GEOMETRY OF CURVES AND SURFACES
2016 SUMMER EXAMINATION
SOLUTIONS

PROBLEM 1.

(a) The map $\phi(x, y) = (x, y, f(x, y))$ is a diffeomorphism. Indeed, it is smooth (since f is smooth), bijective and the vectors $\frac{\partial \phi}{\partial x} = (1, 0, f_x)$ and $\frac{\partial \phi}{\partial y} = (0, 1, f_y)$ are linearly independent for all x and y.

(seen similar, 6 marks).

(b) The curve C can be parametrized by $\gamma(\theta) = (cos(\theta), sin(\theta), f(cos(\theta), sin(\theta))$. The length is then given by $\int_0^{2\pi} |\gamma'(\theta)| d\theta = \int_0^{2\pi} \sqrt{1 + (-sin(\theta)\frac{\partial f}{\partial x} + cos(\theta)\frac{\partial f}{\partial y})^2} d\theta$

(seen similar, 6 marks)

(c) For each regular closed curve $\gamma \in S$ define its turning number to be the turning number of $\phi(\gamma) \subset \mathbb{R}^2$. We claim that two curves in S are regularly homotopic if and only if they have the same turning number. If direction: suppose γ_1 and γ_2 have the same turning number; by Whitney-Graustein theorem there exists a regular homotopy between $\phi(\gamma_1)$ and $\phi(\gamma_2)$; since ϕ is a diffeomorphism we can compose the regular homotopy with ϕ^{-1} to obtain a regular homotopy between γ_1 and γ_2. Only if direction: suppose γ_1 and γ_2 have different turning numbers; if there was a regular homotopy between them then by composing with ϕ we would obtain a regular homotopy between two curves with different turning numbers in the plane, violating Whitney-Graustein theorem.

(seen similar, 8 marks)

PROBLEM 2.

(a) By Sard's lemma there exists a regular value $y \in S_2$. Since f is onto $f^{-1}(y)$ is non-empty. For any point $x \in f^{-1}(y)$ we have that df_x has full rank.

(unseen, 8 marks)

(b) Since $\langle v, w \rangle = \langle df_p(v), df_p(w) \rangle$ we have that df_p has full rank for every p. By the Inverse Function Theorem the restriction of f to a small neighbourhood U of p is a diffeomorophism onto $f(U)$. Then f is also an isometry between U and $f(U)$. By Gauss Theorema Egregium we have $K(p) = K(f(p))$.

(seen, 6 marks)

(c) No. The two surfaces are locally isometric, but not necessarily isometric. Consider for example S_1 a plane, S_2 a cylinder and $f = (cos(2\pi x), sin(2\pi x), y)$. Compute (as we did in class) that $\langle v, w\rangle = \langle df_p(v), df_p(w)\rangle$ for every $v, w \in T_pS_1$. However, this map is not injective and hence is not an isometry.

(seen similar, 6 marks)

PROBLEM 3.

(a) For any curve $\gamma(t)$ in S we have $\langle\gamma'(t), N(\gamma(t))\rangle = 0$. Hence, $\frac{d}{dt}\langle\gamma'(t), N(\gamma(t))\rangle = \langle\gamma''(t), N(\gamma(t))\rangle + \langle\gamma'(t), \frac{d}{dt}N(\gamma(t))\rangle = 0$. Therefore, $\langle\gamma'(t), dN_{\gamma(t)}(\gamma'(t))\rangle = \langle\gamma'(t), \frac{d}{dt}N(\gamma(t))\rangle = -\langle\gamma''(t), N(\gamma(t))\rangle$.

(seen, 7 marks)

(b) Let γ be a curve in S parametrized by the arc length with $\gamma(0) = p$ and $\gamma'(0) = X$. Then by part (a) we have that $\sigma_p(X, X)$ is the component of the curvature $k_\gamma(0)$ in the direction N. In particular, if γ if obtained by intersecting S with a plane P spanned by N and X then its curvature vector must be parallel to N and hence its absolute value is equal to $|\sigma_p(X, X)|$.

(seen, 6 marks)

(c) We showed in class that the second fundamental form is a symmetric operator with principal directions E_1 and E_2 as the eigenvectors. Decompose $\gamma'(0) = a_1E_1 + a_2E_2$. Since γ is a geodesic we have that $|\gamma''(0)| = |\langle\gamma''(0), N(p)\rangle| = (a_1^2\lambda_1 + a^2\lambda_2)$. Since $\lambda_1 \leq \lambda_2$ and $a_1^2 + a_2^2 = 1$ we get $\lambda_1 \leq |\gamma''(0)| \leq \lambda_2$.

(seen similar, 7 marks)

PROBLEM 4.

a) From (STATEMENT 3) it follows that surface S_2 is a totally umbilic surface in $\mathbb{R}^3$. In class we proved that this implies that S_2 is a sphere. A connected sum with the sphere does not change the Euler characteristic of any surface. Hence, S_1 is an orientable compact surface with Euler chrasteristic 0, so by classification of surfaces theorem it is diffeomorphic to a torus. S_3 has Euler characteristic 1, so by classification theorem it is diffeomorphic to $\mathbb{R}P^2$.

Alternatively, one can use (STATEMENT 2) to conclude that surface S_1 has Gaussian curvature 0 everywhere (by Gauss Theorema Egregium) and so by Gauss-Bonnet Theorem it has Euler characteristic 0. Since it's orientable it must be a torus.

(unseen, 8 marks)

b) We showed in class that every surface in $\mathbb{R}^3$ must have a point with $K > 0$. This follows by considering the largest radius r for which the sphere $S_r = \{x^2+y^2+z^2 = r^2\}$ has a non-empty intersection. One notices then that at the point of intersection p the Gaussian curvature of the surface must be larger than $\frac{1}{r^2}$, the Gaussian curvature of S_r.

By Gauss-Bonnet theorem $\int_T KdA = 0$. Hence, T must also contain a point with $K < 0$. Connect these two points by a path. Since K is continuous there will be a point on the path with $K = 0$.
(unseen, 6 marks)

c) Think of an analogy with the circle $S^1 = \{(x,y,0)|x^2 + y^2 = 1\} \subset \mathbb{R}^3$. By Whitney-Graustein theorem there is no regular homotopy turning it inside out while keeping it in the xy plane. However, in 3 dimensions we can rotate the circle around the x axis by 180 degrees. Explicitly, $f_t(x,y) = (x, ycos(\pi t), ysin(\pi t))$ is a homotopy that turns S^1 into the same circle with the opposite orientation through circles of equal length.

Guided by this analogy we define $f_t(x,y,z) = (x,y,zcos(t\pi), zsin(t\pi))$. For each t, $f_t(S^2)$ is isometric to the standard sphere (since it differs from S^2 by a rigid motion in R^4) and $f_1(x,y,z) = (x,y,-z,0)$.
(unseen, 6 marks)

PROBLEM 5. (Mastery question)
a) Let $S_r = \{(x,y,z)|x^2 + y^2 + z^2 = r^2\}$. We showed in class that S_r has constant $K = \frac{1}{r^2} > 0$. But as $r \to 0$ the area $Area(S_r) = 4\pi r^2 \to 0$. It follows that $\mu(S^2) \leq \inf\{Area(S_r)\} = 0$.
(unseen, 7 marks)

b) By Gauss-Bonnet theorem $\int_{\Sigma_g} KdA = 2\pi(2 - 2g)$. Decomposing Σ_g into U_- and $U_+ = \Sigma_g \setminus U_-$ we have $\int_{\Sigma_g} KdA = -\int_{U_-} |K|dA + \int_{U_+} |K|dA$. Since $|K| \leq 1$ on U_- we obtain $\int_{U_-} 1dA \geq \int_{U_-} |K|dA = \int_{U_+} |K| + 4\pi g - 4\pi$. Since $g \geq 2$ the quantity on the right hand side is positive.
(unseen, 8 marks)

c) By part b) we have that $\mu(\Sigma_g) \geq 4\pi g - 4\pi$. On the other hand, we showed in class that for every genus $g \geq 2$ there exists a surface of constant curvature $K = -1$ diffeomorphic to Σ_g. By Gauss-Bonnet theorem the area of such surface is $4\pi g - 4\pi$. We conclude that $\mu(\Sigma_g) = 4\pi g - 4\pi$.
(unseen, 5 marks)