1 Vector Calculus

1.1 Preliminary ideas and some revision of vectors

1.1.1 The Einstein summation convention

In any product of terms, if we have a repeated suffix, then that quantity is considered to be summed over (from 1 to 3, since we will usually be working in three dimensions). For example

$$a_i x_i \text{ is shorthand for } \sum_{i=1}^{3} a_i x_i. \quad \equiv a_j x_j$$

1.1.2 The Kronecker delta

This is the quantity δ_{ij} and is defined such that

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Example

$$\begin{aligned} \delta_{ij} a_j &= \sum_{j=1}^{3} \delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 \\ &= a_i \end{aligned}$$

Note that the left-hand-side had two different subscripts, while the right-hand-side ends up with only one subscript - this is known as a **contraction**.

1.1.3 The permutation symbol

This is the quantity ε_{ijk}, defined as

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two of } i, j, k \text{ are the same;} \\ 1, & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3; \\ -1, & \text{if } i, j, k \text{ is an acyclic permutation of } 1, 2, 3. \end{cases}$$

CYCLIC

1 → 2 → 3 → 1

cyclic	acyclic
123	213
231	321
312	132

For example

$$\varepsilon_{123} = 1, \quad \varepsilon_{321} = -1, \quad \varepsilon_{133} = 0.$$

We can show, by considering the various cases, that the Kronecker delta and the permutation symbol are connected by the formula

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \quad \text{Sum over } k$$

(I will put a proof on blackboard). The quantities δ_{ij} and ε_{ijk} are known as **tensors**.
Exercise: Show this can be rewritten in the alternative form

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \quad \text{Sum over } i$$

1.1.4 Vector product

Recall that this is the multiplication of two vectors which results in a third vector, perpendicular to the first two. It can be written in the form of a determinant as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then the two vectors are parallel. Recall that $(\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a})$. If we just consider the first component of this vector we can write this as

$$\begin{aligned} a_2b_3 - a_3b_2 &= \varepsilon_{123}\, a_2 b_3 + \varepsilon_{132}\, a_3 b_2 \\ &= \varepsilon_{1jk}\, a_j b_k \qquad \left(\text{sum over } j \,\&\, k\right) \end{aligned}$$

since $\varepsilon_{123} = 1, \varepsilon_{132} = -1$, and $\varepsilon_{1ij} = 0$ for all other i and j. In general we can write the ith component of $a \times b$ as

$$[a \times b]_i = \varepsilon_{ijk}\, a_j b_k$$

1.1.5 Scalar product

This is defined as

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= a_i b_i \end{aligned}$$

using the summation convention. Recall that if $\mathbf{a} \cdot \mathbf{b} = 0$ then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

1.1.6 Triple scalar product

This is the quantity

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_i [\underline{b} \times \underline{c}]_i = a_i \varepsilon_{ijk} b_j c_k \\ &= \varepsilon_{ijk}\, a_i b_j c_k \qquad \text{sum over } i, j \,\&\, k \end{aligned}$$

If this quantity is zero then the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar. A useful property of the triple scalar product is that the dot and cross can be swapped without changing the answer, provided the order of the vectors remains unchanged, i.e.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

$$\begin{aligned} & a_i [\underline{b} \times \underline{c}]_i \\ &= \varepsilon_{ijk}\, a_i b_j c_k = (\varepsilon_{kij}\, a_i b_j) c_k = [\underline{a} \times \underline{b}]_k c_k \end{aligned}$$

1.1.7 Triple vector product

This is defined as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

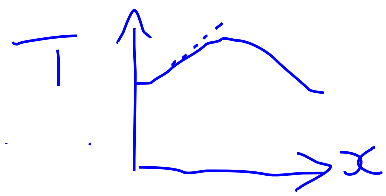
Since $\mathbf{b} \times \mathbf{c}$ is a vector normal to the plane of $\mathbf{b}$ and $\mathbf{c}$, and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is normal to $\mathbf{b} \times \mathbf{c}$, it follows that the triple vector product must lie in the plane of $\mathbf{b}$ and $\mathbf{c}$. In component notation

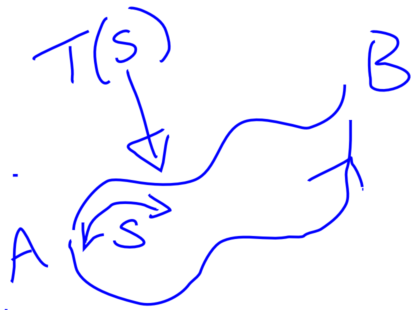
$$\begin{aligned}
[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \varepsilon_{ijk}\, a_j\, [\underline{b} \times \underline{c}]_k \\
&= \varepsilon_{ijk}\, a_j\, \varepsilon_{klm}\, b_l c_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m \\
&= a_j b_i c_j - a_j b_j c_i \\
&= (\underline{a} \cdot \underline{c})\, b_i - (\underline{a} \cdot \underline{b})\, c_i
\end{aligned}$$

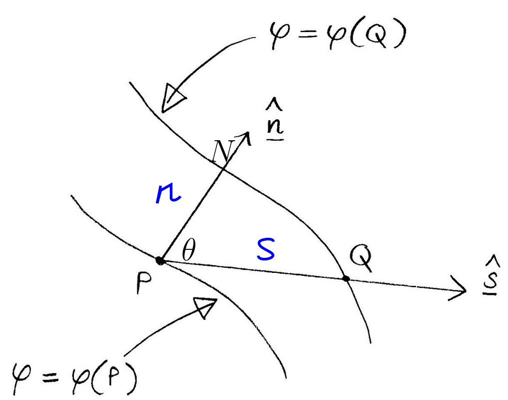
and so we conclude that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

which confirms explicitly that the triple vector product indeed lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.







if $\varphi = x^2+y^2+z^2$ then $\varphi = \text{const}$ is the surface of a sphere

Figure 1: The surface $\phi =$ constant through two neighbouring points.

1.2 Gradient

Let ϕ be a differentiable scalar function of position in three dimensions. If P is a general point, ϕ will depend on the position of P, so we may write $\phi = \phi(P)$. The position of P is defined by reference to a coordinate system e.g. if we consider Cartesian coordinates, then P depends on (x, y, z) and hence $\phi = \phi(x, y, z)$, while if we consider cylindrical polar coordinates (r, θ, z) then $\phi = \phi(r, \theta, z)$.

The equation $\phi =$ constant defines a surface in three dimensions. Varying the constant, we can define a family of surfaces called ‘level surfaces’ or ‘equi-ϕ surfaces’. For example, if ϕ represents pressure, then $\phi =$ constant defines a family of surfaces over which the pressure is constant. The surface through a **specific point** P is $\phi = \phi(P)$. Let Q be a neighbouring point. (See figure 1). The equation of the level surface through Q is $\phi = \phi(Q)$. We draw the normal to $\phi = \phi(P)$ at P. Suppose that it intersects $\phi = \phi(Q)$ at the point N. Since N is on $\phi = \phi(Q)$ we have $\phi(N) = \phi(Q)$. Let s denote the length along PQ and let n denote the length along PN. Introduce unit vectors $\widehat{\mathbf{s}}$ and $\widehat{\mathbf{n}}$ in those directions. We define $\partial\phi/\partial s$ to be the **directional derivative** of ϕ in the direction $\widehat{\mathbf{s}}$:

$$
\begin{aligned}
\frac{\partial \phi}{\partial s} &= \lim_{PQ \to 0} (\varphi(Q) - \varphi(P))/PQ \\
&= \lim_{Q \to P} \frac{(\varphi(N) - \varphi(P))}{PN} \cdot \frac{PN}{PQ} \\
&= \lim_{N \to P} \frac{(\varphi(N) - \varphi(P))}{PN} \lim_{Q \to P} \left(\frac{PN}{PQ}\right) \\
&= \\
&= \frac{\partial \varphi}{\partial n} \cos\theta = \frac{\partial \varphi}{\partial n} (\hat{\underline{n}} \cdot \hat{\underline{s}})
\end{aligned}
$$

Since $\cos\theta \leq 1$, the maximum directional derivative at P occurs along the normal to $\phi = \phi(P)$ at P.

"nabla"

The vector $\widehat{\mathbf{n}}\,\partial\phi/\partial n$ is called the **gradient** of ϕ at P. We write it as grad ϕ or $\nabla\phi$. The operator grad or ∇ is known as the **vector gradient operator**. We have

$$\frac{\partial\phi}{\partial s} = \widehat{\mathbf{s}}\cdot\nabla\phi.$$

1.2.1 Cartesian components of $\nabla\phi$

If $\nabla\phi = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ then $\mathbf{i}\cdot\nabla\phi = A_1$. But, by definition, $\mathbf{i}\cdot\nabla\phi = \partial\phi/\partial x$. Hence $A_1 = \partial\phi/\partial x$. Similarly we find $A_2 = \partial\phi/\partial y, A_3 = \partial\phi/\partial z$ and so we have the result:

$$\nabla\phi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

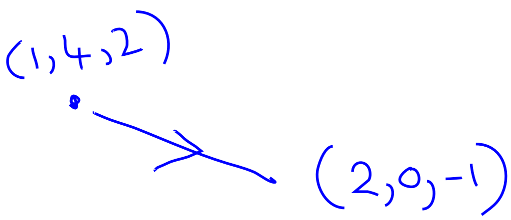
Example

If $\phi = axy^2 + byz + cx^3z^2$, where a, b, c are constants, find $\nabla\phi$. Also find the directional derivative of ϕ at the point $(1, 4, 2)$ in the direction towards the point $(2, 0, -1)$.

$$\underline{\nabla}\varphi = \hat{i}(ay^2 + 3cx^2z^2) + \hat{j}(2axy + bz) + \hat{k}(by + 2cx^3z)$$

$P = (1,4,2)$

$$(\underline{\nabla}\varphi)_P = \hat{i}(16a + 12c) + \hat{j}(8a + 2b) + \hat{k}(4b + 4c)$$



$$\underline{s} = (2,0,-1) - (1,4,2) = (1,-4,-3)$$

$$\hat{\underline{s}} = (\hat{i} - 4\hat{j} - 3\hat{k})/\sqrt{(1^2 + 4^2 + 3^2)}$$

$$= (\hat{i} - 4\hat{j} - 3\hat{k})/\sqrt{26}$$

Directional derivative $= (\underline{\nabla}\varphi \cdot \hat{\underline{s}})_P$

$$= \big((16a + 12c) - 4(8a + 2b) - 3(4b + 4c)\big)/\sqrt{26}$$

$$= (-16a - 20b)/\sqrt{26}$$

$\hat{\theta} \cdot \hat{i} = -\sin\theta$

$\hat{\theta} \cdot \hat{j} = \cos\theta$

$\hat{r} \cdot \hat{i} = \cos\theta$

$\hat{r} \cdot \hat{j} = \sin\theta$

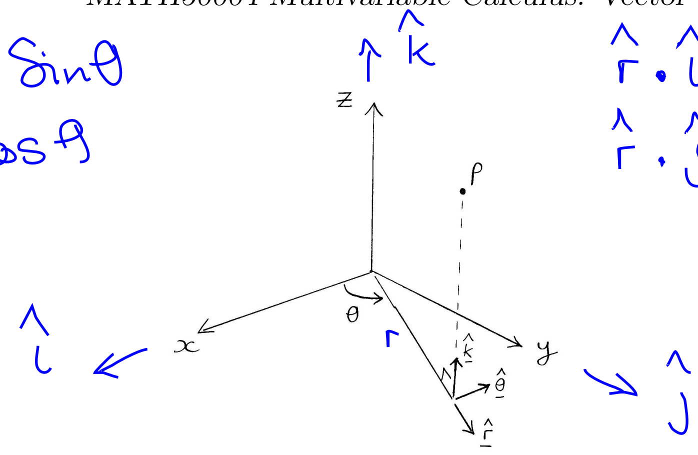


Figure 2: Sketch showing a point P represented by Cartesian coordinates (x, y, z) and cylindrical polar coordinates (r, θ, z). $x = r\cos\theta \quad y = r\sin\theta$

1.2.2 Cylindrical polar components of $\nabla\phi$

The set-up is as shown in figure 2. We write $\nabla\phi = A_1\widehat{\mathbf{r}} + A_2\widehat{\theta} + A_3\mathbf{k}$. Then it follows that

$$\begin{aligned}
A_1 &= \widehat{\mathbf{r}} \cdot \nabla\phi \\
&= \hat{\underline{r}} \cdot \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k} \right) \\
&= \cos\theta \frac{\partial \varphi}{\partial x} + \sin\theta \frac{\partial \varphi}{\partial y} + \text{zero} \\
&= \frac{\partial x}{\partial r}\frac{\partial \varphi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \varphi}{\partial y} \\
&= \partial\varphi/\partial r
\end{aligned}$$

$(\hat{\underline{r}} \cdot \hat{\underline{k}} = 0)$

Similarly, we find

$$\begin{aligned}
A_2 &= \widehat{\theta} \cdot \nabla\phi \\
&= \hat{\underline{\theta}} \cdot \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k} \right) \\
&= -\sin\theta \frac{\partial \varphi}{\partial x} + \cos\theta \frac{\partial \varphi}{\partial y} + \text{zero} \\
&= \frac{1}{r}\frac{\partial x}{\partial \theta}\frac{\partial \varphi}{\partial x} + \frac{1}{r}\frac{\partial y}{\partial \theta}\frac{\partial \varphi}{\partial y} \\
&= \frac{1}{r}\frac{\partial \varphi}{\partial \theta}
\end{aligned}$$

$\hat{\underline{\theta}} \cdot \hat{\underline{k}} = 0$

and $A_3 = \mathbf{k}\cdot\nabla\phi = \partial\phi/\partial z$. Hence

$$\nabla\phi = \widehat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \frac{\widehat{\theta}}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{k}\frac{\partial\phi}{\partial z}.$$

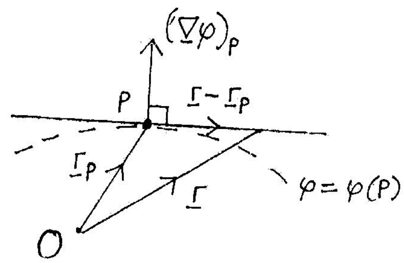


Figure 3: The tangent plane to a surface.

1.2.3 Equation of a tangent plane to $\phi = \phi(P)$

We have that $(\nabla\phi)_P$ is normal to $\phi = \phi(P)$ at P. The equation of the tangent plane is therefore

$$(\mathbf{r} - \mathbf{r}_P) \cdot (\nabla\phi)_P = 0,$$

i.e.

$$\left(\frac{\partial\phi}{\partial x}\right)_P (x - x_P) + \left(\frac{\partial\phi}{\partial y}\right)_P (y - y_P) + \left(\frac{\partial\phi}{\partial z}\right)_P (z - z_P) = 0.$$

constants

Example

Find the tangent plane to the surface

$$z = e^{-(x^2+y^2)^{1/2}}$$

at the point $x = -1, y = 0$.

N.B. choice of φ not unique

e.g. $\varphi = z e^{(x^2+y^2)^{1/2}} = 1$ on S

Let $\varphi = z - e^{-(x^2+y^2)^{1/2}} = 0$ on the surface

Then $\dfrac{\partial \varphi}{\partial x} = \dfrac{x}{(x^2+y^2)^{1/2}} e^{-(x^2+y^2)^{1/2}}$

& $\dfrac{\partial \varphi}{\partial y} = \dfrac{y}{(x^2+y^2)^{1/2}} e^{-(x^2+y^2)^{1/2}}$

& $\dfrac{\partial \varphi}{\partial z} = 1$

P is the point $x = -1 = x_P$, $y = 0 = y_P$ & $z = e^{-1} = z_P$

$\Rightarrow \left.\dfrac{\partial \varphi}{\partial x}\right|_P = -e^{-1}$; $\left.\dfrac{\partial \varphi}{\partial y}\right|_P = 0$; $\dfrac{\partial \varphi}{\partial z} = 1$.

So eqn of tgt plane

$$-e^{-1}(x-(-1)) + 0 + (1)(z - e^{-1}) = 0$$

$$\Rightarrow z = e^{-1}(2+x)$$

1.3 Divergence and Curl

In this section we will assume that $\mathbf{A}$ is a vector function of position in three dimensions, with continuous first partial derivatives.

Since ∇ is a vector operator, we can define formally a scalar product $\nabla \cdot \mathbf{A}$. This is called the **divergence** of the vector $\mathbf{A}$. We can also define the vector product $\nabla \times \mathbf{A}$, which is called the **curl** of $\mathbf{A}$. So to summarize we have

$$\operatorname{div}\mathbf{A} = \nabla \cdot \mathbf{A}, \quad \operatorname{curl}\mathbf{A} = \nabla \times \mathbf{A}.$$

1.3.1 Cartesian form

$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right) \\
&= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \quad (\text{SCALAR}) \\
\operatorname{curl}\mathbf{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} \\
(\text{VECTOR}) \quad &= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)
\end{aligned}$$

Note that these simple forms for div and curl arise because $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are **constant** vectors: this is not so in other coordinate systems.

N.B. Note well!

$$\underline{A}\cdot\underline{\nabla} \neq \underline{\nabla}\cdot\underline{A} = \operatorname{div}\underline{A} \quad (\text{Scalar})$$

$$\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}$$

Scalar operator

Similarly

$$\underline{A}\times\underline{\nabla} \neq \underline{\nabla}\times\underline{A}$$

or $-\underline{\nabla}\times\underline{A}$

vector operator

Examples

(a) If

$$\mathbf{A} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k},$$

find div$\mathbf{A}$ and curl $\mathbf{A}$.

(b) Find div $\mathbf{u}$ and curl $\mathbf{u}$ when (i) $\mathbf{u} = \mathbf{r}$; (ii) $\mathbf{u} = \omega \times \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\omega = \Omega\mathbf{k}$ with Ω constant.

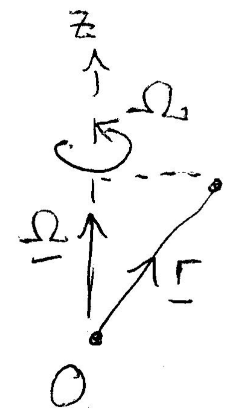
$$\text{div}\,\underline{A} = \frac{\partial}{\partial x}(y^2\cos x + z^3) + \frac{\partial}{\partial y}(2y\sin x - 4) + \frac{\partial}{\partial z}(3xz^2 + 2)$$

$$= -y^2\sin x + 2\sin x + 6xz$$

$$\text{Curl}\,\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix} = \hat{i}(0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2y\cos x - 2y\cos x)$$

$$= \underline{0} \text{ (vector)}$$

We say that $\underline{A}$ is IRROTATIONAL.



(b) (i) $\underline{u} = x\hat{i} + y\hat{j} + z\hat{k}$

$\text{div}\,\underline{u} = 3$ (dimension of space)

$$\text{Curl}\,\underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \underline{0}$$

(ii) $\underline{u} = \underline{\omega} \times \underline{r}$ $\quad \underline{\omega} = \Omega\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = -\Omega y\hat{i} + \Omega x\hat{j}$$

$\text{div}\,\underline{u} = 0$ (SOLENOIDAL FIELD)

$$\text{Curl}\,\underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\Omega y & \Omega x & 0 \end{vmatrix} = \hat{k}(\Omega - (-\Omega)) = 2\Omega\hat{k}$$

Curl $\underline{u}$ is related to ROTATION

1.4 Operations with the gradient operator

1.4.1 Important sum and product formulae

Note that ∇ is a linear operator, and so:

$$\begin{aligned}
\text{(i)}\,\nabla(\phi_1+\phi_2) &= \nabla\phi_1+\nabla\phi_2,\\
\text{(ii)}\,\mathrm{div}(\mathbf{A}+\mathbf{B}) &= \mathrm{div}\mathbf{A}+\mathrm{div}\,\mathbf{B},\\
\text{(iii)}\,\mathrm{curl}\,(\mathbf{A}+\mathbf{B}) &= \mathrm{curl}\,\mathbf{A}+\mathrm{curl}\,\mathbf{B}.
\end{aligned}$$

The proofs of these results follow immediately from the definition of ∇.

Other key results are:

$$\begin{aligned}
\text{(iv)}\,\nabla(\phi\psi) &= \phi\nabla\psi+\psi\nabla\phi,\\
\text{(v)}\,\mathrm{div}(\phi\mathbf{A}) &= \phi\,\mathrm{div}\mathbf{A}+\nabla\phi\cdot\mathbf{A}.
\end{aligned}$$

$$\underline{\nabla}\varphi\cdot\underline{A} \equiv \underline{A}\cdot\underline{\nabla}\varphi = (\underline{A}\cdot\underline{\nabla})\varphi$$

Proof of (v)

$$\begin{aligned}
\mathrm{div}(\phi\mathbf{A}) &= \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\varphi A_1\hat{\imath}+\varphi A_2\hat{\jmath}+\varphi A_3\hat{k}\right)\\
&= \frac{\partial}{\partial x}(\varphi A_1)+\frac{\partial}{\partial y}(\varphi A_2)+\frac{\partial}{\partial z}(\varphi A_3)\\
&= \underbrace{\varphi\left\{\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right\}}_{\varphi\,\mathrm{div}\underline{A}}+\underbrace{A_1\frac{\partial\varphi}{\partial x}+A_2\frac{\partial\varphi}{\partial y}+A_3\frac{\partial\varphi}{\partial z}}_{(\underline{A}\cdot\underline{\nabla})\varphi}
\end{aligned}$$

In writing out these proofs it is easier to use the **summation convention** that we introduced earlier. Rather than write (x, y, z) for Cartesian components, we write (x_1, x_2, x_3) and in place of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ we write $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$. Then we saw earlier that

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_iB_i,\\
\mathbf{A}\times\mathbf{B} &= \varepsilon_{ijk}\widehat{\mathbf{e}}_iA_jB_k
\end{aligned}$$

$$[\underline{A}\times\underline{B}]_i = \varepsilon_{ijk}A_jB_k$$

Also recall the useful result that

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}.$$

Thus, under the summation convention:

$$\begin{aligned}
\mathrm{div}\mathbf{A} &= \partial A_i/\partial x_i\\
[\nabla\phi]_i &= \partial\varphi/\partial x_i\\
[\mathrm{curl}\,\mathbf{A}]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}A_k
\end{aligned}$$

$$\left(\underline{\nabla}\varphi = \hat{e}_i\frac{\partial\varphi}{\partial x_i}\right)$$

where $[\;]_i$ indicates the ith component. Using this approach, the proof of (v) takes the form

$$\begin{aligned}
\mathrm{div}(\phi\mathbf{A}) &= \frac{\partial}{\partial x_i}(\varphi A_i) = \varphi\frac{\partial A_i}{\partial x_i}+A_i\frac{\partial\varphi}{\partial x_i}\\
&= \varphi\,\mathrm{div}\,\underline{A}+(\underline{A}\cdot\underline{\nabla})\varphi
\end{aligned}$$

Other important results are:

$$
\begin{aligned}
\text{(vi)}\ \operatorname{curl}(\phi \mathbf{A}) &= \phi \operatorname{curl} \mathbf{A} + \nabla\phi \times \mathbf{A}, \\
\text{(vii)}\ \operatorname{div}(\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B}, \\
\text{(viii)}\ \operatorname{curl}(\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B} \operatorname{div}\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \operatorname{div}\mathbf{B}, \\
\text{(ix)}\ \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times \operatorname{curl} \mathbf{A} + \mathbf{A} \times \operatorname{curl} \mathbf{B}.
\end{aligned}
$$

Example

Prove relation (ix) above. If we work on the RHS we can write

$\underline{B}\cdot\underline{\nabla}$

$$
\begin{aligned}
&[(\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times \operatorname{curl} \mathbf{A} + \mathbf{A} \times \operatorname{curl}\mathbf{B}]_i \\
&= B_j \frac{\partial}{\partial x_j} A_i + A_j \frac{\partial}{\partial x_j} B_i + \varepsilon_{ijk} B_j (\operatorname{curl} \underline{A})_k + \varepsilon_{ijk} A_j (\operatorname{curl} \underline{B})_k \\
&= \text{---}\ \text{''}\ \text{---} + \varepsilon_{ijk} \left\{ B_j \varepsilon_{klm} \frac{\partial}{\partial x_l} A_m + A_j \varepsilon_{klm} \frac{\partial}{\partial x_l} B_m \right\} \\
&= \text{---}\ \text{''}\ \text{---} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(B_j \frac{\partial A_m}{\partial x_l} + A_j \frac{\partial B_m}{\partial x_l} \right) \\
&= \cancel{B_j \frac{\partial A_i}{\partial x_j}} + \cancel{A_j \frac{\partial B_i}{\partial x_j}} + B_j \left(\frac{\partial A_j}{\partial x_i} - \cancel{\frac{\partial A_i}{\partial x_j}} \right) + A_j \left(\frac{\partial B_j}{\partial x_i} - \cancel{\frac{\partial B_i}{\partial x_j}} \right) \\
&= B_j \frac{\partial A_j}{\partial x_i} + A_j \frac{\partial B_j}{\partial x_i} \\
&= \frac{\partial}{\partial x_i} (A_j B_j) \\
&= \frac{\partial}{\partial x_i} (\underline{A} \cdot \underline{B}) = \left[\underline{\nabla} (\underline{A} \cdot \underline{B}) \right]_i
\end{aligned}
$$

as required.

Note: In the following sections we will assume that our scalar and vector functions possess continuous second derivatives.

1.4.2 The divergence of a gradient: the Laplacian

Consider the operation

repeated subscript

$$
\begin{aligned}
\operatorname{div}(\nabla\phi) &= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\\
&= \partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2+\partial^2\varphi/\partial z^2\\
&\equiv \nabla^2\varphi = \partial^2\varphi/\partial x_i\partial x_i = \partial^2\varphi/\partial x_i^2
\end{aligned}
$$

This is to be read as '**del squared** ϕ' or the **Laplacian** of ϕ. The operator ∇^2 is known as the Laplacian operator. We also define the Laplacian of a vector as

$$\nabla^2\mathbf{A} \equiv \frac{\partial^2\mathbf{A}}{\partial x^2}+\frac{\partial^2\mathbf{A}}{\partial y^2}+\frac{\partial^2\mathbf{A}}{\partial z^2}$$

in Cartesian coordinates, and the equation $\nabla^2\phi = 0$ is known as **Laplace's equation**.

$\nabla^2\varphi = \underbrace{f(x,y,z)}_{\text{given}}$ POISSON'S EQUATION

Example

If $\phi = x^2+y^2$, find $\nabla^2\phi$.

$$\frac{\partial^2\varphi}{\partial x^2} = 2 = \frac{\partial^2\varphi}{\partial y^2}\ ;\ \frac{\partial^2\varphi}{\partial z^2} = 0$$

$\Rightarrow \nabla^2\varphi = 4$ so x^2+y^2 is a soln of Poisson's eqn

Also

$\varphi = x^2-y^2 \Rightarrow \nabla^2\varphi = 0 \Rightarrow x^2-y^2$ is a soln of Laplace's eqn

1.4.3 The curl of a gradient

Consider the operation

$$\left[\text{curl}\,(\nabla\phi)\right]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\nabla\varphi)_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_k}\right)$$

$$\equiv \frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_k}\right) + \frac{1}{2}\varepsilon_{ikj}\frac{\partial}{\partial x_k}\left(\frac{\partial\varphi}{\partial x_j}\right) \qquad \varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$= \frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_k}\right) - \frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_k}\left(\frac{\partial\varphi}{\partial x_j}\right)$$

$$= \frac{1}{2}\varepsilon_{ijk}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial\varphi}{\partial x_j}\right)\right\} = 0$$

(partial derivatives commute).

(This result can also be established by using ~~tensor~~ Subscript notation).

Example

Consider $\phi = axy^2 + byz + cx^3z^2$ and show explicitly that curl $\nabla\phi = 0$.

$$\underline{\nabla}\varphi = \hat{i}\left(ay^2 + 3cx^2z^2\right) + \hat{j}\left(2axy + bz\right) + \hat{k}\left(by + 2cx^3z\right)$$

$$\text{Curl}\,(\underline{\nabla}\varphi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay^2+3cx^2z^2 & 2axy+bz & by+2cx^3z \end{vmatrix}$$

$$= \hat{i}\left(b-b\right) - \hat{j}\left(6cx^2z - 6cx^2z\right) + \hat{k}\left(2ay - 2ay\right)$$

$$= 0 \quad \text{as expected.}$$

1.4.4 The divergence of a curl

This is also always zero, as can be seen from the following argument:

$$\mathrm{div}(\mathrm{curl}\,\mathbf{A}) = \frac{\partial}{\partial x_i}(\mathrm{Curl}\,\underline{A})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}A_k\right)$$

$$\equiv \frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right) + \frac{1}{2}\varepsilon_{jik}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}A_k\right)$$

$$= \frac{1}{2}\varepsilon_{ijk}\left\{\frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial A_k}{\partial x_i}\right)\right\}$$

$$= 0 \quad \text{(Since partial derivatives commute)}$$

Example

Verify that $\mathrm{div}(\mathrm{curl}\,\mathbf{A}) = 0$ for the quantity $\mathbf{A} = y\,e^x\mathbf{i} + (x^2+z)\mathbf{j} + y^3\cos(zx)\mathbf{k}$.

$$\mathrm{Curl}\,\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^x & x^2+z & y^3\cos(zx) \end{vmatrix}$$

$$= \hat{i}\left(3y^2\cos(zx) - 1\right) - \hat{j}\left(-y^3 z\sin(zx)\right) + \hat{k}\left(2x - e^x\right)$$

Then

$\mathrm{div}(\mathrm{Curl}\,\underline{A})$

$$= \cancel{-3y^2 z\sin(zx)} + \cancel{3y^2 z\sin(zx)} + 0$$

$$= 0 \quad \text{as required.}$$

1.4.5 The curl of a curl

This is the vector quantity

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A}).$$

Using tensor notation and the summation convention we can show that

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A}) = \nabla(\mathrm{div}\mathbf{A}) - \nabla^2\mathbf{A}.$$

Proof

$$[\mathrm{Curl}(\mathrm{Curl}\,\underline{A})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\mathrm{Curl}\,\underline{A})_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}\frac{\partial}{\partial x_l}A_m\right)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial^2 A_m}{\partial x_j \partial x_l}\right)$$

$$= \frac{\partial^2 A_j}{\partial x_j \partial x_i} - \frac{\partial^2 A_i}{\partial x_j \partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial A_j}{\partial x_j}\right) - \frac{\partial^2 A_i}{\partial x_j^2}$$

$$= [\underline{\nabla}(\mathrm{div}\,\underline{A})]_i - [\nabla^2 \underline{A}]_i$$

Exercise

Calculate $\mathrm{curl}\,(\mathrm{curl}\,\mathbf{A})$, $\nabla(\mathrm{div}\mathbf{A})$ and $\nabla^2\mathbf{A}$ for $\mathbf{A} = y\,e^x\mathbf{i} + (x^2+z)\mathbf{j} + y^3\cos(zx)\mathbf{k}$.

Answers:

$$\mathrm{Curl}(\mathrm{Curl}\,\underline{A}) = (-y^3\sin zx - y^3 zx\cos zx)\,\hat{i} - \hat{j}(2 - e^x + 3y^2x\sin zx) + \hat{k}(y^3z^2\cos zx - 6y\cos zx)$$

$$\underline{\nabla}(\mathrm{div}\,\underline{A}) = \hat{i}(ye^x - y^3\sin zx - xy^3z\cos zx) + \hat{j}(e^x - 3xy^2\sin zx) + \hat{k}(-x^2y^3\cos zx)$$

$$\nabla^2\underline{A} = \hat{i}(ye^x) + 2\hat{j} + \hat{k}(-y^3z^2\cos zx + 6y\cos zx - y^3x^2\cos zx)$$

1.4.6 Scalar and vector fields

If, at each point of a region V of space, a scalar function ϕ is defined, we say that ϕ is a **scalar field** over the region V. Similarly, if a vector function $\mathbf{A}$ is also defined at all points of V, then $\mathbf{A}$ is a vector field over V. If curl $\mathbf{A} = \mathbf{0}$ we say that A is an **irrotational** vector field. If div $\mathbf{A} = 0$ we say $\mathbf{A}$ is a **solenoidal** vector field. An obvious example of a vector field is the position vector $\mathbf{r}$ of a point in space. In three dimensions:

$$
\begin{aligned}
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,, \\
\operatorname{div}\mathbf{r} &= 3 \\
\operatorname{curl}\mathbf{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \underline{0} \\
|\mathbf{r}| &= r = (x^2 + y^2 + z^2)^{1/2} \\
\nabla r &= \nabla(x^2 + y^2 + z^2)^{1/2} \\
&= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2} \\
&= (x\hat{i} + y\hat{j} + z\hat{k})\,(x^2+y^2+z^2)^{-1/2} \\
&= \underline{r}/r \\
&= \hat{\underline{r}}
\end{aligned}
$$

Example

Find

$$\nabla^2(1/r). \quad (r \neq 0)$$

This is $\underline{\nabla} \cdot \left(\nabla \left(\frac{1}{r} \right) \right)$

First calculate

$$\underline{\nabla}\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)(x^2+y^2+z^2)^{-1/2}$$

$$= -(x\hat{i} + y\hat{j} + z\hat{k})(x^2+y^2+z^2)^{-3/2} = -\frac{\underline{r}}{r^3}$$

Now

$$\underline{\nabla} \cdot \left(\underline{\nabla}\left(\frac{1}{r}\right) \right) = -\left\{ \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{(\quad)^{3/2}} \right) + \frac{\partial}{\partial z}\left(\frac{z}{(\quad)^{3/2}} \right) \right\}$$

$$= -\frac{3}{(x^2+y^2+z^2)^{3/2}} - \left\{ \frac{x\left(-\frac{3}{2}\right)(2x)}{(\quad)^{5/2}} + \frac{y\left(-\frac{3}{2}\right)(2y)}{(\quad)^{5/2}} + \frac{z\left(-\frac{3}{2}\right)(2z)}{(\quad)^{5/2}} \right\}$$

$$= -\frac{3}{(x^2+y^2+z^2)^{3/2}} + \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \underline{\underline{0}}$$

$\therefore\ \nabla^2\left(\frac{1}{r}\right) = 0 \quad (r \neq 0) \Rightarrow \frac{1}{r}$ satisfies Laplace eqn.

In fact $\frac{1}{|\underline{r} - \underline{r_0}|}$ is also a soln. i.e. $\nabla^2\left(\frac{1}{|\underline{r}-\underline{r_0}|}\right) = 0$

$\underline{r_0}$ const vector

IMPORTANT FOR DEVELOPMENT OF PDE THEORY (Partial Diff. Eqns) & GREEN'S FUNCTION

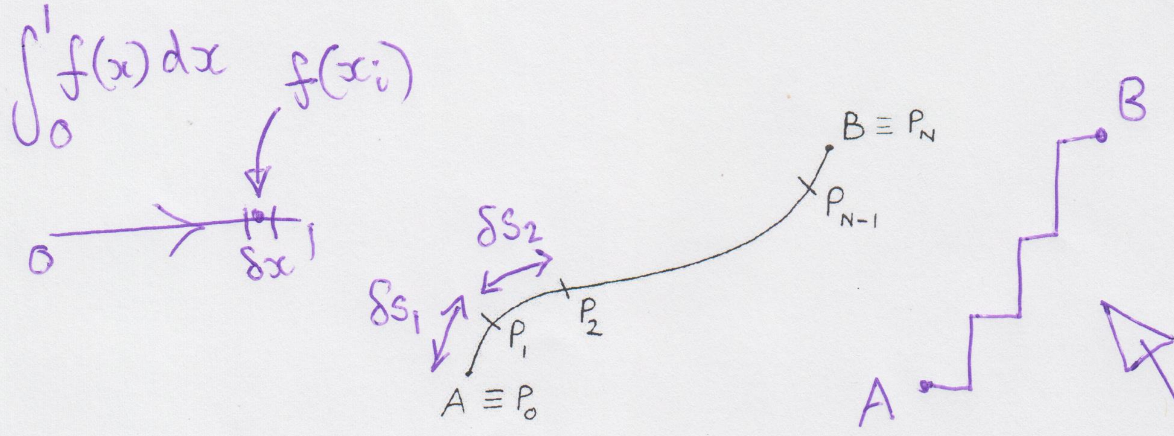


Figure 4: A curve γ joining A to B and divided into N sections.

1.5 Path Integrals

1.5.1 Definition

Consider a curve γ (not necessarily in the plane, and not necessarily smooth) joining the points A and B. (See figure 4). Suppose that the curve is divided into N sections: $AP_1, P_1P_2, \ldots, P_{N-1}B$. Let $AP_1 = \delta s_1, P_1P_2 = \delta s_2, \ldots, P_{N-1}B = \delta s_N$. Next, suppose a function f is defined along this curve γ. We compute the sum

$$f_1\delta s_1 + f_2\delta s_2 + \cdots + f_N\delta s_N,$$

where $f_n = f(P_n)$. On increasing N indefinitely, while letting the maximum $\delta s_n \to 0$, the resulting limit of the sum, if it exists, is called the **path integral of f along γ**, and we write:

$$\int_\gamma f\,ds = \lim_{\substack{N\to\infty \\ \max(\delta s_n)\to 0}} \sum_{n=1}^{N} f_n\,\delta s_n$$

The function f may be a scalar or a vector.

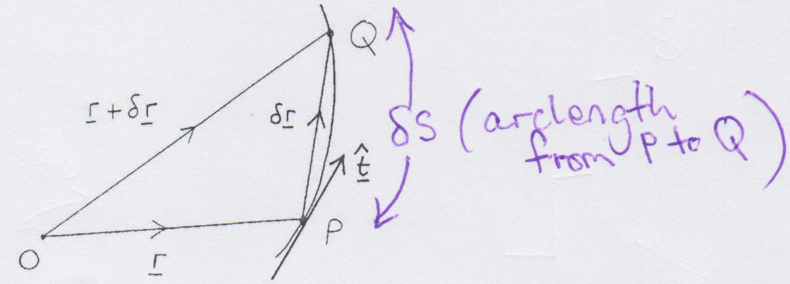


Figure 5: Diagram showing the tangent vector at a point P.

1.5.2 Path element

See figure 5. Let δs represent the arc PQ and suppose that the vector $\overrightarrow{PQ} = \delta\mathbf{r}$. We define the **tangent vector**

$$\widehat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s},$$

and the **path element**

$$d\mathbf{r} = \widehat{\mathbf{t}}\, ds.$$

In Cartesians $d\underline{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Note that $\widehat{\mathbf{t}}$ has length unity because $|\delta\mathbf{r}| \to \delta s$ as $\delta s \to 0$. We can then define the quantity

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_\gamma (\underline{F} \cdot \hat{\underline{t}})\, ds$$

1.5.3 Conservative forces

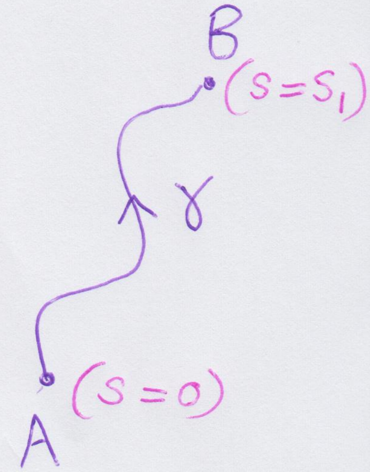
Consider the special case where we have a vector $\mathbf{F}$ of the form

$$\mathbf{F} = \nabla\phi$$

with ϕ a differentiable scalar function. Consider the integral (with γ defined as in figure 3):

$$\begin{aligned}
\int_\gamma \mathbf{F}\cdot d\mathbf{r} &= \int_\gamma (\underline{\nabla}\varphi\cdot\hat{t})\,ds \\
&= \int_\gamma \frac{\partial\varphi}{\partial x_i}\hat{e}_i\cdot\frac{d\underline{r}}{ds}\,ds \\
&= \int_\gamma \frac{\partial\varphi}{\partial x_i}\hat{e}_i\cdot\frac{dx_j}{ds}\hat{e}_j\,ds \\
&= \int_\gamma \frac{\partial\varphi}{\partial x_i}\frac{dx_i}{ds}\,ds \\
&= \int_\gamma \frac{d\varphi}{ds}\,ds \quad \left(\text{Chain rule}\right). \\
&= \int_0^{s_1}\frac{d\varphi}{ds}\,ds = [\varphi]_A^B \\
&= \varphi(B)-\varphi(A)
\end{aligned}$$

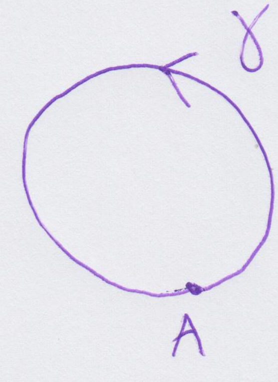
$(\hat{e}_i\cdot\hat{e}_j=\delta_{ij})$



We note that the result is **independent of the path** γ joining A to B. In particular, if γ is a closed curve (i.e. $B \equiv A$), then we have $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$, where we put a circle on the integral to denote the path is closed. We sometimes refer to such an integral as the **circulation** of $\mathbf{F}$ around γ.

Closed curve

$$\oint_\gamma \underline{\nabla}\varphi\cdot d\underline{r} = 0$$



Closed curve $B\equiv A$

CIRCULATION of $\underline{F}$ around closed curve γ is

$$\oint_\gamma \underline{F}\cdot d\underline{r}$$

($\underline{F}$ not necessarily equal to $\underline{\nabla}\varphi$)

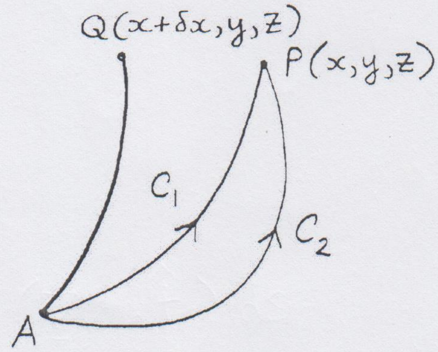


Figure 6: Two curves joining A to P. Q is a neighbouring point.

If a vector field $\mathbf{F}$ has the property that $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$ for **any** closed curve γ, we say that $\mathbf{F}$ is a **conservative field**. Thus, if $\mathbf{F} = \nabla\phi$, then $\mathbf{F}$ is conservative. Conversely, if $\mathbf{F}$ is conservative we can always find a differentiable scalar function ϕ such that $\mathbf{F} = \nabla\phi$. The function ϕ is called the **potential** of the field $\mathbf{F}$.

Proof of this last part

See figure 6. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Since we know that $\mathbf{F}$ is conservative it must be the case that $\int_A^P \mathbf{F}\cdot d\mathbf{r}$ is independent of the path from A to P and hence

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r},$$

where C_1 and C_2 are any two curves drawn from A to P. Suppose that the point A is fixed. Then

$$\begin{aligned}\int_A^P \mathbf{F}\cdot d\mathbf{r} &= G(P), \text{ say}\\ &= G(x,y,z)\end{aligned}$$

Let Q be the point $(x+\delta x, y, z)$ and let P be the point (x,y,z). Consider the quantity

$$\begin{aligned}G(x+\delta x,y,z) - G(x,y,z) &\equiv \int_A^Q \underline{F}\cdot d\underline{r} - \int_A^P \underline{F}\cdot d\underline{r}\\ &= \int_P^Q \underline{F}\cdot d\underline{r}\end{aligned}$$

$\left(-\int_A^P = \int_P^A\right)$

$(dy = dz = 0)$

But we can choose the path from P to Q so that only x varies, in which case $d\mathbf{r} = \mathbf{i}\,dx$. Thus

$$G(x+\delta x,y,z) - G(x,y,z) = \int_x^{x+\delta x} F_1\, dx$$

and hence

$$\boxed{\begin{array}{l}\text{if } g(\delta x) = \int_x^{x+\delta x} F_1(u)\,du \\ \text{then} \\ g'(\delta x) = F_1(x+\delta x) \\ \Rightarrow g'(0) = F_1(x)\end{array}}$$

$$\begin{aligned}\frac{\partial G}{\partial x} &= \lim_{\delta x \to 0} \left[G(x+\delta x, y, z) - G(x,y,z)\right]/\delta x \\ &= \lim_{\delta x \to 0} \left(\int_x^{x+\delta x} F_1\, dx\right)/\delta x \\ &= F_1(x,y,z)\end{aligned}$$

(L'Hopital)

Similarly we can show that

$$F_2 = \frac{\partial G}{\partial y}, \quad F_3 = \frac{\partial G}{\partial z}.$$

Thus, if $\mathbf{F}$ is conservative then a scalar function (G in this case) can be found such that $\mathbf{F} = \nabla G$. $\Rightarrow$ Curl $\underline{F} = 0$

Example

For the vector field

(Can check Curl $\underline{F} = 0$)

$$\mathbf{F} = (3x^2 + yz)\mathbf{i} + (6y^2 + xz)\mathbf{j} + (12z^2 + xy)\mathbf{k}$$

find a scalar function $\phi(x, y, z)$ such that $\mathbf{F} = \nabla\phi$. Hence calculate $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ where $A = (0, 0, 0)$ and $B = (1, 1, 1)$.

$\underline{F} = \underline{\nabla}\varphi$

$\Rightarrow \dfrac{\partial \varphi}{\partial x} = F_1 = 3x^2 + yz \Rightarrow \varphi = x^3 + xyz + f(y,z)$

$\Rightarrow \dfrac{\partial \varphi}{\partial y} = \cancel{xz} + \dfrac{\partial f}{\partial y} = F_2 = 6y^2 + \cancel{xz} \Rightarrow f = 2y^3 + g(z)$

$\Rightarrow \varphi = x^3 + xyz + 2y^3 + g(z)$

$\Rightarrow \dfrac{\partial \varphi}{\partial z} = \cancel{xy} + g'(z) = F_3 = 12z^2 + \cancel{xy}$

$\Rightarrow g(z) = 4z^3 + C$

$\Rightarrow \varphi = x^3 + xyz + 2y^3 + 4z^3 + C$

Then $\int_A^B \underline{F} \cdot d\underline{r} = \left[\varphi\right]_{(0,0,0)}^{(1,1,1)} = (8 + C) - C$

$= 8$ //

1.5.4 Practical evaluation of path integrals

Suppose we wish to evaluate

$$I = \int_\gamma \mathbf{F} \cdot d\mathbf{r}$$

explicitly, where $\mathbf{F}$ is a known function of (x, y, z) and γ is some known curve joining the points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

Along γ we can write

$$x = x(t),\ y = y(t),\ z = z(t) \qquad (t_0 \leqslant t \leqslant t_1)$$

Here, t is a parameter that takes us along γ with $x(t_0) = x_0, x(t_1) = x_1$ and similarly for y and z. Then we can write

$$d\mathbf{r} = \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) dt$$

and hence, with $\mathbf{F} = F_1(t)\mathbf{i} + F_2(t)\mathbf{j} + F_3(t)\mathbf{k}$:

$$I = \int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

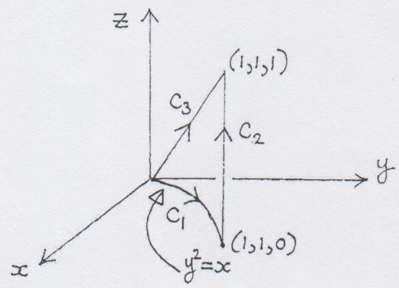


Figure 7: The integration path for this example.

Example (see figure 7)

Evaluate

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} \text{ with } \mathbf{F} = yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$$

when γ joins $(0,0,0)$ to $(1,1,1)$ along

(i) $C_1 + C_2$ with C_1 the curve $x = y^2, z = 0$ from $(0,0,0)$ to $(1,1,0)$ and C_2 is the straight line joining $(1,1,0)$ to $(1,1,1)$;

(ii) C_3 is the straight line joining $(0,0,0)$ to $(1,1,1)$.

(i) On C_1: $z = dz = 0$

$x = y^2$

Let $y = t$ $(0 \leqslant t \leqslant 1)$

$x = t^2$

Along C_1: $\underline{F} = 0\hat{i} + t^3\hat{j} + 0\hat{k}$

$$\int_{C_1} \underline{F}\cdot d\underline{r} = \int_0^1 t^3 \underbrace{\frac{dy}{dt}}_{=1} dt = \frac{1}{4}$$

On C_2: $x = y = 1 \Rightarrow dx = dy = 0$

z goes from 0 to 1 let $z = t$ $(0 \leqslant t \leqslant 1)$

Along C_2: $\underline{F} = z\hat{i} + \hat{j} + z\hat{k} = t\hat{i} + \hat{j} + t\hat{k}$

$$\int_{C_2} \underline{F}\cdot d\underline{r} = \int_0^1 t\underbrace{\frac{dx}{dt}}_{0} + \underbrace{\frac{dy}{dt}}_{0} + t\underbrace{\frac{dz}{dt}}_{1}\, dt = \int_0^1 t\, dt = \frac{1}{2}$$

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

(ii) On C_3: $x = y = z = t$ $(0 \leqslant t \leqslant 1)$

$\underline{F} = t^2\hat{i} + t^2\hat{j} + t^2\hat{k}$ $\quad \int_{C_3} \underline{F}\cdot d\underline{r} = \int_0^1 3t^2\, dt = 1$

Answers not the same $\Rightarrow \underline{F}$ is not conservative

Can check $\operatorname{curl} \underline{F} \neq 0$

1.6 Surface integrals

1.6.1 Definition

To define a surface integral of $f = f(P)$ over a surface S, we divide S into elements of area $\delta S_1, \delta S_2, \ldots, \delta S_N$. Let $f_1, f_2, \ldots, f_N$ be the values of f at typical points $P_1, P_2, \ldots, P_N$ of $\delta S_1, \delta S_2, \ldots, \delta S_N$ respectively. We calculate the quantity

$$\sum_{n=1}^{N} f_n \delta S_n.$$

We now let $N \to \infty$, $\max \delta S_n \to 0$. The resulting limit, if it exists, is called the **surface integral of f over S**, and we write it as

$$\int_S f\, dS = \lim_{\substack{N \to \infty \\ \max(\delta S_n) \to 0}} \sum_{n=1}^{N} f_n\, \delta S_n$$

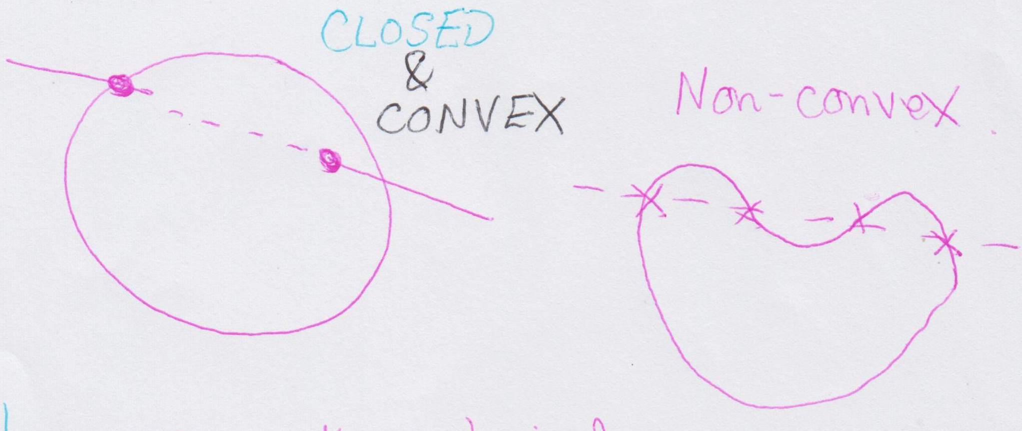
As with the line integral, the function f may be a vector or a scalar.

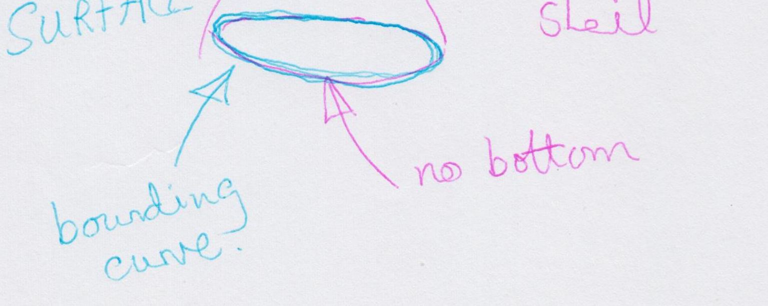
1.6.2 Types of surfaces

Closed surface: this divides three-dimensional space into two non-connected regions - an interior region and an exterior region;

Convex surface: this is a surface which is crossed by a straight line at most twice;

Open surface: this does not divide space into two non-connected regions - it has a rim which can be represented by a closed curve. (A closed surface can be thought of as the sum of two open surfaces).





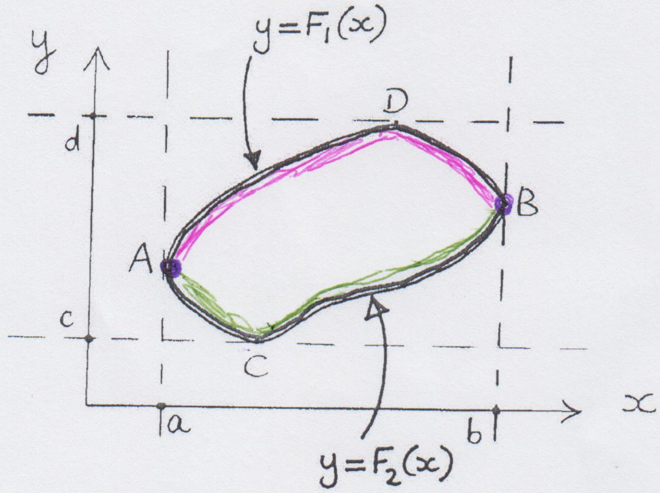


Figure 8: Diagram to illustrate the evaluation of surface integrals.

1.6.3 Evaluation of surface integrals for plane surfaces in the $x - y$ plane

An **areal element** dS is an 'infinitesimally small' element of area of a surface. Even for curved surfaces it can be thought of as approximately plane. The **vector areal element** $d\mathbf{S}$ is the vector $\widehat{\mathbf{n}}\, dS$ where $\widehat{\mathbf{n}}$ is the unit vector normal to dS. For plane surfaces dS can be expressed in Cartesian coordinates (x, y) since we may choose the surface to lie in the plane $z = 0$. Thus we can write $dS = dx\, dy$. (See figure 8). $\hat{\underline{n}} = \hat{\underline{k}}$ (in 2D)

Let the rectangle $x = a, b$ and $y = c, d$ circumscribe S. We will assume for simplicity that S is convex. (If it isn't then we split S up into convex sub-regions). Let the equation of the boundary of S be denoted by

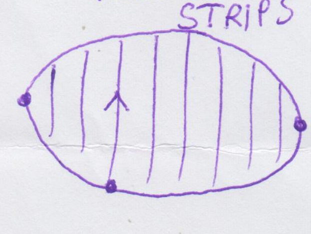
$$y = \begin{cases} F_1(x) & \text{upper half } ADB \\ F_2(x) & \text{lower half } ACB \end{cases}.$$

(n.b. we need to ensure these are single-valued functions, which they will be if S is convex). Then

$$\text{Area of } S = \int_{x=a}^{x=b} \int_{y=F_2(x)}^{y=F_1(x)} 1 .\, dy \; dx$$

$$= \int_a^b (F_1(x) - F_2(x))\, dx .$$

VERTICAL STRIPS.

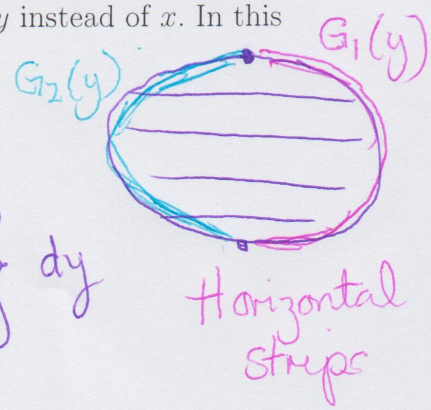


If $f(x, y)$ is any function of position:

$$\int_S f\, dS = \int_{x=a}^{b} \left\{ \int_{y=F_2(x)}^{y=F_1(x)} f(x,y)\, dy \right\} dx$$

e.g. f could represent density (mass/unit area)

In some situations it may be more convenient to do the $x-$integration first. If we want to do this we need to write the boundaries in terms of functions of y instead of x. In this case let the boundary be described by



$$x = \begin{cases} G_1(y) & \text{right half } CBD \\ G_2(y) & \text{left half } CAD \end{cases} .$$

Then

$$\text{Area of } S = \int_{y=c}^{y=d} \left\{ \int_{x=G_2(y)}^{x=G_1(y)} 1.\ dx \right\} dy$$

$$= \int_c^d \left(G_1(y) - G_2(y) \right) dy$$

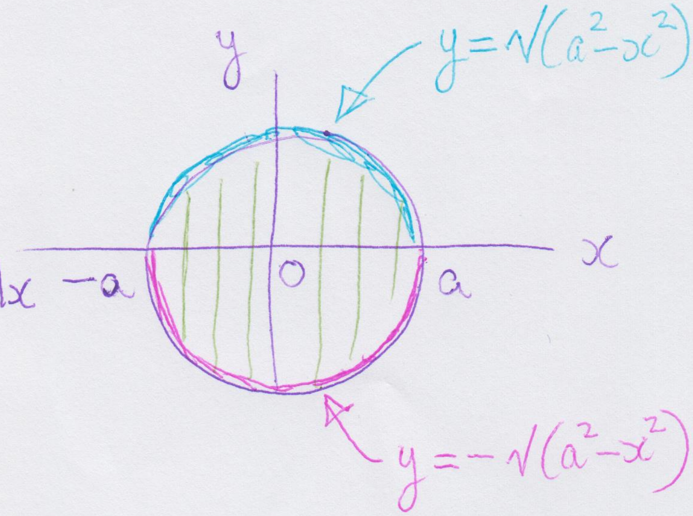
and

$$\int_S f\,dS = \int_{y=c}^{d} \left\{ \int_{x=G_2(y)}^{G_1(y)} f(x,y)\,dx \right\} dy .$$

1.6.4 Example

Find the area of the circle $x^2 + y^2 = a^2$.

$$A = \int_{-a}^{a} \left\{ \int_{y=-\sqrt{(a^2-x^2)}}^{y=+\sqrt{(a^2-x^2)}} 1 \, dy \right\} dx$$



$$= \int_{-a}^{a} \left[y \right]_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx$$

$$= 2\int_{-a}^{a} \sqrt{(a^2-x^2)} \, dx$$

Subst
$x = a \sin u$

$$= \ldots = \pi a^2$$

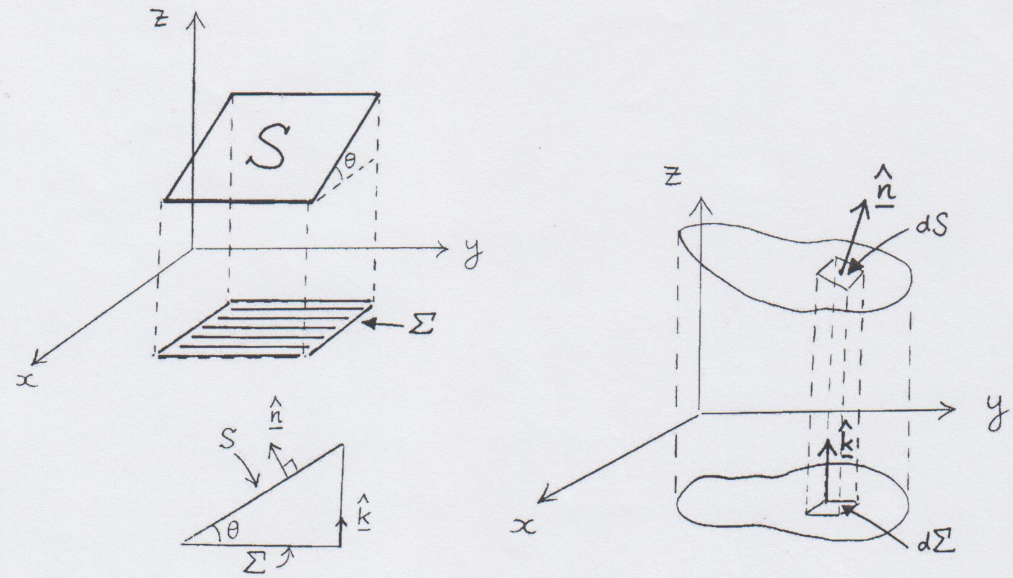


Figure 9: Left: The projection of a plane area S onto the $x-y$ plane. Right: The projection of a curved surface S onto the $x-y$ plane.

1.6.5 Projection of an area onto a plane

Consider first a plane area S (left hand diagram in figure 9). Suppose Σ is the projected area in the $x-y$ plane. Then $\Sigma = S\cos\theta$, where $\cos\theta = |\widehat{\mathbf{n}}\cdot\mathbf{k}\,|$.

Now consider a curved surface. (Right hand diagram in figure 9). If we consider an areal element dS then this will be effectively plane, and so

$$dS = d\Sigma \,/\, |\,\hat{\underline{n}}\cdot\hat{\underline{k}}\,|$$

could depend on position

1.6.6 The projection theorem

Let P denote a general point of a surface S which at no point is orthogonal to the direction $\mathbf{k}$. Then:

$$\int_S f(P)\,dS = \int_\Sigma f(P)\,\frac{dx\,dy}{|\widehat{\mathbf{n}}\cdot\mathbf{k}|},$$

depends on P

Contains the curvature

where Σ is the projection of S onto the plane $z=0$, and $\widehat{\mathbf{n}}$ is normal to S.

Proof

$$\int_S f(P)\,dS = \lim_{\substack{N\to\infty\\ \delta S_r\to 0}} \sum_{r=1}^{N} f(P_r)\,\delta S_r$$

$$= \lim_{\substack{N\to\infty\\ \delta S_r\to 0}} \sum_{r=1}^{N} f(P_r)\left\{\frac{\delta\Sigma_r}{|\hat{\underline{n}}_r\cdot\hat{\underline{k}}|} + \varepsilon_r\right\}$$

(or $\delta\Sigma_r \to 0$)

where $\varepsilon_r \to 0$ as $\delta S_r \to 0$. (Here $\widehat{\mathbf{n}}_r$ is the unit vector normal to S at P_r and $\delta\Sigma_r$ is the projection of δS_r onto the plane $z=0$. It therefore follows that

$$\int_S f(P)\,dS = \int_\Sigma f(P)\,\frac{d\Sigma}{|\hat{\underline{n}}\cdot\hat{\underline{k}}|}$$

Related to Jacobian (see later)

as required. Note that $f(P)$ is evaluated at $P(x,y,z)$ on S in **both integrals**.

If, for example, the equation of S is $z=\phi(x,y)$ then the theorem gives

$$\int_S f(x,y,z)\,dS = \int_\Sigma f(x,y,\phi(x,y))\,\frac{dx\,dy}{|\hat{\underline{n}}\cdot\hat{\underline{k}}|}$$

Alternatively, we may choose to project the surface onto $x=0$ or $y=0$ to give:

$$\int_S f(P)\,dS = \int_{\Sigma_x} f(P)\,\frac{dy\,dz}{|\hat{\underline{n}}\cdot\hat{\underline{i}}|} = \int_{\Sigma_y} f(P)\,\frac{dx\,dz}{|\hat{\underline{n}}\cdot\hat{\underline{j}}|}$$

where Σ_x is the projection of S onto $x=0$ and Σ_y is the projection of S onto $y=0$.

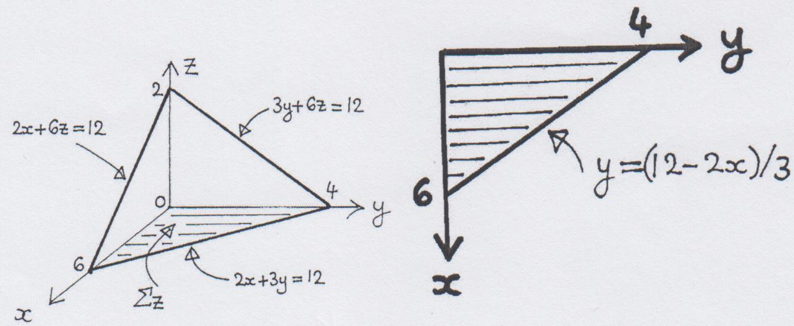


Figure 10: Left: The plane $2x + 3y + 6z = 12$ and its projection onto the $x - y$ plane. Right: The projected region Σ_z viewed from above.

[As an exercise try projecting onto $x=0$ or $y=0$]

Example of using the projection theorem

Evaluate

$$\int_S (y + 2z - 2)\, dS$$

where S is the part of the plane $2x + 3y + 6z = 12$ in the first octant ($x, y, z \geq 0$), by projecting onto the plane $z = 0$.

Normal to plane is $\underline{\nabla}(2x+3y+6z) = 2\hat{i}+3\hat{j}+6\hat{k}$

$\Rightarrow \hat{\underline{n}} = \pm(2\hat{i}+3\hat{j}+6\hat{k})/\sqrt{(2^2+3^2+6^2)} = \pm(2\hat{i}+3\hat{j}+6\hat{k})/7$

$\Rightarrow |\hat{\underline{n}}\cdot\hat{\underline{k}}| = 6/7.$

⚠ Don't set $z=0$

Here $z = (12-2x-3y)/6 = 2 - \frac{1}{3}x - \frac{1}{2}y$

$\Rightarrow y+2z-2 = 2 - \frac{2}{3}x$

Projecting onto $z=0$

$$\int_S (y+2z-2)\, dS = \int_{\Sigma_z} (y+2z-2)\, \frac{dx\,dy}{(6/7)}$$

$$= \frac{7}{6}\int_{\Sigma_z} \left(2-\frac{2}{3}x\right) dx\, dy.$$

$$= \frac{7}{6}\int_{x=0}^{6}\int_{y=0}^{y=(12-2x)/3} \left(2-\frac{2}{3}x\right) dy \quad dx$$

$$= \frac{7}{6}\int_0^6 \left(2-\frac{2}{3}x\right)\frac{(12-2x)}{3}\, dx.$$

$$= \ldots = \frac{14}{27}\int_0^6 (x^2-9x+18)\, dx = \ldots = \underline{\underline{28/3}}$$

1.7 Volume Integrals

1.7.1 Definition

Consider a volume τ and split it up into N subregions $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$. Let $P_1, P_2, \ldots, P_N$ be typical points of $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$.

Consider the sum

$$\sum_{i=1}^{N} f(P_i)\,\delta\tau_i$$

Now let $N \to \infty, \max \delta\tau_i \to 0$. If this sum tends to a limit we call it the volume integral of f over τ and write this as

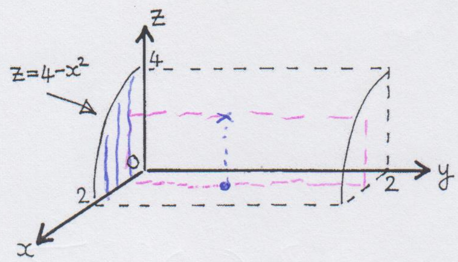
$$\int_\tau f\, d\tau.$$

The function f may be a vector or a scalar.

1.7.2 Volume element

In Cartesian coordinates the volume element

$$d\tau = dx\, dy\, dz.$$



Posh bread bin

Figure 11: The volume τ for the example.

Example

Evaluate

e.g. density

$$\int_\tau (2x + y)\, d\tau$$

when τ is the volume enclosed by the parabolic cylinder $z = 4 - x^2$ and the planes $x = y = z = 0$ and $y = 2$.

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=2} \int_{z=0}^{z=4-x^2} (2x+y)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^2 (2x+y)(4-x^2)\, dy\, dx$$

$$= \int_0^2 \int_0^2 \left(8x - 2x^3 + 4y - x^2 y\right) dy\, dx$$

$$= \int_0^2 \left[8xy - 2x^3 y + 2y^2 - \frac{x^2}{2} y^2 \right]_{y=0}^{y=2} dx$$

$$= \int_0^2 16x - 4x^3 + 8 - 2x^2 \quad dx$$

$$= \ldots = 80/3$$

$$= \int_0^2 (4-x^2) \left[2xy + \frac{y^2}{2} \right]_{y=0}^{2} dx$$

$$= \int_0^2 (4-x^2)(4x+2)\, dx = \ldots = 80/3$$

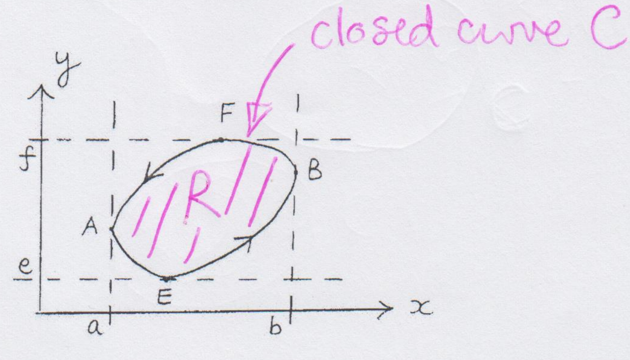


Figure 12: Diagram for proof of Green's theorem.

1.8 Results relating line, surface and volume integrals

1.8.1 Green's theorem in the plane

non-intersecting

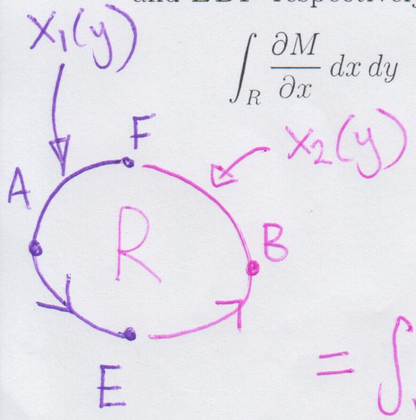
Suppose R is a closed plane region bounded by a simple plane closed convex curve in the $x-y$ plane. Let L, M be continuous functions of x, y having continuous derivatives throughout R. Then:

$$\oint_C (L\,dx + M\,dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy,$$

always keep finite region to the left.

where C is the boundary of R described in the counter-clockwise (positive) sense.

Proof. We draw a rectangle formed by the tangent lines $x = a, b$ and $y = e, f$ (figure 12). This rectangle circumscribes C. Let $x = X_1(y), x = X_2(y)$ be the equations of EAF and EBF respectively. We then can write



$$\int_R \frac{\partial M}{\partial x}\,dx\,dy = \int_{y=e}^{f} \left\{ \int_{X_1(y)}^{X_2(y)} \left(\frac{\partial M}{\partial x}\right) dx \right\} dy$$

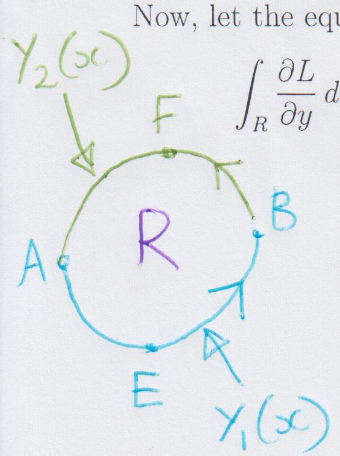
Horiz strips

$$= \int_e^f M(X_2(y),y) - M(X_1(y),y)\,dy$$

$$= \int_e^f M(X_2(y),y)\,dy + \int_f^e M(X_1(y),y)\,dy$$

$$\equiv \oint_C M\,dy$$

Now, let the equations of AEB and AFB be $y = Y_1(x), y = Y_2(x)$. Then



$$\int_R \frac{\partial L}{\partial y}\,dx\,dy = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left(\frac{\partial L}{\partial y}\right) dy \right\} dx$$

vert. strips

$$= \int_a^b L(x,Y_2(x)) - L(x,Y_1(x))\,dx$$

$$= -\left\{ \int_a^b L(x,Y_1(x))\,dx + \int_b^a L(x,Y_2(x))\,dx \right\}$$

$$= -\oint_C L\,dx$$

Hence result.

G-T $\oint_C L\,dx + M\,dy = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$

1.8.2 Vector forms of Green's Theorem

(i) (2D Stokes Theorem). Let $\mathbf{F} = L\mathbf{i} + M\mathbf{j}$, and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$. Then

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathbf{k}.$$

Over the region R we can write $dx\,dy = dS$. Thus using Green's theorem:

$d\underline{S} = \hat{\underline{k}}\,dS$ in 2D; dirn of normal

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_R \mathbf{k}\cdot\operatorname{curl}\mathbf{F}\,dS\\ &= \int_R \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.\end{aligned}$$

This result can be generalized to three dimensions (see **Stokes theorem** later).

(ii)(Divergence theorem in 2D). This time let $\mathbf{F} = M\,\mathbf{i} - L\,\mathbf{j}$. Then

$F_1\hat{i} + F_2\hat{j}$

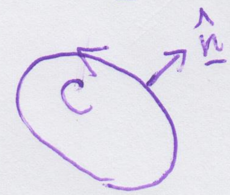
$F_1 = M$

$F_2 = -L$

$$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

and so Green's theorem can be rewritten as

$$\int_R \operatorname{div}\mathbf{F}\,dx\,dy = \oint_C F_1\,dy - F_2\,dx.$$



Now it can be shown (exercise) that

$$\widehat{\mathbf{n}}\,ds = (dy\,\mathbf{i} - dx\,\mathbf{j})$$

where s is arclength along C, and $\widehat{\mathbf{n}}$ is the unit normal to C. Therefore we can rewrite Green's theorem as

$$\int_R \operatorname{div}\mathbf{F}\,dx\,dy = \oint_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds.$$

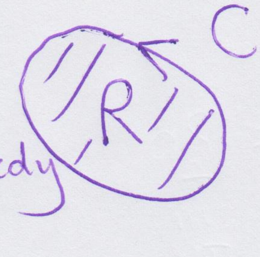
This result also turns out to be true in three dimensions, where it is known as the **Divergence Theorem**.

Example

Show that the area enclosed by a simple closed curve with boundary C can be expressed as

$$\frac{1}{2}\oint_C x\,dy - y\,dx.$$

Use this result to calculate the area of an ellipse.



G-T $\displaystyle\oint_C L\,dx + M\,dy = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$

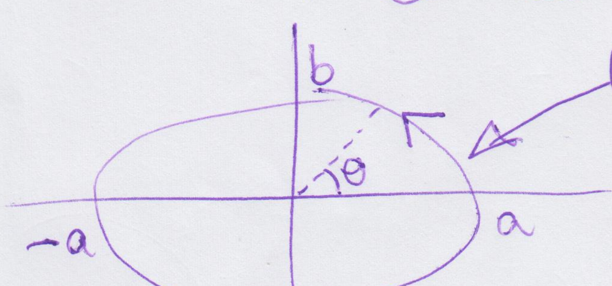
Choose $L = -y$

$M = x$

G-T $\Rightarrow \displaystyle\oint_C -y\,dx + x\,dy = \int_R (1+1)\,dx\,dy$

= twice the area of R

$\therefore$ area of R $= \displaystyle\frac{1}{2}\oint_C x\,dy - y\,dx$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a\cos\theta$ $\quad(0 \leqslant \theta \leqslant 2\pi)$

$y = b\sin\theta$

p'zn of the elliptical boundary

$x\,dy - y\,dx = \left[(a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\right] d\theta$

$= ab\,d\theta$

$\therefore \displaystyle\oint_C x\,dy - y\,dx = \int_0^{2\pi} ab\,d\theta = 2\pi ab$

= twice the area of R

$\therefore$ area of ellipse $= \pi ab$ (as expected)

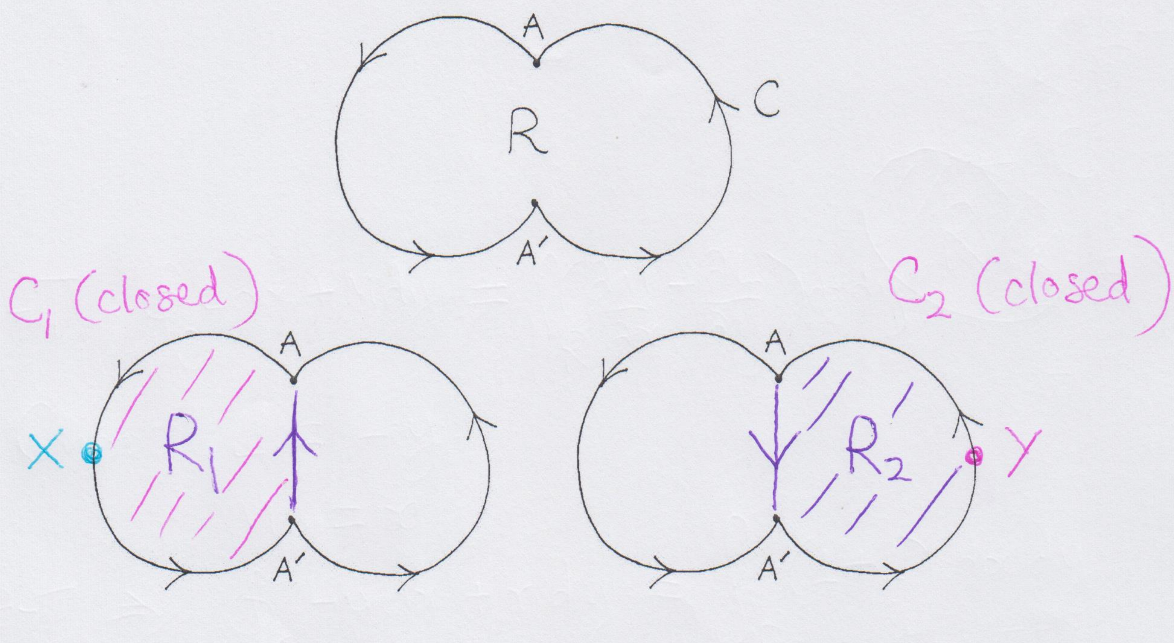


Figure 13: A non-convex boundary.

1.8.3 Extensions of Green's theorem in the plane

Green's theorem is true for more complicated geometries than that assumed in the proof given above. e.g. if C is not convex, but has the shape given in figure 13. We can join the points A, A' so as to form 2 (or more) simple convex closed curves C_1, C_2 enclosing R_1, R_2 where $R_1 + R_2 = R$. Then:

$$\begin{aligned}\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_{R_1} (\mathrm{Curl}\,\underline{F})\cdot d\underline{S} + \int_{R_2} (\mathrm{Curl}\,\underline{F})\cdot d\underline{S} \\ &= \int_{R} (\mathrm{Curl}\,\underline{F})\cdot d\underline{S}\end{aligned}$$

Now

$$\begin{aligned}\oint_{C_1} &= \int_{AXA'} + \int_{A'}^{A} \qquad \left[\int_{A'}^{A} = -\int_{A}^{A'}\right] \\ \oint_{C_2} &= \int_{A'YA} + \int_{A}^{A'}\end{aligned}$$

and so

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_1} \underline{F}\cdot d\underline{r} + \oint_{C_2} \underline{F}\cdot d\underline{r} = \int_R \mathrm{Curl}\,\underline{F}\cdot d\underline{S}$$

We see therefore that the theorem still holds.

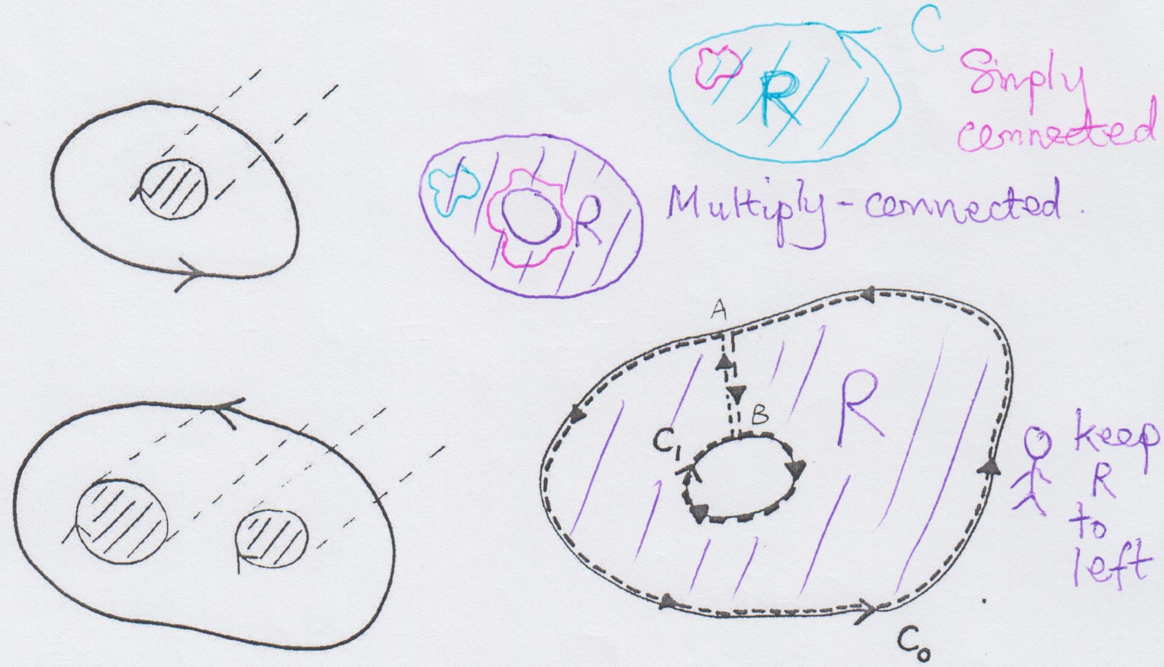


Figure 14: Left: Examples of doubly- and triply-connected regions. Right: Green's theorem in a multiply-connected region.

1.8.4 Green's theorem in multiply-connected regions

A region R is said to be **simply-connected** if any closed curve drawn in R can be shrunk to a point without leaving R. If we restrict ourselves to two dimensions then any region with a hole in it is not simply-connected (left-hand picture in figure 14). A region which is not simply-connected is said to be **multiply-connected**.

If R is multiply-connected, Green's theorem is still true provided C is now interpreted as the entire (outer and inner) boundary, with C described so that the region R is always on the left (right hand picture in figure 14).

For example if we have a doubly-connected region, we describe the outer boundary C_0 in an anti-clockwise fashion and the inner boundary C_1 clockwise. We can then join the point A on C_0 to the point B on C_1 by the line AB. This line then divides R in such a way that it is a simply connected region bounded by the closed curve $C_0 + AB + C_1 + BA$. Then, by Green's theorem:

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \left(\oint_{C_0} + \cancel{\int_A^B} + \oint_{C_1} + \cancel{\int_B^A}\right)(\underline{F} \cdot d\underline{r})$$

and therefore it follows that

$$\int_R \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_0} \underline{F} \cdot d\underline{r} + \oint_{C_1} \underline{F} \cdot d\underline{r}$$

where $C = C_0 + C_1$.

outer boundary (anti-clockwise) inner boundary (clockwise)

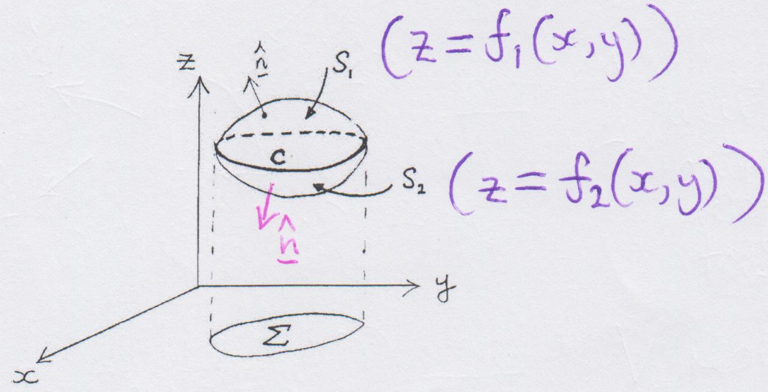


Figure 15: Diagram for the proof of the divergence theorem.

1.8.5 Flux

If S is a surface then the flux of $\mathbf{A}$ across S is defined as

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS.$$

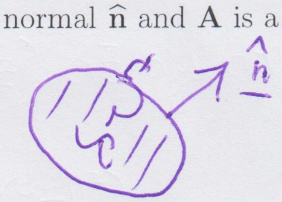
If S is a closed surface then, by convention, we always draw the unit normal $\widehat{\mathbf{n}}$ **out** of S.

1.8.6 The divergence theorem

If τ is the volume enclosed by a closed surface S with unit outward normal $\widehat{\mathbf{n}}$ and $\mathbf{A}$ is a vector field with continuous derivatives throughout τ, then:

denotes closed surface

$$\oint_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_\tau \mathrm{div}\mathbf{A}\, d\tau.$$



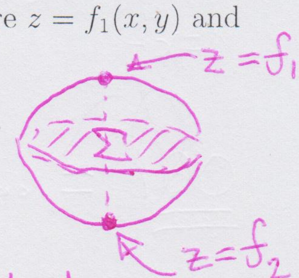
Proof

We will assume that S is convex and that τ is simply connected, with no interior boundaries. Let $\mathbf{A} = (A_1, A_2, A_3)$ and $\widehat{\mathbf{n}} = (l, m, n)$. We have to prove that

$$\int_S (lA_1 + mA_2 + nA_3)\, dS = \int_\tau \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\, dy\, dz$$

Project S onto the plane $z = 0$ (figure 15). The cylinder with normal cross-section Σ and generators parallel to the $z-$axis circumscribes S and it touches S along the curve C which divides S into two open surfaces, S_1 (upper) and S_2 (lower). Both S_1 and S_2 have projection Σ in the plane $z = 0$. Suppose the equations of S_1 and S_2 are $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Then:

$$\begin{aligned}\int_\tau \frac{\partial A_3}{\partial z}\, dx\, dy\, dz &= \int_\tau \frac{\partial A_3}{\partial z}\, dz\, dx\, dy \\ &= \int_\Sigma \Big[A_3(x,y,f_1(x,y)) \\ &\qquad - A_3(x,y,f_2(x,y))\Big]\, dx\, dy\end{aligned}$$



$\hat{\underline{n}} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

Now, using the projection theorem:

$$\int_{S_1} nA_3\, dS = \int_\Sigma nA_3(x, y, f_1(x,y)) \frac{dx\,dy}{|\hat{\underline{n}}.\hat{\underline{k}}|}$$
$$= \int_\Sigma A_3(x, y, f_1(x,y))\, dx\,dy.$$

$|\hat{\underline{n}}.\hat{\underline{k}}| = |n| = n$ since $n > 0$ on S_1

Similarly:

$$\int_{S_2} nA_3\, dS = \int_\Sigma nA_3(x, y, f_2(x,y)) \frac{dx\,dy}{|\hat{\underline{n}}.\hat{\underline{k}}|}$$
$$= -\int_\Sigma A_3(x, y, f_2(x,y))\, dx\,dy$$

$|\hat{\underline{n}}.\hat{\underline{k}}| = |n| = -n$ since $n < 0$ on S_2

Thus:

$$\oint_S nA_3\, dS = \int_\Sigma \left[A_3(x, y, f_1(x,y)) - A_3(x, y, f_2(x,y))\right] dx\,dy$$

and therefore

$$\int_\tau \frac{\partial A_3}{\partial z}\, d\tau = \oint_S nA_3\, dS$$

Similarly, by projecting onto the planes $x = 0$ and $y = 0$: (Check).

$$\int_\tau \frac{\partial A_1}{\partial x}\, d\tau = \oint_S lA_1\, dS$$

and

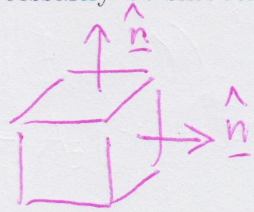
$$\int_\tau \frac{\partial A_2}{\partial y}\, d\tau = \oint_S mA_2\, dS$$

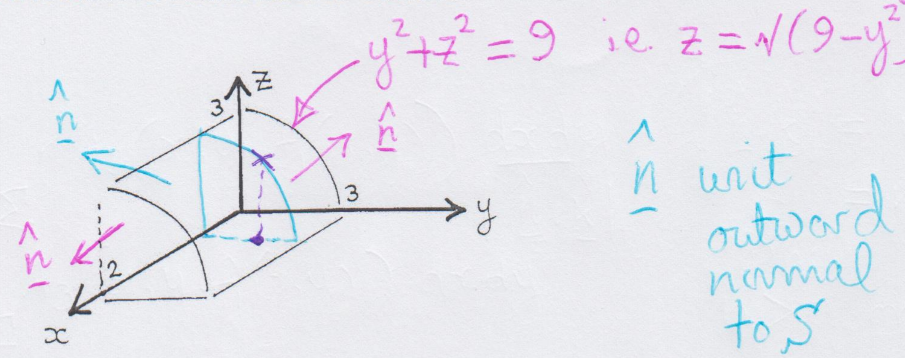
and hence

$$\oint_S \mathbf{A} \cdot \hat{\mathbf{n}}\, dS = \int_\tau \operatorname{div} \underline{A}\, d\tau$$

as required.

Note that the surface S need not necessarily be smooth - it could be, for example, a cube or a tetrahedron.





$\hat{\underline{n}}$ unit outward normal to S

V region enclosed by S.

Figure 16: The surface S in the example.

S has five sides.

Example

Evaluate

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS \text{ if } \mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k},$$

and S is the surface of the region in the first octant bounded by $y^2 + z^2 = 9, x = 2$ and $x = y = z = 0$.

Div thm $\oint_S \underline{A}\cdot\hat{\underline{n}}\, dS = \int_V \operatorname{div}\underline{A}\, dV$

$$= \int_V (4xy - 2y + 8xz)\, dV$$

$$= \int_{x=0}^{2} \int_{y=0}^{y=3} \int_{z=0}^{\sqrt{(9-y^2)}} (4xy-2y+8xz)\, dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \Big[4xyz - 2yz + 4xz^2\Big]_{z=0}^{z=\sqrt{(9-y^2)}} dy\, dx$$

(switched x & y)

$$= \int_0^3 \int_0^2 4xy\sqrt{(9-y^2)} - 2y\sqrt{(9-y^2)} + 4x(9-y^2)\, dx\, dy$$

$$= \int_0^3 \Big[2x^2y\sqrt{(9-y^2)} - 2xy\sqrt{(9-y^2)} + 2x^2(9-y^2)\Big]_{x=0}^{x=2} dy$$

$$= \int_0^3 4y\sqrt{(9-y^2)} + 8(9-y^2)\, dy = \ldots = \underline{\underline{180}}$$

subst. $u = y^2$