MATH50004/MATH50015/MATH50019 Differential Equations
Spring Term 2023/24
Solutions to Quiz 2

Question 1. Correct answer: (a).
Due to the chain rule, we have $f'(x) = \frac{x}{1+x^4}$, which is bounded on $\mathbb{R}$ and Lipschitz continuity follows with arguments explained as in Example 2.6.

Question 2. Correct answer: (c).
Note that

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3}x + \sqrt{2}y \\ -\sqrt{2}x + \sqrt{3}y \end{pmatrix},$$

which implies that

$$\|A(x,y)^\top\| = \sqrt{3x^2 + 2y^2 + \sqrt{6}xy + 2x^2 + 3y^2 - \sqrt{6}xy} = \sqrt{5}\|(x,y)^\top\|.$$

Hence, the operator norm satisfies $\|A\| = \sqrt{5}$.

Question 3. Correct answer: (a).
Consider $A = B = \mathrm{Id}_{2\times 2}$. Then $AB = A = B$ and the statement follows from $\|A\| = \|B\| = \|AB\| = 1$.

Question 4. Correct answer: (a).
Note that the Picard iterates are defined by

$$\lambda_{n+1}(t) := x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s \quad \text{for all } t \in J \text{ and } n \in \mathbb{N}_0.$$

Consider $J = \mathbb{R}$. By induction it follows that all Picard iterates are continuous, since the integral over a continuous function is continuous (even differentiable), and this applies in particular to the continuous function $s \mapsto f(s, \lambda_n(s))$. Note also that the first Picard iterate λ_0 is continuous as a constant function. All this does not depend on the chosen interval J and since f is globally defined, J can be chosen to be $\mathbb{R}$.

Question 5. Correct answer: (c).
Firstly, note that the differential equation is not globally Lipschitz continuous, since we have

$$|e^{t^2}x - e^{t^2}y| = e^{t^2}|x-y|,$$

and $t \mapsto e^{t^2}$ is unbounded. Secondly, it follows from Exercise 1 that every initial value problem has a global solution.