Imperial College London

MATH97066 MATH70009

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Bifurcation Theory

Date: 31 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Consider a system of differential equations in R^4. How many stable equilibria and/or stable periodic orbits can be born at the following bifurcations:

(a) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-\frac{1}{2} \pm i, -1, 0$ and the Lyapunov coefficients $L_2 = L_3 = L_4 = 0$, $L_5 > 0$? (5 marks)

(b) An equilibrium state with the eigenvalues of the linearisation matrix equal to $\pm i, -\frac{1}{2} \pm i$ and the Lyapunov coefficients $L_1 = L_2 = 0$, $L_3 > 0$? (5 marks)

(c) A periodic orbit with the multipliers $1, -\frac{1}{2} \pm i$ and the Lyapunov coefficients $L_2 = L_3 = 0, L_4 < 0$? (5 marks)

(d) A periodic orbit with the multipliers $-1, -2, 2$ and the Lyapunov coefficients $L_1 = 0$, $L_2 < 0$? (5 marks)

(Total: 20 marks)

2. Study bifurcations in the system

$$\begin{cases} \dot{x} = a + y - 2xy, \\ \dot{y} = b + x - y - x^2, \end{cases}$$

as parameters a and b vary. Namely, do the following.

(a) Show that the system cannot have periodic orbits. (6 marks)

(b) Show that the only bifurcations of the equilibria of this system correspond to a single zero eigenvalue of the linearisation matrix. (5 marks)

(c) Draw the bifurcation set in the (a, b) plane. (4 marks)

(d) For each of the regions, into which this curve divides the (a, b)-plane, determine the number of equilibria and their stability. (5 marks)

(Total: 20 marks)

3. Consider the system

$$\begin{cases} \dot{x} = y + \varepsilon x, \\ \dot{y} = -x + (y+u)^2, \\ \dot{u} = -2u + 8y^2. \end{cases}$$

(a) (i) Write the restriction of this system onto the center manifold, up to the terms of the third order, near the zero equilibrium at $\varepsilon = 0$. (8 marks)

(ii) Find the first Lyapunov coefficient. (6 marks)

(b) Determine how many stable and unstable periodic orbits can exist near $(x, y, u) = (0, 0, 0)$ for small $\varepsilon < 0$, for small $\varepsilon > 0$, and for $\varepsilon = 0$. (6 marks)

(Total: 20 marks)

4. (a) Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2, \end{cases}$$

How many periodic orbits can be born at the bifurcations of the zero equilibrium of this system? Find a perturbation that produces most periodic orbits.

(10 marks)

(b) Consider the system

$$\begin{cases} \bar{x} = y + x^3, \\ \bar{y} = -x + \varepsilon y - y^3. \end{cases}$$

How many periodic orbits exist in a small neighbourhood of zero when ε becomes positive? (10 marks)

(Total: 20 marks)

5. Consider the following map defined at $x \geq 0$, x is 1-dimensional:

$$x \mapsto \bar{x} = a - 4\sqrt[4]{x}$$

where a is a parameter.

(a) Show that the map has no periodic points of period larger than 2.

(4 marks)

(b) Study bifurcations of the fixed point of the map. For which values of a is the fixed point stable? (7 marks)

(c) Study bifurcations of points of period 2. Find the set of values of a for which stable orbits of period 2 exist. (9 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

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MATH60009/70009/97066

Bifurcation Theory (Solutions)

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1. (a) There can be no periodic orbits, as the centre manifold is 1-dimensional. Since there are no eigenvalues to the right of the imaginary axis, the equilibria are stable if and only if they are stable on the centre manifold. We have $l_5 > 0$, so up to 5 equilibria can be born, 2 of them are stable.

sim. seen $\Downarrow$

5, A

(b) Up to 3 periodic orbits can be born here, 1 of them will be stable.

sim. seen $\Downarrow$

5, A

(c) There are multipliers outside the unit circle, so no stable periodic orbits can be born.

sim. seen $\Downarrow$

5, A

(d) There are multipliers outside the unit circle, so no stable periodic orbits can be born.

sim. seen $\Downarrow$

5, A

2. (a) This is a gradient system: $\dot{x} = -\partial_x V$, $\dot{y} = -\partial_y V$, where $V = -ax - by - xy + \frac{y^2}{2} + x^2 y$. Gradient systems have no periodic orbits. unseen ⇓

6, D

(b) If (x, y) is an equilibrium, then the linearisation matrix is $\begin{pmatrix} -2y & 1-2x \\ 1-2x & -1 \end{pmatrix}$. meth seen ⇓

If it has a pair of purely imaginary eigenvalues or a double zero eigenvalue, then the trace is zero and the determinant must be non-negative. But for this matrix, if the trace is zero, then $y = -\frac{1}{2}$, and the determinant equals to $2y - (1-2x)^2 = -1 - (1-2x)^2 < 0$. Thus, these bifurcations are impossible. 5, B

(c) The bifurcation set corresponds to a zero eigenvalue of the linearisation matrix, i.e., its determinant $2y - (1-2x)^2$ is zero. This gives the following system of equations for the bifurcation set: sim. seen ⇓

$$\begin{cases} a = 2xy - y, \\ b = y - x + x^2, \\ y = \frac{(1-2x)^2}{2}. \end{cases}$$

This gives the following bifurcation curve:

$$a = \frac{1}{2}(2x-1)^3, \qquad b = \frac{3}{4}(2x-1)^2 - \frac{1}{4},$$

or

$$b = \frac{3}{4}(2a)^{2/3} - \frac{1}{4}.$$

This is a piece-wise smooth curve with 1 singularity (the cusp point) at $a = 0, b = -\frac{1}{4}$. 4, B

meth seen ⇓

(d) The bifurcation curve divides the plane into two regions. Let D_1 be the region which contains the point $(a = 0, b = -1)$. For this value of (a, b), the system has only one equilibrium at $(x = \frac{1}{2}, y = -\frac{3}{4})$. The determinant of the linearisation matrix at this point is negative, so this is a saddle point. It follows that the system has a single saddle point everywhere in D_1. 3, B

When crossing the bifurcation curve into the region D_3 (the inside of the cusp, which contains the positive b axis), a saddle-node bifurcation occurs and two more equilibrium states are born, one is a saddle, and the other is stable (because the trace of the linearisation matrix is negative when the determinant is zero). Thus, for (a, b) in the region D_3, the system has 3 equilibria, 2 saddles and one stable equilibrium. meth seen ⇓

2, C

3. (a) (i) Let us kill the term $8y^2$ in the equations for $\dot{u}$. To do this, introduce the coordinate $\boxed{\text{sim. seen} \Downarrow}$

$$u_{new} = u - x^2 - 3y^2 - 2xy.$$

In the new coordinates we have

$$\dot{u}_{new} = -2u_{new} + O(|x|^3 + |y|^3 + |u_{new}|^3),$$

and the centre manifold, up to the second order terms, is given by

$$u_{new} = 0.$$

Returning to the original coordinates, we find the equation for the centre manifold as

$$u = x^2 + 3y^2 + 2xy + O(|x|^3 + |y|^3 + |z|^3).$$

The system on the centre manifold takes the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + y^2 + 2y(x^2 + 3y^2 + 2xy) + O(x^4 + y^4), \end{cases}$$

at $\varepsilon = 0$. $\boxed{8, \text{B}}$

$\boxed{\text{sim. seen} \Downarrow}$

(ii) Let $z = x - iy$. Then $x = \frac{1}{2}(z + z^*)$, $y = \frac{i}{2}(z - z^*)$, and

$$\dot{z} = iz + \frac{i}{4}(z - z^*)^2 + z^2 z^*(L + i\Omega) + \ldots,$$

where the dots stand for the non-resonant cubic terms and terms of higher order, and

$$L = \frac{5}{2}.$$

Since the coefficients of the quadratic terms are purely imaginary, they do not contribute to the Lyapunov coefficient, so it equals to $5/2$. $\boxed{6, \text{A}}$

(b) Since $L > 0$, and the equilibrium at zero is stable at $\varepsilon < 0$ and unstable at $\varepsilon > 0$, it follows that there exists only one periodic orbit at $\varepsilon < 0$, and this orbit is unstable. There are no periodic orbits at $\varepsilon \geq 0$. $\boxed{\text{meth seen} \Downarrow}$ $\boxed{6, \text{A}}$

4. (a) Infinitely many. Indeed, the perturbed system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\epsilon x + x^2, \end{cases}$$

is Hamiltonian, hence every orbit near zero is periodic for $\varepsilon > 0$.

unseen ⇓

10, D

(b) The system is reversible at $\varepsilon = 0$ (change $x \leftrightarrow y$ and $t \to -t$), so every orbit around zero is periodic at $\varepsilon = 0$, hence no periodic orbits remain at $\varepsilon > 0$ by Hopf theorem.

meth seen ⇓

10, C

5. (a) $\frac{d\bar{x}}{dx} \leq 0$, so this map is monotonically decreasing. Its second iteration is monotonicaly increasing, and such maps cannot have periodic points other than fixed points. Therefore the original map can have only points of period 2 (and one fixed point). unseen ⇓ 4, M

(b) First of all, we must have $a \geq 0$ for a fixed point to exist. The derivative is infinite at the zero fixed point at $a = 0$, so it is unstable, and must remain unstable for all positive a until a bifurcation happens. Bifurcations of the fixed point of the monotonically decreasing map correspond to a multiplier equal to (-1). This gives the following system for the bifurcating fixed point: meth seen ⇓

$$x = a - 4\sqrt[4]{x}, \qquad -1 = -x^{-3/4}.$$

The solution is $x = 1$ at $a = 5$; the fixed point at larger values of a becomes stable. The stability at the bifurcation moment is determined by the Schwartz derivative which equals here $\frac{15}{32}x^{-2} > 0$, so the fixed point is unstable at the bifurcation moment. Thus, the fixed point is stable for $a > 5$. 7, M

(c) The bifurcation set corresponds to a zero eigenvalue of the linearisation matrix, i.e., its determinant $2y - (1-2x)^2$ is zero. This gives the following system of equations for The unstable orbits of period 2 are born when the fixed point becomes stable, i.e., when a becomes larger than 5. The equation for a period-2 orbit is meth seen ⇓

$$x_2 = a - 4\sqrt[4]{x_1}, \qquad x_1 = a - 4\sqrt[4]{x_2}, \qquad x_1 \neq x_2.$$

This gives us

$$z_2^4 - z_1^4 = 4(z_2 - z_1),$$

where we denote $z = \sqrt[4]{x}$. So,

$$(z_2 + z_1)(z_1^2 + z_2^2) = 4,$$

and it is obvious from this equation that z_1 and z_2 cannot grow to infinity, implying that $a = x_2 + 4\sqrt[4]{x_1}$ also must be bounded. This means that the period-2 orbit disappears at sufficiently large a.

One bifurcation which could result in this disappearance corresponds to the multiplier of the orbit of period 2 equal to 1, i.e., $x_1^{-3/4}x_2^{-3/4} = 1$ or, equivalently, $z_1 z_2 = 1$. Thus, we have

$$(z_2 + z_1)(z_1^2 + z_2^2) = (z_2 + z_1)((z_1 + z_2)^2 - 2) = (z_2 + z_1)^3 - 2(z_1 + z_2) = 4,$$

$$(z_1 + z_2 - 2)((z_1 + z_2)^2 + 2(z_1 + z_2) + 2) = 0,$$

implying $z_1 + z_2 = 1$. Along with the condition $z_1 z_2 = 1$, this gives $z_1 = z_2 = 1$, but this contradicts to the condition $z_1 \neq z_2$.

The only other possibility is that the period-2 orbit hits the interval of the definition of the map: this happens when, say, $x_1 = 0$. This gives us $x_2 = a$, hence $0 = a - 4\sqrt[4]{a}$, i.e., $a = 4^{4/3}$. Thus, the unstable orbit of period 2 exists for $5 < a \leq 4^{4/3}$. 9, M

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 100 of 100 marks

Total Mastery marks: 20 of 20 marks

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH60009, MATH97066, MATH70009	1	Most of the students did it well.
MATH60009, MATH97066, MATH70010	2	This question turned out to be the most difficult. In (a), some students did not notice that the system is gradient. Most students found it difficult to draw a bifurcation diagram.
MATH60009, MATH97066, MATH70011	3	Most of the students did it well, but the number of arithmetic errors was unexpected.
MATH60009, MATH97066, MATH70012	4	Most of the students did well in part (b), while in part (a) many students did not notice that the system is Hamiltonian and any Hamiltonian perturbation solves the problem.
MATH60009, MATH97066, MATH70013	5	For Master students, this question was one of the simplest, there were only problems with bifurcations of points of period 2.