										
13–680	0.8372	(0.0165)	0.6762	(0.0168)	0.5145	(0.0222)	0.3547	(0.0337)	0.381	(0.0399)
681–700	0.2365	(0.0201)	0.2847	(0.0139)	0.3598	(0.0146)	0.1942	(0.0224)	0.5168	(0.024)
701–715	–0.0111	(0.0249)	0.0491	(0.0168)	0.1314	(0.0119)	0.1255	(0.0164)	0.2991	(0.0178)
716–725	–0.1647	(0.0345)	–0.1764	(0.0239)	–0.1795	(0.016)	0.0098	(0.0152)	0.0525	(0.0162)
726–high	–0.8979		–0.8336		–0.8262		–0.6842		–1.2494	
Intercept/barrier										
Default	–3.213	(0.0756)	–5.4389	(0.0826)	–5.8904	(0.1285)	–6.011	(0.0967)	–5.1834	(0.0506)
13–680	–0.2078	(0.0734)	–2.179	(0.0657)	–3.2684	(0.1175)	–3.6011	(0.0648)	–3.8213	(0.0436)
681–700	1.022	(0.0736)	–0.3978	(0.0649)	–1.9492	(0.1168)	–2.461	(0.062)	–2.9445	(0.0421)
701–715	1.9941	(0.0746)	0.861	(0.065)	–0.1796	(0.1165)	–1.2049	(0.0611)	–2.06	(0.0415)
716–725	2.7666	(0.0764)	1.6267	(0.0656)	0.7317		0.171	(0.0609)	–1.326	(0.0413)
Likelihood ratio	3661.078		3379.459		4137.587		2838.765		20400.65	
p-value	<0.0001		<0.0001		<0.0001		<0.0001		<0.0001	

but there are significant differences between the transition probabilities of the two matrices. Borrowers who are new on the books are at greater risk of defaulting or of having their behavioural score drop than those who have been with the bank for more than four years.

Again, the final block of Table 5 gives the z-statistic, and the bold values indicate transitions where the differences between the new and mature accounts are statistically significant at the 5% level. This occurs in 27 of the 35 transitions calculated.

## 8. Modeling transition probabilities

Behavioural score segments have a natural ordering structure, with low behavioural scores being associated with high default risks, and vice versa. This is the structure that is exploited when using cumulative (ordered) logistic regression to model borrowers' transition probabilities, as suggested in Section 3 (McElvey & Zavoina, 1975).

The cumulative logistic regression model is appropriate for modelling the movement between the behavioural scorebands and the default state. If one also wished to model whether the borrowers close their accounts or not, one would need to use a two stage model. In the first stage, one would use logistic regression to estimate the probability of the borrower closing the account in the next quarter, given his current state,  $P(\text{Close}|\text{beh.score band})$ . The second stage would be the model presented here, showing the movement between the different scorebands, including default conditional on the borrower not closing the account. To arrive at the final transition probabilities, one would need to multiply the probabilities for each

transition obtained in this second stage by the chance that the account is not closed, as obtained from the first stage  $(1 - P(\text{Close}|\text{beh.score band}))$ . This approach assumes that the residuals of the estimations in the two stages are independent.

Thus, we now fit the cumulative logistic model in order to estimate the transition probabilities of the movement of a borrower's behavioural score, from being in state  $i$  at time  $t - 1$   $B_{t-1} = s_i$  to where the borrower will be at time  $t$ ,  $B_t$ . These transitions depend on the current state  $B_{t-1} = s_i$  (since they are indexed by  $i$ ), the previous state of the borrower,  $B_{t-2}$ , the lagged economic variables and the age of the loan (months on books, or MoB). Thus, one uses the model given by Eqs. (6) and (8), but restricted to the second order case, namely

$$B_t = s_j \Leftrightarrow \mu_j^i \leq U_t^i \leq \mu_{j+1}^i, \quad j = 0, 1, \dots, n$$

$$\text{with } \mu_0 = -\infty, \mu_{n+1} = \infty \quad (11)$$

$$U_t^i = -a_i \text{State}_{t-2} - b_i \text{EcoVar}_{t-1} - c_i \text{MoB}_{t-1} + \varepsilon_t^i.$$

In order to choose which economic variables to include, we recall that Table 3 described the correlations between the variables. To reduce the effects of such correlations (so that the coefficients of the economic variables are understandable), we considered various subsets of the macroeconomic variables as predictors in a cumulative logistic model, where there was little correlation between the variables. In Table 6, we present parameter estimates for the cumulative logistic models for each behavioural score segment with only two macroeconomic variables, namely interest rates and net lending, along with months on books and the previous state. This means that we allow

**Table 7**  
Distribution at the end of each time period for the out-of-sample test period (2005).

Behavioural score segments	1-period				2-period				3-period				4-period			
	Initial distribution matrix	Average	Model predicted	Observed	Average matrix	Model predicted	Observed	Average matrix	Model predicted	Observed	Average matrix	Model predicted	Observed	Average matrix	Model predicted	Observed
13–680	571	520	560	457	498	561	384	475	566	424	457	573	368			
681–700	659	659	696	595	635	702	594	612	711	604	592	719	592			
701–715	1094	1011	1066	982	969	1065	918	935	1073	1007	908	1081	938			
716–725	973	936	1027	952	902	1036	1038	878	1044	971	859	1049	943			
726–high	7436	7535	7304	7666	7589	7208	7644	7627	7098	7511	7647	6989	7612			
Default	0	72	80	81	140	160	155	206	241	216	270	322	280			

the drivers of the dynamics — economic variables and the current duration of the loan — to have different effects on the transitions from different states. The model with these two variables — interest rates and net lending — provided a better fit in terms of the likelihood ratio of the model than other combinations of macroeconomic variables; the next best fit was unemployment and interest rates. We employ stepwise selection, keeping only variables with a 5% significance level for the corresponding regression coefficient to be non-zero. The likelihood ratios and the associated *p*-values show that, for each current behavioural score segment, transitions to other states in the next time period are significantly influenced by current macroeconomic factors, the current months on books and information on the previous state, as represented by the Secstate variable in Table 6. This model fits the data better than the first order average transition matrix. A positive sign on the coefficient in the model is associated with a decrease in creditworthiness, and vice versa. Thus, the creditworthiness of borrowers will decrease in the next time period, given an increase in interest rates in all current behavioural score segments.

Borrowers who have been on the books for between 7 and 18 months have higher default and downgrading risks than the others. This confirms the market presumption that new borrowers have a higher default risk than older borrowers in any given time period, once they have had sufficient time (i.e. at least 3 months) to default. The coefficients of the Secstate variable, with one exception, decrease monotonically in value from the  $s_1 = \{13\text{--}680\}$  category to the  $s_5 = \{726\text{--high}\}$  state. Those with a lower behavioural score last quarter are more likely to have a lower behavioural score next quarter than those with the same current behavioural score but higher previous behavioural scores. Thus, the idea of credit risk continuing in the same direction is not supported.

## 9. Forecasting multi-period transition probabilities

The model with the parameters given in Table 6 was tested by forecasting the future distributions of the scorebands in the portfolio, including those who have defaulted. The forecast uses the Markov assumption, and thus multiplies the probability transition matrix by itself the appropriate number of times to obtain the forecasts. In the first case, we considered all non-defaulted borrowers in December 2004 and used the model to predict their

distribution over the various behavioral score bands and the default state at the end of each quarter of 2005, where closures were dealt with as described in Section 8. In order to avoid adding extra uncertainty to the forecast, the 2005 values of the two economic variables were used. The results are shown in Table 7. The initial distribution column gives the distribution of borrowers across each behavioural score segment in the test sample in December 2004. The observed column gives the observed distribution of borrowers at the end of each quarter of 2005. The other two columns give the expected numbers of borrowers in each segment at the end of each quarter of 2004, as predicted by the second order average transition matrix in Table 2 and the model in Table 6, respectively.

The second order Markov chain model with economic variables gave predictions, particularly for defaults, which were very close to the actual values for the first two quarters, but began to overestimate the risks thereafter. Thus, by the fourth quarter, the average second order Markov chain model which just takes the average of the transition probabilities is superior.

The analysis was repeated on an out-of-time and out-of-sample portfolio. Again, the distribution of the portfolio at the start of the period (April 2005) was given, and estimates for the next three quarters were obtained using the model in Table 6. The results in Table 8 show that the second order model with economic variables and the months on books effect (Table 6) is better at predicting the actual number of defaults than the second order model without these effects (Table 3), even though both approaches underpredict slightly. The model with the extra drivers is better at predicting the numbers in the default and high risk states, while the second order model which just averages over all transitions is better at predicting the numbers in the low risk categories. In this data set it appears that the second order effect is the most important, followed by the months on books effect. However, this could be due to the relative economic stability throughout the periods represented by both the development sample and the out-of-sample test period.

## 10. Conclusions

The paper has developed a pilot scheme for how one could use a Markov chain approach based on behavioural scores to estimate the credit risk of portfolios of consumer loans. This is an attractive approach, since behavioural

Table 8

Distribution at the end of each time period for the out-of-time out-of-sample test period (2005).

Behavioural score segments	1-period				2-period			3-period		
	Initial distribution	Average matrix	Model predicted	Observed	Average matrix	Model predicted	Observed	Average matrix	Model predicted	Observed
13–680	1428	949	1040	1199	879	983	1080	769	889	1043
681–700	1278	1054	1117	1096	978	1061	1076	894	996	1001
701–715	1379	1291	1384	1257	1262	1393	1316	1216	1363	1219
716–725	876	1047	1178	812	1051	1228	774	1044	1234	718
726–high	7514	7994	7621	7968	8059	7535	7943	8208	7596	8074
Default	0	139	134	143	245	274	286	344	397	420

scores are calculated monthly by almost all lenders in consumer finance, both for internal decision purposes and for Basel Accord requirements. The paper emphasises that behavioural scores are dynamic, and since they do have a “systemic” factor – the population odds part of the score – the dynamics depend on changes in the economic conditions. The paper also suggests that one needs to consider carefully the appropriate order of the Markov chain. Table 2 shows the impact of the previous state and the current state on the subsequent transition, and strongly indicates the need for a second order Markov chain.

Unlike for corporate credit risk, one also needs to include the age of the loan in the modelling, as this affects the credit risk. The out-of-sample comparison of second order models with and without economic factors and age in the model is inconclusive about which model is better, but this was at a time when the economic conditions were very stable. In more volatile conditions, or if one wanted to use the model for stress testing, it would be essential to include the economic effects in the modelling.

Such models are relatively easy for banks to develop, since all of the information is readily available. The model would be useful for a number of purposes: debt provisioning estimation, stress testing in the Basel context, and investigating the relationship between point in time behaviour scores and through the cycle probabilities of default by running the model through an economic cycle. The model could also be used by ratings agencies to update their risk estimates of the securitized products based on consumer loan portfolios. This would require them to obtain regular updates of the behavioural scores of the underlying loans, rather than the present approach of only making one initial rating, based on an application or bureau score. This involves extra work, but might avoid the failure of the ratings of the mortgage backed securities (MBS) seen in 2007 and 2008, and would certainly give early warning of the increasing credit risk of such securities.

There are still issues to be resolved in regard to modelling the credit risk of consumer loan portfolios. One important issue is to identify which economic variables affect consumer credit risk most, and hence should be included in such models. One would expect some differences between such a list and the variables which have been recognised in corporate credit risk modelling, and one may even want to use different variables for different types of consumer lending. For example, house price movements will be important for mortgages but may

be less important for credit cards. One also feels that some of the variables in the models should reflect the market conditions as well as the economic conditions, because the tightening in consumer lending which prevented customers from refinancing did exacerbate the problems of 2007 and 2008. This paper has described how such information on economic and market conditions can be used in conjunction with behavioural scores to estimate portfolio-level consumer credit risks. It points out that although Markov chain models based on behavioural scores have been used by the industry, they have not previously appeared in the literature, and there has certainly been no extension of the model to include the maturity of the loan, the economic factors or the need for higher order Markov chains.

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# M345S17 SOLUTIONS

seen ↓

1. (a) Up to four of the following reasons:-

- \* They naturally model the loan default process and incorporate situations when a case has not defaulted in the observation period (censored);
- \* Their use avoids the need to define a fixed period within which default is measured;
- \* They provide a clear approach to assess the expected profitability of a borrower;
- \* Survival estimates provide a forecast of default probability as a function of time from a single equation;
- \* They allow the inclusion of behavioural and economic risk factors over time.

4

- (b) \*
- TVCs are predictor variables indexed by time.
  - In credit risk, TVCs are used to allow inclusion of behavioural and economic predictor variables.

2

(c) Since

$$S(t) = 1 - F(t) \Rightarrow f(t) = -\frac{dS(t)}{dt},$$

$$\int_0^t h(u)du = \int_0^t \frac{f(u)}{S(u)}du = \int_0^t -\frac{1}{S(u)} \frac{dS(u)}{du} du = \int_{S(0)}^{S(t)} -\frac{1}{s} ds$$

with substitution  $s = S(t)$ . Hence,

$$\int_0^t h(u)du = [-\log s]_1^{S(t)} = -\log(S(t))$$

which gives

$$S(t) = \exp\left(-\int_0^t h(u)du\right).$$

2

unseen ↓

- (d) \*
- Being employed is negatively associated with hazard of default.
  - Higher numbers of delinquencies is positively associated with hazard of default.

2

(e) Hazard ratio is  $\exp(\beta_1(1 - 0)) = \exp(-0.5) = 0.607$ .

1

(f) Up to 3 months, since this is the lag on Delinquency.

2

- (g) Expand formula from part (c) with Cox PH model:

$$S(t) = \exp \left( - \int_0^t h_0(u) \exp(\beta \cdot \mathbf{x}) du \right)$$

$$= \exp \left( - \int_3^6 h_0(u) \exp(\beta \cdot \mathbf{x}) du - \int_6^{12} h_0(u) \exp(\beta \cdot \mathbf{x}) du \right)$$

for  $t = 12$  and knowing  $h_0(u) = 0$  for  $u < 3$ . The terms in the integrals are constants, because of fixed values of baseline hazard and  $X_2(t)$  in those intervals, which means

$$S(12) = \exp \left( -(6 - 3) \times 1/30 \times \exp(0 - 0.2 \times 0) - (12 - 6) \times 1/60 \times \exp(0 + 0.1 \times 1) \right)$$

$$= \exp(-0.1 - 0.1 \times \exp(0.1)) = 0.810$$

4

- (h)  $PD = 1 - S(t) = 0.190$ .

1

- (i) Our prior expectation is that an increase in unemployment rate will have an adverse effect on default, hence the credit risk model should have a positive sign on the coefficient for unemployment rate.

Therefore, since the reported model has a negative sign, this suggests either an underlying problem with the model or that it has not been reported properly.

Either way, we would not deploy this model without further investigation.

2

2. (a) \* Traditional lending starts with a fixed price for credit, then assess an applicant's credit risk to decide whether to accept or reject. seen ↓
- \* In contrast, risk-based pricing works by assessing credit risk, then determining the price to cover that risk.
- \* Risk-based pricing allows for a broadening of who can be accepted for credit, but there are limits to a price that can be offered based on risk appetite, regulation and political/market considerations. 3
- (b) \* Since  $x > 0$ , the denominator of  $q$  is always positive. part seen ↓
- \* Hence, an increase/decrease in credit score,  $x$  will lead to a reduction/increase in the response rate.
- \* This is natural since customers with good credit score, can be more choosy in the credit market. 3
- (c) \* Maximizing expected profit is generally the right approach, but maximizing  $P_E(r, x)$  will not work since this will just take  $r \rightarrow \infty$ . seen ↓
- \* The problem is that  $P_E(r, x)$  does not take account of the potential customer possibly rejecting high interest rates. For that reason, the response rate is introduced and hence  $P_R(r, x)$  is maximized instead. 2
- (d) We have  $P_R(r, x) = (1 + rx)^{-2} (a(x)r - b(x))$ . Hence, part seen ↓

$$\frac{dP_R(r, x)}{dr} = a(x)(1 + rx)^{-2} - 2x(a(x)r - b(x))(1 + rx)^{-3}$$

(1 mark)

Taking stationary points  $r_0$ ,  $a(x)(1 + r_0x) - 2x(a(x)r_0 - b(x)) = 0$

$$\Rightarrow xa(x)r_0 = 2xb(x) + a(x) \Rightarrow r_0 = 2\frac{b(x)}{a(x)} + \frac{1}{x}.$$

(1 mark)

Since  $b(x) > 0$ ,  $a(x) > 0$  and  $x > 0$ ,  $r_0 > 0$  as required. Hence  $\hat{r} = r_0$ . 1 mark

It is possible to take second order conditions to show it is a maxima, but it is easier to do this:-

$$* P_R(0, x) = -b(x),$$

\*

$$P_R(r_0, x) = (1 + r_0x)^{-2} \left( a(x) \left[ 2\frac{b(x)}{a(x)} + \frac{1}{x} \right] - b(x) \right)$$

$$= (1 + r_0x)^{-2} \left( 2b(x) + \frac{a(x)}{x} - b(x) \right) > 0$$

since all terms in the equation are positive.

$$* \text{ As } r \rightarrow \infty, P_R(r, x) \rightarrow 0.$$

Since  $r_0$  is the only stationary point and points to either side have a lower value, this shows it is a maxima. (1 mark)

- (e) Since  $b(x) = p_T(x)l_D e_D L_0$ , an increase in  $l_D$  will lead to an increase in  $r_0$ . This makes sense since the increase in interest rate is required to cover higher risk of loss.

2

- (f) Expand on the term from part (d):

$$\begin{aligned} P_R(r_0, x) &= (1 + r_0 x)^{-2} \left( 2b(x) + \frac{a(x)}{x} - b(x) \right) \\ &= \frac{\left( b(x) + \frac{a(x)}{x} \right)}{\left( 1 + \left[ 2\frac{b(x)}{a(x)} + \frac{1}{x} \right] x \right)^2} \\ &= \frac{\left( b(x) + \frac{a(x)}{x} \right)}{4 \left( 1 + \frac{xb(x)}{a(x)} \right)^2} \end{aligned}$$

- (g) We can write the constraint as  $a(x)\hat{r} - b \geq p_{\min}$ .

2

- \* Therefore, if  $r_0 \geq (p_{\min} + b(x))/a(x)$ , then it remains the optimal price.
- \* Alternatively, we have  $r_0 < \hat{r} \geq (p_{\min} + b(x))/a(x)$ .
- \* Since we know that  $P_R(r, x)$  is a continuous function with a single maxima at  $r_0$ , it must be decreasing with  $r > r_0$ , hence  $\hat{r}$  is the smallest value (greater than  $r_0$ ), given its constraint, ie  $\hat{r} = (p_{\min} + b(x))/a(x)$ .

unseen ↓

So, with the given constraint,

$$\hat{r} = \max(r_0, (p_{\min} + b(x))/a(x)).$$

4

3. (a) Given some cumulative distribution on losses  $F_L$  and a level  $q$ ,  $\text{VaR}(q) = F_L^{-1}(q)$ .  
*Note: the student may express  $q$  as a percentage, hence  $q/100$  will be given as the argument.*

seen ↓

2

- (b) Financial institutions want to cover as much loss as possible, up to some percentage of future outcomes, say 99.9%, given some model of future conditions. VaR is a measure that allows them to compute this loss.

1

- (c) Typically financial values and, in particular, credit losses, have a heavy tail distribution towards larger values. This will lead to considerably higher values of VaR compared to the expected value of loss.

2

- (d)  $\text{Regulatory Capital} = \text{VaR}(q) - \text{Expected Loss}$ .

1

- (e) Merton's one factor model assumes there is just one systematic factor which controls correlation between assets. This then leads to the single correlation term.

2

- (f) Merton's one factor model is a model based on changing value of assets,  $R_i$ , for each borrower  $i$ . The  $\rho$  is a correlation term between any two borrowers' change in assets. Precisely,

$$\rho = \frac{\text{Cov}(R_i, R_j)}{\sqrt{\text{Var}(R_i)\text{Var}(R_j)}}$$

for any  $i \neq j$ .

*It turns out that the variance is a constant, so it is acceptable for student to replace this with  $\sigma^2$ .*

3

- (g) If  $\rho = 0$  then this implies that the coefficient on the systematic factor is 0, which means that economic (environmental) conditions have no impact on risk of default.

1

- (h) \* The BIS believes that changes in economic conditions will have a bigger impact on mortgages than on credit card defaults.  
 \* This becomes evident during times of recession (eg the 2008 financial crisis in USA) when mortgage defaults happen together.  
 \* Looking at Formula (1), we see that higher  $\rho$  leads to higher VaR (hence higher Regulatory Capital), hence for the same unit value, mortgages will require more Regulatory Capital than credit cards.

part seen ↓

3

- (i) Formula (1) would become

unseen ↓

$$K(p_i) = \Phi \left( \frac{\Phi^{-1}(p_i) + T_d^{-1}(q)\sqrt{\rho}}{\sqrt{1-\rho}} \right)$$

where  $T_d$  is the cumulative t-distribution with degrees of freedom  $d$ .

2

- (j)
- \* The essence of the discrepancy is that although the t and normal distributions look very similar, the t-distribution is heavy-tailed.
  - \* Consequently, for large values of  $q$ ,  $T^{-1}(q) \gg \Phi^{-1}(q)$  and this will feed into the Vasicek formula to produce a much larger value of VaR.
  - \* This is a problem, since it means that computation of Regulatory Capital is unstable and is sensitive to the underlying distributional assumption of the effect of the economy/environment on changes in assets. It may not be acceptable to assume that this is normal.

*On the last point, there is room for discussion here and other relevant points should be considered for a mark.*

3

4. (a) \* Behavioural scores are dynamic variables which monitor customer behaviour; typically measuring credit risk (Section 2).  
 \* Typically behavioural scores are log-odds scores (Equation (2)).  
 \* They are analogous to credit ratings used in corporate lending; behavioural scores are for consumer credit.

unseen ↓

3

- (b) A latent variable is a random variable that is not directly observed (1) but are important in the expression of the model and may be inferred (1).

2

- (c) We are interested in modelling  $P(B_t = s_j | B_{t-1} = s_i)$ . The condition is introduced just above (8), "given the behavioural score is in state  $s_i$ ". Then (8) tells us that the event  $B_t = s_j$  is equivalent to  $\mu_j^i \leq U_t^i \leq \mu_{j+1}^i$ , so

$$\begin{aligned} P(B_t = s_j | B_{t-1} = s_i) &= P(\mu_j^i \leq U_t^i \leq \mu_{j+1}^i | B_{t-1} = s_i) \\ &= P(U_t^i \leq \mu_{j+1}^i) - P(U_t^i \leq \mu_j^i) \\ &= P(\varepsilon_t^i \leq \mu_{j+1}^i + \beta_i' x_{t-1}) - P(\varepsilon_t^i \leq \mu_j^i + \beta_i' x_{t-1}) \\ &= \text{logit}(\mu_{j+1}^i + \beta_i' x_{t-1}) - \text{logit}(\mu_j^i + \beta_i' x_{t-1}) \end{aligned}$$

since  $\varepsilon_t^i$  is standard logistic distributed and logit is the CDF function for standard logistic. The conditionality is removed, because the condition is implicit in the  $i$  superscript on the parameters.

The boundary cases are given by

$$\begin{aligned} P(B_t = D | B_{t-1} = s_i) &= \text{logit}(\mu_1^i + \beta_i' x_{t-1}) - \text{logit}(\mu_0^i + \beta_i' x_{t-1}) \\ &= \text{logit}(\mu_1^i + \beta_i' x_{t-1}) - \text{logit}(-\infty) \\ &= \text{logit}(\mu_1^i + \beta_i' x_{t-1}) - 0 \end{aligned}$$

and

$$\begin{aligned} P(B_t = S_n | B_{t-1} = s_i) &= \text{logit}(\mu_{n+1}^i + \beta_i' x_{t-1}) - \text{logit}(\mu_n^i + \beta_i' x_{t-1}) \\ &= \text{logit}(\infty) - \text{logit}(\mu_n^i + \beta_i' x_{t-1}) \\ &= 1 - \text{logit}(\mu_n^i + \beta_i' x_{t-1}) \end{aligned}$$

5

- (d) \* General probability of transitioning to default goes down as current state has a higher behavioural score.  
 \* For the lowest current state, transition to default has higher probability if the previous state was also low or also the very highest score. This consequently reflects second order information which was not reflected in Table 1.

2

- (e)
- \* For *Interest rate*, generally, the coefficient is positive which shows that increasing interest rate leads to higher probability of transition to a more risky state, through Equation (10).
  - \* The magnitude of association of *Interest rate* is stronger for higher initial behavioural scores, which suggests that people with better scores are more vulnerable to interest rate changes.
  - \* For *Months on Book*, generally, the coefficient is smaller for longer periods which shows that longer Months on Book leads to lower probability of transition to a more risky state, through Equation (10).
  - \* The magnitude and profile of association of *Months on Book* shows no particular pattern with respect to the initial behavioural score.

4

- (f)
- \* The intercepts correspond to the  $\mu_j$  in Equation (8). The matrix of these values across different initial behavioural scores correspond to the basic transition matrix (on the log-odds scale) before adjustment by the predictor variables.
  - \* We observe generally lower values for intercepts with increasing initial behavioural score. This is natural since higher behavioural scores are less likely to transition to lower ones.

2

- (g)
- \* Higher order Markov chains are evidently appropriate.
  - \* Age of loan is an important risk factor for consumer credit.
  - \* Inclusion of macroeconomic variables is promising but further research is required.

1 mark for each point, to a maximum of 2.

2