Imperial College
London

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

The Geometry of Curves & Surfaces

Date: Thursday, 28 May 2015. Time: 2.00pm – 4.00pm. Time allowed: 2 hours.

This paper has FOUR questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the main book is full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw mark	up to 12	13	14	15	16	17	18	19	20
Extra credit	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4

- Each question carries equal weight.
- Calculators may not be used.

1. (a) Let $\phi : (0,1) \to \mathbb{R}^3$ be a parametrized curve given by $\phi(t) = (2t, t^2, t^3/3)$. Compute the length of the curve $\phi((0,1))$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the curve

$$\phi : \mathbb{R} \to \mathbb{R}^2, \quad \phi(t) = (t, f(t)).$$

Compute the (scalar) curvature $k(t)$ of the curve ϕ.

(c) Prove that if the curvature of a curve in $\mathbb{R}^2$ is constant, then the curve is part of a straight line or a circle.

2. (a) Compute the Gaussian curvature K of the following surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}.$$

(b) Let $\tilde{S}$ be another surface, defined by

$$\tilde{S} = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}.$$

Does there exist a local isometry between $\tilde{S}$ and S in part (a)? If you think so, then construct such a local isometry; otherwise, give reasons for your answer.

(c) Recall that a unit sphere S^2 is given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and we choose N to be the outward unit normal to S^2. Let's define a curve

$$\gamma = S^2 \cap \{(x, y, z) \in \mathbb{R}^3 : z = 1/\sqrt{2}\}.$$

Compute the geodesic curvature k_g of γ in S^2.

3. Let Σ be a compact connected orientable surface possibly with boundary and $\chi(\Sigma)$ be the Euler characteristic of Σ.

(a) State the Gauss-Bonnet Theorem for Σ.

(b) Recall that the mean curvature H of Σ is defined by $(\lambda_1 + \lambda_2)/2$, where λ_1, λ_2 are the two principal curvatures of Σ. If $\chi(\Sigma) = 2$, then prove that

$$\int_\Sigma |H|^2 dA \geq 4\pi.$$

Hint: $(a+b)^2 = (a-b)^2 + 4ab$ for all $a, b \in \mathbb{R}$.

(c) If $\chi(\Sigma) = 2$ and

$$\int_\Sigma |H|^2 dA = 4\pi,$$

show that Σ must be a sphere, i.e.,

$$\Sigma = \{\vec{x} \in \mathbb{R}^3 : |\vec{x} - \vec{x}_0| = r_0\}$$

for some $\vec{x}_0 \in \mathbb{R}^3$ and $r_0 > 0$.

4. For each of the following statements, determine whether it is true or false. Please justify each of your answers; otherwise, you will get zero marks for that part.

(a) There exists a closed curve with winding number equal to $1/2$.

(b) Any surface possibly with boundary which has positive constant Gaussian curvature must be part of a sphere.

(c) If a compact orientable surface Σ without boundary has Gaussian curvature $K(p) \geq 1$ for all $p \in \Sigma$, then the area of Σ is less than or equal to 4π.

(d) Recall that a minimal surface is a surface with vanishing mean curvature everywhere. There exists a compact minimal surface without boundary in $\mathbb{R}^3$.

(e) There is a local isometry between a plane and the cylinder given by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}.$$