MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24

Repetition Material 5: The trace-determinant rule

The trace-determinant rule helps to quickly obtain information about real parts of eigenvalues of 2×2 matrices, which is crucial for stability analysis. It does not require you to look at the characteristic polynomial, and you only need to compute the trace and determinant of the matrix.

Proposition 1 (Trace-determinant rule for two-dimensional linear systems)**.** *Consider a matrix $A \in \mathbb{R}^{2\times 2}$, and let $p := \operatorname{tr} A$ be its trace and $q := \det A$ be its determinant. Then all eigenvalues of A*

(i) have negative real part if and only if $p < 0$ and $q > 0$ (stability),

(ii) are non-real if and only if $p^2 - 4q < 0$ (focus or centre),

(iii) have positive real part if and only if $p > 0$ and $q > 0$ (instability),

(iv) are real and have opposite signs if and only if $q < 0$ (saddle).

Proof. We first show that if λ_1, λ_2 are the eigenvalues of A (i.e. the roots of the characteristic polynomial), then $q = \lambda_1\lambda_2$ and $p = \lambda_1 + \lambda_2$. Here we use that $\det(A - \lambda \operatorname{Id}_2)$ can be written as $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$.

Assume that λ_1 and λ_2 are the eigenvalues of A. Then the characteristic polynomial reads as

$$\chi(\lambda) = \det(A - \lambda \operatorname{Id}) = (\lambda - \lambda_1)(\lambda - \lambda_2) \quad \text{for all } \lambda \in \mathbb{C}\,.$$

For $\lambda = 0$, we get $q = \det(A) = \lambda_1\lambda_2$. To prove $p = \lambda_1 + \lambda_2$, note that the two matrices A and $T^{-1}AT$ have the same trace, and we choose T such that $T^{-1}AT$ is in complex Jordan form, having λ_1 and λ_2 on the diagonal. This implies that $p = \lambda_1 + \lambda_2$. It follows that $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$, and note that either λ_1 and λ_2 are both real, or they are complex conjugate, which implies in both cases that p is real.

(i) If λ_1 and λ_2 have negative real part, then $p = \lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 = q > 0$. Conversely, if $p < 0$, $q > 0$ and λ_1, λ_2 are real, then $q > 0$ implies they have the same sign and $p < 0$ that they are both negative. If $p < 0$, $q > 0$ and λ_1, λ_2 are complex conjugate, then $p < 0$ implies that the real part of λ_1 and λ_2 is negative.

(ii) Since $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$, the roots of this equation are $\frac{1}{2}\big(p \pm \sqrt{p^2 - 4q}\big)$ are therefore non-real if and only if $p^2 - 4q < 0$.

(iii) If λ_1 and λ_2 have positive real part, then $p = \lambda_1 + \lambda_2 > 0$ and $\lambda_1\lambda_2 = q > 0$. Conversely, if $p > 0$, $q > 0$ and λ_1, λ_2 are real, then $q > 0$ implies they have the same sign and $p > 0$ that they are both positive. If $p > 0$, $q > 0$ and λ_1, λ_2 are complex conjugate, then $p > 0$ implies that the real part of λ_1 and λ_2 is positive.

(iv) If λ_1 and λ_2 are real and have opposite signs, then $q = \lambda_1\lambda_2 < 0$. If $q < 0$, then λ_1 and λ_2 cannot be complex conjugate, so they must be both real and have opposite signs due to $q = \lambda_1\lambda_2 < 0$. $\square$