

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Network Science

Date: Friday, 14 May 2021

Time: 09:00 to 11:00

Time Allowed: 2 hours

Upload Time Allowed: 30 minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

- In this question you will work with k -regular Cayley trees with diameter D . Assume that $k > 2$ and that $D > 2$. The nodes with degree=1 are termed leaf nodes, and the *root node* is the node that is equidistant from each leaf node.
 - Set $k = 3$ and $D = 4$.
 - Draw the graph (2 marks)
 - What is the degree distribution for the graph? (2 marks)
 - Assume that x_1 , the Katz centrality for the root node, has been computed and is known. Recall that the Katz centrality for node i in a graph with adjacency matrix A is $x_i = \alpha \sum_{j=1}^N A_{ij}x_j + 1$. Here, α is unknown, but assume that it has been chosen so that the solution is unique. Determine the Katz centralities of the leaf nodes. (3 marks)
 - Let G_0 be a k -regular Cayley tree with $k = 3$ and $D = 4$, and let G_t be the graph generated by the Barabasi-Albert model after t iterations where: 1 link and 1 node are added per iteration, and G_0 is the initial graph prior to the first iteration.
 - What is the probability that a link will be added to one of the leaf nodes in G_0 during the first iteration? (2 marks)
 - What is the expected number of nodes with degree=3 after 2 iterations? (6 marks)
 - Consider the following random graph model. Begin with a k -regular Cayley tree with diameter D . Then, a link is randomly added between each distinct pair of leaf nodes with probability p .
 - What is the number of links in the graph when $p = 1$? (2 marks)
 - For general p , what is the expected degree of a node that is a leaf node in the initial Cayley tree? (3 marks)

(Total: 20 marks)

2. Consider a complete graph, G , with N nodes. Let A be its adjacency matrix, and let L be its Laplacian. You are given that the eigenvalues of L are $0, N, \dots, N$. Consider the ODE

$$\dot{x}_i = \beta(x_i - \alpha) - \alpha \sum_{j=1}^N L_{ij}x_j, \quad i = 1, 2, \dots, N \quad (1)$$

or in vector form

$$\dot{\mathbf{x}} = \beta(\mathbf{x} - \alpha\mathbf{1}) - \alpha L\mathbf{x} \quad (2)$$

where $\mathbf{1}$ is the N -element vector with all components equal to 1, and α and β are positive constants.

- (a) Show that for a complete graph, $A\mathbf{x} = N\bar{\mathbf{x}} - \mathbf{x}$ and $L\mathbf{x} = N(\mathbf{x} - \bar{\mathbf{x}})$ where $\bar{\mathbf{x}}$ is the average of the elements in \mathbf{x} .

(4 marks)

- (b) Show that $\mathbf{v}_1 = \mathbf{1}$ is an eigenvector of L with eigenvalue 0.

(1 mark)

- (c) Let S be a matrix whose columns are the eigenvectors of L , $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$, with $\mathbf{v}_1 = \mathbf{1}$. Let $\mathbf{x} = S\mathbf{w}$, and show that (2) gives

$$\dot{\mathbf{w}} = B\mathbf{w} + \mathbf{c} \quad (3)$$

where B is a diagonal matrix, and \mathbf{c} is a N -element vector whose elements are constants. Clearly describe what the elements of B and \mathbf{c} should be.

(6 marks)

- (d) Given the initial condition, $\mathbf{w}(t = 0)$, find the solution, $\mathbf{w}(t)$, of equation (3).

(4 marks)

- (e) Provide a condition for αN which, when satisfied, leads to synchronization of non-trivial solutions of (1): $|x_i(t) - x_j(t)| \rightarrow 0$ as $t \rightarrow \infty$. Show that there is synchronization when this condition is satisfied.

(5 marks)

(Total: 20 marks)

3. (a) Consider three unweighted, undirected, connected graphs G_1 , G_2 , and G_3 which do not have self-loops or multiedges. Each graph has N nodes and l links, and graph G_i has adjacency matrix A_i . Let G be the graph that consists of G_1 , G_2 , and G_3 collected together (so G is the disjoint union of the three graphs and will have $3N$ nodes and $3l$ links). The nodes in G that correspond to G_1 are numbered from 1 to N ; nodes corresponding to G_2 are numbered from $N + 1$ to $2N$, and the nodes corresponding to G_3 are numbered from $2N + 1$ to $3N$.
- (i) What is the adjacency matrix for G ? Give your answer in terms of a block matrix which includes the three adjacency matrices A_1 , A_2 , A_3 (3 marks)
- (ii) Consider a partition of G into two parts (groups 1 and 2) defined by a vector $v \in \mathbb{R}^{3N}$, where the i th element of v is 1 if node i is in group 1 and is 0 if node i is in group 2. Provide three mutually orthogonal vectors, $\{v_1, v_2, v_3\}$ which each correspond to a partition which minimizes the cut size. Each partition should also have at least one node in each group, and group 2 should have more nodes than group 1.
 For any one of the three vectors you have found, show that $D^{1/2}v$ is an eigenvector of \tilde{A} , the normalized adjacency matrix of G : $\tilde{A} = D^{-1/2}AD^{-1/2}$ where D is the diagonal degree matrix for G . Provide the corresponding eigenvalue for this eigenvector. (7 marks)
- (b) Consider the following definition of a random walk on a unweighted, undirected, connected graph, G , with N nodes which does not have self-loops or multiedges. A walker at node i takes a step to node j with transition probability
- $$\pi_{ij} = A_{ij}k_j / \left(\sum_{l=1}^N A_{il}k_l \right)$$
- where A_{ij} is the adjacency matrix for the graph, and k_j is the degree of node j .
- (i) Provide a concise (1-2 sentence) interpretation of this random walk model. (2 marks)
- (ii) Show that this model is equivalent to the standard model for random walks on unweighted graphs when G is a complete graph. (3 marks)
- (iii) Find the stationary probability distribution, $p_\infty \in \mathbb{R}^N$, for this model. Express your solution in terms of the matrix $M_{ij} = k_i A_{ij} k_j$ (matrix-vector form is not required). (5 marks)

(Total: 20 marks)

4. Let A be the adjacency matrix for an N -node unweighted, undirected, connected graph with no self-loops or multiedges. The network-SIR model equations for this graph are,

$$\begin{aligned}\frac{d \langle s_i \rangle}{dt} &= -\beta \sum_{l=1}^N A_{il} \langle s_i x_l \rangle \quad (\text{susceptible}) \\ \frac{d \langle x_i \rangle}{dt} &= \beta \sum_{l=1}^N A_{il} \langle s_i x_l \rangle - \mu \langle x_i \rangle \quad (\text{infectious}) \\ \frac{d \langle r_i \rangle}{dt} &= \mu \langle x_i \rangle \quad (\text{recovered}),\end{aligned}$$

where $x_i(t) = 1$ if node i is infectious at time t and $x_i(t) = 0$ otherwise; $s_i(t) = 1$ if node i is susceptible at time t and $s_i(t) = 0$ otherwise; and $r_i(t) = 1$ if node i is recovered with immunity at time t and $r_i(t) = 0$ otherwise. The probability that a node which is susceptible at time t becomes infectious at time $t + \Delta t$ via a link to an infectious node is $\beta \Delta t$, and the probability that a node which is infectious at time t becomes recovered at time $t + \Delta t$ is $\mu \Delta t$. Note that $s_i(t) + x_i(t) + r_i(t) = 1$.

- (a) Show that any infection-free state is an equilibrium state for the system. (2 marks)
- (b) Provide a concise (1-3 lines) interpretation of the expression $\langle x_i x_j \rangle / \langle x_j \rangle = 1$ in terms of the states of nodes i and j and how they relate to each other. (2 marks)
- (c) Show that the probability that a node is infectious at both times t and $t + \Delta t$ is $(1 - \mu \Delta t) \langle x_i(t) \rangle$. (4 marks)
- (d) Recall that for the network-SI model, the governing equation for the second moment, $\langle x_i x_j \rangle$ is,

$$\frac{d \langle x_i x_j \rangle}{dt} = \beta \sum_{l=1}^N [A_{jl} \langle x_i s_j x_l \rangle + A_{il} \langle s_i x_j x_l \rangle],$$

Construct a governing equation for $\langle x_i x_j \rangle$ for the network-SIR model and compare the terms on the right-hand-side of your final equation to those in the 2nd-moment equation for the network-SI model.

- (e) Consider the following modification to the network-SIR model above. The probability that a node that is susceptible at time t is vaccinated and becomes recovered at time $t + \Delta t$ is $\gamma \Delta t$. Derive the governing ODEs for $\langle s_i(t) \rangle$ and $\langle r_i(t) \rangle$ for this modified model (note that the equation for $\langle x_i(t) \rangle$ will remain unchanged). (6 marks)

(Total: 20 marks)

TEMPORARY FRONT PAGE

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May – June 2021

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Network Science Solutions

Date: ??

Time: ??

Time Allowed: 2 Hours

This paper has *4 Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

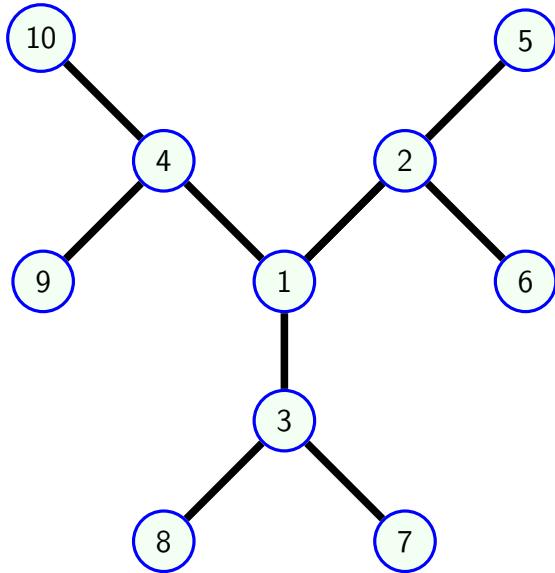
1. In this question you will work with k -regular Cayley trees with diameter D . Assume that $k > 2$ and that $D > 2$. The nodes with degree=1 are termed leaf nodes, and the *root node* is the node that is equidistant from each leaf node.

(a) Set $k = 3$ and $D = 4$.

(i) Draw the graph

(2 marks)

Solution: (Nodes do not need to be numbered)



(ii) What is the degree distribution for the graph?

(2 marks)

Solution: 2/5 of the nodes have degree 3 and 3/5 of the nodes have degree 1

(iii) Assume that x_1 , the Katz centrality for the root node, has been computed and is known. Recall that the Katz centrality for node i in a graph with adjacency matrix A is $x_i = \alpha \sum_{j=1}^N A_{ij}x_j + 1$. Here, α is unknown, but assume that it has been chosen so that the solution is unique. Determine the Katz centralities of the leaf nodes.

(3 marks)

Solution: From the symmetry of the graph, we know that nodes which are the same distance from node 1 will have the same centrality. So, for the graph shown above, $x_1 = 3\alpha x_4 + 1$ and $x_{10} = \alpha x_4 + 1$. From these two equations, we find $x_{10} = (x_1 + 2)/3$

(b) Let G_0 be a k -regular Cayley tree with $k = 3$ and $D = 4$, and let G_t be the graph generated by the Barabasi-Albert model after t iterations where: 1 link and 1 node are added per iteration, and G_0 is the initial graph prior to the first iteration.

- (i) What is the probability that a link will be added to one of the leaf nodes in G_0 during the first iteration? (2 marks)

Solution: There are 6 stubs attached to leaf nodes, and 18 stubs in total so the probability is $6/18=1/3$.

- (ii) What is the expected number of nodes with degree=3 after 2 iterations? (6 marks)

Solution: Let $\langle N_k(t) \rangle$ be the expected number of nodes with degree k after iteration t , and $L(t)$ be the total number of links. For the B-A model with $k > 1$,

$$\langle N_k(t+1) \rangle = \langle N_k(t) \rangle [1 - k/(2L(t))] + \langle N_{k-1}(t) \rangle (k-1)/(2L(t))$$

For this model, $L(t) = 9 + t$, $N_3(0) = 4$, $N_2(0) = 0$, and setting $k = 3$ we find: $\langle N_3(1) \rangle = 4 * (1 - 3/18) = 10/3$ Similarly, $\langle N_2(1) \rangle = 6(1/18) = 1/3$ Finally, $\langle N_3(2) \rangle = 10/3(1 - 3/20) + 1/3(2/20) = 43/15$

(1 point for correctly noting general equation for $\langle N_k(t) \rangle$ and 1 point for correctly computing $L(t)$.)

- (c) Consider the following random graph model. Begin with a k -regular Cayley tree with diameter D . Then, a link is randomly added between each distinct pair of leaf nodes with probability p .

- (i) What is the number of links in the graph when $p = 1$? (2 marks)

Solution: Let $R = D/2$ and $b = k - 1$. Then, the number of leaf nodes is $N_l = b^{R-1}k$ and the total number of links in the Cayley tree is $L_0 = \frac{k(b^R - 1)}{b-1}$. The number of links added to the tree will be the number of distinct pairs of leaf nodes, $\binom{N_l}{2}$, so, $L = L_0 + \binom{N_l}{2}$

- (ii) For general p , what is the expected degree of a node that is a leaf node in the initial Cayley tree? (3 marks)

Solution: Links are assigned to a node via a sequence of $N_l - 1$ Bernoulli trials, so the expected number added is $p(N_l - 1)$ and since a leaf node already has one link, the expected degree is 1 greater than the expected number added, $p(N_l - 1) + 1$.

(Total: 20 marks)

2. Consider a complete graph, G , with N nodes. Let A be its adjacency matrix, and let L be its Laplacian. You are given that the eigenvalues of L are $0, N, \dots, N$. Consider the ODE

$$\dot{x}_i = \beta(x_i - \bar{x}) - \alpha \sum_{j=1}^N L_{ij}x_j, \quad i = 1, 2, \dots, N \quad (1)$$

or in vector form

$$\dot{\mathbf{x}} = \beta(\mathbf{x} - \bar{\mathbf{x}}\mathbf{1}) - \alpha L\mathbf{x} \quad (2)$$

where $\mathbf{1}$ is the N -element vector with all components equal to 1, and α and β are positive constants.

- (a) Show that for a complete graph, $A\mathbf{x} = N\bar{\mathbf{x}} - \mathbf{x}$ and $L\mathbf{x} = N(\mathbf{x} - \bar{\mathbf{x}})$ where $\bar{\mathbf{x}}$ is the average of the elements in \mathbf{x} . (4 marks)

Solution: For a complete graph $A_{ij} = 1 - \delta_{ij}$, so $A\mathbf{x} = \sum_{j=1}^N (1 - \delta_{ij})x_j = \sum_{j=1}^N x_j - \mathbf{x} = N\bar{\mathbf{x}} - \mathbf{x}$ as required. In general, $L = D - A$ where D is the diagonal degree matrix for G . For a complete graph, every element on the diagonal is $N - 1$, so $L\mathbf{x} = (N - 1)\mathbf{x} - A\mathbf{x} = (N - 1)\mathbf{x} - N\bar{\mathbf{x}} + \mathbf{x} = N(\mathbf{x} - \bar{\mathbf{x}})$ as required.

- (b) Show that $\mathbf{v}_1 = \mathbf{1}$ is an eigenvector of L with eigenvalue 0. (1 mark)

Solution: $L\mathbf{v}_1 = N(\mathbf{v}_1 - \bar{\mathbf{v}}_1)$. The average of \mathbf{v}_1 is one, so $L\mathbf{v}_1 = 0$.

- (c) Let S be a matrix whose columns are the eigenvectors of L , $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$, with $\mathbf{v}_1 = \mathbf{1}$. Let $\mathbf{x} = S\mathbf{w}$, and show that (2) gives

$$\dot{\mathbf{w}} = B\mathbf{w} + \mathbf{c} \quad (3)$$

where B is a diagonal matrix, and \mathbf{c} is a N -element vector whose elements are constants. Clearly describe what the elements of B and \mathbf{c} should be. (6 marks)

Solution:

$$\dot{\mathbf{w}} = S^{-1}\dot{\mathbf{x}} = S^{-1}(-\alpha\beta\mathbf{1} + \beta\mathbf{x} - \alpha L\mathbf{x}) = S^{-1}\mathbf{1} + \beta\mathbf{w} - \alpha S^{-1}L\mathbf{x} = -\alpha\beta\mathbf{e}_1 + \beta\mathbf{w} - \alpha S^{-1}LS\mathbf{w}.$$

Here we use $L = SAS^{-1}$ and $S\mathbf{e}_1 = \mathbf{1}$ and conclude, $\mathbf{c} = -\alpha\beta\mathbf{e}_1$ and $B = \beta I - \alpha\Lambda$ with $\Lambda_{11} = 0$ and $\Lambda_{ii} = N$ for $i = 2, 3, \dots, N$.

- (d) Given the initial condition, $\mathbf{w}(t = 0)$, find the solution, $\mathbf{w}(t)$, of equation (3). (4 marks)

Solution: The solution of (3) is $w_1(t) = (w_1(0) - \alpha)e^{\beta t} + \alpha$ and $w_i(t) = w_i(0)e^{(\beta - \alpha N)t}$ for $i = 2, \dots, N$.

- (e) Provide a condition for αN which, when satisfied, leads to synchronization of non-trivial solutions of (1): $|x_i(t) - x_j(t)| \rightarrow 0$ as $t \rightarrow \infty$. Show that there is synchronization when this condition is satisfied. (5 marks)

Solution:

$$\mathbf{x}(t) = S \begin{pmatrix} (w_1(0) - \alpha)e^{\beta t} + \alpha \\ w_1(0)e^{(\beta - \alpha N)t} \\ \vdots \\ w_N(0)e^{(\beta - \alpha N)t} \end{pmatrix} = ((w_1(0) - \alpha)e^{\beta t} + \alpha)\mathbf{1} + w_2(0)e^{(\beta - \alpha N)t}\mathbf{v}_2 + \cdots + w_N(0)e^{(\beta - \alpha N)t}\mathbf{v}_N$$

where we used that the first column of S is equal to $\mathbf{1}$. We require $\alpha N > \beta$. It follows that $x_i(t) - x_j(t)$ will only include terms containing a $e^{(\beta - \alpha N)t}$ factor, and so $|x_i(t) - x_j(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Alternatively, using the result from (a), we have $\dot{x}_i = \beta(x_i - \bar{x}) - \alpha N(x_i - \bar{x})$. Let $z = x_i - \bar{x}$. Then, $\dot{z} = \beta z - \alpha N z$, so $z = z_0 \exp[(\beta - \alpha N)t]$, and $z \rightarrow 0$ as $t \rightarrow \infty$ if $\alpha N > \beta$. (1 point for stating the correct condition)

(Total: 20 marks)

3. (a) Consider three unweighted, undirected, connected graphs G_1 , G_2 , and G_3 which do not have self-loops or multiedges. Each graph has N nodes and l links, and graph G_i has adjacency matrix A_i . Let G be the graph that consists of G_1 , G_2 , and G_3 collected together (so G is the disjoint union of the three graphs and will have $3N$ nodes and $3l$ links). The nodes in G that correspond to G_1 are numbered from 1 to N ; nodes corresponding to G_2 are numbered from $N + 1$ to $2N$, and the nodes corresponding to G_3 are numbered from $2N + 1$ to $3N$.

- (i) What is the adjacency matrix for G ? Give your answer in terms of a block matrix which includes the three adjacency matrices A_1 , A_2 , A_3 (3 marks)

Solution: As there are no links connecting nodes in different G_i s, the adjacency matrix for G is a block-diagonal matrix with the adjacency matrices of the components on the

main diagonal in ascending order:
$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

- (ii) Consider a partition of G into two parts (groups 1 and 2) defined by a vector $v \in \mathbb{R}^{3N}$, where the i th element of v is 1 if node i is in group 1 and is 0 if node i is in group 2. Provide three mutually orthogonal vectors, $\{v_1, v_2, v_3\}$ which each correspond to a partition which minimizes the cut size. Each partition should also have at least one node in each group, and group 2 should have more nodes than group 1.

For any one of the three vectors you have found, show that $D^{1/2}v$ is an eigenvector of \tilde{A} , the normalized adjacency matrix of G : $\tilde{A} = D^{-1/2}AD^{-1/2}$ where D is the diagonal degree matrix for G . Provide the corresponding eigenvalue for this eigenvector.

(7 marks)

Solution: The i th v vector is constructed by setting $v_i = 1$ for all elements corresponding to nodes in G_i and all other elements of v_i should be set to zero (the minimum cut size is zero). (3 marks)

Let L_i be the Laplacian matrix for G_i . Then, since $L_i x = 0$ for any N -element vector where each element is the same, and since L has the same block diagonal structure as A with L_i in place of A_i , we have $L v_i = 0$ where L is the Laplacian matrix for G . Now, $\tilde{L} D^{1/2} v_i = D^{-1/2} L v_i = 0$, so $D^{1/2} v_i$ is an eigenvector of \tilde{L} . Finally, since $I - \tilde{A} = \tilde{L}$, $\tilde{A} D^{1/2} v_i = D^{1/2} v_i$ and $D^{1/2} v_i$ is an eigenvector of \tilde{A} with eigenvalue 1. (4 marks)

- (b) Consider the following definition of a random walk on a unweighted, undirected, connected graph, G , with N nodes which does not have self-loops or multiedges. A walker at node i takes a step to node j with transition probability

$$\pi_{ij} = A_{ij} k_j / \left(\sum_{l=1}^N A_{il} k_l \right)$$

where A_{ij} is the adjacency matrix for the graph, and k_j is the degree of node j .

- (i) Provide a concise (1-2 sentence) interpretation of this random walk model. (2 marks)

Solution: The walker takes a step to a neighbor of its initial location with a linear preference for higher-degree nodes; i.e. a neighbor with twice as many links as another neighbor is twice as likely to be the location that the walker moves to.

- (ii) Show that this model is equivalent to the standard model for random walks on unweighted graphs when G is a complete graph. (3 marks)

Solution: For a complete graph, each node has the same degree, $k = N - 1$. So, $\pi_{ij} = kA_{ij}/(k \sum_{l=1}^N A_{il})$. Since $\sum_{l=1}^N A_{il} = k_i = k$, we have, $\pi_{ij} = A_{ij}/k = A_{ij}/k_i$ which is the transition probability for the standard random walk model.

- (iii) Find the stationary probability distribution, $p_\infty \in \mathbb{R}^N$, for this model. Express your solution in terms of the matrix $M_{ij} = k_i A_{ij} k_j$ (matrix-vector form is not required). (5 marks)

Solution: The stationary density satisfies $p_j = \sum_{i=1}^N \pi_{ij} p_i$, and $\pi_{ij} = M_{ij}/\tilde{k}_i$ where $\tilde{k}_i = \sum_{j=1}^N M_{ij}$. This is in the form that appears for the standard random walk model which means that the stationary density will be $p_i = \tilde{k}_i/\tilde{K}$ where $\tilde{K} = \sum_{i=1}^N \tilde{k}_i$.

(Total: 20 marks)

4. Let A be the adjacency matrix for an N -node unweighted, undirected, connected graph with no self-loops or multiedges. The network-SIR model equations for this graph are,

$$\begin{aligned}\frac{d\langle s_i \rangle}{dt} &= -\beta \sum_{l=1}^N A_{il} \langle s_i x_l \rangle \quad (\text{susceptible}) \\ \frac{d\langle x_i \rangle}{dt} &= \beta \sum_{l=1}^N A_{il} \langle s_i x_l \rangle - \mu \langle x_i \rangle \quad (\text{infectious}) \\ \frac{d\langle r_i \rangle}{dt} &= \mu \langle x_i \rangle \quad (\text{recovered}),\end{aligned}$$

where $x_i(t) = 1$ if node i is infectious at time t and $x_i(t) = 0$ otherwise; $s_i(t) = 1$ if node i is susceptible at time t and $s_i(t) = 0$ otherwise; and $r_i(t) = 1$ if node i is recovered with immunity at time t and $r_i(t) = 0$ otherwise. The probability that a node which is susceptible at time t becomes infectious at time $t + \Delta t$ via a link to an infectious node is $\beta\Delta t$, and the probability that a node which is infectious at time t becomes recovered at time $t + \Delta t$ is $\mu\Delta t$. Note that $s_i(t) + x_i(t) + r_i(t) = 1$.

- (a) Show that any infection-free state is an equilibrium state for the system. (2 marks)

Solution: Setting $x_i = 0$ for all i implies that $\langle x_i \rangle = \langle s_i x_i \rangle = 0$ in the right-hand-sides of the 3 equations, so $\frac{d\langle s_i \rangle}{dt} = \frac{d\langle x_i \rangle}{dt} = \frac{d\langle r_i \rangle}{dt} = 0$.

- (b) Provide a concise (1-3 lines) interpretation of the expression $\langle x_i x_j \rangle / \langle x_j \rangle = 1$ in terms of the states of nodes i and j and how they relate to each other. (2 marks)

Solution: $\langle x_i x_j \rangle = P(x_i = 1, x_j = 1) = P(x_i = 1 | x_j = 1)P(x_j = 1)$, so $\langle x_i x_j \rangle / \langle x_j \rangle = P(x_i = 1 | x_j = 1) = 1$, and this expression means that if node j is infectious, so is node i .

- (c) Show that the probability that a node is infectious at both times t and $t + \Delta t$ is $(1 - \mu\Delta t) \langle x_i(t) \rangle$. (4 marks)

Solution:

$$\begin{aligned}P(x_i(t) = 1, x_i(t + \Delta t) = 1) &= P(x_i(t + \Delta t) = 1 | x_i(t) = 1)P(x_i(t) = 1) = \\ [1 - P(r(t + \Delta t) = 1 | x_i(t) = 1)]P(x_i(t) = 1) &= [1 - \mu\Delta t]P(x_i(t) = 1).\end{aligned}$$

Here, we have used the observations that (1) if a node is infectious at t it will be either recovered or infectious at $t + \Delta t$ and (2) $\mu\Delta t$ is the probability of a node being recovered at $t + \Delta t$ if it is infectious at t . Finally, $\langle x_i(t) \rangle = P(x_i(t) = 1) * 1 + P(x_i(t) = 0) * 0 = P(x_i(t) = 1)$, so $P(x_i(t) = 1, x_i(t + \Delta t) = 1) = (1 - \mu\Delta t) \langle x_i(t) \rangle$.

- (d) Recall that for the network-SI model, the governing equation for the second moment, $\langle x_i x_j \rangle$ is,

$$\frac{d\langle x_i x_j \rangle}{dt} = \beta \sum_{l=1}^N [A_{jl} \langle x_i s_j x_l \rangle + A_{il} \langle s_i x_j x_l \rangle],$$

Construct a governing equation for $\langle x_i x_j \rangle$ for the network-SIR model and compare the terms on the right-hand-side of your final equation to those in the 2nd-moment equation for the network-SI model. (6 marks)

Solution: For the network-SI model, the relevant master equation is: $P(x_i(t + \Delta t) = 1, x_j(t + \Delta t = 1) = P(x_i(t) = 1, x_j(t) = 1) + P(x_i(t) = 1, x_j \rightarrow 1) + P(x_i \rightarrow 1, x_j(t) = 1) + O(\Delta t^2)$.

For the network-SIR model, we have to also consider the possibility of infected nodes recovering giving: $P(x_i(t + \Delta t) = 1, x_j(t + \Delta t) = 1) = P(x_i(t) = 1, x_j(t) = 1, x_i \not\rightarrow 0, x_j \not\rightarrow 0) + P(x_i(t) = 1, x_j \rightarrow 1, x_i \not\rightarrow 0) + P(x_i \rightarrow 1, x_j(t) = 1, x_i \not\rightarrow 0) + O(\Delta t^2)$

. First consider the last term on the RHS:

$$P(x_i \rightarrow 1, x_j(t) = 1, x_i \not\rightarrow 0) = (1 - \mu \Delta t) P(x_i \rightarrow 1, x_j(t) = 1)$$

and since $P(x_i \rightarrow 1, x_j(t) = 1) \sim O(\Delta t)$, we will recover the corresponding term in the network-SI model with an $O(\Delta t^2)$ correction. The same reasoning applies to the 2nd term on the RHS. For, the 1st term on the RHS,

$$\begin{aligned} P(x_i(t + \Delta t) = 1, x_j(t + \Delta t) = 1, x_i \not\rightarrow 0, x_j \not\rightarrow 0) &= \\ P(x_i \not\rightarrow 0, x_j \not\rightarrow 0 | x_i(t) = 1, x_j(t) = 1) P(x_i(t) = 1, x_j(t) = 1) &= \\ (1 - \mu \Delta t)(1 - \mu \Delta t) P(x_i(t) = 1, x_j(t) = 1) &= (1 - 2\mu \Delta t) P(x_i(t) = 1, x_j(t) = 1) + O(\Delta t^2). \end{aligned}$$

Collecting these results, dividing by Δt , and letting $\Delta t \rightarrow 0$, we find that the second moment equation for the network SIR model is the same as for the SI model aside from one additional term on the RHS: $-2\mu \langle x_i x_j \rangle$.

- (e) Consider the following modification to the network-SIR model above. The probability that a node that is susceptible at time t is vaccinated and becomes recovered at time $t + \Delta t$ is $\gamma \Delta t$. Derive the governing ODEs for $\langle s_i(t) \rangle$ and $\langle r_i(t) \rangle$ for this modified model (note that the equation for $\langle x_i(t) \rangle$ will remain unchanged). (6 marks)

Solution: A node is recovered at $t + \Delta t$ if it is recovered at t or recovers during the time step. It can recover via vaccination or “natural” recovery, so the master equation for $r_i(t)$ is

$$P(r_i(t + \Delta t) = 1) = P(r_i(t) = 1) + P(r_i(t + \Delta t) = 1, s_i(t) = 1) + P(r_i(t + \Delta t) = 1, x_i(t) = 1).$$

Then,

$$P(r_i(t + \Delta t) = 1, s_i(t) = 1) = P(r_i(t + \Delta t) = 1 | s_i(t) = 1) P(s_i(t) = 1) = \gamma \Delta t P(s_i(t) = 1),$$

and

$$P(r_i(t + \Delta t) = 1, x_i(t) = 1) = P(r_i(t + \Delta t) = 1 | x_i(t) = 1) P(x_i(t) = 1) = \mu \Delta t P(x_i(t) = 1).$$

So, $P(r_i(t + \Delta t) = 1) = P(r_i(t) = 1) + \gamma \Delta t P(s_i(t) = 1) + \mu \Delta t P(x_i(t) = 1)$. Dividing by Δt and letting $\Delta t \rightarrow 0$ gives,

$$d \langle r_i \rangle / dt = \mu \langle x_i \rangle + \gamma \langle s_i \rangle.$$

where we have used, $\langle s_i(t) \rangle = P(s_i(t) = 1)$, $\langle r_i(t) \rangle = P(r_i(t) = 1)$ and $\langle x_i(t) \rangle = P(x_i(t) = 1)$ (see solution to (c)). Finally, since $d/dt(s_i + x_i + r_i) = 0$ and the equation for $\langle x_i(t) \rangle$ is unchanged, we have,

$$d\langle s_i \rangle / dt = -\beta \sum_{l=1}^N A_{il} \langle s_i x_l \rangle - \gamma \langle s_i \rangle.$$

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH50007	1	Students did fairly well on the most difficult part -- finding the expected number of nodes with degree=3 after 2 iterations with the B-A model, however most attempted to solve this by working out the probabilities of the different possible events. The master equation for the model which we had already derived could have been used instead leading to the solution in a few lines. In an earlier part, students were asked to provide the degree distribution for a small Cayley tree. The values in the degree distribution should be numbers between 0 and 1 which add up to one.
MATH50007	2	There were more difficulties than expected with the solution of the system of ODEs towards the end of the question (part (d)). Since the matrix was diagonal, this was a decoupled system of 1st-order linear constant-coefficient ODEs which could be solved with an elementary separation of variables approach. There seemed to be struggles with time on this question. The results in the first part could have helped with this. The expression for Ax could be used to quickly obtain the required expression for Lx which could be used to (1) show that a vector of ones is an eigenvector of L in a few lines and (2) show that the system synchronizes, again in a few lines.
MATH50007	3	The 2nd part of this question, like the B-A question from Question 1, required the right perspective. There was no need to work with the equation for the cut size since constructing a group of nodes to be one of the 3 disjoint subgraphs ensured the cut size would be zero.

MATH50007	4	It was encouraging to see a number of students provide clear and correct derivations of the 2nd-moment equations for the network-SIR model. The material on 2nd-moment equations was probably the most advanced material in the module. The result from 4(c) on the probability of an infected node staying infected should have been helpful for the subsequent derivations in (d) and (e) as well though it seems many did not notice this.
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