Jordan curves in $\mathbb{R}^2$
Mini Project 2

In this mini-project, we look at *Jordan* curves from analysis point of view.

Consider the set (and notation)

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Also, consider the Euclidean metric $d_2(z_1, z_2) = \|z_1 - z_2\|$, for $z_1, z_2 \in \mathbb{R}^2$. Because S^1 is a subset of $\mathbb{R}^2$, d_2 induces a metric on S^1 as well. That is, for $w_1, w_2 \in S^1$, $d_2(w_1, w_2)$ is well-defined and is a metric on S^1.

The image of any injective continuous map $f : S^1 \to \mathbb{R}^2$ is called a *Jordan curve*. Here the continuity is with respect to the metric d_2 on the domain and range.

For example, the set S^1 is a Jordan curve (with f the identity map). The perimeter of any triangle, rectangle, and polygon is a Jordan curve. What are other examples of Jordan curves? How big a subset of the plane can a Jordan curve be?

Recall that a set $E \subset \mathbb{R}^2$ is called *measure zero*, if there is a collection of points $(z_i)_{i \in I}$ in $\mathbb{R}^2$ and positive real numbers $(r_i)_{i \in I}$ such that

- I is either finite or countable,

- $E \subseteq \cup_{i \in I} B_{r_i}(z_i)$,

- $\sum_{i \in I} \pi r_i^2 < \infty$.

Recall that a map $f : S^1 \to \mathbb{R}^2$ is called Lipschitz if there is $L \in \mathbb{R}$ such that for all $z, w \in S^1$, we have

$$\|f(z) - f(w)\| \leq L \, \|z - w\|.$$

Also, recall that any Lipschitz map is continuous.

Problem 1. Let $f : S^1 \to \mathbb{R}^2$ be an injective Lipschitz map. Show that $f(S^1)$ has measure zero.

A map $f : S^1 \to \mathbb{R}^2$ is called Hölder if are real numbers $K \geq 0$ and $\alpha > 0$ such that for all $z, w \in S^1$, we have

$$\|f(z) - f(w)\| \leq K \, \|z - w\|^\alpha.$$

Problem 2. Show that any Hölder map $f : S^1 \to \mathbb{R}^2$ is uniformly continuous.

Problem 3. Let $f : S^1 \to \mathbb{R}^2$ be an injective Hölder map. Is it true that $f(S^1)$ has measure zero?

Problem 4. Let $f : S^1 \to \mathbb{R}^2$ be an injective continuous map. Is it true that $f(S^1)$ has measure zero?

Problem 5. Is there a Jordan curve γ in the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, such that $([0, 1] \times [0, 1]) \setminus \gamma$ can be covered by a countable collection of balls such that the total area of the balls is less than 0.01? In other words, γ occupies 0.99 percent of the points in the square?

Remark: Jordan curves are also important from topological point of view. There is well-known theorem in topology which states that any Jordan curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two components. That is, if γ is a Jordan curve in $\mathbb{R}^2$, the set $\mathbb{R}^2 \setminus \gamma$ has two connected components. The proof of this statement is not easy.