Question 1

The probability density function f for the χ^2_ν distribution (the chi-squared distribution with ν degrees of freedom) is

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2},$$

where the support is $x \in \mathbb{R}$ and $x > 0$, and the degrees of freedom $\nu \in \{1, 2, \ldots\}$.

(a) Let $Y \sim \chi^2_\nu$. Assuming that we know $\mathrm{E}(Y) = \nu$ and $\mathrm{E}\left(Y^2\right) = \nu(\nu+2)$, find $\mathrm{Var}(Y)$.

(b) Assume that $X_1, X_2, \ldots, X_n \sim \mathrm{N}\left(\mu, \sigma^2\right)$ and are independent. Use Part (a) to show that

$$\mathrm{Var}\left(S^2\right) = \frac{2\sigma^4}{n-1},$$

where S^2 is the sample variance, i.e. $S^2 = \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \overline{X}\right)^2$, as usual.

Question 2

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables following a normal distribution with mean μ and variance σ^2. The value of μ is unknown, but σ^2 is known to be $\sigma^2 = 16$. Suppose we observe $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Given that $\overline{x} = 7$ and $n = 50$, construct a 99% confidence interval for μ.

Question 3

Suppose $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random variables following a normal distribution with mean μ and variance σ^2. The values of μ and σ^2 are both unknown. Suppose we observe $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ as $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Given that the sample mean is $\overline{y} = 11$, the sample variance is $s^2 = 18$ and $n = 8$, construct a 90% confidence interval for μ.

Question 4

Suppose $X_1, X_2, \ldots, X_n$ are the random variables representing the heights of the $n = 300$ students in a particular module, measured in cm. These random variables are observed as $x_1, x_2, \ldots, x_n$, which are plotted below in (a) a scatterplot of the data, (b) a histogram of the data, (c) a Q-Q plot of the data after being standardised by the sample mean and variance. Do these plots suggest that $X_1, X_2, \ldots, X_n$ follow a normal distribution? Justify your answer.