Math40003 Linear Algebra and Groups **Term 2 Unseen 1**

1. Prove that there is some matrix $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = -I_n$ if and only if n is even.

2. A square matrix is a *block upper triangular matrix* if it is of the form

$$\begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{pmatrix}$$

Where $A_1, \dots A_k$ are square matrices, the zeros stand for blocks of square zero matrices, and $*$ can be anything.

(a) Prove $\det \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \det(A) \cdot \det(B)$.

(b) Deduce

$$\det \begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{pmatrix} = \det(A_1) \cdot \det(A_2) \cdot \dots \cdot \det(A_k).$$

3. B is a *submatrix* of A if B is the result of removing any number of rows and columns from A. E.g., if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then all the following matrices are examples of submatrices of A:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 6 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}, (4).$$

Prove that for an arbitrary matrix $A \neq 0$ (not necessarily square):
rank(A) is the maximal natural number n such that A has an $n \times n$ submatrix with non-zero determinant.