Applied Complex Analysis - Lecture Six

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Using contour deformation to evaluate

$$\int_{-\infty}^{\infty} f(z)dz,$$

where f has poles.

(Some) applications

- Statistics, e.g. Cauchy-Lorentz distribution
- Fourier and (inverse) Laplace transforms
- Potential flow theory, poles represent sinks

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Examples

- $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

- $$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx, \quad a, k > 0.$$

- $$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx, \quad k > 0.$$

- $$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1.$$

General strategy

- Add a suitable contour, γ', to $[a, b]$ to get a **closed** contour γ.
- Find a suitable function $g(z)$ which is analytic inside γ except possibly at poles, **and** such that, either $g(x) = f(x)$ for $x \in \mathbb{R}$ **or** there is a simple relation between $g(x)$ and $f(x)$.
- Apply the residue/Cauchy's theorem to evaluate $\oint_\gamma g(z)dz$.
- If $\int_{\gamma'} g(z)dz$ can be computed, or expressed in terms of I (as in example 4) then we're done.

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Analytic Continuation

Analytic continuation

Thm: If f and g are analytic in a connected domain D and $f = g$ in some common open region D' within D, then $f \equiv g$ throughout D.

Example:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad \text{for } D' = \{z \in \mathbb{C} : |z| < 1\}$$

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Branch points and branch cuts

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A point z_0 is called a **branch point** of $f(z)$ if f is not single-valued in a neighbourhood of z_0, i.e., analytically continuing along a path γ around z_0 and back to the same starting point returns a different value of $f(z)$.

A **branch cut** is a line χ such that the multi-valued analytic function $f(z)$ becomes a collection of single-valued analytic functions (each one is called a **branch** of $f(z)$) in a complement to χ.

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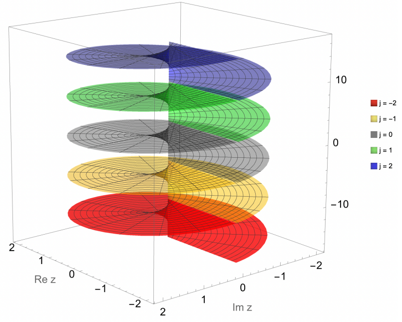
Branch cut example: Complex logarithm



- Traversing a full circuit around 0 gives a different value
- Another branch point at complex infinity
- Infinitely many *branches* - continuing to rotate does not bring us home!
- Possibilities for constructing a single-valued $\log$ - introducing a discontinuity
- Visualisation

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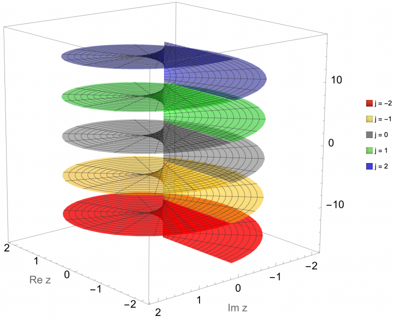

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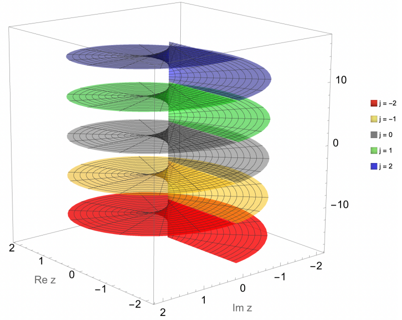

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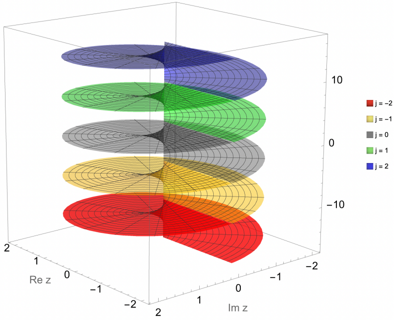

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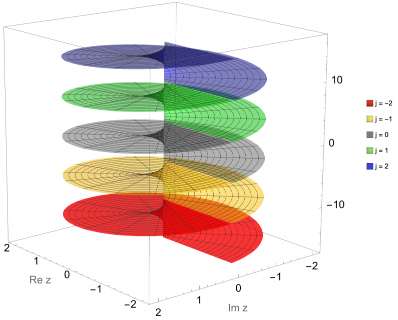

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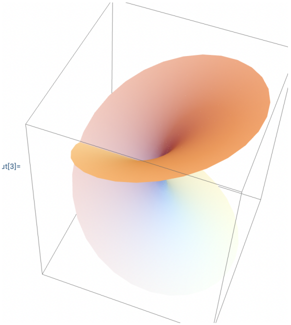

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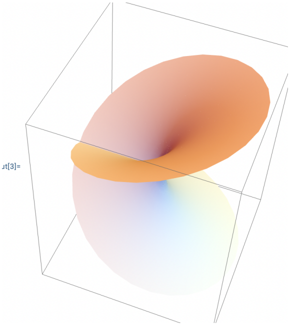

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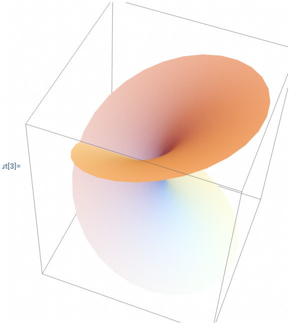

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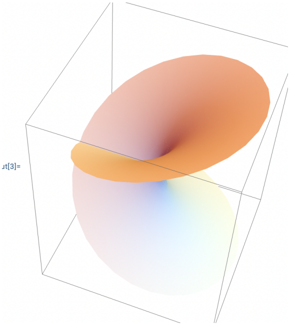

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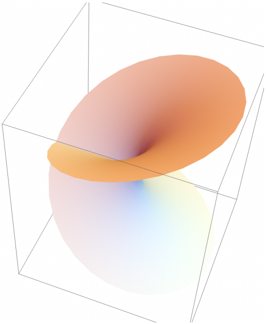

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Branch cut example: two finite branch points

$$f(z) = \sqrt{(z - z_1)(z - z_2)}$$

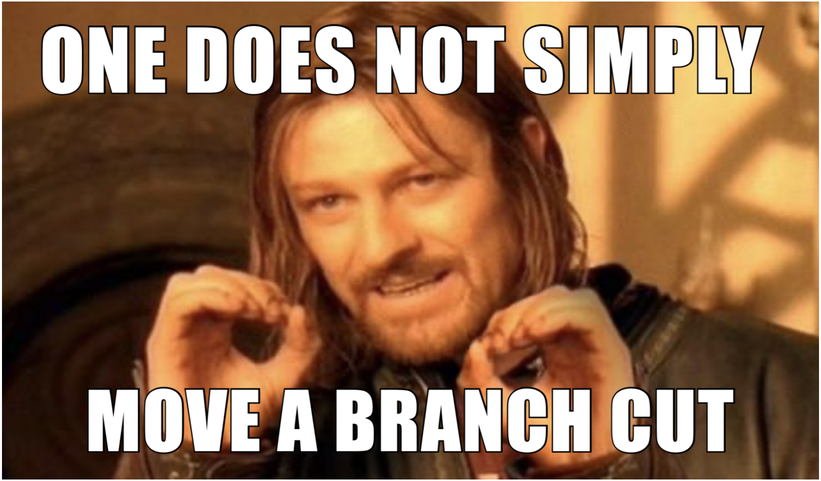
- Introduce local coordinates r_j and θ_j for $j = 1, 2$
- $f(z) = (r_1 r_2)^{1/2} e^{i\frac{\theta_1+\theta_2}{2}}$
- Consider small circuits around z_j
- Consider small circuit around some other finite z
- The point at $z = \infty$
- Choices for branch cuts

Tricky stuff - but how important is it?

Certain aspects of what we've seen are important, and certain aspects we don't need to worry about.

- Locations of branch points (easy)
- Don't worry too much about multi-valued Riemann surface stuff, we will always want to restrict to single-valued functions, by introducing branch cuts.
- (Consequence) we must choose where our function is non-analytic!
- Choice of branch cuts for by-hand calculations (important)
- For example when applying Deformation Theorem, Cauchy's Integral Theorem, etc ...
- Be aware that branch cuts are standardised in most mathematical software packages, e.g. negative real line.

Moving branch cuts in mathematical software



Example problems

- $$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx, \quad \alpha \in (0,1)$$

- $$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

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