

We further note that if the short-run fixed factors \underline{x}_F are fixed at their long-run conditional factor demand for a given output, y^* (i.e. $\underline{n}_F = \underline{n}_F^*(\underline{w}, y^*)$), then

$$LMC(y^*) = SMC(y^*)$$

Proof

The argument is as follows:

$$c^*(\underline{w}, y) = c_s^*(\underline{w}, \underline{n}_F^*(\underline{w}, y), y) \quad \forall y > 0$$

That means, we can take the total derivative on both sides. That is

$$\frac{d c^*(\underline{w}, y)}{dy} = \frac{d c_s^*(\underline{w}, \underline{n}_F^*(\underline{w}, y), y)}{dy}$$

$$= \sum_{j=1}^n \frac{\partial c_s^*(\underline{w}, \underline{n}_F, y)}{\partial (\underline{n}_F)_j} \left| \begin{array}{l} \frac{d(\underline{n}_F^*(\underline{w}, y))}{dy} \\ \underline{n}_F = \underline{n}_F^*(\underline{w}, y) \end{array} \right.$$

" "

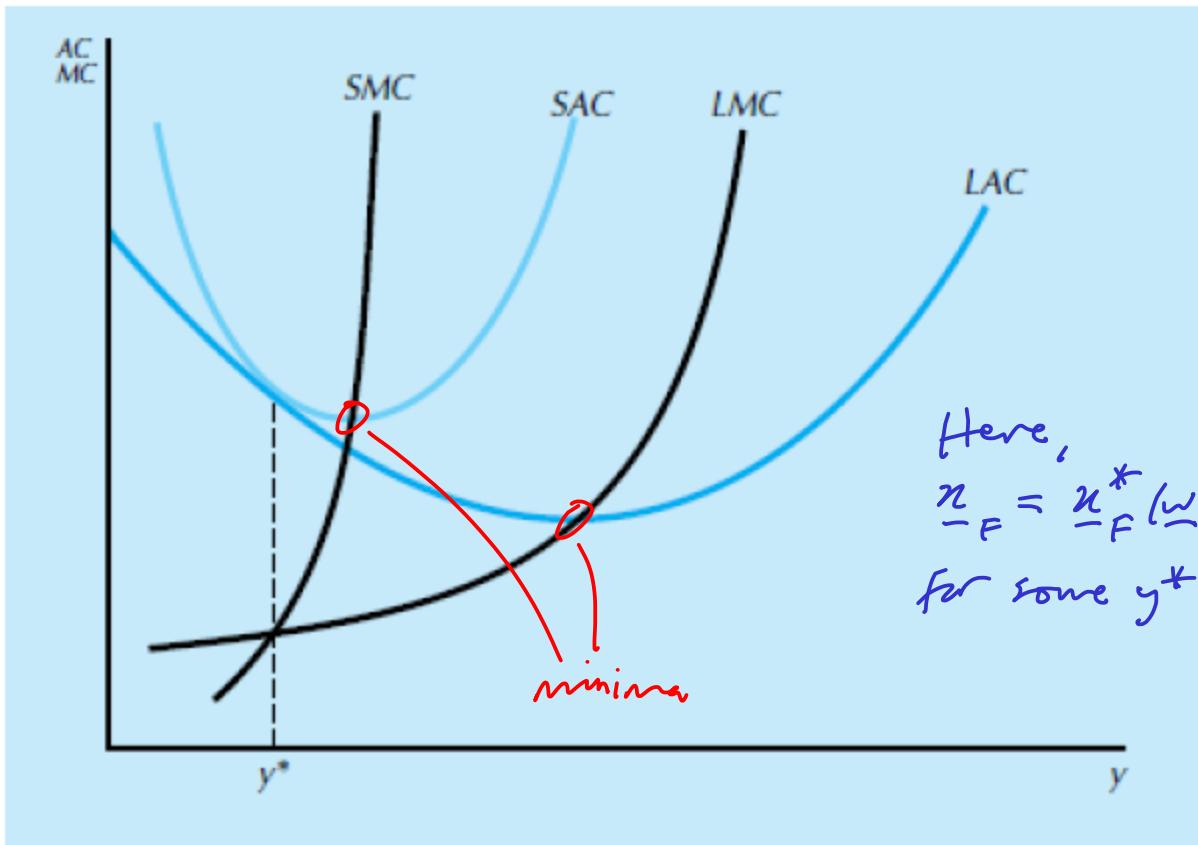
0 since $\underline{n}_F = \underline{n}_F^*(\underline{w}, y)$ minimizes $c_s^*(\underline{w}, \underline{n}_F, y)$

$$+ \frac{\partial c_s^*(\underline{w}, \underline{n}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= \frac{\partial c_s^*(\underline{w}, \underline{n}_F^*(\underline{w}, y), y)}{\partial y}$$

$$= SMC(y)$$

but $\frac{d c^*(\underline{w}, y)}{dy} \equiv \frac{\partial c^*(\underline{w}, y)}{\partial y}$ since \underline{w} is constant. //



So, if $\underline{\pi}_F = \underline{\pi}_F^*(w, y^*)$, then when $y = y^*$ the additional cost for additional unit of output (i.e. the marginal cost) is the same in the short-run as in the long-run (i.e., whether or not any of the inputs are constrained).

Profit maximisation given minimised costs

We have now considered the choices that a firm must make to minimise its costs, given knowledge of its factor prices w and a given level of output y . As mentioned previously, we now consider how a firm should subsequently choose an optimal level of output y in order to maximise profits conditional on minimised costs.

To set up the profit-maximisation question in this conditional framework, we initially maintain the assumption of perfect competition; we also assume to begin with that we are operating in the short-run.

Recall that previously, profit maximisation was framed as a question of how much input to use, and that the output of the firm was specified by the production function f . Now, all of the firm's technical constraints are implicitly specified by the cost function.

We therefore reformulate our profit maximisation problem:

We wish to solve

$$\underset{y \geq 0}{\text{argmax}} \left\{ p y - c_s^*(w, \underline{x}_F, y) \right\}$$

for y . Note that in the short-run, \underline{x}_F is fixed.

And p and w are also fixed (by the assumption of perfect competition). Hence $p y - c_s^*(w, \underline{x}_F, y)$ is simply a function of y .

First- and second-order conditions for the optimal level of output given minimised costs are given by:

$$\frac{\partial}{\partial y} (p y - c_s^*(w, \underline{x}_F, y)) = 0 \Rightarrow p = SMC(y)$$

$$\frac{\partial^2}{\partial y^2} (p y - c_s^*(w, \underline{x}_F, y)) \leq 0 \Rightarrow \frac{\partial^2 c_s^*(w, \underline{x}_F, y)}{\partial y^2} \geq 0$$

i.e., $\frac{\partial SMC(y)}{\partial y} > 0$.

These conditions suggest that, in order to maximise profits, the output should be such that the corresponding short-run marginal cost is increasing and equal to the output price p .

For a cost-minimising competitive firm, this specifies a relationship between the market-defined output price p and the quantity of output that the firm should provide.

Example 1:

Suppose that a firm's short-run cost function for a good is specified as

$$C_S^*(w_1, w_2, \pi_F, y) = 2\sqrt{w_1 w_2} y^2 + \underbrace{FC(w_F, \pi_F)}_1$$

fixed costs

If the market price for the good is £16 and each input costs the firm £4, how many units of the good should the firm produce in the short run, and what is their maximised profit if fixed costs are £12?

Solution:

$$SMC(y) = \frac{\partial C_S^*(w_1, w_2, \pi_F, y)}{\partial y} = 4\sqrt{w_1 w_2} y \quad (\text{a linear function of } y)$$

Then

$$F.O.C.: SMC(y) = p \Rightarrow y = \frac{p}{4\sqrt{w_1 w_2}} = 1, = \hat{y}, \text{ say}$$

$$S.O.C.: \frac{\partial SMC(y)}{\partial y} = 4\sqrt{w_1 w_2} = 16 > 0 \quad \forall y$$

So $\hat{y} = 1$ gives the maximum profit. And this is:

$$\begin{aligned}
 \text{maximum profit} &= p\hat{y} - c_s^*(w_1, w_2, \underline{x}_F, \hat{y}) \\
 &= 16 \cdot 1 - \left(2\sqrt{w_1 w_2} \cdot (1)^2 + 12 \right) \\
 &= -4
 \end{aligned}$$

One may deduce that ...

In the short-run, i.e. when there are fixed costs, the most profitable position for a firm may be one that returns negative profit.

Example 2:

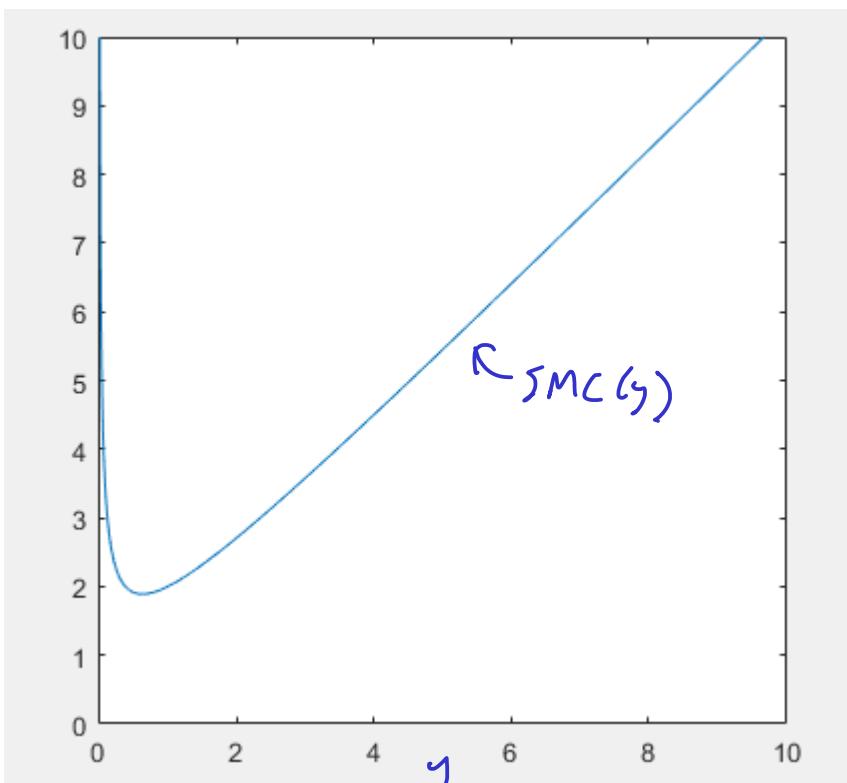
Consider the cost function

$$c_s^*(w, \underline{x}_F, y) = w_1 y^{1/2} + w_2 y^2 + FC(w_F, \underline{x}_F).$$

What is the maximised profit here, when $w_1 = 2$, $w_2 = \frac{1}{2}$, and $p = 2$?

Solution

$$SMC(y) = \frac{\partial c_s^*}{\partial y}(w, \underline{x}_F, y) = \frac{w_1}{2} y^{-1/2} + 2w_2 y = y^{-1/2} + y$$



$$\begin{aligned}
 \text{S.O.C. : } SMC(y) = p &\Rightarrow y^{-\frac{1}{2}} + y = 2 \\
 &\Rightarrow \frac{1}{y} = (2-y)^2 \\
 &\Rightarrow y^3 - 4y^2 + 4y - 1 = 0 \\
 &\Rightarrow (y-1)(y^2 - 3y + 1) = 0 \\
 &\Rightarrow y = 1, \frac{3 \pm \sqrt{5}}{2}
 \end{aligned}$$

Let note (check!), $\frac{3+\sqrt{5}}{2}$ doesn't give $SMC(y) = 2$, so discard it. Next,

$$\begin{aligned}
 S.O.C.: \frac{\partial SMC(y)}{\partial y} &= -\frac{y^{-\frac{3}{2}}}{2} + 1 \\
 &= \begin{cases} \frac{1}{2} & \text{for } y=1 \\ < 0 & \text{for } y = \frac{3-\sqrt{5}}{2} \end{cases}
 \end{aligned}$$

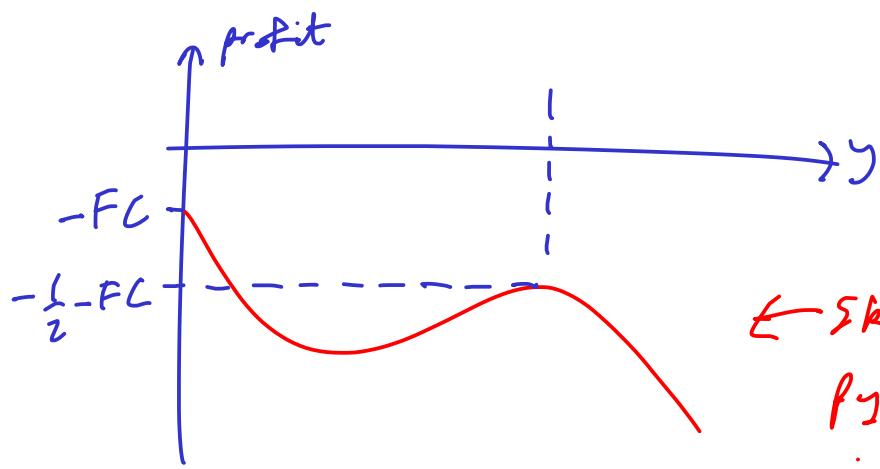
$\Rightarrow y = 1 = \hat{y}$, say, maximizes the profit (for $y \geq 0$).

$$\begin{aligned}
 \Rightarrow \text{max. profit} &= p\hat{y} - C_s^*(w, x_F, \hat{y}) \\
 &= 2 - \left(2 + \frac{1}{2} + FC\right) \\
 &= -\frac{1}{2} - FC
 \end{aligned}$$

$\leftarrow -FC$

" profit if $y=0$.

Our earlier analysis didn't pick up the global maximum of the profit at $y=0$, since the F.O.C. is not satisfied there:



← Sketch of profit curve
 $PY - C_s^*(w, \underline{x}_F, y)$
in this case.

(Note that in Example 1, the profit function $PY - C_s^*(w, \underline{x}_F, y)$ is a quadratic in y , and its value at $y = 1$ is greater than at $y = 0$).

So, as illustrated in Example 2, in some circumstances it may be preferable for a firm to go out of business rather than provide $y > 0$.

(that is, produce no output, i.e., set $y=0$)

(i.e., produce no output)

Indeed, we can generalise: it will be preferable to go out of business when

the profit for $y=0$ exceeds $PY - C_s^*(w, \underline{x}_F, y) \forall y > 0$,
i.e., when

$$\underbrace{-w_F n_F^\top}_{} \Rightarrow p_y = (w_F n_F^\top + w_v n_s^*(w, n_F, y)) \quad \forall y > 0$$

↑ profit for $y=0$ ($n_s^*(w, n_F, 0) = 0$)

$$\Leftrightarrow \frac{w_v n_s^*(w, n_F, y)}{y} > p \quad \forall y > 0$$

$$\Leftrightarrow SAVC(y) > p \quad \forall y > 0$$

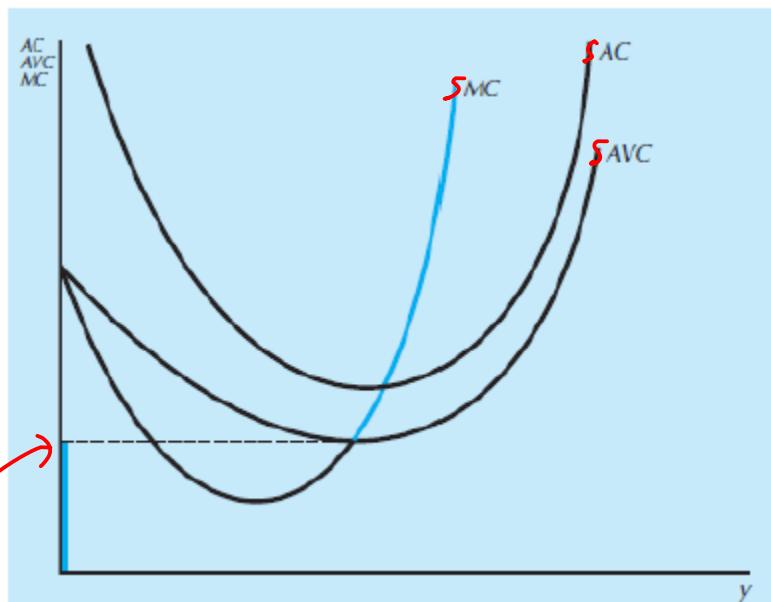
This is known as the **shutdown condition**; when satisfied, it is preferable for the firm to go out of business (i.e., produce nothing).

So we must refine our definition of the firm's chosen short-run supply. The competitive cost-minimising firm should choose a positive level of output y such that:

- $SMC(y) = p$;
- $SMC(y)$ is increasing in y ;
- and $SAVC(y) \leq p$.

If no such $y > 0$ exists for the given p , then the firm should set $y = 0$.

These conditions are satisfied by the portion of the SMC curve that is increasing in y and that lies on or above the $SAVC$ curve:



If p is less than this value
then the firm

should shut down production (i.e., produce nothing) (since in this case there exists no $y > 0$ such that $SMC(y) = p$), $SAVC(y)$).

In the long run, we have a very similar story. Neither the first- nor second-order conditions above explicitly require the costs to be dependent on fixed factors of production; these translate to the long-run scenario as would be expected. The long-run profit-maximising supply for a cost-minimising firm is given by y such that

- $LMC(y) = p$
- $LAC(y)$ must be increasing in y .
- $LAC(y) \leq p$

One arrives at the above by noting that in the long-run, the profit is given by

$$p y - c^*(\underline{w}, y)$$

which is maximised for y s.t.

$$\frac{\partial c^*(\underline{w}, y)}{\partial y} = p \quad \text{and} \quad \frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} > 0$$

$\underbrace{}_{\text{LMC}(y)}$

and $p y - c^*(\underline{w}, y) \geq \underbrace{-c^*(\underline{w}, 0)}$
 profit for $y=0, = 0$ ($c^*(\underline{w}, 0)=0$)

i.e., $\underbrace{\frac{c^*(\underline{w}, y)}{y}}_{\text{"LAC}(y)}$ $\leq p$

Once more, if no such $y > 0$ exists for the given p , then the firm should choose to go out of business. (i.e., produce nothing).

Profit maximisation for a noncompetitive firm

To contrast, we consider the profit maximisation problem for a cost-minimising monopolist. Whilst monopolists have more control over output prices than in a competitive market, they cannot choose price and output independently of one another; they must respect the market demand for their product. We therefore assume that the monopolist chooses the amount of output to provide, y , and the output price is determined according to the market demand for this output, i.e. as a function of y , $p(y)$.

The function $p(y)$ is the inverse of the market's demand function and is referred to as the inverse demand function "facing the firm"; we note that it may be dependent on other determinants, but assume these to be held constant in our analysis.

$$\text{i.e., } p(y) = D^{-1}(y), \text{ or, } y = D(p(y)).$$

To maximise profits, we therefore seek :

$$\arg \max_{y \geq 0} \left\{ p(y)y - c_s^*(w, x_F, y) \right\}.$$

First- and second-order conditions for finding a profit-maximising position for a monopolist facing an inverse demand function are therefore given by

$$\frac{\partial}{\partial y} (p(y)y - c_s^*(w, y)) = 0 \Rightarrow \frac{\partial p(y)}{\partial y} y + p(y) = SMC(y) \quad (\text{FOC})$$

$$\frac{\partial^2}{\partial y^2} (p(y)y - c_s^*(w, y)) \leq 0 \Rightarrow \frac{\partial^2 c_s^*(w, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{\partial y^2} y + 2 \frac{\partial p(y)}{\partial y} \quad (\text{SOC})$$

We can rearrange the FOC as follows :

$$p(y) \left[1 + \frac{1}{\epsilon_D(y)} \right] = SMC(y) \quad \text{X}$$

where $\epsilon_D(y) = \frac{\partial y}{\partial p(y)} \cdot \frac{p(y)}{y}$ is the price elasticity of demand.

But with $y = D(p(y))$ we have (by differentiating wrt y) :

$$(= D'(p(y)) \cdot p'(y)$$

$$\Rightarrow p'(y) = \frac{1}{D'(p(y))}$$

Now note that $\Sigma_D(y) < 0$ (demand y decreases with increasing price p), and $SAC(y) \geq 0$ ($SAC(y) = \frac{\partial c_S^*(w, x_F, y)}{\partial y}$ and $c_S^*(w, x_F, y)$ increases with increasing output y).

Then, it follows from $\textcircled{*}$ that a necessary condition for the firm to maximize profit is that $|\Sigma_D| \geq 1$ (so that the LHS of $\textcircled{*}$ is also ≥ 0), i.e., it should face elastic demand.

Example:

Consider the monopolist faced with a linear inverse demand

$$p(y) = a_1 - a_2y \quad a_1, a_2 > 0$$

and ~~Cobb-Douglas~~ variable costs in the short term :

$$c_S^*(w, x_F, y) = 2\sqrt{w_1 w_2} y^2 + FC(w_F, x_F).$$

What is the maximum profit that this monopolist can achieve?

Solution

First,

$$\frac{1}{\Sigma_D(y)} = \frac{\partial p(y)}{\partial y} \cdot \frac{y}{p(y)} = \frac{-a_2 y}{a_1 - a_2 y} \quad (\text{note this is } \leq 0 \text{ since } a_1, a_2 > 0, 0 \leq y \leq a_1/a_2)$$

Then $|\Sigma_D| \geq 1$ provided

$$|a_2 y| \leq |a_1 - a_2 y| \quad \downarrow \text{since } a_1, a_2 > 0, 0 \leq y \leq a_1/a_2$$

i.e. provided $y \leq \frac{a_1}{2a_2}$.

So any profit-maximising level of output must be below $\frac{a_1}{2a_2}$.

Now solve the FOC:

$$p(\hat{y}) \left[1 + \frac{1}{\varepsilon_p(\hat{y})} \right] = SMC(\hat{y}) \quad (= \frac{\partial C_s^*(w, z_F, \hat{y})}{\partial \hat{y}})$$

$$\Rightarrow a_1 - 2a_2 \hat{y} = 4 \sqrt{w_1 w_2} \hat{y}$$

$$\Rightarrow \hat{y} = \frac{a_1}{2a_2 + 4 \sqrt{w_1 w_2}} \leq \frac{a_1}{2a_2} \text{ as required.}$$

Evaluating the SOC verifies that this is a maximum
(exercise: check).

$$\text{Then } p(\hat{y}) = a_1 - a_2 \hat{y}$$

and the maximum profit is

$$p(\hat{y}) \cdot \hat{y} - C_s^*(w, z_F, \hat{y}) =$$

$$= (a_1 - a_2 \hat{y}) \hat{y} - (2 \sqrt{w_1 w_2} \hat{y}^2 + FC(w_F, z_F))$$

$\Rightarrow \dots$

$$= \frac{a_1^2}{4(a_2 + 2 \sqrt{w_1 w_2})} - FC(w_F, z_F).$$

We can see from this example that it is also possible for profit-maximising monopolists to experience losses in the short-run; this is not a phenomenon unique to competitive markets.

(i.e., the above maximum profit could be < 0 if FC is large enough).

The above optimisation assumes that $y > 0$. Just as for competitive firms, however, we note that the profit-maximising (loss-minimising) position for a monopolist may be to go out of business, i.e. to set $y = 0$. This happens when the losses incurred by setting output according to the above first- and second-order conditions are greater than the fixed costs, i.e. when

$$\Sigma AVC(y) > p(y) \quad \forall y \geq 0.$$

One can see this as follows.

$$\text{profit} = p(y) \cdot y - C_s^*(w, \underline{x}_F, y)$$

For $y=0$, this reduces to minus the fixed costs. So it is best to set $y=0$ if these fixed costs are greater than $p(y) \cdot y - C_s^*(w, \underline{x}_F, y) \quad \forall y \geq 0$, i.e., if

$$0 > p(y) \cdot y - \underline{x}_v \underline{x}_s^*(w, \underline{x}_F, y) \quad \forall y > 0$$

i.e., if $\frac{\underline{x}_v \underline{x}_s^*(w, \underline{x}_F, y)}{y} > p(y) \quad \forall y > 0$.

" $\Sigma AVC(y)$

So, in summary, for a cost-minimising monopolist, the short run profit-maximising output y will satisfy the following conditions:

$$\Rightarrow \rho(y) \left[1 + \frac{1}{\sum_D(y)} \right] = SMC(y)$$

$$\Rightarrow \frac{\partial^2 c^*(w, n_F, y)}{\partial y^2} \geq \frac{\partial^2 \rho(y)}{\partial y^2} y + 2 \frac{\partial \rho(y)}{\partial y}$$

$$\Rightarrow SAC(y) \leq \rho(y)$$

If no such $y > 0$ exists, then the firm should set $y=0$.

We also note that, as for competitive firms, the extension to the long-run is trivial. For a cost-minimising monopolist, the long-run profit-maximising output y will satisfy the following conditions:

$$\Rightarrow \rho(y) \left[1 + \frac{1}{\sum_D(y)} \right] = LMC(y)$$

$$\Rightarrow \frac{\partial^2 c^*(w, y)}{\partial y^2} \geq \frac{\partial^2 \rho(y)}{\partial y^2} y + 2 \frac{\partial \rho(y)}{\partial y}$$

$$\Rightarrow LAC(y) \leq \rho(y)$$