Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

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This paper is also taken for the relevant examination for the Associateship.

M3/4/5 S4

Applied Probability (Solutions)

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1. (a) Either of the following definitions, are appropriate for full marks. A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if $\boxed{\text{seen} \Downarrow}$

   1. $N_0 = 0$.
   2. The increments are independent.
   3. The increments are stationary: for any $0 < s < t$, $k \in \mathbb{Z}_+$

$$\mathbb{P}(\{N_t - N_s = k\}) = \mathbb{P}(\{N_{t-s} = k\})$$

   4. There is a 'single arrival', i.e. for $t \geq 0, \delta > 0$

$$\begin{aligned}
\mathbb{P}(\{N_{t+\delta} - N_t = 0\}) &= 1 - \lambda\delta + o(\delta) \\
\mathbb{P}(\{N_{t+\delta} - N_t = 1\}) &= \lambda\delta + o(\delta) \\
\mathbb{P}(\{N_{t+\delta} - N_t \geq 2\}) &= o(\delta)
\end{aligned}$$

   (Note that you could argue that 4. implies 3. Also, in 4. it is sufficient to give the probabilities for 1 and more than 2 events or for 0 and more than 2 events to obtain full marks.)

   Or: A counting Process, $\{N_t\}_{t\geq 0}$, is a Poisson Process of rate $\lambda > 0$ if

   1. $N_0 = 0$
   2. The increments are independent
   3. For any $0 \leq s < t$, $k \in \mathbb{Z}_+$ we have

$$\mathbb{P}(N_t - N_s = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}.$$

   That is, the number of events in $[s, t]$ is a Poisson random variable, with mean $\lambda(t - s)$. $\boxed{5}$

(b) For $t > 0$ we have $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_1 > t) \;=\; \mathbb{P}(\text{no events in}[0, t]) = \mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

Either you argue that $e^{-\lambda t}$ is the survival function of an exponential random variable with parameter λ, or you compute the density function, which is given by

$$f_{X_1}(t) = \lambda e^{-\lambda t},$$

which is an exponential density function of rate λ. $\boxed{5}$

(c) Using the notation given in the problem, we have $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
p &:= \mathbb{P}(N_{t_1} = x_1, N_{t_2} = x_2, \ldots, N_{t_n} = x_n) \\
&= \mathbb{P}(N_{t_1} = x_1, N_{t_2} - N_{t_1} = x_2 - x_1, N_{t_3} - N_{t_2} = x_3 - x_2 \ldots, N_{t_n} - N_{t_{n-1}} = x_n - x_{n-1}) \\
&= \mathbb{P}(N_{t_1} = x_1)\mathbb{P}(N_{t_2} - N_{t_1} = x_2 - x_1)\mathbb{P}(N_{t_3} - N_{t_2} = x_3 - x_2) \cdots \mathbb{P}(N_{t_n} - N_{t_{n-1}} = x_n - x_{n-1}),
\end{aligned}$$

where we used the independent increment property. $\boxed{3}$

Next, we use the fact that the increments of a Poisson process are stationary and have Poisson distribution. Hence:

$$\begin{aligned}
p &= \frac{e^{-\lambda t_1}(\lambda t_1)^{x_1}}{x_1!} \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2 - t_1))^{x_2-x_1}}{(x_2 - x_1)!} \cdots \frac{e^{-\lambda(t_n-t_{n-1})}(\lambda(t_n - t_{n-1}))^{x_n-x_{n-1}}}{(x_n - x_{n-1})!} \\
&= \exp(-\lambda t_n)\lambda^{x_n} \frac{t_1^{x_1}(t_2 - t_1)^{x_2-x_1} \cdots (t_n - t_{n-1})^{x_n-x_{n-1}}}{x_1!(x_2 - x_1)! \cdots (x_n - x_{n-1})!}.
\end{aligned}$$

(Note that this is the *finite–dimensional distribution* of a Poisson process.) $\boxed{2}$

(d) Either of the following solutions is appropriate for full marks: unseen $\Downarrow$

You can use the property that the exponential distribution is memoryless. Hence, as soon as the first tourist leaves, your service time and the one by the other tourists have the same distribution. By symmetry, you have probability 0.5 of being the last to leave the check–in area.

Alternatively, you can argue as follows: Let X, Y denote the service times for the other two tourists, and let Z denote your service time. Then X, Y, Z are independent and exponentially distributed with parameter $\lambda > 0$. Hence the joint density function is given by

$$f_{X,Y,Z}(x, y, z) = \lambda^3 \exp(-\lambda x) \exp(-\lambda y) \exp(-\lambda z), \qquad x, y, z \geq 0.$$

You need to compute the probability of the event

$$B := \{X < Y < X + Z\} \cup \{Y < X < Y + Z\},$$

which is the union of two disjoint events, which have the same probability (by symmetry). Then

$$\begin{aligned}
\mathbb{P}(X < Y < X + Z) &= \int_0^\infty \int_x^\infty \int_{y-x}^\infty \lambda^3 \exp(-\lambda x) \exp(-\lambda y) \exp(-\lambda z) dz dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \int_x^\infty \lambda \exp(-\lambda y) \int_{y-x}^\infty \lambda \exp(-\lambda z) dz dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \int_x^\infty \lambda \exp(-\lambda y) \exp(-\lambda(y - x)) dy dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \exp(\lambda x) \int_x^\infty \lambda \exp(-\lambda y) \exp(-\lambda y) dy dx \\
&= \int_0^\infty \lambda \int_x^\infty \lambda \exp(-2\lambda y) dy dx.
\end{aligned}$$

Note that

$$\int_x^\infty \lambda \exp(-2\lambda y) dy = \frac{-1}{2} \exp(-2\lambda y) \Big|_x^\infty = \frac{1}{2} \exp(-2\lambda x),$$

and

$$\int_0^\infty \lambda \frac{1}{2} \exp(-2\lambda x) dx = \frac{-1}{4} \exp(-2\lambda x) \Big|_0^\infty = \frac{1}{4},$$

Hence $\mathbb{P}(X < Y < X + Z) = \frac{1}{4} = \mathbb{P}(Y < X < Y + Z)$ and $\mathbb{P}(B) = 0.5$.

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2. (a) A non–homogeneous Poisson process with intensity function $(\lambda(t))$ is a stochastic process $N = \{N_t\}_{t \geq 0}$, which satisfies the following properties: $\boxed{\text{seen} \Downarrow}$

1. $N_0 = 0$.
2. N has independent increments. $\boxed{2}$
3. 'Single arrival' property: For $t \geq 0$, $\delta > 0$:

$$\begin{aligned} \mathbb{P}(N_{t+\delta} - N_t = 0) &= 1 - \lambda(t)\delta + o(\delta), \\ \mathbb{P}(N_{t+\delta} - N_t = 1) &= \lambda(t)\delta + o(\delta), \\ \mathbb{P}(N_{t+\delta} - N_t \geq 2) &= o(\delta), \end{aligned}$$

$\boxed{2}$

(As in Question 1, it is sufficient to state the probability for 0 and more than 2 events or for 1 and more than 2 events. Also you could work with the alternative definition that $N_0 = 0$, N has independent increments and that $N_t - N_s \sim$ Poi$(m(t) - m(s))$ for $0 \leq s < t$, where $m(t) := \int_0^t \lambda(u)du$.) $\boxed{\text{seen} \Downarrow}$

(b) Let $n = 0$. Then

$$\begin{aligned} p_0(t+\delta) &= \mathbb{P}(N_{t+\delta} = 0) = \mathbb{P}(\text{no event in}[0, t+\delta]) \\ &= \mathbb{P}(\{\text{no event in } [0,t]\} \cap \{\text{no event in } (t, t+\delta]\}) \\ &= \mathbb{P}\Big(\text{no event in } [0,t]\Big)\mathbb{P}\Big(\text{no event in } (t, t+\delta]\Big), \end{aligned}$$

where we applied the independent increments property, then

$$p_0(t+\delta) = p_0(t)[1 - \lambda(t)\delta + o(\delta)].$$

Hence we have

$$\frac{p_0(t+\delta) - p_0(t)}{\delta} = -\lambda(t)p_0(t) + \frac{o(\delta)}{\delta}$$

letting $\delta \downarrow 0$ we get

$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t).$$

$\boxed{2}$

For $n \geq 1$ we have

$$\begin{aligned} p_n(t+\delta) &= \mathbb{P}(N_{t+\delta} = n) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(N_{t+\delta} = n | N_t = k)\mathbb{P}(N_t = k) \\ &= \sum_{k=0}^{\infty} \mathbb{P}((n-k) \text{ events in } (t, t+\delta])\mathbb{P}(N_t = k) \quad \text{(by independent increments property)} \\ &= \mathbb{P}(1 \text{ event in } (t, t+\delta])\mathbb{P}(N_t = n-1) \\ &\quad + \mathbb{P}(0 \text{ events in } (t, t+\delta])\mathbb{P}(N_t = n) + o(\delta) \\ &= p_{n-1}(t)\lambda(t)\delta + p_n(t)(1 - \lambda(t)\delta) + o(\delta) \\ &= p_n(t)(1 - \lambda(t)\delta) + p_{n-1}(t)\lambda(t)\delta + o(\delta). \end{aligned}$$

Re-arranging and letting $\delta \downarrow 0$ we have $\boxed{3}$

$$\frac{dp_n(t)}{dt} = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t).$$

$\boxed{1}$

(c) We solve

$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t)$$

with $p_0(0) = \mathbb{P}(N_0 = 0) = 1$. Hence, we obtain

$$p_0(t) = \exp\left(-\int_0^t \lambda(s)ds\right).$$

seen ⇓

Let $n = 1$, then we have the ordinary differential equation (ODE) 1

$$\frac{dp_1(t)}{dt} + \lambda(t)p_1(t) = \lambda(t)\exp\left\{-\int_0^t \lambda(s)ds\right\}.$$

In our case, the integrating factor is

$$M(t) = \exp\left\{\int_0^t \lambda(u)du\right\}.$$

Since $p_1(0) = \mathbb{P}(N_0 = 1) = 0$, the solution is

$$p_1(t) = \left[\int_0^t \lambda(s)ds\right]\exp\left\{-\int_0^t \lambda(s)ds\right\}.$$

3

(d) Let $m(t) := \int_0^t \lambda(s)ds$ for $t \geq 0$.

meth seen ⇓

(i) Let $t > 0$. Then

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in } [0,t]) = \mathbb{P}(N_t = 0) = \exp(-m(t)).$$

Hence the cumulative distribution function is given by $F_{X_1}(t) = 1 - \exp(-m(t))$. Differentiating yields

$$f_{X_1}(t) = -\exp(-m(t))(-1)\lambda(t) = \exp(-m(t))\lambda(t).$$

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(ii) Let $t, t_1 > 0$. Then

$$\begin{aligned}\mathbb{P}(X_2 > t|X_1 = t_1) &= \mathbb{P}(\text{no events in } (t_1, t_1+t]|\text{one event in } [0,t_1]) \\ &= \mathbb{P}(N_{t_1+t} - N_{t_1} = 0|N_{t_1} = 1) = \mathbb{P}(N_{t_1+t} - N_{t_1} = 0) \\ &= \exp\left(-\int_{t_1}^{t_1+t} \lambda(u)du\right)\end{aligned}$$

where we used the independent increment property and the fact that the increments are Poisson distributed. Hence differentiating the conditional cumulative distribution function yields

$$\begin{aligned}f_{X_2|X_1}(t|t_1) &= \exp\left(-\int_{t_1}^{t_1+t} \lambda(u)du\right)\lambda(t_t + t) \\ &= \exp\left(m(t_1) - m(t_1+t)\right)\lambda(t_1+t).\end{aligned}$$

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(iii) Let $t > 0$. Then

unseen ⇓

$$\begin{aligned}f_{X_2}(t) &= \int_0^\infty f_{X_2|X_1}(t|t_1)f_{X_1}(t_1)dt_1 \\ &= \int_0^\infty \exp\left(m(t_1) - m(t_1+t)\right)\lambda(t_1+t)\exp(-m(t_1))\lambda(t_1)dt_1 \\ &= \int_0^\infty \exp\left(-m(t_1+t)\right)\lambda(t_1+t)\lambda(t_1)dt_1.\end{aligned}$$

2

3. (a) A discrete-time stochastic process $\{X_n\}_{n\in\{0,1,2,\dots\}}$ is a Markov chain on E if it satisfies the Markov condition: seen ⇓

$$\mathbb{P}(X_n = s | X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = s | X_{n-1} = x_{n-1})$$

for all integers $n \geq 1$ and for all $s, x_0, \dots, x_{n-1} \in E$. 4

(b) Let D_n be the score of the die at time n. Then D_n is a uniform random variable on $E = \{1, \dots, 6\}$. Then sim. seen ⇓

$$X_n = \min\{D_1, \dots, D_n\} = \min\{X_{n-1}, D_n\}.$$

Clearly X_n depends on $(X_1, \dots, X_{n-1})$ only through X_{n-1} so it is a Markov chain, i.e. $\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_1 = x_1) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1})$ for all integers $n \geq 2$ and for all $x_1, \dots, x_n \in E$. The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 6/6 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 4/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/6 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 2/6 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}.$$

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(c) (i) There are three communicating classes: $C_1 = \{1, 2\}$ and $C_2 = \{4, 5\}$ are closed and hence positive recurrent. The class $T = \{3\}$ is not closed and hence transient. sim. seen ⇓ 2

(ii) The Markov chain is not irreducible since it has more than one communicating class. 1

(iii) A vector $\boldsymbol{\pi}$ is a stationary distribution of the Markov chain on E if:

(1) for each $j \in E$, $\pi_j \geq 0$ and $\sum_{j\in E} \pi_j = 1$.

(2) $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$, that is, for each $j \in E$, $\pi_j = \sum_{i\in E} \pi_i p_{ij}$. 2

(iv) We consider the transition matrices restricted to the essential communicating classes:

$$\mathbf{P}(C_1) := \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{P}(C_2) := \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

Let $\boldsymbol{\pi}(C_1), \boldsymbol{\pi}(C_2)$ denote 2–dimensional row vectors. Solve

$$\boldsymbol{\pi}(C_1)\mathbf{P}(C_1) = \boldsymbol{\pi}(C_1), \qquad \boldsymbol{\pi}(C_2)\mathbf{P}(C_2) = \boldsymbol{\pi}(C_2).$$

Then $\boldsymbol{\pi}(C_1) = (a, 0.5a)$ and $\boldsymbol{\pi}(C_2) = (0.5b, b)$ for constants $a, b \in \mathbb{R}$. Now we define $\boldsymbol{\pi} := (a, 0.5a, 0, 0.5b, b)$ for constants $a, b \geq 0$ such that $1.5a + 1.5b = 1$. Then $\boldsymbol{\pi}$ is a stationary distribution, since $\pi_i \geq 0$ for $i = 1, \dots, 5$ and $\sum_{i=1}^{5} \pi_i = 1$. Also $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. 4

(v) The stationary distribution is not unique since we have two essential communicating classes. 1

4. (a) The process $\{X_t\}_{t\geq 0}$ has to satisfy the Markov property: $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_{t_n} = j | X_{t_1} = i_1, \dots, X_{t_{n-1}} = i_{n-1}) = \mathbb{P}(X_{t_n} = j | X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \dots, i_{n-1} \in E$ and for any sequence $0 \leq t_1 < \cdots < t_n < \infty$ of times (with $n > 1$). $\boxed{4}$

(b) $\boxed{\text{seen} \Downarrow}$

$$\mathbb{P}(X_{t+\delta} = n + m | X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \textit{if } m = 0, \\ \lambda_n \delta + o(\delta) & \textit{if } m = 1 \\ \mu_n \delta + o(\delta) & \textit{if } m = -1 \\ o(\delta) & \textit{if} |m| > 1 \end{cases}.$$

$\boxed{4}$

(c) The generator is $\boxed{\text{seen} \Downarrow}$

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\boxed{3}$

(d) (i) Let $p_n(t) := \mathbb{P}(N_t = n)$ for $n \in \{0, 1, 2, \dots\}$. We derive the forward equations: $\boxed{\text{unseen} \Downarrow}$

$$p_0(t + \delta) = \mathbb{P}(N_{t+\delta} = 0 | N_t = 0)\mathbb{P}(N_t = 0) = (1 - \beta\delta)p_0(t) + o(\delta).$$

Subtract $p_0(t)$ on both sides, divide by δ and consider $\delta \to 0$. Then we get

$$p_0'(t) = -\beta p_0(t).$$

Now, let $n \geq 1$. Then

$$\begin{aligned} p_{2n}(t + \delta) &= \sum_{l=0}^{\infty} \mathbb{P}(N_{t+\delta} = 2n | N_t = l) p_l(t) \\ &= (\alpha\delta + o(\delta))p_{2n-1}(t) + (1 - \beta\delta + o(\delta))p_{2n}(t) + o(\delta). \end{aligned}$$

Subtract $p_{2n}(t)$ on both sides, divide by δ and consider $\delta \to 0$. Then we get

$$p_{2n}'(t) = \alpha p_{2n-1}(t) - \beta p_{2n}(t).$$

Similarly, we get for $n \geq 0$:

$$p_{2n+1}'(t) = -\alpha p_{2n+1}(t) + \beta p_{2n}(t).$$

That is, altogether we have

$$\begin{aligned} p_0'(t) &= -\beta p_0(t) \\ p_n'(t) &= -\alpha p_n(t) + \beta p_{n-1}(t), \qquad n = 1, 3, 5, \dots \\ p_n'(t) &= \alpha p_{n-1}(t) - \beta p_n(t), \qquad n = 2, 4, 6, \dots \end{aligned}$$

$\boxed{5}$

(ii) Then $P_e(t) = \sum_{n=0}^{\infty} p_{2n}(t)$ and

$$\begin{aligned} P_e'(t) &= \sum_{n=0}^{\infty} p_{2n}'(t) = -\beta p_0'(t) + \sum_{n=1}^{\infty} (\alpha p_{2n-1}(t) - \beta p_{2n}(t)) \\ &= \alpha \sum_{n=1}^{\infty} p_{2n-1}(t) - \beta \sum_{n=0}^{\infty} p_{2n}(t) = \alpha P_o(t) - \beta P_e(t). \end{aligned}$$

Using $1 = P_e(t) + P_o(t)$, we also get $P_o'(t) = -\alpha P_o(t) + \beta P_e(t)$.
Now we solve

$$P_e'(t) = \alpha P_o(t) - \beta P_e(t) = \alpha(1 - P_e(t)) - \beta P_e(t) = \alpha - (\alpha + \beta) P_e(t),$$

using the integrating factor approach: Then $M(t) = \exp((\alpha+\beta)t)$. Note that $P_e(0) = P(N_0 \text{ is even}) = 1$. Altogether we have

$$\begin{aligned} P_e(t) &= \left(\int_0^t \alpha \exp((\alpha + \beta)u) du + 1 \right) \exp(-(\alpha + \beta)t) \\ &= \left(\frac{\alpha}{\alpha + \beta} \exp((\alpha + \beta)u) \Big|_0^t + 1 \right) \exp(-(\alpha + \beta)t) \\ &= \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} \exp(-(\alpha + \beta)t). \end{aligned}$$

Then

$$P_o(t) = 1 - P_e(t) = \frac{\beta}{\alpha + \beta} \left(1 - \exp(-(\alpha + \beta)t)\right).$$

$\boxed{4}$