Math40003 **Mid-module test**
Linear Algebra and Groups

1. Let $V = \{a_0 + a_2x^2 + a_4x^4 + a_6x^6 : a_0, a_2, a_4, a_6 \in \mathbb{R}\}$, the vector space over $\mathbb{R}$ with standard addition and scalar multiplication.

   (a) Show that V is a subspace of $\mathbb{R}[x]$ (the set of polynomials in x with coefficients in $\mathbb{R}$).

   Clearly $V \subseteq \mathbb{R}[x]$ so we can use the subspace test:

   S1 Let $a_0 = a_2 = a_4 = a_6 = 0$ then $a_0 + a_2x^2 + a_4x^4 + a_6x^6 = 0 \in V$ so V not empty.

   S2 Let $f(x), g(x) \in V$ then $f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6$, $g(x) = b_0 + a_2x^2 + b_4x^4 + b_6x^6$ for $a_i, b_j \in \mathbb{R}$. Then

$$\begin{aligned} f(x) + g(x) &= (a_0 + a_2x^2 + a_4x^4 + a_6x^6) + (b_0 + b_2x^2 + b_4x^4 + b_6x^6) \\ &= (a_0 + b_0) + (a_2 + b_2)x^2 + (a_4 + b_4)x^4 + (a_6 + b_6)x^6 \in V \end{aligned}$$

   S3 Let $f(x) \in V$, $\lambda \in \mathbb{R}$ then $f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6$. Then

$$\begin{aligned} \lambda f(x) &= \lambda(a_0 + a_2x^2 + a_4x^4 + a_6x^6) \\ &= \lambda a_0 + \lambda a_2x^2 + \lambda a_4x^4 + \lambda a_6x^6 \in V \end{aligned}$$

   (6 marks)

   (b) Find a basis for V. *Justify your answer fully i.e. prove the basis you find is in fact a basis.*

   Let $B = \{1, x^2, x^4, x^6\}$. Claim B is a basis for V:

   i. Spanning. Suppose $f(x) \in V$ then $f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 \in Span(B)$

   ii. Linear Independent. Suppose $a_0 + a_2x^2 + a_4x^4 + a_6x^6 = 0$ then $a_0 = a_2 = a_4 = a_6 = 0$ so B is linearly independent.

   (5 marks)

2. Let $W = \{f(x) \in \mathbb{R}[x] : f(x) = \sum_{i=0}^{m} a_{2i}x^{2i}, a_{2i} \in \mathbb{R}, m \in \mathbb{N} \cup \{0\}\}$, the vector space over $\mathbb{R}$ with standard addition and scalar multiplication.

   (a) Is W finite dimensional?

   (b) Find a basis for W.

In both cases you should justify your answer fully.

(a) **No. Suppose there is a finite set C such that $W \subset Span(C)$. As W is not trivial $C \neq \emptyset$. As C is finite there is $n \in N \cup \{0\}$ such that $n = max\{m : deg(f(x)) = m, f(x) \in C\}$. Now any linear combination of polynomials in C must have degree less than or equal to n. However, $x^{2m+2} \in W$ and $deg(x^{2m+2}) = 2m+2 > m$ so $x^{2m+2} \notin Span(C)$. Contradiction.**

(4 marks)

(b) **Let $D = \{x^{2i} : i \in \mathbb{N} \cup \{0\}\}$. Claim D is a basis for W.**

i. **Spanning. Let $f(x) \in W$ then for some $m \in \mathbb{N} \cup \{0\}$, $f(x) = \sum_{i=0}^{m} a_{2i}x^{2i} \in Span(D)$ as $x^0, ..., x^{2m} \in D$.**

ii. **Linearly independent. Suppose we have a finite linear combination of elements in D that euqals zero, i.e. for some $m \in \mathbb{N} \cup \{0\}$**

$$\sum_{i=0}^{m} a_{2i}x^{2i}$$

Then $a_{2i} = 0$ for all $i \in \{0, ...m\}$

(5 marks)