IMPERIAL

MATH70147

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Statistical Mechanics

Date: Thursday, May 15, 2025

Time: Start time 14:00 – End time 15:30 (BST)

Time Allowed: 1.5 hours

This paper has 3 Questions.

Please Answer All Questions in 1 Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. We consider a variation of the Ising model called the spin-$3/2$ Ising model. In this model, each spin i can take values $s_i \in \{-3/2, -1/2, 1/2, 3/2\}$. The one-dimensional spin-$3/2$ Ising model is described by the Hamiltonian

$$\mathcal{H} = -J \sum_{i=1}^{N} s_i s_{i+1}$$

with $J > 0$ a positive spin-spin coupling. Periodic boundary conditions are imposed, so $s_{N+1} \equiv s_1$.

(a) Write down the partition function $\mathcal{Z}$ for this system. (3 marks)

(b) Show that $\mathcal{Z}$ can be written as a product of Boltzmann weights associated with each interaction term. Define the transfer matrix $\mathbf{T}$ for this system, you will specify its elements $T_{s,s'}$. (4 marks)

(c) Show that the largest eigenvalue of the transfer matrix $\lambda_{\max}$ determines the free energy in the thermodynamic limit. Give an explicit expression for the free energy per spin as a function of $\lambda_{\max}$. (4 marks)

(d) Using the orthogonal matrix

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

show that the transfer matrix can be block diagonalized by an appropriate change of basis. (4 marks)

(e) Use the result of (d) to find the largest eigenvalue $\lambda_{\max}$. (3 marks)

(f) How would you go about studying the existence of a phase transition at finite temperature? (2 marks)

(Total: 20 marks)

2. In this question, we will analyze percolation problems (both site and bond percolation) using real-space renormalization group (RG) techniques. We work on the 2D square lattice.

(a) Denoting the probability of occupation (of a site or a bond) p, explain the renormalization procedure and how to obtain the critical occupation probability p_c and the exponent ν characterizing the divergence of the correlation length ξ at the critical point. (5 marks)

(b) *Site percolation* – First, we will analyze the so-called site percolation in which we consider each site of the lattice to be occupied with probability p and vacant with probability $1-p$. We perform a real-space RG transformation by coarse-graining the lattice into blocks of size 2×2. We use the spanning-cluster rule: the renormalized site occupation probability p' is determined by the probability that a spanning path exists in the 2×2 block, connecting opposite sides.

(i) Show that the renormalization group transformation $R_b(p)$ is given by

$$R_b(p) = p^4 + 4p^3(1-p) + 4p^2(1-p)^2$$

(3 marks)

(ii) Find the critical occupation probability p_c. (3 marks)

(c) *Bond percolation* – Consider now the bond percolation problem on a 2D square lattice, where each bond is occupied with probability p and vacant with probability $1-p$. Similarly to the case of site percolation, the system exhibits a percolation phase transition at a critical threshold p_c, where an infinite connected cluster of bonds first appears.

(i) Describe an appropriate RG transformation which rescales the lattice by length $b = \sqrt{2}$, while respecting its symmetry. (3 marks)

(ii) Show that the rescaled bond occupation probability is given by

$$R_b(p) = 2p^2 - p^4$$

(3 marks)

(iii) Find the critical occupation probability p_c. (3 marks)

(Total: 20 marks)

3. *Mastery question* — In this problem, we consider the random energy model (REM), a simplified model of a disordered system where each configuration i of N Ising spins has an independent random energy E_i drawn from a Gaussian distribution:

$$P(E_i) = \frac{1}{\sqrt{\pi N J^2}} \exp\left(-\frac{E_i^2}{NJ^2}\right),$$

with moments $\langle E_i \rangle = 0$ and $\langle E_i^2 \rangle = NJ^2/2$. Using a saddle-point approximation, one can show that the free energy density of the random energy model is given by

$$f(T) = \begin{cases} -J\sqrt{\ln 2}, & T < T_g \\ -T\ln 2 - J^2/(4T), & T > T_g \end{cases}$$

where T_g is the so-called glass transition. We work throughout in units of energy such that $k_B = 1$.

(a) Compute the entropy density $s(T) = S(T)/N$ and show that it vanishes at a glass temperature T_g which you will compute. Interpret the physical meaning of $s(T) = 0$.

(4 marks)

(b) Compute the specific heat density $c(T)$ and show that it exhibits a discontinuous jump at T_g. Calculate the magnitude of this jump $\Delta c(T_g)$. (3 marks)

(c) At $T = T_g$, the system undergoes a glass transition. Explain why the REM exhibits no true thermodynamic phase transition, but still shows a dramatic change in behavior at T_g.

(2 marks)

(d) As in most disordered systems, we expect the free energy to be self-averaging; that being said, it is often complicated to compute the average of the free energy over the quenched disorder. The REM is luckily solvable using the replica trick. Briefly explain what is the issue with averaging over disorder, and how replicas help circumvent this. (3 marks)

(e) Here, we use the replica trick to rederive the free energy density for $T > T_g$.

(i) By expanding over n independent replicas, show that for integer $n > 1$:

$$\overline{Z^n} \approx 2^{Nn} \exp\left(\frac{\beta^2 N J^2 n}{4}\right).$$

Hint: You can use the fact that for independent Gaussian random variables $E_1, E_2, \ldots, E_n$ with zero mean and variances given by $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, we have

$$\overline{\exp\left(\sum_{j=1}^{n} a_j E_j\right)} = \exp\left(\frac{1}{2}\sum_{j=1}^{n} a_j^2 \sigma_j^2\right)$$

(4 marks)

(ii) Analytically continue this result to $n \to 0$ and verify the free energy given above for $T > T_g$. (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2025

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MATH70147

Statistical Mechanics (Solutions)

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1. (a) The partition function is given by $\boxed{\text{sim. seen} \Downarrow}$

$$\mathcal{Z} = \sum_{\{s_i\}} e^{-\beta\mathcal{H}(\{s_i\})} = \sum_{s_1=-\ell}^{\ell}\sum_{s_2=-\ell}^{\ell}\cdots\sum_{s_N=-\ell}^{\ell} e^{\beta J\sum_{i=1}^{N} s_i s_{i+1}}$$

with $\ell = 3/2$. $\boxed{3, \text{A}}$

(b) The partition function can be written: $\boxed{\text{sim. seen} \Downarrow}$

$$\mathcal{Z} = \sum_{s_1=-\ell}^{\ell}\sum_{s_2=-\ell}^{\ell}\cdots\sum_{s_N=-\ell}^{\ell}\prod_{i=1}^{N} V_i$$

with $V_i = \exp[\beta J s_i s_{i+1}]$; this can be seen as an element of a 4×4 matrix indexed by $\{-3/2, -1/2, 1/2, 3/2\}$. The transfer matrix is then written

$$\mathbf{T} = \begin{bmatrix} T_{-\frac{3}{2},-\frac{3}{2}} & T_{-\frac{3}{2},-\frac{1}{2}} & T_{-\frac{3}{2},\frac{1}{2}} & T_{-\frac{3}{2},\frac{3}{2}} \\ T_{-\frac{1}{2},-\frac{3}{2}} & T_{-\frac{1}{2},-\frac{1}{2}} & T_{-\frac{1}{2},\frac{1}{2}} & T_{-\frac{1}{2},\frac{3}{2}} \\ T_{\frac{1}{2},-\frac{3}{2}} & T_{\frac{1}{2},-\frac{1}{2}} & T_{\frac{1}{2},\frac{1}{2}} & T_{\frac{1}{2},\frac{3}{2}} \\ T_{\frac{3}{2},-\frac{3}{2}} & T_{\frac{3}{2},-\frac{1}{2}} & T_{\frac{3}{2},\frac{1}{2}} & T_{\frac{3}{2},\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} e^{9\beta J/4} & e^{3\beta J/4} & e^{-3\beta J/4} & e^{-9\beta J/4} \\ e^{3\beta J/4} & e^{\beta J/4} & e^{-\beta J/4} & e^{-3\beta J/4} \\ e^{-3\beta J/4} & e^{-\beta J/4} & e^{\beta J/4} & e^{3\beta J/4} \\ e^{-9\beta J/4} & e^{-3\beta J/4} & e^{3\beta J/4} & e^{9\beta J/4} \end{bmatrix}$$

$\boxed{4, \text{A}}$

(c) In the transfer matrix method, we thus have $\boxed{\text{seen} \Downarrow}$

$$\mathcal{Z} = \sum_{s_1=-\ell}^{\ell}\sum_{s_2=-\ell}^{\ell}\cdots\sum_{s_N=-\ell}^{\ell} T_{s_1,s_2}T_{s_2,s_3}\dots T_{s_N,s_1} = \mathrm{Tr}\left[\mathbf{T}^N\right]$$

The free energy is given by

$$F = -k_BT\ln\mathcal{Z} = -k_BT\ln\left(\sum_{i=1}^{4}\lambda_i^N\right)$$

with λ_i the eigenvalues of $\mathbf{T}$. Assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \lambda_3 < \lambda_4 \equiv \lambda_{\max}$ (which is ensured by Perron-Frobenius as all entries of the transfer matrix are positive). Thus, we have

$$f = -k_BT\lim_{N\to\infty}\frac{1}{N}\ln\left[\lambda_4^N\left(1+\sum_{i=1}^{3}\frac{\lambda_i^N}{\lambda_3^N}\right)\right] = -k_BT\ln\lambda_{\max}$$

$\boxed{4, \text{A}}$

(d) Using the orthogonal transformation: $\boxed{\text{unseen} \Downarrow}$

$$\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

we have that

$$\mathbf{Q}^T\mathbf{T}\mathbf{Q} = \begin{pmatrix} e^{9\beta J/4} - e^{-9\beta J/4} & e^{3\beta J/4} - e^{-3\beta J/4} & 0 & 0 \\ e^{3\beta J/4} - e^{-3\beta J/4} & e^{\beta J/4} - e^{\beta J/4} & 0 & 0 \\ 0 & 0 & e^{\beta J/4} + e^{\beta J/4} & e^{3\beta J/4} + e^{-3\beta J/4} \\ 0 & 0 & e^{3\beta J/4} + e^{-3\beta J/4} & e^{9\beta J/4} + e^{-9\beta J/4} \end{pmatrix}$$

showing as requested that an appropriate change of basis block diagonalizes the transfer matrix. We can simplify this matrix into

$$\mathbf{Q}^T\mathbf{T}\mathbf{Q} = 2\begin{pmatrix} \sinh(9\beta J/4) & \sinh(3\beta J/4) & 0 & 0 \\ \sinh(3\beta J/4) & \sinh(\beta J/4) & 0 & 0 \\ 0 & 0 & \cosh(\beta J/4) & \cosh(3\beta J/4) \\ 0 & 0 & \cosh(3\beta J/4) & \cosh(9\beta J/4) \end{pmatrix}$$

$\boxed{3,\ \mathrm{B}}$

(e) In its block diagonal form, it is easy to find the eigenvalues of the transfer matrix. For that, we can diagonalize independently the two 2×2 blocks. Recall that the eigenvalues of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by $\boxed{\text{unseen} \Downarrow}$

$$\lambda_{\pm} = \frac{1}{2}\,\mathrm{Tr}(A) \pm \frac{1}{2}\sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}$$

Here, we find that the largest eigenvalue is given by $\boxed{1,\ \mathrm{B}}$

$$\lambda_{\max} = \cosh(\beta J/4) + \cosh(9\beta J/4) + \sqrt{[\cosh(\beta J/4) + \cosh(9\beta J/4)]^2 - 4\cosh(\beta J/4)\cosh(9\beta J/4) + 4\cosh^2(3\beta J/4)}$$

$\boxed{2,\ \mathrm{D}}$

(f) To study the existence of a phase transition, we could do one of the following: $\boxed{\text{meth seen} \Downarrow}$

* analyse the free energy per spin (using the result of (e)). Phase transitions manifest as non-analyticities (i.e. kinks or discontinuities) in $f(T)$ or its derivatives (including heat capacity for instance or magnetic susceptibiility if one re-introduces an external field h);
* determine the correlation length ξ from the ratio of the two largest eigenvalues and study possible divergences.

$\boxed{3,\ \mathrm{C}}$

2. (a) A renormalization group transformation can be outlined in three steps: $\boxed{\text{seen} \Downarrow}$

1. Divide the lattice into blocks of linear size b with each block containing at least a few sites/bonds;
2. Coarse-graining procedure takes place: the sites in each block are averaged in some way (to be specified) and we replace each block of sites/bonds by a single block site/bond of size b which is occupied with probability $p' = R_b(p)$ according to the renormalization group transformation;
3. Rescale all lengths by the factor b to restore the original lattice spacing.

You need to ensure that your transformation respect the original symmetry of the lattice. Upon the rescaling operation, all lengthscales (and so the correlation length) will be rescaled by b leading to $\xi' = \xi/b$. Recall that close to the critical point, the correlation length is expected to scale as a power-law

$$\xi = C|p - p_c|^{-\nu}$$

From this, we have that

$$\xi' = \frac{\xi}{b} \Rightarrow C|R_b(p) - p_c|^{-\nu} = \frac{C|p - p_c|^{-\nu}}{b} \Rightarrow \nu = \frac{\ln b}{\ln\left(\frac{|R_b(p)-p_c|}{|p-p_c|}\right)}$$

The basic idea is that at the critical point p_c, we have self-similarity, this implies that the correlation length remains unchanged by the RG transformation:

$$\xi = \xi' \Rightarrow C|p - p_c|^{-\nu} = C|R_b(p) - p_c|^{-\nu}$$

with $p' = R_b(p)$. This equation can only be true for $\xi = 0$ which corresponds to $p = 0$ or $p = 1$ and $\xi = \infty$ which corresponds to $p = p_c$. This means that p_c must be a fixed point of the RG transformation and is solution to

$$p_c = R_b(p_c).$$

We can then write

$$\begin{aligned}\frac{|R_b(p) - p_c|}{|p - p_c|} &= \frac{|R_b(p) - R_b(p_c)|}{|p - p_c|} \\ &= \left.\frac{dR_b(p)}{dp}\right|_{p=p_c}, \quad \text{for } p \to p_c\end{aligned}$$

which leads us to the following expression for the critical exponent ν

$$\nu = \frac{\ln b}{\ln\left(\left.\frac{dR_b(p)}{dp}\right|_{p=p_c}\right)}$$

$\boxed{5, \mathrm{A}}$

(b) (i) Here, we consider the RG transformation which uses spin blocks of size 2×2. The spanning cluster rule imposes that the renormalized site occupation probability p' is determined by the probability that a spanning path exists in the 2×2 block, connecting opposite sides. This can happen three ways: sim. seen ⇓

- ☐ All the sites in the block site are occupied, this happens with probabibility p^4 (there is only one way this can happen);
- ☐ Three of the four sites are occupied, this happens with probability $4p^3(1-p)$ (there are four ways to pick the empty site);
- ☐ Two neighboring sites in the block site are occupied, this happens with probability $4p^2(1-p)^2$ (there are $\binom{4}{2} = 6$ ways to have to sites occupied but two of these configurations have the occupied sites on the diagonal).

We conclude that the spanning-cluster rule defines the following RG transformation:

$$R_b(p) = p^4 + 4p^3(1-p) + 4p^2(1-p)^2$$

3, C

(ii) The fixed points are defined as meth seen ⇓

$$p = p^4 + 4p^3(1-p) + 4p^2(1-p)^2$$

which can be factorized as follows

$$p(p-1)(p^2 - 3x + 1) = 0$$

Clearly, $p = 0$ and $p = 1$ are trivial fixed points. The final fixed point is solution to $p^2 - 3p + 1 = 0$. We find

$$p_{\pm} = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$$

only one of these roots is in the $[0, 1]$ interval and so the critical occupation probability is

$$p_c = \frac{1}{2}\left(3 - \sqrt{5}\right)$$

3, B

(c) (i) We define the RG transformation as follows: we scale the original square lattice [Fig. 1(a)] to a square lattice rotated by $\pi/4$ and with a lattice constant larger by a factor of $b = \sqrt{2}$; we do so by removing all sites which are not labelled by the letters A, B, C or D [see Fig. 1(b)]. The rescaled bonds are thus the diagonal of the original square unit cells as can be seen in Fig. 1. unseen ⇓

3, D

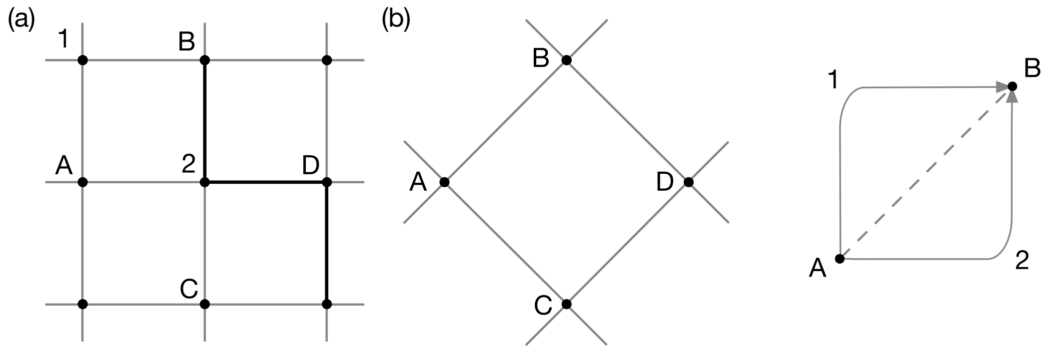

Figure 1: Definition of the RG transformation for our 2D square lattice bond percolation problem.

unseen ⇓

(ii) Here, we use again the spanning-cluster rule which is here defined as follows: a connection (i.e. an occupied bond) between nearest-neighbors on the new lattice (say, A and B) can only exist if there was a path connecting A and B on the original lattice, the two possible paths are shown on Fig. 1(c) and go through sites marked 1 and 2, respectively. Considering only those connection, it is clear that the probability to have an occupied bond on the new lattice emerges from:

- ☐ All the bonds in the block are occupied, this happens with probabibility p^4 (there is only one way this can happen);
- ☐ Three of the four bonds are occupied, this happens with probability $4p^3(1-p)$ (there are four ways to pick the empty site);
- ☐ Two neighboring bonds in the block are occupied created a path from A to B via either site 1 or 2, this happens with probability $2p^2(1-p)^2$.

We conclude that

$$\begin{aligned} R_b(p) &= p^4 + 4p^3(1-p) + 2p^2(1-p)^2 \\ &= p^4 + 4p^3 - 4p^4 + 2p^2 + 2p^4 - 4p^3 \\ &= 2p^2 - p^4 \end{aligned}$$

3, D

(iii) Looking for fixed points of the RG transformation, we write:

unseen ⇓

$$p = 2p^2 - p^4 \Rightarrow p(p-1)(p^2+p-1) = 0$$

We find again the two trivial fixed points $p = 0, 1$. The critical occupation probability is then the root of $p^2 + p - 1 = 0$ which is in the interval $[0, 1]$, that is:

$$p_c = \frac{1}{2}\left(\sqrt{5} - 1\right)$$

3, B

3. (a) The entropy density $s(T)$ is defined as

sim. seen ⇓

$$s(T) = -\frac{\partial f}{\partial T}$$

we thus obtain

$$s(T) = \begin{cases} 0, & T < T_g \\ \ln 2 - J^2/(4T^2), & T > T_g \end{cases}$$

We can see that the entropy density indeed goes to zero when

$$\ln 2 - J^2/(4T_g^2) = 0 \Rightarrow T_g = \frac{J}{2\sqrt{\ln 2}}$$

This is called the entropy crisis; the fact that $s(T_g) = 0$ signals a transition to a frozen state where the system is trapped in a few dominant configurations instead of being able to explore the phase space.

4, M

(b) The specific heat density is defined as

sim. seen ⇓

$$c(T) = -T\frac{\partial^2 f}{\partial T^2} = T\frac{\partial s}{\partial T}$$

We can easily compute it and obtain

$$c(T) = \begin{cases} 0, & T < T_g \\ J^2/(2T^2), & T > T_g \end{cases}$$

In particular, we can see that

$$\lim_{T \to T_g^-} c(T) = 0$$

while

$$\lim_{T \to T_g^+} c(T) = \frac{J^2}{2T_g^2} = 2\ln 2$$

We conclude that the specific heat density exhibits a discontinuous jump in $T = T_g$ and that

$$\Delta c(T_g) = 2\ln 2.$$

3, M

(c) The glass transition observed at T_g is not a true phase transition as there is **no symmetry breaking or no latent heat** associated with it. Nevertheless, we still observe some dramatic changes: the entropy vanishes continuously at the transition, at which point the system freezes into a few low-energy states, which leads to **ergodicity breaking**.

seen ⇓

2, M

(d) Due to the existence of a quenched disorder, the partition function is itself a random quantity. In the case of the random energy model, we need to average the free energy $\overline{F} = -T\overline{\ln \mathcal{Z}}$ over the random energies E_i, but the logarithm is non-linear, making direct averaging intractable, indeed,

seen ⇓

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

The replica trick relies on the following identity

$$\ln \mathcal{Z} = \lim_{n\to 0} \frac{\mathcal{Z}^n - 1}{n}$$

Using the replica trick, taking the averaging of the free energy thus boils down to two key steps:

- ☐ The computation of the moments $\overline{\mathcal{Z}^n}$ for integer n

$$\overline{\mathcal{Z}^n} = \overline{\left(\sum_i e^{-\beta E_i}\right)^n}$$

  which corresponds to the partition function of n non-interacting replicas of the system.
- ☐ The analytic continuation of the result in the limit $n \to 0$.

In the case of the random energy model, the Gaussian nature of the disorder makes the exact computation of $\overline{\mathcal{Z}^n}$ possible. 3, M

unseen ⇓

(e) Here, we use the replica trick to rederive the free energy for $T > T_g$.

(i) First, recall that the partition function is defined as

$$\mathcal{Z} = \sum_{i=1}^{2^N} e^{-\beta E_i}$$

For an integer $n > 1$, we write

$$\overline{\mathcal{Z}^n} = \overline{\left(\sum_{i=1}^{2^N} e^{-\beta E_i}\right)^n}$$

Expanding the product of the sums into a sum of products and using n replicas, we write

$$\overline{\mathcal{Z}^n} = \sum_{i_1, i_2, \ldots, i_n = 1}^{2^N} \overline{\exp\left(-\beta \sum_{\alpha=1}^{n} E_{i_\alpha}\right)}$$

where E_{i_α} is the energy of configuration i in the replica of index α.
At this stage, recall that for a Gaussian random variable $E \sim \mathcal{N}(0, \sigma^2)$, we have the classical result

$$\overline{e^{aE}} = e^{a^2\sigma^2/2}$$

and that for multiple independent Gaussian variables $E_1, E_2, \ldots, E_n$ with zero mean and variances given by $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, we have

$$\overline{\exp\left(\sum_{j=1}^{n} a_j E_j\right)} = \exp\left(\frac{1}{2}\sum_{j=1}^{n} a_j^2 \sigma_j^2\right)$$

As the replicas are independent realizations of the quenched disorder and at high temperature, replicas occupy mostly different configurations with little overlap, i.e. $i_1 \neq i_2 \neq \ldots \neq i_n$, then $\{E_{i_\alpha}\}$ are independent Gaussian variables and we write

$$\overline{\exp\left(-\beta \sum_{\alpha=1}^{n} E_{i_\alpha}\right)} = \exp\left(\frac{\beta^2 N J^2 n}{4}\right)$$

There are a total of 2^{Nn} ways to chose n distinct configurations which leads to

$$\overline{\mathcal{Z}^n} \approx 2^{Nn} \exp\left(\frac{\beta^2 N J^2}{4} n\right)$$

where the approximation stems from the fact that we neglected possible overlap between replicas (independence of the energy levels). $\boxed{4, \mathrm{M}}$

(ii) Finally, we proceed to the analytical continuation in the limit $n \to 0$. Recall that we are after

$$\overline{\ln \mathcal{Z}} = \lim_{n\to 0} \frac{\overline{\mathcal{Z}^n} - 1}{n}$$

From the previous question, we obtained

$$\overline{\ln \mathcal{Z}} = \lim_{n\to 0} \frac{2^{Nn} \exp(\beta^2 N J^2 n/4) - 1}{n}$$

Expanding this expression in small $n \ll 1$, we have: $2^{Nn} \approx 1 + Nn \ln 2$ and $\exp(\beta^2 N J^2 n/4) \approx 1 + \beta^2 N J^2 n/4$ which leads to

$$\begin{aligned}
\overline{\ln \mathcal{Z}} &= \lim_{n\to 0} \frac{(1 + Nn\ln 2)\left(1 + \frac{\beta^2 N J^2 n}{4}\right) - 1}{n} \\
&= \lim_{n\to 0} \frac{1}{n}\left(Nn\ln 2 + \frac{\beta^2 N J^2 n}{4} + \frac{N^2 n^2 \beta^2 J^2 \ln 2}{2}\right) \\
&= N\ln 2 + \frac{\beta^2 N J^2}{4}
\end{aligned}$$

We thus conclude that the free energy density in the regime $T > T_g$ is given by

$$f(T) = -\frac{1}{\beta N}\overline{\ln \mathcal{Z}} = -\frac{\ln 2}{\beta} - \frac{\beta J^2}{4}$$

which is consistent with the result obtained in lectures and cited above. $\boxed{4, \mathrm{M}}$

Review of mark distribution:
Total A marks: 16 of 16 marks
Total B marks: 10 of 10 marks
Total C marks: 6 of 6 marks
Total D marks: 8 of 8 marks
Total Mastery marks: 20 of 20 marks
Total marks: 60 of 60 marks

MATH70147 Statistical Mechanics Markers Comments

Question 1	Core of the question was mostly answered correctly; students too often gave up at the straightforward algebra in later subquestions leading to quite a few marks being taken off. There was a confusion amongst a lot students about the fact that matrix Q is expected to make the transfer matrix T via the operation Q^{-1}TQ (i.e. a change of basis).
Question 2	This questions led to mixed results. The RG procedure was often not described correctly; some more details were often required when deriving the form of the RG transformation. Some issues with simple quadratic polynomials roots, not expected at this stage of a degree in mathematics.
Question 3	This question was seen harder by the students although the first few subquestions only required bookwork or simple algebra. The concept of quenched and annealed was often confused with the replica trick which is what students were asked to carry here.