ExamModuleCode	Question Number	Comments for Students
M45P5	1	This question was generally done well.
M45P5	2	Part (a) was generally done well. In part (b), many candidates assumed that a Frenet frame exists without checking the conditions required (which hold here).
M45P5	3	Candidates generally understood how to proceed here, and computations were mostly accurate.
M45P5	4	This question was generally done well.
M45P5	5	This question was generally done poorly. A distressing number of candidates were unable to prove the Poincare-Hopf theorem.

Imperial College
London

M3P5

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Wednesday 15 May 2019

Time: 10.00 - 12.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College London

M4/5P5

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Geometry of Curves and Surfaces

Date: Wednesday 15 May 2019

Time: 10.00 - 12.30

Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

Candidates should use ONE main answer book.

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1. (a) Let $I \subseteq \mathbb{R}$ be an interval and $\gamma\colon I \to \mathbb{R}^2$ a parametrised curve.

   (i) Define what is meant by the *length* $L(\gamma)$ of γ.

   (ii) Define what is meant by a *reparametrisation* of γ.

   (iii) Show that $L(\gamma)$ is invariant under reparametrisation.

   (b) Let $\phi\colon [a,b] \to \mathbb{R}^2$ be a plane curve parametrised by arc length.

   (i) Define what is meant by the *curvature* $\kappa(t)$ and the *curvature vector* $k(t)$ of ϕ at the point $\phi(t)$. Show that $k(t)$ and $\phi'(t)$ are perpendicular.

   (ii) Suppose that the curve ϕ lies in a disc of radius R, so that $|\phi(t)| \leq R$ for all $t \in [a,b]$. Suppose further that the curve touches the boundary of the disc at some point $t_0 \in (a,b)$, so that $|\phi(t_0)| = R$. Show that

$$\kappa(t_0) \geq \frac{1}{R}.$$

   (iii) Must the conclusion of part (b)(ii) remain true if, instead, t_0 is one of the endpoints of the interval $[a,b]$?

2. (a) Consider the curve H in $\mathbb{R}^3$ parametrised by $\phi\colon \mathbb{R} \to \mathbb{R}^3$ where

$$\phi(t) = (3t, 4\cos t, 4\sin t).$$

   (i) Give an arc-length reparametrisation of H.

   (ii) Compute the *Frenet frame* (T, N, B) for H.

   (iii) Compute the curvature and torsion of H.

   (b) Let C be a regular curve in $\mathbb{R}^3$ such that the curvature of C is never zero and the torsion of C is identically zero. Show that C lies in a plane.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(t) > 0$ for all $t \in \mathbb{R}$. Consider the *surface of revolution* $S \subset \mathbb{R}^3$ defined by f. This is covered by two charts, each with local parametrisation

$$\phi(u,v) = \Big(f(u)\cos v, f(u)\sin v, u\Big)$$

where for the first chart $(u,v) \in \mathbb{R} \times (0, 2\pi)$ and for the second chart $(u,v) \in \mathbb{R} \times (-\pi, \pi)$.

(a) Show that S is a regular surface.

(b) Compute the first and second fundamental forms for S.

(c) Compute the Gaussian and mean curvatures of S.

(d) For which such surfaces S is the Gaussian curvature zero?

4. (a) Let $S \subset \mathbb{R}^3$ be a regular oriented surface and $p \in S$ a point. Show that the Gaussian curvature $K(p)$ at p and the mean curvature $H(p)$ at p satisfy $H(p)^2 \geq K(p)$.

(b) For each of the following either give an example, by sketching S in a neighbourhood of p, or explain why no such example exists:

(i) a regular oriented surface S and a point $p \in S$ with $K(p) > 0$ and $H(p) < 0$;

(ii) a regular oriented surface S and a point $p \in S$ with $K(p) < 0$ and $H(p) > 0$;

(iii) a regular oriented surface S and a point $p \in S$ with $K(p) > 0$ and $H(p) = 0$.

(c) Suppose that $S \subset \mathbb{R}^3$ is a compact, oriented surface. Show that the mean curvature of S cannot be identically zero.

(d) (i) Let $S \subset \mathbb{R}^3$ be a regular oriented surface and $\gamma\colon [a,b] \to S$ a regular curve on S parametrised by arc length. Define what it means for γ to be a geodesic on S.

(ii) Suppose that $S \subset \mathbb{R}^3$ is a regular oriented surface that contains a straight line L. Prove that L is a geodesic on S.

5. (a) Let $S \subset \mathbb{R}^3$ be a closed, oriented surface.

   (i) Define what is meant by a *vector field* on S.

   (ii) Suppose that v is a vector field on S and $p \in S$ is an isolated zero of v, that is, a point such that $v(p) = 0$ and that $v(q) \neq 0$ for all q in a neighbourhood of p. Define what is meant by the *index* of v at p.

(b) State and prove the Poincaré–Hopf Theorem. You may use the Gauss–Bonnet Theorem and related results without proof, provided that you state them accurately.

(c) For each of the following, either sketch an example of the vector field v or explain why no such vector field exists.

   (i) A vector field on the unit disk $D \subset \mathbb{R}^2$ with a single isolated zero at the origin O, and index $I(v, O) = 1$;

   (ii) A vector field on the unit disk $D \subset \mathbb{R}^2$ with a single isolated zero at the origin O, and index $I(v, O) = -2$;

   (iii) A vector field on the unit sphere $S \subset \mathbb{R}^3$ with precisely two isolated zeroes, each of index 1;

   (iv) A vector field on the unit sphere $S \subset \mathbb{R}^3$ with precisely two isolated zeroes, one of index 1 and one of index 2.

M345P5 EXAM SOLUTIONS 2019

TOM COATES

(1) (a) Let $I \subseteq \mathbb{R}$ be an interval and $\gamma\colon I \to \mathbb{R}^2$ a parameterised curve.

(i) $L(\gamma) = \int_I |\gamma'(t)|\, dt$ (seen, 1 marks)

(ii) Let J be an interval and $f\colon J \to I$ a smooth function such that $f'(t) \neq 0$ for all $t \in J$. Then $\eta\colon J \to \mathbb{R}^2$, $\eta = \gamma \circ f$, is a reparametrisation of γ. (seen, 1 marks)

(iii) Apply the change-of-variable formula to the integral defining $L(\gamma)$. (seen, 2 marks)

(b) (i) $\kappa(t) = |\phi''(t)|$ and $k(t) = \phi''(t)$. Since ϕ is an arc length parametrisation, it has unit speed: $\phi'(t) \cdot \phi'(t) = 1$. Differentiating gives that $k(t)$ and $\phi'(t)$ are perpendicular. (seen, 5 marks)

(ii) The quantity $\phi(t) \cdot \phi(t)$ is locally maximised at t_0, so differentiating twice gives

$$\phi(t_0) \cdot \phi''(t_0) + \phi'(t_0) \cdot \phi'(t_0) \leq 0$$

Since ϕ is an arc-length parametrisation, we conclude that

$$\phi(t_0) \cdot \phi''(t_0) \leq -1.$$

The LHS is $|\phi(t_0)|\kappa(t_0) \cos\theta$ where θ is the angle between $\phi(t_0)$ and $\phi''(t_0)$, so $|\kappa(t_0)| \geq 1/R$.

(seen, 7 marks)

(iii) No, because being a local maximum no longer implies that the second derivative is negative. For an explicit counterexample, take

$$\phi(t) = (t, 0)$$

where $t \in [0, R]$. This is a straight line from the origin to the edge of the disc; it has curvature zero. (unseen, 4 marks)

(2) (a) (i) Computation gives $|\phi'(t)| = 5$, so an arc-length parametrisation of H is

$$\psi(t) = (3t/5, 4\cos(t/5), 4\sin(t/5)).$$

(seen similar, 4 marks)

(ii) Straightforward computation gives:

$$T(t) = \psi'(t) = (3/5, -4/5\sin(t/5), 4/5\cos(t/5))$$
$$\psi''(t) = (0, -4/25\cos(t/5), -4/25\sin(t/5))$$
$$N(t) = \psi''(t)/|\psi''(t)| = (0, -\cos(t/5), -\sin(t/5)$$
$$B(t) = T(t) \times N(t) = (4/5, 3/5\sin(t/5), -3/5\cos(t/5))$$

(seen similar, 6 marks)

(iii) Continuing the computation gives curvature $\kappa(t) = |T'(t)| = 4/25$ and torsion $\tau(t)$ where $B'(t) = -\tau(t)N(t)$, so $\tau(t) = 3/25$. Note that curvature and torsion are constant. (seen similar, 4 marks)

(b) Our assumptions imply that C admits a Frenet frame (T, N, B). Let ϕ be an arc-length parametrisation of C. Since curvature and torsion are invariant under rigid motions of $\mathbb{R}^3$, wlog we have that $\phi(0) = (0,0,0)$, $T(0) = (1,0,0)$, $N(0) = (0,1,0)$, $B(0) = (0,0,1)$. We need to show that $\phi(t)$ lies in the (x,y)-plane for all t, that is, that

$$\phi(t) \cdot (0,0,1) = 0 \qquad \text{for all } t.$$

The Frenet formulae imply that $B'(t) = -\tau(t)N(t) = 0$, so $B(t) = (0,0,1)$ for all t. Also $T(0) \cdot (0,0,1) = 0$ by our assumptions, so it suffices to prove that $\frac{d}{dt}\big(T(t) \cdot (0,0,1)\big) = 0$ for all t. This is immediate: $T'(t) \cdot (0,0,1) = N(t) \cdot B(t) = 0$ as the Frenet frame is orthonormal.

(seen, 6 marks)

(3) (a) We have

$$\phi_u \times \phi_v = \begin{vmatrix} i & j & k \\ f'(u)\cos v & f'(u)\sin v & 1 \\ -f(u)\sin v & f(u)\cos v & 0 \end{vmatrix} = \big(-f(u)\cos v, -f(u)\sin v, f'(u)f(u)\big).$$

Our assumptions guarantee that this is never zero, so S is regular.

(seen similar, 4 marks)

(b) Straightforward calculation gives

$$g(u,v) = \begin{pmatrix} \phi_u \cdot \phi_u & \phi_u \cdot \phi_v \\ \phi_v \cdot \phi_u & \phi_v \cdot \phi_v \end{pmatrix} = \begin{pmatrix} 1 + f'(u)^2 & 0 \\ 0 & f(u)^2 \end{pmatrix}$$

To compute the second fundamental form, we choose the normal vector field

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}$$

which from (a) is

$$N = \frac{1}{\sqrt{1 + f'(u)^2}}\big(-\cos v, -\sin v, f'(u)\big).$$

Also

$$\begin{aligned} \phi_{uu} &= \big(f''(u)\cos v, f''(u)\sin v, 0\big) \\ \phi_{uv} &= \big(-f'(u)\sin v, f'(u)\cos v, 0\big) \\ \phi_{vv} &= \big(-f(u)\cos v, -f(u)\sin v, 0\big) \end{aligned}$$

and so the second fundamental form is

$$A(u,v) = \begin{pmatrix} \phi_{uu} \cdot N & \phi_{uv} \cdot N \\ \phi_{vu} \cdot N & \phi_{vv} \cdot N \end{pmatrix} = \frac{1}{\sqrt{1 + f'(u)^2}} \begin{pmatrix} -f''(u) & 0 \\ 0 & f(u) \end{pmatrix}$$

(seen similar, 8 marks)

(c) The Gaussian curvature is

$$K(u,v) = \frac{\det A}{\det g} = -\frac{f''(u)}{f(u)(1 + f'(u)^2)^2}$$

and the mean curvature is

$$H(u,v) = \frac{1}{2}\operatorname{tr} g^{-1}A = -\frac{f''(u)}{2(1+f'(u)^2)^{3/2}} + \frac{1}{2f(u)\sqrt{1+f'(u)^2}}$$

(seen similar, 5 marks)

(d) From (c) we see that $K(u,v) \equiv 0$ if and only if $f''(u)$ is identically zero. This implies that f is an affine-linear function, and since we also assumed that f is defined on all of $\mathbb{R}$ and is positive, this forces f to be constant. That is, S is a cylinder. (unseen, 3 marks)

(4) (a) Let λ_1 and λ_2 be the principal curvatures at p. Then $K(p) = \lambda_1\lambda_2$, $H(p) = (\lambda_1 + \lambda_2)/2$, and the statement that $H^2 \geq K$ is equivalent to the statement $(\lambda_1 - \lambda_2)^2 \geq 0$. (seen, 3 marks)

(b) (i) Any correct example: an elliptic point p, with S oriented correctly. (seen similar, 2 marks)

(ii) Any correct example: a saddle point p with $\lambda_1 > 0$, $\lambda_2 < 0$, and $|\lambda_1| > |\lambda_2|$. (seen similar, 2 marks)

(iii) This is impossible, because we would need $\lambda_1\lambda_2 > 0$, so the λ_i have the same sign, and also $\lambda_1 + \lambda_2 = 0$. (seen similar, 1 marks)

(c) In view of (a) it suffices to show that S has an elliptic point. Since S is compact there is a point $p \in S$ such that $|p|$ is maximal. We proved in class that there is a local parametrisation of S near p of the form

$$\phi(u,v) = p + \phi_u(0,0)u + \phi_v(0,0)v + \frac{1}{2}\big(\phi_{uu}(0,0)u^2 + 2\phi_{uv}(0,0)uv + \phi_{vv}v^2\big) + R(u,v)$$

where $R(u,v)/(u^2+v^2) \to 0$ as $(u,v) \to (0,0)$. The term $p + \phi_u(0,0)u + \phi_v(0,0)v$ here lies in T_pS. Since p maximises the distance to the origin on S we must have that

$$\big(\phi_{uu}(0,0)u^2 + 2\phi_{uv}(0,0)uv + \phi_{vv}v^2\big) \cdot N(p) < 0$$

for (u,v) sufficiently close to $(0,0)$, because $\phi(u,v)$ lies strictly on one side of $T_p(S)$; here N is the unit normal vector field to S that points outwards at p. But this is $A(w,w) < 0$, where A is the second fundamental form and $w = \phi_u(0,0)u + \phi_v(0,0)v$. Thus $\lambda_1 < 0$ and $\lambda_2 < 0$, where λ_1 and λ_2 are the principal curvatures at p. It follows that $K(p) > 0$, so p is an elliptic point. (seen, 7 marks)

(d) (i) γ is a geodesic iff the geodesic curvature $\gamma'' \cdot (N \times \gamma')$ vanishes. Here N is the unit normal vector field to S given by the orientation. (seen, 2 marks)

(ii) Choose a parametrisation of L of the form $\gamma(t) = a + tb$ where b is a unit vector. This is an arc-length parametrisation, and $\gamma''(t) \equiv 0$. Thus the geodesic curvature of γ is zero, and L is a geodesic. (unseen, 3 marks)

(5) (a) (i) A vector field on a regular surface $S \subset \mathbb{R}^3$ is a smooth vector-valued function $v\colon S \to \mathbb{R}^3$ such that $v(p) \in T_pS$ for all $p \in S$. (seen, 1 marks)

(ii) The index $I(v,p)$ of v at p is the winding number of v along a small circle in S about p, oriented anticlockwise. (seen, 1 marks)

(b) Poincaré–Hopf Theorem: Let S be a closed, oriented surface in $\mathbb{R}^3$. Let v be a vector field on S with distinct isolated zeroes $p_1, \ldots, p_n$. Let

$I(v,p)$ denote the index of v at p. Then

$$\sum_{i=1}^{n} I(v,p_i) = \chi(S)$$

Proof. Let D_i be a small disc around p_i, and write $S' = S \setminus \cup_{i=1}^{n} D_i$. The Gauss–Bonnet theorem gives

$$\int_{S'} K\,dA + \sum_{i=1}^{N} \int_{D_i} K\,dA = 2\pi\chi(S)$$

where K is the Gaussian curvature. Let N be the unit normal vector field to S given by the orientation. For $x \in S'$, there is an orthonormal basis for the tangent space $T_x(S)$ given by

$$F_1 = \frac{v(x)}{|v(x)|}, \qquad F_2 = N \times F_1$$

and arguing as in the proof of (local) Gauss–Bonnet gives

$$\int_{S'} K\,dA = -\sum_{i=1}^{n} \int_0^{L_i} F_1(t)\cdot F_2'(t)\,dt$$

where ∂D_i is parametrised by $t \in [0, L_i]$. The minus sign here comes from the fact that the orientation of the boundary circles $\partial S'$ is opposite to their orientation given by regarding them as ∂D_i, $i = 1, 2, \ldots, n$.

On the other hand, if $\phi\colon U \to D_i$ is a parametrisation of $D_i \subset S$ then we have an orthonormal basis for the tangent space T_xS, $x \in D_i$, given by

$$E_1 = \frac{\phi_u}{|\phi_u|}, \qquad E_2 = N \times E_1.$$

and

$$\int_{D_i} K\,dA = \int_0^{L_i} E_1(t)\cdot E_2'(t)\,dt.$$

Thus

$$2\pi\chi(S) = \sum_{i=1}^{N} \int_0^{L_i} E_1(t)\cdot E_2'(t) - F_1(t)\cdot F_1'(t)\,dt$$

Computing the geodesic curvature of ∂D_i in terms of our two bases gives

$$\kappa_g(t) = \theta'(t) - E_1(t)\cdot E_2'(t), \qquad \kappa_g(t) = \phi'(t) - F_1(t)\cdot F_2'(t)$$

where $\theta(t)$, respectively $\phi(t)$, measures the angle between $E_1(t)$, respectively $F_1(t)$, and the tangent vector to ∂D_i at t. Thus

$$\sum_{i=1}^{N} \int_0^{L_i} E_1(t)\cdot E_2'(t) - F_1(t)\cdot F_1'(t)\,dt = \sum_{i=1}^{N} \int_0^{L_i} \theta'(t) - \phi'(t)\,dt$$

is the sum of winding numbers

$$2\pi \sum_{i=1}^{n} I(v,p_i).$$

and therefore

$$\chi(S) = \sum_{i=1}^{N} I(v, p_i)$$

as claimed. (seen in independent study, 10 marks)

□

(c) (i) Any correct example, e.g. $v(x, y) = (-y, x)$.
(seen similar, 2 marks)

(ii) Any correct example, e.g. $v(x, y) = (x^2 - y^2, -2xy)$.
(seen similar, 2 marks)

(iii) Any correct example, e.g. the vector field generating rotation about the z-axis. (seen similar, 2 marks)

(iv) No such vector field exists, because the Euler characteristic of the unit sphere is 2 and so this would violate Poincaré–Hopf.
(seen similar, 2 marks)