Imperial College London

M3B

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

The Mathematics of Business and Economics

Date: **Monday 9th May 2016**

Time: **14.00 – 16.00**

Time Allowed: **2 Hours**

This paper has Four Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

1. Alfred, a consumer, is faced with a choice of two goods with prices $\underline{p} = (p_1, p_2)$. Denote Alfred's consumption set by $X \subseteq \mathbb{R}^2_{\geq 0}$, and suppose that he has preferences defined by

$$x_1^2 + x_2^2 \leq {x'_1}^2 + {x'_2}^2 \Rightarrow \underline{x} \succsim \underline{x}'$$

for consumption bundles $\underline{x}, \underline{x}' \in X$.

a) By stating fundamental properties of the preference relation, show that there must exist a continuous utility function $u_A : X \to \mathbb{R}$ that represents Alfred's preferences.
Note: you do not have to prove any properties of the preference relation.

In fact, Alfred's preferences can be represented by the utility function

$$u_A(x_1, x_2) = \frac{1}{\sqrt{x_1^2 + x_2^2}}$$

b) Find expressions for Alfred's Marshallian demand for each good, $x^*_{A,1}(\underline{p}, m_A)$ and $x^*_{A,2}(\underline{p}, m_A)$, where m_A is his budget for the two goods.
Note: you do not have to check the second-order conditions for any optimisation you perform.

Suppose a second consumer, Barbara, is faced with a choice between the same goods, and that her utility function is given by

$$u_B(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{-3/2}$$

Denote Barbara's budget by m_B.

c) Write down an expression for Barbara's corresponding indirect utility function, $v_B(\underline{p}, m_B)$, justifying your answer.

d) Verify that Barbara's indirect utility function is homogeneous of degree 0 in $(\underline{p}, m_B)$.

e) i) Derive an expression for $\dfrac{\partial x^*_{B,1}(\underline{p}, m_B)}{\partial m_B}$ in terms of $\dfrac{\partial^2 e_B(\underline{p}, u)}{\partial p_1 \partial u}$, where $x^*_{B,1}(\underline{p}, m_B)$ is Barbara's Marshallian demand for good 1 and $e_B(\underline{p}, u)$ is her expenditure function for goods 1 and 2, given fixed utility u.

ii) Hence show that good 1 is a luxury good for Barbara if and only if

$$\frac{\partial^2 e_B(\underline{p}, u)}{\partial p_1 \partial u} < -\frac{m_B^4 p_1}{3\left(p_1^2 + p_2^2\right)^{5/2}}.$$

2. Consider the production function

$$f_1(\underline{x}) = \min\{x_1 + x_2, x_3 + x_4\}$$

a) Describe the relationship between the input factors in this production function.

Consider now the single-output Cobb-Douglas production function

$$f_2(x_1, x_2) = x_1^a x_2^b, \qquad a, b > 0.$$

b) Find the marginal rate of technical substitution $MRTS(x_1, x_2)$ for the firm with production function f_2.

c) For the firm with production function f_2, write down the form of the profit function $\pi(\underline{x}, p, \underline{w})$ and derive an expression for the profit-maximising output $y^*(p, \underline{w})$, where p is the output price and $\underline{w}$ is the vector of unit factor costs.

Note: you do not have to check the second-order conditions for any optimisation you perform.

d) State four properties of the maximised profit function $\pi^*(p, \underline{w})$.

Now consider a generic production function $f(\underline{x})$ for an n-input, single-output production process, i.e. $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$. For this production process, denote the long-run and short-run maximised profits $\pi^*_L(p, \underline{w})$ and $\pi^*_S(p, \underline{w}, \underline{x}_F)$, respectively, where $\underline{x}_F$ denotes the factors of production that are fixed in the short run; denote the long-run profit-maximising demand for these fixed factors $\underline{x}^*_F(p, \underline{w})$.

e) By considering the function

$$h(p) = \pi^*_L(p, \underline{w}) - \pi^*_S(p, \underline{w}, \underline{x}^*_F(p', \underline{w})),$$

where p' is some fixed output price, derive the Le Chatelier principle:

$$\frac{\partial y^*_L(p, \underline{w})}{\partial p} \geq \frac{\partial y^*_S(p, \underline{w}, \underline{x}_F)}{\partial p} \qquad \text{at } p = p', \underline{x}_F = \underline{x}^*_F(p', \underline{w}),$$

where $y^*_L(p, \underline{w})$ and $y^*_S(p, \underline{w})$ are the profit-maximising output functions in the long-run and the short-run, respectively.

3. Consider the market for coffee, and suppose that the market comprises 3 producers and n_C consumers. Suppose that the long-run cost-function of producer $i \in \{1,2,3\}$ is

$$c_i^*(\underline{w}, y) = \frac{2.5}{i}(w_1 + w_2)y^2 + w_1 w_2,$$

where y is the output and $\underline{w}$ is the vector of factor costs. Suppose also that consumer j's demand for coffee is given by

$$x_j^*(p_C, p_T, m_j) = \frac{m_j p_C}{p_C^2 + p_T},$$

where p_C is the price of coffee, p_T is the price of tea, and where m_j is consumer j's combined budget for coffee and tea. You may assume that $w_1, w_2 > 0$ and you may denote the total consumer budget for coffee and tea by $M = \sum_{i=1}^{n_C} m_j$.

a) Find the long-run supply functions for each firm in the coffee market.

b) Find expressions for the market demand and market supply functions and hence determine an expression for the equilibrium price of coffee, $p_{C,0}^{(1)}$.

The government offers a subsidy of 20% on the price of coffee.

c) Describe any effects that this subsidy will have on the market supply and market demand for coffee.

d) i) Find an expression for the new equilibrium price, $p_{C,0}^{(2)}$ in terms of M, $\underline{w}$ and p_T.

ii) Evaluate the benefit to the consumer of this subsidy, when $M = 100$, $w_1 = 0.04$, $w_2 = 0.02$ and $p_T = 1$.

4. Consider two industries, A and B, comprising the same 7 firms, with 2015 annual UK output (in units sold) given in the following table:

Firm	Industry A output (millons of units)	Industry B output (millions of units)
Firm 1	4.5	7.0
Firm 2	4.0	0.5
Firm 3	1.0	0.5
Firm 4	4.5	0.5
Firm 5	1.0	0.5
Firm 6	4.0	0.5
Firm 7	1.0	0.5

Table 1: 2015 UK Outputs for industries A & B

a) Using a suitable concentration ratio, quantify and categorise the level of competition in Industry A.

The Herfindahl index is an alternative measure of the level of competition in a market, and is defined as the sum of the squared market shares of the firms in the chosen industry:

$$H = \sum_{i=1}^{n_F} s_i^2,$$

where s_i is the market share of the i^{th} firm in an industry comprising n_F firms. The Herfindahl index for industry A is $H_A = 0.18875$.

b) By comparing the levels of competition in industries A & B, comment on the relative ability of the Herfindahl index and the measure used in part (a) for establishing the level of competition present in a market.

[Continued on next page.]

Consider Firm 1's short-run operations in industry B. Suppose that their production process uses 4 factors of production, and that their short-run cost function is given by

$$c^*(\underline{w}, y_B) = 2y_B^2\sqrt{w_1 w_2} + w_3 w_4,$$

where $\underline{w} = (w_1, w_2, w_3, w_4)$ is the vector of unit factor costs, and y_B is the firm's output in industry B. Further, suppose that in industry B, the market demand is given by

$$X_B = a_1 - a_2 p_B,$$

where p_B is the price of the good and $a_1, a_2 > 0$ are positive real constants.

c) Treating Firm 1 as a monopolist in industry B, show that a profit-maximising position will only exist in the short-run for this firm for levels of output that satisfy

$$y_B \leq \frac{a_1}{2}.$$

d) Find an expression for the profit-maximising level of output for this monopolist in the short-run.

Note: you do not have to check the second-order conditions for any optimisation you perform.