

## Problem Sheet 1

1. If  $\mathbf{A}$  is a constant vector field, calculate the gradients of the following scalar fields:

- (i)  $\mathbf{A} \cdot \mathbf{r}$ ,
- (ii)  $r^n$ ,
- (iii)  $\mathbf{r} \cdot \nabla(x + y + z)$ ,

where  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $r = |\mathbf{r}|$ .

2. If  $\phi = x^2y + z^2x$  and  $P$  is the point  $(1, 1, 2)$ , find the directional derivative of  $\phi$  at  $P$  in the direction  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

3. If  $\phi = \phi(\mathbf{r})$  with  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $x = x(t), y = y(t), z = z(t)$ , show that

$$\frac{d\phi}{dt} = \mathbf{r}'(t) \cdot \nabla\phi.$$

Verify this relation for  $\phi$  as given in Q2 and  $(x, y, z) = (\cos t, \sin t, t)$ .

Further, if  $\phi = \phi(\mathbf{g}(t))$  with  $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$ , show that

$$\frac{d\phi}{dt} = \mathbf{g}'(t) \cdot \nabla_{\mathbf{g}}\phi.$$

where  $\nabla_{\mathbf{g}} \equiv \mathbf{i} \frac{\partial}{\partial g_1} + \mathbf{j} \frac{\partial}{\partial g_2} + \mathbf{k} \frac{\partial}{\partial g_3}$ .

4. Find the equations of the tangent planes to the following surfaces at the points indicated

- (i)  $x^2 + 2y^2 - z^2 - 8 = 0$  at  $(1, 2, 1)$ ,
- (ii)  $z = 3x^2y \sin(\pi x/2)$  at  $x = 1, y = 1$ .

5. If  $\phi = xr^2, \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $f(r)$  is an arbitrary function of  $r = |\mathbf{r}|$ , evaluate:

- (i)  $\nabla\phi$ ,
- (ii)  $\text{div}(\phi\mathbf{r})$ ,
- (iii)  $\text{curl}(f(r)\mathbf{r})$ .

6. If  $\mathbf{u} = z^2\mathbf{i}, \mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $\psi = |\mathbf{v}|^2$ , verify the identities

- (i)  $\text{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl} \mathbf{u} - \mathbf{u} \cdot \text{curl} \mathbf{v}$ ,
- (ii)  $\text{div}(\psi\mathbf{u}) = (\nabla\psi) \cdot \mathbf{u} + \psi \text{div} \mathbf{u}$ .

7. Use tensor notation and the relation  $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$  to establish the following vector identities:

- (i)  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$ ,
- (ii)  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$ ,
- (iii)  $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d})\mathbf{c} - ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})\mathbf{d}$ ,

8. Simplify the following expressions:

- (i)  $\delta_{ij} \partial x_i / \partial x_j,$
- (ii)  $\delta_{ij} \delta_{ik} x_j x_k,$
- (iii)  $\delta_{ij} \partial^2 \phi / \partial x_i \partial x_j,$
- (iv)  $\delta_{ij} \delta_{jk} \delta_{ki},$
- (v)  $\varepsilon_{ijk} \partial / \partial x_i (\partial A_k / \partial x_j).$

9. Use tensor notation to prove the following identities:

- (i)  $\text{curl}(\phi \mathbf{A}) = \phi \text{curl} \mathbf{A} + \nabla \phi \times \mathbf{A},$
- (ii)  $\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl} \mathbf{A} - \mathbf{A} \cdot \text{curl} \mathbf{B},$
- (iii)  $\mathbf{A} \times \text{curl} \mathbf{A} = \frac{1}{2} \nabla(|\mathbf{A}|^2) - (\mathbf{A} \cdot \nabla) \mathbf{A}.$

## Sheet 1 Answers

1. (i)  $\mathbf{A}$ ; (ii)  $nr^{n-2}\mathbf{r}$ ; (iii)  $\mathbf{i} + \mathbf{j} + \mathbf{k}.$
2.  $20/\sqrt{14}.$
3.  $d\phi/dt = \cos^3 t - 2 \cos t \sin^2 t + 2t \cos t - t^2 \sin t.$
4. (i)  $x + 4y - z = 8$ ; (ii)  $6x + 3y - z = 6.$
5. (i)  $(3x^2 + y^2 + z^2)\mathbf{i} + 2xy\mathbf{j} + 2zx\mathbf{k}$ ; (ii)  $6xr^2$ ; (iii) zero.
8. (i) 3; (ii)  $r^2$ ; (iii)  $\partial^2 \phi / \partial x_1^2 + \partial^2 \phi / \partial x_2^2 + \partial^2 \phi / \partial x_3^2$ ; (iv) 3; (v) zero.