IMPERIAL

MATH70141

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Introduction to Game Theory

Date: Wednesday, May 22, 2024

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5 hours

This paper has 5 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. Throughout this question consider a two-player, simultaneous move game, G, being played between players A and B.

(a) Define what it means to have an **equilibrium** of the game. (3 marks)

(b) Define what it means for a strategy of player A to be **strictly dominated** by another strategy of player A. (2 marks)

(c) Prove that if we delete a strictly dominated strategy from G then the game, say G', with this strategy removed has the same equilibria as G. (5 marks)

(d) Consider the two-player game in which both players, A and B, simultaneously choose to hold up some (at least one) number of fingers on one of their hands. We denote this number by a and b, where $a, b \in \{1, 2, 3, 4, 5\}$, for each player respectively. The players then receive payoffs given by

$$g_A(a, b) = |5 - a - b|,$$

for player A and

$$g_B(a, b) = |1 + a - b|,$$

for player B.

(i) Draw a normal (strategic) form representation of the game. (2 marks)

(ii) Remove all strictly dominated pure strategies from the game. [Hint: consider mixing over suitable combinations of choosing 1 or 5 fingers for each player] (5 marks)

(iii) Determine the equilibrium of the game. (3 marks)

(Total: 20 marks)

2. (a) In a finite two-player zero-sum game the (i, j)th entry of the matrix

$$\begin{bmatrix} 5 & 1 & -1 & -2 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$

gives the payoff to player A when A plays pure strategy a_i and player B plays pure strategy b_j.

(i) Draw a lower-envelope diagram for this game. (3 marks)

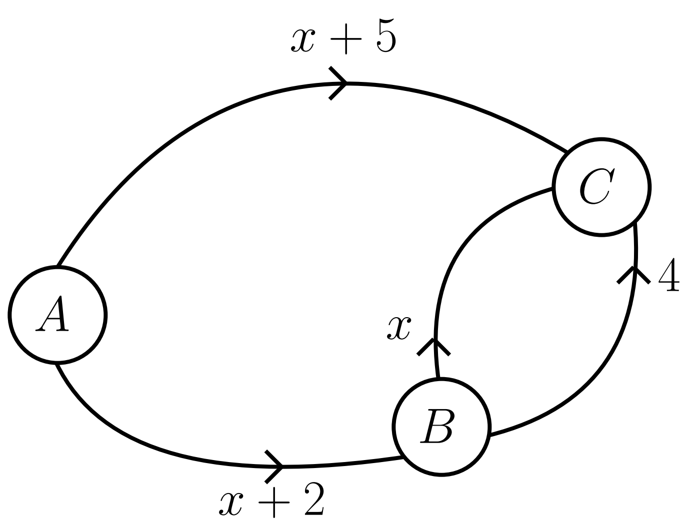
(ii) Identify a max-min strategy for player A in this game. (2 marks)

(iii) Hence find a min-max strategy for player B and determine the value of the game. (3 marks)

(b) In the following question we consider the atomic model of flow through a congestion game.

(i) Define what it means to have an **equilibrium** in a congestion game. (3 marks)

Consider the congestion game in the figure below with three nodes A, B and C and 10 users who all want to travel from node A to node C. Each edge in the diagram shows the cost function for a flow of x users on that edge, so, for example, $x + 2$ is short for $c(x) = x + 2$.



(ii) Find all equilibria of the game. (5 marks)

(iii) Find the socially optimal flow of the game (i.e the distribution of users which minimises the average cost per user of the network). (4 marks)

(Total: 20 marks)

3. The twins Tweedledum (player A) and Tweedledee (player B) are fighting over a cake. In an attempt to resolve the conflict the Mad Hatter (MH) cuts the cake into three pieces and presents A and B with one piece each, keeping the final piece for himself.

However, the piece the MH has given A is one-third of the cake while B has been given only one-sixth of the cake (the MH keeping half of the cake for himself) and a new argument between the twins ensues.

In desperation the MH tells them that each will have to decide either to **accept** their piece of cake or **reject** it. If both accept, the argument is over and each twin keeps their allotted piece of cake. If both reject their pieces of cake the MH will take back their pieces and eat the whole cake himself, leaving the twins with nothing. If one twin accepts and the other rejects their piece of cake then the MH will reassign the twins two pieces of cake at random (but still keeping his largest piece for himself).

(a) Each twin naturally wants to get as much cake as possible and they start to argue about their **joint strategy** by playing the MH's game **cooperatively**.

(i) Determine the threat point for the game. (4 marks)

(ii) Sketch the payoff set and identify the bargaining set. Indicate the pareto-optimal frontier of this set. (4 marks)

(iii) Find the Nash bargaining solution for the game and show how the twins can implement this solution. (6 marks)

The MH thrives on causing mischief, and, being unsatisfied with the reasonable collaborative solution of the twins, decides to offer B the following bargain.

As before A and B can choose to accept/reject their piece of cake with the same outcomes, but now B can also choose the MH's **bargain** instead as a third option. If B takes the bargain and A accepts their piece of cake, then A keeps their piece of cake but B will get to swap their piece of cake with the MH, getting half of the cake! But if B takes the bargain and A rejects their piece of cake, then A gets the MH's half of the cake and B will be assigned either their own piece or A's piece at random.

(b) This succeeds in breaking up the twins coalition and they now play this new game **non-cooperatively**. Find **all** equilibria of the MH's final game.

(6 marks)

(Total: 20 marks)

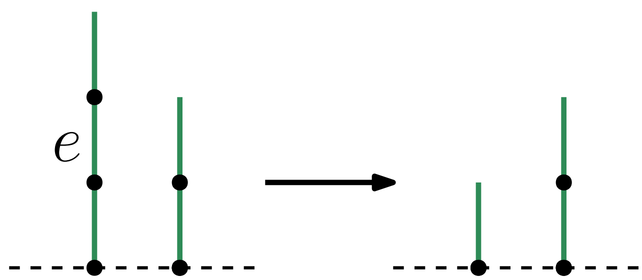
4. [Throughout this question you may assume any results about impartial games and the game of Nim unless you are asked to prove them.]

(a) Define the **Nim value** of an impartial game G. (2 marks)

(b) State, without proof, the **mex rule** for an impartial game G. (3 marks)

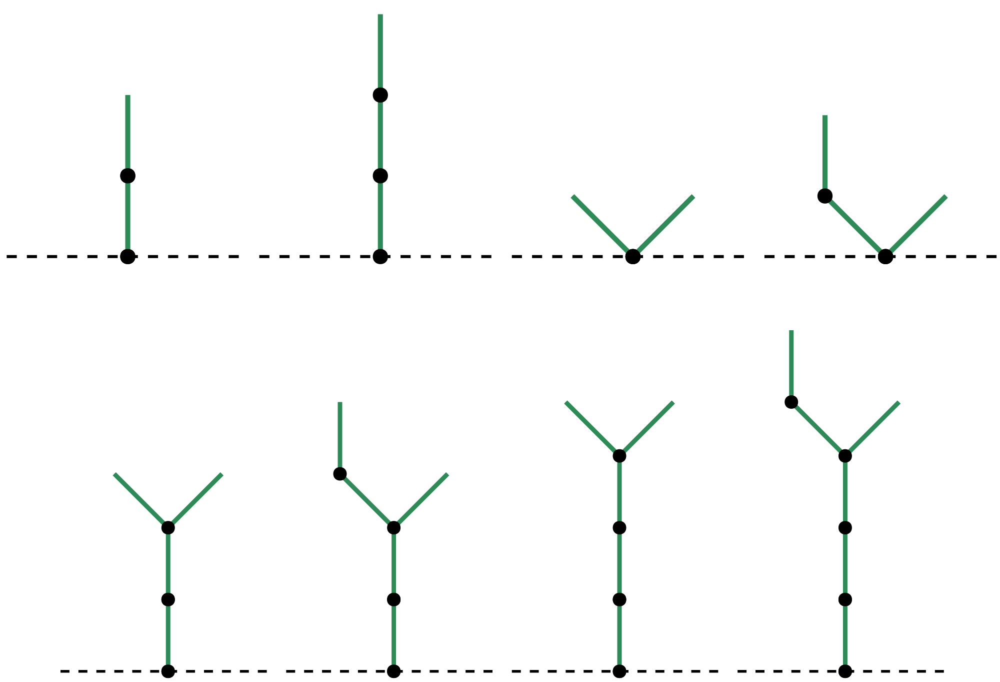
The impartial game **Green Hackenbush** is played on a figure consisting of nodes and edges connected together and to the ground (shown with a dashed line in all following figures).

A move constitutes removing an edge from the figure, and with it all the edges that are then no longer connected to the ground in some way. For example, in the figure below on the left, a player can remove any edge from either of the two 'stems'. Removing the edge labelled e from the 'stem' that consists of three edges also removes the topmost edge with it, leaving only a single edge in that 'stem' (think about floating edges being removed by falling to the ground), see the figure below on the right for the resulting position.

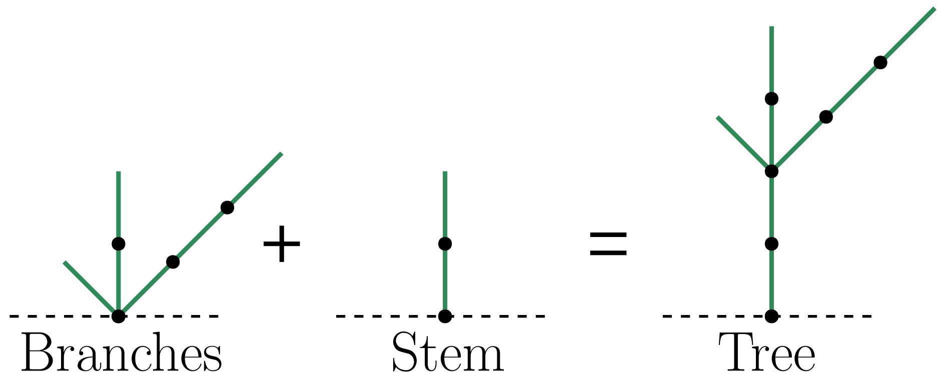


Players alternate removing edges from the figure and the last player able to move wins the game (the normal play convention).

(c) Compute the Nim value of each of the eight Green Hackenbush figures below. (6 marks)

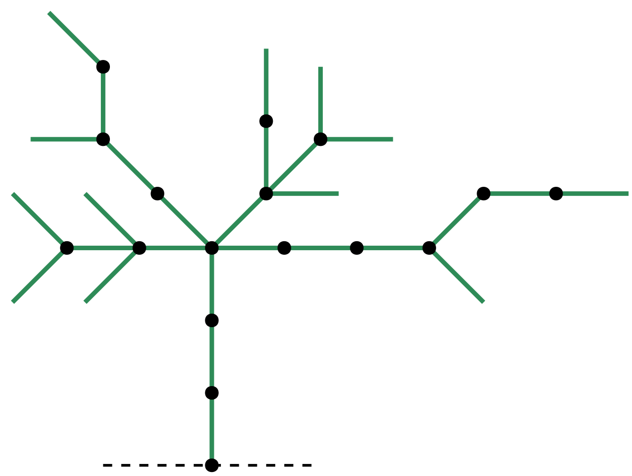


(d) By use of the figures from part (c), conjecture a rule to compute the Nim value of a 'tree' that is constructed by putting several 'branches' with known Nim values on top of a 'stem' of size n, for example $n = 2$ in the figure below where three 'branches' of lengths 1, 2 and 3 are put on top of the 'stem'.



Prove the correctness of your rule. (6 marks)

(e) Compute the Nim value of the Green Hackenbush figure below. (3 marks)



(Total: 20 marks)

5. Consider a finite two-player zero-sum game played between players A and B.

(a) In such a game:

(i) Define what it means for a strategy α^* of player A to be an equaliser strategy. (2 marks)

(ii) Prove that if α^* and β^* are equaliser strategies for A and B respectively, then these strategies are max-min and min-max and form an equilibrium of the game. (4 marks)

Arbitrated by the Mad Hatter over a decadent tea party, Alice and the Cheshire cat play a game. In this game a positive integer, n, is first selected by the Mad Hatter and then each of Alice and the Cheshire cat must simultaneously choose a positive integer less than or equal to n.

If the sum, S, of their choices is **even** then the Cheshire cat must give Alice S of his biscuits. If the sum of their choices is **odd** then Alice must give the Cheshire cat S of her biscuits.

(b) Solve the Mad Hatter's game (find a pair of max-min and min-max strategies in equilibrium and give the value of the game) when:

(i) $n = 2$; (2 marks)

(ii) $n = 3$; (5 marks)

(iii) $n \geq 4$. [Hint: Consider the solution to the subgame corresponding to $n = 3$] (7 marks)

(Total: 20 marks)

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 1	Topic BASICS + DOMINANCE	Marks& seen/unseen

Parts

(a). A pair of strategies, α^* for player A and β^* for player B, are said to be in <u>equilibrium</u> if:

$$g_A(\alpha^*, \beta^*) \geq g_A(\alpha, \beta^*), \quad \forall \alpha \in \mathbb{A}_S$$
$$g_B(\alpha^*, \beta^*) \geq g_B(\alpha^*, \beta), \quad \forall \beta \in \mathbb{B}_S.$$

[Alternatively: (α^*, β^*) are an equilibrium if α^* is a best response to β^* and β^* is a best response to α^*].

Seen definition

3 $\boxed{A}$

(b). A strategy $\alpha \in \mathbb{A}_S$ is <u>strictly dominated</u> by another strategy $\alpha' \in \mathbb{A}_S$ if:

$$g_A(\alpha, \beta) < g_A(\alpha', \beta), \quad \forall \beta \in \mathbb{B}_S.$$

[Note: we can restrict the above to $\forall b \in B_S$].

2 $\boxed{A}$

Seen definition

[Allow for definitions over ~~the~~ pure strategy sets A_S, B_S here as early in exam]

Setter's initials SJB	Checker's initials	Page number 1

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 1	Topic BASICS + DOMINANCE	Marks& seen/unseen

Parts

(c). Suppose α' strictly dominates α for A.

This means that α is never part of an equilibrium since player A could always deviate to α' to do better.

Denoting the original game by G and the game with α removed from A's by G', we can say that any equilibrium in G doesn't contain α, so therefore it is a possible set of strategies in G' where it is also equilibrium.

A 2 Seen proof

On the other hand, any equilibrium in G' is a possible set of strategies in G where it is either an equilibrium, or, if not, then player A must be able to benefit by deviating to strategy α. But if this were the case then they could deviate to α' to do better which is a possible strategy in G', violating the fact we started with an equilibrium of G'.

Seen proof 3 B

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 1	Topic BASICS + DOMINANCE	Marks& seen/unseen

Parts

(d). Rest: Unseen Game

(i).

		B				
		b_1	b_2	b_3	b_4	b_5
A	a_1	3,1	2,0	1,1	0,2	1,3
	a_2	2,2	1,1	0,0	1,1	2,2
	a_3	1,3	0,2	1,1	2,0	3,1
	a_4	0,4	1,3	2,2	3,1	4,0
	a_5	1,5	2,4	3,3	4,2	5,1

$\boxed{A}$ 2 seen similar

(ii). Consider $\alpha_1 = \frac{2}{3}a_1 + \frac{1}{3}a_5$.

Then: $g_A(\alpha_1, b_1) = \frac{7}{3} > 2 = g_A(a_2, b_1)$

$g_A(\alpha_1, b_2) = 2 > 1 = g_A(a_2, b_2)$

$g_A(\alpha_1, b_3) = \frac{5}{3} > 0 = g_A(a_2, b_3)$

$g_A(\alpha_1, b_4) = \frac{4}{3} > 1 = g_A(a_2, b_4)$

$g_A(\alpha_1, b_5) = \frac{7}{3} > 2 = g_A(a_2, b_5)$.

So α_1 dominates a_2 $\Rightarrow$ <u>delete a_2 from the game</u>.

seen similar (method seen)

Similarly, one can show that:

$\alpha_2 = \frac{1}{4}a_1 + \frac{3}{4}a_5$ dominates a_3 $\Rightarrow$ <u>delete a_3</u>

~~α_3~~ a_5 dominates a_4 $\Rightarrow$ <u>delete a_4</u>

next side

1 $\boxed{B}$

Setter's initials SJB	Checker's initials	Page number 3

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 1	Topic BASICS + DOMINANCE	Marks & seen/unseen

Parts (d). (ii). (continued).

Similarly for B: b_1 dominates b_2

$$\beta_1 = \tfrac{1}{3}b_1 + \tfrac{2}{3}b_5 \text{ dominates } b_4$$

$$\beta_2 = \tfrac{3}{4}b_1 + \tfrac{1}{4}b_5 \text{ dominates } b_3$$

$\Rightarrow$ delete b_2, b_3, b_4.

} 1 B

3 D

(2 points for the dominant pure strategies, 3 for the mixed ones)

[Note: many possible answers here – necessary to state the strategy which is dominating the removed pure strategies].

(iii). We are left with the reduced game:

		B: b_1	b_5
A	a_1	3, 1	1, 3
	a_5	1, 5	5, 1

Let $\alpha = (p, 1-p)$
$\beta = (q, 1-q)$

Then: $g_A(a_1, \beta) = 3q + 1 - q = 2q + 1$

$g_A(a_5, \beta) = q + 5 - 5q = 5 - 4q$

Seen similar

Player A indifferent $\Leftrightarrow 2q+1 = 5-4q \Leftrightarrow q = \frac{2}{3}$

$\Rightarrow \beta = \left(\frac{2}{3}, \frac{1}{3}\right)$ is an Equaliser strategy for B

Similarly, $\alpha = \left(\frac{2}{3}, \frac{1}{3}\right)$ is an Equaliser strat. for A.

each ES 1 point. } 2 C

$\Rightarrow \left(\left(\frac{2}{3}, 0, 0, 0, \frac{1}{3}\right), \left(\frac{2}{3}, 0, 0, 0, \frac{1}{3}\right)\right)$ is the equilibrium of the game.

} 1 C extend to full game.

Setter's initials SJB	Checker's initials	Q1: Total: 20	Page number 4

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 2

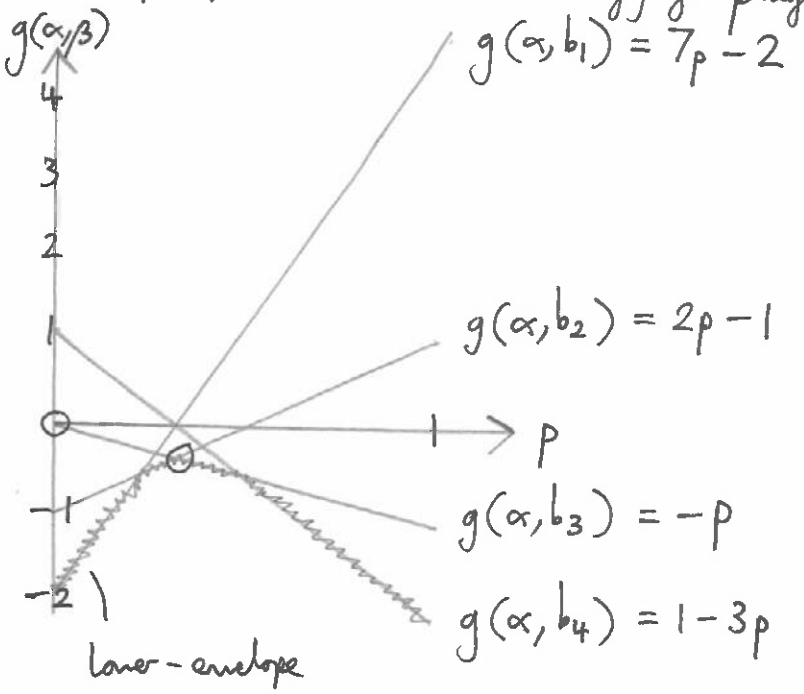
Topic: ZERO-SUM GAMES + CONGESTION GAMES

Marks & seen/unseen

Parts

(a).

(i). Let $\alpha = (p, 1-p)$ be a mixed strategy for player A.



Seen similar — 3 [A]

(ii). We identify the peak of the lower-envelope as corresponding to a max-min strategy for A. This occurs when $2p-1 = -p$, or $\underline{p = \frac{1}{3}}$, i.e: $\underline{\underline{\hat{\alpha} = \left(\frac{1}{3}, \frac{2}{3}\right)}}$ is a max-min strategy for A.

Seen similar — 2 [A]

(iii). When A plays $\hat{\alpha}$, B does best by mixing over b_2 and b_3, so let $\beta = (0, q, 1-q, 0)$, then:

$g(a_1,\beta) = q - (1-q) = 2q-1$

$g(a_2,\beta) = -q$

1 [B] — Seen similar

Setter's initials: SJB

Checker's initials:

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 2	Topic ZERO-SUM GAMES + CONGESTION GAMES	Marks& seen/unseen

Parts

(a).
(iii).
(continued).

So for A's indifference we need $2q-1=-q \iff q=\frac{1}{3}$

$\Rightarrow \hat{\beta}=(0,\frac{1}{3},\frac{2}{3},0)$ is min-max for B.

1 $\boxed{B}$ seen similar

The game has value:

$$V=g(\hat{\alpha},\hat{\beta})=-\frac{1}{3}.$$

1 $\boxed{B}$ seen similar

(b).

(i). In a congestion game with N users, the strategies $P_1,\ldots,P_N$ (paths through the network from origin to destination) of all N users define an equilibrium if each strategy is a best response to the other strategies, i.e. that, for each user i;

$$Cost(P_i) \leq Cost(Q_i),$$

for all possible different paths Q_i of user i.

seen definition

3 $\boxed{A}$

(max 2 if notion of best response has no definition here)

(allow also definitions with potential function + flow defined)

(ii). Suppose that $10-x$ users take the topmost path, y users take the route BC through the middle, $x-y$ users take BC along the bottom and x users take AB.

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 2	Topic ZERO-SUM GAMES + CONGESTION GAMES	Marks& seen/unseen

Parts (b). (ii). (continued).

Then:

$$Cost_{top} = 15 - x$$

$$Cost_{mid} = x + y + 2$$

$$Cost_{bottom} = x + 6$$

In equilibrium we must have $C_{mid} = C_{bottom} (\pm 1)$. This gives $y = 3, 4, 5$. Similarly $C_{top} = C_{bottom} (\pm 1)$ giving $x = 4, 5$. We check each case to see which are in equilibrium, finding all equilibria:

- 6 users on top route, cost = 11 each; 4 users on the middle path, cost = 10 each; 0 users along the bottom path, cost = 10 each.
- 6 users on top route, cost = 11 each; 3 users on the middle, cost = 9 each; 1 user along the bottom path, cost = 10 each.
- 5 users on top route, cost = 10 each; 3 users on the middle, cost = 10 each; 2 users along the bottom, cost = 11 each.
- 5 users on top route, cost = 10 each; 4 users on middle, cost = 11 each; 1 user along the bottom, cost = 11 each.

Seen similar.

Some reasonable attempt at costing routes. 2 B

1 B Some reasonable working

Seen Similar 2 C (if at least 2 correct equilibria award 1 mark; 2 marks only if all four correct)

Setter's initials SJB

Checker's initials

Page number 7

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 2	Topic ZERO-SUM GAMES + CONGESTION GAMES	Marks& seen/unseen

Parts (b). (iii).

Average cost per user

$$= \frac{1}{10}\left(\left(10-x\right)\left(15-x\right) + y^2 + x\left(x+2\right) + 4\left(x-y\right)\right)$$

$$= \frac{1}{10}\left(150 - 25x + x^2 + y^2 + x^2 + 2x + 4x - 4y\right)$$

$$= \frac{1}{10}\left(2x^2 - 19x + y^2 - 4y + 150\right)$$

$$= \frac{1}{10}\left(2\left(x - \frac{19}{4}\right)^2 - 2\left(\frac{19}{4}\right)^2 + \left(y-2\right)^2 - 4 + 150\right)$$

$$= \frac{1}{5}\left(x - \frac{19}{4}\right)^2 + \frac{1}{10}\left(y-2\right)^2 + \frac{1}{10}\left(146 - \frac{19^2}{8}\right)$$

This is minimised when $x = 5, y = 2$; So the social optimal flow is given by 5 users on the top, 2 users on the middle and 3 users along the bottom.

Seen Similar (award up to 2 points for good method) 2 $\boxed{D}$

Seen Similar 2 $\boxed{D}$

Q2 Total: 20

Setter's initials SJB	Checker's initials	Page number 8

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic COOPERATION + DEGENERACY

Marks& seen/unseen

Parts

(a). The payoffs are:

B

A	accept	reject
accept	$\frac{1}{3}, \frac{1}{6}$	$\frac{1}{4}, \frac{1}{4}$
reject	$\frac{1}{4}, \frac{1}{4}$	$0, 0$

Let's multiply all payoffs by 12 to avoid working with fractions:

B

A	accept	reject
accept	4, 2	3, 3
reject	3, 3	0, 0

Game unseen
All method seen similar

NB: No explicit marks given for the payoff matrix.
If incorrect award FT as far as possible

(i). A's payoffs are:

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 4 & \boxed{3} \\ a_2 & 3 & \boxed{0} \end{array}$$

This game has a pure strategy equilibrium at (a_1, b_2), so A's threat level is $\underline{t_A = 3}$.

2 $\boxed{A}$

seen similar

B's payoffs are:

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 2 & \boxed{3} \\ a_2 & \boxed{3} & 0 \end{array}$$

Setter's initials SJB

Checker's initials

Page number 9

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic COOPERATION + DEGENERACY

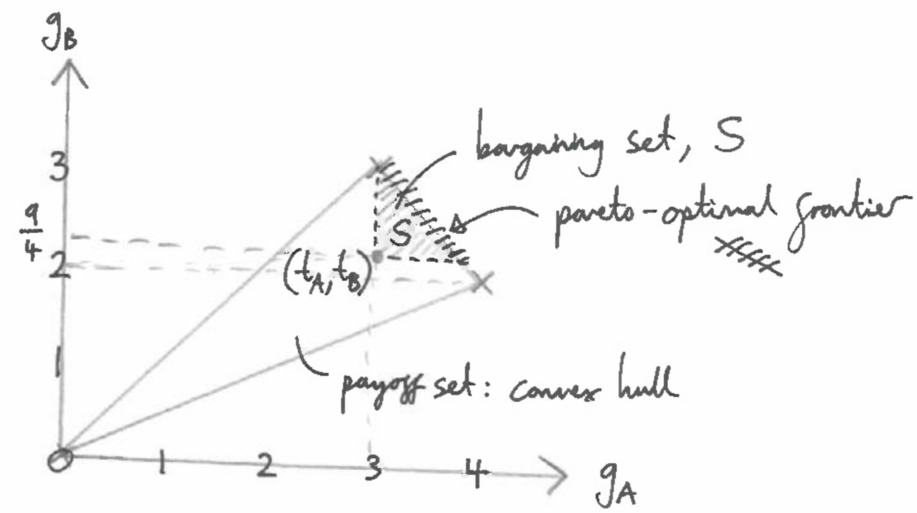
Marks& seen/unseen

Parts (a).

(i). (continued). In this game B's max-min strategy is $\hat{\beta} = (\frac{3}{4}, \frac{1}{4})$, giving B's threat level as: $\underline{t_B = \frac{9}{4}}$.

2 B seen similar

(ii).



seen similar A 4 (2: payoff set 1: bargaining set 1: pareto-frontier)

(iii). We maximise the Nash product $(x-t_A)(y-t_B)$ over the pareto-optimal frontier; which has equation $y = 6-x$:

$$(x-3)(y-\tfrac{9}{4}) = (x-3)(\tfrac{15}{4}-x)$$
$$= -x^2 + \tfrac{27}{4}x - \tfrac{45}{4},$$

which is maximised when $\underline{x = \frac{27}{8}, y = \frac{21}{8}}$.

This is in the bargaining set S, hence it gives the Nash bargaining solution. (÷12 for fractions of the cake)

There are many ways the twins can implement this solution; some of the simplest include:

Seen similar A 2 Nash product.

B 2 $y=6-x$ + quadratic.

1 B solution.

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic COOPERATION + DEGENERACY

Marks& seen/unseen

Parts (a). (iii). (continued).

- A accepts and B plays $\beta = \left(\frac{3}{8}, \frac{5}{8}\right)$. (or vice-versa)
- They assign ~~to~~ the joint strategy

$$\frac{3}{8}(\text{accept}, \text{accept}) + \frac{5}{8}(\text{accept}, \text{reject})$$ (+ variants of this)

seen similar 1 $\boxed{C}$ one correct joint strategy

(b).

		B accept	B reject	B bargain
A	accept	$\frac{1}{3}, \frac{1}{6}$	$\frac{1}{4}, \frac{1}{4}$	$\frac{1}{3}, \frac{1}{2}$
	reject	$\frac{1}{4}, \frac{1}{4}$	0, 0	$\frac{1}{2}, \frac{1}{4}$

Multiply payoffs by 12 to avoid fractions:

		B accept	B ~~reject~~	B bargain
A	accept	4, 2	~~3, 3~~	4, $\boxed{6}$
	reject	3, $\boxed{3}$	~~0, 0~~	6, $\boxed{3}$

unseen game. 1 $\boxed{C}$ payoffs.

reject is strictly dominated by bargain for B, so we delete reject for B.

1 $\boxed{C}$ domination.

Notice that the pure strategy reject for A has two pure best responses (accept, bargain) for B. Hence the game is degenerate owing to B's weakly dominated strategy.

Setter's initials SJB

Checker's initials

Page number 11

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 3

Topic COOPERATION + DEGENERACY

Marks& seen/unseen

Parts (b). (continued).

(reject, bargain) forms a pure strategy equilibrium of the game, however because B is indifferent between accept and bargain when A plays reject, player B can mix. This defines an equilibrium as long as reject stays a best response for A to B's mixing. This happens if and only if

$$g_A(\text{accept}, \beta) \leq g_A(\text{reject}, \beta),$$

for B's mixed strategy $\beta = (q, 0, 1-q)$.

This gives:

$$4 \leq 6 - 3q$$

$$\Leftrightarrow \quad 3q \leq 2$$

$$\Leftrightarrow \quad \underline{q \leq \tfrac{2}{3}}.$$

Therefore:

$$\underline{(\text{reject}, (q, 0, 1-q))},$$

where $\underline{0 \leq q \leq \tfrac{2}{3}}$ form an infinite set of <u>all</u> equilibria of the game.

3 seen similar identification of the pure equilibria award 1 of these marks if some attempt also at noticing degeneracy

2 $\boxed{D}$ Condition for a_2 to remain a best response

seen similar 2 $\boxed{D}$ all equilibria given.

Q3: Total: 20

Setter's initials SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 4	Topic IMPARTIAL GAMES.	Marks& seen/unseen

Parts

(a). If $G \equiv *m$ for an impartial game G, then we call m the Nim value of G. (here $*m$ represents a single Nim pile of size m).

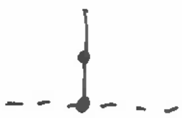
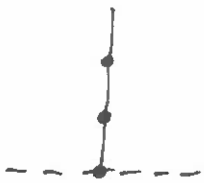
2 $\boxed{A}$ Seen definition

(b). The mex rule states that: Any impartial game G has Nim value m (i.e. $G \equiv *m$), where m is uniquely determined as follows: for each option H of G, let H have Nim value s_H, and let $S = \{s_H : H \text{ is an option of } G\}$. Then $m = \text{mex}(S)$, i.e. $G \equiv *(\text{mex}(S))$. (mex here means the minimum excluded element from the set).

3 $\boxed{A}$ Seen theorem ~~definition~~

unseen game.

(c). By the rules of Green Hackenbush, a single 'stem' of n edges is just a Nim ~~heap~~ pile with n tokens. Thus:

 $\equiv *2$  $\equiv *3$

The third picture is equivalent to two Nim piles of sizes 1, so $\equiv *1 + *1 \equiv 0$, so has Nim value 0.

In (c). award all marks if correct values given with no explanation. If some good explanation but wrong values give partial marks.

1 $\boxed{A}$

Setter's initials SJB	Checker's initials	Page number 13

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 4

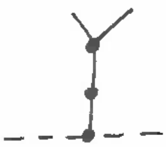
Topic IMPARTIAL GAMES

Parts (c). (continued).

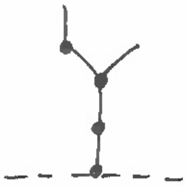
Marks & seen/unseen

The fourth picture is $\equiv *1 + *2 \equiv *3$, so has Nim value 3. Now we use the mex rule for the bottom four figures. 1 $\boxed{A}$

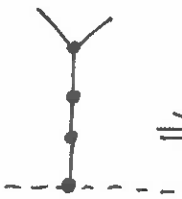
unseen (method seen similar)

Options of  have Nim values: 3, 1, 0, 3

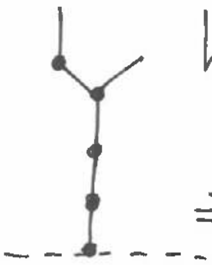
$\Rightarrow$ Nim value $= \text{mex}(3,1,0) = 2$. 1 $\boxed{B}$

Options of  have Nim values: 2, 3, 4, 1, 0

$\Rightarrow$ Nim value $= \text{mex}(2,3,4,1,0) = 5$. 1 $\boxed{B}$

Options of  have Nim values: 4, 4, 2, 1, 0

$\Rightarrow$ Nim value $= \text{mex}(4,2,1,0) = 3$. 1 $\boxed{B}$

Options of  have Nim values:

3, 4, 5, 2, 1, 0

$\Rightarrow$ Nim value $= \text{mex}(3,4,5,2,1,0)$
$= 6$. 1 $\boxed{B}$

Setter's initials SJB

Checker's initials

Page number 14

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 4	Topic IMPARTIAL GAMES	Marks& seen/unseen
Parts		
(d).	Based on the results of (c) we make the following conjecture: "If the Nim sum of the 'branches' is m, and the 'stem' has length n, then the 'tree' obtained by putting the branches on top of the stem has Nim value $m+n$." <u>Proof:</u> Let's start by thinking about the 'branches'. If their Nim Sum is m, then all options from the 'branches' are equivalent to one of $0, *1, \dots, *(m-1)$ (and as for applying the mex rule there may be options equivalent to $*k$ for $k>m$, but importantly their is <u>no</u> option equivalent to $*m$) by the mex rule. Each such option $*k$, for $0 \le k < m$ can be replaced by a single 'stem' of length k, which, when put on top of the ~~stem~~ original 'stem' of length n, are equivalent to 'stems' of length $n+k$ for $0 \le k < m$ because they are extended by the 'stem' length n. Thus the 'tree' has options equivalent to $*n, *(n+1), \dots, *(n+m-1)$ available by removing	unseen 2 $\boxed{C}$ unseen 1 $\boxed{C}$ unseen $\boxed{D}$ 2
	Setter's initials SJB	Checker's initials
		Page number 15

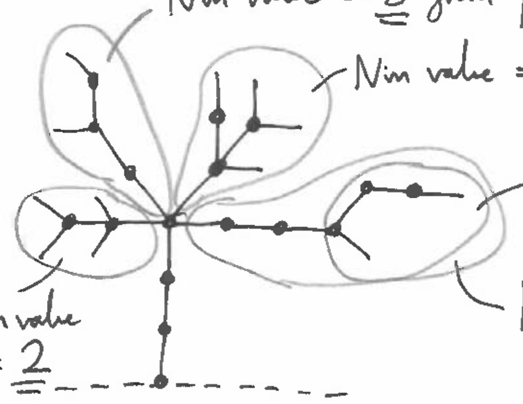
	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 4	Topic IMPARTIAL GAMES	Marks& seen/unseen

Parts

(d). (continued). a 'branches' edge, but also options equivalent to $0, *1, \ldots, *(n-1)$ by removing a 'stem' edge. Thus the tree has all options of Nim values $0, 1, \ldots, n+m-1$, but not $n+m$, and hence by the mex rule, is equivalent to a Nim pile of size $n+m$, proving our conjecture.

1 C unseen.

(e).



Nim sum of all four 'branches' is:

$$2 \oplus 5 \oplus 3 \oplus 5 = 1$$

Thus, by (d), Nim value of tree is $1 + 3 = 4$.

'Stem' length 3.

unseen

2 D

1: correct Nim value each branch

1: correct Nim value of all branches.

1 D (correct final answer)

Q4 Total: 20

Setter's initials SJB	Checker's initials	Page number 16

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic: ZERO-SUM GAMES AND EQUALISER STRATEGIES

Marks & seen/unseen

Parts

(a).

(i). α^* is an equaliser strategy for A if and only if

$$g(\alpha^*, b) = \text{constant}, \quad \forall b \in B_S.$$

~~unseen~~ seen in context of bimatrix games

2

unseen definition in context of zero-sum games

(ii). Assume α^*, β^* are ES for A and B respectively. Then:

$$\inf_\beta \{g(\alpha^*, \beta)\} = g(\alpha^*, \beta^*), \quad \because \alpha^* \text{ ES}$$
$$= g(\alpha, \beta^*), \quad \because \beta^* \text{ ES}$$
$$\geq \inf_\beta \{g(\alpha, \beta)\}, \quad \forall \alpha \in A_S.$$

Hence $\inf_\beta \{g(\alpha^*, \beta)\} \geq \inf_\beta \{g(\alpha, \beta)\}, \forall \alpha,$

So α^* is max-min. Similarly β^* is min-max.

seen proof ideas in context of bimatrix games

3

unseen in context of zero-sum games.

Further, let $g(\alpha^*, b) = c, \forall b \in B_S$, then $g(\alpha^*, \beta^*) = c$ since β^* is a mixture of b's. This means that (α^*, β^*) are in equilibrium since B cannot change to any other mixture β of b's to do better. Similarly for A.

1

Setter's initials: SJB

Checker's initials

	EXAMINATION SOLUTION 23 - 24	Course: Introduction to Game Theory
Question 5	Topic ZERO-SUM GAMES + EQUALISER STRATEGIES	Marks & seen/unseen

Parts

(b).

(i). $\underline{n=2}$:

		Cat 1	Cat 2
Alice	1	2	$\boxed{-3}$
	2	$\boxed{-3}$	4

Seeking a pair of equaliser strategies, one finds:

$$\alpha^* = \left(\frac{7}{12}, \frac{5}{12}\right), \beta^* = \left(\frac{7}{12}, \frac{5}{12}\right)$$

form an equilibrium of the game. } 1

The value is $v = g(\alpha^*, \beta^*) = -\frac{1}{12}$. } 1

unseen game.

seen method

(ii). $\underline{n=3}$:

		Cat 1	Cat 2	Cat 3
Alice	1	2	$\boxed{-3}$	4
	2	-3	4	$\boxed{-5}$
	3	4	$\boxed{-5}$	6

Again we search for a pair of equaliser strategies.

Let:

$\alpha = \beta = (p, q, 1-p-q)$.

owing to the game structure.

unseen game

seen method

Then:

$$g(\alpha, 1) = 2p - 3q + 4(1-p-q)$$
$$= 4 - 2p - 7q \quad ①$$

$$g(\alpha, 2) = -3p + 4q - 5(1-p-q)$$
$$= 2p + 9q - 5 \quad ②$$

$$g(\alpha, 3) = 4p - 5q + 6(1-p-q)$$
$$= 6 - 2p - 11q \quad ③$$

} 2

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic ZERO-SUM GAMES + EQUALISER STRATEGIES

Marks& seen/unseen

Parts (b). (ii). (continued).

Seeking equaliser strategies, set ① = ③

$\Rightarrow 4q = 2$ or $\underline{\underline{q = \frac{1}{2}}}$.

Then ① = ② gives: $\frac{1}{2} - 2p = 2p - \frac{1}{2} \Rightarrow \underline{\underline{p = \frac{1}{4}}}$

Hence $\alpha^* = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) = \beta^*$ gives an equilibrium of the game. It has value: $\underline{\underline{V = 0}}$. } 2 } 1

(iii) For $n \geq 4$ consider the pair of strategies:

$\alpha^* = \beta^* = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \underbrace{0, \ldots, 0}_{n-3}\right)$, i.e. the solution to the subgame corresponding to $n=3$ <u>extended</u> to the full game. unseen } 2

Then clearly:

~~$g(\alpha^*$~~ $g(\alpha^*; 1) = g(\alpha^*; 2) = g(\alpha^*; 3) = 0$, (calculated in part (b).(ii).), } 1

and we have:

$$g(\alpha^*; k) = \frac{(k+1)(-1)^{k-1}}{4} + \frac{(k+2)(-1)^k}{2} + \frac{(k+3)(-1)^{k+1}}{4},$$

for $k > 3$

Setter's initials SJB

Checker's initials

EXAMINATION SOLUTION 23 - 24

Course: Introduction to Game Theory

Question 5

Topic ZERO-SUM GAMES + EQUALISER STRATEGIES

Marks& seen/unseen

Parts (b). (iii). (continued).

$$= \begin{cases} -\frac{(k+1)}{4} + \frac{(k+2)}{2} - \frac{(k+3)}{4}, & k \text{ even} \\ \frac{(k+1)}{4} - \frac{(k+2)}{2} + \frac{(k+3)}{4}, & k \text{ odd} \end{cases}$$

$$= 0, \quad \forall k > 3.$$

3 unseen

Similarly, $g(k, \beta^*) = 0 \quad \forall k$.

So α^*, β^* are a pair of equaliser strategies and therefore form an equilibrium of the game.

The value of the game is $\underline{\underline{0}}$.

1

QS Total: 20

Setter's initials SJB

Checker's initials

Page number 20

MATH60141 Introduction to Game Theory

Question	Marker's comment
3	The majority of students answered part (a.i) and (a.ii) correctly although most diagrams were not detailed enough. Coming up with a strategy for part (a.iii) appeared to be more challenging. For part (b), students that understood how degeneracy appears performed very well although this was not the majority of students.
4	Parts (A) and (B) of the question provided definitions and were generally well answered, with the majority of students earning all 5 marks. Many students answered Part (C) effectively, although including diagrams would have enhanced their responses. Part (D) and (E), being the final questions of the exam, were left incomplete by several students likely due to time constraints. Most students made an effort to answer Part (D), but encountered difficulties in providing comprehensive proofs.

MATH70141 Introduction to Game Theory

Question	Marker's comment
3	The majority of students answered part (a.i) and (a.ii) correctly although most diagrams were not detailed enough. Coming up with a strategy for part (a.iii) appeared to be more challenging. For part (b), students that understood how degeneracy appears performed very well although this was not the majority of students.
4	Parts (A) and (B) of the question provided definitions and were generally well answered, with the majority of students earning all 5 marks. Many students answered Part (C) effectively, although including diagrams would have enhanced their responses. Part (D) and (E), being the final questions of the exam, were left incomplete by several students likely due to time constraints. Most students made an effort to answer Part (D), but encountered difficulties in providing comprehensive proofs.