Mathematical Logic (MATH70132)
Mastery Material Problem Sheet

[1] Suppose $n \in \mathbb{N}$. The first-order language with equality $\mathcal{L}_n^=$ has constant symbols $c_1, \ldots, c_n$ and no other relation, function or constant symbols (apart from equality). Write down a set T_n of closed $\mathcal{L}_n^=$-formulas whose normal models are precisely infinite sets in which the constant symbols $c_1, \ldots, c_n$ are interpreted as distinct elements. Use Vaught's Test (Theorem 8.18 in Cori - Lascar) to prove that T_n is complete.

[2] The language with equality $\mathcal{L}_c^=$ has equality, a single binary relation symbol R and a constant symbols c_1, c_2. How many non-isomorphic countable normal models are there in which R is interpreted as a dense linear ordering without endpoints? How many countable normal models are there which are not elementarily equivalent? What happens if the language has n constant symbols (rather than two), for $n \in \mathbb{N}$?

[3] Suppose $\mathcal{L}^=$ is a language with equality and $\mathcal{M}$ is a normal $\mathcal{L}^=$-structure with domain M. An *automorphism* of $\mathcal{M}$ is an isomorphism $\alpha : \mathcal{M} \to \mathcal{M}$. Note that in this case, if $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}^=$-formula and $a_1, \ldots, a_n \in M$, then

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \Leftrightarrow \mathcal{M} \models \phi[\alpha(a_1), \ldots, \alpha(a_n)].$$

(i) With the above notation, suppose $\mathcal{N}$ is a substructure of $\mathcal{M}$ (with domain N) having the following property. For all $a_1, \ldots, a_n \in N$ and $b \in M$ there is an automorphism α of M with $\alpha(a_i) = a_i$, for $i \leq n$ and $\alpha(b) \in N$. Using the Tarski-Vaught Test, prove that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$.

(ii) With T_n as in [1], show that if $\mathcal{M}$ is a normal model of T_n, then every infinite substructure of $\mathcal{M}$ is an elementary substructure.

(iii) Suppose $\mathcal{M}$ is a vector space (in the usual language of vector spaces over a field F) and $\mathcal{N}$ is a subspace of infinite dimension. Prove that $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$.

[4] Let $\mathcal{L}$ be the usual language of groups and let $(\mathcal{M}_i : i \in I)$ be a family of groups. Suppose $\mathcal{F}$ is a non-principal ultrafilter on I and consider the ultraproduct $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{F}$.

(i) Prove that $\{(a_i : i \in I) \in \prod_{i \in I} \mathcal{M}_i : \{i \in I : a_i = 1\} \in \mathcal{F}\}$ is a normal subgroup of $\prod_{i \in I} \mathcal{M}_i$ and that the quotient group by this is isomorphic to $\mathcal{M}$. (Here we are denoting by 1 the identity element of a group).

(ii) Let I be the set of prime numbers and $\mathcal{M}_i$ the cyclic group of order i. Prove that $\mathcal{M}$ is elementarily equivalent to $\langle \mathbb{Q}; + \rangle$ (the additive group of rational numbers).

[5] Let $\mathcal{R}$ denote the structure $\langle \mathbb{R}; \leq, +, -, \cdot, 0, 1 \rangle$ in the usual language of rings with an ordering. Let $\mathcal{F}$ be a non-principal ultrafilter on ω and consider the ultrapower $\mathcal{R}^* = \mathcal{R}^\omega / \mathcal{F}$. Say why we can regard $\mathcal{R}^*$ as an elementary extension of $\mathcal{R}$. Decide which of the following are true, giving reasons for your answers.

(i) $\mathcal{R}^*$ is a field.

(ii) Every polynomial of odd degree with coefficients in $\mathcal{R}^*$ has a root in $\mathcal{R}^*$.

(iii) For every $r \in \mathcal{R}^*$ there is $n \in \mathbb{N}$ with $r < n$.

(iv) Every non-empty subset of $\mathcal{R}^*$ which is bounded above in $\mathcal{R}^*$ has a least upper bound in $\mathcal{R}^*$.