Imperial College London

Mathematics Pre-arrival course

### Problem Sheet 1 – Language of Mathematics and Real Functions

The starred questions on this problem sheet are for you to think about — we will not be giving solutions to them in the pre-arrival course. Instead these will form the basis of discussion in your first *MATH40001/MATH40009 - Introduction to University Mathematics* session once you arrive at Imperial.

Language of Mathematics

1. Rewrite the following statements formally with quantifiers (don't forget to refer to the mathematical notation handout).

   (a) If x and y are real numbers and y is strictly positive, $x + y$ is always bigger than x.

   (b) Every real number has a complex square root.

   (c) The average of two positive integers is positive.

   (d) The difference of two negative integers is not necessarily negative.

2. Let S be the set of all people living in London, and A a function that associates to every member of S their age.

   (a) Write the function formally.

   (b) Rewrite with quantifiers the following statement: in London, everybody is older than somebody.

3. ★ Let n be an integer. Prove (carefully) that

   (a) if 2 divides n, then 2 divides n^2.

   (b) if 2 divides n^2, then 2 divides n.

4. Show that $\sqrt{2} \notin \mathbb{Q}$.

5. Prove by induction that 3 divides $n^3 - n$ for all integers $n \geq 0$.

Real Functions

6. The following figure shows the graph of a function $f(x)$, its derivative $f'(x)$ and the definite integral $F(x) = \int_0^x f(t)dt$. Can you identify each graph? Explain your reasoning.
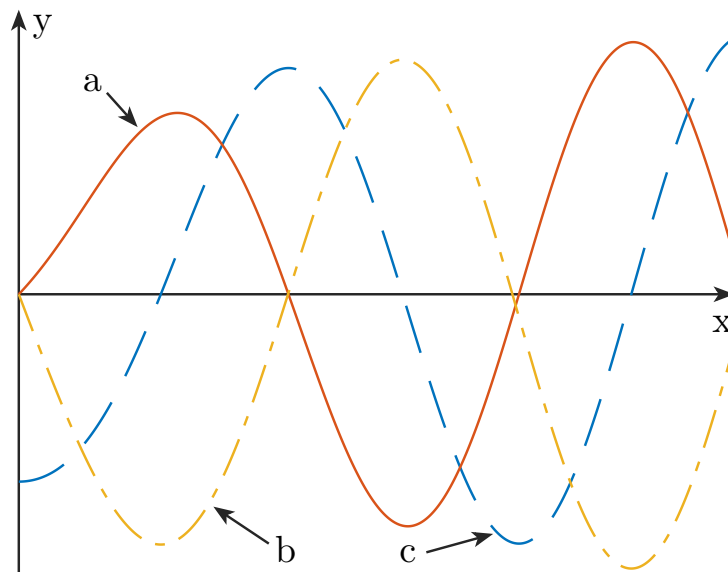
7. Try to sketch by hand the following functions:

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^3 - x^2$$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto e^{-x^2}$$

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$
$$x \mapsto \sin\left(\frac{1}{x}\right)$$

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$
$$x \mapsto x \sin\left(\frac{1}{x}\right)$$

8. Can you think of a **continuous function which is not differentiable**?

9. ★ Can you think of a function which is **discontinuous everywhere**?

10. ★ What about one which is **differentiable everywhere, but with a discontinuous derivative**?

Differential Equations

11. Find the family of solutions to $\frac{dy}{dx} = 2$, and draw them on the plane.

12. Find the solution to the initial value problem:

$$\frac{dy}{dx} = y, \ y(0) = 2$$

13. Check that $y(x) = a\sin(x + \lambda) + b\cos(x + \mu)$ is a solution to:

$$\frac{d^2y}{dx^2} = -y$$

14. ★ Can an initial value problem have more than one solution?

15. ★ Consider a box containing radioactive atoms, at any time t, we denote $x(t)$ the number of radioactive atoms remaining in the box. With time, these atoms decay; as each atom has the same chance to decay, the rate of change of atom number is proportional to the number of atoms remaining. Can you write an ordinary differential equation governing the number of atoms in the box? Find a solution to this problem, knowing that there was initially x_0 atoms in the box.