Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Physics I: Quantum Mechanics

Date: Thursday, 17 May 2012. Time: 10.00am. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Answer all the questions. Each question carries equal weight.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

1. Hermitian operators

(a) Consider a Hermitian matrix $\hat{A} = \hat{A}^\dagger$.

  (i) Show that $\hat{A}$ has a purely real spectrum.

  (ii) Show that eigenvectors of $\hat{A}$ corresponding to different eigenvalues are orthogonal.

(b) Prove that Hermiticity of a matrix is equivalent to the reality of all its expectation values.

(c) Consider now the operators $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.

  (i) Prove that $\hat{x}$ and $\hat{p}$ are Hermitian on the space of differentiable functions in L^2, equipped with the standard inner product $\langle\phi|\psi\rangle = \int_{-\infty}^{+\infty} \phi^*(x)\psi(x)\mathrm{d}x$.

  (ii) Verify that $\hat{x}$ and $\hat{p}$ fulfil the commutation relation satisfied by the position and momentum operator.

  (iii) Calculate the eigenvalues and eigenfunctions of $\hat{p}$. Does $\hat{p}$ have eigenfunctions in L^2?

2. Gaussian wave packets and harmonic oscillator

Consider the Gaussian wave packet

$$\psi(x) = N \exp\left(-\alpha(x-q)^2 + \frac{\mathrm{i}}{\hbar}p(x-q) + \frac{\mathrm{i}}{\hbar}\gamma\right), \tag{1}$$

where $q, p \in \mathbb{R}$ and $\alpha, \gamma \in \mathbb{C}$ are parameters and $N \in \mathbb{R}$ is a normalisation constant .

(a) Calculate the normalisation constant N.

(b) Calculate the expectation values of the position $\langle \hat{x} \rangle$ and the momentum $\langle \hat{p} \rangle$.

(c) Now consider the time evolution generated by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

with an initial wave packet of the form (1). Substitute an ansatz of the form (1) with time dependent parameters $q(t), p(t) \in \mathbb{R}$ and $\alpha(t), \gamma(t) \in \mathbb{C}$ into the time dependent Schrödinger equation and derive differential equations for the parameters $q(t)$, $p(t)$, $\alpha(t)$, $\gamma(t)$.

Hint: You may use the fact that $\int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$, for $\mathrm{Re}(a) > 0$.

3. Transfer matrix for a δ-potential

Consider a potential localised in a finite region $[a, b]$, that is $V(x) = 0$, for $x \notin [a, b]$. The transfer matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{11}^* \end{pmatrix}$$

connects the coefficients of plane waves to the left and to the right of the potential, that is

$$\begin{pmatrix} C \\ D \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix}, \quad \text{with} \quad \psi(x) = \begin{cases} A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}, & x \leq a \\ C\mathrm{e}^{ikx} + D\mathrm{e}^{-ikx}, & x \geq b. \end{cases}$$

(a) Show that $\det M = |M_{11}|^2 - |M_{12}|^2 = 1$ follows from the conservation of probability.

(b) Show that the reflection probability is given by $P_R = \frac{|M_{12}|^2}{|M_{11}|^2}$ and derive an expression for the transmission probability P_T.

(c) Now consider the potential $V(x) = \lambda\delta(x)$ with $\lambda \in \mathbb{R}$.

(i) Show that the transfer matrix is given by

$$M = \begin{pmatrix} 1 - i\beta & -i\beta \\ i\beta & 1 + i\beta \end{pmatrix}, \quad \text{with} \quad \beta = \frac{m\lambda}{\hbar^2 k}. \tag{2}$$

(ii) Calculate the transmission probability P_T as a function of the wave number k. What is the value of $P_T(k)$ in the limit $k \to \infty$? Sketch $P_T(k)$.

Hint: For the derivation of (2) you have to use the continuity of the wave function at $x = 0$. It is useful to further consider the time independent Schrödinger equation, which you should integrate from $x = -a$ to $x = a$ and then take the limit $a \to 0$.

4. Angular Momentum

The matrices

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{L}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

are a representation of the angular momentum operators for the total angular momentum quantum number $l = 1$ in the eigenbasis of $\hat{L}_z$.

(a) Verify that these matrices indeed fulfil the correct commutation relations.

(b) Show that the eigenvalue of $\hat{L}^2$ is really given by $\hbar^2 l(l+1)$ with $l = 1$.

(c) Calculate the normalised eigenvectors of $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ corresponding to the eigenvalue $m = 0$ and show that these vectors form an orthogonal set.

(d) Assume the system is in the state

$$|\phi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(i) Calculate the expectation value and the uncertainty of $\hat{L}_z$.

(ii) Calculate the probabilities that a measurement of $\hat{L}_x$, $\hat{L}_y$ or $\hat{L}_z$ respectively yields the value zero.

M3A4 2012 EXAM — SOLUTIONS

Question 1 All parts seen in lecture

(a) $\hat{A} = \hat{A}^{+}$, that is $(Au, v) = (u, Av)$ $\forall\, u, v$

(i) If $\hat{A}u = a u$

then $(u, \hat{A}u) = a(u,u)$

$= (\hat{A}u, u) = a^*(u,u)$

$a(u,u) = a^*(u,u)$

$\Rightarrow a = a^* \Leftrightarrow a \in \mathbb{R}$ $\square$

(ii) $\hat{A}u = au$ and $\hat{A}v = bv$ with $a \neq b$

Consider $(u, Av) = (Au, v)$

$\Rightarrow b(u,v) = a(u,v)$

and since $a \neq b$

$\Rightarrow (u,v) = 0$ $\square$

— part (a): 6 points —

(b) If $\hat{A} = \hat{A}^{+}$ then $\langle\psi|\hat{A}|\psi\rangle$

$= \langle\psi|\hat{A}|\psi\rangle^* \quad \forall |\psi\rangle$

$\Leftrightarrow \langle\psi|\hat{A}|\psi\rangle \in \mathbb{R}$.

The other way round:

Assume that $\langle\psi|A|\psi\rangle = \langle\psi|A|\psi\rangle^* \quad \forall |\psi\rangle$

Consider $|\psi\rangle = a|\phi_1\rangle + b|\phi_2\rangle$ for arbitrary a, b and $|\phi_1\rangle, |\phi_2\rangle$.

$$|a|^2\langle\phi_1|A|\phi_1\rangle + |b|^2\langle\phi_2|A|\phi_2\rangle + a^* b\langle\phi_1|A|\phi_2\rangle$$
$$+ ab^*\langle\phi_2|A|\phi_1\rangle \in \mathbb{R}$$

part(b): 6 points

First two terms automatically real.

$$a^* b\langle\phi_1|A|\phi_2\rangle + ab^*\langle\phi_2|A|\phi_1\rangle \in \mathbb{R}$$
$$\forall\ a, b \in \mathbb{C}\ \forall\ |\phi_1\rangle, |\phi_2\rangle$$

Consider now the choices (1) $a=b=1$ and (2) $a=1$ and $b=i$

$$(1) \to \langle\phi_1|A|\phi_2\rangle + \langle\phi_2|A|\phi_1\rangle$$
$$= \langle\phi_1|A|\phi_2\rangle^* + \langle\phi_2|A|\phi_1\rangle^*$$

$$(2) \to i\langle\phi_1|A|\phi_2\rangle - i\langle\phi_2|A|\phi_1\rangle$$
$$= -i\langle\phi_1|A|\phi_2\rangle^* + i\langle\phi_2|A|\phi_1\rangle^*$$

add (1) and (2)/i and divide by 2:

$$\Rightarrow \langle\phi_1|A|\phi_2\rangle = \langle\phi_2|A|\phi_1\rangle^* \quad \forall\ |\phi_1\rangle, |\phi_2\rangle$$

$$\Rightarrow A = A^\dagger \quad \square$$

(c)

(i) $\hat{x}$: $\int_{-\infty}^{\infty} \phi^*(x)\, x\, \psi(x)\, dx$

$$= \left(\int_{-\infty}^{\infty} \psi^*(x)\, x\, \phi(x)\, dx\right) \quad \square$$

$\hat{p}$: $\int_{-\infty}^{\infty} \phi^*(x)\, \hat{p}\, \psi(x)$

$$= -i\hbar \int_{-\infty}^{\infty} \phi^*(x) \frac{\partial \psi(x)}{\partial x}\, dx$$

integrate by parts:

$$= -i\hbar \underbrace{\left[\phi^*(x)\psi(x)\right]_{-\infty}^{\infty}}_{=0\ (\phi(x),\psi(x)\in L^2)} + i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{\partial\phi^*}{\partial x}dx$$

$$= \left(-i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial\phi(x)}{\partial x}dx\right)^* \qquad \square$$

part (c): 8 points

(ii) $[\hat{x},\hat{p}]$: apply to test fct $\psi(x)$:

$$\hat{x}\hat{p}\psi(x) - \hat{p}\hat{x}\psi(x)$$

$$= -i\hbar x\frac{\partial\psi}{\partial x} + i\hbar\frac{\partial}{\partial x}(x\psi(x))$$

$$= -i\hbar x\frac{\partial\psi}{\partial x} + i\hbar\psi(x) - i\hbar x\frac{\partial\psi}{\partial x} = i\hbar\psi(x)$$

$$\rightarrow [\hat{x},\hat{p}] = i\hbar\mathbb{1} \qquad \square$$

(iii) The possible eigenfunctions and eigenvalues of $\hat{p}$ are found from the eigenvalue equ:

$$-i\hbar\frac{\partial\phi_p(x)}{\partial x} = p\,\phi_p(x)$$

$$\Rightarrow \phi_p(x) \propto e^{ipx/\hbar}$$

with $p\in\mathbb{C}$.

However, none of those is square integrable as $\phi_p(x)$ does not go to zero for $x\rightarrow\pm\infty$. Thus, the answer to this part of the question is "no".

marking: (a): 6p
(b): 6p
(c): 8p
total 20p

Question 2: Part of homework

(a) Normalisation of ψ:

$$\int_{-\infty}^{\infty}|\psi(x)|^2 dx = N^2\int_{-\infty}^{\infty} \exp\{-2\mathrm{Re}(\alpha)(x-q)^2 - 2\,\mathrm{Im}(\gamma)/\hbar\}\, dx$$

$$= N^2 \exp(-2\,\mathrm{Im}(\gamma)/\hbar)\int_{-\infty}^{\infty} \exp(-2\mathrm{Re}(\alpha)(x-q)^2)\, dx$$

Substitute $y = x-q$

$\downarrow$

$$= N^2 \exp(-2\,\mathrm{Im}(\gamma)/\hbar)\int_{-\infty}^{\infty} \exp(-2\mathrm{Re}(\alpha) y^2)\, dy$$

$$= N^2 \exp(-2\,\mathrm{Im}(\gamma)/\hbar)\sqrt{\frac{\pi}{2\mathrm{Re}(\alpha)}} \overset{!}{=} 1$$

$$\Longrightarrow \boxed{N = \left(\frac{2\mathrm{Re}(\alpha)}{\pi}\right)^{1/4} e^{\mathrm{Im}(\gamma)/\hbar}}$$

part (a): 4 points

b) Expectation values of $\hat{x}$ and $\hat{p}$:

$\hat{x}$: No calculation needed:
$\psi(x)$ is symmetric around maximum at $x = q \Rightarrow \langle\hat{x}\rangle = q$

$\hat{p}$: $-i\hbar\int_{-\infty}^{\infty}\psi^*(x)\psi'(x)\, dx$

$$\psi'(x) = \left(-2\alpha(x-q) + \frac{i}{\hbar}p\right)\psi(x)$$

part (b): 6 points

$$\Rightarrow \langle \hat{p} \rangle = \int_{-\infty}^{\infty} 2i\hbar\alpha(x-q)|\psi(x)|^2 dx + p\underbrace{\int_{-\infty}^{\infty}|\psi(x)|^2 dx}_{=1}$$

$$\int_{-\infty}^{\infty}(x-q)|\psi(x)|^2 dx \propto \underbrace{\int_{-\infty}^{\infty}(x-q)e^{-2\mathrm{Re}(\alpha)(x-q)^2}dx}_{\substack{y=x-q \\ \to \text{antisymm. integrand} \\ =0}}$$

$$\Rightarrow \boxed{\langle \hat{p} \rangle = p}$$

(c) Ansatz:

$$\psi(x,t) = N\exp\{-\alpha(t)(x-q(t))^2 + ip(t)(x-q(t))/\hbar + i\gamma(t)/\hbar\}$$

insert into $i\hbar\dot{\psi} = H\psi$

$$\dot{\psi} = [-\dot{\alpha}(x-q)^2 + 2\alpha(x-q)\dot{q} + i\dot{p}(x-q)/\hbar - ip\dot{q}/\hbar + i\dot{\gamma}/\hbar]\psi$$

$$\psi''(x) = [-2\alpha + (-2\alpha(x-q) + ip/\hbar)^2]\psi(x)$$

rewrite x^2 as $x^2 = (x-q+q)^2 = (x-q)^2 + 2q(x-q) + q^2$

insert into SE:

$$\boxed{\begin{aligned}&\frac{\hbar^2}{m}\alpha - 2\frac{\hbar^2}{m}\alpha^2(x-q)^2 + \frac{1}{2m}p^2 + \frac{2\hbar i\alpha p}{m}(x-q) \\ &+ \frac{1}{2}m\omega^2(x-q)^2 + m\omega^2 q(x-q) + \frac{1}{2}m\omega^2 q^2 \\ &= -i\hbar\dot{\alpha}(x-q)^2 + i\hbar 2\alpha(x-q)\dot{q} - \dot{p}(x-q) + p\dot{q} - \dot{\gamma}\end{aligned}}$$

part (c): 10 points

quadratic term in $(x-q)$:

$$-2\frac{\hbar^2}{m}\alpha^2+\frac{1}{2}m\omega^2=-i\hbar\dot{\alpha}$$

$$\rightarrow \boxed{\dot{\alpha}=-2i\frac{\hbar}{m}\alpha^2+\frac{im}{2\hbar}\omega^2} \quad (1)$$

linear term:

$$2i\hbar\frac{\alpha p}{m}+m\omega^2 q=2i\hbar\alpha\dot{q}-\dot{p}$$

divide into real and imaginary parts:

imaginary:

$$\frac{2\hbar}{m}\mathrm{Re}(\alpha)p=2\hbar\mathrm{Re}(\alpha)\dot{q}$$

$$\rightarrow \boxed{\dot{q}=p/m} \quad (2)$$

real part:

$$-\frac{2\hbar}{m}\mathrm{Im}(\alpha)p+m\omega^2 q=-2\hbar\,\mathrm{Im}(\alpha)\dot{q}-\dot{p}$$

with (2)

$$\Rightarrow \boxed{\dot{p}=-m\omega^2 q} \quad (3)$$

marking:
(a) 4p
(b) 6p
(c) 10p
total: 20p

constant term:

$$\dot{\gamma}=p\dot{q}-\frac{\hbar^2}{m}\alpha-\frac{1}{2m}p^2-\frac{1}{2}m\omega^2 q$$

$$\rightarrow \boxed{\dot{\gamma}=-\frac{\hbar^2}{m}\alpha-\frac{1}{2}m\omega^2 q^2+\frac{1}{2m}p^2} \quad (4)$$

$\boxed{\text{Question 3}}$ Most parts unseen (except part (b); discussed in lecture)

a) conservation of probability:

$$|A|^2 - |B|^2 = |C|^2 - |D|^2 \quad (*)$$

on the other hand: $\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} M_{11} A + M_{12} B \\ M_{12}^* A + M_{11}^* B \end{pmatrix}$

$$\Rightarrow |C|^2 = |M_{11}|^2 |A|^2 + |M_{12}|^2 |B|^2$$
$$+ (M_{11}^* M_{12} A^* B + M_{11} M_{12}^* A B^*)$$
$$|D|^2 = |M_{12}|^2 |A|^2 + |M_{11}|^2 |B|^2$$
$$+ (M_{12} M_{11}^* A^* B + M_{12}^* M_{11} A B^*)$$

$$\rightarrow |C|^2 - |D|^2 = (|M_{11}|^2 - |M_{12}|^2)|A|^2 + (|M_{12}|^2 - |M_{11}|^2)|B|^2$$

inserting this into $(*)$ yields

$$|A|^2(|M_{11}|^2 - |M_{12}|^2) + |B|^2(|M_{12}|^2 - |M_{11}|^2) = |A|^2 - |B|^2$$

$$(|M_{11}|^2 - |M_{12}|^2)(|A|^2 - |B|^2) = |A|^2 - |B|^2$$

$$\Rightarrow (\text{for } |A|^2 - |B|^2 \neq 0) \quad |M_{11}|^2 - |M_{12}|^2 = 1 \quad \square$$

Part (a): 5 points

b) reflection probability from the right:

$$P_R = \left.\frac{|B|^2}{|A|^2}\right|_{D=0}$$

Part (b): 5 points

$D = 0$: $M_{12}^* A + M_{11}^* B = 0 \rightarrow B = -\frac{A M_{12}^*}{M_{11}}$

$$\Rightarrow P_R = \frac{|M_{12}|^2}{|M_{11}|^2} \quad \square \quad (\text{similarly from the left})$$

→ transmission probability:

$$P_T = 1 - P_R = 1 - \frac{|M_{12}|^2}{|M_{11}|^2} = \frac{|M_{11}|^2 - |M_{12}|^2}{|M_{11}|^2}$$

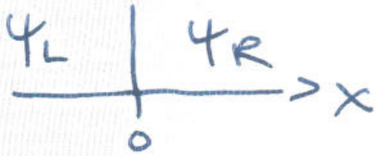
$$= \frac{1}{|M_{11}|^2}$$

c) δ-potential at $x=0$

Contiwuity of $\psi(x)$ at $x=0$:

$$\psi_L(x) = Ae^{ikx} + Be^{-ikx}$$
$$\psi_R(x) = Ce^{ikx} + De^{-ikx}$$

$$A + B = C + D \quad (1)$$



Now for the second relation consider the time independent SE of which ψ is a solution: $E\psi = H\psi$

$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \lambda\delta(x)\psi$$

integrate from $-\varepsilon$ to ε for small ε:

$$\int_{-\varepsilon}^{\varepsilon} E\psi(x)dx = -\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\partial^2}{\partial x^2}\psi(x)\,dx + \lambda\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi(x)dx$$

$$2E\varepsilon\psi(0) = -\frac{\hbar^2}{2m}\left[\psi'(x)\right]_{-\varepsilon}^{\varepsilon} + \lambda\psi(0)$$

take the limit $\varepsilon \to 0$:

$$0 = -\frac{\hbar^2}{2m}(\psi_R'(0) - \psi_L'(0)) + \lambda\psi(0)$$

$$\psi_R'(0) = ikC - ikD \qquad \psi_L'(0) = ikA - ikB$$

$$\Rightarrow \quad ik(C - D - A + B) = \frac{2m\lambda}{\hbar^2}(A+B)$$

$$\left(\frac{2m\lambda}{i\hbar^2 k} + 1\right)A + \left(\frac{2m\lambda}{i\hbar^2 k} - 1\right)B = C - D \qquad (2)$$

(1) and (2):

$$\begin{pmatrix} 1 & 1 \\ \frac{2m\lambda}{i\hbar^2 k}+1 & \frac{2m\lambda}{i\hbar^2 k}-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ \frac{2m\lambda}{i\hbar^2 k}+1 & \frac{2m\lambda}{i\hbar^2 k}-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{-1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \quad M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \frac{2m\lambda}{i\hbar^2 k}+1 & \frac{2m\lambda}{i\hbar^2 k}-1 \end{pmatrix}$$

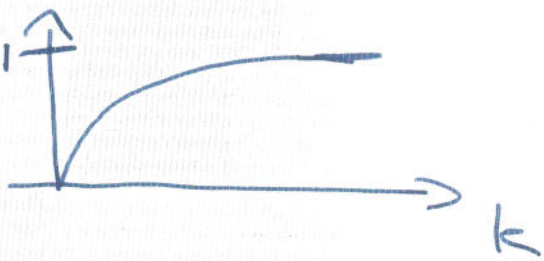
$$= \frac{1}{2}\begin{pmatrix} 2+\frac{2m\lambda}{i\hbar^2 k} & \frac{2m\lambda}{i\hbar^2 k} \\ -\frac{2m\lambda}{i\hbar^2 k} & 2-\frac{2m\lambda}{i\hbar^2 k} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - i\frac{m\lambda}{\hbar^2 k} & -i\frac{m\lambda}{\hbar^2 k} \\ i\frac{m\lambda}{\hbar^2 k} & 1 + i\frac{m\lambda}{\hbar^2 k} \end{pmatrix} \quad \square$$

part (c): 10 points

(ii) $$P_T = \frac{1}{|M_{11}|^2} = \frac{1}{\left|1 - i\frac{m\lambda}{\hbar^2 k}\right|^2} = \frac{1}{1+\frac{m^2\lambda^2}{\hbar^4 k^2}}$$

$$P_T(k\to\infty) \to \underline{\underline{1}}$$



marking: (a): 5p
(b): 5p
(c): 10p
total: 20p

$\boxed{\text{QUESTION 4}}$ Unseen (except part a): homew.)

— part (a): 3 points —

a) $$\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = \frac{\hbar^2}{2}\left\{\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i\end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i\end{pmatrix}\right\}$$

$$= i\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1\end{pmatrix} = i\hbar\hat{L}_z \checkmark$$

$$\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = \frac{\hbar^2}{\sqrt{2}}\left\{\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & +i \\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0\end{pmatrix}\right\}$$

$$= i\frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix} = i\hbar\hat{L}_x \checkmark$$

$$\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z = \frac{\hbar^2}{\sqrt{2}}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0\end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0\end{pmatrix}\right\}$$

$$= \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & +1 \\ 0 & -1 & 0\end{pmatrix} = i\hbar\hat{L}_y \checkmark$$

(b) $$\hat{L}_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1\end{pmatrix} \qquad \hat{L}_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1\end{pmatrix}$$

$$\hat{L}_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{part (b): 3 points}}$

$$\Rightarrow \hat{L}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \text{eigenvalue: } 2\hbar^2 = 1(1+1)\hbar^2$$

$\square$

(c) eigenvectors corresp. to eigenvalues $m=0$:

$$\hat{L}_z: \quad \phi_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{L}_x: \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \phi_x = 0 \qquad \phi_x = \begin{pmatrix} \phi_{x1} \\ \phi_{x2} \\ \phi_{x3} \end{pmatrix}$$

$$\Rightarrow \phi_{x2} = 0 \quad \text{and} \quad \phi_{x1} + \phi_{x3} = 0$$

$$\rightarrow \phi_x \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ normalise:}$$

$$\phi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

part (c): 5 points

$$\hat{L}_y: \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \phi_y = 0$$

$$\Rightarrow \phi_{y2} = 0 \quad \text{and} \quad i\phi_{y1} - i\phi_{y3} = 0$$

$$\Rightarrow \phi_y \propto \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ normalise: } \phi_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

orthonormal : normalised ✓

$$\phi_x \cdot \phi_z = 0 \checkmark \qquad \phi_y \cdot \phi_z = 0 \checkmark$$

$$\phi_x \cdot \phi_y = \frac{1}{2}(1-1) = 0 \checkmark$$

d) $|\phi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$

(i) expectation value of $\hat{L}_z$:

$$\frac{1}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot \hat{L}_z\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{\hbar}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-3\end{pmatrix} = -\frac{8}{14}\hbar = -\frac{4}{7}\hbar$$

uncertainty:

$$\Delta \hat{L}_z = \sqrt{\langle \hat{L}_z^2\rangle - \langle \hat{L}_z\rangle^2}$$

$$\langle \hat{L}_z^2\rangle = \frac{\hbar^2}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{10\hbar^2}{14} = \frac{5}{7}\hbar^2$$

$$\Delta \hat{L}_z = \sqrt{\frac{5}{7} - \frac{4^2}{7^2}}\,\hbar = \frac{\sqrt{35-16}}{7}\hbar = \frac{\sqrt{19}}{7}\hbar$$

(ii) probability that $\hat{L}_k$ measurement yields 0:

$$|\langle \phi_k|\phi\rangle|^2$$

$\hat{L}_z$: $P = \frac{4}{14} = \frac{2}{7}$

$\hat{L}_y$: $P = \left(\frac{1}{\sqrt{28}}(4)\right)^2 = \frac{16}{28} = \frac{4}{7}$

$\hat{L}_x$: $P = \frac{1}{28}(-2)^2 = \frac{4}{28} = \frac{1}{7}$

$\boxed{\text{part (d): 9 points}}$

marking:	(a)	3 p
	(b)	3 p.
	(c)	5 p
	(d)	9 p
	total	20 p