IMPERIAL

MATH70004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Asymptotic Methods

Date: Friday, May 10, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Suppose that $x > 0$ and $y > 1$ satisfy the relationship $y = \mathrm{e}^x + \mathrm{e}^{-x} - 1$.

   (i) Calculate a three-term expansion for $y = y(x)$ in the limit $x \searrow 0$. (4 marks)

   (ii) Hence, calculate a two-term expansion for $x = x(y)$ in the limit $y \searrow 1$. (4 marks)

   (iii) Calculate a three-term expansion for $x = x(y)$ in the limit $y \nearrow \infty$. (4 marks)

   (b) Consider the integral

$$I(\varepsilon) = \int_0^1 \frac{1}{(x - \sin x)^\alpha + \varepsilon}\,\mathrm{d}x,$$

   where $\alpha > 0$ is given and $\varepsilon \searrow 0$.

   (i) Estimate the orders of the local and global contributions to $I(\varepsilon)$. (4 marks)

   (ii) Determine the value of α for which the leading-order behaviour of the integral is of the form $I(\varepsilon) \sim A \ln(1/\varepsilon)$, and obtain the value of A. (You do not need to justify your calculations in detail.) (4 marks)

(Total: 20 marks)

2. Consider the integral

$$I(x) = \int_0^\infty (1+t)^{-2} \mathrm{e}^{\mathrm{i}x(2t - t^2)}\,\mathrm{d}t, \qquad x \nearrow \infty.$$

   (a) Use the method of stationary phase to obtain the leading-order behaviour of $I(x)$.
   [You may use $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, if needed.] (8 marks)

   (b) Use the method of steepest descent to derive a two-term expansion for $I(x)$ with a big-O estimate of the remainder. Sketch the full steepest-descent contour, and the contour you used (if different). (12 marks)

(Total: 20 marks)

3. Suppose that $y(x)$ satisfies

$$\varepsilon y'' + (1+\varepsilon^2 y')y' = -\frac{3}{2}x^{1/2}, \qquad y(0) = y(1) = 0, \qquad \varepsilon \searrow 0.$$

(a) Assuming a boundary layer is at $x = 0$, determine the leading-order outer solution $y(x) \sim y_0(x) + \cdots$. (3 marks)

(b) Determine the width of the boundary layer at $x = 0$ and an appropriate inner variable X, and find the leading-order inner solution $y(x) \sim Y_0(X) + \cdots$. (5 marks)

(c) Obtain an additive composite approximation to $y(x)$, and sketch the solution. (4 marks)

(d) Calculate the $\mathrm{ord}(\varepsilon)$ corrections to the outer and inner solutions. (4 marks)

(e) Determine the order of the next term in the inner expansion, and write down the equation it satisfies together with the required boundary and matching conditions. [You do not need to solve the equation.] (4 marks)

(Total: 20 marks)

4. (a) Let $q(x) > 0$ be given and suppose that $y(x)$ satisfies the equation

$$\varepsilon^2 y'' + q(x)y = 0, \qquad \varepsilon \searrow 0.$$

(i) Use the WKB method to derive two independent leading-order solutions $y(x)$. (4 marks)

(ii) Derive an approximate condition on $q(x)$ for there to exist non-zero solutions $y(x)$ satisfying the boundary conditions $y(0) = y(1) = 0$. (4 marks)

(iii) For $q(x) = \lambda^2(x^2+1)^2$ with $\lambda > 0$, obtain a leading-order approximation for λ. (3 marks)

(b) Let $q(x) > 0$ and $p(x)$ be given and suppose that $y(x)$ satisfies the equation

$$\varepsilon^4 y'' - 2\varepsilon^3 p(x)y' + q(x)y = 0, \qquad \varepsilon \searrow 0.$$

(i) Use the WKB method to derive two independent leading-order solutions $y(x)$. (5 marks)

(ii) Derive an approximate condition for there to exist non-zero solutions $y(x)$ satisfying the boundary conditions $y(0) = y(1) = 0$. (4 marks)

(Total: 20 marks)

5. (a) Consider the solutions $y(x)$ of the differential equation

$$x^4 y y'' = 2.$$

(i) Using the algebraic ansatz $y \sim Ax^{\alpha}$, identify three different possible leading-order behaviours of $y(x)$ as $x \nearrow \infty$, and verify that the approximation is an exact solution in one of the cases. (6 marks)

(ii) For the smaller of the remaining two cases, calculate the next two terms in the asymptotic expansion of $y(x)$.

(6 marks)

(b) Consider the solutions $y(x)$ of the differential equation

$$x^3 y'' = y^2.$$

(i) Identify three different possible leading-order behaviours of $y(x)$ as $x \nearrow x_0$, for any given $x_0 > 0$. (4 marks)

(ii) Identify two different possible leading-order behaviours of $y(x)$ as $x \nearrow \infty$. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60004/70004

Asymptotic Methods (Solutions)

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1. (a)(i) In the limit $x \searrow 0$, we have

unseen $\Downarrow$

$$
\begin{aligned}
y &= \mathrm{e}^{x} + \mathrm{e}^{-x} - 1 \\
&= (1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + O(x^5)) + (1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + O(x^5)) - 1 \\
&\sim 1 + x^2 + \tfrac{1}{12}x^4.
\end{aligned}
$$

4, A

(a)(ii) To expand near $y = 1$, for simplicity we write $y = 1 + z$ and take $z \searrow 0$ instead. Then, the previous result is,

$$z = x^2 + \tfrac{1}{12}x^4 + o(x^4).$$

At leading order we then have $z \sim x^2$ and hence $x \sim z^{1/2}$ (choosing $x > 0$). To find the correction we use iteration,

$$
\begin{aligned}
x = [z - \tfrac{1}{12}x^4 + o(x^4)]^{1/2} = [z - \tfrac{1}{12}z^2 + o(z^2)]^{1/2} &= z^{1/2}(1 - \tfrac{1}{12}z + o(z))^{1/2} \\
&= z^{1/2}(1 - \tfrac{1}{24}z + o(z)) \sim z^{1/2} - \tfrac{1}{24}z^{3/2}.
\end{aligned}
$$

4, B

(a)(iii) For large y, we need at least one term on the right-hand side of

$$y = \mathrm{e}^{x} + \mathrm{e}^{-x} - 1$$

to also be large. The only possibility, with $x > 0$, is that e^{x} is large and $x \nearrow \infty$. Hence, we set up an iteration using

$$x = \ln(y + 1 - \mathrm{e}^{-x}).$$

Then we obtain, since $x \nearrow \infty$,

$$x = \ln(y + O(1)) = \ln y + \ln(1 + O(y^{-1})) = \ln y + O(y^{-1}),$$

and hence

$$
\begin{aligned}
x &= \ln(y + 1 - y^{-1}\mathrm{e}^{O(y^{-1})}) = \ln(y + 1 - y^{-1} + O(y^{-2})) \\
&= \ln y + \ln[1 + y^{-1} - y^{-2} + O(y^{-3})] \\
&= \ln y + \left[y^{-1} - y^{-2} + O(y^{-3})\right] - \frac{1}{2}\left[y^{-1} + O(y^{-2})\right]^2 + O(y^{-3}) \\
&\sim \ln y + y^{-1} - \frac{3}{2}y^{-2}.
\end{aligned}
$$

4, C

(b)(i) For $x = \mathrm{ord}(1)$ the integrand is $\mathrm{ord}(1)$ and hence the global contribution is estimated as $\mathrm{ord}(1)$. unseen $\Downarrow$

We see that the integrand grows as $x \searrow 0$, and expanding in this limit yields

$$(x - \sin x)^\alpha \sim (x^3/6)^\alpha = \mathrm{ord}(x^{3\alpha}).$$

Thus, the two terms in the denominator balance when $x = \mathrm{ord}(\varepsilon^{1/(3\alpha)})$. This is the local scale, and the integrand is $\mathrm{ord}(\varepsilon^{-1})$ so the local contribution is estimated as

$$\mathrm{ord}(\varepsilon^{-1}) \times \mathrm{ord}(\varepsilon^{1/(3\alpha)}) = \mathrm{ord}(\varepsilon^{1/(3\alpha)-1}).$$

4, A

(b)(ii) We expect to obtain an $\mathrm{ord}(\ln \varepsilon)$ result when both local and global contributions are estimated as $\mathrm{ord}(1)$, so $\alpha = 1/3$.

In this case, the dominant contribution is from intermediate scales, and can be estimated as

$$\int_{\mathrm{ord}(\varepsilon)}^{\mathrm{ord}(1)} \frac{1}{(x^3/6)^{1/3}}\,\mathrm{d}x = 6^{1/3}\,[\ln x]_{\mathrm{ord}(\varepsilon)}^{\mathrm{ord}(1)} = 6^{1/3} \ln \frac{1}{\varepsilon}.$$

4, D

Hence, $A = 6^{1/3}$.

2.

$$I(x) = \int_0^\infty t e^{ix(2t-t^2)}\,dt,$$

$\boxed{\text{sim. seen} \Downarrow}$

(a) We identify the phase function ψ and find the point(s) of stationary phase:

$$\psi(t) = 2t - t^2 \quad \Rightarrow \quad \psi'(t) = 2 - 2t \quad \Rightarrow \quad t_* = 1.$$

We expect the dominant contribution to the integral to come from the neighbourhood of the stationary point $t = 1$. Writing $t = 1 + s$ and expanding for small s yields

$$\psi(t) = \psi(1+s) \approx 1 - s^2, \quad t = 1 + s \approx 1$$

$$\Rightarrow \quad J(x) \sim \int_{1-\delta}^{1+\delta} (1+t)^{-2} e^{ix\psi(t)}\,dt \sim \frac{1}{4}\int_{-\delta}^{\delta} e^{ix(1-s^2)}\,ds \overset{s=u/x^{1/2}}{=} \frac{1}{4}\frac{e^{ix}}{x^{1/2}}\int_{-\delta x^{1/2}}^{\delta x^{1/2}} e^{-iu^2}\,du \sim$$

$$\sim \frac{e^{ix}}{2x^{1/2}}\int_0^\infty e^{-iu^2}\,du.$$

We calculate the integral using a variable change $u = w^{1/2}$ followed by a contour deformation, moving the endpoint from $w = +\infty$ to $w = e^{-i\pi/2}\infty$,

$$\int_0^\infty e^{-iu^2}\,du \overset{u=w^{1/2}}{=} \frac{1}{2}\int_0^\infty e^{-iw} w^{-1/2}\,dw \overset{w=re^{-i\pi/2}}{=} \frac{e^{-i\pi/4}}{2}\int_0^\infty e^{-r} r^{-1/2}\,dr = \frac{e^{-i\pi/4}\sqrt{\pi}}{2}.$$

$\boxed{6,\ \text{A}}$

Hence,

$$I(x) \sim e^{ix} e^{-i\pi/4}\frac{\sqrt{\pi}}{4x^{1/2}} = \frac{e^{i(x-\pi/4)}}{4}\sqrt{\frac{\pi}{x}}.$$

$\boxed{2,\ \text{B}}$

(b) We identify the exponent function $h(t) = i(2t - t^2)$. There is only one saddle point, $t = 1$, as calculated in (a), and we expect to have dominant contributions from the saddle and/or the endpoint $t = 0$. $\boxed{2,\ \text{A}}$

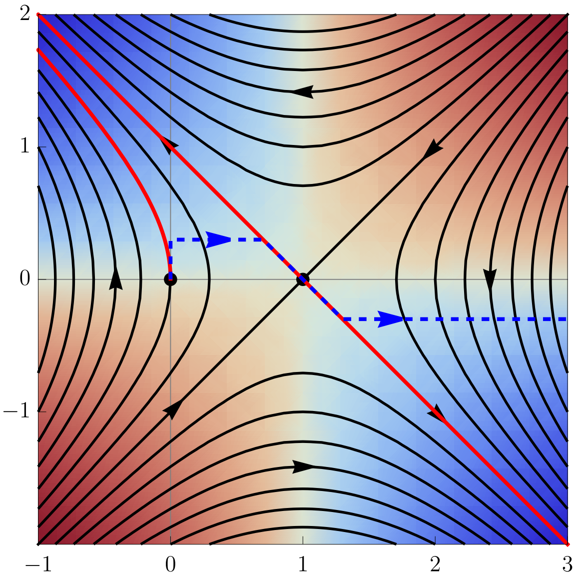
Near the saddle point, we write

$$t = 1 + s \quad \Rightarrow \quad h = i(1 - s^2) = i - is^2.$$

This is a quadratic saddle, so the SD contours are locally hyperbolae, and since this expression is exact the contours are hyperbolae farther away too. We determine the descending directions by seeking a negative change in h, so

$$(2k+1)\pi = \arg(-is^2) = -\frac{\pi}{2} + 2\arg s \quad \Rightarrow \quad \arg s = \frac{(2k+\frac{3}{2})\pi}{2} = (k + \tfrac{3}{4})\pi.$$

Thus, the descending directions are $\arg s = \frac{3}{4}\pi,\ -\frac{1}{4}\pi$, and the ascending directions lie in between at $\arg s = \frac{1}{4}\pi,\ -\frac{3}{4}\pi$. We can then sketch the contours in the whole complex plane.



Alternatively, writing $t = p + \mathrm{i}q$ we find that

$$h = \mathrm{i}(2p + 2\mathrm{i}q - p^2 - 2\mathrm{i}pq + q^2) = 2pq - 2q + \mathrm{i}(2p - p^2 + q^2)$$
$$\Rightarrow \quad \operatorname{Im} h = 2p - p^2 + q^2 = 1 - (p-1)^2 + q^2,$$

3, B

so the SD contours are given by constant $q^2 - (p-1)^2$, i.e. hyperbolae.

From the sketch, we see how to deform the original contour (going from 0 to $+\infty$) onto a full steepest-descent contour: The infinite endpoint $+\infty$ is deformed to the descending direction $\mathrm{e}^{-\mathrm{i}\pi/4}\infty$ (which can be justified by Jordan's lemma), and we reach it by first travelling from 0 upward and to the left to $\mathrm{e}^{\mathrm{i}3\pi/4}\infty$, and then along a straight line through the saddle at 1. The contributions from the end point $t = 0$ and saddle $t = 1$ (which have $\operatorname{Re} h = 0$) are exponentially dominant.

For the calculation it is easier to use a piecewise linear contour that passes through $t = 0$ and $t = 1$ tangent to the SD direction, with the contribution from elsewhere being exponentially smaller. Near the saddle point, we use $t = 1 + \mathrm{e}^{-\mathrm{i}\pi/4}s$ and obtain the contribution

$$\int_{-\delta}^{\delta} (2 + \mathrm{e}^{-\mathrm{i}\pi/4}s)^{-2} \mathrm{e}^{x(\mathrm{i}-\mathrm{i}(\mathrm{e}^{-\mathrm{i}\pi/4}s)^2)} \mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{d}s = \frac{\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{4} \int_{-\delta}^{\delta} (1 + \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/4}s)^{-2} \mathrm{e}^{-xs^2}\,\mathrm{d}s =$$
$$\overset{s=u/x^{1/2}}{=} \frac{\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{4x^{1/2}} \int_{-\infty}^{\infty} (1 - \mathrm{e}^{-\mathrm{i}\pi/4}u/x^{1/2} + O(x^{-1}))\mathrm{e}^{-u^2}\,\mathrm{d}u + \mathrm{EST} =$$
$$= \frac{\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{4x^{1/2}} \left[\sqrt{\pi} + 0/x^{1/2} + O(x^{-1})\right] = \frac{\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{4}\sqrt{\frac{\pi}{x}} + O(x^{-3/2}).$$

Near the endpoint, we use $t = \mathrm{i}s$ and obtain the contribution

$$\int_0^{\delta} (1 + \mathrm{i}s)^{-2} \mathrm{e}^{x(-2s+\mathrm{i}s^2)}\mathrm{i}\,\mathrm{d}s \overset{s=u/x}{=} \frac{\mathrm{i}}{x}\int_0^{x\delta} (1 + \mathrm{i}u/x)^{-2}\mathrm{e}^{-2u+O(x^{-1})}\,\mathrm{d}u = \frac{\mathrm{i}}{2x} + O(x^{-2}).$$

3, C

Hence,

$$I(x) = \frac{\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{4}\sqrt{\frac{\pi}{x}} + \frac{\mathrm{i}}{2x} + O(x^{-3/2}).$$

4, D

3. (a) We solve the outer equations and impose the right-hand boundary condition $y(1) = 0$. The ansatz $y \sim y_0 + \cdots$ yields unseen $\Downarrow$

$$y_0' = -\frac{3}{2}x^{1/2}, \quad y_0(1) = 0 \quad \Rightarrow \quad y_0 = 1 - x^{3/2}.$$

3, A

(b) Near $x = 0$, we seek a boundary-layer rescaling $x = \delta X$ where $\delta \ll 1$, with $y(x) = Y(X)$ (which doesn't need rescaling since $y_0(0) = \mathrm{ord}(1)$) and obtain sim. seen $\Downarrow$

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\delta}(1 + (\varepsilon^2/\delta)Y')Y' = -\frac{3}{2}\delta^{1/2}X^{1/2}.$$

The Y' term is $\gg 1$, and can only be balanced by the Y'' term, so

$$\varepsilon/\delta^2 = 1/\delta \quad \Rightarrow \quad \delta = \varepsilon.$$

Hence, we use

$$x = \varepsilon X, \quad y = Y \quad \Rightarrow \quad Y'' + Y' + \varepsilon(Y')^2 = -\frac{3}{2}\varepsilon^{3/2}X^{1/2}, \quad Y(0) = 0.$$

We now expand $Y \sim Y_0 + \cdots$ and obtain

$$Y_0'' + Y_0' = 0, \quad Y_0(0) = 0 \quad \Rightarrow \quad Y_0 = A_0 + B_0\mathrm{e}^{-X} = A_0(1 - \mathrm{e}^{-X}),$$

We determine A_0 by matching with the outer solution. Equating

$$y_0(0) = 1 \text{ and } Y_0(\infty) = A_0 \quad \Rightarrow \quad A_0 = 1.$$

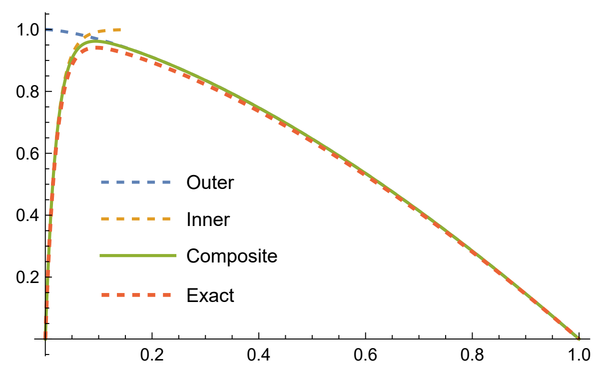
5, A

Hence, $Y_0 = 1 - \mathrm{e}^{-X}$.

(c) The common overlap behaviour is $y \sim 1$, and hence the additive composite is given by unseen $\Downarrow$

$$y \sim (1 - x^{3/2}) + (1 - \mathrm{e}^{-x/\varepsilon}) - 1 = 1 - x^{3/2} - \mathrm{e}^{-x/\varepsilon}.$$

As x decreases from 1 towards 0, the function increases from 0 towards 1, but as x decreases past $\mathrm{ord}(\varepsilon)$ the function rapidly decreases to 0.



4, B

(d) At next order in the outer expansion, $y \sim y_0 + \varepsilon y_1 + \cdots$, we obtain

unseen $\Downarrow$

$$y_0'' + y_1' = 0, \quad y_1(1) = 0 \quad \Rightarrow \quad y_1 = a_1 - y_0' = a_1 + \tfrac{3}{2}x^{1/2} = \tfrac{3}{2}(x^{1/2} - 1).$$

At next order in the inner expansion, $Y \sim Y_0 + \varepsilon Y_1 + \cdots$, we obtain

$$\begin{aligned} Y_1'' + Y_1' &= -(Y_0')^2 = -\mathrm{e}^{-2X}, \quad Y_1(0) = 0 \\ \Rightarrow \quad Y_1 &= A_1 + B_1\mathrm{e}^{-X} - \tfrac{1}{2}\mathrm{e}^{-2X} = A_1 + (\tfrac{1}{2} - A_1)\mathrm{e}^{-X} - \tfrac{1}{2}\mathrm{e}^{-2X}. \end{aligned}$$

We determine A_1 using van Dyke matching. Substituting the inner variable into the outer expansion yields

$$y(\varepsilon X) = (1 - \varepsilon^{3/2}X^{3/2}) + \varepsilon(\tfrac{3}{2}\varepsilon^{1/2}X^{1/2} - \tfrac{3}{2}) + o_x(\varepsilon) = 1 - \tfrac{3}{2}\varepsilon + o_x(\varepsilon) + o_X(\varepsilon),$$

while substituting the outer variable into the inner expansion yields

$$\begin{aligned} Y(x/\varepsilon) = (1 - \mathrm{e}^{-x/\varepsilon}) + \varepsilon(A_1 + (\tfrac{1}{2} - A_1)\mathrm{e}^{-x/\varepsilon} - \tfrac{1}{2}\mathrm{e}^{-2x/\varepsilon}) + o_X(\varepsilon) = \\ = 1 + \varepsilon A_1 + o_X(\varepsilon) + o_x(\varepsilon). \end{aligned}$$

Equating these then yields $A_1 = -\frac{3}{2}$, and hence

$$Y_1 = -\tfrac{3}{2} + 2\mathrm{e}^{-X} - \tfrac{1}{2}\mathrm{e}^{-2X}.$$

4, B

(e) The $\mathrm{ord}(\varepsilon^{3/2})$ term in the inner equation indicates that the next correction is not $\mathrm{ord}(\varepsilon^2)$ but rather $\mathrm{ord}(\varepsilon^{3/2})$, so that $Y \sim Y_0 + \varepsilon Y_1 + \varepsilon^{3/2}Y_{3/2} + \cdots$. At $\mathrm{ord}(\varepsilon^{3/2})$, the equation is

$$Y_{3/2}'' + Y_{3/2}' = -\tfrac{3}{2}X^{1/2}.$$

The unknown coefficients of the homogeneous solutions 1 and e^{-X} require two conditions to be determined. The first is the boundary condition $Y_{3/2}(0) = 0$. The second condition is obtained from matching to the outer solution. In this limit, e^{-X} becomes negligible while 1 yields an $\mathrm{ord}(\varepsilon^{3/2})$ constant term in the outer solution. However, the outer expansion does not have any $\mathrm{ord}(\varepsilon^{3/2})$ term as the equations only have integer powers of ε, so the matching condition is that $Y_{3/2}$ does not have an $\mathrm{ord}(1)$ term as $X \nearrow \infty$. From the local behaviour of the outer solution above, we see that the condition can be written

$$Y_{3/2} = -X^{3/2} + \tfrac{3}{2}X^{1/2} + o(1) \quad \text{as} \quad X \nearrow \infty.$$

4, D

4. (a)(i) We use the WKB ansatz $y(x) = \mathrm{e}^{S(x)/\delta}$ where $\delta \ll 1$. This yields $\boxed{\text{seen} \Downarrow}$

$$y' = \frac{S'}{\delta}\mathrm{e}^{S/\delta}, \quad y'' = \left[\frac{S'^2}{\delta^2} + \frac{S''}{\delta}\right]\mathrm{e}^{S/\delta}$$
$$\Rightarrow \qquad \frac{\varepsilon^2}{\delta^2}\left[S'^2 + \delta S''\right] + q(x) = 0.$$

Balancing the two terms yields $\delta = \varepsilon$, and hence

$$y = \mathrm{e}^{S(x)/\varepsilon}, \quad S'^2 + \varepsilon S'' + q = 0.$$

We expand $S = S_0 + \varepsilon S_1 + \cdots$ and solve order by order to find

$$S_0'^2 + q = 0 \quad \Rightarrow \quad S_0' = \pm\mathrm{i}\sqrt{q} \quad \Rightarrow \quad S_0 = \pm\mathrm{i}\int\sqrt{q}\,\mathrm{d}x,$$
$$2S_0'S_1' + S_0'' = 0 \quad \Rightarrow \quad S_1' = -\frac{S_0''}{2S_0'} = -\frac{q'}{4q} \quad \Rightarrow \quad S_1 = -\frac{1}{4}\ln q.$$

Hence, two independent solutions are

$$y = \exp\left[\pm\mathrm{i}\int\sqrt{q}\,\mathrm{d}x/\varepsilon - \frac{1}{4}\ln q + O(\varepsilon)\right] \sim q^{-1/4}\mathrm{e}^{\pm\mathrm{i}\int\sqrt{q}\,\mathrm{d}x/\varepsilon}.$$

$\boxed{4, \text{A}}$

(a)(ii) Writing

$$\Psi(x) = \frac{1}{\varepsilon}\int_0^x \sqrt{q(x')}\,\mathrm{d}x',$$

we find that the general leading-order approximation is the linear combination

$$y \sim q^{-1/4}\left[A\mathrm{e}^{\mathrm{i}\Psi} + B\mathrm{e}^{-\mathrm{i}\Psi}\right] = q^{-1/4}\left[C\cos\Psi + D\sin\Psi\right].$$

Imposing $y(0) = 0$ yields $C = 0$, and hence

$$y \sim Dq^{-1/4}\sin\Psi(x).$$

For a non-zero solution we require $D \neq 0$, and hence imposing $y(1) = 0$ yields

$$\sin\Psi(1) = 0 \quad \Rightarrow \quad \Psi(1) = n\pi \quad \Rightarrow \quad \frac{1}{\varepsilon}\int_0^1\sqrt{q}\,\mathrm{d}x = n\pi.$$

$\boxed{4, \text{A}}$

Here, n is an integer and since the integral is positive we must have $n = 1, 2, 3, \cdots$.

(a)(iii) Using $q = \lambda^2(x^2+1)^2$ we obtain $\boxed{\text{unseen} \Downarrow}$

$$n\pi = \frac{\lambda}{\varepsilon}\int_0^1 (x^2+1)\,\mathrm{d}x = \frac{4}{3}\frac{\lambda}{\varepsilon} \quad \Rightarrow \quad \lambda = \frac{3\varepsilon n\pi}{4}.$$

$\boxed{3, \text{B}}$

(b)(i) We again use the WKB ansatz $y(x) = \mathrm{e}^{S(x)/\delta}$ and obtain $\boxed{\text{unseen} \Downarrow}$

$$\frac{\varepsilon^4}{\delta^2}\left[S'^2 + \delta S''\right] - \frac{\varepsilon^3}{\delta} 2p(x)S' + q(x) = 0.$$

We seek to find the appropriate value(s) of δ for which there is a dominant balance between at least two terms in the equation. Balancing the first and last terms yields $\delta = \varepsilon^2$, with the first and last terms being $\mathrm{ord}(1)$ while the middle term is $\mathrm{ord}(\varepsilon)$. (Balancing first and second terms yields $\delta = \varepsilon$ with the third term too large, while balancing the second and third terms yields $\delta = \varepsilon^3$ with the first term too large.)

Hence, we obtain

$$y = \mathrm{e}^{S(x)/\varepsilon^2}, \quad S'^2 + \varepsilon^2 S'' - 2\varepsilon p(x)S' + q(x) = 0.$$

We expand

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + O(\varepsilon^3),$$

noting that we need to calculate S_2 as it yields an $\mathrm{ord}(1)$ contribution to the exponent. Substitution into the equation then yields

$$\begin{aligned}
S_0'^2 + q = 0 \quad &\Rightarrow \quad S_0' = \pm \mathrm{i}\sqrt{q} \quad \Rightarrow \quad S_0 = \pm\mathrm{i}\int \sqrt{q}\,\mathrm{d}x,\\
2S_0'S_1' - 2pS_0' = 0 \quad &\Rightarrow \quad S_1' = p \quad \Rightarrow \quad S_1 = \int p\,\mathrm{d}x,\\
2S_0'S_2' + S_1'^2 + S_0'' - 2pS_1' = 0 \quad &\Rightarrow \quad S_2' = \frac{p^2}{2S_0'} - \frac{S_0''}{2S_0'} = \mp\mathrm{i}\frac{p^2}{2\sqrt{q}} - \frac{q'}{4q}\\
&\Rightarrow \quad S_2 = \mp\mathrm{i}\int \frac{p^2}{2\sqrt{q}}\,\mathrm{d}x - \frac{1}{4}\ln q.
\end{aligned}$$

Hence,

$$y \sim q^{-1/4}\exp\left(\pm\frac{\mathrm{i}}{\varepsilon^2}\int\sqrt{q}\,\mathrm{d}x + \frac{1}{\varepsilon}\int p\,\mathrm{d}x \mp \mathrm{i}\int\frac{p^2}{2\sqrt{q}}\,\mathrm{d}x\right)$$ $\boxed{5, \mathrm{C}}$

(b)(ii) The general leading-order approximation is the linear combination

$$\begin{aligned}
y &\sim q^{-1/4}\exp\left(\frac{1}{\varepsilon}\int_0^x p\,\mathrm{d}x\right)(C\cos\Psi + D\sin\Psi),\\
\Psi(x) &= \frac{1}{\varepsilon^2}\int_0^x \sqrt{q(x')}\,\mathrm{d}x' - \int_0^x \frac{p(x')^2}{2\sqrt{q(x')}}\,\mathrm{d}x'.
\end{aligned}$$

Imposing $y(0) = 0$ yields $C = 0$, and $y(1) = 0$ yields

$$\sin\Psi(1) = 0 \quad \Rightarrow \quad \Psi(1) = n\pi \quad \Rightarrow \quad \frac{1}{\varepsilon^2}\int_0^1 \sqrt{q}\,\mathrm{d}x - \int_0^1 \frac{p^2}{2\sqrt{q}}\,\mathrm{d}x = n\pi.$$ $\boxed{4, \mathrm{D}}$

5. (a) $x^4 y y'' = 2$. $\boxed{\text{sim. seen} \Downarrow}$

(a)(i) We consider algebraic behaviour of y as $x \nearrow \infty$, for which we can estimate $y'' = \mathrm{ord}(y/x^2)$. Thus, the left-hand side is $\mathrm{ord}(x^2y^2)$ while the right-hand side is $\mathrm{ord}(1)$. If $y \gg x^{-1}$ then the left-hand side is dominant alone, and we obtain

$$x^4 y y'' \approx 0 \quad \Rightarrow \quad y'' \approx 0 \quad \Rightarrow \quad y \sim Ax \text{ or } y \sim B.$$

Both of these cases satisfy the criterion $y \gg x^{-1}$.

If instead $y = Cx^{-1}$, then the two sides balance and we obtain

$$x^4(Cx^{-1})(2Cx^{-3}) \approx 2 \quad \Rightarrow \quad C = \pm 1 \quad \Rightarrow \quad y \sim \pm x^{-1}.$$

These are exact solutions, since all the approximations are actually equalities in this case.

Finally, if $y \ll x^{-1}$ then the right-hand side is dominant alone so $1 \approx 0$, which is a contradiction.

Hence, the possible leading-order behaviours are

$$y \sim Ax, \qquad y \sim B, \qquad y \sim \pm x^{-1}.$$

$\boxed{6, \mathrm{M}}$

(a)(ii) For $y \sim B$ with $B \neq 0$, we write $y = B + \tilde{y}_1$ with $\tilde{y}_1 \ll 1$ and obtain

$$\begin{aligned}
&\tilde{y}_1'' = \frac{2}{x^4 y} = \frac{2}{B}x^{-4} + o(x^{-4}) \quad \Rightarrow \quad \tilde{y}_1 = \frac{1}{3B}x^{-2} + \tilde{y}_2, \quad \tilde{y}_2 \ll x^{-2} \\
&\Rightarrow \quad \tilde{y}_2'' = \frac{2}{x^4 y} - \frac{2}{Bx^4} = \frac{2}{Bx^4}\left[\left(1 + \frac{1}{3B^2}x^{-2} + o(x^{-2})\right)^{-1} - 1\right] = \\
&\quad = -\frac{2}{3B^3x^6} + o(x^{-6}) \quad \Rightarrow \quad \tilde{y}_2 = -\frac{1}{30B^3}x^{-4} + o(x^{-4}) \\
&\qquad \Rightarrow \quad y = B + \frac{1}{3B}x^{-2} - \frac{1}{30B^3}x^{-4} + o(x^{-4}).
\end{aligned}$$

$\boxed{6, \mathrm{M}}$

(b) $x^3y'' = y^2$.

(b)(i) For $x \nearrow x_0$, we change variables to $X = x_0 - x \searrow 0$ instead, with

$$(x_0 - X)^3 y'' = y^2 \quad \Rightarrow \quad x_0^3 y'' \approx y^2.$$

Estimating $y'' = \mathrm{ord}(y/X^2)$, we see that the left-hand side is dominant alone for $y \ll X^{-2}$, and hence

$$y'' \approx 0 \quad \Rightarrow \quad y \sim A \text{ or } y \sim BX.$$

For $y \gg X^{-2}$ we instead obtain the contradiction $y^2 \approx 0$.

The two sides balance for $y = \mathrm{ord}(X^{-2})$, and we try $y \approx CX^{-2}$ to obtain

$$x_0^3(6CX^{-4}) \approx C^2X^{-4} \quad \Rightarrow \quad C = 6x_0^3.$$

Hence, the possibilities are

$$y \sim A, \quad y \sim B(x - x_0), \quad y \sim 6x_0^3(x - x_0)^{-2} \quad \text{as} \quad x \nearrow x_0.$$

$\boxed{4, \mathrm{M}}$

(b)(ii) Comparing the estimates $\mathrm{ord}(xy)$ and $\mathrm{ord}(y^2)$ for the left- and right-hand sides, we see that the left-hand side is dominant alone for $y \ll x$ which yields

$$y'' \approx 0 \quad \Rightarrow \quad y \sim B.$$

(The case $y \sim Ax$ is excluded by the criterion $y \ll x$.) For $y \gg x$, the right-hand side is dominant alone which yields a contradiction $y^3 \approx 0$.

For $y = \mathrm{ord}(x)$, the order estimate indicates a balance between the two sides, but substituting in the ansatz yields

$$y \approx Ax \quad \Rightarrow \quad x^3 \times 0 \approx A^2x^2.$$

This is an indication that we should adjust our ansatz by a logarithmic factor,

$$\begin{aligned} y \approx Ax(\ln x)^\alpha \quad &\Rightarrow \quad y' \approx A\left[(\ln x)^\alpha + \alpha(\ln x)^{\alpha-1}\right] \\ &\Rightarrow \quad y'' \approx \frac{A}{x}\left[\alpha(\ln x)^{\alpha-1} + \alpha(\alpha-1)(\ln x)^{\alpha-2}\right] \\ \Rightarrow \quad Ax^2\alpha(\ln x)^{\alpha-1} \approx A^2x^2(\ln x)^{2\alpha} \quad &\Rightarrow \quad \alpha = -1, \quad A = -1. \end{aligned}$$

Hence, the possibilities are

$$y \sim B, \quad y \sim -x(\ln x)^{-1}, \quad \text{as} \quad x \nearrow \infty.$$

$\boxed{4, \mathrm{M}}$

Review of mark distribution:
Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks

MATH60004 Asymptotic Methods

Question	Marker's comment
1	The first part consisted of three basic expansions. Everyone solved the direct expansion, but around half struggled with inverting the expansion, and many attempted incorrectly to apply the expansion for y near 1 to the case of large y. The second part concerned the fundamentals of splitting the range of integration, which many struggled with, including assuming that the local scale is epsilon instead of working out the correct value, or attempting to estimate the order of the contributions by evaluating some form of integral instead of just multiplying together the order estimates of the integrand and the integration range.
2	This was a fairly standard combination of stationary phase (which went very well) and steepest descent (which went less well). Most managed to set up the steepest descent problem and sketch the contours, but less than half could obtain the leading-order result (and some made mistakes that led to a different result from the first part, but didn't notice). Only a few realised that there could be a correction from the end point.
3	This was a basic matched asymptotics problem, with the leading-order part handled very well. When solving for the ord(epsilon) corrections, many mistakenly kept the ord(1) term on the right-hand side of the outer equation, or didn't write down the full rescaled inner equation before trying to work out which terms to keep, which makes things harder. Almost nobody spotted the order(epsilon^(3/2)) term in the rescaled equation, which determines the order of the next correction.
4	This question starts with a standard WKB problem, looking at quantisation on a finite interval, and then adds an extra term in the second part. A majority failed to reproduce the basic WKB derivation, missing the fact that two terms must be calculated in the exponent to obtain the correct leading-order solution, or not realising that the given sign choice results in trigonometric solutions that can satisfy the zero boundary conditions at both ends. For the second part, a few managed to calculate two terms in the exponent but none got to the third term correctly.nbsp;

MATH70004 Asymptotic Methods

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2	This was a fairly standard combination of stationary phase (which went very well) and steepest descent (which went less well). Most managed to set up the steepest descent problem and sketch the contours, but less than half could obtain the leading-order result (and some made mistakes that led to a different result from the first part, but didn't notice). Only a few realised that there could be a correction from the end point.
3	This was a basic matched asymptotics problem, with the leading-order part handled very well. When solving for the ord(epsilon) corrections, many mistakenly kept the ord(1) term on the right-hand side of the outer equation, or didn't write down the full rescaled inner equation before trying to work out which terms to keep, which makes things harder. Almost nobody spotted the order(epsilon^(3/2)) term in the rescaled equation, which determines the order of the next correction.
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5	This question covers various cases of local analysis of nonlinear ordinary differential equations. A majority solved all or most of the first half, while half made good progress on the second part. Some were confused about what was fixed and what was variable after shifting to a finite point. For the very last case, a few commented that a logarithm is needed, but none calculated it.