IMPERIAL

MATH70055

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Stochastic Calculus with Applications to non-Linear Filtering

Date: Friday, May 24, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

For the following questions, assume the set-up: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $\mathcal{F}$ and V be a standard one-dimensional $\mathcal{F}_t$-adapted Brownian motion under $\mathbb{P}$. Let f and σ be bounded Lipschitz real valued functions and let X be the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s)\, ds + \int_0^t \sigma(X_s)\, dV_s. \tag{1}$$

Assume that X_0 has distribution π_0 at time 0, is independent of V and $\mathbb{E}\left[(X_0)^2\right] < \infty$. Let W be a standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X, and Y be the process satisfying the following evolution equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. The process $Y = \{Y_t,\ t \geq 0\}$ is called the observation process. Let $\{\mathcal{Y}_t,\ t \geq 0\}$ be the filtration associated with the process Y, that is $\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t])$. The filtering problem consists in determining the conditional distribution π_t of the signal X_t given $\mathcal{Y}_t$. That is, $\pi_t(A) = \mathbb{E}[I_A(X_t) \mid \mathcal{Y}_t]$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$ and I_A is the indicator function of the set A) and $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]$ for any bounded Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$.

1. (a) Give the definition of a d-dimensional continuous semimartingale.

(2 marks)

(b) State Itô's formula as applied to semimartingales [no proof required].

(4 marks)

(c) Suppose that $S = \{S_t,\ t \geq 0\}$ is a solution of the following stochastic differential equation

$$dS_t = \mu\, S_t dt + S_t dV_t, \quad S_0 = e^{\nu}, \tag{3}$$

where $\mu, \nu \in \mathbb{R}$ are constants.

(i) Prove that equation (3) has a unique solution. [You can use without proof any of the results in the lectures.]

(2 marks)

(ii) Using Itô's formula, or otherwise, prove that

$$S_t = \exp\left(\nu + V_t + \left(\mu - \frac{1}{2}\right) t\right).$$

(4 marks)

(iii) Find the evolution equation satisfied by the process $t \mapsto S_t e^{rW_t - \frac{r^2 t}{2}}$, where $r \in \mathbb{R}$ is a constant.

(4 marks)

(iv) Are there any constants ν, μ and r for which the process $t \mapsto S_t e^{rW_t - \frac{r^2 t}{2}}$ is a martingale?

(4 marks)

(Total: 20 marks)

2. (a) Give the definition of a martingale.

(4 marks)

(b) Recall that V and W are standard one-dimensional Brownian motion. Let $a \in \mathbb{R}$ be a constant and Z^a be the process defined as $Z^a_t = V_t(W_t + a)$ for any $t \geq 0$.

(i) Deduce the evolution equation satisfied by the process Z^a.

(5 marks)

(ii) Let $t > 0$ and $b \in \mathbb{R}$. Compute $\mathbb{E}[e^{ibZ^a_t}]$.

(5 marks)

(iii) Find a filtration $(\mathcal{G}_t)_{t\geq 0}$ in $\mathcal{F}$ with respect to which Z^a is a martingale.

(3 marks)

(iv) Is there any value a for which Z^a is a Brownian motion ?

(3 marks)

(Total: 20 marks)

3. Let $z = \{z_t, t > 0\}$ be the process defined by

$$z_t = \exp\left(-\sqrt{2}\int_0^t (h(X_s))dY_s - \int_0^t (h(X_s))^2\, ds\right),\ t \geq 0.$$

Let $\tilde{\mathbb{P}}$ be a probability measure which is absolutely continuous with respect to $\mathbb{P}$ under which Y is a Brownian motion.

(a) State the Novikov condition.

(4 marks)

(b) Using Novikov's condition prove that $z = \{z_t, t > 0\}$ is an $\mathcal{F}_t$-adapted martingale under $\tilde{\mathbb{P}}$.

(5 marks)

(c) Deduce the evolution equation satisfied by the martingale z.

(5 marks)

(d) Prove that $\sup\limits_{t\in[0,1]} \mathbb{E}[z_t^{-4}] < \infty$.

(6 marks)

(Total: 20 marks)

4. Assume that the signal X is an Ornstein-Uhlenbeck process starting from $X_0 = x_0$, satisfying

$$X_t = x_0 - \int_0^t X_s ds + 2V_t. \tag{4}$$

(a) Solve the equation (4).

(5 marks)

(b) Find the distribution p_t of X_t and compute $\lim_{t\to\infty} p_t$.

(6 marks)

(c) Compute

$$\int_0^t \mathbb{E}[(X_s)^2]ds.$$

(3 marks)

(d) Deduce the evolution equation satisfied by $p_t(\varphi)$ for any $\varphi \in C_b^2(\mathbb{R})$.

(6 marks)

You may use any results given in the course without proof, provided that you make it clear which ones you are using.

(Total: 20 marks)

Mastery Question

5. Recall that X is the $\mathcal{F}_t$-adapted process satisfying the stochastic differential equation (1). Let $a : \mathbb{R} \to \mathbb{R}_+$ be the function $a(x) = (\sin(x))^2, x \in \mathbb{R}$ and let $\rho : \mathcal{B}(\mathbb{R}) \mapsto \mathbb{R}$ be the following set function

$$\rho(A) = \mathbb{E}\left[I_A(X_2)a(X_1)\right], \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$.

(a) Prove that ρ is a finite measure.

(6 marks)

(b) Assume that Υ^δ is an equidistant partition of the interval $[0, 2]$:

$$\Upsilon^\delta \quad : \quad 0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots \tau_{2N} = 2$$

with $\tau_{n+1} - \tau_n = \delta = \frac{1}{N}$ for all $n = 0, 1, \ldots, 2N-1$. Describe the *Euler approximation* process X^δ of the signal process associated with the partition Υ^δ.

(3 marks)

(c) Let ρ^δ be the measure $\rho^\delta(A) = \mathbb{E}\left[I_A(X_2^\delta)a(X_1^\delta)\right], \quad A \in \mathcal{B}(\mathbb{R})$. Let $\bar{\rho}$ and $\bar{\rho}^\delta$ be the normalised versions of the measures ρ and ρ^δ, i.e. $\bar{\rho} = \rho/\rho(\mathbb{R})$ and $\bar{\rho}^\delta = \rho^\delta/\rho^\delta(\mathbb{R})$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$. Assume that there exists a constant c such that $\left|\rho(A) - \rho^\delta(A)\right| \le c\delta$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$. Prove that there exists a constant $\bar{c}$ such that $\left|\bar{\rho}(A) - \bar{\rho}^\delta(A)\right| \le \bar{c}\delta$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$.

(4 marks)

(d) Describe the Monte Carlo approximation for $\bar{\rho}^\delta(A)$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ and deduce its rate of convergence of the approximation to $\bar{\rho}^\delta(A)$.

(7 marks)

(Total: 20 marks)

Marking Scheme

Question 1. [20 marks]

(a) [2 marks, seen] Let X_t be an $\{\mathcal{F}_t\}$-adapted d-dimensional process with continuous paths. If X_t can be decomposed as $X_t = M_t + V_t$, $t \geq 0$,where M_t is an $\{\mathcal{F}_t\}$-adapted d-dimensional martingale with continuous paths and the paths of the d-dimensional process V_t are of finite variation, then we call X_t a d-dimensional continuous semimartingale.

(b) [4 marks, seen]

Let $X_t^1, \ldots, X_t^d$ be semi-martingales and $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ a function which is one time continuously differentiable with respect to t and twice with respect to x_i, $i = 1, 2, \ldots, d$. Then

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X_s)dX_s^i$$
$$+ \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)d[X^i, X^j]_s.$$

(c) Suppose that S_t is a solution of the following the stochastic differential equation

$$dS_t = \mu S_t dt + S_t dV_t, \quad S_0 = e^{\nu}. \tag{1}$$

(i) [2 marks, seen similar]

Both the drift and the diffusion coefficients in equation (1) are Lipschitz continuous functions (they are linear) therefore the equation has a unique solution in accordance with one of the theorems in the lectures.

(ii) [4 marks, not seen] Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^x$. We apply Itô's formula for this function to the semimartingale

$$L_t = \nu + \left(\mu - \frac{1}{2}\right)t + V_t, \qquad t \geq 0.$$

Then, by Itô's formula, we get that (we use the fact that $\langle V\rangle_t = t$):

$$\begin{aligned} e^{L_t} &= e^{L_0} + \int_0^t e^{L_s} dL_s + \frac{1}{2}\int_0^t e^{L_s} d\langle L\rangle_s \\ &= e^{\nu} + \int_0^t e^{L_s} d\left(\left(\mu - \frac{1}{2}\right)s + 1V_s\right) + \frac{1}{2}\int_0^t e^{L_s} d\langle V\rangle_s \\ &= e^{\nu} + \int_0^t \mu e^{L_s} ds + \int_0^t e^{L_s} dV_s. \end{aligned}$$

It follows that the process e^{L_t} satisfies equation (1). Since equation (1) has a unique solution it follows that

$$S_t = \exp\left(\nu + \left(\mu - \frac{1}{2}\right) t + V_t\right).$$

(iii) [4 marks, not seen] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x_1, x_2) = e^{rx_1 + x_2}$. We apply Itô's formula for this function to the semimartingale (W_t, L_t) Then, by Itô's formula, we get that (observe that $\langle W, L\rangle = \langle W, V\rangle = 0$ as W and V are independent Brownian motions).

$$\begin{aligned}
\bar{S}_t &= e^{rW_t} S_t \\
&= 2 + \int_0^t r e^{rW_s} S_s dW_s + \int_0^t e^{rW_s} S_s dS_s + \frac{1}{2}\int_0^t r^2 e^{rW_s} S_s d\langle W\rangle_s + \frac{1}{2}\int_0^t e^{rW_s} S_s d\langle L\rangle_s \\
&= 2 + \int_0^t r\bar{S}_s dW_s + \int_0^t \bar{S}_s d\left(\left(\mu - \frac{\nu^2}{2}\right) s + \nu V_s\right) + \frac{1}{2}\int_0^t r^2 \bar{S}_s ds + \frac{1}{2}\int_0^t \nu^2 \bar{S}_s d\langle V\rangle_s \\
&= 2 + r\int_0^t \bar{S}_s dW_s + \nu \int_0^t \bar{S}_s dV_s + \int_0^t \left(\mu + \frac{r^2}{2}\right)\bar{S}_s ds.
\end{aligned}$$

(iv) [4 marks, not seen] Note that the process $t \to e^{rW_t - \frac{r^2 t}{2}}$ is a martingale (one can check this either directly or by using Novikov's condition). It is also independent of S_t as both processes are driven by independent Brownian motions. If S_t is a martingale then also the product $t \to S_t e^{rW_t - \frac{r^2 t}{2}}$ is a martingale. But S_t is a semi-martingale. If we choose $\mu = 0$, then the bounded variation part of S_t is 0 and the stochastic integral

$$\int_0^t e^{L_s} dV_s$$

is a martingale provided the integrability condition on the integrands $s \mapsto e^{L_s}$ is satisfied (as $\int_0^t \mathbb{E}\left[e^{2L_s}\right] ds < \infty$).

Question 2. [20 marks]

(a) [4 marks, seen]
Let M_t be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. The process M_t is a martingale if

- M_t is adapted with respect to the filtration $\mathcal{F}_t$.
- M_t is integrable for all $t \geq 0$.
- For all s, t with $0 \leq s \leq t$ we have

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s, \ P - a.s.$$

(b)
(i) [5 marks, seen similar] By Itô's formula, we get that

$$\begin{aligned} Z_t^a &= V_t W_t + aV_t \\ &= 0 + \int_0^t W_s dV_s + \int_0^t V_s dW_s + \frac{1}{2}\int_0^t 0d\langle W\rangle_s + \frac{1}{2}\int_0^t 0d\langle V\rangle_s + \int_0^t d\langle V, W\rangle_s + \int_0^t adV_s \\ &= \int_0^t W_s dV_s + \int_0^t V_s dW_s + \int_0^t adV_s \end{aligned}$$

Again, we have used that $\langle V, W\rangle = 0$, as W and V are independent Brownian motions.
(ii) [5 marks, not seen]. Since V and W are independent, we have

$$\mathbb{E}[e^{ibV_t(W_t+a)}|V] = \frac{1}{\sqrt{2\pi t}}\int e^{ibV_t(x+a)}e^{-\frac{x^2}{2t}}dx = \frac{e^{iabV_t}}{\sqrt{2\pi t}}\int e^{-\frac{t(bV_t)^2}{2}}e^{-\frac{(x-ibV_tt)^2}{2t}}dx = e^{-\frac{t(bV_t)^2}{2}+iabV_t}.$$

$$\begin{aligned} \mathbb{E}[e^{ibaV_tW_t}] &= \mathbb{E}\left[e^{-\frac{t(bV_t)^2}{2}+iabV_t}\right] \\ &= \frac{1}{\sqrt{2\pi t}}\int e^{-\frac{t(bx)^2}{2}+iabx}e^{-\frac{x^2}{2t}}dx \\ &= \frac{1}{\sqrt{2\pi t}}\int e^{iabx-\frac{x^2}{2}\left(\frac{1}{t}+tb^2\right)}dx \\ &= \frac{1}{\sqrt{2\pi t}}\int e^{-\frac{(ab)^2}{2}\left(\frac{1}{t}+tb^2\right)^{-1}-\left(x-\frac{iab}{\left(\frac{1}{t}+tb^2\right)}\right)^2\frac{\left(\frac{1}{t}+tb^2\right)}{2}}dx \\ &= \frac{\sqrt{\frac{1}{t}+tb^2}}{\sqrt{t}}e^{-\frac{(ab)^2}{2}\left(\frac{1}{t}+tb^2\right)^{-1}} \end{aligned}$$

(iii) [3 marks, not seen] We can either use the original filtration or the filtration $\mathcal{F}_t^{V,W}$ generated by both V, and W. Since V and W are independent, using the properties of the conditional expectation and the independent increments property of both V and W, we get that

$$\begin{aligned}
\mathbb{E}[Z_t^a - Z_s^a|\mathcal{F}_s^{V,W}] &= \mathbb{E}[(V_t - V_s)\,W_t|\mathcal{F}_s^{V,W}] + V_s\mathbb{E}[W_t - W_s|\mathcal{F}_s^{V,W}] \\
&\quad + a\mathbb{E}[(V_t - V_s)\,|\mathcal{F}_s^{V,W}] \\
&= \mathbb{E}[\mathbb{E}[(V_t - V_s)\,W_t|\mathcal{F}_s^{V,W} \vee \sigma\,(V_t - V_s)]|\mathcal{F}_s^{V,W}] \\
&= \mathbb{E}[(V_t - V_s)\,\mathbb{E}[W_t|\mathcal{F}_s^{V,W} \vee \sigma\,(V_t - V_s)]|\mathcal{F}_s^{V,W}] \\
&= \mathbb{E}[(V_t - V_s)\,W_s|\mathcal{F}_s^{V,W}] \\
&= W_s\mathbb{E}[(V_t - V_s)\,|\mathcal{F}_s^{V,W}] \\
&= 0,
\end{aligned}$$

hence Z_t^a is indeed a martingale with respect to the filtration $\mathcal{F}_t^{V,W}$.

Alternatively one can use the fact that Z_t^a is a linear combination of three stochastic integrals which are shown to be martingales via the Itô integral properties.

(iv) [3 marks, not seen] The process Z_t^a cannot be a Brownian motion as that would imply that $Z_t^a \sim N\,(0,t)$, which would mean that

$$\mathbb{E}[e^{ibZ_t^a}] = e^{-\frac{b^2 t}{2}}.$$

which is not the case here no matter what the value of a is.

3.(20 marks)

(a) [4 marks, seen] Novikov's condition states that if $u = \{u_t, t > 0\}$ is a process defined as $u_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ for M a continuous local martingale, then a sufficient condition for u to be a martingale is that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] < \infty, \qquad 0 \le t < \infty.$$

(b) [5 marks, seen] In this case the process $t \to \int_0^t -\sqrt{2}h(X_s)dY_s$ is a local martingale (it is a stochastic integral with respect to a Brownian motion and indeed its quadratic variation process is given by $t \to \int_0^t 2h(X_s)^2 ds$. Moreover, since h is bounded, it follows that $|h(X_s)| \le \|h\|_\infty$ and hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] = \mathbb{E}\left[\exp\left(\int_0^t h(X_s)^2\, ds\right)\right] \le \exp\left(t\,\|h\|_\infty^2\right) < \infty, \qquad 0 \le t < \infty.$$

Hence, by Novikov's condition, the process $z = \{z_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Moreover, since both integrals $t \to \int_0^t -\sqrt{2}h(X_s)dY_s$ and $t \to \int_0^t 2h(X_s)^2 ds$ are $\mathcal{F}_t$-adapted, the property remains true for z as well.

(c) [5 marks, seen] Let $\xi = \{\xi_t, t > 0\}$ be the semimartingale defined by

$$\xi_t = \int_0^t \sqrt{2}h(X_s)dY_s + \int_0^t h(X_s)^2 ds, \qquad t \ge 0.$$

Then, by Itô's formula, we get that

$$\begin{aligned} z_t &= \exp(-\xi_t) \\ &= \exp(-\xi_0) - \int_0^t \exp(-\xi_s)\, d\xi_s + \frac{1}{2}\int_0^t \exp(-\xi_s)\, d\langle\xi\rangle_s \\ &= 1 - \int_0^t z_s\left(\sqrt{2}h(X_s)dY_s + h(X_s)^2 ds\right) + \frac{2}{2}\int_0^t z_s h(X_s)^2 ds \\ &= 1 - \sqrt{2}\int_0^t z_s h(X_s)dY_s. \end{aligned}$$

(d) [6 marks, not seen] Observe that

$$\begin{aligned} z_s^{-4} &= \exp\left(\int_0^t 20h(X_s)^2 ds\right)\bar{z}_s \\ &\le \exp\left(20t\,\|h\|_\infty^2\right)\bar{z}_s, \end{aligned}$$

where $\bar{z} = \{\bar{z}_t, t > 0\}$ is the process defined by

$$\bar{z}_t = \exp\left(\int_0^t 4\sqrt{2}h(X_s)dY_s - \frac{1}{2}\int_0^t \left(4\sqrt{2}h(X_s)\right)^2 ds\right),\ t \geq 0.$$

Again, by Novikov's condition, the process $\bar{z} = \{\bar{z}_t, t > 0\}$ is a martingale under $\tilde{\mathbb{P}}$. Hence

$$E[\bar{z}_s] = E[\bar{z}_0] = 1, \quad s \in [0,1]$$

and

$$\begin{aligned} \sup_{t\in[0,1]} E[z_s^{-4}] &\leq \sup_{t\in[0,1]} \exp\left(20t\left\|h\right\|_\infty^2\right) E[\bar{z}_s] \\ &= \exp\left(20\left\|h\right\|_\infty^2\right) < \infty. \end{aligned}$$

4. (20 marks)

(a) [5 marks, seen similar] We have, by Itô's formula, that

$$e^t X_t = x_0 + \int_0^t X_s de^s + \int_0^t e^s dX_s = x_0 + 2\int_0^t e^s dV_s,$$

hence

$$X_t = x_0 e^{-t} + 2\int_0^t e^{-(t-s)} dV_s.$$

(b) [6 marks, seen similar] We use the fact that if $f : [0,t] \to \mathbb{R}$ be a Borel measurable function such that $v := \int_0^t f(s)^2 ds < \infty$, then the random variable $\int_0^t f(s) dV_s$ has a normal distribution with mean 0 and variance v. It follows that

$$\int_0^t 2e^{-(t-s)} dV_s \sim N\left(0, 4e^{-2t}\int_0^t e^{2s} ds\right) = N\left(0, 2(1-e^{-2t})\right),$$

hence

$$p_t = \mathcal{L}(X_t) = N\left(x_0 e^{-t}, 2(1-e^{-2t})\right).$$

and therefore

$$\lim_{t\to\infty} p_t = N\left(0, 2\right).$$

(c) [3 marks, not seen] We have that

$$E[(X_s)^2] = x_0^2 e^{-2t} + 2(1-e^{-2t})$$

hence

$$\int_0^t E[(X_s)^2] ds = \int_0^t x_0^2 e^{-2t} + 2(1-e^{-2t}) = 2t + \left(x_0^2 - 2\right)\left(e^{-2t} - 1\right)/2.$$

(d) [6 marks, seen] By Itô's formula we get that

$$\varphi(X_t) = \varphi(X_0) + \int_0^t A\varphi ds + \int_0^t \sigma\varphi'(X_s) dV_s, \tag{2}$$

where $A\varphi(x) = -x\varphi'(x) + 2\varphi''(x)$. Since the process $t \to \int_0^t 2\varphi'(X_s) dV_s$ is a genuine martingale (it is local martingale as it is a stochastic integral with respect to a Brownian motion and

$$\int_0^t (2\varphi'(X_s))^2 ds \le 4t||\varphi'||_\infty^2 < \infty$$

it has null expectation. Moreover, by Tonelli's theorem

$$\begin{aligned} E\left[\int_0^t |A\varphi(X_s)|ds\right] &= \int_0^t E[|A\varphi(X_s)|]ds \\ &\leq \|\varphi'\|_\infty \int_0^t E[|X_s|]ds + \frac{t\sigma^2}{2}\|\varphi''\|_\infty < \infty. \end{aligned}$$

as, by Cauchy-Schwarz, we have that

$$\left(\int_0^t E[|X_s|]ds\right)^2 \leq t\int_0^t E[(X_s)^2]ds = t\left(2t + e^{-2t} - 1\right).$$

It follows that we can apply Fubini to get that

$$E\left[\int_0^t A\varphi(X_s)ds\right] = \int_0^t p_s\left(A\varphi\right)ds.$$

Hence, by taking expectation in (2) we get

$$p_t\left(\varphi\right) = E\left[\varphi(X_t)\right] = p_0\left(\varphi\right) + \int_0^t p_s\left(A\varphi\right)ds.$$

Question 5. (20 marks)

(a). [6 marks (1+2+3), seen similar] We need to prove that ρ is a nonnegative, finite and countably additive set function.

i. For any $A \in \mathcal{B}(\mathbb{R})$, $I_A(X_2)a(X_1)$ is a non-negative random variable and therefore also its expectation

$$\rho(A) = \mathbb{E}\left[I_A(X_2)a(X_1)\right]$$

will be non-negative.

ii. Since

$$\rho(\mathbb{R}) = \mathbb{E}\left[I_{\mathbb{R}}(X_2)a(X_1)\right] = \mathbb{E}\left[a(X_1)\right] \leq \|a\|_\infty = 1,$$

it follows that the measure ρ has finite mass.

iii. For mutually disjoint sets $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, 2, \ldots$, we have by the monotone convergence theorem that

$$\begin{aligned}
\rho\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathbb{E}\left[I_{\bigcup_{i=1}^{\infty} A_i}(X_2)a(X_1)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\infty} I_{A_i}(X_2)a(X_1)\right] \\
&= \mathbb{E}\left[\lim_{n\mapsto\infty}\sum_{i=1}^{n} I_{A_i}(X_2)a(X_1)\right] \\
&= \lim_{n\mapsto\infty}\mathbb{E}\left[\sum_{i=1}^{n} I_{A_i}(X_2)a(X_1)\right] = \sum_{i=1}^{\infty}\rho(A_i).
\end{aligned}$$

(a). [3 marks, seen] The *Euler-Maruyama approximation* process $X^\delta = \left\{X^\delta_{\tau_n}, n = 0, 1, \ldots, 2N\right\}$ of the signal equation associated with the partition Υ^δ is defined by the iterative scheme

$$X^\delta_{\tau_{n+1}} = X^\delta_{\tau_n} + f\left(X^\delta_{\tau_n}\right)\delta + \sigma\left(X^\delta_{\tau_n}\right)\left(V_{\tau_{n+1}} - V_{\tau_n}\right), \quad n = 0, 1, \ldots, 2N-1$$

with initial value $X^\delta_0 = X_0$.

(b). [4 marks, see similar] We have the following

$$\begin{aligned}
\bar{\rho}(A) - \bar{\rho}^\delta(A) &= \frac{\rho(A)}{\rho(\mathbb{R})} - \frac{\rho^\delta(A)}{\rho^\delta(\mathbb{R})} \\
&= \frac{\rho(A)}{\rho(\mathbb{R})} - \frac{\rho^\delta(A)}{\rho(\mathbb{R})} + \frac{\rho^\delta(A)}{\rho(\mathbb{R})} - \frac{\rho^\delta(A)}{\rho^\delta(\mathbb{R})} \\
&= \frac{1}{\rho(\mathbb{R})}\left(\rho(A) - \rho^\delta(A)\right) + \frac{\rho^\delta(A)}{\rho^\delta(\mathbb{R})\rho(\mathbb{R})}\left(\rho^\delta(\mathbb{R}) - \rho(\mathbb{R})\right).
\end{aligned}$$

Since $\rho^\delta(A) \leq \rho^\delta(\mathbb{R})$ we deduce from the above that

$$\left|\bar{\rho}(A) - \bar{\rho}^\delta(A)\right| \leq \frac{1}{\rho(\mathbb{R})}\left|\rho(A) - \rho^\delta(A)\right| + \frac{1}{\rho(\mathbb{R})}\left|\rho^\delta(\mathbb{R}) - \rho(\mathbb{R})\right| \leq \bar{c}\delta,$$

where $\bar{c} = \dfrac{2}{\rho(\mathbb{R})}$.

(d). [7 marks, seen similar] A Monte Carlo approximation of $m^{\delta}(A)$ consists in computing M mutually independent copies of the process X^{δ}. We achieve this by starting the copies with M independent samples $X_0^{\delta,j}$ $j = 1, ..., n$ from π_0 (the law of X_0) and then generating the copies using M mutually independent Brownian motions V^j, $j = 1, ..., M$. More precisely

$$X^{\delta,j}_{\tau_{n+1}} = X^{\delta,j}_{\tau_n} + f\left(X^{\delta,j}_{\tau_n}\right)\delta + \sigma\left(X^{\delta,j}_{\tau_n}\right)\left(V^j_{\tau_{n+1}} - V^j_{\tau_n}\right), \quad n = 0, 1, ..., 2N-1.$$

It follows that the random variables

$$\xi^j = I_A(X^{\delta,j}_{\tau_{2N}})a(X^{\delta,j}_{\tau_N}), \quad j = 1, ..., n$$

are mutually independent and identically distributed and have the same distribution as $I_A(X^{\delta}_{\tau_{2N}})a(X^{\delta}_{\tau_N})$. In particular $E\left[\xi^j\right] = \rho^{\delta}(A)$. We then approximate the measure $\rho^{\delta}(A)$ with the (random) measure $\rho^{\delta,n}(A)$ given by

$$\rho^{\delta,n}(A) = \frac{1}{n}\sum_{j=1}^{n} I_A(X^{\delta,j}_{\tau_{2N}})a(X^{\delta,j}_{\tau_N}) = \frac{1}{n}\sum_{j=1}^{n}\xi^j.$$

It follows that

$$E\left[\rho^{\delta,n}(A)\right] = \frac{1}{n}\sum_{j=1}^{n} E\left[\xi^j\right] = \rho^{\delta}(A)$$

and

$$E\left[\left(\rho^{\delta,n}(A) - \rho^{\delta}(A)\right)^2\right] = \frac{1}{n^2}\sum_{j=1}^{n} E\left[\left(\xi^j - \rho^{\delta}(A)\right)^2\right] = \frac{c\,(A)}{n}$$

where

$$c\,(A) = E\left[\left(I_A(X^{\delta,j}_{\tau_{2N}})a(X^{\delta,j}_{\tau_N})\right)^2\right] - \rho^{\delta}(A)^2.$$

MATH70055 Stochastic Calculus with Applications to non-Linear Filtering

Question	Marker's comment
1	A question that the students did very well. Clearly showing understanding this part of the course.
2	A questionnbsp; that was well attempted. Part (b) (ii) prove surprisingly challenging despite being fairly elementary (i..e. no stochastic calculus needed).nbsp;
3	A question that most students did very well. Clearly showing understanding this part of the course.
4	A question that the students attempted reasonably well. Part (d) was less attempted despite being seen before.nbsp; nbsp;
5	The mastery question proved the most challenging. Some very good answer though.