	EXAMINATION SOLUTIONS 2014-15	Course P5
Question		Marks & seen/unseen

Parts

1(a) To calculate the length, we first compute $\phi'(t) = (2, 2t, t^2)$ and so

$$|\phi'(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2.$$

Thus, letting $\gamma = \phi((0,1))$,

$$length(\gamma) = \int_0^1 |\phi'(t)| dt = \int_0^1 (t^2 + 2) dt = \frac{7}{3}.$$

(5 marks, seen similar)

1(b) In class, we derived a general formula to compute curvature of any parametrized planar curve. That is, if $\phi(t) = (x(t), y(t))$ denotes a curve, then the curvature

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

So in this question, substituting $x(t) = t$ and $y(t) = f(t)$ into the formula,

$$k(t) = \frac{f''}{(1 + (f')^2)^{3/2}}.$$

Note that k may differ by a sign upon choices of the normal to the curve.

Alternatively, one may try to first reparametrize the curve by arc-length. Then the length of the second derivative of the curve with arc-length parametrization gives the curvature.

(7 marks, seen similar)

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Parts	1(c) Suppose that $\phi(s)$ is a planar curve parametrized by arc-length. Then, letting N be a unit normal to ϕ, $$k_0 = k(s) = \langle \phi'', N\rangle = -\langle \phi', N'\rangle,$$ where k_0 is a constant and in the second equality we use the fact that $\langle \phi', N\rangle = 0$ along the curve. Observe that $\langle N', N\rangle = 0$, thus we conclude that $N' = -k_0\phi'$, i.e., $N + k_0\phi$ is a constant vector, say $\mathbf{v}_0$. This immediately implies that, if $k_0 = 0$, then $N = \mathbf{v}_0$ and so ϕ is part of a straight line; otherwise, $\|\phi - \|k_0\|^{-1}\mathbf{v}_0\| = 1/\|k_0\|$, i.e., part of a circle. **(8 marks, seen)**	
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Parts

2(a) Choose a chart $\phi(x,y) = (x, y, x^2 - y^2)$. Then

$$\partial_x\phi = (1,0,2x), \quad \partial_y\phi = (0,1,-2y), \quad N = \frac{(-2x, 2y, 1)}{\sqrt{1+4x^2+4y^2}}.$$

Thus the determinant of metric g is

$$\det g = |\partial_x\phi \times \partial_y\phi|^2 = 1 + 4x^2 + 4y^2.$$

Futhermore

$$\partial^2_{xx}\phi = (0,0,2), \quad \partial^2_{xy}\phi = (0,0,0), \quad \partial^2_{yy}\phi = (0,0,-2).$$

Thus the second fundamental form matrix under the chart ϕ is

$$A_{11} = A(\partial_x\phi, \partial_x\phi) = \frac{2}{\sqrt{1+4x^2+4y^2}},$$

$$A_{12} = A_{21} = A(\partial_x\phi, \partial_y\phi) = 0,$$

$$A_{22} = A(\partial_y\phi, \partial_y\phi) = -\frac{2}{\sqrt{1+4x^2+4y^2}}.$$

Hence

$$\det A = -\frac{4}{1+4x^2+4y^2}$$

Therefore the Gaussian curvature of Σ is

$$K = \frac{\det A}{\det g} = -\frac{4}{(1+4x^2+4y^2)^2}.$$

(8 marks, seen similar)

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Parts	2(b) No. From part (a) we saw that S has negative Gaussian curvature everywhere. However, the surface $\tilde{S}$ has positive curvature at the origin. This can be seen either from a direct calculation like in part (a) or from an easy observation that $\tilde{S}$ near the origin lies on one side of its tangent plane at the origin, and hence the Gaussian curvature of $\tilde{S}$ at p must be positive. Since Gaussian curvature is an intrinsic quantity, there does not exist a local isometry between S and $\tilde{S}$. **(6 marks, seen similar)** 2(c) Parametrized curve γ by arc-length: $$\gamma(s) = \frac{1}{\sqrt{2}}(\cos(\sqrt{2}\,s), \sin(\sqrt{2}\,s), 1).$$ Then $$N \times \gamma' = \frac{1}{\sqrt{2}}(-\cos(\sqrt{2}\,s), -\sin(\sqrt{2}\,s), 1),$$ and the curvature of γ in $\mathbb{R}^3$ is $$\vec{k}(s) = -\sqrt{2}(\cos(\sqrt{2}\,s), \sin(\sqrt{2}\,s), 0).$$ Hence the geodesic curvature of γ in S^2 is $$k_g(s) = \langle \vec{k}(s), N \times \gamma' \rangle = 1.$$ **(6 marks, seen similar)**	
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Parts	3(a) Let Σ be a compact orientable surface with boundary $\partial\Sigma$. And K denotes the Gaussian curvature of Σ and k_g denotes the geodesic curvature of $\partial\Sigma$ in Σ. And $\chi(\Sigma)$ is the Euler characteristic of Σ. The Gauss-Bonnet Theorem states that $$\int_\Sigma K dA + \int_{\partial\Sigma} k_g ds = 2\pi\chi(\Sigma).$$ **(6 marks, seen)** 3(b) Let λ_1, λ_2 be the two principal curvatures of Σ. Then the Gaussian curvature $K = \lambda_1\lambda_2$, and using the hint we get that $$\lvert H\rvert^2 = \frac{(\lambda_1+\lambda_2)^2}{4} = \frac{(\lambda_1-\lambda_2)^2}{4} + K.$$ Thus by the Gauss-Bonnet Theorem, if $\chi(\Sigma) = 2$, $$\int_\Sigma \lvert H\rvert^2 dA \geq \int_\Sigma K dA = 2\pi\chi(\Sigma) = 4\pi.$$ **(7 marks, unseen)** 3(c) From part (b) the equality holds if and only if $$\int_\Sigma (\lambda_1-\lambda_2)^2 dA = 0.$$ This implies that $\lambda_1 = \lambda_2$, i.e., Σ is a totally umbilical surface, and the claim follows from the classification for those surfaces we learned in class that Σ must be a sphere. **(7 marks, unseen)**	
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Parts

4(a) False, as the winding number is always an integer.

(4 marks, seen similar)

4(b) False. For instance, in class, we gave such a counter example which is a surface of revolution

$$S = \{(\phi(t)\cos\theta, \phi(t)\sin\theta, \psi(t)) : 0 \le \theta < 2\pi, -\pi/6 \le t \le \pi/6\},$$

where $\phi(t) = \sqrt{2}\cos t$ and

$$\psi(t) = \int \sqrt{1 - 2\sin^2 t}\, dt.$$

Clearly, S has Gaussian curvature $-\phi''/\phi = 1$ but is not part of a sphere.

(4 marks, seen)

4(c) True, since

$$Area(\Sigma) \le \int_\Sigma K dA = 2\pi\chi(\Sigma) \le 4\pi.$$

(4 marks, unseen)

4(d) False. The Gaussian curvature of minimal surfaces is nonpositive. On the other hand, compact surface without boundary contains at least one point with positive Gaussian curvature. This is a contradiction.

(4 marks, seen)

4(e) True. For instance, the following map $(x, y, 0) \to (\cos x, \sin x, y)$ gives a local isometry from the xy-plane to the cylinder with cross section unit circle rotating about z-axis.

(4 marks, seen)

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