3

Continuous-time Markov chains II

This chapter brings together the discrete-time and continuous-time theories, allowing us to deduce analogues, for continuous-time chains, of all the results given in Chapter 1. All the facts from Chapter 2 that are necessary to understand this synthesis are reviewed in Section 3.1. You will require a reasonable understanding of Chapter 1 here, but, given such an understanding, this chapter should look reassuringly familiar. Exercises remain an important part of the text.

3.1 Basic properties

Let I be a countable set. Recall that a Q-*matrix* on I is a matrix $Q = (q_{ij} : i, j \in I)$ satisfying the following conditions:

(i) $0 \leq -q_{ii} < \infty \quad \text{for all} \quad i;$

(ii) $q_{ij} \geq 0 \quad \text{for all} \quad i \neq j;$

(iii) $\displaystyle\sum_{j \in I} q_{ij} = 0 \quad \text{for all} \quad i.$

We set $q_i = q(i) = -q_{ii}$. Associated to any Q-matrix is a *jump matrix* $\Pi = (\pi_{ij} : i, j \in I)$ given by

$$\pi_{ij} = \begin{cases} q_{ij}/q_i & \text{if } j \neq i \text{ and } q_i \neq 0 \\ 0 & \text{if } j \neq i \text{ and } q_i = 0, \end{cases}$$

$$\pi_{ii} = \begin{cases} 0 & \text{if } q_i \neq 0 \\ 1 & \text{if } q_i = 0. \end{cases}$$

Note that Π is a stochastic matrix.

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A *sub-stochastic matrix* on I is a matrix $P = (p_{ij} : i, j \in I)$ with non-negative entries and such that

$$\sum_{j \in I} p_{ij} \leq 1 \quad \text{for all} \quad i.$$

Associated to any Q-matrix is a *semigroup* $(P(t) : t \geq 0)$ of sub-stochastic matrices $P(t) = (p_{ij}(t) : i, j \in I)$. As the name implies we have

$$P(s)P(t) = P(s+t) \quad \text{for all} \quad s, t \geq 0.$$

You will need to be familiar with the following terms introduced in Section 2.2: *minimal right-continuous random process, jump times, holding times, jump chain* and *explosion*. Briefly, a right-continuous process $(X_t)_{t \geq 0}$ runs through a sequence of states $Y_0, Y_1, Y_2, \ldots$, being held in these states for times $S_1, S_2, S_3, \ldots$ respectively and jumping to the next state at times $J_1, J_2, J_3, \ldots$. Thus $J_n = S_1 + \ldots + S_n$. The discrete-time process $(Y_n)_{n \geq 0}$ is the jump chain, $(S_n)_{n \geq 1}$ are the holding times and $(J_n)_{n \geq 1}$ are the jump times. The explosion time ζ is given by

$$\zeta = \sum_{n=1}^{\infty} S_n = \lim_{n \to \infty} J_n.$$

For a minimal process we take a new state ∞ and insist that $X_t = \infty$ for all $t \geq \zeta$. An important point is that a minimal right-continuous process is determined by its jump chain and holding times.

The data for a continuous-time Markov chain $(X_t)_{t \geq 0}$ are a distribution λ and a Q-matrix Q. The distribution λ gives the *initial distribution*, the distribution of X_0. The Q-matrix is known as the *generator matrix* of $(X_t)_{t \geq 0}$ and determines how the process evolves from its initial state. We established in Section 2.8 that there are two different, but equivalent, ways to describe how the process evolves.

The first, in terms of jump chain and holding times, states that

(a) $(Y_n)_{n \geq 0}$ is Markov(λ, Π);

(b) conditional on $Y_0 = i_0, \ldots, Y_{n-1} = i_{n-1}$, the holding times $S_1, \ldots, S_n$ are independent exponential random variables of parameters $q_{i_0}, \ldots, q_{i_{n-1}}$.

Put more simply, given that the chain starts at i, it waits there for an exponential time of parameter q_i and then jumps to a new state, choosing state j with probability π_{ij}. It then starts afresh, forgetting what has gone before.

The second description, in terms of the semigroup, states that the finite-dimensional distributions of the process are given by

(c) for all $n = 0, 1, 2, \dots$, all times $0 \le t_0 \le t_1 \le \dots \le t_{n+1}$ and all states $i_0, i_1, \dots, i_{n+1}$

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n) = p_{i_n i_{n+1}}(t_{n+1} - t_n).$$

Again, put more simply, given that the chain starts at i, by time t it is found in state j with probability $p_{ij}(t)$. It then starts afresh, forgetting what has gone before. In the case where

$$\widetilde{p}_{i\infty}(t) := 1 - \sum_{j \in I} p_{ij}(t) > 0$$

the chain is found at ∞ with probability $\widetilde{p}_{i\infty}(t)$. The semigroup $P(t)$ is referred to as the *transition matrix* of the chain and its entries $p_{ij}(t)$ are the *transition probabilities*. This description implies that for all $h > 0$ the discrete skeleton $(X_{nh})_{n \ge 0}$ is Markov$(\lambda, P(h))$. Strictly, in the explosive case, that is, when $P(t)$ is strictly sub-stochastic, we should say Markov$(\widetilde{\lambda}, \widetilde{P}(h))$, where $\widetilde{\lambda}$ and $\widetilde{P}(h)$ are defined on $I \cup \{\infty\}$, extending λ and $P(h)$ by $\widetilde{\lambda}_\infty = 0$ and $\widetilde{p}_{\infty j}(h) = 0$. But there is no danger of confusion in using the simpler notation.

The information coming from these two descriptions is sufficient for most of the analysis of continuous-time chains done in this chapter. Note that we have not yet said how the semigroup $P(t)$ is associated to the Q-matrix Q, except via the process! This extra information will be required when we discuss reversibility in Section 3.7. So we recall from Section 2.8 that the semigroup is characterized as the minimal non-negative solution of the *backward equation*

$$P'(t) = QP(t), \quad P(0) = I$$

which reads in components

$$p'_{ij}(t) = \sum_{k \in I} q_{ik} p_{kj}(t), \quad p_{ij}(0) = \delta_{ij}.$$

The semigroup is also the minimal non-negative solution of the *forward equation*

$$P'(t) = P(t)Q, \quad P(0) = I.$$

In the case where I is finite, $P(t)$ is simply the matrix exponential e^{tQ}, and is the *unique* solution of the backward and forward equations.

3.2 Class structure

A first step in the analysis of a continuous-time Markov chain $(X_t)_{t\geq 0}$ is to identify its class structure. We emphasise that we deal only with minimal chains, those that die after explosion. Then the class structure is simply the discrete-time class structure of the jump chain $(Y_n)_{n\geq 0}$, as discussed in Section 1.2.

We say that i *leads to* j and write $i \to j$ if

$$\mathbb{P}_i(X_t = j \text{ for some } t \geq 0) > 0.$$

We say i *communicates with* j and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$. The notions of *communicating class, closed class, absorbing state* and *irreducibility* are inherited from the jump chain.

Theorem 3.2.1. *For distinct states i and j the following are equivalent:*

(i) $i \to j$;
(ii) $i \to j$ *for the jump chain;*
(iii) $q_{i_0 i_1} q_{i_1 i_2} \dots q_{i_{n-1} i_n} > 0$ *for some states* $i_0, i_1, \dots, i_n$ *with* $i_0 = i$, $i_n = j$;
(iv) $p_{ij}(t) > 0$ *for all* $t > 0$;
(v) $p_{ij}(t) > 0$ *for some* $t > 0$.

Proof. Implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) are clear. If (ii) holds, then by Theorem 1.2.1, there are states $i_0, i_1, \dots, i_n$ with $i_0 = i$, $i_n = j$ and $\pi_{i_0 i_1} \pi_{i_1 i_2} \dots \pi_{i_{n-1} i_n} > 0$, which implies (iii). If $q_{ij} > 0$, then

$$p_{ij}(t) \geq \mathbb{P}_i(J_1 \leq t, Y_1 = j, S_2 > t) = (1 - e^{-q_i t})\pi_{ij} e^{-q_j t} > 0$$

for all $t > 0$, so if (iii) holds, then

$$p_{ij}(t) \geq p_{i_0 i_1}(t/n) \dots p_{i_{n-1} i_n}(t/n) > 0$$

for all $t > 0$, and (iv) holds. $\square$

Condition (iv) of Theorem 3.2.1 shows that the situation is simpler than in discrete-time, where it may be possible to reach a state, but only after a certain length of time, and then only periodically.

3.3 Hitting times and absorption probabilities

Let $(X_t)_{t\geq 0}$ be a Markov chain with generator matrix Q. The *hitting time* of a subset A of I is the random variable D^A defined by

$$D^A(\omega) = \inf\{t \geq 0 : X_t(\omega) \in A\}$$

with the usual convention that $\inf \emptyset = \infty$. We emphasise that $(X_t)_{t\ge 0}$ is minimal. So if H^A is the hitting time of A for the jump chain, then

$$\{H^A < \infty\} = \{D^A < \infty\}$$

and on this set we have

$$D^A = J_{H^A}.$$

The probability, starting from i, that $(X_t)_{t\ge 0}$ ever hits A is then

$$h_i^A = \mathbb{P}_i(D^A < \infty) = \mathbb{P}_i(H^A < \infty).$$

When A is a closed class, h_i^A is called the *absorption probability*. Since the hitting probabilities are those of the jump chain we can calculate them as in Section 1.3.

Theorem 3.3.1. *The vector of hitting probabilities $h^A = (h_i^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

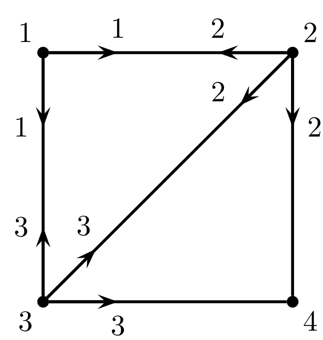
$$\begin{cases} h_i^A = 1 & \text{for } i \in A, \\ \sum_{j\in I} q_{ij} h_j^A = 0 & \text{for } i \notin A. \end{cases}$$

Proof. Apply Theorem 1.3.2 to the jump chain and rewrite (1.3) in terms of Q. $\square$

The average time taken, starting from i, for $(X_t)_{t\ge 0}$ to reach A is given by

$$k_i^A = \mathbb{E}_i(D^A).$$

In calculating k_i^A we have to take account of the holding times so the relationship to the discrete-time case is not quite as simple.



Example 3.3.2

Consider the Markov chain $(X_t)_{t\geq 0}$ with the diagram given on the preceding page. How long on average does it take to get from 1 to 4?

Set $k_i = \mathbb{E}_i(\text{time to get to } 4)$. On starting in 1 we spend an average time $q_1^{-1} = 1/2$ in 1, then jump with equal probability to 2 or 3. Thus

$$k_1 = \tfrac{1}{2} + \tfrac{1}{2}k_2 + \tfrac{1}{2}k_3$$

and similarly

$$k_2 = \tfrac{1}{6} + \tfrac{1}{3}k_1 + \tfrac{1}{3}k_3, \quad k_3 = \tfrac{1}{9} + \tfrac{1}{3}k_1 + \tfrac{1}{3}k_2.$$

On solving these linear equations we find $k_1 = 17/12$.

Here is the general result. The proof follows the same lines as Theorem 1.3.5.

Theorem 3.3.3. *Assume that $q_i > 0$ for all $i \notin A$. The vector of expected hitting times $k^A = (k_i^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ -\sum_{j\in I} q_{ij}k_j^A = 1 & \text{for } i \notin A. \end{cases} \tag{3.1}$$

Proof. First we show that k^A satisfies (3.1). If $X_0 = i \in A$, then $D^A = 0$, so $k_i^A = 0$. If $X_0 = i \notin A$, then $D^A \geq J_1$, so by the Markov property of the jump chain

$$\mathbb{E}_i(D^A - J_1 \mid Y_1 = j) = \mathbb{E}_j(D^A),$$

so

$$k_i^A = \mathbb{E}_i(D^A) = \mathbb{E}_i(J_1) + \sum_{j\neq i}\mathbb{E}(D^A - J_1 \mid Y_1 = j)\mathbb{P}_i(Y_1 = j) = q_i^{-1} + \sum_{j\neq i}\pi_{ij}k_j^A$$

and so

$$-\sum_{j\in I} q_{ij}k_j^A = 1.$$

Suppose now that $y = (y_i : i \in I)$ is another solution to (3.1). Then $k_i^A = y_i = 0$ for $i \in A$. Suppose $i \notin A$, then

$$\begin{aligned} y_i &= q_i^{-1} + \sum_{j\notin A}\pi_{ij}y_j = q_i^{-1} + \sum_{j\notin A}\pi_{ij}\left(q_j^{-1} + \sum_{k\notin A}\pi_{jk}y_k\right) \\ &= \mathbb{E}_i(S_1) + \mathbb{E}_i\big(S_2 1_{\{H^A\geq 2\}}\big) + \sum_{j\notin A}\sum_{k\notin A}\pi_{ij}\pi_{jk}y_k. \end{aligned}$$

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By repeated substitution for y in the final term we obtain after n steps

$$y_i = \mathbb{E}_i(S_1) + \cdots + \mathbb{E}_i\big(S_n 1_{\{H^A \geq n\}}\big) + \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} \pi_{ij_1} \ldots \pi_{j_{n-1}j_n} y_{j_n}.$$

So, if y is non-negative

$$y_i \geq \sum_{m=1}^{n} \mathbb{E}_i\big(S_m 1_{H^A \geq m}\big) = \mathbb{E}_i\Bigg(\sum_{m=1}^{H^A \wedge n} S_m\Bigg)$$

where we use the notation $H^A \wedge n$ for the minimum of H^A and n. Now

$$\sum_{m=1}^{H^A} S_m = D_A$$

so, by monotone convergence, $y_i \geq \mathbb{E}_i(D_A) = k_i^A$, as required. □

Exercise

3.3.1 Consider the Markov chain on $\{1, 2, 3, 4\}$ with generator matrix

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0. \end{pmatrix}$$

Calculate (a) the probability of hitting 3 starting from 1, (b) the expected time to hit 4 starting from 1.

3.4 Recurrence and transience

Let $(X_t)_{t\geq 0}$ be Markov chain with generator matrix Q. Recall that we insist $(X_t)_{t\geq 0}$ be minimal. We say a state i is *recurrent* if

$$\mathbb{P}_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 1.$$

We say that i is *transient* if

$$\mathbb{P}_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 0.$$

Note that if $(X_t)_{t\geq 0}$ can explode starting from i then i is certainly not recurrent. The next result shows that, like class structure, recurrence and transience are determined by the jump chain.

Theorem 3.4.1. *We have:*

(i) *if i is recurrent for the jump chain $(Y_n)_{n\geq 0}$, then i is recurrent for $(X_t)_{t\geq 0}$;*

(ii) *if i is transient for the jump chain, then i is transient for $(X_t)_{t\geq 0}$;*

(iii) *every state is either recurrent or transient;*

(iv) *recurrence and transience are class properties.*

Proof. (i) Suppose i is recurrent for $(Y_n)_{n\geq 0}$. If $X_0 = i$ then $(X_t)_{t\geq 0}$ does not explode and $J_n \to \infty$ by Theorem 2.7.1. Also $X(J_n) = Y_n = i$ infinitely often, so $\{t \geq 0 : X_t = i\}$ is unbounded, with probability 1.

(ii) Suppose i is transient for $(Y_n)_{n\geq 0}$. If $X_0 = i$ then

$$N = \sup\{n \geq 0 : Y_n = i\} < \infty,$$

so $\{t \geq 0 : X_t = i\}$ is bounded by $J(N+1)$, which is finite, with probability 1, because $(Y_n : n \leq N)$ cannot include an absorbing state.

(iii) Apply Theorem 1.5.3 to the jump chain.

(iv) Apply Theorem 1.5.4 to the jump chain. □

The next result gives continuous-time analogues of the conditions for recurrence and transience found in Theorem 1.5.3. We denote by T_i the *first passage time* of $(X_t)_{t\geq 0}$ to state i, defined by

$$T_i(\omega) = \inf\{t \geq J_1(\omega) : X_t(\omega) = i\}.$$

Theorem 3.4.2. *The following dichotomy holds:*

(i) *if $q_i = 0$ or $\mathbb{P}_i(T_i < \infty) = 1$, then i is recurrent and $\int_0^\infty p_{ii}(t)dt = \infty$;*

(ii) *if $q_i > 0$ and $\mathbb{P}_i(T_i < \infty) < 1$, then i is transient and $\int_0^\infty p_{ii}(t)dt < \infty$.*

Proof. If $q_i = 0$, then $(X_t)_{t\geq 0}$ cannot leave i, so i is recurrent, $p_{ii}(t) = 1$ for all t, and $\int_0^\infty p_{ii}(t)dt = \infty$. Suppose then that $q_i > 0$. Let N_i denote the first passage time of the jump chain $(Y_n)_{n\geq 0}$ to state i. Then

$$\mathbb{P}_i(N_i < \infty) = \mathbb{P}_i(T_i < \infty)$$

so i is recurrent if and only if $\mathbb{P}_i(T_i < \infty) = 1$, by Theorem 3.4.1 and the corresponding result for the jump chain.

Write $\pi_{ij}^{(n)}$ for the (i,j) entry in Π^n. We shall show that

$$\int_0^\infty p_{ii}(t)dt = \frac{1}{q_i}\sum_{n=0}^\infty \pi_{ii}^{(n)} \tag{3.2}$$

so that i is recurrent if and only if $\int_0^\infty p_{ii}(t)dt = \infty$, by Theorem 3.4.1 and the corresponding result for the jump chain.

To establish (3.2) we use Fubini's theorem (see Section 6.4):

$$\begin{aligned}\int_0^\infty p_{ii}(t)dt &= \int_0^\infty \mathbb{E}_i(1_{\{X_t=i\}})dt = \mathbb{E}_i \int_0^\infty 1_{\{X_t=i\}}dt \\ &= \mathbb{E}_i \sum_{n=0}^\infty S_{n+1} 1_{\{Y_n=i\}} \\ &= \sum_{n=0}^\infty \mathbb{E}_i(S_{n+1} \mid Y_n = i)\mathbb{P}_i(Y_n = i) = \frac{1}{q_i} \sum_{n=0}^\infty \pi_{ii}^{(n)}. \qquad \square\end{aligned}$$

Finally, we show that recurrence and transience are determined by any discrete-time sampling of $(X_t)_{t\geq 0}$.

Theorem 3.4.3. *Let $h > 0$ be given and set $Z_n = X_{nh}$.*

(i) *If i is recurrent for $(X_t)_{t\geq 0}$ then i is recurrent for $(Z_n)_{n\geq 0}$.*

(ii) *If i is transient for $(X_t)_{t\geq 0}$ then i is transient for $(Z_n)_{n\geq 0}$.*

Proof. Claim (ii) is obvious. To prove (i) we use for $nh \leq t < (n+1)h$ the estimate

$$p_{ii}\big((n+1)h\big) \geq e^{-q_i h} p_{ii}(t)$$

which follows from the Markov property. Then, by monotone convergence

$$\int_0^\infty p_{ii}(t)dt \leq he^{q_i h} \sum_{n=1}^\infty p_{ii}(nh)$$

and the result follows by Theorems 1.5.3 and 3.4.2. $\square$

Exercise

3.4.1 Customers arrive at a certain queue in a Poisson process of rate λ. The single 'server' has two states A and B, state A signifying that he is 'in attendance' and state B that he is having a tea-break. Independently of how many customers are in the queue, he fluctuates between these states as a Markov chain Y on $\{A, B\}$ with Q-matrix

$$\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The total service time for any customer is exponentially distributed with parameter μ and is independent of the chain Y and of the service times of other customers.

Describe the system as a Markov chain X with state-space

$$\{A_0, A_1, A_2, \dots\} \cup \{B_0, B_1, B_2, \dots\},$$

A_n signifying that the server is in state A and there are n people in the queue (including anyone being served) and B_n signifying that the server is in state B and there are n people in the queue.

Explain why, for some θ in $(0,1]$, and $k = 0, 1, 2, \dots$,

$$P(X \text{ hits } A_0 | X_0 = A_k) = \theta^k.$$

Show that $(\theta - 1)f(\theta) = 0$, where

$$f(\theta) = \lambda^2\theta^2 - \lambda(\lambda + \mu + \alpha + \beta)\theta + (\lambda + \beta)\mu.$$

By considering $f(1)$ or otherwise, prove that X is transient if $\mu\beta < \lambda(\alpha+\beta)$, and explain why this is intuitively obvious.

3.5 Invariant distributions

Just as in the discrete-time theory, the notions of invariant distribution and measure play an important role in the study of continuous-time Markov chains. We say that λ is *invariant* if

$$\lambda Q = 0.$$

Theorem 3.5.1. *Let Q be a Q-matrix with jump matrix Π and let λ be a measure. The following are equivalent:*

(i) *λ is invariant;*
(ii) *$\mu\Pi = \mu$ where $\mu_i = \lambda_i q_i$.*

Proof. We have $q_i(\pi_{ij} - \delta_{ij}) = q_{ij}$ for all i, j, so

$$(\mu(\Pi - I))_j = \sum_{i \in I} \mu_i(\pi_{ij} - \delta_{ij}) = \sum_{i \in I} \lambda_i q_{ij} = (\lambda Q)_j. \qquad \square$$

This tie-up with measures invariant for the jump matrix means that we can use the existence and uniqueness results of Section 1.7 to obtain the following result.

Theorem 3.5.2. *Suppose that Q is irreducible and recurrent. Then Q has an invariant measure λ which is unique up to scalar multiples.*

Proof. Let us exclude the trivial case $I = \{i\}$; then irreducibility forces $q_i > 0$ for all i. By Theorems 3.2.1 and 3.4.1, Π is irreducible and recurrent. Then, by Theorems 1.7.5 and 1.7.6, Π has an invariant measure μ, which is unique up to scalar multiples. So, by Theorem 3.5.1, we can take $\lambda_i = \mu_i/q_i$ to obtain an invariant measure unique up to scalar multiples. $\square$

Recall that a state i is recurrent if $q_i = 0$ or $\mathbb{P}_i(T_i < \infty) = 1$. If $q_i = 0$ or the *expected return time* $m_i = \mathbb{E}_i(T_i)$ is finite then we say i is *positive recurrent*. Otherwise a recurrent state i is called *null recurrent*. As in the discrete-time case positive recurrence is tied up with the existence of an invariant distribution.

Theorem 3.5.3. *Let Q be an irreducible Q-matrix. Then the following are equivalent:*

(i) *every state is positive recurrent;*
(ii) *some state i is positive recurrent;*
(iii) *Q is non-explosive and has an invariant distribution λ.*

Moreover, when (iii) holds we have $m_i = 1/(\lambda_i q_i)$ for all i.

Proof. Let us exclude the trivial case $I = \{i\}$; then irreducibility forces $q_i > 0$ for all i. It is obvious that (i) implies (ii). Define $\mu^i = (\mu^i_j : j \in I)$ by

$$\mu^i_j = \mathbb{E}_i \int_0^{T_i \wedge \zeta} 1_{\{X_s = j\}} ds,$$

where $T_i \wedge \zeta$ denotes the minimum of T_i and ζ. By monotone convergence,

$$\sum_{j \in I} \mu^i_j = \mathbb{E}_i(T_i \wedge \zeta).$$

Denote by N_i the first passage time of the jump chain to state i. By Fubini's theorem

$$\begin{aligned}
\mu^i_j &= \mathbb{E}_i \sum_{n=0}^{\infty} S_{n+1} 1_{\{Y_n = j, n < N_i\}} \\
&= \sum_{n=0}^{\infty} \mathbb{E}_i\big(S_{n+1} \mid Y_n = j\big)\mathbb{E}_i\big(1_{\{Y_n = j, n < N_i\}}\big) \\
&= q_j^{-1}\, \mathbb{E}_i \sum_{n=0}^{\infty} 1_{\{Y_n = j, n < N_i\}} \\
&= q_j^{-1}\, \mathbb{E}_i \sum_{n=0}^{N_i - 1} 1_{\{Y_n = j\}} = \gamma^i_j / q_j
\end{aligned}$$

where, in the notation of Section 1.7, γ_j^i is the expected time in j between visits to i for the jump chain.

Suppose (ii) holds, then i is certainly recurrent, so the jump chain is recurrent, and Q is non-explosive, by Theorem 2.7.1. We know that $\gamma^i \Pi = \gamma^i$ by Theorem 1.7.5, so $\mu^i Q = 0$ by Theorem 3.5.1. But μ^i has finite total mass

$$\sum_{j \in I} \mu_j^i = \mathbb{E}_i(T_i) = m_i$$

so we obtain an invariant distribution λ by setting $\lambda_j = \mu_j^i / m_i$.

On the other hand, suppose (iii) holds. Fix $i \in I$ and set $\nu_j = \lambda_j q_j / (\lambda_i q_i)$; then $\nu_i = 1$ and $\nu \Pi = \nu$ by Theorem 3.5.1, so $\nu_j \geq \gamma_j^i$ for all j by Theorem 1.7.6. So

$$\begin{aligned} m_i = \sum_{j \in I} \mu_j^i &= \sum_{j \in I} \gamma_j^i / q_j \leq \sum_{j \in I} \nu_j / q_j \\ &= \sum_{j \in I} \lambda_j / (\lambda_i q_i) = 1/(\lambda_i q_i) < \infty \end{aligned}$$

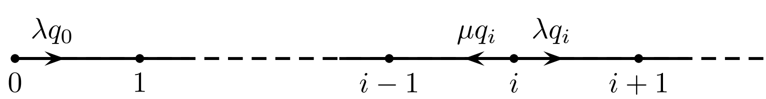
showing that i is positive recurrent.

To complete the proof we return to the preceding calculation armed with the knowledge that Q is recurrent, hence Π is recurrent, $\nu_j = \gamma_j^i$ and $m_i = 1/(\lambda_i q_i)$ for all i. $\square$

The following example is a caution that the existence of an invariant distribution for a continuous-time Markov chain is not enough to guarantee positive recurrence, or even recurrence.

Example 3.5.4

Consider the Markov chain $(X_t)_{t \geq 0}$ on $\mathbb{Z}^+$ with the following diagram, where $q_i > 0$ for all i and where $0 < \lambda = 1 - \mu < 1$:



The jump chain behaves as a simple random walk away from 0, so $(X_t)_{t \geq 0}$ is recurrent if $\lambda \leq \mu$ and transient if $\lambda > \mu$. To compute an invariant measure ν it is convenient to use the *detailed balance equations*

$$\nu_i q_{ij} = \nu_j q_{ji} \qquad \text{for all } i, j.$$

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Look ahead to Lemma 3.7.2 to see that any solution is invariant. In this case the non-zero equations read

$$\nu_i \lambda q_i = \nu_{i+1} \mu q_{i+1} \qquad \text{for all } i.$$

So a solution is given by $\nu_i = q_i^{-1}(\lambda/\mu)^i$. If the jump rates q_i are constant then ν can be normalized to produce an invariant distribution precisely when $\lambda < \mu$.

Consider, on the other hand, the case where $q_i = 2^i$ for all i and $1 < \lambda/\mu < 2$. Then ν has finite total mass so $(X_t)_{t\geq 0}$ has an invariant distribution, but $(X_t)_{t\geq 0}$ is also transient. Given Theorem 3.5.3, the only possibility is that $(X_t)_{t\geq 0}$ is explosive.

The next result justifies calling measures λ with $\lambda Q = 0$ invariant.

Theorem 3.5.5. *Let Q be irreducible and recurrent, and let λ be a measure. Let $s > 0$ be given. The following are equivalent:*

(i) $\lambda Q = 0$;
(ii) $\lambda P(s) = \lambda$.

Proof. There is a very simple proof in the case of finite state-space: by the backward equation

$$\frac{d}{ds}\lambda P(s) = \lambda P'(s) = \lambda Q P(s)$$

so $\lambda Q = 0$ implies $\lambda P(s) = \lambda P(0) = \lambda$ for all s; $P(s)$ is also recurrent, so $\mu P(s) = \mu$ implies that μ is proportional to λ, so $\mu Q = 0$.

For infinite state-space, the interchange of differentiation with the summation involved in multiplication by λ is not justified and an entirely different proof is needed.

Since Q is recurrent, it is non-explosive by Theorem 2.7.1, and $P(s)$ is recurrent by Theorem 3.4.3. Hence any λ satisfying (i) or (ii) is unique up to scalar multiples; and from the proof of Theorem 3.5.3, if we fix i and set

$$\mu_j = \mathbb{E}_i \int_0^{T_i} 1_{\{X_t = j\}} dt,$$

then $\mu Q = 0$. Thus it suffices to show $\mu P(s) = \mu$. By the strong Markov property at T_i (which is a simple consequence of the strong Markov property of the jump chain)

$$\mathbb{E}_i \int_0^s 1_{\{X_t = j\}} dt = \mathbb{E}_i \int_{T_i}^{T_i + s} 1_{\{X_t = j\}} dt.$$

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Hence, using Fubini's theorem,

$$\begin{aligned}
\mu_j &= \mathbb{E}_i \int_s^{s+T_i} 1_{\{X_t=j\}} dt \\
&= \int_0^\infty \mathbb{P}_i(X_{s+t}=j, t<T_i) dt \\
&= \int_0^\infty \sum_{k\in I} \mathbb{P}_i(X_t=k, t<T_i) p_{kj}(s) dt \\
&= \sum_{k\in I} \left(\mathbb{E}_i \int_0^{T_i} 1_{\{X_t=k\}} dt \right) p_{kj}(s) \\
&= \sum_{k\in I} \mu_k p_{kj}(s)
\end{aligned}$$

as required. $\square$

Theorem 3.5.6. *Let Q be an irreducible non-explosive Q-matrix having an invariant distribution λ. If $(X_t)_{t\geq 0}$ is Markov(λ, Q) then so is $(X_{s+t})_{t\geq 0}$ for any $s \geq 0$.*

Proof. By Theorem 3.5.5, for all i,

$$\mathbb{P}(X_s = i) = (\lambda P(s))_i = \lambda_i$$

so, by the Markov property, conditional on $X_s = i$, $(X_{s+t})_{t\geq 0}$ is Markov(δ_i, Q). $\square$

3.6 Convergence to equilibrium

We now investigate the limiting behaviour of $p_{ij}(t)$ as $t \to \infty$ and its relation to invariant distributions. You will see that the situation is analogous to the case of discrete-time, only there is no longer any possibility of periodicity.

We shall need the following estimate of uniform continuity for the transition probabilities.

Lemma 3.6.1. *Let Q be a Q-matrix with semigroup $P(t)$. Then, for all $t, h \geq 0$*

$$|p_{ij}(t+h) - p_{ij}(t)| \leq 1 - e^{-q_i h}.$$

Proof. We have

$$\begin{aligned}
|p_{ij}(t+h) - p_{ij}(t)| &= \left| \sum_{k\in I} p_{ik}(h) p_{kj}(t) - p_{ij}(t) \right| \\
&= \left| \sum_{k\neq i} p_{ik}(h) p_{kj}(t) - (1-p_{ii}(h)) p_{ij}(t) \right| \\
&\leq 1 - p_{ii}(h) \leq \mathbb{P}_i(J_1 \leq h) = 1 - e^{-q_i h}.
\end{aligned}$$

$\square$

Theorem 3.6.2 (Convergence to equilibrium). *Let Q be an irreducible non-explosive Q-matrix with semigroup $P(t)$, and having an invariant distribution λ. Then for all states i, j we have*

$$p_{ij}(t) \to \lambda_j \quad \text{as} \quad t \to \infty.$$

Proof. Let $(X_t)_{t\geq 0}$ be Markov(δ_i, Q). Fix $h > 0$ and consider the *h-skeleton* $Z_n = X_{nh}$. By Theorem 2.8.4

$$\mathbb{P}\big(Z_{n+1} = i_{n+1} \mid Z_0 = i_0, \dots, Z_n = i_n\big) = p_{i_n i_{n+1}}(h)$$

so $(Z_n)_{n\geq 0}$ is discrete-time Markov$(\delta_i, P(h))$. By Theorem 3.2.1 irreducibility implies $p_{ij}(h) > 0$ for all i, j so $P(h)$ is irreducible and aperiodic. By Theorem 3.5.5, λ is invariant for $P(h)$. So, by discrete-time convergence to equilibrium, for all i, j

$$p_{ij}(nh) \to \lambda_j \quad \text{as} \quad n \to \infty.$$

Thus we have a lattice of points along which the desired limit holds; we fill in the gaps using uniform continuity. Fix a state i. Given $\varepsilon > 0$ we can find $h > 0$ so that

$$1 - e^{-q_i s} \leq \varepsilon/2 \qquad \text{for } 0 \leq s \leq h$$

and then find N, so that

$$|p_{ij}(nh) - \lambda_j| \leq \varepsilon/2 \qquad \text{for } n \geq N.$$

For $t \geq Nh$ we have $nh \leq t < (n+1)h$ for some $n \geq N$ and

$$|p_{ij}(t) - \lambda_j| \leq |p_{ij}(t) - p_{ij}(nh)| + |p_{ij}(nh) - \lambda_j| \leq \varepsilon$$

by Lemma 3.6.1. Hence

$$p_{ij}(t) \to \lambda_j \quad \text{as} \quad n \to \infty. \qquad \square$$

The complete description of limiting behaviour for irreducible chains in continuous time is provided by the following result. It follows from Theorem 1.8.5 by the same argument we used in the preceding result. We do not give the details.

Theorem 3.6.3. *Let Q be an irreducible Q-matrix and let ν be any distribution. Suppose that $(X_t)_{t\geq 0}$ is Markov(ν, Q). Then*

$$\mathbb{P}(X_t = j) \to 1/(q_j m_j) \qquad \text{as } t \to \infty \quad \text{for all } j \in I$$

where m_j is the expected return time to state j.

Exercises

3.6.1 Find an invariant distribution λ for the Q-matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

and verify that $\lim_{t\to\infty} p_{11}(t) = \lambda_1$ using your answer to Exercise 2.1.1.

3.6.2 In each of the following cases, compute $\lim_{t\to\infty} \mathbb{P}(X_t = 2 | X_0 = 1)$ for the Markov chain $(X_t)_{t\geq 0}$ with the given Q-matrix on $\{1, 2, 3, 4\}$:

$$\text{(a)} \quad \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{(c)} \quad \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \qquad \text{(d)} \quad \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.6.3 Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process. Write down the generator matrix Q of a continuous-time Markov chain which models this, explaining what the states of the chain represent. Calculate the essentially unique invariant measure for Q, and deduce that the chain is positive recurrent if and only if $\lambda/\mu < 1/2$.

3.7 Time reversal

Time reversal of continuous-time chains has the same features found in the discrete-time case. Reversibility provides a powerful tool in the analysis of Markov chains, as we shall see in Section 5.2. Note in the following result how time reversal interchanges the roles of backward and forward equations. This echoes our proof of the forward equation, which rested on the time reversal identity of Lemma 2.8.5.

A small technical point arises in time reversal: right-continuous processes become left-continuous processes. For the processes we consider, this is unimportant. We could if we wished redefine the time-reversed process to equal its right limit at the jump times, thus obtaining again a right-continuous process. We shall suppose implicitly that this is done, and forget about the problem.

Theorem 3.7.1. *Let Q be irreducible and non-explosive and suppose that Q has an invariant distribution λ. Let $T \in (0,\infty)$ be given and let $(X_t)_{0\le t\le T}$ be Markov(λ, Q). Set $\widehat{X}_t = X_{T-t}$. Then the process $(\widehat{X}_t)_{0\le t\le T}$ is Markov$(\lambda, \widehat{Q})$, where $\widehat{Q} = (\widehat{q}_{ij} : i,j \in I)$ is given by $\lambda_j \widehat{q}_{ji} = \lambda_i q_{ij}$. Moreover, $\widehat{Q}$ is also irreducible and non-explosive with invariant distribution λ.*

Proof. By Theorem 2.8.6, the semigroup $\big(P(t) : t \ge 0\big)$ of Q is the minimal non-negative solution of the forward equation

$$P'(t) = P(t)Q, \quad P(0) = I.$$

Also, for all $t > 0$, $P(t)$ is an irreducible stochastic matrix with invariant distribution λ. Define $\widehat{P}(t)$ by

$$\lambda_j \widehat{p}_{ji}(t) = \lambda_i p_{ij}(t),$$

then $\widehat{P}(t)$ is an irreducible stochastic matrix with invariant distribution λ, and we can rewrite the forward equation transposed as

$$\widehat{P}'(t) = \widehat{Q}\widehat{P}(t).$$

But this is the backward equation for $\widehat{Q}$, which is itself a Q-matrix, and $\widehat{P}(t)$ is then its minimal non-negative solution. Hence $\widehat{Q}$ is irreducible and non-explosive and has invariant distribution λ.

Finally, for $0 = t_0 < \ldots < t_n = T$ and $s_k = t_k - t_{k-1}$, by Theorem 2.8.4 we have

$$\begin{aligned}
\mathbb{P}(\widehat{X}_{t_0} = i_0, \ldots, \widehat{X}_{t_n} = i_n) &= \mathbb{P}(X_{T-t_0} = i_0, \ldots, X_{T-t_n} = i_n) \\
&= \lambda_{i_n} p_{i_n i_{n-1}}(s_n) \ldots p_{i_1 i_0}(s_1) \\
&= \lambda_{i_0} \widehat{p}_{i_0 i_1}(s_1) \ldots \widehat{p}_{i_{n-1} i_n}(s_n)
\end{aligned}$$

so, by Theorem 2.8.4 again, $(\widehat{X}_t)_{0\le t\le T}$ is Markov$(\lambda, \widehat{Q})$. $\square$

The chain $(\widehat{X}_t)_{0\le t\le T}$ is called the *time-reversal* of $(X_t)_{0\le t\le T}$.

A Q-matrix Q and a measure λ are said to be in *detailed balance* if

$$\lambda_i q_{ij} = \lambda_j q_{ji} \qquad \text{for all } i, j.$$

Lemma 3.7.2. *If Q and λ are in detailed balance then λ is invariant for Q.*

Proof. We have $(\lambda Q)_i = \sum_{j\in I} \lambda_j q_{ji} = \sum_{j\in I} \lambda_i q_{ij} = 0$. $\square$

Let $(X_t)_{t\geq 0}$ be Markov(λ, Q), with Q irreducible and non-explosive. We say that $(X_t)_{t\geq 0}$ is *reversible* if, for all $T > 0$, $(X_{T-t})_{0\leq t\leq T}$ is also Markov(λ, Q).

Theorem 3.7.3. *Let Q be an irreducible and non-explosive Q-matrix and let λ be a distribution. Suppose that $(X_t)_{t\geq 0}$ is Markov(λ, Q). Then the following are equivalent:*

(a) *$(X_t)_{t\geq 0}$ is reversible;*

(b) *Q and λ are in detailed balance.*

Proof. Both (a) and (b) imply that λ is invariant for Q. Then both (a) and (b) are equivalent to the statement that $\widehat{Q} = Q$ in Theorem 3.7.1. $\square$

Exercise

3.7.1 Consider a fleet of N buses. Each bus breaks down independently at rate μ, when it is sent to the depot for repair. The repair shop can only repair one bus at a time and each bus takes an exponential time of parameter λ to repair. Find the equilibrium distribution of the number of buses in service.

3.7.2 Calls arrive at a telephone exchange as a Poisson process of rate λ, and the lengths of calls are independent exponential random variables of parameter μ. Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process.

Show that for large t the distribution of the number of lines in use at time t is approximately Poisson with mean λ/μ.

Find the mean length of the busy periods during which at least one line is in use.

Show that the expected number of lines in use at time t, given that n are in use at time 0, is $ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$.

Show that, in equilibrium, the number N_t of calls finishing in the time interval $[0, t]$ has Poisson distribution of mean λt.

Is $(N_t)_{t\geq 0}$ a Poisson process?

3.8 Ergodic theorem

Long-run averages for continuous-time chains display the same sort of behaviour as in the discrete-time case, and for similar reasons. Here is the result.

Theorem 3.8.1 (Ergodic theorem). *Let Q be irreducible and let ν be any distribution. If $(X_t)_{t\geq 0}$ is Markov(ν, Q), then*

$$\mathbb{P}\left(\frac{1}{t}\int_0^t 1_{\{X_s=i\}}ds \to \frac{1}{m_i q_i} \text{ as } t\to\infty\right) = 1$$

where $m_i = \mathbb{E}_i(T_i)$ is the expected return time to state i. Moreover, in the positive recurrent case, for any bounded function $f : I \to \mathbb{R}$ we have

$$\mathbb{P}\left(\frac{1}{t}\int_0^t f(X_s)ds \to \overline{f} \text{ as } t\to\infty\right) = 1$$

where

$$\overline{f} = \sum_{i\in I}\lambda_i f_i$$

and where $(\lambda_i : i \in I)$ is the unique invariant distribution.

Proof. If Q is transient then the total time spent in any state i is finite, so

$$\frac{1}{t}\int_0^t 1_{\{X_s=i\}}ds \leq \frac{1}{t}\int_0^\infty 1_{\{X_s=i\}}ds \to 0 = \frac{1}{m_i}.$$

Suppose then that Q is recurrent and fix a state i. Then $(X_t)_{t\geq 0}$ hits i with probability 1 and the long-run proportion of time in i equals the long-run proportion of time in i after first hitting i. So, by the strong Markov property (of the jump chain), it suffices to consider the case $\nu = \delta_i$.

Denote by M_i^n the length of the nth visit to i, by T_i^n the time of the nth return to i and by L_i^n the length of the nth excursion to i. Thus for $n = 0, 1, 2, \dots$, setting $T_i^0 = 0$, we have

$$\begin{aligned}
M_i^{n+1} &= \inf\{t > T_i^n : X_t \neq i\} - T_i^n \\
T_i^{n+1} &= \inf\{t > T_i^n + M_i^{n+1} : X_t = i\} \\
L_i^{n+1} &= T_i^{n+1} - T_i^n.
\end{aligned}$$

By the strong Markov property (of the jump chain) at the stopping times T_i^n for $n \geq 0$ we find that $L_i^1, L_i^2, \dots$ are independent and identically distributed with mean m_i, and that $M_i^1, M_i^2, \dots$ are independent and identically distributed with mean $1/q_i$. Hence, by the strong law of large numbers (see Theorem 1.10.1)

$$\begin{aligned}
\frac{L_i^1 + \cdots + L_i^n}{n} &\to m_i \quad \text{as } n\to\infty \\
\frac{M_i^1 + \cdots + M_i^n}{n} &\to \frac{1}{q_i} \quad \text{as } n\to\infty
\end{aligned}$$

and hence

$$\frac{M_i^1 + \cdots + M_i^n}{L_i^1 + \cdots + L_i^n} \to \frac{1}{m_i q_i} \quad \text{as } n \to \infty$$

with probability 1. In particular, we note that $T_i^n / T_i^{n+1} \to 1$ as $n \to \infty$ with probability 1. Now, for $T_i^n \le t < T_i^{n+1}$ we have

$$\frac{T_i^n}{T_i^{n+1}} \frac{M_i^1 + \cdots + M_i^n}{L_i^1 + \cdots + L_i^n} \le \frac{1}{t} \int_0^t 1_{\{X_s = i\}} ds \le \frac{T_i^{n+1}}{T_i^n} \frac{M_i^1 + \cdots + M_i^{n+1}}{L_i^1 + \cdots + L_i^{n+1}}$$

so on letting $t \to \infty$ we have, with probability 1

$$\frac{1}{t} \int_0^t 1_{\{X_s = i\}} ds \to \frac{1}{m_i q_i}.$$

In the positive recurrent case we can write

$$\frac{1}{t} \int_0^t f(X_s) ds - \overline{f} = \sum_{i \in I} f_i \left(\frac{1}{t} \int_0^t 1_{\{X_s = i\}} ds - \lambda_i \right)$$

where $\lambda_i = 1/(m_i q_i)$. We conclude that

$$\frac{1}{t} \int_0^t f(X_s) ds \to \overline{f} \quad \text{as } t \to \infty$$

with probability 1, by the same argument as was used in the proof of Theorem 1.10.2. $\square$