Math40003 Linear Algebra and Groups **Problem Sheet 1**
Seen B Solutions

B.1. Use Gaussian elimination to find every solution to the following equation:

$$\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We know how to solve $Ax = 0$, but this is of the form $Ax = x$. We need to transform the given equation into one we know how to handle.

$$\begin{aligned} \begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \left(\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 9 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Then we can apply Gaussian elimination to find the solutions.

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 9 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Giving a solution set

$$\left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

B.2. Let A be an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Suppose $Ax = b$ has at least one solution $x_0 \in \mathbb{R}^m$. Show all solutions are of the form $x = x_0 + h$, where h solves $Ah = 0$.

Suppose that y is a different solution to $Ax = b$ than x_0. Let $h = (y - x_0)$. Then $y = x_0 + h$. Additionally

$$\begin{aligned} Ah &= A(y - x_0) \\ &= Ay - Ax_0 \\ &= b - b \\ &= 0 \end{aligned}$$

Therefore every solution of $Ax = b$ is of the form $x = x_0 + h$, where h solves $Ah = 0$.

B.3. Let A and B be square $n \times n$ matrices with real entries. For each of the following statements, either give a **proof**, or find a **counterexample with** $n = 2$.

(i) If $AB = 0$ then A and B cannot both be invertible.

Suppose that $AB = 0$ and that A is invertible. Then

$$\begin{aligned} B &= A^{-1}AB \\ &= A^{-1}0 \\ &= 0 \end{aligned}$$

and therefore B is not invertible. Very similarly, if $AB = 0$ and B is invertible, then $A = 0$, and is therefore non-invertible.

Thus, if $AB = 0$ then A and B cannot both be invertible.

(ii) If A and B are invertible then $A+B$ is invertible.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $A^{-1} = A$ and $B^{-1} = B$, but $A+B$ is not invertible.

(iii) If A and B are invertible then AB is invertible.

Let A and B be invertible matrices. Then $AB(B^{-1}A^{-1}) = I$, so AB has an inverse, and is hence invertible.

(iv) If A and B are invertible and $(AB)^2 = A^2B^2$, then $AB = BA$. Suppose that A and B are invertible, and that $(AB)^2 = A^2B^2$. Then

$$\begin{array}{ccc} ABAB & = & AABB \\ A^{-1}ABABB^{-1} & = & A^{-1}AABBB^{-1} \\ BA & = & AB \end{array}$$

(v) If $ABA = 0$ and B is invertible then $A^2 = 0$.

Whenever you look for counterexamples involving matrices, especially if the property in question uses multiplication, I would advise you to consider two types of matrix. First, the elementary matrices. These are all invertible, and multiplying by them is very easy to calculate. Second, those with lots and lots of zeros in them, as they to are easy to calculate.

In this case, these two types of matrices give a very nice counterexample.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(vi) If $ABA = I$ then A is invertible and $B = (A^{-1})^2$.

The uniqueness of inverses shows that if $ABA = I$ then $A^{-1} = BA$. This means that if $ABA = I$ then A is invertible. Furthermore

$$\begin{array}{ccl} A^{-1} & = & BA \\ (A^{-1})^2 & = & BAA^{-1} \\ (A^{-1})^2 & = & B \end{array}$$

as desired.

(vii) If A has a left inverse B and a right inverse C then $B = C$.

The key to proving this is to consider BAC. If we multiply BA first, we get C, and if we multiply AC first, we get B. We know that matrix multiplication is associative, and that is doesn't matter what order we do the multiplication in, so we conclude that $B = C$.

B.4. Let $n \geq 2$ and let $A_n = (a_{ij})$ be the $n \times n$ matrix such that

$$a_{i-1,i} = 1 \text{ for } i = 2, \ldots n,$$
$$a_{i+1,i} = 1 \text{ for } i = 1, \ldots n-1,$$

and $a_{ij} = 0$ for all other i, j. Write down A_2, A_3 and A_4. Prove that A_n is invertible for all even values of n, and is not invertible for all odd values of n. Find A_2^{-1} and A_4^{-1}.

For n even you can reduce A_n to the identity by subtracting row n from row $n-2$, then row $n-2$ from row $n-4$ and so on, and then swapping rows; hence A_n is invertible. For n odd you can reduce to a matrix with a bottom row of zeros by subtracting row 1 from row 3, then row 3 from row 5 and so on; hence A_n is not invertible.