Imperial College London

MATH97023 MATH97005

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Tensor Calculus and General Relativity

Date: 1st June 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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Formula Sheet

Christoffel symbol:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

Covariant derivatives:

$$\nabla_c v^a = \partial_c v^a + \Gamma^a_{bc} v^b.$$

$$\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a.$$

Riemann curvature tensor:

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}$$

Symmetries of Riemann tensor:

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab},$$

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0.$$

Ricci tensor and scalar curvature:

$$R_{bd} = R^a_{bad}, \quad R_{bd} = R_{db}, \quad R = g^{bd} R_{bd}.$$

1. (a) The standard Lorentz boost is (here K' is moving in the x direction with speed v relative to the frame K)

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z,$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

A photon in K moves along the y-axis (upwards) at the speed of light c. Determine dx'/dt' and dy'/dt' for the photon.

(5 marks)

(b) What is meant by a contravariant four vector? Define an inner product for such vectors.

(4 marks)

(c) Define the four-acceleration, a^μ, of a particle and show that $u \cdot a = 0$ where u is the four-velocity.

(5 marks)

(d) The motion of a particle is governed by the equation

$$a^\mu = \kappa(c^2 \delta^\mu_\alpha - u^\mu u_\alpha)\frac{da^\alpha}{d\tau}.$$

where κ is a constant and τ denotes proper time (this equation of motion is unusual in that it includes the rate of change of acceleration). How does $a \cdot a = a_\mu a^\mu$ depend on proper time τ?

(6 marks)

(Total: 20 marks)

2. The standard metric on a unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

where θ and ϕ are spherical polar coordinates.

(a) Write down $g_{\theta\theta}$, $g_{\phi\phi}$ and $g_{\theta\phi} = g_{\phi\theta}$. What is $g^{\phi\phi}$?

(4 marks)

(b) Use any method to show that the non-zero Christoffel symbols are

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

(8 marks)

(c) The covariant vector field v has components $v_\theta = 0$ and $v_\phi = \sin^2\theta$. Show that $\nabla_\theta v_\theta$ and $\nabla_\phi v_\phi$ are zero. Compute the covariant derivatives $\nabla_\theta v_\phi$ and $\nabla_\phi v_\theta$. Comment on the results.

(8 marks)

(Total: 20 marks)

3. (a) Use the equation

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma_{bac} - \Gamma_{abc} = 0,$$

where

$$\Gamma_{abc} = g_{ad}\Gamma^d_{bc}$$

and the zero torsion condition

$$\Gamma_{abc} = \Gamma_{acb}$$

to show that

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}\right).$$

(7 marks)

(b) The type $(0,4)$ curvature tensor can be written in the form

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} - g^{ef}\left(\Gamma_{fac}\Gamma_{ebd} - \Gamma_{fad}\Gamma_{ebc}\right).$$

Establish the symmetry properties

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = R_{cdab}.$$

Hint: for any given point the coordinate system can be chosen so that $\Gamma_{abc} = 0$ and

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc}.$$

(8 marks)

(c) Show that the Ricci tensor is symmetric. (5 marks)

(Total: 20 marks)

4. The Schwarzschild metric is

$$ds^2 = c^2\left(1-\frac{R}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{R}{r}\right)} - r^2 d\phi^2,$$

where R is the Schwarzschild radius and $\theta = \pi/2$.

(a) Show that $h = r^2 d\phi/ds$ is constant along geodesics and

$$c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1. \tag{1}$$

(4 marks)

(b) The geodesic equation for the r coordinate is

$$\left(1-\frac{R}{r}\right)^{-1}\frac{d^2r}{ds^2} = r\left(\frac{d\phi}{ds}\right)^2 - \frac{c^2R}{2r^2}\left(\frac{dt}{ds}\right)^2 + \frac{R}{2r^2}\left(1-\frac{R}{r}\right)^{-2}\left(\frac{dr}{ds}\right)^2. \tag{2}$$

Use this together with the results quoted in part (a) to show that there are circular orbits ($r =$constant) for $r > \frac{3}{2}R$. What happens if $r = \frac{3}{2}R$?
Hint: use (2) to calculate h^2 and insert the result into (1) to determine $(dt/ds)^2$ for circular orbits.

(6 marks)

(c) Show that $k = (1 - R/r)dt/ds$ is constant along geodesics and use this result to rewrite (1) in the form

$$\left(\frac{dr}{ds}\right)^2 + U(r,h) = \text{constant}, \tag{3}$$

where U is an effective potential.

(5 marks)

(d) Show that circular orbits are stable if $r > 3R$.
Hint: consider $U''(r,h) + 2U'(r,h)/r$. The prime denotes differentiation with respect to r.

(5 marks)

(Total: 20 marks)

5. A two dimensional space with coordinates r and θ has the metric

$$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\theta^2.$$

The non-zero Christoffel symbols are

$$\Gamma^r_{rr} = -\frac{r}{1+r^2} \qquad \Gamma^r_{\theta\theta} = -r(1+r^2), \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}.$$

Here $h = r^2 d\theta/ds$ is constant along geodesics.

(a) Use

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 1,$$

to find all geodesics for which $h = 1$.

Hints: Express s as an integral with respect to r. Use the substitution $r^2 = \cosh w$ to compute the integral. To determine θ as a function of s use the integral

$$\int \frac{du}{\cosh u} = \tan^{-1}(\sinh u) + c.$$

(10 marks)

(b) Determine the form of the Laplacian. That is compute $g^{ab}\nabla_a\nabla_b\phi$ where ϕ is a scalar field.

(10 marks)

(Total: 20 marks)

Tensor Calculus and General Relativity **Summer Term 2020**

Answers to Summer Examination

1. (a) $dx' = \gamma(dx - vdt)$, $dt' = \gamma(dt - vdx/c^2)$, $dy' = dy$, $dz' = dz$.

$$\frac{dx'}{dt'} = \frac{dx - vdt}{dt - vdx/c^2} = \frac{\dot{x} - v}{1 - v\dot{x}/c^2}, \quad \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - vdx/c^2)} = \frac{\dot{y}}{\gamma(1 - v\dot{x}/c^2)}.$$

For the photon $\dot{x} = 0$, $\dot{y} = c$ so that

$$\frac{dx'}{dt'} = -v, \quad \frac{dy'}{dt'} = \frac{c}{\gamma} = \sqrt{c^2 - v^2}.$$

(5 marks, seen similar A)

(b) A contravariant four vector is four numbers v^μ ($\mu = 0, 1, 2, 3$) with the transformation property

$$v^{\mu'} = \Lambda^{\mu'}_{\nu} v^\nu$$

under the Lorentz transformation

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu.$$

The inner product of two four vectors u and v is $u \cdot v = \eta_{\mu\nu} u^\mu v^\nu$ where $\eta_{\mu\nu}$ is the metric.

(4 marks, bookwork A)

(c) The four-acceleration of a particle is defined through

$$a^\mu = \frac{du^\mu}{d\tau},$$

where u^μ is the four velocity and τ denotes proper-time. Using the property $u \cdot u = \eta_{\mu\nu} u^\mu u^\nu = c^2$

$$0 = \frac{d}{d\tau}\eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu + \eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} = 2u \cdot a,$$

giving $u \cdot a = 0$.

(5 marks, bookwork B)

(d) Taking the dot product of a and the equation of motion gives

$$a_\mu a^\mu = \kappa a_\mu (c^2 \delta^\mu_\alpha - u^\mu u_\alpha) \frac{da^\alpha}{d\tau} = \kappa c^2 a_\alpha \frac{da^\alpha}{d\tau}$$

as $a_\mu u^\mu = 0$. This can be written in the form

$$a_\mu a^\mu = \frac{\kappa c^2}{2} \frac{d}{d\tau} a_\mu a^\mu,$$

giving

$$a \cdot a = A \exp\left(\frac{2\tau}{\kappa c^2}\right),$$

where A is an arbitrary constant.

(6 marks, unseen D)

2. (a) $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$, $g_{\theta\phi} = g_{\phi\theta} = 0$. $g^{\phi\phi} = 1/\sin^2\theta$.

(4 marks, seen similar A)

(b)

$$\Gamma^{\theta}_{bc} = \frac{g^{\theta d}}{2}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right).$$

To get a non-zero result requires $d = \theta$ and $b = c = \phi$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}\Big(0 + 0 - \partial_\theta \sin^2\theta\Big) = -\sin\theta\cos\theta.$$

$$\Gamma^{\phi}_{bc} = \frac{g^{\phi d}}{2}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right).$$

To get a non-zero result requires $d = \phi$ and $(b, c) = (\theta, \phi)$ or $(b.c) = (\phi, \theta)$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

Alternatively use the Lagrangian $L = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$.

(8 marks, seen A)

(c) Use $\nabla_c v_b = \partial_c v_b - \Gamma^a_{bc} v_a$.

$\nabla_\theta v_\theta = \partial_\theta v_\theta - \Gamma^a_{\theta\theta} v_a = 0 - \Gamma^{\phi}_{\theta\theta} v_\phi = 0 - 0 = 0.$

$\nabla_\phi v_\phi = \partial_\phi v_\phi - \Gamma^a_{\phi\phi} v_a = 0 - \Gamma^{\phi}_{\phi\phi} v_\phi = 0.$

$\nabla_\theta v_\phi = \partial_\theta v_\phi - \Gamma^a_{\phi\theta} v_a = \partial_\theta \sin^2\theta - \Gamma^{\phi}_{\phi\theta} v_\phi = 2\sin\theta\cos\theta - \cot\theta\sin^2\theta = \sin\theta\cos\theta$

$\nabla_\phi v_\theta = \partial_\phi v_\theta - \Gamma^a_{\theta\phi} v_a = 0 - \Gamma^{\phi}_{\theta\phi} v_\phi = -\cot\theta \cdot \sin^2\theta = -\sin\theta\cos\theta.$

v is a Killing vector as $\nabla_a v_b + \nabla_b v_a = 0$. Therefore $v_a dx^a/ds = \sin^2\theta d\phi/ds$ is constant along geodesics.

(8 marks, unseen C)

3. (a)

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma_{bac} - \Gamma_{abc} = 0,$$

Cycling the indices:

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma_{cba} - \Gamma_{bca} = 0,$$

$$\nabla_b g_{ca} = \partial_b g_{ca} - \Gamma_{acb} - \Gamma_{cab} = 0$$

Take the second equation and subtract the other two:

$$\partial_a g_{bc} - \partial_b g_{ca} - \partial_c g_{ab} = \Gamma_{cba} + \Gamma_{bca} - \Gamma_{acb} - \Gamma_{cab} - \Gamma_{bac} - \Gamma_{abc} = -2\Gamma_{abc},$$

using the zero torsion condition. Hence the result. (7 marks, seen A)

(b)

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc}$$
$$= \frac{1}{2}\partial_c(\partial_b g_{ad} + \partial_d g_{ab} - \partial_a g_{bd}) - \frac{1}{2}\partial_d(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc})$$
$$= \frac{1}{2}\left(\partial_b\partial_c g_{ad} - \partial_a\partial_c g_{bd} - \partial_b\partial_d g_{ac} + \partial_a\partial_d g_{bc}\right).$$

$R_{bacd} = \frac{1}{2}\left(\partial_a\partial_c g_{bd} - \partial_b\partial_c g_{ad} - \partial_a\partial_d g_{bc} + \partial_b\partial_d g_{ac}\right) = -R_{abcd}$.

Exchange a and c and also exchange b and d. Under these swaps the middle two terms are unchanged while the first and last terms are exchanged. Hence $R_{abcd} = R_{cdab}$.

(8 marks, unseen C)

(c) $R_{bd} = R^a_{\ bad} = g^{ae}R_{ebad} = g^{ae}R_{adeb} = R^e_{\ deb} = R_{db}$

(5 marks, seen B)

4. (a) The component of the metric do not depend on ϕ. Hence $g_{\phi\mu}dx^\mu/ds = -r^2 d\phi/ds = -h$ is constant along geodesics.

Alternatively, ϕ is cyclic in a Lagrangian approach.

The equation $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1$ yields

$$c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2$$

$$= c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

(4 marks, seen similar A)

(b) Setting r =constant in (2) gives

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{c^2R}{2r^3}\left(\frac{dt}{ds}\right)^2.$$

or

$$h^2 = \frac{c^2Rr}{2}\left(\frac{dt}{ds}\right)^2$$

Inserting this into (1) yields

$$c^2\left(1-\frac{R}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{c^2R}{2r}\left(\frac{dt}{ds}\right)^2 = 1,$$

or

$$c^2\left(1-\frac{3R}{2r}\right)\left(\frac{dt}{ds}\right)^2 = 1.$$

Therefore

$$c^2\left(\frac{dt}{ds}\right)^2 = \frac{1}{\left(1-\frac{3R}{2r}\right)}.$$

As the left hand side is positive $r > \frac{3}{2}R$. At $r = \frac{3}{2}R$ the geodesic is null.

(6 marks, seen similar B)

(c) As the components of the metric do not depend on t $g_{t\mu}dx^\mu/ds = c^2(1-R/r)^{-1}dt/ds = c^2k$ is constant along geodesics

$$c^2k^2\left(1-\frac{R}{r}\right)^{-1} - \left(1-\frac{R}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} = 1.$$

Multiplying by $(1 - R/r)$ gives

$$c^2k^2 - \left(\frac{dr}{ds}\right)^2 - h^2\left(\frac{1}{r^2} - \frac{R}{r^3}\right) = 1 - \frac{R}{r}.$$

Therefore

$$\left(\frac{dr}{ds}\right)^2 + U(r,h) = \text{constant},$$

where

$$U(r,h) = h^2\left(\frac{1}{r^2} - \frac{R}{r^3}\right) - \frac{R}{r}.$$

(5 marks, seen similar B)

(d)

$$U'(r,h) = h^2\left(-\frac{2}{r^3} + \frac{3R}{r^4}\right) + \frac{R}{r^2}$$

$$U''(r,h) = h^2\left(\frac{6}{r^4} - \frac{12R}{r^5}\right) - \frac{2R}{r^3}.$$

Therefore

$$U''(r,h) + \frac{2}{r}U'(r) = h^2\left(\frac{2}{r^4} - \frac{6R}{r^5}\right).$$

This is equal to $U''(r,h)$ at a stationary point. This is positive for $r > 3$

(5 marks, unseen D)

5. (a)

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = \frac{1}{1+r^2}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{1+r^2}\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} = 1.$$

Setting $h = 1$ gives

$$\frac{1}{1+r^2}\left(\frac{dr}{ds}\right)^2 = 1 - \frac{1}{r^2} = \frac{r^2-1}{r^2}$$

or

$$\int \frac{r}{\sqrt{r^4-1}}dr = \int ds = s + c.$$

Using the substitution $r^2 = \cosh w$, $2rdr = \sinh w dw$

$$\frac{1}{2}\int dw = \frac{w}{2} = s + c$$

giving $r^2 = \cosh(2s + C)$ To obtain θ use $h = r^2 d\phi/ds = 1$:

$$d\phi = \frac{1}{\cosh(2s+C)}ds$$

$$\phi = \int \frac{ds}{\cosh(2s+C)} = \frac{1}{2}\tan^{-1}[\sinh(2s+C)] + D.$$

(10 marks, seen similar)

(b)

$$\begin{aligned}
g^{ab}\nabla_a\nabla_b\phi &= g^{ab}\nabla_a\partial_b\phi = g^{ab}(\partial_a\partial_b - \Gamma^d_{ba}\partial_d)\phi \\
&= g^{rr}(\partial_r^2 - \Gamma^d_{rr}\partial_d)\phi + g^{\theta\theta}(\partial_\theta^2 - \Gamma^d_{\theta\theta}\partial_d)\phi \\
& g^{rr}(\partial_r^2 - \Gamma^r_{rr}\partial_r)\phi + g^{\theta\theta}(\partial_\theta^2 - \Gamma^r_{\theta\theta}\partial_r)\phi \\
&= (1+r^2)\left(\partial_r^2 + \frac{r}{1+r^2}\partial_r\right)\phi + \frac{1}{r^2}(\partial_\theta^2 + r(1+r^2)\partial_r)\phi \\
&= \left[(1+r^2)\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r} + 2r\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi.
\end{aligned}$$

(10 marks, seen similar)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97023 MATH97005	1	The first 3 parts were standard and very well answered. Students found the unseen part (d) difficult.	
MATH97023 MATH97005	2	This was well answered. A few students didn't recognise that part (c) was leading to the Killing equations. However, this did not affect the generally high marks for this question.	
MATH97023 MATH97005	3	In this question part (b) was problematic for some.	
MATH97023 MATH97005	4	This question contained fairly standard material about the Schwarzschild metric. The last unseen part on stability of orbits was handled well by a majority of candidates.	
MATH97023 MATH97005	5	This question concerned the space of negative curvature considered in the second in-class test. The Laplacian computation in part (b) was well done. Although (a) is straightforward the number of steps involved meant that most students didn't quite get to the end.	