Imperial College London

MATH40003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Linear Algebra and Groups

Date: 9 May 2023

Time: 14:00 – 17:00 (BST)

Time Allowed: 3hrs

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklets

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Is the following matrix invertible? If so find its inverse.

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

(3 marks)

(b) Let $n \in \mathbb{N}$. Show, from the definition of matrix multiplication, that for any matrices $A, B, C \in M_{n\times n}(\mathbb{R})$ we have $A(B+C) = AB + AC$. (4 marks)

(c) A set R with two binary operations $\oplus, \otimes$ is called a *ring* if:

- $(R, \oplus)$ forms an abelian group.
- $\otimes$ is associative on R.
- $\otimes$ is distributive, that is:

$$\begin{array}{ll} \text{left distributibe} & a \otimes (b \oplus c) = (a \otimes c) \oplus (a \otimes b), \\ \text{right distributibe} & (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c). \end{array}$$

Let $n \in \mathbb{N}$. Show that $M_{n\times n}(\mathbb{R})$ with standard matrix multiplication for $\otimes$ and matrix addition for $\oplus$ is a ring (you may use—if stated clearly—any results from lectures around properties of matrix addition and multiplication). (6 marks)

(d) Which of the following are rings (see part (c))?

(i) $\mathbb{Z}$ with $x \oplus y = |x+y|$ and $x \otimes y = |xy|$.

(ii) $M_{n\times n}(\mathbb{R})$ (for $n \in \mathbb{N}$) with $A \oplus B = A + B$ (standard matrix addition) and $A \otimes B = (\det A)B$.

(4 marks)

(e) Find a set R and two binary operations $\oplus$ and $\otimes$ on that set that fail both of the left distributive and right distributive laws (see part (c)). (3 marks)

(Total: 20 marks)

2. (a) Determine for which values of a the following matrix is invertible, and provide the inverse for those values of a.

$$\begin{pmatrix} 2-a & 3 & 2 \\ 0 & 1 & 5-a \\ 2a-4 & -5 & -4 \end{pmatrix}.$$

(6 marks)

(b) In a school, where there are only teachers and students, everyone has one of three colours of eyes: blue, brown or green. Furthermore, we have the following information.

- The ratio of blue eyes to brown eyes is $3:5$.
- There are twice as many students as there are brown-eyed people.
- There are twice as many people in total as there are people with green eyes.
- There are 130 people that do not have blue eyes.

Determine how many members there are in each of the following groups: teachers (t), students (s), blue-eyed people (a), brown-eyed people (b) and green-eyed people (c). (6 marks)

(c) Let G be the group $(V,+)$, where V is a vector space over the field $\mathbb{R}$ and $+$ is the addition in V.

(i) Find all elements of finite order in G. Remember to prove that these are in fact all the elements of finite order. (2 marks)

(ii) Prove or disprove: every subgroup of G is in fact a subspace of the vector space V.

(3 marks)

(d) Let G be a group. Suppose that $H, K \leq G$ are such that $H \not\subseteq K$ and $K \not\subseteq H$. Prove that $H \cup K$ is not a subgroup of G. (3 marks)

(Total: 20 marks)

3. Let

- $V = \mathbb{R}^4$ be the usual four-dimensional real vector space,
- Let U be the 3 dimensional subspace of V given by $U = \{(x, y, z, w)^T \in V : x+2y-z = 0\}$,
- Let W be the 3 dimensional subspace of V given by $W = \{(x, y, z, w)^T \in V : 2x+y-z = 0\}$.

(a) Show that $U \cap W$ is a subspace of V. (3 marks)

(b) Show that $B = \{b_1 = (1, 1, 3, 0)^T, b_2 = (0, 0, 0, 1)^T\}$ is a basis for $U \cap W$. (3 marks)

(c) Find vectors $\mathbf{b}_U$ and $\mathbf{b}_W$ such that $B \cup \{\mathbf{b}_U\}$ is a basis for U and $B \cup \{\mathbf{b}_W\}$ is a basis for W. Prove that these do indeed form bases. (6 marks)

(d) Prove that $B \cup \{\mathbf{b}_U, \mathbf{b}_W\}$ is a basis for V, for any vectors $\mathbf{b}_U$ and $\mathbf{b}_W$ such that $B \cup \{\mathbf{b}_U\}$ is a basis for U and $B \cup \{\mathbf{b}_W\}$ is a basis for W. (2 marks)

(e) For each of f, g, h, below, either find a linear transformation fitting the description or prove that no such linear transformation can exist.

(i) $f : V \to U \cap W$ such that the nullity of f is 2 and $\dim(f(U \cap W)) \neq 2$.

(ii) $g : V \to V$ such that g is surjective and $g(U) \cap g(W) = \{\mathbf{0}\}$.

(iii) $h : V \to U \cap W$ such that the rank of h is 2 and $h(U) \cap h(W) = \{\mathbf{0}\}$.

Justify your answer fully, stating any result you use fully. (6 marks)

(Total: 20 marks)

4. (a) (i) Show that no pair of the following vectors in $\mathbb{R}^3$ is orthogonal:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}.$$

Show the details of your computations. (3 marks)

(ii) Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. (1 mark)

(iii) Use the Gram-Schmidt process on $\{v_1, v_2, v_3\}$ to find an orthogonal basis of $\mathbb{R}^3$ whose first element is v_1. (4 marks)

(b) Let G be a group with identity element e. Suppose that there is an $x \in G$ with $x \neq e$. Determine which of the following maps is a homomorphism. If it is a homomorphism, determine whether it is injective, surjective, neither, or both. Justify your answers.

(i) $\varphi_1 : G \to G$ defined as $g \mapsto x^{-1}gx$ for all $g \in G$. (2 marks)

(ii) $\varphi_2 : G \to G$ defined as $g \mapsto xg$ for all $g \in G$. (2 marks)

(iii) $\varphi_3 : G \to G$ defined as $g \mapsto e$ for all $g \in G$. (1 mark)

(c) Let G, H be groups and let $\varphi : G \to H$ be a surjective group homomorphism. Prove that, for all $h \in H$, the set

$$\varphi^{-1}(h) = \{g \in G \mid \varphi(g) = h\}$$

is a left coset of $\ker \varphi$. (3 marks)

(d) Let $p, q \in \mathbb{N}$ be two distinct prime numbers and let U_p be the subgroup of $\mathbb{C}^\times$ of all the p-th roots of unity, that is

$$U_p = \{z \in \mathbb{C}^\times \mid z^p = 1\}.$$

Show that $z \mapsto z^q$ defines an isomorphism $U_p \to U_p$. (4 marks)

(Total: 20 marks)

5. (a) For each of the following definitions of the group G and of $H \subseteq G$ determine if H is a subgroup of G. Justify your answers.

(i) $G = (\mathbb{Z}, +)$, $H = 2\mathbb{Z}$. (2 marks)

(ii) $G = (M_{3\times 3}(\mathbb{R}), +)$,

$$H = \{A \in M_{3\times 3}(\mathbb{R}) \mid \det(A) \neq 0\}.$$

(2 marks)

(iii) $G = S_3$ (the symmetric group on $\{1, 2, 3\}$),

$$H = \{\sigma \in S_3 \mid \sigma^3 = id\}.$$

(Recall that in this context id is the identity function from $\{1, 2, 3\}$ to itself.)

(2 marks)

(iv) $G = S_4$ (the symmetric group on $\{1, 2, 3, 4\}$),

$$H = \{\sigma \in S_4 \mid \sigma^4 = id\}.$$

(Recall that in this context id is the identity function from $\{1, 2, 3, 4\}$ to itself.)

(2 marks)

(b) Let σ be the permutation in S_9 given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 5 & 4 & 7 & 3 & 6 & 9 & 8 \end{pmatrix}.$$

Answer the questions below, justifying your answers.

(i) Decompose σ into disjoint cycles. (2 marks)

(ii) Decompose σ into 2-cycles. (2 marks)

(iii) Compute $\operatorname{sgn}(\sigma)$. (1 mark)

(c) Determine the cosets of $H = \langle (12) \rangle$ in S_3. (3 marks)

(d) Let $n \in \mathbb{N}$. Let $a, b \in S_n$ be defined as

$$a = (1)(2, n)(3, n-1)\ldots \qquad b = (1, n)(2, n-1)\ldots$$

Recall that we defined the *dihedral group* D_{2n} as the subgroup of the symmetric group S_n generated by a and b, that is, $D_{2n} := \langle a, b \rangle \leq S_n$. Let r be the n-cycle $(1, \ldots, n)$. Show that $ara = r^{-1}$ and that $ab = r$. Deduce that $D_{2n} = \langle a, r \rangle$. (4 marks)

(Total: 20 marks)

6. (a) Let $(G_1, *_1)$ and $(G_2, *_2)$ be groups. Define a binary operation $*$ on $G_1 \times G_2$ by $(x, y) * (x', y') = (x *_1 x', y *_2 y')$. Prove that $(G_1 \times G_2, *)$ is a group. (3 marks)

(b) Set $(G_1, *_1) = (G_2, *_2) = (\mathbb{Z}, +)$ and let $(\mathbb{Z} \times \mathbb{Z}, +)$ be the result of the construction from part (a). Prove that $(\mathbb{Z}, +)$ and $(\mathbb{Z} \times \mathbb{Z}, +)$ are not isomorphic. (5 marks)

(c) Let G be any group.

(i) For $a, b \in G$, define

$$[a, b] = aba^{-1}b^{-1}.$$

Prove that, for every $g \in G$, we have that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$. (2 marks)

(ii) Define $[G, G] \subseteq G$ as

$$[G, G] = \{[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] : \text{for some } n \in \mathbb{N} \text{ and } a_1, b_1, a_2, b_2, \ldots, a_n, b_n \in G\}.$$

Prove that $[G, G]$ is a subgroup of G and that, if $x \in [G, G]$ and $g \in G$, then $g^{-1}xg \in [G, G]$. (6 marks)

(iii) Prove that, for all $x, y \in G$, the cosets $(xy)[G, G]$ and $(yx)[G, G]$ are equal. (4 marks)

(Total: 20 marks)

Solutions

1. (a) Use row reduction method:

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

As row reduction preserves invertibility and $\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$ is not invertible (row of 0's) we must have that $\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$ is not invertible.

(3 marks) **Cat A**

(b) For $A(B+C)$ to be defined, we require the respective sizes of the matrices to be $m \times n, n \times p, n \times p$ in which case $AB + AC$ is also defined.

Calculating the (i,j)th element of this product, we obtain,

$$\begin{aligned} [A(B+C)]_{ij} &= \Sigma_{k=1}^{n} a_{ik}[B+C]_{kj} \\ &= \Sigma_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \end{aligned}$$

If we now calculate the (i,j)th element of $AB + AC$ we obtain the same result:

$$\begin{aligned} [AB+AC]_{ij} &= \Sigma_{k=1}^{n} a_{ik}b_{kj} + \Sigma_{k=1}^{n} a_{ik}c_{kj} \\ &= \Sigma_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \end{aligned}$$

Consequently, we see that $A(B+C) = AB + AC$.

(4 marks) **Cat A**

(c) In lectures we saw that matrix addition is associative and commutative the identity element is the zero matrix and the additive inverse of $A \in M_{n\times n}(\mathbb{R})$ is $(-1)A$ as $A + (-1)A = (1-1)A = 0A = \bar{0}$. Thus $(R, \oplus)$ forms a commutative group.

(2 marks) **Cat B**

In lectures we proved matrix multiplication is associative, thus $\otimes$ is associative on R.

(1 mark) **Cat A**

In part (b) we showed left distributivity. Right distributivity is essentially the same proof, and follows from right distributivity of the reals. For full marks this should be shown.

(3 marks) **Cat A**

(d) i. This is not a ring because $(R, \oplus)$ is not a commutative group, there is no identity element. Let $-2 \in \mathbb{Z}$, then there is no $x \in \mathbb{Z}$ such that $|-2+x| = -2$.

(2 marks) **Cat D**

ii. This is not a ring because $\otimes$ is not associative (right distributivity also fails).Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\det(\det(A)B) = 4$ and $\det(A)\det(B) = 2$ thus $A \otimes (B \otimes C) = \det(A)(\det(B)C) = (\det(A)\det(B))C = 2C$ whereas $(A \otimes B) \otimes C = \det(\det(A)B)C = 4C$ so for any $C \neq \bar{0}$ we have $A \otimes (B \otimes C) \neq (A \otimes B) \otimes C$.

(2 marks) **Cat D**

(e) There are a lot of possibilities here. Here is one: $R = \mathbb{Z}$, $x \oplus y = x + y$ standard addition $x \otimes y = x^2 + y^2$ then letting $A = 1$, $B = 2$ $C = -3$ we get:

- $A \otimes (B \oplus C) = 1^2 + (2-3)^2 = 2$ and $(A \otimes B) \oplus (A \otimes C) = (1^2 + 2^2) + (1^2 + (-3)^2) = 5 + 10 = 15$ thus this is not left distributive.
- $(A \oplus B) \otimes C = 3^2 + (-3)^2 = 9 + 9 = 18$ and $(A \otimes C) \oplus (B \otimes C) = (1^2 + (-3)^2) + (2^2 + (-3)^2) = 10 + 13 = 23$ thus this is not right distributive.

(3 marks) **Cat D**

(Total: 20 marks)

2. (a) To determine the values for which a is invertible we calculate the determinant.

(1 mark) **Cat A**

In the calculation below we expand along the second row and use the known formula for the determinant of a 2×2 matrix.

$$\begin{aligned}
&\det\begin{pmatrix} 2-a & 3 & 2 \\ 0 & 1 & 5-a \\ 2a-4 & -5 & -4 \end{pmatrix} &&= \\
&\det\begin{pmatrix} 2-a & 2 \\ 2a-4 & -4 \end{pmatrix} - (5-a)\det\begin{pmatrix} 2-a & 3 \\ 2a-4 & -5 \end{pmatrix} &&= \\
&-4(2-a) - 2(2a-4) - (5-a)(-5(2-a) - 3(2a-4)) &&= \\
&-(5-a)(2-a).
\end{aligned}$$

(1 mark) **Cat A**

So the determinant is non-zero whenever $a \neq 5$ and $a \neq 2$, so these are the values of a for which the matrix is invertible.

(1 mark) **Cat A**

Now assuming $a \neq 5$ and $a \neq 2$, we calculate the inverse using augmented matrices. In what follows $R_{ij}(\lambda)$ means adding λ times Row j to row i.

$$\left(\begin{array}{ccc|ccc} 2-a & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5-a & 0 & 1 & 0 \\ 2a-4 & -5 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_{13}(2)} \left(\begin{array}{ccc|ccc} 2-a & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5-a & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_{23}}$$

$$\left(\begin{array}{ccc|ccc} 2-a & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 5-a & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_{23}(-1)} \left(\begin{array}{ccc|ccc} 2-a & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 5-a & -2 & 1 & -1 \end{array}\right) \xrightarrow{R_3(\frac{1}{5-a})}$$

$$\left(\begin{array}{ccc|ccc} 2-a & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{-2}{5-a} & \frac{1}{5-a} & \frac{-1}{5-a} \end{array}\right) \xrightarrow{R_{31}(-2)} \left(\begin{array}{ccc|ccc} 2-a & 0 & 0 & \frac{4}{5-a}+1 & \frac{-2}{5-a} & \frac{2}{5-a} \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{-2}{5-a} & \frac{1}{5-a} & \frac{-1}{5-a} \end{array}\right) \xrightarrow{R_{21}(-3)}$$

$$\left(\begin{array}{ccc|ccc} 2-a & 0 & 0 & \frac{4}{5-a}-5 & \frac{-2}{5-a} & \frac{2}{5-a}-3 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{-2}{5-a} & \frac{1}{5-a} & \frac{-1}{5-a} \end{array}\right) \xrightarrow{R_1(\frac{1}{2-a})}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{(5-a)(2-a)} - \frac{5}{2-a} & \frac{-2}{(5-a)(2-a)} & \frac{2}{(5-a)(2-a)} - \frac{3}{2-a} \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{-2}{5-a} & \frac{1}{5-a} & \frac{-1}{5-a} \end{array}\right)$$

Note that in steps 4 and 7 we used the fact that $a \neq 5$ and $a \neq 2$ respectively. We can now read off the inverse as:

$$\begin{pmatrix} \frac{4}{(5-a)(2-a)} - \frac{5}{2-a} & \frac{-2}{(5-a)(2-a)} & \frac{2}{(5-a)(2-a)} - \frac{3}{2-a} \\ 2 & 0 & 1 \\ \frac{-2}{5-a} & \frac{1}{5-a} & \frac{-1}{5-a} \end{pmatrix}.$$

(3 marks) **Cat A**

(b) Using the suggested variables for each group of people, we can extract the following equations from the text:

- $t + s = a + b + c$, this corresponds to two different ways of dividing the total group of people, namely in students and teachers or by eye colour;
- $5a = 3b$, as the ratio blue to brown eyes is $3 : 5$;
- $2b = s$, as there are twice as many students as brown-eyed people;
- $t + s = 2c$, as there twice as many people in total as there are green-eyed people;
- $b + c = 130$, as there are 130 people that do not have blue eyes.

(2 marks) **Cat B**

We can represent this system of equations using a matrix as follows:

$$\begin{pmatrix} 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 5 & -3 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ s \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 130 \end{pmatrix}.$$

(1 mark) **Cat A**

To find the values for t, s, a, b, c we solve this system using an augmented matrix.

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_{14}(-1)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_{23}}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_2(-1)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_{34}}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_{34}(-5)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -8 & 5 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \end{array}\right) \xrightarrow{R_{45}}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \\ 0 & 0 & 0 & -8 & 5 & 0 \end{array}\right) \xrightarrow{R_{45}(8)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \\ 0 & 0 & 0 & 0 & 13 & 1040 \end{array}\right) \xrightarrow{R_5(1/13)}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 130 \\ 0 & 0 & 0 & 0 & 1 & 80 \end{array}\right) \xrightarrow{R_{54}(-1), R_{53}(1), R_{51}(1)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 0 & 80 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 80 \\ 0 & 0 & 0 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 1 & 80 \end{array}\right)$$

$$\xrightarrow{R_{43}(-1), R_{42}(2), R_{41}(1)} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 130 \\ 0 & 1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 0 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 1 & 80 \end{array}\right) \xrightarrow{R_{31}(1), R_{21}(-1)} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 0 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 1 & 80 \end{array}\right).$$

We conclude that $t = 60$, $s = 100$, $a = 30$, $b = 50$ and $c = 80$.

(3 marks) **Cat A**

(c) i. The only element of finite order is $\mathbf{0}_V$. This is the identity element of V, when we view V as a group; therefore, it has finite order (namely, it has order 1).

(1 mark) **Cat A**

Now let $\mathbf{v} \neq \mathbf{0}_V$ be any other element of V. Then $\mathbf{v}^n$ as computed in the group V is just $n\mathbf{v}$ as computed in the vector space V. So $\mathbf{v}^n \neq \mathbf{0}_V$ for all $n > 0$, and we conclude that $\mathbf{v}$ has infinite order, because the additive group $(\mathbb{R}, +)$ does not contain elements of finite order except for $0_{\mathbb{R}}$.

(1 mark) **Cat B**

ii. This is false.

(1 mark) **Cat B**

For example, consider $V = \mathbb{R}$. Then $\mathbb{Z} \subseteq \mathbb{R}$ is a subgroup but not a subspace. We quickly verify both claims. First, $\mathbb{Z}$ is a subgroup as it contains the identity 0, is closed under inverses (if $x \in \mathbb{Z}$ then $-x \in \mathbb{Z}$) and is closed under the group operation (if $x, y \in \mathbb{Z}$ then $x + y \in \mathbb{Z}$).

(1 mark) **Cat B**

However, $\mathbb{Z}$ is not a subspace, because for example $1 \in \mathbb{Z}$ but $\frac{1}{2} \cdot 1 \notin \mathbb{Z}$.

(1 mark) **Cat C**

(d) As $H \nsubseteq K$ there is $h \in H$ such that $h \notin K$. Similarly, there is $k \in K$ such that $k \notin H$. If $H \cup K$ were a subgroup then it must contain hk, because $h, k \in H \cup K$.

(1 mark) **Cat A**

We distinguish two cases.

- The case where $hk \in H$. As $h^{-1} \in H$ we must then have that $k = h^{-1}hk \in H$, a contradiction.
- The case where $hk \in K$. As $k^{-1} \in K$ we must then have that $h = hkk^{-1} \in K$, a contradiction.

In both cases we reach a contradiction and we conclude that $H \cup K$ is not a subgroup.

(2 marks) **Cat B**

(Total: 20 marks)

3. (a) *One approach is that $U \cap W$ is always a subspace, if U and W are. However, I think it is probably more likely students will find $U \cap W$ then show it is a subspace.*

$$U \cap W = \{(x,y,z,w) \in V : 2x + y - z = 0 \text{ and } x + 2y - z = 0\}$$

$(0,0,0,0) \in U \cap W$ so it is non-empty.

Let $v_1, v_2 \in U \cap W$ then $v_1 = (x_1, y_1, z_1, w_1)$ with $x_1 + 2y_1 - z_1 = 0$ and $2x_1 + y_1 - z_1 = 0$, $v_2 = (x_2, y_2, z_2, w_2)$ with $x_2 + 2y_2 - z_2 = 0$ and $2x_2 + y_2 - z_2 = 0$, let $\lambda \in \mathbb{R}$

$$v_1 + \lambda v_2 = (x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, w_1 + \lambda w_2)$$

Now

$$\begin{aligned}(x_1 + \lambda x_2) + 2(y_1 + \lambda y_2) - (z_1 + \lambda z_2) &= (x_1 + 2y_1 - z_1) + \lambda x_2 + 2\lambda y_2 - \lambda z_2 \\ &= (x_1 + 2y_1 - z_1) + \lambda(x_2 + 2y_2 - z_2) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

And

$$\begin{aligned}2(x_1 + \lambda x_2) + (y_1 + \lambda y_2) - (z_1 + \lambda z_2) &= (2x_1 + y_1 - z_1) + 2\lambda x_2 + \lambda y_2 - \lambda z_2 \\ &= (2x_1 + y_1 - z_1) + \lambda(2x_2 + y_2 - z_2) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

So $v_1 + \lambda v_2 \in U \cap W$.

(3 marks) **Cat A**

(b) You could show spanning, but you could also argue as follows:
$dim(U \cap W) = dim(U) + dim(W) - dim(V) = 2$
Now $(1,1,3,0), (0,0,0,1) \in U \cap W$ and they are linearly independent:
$\alpha(1,1,3,0) + \beta(0,0,0,1) = (0,0,0,0)$ implies
$\alpha = 0$ and $\beta = 0$
So we have a linearly independent set of size 2 in a vector space of dimension 2 so it must be spanning.

(3 marks) **Cat A**

(c) There are several posibilities here, but here is an example:
$b_U = (1,0,1,0)$ $b_W = (0,1,1,0)$

(2 marks) **Cat B**

It is sufficent to show that $B \cup \{b_U\}$ and $B \cup \{b_W\}$ are both linearly independent sets as they are both sets of size three, and we know both U and W have dimension three.
$B \cup \{b_U\} = \{(1,1,3,0), (0,0,0,1), (1,0,1,0)\}$
Suppose $\alpha(1,1,3,0) + \beta(0,0,0,1) + \gamma(1,0,1,0) = (0,0,0,0)$

Then $(\alpha+\gamma, \alpha, 3\alpha+\gamma, \beta) = (0,0,0,0)$

Thus $\beta = \alpha = \gamma = 0$.

$B \cup \{b_W\} = \{(1,1,3,0), (0,0,0,1), (0,1,1,0)\}$

Suppose $\alpha(1,1,3,0) + \beta(0,0,0,1) + \gamma(0,1,1,0) = (0,0,0,0)$

Then $(\alpha, \alpha+\gamma, 3\alpha+\gamma, \beta) = (0,0,0,0)$

Thus $\beta = \alpha = \gamma = 0$.

(4 marks) **Cat B**

(d) As V has dimension 4 it is sufficient to show that this size 4 set is linearly independent.

Now, from above we know $B \cup \{b_U\}$ is lin. ind., but if $b_W \in Span(B \cup \{b_U\})$ then we would have $W \subseteq U$, but $(0,1,1,0) \in W \setminus U$ so we must have that $b_W \notin Span(B \cup \{b_U\})$, thus $B \cup \{b_U, b_W\}$ is linearly independent.

(2 marks) **Cat C**

(e) i. Linear transformations are determined by what they do to the basis elements.
So let: $f(b_1) = f(b_2) = (0,0,0,0)$, $f(b_U) = b_U$, and $f(b_W) = b_W$
then $dim(f(U \cap W)) = 0 \neq 2$ and $\ker f = Span(b_1, b_2)$ which has dimension 2

(2 marks) **Cat C**

ii. This is impossible. As g is onto, its rank must be 4, and as $\dim(V) = 4$ we must have that the nullity of g is 0 (by the rank-nulity theorem ($\dim V =$ rank(g) + nullity(g))).
Thus $\ker g = \{0\}$.
Now for $v \in U \cap W$ then $g(v) \in g(U)$ and $g(v) \in g(W)$ thus $g(v) \in g(U) \cap g(W)$. So if $g(U) \cap g(W) = \{0\}$ we would have $g(b_1) = 0$ thus $b_1 \in \ker g$. Contradiction.

(2 marks) **Cat D**

iii. Linear transformations are determined by what they do to the basis elements.
So let: $h(b_1) = h(b_2) = (0,0,0,0)$, $h(b_U) = b_1$, and $h(b_W) = b_2$
then Imh = Span(b_U, b_W) so h has rank 2, and $f(U) =$ Span(b_1), $f(W) =$ Span(b_2) so $f(U) \cap f(W) = \{0\}$.

(2 marks) **Cat D**

Part (a),(b) and (c) are seen - the rest are unseen.

(Total: 20 marks)

4. (a) i. We compute

$$v_1 \cdot v_2 = 5 + 10 = 15, \quad v_1 \cdot v_3 = 1 + 4 = 5, \quad v_2 \cdot v_3 = 5 + 6 + 10 = 21.$$

Therefore no pair chosen among the given vectors is orthogonal.

(3 marks) **Cat A** Seen

ii. We show that $\{v_1, v_2, v_3\}$ is a basis. Indeed

$$\begin{vmatrix} 1 & 5 & 1 \\ 0 & 1 & 6 \\ 2 & 5 & 2 \end{vmatrix} = 2 + 60 - 2 - 30 = 30.$$

Thus $\{v_1, v_2, v_3\}$ is a basis.

(1 mark) **Cat A** Seen

iii. By the Gram-Schmidt process we define

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = v_2 - 3w_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(2 marks) **Cat A** Seen

$$w_3 = v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 = v_3 - w_1 - w_2 = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}.$$

(2 marks) **Cat A** Seen

(b) i. φ_1 is a homomorphism. Indeed, let $g, h \in G$. Then $x^{-1}gxx^{-1}hx = x^{-1}ghx$. This homomorphism is bijective because $g \mapsto xgx^{-1}$ is its inverse map.

(2 marks) **Cat B** Seen

ii. φ_2 is not a homomorphism. Indeed, let $g, h \in G$. Then $xgxh = xgh$ if and only if. $x = e$.

(2 marks) **Cat B** Seen

iii. φ_3 is a homomorphism. Indeed, let $g, h \in G$. Then $\varphi_3(gh) = e = ee = \varphi_3(g)\varphi_3(h)$. Since G is a non-trivial group, this homomorphism is neither injective nor surjective.

(1 mark) **Cat B** Seen

(c) We show that the set $\varphi^{-1}(h)$ is a left coset of $\ker \varphi$ (it is also possible to show it is a right coset, so both solutions should be accepted). Let $g \in G$ such that $\varphi(g) = h$ (such g exists because φ is surjective).

(1 mark) **Cat C** Seen

Then $\varphi(g') = \varphi(g) = h$ if and only if $\varphi(g)^{-1}\varphi(g') = e_H$.

(1 mark) **Cat C** Unseen

Since φ is a homomorphism $\varphi(g)^{-1} = \varphi(g^{-1})$ and $\varphi(g^{-1})\varphi(g') = \varphi(g^{-1}g')$. Hence $\varphi(g)^{-1}\varphi(g') = e_H$ if and only if $g^{-1}g' \in \ker\varphi$.

(1 mark) **Cat C** Unseen

(d) We show first that $z^q \in U^p$ for all $z \in U_p$. This is immediate, because $(z^q)^p = (z^p)^q$ by the associativity of the group operation; therefore, $(z^q)^p = 1^q = 1$ and $z^q \in U^p$. (This first part is not strictly necessary, accept solutions that are not showing this initial check, as it is rather obvious that raising to an integer power is a function $G \to G$ for any group G).

Let $\varphi : U_p \to U_p$ be the map defined by $z \mapsto z^p$. We check that φ defines a homomorphism. Indeed, let $a, b \in U_p$. Then, since U_p is abelian, $a^q b^q = (ab)^q$.

(1 mark) **Cat D** Seen

Next we show that φ is injective. Every element in U_p that is not the identity has order p (because p is a prime number and the order of an element of U_p has to divide p, which is the order of the group). Hence $\ker\varphi = \{1\}$ and φ is injective.

(2 marks) **Cat D** Unseen

The map φ is also surjective by the pigeon-hole principle, because U_p is a finite group.

(1 mark) **Cat D** Unseen

(Total: 20 marks)

5. (a) i. H is a subgroup. Indeed, $2 \in 2\mathbb{Z}$ thus $2\mathbb{Z} \neq \emptyset$ for all $a, b \in \mathbb{Z}$ $2a + 2b = 2(a+b) \in 2\mathbb{Z}$ and $-2a = 2(-a) \in 2\mathbb{Z}$.

(2 marks) **Cat A** Seen

ii. H is a non-empty subset of G but it is not closed under the binary operation $+$. One example to show it is not closed is indeed $A = I_3$, $B = -I_3$. In this case $\det(A) \neq 0 \neq \det(B)$ but $\det(A+B) = 0$.

(2 marks) **Cat A** Seen

iii. H is a subgroup: $H = \{id, (123), (132)\}$, so $H = \langle(123)\rangle$. Alternatively, the product of the non-trivial elements in H is id; hence, they are one the inverse of the other and H is closed under multiplication.

(2 marks) **Cat A** Seen

iv. H is not a subgroup because (even if it contains all the inverses) it is not closed under the multiplication: $(1234)(1324) = (142)$ which has order 3 and hence is not in H.

(2 marks) **Cat A** Seen

(b) i. We pick 1 in the support of σ, its image is 2 and $\sigma(2) = 1$.
We pick $3 \in \operatorname{supp}(\sigma) \smallsetminus \{1, 2\}$, we have the following:

$$\sigma(3) = 5,\ \sigma(5) = 7,\ \sigma(7) = 6, \sigma(6) = 3.$$

Finally, choosing $8 \in \operatorname{supp}(\sigma) \smallsetminus \{1, 2, 3, 5, 7, 6\}$, we see $\sigma(8) = 9$ (and $\sigma(9) = 8$). Thus

$$\sigma = (12)(3576)(89).$$

(2 marks) **Cat B** Seen

ii. (12) and (89) are already 2-cycles, we decompose (3576) as $(35)(57)(76)$; so,

$$\sigma = (12)(35)(57)(76)(89).$$

(2 marks) **Cat B** Seen

iii. The decomposition given in the previous point has 5 transpositions. It follows that

$$\operatorname{sgn}(\sigma) = (-1)^5 = -1.$$

(1 mark) **Cat B** Seen

(c) Let $\tau = (12)$. The subgroup H has order 2 so it has 3 cosets by Lagrange's theorem. One of them is $H = \{id, \tau\}$, of course. Let $\alpha = (23)$, clearly $\alpha \notin H$. We compute $\alpha H = \{\alpha id, \alpha\tau\} = \{(23), (132)\}$.

(2 marks) **Cat C** Similar to seen

The remaining elements in S_3 form another coset, namely βH where $\beta = (13)$. More explicitly, $\beta H = \{(13), (123)\}$.

(1 mark) **Cat C** Similar to seen

(d) To prove that $ara = r^{-1}$ and that $ab = r$, we use the following properties of a and b:

$$\begin{aligned}
a(1) &= 1, \qquad & a(2+i) &= n-i \qquad & i &= 0,\dots,n-2;\\
b(1+i) &= n-i & & & i &= 0,\dots,n-1.
\end{aligned}$$

Let $r' = ara$. Then

$$\begin{aligned}
r'(1) &= ara(1) = ar(a(1)) = ar(1) = a(2) = n,\\
r'(n-i) &= ar(2+i) = a(2+i+1) = n-i-1 \qquad & i &= 0,\dots,n-3,\\
r'(2) &= ar(n) = a(1) = 1.
\end{aligned}$$

This shows that $r' = r^{-1}$.

(2 marks) **Cat D** Unseen

Similarly we compute that

$$\begin{aligned}
ab(1+i) &= a(n-i) = 2+i = 1+i+1 \qquad & i &= 0,\dots,n-2,\\
ab(n) &= a(1) = 1.
\end{aligned}$$

Hence $ab = r$.

(1 mark) **Cat D** Unseen

Since $ab = r$, we immediately see that $\langle a, r\rangle \leq \langle a, r\rangle = D_{2n}$, because D_{2n} is a subgroup of S_n that contains both a and r. To show the other inclusion, it suffices to note that $ab = r$ implies $b = ar$; therefore, $\langle a, b\rangle \leq \langle a, r\rangle$, because the latter is a subgroup of S_n that contains a and b.

(1 mark) **Cat D** Unseen

(Total: 20 marks)

6. (a) We check the three group axioms.

- *Associativity.* Let $(x, y), (x', y'), (x'', y'') \in G_1 \times G_2$. We calculate:

$$\begin{aligned}((x, y) * (x', y')) * (x'', y'') &= (x *_1 x', y *_2 y') * (x'', y'') \\ &= ((x *_1 x') *_1 x'', (y *_2 y') *_2 y'') \\ &= (x *_1 (x' *_1 x''), y *_2 (y' *_2 y'')) \\ &= (x, y) * (x' *_1 x'', y' *_2 y'') \\ &= (x, y) * ((x', y') * (x'', y'')).\end{aligned}$$

So we see that the operation is indeed associative.

(1 mark) **Cat A**

- *Identity.* Let $e_1 \in G_1$ and $e_2 \in G_2$ be the respective identity elements. We claim that (e_1, e_2) is the identity element for $*$ in $G_1 \times G_2$. Indeed, for any $(x, y) \in G_1 \times G_2$ we have:

$$(x, y) * (e_1, e_2) = (x *_1 e_1, y *_2 e_2) = (x, y) = (e_1 *_1 x, e_2 *_2 y) = (e_1, e_2) * (x, y).$$

(1 mark) **Cat A**

- *Inverses.* Let $(x, y) \in G_1 \times G_2$, we claim that (x^{-1}, y^{-1}) is the inverse of (x, y) for the operation $*$. Indeed:

$$(x, y)*(x^{-1}, y^{-1}) = (x*_1x^{-1}, y*_2y^{-1}) = (e_1, e_2) = (x^{-1}*_1x, y^{-1}*_2y) = (x^{-1}, y^{-1})*(x, y).$$

(1 mark) **Cat A**

Having checked the three group axioms, we conclude that $(G_1 \times G_2, *)$ is indeed a group.

(b) Suppose for a contradiction that $(\mathbb{Z}, +)$ and $(\mathbb{Z} \times \mathbb{Z}, +)$ are isomorphic. So there is an isomorphism $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

(1 mark) **Cat A**

Write $n = f((1, 0))$ and $k = f((0, 1))$. Since f is an isomorphism, it must be injective and so $\ker f = \{(0, 0)\}$. In particular this means that $n \neq 0$ and $k \neq 0$.

(2 marks) **Cat C**

As f is in particular a group homomorphism, we have

$$f((k, 0)) = f(\underbrace{(1, 0) + \ldots + (1, 0)}_{k \text{ times}}) = \underbrace{f((1, 0)) + \ldots + f((1, 0))}_{k \text{ times}} = kn.$$

Similarly we find $f((0, n)) = nk$. However, now we have $f((k, 0)) = f((0, n))$, contradicting injectivity of f (here it is important that $n \neq 0$ and $k \neq 0$). So we conclude that no such isomorphism f can exist and hence that $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are not isomorphic.

(2 marks) **Cat C**

Alternative solution. Assume, by contradiction, that there is an isomorphism. Since $\mathbb{Z}$ is cyclic, it follows that $\mathbb{Z} \times \mathbb{Z}$ is also cyclic. 1 mark **Cat A**

This implies that there is an $(a, b) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and $n, k \in \mathbb{N}$ such that $(1, 0) = n(a, b)$ and $(0, 1) = k(a, b)$. 2 marks **Cat C**

From $na = 1$ it follows that $a \neq 0$. Similarly $b \neq 0$. Looking at the second coordinate, however, we see that $nb = ka = 0$. This implies that $n = k = 0$, which is a contradiction ($0 \notin \mathbb{N}$ for us). 2 marks **Cat C**

(c) i. This follows by direct computation:

$$\begin{aligned} g[a,b]g^{-1} &= gaba^{-1}b^{-1}g^{-1} \\ &= ga(g^{-1}g)b(g^{-1}g)a^{-1}(g^{-1}g)b^{-1}g^{-1} \\ &= (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) \\ &= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} \\ &= [gag^{-1}, gbg^{-1}]. \end{aligned}$$

We have put in some brackets to clarify the steps, although this is not necessary.

(2 marks) **Cat B**

ii. As $[e, e] \in [G, G]$ we see that $[G, G]$ is nonempty. We can thus apply the subgroup test. For that, the following equality will be useful:

$$[a, b]^{-1} = (aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1} = [b, a].$$

(1 mark) **Cat B**

Now let $x, y \in [G, G]$, so there are $n, k \in \mathbb{N}$ and $a_1, b_1, \ldots, a_n, b_n \in G$ and $c_1, d_1, \ldots, c_k, d_k \in G$ such that $x = [a_1, b_1] \cdots [a_n, b_n]$ and $y = [c_1, d_1] \cdots [c_k, d_k]$. We use the equality we found above to compute

$$\begin{aligned} xy^{-1} &= [a_1, b_1] \cdots [a_n, b_n]([c_1, d_1] \cdots [c_k, d_k])^{-1} \\ &= [a_1, b_1] \cdots [a_n, b_n][c_k, d_k]^{-1} \cdots [c_1, d_1]^{-1} \\ &= [a_1, b_1] \cdots [a_n, b_n][d_k, c_k] \cdots [d_1, c_1]. \end{aligned}$$

We thus see that $xy^{-1} \in [G, G]$ and conclude that $[G, G]$ is a subgroup by the subgroup test.

(3 marks) **Cat B**

To see the second request, let $x \in [G, G]$ and $g \in G$. Then we can write $x = [a_1, b_1] \cdots [a_n, b_n]$ for some $a_1, b_1, \ldots, a_n, b_n \in G$. Using part (c) we find

$$\begin{aligned} gxg^{-1} &= g[a_1, b_1] \cdots [a_n, b_n]g^{-1} \\ &= g[a_1, b_1]g^{-1}g[a_2, b_2]g^{-1} \cdots g[a_{n-1}, b_{n-1}]g^{-1}g[a_n, b_n]g^{-1} \\ &= [ga_1g^{-1}, gb_1g^{-1}][ga_2g^{-1}, gb_2g^{-1}] \cdots [ga_{n-1}g^{-1}, gb_{n-1}g^{-1}][ga_ng^{-1}, gb_ng^{-1}]. \end{aligned}$$

So $gxg^{-1} \in [G, G]$.

(2 marks) **Cat B**

iii. Let $x, y \in G$. We want to show that

$$xy[G, G] = yx[G, G].$$

By the definition of (left) coset, this is equivalent to showing that

$$(yx)^{-1}xy \in [G, G].$$

This is true, because $(yx)^{-1}xy = x^{-1}y^{-1}xy = [a, b]$ for $a = x^{-1}$ and $b = y^{-1}$. Hence $xy[G, G] = yx[G, G]$.

(4 marks) **Cat D**

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH40003	1	(a) Answered well (b) Some of you used wrong indices for matrix multiplication and many of you forgot to indicate the (i,j)th element of the matrix when proving the equality. (c) Many of you forgot to check the group axioms (just checked commutativity of addition) and did not show right-distributivity explicitly. (d) (i) Many of you answered correctly that it's not a ring, but said that it's an abelian group just because addition is commutative. (d) (ii) Many of you answered correctly that it's not a ring by showing that right-distributivity fails, but said that multiplication is associative. (e) Some of you gave examples of mappings that are not even binary operations on the chosen set, or just wrote two binary operations without checking the distributive laws. In general, in (d) and (e) many of you did not give an explicit counterexample by choosing a suitable triple of elements which shows the given property does not hold.
MATH40003	2	Overall the question was well answered. a) This was an easy part and most students computed the determinant and did the Gauss algorithm correctly. b) Most students found the correct answer here as well, although mostly not using matrices. c) i) was well answered. Almost everyone figured that 0 was the only element, though sometimes the proof for this was missing or incorrect (especially several students wrote the proof for R^n and not for an arbitrary abstract vector space). ii) Many students didn't realize that it was not a subspace and attempted to show it was, mixing up the definition of a subgroup and of a subspace. e) This part was not well answered. Few people were able give a proper proof of the statement using a proof by contradiction.
MATH40003	3	(a) Most got it correctly. Though, instead of giving the straightforward proof that U and W being a subspace implies U \cap W is a subspace, most of them went into giving an explicit description of U \cap W.(b) and (c), generally more or less correct but there was a significant number of them who instead of checking that given vectors span the vector space, they merely showed that the vectors belong to the vector space (and called this the spanning property!). In a lot of cases, people checked either Linear Independence or Spanning but not both.(d) Most of them got this wrong. They only verified the basis property for the specific b_U and b_W found in the previous parts.(e) Mostly incorrect or unanswered. Sometimes they even just put piecewise linear maps.

MATH40003	4	- 4a this was mostly done correctly by everyone. ii most people got it right and most of them have justified their answer sufficiently. One thing that only a few people remembered: at this point, checking that 3 vectors in R^3 form a basis should be dealt with more swiftly using the determinant, instead of wasting time and effort solving linear systems to prove they are independent or expressing the vectors of the standard basis. iii mostly done well, if there were mistakes in most cases they were only in the arithmetic. A few people forgot the square of the norm (or to square the norm) in the denominators in the Gram-Schmidt process. - 4b i was done well by most people, some people said that the function was surjective by definition, which is not acceptable as a justification. ii you need to justify the inequality xg_1g_2 \neq xg_1 x g_2 in a group theory exam (that is because x \neq e). Most people did not write the quantifiers in the proof, which, technically, is a mistake. (xg_1g_2 \neq xg_1 x g_2 holds if g_1 = g_2 = e, but it was enough to find a pair for which it did not...) I haven't penalised people for not using the quantifiers. iii this was the trivial homomorphism, which is never injective (unless G = \{ e\}) and never surjective (unless H = \{e\}). Some people claimed it was surjective as it hit every element in its image, but if surjectivity were defined that way, every map would be surjective. -4c A lot of people forgot to show one of the two inclusions. -4d A lot of people claimed z_1^q z_2^q = (z_1 _2)^q without saying that this holds because U_p is an abelian group, which you have to keep in mind in a group theory exam.

MATH40003	5	a i) Almost everyone got this right, a few people have forgotten to check H was not empty (1/2) or did not give enough details for the proof (1/2). There were also a few instances of people taking the wrong inverse or not justifying things at all. ii) Again, almost everyone recognised it was not a subgroup. Some people got confused and used matrix multiplication instead of the given operation (addition) (0/2) for that. iii) a lot have argued (Σ_1 Σ_2)^3 = Σ_1^3 Σ_2^3 without specifying them. If you write something like this the implicit assumption is that this holds in S_3, but this isn't true because S_3 is not abelian. It is true if the Σ's have order 3 or 1, but you need to justify why this is. iv) this wasn't a subgroup. Many that made the mistake of assuming S_3 was abelian have repeated it here and got it was a subgroup (be careful with these things). b i) very few people justified the decomposition, but I gave full marks anyway ii) a large proportion did not justify the decomposition in 2-cycles. It sufficed to show a 2-cycle decomposition of the 4-cycle. iii) 0/1 for a wrong sing even if the mistake is consistent with the wrong decomposition at (ii). This is because it is possible to compute the sign directly form the disjoint cycle decomposition at (i) and the two computations should agree.c a significant proportion of the students did not justify why there are only 3 cosets (either by exhaustion of H or using Lagrange). Not everyone has justified the choice of coset representatives. Giving the wrong cosets without justification resulted in no marks. Some people listed H (13)H and (23)H as cosets because it does not give enough info to pin them down inside S_3. Some have just listed all 6 possibilities for Σ H (which was a valid method) but in the end did not recognised those that were equal (one should end up with 3 cosets, not more).d most proofs were not perfect for one reason or another. Many started a proof by computing ara(1) ara(2) etc without justifying it. The correct way of justifying it was to start by computing a(1) a(2)... r(1), r(2) etc. and then combine the results. The correct way of justifying <a,b> = <a,r> was with the double inclusion: we did not prove results saying <a,b> = <a,r> if ab = r etc. so an ad hoc argument was required. Some people have used
MATH40003	6	This was a pretty easy problem, but not many students did all parts well.Part (a) only required to check the definitions, but many students confused the axioms for groups with the axioms for subgroups. The latter does not make sense, since the direct product is not a subgroup of anything.Part (b) needed the student to realise that Z is cyclic, while Z^2 is not. Many students made the elementary error of confusing the need for showing that one particular homomorphism is not an isomorphism with the need for showing this for all homomorphisms.Part (c) were checking the basic properties of the derived subgroup, where essentially all nontrivial ideas were already provided in the problem itself. However many students committed the elementary error of not checking the axioms for products of commutators, only for individual commutators.