1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(\mathbb{R}) \subset \mathbb{Q}$. Prove that f is constant.

   Solution. Suppose that $f(x) \neq f(y)$ for some $x < y$. Then the interval $[f(x), f(y)]$ contains at least one irrational number r (in fact, uncountably many), say $r = f(x) + \frac{f(y)-f(x)}{\sqrt{2}}$ for concreteness. The intermediate value theorem says that there is some $c \in [x, y]$ such that $f(c) = r$, but $r \notin \mathbb{Q}$, contradiction.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Is this still true if we only assume that $f(x) = g(x)$ for $x \in \mathbb{Z}$?

   Solution. We fix $x \in \mathbb{R}$ and take a sequence of rational numbers $y_1, y_2, \dots \to y$. Since f and g are continuous we have $f(y_n) \to f(y)$ and $g(y_n) \to g(y)$, but the sequences $\big(f(y_n)\big)$ and $\big(g(y_n)\big)$ are identical, so their limits $f(y)$ and $g(y)$ must be the same.

   On the other hand, let $f(x) = \sin(\pi x)$ and $g(x) = 0$. Then $f(x) = g(x)$ for all $x \in \mathbb{Z}$, but $f(\frac{\pi}{2}) = 1 \neq g(\frac{\pi}{2})$.

3. Prove the following:

   (a) $|\sin(x)| \leq |x|$, for all $x \in \mathbb{R}$.

   (b) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is uniformly continuous. (Hint: first prove the identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$.)

   Solution. (a) During the lectures we gave a geometric argument for $\sin(x) \leq x$, when $x \in [0, \pi/2]$. Since $\sin(x) = \sin(-x)$, this implies the claim also for $x \in [-\pi/2, 0]$. Finally, the rest follows since $-1 \leq \sin(x) \leq 1$.

   (b) The identity can be proved by writing

$$\begin{aligned}
\sin(\alpha) - \sin(\beta) &= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \\
&= \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\
&\quad - \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\
&= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).
\end{aligned}$$

   With it in hand, we have for any $x, y \in \mathbb{R}$ an inequality

$$\begin{aligned}
|f(x) - f(y)| &= \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right|,
\end{aligned}$$

since $|\cos(\theta)| \leq 1$ for all θ. Now we apply $|\sin(\theta)| \leq |\theta|$ from part (a) to get

$$|f(x) - f(y)| \leq 2\left|\frac{x-y}{2}\right| = |x-y|$$

for all $x, y \in \mathbb{R}$. Thus if we are given any $\epsilon > 0$, we can set $\delta = \epsilon > 0$, and we have

$$|x-y| < \delta \implies |f(x) - f(y)| \leq |x-y| < \delta = \epsilon$$

for all $x, y \in \mathbb{R}$, proving that f is indeed uniformly continuous.

4. Consider the function $f : [1,2] \cap \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = \left|x - \sqrt{2}\right|$. Prove that f does *not* have a minimum value. Why doesn't the extreme value theorem apply?

   Solution. We have $f(x) > 0$ for all x in the domain, since $\sqrt{2}$ is irrational. If we take a sequence of rational numbers $x_n \to \sqrt{2}$ then $f(x_n) \to 0$, so $\inf f(x) = 0$ and hence the infimum is not achieved anywhere on the domain. The extreme value theorem fails here because the domain is not closed, even though it's bounded and contains both a minimum and a maximum.

5. (*) Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n. \end{cases}$$

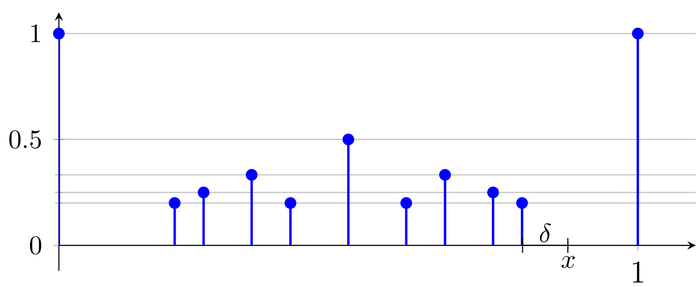
   Here all rational numbers $x = \frac{m}{n}$ are written in lowest terms, with $n > 0$.

   (a) Prove that if x is rational, then f is not continuous at x.

   (b) Prove that if x is irrational, then f is continuous at x.

   Solution. (a) If $x = \frac{m}{n}$, then we take $\epsilon = \frac{1}{2n}$ and find that for any $\delta > 0$ there are irrational y with $|y - x| < \delta$, and these satisfy $|f(y) - f(x)| = \frac{1}{n} > \epsilon$.

   (b) Suppose that $x \notin \mathbb{Q}$ and fix $\epsilon > 0$. There are only finitely many rational numbers $q_1, \ldots, q_k$ with denominator at most $\frac{1}{\epsilon}$ between the integers $\lfloor x \rfloor$ and $\lceil x \rceil$, inclusive, because no more than $d+1$ of them can have a given denominator d. For example, if $\frac{1}{\epsilon} = 5$ and $0 < x < 1$, we might have the following picture, where $f(\frac{m}{n})$ is shown for all rational numbers in $[0,1]$ with $n \leq 5$:



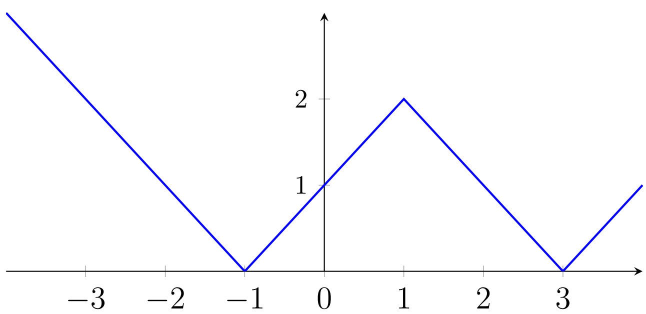
We let
$$\delta = \min_j |x - q_j| > 0,$$
and then if $|y - x| < \delta$, it follows that y is either irrational ($\Rightarrow\ f(y) = 0$) or has denominator greater than $\frac{1}{\epsilon}$ ($\Rightarrow\ f(y) < \epsilon$). So for all such y we have $|f(y) - f(x)| = |f(y)| < \epsilon$, and this proves continuity at x.

6. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that $f(a) \le y \le f(b)$.
   (a) Let $(a_0, b_0) = (a, b)$, and for all $n \ge 0$, define $m_n = \frac{a_n+b_n}{2}$ and
   $$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, m_n), & f(m_n) > y \\ (m_n, b_n), & f(m_n) \le y. \end{cases}$$
   Prove that the sequences $\big(a_n\big)$ and $\big(b_n\big)$ converge to the same limit $L \in [a, b]$.
   (b) Prove that $f(L) = y$, concluding a new proof of the intermediate value theorem.

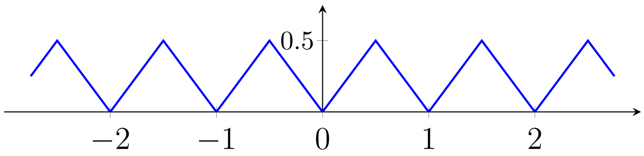
Solution. (a) We have $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n, so the sequence $\big(a_n\big)$ is increasing and bounded above by b, while the sequence $\big(b_n\big)$ is decreasing and bounded below by a. Thus $\big(a_n\big) \to L_a$ and $\big(b_n\big) \to L_b$ for some $L_a \le b$ and $L_b \ge a$. We also have
$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}\ \forall n \quad \Longrightarrow \quad \lim_{n\to\infty}(b_n - a_n) = 0,$$
but by the algebra of limits this means that $L_b - L_a = 0$, so $a \le L_b = L_a \le b$.
(b) By induction we see that $f(a_n) \le y$ for all n and also $f(b_n) \ge y$ for all n. Since f is continuous, we have $f(a_n) \to f(L)$, which now implies that $f(L) \le y$, and likewise $f(b_n) \to f(L)$ gives us $f(L) \ge y$. We combine these to conclude that $f(L) = y$.

7. For any nonempty set $S \subset \mathbb{R}$, define $d_S : \mathbb{R} \to \mathbb{R}$ by $d_S(x) = \inf_{s\in S} |x - s|$.
   (a) Describe or draw graphs of d_S when S is each of $\{0\}$, $\{-1, 3\}$, $\mathbb{Z}$, $\mathbb{Q}$.
   (b) Prove that $|d_S(y) - d_S(x)| \le |y - x|$ for all $x, y \in \mathbb{R}$, and conclude that d_S is continuous.

Solution. (a) We have $d_{\{0\}}(x) = |x|$. Here's a graph of $d_{\{-1,3\}}$:



And of $d_{\mathbb{Z}}$:



We have $d_{\mathbb{Q}}(x) = 0$, since there are rational numbers arbitrarily close to x.

(b) By the definition of $d_S(x)$, for all $n \geq 1$ there's an $s_n \in S$ such that $|x - s_n| < d_S(x) + \frac{1}{n}$, and the triangle inequality tells us that

$$|y - s_n| < |y - x| + |x - s_n| < |y - x| + d_S(x) + \frac{1}{n}.$$

Taking limits as $n \to \infty$ gives $d_S(y) \leq \inf_n |y - s_n| \leq |y - x| + d_S(x)$, hence

$$d_S(y) - d_S(x) \leq |y - x|.$$

We repeat this argument with x and y swapped to get $d_S(x) - d_S(y) \leq |y - x|$ as well, so $|d_S(y) - d_S(x)| \leq |y - x|$ as claimed.

Now for any $x \in \mathbb{R}$ and $\epsilon > 0$ we have $|y - x| < \epsilon \implies |d_S(y) - d_S(x)| \leq |y - x| < \epsilon$, so the definition of continuity at x is satisfied by taking $\delta = \epsilon$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, not necessarily continuous. Define $S(x) = \sup_{y<x} f(y)$ and $I(x) = \inf_{y>x} f(y)$.

   (a) Prove for all $x \in \mathbb{R}$ that $S(x) \leq f(x) \leq I(x)$.

   (b) Prove for all $x \in \mathbb{R}$ that $S(x) = I(x)$ if and only if f is continuous at x.

   (c) Find an injective mapping

   $$\{x \in \mathbb{R} \mid f \text{ is not continuous at } x\} \to \mathbb{Q}.$$

   Conclude that f is continuous at all but at most countably many real numbers.

Solution. (a) By monotonicity, we have $f(y) \leq f(x)$ for all $y < x$, so $f(x)$ is an upper bound for $\{f(y) \mid y < x\}$ and hence $f(x) \geq S(x)$. The proof that $f(x) \leq I(x)$ is the same.

(b) ($\implies$) Suppose that $S = f(x) = I$, and fix $\epsilon > 0$. We can find $z < x$ such that $f(z) > f(x) - \epsilon$, since $f(x) = \sup_{z<x} f(z)$, and likewise $w > x$ such that $f(w) < f(x) + \epsilon$. If we let $\delta = \min(x - z, w - x) > 0$, then $|y - x| < \delta$ implies $z < y < w$, and hence implies

$$f(y) \in [f(z), f(w)] \subset (f(x) - \epsilon, f(x) + \epsilon)$$

by the monotonicity of f. So $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$ and the continuity of f at x follows.

($\Longleftarrow$) Suppose that f is continuous at x. Then for any $\epsilon > 0$ we can find $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, and if we take $y = x \pm \frac{\delta}{2}$ then

$$S \geq f(x - \tfrac{\delta}{2}) > f(x) - \epsilon \qquad \text{and} \qquad I \leq f(x + \tfrac{\delta}{2}) < f(x) + \epsilon.$$

Now $f(x)$ is an upper bound for $\{f(y) \mid y < x\}$ by monotonicity, so we have

$$f(x) \geq S > f(x) - \epsilon \text{ for all } \epsilon > 0 \quad \Longrightarrow S = f(x)$$

and similarly $f(x) \leq I < f(x) + \epsilon$ leads to $f(x) = I$.

(c) Parts (a) and (b) say that if f is discontinuous at x then the open interval $(S(x), I(x))$ is nonempty, so we can pick a rational number q_x in this interval. If $x < y$ are two points of discontinuity, then we have $I(x) \leq f(\frac{x+y}{2}) \leq S(y)$, so the intervals $(S(x), I(x))$ and $(S(y), I(y))$ are disjoint and thus $q_x \neq q_y$. Therefore the mapping $x \mapsto q_x$ is injective.

9. Remember that a set U is open if for all $x \in U$, there exist $\delta = \delta(x) > 0$, such that $(x - \delta, x + \delta) \subset U$. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage

$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$

is open.

Solution. ($\Longrightarrow$): Suppose that f is continuous, and fix an open set $U \subset \mathbb{R}$. Let x be a point of $f^{-1}(U)$; then $f(x) \in U$ by definition, and since U is open, there is some $\epsilon > 0$ such that the whole open interval $(f(x) - \epsilon, f(x) + \epsilon)$ is a subset of U. Since f is continuous at x, there is $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, hence

$$f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset U.$$

But then $y \in f^{-1}(U)$ for all such y, so $(x - \delta, x + \delta) \subset U$. Since we can find such a neighborhood for any $x \in f^{-1}(U)$, it follows that $f^{-1}(U)$ is open.

($\Longleftarrow$): We will show that f is continuous at any $x \in \mathbb{R}$. Fix $\epsilon > 0$ and let $U = (f(x) - \epsilon, f(x) + \epsilon)$. Then $f^{-1}(U)$ contains x by definition, and since U is open, so is $f^{-1}(U)$. This means that $f^{-1}(U)$ contains an open neighborhood $(x - \delta, x + \delta)$ of x for some $\delta > 0$. Now if $|y - x| < \delta$ then

$$y \in f^{-1}(U) \implies f(y) \in U = (f(x) - \epsilon, f(x) + \epsilon) \implies |f(y) - f(x)| < \epsilon,$$

and we can do this for any $\epsilon > 0$, so f is continuous at x.

10. Give an example of a bounded open set $S \subset \mathbb{R}$ and a continuous function $f : S \to \mathbb{R}$ which does *not* satisfy the intermediate value theorem: in other words, there are points $a < b$ in S and some x between $f(a)$ and $f(b)$ such that $f(c) \neq x$ for all $c \in S$.

Solution. Let $S = (0, 1) \cup (3, 4)$. The function $f : S \to \mathbb{R}$ given by $f(x) = x$ is continuous, and it satisfies $f(1) = 1$ and $f(3) = 3$, but there is no $c \in S$ such that $f(c) = 2$.

11. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is closed.

    Solution. Let $(x_n) \subset f^{-1}(c)$ be a sequence which converges to a limit $x \in \mathbb{R}$. By sequential continuity we have $f(x_n) \to f(x)$, but $f(x_n) = c$ for all n, so $f(x) = c$ as well and thus $x \in f^{-1}(c)$. It follows that the limit of any convergent sequence in $f^{-1}(c)$ also lies in $f^{-1}(c)$, so $f^{-1}(c)$ is closed.

12. Let $(S_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of nonempty subsets of $\mathbb{R}$, meaning that

    $$S_1 \supset S_2 \supset S_3 \supset \dots .$$

    Let $S = \bigcap_{n=1}^{\infty} S_n$ be their intersection.

    (a) Give an example where all of the S_n are open and S is empty.

    (b) Prove that if all of the $S_n = [a_n, b_n]$ are closed intervals, then S is nonempty. (Hint: consider the sequence a_n.)

    Solution. (a) Take $S_n = (0, \frac{1}{n})$ for all $n \geq 1$.

    (b) Since $S_1 \supset S_2 \supset S_3 \cdots$, the sequence a_n is monotone increasing and bounded above by b_1, so it must converge, say to $a_n \to a$. We claim that $a \in S$. In fact, fix any $N \in \mathbb{N}$, then for all $n \geq N$, we have $a_N \leq a_n \leq b_N$. Taking the limit when $n \to \infty$, one concludes $a_N \leq a \leq b_N$. This shows that $a \in S_N$, for all $N \in \mathbb{N}$.

13. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}$ is a closed and bounded interval, then the image $f(S)$ is also a closed bounded interval. Is the same true for open intervals?

    Solution. By the extreme value theorem we have $\min_S f \in f(S)$ and $\max_S f \in f_(S)$. Clearly, $f(S) \subset [\min_S f, \max_S f]$. By the intermediate value theorem, for all $y \in [\min_S f, \max_S f]$, there exists $x \in S$ such that $f(x) = y$. This implies $[\min_S f, \max_S f] \subset f(S)$. The statement is false for open intervals: take any constant function on an open bounded interval. Its image is not open.

14. (a) Show that $f(x) = x^{1/2}$ is differentiable on $(0, \infty)$, and compute its derivative.

    (b) Do the same for $f(x) = x^{1/n}$, where n is any positive integer.

    (c) Now do the same for $f(x) = x^{m/n}$, where m and n are positive integers.

Solution. (a) We observe for $x, a > 0$ that $x - a = (x^{1/2} - a^{1/2})(x^{1/2} + a^{1/2})$, so

$$\lim_{x\to a} \frac{x^{1/2} - a^{1/2}}{x - a} = \lim_{x\to a} \frac{1}{x^{1/2} + a^{1/2}} = \frac{1}{2a^{1/2}}.$$

The last step implicitly uses the fact that $x^{1/2}$ is continuous for $x > 0$, but this follows from it being the inverse of the continuous function x^2 on the same interval. Therefore f is differentiable for all $x > 0$, with $f'(x) = \frac{1}{2}x^{-1/2}$.

(b) Each $x^{1/n}$ is continuous for $x > 0$, since it is the inverse of the continuous function x^n. We apply the identity

$$\begin{aligned} x - a &= (x^{1/n})^n - (a^{1/n})^n \\ &= (x^{1/n} - a^{1/n})(x^{(n-1)/n} + x^{(n-2)/n}a^{1/n} + \cdots + x^{1/n}a^{(n-2)/n} + a^{(n-1)/n}) \end{aligned}$$

to write

$$\lim_{x\to a} \frac{x^{1/n} - a^{1/n}}{x - a} = \lim_{x\to a} \frac{1}{\sum_{i=0}^{n-1} x^{(n-1-i)/n}a^{i/n}} = \frac{1}{na^{(n-1)/n}} = \frac{a^{(1-n)/n}}{n}.$$

So $f(x) = x^{1/n}$ has derivative $f'(x) = \frac{1}{n}x^{1/n-1}$.

(c) At this point we might well guess that $x^{m/n}$ should have derivative $\frac{m}{n}x^{m/n-1}$, and we can prove it by induction. We've already done the case $m = 1$. If the claim holds for exponent $\frac{m-1}{n}$ then we let $f(x) = x^{(m-1)/n}$ and $g(x) = x^{1/n}$, and apply the product rule to $h(x) = f(x)g(x) = x^{m/n}$ to see that $x^{m/n}$ is differentiable on $(0, \infty)$ with derivative

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \left(\frac{m-1}{n}x^{(m-1)/n-1}\right) x^{1/n} + x^{\frac{m-1}{n}} \left(\frac{1}{n}x^{1/n-1}\right) \\ &= \frac{m-1}{n}x^{m/n-1} + \frac{1}{n}x^{m/n-1} \\ &= \frac{m}{n}x^{m/n-1}. \end{aligned}$$

This proves the claim by induction on m.

15. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Hölder continuous* with exponent $\alpha > 0$ if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbb{R}$. Show that if $\alpha > 1$ then f is differentiable, and $f'(x) = 0$.

Remark: We will see in lecture soon that if $f' \equiv 0$ then f must be constant.

Solution. For $x \neq y$ we can write

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq C|x - y|^\beta,$$

where $\beta = \alpha - 1$ is strictly positive. But then $\lim_{x\to y} |x-y|^\beta = 0$ for any fixed y, so on the left side we must have

$$\lim_{x\to y} \left| \frac{f(x)-f(y)}{x-y} \right| = 0$$

and this imeans that $f'(y)$ exists and is zero.

16. Find all $x \in \mathbb{R}$ where $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ x^2, & x \in \mathbb{Q} \end{cases}$ is differentiable and compute its derivative.

Solution. We know that $f(x)$ is not continuous at any nonzero $x = a$, because we can find a sequence of rationals $r_n \to a$ with $f(r_n) = r_n^2 \to a^2$ and a sequence of irrationals $s_n \to a$ with $f(s_n) = 0 \to 0$, and these limits are not equal, whereas if f were continuous at a then they would have both been equal to $f(a)$. Since f is continuous at all points where it is differentiable, it cannot be differentiable anywhere except possibly at $x = 0$.

On the other hand, for nonzero x we have

$$\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \begin{cases} 0, & x \notin \mathbb{Q} \\ x, & x \in \mathbb{Q} \end{cases}.$$

Regardless of whether $x \neq 0$ is rational or irrational, it follows that

$$\left| \frac{f(x)-f(0)}{x-0} \right| \leq |x| \text{ for all } x \neq 0 \implies \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0.$$

So f is differentiable at x if and only if $x = 0$, and $f'(0) = 0$.

17. $(*)$ Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. We will prove that $f'(x)$ has the *intermediate value property* even though it may not be continuous. In both parts we will suppose that $f'(a) < f'(b)$ and fix some t such that $f'(a) < t < f'(b)$.

   (a) Let $g(x) = f(x) - tx$. Prove that there is some $c \in (a,b)$ such that $g(c) < g(a)$. (Hint: what is $g'(a)$?) Similarly, prove that there is some $d \in (a,b)$ such that $g(d) < g(b)$. In other words, $g(x)$ is not minimized at $x = a$ or at $x = b$.

   (b) Show that $g'(y) = 0$ for some $y \in (a,b)$, and deduce that $f'(y) = t$.

Solution. (a) We know that $g(x)$ is differentiable, with $g'(x) = f'(x) - t$, so in particular $g'(a) < 0$. Fixing $\epsilon = |g'(a)| > 0$, there is $\delta > 0$ such that

$$a < x < a + \delta \Rightarrow \left| \frac{g(x)-g(a)}{x-a} - g'(a) \right| < \epsilon = -g'(a),$$

and since $x - a$ is positive, this implies that

$$\frac{g(x)-g(a)}{x-a} - g'(a) < -g'(a) \Rightarrow g(x) - g(a) < 0.$$

Thus $g(x) < g(a)$ for all $x \in (a, a+\delta)$ and we can take $c = \min(a + \frac{\delta}{2}, \frac{a+b}{2})$. (The point of taking this minimum is just to make sure that $c \in (a,b)$.)
Similarly, we have $g'(b) > 0$, so for $\epsilon = g'(b)$ we can find $\delta > 0$ such that

$$b - \delta < x < b \;\Rightarrow\; \left| \frac{g(x) - g(b)}{x - b} - g'(b) \right| < \epsilon = g'(b).$$

We deduce from this and the fact that $x - b < 0$ that

$$\frac{g(x) - g(b)}{x - b} - g'(b) \geq -g'(b) \;\Rightarrow\; g(x) - g(b) \leq 0,$$

so $g(x) < g(b)$ for $x \in (b - \delta, b)$ and we can take $d = \max(b - \frac{\delta}{2}, \frac{a+b}{2})$.

(b) We know that g is continuous since it is differentiable, so the extreme value theorem says that $g(x)$ achieves a minimum at some $y \in [a,b]$. By part (a) we know that $y \neq a$ and $y \neq b$, so $y \in (a,b)$, and since y is a local minimum of g it follows that $g'(y) = 0$. Since $g'(x) = f'(x) - t$, we must have $f'(y) = t$.