Thm 1 (Identity principle linear maps)
Let V, W be vect. spaces

$$\varphi, \psi : V \longrightarrow W \quad \text{linear}$$

Suppose B is a basis of V and

$$\varphi(u) = \psi(u) \quad \forall\, u \in B.$$

Then $\varphi = \psi$.

Proof: Exercise

Thm 2 (Extending by linearity)
V, W, B as in Thm 1. Let

$$f : B \to W \quad (\text{that is } b_i \mapsto w_i \ \forall\, i = 1, \dots, n)$$

$$B = \{b_1, \dots, b_n\}$$

i) Then there is a unique linear map

$$\varphi : V \to W \quad \text{s.t.} \quad \varphi(b) = f(b) \ \forall\, b \in B.$$

ii) φ is bijective (φ is an isomorphism) iff $\{f(b_1), \dots, f(b_n)\}$ is a basis of W.

Coordinates

Thm 3

V n-dim v.s over a field F

$B = \{v_1, \ldots, v_n\}$ basis of V

E standard basis of F^n

Then

$$f_B : v_i \longmapsto e_i \qquad i = 1, \ldots, n$$

extends to a linear isomorphism

$$\varphi_B : V \longrightarrow F^n.$$

Def 1 For all $v \in V$ we write

$$[v]_B := \varphi_B(v) \qquad \left(\text{where } \varphi_B \text{ is as in Thm 2}\right)$$

Concretely

$$v = \sum_{i=1}^{n} \alpha_i b_i$$

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Example:

$\mathbb{R}[x]_{\leq 2}$ v.s. $/\mathbb{R}$

$E = \{1, x, x^2\}$

$B = \{1+x, 1-x, x^2\}$

$x^2 + 1$ $\qquad [x^2+1]_E = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad [x^2+1]_B = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$

3. Matrices

V is m-dim
W is n-dim
B basis of V
C " of W

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_B}"'] & W \arrow[d, "{[\cdot]_C}"] \\ F^m \arrow[r] & F^n \end{tikzcd}$$

Understanding $T: F^m \longrightarrow F^n$ that is defined as

$$T([v]_B) = [\varphi(v)]_C$$

Prop: For every linear map
$$T: F^m \longrightarrow F^n$$
there is a matrix (unique)
$$A \in M_{n \times m}(F)$$
s.t. $\quad T(v) = Av \quad \forall\, v \in F^m$.

Def 2: (Matrix of a linear map)
Let $\varphi: V \longrightarrow W$ lin map. We define

$${}_C[\varphi]_B := \left([\varphi(v_1)]_C \;\middle|\; \cdots \;\middle|\; [\varphi(v_m)]_C \right)$$

$$[\varphi(v)]_C = {}_C[\varphi]_B\,[v]_B \qquad \forall\, v \in V$$

Prop: Let U be another v.s. $/F$ with basis D. Let $\psi : W \longrightarrow U$. Then

$$ {}_D[\psi \circ \varphi]_B = {}_D[\psi]_C \; {}_C[\varphi]_B . $$

Example

$V = \mathbb{R}^2 \qquad T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$ T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_1 + 3x_2 \end{pmatrix} $$

$$ B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} $$

$$ E = \{ e_1, e_2 \} $$

$$ {}_E[T]_E = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} $$

$$ {}_E[T]_B = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} $$

$$ {}_B[T]_B \quad ? $$

Basis Changes

For This $V = W$ $(n = m)$

$$B = \{v_1, \dots, v_n\}$$

$$C = \{w_1, \dots, w_n\}$$

$\varphi : V \to V$ linear map

Not:

$${}_B[\varphi]_B := [\varphi]_B$$

Quest: How do we get $[\varphi]_C$ if we know $[\varphi]_B$?

Let $v \in V$. Then by Def 2

$$[v]_C = [\mathrm{id}(v)]_C = {}_C[\mathrm{id}]_B \, [v]_B$$

Basis Change Matrix from B to C.

$$[\varphi]_C [v]_C$$
$$\|$$
$$[\varphi(v)]_C = {}_C[\mathrm{id}]_B \, \underbrace{[\varphi(v)]_B}_{[\varphi]_B \underbrace{[v]_B}_{{}_B[\mathrm{id}]_C [v]_C}}$$

$$[\varphi]_C = {}_C[\mathrm{id}]_B \, [\varphi]_B \, {}_B[\mathrm{id}]_C$$

Rem:

$$ {}_C[id]_B \; {}_B[id]_C = I_n $$

Therefore

$$ P = {}_B[id]_C \implies {}_C[id]_B = P^{-1} $$

Thm (Basis Change Formula)

$\varphi: V \to V$ $\quad$ B, C are bases of V

Then

i) $$ [v]_C = {}_C[id]_B \, [v]_B $$

ii) if we write $P = {}_B[id]_C$, then

$$ [\varphi]_C = P^{-1} [\varphi]_B P . $$