Imperial College London

MATH70061

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Commutative Algebra

Date: 26 May 2023

Time: 10:00 – 12:30 (BST)

Time Allowed: 2.5hrs

This paper has 5 Questions.

Please Answer All Questions in 1 Answer Booklet

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

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1. Let k be a field. Suppose that we have n points $V = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subset k^2$ where $a_1, \ldots, a_n$ are distinct. Let

$$f(X) = \sum_{i=1}^{n} b_i \prod_{j \neq i} \frac{X - a_j}{a_i - a_j} \in k[X]$$

be the Lagrange interpolation polynomial.

(a) Show that $f(X)$ is the unique polynomial of degree $\leq n-1$ satisfying $f(a_i) = b_i$ for $i = 1, \ldots, n$. (6 marks)

(b) Prove that $\mathcal{I}(V) = (Y - f(X), g(X)) \subset k[X, Y]$ where $g(X) = \prod_{i=1}^{n}(X - a_i)$. (7 marks)

(c) Prove that $\{Y - f(X), g(X)\}$ is the reduced Gröbner basis for $\mathcal{I}(V)$ for the lex order with $Y > X$. (7 marks)

(Total: 20 marks)

2 Let A be a commutative ring with unit.

(a) Suppose that all the prime ideals of A are finitely generated. Show that A is Noetherian.

[*Hint: Let Γ be the set of all ideals of A which are not finitely generated. If $\Gamma \neq \emptyset$, use Zorn's lemma to show that Γ contains a maximal element I. As I is not a prime ideal, there exist $x, y \in A$ such that $x \notin I$, $y \notin I$ but $xy \in I$. Consider the ideals $I + (y)$ and $I : y$. Show that these two ideals are finitely generated and deduce that I is finitely generated to arrive at a contradiction.*] (10 marks)

(b) Suppose that A is an integral domain and all the prime ideals of A are principal. Show that A is a principal ideal domain (PID).

[*Hint: Follow a similar approach to the one in Part (a).*]

(10 marks)

(Total: 20 marks)

3. (a) Define the **Krull dimension** of a ring. (3 marks)

(b) What is the Krull dimension of $\mathbb{Z}[i]$? Justify your answer. (5 marks)

(c) Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Show that if there exist elements $x_1, \dots, x_n \in \mathfrak{m}$ such that $R/(x_1, \dots, x_n)$ is Artinian, then $\dim R \leq n$. (5 marks)

(d) Conversely, let R be a local Noetherian ring and $\mathfrak{m}$ be the maximal ideal of height $n = \dim R$. Show that there exist elements $x_1, \dots, x_n \subset \mathfrak{m}$ such that $\mathfrak{m}$ is minimal among prime ideals containing $(x_1, \dots, x_n)$.

[*Hint: To argue by induction, suppose that you have constructed elements* $x_1, \dots, x_k$ *such that any prime minimal among prime ideals containing* $(x_1, \dots, x_k)$ *has height* k.]

(7 marks)

(Total: 20 marks)

4. (a) Describe the set of all polynomials in $\mathbb{C}[X, Y]$ that vanish on $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 - 1 = 0\}$. Justify your answer. (2 marks)

(b) (i) Let k be a field that is not algebraically closed and $I \subset k[X_1, \dots, X_n]$ an arbitrary ideal. Show that the variety $\mathcal{V}(I) \subset k^n$ can be written as the zero set of a single polynomial in $k[X_1, \dots, X_n]$.

[*Hint: If* $I = (f_1, \dots, f_m)$, *consider* $\phi_m(f_1, \dots, f_m)$ *where* $\phi_m \in k[X_1, \dots, X_m]$ *whose only zero is* $(0, \dots, 0) \in k^m$. *To find such* ϕ_m, *first construct* $\phi_2(X, Y) \in k[X, Y]$ *which only vanishes at* $(0, 0)$ *and then apply induction by defining* $\phi_m(X_1, \dots, X_m) = \phi_2(\phi_{m-1}(X_1, \dots, X_{m-1}), X_m)$.] (6 marks)

(ii) Give an example of an affine variety in $\mathbb{C}^2$ that cannot be written as the zero set of a single polynomial. Justify your answer. (6 marks)

(c) Let $\mathfrak{m} \subset \mathbb{R}[X, Y]$ be a maximal ideal, what are the possible fields that one can get as $\mathbb{R}[X, Y]/\mathfrak{m}$? Justify your answer. For each such field, give a specific example of a maximal ideal. (6 marks)

(Total: 20 marks)

5. (a) (i) Define what it means for a ring to be a **valuation ring**. (2 marks)

(ii) Let K be a field, define the notion of a **valuation** of K. What is the valuation ring R associated to such a valuation? In which case is R called a **discrete valuation ring (DVR)**? (2 marks)

(iii) Show that a DVR is normal. (3 marks)

(b) Which of the following sets is a valuation ring of the field $\mathbb{C}(X,Y)$? Justify your answer.

(i) The subring $R = \mathbb{C}[X,Y]_{(Y)}$. (3 marks)

(ii) The subring $\mathbb{C}[X,Y]_{(X,Y)}$. (3 marks)

(c) If R is a valuation ring with field of fractions K and $\dim R = 1$ then show that R is maximal as a subring of K. Conversely, show that a maximal proper subring of a field is a valuation ring of dimension 1.

[*Hint: To show the converse, first prove that a maximal proper subring R of a field K is integrally closed. Now take $x \in K \setminus R$, and show that $x^{-1} \in R$.*]

(7 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2023

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MATH70061

Commutative Algebra (Solutions)

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1. (a) It is clear that $f(a_i) = b_i$. Consider any other polynomial $h(X)$ such that $h(a_i) = b_i$. Now, the difference $f(X) - h(X)$ vanishes on a_i, hence we can write

$$f(X) - h(X) = (X - a_1)(X - a_2) \dots (X - a_n)g(X)$$

If both $f(X)$ and $h(X)$ are of degree $\leq n-1$, then $g(X) = 0$, hence $f(X) = h(X)$. $\boxed{6, \text{A}}$

(b) Consider $p(X,Y) \in \mathcal{I}(V)$ divide it by $Y - f(X)$ and $g(X)$ with remainder, we get

$$p(X,Y) = h_1(X,Y)(Y - f(X)) + h_2(X,Y)g(X) + r(X,Y)$$

Now, none of the terms of $r(X,Y)$ are divisible by $Y = LT(Y - f(X))$ and $X^n = LT(g(X)$ hence $r(X,Y) = r(X) \in k[X]$ a polynomial of degree $\leq n - 1$. But, $r(a_i) = 0$ for all $i = 1, \dots, n$, hence $r(X) = 0$. $\boxed{7, \text{B}}$

(c) We compute the S-polynomial

$$S(Y - f(X), g(X)) = X^n(Y - f(X)) - Yg(X) = -X^n f(X) - Yh(X)$$

where $h(X) = g(X) - X^n$. Now, we divide by $Y - f(X), g(X)$, we get

$$S(Y-f(X), g(X) = h(X)(Y-f(X)-(X^n f(X)-h(X)f(X)) = h(X)(Y-f(X))-f(X)g(X)$$

so the remainder is zero. $\boxed{7, \text{B}}$

2. (a) Write Γ for the set of ideals of A which are not finitely generated. If $\Gamma \neq \emptyset$, let $\mathcal{T} \subset \Gamma$ be a totally ordered set, then the ideal $\mathfrak{b} = \bigcup_{\lambda \in \mathcal{T}} I_\lambda$ is in Γ. Indeed, if $\mathfrak{b} = (x_1, \ldots, x_s)$, then $\{x_1, \ldots, x_s\} \subset I_\lambda$ for some λ, so that $\mathfrak{b} \subset I_\lambda$ which implies $\mathfrak{b} = I_\lambda$ is finitely generated, contradiction. Hence, $\mathfrak{b}$ is an upperbound for $\mathcal{T}$. By Zorn's lemma Γ contains a maximal element I. Then I is not a prime ideal, so there are elements $x, y \in A$ with $x \notin I, y \notin I$ but $xy \in I$. Now, $I + (y)$ is bigger than I, and hence is finitely generated, so that we can choose $u_1, \ldots, u_n \in I$ such that $I + (y) = (u_1, \ldots, u_n, y)$. Moreover $I : y = \{a \in A : ay \in I\}$ contains x, and is thus bigger than I, so it has a finite system of generators $v_1, \ldots, v_m$. Finally, it is easy to check that $I = (u_1, \ldots, u_n, v_1 y, v_2 y, \ldots, v_m y)$, hence $I \notin \Gamma$, which is a contradiction. $\boxed{10, \mathrm{C}}$

(b) Write Γ for the set of ideals of A that are not principal. Suppose Γ is non-empty. Let $\mathcal{T} \subset \Gamma$ be a totally ordered set, then the ideal $\mathfrak{b} = \bigcup_{\lambda \in \mathcal{T}} I_\lambda$ is in Γ. Indeed, if $\mathfrak{b} = (x)$ for some $x \in A$, then $x \in I_\lambda$ for some λ, so that $\mathfrak{b} \subset I_\lambda$ which implies $\mathfrak{b} = I_\lambda$ is principal, contradiction. Hence, $\mathfrak{b}$ is an upperbound for $\mathcal{T}$. By Zorn's lemma, the set Γ has a maximal element, call it I. By assumption I is not prime, so there exists $x, y \in A$ with $x \notin I, y \notin I$ but $xy \in I$. Now, $I + (y)$ is bigger than I and so it is principal, let $I + (y) = (a)$. Similarly, $I : y$ contains I and x hence is also principal, say $I : y = (b)$. We claim that $I = (ab)$. Indeed, let $c \in I \subset I + (y)$, then $c = am$ for some $m \in I$. Then, $m \in I : y$, hence $m = bn$ for some n, hence $c = abn$, which shows $I \subset (ab)$. Conversely, if $b \in I : y$ implies $by \in I$ so, $b(a) \subset I$, hence $ab \in I$. It follows that I is principal, which is a contradiction. $\boxed{10, \mathrm{C}}$

3. (a) For a ring A, the **Krull dimension** is defined to be

$$\dim A := \dim \operatorname{Spec}(A) = \sup\{n \geq 0 : \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \ldots \subsetneq \mathfrak{p}_n \subsetneq A \text{ with } \mathfrak{p}_i \text{ prime ideal } \}$$

$\boxed{3, \mathrm{A}}$

(b) $\mathbb{Z}[i]$ is isomorphic to $\mathbb{Z}[x]/(x^2+1)$ hence is an integral extension of $\mathbb{Z}$. Therefore, by Cohen-Seidenberg theorems, $\dim\mathbb{Z}[i] = \dim\mathbb{Z}$. In $\mathbb{Z}$ every non-zero prime ideal is maximal and is given by (p) for some prime p, hence the longest chain of prime ideals are all of the form

$$(0) \subsetneq (p)$$

for some prime number p. Therefore, $\dim\mathbb{Z}[i] = \dim\mathbb{Z} = 1$. $\boxed{5, \mathrm{A}}$

(c) Let $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\overline{R} = R/(x_1, \ldots, x_n)$ is Artinian. Suppose that there exists a prime ideal $\mathfrak{p} \subset R$ such that

$$(x_1, \ldots, x_n) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$$

Then, in $\overline{R}$, we would have a chain

$$\mathfrak{p}/(x_1, \ldots, x_n) \subsetneq \mathfrak{m}/(x_1, \ldots, x_n)$$

of prime ideals, which implies $\dim\overline{R} \geq 1$, hence $\overline{R}$ cannot be Artinian, contradiction. Therefore, $\mathfrak{m}$ is a minimal prime ideal containing $(x_1, \ldots, x_n)$ then it follows from Krull's Height Theorem that $\dim R = \operatorname{ht}\mathfrak{m} \leq n$. $\boxed{5, \mathrm{B}}$

(d) There are finitely many minimal prime ideals in a Noetherian ring, these are all the prime ideals of height 0 and we can view them as containing the empty set. Suppose now by induction that for $0 \leq k < n$, we have constructed elements $x_1, \ldots, x_k \in \mathfrak{m}$ such that every minimal prime ideal containing $x_1, \ldots, x_k$ has height k. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be these minimal primes containing $(x_1, \ldots, x_k)$ with height k. We see that $\mathfrak{m} \nsubseteq \bigcup_{i=1}^{s} \mathfrak{q}_i$ since, by prime avoidance, this would imply $\mathfrak{m} \subset \mathfrak{q}_i$ for some i but height of $\mathfrak{q}_i$ is $k < n = h(\mathfrak{m})$ which is a contradiction. Hence, we can find $x_{k+1} \in \mathfrak{m} \setminus \bigcup_{i=1}^{s} \mathfrak{q}_i$. Now, let $\mathfrak{p}$ be a minimal prime ideal containing $(x_1, \ldots, x_k, x_{k+1})$. We have $h(\mathfrak{p}) \leq k+1$ by Krull's height theorem. On the other hand, since $\mathfrak{p} \supset (x_1, \ldots, x_n)$ and $\mathfrak{q}_i$ are minimal primes containing $(x_1, \ldots, x_n)$, we have $\mathfrak{p} \supset \sqrt{(x_1, \ldots, x_n)} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s \supset \mathfrak{q}_1 \ldots \mathfrak{q}_s$. Hence $\mathfrak{p} \supset \mathfrak{q}_i$ for some i, hence $h(\mathfrak{p}) = k+1$. Now, by induction, we can continue this until we produce elements $x_1, \ldots, x_n$. The minimal prime ideal containing $(x_1, \ldots, x_n)$ has height n, and thus has to coincide with $\mathfrak{m}$. $\boxed{7, \mathrm{D}}$

4. (a) Let $I = (X^2 + Y^2 - 1) \subset \mathbb{C}[X,Y]$. By Hilbert's Nullstellensatz, we have $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$. But, X^2+Y^2-1 is an irreducible polynomial, hence I is prime, therefore $\sqrt{I} = I$. So, the polynomials vanishing on $\mathcal{V}(I)$ are exactly given by I, hence are of the form $P(X,Y)(X^2+Y^2-1)$ where $P(X,Y) \in \mathbb{C}[X,Y]$ is an arbitrary polynomial. $\boxed{2, \text{A}}$

(b) (i) Since k is not algebraically closed, we can find a non-trivial polynomial $p(X) \in k[X]$ with no zero. Write

$$p(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$$

Now, consider the homogenization $\phi_2(X,Y) \in k[X,Y]$ given by

$$\phi_2(X,Y) = a_n X^n + a_{n-1} X^{n-1} Y + \ldots + a_1 X Y^{n-1} + a_0 Y^n.$$

If $\phi_2(X,Y)$ has a root with $Y \neq 0$, then we would get a root of $\phi_2(X,1) = p(X)$ which is not the case. Therefore, the only zero of $\phi_2(X,Y)$ is at $(0,0)$. Now, we define recursively

$$\phi_m(X_1, \ldots, X_m) = \phi_2(\phi_{m-1}(X_1, \ldots, X_{m-1}), X_m)$$

It is clear that the only zero of ϕ_m is at $(0, \ldots, 0) \in k^m$ as required. $\boxed{6, \text{B}}$

(ii) Consider the maximal ideal $(X,Y) \in \mathbb{C}[X,Y]$, the variety associated to this ideal is the point $(0,0) \in \mathbb{C}^2$. Suppose $(0,0) = \mathcal{V}((f))$ for $f \in \mathbb{C}[X,Y]$, then by Nullstellensatz, we would have $\sqrt{(X,Y)} = (X,Y) = \sqrt{(f)}$. Thus, there exists, n, m such that f divides X^n and Y^m, but this implies f has to be constant (since $\mathbb{C}[X,Y]$ is a UFD) which is a contradiction. $\boxed{6, \text{A}}$

(c) By Zariski's lemma the field $\mathbb{R}[X,Y]/\mathfrak{m}$ is a finite field extension of $\mathbb{R}$. There are two such fields $\mathbb{R}$ and $\mathbb{C}$. One can take (X,Y) and (X^2+1, Y) as examples of maximal ideals such that $\mathbb{R}[X,Y]/\mathfrak{m}$ is $\mathbb{R}$ and $\mathbb{C}$. In the first case the isomorphism is given by sending X and Y to 0 and in the second case, X to i and Y to 0. $\boxed{6, \text{B}}$

5. (a) (i) A valuation ring is an integral domain R such that for all $x \in K \setminus \{0\}$, where K is the field of fractions of R, either $x \in R$ or $x^{-1} \in R$. $\boxed{2, \text{A}}$

(ii) A valuation on K is given by map $\nu : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, satisfying

(1) $\nu(xy) = \nu(x) + \nu(y)$, for all $x, y \in K$,

(2) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$,

(3) $\nu(x) = \infty$ if and only if $x = 0$.

The valuation ring associated to ν is defined by $R = \{x \in K : \nu(x) \geq 0\}$. R is called a discrete valuation ring (DVR) if $\Gamma = \mathbb{Z} \cup \{\infty\}$. $\boxed{2, \text{A}}$

(iii) If $x \in K$ satisfies an equation $x^n + a_1x^{n-1} + \ldots + a_n = 0$ with $a_i \in R$. Thus, we have $\nu(a_i) \geq 0$. Now, if $\nu(x) < 0$, then we have $\nu(x^n) = n\nu(x) < \nu(a_ix^{n-i}) = (n - i)\nu(x) + \nu(a_i)$ for all $i = 1, \ldots, n$. Hence, this violates condition (2) of the valuation. $\boxed{3, \text{A}}$

(b) (i) Every element of $K = \mathbb{C}(X, Y)$ we can express is as $Y^n \frac{f}{g}$ for $f, g \in \mathbb{C}[X, Y]$ with $n \in \mathbb{Z}$ and Y not dividing f or g. The elements that are in R correspond precisely to the ones with $n \geq 0$. Hence, it is clear that if $x \in K \setminus R$, then $x^{-1} \in R$. Equivalently, one could define a valuation by letting $\nu(Y^n \frac{f}{g}) = n$. $\boxed{3, \text{A}}$

(ii) Both $\frac{X}{Y}$ and $\frac{Y}{X}$ are not in R, hence this is not a valuation ring. $\boxed{3, \text{A}}$

(c) Any ring R' with $R \subset R' \subset K$ is given by $R_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R$. But, R is a local ring and an integral domain of dimension 1, hence the only prime ideals it has are (0) and $\mathfrak{m}$ (the maximal ideal in R), and localisation on these ideals gives R and K. So, R is maximal as a subring of K.

Conversely, suppose R is maximal proper subring of K. The integral closure of R is not the whole of K (as this would imply R is a field), hence R is integrally closed. On the other hand, let $x \in K \setminus R$. Then we have $R[x] = K$, so $x^{-1} \in R[x]$. Hence, we can write

$$x^{-1} = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, \quad a_i \in R$$

which implies

$$x^{-n-1} - a_0x^{-n} - \ldots - a_n = 0$$

Thus, x^{-1} is integral over R, hence $x^{-1} \in R$. Thus R is a valuation ring. It is of dimension 1, since otherwise we would have a prime ideal $\mathfrak{p} \neq (0), \mathfrak{m}$ and we would have $R \subsetneq R_\mathfrak{p} \subsetneq K$. $\boxed{7, \text{D}}$

Review of mark distribution:

Total A marks: 35 of 35 marks

Total B marks: 31 of 31 marks

Total C marks: 20 of 20 marks

Total D marks: 14 of 14 marks

Total marks: 100 of 100 marks

Total Mastery marks: 0 of 0 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.		
ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH70061	1	Generally the first two parts were answered correctly. In the last part, they are supposed to show that division of the syzygy polynomial of Y-f(X) and g(X) by Y-f(X) and g(X) gives zero. Most people who attempted this applied the division algorithm incorrectly.
MATH70061	2	Most people got a good start with the induction. A few of them failed to check the hypothesis of Zorn's lemma before applying it. Most people couldn't complete the proof in the part where one is supposed to deduce that I is finitely generated (or principal in part (b)) from the fact that I+(y) and I:y are finitely generated (or principal). Generous partial credit was given if only this part is left.
MATH70061	3	(a) and (b) are straightforward and were generally answered correctly. Though, I have seen wrong claims in the case of (b). Some of them answered (c) correctly. Almost no one got (d) completely. This is expected as part (d) of this problem is probably the hardest problem of the exam. Nonetheless, there was one student who gave a perfect solution and a couple who got close to perfect.
MATH70061	4	(a) was generally answered correctly. In (b, i), surprisingly many of them failed to give a correct construction of phi_2. But, given phi_2, the rest was easy. (b,ii) was generally answered correctly. Again surprisingly few of them, answered (c) correctly even though this is essentially a one-line application of Zariski's lemma.
MATH70061	5	Part (a) was generally answered correctly. In part (a,iii), some people quoted results from the lecture notes that directly imply this statement. Even though, that was not the expected way of providing a solution here, I did not cut any points as I have told students beforehand that they can use any result from the lecture notes. A lot of them failed in part (b) but some did get it right. Part (c) was answered correctly by very few (which is expected).